1

General Theorems

1.1 INTRODUCTION

The students are already familiar with the concept of limit, continuity and differentiability of a function of single variable. Now we shall study some fundamental theorems of calculus. That needs a revision and enlargement of the concept of successive differentiation.

1.2 SUCCESSIVE DIFFERENTIATION

The process of differentiating a function again and again is called **Successive Differentiation.**

Notation. If y be a function of x, then its successive derivatives are denoted by

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots, \frac{d^ny}{dx^n}$$

or $$y_1, y_2, y_3, \ldots, y_n$$

or $$y', y'', y''', \ldots, y^{(n)}$$

ILLUSTRATIVE EXAMPLES

Example 1. *(a) If* $y = \log(\cos x)$, *find* y_3

(b) If $y = \frac{x}{\sqrt{1-x^2}}$, *find* $\frac{d^3y}{dx^3}$

(c) If $y = e^{-x} \cos x$, *show that* $y_4 + 4y = 0$

(d) If $y = \cot x$, *find* y_4.

Sol. (*a*) We have $\quad y = \log(\cos x)$

$$y_1 = \frac{1}{\cos x} \cdot \frac{d}{dx}(\cos x) = \frac{-\sin x}{\cos x} = -\tan x$$

$$y_2 = -\sec^2 x$$

$$y_3 = -2 \sec x \cdot \frac{d}{dx}(\sec x) = -2 \sec x \sec x \tan x$$

$$= -2 \sec^2 x \tan x.$$

(b) We have $y = \dfrac{x}{\sqrt{1-x^2}}$ | Form $y = \dfrac{u}{v}$

$$\frac{dy}{dx} = \frac{\sqrt{1-x^2}\cdot 1 - x\cdot\frac{d}{dx}(\sqrt{1-x^2})}{1-x^2} = \frac{\sqrt{1-x^2} - x\,\frac{1}{2}\cdot(1-x^2)^{-1/2}\,(-2x)}{1-x^2}$$

$$= \frac{\sqrt{1-x^2} + \frac{x^2}{\sqrt{1-x^2}}}{1-x^2} = \frac{1-x^2+x^2}{(1-x^2)^{3/2}} = \frac{1}{(1-x^2)^{3/2}} = (1-x^2)^{-3/2}$$

$$\frac{d^2y}{dx^2} = -\frac{3}{2}(1-x^2)^{-5/2}(-2x) = 3x(1-x^2)^{-5/2}$$

$$\frac{d^3y}{dx^3} = 3x\left(-\frac{5}{2}\right)(1-x^2)^{-7/2}(-2x) + 3(1-x^2)^{-5/2}$$

$$= 15x^2(1-x^2)^{-7/2} + 3(1-x^2)^{-5/2}$$

$$= \frac{15x^2}{(1-x^2)^{7/2}} + \frac{3}{(1-x^3)^{5/2}} = \frac{15x^2+3(1-x^2)}{(1-x^2)^{7/2}} = \frac{12x^2+3}{(1-x^2)^{7/2}} = \frac{3(4x^2+1)}{(1-x^2)^{7/2}}.$$

(c) We have

$$y = e^{-x}\cos x \qquad \text{...(1)}$$

$$y_1 = -e^{-x}\cos x + e^{-x}(-\sin x)$$

$$= -y - e^{-x}\sin x \qquad \text{...(2) [Using (1)]}$$

$$y_2 = -y_1 - (-e^{-x}\sin x + e^{-x}\cos x)$$

$$= -y_1 + e^{-x}\sin x - e^{-x}\cos x$$

$$= -y_1 + (-y - y_1) - y \qquad \text{[Using (1) and (2)]}$$

$$= -2y - 2y_1 \qquad \text{...(3)}$$

$$y_3 = -2y_1 - 2y_2$$

$$= -2y_1 - 2(-2y - 2y_1) \qquad \text{[Using (3)]}$$

$$= 4y + 2y_1$$

$$y_4 = 4y_1 + 2y_2$$

$$= 4y_1 + 2(-2y - 2y_1) \qquad \text{[Using (3)].}$$

$$= -4y$$

$$\therefore \quad y_4 + 4y = 0.$$

(d)

$$y = \cot x$$

$$y_1 = -\text{cosec}^2 x = -(1+\cot^2 x) = -1 - \cot^2 x$$

$$y_2 = 0 - 2\cot x\,(-\text{cosec}^2 x)$$

$$= 2\cot x\,\text{cosec}^2 x = 2\cot x\,(1+\cot^2 x) = 2\cot x + 2\cot^3 x$$

$$y_3 = 2(-\text{cosec}^2 x) + 2.3\cot^2 x\,(-\text{cosec}^2 x)$$

$$= -2\,\text{cosec}^2 x - 6\cot^2 x\,\text{cosec}^2 x$$

$$= -2(1+\cot^2 x) - 6\cot^2 x\,(1+\cot^2 x)$$

$$= -2 - 2\cot^2 x - 6\cot^2 x - 6\cot^4 x$$
$$= -2 - 8\cot^2 x - 6\cot^4 x$$
$$y_4 = 0 - 8 \cdot 2\cot x \cdot (-\operatorname{cosec}^2 x) - 6 \cdot 4\cot^3 x \cdot (-\operatorname{cosec}^2 x)$$
$$= 16\cot x \operatorname{cosec}^2 x + 24\cot^3 x \operatorname{cosec}^2 x$$
$$= 16\cot x(1 + \cot^2 x) + 24\cot^3 x(1 + \cot^2 x)$$
$$= 16\cot x + 40\cot^3 x + 24\cot^5 x.$$

Example 2. *If $y = e^{ax} \sin bx$, prove that $\frac{d^2y}{dx^2} - 2a \cdot \frac{dy}{dx} + (a^2 + b^2)y = 0$*

Sol.
$$y = e^{ax} \sin bx \qquad ...(i)$$
$$\frac{dy}{dx} = e^{ax} \cdot b\cos bx + ae^{ax} \sin bx$$
$$= b \cdot e^{ax} \cos bx + ay \qquad [\because \text{ of } (i)] \qquad ...(ii)$$
$$\frac{d^2y}{dx^2} = b[e^{ax} \cdot (-b \sin bx) + ae^{ax} \cdot \cos bx] + a \cdot \frac{dy}{dx}$$
$$= -b^2 \boxed{e^{ax} \sin bx} + a \boxed{be^{ax} \cdot \cos bx} + a\frac{dy}{dx}$$
$$= -b^2 \cdot y + a\left(\frac{dy}{dx} - ay\right) + a\frac{dy}{dx} \qquad [\because \text{ of } (i) \text{ and } (ii)]$$
$$= -b^2y + a \cdot \frac{dy}{dx} - a^2y + a \cdot \frac{dy}{dx} = 2a\frac{dy}{dx} - (a^2 + b^2)y$$
$$\therefore \quad \frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2 + b^2)y = 0.$$

Example 3. *(a) If $y = \frac{ax + b}{cx + d}$, prove that $2y_1y_3 = 3y_2^2$*

(b) If $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, show that $\frac{d^2y}{dx^2} = -\frac{b^4}{a^2y^3}$

Sol. (*a*)
$$y = \frac{ax + b}{cx + d} \Rightarrow (cx + d)y = ax + b$$

Differentiating w.r.t. x
$$(cx + d)y_1 + cy = a \qquad \therefore \quad (cx + d)y_1 = a - cy$$

Differentiating again w.r.t. x.
$$(cx + d)y_2 + cy_1 = -cy_1 \qquad \therefore \quad (cx + d)y_2 = -2cy_1 \qquad ...(i)$$

Differentiating again w.r.t. x
$$(cx + d)y_3 + cy_2 = -2cy_2 \qquad \therefore \quad (cx + d)y_3 = -3cy_2 \qquad ...(ii)$$

Dividing (*i*) by (*ii*),
$$\frac{y_2}{y_3} = \frac{2y_1}{3y_2} \qquad \therefore \qquad 2y_1y_3 = 3y_2^2.$$

(*b*) $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Differentiating w.r.t. x

$$\frac{2x}{a^2}+\frac{2y}{b^2}\cdot\frac{dy}{dx}=0 \qquad \therefore \quad \frac{dy}{dx}=-\frac{b^2x}{a^2y} \qquad \ldots(i)$$

Differentiating again w.r.t. x

$$\frac{d^2y}{dx^2}=-\frac{b^2}{a^2}\cdot\frac{d}{dx}\left(\frac{x}{y}\right)=-\frac{b^2}{a^2}\cdot\frac{y\cdot 1-x\cdot\frac{dy}{dx}}{y^2}$$

$$=-\frac{b^2}{a^2}\cdot\frac{y-x\left(-\frac{b^2x}{a^2y}\right)}{y^2} \qquad [\because \text{ of } (i)]$$

$$=-\frac{b^2}{a^2}\cdot\frac{a^2y^2+b^2x^2}{a^2y^3}=-\frac{b^2}{a^2}\cdot\frac{a^2b^2}{a^2y^3}$$

$$\left|\ \because \frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow a^2y^2+b^2x^2=a^2b^2\right.$$

$$\therefore \qquad \frac{d^2y}{dx^2}=-\frac{b^4}{a^2y^3}.$$

Example 4. (*a*) *If* $x=a(\theta+\sin\theta)$, $y=a(1+\cos\theta)$, *find the value of* $\frac{d^2y}{dx^2}$ *at* $\theta=\frac{\pi}{2}$.

(*b*) *If* $x=\sin t$, $y=\sin pt$, *prove that* $(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+p^2y=0$.

(*c*) *If* $x=a(\cos\theta+\theta\sin\theta)$, $y=a(\sin\theta-\theta\cos\theta)$, *prove that* $a\theta\frac{d^2y}{dx^2}=\sec^3\theta$.

Sol. (*a*) $\quad x=a(\theta+\sin\theta);\ y=a(1+\cos\theta)$

$$\frac{dx}{d\theta}=a(1+\cos\theta);\ \frac{dy}{d\theta}=-a\sin\theta$$

$$\frac{dy}{dx}=\frac{dy}{d\theta}\Big/\frac{dx}{d\theta}=\frac{-a\sin\theta}{a(1+\cos\theta)}=\frac{-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}=-\tan\frac{\theta}{2}$$

Differentiating again w.r.t. x,

$$\frac{d^2y}{dx^2}=-\frac{1}{2}\sec^2\frac{\theta}{2}\cdot\frac{d\theta}{dx}=-\frac{1}{2}\sec^2\frac{\theta}{2}\cdot\frac{1}{a(1+\cos\theta)}$$

$$=-\frac{1}{2a}\sec^2\frac{\theta}{2}\cdot\frac{1}{2\cos^2\frac{\theta}{2}}=-\frac{1}{4a}\sec^4\frac{\theta}{2}$$

$$\left[\frac{d^2y}{dx^2}\right]_{\theta=\frac{\pi}{2}}=-\frac{1}{4a}\sec^4\frac{\pi}{4}=-\frac{1}{4a}(\sqrt{2})^4=-\frac{1}{a}.$$

(b) $x = \sin t;\ y = \sin pt$

$$\frac{dx}{dt} = \cos t;\ \frac{dy}{dt} = p\cos pt \qquad ...(1)$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dt}\bigg/\frac{dx}{dt} = \frac{p\cos pt}{\cos t}$$

Differentiating both sides w.r.t. x, ...(2)

$$\frac{d^2y}{dx^2} = \frac{\cos t\,(-p^2 \sin pt) - p\cos pt\,(-\sin t)}{\cos^2 t}\cdot\frac{dt}{dx}$$

$$= \frac{-p^2\cos t \sin pt + p\cos pt \sin t}{1-\sin^2 t}\cdot\frac{1}{\cos t}$$

or $$(1-\sin^2 t)\frac{d^2y}{dx^2} = -p^2 \sin pt + \frac{p\cos pt}{\cos t}\sin t$$

or $$(1-x^2)\frac{d^2y}{dx^2} = -p^2 y + \frac{dy}{dx}\cdot x$$ [Using (1) and (2)]

$$\therefore\ (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2 y = 0.$$

(c) $x = a(\cos\theta + \theta\sin\theta),\ y = a(\sin\theta - \theta\cos\theta)$

$$\frac{dx}{d\theta} = a[-\sin\theta + \theta\cdot\cos\theta + \sin\theta] = a\theta\cos\theta$$

$$\frac{dy}{d\theta} = a\,[\cos\theta - (-\theta\sin\theta + \cos\theta)] = a\theta\sin\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta}\bigg/\frac{dx}{d\theta} = \tan\theta$$

Differentiating again w.r.t. x,

$$\frac{d^2y}{dx^2} = \sec^2\theta\,\frac{d\theta}{dx} = \sec^2\theta\cdot\frac{1}{a\theta\cos\theta} \quad \therefore \quad a\theta\cdot\frac{d^2y}{dx^2} = \sec^3\theta.$$

TEST YOUR KNOWLEDGE

1. If $y = x^3 \log x$, prove that $y_4 = \dfrac{6}{x}$.
2. Find the fourth derivative of $\tan x$ at $x = \dfrac{\pi}{4}$.
3. If $y = \dfrac{\log x}{x}$, prove that $\dfrac{d^2y}{dx^2} = \dfrac{2\log x - 3}{x^3}$.
4. If $y = A\sin mx + B\cos mx$, show that $\dfrac{d^2y}{dx^2} + m^2 y = 0$.
5. If $y = ae^{mx} + be^{-mx}$, show that $y_2 = m^2 y$.
6. If $y = A\,e^{mx} + B\,e^{nx}$, prove that $\dfrac{d^2y}{dx^2} - (m+n)\dfrac{dy}{dx} + mny = 0$.

7. If $x = 2\cos t - \cos 2t$, $y = 2\sin t - \sin 2t$, find the value of $\dfrac{d^2y}{dx^2}$, when $t = \dfrac{\pi}{2}$.

8. If $y = (\sin^{-1} x)^2$, prove that $(1 - x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} = 2$.

9. If $y = e^{m\sin^{-1}x}$, prove that $(1 - x^2)y_2 - xy_1 = m^2y$.

10. If $\sqrt{x} + \sqrt{y} = \sqrt{a}$, find the value of $\dfrac{d^2y}{dx^2}$ for $x = a$.

Answers

2. 80 7. $-3/2$ 10. $\dfrac{1}{2a}$.

1.3 STANDARD RESULTS

• **(*i*) To find the n^{th} differential co-efficient of log $(ax + b)$.**

Let $y = \log(ax + b)$

$$y_1 = \frac{1}{ax+b} \cdot a = a(ax+b)^{-1}$$

$$y_2 = (-1)a^2(ax+b)^{-2}$$

$$y_3 = (-1)(-2)a^3(ax+b)^{-3} = (-1)^2\, 2\,!\, a^3(ax+b)^{-3}$$

$$y_4 = (-1)^3\, 3 \cdot 2\,! \cdot a^4(ax+b)^{-4} = (-1)^3 \cdot 3\,!\, a^4(ax+b)^{-4}$$

....

....

$$y_n = (-1)^{n-1} \cdot (n-1)\,!\, a^n (ax+b)^{-n}$$

Hence $$y_n = \frac{(-1)^{n-1}(n-1)\,! \cdot a^n}{(ax+b)^n}.$$

ILLUSTRATIVE EXAMPLES

Example 1. *Find the nth differential co-efficient of log* $\sqrt{\dfrac{2x+1}{x-2}}$.

Sol. Let $$y = \log\sqrt{\frac{2x+1}{x-2}} = \frac{1}{2}\log\frac{2x+1}{x-2}$$

$$= \frac{1}{2}[\log(2x+1) - \log(x-2)]$$

$\therefore$ $$y_n = \frac{1}{2}\left[\frac{(-1)^{n-1}\cdot(n-1)\,!\cdot 2^n}{(2x+1)^n} - \frac{(-1)^{n-1}(n-1)\,!}{(x-2)^n}\right]$$

$$= \frac{1}{2}\cdot(-1)^{n-1}\cdot(n-1)\,!\left[\frac{2^n}{(2x+1)^n} - \frac{1}{(x-2)^n}\right].$$

(ii) To find the nth differential co-efficient of a^{mx}.

Let
$$y = a^{mx}$$
$$y_1 = ma^{mx} \cdot (\log a)$$
$$y_2 = m^2 \cdot a^{mx} \cdot (\log a)^2$$
$$y_3 = m^3 \cdot a^{mx} \cdot (\log a)^3$$
....

....
$$y_n = m^n \cdot a^{mx} \cdot (\log a)^n$$

Cor. If $y = a^x$, $m = 1$ $\therefore$ $y_n = a^x (\log a)^n$.

(iii) To find the nth differential co-efficient of e^{mx}.

Let
$$y = e^{mx}$$
$$y_1 = me^{mx}$$
$$y_2 = m^2 \cdot e^{mx}$$
$$y_3 = m^3 \cdot e^{mx}$$
....

....
$$y_n = m^n \cdot e^{mx}.$$

Cor. If $y = e^x$, $m = 1$ $\therefore$ $y_n = e^x$.

(iv) To find the nth differential co-efficient of sin $(ax + b)$.

Let
$$y = \sin (ax + b)$$
$$y_1 = a \cos (ax + b)$$
$$= a \sin \left(ax + b + \frac{\pi}{2}\right) \qquad | \because \sin (\pi/2 + \theta) = \cos \theta$$
$$y_2 = a^2 \cos \left(ax + b + \frac{\pi}{2}\right) = a^2 \sin \left(ax + b + \frac{\pi}{2} + \frac{\pi}{2}\right)$$
$$= a^2 \sin \left(ax + b + 2 \cdot \frac{\pi}{2}\right)$$
$$y_3 = a^3 \cos \left(ax + b + 2 \cdot \frac{\pi}{2}\right) = a^3 \sin \left(ax + b + 2 \cdot \frac{\pi}{2} + \frac{\pi}{2}\right)$$
$$= a^3 \sin \left(ax + b + 3 \cdot \frac{\pi}{2}\right)$$
....

....
$$\boldsymbol{y_n = a^n \sin \left(ax + b + n \cdot \frac{\pi}{2}\right)}$$

(v) Similarly, the nth differential co-efficient of
$$\boldsymbol{\cos (ax + b) = a^n \cos \left(ax + b + n \frac{\pi}{2}\right)}.$$

Example 2. *Find the nth differential co-efficient of*
(a) $sin^3 x$ *(b) $sin^4 x$.*

Sol. (a) Let $y = \sin^3 x$

Now $\sin 3x = 3 \sin x - 4 \sin^3 x$

$\Rightarrow$ $4 \sin^3 x = 3 \sin x - \sin 3x$

$$\Rightarrow \quad \sin^3 x = \frac{1}{4}(3 \sin x - \sin 3x)$$

$$\Rightarrow \quad y = \frac{3}{4} \sin x - \frac{1}{4} \sin 3x$$

$$\therefore \quad y_n = \frac{3}{4} \sin\left(x + n \cdot \frac{\pi}{2}\right) - \frac{1}{4} \cdot 3^n \sin\left(3x + n \cdot \frac{\pi}{2}\right)$$

(*b*) Let $\quad y = \sin^4 x = (\sin^2 x)^2$

$$= \left[\frac{1 - \cos 2x}{2}\right]^2 = \frac{1}{4}[1 - 2\cos 2x + \cos^2 2x]$$

$$= \frac{1}{4}\left[1 - 2\cos 2x + \frac{1 + \cos 4x}{2}\right]$$

$$= \frac{1}{8}[2 - 4\cos 2x + 1 + \cos 4x] = \frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x$$

$$\therefore \quad y_n = 0 - \frac{1}{2} \cdot 2^n \cos\left[2x + n \cdot \frac{\pi}{2}\right] + \frac{1}{8} \cdot 4^n \cdot \cos\left[4x + n \cdot \frac{\pi}{2}\right]$$

$$= -2^{n-1} \cos\left[2x + n \cdot \frac{\pi}{2}\right] + \frac{1}{2^3} \cdot 2^{2n} \cos\left[4x + n \cdot \frac{\pi}{2}\right]$$

$$= -2^{n-1} \cos\left[2x + n \cdot \frac{\pi}{2}\right] + 2^{2n-3} \cos\left[4x + n \cdot \frac{\pi}{2}\right].$$

Example 3. *Find the nth differential co-efficient of*

(*a*) $\cos 2x \sin 3x$ (*b*) $\sin^3 x \cos^2 x$

(*c*) $\cos x \cos 2x \cos 3x$

Sol. (*a*) Let $\quad y = \cos 2x \sin 3x$

$$= \frac{1}{2}(2 \sin 3x \cos 2x) \qquad \left| \begin{array}{l} 2 \sin A \cos B \\ = \sin(A + B) + \sin(A - B) \end{array} \right.$$

$$= \frac{1}{2}[\sin 5x + \sin x]$$

$$\therefore \quad y_n = \frac{1}{2}\left[5^n \sin\left(5x + \frac{n\pi}{2}\right) + \sin\left(x + \frac{n\pi}{2}\right)\right]$$

(*b*) Let $\quad y = \sin^3 x \cos^2 x$

$$= \frac{1}{4} \cdot (4 \sin^2 x \cos^2 x) \sin x = \frac{1}{4} \sin^2 2x \sin x$$

$$= \frac{1}{4} \cdot \frac{1 - \cos 4x}{2} \cdot \sin x = \frac{1}{8}(\sin x - \cos 4x \sin x)$$

$$= \frac{1}{16}(2 \sin x - 2 \cos 4x \sin x)$$

$$= \frac{1}{16}[2 \sin x - (\sin 5x - \sin 3x)]$$

$$= \frac{1}{16}(2 \sin x + \sin 3x - \sin 5x)$$

$$\therefore \quad y_n = \frac{1}{16}\left[2 \sin\left(x + n \cdot \frac{\pi}{2}\right) + 3^n \sin\left(3x + n \cdot \frac{\pi}{2}\right) - 5^n \sin\left(5x + n \cdot \frac{\pi}{2}\right)\right].$$

(c) Let $y = \cos x \cos 2x \cos 3x$

$$= \frac{1}{2}(2\cos 3x \cos x)\cdot \cos 2x = \frac{1}{2}(\cos 4x + \cos 2x)\cos 2x$$

$$= \frac{1}{4}[2\cos 4x \cos 2x + 2\cos^2 2x] = \frac{1}{4}[\cos 6x + \cos 2x + 1 + \cos 4x]$$

$$= \frac{1}{4}[1 + \cos 2x + \cos 4x + \cos 6x]$$

$$\therefore \quad y_n = \frac{1}{4}\left[2^n \cos\left(2x + n\cdot\frac{\pi}{2}\right) + 4^n\cdot\cos\left(4x + n\,.\,\frac{\pi}{2}\right) + 6^n \cos\left(6x + n\cdot\frac{\pi}{2}\right)\right].$$

(vi) To find the *n*th differential co-efficient of $e^{ax}\sin(bx + c)$.

Let $y = e^{ax}\sin(bx + c)$

$y_1 = e^{ax}\cdot\cos(bx + c)\cdot b + \sin(bx + c)\cdot e^{ax}\cdot a$ | Form $y = uv$

$= e^{ax}[a\sin(bx + c) + b\cos(bx + c)]$

Put $a = r\cos\theta,\ b = r\sin\theta$, so that $a^2 + b^2 = r^2(\cos^2\theta + \sin^2\theta) = r^2$

$$\therefore \quad r = \sqrt{a^2 + b^2}$$

and

$$\tan\theta = \frac{b}{a} \quad \therefore \quad \theta = \tan^{-1}\frac{b}{a}$$

$$\therefore \quad y_1 = e^{ax}[r\cos\theta\sin(bx + c) + r\sin\theta\cos(bx + c)]$$

$$= r\cdot e^{ax}[\sin(bx + c)\cos\theta + \cos(bx + c)\sin\theta]$$

$$= r\cdot e^{ax}\sin(bx + c + \theta).$$

Thus, y_1 has been obtained from y by multiplying it by r and increasing the angle by θ. Applying the same rule for the differentiation of y_1, we have

$$y_2 = r^2 \times e^{ax}\sin(bx + c + 2\theta).$$

Similarly, $y_3 = r^3\cdot e^{ax}\sin(bx + c + 3\theta)$

....

....

$$y_n = r^n e^{ax}\sin(bx + c + n\theta)$$

Hence $y_n = (a^2 + b^2)^{n/2}\cdot e^{ax}\cdot\sin\left[bx + c + n\tan^{-1}\frac{b}{a}\right]$

(vii) Similarly, the *n*th differential co-efficient of

$$\boldsymbol{e^{ax}\cos(bx + c) = (a^2 + b^2)^{n/2}\cdot e^{ax}\cos\left(bx + c + n\tan^{-1}\frac{b}{a}\right)}.$$

Example 4. *Find the nth differential co-efficient of*

(a) $e^{3x}\sin^2(2x)$ *(b)* $e^x\cos x\cos 2x$

Sol. (*a*) Let $y = e^{3x}\sin^2(2x) = e^{3x}\times\frac{1 - \cos 4x}{2} = \frac{1}{2}[e^{3x} - e^{3x}\cos 4x]$

$$\therefore \quad y_n = \frac{1}{2}\left[3^n e^{3x} - (3^2 + 4^2)^{n/2} e^{3x}\cos\left(4x + n\tan^{-1}\frac{4}{3}\right)\right]$$

$$= \frac{1}{2}e^{3x}\left[3^n - 5^n\cos\left(4x + n\tan^{-1}\frac{4}{3}\right)\right]$$

(b) Let $y = e^x \cos x \cos 2x = \dfrac{1}{2}\, e^x\, (2 \cos 2x \cos x)$

$$= \frac{1}{2}\, e^x\, (\cos 3x + \cos x) = \frac{1}{2}\, (e^x \cos x + e^x \cos 3x)$$

$$y_n = \frac{1}{2}\, [(1^2 + 1^2)^{n/2} \cdot e^x \cos (x + n \tan^{-1} \frac{1}{1})]$$

$$+ (1^2 + 3^2)^{n/2} \cdot e^x \cos (3x + n \tan^{-1} \frac{3}{1})]$$

$$= \frac{1}{2} \cdot e^x\, [2^{n/2} \cdot \cos (x + n \cdot \pi/4) + 10^{n/2} \cdot \cos (3x + n \tan^{-1} 3)].$$

TEST YOUR KNOWLEDGE

Find the nth derivative of

1. $\log (ax + x^2)$
2. $\log (x^2 - a^2)$
3. $\cos^2 x$
4. $\cos^3 x$
5. $\cos^4 x$
6. $\sin x \sin 3x$
7. $\sin x \sin 2x \sin 3x$
8. $\sin^2 x \cos^3 x$
9. Find the nth differential co-efficient of $e^x \sin x \cos x$.

Answers

1. $(-1)^{n-1} \cdot (n-1)!\left[\dfrac{1}{x^n} + \dfrac{1}{(x+a)^n}\right]$

2. $(-1)^{n-1} \cdot (n-1)!\left[\dfrac{1}{(x+a)^n} + \dfrac{1}{(x-a)^n}\right]$

3. $2^{n-1} \cos\left(2x + n \cdot \dfrac{\pi}{2}\right)$

4. $\dfrac{1}{4} \cdot 3^n \cos\left(3x + \dfrac{n\pi}{2}\right) + \dfrac{3}{4} \cos\left(x + \dfrac{n\pi}{2}\right)$

5. $2^{n-1} \cos\left(2x + \dfrac{n\pi}{2}\right) + 2^{2n-3} \cos\left(4x + \dfrac{n\pi}{2}\right)$

6. $\dfrac{1}{2}\left[2^n \cos\left(2x + \dfrac{n\pi}{2}\right) - 4^n \cos\left(4x + \dfrac{n\pi}{2}\right)\right]$

7. $\dfrac{1}{4}\left[2^n \sin\left(2x + \dfrac{n\pi}{2}\right) + 4^n \sin\left(4x + \dfrac{n\pi}{2}\right) - 6^n \sin\left(6x + \dfrac{n\pi}{2}\right)\right]$

8. $\dfrac{1}{16}\left[2 \cos\left(x + \dfrac{n\pi}{2}\right) - 3^n \cos\left(3x + \dfrac{n\pi}{2}\right) - 5^n \cos\left(5x + \dfrac{n\pi}{2}\right)\right]$

9. $\dfrac{1}{2} \cdot 5^{n/2} \cdot e^x \sin (2x + n \tan^{-1} 2)$

1.4 LEIBNITZ'S THEOREM

Leibnitz's Theorem helps us to find the nth derivative of the product of two functions.

Statement. *If $y = uv$, where u and v are functions of x, having derivatives of nth order, then*

$$y_n = {}^nC_0 u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \cdots + {}^nC_r u_{n-r} v_r + \cdots + {}^nC_n u v_n$$

where suffixes of u and v denote the number of times they are differentiated.

Note. If x^m, where m is a +ve integer, is one of the factors, taking $v = x^m$ simplifies the process of writing the nth derivative.

ILLUSTRATIVE EXAMPLES

Example 1. *Find the nth derivative of*

(a) $x^3 e^{ax}$ *(b)* $x^2 \sin x$

Sol. *(a)* Let

$$u = e^{ax} \qquad v = x^3$$
$$u_n = a^n e^{ax} \qquad v_1 = 3x^2$$
$$u_{n-1} = a^{n-1} e^{ax} \qquad v_2 = 6x$$
$$u_{n-2} = a^{n-2} e^{ax} \qquad v_3 = 6$$
$$u_{n-3} = a^{n-3} e^{ax} \qquad v_4 = 0$$

Now by Leibnitz's Theorem, we have

$$\frac{d^n}{dx^n}(uv) = {}^nC_0 u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \cdots + {}^nC_n u v_n.$$

$$\therefore \frac{d^n}{dx^n}(e^{ax} \cdot x^3) = {}^nC_0 a^n e^{ax} x^3 + {}^nC_1\, a^{n-1} e^{ax} \cdot 3x^2 + {}^nC_2 a^{n-2} e^{ax} \cdot 6x + {}^nC_3 \cdot a^{n-3} e^{ax} \cdot 6$$

$$= e^{ax} \cdot a^{n-3}\left[a^3x^3 + 3na^2x^2 + 6\frac{n(n-1)}{2!} \cdot ax + 6 \cdot \frac{n(n-1)(n-2)}{3!}\right]$$

$$= e^{ax} \cdot a^{n-3}\,[a^3x^3 + 3na^2x^2 + 3n(n-1) \cdot ax + n(n-1)(n-2)].$$

(b) Let

$$u = \sin x \qquad v = x^2$$
$$u_n = \sin\left(x + \frac{n\pi}{2}\right) \qquad v_1 = 2x$$
$$u_{n-1} = \sin\left[x + (n-1)\frac{\pi}{2}\right] \qquad v_2 = 2$$
$$u_{n-2} = \sin\left[x + (n-2)\frac{\pi}{2}\right] \qquad v_3 = 0$$

$\therefore$ By Leibnitz's Theorem,

$$\frac{d^n}{dx^n}(\sin x \cdot x^2) = {}^nC_0 \sin\left(x + \frac{n\pi}{2}\right) x^2$$
$$+ {}^nC_1 \cdot \sin\left(x + (n-1)\frac{\pi}{2}\right) \cdot 2x + {}^nC_2 \sin\left[x + (n-2)\frac{\pi}{2}\right] 2$$

$$= x^2 \sin\left(x + \frac{n\pi}{2}\right) + 2nx \sin\left(x + (n-1)\frac{\pi}{2}\right) + 2 \cdot \frac{n(n-1)}{2 \cdot 1} \sin\left[x + (n-2)\frac{\pi}{2}\right]$$

$$= x^2 \sin\left(x + \frac{n\pi}{2}\right) + 2nx \sin\left(x + (n-1)\frac{\pi}{2}\right) + n(n-1) \sin\left(x + (n-2)\frac{\pi}{2}\right).$$

Example 2. *Find the nth derivative of $x^2 \log x$.*

Sol. Let

$u = \log x$	$v = x^2$
$u_n = \dfrac{(-1)^{n-1}(n-1)!}{x^n}$	$v_1 = 2x$
$u_{n-1} = \dfrac{(-1)^{n-2}(n-2)!}{x^{n-1}}$	$v_2 = 2$
$u_{n-2} = \dfrac{(-1)^{n-3}(n-3)!}{x^{n-2}}$	$v_3 = 0$

∴ By Leibnitz's Theorem,

$$\frac{d^n}{dx^n}(\log x \cdot x^2) = {}^nC_0 \cdot \frac{(-1)^{n-1}(n-1)!}{x^n} x^2 + {}^nC_1 \cdot \frac{(-1)^{n-2}(n-2)!}{x^{n-1}} \cdot 2x + {}^nC_2 \cdot \frac{(-1)^{n-3}(n-3)!}{x^{n-2}} \cdot 2$$

$$= \frac{(-1)^{n-1}(n-1)!}{x^{n-2}} + 2n \frac{(-1)^{n-2}(n-2)!}{x^{n-2}} + n(n-1) \cdot \frac{(-1)^{n-3}(n-3)!}{x^{n-2}}$$

$$= \frac{(-1)^{n-1} n!}{nx^{n-2}} + 2 \cdot \frac{(-1)^{n-2} . n!}{(n-1)x^{n-2}} + \frac{(-1)^{n-3} n!}{(n-2)x^{n-2}}$$

$$= \frac{(-1)^{n-1} n!}{x^{n-2}} \left[\frac{1}{n} - \frac{2}{n-1} + \frac{1}{n-2}\right].$$

Example 3. *If $y = x^n \log x$, prove that $y_{n+1} = \dfrac{n!}{x}$.*

Sol. $$y = x^n \log x \quad \text{...(i)}$$

∴ $$y_1 = x^n \cdot \frac{1}{x} + nx^{n-1} \cdot \log x$$

or $$xy_1 = x^n + nx^n \log x$$

or $$xy_1 = x^n + ny \quad [\because \text{ of } (i)]$$

Differentiating n times by Leibnitz's Theorem,

$$y_{n+1} x + n \cdot y_n \cdot 1 = n! + ny_n$$

∴ $$xy_{n+1} = n! \quad \text{or} \quad y_{n+1} = \frac{n!}{x}.$$

Note. To differentiate $(1 - x^2)\, y_2$, n times by Leibnitz's Theorem.

Let

$u = y_2$	$v = 1 - x^2$
$u_n = y_{n+2}$	$v_1 = -2x$
$u_{n-1} = y_{n+1}$	$v_2 = -2$
$u_{n-2} = y_n$	$v_3 = 0$

$$\therefore \quad \frac{d^n}{dx^n}[(1-x^2)\,y_2] = y_{n+2}\,(1-x^2) + n\,y_{n+1}\,(-2x) + \frac{n(n-1)}{2!}\,y_n\,(-2)$$

$$= (1-x^2)\,y_{n+2} - 2nxy_{n+1} - n(n-1)\,y_n.$$

Example 4. *Differentiate n times the equation* $(1-x^2)\,y_2 - xy_1 + a^2y = 0$

Sol. $\quad (1-x^2)\,y_2 - xy_1 + a^2y = 0$

Differentiating every term n times by Leibnitz's Theorem

$$\left[y_{n+2}\,(1-x^2) + ny_{n+1}\,(-2x) + \frac{n(n-1)}{2!}\cdot y_n\,(-2)\right]$$

$$- [y_{n+1}\cdot x + ny_n \cdot 1] + a^2\,y_n = 0$$

or $\quad (1-x^2)\,y_{n+2} + (-2nx - x)\,y_{n+1} + (-n^2 + n - n + a^2)\,y_n = 0$

or $\quad (1-x^2)\,y_{n+2} - (2n+1)\,xy_{n+1} - (n^2 - a^2)\,y_n = 0.$

Example 5. *If* $y = a \cos(\log x) + b \sin(\log x)$, *show that*

$$x^2y_{n+2} + (2n+1)\,xy_{n+1} + (n^2+1)\,y_n = 0.$$

Sol. $\quad y = a\cos(\log x) + b\sin(\log x) \quad \text{...}(i)$

$\Rightarrow \quad y_1 = -a\sin(\log x)\cdot\frac{1}{x} + b\cos(\log x)\cdot\frac{1}{x}$

or $\quad xy_1 = -a\sin(\log x) + b\cos(\log x)$

Differentiating again,

$$xy_2 + y_1 \cdot 1 = -a\cos(\log x)\times\frac{1}{x} - b\sin(\log x)\times\frac{1}{x}$$

or $\quad x^2y_2 + xy_1 = -[a\cos(\log x) + b\sin(\log x)]$

or $\quad x^2y_2 + xy_1 = -y \quad | \because$ of (i)

or $\quad x^2y_2 + xy_1 + y = 0.$

Differentiating every term n times by Leibnitz's Theorem,

$$\left[y_{n+2}\,x^2 + ny_{n+1}\,2x + \frac{n(n-1)}{2!}\,y_n\cdot 2\right] + [y_{n+1}\cdot x + ny_n\cdot 1] + y_n = 0$$

or $\quad x^2y_{n+2} + (2nx + x)\,y_{n+1} + (n^2 - n + n + 1)\,y_n = 0$

or $\quad x^2y_{n+2} + (2n+1)\,xy_{n+1} + (n^2+1)\,y_n = 0.$

Example 6. *If* $y = (\sin^{-1} x)^2$, *prove that* $(1-x^2)\,y_{n+2} - (2n+1)\,xy_{n+1} - n^2y_n = 0.$

Sol. $\quad y = (\sin^{-1} x)^2 \quad \text{...}(i)$

$\Rightarrow \quad y_1 = 2\sin^{-1} x \cdot \frac{1}{\sqrt{1-x^2}}$

Squaring and cross-multiplying,

$$(1 - x^2)\, y_1^2 = 4(\sin^{-1} x)^2 = 4y \qquad [\because \text{ of } (i)]$$

Differentiating again,

$$(1 - x^2) \cdot 2y_1 y_2 + y_1^2 (-2x) = 4y_1$$

Dividing both sides by $2y_1$, $(1 - x^2)\, y_2 - xy_1 = 2$

Differentiating every term n times by Leibnitz's Theorem,

$$\left[y_{n+2}(1-x^2) + ny_{n+1}(-2x) + \frac{n(n-1)}{2!} \cdot y_n(-2)\right] - (y_{n+1} \cdot x + ny_n \cdot 1) = 0$$

or $$(1 - x^2)\, y_{n+2} - (2n + 1)\, xy_{n+1} + (-n^2 + n - n)\, y_n = 0$$

or $$(1 - x^2)\, y_{n+2} - (2n + 1)\, xy_{n+1} - n^2 y_n = 0.$$

Example 7. *If $y = e^{a \sin^{-1} x}$, show that $(1 - x^2)\, y_{n+2} - (2n + 1)\, xy_{n+1} - (n^2 + a^2)\, y_n = 0$.*

(M.D.U. Dec. 2010; Kerala 2010)

Sol. $$y = e^{a \sin^{-1} x} \qquad ...(i)$$

$\Rightarrow$ $$y_1 = e^{a \sin^{-1} x} \cdot \frac{a}{\sqrt{1-x^2}} = \frac{ay}{\sqrt{1-x^2}} \qquad [\because \text{ of } (i)]$$

Squaring and cross-multiplying, $(1 - x^2)\, y_1^2 = a^2 y^2$

Differentiating again, $(1 - x^2) 2y_1 y_2 + y_1^2 (-2x) = a^2 \cdot 2yy_1$

Dividing both sides by $2y_1$, $(1 - x^2) y_2 - xy_1 = a^2 y$

Differentiating every term n times by Leibnitz's Theorem,

$$\left[y_{n+2}(1-x^2) + ny_{n+1}(-2x) + \frac{n(n-1)}{2!}\, y_n(-2)\right] - [y_{n+1} \cdot x + ny_n \cdot 1] = a^2 y_n$$

or $$(1 - x^2) y_{n+2} - (2n + 1) xy_{n+1} + (-n^2 + n - n - a^2)\, y_n = 0$$

or $$(1 - x^2)\, y_{n+2} - (2n + 1)\, xy_{n+1} - (n^2 + a^2) y_n = 0.$$

Example 8. *If $y = \sin (m \sin^{-1} x)$, prove that $(1 - x^2)\, y_{n+2} = (2n + 1)\, xy_{n+1} + (n^2 - m^2)\, y_n$.*

Sol. $$y = \sin (m \sin^{-1} x) \qquad ...(i)$$

$\Rightarrow$ $$y_1 = \cos (m \sin^{-1} x) \frac{m}{\sqrt{1-x^2}}$$

Squaring and cross-multiplying,

$$(1 - x^2)\, y_1^2 = m^2 \cos^2 (m \sin^{-1} x) = m^2 [1 - \sin^2 (m \sin^{-1} x)]$$

$\therefore$ $$(1 - x^2)\, y_1^2 = m^2 (1 - y^2) \qquad [\because \text{ of } (i)]$$

Differentiating again, $(1 - x^2)\, 2y_1 y_2 + y_1^2 (-2x) = m^2 (-2yy_1)$

Dividing both sides by $2y_1$, $(1 - x^2)\, y_2 - xy_1 + m^2 y = 0$

Differentiating every term n times by Leibnitz's Theorem,

$$\left[y_{n+2}(1-x^2)+ny_{n+1}(-2x)+\frac{n(n-1)}{2!}y_n(-2)\right]-[y_{n+1}x+ny_n\cdot 1]+m^2y_n=0$$

or
$$(1-x^2)\,y_{n+2}-(2n+1)\,xy_{n+1}+(-n^2+n-n+m^2)\,y_n=0$$

or
$$(1-x^2)\,y_{n+2}-(2n+1)\,xy_{n+1}-(n^2-m^2)\,y_n=0$$

or
$$(1-x^2)\,y_{n+2}=(2n+1)\,xy_{n+1}+(n^2-m^2)\,y_n.$$

1.5 DETERMINATION OF THE VALUE OF THE *n*th DERIVATIVE OF A FUNCTION FOR x = 0

Sometimes, it is required to find the nth derivative of a function for $x = 0$. This is illustrated by the following solved examples.

Example 9. *Find the value of the nth derivative of $e^{m\sin^{-1}x}$ for $x = 0$.*

Sol. Let
$$y=e^{m\sin^{-1}x} \qquad \text{...(i)}$$

$$y_1=e^{m\sin^{-1}x}\cdot\frac{m}{\sqrt{1-x^2}}=\frac{my}{\sqrt{1-x^2}} \qquad \text{...(ii)}$$

Squaring and cross-multiplying, $(1-x^2)\,y_1^{\,2}=m^2y^2$

Differentiating again, $(1-x^2)\,2y_1y_2+y_1^2\,(-2x)=m^2\cdot 2yy_1$

Dividing both sides by $2y_1$, $(1-x^2)\,y_2-xy_1=m^2y$ ...(iii)

Differentiating every term n times by Leibnitz's Theorem,

$$\left[y_{n+2}\cdot(1-x^2)+ny_{n+1}(-2x)+\frac{n(n-1)}{2!}\cdot y_n(-2)\right]$$
$$-[y_{n+1}\cdot x+ny_n\cdot 1]=m^2y_n$$

or
$$(1-x^2)y_{n+2}-(2n+1)xy_{n+1}=(m^2+n^2)y_n \qquad \text{...(iv)}$$

Putting $x = 0$ in (*i*), (*ii*), (*iii*) and (*iv*)

$$y(0)=e^{m\sin^{-1}0}=e^0=1$$
$$y_1(0)=my(0)=m\cdot 1=m$$
$$y_2(0)=m^2y(0)=m^2$$
$$y_{n+2}(0)=(m^2+n^2)\,y_n(0) \qquad \text{...(v)}$$

Putting $n = 1, 2, 3, 4, \ldots$ in (*v*)

$$y_3(0)=(m^2+1^2)\,y_1(0)=(m^2+1^2)\,m$$
$$y_4(0)=(m^2+2^2)\,y_2(0)=(m^2+2^2)\,m^2$$
$$y_5(0)=(m^2+3^2)\,y_3(0)=(m^2+3^2)\,(m^2+1^2)\,m$$
$$y_6(0)=(m^2+4^2)\,y_4(0)=(m^2+4^2)\,(m^2+2^2)\,m^2$$

....

....

Hence when n is odd,

$$y_n(0) = m[m^2 + 1^2]\,[m^2 + 3^2] \,...\, [m^2 + (n-2)^2]$$

and when n is even,

$$y_n(0) = m^2[m^2 + 2^2]\,[m^2 + 4^2] \,...\, [m^2 + (n-2)^2].$$

Example 10. *If $y = (\sin^{-1} x)^2$, prove that*

$$(1 - x^2)\, y_{n+2} - (2n + 1)\, xy_{n+1} - n^2 y_n = 0.$$

Hence find the value of y_n at $x = 0$.

Sol.
$$y = (\sin^{-1} x)^2 \quad ...(i)$$

$$y_1 = 2 \sin^{-1} x \cdot \frac{1}{\sqrt{1-x^2}} \quad ...(ii)$$

Squaring and cross-multiplying,

$$(1 - x^2)\, y_1^2 = 4(\sin^{-1} x)^2 = 4y$$

Differentiating again, $(1 - x^2)\, 2y_1y_2 + y_1^2(-2x) = 4y_1$

Dividing by $2y_1$,
$$(1 - x^2)\, y_2 - xy_1 = 2 \quad ...(iii)$$

Differentiating every term n times by Leibnitz's Theorem,

$$\left[y_{n+2}\,(1-x^2) + ny_{n+1}\,(-2x) + \frac{n(n-1)}{2!} \cdot y_n\,(-2)\right] - [y_{n+1}\, x + n \cdot y_n \cdot 1] = 0$$

or
$$(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} - n^2y_n = 0 \quad ...(iv)$$

Putting $x = 0$ in (*i*), (*ii*), (*iii*) and (*iv*),

$$y(0) = 0,\ y_1(0) = 0,\ y_2(0) = 2,\ y_{n+2}(0) = n^2 y_n(0) \quad ...(v)$$

Putting $n = 1, 2, 3, 4, ...$ in (*v*)

$$y_3(0) = 1^2 \cdot y_1(0) = 1^2 \cdot (0) = 0$$
$$y_4(0) = 2^2 \cdot y_2(0) = 2^2 \cdot 2$$
$$y_5(0) = 3^2 \cdot y_3(0) = 3^2 \cdot 0 = 0$$
$$y_6(0) = 4^2 \cdot y_4(0) = 4^2 \cdot 2^2 \cdot 2$$

....

....

Hence, when n is odd, $y_n(0) = 0$

When n is even, $\quad y_n(0) = (n-2)^2 \,...\, 4^2 \cdot 2^2 \cdot 2.$

TEST YOUR KNOWLEDGE

1. Find the nth derivative of
   (*i*) $x^3 \cos x$ (*ii*) $x^3 \log x$
2. If $y = \sin^{-1} x$, prove that $(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} - n^2y_n = 0.$
3. If $y = e^{m \cos^{-1} x}$, prove that $(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} - (n^2 + m^2)y_n = 0.$
4. If $y = [\log (x + \sqrt{1+x^2})]^2$, prove that $(1 + x^2)y_{n+2} + (2n + 1)xy_{n+1} + n^2y_n = 0.$

5. If $y = \dfrac{\sin^{-1} x}{\sqrt{1-x^2}}$, show that $(1 - x^2)y_{n+2} - (2n + 3)xy_{n+1} - (n + 1)^2 y_n = 0$.

6. If $y = e^{\tan^{-1} x}$, prove that $(1 + x^2)y_{n+2} + [2(n + 1)x - 1]y_{n+1} + n(n + 1)y_n = 0$.

7. If $y = \sin^{-1} x$, prove that $(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} - n^2y_n = 0$.
Hence find the value of y_n when $x = 0$.

Answers

1. (i) $x^3 \cos\left(x + \dfrac{n\pi}{2}\right) + 3nx^2 \cos\left(x + (n-1)\dfrac{\pi}{2}\right) + 3n(n-1)x \cos\left(x + (n-2)\dfrac{\pi}{2}\right)$
$+ n(n-1)(n-2) \cos\left(x + (n-3)\dfrac{\pi}{2}\right)$

(ii) $\dfrac{(-1)^{n-1} n!}{x^{n-3}}\left[\dfrac{1}{n} - \dfrac{3}{n-1} + \dfrac{3}{n-2} - \dfrac{1}{n-3}\right]$

7. When n is odd,
$y_n(0) = (n-2)^2 (n-4)^2 \ldots 3^2 \cdot 1^2 \cdot 1$ and when n is even, $y_n(0) = 0$.

GENERAL THEOREMS

1.6 ROLLE'S THEOREM

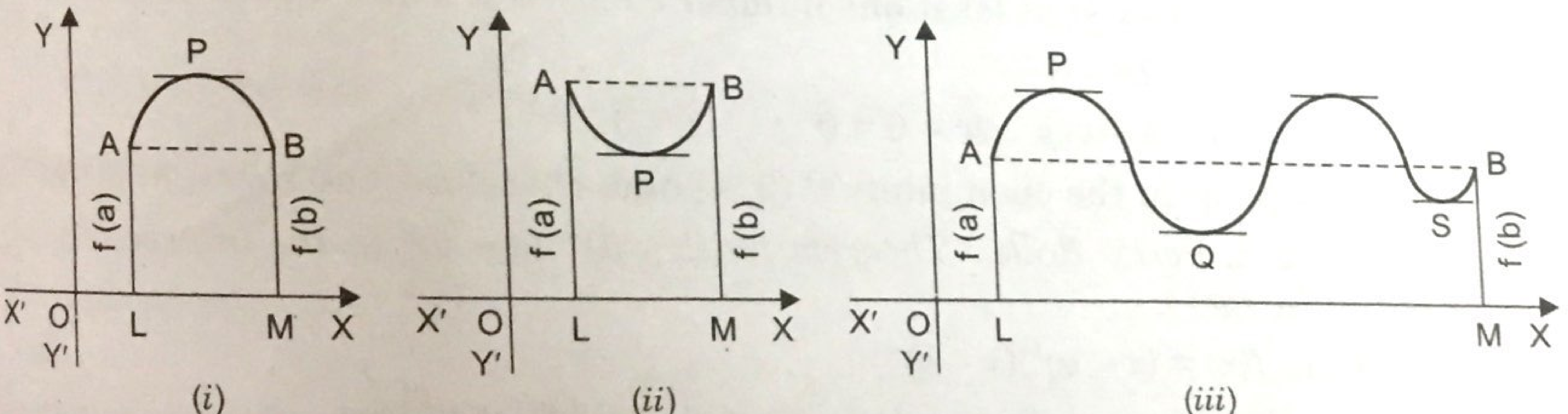
If a function $f(x)$ is such that

(i) it is continuous in the closed interval $[a, b]$

(ii) it is derivable in the open interval (a, b)

(iii) $f(a) = f(b)$ then there exists at least one value 'c' of x in the open interval (a, b) such that $f'(c) = 0$

Geometrical Proof. Let AB be the graph of function $y = f(x)$ such that the points A and B of the graph correspond to the numbers a and b of the interval $[a, b]$.



1. As $f(x)$ is continuous in the interval $a \le x \le b$

$\therefore$ its graph is a continuous curve from A to B.

2. As $f(x)$ is derivable in the interval $a < x < b$

$\therefore$ The graph of $f(x)$ has a unique tangent at every point between A and B.

3. As $f(a) = f(b)$.

$\therefore$ AL and BM, the ordinates of A and B are equal.

It is evident from the figures that there is **at least one** point P on the curve between A and B, the tangent at which is parallel to x-axis.

$\because$ tangent at P is parallel to x-axis.

$\therefore$ slope of tangent at P = 0.

If 'c' is the abscissa of P, then $f'(c) = 0$ where $a < c < b$.

Note 1. Rolle's Theorem fails to hold good for a function which does not satisfy even one of the three conditions stated above.

Note 2. Every polynomial is a continuous function of x for every value of x. $\sin x$, $\cos x$, e^x are continuous for all values of x. $\log x$ is continuous for all $x > 0$.

Note 3. Alternative Form of Rolle's Theorem. If a function $f(x)$ is such that

(*i*) it is continuous in the interval $a \le x \le a + h$

(*ii*) it is derivable in the interval $a < x < a + h$

(*iii*) $f(a) = f(a + h)$

then there exists at least one number θ such that $f'(a + \theta h) = 0, \quad 0 < \theta < 1$

($\because$ the number c which lies between a and $a + h$ must be greater than a by a fraction of h and may be written as $c = a + \theta h$ where $0 < \theta < 1$)

ILLUSTRATIVE EXAMPLES

Example 1. *Verify Rolle's Theorem for the function* $f(x) = x^2 - 6x + 8$ *in the interval* [2, 4].

Sol. Here $\quad a = 2, \quad b = 4$

1. $f(x) = x^2 - 6x + 8$

$\therefore$ $f(x)$ is a polynomial.

Since every polynomial is a continuous function of x for every value of x.

$\therefore$ $f(x)$ is continuous in the closed interval [2, 4]

2. $f'(x) = 2x - 6$ which exists in the open interval (2, 4)

3. $f(2) = 4 - 12 + 8 = 0$

$f(4) = 16 - 24 + 8 = 0 \quad \therefore \quad f(2) = 0 = f(4)$

$\therefore$ $f(x)$ satisfies all the three conditions of Rolle's Theorem.

$\therefore$ there must exist at least one number c between 2 and 4 such that $f'(c) = 0$.

Now $\quad f'(x) = 2x - 6$

$\therefore \quad f'(c) = 0$ gives $\quad 2c - 6 = 0 \qquad c = 3$

This is a point in the open interval (2, 4) and, therefore, the theorem is verified.

Example 2. *Verify Rolle's Theorem for* $(x - a)^m (x - b)^n$ *in the interval* $[a, b]$; m, n *being positive integers.*

Sol. Here $f(x) = (x - a)^m (x - b)^n$

As m and n are + ve integers, $(x - a)^m$ and $(x - b)^n$ are polynomials on expansion by Binomial Theorem and consequently $f(x)$ is a polynomial of degree $(m + n)$.

1. Since every polynomial is a continuous function of x for every value of x.

$\therefore$ $f(x)$ is continuous in the closed interval $[a, b]$.

2. $f'(x) = m(x - a)^{m-1}(x - b)^n + n(x - a)^m (x - b)^{n-1}$

$= (x - a)^{m-1}(x - b)^{n-1}[m(x - b) + n(x - a)]$

which exists in the open interval (a, b).

3. $f(a) = 0 = f(b)$

$\therefore$ $f(x)$ satisfies all the three conditions of Rolle's Theorem.

$\therefore$ there must exists **at least one** number 'c' between a and b such that $f'(c) = 0$

$\because$ $$f'(x) = (x-a)^{m-1}(x-b)^{n-1}[m(x-b)+n(x-a)].$$

$\therefore$ $$f'(c) = 0 \text{ gives } (c-a)^{m-1}(c-b)^{n-1}[m(c-b)+n(c-a)] = 0$$

or $$m(c-b)+n(c-a) = 0 \qquad [\because \ c \neq a, c \neq b \text{ as } a < c < b]$$

or $$(m+n)c = mb + na$$

or $$c = \frac{mb+na}{m+n},$$

which is a point within the interval (a, b) dividing it in the ratio $m : n$ internally.

Hence, the verification.

Example 3. *Verify Rolle's Theorem for $x^3 - 4x$ in $[-2, 2]$.*

Sol. Here $f(x) = x^3 - 4x$, a polynomial.

1. Since every polynomial is a continuous function of x for every value of x.

$\therefore$ $f(x)$ is continuous in $[-2, 2]$

2. $f'(x) = 3x^2 - 4$ which exists in $(-2, 2)$

3. $f(-2) = 0 = f(2)$

$\therefore$ $f(x)$ satisfies all the three conditions of Rolle's Theorem.

$\therefore$ There must exist **at least one** number 'c' in $(-2, 2)$ such that $f'(c) = 0$

i.e. $$3c^2 - 4 = 0 \quad \text{or} \quad c = \pm\frac{2}{\sqrt{3}} = \pm 1.155 \text{ (nearly)}$$

Both the values of 'c' lie in $(-2, 2)$. Hence the verification.

Example 4. *Verify Rolle's Theorem for $2 + (x-1)^{2/3}$ in $[0, 2]$.*

Sol. Here $$f(x) = 2 + (x-1)^{2/3}$$

$$f'(x) = \frac{2}{3}(x-1)^{-1/3} = \frac{2}{3(x-1)^{1/3}}$$

which does not exist at $x = 1 \in (0, 2)$

$\therefore$ Rolle's Theorem is not applicable to $f(x)$ in $[0, 2]$.

Example 5. *Verify Rolle's Theorem for*

$$f(x) = x(x+3)\,e^{-x/2} \text{ in } [-3, 0].$$

Sol. Here $f(x) = (x^2 + 3x)^{-x/2}$

Since $x(x+3) = x^2 + 3x$ and $e^{-x/2}$ are continuous for all x

$\therefore$ their product $= x(x+3)e^{-x/2} = f(x)$ is continuous for all x.

$\Rightarrow$ $f(x)$ is continuous in $[-3, 0]$.

Also, $$f'(x) = (2x+3)e^{-x/2} + (x^2+3x)e^{-x/2}\cdot\left(-\frac{1}{2}\right) = -\frac{1}{2}(x^2 - x - 6)e^{-x/2}$$

which exists in $(-3, 0)$

$\Rightarrow$ $f(x)$ is derivable in $(-3, 0)$

and $$f(-3) = 0 = f(0)$$

$\therefore$ $f(x)$ satisfies all the three conditions of Rolle's Theorem.

$\therefore$ there must exist at least one number 'c' in $(-3, 0)$ such that $f'(c) = 0$

i.e. $$-\frac{1}{2}(c^2 - c - 6)e^{-c/2} = 0$$

or $$c^2 - c - 6 = 0 \qquad [\because\ e^x \neq 0 \text{ for any finite value of } x]$$

or $$c = 3, -2$$

But $c = 3 \notin (-3, 0)$ while $c = -2 \in (-3, 0)$

Hence the verification.

Example 6. *Verify Rolle's Theorem for the following functions:*

(*i*) $f(x) = \sin x$ *in* $[-\pi, \pi]$

(*ii*) $f(x) = e^x \sin x$ *in* $[0, \pi]$

(*iii*) $f(x) = \dfrac{\sin x}{e^x}$ *in* $[0, \pi]$

(*iv*) $f(x) = e^x (\sin x - \cos x)$ *in* $\left[\dfrac{\pi}{4}, \dfrac{5\pi}{4}\right]$

(*v*) $f(x) = \log\left[\dfrac{x^2 + ab}{(a+b)x}\right]$ *in* $[a, b]$

(*vi*) $f(x) = \sqrt{4 - x^2}$ *in* $[-2, 2]$

Sol. (*i*) $f(x) = \sin x$ is continuous for all x

$\Rightarrow$ $f(x)$ is continuous in $[-\pi, \pi]$

$f'(x) = \cos x$ which exists in $(-\pi, \pi)$

$\Rightarrow$ $f(x)$ is derivable in $(-\pi, \pi)$

Also, $f(-\pi) = 0 = f(\pi)$

$\therefore$ $f(x)$ satisfies all the three conditions of Rolle's Theorem

$\therefore$ there exists at least one number 'c' in $(-\pi, \pi)$ such that

$$f'(c) = 0$$

i.e. $$\cos c = 0 \quad \text{or} \quad c = \pm\frac{\pi}{2}$$

Both these values of c lie in $(-\pi, \pi)$.

Hence the verification.

(*ii*) $f(x) = e^x \sin x$

Since e^x and $\sin x$ are continuous for all x.

$\therefore$ their product $= e^x \sin x = f(x)$ is continuous in $[0, \pi]$

$$f'(x) = e^x \sin x + e^x \cos x = e^x (\sin x + \cos x)$$

which exists for all x

$\therefore$ $f(x)$ is derivable in $(0, \pi)$

Also, $f(0) = 0 = f(\pi)$

$\therefore$ $f(x)$ satisfies all the three conditions of Rolle's Theorem.

$\therefore$ there exists at least one number 'c' in $(0, \pi)$ such that

$$f'(c) = 0$$

i.e. $$e^c (\sin c + \cos c) = 0$$

or $$\sin c + \cos c = 0 \qquad (\because\ e^x \neq 0 \text{ for any finite value of } x)$$

$$\tan c = 1 \quad \text{or} \quad c = \frac{3\pi}{4}$$

This value of c lies in $(0, \pi)$. Hence the verification.

(iii) $f(x) = \dfrac{\sin x}{e^x}$

Since $\sin x$ and e^x are continuous for all x and $e^x \neq 0$ for any (finite) values of x

$\therefore$ their quotient $= \dfrac{\sin x}{e^x} = f(x)$ is continuous in $[0, \pi]$

$$f'(x) = \frac{e^x \cdot \cos x - \sin x \cdot e^x}{e^{2x}} = \frac{\cos x - \sin x}{e^x} \qquad (\because \; e^x \neq 0)$$

which exists for all x.

$\therefore$ $f(x)$ is derivable in $(0, \pi)$

Also, $\quad f(0) = 0 = f(\pi)$

$\therefore$ $f(x)$ satisfies all the three conditions of Rolle's Theorem.

$\therefore$ there exists at least one number 'c' in $(0, \pi)$ such that

$$f'(c) = 0$$

$$\Rightarrow \quad \frac{\cos c - \sin c}{e^c} = 0$$

or, $\quad \tan c = 1 \quad \text{or} \quad c = \dfrac{\pi}{4}$

This value of c lies in $(0, \pi)$. Hence the verification.

(iv) $f(x) = e^x(\sin x - \cos x)$

Since $\sin x$ and $\cos x$ are continuous for all x

$\therefore$ their difference $= \sin x - \cos x$ is continuous for all x. Also e^x is continuous for all x.

$\therefore$ their product $= e^x(\sin x - \cos x) = f(x)$ is continuous for all x.

$\Rightarrow$ $f(x)$ is continuous in $\left[\dfrac{\pi}{4}, \dfrac{5\pi}{4}\right]$

$$f'(x) = e^x(\sin x - \cos x) + e^x(\cos x + \sin x) = 2e^x \sin x \text{ which exists for all } x.$$

$\therefore$ $f(x)$ is derivable in $\left(\dfrac{\pi}{4}, \dfrac{5\pi}{4}\right)$

Also,

$$f\left(\frac{\pi}{4}\right) = e^{\pi/4}\left(\sin\frac{\pi}{4} - \cos\frac{\pi}{4}\right) = e^{\pi/4}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) = 0$$

$$f\left(\frac{5\pi}{4}\right) = e^{5\pi/4}\left(\sin\frac{5\pi}{4} - \cos\frac{5\pi}{4}\right) = e^{5\pi/4}\left[-\frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}}\right)\right] = 0$$

$$\Rightarrow \quad f\left(\frac{\pi}{4}\right) = f\left(\frac{5\pi}{4}\right)$$

$\therefore$ $f(x)$ satisfies all the three conditions of Rolle's Theorem.

$\therefore$ there exists at least one number 'c' in $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$ such that

$$f'(c) = 0$$

i.e. $$2e^c \sin c = 0 \quad \text{or} \quad \sin c = 0 \qquad (\because e^c \neq 0)$$

or $$c = \pi$$

This value of c lies in $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$. Hence the verification.

(v) $$f(x) = \log\left[\frac{x^2 + ab}{(a+b)x}\right] \text{ in } [a, b]$$

$$= \log(x^2 + ab) - \log(a+b) - \log x.$$

Since $\log x$ is defined only for $x > 0$, $\quad \therefore \quad 0 < a < b$

$\therefore$ $f(x)$ has a unique and definite value for each x in $[a, b]$

$\therefore$ $f(x)$ is continuous in $[a, b]$

Also, $$f'(x) = \frac{2x}{x^2 + ab} - 0 - \frac{1}{x} = \frac{x^2 - ab}{x(x^2 + ab)}$$

which exists for all x in (a, b)

$\therefore$ $f(x)$ is derivable in (a, b)

$$f(a) = \log \frac{a(a+b)}{(a+b)a} = \log 1 = 0$$

$$f(b) = \log \frac{b(a+b)}{(a+b)b} = \log 1 = 0$$

$\therefore$ $$f(a) = f(b)$$

Thus $f(x)$ satisfies all the three conditions of Rolle's Theorem.

$\therefore$ there exists all least one number 'c' in (a, b) such that

$$f'(c) = 0$$

i.e. $$\frac{c^2 - ab}{c(c^2 + ab)} = 0 \quad \text{or} \quad c^2 = ab$$

or $$c = \sqrt{ab} \quad \text{which lies in } (a, b)$$

Hence, the verification.

(vi) $f(x) = \sqrt{4 - x^2}$ has a unique and definite value for each x in $[-2, 2]$

$\therefore$ $f(x)$ is continuous in $[-2, 2]$

$$f'(x) = \frac{1}{2}(4 - x^2)^{-1/2} \cdot (-2x) = \frac{-x}{\sqrt{4 - x^2}}$$

which exists for all x in $(-2, 2)$ $\quad \therefore \quad$ for $-2 < x < 2$

$\therefore$ $f(x)$ is derivable in $(-2, 2)$

Also $f(-2) = 0 = f(2)$

$\therefore$ $f(x)$ satisfies all the three conditions of Rolle's Theorem.

$\therefore$ there exists at least one number 'c' in $(-2, 2)$ such that

$$f'(c) = 0$$

$$\frac{-c}{\sqrt{4-c^2}} = 0 \qquad \text{or} \qquad c = 0$$

or

which lies in $(-2, 2)$.

Hence the verification.

Example 7. *Can Rolle's Theorem be applied to*

(i) $f(x) = \tan x$ *in* $[0, \pi]$

(ii) $f(x) = \sec x$ *in* $[0, 2\pi]$.

Sol. *(i)* $f(x) = \tan x$ is discontinuous at $x = \frac{\pi}{2} \in [0, \pi]$

$\therefore$ Rolle's Theorem cannot be applied.

(ii) $f(x) = \sec x$ is discontinuous at $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$, both of which lie in $[0, 2\pi]$

$\therefore$ Rolle's Theorem cannot be applied.

Example 8. *Discuss the applicability of Rolle's Theorem to the function*

$$f(x) = |x| \text{ in } [-1, 1]$$

Sol.
$$f(x) = |x| = \begin{cases} x & \text{for} \quad 0 \le x \le 1 \\ -x & \text{for} \quad -1 \le x < 0 \end{cases}$$

$f(x)$ being a linear function is continuous and derivable for every value of x, except the partitioning value $x = 0$.

Continuity at $x = 0$

$$\text{Right Limit} = \underset{x \to 0+}{\text{Lt}} f(x) = \underset{x \to 0+}{\text{Lt}} x = 0 \qquad (\because \ x \to 0+ \Rightarrow x > 0)$$

$$\text{Left Limit} = \underset{x \to 0-}{\text{Lt}} f(x) = \underset{x \to 0-}{\text{Lt}} (-x) = 0 \qquad (\because \ x \to 0- \Rightarrow x < 0)$$

$$\therefore \qquad \underset{x \to 0}{\text{Lt}} f(x) = 0 \qquad \text{Also } f(0) = 0$$

$\Rightarrow$ $f(x)$ is continuous at (the doubtful point) $x = 0$

$\Rightarrow$ $f(x)$ is continuous in $[-1, 1]$.

Derivability at $x = 0$

$$\text{Right-hand Derivative} = \underset{x \to 0+}{\text{Lt}} \frac{f(x) - f(0)}{x - 0} = \underset{x \to 0+}{\text{Lt}} \frac{x - 0}{x} = \underset{x \to 0+}{\text{Lt}} (1) = 1$$

$$\text{Left-hand Derivative} = \underset{x \to 0-}{\text{Lt}} \frac{f(x) - f(0)}{x - 0} = \underset{x \to 0-}{\text{Lt}} \frac{-x - 0}{x} = \underset{x \to 0-}{\text{Lt}} (-1) = -1$$

$\because$ Right-hand Derivative $\neq$ Left-hand Derivative:

$\therefore$ $f(x)$ is not derivable at $x = 0$

$\Rightarrow$ $f(x)$ is not derivable in $(-1, 1)$

$\therefore$ Rolle's Theorem cannot be applied to $f(x) = |x|$ in $[-1, 1]$

TEST YOUR KNOWLEDGE

1. Verify Rolle's Theorem for the following functions:
   (i) $f(x) = x^2 + x - 6$ in $[-3, 2]$
   (ii) $f(x) = (x - 1)(x - 2)^2$ in $[1, 2]$
   (iii) $f(x) = (x^2 - 1)(x - 2)$ in $[-1, 2]$
   (iv) $f(x) = (x - 2)^3 (x - 4)^4$ in $[2, 4]$
   (v) $f(x) = 8x - x^2$ in $[0, 8]$
2. Verify Rolle's theorem for the following functions:
   (i) $f(x) = \sin x + \cos x$ in $\left[0, \frac{\pi}{2}\right]$
   (ii) $f(x) = \sin^4 x + \cos^4 x$ in $\left[0, \frac{\pi}{2}\right]$
   (iii) $f(x) = e^x \cos x$ in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
3. Apply Rolle's Theorem to find a point (or points) on the curve $y = -1 + \cos x$ on $[0, 2\pi]$ where the tangent is parallel to x-axis.
4. If Rolle's Theorem holds for the function $f(x) = x^3 + ax^2 + bx + 5$ in $[1, 3]$ at the point $x = 2 + \frac{1}{\sqrt{3}}$, find the values of a and b.

Answers

3. $(\pi, -2)$ **4.** $a = -6, b = 11$

1.7 LAGRANGE'S MEAN VALUE THEOREM

If a function $f(x)$ is such that

(i) it is continuous in the closed interval $[a, b]$

(ii) it is derivable in the open interval (a, b), then there exists at **least one** *value 'c' of x in the open interval (a, b) such that*

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

Proof. Consider the function $F(x) = f(x) + Ax$ ...(i)

where A is a constant to be determined such that

$$F(a) = F(b)$$

Now $F(a) = f(a) + Aa,$

$F(b) = f(b) + Ab$

Since $F(a) = F(b)$

$\therefore$ $f(a) + Aa = f(b) + Ab$

or $F(b) - f(a) = -A(b - a)$

$$\therefore \quad -A = \frac{f(b) - f(a)}{b - a} \quad \text{...(ii)}$$

($\because$ $b - a$ = length of interval $\neq 0$)

Now $f(x)$ is given to be continuous in $a \le x \le b$ and derivable in $a < x < b$.

Aslo, A being a constant, Ax is also continuous in $a \le x \le b$ and derivable in $a < x < b$.

$\therefore$ $F(x) = f(x) + Ax$ is

1. continuous in the interval $a \le x \le b$
2. derivable in the interval $a < x < b$
3. $F(a) = F(b)$

$\therefore$ F(x) satisfies **all the three** conditions of Rolle's Theorem.

$\therefore$ there must exist **at least one** value 'c' of x in the open interval (a, b) such that $F'(c) = 0$

Now $\qquad F'(c) = f'(x) + A$

$\qquad F'(c) = 0$ gives $f'(c) + A = 0$

$\therefore$ or $\qquad -A = f'(c)$ ...(*iii*)

From (*ii*) and (*iii*)

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

1.8 ALTERNATIVE FORM

If a function $f(x)$ is such that

1. it is continuous in the closed interval $[a, a + h]$
2. it is derivable in the open interval $(a, a + h)$ then there exists at least one number θ such that $f(a + h) = f(a) + hf'(a + \theta h)$ where $0 < \theta < 1$.

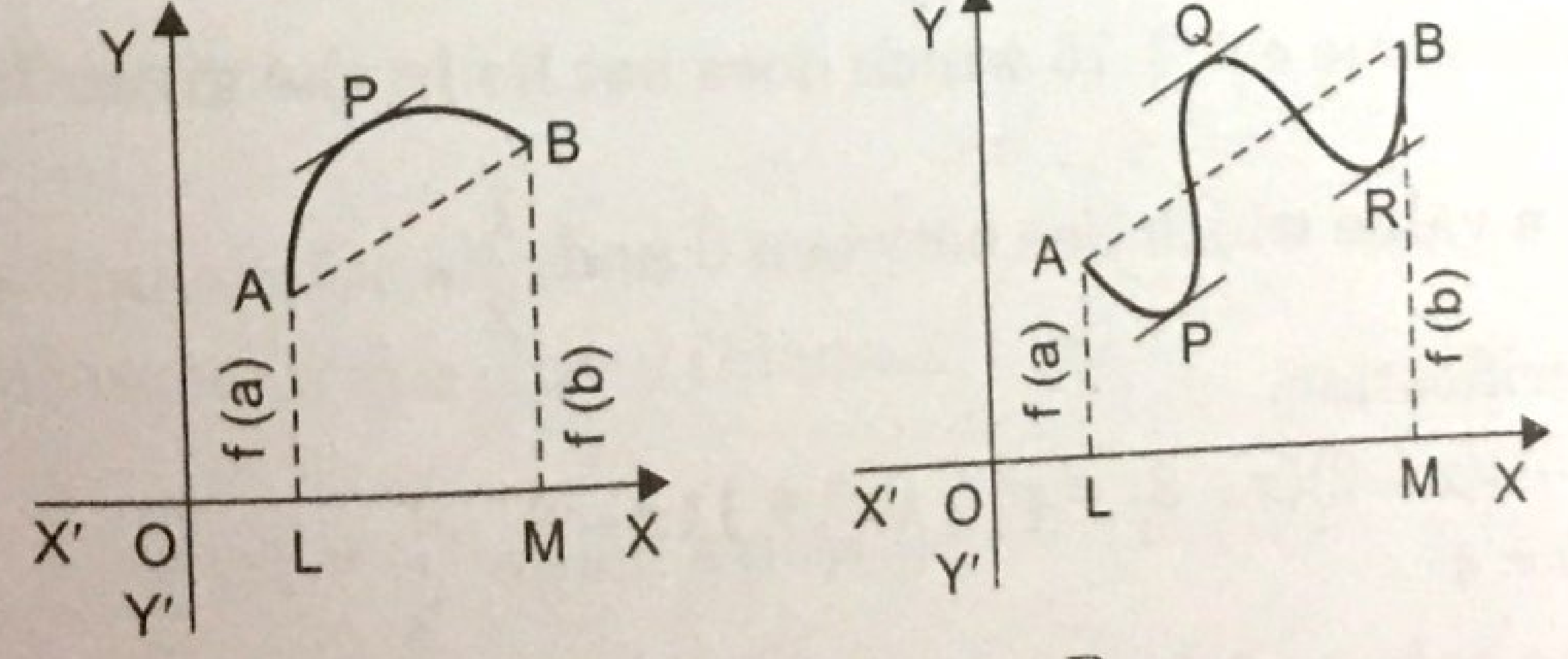
1.9 GEOMETRICAL INTERPRETATION OF LAGRANGE'S MEAN VALUE THEOREM

Let A and B be points on the graph of the function $y = f(x)$ corresponding to $x = a$ and $x = b$.

$\therefore$ the coordinates of the points A and B are $[a, f(a)]$ and $[b, f(b)]$ respectively.

$$\text{Slope of chord AB} = \frac{\text{difference of ordinates}}{\text{difference of abscissae}} = \frac{f(b) - f(a)}{b - a}$$

Now 1. Since $f(x)$ is continuous in the interval $a \le x \le b$.



$\therefore$ its graph is a continuous curve from A to B.

2. Since $f(x)$ is derivable in the interval $a < x < b$.

$\therefore$ it possesses a unique tangent at every point between A and B.

It is evident from the figures that there is **at least one** point P between A and B, the tangent at which is parallel to chord AB.

If c be the abscissa of this point, then slope of tangent there at is $f'(c)$.

Hence $$\frac{f(b)-f(a)}{b-a} = f'(c) \qquad | \because \ m_1 = m_2 \text{ for parallelism} |$$

ILLUSTRATIVE EXAMPLES

Example 1. *Verify mean value theorem for the following functions and find c if possible.*

(*a*) $f(x) = x(x-1)(x-2)$ *in* $\left[0 - \frac{1}{2}\right]$ (*b*) $f(x) = (x-1)(x-2)(x-3)$ *in* $[0, 4]$

Sol. (*a*) $f(x) = x(x-1)(x-2) = x(x^2 - 3x + 2) = x^3 - 3x^2 + 2x$

$$a = 0,\ b = \frac{1}{2}$$

1. $f(x)$ being a polynomial is continuous in the interval $0 \le x \le \frac{1}{2}$

2. $f'(x) = 3x^2 - 6x + 2$ which exists in the interval $0 < x < \frac{1}{2}$.

$\therefore$ By the mean value theorem, we have

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

i.e. $$3c^2 - 6c + 2 = \frac{\left(\frac{1}{8} - \frac{3}{4} + 1\right) - (0)}{\frac{1}{2} - 0} = \frac{3}{4}$$

i.e. $$12c^2 - 24c + 5 = 0.$$

$\therefore$ $$c = \frac{24 \pm \sqrt{576 - 240}}{24} = 1.76,\ 0.24$$

Discarding the value $c =$ 1.76 which does not lie in the given interval (0, 5).

$\therefore$ $c = 0.24$, a value which lies between 0 and $\frac{1}{2}$.

Hence the verification.

(*b*) $f(x) = (x-1)(x-2)(x-3) = x^3 - 6x^2 + 11x - 6$

$a = 0,\ b = 4$

1. $f(x)$ being a polynomial, is continuous in the interval $0 \le x \le 4$
2. $f'(x) = 3x^2 - 12x + 11$ which exists in the interval $0 < x < 4$

$\therefore$ By mean value theorem, we have

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$3c^2 - 12c + 11 = \frac{(64 - 96 + 44 - 6) - (-6)}{4 - 0} = 3$$

i.e. $$3c^2 - 12c + 8 = 0$$

or $$c = \frac{12 \pm \sqrt{144 - 96}}{6} = 3.155, 0.845$$

$\therefore$

Both these values lie within the given interval.
Hence the verification.

Example 2. *Verify Lagrange's Mean Value Theorem for the following functions:*

(*i*) $f(c) = \sqrt{x^2 - 4}$ *in* $[2, 4]$ (*ii*) $f(x) = \log x$ *in* $[1, e]$

Sol. (*i*) $f(x) = \sqrt{x^2 - 4}$, $a = 2, b = 4$

$\because$ $f(x)$ has a unique and definite value for each x in the closed interval $[2, 4]$

$\therefore$ $f(x)$ is continuous in $[2, 4]$

$$f'(x) = \frac{1}{2}(x^2 - 4)^{-\frac{1}{2}} \cdot 2x = \frac{x}{\sqrt{x^2 - 4}}$$

which exists for all x in the open interval $(2, 4)$

i.e. for $2 < x < 4$.

$\therefore$ $f(x)$ is derivable in $(2, 4)$

$\therefore$ By Lagrange's Mean Value Theorem, we have

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{f(4) - f(2)}{4 - 2}$$

i.e. $$\frac{c}{\sqrt{c^2 - 4}} = \frac{\sqrt{12} - 0}{2} = \sqrt{3}$$

or $$c^2 = 3(c^2 - 4) \quad \text{or} \quad 2c^2 = 12 \Rightarrow c = \pm\sqrt{6}$$

Discarding the value $c = -\sqrt{6}$ which does not lie in $(2, 4)$.

$\therefore$ $c = \sqrt{6}$, a value which lies in $(2, 4)$.

Hence the verification.

(*ii*) $f(x) = \log x$, $a = 1, b = e$

$\because$ $\log x$ is continuous for all $x > 0$

$\therefore$ $f(x)$ is continuous in the closed interval $[1, e]$.

$$f'(x) = \frac{1}{x} \text{ which exists for all } x \text{ in } (1, e)$$

$\therefore$ $f(x)$ is derivable in the open interval $(1, e)$.

$\therefore$ By Lagrange's Mean Value Theorem, we have

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{f(e) - f(1)}{e - 1}$$

i.e. $$\frac{1}{c} = \frac{\log e - \log 1}{e-1} = \frac{1-0}{e-1}$$

or $$c = e - 1$$

Since, $2 < e < 3,\ \ 1 < e - 1 < 2 < e$

$\therefore$ $c = e - 1$ lies in $(1, e)$

Hence, the verification.

Example 3. *The function* $f(x) = \dfrac{2x-1}{3x-4}$; $a = 1, b = 2$; *prove that there is no number c in the open interval (a, b) that satisfies the conclusion of Mean value Theorem. Determine the conditions of the theorem which fail to hold.*

Sol. $f(x) = \dfrac{2x-1}{3x-4}$ is discontinuous at $x = \dfrac{4}{3}$ and $1 < \dfrac{4}{3} < 2$

$\therefore$ $f(x)$ is not continuous in the closed interval $[1, 2]$

$$f'(x) = \frac{(3x-4)\cdot 2 - (2x-1)\cdot 3}{(3x-4)^2} = \frac{-5}{(3x-4)^2}$$

which does not exist at $x = \dfrac{4}{3}$

$\therefore$ $f(x)$ is not derivable in the open interval $(1, 2)$

Both the conditions of mean value theorem fail to hold.

1.10 IMPORTANT DEDUCTIONS FROM LAGRANGE'S MEAN VALUE THEOREM

1. *If a function f is (i) continuous on [a, b] and (ii) derivable on (a, b), then*

$$f'(x) = 0\ \forall\ x \in [a, b] \Rightarrow\ f \text{ is constant on } [a, b].$$

Proof. Let x_1, x_2 (where $x_1 < x_2$) be any two distinct points of $[a, b]$ so that

$$[x_1, x_2] \subset [a, b].$$

Then f satisfies both the conditions of Lagrange's Mean Value Theorem on $[x_1, x_2]$.

$$\therefore\ \exists\ c \in (x_1, x_2) \text{ such that } \frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) \qquad ...(1)$$

But $\quad f'(x) = 0\ \forall\ x \in [a, b]$ and $x_1 < c < x_2 \Rightarrow f'(c) = 0$

$\therefore$ From (1), $f(x_2) - f(x_1) = 0 \Rightarrow f(x_1) = f(x_2)$.

Since x_1 and x_2 are any two distinct points of $[a, b]$, it follows that f keeps the same value for every $x \in [a, b]$.

Hence f is constant on $[a, b]$.

2. *If two functions f and g are (i) continuous on [a, b] and (ii) derivable on (a, b), then*

$$f'(x) = g'(x)\ \forall\ x \in [a, b] \Rightarrow\ f - g \text{ is constant on } [a, b].$$

Proof. Consider $\quad \phi(x) = f(x) - g(x)\ \forall\ x \in [a, b]$

Clearly, ϕ is continuous on $[a, b]$; ϕ is derivable on (a, b)

$$\phi'(x) = f'(x) - g'(x) = 0\ \forall\ x \in [a, b]$$

$\therefore$ By deduction 1 above, ϕ is constant on $[a, b]$

$\Rightarrow$ $f - g$ is constant on $[a, b]$.

3. *If a function f is (i) continuous on [a, b] and (ii) derivable on (a, b), then*

$$f'(x) > 0 \ \forall \ x \in [a, b] \ \Rightarrow \ f \text{ is strictly increasing on } [a, b].$$

Proof. Let x_1, x_2 (where $x_1 < x_2$) be any two distinct points of $[a, b]$ so that

$$[x_1, x_2] \subset [a, b].$$

Then f satisfies both the conditions of Lagrange's mean value theorem on $[x_1, x_2]$

$$\therefore \quad \exists \, c \in (x_1, x_2) \text{ such that } \frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) \text{ or } f(x_2) - f(x_1) = (x_2 - x_1) f'(c)$$

Now $\qquad x_1 < x_2 \Rightarrow x_2 - x_1 > 0 \qquad$ ...(1)

$$f'(x) > 0 \ \forall \ x \in [a, b] \text{ and } x_1 < c < x_2 \Rightarrow f'(c) > 0$$

$\therefore$ From (1), $f(x_2) - f(x_1) > 0$ or $f(x_1) < f(x_2)$

Since $\qquad x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

$\therefore$ f is strictly increasing on $[a, b]$.

4. *If a function f is (i) continuous on [a, b] and (ii) derivable on (a, b) then*

$$f'(x) < 0 \ \forall \ x \in [a, b] \ \Rightarrow \ f \text{ is strictly decreasing on } [a, b].$$

Proof. Proceeding as in deduction 3 above, we have

$$f(x_2) - f(x_1) = (x_2 - x_1) f'(c) \qquad \text{...(1)}$$

Now $\qquad x_1 < x_2 \Rightarrow x_2 - x_1 > 0$

$$f'(x) < 0 \ \forall \ x \in [a, b] \text{ and } x_1 < c < x_2 \Rightarrow f'(c) < 0$$

$\therefore$ From (1), $f(x_2) - f(x_1) < 0$ or $f(x_1) > f(x_2)$

Since $\qquad x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$

$\therefore$ f is strictly decreasing on $[a, b]$.

Example 4. *Show that if $x > 0$, $\log(1 + x) > \dfrac{x}{1+x}$.*

Sol. Let $\qquad f(x) = \log(1 + x) - \dfrac{x}{1+x}$

$$\therefore \quad f'(x) = \frac{1}{1+x} - \frac{(1+x)\cdot 1 - x \cdot 1}{(1+x)^2} = \frac{1}{1+x} - \frac{1}{(1+x)^2} = \frac{1+x-1}{(1+x)^2} = \frac{x}{(1+x)^2}$$

which is positive, because $x > 0$

$\therefore$ $f(x)$ is monotonic increasing when $x > 0 \Rightarrow f(x) > f(0)$

Now, $\qquad f(0) = \log 1 - 0 = 0$

$$\therefore \quad f(x) > 0 \Rightarrow \log(1 + x) - \frac{x}{1-x} > 0 \Rightarrow \log(1 + x) > \frac{x}{1+x}$$

Example 5. *If $x > 0$, show that* $x - \dfrac{x^2}{2} < \log(1+x) < x - \dfrac{x^2}{2(1+x)}$.

Sol. Let $\quad f(x) = x - \dfrac{x^2}{2} - \log(1+x)$

$\therefore \quad f'(x) = 1 - x - \dfrac{1}{1+x} = \dfrac{1 - x^2 - 1}{1+x} = -\dfrac{x^2}{1+x} < 0$ for $x > 0$

$\Rightarrow$ $f(x)$ is monotonic decreasing for $x > 0 \Rightarrow f(x) < f(0)$

But $\quad f(0) = 0 - 0 - \log 1 = 0$

$\therefore \quad f(x) < 0 \Rightarrow x - \dfrac{x^2}{2} - \log(1+x) < 0 \Rightarrow x - \dfrac{x^2}{2} < \log(1+x)$ ...(1)

Now let $\quad g(x) = \log(1+x) - x + \dfrac{x^2}{2(1+x)}$

$$\therefore \quad g'(x) = \frac{1}{1+x} - 1 + \frac{1}{2}\cdot\frac{(1+x).2x - x^2}{(1+x)^2} = \frac{1-1-x}{1-x} + \frac{1}{2}\cdot\frac{2x+x^2}{(1+x)^2}$$

$$= -\frac{x}{1+x} + \frac{2x+x^2}{2(1+x)^2}$$

$$= \frac{-2x(1+x) + 2x + x^2}{2(1+x)^2} = -\frac{x^2}{2(1+x)^2} < 0 \text{ for } x > 0$$

$\Rightarrow$ $g(x)$ is monotonic decreasing for $x > 0 \Rightarrow g(x) < g(0)$

But $\quad g(0) = 0 \quad \therefore \quad g(x) < 0 \Rightarrow \log(1+x) - x + \dfrac{x^2}{2(1+x)} < 0$

$\therefore \quad \log(1+x) < x - \dfrac{x^2}{2(1+x)}$ ...(2)

Combining (1) and (2), $x - \dfrac{x^2}{2} < \log(1+x) < x - \dfrac{x^2}{2(1+x)}$.

TEST YOUR KNOWLEDGE

1. Verify Lagrange's mean value theorem for the following functions in the given interval:

(*i*) $f(x) = x^2 + 2x + 3$ in $[4, 6]$

(*ii*) $f(x) = x^3 - 5x^2 - 3x$ in $[1, 3]$

(*iii*) $f(x) = (x-3)(x-6)(x-9)$ in $[3, 5]$

(*iv*) $f(x) = \dfrac{1}{4x-1}$ in $[1, 4]$

(*v*) $f(x) = \sqrt{25 - x^2}$ in $[1, 5]$

(*vi*) $f(x) = 2\sin x + \sin 2x$ in $[0, \pi]$

2. Using Lagrange's mean value theorem, prove that there is a point on the curve $y = 2x^2 - 5x + 3$ between the points A(1, 0) and B(2, 1), where the tangent is parallel to the chord AB. Also find that point.

3. Show that $x - \frac{x^2}{2} + \frac{x^3}{3(1+x)} < \log(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3}$ for $x > 0$.

Answers

2. $\left(\frac{3}{2}, 0\right)$

1.11 CAUCHY'S MEAN VALUE THEOREM

If two functions $f(x)$ and $\phi(x)$ are such that

(i) both are continuous in the closed interval $[a, b]$

(ii) both are derivable in the open interval (a, b)

(iii) $\phi'(x) \neq 0$ for any value of x in the open interval (a, b) then there exists at least one value c of x in the open interval (a, b) such that

$$\frac{f(b) - f(a)}{\phi(b) - \phi(a)} = \frac{f'(c)}{\phi'(c)}.$$

Proof. Consider the function $F(x) = f(x) + A\phi(x)$ ...(i)

where A is a constant to be determined such that

$$F(a) = F(b)$$

Now $F(a) = f(a) + A\phi(a)$

$F(b) = f(b) + A\phi(b)$

Since $F(a) = F(b)$

$\therefore$ $f(a) + A\phi(a) = f(b) + A\phi(b)$

$f(b) - f(a) = -A[\phi(b) - \phi(a)]$

$\therefore$ $$-A = \frac{f(b) - f(a)}{\phi(b) - \phi(a)} \quad [\text{where } \phi(b) - \phi(a) \neq 0] \quad \text{...(ii)}$$

[If $\phi(b) - \phi(a) = 0$, then $\phi(a) = \phi(b)$

$\therefore$ $\phi(x)$ satisfies all the three conditions of Rolle's Theorem.

$\therefore$ $\phi'(x) = 0$ for at least one value of x in the interval $a < x < b$ which is contrary to the given condition that $\phi'(x) \neq 0$ for any value of x in the interval $a < x < b$].

Since $f(x)$ and $\phi(x)$ are both given to be continuous in the interval $a \le x \le b$ and derivable in the interval $a < x < b$.

$\therefore$ $F(x) = [f(x) + A\phi(x)]$ is

1. continuous in the interval $a \le x \le b$
2. derivable in the interval $a < x < b$.
3. $F(a) = F(b)$

$\therefore$ $F(x)$ staisfies **all the three** conditions of Rolle's Theorem.

$\therefore$ there exists **at least one** value c of x in the interval $0 < x < b$ such that $F'(c) = 0$

Now, $F'(x) = f'(x) + A\phi'(x)$

$\therefore$ $F'(c) = 0$ gives $f'(c) + A\phi'(c) = 0$. $\therefore \quad -A = \dfrac{f'(c)}{\phi'(c)}$ ...(*iii*)

From (*ii*) and (*iii*),

$$\frac{f(b)-f(a)}{\phi(b)-\phi(a)} = \frac{f'(c)}{\phi'(c)}.$$

1.12 DEDUCTION

If $\phi(x) = x$, then $\phi(b) = b$, $\phi(a) = a$ and $\phi'(x) = 1$ for all x.

$\therefore$ The result of Cauchy's Mean Value Theorem

viz. $$\frac{f(b)-f(a)}{\phi(b)-\phi(a)} = \frac{f'(c)}{\phi'(c)}$$

reduces to $$\frac{f(b)-f(a)}{b-a} = \frac{f'(c)}{1} = f'(c)$$

which is Lagrange's Mean Value Theorem.

1.13 ALTERNATIVE FORM

If two functions $f(x)$ and $\phi(x)$ are such that

(*i*) both are continuous in the closed interval $[a, a+h]$

(*ii*) both are derivable in the open interval $(a, a+h)$

(*iii*) $\phi'(x) \neq 0$ for any value of x in the open interval $(a, a+h)$, then there exists at least one number θ such that

$$\frac{f(a+h)-f(a)}{\phi(a+h)-\phi(a)} = \frac{f'(a+\theta h)}{\phi'(a+\theta h)}. \qquad \text{where } 0 < \theta < 1$$

Example. *Find 'c' of Cauchy's Mean Value Theorem for the following pairs of functions in* $[a, b]$:

(*i*) $f(x) = \sqrt{x}$, $\phi(x) = \dfrac{1}{\sqrt{x}}$ (*ii*) $f(x) = e^x$, $\phi(x) = e^{-x}$ (*iii*) $f(x) = \sin x$, $\phi(x) = \cos x$.

Sol. (*i*) $$f(x) = \sqrt{x}, \ \phi(x) = \frac{1}{\sqrt{x}} \qquad \text{(Assuming } 0 < a < b)$$

$$f'(x) = \frac{1}{2\sqrt{x}}, \ \phi'(x) = -\frac{1}{2x\sqrt{x}}$$

Both $f(x)$ and $\phi(x)$ are continuous in $[a, b]$ and derivable in (a, b)

$\therefore$ By Cauchy's Mean Value Theorem, we have

$$\frac{f(b)-f(a)}{\phi(b)-\phi(a)} = \frac{f'(c)}{\phi'(c)}$$

or $$\frac{\sqrt{b}-\sqrt{a}}{\dfrac{1}{\sqrt{b}}-\dfrac{1}{\sqrt{a}}} = \frac{\dfrac{1}{2\sqrt{c}}}{-\dfrac{1}{2c\sqrt{c}}} \quad \text{or} \quad (\sqrt{b}-\sqrt{a})\cdot\frac{\sqrt{ab}}{\sqrt{a}-\sqrt{b}} = -c \quad \therefore \quad c = \sqrt{ab}$$

(ii)
$$f(x) = e^x,\ \phi(x) = e^{-x}$$
$$f'(x) = e^x,\ \phi'(x) = -e^{-x}$$

Both $f(x)$ and $\phi(x)$ are continuous in $[a, b]$ and derivable in (a, b)

$\therefore$ By Cauchy's Mean Value Theorem, we have

$$\frac{f(b) - f(a)}{\phi(b) - \phi(a)} = \frac{f'(c)}{\phi'(c)}$$

or
$$\frac{e^b - e^a}{e^{-b} - e^{-a}} = \frac{e^c}{-e^{-c}}$$

or
$$(e^b - e^a) \cdot \frac{e^a \cdot e^b}{e^a - e^b} = -e^{2c}$$

or
$$-e^{a+b} = -e^{2c}$$

or
$$a + b = 2c \qquad \therefore \quad c = \frac{a+b}{2}$$

(iii) $f(x) = \sin x,\ \phi(x) = \cos x$

$f'(x) = \cos x,\ \phi'(x) = -\sin x$

Both $f(x)$ and $\phi(x)$ are continuous in $[a, b]$ and derivable in (a, b)

$\therefore$ By Cauchy's Mean Value Theorem, we have

$$\frac{f(b) - f(a)}{\phi(b) - \phi(a)} = \frac{f'(c)}{\phi'(c)}$$

or
$$\frac{\sin b - \sin a}{\cos b - \cos a} = \frac{\cos c}{-\sin c}$$

or
$$\frac{2 \cos \frac{b+a}{2} \sin \frac{b-a}{2}}{2 \sin \frac{b+a}{2} \sin \frac{a-b}{2}} = -\cot c$$

or
$$-\cot \frac{b+a}{2} = -\cot c$$

$\therefore$
$$c = \frac{a+b}{2}.$$

TEST YOUR KNOWLEDGE

1. Verify Cauchy's mean value theorem for the functions x^2 and x^3 in the interval $[1, 2]$.
2. Find 'c' of Cauchy's mean value theorem for the following pairs of functions:

   (i) $f(x) = x^2,\ g(x) = x$ in $[a, b]$

   (ii) $f(x) = \dfrac{1}{x^2},\ g(x) = \dfrac{1}{x}$ in $[a, b]$

Answers

2. (i) $\dfrac{a+b}{2}$ (ii) $\dfrac{2ab}{a+b}$

1.14 TAYLOR'S THEOREM WITH LAGRANGE'S FORM OF REMAINDER AFTER n TERMS

Statement. If a function $f(x)$ is such that

1. $f(x), f'(x), f''(x), \ldots\ldots, f^{n-1}(x)$ are continuous in the closed interval $a \le x \le a+h$.

2. $f^n(x)$ exist in the open interval $a < x < a+h$, then there exists at least one number θ between 0 and 1 such that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3} f'''(a) + \ldots\ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{n!} f^n(a+\theta h).$$

The $(n+1)^{\text{th}}$ term $\frac{h^n}{n!} f^n(a+\theta h)$ is called Lagrange's form of remainder after n terms and is denoted by R_n.

Note: Alternative form of **Taylor's** Theorem with Lagrange's Form of remainder after n terms.

If we put $a + h = b$, then the interval $[a, a+h]$ becomes $[a, b]$.

Since $a + \theta h$ is a number between a and $a + h$ *i.e.*, between a and b, let $a + \theta h = c$ where $a < c < b$

Then the above theorem becomes

$$f(b) = f(a) + (b-a) f'(a) + \frac{(b-a)^2}{2!} f''(a) + \ldots\ldots + \frac{(b-a)^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{(b-a)^n}{n!} f^n(c)$$

where $a < c < b$

1.15 MACLAURIN'S THEOREM WITH LAGRANGE'S FORM OF REMAINDER AFTER n TERMS

Statement. If a function $f(x)$ is such that

1. $f(x), f'(x), f''(x), \ldots\ldots, f^{n-1}(x)$ are continuous in the closed interval $[0, x]$

2. $f^n(x)$ exists in the open interval $(0, x)$ then there exists at least one number θ between 0 and 1 such that

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots\ldots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + \frac{x^n}{n!} f^n(\theta x)$$

Thus we can get by putting $a = 0$ and $h = x$ in Taylor's Theorem.

1.16 TAYLOR'S THEOREM WITH CAUCHY'S FORM OF REMAINDER

Statement. If a function $f(x)$ be such that

1. $f(x), f'(x), f'(x), f''(x), \ldots\ldots f^{n-1}(x)$ are continuous in the closed interval $a \le x \le a+h$.

2. $f^n(x)$ exists in the open interval $a < x < a + h$, then there exists at least one number θ, between 0 and 1, such that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots\ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{(n-1)!} (1-\theta)^{n-1} f^n(a+\theta h)$$

The $(n + 1)$th term $\dfrac{h^n}{(n-1)}(1-\theta)^{n-1} f^n(a+\theta h)$ is called Cauchy's form of remainder after n terms.

1.17 MACLAURIN'S THEOREM WITH CAUCHY'S FORM OF REMAINDER AFTER *n* TERMS

Statement. If a function $f(x)$ be such that

1. $f(x), f'(x), f''(x), \ldots\ldots, f^{n-1}(x)$ are continuous in the closed interval $[0, x]$
2. $f^n(x)$ exists in the open interval $(0, x)$ then there exists at least one number θ between 0 and 1 such that

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots\ldots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + \frac{x^n}{(n-1)!}(1-\theta)^{n-1} f^n(\theta x)$$

ILLUSTRATIVE EXAMPLES

Example 1. *Expand a^x by Maclaurin's theorem with Lagrange's form of remainder after n terms.*

Solution. Here $f(x) = a^x$

$\therefore$ $f^n(x) = a^x(\log a)^n$

Putting $x = 0, f^n(0) = (\log a)^n$

$\therefore$ $f(0) = 1, f'(0) = \log a, f''(0) = (\log a)^2, \ldots\ldots, f^{n-1}(0) = (\log a)^{n-1}$

and $f^n(\theta x) = a^{\theta x}(\log a)^n$.

By Maclaurin's Theorem with Lagrange's Form of remainder after n terms, we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots\ldots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + \frac{x^n}{n!} f^n(\theta x) (0 < \theta < 1)$$

$$\therefore \quad a^x = 1 + x \cdot \log a + \frac{x^2}{2!}(\log a)^2 + \ldots\ldots + \frac{x^{n-1}}{(n-1)!}(\log a)^{n-1} + \frac{x^n}{n!} \cdot a^{\theta x}(\log a)^n$$

Here Lagrange's remainder after n terms $= \dfrac{x^n}{n!} \cdot a^{\theta x}(\log a)^n$, where $0 < \theta < 1$

Example 2. *Show that:*

(i) $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} \ldots\ldots + (-1)^{n-1} \cdot \dfrac{x^{2n-1}}{(2n-1)!} + (-1)^n \cdot \dfrac{x^{2n}}{2n!} \sin(\theta x)$ *for every value of x.*

(ii) $\log(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \ldots\ldots + (-1)^{n-2} \cdot \dfrac{x^{n-1}}{n-1} + (-1)^{n-1} \cdot \dfrac{x^n}{n(1+\theta x)^n}$ *for $x > -1$.*

Solution.

(i) Here $f(x) = \sin x$, $\therefore$ $f^n(x) = \sin\left(x + \dfrac{n\pi}{2}\right)$

Putting $x = 0$, $\quad f^n(0) = \sin\dfrac{n\pi}{2}$

$$\therefore \quad f(0) = 0, \quad f'(0) = \sin\frac{\pi}{2} = 1, \quad f''(0) = \sin\pi = 0$$

$$f'''(0) = \sin\frac{3\pi}{2} = -1, \quad f^{iv}(0) = \sin 2\pi = 0$$

$$f^{v}(0) = \sin\frac{5\pi}{2} = \sin\left(2\pi + \frac{\pi}{2}\right) = \sin\frac{\pi}{2} = 1$$

$$.....f^{2n-1}(0) = \sin\frac{(2n-1)\pi}{2} = \sin\left(\pi n - \frac{\pi}{2}\right) = (-1)^n \sin\left(-\frac{\pi}{2}\right)$$

$$= -(-1)^n \qquad [\because \sin(n\pi + \theta) = (-1)^n \sin\theta]$$

$$= \frac{(-1)^n}{-1} = (-1)^{n-1}$$

Also, $\quad f^{2n}(\theta x) = \sin(\theta x + n\pi) = (-1)^n \sin(\theta x)$

By Maclaurin's Theorem with Lagrange's form of remainder after $2n$ terms, we have,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + ... + \frac{x^{2n-1}}{(2n-1)!}f^{2n-1}(0) + \frac{x^{2n}}{2n!}f^{2n}(\theta x)$$

$$\therefore \quad \sin x = \frac{x^3}{3!} + \frac{x^5}{5!} ..... + (-1)^{n-1}\cdot\frac{x^{2n-1}}{(2n-1)!} + (-1)^n\frac{x^{2n}}{2n!}\sin(\theta x) \qquad [0 < \theta < 1]$$

(*ii*) Here $\quad f(x) = \log(1 + x)$

$$\therefore \quad f^n(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$$

Putting $\quad x = 0 \quad f^n(0) = (-1)^{n-1}(n-1)!$

$$\therefore \quad f(0) = 0, \quad f'(0) = 0! = 1,$$

$$f''(0) = -1 \quad f'''(0) = 2!$$

$$.....f^{n-1}(0) = (-1)^{n-2}(n-2)!, f^n(\theta x) = \frac{(-1)^{n-1}(n-1)!}{(1+\theta x)^n}$$

By Maclaurin's Theorem with Lagrange's form of remainder after n terms, we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + ... + \frac{x^{n-1}}{(n-1)!}f^{n-1}(0) + \frac{x^n}{n!}f^n(\theta x)$$

$$\therefore \quad \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} \ldots\ldots + \frac{(-1)^{n-2}\cdot(n-2)!\,x^{n-1}}{(n-1)!} + \frac{(-1)^{n-1}(n-1)!}{(1+\theta x)^n}\cdot\frac{x^n}{n!}$$

$$= x - \frac{x^2}{2} + \frac{x}{3} + \ldots\ldots + (-1)^{n-2}\cdot\frac{x^{n-1}}{n-1} + (-1)^{n-1}\cdot\frac{x^n}{n(1+\theta x)^n}.$$

TEST YOUR KNOWLEDGE

1. Show that $\log(x+h) = \log x + \dfrac{h}{x} - \dfrac{h^2}{2x^2} + \ldots\ldots + (-1)^{n-1}\dfrac{h^n}{n(x+\theta h)^n}$.

   [**Hint.** $f(x) = \log(x+h)$. Put $h = 0$, then $f(x) = \log x$]

2. Show that: $e^x = 1 + x + \dfrac{x^2}{2!} + \ldots\ldots + \dfrac{x^{n-1}}{(n-1)!} + \dfrac{x^n}{n!}e^{\theta x}$, $[0 < \theta < 1]$

3. Show that:

   (*i*) $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \ldots\ldots + (-1)^n\cdot\dfrac{x^{2n}}{2n!} + (-1)^{n+1}\dfrac{x^{2n+1}}{(2n+1)!}\sin(\theta x)$

   (*ii*) $\sin x + \cos x = 1 + x - \dfrac{x^2}{2!} - \dfrac{x^3}{3!} + \dfrac{x^4}{4!}(\sin\theta x + \cos\theta x)$

   (*iii*) $e^{ax}\sin bx = bx + 2ab\cdot\dfrac{x^2}{2!} + b(3a^2 - b^2)\dfrac{x^3}{3!} + \ldots\ldots$

$$+ (a^2+b^2)^{n/2}(1-\theta)^{n-1}\frac{x^n}{(n-1)!}e^{a\theta x}\sin\left(b\theta x + n\tan^{-1}\frac{b}{a}\right).$$

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