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Ordinary Differential Equations of First Order

1.1. DEFINITIONS

(*i*) A differential equation is an equation involving differentials or differential coefficients. Thus

$$\frac{dy}{dx} = x^2 - 1 \quad \ldots(1) \qquad \frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + y = 0 \quad \ldots(2)$$

$$(x + y^2 - 3y)\, dx = (x^2 + 3x + y)\, dy \quad \ldots(3) \qquad y = x\frac{dy}{dx} + \frac{c}{\frac{dy}{dx}} \quad \ldots(4)$$

$$\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} \cdot \frac{dy}{dx} + x^2\left(\frac{dy}{dx}\right)^3 = 0 \quad \ldots(5) \qquad \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}} = k \cdot \frac{d^2y}{dx^2} \quad \ldots(6)$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu \quad \ldots(7) \qquad \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = x + y \quad \ldots(8)$$

are all differential equations.

(*ii*) Differential equations which involve only one independent variable and the differential coefficients with respect to it are called **ordinary differential equations.**

Thus equations (1) to (6) are all ordinary differential equations.

(*iii*) Differential equations which involve two or more independent variables and partial derivatives with respect to them are called **partial differential equations.**

Thus equations (7) and (8) are partial differential equations.

(*iv*) The **order** of a differential equation is the order of the highest order derivative occurring in the differential equation. **(P.T.U., Jan. 2009)**

Thus equations (1), (3) and (4) are of first order ; equations (2) and (6) are of the second order while equation (5) is of the third order.

(*v*) The **degree** of a differential equation is the degree of the highest order derivative which occurs in the differential equation provided the equation has been made free of the radicals and fractions as far as the derivatives are concerned. **(P.T.U., Jan. 2009)**

Thus, equations (1), (2), (3) and (5) are of the first degree.

Equation (4) is $$y\frac{dy}{dx} = x\left(\frac{dy}{dx}\right)^2 + c$$

It is of the second degree.

Equation (6) is $$\left[1 + \left(\frac{dy}{dx}\right)^2\right]^3 = k^2\left(\frac{d^2y}{dx^2}\right)^2$$

It is of the second degree.

(*vi*) **Solution of a Differential Equation.** A solution (or integral) of a differential equation is a relation, free from derivatives, between the variables which satisfies the given equation.

Thus if $y = f(x)$ be the solution, then by replacing y and its derivatives with respect to x, the given differential equation will reduce to an identity.

For example, $$y = c_1 \cos x + c_2 \sin x$$

is the solution of the differential equation $\frac{d^2y}{dx^2} + y = 0$

Since $$\frac{dy}{dx} = -c_1 \sin x + c_2 \cos x$$

$$\frac{d^2y}{dx^2} = -c_1 \cos x - c_2 \sin x = -y$$

The **general (or complete) solution** of a differential equation is that in which the number of independent arbitrary constants is equal to the order of the differential equation. **(P.T.U., Dec. 2005)**

Thus, $y = c_1 \cos x + c_2 \sin x$ (involving two arbitrary constants c_1, c_2) is the general solution of the differential equation $\frac{d^2y}{dx^2} + y = 0$ of second order.

A **particular solution** of a differential equation is that which is obtained from its general solution by giving particular values to the arbitrary constants.

For example, $y = c_1 e^x + c_2 e^{-x}$ is the general solution of the differential equation $\frac{d^2y}{dx^2} - y = 0$, whereas $y = e^x - e^{-x}$ or $y = e^x$ are its particular solutions.

The solution of a differential equation of nth order is its particular solution if it contains less than n arbitrary constants.

A singular solution of a differential equation is that solution which satisfies the equation but cannot be derived from its general solution.

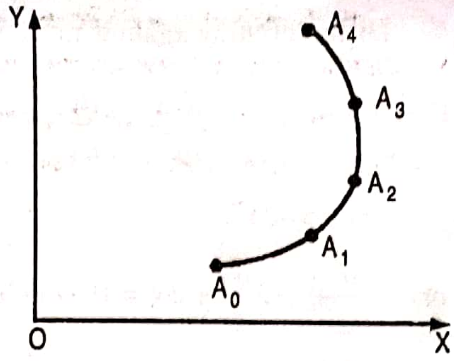
1.2. GEOMETRICAL MEANING OF A DIFFERENTIAL EQUATION OF THE FIRST ORDER AND FIRST DEGREE

Let $$f\left(x, y, \frac{dy}{dx}\right) = 0 \quad \text{...(1)}$$

be a differential equation of the first order and first degree.

We know that the direction of a curve at a particular point is determined by drawing a tangent line at that point, *i.e.*, its slope is given by $\frac{dy}{dx}$ at that particular point.

Let $A_0 (x_0, y_0)$ be any point in the plane. Let $m_0 = \frac{dy_0}{dx_0}$ be the slope of the curve at A_0 derived from (1). Take a neighbouring point $A_1 (x_1, y_1)$ such that the slope of $A_0 A_1$ is m_0. Let $m_1 = \frac{dy_1}{dx_1}$ be the slope of the curve at A_1 derived from (1). Take a neighbouring point $A_2(x_2, y_2)$ such that the slope of A_1A_2 is m_1. Continuing like this, we get a succession of points. If the points are taken sufficiently close to each other, they approximate a smooth curve $C: \ y = \phi(x)$ which is a solution of (1) corresponding to the initial point $A_0(x_0, y_0)$. Any point on C and the slope of the tangent at that point satisfy (1). If the moving point starts at any other point, not on C and moves as before, it will describe another curve. The equation of each such curve is a *particular solution* of the differential equation (1). The equation of the system of all such curves is the general solution of (1).



1.3. FORMATION OF A DIFFERENTIAL EQUATION

Differential equations are formed by elimination of arbitrary constants. To eliminate two arbitrary constants, we require two more equations besides the given relation, leading us to second order derivatives and hence a differential equation of the second order. Elimination of n arbitrary constants leads us to nth order derivatives and hence a differential equation of the nth order.

Let $$f(x, y, c_1, c_2, ..., c_n) = 0 \quad ...(1)$$

be an equation containing n arbitrary constants $c_1, c_2, ..., c_n$ (sometimes called parameters)

Differentiating (1) w.r.t. x successively n times, we get

$$\left.\begin{aligned} f_1\left(x, y, c_1, c_2, \cdots, c_n, \frac{dy}{dx}\right) &= 0 \\ f_2\left(x, y, c_1, c_2, \cdots, c_n, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) &= 0 \\ \text{and } f_n\left(x, y, c_1, c_2, \cdots, c_n, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \cdots, \frac{d^ny}{dx^n}\right) &= 0 \end{aligned}\right\} \quad ...(2)$$

Eliminating $c_1, c_2, \cdots, c_n$ from (1) and (2), we get

$$\phi\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \cdots, \frac{d^ny}{dx^n}\right) = 0$$

which is required nth order differential equation.

Hence an nth order differential equation has exactly n arbitrary constants in its general solution.

ILLUSTRATIVE EXAMPLES

$\frac{d^n y}{dx^n}$

Example 1. *Eliminate the constants from the following equations*

(*i*) $y = e^x (A \cos x + B \sin x)$ ...(1) **(P.T.U., June 2003)**

(*ii*) $y = cx + c^2$ **(P.T.U., Dec. 2003)**

(*iii*) $y = Ae^x + Be^{-x} + C$ **(P.T.U., May 2004)**

and obtain the differential equation.

Sol. (*i*) There are two arbitrary constants A and B in equation (1).

Differentiating (1) w.r.t. x, we have

$$\frac{dy}{dx} = \underbrace{e^x(A\cos x + B\sin x)}_{y} + e^x(-A\sin x + B\cos x) = y + e^x(-A\sin x + B\cos x) \quad ...(2)$$

Differentiating again w.r.t. x, we have

$$\frac{d^2y}{dx^2} = \frac{dy}{dx} + e^x(-A\sin x + B\cos x) + e^x(-A\cos x - B\sin x) = \frac{dy}{dx} + \left(\frac{dy}{dx} - y\right) - y$$

[Using (1) and (2)]

or $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$, which is the required differential equation.

(*ii*) $y = cx + c^2$ ...(1)

Equation has only one parameter 'c'

$$\frac{dy}{dx} = c \quad ...(2)$$

Eliminate c from (1) and (2), we get

$$y = x.\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2$$

or $\left(\frac{dy}{dx}\right)^2 + x\frac{dy}{dx} - y = 0$; required differential equation.

(*iii*) $y = Ae^x + Be^{-x} + C$ ...(1)

Equation has three arbitrary constants so differentiate (1) thrice

$$\frac{dy}{dx} = Ae^x - Be^{-x} \quad ...(2)$$

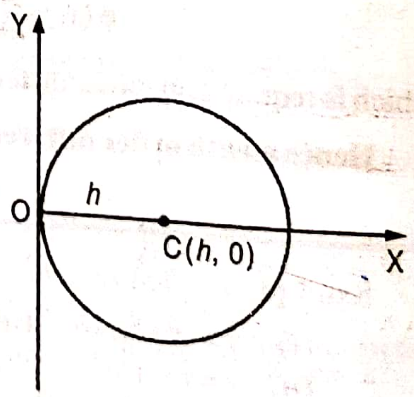
$$\frac{d^2y}{dx^2} = Ae^x + Be^{-x}$$

$$\frac{d^3y}{dx^3} = Ae^x - Be^{-x} = \frac{dy}{dx} \quad \text{[From (2)]}$$

∴ Required differential equation is

$$\frac{d^3y}{dx^3} = \frac{dy}{dx}.$$

Example 2. *Find the differential equation of all circles passing through the origin and havingcentres on the axis of x.*



Sol. The equation of such a circle is $(x-h)^2 + y^2 = h^2$

or $x^2 + y^2 - 2hx = 0$ ...(1)

where h is the only arbitrary constant.

Differentiating (1) w.r.t. x, we have $2x + 2y\frac{dy}{dx} - 2h = 0$

or $h = x + y\frac{dy}{dx}$

Substituting the value of h in (1), we have $x^2 + y^2 - 2x\left(x + y\frac{dy}{dx}\right) = 0$

or $2xy\frac{dy}{dx} + x^2 - y^2 = 0$

which is the required differential equation.

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Example 3. *Form the differential equation of all circles of radius a.*

Sol. The equation of any circle of radius a is $(x-h)^2+(y-k)^2=a^2$ ...(1)

where (h, k), the coordinates of the centre are arbitrary.

Differentiating (1) w.r.t. x, we have $2(x-h)+2(y-k)\dfrac{dy}{dx}=0$

or
$$(x-h)+(y-k)\frac{dy}{dx}=0 \qquad ...(2)$$

Differentiating again, we have $1+(y-k)\dfrac{d^2y}{dx^2}+\left(\dfrac{dy}{dx}\right)^2=0$ ...(3)

From (3),
$$y-k=-\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}$$

$1+(y-k)\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=0$

and from (2),
$$x-h=-(y-k)\frac{dy}{dx}=\frac{\frac{dy}{dx}\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}}$$

Substituting the values of $(x-h)$ and $(y-k)$ in (1), we get

$$\frac{\left(\frac{dy}{dx}\right)^2\left[1+\left(\frac{dy}{dx}\right)^2\right]^2}{\left(\frac{d^2y}{dx^2}\right)^2}+\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^2}{\left(\frac{d^2y}{dx^2}\right)^2}=a^2$$

or
$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^2\left[\left(\frac{dy}{dx}\right)^2+1\right]=a^2\left(\frac{d^2y}{dx^2}\right)^2 \quad \text{or} \quad \left[1+\left(\frac{dy}{dx}\right)^2\right]^3=a^2\left(\frac{d^2y}{dx^2}\right)^2$$

which is the required differential equation.

Example 4. *Find the differential equations of all parabolas whose axes are parallel to y-axis.*

(P.T.U., May 2002)

Sol. Equations of the parabolas whose axes are parallel to y-axis is $(x-h)^2=4a(y-k)$ ...(1)

where a, h, k are three parameters.

Differentiating (1) w.r.t. x three times, we get

$$2(x-h)=4a\frac{dy}{dx}$$

or
$$x-h=2a\frac{dy}{dx}$$

Differentiate again
$$1=2a\frac{d^2y}{dx^2}$$

Differentiate third time, we get

$$0 = 2a\frac{d^3y}{dx^3} \quad \text{or} \quad \frac{d^3y}{dx^3} = 0 \qquad (\because \ a \neq 0)$$

Hence differential equation of given parabolas is

$$\frac{d^3y}{dx^3} = 0 \text{ or } y_3 = 0.$$

TEST YOUR KNOWLEDGE

Eliminate the arbitrary constants and obtain the differential equations :

1. $y = cx + c^2$
2. $y = A + Bx + Cx^2$
3. $y = A\cos 2t + B\sin 2t$
4. $y = Ae^{3x} + Be^{2x}$
5. $y = Ae^x + Be^{-x} + C$
6. $y = ax^3 + bx^2$
7. $xy = Ae^x + Be^{-x} + x^2$
8. $x = A\cos(nt + \alpha)$
9. $y = ae^{2x} + be^{-3x} + ce^x$
10. $Ax^2 + By^2 = 1$
11. $y^2 - 2ay + x^2 = a^2$
12. $e^{2y} + 2ax\,e^y + a^2 = 0$

Find the differential equations of:

13. All straight lines in a plane. **[Hint:** Equation of the lines are $y = mx + c$]
14. All circles of radius r whose centres lie on the x-axis. **[Hint:** $(x-a)^2 + y^2 = r^2$ only a is parameter]
15. All parabolas with x-axis as the axis and $(a, 0)$ as focus. **[Hint:** $y^2 = 4ax$]
16. All conics whose axes coincide with the axes of co-ordinates. **[Hint:** $ax^2 + by^2 = 1$]
17. All circle in a plane.
18. All circles in the first quadrant which touch the co-ordinate axes **[Hint:** $(x-a)^2 + (y-a)^2 = a^2$]
19. All circles touching the axis of y at the origin and having centres on the x-axis. **[Hint:** Same as solved example 2]
20. All parabolas with latus rectum '$4a$' and axis parallel to the x-axis. **[Hint:** $(y-k)^2 = 4a(x-4)$ two parameters h and k]

ANSWERS

1. $x\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2 = y$
2. $\frac{d^3y}{dx^3} = 0$
3. $\frac{d^2y}{dt^2} + 4y = 0$
4. $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$
5. $\frac{d^3y}{dx^3} - \frac{dy}{dx} = 0$
6. $x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = 0$
7. $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + x^2 - xy - 2 = 0$
8. $\frac{d^2x}{dt^2} + n^2x = 0$
9. $\frac{d^3y}{dx^3} - 7\frac{dy}{dx} + 6y = 0$
10. $xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$
11. $\left(x^2 - 2y^2\right)\left(\frac{dy}{dx}\right)^2 - 4xy\frac{dy}{dx} - x^2 = 0$
12. $\left(1 - x^2\right)\left(\frac{dy}{dx}\right)^2 + 1 = 0$
13. $\frac{d^2y}{dx^2} = 0$
14. $y^2\left[1 + \left(\frac{dy}{dx}\right)^2\right] = r^2$
15. $y\frac{dy}{dx} = 2a$
16. $xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 = y\frac{dy}{dx}$
17. $\left[1 + \left(\frac{dy}{dx}\right)^2\right]\frac{d^3y}{dx^3} - 3\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)^2 = 0$
18. $(x-y)^2\left[1 + \left(\frac{dy}{dx}\right)^2\right] = \left(x + y\frac{dy}{dx}\right)^2$
19. $x^2 - y^2 + 2xy\frac{dy}{dx} = 0$
20. $2a\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = 0.$

1.4. SOLUTION OF DIFFERENTIAL EQUATIONS OF THE FIRST ORDER AND FIRST DEGREE

All differential equations of the first order and first degree cannot be solved. Only those among them which belong to (or can be reduced to) one of the following categories can be solved by the standard methods.

(*i*) Equations in which variables are separable.

(*ii*) Differential equation of the form $\frac{dy}{dx} = f(ax+by+c)$.

(*iii*) Homogeneous equations. (*iv*) Linear equations. (*v*) Exact equations.

1.4(a). VARIABLES SEPARABLE FORM

If a differential equation of the first order and first degree can be put in the form where dx and all terms containing x are at one place, also dy and all terms containing y are at one place, then the variables are said to be separable.

Thus the general form of such an equation is $f(x)\, dx + \phi(y)\, dy = 0$

Integrating, we get $\int f(x)\, dx + \int \phi(y)\, dy = c$ which is the general solution, c being an arbitrary constant.

Note. Any equation of the form $f_1(x)\, \phi_2(y)\, dx + f_2(x)\, \phi_1(y)\, dy = 0$ can be expressed in the above form by dividing throughout by $f_2(x)\, \phi_2(y)$.

Thus $$\frac{f_1(x)}{f_2(x)}\, dx + \frac{\phi_1(y)}{\phi_2(y)}\, dy = 0 \quad \text{or} \quad f(x)\, dx + \phi(y)\, dy = 0.$$

1.4(b). DIFFERENTIAL EQUATIONS OF THE FORM $\frac{dy}{dx} = f(ax + by + c)$

It is a differential equation of the form

$$\frac{dy}{dx} = f(ax+by+c) \qquad \text{...(1)}$$

It can be reduced to a form in which the variables are separable by the substitution $ax + by + c = t$.

so that $$a + b\frac{dy}{dx} = \frac{dt}{dx} \quad \text{or} \quad \frac{dy}{dx} = \frac{1}{b}\left(\frac{dt}{dx} - a\right)$$

$\therefore$ Equation (1) becomes $$\frac{1}{b}\left(\frac{dt}{dx} - a\right) = f(t) \quad \text{or} \quad \frac{dt}{dx} = a + b\,f(t).$$

or $$\frac{dt}{a + b\,f(t)} =/ 2$$

After integrating both sides, t is to be replaced by its value.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve :* $y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$.

Sol. The given equation can be written as $y(1 - ay) - (x + a)\frac{dy}{dx} = 0$

or
$$\frac{dx}{x+a} = \frac{dy}{y(1-ay)}$$

Integrating both sides, we have $\int \frac{dx}{x+a} = \int \left(\frac{1}{y} + \frac{a}{1-ay}\right) dy + c$ [Partial Fractions]

$\Rightarrow$
$$\log(x+a) = \left[\log y + a \cdot \frac{\log(1-ay)}{-a}\right] + c$$

$\Rightarrow$
$$\log(x+a) - \log y + \log(1-ay) = \log C, \text{ where } c = \log C$$

$\Rightarrow$
$$\log \frac{(x+a)(1-ay)}{y} = \log C \Rightarrow (x+a)(1-ay) = Cy$$

which is the general solution of the given equation.

Note. Here c is replaced by $\log C$ to get a neat form of the solution.

Example 2. *Solve : $3e^x \tan y\, dx + (1+e^x)\sec^2 y\, dy = 0$, given $y = \frac{\pi}{4}$ when $x = 0$.*

Sol. The given equation can be written as $\frac{3e^x}{1+e^x}\, dx + \frac{\sec^2 y}{\tan y}\, dy = 0$

Integrating, we have $3\log(1+e^x) + \log \tan y = \log c$

$\Rightarrow$ $\log(1+e^x)^3 \tan y = \log c$

$\Rightarrow$ $(1+e^x)^3 \tan y = c$ ...(1)

which is the general solution of the given equation.

Since $y = \frac{\pi}{4}$ when $x = 0$, we have from (1)

$(1+1)^3 \times 1 = c \quad \Rightarrow \quad c = 8$

$\therefore$ The required particular solution is $(1+e^x)^3 \tan y = 8$.

Example 3. *Solve $x \cos x \cos y + \sin y \frac{dy}{dx} = 0$.* **(P.T.U., Dec. 2002)**

Sol.
$$x\cos x \cos y + \sin y \frac{dy}{dx} = 0$$

or
$$x \cos x \cos y = -\sin y \frac{dy}{dx}$$

or
$$x\cos x\, dx = -\tan y\, dy$$

Integrating both sides,
$$\int x\cos x\, dx = -\int \tan y\, dy + c$$

$$x\sin x - \int 1 . \sin x\, dx = \log \cos y + c$$

or
$$x \sin x + \cos x = \log \cos y + c.$$

Example 4. *Solve* $xy\dfrac{dy}{dx} = 1 + x + y + xy.$ **(P.T.U., Dec. 2003)**

Sol.
$$xy\frac{dy}{dx} = (1+x) + y(1+x)$$
$$= (1+x)(1+y)$$
$$\frac{y\,dy}{1+y} = \frac{1+x}{x}dx$$

Integrating both sides,
$$\int \frac{y}{1+y}dy = \int \frac{1+x}{x}dx + c$$
$$\int\left(1 - \frac{1}{1+y}\right)dy = \int\left(\frac{1}{x} + 1\right)dx + c$$

or $$y - \log(1+y) = \log x + x + c$$

or $$x - y + \log x(1+y) = -c = c'.$$

Example 5. *Solve* $(x + y + 1)^2\dfrac{dy}{dx} = 1.$

Sol. Putting $x + y + 1 = t$, we get $\quad 1 + \dfrac{dy}{dx} = \dfrac{dt}{dx}$ or $\dfrac{dy}{dx} = \dfrac{dt}{dx} - 1$

$\therefore$ The given equation becomes $\quad t^2\left(\dfrac{dt}{dx} - 1\right) = 1$ or $\dfrac{dt}{dx} = \dfrac{1+t^2}{t^2}$

$$\Rightarrow \qquad \frac{t^2}{1+t^2}dt = dx$$

Integrating, we have $\quad \displaystyle\int\left(1 - \frac{1}{1+t^2}\right)dt = \int dx + c = dx$ or $t - \tan^{-1} t = x + c$

or $$(x + y + 1) - \tan^{-1}(x + y + 1) = x + c$$

or $$y = \tan^{-1}(x + y + 1) + C, \text{ where } C = c - 1.$$

Example 6. *Solve* $\dfrac{dy}{dx} = \sin(x + y).$ **(P.T.U., May 2006)**

Sol. Put $x + y = t$

$$\therefore \qquad 1 + \frac{dy}{dx} = \frac{dt}{dx}$$

$$\therefore \qquad \frac{dy}{dx} = \frac{dt}{dx} - 1$$

$\therefore$ Given equation changes to
$$\frac{dt}{dx} - 1 = \sin t$$
$$\frac{dt}{dx} = 1 + \sin t$$

or $$\frac{dt}{1 + \sin t} = dx$$

Integrate both sides,

$$\int \frac{dt}{1+\sin t} = \int dx + c$$

or $$\int \frac{1-\sin t}{\cos^2 t}\, dt = x + c$$

or $$\int \left(\sec^2 t - \tan t \sec t\right) dt = x + c$$

or $$\tan t - \sec t = x + c$$

or $$\sin t - 1 = (x + c)\cos t$$

Substitute back the value of t

$$\sin(x+y) - 1 = (x+c)\cos(x+y).$$

TEST YOUR KNOWLEDGE

Solve the following differential equations :

1. $\frac{dy}{dx} = e^{2x+3y}$
2. $\frac{dy}{dx} = \frac{y}{x}$ **(P.T.U., Dec. 2005)**
3. (a) $(x+y)(dx-dy) = dx+dy$ (b) $\frac{dy}{dx} = \frac{x(2\log x + 1)}{\sin y + y\cos y}$
4. $x\frac{dy}{dx} + \cot y = 0$ if $y = \frac{\pi}{4}$ when $x = \sqrt{2}$
5. $\frac{dy}{dx} + \sqrt{\frac{1-y^2}{1-x^2}} = 0$
6. $y\sqrt{1-x^2}\, dy + x\sqrt{1-y^2}\, dx = 0$
7. $\frac{y}{x}\cdot\frac{dy}{dx} = \sqrt{1+x^2+y^2+x^2y^2}$
8. $e^y\left(1+x^2\right)\frac{dy}{dx} - 2x\left(1+e^y\right) = 0$
9. $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$ **(P.T.U., Dec. 2002)**

   [Hint: $\frac{\sec^2 x}{\tan x}dx = -\frac{\sec^2 y}{\tan y}dy$ Integrate, $\log\tan x = -\log\tan y + \log c \therefore \tan x\tan y = c$]
10. $\left(x^2 - yx^2\right)\frac{dy}{dx} + y^2 + xy^2 = 0$
11. $\left(1+x^3\right)dy - x^2y\, dx = 0$, if $y = 2$ when $x = 1$
12. $a(x\,dy + y\,dx) = xy\,dy$
13. $\frac{dy}{dx} = e^{x-y} + x^2e^{-y}$
14. $\frac{dy}{dx} = (4x+y+1)^2$
15. $(x+y)^2\frac{dy}{dx} = a^2$
16. $\sin(x+y)\,dy = dx$
17. $\frac{dy}{dx} = \cos(x+y)$

    [Hint: Consult S.E. 7] **(P.T.U., June 2003)**
18. $\frac{dy}{dx} = \sin(x+y) + \cos(x+y)$
19. $\tan y\frac{dy}{dx} = \sin(x+y) + \sin(x-y)$

    [Hint: $\tan y\frac{dy}{dx} = 2\sin x\cos y$ $\int \tan y \sec y\, dy = \int 2\sin x\, dx + c$, $\sec y = -2\cos x + c$]
20. $\frac{dy}{dx} - x\tan(y-x) = 1$.

    [Hint: Put $y - x = t$]

ANSWERS

1. $3e^{2x} + 2e^{-3y} = c$
2. $y = cx$
3. (a) $x + y = c\,e^{x-y}$, (b) $y \sin y = x^2 \log x + c$
4. $x \sec y = 2$
5. $y\sqrt{1-x^2} + x\sqrt{1-y^2} = c$
6. $\sqrt{1-x^2} + \sqrt{1-y^2} = c$
7. $\sqrt{1+y^2} = \frac{2}{3}\left(1+x^2\right)^{\frac{3}{2}} + c$
8. $1 + e^y = c\left(1+x^2\right)$
9. $\tan x \tan y = c$
10. $\log\left(\frac{x}{y}\right) - \frac{1}{x} - \frac{1}{y} = c$
11. $y^3 = 4\,(x^3 + 1)$
12. $y = a \log (xy) + c$
13. $e^y = e^x + \frac{x^3}{3} + c$
14. $4x + y + 1 = 2 \tan (2x + c)$
15. $x + y = a \tan\left(\frac{y-c}{a}\right)$
16. $\tan (x + y) - \sec (x + y) = y + c$
17. $x + c = \tan\left(\frac{x+y}{2}\right)$
18. $\log\left[1 + \tan\left(\frac{x+y}{2}\right)\right] = x + c$
19. $2 \cos x + \sec y = c$
20. $\log \sin (y - x) = \frac{x^2}{2} + c$.

1.5. HOMOGENEOUS DIFFERENTIAL EQUATION AND ITS SOLUTION

A differential equation of the form $\dfrac{dy}{dx} = \dfrac{f_1(x, y)}{f_2(x, y)}$ ...(1)

is called a homogeneous differential equation if $f_1(x, y)$ and $f_2(x, y)$ are homogeneous functions of the same degree in x and y.

If $f_1(x, y)$ and $f_2(x, y)$ are homogeneous functions of degree r in x and y, then

$$f_1(x, y) = x^r\,\phi_1\left(\frac{y}{x}\right) \text{ and } f_2(x, y) = x^r\,\phi_2\left(\frac{y}{x}\right)$$

$\therefore$ Equation (1) reduces to $\dfrac{dy}{dx} = \dfrac{\phi_1\left(\frac{y}{x}\right)}{\phi_2\left(\frac{y}{x}\right)} = F\left(\dfrac{y}{x}\right)$ ...(2)

Putting $\dfrac{y}{x} = v$ *i.e.*, $y = vx$ so that $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

Equation (2) becomes $v + x\dfrac{dv}{dx} = F(v)$

Separating the variables, $\dfrac{dv}{F(v) - v} = \dfrac{dx}{x}$

Integrating, we get the solution in terms of v and x. Replacing v by $\dfrac{y}{x}$, we get the required solution.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve :* (*i*) $x\,dy - y\,dx = \sqrt{x^2+y^2}\,dx$. **(P.T.U., June 2003)**

(*ii*) $(3x^2y - y^3)dx - (2x^2y - xy^2)dy = 0.$ **(P.T.U., May 2007)**

Sol. (*i*) The given equation can be written as $\dfrac{dy}{dx} = \dfrac{y+\sqrt{x^2+y^2}}{x}$ ...(1)

The numerator and denominator on R.H.S. of (1) are homogeneous functions of degree one.

Putting $y = vx$, so that $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

Equation (1) becomes $v + x\dfrac{dv}{dx} = v + \sqrt{1+v^2}$ or $x\dfrac{dv}{dx} = \sqrt{1+v^2}$

Separating the variables, $\dfrac{dv}{\sqrt{1+v^2}} = \dfrac{dx}{x}$

Integrating both sides $\log\left(v+\sqrt{1+v^2}\right) = \log x + \log c$ $\left[\because \int \dfrac{1}{\sqrt{1+v^2}}dv = \cosh^{-1}v = \log\left(v+\sqrt{1+v^2}\right)\right]$

or $\log\left(v+\sqrt{1+v^2}\right) = \log(cx)$ or $v + \sqrt{1+v^2} = cx$

or $\dfrac{y}{x} + \sqrt{1+\dfrac{y^2}{x^2}} = cx$ $\left[\because v = \dfrac{y}{x}\right]$

or $y + \sqrt{x^2+y^2} = cx^2$

which is the required solution.

(*ii*) $$\frac{dy}{dx} = \frac{3xy^2 - y^3}{2x^2y - xy^2} = \frac{3\dfrac{y^2}{x^2} - \dfrac{y^3}{x^3}}{2\dfrac{y}{x} - \dfrac{y^2}{x^2}}$$

Put $\dfrac{y}{x} = v$ $\quad\therefore\quad y = vx;\ \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$$\therefore\quad v + x\frac{dv}{dx} = \frac{3v^2 - v^3}{2v - v^2} = \frac{3v - v^2}{2 - v}$$

$$\therefore\quad x\frac{dv}{dx} = \frac{3v - v^2}{2 - v} - v = \frac{3v - v^2 - 2v + v^2}{2 - v} = \frac{v}{2 - v}$$

$$\therefore\quad \frac{2 - v}{v}dv = \frac{1}{x}dx$$

Integrate both sides :

$$\int\left(\frac{2}{v} - 1\right)dv = \int \frac{1}{x}dx + c$$

or $$2\log|v| - v = \log|x| + c$$

or $$2\log|v| - \log|x| = v + c$$

or $$\log v^2 - \log x = v + c$$

or $$\log\frac{v^2}{x} = v + c \quad \text{or} \quad \log\frac{y^2}{x^2 . x} = \frac{y}{x} + c$$

or $$\frac{y^2}{x^3} = e^{\frac{y}{x}+c} = e^c . e^{\frac{y}{x}} = Ae^{\frac{y}{x}}, \text{ where } A = e^c$$

$$\therefore \quad y^2 = Ax^3\, e^{\frac{y}{x}}.$$

Example 2. *Solve :* $\left(x\tan\frac{y}{x} - y\sec^2\frac{y}{x}\right)dx + x\sec^2\frac{y}{x}dy = 0.$ **(P.T.U., Dec. 2003)**

Sol. The given equation can be written as $\frac{dy}{dx} = \frac{y\sec^2\frac{y}{x} - x\tan\frac{y}{x}}{x\sec^2\frac{y}{x}} = \frac{y}{x} - \frac{\tan\frac{y}{x}}{\sec^2\frac{y}{x}}$ ...(1)

Putting $\frac{y}{x} = v$ *i.e.*, $y = vx$ so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Equation (1) becomes $v + x\frac{dv}{dx} = v - \frac{\tan v}{\sec^2 v}$ or $\frac{\sec^2 v}{\tan v}dv + \frac{dx}{x} = 0$

Integrating, we get $\log\tan v + \log x = \log c$

or $$\log(x\tan v) = \log c \text{ or } x\tan v = c$$

or $$x\tan\frac{y}{x} = c \qquad \left[\because v = \frac{y}{x}\right]$$

which is the required solution.

Example 3. *Find the general solution of the differential equation* $(2xy + x^2)\,y' = 3y^2 + 2xy.$ **(P.T.U., May 2006)**

Sol. Given equation is $(2xy + x^2)\,y' = 3y^2 + 2xy$

or $$\frac{dy}{dx} = \frac{3y^2 + 2xy}{2xy + x^2} = \frac{3\left(\frac{y}{x}\right)^2 + 2\left(\frac{y}{x}\right)}{2.\frac{y}{x} + 1}$$, which is homogeneous equation of order 2.

Put $\frac{y}{x} = v \quad \therefore \quad y = vx \quad \therefore \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\therefore \quad v + x\frac{dv}{dx} = \frac{3v^2 + 2v}{2v+1} \quad \text{or} \quad x\frac{dv}{dx} = \frac{3v^2 + 2v}{2v+1} - v$$

or $$x\frac{dv}{dx} = \frac{v^2 + v}{2v+1} \quad \text{or} \quad \frac{2v+1}{v(v+1)}dv = \frac{1}{x}dx$$

Integrate both sides,

$$\int \frac{2v+1}{v(v+1)}\,dv = \int \frac{1}{x}\,dx + \log c$$

or $$\int \left(\frac{1}{v} + \frac{1}{v+1}\right) dv = \log x + \log c \qquad \text{(By partial fractions)}$$

or $$\log v + \log (v+1) - \log x = \log c \quad \text{or} \quad \log \frac{v(v+1)}{x} = \log c$$

or $$v(v+1) = cx \text{ or } \frac{y}{x}\left(\frac{y}{x}+1\right) = cx$$

or $$y(y+x) = cx^3.$$

Example 4. *Solve* $(1 + e^{x/y})\,dx + e^{x/y}\left(1 - \frac{x}{y}\right) dy = 0.$ **(P.T.U., Dec. 2006)**

Sol. $(1 + e^{x/y})\,dx + e^{x/y}\left(1 - \frac{x}{y}\right) dy = 0$

or $$\frac{dx}{dy} = -\frac{e^{x/y}\left(1 - \frac{x}{y}\right)}{1 + e^{x/y}}$$, which is homogeneous equation in $\frac{x}{y}$ form

$\therefore$ Put $$\frac{x}{y} = v \quad i.e., x = vy \quad \therefore \quad \frac{dx}{dy} = v + y\frac{dv}{dy}$$

$$v + y\frac{dv}{dy} = -\frac{e^v(1-v)}{1+e^v} \quad \text{or} \quad y\frac{dv}{dy} = -\frac{e^v(1-v)}{1+e^v} - v$$

or $$y\frac{dv}{dy} = \frac{-(e^v + v)}{1+e^v} \quad \text{or} \quad \frac{1+e^v}{v+e^v}\,dv = -\frac{1}{y}\,dy$$

Integrate both sides,

$$\log (v + e^v) = -\log y + \log c$$

$\therefore$ $$\log (v + e^v)\,y = \log c \qquad \because \; y(v + e^v) = c$$

$\therefore$ $$y\left(\frac{x}{y} + e^{x/y}\right) = c \text{ or } x + ye^{x/y} = c.$$

TEST YOUR KNOWLEDGE

Solve the following differential equations :

1. $(x + y)\,dx + (y - x)\,dy = 0$ **(P.T.U., May 2004, May 2011)**
2. $x\frac{dy}{dx} + \frac{y^2}{x} = 2xy$
3. $(x^2 - y^2)\,dx = 2xy dy$
4. $(y^2 - 2xy)\,dx = (x^2 - 2xy)\,dy$
5. $x(x - y)\frac{dy}{dx} = y(x + y)$
6. $(\sqrt{xy} - x)\,dy + y\,dx = 0$
7. $(y^2 + 2xy)\,dx + (2x^2 + 3xy)\,dy = 0$
8. $x^2y\,dx - (x^3 + y^3)\,dy = 0$
9. $(x^2 + 2y^2)\,dx - xy\,dy = 0$, given that $y = 0$ when $x = 1$
10. $x\frac{dy}{dx} = y(\log y - \log x)$

11. $x\dfrac{dy}{dx} = y + x\cos^2\dfrac{y}{x}$ 12. $\dfrac{dy}{dx} = \dfrac{y}{x} + \sin\dfrac{y}{x}$

13. $\left(x\cos\dfrac{y}{x} + y\sin\dfrac{y}{x}\right)y - \left(y\sin\dfrac{y}{x} - x\cos\dfrac{y}{x}\right)x\dfrac{dy}{dx} = 0$ 14. $(1 + e^{x/y})\,dx + e^{x/y}\left(1 - \dfrac{x}{y}\right)dy = 0$

15. $ye^{x/y}\,dx = (xe^{x/y} + y)\,dy$ 16. $xy\log\left(\dfrac{x}{y}\right)dx + \left[y^2 - x^2\log\left(\dfrac{x}{y}\right)\right]dy = 0.$

ANSWERS

1. $\log(x^2 + y^2) = 2\tan^{-1}\dfrac{y}{x} + c$ 2. $cx = e^{x/y}$ 3. $x(x^2 - 3y^2) = c$

4. $xy(x - y) = c$ 5. $\dfrac{x}{y} + \log\dfrac{x}{y} = c$ 6. $2\sqrt{\dfrac{x}{y}} + \log y = c$ 7. $xy^2(x + y) = c$

8. $y = ce^{\frac{x^2}{3y^3}}$ 9. $x^2 + y^2 = cx^4$ 10. $y = xe^{1+cx}$

11. $\tan\dfrac{y}{x} = \log(cx)$ 12. $y = 2x\tan^{-1}(cx)$ 13. $xy\cos\dfrac{y}{x} = c$

14. $x + ye^{x/y} = c$ 15. $e^{x/y} = \log y + c$ 16. $\log y - \dfrac{x^2}{4y^2}\left(2\log\dfrac{y}{x} + 1\right) = c.$

1.6. EQUATIONS REDUCIBLE TO HOMOGENEOUS FORM

A differential equation of the form $\dfrac{dy}{dx} = \dfrac{ax + by + c}{a'x + b'y + c'}$ ...(1)

can be reduced to the homogeneous from as follows :

Case I. When $\dfrac{\mathbf{a}}{\mathbf{a'}} \neq \dfrac{\mathbf{b}}{\mathbf{b'}}$ **(P.T.U., Dec. 2002)**

Putting $x = X + h,\ y = Y + k$ (h, k are constants)

so that $dx = dX,\ dy = dY$

Equation (1) becomes
$$\frac{dY}{dX} = \frac{a(X + h) + b(Y + k) + c}{a'(X + h) + b'(Y + k) + c'}$$
$$= \frac{aX + bY + (ah + bk + c)}{a'X + b'Y + (a'h + b'k + c')} \qquad ...(2)$$

Choose h and k such that (2) becomes homogeneous.

This requires $ah + bk + c = 0$ and $a'h + b'k + c' = 0$

so that $\dfrac{h}{bc' - b'c} = \dfrac{k}{ca' - c'a} = \dfrac{1}{ab' - a'b}$ or $h = \dfrac{bc' - b'c}{ab' - a'b}, k = \dfrac{ca' - c'a}{ab' - a'b}$

Since $\dfrac{a}{a'} \neq \dfrac{b}{b'}$ $\therefore$ $ab' - a'b \neq 0$ so that h, k are finite.

$\therefore$ Equation (2) becomes $\dfrac{dY}{dX} = \dfrac{aX + bY}{a'X + b'Y}$

which is homogeneous in X, Y and can be solved by putting $Y = vX$.

Case II. When $\dfrac{a}{a'} = \dfrac{b}{b'}, ab' - a'b = 0$ **and the above method fails**

Now, $\dfrac{a}{a'} = \dfrac{b}{b'} = \dfrac{1}{m}$ (say) so that $a' = ma,\ b' = mb$

Equation (1) becomes $\dfrac{dy}{dx} = \dfrac{(ax+by)+c}{m(ax+by)+c'} = f(ax+by)$

which can be solved by putting $ax + by = t$.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve : $(3y - 7x + 7)dx + (7y - 3x + 3)\,dy = 0$.*

Sol. The given equation can be written as $\dfrac{dy}{dx} = -\dfrac{3y-7x+7}{7y-3x+3}$ $\left[\text{Here } \dfrac{a}{a'} \neq \dfrac{b}{b'}\right]$ ...(1)

Putting $x = X + h,\ y = Y + k$ so that $dx = dX,\ dy = dY$ (h, k are constants)

Equation (1) becomes $\dfrac{dY}{dX} = -\dfrac{3(Y+k)-7(X+h)+7}{7(Y+k)-3(X+h)+3} = -\dfrac{3Y-7X+(-7h+3k+7)}{7Y-3X+(-3h+7k+3)}$ ...(2)

Now, choosing, h, k such that $-7h + 3k + 7 = 0$ and $-3h + 7k + 3 = 0$

Solving these equations $h = 1,\ k = 0$.

With these values of h, k equation (2) reduces to $\dfrac{dY}{dX} = -\dfrac{3Y-7X}{7Y-3X}$ ...(3)

Putting $Y = vX$ so that $\dfrac{dY}{dX} = v + X\dfrac{dv}{dX}$

Equation (3) becomes $v + X\dfrac{dv}{dX} = -\dfrac{3vX-7X}{7vX-3X}$ or $X\dfrac{dv}{dX} = \dfrac{7-3v}{7v-3} - v = \dfrac{7-7v^2}{7v-3}$

Separating the variables $\dfrac{7v-3}{1-v^2}\,dv = 7\dfrac{dX}{X}$ or $\left(\dfrac{2}{1-v} - \dfrac{5}{1+v}\right)dv = 7\dfrac{dX}{X}$

Integrating $-2\log(1-v) - 5\log(1+v) = 7\log X + c$

or $7\log X + 2\log(1-v) + 5\log(1+v) = -c$

or $\log[X^7(1-v)^2(1+v)^5] = -c$ or $X^7\left(1-\dfrac{Y}{X}\right)^2\left(1+\dfrac{Y}{X}\right)^5 = e^{-c}$

or $(X-Y)^2(X+Y)^5 = C$, where $C = e^{-c}$ ...(4)

Putting $X = x - h = x - 1,\ Y = y - k = y$

Equation (4) becomes $(x-y-1)^2(x+y-1)^5 = C$, which is the required solution.

Example 2. *Solve : $(3y + 2x + 4)\,dx - (4x + 6y + 5)\,dy = 0$.* **(P.T.U., May 2011)**

Sol. The given equation can be written as $\dfrac{dy}{dx} = \dfrac{(2x+3y)+4}{2(2x+3y)+5}$ ...(1)

Here, $\dfrac{a}{a'} = \dfrac{b}{b'}$

Putting $2x + 3y = t$ so that $2 + 3\dfrac{dy}{dx} = \dfrac{dt}{dx}$

or $\dfrac{dy}{dx} = \dfrac{1}{3}\left(\dfrac{dt}{dx} - 2\right)$

Equation (1) becomes $\frac{1}{3}\left(\frac{dt}{dx}-2\right)=\frac{t+4}{2t+5}$

or $$\frac{dt}{dx}=\frac{3t+12}{2t+5}+2=\frac{7t+22}{2t+5}$$

Separating the variables $\frac{2t+5}{7t+22}dt=dx$ or $\left(\frac{2}{7}-\frac{9}{7}\cdot\frac{1}{7t+22}\right)dt=dx$

Integrating both sides $\frac{2}{7}t-\frac{9}{49}\log(7t+22)=x+c$

or $$14t-9\log(7t+22)=49x+49c$$

Putting $t=2x+3y$, we have $14(2x+3y)-9\log(14x+21y+22)=49x+49c$

or $$21x-42y+9\log(14x+21y+22)=-49c$$

or $$7(x-2y)+3\log(14x+21y+22)=C \qquad \left(\text{where } C=-\frac{49}{3}c\right)$$

which is the required solution.

TEST YOUR KNOWLEDGE

Solve the following differential equations:

1. $\frac{dy}{dx}=\frac{y+x-2}{y-x-4}$

2. $\frac{dy}{dx}=\frac{x+2y-3}{2x+y-3}$

3. $\frac{dy}{dx}+\frac{2x+3y+1}{3x+4y-1}=0$

4. $(x+2y+3)\,dx-(2x-y+1)\,dy=0$

5. $(2x-2y+5)\frac{dy}{dx}=x-y+3$

6. $\frac{dy}{dx}=\frac{6x-4y+3}{3x-2y+1}$

7. $(2x+y+1)\,dx+(4x+2y-1)\,dy=0$

8. $(x+y)(dx-dy)=dx+dy.$

ANSWERS

1. $x^2+2xy-y^2-4x+8y=C$

2. $(x-y)^3=C(x+y-2)$

3. $x^2+3xy+2y^2+x-y=C$

4. $\log[(x+1)^2+(y+1)^2]=4\tan^{-1}\frac{y+1}{x+1}+C$

5. $x-2y+\log(x-y+2)=C$

6. $2x-y=\log(3x-2y+3)+C$

7. $x+2y+\log(2x+y-1)=C$

8. $x-y=\log(x+y)+C.$

1.7. EXACT DIFFERENTIAL EQUATIONS (P.T.U., Jan. 2009)

Definition : A differential equation obtained from its primitive directly by differentiation, without any operation of multiplication, elimination or reduction etc. is said to be **an exact differential equation.**

Thus a differential equation of the form $M(x,y)\,dx+N(x,y)\,dy=0$ is an exact differential equation if it can be obtained directly by differentiating the equation $u(x,y)=C$ which is its primitive *i.e.*, if $du=M\,dx+N\,dy$.

1.8. THEOREM

The necessary and sufficient condition for the differential equation $Mdx + Ndy = 0$ to be exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. **(P.T.U., 2002, Jan. 2010)**

Proof. The condition is necessary

The equation $Mdx + Ndy = 0$ will be exact, if $du = Mdx + Ndy$

But $$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

$\therefore$ $$Mdx + Ndy = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

Equating coefficients of dx and dy, we get

$$M = \frac{\partial u}{\partial x} \quad \text{and} \quad N = \frac{\partial u}{\partial y}$$

$\therefore$ $$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

But $$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

$\therefore$ $$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

which is the necessary condition of exactness.

The condition is sufficient.

Let $$u = \int_{y\,\text{constant}} M\,dx$$

$\therefore$ $$\frac{\partial u}{\partial x} = M \quad \text{and} \quad \frac{\partial^2 u}{\partial y\,\partial x} = \frac{\partial M}{\partial y}$$

But $$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y} \quad \text{and} \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ $$\frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$$

Integrating both sides w.r.t. x treating y as constant, we have $N = \frac{\partial u}{\partial y} + f(y)$

$\therefore$ $$Mdx + Ndy = \frac{\partial u}{\partial x} dx + \left\{\frac{\partial u}{\partial y} + f(y)\right\} dy \qquad \left[\because M = \frac{\partial u}{\partial x},\ N = \frac{\partial u}{\partial y} + f(y)\right]$$

$$= \left(\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy\right) + f(y)\,dy = du + f(y)\,dy = d\left[u + \int f(y)\,dy\right]$$

which shows that $Mdx + Ndy$ is an exact differential and hence $Mdx + Ndy = 0$ is an exact differential equation.

Note. Since $\quad M dx + N dy = d\,[u + \int f(y)\,dy]$

$\therefore \quad M dx + N dy = 0 \Rightarrow d\,[u + \int f(y)\,dy] = 0$

Integrating $\quad u + \int f(y)\,dy = c$

But $\quad u = \int_{y\,\text{constant}} M dx$ and $f(y) =$ terms of N not containing x

Hence the solution of Mdx + Ndy = 0 is $\displaystyle\int_{y\,\text{constant}} \mathbf{M}\,dx + \int (\textbf{terms of N not containing } x)\,dy = \mathbf{c}.$

ILLUSTRATIVE EXAMPLES

Example 1. *Solve :* $(5x^4 + 3x^2y^2 - 2xy^3)\,dx + (2x^3y - 3x^2y^2 - 5y^4)\,dy = 0.$

Sol. Here $M = 5x^4 + 3x^2y^2 - 2xy^3$ and $N = 2x^3y - 3x^2y^2 - 5y^4$

$$\therefore \quad \frac{\partial M}{\partial y} = 6x^2y - 6xy^2, \frac{\partial N}{\partial x} = 6x^2y - 6xy^2 \quad \therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Thus the given equation is exact and its solution is

$$\int_{y\,\text{constant}} M dx + \int (\text{terms of N not containing } x)\,dy = c$$

i.e., $$\int_{y\,\text{constant}} (5x^4 + 3x^2y^2 - 2xy^3)\,dx + \int -5y^4\,dy = c$$

or $$x^5 + x^3y^2 - x^2y^3 - y^5 = c$$

Example 2. *Solve the initial value problem* $e^x(\cos y\,dx - \sin y\,dy) = 0\,;\ y(0) = 0.$ **(P.T.U., May 2008)**

Sol. $\quad e^x \cos y\,dx - e^x \sin y\,dy = 0$

Compare it with $M\,dx + N\,dy = 0$

$$M = e^x \cos y,\ N = -e^x \sin y$$

$$\frac{\partial M}{\partial y} = -e^x \sin y\,;\ \frac{\partial N}{\partial x} = -e^x \sin y$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ Given equation is exact and its solution is $\displaystyle\int_{y\,\text{constant}} M\,dx + \int (\text{terms of N not containing } x)\,dy = c$

or $$\int_{y\,\text{constant}} e^x \cos y\,dx + \int 0 \cdot dy = c$$

or $$\cos y \cdot e^x = c \qquad \ldots(1)$$

Given $\quad y(0) = 0$, *i.e.*, $y = 0$ when $x = 0$

Substituting in (1), we get $\quad 1 = c$

$\therefore$ Solution of the given equation is

$$e^x \cos y = 1$$

Example 3. *Solve :* $\quad (x^2 - ay)\,dx = (ax - y^2)\,dy.$ **(P.T.U., 2005)**

Sol. $\quad (x^2 - ay)\,dx - (ax - y^2)\,dy = 0 \qquad \ldots(1)$

Compare it with $\quad M dx + N dy = 0$

$$M = x^2 - ay, \quad N = -ax + y^2$$

$$\frac{\partial M}{\partial y} = -a, \quad \frac{\partial N}{\partial x} = -a$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ (1) is exact differential equation. Therefore, its solution is

$$\int_{y} M\,dx + \int (\text{terms of N not containing } x)\, dy = c$$

or $$\int_{y} (x^2 - ay)\, dx + \int y^2\, dy = c$$

or $$\frac{x^3}{3} - ayx + \frac{y^3}{3} = c$$

or $x^3 + y^3 - 3axy = c'$, where $c' = 3c$.

Example 4. *Solve : $(\sec x \tan x \tan y - e^x)\, dx + \sec x \sec^2 y\, dy = 0$.* **(P.T.U., Dec. 2005)**

Sol. $(\sec x \tan x \tan y - e^x)\, dx + \sec x \sec^2 y\, dy = 0$ ...(1)

Compare it with $M\,dx + N\,dy = 0$

$$M = \sec x \tan x \tan y - e^x \text{ and } N = \sec x \sec^2 y$$

$\therefore$ $$\frac{\partial M}{\partial y} = \sec x \tan x \sec^2 y$$

$$\frac{\partial N}{\partial x} = \sec x \tan x \sec^2 y$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ Equation (1) is exact.

$\therefore$ Its solution is

$$\int_{y\text{ constant}} (\sec x \tan x \tan y - e^x)\, dx + \int (\text{terms of N not containing } x)\, dy = c$$

$$\tan y \int (\sec x \tan x - e^x)\, dx + \int 0\, dy = c$$

$\therefore$ $$\tan y\,(\sec x) - e^x = c.$$

Example 5. *For what value of k, the differential equation $\left(1 + e^{\frac{kx}{y}}\right) dx + e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right) dy = 0$ is exact?* **(P.T.U., May 2010)**

Sol. Given differential equation is

$$\left(1 + e^{\frac{kx}{y}}\right) dx + e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right) dy = 0 \quad ...(1)$$

Compare it with $\quad M dx + N dy = 0$

$$M = 1 + e^{\frac{kx}{y}} \quad N = e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)$$

(1) will be exact if $\quad \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

i.e.,
$$\frac{\partial}{\partial y}\left(1 + e^{\frac{kx}{y}}\right) = \frac{\partial}{\partial x}\left\{e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)\right\}$$

or
$$e^{\frac{kx}{y}}\left(\frac{-kx}{y^2}\right) = e^{\frac{x}{y}}\frac{1}{y}\left(1 - \frac{x}{y}\right) + e^{\frac{x}{y}}\left(-\frac{1}{y}\right) = e^{\frac{x}{y}}\left(\frac{1}{y} - \frac{x}{y^2} - \frac{1}{y}\right) = e^{\frac{x}{y}}\left(-\frac{x}{y^2}\right)$$

or
$$k e^{\frac{kx}{y}} = e^{\frac{x}{y}}, \text{ which holds when } k = 1$$

$\therefore$ For $k = 1$, (1) is an exact differential equation.

TEST YOUR KNOWLEDGE

Solve the following differential equations :

1. $(x^2 - 4xy - 2y^2)\, dx + (y^2 - 4xy - 2x^2)\, dy = 0$
2. $(3x^2 + 6xy^2)\, dx + (6x^2 y + 4y^3)\, dy = 0$ **(P.T.U., May 2009)**
3. $(x^2 + y^2 - a^2)\, x\, dx + (x^2 - y^2 - b^2)\, y\, dy = 0$
4. $\left(1 + e^{\frac{x}{y}}\right) dx + \left(1 - \frac{x}{y}\right) e^{\frac{x}{y}}\, dy = 0$
5. $(y \cos x + 1)\, dx + \sin x\, dy = 0$
6. $(\sec x \tan x \tan y - e^x)\, dx + \sec x \sec^2 y\, dy = 0$
7. $ye^{xy}\, dx + (xe^{xy} + 2y)\, dy = 0$
8. $\left(y^2 e^{xy^2} + 4x^3\right) dx + \left(2xye^{xy^2} - 3y^2\right) dy = 0$
9. $(2xy \cos x^2 - 2xy + 1)\, dx + (\sin x^2 - x^2)\, dy = 0$
10. $(\sin x \cos y + e^{2x})\, dx + (\cos x \sin y + \tan y)\, dy = 0$
11. $\left[y\left(1 + \frac{1}{x}\right) + \cos y\right] dx + (x + \log x - x \sin y)\, dy = 0$
12. $\dfrac{dy}{dx} + \dfrac{y \cos x + \sin y + y}{\sin x + x \cos y + x} = 0$ **(P.T.U., May 2011)**
13. $x\,dy + y\,dx + \dfrac{x\,dy - y\,dx}{x^2 + y^2} = 0$
14. $[\cos x \tan y + \cos (x + y)]\, dx + [\sin x \sec^2 y + \cos (x + y)]\, dy = 0.$

ANSWERS

1. $x^3 - 6x^2 y - 6xy^2 + y^3 = c$
2. $x^3 + 3x^2 y^2 + y^4 = c$
3. $x^4 + 2x^2 y^2 - 2a^2 x^2 - y^4 - 2b^2 y^2 = c$
4. $x + ye^{x/y} = c$
5. $y \sin x + x = c$
6. $\sec x \tan y - e^x = c$

7. $e^{xy}+y^2=c$

8. $e^{xy^2}+x^4-y^3=c$

9. $y\sin x^2-x^2y+x=c$

10. $-\cos x\cos y+\frac{1}{2}e^{2x}+\log\sec y=c$

11. $y(x+\log x)+x\cos y=c$

12. $y\sin x+(\sin y+y)x=c$

13. $xy+\tan^{-1}\frac{y}{x}=c$ $\left[\textbf{Hint:} \text{ The given equation is } d(xy)+d\left(\tan^{-1}\frac{y}{x}\right)=0.\right]$

14. $\sin x\tan y+\sin(x+y)=c$

1.9. EQUATIONS REDUCIBLE TO EXACT EQUATIONS

Integrating factor

Differential equations which are not exact can sometimes be made exact after multiplying by a suitable factor (a function of x or y or both) called the **integrating factor**.

For example, consider the equation $y\,dx-x\,dy=0$ ...(1)

Here, $M=y$ and $N=-x$

$\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, therefore the equation is not exact.

(*i*) Multiplying the equation by $\frac{1}{y^2}$, it becomes $\frac{y\,dx-x\,dy}{y^2}=0$ or $d\left(\frac{x}{y}\right)=0$ which is exact.

(*ii*) Multiplying the equation by $\frac{1}{x^2}$, it becomes $\frac{y\,dx-x\,dy}{x^2}=0$ or $d\left(\frac{y}{x}\right)=0$ which is exact.

(*iii*) Multiplying the equation by $\frac{1}{xy}$, it becomes $\frac{dx}{x}-\frac{dy}{y}=0$ or $d(\log x-\log y)=0$ which is exact.

$\therefore$ $\frac{1}{y^2},\frac{1}{x^2}$ and $\frac{1}{xy}$ are integrating factors of (1).

Note. *If a differential equation has one integrating factor, it has an infinite number of integrating factors.*

1.9(a). I.F. FOUND BY INSPECTION

In a number of problems, a little analysis helps to find the integrating factor. The following differentials are useful in selecting a suitable integrating factor.

(*i*) $y\,dx+x\,dy=d(xy)$

(*ii*) $\frac{x\,dy-y\,dx}{x^2}=d\left(\frac{y}{x}\right)$

(*iii*) $\frac{y\,dx-x\,dy}{y^2}=d\left(\frac{x}{y}\right)$

(*iv*) $\frac{x\,dy-y\,dx}{x^2+y^2}=d\left(\tan^{-1}\frac{y}{x}\right)$

(*v*) $\frac{x\,dy-y\,dx}{xy}=d\left[\log\left(\frac{y}{x}\right)\right]$

(*vi*) $\frac{y\,dx+x\,dy}{xy}=d[\log(xy)]$

(*vii*) $\frac{x\,dy+y\,dy}{x^2+y^2}=d\left[\frac{1}{2}\log\left(x^2+y^2\right)\right]$

(*viii*) $\frac{x\,dy-y\,dx}{x^2-y^2}=d\left(\frac{1}{2}\log\frac{x+y}{x-y}\right)$

ILLUSTRATIVE EXAMPLES

Example 1. *Solve* $y\,dx - x\,dy + 3x^2y^2e^{x^3}\,dx = 0.$ **(P.T.U., Dec. 2003)**

Sol. Since $3x^2e^{x^3} = d\left(e^{x^3}\right)$, the term $3x^2y^2e^{x^3}\,dx$ should not involve y^2.

This suggests that $\dfrac{1}{y^2}$ may be an I.F.

Multiplying throughout by $\dfrac{1}{y^2}$, we have $\dfrac{y\,dx - x\,dy}{y^2} + 3x^2e^{x^3}\,dx = 0$

$$d\left(\frac{x}{y}\right) + d\left(e^{x^3}\right) = 0, \text{ which is exact.}$$

Integrating, we get $\dfrac{x}{y} + e^{x^3} = c$, which is the required solution.

Example 2. *Find the integrating factor of the differential equation* $(y-1)\,dx - x\,dy = 0$ *and hence solve it.* **(P.T.U., May 2006)**

Sol. Given equation is

$$(y-1)\,dx - x\,dy = 0 \quad \text{...(1)}$$

Compare it with $M\,dx + N\,dy = 0$

$$M = y - 1,\ N = -x$$

$$\frac{\partial M}{\partial y} = 1,\ \frac{\partial N}{\partial x} = -1;\ \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

∴ Equation (1) is not exact.

∴ Write (1) as $y\,dx - x\,dy = dx$

Multiply it by $\dfrac{1}{x^2}$, we get

$$\frac{y\,dx - x\,dy}{x^2} = \frac{1}{x^2}\,dx$$

or

$$d\left(-\frac{y}{x}\right) = d\left(-\frac{1}{x}\right) \quad \text{...(2)}$$

∴ Factor $\dfrac{1}{x^2}$ makes the equation (1) exact differential equation

∴ $\dfrac{1}{x^2}$ is the I.F.

Integrate (2);

$$\frac{-y}{x} = -\frac{1}{x} + c$$

or $y = 1 - cx$ is the required solution.

Example 3. *Solve* : $x\,dy - y\,dx = x\sqrt{x^2 - y^2}\,dx$. **(P.T.U., Jan. 2009)**

Sol. The given equation is $x dy - y\,dx = x^2\sqrt{1-\left(\frac{y}{x}\right)^2}\,dx$ or $\dfrac{\frac{x\,dy - y\,dx}{x^2}}{\sqrt{1-\left(\frac{y}{x}\right)^2}} = dx$.

or $\quad d\left(\sin^{-1}\dfrac{y}{x}\right) = dx$, which is exact.

Integrating, we get $\sin^{-1}\dfrac{y}{x} = x + c$ or $y = x\sin(x+c)$, which is the required solution.

1.9(b). I.F. FOR A HOMOGENEOUS EQUATION

If $Mdx + Ndy = 0$ is a homogeneous equation in x and y, then $\dfrac{1}{Mx+Ny}$ is an I.F. provided $Mx + Ny \neq 0$.

Proof. If $\dfrac{1}{Mx+Ny}$ is an integrating factor of $Mdx + Ndy = 0$ ...(1)

Then $\dfrac{M\,dx}{Mx+Ny} + \dfrac{Ndy}{Mx+Ny} = 0$ is an exact equation.

$$\therefore \quad \frac{\partial}{\partial y}\left[\frac{M}{Mx+Ny}\right] = \frac{\partial}{\partial x}\left[\frac{N}{Mx+Ny}\right]$$

$$\frac{(Mx+Ny)\dfrac{\partial M}{\partial y} - M\left[x\dfrac{\partial M}{\partial y} + N + y\dfrac{\partial N}{\partial y}\right]}{(Mx+Ny)^2} - \frac{(Mx+Ny)\dfrac{\partial N}{\partial x} - N\left[M + x\dfrac{\partial M}{\partial x} + y\dfrac{\partial N}{\partial x}\right]}{(Mx+Ny)^2} = 0$$

or $$Mx\frac{\partial M}{\partial y} + Ny\frac{\partial M}{\partial y} - Mx\frac{\partial M}{\partial y} - MN - My\frac{\partial N}{\partial y} - Mx\frac{\partial N}{\partial x} - Ny\frac{\partial N}{\partial x} + NM + Nx\frac{\partial M}{\partial x} + Ny\frac{\partial N}{\partial x} = 0$$

or $$Ny\frac{\partial M}{\partial y} - My\frac{\partial N}{\partial y} - Mx\frac{\partial N}{\partial x} + Nx\frac{\partial M}{\partial x} = 0$$

or $$N\left\{x\frac{\partial M}{\partial x} + y\frac{\partial M}{\partial y}\right\} - M\left\{x\frac{\partial N}{\partial x} + y\frac{\partial N}{\partial y}\right\} = 0 \qquad \text{...(2)}$$

Now $\because$ M, N are homogeneous functions of x and y of order n therefore, we have

$x\dfrac{\partial M}{\partial x} + y\dfrac{\partial M}{\partial y} = nM$ and $x\dfrac{\partial N}{\partial x} + y\dfrac{\partial N}{\partial y} = nN$ (By Euler's theorem of homogeneous partial differential equations)

$\therefore$ From (2) $N . nM - M . nN = 0$ *i.e.*, $0 = 0$, which is true.

Hence $\dfrac{1}{Mx+Ny}$ is the I.F. of (1)

Case of failure. When $Mx + Ny = 0 \quad \therefore \quad N = -\dfrac{Mx}{y}$

From (1) $Mdx - \frac{M\,x\,dy}{y} = 0$ or $\frac{dx}{x} = \frac{dy}{y}$; Integrate $\log x = \log y + \log c \quad \therefore \quad x = cy$.

Note. If $Mx + Ny$ consists of only one term, use the above method of I.F. otherwise, proceed by putting $y = vx$.

Another Method. Consider $Mdx + Ndy = \frac{1}{2}\left[(Mx + Ny)\left(\frac{dx}{x} + \frac{dy}{y}\right) + (Mx - Ny)\left(\frac{dx}{x} - \frac{dy}{y}\right)\right]$

Divide by $Mx + Ny$ $(\neq 0)$

$$\frac{Mdx + Ndy}{Mx + Ny} = \frac{1}{2}\left[d(\log|xy| + \frac{Mx - Ny}{Mx + Ny}\, d\log\left|\frac{x}{y}\right|\right]$$

Now, $$\frac{Mx - Ny}{Mx + Ny} = \frac{M\frac{x}{y} - N}{M\frac{x}{y} + N} \quad \because \text{ M, N are homogeneous functions of } x \text{ and } y$$

$\therefore$ They can also be expressed in the form $\frac{x}{y}$.

$$\therefore \quad \frac{Mx - Ny}{Mx + Ny} = \phi\left(\frac{x}{y}\right)$$

$$\therefore \quad \frac{Mdx + Ndy}{Mx + Ny} = \frac{1}{2}\left[d\,\{\log|xy|\} + \phi\left(\frac{x}{y}\right) d\left\{\log\left|\frac{x}{y}\right|\right\}\right]$$

$\frac{\partial N}{\partial y} = -\frac{1}{y^2}$

$\frac{\partial N}{\partial x} = -\frac{1}{y^2}$

which is an exact derivative

Hence $\frac{1}{Mx + Ny}$ is the I.F.

Example 4. *Solve* $(x^2 y - 2xy^2)\, dx - (x^3 - 3x^2 y)\, dy = 0.$ **(P.T.U., Dec. 2003, Dec. 2010)**

Sol. The given equation is homogeneous in x and y with $M = x^2 y - 2xy^2$ and $N = -x^3 + 3x^2 y$

Now, $\quad Mx + Ny = \cancel{x^3 y} - 2x^2 y^2 - \cancel{x^3 y} + 3x^2 y^2 = x^2 y^2 \neq 0$

$$\therefore \quad \boxed{\text{I.F.} = \frac{1}{Mx + Ny} = \frac{1}{x^2 y^2}}$$

Multiplying throughout by $\frac{1}{x^2 y^2}$, the given equation becomes $\left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy = 0$, which

is exact. The solution is $\displaystyle\int_{y\,\text{constant}} \left(\frac{1}{y} - \frac{2}{x}\right) dx + \int \frac{3}{y}\, dy = c$

or $\quad \frac{x}{y} - 2\log x + 3\log y = c$.

1.9(c). I.F. FOR AN EQUATION OF THE FORM $y\, f_1(xy)\, dx + x\, f_2(xy)\, dy = 0$

(P.T.U., Dec. 2004)

Let the given differential equation be of the form

$$M\,dx + N\,dy = 0, \text{ where } M = y f_1(xy),\ N = x f_2(xy) \qquad \ldots(1)$$

$\frac{1}{Mx - Ny}$ will be the I.F of (1) if $\frac{M}{Mx-Ny}dx + \frac{N}{Mx-Ny}dy = 0$ is an exact equation

$$\therefore \quad \frac{\partial}{\partial y}\left[\frac{M}{Mx-Ny}\right] = \frac{\partial}{\partial x}\left[\frac{N}{Mx-Ny}\right]$$

i.e.,
$$\frac{(Mx-Ny)\frac{\partial M}{\partial y} - M\left[x\frac{\partial M}{\partial y} - N - y\frac{\partial N}{\partial y}\right]}{(Mx-Ny)^2} - \frac{(Mx-Ny)\frac{\partial N}{\partial x} - N\left[M + x\frac{\partial M}{\partial x} - y\frac{\partial N}{\partial x}\right]}{(Mx-Ny)^2} = 0$$

or
$$Mx\frac{\partial M}{\partial y} - Ny\frac{\partial M}{\partial y} - Mx\frac{\partial M}{\partial y} + MN + My\frac{\partial N}{\partial y} - Mx\frac{\partial N}{\partial x} + Ny\frac{\partial N}{\partial x} + NM + Nx\frac{\partial M}{\partial x} - Ny\frac{\partial N}{\partial x} = 0$$

or
$$-Ny\frac{\partial M}{\partial y} + My\frac{\partial N}{\partial y} - Mx\frac{\partial N}{\partial x} + Nx\frac{\partial M}{\partial x} + 2MN = 0$$

or
$$N\left\{x\frac{\partial M}{\partial x} - y\frac{\partial M}{\partial y}\right\} - M\left[x\frac{\partial N}{\partial x} - y\frac{\partial N}{\partial y}\right] + 2MN = 0 \qquad \ldots(2)$$

$\because \quad M = y f_1(xy)$

$$\therefore \quad \frac{\partial M}{\partial x} = y f_1'(xy).\, y$$

$$\frac{\partial M}{\partial y} = f_1(xy).1 + y f_1'(xy).\, x$$

$$\therefore \quad x\frac{\partial M}{\partial x} - y\frac{\partial M}{\partial y} = xy^2 f_1' - y f_1 - xy^2 f_1'$$

or
$$x\frac{\partial M}{\partial x} - y\frac{\partial M}{\partial y} = - f_1\, y = -M$$

Similarly, $N = x f_2(xy)$

$$\frac{\partial N}{\partial x} = f_2(xy) + x.\, f_2'(xy)\, y$$

$$\frac{\partial N}{\partial y} = x.\, f_2'(xy)\, x$$

$$x\frac{\partial N}{\partial x} - y\frac{\partial N}{\partial y} = x f_2 + x^2 y f_2' - x^2 y f_2'$$

$$= x f_2 = f_2(xy)\, x = M$$

$\therefore$ From (2), $N(-M) - M(N) + 2MN = 0$

or $\quad -2MN + 2MN = 0 \Rightarrow 0 = 0$ which is true. Hence $\frac{1}{Mx-Ny}$ is the I.F. of (1)

Case of failure. I.F. fails when $Mx - Ny = 0$ *i.e.*, $N = \frac{Mx}{y}$

$\therefore$ From (1), $\quad M dx + \frac{Mx}{y} dy = 0$

or
$$\frac{dx}{x} + \frac{dy}{y} = 0$$

Integrating both sides $\log x + \log y = c_1$

or $\quad \log(xy) = c_1$ or $xy = c$

If $Mdx + Ndy = 0$ is of the form $f_1(xy)\,y\,dx + f_2(xy)\,x\,dy = 0$, then $\dfrac{1}{Mx - Ny}$ is an I.F. provided

$Mx - Ny \neq 0$.

Example 5. *Solve :* $y(xy + 2x^2y^2)\,dx + x(xy - x^2y^2)\,dy = 0$.

Sol. The given equation is of the form $y f_1(xy)\,dx + x f_2(xy)\,dy = 0$

Here $M = xy^2 + 2x^2y^3$ and $N = x^2y - x^3y^2$

Now, $Mx - Ny = x^2y^2 + 2x^3y^3 - x^2y^2 + x^3y^3 = 3x^3y^3 \neq 0$

$$\therefore \quad \text{I.F.} = \frac{1}{Mx - Ny} = \frac{1}{3x^3y^3}$$

Multiplying throughout by $\dfrac{1}{3x^3y^3}$, the given equation becomes

$$\left(\frac{1}{3x^2y} + \frac{2}{3x}\right)dx + \left(\frac{1}{3xy^2} - \frac{1}{3y}\right)dy = 0$$

which is exact. The solution is $\displaystyle\int_{y\text{ constant}} \left(\frac{1}{3x^2y} + \frac{2}{3x}\right)dx + \int -\frac{1}{3y}\,dy = c$

or $$-\frac{1}{3xy} + \frac{2}{3}\log x - \frac{1}{3}\log y = c$$

or $$-\frac{1}{xy} + 2\log x - \log y = C, \text{ where } C = 3c.$$

1.9(d). FOR THE EQUATION $Mdx + Ndy = 0$

(*i*) If $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N}$ is a function of x only, say $f(x)$, then $e^{\int f(x)\,dx}$ is an I.F.

(*ii*) If $\dfrac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M}$ is a function of y only, we say $g(y)$, then $e^{\int g(y)\,dy}$ is an I.F.

Proof. $Mdx + Ndy = 0$ ...(1)

If $\dfrac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M} = g(y)$ *i.e.*, a function of y alone then $e^{\int g(y)\,dy}$ is an I.F of (1).

Now, $e^{\int g(y)\,dy}$ will be I.F. of (1)

If $M\,e^{\int g(y)dy}\,dx + N\,e^{\int g(y)\,dy}\,dy = 0$ is exact

i.e., $$\frac{\partial}{\partial y}\left\{M e^{\int g(y)\,dy}\right\} = \frac{\partial}{\partial x}\left\{N e^{\int g(y)\,dy}\right\}$$

$$M.\,e^{\int g(y)dy}\cdot g(y) + \frac{\partial M}{\partial y}\,e^{\int g(y)dy} = e^{\int g(y)dy}\,\frac{\partial N}{\partial x}$$

or
$$Mg(y) = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

or
$$\frac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M} = g(y), \text{ which is true}$$

Hence $e^{\int g(y)\,dy}$ is the I.F. of (1).
Students can easily prove (*i*) part themselves.

Example 6. *Solve :* $\left(xy^2 - e^{1/x^3}\right)dx - x^2 y\,dy = 0$. **(P.T.U., Dec. 2003, May 2011)**

Sol. Here $M = xy^2 - e^{1/x^3}$ and $N = -x^2 y$

$$\frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = \frac{2xy-(-2xy)}{-x^2 y} = -\frac{4}{x}, \text{ which is a function of } x \text{ only.}$$

$$\therefore \quad \text{I.F.} = e^{\int -\frac{4}{x}dx} = e^{-4\log x} = \frac{1}{x^4}$$

Multiplying throughout by $\frac{1}{x^4}$, we have $\left(\frac{y^2}{x^3} - \frac{1}{x^4}e^{1/x^3}\right)dx - \frac{y}{x^2}\,dy = 0$

which is exact. The solution is $\displaystyle\int_{y\text{ constant}} \left(\frac{y^2}{x^3} - \frac{1}{x^4}e^{1/x^3}\right)dx + \int 0.dy = c$

or $\displaystyle -\frac{y^2}{2x^2} + \frac{1}{3}\int -\frac{3}{x^4}e^{1/x^3}dx = c$ or $\displaystyle -\frac{y^2}{2x^2} + \frac{1}{3}\int e^t\,dt = c$, where $t = \frac{1}{x^3}$ $\therefore$ $dt = -\frac{3}{x^4}dx$

or $\displaystyle -\frac{y^2}{2x^2} + \frac{1}{3}e^t = c$ or $\displaystyle -\frac{3y^2}{x^2} + 2e^{1/x^3} = C$, where $C = 6c$.

Example 7. *Find the integrating factor of the equation* $(x^2 + y^2 + x)\,dx + xy\,dy = 0$.

Sol. Compare given equation with $Mdx + Ndy = 0$, we have

$$M = x^2 + y^2 + x, N = xy$$

$$\frac{\partial M}{\partial y} = 2y, \frac{\partial N}{\partial x} = y; \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\therefore$ Given equation is not exact

$$\frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = \frac{2y-y}{xy} = \frac{1}{x}, \text{ which is a function of } x \text{ only}$$

$\therefore$ I.F. of given equation is

$$e^{\int \frac{1}{x}dx} = e^{\log x} = x.$$

Example 8. *Solve :* $(xy^3 + y)\,dx + 2\,(x^2y^2 + x + y^4)\,dy = 0$. **(P.T.U., Dec. 2011)**

Sol. Here $M = xy^3 + y$ and $N = 2x^2y^2 + 2x + 2y^4$, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ $\therefore$ The given equation is not exact

$$\frac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M} = \frac{4xy^2 + 2 - 3xy^2 - 1}{xy^3 + y} = \frac{xy^2 + 1}{y\left(xy^2+1\right)} = \frac{1}{y}, \text{ which is a function of } y \text{ only.}$$

$$\therefore \quad \text{I.F.} = e^{\int \frac{1}{y} dy} = e^{\log y} = y$$

Multiplying throughout by y, we have $(xy^4 + y^2)\,dx + 2(x^2 y^3 + xy + y^5)\,dy = 0$

which is exact. The solution is $\displaystyle\int_{y\text{ constant}} (xy^4 + y^2)\,dx + \int 2y^5\,dy = c$

or $$\frac{x^2 y^4}{2} + xy^2 + \frac{y^6}{3} = c.$$

1.9(e). I.F. FOR THE EQUATION OF THE FORM $x^a y^b (my\,dx + nx\,dy) + x^c y^d (py\,dx + qx\,dy) = 0$ (where a, b, c, d, m, n, p, q are all CONSTANT)

If it is $x^h y^k$, then h, k are so chosen that after multiplication by $x^h y^k$ the equation becomes exact.

Example 9. *Solve :* $(2x^2 y^2 + y)\,dx + (3x - x^3 y)\,dy = 0.$

Sol. The equation can be written as $2(x^2 y^2\,dx - x^3 y\,dy) + (y\,dx + 3x\,dy) = 0$

or $$x^2 y(2y\,dx - x\,dy) + x^0 y^0 (y\,dx + 3x\,dy) = 0$$

which is of the form $x^a y^b (my\,dx + nx\,dy) + x^c y^d (py\,dx + qx\,dy) = 0$. Therefore, it has an I.F. of the form $x^h y^k$.

Multiplying the given equation by $x^h y^k$, we have

$$(2x^{h+2} y^{k+2} + x^h y^{k+1})\,dx + (3x^{h+1} y^k - x^{h+3} y^{k+1})\,dy = 0$$

For this equation to be exact, we must have

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad i.e., \quad 2(k+2)x^{h+2} y^{k+1} + (k+1)x^h y^k = 3(h+1)x^h y^k - (h+3)x^{h+2} y^{k+1}$$

which holds when $2(k+2) = -(h+3)$ and $k+1 = 3(h+1)$

i.e., when $h + 2k + 7 = 0$ and $3h - k + 2 = 0$

Solving these equation, we have $h = -\dfrac{11}{7}$, $k = -\dfrac{19}{7}$

$$\therefore \quad \text{I.F.} = x^{-\frac{11}{7}} y^{-\frac{19}{7}}$$

Multiplying the given equation by $x^{-\frac{11}{7}} y^{-\frac{19}{7}}$, we have

$$\left(2x^{\frac{3}{7}} y^{-\frac{5}{7}} + x^{-\frac{11}{7}} y^{-\frac{12}{7}}\right)dx + \left(3x^{-\frac{4}{7}} y^{-\frac{19}{7}} - x^{\frac{10}{7}} y^{-\frac{12}{7}}\right)dy = 0$$

which is exact. The solution is $\displaystyle\int_{y\text{ constant}} \left(2x^{\frac{3}{7}} y^{-\frac{5}{7}} + x^{-\frac{11}{7}} y^{-\frac{12}{7}}\right)dx = c$

or $$\frac{7}{5} x^{\frac{10}{7}} y^{-\frac{5}{7}} - \frac{7}{4} x^{-\frac{4}{7}} y^{-\frac{12}{7}} = c \quad \text{or} \quad 4x^{\frac{10}{7}} y^{-\frac{5}{7}} - 5x^{-\frac{4}{7}} y^{-\frac{12}{7}} = C$$

where $C = \dfrac{20}{7} c$.

Note. The values of h and k can be determined from the relations

$$\frac{a+h+1}{m} = \frac{b+k+1}{n} \quad \text{and} \quad \frac{c+h+1}{p} = \frac{d+k+1}{q}.$$

Example 10. *Solve:* $(2y^2 + 4x^2y)\,dx + (4xy + 3x^3)\,dy = 0.$ **(P.T.U., May 2011)**

Sol. Given equation is

$$(2y^2 + 4x^2y)\,dx + (4xy + 3x^3)\,dy = 0 \qquad \ldots(1)$$

The equation in not exact

$\therefore$ Rewrite the equation in the form:

$$x^2(4y\,dx + 3x\,dy) + y(2y\,dx + 4x\,dy) = 0 \qquad \ldots(2)$$

which is of the form

$$x^a y^b (my\,dx + nx\,dy) + x^c y^d (py\,dx + qx\,dy) = 0$$

Let $x^h y^k$ be the I.F. of (1)

$$\therefore \quad (2x^h y^{k+2} + 4x^{h+2} y^{k+1})dx + (4x^{h+1} y^{k+1} + 3x^{h+3} y^k)\,dy = 0$$

is an exact equation

$$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

i.e.,
$$\frac{\partial}{\partial y}\left(2x^h y^{k+2} + 4x^{h+2} y^{k+1}\right) = \frac{\partial}{\partial x}\left(4x^{h+1} y^{k+1} + 3x^{h+3} y^k\right)$$

or
$$2x^h (k+2)y^{k+1} + 4x^{h+2}(k+1)y^k = 4(h+1)x^h y^{k+1} + 3(h+3)x^{h+2} y^k$$

which holds when

$$2(k+2) = 4(h+1)$$

and
$$4(k+1) = 3(h+3)$$

i.e.,
$$k + 2 = 2h + 2 \quad \text{or} \quad 2h = k$$

and
$$4k = 3h + 5 \quad \text{or} \quad 8h - 3h = 5$$

$$\therefore \quad h = 1, \quad k = 2$$

$\therefore$ xy^2 is the I.F. of (1)

Multiply (1) by xy^2, we get

$$(2xy^4 + 4x^3y^3)dx + (4x^2y^3 + 3x^4y^2)dx = 0$$

which is an exact solution

$\therefore$ Its solution is

$$\int_y \left(2xy^4 + 4x^3y^3\right) dx + \int 0\,dy = C$$

or $\quad x^2y^4 + x^4y^3 = C$ is the solution of the given equation.

TEST YOUR KNOWLEDGE

Solve the following differential equations:

1. $x\,dy - y\,dx = (x^2 + y^2)\,dx$ **(P.T.U., Jan. 2010)**
2. $y(2xy + e^x)\,dx - e^x\,dy = 0$
3. $(1 + xy)\,y\,dx + (1 - xy)\,x\,dy = 0$
4. $x\,dy - y\,dx = xy^2\,dx$
5. $\left(xye^{\frac{x}{y}} + y^2\right)dx - x^2 e^{\frac{x}{y}}\,dy = 0$
6. $x^2 y\,dx - (x^3 + y^3)\,dy = 0$
7. $(3xy^2 - y^3)\,dx - (2x^2y - xy^2)\,dy = 0$
8. $(x^2y^2 + xy + 1)\,y\,dx + (x^2y^2 - xy + 1)\,x\,dy = 0$ **(P.T.U., May 2010)**
9. $y(2xy + 1)\,dx + x(1 + 2xy - x^3y^3)\,dy = 0$
10. $(x^2 + y^2 + 2x)\,dx + 2y\,dy = 0$

11. $(x^2+y^2+1)\,dx-2xy\,dy=0$

12. $\left(y+\dfrac{y^3}{3}+\dfrac{x^2}{2}\right)dx+\dfrac{1}{4}\left(x+xy^2\right)dy=0$

13. $(y^4+2y)\,dx+(xy^3+2y^4-4x)\,dy=0$

14. $(3xy-2ay^2)\,dx+(x^2-2axy)\,dy=0$

15. $y\,dx-x\,dy+\log x\,dx=0$

16. $(xy^2+2x^2y^3)\,dx+(x^2y-x^3y^2)\,dy=0$

17. $(2x^2y-3y^4)\,dx+(3x^3+2xy^3)\,dy=0.$

ANSWERS

1. $y=x\tan(x+c)$
2. $x^2+\dfrac{e^x}{y}=c$
3. $-\dfrac{1}{xy}+\log\dfrac{x}{y}=c$
4. $\dfrac{x^2}{2}+\dfrac{x}{y}=c$
5. $e^{\frac{x}{y}}+\log x=c$
6. $\log y-\dfrac{x^3}{3y^3}=c$
7. $3\log x-2\log y+\dfrac{y}{x}=c$
8. $xy+\log\dfrac{x}{y}-\dfrac{1}{xy}=c$
9. $\dfrac{1}{x^2y^2}+\dfrac{1}{3x^3y^3}+\log y=c$
10. $e^x(x^2+y^2)=c$
11. $x-\dfrac{y^2}{x}-\dfrac{1}{x}=c$
12. $x^4y+x^4y^3+x^6=c$
13. $y+\dfrac{2}{y^2}x+y^2=c$
14. $x^2(ay^2-xy)=c$
15. $cx+y\log x+1=0$
16. $-\dfrac{1}{xy}+2\log x-\log y=c$
17. $5x^{-\frac{36}{13}}y^{\frac{24}{13}}-12x^{-\frac{10}{13}}y^{-\frac{15}{13}}=c.$

1.10. DIFFERENTIAL EQUATIONS OF THE FIRST ORDER AND HIGHER DEGREE

So far, we have discussed differential equations of the first order and first degree. Now we shall study differential equations of the first order and degree higher than the first. For convenience, we denote $\dfrac{dy}{dx}$ by p.

A differential equation of the first order and nth degree is of the form

$$p^n+P_1p^{n-1}+P_2p^{n-2}+\dots\dots\dots+P_n=0 \qquad \dots(1)$$

where $P_1, P_2, \dots\dots\dots, P_n$ are functions of x and y.

Since it is a differential equation of the first order, its general solution will contain only one arbitrary constant.

In the various cases which follow, the problem is reduced to that of solving one or more equations of the first order and first degree.

1.11. EQUATIONS SOLVABLE FOR p

Resolving the left hand side of (1) into n linear factors, we have

$$[p-f_1(x,y)]\,[p-f_2(x,y)],\dots\dots\dots,[p-f_n(x,y)]=0$$

which is equivalent to $p-f_1(x,y)=0,\ p-f_2(x,y)=0,\ \dots\dots\dots,\ p-f_n(x,y)=0$

Each of these equations is of the first order and first degree and can be solved by the methods already discussed.

If the solutions of the above n component equations are

$$F_1(x,y,c)=0,\ F_2(x,y,c)=0,\ \dots\dots\dots,F_n(x,y,c)=0$$

then the general solution of (1) is given by $F_1(x,y,c).\ F_2(x,y,c)\dots\dots\dots F_n(x,y,c)=0.$

ILLUSTRATIVE EXAMPLES

Example 1. *Solve :* $x^2\left(\frac{dy}{dx}\right)^2 + xy\frac{dy}{dx} - 6y^2 = 0.$

Sol. The given equation is $x^2p^2 + xyp - 6y^2 = 0$ where $p = \frac{dy}{dx}$

Factorising $(xp + 3y)(xp - 2y) = 0 \Rightarrow xp + 3y = 0$ or $xp - 2y = 0$

Now, $xp + 3y = 0 \Rightarrow x\frac{dy}{dx} + 3y = 0$ or $\frac{dy}{y} + 3\frac{dx}{x} = 0$

Integrating $\log y + 3\log x = \log c$ or $x^3 y = c$

Also, $xp - 2y = 0 \Rightarrow x\frac{dy}{dx} - 2y = 0$ or $\frac{dy}{y} - 2\frac{dx}{x} = 0$

Integrating $\log y - 2\log x = \log c$ or $\frac{y}{x^2} = c$ or $y = cx^2$

$\therefore$ The general solution of the given equation is $(x^3 y - c)(y - cx^2) = 0.$

Example 2. *Solve: $p(p + y) = x(x + y)$.* **(P.T.U., May 2007)**

Sol. $p^2 + py = x^2 + xy$ ...(1)

or $p^2 + py - (x^2 + xy) = 0$, which is quadratic in p

$$\therefore \quad p = \frac{-y \pm \sqrt{y^2 + 4(x^2 + xy)}}{2} = \frac{-y \pm \sqrt{(y + 2x)^2}}{2}$$

$\therefore \quad p = \frac{-y + y + 2x}{2}$

or $p = x$

or $\frac{dy}{dx} = x$

Integrating both sides,

$$y = \frac{x^2}{2} + c$$

or $y - \frac{x^2}{2} - c = 0$ ...(2)

and $p = \frac{-y - y - 2x}{2}$

or $p = -y - x$

or $\frac{dy}{dx} = -y - x$

or $\frac{dy}{dx} + y = -x$

which is linear equation in y

Its I.F. $= e^{\int 1\,dx} = e^x$

$\therefore$ Its solution is

$$y\,e^x = \int e^x(-x)\,dx + c$$

or $y\,e^x = -(x - 1)\,e^x + c$ [Integrating by parts]

or $y = -(x - 1) + ce^{-x}.$

or $y + x - 1 - ce^{-x} = 0$ ...(3)

Combining (2) and (3), general solution is

$$\left(y-\frac{x^2}{2}-c\right)\left(y+x-1-ce^{-x}\right)=0.$$

Note. $\frac{dy}{dx}=-(x+y)$ can also be solved by putting $x+y=t$, but that is a lengthy solution.

Example 3. *Solve* : $p^2+2py\cot x=y^2$. **(P.T.U., Jan. 2009)**

Sol. The given equation can be written as $(p+y\cot x)^2=y^2(1+\cot^2 x)$

or $\quad p+y\cot x=\pm\, y\operatorname{cosec} x$

$\therefore$ The component equations are

$$p=y(-\cot x+\operatorname{cosec} x) \quad \ldots(1)$$

and
$$p=y(-\cot x-\operatorname{cosec} x) \quad \ldots(2)$$

From (1), $\frac{dy}{dx}=y(-\cot x+\operatorname{cosec} x)$

or $\quad \frac{dy}{y}=(-\cot x+\operatorname{cosec} x)\,dx$

Integrating $\quad \log y=-\log\sin x+\log\tan\frac{x}{2}$

$$+\log c=\log\frac{c\tan\frac{x}{2}}{\sin x}$$

or $\quad y=\frac{c\tan\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}=\frac{c}{2\cos^2\frac{x}{2}}$

or $\quad y\cos^2\frac{x}{2}=C,\quad$ where $C=\frac{c}{2}$

From (2), $\frac{dy}{dx}=y(-\cot x-\operatorname{cosec} x)$

or $\quad \frac{dy}{y}=(-\cot x-\operatorname{cosec} x)\,dx$

Integrating $\quad \log y=-\log\sin x-\log\tan\frac{x}{2}$

$$+\log c=\log\frac{c}{\sin x\tan\frac{x}{2}}$$

or $\quad y=\frac{c}{2\sin^2\frac{x}{2}}\quad$ or $\quad y\sin^2\frac{x}{2}=C$

$\therefore$ The general solution of the given equation is $\left(y\cos^2\frac{x}{2}-C\right)\left(y\sin^2\frac{x}{2}-C\right)=0$

TEST YOUR KNOWLEDGE

Solve the following equations :

1. $p^2-7p+12=0$ **(P.T.U., Dec. 2006)**

   [**Hint:** Solve $p=3,\ p=4$]

2. $xy\left(\frac{dy}{dx}\right)^2-(x^2+y^2)\frac{dy}{dx}+xy=0$

3. $yp^2+(x-y)\,p-x=0$

4. $x^2\left(\frac{dy}{dx}\right)^2+3xy\frac{dy}{dx}+2y^2=0$ **(P.T.U., Dec. 2011)**

5. $\frac{dy}{dx}-\frac{dx}{dy}=\frac{x}{y}-\frac{y}{x}$

6. $p^2-2p\sinh x-1=0$

7. $xyp^2+p(3x^2-2y^2)-6xy=0$

   [**Hint:** Quadratic in p and values of p are $\frac{2y}{x},-\frac{3x}{y}$.]

8. $4y^2p^2+2pxy(3x+1)+3x^3=0$.

ANSWERS

1. $(y-4x-c)(y-3x-c)=0$
2. $(y^2-x^2-c)(y-cx)=0$
3. $(y-x-c)(x^2+y^2-c)=0$
4. $(xy-c)(x^2y-c)=0$
5. $(xy-c)(x^2-y^2-c)=0$
6. $(y-e^x-c)(y-e^{-x}-c)=0$
7. $(y-cx^2)(y^2+3x^2-c)=0$
8. $(y^2+x^3-c)\left(y^2+\frac{1}{2}x^2-c\right)=0.$

1.12. EQUATIONS SOLVABLE FOR y

If the equation is solvable for y, we can express y explicitly in terms of x and p. Thus, the equations of this type can be put as $y = f(x, p)$ ...(1)

Differentiating (1) w.r.t. x, we get $\frac{dy}{dx} = p = F\left(x, p, \frac{dp}{dx}\right)$ ...(2)

Equation (2) is a differential equation of first order in p and x.

Suppose the solution of (2) is $\phi(x, p, c) = 0$ ...(3)

Now, elimination of p from (1) and (3) gives the required solution.

If p cannot be easily eliminated, then we solve equations (1) and (3) for x and y to get

$$x = \phi_1(p, c), \; y = \phi_2(p, c)$$

These two relations together constitute the solution of the given equation with p as parameter.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve :* $y + px = x^4 p^2$.

Sol. Given equation is $y = -px + x^4 p^2$ ...(1)

Differentiating both sides w.r.t. x,

$$\frac{dy}{dx} = p = -p - x\frac{dp}{dx} + 4x^3p^2 + 2x^4p\frac{dp}{dx}$$

or $$2p + x\frac{dp}{dx} - 2px^3\left(2p + x\frac{dp}{dx}\right) = 0 \quad \text{or} \quad \left(2p + x\frac{dp}{dx}\right)\left(1 - 2px^3\right) = 0$$

Discarding the factor $(1 - 2px^3)$, we have $2p + x\frac{dp}{dx} = 0$ or $\frac{dp}{p} + 2\frac{dx}{x} = 0$

Integrating $\log p + 2\log x = \log c$ or $\log px^2 = \log c$ or $px^2 = c$

or $$p = \frac{c}{x^2}.$$

Putting this value of p in (1), we have $y = -\frac{c}{x} + c^2$, which is the required solution.

Example 2. *Solve :* $y = 2px - p^2$.

Sol. The given equation is $y = 2px - p^2$ ...(1)

Differentiating both sides w.r.t. x, $\frac{dy}{dx} = 2p + 2x\frac{dp}{dx} - 2p\frac{dp}{dx}$

or $$p = 2p + 2x\frac{dp}{dx} - 2p\frac{dp}{dx}$$

or $$p + (2x - 2p)\frac{dp}{dx} = 0 \quad \text{or} \quad p\frac{dx}{dp} + 2x - 2p = 0$$

or $$\frac{dx}{dp} + \frac{2}{p}x = 2 \qquad \text{...(2)}$$

which is a linear equation.

$$\text{I.F.} = e^{\int \frac{2}{p} dp} = e^{2\log p} = p^2$$

$\therefore$ The solution of (2) is $x\,(\text{I.F.}) = \int 2(\text{I.F.})\,dp + c$ or $xp^2 = \int 2p^2\,dp + c$

or $$xp^2 = \frac{2}{3}p^3 + c \text{ or } x = \frac{2}{3}p + cp^{-2} \quad ...(3)$$

Putting this value of x in (1), we have $y = 2p\left(\frac{2}{3}p + cp^{-2}\right) - p^2$

or $$y = \frac{1}{3}p^2 + 2c\,p^{-1} \quad ...(4)$$

Equations (3) and (4) together constitute the general solution of (1).

TEST YOUR KNOWLEDGE

Solve the following equations :

1. $xp^2 - 2yp + ax = 0$ **(P.T.U., May 2011)**
2. $y - 2px = \tan^{-1}(xp^2)$ **(P.T.U., May 2010)**
3. $16x^2 + 2p^2y - p^3x = 0$
4. $y = x + 2\tan^{-1}p$
5. $y = 3x + \log p$
6. $x - yp = ap^2$
7. $x^2\left(\frac{dy}{dx}\right)^4 + 2x\frac{dy}{dx} - y = 0$
8. $3x^4p^2 - px - y = 0$ **(P.T.U., May 2010)**

    [Hint: See S.E. 1]

ANSWERS

1. $2y = cx^2 + \frac{a}{c}$
2. $y = 2\sqrt{cx} + \tan^{-1}c$
3. $16 + 2c^2y - c^3x^2 = 0$
4. $x = \log\frac{p-1}{\sqrt{p^2+1}} - \tan^{-1}p + c, y = \log\frac{p-1}{\sqrt{p^2+1}} + \tan^{-1}p + c$
5. $y = 3x + \log\frac{3}{1-ce^{3x}}$
6. $x = \frac{p}{\sqrt{1-p^2}}\left(c + a\sin^{-1}p\right), \; y = \frac{1}{\sqrt{1-p^2}}\left(c + a\sin^{-1}p\right) - ap$
7. $y = c^2 + 2\sqrt{cx}$
8. $y = 3c^2 - \frac{c}{x}$.

1.13. EQUATIONS SOLVABLE FOR x

If the equation is solvable for x, we can express x explicitly in terms of y and p. Thus, the equations of this type can be put as $x = f(y, p)$ ...(1)

Differentiating (1) w.r.t. y, we get $\frac{dx}{dy} = \frac{1}{p} = F\left(y, p, \frac{dp}{dy}\right)$ ...(2)

Equation (2) is a differential equation of first order in p and y.

Suppose the solution of (2) is $\phi(y, p, c) = 0$ ...(3)

Now, elimination of p from (1) and (3) gives the required solution.

If p cannot be easily eliminated, then we solve equations (1) and (3) for x and y to get

$$x = \phi_1(p, c), \; y = \phi_2(p, c)$$

These two relations together constitute the solution of the given equation with p as parameter.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve : $y = 2px + y^2p^3$.*

Sol. Solving for x, we have $x = \frac{1}{2}\left(\frac{y}{p} - y^2p^2\right)$

Differentiating both sides w.r.t. y

$$\frac{dx}{dy}=\frac{1}{p}=\frac{1}{2}\left(\frac{1}{p}-\frac{y}{p^2}\cdot\frac{dp}{dy}-2yp^2-2y^2\,p\,\frac{dp}{dy}\right)$$

or $$2p=p-y\frac{dp}{dy}-2yp^4-2y^2p^3\frac{dp}{dy}$$

or $$p+2yp^4+y\frac{dp}{dy}+2y^2p^3\frac{dp}{dy}=0 \quad \text{or} \quad p(1+2yp^3)+y\frac{dp}{dy}\left(1+2yp^3\right)=0$$

or $$\left(p+y\frac{dp}{dy}\right)\left(1+2yp^3\right)=0$$

Discarding the factor $(1+2yp^3)$, we have $p+y\dfrac{dp}{dy}=0$ or $\dfrac{dy}{y}+\dfrac{dp}{p}=0$

Integrating $\log y+\log p=\log c$ or $py=c$ or $p=\dfrac{c}{y}$

Putting this value of p in the given equation, we have $y=\dfrac{2cx}{y}+\dfrac{c^3}{y}$ or $y^2=2cx+c^3$

which is the required solution.

Example 2. *Solve :* $p=\tan\left(x-\dfrac{p}{1+p^2}\right)$.

Sol. Solving for x, we have $x=\tan^{-1}p+\dfrac{p}{1+p^2}$ ...(1)

Differentiating both sides w.r.t. y, $\dfrac{dx}{dy}=\dfrac{1}{p}=\dfrac{1}{1+p^2}\cdot\dfrac{dp}{dy}+\dfrac{(1+p^2)-2p^2}{(1+p^2)^2}\cdot\dfrac{dp}{dy}$

or $$\frac{1}{p}=\frac{2(1+p^2)-2p^2}{(1+p^2)^2}\,\frac{dp}{dy} \quad \text{or} \quad dy=\frac{2p}{(1+p^2)^2}\,dp$$

Integrating $y=c-\dfrac{1}{1+p^2}$ ...(2)

Equations (1) and (2) together constitute the general solution.

TEST YOUR KNOWLEDGE

Solve the following equations:

1. $y=3px+6p^2y^2$
2. $y=2px+p^2y$
3. $p^3-4xyp+8y^2=0$
4. $y^2\log y=xyp+p^2$.

ANSWERS

1. $y^3=3cx+6c^2$
2. $y^2=2cy+c^2$
3. $64y=c(c-4x)^2$
4. $\log y=cx+c^2$.

1.14. CLAIRAUT'S EQUATION

(P.T.U., May 2007, Jan. 2009)

An equation of the form $y = px + f(p)$ ...(1)
is known as Clairaut's equation.

Differentiating (1) w.r.t. x, we get $p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$ or $[x + f'(p)]\frac{dp}{dx} = 0$

Discarding the factor $[x + f'(p)]$, we have $\frac{dp}{dx} = 0$

Integrating $p = c$

Putting $p = c$ in (1), the required solution is $y = cx + f(c)$

Thus, the solution of Clairaut's equation is obtained by writing c for p.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve the following equations :*

(i) $p = \log(px - y)$ **(P.T.U., Dec. 2005, Dec. 2011)**

(ii) $\sin px \cos y = \cos px \sin y + p$. **(P.T.U., Dec. 2006)**

Sol. *(i)* $p = \log(px - y)$

or $e^p = px - y$ or $y = px - e^p$, which is Clairaut's equation where $f(p) = -e^p$

$\therefore$ Its solution is obtained by putting $p = c$

$\therefore$ solution is $y = cx - e^c$.

(ii) $\sin px \cos y = \cos px \sin y + p$

or $\sin px \cos y - \cos px \sin y = p$

or $\sin(px - y) = p$

or $px - y = \sin^{-1} p$

or $y = px - \sin^{-1} p$, which is Clairaut's form

$\therefore$ Its solution is (put $p = c$) $y = cx - \sin^{-1} c$.

Note. Many differential equations can be reduced to Clairaut's form by suitably changing the variables.

Example 2. *Solve :* $e^{4x}(p - 1) + e^{2y}p^2 = 0$.

Sol. [In problems involving e^{lx} and e^{my}, put $X = e^{kx}$ and $Y = e^{ky}$, where k is the H.C.F. of l and m].

Put $X = e^{2x}$ and $Y = e^{2y}$

so that $dX = 2e^{2x}\,dx$ and $dY = 2e^{2y}\,dy$

$$\therefore \quad p = \frac{dy}{dx} = \frac{e^{2x}}{e^{2y}}\frac{dY}{dX} = \frac{X}{Y}P, \text{ where } P = \frac{dY}{dX}$$

The given equation becomes $X^2\left(\frac{X}{Y}P - 1\right) + Y \cdot \frac{X^2}{Y^2}P^2 = 0$

or $XP - Y + P^2 = 0$ or $Y = PX + P^2$, which is of Clairaut's form

$\therefore$ Its solution is $Y = cX + c^2$ and hence $e^{2y} = ce^{2x} + c^2$.

Example 3. *Solve :* $(px - y)(py + x) = 2p$. **(P.T.U., Jan. 2009, May 2009)**

Sol. Put $X = x^2$ and $Y = y^2$

so that $dX = 2x\,dx$ and $dY = 2y\,dy$

$$\therefore \quad p = \frac{dy}{dx} = \frac{x}{y}\frac{dY}{dX} = \frac{\sqrt{X}}{\sqrt{Y}}P, \text{ where } P = \frac{dY}{dX}$$

The given equation becomes $\left(\frac{\sqrt{X}}{\sqrt{Y}}P.\sqrt{X}-\sqrt{Y}\right)\left(\frac{\sqrt{X}}{\sqrt{Y}}P.\sqrt{Y}+\sqrt{X}\right)=2\frac{\sqrt{X}}{\sqrt{Y}}P$

or $\quad (PX-Y)(P+1)=2P$ or $PX-Y=\dfrac{2P}{P+1}$

or $$Y=PX-\frac{2P}{P+1}\text{, which is of Clairaut's form.}$$

$\therefore$ Its solution is $Y=cX-\dfrac{2c}{c+1}$ and hence $y^2=cx^2-\dfrac{2c}{c+1}$.

Example 4. *Solve:* $(x^2+y^2)(1+p)^2=2(x+y)(1+p)(x+yp)-(x+yp)^2$ **(P.T.U., May 2011)**

Sol. Given equation can be written as:

$$x^2+y^2=\frac{2(x+y)(x+py)}{1+p}-\left(\frac{x+py}{1+p}\right)^2 \quad ...(1)$$

Put $\quad X=x+y,\ Y=x^2+y^2$

and $\quad$ Let $P=\dfrac{dY}{dX}=\dfrac{dY}{dx}\Big/\dfrac{dX}{dx}$

$$\therefore\quad P=\frac{2x+2y\dfrac{dy}{dx}}{1+\dfrac{dy}{dx}}=\frac{2(x+py)}{1+p}$$

Substituting in (1)

$$Y=2X\frac{P}{2}-\left(\frac{P}{2}\right)^2$$

or $\quad Y=PX-\dfrac{P^2}{4}$, which is Clairaut's differential equation

$\therefore\quad$ Its solution is $Y=CX-\dfrac{C^2}{4}$

or $\quad x^2+y^2=C(x+y)-\dfrac{C^2}{4}$.

TEST YOUR KNOWLEDGE

Solve the following equations :

1. $y=xp+\dfrac{a}{p}$
2. (*a*) $y=px+\sqrt{a^2p^2+b^2}$

   (*b*) $(y-px)(p-1)=p$
3. $p=\log(px-y)$
4. $p=\sin(y-px)$ **(P.T.U., May 2007)**

   [**Hint:** See Example 1 (*ii*)]
5. $p^2(x^2-1)-2pxy+y^2-1=0$
6. $e^{3x}(p-1)+p^3e^{2y}=0$
7. $x^2(y-px)=yp^2$

   [**Hint:** Put $x^2=X$, $y^2=Y$]
8. $(y+px)^2=x^2p$.

   [**Hint:** Put $xy=v$]

ANSWERS

1. $y=cx+\dfrac{a}{c}$
2. (*a*) $y=cx+\sqrt{a^2c^2+b^2}$, (*b*) $y=cx+\dfrac{c}{c-1}$
3. $y=cx-e^c$
4. $y=cx+\sin^{-1}c$
5. $(y-cx)^2=1+c^2$
6. $e^y=ce^x+c^2$
7. $y^2=cx^2+c^2$
8. $xy=cx-c^2$.

1.15. DEFINITION OF LEIBNITZ'S LINEAR DIFFERENTIAL EQUATION

A differential equation is said to be ***Leibnitz's linear*** *or simply linear if the dependent variable and its derivatives occur only in the first degree and are not multiplied together.*

(P.T.U., June 2003, May 2005, May 2007)

The general form of a linear differential equation of the first order is $\frac{dy}{dx} + Py = Q$ ...(1)

where P and Q are functions of x only or may be constants.

1.16. SOLVE THE LINEAR DIFFERENTIAL EQUATION $\frac{dy}{dx} + Py = Q$

(P.T.U., Dec. 2006)

To solve it, we multiply both sides by $e^{\int P dx}$, we get

$$\frac{dy}{dx} e^{\int P dx} + y\left(e^{\int P dx} P\right) = Q e^{\int P dx}$$

or

$$\frac{d}{dx}\left(y e^{\int P dx}\right) = Q e^{\int P dx}$$

Integrating both sides, we have $y e^{\int P dx} = \int Q e^{\int P dx} dx + c$

which is the required solution.

Note 1. In the general form of a linear differential equation, the coefficient of $\frac{dy}{dx}$ is unity.

The equation $R \frac{dy}{dx} + Sy = T$, where R, S and T are functions of x only or constants, must be divided by R to bring it to the general linear form.

Note 2. The factor $e^{\int P dx}$ on multiplying by which the L.H.S. of (1) becomes the differential coefficient of a single function is called the integrating factor (briefly written as I.F.) of (1).

Thus I.F. $= e^{\int P dx}$ and the solution is $y\,(\text{I.F.}) = \int Q(\text{I.F.})\, dx + c$.

Note 3. Sometimes a differential equation takes linear form if we regard x as dependent variable and y as independent variable. The equation can then be put as $\frac{dx}{dy} + Px = Q$, where P, Q are functions of y only or constants.

The integrating factor in this case is $e^{\int P dy}$ and the solution is $x\ (\text{I.F.}) = \int Q\ (\text{I.F.})\, dy + c$.

Note 4. $e^{\log f(x)} = f(x)$

ILLUSTRATIVE EXAMPLES

Example 1. *Solve* : $x\left(1 - x^2\right)\frac{dy}{dx} + \left(2x^2 - 1\right) y = x^3$.

Sol. Dividing by $x\,(1 - x^2)$ to make the coefficient of $\frac{dy}{dx}$ unity, the given equation becomes

$$\frac{dy}{dx} + \frac{2x^2 - 1}{x\left(1 - x^2\right)} y = \frac{x^2}{1 - x^2}$$

Comparing it with $\frac{dy}{dx} + Py = Q$, we have $P = \frac{2x^2 - 1}{x\left(1 - x^2\right)}$, $Q = \frac{x^2}{1 - x^2}$

Now, $\quad P = \dfrac{2x^2-1}{x(1-x)(1+x)} = -\dfrac{1}{x} + \dfrac{1}{2(1-x)} - \dfrac{1}{2(1+x)}$ by partial fractions

$$\int P\,dx = -\log x - \frac{1}{2}\log(1-x) - \frac{1}{2}\log(1+x) = -\log\left[x(1-x)^{\frac{1}{2}}(1+x)^{\frac{1}{2}}\right]$$

$$= -\log\left[x\left(1-x^2\right)^{\frac{1}{2}}\right] = \log\frac{1}{x\sqrt{1-x^2}}$$

$$\text{I.F.} = e^{\int P\,dx} = e^{\log\frac{1}{x\sqrt{1-x^2}}} = \frac{1}{x\sqrt{1-x^2}}$$

Thus the solution is

$$y\,(\text{I.F.}) = \int Q\,(\text{I.F.})\,dx + c \quad \text{or} \quad y.\frac{1}{x\sqrt{1-x^2}} = \int \frac{x^2}{1-x^2} \times \frac{1}{x\sqrt{1-x^2}}\,dx + c = \int \frac{x}{\left(1-x^2\right)^{\frac{3}{2}}}\,dx + c$$

$$= -\frac{1}{2}\int\left(1-x^2\right)^{-\frac{3}{2}}(-2x)\,dx + c = \left(1-x^2\right)^{-\frac{1}{2}} + c \qquad \because \int [f(x)]^n\, f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1}\ (n \neq -1)$$

or $\quad y = x + cx\sqrt{1-x^2}$.

Example 2. *Solve :* $\left(\dfrac{e^{-2\sqrt{x}}}{\sqrt{x}} - \dfrac{y}{\sqrt{x}}\right)\dfrac{dx}{dy} = 1$.

Sol. The given equation can be written as $\quad \dfrac{dy}{dx} + \dfrac{y}{\sqrt{x}} = \dfrac{e^{2\sqrt{x}}}{\sqrt{x}}$

Comparing it with $\dfrac{dy}{dx} + Py = Q$, we have $P = \dfrac{1}{\sqrt{x}}$, $Q = \dfrac{e^{-2\sqrt{x}}}{\sqrt{x}}$

$$\text{I.F.} = e^{\int P\,dx} = e^{\int \frac{1}{\sqrt{x}}dx} = e^{\frac{x^{1/2}}{1/2}} = e^{2\sqrt{x}}$$

$\therefore$ The solution is $\quad y\,(\text{I.F.}) = \int Q\,(\text{I.F.})\,dx + c \quad$ or $\quad ye^{2\sqrt{x}} = \int \dfrac{e^{2\sqrt{x}}}{\sqrt{x}} \,.\, e^{2\sqrt{x}}\,dx + c$

or $$ye^{2\sqrt{x}} = \int x^{\frac{1}{2}}\,dx + c = 2\sqrt{x} + c$$

Example 3. *Solve :* $\left(1+y^2\right)dx = \left(\tan^{-1} y - x\right)dy$. **(P.T.U., Dec. 2011)**

Sol. The given equation can be written as $\quad \left(1+y^2\right)\dfrac{dx}{dy} + x = \tan^{-1} y$

Dividing by $(1+y^2)$, we get $\quad \dfrac{dx}{dy} + \dfrac{x}{1+y^2} = \dfrac{\tan^{-1} y}{1+y^2}$

which is of the form $\frac{dx}{dy} + Px = Q$

Here $P = \frac{1}{1+y^2}$, $Q = \frac{\tan^{-1} y}{1+y^2}$

$$\text{I.F.} = e^{\int P\,dy} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$$

$\therefore$ The solution is $x\,(\text{I.F.}) = \int Q\,(\text{I.F.})\,dy + c$

or $$xe^{\tan^{-1} y} = \int \frac{\tan^{-1} y}{1+y^2} e^{\tan^{-1} y}\,dy + c = \int t e^t\,dt + c, \text{ where } t = \tan^{-1} y$$

$$= te^t - \int 1 . e^t\,dt + c = te^t - e^t + c = \left(\tan^{-1} y - 1\right) e^{\tan^{-1} y} + c$$

or $$x = \tan^{-1} y - 1 + ce^{-\tan^{-1} y}.$$

TEST YOUR KNOWLEDGE

Solve the following differential equations :

1. $\frac{dy}{dx} + \frac{y}{x} = x^3 - 3$
2. $x \log x \frac{dy}{dx} + y = 2 \log x$
3. $(x+1)\frac{dy}{dx} - y = e^x (x+1)^2$
4. $\left(x^2+1\right)\frac{dy}{dx} + 2xy = x^2$
5. $\cos^2 x \frac{dy}{dx} + y = \tan x$
6. $\left(1+x^3\right)\frac{dy}{dx} + 6x^2 y = 1 + x^2$
7. $\frac{dy}{dx} + y \cot x = \cos x$
8. $\left(1+x^2\right)\frac{dy}{dx} + y = e^{\tan^{-1} x}$
9. $\frac{dy}{dx} + y \cot x = 4x \operatorname{cosec} x$, if $y = 0$ when $x = \frac{\pi}{2}$
10. $\frac{dy}{dx} + y \cot x = 5e^{\cos x}$, if $y = -4$ when $x = \frac{\pi}{2}$
11. $\frac{dy}{dx} - y \tan x = 3e^{-\sin x}$, if $y = 4$ when $x = 0$
12. $\left(1-x^2\right)\frac{dy}{dx} + 2xy = x\sqrt{1-x^2}$
13. $x\frac{dy}{dx} + y = e^x - xy$
14. $\left(1+y^2\right) + \left(x - e^{-\tan^{-1} y}\right)\frac{dy}{dx} = 0$
15. $e^{-y} \sec^2 y\,dy = dx + x\,dy$
16. $\left(x + 2y^3\right)\frac{dy}{dx} = y$
17. $ye^y\,dx = \left(y^2 + 2xe^y\right) dy$
18. $\frac{dy}{dx} = \frac{y}{2y \log y + y - x}$
19. $x^2 \frac{dy}{dx} = 3x^2 - 2xy + 1$
20. $\sqrt{1-y^2}\,dx = \left(\sin^{-1} y - x\right) dy$.

ANSWERS

1. $10xy = 2x^5 - 15x^2 + c$
2. $y \log x = (\log x)^2 + c$
3. $y = (x+1)(e^x + c)$
4. $y(x^2+1) = \frac{x^3}{3} + c$
5. $y = \tan x - 1 + c\,e^{-\tan x}$
6. $y\left(1+x^3\right)^2 = y + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^6}{6} + c$
7. $y \sin x = \frac{1}{2}\sin^2 x + c$
8. $2ye^{\tan^{-1}x} = e^{2\tan^{-1}x} + c$
9. $y \sin x = 2x^2 - \frac{\pi^2}{2}$
10. $y \sin x = 5e^{\cos x} - 9$
11. $y \cos x = 7 - 3e^{-\sin x}$
12. $y = \sqrt{1-x^2} + c\left(1-x^2\right)$
13. $xy = \frac{1}{2}e^x + ce^{-x}$
14. $x = e^{-\tan^{-1}y}\left(\tan^{-1}y + c\right)$
15. $xe^y = \tan y + c$
16. $x + y^3 + cy$
17. $x = y^2(c - e^{-y})$
18. $x = \frac{c}{y} + y \log y$
19. $y = x + x^{-1} + cx^{-2}$
20. $x = \sin^{-1}y - 1 + ce^{-\sin^{-1}y}$.

1.17. EQUATIONS REDUCIBLE TO THE LINEAR FORM (Bernoulli's Equation)

(P.T.U., May 2007, Jan 2009)

(a) An equation of the form $\frac{dy}{dx} + Py = Qy^n$ ...(1)

where P and Q are functions of x only or constants is known as *Bernoulli's* equation. Though not linear, it can be made linear.

Dividing both sides of (1) by y^n, we have $y^{-n}\frac{dy}{dx} + Py^{1-n} = Q$ ...(2)

Putting $y^{1-n} = z$ so that $(1-n)\,y^{-n}\frac{dy}{dx} = \frac{dz}{dx}$

or $$y^{-n}\frac{dy}{dx} = \frac{1}{1-n}\frac{dz}{dx}$$

Equation (2) becomes $\frac{1}{1-n}\frac{dz}{dx} + Pz = Q$ or $\frac{dz}{dx} + (1-n)Pz = (1-n)Q$

which is a linear differential equation with z as the dependent variable.

(b) General equation reducible to linear form is $f'(y)\frac{dy}{dx} + Pf(y) = Q$ ...(1)

where P and Q are functions of x only or constants.

Putting $f(y) = z$ so that $f'(y)\dfrac{dy}{dx} = \dfrac{dz}{dx}$

Equation (1) becomes $\dfrac{dz}{dx} + Pz = Q$, which is linear.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve the following differential equations.*

(i) $x\dfrac{dy}{dx} + y = x^3y^6$ **(P.T.U., May 2007, May 2011)**

(ii) $y' + y = y^2$ **(P.T.U., May 2008)**

(iii) $\left(xy^2 - e^{1/x^3}\right)dx - x^2y\,dy = 0.$

Sol. (*i*) $x\dfrac{dy}{dx} + y = x^3y^6$

Dividing by xy^6, we get

$$y^{-6}\frac{dy}{dx} + \frac{1}{x}y^{-5} = x^2 \quad \text{...(1)}$$

Put $y^{-5} = z$ so that

$$-5y^{-6}\frac{dy}{dx} = \frac{dz}{dx}$$

Equation (1) becomes

$$-\frac{1}{5}\frac{dz}{dx} + \frac{1}{x}z = x^2$$

or
$$\frac{dz}{dx} - \frac{5}{x}z = -5x^2$$

which is linear in z, where $P = -\dfrac{5}{x}$, $Q = -5x^2$

$$\text{I.F.} = e^{\int -\frac{5}{x}dx} = e^{-5\log x} = e^{\log x^{-5}} = x^{-5} = \frac{1}{x^5}$$

$\therefore$ Solution in z is

$$z(\text{I.F.}) = \int Q\cdot \text{I.F.}\,dx + c$$

$$z\cdot\frac{1}{x^5} = \int -5x^2\cdot\frac{1}{x^5}dx + c = -5\int x^{-3}dx + c = -5\frac{x^{-2}}{-2} + c = \frac{5}{2x^2} + c$$

Substituting the value of z, we get

$$\frac{1}{y^5}\cdot\frac{1}{x^5} = \frac{5}{2x^2} + c$$

or $\quad 1 \quad 5x^3 \quad 5$

(ii) $$y' + y = y^2$$

or $$\frac{dy}{dx} + y = y^2$$

Divide by y^2; $$y^{-2}\frac{dy}{dx} + \frac{1}{y} = 1 \quad ...(1)$$

Put $$\frac{1}{y} = z \quad \therefore \quad -\frac{1}{y^2}\frac{dy}{dx} = \frac{dz}{dx}$$

Substituting in (1); $$-\frac{dz}{dx} + z = 1 \quad \text{or} \quad \frac{dz}{dx} - z = -1$$

which is linear differential equation in z
where $P = -1, Q = -1$

$$\text{I.F.} = e^{\int -1\,dx} = e^{-x}$$

Solution in z is $$z(\text{I.F.}) = \int Q\cdot(\text{I.F.})\,dx + c$$

or $$z e^{-x} = \int (-1)\, e^{-x}\,dx + c = e^{-x} + c$$

or $$z = 1 + ce^{x}$$

Substituting the value of z; $$\frac{1}{y} = 1 + ce^{x}$$

or $$y = \frac{1}{1 + ce^{x}}$$

(iii) $$\left(xy^2 - e^{\frac{1}{x^3}}\right)dx - x^2 y\, dy = 0$$

or $$x^2 y\frac{dy}{dx} - xy^2 + e^{\frac{1}{x^3}} = 0$$

or $$x^2 y\frac{dy}{dx} - xy^2 = -e^{\frac{1}{x^3}}$$

or $$y\frac{dy}{dx} - \frac{1}{x}y^2 = -\frac{1}{x^2}e^{\frac{1}{x^3}}$$

Put $y^2 = z$; $$2y\frac{dy}{dx} = \frac{dz}{dx}$$

$$\frac{1}{2}\frac{dz}{dx} - \frac{1}{x}z = -\frac{1}{x^2}e^{\frac{1}{x^3}}$$

or $$\frac{dz}{dx} - \frac{2}{x}z = -\frac{2}{x^2}e^{\frac{1}{x^3}}$$

which is linear differential equation in z, where $P = -\frac{2}{x}, Q = -\frac{2}{x^2} e^{\frac{1}{x^3}}$

$$\text{I.F.} = e^{\int \frac{-2}{x} dx} = e^{-2\log x} = e^{\log x^{-2}} = \frac{1}{x^2}$$

Solution in z is; $\quad z(\text{I.F.}) = \int Q \cdot (\text{I.F.})\, dx + c$

or $$z\left(\frac{1}{x^2}\right) = \int \frac{1}{x^2}\left(-\frac{2}{x^2} e^{\frac{1}{x^3}}\right) dx + c = -2\int \frac{1}{x^4} e^{\frac{1}{x^3}}\, dx + c$$

Put $\frac{1}{x^3} = t \quad \therefore \quad \frac{-3}{x^4} dx = dt$

$$\therefore \quad z \cdot \frac{1}{x^2} = (-2)\int e^t \frac{dt}{-3} + c = \frac{2}{3} e^t + c = \frac{2}{3} e^{\frac{1}{x^3}} + c$$

or $$\frac{y^2}{x^2} = \frac{2}{3} e^{\frac{1}{x^3}} + c \quad \text{or} \quad 3y^2 = 2x^2 e^{\frac{1}{x^3}} + cx^2$$

Example 2. *Solve :* $xy\left(1 + xy^2\right)\frac{dy}{dx} = 1.$ **(P.T.U., May 2009)**

Sol. The given equation can be written as $\frac{dx}{dy} - yx = y^3 x^2$

Dividing by x^2, we have $\quad x^{-2}\frac{dx}{dy} - y x^{-1} = y^3 \quad$ ...(1)

Putting $\quad x^{-1} = z$ so that $\quad -x^{-2}\frac{dx}{dy} = \frac{dz}{dy} \quad$ or $\quad x^{-2}\frac{dx}{dy} = -\frac{dz}{dy}$

Equation (1) becomes $-\frac{dz}{dy} - yz = y^3$

or $\quad \frac{dz}{dy} + yz = -y^3$, which is linear in z.

$$\text{I.F.} = e^{\int y\, dy} = e^{\frac{1}{2}y^2}$$

$\therefore$ The solution is $z\,(\text{I.F.}) = \int -y^3\,(\text{I.F.})\, dy + c$

or $$z\,.e^{\frac{1}{2}y^2} = \int -y^3 e^{\frac{1}{2}y^2}\, dy + c$$

or $$z.e^{\frac{1}{2}y^2} = -\int y^2 e^{\frac{1}{2}y^2}.y\, dy + c = -\int 2te^t dt + c, \text{ where } t = \frac{1}{2}y^2$$

or $$z.e^{\frac{1}{2}y^2} = -2\left[t\,e^t - \int 1 - e^t dt\right] + c = -2\left(te^t - e^t\right) + c = -2e^{\frac{1}{2}y^2}\left(\frac{1}{2}y^2 - 1\right) + c$$

or $$z = -2\left(\frac{1}{2}y^2 - 1\right) + ce^{-\frac{1}{2}y^2}$$

or $$\frac{1}{x} = 2 - y^2 + ce^{-\frac{1}{2}y^2}. \qquad \left(\because\ z = \frac{1}{x}\right)$$

Example 3. *Solve :* $\dfrac{dy}{dx} + x\sin 2y = x^3\cos^2 y$. **(P.T.U., May 2002)**

Sol. Dividing by $\cos^2 y$, we have $\sec^2 y\dfrac{dy}{dx} + x\dfrac{2\sin y\cos y}{\cos^2 y} = x^3$

or $$\sec^2 y\frac{dy}{dx} + 2x\tan y = x^3 \qquad ...(1)$$

Putting $\tan y = z$ so that $\sec^2 y\dfrac{dy}{dx} = \dfrac{dz}{dx}$

Equation (1) becomes $\dfrac{dz}{dx} + 2xz = x^3$, which is linear in z.

$$\text{I.F.} = e^{\int 2x\,dx} = e^{x^2}$$

$\therefore$ The solution is $z\,.\,e^{x^2} = \int x^3\,.\,e^{x^2}dx + c = \int x^2 e^{x^2}\,.\,x\,dx + c$

$$= \frac{1}{2}\int te^t\,dt + c, \quad \text{where } t = x^2$$

$$= \frac{1}{2}(t-1)e^t + c = \frac{1}{2}\left(x^2 - 1\right)e^{x^2} + c$$

or $$z = \frac{1}{2}\left(x^2 - 1\right) + ce^{-x^2}$$

or $$\tan y = \frac{1}{2}\left(x^2 - 1\right) + ce^{x^2}. \qquad (\because\ z = \tan y)$$

Example 4. *Solve :* $e^y y' = e^x(e^x - e^y)$. **(P.T.U., May 2004)**

Sol. $$e^y y' = e^x(e^x - e^y) \qquad ...(1)$$

Put $$e^y = z$$

Differentiating w.r.t. x $$e^y\frac{dy}{dx} = \frac{dz}{dx}$$

i.e., $$e^y y' = \frac{dz}{dx} \qquad ...(2)$$

Substituting in (1)

$$\frac{dz}{dx} = e^{2x} - e^x\,.\,z$$

or $\dfrac{dz}{dx} + e^x\,.\,z = e^{2x}$, which is a linear differential equation in z.

$$\text{I.F.} = e^{\int e^x dx} = e^{e^x}$$

Solution is $\quad z \cdot e^{e^x} = \int e^{2x} \cdot e^{e^x} dx \; c$

Put $\quad e^x = t \qquad \therefore \; e^x dx = dt$

$\therefore \quad z \cdot e^{e^x} = \int t e^t dt + c$ Integrate by parts

$$= (t-1)e^t + c$$

$\therefore \quad e^y \cdot e^{e^x} = \left(e^x - 1\right) e^{e^x} + c$ or $e^{e^x}\left(1 - e^x + e^y\right) = c$.

Example 5. *Solve : $(2x \log x - xy)\, dy = -2y\, dx$.* **(P.T.U., Dec. 2004)**

Sol. $\quad 2x \log x - xy = -2y \dfrac{dx}{dy}$

or $\quad 2y \dfrac{dx}{dy} - y \cdot x + 2x \log x = 0$

or $\quad \dfrac{dx}{dy} - \dfrac{1}{2} x + \dfrac{x}{y} \log x = 0$

Divide by x; $\quad \dfrac{1}{x}\dfrac{dy}{dx} + \dfrac{1}{y} \log x = \dfrac{1}{2}$

Put $\log x = z \quad \therefore \quad \dfrac{1}{x}\dfrac{dx}{dy} = \dfrac{dz}{dy}$

or $\quad \dfrac{dz}{dy} + \dfrac{1}{y} z = \dfrac{1}{2}$, which is linear differential equation in z.

$$\text{I.F.} = e^{\int \frac{1}{y} dy} = e^{\log y} = y$$

$\therefore$ Solution is $\quad z \cdot y = \int y \cdot \dfrac{1}{2} dy + c = \dfrac{y^2}{4} + c$

or $\quad y \log x = \dfrac{y^2}{4} + c.$

Example 6. *Solve : $\dfrac{dy}{dx} - \tan xy = -y^2 \sec^2 x$.* **(P.T.U., Dec. 2004)**

Sol. $\dfrac{dy}{dx} - \tan x \cdot y = -y^2 \sec^2 x$

Divide by y^2; $\quad \dfrac{1}{y^2}\dfrac{dy}{dx} - \tan x \dfrac{1}{y} = -\sec^2 x$

Put $\quad \dfrac{1}{y} = z \qquad \therefore \; -\dfrac{1}{y^2}\dfrac{dy}{dx} = \dfrac{dz}{dx}$

$$\therefore \qquad -\frac{dz}{dx} - \tan x \,.\, z = -\sec^2 x$$

or
$$\frac{dz}{dx} + \tan x \,.\, z = \sec^2 x$$

which is linear differential equation in z

$$\text{I.F.} = e^{\int \tan x\, dx} = e^{-\log \cos x} = e^{\log \sec x} = \sec x$$

$\therefore$ Its solution is

$$z \sec x = \int \sec^2 x \,.\, \sec x\, dx + c = \int \sec^3 x\, dx + c \qquad \text{...(1)}$$

Let
$$\text{I} = \int \sec^3 x\, dx = \int \sec x \sec^2 x\, dx \text{ Integrate by parts}$$

$$= (\sec x)(\tan x) - \int \sec x \tan x \tan x\, dx$$

$$= \sec x \tan x - \int \sec x (\sec^2 x - 1)\, dx$$

$$= \sec x \tan x - \text{I} + \int \sec x\, dx$$

$$\therefore \qquad 2\text{I} = \sec x \tan x + \log(\sec x + \tan x)$$

$$\therefore \qquad \text{I} = \frac{1}{2}\left[\sec x \tan x + \log(\sec x + \tan x)\right]$$

Substituting in equation (1),

$$z \,.\, \sec x = \frac{1}{2}\left[\sec x \tan x + \log(\sec x + \tan x)\right] + c$$

Substitute the value of z,

$$\frac{1}{y} \sec x = \frac{1}{2}\left[\sec x \tan x + \log(\sec x + \tan x)\right] + c.$$

TEST YOUR KNOWLEDGE

Solve the following differential equations:

1. $2\dfrac{dy}{dx} = \dfrac{y}{x} + \dfrac{y^2}{x^2}$

2. $\dfrac{dy}{dx} - x^2 y = y^2 e^{-\frac{1}{3}x^3}$

3. $(x^3 y^2 + xy)\, dx = dy$

4. (*a*) $e^y\left(\dfrac{dy}{dx} + 1\right) = e^x$ **(P.T.U., May 2002)**

[**Hint:** Put $e^y = z$]

(*b*) $(x+1)\dfrac{dy}{dx} + 1 = 2e^{-y}$

[**Hint:** Divide by $(x + 1)e^{-y}$ and Put $e^y = z$]

5. $\dfrac{dy}{dx} - \dfrac{\tan y}{1+x} = (1 + x)\, e^x \sec y$

6. $\dfrac{dy}{dx} + y \tan x = y^3 \cos x$

7. $\frac{dy}{dx}+\frac{x}{1-x^2}y=x\sqrt{y}$

8. $\frac{dy}{dx}+\frac{y\log y}{x}=\frac{y(\log y)^2}{x^2}$

9. $y-\cos x\frac{dy}{dx}=y^2(1-\sin x)\cos x$, given that $y=2$ when $x=0$.

10. $y\left(2xy+e^x\right)dx=e^x\,dy$

11. $\tan y\frac{dy}{dx}+\tan x=\cos y\cos^2 x$

ANSWERS

1. $\frac{x}{y}=1+c\sqrt{x}$

2. $y(c-x)=e^{\frac{1}{3}x^3}$

3. $\frac{1}{y}=-x^2+2+ce^{-\frac{1}{2}x^2}$

4. (a) $e^{x+y}=\frac{1}{2}e^{2x}+c$ (b) $(x+1)e^y=2x+c$

5. $\sin y=(1+x)(e^x+c)$

6. $\cos^2 x=y^2\left(c-2\sin x+\frac{2}{3}\sin^3 x\right)$

7. $\sqrt{y}=-\frac{1}{3}\left(1-x^2\right)+c\left(1-x^2\right)^{\frac{1}{4}}$

8. $\frac{1}{x\log y}=\frac{1}{2x^2}+c$

9. $2(\tan x+\sec x)=y(2\sin x+1)$

10. $e^x=y\left(c-x^2\right)$

11. $\sec y=(c+\sin x)\cos x$.

REVIEW OF THE CHAPTER

1. **Ordinary Differential Equation:** Differential equations which involve only one independent variable and the differential co-efficients w.r.t. it are called ordinary differential equations.
2. **Order and Degree of a Differential Equations: The order** of a differential equation is the order of the highest order derivative occurring in the differential equation. **The degree** of a differential equation is the degree of the highest order derivative which occurs in the differential equation.
3. **The general solution, the particular and the singular solution of a differential equation.**

   The general solution of a differential equation is that in which the number of independent arbitrary constants is equal to the order of differential equation.

   The particular solution of a differential equation is that which is obtained from the general solution by giving particular values to the arbitrary constants.

   The singular solution of a differential equation is that which satisfies the equation but cannot be derived from its general solution.
4. **Solution of differential equations of first order and first degree.**

   (a) **Variable separable form:** Put dx and all the terms containing x on one side, also dy and all the terms containing y on other side and integrate.

   (b) If $\frac{dy}{dx}=-f(ax+by+c), f$ then put $ax+by+c=t$ equation will be changed to variable separable form.

5. **Homogeneous Differential Equation:** A differential equation of the form $\frac{dy}{dx} = \frac{f_1(x, y)}{g_1(x, y)}$ is called a homogeneous differential equation if $f_1(x, y)$ and $g_1(x, y)$ are homogeneous functions of the same degree in x and y. To solve homogeneous differential equation put $\frac{y}{x}$ or $\frac{x}{y} = v$, equation will be changed to variable separable form.

6. For solution of the differential equation of the form $\frac{dy}{dx} = \frac{ax + by + c}{a'x + b'y + c'}$

Case I. If $\frac{a}{a'} \neq \frac{b}{b'}$, put $x = X + h$, $y = Y + k$ such that $ah + bk + c = 0$, $a'h + b'k + c' = 0$, equation will change to homogeneous form, then put $\frac{Y}{X} = V$ and in the end change X, Y to x, y.

Case II. If $\frac{a}{a'} = \frac{b}{b'}$ then put $ax + by = t$ differential equation is changed to variable separable form.

7. **Exact Differential Equation:** A differential equation obtained from its primitive directly by differentiation, without any operation of multiplication, elimination or reduction, etc., is called an exact differential equations.

8. **Necessary and Sufficient Condition for the exactness of a differential equation:** The necessary and sufficient conditions for the exactness of $Mdx + Ndy = 0$ is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and the solution is $\int_{y \text{ constant}} M dx + \int (\text{terms of N not containing } x)\, dy = c$.

9. **Integrating Factor:** If a differential equation is not exact but can be made exact after multiplying by a suitable function of (x or y or both) then that function is called integrating factor (I.F.). If a differential equation has one I.F., it has an infinite number of integrating factors.

10. The I.F. of $Mdx + Ndy = 0$ are

(*a*) If $Mdx + Ndy = 0$ is homogeneous differential equation then I.F. $= \frac{1}{Mx + Ny}$ provided $Mx + Ny \neq 0$.

If $Mx + Ny = 0$, then equation can be reduced to variable separable from by putting $N = -\frac{Mx}{y}$

(*b*) If $Mdx + Ndy = 0$ is of the form $y f_1(xy)\, dx + x f_2(xy)\, dy = 0$, then I.F. $= \frac{1}{Mx - Ny}$ provided $Mx - Ny \neq 0$.

If $Mx - Ny = 0$, then it reduces to variable separable form.

(*c*) If $Mdx + Ndy = 0$; $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$ is a function of x say $f(x)$ then I.F. $= e^{\int f(x)\, dx}$ and if $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$ is a function of y say $g(y)$, then I.F. $= e^{\int g(y)\, dy}$

(*d*) I.F. of the differential equation of the form $x^a y^b (mydx + nxdy) + x^c y^d (pydx + qxdy) = 0$ is $x^h y^k$, where h, k as so chosen that when multiplied with the given equation, changes the equation to exact equation.

11. If the differential equation is of first order and higher degree then to solve the equation replace $\frac{dy}{dx}$ by p

(*a*) If equation is solvable for p then find different values of p *i.e.*, $\frac{dy}{dx}$ and integrate each separately (all solutions having one arbitrary constant only) multiply all the factors formed by different solutions. That is the solution of the given differential equation.

(b) If equation is solvable for y: Express y as a function of x and p *i.e.*, $y=f(x, p)$ then differentiate w.r.t. x, equation will reduce to differential equation of first order in x and p. Solve and eliminate p with the help of given equation.

(c) If equation is solvable for x: Express x as a function of y and p *i.e.*, $x=f(y, p)$; then differentiate w.r.t. y, then equation will reduce to first order equation is y and p. Solve and eliminate p.

12. **Clairaut's Equation:** An equation of the form $y = px + f(p)$ is known as Clairaut's equation.

Its solution is $y = cx + f(c)$ *i.e.*, replace p by an arbitrary constant c.

13. **Leibnitz's Linear Equation:** A differential equation is said to the linear if the dependent variable and its derivatives occur only in the first degree and are not multiplied together.

The general form of linear differential equation is $\frac{dy}{dx} + \text{P}y = \text{Q}$, where P, Q are functions of x or constants.

Solution of linear differential equation is $y\,(\text{I.F.}) = \int \text{Q}\,(\text{I.F.})\,dx + c$, where I.F. $= e^{\int \text{P}\,dx}$

Similar method for $\frac{dy}{dx} + \text{P}x = \text{Q}$, where P, Q are functions of y or constant.

14. **Bernoulli's Form:** Any equation of the form $\frac{dy}{dx} + \text{P}y = \text{Q}y^n$, where P, Q are functions of x only is called Bernoulli's equation. To solve it, divide by y^n and put $y^{1-n} = z\,j$; it will reduce to linear equation in x and z whose solution is

$$z\,\text{I.F.} = \int \text{Q}\,\text{I.F.}\,dx + c, \quad \text{where I.F.} = e^{\int \text{P}\,dx}$$

Replace z by y^{1-n}

15. Differential equation $f'(y)\frac{dy}{dx} + \text{P}f(y) = \text{Q}$ can be reduced to linear differential equation by putting $f(y) = z$.

SHORT ANSWER TYPE QUESTIONS

1. Distinguish between order and degree of a differential equation. **(P.T.U., Jan. 2010)**

[**Hint :** See art. 1.1 (*iv*, *v*)]

2. Define complete solution of a differential equation.

Or

When a solution of a differential equation is called its general solution. **(P.T.U., Dec. 2005)**

[**Hint :** See art 1.1 (*vi*)]

3. How will you form a differential equation whose solution contains n parameters ? What will be the order of that differential equation ?

4. Verify that $y = cx + \frac{a}{c}$ and $y^2 = 4ax$ both are solutions of the same differential equation;

$$y = x\frac{dy}{dx} + a\frac{dx}{dy}.$$

5. Define a singular solution of a differential equation.

[**Hint :** Consult art. 1.1 (*vi*)]

6. Show that $y = x\,e^{2x}$ is a solution of $\dfrac{dy}{dx} = y\left(2+\dfrac{1}{x}\right)$.

7. Obtain the differential equations from the following equations:

(i) $y = Cx + C - C^2$

(ii) $y = A\cos mx + B\sin mx$, where m is fixed ; A, B are parameters.

(iii) $y = Ae^{x} + Be^{-x} + C$ **(P.T.U., May 2004)**

(iv) $y = e^{x}(A\cos x + B\sin x)$ **(P.T.U., June 2003)**

(v) $y = cx + c^2$ **(P.T.U., Dec. 2003)**

[**Hint:** See S.E. 1 (*i, ii, iii*) art. 1.3]

8. Find the differential equation of all circles passing through the origin and having centres on x-axis.

[**Hint:** See S.E. 2 art. 1.3]

9. Find the differential equation of all parabolas whose axes are parallel to y-axis.

[**Hint:** See S.E. 4 art. 1.3]

10. Solve the following differential equations:

(i) $e^{y}\left(1+x^2\right)\dfrac{dy}{dx} - 2x\left(1+e^{y}\right) = 0$ (ii) $\dfrac{dy}{dx} + \sqrt{\dfrac{1-y^2}{1-x^2}} = 0$

(iii) $xy\dfrac{dy}{dx} = 1 + x + y + xy$ [**Hint:** See S.E. 4 art. 1.4] **(P.T.U., Dec. 2003)**

(iv) $(1+x^3)\,dy - x^2y\,dx = 0$

(v) $x\cos x\cos y + \sin y\dfrac{dy}{dx} = 0$ [**Hint:** See S.E. 3 art. 1.4] **(P.T.U., May 2003)**

(vi) $\dfrac{dy}{dx} - x\tan(y-x) = 1$ [**Hint:** Put $y - x = t$]

(vii) $\sec^2 x\tan y\,dx + \sec^2 y\tan x\,dy = 0$ [**Hint:** Separate variables and integrate]

(viii) $\dfrac{dy}{dx} = \dfrac{y}{x}$. **(P.T.U., Dec. 2005)** (ix) $(y+x)\,dy = (y-x)\,dx$ **(P.T.U., May 2011)**

11. Explain briefly how to solve the differential equation:

(i) $\dfrac{dy}{dx} = \dfrac{ax+by+c}{a_1x+b_1y+c_1}$, where $\dfrac{a}{a_1} \neq \dfrac{b}{b_1}$ **(P.T.U., May 2003)**

(ii) $\dfrac{dy}{dx} = \dfrac{ax+by+c}{a_1x+b_1y+c_1}$, where $\dfrac{a}{a_1} = \dfrac{b}{b_1}$

12. (*a*) What is an exact differential equation? Check the exactness of the equation $(3x^2+2e^{y})\,dx + (2xe^{y}+3y^2)\,dy = 0$.

(P.T.U., Jan. 2009, May 2010)

(*b*) State necessary and sufficient conditions for the differential equation $M\,dx + N\,dy = 0$ to be exact.

(P.T.U., Jan. 2009)

13. Solve the following differential equations:

(i) $(x^2 - ay)\,dx = (ax - y^2)\,dy$ [**Hint:** See S.E. 3 art. 1.8] **(P.T.U., May 2005)**

(ii) $(y\cos x + 1)\,dx + \sin x\,dy = 0$

(iii) $\dfrac{dy}{dx} + \dfrac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$. **(P.T.U., May 2011)**

14. Define integrating factor of a differential equation and find I.F. of $(y-1)\,dx - x\,dy = 0$. **(P.T.U., May 2006)**
[**Hint:** See S.E. 2 art. 1.9(*a*)]

15. Find I.F. of the following differential equations:
(*i*) $y\,dx - x\,dy + 3x^2y^2 e^{x^3}\,dx = 0$ [**Hint:** See S.E. 1 art. 1.9(*a*)] **(P.T.U., Dec. 2003)**
(*ii*) $(x^2 + y^2 + x)\,dx + xy\,dy = 0$. [**Hint:** See S.E. 7 art. 1.9(*d*)]

16. Define Clairaut's equation and write its solution. **(P.T.U., May 2007)**

17. Solve the following differential equations:
(*i*) $(y - px)(p - 1) = p$
(*ii*) $p^2 - 7p + 12 = 0$ **(P.T.U., Dec. 2006)**
(*iii*) $p = \log(px - y)$ [**Hint:** See S.E. 1(*i*) art. 1.14] **(P.T.U., Dec. 2005, Dec. 2011)**
(*iv*) $\sin px \cos y = \cos px \sin y + p$. [**Hint:** See S.E. 1(*ii*) art. 1.14] **(P.T.U., May 2007)**
(*v*) $p = \sin(y - px)$. [**Hint:** Same as (*iv*) part] **(P.T.U., May 2011)**

18. (*i*) For the differential equation of the type $yf(xy)\,dx + xg(xy)\,dy = 0$, the I.F. is $\dfrac{1}{xy\left[f(xy) - g(xy)\right]}$. Justify it. **(P.T.U., Dec. 2004)**

(*ii*) For the differential equation $M\,dx + N\,dy = 0$; where M, N are homogeneous functions of x and y, the I.F. is $\dfrac{1}{Mx + Ny}$ $(Mx + Ny \neq 0)$. Justify it.

Also reduce $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)dy = 0$ to exact differential equation. [**Hint:** See S.E. 1.art. 1.9(*b*)] **(P.T.U., Dec. 2009)**

(*iii*) For the differential equation $Mdx + Ndy = 0$ if $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = f(x)$ then $e^{\int f(x)\,dx}$ is the I.F. Justify it.

(*iv*) For the differential equation $Mdx + Ndy = 0$ if $\dfrac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M} = g(y)$ then $e^{\int g(y)\,dy}$ is the I.F. Justify it.

19. For what value of k, the differential equation $\left(1 + e^{\frac{kx}{y}}\right)dx + e^{\frac{x}{y}}\left(1 - \dfrac{x}{y}\right)dy = 0$ is exact. **(P.T.U., May 2010)**

[**Hint:** See Solved Example 5 art. 1.8]

20. Define Leibnitz's linear differential equation of first order. Also give an example. **(P.T.U., May 2005, 2007)**

21. Solve $\dfrac{dy}{dx} + Py = Q$, where P, Q are functions of x or constants. **(P.T.U., June 2003, Dec. 2006)**

22. Define Bernoulli's linear differential equation and write its standard form. **(P.T.U., May 2007, Jan. 2009)**

23. How will you reduce $f'(y)\dfrac{dy}{dx} + P\,f(y) = Q$ to linear differential equation where, P, Q are function of x or constant?

24. Solve the following differential equations:

(*i*) $(x+1)\dfrac{dy}{dx} - y = e^x(x+1)^2$ (*ii*) $2\dfrac{dy}{dx} = \dfrac{y}{x} + \dfrac{y^2}{x^2}$ **(P.T.U., May 2007)** (*iii*) $x\frac{dy}{dx} + y = x^3y^6$ **(P.T.U., May 2011)** **[Hint:** See S.E. 1(*i*) art. 1.17]

(*iv*) $(x+1)\frac{dy}{dx} + 1 = 2e^{-y}$ **[Hint:** Divide by $(x+1)e^{-y}$; $e^y\frac{dy}{dx} + \frac{1}{x+1}e^y = \frac{2}{x+1}$; put $e^y = t$]

(*v*) $y' + y = y^2$ **(P.T.U., May 2008)** **[Hint:** See S.E. 1(*ii*) art. 1.17]

(*vi*) $(1+y^2)\,dx = (\tan^{-1}y - x)dy$ **(P.T.U., Dec. 2011)** **[Hint:** S.E. 3 art. 1.16]

ANSWERS

3. By differentiating the equation *n* times and then eliminating *n* parameters from *n* + 1 equations. (One is given equation and remaining *n* are the differential equations obtained by differentiating given equation *n* times)

7. (*i*) $y = (x+1)\frac{dy}{dx} - \left(\frac{dy}{dx}\right)^2$ (*ii*) $y_2 + m^2y = 0$ (*iii*) $y_3 = y_1$

(*iv*) $y_2 - 2y_1 + 2y = 0$ (*v*) $y = x\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2$

8. $x^2 - y^2 + 2xy\frac{dy}{dx} = 0$ **9.** $y_3 = 0$

10. (*i*) $(1+e^y) = A(1+x^2)$ (*ii*) $y\sqrt{1-x^2} + x\sqrt{1-y^2} = c$

(*iii*) $y = x + \log[x(1+y)] + c$ (*iv*) $y^3 = 4(x^3+1)$

(*v*) $x\sin x + \cos x = \log(\cos y) + c$ (*vi*) $\log\sin(y-x) = \frac{x^2}{2} + c$

(*vii*) $\tan x \tan y = c$ (*viii*) $y = cx$

(*ix*) $\log(x^2+y^2) = 2\tan^{-1}\frac{y}{x} + c$

12. (*a*) Exact equation

13. (*i*) $x^3 + y^3 - 3axy = c$ (*ii*) $y\sin x + x = c$

(*iii*) $y\sin x + (\sin y + y)x = c$

14. $\frac{1}{x^2}$

15. (*i*) $\frac{1}{y^2}$ (*ii*) x

17. (*i*) $y = cx + \frac{c}{c-1}$ (*ii*) $(y-4x-c)(y-3x-c) = 0$

(*iii*) $y = cx - e^c$ (*iv*) $y = cx - \sin^{-1}c$.

(*v*) $y = cx + \sin^{-1}c$.

18. $d\left(\frac{x}{y}\right) - d(2\log x) + d(3\log y) = 0$ or $d\left(\frac{x}{y} - \log x^2 + \log y^3\right) = 0$

24. (*i*) $y = (x+1)(e^x + c)$ (*ii*) $\frac{x}{y} = 1 + c\sqrt{x}$ (*iii*) $\frac{1}{y^5} = \frac{5}{2}x^3 + cx^5$