4

Linear Algebra

4.1. WHAT IS A MATRIX?

A set of mn numbers (real or complex) arranged in a rectangular array having m rows (horizontal lines) and n columns (vertical lines), the numbers being enclosed by brackets [ ] or ( ) is called $m \times n$ matrix (read as "m and n" matrix). An $m \times n$ is usually written as

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ a_{41} & a_{42} & a_{43} & \cdots & a_{4n} \\ - & - & - & - & - \\ - & - & - & - & - \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

. Here each element has two suffixes. The first suffix indicates the row and second suffix indicates the column in which element lies.

(*i*) **Square Matrix :** A matrix in which number of rows is equal to the number of columns is called a square matrix.

(*ii*) **Multiplication of a Matrix by a Scalar :** When each element of a matrix A (say) is multiplied by a scalar k (say) then k A is defined as multiplication of A by a scalar k.

(*iii*) **Matrix Multiplication :** Two matrices A and B are said to be conformable for the product AB if number of columns of A is equal to the number of rows of B.

Thus if $A = [a_{ij}]_{m \times n}$ and $B = [b_{jk}]_{n \times p}$, where $1 \le i \le m$, $1 \le j \le n$, $1 \le k \le p$. Then AB is defined as the matrix $C = [c_{ik}]_{m \times p}$.

where $c_{ik} = a_{i1} b_{1k} + a_{i2} b_{2k} + a_{i3} b_{3k} + \dots\dots\dots + a_{in} b_{nk}$.

$$= \sum_{j=1}^{n} a_{ij} b_{jk}$$

(*iv*) **Properties of Matrix Multiplication :**

(*a*) Matrix multiplication is not commutative in general *i.e.*, $AB \neq BA$

(*b*) Matrix multiplication is associative *i.e.*, $A(BC) = (AB)C$.

(*c*) Matrix multiplication is distributive w.r.t. matrix addition *i.e.*, $A(B + C) = AB + AC$

(*d*) If A, I are square matrices of the same order then $AI = IA = A$

(*e*) If A is a square matrix of order n then $A \times A = A^2$; $A \times A \times A = A^3$; ....; $A \times A \times A \times \dots\dots\dots \times A$ (m times) $= A^m$

(*v*) **Transpose of a Matrix :** Given a matrix A then matrix obtained from A by changing its rows into columns and columns into rows is called the transpose of A and is denoted by A' or A^t.

(*vi*) **Properties of Transpose of Matrix :**

(*a*) $(A')' = A$,

(*b*) $(A + B)' = A' + B'$

(*c*) $(AB)' = B'A'$ known as **Reversal Law of transposes.**

(vii) *(a)* **Symmetric Matrix :** A square matrix is said to symmetric if $A' = A$ *i.e.*, if $A = [a_{ij}]$ then $a_{ij} = a_{ji}$ $\forall$ i, j

(b) **Skew symmetric Matrix :** A square matrix is said to be skew symmetric if $A' = -A$ *i.e.*, if $A = [a_{ij}]$ then $a_{ij} = -a_{ji}$ $\forall$ i, j and when $i = j$, then $a_{ii} = 0$ for all values of i.

Thus in a skew symmetric matrix all diagonal elements are zero.

(viii) **Involutory Matrix :** A square matrix A is said to be involutory if $A^2 = I$

(ix) **Adjoint of a Square Matrix :** The adjoint of a square matrix is the transpose of the matrix obtained by replacing each element of A by its co-factors in $|A|$

$$A\,(\text{Adj }A) = (\text{Adj }A)\,A = |A|\,I_n \text{ ; } n \text{ being the order of matrix } A$$

(x) **Singular and Non-Singular Matrices :** A square matrix is said to be singular if $|A| = 0$ and non-singular if $|A| \neq 0$.

(xi) *(a)* **Inverse of a Square Matrix :** Let A be a square matrix of order n. If there exists another matrix B of the same order such that $AB = BA = I$ then matrix **A is said to be invertible** and B is called **inverse of A.** Inverse of A is denoted by A^{-1}. Thus, $B = A^{-1}$ and $A\,A^{-1} = A^{-1}\,A = I$. From *(xv)* we see that $A\,(\text{Adj }A) = |A|\,I$

$$\therefore \qquad A^{-1} = \frac{\text{Adj }A}{|A|}.$$

(b) The inverse of a square matrix, if it exists, is unique.

(c) The necessary and sufficient condition for a square matrix A to posses inverse is that $|A| \neq 0$ *i.e.*, A is non singular.

(d) If A is invertible, then so is A^{-1} and $(A^{-1})^{-1} = A$

(xii) **Reversal Law of Inverses :** If A and B are two non-singular matrices of the same order, then $(AB)^{-1} = B^{-1}\,A^{-1}$.

(xiii) **Reversal Law of Adjoints :** If A, B are two square matrices of the same order, then $\text{Adj }(AB) = (\text{Adj }B)\,(\text{Adj }A)$.

4.2. ELEMENTARY TRANSFORMATIONS (OR OPERATIONS) (P.T.U., Dec. 2004)

Let $A = [a_{ij}]$ be any matrix of order $m \times n$ *i.e.*, $1 \le i \le m$, $1 \le j \le n$, then anyone of the following operations on the matrix is called an elementary transformation (or E-operation).

(i) **Interchange of two rows or columns.**

The interchange of i^{th} and j^{th} rows is denoted by R_{ij}.

The interchange of i^{th} and j^{th} columns is denoted by C_{ij}.

(ii) **Multiplication of (each element of) a row or column by a non-zero number k.**

The multiplication of i^{th} row by k is denoted by kR_i.

The multiplication of i^{th} column by k is denoted by kC_i.

(iii) **Addition of k times the elements of a row (column) to the corresponding elements of another row (or column), $k \neq 0$.**

The addition of k times the j^{th} row to the i^{th} row is denoted by $R_i + kR_j$.

The addition of k times the j^{th} column to the i^{th} column is denoted by $C_i + kC_j$.

If a matrix B is obtained from a matrix A by one or more E-operations, then B is said to be equivalent to A. Two equivalent matrices A and B are written as $A \sim B$.

4.3. ELEMENTARY MATRICES

The matrix obtained from a unit matrix I by subjecting it to one of the E-operations is called an elementary matrix.

For example, let $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(*i*) Operating R_{23} or C_{23} on I, we get the same elementary matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

It is denoted by E_{23}. Thus, the E-matrix obtained by either of the operations R_{ij} or C_{ij} on I is denoted by E_{ij}.

(*ii*) Operating $5R_2$ or $5C_2$ on I, we get the same elementary matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

It is denoted by $5E_2$. Thus, the E-matrix obtained by either of the operations kR_i or kC_i is denoted by kE_i.

(*iii*) Operating $R_2 + 4R_3$ on I, we get the elementary matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$.

It is denoted by $E_{23}(4)$. Thus, the E-matrix obtained by the operation $R_i + kR_j$ is denoted by $E_{ij}(k)$.

(*iv*) Operating $C_2 + 4C_3$ in I, we get the elementary matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}$, which is the transpose of

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} = E_{23}(4)$ and is, therefore, denoted by $E'_{23}(4)$. Thus, the E-matrix obtained by the operation $C_i + kC_j$ is denoted by $E_{ij}'(k)$.

4.4. THE FOLLOWING THEOREMS ON THE EFFECT OF E-OPERATIONS ON MATRICES HOLD GOOD

(*a*) *Any E-row operation on the product of two matrices is equivalent to the same E-row operation on the pre-factor.*

If the E-row operation is denoted by R, then $R(AB) = R(A). B$

(*b*) *Any E-column operation on the product of two matrices is equivalent to the same E-column operation on the post-factor.*

If the E-column operation is denoted by C, then $C(AB) = A. C(B)$

(*c*) *Every E-row operation on a matrix is equivalent to pre-multiplication by the corresponding E-matrix.*

Thus the effect of E-row operation R_{ij} on $A = E_{ij}.A$

The effect of E-row operation kR_i on $A = kE_i \,.\, A$

The effect of E-row operation $R_i + kR_j$ on $A = E_{ij}(k).\, A$

(*d*) *Every E-column operation on a matrix is equivalent to post-multiplication by the corresponding E-matrix.*

Thus, the effect of E-column operation C_{ij} on $A = A \,.\, E_{ij}$

The effect of E-column operation kC_i on $A = A \,.\, (kE_i)$

The effect of E-column operation $C_i + kC_j$ on $A = A \,.\, E'_{ij}(k)$.

4.5. INVERSE OF MATRIX BY E-OPERATIONS (Gauss-Jordan Method)

The elementary row transformations which reduce a square matrix A to the unit matrix, when applied to the unit matrix, gives the inverse matrix A^{-1}.

Let A be a non-singular square matrix. Then $A = IA$

Apply suitable E-row operations to A on the left hand side so that A is reduced to I.

Simultaneously, apply the same E-row operations to the pre-factor I on right hand side. Let I reduce to B, so that

$$I = BA$$

Post-multiplying by A^{-1}, we get

$$IA^{-1} = BAA^{-1} \Rightarrow A^{-1} = B(AA^{-1}) = BI = B$$

$$\therefore \quad B = A^{-1}.$$

Note. In practice, to find the inverse of A by E-row operations, we write A and I side by side and the same operations are performed on both. As soon as A is reduced to I, I will reduce to A^{-1}.

4.6. WORKING RULE TO REDUCE A SQUARE MATRIX TO A UNIT MATRIX I BY ELEMENTARY TRANSFORMATIONS (For Convenience We can Consider a Matrix A of Order 4 × 4)

(*i*) If in the first column, the principal element (*i.e.*, a_{11}) is not 'one' but 'one' is present some where else in the first column then first of all make 'one' as principal element (by applying row transformations R_{ij}).

(*ii*) Operate R_1 on R_2, R_3, R_4 to make elements of C_1 all zero except first element.

(*iii*) Then operate R_2 on R_3, R_4 to make elements of C_2 all zero except first and second elements. Similarly, operate R_3 on R_4 to make elements of C_3 all zero except 1st, 2nd and 3rd.

(*iv*) Reduce each diagonal element to element 'one'.

Then reverse process starts :

(*v*) Operate R_4 on $R_1, R_2 R_3$ to make all elements of C_4 zero except last.

(*vi*) Then operate R_3 on R_1 and R_2 to make all elements of C_3 zero except last but one. Similarly operate R_2 on R_1 to make all elements of C_2 zero except last but second and the matrix is reduced to unit matrix.

Note. We can apply the above rule to any square matrix of any order.

ILLUSTRATIVE EXAMPLES

Example 1. *If* $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ *then show that*

$$A^n = A^{n-2} + A^2 - I \text{ for } n \geq 3. \text{ Hence find } A^{50}.$$ **(P.T.U., Jan. 2008)**

Sol.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

We will use induction method to prove $A^n = A^{n-2} + A^2 - I$ for $n \geq 3$

$\therefore$ for $n = 3$ we will prove $\quad A^3 = A + A^2 - I$

Now,
$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Now,
$$A + A^2 - I = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = A^3$$

$\therefore$ $$A^3 = A + A^2 - I \qquad \text{...(1)}$$

$\therefore$ Result is true for $n = 3$

Let us assume that the result is true for $n = k$

i.e., $$A^k = A^{k-2} + A^2 - I \qquad \text{...(2)}$$

To prove result is true for $n = k + 1$ *i.e.*, to prove
$$A^{k+1} = A^{k-1} + A^2 - I$$

Now, $$A^{k+1} = A^k \cdot A = (A^{k-2} + A^2 - I)\,A \qquad \text{[Using (2)]}$$
$$= A^{k-1} + A^3 - A$$

$\therefore$ $$A^{k+1} = A^{k-1} + A^2 - I \qquad \text{[Using (1)]}$$

Hence the result is true for $n = k + 1$, so the result is true for all values of $n \geq 3$

Now, $$A^{50} = A^{48} + A^2 - I, \qquad \text{[Using (2)]}$$

or $$A^{50} - A^{48} = A^2 - I$$

or $$A^{48}(A^2 - I) = (A^2 - I)$$

or $$A^{48}(A^2 - I) = I(A^2 - I)$$

$\therefore$ $$A^{48} = I$$

$\therefore$ $$A^{50} = A^{48} \cdot A^2 = IA^2 = A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Example 2. *Reduce the following matrix to upper triangular form :* $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 1 & 2 \end{bmatrix}$.

Sol. (*Upper triangular matrix*) If in a square matrix, all the elements below the principal diagonal are zero, the matrix is called upper triangular.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -5 & -7 \end{bmatrix} \text{ by operations } \begin{matrix} R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix} \text{ by operation } R_3 + 5R_2$$

which is the upper triangular form of the given matrix.

Example 3. *Use Gauss-Jordan method to find inverse of the following matrices :*

(*i*) $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$ **(P.T.U., Dec. 2010)** (*ii*) $\begin{bmatrix} 2 & -6 & -2 & -3 \\ 5 & -13 & -4 & -7 \\ -1 & 4 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix}$.

Sol. (*i*) Let $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$

Consider $A = IA$

i.e.,
$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

To reduce L.H.S. to a unit matrix

Operate $R_2 - R_1$, $R_3 + 2R_1$

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & -6 \\ 0 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} A$$

Operate $R_3 + R_2$

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & -6 \\ 0 & 0 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} A$$

Operate $R_2\left(\frac{1}{2}\right)$, $R_3\left(-\frac{1}{4}\right)$

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix} A$$

Operate $R_1 - 3R_3$, $R_2 + 3R_3$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{7}{4} & \frac{3}{4} & \frac{3}{4} \\ -\frac{5}{4} & -\frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix} A$$

Operate $R_1 - R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 & \frac{3}{2} \\ -\frac{5}{4} & -\frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix} A$$

$\therefore$ $\quad I = BA$, where

$$B = \begin{bmatrix} 3 & 1 & \frac{3}{2} \\ -\frac{5}{4} & -\frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

Hence

$$A^{-1} = \begin{bmatrix} 3 & 1 & \frac{3}{2} \\ -\frac{5}{4} & -\frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

(ii) Let

$$A = \begin{bmatrix} 2 & -6 & -2 & -3 \\ 5 & -13 & -4 & -7 \\ -1 & 4 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$

Operate R_{13} then R_{23} ;

$$\begin{bmatrix} -1 & 4 & 1 & 2 \\ 2 & -6 & -2 & -3 \\ 5 & -13 & -4 & -7 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$

Operate $R_2 + 2R_1, R_3 + 5R_1$;

$$\begin{bmatrix} -1 & 4 & 1 & 2 \\ 0 & 2 & 0 & 1 \\ 0 & 7 & 1 & 3 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$

Operate $R_1(-1), R_2\left(\frac{1}{2}\right)$;

$$\begin{bmatrix} 1 & -4 & -1 & -2 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 7 & 1 & 3 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ \frac{1}{2} & 0 & 1 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$

Operate $R_3 - 7R_2, R_4 - R_2$;

$$\begin{bmatrix} 1 & -4 & -1 & -2 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ \frac{1}{2} & 0 & 1 & 0 \\ -\frac{7}{2} & 1 & -2 & 0 \\ -\frac{1}{2} & 0 & -1 & 1 \end{bmatrix} A$$

Operate $R_4(2)$;

$$\begin{bmatrix} 1 & -4 & -1 & -2 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ \frac{1}{2} & 0 & 1 & 0 \\ -\frac{7}{2} & 1 & -2 & 0 \\ -1 & 0 & -2 & 2 \end{bmatrix} A$$

Operate $R_1 + 2R_4$, $R_2 - \frac{1}{2}R_4$, $R_3 + \frac{1}{2}R_4$;

$$\begin{bmatrix} 1 & -4 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 & -5 & 4 \\ 1 & 0 & 2 & -1 \\ -4 & 1 & -3 & 1 \\ -1 & 0 & -2 & 2 \end{bmatrix} A$$

Operate $R_1 + R_3$;

$$\begin{bmatrix} 1 & -4 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -6 & 1 & -8 & 5 \\ 1 & 0 & 2 & -1 \\ -4 & 1 & -3 & 1 \\ -1 & 0 & -2 & 2 \end{bmatrix} A$$

Operate $R_1 + 4R_2$;

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 & 1 \\ 1 & 0 & 2 & -1 \\ -4 & 1 & -3 & 1 \\ -1 & 0 & -2 & 2 \end{bmatrix} A$$

$\therefore$

$$I = BA \text{ and } B = A^{-1} = \begin{bmatrix} -2 & 1 & 0 & 1 \\ 1 & 0 & 2 & -1 \\ -4 & 1 & -3 & 1 \\ -1 & 0 & -2 & 2 \end{bmatrix}$$

4.7. NORMAL FORM OF A MATRIX

Any non-zero matrix $A_{m \times n}$ can be reduced to anyone of the following forms by performing elementary (row, column or both) transformations :

(*i*) I_r (*ii*) $[I_r \;\; 0]$ (*iii*) $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$ (*iv*) $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$, where I_r is a unit matrix of order r

All those forms are known as Normal forms of the matrix

Note. The form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ is called **First Canonical Form of A.**

4.8. FOR ANY MATRIX A OF ORDER $m \times n$, FIND TWO SQUARE MATRICES P AND Q OF ORDERS m AND n RESPECTIVELY SUCH THAT PAQ IS IN THE NORMAL FORM $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

Method : Write $A = IAI$

Reduce the matrix A on the L.H.S. to normal form by performing elementary row and column transformations. Every row transformation on A must be accompanied by the same row transformation on the pre-factor on R.H.S.

Every column transformation on A must be accompanied by the same column transformation on the post factor on R.H.S.

Hence $A = IAI$ will transform to $I = PAQ$

Note. $A^{-1} = QP \quad \because P(AQ) = I \therefore AQ = P^{-1}$ or $(AQ)^{-1} = P$ *i.e.,* $Q^{-1} A^{-1} = P \quad \therefore \quad A^{-1} = QP.$

4.9. RANK OF A MATRIX

(P.T.U., May 2007, May 2008, Dec. 2010, May 2011

Let A be any $m \times n$ matrix. It has square sub-matrices of different orders. The determinants of thes square sub-matrices are called **minors of A. If all** minors of order $(r + 1)$ are zero but there **is at least one non zero minor** of order r, then r is called **the rank of A.** Symbolically, rank of $A = r$ is written as $\rho(\mathbf{A}) = \boldsymbol{r}$.

From the definition of the rank of a matrix A, it follows that :

(*i*) If A is a null matrix, then $\rho(A) = 0$

[$\because$ every minor of A has zero value.]

(*ii*) If A is not a null matrix, then $\rho(A) \geq 1$.

(*iii*) If A is non-singular $n \times n$ matrix, then $\rho(A) = n$

[$\because |A| \neq 0$ is largest minor of A.]

If I_n is the $n \times n$ unit matrix, then $|I_n| = 1 \neq 0 \Rightarrow \rho(I_n) = n$.

(*iv*) If A is an $m \times n$ matrix, then $\rho(A) \leq$ minimum of m and n.

(*v*) If all minors of order r are equal to zero, then $\rho(A) < r$.

To determine the rank of a matrix A, we adopt the following different methods :

4.10. WORKING RULE TO DETERMINE THE RANK OF A MATRIX

Method I : Start with the highest order minor (or minors) of A. Let their order be r. **If anyone of them** is no zero, then $\rho(A) = r$.

If **all of them** are zero, start with minors of next lower order $(r - 1)$ and continue this process till you ge non-zero minor. The order of that minor is the rank of A.

This method usually involves a lot of computational work since we have to evaluate several determinan

Method II : Reduce the matrix to the upper triangular form of the matrix by elementary row transformatioı then number of non-zero rows of triangular matrix is equal to rank of the matrix.

Method III : Reduce the matrix to the normal form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ by performing elementary transformatio (row and column both) then r is the rank of the matrix. [$\because$ r^{th} order minor $|I_r| = 1$ ¹ 0 and each $(r + 1$ order minor =

4.11. PROPERTIES OF THE RANK OF A MATRIX

(*i*) Elementary transformations of a matrix do not alter the rank of the matrix.

(*ii*) $\rho(A') = \rho(A)$; $\rho(A^{\theta}) = \rho(A)$

(*iii*) $\rho(A) =$ number of non-zero rows in upper triangular form of the matrix A.

Example 4. *If A is a non-zero column matrix and B is a non-zero row matrix then* $\rho(AB) = 1$.

Sol. Let A be a non-zero column matrix

Let $A = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix}_{m \times 1}$ and B be a non-zero row matrix

Let $B = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}_{1 \times n}$, where at least one of a's and at least one of b's is non-zero

Now AB will be a matrix of order $m \times n$

$$\therefore \quad AB = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix} = \begin{bmatrix} a_1 b_1 & a_1 b_2 & \cdots & a_1 b_n \\ a_2 b_1 & a_2 b_2 & \cdots & a_2 b_n \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_m b_1 & a_m b_2 & \cdots & a_{mn} \end{bmatrix}$$

$$= b_1 b_2 \cdots b_n \begin{bmatrix} a_1 \; a_1 & \cdots & a_1 \\ a_2 \; a_2 & \cdots & a_2 \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ a_m \; a_m & \cdots & a_m \end{bmatrix}$$

AB has at least one element non-zero and all minors of order ≥ 2 are zero because all lines are identical

$\therefore \quad \rho(AB) = 1.$

Example 5. *Find the rank of the following matrices:*

(*i*) $A = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 6 & 8 \\ 3 & 7 & 22 \end{bmatrix}$ **(P.T.U., Dec. 2004)** (*ii*) *Diag. matrix* $[-1 \;\; 0 \;\; 1 \;\; 0 \;\; 0 \;\; 4]$.

Sol. (*i*) Let $A = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 6 & 8 \\ 3 & 7 & 22 \end{bmatrix}$

A is of order 3×3 $\quad \therefore \quad \rho(A) \leq 3$

Reduce the matrix to triangular form

Operate $R_2 - 2R_1, R_3 - 3R_1$, $A \sim \begin{bmatrix} 1 & 4 & 5 \\ 0 & -2 & -2 \\ 0 & -5 & 7 \end{bmatrix}$

Operate $\quad R_2\left(-\dfrac{1}{2}\right) \sim \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 1 \\ 0 & -5 & 7 \end{bmatrix}$

Operate $\quad R_3 + 5R_1 \sim \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & 12 \end{bmatrix}$

Now, minor of order $3 = \begin{vmatrix} 1 & 4 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & 12 \end{vmatrix} = 12 \neq 0$

$\therefore \quad \rho(A) = 3$

(*ii*) Let A = diag. matrix [–1 0 1 0 0 4]

$$= \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$, which is a square matrix of order 6×6 $\therefore$ $\rho(A) \le 6$

Also, it is a diagonal matrix so triangular matrix $\therefore$ $\rho(A) =$ Number of non zero rows of triangular matrix = 3 hence $\rho(A) = 3$.

Example 6. *Reduce the following matrices to normal form and find their ranks.*

(*i*) $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ **(P.T.U., May 2012, Dec. 2012)** (*ii*) $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$ **(P.T.U., May 2007)**

Sol. (*i*) Let $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

Operate R_{12} ; $\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

Operate $R_3 - 3R_1, R_4 - R_1$; $\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix}$

Operate $R_3 - R_2, R_4 - R_2$; $\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Operate $C_3 - C_1, C_4 - C_1$; $\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Operate $C_3 + 3C_2, C_4 + C_2$; $\sim \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$$= \begin{bmatrix} I_{2\times 2} & O_{2\times 2} \\ O_{2\times 2} & O_{2\times 2} \end{bmatrix}$$

which is the repuired normal form and $\rho(A) = 2$

$$(ii) \text{ Let } A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

Operate R_{12}; $\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

Operate $R_2 - 2R_1, R_3 - 3R_1, R_4 - 6R_1$; $\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$

Operate $C_2 + C_1, C_3 + 2C_1, C_4 + 4C_1$; $\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$

[Now to change 5 to 1, instead of operating by $R_2\left(\frac{1}{5}\right)$, operate $R_2 - R_3$ and similarly to change 9 (in 2nd column) to 1 operate $R_4 - 2R_3$]

Operate $R_2 - R_3, R_4 - 2R_3$; $\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 4 & 9 & 10 \\ 0 & 1 & -6 & -3 \end{bmatrix}$

Operate $R_3 - 4R_2, R_4 - R_2$; $\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Operate $C_3 + 6C_2, C_4 + 3C_2$; $\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Operate $C_3\left(\frac{1}{33}\right), C_4\left(\frac{1}{22}\right)$; $\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Operate $C_4 - C_3$;
$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] \sim \begin{bmatrix} I_{3\times 3} & O_{3\times 1} \\ O_{1\times 3} & O_{1\times 1} \end{bmatrix} = \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$

which is the required normal form and rank of $A = 3$.

Example 7. *Reduce the following matrix to normal form and hence find its rank* $\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -2 & 1 \\ 1 & -1 & 4 & 0 \\ -2 & 2 & 8 & 0 \end{bmatrix}$.

(P.T.U., May 2004)

Sol. Let
$$A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -2 & 1 \\ 1 & -1 & 4 & 0 \\ -2 & 2 & 8 & 0 \end{bmatrix}$$

Operate $R_3 - R_1, R_4 + 2R_1$;
$$\sim \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & -1 & 2 & -1 \\ 0 & 2 & 12 & 2 \end{bmatrix}$$

Operate $C_3 - 2C_1, C_4 - C_1$;
$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & -1 & 2 & -1 \\ 0 & 2 & 12 & 2 \end{bmatrix}$$

Operate $R_3 + R_2, R_4 - 2R_2$;
$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 16 & 0 \end{bmatrix}$$

Operate $C_3 + 2C_2, C_4 - C_2$;
$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 16 & 0 \end{bmatrix}$$

Operate R_{34};
$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Operate $R_3\left(\frac{1}{16}\right)$;
$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} I_3 & O_{3\times 1} \\ O_{3\times 1} & O_{1\times 1} \end{bmatrix}$$

$= \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$, which is the required normal form and $\rho(A) = 3$.

Example 8. *Find non singular matrices P and Q such that PAQ is in the normal form for the matrix*

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}. \text{ Also find } A^{-1} \text{ (if it exists).}$$

(P.T.U., Dec. 2003)

Sol. Consider A = IAI

$$\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operate $R_2 - R_1, R_3 - 3R_1$ (Subjecting prefactor the same operations)

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 2 \\ 0 & 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operate $C_2 + C_1, C_3 + C_1$ (Subjecting post factor the same operations)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operate $R_3 - 2R_2$ (Same operation on prefactor on R.H.S.)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operate $C_3 - C_2$ (Same operation on post factor on R.H.S.)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Operating $R_2\left(\frac{1}{2}\right)$ (Same operation on prefactor)

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} = PAQ, \text{ where } P = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -1 & -2 & 1 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

As $\rho(A) = 2$ $\therefore$ A is a singular matrix and A^{-1} does not exist.

TEST YOUR KNOWLEDGE

1. (*a*) Reduce to triangular form $\begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix}$.

(*b*) If $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$, find $|AB|$. **(P.T.U., May 2009)**

2. Use Gauss Jordan method to find the inverse of the following:

(*i*) $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$

(*iii*) $\begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$ (*iv*) $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ **(P.T.U., May 2012)**

(*v*) $\begin{bmatrix} 2 & 4 & 3 & 2 \\ 3 & 6 & 5 & 2 \\ 2 & 5 & 2 & -3 \\ 4 & 5 & 14 & 14 \end{bmatrix}$ (*vi*) $\begin{bmatrix} 2 & 1 & -1 & 2 \\ 1 & 3 & 2 & -3 \\ -1 & 2 & 1 & -1 \\ 2 & -3 & -1 & 4 \end{bmatrix}$.

3. Find rank of the following matrices:

(*i*) $\begin{bmatrix} 2 & -1 & 0 & 5 \\ 0 & 3 & 1 & 4 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{bmatrix}$

(*iii*) $\begin{bmatrix} 2 & 4 & 3 & -2 \\ -3 & -2 & -1 & 4 \\ 6 & -1 & 7 & 2 \end{bmatrix}$ (*iv*) $\begin{bmatrix} 2 & 5 & 3 & 3 \\ 2 & 3 & 3 & 4 \\ 3 & 6 & 3 & 2 \\ 4 & 12 & 0 & 8 \end{bmatrix}$

(*v*) $\begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ 2 & 1 & 3 \end{bmatrix}$ **(P.T.U., June 2003)** (*vi*) $\begin{bmatrix} 1 & 3 & 4 & 5 \\ 1 & 2 & 6 & 7 \\ 1 & 5 & 0 & 10 \end{bmatrix}$

(*vii*) $\begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$ **(P.T.U., May 2009)** (*viii*) $\begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}$ **(P.T.U., Dec. 2011)**

4. Reduce the following matrices to normal form and hence find their ranks:

(*i*) $\begin{bmatrix} 2 & 2 & 2 \\ 1 & 2 & 1 \\ 3 & 4 & 3 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$ **(P.T.U., Dec. 2010)**

(*iii*) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$ (*iv*) $\begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$

(*v*) $\begin{bmatrix} 2 & 2 & 2 \\ 1 & 2 & 1 \\ 3 & 4 & 3 \end{bmatrix}$ **(P.T.U., May 2012)** (*vi*) $\begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix}$ **(P.T.U., Dec. 2012)**

5. If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$; find two non-singular matrices P and Q such that PAQ = I. Hence find A^{-1}.

[**Hint:** $\rho(A) = 3 \therefore A^{-1}$ exists = QP]

6. For a matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$, find non-singular matrices P and Q such that PAQ is in the normal form. Also find A^{-1} (if it exists).

ANSWERS

1. (*a*) $\begin{bmatrix} 3 & 4 & -5 \\ 0 & 13 & -11 \\ 0 & 0 & 1 \end{bmatrix}$; $\begin{bmatrix} 3 & 0 & 0 \\ -9 & 13 & 0 \\ -15 & 29 & 11 \end{bmatrix}$ (*b*) 16

Upper triangular Lower triangular

2. (*i*) $\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix}$

(*iii*) $\begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{5}{3} & \frac{2}{3} \\ -1 & 4 & 0 \end{bmatrix}$ (*iv*) $\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

(*v*) $\frac{1}{25}\begin{bmatrix} -23 & 29 & -64 & -18 \\ 10 & -12 & 26 & 7 \\ 1 & -2 & 6 & 2 \\ 2 & -2 & 3 & 1 \end{bmatrix}$ (*vi*) $\frac{1}{18}\begin{bmatrix} 2 & 5 & -7 & 1 \\ 5 & -1 & 5 & -2 \\ -7 & 5 & 11 & 10 \\ -1 & -2 & 10 & 5 \end{bmatrix}$

3. (*i*) 2 (*ii*) 2 (*iii*) 3 (*iv*) 4 (*v*) 3 (*vi*) 3 (*vii*) 2 (*viii*) 2

4. (*i*) $\begin{bmatrix} I_2 & O_{2\times1} \\ O_{1\times2} & O_{1\times1} \end{bmatrix}$; rank = 2 (*ii*) $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}; \rho = 2$

(*iii*) $[I_3 \times O_{3\times1}]$; $\rho = 3$ (*iv*) $\begin{bmatrix} I_2 & O_{2\times2} \\ O_{2\times2} & O_{2\times2} \end{bmatrix}; \rho = 2$

(*v*) $\begin{bmatrix} I_2 & O_{2\times1} \\ O_{1\times2} & O_{1\times1} \end{bmatrix}; \rho = 2$. (*vi*) $\begin{bmatrix} I_{3\times3} & O_{3\times1} \end{bmatrix}; \rho = 3$

5. $P = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ -2 & 3 & -3 \end{bmatrix}; Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}; A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$.

6. $P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}; Q = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$; A^{-1} does not exist as $\rho(A) = 2$.

4.12. CONSISTENCY AND SOLUTION OF LINEAR ALGEBRAIC EQUATIONS

Proof. Consider the system of equations $\left.\begin{aligned} a_1x+b_1y+c_1z &= d_1 \\ a_2x+b_2y+c_2z &= d_2 \\ a_3x+b_3y+c_3z &= d_3 \end{aligned}\right\}$ (3 equations in 3 unknowns)

In matrix notation, these equations can be written as

$$\begin{bmatrix} a_1x+b_1y+c_1z \\ a_2x+b_2y+c_2z \\ a_3x+b_3y+c_3z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \text{ or } \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

or

$$AX = B$$

where $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ is called the coefficient matrix, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is the column matrix of unknowns

$B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$ is the column of constants.

If $d_1 = d_2 = d_3 = 0$, then B= O and the matrix equation AX = B reduce to AX = O.
Such a system of equation is called a system of **homogeneous linear equation.**
If at least one of d_1, d_2, d_3 is non-zero, then $B \neq O$.
Such a system of equation is called a system of **non-homogeneous linear equation.**

Solving the matrix equation AX = B means finding X, *i.e.*, finding a column matrix $\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$ such that

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}. \text{ Then } x = \alpha, y = \beta, z = \gamma.$$

The matrix equation AX = B need not always have a solution. It may have no solution or a unique solution or an infinite number of solutions.

(*a*) Consistent Equations : A system of equations having **one or more solutions** is called a **consistent** system of equations.

(*b*) Inconsistent Equations : A system of equations having **no solutions** is called an **inconsistent** system of equations.

(*c*) State the conditions in terms of rank of the coefficient matrix and rank of the augmented matrix for a unique solution; no solution ; infinite number of solutions of a system of linear equations.

(P.T.U., May 2005, Dec. 2010)

For a system of non-homogeneous linear equation AX = B.

(*i*) if $\rho\,[A \,\vdots\, B] \neq \rho(A)$, the system is inconsistent.
(*ii*) if $\rho\,[A \,\vdots\, B] = \rho(A)$ = number of the unknowns, the system has a unique solution.
(*iii*) if $\rho\,[A \,\vdots\, B] = \rho(A) <$ number of unknowns, the system has an infinite number of solutions.

The matrix $[A \,\vdots\, B]$ in which the elements of A and B are written side by side is called the **augmented matrix.**

For a system of homogeneous linear equations AX = O.

(*i*) X = O is always a solution. This solution in which each unknown has the value zero is called the **Null Solution** or the **Trivial Solution.** Thus a homogeneous system is always consistent.
(*ii*) if $\rho(A)$ = number of unknown, the system has only the trivial solution.
(*iii*) if $\rho(A) <$ number of unknown, the system has an infinite number of non-trivial solutions.

4.13. IF A IS A NON-SINGULAR MATRIX, THEN THE MATRIX EQUATION AX = B HAS A UNIQUE SOLUTION

The given equation is $\quad AX = B \qquad\qquad ...(1)$

$\because$ A is a non-singular matrix, $\therefore$ A^{-1} exists.

Pre-multiplying both sides of (1) by A^{-1}, we get

$$A^{-1}(AX) = A^{-1}B \qquad \text{or} \quad (A^{-1}A)X = A^{-1}B$$

or
$$IX = A^{-1}B \qquad \text{or} \quad X = A^{-1}B$$

which is the required unique solution (since A^{-1} is unique).

Another Method to find the solution of AX = B:

Write the augmented matrix [A ⋮ B]. By **E-row operations** on A and B, reduce A to a diagonal matrix thus getting

$$[A \vdots B] \sim \begin{bmatrix} p_1 & 0 & 0 & \vdots & q_1 \\ 0 & p_2 & 0 & \vdots & q_2 \\ 0 & 0 & p_3 & \vdots & q_3 \end{bmatrix}$$

Then $\quad p_1x = q_1,\ p_2y = q_2,\ p_3z = q_3$ gives the solution of AX = B.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve the system of equations :*

$$5x + 3y + 7z = 4, \qquad 3x + 26y + 2z = 9, \qquad 7x + 2y + 11z = 5$$

with the help of matrix inversion. **(P.T.U., Dec. 2004, May 2007, Jan. 2010)**

Sol. In matrix notation, the given system of equations can be written as

$$AX = B \qquad ...(1)$$

where
$$A = \begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 11 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}$$

Let
$$A = \begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 11 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 11 \end{vmatrix} = 5\begin{vmatrix} 26 & 2 \\ 2 & 11 \end{vmatrix} - 3\begin{vmatrix} 3 & 2 \\ 7 & 11 \end{vmatrix} + 7\begin{vmatrix} 3 & 26 \\ 7 & 2 \end{vmatrix}$$

$$= 5(286 - 4) - 3(33 - 14) + 7(6 - 182) = 1410 - 57 - 1232 = 121 \neq 0$$

$\Rightarrow$ A is non-singular $\therefore$ A^{-1} exists and the unique solution of (1) is

$$X = A^{-1}B \qquad ...(2)$$

Now cofactors of the elements of | A | are as follows:

$$A_1 = \begin{vmatrix} 26 & 2 \\ 2 & 11 \end{vmatrix} = 282,\ A_2 = -\begin{vmatrix} 3 & 7 \\ 2 & 11 \end{vmatrix} = -19,\ A_3 = \begin{vmatrix} 3 & 7 \\ 26 & 2 \end{vmatrix} = -176$$

$$B_1 = \begin{vmatrix} 3 & 2 \\ 7 & 11 \end{vmatrix} = -19,\ B_2 = \begin{vmatrix} 5 & 7 \\ 7 & 11 \end{vmatrix} = 6,\ B_3 = -\begin{vmatrix} 5 & 7 \\ 3 & 2 \end{vmatrix} = 11$$

$$C_1 = \begin{vmatrix} 3 & 26 \\ 7 & 2 \end{vmatrix} = -176,\ C_2 = -\begin{vmatrix} 5 & 3 \\ 7 & 2 \end{vmatrix} = 11,\ C_3 = \begin{vmatrix} 5 & 3 \\ 3 & 26 \end{vmatrix} = 121$$

$$\therefore \quad \text{adj } A = \text{transpose of} \begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} = \begin{bmatrix} 282 & -19 & -176 \\ -19 & 6 & 11 \\ -176 & 11 & 121 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{121}\begin{bmatrix} 282 & -19 & -176 \\ -19 & 6 & 11 \\ -176 & 11 & 121 \end{bmatrix}$$

From (2), $$X = A^{-1} B = \frac{1}{121}\begin{bmatrix} 282 & -19 & -176 \\ -19 & 6 & 11 \\ -176 & 11 & 121 \end{bmatrix}\begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix} = \frac{1}{121}\begin{bmatrix} 282(4)-19(9)-176(5) \\ -19(4)+6(9)+11(5) \\ -176(4)+11(9)+121(5) \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{121}\begin{bmatrix} 77 \\ 33 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{77}{121} \\ \frac{33}{121} \\ \frac{0}{121} \end{bmatrix} = \begin{bmatrix} \frac{7}{11} \\ \frac{3}{11} \\ 0 \end{bmatrix}$$

Hence $\quad x = \frac{7}{11},\ y = \frac{3}{11},\ z = 0.$

Example 2. *Use the rank method to test the consistency of the system of equations $4x - y = 12$, $-x - 5y - 2z = 0$, $-2y + 4z = -8$* **(P.T.U., Dec. 2012)**

Sol. In matrix notation, the given equations can be written as

$$AX = B$$

where $$A = \begin{bmatrix} 4 & -1 & 0 \\ -1 & -5 & -2 \\ 0 & -2 & 4 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 12 \\ 0 \\ -8 \end{bmatrix}$$

Augmented matrix $[A \vdots B]$ $$= \left[\begin{array}{ccc:c} 4 & -1 & 0 & 12 \\ -1 & -5 & -2 & 0 \\ 0 & -2 & 4 & -8 \end{array}\right]$$

Operate R_{12}; $$\sim \left[\begin{array}{ccc:c} -1 & -5 & -2 & 0 \\ 4 & -1 & 0 & 12 \\ 0 & -2 & 4 & -8 \end{array}\right]$$

Operate $R_2 + 4R_1$; $$\sim \left[\begin{array}{ccc:c} -1 & -5 & -2 & 0 \\ 0 & -21 & -8 & 12 \\ 0 & -2 & 4 & -8 \end{array}\right]$$

Operate $R_1(-1), R_2\left(-\frac{1}{21}\right), R_3\left(-\frac{1}{2}\right)$;

$$\sim \left[\begin{array}{ccc:c} 1 & 5 & 2 & 0 \\ 0 & 1 & \frac{8}{21} & -\frac{4}{7} \\ 0 & 1 & -2 & 4 \end{array}\right]$$

Operate $R_3 - R_2$; $\sim \left[\begin{array}{ccc:c} 1 & 5 & 2 & 0 \\ 0 & 1 & \frac{8}{21} & -\frac{4}{7} \\ 0 & 0 & -\frac{50}{21} & \frac{32}{7} \end{array}\right]$

Operate $R_3\left(-\frac{21}{50}\right)$; $\sim \left[\begin{array}{ccc:c} 1 & 5 & 2 & 0 \\ 0 & 1 & \frac{8}{21} & -\frac{4}{7} \\ 0 & 0 & 1 & -\frac{48}{25} \end{array}\right]$

Operate $R_2 - \frac{8}{21} R_3, R_1 - 2R_3$; $\sim \left[\begin{array}{ccc:c} 1 & 5 & 0 & \frac{96}{25} \\ 0 & 1 & 0 & \frac{4}{25} \\ 0 & 0 & 1 & -\frac{48}{25} \end{array}\right]$

Operate $R_1 - 5R_2$; $\sim \left[\begin{array}{ccc:c} 1 & 0 & 0 & \frac{76}{25} \\ 0 & 1 & 0 & \frac{4}{25} \\ 0 & 0 & 1 & -\frac{48}{25} \end{array}\right]$

$\therefore$ $\rho(A) = 3 = \rho(A \vdots B) = \text{number of unknowns}$

$\therefore$ The given system of equations is consistent and have a unique solution

Hence the solution is $x = \frac{76}{25}, y = \frac{4}{25}, z = -\frac{48}{25}$

Example 3. *For what values of λ and μ do the system of equations : $x + y + z = 6$, $x + 2y + 3z = 10$, $x + 2y + \lambda z = \mu$ have (i) no solution (ii) unique solution (iii) more than one solution ?*

(P.T.U., Dec. 2002, May 2010)

Sol. In matrix notation, the given system of the equations can be written as

$$AX = B$$

where $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$

Augmented matrix $[A \vdots B]$

$$= \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right] \text{operating } R_2 - R_1, R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & \vdots & 6 \\ 0 & 1 & 2 & \vdots & 4 \\ 0 & 1 & \lambda-1 & \vdots & \mu-6 \end{bmatrix} \text{operating } R_1 - R_2, R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & \vdots & 2 \\ 0 & 1 & 2 & \vdots & 4 \\ 0 & 0 & \lambda-3 & \vdots & \mu-10 \end{bmatrix}$$

Case I. If $\lambda = 3, \mu \neq 10$

$$\rho(A) = 2, \rho(A \vdots B) = 3$$

$\because$ $\rho(A) \neq \rho(A \vdots B)$

$\therefore$ The system has no solution.

Case II. If $\lambda \neq 3$, μ may have any value

$\rho(A) = \rho(A \vdots B) = 3 =$ number of the unknowns

$\therefore$ System has unique solution.

Case III. If $\lambda = 3, \mu = 10$

$\rho(A) = \rho(A \vdots B) = 2 <$ number of the unknowns

$\therefore$ The system has an infinite number of solution.

Example 4. *(a) Solve the equations $x_1 + 3x_2 + 2x_3 = 0$, $2x_1 - x_2 + 3x_3 = 0$, $3x_1 - 5x_2 + 4x_3 = 0$, $x_1 + 17x_2 + 4x_3 = 0$.*

(b) Find the real value of λ for which the system of equations $x + 2y + 3z = \lambda x$, $3x + y + 2z = \lambda y$, $2x + 3y + z = \lambda z$ have non-trivial solution. **(P.T.U., May 2010, Dec. 2012)**

Sol. *(a)* In matrix notation, the given system of equations can be written as

$$AX = O$$

where $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \\ 3 & -5 & 4 \\ 1 & 17 & 4 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

Operating $R_2 - 2R_1, R_3 - 3R_1, R_4 - R_1$ $A \sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & -7 & -1 \\ 0 & -14 & -2 \\ 0 & 14 & 2 \end{bmatrix}$

Operating $R_3 - 2R_2, R_4 + 2R_2$ $\sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & -7 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Operating $R_1 + 2R_2$ $\sim \begin{bmatrix} 1 & -11 & 0 \\ 0 & -7 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\rho(A) = 2 <$ number of unknowns

$\Rightarrow$ The system has an infinite number of non-trivial solutions given by

$$x_1 - 11x_2 = 0, \quad -7x_2 - x_3 = 0$$

i.e., $$x_1 = 11k, \quad x_2 = k, x_3 = -7k, \text{ where } k \text{ is any number.}$$

Different values of k give different solutions.

(*b*) Given equations are

$$x + 2y + 3z = \lambda x$$
$$3x + y + 2z = \lambda y$$
$$2x + 3y + z = \lambda z$$

or $$(1-\lambda)x + 2y + 3z = 0$$
$$3x + (1-\lambda)y + 2z = 0$$
$$2x + 3y + (1-\lambda)z = 0$$

These equations are homogeneous in x, y, z and will have a non-trivial solution if $\rho(A) < 3$ (the number of unknowns)

i.e., determinant of order 3 = 0

i.e., $$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0$$

or $$(1-\lambda)\left[(1-\lambda)^2 - 6\right] - 2\left[3(1-\lambda) - 4\right] + 3\left[9 - 2(1-\lambda)\right] = 0$$

or $$(1-\lambda)^3 - 6 + 6\lambda - 6 + 6\lambda + 8 + 27 - 6 + 6\lambda = 0$$

or $$1 - 3\lambda + 3\lambda^2 - \lambda^3 + 18\lambda + 17 = 0$$

or $$\lambda^3 - 3\lambda^2 - 15\lambda - 18 = 0$$

or $$(\lambda - 6)(\lambda^2 + 3\lambda + 3) = 0$$

$\therefore$ either $\lambda = 6$ or $$\lambda^2 + 3\lambda + 3 = 0$$

or $$\lambda = \frac{-3 \pm \sqrt{9-12}}{2} = \frac{-3 \pm i\sqrt{3}}{2}$$

$\therefore$ The only real value of λ is 6.

Example 5. *Discuss the consistency of the following system of equations. Find the solution if consistent :*

(*i*) $$x + y + z = 4$$
$$2x + 5y - 2z = 3$$

(*ii*) $$5x + 3y + 7z = 4$$
$$3x + 26y + 2z = 9$$
$$7x + 2y + 10z = 5.$$

(P.T.U., May 2005)

Sol. (*i*) $$x + y + z = 4$$
$$2x + 5y - 2z = 3, \text{ which can be written as}$$

$$AX = B, \text{ where } A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & -2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

consider augmented matrix $[A \vdots B]$

$$= \begin{bmatrix} 1 & 1 & 1 & \vdots & 4 \\ 2 & 5 & -2 & \vdots & 3 \end{bmatrix}$$

Operating $R_2 - 2R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & \vdots & 4 \\ 0 & 3 & -4 & \vdots & -5 \end{bmatrix}$$

Operating $R_2\left(\frac{1}{3}\right)$

$$\sim\begin{bmatrix} 1 & 1 & 1 & \vdots & 4 \\ 0 & 1 & \frac{-4}{3} & \vdots & \frac{-5}{3} \end{bmatrix}$$

Operating $R_1 - R_2$

$$\begin{bmatrix} 1 & 0 & \frac{7}{3} & \vdots & \frac{17}{3} \\ 0 & 1 & \frac{-4}{3} & \vdots & \frac{-5}{3} \end{bmatrix}$$

$$\rho(A) = 2;\ \rho(A : B) = 2$$

$$\rho(A) = \rho(A \vdots B) < \text{number of unknowns}$$

$\therefore$ Given system of equations are consistent and have infinite number of solutions given by

$$x + \frac{7}{3}z = \frac{17}{3}$$

$$y - \frac{4}{3}z = -\frac{5}{3}$$

Take $z = k$ we have $x = \dfrac{17 - 7k}{3}$, $y = \dfrac{4k - 5}{3}$

Hence solutions is $x = \dfrac{17 - 7k}{3}$, $y = \dfrac{4k - 5}{3}$, $z = k$, where k is any arbitrary constant.

(ii) Given equations can be put into the form $AX = B$

where $A = \begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}$.

Consider Augmented matrix

$$[A \vdots B] = \begin{bmatrix} 5 & 3 & 7 & \vdots & 4 \\ 3 & 26 & 2 & \vdots & 9 \\ 7 & 2 & 10 & \vdots & 5 \end{bmatrix}$$

Operating $R_2 - \frac{3}{5}R_1, R_3 - \frac{7}{5}R_1$

$$\sim\begin{bmatrix} 5 & 3 & 7 & \vdots & 4 \\ 0 & \frac{121}{5} & -\frac{11}{5} & \vdots & \frac{33}{5} \\ 0 & -\frac{11}{5} & \frac{1}{5} & \vdots & -\frac{3}{5} \end{bmatrix}$$

Operate $R_3 + \frac{1}{11}R_2$; $\sim\begin{bmatrix} 5 & 3 & 7 & \vdots & 4 \\ 0 & \frac{121}{5} & -\frac{11}{5} & \vdots & \frac{33}{5} \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$

Operate $R_2\left(\frac{5}{11}\right)$ $\sim\begin{bmatrix} 5 & 3 & 7 & \vdots & 4 \\ 0 & 11 & -1 & \vdots & 3 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$

$$\rho(A) = 2, \rho(A:B) = 2$$
$$\rho(A) = \rho(A:B) = 2 < \text{Number of unknowns}$$

$\therefore$ Given equations are consistent and have infinite number of solutions given by

$$5x + 3y + 7z = 4$$
$$11y - z = 3$$

Let $z = k\,;\, y = \dfrac{k+3}{11}$ $\quad\therefore\quad x = \dfrac{4 - \dfrac{3(k+3)}{11} - 7k}{5} = \dfrac{7-16k}{11}$

Hence solution is

$$x = \frac{7-16k}{11},\ y = \frac{k+3}{11},\ z = k.$$

Example 6. *For what value of k, the equations $x + y + z = 1$, $2x + y + 4z = k$, $4x + y + 10z = k^2$ have a solution and solve them completely in each case?* **(P.T.U., Dec. 2005)**

Sol.
$$x + y + z = 1$$
$$2x + y + 4z = k$$
$$4x + y + 10z = k^2$$

which can be put into matrix form AX = B

where $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \\ 4 & 1 & 10 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 1 \\ k \\ k^2 \end{bmatrix}$

Consider the Augmented matrix

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & \vdots & 1 \\ 2 & 1 & 4 & \vdots & k \\ 4 & 1 & 10 & \vdots & k^2 \end{bmatrix}$$

Operate $R_2 - 2R_1, R_3 - 4R_1$; $\sim \begin{bmatrix} 1 & 1 & 1 & \vdots & 1 \\ 0 & -1 & 2 & \vdots & k-2 \\ 0 & -3 & 6 & \vdots & k^2-4 \end{bmatrix}$

Operate $R_3 - 3R_2$; $\sim \begin{bmatrix} 1 & 1 & 1 & \vdots & 1 \\ 0 & -1 & 2 & \vdots & k-2 \\ 0 & 0 & 0 & \vdots & k^2-3k+2 \end{bmatrix}$

$$\rho(A) = 2 < \text{number of unknowns}$$

$\therefore$ System of equations cannot have a unique solution.

These will have an infinite number of solution only if $\rho(A:B) = \rho(A) = 2$ which is only possible if $k^2 - 3k + 2 = 0$ *i.e.*, $k = 1$ or $k = 2$.

when $k = 1$; the augmented matrix $= \begin{bmatrix} 1 & 1 & 1 & \vdots & 1 \\ 0 & -1 & 2 & \vdots & -1 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$

$\therefore$ Equations are $x + y + z = 1$
$$-y + 2z = -1$$

Let $$z = \lambda, y = 1 + 2\lambda, x = -3\lambda, \text{ where } \lambda \text{ is an arbitrary constant}$$

when $$k = 2 \text{ ; Augmented matrix} = \begin{bmatrix} 1 & 1 & 1 & \vdots & 1 \\ 0 & -1 & 2 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$$

Equations are $x + y + z = 1$ and $-y + 2z = 0$.

Take $$z = \lambda', y = 2\lambda', x = 1 - 3\lambda'$$

where λ' is any arbitrary constant.

Example 7. *Find the values of a and b for which the equations $x + ay + z = 3$; $x + 2y + 2z = b$, $x + 5y + 3z = 9$ are consistent. When will these equations have a unique solution?* **(P.T.U., Dec. 2003)**

Sol. Given equations are

$$x + ay + z = 3$$
$$x + 2y + 2z = b$$
$$x + 5y + 3z = 9.$$

The matrix equation is AX = B, where

$$A = \begin{bmatrix} 1 & a & 1 \\ 1 & 2 & 2 \\ 1 & 5 & 3 \end{bmatrix}, B = \begin{bmatrix} 3 \\ b \\ 9 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Augmented matrix is $\begin{bmatrix} 1 & a & 1 & \vdots & 3 \\ 1 & 2 & 2 & \vdots & b \\ 1 & 5 & 3 & \vdots & 9 \end{bmatrix}$

Operate $R_2 - R_1, R_3 - R_1 \sim \begin{bmatrix} 1 & a & 1 & \vdots & 3 \\ 0 & 2-a & 1 & \vdots & b-3 \\ 0 & 5-a & 2 & \vdots & 6 \end{bmatrix}$

Operate $R_3 - \dfrac{5-a}{2-a} R_2 \sim \begin{bmatrix} 1 & a & 1 & \vdots & 3 \\ 0 & 2-a & 1 & \vdots & b-3 \\ 0 & 0 & 2 - \frac{5-a}{2-a} & \vdots & 6 - \frac{5-a}{2-a}(b-3) \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & a & 1 & \vdots & 3 \\ 0 & 2-a & 1 & \vdots & b-3 \\ 0 & 0 & \frac{-1-a}{2-a} & \vdots & 6 - \frac{5-a}{2-a}(b-3) \end{bmatrix}$$

Case I. If $\dfrac{-1-a}{2-a} = 0$ and $6 - \dfrac{5-a}{2-a}(b-3) = 0$

then $a = -1 \therefore b = 6$ *i.e.*, $a = -1, b = 6$

$\rho(A) = \rho(A : B) = 2 <$ number of unknowns

$\therefore$ Given equations have infinite number of solutions given by

$$\begin{bmatrix} 1 & -1 & 1 & \vdots & 3 \\ 0 & 3 & 1 & \vdots & 3 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$$

Equations are $x - y + z = 3$

$$3y + z = 3$$

Let $z = k\,;\, y = \dfrac{3-k}{3} \quad \therefore\ x = 3 + \dfrac{3-k}{3} - k = 4 - \dfrac{4}{3}k$

$\therefore \quad x = 4 - \dfrac{4}{3}k\,,\ y = \dfrac{3-k}{3}\,,\ z = k$ is the solution.

Case II. If $a = -1$, but $b \neq 6$

Then $\rho(A) = 2$, but $\rho(A : B) = 3$

$\rho(A) \neq \rho(A : B)$ $\therefore$ Equations are inconsistent, *i.e.*, having no solution.

Case III. If $a \neq -1$, b can have any value then $\rho(A) = 3 = \rho(A : B)$

$\therefore$ Given equations have a unique solution which is given by the equation.

$$x + ay + z = 3$$

$$(2-a)\,y + z = b - 3$$

$$\frac{-(1+a)}{2-a}\, z = 6 - \frac{5-a}{2-a}\,(b-3)$$

$$\therefore \quad z = \frac{2-a}{-(1+a)}\left\{6 - \frac{(5-a)(b-3)}{2-a}\right\} = \frac{6(2-a)-(5-a)(b-3)}{-(1+a)}$$

$$y = \frac{1}{2-a}\left[(b-3) - \frac{6(2-a)-(5-a)(b-3)}{-(1+a)}\right]$$

$$= \frac{1}{(2-a)(1+a)}\left[(b-3)(1+a) + 6(2-a) - (5-a)(b-3)\right]$$

$$\therefore \quad y = \frac{2(6-b)}{1+a}\,;\ x = \frac{(5-3a)(6-b)}{1+a}.$$

TEST YOUR KNOWLEDGE

1. Write the following equations in matrix form AX = B and solve for X by finding A^{-1}.

   (*i*) $2x - 2y + z = 1$, $x + 2y + 2z = 2$, $2x + y - 2z = 7$

   (*ii*) $2x_1 - x_2 + x_3 = 4$, $x_1 + x_2 + x_3 = 1$, $x_1 - 3x_2 - 2x_3 = 2$

2. Using the loop current method on a circuit, the following equations were obtained :

   $7i_1 - 4i_2 = 12,\ -4i_1 + 12i_2 - 6i_3 = 0,\ 6i_2 + 14i_3 = 0.$

   By matrix method, solve for i_1, i_2 and i_3

3. Solve the following system of equations by matrix method :

   (*i*) $x + y + z = 8,\quad x - y + 2z = 6,\quad 3x + 5y - 7z = 14$

   (*ii*) $x + y + z = 6,\quad x - y + 2z = 5,\quad 3x + y + z = 8$

   (*iii*) $x + 2y + 3z = 1,\quad 2x + 3y + 2z = 2,\quad 3x + 3y + 4z = 1.$

   (*iv*) $3x + 3y + 2z = 1,\quad x + 2y = 4,\quad 10y + 3z = -2,\ 2x - 3y - z = 5$

4. Show that the equations $x + 2y - z = 3,\ 3x - y + 2z = 1,\ 2x - 2y + 3z = 2,\ x - y + z = -1$ are consistent and solve them.

5. Test for consistency the equations $2x - 3y + 7z = 5,\ 3x + y - 3z = 13,\ 2x + 19y - 47z = 32.$

   (P.T.U., May 2012, Dec. 2012)

6. Solve the equations $x + 3y + 2z = 0,\ 2x - y + 3z = 0,\ 3x - 5y + 4z = 0,\ x + 17y + 4z = 0.$

7. (*a*) For what values of a and b do the equations $x + 2y + 3z = 6,\ x + 3y + 5z = 9,\ 2x + 5y + az = b$ have

   (*i*) no solution (*ii*) a unique solution (*iii*) more than one solution ?

   (*b*) For what value of k the system of equations $x + y + z = 2,\ x + 2y + z = -2,\ x + y + (k-5)z = k$ has no solution ?

   (P.T.U., May 2012)

8. (a) Find the value of k so that the equations $x+y+3z=0, 4x+3y+kz=0, 2x+y+2z=0$ have a non-trivial solution.
(b) For what values of λ do the equations $ax + by = \lambda x$ and $cx + dy = \lambda y$ have a solution other than $x = 0, y = 0$. **(P.T.U., May 2003)**

9. Show that the equations $3x + 4y + 5z = a, 4x + 5y + 6z = b, 5x + 6y + 7z = c$ do not have a solution unless $a + c = 2b$. **(P.T.U., Dec. 2011)**

10. Investigate the value of λ and μ so that the equations $2x + 3y + 5z = 9, 7x + 3y - 2z = 8, 2x + 3y + \lambda z = \mu$ have (i) no solution, (ii) a unique solution and (iii) an infinite number of solution.

11. Determine the value of λ for which the following set of equations may possess non-trivial solution.
$3x_1 + x_2 - \lambda x_3 = 0, 4x_1 - 2x_2 - 3x_3 = 0, 2\lambda x_1 + 4x_2 + \lambda x_3 = 0$. For each permissible value of λ, determine the general solution.

12. Investigate for consistency of the following equations and if possible find the solutions.
$4x - 2y + 6z = 8,\ x + y - 3z = -1,\ 15x - 3y + 9z = 21.$ **(P.T.U., Jan. 2009)**

13. Show that if $\lambda \neq -5$, the system of equations $3x - y + 4z = 3, x + 2y - 3z = -2, 6x + 5y + \lambda z = -3$ have a unique solution. If $\lambda = -5$, show that the equations are consistent. Determine the solutions in each case.

14. Show that the equations $2x + 6y + 11 = 0, 6x + 20y - 6z + 3 = 0, 6y - 18z + 1 = 0$ are not consistent.
[**Hint:** To prove $\rho[A : B] \neq \rho(A)$] **(P.T.U., Dec. 2003)**

15. Solve the system of equations $2x_1 + x_2 + 2x_3 + x_4 = 6;\ 6x_1 - 6x_2 + 6x_3 + 12x_4 = 36$
$4x_1 + 3x_2 + 3x_3 - 3x_4 = -1;\ 2x_1 + 2x_2 - x_3 + x_4 = 10.$

ANSWERS

1. (i) $x = 2, y = 1, z = -1$ (ii) $x_1 = 1, x_2 = -1, x_3 = 1$
2. $i_1 = \frac{396}{175}, i_2 = \frac{24}{25}, i_3 = \frac{72}{175}$
3. (i) $x = 5, y = \frac{5}{3}, z = \frac{4}{3}$ (ii) $x = 1, y = 2, z = 3$ (iii) $x = -\frac{3}{7}, y = \frac{8}{7}, z = -\frac{2}{7}$
(iv) $x = 2, y = 1, z = -4$
4. $x = -1, y = 4, z = 4$
5. Inconsistent
6. $x = 11k, y = k, z = -7k$, where k is arbitrary
7. (a) (i) $a = 8, b \neq 15$ (ii) $a \neq 8$, b may have any value (iii) $a = 8, b = 15$
(b) $k = 6$
8. (a) $k = 8$, (b) $\lambda = a, b = 0, \lambda = d, c = 0$
10. (i) $\lambda = 5, \mu \neq 9$ (ii) $\lambda \neq 5$, μ arbitrary (iii) $\lambda = 5, \lambda = 9$
11. $\lambda = 1, -9$ for $\lambda = 1$ solution is $x = k, y = -k, z = 2k$. For $\lambda = -9$ solution is $x = 3k, y = 9k, z = -2k$
12. Consistent: $x = 1, y = 3k - 2, z = k$, where k is arbitrary
13. $\lambda \neq -5, x = \frac{4}{7}, y = -\frac{9}{7}, z = 0; \lambda = -5, x = \frac{4-5k}{7}, y = \frac{13k-9}{7}, z = k$, where k is arbitrary
15. $x_1 = 2, x_2 = 1, x_3 = -1, x_4 = 3.$

4.14. VECTORS

Any ordered n-tuple of numbers is called an n-vector. By an ordered n-tuple, we mean a set consisting of n numbers in which the place of each number is fixed. If $x_1, x_2, \ldots\ldots, x_n$ be any n numbers then the ordered n-tuple $X = (x_1, x_2, \ldots\ldots, x_n)$ is called an n-vector. Thus the co-ordinates of a point in space can be represented by a 3-vector (x, y, z). Similarly $(1, 0, 2, -1)$ and $(2, 7, 5, -3)$ are 4-vectors. The n numbers $x_1, x_2, \ldots\ldots, x_n$ are called the components of the n-vector $X = (x_1, x_2, \ldots, x_n)$. A vector may be written either as a *row vector* or as a *column vector*. If A be a matrix of order $m \times n$, then each row of A will be an n-vector and each column of A will be an m-vector. A vector whose components are all zero is called a zero vector and is denoted by O. Thus $O = (0, 0, 0, \ldots, 0)$.

Let $X = (x_1, x_2, \ldots\ldots\ldots, x_n)$ and $Y = (y_1, y_2, \ldots\ldots\ldots, y_n)$ be two vectors.

Then X = Y if and only if their corresponding components are equal.

i.e., If $x_i = y_i$ for $i = 1, 2, \ldots\ldots\ldots, n$

If k be a scalar, then $kX = (kx_1, kx_2, \ldots\ldots\ldots, kx_n)$.

4.15. LINEAR DEPENDENCE AND LINEAR INDEPENDENCE OF VECTORS

(P.T.U., May 2004, 2006, Jan. 2009)

imp

A set of r, n-tuple vectors $X_1, X_2, \ldots\ldots\ldots, X_r$ is said to be *linearly dependent* if there exists r scalars (numbers) $k_1, k_2, \ldots\ldots\ldots, k_r$ not all zero, such that

$$k_1 X_1 + k_2 X_2 + \ldots\ldots + k_r X_r = O$$

A set of r, n-tuple vectors $X_1, X_2, \ldots\ldots\ldots, X_r$ is said to be *linearly independent* if every relation of the type

$$k_1 X_1 + k_2 X_2 + \ldots\ldots + k_r X_r = O \text{ implies } k_1 = k_2 = \ldots\ldots = k_r = 0$$

Note. If a set of vectors is linearly dependent, then at least one member of the set can be expressed as a linear combination of the remaining vectors.

imp **Example 1.** *Show that the vectors* $x_1 = (1, 2, 4)$, $x_2 = (2, -1, 3)$, $x_3 = (0, 1, 2)$ *and* $x_4 = (-3, 7, 2)$ *are linearly dependent and find the relation between them.*

Sol. Consider the matrix equation

$$k_1 x_1 + k_2 x_2 + k_3 x_3 + k_4 x_4 = 0$$

i.e., $$k_1 (1, 2, 4) + k_2 (2, -1, 3) + k_3 (0, 1, 2) + k_4 (-3, 7, 2) = 0$$

i.e.,
$$\begin{aligned} k_1 + 2k_2 + 0 \cdot k_3 - 3k_4 &= 0 \\ 2k_1 - k_2 + k_3 + 7k_4 &= 0 \\ 4k_1 + 3k_2 + 2k_3 + 2k_4 &= 0 \end{aligned}$$

which is a system of homogeneous linear equations and can be put in the form AX = 0.

i.e.,
$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 2 & -1 & 1 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operate $R_2 - 2R_1, R_3 - 4R_1$;

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & -5 & 2 & 14 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operate $R_3 - R_2$;

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$
$$\begin{aligned} k_1 + 2k_2 - 3k_4 &= 0 \\ -5k_2 + k_3 + 13k_4 &= 0 \\ k_3 + k_4 &= 0 \end{aligned}$$

$\therefore$ $$k_4 = -k_3$$

From (ii), $\quad 5k_2 = k_3 - 13k_3 = -12k_3 \therefore k_2 = -\frac{12}{5}k_3$

From (i), $\quad k_1 = +3k_4 - 2k_2 = -3k_3 + \frac{24}{5}k_3 = \frac{9}{5}k_3$

Let $\quad k_3 = t$

$\therefore \quad k_1 = \frac{9}{5}t,\ k_2 = -\frac{12}{5}t,\ k_3 = t,\ k_4 = -t$

$\therefore$ Given vectors are L.D. and the relation between them is $\frac{9}{5}tx_1 - \frac{12}{5}tx_2 + tx_3 - tx_4 = 0$

or $9x_1 - 12x_2 + 5x_3 - 5x_4 = 0.$

Example 2. *Show that the column vectors of the matrix*

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 1 & 2 \end{bmatrix}$$ *are linearly dependent.* **(P.T.U., Dec. 2002)**

Sol. Let $\quad X_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, X_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, X_3 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

Consider the matrix equation

$$k_1 X_1 + k_2 X_2 + k_3 X_3 = 0$$

$$k_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + k_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + k_3 \begin{bmatrix} 3 \\ 2 \end{bmatrix} = 0$$

i.e.,
$$k_1 + 2k_2 + 3k_3 = 0$$
$$-2k_1 + k_2 + 2k_3 = 0$$

$$\begin{bmatrix} 1 & 2 & 3 \\ -2 & 1 & 2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = 0$$

Operate $R_2 + 2R_1$; $$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 8 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = 0$$

or
$$k_1 + 2k_2 + 3k_3 = 0$$
$$5k_2 + 8k_3 = 0$$

$\therefore \quad k_2 = -\frac{8}{5}k_3 ;\ k_1 = \frac{16}{5}k_3 - 3k_3 = \frac{1}{5}k_3$

Let $\quad k_3 = \lambda \neq 0$

$\therefore \quad k_1 = \frac{1}{5}\lambda,\ k_2 = -\frac{8}{5}\lambda,\ k_3 = \lambda$

$\therefore$ Given column vectors are L.D.

Example 3. *Determine whether the vectors $(3, 2, 4)'$, $(1, 0, 2)'$, $(1, -1, -1)'$ are linearly dependent or not.* **(P.T.U., May 2006)**
(where 't' denotes transpose)

Sol. Let $X_1 = (3, 2, 4)' = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}, X_2 = (1, 0, 2)' = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, X_3 = (1, -1, -1)' = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

Consider $k_1 X_1 + k_2 X_2 + k_3 X_3 = 0$

i.e., $$k_1 \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} + k_2 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + k_3 \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = 0$$

or $$\begin{aligned} 3k_1 + k_2 + k_3 &= 0 \\ 2k_1 + 0.k_2 - k_3 &= 0 \\ 4k_1 + 2k_2 - k_3 &= 0 \end{aligned}$$

or $$\begin{bmatrix} 3 & 1 & 1 \\ 2 & 0 & -1 \\ 4 & 2 & -1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operate $R_1 - R_2$; $$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 0 & -1 \\ 4 & 2 & -1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operate $R_2 - 2R_1, R_3 - 4R_1$;
$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & -2 & -5 \\ 0 & -2 & -9 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operate $R_3 - R_2$;
$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & -2 & -5 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ $$\left.\begin{aligned} k_1 + k_2 + 2k_3 &= 0 \\ -2k_2 - 5k_3 &= 0 \\ -4k_3 &= 0 \end{aligned}\right] \Rightarrow k_3 = 0, k_2 = 0, k_1 = 0$$

$\therefore$ Given vectors are not linearly dependent. These are linearly independent.

4.16. LINEAR TRANSFORMATIONS

Let a point $P(x, y)$ in a plane transform to the point $P'(x', y')$ under reflection in the co-ordinate axes, or reflection in the line $y = x \tan \theta$ or rotation of OP through an angle θ about the origin or rotation of axes, through an angle θ etc. Then the co-ordinates of P' can be expressed in terms of those of P by the linear relations of the form

$$\left.\begin{aligned} x' &= a_1 x + b_1 y \\ y' &= a_2 x + b_2 y \end{aligned}\right]$$

which in matrix notation is $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ or $X' = AX$

such transformations are called linear transformation in two dimensions.

Similarly, relations of the form

$$\begin{aligned} x' &= a_1 x + b_1 y + c_1 z \\ y' &= a_2 x + b_2 y + c_2 z \\ z' &= a_3 x + b_3 y + c_3 z \end{aligned}$$

which in matrix notation is $\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ or $X' = AX$ gives a linear transformation $(x, y, z) \rightarrow (x', y', z')$ in three dimensions.

In general, the relation $Y = AX$, where $Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$, $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

defines a linear transformation which carries any vector X into another vector Y over the matrix A which is called the linear operator of the transformation.

This transformation is called linear because $Y_1 = AX_1$ and $Y_2 = AX_2$ implies $aY_1 + bY_2 = A(aX_1 + bX_2)$ for all values of a and b.

Thus, if $X = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ then $Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \end{bmatrix}$

so that $(2, -3) \rightarrow (5, -5)$ under the transformation defined by A.

If the transformation matrix A is non-singular, *i.e.*, if $|A| \neq 0$, then the linear transformation is called **non-singular or regular.**

If the transformation matrix A is singular, *i.e.*, if $|A| = 0$, then the linear transformation is also called **singular.**

For a non-singular transformation $Y = AX$, since A is non-singular, A^{-1} exists and we can write the inverse transformation, which carries the vector Y back into the vector X, as $X = A^{-1} Y$.

Note. If a transformation from $(x_1, x_2, \ldots\ldots\ldots, x_n)$ to $(y_1, y_2, \ldots\ldots\ldots, y_n)$ is given by $Y = AX$ and another transformation from $(y_1, y_2, \ldots\ldots\ldots, y_n)$ to $(z_1, z_2, \ldots\ldots\ldots, z_n)$ is given by $Z = BY$, then the transformation from $(x_1, x_2, \ldots\ldots\ldots, x_n)$ to, $(z_1, z_2, \ldots\ldots\ldots, z_n)$ is given by $Z = BY = B(AX) = (BA)X$.

4.17. ORTHOGONAL TRANSFORMATION (P.T.U., Dec. 2012)

The linear transformation $Y = AX$, where

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

is said to be *orthogonal* if it transforms $y_1^2 + y_2^2 + \ldots\ldots + y_n^2$ into $x_1^2 + x_2^2 + \ldots\ldots + x_n^2$.

4.18(a). ORTHOGONAL MATRIX (P.T.U., Jan. 2009)

The matrix A of the above transformation is called ***an orthogonal matrix.***

Now, $$X'X = [x_1 x_2 \ldots\ldots x_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1^2 + x_2^2 + \ldots\ldots + x_n^2$$

and similarly $\quad Y'Y = y_1^2 + y_2^2 + \ldots\ldots + y_n^2.$

$\therefore$ If $Y = AX$ is an orthogonal transformation, then

$$X'X = x_1^2 + x_2^2 + \ldots\ldots + x_n^2 = y_1^2 + y_2^2 + \ldots\ldots + y_n^2.$$
$$= Y'Y = (AX)'(AX) = (X'A')(AX)$$
$$= X'(A'A)X \qquad [\because (AB)' = B'A']$$

which holds only when $A'A = I$ or when $A'A = A^{-1}A$

or when $\quad A' = A^{-1} \qquad [\because A^{-1}A = I]$

Hence *a real square matrix A is said to be orthogonal if* $AA' = A'A = I$

Also, *for an orthogonal matrix* A, $A' = A^{-1}$.

4.18(b). PROPERTIES OF AN ORTHOGONAL MATRIX

(*i*) **The transpose of an orthogonal matrix is orthogonal.**

Proof. Let A be an orthogonal matrix

$\therefore \quad AA' = I = A'A$

Taking transpose of both sides of $AA' = I$

$(AA')' = I'$ or $(A')'\,A' = I$ *i.e.*, product of A' and its transpose *i.e.*, $(A')' = I$

$\therefore$ A' is an orthogonal matrix

(*ii*) **The inverse of an orthogonal matrix is orthogonal**

Proof. Let A be an orthogonal matrix $\therefore AA' = I$

Take inverse of both sides $(AA')^{-1} = I^{-1}$

or $\quad (A')^{-1}A^{-1} = I \quad$ or $(A^{-1})'(A^{-1}) = I$

i.e., Product of A^{-1} and its transpose *i.e.*, $(A^{-1})'$ is I

$\therefore$ A^{-1} is orthogonal.

(*iii*) **If A is an orthogonal matrix then $|A| = \pm 1$**

Proof. A is an orthogonal matrix

$\therefore \quad AA' = I$

Take determinant of both sides

$|AA'| = |I| \quad$ or $|A|\,|A'| = 1 \because |I| = 1$

i.e., $\quad |A|^2 = 1 \quad \because |A'| = |A|$

i.e., $\quad |A| = \pm 1$

Note. An orthogonal matrix A is called proper or improper according as $|A| = 1$ or -1.

(*iv*) **The product of two orthogonal matrices of the same order is orthogonal**

Proof. Let A, B be two orthogonal matrices of the same order so that

$$AA' = BB' = I$$

Now, $\quad (AB)(AB)' = (AB)(B'A') = A(BB')A'$
$$= A(I)A' = (AI)A' = AA' = I$$

$\therefore$ AB is also an orthogonal matrix.

Example 1. *Let T be the transformation from R^1 to R^3 defined by $T(x) = (x, x^2, x^3)$. Is T linear or not.* **(P.T.U., May 2010)**

Sol. Given $\quad T(x) = (x, x^2, x^3)$

$$T(x_1) = (x_1, x_1^2, x_1^3)$$
$$T(x_2) = (x_2, x_2^2, x_2^3)$$
$$\alpha T(x_1) + \beta T(x_2) = \alpha(x_1, x_1^2, x_1^3) + \beta(x_2, x_2^2, x_2^3)$$
$$= (\alpha x_1 + \beta x_2, \alpha x_1^2 + \beta x_2^2, \alpha x_1^3 + \beta x_2^3)$$

Now, $T(\alpha x_1 + \beta x_2) = [(\alpha x_1 + \beta x_2), (\alpha x_1 + \beta x_2)^2, (\alpha x_1 + \beta x_2)^3]$

$$\neq (\alpha x_1 + \beta x_2, \alpha x_1^2 + \beta x_2^2, \alpha x_1^3 + \beta x_2^3)$$

$\therefore$ $\alpha T(x_1) + \beta T(x_2) \neq T(\alpha x_1 + \beta x_2)$

$\therefore$ T is not linear

Example 2. *Show that the transformation* $y_1 = x_1 + 2x_2 + 5x_3$; $y_2 = -x_2 + 2x_3$; $y_3 = 2x_1 + 4x_2 + 11x_3$ *is regular. Write down the inverse transformation.* **(P.T.U., May 2011)**

Sol. The given transformation in the matrix form is Y = AX where

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 0 & -1 & 2 \\ 2 & 4 & 11 \end{bmatrix}; X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}; Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 2 & 5 \\ 0 & -1 & 2 \\ 2 & 4 & 11 \end{vmatrix}$$
$$= 1(-11-8) + 2(4+5)$$
$$= -19 + 18 = -1 \neq 0$$

$\therefore$ Matrix A is non-singular.

Hence given transformation is non-singular or regular.

The inverse transformation of $Y = AX$ is $X = A^{-1}Y$

[To find A^{-1}]

Consider A = IA

$$\begin{bmatrix} 1 & 2 & 5 \\ 0 & -1 & 2 \\ 2 & 4 & 11 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Operate $R_3 - 2R_1$;

$$\begin{bmatrix} 1 & 2 & 5 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} A$$

Operate $R_1 - 5R_3$, $R_2 - 2R_3$;

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 0 & -5 \\ 4 & 1 & -2 \\ -2 & 0 & 1 \end{bmatrix} A$$

Operate $R_1 + 2R_2$;

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 19 & 2 & -9 \\ 4 & 1 & -2 \\ -2 & 0 & 1 \end{bmatrix} A$$

Operate $R_2(-1)$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 19 & 2 & -9 \\ -4 & -1 & 2 \\ -2 & 0 & 1 \end{bmatrix} A$

$$I = BA \text{ where } B = \begin{bmatrix} 19 & 2 & -9 \\ -4 & -1 & 2 \\ -2 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad A^{-1} = B$$

$$\therefore \quad X = \begin{bmatrix} 19 & 2 & -9 \\ -4 & -1 & 2 \\ -2 & 0 & 1 \end{bmatrix} Y$$

$$\therefore \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 19 & 2 & -9 \\ -4 & -1 & 2 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$x_1 = 19y_1 + 2y_2 - 9y_3$$
$$x_2 = -4y_1 - y_2 + 2y_3$$
$$x_3 = -2y_1 + y_3$$

Example 3. *(a) Prove that the following matrix is orthogonal*

$$\frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}.$$ **(P.T.U., May 2007)**

(b) Find the values of a, b, c if the matrix

$$A = \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix} \text{ is orthogonal.}$$ **(P.T.U., May 2009)**

Sol. (*a*) Denoting the given matrix by A, we have

$$A' = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

Now,
$$AA' = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \times \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$
$$= \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Since $AA' = I$, A is an orthogonal matrix.

(*b*) Matrix A will be orthogonal if $AA' = I$

i.e.,
$$\begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}\begin{bmatrix} 0 & a & a \\ 2b & b & -b \\ c & -c & c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

or
$$\begin{bmatrix} 4b^2 + c^2 & 2b^2 - c^2 & -2b^2 + c^2 \\ 2b^2 - c^2 & a^2 + b^2 + c^2 & a^2 - b^2 - c^2 \\ -2b^2 + c^2 & a^2 - b^2 - c^2 & a^2 + b^2 + c^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$4b^2 + c^2 = 1 \quad \text{and} \quad 2b^2 - c^2 = 0$$

or

$$a^2 + b^2 + c^2 = 1 \qquad a^2 - b^2 - c^2 = 0$$

Solving:

$$c^2 = 2b^2 \quad \therefore \quad 4b^2 + 2b^2 = 1 \quad \text{or} \quad b^2 = \frac{1}{6}$$

$$\text{or} \quad b = \pm\frac{1}{\sqrt{6}}$$

$\therefore$

$$c^2 = 2\cdot\frac{1}{6} = \frac{1}{3} \quad \therefore \quad c = \pm\frac{1}{\sqrt{3}}$$

$$a^2 = b^2 + c^2 = \frac{1}{6} + \frac{1}{3} = \frac{1}{2} \quad \therefore \quad a = \pm\frac{1}{\sqrt{2}}$$

Hence

$$a = \pm\frac{1}{\sqrt{2}}, b = \pm\frac{1}{\sqrt{6}}, c = \pm\frac{1}{\sqrt{3}}.$$

TEST YOUR KNOWLEDGE

1. Are the following vectors linearly dependent ? If so, find a relation between them.
   (i) $x_1 = (1, 2, 1), x_2 = (2, 1, 4), x_3 = (4, 5, 6), \; x_4 = (1, 8, -3)$ **(P.T.U., Jan. 2010)**
   (ii) $x_1 = (2, -1, 4), x_2 = (0, 1, 2), x_3 = (6, -1, 16), \; x_4 = (4, 0, 12)$
   (iii) $x_1 = (2, -1, 3, 2), x_2 = (1, 3, 4, 2), x_3 = (3, -5, 2, 2)$
   (iv) $x_1 = (2, 3, 1, -1), x_2 = (2, 3, 1, -2), x_3 = (4, 6, 2, 1)$
   (v) $x_1 = (2, 2, 1)^t, x_2 = (1, 3, 1)^t, x_3 = (1, 2, 2)^t$, where '$t$' stands for transpose. **[Hint: See S.E.3]**
   (vi) $x_1 = (1, 1, 1), x_2 = (1, -1, 1), x_3 = (3, -1, 3)$ **(P.T.U., Dec. 2012)**
2. For what value(s) of k, do the set of vectors $(k, 1, 1), (0, 1, 1), (k, 0, k)$ in R^3 are linearly independent. **(P.T.U., May 2010, May 2012)**
3. (a) Show that the transformation $y_1 = x_1 - x_2 + x_3, y_2 = 3x_1 - x_2 + 2x_3, \; y_3 = 2x_1 - 2x_2 + 3x_3$ is non-singular. Find the inverse transformation.
   (b) Show that the transformation $y_1 = 2x_1 + x_2 + x_3; y_2 = x_1 + x_2 + 2x_3; y_3 = x_1 - 2x_3$ is regular. Write down the inverse transformation.
4. Represent each of the transformation $x_1 = 3y_1 + 2y_2, \; y_1 = z_1 + 2z_2, x_2 = -y_1 + 4y_2$ and $y_2 = 3z_1$ by the use of matrices and find the composite transformation which expresses x_1, x_2 in terms of z_1, z_2.
5. A transformation from the variables x_1, x_2, x_3 to y_1, y_2, y_3 is given by Y = AX, and another transformation from y_1, y_2, y_3 to z_1, z_2, z_3 is given by Z = BY, where $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \\ -1 & 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 1 & 3 & 5 \end{bmatrix}$. Obtain the transformation from x_1, x_2, x_3 to z_1, z_2, z_3.
6. Which of the following matrices are orthogonal:
   (i) $\frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & -3 & 1 \\ 4 & 3 & 1 \\ -3 & 1 & 9 \end{bmatrix}$. **(P.T.U., Jan. 2009)**
7. Prove that the following matrix is orthogonal:
   $\begin{bmatrix} -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{bmatrix}$ **(P.T.U., May 2011)**

ANSWERS

1. (*i*) Yes ; $x_3 = 2x_1 + x_2$ and $x_4 = 5x_1 - 2x_2$ (*ii*) Yes ; $x_3 = 3x_1 + 2x_2$ and $x_4 = 2x_1 + x_2$
(*iii*) Yes ; $2x_1 - x_2 - x_3 = 0$ (*iv*) Yes ; $5x_1 - 3x_2 - x_3 = 0$
(*v*) No; L.I. (*vi*) No; L.I.
2. For all non zero values of k
3. (*a*) $x_1 = \frac{1}{2}(y_1 + y_2 - y_3),\ x_2 = \frac{1}{2}(-5y_1 + y_2 + y_3),\ x_3 = -4y_1 + 2y_3$
(*b*) $x_1 = 2y_1 - 2y_2 - y_3,\ x_2 = -4y_1 + 5y_2 + 3y_3,\ x_3 = y_1 - y_2 - y_3$
4. $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 9 & 6 \\ 11 & -2 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$
5. $Z = (BA)\,X$, where $BA = \begin{bmatrix} 1 & 4 & -1 \\ -1 & 9 & -1 \\ -3 & 14 & -1 \end{bmatrix}$
6. (*i*) Orthogonal (*ii*) Not orthogonal.

4.19. COMPLEX MATRICES

If *all* the elements of a matrix are real numbers, then it is called *a real matrix* or a matrix over R. On the other hand, if *at least one* element of a matrix is a complex number $a + ib$, where a, b are real and $i = \sqrt{-1}$, then the matrix is called a *complex matrix.*

4.20(a). CONJUGATE OF A MATRIX

The matrix obtained by replacing the elements of a complex matrix A by the corresponding conjugate complex numbers is called the *conjugate of the matrix A* and is denoted by $\bar{A}$.

Thus, if $$A = \begin{bmatrix} 2+3i & -7i \\ 5 & 1-i \end{bmatrix}, \text{ then } \bar{A} = \begin{bmatrix} 2-3i & 7i \\ 5 & 1+i \end{bmatrix}.$$

4.20(b). CONJUGATE TRANSPOSE OF A MATRIX

It is easy to see the *conjugate of the transpose of A i.e.,* $(\overline{A'})$ and the *transpose conjugate of A i.e.,* $(\bar{A})'$ are equal. Each of them is denoted by A^θ.

Thus $$(\overline{A'}) = (\bar{A})' = A^\theta.$$

4.21. HERMITIAN AND SKEW HERMITIAN MATRIX (P.T.U., May 2002, 2007, Dec. 2010)

A square matrix A is said to be **Hermitian if $A^\theta = A$.** *i.e.,* if $A = [a_{ij}]$, then $\overline{a_{ij}} = a_{ji}\ \forall\ i, j$ and when $i = j$, then $\overline{a_{ii}} = a_{ii} \Rightarrow a_{ii}$ is purely real *i.e.,* **all diagonal elements of a Hermitian matrix are purely real** while every other element is the conjugate complex of the element in the transposed position.

For example, $A = \begin{bmatrix} 5 & 2+i & -3i \\ 2-i & -3 & 1-i \\ 3i & 1+i & 0 \end{bmatrix}$ is a Hermitian matrix.

A square matrix A is said to be **Skew Hermitian** if $A^\theta = -A$ *i.e.,* if $A = [a_{ij}]$, then $\overline{a_{ij}} = -a_{ji}\ \forall\ i, j$ and when $i = j$, then $\overline{a_{ii}} = -a_{ii}$ *i.e.,* if $a_{ii} = a + ib$, then $\overline{a_{ii}} = a - ib$ and $\overline{a_{ii}} = -a_{ii}$

$\Rightarrow$ $a - ib = -(a + ib) \Rightarrow a = 0$

$\therefore$ a_{ii} is either purely imaginary or zero.

In a skew Hermitian matrix, **the diagonal elements are zero or purely imaginary number** of the form $i\beta$, where β is real. Every other element is the negative of the conjugate complex of the element in the transposed position.

For example, $B = \begin{bmatrix} 3i & 1+i & 7 \\ -1+i & 0 & -2-i \\ -7 & 2-i & -i \end{bmatrix}$ is a skew Hermitian matrix.

Note. The following result hold :

(i) $\overline{(\bar{A})} = A$ (ii) $\overline{A+B} = \bar{A}+\bar{B}$ (iii) $\overline{\lambda A} = \bar{\lambda}\bar{A}$ (iv) $\overline{AB} = \bar{A}\bar{B}$

(v) $(A^\theta)^\theta = A$ (vi) $(A+B)^\theta = A^\theta + B^\theta$ (vii) $(\lambda A)^\theta = \bar{\lambda} A^\theta$ (viii) $(AB)^\theta = B^\theta A^\theta$.

4.22(a). UNITARY MATRIX

A complex square matrix A is said to unitary if $A^\theta A = I$

or we can say $(\bar{A}')A = I$

Taking conjugate of both sides $A'\bar{A} = I$

Incase of real matrices : If A is a real matrix there $\bar{A} = A$, then A will be unitary if $A'\bar{A} = I \Rightarrow A'A = I$ which clearly shows that A is also an orthogonal matrix.

Hence every orthogonal matrix is unitary.

4.22(b). PROPERTIES OF A UNITARY MATRIX

(i) **Determinant of a Unitary Matrix is of Modulus Unity**

Proof. Let A be a unitary matrix

Then $AA^\theta = I$

Taking determinant of both sides $|AA^\theta| = |I|$

or $|A|\,|\bar{A}'| = 1$ or $|A|\,|\bar{A}| = 1 \quad \because |A'| = |A|$

or $|A^2| = 1$ hence the result.

(ii) **The Product of Two Unitary Matrices of the Same Order is Unitary**

Proof. Let A, B be two unitary matrices $\therefore AA^\theta = A^\theta A = I$ and $BB^\theta = B^\theta B = I$

Now, $(AB)(AB)^\theta = AB(B^\theta A^\theta) = A(BB^\theta)A^\theta = (AI)A^\theta = AA^\theta = I$

Hence AB is unitary matrix.

(iii) **The Inverse of a Unitary Matrix is Unitary** **(P.T.U., May 2012)**

Proof. Let A be a unitary matrix $\therefore AA^\theta = A^\theta A = 1$

$AA^\theta = I$

Take inverse of both sides $(AA^\theta)^{-1} = I$ or $(A^\theta)^{-1}.\ A^{-1} = I$

or $(A^{-1})^\theta (A^{-1}) = I \quad \therefore A^{-1}$ is also unitary.

ILLUSTRATIVE EXAMPLES

Example 1. *If* $A = \begin{bmatrix} 2+i & 3 & -1+3i \\ -5 & i & 4-2i \end{bmatrix}$, *verify that $A^\theta A$ is a Hermitian matrix.*

Sol. $A' = \begin{bmatrix} 2+i & -5 \\ 3 & i \\ -1+3i & 4-2i \end{bmatrix}$

$$A^\theta = \overline{(A')} = \begin{bmatrix} 2-i & -5 \\ 3 & -i \\ -1-3i & 4+2i \end{bmatrix}$$

$$\therefore \qquad A^\theta A = \begin{bmatrix} 2-i & -5 \\ 3 & -i \\ -1-3i & 4+2i \end{bmatrix} \begin{bmatrix} 2+i & 3 & -1+3i \\ -5 & i & 4-2i \end{bmatrix}$$

$$= \begin{bmatrix} 30 & 6-8i & -19+17i \\ 6+8i & 10 & -5+5i \\ -19-17i & -5-5i & 30 \end{bmatrix} = B(\text{say})$$

Now, $$B' = \begin{bmatrix} 30 & 6+8i & -19-17i \\ 6-8i & 10 & -5-5i \\ -19-17i & -5-5i & 30 \end{bmatrix}$$

$$B^\theta = \overline{(B')} = \begin{bmatrix} 30 & 6-8i & -19+17i \\ 6+8i & 10 & -5+5i \\ -19-17i & -5-5i & 30 \end{bmatrix} = B$$

$\Rightarrow$ $B\ (= A^\theta A)$ is a Hermitian matrix.

Example 2. *If A and B are Hermitian, show that AB – BA is skew Hermitian.*

Sol. A and B are Hermitian. $\Rightarrow A^\theta = A$ and $B^\theta = B$

Now, $$(AB - BA)^\theta = (AB)^\theta - (BA)^\theta$$
$$= B^\theta A^\theta - A^\theta B^\theta = BA - AB = -(AB - BA)$$

$\Rightarrow$ AB – BA is skew Hermitian.

Example 3. *(a) If A is a skew Hermitian matrix, then show that iA is Hermitian.* **(P.T.U., May 2007)**

(b) If A is Hermitian, then $A^\theta A$ is also Hermitian.

Sol. (*a*) A is a skew Hermitian matrix $\Rightarrow A^\theta = -A$

Now, $$(iA)^\theta = \bar{i}A^\theta = (-i)(-A) = iA$$

$\Rightarrow$ iA is a Hermitian matrix.

(*b*) A is a Hermitian Matrix $\therefore A^\theta = A$

$A^\theta A$ will be Hermitian if $(A^\theta A)^\theta = A^\theta A$

Now, $$(A^\theta A)^\theta = A^\theta (A^\theta)^\theta = A^\theta \cdot A$$

Hence $A^\theta A$ is Hermitian

Example 4. *If* $N = \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$, *obtain the matrix* $(I - N)(I + N)^{-1}$, *and show that it is unitary.*

Sol. $$I - N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$$

$$I + N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1+2i \\ -1+2i & 1 \end{bmatrix}$$

$$|I + N| = \begin{vmatrix} 1 & 1+2i \\ -1+2i & 1 \end{vmatrix} = 1 - (4i^2 - 1) = 6 \; \therefore \; I + N \text{ is non-singular and } (I+N)^{-1} \text{ exists}$$

$$\text{adj}\,(I + N) = \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$$

$$(I+N)^{-1} = \frac{1}{|I+N|}\text{adj}(I+N) = \frac{1}{6}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$$

$$\therefore \quad (I-N)(I+N)^{-1} = \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}\frac{1}{6}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix} = A \text{ (say)}$$

$$A' = \frac{1}{6}\begin{bmatrix} -4 & 2-4i \\ -2-4i & -4 \end{bmatrix}$$

$$\overline{(A)'} = A^{\theta} = \frac{1}{6}\begin{bmatrix} -4 & 2+4i \\ -2+4i & -4 \end{bmatrix}$$

$$A^{\theta}A = \frac{1}{6}\begin{bmatrix} -4 & 2+4i \\ -2+4i & -4 \end{bmatrix}\frac{1}{6}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix} = \frac{1}{36}\begin{bmatrix} 36 & 0 \\ 0 & 36 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

$A = (I-N)(I+N)^{-1}$ is unitary.

Example 5. *Prove that every Hermitian matrix can be written as A + iB, where A is real and symmetric and B is real and skew-symmetric.*

Sol. Let P be any Hermitian matrix.

Then $\qquad P^{\theta} = P$

Consider, $\qquad P = \dfrac{P+\overline{P}}{2} + i\,\dfrac{P-\overline{P}}{2i} = A + iB$, where

$$A = \frac{P+\overline{P}}{2}, \; B = \frac{P-\overline{P}}{2i}$$

To prove A and B are real.

We know that $\qquad z = x+iy \therefore \bar{z} = x-iy$, then $\dfrac{z+\bar{z}}{2} = 2x$ (real)

and $\qquad \dfrac{z-\bar{z}}{2i} = \dfrac{2iy}{2i} = y$ (real)

Similarly,

$\dfrac{P+\overline{P}}{2}$ is a real matrix and $\dfrac{P-\overline{P}}{2i}$ is also real

$\therefore$ A, B are real.

To prove A is symmetric

$$A' = \left(\frac{P+\overline{P}}{2}\right)' = \frac{P'+P^{\theta}}{2} = \frac{P'+P}{2} = A \qquad (\because \; P^{\theta} = P$$

$\therefore$ A is symmetric.

Similarly, $B' = \left(\dfrac{P-\overline{P}}{2i}\right)' = \dfrac{P'-P^{\theta}}{2i} = \dfrac{P'-P}{2i} = -\dfrac{P-P'}{2i} = -B \therefore$ B is skew-symmetric.

TEST YOUR KNOWLEDGE

1. If $A = \begin{bmatrix} -1 & 2+i & 5-3i \\ 2-i & 7 & 5i \\ 5+3i & -5i & 2 \end{bmatrix}$, show that A is a Hermitian matrix and iA is a skew-Hermitian matrix.

(P.T.U., May 2009, Jan. 201

2. If A is any square matrix, prove that $A + A^\theta$, AA^θ, $A^\theta A$ are all Hermitian and $A - A^\theta$ is skew-Hermitian.
3. If A, B are Hermitian or skew-Hermitian, then so is A + B.
4. Show that the matrix $B^\theta AB$ is Hermitian or skew-Hermitian according as A is Hermitian or skew-Hermitian.
5. Prove that $\frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$ is a unitary matrix. **(P.T.U., Jan. 2009)**
6. If A is a Hermitian matrix, then show that iA is a skew-Hermitian matrix.
7. Show that every square matrix is uniquely expressible as the sum of a Hermitian and a skew-Hermitian matrix.

[**Hint:** Let $A = \frac{A + A^\theta}{2} + \frac{A - A^\theta}{2} = P + Q$, prove $P^\theta = P$ and $Q^\theta = -Q$ (*ii*) to prove uniqueness : Let A = R + S where $R^\theta = R$, $S^\theta = -S$ to prove R = P, S = Q]

4.23. CHARACTERISTIC EQUATION, CHARACTERISTIC ROOTS OR EIGEN VALUES, TRACE OF A MATRIX

If A is square matrix of order n, we can form the matrix $A - \lambda I$, where λ is a scalar and I is the unit matrix of order n. The determinant of this matrix equated to zero, *i.e.*,

$$|A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0$$ is called the ***characteristics equation*** of A.

On expanding the determinant, the characteristic equation can be written as a polynomial equation of degree n in λ of the form $(-1)^n \lambda^n + k_1 \lambda^{n-1} + k_2 \lambda^{n-2} + \ldots\ldots\ldots + k_n = 0$.

The roots of this equation are called the ***characteristic roots or latent roots or eigen-values* of A.**

(P.T.U., Jan. 2009)

Note. The sum of the eigen-values of a matrix **A is equal to trace of A.**

[The trace of a square matrix is the sum of its diagonal elements].

4.24. EIGEN VECTORS **(P.T.U., Jan. 2009)**

Consider the linear transformation $Y = AX$ ...(1)

which transforms the column vector X into the column vector Y. In practice, we are often required to find those vectors X which transform into scalar multiples of themselves.

Let X be such a vector which transforms into λX (λ being a non-zero scalar) by the transformation (1).

Then $Y = \lambda X$ ...(2)

From (1) and (2), $AX = \lambda X \Rightarrow AX - \lambda IX = O \Rightarrow (A - \lambda I)X = O$ ...(3)

This matrix equation gives n homogeneous linear equations

$$\left.\begin{array}{llll} (a_{11} - l)x_1 & + a_{12}x_2 + & \ldots\ldots + a_{1n} & x_n = 0 \\ a_{21}x_1 & + (a_{22} - l)x_2 + & \ldots\ldots + a_{2n} & x_n = 0 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_{n1}x_1 & + a_{n2}x_2 + & \ldots\ldots + (a_{nn} - l) & x_n = 0 \end{array}\right] \quad \ldots(4)$$

These equations will have a non-trivial solution only if the coefficient matrix $|A - \lambda I|$ is singular *i.e.*, if $\quad |A - \lambda I| = 0 \quad ...(5)$

This is the characteristic equation of the matrix A and has n roots which are the eigen-values of A. Corresponding to each root of (5), the homogeneous system (3) has a non-zero solution

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$ which is called an ***eigen vector or latent vector.***

Note. If X is a solution of (3), then so is kX, where k is an arbitrary constant. Thus, the eigen vector corresponding to an eigen-value is not unique.

4.25. PROPERTIES OF EIGEN VALUES AND EIGEN VECTORS **(P.T.U., May 2008)**

If λ is an eigen value of A and X be its corresponding eigen vector then we have the following properties:

(*i*) $\alpha\lambda$ is an eigen value of αA and the corresponding eigen vector remains the same.

$$AX = \lambda X \Rightarrow \alpha(AX) = \alpha(\lambda X) \Rightarrow (\alpha A) X = (\alpha\lambda) X$$

$\therefore$ $\alpha\lambda$ is an eigen value of αA and eigen vector is X.

(*ii*) λ^m is an eigen value of A^m and corresponding eigen vector remains the same **(P.T.U., Dec. 2004)**

$$AX = \lambda X \Rightarrow A(AX) = A(\lambda X) \Rightarrow (AA) X = \lambda (AX)$$

$\therefore$ $$A^2X = \lambda(\lambda X) = \lambda^2 X \Rightarrow \lambda^2 \text{ is an eigen value of } A^2$$

and eigen vector is X.

Pre-multiply successively m times by A, we get the result.

(*iii*) $\lambda - k$ is an eigen-value of $A - kI$ and corresponding eigen vector is X.

$AX = \lambda X \Rightarrow AX - kIX = \lambda X - kIX$

or $$(A - kI) X = (\lambda - k) X \Rightarrow \lambda - k \text{ is the eigen vector of } A - kI \text{ and eigen vector is X.}$$

(*iv*) $\dfrac{1}{\lambda}$ is an eigen value of A^{-1} (if it exists) and the corresponding eigen vector is X. **(P.T.U., May 2005)**

$$AX = \lambda X \text{ ; Pre-multiply by } A^{-1}$$

$$A^{-1}(AX) = A^{-1}(\lambda X) \Rightarrow (A^{-1}A) X = \lambda (A^{-1}X)$$

or $$IX = \lambda(A^{-1}X) \quad \text{or} \quad A^{-1}X = \frac{1}{\lambda} X$$

$\therefore$ $\dfrac{1}{\lambda}$ is an eigen value of A^{-1} and eigen vector is X.

(*v*) $\dfrac{1}{\lambda - k}$ is an eigen value of $(A - kI)^{-1}$ and corresponding eigen vector is X

$$AX = \lambda X \Rightarrow (A - kI) X = (\lambda - k) X$$

Pre-multiply by $(A - kI)^{-1}$, we get

$$X = (A - kI)^{-1}(\lambda - k) X$$

Divide by $\lambda - k$, we get $\dfrac{1}{\lambda - k} X = (A - kI)^{-1} X$

$\therefore$ $$(A - kI)^{-1} X = \left(\frac{1}{\lambda - k}\right) X$$

$\therefore$ $\dfrac{1}{\lambda - k}$ is an eigen value of $(A - kI)^{-1}$ and the eigen vector is X.

(*vi*) $\dfrac{|A|}{\lambda}$ is an eigen value of adj A. **(P.T.U., Dec. 2003)**

$$AX = \lambda X \text{ ; Pre-multiply both sides by adj A}$$

$$(\text{adj A})(AX) = (\text{adj A})\lambda X$$

$$\Rightarrow \quad [(\text{adj A})\,A]\,X = \lambda\,[(\text{adj A})\,X] \qquad |\because A\,(\text{adj A}) = (\text{adj A})\,A = |A|\,I$$

$$\Rightarrow \quad |A|\,X = \lambda\,[(\text{adj A})\,X]$$

$$\Rightarrow \quad (\text{adj A})\,X = \left[\frac{|A|}{\lambda}\right] X$$

$\Rightarrow \dfrac{|A|}{\lambda}$ is an eigen value of adj A and eigen vector is X.

(*vii*) A and A^T have the same eigen values

$\because$ eigen values of A are given by $|A - \lambda I| = 0$

We know that $|A| = |A^T|$

$\therefore \quad |A - \lambda I| = |(A - \lambda I)^T| = |A^T - (\lambda I)^T| = |A^T - \lambda I|$

$\therefore$ eigen values of A and A^T are same.

(*viii*) For a real matrix A, if $\alpha + i\beta$ is an eigen value, then its conjugate $\alpha - i\beta$ is also an eigen value of A.

Since eigen values of A are given by its characteristic equation $|A - \lambda I| = 0$ and if A is real, then characteristic equation is also a real polynomial equation and in a real polynomial equation, imaginary roots always occur in conjugate pairs. If $\alpha + i\beta$ is an eigen value then $\alpha - i\beta$ is also an eigen value.

ILLUSTRATIVE EXAMPLES

Example 1. (*i*) *Find the eigen values and eigen vectors of the matrix* $A = \begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}$.

2a

(*ii*) *Find the eigen values of the matrix* $\begin{bmatrix} 1 & 2 & 2 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{bmatrix}$. **(P.T.U., Dec. 2006)**

Sol. (*i*) The characteristic equation of the given matrix is

$$|A - \lambda I| = 0 \quad \text{or} \quad \begin{vmatrix} 1-\lambda & -2 \\ -5 & 4-\lambda \end{vmatrix} = 0$$

$$\text{or} \quad (1-\lambda)(4-\lambda) - 10 = 0 \quad \text{or} \quad \lambda^2 - 5\lambda - 6 = 0$$

$$\text{or} \quad (\lambda - 6)(\lambda + 1) = 0 \quad \therefore \quad \lambda = 6, -1.$$

Thus, the eigen values of A are 6, – 1

Corresponding to $\lambda = 6$, the eigen vectors are given by $(A - 6I)\,X = O$

$$\text{or} \quad \begin{bmatrix} 1-6 & -2 \\ -5 & 4-6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = O \quad \text{or} \quad \begin{bmatrix} -5 & -2 \\ -5 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = O$$

we get only one independent equation $-5x_1 - 2x_2 = 0$

$\therefore \quad \dfrac{x_1}{2} = \dfrac{x_2}{-5}$ gives the eigen vector (2, – 5)

Corresponding to $\lambda = -1$, the eigen vectors are given by $\begin{bmatrix} 2 & -2 \\ -5 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = O$

We get only one independent equation $2x_1 - 2x_2 = 0$.

$\therefore \quad x_1 = x_2$ gives the eigen vector (1, 1).

(*ii*) The characteristic equation of the given matrix is $|A - \lambda I| = 0$

i.e., $\begin{vmatrix} 1-\lambda & 2 & 2 \\ 0 & -4-\lambda & 2 \\ 0 & 0 & 7-\lambda \end{vmatrix} = 0$, expand w.r.t. 1st column, we get

$(1-\lambda)(-4-\lambda)(7-\lambda) = 0$, *i.e.,* $\lambda = 1, \lambda = -4, \lambda = 7$.

Hence the eigen values are – 4, 1, 7.

2012 **Example 2.** *Find the eigen values and eigen vectors of the following matrices*

$(i)\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ **(P.T.U., May 2012)** $(ii)\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$. **(P.T.U., Dec. 2012)**

Sol. (*i*) The characteristic equation of the given matrix is $|A - \lambda I| = 0$

or $$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$$

or $$(-2-\lambda)[-\lambda(1-\lambda)-12]-2[-2\lambda-6]-3[-4+1(1+\lambda)=0$$

or $$\lambda^3+\lambda^2-21\lambda-45=0$$

By trial, $\lambda = -3$ satisfies it.

$\therefore$ $$(\lambda+3)(\lambda^2-2\lambda-15)=0 \Rightarrow (\lambda+3)(\lambda+3)(\lambda-5)=0 \Rightarrow \lambda=-3,-3,5$$

Thus, the eigen values of A are $-3, -3, 5$.

Corresponding to $\lambda = -3$, eigen vectors are given by

$$(A+3I)X = O \qquad \text{or} \qquad \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = O$$

We get only one independent equation $x_1 + 2x_2 - 3x_3 = 0$

Choosing $x_2 = 0$, we have $x_1 - 3x_3 = 0$

$\therefore$ $\frac{x_1}{3} = \frac{x_2}{0} = \frac{x_3}{1}$ giving the eigen vector (3, 0, 1)

Choosing $x_3 = 0$, we have $x_1 + 2x_2 = 0$

$\therefore$ $\frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{0}$ giving the eigen vector (2, –1, 0)

Any other eigen vector corresponding to $\lambda = -3$ will be a linear combination of these two.

Corresponding to $\lambda = 5$, the eigen vectors are given by $\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = O$

$$\Rightarrow \begin{aligned} -7x_1 + 2x_2 - 3x_3 &= 0 \\ 2x_1 - 4x_2 - 6x_3 &= 0 \\ -x_1 - 2x_2 - 5x_3 &= 0 \end{aligned}$$

From first two equations, we have $\frac{x_1}{-12-12} = \frac{x_2}{-6-42} = \frac{x_3}{28-4}$

or $\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{-1}$ giving the eigen vector (1, 2, – 1).

(*ii*) The characteristic equation of the given matrix is $|A - \lambda I| = 0$

or $$\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$

or $$(1-\lambda)\{(5-\lambda)(1-\lambda)-1\}-1\{1-\lambda-3\}+3\{1-15+3\lambda\}=0$$

or $$(1-\lambda)\{4-6\lambda+\lambda^2\}+\lambda+2-42+9\lambda=0$$

or $$4-10\lambda+7\lambda^2-\lambda^3+10\lambda-40=0$$

or $$\lambda^3-7\lambda^2+36=0$$

or $$(\lambda+2)(\lambda^2-9\lambda+18)=0$$

$\therefore$ $$(\lambda+2)(\lambda-6)(\lambda-3)=0$$

$$\lambda=-2,3,6$$

Thus the eigen values of A are $-2, 3, 6$

Corresponding to $\lambda = -2$, eigen vectors are given by $(A + 2I)X = 0$

or $$\begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

we get two independent equations

$$3x_1 + x_2 + 3x_3 = 0$$
$$x_1 + 7x_2 + x_3 = 0$$
$$\frac{x_1}{-20} = \frac{x_2}{0} = \frac{x_3}{20}$$
$$\frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{1}$$

$\therefore$ Eigen vector corresponding to $\lambda = -2$ is $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Eigen vector corresponding to $\lambda = 3$ is given by $\begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$$-2x_1 + x_2 + 3x_3 = 0$$
$$x_1 + 2x_2 + x_3 = 0$$
$$3x_1 + x_2 - 2x_3 = 0$$

From first two equations $$\frac{x_1}{-5} = \frac{x_2}{5} = \frac{x_3}{-5}$$

or $$\frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{1}$$

It satisfies third equation

$\therefore$ Eigen vector corresponding to $\lambda = 3$ is $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

Eigen vector corresponding to $\lambda = 6$ is given by $\begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$$-5x_1 + x_2 + 3x_3 = 0$$
$$x_1 - x_2 + x_3 = 0$$
$$3x_1 + x_2 - 5x_3 = 0$$

From first two equations

$$\frac{x_1}{4} = \frac{x_2}{8} = \frac{x_3}{4}$$

or $$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{1}$$

The values of x_1, x_2, x_3 satisfy third equation

$\therefore$ Eigen vector corresponding to $\lambda = 6$ is $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Hence the eigen vectors are $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Example 3. *If λ is an eigen value of the matrix A, then prove that $g(\lambda)$ is an eigen value of $g(A)$, where g is polynomial.* **(P.T.U., May 2010)**

Sol. Given λ is an eigen value of matrix A

$\therefore$ There exists a non zero vector X such that

$$AX = \lambda X \quad \ldots(1)$$

Now,

$$A(AX) = A(\lambda X) \Rightarrow A^2X = \lambda(AX) = \lambda(\lambda X) = \lambda^2 X \quad \ldots(2)$$

$\therefore$ λ^2 is an eigen value of matrix A^2

Again

$$A(A^2X) = A(\lambda^2 X)$$

$$A^3X = \lambda^2(AX) = \lambda^2(\lambda X) = \lambda^3 X \quad \ldots(3)$$

$\Rightarrow$

$\therefore$ λ^3 is an eigen value of matrix A^3

Continue this process we can prove that

$$A^nX = \lambda^n X \text{ i.e., } \lambda^n \text{ is an eigen value of } A^n \quad \ldots(4)$$

As g is a polynomial

Let

$$g(\lambda) = a_0 + a_1\lambda + a_2\lambda^2 + \ldots a_n\lambda^n$$

$$g(A) = a_0I + a_1A + a_2A^2 + \ldots a_nA^n$$

$\Rightarrow$

$$g(A)X = [a_0I + a_1A + a_2A^2 + \ldots a_nA^n]X$$

$$= a_0(IX) + a_1(AX) + a_2(A^2X) + \ldots a_n(A^nX)$$

$$= a_0X + a_1(\lambda X) + a_2(\lambda^2 X) + \ldots a_n(\lambda^n X) \quad \text{[By using (1), (2), (3), (4)]}$$

$\therefore$

$$g(A)X = (a_0 + a_1\lambda + a_2\lambda^2 + \ldots a_n\lambda^n)X$$

i.e.,

$$g(A)X = g(\lambda)X$$

$\Rightarrow$ $g(\lambda)$ is an eigen value of $g(A)$

Example 4. *Show that eigen values of a Skew Hermitian matrix are either zero or purely imaginary.* **(P.T.U., Dec. 2012)**

Sol. Let A be a Skew Hermitian matrix

$\therefore$

$$A^\theta = -\hat{A} \quad \ldots(1)$$

Let λ be an eigen value of A, then there exists a non-zero vector X such that

$$AX = \lambda X$$

$$(AX)^\theta = (\lambda X)^\theta \text{ or } X^\theta A^\theta = \bar{\lambda}\, X^\theta$$

or

$$-X^\theta A = \bar{\lambda}\, X^\theta \quad [\because \text{ By using (1)}]$$

Post multiply both sides by X

$$-(X^\theta A)X = \left(\bar{\lambda} X^\theta\right) X$$

or

$$-X^\theta (AX) = \bar{\lambda}\, (X^\theta X)$$

$$-X^\theta (\lambda X) = \bar{\lambda}\, (X^\theta X)$$

or

$$-\lambda (X^\theta X) = \bar{\lambda}\, (X^\theta X) \Rightarrow \bar{\lambda} = -\lambda$$

$\Rightarrow \qquad \lambda + \bar{\lambda} = 0$

Now if $\qquad \lambda = a + ib$

then $\qquad \bar{\lambda} = a - ib$

$\lambda + \bar{\lambda} = 0 \quad \Rightarrow \quad a + ib + a - ib = 0$ or $a = 0$

i.e., $\qquad \lambda = ib$, *i.e.*, λ is purely imaginary.

Hence either eigen values are zero or purely imaginary.

4.26. CAYLEY HAMILTON THEOREM

(P.T.U., May 2004, May 2006, May 2007, Jan. 2009, May 2011)

Every square matrix satisfies its characteristic equation.

i.e., if the characteristic equation of the nth order square matrix A is

$$|A - \lambda I| = (-1)^n \lambda^n + k_1 \lambda^{n-1} + k_2 \lambda^{n-2} + \ldots\ldots + k_n = 0 \qquad \ldots(1)$$

then

$$(-1)^n A^n + k_1 A^{n-1} + k_2 A^{n-2} + \ldots\ldots + k_n I = O \qquad \ldots(2)$$

Let $\qquad P = \text{adj}\,(A - \lambda I)$

Since the elements of $A - \lambda I$ are at most of first degree in λ, the elements of $P = \text{adj}\,(A - \lambda I)$ are polynomials in λ of degree $(n - 1)$ or less. We can, therefore, split up P into a number of matrices each containing the same power of λ and write

$$P = P_1 \lambda^{n-1} + P_2 \lambda^{n-2} + \ldots\ldots + P_{n-2} \lambda^2 + P_{n-1} \lambda + P_n$$

Also, we know that if M is a square matrix, then $M\,(\text{adj}\, M) = |M| \times I$

$\therefore \qquad (A - \lambda I)\,P = |A - \lambda I| \times I$

By (1) and (2), we have

$$(A - \lambda I)\,(P_1 \lambda^{n-1} + P_2 \lambda^{n-2} + \ldots\ldots + P_{n-2} \lambda^2 + P_{n-1} \lambda + P_n) = [(-1)^n \lambda^n + k_1 \lambda^{n-1} + k_i \lambda^{n-2} + \ldots\ldots + k_{n-2} \lambda^2 + k_{n-1} \lambda + k_n]\, I$$

Equating coefficients of like powers of λ on both sides, we have

$$-P_1 = (-1)^n I \qquad [\because IP_1 = P_1]$$

$$AP_1 - P_2 = k_1 I$$

$$AP_2 - P_3 = k_2 I$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$AP_{n-2} - P_{n-1} = k_{n-2} I$$

$$AP_{n-1} - P_n = k_{n-1} I$$

$$AP_n = k_n I$$

Pre-multiplying these equations by $A^n, A^{n-1}, A^{n-2}, \ldots\ldots\ldots, A^2, A, I$ respectively and adding, we get

$O = (-1)^n A^n + k_1 A^{n-1} + k_2 A^{n-2} + \ldots\ldots\ldots + k_{n-2} A^2 + k_{n-1} A + k^n I$ terms on the L.H.S. Cancel in pairs

or

$$(-1)^n A^n + k_1 A^{n-1} + k_2 A^{n-2} + \ldots\ldots\ldots + k^{n-1} A + k_n I = O \qquad \ldots(3)$$

which proves the theorem.

Note 1. Multiplying (3) by A^{-1}, we have $(-1)^n A^{n-1} + k_1 A^{n-2} + \ldots\ldots + k_{n-1} I + k_n A^{-1} = O$

$$\Rightarrow \qquad A^{-1} = -\frac{1}{k_n}[(-1)^n A^{n-1} + k_1 A^{n-2} + \ldots\ldots\ldots + k_{n-1} I]$$

Thus Cayley Hamilton theorem gives another method for computing the inverse of a matrix. Since this method express the inverse of a matrix of order n in terms of $(n - 1)$ powers of A, it is most suitable for computing inverses of large matrices.

Note 2. If m be a positive integer such that $m > n$, then multiplying (3) by A^{m-n}, we get

$$(-1)^n A^m + k_1 A^{m-1} + k_2 A^{m-2} + \ldots\ldots\ldots + k_{n-1} A^{m-n+1} + k_n A^{m-n} = O$$

showing that any positive integral power A^m $(m > n)$ of A is linearly expressible in terms of those of lower degree.

Example 5. *Verify Cayley Hamilton Theorem for the following matrices :*

(*i*) $\begin{bmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ **(P.T.U., May 2006)** (*ii*) $\begin{bmatrix} 3 & 2 & 4 \\ 4 & 3 & 2 \\ 2 & 4 & 3 \end{bmatrix}$. **(P.T.U., Dec. 2006)**

Sol. (*i*) Let $A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

Characteristic equation of A is $|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & 3 & 1 \\ 3 & 1-\lambda & 2 \\ 1 & 2 & 3-\lambda \end{vmatrix} = 0$$

or $$(2-\lambda)\{(1-\lambda)(3-\lambda)-4\} - 3\{3(3-\lambda)-2\} + 1\{6-1+\lambda\} = 0$$

or $$\lambda^3 - 6\lambda^2 - 3\lambda + 18 = 0$$

or To verify Cayley Hamilton Theorem, we have to show that $A^3 - 6A^2 - 3A + 18I = 0$

Now,
$$A^2 = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 14 & 11 & 11 \\ 11 & 14 & 11 \\ 11 & 11 & 14 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 14 & 11 & 11 \\ 11 & 14 & 11 \\ 11 & 11 & 14 \end{bmatrix}\begin{bmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 72 & 75 & 69 \\ 75 & 69 & 72 \\ 69 & 72 & 75 \end{bmatrix}$$

$$A^3 - 6A^2 - 3A + 18I = \begin{bmatrix} 72 & 75 & 69 \\ 75 & 69 & 72 \\ 69 & 72 & 75 \end{bmatrix} - 6\begin{bmatrix} 14 & 11 & 11 \\ 11 & 14 & 11 \\ 11 & 11 & 14 \end{bmatrix} - 3\begin{bmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 18 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

Hence theorem is proved.

(*ii*) Let $A = \begin{bmatrix} 3 & 2 & 4 \\ 4 & 3 & 2 \\ 2 & 4 & 3 \end{bmatrix}$

Characteristic equation of A is $|A - \lambda I| = 0$

or $$\begin{vmatrix} 3-\lambda & 2 & 4 \\ 4 & 3-\lambda & 2 \\ 2 & 4 & 3-\lambda \end{vmatrix} = 0 \text{ or } (3-\lambda)\{(3-\lambda)^2 - 8\} - 2\{12 - 4\lambda - 4\} + 4\{16 - 6 + 2\lambda\} = 0$$

or $$\lambda^3 - 9\lambda^2 + 3\lambda - 27 = 0$$

To verify Cayley Hamilton Theorem, we have to prove

$A^3 - 9A^2 + 3A - 27I = 0$

$$A^2 = \begin{bmatrix} 3 & 2 & 4 \\ 4 & 3 & 2 \\ 2 & 4 & 3 \end{bmatrix}\begin{bmatrix} 3 & 2 & 4 \\ 4 & 3 & 2 \\ 2 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 25 & 28 & 28 \\ 28 & 25 & 28 \\ 28 & 28 & 25 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 25 & 28 & 28 \\ 28 & 25 & 28 \\ 28 & 28 & 25 \end{bmatrix}\begin{bmatrix} 3 & 2 & 4 \\ 4 & 3 & 2 \\ 2 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 243 & 246 & 240 \\ 240 & 243 & 246 \\ 246 & 240 & 243 \end{bmatrix}$$

$$A^3-9A^2+3A-27I=\begin{bmatrix}243&246&240\\240&243&246\\246&240&243\end{bmatrix}-9\begin{bmatrix}25&28&28\\28&25&28\\28&28&25\end{bmatrix}+3\begin{bmatrix}3&2&4\\4&3&2\\2&4&3\end{bmatrix}-27\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}=O$$

Hence theorem is verified.

Example 6. *If* $A=\begin{bmatrix}2&3\\3&5\end{bmatrix}$, *then use Cayley Hamilton Theorem to find the matrix represented by* A^5.

Sol. Characteristic equation of A is

$$|A-\lambda I|=0$$

i.e., $$\begin{vmatrix}2-\lambda&3\\3&5-\lambda\end{vmatrix}=0 \text{ or } \lambda^2-7\lambda+1=0$$

By Cayley Hamilton Theorem $A^2-7A+I=0$

$$\therefore\quad A^2=7A-I \quad ...(1)$$

$$A^4=49A^2-14A+I$$

$$=49(7A-I)-14A+I \quad \text{[Using (1)]}$$

$$=329A-48I$$

$$A^5=A^4\cdot A=(329A-48I)A$$

$$=329A^2-48A=329(7A-I)-48A=2255A-329I$$

$$=2255\begin{bmatrix}2&3\\3&5\end{bmatrix}-329\begin{bmatrix}1&0\\0&1\end{bmatrix}=\begin{bmatrix}4181&6765\\6765&10946\end{bmatrix}.$$

Example 7. *Verify Cayley Hamilton Theorem for the matrix* $A=\begin{bmatrix}1&2&3\\2&4&5\\3&5&6\end{bmatrix}$ *and hence find* $B=A^8-11A^7-4A^6+A^5+A^4-11A^3-3A^2+2A+I$; *Also find* A^{-1} *and* A^4. **(P.T.U., May 2011)**

Sol. The characteristic equation of A is

$$|A-\lambda I|=0;\quad \begin{vmatrix}1-\lambda&2&3\\2&4-\lambda&5\\3&5&6-\lambda\end{vmatrix}=0$$

or $$(1-\lambda)\{(4-\lambda)(6-\lambda)-25\}-2\{2(6-\lambda)-15\}+3\{10-3(4-\lambda)\}=0$$

or $$\lambda^3-11\lambda^2-4\lambda+1=0 \quad ...(1)$$

Cayley Hamilton Theorem is verified if A satisfies the characteristic equation *i.e.*, (1)

$$\therefore\quad A^3-11A^2-4A+I=0 \quad ...(2)$$

Now, $$A^2=\begin{bmatrix}1&2&3\\2&4&5\\3&5&6\end{bmatrix}\begin{bmatrix}1&2&3\\2&4&5\\3&5&6\end{bmatrix}$$

i.e., $$A^2=\begin{bmatrix}14&25&31\\25&45&56\\31&56&70\end{bmatrix}$$

$$A^3 = \begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 157 & 283 & 353 \\ 283 & 510 & 636 \\ 353 & 636 & 793 \end{bmatrix}$$

Verification: $$\begin{bmatrix} 157 & 283 & 353 \\ 283 & 510 & 636 \\ 353 & 636 & 793 \end{bmatrix} - 11 \begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix} - 4 \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

$\therefore$ Cayley Hamilton Theorem is satisfied.

Now, $$B = A^5 (A^3 - 11A^2 - 4A + I) + A (A^3 - 11A^2 - 4A + I) + A^2 + A + I$$

$$= A^5.0 + A.0 + A^2 + A + I \qquad \text{(Using 2)}$$

$$= A^2 + A + I$$

$$= \begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 16 & 27 & 34 \\ 27 & 50 & 61 \\ 34 & 61 & 77 \end{bmatrix}$$

From (2), $$A^{-1} = -A^2 + 11A + 4I$$

$\therefore$ $$A^{-1} = -\begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix} + 11 \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} + 4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$$

From (2), $$A^4 = 11A^3 + 4A^2 - A$$

$$= 11 \begin{bmatrix} 157 & 283 & 353 \\ 283 & 510 & 636 \\ 353 & 636 & 793 \end{bmatrix} + 4 \begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

$\therefore$ $$A^4 = \begin{bmatrix} 1782 & 3211 & 4004 \\ 3211 & 5786 & 7215 \\ 4004 & 7215 & 8997 \end{bmatrix}.$$

Example 8. *Using Cayley Hamilton Theorem find the inverse of* $\begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix}$. **(P.T.U., Dec. 2012)**

Sol. Let $A = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix}$

Characteristics equation of A is $|A - \lambda I| = 0$

or $$\begin{vmatrix} 4-\lambda & 3 & 1 \\ 2 & 1-\lambda & -2 \\ 1 & 2 & 1-\lambda \end{vmatrix} = 0$$

or $$(4-\lambda)\{(1-\lambda)^2 + 4\} - 3\{2(1-\lambda) + 2\} + 1\{4 - 1 + \lambda\} = 0$$

or $\quad (4-\lambda)(5-2\lambda+\lambda^2)-3(4-2\lambda)+(3+\lambda)=0$

or $\quad 20-13\lambda+6\lambda^2-\lambda^3-12+6\lambda+3+\lambda=0$

or $\quad \lambda^3-6\lambda^2+6\lambda-11=0$

By Cayley Hamilton Theorem

$$A^3-6A^2+6A-11I=0$$

or $$11I=A^3-6A^2+6A$$

Operate both sides by A^{-1}

$$11A^{-1}=A^2-6A+6I$$

$$A^2=\begin{bmatrix}4&3&1\\2&1&-2\\1&2&1\end{bmatrix}\begin{bmatrix}4&3&1\\2&1&-2\\1&2&1\end{bmatrix}$$

$$=\begin{bmatrix}23&17&-1\\8&3&-2\\9&7&-2\end{bmatrix}$$

$\therefore$ $$11A^{-1}=\begin{bmatrix}23&17&-1\\8&3&-2\\9&7&-2\end{bmatrix}-6\begin{bmatrix}4&3&1\\2&1&-2\\1&2&1\end{bmatrix}+\begin{bmatrix}6&0&0\\0&6&0\\0&0&6\end{bmatrix}$$

$$=\begin{bmatrix}5&-1&-7\\-4&3&10\\3&-5&-2\end{bmatrix}$$

$\therefore$ $$A^{-1}=\frac{1}{11}\begin{bmatrix}5&-1&-7\\-4&3&10\\3&-5&-2\end{bmatrix}$$

TEST YOUR KNOWLEDGE

1. Find the eigen values and eigen vectors of the matrix $\begin{bmatrix}5&4\\1&2\end{bmatrix}$.
2. Find the eigen values and eigen vectors of the matrices

   (i) $\begin{bmatrix}8&-6&2\\-6&7&-4\\2&-4&3\end{bmatrix}$ **(P.T.U., June 2003, Jan. 2010)** (ii) $\begin{bmatrix}6&-2&2\\-2&3&-1\\2&-1&3\end{bmatrix}$

   (iii) $\begin{bmatrix}2&0&1\\0&2&0\\1&0&2\end{bmatrix}$ (iv) $\begin{bmatrix}2&1&1\\1&2&1\\0&0&1\end{bmatrix}$ (v) $\begin{bmatrix}3&1&4\\0&2&0\\0&0&5\end{bmatrix}$

   (vi) $\begin{bmatrix}1&0&-1\\1&2&1\\2&2&3\end{bmatrix}$ **(P.T.U., May 2006)** (vii) $\begin{bmatrix}8&-8&-2\\4&-3&-2\\3&-4&1\end{bmatrix}$ **(P.T.U., May 2012)**
3. Prove that the characteristic roots of a diagonal matrix are the diagonal elements of the matrix.
4. Show that 0 is a characteristic root of a matrix if and only if the matrix is singular.
5. Show that if λ is a characteristic root of the matrix A, then $\lambda+k$ is a characteristic root of the matrix $A+kI$.
6. If $\lambda_1, \lambda_2, ..., \lambda_n$ are the given values of a matrix A, then A^m has the eigen values $\lambda_1^m, \lambda_2^m, ..., \lambda_n^m$ (*m* being a positive integer).
7. Show that eigen values of a Hamilton matrix are real.

8. Find the characteristic equation of the matrix $A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$. Show that the equation is satisfied by A and hence obtain the inverse of the given matrix.

9. Find the characteristic equation of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$. Show that the equation is satisfied by A.

10. If $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ use Cayley Hamilton Theorem to find A^8. **[Hint: $A^2 = 5I$] (P.T.U., Dec. 2003, May 2010)**

11. Using Cayley Hamilton Theorem, find the inverse of

(*i*) $\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 7 & -1 & 3 \\ 6 & 1 & 4 \\ 2 & 4 & 8 \end{bmatrix}$ (*iii*) $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

(P.T.U., Dec. 2005, Jan. 2009)

(*iv*) $\begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$ **(P.T.U., May 2010)** (*v*) $\begin{bmatrix} 2 & 5 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$ **(P.T.U., Dec. 2005)**

12. Find the characteristic equation of matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ and hence find the matrix represented by $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$.

ANSWERS

1. 1, 6 ; (1, – 1), (1, 4)

2. (*i*) 0, 3, 15 ; (1, 2, 2), (2, 1 , –2) (2, –2, 1) (*ii*) 2, 2, 8 ; (1, 0, –2), (1, 2, 0), (2, –1, 1)
(*iii*) 1, 2, 3 ; (1, 0, 1), (1, 0, –1), (0, 1, 0) (*iv*) 1, 1, 3 ; (1, –2, 1) (1, 1, 0)
(*v*) 2, 3, 5 ; (1, –1, 0), (1, 0, 0), (2, 0, 1) (*vi*) 1, 2, 3 ; (1, – 1, 0), (– 2, 1, 2), (1, –1, – 2)
(*vii*) 1, 2, 3 ; (4, 3, 2), (3, 2, 1), (2, 1, 1)

8. $\lambda^3 - 4\lambda^2 - 20\lambda - 35 = 0$, $\dfrac{1}{35}\begin{bmatrix} -4 & 11 & -5 \\ -1 & -6 & 25 \\ -6 & 1 & -10 \end{bmatrix}$

9. $\lambda^3 - \lambda^2 - 18\lambda - 40 = 0$

10. $625I$

11. (*i*) $\begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}$ (*ii*) $\begin{bmatrix} \frac{2}{65} & \frac{2}{13} & -\frac{9}{130} \\ \frac{21}{65} & \frac{5}{13} & -\frac{3}{130} \\ \frac{2}{13} & -\frac{3}{13} & \frac{2}{13} \end{bmatrix}$ (*iii*) $\dfrac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$ (*iv*) $\dfrac{1}{3}\begin{bmatrix} -3 & -2 & 4 \\ 3 & 1 & -2 \\ -3 & 0 & 3 \end{bmatrix}$

(*v*) $\dfrac{1}{4}\begin{bmatrix} -3 & 1 & 7 \\ -1 & -1 & 5 \\ 5 & 1 & -13 \end{bmatrix}$

12. $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$; $\begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$.

4.27. DIAGONALIZABLE MATRICES

A matrix A is said to be diagonalizable if there exists an invertible matrix B. Such that $B^{-1} AB = D$, where D is a diagonal matrix and the diagonal elements of D are the eigen values of A.

Theorem. *A square matrix A of order n is diagonalizable if and only if it has n linearly independent eigen vectors.*

Proof. Let $X_1, X_2, \ldots\ldots, X_n$ be n linearly independent eigen vectors corresponding to the eigen values $\lambda_1, \lambda_2, \ldots\ldots, \lambda_n$ (not necessarily distinct) of matrix A

$$\therefore \quad AX_1 = \lambda_1 X_1, AX_2 = \lambda_2 X_2, \ldots\ldots, AX_n = \lambda_n X_n$$

Let $B = [X_1, X_2, \ldots\ldots, X_n]$ and $D = \text{Diag.} [\lambda_1, \lambda_2, \ldots\ldots, \lambda_n]$ formed by eigen values of A.

then
$$\begin{aligned} AB &= A [X_1, X_2, \ldots\ldots, X_n] = [AX_1, AX_2, \ldots\ldots, AX_n] \\ &= [\lambda_1 X_1, \lambda_2 X_2, \ldots\ldots, \lambda_n X_n] \\ &= [X_1 X_2, \ldots\ldots, X_n] \text{ Diag } [\lambda_1 \lambda_2, \ldots\ldots, \lambda_n] \\ AB &= BD \end{aligned} \quad \ldots(1)$$

Since columns of B and L.I. $\therefore p(B) = n \quad \therefore$ B is invertible

Pre-multiply both sides by B^{-1}

$$\therefore \quad B^{-1} AB = (B^{-1} B) D = D$$

$\therefore$ The matrix B, formed by eigen vectors of A, reduces the matrix A to its diagonal form.

Post multiply (1) by B^{-1}

$$A (BB^{-1}) = BDB^{-1} \quad \text{or} \quad A = BDB^{-1}.$$

Note 1. The matrix B which diagonalises A is called the **Modal Matrix of A,** obtained by grouping the eigen values of A into a square matrix and matrix D is called **Spectral Matrix of A.**

Note 2. We have
$$A = BDB^{-1}$$
$$\therefore \quad A^2 = A \cdot A = (BDB^{-1}) (BDB^{-1}) = BD (B^{-1} B) DB^{-1}$$
$$= B (DID) B^{-1} = BD^2 B^{-1}$$

Repeating this process m times, we get

$$A^m = BD^m B^{-1} \ (m, a \text{ +ve integer}).$$

$\therefore$ If A is diagonalizable so is A^m.

Note 3. If D is a diagonal matrix of order n and

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & \ldots\ldots & 0 \\ 0 & \lambda_2 & 0 & \ldots\ldots & 0 \\ 0 & 0 & \lambda_3 & \ldots\ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots\ldots & \lambda_n \end{bmatrix}, \text{ then } D^m = \begin{bmatrix} \lambda_1^m & 0 & 0 & \ldots\ldots & 0 \\ 0 & \lambda_2^m & 0 & \ldots\ldots & 0 \\ 0 & 0 & \lambda_3^m & \ldots\ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots\ldots & \lambda_n^m \end{bmatrix}$$

$$\therefore \quad A^m = BD^m B^{-1}$$

Similarly if Q (D) is a polynomial in D, then

$$Q(D) = \begin{bmatrix} Q(\lambda_1) & 0 & 0 & \ldots\ldots & 0 \\ 0 & Q(\lambda_2) & 0 & \ldots\ldots & 0 \\ 0 & 0 & Q(\lambda_3) & \ldots\ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots\ldots & Q(\lambda_n) \end{bmatrix}$$

$$\therefore \quad Q(A) = B [Q(D)] B^{-1}$$

ILLUSTRATIVE EXAMPLES

Example 1. *Show that the matrix* $A = \begin{bmatrix} 3 & 1 & -1 \\ -2 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$ *is diagonalizable. Hence find P such that* $P^{-1}AP$ *is a diagonal matrix, then obtain the matrix* $B = A^2 + 5A + 3I$. **(P.T.U., May 2008, May 2012)**

Sol. Characteristic equation of A is

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 1 & -1 \\ -2 & 1-\lambda & 2 \\ 0 & 1 & 2-\lambda \end{vmatrix} = \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

or $\lambda = 1, 2, 3.$

Since the matrix has three distinct eigen values

∴ It has three linearly independent eigen values and hence A is diagonalizable.

The eigen vector corresponding to $\lambda = 1$ is given by

$$(A - \lambda I)\,X = \begin{bmatrix} 2 & 1 & -1 \\ -2 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

i.e., $2x_1 + y_1 - z_1 = 0;\ -2x_1 + 2z_1 = 0$ and $y_1 + z_1 = 0$

which gives the solution.

$$x_1 = 1,\quad y_1 = -1,\quad z_1 = 1$$

The eigen vector corresponding to $\lambda = 2$ is

$$(A - 2I)\,X = \begin{bmatrix} 1 & 1 & -1 \\ -2 & -1 & 2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

i.e.,

$$x_1 + y_1 - z_1 = 0$$
$$-2x_1 - y_1 + 2z_1 = 0$$
$$y_1 = 0$$

which gives the solution

$$x_1 = 1,\quad y_1 = 0,\quad z_1 = 1.$$

Eigen vector corresponding to $\lambda = 3$ is given by

$$(A - 3I)\,X = \begin{bmatrix} 0 & 1 & -1 \\ -2 & -2 & 2 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

i.e.,

$$y_1 - z_1 = 0$$
$$-2x_1 - 2y_1 + 2z_1 = 0$$
$$y_1 - z_1 = 0$$

which gives the solution

$$x_1 = 0,\quad y_1 = 1,\quad z_1 = 0.$$

∴ This Modal Matrix $P = [X_1, X_2, X_3]$

$$= \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Now,
$$P^{-1} = \frac{\text{Adj. P}}{|P|} = \frac{1}{1}\begin{bmatrix} -1 & -1 & 1 \\ 2 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad P^{-1}AP = \begin{bmatrix} -1 & -1 & 1 \\ 2 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 1 & -1 \\ -2 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \text{diag}\,[1, 2, 3]$$

Hence A is diagonalizable and its diagonal form matrix contains the eigen values only as its diagonal elements.

Now to obtain $B = A^2 + 5A + 3I$ we use $Q(A) = P[Q(D)]P^{-1}$ [Art. 4.27 Note 3]

$$D = \text{diag}\,(1, 2, 3)$$
$$D^2 = \text{diag}\,(1, 4, 9)$$
$$A^2 + 5A + 3I = P(D^2 + 5D + 3I)P^{-1}$$

$$D^2 + 5D + 3I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 9 \\ 0 & 0 & 9 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 15 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 0 & 0 \\ 0 & 17 & 0 \\ 0 & 0 & 27 \end{bmatrix}$$

$$\therefore \quad B = A^2 + 5A + 3I = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 17 & 0 \\ 0 & 0 & 27 \end{bmatrix}\begin{bmatrix} -1 & -1 & 1 \\ 2 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 25 & 8 & -8 \\ -18 & 9 & 18 \\ -2 & 8 & 19 \end{bmatrix}$$

Example 2. *Find a matrix P which tranforms the matrix* $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ *into a diagonal form.*

(P.T.U., Dec. 2003)

Sol. Let
$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

Characteristic equation of A is $|A - \lambda I| = 0$

i.e.,
$$\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0 \text{ or } \lambda^3 - 7\lambda^2 + 36 = 0$$

or $\qquad (\lambda + 2)(\lambda^2 - 9\lambda + 18) = 0$

or $\qquad (\lambda + 2)(\lambda - 3)(\lambda - 6) = 0$ *i.e.,* $\lambda = -2, 3, 6$ are the eigen values.

When $\lambda = -2$; eigen vectors are given by

$$3x_1 + y_1 + 3z_1 = 0$$
$$x_1 + 7y_1 + z_1 = 0$$
$$3x_1 + y_1 + 3z_1 = 0$$

Solving first and second equations (3^{rd} is same as first)

$$\frac{x_1}{-20} = \frac{y_1}{0} = \frac{z_1}{20} \quad \therefore \quad X_1 = k(-1, 0, 1)$$

When $\lambda = 3$; eigen vectors are given by

$$2x_1 + y_1 + 3z_1 = 0$$
$$x_1 + 2y_1 + z_1 = 0$$
$$3x_1 + y_1 - 2z_1 = 0$$

Solving first and second equations :

$$\frac{x_1}{-5} = \frac{y_1}{5} = \frac{z_1}{-5} \quad \therefore \quad X_2 = k(-1, 1, -1)$$

When $\lambda = 6$; eigen vectors are given by

$$-5x_1 + y_1 + 3z_1 = 0$$
$$x_1 - y_1 + z_1 = 0$$
$$3x_1 + y_1 - 5z_1 = 0$$

Solving first and second equations

$$\frac{x_1}{4} = \frac{y_1}{8} = \frac{z_1}{4} \quad \therefore \quad X_3 = k(1, 2, 1).$$

Modal Matrix
$$P = \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 1 \\ 0 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$$

$$P^{-1} = \frac{\text{Adj. P}}{|P|} = \frac{-1}{6}\begin{bmatrix} 3 & 2 & -1 \\ 0 & -2 & -2 \\ -3 & 2 & -1 \end{bmatrix} = -\frac{1}{6}\begin{bmatrix} 3 & 0 & -3 \\ 2 & -2 & 2 \\ -1 & -2 & -1 \end{bmatrix}$$

$\therefore$ Required diagonal form $D = P^{-1} AP$

$$= -\frac{1}{6}\begin{bmatrix} 3 & 0 & -3 \\ 2 & -2 & 2 \\ -1 & -2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & -1 & 1 \\ 0 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$$

$$= -\frac{1}{6}\begin{bmatrix} 12 & 0 & 0 \\ 0 & -18 & 0 \\ 0 & 0 & -36 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}, \text{ which is formed by the eigen values of A.}$$

Example 3. *Diagonalise the matrix* $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ *and obtain its Modal Matrix.*

Sol. Let $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

Characteristic equation of A is $|A - \lambda I| = 0$ *i.e.*, $\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$

or $(-2-\lambda)[-\lambda(1-\lambda)-12]-2[-2\lambda-6]-3[-4+1-\lambda] = 0$

or $-(2+\lambda)(\lambda^2-\lambda-12)+4(\lambda+3)+3(\lambda+3) = 0$

or $-(2+\lambda)(\lambda+3)(\lambda-4)+7(\lambda+3) = 0$

or $(\lambda+3)(-\lambda^2+2\lambda+8+7) = 0$

or $-(\lambda+3)(\lambda+3)(\lambda-5) = 0$

$\therefore$ $\lambda = -3, -3, 5.$

Characteristic vectors corresponding to $\lambda = -3$ are

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Operating $R_2 - 2R_1$, $R_3 + R_1$, we get

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\therefore$ $x_1 + 2x_2 - 3x_3 = 0$

$\therefore$ $x_1 = -2x_2 + 3x_3$

$x_2 = 1 \cdot x_2 + 0 \cdot x_3$

$x_3 = 0 \cdot x_2 + 1 \cdot x_3$

$\therefore$ $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ Eigen vectors are $X_1 = (-2, 1, 0)$ and $X_2 = (3, 0, 1)$.

Characteristic vector corresponding to $\lambda = 5$ is

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

or $\begin{bmatrix} -1 & -2 & -5 \\ -7 & 2 & -3 \\ 2 & -4 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$ by operating R_{32} and R_{21}

Operate $R_2 - 7R_1$, $R_3 + 2R_1$, we get

$$\begin{bmatrix} -1 & -2 & -5 \\ 0 & 16 & 32 \\ 0 & -8 & -16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Operating $R_3 + \frac{1}{2} R_2$, we get

$$\begin{bmatrix} -1 & -2 & -5 \\ 0 & 16 & 32 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\therefore$
$$-x_1 - 2x_2 - 5x_3 = 0$$
$$16x_2 + 32x_3 = 0$$

or
$$x_1 + 2x_2 + 5x_3 = 0$$
$$x_2 + 2x_3 = 0$$
$$x_1 = -x_3$$

or
$$x_2 = -2x_3$$

$\therefore$
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = -x_3\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$\therefore$ Eigen vector $\quad X_3 = (1, 2, -1)$

$\therefore$ Modal Matrix $P = \begin{bmatrix} -2 & 3 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$; $|P| = 8$

$$P^{-1} = \frac{\text{Adj. } P}{|P|} = \frac{1}{8}\begin{bmatrix} -2 & 4 & 6 \\ 1 & 2 & 5 \\ 1 & 2 & -3 \end{bmatrix}$$

Now Diagonal Matrix $D = P^{-1}AP$

$$= \frac{1}{8}\begin{bmatrix} -2 & 4 & 6 \\ 1 & 2 & 5 \\ 1 & 2 & -3 \end{bmatrix}\begin{bmatrix} -2 & 2 & 3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}\begin{bmatrix} -2 & 3 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} -24 & 0 & 0 \\ 0 & -24 & 0 \\ 0 & 0 & 40 \end{bmatrix} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$ *i.e.*, the diagonal matrix formed by eigen values of A.

Example 4. *Diagonalize* $A = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ *and hence find* A^8. *Find the Modal Matrix.* **(P.T.U., May 2011)**

Sol. The characteristic equation of A is $|A - \lambda I| = 0$

i.e.,
$$\begin{vmatrix} 1-\lambda & 6 & 1 \\ 1 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

Expand the determinant w.r.t. R_3

$$(3-\lambda)\{(1-\lambda)(2-\lambda-6\} = 0$$

or
$$(3-\lambda)\{(\lambda^2 - 3\lambda - 4\} = 0$$

or
$$(3-\lambda)(\lambda-4)(\lambda+1)\} = 0$$

$\therefore$ Eigen values are $\quad \lambda = -1, 3, 4$

For $\lambda = -1$; the eigen vector is given by

$$\begin{bmatrix} 2 & 6 & 1 \\ 1 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = O$$

or
$$2x_1 + 6x_2 + x_3 = 0$$
$$x_1 + 3x_2 = 0$$
$$4x_3 = 0$$

$\therefore$
$$x_3 = 0, x_1 = -3x_2$$

$\therefore$
$$X_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$$

For $\lambda = 3$; eigen vector is given by

$$\begin{bmatrix} -2 & 6 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = O$$

or
$$-2x_1 + 6x_2 + x_3 = 0$$
$$x_1 - x_2 = 0$$

$\therefore$
$$x_2 = x_1 \text{ and } x_3 = -4x_1$$

$\therefore$
$$X_2 = \begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix}$$

For $\lambda = 4$; eigen vector is

$$\begin{bmatrix} -3 & 6 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = O$$

$\therefore$
$$-3x_1 + 6x_2 + x_3 = 0$$
$$x_1 - 2x_2 = 0$$
$$-x_3 = 0$$
$$x_1 = 2x_2$$

$\therefore$
$$x_3 = 0, x_1 = 2x_2$$

$\therefore$
$$X_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

Thus the Modal Matrix P is

$$P = \begin{bmatrix} -3 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & -4 & 0 \end{bmatrix}$$

$$P^{-1} = \frac{\text{Adj } P}{|P|}$$

$$|P| = \begin{vmatrix} -3 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & -4 & 0 \end{vmatrix}$$

Expand w.r.t. R_3 we get
$$|P| = -20$$

$$\text{Adj } P = \begin{bmatrix} -4 & 0 & -4 \\ -8 & 0 & -12 \\ -1 & 5 & -4 \end{bmatrix}'$$

$$\therefore \quad P^{-1} = -\frac{1}{20}\begin{bmatrix} 4 & -8 & -1 \\ 0 & 0 & 5 \\ -4 & -12 & -4 \end{bmatrix}$$

$$= \frac{1}{20}\begin{bmatrix} -4 & 8 & 1 \\ 0 & 0 & -5 \\ 4 & 12 & 4 \end{bmatrix}$$

Now the diagonal matrix D is given by

$$D = P^{-1}AP = \frac{1}{20}\begin{bmatrix} -4 & 8 & 1 \\ 0 & 0 & -5 \\ 4 & 12 & 4 \end{bmatrix}\begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} -3 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & -4 & 0 \end{bmatrix}$$

$$= \frac{1}{20}\begin{bmatrix} 4 & -8 & -1 \\ 0 & 0 & -15 \\ 16 & 48 & 16 \end{bmatrix}\begin{bmatrix} -3 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & -4 & 0 \end{bmatrix}$$

$$= \frac{1}{20}\begin{bmatrix} -20 & 0 & 0 \\ 0 & 60 & 0 \\ 0 & 0 & 80 \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

The diagonal matrix formed by eigen values of A

To find A^8; $A = PDP^{-1}$

$$\therefore \quad A^8 = PD^8P^{-1}$$

$$= \frac{1}{20}\begin{bmatrix} -3 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & -4 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 6561 & 0 \\ 0 & 0 & 65536 \end{bmatrix}\begin{bmatrix} -4 & 8 & 1 \\ 0 & 0 & -5 \\ 4 & 12 & 4 \end{bmatrix}$$

$$= \frac{1}{20}\begin{bmatrix} -3 & 6561 & 131072 \\ 1 & 6561 & 65536 \\ 0 & -26244 & 0 \end{bmatrix}\begin{bmatrix} -4 & 8 & 1 \\ 0 & 0 & -5 \\ 4 & 12 & 4 \end{bmatrix}$$

$$= \frac{1}{20}\begin{bmatrix} 524300 & 1572840 & 491480 \\ 262140 & 786440 & 229340 \\ 0 & 0 & 131220 \end{bmatrix}$$

$$A^8 = \begin{bmatrix} 26215 & 78642 & 24574 \\ 13107 & 39322 & 11467 \\ 0 & 0 & 6561 \end{bmatrix}.$$

4.28. SIMILAR MATRICES **(P.T.U., May 2007)**

Let A and B be square matrices of the same order. The matrix A is said to be similar to B if there exists an invertible matrix P such that $A = P^{-1}BP$ or $PA = BP$

Post multiply both sides by P^{-1}, we have

$$PAP^{-1} = B(PP^{-1}) = BI = B \quad \therefore B = PAP^{-1}$$

$\therefore$ A is similar to B if and only if B is similar to A. The matrix P is called the **similarity matrix.**

4.29. Theorem, Similar Matrices have the same Characteristic Equation (and hence the same Eigen Values). Also if X is an Eigen Vector of A corresponding to Eigen Value λ then $P^{-1}X$ is an Eigen Vector of B Corresponding to the Eigen Value λ, where P is Similarity Matrix

Proof. $\because$ B is similar to A and P is similarity matrix. $\therefore$ $AP = PB$ or $P^{-1}AP = B$

Let λ be the eigen value and X be the corresponding eigen vector of A

$$\therefore \quad AX = \lambda X \qquad \ldots(1)$$

Now,
$$B - \lambda I = P^{-1}AP - \lambda I = P^{-1}AP - P^{-1}(\lambda IP) = P^{-1}(A - \lambda I)P$$

$$\therefore \quad |B - \lambda I| = |P^{-1}(A - \lambda I)P| = |P^{-1}||A - \lambda I||P|$$
$$= |A - \lambda I||P^{-1}P| = |A - \lambda I||I|$$
$$= |A - \lambda I|$$

$\therefore$ Similar matrices have same characteristic polynomials.

Pre-multiply (1) both sides by an invertible matrix P^{-1}.

$$\therefore \quad P^{-1}(AX) = P^{-1}(\lambda X) = \lambda P^{-1}X$$

Let
$$X = PY \quad \therefore \quad P^{-1}(APY) = \lambda P^{-1}(PY)$$

or
$$(P^{-1}AP)Y = \lambda(P^{-1}P)Y$$

or
$$BY = \lambda Y \quad \text{where } B = P^{-1}AP \qquad \ldots(2)$$

$\therefore$ B has the same eigen value λ as that of A which shows that eigen values of similar matrices are same.

$\therefore$ Similar matrices have the same characteristic equation and hence the same eigen values.

Now from (2) Y is an eigen vector of B corresponding to λ, the eigen value of B.

$\therefore$ Eigen vector of $B = Y = P^{-1}X$

Hence, the result.

Note 1. Converse of the above theorem is not always true *i.e.*, two matrices which have the same characteristic equation need not always be similar.

Note 2. If A is similar to B, B is similar to C, then A is similar to C

Let there be two invertible matrices P and Q.

Such that
$$A = P^{-1}BP \text{ and } B = Q^{-1}CQ$$

Thus
$$A = P^{-1}(Q^{-1}CQ)P = (P^{-1}Q^{-1})C(QP) = (QP)^{-1}C(QP)$$

$$\therefore \quad A = R^{-1}CR, \text{ where } R = QP$$

Hence A is similar to C.

4.30. The Necessary and Sufficient Condition for an *n* Rowed Square Matrix A to be Similar to a Diagonal Matrix is that the Set of Characteristic Vectors of A Includes a Set of *n* Linearly Independent Vectors

Proof. Necessary Condition : A is similar to a diagonal matrix D(say) $\therefore$ there exists a non-singular matrix P such that $P^{-1}AP = D = \text{diag}(\lambda_1, \lambda_2, \ldots\ldots\ldots, \lambda_n)$

$$\therefore \quad AP = PD$$

Let
$$P = [C_1, C_2, \ldots\ldots\ldots, C_n]$$

$$\therefore \quad A[C_1, C_2, \ldots\ldots\ldots, C_n] = [C_1, C_2, \ldots\ldots\ldots, C_n] \text{ Diag } [\lambda_1, \lambda_2, \ldots\ldots\ldots, \lambda_n]$$

$$\therefore \quad AC_1 = \lambda_1 C_1 ; AC_2 = \lambda_2 C_2 ; AC_3 = \lambda_3 C_3, \ldots\ldots\ldots, AC_n = \lambda_n C_n$$

which shows that $C_1, C_2, \ldots\ldots\ldots, C_n$ are n characteristic vectors corresponding to eigen values $\lambda_1, \lambda_2, \ldots\ldots\ldots, \lambda_n$ of A. As $C_1, C_2, \ldots\ldots\ldots, C_n$ are columns of a non-singular matrix $\therefore$ they form a L.I. set of vectors.

Sufficient Conditions : Let $C_1, C_2, \ldots\ldots\ldots, C_n$ be n L.I. set of n characteristic vectors and let $\lambda_1, \lambda_2, \ldots\ldots\ldots, \lambda_n$ be the corresponding characteristic roots.

We have $AC_1 = \lambda_1 C_1, AC_2 = \lambda_2 C_2, \ldots\ldots\ldots, AC_n = \lambda_n C_n \qquad \ldots(1)$

If we take $P = [C_1, C_2, \ldots\ldots\ldots, C_n]$

Then system (1) is equivalent to $AP = PD$ ...(2)

where $D = \text{Diag}\,[\lambda_1, \lambda_2, \ldots\ldots\ldots, \lambda_n]$

Also matrix P is non-singular as its columns are L.I. $\therefore$ P^{-1} exists and we may write (2) as

$$P^{-1}AP = D$$

Hence A to similar to D.

Example 5. *Examine whether A is similar to B where*

(i) $A = \begin{bmatrix} 5 & 5 \\ -2 & 0 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. **(P.T.U., May 2010)**

Sol. We know that A will be similar to B if there exists a non-singular matrix P such that $A = P^{-1}BP$ or $PA = BP$

Let $$P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

(i) $$PA = BP \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 5 & 5 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

or $$\begin{bmatrix} 5a-2b & 5a \\ 5c-2d & 5c \end{bmatrix} = \begin{bmatrix} a+2c & b+2d \\ -3a+4c & -3b+4d \end{bmatrix}$$

$\therefore$ $5a - 2b = a + 2c$ $\quad$ $5a = b + 2d$

$5c - 2d = -3a + 4c$ $\quad$ $5c = -3b + 4d$

or $4a = 2b + 2c$ $\quad$ *i.e.*, $2a = b + c$

$3a = -c + 2d$ $\quad$ *i.e.*, $3a = -c + 2d$

or equations are
$$2a - b - c + 0.d = 0$$
$$3a + 0.b + c - 2d = 0$$
$$5a - b + 0.c - 2d = 0$$
$$0.a + 3b + 5c - 4d = 0$$

which is a set of homogeneous equation

$$\therefore \quad \begin{bmatrix} 2 & -1 & -1 & 0 \\ 3 & 0 & 1 & -2 \\ 5 & -1 & 0 & -2 \\ 0 & 3 & 5 & -4 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0 \text{ or } \begin{bmatrix} -1 & 2 & -1 & 0 \\ 0 & 3 & 1 & -2 \\ -1 & 5 & 0 & -2 \\ 3 & 0 & 5 & -4 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0 \text{ by operating } C_{12}$$

Operate $R_3 - R_1, R_4 + 3R_1$; $$\begin{bmatrix} -1 & 2 & -1 & 0 \\ 0 & 3 & 1 & -2 \\ 0 & 3 & 1 & -2 \\ 0 & 6 & 2 & -4 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0$$

Operate $R_3 - R_2, R_4 - 2R_2$; $$\begin{bmatrix} -1 & 2 & -1 & 0 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0$$

$\therefore$ $-a + 2b - c = 0$

$3b + c - 2d = 0$

$\therefore$ If $a = 1, b = 1$, we get $c = 1$ and $d = 2$

$\therefore$ $$P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \text{ which is non-singular}$$

Hence A, B are similar

(ii) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

or $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ c & d \end{bmatrix}$

$\therefore \quad a = a + c \Rightarrow c = 0$

$b = b + d \Rightarrow d = 0$

$\therefore \quad P = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$, which is a singular matrix

$\therefore$ A, B are not similar matrices.

Example 6. *Examine which of the following matrices are similar to diagonal matrices*

(i) $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$.

Sol. (*i*) Characteristic equation of $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ is $|\lambda I - A| = 0$

i.e., $\begin{vmatrix} \lambda - 8 & 6 & -2 \\ 6 & \lambda - 7 & 4 \\ -2 & 4 & \lambda - 3 \end{vmatrix} = 0$ *i.e.,* $\lambda^3 - 18\lambda^2 + 45\lambda = 0;\quad \lambda = 0, 3, 15$

Characteristic vectors corresponding to $\lambda = 0$ is given by $(\lambda I - A)X = 0$ Put $\lambda = 0$

$$\begin{bmatrix} -8 & 6 & -2 \\ 6 & -7 & 4 \\ -2 & 4 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

Operate R_{13}
$$\begin{bmatrix} -2 & 4 & -3 \\ 6 & -7 & 4 \\ -8 & 6 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

Operate $R_2 + 3R_1, R_3 - 4R_1$
$$\begin{bmatrix} -2 & 4 & -3 \\ 0 & 5 & -5 \\ 0 & -10 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0;$$

Operate $R_3 + 2R_2$;
$$\begin{bmatrix} -2 & 4 & -3 \\ 0 & 5 & -5 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

or
$$-2x + 4y - 3z = 0$$
$$5y - 5z = 0$$

$\therefore$
$$y = z, \quad x = \frac{z}{2}$$

$\therefore$
$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{z}{2} \\ z \\ z \end{bmatrix} = \frac{z}{2}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \Rightarrow X = \frac{z}{2}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$\therefore$ We may take single L.I. solution $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

Similarly for $\lambda = 3$; $\begin{bmatrix} -5 & 6 & -2 \\ 6 & -4 & +4 \\ -2 & 4 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

Operate R_{13}; $\begin{bmatrix} -2 & 4 & 0 \\ 6 & -4 & 4 \\ -5 & 6 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

Operate $R_1\left(-\frac{1}{2}\right), R_2\left(\frac{1}{2}\right)$; $\begin{bmatrix} 1 & -2 & 0 \\ 3 & -2 & 2 \\ -5 & 6 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

Operate $R_2 - 3R_1,\ R_3 + 5R_1$; $\begin{bmatrix} 1 & -2 & 0 \\ 0 & 4 & 2 \\ 0 & -4 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

Operate $R_3 + R_2$; $\begin{bmatrix} 1 & -2 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

i.e., $x - 2y = 0$ or $x = 2y$

$4y + 2z = 0$ $\quad z = -2y$

$$\therefore \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2y \\ y \\ -2y \end{bmatrix} = y\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

$\therefore$ Eigen vector corresponding to $\lambda = 3$ is $\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

For $\lambda = 15$ $\begin{bmatrix} 7 & 6 & -2 \\ 6 & 8 & 4 \\ -2 & 4 & 12 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$ or $\begin{bmatrix} -1 & 2 & 6 \\ 3 & 4 & 2 \\ 7 & 6 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$ by operating $R_3\left(\frac{1}{2}\right); R_2\left(\frac{1}{2}\right); R_{13}$

Operate $R_2 + 3R_1, R_3 + 7R_1$; $\begin{bmatrix} -1 & 2 & 6 \\ 0 & 10 & 20 \\ 0 & 20 & 40 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

Operate $R_3 - 2R_2,\ R_2\left(\frac{1}{10}\right)$; $\begin{bmatrix} -1 & 2 & 6 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

$\therefore \quad -x + 2y + 6z = 0,\ y + 2z = 0$

$$\therefore \qquad y = -2z, \quad x = 2z \qquad \therefore \; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2z \\ -2z \\ z \end{bmatrix} = z\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

Eigen vector corresponding to $\lambda = 15$ is $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

$\therefore$ Set of L.I. characteristic vectors is

$$P = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

Now, $$P^{-1} = -\frac{1}{27}\begin{bmatrix} -3 & -6 & -6 \\ -6 & -3 & 6 \\ -6 & 6 & -3 \end{bmatrix} \qquad \left(\because P^{-1} = \frac{\text{Adj}\, P}{|P|}\right)$$

$$= \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$$

= diag (0, 3, 15) *i.e.*, diagonal matrix formed by eigen values

Hence A is similar to diagonal matrix.

(ii) $$A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Characteristic roots of A are $|\lambda I - A| = 0$

$$\begin{vmatrix} \lambda - 2 & -3 & -4 \\ 0 & 2 - \lambda & 1 \\ 0 & 0 & \lambda - 1 \end{vmatrix} = 0 \quad \text{or} \quad (\lambda - 2)^2 (\lambda - 1) = 0$$

$$\lambda = 1, 2, 2$$

Eigen vector corresponding to $\lambda = 1$ is $\begin{bmatrix} -1 & -3 & -4 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

i.e.,
$$-x - 3y - 4z = 0$$
$$-y + z = 0$$

$$\therefore \qquad y = z, \; x = -3y - 4z = -7z$$

$$\therefore \qquad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -7z \\ z \\ z \end{bmatrix} = z\begin{bmatrix} -7 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore$ Single eigen vector corresponding to $\lambda = 1$ is $\begin{bmatrix} -7 \\ 1 \\ 1 \end{bmatrix}$

For $\lambda = 2$,
$$\begin{bmatrix} 0 & -3 & -4 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$
$$-3y - 4z = 0$$
$$z = 0$$
$$y = 0$$

$\therefore$

$\therefore$
$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ Corresponding to $\lambda = 2$ we get only one vector $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

As there are only two L.I. eigen vectors corresponding to three eigen values.

$\therefore$ There does not exist any non-singular matrix P.

Hence A is not similar to diagonal matrix.

Example 7. *Prove that if A is similar to a diagonal matrix, then A' is similar to A.*

Sol. Let A be similar to diagonal matrix D then there exists a non-singular matrix P such that
$$P^{-1}AP = D$$
or
$$A = PDP^{-1}$$
$$A' = (PDP^{-1})' = (P^{-1})' D'P' = (P')^{-1} DP' \qquad (\because \text{ D is a diagonal matrix } \therefore D' = D)$$

$\Rightarrow$ A' is similar to D

$\Rightarrow$ D is similar to A'

Now A is similar to D ; D is similar to A'

$\Rightarrow$ A is similar to A'

i.e., A' is similar to A.

Example 8. *Show that the rank of every matrix similar to A is the same as that of A.*

Sol. Let B be similar to A. Then there exists a non-singular matrix P such that
$$B = P^{-1}AP$$
Now, rank of B = rank of $(P^{-1}AP)$
= rank of A

$\because$ We know that rank of a matrix does not change on multiplication by a non-singular matrix.

Hence rank of B = rank of A.

4.31. MUTUAL RELATIONS BETWEEN CHARACTERISTIC VECTORS CORRESPONDING TO DIFFERENT CHARACTERISTIC ROOTS OF SOME SPECIAL MATRICES

Before discussing these relations, we first give some definitions.

(*a*) **Inner Product of two Vectors :** We consider the vector space V_n (C) of n-tuples over the field C of complex numbers.

Let X, Y be any two members of V_n (C) written as column vectors then the scalar $X^{\theta}Y$ is called inner product of vectors X and Y.

Thus if $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$

Then $X^\theta Y = [\bar{x}_1, \bar{x}_2, \ldots\ldots\ldots, \bar{x}_n] \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = [\bar{x}_1 y_1 + \bar{x}_2 y_2 + \ldots\ldots\ldots + \bar{x}_n y_n]$, which is a single element matrix

Hence inner product of X and Y is $\bar{x}_1 y_1 + \bar{x}_2 y_2 + \ldots\ldots + \bar{x}_n y_n$.

Note 1. $X^\theta Y \neq XY^\theta$; infact one side is complex conjugate of other side.

Note 2. In case vectors are real, then we have $X^\theta Y = X'Y = XY' = XY^\theta \therefore$ inner product concides.

Hence inner product of two real n-tuple vectors is

$$x_1 y_1 + x_2 y_2 + \ldots\ldots\ldots + x_n y_n$$

(*b*) **Length of a Vector :** The positive square root of the inner product $X^\theta X$ is called length of X. Thus the length of an n-vector with components $x_1, x_2, \ldots\ldots\ldots, x_n$ is positive square root of $\bar{x}_1 x_1 + \bar{x}_2 x_2 + \bar{x}_3 x_3 \ldots\ldots \bar{x}_n x_n$, which is always positive except when X = 0 and when X = 0 then length is also zero.

In case of real vectors length of the vector $= x_1^2 + x_2^2, \ldots\ldots\ldots, x_n^2$.

(*c*) **Normal Vector:** A vector whose length is 1, is called a normal vector.

(*d*) **Orthogonal Vectors:** A vector X is said to be orthogonal to a vector Y, if the inner product of X and Y is 0 *i.e.*, $X^\theta Y = 0 \Leftrightarrow XY^\theta = 0$

i.e., $\bar{x}_1 y_1 + \bar{x}_2 y_2 \ldots\ldots \bar{x}_n y_n = 0$

or $x_1 \bar{y}_1 + x_2 \bar{y}_2 \ldots\ldots x_n \bar{y}_n = 0$

In case of real vectors the condition of orthogonality becomes $x_1 y_1 + x_2 y_2 + \ldots\ldots\ldots + x_n y_n = 0$.

(*e*) **Condition for a Linear Transformation X = PY to Preserve length is that $P^\theta P = I$:**

[Lengths of the vectors preserved means length of vector X = length of vector Y]

We have $X = PY$

$\Rightarrow$ $X^\theta = (PY)^\theta = Y^\theta P^\theta$

$\therefore$ $X^\theta X = (Y^\theta P^\theta)(PY)$

$= Y^\theta (P^\theta P) Y$

Given $P^\theta P = I$ $\therefore$ $X^\theta X = Y^\theta Y$

$\therefore$ Length of the vectors is preseved.

(*f*) **Every Unitary Transformation X = PY Preserves Inner Products:**

$\because$ X = PY is unitary transformation

$\therefore$ P is a unitary matrix $\therefore PP^\theta = I$

If $X_1 = PY_1$

and $X_2 = PY_2$, then $X_2^\theta = (PY_2)^\theta = Y_2^\theta P^\theta$

$\therefore$ $X_2^\theta X_1 = (Y_2^\theta P^\theta)(PY_1) = Y_2^\theta (P^\theta P) Y_1 = Y_2^\theta I Y_1$

or $X_2^\theta X_1 = Y_2^\theta Y_1$

Hence inner product is preserved.

4.32. COLUMN VECTORS OF A UNITARY MATRIX ARE NORMAL AND ORTHOGONAL IN PAIRS

Proof. Let $P = [X_1, X_2, \ldots\ldots\ldots, X_n]$ be a unitary matrix (where $X_1, X_2, \ldots\ldots\ldots, X_n$ represent columns of P)

$$P^\theta P = \begin{bmatrix} X_1^\theta \\ X_2^\theta \\ \vdots \\ X_n^\theta \end{bmatrix} [X_1, X_2, \ldots\ldots, X_n]$$

$$= \begin{bmatrix} X_1^\theta X_1 & X_1^\theta X_2 & \ldots\ldots & X_1^\theta X_n \\ X_2^\theta X_1 & X_2^\theta X_2 & \ldots\ldots & X_2^\theta X_n \\ \ldots\ldots & & & \\ \ldots\ldots & & & \\ X_n^\theta X_1 & X_n^\theta X_2 & \ldots\ldots & X_n^\theta X_n \end{bmatrix}$$

Now,
$$P^\theta P = I \Rightarrow \begin{bmatrix} X_1^\theta X_1 & X_1^\theta X_2 & \ldots\ldots & X_1^\theta X_n \\ X_2^\theta X_1 & X_2^\theta X_2 & \ldots\ldots & X_2^\theta X_n \\ \ldots\ldots & & & \\ \ldots\ldots & & & \\ X_n^\theta X_1 & X_n^\theta X_2 & \ldots\ldots & X_n^\theta X_n \end{bmatrix} = \begin{bmatrix} I & 0 & \ldots\ldots & 0 \\ 0 & I & \ldots\ldots & 0 \\ \ldots\ldots & & & \\ \ldots\ldots & & & \\ 0 & 0 \; 0 & \ldots\ldots & I \end{bmatrix}$$

$$\Rightarrow \quad X_1^\theta X_1 = X_2^\theta X_2 = X_3^\theta X_3, \ldots\ldots, X_n^\theta X_n = I$$

whereas all other sub matrices are zero matrices

i.e.,
$$X_i^\theta X_j = 0 \qquad i \neq j$$
$$= I \qquad i = j$$

which shows that column vectors $X_1, X_2, \ldots\ldots, X_n$ of P are normal $\left(\because X_i^\theta X_j = 1 \; ; \; \forall\, i = j\right)$ and orthogonal $\left(\because X_i^\theta X_j = 0 \; ; \; i \neq j\right)$

Cor. Similarity we can prove that the row vectors of a unitary matrix are also normal and orthogonal in pairs we will write $P = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix}$ and employ $PP^\theta = I$.

4.33(a). ORTHONORMAL SYSTEM OF VECTORS

A set of normal vectors which are orthogonal in pairs is called an orthonormal set.

4.33(b). EVERY ORTHONORMAL SET OF VECTORS IS LINEARLY INDEPENDENT

Proof. Let $X_1, X_2, \ldots\ldots, X_k$ be the set of orthonormal vectors of n-tuple

Consider the relation. $a_1 X_1 + a_2 X_2 + \ldots\ldots + a_k X_k = 0$

Pre-multiply by X_1^θ, we get

$$a_1\left(X_1^\theta X_1\right) + a_2\left(X_1^\theta X_2\right) + a_3\left(X_1^\theta X_3\right) + \ldots\ldots + a_k\left(X_1^\theta X_k\right) = 0$$

$\Rightarrow \quad a_1\left(X_1^\theta X_1\right) = 0$ [$\because$ the set is that of orthogonal vectors]

$\Rightarrow \quad a_1 I = 0$ [$\because$ set of vectors is normal]

$\Rightarrow \quad a_1 = 0$

Similarly, by pre-multiplying by $X_2^\theta, X_3^\theta, \ldots\ldots, X_k^\theta$ successively, we get $a_2 = 0, a_3 = 0, \ldots\ldots, a_k = 0$ hence the set is linearly independent.

4.34. ANY TWO CHARACTERISTIC VECTORS CORRESPONDING TO TWO DISTINCT CHARACTERISTIC ROOTS OF A HERMITIAN MATRIX ARE ORTHOGONAL

Proof. Let X_1, X_2 be any two characteristic vectors corresponding to two distinct characteristic roots λ_1 and λ_2 respectively of a Hermitian matrix

$$\therefore \quad AX_1 = \lambda_1 X_1 \qquad \lambda_1, \lambda_2 \text{ being real scalars} \qquad \ldots(1)$$

$$AX_2 = \lambda_2 X_2 \qquad \ldots(2)$$

Pre-multiply (1) by X_2^θ and (2) by X_1^θ we have

$$X_2^\theta AX_1 = \lambda_1 X_2^\theta X_1 \qquad \ldots(3)$$

$$X_1^\theta AX_2 = \lambda_2 X_1^\theta X_2 \qquad \ldots(4)$$

Take conjugate transpose of (3)

$$X_1^\theta A^\theta X_2 = \lambda_1 X_1^\theta X_2 \quad \textit{i.e.,} \quad X_1^\theta AX_2 = \lambda_1 X_1^\theta X_2 \qquad [\because \text{ A is Hermitian } \therefore A^\theta = A]$$

or $$\lambda_1 X_1^\theta X_2 = X_1^\theta (AX_2) = X_1^\theta (\lambda_2 X_2) \qquad \because \text{ of (2)}$$

$\therefore$ $$\lambda_1 X_1^\theta X_2 = \lambda_2 X_1^\theta X_2$$

or $$(\lambda_1 - \lambda_2)(X_1^\theta X_2) = 0 \text{ but } \lambda_1 - \lambda_2 \neq 0 \because \lambda_1, \lambda_2 \text{ are distinct}$$

$\therefore$ $$X_1^\theta X_2 = 0 \qquad \Rightarrow X_1, X_2 \text{ are orthogonal vectors.}$$

Cor. Any two characteristic vectors corresponding to two distinct characteristic roots of a real symmetric matrix are orthogonal.

4.35. ANY TWO CHARACTERISTIC VECTORS CORRESPONDING TO TWO DISTINCT CHARACTERISTIC ROOTS OF A UNITARY MATRIX ARE ORTHOGONAL

(P.T.U., May 2004)

Proof. Let X_1, X_2 be two characteristic vectors corresponding to two distinct characteristic roots λ_1 and λ_2

$$\therefore \quad AX_1 = \lambda_1 X_1, \qquad \text{where A is a unitary matrix } \therefore A^\theta A = I \qquad \ldots(1)$$

$$AX_2 = \lambda_2 X_2 \qquad \ldots(2)$$

Take conjugate transpose of (2)

$$(AX_2)^\theta = (\lambda_2 X_2)^\theta \text{ or } X_2^\theta A^\theta = \bar{\lambda}_2 X_2^\theta \qquad \ldots(3)$$

From (1) and (3) by multiplication

$$\left(X_2^\theta A^\theta\right)(AX_1) = \left(\bar{\lambda}_2 X_2^\theta\right)(l_1 X_1)$$

or $$X_2^\theta \left(A^\theta A\right) X_1 = \bar{\lambda}_2 \lambda_1 \left(X_2^\theta X_1\right)$$

or $$X_2^\theta (IX_1) = \bar{\lambda}_2 \lambda_1 \left(X_2^\theta X_1\right) \qquad (\because \text{ A is unitary matrix})$$

or $$X_2^\theta X_1 = \bar{\lambda}_2 \lambda_1 \left(X_2^\theta X_1\right)$$

or $$\left(1 - \bar{\lambda}_2 \lambda_1\right) X_2^\theta X_1 = 0 \qquad \ldots(4)$$

Since in a unitary matrix modulus of each of the characteristic roots is unity $\therefore \ \bar{\lambda}_2\lambda_2 = 1$

$\therefore$ From (4), $\left(\bar{\lambda}_2 \lambda_2 - \bar{\lambda}_2 \lambda_1\right) X_2^{\theta} X_1 = 0$

or $$\bar{\lambda}_2 (\lambda_2 - \lambda_1) X_2^{\theta} X_1 = 0$$

Now, $$\bar{\lambda}_2 (\lambda_2 - \lambda_1) \neq 0$$

$\because \ \lambda_1, \lambda_2$ are distinct

$\therefore \ \lambda_2 - \lambda_1 \neq 0$ also $\lambda_2 \neq 0$

$\therefore$ $$X_2^{\theta} X_1 = 0$$

Hence X_1, X_2 are orthogonal.

TEST YOUR KNOWLEDGE

1. Diagonalise the following matrices and obtain the modal matrix in each case

(*i*) $\begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 9 & -1 & -9 \\ 3 & -1 & 3 \\ -7 & 1 & -7 \end{bmatrix}$

2. Show that the following matrices are similar to diagonal matrices. Also find the transforming matrices and the diagonal matrices

(*i*) $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ (*ii*) $\begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$

3. Show that the following matrices are not similar to diagonal matrices

(*i*) $\begin{bmatrix} 2 & -1 & 1 \\ 2 & 2 & -1 \\ 1 & 1 & -1 \end{bmatrix}$ (*ii*) $\begin{bmatrix} -1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 2 & -1 \\ 0 & 5 & 3 & -1 \end{bmatrix}$

4. If A and B are non-singular matrices of order n, show that the matrices AB and BA are similar.
5. Prove that every orthogonal set of vectors is linearly independent.
6. Prove that any two characteristic vectors corresponding to two distinct characteristic roots of a real symmetric matrix are orthogonal.
7. Show that characteristic vectors corresponding to different characteristic roots of a normal matrix are orthogonal.
8. If X is characteristic vector of a normal matrix A corresponding to characteristic root λ then X is also a characteristic vector of A^{θ}, the corresponding characteristic root being $\bar{\lambda}$.

ANSWERS

1. (*i*) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & -2 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 0 & 1 & 4 \\ 1 & 0 & 1 \\ -1 & -1 & -3 \end{bmatrix}$

2. (*i*) $P = \begin{bmatrix} -1 & 1 & 2 \\ 0 & 2 & -1 \\ 2 & 0 & 1 \end{bmatrix}, D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$ (*ii*) $P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix}, D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

4.36. QUADRATIC FORM

Definition. A homogeneous polynomial of second degree in any number of variables is called a quadratic form. For example,

(i) $ax^2 + 2hxy + by^2$ (ii) $ax^2 + by^2 + cz^2 + 2hxy + 2gyz + 2fzx$ and

(iii) $ax^2 + by^2 + cz^2 + dw^2 + 2hxy + 2gyz + 2fzx + 2lxw + 2myw + 2nzw$

are quadratic forms in two, three and four variables.

In n-variables $x_1, x_2, \ldots\ldots\ldots, x_n$, the general quadratic form is $\sum_{j=1}^{n}\sum_{i=1}^{n} b_{ij}x_ix_j$, where $b_{ij} \neq b_{ji}$

In the expansion, the coefficient of $x_ix_j = (b_{ij} + b_{ji})$.

Suppose $2a_{ij} = b_{ij} + b_{ji}$, where $a_{ij} = a_{ji}$ and $a_{ii} = b_{ii}$.

$$\therefore \qquad \sum_{j=1}^{n}\sum_{i=1}^{n} b_{ij}x_ix_j = \sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}x_ix_j \text{ , where } a_{ij} = \frac{1}{2}\left(b_{ij} + b_{ji}\right).$$

Hence every quadratic form can be written as $\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}x_ix_j$ = X′AX, so that the **matrix A is always symmetric**, where A = $[a_{ij}]$ and X = $[x_1\ x_2 \ldots x_n]$.

Now, writing the above examples of quadratic forms in matrix form, we get

$$(i)\quad ax^2 + 2hxy + by^2 = [x\ \ y]\begin{bmatrix} a & h \\ h & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$(ii)\quad ax^2 + by^2 + cz^2 + 2hxy + 2gyz + 2fzx = [x\ \ y\ \ z]\begin{bmatrix} a & h & f \\ h & b & g \\ f & g & c \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

and (iii) $ax^2 + by^2 + cz^2 + dw^2 + 2hxy + 2gyz + 2fzx + 2lxw + 2myw + 2nzw$

$$= [x\ \ y\ \ z\ \ w]\begin{bmatrix} a & h & f & l \\ h & b & g & m \\ f & g & c & n \\ l & m & n & d \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}.$$

Example. *Find a real symmetric matrix C such that* $Q = X'CX$ *equals* : $(x_1 + x_2)^2 - x_3^2$.

(P.T.U., Dec. 2002)

Sol.

$$Q = X'CX = (x_1 + x_2)^2 - x_3^2 = x_1^2 + x_2^2 - x_3^2 + 2x_1x_2$$

$$= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\therefore \qquad C = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \text{ which is a real symmetric matrix.}$$

4.37. LINEAR TRANSFORMATION OF A QUADRATIC FORM

Let $X'AX$ be a quadratic form in n-variables and let $X = PY$ ...(1)

(where P is a non-singular matrix) be the non-singular transformation.

$$X' = (PY)' = Y'P' \text{ and hence}$$

From (1),

$$X'AX = Y'P'APY = Y'(PAP)Y = Y'BY \quad \ldots(2)$$

where $B = P'AP$. Therefore $Y'BY$ is also a quadratic form in n-variables. Hence it is a linear transformation of the quadratic form $X'AX$ under the linear transformation $X = PY$ and $B = P'AP$.

Note. (*i*) Here $B' = (P'AP)' = P'AP = B$ (*ii*) $\rho(B) = \rho(A)$.

$\therefore$ **A and B are congruent matrices.**

4.38. CANONICAL FORM

If a real quadratic form be expressed as a sum or difference of the squares of new variables by means of any real non-singular linear transformation, then the later quadratic expression is called a *canonical form* of the given quadratic form.

i.e., if the quadratic form $X'AX = \sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}x_i x_j$ can be reduced to the quadratic form

$Y'BY = \sum_{i=1}^{n} \lambda_i y_i^2$ by a non-singular linear transformation $X = PY$ then

$Y'BY$ is called the canonical form of the given one.

$\therefore$ If $B = P'AP = \text{diag}(\lambda_1, \lambda_2, \ldots\ldots\ldots, \lambda_n)$ then $X'AX = Y'BY = \sum_{j=1}^{n} \lambda_i y_i^2$.

Note. (*i*) Here some of λ_i (eigen values) may be positive or negative or zero.

(*ii*) A quadratic form is said to be real if the elements of the symmetric matrix are real.

(*iii*) If $\rho(A) = r$, then the quadratic form $X'AX$ will contain only r terms.

4.39. INDEX AND SIGNATURE OF THE QUADRATIC FORM

The number p of positive terms in the canonical form is called the *index* of the quadratic form.

(The number of positive terms) – (The number of negative terms) *i.e.,* $p - (r - p) = 2p - r$ is called *signature* of the quadratic form, where $\rho(A) = r$.

4.40. DEFINITE, SEMI-DEFINITE AND INDEFINITE REAL QUADRATIC FORMS

Let $X'AX$ be a real quadratic form in n-variables $x_1, x_2, \ldots\ldots\ldots, x_n$ with rank r and index p. Then we say that the quadratic form is

(*i*) *Positive definite* if $r = n, p = r$ (*ii*) *negative definite* if $r = n, p = 0$

(*iii*) *Positive semi-definite* if $r < n$, $p = r$ and (*iv*) *negative semi-definite* if $r < n, p = 0$

If the canonical form has both positive and negative terms, the quadratic form is said to be *indefinite*.

Note. If $X'AX$ is positive definite then $|A| > 0$.

4.41. LAW OF INERTIA OF QUADRATIC FORM

The index of a real quadratic form is invariant under real non-singular transformations.

4.42. LAGRANGE'S METHOD OF REDUCTION OF A QUADRATIC FORM TO DIAGONAL FORM

Let the quadratic form be in three variables x, y, z.

Step 1: Reduce the quadratic form to X'AX and find matrix A where $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Step 2: In the quadratic form *i.e., in* Q collect all the terms of x and express them as a perfect square in x, y, z by adding or subtracting the terms of y and z.

Step 3: In the next group collect all terms of y and express them as a perfect square by adding or subtracting the terms of z. Now only terms of z^2 will be left which will form 3rd group.

Step 4: Equating terms on the R.H.S. to $\lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2$ and write down the values of y_1, y_2, y_3 in terms of x, y, z.

Step 5: Then express x, y, z in terms of y_1, y_2, y_3 and the linear transformation X = PY is known where P will be formed by coefficients of y_1, y_2, y_3 and $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

Step 6: Then Q will be transformed to the diagonal matrix $\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$, which will be same as P'AP.

ILLUSTRATIVE EXAMPLES

Example 1. *Reduce $3x^2 + 3z^2 + 4xy + 8xz + 8yz$ into canonical form.*

Sol. The given quadratic form can be written as X' AX where X' = $[x, y, z]'$ and the symmetric matrix

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 4 \\ 4 & 4 & 3 \end{bmatrix}.$$

Let us reduce A into diagonal matrix. We know that $A = I_3 A I_3$

i.e.,
$$\begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 4 \\ 4 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 4 \\ 4 & 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operating $R_2 - \frac{2}{3}R_1$, $R_3 - \frac{4}{3}R_1$ on A and pre-factor on R.H.S., we get

$$\begin{bmatrix} 3 & 2 & 4 \\ 0 & -\frac{4}{3} & \frac{4}{3} \\ 0 & \frac{4}{3} & -\frac{7}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ -\frac{4}{3} & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operating $C_2 - \frac{2}{3}C_1$, $C_3 - \frac{4}{3}C_1$ on A and the post-factor on R.H.S., we get

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & -\frac{4}{3} & \frac{4}{3} \\ 0 & \frac{4}{3} & -\frac{7}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ -\frac{4}{3} & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -\frac{2}{3} & -\frac{4}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operating $R_3 + R_2$ on L.H.S. and pre-factor on R.H.S.

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & -\frac{4}{3} & \frac{4}{3} \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -\frac{2}{3} & -\frac{4}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operating $C_3 + C_2$ on L.H.S. and post-factor on R.H.S.

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & -\frac{4}{3} & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -\frac{2}{3} & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

or $$\text{Diag}\left(3, -\frac{4}{3}, -1\right) = P'AP$$

$\therefore$ The canonical form of the given quadratic form is

$$Y'(P'AP)Y = [y_1, y_2, y_3]\begin{bmatrix} 3 & 0 & 0 \\ 0 & -\frac{4}{3} & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 3y_1^2 - \frac{4}{3}y_2^2 - y_3^2.$$

Here $\rho(A) = 3$, index $= 1$, signature $= 1 - (2) = -1$.

Note 1. In this problem the non-singular transformation which reduces the given quadratic form into the canonical form is X = PY *i.e.*, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -\frac{2}{3} & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$.

Note 2. The above example can also be questioned as 'Diagonalise' the quadratic form $3x^2 + 3z^2 + 4xy + 8xz + 8yz$ by linear transformations and write the linear transformation.

Or

Reduce the quadratic form $3x^2 + 3z^2 + 4xy + 8xz + 8yz$ into the sum of squares.

Example 2. *Reduce the quadratic form $x^2 - 4y^2 + 6z^2 + 2xy - 4xz + 2w^2 - 6zw$ into sum of squares.*

Sol. The matrix form of the given quadratic is X'AX, where $X = (x, y, z, w)'$

and $$A = \begin{bmatrix} 1 & 1 & -2 & 0 \\ 1 & -4 & 0 & 0 \\ -2 & 0 & 6 & -3 \\ 0 & 0 & -3 & 2 \end{bmatrix}$$

Let us reduce A to the diagonal matrix. We know that $A = I_4 A I_4$.

$$\begin{bmatrix} 1 & 1 & -2 & 0 \\ 1 & -4 & 0 & 0 \\ -2 & 0 & 6 & -3 \\ 0 & 0 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operating $R_2 - R_1, R_3 + 2R_1$, also on pre-factor on R.H.S.

$$\begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & -5 & 2 & 0 \\ 0 & 2 & 2 & -3 \\ 0 & 0 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operating $C_2 - C_1, C_3 + 2C_1$, also on post-factor on R.H.S.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & 2 & 0 \\ 0 & 2 & 2 & -3 \\ 0 & 0 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operating $R_3 + \frac{2}{5}R_2$, also on pre-factor on R.H.S.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & 2 & 0 \\ 0 & 0 & \frac{14}{5} & -3 \\ 0 & 0 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ \frac{8}{5} & \frac{2}{5} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operating $C_3 + \frac{2}{5}C_2$, also on post-factor on R.H.S.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & \frac{14}{5} & -3 \\ 0 & 0 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ \frac{8}{5} & \frac{2}{5} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & \frac{8}{5} & 0 \\ 0 & 1 & \frac{2}{5} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operating $R_4 + \frac{15}{14}R_3$, also on pre-factor on R.H.S.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & \frac{14}{5} & -3 \\ 0 & 0 & 0 & -\frac{17}{14} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ \frac{8}{5} & \frac{2}{5} & 1 & 0 \\ \frac{12}{7} & \frac{3}{7} & \frac{15}{14} & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & \frac{8}{5} & 0 \\ 0 & 1 & \frac{2}{5} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operating $C_4 + \frac{15}{14}C_3$, also on post-factor on R.H.S.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & \frac{14}{5} & 0 \\ 0 & 0 & 0 & -\frac{17}{14} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ \frac{8}{5} & \frac{2}{5} & 1 & 0 \\ \frac{12}{7} & \frac{3}{7} & \frac{15}{14} & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & \frac{8}{5} & \frac{12}{7} \\ 0 & 1 & \frac{2}{3} & \frac{3}{7} \\ 0 & 0 & 1 & \frac{15}{14} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{diag.}\left(1, -5, \frac{14}{5}, -\frac{17}{14}\right) = P'AP$$

i.e.,

$\therefore$ The canonical form of the given quadratic form is

$$Y'(P'AP)Y = Y'\text{diag}\left(1, -5, \frac{14}{5}, -\frac{17}{14}\right)Y$$

$$= \begin{bmatrix} y_1 & y_2 & y_3 & y_4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & \frac{14}{5} & 0 \\ 0 & 0 & 0 & -\frac{17}{14} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

$$= y_1^2 - 5y_2^2 + \frac{14}{5}y_3^2 - \frac{17}{14}y_4^2, \text{ which is the sum of the squares.}$$

Example 3. *Show that the form $5x_1^2 + 26x_2^2 + 10x_3^2 + 4x_2x_3 + 14x_3x_1 + 6x_1x_2$ is a positive semi-definite and find a non-zero set of values of x_1, x_2, x_3 which makes the form zero.* **(P.T.U., Dec. 2002)**

Sol. The matrix form of the given quadratic is X′AX, where

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } A = \begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix}$$

Let us reduce A to the diagonal form

$$A = IAI$$

$\therefore$

$$\begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

[To avoid fractions first of all operate $R_2(5)$, $R_3(5)$ then $C_2(5)$, $C_3(5)$] Note that row transformations will effect pre-factor and column transformation will affect post-factor on R.H.S.

$$\begin{bmatrix} 5 & 15 & 35 \\ 15 & 650 & 50 \\ 35 & 50 & 250 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Operate $R_2 - 3R_1$, $R_3 - 7R_1$

$$\begin{bmatrix} 5 & 15 & 35 \\ 0 & 605 & -55 \\ 0 & -55 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 5 & 0 \\ -7 & 0 & 5 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Operate $C_2 - 3C_1$, $C_3 - 7C_1$

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 605 & -55 \\ 0 & -55 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 5 & 0 \\ -7 & 0 & 5 \end{bmatrix} A \begin{bmatrix} 1 & -3 & -7 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Operate $R_3 + \frac{1}{11}R_2$, $C_3 + \frac{1}{11}C_2$

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 605 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 5 & 0 \\ -\frac{80}{11} & \frac{5}{11} & 5 \end{bmatrix} A \begin{bmatrix} 1 & -3 & -\frac{80}{11} \\ 0 & 5 & \frac{5}{11} \\ 0 & 0 & 5 \end{bmatrix}$$

$\therefore$ Diagonal matrix $(5, 605, 0) = P'AP$

The quadratic form reduces to the diagonal form $5y_1^2 + 605y_2^2$

$$\rho(A) = 2;$$

Index p = Number of positive terms in the diagonal form = 2

n = the number of variables in quadratic form = 3

$$\rho(A) < 3 \text{ and } p = \rho(A)$$

$\therefore$ Given quadratic form is positive semi-definite

Now, $$X = PY \text{ gives } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & -3 & -\frac{80}{11} \\ 0 & 5 & \frac{5}{11} \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$\therefore$ $$x_1 = y_1 - 3y_2 - \frac{80}{11} y_3$$
$$x_2 = 5y_2 + \frac{5}{11} y_3$$
$$x_3 = 5y_3$$

Assume, $y_1 = 0, y_2 = 0, y_3 = 11$, we get

$x_1 = -80, x_2 = 5, x_3 = 55$; Clearly this set of values of x_1, x_2, x_3 makes the given form zero.

Example 4. *Use Lagrange's method to diagonalize the quadratic form :* $2x^2 + 2y^2 + 3z^2 - 4yz + 2xy - 4xz$

(P.T.U., May 2002)

Sol. Step I: The given quadratic form is $Q = 2x^2 + 2y^2 + 3z^2 - 4yz + 2xy - 4xz$ which can be expressed as

$$X'AX = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 2 & 1 & -2 \\ 1 & 2 & -2 \\ -2 & -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \text{ where } A = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 2 & -2 \\ -2 & -2 & 3 \end{bmatrix}$$

Step II: In Q collect all the terms of x and express them as a perfect square

$$= 2(x^2 + xy - 2xz) + 2y^2 + 3z^2 - 4yz$$
$$= 2[x^2 + x(y - 2z)] + 2y^2 + 3z^2 - 4yz$$
$$= 2\left\{\left(x + \frac{y-2z}{2}\right)^2\right\} + 2y^2 + 3z^2 - 4yz - \frac{(y-2z)^2}{2}$$
$$= 2\left\{x + \frac{1}{2}y - z\right\}^2 + 2y^2 + 3z^2 - 4yz - \frac{y^2 - 4yz + 4z^2}{2}$$

Step III: Collect the terms of y and express them as a perfect square

$$= 2\left\{x + \frac{1}{2}y - z\right\}^2 + \left(\frac{3y^2}{2} - 2yz\right) + z^2$$
$$= 2\left\{x + \frac{1}{2}y - z\right\}^2 + \frac{3}{2}\left\{y^2 - \frac{4}{3}yz + \frac{4}{9}z^2\right\} + z^2 - \frac{2}{3}z^2$$
$$= 2\left\{x + \frac{1}{2}y - z\right\}^2 + \frac{3}{2}\left\{y - \frac{2}{3}z\right\}^2 + \frac{1}{3}z^2$$
$$= 2y_1^2 + \frac{3}{2}y_2^2 + \frac{1}{3}y_3^2, \text{ where } y_1 = x + \frac{1}{2}y - z,\ y_2 = y - \frac{2}{3}z,\ y_3 = z$$

Step IV: Express x, y, z, in terms of y_1, y_2, y_3, we get $z = y_3$; $y = y_2 + \frac{2}{3}y_3$, $x = y_1 - \frac{1}{2}y_2 + \frac{2}{3}y_3$

$$\therefore \left.\begin{aligned} x &= y_1 - \frac{1}{2}y_2 + \frac{2}{3}y_3 \\ y &= y_2 + \frac{2}{3}y_3 \\ z &= y_3 \end{aligned}\right\}$$

Step V: Express these in the matrix form X = PY

where $$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} ; Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \text{ and } P = \begin{bmatrix} 1 & -\frac{1}{2} & \frac{2}{3} \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ The linear transformation is X = PY

Step VI: Reduces the quadratic form Q to the diagonal form.

$$B = P'AP = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ \frac{2}{3} & \frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -2 \\ 1 & 2 & -2 \\ -2 & -2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & \frac{2}{3} \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & -2 \\ 0 & \frac{3}{2} & -1 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & \frac{2}{3} \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

$$= \text{diag.} \left(2, \frac{3}{2}, \frac{1}{3}\right)$$

Note. Here $\rho(A) = 4$, index = 2, signature = 2 – 2 = 0.

4.43. REDUCTION TO CANONICAL FORM BY ORTHOGONAL TRANSFORMATION

Let X′AX be a given quadratic form. The modal matrix B of A is that matrix whose columns are characteristic vectors of A. If B represents the orthogonal matrix of A (the normalised modal matrix of A whose column vectors are pairwise orthogonal) then X = BY will reduce X′AX to Y′ diag $(\lambda_1, \lambda_2, \ldots\ldots\ldots, \lambda_n)$ Y, where $\lambda_1, \lambda_2, \ldots\ldots\ldots, \lambda_n$ are characteristic values of A.

Note. This method works successfully if the characteristic vectors of A are linearly independent which are pairwise orthogonal.

Example 5. *Reduce $8x^2 + 7y^2 + 3z^2 + 12xy + 4xz - 8yz$ into canonical form by orthogonal reduction.*

Sol. The matrix of the quadratic form is $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

The characteristic roots of A are given by $|A - \lambda I| = 0$

i.e., $$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \lambda(\lambda-3)(\lambda-15)=0$$

$\therefore$ $$\lambda = 0, 3, 15$$

Characteristic vector for $\lambda = 0$ is given by $[A-(0)I]X = 0$

i.e., $$8x_1 - 6x_2 + 2x_3 = 0$$
$$-6x_1 + 7x_2 - 4x_3 = 0$$
$$2x_1 - 4x_2 + 3x_3 = 0$$

Solving first two, we get $\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2}$ giving the eigen vector $X_1 = (1, 2, 2)$

When $\lambda = 3$, the corresponding characteristic vector is given by $[A-3I]X = 0$

i.e., $$5x_1 - 6x_2 + 2x_3 = 0$$
$$-6x_1 + 4x_2 - 4x_3 = 0$$
$$2x_1 - 4x_2 = 0$$

Solving any two equations, we get $X_2 = (2, 1, -2)$.

Similarly characteristic vector corresponding to $\lambda = 15$ is $X_3 = (2, -2, 1)$.

Now, X_1, X_2, X_3 are pairwise orthogonal *i.e.*, $X_1.X_2 = X_2.X_3 = X_3.X_1 = 0$.

$\therefore$ The normalised modal matrix is

$$B = \left[\frac{X_1}{\|X_1\|}, \frac{X_2}{\|X_2\|}, \frac{X_3}{\|X_3\|}\right] = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

$$BB' = I$$

Now, B is orthogonal matrix and $|B| = -I$

i.e., $B^{-1} = B'$ and $B^{-1}AB = D = \text{diag}\{3, 0, 15\}$

i.e., $$\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix} A \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 15 \end{bmatrix}$$

$$X'AX = Y'(B^{-1}AB)Y = Y'DY$$

$$= [y_1, y_2, y_3]\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 15 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 3y_1^2 + 0.y_2^2 + 15y_3^2$$, which is the required canonical form.

Note. Here the orthogonal transformation is X = BY, rank of the quadratic form = 2 ; index = 2, signature = 2. It is positive definite.

Example 6. *Reduce $6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 - 2x_2x_3 + 4x_3x_1$ into canonical form.*

Sol. The matrix of the quadratic from is $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

The characteristic values are given by $|A-\lambda I|=0$

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

i.e.,

$$\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0, \text{ which on solving gives } \lambda = 2, 2, 8.$$

or The characteristic vector for $\lambda = 2$ is given by $[A-2I]X = 0$, which reduces to single equation

$$2x_1 - x_2 + x_3 = 0.$$

Putting $x_2 = 0$, we get $\frac{x_1}{1} = \frac{x_3}{-2}$ or the vector is $[1, 0, -2]$. Again by putting $x_1 = 0$, we get $\frac{x_2}{1} = \frac{x_3}{1}$ or the vector is $[0, 1, 1]$.

The vector corresponding to $\lambda = 8$ is given by $[A-8I]X = 0$

i.e.,

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$-2x_1 - 2x_2 + 2x_3 = 0$$
$$-2x_1 - 5x_2 - x_3 = 0$$
$$2x_1 - x_2 - 5x_3 = 0$$

Solving any two of the equations, we get the vector as $[2, -1, 1]$.

Now, $X_1 = [2, -1, 1]$; $X_2 = [0, 1, 1]$ and $X_3 = [1, 0, -2]$

Here X_1, X_2, X_3 are not pairwise orthogonal

$\because$ $X_1 \cdot X_2 = 0$; $X_2 \cdot X_3 \neq 0$ and $X_3 \cdot X_1 = 0$

To get X_3 orthogonal to X_2 assume a vector $[u, v, w]$ orthogonal to X_2 also satisfying $2x_1 - x_2 + x_3 = 0$; *i.e.*, $2u - v + w = 0$ and $0.u + 1.v + 1.w = 0$

Solving $[u, v, w] = [1, 1, -1] = X_3$ so that $X_1 \cdot X_2 = X_2 \cdot X_3 = X_3 \cdot X_1 = 0$

$\therefore$ The normalised modal matrix is
$$B = \begin{bmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \end{bmatrix}$$

Now B is orthogonal matrix and $|B| = 1$

i.e.,
$$B' = B^{-1} \text{ and } B^{-1}AB = D \text{ where } D = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$\therefore$
$$X'AX = Y'(B^{-1}AB)Y = Y'DY$$

$$= \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}\begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 8y_1^2 + 2y_2^2 + 2y_3^2$$

which is the required canonical form.

Note. In the above form rank of the quadratic form is 3, index = 3, signature = 3. It is positive definite.

TEST YOUR KNOWLEDGE

1. Write down the matrices of the following quadratic forms:

(*i*) $2x^2 + 3y^2 + 6xy$ (*ii*) $2x^2 + 5y^2 - 6z^2 - 2xy - yz + 8zx$

(*iii*) $x_1^2 + 2x_2^2 + 3x_3^2 + 4x_4^2 + 2x_1x_2 + 4x_1x_3 - 6x_1x_4 - 4x_2x_3 - 8x_2x_4 + 12x_3x_4$ **(P.T.U., May 2010)**

(*iv*) $x_1^2 + 2x_1x_2 - 4x_1x_3 + 6x_2x_3 - 5x_2^2 + 4x_3^2$ **(P.T.U., Dec. 2011)**

(*v*) $3x^2 + 7y^2 - 8z^2 - 4yz + 3xz$.

2. Write down the quadratic form corresponding to the following matrices:

(*i*) $\begin{bmatrix} 2 & 4 & 5 \\ 4 & 3 & 1 \\ 5 & 1 & 1 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 1 & 1 & -2 & 0 \\ 1 & -4 & 0 & 0 \\ -2 & 0 & 6 & -3 \\ 0 & 0 & -3 & 2 \end{bmatrix}$.

3. Reduce the following quadratic forms to canonical forms or to sum of squares by linear transformation. Write also the rank, index and signature:

(*i*) $2x^2 + 2y^2 + 3z^2 + 2xy - 4yz - 4xz$ (*ii*) $12x_1^2 + 4x_2^2 + 5x_3^2 - 4x_2x_3 + 6x_1x_3 - 6x_1x_2$

(*iii*) $2x^2 + 6y^2 + 9z^2 + 2xy + 8yz + 6xz$ (*iv*) $x^2 + 4y^2 + z^2 + 4xy + 6yz + 2zx$.

4. Reduce the following quadratic forms to canonical forms or to sum of square by orthogonal transformation. Write also rank, index, signature:

(*i*) $3x^2 + 5y^2 + 3z^2 - 2xy - 2yz + 2zx$ (*ii*) $2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_3 + 2x_1x_3 - 2x_2x_3$

(*iii*) $3x^2 - 2y^2 - z^2 - 4xy + 8xz + 12yz$ (*iv*) $x^2 + 3y^2 + 3z^2 - 2xy$.

5. Use Lagranges method to diagonalize the quadratic form $6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 - 2x_2x_3 + 4x_3x_1$.

ANSWERS

1. (*i*) $\begin{bmatrix} 2 & 3 \\ 3 & 3 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 2 & -1 & 4 \\ -1 & 5 & -\frac{1}{2} \\ 4 & -\frac{1}{2} & -6 \end{bmatrix}$ (*iii*) $\begin{bmatrix} 1 & 1 & 2 & -3 \\ 1 & 2 & -2 & -4 \\ -2 & -2 & 3 & 6 \\ -3 & -4 & 6 & 4 \end{bmatrix}$

(*iv*) $\begin{bmatrix} 1 & 1 & -2 \\ 1 & -5 & 3 \\ -2 & 3 & 4 \end{bmatrix}$ (*v*) $\begin{bmatrix} 3 & 0 & \frac{3}{2} \\ 0 & 7 & -2 \\ \frac{3}{2} & -2 & -8 \end{bmatrix}$.

2. (*i*) $2x^2 + 3y^2 + z^2 + 8xy + 2yz + 10xz$ (*ii*) $x_1^2 - 4x_2^2 + 6x_3^2 + 2x_4^2 + 2x_1x_2 - 4x_1x_3 - 6x_3x_4$.

3. (*i*) $3y_1^2 + \frac{3}{2}y_2^2 + \frac{1}{3}y_3^2$; Rank = 3, index = 3, Sig. = 3

(*ii*) $12y_1^2 + \frac{13}{4}y_2^2 + \frac{49}{13}y_3^2$; Rank = 3, index = 3, Sig. = 3

(*iii*) $2y_1^2 - 7y_2^2 - \frac{13}{14}y_3^2$; Rank = 3, index = 1, Sig. = – 1 (*iv*) $y_1^2 + y_2^2 + y_3^2$; Rank = 3 , index = 3, Sig. = 3.

4. (*i*) $2y_1^2 + 2y_2^2 + 6y_3^2$; Rank = 3, index = 3 , Sig. = 3

(*ii*) $4y_1^2 + y_2^2 + y_3^2$; Rank = 3, index = 3, Sig. = 3

(*iii*) $3y_2^2 + 6y_2^2 - 9y_3^2$; Rank = 3, index = 2, Sig. = 1

(*iv*) $y_1^2 + 2y_2^2 - 4y_3^2$; Rank = 3, index = 3, Sig. = 3.

5. diag. $\left(6, \frac{7}{3}, \frac{16}{7}\right)$.

REVIEW OF THE CHAPTER

1. **Matrix:** A set of *mn* numbers (real or complex) arranged in a rectangular array having *m* rows and *n* columns, enclosed by brackets [ ] or ( ) is called a $m \times n$ matrix.
2. **Elementary Transformations:** The following operations on a matrix are called elementary transformations:
   (*a*) Interchange of two rows or columns (R_{ij} or C_{ij})]
   (*b*) Multiplication of each element of a row or column by a non-zero number *k* (kR_i or kC_i)
   (*c*) Addition of *k* times the elements of a row (column) to the corresponding elements of another row (column) $(k \neq 0)$ ($R_i + kR_j$ or $C_i + kC_j$).
3. **Elementary Matrix:** The matrix obtained from a unit matrix I by subjecting it to one of the E-operations is called an elementary matrix.
4. **Gauss-Jordan Method:** It is the method to find inverse of a matrix by E-operations.
5. **Normal Form of a Matrix:** Any non-zero matrix $m \times n$ can be reduced to anyone of the following forms by performing E-operations (row, column or both)

   (*i*) I_r (*ii*) $\begin{bmatrix} I_r & 0 \end{bmatrix}$ (*iii*) $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$ (*iv*) $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

   where I_r is a unit matrix of order *r*. All these forms are known as normal forms of the matrix.
6. **Rank of a Matrix:** Let A be any $m \times n$ matrix. If all minors of order $(r + 1)$ are zero but there is at least one non-zero minor of order *r*, then *r* is called the rank of A, and is written as $\rho(A) = r$.
7. **Consistent, Inconsistent Equations:** A system of equations having one or more number of solutions is called a consistent system of equations. A system of equations having no solution is called inconsistent system of equations.
   For a system of *non-homogeneous linear equation* AX = B
   (*i*) If $\rho[A : B] \neq \rho(A)$, the system is inconsistent.
   (*ii*) If $\rho[A : B] = \rho(A)$ = number of unknowns, the system has a unique solution.
   (*iii*) If $\rho[A : B] = \rho(A)$ < number of unknowns, the system has an infinite number of solutions.
   The matrix [A : B] is called Augmented Matrix.
   For a system of **Homogeneous Linear Equations** AX = 0.
   (*i*) X = 0 is always a solution; called *trivial solution*
   (*ii*) If $\rho(A)$ = number of unknowns, then the system has only the trivial solution
   (*iii*) IF $\rho(A)$ < number of unknowns, in system has an infinite number of non-trivial solutions.
   Homogeneous Equations are always consistent.
8. **Linear Dependence and Linear Independence of Vectors:** A set of *n*-tuple vectors $x_1, x_2, \ldots, x_r$ is said to be:
   (*i*) Linearly dependent if $\exists$s *r* scalar $k_1, k_2, \ldots, k_r$ ***Not All Zero*** such that $k_1x_1 + k_2x_2 + \ldots + k_rx_r = 0$
   (*ii*) Linearly independent if each one of $k_1, k_2, \ldots, k_r$ is zero *i.e.*, $k_1 = k_2 = \ldots\ldots = k_r = 0$.
   If a set of vectors is linearly dependent then at least one member of the set can be expressed as a linear combination of the remaining vectors.
9. **Linear Transformation:** A transformation Y = AX is said to be linear transformation if $Y_1 = AX_1$ and $Y_2 = AX_2$ $\Rightarrow aY_1 + bY_2 = A(aX_1 + bX_2) \; \forall \; a, b$
   If the transformation matrix A is nonsingular then the linear transformation is called non-singular or regular and if A is singular then linear transformation is also singular.

10. Orthogonal Transformation: The linear transformation Y = AX is orthogonal transformation if it transforms $y_1^2 + y_2^2 + \ldots + y_n^2$ to $x_1^2 + x_2^2 + \ldots + x_n^2$.

11. Orthogonal Matrix: A real square matrix A is said to be orthogonal if $AA' = A'A = I$

For an orthogonal matrix $A' = A^{-1}$

12. Properties of an Orthogonal Matrix:

(*i*) The transpose of an orthogonal matrix is orthogonal

(*ii*) The inverse of an orthogonal matrix is orthogonal

(*iii*) If A is orthogonal matrix then $|A| = \pm 1$

(*iv*) Product of two orthogonal matrices of the same order is an orthogonal matrix.

13. Hermitian and Skew Hermitian Matrix: A square matrix A is said to be Hermitian if $A^\theta = A$. All diagonal elements of a Hermitian matrix are purely real.

A square matrix A is said to be Skew Hermitian. If $A^\theta = -A$. All diagonal elements of a Skew Hermitian Matrix are zero or purely imaginary of the form $i\beta$.

14. Unitary Matrix: A complex matrix A is said to be unitary matrix if $A^\theta A = I$

15. Properties of a Unitary Matrix:

(*i*) Determinant of a unitary matrix is of modulus unity

(*ii*) The product of two unitary matrices of the same order is unitary

(*iii*) The inverse of a unitary matrix is unitary.

16. Characteristic Equation, Characteristic Roots or Eigen Values, Trace of a Matrix: If A is a square matrix of order, n is a scalar and I is a unit matrix of order n, then $|A - \lambda I| = 0$ is called characteristic equation of A. The roots of the characteristic equation are called characteristic roots or Eigen values of A.

The sum of Eigen values of A is equal to trace of A.

17. Eigen Vectors: Let A be a square matrix of order n, λ is a scalar. Consider the linear transformation Y = AX, where X be such a vector which transforms in λX. Then $Y = \lambda X$ and $\therefore$ we have $AX = \lambda X$ or $(A - \lambda I)X = 0$ which gives n homogeneous linear equations and for non trivial solutions of these linear homogeneous equation we must have $|A - \lambda I| = 0$, which gives n eigen values of A. Corresponding to each eigen value $(A - \lambda I)X = 0$ has a non-zero solution called eigen vector or latent vector.

18. Cayley Hamilton Theorem: Every square matrix satisfies its characteristic equation.

19. Diagonalizable Matrices: A matrix A is said to be diagonalizable if $\exists$s an invertible matrix B such that $\overline{B}\ AB = D$, where D is a diagonal matrix and the diagonal elements of D are the eigen values of A.

Note: A square matrix A is diagonalizable iff it has n linearly independent eigen vectors.

20. Similar Matrices: Let A and B be two square matrices of the same order. The matrix A is said to be similar to matrix B if $\exists$s an invertible matrix P such that PA = BP or $A = \overline{P}\, BP$

Similar matrices have the same characteristic equation.

21. Column vectors of a unitary matrix are normal and orthogonal.

22. Every orthonormal set of vectors is L.I.

23. Any two characteristic vectors corresponding to two distinct characteristic roots of a Hermitian/unitary matrix are orthogonal.

24. Quadratic Form: A homogeneous polynomial of second degree in any number of variables is called quadratic form. Every quadratic form can be expressed in the form $X'AX$, where A is a symmetric matrix.

25. Linear Transformation of a Quadratic form: Let $X'AX$ be a quadratic form in n-variables and Let $X = PY$ be a non-singular transformation then $X'AX = Y'BY$, where $B = P'AP$. Then $Y'BY$ is a linear transformation of the quadratic form $X'AX$ under the linear transformation $X = PY$. A and B are congruent matrices.

26. Canonical Form: If a real quadratic form $X'AX$ be expressed as a sum or difference of the squares of new variables by means of any real linear transformation then the later quadratic expression is called a canonical form of the given quadratic form non-singular. If $\rho(A) = r$, then quadratic form $X'AX$ will contain only r terms.

27. Index and Signature of the Quadratic form: The number p of the positive terms in the canonical form is called the index of the quadratic form.

The number of positive terms minus number of negative terms is called signature of the quadratic form *i.e.*, signature $= p - (r - p) = 2p - r$, where $\rho(A) = r$.

28. Definite, Semi Definite and Indefinite Real Quadratic Forms: Let X′ AX be a real quadratic form then it will be

(i) positive definite if $r = n, p = r$

(ii) negative definite if $r = n, p = 0$

(iii) positive semi-definite if $r < n, p = r$

(iv) negative semi-definite if $r < n, p = 0$

SHORT ANSWER TYPE QUESTIONS

1. (a) If $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$; $B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$ prove that $|AB| = 16$ **(P.T.U., May 2009)**

(b) Prove by an example that AB can be zero matrix when neither A nor B is zero.

2. Explain elementary transformations on a matrix. **(P.T.U., Dec. 2004)**
3. What is Gauss Jordan Method of finding inverse of a non-singular matrix? Hence find the inverse of the matrix

$\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ **(P.T.U., May 2012)**

4. Reduce the matrix $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ to diagonal form. **(P.T.U., Dec. 2012)**
5. (a) Define rank of a matrix and give one example. **(P.T.U., Dec. 2005, Dec. 2006, May 2007, Jan. 2008, May 2011)**

(b) What is the rank of a non-singular matrix of order n. **(P.T.U., Dec. 2010)**

(c) Find rank of $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \end{bmatrix}$.

6. Find rank of the following matrices:

(i) $\begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & 1 & 2 \\ 3 & 2 & 1 & 6 \\ 2 & 4 & 2 & 4 \end{bmatrix}$ (iii) $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & 2 & 0 \end{bmatrix}$ **(P.T.U., Dec. 2012)**

(iv) $\begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$ **(P.T.U., May 2009)** (v) $\begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix}$ **(P.T.U., Dec. 2011)**

7. If A is a non-zero column and B is a non-zero row matrix, show that rank AB = 1.
8. State the conditions in terms of rank of the coefficient matrix and rank of the augmented matrix for a unique solution, no solution, infinite number of solutions of a system of linear equations. **(P.T.U., May 2005, Dec 2010)**
9. (a) For what values of λ do the equations $ax + by = \lambda x$ and $cx + dy = \lambda y$ have a solution other than $x = 0, y = 0$. [**Hint:** Consult S.E. 4(*b*) art. 4.13] **(P.T.U., May 2003)**

(b) Show that the equations $2x + 6y + 11 = 0$; $6x + 20y - 6z + 3 = 0$; $6y - 18z + 1 = 0$ are not consistent. [**Hint:** To prove $\rho(A) = 2$, $\rho(A : B) = 3$ as $\rho(A : B) \neq \rho(A)$ $\therefore$ equations are inconsistent] **(P.T.U., Dec. 2003)**

(c) For what value of K, the system of equations $x + y + z = 2$; $x + 2y + z = -2$; $x + y + (K - 5)z = K$ has no solution. **(P.T.U., May 2012)**

10. If A is a non-singular matrix then the matrix equation AX = B has a unique solution.
11. (a) Define linear dependence and linear independence of vectors and give one example of each. **(P.T.U., May 2004, May 2006, Jan. 2009)**

(b) Test whether the subset S of R^3 is L.I. or L.D., given $S = \{(1, 0, 1), (1, 1, 0), (-1, 0, -1)\}$. **(P.T.U., May 2010)**

(c) Define linear dependence of vectors and determine whether the vectors $(3, 2, 4)$, $(1, 0, 2)$, $(1, -1, -1)$ are linear dependent or not, where 't' denotes transpose. **(P.T.U., May 2006)**

[**Hint:** Consult S.E. 3 art. 4.15]

12. (a) Prove that $X_1 = (1, 1, 1)$, $X_2 = (1, -1, 1)$, $X_3 = (3, -1, 3)$ are linearly independent vectors. **(P.T.U., Dec. 2012)**

(b) Are these vectors $x_1 = (1, 2, 1)$, $x_2 = (2, 1, 4)$, $x_3 = (4, 5, 6)$, $x_4 = (1, 8, -3)$ L.D? **(P.T.U., Jan. 2010)**

(c) For what value(s) of K do the set of vectors (K, 1, 1), (0, 1, 1), (K, 0, K) in R^3 are linearly independent. **(P.T.U., May 2010, May 2012)**

13. Show that column vectors of the matrix $A = \begin{bmatrix} -2 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ are linearly dependent. **(P.T.U., May 2003)**

[**Hint:** S.E. 2 art. 4.15]

14. Define an orthogonal transformation. Derive the condition for the linear transformation on $Y = AX$ to be orthogonal. **(P.T.U., May 2012)**

[**Hint:** art 4.17]

15. (a) Define an orthogonal matrix and prove that

$$A = \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{bmatrix} \text{ is orthogonal.}$$ **(P.T.U., Jan. 2009, May 2011)**

(b) Prove that the matrix $\frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$ is orthogonal. **(P.T.U., June 2003, May 2007, Jan. 2009)**

(c) Find the values of a, b, c if the matrix $A = \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}$ is orthogonal. **(P.T.U., May. 2009)**

[**Hint:** S.E. 2 art. 4.17]

16. Prove that transpose of an orthogonal matrix is orthogonal.

17. Prove that inverse of an orthogonal matrix is orthogonal.

18. State the properties of an orthogonal matrix.

19. (a) Show that the transformation $y_1 = x_1 - x_2 + x_3$; $y_2 = 3x_1 - x_2 + 2x_3$ and $y_3 = 2x_1 - 2x_2 + 3x_3$ is non-singular (regular).

(b) Find the inverse transformation of $y_1 = x_1 + 2x_2 + 5x_3$; $y_2 = -x_2 + 2x_3$ and $y_3 = 2x_1 + 4x_2 + 11x_3$. **(P.T.U., May 2011)**

[**Hint:** S.E. 2 art. 4.18]

20. Prove that determinant of an orthogonal matrix is of modulus unity.

21. Define symmetric matrix and prove that inverse of a non-singular matrix is symmetric.

22. Define symmetric and skew symmetric matrix and express a square matrix A as the sum of a symmetric and a skew symmetric matrix.

23. Define a Hermitian matrix and prove that if A is Hermitian then $A^{\theta} A$ is also Hermitian.

[**Hint:** S.E. 3(b) art. 4.22] **(P.T.U., May 2007, Dec. 2010)**

24. (a) Define a skew Hermitian matrix and prove that if A is Hermitian then iA is skew Hermitian.

(b) Show that if $A = \begin{bmatrix} -1 & 2+i & 5-3i \\ 2-i & 7 & 5i \\ 5+3i & -5i & 2 \end{bmatrix}$, then iA is skew-Hermitian. **(P.T.U., Jan. 2010)**

25. (a) Define a unitary matrix and give one example of a unitary matrix.

(b) Show that $A = \frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$ is unitary. **(P.T.U., Jan. 2009)**

26. State properties of a unitary matrix.
27. Prove that inverse of a unitary matrix is unitary. **(P.T.U., May 2012)**
28. Prove that product of two unitary matrices of the same order is again a unitary matrix.
29. Prove that determinant of a unitary matrix is of modulus unity.
30. Define the following :
    (*i*) Characteristic equation of a square matrix
    (*ii*) Characteristic roots or latent roots or eigen values of a matrix **(P.T.U., Jan. 2009)**
    (*iii*) Eigen vectors of a square matrix. **(P.T.U., Jan. 2009)**
31. (*a*) Find eigen values of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{bmatrix}$. **(P.T.U., Dec. 2006)**

    (*b*) Prove that eigen values of a diagonal matrix are given by its diagonal elements.

    [**Hint:** Let $A = \begin{bmatrix} a_{11} & a_{22} & \ldots\ldots & a_{nn} \end{bmatrix}$; $|A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & 0 & 0 & \ldots\ldots & 0 \\ 0 & a_{22}-\lambda & 0 & \ldots\ldots & 0 \\ \vdots & \vdots & \vdots & \ldots\ldots & \vdots \\ \vdots & \vdots & \vdots & \ldots\ldots & \vdots \\ 0 & 0 & 0 & \vdots & a_{nn}-\lambda \end{vmatrix} = 0$

    i.e., $(a_{11} - \lambda)(a_{22} - \lambda) \ldots\ldots (a_{nn} - \lambda) = 0$ *i.e.*, $\lambda = a_{11}, a_{22}, \ldots\ldots, a_{nn}$]
32. Show that if $\lambda_1, \lambda_2, \lambda_3, \ldots\ldots, \lambda_n$ are the latent roots of the matrix A, then A^3 has latent roots. $\lambda_1^3, \lambda_2^3, \ldots\ldots, \lambda_n^3$.
    [**Hint:** S.E. 3 art. 4.25]
33. (*a*) Show that the eigen values of a Hermitian matrix are real.
    (*b*) Show that eigen values of a Skew Hermitian matrix are either zero or purely imaginary. **(P.T.U., Dec. 2012)**
    [**Hint:** S.E. 4 art. 4.25]
34. Prove that matrix A and its transpose A′ have the same characteristic roots.
    [**Hint:** Characteristic roots of A are $|A - \lambda I| = 0$ we have $|A - \lambda I| = |(A - \lambda I)'| = |A' - \lambda I'| = |A' - \lambda I|$
    $\therefore$ A and A′ have same eigen roots]
35. (*a*) If λ is an eigen value of a non-singular matrix A prove the following :
    (*i*) λ^{-1} is an eigen value of A^{-1} **(P.T.U., May 2005)**
    (*ii*) $\dfrac{|A|}{\lambda}$ is an eigen value of Adj. A **(P.T.U., Dec. 2003)**
    (*iii*) λ^2 is an eigen value of A^2. **(P.T.U., Dec. 2004)**
    [**Hint:** See art. 4.25]
    (*b*) Write four properties of eigen values **(P.T.U., May 2008)**
    [**Hint:** See art. 4.25]
36. (*a*) State Cayley Hamilton Theorem. **(P.T.U., Dec. 2003, Jan. 2010, May 2011)**
    (*b*) Use Cayley Hamilton Theorem to find A^8, where $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$. **(P.T.U., Dec. 2003, May 2010)**
    (*c*) If $A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$ then use Cayley Hamilton Theorem to find the matrix represented by A^5.
    [**Hint:** S.E. 6 art. 4.26]
37. Find A^{-1} from $A^2 + A + I = 0$, where $A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$.

38. Test whether the matrix $\begin{bmatrix} 3 & 1 & -1 \\ -2 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$ is diagonalisable or not? **(P.T.U., May 2012)**

[**Hint:** S.E. 1 art 4.27]

39. (*a*) Define similar matrices. **(P.T.U., May 2007)**

(*b*) Examine whether the matrix A is similar to matrix B, where A = $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and B = $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. **(P.T.U., May 2010)**

[**Hint:** See Solved Example 4(*ii*) art 4.30]

40. Prove that if A is similar to a diagonal matrix B, then A′ is similar to A.

[**Hint:** See Solved Example 6 Art. 4.30]

41. Show that rank of every matrix similar to A is same as that of A.

[**Hint:** S.E. 7 art. 4.30]

42. Show that two similar matrices have the same characteristic roots. **(P.T.U., May 2003)**

[**Hint:** Let A and B be two similar matrices $\therefore$ $A = P^{-1}BP$; $|A - \lambda I| = |P^{-1}BP - \lambda I| = |P^{-1}BP - P^{-1}\lambda P|$ $= |P^{-1}(B - \lambda I)P| = |P^{-1}||B - \lambda I||P| = |B - \lambda I||P^{-1}P| = |B - \lambda I|.1$ $\therefore$ A, B have same characteristic roots]

43. Define index and signature of the quadratic form.

44. Find a real symmetric matrix such that $Q = X'CX$ equals ; $(x_1 + x_2)^2 - x_3^2$.

45. (*a*) Express the quadratic form $x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3$ as the product of matrices.

(*b*) Obtain the symmetric matrix A for the quadratic forms

(*i*) $x_1^2 + 2x_1x_2 - 4x_1x_3 + 6x_2x_3 - 5x_2^2 + 4x_3^2$ **(P.T.U., May 2010)**

(*ii*) $3x^2 + 7y^2 - 8z^2 - 4yz + 3xz$. **(P.T.U., Dec. 2011)**

46. Write down quadratic form corresponding to the matrix $\begin{bmatrix} 2 & -1 & 4 \\ -1 & 5 & -\frac{1}{2} \\ 4 & -\frac{1}{2} & -6 \end{bmatrix}$.

47. Define orthogonal set of vectors.

[**Hint:** art. 4.33(*a*)]

48. Prove that every orthonormal set of vectors is linearly independent.

[**Hint:** art. 4.33(*b*)]

49. Let T be a transformation from R^1 to R^3 defined by $T(x) = (x, x^2, x^3)$. Is T linear or not?

[**Hint:** See S.E. 1 art. 4.16] **(P.T.U., May 2010)**

ANSWERS

1. (*b*) $A = \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}$

3. $\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

5. (*b*) n (*c*) 2

6. (*i*) 1 (*ii*) 2 (*iii*) 2 (*iv*) 2 (*v*) 2

9. (*a*) $\lambda = a, b = 0$; $\lambda = d, c = 0$, (*c*) K = 6

11. (*b*) L.D. (*c*) Not dependent

12. (*b*) yes (*c*) for all non zero values of K

15. (*c*) $a = \pm\frac{1}{\sqrt{2}}, b = \pm\frac{1}{\sqrt{6}}, c = \pm\frac{1}{\sqrt{3}}$

19. (*b*) $x_1 = 19y_1 + 2y_2 - 9y_3; x_2 = -4y_1 - y_2 + 2y_3; x_3 = -2y_1 + y_3$

31. (*a*) 1, – 4, 7

36. (*b*) 6251 (*c*) $\begin{bmatrix} 4181 & 6765 \\ 6765 & 10946 \end{bmatrix}$

37. $\begin{bmatrix} 3 & 3 \\ 3 & 6 \end{bmatrix}$

38. Diagonalisable

39. (*b*) not similar

44. $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

45. (*a*) $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

(*b*) (*i*) $\begin{bmatrix} 1 & 1 & -2 \\ 1 & -5 & 3 \\ -2 & 3 & 4 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 3 & 0 & \frac{3}{2} \\ 0 & 7 & -2 \\ \frac{3}{2} & -2 & -8 \end{bmatrix}$

46. $2x^2 + 5y^2 - 6z^2 - 2xy - yz + 8zx$

49. not linear.