5

Infinite Series

5.1. SEQUENCE

A sequence is a function whose domain is the set N of all natural numbers whereas the range may be any set S. In other words, a sequence in a set S is a rule which assigns to each natural number a unique element of S.

5.2. REAL SEQUENCE

A real sequence is a function whose domain is the set N of all natural numbers and range a subset of the set R of real numbers.

Symbolically $\quad f : N \to R \quad$ or $\quad (x : N \to R$ or $a : N \to R)$

is a real sequence.

Note. If $x : N \to R$ be a sequence, the image of $n \in N$ instead of denoting it by $x(n)$, we shall generally denote it by x_n. Thus x_1, x_2, x_3 etc. are the real numbers associated to 1, 2, 3 etc. by this mapping. Also, the sequence $x : N \to R$ is denoted by $\{x_n\}$ or $\langle x_n \rangle$.

$x_1, x_2, \ldots\ldots$ are called the first, second terms of the sequence. The mth and nth terms x_m and x_n for $m \neq n$ are treated as distinct terms if $x_m = x_n$ *i.e.*, the terms occurring at different positions are treated as distinct terms even if they have the same value.

5.3. RANGE OF A SEQUENCE

The set of all **distinct** terms of a sequence is called its range.

Note. In a sequence $\{x_n\}$, since $n \in N$ and N is an infinite set, **the number of terms of a sequence is always infinite.** The range of a sequence may be a finite set. *e.g.*, if $x_n = (-1)^n$, then $\{x_n\} = \{-1, 1, -1, 1, \ldots\ldots\}$

The range of sequence $\{x_n\} = \{-1, 1\}$, which is a finite set.

5.4. CONSTANT SEQUENCE

A sequence $\{x_n\}$ defined by $x_n = c \in R \quad \forall \; n \in N$ is called a constant sequence.

e.g. $\quad \{x_n\} = \{c, c, c, \ldots\ldots\}$ is a constant sequence with range $= \{c\}$.

5.5. BOUNDED AND UNBOUNDED SEQUENCES

Bounded above sequence. A sequence $\{a_n\}$ is said to be bounded above if $\exists$ a real number K such that $a_n \le K \quad \forall \; n \in N$.

Bounded below sequence. A sequence $\{a_n\}$ is said to be bounded below if $\exists$ a real number k such that $a_n \ge k \quad \forall \; n \in N$.

Bounded sequence. A sequence $\{a_n\}$ is said to be bounded when it is bounded both above and below.

$\Rightarrow$ A sequence $\{a_n\}$ is bounded if $\exists$ two real numbers k and K ($k \le$ K) such that

$$k \le a_n \le K \quad \forall \; n \in N.$$

Choosing M = maximum $\{|k|, |K|\}$, we can also define a sequence $\{a_n\}$ to be bounded if

$$|a_n| \le M \quad \forall \quad n \in N.$$

Unbounded sequence. If $\exists$ no real number M such that $|a_n| \le M \; \forall \; n \in N$, then the sequence $\{a_n\}$ is said to be unbounded.

For examples (1). Consider the sequence $\{a_n\}$ defined by $a_n = \dfrac{1}{n}$.

$$\{a_n\} = \left\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots\ldots\right\}$$

Here

$$0 < a_n \le 1 \quad \forall \quad n \in N$$

$\therefore$ $\{a_n\}$ is bounded.

(2) Consider the sequence $\{a_n\}$ defined by $a_n = 2^{n-1}$

$$\{a_n\} = \{1, 2, 2^2, 2^3, \ldots\ldots\}.$$

Here

Although $a_n \ge 1, \quad \forall \quad n \in N$, $\exists$ no real number K such that $a_n \le K$.

$\therefore$ The sequence is unbounded above.

5.6. CONVERGENT, DIVERGENT, OSCILLATING SEQUENCES

Convergent sequence. A sequence $\{a_n\}$ is said to be convergent if $\underset{n \to \infty}{\text{Lt}} a_n$ is finite.

For example, consider the sequence $\dfrac{1}{2}, \dfrac{1}{2^2}, \dfrac{1}{2^3}, \ldots\ldots, \dfrac{1}{2^n}, \ldots\ldots$

Here $\quad a_n = \dfrac{1}{2^n}, \; \underset{n \to \infty}{\text{Lt}} a_n = \underset{n \to \infty}{\text{Lt}} \dfrac{1}{2^n} = 0$, which is finite.

$\Rightarrow$ The sequence $\{a_n\}$ is convergent.

Divergent sequence. A sequence $\{a_n\}$ is said to be divergent if $\underset{n \to \infty}{\text{Lt}} a_n$ is not finite, *i.e.*, if

$$\underset{n \to \infty}{\text{Lt}} a_n = +\infty \text{ or } -\infty.$$

For examples

(i) Consider the sequence $\{n^2\}$

Here $\quad a_n = n^2, \quad \underset{n \to \infty}{\text{Lt}} a_n = +\infty \quad \Rightarrow \quad$ The sequence $\{n^2\}$ is divergent.

(ii) Consider the sequence $\{-2^n\}$.

Here

$$a_n = -2^n, \quad \underset{n \to \infty}{\text{Lt}} a_n = \underset{n \to \infty}{\text{Lt}} (-2^n) = -\infty$$

$\Rightarrow$ The sequence $\{-2^n\}$ is divergent.

Oscillatory sequence. If a sequence $\{a_n\}$ neither converges to a finite number nor diverges to $+\infty$ or $-\infty$, it is called an oscillatory sequence. Oscillatory sequences are of two types :

(i) A bounded sequence which does not converge is said to **oscillate finitely.**

For example, consider the sequence $\{(-1)^n\}$.

Here

$$a_n = (-1)^n$$

It is a bounded sequence. $\quad \underset{n \to \infty}{\text{Lt}} a_{2n} = \underset{n \to \infty}{\text{Lt}} (-1)^{2n} = 1$

$$\underset{n \to \infty}{\text{Lt}} a_{2n+1} = \underset{n \to \infty}{\text{Lt}} (-1)^{2n+1} = -1.$$

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for Non

Thus $\underset{n\to\infty}{\text{Lt}}\ a_n$ does not exist $\Rightarrow$ the sequence does not converge.

Hence this sequence oscillates finitely.

(*ii*) An unbounded sequence which does not diverge is said to **oscillate infinitely.**

For example, consider the sequence $\{(-1)^n n\}$.

Here $\quad a_n = (-1)^n n.$

It is an unbounded sequence.

$$\underset{n\to\infty}{\text{Lt}}\ a_{2n} = \underset{n\to\infty}{\text{Lt}}\ (-1)^{2n} . 2n = \underset{n\to\infty}{\text{Lt}}\ 2n = +\infty$$

$$\underset{n\to\infty}{\text{Lt}}\ a_{2n+1} = \underset{n\to\infty}{\text{Lt}}\ (-1)^{2n+1}(2n+1) = \underset{n\to\infty}{\text{Lt}}\ -(2n+1) = -\infty.$$

Thus the sequence does not diverge.

Hence this sequence oscillates infinitely.

Note. When we say $\underset{n\to\infty}{\text{Lt}}\ a_n = l$, it means $\underset{n\to\infty}{\text{Lt}}\ a_{2n} = \underset{n\to\infty}{\text{Lt}}\ a_{2n+1} = l$

Similarly $\quad \underset{n\to\infty}{\text{Lt}}\ a_n = +\infty$ means $\underset{n\to\infty}{\text{Lt}}\ a_{2n} = \underset{n\to\infty}{\text{Lt}}\ a_{2n+1} = +\infty.$

5.7. MONOTONIC SEQUENCES

(*i*) A sequence $\{a_n\}$ is said to be **monotonically increasing** if $a_{n+1} \geq a_n \ \forall \ n \in N$.

i.e., if $\quad a_1 \leq a_2 \leq a_3 \leq \ldots\ldots \leq a_n \leq a_{n+1} \leq \ldots\ldots$

(*ii*) A sequence $\{a_n\}$ is said to be **monotonically decreasing** if $a_{n+1} \leq a_n \ \forall \ n \in N$.

i.e., if $\quad a_1 \geq a_2 \geq a_3 \geq \ldots\ldots \geq a_n \geq a_{n+1} \geq \ldots\ldots$

(*iii*) A sequence $\{a_n\}$ is said to be **monotonic** if it is either monotonically increasing or monotonically decreasing.

(*iv*) A sequence $\{a_n\}$ is said to be **strictly monotonically increasing** if

$$a_{n+1} > a_n \ \forall \ n \in N.$$

(*v*) A sequence $\{a_n\}$ is said to be **strictly monotonically decreasing** if

$$a_{n+1} < a_n \ \forall \ n \in N.$$

(*vi*) A sequence $\{a_n\}$ is said to be **strictly monotonic** if it is either strictly monotonically increasing or strictly monotonically decreasing.

5.8. LIMIT OF A SEQUENCE

A sequence $\{a_n\}$ is said to approach the limit l (say) when $n \to \infty$, if for each $\varepsilon > 0$, $\exists$ a +ve integer m (depending upon ε) such that $|a_n - l| < \varepsilon \ \forall \ n \geq m$.

In symbols, we write $\underset{n\to\infty}{\text{Lt}}\ a_n = l$.

Note. $|a_n - l| < \varepsilon \ \forall n \geq m \Rightarrow l - \varepsilon < a_n < l + \varepsilon$ for $n = m, m+1, m+2, \ldots\ldots$

5.9. EVERY CONVERGENT SEQUENCE IS BOUNDED

Let the sequence $\{a_n\}$ be convergent. Let it tend to the limit l.

Then given $\varepsilon > 0$, $\exists$ a +ve integer m, such that

$$|a_n - l| < \varepsilon \ \forall \ n \geq m$$

$$\Rightarrow \quad l - \varepsilon < a_n < l + \varepsilon \ \forall \ n \geq m.$$

Let k and K be the least and the greatest of $a_1, a_2, a_3, \ldots\ldots, a_{m-1}, l - \varepsilon, l + \varepsilon$

Then $\quad k \leq a_n \leq K \ \forall \ n \in N,$

$\Rightarrow$ the sequence $\{a_n\}$ is bounded.

The converse is **not always true** *i.e.*, a sequence may be bounded, yet it may not be convergent. *e.g.*, Consider $a_n = (-1)^n$, then the sequence $\{a_n\}$ is bounded but not convergent since it does not have a unique limiting point.

5.10. CONVERGENCE OF MONOTONIC SEQUENCES

Theorem I. *The necessary and sufficient condition for the convergence of a monotonic sequence is that it is bounded.*

A monotonic increasing sequence which is bounded above converges.

A monotonic decreasing sequence which is bounded below converges.

Theorem II. *If a monotonic increasing sequence is not bounded above, it diverges to* $+\infty$.

Theorem III. *If a monotonic decreasing sequence is not bounded below, it diverges to* $-\infty$.

Theorem IV. *If* $\{a_n\}$ *and* $\{b_n\}$ *are two convergent sequences, then sequence* $\{a_n + b_n\}$ *is also convergent.*

Or

If $\text{Lt}\ a_n = A$ *and* $\text{Lt}\ b_n = B$, *then* $\text{Lt}\ (a_n + b_n) = A + B$.

Theorem V. *If* $\{a_n\}$ *and* $\{b_n\}$ *are two convergent sequences such that* $\text{Lt}\ a_n = A$ *and* $\text{Lt}\ b_n = B$, *then*

(*i*) *sequence* $\{a_n b_n\}$ *is also convergent and converges to AB.*

(*ii*) *sequence* $\left\{\dfrac{a_n}{b_n}\right\}$ *is also convergent and converges to* $\dfrac{A}{B}$, $(B \neq 0)$.

Theorem VI. *The sequence* $\{|a_n|\}$ *converges to zero if and only if the sequence* $\{a_n\}$ *converges to zero.*

Theorem VII. *If a sequence* $\{a_n\}$ *converges to a and* $a_n \geq 0 \ \forall n$, *then* $a \geq 0$.

Theorem VIII. *If* $a_n \to a$, $b_n \to b$ *and* $a_n \leq b_n \ \forall n$, *then* $a \leq b$.

Theorem IX. *If* $a_n \to l$, $b_n \to l$, *and* $a_n \leq c_n \leq b_n$, $\forall n$, *then* $c_n \to l$. *(Squeeze Principle)*

ILLUSTRATIVE EXAMPLES

Example 1. *Give an example of a monotonic increasing sequence which is (i) convergent, (ii) divergent.* **(P.T.U., Dec. 2004)**

Sol. (*i*) Consider the sequence $\dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \ldots\ldots, \dfrac{n}{n+1}, \ldots\ldots$

Since $\dfrac{1}{2} < \dfrac{2}{3} < \dfrac{3}{4} < \ldots\ldots$ the sequence is monotonic increasing.

$$a_n = \frac{n}{n+1}, \quad \underset{n\to\infty}{\text{Lt}}\ a_n = \underset{n\to\infty}{\text{Lt}}\ \frac{n}{n+1} = \underset{n\to\infty}{\text{Lt}}\ \frac{1}{1+\frac{1}{n}} = 1$$

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which is finite.

$\therefore$ The sequence is convergent.

(ii) Consider the sequence 1, 2, 3, ..., n, ...

Since $1 < 2 < 3 < ......$, the sequence is monotonic increasing,

$$a_n = n, \quad \underset{n\to\infty}{\text{Lt}}\ a_n = \underset{n\to\infty}{\text{Lt}}\ n = \infty$$

$\therefore$ The sequence diverges to $+\infty$.

Example 2. *Give an example of a monotonic decreasing sequence which is*

(i) *convergent,* (ii) *divergent.*

Sol. (i) Consider the sequence $1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{n}, ...$

Since $1 > \frac{1}{2} > \frac{1}{3} > ......$, the sequence is monotonic decreasing.

$$a_n = \frac{1}{n}, \quad \underset{n\to\infty}{\text{Lt}}\ a_n = \underset{n\to\infty}{\text{Lt}}\ \frac{1}{n} = 0.$$

$\therefore$ The sequence converges to 0.

(ii) Consider the sequence $-1, -2, -3, ..., -n, ...$

Since $-1 > -2 > -3 > ......$, the sequence is monotonic decreasing.

$$a_n = -n, \quad \underset{n\to\infty}{\text{Lt}}\ a_n = \underset{n\to\infty}{\text{Lt}}\ (-n) = -\infty$$

$\therefore$ The sequence diverges to $-\infty$.

Example 3. *Discuss the convergence of the sequence* $\{a_n\}$ *where*

(i) $a_n = \dfrac{n+1}{n}$ (ii) $a_n = \dfrac{n}{n^2+1}$ (iii) $a_n = 1 + \dfrac{1}{3} + \dfrac{1}{3^2} + ... + \dfrac{1}{3^n}$.

Sol. (i)
$$a_n = \frac{n+1}{n}$$

$$a_{n+1} - a_n = \frac{n+2}{n+1} - \frac{n+1}{n} = \frac{-1}{n(n+1)} < 0\ \forall\ n$$

$\Rightarrow$ $a_{n+1} < a_n \quad \forall\ n$

$\Rightarrow$ $\{a_n\}$ is a decreasing sequence.

Also
$$a_n = \frac{n+1}{n} = 1 + \frac{1}{n} > 1 \quad \forall\ n$$

$\Rightarrow$ $\{a_n\}$ is bounded below by 1.

$\because$ $\{a_n\}$ is decreasing and bounded below, it is convergent.

$$\underset{n\to\infty}{\text{Lt}}\ a_n = \underset{n\to\infty}{\text{Lt}}\left(1 + \frac{1}{n}\right) = 1.$$

(ii)
$$a_n = \frac{n}{n^2+1}$$

$$a_{n+1} - a_n = \frac{n+1}{(n+1)^2+1} - \frac{n}{n^2+1} = \frac{(n+1)(n^2+1) - n(n^2+2n+2)}{(n^2+2n+2)(n^2+1)}$$

$$= \frac{-n^2-n+1}{(n^2+2n+2)(n^2+1)} < 0 \quad \forall\ n \quad \Rightarrow \quad a_{n+1} < a_n \quad \forall\ n$$

$\Rightarrow$ $\{a_n\}$ is a decreasing sequence.

Also
$$a_n = \frac{n}{n^2+1} > 0 \quad \forall\ n \quad \Rightarrow \quad \{a_n\} \text{ is bounded below by 0.}$$

$\because$ $\{a_n\}$ is decreasing and bounded below, it is convergent.

$$\underset{n\to\infty}{\text{Lt}}\ a_n = \underset{n\to\infty}{\text{Lt}}\ \frac{n}{n^2+1} = \underset{n\to\infty}{\text{Lt}}\ \frac{\frac{1}{n}}{1+\frac{1}{n^2}} = 0.$$

(iii) $a_n = 1 + \dfrac{1}{3} + \dfrac{1}{3^2} + ... + \dfrac{1}{3^n}$

= sum of $(n+1)$ terms of a G.P. whose first term is 1 and common ratio is $\frac{1}{3}$

$$= \frac{1\left(1 - \frac{1}{3^{n+1}}\right)}{1 - \frac{1}{3}} \qquad \left|\ S_n = \frac{a(1-r^n)}{1-r}\right.$$

$$= \frac{3}{2}\left(1 - \frac{1}{3^{n+1}}\right)$$

Now,
$$a_{n+1} = 1 + \frac{1}{3} + \frac{1}{3^2} + ... + \frac{1}{3^n} + \frac{1}{3^{n+1}}$$

$\therefore$
$$a_{n+1} - a_n = \frac{1}{3^{n+1}} > 0 \quad \forall\ n \quad \Rightarrow \quad a_{n+1} > a_n \quad \forall\ n$$

$\Rightarrow$ $\{a_n\}$ is an increasing sequence.

Also
$$a_n = \frac{3}{2}\left(1 - \frac{1}{3^{n+1}}\right) < \frac{3}{2} \quad \forall\ n \quad \Rightarrow \quad \{a_n\} \text{ is bounded above by } \frac{3}{2}.$$

$\because$ $\{a_n\}$ is increasing and bounded above, it is convergent.

$$\underset{n\to\infty}{\text{Lt}}\ a_n = \underset{n\to\infty}{\text{Lt}}\ \frac{3}{2}\left(1 - \frac{1}{3^{n+1}}\right) = \frac{3}{2}.$$

5.11. INFINITE SERIES

If $\{u_n\}$ is a sequence of real numbers, then the expression $u_1 + u_2 + u_3 + ... + u_n ......$ [i.e., the sum of the terms of the sequence, which are infinite in number] is called an infinite series.

The infinite series $u_1 + u_2 + ... + u_n + ...$ is usually denoted by $\sum_{n=1}^{\infty} u_n$ or more briefly, by Σu_n.

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5.12. SERIES OF POSITIVE TERMS

If all the terms of the series $\Sigma u_n = u_1 + u_2 + ... + u_n + ...$ are positive *i.e.*, if $u_n > 0, \forall n$, then the series Σu_n is called a series of positive terms.

5.13. ALTERNATING SERIES

A series in which the terms are alternate positive and negative is called an alternating series. Thus, the series $\Sigma (-1)^{n-1} u_n = u_1 - u_2 + u_3 - u_4 + ... + (-1)^{n-1} u_n + ...$, where $u_n > 0, \forall n$, is an alternating series.

5.14. PARTIAL SUMS

If $\Sigma u_n = u_1 + u_2 + u_3 + ... + u_n + ...$ is an infinite series, where the terms may be +ve or –ve, then $S_n = u_1 + u_2 + ... + u_n$ is called the nth partial sum of Σu_n. Thus, the nth partial sum of an infinite series is the sum of its first n terms.

$S_1, S_2, S_3, ...$ are the first, second, third, ... partial sums of the series.

Since $n \in N$, $\{S_n\}$ is a sequence called the **sequence of partial sums** of the infinite series Σu_n.

$\therefore$ To every infinite series Σu_n, there corresponds a sequence $\{S_n\}$ of its partial sums.

5.15. CONVERGENCE, DIVERGENCE AND OSCILLATION OF AN INFINITE SERIES (Behaviour of an Infinite Series) (P.T.U. Dec. 2007)

An infinite series Σu_n converges, diverges or oscillates (finitely or infinitely) according as the sequence $\{S_n\}$ of its partial sums converges, diverges or oscillates (finitely or infinitely).

(*i*) The series Σu_n converges (or is said to be convergent) if the sequence $\{S_n\}$ of its partial sums converges.

Thus, Σu_n is convergent if $\quad \underset{n\to\infty}{\text{Lt}} S_n = \text{finite}.$

(*ii*) The series Σu_n diverges (or is said to be divergent) if the sequence $\{S_n\}$ of its partial sums diverges.

Thus, Σu_n is divergent if $\quad \underset{n\to\infty}{\text{Lt}} S_n = +\infty \text{ or } -\infty$

(*iii*) The series Σu_n oscillates finitely if the sequence $\{S_n\}$ of its partial sums oscillates finitely.

Thus, Σu_n oscillates finitely if $\{S_n\}$ is bounded and neither converges nor diverges.

(*iv*) The series Σu_n oscillates infinitely if the sequence $\{S_n\}$ of its partial sums oscillates infinitely.

Thus, Σu_n oscillates infinitely if $\{S_n\}$ is unbounded and neither converges nor diverges.

Example 4. *Discuss the convergence or otherwise of the series*

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + ... + \frac{1}{n(n+1)} + ... \text{ to } \infty.$$

Sol. Here $\quad u_n = \dfrac{1}{n(n+1)} = \dfrac{1}{n} - \dfrac{1}{n+1}$

Putting $\quad n = 1, 2, 3, ..., n$

$$u_1 = \frac{1}{1} - \frac{1}{2}$$

$$u_2 = \frac{1}{2} - \frac{1}{3}$$

$$u_3 = \frac{1}{3} - \frac{1}{4}$$

....

....

$$u_n = \frac{1}{n} - \frac{1}{n+1}$$

Adding $\quad S_n = 1 - \dfrac{1}{n+1}$

$$\underset{n\to\infty}{\text{Lt}} S_n = 1 - 0 = 1$$

$\Rightarrow \{S_n\}$ converges to 1 $\Rightarrow$ Σu_n converges to 1.

Note. For another method, see solved example 8(*iii*) art 5.21.

Example 5. *Show that the series $1^2 + 2^2 + 3^2 + ... + n^2 + ...$ diverges to $+\infty$.*

Sol. $\quad S_n = 1^2 + 2^2 + 3^2 + ... + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

$$\underset{n\to\infty}{\text{Lt}} S_n = +\infty$$

$\Rightarrow \{S_n\}$ diverges to $+\infty$

$\Rightarrow$ The given series diverges to $+\infty$.

5.16. NATURE OF GEOMETRIC SERIES $1 + x + x^2 + x^3 + ...$ to ∞

(*i*) *Converges if* $-1 < x < 1$ *i.e.*, $|x| < 1$ (*ii*) *Diverges if* $x \geq 1$

(*iii*) *Oscillates finitely if* $x = -1$

(*iv*) *Oscillates infinitely if* $x < -1$

Proof. (*i*) **When $|x| < 1$**

Since $\quad |x| < 1, x^n \to 0$ as $n \to \infty$

$$S_n = 1 + x + x^2 + ... \text{ to } n \text{ terms} = \frac{1(1-x^n)}{1-x} = \frac{1}{1-x} - \frac{x^n}{1-x}$$

$\underset{n\to\infty}{\text{Lt}} S_n = \dfrac{1}{1-x} \quad \Rightarrow$ the sequence $\{S_n\}$ is convergent

$\Rightarrow$ the given series is convergent.

(*ii*) **When $x \geq 1$**

Sub-case I. When $\quad x = 1$

$$S_n = 1 + 1 + 1 + ... \text{ to } n \text{ terms} = n$$

$\underset{n\to\infty}{\text{Lt}} S_n = \infty \quad \Rightarrow$ the sequence $\{S_n\}$ diverges to ∞.

$\Rightarrow$ the given series diverges to ∞.

Sub-case II. When $x > 1$, $x^n \to \infty$ as $n \to \infty$

$$S_n = 1 + x + x^2 + ... \text{ to } n \text{ terms} = \frac{1(x^n - 1)}{x - 1}$$

$\underset{n\to\infty}{\text{Lt}}\ S_n = \infty \Rightarrow$ the sequence $\{S_n\}$ diverges to ∞

$\Rightarrow$ the given series diverges to ∞.

(*iii*) **When** $x = -1$

$S_n = 1 - 1 + 1 - 1 + \ldots\ldots$ to n terms

$= 1$ or 0 according as n is odd or even.

$\Rightarrow \underset{n\to\infty}{\text{Lt}}\ S_n = 1$ or $0 \Rightarrow$ the sequence $\{S_n\}$ oscillates finitely.

$\Rightarrow$ the given series oscillates finitely.

(*iv*) **When** $x < -1$

$x < -1 \Rightarrow -x > 1$

Let $r = -x$, then $r > 1$

$\therefore r^n \to \infty$ as $n \to \infty$

$$S_n = 1 + x + x^2 + x^3 + \ldots\ldots \text{ to } n \text{ terms} = \frac{1-x^n}{1-x} = \frac{1-(-r)^n}{1+r} \qquad [\because\ x = -r]$$

$$= \frac{1-r^n}{1+r} \quad \text{or} \quad \frac{1+r^n}{1+r} \text{ according as } n \text{ is even or odd}$$

$$\underset{n\to\infty}{\text{Lt}}\ S_n = \frac{1-\infty}{1+r} \quad \text{or} \quad \frac{1+\infty}{1+r} = -\infty \text{ or } +\infty$$

$\Rightarrow$ the sequence $\{S_n\}$ oscillates infinitely.

$\Rightarrow$ the given series oscillates infinitely.

5.17. NECESSARY CONDITION FOR CONVERGENCE OF A POSITIVE TERM SERIES

(P.T.U., Dec. 2002, May 2003, Dec. 2003, May 2004, Dec. 2005, Jan. 2009)

If a positive term series Σu_n is convergent, then $\underset{n\to\infty}{Lt}\ u_n = 0$

Proof. Let S_n denote the nth partial sum of the series Σu_n.

Then Σu_n is convergent $\Rightarrow$ $\{S_n\}$ is convergent

$\Rightarrow \underset{n\to\infty}{\text{Lt}}\ S_n$ is finite and unique $= s$ (say). $\Rightarrow \underset{n\to\infty}{\text{Lt}}\ S_{n-1} = s$

Now, $S_n - S_{n-1} = u_n$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}}\ u_n = \underset{n\to\infty}{\text{Lt}}\ (S_n - S_{n-1}) = \underset{n\to\infty}{\text{Lt}}\ S_n - \underset{n\to\infty}{\text{Lt}}\ S_{n-1} = s - s = 0.$$

Hence Σu_n is convergent $\Rightarrow \underset{n\to\infty}{\text{Lt}}\ u_n = 0$.

The converse of the above theorem is not always true, *i.e.*, the nth term may tend to zero as $n \to \infty$ even if the series is not convergent.

For example, the series $1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots + \frac{1}{n} + \ldots\ldots$ diverges, though

$$\underset{n\to\infty}{\text{Lt}}\ u_n = \underset{n\to\infty}{\text{Lt}}\ \frac{1}{n} = 0.$$

Hence $\underset{n\to\infty}{Lt}\ u_n = 0 \Rightarrow$ **Σu_n may or may not be convergent.**

Note. $\underset{n\to\infty}{Lt}\ u_n \neq 0 \Rightarrow$ **Σu_n is not convergent.**

5.18. A POSITIVE TERM SERIES EITHER CONVERGES OR DIVERGES TO $+\infty$

Proof. Let Σu_n be a positive term series and S_n be its nth partial sum.

Then $S_{n+1} = u_1 + u_2 + \ldots\ldots + u_n + u_{n+1} = S_n + u_{n+1}$

$\Rightarrow \quad S_{n+1} - S_n = u_{n+1} > 0 \quad \forall \ n \qquad [\because\ u_n > 0\ \forall\ n]$

$\Rightarrow \quad S_{n+1} > S_n \quad \forall \ n$

$\Rightarrow \{S_n\}$ is a monotonic increasing sequence.

Two cases arise. The sequence $\{S_n\}$ may be bounded or unbounded above.

Case I. When $\{S_n\}$ is bounded above.

Since $\{S_n\}$ is monotonic increasing and bounded above, it is convergent $\Rightarrow \Sigma u_n$ is convergent.

Case II. When $\{S_n\}$ is not bounded above.

Since $\{S_n\}$ is monotonic increasing and not bounded above, it diverges to $+\infty \Rightarrow \Sigma u_n$ diverges to $+\infty$.

Hence a positive term series either converges or diverges to $+\infty$.

Cor. If $u_n > 0\ \forall\ n$ and $\underset{n\to\infty}{\text{Lt}}\ u_n \neq 0$, then the series Σu_n diverges to $+\infty$.

Proof. $u_n > 0\ \forall\ n \Rightarrow \Sigma u_n$ is a series of +ve terms.

$\Rightarrow \Sigma u_n$ either converges or diverges to $+\infty$

Since $\underset{n\to\infty}{\text{Lt}}\ u_n \neq 0$ (given)

$\therefore \Sigma u_n$ does not converge.

Hence Σu_n diverges to $+\infty$.

5.19 (a). THE NECESSARY AND SUFFICIENT CONDITION FOR THE CONVERGENCE OF A POSITIVE TERM SERIES Σu_n IS THAT THE SEQUENCE $\{S_n\}$ OF ITS PARTIAL SUMS IS BOUNDED ABOVE

Proof. Necessary Condition. Suppose the sequence $\{S_n\}$ is bounded above. Since the series Σu_n is of positive terms, the sequence $\{S_n\}$ is monotonically increasing. Since every monotonically increasing sequence which is bounded above, converges, therefore $\{S_n\}$ and hence Σu_n converges.

Sufficient Condition. Suppose Σu_n converges. Then the sequence $\{S_n\}$ of its partial sums also converges. Since every convergent sequence is bounded, $\{S_n\}$ is bounded. In particular, $\{S_n\}$ is bounded above.

5.19 (b). CAUCHY'S GENERAL PRINCIPLE OF CONVERGENCE OF SERIES

The necessary and sufficient condition for the infinite series $\sum_{n=1}^{\infty} u_n$ to converge is that given $\varepsilon > 0$, however small, there exists a positive integer p such that $|S_{n+p} - S_n| < \varepsilon\ \forall\ n \geq m$; m and $p \in N$ *i.e.*, $|u_{n+1} + u_{n+2} + \ldots + u_{n+p}| < \varepsilon$ for $n \geq m$, $p, m \in N$.

Necessary Condition. Given $\sum_{n=1}^{\infty} u_n$ is convergent.

$\therefore$ $\underset{n\to\infty}{\text{Lt}}\ S_n =$ finite, where $\{S_n\}$ is the sequence of its partial sums

Let $\underset{n\to\infty}{\text{Lt}}\ S_n = l$, where l is a finite number.

$\therefore$ Given $\varepsilon > 0$, however small, $\exists\, m \in N$ such that $|S_n - l| < \varepsilon/2\ \forall\, n \ge m$ ...(1)

If $p \in N$ and $n \ge m$ then $n + p \ge m$

$\therefore$ From (1), $|S_{n+p} - l| < \dfrac{\varepsilon}{2}\quad \forall\, n \ge m$ ...(2)

Now,
$$|S_{n+p} - S_n| = |(S_{n+p} - l) - (S_n - l)|$$
$$\le |S_{n+p} - l| + |S_n - l|$$
$$< \varepsilon/2 + \varepsilon/2 \quad \text{for} \quad n \ge m, p \in N$$

Hence $|S_{n+p} - S_n| < \varepsilon$ for $n \ge m, p \in N$.

Sufficient Condition. Given $|S_{n+p} - S_n| < \varepsilon\ \forall\, n \ge m, p \in N$

In particular $|S_{m+p} - S_m| < \varepsilon\quad \forall\, p \in N$

Now S_m, being the sum of first m terms of the sequence $\{S_n\}$ and S_{m+p} differs from S_m by a number $< \varepsilon\ \forall$ $p \in N$.

$\therefore$ S_{m+p} cannot be infinite when $p \to \infty$ *i.e.*, $\underset{p\to\infty}{\text{Lt}}\ S_{m+p} \ne \infty$.

$\therefore$ $\underset{n\to\infty}{\text{Lt}}\ S_n \ne \infty$ (replace $m + p$ by n)

Also $\underset{n\to\infty}{\text{Lt}}\ S_n$ and $\underset{n\to\infty}{\text{Lt}}\ S_{n+p}$ have the same value S

Now, $|S_{n+p} - S_n| < \varepsilon\ \forall\, p \in N$

$\Rightarrow \quad \underset{n\to\infty}{\text{Lt}}\ S_{n+p} = \underset{n\to\infty}{\text{Lt}}\ S_n = l$ (say) $\forall\, p \in N$

$\therefore$ $\sum_{n=1}^{\infty} u_n$ is convergent.

5.19(c). IF m IS A GIVEN POSITIVE INTEGER, THEN THE TWO SERIES $u_1 + u_2 + \ldots\ldots + u_{m+1} + u_{m+2} + \ldots\ldots u_n$ AND $u_{m+1} + u_{m+2} + \ldots\ldots u_n$ CONVERGE OR DIVERGE TOGETHER

Proof. Let S_n and s_n denote the nth partial sums of the two series.

Then
$$S_n = u_1 + u_2 + \ldots\ldots + u_n$$
$$s_n = u_{m+1} + u_{m+2} + \ldots\ldots + u_n$$
$$= (u_1 + u_2 + \ldots\ldots + u_n) - (u_1 + u_2 + \ldots\ldots + u_m) \qquad \ldots(1)$$
$$= S_n - S_m \;\Rightarrow\; s_n = S_n - S_m$$

S_m being the sum of a finite number of terms of Σu_n is a fixed finite quantity.

(i) If $S_n \to$ a finite limit as $n \to \infty$, then from (1), so does s_n.

(ii) If $S_n \to +\infty$ as $n \to \infty$, so does s_n.

(iii) If $S_n \to -\infty$ as $n \to \infty$, so does s_n.

(iv) If S_n does not tend to any limit (finite or infinite), so does s_n.

$\Rightarrow$ The sequences $\{S_n\}$ and $\{s_n\}$ converge or diverge together.

$\Rightarrow$ The two given series converge or diverge together. Hence the result.

Note. The above theorem shows that the convergence, divergence or oscillation of a series is not affected by addition or omission of a finite number of its terms.

Example 6. *Prove that the series* $\sum_{n=1}^{\infty} \dfrac{1}{n}$ *does not converge (by applying Cauchy's general principle of convergence).*

Doubt

Sol. If possible suppose $\sum_{n=1}^{\infty} \dfrac{1}{n}$ is convergent.

$\therefore$ By Cauchy's general principle of convergence
$$|S_{n+p} - S_n| < \varepsilon \quad \forall\, n \ge m, p \in N$$

Take $\varepsilon = \dfrac{1}{2} \quad \therefore \quad |S_{n+p} - S_n| < \dfrac{1}{2} \quad \forall\, n \ge m, p \in N$

Put $n = m$; $\quad |S_{m+p} - S_m| < \dfrac{1}{2} \quad \forall \quad p \in N$

$\Rightarrow \left|\dfrac{1}{m+1} + \dfrac{1}{m+2} + \ldots + \dfrac{1}{m+p}\right| < \dfrac{1}{2} \quad \forall \quad p \in N \left(\because\ S_n = \dfrac{1}{n}\right)$

or $\dfrac{1}{m+1} + \dfrac{1}{m+2} + \ldots + \dfrac{1}{m+p} < \dfrac{1}{2} \quad \forall \quad p \in N$

Put $p = m$, $\dfrac{1}{m+1} + \dfrac{1}{m+2} + \ldots + \dfrac{1}{2m} < \dfrac{1}{2}$ ...(1)

But $\dfrac{1}{m+1} + \dfrac{1}{m+2} + \ldots + \dfrac{1}{2m} > \dfrac{1}{m+m} + \dfrac{1}{m+m} + \ldots + \dfrac{1}{2m} = \dfrac{m}{2m} = \dfrac{1}{2}$

$\therefore$ $\dfrac{1}{m+1} + \dfrac{1}{m+2} + \ldots + \dfrac{1}{2m} > \dfrac{1}{2}$ ...(2)

(2) contradicts (1)

$\therefore$ Our supposition is wrong

$\therefore$ Given series does not converge.

Example 7. *Prove that the series* $\sum_{n=1}^{\infty} \dfrac{1}{n^2}$ *is convergent (by applying Cauchy's general principle of convergence).*

Sol. Let
$$S_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2}$$

$$S_{n+p} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} + \frac{1}{(n+1)^2} + \ldots \frac{1}{(n+p)^2}$$

$$|S_{n+p} - S_n| = \left| \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \ldots + \frac{1}{(n+p)^2} \right|$$

$$= \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \ldots + \frac{1}{(n+p)^2}$$

$$< \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} + \ldots + \frac{1}{(n+p-1)(n+p)}$$

$$< \left(\frac{1}{n} - \frac{1}{n+1}\right) + \left(\frac{1}{n+1} - \frac{1}{n+2}\right) + \ldots + \left(\frac{1}{n+p-1} - \frac{1}{n+p}\right)$$

$$= \frac{1}{n} - \frac{1}{n+p}$$

$$< \frac{1}{n}.$$

$\therefore \quad |S_{n+p} - S_n| < \dfrac{1}{n}.$

Let us choose m such that $m > \dfrac{1}{\varepsilon}$

$\therefore \quad$ for $n \geq m > \dfrac{1}{\varepsilon}$, we have $\dfrac{1}{n} < \varepsilon$

$\therefore \quad |S_{n+p} - S_n| < \varepsilon \quad$ for $\quad n \geq m, p \in N$

$\therefore \quad$ by Cauchy's general principle of convergence $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$ is convergent.

Note. These examples can be proved by applying p series test art. 5.21.

5.19 (d). IF Σu_n AND Σv_n CONVERGE TO u AND v RESPECTIVELY, THEN $\Sigma(u_n + v_n)$ CONVERGES TO (u + v)

Proof. Let $\quad U_n = u_1 + u_2 + \ldots\ldots + u_n$

$V_n = v_1 + v_2 + \ldots\ldots + v_n$

and $\quad S_n = (u_1 + v_1) + (u_2 + v_2) + \ldots\ldots + (u_n + v_n)$

Then $\quad S_n = (u_1 + u_2 + \ldots\ldots + u_n) + (v_1 + v_2 + \ldots\ldots + v_n) = U_n + V_n.$

Since Σu_n converges to u, $\underset{n\to\infty}{\text{Lt}}\ U_n = u$

Σv_n converges to v, $\underset{n\to\infty}{\text{Lt}}\ V_n = v$

$\therefore \quad \underset{n\to\infty}{\text{Lt}}\ S_n = \underset{n\to\infty}{\text{Lt}}\ (U_n + V_n) = \underset{n\to\infty}{\text{Lt}}\ U_n + \underset{n\to\infty}{\text{Lt}}\ V_n = u + v.$

$\Rightarrow \quad \Sigma(u_n + v_n)$ converges to $(u + v)$.

5.20. COMPARISON TESTS

Test I. (a) If Σu_n and Σv_n are series of positive terms and Σv_n is convergent and there is a positive constant k such that $u_n \leq kv_n$, $\forall\ n > m$, then Σu_n is also convergent.

Proof. Let $\quad U_n = u_1 + u_2 + \ldots\ldots + u_n \quad$ and $\quad V_n = v_1 + v_2 + \ldots\ldots + v_n$

$u_n \leq kv_n\ \forall\ n > m$

Now

$\Rightarrow \quad u_{m+1} \leq kv_{m+1}$

$u_{m+2} \leq kv_{m+2}$

....

Adding $\quad u_{m+1} + u_{m+2} + \ldots\ldots + u_n \leq k(v_{m+1} + v_{m+2} + \ldots\ldots + v_n)$

$\Rightarrow \quad U_n - U_m \leq k(V_n - V_m) \quad \forall \quad n > m$

$\Rightarrow \quad U_n \leq kV_n + (U_m - kV_m) \quad \forall \quad n > m$

$\Rightarrow \quad U_n \leq kV_n + k_0 \quad \forall \quad n > m \qquad$ ...(1)

where $k_0 = U_m - kV_m$ is a fixed number.

Since Σv_n is convergent, the sequence $\{V_n\}$ is convergent and hence bounded above.

$\therefore$ From (1), the sequence $\{U_n\}$ is bounded above.

$\because$ Σu_n is a series of +ve terms.

$\therefore$ $\{U_n\}$ is monotonic increasing.

$\because$ $\{U_n\}$ is monotonic increasing sequence and is bounded above.

$\therefore$ It is convergent.

$\Rightarrow$ Σu_n is convergent.

Test I. (b) If Σu_n and Σv_n are two series of positive terms and Σv_n is divergent and there is a positive constant k such that $u_n > kv_n$, $\forall\ n > m$, then Σu_n is also divergent.

Proof. Let $\quad U_n = u_1 + u_2 + \ldots\ldots + u_n$

and $\quad V_n = v_1 + v_2 + \ldots\ldots + v_n$

Now, $\quad u_n > kv_n \quad \forall \quad n > m$

$\Rightarrow \quad u_{m+1} > kv_{m+1}$

$u_{m+2} > kv_{m+2}$

....

....

Adding $\quad u_{m+1} + u_{m+2} + \ldots\ldots + u_n > k(v_{m+1} + v_{m+2} + \ldots\ldots + v_n)$

$\Rightarrow \quad U_n - U_m > k(V_n - V_m) \quad \forall \quad n > m$

$\Rightarrow \quad U_n > kV_n + (U_m - kV_m) \quad \forall \quad n > m \qquad$ ...(1)

$\Rightarrow \quad U_n > kV_n + k_0 \quad \forall \quad n > m$

where $k_0 = U_m - kV_m$ is a fixed number.

Since Σv_n is divergent, the sequence $\{V_n\}$ is divergent.

$\Rightarrow$ For each positive real number k_1, however large, there exists a +ve integer m' such that

$V_n > k_1 \quad \forall \quad n > m'$

Let $m^* = \max.\{m, m'\}$, then $V_n > k_1 \quad \forall \quad n > m^*$

From (1),

$\Rightarrow \quad U_n > kk_1 + k_0 = K \quad \forall \quad n > m^*$

$\Rightarrow \quad \{U_n\}$ is divergent

Σu_n is divergent.

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Test II. If Σu_n and Σv_n are two positive term series and there exist two positive constants H and K (independent of n) and a positive integer m such that $H < \frac{u_n}{v_n} < K \; \forall\, n > m$, then the two series Σu_n and Σv_n converge or diverge together.

Proof. Since Σv_n is a series of + ve terms, $v_n > 0, \forall\, n$

$$\therefore \quad H < \frac{u_n}{v_n} < K \qquad \forall\, n > m$$

$$\Rightarrow \quad Hv_n < u_n < Kv_n \qquad \forall\, n > m \qquad ...(1)$$

Case I. When Σv_n is convergent

From (1), $\quad u_n < Kv_n \qquad \forall\, n > m$ and Σv_n is convergent.

$\Rightarrow \Sigma u_n$ is convergent. [See test I(a)]

Case II. When Σv_n is divergent

From (1), $\quad u_n > Hv_n \qquad \forall\, n > m$ and Σv_n is divergent.

$\Rightarrow \Sigma u_n$ is divergent. [See test I(b)]

Case III. When Σu_n is convergent

From (1), $\quad Hv_n < u_n \qquad \forall\, n > m$

$$\Rightarrow \quad v_n < \frac{1}{H} u_n \qquad \forall\, n > m \qquad (\because H > 0)$$

Since Σu_n is convergent $\therefore$ Σv_n is convergent. [See test I(a)]

Case IV. When Σu_n is divergent

From (1), $\quad Kv_n > u_n \qquad \forall\, n > m$

$$\Rightarrow \quad v_n > \frac{1}{K} u_n. \qquad \forall\, n > m \qquad (\because K > 0)$$

Since Σu_n is divergent $\therefore$ Σv_n is divergent. [See test I(b)]

Particular Case of Test II (When m = 0)

If Σu_n and Σv_n are two positive term series and there exist two positive constants H and K (independent of n) such that $\quad H < \frac{u_n}{v_n} < K \qquad \forall\, n,$

then the two series Σu_n and Σv_n converge or diverge together.

Test III. (limit comparison test) Let Σu_n and Σv_n be two positive term series.

(i) If $\lim\limits_{n\to\infty} \frac{u_n}{v_n} = l$ (finite and non-zero), then Σu_n and Σv_n both converge or diverge together.

(ii) If $\lim\limits_{n\to\infty} \frac{u_n}{v_n} = 0$ and Σv_n converges, then Σu_n also converges.

(iii) If $\lim\limits_{n\to\infty} \frac{u_n}{v_n} = \infty$ and Σv_n diverges, then Σu_n also diverges.

(iv) If $\lim\limits_{n\to\infty} \frac{u_n}{v_n} = \infty$ and Σu_n converges then Σv_n also converges. (P.T.U., Dec. 2004)

Proof. (i) Since $u_n > 0, v_n > 0 \quad \therefore \quad \frac{u_n}{v_n} > 0$

$$\therefore \quad \lim_{n\to\infty} \frac{u_n}{v_n} \geq 0$$

But $$\lim_{n\to\infty} \frac{u_n}{v_n} = l \neq 0 \quad \Rightarrow \quad l > 0$$

Now $$\lim_{n\to\infty} \frac{u_n}{v_n} = l$$

$\Rightarrow$ Given $\varepsilon > 0$, there exists a +ve integer m such that $\left| \frac{u_n}{v_n} - l \right| < \varepsilon \;\; \forall\; n > m$

$$\Rightarrow \quad l - \varepsilon < \frac{u_n}{v_n} < l + \varepsilon \qquad \forall\, n > m$$

$$\Rightarrow \quad (l - \varepsilon)\, v_n < u_n < (l + \varepsilon)\, v_n \qquad \forall\, n > m \qquad (\because v_n > 0)$$

Choose $\varepsilon > 0$ such that $l - \varepsilon > 0$.

Let $l - \varepsilon = H, l + \varepsilon = K$, where H, K are > 0

$$\therefore \quad Hv_n < u_n < Kv_n \qquad \forall\, n > m \qquad ...(1)$$

Case I. When Σu_n is convergent

From (1), $\quad Hv_n < u_n \qquad \forall\, n > m$

$$\Rightarrow \quad v_n < \frac{1}{H} u_n \qquad \forall\, n > m \qquad (\because H > 0)$$

Since Σu_n is convergent, Σv_n is also convergent.

Case II. When Σu_n is divergent.

From (1), $\quad Kv_n > u_n \qquad \forall\, n > m$

$$\Rightarrow \quad v_n > \frac{1}{K} u_n \qquad \forall\, n > m \qquad (\because K > 0)$$

Since Σu_n is divergent, Σv_n is also divergent.

Case III. When Σv_n is convergent.

From (1), $\quad u_n < Kv_n \qquad \forall\, n > m$

Since Σv_n is convergent, Σu_n is also convergent.

Case IV. When Σv_n is divergent.

From (1), $\quad u_n > Hv_n \qquad \forall\, n > m$

Since Σv_n is divergent, Σu_n is also divergent.

Hence Σu_n and Σv_n converge or diverge together.

(ii) Here $\quad \lim\limits_{n\to\infty} \frac{u_n}{v_n} = 0$

$\therefore$ Given $\varepsilon > 0$, there exists a +ve integer m such that $\left| \frac{u_n}{v_n} - 0 \right| < \varepsilon \quad \forall\, n > m$

$$-\varepsilon < \frac{u_n}{v_n} < \varepsilon \qquad \forall\, n > m$$

$\Rightarrow$ $$u_n < \varepsilon\, v_n \qquad \forall\, n > m \qquad (\because v_n > 0)$$

$\Rightarrow$ Since Σv_n is convergent, Σu_n is also convergent.

(iii) Here $\displaystyle \operatorname*{Lt}_{n \to \infty} \frac{u_n}{v_n} = \infty$

$\therefore$ Given M > 0, however large, $\exists$s a +ve integer m such that $\dfrac{u_n}{v_n} > M \;\forall\, n > m$

$\Rightarrow$ $$u_n > Mv_n \qquad \forall\, n > m$$

Since Σv_n is divergent, Σu_n is also divergent.

(iv) Given $\displaystyle \operatorname*{Lt}_{n \to \infty} \frac{u_n}{v_n} = \infty$

$\therefore$ Given M > 0, however large $\exists$s a positive integer m such that $\dfrac{u_n}{v_n} > M$ for $n > m$

$\therefore$ $$u_n > M\, v_n$$

or $$Mv_n < u_n \quad \text{or} \quad v_n < \frac{1}{M} u_n$$

As M is a large $\therefore$ $\dfrac{1}{M}$ is small.

Given $\displaystyle\sum_{n=1}^{\infty} u_n$ is convergent

$\therefore$ By comparison test $\Sigma\, v_n$ is also convergent.

Test IV. Let Σu_n and Σv_n be two positive term series.

(i) **If $\dfrac{u_n}{u_{n+1}} > \dfrac{v_n}{v_{n+1}}$ $\forall\, n > m$ and Σv_n is convergent, then Σu_n is also convergent.**

(ii) **If $\dfrac{u_n}{u_{n+1}} < \dfrac{v_n}{v_{n+1}}$ $\forall\, n > m$ and Σv_n is divergent, then Σu_n is also divergent.**

Proof. *(i)* $$\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}} \qquad \forall\, n > m$$

$\Rightarrow$
$$\frac{u_{m+1}}{u_{m+2}} > \frac{v_{m+1}}{v_{m+2}}$$
$$\frac{u_{m+2}}{u_{m+3}} > \frac{v_{m+2}}{v_{m+3}}$$
$$\frac{u_{m+3}}{u_{m+4}} > \frac{v_{m+3}}{v_{m+4}}$$
....

....
$$\frac{u_{n-1}}{u_n} > \frac{v_{n-1}}{v_n}$$

Multiplying the corresponding sides of the above inequalities, we have

$$\frac{u_{m+1}}{u_n} > \frac{v_{m+1}}{v_n} \qquad \forall\, n > m$$

$\Rightarrow$ $$u_n < \left(\frac{u_{m+1}}{v_{m+1}}\right) v_n \qquad \forall\, n > m$$

$\Rightarrow$ $$u_n < kv_n \qquad \forall\, n > m,$$

where $k = \dfrac{u_{m+1}}{v_{m+1}}$ is a fixed +ve quantity.

Since Σv_n is convergent, so is Σu_n.

(ii) Using $$\frac{u_n}{u_{n+1}} < \frac{v_n}{v_{n+1}} \qquad \forall\, n > m$$

and proceeding as in part *(i)*, we have $$\frac{u_{m+1}}{u_n} < \frac{v_{m+1}}{v_n} \qquad \forall\, n > m$$

$\Rightarrow$ $$u_n > \left(\frac{u_{m+1}}{v_{m+1}}\right) v_n \qquad \forall\, n > m$$

$\Rightarrow$ $$u_n > kv_n \qquad \forall\, n > m,$$

where $k = \dfrac{u_{m+1}}{v_{m+1}}$ is a fixed +ve quantity.

Since Σv_n is divergent, so is Σu_n.

5.21. AN IMPORTANT TEST FOR COMPARISON KNOWN AS p-SERIES TEST FOR THE SERIES $\sum \frac{1}{n^p}$. [HYPER HARMONIC SERIES OR p-SERIES]

The series $\displaystyle\sum \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots\ldots + \frac{1}{n^p} + \ldots\ldots$ to ∞ converges if $p > 1$ and diverges if $p \le 1$.

Proof. Case I. *When $p > 1$*

$$\frac{1}{1^p} = 1$$

$$\frac{1}{2^p} + \frac{1}{3^p} < \frac{1}{2^p} + \frac{1}{2^p} = \frac{2}{2^p} = \frac{1}{2^{p-1}} \qquad \left[\because \frac{1}{3^p} < \frac{1}{2^p}\right]$$

$$\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} < \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p}$$

$$= \frac{4}{4^p} = \frac{1}{4^{p-1}} = \frac{1}{(2^{p-1})^2} \qquad \left[\because \frac{1}{5^p} < \frac{1}{4^p}, \frac{1}{6^p} < \frac{1}{4^p} \text{ etc.}\right]$$

Similarly, the sum of next eight terms

$$= \frac{1}{8^p} + \frac{1}{9^p} + \ldots\ldots + \frac{1}{15^p} < \frac{1}{8^p} + \frac{1}{8^p} + \ldots\ldots + \frac{1}{8^p} = \frac{8}{8^p} = \frac{1}{8^{p-1}} = \frac{1}{(2^{p-1})^3}$$

and so on.

Now, $\sum \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots\ldots + \frac{1}{n^p} + \ldots\ldots$

$$= \frac{1}{1^p} + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right) + \left(\frac{1}{8^p} + \frac{1}{9^p} + \ldots\ldots + \frac{1}{15^p}\right) + \ldots\ldots$$

$$< 1 + \frac{1}{2^{p-1}} + \frac{1}{(2^{p-1})^2} + \frac{1}{(2^{p-1})^3} + \ldots\ldots \quad \ldots(1)$$

$\ldots(2)$

But (2) is a G.P. whose common ratio $= \frac{1}{2^{p-1}} < 1$ $\quad (\because p > 1)$

$\therefore$ (2) is convergent $\Rightarrow$ (1) is convergent.

Hence the given series is convergent.

Case II. *When* $p = 1$

$$\sum \frac{1}{n^p} = \sum \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots$$

$$1 + \frac{1}{2} = 1 + \frac{1}{2}$$

$$\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$$

$$\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{4}{8} = \frac{1}{2} \quad \text{and so on.}$$

Now, $\sum \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots + \frac{1}{n} + \ldots\ldots$

$$= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \ldots\ldots \quad \ldots(1)$$

$$> 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots\ldots = 1 + \left(\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots\ldots \infty\right) \quad \ldots(2)$$

But $\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots\ldots \infty$ is a G.P. whose common ratio $= 1$.

$\therefore$ (2) is divergent. $\Rightarrow$ (1) is divergent.

Hence the given series is divergent.

Case III. *When* $p < 1$

$$p < 1 \Rightarrow n^p < n \Rightarrow \frac{1}{n^p} > \frac{1}{n} \quad \forall \ n$$

But the series $\sum \frac{1}{n}$ is divergent (Case II).

Hence $\sum \frac{1}{n^p}$ is also divergent.

Example 8. *Examine the convergence of the series:*

(i) $\frac{3}{5} + \frac{4}{5^2} + \frac{3}{5^3} + \frac{4}{5^4} + \ldots\ldots$ to ∞ $\qquad$ (ii) $1 + \frac{1}{4^{2/3}} + \frac{1}{9^{2/3}} + \frac{1}{16^{2/3}} + \ldots\ldots$ to ∞

(iii) $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots \infty$

Sol. (i) $\frac{3}{5} + \frac{4}{5^2} + \frac{3}{5^3} + \frac{4}{5^4} + \ldots\ldots$ to ∞

$$= \left(\frac{3}{5} + \frac{3}{5^3} + \ldots\ldots \text{ to } \infty\right) + \left(\frac{4}{5^2} + \frac{4}{5^4} + \ldots\ldots \text{ to } \infty\right) = \Sigma u_n + \Sigma v_n \text{ (say)}$$

Now Σu_n is a G.P. with common ratio $= \frac{1}{5^2}$, which is numerically less than 1,

$\therefore$ Σu_n is convergent.

Σv_n is also a G.P. with common ratio $= \frac{1}{5^2}$, which is numerically less than 1.

$\therefore$ Σv_n is convergent.

$\therefore$ The given series *viz.* $\Sigma(u_n + v_n)$ is also convergent.

(ii) $1 + \frac{1}{4^{2/3}} + \frac{1}{9^{2/3}} + \frac{1}{16^{2/3}} + \ldots\ldots$ to $\infty = 1 + \frac{1}{(2^2)^{2/3}} + \frac{1}{(3^2)^{2/3}} + \frac{1}{(4^2)^{2/3}} + \ldots\ldots$ to ∞

$$= \frac{1}{1^{4/3}} + \frac{1}{2^{4/3}} + \frac{1}{3^{4/3}} + \frac{1}{4^{4/3}} + \ldots\ldots \text{ to } \infty = \sum \frac{1}{n^{4/3}} = \sum \frac{1}{n^p} \text{ with } p = \frac{4}{3} > 1$$

$\therefore$ By p-series test, the given series is convergent.

(iii) $\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \quad \therefore \quad u_n = \frac{1}{n(n+1)} = \frac{1}{n^2\left(1 + \frac{1}{n}\right)}$

Let $v_n = \frac{1}{n^2}$

Compare Σu_n with Σv_n, we have $\frac{u_n}{v_n} = \frac{1}{1 + \frac{1}{n}}$.

$$\operatorname*{Lt}_{n \to \infty} \frac{u_n}{v_n} = \operatorname*{Lt}_{n \to \infty} \frac{1}{1 + \frac{1}{n}} = 1, \text{ which is finite and } \neq 0.$$

$\therefore$ Σu_n and Σv_n behave together $\Sigma v_n = \Sigma \frac{1}{n^2}$ is of the form $\Sigma \frac{1}{n^p}$, where $p = 2 > 1$

$\therefore$ by p-series test $\Sigma \frac{1}{n^2}$ converges.

By limit comparison test (5.20 Test III)

Σu_n also converges *i.e.*, $\Sigma \frac{1}{n(n+1)}$ converges.

Hence $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots \infty$ converges.

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Example 9. *Test the convergence of the series :* $\frac{1}{1.2.3}+\frac{3}{2.3.4}+\frac{5}{3.4.5}+\ldots\ldots$ **(P.T.U. May 2009)**

Sol. Here $$u_n=\frac{T_n \text{ of } 1,3,5,\ldots\ldots}{n(n+1)(n+2)}=\frac{2n-1}{n(n+1)(n+2)}$$

(As 1, 3, 5, ... form an A.P. with $a=1, d=2$

$\therefore$ n^{th} term $T_n=1+(n-1)2=2n-1$)

$$=\frac{n\left(2-\frac{1}{n}\right)}{n^3\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}=\frac{2-\frac{1}{n}}{n^2\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$

Let $$v_n=\frac{1}{n^2}$$

Let us compare Σu_n with Σv_n.

$$\frac{u_n}{v_n}=\frac{2-\frac{1}{n}}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$

$$\underset{n\to\infty}{\text{Lt}}\frac{u_n}{v_n}=\underset{n\to\infty}{\text{Lt}}\frac{2-\frac{1}{n}}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}=\frac{2}{(1)(1)}=2, \text{ which is finite and } \neq 0.$$

$\therefore$ Σu_n and Σv_n converge or diverge together.

Since $\Sigma v_n=\Sigma\frac{1}{n^2}$ is of the form $\Sigma\frac{1}{n^p}$ with $p=2>1$.

$\therefore$ Σv_n is convergent $\Rightarrow$ Σu_n is convergent.

Example 10. *Test the convergence of the following series:*

(*i*) $\frac{1}{1^p}+\frac{1}{3^p}+\frac{1}{5^p}+\ldots\ldots\infty$ (*ii*) $\frac{2}{1^p}+\frac{3}{2^p}+\frac{4}{3^p}+\ldots\ldots\infty.$

Sol. (*i*) Here $$u_n=\frac{1}{(2n-1)^p}=\frac{1}{n^p\left(2-\frac{1}{n}\right)^p}$$

($\because$ 1, 3, 5, ... are in AP and n^{th} term $=1+(n-1)2=2n-1$)

Let $$v_n=\frac{1}{n^p} \quad \therefore \quad \frac{u_n}{v_n}=\frac{1}{\left(2-\frac{1}{n}\right)^p}$$

$$\underset{n\to\infty}{\text{Lt}}\frac{u_n}{v_n}=\frac{1}{2^p}, \text{ which is finite and } \neq 0.$$

$\therefore$ Σu_n and Σv_n behave alike

$$\Sigma v_n=\Sigma\frac{1}{n^p}, \text{ which converges if } p>1 \text{ and diverges if } p\leq 1.$$

$\therefore$ Given series converges for $p>1$ and diverges for $p\leq 1$

(*ii*) Here $$u_n=\frac{n+1}{n^p}=\frac{1}{n^{p-1}}\left(1+\frac{1}{n}\right)$$

Let $$v_n=\frac{1}{n^{p-1}}; \quad \frac{u_n}{v_n}=1+\frac{1}{n}$$

$$\underset{n\to\infty}{\text{Lt}}\frac{u_n}{v_n}=1, \text{ which is finite and } \neq 0$$

$\therefore$ Σu_n and Σv_n behave alike

$$\Sigma v_n=\frac{1}{n^{p-1}}$$

converges if $p-1>1$ *i.e.*, $p>2$ and diverges if $p-1\leq 1$ *i.e.*, $p\leq 2$

$\therefore$ Given series converges for $p>2$ and diverges for $p\leq 2$.

Example 11. *Test the convergence of the following series :*

(*i*) $\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}+\ldots\ldots$ (*ii*) $\sqrt{\frac{1}{4}}+\sqrt{\frac{2}{6}}+\sqrt{\frac{3}{8}}+\ldots\ldots+\sqrt{\frac{n}{2(n+1)}}+\ldots$

(P.T.U., Dec. 2003)

Sol. (*i*) Here $$u_n=\frac{1}{\sqrt{n}+\sqrt{n+1}}=\frac{1}{\sqrt{n}\left[1+\sqrt{1+\frac{1}{n}}\right]}$$

Let us compare Σu_n with Σv_n, where $v_n=\frac{1}{\sqrt{n}}$

$$\frac{u_n}{v_n}=\frac{1}{1+\sqrt{1+\frac{1}{n}}}$$

an

$$\underset{n\to\infty}{\text{Lt}}\frac{u_n}{v_n}=\underset{n\to\infty}{\text{Lt}}\frac{1}{1+\sqrt{1+\frac{1}{n}}}=\frac{1}{1+1}=\frac{1}{2}, \text{ which is finite and } \neq 0.$$

bn

$\therefore$ Σu_n and Σv_n converge or diverge together.

Since $\Sigma v_n=\Sigma\frac{1}{n^{1/2}}$ is of the form $\Sigma\frac{1}{n^p}$ with $p=\frac{1}{2}<1$

$\therefore$ Σv_n is divergent $\Rightarrow$ Σu_n is divergent.

(*ii*) Here $$u_n=\sqrt{\frac{n}{2(n+1)}}=\sqrt{\frac{1}{2\left(1+\frac{1}{n}\right)}}$$

$$\underset{n\to\infty}{\text{Lt}}\ u_n = \underset{n\to\infty}{\text{Lt}}\ \frac{1}{\sqrt{2\left(1+\frac{1}{n}\right)}} = \frac{1}{\sqrt{2}} \neq 0$$

$\Rightarrow$ Σu_n does not converge.

Since the given series is a series of +ve terms, it either converges or diverges. Since it does not converge, it must diverge.

Hence the given series is divergent.

Example 12. *Test the convergence of* $\frac{1}{5} + \frac{\sqrt{2}}{7} + \frac{\sqrt{3}}{9} + \frac{\sqrt{4}}{11} + \ldots \infty$

Sol. Here $\quad u_n = \dfrac{\sqrt{n}}{2n+3}$

($\because$ $5, 7, 9, 11, \ldots$ are in A.P. and nth term of A.P. $= 5 + (n-1)2 = 2n+3$)

$$= \frac{\sqrt{n}}{n\left(2+\frac{3}{n}\right)} = \frac{1}{\sqrt{n}\left(2+\frac{3}{n}\right)}$$

Let $\quad v_n = \dfrac{1}{\sqrt{n}} \quad \therefore \quad \dfrac{u_n}{v_n} = \dfrac{1}{2+\frac{3}{n}}$

$\underset{n\to\infty}{\text{Lt}}\ \dfrac{u_n}{v_n} = \dfrac{1}{2}$, which is finite and $\neq 0$

$\therefore$ $\quad \Sigma u_n$ and Σv_n behave alike

$\Sigma v_n = \dfrac{1}{n^{1/2}}$, which is p series, where $p = \dfrac{1}{2} < 1$

$\therefore$ $\quad \Sigma v_n$ diverges

Hence $\Sigma u_n = \dfrac{1}{5} + \dfrac{\sqrt{2}}{7} + \dfrac{\sqrt{3}}{9} + \ldots \infty$ also diverges.

Example 13. *Test the convergence of the following series :*

(i) $1 + \dfrac{1}{2^2} + \dfrac{2^2}{3^3} + \dfrac{3^3}{4^4} + \dfrac{4^4}{5^5} + \ldots\ldots$ $\qquad$ (ii) $\Sigma\, \dfrac{n^2+1}{n^3+1}$. $\qquad$ (P.T.U., May 2006)

Sol. (i) Leaving aside the first term ($\because$ *Addition or deletion of a finite number of terms does not alter the nature of the series*), we have

$$u_n = \frac{n^n}{(n+1)^{n+1}} = \frac{n^n}{n^{n+1}\left(1+\frac{1}{n}\right)^{n+1}} = \frac{1}{n\left(1+\frac{1}{n}\right)^{n+1}}$$

Take $\quad v_n = \dfrac{1}{n}$.

$$\underset{n\to\infty}{\text{Lt}}\ \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}}\ \frac{1}{\left(1+\frac{1}{n}\right)^{n+1}} = \underset{n\to\infty}{\text{Lt}}\ \frac{1}{\left(1+\frac{1}{n}\right)^{n}} \times \frac{1}{\left(1+\frac{1}{n}\right)}$$

$$= \frac{1}{e} \cdot \frac{1}{1} \qquad \left[\because \underset{n\to\infty}{\text{Lt}}\left(1+\frac{1}{n}\right)^n = e\right]$$

$= \dfrac{1}{e}$, which is finite and $\neq 0$.

$\therefore$ Σu_n and Σv_n converge or diverge together.

Since $\Sigma v_n = \Sigma\, \dfrac{1}{n}$ is of the form $\Sigma\, \dfrac{1}{n^p}$ with $p = 1$

$\therefore$ Σv_n is divergent. $\Rightarrow$ Σu_n is divergent.

(ii) $$u_n = \frac{n^2+1}{n^3+1} = \frac{n^2\left(1+\frac{1}{n^2}\right)}{n^3\left(1+\frac{1}{n^3}\right)} = \frac{1}{n} \cdot \frac{1+\frac{1}{n^2}}{1+\frac{1}{n^3}}$$

Let $\quad v_n = \dfrac{1}{n}$

$\therefore$ $\quad \dfrac{u_n}{v_n} = \dfrac{1+\frac{1}{n^2}}{1+\frac{1}{n^3}}$

When $n \to \infty$ $\quad \dfrac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}}\ \dfrac{u_n}{v_n} = \dfrac{1+0}{1+0} = 1$, which is finite and non-zero

$\therefore$ Σu_n and Σv_n both converge or diverge together.

Since $\Sigma v_n = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is of the form $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ with $p = 1$.

$\therefore$ Σv_n is divergent

$\Rightarrow$ Σu_n is also divergent.

Hence $\displaystyle\sum \frac{n^2+1}{n^3+1}$ is divergent series.

Example 14. *Discuss the convergence or divergence of the following series:*

(i) $\displaystyle\sum \frac{1}{\sqrt{n}} \sin\frac{1}{n}$ $\qquad$ (ii) $\Sigma \cot^{-1} n^2$.

Sol. (i) Here $\quad u_n = \dfrac{1}{\sqrt{n}} \sin\dfrac{1}{n} = \dfrac{1}{\sqrt{n}} \cdot \dfrac{\sin\frac{1}{n}}{\frac{1}{n}} \times \dfrac{1}{n}$

$\therefore \quad u_n = \dfrac{1}{n^{3/2}} \left(\dfrac{\sin 1/n}{\dfrac{1}{n}} \right)$

Let $\quad v_n = \dfrac{1}{n^{3/2}} \quad \therefore \quad \dfrac{u_n}{v_n} = \dfrac{\sin \dfrac{1}{n}}{1/n}.$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}} \frac{\sin \dfrac{1}{n}}{\dfrac{1}{n}} = \underset{\frac{1}{n}\to 0}{\text{Lt}} \frac{\sin \dfrac{1}{n}}{\sin \dfrac{1}{n}} = 1 \qquad \left| \underset{\theta\to 0}{\text{Lt}} \frac{\sin\theta}{\theta} = 1 \right.$$

which is finite and non-zero

$\therefore \quad \Sigma u_n$ and Σv_n behave alike

$\Sigma v_n = \sum \dfrac{1}{n^{3/2}}$ is p-series, where $p = 3/2 > 1$

$\therefore \quad \Sigma v_n$ converges

and hence $\Sigma u_n = \sum \dfrac{1}{\sqrt{n}} \sin \dfrac{1}{n}$ converges.

(*ii*) Here $\quad u_n = \cot^{-1} n^2 = \tan^{-1} \dfrac{1}{n^2} = \dfrac{1}{n^2} \cdot \dfrac{\tan^{-1} 1/n^2}{1/n^2}.$

Take $\quad v_n = \dfrac{1}{n^2} \quad \therefore \quad \dfrac{u_n}{v_n} = \dfrac{\tan^{-1} \dfrac{1}{n^2}}{1/n^2}.$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}} \frac{\tan^{-1} \dfrac{1}{n^2}}{1/n^2} = \underset{h\to 0}{\text{Lt}} \frac{\tan^{-1} h}{h} \left(\text{where } h = \frac{1}{n^2} \right)$$

$= 1 \neq 0$, which is finite and non-zero

$\therefore \quad \Sigma u_n$ and Σv_n behave alike

$\Sigma v_n = \sum \dfrac{1}{n^2}$ is p-series, where $p = 2$

$\therefore \quad \Sigma v_n$ is convergent

So $\Sigma u_n = \Sigma \cot^{-1} n^2$ is also convergent.

Example 15. *Examine the convergence of the series:* $\dfrac{\sqrt{2}-1}{3^3-1} + \dfrac{\sqrt{3}-1}{4^3-1} + \dfrac{\sqrt{4}-1}{5^3-1} + \ldots\ldots$

Sol. Here $u_n = \dfrac{\sqrt{n+1}-1}{(n+2)^3-1} = \dfrac{\sqrt{n}\left(\sqrt{1+\dfrac{1}{n}} - \dfrac{1}{\sqrt{n}}\right)}{n^3\left[\left(1+\dfrac{2}{n}\right)^3 - \dfrac{1}{n^3}\right]} = \dfrac{\sqrt{1+\dfrac{1}{n}} - \dfrac{1}{\sqrt{n}}}{n^{5/2}\left[\left(1+\dfrac{2}{n}\right)^3 - \dfrac{1}{n^3}\right]}$

Take $\quad v_n = \dfrac{1}{n^{5/2}}$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}} \frac{\sqrt{1+\dfrac{1}{n}} - \dfrac{1}{\sqrt{n}}}{\left(1+\dfrac{2}{n}\right)^3 - \dfrac{1}{n^3}} = \frac{\sqrt{1+0}-0}{(1+0)^3-0} = 1, \text{ which is finite and} \neq 0.$$

$\therefore \quad \Sigma u_n$ and Σv_n converge or diverge together.

Since $\Sigma v_n = \Sigma \dfrac{1}{n^{5/2}}$ is of the form $\Sigma \dfrac{1}{n^p}$ with $p = \dfrac{5}{2} > 1.$

Σv_n is convergent. $\Rightarrow \quad \Sigma u_n$ is convergent.

Example 16. *Examine the convergence of the series:*

(*i*) $\sum \dfrac{\sqrt{n+1}-\sqrt{n}}{n^p}$ $\qquad$ (*ii*) $\sum \left(\sqrt[3]{n^3+1} - n\right).$

(P.T.U. May 2007, Jan. 2010)

Sol. (*i*) Here $\quad u_n = \dfrac{\sqrt{n+1}-\sqrt{n}}{n^p} = \dfrac{\sqrt{n+1}-\sqrt{n}}{n^p} \times \dfrac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$ $\qquad$ (Rationalising)

$$= \frac{(n+1)-n}{n^p \cdot \sqrt{n}\left(\sqrt{1+\dfrac{1}{n}}+1\right)} = \frac{1}{n^{p+1/2}\left(\sqrt{1+\dfrac{1}{n}}+1\right)}$$

Take $\quad v_n = \dfrac{1}{n^{p+1/2}}$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}} \frac{1}{\sqrt{1+\dfrac{1}{n}}+1} = \frac{1}{1+1} = \frac{1}{2}, \text{ which is finite and} \neq 0.$$

$\therefore \quad \Sigma u_n$ and Σv_n converge or diverge together.

Since $\sum v_n = \sum \dfrac{1}{n^{p+1/2}}$ is convergent if $p + \dfrac{1}{2} > 1$ and divergent if $p + \dfrac{1}{2} \leq 1.$

convergent if $p > \frac{1}{2}$ and divergent if $p \leq \frac{1}{2}$.

$\therefore \quad \Sigma u_n$ is convergent if $p > \frac{1}{2}$ and divergent if $p \leq \frac{1}{2}$.

(*ii*) Here

$$u_n = (n^3+1)^{1/3} - n = \left[n^3\left(1+\frac{1}{n^3}\right)\right]^{1/3} - n$$

$$= n\left(1+\frac{1}{n^3}\right)^{1/3} - n = n\left[\left(1+\frac{1}{n^3}\right)^{1/3} - 1\right]$$

$$= n\left[1 + \frac{1}{3}\cdot\frac{1}{n^3} + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\cdot\left(\frac{1}{n^3}\right)^2 + \ldots\ldots - 1\right]$$

$$= \frac{n}{n^3}\left[\frac{1}{3}-\frac{1}{9}\cdot\frac{1}{n^3}\ldots\ldots\right] = \frac{1}{n^2}\left[\frac{1}{3}-\frac{1}{9}\cdot\frac{1}{n^3}\ldots\ldots\right]$$

Take $$v_n = \frac{1}{n^2}.$$

$$\underset{n\to\infty}{\text{Lt}}\frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}}\left(\frac{1}{3}-\frac{1}{9}\cdot\frac{1}{n^3}\ldots\ldots\right) = \frac{1}{3}, \text{ which is finite and } \neq 0.$$

$\therefore$ Σu_n and Σv_n converge or diverge together.

Since $\sum v_n = \sum \frac{1}{n^2}$ is of the form $\sum \frac{1}{n^p}$ with $p = 2 > 1$

$\therefore$ Σv_n is convergent $\Rightarrow$ Σu_n is convergent.

Note. Rationalisation is effective only when square roots are involved whereas. Binomial Expansion is the general method.

Example 17. *Discuss the convergence or divergence of the following series:*

(*i*) $\sum\left(\frac{1}{n}-\log\frac{n+1}{n}\right)$ (*ii*) $\frac{1}{\log 2}+\frac{1}{\log 3}+\ldots+\frac{1}{\log n}+\ldots\infty.$

Sol. (*i*)
$$u_n = \frac{1}{n}-\log\frac{n+1}{n} = \frac{1}{n}-\log\left(1+\frac{1}{n}\right)$$
$$= \frac{1}{n}-\left[\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\frac{1}{4n^4}+\ldots\infty\right]$$
$$= \frac{1}{2n^2}-\frac{1}{3n^3}+\frac{1}{4n^4}\ldots\ldots\infty$$
$$= \frac{1}{n^2}\cdot\left\{\frac{1}{2}-\frac{1}{3n}+\frac{1}{4n^2}\ldots\ldots\infty\right\}$$

Let $$v_n = \frac{1}{n^2} \quad \therefore \quad \frac{u_n}{v_n} = \frac{1}{2}-\frac{1}{3n}+\frac{1}{4n^2}\ldots\infty$$

$$\underset{n\to\infty}{\text{Lt}}\frac{u_n}{v_n} = \frac{1}{2}, \text{ which is finite and } \neq 0$$

$\therefore$ Σu_n and Σv_n behave alike

$$\Sigma v_n = \sum\frac{1}{n^2} \text{ is } p\text{-series where } p = 2 > 1$$

$\therefore$ Σv_n converges and so given series Σu_n converges.

(*ii*) Given series is $$\sum_{n=2}^{\infty} u_n = \sum_{n=2}^{\infty}\frac{1}{\log n}$$

$$\therefore \quad u_n = \frac{1}{\log n}$$

We know that $\log n < n$ $\therefore$ $\frac{1}{\log n} > \frac{1}{n}$

$$u_n > \frac{1}{n}. \text{ Take } v_n = \frac{1}{n}$$

$\therefore$ $u_n > v_n$ and $\Sigma v_n = \sum\frac{1}{n}$ is of the type $\sum\frac{1}{n^p}$, where $p = 1$.

$\therefore$ Σv_n divergent

$\therefore$ By comparison test 5.20 I(*b*)

$\therefore$ Σu_n is also divergent.

Hence $\sum_{n=2}^{\infty}\frac{1}{\log n}$ is divergent.

Example 18. *Test the convergence or divergence of the following series:*

(*i*) $1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\ldots\infty$ (*ii*) $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n!}}.$

Sol. (*i*) $u_n = \frac{1}{n!}$

$n! = 1.2.3.\ldots n \geq 1.2.2.2\ldots(n-1)$ times $= 2^{n-1}$

$$\therefore \quad \frac{1}{n!} < \frac{1}{2^{n-1}} = v_n \text{ (say)}$$

$$\therefore \quad u_n = \frac{1}{n!} < v_n, \text{ where } v_n = \frac{1}{2^{n-1}}$$

Σv_n is a G.P. series with common ratio $\frac{1}{2} < 1$

$\therefore$ Σv_n is convergent

$\therefore$ By comparison test 5.20 I(*a*) Σu_n is also convergent

Hence $\Sigma u_n = \sum\frac{1}{n!}$ is convergent.

(*ii*) $\Sigma u_n = \sum\frac{1}{\sqrt{n!}}$ $\therefore$ $u_n = \frac{1}{\sqrt{n!}}$

As proved in (*i*) part $\frac{1}{n!} < \frac{1}{2^{n-1}}$ $\therefore$ $\frac{1}{\sqrt{n!}} < \frac{1}{2^{\frac{n-1}{2}}}$

$\therefore$ $u_n < v_n$ where $v_n = \frac{1}{2^{\frac{n-1}{2}}}$

Σv_n is an infinite G.P. with common ratio $\frac{1}{\sqrt{2}} < 1$

$\therefore$ Σv_n is convergent. Hence Σu_n *i.e.*, $\sum\frac{1}{\sqrt{n!}}$ is convergent

TEST YOUR KNOWLEDGE

Test the convergence or divergence of the following series :

1. $\frac{1}{2}+\frac{1}{3^2}+\frac{1}{2^3}+\frac{1}{3^4}+ \ldots\ldots$ to ∞

2. $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+ \ldots\ldots$ to ∞

3. $\frac{1}{1.4}+\frac{1}{2.5}+\frac{1}{3.6}+ \ldots\ldots$ to ∞

4. $\frac{1}{1.2}+\frac{2}{3.4}+\frac{3}{5.6}+ \ldots\ldots$ to ∞

5. $\frac{1}{1.3}+\frac{2}{3.5}+\frac{3}{5.7}+ \ldots\ldots$ to ∞ **(P.T.U., Jan. 2010)**

6. $\frac{1}{\sqrt{1.2}}+\frac{1}{\sqrt{2.3}}+\frac{1}{\sqrt{3.4}}+ \ldots\ldots$ to ∞

7. $\sum_{n=1}^{\infty} \frac{n+1}{n(2n-1)}$

8. $\sum_{n=1}^{\infty} \frac{1}{n^p(n+1)^p}$

9. $\frac{2^p}{1^q}+\frac{3^p}{2^q}+\frac{4^p}{3^q}+ \ldots\ldots$ to ∞

   (p and q are positive numbers)

10. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$

11. $\sum \frac{2n^3+5}{4n^5+1}$

12. $\sum \frac{\sqrt{n^2-1}}{n^3+1}$

13. $\sum\left(\sqrt{n^2+1}-n\right)$ **(P.T.U., Dec. 2006)**

14. $\sum\left(\sqrt{n^3+1}-\sqrt{n^3}\right)$

$\left(\sqrt{n^4+1}-\sqrt{n-1}\right)$

16. $\frac{\sqrt{2}-\sqrt{1}}{1}+\frac{\sqrt{3}-\sqrt{2}}{2}+\frac{\sqrt{4}-\sqrt{3}}{3}+\ldots\ldots$

17. $\sum\left\{\left(\sqrt[3]{n+1}-\sqrt[3]{n}\right)\right\}$

ANSWERS

1. Convergent	2. Convergent	3. Convergent	4. Divergent
5. Divergent	6. Divergent	7. Divergent	
8. Convergent for $p > \frac{1}{2}$, divergent for $p \le \frac{1}{2}$		9. Convergent for $q > p + 1$, divergent for $q \le p+1$	
10. Convergent	11. Convergent	12. Convergent	13. Divergent
14. Convergent	15. Convergent	16. Convergent	17. Divergent.

5.22. D' ALEMBERT'S RATIO TEST

(P.T.U., Dec. 2004)

Statement. *If Σu_n is a positive term series, and* $\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = l$, *then*

(*i*) *Σu_n is convergent if $l > 1$.* (*ii*) *Σu_n is divergent if $l < 1$.*

Note. If $l = 1$, the test fails, *i.e.*, no conclusion can be drawn about the convergence or divergence of the series. The series may converge, it may diverge.

Proof. Since $\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = l,$

$\therefore$ Given $\varepsilon > 0$, however small, there exists a positive integer m such that

$$\left|\frac{u_n}{u_{n+1}} - l\right| < \varepsilon \quad \forall \quad n \ge m$$

$$\Rightarrow \quad l - \varepsilon < \frac{u_n}{u_{n+1}} < l + \varepsilon \quad \forall \quad n \ge m \qquad \ldots(1)$$

Case I. When $l > 1$, choose $\varepsilon > 0$ such that $l - \varepsilon = r > 1$

$$\therefore \quad \text{for } n \ge m; \quad \frac{u_n}{u_{n+1}} > l - \varepsilon = r \quad i.e., \quad \frac{u_n}{u_{n+1}} > r \text{ for } n \ge m$$

Put $n = m, m+1, m+2, \ldots n-1$ (*i.e.*, $n - m$ terms)

$$\therefore \quad \frac{u_m}{u_{m+1}} > r$$

$$\frac{u_{m+1}}{u_{m+2}} > r$$

$$\frac{u_{m+2}}{u_{m+3}} > r$$

....

....

$$\frac{u_{n-1}}{u_n} > r$$

Multiply these inequalities ; $\frac{u_m}{u_n} > r^{n-m}$

$$\therefore \quad \frac{u_n}{u_m} < \frac{1}{r^{n-m}} \quad \text{or} \quad u_n < \frac{u_m}{r^{n-m}} = (r^m u_m)\frac{1}{r^n}$$

$$\therefore \quad u_n < k.\frac{1}{r^n} \quad \forall\, n \ge m \text{ (where } k = r^m.u_m)$$

Let $v_n = \frac{1}{r^n}$, where $r > 1 \quad \therefore \quad \frac{1}{r} < 1$

Σv_n is a geometric series with common ratio < 1

$\therefore$ Σv_n is convergent and by comparison test 5.20 I(*i*)

$\sum_{n=1}^{\infty} u_n$ is also convergent.

Case II. *When $l < 1$;* choose $\varepsilon > 0$ such that $l + \varepsilon = R < 1$

$$\therefore \quad \text{From (1)} \quad \frac{u_n}{u_{n+1}} < R \quad \forall\, n \ge m$$

Put $n = m, m+1, m+2, \ldots n-1$ and multiply (as in case I)

we get $\quad \frac{u_m}{u_n} < R^{n-m}$ or $u_n > \frac{u_m}{R^{n-m}} = (R^m u_m) \cdot \frac{1}{R^n} = k' v_n$

where $\quad k' = R^m u_m$ and $v_n = \frac{1}{R^n}$

$\Sigma v_n = \sum \frac{1}{R^n}$, which is G.P. with common ratio $\frac{1}{R} > 1$

$\therefore \quad \Sigma v_n$ is divergent.

$\therefore \quad \Sigma u_n$ is also divergent.

Hence if Σu_n is a positive term series, and $\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = l$, then

(i) Σu_n is convergent if $l > 1$ $\qquad$ (ii) Σu_n is divergent if $l < 1$.

ILLUSTRATIVE EXAMPLES

Example 1. *Discuss the convergence of the following series:*

(i) $1 + \frac{2^p}{2!} + \frac{3^p}{3!} + \frac{4^p}{4!} + \ldots\ldots, (p > 0)$

(ii) $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \ldots\ldots + \frac{1}{2^{n-1}+1} + \ldots\ldots$

(iii) $\frac{2}{1.} + \frac{2^2 . 3^2}{2!} + \frac{3^2 . 4^2}{3!} + \frac{4^2 . 5^2}{4!} + \ldots\ldots$

Sol. (i) Here $\qquad u_n = \frac{n^p}{n!} \qquad \left[\because\ 1 = \frac{1^p}{1!}\right]$

$\therefore \qquad u_{n+1} = \frac{(n+1)^p}{(n+1)!}$

$$\frac{u_n}{u_{n+1}} = \frac{n^p}{n!} \cdot \frac{(n+1)!}{(n+1)^p} = \frac{n^p \cdot (n+1)\, n!}{n!\,(n+1)^p} = \frac{n^p}{(n+1)^{p-1}}$$

$$= \frac{n^p}{n^{p-1}\left(1+\frac{1}{n}\right)^{p-1}} = \frac{n}{\left(1+\frac{1}{n}\right)^{p-1}}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}} \frac{n}{\left(1+\frac{1}{n}\right)^{p-1}} = \infty > 1$$

$\therefore$ By D'Alembert's Ratio Test, Σu_n is convergent.

(ii) Here $\qquad u_n = \frac{1}{2^{n-1}+1} \qquad \therefore \quad u_{n+1} = \frac{1}{2^n+1}$

$$\frac{u_n}{u_{n+1}} = \frac{2^n+1}{2^{n-1}+1} = \frac{2^n\left(1+\frac{1}{2^n}\right)}{2^{n-1}\left(1+\frac{1}{2^{n-1}}\right)} = 2 \cdot \frac{1+\frac{1}{2^n}}{1+\frac{1}{2^{n-1}}}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}}\ 2 \cdot \frac{1+\frac{1}{2^n}}{1+\frac{1}{2^{n-1}}} = 2 > 1$$

$\therefore$ By D'Alembert's Ratio Test, Σu_n is convergent.

(iii) Here $\qquad u_n = \frac{n^2 (n+1)^2}{n!} \qquad \therefore \quad u_{n+1} = \frac{(n+1)^2 (n+2)^2}{(n+1)!}$

$$\frac{u_n}{u_{n+1}} = \frac{n^2 (n+1)^2}{n!} \cdot \frac{(n+1)!}{(n+1)^2 (n+2)^2} = \frac{n^2 (n+1)}{(n+2)^2}$$

$$= \frac{n^3\left(1+\frac{1}{n}\right)}{n^2\left(1+\frac{2}{n}\right)^2} = n \cdot \frac{1+\frac{1}{n}}{\left(1+\frac{2}{n}\right)^2}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}}\ n \cdot \frac{1+\frac{1}{n}}{\left(1+\frac{2}{n}\right)^2} = \infty > 1$$

$\therefore$ By D'Alembert's Ratio Test, Σu_n is convergent.

Example 2. *Discuss the convergence of the following series:*

(i) $\frac{2}{1} + \frac{2.5.8}{1.5.9} + \frac{2.5.8.11}{1.5.9.13} + \ldots \infty$ $\qquad$ (ii) $\left(\frac{1}{3}\right)^2 + \left(\frac{1.2}{3.5}\right)^2 + \left(\frac{1.2.3.}{3.5.7}\right)^2 + \ldots \infty$

Sol. (i) $\qquad u_n = \frac{2.5.8.11 \ldots (3n-1)}{1.5.9.13 \ldots (4n-3)}$

(As 2, 5, 8, ... are in A.P
$\therefore$ its nth term $= 2 + (n-1)\,3 = 3n - 1$.
Also 1, 5, 9, ... are in A.P
$\therefore$ its nth term $= 1 + (n-1)\,4 = 4n - 3$)

$$u_{n+1} = \frac{2.5.8.11 \ldots (3n-1)(3n+2)}{1.5.9.13 \ldots (4n-3)(4n+1)}$$

$$\frac{u_n}{u_{n+1}} = \frac{4n+1}{3n+2}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}} \frac{4+\frac{1}{n}}{3+\frac{2}{n}} = \frac{4}{3} > 1$$

$\therefore$ By D'Alembert's Ratio Test Σu_n is convergent.

(ii)

$$u_n = \left(\frac{1.2.3\ldots n}{3.5.7\ldots 2n+1}\right)^2$$

$$u_{n+1} = \left[\frac{1.2.3\ldots n(n+1)}{3.5.7\ldots(2n+1)(2n+3)}\right]^2$$

$$\frac{u_n}{u_{n+1}} = \left(\frac{2n+3}{n+1}\right)^2 = \left(\frac{2+\frac{3}{n}}{1+\frac{1}{n}}\right)^2$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \frac{2}{1} = 2 > 1$$

$\therefore$ By D'Alembert's Ratio test Σu_n is convergent

Example 3. *Test the convergence of the following series:*

(i) $\sum \frac{n^3+a}{2^n+a}$ *(ii)* $\sum \frac{n!\,2^n}{n^n}$

(iii) $\sum \frac{2^{n-1}}{3^n+1}$ *(iv)* $\sum \frac{n^2(n+1)^2}{n!}$.

Sol. *(i)* Here $u_n = \frac{n^3+a}{2^n+a}$ $\therefore$ $u_{n+1} = \frac{(n+1)^3+a}{2^{n+1}+a}$

$$\frac{u_n}{u_{n+1}} = \frac{n^3+a}{(n+1)^3+a}\cdot\frac{2^{n+1}+a}{2^n+a}$$

$$= \frac{n^3\left(1+\frac{a}{n^3}\right)}{n^3\left[\left(1+\frac{1}{n}\right)^3+\frac{a}{n^3}\right]}\cdot\frac{2^{n+1}\left(1+\frac{a}{2^{n+1}}\right)}{2^n\left(1+\frac{a}{2^n}\right)} = \frac{1+\frac{a}{n^3}}{\left(1+\frac{1}{n}\right)^3+\frac{a}{n^3}}\cdot\frac{2\left(1+\frac{a}{2^{n+1}}\right)}{1+\frac{a}{2^n}}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \frac{1+0}{1+0}\cdot 2\cdot\frac{1+0}{1+0} = 2 > 1$$

$\therefore$ By D'Alembert's Ratio Test, Σu_n is convergent.

(ii) Here $u_n = \frac{n!\,2^n}{n^n}$ $\therefore$ $u_{n+1} = \frac{(n+1)!\,2^{n+1}}{(n+1)^{n+1}}$

$$\frac{u_n}{u_{n+1}} = \frac{n!\,2^n}{(n+1)!\,2^{n+1}}\cdot\frac{(n+1)^{n+1}}{n^n} = \frac{1}{2(n+1)}\cdot\frac{(n+1)^{n+1}}{n^n}$$

$$= \frac{1}{2}\cdot\frac{(n+1)^n}{n^n} = \frac{1}{2}\cdot\left(\frac{n+1}{n}\right)^n = \frac{1}{2}\left(1+\frac{1}{n}\right)^n$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}} \frac{1}{2}\left(1+\frac{1}{n}\right)^n = \frac{e}{2}$$

$$2 < e < 3 \Rightarrow 1 < \frac{e}{2} < \frac{3}{2}$$

Now, $\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \frac{e}{2} > 1 \Rightarrow \Sigma u_n$ is convergent.

(iii) $u_n = \frac{2^{n-1}}{3^n+1}$; $u_{n+1} = \frac{2^n}{3^{n+1}+1}$

$$\frac{u_n}{u_{n+1}} = \frac{2^{n-1}}{3^n+1}\times\frac{3^{n+1}+1}{2^n} = \frac{1}{2}\frac{3^{n+1}+1}{3^n+1} = \frac{1}{2}\frac{3^n\left(3+\frac{1}{3^n}\right)}{3^n\left(1+\frac{1}{3^n}\right)}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}} \frac{1}{2}\frac{3+\frac{1}{3^n}}{1+\frac{1}{3^n}} = \frac{3}{2} > 1$$

$\therefore$ By Ratio test Σu_n is convergent.

(iv) $u_n = \frac{n^2(n+1)^2}{n!}$, $u_{n+1} = \frac{(n+1)^2(n+2)^2}{(n+1)!}$

$$\frac{u_n}{u_{n+1}} = \frac{n^2(n+1)^2}{n!}\cdot\frac{(n+1)!}{(n+1)^2(n+2)^2}$$

$$= \frac{n^2}{(n+2)^2}(n+1) = (n+1)\left(\frac{n}{n+2}\right)^2 = (n+1)\left(\frac{1}{1+\frac{2}{n}}\right)^2$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \infty\,.\,1 = \infty > 1$$

$\therefore$ By Ratio test Σu_n is convergent

Example 4. *Discuss the convergence of the series:*

(i) $\sum \frac{\sqrt{n}}{\sqrt{n^2+1}}x^n$ *(ii)* $\sum \frac{x^n}{3^n.n^2}, x>0$

(iii) $\sum_{n=1}^{\infty} \frac{x^n}{2n!}$ *(iv)* $\sum \frac{3^n-2}{3^n+1}x^{n-1}, x>0.$

Sol. *(i)* Here

$$u_n = \sqrt{\frac{n}{n^2+1}}\,x^n$$

$\therefore$

$$u_{n+1} = \sqrt{\frac{n+1}{(n+1)^2+1}} \cdot x^{n+1}$$

$$\frac{u_n}{u_{n+1}} = \sqrt{\frac{n}{n+1} \cdot \frac{n^2+2n+2}{n^2+1}} \cdot \frac{1}{x} = \sqrt{\frac{1}{1+\frac{1}{n}} \cdot \frac{1+\frac{2}{n}+\frac{2}{n^2}}{1+\frac{1}{n^2}}} \cdot \frac{1}{x}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}} \sqrt{\frac{1}{1+\frac{1}{n}} \cdot \frac{1+\frac{2}{n}+\frac{2}{n^2}}{1+\frac{1}{n^2}}} \cdot \frac{1}{x} = \frac{1}{x}$$

$\therefore$ By D'Alembert's Ratio Test, $\sum u_n$ converges if $\frac{1}{x} > 1$ *i.e.*, $x < 1$

and diverges if $\frac{1}{x} < 1$ *i.e.*, $x > 1$

When $x = 1$, the Ratio Test fails.

$\therefore$ for $x = 1$,
$$u_n = \sqrt{\frac{n}{n^2+1}} = \sqrt{\frac{n}{n^2\left(1+\frac{1}{n^2}\right)}} = \frac{1}{\sqrt{n}} \cdot \frac{1}{\sqrt{1+\frac{1}{n^2}}}$$

Take
$$v_n = \frac{1}{\sqrt{n}}, \quad \underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}} \frac{1}{\sqrt{1+\frac{1}{n^2}}} = 1, \text{ which is finite and} \neq 0.$$

$\therefore$ By Comparison Test, Σu_n and Σv_n converge or diverge together.

Since $\sum v_n = \sum \frac{1}{\sqrt{n}}$ is of the form $\sum \frac{1}{n^p}$ with $p = \frac{1}{2} < 1$

Σv_n diverges $\Rightarrow$ Σu_n diverges.

Hence the given series Σu_n converges if $x < 1$ and diverges if $x \geq 1$.

(ii)
$$u_n = \frac{x^n}{3^n . n^2}\ ; u_{n+1} = \frac{x^{n+1}}{3^{n+1}(n+1)^2} \quad \therefore \quad \frac{u_n}{u_{n+1}} = \frac{x^n}{3^n . n^2} \cdot \frac{3^{n+1}(n+1)^2}{x^{n+1}}$$

$\therefore$
$$\frac{u_n}{u_{n+1}} = 3\left(\frac{n+1}{n}\right)^2 \cdot \frac{1}{x} = 3\left(1+\frac{1}{n}\right)^2 \cdot \frac{1}{x}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \frac{3}{x}.$$

$\therefore$ By ratio test Σu_n converges if $\frac{3}{x} > 1$ and diverges *i.e.*, converges for $x < 3$ and diverges for $x > 3$

$= 3$,
$$u_n = \frac{3^n}{3^n . n^2} = \frac{1}{n^2}$$

$$\Sigma u_n = \sum \frac{1}{n^2}, \text{ which is of the type } \sum \frac{1}{n^p}, \text{ where } p = 2 > 1$$

$\therefore$ Σu_n converges for $x = 3$

Hence u_n is convergent for $x \leq 3$ and diverges for $x > 3$.

(iii)
$$u_n = \frac{x^n}{2n\,!},\ u_{n+1} = \frac{x^{n+1}}{2\,(n+1)\,!}$$

$$\frac{u_n}{u_{n+1}} = \frac{x^n}{2n\,!} \times \frac{(2n+2)\,!}{x^{n+1}} = \frac{(2n+2)(2n+1)}{x}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}} \frac{(2n+2)(2n+1)}{x} \to \infty > 1$$

$\therefore$ By Ratio test Σu_n is convergent.

(iv)
$$u_n = \frac{3^n - 2}{3^n + 1} x^{n-1};\ u_{n+1} = \frac{3^{n+1} - 2}{3^{n+1} + 1} x^n$$

$$\frac{u_n}{u_{n+1}} = \frac{3^n - 2}{3^n + 1} x^{n-1} \cdot \frac{3^{n+1}+1}{3^{n+1}-2} \cdot \frac{1}{x^n} = \frac{(3^n-2)(3^{n+1}+1)}{(3^n+1)(3^{n+1}-2)} \cdot \frac{1}{x}$$

$$= \frac{3^n\left(1-\frac{2}{3^n}\right) . 3^{n+1}\left(1+\frac{1}{3^{n+1}}\right)}{3^n\left(1+\frac{1}{3^n}\right)3^{n+1}\left(1-\frac{2}{3^{n+1}}\right)} \cdot \frac{1}{x}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \frac{1}{x}.$$

$\therefore$ By Ratio test Σu_n converges for $\frac{1}{x} > 1$ *i.e.*, for $x < 1$ and diverges for $x > 1$.

for $x = 1$
$$u_n = \frac{3^n - 2}{3^n + 1} = \frac{1 - \frac{2}{3^n}}{1 + \frac{2}{3^n}}$$

$$\underset{n\to\infty}{\text{Lt}}\ u_n = 1 \neq 0 \quad \therefore \quad \Sigma u_n \text{ is divergent.}$$

$\therefore$ Σu_n is convergent for $x < 1$ and divergent for $n \geq 1$.

Example 5. *Examine the convergence or divergence of the following series :*

(i) $\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \frac{x^6}{5\sqrt{4}} + \ldots\ldots$ **(P.T.U., Dec. 2002)**

(ii) $\sum \frac{x^{n+1}}{(n+1)\sqrt{n}}$. **(P.T.U. Dec. 2007)**

Sol. *(i)* Here
$$u_n = \frac{x^{2n-2}}{(n+1)\sqrt{n}} \quad \therefore \quad u_{n+1} = \frac{x^{2n}}{(n+2)\sqrt{n+1}}$$

$$\frac{u_n}{u_{n+1}} = \frac{(n+2)\sqrt{n+1}}{(n+1)\sqrt{n}} \cdot \frac{1}{x^2} = \frac{1+\frac{2}{n}}{1+\frac{1}{n}} \cdot \sqrt{1+\frac{1}{n}} \cdot \frac{1}{x^2}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \frac{1}{x^2}$$

By D'Alembert's Ratio Test, $\sum u_n$ converges if $\frac{1}{x^2} > 1$, *i.e.,* $x^2 < 1$

and diverges if $\frac{1}{x^2} < 1$ *i.e.,* $x^2 > 1$.

When $x^2 = 1$, $\quad u_n = \frac{1}{(n+1)\sqrt{n}} = \frac{1}{n^{3/2}\left(1+\frac{1}{n}\right)}$

Take $\quad v_n = \frac{1}{n^{3/2}}$; $\frac{u_n}{v_n} = \frac{1}{1+\frac{1}{n}}$; $\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = 1$, which is finite and $\neq 0$

Σv_n is convergent by p-series test $\because$ here $p = 3/2 > 1$

$\therefore$ By comparison test Σu_n is also convergent $\quad (\because\ p>1)$

Hence Σu_n is convergent if $x^2 \leq 1$ and divergent if $x^2 > 1$.

(*ii*) $\quad u_n = \frac{x^{n+1}}{(n+1)\sqrt{n}}$; $u_{n+1} = \frac{x^{n+2}}{(n+2)\sqrt{n+1}}$

$$\frac{u_n}{u_{n+1}} = \frac{(n+2)\sqrt{n+1}}{(n+1)\sqrt{n}} \cdot \frac{1}{x} = \frac{\left(1+\frac{2}{n}\right)}{\left(1+\frac{1}{n}\right)} \sqrt{1+\frac{1}{n}} \cdot \frac{1}{x}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \frac{1}{x}$$

By D' Alembert's Ratio test Σu_n converges for $\frac{1}{x} > 1$ *i.e.,* $x < 1$ and diverges for $x > 1$

when $x = 1$, Ratio test fails

$\therefore$ For $x = 1$, $u_n = \frac{1}{(n+1)\sqrt{n}} = \frac{1}{n^{3/2}\left(1+\frac{1}{n}\right)}$; Take $v_n = \frac{1}{n^{3/2}}$

$\therefore \frac{u_n}{v_n} = \frac{1}{1+\frac{1}{n}}$; $\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = 1$, which is finite and non-zero $\therefore$ Σu_n and Σv_n behave alike Σv_n is a cgt series (by p-test ; $p > 1$) $\therefore$ By comparison test Σu_n is also cgt. Hence Σu_n converges for $x \leq 1$ and diverges for $x > 1$

Example 6. *Discuss the convergence or divergence of the following series:*

(*i*) $x + \frac{3}{5}x^2 + \frac{8}{10}x^3 + \frac{15}{17}x^4 + \ldots + \frac{n^2-1}{n^2+1}x^n + \ldots \infty$

(*ii*) $\frac{x}{2\sqrt{3}} + \frac{x^2}{3\sqrt{4}} + \frac{x^3}{4\sqrt{5}} + \ldots\ldots \infty.$

Sol. (*i*) $u_n = \frac{n^2-1}{n^2+1}x^n$; $u_{n+1} = \frac{(n+1)^2-1}{(n+1)^2+1}x^{n+1}$

$$\frac{u_n}{u_{n+1}} = \frac{n^2-1}{n^2+1} \frac{(n+1)^2+1}{(n+1)^2-1} \cdot \frac{1}{x}$$

$$= \frac{n^2\left(1-\frac{1}{n^2}\right)}{n^2\left(1+\frac{1}{n^2}\right)} \frac{n^2\left\{\left(1+\frac{1}{n}\right)^2+\frac{1}{n^2}\right\}}{n^2\left\{\left(1+\frac{1}{n}\right)^2-\frac{1}{n^2}\right\}} \cdot \frac{1}{x}$$

$$= \frac{1-\frac{1}{n^2}}{1+\frac{1}{n^2}} \cdot \frac{\left(1+\frac{1}{n}\right)^2+\frac{1}{n^2}}{\left(1+\frac{1}{n}\right)^2-\frac{1}{n^2}} \cdot \frac{1}{x}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \frac{1}{x}$$

$\therefore$ By Ratio test Σu_n converges if $\frac{1}{x} > 1$ *i.e.,* $x < 1$ and diverges if $x > 1$

When $x = 1$, $u_n = \frac{n^2-1}{n^2+1}$

$$\underset{n\to\infty}{\text{Lt}}\ u_n = \underset{n\to\infty}{\text{Lt}} \frac{1-\frac{1}{n^2}}{1+\frac{1}{n^2}} = 1 \neq 0$$

$\therefore$ Σu_n diverges.

$\therefore$ Σu_n converges for $x < 1$ and diverges for $x \geq 1$.

(*ii*) $\quad u_n = \frac{x^n}{(n+1)\sqrt{n+2}}$; $u_{n+1} = \frac{x^{n+1}}{(n+2)\sqrt{n+3}}$

$$\frac{u_n}{u_{n+1}} = \frac{x^n}{(n+1)\sqrt{n+2}} \times \frac{(n+2)\sqrt{n+3}}{x^{n+1}} = \frac{n+2}{n+1}\sqrt{\frac{n+3}{n+2}} \cdot \frac{1}{x}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}} \frac{1+\frac{2}{n}}{1+\frac{1}{n}} \sqrt{\frac{1+3/n}{1+2/n}} \cdot \frac{1}{n} = \frac{1}{x}$$

Σu_n converges for $\frac{1}{x} > 1$ *i.e.*, $x < 1$ and diverges for $x > 1$.

When $x = 1$, $u_n = \frac{1}{(n+1)\sqrt{n+2}}$; $\underset{n\to\infty}{\text{Lt}}\ u_n = \underset{n\to\infty}{\text{Lt}}\ \frac{1}{(n+1)\sqrt{n+2}} = 0$

As Σu_n is a +ve term series and $\underset{n\to\infty}{\text{Lt}}\ u_n = 0$

$\therefore$ Σu_n converges. Hence Σu_n converges for $x \le 1$, diverges for $x > 1$.

Example 7. *Examine the convergence or divergence of the following series :*

$$1 + \frac{2}{5}x + \frac{6}{9}x^2 + \frac{14}{17}x^3 + \ldots\ldots + \frac{2^{n+1}-2}{2^{n+1}+1}x^n + \ldots\ldots (x > 0).$$

(P.T.U. Jan. 2009)

Sol. Here, leaving the first term, $u_n = \frac{2^{n+1}-2}{2^{n+1}+1}x^n$

$$\therefore \qquad u_{n+1} = \frac{2^{n+2}-2}{2^{n+2}+1}x^{n+1}$$

$$\frac{u_n}{u_{n+1}} = \frac{2^{n+1}-2}{2^{n+1}+1}\cdot\frac{2^{n+2}+1}{2^{n+2}-2}\cdot\frac{1}{x} = \frac{2^{n+1}\left(1-\frac{2}{2^{n+1}}\right)}{2^{n+1}\left(1+\frac{1}{2^{n+1}}\right)}\cdot\frac{2^{n+2}\left(1+\frac{1}{2^{n+2}}\right)}{2^{n+2}\left(1-\frac{2}{2^{n+2}}\right)}\cdot\frac{1}{x}$$

$$\underset{n\to\infty}{\text{Lt}}\ \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}}\ \frac{1-\frac{1}{2^n}}{1+\frac{1}{2^{n+1}}}\cdot\frac{1+\frac{1}{2^{n+2}}}{1-\frac{1}{2^{n+1}}}\cdot\frac{1}{x} = \frac{1}{x}$$

$\therefore$ By D'Alembert's Ratio Test, $\sum u_n$ converges if $\frac{1}{x} > 1$ *i.e.*, $x < 1$

and diverges if $\frac{1}{x} < 1$ *i.e.*, $x > 1$.

When $x = 1$,
$$u_n = \frac{2^{n+1}-2}{2^{n+1}+1} = \frac{2^{n+1}\left(1-\frac{2}{2^{n+1}}\right)}{2^{n+1}\left(1+\frac{1}{2^{n+1}}\right)} = \frac{1-\frac{1}{2^n}}{1+\frac{1}{2^{n+1}}}$$

$\underset{n\to\infty}{\text{Lt}}\ u_n = 1 \neq 0 \Rightarrow \Sigma u_n$ does not converge. Being a series of +ve terms, it must diverge.

Hence Σu_n is convergent if $x < 1$ and divergent if $x \ge 1$.

Example 8. *Test for convergence the positive term series :*

$$1 + \frac{\alpha+1}{\beta+1} + \frac{(\alpha+1)(2\alpha+1)}{(\beta+1)(2\beta+1)} + \frac{(\alpha+1)(2\alpha+1)(3\alpha+1)}{(\beta+1)(2\beta+1)(3\beta+1)} + \ldots\ldots$$

Sol. Leaving the first term $u_n = \frac{(\alpha+1)(2\alpha+1)\ldots\ldots(n\alpha+1)}{(\beta+1)(2\beta+1)\ldots\ldots(n\beta+1)}$

$$u_{n+1} = \frac{(\alpha+1)(2\alpha+1)\ldots\ldots(n\alpha+1)[(n+1)\alpha+1]}{(\beta+1)(2\beta+1)\ldots\ldots(n\beta+1)[(n+1)\beta+1]}$$

$\therefore$
$$\frac{u_n}{u_{n+1}} = \frac{(n+1)\beta+1}{(n+1)\alpha+1} = \frac{\left(1+\frac{1}{n}\right)\beta+\frac{1}{n}}{\left(1+\frac{1}{n}\right)\alpha+\frac{1}{n}}$$

$$\underset{n\to\infty}{\text{Lt}}\ \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}}\ \frac{\left(1+\frac{1}{n}\right)\beta+\frac{1}{n}}{\left(1+\frac{1}{n}\right)\alpha+\frac{1}{n}} = \frac{\beta}{\alpha}$$

$\therefore$ By D'Alembert's Ratio Test, Σu_n converges if $\frac{\beta}{\alpha} > 1$ *i.e.* $\beta > \alpha > 0$

and diverges if $\frac{\beta}{\alpha} < 1$ *i.e.*, $\beta < \alpha$ or $\alpha > \beta > 0$

When $\alpha = \beta$, the Ratio Test fails.

When $\alpha = \beta$, $u_n = 1$ $\underset{n\to\infty}{\text{Lt}}\ u_n = 1 \neq 0$

$\Rightarrow \Sigma u_n$ does not converge. Being a series of +ve terms, it must diverge.

Hence the given series is convergent if $\beta > \alpha > 0$ and divergent if $\alpha \ge \beta > 0$.

TEST YOUR KNOWLEDGE

Discuss the convergence of the following series :

1. $1 + \frac{2^2}{2!} + \frac{3^2}{3!} + \frac{4^2}{4!} + \ldots\ldots$ to ∞
2. $1 + \frac{2!}{2^2} + \frac{3!}{3^3} + \frac{4!}{4^4} + \ldots\ldots$ to ∞
3. $\frac{1}{1+2} + \frac{2}{1+2^2} + \frac{3}{1+2^3} + \ldots\ldots$ to ∞
4. (i) $\frac{2!}{3} + \frac{3!}{3^2} + \frac{4!}{3^3} + \ldots\ldots$ to ∞ (ii) $\sum \frac{n^2}{3^n}$
5. (i) $\sum \frac{1}{n!}$ (ii) $\sum \frac{n!}{n^n}$
6. $\sum \frac{x^n}{n}, x > 0$
7. $\sum \frac{n}{n^2+1}x^n, x > 0$
8. $\sum \sqrt{\frac{n+1}{n^3+1}}\cdot x^n, x > 0$
9. $x + 2x^2 + 3x^3 + 4x^4 + \ldots\ldots$ to ∞
10. $1 + \frac{x}{2} + \frac{x^2}{5} + \frac{x^3}{10} + \ldots\ldots \frac{x^n}{n^2+1} + \ldots\ldots$ to ∞
11. $\frac{x}{1.3} + \frac{x^2}{3.5} + \frac{x^3}{5.7} + \ldots\ldots$ to ∞

ANSWERS

1. Convergent
2. Convergent
3. Convergent
4. (i) Divergent (ii) Convergent
5. (i) Convergent (ii) Convergent
6. Convergent for $x < 1$, divergent for $x \geq 1$
7. Convergent for $x < 1$, divergent for $x \geq 1$
8. Convergent for $x < 1$, divergent for $x \geq 1$
9. Convergent for $x < 1$, divergent for $x \geq 1$
10. Convergent for $x \leq 1$, divergent for $x > 1$
11. Convergent for $x \leq 1$, divergent for $x > 1$.

5.23. RAABE'S TEST

(P.T.U. May 2007)

Statement. *If Σu_n is a series of positive terms and* $\underset{n\to\infty}{Lt}\ n\left(\frac{u_n}{u_{n+1}} - 1\right) = l$, *then the series is convergent if $l > 1$ and divergent if $l < 1$.*

Proof. Let us compare the given series Σu_n with an auxiliary series $\Sigma v_n = \Sigma \frac{1}{n^p}$, which we know converges if $p > 1$ and diverges if $p \leq 1$.

Now,
$$\frac{v_n}{v_{n+1}} = \frac{\frac{1}{n^p}}{\frac{1}{(n+1)^p}} = \left(\frac{n+1}{n}\right)^p = \left(1 + \frac{1}{n}\right)^p = 1 + \frac{p}{n} + \frac{p(p-1)}{2!} \cdot \frac{1}{n^2} + \ldots\ldots$$

Case I. *Let $\Sigma v_n = \Sigma \frac{1}{n^p}$ be convergent,* so that $p > 1$.

Then Σu_n will also converge if $\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}}$

or if
$$\frac{u_n}{u_{n+1}} > 1 + \frac{p}{n} + \frac{p(p-1)}{2!} \cdot \frac{1}{n^2} + \ldots\ldots$$

or if
$$n\left(\frac{u_n}{u_{n+1}} - 1\right) > p + \frac{p(p-1)}{2!} \cdot \frac{1}{n} + \ldots\ldots$$

or if
$$\underset{n\to\infty}{\text{Lt}}\ n\left(\frac{u_n}{u_{n+1}} - 1\right) > p$$

or if
$$l > p$$

But p is itself greater than 1, $\therefore$ Σu_n is convergent if $l > 1$.

Case II. *Let Σv_n be divergent,* so that $p \leq 1$.

Then Σu_n will also diverge if $\frac{u_n}{u_{n+1}} < \frac{v_n}{v_{n+1}}$

or if
$$\frac{u_n}{u_{n+1}} < 1 + \frac{p}{n} + \frac{p(p-1)}{2!} \cdot \frac{1}{n^2} + \ldots\ldots$$

or if
$$n\left(\frac{u_n}{u_{n+1}} - 1\right) < p + \frac{p(p-1)}{2!} \cdot \frac{1}{n} + \ldots\ldots$$

$$\underset{n\to\infty}{\text{Lt}}\ n\left(\frac{u_n}{u_{n+1}} - 1\right) < p$$

or if
$$l < p.$$

But p itself ≤ 1. Thus the given series Σu_n diverges if $l < 1$. This proves the result.

Note 1. If $\underset{n\to\infty}{\text{Lt}}\ n\left(\frac{u_n}{u_{n+1}} - 1\right) = 1$, then Raabe's test fails.

Note 2. Raabe's test is used when D'Alembert's Ratio test fails and when in the ratio test, $\frac{u_n}{u_{n+1}}$ does not involve the number e. When $\frac{u_n}{u_{n+1}}$ involves e, we apply logarithmic test after the ratio test and not Raabe's test.

5.24. LOGARITHMIC TEST

(P.T.U. May 2007)

Statement. *A positive term series Σu_n converges or diverges according as*

$$\underset{n\to\infty}{Lt}\ n \log \frac{u_n}{u_{n+1}} > 1 \text{ or } < 1.$$

Proof. Let us compare the given series Σu_n with an auxiliary series $\Sigma v_n = \Sigma \frac{1}{n^p}$, which we know converges if $p > 1$ and diverges if $p \leq 1$.

Now,
$$\frac{v_n}{v_{n+1}} = \frac{(n+1)^p}{n^p} = \left(1 + \frac{1}{n}\right)^p.$$

Case I. *Let Σv_n be convergent,* so that $p > 1$.

Then Σu_n will also be convergent if $\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}}$

or if
$$\frac{u_n}{u_{n+1}} > \left(1 + \frac{1}{n}\right)^p$$

or if
$$\log \frac{u_n}{u_{n+1}} > \log\left(1 + \frac{1}{n}\right)^p = p \log\left(1 + \frac{1}{n}\right)$$

or if
$$\log \frac{u_n}{u_{n+1}} > p\left[\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \ldots\ldots\right] \qquad \left[\because \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots\ldots\right]$$

or if
$$n \log \frac{u_n}{u_{n+1}} > p\left[1 - \frac{1}{2n} + \frac{1}{3n^2} - \ldots\ldots\right]$$

or if
$$\underset{n\to\infty}{\text{Lt}}\ n \log \frac{u_n}{u_{n+1}} > p$$

or if
$$\underset{n\to\infty}{\text{Lt}}\ n \log \frac{u_n}{u_{n+1}} > 1 \qquad | \because p > 1$$

Case II. *Let Σv_n be divergent*, so that $p \le 1$.

Then Σu_n also diverges if $\dfrac{u_n}{u_{n+1}} < \dfrac{v_n}{v_{n+1}}$

or if $\quad \log \dfrac{u_n}{u_{n+1}} < \log \dfrac{v_n}{v_{n+1}}$

or if $\quad \log \dfrac{u_n}{u_{n+1}} < \log \left(1+\dfrac{1}{n}\right)^p = p \log \left(1+\dfrac{1}{n}\right)$

or if $\quad \log \dfrac{u_n}{u_{n+1}} < p\left(\dfrac{1}{n} - \dfrac{1}{2n^2} + \dfrac{1}{3n^3} - \ldots\ldots\right)$

or if $\quad n \log \dfrac{u_n}{u_{n+1}} < p\left[1 - \dfrac{1}{2n} + \dfrac{1}{3n^2} - \ldots\ldots\right]$

or if $\quad \underset{n\to\infty}{\text{Lt}}\; n \log \dfrac{u_n}{u_{n+1}} < p$

or if $\quad \underset{n\to\infty}{\text{Lt}}\; n \log \dfrac{u_n}{u_{n+1}} < 1 \qquad [\because p \le 1$

Thus the series Σu_n converges or diverges according as $\underset{n\to\infty}{\text{Lt}}\; n \log \dfrac{u_n}{u_{n+1}} > 1$ or < 1.

Note 1. The test fails if $\underset{n\to\infty}{\text{Lt}}\; n \log \dfrac{u_n}{u_{n+1}} = 1$.

Note 2. The test is applied after the failure of Ratio test and generally when in Ratio test, $\dfrac{u_n}{u_{n+1}}$ involves 'e'.

5.25. GAUSS TEST

Statement. *If for the series Σu_n of positive terms, $\dfrac{u_n}{u_{n+1}}$ can be expanded in the form*

$$\frac{u_n}{u_{n+1}} = 1 + \frac{\lambda}{n} + O\left(\frac{1}{n^2}\right)$$

then Σu_n converges if $\lambda > 1$ and diverges if $\lambda \le 1$.

Note. The test never fails as we know that the series diverges for $\lambda = 1$. Moreover the test is applied after the failure of ratio test and when it is possible to expand $\dfrac{u_n}{u_{n+1}}$ in powers of $\dfrac{1}{n}$ by Binomial Theorem or by any other method.

[Binomial theorem is $(1+x)^n = 1 + nx + \dfrac{n(n-1)}{2!}x^2 + \ldots\ldots \infty$, where $|x| < 1$]

ILLUSTRATIVE EXAMPLES

Example 1. *Discuss the convergence of the series :* $\dfrac{1}{2} + \dfrac{1.3}{2.4} + \dfrac{1.3.5}{2.4.6} + \ldots\ldots$

Sol. Here
$$u_n = \frac{1.3.5\ldots\ldots(2n-1)}{2.4.6.\ldots\ldots 2n}.$$

$\therefore$
$$u_{n+1} = \frac{1.3.5\ldots\ldots(2n-1)(2n+1)}{2.4.6.\ldots\ldots(2n)(2n+2)}$$

$\therefore$
$$\frac{u_n}{u_{n+1}} = \frac{1.3.5.\ldots\ldots(2n-1)}{2.4.6\ldots\ldots 2n} \times \frac{2.4.6.\ldots\ldots 2n(2n+2)}{1.3.5\ldots\ldots(2n-1)(2n+1)}$$

$$= \frac{2n+2}{2n+1} = \frac{1+\frac{1}{n}}{1+\frac{1}{2n}} \to 1 \text{ as } n \to \infty.$$

$\therefore$ D'Alembert's Ratio test fails.

$$n\left[\frac{u_n}{u_{n+1}} - 1\right] = n\left[\frac{2n+2}{2n+1} - 1\right] = \frac{n}{2n+1} = \frac{1}{2+\frac{1}{n}}$$

$\therefore$
$$\underset{n\to\infty}{\text{Lt}}\; n\left[\frac{u_n}{u_{n+1}} - 1\right] = \frac{1}{2} < 1.$$

$\therefore$ By Raabe's test, Σu_n diverges.

Example 2. *Discuss the convergence of the following series :*

(i) $\dfrac{1^2}{2^2} + \dfrac{1^2.2^2}{2^2.4^2} + \dfrac{1^2.3^2.5^2}{2^2.4^2.6^2} + \ldots\ldots$

(ii) $1 + \dfrac{2^2}{3^2} + \dfrac{2^2.4^2}{3^2.5^2} + \dfrac{2^2.4^2.6^2}{3^2.5^2.7^2} + \ldots\ldots \infty.$ **(P.T.U., May 2004)**

Sol. (i) Here
$$u_n = \frac{1^2.3^2.5^2\ldots\ldots(2n-1)^2}{2^2.4^2.6^2\ldots\ldots(2n)^2}$$

$$u_{n+1} = \frac{1^2.3^2.5^2\ldots\ldots(2n-1)^2(2n+1)^2}{2^2.4^2.6^2\ldots\ldots(2n)^2(2n+2)^2}$$

$\therefore$
$$\frac{u_n}{u_{n+1}} = \frac{(2n+2)^2}{(2n+1)^2} = \frac{4n^2\left(1+\frac{1}{n}\right)^2}{4n^2\left(1+\frac{1}{2n}\right)^2} = \frac{\left(1+\frac{1}{n}\right)^2}{\left(1+\frac{1}{2n}\right)^2}$$

$\therefore$
$$\underset{n\to\infty}{\text{Lt}}\; \frac{u_n}{u_{n+1}} = 1.$$

Hence the ratio test fails

$$n\left(\frac{u_n}{u_{n+1}}-1\right)=n\left[\frac{(2n+2)^2}{(2n+1)^2}-1\right]$$

$$=n\left[\frac{\cancel{4n^2}+8n+4-(\cancel{4n^2}+4n+1)}{(2n+1)^2}\right]=n\,\frac{(4n+3)}{(2n+1)^2}=\frac{4n^2+3n}{(2n+1)^2}$$

$$=\frac{1+\frac{3}{4n}}{\left(1+\frac{1}{2n}\right)^2}\to 1 \text{ as } n\to\infty$$

$\therefore$ Raabe's test also fails.

Now,
$$\frac{u_n}{u_{n+1}}=\frac{(2n+2)^2}{(2n+1)^2}=\frac{\left(1+\frac{1}{n}\right)^2}{\left(1+\frac{1}{2n}\right)^2}=\left(1+\frac{1}{n}\right)^2\left(1+\frac{1}{2n}\right)^{-2}$$

$$=\left(1+\frac{2}{n}+\frac{1}{n^2}\right)\left(1-\frac{2}{2n}+\frac{3}{4n^2}-\ldots\ldots\right)=1+\frac{1}{n}+\frac{1}{n^2}\left(1-2+\frac{3}{4}\right)+\ldots$$

$$=1+\frac{1}{n}-\frac{1}{4n^2}+\ldots\ldots=1+\frac{1}{n}+O\left(\frac{1}{n^2}\right)$$

Comparing it with $\frac{u_n}{u_{n+1}}=1+\frac{\lambda}{n}+O\left(\frac{1}{n^2}\right)$

we have $\lambda = 1$. Thus by Gauss test, the series Σu_n diverges.

Note that when D'Alembert ratio test fails, we can directly apply Gauss test.

(*ii*) Here
$$u_n=\frac{2^2.4^2.6^2\ldots\ldots(2n)^2}{3^2.5^2.7^2\ldots\ldots(2n+1)^2}$$

$$u_{n+1}=\frac{2^2.4^2.6^2\ldots\ldots(2n)^2(2n+2)^2}{3^2.5^2.7^2\ldots\ldots(2n+1)^2(2n+3)^2}$$

$$\frac{u_n}{u_{n+1}}=\frac{2^2.4^2.6^2\ldots\ldots(2n)^2}{3^2.5^2.7^2\ldots\ldots(2n+1)^2}\times\frac{3^2.5^2.7^2\ldots\ldots(2n+1)^2(2n+3)^2}{2^2.4^2.6^2\ldots\ldots(2n)^2(2n+2)^2}$$

$$=\frac{(2n+3)^2}{(2n+2)^2}=\frac{\left(1+\frac{3}{2n}\right)^2}{\left(1+\frac{2}{2n}\right)^2}=\frac{\left(1+\frac{3}{2n}\right)^2}{\left(1+\frac{1}{n}\right)^2}$$

$$\underset{n\to\infty}{\text{Lt}}\frac{u_n}{u_{n+1}}=\underset{n\to\infty}{\text{Lt}}\frac{\left(1+\frac{3}{2n}\right)^2}{\left(1+\frac{1}{n}\right)^2}=1$$

$\therefore$ D'Alemberts' Ratio Test cannot be applied

Apply Gauss Test

$$\frac{u_n}{u_{n+1}}=\left(1+\frac{3}{2n}\right)^2\left(1+\frac{1}{n}\right)^{-2}=\left(1+\frac{3}{n}+\frac{9}{4n^2}\right)\left[1-\frac{2}{n}+O\left(\frac{1}{n^2}\right)\right]$$

$$=1-\frac{2}{n}+\frac{3}{n}+O\left(\frac{1}{n^2}\right)=1+\frac{1}{n}+O\left(\frac{1}{n^2}\right)$$

Compare it with $\frac{u_n}{u_{n+1}}=1+\frac{\lambda}{n}+O\left(\frac{1}{n^2}\right)$

We have $\lambda=1$

$\therefore$ By Gauss test given series is divergent.

Example 3. *Discuss the convergence of the series:*

$$x+\frac{2^2x^2}{2!}+\frac{3^3x^3}{3!}+\frac{4^4x^4}{4!}+\ldots\infty.$$ **(P.T.U. May 2007, May 2008)**

Sol. Here
$$u_n=\frac{n^nx^n}{n!}\ ;\ u_{n+1}=\frac{(n+1)^{n+1}x^{n+1}}{(n+1)!}$$

$\therefore$
$$\frac{u_n}{u_{n+1}}=\frac{n^nx^n}{n!}\cdot\frac{(n+1)!}{(n+1)^{n+1}x^{n+1}}=\frac{n^n}{(n+1)^n}\cdot\frac{1}{x}=\frac{1}{\left(1+\frac{1}{n}\right)^n}\cdot\frac{1}{x}$$

$$\underset{n\to\infty}{\text{Lt}}\frac{u_n}{u_{n+1}}=\underset{n\to\infty}{\text{Lt}}\frac{1}{\left(1+\frac{1}{n}\right)^n}\cdot\frac{1}{x}=\frac{1}{e}\cdot\frac{1}{x}\qquad\because\ \underset{n\to\infty}{\text{Lt}}\left(1+\frac{1}{n}\right)^n=e$$

$\therefore$ By Ratio test Σu_n converges if $\frac{1}{ex}>1$ or $x<\frac{1}{e}$ and diverges for $x>\frac{1}{e}$

for $x=\frac{1}{e}$
$$\frac{u_n}{u_{n+1}}=\frac{1}{\left(1+\frac{1}{n}\right)^n}.e.$$

As $\frac{u_n}{u_{n+1}}$ involves e

$\therefore$ Apply logarithmic test

$$n\log\frac{u_n}{u_{n+1}}=n.\log\frac{e}{\left(1+\frac{1}{n}\right)^n}$$

$$=n\left\{\log e-\log\left(1+\frac{1}{n}\right)^n\right\}$$

$$=n\left\{1-n\log\left(1+\frac{1}{n}\right)\right\}$$

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$$= n - n^2\left\{\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \frac{1}{4n^4} + \dots \infty\right\}$$

$$\left[\because \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \infty\right]$$

$$= n - n + \frac{1}{2} - \frac{1}{3n} + \frac{1}{4n^2} \dots \infty$$

$$= \frac{1}{2} - \frac{1}{3n} + \frac{1}{4n} - \frac{1}{5n^3} + \dots \infty$$

$$\underset{n\to\infty}{\text{Lt}}\ n\frac{u_n}{u_{n+1}} = \frac{1}{2} < 1$$

$\therefore$ Σu_n diverges for $x = \frac{1}{e}$.

Hence Σu_n converges for $x < \frac{1}{e}$ and diverges for $x \geq \frac{1}{e}$.

Example 4. *Discuss the convergence of the series :*

$$\frac{x}{1} + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1.3}{2.4}\cdot\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\cdot\frac{x^7}{7} + \dots\dots (x > 0).$$

Sol. Neglecting the first term, we have $u_n = \frac{1.3.5\dots\dots(2n-1)}{2.4.6\dots\dots(2n)}\cdot\frac{x^{2n+1}}{2n+1}$

and
$$u_{n+1} = \frac{1.3.5\dots\dots(2n-1)(2n+1)}{2.4.6\dots\dots(2n)(2n+2)}\cdot\frac{x^{2n+3}}{2n+3}$$

$\therefore$
$$\frac{u_n}{u_{n+1}} = \frac{2n+2}{2n+1}\cdot\frac{2n+3}{2n+1}\cdot\frac{1}{x^2} = \frac{2n\left(1+\frac{1}{n}\right).2n\left(1+\frac{3}{2n}\right)}{2n\left(1+\frac{1}{2n}\right).2n\left(1+\frac{1}{2n}\right)}\cdot\frac{1}{x^2}$$

$$= \frac{\left(1+\frac{1}{n}\right)\left(1+\frac{3}{2n}\right)}{\left(1+\frac{1}{2n}\right)^2}\cdot\frac{1}{x^2}$$

$$\underset{n\to\infty}{\text{Lt}}\frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}}\frac{\left(1+\frac{1}{n}\right)\left(1+\frac{3}{2n}\right)}{\left(1+\frac{1}{2n}\right)^2}\cdot\frac{1}{x^2} = \frac{1}{x^2}$$

$\therefore$ By Ratio Test, Σu_n is convergent if $\frac{1}{x^2} > 1$ *i.e.,* $x^2 < 1$ and divergent if $\frac{1}{x^2} < 1$ *i.e.,* $x^2 > 1$.

If $x^2 = 1$, then Ratio Test fails.

When $x^2 = 1$, we have $\frac{u_n}{u_{n+1}} = \frac{(2n+2)(2n+3)}{(2n+1)^2} = \frac{4n^2+10n+6}{4n^2+4n+1}$

$$\underset{n\to\infty}{\text{Lt}}\ n\left(\frac{u_n}{u_{n+1}} - 1\right) = \underset{n\to\infty}{\text{Lt}}\ n\left(\frac{4n^2+10n+6}{4n^2+4n+1} - 1\right)$$

$$= \underset{n\to\infty}{\text{Lt}}\frac{6n^2+5n}{4n^2+4n+1} = \underset{n\to\infty}{\text{Lt}}\frac{6+\frac{5}{n}}{4+\frac{4}{n}+\frac{1}{n^2}} = \frac{6}{4} = \frac{3}{2} > 1.$$

$\therefore$ By Raabe's Test, the series converges.

Hence Σu_n is convergent if $x^2 \leq 1$ and divergent if $x^2 > 1$.

Example 5. *Test for convergence the following series :*

$$1 + \frac{2x}{2!} + \frac{3^2x^2}{3!} + \frac{4^3x^3}{4!} + \frac{5^4x^4}{5!} + \dots\dots + \infty +$$ **(P.T.U., May 2005, May 2006)**

Sol. Neglecting first term we have

$$u_n = \frac{(n+1)^n}{(n+1)!}x^n$$

$$u_{n+1} = \frac{(n+2)^{n+1}}{(n+2)!}x^{n+1} \quad \therefore \quad \frac{u_n}{u_{n+1}} = \frac{(n+1)^n}{(n+1)!}x^n\,\frac{(n+2)!}{(n+2)^{n+1}.x^{n+1}}$$

$$\frac{u_n}{u_{n+1}} = \frac{(n+1)^n}{x}\cdot\frac{(n+2)}{(n+2)^{n+1}} = \frac{(n+1)^n}{(n+2)^n}\cdot\frac{1}{x} = \frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{2}{n}\right)^n}\cdot\frac{1}{x}$$

$$\underset{n\to\infty}{\text{Lt}}\frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}}\frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{2}{n}\right)^n}\cdot\frac{1}{x} = \underset{n\to\infty}{\text{Lt}}\frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{2}{n}\right)^{\frac{n}{2}\cdot 2}}\cdot\frac{1}{x}$$

$$= \frac{e}{e^2}\cdot\frac{1}{x} = \frac{1}{ex} \quad \because \quad \underset{n\to\infty}{\text{Lt}}\left(1+\frac{a}{n}\right)^n = e^a$$

$\therefore$ By D'Alembert's Ratio test the given series is convergent if $\frac{1}{ex} > 1$ *i.e.,* $x < \frac{1}{e}$ and divergent if

When $x = \frac{1}{e}$, Ratio test fails

$$\frac{u_n}{u_{n+1}} = \frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{2}{n}\right)^n}\cdot e$$

Since $\frac{u_n}{u_{n+1}}$ involves e $\therefore$ Apply logarithm test

$$\log \frac{u_n}{u_{n+1}} = n\log\left(1+\frac{1}{n}\right) - n\log\left(1+\frac{2}{n}\right) + \log e$$

$$= n\left[\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \frac{1}{4n^4} + \ldots\ldots\right] - n\left[\frac{2}{n} - \frac{1}{2}\frac{4}{n^2} + \frac{1}{3}\cdot\frac{8}{n^3}\ldots\ldots + 1\right] + 1$$

$$= \left[1 - \frac{1}{2n} + \frac{1}{3n^2} - \frac{1}{4n^3} + \ldots\ldots\infty\right] + \left[-2 + \frac{2}{n} - \frac{8}{3n^2} + \frac{4}{n^3} - \ldots\ldots\infty\right] + 1$$

$$= \frac{3}{2n} - \frac{7}{3n^2} + \frac{15}{4n^3}\ \ldots\ldots\infty$$

$$n\log\frac{u_n}{u_{n+1}} = \frac{3}{2} - \frac{7}{3n} + \frac{15}{4n^2}\ \ldots\ldots\infty$$

$$\underset{n\to\infty}{\text{Lt}}\ n\log\frac{u_n}{u_{n+1}} = \frac{3}{2} > 1$$

$\therefore$ Series is convergent

$\therefore$ By logarithm test the given series is convergent when $x \le \frac{1}{e}$ and divergent for $x > \frac{1}{e}$.

Example 6. *Test for convergence the series:*

$$\frac{a+x}{1!} + \frac{(a+2x)^2}{2!} + \frac{(a+3x)^3}{3!} + \ldots\infty.$$

Sol.
$$u_n = \frac{(a+nx)^n}{n!};\ u_{n+1} = \frac{[a+(n+1)x]^{n+1}}{(n+1)!}$$

$$\frac{u_n}{u_{n+1}} = \frac{(a+nx)^n}{n!}\cdot\frac{(n+1)!}{[a+(n+1)x]^{n+1}} = (n+1)\frac{(nx)^n\left(1+\frac{a}{nx}\right)^n}{[(n+1)x]^{n+1}\left[1+\frac{a}{(n+1)x}\right]^{n+1}}$$

$$= \frac{n^n}{(n+1)^n}\cdot\frac{1}{x}\,\frac{\left(1+\frac{a}{nx}\right)^{\frac{nx}{a}\cdot\frac{a}{x}}}{\left[1+\frac{a}{(n+1)x}\right]^{\frac{(n+1)x}{a}\cdot\frac{a}{x}}}$$

$$= \frac{1}{\left(1+\frac{1}{n}\right)^n}\,\frac{\left[\left(1+\frac{a}{nx}\right)^{\frac{nx}{a}}\right]^{a/x}}{\left[\left(1+\frac{a}{(n+1)x}\right)^{\frac{(n+1)x}{a}}\right]^{a/x}}\cdot\frac{1}{x}$$

$$\underset{n\to\infty}{\text{Lt}}\ \frac{u_n}{u_{n+1}} = \frac{1}{e}\,\frac{e^{a/x}}{e^{a/x}}\cdot\frac{1}{x} = \frac{1}{ex} \qquad \because\ \underset{n\to\infty}{\text{Lt}}\left(1+\frac{1}{n}\right)^n = e.$$

$\therefore$ By Ratio test $\Sigma\, u_n$ converges for $\frac{1}{ex} > 1$ i.e., $x < \frac{1}{e}$ and diverges for $x > \frac{1}{e}$, where $x = \frac{1}{e}$;

$\frac{u_n}{u_{n+1}}$ involves e $\therefore$ We apply logarithmic test)

$\therefore$
$$n\log\frac{u_n}{u_{n+1}} = n\log\left\{\frac{1}{\left(1+\frac{1}{n}\right)^n}\,\frac{\left(1+\frac{ae}{n}\right)^{\frac{n}{ae}ae}}{\left(1+\frac{ae}{n+1}\right)^{\frac{n+1}{ae}.ae}}\cdot e\right\}$$

$$= n\log\left\{\frac{1}{\left(1+\frac{1}{n}\right)^n}\,\frac{\left(1+\frac{ae}{n}\right)^n}{\left(1+\frac{ae}{n+1}\right)^{n+1}}\cdot e\right\}$$

$$= n\left\{\log\left(1+\frac{1}{n}\right)^{-n} + \log\left(1+\frac{ae}{n}\right)^n - \log\left(1+\frac{ae}{n+1}\right)^{(n+1)} + \log e\right\}$$

$$= n\left\{-n\log\left(1+\frac{1}{n}\right) + n\log\left(1+\frac{ae}{n}\right) - (n+1)\log\left(1+\frac{ae}{n+1}\right) + 1\right\}$$

$$= n\left\{-n\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} + \ldots\infty\right) + n\left(\frac{ae}{n} - \frac{a^2e^2}{2n^2} + \frac{a^3e^3}{3n^3}\ldots\right)\right.$$
$$\left. - (n+1)\left(\frac{ae}{n+1} - \frac{a^2e^2}{2(n+1)^2} + \frac{a^3e^3}{3(n+1)^3}\ldots\infty\right)\right\}$$

$$= \left(-n + \frac{1}{2} - \frac{1}{3n} + \ldots\infty\right) + \left(aen - \frac{a^2e^2}{2} + \frac{a^3e^3}{3n}\ldots\right)$$
$$- nae + \frac{na^2e^2}{2(n+1)} - \frac{na^3e^3}{3(n+1)^2} + \ldots + n$$

$$\underset{n\to\infty}{\text{Lt}}\ n\log\frac{u_n}{u_{n+1}} = \frac{1}{2} - \frac{a^2e^2}{2} + \frac{a^2e^2}{2} = \frac{1}{2} < 1 \quad \therefore\ \Sigma\, u_n \text{ is divergent at } x = \frac{1}{e}$$

$\therefore$ $\Sigma\, u_n$ converges for $x < \frac{1}{e}$ and diverges for $x \ge \frac{1}{e}$.

Example 7. *Discuss the convergence of the series :* $1 + \frac{x}{2} + \frac{2!}{3^2}x^2 + \frac{3!}{4^3}x^3 + \frac{4!}{5^4}x^4 + \ldots\ldots$

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Sol. Neglecting the first term, we have

$$u_n = \frac{n!}{(n+1)^n}\, x^n \quad \text{and} \quad u_{n+1} = \frac{(n+1)!}{(n+2)^{n+1}} \cdot x^{n+1}$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}} \frac{n!}{(n+1)^n} \cdot \frac{(n+2)^{n+1}}{(n+1)!} \cdot \frac{1}{x}$$

$$= \underset{n\to\infty}{\text{Lt}} \frac{1}{n^n\left(1+\frac{1}{n}\right)^n} \cdot \frac{n^{n+1}\left(1+\frac{2}{n}\right)^{n+1}}{(n+1)} \cdot \frac{1}{x}$$

$$= \underset{n\to\infty}{\text{Lt}} \frac{\left(1+\frac{2}{n}\right)^n\left(1+\frac{2}{n}\right)}{\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right)} \cdot \frac{1}{x} = \frac{e^2}{e} \cdot \frac{1}{x} = \frac{e}{x}$$

$$\left[\because \underset{n\to\infty}{\text{Lt}} \left(1+\frac{a}{n}\right)^n = \underset{n\to\infty}{\text{Lt}} \left[\left(1+\frac{a}{n}\right)^{n/a}\right]^a = e^a\right]$$

$\therefore$ By D'Alembert's ratio test, the series converges if $\frac{e}{x} > 1$ or if $x < e$ and diverges if $\frac{e}{x} < 1$ or if $x > e$.

If $x = e$, the ratio test fails, $\because \underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = 1$.

Now, when $x = e$

$$\frac{u_n}{u_{n+1}} = \frac{\left(1+\frac{2}{n}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^{n+1}} \cdot \frac{1}{e}.$$

Since the expression $\frac{u_n}{u_{n+1}}$ involves the number e, so we do not apply Raabe's test but apply logarithmic test.

$$\therefore \quad \log \frac{u_n}{u_{n+1}} = (n+1)\log\left(1+\frac{2}{n}\right) - (n+1)\log\left(1+\frac{1}{n}\right) - \log e$$

$$= (n+1)\left[\log\left(1+\frac{2}{n}\right) - \log\left(1+\frac{1}{n}\right)\right] - 1$$

$$= (n+1)\left[\left(\frac{2}{n} - \frac{1}{2}\cdot\frac{4}{n^2} + \frac{1}{3}\cdot\frac{8}{n^3} - \ldots\ldots\right) - \left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} + \ldots\ldots\right)\right] - 1$$

$$= (n+1)\left[\frac{1}{n} - \frac{3}{2n^2} + \ldots\ldots\right] - 1$$

$$= 1 - \frac{3}{2n} + \frac{1}{n} - \frac{3}{2n^2} + \ldots\ldots - 1 = -\frac{1}{2n} - \frac{3}{2n^2} + \ldots\ldots$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}}\; n \log \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}}\; n\left[-\frac{1}{2n} - \frac{3}{2n^2} + \ldots\ldots\right] = \underset{n\to\infty}{\text{Lt}}\left(-\frac{1}{2} - \frac{3}{2n} + \ldots\ldots\right) = -\tfrac{1}{2} < 1$$

$\therefore$ By log test, the series *diverges.*

Hence the given series Σu_n converges if $x < e$ and diverges if $x \geq e$.

Example 8. *Discuss the convergence of the series:*

$$\frac{a}{b} + \frac{a(a+d)}{b(b+d)}x + \frac{a(a+d)(a+2d)}{b(b+d)(b+2d)}x^2 + \ldots\infty. \qquad (a > 0, b > 0, x > 0)$$

Sol.

$$u_n = \frac{a(a+d)(a+2d)\ldots(a+\overline{n-1}\,d)}{b(b+d)(b+2d)\ldots(b+\overline{n-1}\,d)}\, x^n$$

$$u_{n+1} = \frac{a(a+b)(a+2d)\ldots(a+\overline{n-1}\,d)(a+nd)}{b(b+d)(b+2d)\ldots(b+\overline{n-1}\,d)(b+nd)}\, x^{n+1}$$

$$\frac{u_n}{u_{n+1}} = \frac{b+nd}{a+nd} \cdot \frac{1}{x}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}} \frac{nd\left(1+\frac{b}{nd}\right)}{nd\left(1+\frac{a}{nd}\right)} \cdot \frac{1}{x} = \underset{n\to\infty}{\text{Lt}} \frac{1+\frac{b}{nd}}{1+\frac{a}{nd}} \cdot \frac{1}{x} = \frac{1}{x}.$$

By Ratio test Σu_n converges if $\frac{1}{x} > 1$ *i.e.*, $x < 1$ and diverges if $x > 1$.

when $x = 1$, Ratio test fails

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{1+\frac{b}{nd}}{1+\frac{a}{nd}} = \left(1+\frac{b}{nd}\right)\left(1+\frac{a}{nd}\right)^{-1} = \left(1+\frac{b}{nd}\right)\left(1-\frac{a}{nd}+\ldots\right)$$

$$= 1 + \frac{b-a}{d} \cdot \frac{1}{n} + O\left(\frac{1}{n^2}\right)$$

By Gauss test, Σu_n converges if $\frac{b-a}{d} > 1$ *i.e.*, $b > a + d$ and diverges if $b \leq a + d$.

$\therefore$ Σu_n converges if $x > 1$ and diverges when $x < 1$ when $x = 1$ then Σu_n converges if $b > a + d$ and diverges if $b \leq a + d$.

Example 9. *Discuss the convergence of the series :*

$$1 + \frac{\alpha\beta}{1.\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}x^2 + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1.2.3.\gamma(\gamma+1)(\gamma+2)}x^3 + \ldots\ldots$$

(P.T.U., Dec. 2003)

Sol. Neglecting the first term,

$$u_n = \frac{\alpha(\alpha+1)\ldots\ldots(\alpha+n-1)\,\beta(\beta+1)\ldots\ldots(\beta+n-1)}{1.2.3\ldots\ldots n.\gamma(\gamma+1)\ldots\ldots(\gamma+n-1)} \cdot x^n$$

$$u_{n+1} = \frac{\alpha(\alpha+1)\ldots\ldots(\alpha+n-1)(\alpha+n)\,\beta(\beta+1)\ldots\ldots(\beta+n-1)(\beta+n)}{1.2.3\ldots\ldots n(n+1)\,\gamma(\gamma+1)(\gamma+2)\ldots\ldots(\gamma+n-1)(\gamma+n)} \cdot x^{n+1}$$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{(n+1)(\gamma+n)}{(\alpha+n)(\beta+n)} \cdot \frac{1}{x} = \frac{\left(1+\frac{1}{n}\right)\left(1+\frac{\gamma}{n}\right)}{\left(1+\frac{\alpha}{n}\right)\left(1+\frac{\beta}{n}\right)} \cdot \frac{1}{x}$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \frac{1}{x}.$$

$\therefore$ By D'Alembert's Ratio test the series Σu_n converges if $\frac{1}{x} > 1$

i.e., if $x < 1$ and diverges if $\frac{1}{x} < 1$ *i.e.*, if $x > 1$.

If $x = 1$, $\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = 1$ $\therefore$ Ratio test fails.

Putting $x = 1$ in $\frac{u_n}{u_{n+1}}$, we have $\frac{u_n}{u_{n+1}} = \frac{\left(1+\frac{1}{n}\right)\left(1+\frac{\gamma}{n}\right)}{\left(1+\frac{\alpha}{n}\right)\left(1+\frac{\beta}{n}\right)}$

$$= \left(1+\frac{1}{n}\right)\left(1+\frac{\gamma}{n}\right)\left(1+\frac{\alpha}{n}\right)^{-1}\left(1+\frac{\beta}{n}\right)^{-1} \quad \text{[Expand by Binomial Theorem]}$$

$$= \left(1+\frac{1}{n}\right)\left(1+\frac{\gamma}{n}\right)\left(1-\frac{\alpha}{n}+\frac{\alpha^2}{n^2}+\ldots\ldots\right)\left(1-\frac{\beta}{n}+\frac{\beta^2}{n^2}\ldots\ldots\right)$$

$$= \left(1+\frac{1}{n}+\frac{\gamma}{n}+\frac{\gamma}{n^2}\right)\left(1-\frac{\alpha}{n}-\frac{\beta}{n}+\frac{\alpha\beta}{n^2}+\frac{\alpha^2}{n^2}+\frac{\beta^2}{n^2}\ldots\ldots\right)$$

$$= 1 + \frac{1}{n}(1+\gamma-\alpha-\beta) + O\left(\frac{1}{n^2}\right).$$

$\therefore$ By Gauss test, the series Σu_n converges if $1+\gamma-\alpha-\beta > 1$ *i.e.*, if $\gamma > \alpha+\beta$ and diverges if $1+\gamma-\alpha-\beta \le 1$ *i.e.*, if $\gamma \le \alpha+\beta$.

Thus the given series *converges if $x < 1$ and diverges if $x > 1$. If $x = 1$, then the series converges if $\gamma > \alpha+\beta$ and diverges if $\gamma \le \alpha+\beta$.*

Example 10. *Discuss the convergence of the series*

$$\sum \frac{n!}{x(x+1)(x+2)\ldots(x+n-1)}.$$

Sol.
$$u_n = \frac{n!}{x(x+1)(x+2)\ldots(x+n-1)}$$

$$u_{n+1} = \frac{(n+1)!}{x(x+1)(x+2)\ldots(x+n-1)(x+n)}$$

$$\frac{u_n}{u_{n+1}} = \frac{x+n}{n+1} = \frac{1+\frac{x}{n}}{1+\frac{1}{n}}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}} \frac{1+x/n}{1+\frac{1}{n}} = 1$$

Ratio test fails. We apply Gauss test

$$\therefore \quad \frac{u_n}{u_{n+1}} = \left(1+\frac{x}{n}\right)\left(1+\frac{1}{n}\right)^{-1} = \left(1+\frac{x}{n}\right)\left(1-\frac{1}{n}+\ldots\right)$$

$$\therefore \quad = 1 + \frac{x-1}{n} + O\left(\frac{1}{n^2}\right).$$

$$= 1 + (x-1)\cdot\frac{1}{n} + O\left(\frac{1}{n^2}\right).$$

By Gauss test Σu_n converges if $x-1 > 1$ and diverges for $x-1 \le 1$ *i.e.*, Σu_n converges for $x > 2$ and diverges for $x \le 2$.

Example 11. *Verify the series* $\sum \frac{4.7.10\ldots\ldots(3n+1)}{1.2.3\ldots\ldots n} x^n$ *is convergent or divergent.*

(P.T.U., Dec. 2006, May 2006)

Sol. Here
$$u_n = \frac{4.7.10\ldots\ldots(3n+1)}{1.2.3\ldots\ldots n} x^n$$

$$u_{n+1} = \frac{4.7.10\ldots\ldots(3n+1)(3n+4)}{1.2.3\ldots\ldots n.(n+1)} x^{n+1}$$

$$\frac{u_n}{u_{n+1}} = \frac{n+1}{3n+4}\cdot\frac{1}{x} = \frac{1+\frac{1}{n}}{3+\frac{4}{n}}\cdot\frac{1}{x}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}} \frac{1+\frac{1}{n}}{3+\frac{4}{n}}\cdot\frac{1}{x} = \frac{1}{3x}$$

$\therefore$ By D'Alembert's ratio test, series converges if $\frac{1}{3x} > 1$

i.e. if $x < \frac{1}{3}$ and diverges for $x > \frac{1}{3}$.

When $x = \frac{1}{3}$ Ratio test fails, so apply Gauss test

$$\frac{u_n}{u_{n+1}} = \frac{\left(1+\frac{1}{n}\right)}{3+\frac{4}{n}}.3 = \frac{1+\frac{1}{n}}{3\left(1+\frac{4}{3n}\right)}.3 = \left(1+\frac{1}{n}\right)\left(1+\frac{4}{3n}\right)^{-1}$$

$$= \left(1+\frac{1}{n}\right)\left(1-\frac{4}{3n}+O\left(\frac{1}{n^2}\right)\right) = 1 - \frac{1}{3n} + O\left(\frac{1}{n^2}\right)$$

Compare it with $\frac{u_n}{u_{n+1}} = 1 + \lambda \cdot \frac{1}{n} + O\left(\frac{1}{n^2}\right)$

$\lambda = -\frac{1}{3} < 1$ ∴ Series is divergent.

Hence the given series is convergent for $x < \frac{1}{3}$ and divergent for $x \geq \frac{1}{3}$.

Example 12. *Verify the series:* $\frac{1}{2} + \frac{2}{3}x + \left(\frac{3}{4}\right)^2 x^2 + \left(\frac{4}{5}\right)^3 x^3 + \ldots \infty \, (x > 0)$.

Sol.

$$u_n = \left(\frac{n+1}{n+2}\right)^n x^n \text{ (neglecting 1st term)}$$

$$u_{n+1} = \left(\frac{n+2}{n+3}\right)^{n+1} x^{n+1}$$

$$\frac{u_n}{u_{n+1}} = \frac{(n+1)^n}{(n+2)^n} \cdot \frac{(n+3)^{n+1}}{(n+2)^{n+1}} \cdot \frac{1}{x}$$

$$= \frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{2}{n}\right)^n} \cdot \frac{(n+3)^n}{(n+2)^n} \cdot \frac{n+3}{n+2} \cdot \frac{1}{x}$$

$$= \frac{\left(1+\frac{1}{n}\right)^n}{\left[\left(1+\frac{2}{n}\right)^{\frac{n}{2}}\right]^2} \cdot \frac{\left[\left(1+\frac{3}{n}\right)^{\frac{n}{3}}\right]^3}{\left[\left(1+\frac{2}{n}\right)^{\frac{n}{2}}\right]^2} \cdot \frac{1+\frac{3}{n}}{1+\frac{2}{n}} \cdot \frac{1}{x}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \frac{e}{e^2} \cdot \frac{e^3}{e^2} \cdot \frac{1}{x} \qquad \left| \because \underset{n\to\infty}{\text{Lt}} \left(1+\frac{1}{n}\right)^n = e \right.$$

$$= \frac{1}{x}.$$

By Ratio test Σu_n converges when $\frac{1}{x} > 1$ *i.e.* if $x < 1$ and diverges if $x > 1$

When $x = 1$, Ratio test fails

∴ For $x = 1$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}} u_n = \underset{n\to\infty}{\text{Lt}} \frac{(n+1)^n}{(n+2)^n} = \underset{n\to\infty}{\text{Lt}} \frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{2}{n}\right)^n} = \underset{n\to\infty}{\text{Lt}} \frac{\left(1+\frac{1}{n}\right)^n}{\left[\left(1+\frac{2}{n}\right)^{\frac{n}{2}}\right]^2}$$

$$= \frac{e}{e^2} = \frac{1}{e} \neq 0.$$

for $x = 1$, as Σu_n is a +ve term series and $\underset{n\to\infty}{\text{Lt}} u_n \neq 0$

∴ Σu_n diverges.

∴ Σu_n converges for $x < 1$ and diverges for $x \geq 1$.

Note. This question can be done by applying Cauchy's root test see Example 3 art. 5.26.

Example 13. *Discuss for what values of x does the series* $\sum_{x=1}^{\infty} \frac{(n!)^2}{(2n)!} x^{2n}$ *converge/diverge ?*

(P.T.U., May 2002, May 2003)

Sol. Here $u_n = \frac{(n!)^2}{2n!} x^{2n}$, $u_{n+1} = \frac{((n+1)!)^2}{(2n+2)!} x^{2n+2}$

$$\frac{u_n}{u_{n+1}} = \frac{(n!)^2}{(2n)!} \cdot x^{2n} \cdot \frac{(2n+2)!}{((n+1)!)^2} \cdot \frac{1}{x^{2n+2}} = \frac{1}{(n+1)^2}(2n+2)(2n+1)\frac{1}{x^2}$$

$$= \frac{2(2n+1)}{n+1} \cdot \frac{1}{x^2} = \frac{4\left(1+\frac{1}{2n}\right)}{1+\frac{1}{n}} \cdot \frac{1}{x^2}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \frac{4}{x^2}.$$

∴ By D'Alembert's ratio test the series Σu_n converges if $\frac{4}{x^2} > 1$ or $x^2 < 4$ or $|x| < 2$ or $-2 < x < 2$ and diverges for $x^2 > 4$ *i.e.*, for either $x > 2$ or $x < -2$.

When $x^2 = 4$ Ratio test fails ∴ Apply Gauss test

$$\frac{u_n}{u_{n+1}} = 4\left(1+\frac{1}{2n}\right)\left(1+\frac{1}{n}\right)^{-1} \cdot \frac{1}{4} = \left(1+\frac{1}{2n}\right)\left(1-\frac{1}{n}+0\left(\frac{1}{n^2}\right)\right)$$

$$= 1 - \frac{1}{2n} + 0\left(\frac{1}{n^2}\right)$$

Compare it with $\frac{u_n}{u_{n+1}} = 1 + \lambda \cdot \frac{1}{n} + 0 \cdot \left(\frac{1}{n^2}\right)$

∴ $\lambda = -\frac{1}{2} < 1$ ∴ Given series is divergent for $x^2 = 4$ *i.e.*, $x = \pm 2$

Hence given series converges for $x^2 < 4$ *i.e.*, $-2 < x < 2$ and diverges for $x^2 \geq 4$ *i.e.*, for either $x \geq 2$ or $x \leq -2$.

Example 14. *Test the convergence of the series* $x^2 (\log 2)^q + x^3 (\log 3)^q + x^4 (\log 4)^q + \ldots \infty$.

Sol.

$$u_n = x^{n+1} [\log (n+1)]^q$$

$$u_{n+1} = x^{n+2} [\log (n+2)]^q$$

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$$\frac{u_n}{u_{n+1}} = \left[\frac{\log(n+1)}{\log(n+2)}\right]^q \cdot \frac{1}{x} = \left[\frac{\log(n+1)}{\log(1+n+1)}\right]^q \cdot \frac{1}{x}$$

$$= \left[\frac{\log(n+1)}{\log(n+1)\left[1+\frac{1}{n+1}\right]}\right]^q \cdot \frac{1}{x} = \left[\frac{\log(n+1)\left(1+\frac{1}{n+1}\right)}{\log(n+1)}\right]^{-q} \cdot \frac{1}{x}$$

$$= \left[\frac{\log(n+1)+\log\left[1+\frac{1}{n+1}\right]}{\log(n+1)}\right]^{-q} \cdot \frac{1}{x}$$

$$= \left[1+\frac{1}{\log(n+1)}\left(\frac{1}{n+1}-\frac{1}{2(n+1)^2}+\frac{1}{3(n+1)^3}\cdots\infty\right)\right]^{-q} \cdot \frac{1}{x}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \frac{1}{x}.$$

By ratio test Σu_n converges if $\frac{1}{x} > 1$ or $x < 1$ diverges if $x > 1$

When $x = 1$; $\quad u_n = [\log(n+1)]^q$

$$\underset{n\to\infty}{\text{Lt}} u_n = \infty \neq 0$$

Σu_n is a +ve term series and $\underset{n\to\infty}{\text{Lt}} u_n \neq 0$

$\therefore \quad \Sigma u_n$ is divergent for $x = 1$

$\therefore \quad \Sigma u_n$ is convergent for $x < 1$

and divergent for $x \geq 1$.

TEST YOUR KNOWLEDGE

Discuss the convergence of the following series :

1. $\frac{1}{1^2}+\frac{1+2}{1^2+2^2}+\frac{1+2+3}{1^2+2^2+3^2}+ \ldots\ldots$ to ∞
2. $\frac{1^2}{4^2}+\frac{1^2.5^2}{4^2.8^2}+\frac{1^2.5^2.9^2}{4^2.8^2.12^2}+ \ldots\ldots$ to ∞
3. $1+\frac{3}{7}x+\frac{3.6}{7.10}x^2+\frac{3.6.9}{7.10.13}x^3+ \ldots\ldots$ to ∞
4. $1+\frac{2}{1}\cdot\frac{1}{2}+\frac{2.4}{1.3}\cdot\frac{1}{3}+\frac{2.4.6}{1.3.5}\cdot\frac{1}{4}+ \ldots\ldots$ to ∞ (P.T.U., Dec. 2010)
5. $\frac{1^2}{2^2}+\frac{1^2.3^2}{2^2.4^2}x+\frac{1^2.3^2.5^2}{2^2.4^2.6^2}x^2+ \ldots\ldots$ to ∞
6. $1^p+\left(\frac{1}{2}\right)^p+\left(\frac{1.3}{2.4}\right)^p+\left(\frac{1.3.5}{2.4.6}\right)^p+ \ldots\ldots$ to ∞
7. $\frac{a}{b}+\frac{a(a+1)}{b(b+1)}+\frac{a(a+1)(a+2)}{b(b+1)(b+2)}+ \ldots\ldots$ to ∞
8. $1+\frac{1}{2}x+\frac{1.3}{2.4}x^2+\frac{1.3.5}{2.4.6}x^3+ \ldots\ldots$ to ∞ $(x>0)$

9. $\frac{x}{1.2}+\frac{x^2}{3.4}+\frac{x^3}{5.6}+\frac{x^4}{7.8}+ \ldots\ldots$ to ∞ $(x > 0)$
10. $\frac{x}{1.2}+\frac{x^2}{2.3}+\frac{x^3}{3.4}+\frac{x^4}{4.5}+ \ldots\ldots$ to ∞
11. $\sum \frac{1.3.5\ldots\ldots(2n-1)}{2.4.6\ldots\ldots(2n)}\cdot\frac{x^{2n}}{2n}$
12. $\sum \left[\frac{1.3.5\ldots\ldots(2n-1)}{2.4.6\ldots\ldots2n}\right]^p x^n$
13. $1+\frac{(1!)^2}{2!}x^2+\frac{(2!)^2}{4!}x^4+\frac{(3!)^2}{6!}$ then $x^6 + \ldots\ldots$ to ∞ $(x > 0)$

[Hint: Neglect 1st term and see solved example 13]

ANSWERS

1. Divergent
2. Convergent
3. Convergent for $x \leq 1$, divergent for $x > 1$
4. Divergent
5. Convergent for $x < 1$, divergent for $x \geq 1$.
6. Convergent for $p > 2$, divergent for $p \leq 2$
7. Convergent for $b > a + 1$, divergent for $b \leq a + 1$
8. Convergent for $x < 1$, divergent for $x \geq 1$
9. Convergent for $x \leq 1$, divergent for $x > 1$
10. Convergent for $x \leq 1$, divergent for $x > 1$
11. Convergent for $x^2 \leq 1$, divergent for $x^2 > 1$
12. Convergent for $x \leq 1, p > 2$, divergent for $x > 1, p \leq 2$
13. Convergent for $x^2 < 4$, divergent for $x^2 \geq 4$.

5.26. CAUCHY'S ROOT TEST (P.T.U., Dec. 2002)

Statement. *If Σu_n is a positive term series and $\underset{n\to\infty}{Lt} (u_n)^{1/n} = l$, then*

(i) *Σu_n is convergent if $l < 1$* (ii) *Σu_n is divergent if $l > 1$.*

Note. If $l = 1$, the test fails *i.e.*, no conclusion can be drawn about the convergence or divergence of the series. The series may converge, it may diverge.

Proof. Since $\underset{n\to\infty}{\text{Lt}} (u_n)^{1/n} = l$,

$\therefore$ Given $\varepsilon > 0$, however small, there exists a +ve integer m such that

$$|(u_n)^{1/n} - l| < \varepsilon \quad \forall \; n \geq m$$

$$\Rightarrow \quad l - \varepsilon < (u_n)^{1/n} < l + \varepsilon \quad \forall \; n \geq m$$

$$\Rightarrow \quad (l-\varepsilon)^n < u_n < (l+\varepsilon)^n \quad \forall \; n \geq m \qquad \ldots(1)$$

(i) **When $l < 1$**

Choose $\varepsilon > 0$ such that $\quad l < l + \varepsilon < 1$

Put $\quad l + \varepsilon = r$, then $0 < r < 1$

From (1), $\quad u_n < r^n \quad \forall \; n \geq m$

Putting $n = m, m + 1, m + 2, \ldots\ldots$, we get $u_m < r^m, u_{m+1} < r^{m+1}, u_{m+2} < r^{m+2}, \ldots\ldots$ and so on.

Adding

$$u_m + u_{m+1} + u_{m+2} + \ldots\ldots + < r^m + r^{m+1} + r^{m+2} + \ldots\ldots$$

$\Rightarrow$ each term of the given series Σu_n after leaving the first $(m-1)$ terms, (*i.e.*, a finite number of terms) is less than the corresponding term of a geometric series which is convergent ($\because$ its common ratio $r < 1$). Hence the given series is also convergent.

(ii) **When $l > 1$**

Choose $\varepsilon > 0$ such that $l - \varepsilon > 1$

Put $l - \varepsilon = R$, then $R > 1$

From (1), $u_n > R^n \quad \forall \ n \geq m$

Putting $n = m, m+1, m+2, \ldots\ldots$, we get $u_m > R^m, u_{m+1} > R^{m+1}, u_{m+2} > R^{m+2}, \ldots\ldots$ and so on.

Adding $u_m + u_{m+1} + u_{m+2} + \ldots\ldots > R^m + R^{m+1} + R^{m+2} + \ldots\ldots$

$\Rightarrow$ each term of the series Σu_n after leaving the first $(m-1)$ terms, (*i.e.*, a finite number of terms) is greater than the corresponding term of a geometric series which is divergent. ($\because$ its common ratio $R > 1$). Hence the given series is also divergent.

ILLUSTRATIVE EXAMPLES

Example 1. *Test the convergence of the following series :*

(i) $\sum \left(\frac{n}{n+1}\right)^{n^2}$ *or* $\sum \left(1+\frac{1}{n}\right)^{-n^2}$ (P.T.U., May 2009) *(ii)* $\sum_{n=2}^{\infty} \frac{1}{(\log n)^n}$ (P.T.U., Dec. 2003)

(iii) $\sum \frac{(n-\log_n)^n}{2^n n^n}$.

Sol. *(i)* Here $u_n = \left(\frac{n}{n+1}\right)^{n^2}$

$$\therefore \quad (u_n)^{1/n} = \left[\left(\frac{n}{n+1}\right)^{n^2}\right]^{1/n} = \left(\frac{n}{n+1}\right)^n = \left(\frac{n+1}{n}\right)^{-n} = \left[\left(1+\frac{1}{n}\right)^n\right]^{-1}$$

$$\underset{n\to\infty}{\text{Lt}} (u_n)^{1/n} = \underset{n\to\infty}{\text{Lt}} \left[\left(1+\frac{1}{n}\right)^n\right]^{-1} = e^{-1} = \frac{1}{e} < 1 \quad (\because e = 2.7)$$

$\therefore$ By Cauchy's Root Test, the given series Σu_n is convergent.

(ii)
$$u_n = \frac{1}{(\log n)^n}$$

$$(u_n)^{\frac{1}{n}} = \frac{1}{\log n}$$

$$\underset{n\to\infty}{\text{Lt}} (u_n)^{\frac{1}{n}} = \underset{n\to\infty}{\text{Lt}} \frac{1}{\log n} = \frac{1}{\log \infty} = \frac{1}{\infty} = 0 < 1$$

$\therefore$ By Cauchy's root test the given series is convergent

Hence $\Sigma u_n = \sum_{n=2}^{\infty} \frac{1}{(\log n)^n}$ is convergent.

$$u_n = \frac{(n-\log n)^n}{2^n n^n}$$

$$(u_n)^{\frac{1}{n}} = \frac{n-\log n}{2n} = \frac{1}{2} - \frac{1}{2}\frac{\log n}{n}$$

$$\underset{n\to\infty}{\text{Lt}} (u_n)^{\frac{1}{n}} = \frac{1}{2} - \frac{1}{2} \underset{n\to\infty}{\text{Lt}} \frac{\log n}{n} = \frac{1}{2} - 0$$

$$\because \quad \underset{n\to\infty}{\text{Lt}} \frac{\log n}{n} = \underset{n\to\infty}{\text{Lt}} \frac{\frac{1}{n}}{0} = 0 \text{ (by L' Hospital rule)}$$

$$\underset{n\to\infty}{\text{Lt}} (u_n)^{\frac{1}{n}} = \frac{1}{2} < 1$$

By Cauchy's root test Σu_n is convergent.

Example 2. *Examine the convergence of the series :*

$$\left(\frac{2^2}{1^2} - \frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3} - \frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4} - \frac{4}{3}\right)^{-3} + \ldots\ldots$$

Sol. Here
$$u_n = \left[\left(\frac{n+1}{n}\right)^{n+1} - \frac{n+1}{n}\right]^{-n}$$

$$(u_n)^{1/n} = \left[\left(\frac{n+1}{n}\right)^{n+1} - \frac{n+1}{n}\right]^{-1} = \left[\left(1+\frac{1}{n}\right)^{n+1} - \left(1+\frac{1}{n}\right)\right]^{-1}$$

$$= \left[\left(1+\frac{1}{n}\right)^n \cdot \left(1+\frac{1}{n}\right) - \left(1+\frac{1}{n}\right)\right]^{-1}$$

$$\underset{n\to\infty}{\text{Lt}} (u_n)^{1/n} = \underset{n\to\infty}{\text{Lt}} \left[\left(1+\frac{1}{n}\right)^n \left(1+\frac{1}{n}\right) - \left(1+\frac{1}{n}\right)\right]^{-1}$$

$$= (e.1 - 1)^{-1} = \frac{1}{e-1} < 1 \quad (\because e = 2.7)$$

$\therefore$ By Cauchy's Root Test, Σu_n is convergent.

Example 3. *Discuss the convergence of the following series:*

(i) $\frac{1}{2} + \frac{2}{3}x + \left(\frac{3}{4}\right)^2 x^2 + \left(\frac{4}{5}\right)^3 x^3 + \ldots \infty$

(ii) $\sum \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$. (P.T.U. May 2008)

Sol. *(i)* Neglecting first term

$$u_n = \left(\frac{n+1}{n+2}\right)^n x^n; \ (u_n)^{\frac{1}{n}} = \frac{n+1}{n+2} x.$$

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$$\underset{n\to\infty}{\text{Lt}} (u_n)^{\frac{1}{n}} = \underset{n\to\infty}{\text{Lt}} \frac{1+\frac{1}{n}}{1+\frac{2}{n}} x = x.$$

By Cauchy's root test Σu_n converges for $x < 1$ and diverges for $x > 1$
When $x = 1$, Cauchy's root test fail

Then
$$u_n = \left(\frac{n+1}{n+2}\right)^n = \frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{2}{n}\right)^n}$$

$$\underset{n\to\infty}{\text{Lt}}\ u_n = \underset{n\to\infty}{\text{Lt}} \frac{\left(1+\frac{1}{n}\right)^n}{\left[\left(1+\frac{2}{n}\right)^{n/2}\right]^2} = \frac{e}{e^2} \qquad \left|\ \because \underset{n\to\infty}{\text{Lt}}\left(1+\frac{1}{n}\right)^n = e\right.$$

$$= \frac{1}{e} \neq 0$$

$\because$ Σu_n is +ve term series and $\underset{n\to\infty}{\text{Lt}}\ u_n \neq 0$

$\therefore$ for $x = 1$; Σu_n diverges

$\therefore$ Σu_n converges for $x < 1$ and diverges for $x \geq 1$.

(*ii*) $$u_n = \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}} ;\ (u_n)^{\frac{1}{n}} = \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{1/2}} = \frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{n^{1/2}}}$$

$\therefore$ $$\underset{n\to\infty}{\text{Lt}} (u_n)^{\frac{1}{n}} = \underset{n\to\infty}{\text{Lt}} \frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}} = \frac{1}{e} < 1$$

$\therefore$ Σu_n converges.

TEST YOUR KNOWLEDGE

Discuss the convergence of the following series :

1. $\sum \frac{1}{n^n}$
2. $\sum_{n=2}^{\infty} \frac{n}{(\log n)^n}$
3. $\sum \left(\frac{n+1}{3n}\right)^n$
4. $\sum \left(\frac{nx}{n+1}\right)^n$
5. $\sum 5^{-n-(-1)^n}$
6. $\sum \frac{(1+nx)^n}{n^n}$
7. $\sum \frac{(n+1)^n x^n}{n^{n+1}}$.

ANSWERS

1. Convergent
2. Convergent
3. Convergent
4. Convergent for $x < 1$, divergent for $x \geq 1$.
5. Convergent
6. Convergent for $x < 1$, divergent for $x \geq 1$
7. Convergent for $x < 1$, divergent for $x \geq 1$.

CAUCHY'S INTEGRAL TEST

(P.T.U., May 2002, Dec. 2005, Jan. 2009)

Statement. *If for $x \geq 1$, $f(x)$ is a non-negative, monotonic decreasing function of x such that $f(n) = u_n$, for positive integral values of n, then the series Σu_n and the integral $\int_1^{\infty} f(x)\,dx$ converge or diverge together.*

Proof. Let r be a +ve integer. Choose x such that $r + 1 \geq x \geq r \geq 1$

Since $f(x)$ is a monotonic decreasing function of x.

$$f(r+1) \leq f(x) \leq f(r) \quad \Rightarrow \quad u_{r+1} \leq f(x) \leq u_r \qquad [\because f(n) = u_n, n \in N]$$

$$\Rightarrow \int_r^{r+1} u_{r+1}\,dx \leq \int_r^{r+1} f(x)\,dx \leq \int_r^{r+1} u_r\,dx$$

$$\Rightarrow u_{r+1}\int_r^{r+1} dx \leq \int_r^{r+1} f(x)\,dx \leq u_r \int_r^{r+1} dx$$

$$\Rightarrow u_{r+1}\Big[x\Big]_r^{r+1} \leq \int_r^{r+1} f(x)\,dx \leq u_r \Big[x\Big]_r^{r+1}$$

$$\Rightarrow u_{r+1} \leq \int_r^{r+1} f(x)\,dx \leq u_r \qquad ...(1)$$

Putting $r = 1, 2, 3, \ldots\ldots, n$ in succession in (1), we have

$$u_2 \leq \int_1^2 f(x)\,dx \leq u_1$$

$$u_3 \leq \int_2^3 f(x)\,dx \leq u_2$$

....

....

$$u_{n+1} \leq \int_n^{n+1} f(x)\,dx \leq u_n$$

Adding the above inequalities, we have

$$u_2 + u_3 + \ldots\ldots + u_{n+1} \leq \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx + \ldots\ldots + \int_n^{n+1} f(x)\,dx \leq u_1 + u_2 + \ldots\ldots + u_n$$

$$\Rightarrow S_{n+1} - u_1 \leq \int_1^{n+1} f(x)\,dx \leq S_n \quad \text{where} \quad S_n = \sum_1^n u_n = u_1 + u_2 + \ldots\ldots + u_n.$$

Proceeding to the limit as $n \to \infty$

$$\Rightarrow \underset{n\to\infty}{\text{Lt}}\ S_{n+1} - u_1 \leq \underset{n\to\infty}{\text{Lt}} \int_1^{n+1} f(x)\,dx \leq \underset{n\to\infty}{\text{Lt}}\ S_n$$

$$\underset{n\to\infty}{\text{Lt}}\ S_{n+1} - u_1 \leq \int_1^{\infty} f(x)\,dx \leq \underset{n\to\infty}{\text{Lt}}\ S_n \qquad ...(2)$$

(i) If $\int_1^{\infty} f(x)\,dx$ converges, then $\int_1^{\infty} f(x)\,dx$ = a fixed finite number = I (say).

Then from (2), we have $\lim\limits_{n\to\infty} S_{n+1} - u_1 \le I$

$\Rightarrow$ $\lim\limits_{n\to\infty} S_{n+1} \le I + u_1$ = a fixed finite number

$\Rightarrow$ $\{S_n\}$ is a convergent sequence

$\Rightarrow$ the series Σu_n is convergent.

(ii) If $\int_1^{\infty} f(x)\,dx$ diverges, then $\int_1^{\infty} f(x)\,dx = +\infty$

From (2), $\lim\limits_{n\to\infty} S_n \ge \lim\limits_{n\to\infty} \int_1^{\infty} f(x)\,dx = +\infty$

$\Rightarrow$ $\{S_n\}$ is a divergent sequence

$\Rightarrow$ the series Σu_n is divergent.

Hence Σu_n and $\int_1^{\infty} f(x)\,dx$ converge or diverge together.

Note. If $x \ge k$, then Σu_n and $\int_k^{\infty} f(x)\,dx$ converge or diverge together.

ILLUSTRATIVE EXAMPLES

Example 1. *Test for convergence the series :* $\sum \dfrac{1}{n^2+1}$.

Sol. Here $u_n = \dfrac{1}{n^2+1} = f(n)$

$\therefore$ $f(x) = \dfrac{1}{x^2+1}$

For $x \ge 1$, $f(x)$ is +ve and monotonic decreasing.

$\therefore$ Cauchy's Integral Test is applicable.

Now, $$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{dx}{x^2+1} = \left[\tan^{-1} x\right]_1^{\infty} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} = \text{finite}$$

$\Rightarrow$ $\int_1^{\infty} f(x)\,dx$ converges and hence by Integral Test, Σu_n also converges.

Example 2. *Using integral test discuss the convergence of* $\sum\limits_{n=2}^{\infty} \dfrac{1}{n\sqrt{n^2-1}}$.

Sol. $u_n = \dfrac{1}{n\sqrt{n^2-1}} = f(n)$

$\therefore$ $f(x) = \dfrac{1}{x\sqrt{x^2-1}}$

for $x \ge 2$, $f(x)$ is +ve and monotonic decreasing.

$\therefore$ Cauchy's Integral test is applicable.

Now, $$\int_2^{\infty} f(x)\,dx = \int_2^{\infty} \frac{1}{x\sqrt{x^2-1}}\,dx \qquad ...(1)$$

Put $\sqrt{x^2-1} = t$ $\therefore$ $x^2 = t^2 + 1$; Differentiate $\dfrac{x}{\sqrt{x^2-1}}\,dx = dt$

$\therefore$ $\dfrac{dx}{x\sqrt{x^2-1}} = \dfrac{dt}{x^2} = \dfrac{dt}{t^2+1}$; when $x = 2$, $t = \sqrt{3}$; when $x = \infty$, $t = \infty$

$\therefore$ From (1) $$\int_2^{\infty} f(x)\,dx = \int_{\sqrt{3}}^{\infty} \frac{1}{t}\cdot\frac{dt}{t^2+1} = \int_{\sqrt{3}}^{\infty} \frac{1}{t\,(t^2+1)}\,dt$$

By partial fraction, let
$$\frac{1}{t\,(t^2+1)} = \frac{A}{t} + \frac{Bt+C}{t^2+1}$$
$$1 = A\,(t^2+1) + t\,(Bt+C)$$

Put $t = 0$ on both sides, we get $1 = A$

Comparing coefficients of t^2 and t on both sides; $0 = A + B$ $\therefore$ $B = -1, 0 = C$

$\therefore$ $$\frac{1}{t\,(t^2+1)} = \frac{1}{t} - \frac{t}{t^2+1}$$

$\therefore$ $$\int_2^{\infty} f(x)\,dx = \int_{\sqrt{3}}^{\infty}\left(\frac{1}{t} - \frac{t}{t^2+1}\right)dt = \log t - \frac{1}{2}\log(t^2+1)\Big|_{\sqrt{3}}^{\infty}$$
$$= \log \frac{t}{\sqrt{t^2+1}}\Big|_{\sqrt{3}}^{\infty} = \log \frac{1}{\sqrt{1+\dfrac{1}{t^2}}}\Bigg|_{\sqrt{3}}^{\infty}$$
$$= \log 1 - \log \frac{\sqrt{3}}{2} = -\log\frac{\sqrt{3}}{2} = \text{finite}$$

$\therefore$ $\int_2^{\infty} f(x)\,dx$ converges and hence Σu_n converges.

Example 3. *Show that the series* $\sum\limits_{1}^{\infty} \dfrac{1}{n^p}$ *converges if* $p > 1$ *and diverges if* $0 < p \le 1$.

Sol. Here $u_n = \dfrac{1}{n^p} = f(n)$

$\therefore$ $f(x) = \dfrac{1}{x^p}$

For $x \ge 1$, $f(x)$ is +ve and monotonic decreasing.

$\therefore$ Cauchy's Integral Test is applicable.

Case I. *When* $p \ne 1$

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{1}{x^p}\,dx = \int_1^{\infty} x^{-p}\,dx = \left[\frac{x^{-p+1}}{-p+1}\right]_1^{\infty}$$

Sub-Case 1. *When $p > 1$, $p-1$ is +ve, so that* $\int_1^\infty f(x)\,dx = -\frac{1}{p-1}\left[\frac{1}{x^{p-1}}\right]_1^\infty$

$$= -\frac{1}{p-1}[0-1] = \frac{1}{p-1} = \text{finite}$$

$\Rightarrow \int_1^\infty f(x)\,dx$ converges $\Rightarrow \Sigma u_n$ is convergent.

Sub-Case 2. *When $0 < p < 1$*, $1-p$ is +ve, so that

$$\int_1^\infty f(x)\,dx = \frac{1}{1-p}\left[x^{1-p}\right]_1^\infty = \frac{1}{1-p}(\infty - 1) = \infty$$

$\Rightarrow \int_1^\infty f(x)\,dx$ diverges $\Rightarrow \Sigma u_n$ is divergent.

Case II. *When $p = 1$, $f(x) = \frac{1}{x}$*

$$\int_1^\infty f(x)\,dx = \int_1^\infty \frac{1}{x}\,dx = \left[\log x\right]_1^\infty = \infty - \log 1 = \infty - 0 = \infty$$

$\Rightarrow \int_1^\infty f(x)\,dx$ diverges $\Rightarrow \Sigma u_n$ is divergent.

Hence Σu_n converges if $p > 1$ and diverges if $p \le 1$.

Example 4. *Discuss the convergence of* $\Sigma\, ne^{-n^2}$.

Sol. Here $u_n = ne^{-n^2} = f(n)$

$\therefore$ $f(x) = xe^{-x^2}$

For $x \ge 1$, $f(x)$ is +ve and monotonic decreasing

$\therefore$ Cauchy's Integral Test is applicable.

Now $\int_1^\infty f(x)\,dx = \int_1^\infty xe^{-x^2}\,dx.$

Put $x^2 = t$ $\therefore$ $2x\,dx = dt$

$$= \int_1^\infty e^{-t}\frac{dt}{2} = \left.\frac{e^{-t}}{-2}\right|_1^\infty = 0 + \frac{1}{2e} = \frac{1}{2e} = \text{finite}$$

$\Rightarrow$ $\int_1^\infty f(x)\,dx$ converges and hence by Integral Test $\Sigma\, u_n$ converges.

Example 5. *Discuss the convergence of the series :* $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$, $(p > 0)$.

Hence show that $\int_2^\infty \frac{dx}{x(\log x)^p}$ $(p > 0)$ *converges if and only if $p > 1$.* **(P.T.U., Dec. 2004)**

Sol. Here $u_n = \frac{1}{n(\log n)^p} = f(x)$ $\therefore$ $f(x) = \frac{1}{x(\log x)^p}$

For $x \ge 2$, $p > 0$, $f(x)$ is +ve and monotonic decreasing.

$\therefore$ By Cauchy's Integral Test $\sum_{n=2}^{\infty} u_n$ and $\int_2^\infty f(x)\,dx$ converge or diverge together.

Case I. *When $p \ne 1$*

$$\int_2^\infty f(x)\,dx = \int_2^\infty (\log x)^{-p}\cdot\frac{1}{x}\,dx = \left[\frac{(\log x)^{-p+1}}{-p+1}\right]_2^\infty$$

$$\left[\because \int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1}, n \ne -1\right]$$

Sub-Case 1. *When $p > 1$*, $p-1$ is +ve, so that $\int_2^\infty f(x)\,dx = -\frac{1}{p-1}\left[\frac{1}{(\log x)^{p-1}}\right]_2^\infty$

$$= -\frac{1}{p-1}\left[0 - \frac{1}{(\log 2)^{p-1}}\right] = \frac{1}{(p-1)(\log 2)^{p-1}} = \text{finite}$$

$\Rightarrow \int_2^\infty f(x)\,dx$ converges $\Rightarrow \sum_{n=2}^{\infty} u_n$ converges.

Sub-Case 2. *When $p < 1$*, $1-p$ is +ve, so that

$$\int_2^\infty f(x)\,dx = \frac{1}{1-p}\left[(\log x)^{1-p}\right]_2^\infty = \frac{1}{1-p}[\infty - (\log 2)^{1-p}] = \infty$$

$\Rightarrow \int_2^\infty f(x)\,dx$ diverges $\Rightarrow \sum_{n=2}^{\infty} u_n$ diverges.

Case II. *When $p = 1$*, $f(x) = \frac{1}{x\log x}$

$$\int_2^\infty f(x)\,dx = \int_2^\infty \frac{dx}{x\log x} = \int_2^\infty \frac{\frac{1}{x}}{\log x}\,dx = \left[\log\log x\right]_2^\infty \quad \left[\because \int \frac{f'(x)}{f(x)}\,dx = \log f(x)\right]$$

$$= \infty - \log\log 2 = \infty$$

$\Rightarrow \int_2^\infty f(x)\,dx$ diverges $\Rightarrow \sum_{n=2}^{\infty} u_n$ diverges.

Hence Σu_n converges if $p > 1$ and diverges if $0 < p \le 1$.

By Cauchy's integration test we know that

$\int_2^\infty \frac{dx}{x(\log x)^p}$ and $\sum_{n=2}^{\infty} u_n = \sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converge or diverge together

For Non

Since $\sum_{n=2}^{\infty} u_n$ converges for $p > 1$ as discussed in case I $\therefore \int_2^{\infty} \frac{dx}{x(\log x)^p}$ also converges for $p > 1$.

Example 6. *Using the integral test, discuss the convergence of* $\sum \frac{1}{(n \log n)(\log \log n)^p}$ *; $p > 0$.*

Sol.
$$u_n = \frac{1}{(n \log n)(\log \log n)^p} = f(n)$$

$$\therefore \quad f(x) = \frac{1}{(x \log x)(\log \log x)^p}$$

Clearly, for $x \geq 2$, $f(x)$ is +ve and monotonic decreasing $\therefore$ Integral test is applicable

$\therefore$ By Cauchy's Integral Test $\sum_{n=2}^{\infty} u_n$ and $\int_2^{\infty} f(x)\, dx$ behave alike

$$\therefore \quad \int_2^{\infty} f(x)\, dx = \int_2^{\infty} \frac{1}{(x \log x)(\log \log x)^p}\, dx \qquad ...(1)$$

$$= \int_2^{\infty} (\log \log x)^{-p} \left(\frac{1}{x \log x}\right) dx \quad \left| \because \int [f(x)]^n f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1} \right.$$

$$= \left. \frac{(\log \log x)^{-p+1}}{-p+1} \right|_2^{\infty} \text{ when } p \neq 1.$$

Case 1. *When $p < 1$*; $1 - p > 0$, $\int_2^{\infty} f(x)\, dx = \infty - \frac{(\log \log 2)^{1-p}}{1-p} = \infty$ = not finite

$\therefore \quad \Sigma u_n$ diverges for $p < 1$.

Case 2. *When $p > 1$*; $\therefore$ $p - 1$ is +ve

$$\int_2^{\infty} f(x)\, dx = \left. \frac{(\log \log x)^{-(p-1)}}{-(p-1)} \right|_2^{\infty} = -\frac{1}{p-1}\left\{ \left. \frac{1}{(\log \log x)^{p-1}} \right|_2^{\infty} \right\}$$

$$= -\frac{1}{p-1}\left\{\frac{1}{\infty} - \frac{1}{(\log \log 2)^{p-1}}\right\} = \frac{1}{(p-1)(\log \log 2)^{p-1}} = \text{finite}$$

$\therefore \quad \Sigma u_n$ converges for $p > 1$.

Case 3. *When $p = 1$*, from (1), $\int_2^{\infty} f(x)\, dx = \int_2^{\infty} \frac{\frac{1}{x \log x}}{\log \log x}\, dx$, which is of the type

$$\int \frac{f'(x)}{f(x)}\, dx = \log f(x) \quad \therefore \quad \int_2^{\infty} f(x)\, dx = \log \log \log x \,\Big|_2^{\infty} = \infty \text{ (not finite)}$$

$\therefore \ \Sigma u_n$ diverges.

Hence $\quad \Sigma u_n$ converges for $p > 1$ and diverges for $p \leq 1$.

Example 7. *Test the convergence of the series* $\sum_{n=1}^{\infty} \frac{8 \tan^{-1} n}{1 + n^2}$. (P.T.U., May 2002)

Sol. Here
$$u_n = \frac{8 \tan^{-1} n}{1 + n^2} = f(x)$$

$$\therefore \quad f(x) = \frac{8 \tan^{-1} x}{1 + x^2}$$

Clearly for $x \geq 1$ $f(x)$ is +ve and monotonic decreasing

$\therefore$ Cauchy's Integral Test is applicable

Now,
$$\int_1^{\infty} f(x)\, dx = \int_1^{\infty} \frac{8 \tan^{-1} x}{1 + x^2}\, dx = \int_1^{\infty} 8 \tan^{-1} x \left(\frac{1}{1+x^2}\right) dx$$

$$= \left. \frac{8 (\tan^{-1} x)^2}{2} \right|_1^{\infty} \quad \left| \because \int f(x)\, f'(x)\, dx = \frac{[f(x)]^2}{2} \right.$$

$$= 4 \,.\, \{(\tan^{-1} \infty)^2 - (\tan^{-1} 1)^2\} = 4 \,.\, \left\{\left(\frac{\pi}{2}\right)^2 - \left(\frac{\pi}{4}\right)^2\right\}$$

$$= 4\pi^2 \left[\frac{1}{4} - \frac{1}{16}\right] = 4\pi^2 \,.\, \frac{12}{16} = 3\pi^2, \text{ which is finite}$$

$\therefore$ By Cauchy's Integral Test

$\int_1^{\infty} f(x)\, dx$ and hence Σu_n *i.e.*, $\sum_{n=1}^{\infty} \frac{8 \tan^{-1} n}{1 + n^2}$ converges.

TEST YOUR KNOWLEDGE

Using the integral test, discuss the convergence of the following series :

1. $\sum \frac{1}{2n+3}$
2. $\sum \frac{1}{n(n+1)}$
3. $\sum \frac{1}{\sqrt{n}}$
4. $\sum \frac{1}{(n+1)^2}$
5. $\sum \frac{2n^3}{n^4+3}$
6. $\sum \frac{n}{(n^2+1)^2}$.

ANSWERS

1. Divergent
2. Convergent
3. Divergent
4. Convergent
5. Divergent
6. Convergent.

5.28. LEIBNITZ'S TEST ON ALTERNATING SERIES (P.T.U. Dec. 2007)

Statement. *The alternating series* $\Sigma(-1)^{n-1} u_n = u_1 - u_2 + u_3 - u_4 + \ldots\ldots$ $(u_n > 0\ \forall\ n)$ *converges if*

(i) $u_n > u_{n+1}\ \forall\ n$ (ii) $\lim\limits_{n \to \infty} u_n = 0$.

(ii)
$$\underset{n\to\infty}{\text{Lt}}\, u_n = \underset{n\to\infty}{\text{Lt}} \frac{n}{2(n+1)^2} = \underset{n\to\infty}{\text{Lt}} \frac{1}{2\left(1+\frac{1}{n}\right)^2} = 0.$$

Since both the conditions of Leibnitz's Test are satisfied, the given series is convergent.

Example 3. *Test the convergence of the following series :* $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\cdot n}{2n-1}$.

Sol. The given series is $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\cdot n}{2n-1} = \frac{1}{1} - \frac{2}{3} + \frac{3}{5} - \frac{4}{7} + \ldots\ldots$

It is an alternating series

(i)
$$u_n = \frac{n}{2n-1}, \quad u_{n+1} = \frac{n+1}{2n+1}$$

$$u_n - u_{n+1} = \frac{1}{4n^2-1} > 0 \quad \forall n \Rightarrow u_n > u_{n+1} \quad \forall n$$

(ii)
$$\underset{n\to\infty}{\text{Lt}}\, u_n = \underset{n\to\infty}{\text{Lt}} \frac{n}{2n-1} = \underset{n\to\infty}{\text{Lt}} \frac{1}{2-\frac{1}{n}} = \frac{1}{2} \neq 0.$$

Here the second condition of Leibnitz's Test is not satisfied. Hence the given series is not convergent.

Example 4. *Examine the convergence of the series*

$$1 + \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} - \frac{1}{7^2} - \frac{1}{8^2} + \ldots \infty.$$

Sol. The given series can be rearranged in the form

$$\left(1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \ldots \infty\right) + \left(\frac{1}{2^2} - \frac{1}{4^2} + \frac{1}{6^2} \ldots \infty\right)$$
$$= \Sigma(-1)^{n-1} u_n + \Sigma(-1)^{n-1} v_n$$

Consider $\Sigma(-1)^{n-1} u_n$, which is an alternating series

where $u_n = \frac{1}{(2n-1)^2}$

($\because$ 1, 3, 5, ... are in A.P. and its nth term $= 2n-1$)

$$\frac{1}{(2n-1)^2} > \frac{1}{(2n+1)^2} \quad \therefore \quad u_n > u_{n+1}$$

and
$$\underset{n\to\infty}{\text{Lt}}\, u_n = \underset{n\to\infty}{\text{Lt}} \frac{1}{(2n-1)^2} = 0$$

$\therefore$ Both conditions of Leibnitz's test are satisfied

$\therefore$ $\Sigma(-1)^{n-1} u_n$ is convergent.

Now, for $\Sigma(-1)^{n-1} v_n$; $v_n = \frac{1}{(2n)^2}$

As $\frac{1}{(2n)^2} > \frac{1}{(2n+2)^2}$ $\quad \therefore \quad v_n > v_{n+1}$ and $\underset{n\to\infty}{\text{Lt}}\, v_n = \underset{n\to\infty}{\text{Lt}} \frac{1}{(2n)^2} = 0$

$\therefore$ Both conditions of Leibnitz's test are satisfied by $\Sigma(-1)^{n-1} v_n$. Therefore, both $\Sigma(-1)^{n-1} u_n$ and $\Sigma(-1)^{n-1} v_n$ are convergent $\quad \therefore \quad$ Their sum is also convergent.

Hence given series is convergent.

Example 5. *The series* $\frac{1}{3} - \frac{1}{2} + \frac{1}{9} - \frac{1}{4} + \frac{1}{27} - \frac{1}{8} + \ldots\ldots \infty$, *does not meet one of the conditions of Leibnitz's test which one ? Find the sum of this series.* **(P.T.U., Dec. 2004)**

Sol. The given series is

$$\left(\frac{1}{3} - \frac{1}{2}\right) + \left(\frac{1}{9} - \frac{1}{4}\right) + \left(\frac{1}{27} - \frac{1}{8}\right) + \ldots + \left(\frac{1}{3^n} - \frac{1}{2^n}\right) + \ldots \infty$$

Here
$$u_n = \frac{1}{3^n} - \frac{1}{2^n}$$

$$u_{n+1} = \frac{1}{3^{n+1}} - \frac{1}{2^{n+1}}$$

$$u_n - u_{n+1} = \left(\frac{1}{3^n} - \frac{1}{3^{n+1}}\right) - \left(\frac{1}{2^n} - \frac{1}{2^{n+1}}\right) = \frac{2}{3^{n+1}} - \frac{1}{2^{n+1}}$$
$$= \frac{2^{n+2} - 3^{n+1}}{6^{n+1}} < 0 \quad \forall n$$

$\therefore$
$$u_n < u_{n+1} \quad \forall n \quad i.e., u_n \not> u_{n+1} \; \forall n$$

$\therefore$ First condition of Leibnitz's test is not satisfied

where as second condition *i.e.*, $\underset{n\to\infty}{\text{Lt}}\, u_n = \underset{n\to\infty}{\text{Lt}} \frac{1}{3^n} - \frac{1}{2^n} = 0$ is satisfied.

Now sum of the series

$$= \left(\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots\ldots \infty\right) - \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\ldots \infty\right)$$

both are infinite G.Ps and sum of an infinite G.P. $= \frac{a}{1-r}$

$\therefore$ Sum of the given series $= \frac{\frac{1}{3}}{1-\frac{1}{3}} - \frac{\frac{1}{2}}{1-\frac{1}{2}} = \frac{1}{2} - 1 = -\frac{1}{2}$.

TEST YOUR KNOWLEDGE

Examine the convergence of the following series :

1. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\ldots \infty$
2. $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots$ to ∞
3. $1 - \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} - \frac{1}{4\sqrt{4}} + \ldots\ldots$ to ∞
4. $\frac{1}{a} - \frac{1}{a+b} + \frac{1}{a+2b} - \frac{1}{a+3b} + \ldots\ldots$ to ∞ $\quad (a > 0, b > 0)$
5. $\frac{1}{\log 2} - \frac{1}{\log 3} + \frac{1}{\log 4} - \frac{1}{\log 5} + \ldots\ldots$ to ∞
6. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!}$

7. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3n-2}$

8. $\sum_{n=1}^{\infty} \frac{n(-1)^{n-1}}{5^n}$

9. $\frac{1}{\sqrt{2}+1} - \frac{1}{\sqrt{3}+1} + \frac{1}{\sqrt{4}+1} - \frac{1}{\sqrt{5}+1} + \ldots\ldots$ to ∞

10. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+1}$

11. $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2+1}$ **[Hint:** $\cos n\pi = (-1)^n$]

ANSWERS

1. Convergent	2. Convergent	3. Convergent	4. Convergent
5. Convergent	6. Convergent	7. Convergent	8. Convergent
9. Convergent	10. Convergent	11. Convergent.	

5.29(a). ABSOLUTE CONVERGENCE OF A SERIES (P.T.U., Dec. 2003)

Def. *If a convergent series whose terms are not all positive, remains convergent when all its terms are made positive, then it is called an* **absolutely convergent series**, *i.e.*,

The series Σu_n is said to be absolutely convergent if $\Sigma\, |u_n|$ is a convergent series.

5.29(b). CONDITIONAL CONVERGENCE OF A SERIES

A series is said to be **conditionally convergent** *if it is convergent but does not converge absolutely.*

Example 1. *Test whether the following series are absolutely convergent or conditionally convergent.*

(*a*) $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots$ **(P.T.U., Dec. 2006)** (*b*) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$.

Sol. (*a*) The series is $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots$

(*i*) In this alternating series, each term is less than the preceeding term numerically.

(*ii*) Moreover $u_n = \frac{1}{n^2}$, which $\to 0$ as $n \to \infty$.

Hence the series satisfies both the conditions of the test on alternating series and so the given series *converges.*

Again when all the term of the series are made positive, the series becomes

$$\sum |u_n| = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots = \Sigma \frac{1}{n^2}, \text{ which is a } p\text{-series, where } p = 2 > 1$$

$\therefore$ $\Sigma\, |u_n|$ is a convergent series.

Thus the given series converges absolutely.

(*b*) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} = \Sigma (-1)^{n-1} u_n$ (say), where $u_n = \frac{1}{2n-1}$

Putting $n = 1, 2, 3, \ldots\ldots$, the series becomes $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots$

The series is clearly an alternating series.

The terms go on decreasing numerically and

$$\operatorname*{Lt}_{n\to\infty} u_n = \operatorname*{Lt}_{n\to\infty} \frac{1}{2n-1} = 0$$

$\therefore$ By Leibnitz's Test, the series *converges.*

But when all terms are made positive, the series becomes,

$$\sum |u_n| = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots\ldots$$

Here $u_n = \frac{1}{2n-1}$. Take $v_n = \frac{1}{n}$.

$$\therefore \quad \operatorname*{Lt}_{n\to\infty} \frac{u_n}{v_n} = \operatorname*{Lt}_{n\to\infty} \frac{n}{2n-1} = \operatorname*{Lt}_{n\to\infty} \left[\frac{1}{2-\frac{1}{n}}\right] = \frac{1}{2} = \text{finite} \neq 0$$

Hence by comparison test series Σu_n and Σv_n behave alike.

But $\Sigma v_n = \Sigma \frac{1}{n}$ is a divergent series ($\because$ here $p = 1$), $\therefore$ Σu_n also diverges.

Hence the given series converges, and the series of absolute terms diverges, therefore the given series converges conditionally.

5.30. EVERY ABSOLUTELY CONVERGENT SERIES IS CONVERGENT OR IF $\sum_{n=1}^{\infty} |u_n|$ IS CONVERGENT THEN $\sum_{n=1}^{\infty} u_n$ IS CONVERGENT

Proof. Let $\sum_{n=1}^{\infty} u_n$ be an absolutely convergent series.

$\therefore$ $\sum_{n=1}^{\infty} |u_n|$ is convergent.

By Cauchy's general principle of convergence, given $\varepsilon > 0$, $\exists$s a positive integer m such that

$$\big|\, |u_{m+1}| + |u_{m+2}| \ldots\ldots |u_n| \,\big| < \varepsilon \quad \forall \quad n > m$$

$$|u_{m+1}| + |u_{m+1}| + \ldots\ldots + |u_n| < \varepsilon \quad \forall \quad n > m \quad \ldots(1)$$

Now, by triangle inequality, we have

$$|u_{m+1} + u_{m+2} + \ldots\ldots + u_n| \le |u_{m+1}| + |u_{m+2}| + \ldots\ldots + |u_n| < \varepsilon \quad \forall \quad n > m \quad \text{[Using (1)]}$$

$\therefore$ By Cauchy's general principle of convergence, the series $\sum_{n=1}^{\infty} u_n$ is convergent.

Hence $\sum |u_n|$ is convergent $\Rightarrow$ Σu_n is convergent.

Note 1. Absolute convergence $\Rightarrow$ Convergence, but convergence need not imply absolute convergence *i.e.*, the converse of above theorem need not be true.

For example, consider the series $\sum \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\ldots$

It is convergent. [See Example 1 with Leibnitz's Test]

But the series $\sum \left| \frac{(-1)^{n-1}}{n} \right| = \sum \frac{1}{n}$ is divergent.

Note 2. The divergence of $\Sigma\,|u_n|$ does not imply the divergence of Σu_n.

For example, $\sum \left| \frac{(-1)^{n-1}}{n} \right| = \sum \frac{1}{n}$ is divergent whereas $\sum \frac{(-1)^{n-1}}{n}$ is convergent.

5.31. POWER SERIES

Def. A series of the form $a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots \infty$, where a's are independent of x, is called a power series in x. Such a series may converge for some or all values of x.

Interval of Convergence. If the power series is $\sum_{n=0}^{\infty} a_n x^n$ then take $u_n = a_n x^n$

$\therefore$ Series become $\sum_{n=0}^{\infty} u_n$, where $u_n = a_n x^n$.

As in power series a's can be +ve as well as –ve $\therefore$ for convergence of $\Sigma\, u_n$ we test the convergence of $\Sigma\,|u_n|$ $\because$ every absolutely convergent series is a convergent series.

$$\therefore \quad \left| \frac{u_n}{u_{n+1}} \right| = \left| \frac{a_n\, x^n}{a_{n+1}\, x^{n+1}} \right| = \left| \frac{a_n}{a_{n+1}} \cdot \frac{1}{x} \right| = \left| \frac{a_n}{a_{n+1}} \right| \cdot \frac{1}{|x|}$$

$$\text{Let} \quad \operatorname*{Lt}_{n\to 0} \left| \frac{a_n}{a_{n+1}} \right| = l$$

$$\therefore \quad \operatorname{Lt} \left| \frac{u_n}{u_{n+1}} \right| = \frac{l}{|x|}$$

$\therefore$ $\Sigma\,|u_n|$ converges if $\frac{l}{|x|} > 1$ (By Ratio test)

$\therefore$ $\Sigma\, u_n$ converges for $|x| < l$ *i.e.,* for $-l < x < l$

$\therefore$ The power series converges in the interval $(-l, l)$ and diverges outside this interval.

Interval $(-l, l)$ is called the Interval of Convergence of the power series.

Example 2. *Prove that the series* $\frac{\sin x}{1^3} - \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} - \ldots\ldots$ *converges absolutely.*

Sol. The given series is $\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \sin nx}{n^3}$

Since $|u_n| = \frac{|\sin nx|}{n^3} \le \frac{1}{n^3}$ $\forall\, n$ and $\sum \frac{1}{n^3}$ converges by p-series test $\because$ here $p = 3 > 1$.

$\therefore$ By comparison test, the series $\sum |u_n|$ converges.

$\Rightarrow$ The given series converges absolutely.

Example 3. *For what value of x does the series $\sum_{n=0}^{\infty} (-1)^n (4x+1)^n$ converges absolutely.* (P.T.U., May 2003)

Sol. The series is $\sum_{n=0}^{\infty} u_n = \sum_{n=0}^{\infty} (-1)^n (4x+1)^n$

$$|u_n| = |(-1)^n (4x+1)^n| = |(4x+1)^n|$$

$$|u_{n+1}| = |(-1)^{n+1} (4x+1)^{n+1}| = |(4x+1)^{n+1}|$$

$$\frac{|u_n|}{|u_{n+1}|} = \frac{|(4x+1)^n|}{|(4x+1)^{n+1}|} = \frac{1}{|4x+1|}$$

$$\operatorname*{Lt}_{n\to\infty} \left| \frac{u_n}{u_{n+1}} \right| = \frac{1}{|4x+1|}$$

$\therefore$ By ratio test $\Sigma\,|u_n|$ converges if $\frac{1}{|4x+1|} > 1$

$$|4x+1| < 1 \quad \text{or} \quad |4x-(-1)| < 1$$

$$-1-1 < 4x < -1+1$$

$$-2 < 4x < 0 \qquad \Big|\; \because\ |x-a| < l \Rightarrow a-l < x < a+l$$

$$-\frac{1}{2} < x < 0$$

Hence the given series converges absolutely for $-\frac{1}{2} < x < 0$ *i.e.,* when $x \in \left(-\frac{1}{2}, 0\right)$.

Example 4. *Discuss the convergence of the series $\sum_{n=1}^{\infty} (-1)^n \frac{\tan^{-1} n}{1+n^2}$.* (P.T.U., May 2003)

Sol. Here $u_n = \frac{(-1)^n \tan^{-1} n}{1+n^2}$; $|u_n| = \left| \frac{(-1)^n \tan^{-1} n}{1+n^2} \right| = \frac{\tan^{-1} n}{1+n^2} = f(n)$ (say)

Apply Cauchy's integral test:

Now,
$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{\tan^{-1} x}{1+x^2}\,dx = \left. \frac{(\tan^{-1} x)^2}{2} \right|_1^{\infty} = \frac{1}{2}\left[\left(\frac{\pi}{2}\right)^2 - \left(\frac{\pi}{4}\right)^2 \right]$$

$= \frac{3}{8}\pi^2$, which is finite $\therefore$ $\int_1^{\infty} f(x)\,dx$ is convergent.

$\therefore$ By Cauchy's integral test $\sum_{n=1}^{\infty} |u_n|$ and $\int_1^{\infty} f(x)\,dx$. Converge or diverge together and

$\because \int_1^{\infty} f(x)\,dx$ is convergent $\therefore \sum_{n=1}^{\infty} |u_n|$ is also convergent.

As every absolutely convergent series is convergent

$\therefore \sum_{n=1}^{\infty} u_n$ is also convergent.

Hence $\sum_{n=1}^{\infty} (-1)^n \frac{\tan^{-1} n}{1+n^2}$ is convergent series.

Example 5. *Prove that the series* $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \infty$ *is convergent for* $-1 < x \le 1$. *Also write the interval of convergence.*

Sol. The given series is

$$\Sigma u_n = \Sigma (-1)^{n-1} \frac{x^n}{n}$$

$$|u_n| = \left|\frac{x^n}{n}\right| = \frac{|x|^n}{n};\ |u_{n+1}| = \frac{|x|^{n+1}}{n+1}$$

$$\frac{|u_n|}{|u_{n+1}|} = \frac{|x|^n}{n} \cdot \frac{n+1}{|x|^{n+1}} = \frac{n+1}{n} \cdot \frac{1}{|x|}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{|u_n|}{|u_{n+1}|} = \underset{n\to\infty}{\text{Lt}} \left(1+\frac{1}{n}\right) \cdot \frac{1}{|x|} = \frac{1}{|x|}$$

By Ratio test $\Sigma |u_n|$ converges when

$$\frac{1}{|x|} > 1 \quad i.e., \quad |x| < 1 \quad i.e., \quad -1 < x < 1$$

and diverges when $\frac{1}{|x|} < 1$ *i.e.*, $|x| > 1$ *i.e.*, for $x > 1$ or $x < -1$

Ratio test fails when $|x| = 1$ *i.e.*, when $x = \pm 1$.

$\therefore$ When $x = 1$, the series becomes $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots \infty$, which is an alternating series and is convergent

(see S.E. 1 art. 5.28)

When $x = -1$, the series becomes

$$-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \ldots \infty$$

$$= -\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots \infty\right)$$

$$= -\sum \frac{1}{n}, \text{ which is of the type } \sum \frac{1}{n^p} \text{ where } p = 1$$

$\therefore$ by *p*-series test; it is divergent.

$\therefore$ Given series converges for $-1 < x \le 1$ and diverges for $x > 1$ or $x \le -1$

$\therefore$ The Interval of convergence $(-1, 1]$.

Example 6. *For what value of x the power series*

$$1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 - \frac{1}{8}(x-2)^3 + \ldots + \left(-\frac{1}{2}\right)^n (x-2)^n + \ldots \infty$$

converges? What is its sum? (P.T.U., May 2002)

Sol. The given series is an alternating series

Here
$$u_n = \left(-\frac{1}{2}\right)^n (x-2)^n$$

$$|u_n| = \frac{1}{2^n} |(x-2)^n|;\ |u_{n+1}| = \frac{1}{2^{n+1}} |(x-2)^{n+1}|$$

$$\left|\frac{u_n}{u_{n+1}}\right| = \left|\frac{1}{2^n}(x-2)^n \cdot \frac{2^{n+1}}{(x-2)^{n+1}}\right| = \frac{2}{|x-2|}$$

$$\underset{n\to\infty}{\text{Lt}} \left|\frac{u_n}{u_{n+1}}\right| = \frac{2}{|x-2|}$$

$\therefore$ By Ratio's test the series $\Sigma |u_n|$ is convergent

$$\frac{2}{|x-2|} > 1 \quad \text{or} \quad |x-2| < 2 \quad \text{or} \quad 2-2 < x < 2+2 \quad \text{or} \quad 0 < x < 4$$

$\therefore \Sigma |u_n|$ is convergent for $0 < x < 4$.

As every absolutely convergent series is convergent

$$\therefore \quad \sum u_n = \sum \left(-\frac{1}{2}\right)^n (x-2)^n \text{ is convergent for } 0 < x < 4$$

Now sum of the series $= 1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 + \ldots\ldots \infty$ is an infinite G.P. with first term 1 and common ratio $-\frac{1}{2}(x-2)$ and $|C.R| = \left|-\frac{1}{2}(x-2)\right| < \frac{1}{2}|x-2| < 1$

$$\therefore \quad \text{Sum of the series} = \frac{1}{1 - \left[-\frac{1}{2}(x-2)\right]} = \frac{1}{1 + \frac{x-2}{2}} \qquad \left[S = \frac{a}{1-r}\right]$$

$$= \frac{2}{2+x-2} = \frac{2}{x} \quad \text{for } 0 < x < 4.$$

TEST YOUR KNOWLEDGE

1. Prove that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\cos^2 nx}{n\sqrt{n}}$ converges absolutely.

[Hint: See S.E.1**]**

2. For what values of x are the following series convergent.

(*i*) $x - \frac{x^2}{\sqrt{2}} + \frac{x^3}{\sqrt{3}} - \frac{x^4}{\sqrt{4}} + \ldots\ldots$ (*ii*) $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots$

(P.T.U., Jan. 2010)

$\left[\textbf{Hint :} (ii)\ u_n = \frac{x^n}{n!},\ u_{n+1} = \frac{x^{n+1}}{(n+1)!};\ \underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} \to \infty \text{ irrespective of the values of } x\right.$

$\left. \therefore\ \text{Lt} \frac{u_n}{u_{n+1}} > 1\ \forall\ x\ \therefore\ \Sigma u_n \text{ converges for all values of } x\right]$

(*iii*) $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\ldots$ (*iv*) $x - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \frac{x^4}{4^2} + \ldots\ldots$

ANSWERS

2. (*i*) $-1 < x \le 1$ (*ii*) all x (*iii*) $-1 < x < 1$ (*iv*) $-1 < x \le 1$.

5.32. UNIFORM CONVERGENCE OF SERIES OF FUNCTIONS

(P.T.U., May 2004, May 2005)

Let $u_n(x)$ be a real valued function defined on an interval I and for each $n \in$ N. Then $u_1(x) + u_2(x) + u_3(x) + \ldots\ldots = \sum_{n=1}^{\infty} u_n(x)$ is called an infinite series of functions each of which is defined on the interval I.

Let $S_n(x) = u_1(x) + u_2(x) + \ldots\ldots + u_n(x)$ be the nth partial sum of $\sum u_n(x)$.

Let $\alpha \in$ I and $\lim_{n\to\infty} S_n(\alpha) = S(\alpha)$ then the series $\sum u_n(x)$ is said to converge to $S(\alpha)$ at $x = \alpha$.

Thus, given $\varepsilon > 0$, there exists a positive integer m such that

$$|S_n(\alpha) - S(\alpha)| < \varepsilon \quad \forall \quad n > m.$$

The positive integer m depends on $\alpha \in$ I and the given value of $\varepsilon > 0$, *i.e.* $m = m(\alpha, \varepsilon)$. It is not always possible to find an m which works for each $x \in$ I. If we can find an m which depends only on ε and not on $x \in$ I, we say $\sum u_n(x)$ is uniformly convergent.

Definition. A series $\sum_{n=1}^{\infty} u_n(x)$ converges uniformally to a function S(x) if for a given $\varepsilon > 0$, there exists a positive integer m depending only on ε and independent of x such that for every $x \in$ I, $|S_n(x) - S(x)| < \varepsilon$ $\forall$ $n > m$.

Note. The method of testing the uniform convergence of a series $\sum u_n(x)$, by definition, involves finding S_n which is not always easy. The following test avoids $S_n(x)$.

WEIRSTRASS'S M-TEST

(P.T.U., May 2003)

Statement. *A series $\sum_{n=1}^{\infty} u_n(x)$ of functions converges uniformly and absolutely on an interval I if there is a convergent series $\sum_{n=1}^{\infty} M_n$ of positive constants such that $|u_n(x)| \le M_n$ $\forall$ $n \in N$ and $\forall$ $x \in I$.*

Proof. Since $\sum_{n=1}^{\infty} M_n$ is convergent, by Cauchy's general principle of convergence, for each $\varepsilon > 0$, there exists a positive integer m such that

$$|M_{m+1} + M_{m+2} + \ldots\ldots + M_n| < \varepsilon \quad \forall \quad n > m$$

$$M_{m+1} + M_{m+2} + \ldots\ldots + M_n < \varepsilon \quad \forall \quad n > m \qquad \ldots(1)$$

Now, for all $x \in$ I, $|u_n(x)| \le M_n$ ...(2)

$$\therefore\ |u_{m+1}(x) + u_{m+2}(x) + \ldots\ldots + u_n(x)|$$
$$\le |u_{m+1}(x)| + |u_{m+2}(x)| + \ldots\ldots + |u_n(x)|$$
$$\le M_{m+1} + M_{m+2} + \ldots\ldots + M_n \qquad [\text{by } (2)]$$
$$< \varepsilon \quad \forall \quad n > m \qquad [\text{by } (1)]$$

$\Rightarrow$ By Cauchy's criterion, the series $\sum_{n=1}^{\infty} u_n(x)$ is uniformly convergent on I.

Also, $|u_{m+1}(x)| + |u_{m+2}(x)| + \ldots\ldots + |u_n(x)| < \varepsilon \quad \forall \quad n > m$

$\Rightarrow$ $||u_{m+1}(x)| + |u_{m+2}(x)| + \ldots\ldots + |u_n(x)|| < \varepsilon \quad \forall \quad n > m$

$\Rightarrow$ The series $\sum_{n=1}^{\infty} |u_n(x)|$ is uniformly convergent on I.

Hence the series $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly and absolutely on I.

Example 1. *Show that the following series are uniformly convergent :*

(*i*) $\sum_{n=1}^{\infty} \frac{\sin(x^2 + nx)}{n(n+2)}$ *for all real x.* **(P.T.U., May 2009)**

(*ii*) $\sum_{n=1}^{\infty} \frac{\cos nx}{n^p}$ *for all real x and p > 1.* **(P.T.U., Dec. 2005)**

(*iii*) $\sum_{n=1}^{\infty} \frac{1}{n^p + n^q x^2}$ *for all real x and p > 1.*

Sol. (*i*) Here $u_n(x) = \frac{\sin(x^2 + nx)}{n(n+2)}$

$$\therefore\ |u_n(x)| = \left|\frac{\sin(x^2 + nx)}{n(n+2)}\right| = \frac{|\sin(x^2 + nx)|}{n(n+2)} \le \frac{1}{n(n+2)} < \frac{1}{n^2}\ (= M_n)\ \forall\ x \in R$$

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Since $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, therefore, by M-test, the given series is uniformly convergent for all real x.

(ii) Here
$$u_n(x) = \frac{\cos nx}{n^p}$$

$$\therefore \quad |u_n(x)| = \left| \frac{\cos nx}{n^p} \right| = \frac{|\cos nx|}{n^p} \le \frac{1}{n^p} \; (= M_n) \quad \forall \quad x \in R.$$

Since $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent for $p > 1$, therefore, by M-test, the given series is uniformly convergent for all real x and $p > 1$.

(iii) Here
$$u_n(x) = \frac{1}{n^p + n^q x^2}$$

Since $x^2 \ge 0$ for all real x

$$\therefore \quad n^q x^2 \ge 0 \;\Rightarrow\; n^p + n^q x^2 \ge n^p \;\Rightarrow\; \frac{1}{n^p + n^q x^2} \le \frac{1}{n^p}$$

$$\therefore \quad |u_n(x)| = \frac{1}{n^p + n^q x^2} \le \frac{1}{n^p} \; (= M_n) \quad \forall \quad x \in R.$$

Since $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent for $p > 1$.

therefore, by M-test, the given series is uniformly convergent for all real x and $p > 1$.

TEST YOUR KNOWLEDGE

Test for uniform convergence the series :

1. $\frac{\sin x}{1^2} + \frac{\sin 2x}{2^2} + \frac{\sin 3x}{3^2} + \frac{\sin 4x}{4^2} + \ldots\ldots$ **(P.T.U. May 2003)**
   [Hint: See S.E.1. *(ii)*]
2. $\sin x - \frac{\sin 2x}{2\sqrt{2}} + \frac{\sin 3x}{3\sqrt{3}} - \frac{\sin 4x}{4\sqrt{4}} + \ldots\ldots$
3. $\sum_{n=1}^{\infty} \frac{\cos(x^2 + n^2 x)}{m(n^2 + 2)}$
4. $\sum_{n=1}^{\infty} \frac{1}{n^4 + n^3 x^2}$
5. Show that if $0 < r < 1$, then each of the following series is uniformly convergent on R;
   (i) $\sum_{n=1}^{\infty} r^n \cos nx$ *(ii)* $\sum_{n=1}^{\infty} r^n \sin nx$
   (iii) $\sum_{n=1}^{\infty} r^n \cos n^2 x$ *(iv)* $\sum_{n=1}^{\infty} r^n \sin a^n x$.

ANSWERS

1. Uniformly convergent for all real x
2. Uniformly convergent for all real x
3. Uniformly convergent for all real x
4. Uniformly convergent for all real x.

14. HOW TO TEST A SERIES FOR CONVERGENCE?

We have four types of infinite series,

(1) *Positive term series*
(2) *Geometric series*
(3) *Alternating series*
(4) *Power series*

(i) For positive terms series first find u_n and if possible evaluate $\underset{n\to\infty}{\text{Lt}}\ u_n$. If limit of u_n is $\ne 0$, the series is divergent. If limit of $u_n = 0$, then compare $\sum u_n$ with $\sum v_n$, where v_n is always of the type $\frac{1}{n^p}$; compare $\sum u_n$ with $\sum \frac{1}{n^p}$ and

apply comparison test ; if comparison test fails
apply Ratio Test ; if ratio test fails
apply Raabe's Test ; if Raabe's test fails
apply logarithmic Test ; If logarithmic test fails
apply Gauss Test

Special cases : *(a)* If in Ratio test $\frac{u_n}{u_{n+1}}$ involves e, we directly apply logarithmic test.

(b) If in Ratio test it is possible to expand $\frac{u_n}{u_{n+1}}$ in powers of $\frac{1}{n}$ then directly apply Gauss Test.

(ii) For alternating series apply Leibnitz's rule.

(iii) The geometric series $1 + x + x^2 \ldots\ldots \infty$
converges if $-1 < x < 1$ *i.e.*, $|x| < 1$
diverges if $x \ge 1$
oscillates finitely if $x = -1$
oscillates infinitely if $x < -1$

(iv) For power series apply the Ratio test. If Ratio test fails then apply the same tests as applied in *(i)* case.

REVIEW OF THE CHAPTER

1. A sequence $\{a_n\}$ is said to be bounded if $\exists$s two real numbers, k and K such that
$$k \le a_n \le K \quad \forall\ n \in N.$$
2. A sequence $\{a_n\}$ is said to be:
   (i) convergent if $\underset{n\to\infty}{\text{Lt}}\ a_n$ is finite
   (ii) divergent if $\underset{n\to\infty}{\text{Lt}}\ a_n$ is not finite *i.e.*, $\underset{n\to\infty}{\text{Lt}}\ a_n = +\infty$ or $-\infty$
   (iii) oscillary if $\{a_n\}$ neither converges to a finite number nor diverges to $+\infty$ or $-\infty$.
3. A sequence $\{a_n\}$ is said to:
   (i) monotonic increasing if $a_{n+1} \ge a_n \quad \forall\ n \in N$
   (ii) monotonic decreasing if $a_{n+1} \le a_n \quad \forall\ n \in N$
   (iii) a sequence is said to be monotonic if it is either monotonic increasing or monotonic decreasing.
4. Every convergent sequence is bounded.

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5. Necessary and sufficient condition for the convergence of monotonic sequence is that it is bounded.

6. (*i*) A monotonic increasing sequence which is bounded above converges and if it is not bounded above it diverges to $+\infty$.

(*ii*) A monotonic decreasing sequence which is bounded below converges and if it not bounded below diverges to $-\infty$.

7. **Infinite series:** If $\{u_n\}$ is a sequence of real numbers then the expression $\sum_{n=1}^{\infty} u_n = u_1 + u_2 + \ldots + u_n + \ldots \infty$ is called an infinite series.

8. **Behaviour of infinite series:** An infinite series $\Sigma\, u_n$ converges, diverges or oscillates (finitely or infinitely) according as the sequence $\{S_n\}$ of its partial sums converges, diverges or oscillates.

Thus (*i*) If series $\sum_{n=1}^{\infty} u_n$ is convergent, then $\operatorname{Lt}_{n\to\infty} u_n = 0$ but converse is not true.

(*ii*) If $\operatorname{Lt}_{n\to\infty} u_n \neq 0$ then $\sum_{n=1}^{\infty} u_n$ is not convergent. But if $\sum_{n=1}^{\infty} u_n$ is a +ve term series then if $\operatorname{Lt}_{n\to\infty} u_n \neq 0$ then $\sum_{n=1}^{\infty} u_n$ diverges to $+\infty$ $\because$ a +ve term series either converges or diverges to $+\infty$.

9. **Cauchy's general principle of convergence:** The necessary and sufficient condition for the convergence of infinite series is that given $\varepsilon > 0$, however small $\exists$s a +ve integer m such that $|S_{n+p} - S_n| < \varepsilon$ for $n \geq m$ and $p \in N$.
i.e., $|u_{n+1} + u_{n+2} \cdots u_{n+p}| < \varepsilon$ for $n \geq m$ and $p \in N$

10. **Comparison tests:** If $\Sigma\, u_n$ and $\Sigma\, v_n$ are two +ve term series then:

Test 1. (*a*) If $u_n \leq K\, v_n$ $(K > 0)$ $\forall\, n > m$ and $\Sigma\, v_n$ is convergent then $\Sigma\, u_n$ is also convergent

(*b*) $u_n \geq k\, v_n$ $(k > 0)$ $\forall\, n > m$ and $\Sigma\, v_n$ is divergent then $\Sigma\, u_n$ is also divergent.

Test 2. $h < \dfrac{u_n}{v_n} < k$ $(h, k > 0)$ $\forall\, n > m$ both $\Sigma\, u_n$ and $\Sigma\, v_n$ converge and diverge together.

Test 3. (*i*) If $\operatorname{Lt}_{n\to\infty} \dfrac{u_n}{v_n} = l$ (finite and non-zero) then $\Sigma\, u_n$ and $\Sigma\, v_n$ both converge and diverge together.

(*ii*) If $\operatorname{Lt}_{n\to\infty} \dfrac{u_n}{v_n} = 0$ and $\Sigma\, v_n$ converges then $\Sigma\, u_n$ also converges.

(*iii*) If $\operatorname{Lt}_{n\to\infty} \dfrac{u_n}{v_n} = \infty$ and $\Sigma\, v_n$ diverges then $\Sigma\, u_n$ also diverges.

(*iv*) If $\operatorname{Lt}_{n\to\infty} \dfrac{u_n}{v_n} = \infty$ and $\Sigma\, u_n$ converges then $\Sigma\, v_n$ also converges.

Test 4. (*i*) If $\dfrac{u_n}{u_{n+1}} > \dfrac{v_n}{v_{n+1}}$ for $n > m$ and $\Sigma\, v_n$ is convergent then $\Sigma\, u_n$ is also convergent.

(*ii*) If $\dfrac{u_n}{u_{n+1}} < \dfrac{v_n}{v_{n+1}}$ for $n > m$ and $\Sigma\, v_n$ is divergent then $\Sigma\, u_n$ is also divergent.

p-Series or Hyper-Harmonic Series: The series of the type $\sum \dfrac{1}{n^p}$ is known as p- series of hyper harmonic series and it converges if $p > 1$ and diverges if $p \leq 1$.

D'Alembert's Ratio Test: If $\Sigma\, u_n$ is a +ve term series and $\operatorname{Lt}_{n\to\infty} \dfrac{u_n}{u_{n+1}} = l$. Then $\Sigma\, u_n$ is convergent if $l > 1$ and divergent if $l < 1$ when $l = 1$; Ratio test fails.

Raabe's Test: If $\Sigma\, u_n$ is a +ve terms series and $\operatorname{Lt}_{n\to\infty} n\left[\dfrac{u_n}{u_{n+1}} - 1\right] = l$ then $\Sigma\, u_n$ converges if $l > 1$ and diverges if $l < 1$ Raabe's test fails when $l = 1$.

Logarithmic Test: If $\Sigma\, u_n$ is a +ve term series and $\operatorname{Lt}_{n\to\infty} n \log \dfrac{u_n}{u_{n+1}} = l$ then $\Sigma\, u_n$ converges if $l > 1$ and diverges if $l < 1$

Logarithmic test fails when $l = 1$.

Gauss Test: If $\Sigma\, u_n$ is a +ve term series $\dfrac{u_n}{u_{n+1}}$ can be expressed as $\dfrac{u_n}{u_{n+1}} = 1 + \dfrac{\lambda}{n} + 0\left(\dfrac{1}{n^2}\right)$ then $\Sigma\, u_n$ converges if $\lambda > 1$ and diverges if $\lambda \leq 1$.

Cauchy's root Test: If $\Sigma\, u_n$ is a +ve term series and $\operatorname{Lt}_{n\to\infty} (u_n)^{\frac{1}{n}} = l$ then $\Sigma\, u_n$ is a convergent series if $l < 1$ and divergent if $l > 1$. Cauchy's root test fails where $l = 1$.

Cauchy's Integral Test: If $f(x)$ is a non-negative, monotonic decreasing function of x such that $f(n) = u_n$ for all positive integral values of n, then $\Sigma\, u_n$ and $\int_1^{\infty} f(x)\, dx$ converge or diverge together.

Leibnitz's Test on Alternating Series: The alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} u_n$ converges if

(*i*) $u_n > u_{n+1}$ (*ii*) $\operatorname{Lt}_{n\to\infty} u_n = 0$.

Absolute Convergence of a series:

Σu_n is absolutely convergent if $\Sigma\, |u_n|$ is convergent

Conditional Convergence of a series:

Σu_n is conditionally convergent if it is convergent but does not converge absolutely

Every absolutely convergent series is convergent.

If in power series $\sum_{n=1}^{\infty} a_n x^n$; $\operatorname{Lt}_{n\to\infty} \dfrac{a_n}{a_{n+1}} = l$, then interval of convergence of the power series is $(-l, l)$.

Uniform Convergence of Series of functions: A series $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly to a function $S(x)$ if for $\varepsilon > 0$; $\exists$s a +ve integer m depending on ε and independent of x such that for every $x \in I$, $|S_n(x) - S(x)| < \varepsilon$ $\forall\, n > m$.

Weierstrass's M-test: A series $\Sigma\, u_n(x)$ of functions converges uniformly and absolutely on an interval I if $\exists$s a convergent series $\sum_{n=1}^{\infty} M_n$ of +ve constants such that, $|u_n(x)| \leq M_n$ $\forall\, n \in N$ and $\forall\, x \in I$

SHORT ANSWER TYPE QUESTIONS

1. Prove that every convergent sequence is bounded.
[**Hint:** See art. 5.9]
2. Define monotonic increasing and decreasing sequence.
[**Hint:** See art. 5.7]
3. Give an example of a monotonic increasing sequence which is (*i*) convergent (*ii*) divergent.
[**Hint:** S.E. 1 art. 5.10]
4. Give an example of a monotonic decreasing sequence which is (*i*) convergent (*ii*) divergent.
[**Hint:** S.E. 2 art. 5.10]
5. Define convergence, divergence, oscillation of an infinite series. **(P.T.U., Dec. 2007)**
[**Hint:** See art. 5.15]
6. Prove that a positive term series either converges or diverges to $+\infty$. [**Hint:** See art. 5.18]
7. State Cauchy's general principle of convergence. [**Hint:** See art. 5.19(*b*)]
8. Prove that sequence $\{a_n\}$, where $a_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \dots \frac{1}{2^n} + \dots \infty$ is convergent.
[**Hint:** S.E. 3(*iii*) art. 5.10]
9. If $\sum_{n=1}^{\infty} a_n$ is convergent then $\underset{n\to\infty}{\text{Lt}}\, a_n = 0$. Give an example to show that converse is not true.
[**Hint:** See art. 5.17] **(P.T.U., Dec. 2003, May 2004, Jan. 2009)**
10. Show that $\sum_{n=1}^{\infty} \sqrt{\frac{n}{2(n+1)}}$ is divergent. [**Hint:** S.E. 11(*ii*) art. 5.21]
11. Test the convergence of the following series:

(*i*) $\sum \frac{1}{n(n+1)}$ [**Hint:** S.E. 8 (*iii*) art. 5.21]

(*ii*) $\sum \frac{1}{n \log n}$ [**Hint:** S.E. 12 (*ii*) art. 5.21]

(*iii*) $\sum \frac{n^2+1}{n^3+1}$ [**Hint:** S.E. 13 (*ii*) art. 5.21] **(P.T.U., May 2006)**

(*iv*) $\sum_{n=1}^{\infty} \sin \frac{1}{n}$ **(P.T.U., Dec. 2010)**

$$\left[\textbf{Hint: } \text{Let } u_n = \sin\frac{1}{n};\ v_n = \frac{1}{n};\ \frac{u_n}{v_n} = \frac{\sin\frac{1}{n}}{\frac{1}{n}};\ \underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}} \frac{\sin\frac{1}{n}}{\frac{1}{n}} = 1 \text{ and } \sum v_n = \sum \frac{1}{n} \text{ diverges} \neq 0 \right.$$

$$\left. \therefore \sum u_n \text{ diverges} \right]$$

(*v*) $\sum \frac{n+1}{n^p}$ [**Hint:** S.E. 10 (*ii*) art. 5.21]

(*vi*) $\frac{1}{2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^4} + \dots \infty.$

$$\left[\textbf{Hint: } \left(\frac{1}{2} + \frac{1}{2^3} + \dots \infty\right) + \left(\frac{1}{3^2} + \frac{1}{3^4} + \dots \infty\right) \text{ both and G.Ps with C.R.} < 1 \ \therefore \text{ convergent}\right]$$

(*vii*) $\frac{1}{1.4} + \frac{1}{2.5} + \frac{1}{3.6} + \dots \infty$; $\left[\textbf{Hint: } u_n = \frac{1}{n(n+3)} = \frac{1}{n^2\left(1+\frac{3}{n}\right)}. \text{ Take } \Sigma v_n = \sum \frac{1}{n^2} \text{ which is convergent}\right]$

(*viii*) $\sum \frac{2n^3+5}{4n^5+1}$; $\left[\textbf{Hint: } \text{Take } v_n = \frac{1}{n^2}\right]$

(*ix*) $\sum_{n=1}^{\infty} \frac{1}{n^p (n+1)^p}$; $\left[\textbf{Hint: } u_n = \frac{1}{n^{2p}\left(1+\frac{1}{n}\right)^p}. \text{ Take } v_n = \frac{1}{n^{2p}}\right]$

(*x*) $\sum_{n=1}^{\infty} \frac{n+1}{n(2n-1)}$; [**Hint:** $u_n = \frac{n\left(1+\frac{1}{n}\right)}{n^2\left(2-\frac{1}{n}\right)} = \frac{1}{n}\frac{1+\frac{1}{n}}{2-\frac{1}{n}}$; Take $v = \frac{1}{n}$]

12. If Σu_n and Σv_n are two +ve term series then

(*i*) If $\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = l$ (finite and non-zero) then Σu_n and Σv_n both converge and diverge together.
[**Hint:** See art. 5.20 Test III(*i*)]

(*ii*) If $\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \infty$ and Σv_n diverges then Σu_n also diverges. [**Hint:** See art. 5.20 Test III(*iii*)]

(*iii*) $\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = 0$ and Σv_n converges then Σu_n also converges. [**Hint:** See art. 5.20 Test III(*ii*)]

13. Suppose $a_n > 0, b_n > 0 \quad \forall n \in \mathbb{N}$. If $\underset{n\to\infty}{\text{Lt}} \frac{a_n}{b_n} = \infty$ and Σa_n converges. Can anything be said about Σb_n? Give reason for your answer. [See art. 5.20 Test III (*iv*)]. **(P.T.U., Dec. 2004)**
14. Test for convergence of the series:

(*i*) $\sum \frac{n^2+1}{n^3+1}$ **(P.T.U., May 2006)**
[**Hint:** S.E. 13 (*ii*) art. 5.21]

(*ii*) $\sum \frac{1}{\sqrt{n}+\sqrt{n+1}}$ **(P.T.U., Dec. 2003)**
[**Hint:** S.E. 11 (*i*) art. 5.21]

(*iii*) $\sum \frac{1}{n \log n}$ **(P.T.U., Dec. 2002)**
[**Hint:** S.E. 12 (*ii*) art. 5.21]

(*iv*) $\sum \sqrt[3]{n^3+1} - \sqrt[3]{n^3}$ **(P.T.U., May 2007)**
[**Hint:** S.E. 16 (*ii*) art. 5.21]

15. Show that $\sum \sqrt{\frac{n}{n+1}}$ is divergent.
16. Show that $\sum \frac{n}{2n+1}$ is not convergent. $\left[\textbf{Hint: } \underset{n\to\infty}{\text{Lt}} \frac{n}{2n+1} = \frac{1}{2} \neq 0 \ \therefore \text{ not convergent}\right]$

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17. Show that $\sum \frac{n}{1+2^{-n}}$ is divergent.

$$\left[\textbf{Hint: } \underset{n\to\infty}{\text{Lt}} \frac{n}{1+2^{-n}} = \frac{\infty}{1+0} = \infty \neq 0 \text{ and } \Sigma\, u_n \text{ is +ve term series } \therefore \text{ divergent}\right]$$

18. Show that $\sum \left(\frac{n}{n+1}\right)^n$ is divergent.

$$\left[\textbf{Hint: } \underset{n\to\infty}{\text{Lt}} \left(\frac{n}{n+1}\right)^n = \underset{n\to\infty}{\text{Lt}} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} \neq 0. \text{ Also } \Sigma\, u_n \text{ is +ve term series } \therefore \Sigma\, u_n \text{ is divergent}\right].$$

19. Is the series $1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots \infty$, summable ? If so find its sum. [**Hint:** See S.E. 3 (*iii*) art. 5.10]

20. State Cauchy's root test and prove the following:

(*i*) $\sum \frac{1}{n^n}$ is convergent

(*ii*) $\sum 5^{-n-(-1)^n}$ is convergent

$$\left[\textbf{Hint: } \underset{n\to\infty}{\text{Lt}} (u_n)^{\frac{1}{n}} = 5^{-1} \times \frac{1}{5^{\frac{(-1)^n}{n}}} = \frac{1}{5} \times 1 = \frac{1}{5} < 1 \;\therefore\; \text{convergent} \left(\because \underset{n\to\infty}{\text{Lt}} \frac{(-1)^n}{n} = 0\right)\right]$$

(*iii*) $\sum \left(\frac{n+1}{3n}\right)^n$ is convergent

(*iv*) $\sum \left(1 + \frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$ is convergent. [**Hint:** S.E. 3 (*ii*) art. 5.26] **(P.T.U., May 2008)**

(*v*) $\sum_{n=2}^{\infty} \frac{1}{(\log n)^n}$. [**Hint:** S.E. 1 (*iii*) art. 5.26] **(P.T.U., Dec. 2002)**

(*vi*) $\sum \left(\frac{n}{n+1}\right)^{n^2}$. [**Hint:** S.E. 1 (*i*) art. 5.26] **(P.T.U., May 2009)**

21. State Cauchy's Integral test and prove the following:

(*i*) $\sum \frac{1}{n^2+1}$ is convergent. [**Hint:** See S.E. 1 art. 5.27] **(P.T.U., Dec. 2002)**

(*ii*) $\sum \frac{8\tan^{-1} n}{1+n^2}$ is convergent. [**Hint:** See S.E. 7 art. 5.27] **(P.T.U., May 2002, Dec. 2005)**

22. Apply Cauchy's Integral test to test the convergence of the series $\sum_{n=1}^{\infty} 1/n^p$. [**Hint:** S.E. 3 art. 5.27]

23. Test the convergence of the series $\sum \frac{x^{n+1}}{(n+1)\sqrt{n}}$. [**Hint:** S.E. 10 art. 5.22]

24. The series $\frac{1}{3} - \frac{1}{2} + \frac{1}{9} - \frac{1}{4} + \frac{1}{27} - \frac{1}{8} + \ldots \frac{1}{3^n} - \frac{1}{2^n} + \ldots \infty$ does not meet one of the conditions of Leibnitz's test, which one ? Find the sum of the series. [**Hint:** S.E. 5 art. 5.28] **(P.T.U., Dec. 2004)**

25. Examine the convergence of the following series: **(P.T.U., May 2010)**

(*i*) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots \infty$; [**Hint:** S.E. 1 (*ii*) art. 5.28]

(*ii*) $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \ldots \infty$; [**Hint:** S.E. 1 (*i*) art. 5.28]

(*iii*) $2 - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \ldots \infty$; [**Hint:** S.E. 2 (*i*) art. 5.28]

(*iv*) $1 + \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} - \frac{1}{7^2} - \frac{1}{8^2} + \ldots \infty$; [**Hint:** S.E. 4 art. 5.28]

(*v*) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\, n}{2n-1}$; [**Hint:** S.E. 3 art. 5.28]

(*vi*) $\sum_{n=1}^{\infty} \frac{1}{\log(n+1)}$

$$\left[\textbf{Hint: } u_n = \frac{1}{\log(n+1)},\; u_{n+1} = \frac{1}{\log(n+2)},\; \log(n+1) < \log(n+2): \frac{1}{\log(n+1)} > \frac{1}{\log(n+2)}\right.$$
$$\left. u_n > u_{n+1}\; \forall\, n \;\; \underset{n\to\infty}{\text{Lt}}\; u_n = \underset{n\to\infty}{\text{Lt}} \frac{1}{\log(n+1)} = 0. \therefore \text{ Convergent.}\right]$$

(*vii*) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+1}$ (*viii*) $\sum_{n=1}^{\infty} \frac{n\,(-1)^{n-1}}{5^n}$.

26. Define conditional convergence of a series.

27. Prove that every absolutely convergent series is convergent. [**Hint:** art. 5.30]

28. For what value of x does the series $\sum_{n=0}^{\infty} (-1)^n (4x+1)^n$ converges absolutely? **(P.T.U., May 2003)**

[**Hint.** S.E. 3 art. 5.30]

29. Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^n \tan^{-1} n}{1+n^2}$ [**Hint:** S.E. 4 art. 5.30] **(P.T.U., May 2003)**

30. Test whether the following series are absolutely convergent or conditionally convergent

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+ve deviation	-ve deviation
Pobs > Pcal	Pobs < Pcal

(i) $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots \infty$. **[Hint:** S.E. 1(*a*) art. 5.29] **(P.T.U., Dec. 2006)**

(ii) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$. **[Hint:** S.E. 1(*b*) art. 5.29]

(iii) $\sum (-1)^{n-1} \frac{n+1}{n^2}$. **(P.T.U., Dec. 2010)**

31. Prove that the series $\frac{\sin x}{1} - \frac{\sin 2x}{2^2} + \frac{\sin 3x}{3^2} \ldots \infty$ converges absolutely. **[Hint:** S.E. 2 art. 5.30]

32. Prove that $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \infty$ converges $\forall x$

$$\left[\textbf{Hint}: u_n = \frac{x^n}{n!}, u_{n+1} = \frac{x^{n+1}}{(n+1)!}; \frac{u_n}{u_{n+1}} = \frac{(n+1)}{x}; \operatorname*{Lt}_{n \to \infty} \frac{u_n}{u_{n+1}} = \infty \ \forall\, x \ \therefore \ \Sigma u_n \text{ converges } \forall n\right].$$

33. Find the interval of convergence of $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \infty$.

[Hint: S.E. 5 art. 5.31]

34. For what values of x, the power series $1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 \ldots + \left(\frac{-1}{2}\right)^n (x-2)^n + \ldots \infty$ converges?

[Hint: S.E. 6 art. 5.31]

35. What do you understand by uniform convergence of a series? Explain with the help of an example.
[Hint: See art. 5.32] **(P.T.U., May 2004, May 2005)**

36. State Weierstrass's M-Test for uniform convergence of $\Sigma u_n(x)$ in an interval and apply it to show that $\sum_{n=1}^{\infty} \frac{\sin nx}{n^p}$ $(p > 1)$ converges uniformly for all values of x. **[Hint:** S.E. 1 (*ii*) art. 5.33] **(P.T.U., May 2003)**

37. State the following:

(i) *p*-series test or Hyper Harmonic test **(P.T.U., Dec. 2004)**
(ii) D'Alembert's Ratio test **(P.T.U., May 2007)**
(iii) Raabe's test **(P.T.U., May 2007)**
(iv) Logarithmic test
(v) Gauss test **(P.T.U., Dec. 2002)**
(vi) Cauchy's root test **(P.T.U., May 2003, Dec. 2005, Jan. 2009)**
(vii) Cauchy's Integral test **(P.T.U., Dec. 2007)**
(viii) Leibnitz's test on alternating series
(ix) Power series **(P.T.U., Dec. 2003)**
(x) Asolute convergence of a series **(P.T.U., May 2004, May 2005)**
(xi) Uniform convergence of a series of functions
(xii) Weierstrass's M-test.

ANSWERS

11. (i) Convergent (ii) Divergent (iii) Divergent
(iv) Divergent (v) Convergent for $p > 2$, divergent for $p \le 2$
(vi) Convergent (vii) Convergent (viii) Convergent
(ix) Convergent for $p > \frac{1}{2}$; Divergent for $p \le \frac{1}{2}$ (x) Divergent
14. (i) Divergent (ii) Divergent (iii) Divergent (iv) Convergent
19. Convergent
23. Convergent for $x \le 1$; Divergent for $x > 1$
25. (i) Convergent (ii) Convergent (iii) not Convergent (iv) Convergent
(v) not Convergent (vi) Convergent (vii) Convergent (viii) Convergent
28. $x \in \left(-\frac{1}{2}, 0\right)$ **29.** Convergent
30. (i) Converges absolutely (ii) Converges conditionally (iii) Converges conditionally
33. $(-1, 1)$ **34.** $0 < x < 4$.