CHAPTER 4

Differential Calculus & Its Applications

4.1 (1) SUCCESSIVE DIFFERENTIATION

The reader is already familiar with the process of differentiating a function $y = f(x)$. For ready reference, a list of derivatives of some standard functions is given in the beginning.

The derivative dy/dx is, in general, another function of x which can be differentiated. The derivative of dy/dx is called the *second derivative* of y and is denoted by d^2y/dx^2. Similarly, the derivative of d^2y/dx^2 is called the *third derivative* of y and is denoted by d^3y/dx^3. In general, the nth derivative of y is denoted by d^ny/dx^n.

Alternative notations for the successive derivatives of $y = f(x)$ are

$$Dy, D^2y, D^3y, \ldots, D^ny\ ;$$

or $$y_1, y_2, y_3, \ldots, y_n\ ;$$

or $$f'(x), f''(x), f'''(x) \ldots, f^n(x).$$

The nth derivative of $y = f(x)$ at $x = a$ is denoted by $(d^ny/dx^n)_a$, $(y_n)_a$ or $f^n(a)$.

Example 4.1. *If $y = e^{ax} \sin bx$, prove that $y_2 - 2ay_1 + (a^2 + b^2)y = 0$.* (Cochin, 2005)

Solution. We have $$y = e^{ax} \sin bx \qquad \ldots(i)$$

$\therefore$ $$y_1 = e^{ax} (\cos bx \,.\, b) + \sin bx\, (e^{ax} \,.\, a) = be^{ax} \cos bx + ay \qquad \text{[By } (i)]$$

or $$y_1 - ay = be^{ax} \cos bx \qquad \ldots(ii)$$

Again differentiating both sides,

$$y_2 - ay_1 = be^{ax} (-\sin bx \,.\, b) + b \cos bx\, (e^{ax} \,.\, a) = -b^2y + a(y_1 - ay)$$

or $$y_2 - 2ay_1 + (a^2 + b^2)y = 0.$$

Example 4.2. *If $x = a (\cos t + t \sin t)$, $y = a (\sin t - t \cos t)$, find d^2y/dx^2.*

Solution. We have $$\frac{dx}{dt} = a(-\sin t + t \cos t + \sin t) = at \cos t$$

and $$\frac{dy}{dt} = a(\cos t + t \sin t - \cos t) = at \sin t$$

$$\therefore \qquad \frac{dy}{dx} = \frac{dy}{dt}\bigg/\frac{dx}{dt} = \frac{at \sin t}{at \cos t} = \tan t$$

$$\therefore \qquad \frac{d^2y}{dx^2} = \frac{d}{dt}(\tan t) \cdot \frac{dt}{dx} = \sec^2 t \cdot \frac{1}{at \cos t} = 1/at \cos^3 t.$$

Example 4.3. *Given $y^2 = f(x)$, a polynomial of third degree, then evaluate* $\frac{d}{dx}\left(y^3 \frac{d^2y}{dx^2}\right)$.

Solution. Differentiating $y^2 = f(x)$ w.r.t. x, we get

$$2y \frac{dy}{dx} = f'(x) \qquad \text{...}(i)$$

Differentiating (i) w.r.t. x again, we obtain

$$2\left(\frac{dy}{dx} \cdot \frac{dy}{dx} + y \frac{d^2y}{dx^2}\right) = f''(x) \quad \text{or} \quad 2\left(\frac{dy}{dx}\right)^2 + 2y \frac{d^2y}{dx^2} = f''(x)$$

Again differentiating, we get

$$4 \cdot \frac{dy}{dx} \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} \frac{d^2y}{dx^2} + 2y \frac{d^3y}{dx^3} = f'''(x)$$

or

$$3y^2 \frac{dy}{dx} \frac{d^2y}{dx^2} + y^3 \frac{d^3y}{dx^3} = \frac{1}{2} y^2 f'''(x) \qquad \text{[Multiplying by } y^2]$$

Hence

$$\frac{d}{dx}\left(y^3 \frac{d^2y}{dx^2}\right) = \frac{1}{2} f(x) f'''(x). \qquad [\because \; y^2 = f(x)]$$

Example 4.4. *If $ax^2 + 2hxy + by^2 = 1$, prove that* $\frac{d^2y}{dx^2} = \frac{h^2 - ab}{(hx + by)^3}$.

Solution. Differentiating the given equation w.r.t. x,

$$2ax + 2h\left(x \frac{dy}{dx} + y\right) + 2by \frac{dy}{dx} = 0 \quad \text{or} \quad \frac{dy}{dx} = -\frac{ax + hy}{hx + by} \qquad \text{...}(i)$$

Differentiating both sides of (i) w.r.t. x,

$$\frac{d^2y}{dx^2} = -\frac{(hx + by)(a + h\,dy/dx) - (ax + hy)(h + b\,dy/dx)}{(hx + by)^2}$$

[Substituting the value of dy/dx from (i)]

$$= -\frac{(hx + by)\left(a - h \cdot \frac{ax + hy}{hx + by}\right) - (ax + hy)\left(h - b \cdot \frac{ax + hy}{hx + by}\right)}{(hx + by)^2}$$

$$= \frac{(h^2 - ab)(ax^2 + 2hxy + by^2)}{(hx + by)^3}$$

$$= (h^2 - ab)/(hx + by)^3 \qquad [\because \; ax^2 + 2hxy + by^2 = 1]$$

PROBLEMS 4.1

1. If $y = (ax + b)/(cx + d)$, show that $2y_1 y_3 = 3y_2^2$.
2. If $y = \sin(\sin x)$, prove that $\frac{d^2y}{dx^2} + \tan x \frac{dy}{dx} + y \cos^2 x = 0$.
3. If $y = e^{-kt} \cos(lt + c)$, show that $\frac{d^2y}{dt^2} + 2k \frac{dy}{dt} + n^2 y = 0$, where $n^2 = k^2 + l^2$.

4. If $y = \sinh [m \log \{x + \sqrt{(x^2+1)}\}]$, show that $(x^2+1)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} = m^2 y$.

5. If $y = \sin^{-1} x$, show that $(1-x^2)\, y_5 - 7xy_4 - 9y_3 = 0$. *(Madras, 2000 S)*

6. If $x = \dfrac{1}{2}\left(t + \dfrac{1}{t}\right), y = \dfrac{1}{2}\left(t - \dfrac{1}{t}\right)$, find $\dfrac{d^2y}{dx^2}$. *(Cochin, 2005)*

7. If $x = 2\cos t - \cos 2t$, $y = 2\sin t - \sin 2t$, find the value of d^2y/dx^2 when $t = \pi/2$.

8. If $x = a(\cos t + \log \tan t/2)$, $y = a \sin t$, find d^2y/dx^2.

9. If $x = \sin t$, $y = \sin pt$, prove that $(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + p^2 y = 0$.

10. If $x^3 + y^3 = 3axy$, prove that $\dfrac{d^2y}{dx^2} = -\dfrac{2a^2xy}{(y^2-ax)^3}$.

(2) Standard Results

We have (1) $\mathbf{D^n (ax+b)^m = m(m-1)(m-2) \ldots (m-n+1)\, a^n (ax+b)^{m-n}}$

(2) $\mathbf{D^n \left(\dfrac{1}{ax+b}\right) = \dfrac{(-1)^n (n!)\, a^n}{(ax+b)^{n+1}}}$

(3) $\mathbf{D^n \log (ax+b) = \dfrac{(-1)^{n-1} (n-1)!\, a^n}{(ax+b)^n}}$

(4) $\mathbf{D^n (a^{mx}) = m^n (\log a)^n \,.\, a^{mx}}$

(5) $\mathbf{D^n (e^{mx}) = m^n e^{mx}}$

(6) $\mathbf{D^n \sin (ax+b) = a^n \sin (ax+b+n\pi/2)}$

(7) $\mathbf{D^n \cos (ax+b) = a^n \cos (ax+b+n\pi/2)}$

(8) $\mathbf{D^n [e^{ax} \sin (bx+c)] = (a^2+b^2)^{n/2} e^{ax} \sin (bx+c+n \tan^{-1} b/a)}$

(9) $\mathbf{D^n [e^{ax} \cos (bx+c)] = (a^2+b^2)^{n/2} e^{ax} \cos (bx+c+n \tan^{-1} b/a)}$

To prove (1), let $\quad y = (ax+b)^m$

$$y_1 = m \,.\, a(ax+b)^{m-1}$$
$$y_2 = m(m-1)a^2 (ax+b)^{m-2}$$
$$y_3 = m(m-1)(m-2)\, a^3 (ax+b)^{m-3}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

Hence $\quad y_n = m(m-1)(m-2) \ldots (m-n+1)\, a^n (ax+b)^{m-n}$

In particular, $\quad \mathbf{D^n (x^n) = n\,!}$.

(2) follows from (1) by taking $m = -1$. The proof of (3) is left as an exercise for the student.

To prove (4), let $\quad y = a^{mx}$

$$y_1 = m \log a \,.\, a^{mx},\ y_2 = (m \log a)^2 a^{mx}, \text{ etc.}$$

In general $\quad y_n = (m \log a)^n a^{mx}$.

(5) follows from (4) by taking $a = e$.

To prove (6), let $\quad y = \sin (ax+b)$

$\therefore$
$$y_1 = a \cos (ax+b) = a \sin (ax+b+\pi/2)$$
$$y_2 = a^2 \cos (ax+b+\pi/2) = a^2 \sin (ax+b+2\pi/2)$$
$$y_3 = a^3 \cos (ax+b+2\pi/2) = a^3 \sin (ax+b+3\pi/2)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

In general, $\quad y_n = a^n \sin (ax+b+n\pi/2)$.

The proof of (7), is left as an exercise for the reader.

To prove (8), let $\quad y = e^{ax} \sin (bx+c)$

$\therefore$
$$y_1 = e^{ax} \cos (bx+c) \,.\, b + ae^{ax} \sin (bx+c)$$
$$= e^{ax} [a \sin (bx+c) + b \cos (bx+c)]$$

Put $a = r\cos\alpha$, $b = r \sin \alpha$ so that $r = \sqrt{(a^2+b^2)}$, $\alpha = \tan^{-1} b/a$

$\therefore$
$$y_1 = re^{ax} [\sin (bx+c) \cos \alpha + \cos (bx+c) \sin \alpha]$$
$$= re^{ax} \sin (bx+c+\alpha)$$

Similarly,
$$y_2 = r^2 e^{ax} \sin (bx+c+2\alpha)$$
$$y_3 = r^3 e^{ax} \sin (bx+c+3\alpha)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

In general, $y_n = r^n e^{ax} \sin(bx + c + n\alpha)$ (V.T.U., 2000)

where $r = \sqrt{(a^2 + b^2)}$ and $\alpha = \tan^{-1} b/a$.

Proceeding as in (8), the student should prove (9) himself.

(3) **Preliminary transformations.** Quite often preliminary simplification reduces the given function to one of the above standard forms and then the nth derivative can be written easily.

To find the nth derivative of the powers of sines or cosines or their products, we first express each of these as a series of sines or cosines of multiple angles and then use the above formulae (6) and (7).

Example 4.5. *If* $y = x \log \dfrac{x-1}{x+1}$, *show that* $y_n = (-1)^{n-2}(n-2)!\left[\dfrac{x-n}{(x-1)^n} - \dfrac{x+n}{(x+1)^n}\right]$. (U.P.T.U., 2003)

Solution. Differentiating y w.r.t. x, we have

$$y_1 = \log \frac{x-1}{x+1} + x\left[\frac{1}{x-1} - \frac{1}{x+1}\right]$$

$$= \log(x-1) - \log(x+1) + \frac{1}{x-1} + \frac{1}{x+1} \qquad ...(i)$$

Now differentiating (i) $(n-1)$ times w.r.t. x,

$$y_n = \frac{(-1)^{n-2}(n-2)!}{(x-1)^{n-1}} - \frac{(-1)^{n-2}(n-2)!}{(x+1)^{n-1}} + \frac{(-1)^{n-1}(n-1)!}{(x-1)^n} + \frac{(-1)^{n-1}(n-1)!}{(x+1)^n}$$

$$= (-1)^{n-2}(n-2)!\left\{\frac{x-1}{(x-1)^n} - \frac{x+1}{(x+1)^n} + \frac{-(n-1)}{(x-1)^n} + \frac{-(n-1)}{(x+1)^n}\right\}$$

$$= (-1)^{n-2}(n-2)!\left[\frac{x-n}{(x-1)^n} - \frac{x+n}{(x+1)^n}\right].$$

Example 4.6. *Find the nth derivative of (i)* $\cos x \cos 2x \cos 3x$ (S.V.T.U., 2009)
(ii) $e^{2x} \cos^2 x \sin x$.

Solution. (i) $y = \cos x \cos 2x \cos 3x = \frac{1}{2}\cos x(\cos 5x + \cos x)$

$$= \tfrac{1}{4}(2\cos x \cos 5x + 2\cos^2 x) = \tfrac{1}{4}[(\cos 6x + \cos 4x) + (1 + \cos 2x)]$$

$$= \tfrac{1}{4}(1 + \cos 2x + \cos 4x + \cos 6x)$$

$$\therefore \quad y_n = \tfrac{1}{4}[2^n \cos(2x + n\pi/2) + 4^n \cos(4x + n\pi/2 + 6^n \cos(6x + n\pi/2)]$$

(ii) $\cos^2 x \sin x = \cos x(\sin x \cos x) = \cos x \cdot \frac{1}{2}\sin 2x$

$$= \tfrac{1}{4}(2\sin 2x \cos x) = \tfrac{1}{4}(\sin 3x + \sin x)$$

$$\therefore \quad D^n(e^{2x}\cos^2 x \sin x) = \tfrac{1}{4}[D^n(e^{2x}\sin 3x) + D^n(e^{2x}\sin x)]$$

$$= \tfrac{1}{4}[(2^2 + 3^2)^{n/2}\sin(3x + n\tan^{-1}3/2) + (2^2 + 1^2)^{n/2}\sin(x + n\tan^{-1}\tfrac{1}{2})]$$

$$= \tfrac{1}{4}[(13)^{n/2}\sin(3x + n\tan^{-1}3/2) + (5)^{n/2}\sin(x + n\tan^{-1}\tfrac{1}{2})].$$

(4) Use of partial fractions. To find the nth derivative of any rational algebraic fraction, we first split it up into partial fractions. Even when the denominator cannot be resolved into real factors, the method of partial fractions can still be used after breaking the denominator into complex linear factors. Then to put the result back in a real form, we apply De Moivre's theorem (p. 647).

Example 4.7. *Find the nth derivative of* $\dfrac{x}{(x-1)(2x+3)}$.

Solution. $$\frac{x}{(x-1)(2x+3)} = \frac{1}{(x-1)(2.1+3)} + \frac{-3/2}{(-3/2-1)(2x+3)}$$

$$= \frac{1}{5}\cdot\frac{1}{x-1} + \frac{3}{5}\cdot\frac{1}{2x+3}$$

Hence $$D^n\left[\frac{x}{(x-1)(2x+3)}\right] = \frac{1}{5}\cdot\frac{(-1)^n\, n!}{(x-1)^{n+1}} + \frac{3}{5}\cdot\frac{(-1)^n\,(n!)2^n}{(2x+3)^{n+1}}$$

$$= \frac{(-1)^n\, n!}{5}\left\{\frac{1}{(x-1)^{n+1}} + \frac{3.2^n}{(2x+3)^{n+1}}\right\}.$$

Example 4.8. *Find the nth derivative of* $\dfrac{1}{x^2+a^2}$.

Solution. We have $$y = \frac{1}{x^2+a^2} = \frac{1}{(x+ia)(x-ia)} = \frac{1}{2ia}\left(\frac{1}{x-ia} - \frac{1}{x+ia}\right)$$

$\therefore$ $$y_n = \frac{1}{2ia}\left\{\frac{(-1)^n\, n!}{(x-ia)^{n+1}} - \frac{(-1)^n\, n!}{(x+ia)^{n+1}}\right\}$$

[Put $x = r\cos\theta$, $a = r\sin\theta$ so that $r = \sqrt{(x^2+a^2)}$, $\theta = \tan^{-1}(a/x)$]

$$= \frac{(-1)^n\, n!}{2ia}\left\{\frac{1}{r^{n+1}(\cos\theta - i\sin\theta)^{n+1}} - \frac{1}{r^{n+1}(\cos\theta + i\sin\theta)^{n+1}}\right\}$$

$$= \frac{(-1)^n\, n!}{2iar^{n+1}}\{(\cos\theta - i\sin\theta)^{-(n+1)} - (\cos\theta + i\sin\theta)^{-(n+1)}\}$$

$$= \frac{(-1)^n\, n!}{2iar^{n+1}}[\cos(n+1)\theta + i\sin(n+1)\theta - [\cos(n+1)\theta - i\sin(n+1)\theta]]$$

[By De Moivre's theorem]

$$= \frac{(-1)^n\, n!}{2iar^{n+1}}\,.\,2i\sin(n+1)\theta \qquad \left[\text{Put } \frac{1}{r} = \frac{\sin\theta}{a}\right]$$

$$= \frac{(-1)^n\, n!}{a^{n+2}}\sin(n+1)\theta\,\sin^{n+1}\theta.$$

PROBLEMS 4.2

Find the nth derivative of (1 to 11) :

1. $\log(4x^2-1)$ (V.T.U., 2010)
2. $\dfrac{x+2}{x+1} + \log\dfrac{x+2}{x+1}$
3. $\sin^3 x\cos^2 x$ (V.T.U., 2006)
4. $\cos^9 x$ (Mumbai, 2008)
5. $\sinh 2x\sin 4x$ (V.T.U., 2010 S)
6. $e^{5x}\cos x\cos 3x$ (Mumbai, 2007)
7. $\dfrac{x+3}{(x-1)(x+2)}$ (V.T.U., 2009)
8. $\dfrac{x^2}{2x^2+7x+6}$ (V.T.U., 2005)
9. $\dfrac{1}{1+x+x^2+x^3}$ (Mumbai, 2009)
10. $\dfrac{x}{x^2+a^2}$ (Mumbai, 2007)
11. Find the nth derivative of $\tan^{-1}\dfrac{2x}{1-x^2}$ in terms of r and θ. (U.P.T.U., 2002)

4.2 LEIBNITZ'S THEOREM for the *n*th Derivative of the product of two functions*

If u, v be two function of x possessing derivatives of the nth order, then

$$(\mathbf{uv})_n = \mathbf{u}_n\,\mathbf{v} + {}^n\mathbf{C}_1\,\mathbf{u}_{n-1}\,\mathbf{v}_1 + {}^n\mathbf{C}_2\,\mathbf{u}_{n-2}\,\mathbf{v}_2 + \ldots + {}^n\mathbf{C}_r\mathbf{u}_{n-r}\,\mathbf{v}_r + \ldots + {}^n\mathbf{C}_n\,\mathbf{uv}_n$$

We shall prove this theorem by mathematical induction.

Step I. By actual differentiation,

$$(uv)_1 = u_1 v + uv_1$$

$$(uv)_2 = (u_2 v + u_1 v_1) + (u_1 v_1 + uv_2)$$

$$= u_2 v + {}^2C_1\,u_1 v_1 + {}^2C_2\,uv_2 \qquad [\because\ 2 = {}^2C_1,\ 1 = {}^2C_2]$$

Thus we see that the theorem is true for $n = 1, 2$.

Step II. Assume the theorem to be true for $n = m$ (say) so that

$$(uv)_m = u_m v + {}^mC_1\,u_{m-1} v_1 + {}^mC_2\,u_{m-2} v_2 + \ldots + {}^mC_{r-1} u_{m-r+1} v_{r-1} + {}^mC_r\,u_{m-r}\,v_r + \ldots + {}^mC_m\,uv_m$$

Differentiating both sides,

$$(uv)_{m+1} = (u_{m+1} v + u_m v_1) + {}^mC_1\,(u_m v_1 + u_{m-1} v_2) + {}^mC_2\,(u_{m-1} v_2 + u_{m-2} v_3) + \ldots$$
$$+ {}^mC_{r-1}\,u_{m-r+2}\,v_{r-1} + u_{m-r+1}\,v_r) + {}^mC_r\,(u_{m-r+1}\,v_r + u_{m-r} v_{r+1}) + \ldots$$
$$+ {}^mC_m\,(u_1 v_m + uv_{m+1})$$
$$= u_{m+1} v + (1 + {}^mC_1)\,u_m v_1 + ({}^mC_1 + {}^mC_2)\,u_{m-1} v_2 + \ldots$$
$$+ ({}^mC_{r-1} + {}^mC_r)\,u_{m-r+1}\,v_r + \ldots + {}^mC_m\,uv_{m+1}$$

But $\quad 1 + {}^mC_1 = {}^mC_0 + {}^mC_1 = {}^{m+1}C_1,\ {}^mC_1 + {}^mC_2 = {}^{m+1}C_2 \ldots$

$${}^mC_{r-1} + {}^mC_r = {}^{m+1}C_r, \ldots \quad \text{and} \quad {}^mC_m = 1 = {}^{m+1}C_{m+1}$$

$$\therefore \quad (uv)_{m+1} = u_{m+1} v + {}^{m+1}C_1\,u_m v_1 + {}^{m+1}C_2\,u_{m-1} v_2 + \ldots + {}^{m+1}C_r\,u_{m+1-r} v_r + \ldots + {}^{m+1}C_{m+1} uv_{m+1}$$

which is of exactly the same form as the given formula with n replaced by $m + 1$. Hence if the theorem is true for $n = m$, it is also true for $n = m + 1$.

Step III. In step I, the theorem has been seen to be true for $n = 2$, and by step II, it must be true for $n = 2 + 1$ *i.e.*, 3 and so for $n = 3 + 1$ *i.e.*, 4 and so on.

Hence the theorem is true for all positive integral values of n.

Example 4.9. *Find the nth derivative of $e^x (2x + 3)^3$.*

Solution. Take $u = e^x$ and $v = (2x + 3)^3$, so that $u_n = e^x$ for all integral values of n, and $v_1 = 6\,(2x + 3)^2$, $v_2 = 24(2x + 3)$, $v_3 = 48$, v_4, v_5 etc. are all zero.

$\therefore$ By Leibnitz's theorem,

$$(uv)_n = u_n v + {}^nC_1\,u_{n-1} v_1 + {}^nC_2\,u_{n-2} v_2 + {}^nC_3\,u_{n-3} v_3$$

i.e.,

$$[e^x (2x + 3)^3]_n = e^x (2x + 3)^3 + ne^x[6\,(2x + 3)^2]$$
$$+ \frac{n(n-1)}{1\,.\,2} e^x\,[24(2x + 3)] + \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}\,e^x\,[48]$$
$$= e^x\,\{(2x + 3)^3 + 6n(2x + 3)^2 + 12n\,(n - 1)\,(2x + 3) + 8n(n - 1)\,(n - 2)\}.$$

Example 4.10. *If $y = (\sin^{-1} x)^2$, show that $(1 - x^2)\,y_{n+2} - (2n + 1)\,xy_{n+1} - n^2\,y_n = 0$. Hence find $(y_n)_0$.* (*U.P.T.U., 2005*)

Solution. We have $\quad y = (\sin^{-1} x)^2$

Differentiating, $\quad y_1 = \dfrac{2 \sin^{-1} x}{\sqrt{(1 - x^2)}}$ or $(1 - x^2)\,y_1{}^2 = 4\,(\sin^{-1} x)^2 = 4y \qquad \ldots(i)$

Again differentiating, $\quad (1 - x^2)\,2y_1 y_2 + (-2x)\,y_1{}^2 = 4y_1 \qquad \ldots(ii)$

Dividing by $2y_1$, $(1 - x^2)\,y_2 - xy_1 - 2 = 0$

Differentiating it n times by Leibnitz's theorem,

*Named after the German mathematician and philosopher *Gottfried Wilhelm Leibnitz* (1646–1716) who invented the differential and integral calculus independent of Sir Issac Newton.

$$(1-x^2)\,y_{n+2} + n(-2x)\,y_{n+1} + \frac{n(n-1)}{2}\,(-2)y_n - [xy_{n+1} + n(1)\,y_n] = 0$$

or $$(1-x^2)\,y_{n+2} - (2n+1)\,xy_{n+1} - n^2\,y_n = 0$$

which is the required result.

Putting $x = 0$, $(y_{n+2})_0 = n^2(y_n)_0$ ...(*iii*)

From (*i*), $(y_1)_0 = 0$. From (*ii*), $(y_2)_0 = 2$.

Putting $n = 1, 3, 5, 7, \ldots$ in (*iii*), $0 = y_1 = y_3 = y_5 = y_7 = \ldots$

i.e., if n is odd, $(y_n)_0 = 0$

Again putting $n = 2, 4, 6, \ldots$ in (*iii*)

$$(y_4)_0 = 2^2\,(y_2)_0 = 2\,.\,2^2$$
$$(y_6)_0 = 4^2\,(y_6)_0 = 2.\,2^2\,.\,4^2$$
$$(y_8)_0 = 6^2\,(y_6)_0 = 2.\,2^2\,.\,4^2\,.\,6^2$$

In general, if n is even, $(y_n)_0 = 2\,.\,2^2\,.\,4^2\,.\,6^2 \ldots (n-2)^2, (n \neq 2)$.

Example 4.11. *If $y = e^{a \sin^{-1} x}$, prove that $(1-x^2)\,y_{n+2} - (2n+1)\,x\,y_{n+1} - (n^2+a^2)y_n = 0$. Hence find the value of y_n when $x = 0$.* (V.T.U., 2003)

Solution. We have $y = e^{a \sin^{-1} x}$ ...(*i*)

Differentiating, $y_1 = e^{a\sin^{-1} x}\,\dfrac{a}{\sqrt{(1-x^2)}} = \dfrac{ay}{\sqrt{(1-x^2)}}$ ...(*ii*)

or $$(1-x^2)\,y_1^{\,2} = a^2\,y^2.$$

Again differentiating, $(1-x^2)\,2y_1y_2 + (-2x)y_1^{\,2} = 2a^2\,yy_1$.

Dividing by $2y_1$, $(1-x^2)y_2 - xy_1 - a^2y = 0$ ...(*iii*)

Differentiating it n times by Leibnitz's theorem,

$$(1-x^2)\,y_{n+2} + n\,.\,(-2x)\,y_{n+1} + \frac{n(n-1)}{2}\,.\,(-2)\,y_n - [xy_{n+1} + n\,.\,1\,.\,y_n] - a^2\,y_n = 0$$

or $$(1-x^2)y_{n+2} - (2n+1)\,xy_{n+1} - (n^2+a^2)\,y_n = 0$$

which is the required result.

Putting $x = 0$, $(y_{n+2})_0 = (n^2+a^2)\,(y_n)_0$ ...(*iv*)

From (*i*), (*ii*), (*iii*) : $(y)_0 = 1, (y_1)_0 = a, (y_2)_0 = a^2$

Putting $n = 1, 2, 3, 4 \ldots$ in (*iv*),

$$(y_3)_0 = (1^2+a^2)\,(y_1)_0 = a(1^2+a^2)$$
$$(y_4)_0 = (2^2+a^2)\,(y_2)_0 = a^2(2^2+a^2)$$
$$(y_5)_0 = (3^2+a^2)\,(y_3)_0 = a(1^2+a^2)\,(3^2+a^2)$$
$$(y_6)_0 = (4^2+a^2)\,(y_4)_0 = a^2(2^2+a^2)\,(4^2+a^2).$$

Hence in general, $(y_n)_0 = a(1^2+a^2)\,(3^2+a^2) \ldots [(n-2)^2+a^2]$, when n is odd.

$= a^2(2^2+a^2)\,(4^2+a^2) \ldots\ldots [(n-2)^2+a^2]$, when n is even.

Example 4.12. *If $y^{1/m} + y^{-1/m} = 2x$, prove that*

$$(x^2-1)y_{n+2} + (2n+1)\,xy_{n+1} + (n^2-m^2)\,y_n = 0.$$

(V.T.U., 2008 S ; Mumbai, 2007 ; S.V.T.U., 2007)

Solution. We have $y^{1/m} + \dfrac{1}{y^{1/m}} = 2x$

or $$(y^{1/m})^2 - 2x(y^{1/m}) + 1 = 0$$

$$\therefore \quad y^{1/m} = \frac{2x \pm \sqrt{(4x^2-4)}}{2} = x \pm \sqrt{(x^2-1)}$$

Hence $$y = [x \pm \sqrt{(x^2-1)}\,]^m$$

Taking logarithm, $\log y = m \log [x \pm \sqrt{(x^2-1)}\,]$

Differentiating both sides w.r.t. x,

$$\frac{1}{y}y_1 = m \cdot \frac{1}{x \pm \sqrt{(x^2-1)}} \cdot \left\{1 \pm \frac{x}{\sqrt{(x^2-1)}}\right\} = \pm \frac{m}{\sqrt{(x^2-1)}}$$

Squaring, $\quad y_1^2 (x^2-1) = m^2 y^2$

Again differentiating, $\quad (x^2-1)\, 2y_1 y_2 + y_1^2 (2x) = m^2 \cdot 2y \cdot y_1$

Dividing by $2y_1$, $\quad (x^2-1) y_2 + xy_1 - m^2 y = 0$

Differentiating it n times by Leibnitz's theorem,

$$(x^2-1) y_{n+2} + ny_{n+1}(2x) + \frac{n(n-1)}{2} y_n (2) + xy_{n+1} + n \cdot y_n (1) - m^2 y_n = 0$$

or $$(x^2-1) y_{n+2} + (2n+1) xy_{n+1} + (n^2 - m^2) y_n = 0.$$

PROBLEMS 4.3

1. Find the nth derivative of (*i*) $x^2 \log 3x$. (*ii*) $2^x \cos^9 x$. *(Mumbai, 2009)*
2. If $y = a \cos(\log x) + b \sin(\log x)$, show that $x^2 y_2 + xy_1 + y = 0$ and $x^2 y_{n+2} + (2n+1) xy_{n+1} + (n^2+1) y_n = 0$. *(U.P.T.U., 2004 ; Madras, 2000)*
3. If $y = \sin^{-1} x$, prove that $(1-x^2) y_{n+2} - (2n+1) xy_{n+1} - n^2 y_n = 0$. Also find (y_n). *(S.V.T.U., 2009)*
4. If $\cos^{-1}(y/b) = \log(x/n)^n$, prove that $x^2 y_{n+2} + (2n+1) xy_{n+1} + 2n^2 y_n = 0$. *(U.P.T.U., 2006)*
5. If $y = \tan^{-1} x$, prove that $(1+x^2) y_{n+1} + 2nxy_n + n(n-1) y_{n-1} = 0$. Find $y_{n(0)}$.
6. If $y = \cos(m \sin^{-1} x)$, prove that $(1-x^2) y_{n+2} - (2n+1) xy_{n+1} + (m^2 - n^2) y_n = 0$. *(Mumbai, 2008 S)*
7. If $y = \sin(m \sin^{-1} x)$, prove that $(1-x^2) y_2 - xy_1 + m^2 y = 0$
   and $\quad (1-x^2) y_{n+2} - 2(n+1) xy_{n+1} + (m^2 - n^2) y_n = 0$. *(V.T.U., 2009 ; Cochin, 2005)*
   Also find $(y_n)_0$. *(U.P.T.U., 2005)*
8. If $y = e^{m \cos^{-1} x}$, prove that (*i*) $(1-x^2) y_2 - xy_1 = m^2 y$
   (*ii*) $(1-x^2) y_{n+2} - (2n+1) xy_{n+1} - (n^2 + m^2) y_n = 0$. Also find $(y_n)_0$.
9. If $y = (x^2-1)^n$, prove that $(x^2-1) y_{n+2} + 2xy_{n+1} - n(n+1) y_n = 0$. *(U.T.U., 2010)*
10. If $\sin^{-1} y = 2 \log(x+1)$, prove that $(x+1)^2 y_{n+2} + (2n+1)(x+1) y_{n+1} + (n^2+4) y_n = 0$. *(V.T.U., 2003)*
11. If $y = x^n \log x$, prove that $y_{n+1} = n!/x$. *(Mumbai, 2008)*
12. If $V_n = \dfrac{d^n}{dx^n}(x^n \log x)$, show that $V_n = nV_{n-1} + (n-1)!$

    Hence, show that $V_n = n!\left(\log x + 1 + \dfrac{1}{2} + \dfrac{1}{3} + \ldots + \dfrac{1}{n}\right)$. *(V.T.U., 2001)*
13. Show that $\dfrac{d^n}{dx^n}\left(\dfrac{\log x}{x}\right) = \dfrac{(-1)^n n!}{x^{n+1}}\left\{\log x - 1 - \dfrac{1}{2} - \dfrac{1}{3} - \ldots - \dfrac{1}{n}\right\}$. *(V.T.U., 2006)*
14. If $y = x \log\left(\dfrac{x-1}{x+1}\right)$, show that $y_n = (-1)^{n-2}(n-2)!\left[\dfrac{x-n}{(x-1)^n} - \dfrac{x+n}{(x+1)^n}\right]$. *(U.P.T.U., 2003)*
15. If $x = \sin t$, $y = \cos pt$, show that $(1-x^2) y_2 - xy_1 + p^2 y = 0$. Hence prove that
    $(1-x^2) y_{n+2} - (2n+1) xy_{n+1} - (n^2 - p^2) y_n = 0$. *(Raipur, 2005 ; V.T.U., 2005)*
16. If $y = \log[x + \sqrt{(1+x^2)}]^2$, prove that $(1+x^2) y_{n+2} + (2n+1) xy_{n+1} + n^2 y_n = 0$. *(V.T.U., 2007 ; Bhillai, 2005)*
    Hence show that $(y_{2k})_0 = (-1)^{k-1} \cdot 2^k \cdot [(k-1)!]^2$, where k is positive integer.
17. If $y = [x + \sqrt{(x^2+1)}]^m$, prove that (*i*) $(x^2+1) y_2 + xy_1 - m^2 y = 0$, (*ii*) $y_{n+2} + (n^2 - m^2) y_n = 0$ at $x = 0$. *(V.T.U., 2009 S)*
    Hence find $y_n(0)$. *(Madras, 2000)*
18. If $y = \sin \log(x^2 + 2x + 1)$, prove that (*i*) $(x+1)^2 y_2 + (x+1) y_1 + 4y = 0$
    (*ii*) $(x+1)^2 y_{n+2} + (2n+1)(x+1) y_{n+1} + (n^2+4) y_n = 0$. *(U.P.T.U., 2006)*

19. If $y = \dfrac{\sinh^{-1} x}{\sqrt{(1+x^2)}}$, show that $(1 + x^2)\, y_{n+2} + (2n + 3)\, xy_{n+1} + (n + 1)^2\, y_n = 0.$ (V.T.U., 2010)

20. If $y = \sinh [m \log (x + \sqrt{x^2 + 1})]$, prove that $(x^2 + 1)\, y_{n+2} + (2n + 1)\, xy_{n+1} + (n^2 - m^2)\, y_n = 0.$ (V.T.U., 2010 S)

4.3 FUNDAMENTAL THEOREMS

(1) Rolle's Theorem

If (*i*) *f(x) is continuous in the closed interval [a, b], (ii) f ′(x) exists for every value of x in the open interval (a, b) and (iii) f (a) = f (b), then there is at least one value c of x in (a, b) such that f ′(c) = 0.*

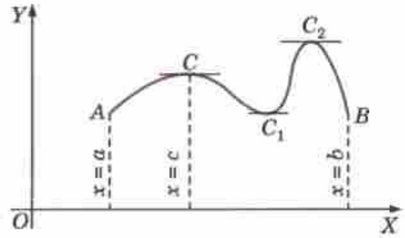

Fig. 4.1

Consider the portion AB of the curve $y = f(x)$, lying between $x = a$ and $x = b$, such that

(*i*) it goes continuously from A to B,

(*ii*) it has a tangent at every point between A and B, and

(*iii*) ordinate of A = ordinate of B.

From the Fig. 4.1, it is self-evident that there is at least one point C (may be more) of the curve at which the tangent is parallel, to the x-axis.

i.e., slope of the tangent at C $(x = c) = 0$

But the slope of the tangent at C is the value of the differential coefficient of $f(x)$ w.r.t. x thereat, therefore $f'(c) = 0$.

Hence the theorem is proved.

Example 4.13. *Verify Rolle's theorem for (i) $\sin x / e^x$ in $(0, \pi)$.* (J.N.T.U., 2003)
(ii) $(x - a)^m (x - b)^n$ where m, n are positive integers in [a, b]. (V.T.U., 2010 ; Nagarjuna, 2008)

Solution. (*i*) Let $f(x) = \sin x/e^x$.

$f(x)$ is derivable in $(0, \pi)$.

Also $f(0) = f(\pi) = 0$.

Hence the conditions of Rolle's theorem are satisfied.

$$\therefore \quad f'(x) = \frac{e^x \cos x - e^x \sin x}{e^{2x}} \quad \text{vanishes where } e^x (\cos x - \sin x) = 0$$

or $\tan x = 1$ *i.e.*, $x = \pi/4$.

The value $x = \pi/4$ lies in $(0, \pi)$, so that Rolle's theorem is verified.

(*ii*) Let $f(x) = (x - a)^m (x - b)^n$.

Since every polynomial is continuous for all values, $f(x)$ is also continuous in $[a, b]$.

$$f'(x) = m(x - a)^{m-1} (x - b)^n + (x - a)^m \,.\, n(x - b)^{n-1}$$
$$= (x - a)^{m-1} (x - b)^{n-1} [(m + n)\, x - (mb + na)]$$

which exists, *i.e.*, $f(x)$ is derivable in (a, b).

Also $f(a) = 0 = f(b)$.

Thus all the conditions of Rolle's theorem are satisfied and there exists c in (a, b) such that $f'(c) = 0$.

$$\therefore \quad (c - a)^{m-1} (c - b)^{n-1} [(m + n)\, c - (mb + na)] = 0 \quad \text{or} \quad c = (mb + na)/(m + n).$$

Hence, Rolle's theorem is verified.

(2) Lagrange's Mean-Value Theorem*

First form. *If (i) f(x) is continuous in the closed interval [a, b], and*

(ii) f′(x) exists in the open interval (a, b), then there is at least one value c of x in (a, b), such that

$$\mathbf{\frac{f(b) - f(a)}{b - a} = f'(c).}$$

*Named after the great French mathematician *Joseph Louis Lagrange* (1736–1813) who became professor at Military Academy, Turin when he was just 19 and director of Berlin Academy in 1766. His important contribution are to algebra, number theory, differential equations, mechanics, approximation theory and calculus of variations.

Consider the function $\phi(x) = f(x) - \dfrac{f(b) - f(a)}{b - a}\, x$

Since $f(x)$ is continuous in $[a, b]$; $\therefore$ $\phi(x)$ is also continuous in $[a, b]$.

Since $f'(x)$ exists in (a, b) ;

$\therefore$ $\phi'(x)$ also exists in (a, b) and $= f'(x) - \dfrac{f(b) - f(a)}{b - a}$ ...(i)

Clearly, $$\phi(a) = \frac{b\, f(a) - a\, f(b)}{b - a} = \phi(b).$$

Thus $\phi(x)$ satisfies all the conditions of Rolle's theorem.

$\therefore$ There is at least one value c of x between a and b such that $\phi'(c) = 0$. Substituting $x = c$ in (1), we get

$$f'(c) = \frac{f(b) - f(a)}{b - a} \qquad ...(2)$$

which proves the theorem.

Second form. If we write $b = a + h$, then since $a < c < b$,

$$c = a + \theta h \text{ where } 0 < \theta < 1.$$

Thus the mean value theorem may be stated as follows :

If (i) f(x) is continuous in the closed interval [a, a + h] and (ii) f′ (x) exists in the open interval (a, a + h), then there is at least one number θ(0 < θ < 1) such that

$$\mathbf{f(a + h) = f(a) + hf'(a + \theta h)}$$

Geometrical Interpretation. Let A, B be the points on the curve $y = f(x)$ corresponding to $x = a$ and $x = b$ so that $A = [a, f(a)]$ and $B = [b, f(b)]$. (Fig. 4.2)

$\therefore$ Slope of chord $AB = \dfrac{f(b) - f(a)}{b - a}$

By (2), the slope of the chord $AB = f'(c)$, the slope of the tangent of the curve at $C(x = c)$.

Hence the Lagrange's mean value theorem asserts that *if a curve AB has a tangent at each of its points, then there exists at least one point C on this curve, the tangent at which is parallel to the chord AB.*

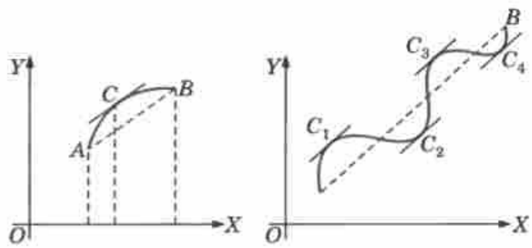

Fig. 4.2

Cor. *If f′(x) = 0 in the interval (a, b) then f (x) is constant in [a, b].* For, if x_1, x_2 be any two values of x in (a, b), then by (2),

$$f(x_2) - f(x_1) = (x_2 - x_1)\, f'(c) = 0 \quad (x_1 < c < x_2)$$

Thus, $f(x_1) = f(x_2)$ *i.e.*, $f(x)$ has the same value for every value of x in (a, b).

Example 4.14. *In the Mean value theorem $f(b) - f(a) = (b - a) f'(c)$,* ...(i)
determine c lying between a and b, if $f(x) = x(x - 1)(x - 2)$, $a = 0$ and $b = 1/2$. (Gorakhpur, 1999)

Solution. $f(a) = 0,$ $\qquad f(b) = \dfrac{1}{2}\left(-\dfrac{1}{2}\right)\left(-\dfrac{3}{2}\right) = \dfrac{3}{8}$

$f'(x) = 3x^2 - 6x + 2,$ $\qquad f'(c) = 3c^2 - 6c + 2$

Substituting in (i), $$\frac{3}{8} - 0 = \left(\frac{1}{2} - 0\right)(3c^2 - 6c + 2)$$

or $$12c^2 - 24c + 5 = 0$$

whence $$c = \frac{24 \pm \sqrt{(24)^2 - 12 \times 5 \times 4}}{24} = 1 \pm 0.764 = 1.764 ; 0.236.$$

Hence $c = 0.236$, since it only lies between 0 and 1/2.

Example 4.15. *Prove that (if $0 < a < b < 1$), $\dfrac{b - a}{1 + b^2} < \tan^{-1} b - \tan^{-1} a < \dfrac{b - a}{1 + a^2}$.*

Hence show that $\dfrac{\pi}{4} + \dfrac{3}{25} < \tan^{-1}\dfrac{4}{3} < \dfrac{\pi}{4} + \dfrac{1}{6}$. (Mumbai, 2009 ; V.T.U., 2006)

Solution. Let $f(x) = \tan^{-1} x$, so that $f'(x) = \dfrac{1}{1+x^2}$.

By Mean value theorem, $\dfrac{\tan^{-1} b - \tan^{-1} a}{b-a} = \dfrac{1}{1+c^2}, a < c < b$ ...(*i*)

Now $a < c < b$, $\therefore$ $1 + a^2 < 1 + c^2 < 1 + b^2$.

$$\therefore \quad \frac{1}{1+a^2} > \frac{1}{1+c^2} > \frac{1}{1+b^2} \text{ i.e., } \frac{1}{1+b^2} < \frac{1}{1+c^2} < \frac{1}{1+a^2}$$

i.e.,
$$\frac{1}{1+b^2} < \frac{\tan^{-1} b - \tan^{-1} a}{b-a} < \frac{1}{1+a^2} \qquad [\text{By } (i)]$$

Hence
$$\frac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \frac{b-a}{1+a^2}$$

Now let $a = 1, b = 4/3$.

Then
$$\frac{1/3}{1+16/9} < \tan^{-1}\frac{4}{3} - \frac{\pi}{4} < \frac{1/3}{1+1}$$

i.e.,
$$\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}.$$

Example 4.16. *Prove that $\log(1+x) = x/(1+\theta x)$, where $0 < \theta < 1$ and hence deduce that*

$$\frac{x}{1+x} < \log(1+x) < x, x > a$$ *(Mumbai, 2008)*

Solution. Let $f(x) = \log(1+x)$, then by second form of Lagrange's mean value theorem

$$f(a+h) = f(a) + h f'(a+\theta h), \qquad (0 < \theta < 1)$$

we have
$$f(x) = f(0) + x f'(\theta x) \qquad [\text{Taking } a = 0, h = x]$$

or
$$\log(1+x) = \log(1) + x \,.\, 1/(1+\theta x) \qquad [\because \; f'(x) = 1/1+x)$$

Hence
$$\log(1+x) = x/(1+\theta x) \qquad ...(i)\ [\because \; \log(1) = 0$$

Since $0 < \theta < 1$, $\therefore$ $0 < \theta x < x$ for $x > 0$.

or
$$1 < 1 + \theta x < 1 + x \quad \text{or} \quad 1 > \frac{1}{1+\theta x} > \frac{1}{1+x}$$

or
$$x > \frac{x}{1+\theta x} > \frac{x}{1+x}$$

or
$$\frac{x}{1+x} < \log(1+x) < x, x > 0. \qquad [\text{By } (i)]$$

(3) Cauchy's Mean-value Theorem*

If (i) $f(x)$ and $g(x)$ be continuous in $[a, b]$

(*ii*) $f'(x)$ and $g'(x)$ exist in (a, b)

and (iii) $g'(x) \neq 0$ for any value of x in (a, b),

then there is at least one value c of x in (a, b), such that $\dfrac{f(b)-f(a)}{g(b)-g(a)} = \dfrac{f'(c)}{f'(c)}$

Consider the function $\phi(x) = f(x) - \dfrac{f(b)-f(a)}{g(b)-g(a)} g(x)$

Since $f(x)$ and $g(x)$ are continuous in $[a, b]$

$\therefore$ $\phi(x)$ is also continuous in $[a, b]$.

Again since $f'(x)$ and $g'(x)$ exist in (a, b).

*Named after the great French mathematician *Augustin-Louis Cauchy* (1789–1857) who is considered as the father of modern analysis and creator of complex analysis. He published nearly 800 research papers of basic importance. Cauchy is also well known for his contributions to differential equations, infinite series, optics and elasticity.

$\therefore$ $\phi'(x)$ also exists in (a, b) and $= f'(x) - \dfrac{f(b) - f(a)}{g(b) - g(a)} g'(x)$

Clearly, $\phi(a) = \phi(b)$.

Thus, $\phi(x)$ satisfies all the conditions of Rolle's theorem. There is therefore, at least one value c of x between a and b, such that $\phi'(c) = 0$

i.e., $$0 = f'(c) - \frac{f(b) - f(a)}{g(b) - g(a)} g'(c) \text{ whence follows the result.}$$

(*P.T.U., 2007 S ; V.T.U., 2006*)

Obs. Cauchy's mean value theorem is a generalisation of Lagrange's mean value theorem, where $g(x) = x$.

Example 4.17. *Verify Cauchy's Mean-value theorem for the functions e^x and e^{-x} in the interval (a, b).*

Solution. $f(x) = e^x$ and $g(x) = e^{-x}$ are both continuous in $[a, b]$ and both functions are differentiable in (a, b).

$\therefore$ $$f'(x) = e^x, g'(x) = -e^{-x}$$

By Cauchy's mean value theorem.

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

$\therefore$ $$\frac{e^b - e^a}{e^{-b} - e^{-a}} = \frac{e^c}{-e^{-c}} \quad i.e., c = \frac{1}{2}(a + b)$$

Thus c lies in (a, b) which verifies the Cauchy's Mean value theorem.

(4) Taylor's Theorem* (Generalised mean value theorem)

If (i) f(x) and its first (n – 1) derivatives be continuous in [a, a + h], and (ii) $f^n(x)$ exists for every value of x in (a, a + h), then there is at least one number θ (0 < θ < 1), such that

$$\mathbf{f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + ... + \frac{h^n}{n!} f^n(a + \theta h)} \quad ...(1)$$

which is called Taylor's theorem with Lagrange's form remainder, the remainder R_n being $\dfrac{h^n}{n!} f^n(a + \theta h)$.

Proof. Consider the function

$$\phi(x) = f(x) + (a + h - x) f'(x) + \frac{(a + h - x)^2}{2!} f''(x) + ... + \frac{(a + h - x)^n}{n!} K$$

where K is defined by

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + ... + \frac{h^n}{n!} K \quad ...(2)$$

(*i*) Since $f(x), f'(x), ..., f^{n-1}(x)$ are continuous in $[a, a + h]$, therefore $\phi(x)$ is also continuous in $[a, a + h]$,

(*ii*) $\phi'(x)$ exists and $= \dfrac{(a + h - x)^{n-1}}{(n - 1)!} [f^n(x) - K]$

(*iii*) Also $\phi(a) = \phi(a + h)$. [By (2)]

Hence $\phi(x)$ satisfies all the conditions of Rolle's theorem, and therefore, there exists at least one number $\theta (0 < \theta < 1)$, such that $\phi'(a + \theta h) = 0$ *i.e.,* $K = f^n(a + \theta h)\ (0 < \theta < 1)$

Substituting this value of K in (2), we get (1).

Cor. 1. Taking $n = 1$ in (1), *Taylor's theorem reduces to Lagrange's Mean-value theorem.*

Cor. 2. Putting $a = 0$ and $h = x$ in (1), we get

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + ... + \frac{x^n}{n!} f^n(\theta x). \quad ...(3)$$

which is known as *Maclaurin's theorem with Lagrange's form of remainder.*

*Named after an English mathematician, *Brooke Taylor* (1685–1731).

Example 4.18. *Find the Maclaurin's theorem with Lagrange's form of remainder for $f(x) = \cos x$.* (J.N.T.U., 2003)

Solution. $f^n(x) = \dfrac{d^n}{dx^n}(\cos x) = \cos\left(\dfrac{n\pi}{2} + x\right)$ so that $f^n_{(0)} = \cos(n\pi/2)$

Thus $f(0) = 1$,

$$f^{2n}(0) = \cos(2n\pi/2) = (-1)^n$$

$$f^{2n+1}(0) = \cos[(2n+1)\pi/2] = 0$$

Substituting these values in the Maclaurin's theorem with Lagrange's form of remainder *i.e.*,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + ... + \frac{x^{2n}}{(2n)!}f^{2n}(0) + \frac{x^{2n+1}}{(2n+1)!}f^{2n+1}(\theta x)$$

We get
$$\cos x = 1 + 0 + \frac{x^2}{2!}(-1) + 0 + ... + \frac{x^{2n}}{(2n)!}(-1)^n + \frac{x^{2n+1}}{(2n+1)!}(-1)^n(-1)\cos(\theta x)$$

i.e.,
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + ... + (-1)^n\frac{x^{2n}}{(2n)!} + \frac{(-1)^{n+1}x^{2n+1}}{(2n+1)!}\cos(\theta x)$$

Example 4.19. *If $f(x) = \log(1 + x)$, $x > 0$, using Maclaurin's theorem, show that for $0 < \theta < 1$,*

$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3(1+\theta x)^3}.$$

Deduce that $\log(1 + x) < x - \dfrac{x^2}{2} + \dfrac{x^3}{3}$ for $x > 0$. (J.N.T.U., 2005)

Solution. By Maclaurin's theorem with remainder R_3, we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(\theta x) \qquad ...(i)$$

Here $f(x) = \log(1 + x)$, $\quad f(0) = 0$

$\therefore$ $f'(x) = \dfrac{1}{1+x}$, $\quad f'(0) = 1$

$f''(x) = \dfrac{-1}{(1+x)^2}$, $\quad f''(0) = -1$

and $f'''(x) = \dfrac{2}{(1+x)^3}$, $\quad f'''(\theta x) = \dfrac{2}{(1+\theta x)^3}$

Substituting in (*i*), we get $\log(1 + x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3(1+\theta x)^3}$ ...(*ii*)

Since $x > 0$ and $\theta > 0$, $\theta x > 0$

or
$$(1 + \theta x)^3 > 1 \quad i.e., \quad \frac{1}{(1+\theta x)^3} < 1$$

$\therefore$
$$x - \frac{x^2}{2} + \frac{x^3}{3(1+\theta x)^3} < x - \frac{x^2}{2} + \frac{x^3}{3}$$

Hence $\log(1 + x) < x - \dfrac{x^2}{2} + \dfrac{x^3}{3}$ [By (*ii*)]

PROBLEMS 4.4

1. Verify Rolle's theorem for (*i*) $f(x) = (x + 2)^3 (x - 3)^4$ in $(-2, 3)$.
(*ii*) $y = e^x(\sin x - \cos x)$ in $(\pi/4, 5\pi/4)$. (*iii*) $f(x) = x(x + 3)e^{-1/2x}$ in $(-3, 0)$.
(*iv*) $f(x) = \log\left\{\dfrac{x^2 + ab}{x(a+b)}\right\}$ in (a, b). (V.T.U., 2005)

2. Using Rolle's theorem for $f(x) = x^{2m-1}(a-x)^{2n}$, find the value of x between o and a where $f'(x) = 0$.
3. Verify Lagrange's Mean value theorem for the following functions and find the appropriate value of c in each case :
   (i) $f(x) = (x-1)(x-2)(x-3)$ in $(0, 4)$ *(V.T.U., 2009)*
   (ii) $f(x) = \sin x$ in $[0, \pi]$ *(Nagpur, 2008)*
   (iii) $f(x) = \log_e x$ in $[1, e]$. *(Burdwan, 2003)*
   (iv) $f(x) = e^x$ in $[0, 1]$. *(V.T.U., 2007)*
4. By applying Mean value theorem to
$$f(x) = \log 2 \,.\, \sin \frac{\pi x}{2} + \log x, \text{ prove that } \frac{\pi}{2} \log 2 \,.\, \cos \frac{\pi x}{2} + \frac{1}{x} = 0 \text{ for some } x \text{ between 1 and 2.}$$
5. In the Mean value theorem : $f(x+h) = f(x) + h f'(x + \theta h)$,
   show that $\theta = 1/2$ for $f(x) = ax^2 + bx + c$ in $(0, 1)$.
6. If $f(h) = f(0) + h f'(0) + \dfrac{h^2}{2!} f''(\theta h)$, $0 < \theta < 1$, find θ when $h = 1$ and $f(x) = (1-x)^{5/2}$.
7. If x is positive, show that $x > \log(1+x) > x - \dfrac{1}{2}x^2$. *(V.T.U., 2000)*
8. If $f(x) = \sin^{-1} x$, $0 < a < b < 1$, use Mean value theorem to prove that
$$\frac{b-a}{\sqrt{(1-a^2)}} < \sin^{-1} b - \sin^{-1} a < \frac{b-a}{\sqrt{(1-b^2)}}$$
9. Prove that $\dfrac{b-a}{b} < \log\left(\dfrac{b}{a}\right) < \dfrac{b-a}{b}$ for $0 < a < b$.

   Hence show that $\dfrac{1}{4} < \log \dfrac{4}{3} < \dfrac{1}{3}$. *(Mumbai, 2008)*
10. Verify the result of Cauchy's mean value theorem for the functions
    (i) $\sin x$ and $\cos x$ in the interval $[a, b]$. *(J.N.T.U., 2006 S)*
    (ii) $\log_e x$ and $1/x$ in the interval $[1, e]$.
11. If $f(x)$ and $g(x)$ are respectively e^x and e^{-x}, prove that 'c' of Cauchy's mean value theorem is the arithmetic mean between a and b. *(Mumbai, 2008)*
12. Verify Maclaurin's theorem $f(x) = (1-x)^{5/2}$ with Lagrange's form of remainder upto 3 terms where $x = 1$.
13. Using Taylor's theorem, prove that
$$x - \frac{x^3}{6} < \sin x < x - \frac{x^3}{6} + \frac{x^5}{120}, \quad \text{for } x > 0.$$

4.4 EXPANSIONS OF FUNCTIONS

(1) Maclaurin's series. *If $f(x)$ can be expanded as an infinite series, then*

$$\mathbf{f(x) = f(0) + x\, f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots \infty} \qquad \ldots(1)$$

If $f(x)$ possess derivatives of all orders and the remainder R_n in (3) on page 145 tends to zero as $n \to \infty$, then the Maclaurin's theorem becomes the Maclaurin's series (1).

Example 4.20. *Using Maclaurin's series, expand tan x upto the term containing x^5.* (V.T.U., 2006)

Solution. Let $\qquad f(x) = \tan x \qquad\qquad f(0) = 0$

$\therefore \qquad f'(x) = \sec^2 x = 1 + \tan^2 x \qquad\qquad f'(0) = 1$

$$f''(x) = 2 \tan x \sec^2 x = 2 \tan x (1 + \tan^2 x)$$
$$= 2 \tan x + 2 \tan^3 x \qquad\qquad f''(0) = 0$$
$$f'''(0) = 2 \sec^2 x + 6 \tan^2 x \sec^2 x$$
$$= 2(1 + \tan^2 x) + 6 \tan^2 x (1 + \tan^2 x)$$
$$= 2 + 8 \tan^2 x + 6 \tan^4 x \qquad\qquad f'''(0) = 2$$
$$f^{iv}(0) = 16 \tan x \sec^2 x + 24 \tan^3 x \sec^2 x$$

$$= 16 \tan x\,(1 + \tan^2 x) + 24 \tan^3 x\,(1 + \tan^2 x)$$

$$= 16 \tan x + 40 \tan^3 x + 24 \tan^5 x \qquad f^{iv}(0) = 0$$

$$f^{v}(0) = 16 \sec^2 x + 120 \tan^2 x \sec^2 x + 120 \tan^4 x \sec^2 x. \qquad f^{v}(0) = 16$$

and so on.

Substituting the values of $f(0)$, $f'(0)$, etc. in the Maclaurin's series, we get

$$\tan x = 0 + x\,.\,1 + \frac{x^2}{2!} + \frac{x^3}{3!}\,.\,2 + \frac{x^4}{4!}\,.\,0 + \frac{x^5}{5!}\,.\,16 + \dots = x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots$$

(2) Expansion by use of known series. When the expansion of a function is required only upto first few terms, it is often convenient to employ the following well-known series :

1. $\sin\theta = \theta - \dfrac{\theta^3}{3!} + \dfrac{\theta^5}{5!} - \dfrac{\theta^7}{7!} + \dots$

2. $\sinh\theta = \theta + \dfrac{\theta^3}{3!} + \dfrac{\theta^5}{5!} + \dfrac{\theta^7}{7!} + \dots$

3. $\cos\theta = 1 - \dfrac{\theta^2}{2!} + \dfrac{\theta^4}{4!} - \dfrac{\theta^6}{6!} + \dots$

4. $\cosh\theta = 1 + \dfrac{\theta^2}{2!} + \dfrac{\theta^4}{4!} + \dfrac{\theta^6}{6!} + \dots$

5. $\tan\theta = \theta + \dfrac{\theta^3}{3} + \dfrac{2}{15}\theta^5 + \dots$

6. $\tan^{-1} x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dots$

7. $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \dots$

8. $\log(1 + x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \dots$

9. $\log(1 - x) = -\left(x + \dfrac{x^2}{2} + \dfrac{x^3}{3} + \dfrac{x^4}{4} + \dots\right)$

10. $(1 + x)^n = 1 + nx + \dfrac{n(n-1)}{2!}x^2 + \dfrac{n(n-1)(n-2)}{3!}x^3 + \dots$

Example 4.21. *Expand $e^{\sin x}$ by Maclaurin's series or otherwise upto the term containing x^4.* *(Bhopal, 2009; V.T.U., 2011)*

Solution. We have $e^{\sin x} = 1 + \sin x + \dfrac{(\sin x)^2}{2!} + \dfrac{(\sin x)^3}{3!} + \dfrac{(\sin x)^4}{4!} + \dots$

$$= 1 + \left(x - \frac{x^3}{3!} + \dots\right) + \frac{1}{2!}\left(x - \frac{x^3}{3!} + \dots\right)^2 + \frac{1}{3!}\left(x - \frac{x^3}{3!} + \dots\right)^3 + \frac{1}{4!}(x - \dots)^4 + \dots$$

$$= 1 + \left(x - \frac{x^3}{6} + \dots\right) + \frac{1}{2}\left(x^2 - \frac{x^4}{3} + \dots\right) + \frac{1}{6}\left(x^3 - \dots\right) + \frac{1}{24}(x^4 + \dots) + \dots$$

$$= 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \dots$$

Otherwise, let $f(x) = e^{\sin x}$ $\qquad f(0) = 1$

$\therefore$ $f'(x) = e^{\sin x}\cos x = f(x)\,.\,\cos x$ $\qquad f'(0) = 1$

$f''(x) = f'(x)\cos x - f(x)\sin x,$ $\qquad f''(0) = 1$

$f'''(x) = f''(x)\cos x - 2f'(x)\sin x - f(x)\cos x,$ $\qquad f'''(0) = 0$

$f^{iv}(x) = f'''(x)\cos x - 3f''(x)\sin x - 3f'(x)\cos x + f(x)\sin x,$ $\qquad f^{iv}(0) = -3$

and so on.

Substituting the values of $f(0)$, $f'(0)$ etc., in the Maclaurin's series, we obtain

$$e^{\sin x} = 1 + x\,.\,1 + \frac{x^2}{2!}\,.\,1 + \frac{x^3}{3!}\,.\,0 + \frac{x^4}{4!}\,.\,(-3) + \dots$$

$$= 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \dots$$

Example 4.22. *Expand log $(1 + \sin^2 x)$ in powers of x as far as the term in x^6.* (Hissar, 2005 S)

Solution. We have $\sin^2 x = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right)^2 = \left[x - \left(\frac{x^3}{6} - \frac{x^5}{120} + \ldots\right)\right]^2$

$$= x^2 - 2x\left(\frac{x^3}{6} - \frac{x^5}{120} + \ldots\right) + \left(\frac{x^3}{6} - \frac{x^5}{120} + \ldots\right)^2$$

$$= x^2 - \frac{x^4}{3} + \frac{x^6}{60} + \frac{x^6}{36} + \ldots = x^2 - \frac{x^4}{3} + \frac{2x^6}{45} + \ldots = t, \text{ say.}$$

Now $\log(1 + \sin^2 x) = \log(1 + t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \ldots$

Substituting the value of t, we get

$$\log(1 + \sin^2 x) = x^2 - \frac{x^4}{3} + \frac{2x^6}{45} + \ldots - \frac{1}{2}\left(x^2 - \frac{x^4}{3} + \ldots\right)^2 - \frac{1}{3}(x^2 - \ldots)^3 - \ldots$$

$$= x^2 - \frac{x^4}{3} + \frac{2x^6}{45} - \frac{1}{2}\left(x^4 - \frac{2x^6}{3} + \ldots\right) + \frac{1}{3}(x^6 + \ldots) + \ldots$$

$$= x^2 - \frac{5}{6}x^4 + \frac{32}{45}x^6 + \ldots$$

Obs. As it is very cumbersome to find the successive derivatives of $\log(1 + \sin^2 x)$, therefore the above method is preferable to Maclaurin's series method.

Example 4.23. *Expand $e^{a\sin^{-1} x}$ in ascending powers of x.*

Solution. Let $y = e^{a\sin^{-1} x}$. In Ex. 4.9, we have shown that
$(y)_0 = 1, (y_1)_0 = a, (y_2)_0 = a^2, (y_3)_0 = a(1 + a^2), (y_4)_0 = a^2(2^2 + a^2)$
and so on.

Substituting these values in the Maclaurin's series

$$y = (y)_0 + \frac{(y_1)_0}{1!}x + \frac{(y_2)_0}{2!}x^2 + \frac{(y_3)_0}{3!}x^3 + \frac{(y_4)_0}{4!}x^4 + \ldots$$

we get $$e^{a\sin^{-1} x} = 1 + ax + \frac{a^2}{2!}x^2 + \frac{a(1^2 + a^2)}{3!}x^3 + \frac{a^2(2^2 + a^2)}{4!}x^4 + \ldots$$

(3) Taylor's series. *If $f(x + h)$ can be expanded as an infinite series, then*

$$f(x + h) = f(x) + h f'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots \infty \quad \ldots(1)$$

If $f(x)$ possesses derivatives of all orders and the remainder R_n in (1) on page 147, tends to zero as $n \to \infty$, then the Taylor's theorem becomes the *Taylor's series* (1).

Cor. Replacing x by a and h by $(x - a)$ in (1), we get

$$f(x) = f(a) + (x - a)f'(a) + \frac{(x - a)^2}{2!}f''(a) + \frac{(x - a)^3}{3!}f'''(a) + \ldots \infty$$

Taking $a = 0$, we get *Maclaurin's series.*

Example 4.24. *Expand $\log_e x$ in powers of $(x - 1)$ and hence evaluate $\log_e 1{\cdot}1$ correct to 4 decimal places.* (Bhopal, 2007 ; Kurukshetra 2006)

Solution. Let $f(x) = \log_e x$ $\qquad f(1) = 0$

$\therefore$ $f'(x) = \frac{1}{x}$, $\qquad f'(1) = 1$

$f''(x) = -\frac{1}{x^2}$, $\qquad f''(1) = -1$

$$f'''(x) = \frac{2}{x^3}, \qquad f'''(1) = 2$$

$$f^{iv}(x) = -\frac{6}{x^4}, \qquad f^{iv}(0) = -6$$

etc. etc.

Substituting these values in the Taylor's series

$$f(x) = f(1) + (x-1)f'(1) + \frac{(x-1)^2}{2!}f''(1) + \frac{(x-1)^3}{3!}f'''(1) + \ldots,$$

we get $$\log x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \ldots$$

Now putting $x = 1.1$, so that $x - 1 = 0.1$, we have

$$\log(1.1) = 1.1 - \tfrac{1}{2}(0.1)^2 + \tfrac{1}{3}(0.1)^3 - \tfrac{1}{4}(0.1)^4 + \ldots$$

$$= 0.1 - 0.005 + 0.0003 - 0.00002 + \ldots = 0.0953.$$

Example 4.25. *Use Taylor's series, to prove that*

$$\tan^{-1}(x+h) = \tan^{-1}x + (h\sin z) \cdot \frac{\sin z}{1} - (h\sin z)^2 \cdot \frac{\sin 2z}{2} + (h\sin z)^3 \cdot \frac{\sin 3z}{3} - \ldots$$

where $z = \cot^{-1}x$. *(Bhillai, 2005)*

Solution. We have $\cot z = x$ ...(*i*)

$\therefore$ $-\text{cosec}^2 z . dz/dx = 1$ or $dz/dx = -\sin^2 z$ ...(*ii*)

Now let $f(x+h) = \tan^{-1}(x+h)$, so that $f(x) = \tan^{-1}x$

$\therefore$ $$f'(x) = \frac{1}{1+x^2} = \frac{1}{1+\cot^2 z} = \sin^2 z \qquad \text{[By } (i)]$$

$$f''(x) = 2\sin z\cos z\,\frac{dz}{dx} = \sin 2z \cdot (-\sin^2 z) \qquad \text{[By } (ii)]$$

$$f'''(x) = -[2\cos 2z \cdot \sin^2 z + \sin 2z \cdot 2\sin z\cos z]\,\frac{dz}{dx}$$

$$= -2\sin z\,[\sin z\cos 2z + \sin 2z\cos z]\,(-\sin^2 z) = 2\sin^3 z\sin 3z$$

and so on.

Substituting these values in the Taylor's series

$$f(x+h) = f(x) + hf(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots,$$

we get the required result.

PROBLEMS 4.5

Using Maclaurin's series, expand the following functions :

1. $\log(1+x)$. Hence deduce that $\log\sqrt{\frac{1+x}{1-x}} = x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots$
2. $\sin x$ *(P.T.U., 2005)*
3. $\sqrt{(1+\sin 2x)}$ *(V.T.U., 2010)*
4. $\sin^{-1}x$ *(Mumbai, 2007)*
5. $\tan^{-1}x$
6. $\log\sec x$ *(Mumbai, 2009 S ; V.T.U., 2009)*

Prove that :

7. $\sec x = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \ldots$
8. $x\,\text{cosec}\,x = 1 + \frac{x^2}{6} + \frac{7x^4}{360} + \ldots$ *(Mumbai, 2007)*
9. $\sin^{-1}\frac{2x}{1+x^2} = 2\left\{x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\right\}$
10. $\tan^{-1}\frac{\sqrt{(1+x^2)}-1}{x} = \frac{1}{2}\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\right)$

11. $\sin^{-1}(3x-4x^3) = 3\left(x+\frac{x^3}{3}+\frac{3x^5}{40}+\ldots\right)$ 12. $e^x \cos x = 1+x-\frac{2x^3}{3!}-\frac{2^2x^4}{4!}\ \ldots$ (*Raipur, 2005*)

13. $e^{x \sin x} = 1+x^2+\frac{x^4}{3}+\frac{x^6}{120}+\ldots$ (*Kurukshetra, 2009*)

14. $e^{\cos^{-1}x} = e^{\pi/2}\left(1-x+\frac{x^2}{3}-\frac{x^3}{3}+\ldots\right)$ (*Mumbai, 2008*) 15. $\log\frac{\sin x}{x} = -\left(\frac{x^2}{6}+\frac{x^4}{180}+\frac{x^6}{2835}+\ldots\right)$

16. $\log(1+\sin x) = x-\frac{x^2}{2}+\frac{x^3}{6}-\frac{x^4}{12}+\ldots$ (*S.V.T.U. 2009 ; J.N.T.U., 2006 S*)

17. $\sqrt{(1+\sin x)} = 1+\frac{x}{2}-\frac{x^2}{2}-\frac{x^3}{48}+\frac{x^4}{384}+\ldots$ (*V.T.U., 2006*)

18. $\log(1+e^x) = \log 2+\frac{x}{2}+\frac{x^2}{8}-\frac{x^4}{192}+\ldots$ (*Bhopal, 2008*)

19. $\frac{e^x}{e^x+1} = \frac{1}{2}+\frac{x}{4}-\frac{x^3}{48}+\ldots$ (*Bhopal, 2008 S*) 20. $\frac{x}{2}\left(\frac{e^x+1}{e^x-1}\right) = 1+\frac{1}{6}\cdot\frac{x^2}{2!}-\frac{1}{30}\cdot\frac{x^4}{4!}+\ldots$ (*Mumbai, 2007*)

21. $\sin x \cosh x = x+\frac{x^3}{3}-\frac{x^5}{30}+\ldots$

By forming a differential equation, show that

22. $(\sin^{-1}x)^2 = 2\frac{x^2}{2!}+2\cdot 2^2\frac{x^4}{4!}+2\cdot 2^2\cdot 4^2\cdot\frac{x^6}{6!}+\ldots$

23. $\log[1+\sqrt{(1+x^2)}] = x-\frac{1}{2}\frac{x^3}{3}+\frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5}-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7}+\ldots$

24. If $y = \sin(m\sin^{-1}x)$, show that $(1-x^2)y_2 - xy_1 + m^2y = 0$
Hence expand $\sin m\theta$ in powers of $\sin\theta$. (*S.V.T.U., 2008*)

25. Using Taylor's theorem, express the polynomial $2x^3+7x^2+x-6$ in powers of $(x-1)$ (*Burdwan, 2003*)

26. Expand (*i*) e^x (*Cochin., 2005*) (*ii*) $\tan^{-1}x$, in powers of $(x-1)$ upto four terms.

27. Expand $\sin x$ in powers of $(x-\pi/2)$. Hence find the value of $\sin 91°$ correct to 4 decimal places. (*Rohtak, 2003*)

28. Prove that $\log\sin x = \log\sin a+(x-a)\cot a-\frac{1}{2}(x-a)^2\operatorname{cosec}^2 a+\ldots$

29. Find the Taylor's series expansion for $\log\cos x$ about the point $\pi/3$.

30. Compute to four decimal places, the value of $\cos 32°$, by the use of Taylor's series. (*Kurukshetra, 2006*)

31. Calculate approximately (*i*) $\log_{10}404$, given $\log 4 = 0.6021$. (*Rohtak, 2005 S*)
(*ii*) $(1.04)^{3.01}$ (*Mumbai, 2007*)

4.5 INDETERMINATE FORMS

In general $\underset{x\to a}{\text{Lt}}[f(x)/\phi(x)] = \underset{x\to a}{\text{Lt}} f(x)/\underset{x\to a}{\text{Lt}}\phi(x)$. But when $\underset{x\to a}{\text{Lt}} f(x)$ and $\underset{x\to a}{\text{Lt}}\phi(x)$ are both zero, then the quotient reduces to the indeterminate form 0/0. This does not imply that $\underset{x\to a}{\text{Lt}}[f(x)/\phi(x)]$ is meaningless or it does not exist. In fact, in many cases, it has a finite value. We shall now, study the methods of evaluating the limits in such and similar other cases :

(1) Form 0/0. *If* $f(a) = \phi(a) = 0$, *then*

$$\underset{x\to a}{\text{Lt}}\frac{f(x)}{\phi(x)} = \underset{x\to a}{\text{Lt}}\frac{f'(x)}{\phi'(x)}$$

By Taylor's series,

$$\underset{x\to a}{\text{Lt}}\frac{f(x)}{\phi(x)} = \underset{x\to a}{\text{Lt}}\frac{f(a)+(x-a)f'(a)+\frac{1}{2!}(x-a)^2f''(a)+\ldots}{\phi(a)+(x-a)\phi'(a)+\frac{1}{2!}(x-a)^2\phi''(a)+\ldots}$$

$$= \underset{x \to a}{\text{Lt}} \frac{f'(a) + \frac{1}{2}(x-a) f'(a) + \ldots}{\phi'(a) + \frac{1}{2}(x-a)\phi'(a) + \ldots}$$

$$= \frac{f'(a)}{\phi'(a)} = \underset{x \to a}{\text{Lt}} \frac{f'(x)}{\phi'(x)} \quad \ldots(1)$$

This is known as *L'Hospital's rule.*

In general, if

$$f(a) = f'(a) = f''(a) = \ldots = f^{n-1}(a) = 0, \text{ but } f^n(a) \neq 0,$$

and

$$\phi(a) = \phi'(a) = \phi''(a) = \ldots = \phi^{n-1}(a) = 0, \text{ but } \phi^n(a) \neq 0,$$

then from (1),

$$\underset{x \to a}{\textbf{Lt}} \frac{\mathbf{f(x)}}{\boldsymbol{\phi}\mathbf{(x)}} = \frac{\mathbf{f^n(a)}}{\boldsymbol{\phi}^\mathbf{n}\mathbf{(a)}} = \underset{x \to a}{\textbf{Lt}} \frac{\mathbf{f^n(x)}}{\boldsymbol{\phi}^\mathbf{n}\mathbf{(x)}}$$

[Rule to evaluate Lt $[f(x)/\phi(x)]$ in 0/0 form :

Differentiating the numerator and denominator separately as many times as would be necessary to arrive a determinate form].

Example 4.26. *Evaluate (i)* $\underset{x \to 0}{Lt} \frac{xe^x - \log(1+x)}{x^2}$. *(V.T.U., 2004 ; Osmania, 2000 S)*

(ii) $\underset{x \to 1}{Lt} \frac{x^x - x}{x - 1 - \log x}$

Solution. (*i*)

$$\underset{x \to 0}{\text{Lt}} \frac{xe^x - \log(1+x)}{x^2} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{(xe^x + e^x \cdot 1) - 1/(1+x)}{2x} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{xe^x + e^x + e^x + 1/(1+x)^2}{2} = \frac{0+1+1+1}{2} = 1\frac{1}{2}.$$

(*ii*)

$$\underset{x \to 1}{\text{Lt}} \frac{x^x - x}{x - 1 - \log x} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \underset{x \to 1}{\text{Lt}} \frac{d(x^x)/dx - 1}{1 - 0 - 1/x}$$

$$= \underset{x \to 1}{\text{Lt}} \frac{x^x(1 + \log x) - 1}{1 - 1/x}$$

$$\left(\text{form } \frac{0}{0}\right)$$

Let $y = x^x$ so that

$\log y = x \log x$

$$\therefore \quad \frac{1}{y}\frac{dy}{dx} = x \cdot \frac{1}{x} + 1 \,.\, \log x$$

or $$\frac{d}{dx}(x^x) = x^x(1 + \log x) \quad \ldots(i)$$

$$= \underset{x \to 1}{\text{Lt}} \frac{d(x^x)/dx \,.\, (1 + \log x) + x^x(1/x) - 0}{1/x^2}$$

$$= \underset{x \to 1}{\text{Lt}} \frac{x^x(1 + \log x)^2 + x^x(1/x)}{x^{-2}} \qquad \text{[By } (i)\text{]}$$

$$= \frac{1(1+0)^2 + 1 \cdot 1}{1} = 2.$$

Example 4.27. *Find the values of a and b such that* $\underset{x \to 0}{Lt} \frac{x(a + b\cos x) - c\sin x}{x^5} = 1.$ *(Mumbai., 2007)*

Solution. $\underset{x \to 0}{\text{Lt}} \dfrac{x(a + b\cos x) - c\sin x}{x^5}$ $\left(\text{form } \dfrac{0}{0}\right)$

$$= \underset{x \to 0}{\text{Lt}} \frac{a + b\cos x - bx\sin x - c\cos x}{5x^4} \quad ...(i)$$

As the denominator is 0 for $x = 0$, (i) will tend to a finite limit if and only if the numerator also becomes 0 for $x = 0$. This requires $a + b - c = 0$ ...(ii)

With this condition, (i) assumes the form 0/0.

$$\therefore \quad (i) = \underset{x \to 0}{\text{Lt}} \frac{-b\sin x - b(\sin x + x\cos x) + c\sin x}{20x^3}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{(c - 2b)\sin x - bx\cos x}{20x^3} \quad \left(\text{form } \frac{0}{0}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{(c - 2b)\cos x - b(\cos x - x\sin x)}{60x^2} \quad ...(iii)$$

$$= \frac{c - 2b - b}{0} = \frac{c - 3b}{0} = 1 \quad \text{(Given)}$$

$\therefore \quad c - 3b = 0 \quad$ *i.e.*, $\quad c = 3b$.

Now (iii) $$= \underset{x \to 0}{\text{Lt}} \frac{b\cos x - b\cos x + bx\sin x}{60x^2}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{b\sin x}{60x} = \frac{b}{60} \underset{x \to 0}{\text{Lt}} \left(\frac{\sin x}{x}\right) = \frac{b}{60} = 1.$$

i.e., $b = 60$, and $\therefore \quad c = 180$.

Form (ii), $a = 120$.

(2) Form ∞/∞. It can be shown that L'Hospital's rule can also be applied to this case by differentiating the numerator and denominator separately as many times as would be necessary.

Example 4.28. *Evaluate* $\underset{x \to 0}{Lt} \dfrac{\log x}{\cot x}$.

Solution. $\underset{x \to 0}{\text{Lt}} \dfrac{\log x}{\cot x} = \underset{x \to 0}{\text{Lt}} \dfrac{1/x}{-\operatorname{cosec}^2 x} = -\underset{x \to 0}{\text{Lt}} \dfrac{\sin^2 x}{x}$ $\left(\text{form } \dfrac{0}{0}\right)$

$$= -\underset{x \to 0}{\text{Lt}} \frac{2\sin x\cos x}{1} = 0$$

Obs. Use of known series and standard limits. In many cases, it would be found more convenient to use expansions of known functions and standard limits for evaluating the indeterminate forms. For this purpose, remember the series of § 4.4 (2) and the following limits :

$$\underset{x \to 0}{\text{Lt}} \frac{\sin x}{x} = 1, \qquad \underset{x \to 0}{\text{Lt}} (1 + x)^{1/x} = e$$

Example 4.29. *Evaluate* $\underset{x \to 0}{Lt} \dfrac{e^x \sin x - x - x^2}{x^2 + x\log(1 - x)}$.

Solution. Using the expansions of e^x, $\sin x$ and $\log(1 - x)$, we get

$$\underset{x \to 0}{\text{Lt}} \frac{e^x\sin x - x - x^2}{x^2 + x\log(1 - x)}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + ...\right)\left(x - \frac{1}{3!}x^3 + ...\right) - x - x^2}{x^2 + x\left(-x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - ...\right)}$$

$$= \underset{x\to 0}{\text{Lt}} \frac{\left(x + x^2 + \frac{1}{3}x^3 - 0 \cdot x^4 + \ldots\right) - x - x^2}{x^2 - \left(x^2 + \frac{1}{2}x^3 + \frac{1}{3}x^4 + \ldots\right)} = \underset{x\to 0}{\text{Lt}} \frac{\frac{1}{3}x^3 - 0 \cdot x^4 + \ldots}{-\frac{1}{2}x^3 - \frac{1}{3}x^4 - \ldots} = \underset{x\to 0}{\text{Lt}} \frac{\frac{1}{3} + \ldots}{-\frac{1}{2} - \frac{1}{3}x - \ldots} = -\frac{2}{3}.$$

Example 4.30. *Evaluate* $\underset{x\to 0}{Lt} \dfrac{(1+x)^{1/x} - e}{x}$.

Solution. Let $y = (1 + x)^{1/x}$

$\therefore$ $$\log y = \frac{1}{x} \log (1 + x) = \frac{1}{x}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\right) = 1 - \frac{x}{2} + \frac{x^2}{3} - \ldots$$

or $$y = e^{1 - \frac{1}{2}x + \frac{1}{3}x^2 - \ldots} = e \cdot e^{-\frac{1}{2}x + \frac{1}{3}x^2 - \ldots}$$

$$= e\left[1 + \left(-\frac{1}{2}x + \frac{1}{3}x^2 - \ldots\right) + \frac{1}{2!}\left(-\frac{1}{2}x + \frac{1}{3}x^2 - \ldots\right)^2 + \ldots\right] = e\left(1 - \frac{x}{2} + \frac{11}{24}x^2 + \ldots\right)$$

$\therefore$ $$\underset{x\to 0}{\text{Lt}} \frac{(1+x)^{1/x} - e}{x} = \underset{x\to 0}{\text{Lt}} \frac{e\left(1 - \frac{x}{2} + \frac{11}{24}x^2 + \ldots\right) - e}{x}$$

$$= \underset{x\to 0}{\text{Lt}} \frac{e\left(-\frac{1}{2}x + \frac{11}{24}x^2 + \ldots\right)}{x} = \underset{x\to 0}{\text{Lt}} \left(\frac{-e}{2} + \frac{11}{24}ex + \ldots\right) = -\frac{e}{2}.$$

PROBLEMS 4.6

Evaluate the following limits :

1. $\underset{x\to 0}{\text{Lt}} \dfrac{a^x - b^x}{x}$ *(V.T.U., 2008)*
2. $\underset{x\to 0}{\text{Lt}} \dfrac{x \cos x - \sin x}{x^2 \sin x}$ *(J.N.T.U., 2006 S)*
3. $\underset{\theta\to 0}{\text{Lt}} \dfrac{\theta - \sin\theta}{\sin\theta\,(1 - \cos\theta)}$
4. $\underset{x\to \pi/2}{\text{Lt}} \dfrac{a^{\sin x} - a}{\log_e \sin x}$
5. $\underset{x\to 0}{\text{Lt}} \dfrac{2\sin x - \sin 2x}{x^3}$
6. $\underset{x\to 0}{\text{Lt}} \dfrac{\sin x \sin^{-1} x - x^2}{x^6}$
7. $\underset{x\to 0}{\text{Lt}} \dfrac{x\cos x - \log(1+x)}{x^2}$
8. $\underset{x\to 0}{\text{Lt}} \dfrac{\log \sec x - \frac{1}{2}x^2}{x^4}$
9. $\underset{x\to 0}{\text{Lt}} \dfrac{x e^x - \log(1+x)}{\cosh x - \cos x}$
10. $\underset{x\to 0}{\text{Lt}} \dfrac{\cos x - \log(1+x) - 1 + x}{\sin^2 x}$
11. $\underset{x\to 0}{\text{Lt}} \dfrac{e^x + 2\sin x - e^{-x} - 4x}{x^5}$
12. $\underset{x\to 0}{\text{Lt}} \dfrac{\log(x-a)}{\log(e^x - e^a)}$
13. $\underset{x\to 0}{\text{Lt}} \dfrac{\tan x - x}{x^2 \tan x}$
14. $\underset{x\to 0}{\text{Lt}} \dfrac{e^x - e^{-x} - 2x}{x - \sin x}$
15. $\underset{x\to 0}{\text{Lt}} \dfrac{e^x - e^{\sin x}}{x - \sin x}$
16. $\underset{x\to 0}{\text{Lt}} \dfrac{\sin\{(\log(1+x)\}}{\log(1+\sin x)}$
17. $\underset{x\to 0}{\text{Lt}} \dfrac{e^x + \sin x - 1}{\log(1+x)}$
18. $\underset{x\to 0}{\text{Lt}} \dfrac{(1+x)^{1/x} - e + \frac{1}{2}ex}{x^2}$
19. If $\underset{x\to 0}{\text{Lt}} \dfrac{\sin 2x + a\sin x}{x^3}$ is finite, find the value of a and the limit. *(Nagpur, 2009)*
20. Find a, b if $\underset{x\to 0}{\text{Lt}} \dfrac{a\sinh x + b\sin x}{x^3} = \dfrac{5}{3}$. *(Mumbai, 2009)*
21. Find a, b, c so that $\underset{x\to 0}{\text{Lt}} \dfrac{ae^x - b\cos x + ce^{-x}}{x \sin x} = 2$. *(Mumbai, 2008)*

(3) Forms reducible to 0/0 form. Each of the following indeterminate forms can be easily reduced to the form 0/0 (or ∞/∞) by suitable transformation and then the limits can be found as usual.

I. Form 0 × ∞. If $\underset{x \to 0}{Lt} f(x) = 0$ and $\underset{x \to \infty}{Lt} \phi(x) = \infty$, then

$\underset{x \to a}{\text{Lt}} [f(x) \cdot \phi(x)]$ assumes the form $0 \times \infty$.

To evaluate this limit, we write

$$f(x) \cdot \phi(x) = f(x)/[1/\phi(x)] \text{ to take the form } 0/0.$$
$$= \phi(x)/[1/f(x)] \text{ to take the form } \infty/\infty.$$

Example 4.31. *Evaluate* $\underset{x \to 0}{Lt} (\tan x \log x)$ *(V.T.U., 2009)*

Solution.
$$\underset{x \to 0}{\text{Lt}} (\tan x \log x) = \underset{x \to 0}{\text{Lt}} \left(\frac{\log x}{\cot x}\right) \qquad \left(\text{form } \frac{\infty}{\infty}\right)$$
$$= \underset{x \to 0}{\text{Lt}} \left(\frac{1/x}{-\text{cosec}^2 x}\right) = -\underset{x \to 0}{\text{Lt}} \left(\frac{\sin^2 x}{x}\right) \qquad \left(\text{form } \frac{0}{0}\right)$$
$$= \underset{x \to 0}{\text{Lt}} \frac{2 \sin x \cos x}{1} = 0.$$

II. Form ∞ − ∞. If $\underset{x \to a}{Lt} f(x) = \infty = \underset{x \to a}{Lt} \phi(x)$, then $\underset{x \to a}{\text{Lt}} [f(x) - \phi(x)]$ assumes the form $\infty - \infty$.

It can be reduced to the from 0/0 by writing

$$f(x) - \phi(x) = \left[\frac{1}{\phi(x)} - \frac{1}{f(x)}\right] \Big/ \frac{1}{f(x)\phi(x)}$$

Example 4.32. *Evaluate* $\underset{x \to 0}{Lt} \left(\frac{1}{\sin x} - \frac{1}{x}\right)$.

Solution.
$$\underset{x \to 0}{\text{Lt}} \left(\frac{1}{\sin x} - \frac{1}{x}\right) = \underset{x \to 0}{\text{Lt}} \frac{x - \sin x}{x \sin x} = \underset{x \to 0}{\text{Lt}} \frac{1 - \cos x}{x \cos x + \sin x} \qquad \left(\text{form } \frac{0}{0}\right)$$
$$= \underset{x \to 0}{\text{Lt}} \frac{\sin x}{x(-\sin x) + \cos x + \cos x} = \frac{0}{0 + 1 + 1} = 0.$$

III. Forms $0^0, 1^\infty, \infty^0$. If $y = \underset{x \to a}{\text{Lt}} [f(x)]^{\phi(x)}$ assumes one of these forms, then $\log y = \underset{x \to a}{\text{Lt}} \phi(x) \log f(x)$ takes the form $0 \times \infty$, which can be evaluated by the method given in I above. If $\log y = l$, then $y = e^l$.

Example 4.33. *Evaluate (i)* $\underset{x \to \pi/2}{Lt} (\sin x)^{\tan x}$ *(ii)* $\underset{x \to 0}{Lt} \left(\frac{a^x + b^x + c^x}{3}\right)^{1/x}$ *(V.T.U., 2011)*

(iii) $\underset{x \to 0}{Lt} \left(\frac{\tan x}{3}\right)^{1/x^2}$

Solution. (*i*) Let $y = \underset{x \to \pi/2}{\text{Lt}} (\sin x)^{\tan x}$.

$$\therefore \quad \log y = \underset{x \to \pi/2}{\text{Lt}} \tan x \log \sin x = \underset{x \to \pi/2}{\text{Lt}} \frac{\log \sin x}{\cot x} \qquad \left(\text{form } \frac{0}{0}\right)$$
$$= \underset{x \to \pi/2}{\text{Lt}} \frac{(1/\sin x)\cos x}{-\text{cosec}^2 x} = -\underset{x \to \pi/2}{\text{Lt}} (\sin x \cos x) = 0$$

Hence $y = e^0 = 1.$

(*ii*) Let $$y = \underset{x \to 0}{\text{Lt}} \left(\frac{a^x + b^x + c^x}{3} \right)^{1/x}$$

so that $$\log y = \underset{x \to 0}{\text{Lt}} \frac{\log (a^x + b^x + c^x) - \log 3}{x} \qquad \text{form } \left(\frac{0}{0}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{(a^x + b^x + c^x)^{-1} (a^x \log a + b^x \log b + c^x \log c)}{1}$$

$$= (1 + 1 + 1)^{-1} (\log a + \log b + \log c) = \frac{1}{3} \log (abc) = \log (abc)^{1/3}.$$

$\therefore$ $$y = (abc)^{1/3}$$

(*iii*) $$\underset{x \to 0}{\text{Lt}} \left(\frac{\tan x}{x} \right)^{1/x^2} = \underset{x \to 0}{\text{Lt}} \left(\frac{x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots}{x} \right)^{1/x^2}$$

$$= \underset{x \to 0}{\text{Lt}} \left(1 + \frac{x^2}{3} + \frac{2}{15}x^4 + \ldots \right)^{1/x^2}$$

$$= \underset{x \to 0}{\text{Lt}} (1 + tx^2)^{1/x^2} \qquad \text{where } t = \frac{1}{3} + \frac{2}{15}x^2 + \ldots$$

$$= \underset{x \to 0}{\text{Lt}} [(1 + tx^2)^{1/tx^2}]^t = \underset{x \to 0}{\text{Lt}} e^t = e^{1/3}. \qquad \left[\because \underset{z \to 0}{\text{Lt}} (1 + z)^{1/z} = e\right]$$

PROBLEMS 4.7

Evaluate the following limits :

1. $\underset{x \to 0}{\text{Lt}} \left(\frac{1}{x^2} - \frac{1}{\sin^2 x} \right)$

2. $\underset{x \to 0}{\text{Lt}} \left(\frac{1}{x} - \frac{1}{e^x - 1} \right)$ (*Burdwan, 2003*)

3. $\underset{x \to 1}{\text{Lt}} (2x \tan x - \pi \sec x)$ (*V.T.U., 2008*)

4. $\underset{x \to 0}{\text{Lt}} \left(\frac{\cot x - 1/x}{x} \right)$

5. $\underset{x \to 0}{\text{Lt}} \left(\frac{1}{x^2} - \cot^2 x \right)$

6. $\underset{x \to 1}{\text{Lt}} (x)^{1/(1-x)}$

7. $\underset{x \to 0}{\text{Lt}} (a^x + x)^{1/x}$ (*V.T.U., 2007*)

8. $\underset{x \to \pi/2}{\text{Lt}} (\sec x)^{\cot x}$

9. $\underset{x \to 0}{\text{Lt}} (1 + \sin x)^{\cot x}$

10. $\underset{x \to 0}{\text{Lt}} (\cos x)^{1/x^2}$

11. $\underset{x \to \pi/2}{\text{Lt}} (\tan x)^{\tan 2x}$ (*V.T.U., 2004*)

12. $\underset{x \to 0}{\text{Lt}} (\cot x)^{1/\log x}$

13. $\underset{x \to \pi/2}{\text{Lt}} (\cos x)^{\frac{\pi}{2} - x}$

14. $\underset{x \to 0}{\text{Lt}} \left(\frac{\sin x}{x} \right)^{1/x}$

15. $\underset{x \to 0}{\text{Lt}} \left(\frac{\sin x}{x} \right)^{1/x^2}$ (*V.T.U., 2001*)

16. $\underset{x \to 1}{\text{Lt}} (1 - x^2)^{1/\log(1-x)}$

17. $\underset{x \to a}{\text{Lt}} \left(2 - \frac{x}{a} \right)^{\tan (\pi x/2a)}$ (*V.T.U., 2010 ; Nagpur, 2009*)

18. $\underset{x \to 0}{\text{Lt}} \left\{ \frac{2 (\cosh x - 1)^{1/x^2}}{x^2} \right\}$

19. $\underset{x \to 2}{\text{Lt}} \left\{ \frac{1}{x - 2} - \frac{1}{\log (x - 1)} \right\}$ (*Osmania, 2000 S*)

20. $\underset{x \to 0}{\text{Lt}} \left(\frac{1^x + 2^x + 3^x}{3} \right)^{1/x}$. (*V.T.U., 2008*)

4.6 TANGENTS AND NORMALS – CARTESIAN CURVES

(1) Equation of the tangent *at the point (x, y) of the curve $y = f(x)$ is*

$$\mathbf{Y - y = \frac{dy}{dx}(X - x).}$$

The equation of any line through $P(x, y)$ is

$$Y - y = m(X - x)$$

where X, Y are the current coordinates of any point on the line (Fig. 4.3).

If this line is the tangent PT, then

$$m = \tan\psi = dy/dx$$

Hence the equation of the tangent at (x, y) is

$$Y - y = \frac{dy}{dx}(X - x) \qquad ...(2)$$

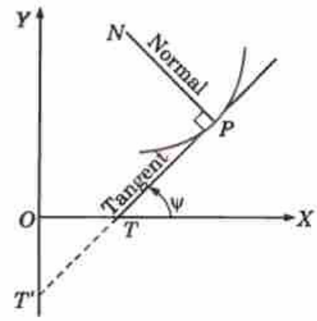

Fig. 4.3

Cor. Intercepts. Putting $Y = 0$ in (2)

$$-y = \frac{dy}{dx}(X - x) \quad \text{or} \quad X = x - y/\frac{dy}{dx}$$

∴ *Intercept which the tangent cuts off from x-axis (= OT)* = $\mathbf{x - y\, dx/dy}$

Similarly putting $X = 0$ in (2), we see that
the intercept which the tangent cuts off from the y-axis

$$(= OT') = \mathbf{y - x\frac{dy}{dx}}$$

(2) Equation of the normal *at the point (x, y) of the curve $y = f(x)$ is*

$$\mathbf{Y - y = -\frac{dx}{dy}(X - x)}$$

A normal *to the curve $y = f(x)$ at $P(x, y)$ is a line through P perpendicular to the tangent there at.*

∴ Its equation is $Y - y = m'(X - x)$

where $$m' \cdot dy/dx = -1 \quad \text{or} \quad m' = -1/\frac{dy}{dx} = -dx/dy$$

Hence the equation of the normal at (x, y) is $Y - y = -\dfrac{dx}{dy}(X - x)$.

Example 4.34. *Find the equation of the tangent at any point (x, y) to the curve $x^{2/3} + y^{2/3} = a^{2/3}$. Show that the portion of the tangent intercepted between the axes is of constant length.*

Solution. Equation of the curve is $x^{2/3} + y^{2/3} = a^{2/3}$. ...(*i*)

Differentiating (*i*) w.r.t. x,

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$$

∴ Slope of the tangent at $(x, y) = \dfrac{dy}{dx} = -\left(\dfrac{y}{x}\right)^{1/3}$

∴ Equation of the tangent at (x, y) is

$$Y - y = -(y/x)^{1/3}(X - x) \qquad ...(ii)$$

Put $Y = 0$ in (*ii*). Then $X = x + x^{1/3} \cdot y^{2/3}$

i.e., Intercept on x-axis $= (x^{2/3} + y^{2/3})x^{1/3} = a^{2/3} \cdot x^{1/3}$ [By (*i*)]

Put $X = 0$ in (*ii*). Then $Y = y + y^{1/3} \cdot x^{2/3}$

i.e., Intercept on y-axis $= (x^{2/3} + y^{2/3})\, y^{1/3} = a^{2/3} \cdot y^{1/3}$ [By (*i*)]

Thus the portion of the tangent intercepted between the axes

$$= \sqrt{[(\text{Intercept on } x\text{-axis})^2 + (\text{Intercept on } y\text{-axis})^2]}$$

$$= \sqrt{[(a^{2/3} \cdot x^{1/3})^2 + (a^{2/3} \cdot y^{1/3})^2]}$$

$$= \sqrt{[a^{4/3}(x^{2/3} + y^{2/3})]} = a^{2/3}\sqrt{(a)^{2/3}} \qquad \text{[By } (i)\text{]}$$

$= a$, which is a constant length.

Example 4.35. *Show that the conditions for the line $x \cos \alpha + y \sin \alpha = p$ to touch the curve $(x/a)^m + (y/b)^m = 1$ is $(a \cos \alpha)^{m/(m-1)} + (b \sin \alpha)^{m/(m-1)} = p^{m/(m-1)}$.*

Solution. Equation of the curve is $\dfrac{x^m}{a^m} + \dfrac{y^m}{b^m} = 1$ ...(*i*)

Differentiating (*i*) w.r.t. x, $\dfrac{mx^{m-1}}{a^m} + \dfrac{my^{m-1}}{b^m}\dfrac{dy}{dx} = 0$

$\therefore$ Slope of the tangent at $(x, y) = \dfrac{dy}{dx} = -\left(\dfrac{b}{a}\right)^m \left(\dfrac{x}{y}\right)^{m-1}$

$\therefore$ Equation of the tangent at (x, y) is

$$Y - y = -\left(\frac{b}{a}\right)^m \left(\frac{x}{y}\right)^{m-1} (X - x)$$

or $$\frac{x^{m-1}X}{a^m} + \frac{y^{m-1}Y}{b^m} = \frac{x^m}{a^m} + \frac{y^m}{b^m} = 1 \qquad ...(ii) \text{ [By } (i)\text{]}$$

If the given line touches (*i*) at (x, y) then (*ii*) must be same as $X \cos \alpha + Y \sin \alpha = p$ ...(*iii*)

Comparing coefficients in (*ii*) and (*iii*),

$$\frac{x^{m-1}}{a^m}\Big/\cos \alpha = \frac{y^{m-1}}{b^m}\Big/\sin \alpha = \frac{1}{p}$$

or $$\left(\frac{x}{a}\right)^{m-1} = \frac{a \cos \alpha}{p}, \left(\frac{y}{b}\right)^{m-1} = \frac{b \sin \alpha}{p}$$

or $$\left(\frac{a \cos \alpha}{p}\right)^{\frac{m}{m-1}} + \left(\frac{b \sin \alpha}{p}\right)^{\frac{m}{m-1}} = \left(\frac{x}{a}\right)^m + \left(\frac{y}{b}\right)^m = 1 \qquad \text{[By } (i)\text{]}$$

whence follows the required condition.

Example 4.36. *Find the equation of the normal at any point θ to the curve $x = a (\cos \theta + \theta \sin \theta)$, $y = a (\sin \theta - \theta \cos \theta)$. Verify that these normals touch a circle with its centre at the origin and whose radius is constant.*

Solution. We have $$\frac{dx}{d\theta} = a(-\sin \theta + \sin \theta + \theta \cos \theta) = a\theta \cos \theta$$

$$\frac{dy}{d\theta} = a(\cos \theta - \cos \theta + \theta \sin \theta) = a\theta \sin \theta$$

$\therefore$ $$\frac{dy}{dx} = \frac{dy}{d\theta}\Big/\frac{dx}{d\theta} = \frac{\sin \theta}{\cos \theta}$$

$\therefore$ Slope of the normal at $\theta = -\dfrac{\cos \theta}{\sin \theta}$

Hence the equation of the normal at θ

$$y - a(\sin \theta - \theta \cos \theta) = -\frac{\cos \theta}{\sin \theta}[x - a(\cos \theta + \theta \sin \theta)]$$

i.e., $$y \sin \theta - a \sin^2 \theta + a\,\theta \sin \theta \cos \theta = -x \cos \theta + a \cos^2 \theta + a\,\theta \sin \theta \cos \theta$$

i.e., $$x \cos \theta + y \sin \theta = a(\cos^2 \theta + \sin^2 \theta) = a.$$

Now the perpendicular distance of this normal from $(0, 0) = a$, which is a constant. Hence it touches a circle of radius a having its centre at $(0, 0)$.

(3) Angle of intersection of two curves *is the angle between the tangents to the curves at their point of intersection.*

To find this angle θ, proceed as follows :

(*i*) Find P, the point of intersection of the curves by solving their equations simultaneously.

(*ii*) Find the values of dy/dx at P for the two curves (say : m_1, m_2).

(*iii*) Find $\angle\theta$, using the $\tan\theta = \dfrac{m_1 - m_2}{1 + m_1 m_2}$.

When $m_1 m_2 = -1$, $\theta = 90°$ *i.e.*, the curves cut orthogonally.

Example 4.37. *Find the angle of intersection of the curves* $x^2 = 4y$ ...(*i*)

and $y^2 = 4x.$ ...(*ii*)

Solution. We have $x^4 = 16y^2 = 16.4\,x = 64x$

or $x(x^3 - 64) = 0$ whence $x = 0$ and 4.

Substituting these values in (*i*), $y = 0$ and 4.

∴ The curves intersect at (0, 0) and (4, 4).

For the curve (*i*), $dy/dx = x/2$. For the curve (*ii*), $dy/dx = 2/y$

At (0, 0), slope of tangent to (*i*) $(= m_1) = 0/2 = 0$ and slope of tangent to (*ii*) $(= m_2) = 2/0 = \infty$.

Evidently the curves intersect at right angles.

At (4, 4), slope of tangent to (*i*) $(= m_1) = 4/2 = 2$ and slope of tangent to (*ii*) $(= m_2) = 2/4 = \frac{1}{2}$

∴ Angle of intersection of the curves

$$= \tan^{-1}\frac{m_1 - m_2}{1 + m_1 m_2} = \tan^{-1}\frac{2 - \frac{1}{2}}{1 + 2\cdot\frac{1}{2}} = \tan^{-1}\frac{3}{4}.$$

Example 4.38. *Show that the condition that the curves* $ax^2 + by^2 = 1$ *and* $a'x^2 + b'y^2 = 1$ *should intersect orthogonally is that*

$$\frac{1}{a} - \frac{1}{b} = \frac{1}{a'} - \frac{1}{b'}.$$

Solution. Given curves are $ax^2 + by^2 = 1$ ...(*i*) and $a'x^2 + b'y^2 = 1$ ...(*ii*)

Let $P(h, k)$ be a point of intersection of (*i*) and (*ii*) so that

$$ah^2 + bk^2 = 1 \quad \text{and} \quad a'h^2 + b'k^2 = 1$$

$$\therefore \quad \frac{h^2}{-b + b'} = \frac{k^2}{-a' + a} = \frac{1}{ab' - a'b}$$

or $$h^2 = (b' - b)/(ab' - a'b),\ k^2 = (a - a')/(ab' - a'b) \quad ...(iii)$$

Differentiating (*i*) w.r.t. x,

$$2ax + 2by\,dy/dx = 0 \quad \text{or} \quad dy/dx = -ax/by.$$

Similarly for (*ii*), $dy/dx = -a'x/b'y$

∴ m_1 = slope of tangent to (*i*) at $P = -ah/bk$; m_2 = slope of tangent to (*ii*) at $P = -a'h/b'k$

For orthogonal intersection, we should have $m_1 m_2 = -1$.

i.e., $$\frac{-ah}{bk} \times \frac{-a'h}{b'k} = 1 \text{ i.e., } aa'h^2 + bb'k^2 = 0$$

Substituting the values of h^2 and k^2 from (*iii*),

$$\frac{aa'(b' - b)}{ab' - a'b} + \frac{bb'(a - a')}{ab' - a'b} = 0 \quad \text{or} \quad \frac{b' - b}{bb'} + \frac{a - a'}{aa'} = 0$$

i.e., $\dfrac{1}{b} - \dfrac{1}{b'} = \dfrac{1}{a} - \dfrac{1}{a'}$ which leads to the required condition.

(4) Lengths of tangent, normal, subtangent and subnormal. Let the tangent and the normal at any point $P(x, y)$ of the curve meet the x-axis at T and N respectively. (Fig. 4.4). Draw the ordinate PM. Then PT *and PN are called the lengths of the tangent and the normal respectively. Also TM and MN are called the subtangent and subnormal respectively.*

Let $\angle MTP = \psi$ so that $\tan \psi = dy/dx$.

Clearly, $\angle MPN = \psi$.

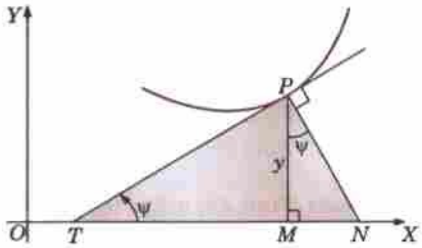

Fig. 4.4

(1) **Tangent** $= TP = MP \operatorname{cosec} \psi = y\sqrt{(1+\cot^2 \psi)} = \mathbf{y\sqrt{[1+(dx/dy)^2]}}$

(2) **Normal** $= NP = MP \sec \psi = y\sqrt{(1+\tan^2 \psi)} = \mathbf{y\sqrt{[1+(dy/dx)^2]}}$

(3) **Subtangent** $= TM = y \cot \psi = \mathbf{y\ dx/dy}$

(4) **Subnormal** $= MN = y \tan \psi = \mathbf{y\ dy/dx}$.

Example 4.39. *For the curve $x = a(\cos t + \log \tan t/2)$, $y = a \sin t$, prove that the portion of the tangent between the curve and x-axis is constant.*

Also find its subtangent.

Solution. Differentiating with respect to t,

$$\frac{dx}{dt} = a\left(-\sin t + \frac{1}{\tan t/2} \cdot \sec^2 \frac{t}{2} \cdot \frac{1}{2}\right) = a\left(-\sin t + \frac{\cos t/2}{2 \sin t/2} \cdot \frac{1}{\cos^2 t/2}\right)$$

$$= a\left(-\sin t + \frac{1}{\sin t}\right) = \frac{a(1-\sin^2 t)}{\sin t} = a \cos^2 t/\sin t\ ; \quad \frac{dy}{dt} = a \cos t\ .$$

$$\therefore \qquad \frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = a \cos t \cdot \frac{\sin t}{a \cos^2 t} = \tan t.$$

Thus *length of the tangent between the curve and x-axis*

$$= y\sqrt{[1+(dx/dy)^2]} = a \sin t \cdot \sqrt{(1+\cot^2 t)} = a \sin t\ .\ \operatorname{cosec} t = a \text{ which is a constant.}$$

Also *subtangent* $= y\dfrac{dx}{dy} = a \sin t \cdot \cot t = a \cos t$.

PROBLEMS 4.8

1. Find the equation of the tangent and the normal to the curve $y(x-2)(x-3) - x + 7 = 0$ at the point where it cuts the x-axis.
2. The straight line $x/a + y/b = 2$ touches the curve $(x/a)^n + (y/b)^n = 2$ for all values of n. Find the point of contact. (*Bhopal, 2008*)
3. Prove that $\dfrac{x}{a} + \dfrac{y}{b} = 1$ touches the curve $y = be^{-x/a}$ at the point where the curve crosses the axis of y. (*Bhopal, 2009*)
4. If $p = x \cos \alpha + y \sin \alpha$, touches the curve $(x/a)^{n/(n-1)} + (y/b)^{n/(n-1)} = 1$, prove that
$$p^n = (a \cos \alpha)^n + (b \sin \alpha)^n.$$
5. Prove that the condition for the line $x \cos \alpha + y \sin \alpha = p$ to touch the curve $x^m y^n = a^{m+n}$, is
$$p^{m+n} \cdot m^m \cdot n^n = (m+n)^{m+n} a^{m+n} \cos^m \alpha \sin^n \alpha.$$
6. Show that the sum of the intercepts on the axes of any tangent to the curve $\sqrt{x} + \sqrt{y} = a$ is a constant.
7. If x, y be the parts of the axes of x and y intercepted by the tangent at any point (x, y) on the curve $(x/a)^{2/3} + (x/b)^{2/3} = 1$, then show that $(x_1/a)^2 + (y_1/b)^2 = 1$. (*Bhopal, 2008*)
8. If the tangent at (x_1, y_1) to the curve $x^3 + y^3 = a^3$ meets the curve again in (x_2, y_2), show that
$$\frac{x_2}{x_1} + \frac{y_2}{y_1} = -1.$$

9. If the normal to the curve $x^{2/3} + y^{2/3} = a^{2/3}$ makes an angle ϕ with the axis of x, show that its equation is $y \cos \phi - x \sin \phi = a \cos 2\phi$.
10. Find the angle of intersection of the curves $x^2 - y^2 = a^2$ are $x^2 + y^2 = a^2\sqrt{2}$.
11. Show that the parabolas $y^2 = 4ax$ and $2x^2 = ay$ intersect at an angle $\tan^{-1}(3/5)$.
12. Prove that the curves $\frac{x^2}{a} + \frac{y^2}{b} = 1$ and $\frac{x^2}{a'} + \frac{y^2}{b'} = 1$ will cut orthogonally if $a - b = a' - b'$.
13. Show that in the exponential curve $y = be^{x/a}$, the subtangent is of constant length and that the subnormal varies as the square of the ordinate. (*Madras, 2000 S*)
14. Find the lengths of the tangent, normal, subtangent and subnormal for the cycloid:
$$x = a(t + \sin t),\ y = a(1 - \cos t),$$
15. For the curve $x = a\cos^3\theta, y = a\sin^3\theta$, show that the portion of the tangent intercepted between the point of contact and the x-axis is $y \operatorname{cosec}\theta$. Also find the length of the subnormal.

4.7 POLAR CURVES

(1) Angle between radius vector and tangent. *If ϕ be the angle between the radius vector and the tangent at any point of the curve $r = f(\theta)$,* $\tan\theta = r\dfrac{d\theta}{dr}$.

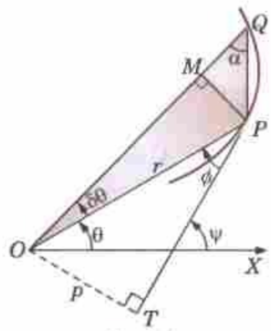

Fig. 4.5

Let $P(r, \theta)$ and $Q(r + \delta r, \theta + \delta\theta)$ be two neighbouring points on the curve (Fig. 4.5). Join PQ and draw $PM \perp OQ$. Then from the rt. angled ΔOMP, $MP = r\sin\delta\theta$, $OM = r\cos\delta\theta$.

$$\therefore \quad MQ = OQ - OM = r + \delta r - r\cos\delta\theta$$
$$= \delta r + r(1 - \cos\delta\theta) = \delta r + 2r\sin^2\delta\theta/2.$$

If $\angle MQP = \alpha$, then
$$\tan\alpha = \frac{MP}{MQ} = \frac{r\sin\delta\theta}{\delta r + 2r\sin^2\delta\theta/2}$$

In the limit as $Q \to P$ (*i.e.*, $\delta\theta \to 0$), the chord PQ turns about P and becomes the tangent at P and $\alpha \to \phi$.

$$\therefore \quad \tan\phi = \underset{Q\to P}{\text{Lt}}(\tan\alpha) = \underset{\delta\theta\to 0}{\text{Lt}} \frac{r\sin\delta\theta}{\delta r + 2r\sin^2\delta\theta/2}$$
$$= \underset{\delta\theta\to 0}{\text{Lt}} \frac{r(\sin\delta\theta/\delta\theta)}{(\delta r/\delta\theta) + r\sin\delta\theta/2\cdot(\sin\delta\theta/2 \div \delta\theta/2)}$$
$$= \frac{r\cdot 1}{(dr/d\theta) + r\cdot 0\cdot 1} = r\frac{d\theta}{dr}$$

Cor. Angle of intersection of two curves. *If ϕ_1, ϕ_2 be the angles between the common radius vector and the tangents to the two curves at their point of intersection, then the angle of intersection of these curves is $\phi_1 \sim \phi_2$.*

(2) Length of the perpendicular from pole on the tangent. *If p be the perpendicular from the pole on the tangent, then*

(*i*) $\mathbf{p = r\sin\phi}$ (*ii*) $\dfrac{1}{p^2} = \dfrac{1}{r^2} + \dfrac{1}{r^4}\left(\dfrac{dr}{d\theta}\right)^2$

From the rt. $\angle$ed ΔOTP, $p = r\sin\phi$

$$\therefore \quad \frac{1}{p^2} = \frac{1}{r^2}\operatorname{cosec}^2\phi = \frac{1}{r^2}(1 + \cot^2\phi)$$
$$= \frac{1}{r^2}\left[1 + \frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right] = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 \quad \text{[By (1)]}$$

(3) Polar subtangent and subnormal. Let the tangent and the normal at any point $P(r, \theta)$ of a curve meet the line through the pole perpendicular to the radius vector OP in T and N respectively (Fig. 4.6). Then OT is called the *polar subtangent* and ON the *polar subnormal*.

Let $\angle OTP = \phi$ so that $\tan\phi = r d\theta/dr$

Clearly, $\angle PNO = \phi$.

$\therefore$ (*i*) **Polar subtangent**

$$= OT = r\tan\phi = r \cdot r d\theta/dr = \mathbf{r^2 \frac{d\theta}{dr}}$$

(*ii*) **Polar subnormal**

$$= ON = r\cot\phi = r \cdot \frac{1}{r}\frac{dr}{d\theta} = \mathbf{\frac{dr}{d\theta}}$$

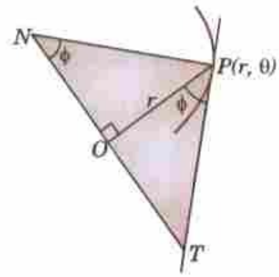

Fig. 4.6

Example 4.40. *For the cardioid $r = a(1 - \cos\theta)$, prove that*

(i) $\phi = \theta/2$ *(ii) $p = 2a\sin^3\theta/2$*

(iii) polar subtangent $= 2a\sin^2\dfrac{\theta}{2}\tan\dfrac{\theta}{2}$.

Solution. We have $\dfrac{dr}{d\theta} = a\sin\theta$

$$\therefore \quad \tan\phi = r\frac{d\theta}{dr} = a(1-\cos\theta)\cdot\frac{1}{a\sin\theta}$$

$$= 2\sin^2\theta/2 \div 2\sin\theta/2\cos\theta/2 = \tan\theta/2. \text{ Thus } \phi = \theta/2 \qquad \text{...}(i)$$

Also $p = r\sin\phi = a(1-\cos\theta)\cdot\sin\theta/2 = a\cdot 2\sin^2\theta/2\cdot\sin\theta/2$

$$= 2a\sin^3\theta/2 \qquad \text{...}(ii)$$

Polar subtangent $= r^2 d\theta/dr = [a(1-\cos\theta)]^2 \div a\sin\theta$

$$= 4a\sin^4\theta/2 \div 2\sin\theta/2\cos\theta/2 = 2a\sin^2\theta/2\tan\theta/2. \qquad \text{...}(iii)$$

Example 4.41. *Find the angle of intersection of the curves $r = \sin\theta + \cos\theta$, $r = 2\sin\theta$.*

Solution. To find the point of intersection of the curves $r = \sin\theta + \cos\theta$

and $r = 2\sin\theta$, ...(*ii*), we eliminate r.

Then $2\sin\theta = \sin\theta + \cos\theta$ or $\tan\theta = 1$ *i.e.*, $\theta = \pi/4$.

For (*i*), $\dfrac{dr}{d\theta} = \cos\theta - \sin\theta$

$$\therefore \quad \tan\phi = r\frac{d\theta}{dr} = \frac{\sin\theta + \cos\theta}{\cos\theta - \sin\theta} \text{ which} \to \infty \text{ at } \theta = \pi/4. \text{ Thus } \phi = \pi/2.$$

For (*ii*), $dr/d\theta = 2\cos\theta$ $\therefore$ $\tan\phi' = r\dfrac{d\theta}{dr} = \dfrac{2\sin\theta}{2\cos\theta} = 1$ at $\theta = \pi/4$. Thus $\phi' = \pi/4$

Hence the angle of intersection of (*i*) and (*ii*) $= \phi - \phi' = \pi/4$.

PROBLEMS 4.9

1. For a curve in Cartesian form, show that $\tan\phi = \dfrac{xy' - y}{x + yy'}$.
2. Show that in the equiangular spiral $r = ae^{\theta\cot\alpha}$, the tangent is inclined at a constant angle to the radius vector.
3. Show that the tangent to the cardioid $r = a(1 + \cos\theta)$ at the points $\theta = \pi/3$ and $\theta = 2\pi/3$ are respectively parallel and perpendicular to the initial line. (*V.T.U., 2006*)
4. Prove that, in the parabola $2a/r = 1 - \cos\theta$,

   (*i*) $\phi = \pi - \theta/2$ (*ii*) $\pi = a\,\text{cosec}\,\theta/2$, and (*iii*) polar subtangent $= 2a\,\text{cosec}\,\theta$.
5. Show that the angle between the tangent at any point P and the line joining P to the origin is the same at all points of the curve

$$\log(x^2 + y^2) = k\tan^{-1}(y/x).$$

6. Show that in the curve $r = a\theta$, the polar subnormal is constant and in the curve $r\,\theta = a$ the polar subtangent is constant.
7. Find the angle of intersection of the curves
   (*i*) $r = 2 \sin \theta$, and $r = 2 \cos \theta$ (*Bhopal, 1991*)
   (*ii*) $r = a/(1 + \cos \theta)$ and $r = b/(1 - \cos \theta)$. (*V.T.U., 2008 S*)
8. Prove that the curves $r = a(1 + \cos \theta)$ and $r = b(1 - \cos \theta)$ intersect at right angles. (*V.T.U., 2011 S*)
9. Show that the curves $r^n = a^n \cos n\theta$ and $r^n = b^n \sin n\theta$ cut each other orthogonally.
10. Show that the angle of intersection of the curves $r = a \log \theta$ and $r = a/\log \theta$ is $\tan^{-1} [2e/(1 - e^2)]$. (*V.T.U., 2005*)

4.8 PEDAL EQUATION

If r be the radius vector of any point on the curve and p, the length of the perpendicular from the pole on the tangent at that point, them *the relation between p and r is called pedal equation of the curve.*

Given the cartesian or polar equation of a curve, we can derive its pedal equation. The method is explained through the following examples.

Example 4.42. *Find the pedal equation of the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$ ...(*i*)

Solution. Equation of the tangent at (x, y) is $\frac{Xx}{a^2} + \frac{Yy}{b^2} = 1$ ...(*ii*)

p, length of $\perp$ from (0, 0) on (*ii*) $= \frac{-1}{\sqrt{[(x/a^2)^2 + (y/b^2)^2]}}$

or $$\frac{1}{p^2} = \frac{x^2}{a^4} + \frac{y^2}{b^4} \quad ...(iii)$$

Also $$r^2 = x^2 + y^2 \quad ...(iv)$$

Substituting the value of y^2 from (*iv*) in (*i*),

$$\frac{x^2}{a^2} = \frac{r^2 - b^2}{a^2 - b^2}.$$

Then from (*i*), $$\frac{y^2}{b^2} = \frac{a^2 - r^2}{a^2 - b^2}$$

Now substituting these values of x^2/a^2 and y^2/b^2 in (*iii*),

$$\frac{1}{p^2} = \frac{1}{a^2}\left(\frac{r^2 - b^2}{a^2 - b^2}\right) + \frac{1}{b^2}\left(\frac{a^2 - r^2}{a^2 - b^2}\right)$$

or $$\frac{a^2 b^2}{p^2} = \frac{r^2 b^2 - b^4 + a^4 - a^2 r^2}{a^2 - b^2} = a^2 + b^2 - r^2$$

Her[illegible] e required pedal equation.

Example 4.43. [illegible] *of the curves*
(*i*) $2a/r = $ [illegible] $r^n = a^n \cos n\,\theta$ (*V.T.U., 2010*)

Solution.
Takin[illegible]

log $2a$ – [illegible]

Differentiating both sides with respect to θ, we get

$$-\frac{1}{r}\frac{dr}{d\theta} = \frac{1}{1-\cos\theta}\cdot\sin\theta = \cot\frac{\theta}{2}$$

∴ $$\tan\phi = r\frac{d\theta}{dr} = -\tan\theta/2 = \tan(\pi-\theta/2)\ i.e.,\ \phi = \pi - \theta/2$$

Also $$p = r\sin\phi = r\sin(\pi-\theta/2)\ i.e.,\ p = r\sin\theta/2$$

or $$p^2 = r^2\sin^2\theta/2 = r^2\left(\frac{1-\cos\theta}{2}\right) = r^2.\,a/r \qquad \text{[By } (i)]$$

Hence $p^2 = ar$, which is the required pedal equation.

(*ii*) From the given equation, $nr^{n-1}\dfrac{dr}{d\theta} = -na^n\sin n\theta$

so that $$\tan\phi = r\,d\theta/dr = r\frac{nr^{n-1}}{-na^n\sin n\theta} = -\cot n\theta = \tan\left(\frac{\pi}{2}+n\theta\right)$$

i.e., $$\phi = \pi/2 + n\theta$$

∴ $$p = r\sin\phi = r\sin\left(\frac{\pi}{2}+n\theta\right) = r\cos n\theta = r\,.\,(r^n/a^n) = r^{n+1}/a^n.$$

Hence $p\,a^n = r^{n+1}$, which is the required pedal equation.

4.9 DERIVATIVE OF ARC

(1) *For the curve $y = f(x)$, we have*

$$\mathbf{\frac{ds}{dx} = \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}}$$

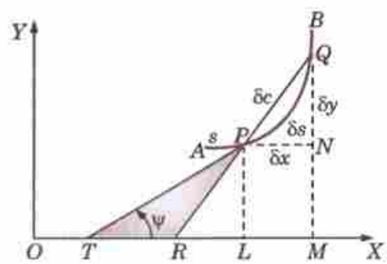

Fig. 4.7

Let $P(x, y)$, $Q(x+\delta x, y+\delta y)$ be two neighbouring points on the curve AB (Fig. 4.7). Let arc $AP = s$, arc $PQ = \delta s$ and chord $PQ = \delta c$.

Draw PL, QM ⊥s on the x-axis and $PN \perp QM$.

∴ From the rt. ∠ed ΔPNQ,

$$PQ^2 = PN^2 + NQ^2$$

i.e., $$\delta c^2 = \delta x^2 + \delta y^2$$

or $$\left(\frac{\delta c}{\delta x}\right)^2 = 1 + \left(\frac{\delta y}{\delta x}\right)^2$$

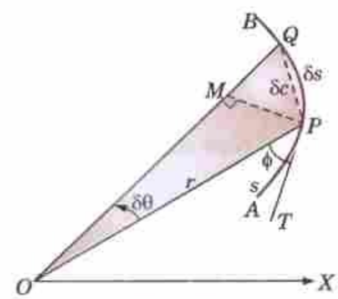

Fig. 4.8

∴ $$\left(\frac{\delta s}{\delta x}\right)^2 = \left(\frac{\delta s}{\delta c}\cdot\frac{\delta c}{\delta x}\right)^2 = \left(\frac{\delta s}{\delta c}\right)^2\left[1+\left(\frac{\delta y}{\delta x}\right)^2\right]$$

Taking limits as $Q \to P$ (*i.e.,* $\delta c \to 0$),

$$\left(\frac{ds}{dx}\right)^2 = 1\cdot\left[1+\left(\frac{dy}{dx}\right)^2\right] \qquad \left.\frac{\delta s}{\delta c} = 1\right]$$

If s increases with x as in Fig. 4.7, dy/dx is positive.

Thus $$\frac{ds}{dx} = \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]},\ \text{taking positive sign before the radical.} \qquad ...(1)$$

Cor. 1. *If the equation of the curve is $x = f(y)$, then*

$$\frac{ds}{dy} = \frac{ds}{dx}\cdot\frac{dx}{dy} = \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}\cdot\frac{dx}{dy}$$

$\therefore$ $$\mathbf{\frac{ds}{dy} = \sqrt{\left[1+\left(\frac{dx}{dy}\right)^2\right]}} \quad ...(2)$$

Cor. 2. *If the equation of the curve is in parametric form $x = f(t)$, $y = \phi(t)$, then*

$$\frac{ds}{dt} = \frac{ds}{dx}\cdot\frac{dx}{dt} = \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}\cdot\frac{dx}{dt}$$

$$= \sqrt{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dx}\cdot\frac{dx}{dt}\right)^2\right]}$$

$\therefore$ $$\mathbf{\frac{ds}{dt} = \sqrt{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]}} \quad ...(3)$$

Cor. 3. We have $$\frac{ds}{dx} = \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]} = \sqrt{(1+\tan^2\psi)} = \sec\psi$$

$\therefore$ $$\mathbf{\cos\psi = \frac{dx}{ds}.} \quad ...(4)$$

Also $$\sin\psi = \tan\psi\cos\psi = \frac{dy}{dx}\cdot\frac{dx}{ds}$$

$\therefore$ $$\mathbf{\sin\psi = \frac{dy}{ds}} \quad ...(5)$$

(2) *For the curve $r = f(\theta)$, we have* $\mathbf{\dfrac{ds}{d\theta} = \sqrt{\left[r^2 + \left(\dfrac{dr}{d\theta}\right)^2\right]}}$.

Let $P(r, \theta)$, $Q(r + \delta r, \theta + \delta\theta)$ be two neighbouring points on the curve AB (Fig. 4.8). Let arc $AP = s$, arc $PQ = \delta s$ and chord $PQ = \delta c$.

Draw $PM \perp OQ$, then

$$PM = r\sin\delta\theta \text{ and } MQ = OQ - OM = r + \delta r - r\cos\delta\theta = \delta r + 2r\sin^2\delta\theta/2$$

From the rt. $\angle$ed ΔPMQ,

$$PQ^2 = PM^2 + MQ^2$$

or $$\delta c^2 = (r\sin\delta\theta)^2 + (\delta r + 2r\sin^2\delta\theta/2)^2$$

$\therefore$ $$\left(\frac{\delta s}{\delta\theta}\right)^2 = \left(\frac{\delta s}{\delta c}\cdot\frac{\delta c}{\delta\theta}\right)^2 = \left(\frac{\delta s}{\delta c}\right)^2\left[\left(\frac{r\sin\delta\theta}{\delta\theta}\right)^2 + \left(\frac{\delta r}{\delta\theta} + \frac{2r\sin^2\delta\theta/2}{\delta\theta}\right)^2\right]$$

$$= \left(\frac{\delta s}{\delta c}\right)^2\left[r^2\left(\frac{\sin\delta\theta}{\delta\theta}\right)^2 + \left(\frac{\delta r}{\delta\theta} + r\sin\frac{\delta\theta}{2}\cdot\frac{\sin\delta\theta/2}{\delta\theta/2}\right)^2\right]$$

Taking limits as $Q \to P$

$$\left(\frac{ds}{d\theta}\right)^2 = 1^2\cdot\left[r^2\cdot 1^2 + \left(\frac{dr}{d\theta} + r\cdot 0\cdot 1\right)^2\right] = r^2 + \left(\frac{dr}{d\theta}\right)^2$$

As s increases with the increase of θ, $ds/d\theta$ is positive. Thus

$$\frac{ds}{d\theta} = \sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]} \quad ...(1)$$

Cor. 1. If the equation of the curve is $\theta = f(r)$, then

$$\frac{ds}{dr} = \frac{ds}{d\theta}\cdot\frac{d\theta}{dr} = \sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]}\cdot\frac{d\theta}{dr}$$

$$\therefore \qquad \frac{ds}{dr} = \sqrt{\left[1 + r^2\left(\frac{d\theta}{dr}\right)^2\right]} \qquad ...(2)$$

Cor. 2. We have $\dfrac{ds}{dr} = \sqrt{\left[1+\left(r\dfrac{d\theta}{dr}\right)^2\right]} = \sqrt{[1+\tan^2\phi]}\ \dfrac{ds}{dr} = \sqrt{\left[1+\left(r\dfrac{d\theta}{dr}\right)^2\right]} = \sqrt{[1+\tan^2\phi]} = \sec\phi$

$$\therefore \qquad \cos\phi = \frac{dr}{ds} \qquad ...(3)$$

Also $\sin\phi = \tan\phi \,.\, \cos\phi = r\dfrac{d\theta}{dr}\cdot\dfrac{dr}{ds}$

$$\therefore \qquad \sin\phi = r\frac{d\theta}{ds} \qquad ...(4)$$

PROBLEMS 4.10

Prove that the pedal equation of :

1. the parabola $y^2 = 4a(x + a)$ is $p^2 = ar$.
2. the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ is $a^2b^2/p^2 = r^2 - a^2 + b^2$.
3. the astroid $x = a\cos^3 t, y = a\sin^3 t$ is $r^2 = a^2 - 3p^2$.

Find the pedal equations of the following curves :

4. $r = a(1 + \cos\theta)$ (*V.T.U., 2009*)
5. $r^2 = a^2\sin^2\theta$
6. $r^m \cos m\theta = a^m$. (*V.T.U., 2004*)
7. $r^m = a^m(\cos m\theta + \sin m\theta)$ (*V.T.U., 2010*)
8. $r = ae^{m\theta}$. (*V.T.U., 2007*)
9. Calculate ds/dx for the following curves :
   (*i*) $ay^2 = x^3$. (*ii*) $y = c\cosh x/c$.
10. Find $ds/d\theta$ for the curve $x = a(\cos\theta + \theta\sin\theta), y = a(\sin\theta - \theta\cos\theta)$ (*V.T.U., 2007*)
11. Find $ds/d\theta$ for the following curves :
    (*i*) $r = a(1 - \cos\theta)$ (*V.T.U., 2004*) (*ii*) $r^2 = a^2\cos^2 2\theta$
    (*iii*) $r = \frac{1}{2}\sec^2\theta$ (*V.T.U., 2007*)
12. For the curves $\theta = \cos^{-1}(r/k) - \sqrt{(k^2 - r^2)}/r$, prove that $r\dfrac{ds}{dr}$ = constant. (*V.T.U., 2005*)
13. With the usual meanings for r, s, θ and ϕ for the polar curve $r = f(\theta)$, show that $\dfrac{d\phi}{d\theta} + r\,\mathrm{cosec}^2\theta\,\dfrac{d^2r}{ds^2} = 0$. (*V.T.U., 2000*)

4.10 CURVATURE

Let P be any point on a given curve and Q a neighbouring point. Let arc $AP = s$ and arc $PQ = \delta s$. Let the tangents at P and Q make angle ψ and $\psi + \delta\psi$ with the x-axis, so that the angle between the tangents at P and $Q = \delta\psi$ (Fig. 4.9).

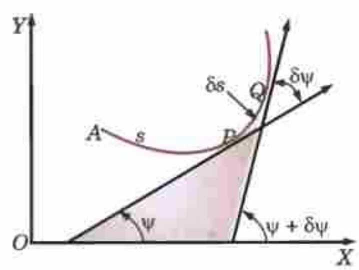

Fig. 4.9

In moving from P to Q through a distance δs, the tangent has turned through the angle $\delta\psi$. This is called the *total bending or total curvature* of the arc PQ.

$\therefore$ The *average curvature* of arc $PQ = \dfrac{\delta\psi}{\delta s}$

The limiting value of average curvature when Q approaches P (i.e., $\delta s \to 0$) is defined as the curvature of the curve at P.

Thus **curvature** K (at P) $= \dfrac{d\psi}{ds}$

Obs. Since $\delta\psi$ is measured in radians, the *unit of curvature is radians per unit length e.g.*, radians per centimetre.

(2) Radius of curvature. *The reciprocal of the curvature of a curve at any point P is called the* **radius of curvature** *at P and is denoted by* ρ, so the $\rho = \mathbf{ds/d\psi}$.

(3) Centre of curvature. *A point C on the normal at any point P of a curve distant* ρ *from it, is called the* **centre of curvature** *at P.*

(4) Circle of curvature. *A circle with centre C (centre of curvature at P) and radius* ρ *is called the* **circle of curvature** *at P.*

4.11 (1) RADIUS OF CURVATURE FOR CARTESIAN CURVE y = f (x), is given by

$$\rho = \frac{(1+\mathbf{y_1}^2)^{3/2}}{\mathbf{y_2}}$$

We know that $\tan\psi = dy/dx = y_1$ or $\psi = \tan^{-1}(y_1)$

Differentiating both sides w.r.t. x,

$$\frac{d\psi}{dx} = \frac{1}{1+y_1^2}\cdot\frac{d(y_1)}{dx} = \frac{y_2}{1+y_1^2}$$

$$\therefore \qquad \rho = \frac{ds}{d\psi} = \frac{ds}{dx}\cdot\frac{dx}{d\psi} = \sqrt{(1+y_1^2)}\,.\,\frac{1+y_1^2}{y_2} = \frac{(1+y_1^2)^{3/2}}{y_2} \qquad ...(1)$$

(2) Radius of curvature for parametric equations

$$x = f(t), \qquad y = \phi(t).$$

Denoting differentiations with respect to t by dashes,

$$y_1 = \frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = y'/x'.$$

$$y_2 = \frac{d}{dx}(y_1) = \frac{d}{dt}\left(\frac{y'}{x'}\right)\cdot\frac{dt}{dx} = \frac{x'y'' - y'x''x}{(x')^2}\cdot\frac{1}{x'}$$

Substituting the values of y_1 and y_2 in (1)

$$\rho = \left[1+\left(\frac{y'}{x'}\right)^2\right]^{3/2}\Big/\left[\frac{x'y''-y'x''}{(x')^3}\right] = \frac{(x'^2+y'^2)^{3/2}}{x'y''-y'x''} \qquad \textit{(Rajasthan, 2005)}$$

(3) Radius of curvature at the origin. Newton's formulae*

(*i*) *If x-axis is tangent to a curve at the origin, then*

$$\rho \text{ at } (0, 0) = \mathbf{\underset{x\to 0}{Lt}}\left(\frac{\mathbf{x}^2}{\mathbf{2y}}\right)$$

Since x-axis is a tangent at (0, 0), $(dy/dx)_0$ or $(y_1)_0 = 0$

$$\text{Also} \qquad \underset{x\to 0}{\text{Lt}}\left(\frac{x^2}{2y}\right) = \underset{x\to 0}{\text{Lt}}\left(\frac{2x}{2dy/dx}\right) = \underset{x\to 0}{\text{Lt}}\frac{1}{d^2y/dx^2} = \frac{1}{(y_2)_0} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$\therefore \qquad \rho \text{ at } (0, 0) = \frac{\{1+(y_1^2)_0\}^{3/2}}{(y_2)_0} = \frac{1}{(y_2)_0} = \underset{x\to 0}{\text{Lt}}\frac{x^2}{2y} \qquad \text{[From (1)]}$$

(*ii*) Similarly, *if y-axis is tangent to a curve at the origin, then*

$$\rho \text{ at } (0, 0) = \underset{x\to 0}{\text{Lt}}\left(\frac{\mathbf{y}^2}{\mathbf{2x}}\right)$$

* Named after the great English mathematician and physicist *Sir Issac Newton* (1642–1727) whose contributions are of utmost importance. He discovered many physical laws, invented Calculus alongwith *Leibnitz* (see footnote p. 139) and created analytical methods of investigating physical problems. He became professor at Cambridge in 1699, but his '*Mathematical Principles of Natural Philosophy*' containing development of classical mechanics had been completed in 1687.

(*iii*) In case the curve passes through the origin but neither x-axis nor y-axis is tangent at the origin, we write the equation of the curve as

$$y = f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots \qquad \text{[By Maclaurin's series]}$$

$$= px + qx^2/2 + \ldots \qquad [\because \ f(0) = 0]$$

where $p = f'(0)$ and $q = f''(0)$

Substituting this in the equation $y = f(x)$, we find the values of p and q by equating coefficients of like powers of x. Then $\rho\ (0, 0) = (1 + p^2)^{3/2}/q$.

Obs. *Tangents at the origin to a curve are found by equating to zero the lowest degree terms in its equation.*

Example 4.44. *Find the radius of curvature at the point* (*i*) $(3a/2, 3a/2)$ *of the Folium* $x^3 + y^3 = 3axy$.
(*Anna, 2009 ; Kurukshetra, 2009 S ; V.T.U., 2008*)
(*ii*) $(a, 0)$ *on the curve* $xy^2 = a^3 - x^3$. (*Anna, 2009 ; Kerala, 2005*)

Solution. (*i*) Differentiating with respect to x, we get

$$3x^2 + 3y^2 \frac{dy}{dx} = 3a\left(y + x\frac{dy}{dx}\right)$$

or $$(y^2 - ax)\frac{dy}{dx} = ay - x^2 \qquad \ldots(i) \qquad \therefore \ \frac{dy}{dx} \text{ at } (3a/2, 3a/2) = -1$$

Differentiating (*i*),

$$\left(2y\frac{dy}{dx} - a\right)\frac{dy}{dx} + (y^2 - ax)\frac{d^2y}{dx^2} = a\frac{dy}{dx} - 2x \quad \therefore \ \frac{d^2y}{dx^2} \text{ at } (3a/2, 3a/2) = -32/3a$$

Hence ρ at $(3a/2, 3a/2)$ $$= \frac{[1 + (dy/dx)^2]^{3/2}}{d^2y/dx^2} = \frac{[1 + (-1)^2]^{3/2}}{-32/3a} = \frac{3a}{8\sqrt{2}} \text{ (in magnitude).}$$

(*ii*) We have $$y^2 = a^3 x^{-1} - x^2$$

$\therefore$ $$2y\, dy/dx = -a^3x^{-2} - 2x \quad \text{or} \quad dy/dx = -a^3/(2x^2y) - x/y$$

At $(a, 0)$, $dy/dx \to \infty$, so we find dx/dy from $xy^2 = a^3 - x^3$

$\therefore$ $$x - 2y + y^2\frac{dx}{dy} = -3x^2\frac{dx}{dy}$$

or $$\frac{dx}{dy} = \frac{-2xy}{3x^2 + y^2} \quad \text{or} \quad \frac{dx}{dy} \text{ at } (a, 0) = 0.$$

$\therefore$ $$\frac{d^2x}{dy^2} = \frac{(3x^2 + y^2)\left(-2y\frac{dx}{dy} - 2x\right) - (-2xy)\left(6x\frac{dx}{dy} + 2y\right)}{(3x^2 + y^2)^2}$$

or $$\frac{d^2x}{dy^2} \text{ at } (a, 0) = \frac{(3a^2 + 0)(0 - 2a) - 0}{(3a^2 + 0)^2} = \frac{-2}{3a}$$

Hence $$\rho \text{ at } (a, 0) = \frac{\left[1 + \left(\frac{dx}{dy}\right)_{(a,0)}\right]^{3/2}}{\left(\frac{d^2x}{dy^2}\right)_{(a,0)}} = \frac{(1 + 0)^{3/2}}{(-2/3a)} = -\frac{3a}{2}.$$

Example 4.45. *Show that the radius of curvature at any point of the cycloid* $x = a\ (\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ *is* $4a \cos\theta/2$. (*V.T.U., 2011 ; P.T.U., 2006*)

Solution. We have $$\frac{dx}{d\theta} = a\ (1 + \cos\theta), \ \frac{dy}{d\theta} = a \sin\theta.$$

$$\therefore \qquad \frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = \frac{a \sin \theta}{a(1+\cos\theta)} = \frac{2\sin\theta/2 \cos\theta/2}{2\cos^2 \theta/2} = \tan \theta/2$$

$$\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{dy}{dx}\right) \cdot \frac{d\theta}{dx} = \frac{1}{2}\sec^2\frac{\theta}{2} \cdot \frac{1}{a(1+\cos\theta)}$$

$$= \frac{1}{2}\sec^2\frac{\theta}{2} \cdot \frac{1}{2a\cos^2\theta/2} = \frac{1}{4a}\sec^4\frac{\theta}{2}.$$

$$\therefore \qquad \rho = \frac{[1+(dy/dx)^2]^{3/2}}{d^2y/dx^2} = \frac{4a(1+\tan^2\theta/2)^{3/2}}{\sec^4\theta/2}$$

$$= 4a \cdot (\sec^2 \theta/2)^{3/2} \cdot \cos^4 \theta/2 = 4a \cos \theta/2.$$

Example 4.46. *Prove that the radius of curvature at any point of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$, is three times the length of the perpendicular from the origin to the tangent at that point.*

(J.N.T.U., 2005 ; Bhopal, 2002 S)

Solution. The parametric equation of the curve is

$$x = a \cos^3 t, y = a \sin^3 t.$$

$$\therefore \qquad x' (= dx/dt) = -3a \cos^2 t \sin t, \; y' = 3a \sin^2 t \cos t.$$

$$x'' = -3a(\cos^3 t - 2\cos t \sin^2 t) = 3a \cos t(2\sin^2 t - \cos^2 t)$$

$$y'' = 3a(2\sin t \cos^2 t - \sin^3 t) = 3a \sin t(2\cos^2 t - \sin^2 t)$$

$$x'^2 + y'^2 = 9a^2(\cos^4 t \sin^2 t + \sin^4 t \cos^2 t) = 9a^2 \sin^2 t \cos^2 t$$

$$x'y'' - y'x'' = -9a^2 \cos^2 t \sin^2 t(2\cos^2 t - \sin^2 t)$$

$$-9a^2\cos^2 t \sin^2 t(2\sin^2 t - \cos^2 t) = -9a^2 \sin^2 t \cos^2 t$$

$$\therefore \qquad \rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''} = \frac{27a^3 \sin^3 t \cos^3 t}{-9a^2 \sin^2 t \cos^2 t} = -3a \sin t \cos t.$$

Since $\qquad dy/dx = y'/x' = -\tan t,$

$\therefore$ Equation of the tangent at $(a\cos^3 t, a \sin^3 t)$ is $y - a\sin^3 t = -\tan t\,(x - a\cos^3 t)$

i.e., $\qquad x \tan t + y - a \sin t = 0 \qquad$ ...(*i*)

p, length of $\perp$ from (0, 0) on (*i*) $= \dfrac{0+0-a\sin t}{\sqrt{(\tan^2 t + 1)}} = -a \sin t \cos t$. Thus $\rho = 3p$.

EExample 4.47. *If ρ_1 and ρ_2 be the radii of curvature at the ends of a focal chord of the parabola $y^2 = 4ax$, then show that $\rho_1^{-2/3} + \rho_2^{-2/3} = (2a)^{-2/3}$.*

(Rohtak, 2006 S ; Kurukshetra, 2005)

Solution. Given parabola is $y^2 = 4ax$ or $x = at^2, y = 2at$. If dashes denote differentiation w.r.t. t, then

$$x' = 2at, y' = 2a \;;\; x'' = 2a, y'' = 0.$$

$\therefore \quad \rho$ at $\quad (at^2, 2at) = \dfrac{(x'^2 + y'^2)^{3/2}}{x'y'' - x''y'} = \dfrac{(4a^2t^2 + 4a^2)^{3/2}}{0 - 4a^2} = 2a(1+t^2)^{3/2} \qquad$ (Numerically)

If $P(t_1)$ and $Q(t_2)$ be the extremities of the focal chord of the parabola, then

$$t_1 t_2 = -1 \quad i.e., \quad t_2 = -1/t_1 \qquad ...(i)$$

$\therefore \qquad \rho_1$ at $P(t_1) = 2a(1+t_1^2)^{3/2}$; ρ_2 at $Q(t_2) = 2a(1+t_2^2)^{3/2}$

Thus $\qquad \rho_1^{-2/3} + \rho_2^{-2/3} = (2a)^{-2/3} = [(1+t_1^2)^{-1} + (1+t_2^2)^{-1}]$

$$= (2a)^{-2/3}\left[\frac{1}{1+t_1^2} + \frac{t_1^2}{1+t_1^2}\right] \qquad \text{[By } (i)]$$

$$= (2a)^{-2/3}$$

Example 4.48. *Show that the radius of curvature of P on an ellipse $x^2/a^2 + y^2/b^2 = 1$ is CD^3/ab where CD is the semi-diameter conjugate to CP.*

(J.N.T.U., 2002)

Solution. Two diameters of an ellipse are said to be conjugate if each bisects chords parallel to the other. If CP and CD are two semi-conjugate diameters and P is $(a \cos \theta, b \sin \theta)$ then D is $a \cos \left(\theta + \frac{\pi}{2}\right), b \sin \left(\theta + \frac{\pi}{2}\right)$ *i.e.*, $(- a \sin \theta, b \cos \theta)$.

Also $C(0, 0)$ is the centre of the ellipse.

$$\therefore \qquad CD = \sqrt{(a^2 \sin^2 \theta + b^2 \cos^2 \theta)}$$

At P, we have $x = a \cos \theta, y = b \sin \theta$.

$$\therefore \qquad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{b \cos \theta}{- a \sin \theta} = \frac{-b}{a} \cot \theta \,; \frac{d^2y}{dx^2} = \frac{b}{a} \operatorname{cosec}^2 \theta \,. \frac{d\theta}{dx} = \frac{-b}{a^2} \operatorname{cosec}^3 \theta .$$

$$\therefore \qquad \rho = \frac{[1 + (dy/dx)^2]^{3/2}}{d^2y/dx^2} = \frac{\left(1 + \frac{b^2}{a^2} \cot^2 \theta\right)^{3/2}}{-\frac{b}{a^2} \operatorname{cosec}^3 \theta}$$

$$= \frac{a^2}{b \operatorname{cosec}^3 \theta} \frac{(a^2 \sin^2 \theta + b^2 \cos^2 \theta)^{3/2}}{a^3 \sin^3 \theta} \qquad \text{(Numerically)}$$

$$= \frac{(a^2 \sin^2 \theta + b^2 \cos^2 \theta)^{3/2}}{ab} = \frac{CD^3}{ab}.$$

Example 4.49. *Find ρ at the origin for the curves*
(i) $y^4 + x^3 + a(x^2 + y^2) - a^2 y = 0$ *(ii)* $y - x = x^2 + 2xy + y^2$

Solution. (*i*) Equating to zero the lowest degree terms, we get $y = 0$.

$\therefore$ x-axis is the tangent at the origin. Dividing throughout by y, we have

$$y^3 + x \,. \frac{x^2}{y} + a\left(\frac{x^2}{y} + y\right) - a^2 = 0$$

Let $x \to 0$, so that $\underset{x \to \infty}{\text{Lt}} (x^2/2y) = \rho$.

$$\therefore \qquad 0 + 0.2\rho + a(2\rho + 0) - a^2 = 0 \quad \text{or} \quad \rho = a/2.$$

(*ii*) Equating to zero the lowest degree terms, we get $y = x$, as the tangent at the origin, which is neither of the coordinates axes.

$\therefore$ Putting $y = px + qx^2/2 + \ldots$ in the given equation, we get

$$px + qx^2/2 + \ldots - x = x^2 + 2x\,(px + qx^2/2 + \ldots) + (px + qx^2/2 + \ldots)^2$$

Equating coefficients of x and x^2,

$$p - 1 = 0, q/2 = 1 + 2p + p^2 \quad i.e., \quad p = 1 \text{ and } q = 2 + 4 \cdot 1 + 2 \cdot 1^2 = 8.$$

$$\therefore \qquad \rho\,(0, 0) = (1 + p^2)^{3/2}/q = (1 + 1)^{3/2}/8 = 1/2\sqrt{2}.$$

(4) Radius of curvature for polar curve $r = f(\theta)$ is given by

$$\rho = \frac{(\mathbf{r}^2 + \mathbf{r}_1^2)^{3/2}}{\mathbf{r}^2 + 2\mathbf{r}_1^2 - \mathbf{r}\mathbf{r}_2}$$

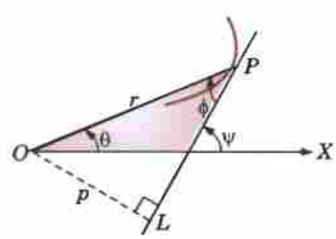

Fig. 4.10

With the usual notations, we have from Fig. 4.10.

$$\psi = \theta + \phi$$

Differentiating w.r.t. s,

$$\frac{1}{\rho} = \frac{d\psi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{d\theta} \cdot \frac{d\theta}{ds}$$

$$= \frac{d\theta}{ds}\left(1 + \frac{d\phi}{d\theta}\right) \qquad \ldots(1)$$

Also we know that

$$\tan\phi = r\frac{d\theta}{dr} = \frac{r}{r_1} \quad \text{or} \quad \phi = \tan^{-1}\left(\frac{r}{r_1}\right) \qquad \text{where } r_1 = \frac{dr}{d\theta}$$

Differentiating w.r.t. θ,

$$\frac{d\phi}{d\theta} = \frac{1}{1+(r/r_1)^2}\cdot\frac{r_1\cdot r_1 - rr_2}{r_1^2} = \frac{r_1^2 - rr_2}{r^2+r_1^2} \qquad ...(2)$$

Also, $$\frac{ds}{d\theta} = \sqrt{(r^2+r_1^2)} \qquad ...(3)$$

Substituting the value from (2) and (3) in (1),

$$\frac{1}{\rho} = \frac{1}{\sqrt{r^2+r_1^2}}\cdot\left(1+\frac{r_1^2-rr_2}{r^2+r_1^2}\right)$$

Hence $$\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}$$

(5) Radius of curvature for pedal curve $p = f(r)$ *is given by*

$$\rho = \mathbf{r}\frac{\mathbf{dr}}{\mathbf{dp}}$$

With the usual notation (Fig. 4.10), we have $\psi = \theta + \phi$

Differentiating w.r.t. s,

$$\frac{1}{\rho} = \frac{d\psi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{ds} \qquad ...(1)$$

Also we know that $p = r\sin\phi$

$$\therefore \quad \frac{dp}{dr} = \sin\phi + r\cos\phi\,\frac{d\phi}{ds}$$

$$= r\frac{d\theta}{ds} + r\frac{dr}{ds}\cdot\frac{d\phi}{dr} \qquad \text{[By (3) and (4) of § 4.9 (2)]}$$

$$= r\left(\frac{d\theta}{ds}+\frac{d\phi}{ds}\right) = \frac{r}{\rho} \qquad \text{[By (1)]}$$

Hence $$\rho = r\frac{dr}{dp}.$$

Example 4.50. *Show that the radius of curvature at any point of the cardioid $r = a\,(1-\cos\theta)$ varies as $\sqrt{r}$.* *(V.T.U., 2003)*

Solution. Differentiating w.r.t. θ, we get

$$r_1 = a\sin\theta,\ r_2 = a\cos\theta$$

$$\therefore \quad (r^2+r_1{}^2)^{3/2} = [a^2(1-\cos\theta)^2 + a^2\sin^2\theta]^{3/2} = a^3[2\,(1-\cos\theta)]^{3/2}$$

$$r^2 - rr_2 + 2r_1{}^2 = a^2(1-\cos\theta)^2 - a^2\,(1-\cos\theta)\cos\theta + 2a^2\sin^2\theta = 3a^2\,(1-\cos\theta)$$

Thus $$\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2-rr_2+2r_1^2} = \frac{a^3\,2\sqrt{2}(1-\cos\theta)^{3/2}}{3a^2(1-\cos\theta)}$$

$$= \frac{2\sqrt{2}}{3}a\,(1-\cos\theta)^{1/2} = \frac{2\sqrt{2}a}{3}\left(\frac{r}{a}\right)^{1/2} \propto \sqrt{r}.$$

Otherwise. The pedal equation of this cardioid is $2ap^2 = r^3$ ...(*i*)

Differentiating w.r.t. p, we get

that $$4ap = 3r^2\frac{dr}{dp} \text{ whence } \rho = r\frac{dr}{dp} = \frac{4ap}{3r} = \frac{4ar^{3/2}}{3r\cdot\sqrt{(2a)}} \propto \sqrt{r}.$$

$[\because \ p = r^{3/2}/\sqrt{(2a)}$ from $(i)]$

PROBLEMS 4.11

1. Find the radius of curvature at any point
   (*i*) $(at^2, 2at)$ of the parabola $y^2 = 4ax$.
   (*ii*) (o, c) of the catenary $y = c \cosh x/c$.
   (*iii*) $(a, 0)$ of the curve $y = x^3(x-a)$. (*V.T.U., 2010*)
2. Show that for (*i*) the rectangular hyperbola $xy = c^2$, $\rho = \dfrac{(x^2+y^2)^{3/2}}{2c^2}$. (*Rohtak, 2005 ; Madras, 2000*)
   (*ii*) the curve $y = ae^{x/a}$, $\rho = a \sec^2\theta \operatorname{cosec}\theta$ where $\theta = \tan^{-1}(y/a)$. (*Rajasthan, 2006*)
3. Show that the radius of curvature at
   (*i*) $(a, 0)$ on the curve $y^2 = a^2(a-x)/x$ is $a/2$. (*V.T.U., 2000 S*)
   (*ii*) $(a/4, a/4)$ on the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ is $a/\sqrt{2}$. (*J.N.T.U., 2006 S*)
   (*iii*) $x = \pi/2$ of the curve $y = 4\sin x - \sin 2x$ is $5\sqrt{5}/4$. (*V.T.U., 2009 S*)
4. For the curve $y = \dfrac{ax}{a+x}$, show that $\left(\dfrac{2\rho}{a}\right)^{2/3} = \left(\dfrac{x}{y}\right)^2 + \left(\dfrac{y}{x}\right)^2$. (*V.T.U., 2008*)
5. Find the radius of curvature at any point on the
   (*i*) *ellipse* : $x = a\cos\theta$, $y = b\sin\theta$. (*V.T.U., 2003*)
   (*ii*) *cycloid* : $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$.
   (*iii*) *curve* : $x = a(\cos t + t\sin t)$, $y = a(\sin t - t\cos t)$.
6. Show that the radius of curvature (*i*) at the point $(a\cos^3\theta, a\sin^3\theta)$ on the curve
   $x^{2/3} + y^{2/3} = a^{2/3}$ is $3a\sin\theta\cos\theta$. (*Anna, 2009*)
   (*ii*) at the point t on the curve $x = e^t\cos t$, $y = e^t\sin t$ is $\sqrt{2}e^t$. (*Calicut, 2005*)
7. If ρ be the radius of curvature at any point P on the parabola, $y^2 = 4ax$ and S be its focus, then show that ρ^2 varies as $(SP)^3$. (*Kurukshetra, 2006*)
8. Prove that for the ellipse in pedal form $\dfrac{1}{p^2} = \dfrac{1}{a^2} + \dfrac{1}{b^2} - \dfrac{1}{a^2b^2}$, the radius of curvature at the point (p, r) is $\rho = a^2b^2/p^3$. (*V.T.U., 2010 S*)
9. Show that the radius of curvature at an end of the major axis of the ellipse $x^2/a^2 + y^2/b^2 = 1$ is equal to the semi-latus rectum. (*Osmania, 2000 S*)
10. Show that the radius of curvature at each point of the curve $x = a(\cos t + \log\tan t/2)$, $y = a\sin t$, is inversely proportional to the length of the normal intercepted between the point on the curve and the x-axis. (*J.N.T.U., 2003*)
11. Find the radius of curvature at the origin for
    (*i*) $x^3 + y^3 - 2x^2 + 6y = 0$ (*Burdwan, 2003*)
    (*ii*) $2x^4 + 3y^4 + 4x^2y + xy - y^2 + 2x = 0$
    (*iii*) $y^2 = x^2(a+x)/(a-x)$.
12. Find the radius of the curvature at the point (r, θ) on each of the curves :
    (*i*) $r = a(1-\cos\theta)$ (*Kurukshetra, 2005*)
    (*ii*) $r^n = a^n\cos n\theta$. (*P.T.U., 2010; J.N.T.U., 2006*)
13. For the cardioid $r = a(1+\cos\theta)$, show that ρ^2/r is constant. (*P.T.U., 2005*)
14. Find the radius of curvature for the parabola $2a/r = 1 + \cos\theta$. (*Kurukshetra, 2006*)
15. If ρ_1, ρ_2 be the radii of curvature at the extremities of any chord of the cardioid $r = a(1+\cos\theta)$ which passes through the pole, show that $\rho_1^2 + \rho_2^2 = 16a^2/9$.
16. For any curve $r = f(\theta)$, prove that $\dfrac{r}{\rho} = \sin\phi\left(1 + \dfrac{d\phi}{d\theta}\right)$.

4.12 (1) CENTRE OF CURVATURE at any point P (x, y) on the curve y = f (x) is given by

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}, \quad \bar{y} = y + \frac{1+y_1^2}{y_2}.$$

Let $C(\bar{x}, \bar{y})$ be the centre of curvature and ρ the radius of curvature of the curve at $P(x, y)$ (Fig. 4.11). Draw PL and CM ⊥s to OX and $PN \perp CM$. Let the tangent at P make an $\angle\psi$ with the x-axis. Then $\angle NCP = 90° - \angle NPC = \angle NPT = \psi$

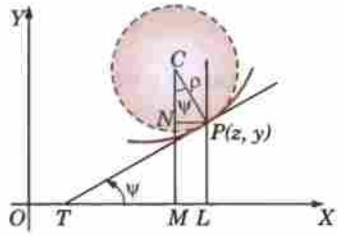

Fig. 4.11

$$\therefore \quad \bar{x} = OM = OL - ML = OL - NP$$

$$= x - \rho \sin\psi = x - \frac{(1 + y_1^2)^{3/2}}{y_2} \cdot \frac{y_1}{\sqrt{(1 + y_1^2)}}$$

$$\left[\because \tan\psi = y_1, \quad \therefore \sin\psi = \frac{y_1}{\sqrt{(1 + y_1^2)}}\right.$$

$$= x - \frac{y_1(1 + y_1^2)}{y_2}$$

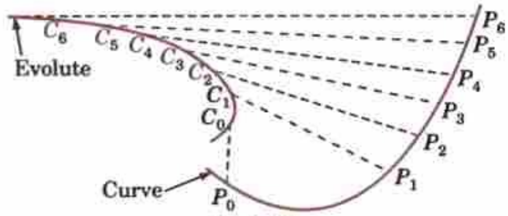

Fig. 4.12

and $$\bar{y} = MC = MN + NC = LP + \rho\cos\psi$$

$$\left[\because \sec\psi = \sqrt{(1 + \tan^2\psi)} = \sqrt{(1 + y_1^2)}\right.$$

$$= y + \frac{(1 + y_1^2)^{3/2}}{y_2} \cdot \frac{1}{\sqrt{(1 + y_1^2)}} = y + \frac{1 + y_1^2}{y_2}$$

Cor. Equation of the circle of curvature at P is $(x - \bar{x})^2 + (y - \bar{y})^2 = \rho^2$.

(2) Evolute. *The locus of the centre of curvature for a curve is called its* **evolute** *and the curve is called an* **involute** *of its evolute.* (*Fig. 4.12*)

Example 4.51. *Find the coordinates of the centre of curvature at any point of the parabola $y^2 = 4ax$. Hence show that its evolute is*

$$27ay^2 = 4(x - 2a)^3. \qquad \text{(V.T.U., 2000)}$$

Solution. We have $\quad 2yy_1 = 4a \quad$ *i.e.*, $\quad y_1 = 2a/y$

and $$y_2 = -\frac{2a}{y^2} \cdot y_1 = -\frac{4a^2}{y^3}$$

If $(\bar{x}, \bar{y})$ be the centre of curvature, then

$$\bar{x} = x - \frac{y_1(1 + y_1^2)}{y_2} = x - \frac{2a/y(1 + 4a^2/y^2)}{-4a^2/y^3}$$

$$= x + \frac{y^2 + 4a^2}{2a} = x + \frac{4ax + 4a^2}{2a} = 3x + 2a \qquad [\because y^2 = 4ax] \qquad ...(i)$$

and $$\bar{y} = y + \frac{1 + y_1^2}{y_2} = y + \frac{1 + 4a^2/y^2}{-4a^2/y^2}$$

$$= y - \frac{y(y^2 + 4a^2)}{4a^2} = \frac{-y^3}{4a^2} = -\frac{2x^{3/2}}{\sqrt{a}} \qquad ...(ii)$$

To find the evolute, we have to eliminate x from (i) and (ii)

$$\therefore \quad (\bar{y})^2 = \frac{4x^3}{a} = \frac{4}{a}\left(\frac{\bar{x} - 2a}{3}\right)^3 \quad \text{or} \quad 27a(\bar{y})^2 = 4(\bar{x} - 2a)^3.$$

Thus the locus of $(\bar{x}, \bar{y})$ *i.e.*, evolute, is $27ay^2 = 4(x - 2a)^3$.

Example 4.52. *Show that the evolute of the cycloid $x = a(\theta - \sin\theta), y = a(1 - \cos\theta)$ is another equal cycloid.* (*Madras, 2006*)

Solution. We have $y_1 = \dfrac{dy}{d\theta} \div \dfrac{dx}{d\theta} = \dfrac{a\sin\theta}{a(1-\cos\theta)} = \cot\dfrac{\theta}{2}$.

$$y^2 = \frac{d}{dx}(y_1) = \frac{d}{d\theta}\left(\cot\frac{\theta}{2}\right)\cdot\frac{d\theta}{dx}$$

$$= -\operatorname{cosec}^2\frac{\theta}{2}\cdot\frac{1}{2}\cdot\frac{1}{a(1-\cos\theta)} = -\frac{1}{4a\sin^4\theta/2}$$

If $(\bar{x}, \bar{y})$ be the centre of curvature, then

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2} = a(\theta - \sin\theta) + \cot\frac{\theta}{2}\left(-4a\sin^4\frac{\theta}{2}\right)\left(1+\cot^2\frac{\theta}{2}\right)$$

$$= a(\theta - \sin\theta) + \frac{\cos\theta/2}{\sin\theta/2}\cdot 4a\sin^4\frac{\theta}{2}\cdot\operatorname{cosec}^2\frac{\theta}{2}$$

$$= a(\theta - \sin\theta) + 4a\sin\theta/2\cos\theta/2 = a(\theta - \sin\theta) + 2a\sin\theta = a(\theta + \sin\theta)$$

$$\bar{y} = y + \frac{1+y_1^2}{y_2} = a(1-\cos\theta) + \left(1+\cot^2\frac{\theta}{2}\right)\left(-4a\sin^4\frac{\theta}{2}\right)$$

$$= a(1-\cos\theta) - 4a\sin^4\theta/2\,.\operatorname{cosec}^2\theta/2$$

$$= a(1-\cos\theta) - 4a\sin^2\theta/2$$

$$= a(1-\cos\theta) - 2a(1-\cos\theta) = -a(1-\cos\theta)$$

Hence the locus of $(\bar{x}, \bar{y})$ *i.e.*, the evolute, is given by

$x = a(\theta + \sin\theta), y = -a(1-\cos\theta)$ which is another equal cycloid.

(3) Chord or curvature *at a given point of a curve*

(*i*) parallel to x-axis $= 2\rho\sin\psi$

(*ii*) parallel to y-axis $= 2\rho\cos\psi$

Consider the circle of curvature at a given point P on a curve. Let C be the centre and ρ the radius of curvature at P so that $PQ = 2\rho$. (Fig. 4.13)

Let PL, PM be the chords of curvature parallel to the axes of x and y respectively. Let the tangent PT make an $\angle\psi$ with the x-axis so that $\angle LQP = \angle QPM = \psi$.

Then from the rt. $\angle$ed ΔPLQ,

$$PL = 2\rho\sin\psi\sigma$$

and $$PM = 2\rho\cos\psi.$$

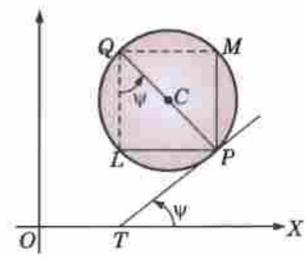

Fig. 4.13

4.13 (1) ENVELOPE

The equation $x\cos\alpha + y\sin\alpha = 1$ ...(1)

represents a straight line for a given value of α. If different values are given to α, we get different straight lines. All these straight lines thus obtained are said to constitute a family of straight lines.

In general, the *curves corresponding to the equation $f(x, y, \alpha) = 0$ for different values of α, constitute a* **family of curves** *and α is called the* **parameter** *of the family.*

The envelope of a family of curves is the curve which touches each member of the family. For example, we know that all the straight lines of the family (1) touch the circle

$$x^2 + y^2 = 1 \quad ...(2)$$

i.e., the envelope of the family of lines (1) is the circle (2)—Fig. 4.14, which may also be seen as the locus of the ultimate points of intersection of the consecutive members of the family of lines (1). This leads to the following :

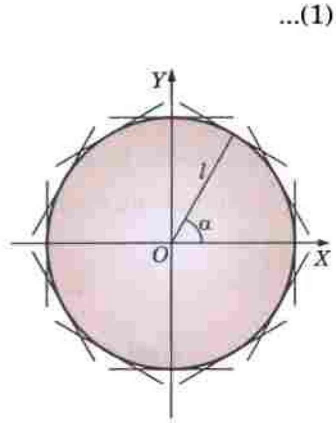

Fig. 4.14

Def. *If $f(x, y, \alpha) = 0$ and $f(x, y, \alpha + \delta\alpha) = 0$ be two consecutive members of a family of curves, then the locus of their ultimate points of intersection is called the* **envelope** *of that family.*

(2) Rule to find the envelope of the family of curves $f(x, y, \alpha) = 0$:

Eliminate α from $f(x, y, \alpha) = 0$ and $\dfrac{\partial f(x, y, \alpha)}{\partial \alpha} = 0$.

Example 4.53. *Find the envelope of the family of lines $y = mx + \sqrt{(1 + m^2)}$, m being the parameter.*

Solution. We have $(y - mx)^2 = 1 + m^2$ ...(*i*)

Differentiating (*i*) partially with respect to m,

$$2(y - mx)(-x) = 2m \quad \text{or} \quad m = xy/(x^2 - 1) \qquad ...(ii)$$

Now eliminating m from (*i*) and (*ii*)

Substituting the value of m in (*i*), we get

$$\left(y - \frac{x^2 y}{x^2 - 1}\right)^2 = 1 + \left(\frac{xy}{x^2 - 1}\right)^2 \quad \text{or} \quad y^2 = (x^2 - 1)^2 + x^2y^2$$

or $\quad x^2 + y^2 = 1 \quad$ which is the required equation of the envelope.

Obs. Sometimes the equation to the family of curves contains two parameters which are connected by a relation. In such cases, we eliminate one of the parameters by means of the given relation, then proceed to find the envelope.

Example 4.54. *Find the envelope of a system of concentric and coaxial ellipses of constant area.*

Solution. Taking the common axes of the system of ellipses as the coordinate axes, the equation to an ellipse of the family is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ where } a \text{ and } b \text{ are the parameters.} \qquad ...(i)$$

The area of the ellipse = πab which is given to be constant, say = πc^2.

$$\therefore \qquad ab = c^2 \quad \text{or} \quad b = c^2/a. \qquad ...(ii)$$

Substituting in (*i*), $\dfrac{x^2}{a^2} + \dfrac{y^2}{(c^2/a^2)} = 1 \quad$ or $\quad x^2 a^{-2} + (y^2/c^4)\, a^2 = 0$ ...(*iii*)

which is the given family of ellipses with a as the only parameter.

Differentiating partially (*iii*) with respect to a,

$$-2x^2 a^{-3} + 2(y^2/c^4)\, a = 0 \quad \text{or} \quad a^2 = c^2\, x/y \qquad ...(iv)$$

Eliminate a from (*iii*) and (*iv*).

Substituting the value of a^2 in (*iii*), we get

$$x^2(y/c^2x) + (y^2/c^4)(c^2x/y) = 1 \text{ or } 2xy = c^2$$

which is the required equation of the envelope. P

(3) Evolute *of a curve is the envelope of the normals to that curve* (*Fig. 4.12*)

Example 4.55. *Find the evolute of the parabola $y^2 = 4ax$.* (*Madras, 2003*)

Solution. Any normal to the parabola is $y = mx - 2am - am^3$ ...(*i*)

Differentiating it with respect to m partially,

$$0 = x - 2a - 3am^2 \text{ or } m = [(x - 2a)/3a]^{1/2}$$

Substituting this value of m in (*i*),

$$y = \left(\frac{x - 2a}{3a}\right)^{1/2} \left[x - 2a - a \cdot \frac{x - 2a}{3a}\right]$$

Squaring both sides, we have

$$27ay^2 = 4(x - 2a)^3$$

which is the evolute of the parabola. (*cf.* Example 4.51).

PROBLEMS 4.12

1. Find the coordinates of the centre of curvature at $(at^2, 2at)$ on the parabola $y^2 = 4ax$. (V.T.U., 2000 S)
2. If the centre of curvature of the ellipse $x^2/a^2 + y^2/b^2 = 1$ at one end of the minor axis lies at the other end, then show that the eccentricity of the ellipse is $1/\sqrt{2}$. (Anna, 2005 S ; Madras, 2003)
3. Show that the equation of the evolute of the
   (i) *parabola* $x^2 = 4ay$ is $4(y - 2a)^3 = 27ax^2$. (Anna, 2009)
   (ii) *ellipse* $x = a \cos \theta, y = b \sin \theta$ (*i.e.*, $x^2/a^2 + y^2/b^2 = 1$) is
   $$(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}.$$
   (iii) *rectangular hyperbola* $xy = c^2$, (*i.e.*, $x = ct, y = c/t$) is
   $$(x + y)^{2/3} - (x - y)^{2/3} = (4c)^{2/3}.$$ (Anna, 2003)
4. Find the evolute of (i) cycloid $x = a(t + \sin t), y = a(1 - \cos t)$
   (ii) the curve $x = a(\cos \theta + \theta \sin \theta), y = a(\sin \theta - \theta \cos \theta)$. (Anna, 2009 S)
5. Find the evolute of the curve $x = a \cos^3 \theta, y = a \sin^3 \theta$ *i.e.*, $x^{2/3} + y^{2/3} = a^{2/3}$. (Osmania, 2002)
6. Show that the evolute of the curve $x = a(\cos t + \log \tan t/2), y = a \sin t$ is $y = a \cosh x/a$. (Anna, 2005 S)
7. Find the circle of curvature at the point (i) $(a/4, a/4)$ of the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$.
   (ii) $(3/2, 3/2)$ of the curve $x^3 + y^3 = 3xy$ (Anna, 2009 ; Madras, 2006 ; Calicut, 2005)
8. Show that the circle of curvature at the origin for the curve $x + y = ax^2 + by^2 + cx^3$ is
   $$(a + b)(x^2 + y^2) = 2(x + y).$$ (Nagpur, 2009)
9. If C_x, C_y be the chords of curvature parallel to the axes at any point on the curve $y = ae^{x/a}$, prove that
   $$\frac{1}{C_x^2} + \frac{1}{C_y^2} = \frac{1}{2aC_x}.$$
10. In the curve $y = a \cosh x/a$, prove that the chord of curvature parallel to y-axis is the double the ordinate.

Find the envelope of the following family of lines :

11. $y = mx + a/m$, m being the parameter. (Madras, 2006)
12. $\dfrac{x}{a} \cos \alpha + \dfrac{y}{b} \sin \alpha = 1$, α being the parameter.
13. $y = mx - 2am - am^3$.
14. $y = mx + \sqrt{(a^2m^2 + b^2)}$, m being the parameter. (Anna, 2009)
15. Find the envelope of the family of parabolas $y = x \tan \alpha - \dfrac{gx^2}{2u^2 \cos \alpha}$, α being the parameter.
16. Find the envelope of the straight line $x/a + y/b = 1$, where the parameters a and b are connected by the relation :
   (i) $a + b = c$. (ii) $ab = c^2$
   (iii) $a^2 + b^2 = c^2$.
17. Find the envelope of the family of ellipses $x^2/a^2 + y^2/b^2 = 1$ for which $a + b = c$. (Madras, 2006)

Prove that the evolute of the

18. *ellipse* $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is $(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$. (J.N.T.U., 2006 ; Anna, 2005)
19. *hyperbola* $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ is $(ax)^{2/3} - (by)^{2/3} = (a^2 + b^2)^{2/3}$. (Anna, 2009)
20. *parabola* $x^2 = 4by$ is $27bx^2 = 4(y - 2b)^3$.

4.14 (1) INCREASING AND DECREASING FUNCTIONS

In the function $y = f(x)$, *if y increases as x increases* (as at A), *it is called an* **increasing function of** x. On the contrary, *if y decreases as x increases* (as at C), *it is called a* **decreasing function** *of* x.

Let the tangent at any point on the graph of the function make an $\angle\psi$ with the x-axis (Fig. 4.15) so that

$$dy/dx = \tan \psi$$

At any point such as A, where the function is increasing $\angle\psi$ is acute *i.e.*, dy/dx is positive. At a point such as C, where the function is decreasing $\angle\psi$ is obtuse *i.e.*, dy/dx is negative.

Hence the *derivative of an increasing function is + ve*, and *the derivative of a decreasing function is – ve.*

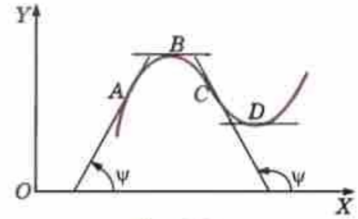

Fig. 4.15

Obs. *If the derivative is zero* (as at B or D), *then y is neither increasing nor decreasing.* In such cases, we say *that the function is* **stationary.**

(2) Concavity, Convexity and Point of Inflexion

(*i*) *If a portion of the curve on both sides of a point, however small it may be, lies above the tangent* (*as at D*), *then the curve is said to be* **concave upwards** at D where d^2y/dx^2 is positive.

(*ii*) *If a portion of the curve on both sides of a point lies below the tangent* (as at *B*), *then the curve is said to be* **Convex upwards** at B where d^2y/dx^2 is negative.

(*iii*) *If the two portions of the curve lie on different sides of the tangent thereat* (*i.e.*, the curve crosses the tangent (as at *C*), *then the point C* is said to be a **point of inflexion** of the curve.

At a point of inflexion $\dfrac{d^2y}{dx^2} = 0$ and $\dfrac{d^3y}{dx^3} \neq 0$.

4.15 (1) MAXIMA AND MINIMA

Consider the graph of the continuous function $y = f(x)$ in the interval (x_1, x_2) (Fig. 4.16). Clearly the point P_1 is the highest in its own immediate neighbourhood. So also is P_3. At each of these points P_1, P_3 the function is said to have a *maximum* value.

On the other hand, the point P_2 is the lowest in its own immediate neighbourhood. So also is P_4. At each of these points P_2, P_4 the function is said to have a *minimum* value.

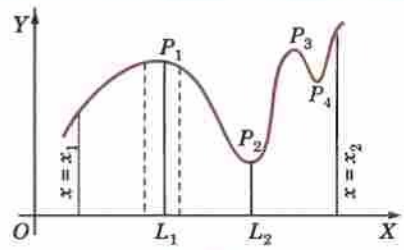

Fig. 4.16

Thus, we have

Def. *A function f(x) is said to have a* **maximum** *value at x = a, if there exists a small number h, however small, such that f(a) > both f(a – h) and f(a + h).*

A function f(x) is said to have a **minimum** *value at x = a, if there exists a small number h, however small, such that f(a) < both f(a – h) and f(a + h).*

Obs. 1. *The maximum and minimum values of a function taken together are called its* **extreme** *values and the points at which the function attains the extreme values are called the* **turning points** *of the function.*

Obs. 2. *A maximum or minimum value of a function is not necessarily the greatest or least value of the function in any finite interval.* The maximum value is simply the greatest value in the immediate neighbourhood of the maxima point or the minimum value is the least value in the immediate neighbourhood of the minima point. In fact, *there may be several maximum and minimum values of a function in an interval and a minimum value may be even greater than a maximum value.*

Obs. 3. It is seen from the Fig. 4.16 that *maxima and minima values occur alternately.*

(2) Conditions for maxima and minima. At each point of extreme value, it is seen from Fig. 4.16 that the tangent to the curve is parallel to the x-axis, *i.e.*, its slope (= dy/dx) is zero. Thus if *the function is maximum or minimum at* $x = a$, *then* $(dy/dx)_a = 0$.

Around a maximum point say, P_1 $(x = a)$, the curve is increasing in a small interval $(a - h, a)$ before L_1 and decreasing in $(a, a + h)$ after L_1 where h is positive and small.

i.e., in $(a - h, a)$, $dy/dx \geq 0$; at $x = a$, $dy/dx = 0$ and in $(a, a + h)$, $dy/dx \leq 0$.

Thus dy/dx (which is a function of x) changes sign from positive to negative in passing through P_1, *i.e.*, it is a decreasing function in the interval $(a - h, a + h)$ and therefore, its derivative d^2y/dx^2 is negative at $P_1(x = a)$.

Similarly, around a minimum point say P_2, dy/dx changes sign from negative to positive in passing through P_2, *i.e.*, it is an increasing function in the small interval around L_2 and therefore its derivative d^2y/dx^2 is positive at P_2.

Hence (*i*) *f(x) is maximum at x = a if f′(a) = 0 and f″(a) is – ve [i.e., f′(a) changes sign from + ve to – ve]*
(*ii*) *f(x) is minimum at x = a, if f′(a) = 0 and f″(a) is + ve [i.e., f′(a) changes sign from – ve to + ve]*

Obs. A maximum or a minimum value is a stationary value but a stationary value may neither be a maximum nor a minimum value.

(3) Procedure for finding maxima and minima

(*i*) *Put the given function = f(x)*
(*ii*) *Find f′(x) and equate it to zero. Solve this equation and let its roots be a, b, c, ...*
(*iii*) *Find f″(x) and substitute in it by turns x = a, b, c, ...*
If f″(a) is – ve, f(x) is maximum at x = a.
If f″(a) is + ve, f″(x) is minima at x = a.
(*iv*) Sometimes $f''(x)$ may be difficult to find out or $f''(x)$ may be zero at $x = a$. In such cases, see if $f'(x)$ *changes sign from + ve to – ve as x passes through a, then f(x) is maximum at x = a.*
If f′(x) changes sign from – ve to + ve as x passes through a, f(x) is minimum at x = a.
If f′(x) does not change sign while passing through x = a, f(x) is neither maximum nor minimum at x = a.

Example 4.56. *Find the maximum and minimum values of $3x^4 - 2x^3 - 6x^2 + 6x + 1$ in the interval (0, 2).*

Solution. Let $f(x) = 3x^4 - 2x^3 - 6x^2 + 6x + 1$
Then $f'(x) = 12x^3 - 6x^2 - 12x + 6 = 6(x^2 - 1)(2x - 1)$
$\therefore$ $f'(x) = 0$ when $x = \pm 1, \frac{1}{2}$.

So in the interval (0, 2) $f(x)$ can have maximum or minimum at $x = \frac{1}{2}$ or 1.

Now $f''(x) = 36x^2 - 12x - 12 = 12(3x^2 - x - 1)$ so that $f''\left(\frac{1}{2}\right) = -9$ and $f''(1) = 12$.

$\therefore$ $f(x)$ has a maximum at $x = \frac{1}{2}$ and a minimum at $x = 1$.

Thus the maximum value $= f\left(\frac{1}{2}\right) = 3\left(\frac{1}{2}\right)^4 - 2\left(\frac{1}{2}\right)^3 - 6\left(\frac{1}{2}\right)^2 + 6\left(\frac{1}{2}\right) + 1 = 2\frac{7}{16}$

and the minimum value $= f(1) = 3(1)^4 - 2(1)^3 - 6(1)^2 + 6(1) + 1 = 2$.

Example 4.57. *Show that sin x (1 + cos x) is a maximum when x = π/3.* (*Bhopal, 2009 ; Rajasthan, 2005*)

Solution. Let $f(x) = \sin x\,(1 + \cos x)$
Then $f'(x) = \cos x\,(1 + \cos x) + \sin x\,(-\sin x)$
$= \cos x\,(1 + \cos x) - (1 - \cos^2 x) = (1 + \cos x)(2\cos x - 1)$
$\therefore$ $f'(x) = 0$ when $\cos x = \frac{1}{2}$ or -1 *i.e.*, when $x = \pi/3$ or π.
Now $f''(x) = -\sin x\,(2\cos x - 1) + (1 + \cos x)(-2\sin x) = -\sin x(4\cos x + 1)$
so that $f''(\pi/3) = -3\sqrt{2}/2$ and $f''(\pi) = 0$.

Thus $f(x)$ has a maximum at $x = \pi/3$.

Since $f''(\pi)$ is 0, let us see whether $f'(x)$ changes sign or not.

When x is slightly $< \pi$, $f'(x)$ is – ve, then when x is slightly $> \pi$, $f'(x)$ is again – ve *i.e.*, $f'(x)$ does not change sign as x passes through π. So $f(x)$ is neither maximum nor minimum at $x = \pi$.

(4) Practical Problems

In many problems, the function (whose maximum or minimum value is required) is not directly given. It has to be formed from the given data. If the function contains two variables, one of them has to be eliminated with the help of the other conditions of the problem. A number of problems deal with triangles, rectangles, circles, spheres, cones, cylinders etc. The student is therefore, advised to remember the formulae for areas, volumes, surfaces etc. of such figures.

Example 4.58. *A window has the form of a rectangle surmounted by a semi-circle. If the perimeter is 40 ft., find its dimensions so that the greatest amount of light may be admitted.* (*Madras, 2000 S*)

Solution. The greatest amount of light may be admitted means that the area of the window may be maximum.

Let x ft. be the radius of the semi-circle so that one side of the rectangle is $2x$ ft. (Fig. 4.17). Let the other side of the rectangle y ft. Then the perimeter of the whole figure

$$= \pi x + 2x + 2y = 40 \text{ (given) and the area } A = \tfrac{1}{2}\pi x^2 + 2xy. \qquad ...(i)$$

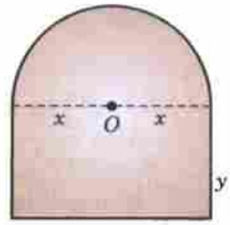

Fig. 4.17

Here A is a function of two variables x and y. To express A in terms of one variable x (say), we substitute the value of y from (i) in it.

$$\therefore \quad A = \tfrac{1}{2}\pi x^2 + x[40 - (\pi + 2)x] = 40\,x - \left(\frac{\pi}{2} + 2\right)x^2$$

Then $$\frac{dA}{dx} = 40 - (\pi + 4)x$$

For A to be maximum or minimum, we must have $dA/dx = 0$ *i.e.*, $40 - (\pi + 4)x = 0$

or $$x = 40/(\pi + 4)$$

$\therefore$ From (i), $$y = \tfrac{1}{2}[40 - (\pi + 2)x] = \tfrac{1}{2}[40 - (\pi + 2)\,40/(\pi + 4)] = 40/(\pi + 4) \text{ i.e., } x = y$$

Also $$\frac{d^2A}{dx^2} = -(\pi + 4), \text{ which is negative.}$$

Thus the area of the window is maximum when the radius of the semi-circle is equal to the height of the rectangle.

Example 4.59. *A rectangular sheet of metal of length 6 metres and width 2 metres is given. Four equal squares are removed from the corners. The sides of this sheet are now turned up to form an open rectangular box. Find approximately, the height of the box, such that the volume of the box is maximum.*

Solution. Let the side of each of the squares cut off be x m so that the height of the box is x m and the sides of the base are $6 - 2x$, $2 - 2x$ m (Fig. 4.18).

Fig. 4.18

$\therefore$ Volume V of the box

$$= x(6 - 2x)(2 - 2x) = 4(x^3 - 4x^2 + 3x)$$

Then $$\frac{dV}{dx} = 4(3x^2 - 8x + 3)$$

For V to be maximum or minimum, we must have

$$dV/dx = 0 \text{ i.e., } 3x^2 - 8x + 3 = 0$$

$$\therefore \quad x = \frac{8 \pm \sqrt{[64 - 4 \times 3 \times 3]}}{6} = 2.2 \text{ or } 0.45 \text{ m.}$$

The value $x = 2.2$ m is inadmissible, as no box is possible for this value.

Also $\dfrac{d^2V}{dx^2} = 4(6x - 8)$, which is – ve for $x = 0.45$ m.

Hence the volume of the box is maximum when its height is 45 cm.

Example 4.60. *Show that the right circular cylinder of given surface (including the ends) and maximum volume is such that its height is equal to the diameter of the base.*

Solution. Let r be the radius of the base and h, the height of the cylinder.

Then given surface $S = 2\pi rh + 2\pi r^2 \quad ...(i)$ and the volume $V = \pi r^2 h \quad ...(ii)$

Hence V is a function of two variables r and h. To express V in terms of one variable only (say r), we substitute the value of h from (i) in (ii).

Then $$V = \pi r^2\left(\frac{S - 2\pi r^2}{2\pi r}\right) = \frac{1}{2}Sr - \pi r^3 \quad \therefore \quad \frac{dV}{dr} = \frac{1}{2}S - 3\pi r^2.$$

For V to be maximum or minimum, we must have $dV/dr = 0$,

i.e., $$\tfrac{1}{2}S - 3\pi r^2 = 0 \quad \text{or} \quad r = \sqrt{(S/6\pi)}.$$

Also $\dfrac{d^2V}{dr^2} = -6\pi r$, which is negative for $r = \sqrt{(S/6\pi)}$.

Hence V is maximum for $r = \sqrt{(S/6\pi)}$.

i.e., for $6\pi r^2 = S = 2\pi rh + 2\pi r^2$ *i.e.*, for $h = 2r$, which proves the required result. [By (*i*)]

Example 4.61. *Show that the diameter of the right circular cylinder of greatest curved surface which can be inscribed in a given cone is equal to the radius of the cone.*

Solution. Let r be the radius OA of the base and α the semi-vertical angle of the given cone (Fig. 4.19). Inscribe a cylinder in it with base-radius $OL = x$.

Then the height of the cylinder LP

$$= LA \cot\alpha = (r - x)\cot\alpha$$

$\therefore$ The curved surface S of the cylinder

$$= 2\pi x.\, LP = 2\pi x(r - x)\cot\alpha$$
$$= 2\pi\cot\alpha\,(rx - x^2)$$

$\therefore$ $$\frac{dS}{dx} = 2\pi\cot\alpha\,(r - 2x) = 0 \text{ for } x = r/2.$$

and $$\frac{d^2S}{dx^2} = -4\pi\cot\alpha$$

Hence S is maximum when $x = r/2$.

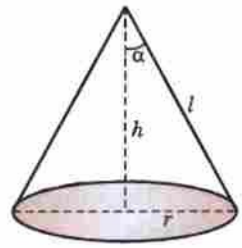

Fig. 4.19

Example 4.62. *Find the altitude and the semi-vertical angle of a cone of least volume which can be circumscribed to a sphere of radius a.*

Solution. Let h be the height and α the semi-vertical angle of the cone so that its radius $BD = h\tan\alpha$ (Fig. 4.20).

$\therefore$ The volume V of the cone is given by

$$V = \frac{1}{3}\pi(h\tan\alpha)^2 h = \frac{1}{3}\pi h^3\tan^2\alpha.$$

Now we must express $\tan\alpha$ in terms of h.

In the rt. $\angle d\ \Delta AEO$,

$$EA = \sqrt{(OA^2 - a^2)} = \sqrt{[(h-a)^2 - a^2]} = \sqrt{(h^2 - 2ha)}$$

$\therefore$ $$\tan\alpha = \frac{EO}{EA} = \frac{a}{\sqrt{(h^2 - 2ha)}}$$

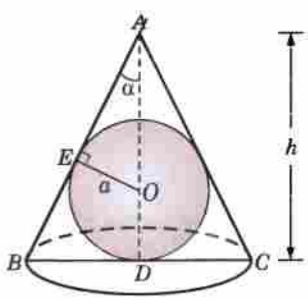

Fig. 4.20

Thus $$V = \frac{1}{3}\pi h^3 \cdot \frac{a^2}{h^2 - 2ha} = \frac{1}{3}\pi a^3 \cdot \frac{h^2}{h - 2a}$$

$\therefore$ $$\frac{dV}{dh} = \frac{1}{3}\pi a^2 \cdot \frac{(h-2a)2h - h^2\cdot 1}{(h-2a)^2} = \frac{1}{3}\pi a^2 \cdot \frac{h(h-4a)}{(h-2a)^2}$$

Thus $\dfrac{dV}{dh} = 0$ for $h = 4a$, the other value $(h = 0)$ being not possible.

Also dV/dh is – ve when h is slightly $< 4a$, and it is + ve when h is slightly $> 4a$.

Hence V is minimum (*i.e.* least) when $h = 4a$

and $$\alpha = \sin^{-1}\left(\frac{a}{OA}\right) = \sin^{-1}\left(\frac{a}{3a}\right) = \sin^{-1}\frac{1}{3}.$$

Example 4.63. *Find the volume of the largest possible right-circular cylinder that can be inscribed in a sphere of radius a.*

Solution. Let O be the centre of the sphere of radius a. Construct a cylinder as shown in Fig. 4.21. Let $OA = r$.

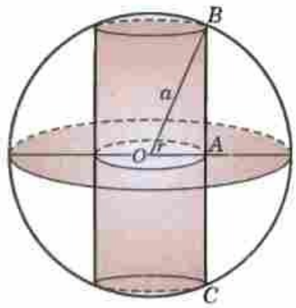

Fig. 4.21

Then $$AB = \sqrt{(OB^2 - OA^2)} = \sqrt{(a^2 - r^2)}$$

$\therefore$ Height h of the cylinder $= 2 \cdot AB = 2\sqrt{(a^2 - r^2)}$.

Thus volume V of the cylinder

$$= \pi r^2 h = 2\pi r^2 \sqrt{(a^2 - r^2)}$$

$$\therefore \quad \frac{dV}{dr} = 2\pi \{2r\sqrt{(a^2 - r^2)} + r^2 \cdot \frac{1}{2}(a^2 - r^2)^{-1/2}(-2r)\}$$

$$= \frac{2\pi r(2a^2 - 3r^2)}{\sqrt{(a^2 - r^2)}}$$

The $dV/dr = 0$ when $r^2 = 2a^2/3$, the other value $(r = 0)$ being not admissible.

Now $$\frac{d^2V}{dr^2} = 2\pi \frac{\sqrt{(a^2 - r^2)}(2a^2 - 9r^2) - r(2a^2 - 3r^2) \times \frac{1}{2}(a^2 - r^2)^{-1/2}.(-2r)}{(a^2 - r^2)}$$

$$= 2\pi \frac{(a^2 - r^2)(2a^2 - 9r^2) + r^2(2a^2 - 3r^2)}{(a^2 - r^2)^{3/2}}$$ which is – ve for $r^2 = 2a^2/3$.

Hence V is maximum for $r^2 = 2a^2/3$ and maximum volume

$$= 2\pi r^2 \sqrt{(a^2 - r^2)} = 4\pi a^3/3\sqrt{3}.$$

Example 4.64. *Assuming that the petrol burnt (per hour) in driving a motor boat varies as the cube of its velocity, show that the most economical speed when going against a current of c miles per hour is $\frac{3}{2}c$ miles per hour.*

Solution. Let v m.p.h. be the velocity of the boat so that its velocity relative to water (when going against the current) is $(v - c)$ m.p.h.

$\therefore$ Time required to cover a distance of s miles $= \dfrac{s}{v - c}$ hours.

Since the petrol burnt per hour $= kv^3$, k being a constant.

$\therefore$ The total petrol burnt, y, is given by

$$y = k\frac{v^3 s}{v - c} = ks\frac{v^3}{v - c} \quad \therefore \quad \frac{dy}{dv} = ks \cdot \frac{(v - c)3v^2 - v^3 \cdot 1}{(v - c)^2}$$

$$= ks \cdot \frac{v^2(2v - 3c)}{(v - c)^2}$$

Thus $dy/dv = 0$ for $v = 3c/2$, the other value $(v = 0)$ is inadmissible.

Also dy/dv is $-ve$, when v is slightly $< 3c/2$ and it is $+ve$, when v is slightly $> 3c/2$.

Hence y is minimum for $v = 3c/2$.

PROBLEMS 4.13

1. (*i*) Test the curve $y = x^4$ for points of inflexion ? (*Burdwan, 2003*)

   (*ii*) Show that the points of inflexion of the curve $y^2 = (x - a)^2(x - b)$ lie on the straight line $3x + a = 4b$. (*Rajasthan, 2005*)

2. The function $f(x)$ defined by $f(x) = a/x + bx$, $f(2) = 1$, has an extremum at $x = 2$. Determine a and b. Is this point (2, 1), a point of maximum or minimum on the graph of $f(x)$?
3. Show that $\sin^p\theta \cos^q\theta$ attains a maximum when $\theta = \tan^{-1}(p/q)$. *(Rajasthan, 2006)*
4. If a beam of weight w per unit length is built-in horizontally at one end A and rests on a support O at the other end, the deflection y at a distance x from O is given by

$$EIy = \frac{w}{48}(2x^4 - 3lx^3 + l^3x),$$

where l is the distance between the ends. Find x for y to be maximum.
5. The horse-power developed by an aircraft travelling horizontally with velocity v feet per second is given by

$$H = \frac{aw^2}{v} + bv,$$

where a, b and w are constants. Find for what value of v the horse-power is maximum.
6. The velocity of waves of wave-length λ on deep water is proportional to $\sqrt{(\lambda/a + a/\lambda)}$, where a is a certain constant, prove that the velocity is minimum when $\lambda = a$.
7. In a submarine telegraph cable, the speed of signalling varies as $x^2 \log_e (1/x)$, where x is the ratio of the radius of the core to that of the covering. Show that the greatest speed is attained when this ratio is $1/\sqrt{e}$.
8. The efficiency e of a screw-jack is given by $e = \tan\theta/\tan(\theta + \alpha)$, where α is a constant. Find θ if this efficiency is to be maximum. Also find the maximum efficiency.
9. Show that of all rectangles of given area, the square has the least parameter.
10. Find the rectangle of greatest perimeter that can be inscribed in a circle of radius a.
11. A gutter of rectangular section (open at the top) is to be made by bending into shape of a rectangular strip of metal. Show that the capacity of the gutter will be greatest if its width is twice its depth.
12. Show that the triangle of maximum area that can be inscribed in a given circle is an equilateral triangle.
13. An open box is to be made from a rectangular piece of sheet metal 12 cms × 18 cms, by cutting out equal squares from each corner and folding up the sides. Find the dimensions of the box of largest volume that can be made in this manner.
14. An open tank is to be constructed with a square base and vertical sides to hold a given quantity of water. Find the ratio of its depth to the width so that the cost of lining the tank with lead is least.
15. A corridor of width b runs perpendicular to a passageway of width a. Find the longest beam which can be moved in a horizontal plane along the passageway into the corridor ?
16. One corner of a rectangular sheet of paper of width a is folded so as to reach the opposite edge of the sheet. Find the minimum length of the crease.
17. Show that the height of closed cylinder of given volume and least surface is equal to its diameter.
18. Prove that a conical vessel of a given storage capacity requires the least material when its height is $\sqrt{2}$ times the radius of the base. *(Warangal, 1996)*
19. Show that the semi-vertical angle of a cone of maximum volume and given slant height is $\tan^{-1}\sqrt{2}$.
20. The shape of a hole bored by a drill is cone surmounting a cylinder. If the cylinder be of height h and radius r and the semi-vertical angle of the cone be α where $\tan\alpha = h/r$, show that for a total fixed depth H of the hole, the volume removed is maximum if $h = \frac{H}{6}(1 + \sqrt{7})$. *(Raipur, 2005)*
21. A cylinder is inscribed in a cone of height h. If the volume of the cylinder is maximum, show that its height is $h/3$.
22. Show that the volume of the biggest right circular cone that can be inscribed in a sphere of given radius is 8/27 times that of the sphere.
23. A given quantity of metal is to be cast into a half-cylinder with a rectangular base and semi-circular ends. Show that in order that the total surface area may be a minimum, the ratio of the length of the cylinder to the diameter of its semi-circular ends is $\pi/(\pi + 2)$.
24. A person being in a boat a miles from the nearest point of the beach, wishes to reach as quickly as possible a point b miles from that point along the shore. The ratio of his rate of walking to his rate of rowing is $\sec\alpha$. Prove that he should land at a distance $b - a\cot\alpha$ from the place to be reached.
25. The cost per hour of propelling a steamer is proportional to the cube of her speed through water. Find the relative speed at which the steamer should be run against a current of 5 km per hour to make a given trip at the least cost.

4.16 ASYMPTOTES

(1) Def. *An asymptote of a curve is a straight line at a finite distance from the origin, to which a tangent to the curve tends as the point of contact recedes to infinity.*

In other words, an asymptote is a straight line which cuts a curve on two points, at an infinite distance from the origin and yet is not itself wholly at infinity.

(2) Asymptotes parallel to axes. Let the equation of the curve arranged according to powers of x be

$$a_0x^n + (a_1y + b_1)x^{n-1} + (a_2y^2 + b_2y + c_2)x^{n-2} + \dots 0 \qquad \dots(1)$$

If $a_0 = 0$ and y be so chosen that $a_1y + b_1 = 0$, then the coefficients of two highest powers of x in (1) vanish and therefore, two of its roots are infinite. Hence $a_1y + b_1 = 0$ is an asymptote of (1) which is parallel to x-axis.

Again if a_0, a_1, b_1 are all zero and if y be so chosen that $a_2y^2 + b_2y + c_2 = 0$, then three roots of (1) become infinite. Therefore, the two lines represented by $a_2y^2 + b_2y + c_2 = 0$ are the asymptotes of (1) which are parallel to x-axis, and so on.

Similarly, for asymptotes parallel to y-axis.

Thus we have the following **rules :**

I. *To find the asymptotes parallel to x-axis, equate to zero the coefficient of the highest power of x in the equation,* provided this is not merely a constant.

II. *To find the asymptotes parallel to y-axis, equate to zero the coefficient of the highest power of y in the equation,* provided this is not merely a constant.

Example 4.65. *Find the asymptotes of the curve*

$$x^2y^2 - x^2y - xy^2 + x + y + 1 = 0.$$

Solution. The highest power of x is x^2 and its coefficient is $y^2 - y$.

$\therefore$ The asymptotes parallel to the x-axis are given by

$$y(y-1) = 0 \text{ i.e., by } y = 0 \text{ and } y = 1.$$

The highest power of y is y^2 and its coefficient is $x^2 - x$.

$\therefore$ The asymptotes parallel to the y-axis are given by

$$x(x-1) = 0 \text{ i.e., by } x = 0 \text{ and } x = 1.$$

Hence the asymptotes are $x = 0$, $x = 1$, $y = 0$ and $y = 1$.

(3) Inclined asymptotes. Let the equation of the curve be of the form

$$x^n\phi_n(y/x) + x^{n-1}\phi_{n-1}(y/x) + x^{n-2}\phi_{n-2}(y/x) + \dots = 0 \qquad \dots(1)$$

where $\phi_r(y/x)$ is an expression of degree r is y/x.

To find where this curve is cut by the line $y = mx + c$, ...(2)

put $y/x = m + c/x$ in (1). The resulting equation is

$$x^n\phi_n(m + c/x) + x^{n-1}\phi_{n-1}(m + c/x) + x^{n-2}\phi_{n-2}(m + c/x) + \dots = 0$$

which gives the abscissae of the points of intersection.

Expanding each of the ϕ-functions by Taylor's series,

$$x^n\left\{\phi_n(m) + \frac{c}{x}\phi_n'(m) + \frac{c^2}{2!\,x^2}\phi_n''(m) + \dots\right\} + x^{n-1}\left\{\phi_{n-1}(m) + \frac{c}{x}\phi'_{n-1}(m) + \dots\right\}$$
$$+ x^{n-2}\{\phi_{n-2}(m) + \dots\} = 0$$

or

$$x^n\phi_n(m) + x^{n-1}\{c\phi'_n(m) + \phi_{n-1}(m)\}$$
$$+ x^{n-2}\left\{\frac{c^2}{2!}\phi_n''(m) + c\phi'_{n-1}(m) + \phi_{n-2}(m)\right\} + \dots = 0 \qquad \dots(3)$$

If the line (2) is an asymptote to the curve, it cuts the curve in two points at infinity *i.e.*, the equation (3) has two infinite roots for which the coefficients of two highest terms should be zero.

i.e., $\phi_n(m) = 0$ ...(4) and $c\,\phi'_n(m) + \phi_{n-1}(m) = 0$ ...(5)

If the roots of (4) be $m_1, m_2, \dots m_n$, then the corresponding values of c (*i.e.* $c_1, c_2, \dots\dots c_n$) are given by (5). Hence the asymptotes are

$$y = m_1x + c_1, y = m_2x + c_2, \dots\dots, y = m_nx + c_n.$$

Obs. If (4) gives two equal values of m, then the corresponding values of c cannot be found from (5). Then c is determined by equating to zero the coefficient of x^{n-2} *i.e.*, from

$$\frac{c^2}{2!}\phi''_n(m) + c\,\phi'_{n-1}(m) + \phi_{n-2}(m) = 0 \qquad \text{...(6)}$$

In this case, *there will be two parallel asymptotes.*

Working rule :

1. Put $x = 1$, $y = m$ in the highest degree terms, thus getting $\phi_n(m)$. Equate it to zero and solve for m. Let its roots be $m_1, m_2, \ldots\ldots$
2. Form $\phi_{n-1}(m)$ by putting $x = 1$ and $y = m$ in the $(n-1)$th degree terms.
3. Find the values of c (*i.e.* $c_1, c_2, \ldots\ldots$) by substituting $m = m_1, m_2, \ldots\ldots$ in turn in the formula
$$c = -\phi_{n-1}(m)/\phi'_n(m)$$
[Sometimes it take (0/0) form, then find c from (6).]
4. Substitute the values of m and c in $y = mx + c$ in turn.

Example 4.66. *Find the asymptotes of the curve*
(i) $y^3 - 2xy^2 - x^2y + 2x^3 + 3y^2 - 7xy + 2y^2 + 2y + 2x + 1 = 0.$
(ii) $x^3 + 3x^2y - 4y^3 - x + y + 3 = 0.$
(iii) $(x + y)^2(x + y + 2) = x + 9y - 2.$ *(Rohtak, 2005)*

Solution. (*i*) Putting $x = 1$ and $y = m$ in the third degree terms,

$$\phi_3(m) = m^3 - 2m^2 - m + 2, \quad \therefore \quad \phi_3(m) = 0 \text{ gives } m^3 - 2m^2 - m + 2 = 0$$

or $\qquad (m^2 - 1)(m - 2) = 0$ whence $m = 1, -1, 2.$

Also putting $x = 1$ and $y = m$ in the 2nd degree terms, $\phi_2(m) = 3m^2 - 7m + 2$

$$\therefore \qquad c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{3m^2 - 7m + 2}{3m^2 - 4m - 1}$$

$$= -1 \text{ when } m = 1, = -2 \text{ when } m = -1, = 0 \text{ when } m = 2.$$

Hence the asymptotes are $y = x - 1$, $y = -x - 2$ and $y = 2x$.

(*ii*) Putting $x = 1$ and $y = m$ in the third degree terms,

$$\phi_3(m) = 1 + 3m - 4m^3$$

$$\therefore \qquad \phi_3(m) = 0 \text{ gives } 4m^3 - 3m - 1 = 0, \quad \text{or} \quad (m - 1)(2m + 1)^2 = 0$$

whence $\qquad m = 1, -1/2, -1/2.$

Similarly, $\qquad \phi_2(m) = 0$

$$\therefore \qquad c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{0}{3 - 12m^2}$$

$$= 0 \text{ when } m = 1, = \frac{0}{0} \text{ form when } m = -\frac{1}{2}.$$

Thus $\left(\text{when } m = -\tfrac{1}{2}\right)$ c is to be obtained from

$$\frac{c^2}{2!}\phi''_3(m) + c\,\phi'_2(m) + \phi_1(m) = 0$$

or
$$\frac{c^2}{2}(-24m) + c\,.\,0 + (-1 + m) = 0$$

Putting $m = -1/2$, $6c^2 - 3/2 = 0$ whence $c = \pm 1/2$.

Hence the asymptotes are $y = x$, $y = -\frac{1}{2}x + \frac{1}{2}$, $y = -\frac{1}{2}x - \frac{1}{2}$.

(*iii*) Putting $x = 1$ and $y = m$ in the third degree terms, $\phi_3(m) = (1 + m)^3$.

$$\therefore \qquad \phi_3(m) = 0 \text{ gives } (m + 1)^3 = 0 \text{ whence } m = -1, -1, -1.$$

Similarly, $\qquad \phi_2(m) = 2(1 + m)^2$, $\phi_1(m) = -1 - 9m$, $\phi_0(m) = 2$.

For these three equal values of $m = -1$, values of c are obtained from

$$\frac{c^3}{3!}\phi_3'''(m) + \frac{c^2}{2!}\phi_2''(m) + c\,\phi_1'(m) + \phi_0(m) = 0$$

or $$\frac{c^3}{6}(6) + \frac{c^2}{2}(4) + c(-9) + 2 = 0 \quad \text{or} \quad c^3 + 2c^2 - 9c + 2 = 0.$$

Solving for c, we have $c = 2, -2 \pm \sqrt{5}$.

Hence the three asymptotes are

$$y = -x + 2,\ y = -x - 2 + \sqrt{5},\ y = -x - 2 - \sqrt{5}.$$

4. Asymptotes of polar curves. It can be shown that an *asymptote of the curve $1/r = f(\theta)$ is*

$$r \sin(\theta - \alpha) = 1/f'(\alpha),$$

where α is a root of the equation $f(\theta) = 0$
and $f'(\alpha)$ is the derivative of $f(\theta)$ w.r.t. θ at $\theta = \alpha$.

Example 4.67. *Find the asymptote of the spiral $r = a/\theta$.*

Equation of the curve can be written as $1/r = \theta/a = f(\theta)$, say.

$$f(\theta) = 0, \text{ if } \theta = 0\ (= \alpha). \text{ Also } f'(\theta) = 1/a \quad \therefore \quad f'(\alpha) = 1/a.$$

$\therefore$ The asymptote is $r \sin(\theta - 0) = 1/f'(0)$ or $r \sin\theta = a$.

PROBLEMS 4.14

Find the asymptotes of

1. $x^3 + y^3 = 3axy$ *(Agra, 2002)*
2. $(x^2 - a^2)(y^2 - b^2) = a^2 b^2$ *(Osmania, 2002)*
3. $(a/x)^2 + (b/y)^2 = 1$ *(Burdwan, 2003)*
4. $x^2y + xy^2 + xy + y^2 + 3x = 0$.
5. $4x^3 + 2x^2 - 3xy^2 - y^3 - 1 - xy - y^2 = 0$. *(U.P.T.U., 2001)*
6. $x^2(x - y)^2 - a^2(x^2 + y^2) = 0$ *(Kurukshetra, 2006)*
7. $(x + y)^2(x + 2y + 2) = (x + 9y - 2)$ *(Rajasthan, 2006)*
8. Show that the asymptotes of the curve $x^2y^2 = a^2(x^2 + y^2)$ form a square of side $2a$.
9. Find the asymptotes of the curve $x^2y - xy^2 + xy + y^2 + x - y = 0$ and show that they cut the curve again in three points which lie on the line $x + y = 0$. *(Kurukshetra, 2006)*

Find the asymptotes of the following curves :

10. $r = a \tan\theta$. *(Rohtak, 2006 S)*
11. $r = a(\sec\theta + \tan\theta)$
12. $r \sin\theta = 2\cos 2\theta$. *(Kurukshetra, 2009 S)*
13. $r \sin n\theta = a$.

4.17 (1) CURVE TRACING

In many practical applications, a knowledge about the shapes of given equations is desirable. On drawing a sketch of the given equation, we can easily study the behaviour of the curve as regards its symmetry asymptotes, the number of branches passing through a point etc.

A point through which two branches of a curve pass is called a **double point.** At such a point P, the curve has two tangents, one for each branch.

If the tangents are real and distinct, the double point is called a **node** [Fig. 4.22 (a)].

If the tangents are real and coincident, the double point is called a **cusp** [Fig. 4.22 (b)].

If the tangents are imaginary, the double point is called a *conjugate point* (or an *isolated point*). Such a point cannot be shown in the figure.

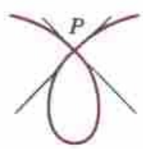

Fig. 4.22 (*a*)

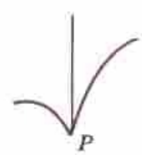

Fig. 4.22 (*b*)

(2) Procedure for tracing cartesian curves.

1. Symmetry. *See if the curve is symmetrical about any line.*

(*i*) *A curve is symmetrical about the x-axis, if only even powers of y occur in its equation.* (*e.g.*, $y^2 = 4ax$ is symmetrical about x-axis).

(*ii*) *A curve is symmetrical about the y-axis, if only even powers of x occur in its equation.*
(*e.g.*, $x^2 = 4ay$ is symmetrical about y-axis).

(*iii*) *A curve is symmetrical about the line* $y = x$, *if on interchanging* x and y *its equation remains unchanged,* (*e.g.*, $x^3 + y^3 = 3axy$ *is symmetrical about the line* $y = x$).

2. Origin. (*i*) *See if the curve passes through the origin.*

(A curve passes through the origin if there is no constant term in its equation).

(*ii*) *If it does, find the equation of the tangents thereat,* by equating to zero the lowest degree terms.

(*iii*) *If the origin is a double point, find whether the origin is a node, cusp or conjugate point.*

3. Asymptotes. (*i*) *See if the curve has any asymptote parallel to the axes* (p. 183).

(*ii*) *Then find the inclined asymptotes,* if need be. (p. 183).

4. Points. (*i*) *Find the points where the curve crosses the axes and the asymptotes.*

(*ii*) *Find the points where the tangent is parallel or perpendicular to the x-axis,*
(*i.e.* the points where $dy/dx = 0$ or ∞).

(*iii*) *Find the region (or regions) in which no portion of the curve exists.*

Example 4.68. *Trace the curve* $y^2(2a - x) = x^3$.
(*P.T.U., 2010, V.T.U., 2008 ; Rajasthan, 2006 ; U.P. T.U., 2005*)

Solution. (*i*) *Symmetry*: The curve is symmetrical about the x-axis.
[$\because$ only even powers of y occur in the equation.

(*ii*) *Origin :* The curve passes through the origin
[$\because$ there is no constant term in its equation.

The tangents at the origin are $y = 0, y = 0$ [Equating to zero the lowest degree terms.]

$\therefore$ Origin is a *cusp*

(*iii*) *Asymptotes :* The curve has an asymptote $x = 2a$.
[$\because$ co-eff. of y^3 is absent, co-eff. of y^2 is an asymptote.

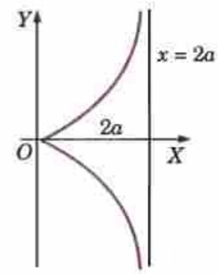

Fig. 4.23

(*iv*) *Points* : (*a*) curve meets the axes at (0, 0) only. (*b*) $y^2 = x^3/(2a - x)$

When x is –ve, y^2 is –ve (*i.e.* y is imaginary) so that no portion of the cuve lies to the left of the y-axis. Also when $x > 2a$, y^2 is again –ve, so that no portion of the curve lies to the right of the line $3x = 2a$.

Hence, the shape of the curve is as shown in Fig. 4.23. This curve is known as *Cissoid.*

Example 4.69. *Trace the curve* $y^2(a - x) = x^2(a + x)$. (*V.T.U., 2010 ; B.P.T.U., 2005*)

Solution. (*i*) *Symmetry* : The curve is symmetrical about the x-axis.

(*ii*) *Origin* : The curve passes through the origin and the tangents at the origin are $y^2 = x^2$,

i.e. $y = x$ and $y = -x$. $\therefore$ Origin is a *node.*

(*iii*) *Asymptotes* : The curve has an asymptote $x = a$

(*iv*) *Points* : (*a*) When $x = 0, y = 0$; when $y = 0, x = 0$ or $-a$.

$\therefore$ The curve crosses the axes at (0, 0) and $(-a, 0)$.

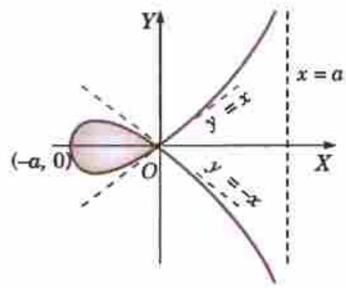

Fig. 4.24

We have $y = \pm x\sqrt{\left(\dfrac{a + x}{a - x}\right)}$

When $x > a$ or $< -a$, y is imaginary.

$\therefore$ No portion of the curve lies to the right of the line $x = a$ or to the left of the line $x = -a$.

Hence the shape of the curve is as shown in Fig. 4.24. This curve is known as *Strophoid.*

Example 4.70. *Trace the curve* $y = x^2/(1 - x^2)$.

Solution. (*i*) *Symmetry* : The curve is symmetrical about y-axis.

(*ii*) *Origin* : It passes through the origin and the tangent at the origin is $y = 0$ (*i.e.*, x-axis).

(*iii*) *Asymptotes* : The asymptotes are given by $1 - x^2 = 0$ or $x = \pm 1$ and $y = -1$.

(*iv*) *Points* : (*a*) The curve crosses the axes at the origin only. (*b*) When $x \to 1$ from left, $y \to \infty$

When $x \to 1$ from right $y \to -\infty$

When $x > 1$, y is – ve

Hence the curve is as shown in Fig. 4.25.

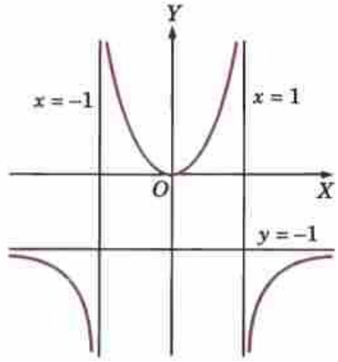

Fig. 4.25

Example 4.71. *Trace the curve* $a^2y^2 = x^2(a^2 - x^2)$. (*P.T.U., 2009 ; V.T.U., 2008 S*)

Solution. (*i*) *Symmetry*. The curve is symmetrical about x-axis, y-axis and origin.

(*ii*) *Origin*. The curve passes through the origin and the tangents at the origin are $a^2y^2 = a^2x^2$ *i.e.*, $y = \pm x$.

(*iii*) *Asymptotes*. The curve has no asymptote.

(*iv*) *Points*. (*a*) The curve cuts x-axis $(y = 0)$ at $x = 0, \pm a$. and cuts y-axis $(x = 0)$ at $y = 0$ *i.e.*, $(0, 0)$ only.

(*b*) $\dfrac{dy}{dx} = \dfrac{x(a^2 - 2x^2)}{a^2 y} \to \infty$ at $(a, 0)$

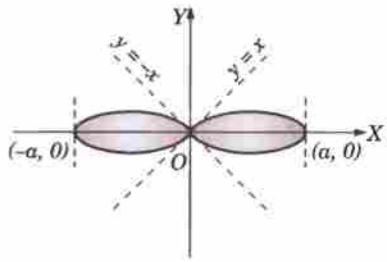

Fig. 4.26

i.e., tangent to the curve at $(a, 0)$ is parallel to y-axis. Similarly the tangent at $(-a, 0)$ is parallel to y-axis.

(*c*) We have $y = \dfrac{x}{a}\sqrt{a^2 - x^2}$ which is real for $x^2 < a^2$ *i.e.*, $-a < x < a$.

$\therefore$ The curve lies between $x = a$ and $x = -a$

Hence the shape of the curve is as shown is Fig. 4.26.

Example 4.72. *Trace the curve* $y = x^3 - 12x - 16$. (*P.T.U., 2008*)

Solution. (*i*) *Symmetry*. The curve has no symmetry.

(*ii*) *Origin*. It doesn't pass through the origin.

(*iii*) *Asymptotes* : The curve has no asymptote.

(*iv*) *Points*. (*a*) The curve cuts x-axis $(y = 0)$ at $(-2, 0)$, $(4, 0)$ and cuts y-axis $(x = 0)$ at $(0, -16)$.

(*b*) $\dfrac{dy}{dx} = 3x^2 - 12$

At (– 2, 0), $\dfrac{dy}{dx} = 0$ *i.e.*, tangent is parallel to x-axis at $(-2, 0)$.

At (4, 0), $\dfrac{dy}{dx} = 36$ *i.e.*, $\tan\theta = 36$ *i.e.*, tangent makes an acute angle $\tan^{-1} 36$ with x-axis at $(4, 0)$.

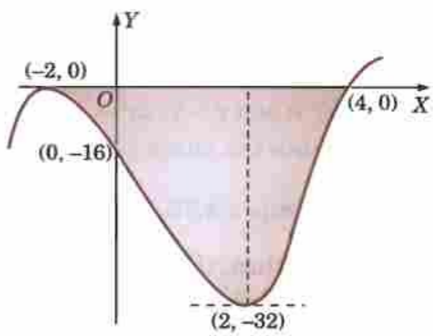

Fig. 4.27

Also $\dfrac{dy}{dx} = 0$ at $3x^2 - 12 = 0$ or $x = \pm 2$ *i.e.*, tangent is also parallel to x-axis at $(2, -32)$.

(*c*) $y \to \infty$ as $x \to \infty$ and $y \to -\infty$ as $x \to -\infty$; y is + *ve* for $x > 4$ and y is – *ve* for $x < 4$.

Hence the shape of the curve is as shown in Fig. 4.27.

Example 4.73. *Trace the curve* $9ay^2 = (x - 2a)(x - 5a)^2$ (*J.N.T.U., 2008*)

Solution. (*i*) *Symmetry*. The curve is symmetrical about the x-axis.

(*ii*) *Origin*. The curve does'nt pass through the origin.

(*iii*) *Asymptotes*. It has no asymptotes.

(*iv*) *Points.* (*a*) The curve cuts the x-axis ($y = 0$) at $x = 2a$, and $x = 5a$. *i.e.*, at A $(2a, 0)$ and $B(5a, 0)$.

It cuts the y-axis ($x = 0$) at $y^2 = -50\,a^2/9$, *i.e.*, y is imaginary.

So the curve doesn't cut the y-axis.

(*b*) $y = \dfrac{(x-5a)\sqrt{(x-2a)}}{3\sqrt{a}}$ *i.e.*, y is imaginary for $x < 2a$. So the curve exists only for $x \geq 2a$.

(*c*) $\dfrac{dy}{dx} = \pm \dfrac{x-3a}{2\sqrt{a}\,\sqrt{(x-2a)}}$

At A $(2a, 0)$, $\dfrac{dy}{dx} \to \infty$ *i.e.*, tangent is parallel to y-axis.

At B $(5a, 0)$, $\dfrac{dy}{dx} = \pm \dfrac{1}{\sqrt{3}}$ *i.e.*, there are two distinct tangents.

So there is a node at B $(5a, 0)$.

Hence the shape of the curve is as shown in Fig. 4.28.

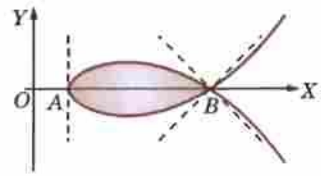

Fig. 4.28

Example 4.74. *Trace the curve* $x^3 + y^3 = 3axy$ (*Kurukshetra, 2005 ; U.P.T.U., 2003*)

or $$r = \frac{3a \sin\theta \cos\theta}{\sin^3\theta + \cos^3\theta}.$$

Solution. (*i*) *Symmetry* : The curve is symmetrical about the line $y = x$.

(*ii*) *Origin* : It passes through the origin and tangents at the origin are $xy = 0$, *i.e.*, $x = 0, y = 0$.

$\therefore$ Origin is a *node*.

(*iii*) *Asymptotes* : (*a*) It has no asymptote parallel to the axes.

(*b*) Putting $y = m$ and $x = 1$ in the third degree terms,

$$\phi_3(m) = 1 + m^3,\ \phi_3(m) = 0 \text{ gives } m = -1.$$

$$\therefore \quad c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\left(\frac{-3am}{3m^2}\right) = \frac{a}{m}.$$

$$= -a, \text{ when } m = -1.$$

Hence $y = -x - a$ $\left(i.e., \ \dfrac{x}{-a} + \dfrac{y}{-a} = 1\right)$ is an asymoptote.

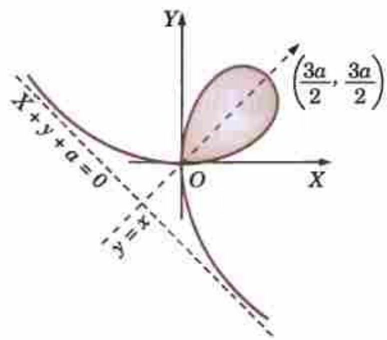

Fig. 4.29

(*iv*) *Points* : (*a*) It meets the axes at the origin only.

(*b*) When $y = x$, $2x^3 = 3ax^2$, *i.e.* $x = 0$ or $3a/2$. *i.e.*, the curve crosses the line $y = x$ at $(3a/2, 3a/2)$.

Hence the shape of the curve is as shown in Fig. 4.29. This curve is known as *Folium of Descartes*.

Example 4.75. *Trace the curve* $x^3 + y^3 = 3ax^2$.

Solution. (*i*) *Symmetry* : The curve has no symmetry.

(*ii*) *Origin* : The curve passes through the origin and the tangents at the origin are $x = 0$ and $x = 0$.

$\therefore$ The origin is a *cusp*.

(*iii*) *Asymptotes* : (*a*) The curve has no asymptote parallel to the axes.

(*b*) Putting $x = 1, y = m$ in the third degree terms, we get

$$\phi_3(m) = m^3 + 1\ ; \quad \therefore \quad \phi_3(m) = 0, \text{ gives } m = -1.$$

$$\therefore \quad c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{-3a}{3m^2} = a \text{ for } m = 1.$$

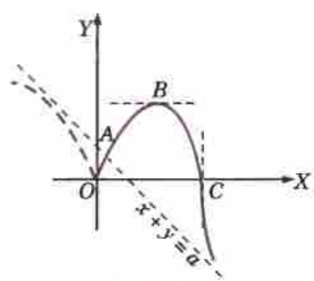

Fig. 4.30

Thus $x + y = a$ is the only asymptote.

The curve lies above the asymptote when x is positive and large and it lies below the asymptote when x is negative.

(*iv*) *Points.* (*a*) The curve crosses the axes at $O\,(0, 0)$ and $C(3a, 0)$. It crosses the asymptote at $A(a/3,\ 2a/3)$.

(*b*) Since $y^2\, dy/dx = x(2a - x)$. $\therefore$ $dy/dx = 0$ for $x = 2a$.

(*c*) Now $y = [x^2\,(3a - x)]^{1/3}$.

When $0 < x < 3a$, y is positive. As x increases from 0, y also increases till $x = 2a$ where the tangent is parallel to the x-axis. As x increases from $2a$ to $3a$, y constantly decreases to zero.

When $x > 3a$, y is negative.

When $x < 0$, y is positive and constantly increases as x varies from 0 to $-\infty$.

Combining all these facts we see that the shape of the curve is as shown in Fig. 4.30.

Example 4.76. *Trace the curve* $y^2\,(x - a) = x^2(x + a)$.

Solution. (*i*) *Symmetry* : The curve is symmetrical about the x-axis.

(*ii*) *Origin* : The curve passes through the origin and the tangents at the origin are $y^2 = -x^2$ *i.e.*, $y = \pm\, ix$, which are imaginary lines. $\therefore$ The origin is an *isolated point*.

(*iii*) *Asymptotes* : (*a*) $x = a$ is the only asymptote parallel to the y-axis.

(*b*) Putting $x = 1$ and $y = m$ in the third degree terms, we get

$$\phi_3\,(m) = m^2 - 1.$$

$$\therefore \quad \phi_3\,(m) = 0 \text{ gives } m = \pm\, 1$$

$$\therefore \quad c = \frac{\phi_2(m)}{\phi_3'(m)} = -\frac{-\,a(m^2+1)}{2m} = \pm\, a \text{ for } m = \pm\, 1.$$

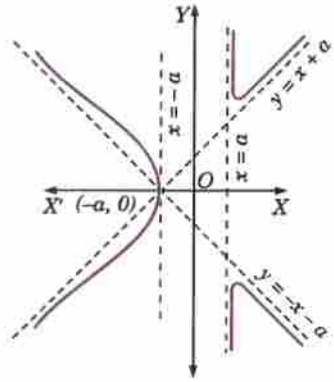

Fig. 4.31

Thus the other two asymptotes are $y = x + a$; $y = -\,x - a$.

(*iv*) *Points* : (*a*) The curve crosses the axes at $(-\,a, 0)$ and $(0, 0)$.

It crosses the asymptotes $y = x + a$ and $y = -\,x - a$ at $(-\,a, 0)$.

(*b*) $y = \pm\, x\sqrt{\left(\dfrac{x+a}{x-a}\right)}$

When $x < a$ and $x > -\,a$, y is imaginary.

$\therefore$ no portion of the curve lies between the lines $x = a$ and $x = -\,a$. Thus the vertical asymptote must be approached from the right.

(*c*) $\dfrac{dy}{dx} = \pm\,\dfrac{x^2 - ax + a^2}{(x-a)^{3/2}\,(x+a)^{1/2}}$

$\therefore$ $dy/dx = 0$, when $x = \frac{1}{2}\,(1 + \sqrt{5}\,)\,a = 1.6a$ approx.

[rejecting the value $\frac{1}{2}(1 - \sqrt{5})\,a$ which lies between $-\,a$ and a]

and $dy/dx \to \infty$, when $x = \pm\, a$.

Thus the tangent is parallel to the x-axis at $x = 1.6a$ and perpendicular to the x-axis at $x = \pm\, a$.

Hence the shape of the curve is as shown in Fig. 4.31.

4.17 (3) PROCEDURE FOR TRACING CURVES IN PARAMETRIC FORM : $x = f(t)$ and $y = \phi(t)$

1. **Symmetry.** *See if the curve has any symmetry.*

(*i*) *A curve is symmetrical about the x-axis, if on replacing t by – t, f(t) remains unchanged and φ(t) changes to – φ(t).*

(*ii*) *A curve is symmetrical about the y-axis if on replacing t by – t, f(t) changes to – f(t) and φ(t) remains unchanged.*

(*iii*) *A curve is symmetrical in the opposite quadrants, if on replacing t by – t, both f(t) and φ(t) remains unchanged.*

2. **Limits.** *Find the greatest and least values of x and y so as to determine the strips, parallel to the axes, within or outside which the curve lies.*
3. **Points.** (*a*) *Determine the points where the curve crosses the axes.*

The points of intersection of the curve with the x-axis given by the roots of $\phi(t) = 0$, while those with the y-axis are given by the roots of $f(t) = 0$.

(*b*) *Giving t a series of value, plot the corresponding values of x and y, noting whether x and y increase or decrease for the intermediates values of t.* For this purpose, we consider the sign of dx/dt and dy/dt for the different values of t.

(*c*) *Determine the points where the tangent is parallel or perpendicular to the x-axis, (i.e.,* where $dy/dx = 0$ or $\to \infty$).

(*d*) *When x and y are periodic functions of t with a common period, we need to study the curve only for one period,* because the other values of t will repeat the same curve over and over again.

Obs. Sometimes it is convenient to eliminate t between the given equations and use the resulting cartesian equation to trace the curve.

Example 4.77. *Trace the curve* $x = a \cos^3 t$, $y = a \sin^3 t$ *or* $x^{2/3} + y^{2/3} = a^{2/3}$.
(*P.T.U., 2009 S ; U.P.T.U., 2005 ; V.T.U., 2003*)

Solution. (*i*) *Symmetry.* The curve is symmetrical about the x-axis.

[$\because$ On changing t to $-t$, x remains unchanged but y changed to $-y$]

(*ii*) *Limits.* $\because$ $|x| \le a$ and $|y| \le a$.

$\therefore$ The curve lies entirely within the square bounded by the lines $x = \pm a$, $y = \pm a$.

(*iii*) *Points* : We have $\dfrac{dx}{dt} = -3a \cos^2 t \sin t$,

$$\frac{dy}{dt} = 3a \sin^2 t \cos t,\ \frac{dy}{dx} = -\tan t.$$

$\therefore$ $dy/dx = 0$ when $t = 0$ or π

and $dy/dx \to \infty$, when $t = \pi/2$.

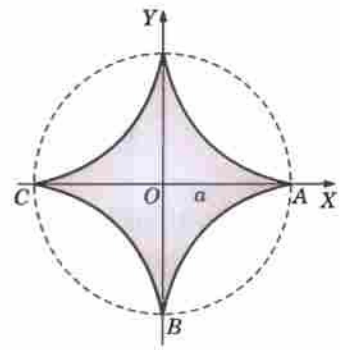

Fig. 4.32

The following table gives the corresponding values of t, x, y and dy/dx.

As t increases	*x*	*y*	*dy/dx varies*	*Portion traced*
from 0 to $\pi/2$	+ve and decreases from a to 0	+ve and increases from 0 to a	from 0 to ∞	*A* to *B*
from $\pi/2$ to π	+ ve and increases numerically from 0 to $-a$	+ve and decreases from a to 0	from ∞ to 0	*B* to *C*

As t increases from π to 2π, we get the reflection of the curve ABC in the x-axis. The values of $t > 2\pi$ give no new points.

Hence the shape of the curve is as shown in Fig. 4.32 and is known as **Astroid.**

Example 4.78. *Trace the curve* $x = a(\theta + \sin\theta)$, $y = a(1 + \cos\theta)$. (*J.N.T.U., 2009 S*)

Solution. (*i*) *Symmetry.* The curve is symmetrical about the y-axis.

[$\because$ On changing θ to $-\theta$, x changes to $-x$ and y remains unchanged]

Thus we may consider the curve only for positive value of x, *i.e.*, for $\theta > 0$.

(*ii*) *Limits.* The greatest value of y is $2a$ and the least value is zero.

Hence the curve lies entirely between the lines $y = 2a$ and $y = 0$.

(*iii*) *Points.* We have

$$\frac{dx}{d\theta} = a(1 + \cos\theta),\ \frac{dy}{d\theta} = a \sin\theta \text{ and } \frac{dy}{dx} = -\tan\theta/2.$$

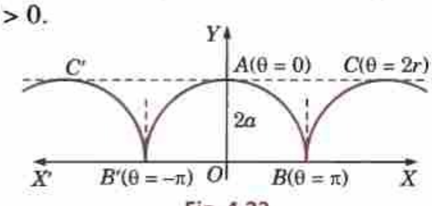

Fig. 4.33

$\therefore \quad dy/dx = 0$ when $\theta = 0$ or 2π and $dy/dx \to \infty$ when $\theta = \pi$.

The following table gives the corresponding values of θ, x, y and $\dfrac{dy}{dx}$:

As θ increases	x	y	*dy / dx varies*	*Portion traced*
from 0 to π	increases from 0 to $a\pi$	decreases from $2a$ to 0	from 0 to ∞	A to B
from π to 2π	increases from $a\pi$ to $2a\pi$	increases from 0 to $2a$	from ∞ to 0	B to C

As θ decreases from 0 to -2π, we get the reflection of the curve ABC in the y-axis.

The curve consists of congruent arches extending to infinity in both the directions of the x-axis in the intervals ... $(-3\pi, -\pi)$ $(-\pi, \pi)$ $(\pi, 3\pi)$, ...

Hence the shape of the curve is as shown in Fig. 4.33 and is known as **Cycloid.**

Obs. 1. Cycloid is the curve described by a point on the circumference of a circle which rolls without sliding on a fixed straight line. This fixed line (x-axis) is called the *base* and the farthest point (A) from it the *vertex* of the cycloid.

The complete cycloid consists of the arch $B'AB$ and its endless repetitions on both sides.

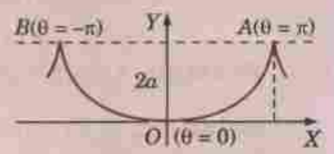

Fig. 4.34

2. Inverted cycloid: $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$.

The complete inverted cycloid consists of the arch BOA and an endless repetitions of the same on both sides. Here AB is the base and O the *vertex* of this cycloid. (Fig. 4.34).

4.17 (4) PROCEDURE FOR TRACING POLAR CURVES

1. **Symmetry.** *See if the curve is symmetrical about any line.*

(*i*) *A curve is symmetrical about the initial line OX, if only* $\cos\theta$ *(or* $\sec\theta$*) occur in its equation.* (*i.e.*, it remains unchanged when θ is changed to $-\theta$) *e.g.*, $r = a(1 + \cos\theta)$ is symmetrical about the initial line.

(*ii*) *A curve is symmetrical about the line through the pole* $\perp$ *to the initial line (i.e., OY), if only* $\sin\theta$ *(or* $\operatorname{cosec}\theta$*) occur in its equation.* (*i.e.*, it remains unchanged when θ is changed to $\pi - \theta$) *e.g.*, $r = a\sin 3\theta$ is symmetrical about OY.

(*iii*) *A curve is symmetrical about the pole, if only even powers of r occur in the equation* (*i.e.*, it remains unchanged when r is changed to $-r$) *e.g.*, $r^2 = a^2\cos 2\theta$ is symmetrical about the pole.

2. **Limits.** See if r and θ are confined between certain limits.

(*i*) *Determine the numerically greatest value of r, so as to notice whether the curve lies within a circle or not e.g.*, $r = a\sin 3\theta$ lies wholly within the circle $r = a$.

(*ii*) *Determine the region in which no portion of the curve lies by finding those values of* θ *for which r is imaginary e.g.*, $r^2 = a^2\cos 2\theta$ does not lie between the lines $\theta = \pi/4$ and $\theta = 3\pi/4$.

3. **Asymptotes.** *If the curve possesses an infinite branch, find the asymptotes* *(p. 183)*.

4. **Points.** (*i*) *Giving successive values to* θ*, find the corresponding values of r.*

(*ii*) *Determine the points where the tangent coincides with the radius vector or is perpendicular to it* (*i.e.*, the points where $\tan\phi = r\,d\theta/dr = 0$ or ∞).

Example 4.79. *Trace the curve* $r = a\sin 3\theta$. *(U.P.T.U., 2002)*

Solution. (*i*) *Symmetry.* The curve is symmetrical about the line through the pole $\perp$ to the initial line.

(*ii*) *Limits.* The curve wholly lies within the curve $r = a$. ($\because$ r is never $> a$)

(*iii*) *Asymptotes.* It has no asymptotes.

(*iv*) *Points.* (*a*) $\tan\phi = r\dfrac{d\theta}{dr} = \dfrac{a\sin 3\theta}{3a\cos 3\theta} = \dfrac{1}{3}\tan 3\theta$

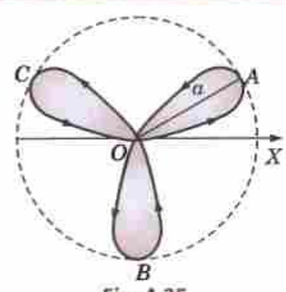

Fig. 4.35

$\therefore$ $\phi = 0$, when $\theta = 0, \pi/3, \ldots\ldots$

$\phi = \pi/2$, when $\theta = \pi/6, \pi/2, \ldots\ldots$

Hence the curve of the curve

(*b*) The following table gives the variations of r, θ and ϕ :

As θ varies from	*r varies from*	*φ varies from*	*Portion traced from*
0 to $\pi/6$	0 to a	0 to $\pi/2$	O to A
$\pi/6$ to $\pi/3$	a to 0	$\pi/2$ to 0	A to O
$\pi/3$ to $\pi/2$	0 to $-a$	0 to $\pi/2$	O to B

As θ increases from $\pi/2$ to π, portions of the curve from B to O, O to C and C to O are traced by symmetry about the line $\theta = \pi/2$.

Hence the curve consists of three loops as shown in Fig. 4.35 and is known as *three-leaved rose.*

Obs. *The curves of the form $r = a \sin n\theta$ or $r = a \cos n\theta$ are called* ***Roses*** *having*
(*i*) n leaves (loops) when n is odd,
(*ii*) $2n$ leaves (loops) when n is even.

Example 4.80. *Trace the curve $r = a \sin 2\theta$.(Four Leaved Rose)* (*V.T.U., 2009*)

Solution. (*i*) *Symmetry.* The curve is symmetrical about the line through the pole,⊥ to the initial line.

(*ii*) *Limits* : The curve lies wholly within the circle $r = a$ ($\because$ r is never $> a$)

(*iii*) *Points* : (*a*)

As θ increases from	*r varies from*	*Loop*	
0 to $\frac{\pi}{4}$	0 to a	no : 1,	
$\frac{\pi}{4}$ to $\frac{\pi}{2}$	a to 0		
$\frac{\pi}{2}$ to $\frac{3\pi}{4}$	0 to $-a$	no : 2,	
$\frac{3\pi}{4}$ to $\frac{\pi}{2}$	$-a$ to 0		etc. etc.

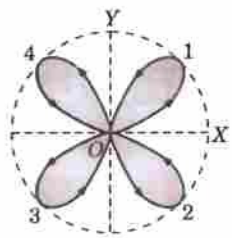

Fig. 4.36

(*b*) $$\tan \phi = r\frac{d\theta}{dr} = \frac{1}{2} \tan 2\theta \,;$$

$\therefore$ $$\phi = 0, \text{ when } \theta = 0, \frac{\pi}{2}, \pi, 3\frac{\pi}{2}, 2\pi \ldots$$

$$\phi = \frac{\pi}{2}, \text{ when } \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \ldots$$

Hence, the shape of the curve is as shown in Fig. 4.36.

Example 4.81. *Trace the curve $r^2 = a^2 \cos 2\theta$.* (*V.T.U., 2007 ; Kurukshetra, 2006 ; B.P.T.U., 2005*)

Solution. (*i*) *Symmetry.* The curve is symmetrical about the pole.

(*ii*) *Limits* : (*a*) The curve lies wholly within the circle $r = a$.

(*b*) No portion of the curve lies between the lines $\theta = \pi/4$ and $\theta = 3\pi/4$.

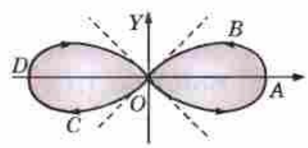

Fig. 4.37

(*iii*) *Points* : (*a*) $\tan \phi = r \dfrac{d\theta}{dr} = -\cot 2\theta = \tan\left(\dfrac{\pi}{2} + 2\theta\right)$

i.e., $\phi = \dfrac{\pi}{2} + 2\theta$ $\therefore$ $\phi = 0$, when $\theta = -\pi/4$; $\phi = \pi/2$ when $\theta = 0$.

Thus, the tangent at O is $\theta = -\pi/4$ and the tangent at A is ⊥ to the intial line.

(*b*) The variations of r and θ are given below :

As θ varies from	*r varies from*	*Portion traced*
0 to $\pi/4$	a to 0	ABO
$3\pi/4$ to π	0 to a	OCD

As θ increase from π to 2π, we get the reflection of the arc $ABOCD$ in the initial line. Hence the shape of the curve is as shown in Fig. 4.37. This curve is known as *Lemniscate of Bernoulli.*

Example 4.82. *Trace the curve* $r = a + b \cos \theta$ *(Limacon)*

Solution. (*i*) *Symmetry.* It is symmetrical about the initial line.

(*ii*) *Limits* : The curve wholly lies within the circle $r = a + b$ ($\because$ r is never $> a + b$)

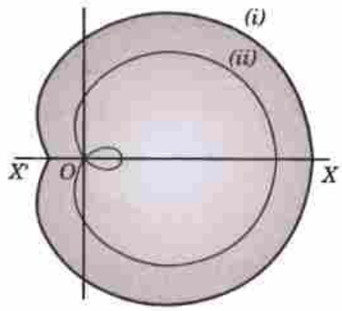

Fig. 4.38

(*iii*) *Points* : (α) when $a > b$.

As θ increases from 0 to $\pi/2$; r decreases from $a + b$ to a

As θ increases from $\pi/2$ to π ; r decreases from a to $a - b$

The shape of the curve is as shown in Fig. 4.38 (*i*).

(β) when $a < b$.

As θ increases from 0 to $\pi/2$; r decreases from $a + b$ to a

As θ increases from $\pi/2$ to α ; r decreases from a to 0

As θ increases from α to π ; r decreases from 0 to $a - b$

when $\alpha = \cos^{-1}\left(-\dfrac{a}{b}\right)$

In this case, the curve consists of two parts, one of which forms a loop within the other and the shape is as shown in Fig. 4.38 (*ii*).

Example 4.83. *Trace the curve* $r\theta = a$. *(Spiral)*

Solution. (*i*) *Symmetry.* There is no symmetry.

(*ii*) *Limits* : There are no limits to the values of r.

The curve does not pass through the pole for r does not become zero for any real value of θ.

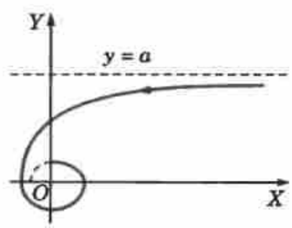

Fig. 4.39

(*iii*) *Asymptotes* : $\dfrac{1}{r} = \dfrac{\theta}{a} = f(\theta)$

$$f(\theta) = 0 \text{ for } \theta = 0 \,;\, f'(\theta) = 1/a,\, f'(0) = 1/a.$$

$\therefore$ Asymptote is $r \sin(\theta - 0) = 1/f'(0)$

i.e., $y = r \sin \theta = a$ is an asymptote.

(*iv*) *Points* : As θ increases from 0 to ∞, r to positive and decreases from ∞ to 0.

Hence the space of the curve is as shown in Fig. 4.39.

Example 4.84. *Trace the curve* $x^5 + y^5 = 5ax^2y^2$.

Solution. (*i*) *Symmetry.* The curve is symmetrical about the line $y = x$.

$\because$ On interchanging x and y, it remains unchanged.]

(*ii*) *Origin* : It passes through the origin and the tangents at the origin are given by

$$x^2 y^2 = 0,\ i.e.,\ x = 0, x = 0 \,;\, y = 0, y = 0.$$

Hence the curve has both *node* and the *cusp* at the origin.

(*iii*) *Asymptotes* : (*a*) It has no asymptotes parallel to the axes.

(*b*) Putting $x = 1, y = m$ in the fifth degree terms, we get

$$\phi_5(m) = 1 + m^5. \quad \therefore \quad \phi_5(m) = 0 \text{ gives } m = -1.$$

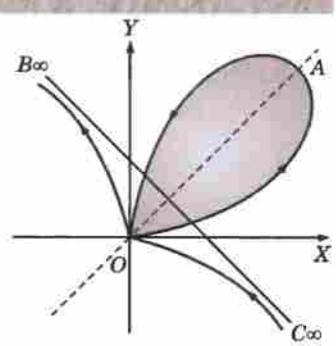

Fig. 4.40

$$\therefore \qquad c = -\frac{\phi_4(m)}{\phi_5'(m)} = -\frac{-5am^2}{5m^4} = a \text{ for } m = -1.$$

Hence $y = -x + a$ or $x + y = a$ is an asymptote.

(*iv*) *Points* : Since it is not convenient to express y as a function of x or *vice versa*, hence we change the equation into polar coordinates by putting, $x = r\cos\theta$ and $y = r\sin\theta$. The equation of the curve becomes :

$$r = \frac{5a\sin^2\theta\cos^2\theta}{\cos^5\theta + \sin^5\theta} = \frac{5a}{4}\cdot\frac{\sin^5 2\theta}{\cos^5\theta + \sin^5\theta}$$

As θ increases from	*r*	*Portion traced from*
0 to $\pi/4$	is +ve and increases from 0 to $\frac{5\sqrt{2}}{2}a$	0 to A
$\pi/4$ to $\pi/2$	is +ve and decreases from $\frac{5\sqrt{2}}{2}a$ to 0	A to 0
$\pi/2$ to $3\pi/4$	is +ve and increases from 0 to ∞	0 to B_∞
$3\pi/4$ to π	is –ve and decreases from ∞ to 0	C_∞ to 0

As θ increases from π to 2π, the curve will retraced.

Hence the shape of the curve is as shown in Fig. 4.40.

PROBLEMS 4.15

Trace the following curves :

1. $y^2(a + x) = x^2(a - x)$. (*S.V.T.U., 2008 ; U.P.T.U., 2006 ; Rajasthan, 2005*)
2. $y^2(a^2 + x^2) = x^2(a^2 - x^2)$ (*V.T.U., 2010*)
3. $y = (x^2 + 1)/(x^2 - 1)$ (*Kuruksehtra, 2009 S ; V.T.U., 2004*)
4. $ay^2 = x^2(a - x)$
5. $x^2y^2 = a^2(y^2 - x^2)$
6. $x = a\cos^3\theta, y = b\sin^3\theta$
7. $x = a(\theta - \sin\theta), y = a(1 - \cos\theta)\ (0 < \theta < 2\pi)$
8. $x = (a\cos t + \log\tan t/2), y = a\sin t$.
9. $r = a\cos 2\theta$
10. $r = a\cos 3\theta$
11. $r = a(1 - \cos\theta)$
12. $r = 2 + 3\cos\theta$
13. $r^2\cos 2\theta = a^2$. (*S.V.T.U., 2009*)

[**Hint.** Changing to Cartesian form $x^2 - y^2 = a^2$. This is a rectangular hyperbola with asymptotes $x + y = 0$ and $x - y = 0$]

4.18 OBJECTIVE TYPES OF QUESTIONS

PROBLEMS 4.16

Select the correct answer or fill up the blanks in each of the following questions :

1. The radius of curvature of the catenary $y = c\cosh x/c$ at the point where it crosses the y-axis is ......
2. The envelope of the family of straight lines $y = mx + am^2$, (m being the parameter) is ......
3. The curvature of the circle $x^2 + y^2 = 25$ at the point (3, 4) is ......
4. The value of $\underset{x\to\pi/2}{\text{Lt}} \frac{\log\sin x}{(\pi/2 - x)^2}$ is

   (*a*) zero (*b*) 1/2 (*c*) – 1/2 (*d*) – 2. (*V.T.U., 2010*)
5. Taylor's expansion of the function $f(x) = \frac{1}{1 + x^2}$ is

(a) $\sum_{n=0}^{\infty} (-1)^n x^{2n}$ for $-1 < x < 1$ (b) $\sum_{n=0}^{\infty} x^{2n}$ for $-1 < x < 1$

(c) $\sum_{n=0}^{\infty} (-1)^n x^{2n}$ for any real x (d) $\sum_{n=0}^{\infty} (-1)^n x^n$ for $-1 < x \le 1$.

6. A triangle of maximum area inscribed in a circle of radius r
   (a) is a right angled triangle with hypotenuse measuring $2r$
   (b) is an equilateral triangle
   (c) is an isosceles triangle of height r
   (d) does *not* exist.
7. The extreme value of $(x)^{1/x}$ is
   (a) e (b) $(1/e)^e$ (c) $(e)^{1/e}$ (d) 1.
8. The percentage error in computing the area of an ellipse when an error of 1 per cent is made in measuring the major and minor axes is
   (a) 0.2% (b) 2% (c) 0.02%.
9. The length of subtangent of the rectangular hyperbola $x^2 - y^2 = a^2$ at the point $(a, \sqrt{2}a)$ is
   (a) $\sqrt{2}\,a$ (b) $2a$ (c) $\frac{1}{2a}$ (d) $\frac{a^{3/2}}{\sqrt{2}}$.
10. The length of subnormal to the curve $y = x^2$ at (2, 8) is
   (a) 2/3 (b) 32 (c) 96 (d) 64.
11. If the normal to the curve $y^2 = 5x - 1$ at the point $(1, -2)$ is of the form $ax - 5y + b = 0$, then a and b are
   (a) 4, 14 (b) 4, − 14 (c) − 4, 14 (d) − 4, − 14.
12. The radius of curvature of the curve $y = e^x$ at the point where it crosses the y-axis is
   (a) 2 (b) $\sqrt{2}$ (c) $2\sqrt{2}$ (d) $\frac{1}{2}\sqrt{2}$.
13. The equation of the asymptotes of $x^3 + y^3 = 3axy$, is
   (a) $x + y - a = 0$ (b) $x - y + a = 0$ (c) $x + y + a = 0$ (d) $x - y - a = 0$.
14. If ϕ be the angle between the tangent and radius vector at any point on the curve $r = f(\theta)$, then $\sin\phi$ equals to
   (a) $\frac{dr}{ds}$ (b) $r\frac{d\theta}{ds}$ (c) $r\frac{d\theta}{dr}$.
15. Envelope of the family of lines $x = my + 1/m$ is ...
16. The chord of curvature parallel to y-axis for the curve $y = a \log \sec x/a$ is ....
17. $\sinh x = \ldots x + \ldots x^3 + \ldots x^5 + \ldots$
18. The nth derivative of $(\cos x \cos 2x \cos 3x)$ = ...
19. If $x^3 + y^3 - 3axy = 0$, then d^2y/dx^2 at $(3a/2, 3a/2)$ = ...
20. When the tangent at a point on a curve is parallel to x-axis, then the curvature at that point is same as the second derivative at that point. *(True or False)*
21. If $x = at^2, y = 2at$, t being the parameter, then $xy\, d^2y/dx^2$ = ...
22. The radius of curvature for the parabola $x = a, y = 2at$ at any point t = ...
23. If (a, b) are the coordinates of the centre of curvature whose curvature is k, then the equation of the circle of curvature is ....
24. Evolute is defined as the .... of the normals for a given curve.
25. Envelope of the family of lines $\frac{x}{t} + yt = 2c$ (where t is the parameter) is ....
26. The angle between the radius vector and tangent for the curve $r = ae^{\theta \cot \alpha}$ is ....
27. The subnormal of the parabola $y^2 = 4ax$ is ....
28. The fourth derivative of $(e^{-x} x^3)$ is ....

29. If $y^2 = P(x)$, a polynomial of degree 3, then $\frac{2d}{dx}\left(y^3 \frac{d^2y}{dx^2}\right)$ equals ....

(a) $P'''(x) + P'(x)$ (b) $P''(x) + P'''(x)$ (c) $P(x)\,P'''(x)$.

30. The envelope of the family of straight line $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$.

31. Curvature of a straight line is

(A) ∞ (B) zero (C) Both (A) and (B) (D) None of these.

32. The value of 'c' of the Cauchy's Mean value theorem for $f(x) = e^x$ and $g(x) = e^{-x}$ in [2, 3] is ....

33. If the equation of a curve remains unchanged when x and y are interchanged, then the curve is symmetrical about ....

34. For the curve $y^2(1 + x) = x^2(1 - x)$, the origin is a .... (node/cusp/conjugate point).

35. The number of loops of $r = a \sin 2\theta$ are .... and these of $r = a \cos 3\theta$ are ....

36. Tangents at the origin for the curve $y^2(x^2 + y^2) + a^2(x^2 - y^2) = 0$ are ....

37. The asymptote to the curve $y^2(4 - x) = x^3$ is ....

38. The points of inflexion of the curve $y^2 = (x - a)^2(x - b)$ lie on the line $3x + a =$ ....

39. The curve $r = a/(1 + \cos\theta)$ intersects orthogonally with the curve

(A) $r = b/(1 - \cos\theta)$ (B) $r = b/(1 + \sin\theta)$ (C) $r = b/(1 + \sin^2\theta)$ (D) $r = b/(1 + \cos^2\theta)$. (V.T.U., 2010)

40. The region where the curve $r = a \sin\theta$ does not lie is ....

41. If $f(x)$ is continuous in the closed interval $[a, b]$, differentiable in (a, b) and $f(a) = f(b)$, then there exists at least one value c of x in (a, b) such that $f'(c)$ is equal to

(A) 1 (B) -1 (C) 2 (D) 0. (V.T.U., 2009)

42. If two curves intersect orthogonally in cartesian form, then the angle between the same two curves in polar form is

(A) $\pi/4$ (B) Zero (C) 1 radian (D) None of these.

43. If the angle between the radius vector and the tangent is constant, then the curve is,

(A) $r = a\cos\theta$ (B) $r^2 = a^2\cos^2\theta$ (C) $r = ae^{b\theta}$. (V.T.U., 2009)

CHAPTER

5

Partial Differentiation and Its Applications

1. Functions of two or more variables. 2. Partial derivatives. 3. Which variable is to be treated as constant. 4. Homogeneous functions—Euler's theorem. 5. Total derivative—Diff. of implicit functions. 6. Change of variables. 7. Jacobians. 8. Geometrical interpretation—Tangent plane and normal to a surface. 9. Taylor's theorem for functions of two variables. 10. Errors and approximations; Total differential. 11. Maxima and minima of functions of two variables. 12. Lagrange's method of undetermined multipliers. 13. Differentiation under the integral sign—Leibnitz Rule. 14. Objective Type of Questions.

5.1 (1) FUNCTIONS OF TWO OR MORE VARIABLES

We often come across quantities which depend on two or more variables. For example, the area of a rectangle of length x and breadth y is given by $A = xy$. For a given pair of values of x and y, A has a definite value. Similarly, the volume of a parallelopiped $(= xyh)$ depends on the three variables x(= length), y(= breadth) and h(=height).

Def. *A symbol z which has a definite value for every pair of values of x and y is called a function of two independent variables x and y and we write $z = f(x, y)$ or $\phi(x, y)$.*

We may interpret (x, y) as the coordinates of a point in the XY-plane and z as the height of the surface $z = f(x, y)$. We have come across several examples of such surfaces in Chapter 4.

The set R of points (x, y) such that any two points P_1 and P_2 of R can be so joined that any arc P_1P_2 wholly lies in R, is called as *region* in the XY-plane. A region is said to be a *closed region* if it includes all the points of its boundary, otherwise it is called an *open region.*

A set of points lying within a circle having centre at (a, b) and radius $\delta > 0$, is said to be *neighbourhood* of (a, b) in the circular region $R : (x - a)^2 + (y - b)^2 < \delta^2$.

When z is a function of three or more variables $x, y, t, \ldots$, we represent the relation by writing $z = f(x, y, t, \ldots)$. For such functions, no geometrical representation is possible. However, the concepts of a region and neighbourhood can easily be extended to functions of three or more variables.

(2) Limits. *The function $f(x, y)$ is said to tend to the limit l as $x \to a$ and $y \to b$ if and only if the limit l is independent of the path followed by the point (x, y) as $x \to a$ and $y \to b$ and we write*

$$\underset{\substack{x \to a \\ y \to b}}{Lt} f(x,y) = l$$

In terms of a circular neighbourhood, we have the following *definition of the limit :*

The function $f(x, y)$ defined in a region R, is said to tend to the limit l as $x \to a$ and $y \to b$ if and only if corresponding to a positive number ε, there exists another positive number δ such that $| f(x, y) - l | < \varepsilon$ for $0 < (x - a)^2 + (y - b)^2 < \delta^2$ for every point (x, y) in R.

(3) Continuity. *A function $f(x, y)$ is said to be continuous at the point (a, b) if*

$$\underset{\substack{x \to a \\ y \to b}}{Lt} f(x, y) \text{ exists and } = f(a, b)$$

If a function is continuous at all points of a region, then it is said to be *continuous in that region.* A function which is not continuous at a point is said to be *discontinuous* at that point.

Obs. *Usually, the limit is the same irrespective of the path along which the point (x, y) approaches (a, b) and*

$$\underset{x\to a}{Lt}\left\{\underset{y\to b}{Lt} f(x,y)\right\} = \underset{y\to b}{Lt}\left\{\underset{x\to a}{Lt} f(x,y)\right\}$$

But it is not always so, as the following examples show :

$$\text{Lt}\left(\frac{x-y}{x+y}\right) \text{ as } (x,y)\to(0,0) \text{ along the line } y = mx$$

$$= \underset{x\to 0}{\text{Lt}} \frac{x-mx}{x+mx} = \frac{1-m}{1+m} \text{ which is different for lines with different slopes.}$$

Also $\underset{x\to 0}{\text{Lt}}\left[\underset{y\to 0}{\text{Lt}}\left(\frac{x-y}{x+y}\right)\right] = \underset{x\to 0}{\text{Lt}}\left(\frac{x}{x}\right) = 1$, whereas $\underset{y\to 0}{\text{Lt}}\left[\underset{x\to 0}{\text{Lt}}\left(\frac{x-y}{x+y}\right)\right] = \underset{y\to 0}{\text{Lt}}\left(\frac{-y}{y}\right) = 1.$

$\therefore$ As (x, y) is made to approach $(0, 0)$ along different paths, $f(x, y)$ approaches different limits. Hence the two repeated limits are not equal and $f(x, y)$ is discontinuous at the origin.

Also the function is not defined at $(0, 0)$ since $f(x, y) = 0/0$ for $x = 0, y = 0$.

(4) *As in the case of functions of one variable, the following results hold :*

I. *If* $\underset{\substack{x\to a\\ y\to b}}{Lt} f(x,y) = l$ and $\underset{\substack{x\to a\\ y\to b}}{Lt} g(x,y) = m$,

then (*i*) *If* $\underset{\substack{x\to a\\ y\to b}}{Lt} [f(x,y) \pm g(x,y)] = l \pm m$ (*ii*) $\underset{\substack{x\to a\\ y\to b}}{Lt} [f(x,y) \,.\, g(x,y)] = l \,.\, m$

(*iii*) $\underset{\substack{x\to a\\ y\to b}}{Lt} [f(x,y)/g(x,y)] = l/m$ $(m \neq 0)$

II. *If* $f(x, y), g(x, y)$ *are continuous at* (a, b) *then so also are the functions*

$f(x, y) \pm g(x, y), f(x, y) \,.\, g(x, y)$ and $f(x, y)/g(x, y)$

provided $g(x, y) \neq 0$ in the last case.

PROBLEMS 5.1

Evaluate the following limits :

1. $\underset{\substack{x\to 1\\ y\to 2}}{\text{Lt}} \dfrac{2x^2y}{x^2+y^2+1}$ 2. $\underset{\substack{x\to 0\\ y\to 0}}{\text{Lt}} \dfrac{xy}{x^2+y^2}$ 3. $\underset{\substack{x\to \infty\\ y\to 2}}{\text{Lt}} \dfrac{xy+1}{x^2+2y^2}$ 4. $\underset{\substack{x\to 1\\ y\to 1}}{\text{Lt}} \dfrac{x(y-1)}{y(x-1)}$

5. If $f(x,y) = \dfrac{x-y}{2x+y}$, show that $\underset{x\to 0}{\text{Lt}}\left[\underset{y\to 0}{\text{Lt}} f(x,y)\right] \neq \underset{y\to 0}{\text{Lt}}\left[\underset{x\to 0}{\text{Lt}} f(x,y)\right]$

Also show that the function is discountinuous at the origin.

6. Show that the function $f(x, y) = x^2 + 2y$, $(x, y) \neq (1, 2)$

$3(x, y) = (1, 2)$ $= 0$

is discontinuous at $(1, 2)$.

7. Investigate the continuity of the function

$f(x, y) = xy/(x^2 + y^2)$, $(x, y) \neq (0, 0)$

$= 0$ $(x, y) = (0, 0)$

at the origin.

Note. In whatever follows, all the functions considered are continuous and their partial derivatives (as defined below) exist.

5.2 PARTIAL DERIVATIVES

Let $z = f(x, y)$ be a function of two variables x and y.

If we keep y as constant and vary x alone, then z is a function of x only. The derivative of z with respect to x, treating y as constant, is called the *partial derivative of z with respect to x* and is denoted by one of the symbols

$$\frac{\partial z}{\partial x}, \frac{\partial f}{\partial x}, f_x(x, y), D_x f. \qquad \text{Thus} \quad \frac{\partial z}{\partial x} = \underset{\delta x \to 0}{\text{Lt}} \frac{f(x + \delta x, y) - f(x, y)}{\delta x}$$

Similarly, the derivative of z with respect to y, keeping x as constant, is called the *partial derivative of z with respect to y* and is denoted by one of the symbols.

$$\frac{\partial z}{\partial y}, \frac{\partial f}{\partial y}, f_y(x, y), D_y f. \qquad \text{Thus} \quad \frac{\partial z}{\partial y} = \underset{\partial y \to 0}{\text{Lt}} \frac{f(x, y + \delta y) - f(x, y)}{\delta y}$$

Similarly, if z is a function of three or more variables $x_1, x_2, x_3, \ldots$ *the partial derivative of z with respect to x_1, is obtained by differentiating z with respect to x_1, keeping all other variables constant* and is written as $\partial z/\partial x_1$.

In general f_x and f_y are also functions of x and y and so these can be differentiated further partially with respect to x and y.

Thus $$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} \text{ or } \frac{\partial^2 z}{\partial x^2} \text{ or } f_{xx}, \ \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} \text{ or } \frac{\partial^2 f}{\partial x \partial y} \text{ or } f_{yx}{}^{*}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} \text{ or } \frac{\partial^2 f}{\partial y \partial x} \text{ or } f_{xy}, \text{ and } \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} \text{ or } \frac{\partial^2 f}{\partial y^2} \text{ or } f_{yy}.$$

It can easily be verified that, in all ordinary cases,

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}.$$

Sometimes we use the following notation

$$\frac{\partial z}{\partial x} = p, \ \frac{\partial z}{\partial y} = q, \ \frac{\partial^2 z}{\partial x^2} = r, \ \frac{\partial^2 z}{\partial x \partial y} = s, \ \frac{\partial^2 z}{\partial y^2} = t.$$

Example 5.1. *Find the first and second partial derivatives of $z = x^3 + y^3 - 3axy$.*

Solution. We have $z = x^3 + y^3 - 3axy$.

$\therefore$ $$\frac{\partial z}{\partial x} = 3x^2 + 0 - 3ay(1) = 3x^2 - 3ay, \text{ and } \frac{\partial z}{\partial y} = 0 + 3y^2 - 3ax(1) = 3y^2 - 3ax$$

Also $$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(3x^2 - 3ay) = 6x, \ \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}(3x^2 - 3ay) = -3a$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}(3y^2 - 3ax) = 6y, \text{ and } \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}(3y^2 - 3ax) = -3a.$$

We observe that $\dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{\partial^2 z}{\partial x \partial y}$.

Example 5.2. *If $u = x^2 \tan^{-1} \dfrac{y}{x} - y^2 \tan^{-1} \dfrac{x}{y}$,*

show that $$\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2 - y^2}{x^2 + y^2}.$$ *(Mumbai, 2008 S)*

and $$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$ *(Madras, 2000)*

Solution. We have $$\frac{\partial u}{\partial y} = x^2 \cdot \frac{1}{1 + (y/x)^2} \cdot \frac{1}{x} - \left\{2y \cdot \tan^{-1} \frac{x}{y} + y^2 \cdot \frac{1}{1 + (x/y)^2} \cdot \left(-\frac{x}{y^2}\right)\right\}$$

$$= \frac{x^3}{x^2 + y^2} - 2y \tan^{-1} \frac{x}{y} + \frac{xy^2}{x^2 + y^2} = x - 2y \tan^{-1} \frac{x}{y}.$$

*It is important to note that in the subscript notation the subscripts are written in the same order in which we differentiate whereas in the '∂' notation the order is opposite.

$\therefore \qquad \dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left\{x - 2y \tan^{-1}\dfrac{x}{y}\right\} = 1 - 2y \cdot \dfrac{1}{1+(x/y)^2} \cdot \dfrac{1}{y} = 1 - \dfrac{2y^2}{x^2+y^2} = \dfrac{x^2-y^2}{x^2+y^2}.$

Similarly, $\qquad \dfrac{\partial u}{\partial x} = 2x \tan^{-1} y/x - y$

and $\qquad \dfrac{\partial^2 u}{\partial y \partial x} = \dfrac{\partial}{\partial y}\left\{2x \tan^{-1}\dfrac{y}{x} - y\right\} = \dfrac{x^2-y^2}{x^2+y^2}$. Hence the result.

Example 5.3. *If $z = f(x+ct) + \phi(x-ct)$, prove that*

$$\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$$

(*J.N.T.U., 2006 ; V.T.U., 2003 S*)

Solution. We have $\dfrac{\partial z}{\partial x} = f'(x+ct) \cdot \dfrac{\partial}{\partial x}(x+ct) + \phi'(x-ct)\dfrac{\partial}{\partial x}(x-ct) = f'(x+ct) + \phi'(x-ct)$

and $\qquad \dfrac{\partial^2 z}{\partial x^2} = f''(x+ct) + \phi''(x-ct) \qquad ...(i)$

Again $\qquad \dfrac{\partial z}{\partial t} = f'(x+ct)\dfrac{\partial}{\partial t}(x+ct) + \phi'(x-ct)\dfrac{\partial}{\partial t}(x-ct) = cf'(x-ct) - c\phi'(x-ct)$

and $\qquad \dfrac{\partial^2 z}{\partial t^2} = c^2 f''(x+ct) + c^2\phi''(x-ct) = c^2[f''(x+ct) + \phi''(x-ct)] \qquad ...(ii)$

From (*i*) and (*ii*), it follows that $\dfrac{\partial^2 z}{\partial t^2} = c^2\dfrac{\partial^2 z}{\partial x^2}$.

Obs. This is an important partial differential equation, known as *wave equation* (§ 18.4).

Example 5.4. *If $\theta = t^n e^{-r^2/4t}$, what value of n will make $\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial\theta}{\partial r}\right) = \dfrac{\partial\theta}{\partial t}$?*

(*Nagpur, 2009 ; Kurukshetra, 2006 ; U.P.T.U., 2006*)

Solution. We have $\dfrac{\partial\theta}{\partial r} = t^n \cdot e^{-r^2/4t} \cdot \left(\dfrac{-2r}{4t}\right) = -\dfrac{r}{2}t^{n-1}e^{-r^2/4t}$

$\therefore \qquad r^2\dfrac{\partial\theta}{\partial r} = -\dfrac{r^3}{2}t^{n-1} \cdot e^{-r^2/4t}$

and $\qquad \dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial\theta}{\partial r}\right) = -\dfrac{3r^2}{2}t^{n-1}e^{-r^2/4t} - \dfrac{r^3}{2}t^{n-1} \cdot e^{-r^2/4t}\left(-\dfrac{2r}{4t}\right)$

$\therefore \qquad \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial\theta}{\partial r}\right) = \left(-\dfrac{3}{2}t^{n-1} + \dfrac{r^2}{4}t^{n-2}\right)e^{-r^2/4t}$

Also $\qquad \dfrac{\partial\theta}{\partial t} = nt^{n-1} \cdot e^{-r^2/4t} + t^n \cdot e^{-r^2/4t} \cdot \dfrac{r^2}{4t^2} = \left(nt^{n-1} + \dfrac{1}{4}r^2t^{n-2}\right)e^{-r^2/4t}$

Since $\quad \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial\theta}{\partial r}\right) = \dfrac{\partial\theta}{\partial t}$,

$\therefore \quad \left(-\dfrac{3}{2}t^{n-1} + \dfrac{1}{4}r^2t^{n-2}\right)e^{-r^2/4t} = \left(nt^{n-1} + \dfrac{1}{4}r^2t^{n-2}\right)e^{-r^2/4t}$ or $\left(n+\dfrac{3}{2}\right)t^{n-1}e^{-r^2/4t} = 0.$

Hence $\qquad n = -3/2.$

Example 5.5. *If $v = (x^2+y^2+z^2)^{-1/2}$, prove that*

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0. \text{ (Laplace equation)*}$$

(*V.T.U., 2006 ; Osmania, 2003 S*)

*See footnote p. 18.

Solution. We have $\dfrac{\partial v}{\partial x} = -\dfrac{1}{2}(x^2+y^2+z^2)^{-3/2} \cdot 2x = -x(x^2+y^2+z^2)^{-3/2}$

and
$$\frac{\partial^2 v}{\partial x^2} = -1[1 \cdot (x^2+y^2+z^2)^{-3/2} + x(-3/2)(x^2+y^2+z^2)^{-5/2} \cdot 2x]$$
$$= -(x^2+y^2+z^2)^{-5/2}[x^2+y^2+z^2-3x^2] = (x^2+y^2+z^2)^{-5/2}(2x^2-y^2-z^2)$$

Similarly, $\dfrac{\partial^2 v}{\partial y^2} = (x^2+y^2+z^2)^{-5/2}(-x^2+2y^2-z^2)$ and $\dfrac{\partial^2 v}{\partial z^2} = (x^2+y^2+z^2)^{-5/2}(-x^2-y^2+2z^2)$

Hence $\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} + \dfrac{\partial^2 v}{\partial z^2} = (x^2+y^2+z^2)^{-5/2} \cdot (0) = 0.$

Obs. *A function v satisfying the Laplace equation is said to be a* **harmonic function.**

Example 5.6. *If $u = \log(x^3+y^3+z^3-3xyz)$, show that* $\left(\dfrac{\partial}{\partial x}+\dfrac{\partial}{\partial y}+\dfrac{\partial}{\partial z}\right)^2 u = \dfrac{-9}{(x+y+z)^2}$.

(P.T.U., 2010 ; Anna, 2009 ; Bhopal, 2008 ; U.P.T.U., 2006)

Solution. We have $\dfrac{\partial u}{\partial x} = \dfrac{3x^2-3yz}{x^3+y^3+z^3-3xyz}$, $\dfrac{\partial u}{\partial y} = \dfrac{3y^2-3zx}{x^3+y^3+z^3-3xyz}$, $\dfrac{\partial u}{\partial z} = \dfrac{3z^2-3xy}{x^3+y^3+z^3-3xyz}$

$$\therefore \quad \frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = \frac{3(x^2+y^2+z^2-xy-yz-zx)}{x^3+y^3+z^3-3xyz}$$
$$= \frac{3(x^2+y^2+z^2-xy-yz-zx)}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)} = \frac{3}{x+y+z} \qquad \textit{(V.T.U., 2009)}$$

Now
$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$
$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right) = \left(\frac{3}{x+y+z}\right)$$
$$= -\frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} = -\frac{9}{(x+y+z)^2}.$$

Example 5.7. *If $\dfrac{x^2}{a^2+u}+\dfrac{y^2}{b^2+u}+\dfrac{z^2}{c^2+u} = 1$, prove that*

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 2\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right) \qquad \textit{(U.P.T.U., 2003)}$$

Solution. We have $x^2(a^2+u)^{-1} + y^2(b^2+u)^{-1} + z^2(c^2+u)^{-1} = 1$ ...(*i*)

Differentiating (*i*) partially w.r.t. x, we get

$$2x(a^2+u)^{-1} - x^2(a^2+u)^{-2}\frac{\partial u}{\partial x} - y^2(b^2+u)^{-2}\frac{\partial u}{\partial y} - z^2(c^2+u)^{-2}\frac{\partial u}{\partial z} = 0$$

or
$$\frac{2x}{a^2+u} = \left\{\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{y^2}{(c^2+u)^2}\right\}\frac{\partial u}{\partial x}$$

or
$$\frac{\partial u}{\partial x} = \frac{2x}{(a^2+u)v} \text{ where } v = \frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{y^2}{(c^2+u)^2}$$

Similarly differentiating (*i*) partially w.r.t. y, we get

$$\frac{2y}{b^2+u} = \left\{\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right\}\frac{\partial u}{\partial y} \text{ or } \frac{\partial u}{\partial y} = \frac{2y}{(b^2+u)v}$$

Similarly, differentiating (i) partially w.r.t. z, we get

$$\frac{2z}{(b^2+u)} = \left\{\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right\}\frac{\partial u}{\partial z} \text{ or } \frac{\partial u}{\partial z} = \frac{2z}{(c^2+u)v}$$

$$\therefore \quad \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = \frac{4}{v^2}\left\{\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{y^2}{(c^2+u)^2}\right\} = \frac{4}{v} \quad ...(ii)$$

Also $$2\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right) = 2\left\{\frac{2x^2}{(a^2+u)v} + \frac{2y^2}{(b^2+u)v} + \frac{2z^2}{(c^2+u)v}\right\}$$

$$= \frac{4}{v}\left\{\frac{x^2}{(a^2+u)} + \frac{y^2}{(b^2+u)} + \frac{z^2}{(c^2+u)}\right\} = \frac{4}{v} \quad \text{[By } (i)\text{] } ...(iii)$$

Hence the equality of (ii) and (iii) proves the result.

Example 5.8. *If $u = x^y$, show that* $\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial^3 u}{\partial x \partial y \partial x}$. *(Anna, 2009)*

Solution. We have $\frac{\partial u}{\partial y} = x^y \log_e x$ and $\frac{\partial^2 u}{\partial x \partial y} = yx^{y-1} . \log x + x^y . \frac{1}{x} = x^{y-1}(y \log x + 1)$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2 \partial y} = \frac{\partial}{\partial x}[x^{y-1}(y \log x + 1)] \quad ...(i)$$

Again $\frac{\partial u}{\partial x} = yx^{y-1}$ and $\frac{\partial^2 u}{\partial y \partial x} = 1 . x^{y-1} + y\left(\frac{1}{x}x^y \log x\right) = x^{y-1}(1 + y \log x)$

$$\therefore \quad \frac{\partial^3 u}{\partial x \partial y \partial x} = \frac{\partial}{\partial x}[x^{y-1}(y \log x + 1)] \quad ...(ii)$$

From (i) and (ii) follows the required result.

PROBLEMS 5.2

1. Evaluate $\partial z/\partial x$ and $\partial z/\partial y$, if
   (i) $z = x^2 y - x \sin xy$; (ii) $z = \log(x^2 + y^2)$;
   (iii) $z = \tan^{-1}[(x^2+y^2)/(x+y)]$; (iv) $x + y + z = \log z$.
2. If $z(x+y) = x^2 + y^2$, show that $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$. (V.T.U., 2003)
3. If $z = e^{ax+by} f(ax - by)$, prove that $b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abz$. (V.T.U., 2010)
4. Given $u = e^{r\cos\theta}\cos(r\sin\theta)$, $v = e^{r\cos\theta}\sin(r\sin\theta)$; prove that $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$.
5. If $z = \tan(y + ax) - (y - ax)^{3/2}$, show that $\frac{\partial^2 z}{\partial x^2} = a^2\frac{\partial^2 z}{\partial y^2}$. (Mumbai, 2009)
6. Verify that $f_{xy} = f_{yx}$, when f is equal to (i) $\sin^{-1}(y/x)$; (ii) $\log x \tan^{-1}(x^2 + y^2)$.
7. If $f(x, y) = (1 - 2xy + y^2)^{-1/2}$, show that $\frac{\partial}{\partial x}\left[(1 - x^2)\frac{\partial f}{\partial x}\right] + \frac{\partial}{\partial y}\left[y^2\frac{\partial f}{\partial y}\right] = 0$. (Rohtak, 2006 S)
8. Prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ if (i) $u = \tan^{-1}\left[\frac{2xy}{x^2 - y^2}\right]$, (ii) $u = \log(x^2 + y^2) + \tan^{-1}(y/x)$. (Anna, 2009)
9. If $v = \frac{1}{\sqrt{t}}e^{-x^2/4a^2 t}$, prove that $\frac{\partial v}{\partial t} = a^2\frac{\partial^2 v}{\partial x^2}$.

10. The equation $\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2}$ refers to the conduction of heat along a bar without radiation, show that if $u = Ae^{-gx} \sin (nt - gx)$, where A, g, n are positive constants then $g = \sqrt{(n/2\mu)}$.
11. Find the value of n so that the equation $V = r^n (3 \cos^2 \theta - 1)$ satisfies the relation
$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta}\left(\sin \theta \frac{\partial V}{\partial \theta}\right) = 0.$$
12. If $z = \log (e^x + e^y)$, show that $rt - s^2 = 0$ where $r = \partial^2 z/\partial x^2$, $s = \partial^2 z/\partial x \partial y$, $t = \partial^2 z/\partial y^2$.
13. If $u = \frac{y}{z} + \frac{z}{x}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$.
14. Let $r^2 = x^2 + y^2 + z^2$ and $V = r^m$, prove that $V_{xx} + V_{yy} + V_{zz} = m(m+1)r^{m-2}$. (*Raipur, 2005*)
15. If $v = \log (x^2 + y^2 + z^2)$, prove that $(x^2 + y^2 + z^2)\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) = 2$.
16. If $v = x^y \cdot y^x$, prove that $x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = v(x + y + \log v)$. (*Anna, 2005*)
17. If $x^x y^y z^z = c$, show that at $x = y = z$, $\frac{\partial^2 z}{\partial x \partial y} = -(x \log ex)^{-1}$. (*Bhopal, 2008*)
18. If $u = e^{xyz}$, find the value of $\frac{\partial^3 u}{\partial x \partial y \partial z}$. (*Rajasthan, 2005 ; Osmania, 2003 S*)

5.3 WHICH VARIABLE IS TO BE TREATED AS CONSTANT

(1) Consider the equation $x = r \cos \theta$, $y = r \sin \theta$ ...(1)

To find $\partial r/\partial x$, we need a relation between r and x. Such a relation will contain one more variable θ or y, for we can eliminate only one variable out of four from the relations (1). Thus the two possible relations are

$$r = x \sec \theta \quad ...(2) \qquad \text{and} \qquad r^2 = x^2 + y^2 \quad ...(3)$$

Now we can find $\partial r/\partial x$ either from (2) by treating θ as constant or from (3) by regarding y as constant. And there is no reason to suppose that the two values of $\partial r/\partial x$ so found, are equal. *To avoid confusion as to which variable is regarded constant, we introduce the following :*

Notation : $(\partial r/\partial x)_\theta$ *means the partial derivative of r with respect to x keeping θ constant in* a relation expressing r as a function of x and θ.

Thus from (2), $(\partial r/\partial x)_\theta = \sec \theta$.

When no indication is given regarding the variable to be kept constant, then according to convention $(\partial/\partial x)$ always means $(\partial/\partial x)_y$ and $\partial/\partial y$ means $(\partial/\partial y)_x$. Similarly, $\partial/\partial r$ means $(\partial/\partial r)_\theta$ and $\partial/\partial \theta$ means $(\partial/\partial \theta)_r$.

(2) In thermodynamics, we come across ten variables such as p (pressure), v (volume), T (temperature), W (work), ϕ (entropy) etc. Any one of these can be expressed as a function of other two variables *e.g.*, $T = f(p, v)$, $T = g(p, \phi)$

As we shall see, these respectively give rise to the following results :

$$dT = \frac{\partial T}{\partial p} dp + \frac{\partial T}{\partial v} dv \qquad ...(i)$$

$$dT = \frac{\partial T}{\partial p} dp + \frac{\partial T}{\partial \phi} d\phi \qquad ...(ii)$$

Now, $\partial T/\partial p$ appearing in (*i*), has been obtained from T as function of p and v, treating v as constant, we write it as $(\partial T/\partial p)_v$.

Similarly, $\partial T/\partial p$ occurring in (*ii*), is written as $(\partial T/\partial p)_\phi$.

Example 5.9. *If $u = f(r)$ and $x = r \cos \theta$, $y = r \sin \theta$, prove that*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r).$$ (*S.V.T.U., 2008 ; Rajasthan, 2006 ; U.P.T.U., 2005*)

Solution. We have $\frac{\partial u}{\partial x} = f'(r) \cdot \frac{\partial r}{\partial x}$ and $\frac{\partial^2 u}{\partial x^2} = f''(r) \cdot \left(\frac{\partial r}{\partial x}\right)^2 + f'(r) \cdot \frac{\partial^2 r}{\partial x^2}$

Similarly, $$\frac{\partial^2 u}{\partial y^2} = f''(r) \cdot \left(\frac{\partial r}{\partial y}\right)^2 + f'(r) \cdot \frac{\partial^2 r}{\partial y^2}$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) \cdot \left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right] + f'(r) \left[\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2}\right]$$

Now to find $\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}$ etc., we write $r = (x^2 + y^2)^{1/2}$

$$\therefore \quad \frac{\partial r}{\partial x} = \frac{1}{2}(x^2 + y^2)^{-1/2} \cdot 2x = \frac{x}{r} \quad \text{and} \quad \frac{\partial^2 r}{\partial x^2} = \frac{r \cdot 1 - x \cdot \partial r/\partial x}{r^2} = \frac{r - x^2/r}{r^2} = \frac{y^2}{r^3}.$$

Similarly, $$\frac{\partial r}{\partial y} = \frac{y}{x} \text{ and } \frac{\partial^2 r}{\partial y^2} = \frac{x^2}{r^3}.$$

Substituting the values of $\partial r/\partial x$ etc. in (*i*), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r)\left[\frac{x^2}{r^2} + \frac{y^2}{r^2}\right] + f'(r)\left[\frac{y^2}{r^3} + \frac{x^2}{r^3}\right] = f''(r) + \frac{1}{r}f'(r).$$

Example 5.10. *If $x = e^{r\cos\theta}\cos(r\sin\theta)$ and $y = e^{r\cos\theta}\sin(r\sin\theta)$, prove that $\frac{\partial x}{\partial \theta} = -r\frac{\partial y}{\partial r}, \frac{\partial y}{\partial \theta} = r\frac{\partial x}{\partial r}$.*

Hence show that $\frac{\partial^2 x}{\partial \theta^2} + r\frac{\partial x}{\partial r} + r^2\frac{\partial^2 x}{\partial r^2} = 0.$

Solution. We have $x = e^{r\cos\theta}\cos(r\sin\theta)$

$$\therefore \quad \frac{\partial x}{\partial \theta} = e^{r\cos\theta}(-r\sin\theta) \cdot \cos(r\sin\theta) + e^{r\cos\theta}[-\sin(r\sin\theta)] \cdot r\cos\theta$$
$$= -re^{r\cos\theta}[\sin\theta\cos(r\sin\theta) + \cos\theta\sin(r\sin\theta)]$$
$$= -re^{r\cos\theta}\sin(\theta + r\sin\theta) \quad \text{...}(i)$$

and $$\frac{\partial x}{\partial r} = e^{r\cos\theta} \cdot \cos\theta \cdot \cos(r\sin\theta) - e^{r\cos\theta}\sin\theta\,(r\sin\theta)\sin\theta$$
$$= e^{r\cos\theta}\cos(\theta + r\sin\theta) \quad \text{...}(ii)$$

Similarly, $y = e^{r\cos\theta}\sin(r\sin\theta)$ gives

$$\frac{\partial y}{\partial \theta} = re^{r\cos\theta}\cos(\theta + r\sin\theta) \quad \text{...}(iii)$$

and $$\frac{\partial y}{\partial r} = e^{r\cos\theta}\sin(\theta + r\sin\theta) \quad \text{...}(iv)$$

From (*i*) and (*iv*), $\frac{\partial x}{\partial \theta} = -r\frac{\partial y}{\partial r}$ ...(*v*)

From (*ii*) and (*iii*), $\frac{\partial y}{\partial \theta} = r\frac{\partial x}{\partial r}$ ...(*vi*)

From (*v*), $$\frac{\partial^2 x}{\partial \theta^2} = -r\frac{\partial^2 y}{\partial \theta \partial r} = -r\frac{\partial^2 y}{\partial r \partial \theta}$$

From (*vi*), $$\frac{\partial x}{\partial r} = \frac{1}{r}\frac{\partial y}{\partial \theta} \quad \text{which gives} \quad \frac{\partial^2 x}{\partial r^2} = -\frac{1}{r^2}\frac{\partial y}{\partial \theta} + \frac{1}{r}\frac{\partial^2 y}{\partial r \partial \theta}$$

$$\therefore \quad \frac{\partial^2 x}{\partial \theta^2} + r\frac{\partial x}{\partial r} + r^2\frac{\partial^2 x}{\partial r^2} = -r\frac{\partial^2 y}{\partial r \partial \theta} + \frac{\partial y}{\partial \theta} - \frac{\partial y}{\partial \theta} + r\frac{\partial^2 y}{\partial r \partial \theta} = 0.$$

PROBLEMS 5.3

1. If $x = r\cos\theta$, $y = r\sin\theta$, show that (i) $\dfrac{\partial r}{\partial x} = \dfrac{\partial x}{\partial r}$ (ii) $\dfrac{1}{r}\dfrac{\partial x}{\partial \theta} = r\dfrac{\partial \theta}{\partial x}$, (iii) $\left(\dfrac{\partial r}{\partial x}\right)^2 + \left(\dfrac{\partial r}{\partial y}\right)^2 = 1.$ *(Burdwan, 2003)*

2. If $x^2 = au + bv$, $y^2 = au - bv$, prove that $\left(\dfrac{\partial u}{\partial x}\right)_y \cdot \left(\dfrac{\partial x}{\partial u}\right)_v = \dfrac{1}{2} = \left(\dfrac{\partial v}{\partial y}\right)_x \cdot \left(\dfrac{\partial y}{\partial v}\right)_u$.

3. If $u = lx + my$, $v = mx - ly$, show that $\left(\dfrac{\partial u}{\partial x}\right)_y \cdot \left(\dfrac{\partial x}{\partial u}\right)_v = \dfrac{l^2}{l^2 + m^2}, \left(\dfrac{\partial y}{\partial v}\right)_x \left(\dfrac{\partial v}{\partial y}\right)_u = \dfrac{l^2 + m^2}{l^2}$.

4. If $x = r\cos\theta$, $y = r\sin\theta$, prove that

   (i) $\dfrac{\partial^2 r}{\partial x^2} + \dfrac{\partial^2 r}{\partial y^2} = \dfrac{1}{r}\left[\left(\dfrac{\partial r}{\partial x}\right)^2 + \left(\dfrac{\partial r}{\partial y}\right)^2\right]$ (ii) $\dfrac{\partial^2 \theta}{\partial x^2} + \dfrac{\partial^2 \theta}{\partial y^2} = 0 \quad (x \neq 0, y \neq 0).$

5. If $z = x\log(x + r) - r$ where $r^2 = x^2 + y^2$, prove that $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = \dfrac{1}{x + y}, \dfrac{\partial^3 z}{\partial x^3} = -\dfrac{x}{r^3}$. *(Mumbai, 2008)*

6. If $u = f(r)$ where $r = \sqrt{(x^2 + y^2 + z^2)}$, show that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = f''(r) + \dfrac{2}{r}f'(r)$.

5.4 (1) HOMOGENEOUS FUNCTIONS

An expression of the form $a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \ldots + a_ny^n$ in which every term is of the nth degree, is called a homogeneous function of degree n. This can be rewritten as

$$x^n[a_0 + a_1(y/x) + a_2(y/x)^2 + \ldots + a_n(y/x)^n].$$

Thus *any function $f(x, y)$ which can be expressed in the form $x^n\phi(y/x)$, is called a* **homogeneous function** of degree n in x and y.

For instance, $x^3\cos(y/x)$ is a homogeneous function of degree 3, in x and y.

In general, a function $f(x, y, z, t, \ldots)$ is said to be a homogeneous function of degree n in $x, y, z, t, \ldots$, if it can be expressed in the form $x^n\phi(y/x, z/x, t/x, \ldots)$.

(2) Euler's theorem on homogeneous functions*. *If u be a homogeneous function of degree n in x and y, then*

$$\mathbf{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.}$$

Since u is a homogeneous function of degree n in x and y, therefore,

$$u = x^n f(y/x)$$

$$\therefore \quad \frac{\partial u}{\partial x} = nx^{n-1}f\left(\frac{y}{x}\right) + x^n f'\left(\frac{y}{x}\right) \cdot y\left(-\frac{1}{x^2}\right) = nx^{n-1}f\left(\frac{y}{x}\right) - yx^{n-2}f'\left(\frac{y}{x}\right)$$

and $$\frac{\partial u}{\partial y} = x^n f'\left(\frac{y}{x}\right) \cdot \frac{1}{x} = x^{n-1}f'\left(\frac{y}{x}\right). \text{ Hence } x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^n f\left(\frac{y}{x}\right) = nu.$$

In general, if u be a homogeneous function of degree n in $x, y, z, t, \ldots$, then,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} + t\frac{\partial u}{\partial t} \ldots = nu.$$

Example 5.11. *Show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2u\log u$ where $\log u = (x^3 + y^3)/(3x + 4y)$.*

Solution. Since $$z = \log u = \frac{x^3 + y^3}{3x + 4y} = x^2 \cdot \frac{1 + (y/x)^3}{3 + 4(y/x)},$$

* After an enormously creative Swiss mathematician *Leonhard Euler* (1707–1783). He studied under John *Bernoulli* and became a professor of mathematics in St. Petersburg, Russia. Even after becoming totally blind in 1771, he contributed to almost all branches of mathematics.

$\therefore$ z is a homogeneous function of degree 2 in x and y.

By Euler's theorem, we get

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z \qquad ...(i)$$

But $$\frac{\partial z}{\partial x} = \frac{1}{u}\frac{\partial u}{\partial x} \text{ and } \frac{\partial z}{\partial y} = \frac{1}{u}\frac{\partial u}{\partial y}$$

Hence (i) becomes

$$x \cdot \frac{1}{u}\frac{\partial u}{\partial x} + y \cdot \frac{1}{u}\frac{\partial u}{\partial y} = 2 \log u \quad \text{or} \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u \log u.$$

Example 5.12. *If* $u = \sin^{-1} \dfrac{x + 2y + 3z}{x^8 + y^8 + z^8}$, *find the value of* $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z}$. *(U.P.T.U., 2004)*

Solution. Here u is not a homogeneous function. We therefore, write

$$\omega = \sin u = \frac{x + 2y + 3z}{x^8 + y^8 + z^8} = x^{-7} \cdot \frac{1 + 2(y/x) + 3(z/x)}{1 + (y/x)^8 + (z/x)^8}$$

Thus ω is a homogeneous function of degree -7 in x, y, z. Hence by Euler's theorem

$$x\frac{\partial \omega}{\partial x} + y\frac{\partial \omega}{\partial y} + z\frac{\partial \omega}{\partial z} = (-7)\, \omega \qquad ...(i)$$

But $$\frac{\partial \omega}{\partial x} = \cos u \frac{\partial u}{\partial x}, \frac{\partial \omega}{\partial y} = \cos u \frac{\partial u}{\partial y}, \frac{\partial \omega}{\partial z} = \cos u \frac{\partial u}{\partial z}$$

$\therefore$ (i) becomes $\quad x \cos u\dfrac{\partial u}{\partial x} + y \cos u\dfrac{\partial u}{\partial y} + z \cos u\dfrac{\partial u}{\partial z} = -7 \sin u \quad$ or $\quad x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = -7 \tan u.$

Example 5.13. *If* $u = \dfrac{x^3y^3z^3}{x^3 + y^3 + z^3} + \log\left(\dfrac{xy + yz + zx}{x^2 + y^2 + z^2}\right)$, *find the value of* $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z}$. *(Mumbai, 2009)*

Solution. Let $\quad v = \dfrac{x^3y^3z^3}{x^3 + y^3 + z^3}$ and $w = \log\left(\dfrac{xy + yz + zx}{x^2 + y^2 + z^2}\right)$ ...(i)

so that $\quad u = v + w$

Since $v = x^6 \dfrac{(y/x)^3(z/x)^3}{1 + (y/x)^3 + (z/x)^3}$, therefore v is a homogeneous function of degree 6 in x, y, z.

Hence by Euler's theorem $\quad x\dfrac{\partial v}{\partial x} + y\dfrac{\partial v}{\partial y} + z\dfrac{\partial v}{\partial z} = 6v$ ...(ii)

Since $w = \log\left\{\dfrac{\frac{y}{x} + \frac{y}{x}.\frac{z}{x} + \frac{z}{x}}{1 + \left(\frac{y}{x}\right)^2 + \left(\frac{z}{x}\right)^2}\right\}$ therefore w is a homogeneous function of degree zero in x, y, z.

Hence by Euler's theorem $\quad x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} + z\dfrac{\partial w}{\partial z} = 0$ ...(iii)

Addint (ii) and (iii), we obtain

$$x\left(\frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\right) + y\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial y}\right) + z\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial z}\right) = 6v$$

or $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 6\, \frac{x^3y^3z^3}{x^3 + y^3 + z^3} \qquad \text{[By } (i)\text{]}$$

Example 5.14. *If z is a homogeneous function of degree n in x and y, show that*

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial z^2}=n(n-1)\,z.$$ *(Anna, 2009 ; V.T.U., 2007 ; U.P.T.U., 2006)*

Solution. By Euler's theorem, $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz$ ...(*i*)

Differentiating (*i*) partially w.r.t. *x*, we get $x\frac{\partial^2 z}{\partial x^2}+\frac{\partial z}{\partial x}+y\frac{\partial^2 z}{\partial x\partial y}=n\frac{\partial z}{\partial x}$

i.e., $$x\frac{\partial^2 z}{\partial x^2}+y\frac{\partial^2 z}{\partial x\partial y}=(n-1)\frac{\partial z}{\partial x}$$ ...(*ii*)

Again differentiating (*i*) partially w.r.t. *y*, we get $x\frac{\partial^2 z}{\partial y\partial x}+\frac{\partial z}{\partial y}+y\frac{\partial^2 z}{\partial y^2}=n\frac{\partial z}{\partial y}$

i.e., $$x\frac{\partial^2 z}{\partial x\partial y}+y\frac{\partial^2 z}{\partial y^2}=(n-1)\frac{\partial z}{\partial y}$$ ...(*iii*)

Multiplying (*ii*) by *x* and (*iii*) by *y* and adding, we get

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=(n-1)\left(x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}\right)=n(n-1)\,z.$$ [By (*i*)]

Example 5.15. *If* $u=\sin^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$, *prove that* $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}\tan u.$

(Rajasthan, 2006 ; Calicut, 2005)

and $$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}==-\frac{\sin u\cos 2u}{4\cos^3 u}.$$ *(P.T.U., 2006)*

Solution. Here *u* is not a homogeneous function but $z=\sin u=\frac{x+y}{\sqrt{x}+\sqrt{y}}$ is a homogeneous function of degree 1/2 in *x* and *y*.

∴ By Euler's theorem, $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=\frac{1}{2}z$

or $$x\cos u\frac{\partial u}{\partial x}+y\cos u\frac{\partial u}{\partial y}=\frac{1}{2}\sin u$$

Thus $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}\tan u$$ ...(*i*)

Differentiating (*i*) w.r.t. *x* partially, we get

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\partial y}=\frac{1}{2}\sec^2 u\frac{\partial u}{\partial x}\quad\text{or}\quad x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=\left(\frac{1}{2}\sec^2 u-1\right)\frac{\partial u}{\partial x}$$ ...(*ii*)

Again differentiating (*i*) w.r.t. *y* partially, we get

$$x\frac{\partial^2 u}{\partial y\partial x}+y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}=\frac{1}{2}\sec^2 u\frac{\partial u}{\partial y}\quad\text{or}\quad x\frac{\partial^2 u}{\partial x\partial y}+y\frac{\partial^2 u}{\partial y^2}=\left(\frac{1}{2}\sec^2 u-1\right)\frac{\partial u}{\partial x}$$ ...(*iii*)

Multiplying (*ii*) by *x* and (*iii*) by *y* and adding, we obtain

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\left(\frac{1}{2}\sec^2 u-1\right)\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

or $$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\left(\frac{1}{2}\sec^2 u-1\right)\left(\frac{1}{2}\tan u\right)$$ [By (*i*)]

$$=\frac{1}{4}\frac{\sin u}{\cos^3 u}-\frac{1}{2}\frac{\sin u}{\cos u}=-\frac{\sin u\,(2\cos^2 u-1)}{4\cos^3 u}$$

Hence $$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=-\frac{\sin u\cos 2u}{4\cos^3 u}.$$

PROBLEMS 5.4

1. Verify Euler's theorem, when (*i*) $f(x, y) = ax^2 + 2hxy + by^2$
(*ii*) $f(x, y) = x^2(x^2 - y^2)^3/(x^2 + y^2)^3$.
(*iii*) $f(x, y) = 3x^2yz + 5xy^2z + 4z^4$ (*J.N.T.U., 1999*)

2. If $u = \sin^{-1}\dfrac{x}{y} + \tan^{-1}\dfrac{y}{x}$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$. (*Hazaribagh, 2009 ; Osmania, 2003 S*)

3. If $u = \sin^{-1}\dfrac{x^2 + y^2}{x + y}$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \tan u$ (*Bhopal, 2009 ; V.T.U., 2003*)

4. If $\sin u = \dfrac{x^2y^2}{x + y}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 3\tan u$. (*Kottayam, 2005 ; V.T.U., 2003 S*)

5. If $u = \cos^{-1}\dfrac{x + y}{\sqrt{x} + \sqrt{y}}$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = -\dfrac{1}{2}\cot u$. (*V.T.U., 2004*)

6. Show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2u\log u$, where $u = e^{x^2 + y^2}$ (*P.T.U., 2010*)

7. If $z = f(y/x) + \sqrt{(x^2 + y^2)}$, show that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = \sqrt{x^2 + y^2}$. (*Mumbai, 2008*)

8. If $u = \dfrac{x}{y + z} + \dfrac{y}{z + x} + \dfrac{z}{x + y}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 0$. (*V.T.U., 2000 S*)

9. If $\sin u = \dfrac{x + 2y + 3z}{\sqrt{(x^8 + y^8 + z^8)}}$, show that $xu_x + yu_y + zu_z + 3\tan u = 0$. (*S.V.T.U., 2009 ; U.T.U., 2009*)

10. If $z = x\phi\left(\dfrac{y}{x}\right) + \psi\left(\dfrac{y}{x}\right)$, prove that $x^2\dfrac{\partial^2 z}{\partial x^2} + 2xy\dfrac{\partial^2 z}{\partial x\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = 0$. (*S.V.T.U., 2009 ; U.P.T.U., 2006*)

11. If $u = \tan^{-1}\dfrac{x^3 + y^3}{x + y}$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \sin 2u$. (*P.T.U., 2009 S*)

and $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = 2\cos 3u\sin u$. (*Mumbai, 2009 ; Bhopal, 2008 ; S.V.T.U., 2007*)

12. Given $z = x^n f_1(y/x) + y^{-n} f_2(x/y)$, prove that $x^2\dfrac{\partial^2 z}{\partial x^2} + 2xy\dfrac{\partial^2 z}{\partial x\partial y} + y^2\dfrac{\partial^2 z}{\partial y^2} + x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = n^2 z$.
(*Kurukshetra, 2009 S ; Rohtak, 2003*)

13. If $u = x^2\tan^{-1}(y/x) - y^2\tan^{-1}(x/y)$, evaluate $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2}$. (*U.T.U., 2009 ; Hissar, 2005 S*)

14. If $u = \tan^{-1}(y^2/x)$, prove that $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = -\sin^2 u \,.\, \sin 2u$. (*Bhillai, 2005 ; P.T.U., 2005*)

15. If $u = \operatorname{cosec}^{-1}\left(\dfrac{x^{1/2} + y^{1/2}}{x^{1/3} + y^{1/3}}\right)^{1/2}$, prove that $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = \dfrac{\tan u}{12}\left(\dfrac{13}{12} + \dfrac{\tan^2 u}{12}\right)$
(*Mumbai, 2008 ; Rohtak, 2006 S*)

5.5 (1) TOTAL DERIVATIVE

If $u = f(x, y)$, where $x = \phi(t)$ and $y = \psi(t)$, then we can express u as a function of t alone by substituting the values of x and y in $f(x, y)$. Thus we can find the ordinary derivative du/dt which is called the *total derivative* of u to distinguish it from the partial derivatives $\partial u/\partial x$ and $\partial u/\partial y$.

Now to find du/dt without actually substituting the values of x and y in $f(x, y)$, we establish the following **Chain rule :**

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} \qquad ...(i)$$

Proof. We have $u = f(x, y)$

Giving increment δt to t, let the corresponding increments of x, y and u be δx, δy and δu respectively.

Then $\quad u + \delta u = f(x + \delta x, y + \delta y)$

Subtracting, $\quad \delta u = f(x + \delta x, y + \delta y) - f(x, y)$

$$= \{f(x + \delta x, y + \delta y) - f(x, y + \delta y)\} + \{f(x, y + \delta y) - f(x, y)\}$$

$$\therefore \quad \frac{\delta u}{\delta t} = \frac{f(x + \delta x, y + \delta y) - f(x, y + \delta y)}{\delta x} \cdot \frac{\delta x}{\delta t} + \frac{f(x, y + \delta y) - f(x, y)}{\delta y} \cdot \frac{\delta y}{\delta t}$$

Taking limits as $\delta t \to 0$, δx and δy also $\to 0$, we have

$$\frac{du}{dt} = \underset{\delta y \to 0}{\text{Lt}} \left[\underset{\delta y \to 0}{\text{Lt}} \left\{ \frac{f(x + \delta x, y + \delta y) - f(x, y + \delta y)}{\delta x} \right\} \right] \frac{dx}{dt} + \underset{\delta y \to 0}{\text{Lt}} \left\{ \frac{f(x, y + \delta y) - f(x, y)}{\delta y} \right\} \frac{dy}{dt}$$

$$= \underset{\delta y \to 0}{\text{Lt}} \left\{ \frac{\partial f(x, y + \delta y)}{\delta y} \right\} \cdot \frac{dx}{dt} + \frac{\partial f(x, y)}{\partial y} \cdot \frac{dy}{dt}$$

[Supposing $\partial f(x, y)/\partial x$ to be a continuous function of y]

$$= \frac{\partial f(x, y)}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f(x, y)}{\partial y} \cdot \frac{dy}{dt} \text{ which is the desired formula.}$$

Cor. Taking $t = x$, (*i*) becomes, $\dfrac{du}{dx} = \dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} \cdot \dfrac{dy}{dx}$ ...(*ii*)

Obs. If $u = f(x, y, z)$, where x, y, z are all functions of a variable t, then **Chain rule** is

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt} \quad \text{...(iii)}$$

(2) Differentiation of implicit functions. If $f(x, y) = c$ be an implicit relation between x and y which defines as a differentiable function of x, then (*ii*) becomes

$$0 = \frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}$$

This gives the *important formula* $\dfrac{\mathbf{dy}}{\mathbf{dx}} = -\dfrac{\partial \mathbf{f}}{\partial \mathbf{x}} \Big/ \dfrac{\partial \mathbf{f}}{\partial \mathbf{y}}$ $\quad \left[\dfrac{\partial f}{\partial y} \neq 0\right]$

for the first differential coefficient of an implicit function.

Example 5.16. *Given $u = \sin(x/y)$, $x = e^t$ and $y = t^2$, find du/dt as a function of t. Verify your result by direct substitution.*

Solution. We have $\dfrac{du}{dt} = \dfrac{\partial u}{\partial x} \cdot \dfrac{dx}{dt} + \dfrac{\partial u}{\partial y} \cdot \dfrac{dy}{dt} = \left(\cos \dfrac{x}{y}\right)\dfrac{1}{y} \cdot e^t + \left(\cos \dfrac{x}{y}\right)\left(-\dfrac{x}{y^2}\right) 2t$

$$= \cos(e^t/t^2) \cdot e^t/t^2 - 2\cos(e^t/t^2) \cdot e^t/t^3 = \{(t - 2)/t^3\} e^t \cos(e^t/t^2)$$

Also $u = \sin(x/y) = \sin(e^t/t^2)$

$$\therefore \quad \frac{du}{dt} = \cos\left(\frac{e^t}{t^2}\right) \cdot \frac{t^2 e^t - e^t \cdot 2t}{t^4} = \frac{t-2}{t^3} e^t \cos\left(\frac{e^t}{t^2}\right) \text{ as before.}$$

Example 5.17. *If x increases at the rate of 2 cm/sec at the instant when $x = 3$ cm. and $y = 1$ cm., at what rate must y be changing in order that the function $2xy - 3x^2y$ shall be neither increasing nor decreasing ?*

Solution. Let $\quad u = 2xy - 3x^2y$, so that

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} = (2y - 6xy)\frac{dx}{dt} + (2x - 3x^2)\frac{dy}{dt} \quad \text{...(i)}$$

when $x = 3$ and $y = 1$, $dx/dt = 2$, and u is neither increasing nor decreasing, *i.e.*, $du/dt = 0$.

$\therefore$ (*i*) becomes $\quad 0 = (2 - 6 \times 3)\, 2 + (2 \times 3 - 3 \times 9)\dfrac{dy}{dt}$

or $\quad \dfrac{dy}{dt} = -\dfrac{32}{21}$ cm/sec. Thus y is decreasing at the rate of 32/21 cm/sec.

Example 5.18. *If $u = x \log xy$ where $x^3 + y^3 + 3xy = 1$, find du/dx.* (V.T.U., 2009)

Solution. From $f(x, y) = x^3 + y^3 + 3xy - 1$, we have

$$\frac{dy}{dx} = -\frac{\partial f}{\partial x}\bigg/\frac{\partial f}{\partial y} = -\frac{3x^2 + 3y}{3y^2 + 3x} = -\frac{x^2 + y}{y^2 + x} \quad ...(i)$$

Also $$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx} = (1 \cdot \log xy + x \cdot 1/x) + (x/y) \cdot dy/dx.$$

Hence $du/dx = 1 + \log xy - x(x^2 + y)/y(y^2 + x)$ [By (*i*)]

Example 5.19. *If $u = u\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$, show that $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0$.* (U.P.T.U., 2005)

Solution. Let $$v = \frac{y-x}{xy} = \frac{1}{x} - \frac{1}{y} \quad \text{and} \quad w = \frac{z-x}{xz} = \frac{1}{x} - \frac{1}{z} \quad ...(i)$$

so that $u = u(v, w)$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial u}{\partial w}\cdot\frac{\partial w}{\partial x} = \frac{\partial u}{\partial v}\left(-\frac{1}{x^2}\right) + \frac{\partial u}{\partial w}\left(-\frac{1}{x^2}\right) \quad \text{[Using (i)]}$$

or $$x^2\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial v} - \frac{\partial u}{\partial w} \quad ...(ii)$$

Also $$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial v}\cdot\frac{\partial u}{\partial y} + \frac{\partial u}{\partial w}\cdot\frac{\partial w}{\partial y} = \frac{\partial u}{\partial v}\left(\frac{1}{y^2}\right) + \frac{\partial u}{\partial w}(0) \quad \text{[Using (i)]}$$

or $$y^2\frac{\partial u}{\partial y} = \frac{\partial u}{\partial v} \quad ...(iii)$$

Similarly $$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial v}\ \frac{\partial v}{\partial z} + \frac{\partial u}{\partial w}\ \frac{\partial w}{\partial z} = \frac{\partial u}{\partial v}(0) + \frac{\partial u}{\partial w}\left(\frac{1}{z^2}\right) \quad \text{[Using (i)]}$$

or $$z^2\frac{\partial u}{\partial z} = \frac{\partial u}{\partial w} \quad ...(iv)$$

Adding (*ii*), (*iii*) and (*iv*), we have

$$x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0.$$

Example 5.20. *Formula for the second differential coefficient of an implicit function. If $f(x, y) = 0$, show that*

$$\frac{d^2y}{dx^2} = -\frac{q^2r - 2pqs + p^2t}{q^3}$$ (Kurukshetra, 2006)

Solution. We have $$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{p}{q} \quad ...(i)$$

$$\therefore \quad \frac{d^2y}{dx^2} = -\frac{d}{dx}\left(\frac{dy}{dx}\right) = -\frac{d}{dx}\left(\frac{p}{q}\right) = -\frac{q(dp/dx) - p(dq/dx)}{q^2} \quad ...(ii)$$

Using the notations : $r = \frac{\partial^2 f}{\partial x^2} = \frac{\partial p}{\partial x}$, $s = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial q}{\partial x}$, $t = \frac{\partial^2 f}{\partial y^2} = \frac{\partial q}{\partial y}$,

we have $$\frac{dp}{dx} = \frac{\partial p}{\partial x} + \frac{\partial p}{\partial y}\cdot\frac{dy}{dx} = r + s\,(-p/q) = -\frac{qr - ps}{q}$$

and $$\frac{dq}{dx} = \frac{\partial q}{\partial x} + \frac{\partial q}{\partial y}\cdot\frac{dy}{dx} = s + t\,(-p/q) = \frac{qs - pt}{q}$$

Substituting the values of dp/dx and dq/dx in (ii), we get

$$\frac{d^2y}{dx^2} = -\frac{1}{q^2}\left[q\left(\frac{qr-ps}{q}\right) - p\left(\frac{qs-pt}{q}\right)\right] = -\frac{q^2r - 2pqs + p^2t}{q^3}.$$

PROBLEMS 5.5

1. If $z = u^2 + v^2$ and $u = at^2$, $v = 2at$, find dz/dt. *(P.T.U., 2005)*
2. If $u = \tan^{-1}(y/x)$ where $x = e^t - e^{-t}$, and $y = e^t + e^{-t}$, find du/dt. *(V.T.U., 2003)*
3. Find the value of $\dfrac{du}{dt}$ given $u = y^2 - 4ax$, $x = at^2$, $y = 2at$. *(Anna, 2009)*
4. At a given instant the sides of a rectangle are 4 ft. and 3 ft. respectively and they are increasing at the rate of 1.5 ft./sec. and 0.5 ft./sec. respectively, find the rate at which the area is increasing at that instant.
5. If $z = 2xy^2 - 3x^2y$ and if x increases at the rate of 2 cm. per second and it passes through the value $x = 3$ cm., show that if y is passing through the value $y = 1$ cm., y must be decreasing at the rate of $2\frac{2}{15}$ cm. per second, in order that z shall remain constant.
6. If $u = x^2 + y^2 + z^2$ and $x = e^{2t}$, $y = e^{2t}\cos 3t$, $z = e^{2t}\sin 3t$. Find $\dfrac{du}{dt}$ as a total derivative and verify the result by direct substitution.
7. If $\phi(cx - az, cy - bz) = 0$, show that $\dfrac{a\partial z}{\partial x} + \dfrac{b\partial z}{\partial y} = c$.
8. If $f(x, y) = 0$, $\phi(y, z) = 0$, show that $\dfrac{\partial f}{\partial y}\cdot\dfrac{\partial \phi}{\partial z}\cdot\dfrac{dz}{dx} = \dfrac{\partial f}{\partial x}\cdot\dfrac{\partial \phi}{\partial y}$.
9. If the curves $f(x, y) = 0$ and $\phi(y, z) = 0$ touch, show that at the point of contact, $\dfrac{\partial f}{\partial x}\cdot\dfrac{\partial \phi}{\partial y} = \dfrac{\partial f}{\partial y}\cdot\dfrac{\partial \varphi}{\partial \gamma}$.
10. If $f(x, y) = 0$, show that $\left(\dfrac{\partial f}{\partial y}\right)^3\dfrac{d^2y}{dx^2} = 2\left(\dfrac{\partial f}{\partial x}\right)\left(\dfrac{\partial f}{\partial y}\right)\left(\dfrac{\partial^2 f}{\partial x\partial y}\right) - \left(\dfrac{\partial f}{dy}\right)^2\dfrac{\partial^2 f}{\partial x^2} - \left(\dfrac{\partial f}{\partial x}\right)^2\left(\dfrac{\partial^2 f}{\partial y^2}\right)$.

5.6 CHANGE OF VARIABLES

If $\qquad u = f(x, y)$ ...(1)

where $\qquad x = \phi(s, t)$ and $y = \Psi(s, t)$ ...(2)

it is often necessary to change expressions involving $u, x, y, \partial u/\partial x, \partial u/\partial y$ etc. to expressions involving $u, s, t, \partial u/\partial s, \partial u/\partial t$ etc.

The necessary formulae for the change of variables are easily obtained. If t is regarded as a constant, then x, y, u will be functions of s alone. Therefore, by (i) of page 208, we have

$$\frac{\partial \mathbf{u}}{\partial \mathbf{s}} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}}\cdot\frac{\partial \mathbf{x}}{\partial \mathbf{s}} + \frac{\partial \mathbf{u}}{\partial \mathbf{y}}\cdot\frac{\partial \mathbf{y}}{\partial \mathbf{s}} \qquad ...(3)$$

where the ordinary derivatives have been replaced by the partial derivatives because x, y are functions of two variables s and t.

$\because$ Similarly, regarding s as constant, we obtain $\dfrac{\partial \mathbf{u}}{\partial \mathbf{t}} = \dfrac{\partial \mathbf{u}}{\partial \mathbf{x}}\cdot\dfrac{\partial \mathbf{x}}{\partial \mathbf{t}} + \dfrac{\partial \mathbf{u}}{\partial \mathbf{y}}\cdot\dfrac{\partial \mathbf{y}}{\partial \mathbf{t}}$ ...(4)

On solving (3) and (4) as simultaneous equations in $\partial u/\partial x$ and $\partial u/\partial y$, we get their values in terms of $\partial u/\partial s$, $\partial u/\partial t$, u, s, t.

If instead of the equations (2), s and t are given in terms of x and y, say: $s = \xi(x, y)$ and $t = \eta(x, y)$, ...(5)

then it is easier to use the formulae $\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial x}\cdot\dfrac{\partial s}{\partial x} + \dfrac{\partial u}{\partial t}\cdot\dfrac{\partial t}{\partial x}$ ...(6)

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y}+\frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y} \qquad ...(7)$$

The higher derivatives of u can be found by repeated application of formulae (3) and (4) or of (6) and (7).

Example 5.21. *If $u = F(x - y, y - z, z - x)$, prove that*

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$$

(V.T.U., 2010 ; U.T.U. 2009 ; U.P.T.U., 2003)

Solution. Put $x - y = r$, $y - z = s$ and $z - x = t$, so that $u = f(r, s, t)$.

$$\therefore \quad \frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x}+\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x}+\frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x}$$

$$=\frac{\partial u}{\partial r}.(1)+\frac{\partial x}{\partial s}.(0)+\frac{\partial u}{\partial t}.(-1)=\frac{\partial u}{\partial r}-\frac{\partial u}{\partial t} \qquad ...(i)$$

Similarly,
$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y}+\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y}+\frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y}=-\frac{\partial u}{\partial r}+\frac{\partial u}{\partial s} \qquad ...(ii)$$

and
$$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z}+\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z}+\frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial z}=-\frac{\partial u}{\partial s}+\frac{\partial u}{\partial t} \qquad ...(iii)$$

Adding (*i*), (*ii*) and (*iii*), we get the required result.

Example 5.22. *If $z = f(x, y)$ and $x = e^u \cos v$, $y = e^u \sin v$, prove that* $x\dfrac{\partial z}{\partial v}+y\dfrac{\partial z}{\partial u}=e^{2u}\dfrac{\partial z}{\partial y}$

and
$$\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2=e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2+\left(\frac{\partial u}{\partial v}\right)^2\right]$$

(Mumbai, 2009)

Solution. We have $\dfrac{\partial z}{\partial u}=\dfrac{\partial z}{\partial x}\cdot\dfrac{\partial x}{\partial u}+\dfrac{\partial z}{\partial y}\cdot\dfrac{\partial y}{\partial u}$

$$=\frac{\partial z}{\partial x}(e^u\cos v)+\frac{\partial z}{\partial y}(e^u\sin v) \qquad ...(i)$$

$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$=\frac{\partial z}{\partial x}(-e^u\sin v)+\frac{\partial x}{\partial y}(e^u\cos v) \qquad ...(ii)$$

$$\therefore \quad x\frac{\partial x}{\partial v}+y\frac{\partial z}{\partial u}=(e^u\cos v)\left[-e^u\sin v\frac{\partial z}{\partial x}+e^u\cos v\frac{\partial z}{\partial y}\right]+(e^u\sin v)\left[e^u\cos v\frac{\partial z}{\partial x}+e^u\sin v\frac{\partial z}{\partial y}\right]$$

$$=(e^{2u}\cos^2 v+e^{2u}\sin^2 v)\frac{\partial z}{\partial y}=e^{2u}\frac{\partial z}{\partial y}$$

Now squaring (*i*) and (*ii*) and adding, we get

$$\left(\frac{\partial z}{\partial u}\right)^2+\left(\frac{\partial z}{\partial v}\right)^2=e^{2u}\left(\cos v\frac{\partial z}{\partial x}+\sin v\frac{\partial z}{\partial y}\right)^2+e^{2u}\left(-\sin v\frac{\partial z}{\partial x}+\cos v\frac{\partial z}{\partial y}\right)^2$$

or
$$e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2+\left(\frac{\partial z}{\partial v}\right)^2\right]=\cos^2 v\left(\frac{\partial z}{\partial x}\right)^2+\sin^2 v\left(\frac{\partial z}{\partial y}\right)^2+2\sin v\cos v\,\frac{\delta z}{\delta z}\,\frac{\delta z}{\delta y}$$

$$+\sin^2 v\left(\frac{\partial z}{\partial x}\right)^2+\cos^2 v\left(\frac{\partial z}{\partial y}\right)^2-2\sin v\cos v\,\frac{\partial z}{\partial x}\,\frac{\partial z}{\partial y}$$

$$=(\cos^2 v+\sin^2 v)\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2$$

Hence $$\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2=e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2+\left(\frac{\partial z}{\partial v}\right)^2\right].$$

Example 5.23. *If $x + y = 2e^{\theta}\cos\phi$ and $x - y = 2ie^{\theta}\sin\phi$, show that*

$$\frac{\partial^2 u}{\partial\theta^2}+\frac{\partial^2 u}{\partial\phi^2}=4xy\frac{\partial^2 u}{\partial x\partial y}$$ *(Nagpur, 2009 ; U.P.T.U., 2002)*

Solution. We have $x = e^{\theta}(\cos\phi + i\sin\phi) = e^{\theta}\,.\,e^{i\phi}$ [See p. 205]

and $y = e^{\theta}(\cos\phi - i\sin\phi) = e^{\theta}\,.\,e^{-i\phi}$

Here u is a composite function of θ and ϕ.

$$\therefore \quad \frac{\partial u}{\partial\theta}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial\theta}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial\theta}$$

$$=\frac{\partial u}{\partial x}\cdot(e^{\theta}\,.\,e^{i\phi})+\frac{\partial u}{\partial y}(e^{\theta}\,.\,e^{-i\phi})=x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}$$

or $$\frac{\partial}{\partial\theta}=x\frac{\partial}{\partial u}+y\frac{\partial}{\partial y} \qquad ...(i)$$

Also $$\frac{\partial u}{\partial\phi}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial\phi}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial\phi}=\frac{\partial u}{\partial x}\cdot(e^{\theta}\,.\,ie^{i\phi})+\frac{\partial u}{\partial y}(e^{\theta}\,.-ie^{-i\phi})=ix\frac{\partial u}{\partial x}-iy\frac{\partial u}{\partial y}$$

$$\frac{\partial}{\partial\phi}=ix\frac{\partial}{\partial x}-iy\frac{\partial}{\partial y} \qquad ...(ii)$$

Using the operator (i), we have

$$\frac{\partial^2 u}{\partial\theta^2}=\frac{\partial}{\partial\theta}\left(\frac{\partial u}{\partial\theta}\right)=\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

$$=x\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right)+x\frac{\partial}{\partial x}\left(y\frac{\partial u}{\partial y}\right)+y\frac{\partial}{\partial y}\left(x\frac{\partial u}{\partial x}\right)+y\frac{\partial}{\partial y}\left(y\frac{\partial u}{\partial y}\right)$$

$$=x\left(x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}\right)+xy\frac{\partial^2 u}{\partial x\partial y}+yx\frac{\partial^2 u}{\partial y\partial x}+y\left(y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}\right)$$

$$=x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} \qquad ...(iii)$$

Similarly using (ii), $$\frac{\partial^2 u}{\partial\phi^2}=\frac{\partial}{\partial\phi}\left(\frac{\partial u}{\partial\phi}\right)=\left(ix\frac{\partial}{\partial x}-iy\frac{\partial}{\partial y}\right)\left(ix\frac{\partial u}{\partial x}-iy\frac{\partial u}{\partial y}\right)$$

$$=-x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}-y^2\frac{\partial^2 u}{\partial y^2}-x\frac{\partial u}{\partial x}-y\frac{\partial u}{\partial y} \qquad ...(iv)$$

Adding (iii) and (iv), we get $$\frac{\partial^2 u}{\partial\theta^2}+\frac{\partial^2 u}{\partial\phi^2}=4xy\frac{\partial^2 u}{\partial x\partial y}$$

Example 5.24. *Transform the equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ into polar coordinates.* *(P.T.U., 2010)*

Solution. We have $x = r\cos\theta$, $y = r\sin\theta$ and $r=\sqrt{(x^2+y^2)}$, $\theta=\tan^{-1}(y/x)$

$$\frac{\partial r}{\partial x}=\frac{x}{\sqrt{(x^2+y^2)}}=\cos\theta \text{ and } \frac{\partial\theta}{\partial x}=-\frac{y}{x^2+y^2}=-\frac{\sin\theta}{r}$$

Thus, $$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x}+\frac{\partial u}{\partial\theta}\cdot\frac{\partial\theta}{\partial x}=\cos\theta\frac{\partial u}{\partial r}-\frac{\sin\theta}{r}\frac{\partial u}{\partial\theta}$$

i.e., $$\frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta} \qquad \text{Similarly, } \frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}.$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial\theta}\right)$$

$$= \cos^2\theta\frac{\partial^2 u}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\sin^2\theta}{r}\frac{\partial u}{\partial r} + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial\theta} \qquad ...(i)$$

and $$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial\theta}\right)$$

$$= \sin^2\theta\frac{\partial^2 u}{\partial r^2} + \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\cos^2\theta}{r}\frac{\partial u}{\partial r} - \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial r} \qquad ...(ii)$$

Adding (*i*) and (*ii*), we get $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial\theta^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r}$.

Hence the transformed equation is $\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial\theta^2} = 0$.

PROBLEMS 5.6

1. If $z = f(x, y)$ and $x = e^u + e^{-v}$, $y = e^{-u} - e^v$, prove that $\dfrac{\partial z}{\partial u} - \dfrac{\partial z}{\partial v} = x\dfrac{\partial z}{\partial x} - y\dfrac{\partial z}{\partial y}$. (*V.T.U., 2006*)
2. If $u = f(r, s)$, $r = x + at$, $s = y + bt$ and x, y, t are independent variables, show that $\dfrac{\partial u}{\partial t} = a\dfrac{\partial u}{\partial x} + b\dfrac{\partial u}{\partial y}$.
3. If $\phi(z/x^3, y/x) = 0$, prove that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 3z$. (*Mumbai, 2007*)
4. If $u = f(x, y)$ and $x = r\cos\theta$, $y = r\sin\theta$, prove that $\left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial u}{\partial y}\right)^2 = \left(\dfrac{\partial u}{\partial r}\right)^2 + \dfrac{1}{r^2}\left(\dfrac{\partial u}{\partial\theta}\right)^2$.
(*V.T.U., 2010 ; Madras 2006 ; Rohtak, 2005*)
5. If $u = f(2x - 3y, 3y - 4z, 4z - 2x)$, prove that $\dfrac{1}{2}\dfrac{\partial u}{\partial x} + \dfrac{1}{3}\dfrac{\partial u}{\partial y} + \dfrac{1}{4}\dfrac{\partial u}{\partial z} = 0$. (*U.P.T.U., 2006 ; Raipur, 2005*)
6. If $u = f(e^{y-z}, e^{z-x}, e^{x-y})$, prove that $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = 0$ (*Mumbai, 2008 S*)
7. If $u = f(r, s, t)$ and $r = x/y$, $s = y/z$, $t = z/x$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 0$. (*Anna, 2009 ; Kurukshetra, 2006*)
8. If $x = u + v + w$, $y = vw + wu + uv$, $z = uvw$ and F is a function of x, y, z, show that
$$u\frac{\partial F}{\partial u} + v\frac{\partial F}{\partial v} + w\frac{\partial F}{\partial w} = x\frac{\partial F}{\partial x} + 2y\frac{\partial F}{\partial y} + 3z\frac{\partial F}{\partial z}.$$
9. Given that $u(x, y, z) = f(x^2 + y^2 + z^2)$ where $x = r\cos\theta\cos\phi$, $y = r\cos\theta\sin\phi$ and $z = r\sin\theta$, find $\dfrac{\partial u}{\partial\theta}$ and $\dfrac{\partial u}{\partial\phi}$.
10. If the three thermodynamic variables P, V, T are connected by a relation $f(P, V, T) = 0$, show that
$$\left(\frac{\partial P}{\partial T}\right)_V\left(\frac{\partial T}{\partial V}\right)_P\left(\frac{\partial V}{\partial P}\right)_T = -1.$$
11. If by the substitution $u = x^2 - y^2$, $v = 2xy$, $f(x, y) = \theta(u, v)$, show that
$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 4(x^2 + y^2)\left(\frac{\partial^2\theta}{\partial u^2} + \frac{\partial^2\theta}{\partial v^2}\right).$$ (*Anna, 2003*)
12. Transform $\dfrac{\partial^2 z}{\partial x^2} + 2xy^2\dfrac{\partial z}{\partial x} + 2(y - y^3)\dfrac{\partial z}{\partial y} + x^2y^2z = 0$ by the substitution $x = uv$, $y = 1/v$. Hence show that z is the same function of u and v as of x and y.

5.7 (1) JACOBIANS

If u and v are functions of two independent variables x and y, then the determinant

$$\begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix}$$ is called the *Jacobian** of u, v with respect to x, y

and is written as $\dfrac{\partial(u, v)}{\partial(x, y)}$ or $J\left(\dfrac{u, v}{x, y}\right)$.

Similarly the Jacobian of u, v, w with respect to x, y, z is

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y & \partial u/\partial z \\ \partial v/\partial x & \partial v/\partial y & \partial v/\partial z \\ \partial w/\partial x & \partial w/\partial y & \partial w/\partial z \end{vmatrix}$$

Likewise, we can define Jacobians of four or more variables. An important application of Jacobians is in connection with the change of variables in multiple integrals (§ 7.7).

(2) Properties of Jacobians. We give below two of the important properties of Jacobians. For simplicity, the properties are stated in terms of two variables only, but these are evidently true in general.

I. If $J = \partial(u, v)/\partial(x, y)$ and $J' = \partial(x, y)/\partial(u, v)$ then $JJ' = 1$.

Let $u = f(x, y)$ and $v = g(x, y)$.

Suppose, on solving for x and y, we get $x = \phi(u, v)$ and $y = \psi(u, v)$.

Then
$$\left.\begin{aligned} \frac{\partial u}{\partial u} &= 1 = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial u}, \\ \frac{\partial u}{\partial v} &= 0 = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial v}, \\ \frac{\partial v}{\partial u} &= 0 = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial u}, \\ \text{and}\quad \frac{\partial v}{\partial v} &= 1 = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial v}, \end{aligned}\right\} \quad \text{...}(i)$$

$$\therefore \quad JJ' = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} \times \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial y/\partial x & \partial v/\partial y \end{vmatrix} \times \begin{vmatrix} \partial x/\partial u & \partial y/\partial u \\ \partial x/\partial v & \partial y/\partial v \end{vmatrix}$$

(Interchanging rows and columns of the 2nd determinant).

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1.$$ [By virtue of (i)]

II. **Chain rule for Jacobians.** *If u, v are functions of r, s and r, s are functions of x, y, then*

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial(u, v)}{\partial(r, s)} \cdot \frac{\partial(r, s)}{\partial(x, y)}.$$

$$\frac{\partial(u, v)}{\partial(r, s)} \cdot \frac{\partial(r, s)}{\partial(x, y)} = \begin{vmatrix} \partial u/\partial r & \partial u/\partial s \\ \partial v/\partial r & \partial v/\partial s \end{vmatrix} \times \begin{vmatrix} \partial r/\partial x & \partial s/\partial x \\ \partial r/\partial y & \partial s/\partial y \end{vmatrix}$$

[Interchanging rows and columns of the 2nd det.]

$$= \begin{vmatrix} \dfrac{\partial u}{\partial r}\cdot\dfrac{\partial r}{\partial x} + \dfrac{\partial u}{\partial s}\cdot\dfrac{\partial s}{\partial x} & \dfrac{\partial u}{\partial r}\cdot\dfrac{\partial r}{\partial y} + \dfrac{\partial u}{\partial s}\cdot\dfrac{\partial s}{\partial y} \\ \dfrac{\partial v}{\partial r}\cdot\dfrac{\partial r}{\partial x} + \dfrac{\partial v}{\partial s}\cdot\dfrac{\partial s}{\partial x} & \dfrac{\partial v}{\partial r}\cdot\dfrac{\partial r}{\partial y} + \dfrac{\partial v}{\partial s}\cdot\dfrac{\partial s}{\partial y} \end{vmatrix} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \frac{\partial(u, v)}{\partial(x, y)}.$$

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* Called after the German mathematician *Carl Gustav Jacob Jacobi* (1804–1851), who made significant contributions to mechanics, partial differential equations, astronomy, elliptic functions and the calculus of variations.

Example 5.25. *(i) In polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$, show that*

$$\frac{\partial(x, y)}{\partial(r, \theta)} = r.$$ *(U.P.T.U., 2006 ; V.T.U., 2004 ; Andhra, 2000)*

(ii) In cylindrical coordinates (Fig. 8.28), $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$, show that

$$\frac{\partial(x, y, z)}{\partial(\rho, \phi, z)} = \rho.$$

(iii) In spherical polar coordidnates (Fig. 8.29), $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, show that

$$\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2\sin\theta.$$ *(Anna, 2009 ; Hazaribagh, 2009 ; Rohtak, 2003)*

Solution. (*i*) We have $\quad \frac{\partial x}{\partial r} = \cos\theta,\ \frac{\partial x}{\partial \theta} = -r\sin\theta$ and $\frac{\partial y}{\partial r} = \sin\theta,\ \frac{\partial y}{\partial \theta} = -r\cos\theta$

$$\therefore \quad \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

(*ii*) We have $\quad \frac{\partial x}{\partial \rho} = \cos\phi,\ \frac{\partial x}{\partial \phi} = -\rho\sin\phi,\ \frac{\partial x}{\partial z} = 0,$

$$\frac{\partial y}{\partial \rho} = \sin\phi,\ \frac{\partial y}{\partial \phi} = \rho\cos\phi,\ \frac{\partial y}{\partial z} = 0 \text{ and } \frac{\partial z}{\partial \rho} = 0,\ \frac{\partial z}{\partial \phi} = 0,\ \frac{\partial z}{\partial z} = 1$$

$$\therefore \quad \frac{\partial(x, y, z)}{\partial(\rho, \phi, z)} = \begin{vmatrix} \cos\phi & -\rho\sin\theta & 0 \\ \sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} = \rho.$$

(*iii*) We have $\quad \frac{\partial x}{\partial r} = \sin\theta\cos\theta,\ \frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi,\ \frac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi,$

$$\frac{\partial y}{\partial r} = \sin\theta\sin\phi,\ \frac{\partial y}{\partial \theta} = r\cos\theta\sin\phi,\ \frac{\partial y}{\partial \phi} = r\sin\theta\cos\phi,$$

and $$\frac{\partial z}{\partial r} = \cos\theta,\ \frac{\partial z}{\partial \theta} = -r\sin\theta,\ \frac{\partial z}{\partial \phi} = 0.$$

$$\therefore \quad \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin\theta.$$

Example 5.26. *If $y_1 = \frac{x_2x_3}{x_1}$, $y_2 = \frac{x_3x_1}{x_2}$, $y_3 = \frac{x_1x_2}{x_3}$, show that the Jacobian of y_1, y_2, y_3 with respect to x_1, x_2, x_3 is 4.* *(U.P.T.U., 2006)*

Solution. We have $\frac{\partial y_1}{\partial x_1} = -\frac{x_2x_3}{x_1^2},\ \frac{\partial y_1}{\partial x_2} = \frac{x_3}{x_1},\ \frac{\partial y_1}{\partial x_3} = \frac{x_2}{x_1}$

$$\frac{\partial y_2}{\partial x_1} = \frac{x_3}{x_2},\ \frac{\partial y_2}{\partial x_2} = -\frac{x_3x_1}{x_2^2},\ \frac{\partial y_2}{\partial x_3} = \frac{x_1}{x_2}$$

and $$\frac{\partial y_3}{\partial x_1} = \frac{x_2}{x_3},\ \frac{\partial y_3}{\partial x_2} = \frac{x_1}{x_3},\ \frac{\partial y_3}{\partial x_3} = -\frac{x_1x_2}{x_3^2}$$

$$\therefore \quad \frac{\partial(y_1y_2y_3)}{\partial(x_1x_2x_3)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix} \begin{vmatrix} \frac{-x_2x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ \frac{x_3}{x_2} & \frac{-x_3x_1}{x_2^2} & \frac{x_1}{x_1} \\ \frac{x_2}{x_3} & \frac{x_1}{x_3} & \frac{-x_1x_2}{x_3^2} \end{vmatrix}$$

$$= \frac{1}{x_1^2x_2^2x_3^2}\begin{vmatrix} -x_2x_3 & x_3x_1 & x_1x_2 \\ x_2x_3 & -x_3x_1 & x_1x_2 \\ x_2x_3 & x_3x_1 & -x_1x_2 \end{vmatrix} = -\frac{x_1^2x_2^2x_3^2}{x_1^2x_2^2x_3^2}\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= -1(1-1) - 1(-1-1) + 1(1+1) = 0 + 2 + 2 = 4.$$

Example 5.27. *In $u = x + 3y^2 - z^3$, $v = 4x^2yz$, $w = 2z^2 - xy$, evaluate $\partial(u, v, w)/\partial(x, y, z)$ at $(1, -1, 0)$.* (V.T.U., 2006)

Solution. $$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 6y & -3z^2 \\ 8xyz & 4x^2z & 4x^2y \\ -y & -x & 4z \end{vmatrix}$$

$\therefore$ At the point $(1, -1, 0)$ $$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} 1 & -6 & 0 \\ 0 & 0 & -4 \\ 1 & -1 & 0 \end{vmatrix} = 4(-1+6) = 20.$$

Example 5.28. *If $u = x^2 - y^2$, $v = 2xy$ and $x = r\cos\theta$, $y = r\sin\theta$, find $\frac{\partial(u, v)}{\partial(r, \theta)}$.* (V.T.U., 2009 ; Madras, 2006)

Solution. We have $$\frac{\partial(u, v)}{\partial(r, \theta)} = \frac{\partial(u, v)}{\partial(x, y)} \times \frac{\partial(x, y)}{\partial(r, \theta)}$$

Since $u = x^2 - y^2$, $u = 2xy$

$$\therefore \quad \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4(x^2 + y^2) \qquad ...(ii)$$

Since $x = r\cos\theta$, $y = r\sin\theta$,

$$\therefore \quad \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r \qquad ...(iii)$$

Hence, $$\frac{\partial(u, v)}{\partial(r, \theta)} = 4(x^2 + y^2).\, r = 4(r^2\cos^2\theta + r^2\sin^2\theta).\, r = 4r^3 \qquad \text{[Using (ii) \& (iii)]}$$

(3) Jacobian of Implicit functions. *If u_1, u_2, u_3 instead of being given explicitly in terms x_1, x_2, x_3, be connected with them equations such as*

$$f_1(u_1, u_2, u_3, x_1, x_2, x_3) = 0,\ f_2(u_1, u_2, u_3, x_1, x_2, x_3) = 0,\ f_3(u_1, u_2, u_3, x_1, x_2, x_3) = 0, \text{ then}$$

$$\frac{\partial(u_1, u_2, u_3)}{\partial(x_1, x_2, x_3)} = (-1)^3 \frac{\partial(f_1, f_2, f_3)}{\partial(x_1, x_2, x_3)} \div \frac{\partial(f_1, f_2, f_3)}{\partial(u_1, u_2, u_3)}$$

Obs. This result can be easily generalised. It bears analogy to the result $\frac{dy}{dx} = -\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial y}$, where x, y are connected by the relation $f(x, y) = 0$.

Example 5.29. *If $u = x, y, z$, $v = x^2 + y^2 + z^2$, $w = x + y + z$, find $\partial(x, y, z)/\partial(u, v, w)$.* (U.P.T.U., 2003)

Solution. Let $\quad f_1 = u - x\,y\,z,\ f_2 = v - x^2 - y^2 - z^2,\ f_3 = w - x - y - z.$

We have $$\frac{\partial(x, y, z)}{\partial(u, v, w)} = (-1)^3 \frac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)} \div \frac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)} \quad ...(i)$$

Now, $$\frac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)} = \begin{vmatrix} \partial f_1/\partial x & \partial f_1/\partial y & \partial f_1/\partial z \\ \partial f_2/\partial x & \partial f_2/\partial y & \partial f_2/\partial z \\ \partial f_3/\partial x & \partial f_3/\partial y & \partial f_3/\partial z \end{vmatrix} = \begin{vmatrix} -yz & -xz & -xy \\ -2x & -2y & -2z \\ -1 & -1 & -1 \end{vmatrix}$$
$$= -2(x-y)(y-z)(z-x) \quad ...(ii)$$

and $$\frac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \quad ...(iii)$$

Substituting values from (*ii*) and (*iii*) in (*i*), we get

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = (-1) \times 1/[-2(x-y)(y-z)(z-x)] = 1/2(x-y)(y-z)(z-x).$$

(4) Functional relationship. *If u_1, u_2, u_3 be functions of x_1, x_2, x_3 then the necessary and sufficient condition for the existence of a functional relationship of the form $f(u_1, u_2, u_3) = 0$, is*

$$J\left(\frac{u_1, u_2, u_3}{x_1, x_2, x_3}\right) = 0.$$

Example 5.30. *If $u = x\sqrt{(1-y^2)} + y\sqrt{(1-x^2)}$, $v = \sin^{-1} x + \sin^{-1} y$, show that u, v are functionally related and find the relationship.* (*Kurukshetra, 2006*)

Solution. We have $$\frac{\partial u}{\partial x} = \sqrt{(1-y^2)} - \frac{xy}{\sqrt{(1-x^2)}}, \quad \frac{\partial u}{\partial y} = \frac{-xy}{\sqrt{(1-y^2)}} + \sqrt{(1-x^2)}$$

and $$\frac{\partial v}{\partial x} = \frac{1}{\sqrt{(1-x^2)}}, \quad \frac{\partial v}{\partial y} = \frac{1}{\sqrt{(1-y^2)}}$$

$\therefore$ $$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \sqrt{(1-y^2)} - \frac{xy}{\sqrt{(1-x^2)}}, & \sqrt{(1-x^2)} - \frac{xy}{\sqrt{(1-y^2)}} \\ \frac{1}{\sqrt{(1-x^2)}}, & \frac{1}{\sqrt{(1-y^2)}} \end{vmatrix}$$

$$= 1 - \frac{xy}{\sqrt{[(1-x^2)(1-y^2)]}} - 1 + \frac{xy}{\sqrt{[(1-x^2)(1-y^2)]}} = 0$$

Hence u and v are functionally related *i.e.*, they are not independent.

We have $$v = \sin^{-1} x + \sin^{-1} y = \sin^{-1} [x\sqrt{(1-y^2)} + y\sqrt{(1-x^2)}]$$

i.e., $$u = \sin v$$

which is the required relationship between u and v.

PROBLEMS 5.7

1. If $x = r \cos\theta$, $y = r \sin\theta$, evaluate $\partial(r, \theta)/\partial(x, y)$, $\partial(x, y)/\partial(r, \theta)$ and prove that $[\partial(r, \theta)\,\partial(x, y)]$. $[\partial(x, y)/\partial(r, \theta)] = 1$. (*V.T.U., 2010*)
2. If $x = u(1-v)$, $y = uv$, prove that $JJ' = 1$. (*V.T.U., 2000 S*)
3. If $x = a \cosh\xi \cos\eta$, $y = a \sinh\xi \sin\eta$, show that $\partial(x, y)/\partial(\xi, \eta) = \frac{1}{2} a^2 (\cosh 2\xi - \cos 2\eta)$ (*S.V.T.U., 2007*)
4. If $x = e^u \sec v$, $y = e^u \tan v$, find $J = \partial(u, v)/\partial(x, y)$, $J' = \partial(x, y)/\partial(u, v)$. Hence show $JJ' = 1$. (*V.T.U., 2007 S*)
5. If $u = x^2 - 2y^2$, $v = 2x^2 - y^2$ where $x = r\cos\theta$, $y = r\sin\theta$, show that $\frac{\partial(u, v)}{\partial(r, \theta)} = 6r^3 \sin 2\theta$.
6. If $u = x^2 + y^2 + z^2$, $v = xy + yz + zx$, $w = x + y + z$, find $\frac{\partial(u, v, w)}{\partial(x, y, z)}$. (*U.T.U., 2009 ; V.T.U., 2008*)

7. If $F = xu + v - y$, $G = u^2 + vy + w$, $H = zu - v + vw$, compute $\partial(F, G, H)/\partial(u, v, w)$.
8. If $u = x + y + z$, $uv = y + z$, $uvw = z$, show that $\partial(x, y, z)/\partial(u, v, w) = u^2v$.
(*Kurukshetra, 2009; P.T.U., 2009 S; V.T.U., 2003*)
9. If $u^3 + v^3 = x + y$ and $u^2 + v^2 = x^3 + y^3$, show that $\dfrac{\partial(u, v)}{\partial(x, y)} = \dfrac{1}{2}\dfrac{y^2 - x^2}{uv(u - v)}$ (*U.P.T.U., 2006 MCA*)
10. If $u = \dfrac{x + y}{1 - xy}$ and $v = \tan^{-1} x + \tan^{-1} y$, find $\dfrac{\partial(u, v)}{\partial(x, y)}$. Are u and v functionally related. If so, find this relationship.
(*Nagpur, 2008*)
11. If $u = 3x + 2y - z$, $v = x - 2y + z$ and $w = x(x + 2y - z)$, show that they are functionally related, and find the relation.
(*Nagpur, 2009*)

5.8 (1) GEOMETRICAL INTERPRETATION

If $P(x, y, z)$ be the coordinates of a point referred to axes OX, OY, OZ then the equation $z = f(x, y)$ represents a surface. (Fig. 5.1)

Let a plane $y = b$ parallel to the XZ-plane pass through P cutting the surface along the curve APB given by

$$z = f(x, b).$$

As y remains equal to b and x varies then P moves along the curve APB and $\partial z/\partial x$ is the ordinary derivative of $f(x, b)$ w.r.t. x.

Hence *$\partial z/\partial x$ at P is the tangent of the angle which the tangent at P to the section of the surface $z = f(x, y)$ by a plane through P parallel to the plane XOZ, makes with a line parallel to the x-axis.*

Similarly, *$\partial z/\partial y$ at P is the tangent of the angle which the tangent at P to the curve of intersection of the surface $z = f(x, y)$ and the plane $x = a$, makes with a line parallel to the y-axis.*

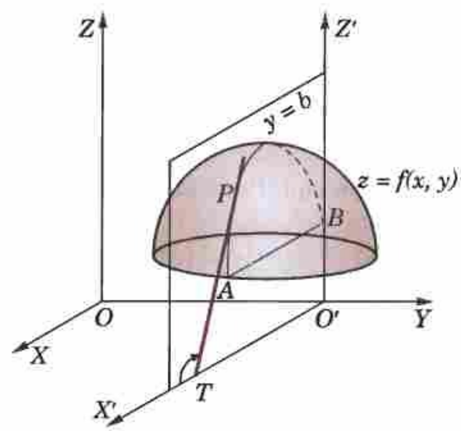

Fig. 5.1

(2) Tangent plane and Normal to a surface. Let $P(x, y, z)$ and $Q(x + \delta x, y + \delta y, z + \delta z)$ be two neighbouring points on the surface $F(x, y, z) = 0$. (Fig. 5.2) ...(*i*)

Let the arc PQ be δs and the chord PQ be δc, so that (as for plane curves)

$$\underset{Q \to P}{\text{Lt}} (\delta s/\delta c) = 1.$$

The direction cosines of PQ are $\dfrac{\delta x}{\delta c}, \dfrac{\delta y}{\delta c}, \dfrac{\delta z}{\delta c}$ *i.e.*, $\dfrac{\delta x}{\delta s} \cdot \dfrac{\delta s}{\delta c}, \dfrac{\delta y}{\delta s} \cdot \dfrac{\delta s}{\delta c}, \dfrac{\delta z}{\delta s} \cdot \dfrac{\delta s}{\delta c}$

When $\delta s \to 0$, $Q \to P$ and PQ tends to tangent line PT. Then noting that the coordinates of any point on arc PQ are functions of s only, the direction cosines of PT are

$$\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds} \qquad ...(ii)$$

Differentiating (*i*) with respect to s, we obtain $\dfrac{\partial F}{\partial x} \cdot \dfrac{dx}{ds} + \dfrac{\partial F}{\partial y} \cdot \dfrac{dy}{ds} + \dfrac{\partial F}{\partial z} \cdot \dfrac{dz}{ds} = 0.$

This shows that the tangent line whose direction cosines are given by (*ii*), is perpendicular to the line having direction ratios

$$\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \qquad ...(iii)$$

Since we can take different curves joining Q to P, we get a number of tangent lines at P and the line having direction ratios (*iii*) will be perpendicular to *all* these tangent lines at P. Thus all the tangent lines at P lie in a plane through P perpendicular to line (*iii*).

Hence *the equation of the tangent plane to (i) at the point P is*

$$\frac{\partial \mathbf{F}}{\partial \mathbf{x}}(\mathbf{X} - \mathbf{x}) + \frac{\partial \mathbf{F}}{\partial \mathbf{y}}(\mathbf{Y} - \mathbf{y}) + \frac{\partial \mathbf{F}}{\partial \mathbf{z}}(\mathbf{Z} - \mathbf{z}) = \mathbf{0}$$

where (X, Y, Z) are the current coordinates of any point on this tangent plane.

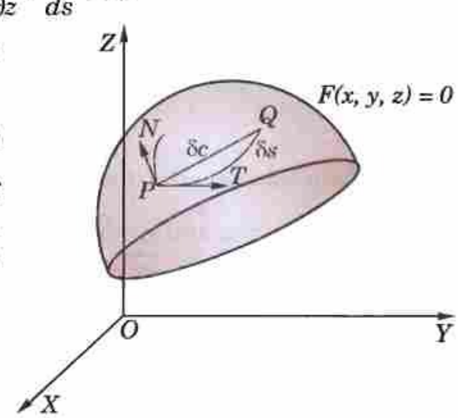

Fig. 5.2

Also *the equation of the normal to the surface at P (i.e., the line through P,* perpendicular to the tangent plane at *P*) is

$$\frac{\mathbf{X-x}}{\partial \mathbf{F}/\partial \mathbf{x}} = \frac{\mathbf{Y-y}}{\partial \mathbf{F}/\partial \mathbf{y}} = \frac{\mathbf{Z-z}}{\partial \mathbf{F}/\partial \mathbf{z}}.$$

Example 5.31. *Find the equations of the tangent plane and the normal to the surface $z^2 = 4(1 + x^2 + y^2)$ at (2, 2, 6).*

Solution. We have $F(x, y, z) = 4x^2 + 4y^2 - z^2 + 4$.

$\therefore$ $\partial F/\partial x = 8x$, $\partial F/\partial y = 8y$, $\partial F/\partial z = -2z$, and at the point (2, 2, 6)

$\partial F/\partial x = 16$, $\partial F/\partial y = 16$, $\partial F/\partial z = -12$

Hence the equation of the tangent plane at (2, 2, 6) is $16(X-2) + 16(Y-2) - 12(Z-6) = 0$

i.e., $4X + 4Y - 3Z + 2 = 0$ ...(*i*)

Also the equation of the normal at (2, 2, 6) [being perpendicular to (*i*)] is

$$\frac{X-2}{4} = \frac{Y-2}{4} = \frac{Z-6}{-3}.$$

PROBLEMS 5.8

Find the equations of the tangent plane and normal to each of the following surfaces at the given points :

1. $2x^2 + y^2 = 3 - 2z$ at $(2, 1, -3)$ (*Assam, 1998*)
2. $x^3 + y^3 + 3xyz = 3$ at $(1, 2, -1)$ (*Osmania, 2003 S*)
3. $xyz = a^2$ at (x_1, y_1, z_1).
4. $2xz^2 - 3xy - 4x = 7$ at $(1, -1, 2)$.
5. Show the plane $3x + 12y - 6z - 17 = 0$ touches the conicoid $3x^2 - 6y^2 + 9z^2 + 17 = 0$. Find also the point of contact.
6. Show that the plane $ax + by + cz + d = 0$ touches the surface $px^2 + qy^2 + 2z = 0$, if $\frac{a^2}{p} + \frac{b^2}{q} + 2cd = 0$.
7. Find the equation of the normal to the surface $x^2 + y^2 + z^2 = a^2$. (*P.T.U., 2009 S*)

5.9 TAYLOR'S THEOREM FOR FUNCTIONS OF TWO VARIABLES

Considering $f(x + h, y + k)$ as a function of a single variable x, we have by Taylor's theorem*

$$f(x+h, y+k) = f(x, y+k) + h\,\frac{\partial f(x, y+k)}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 f(x, y+k)}{\partial x^2} + \ldots \qquad \ldots(i)$$

Now expanding $f(x, y + k)$ as a function of y only,

$$f(x, y+k) = f(x, y) + k\,\frac{\partial f(x, y)}{\partial y} + \frac{k^2}{2!}\frac{\partial^2 f(x, y)}{\partial y^2} + \ldots$$

$\therefore$ (*i*) takes the form $f(x+h, y+k) = f(x, y) + k\,\frac{\partial f(x, y)}{\partial y} + \frac{k^2}{2!}\frac{\partial^2 f(x, y)}{\partial y^2} + \ldots$

$$+ h\,\frac{\partial}{\partial x}\left\{f(x, y) + k\,\frac{\partial f(x, y)}{\partial y} + \frac{k^2}{2!}\frac{\partial^2 f(x, y)}{\partial y^2} + \ldots\right\} + \frac{h^2}{2!}\frac{\partial^2}{\partial x^2}\left\{f(x, y) + k\,\frac{\partial (x, y)}{\partial y} + \ldots\right\}$$

Hence, $$f(x+h, y+k) = f(x, y) + h\,\frac{\partial f}{\partial x} + k\,\frac{\partial f}{\partial y} + \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\,\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right) + \ldots \qquad \ldots(1)$$

In symbols we write it as $f(x+h, y+k) = f(x, y) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f + \ldots$

Taking $x = a$ and $y = b$, (1) becomes

$$f(a+h, b+k) = f(a, b) + [hf_x(a, b) + kf_y(a, b)] + \frac{1}{2!}[h^2 f_{xx}(a, b) + 2hkf_{xy}(a, b) + k^2 f_{yy}(a, b)] + \ldots$$

*See footnote on page 145.

Putting $a + h = x$ and $b + k = y$ so that $h = x - a$, $k = y - b$, we get

$$\mathbf{f(x, y) = f(a, b) + [(x - a)\, f_x\,(a, b) + (y - b)\, f_y\,(a, b)]}$$
$$\mathbf{+ \frac{1}{2!}\,[(x - a)^2 f_{xx}\,(a, b) + 2(x - a)(y - b)\, f_{xy}\,(a, b) + (y - b)^2 f_{yy}(a, b)] + ...} \quad ...(2)$$

This is Taylor's expansion of f(x, y) in powers of (x − a) and (y − b). It is used to expand f(x, y) in the neighbourhood of (a, b).

Cor. Putting $a = 0$, $b = 0$, in (2), we get

$$\mathbf{f(x, y) = f(0, 0) + [xf_x(0, 0) + yf_y\,(0, 0)] + \frac{1}{2!}\,[x^2 f_{xx}\,(0, 0) + 2xy\, f_{xy}\,(0, 0) + y^2 f_{yy}\,(0, 0)] + ...} \quad ...(3)$$

This is Maclaurin's expansion of $f(x, y)$.

Example 5.32. *Expand $e^x \log (1 + y)$ in powers of x and y upto terms of third degree.*

(V.T.U., 2010 ; P.T.U., 2009 ; J.N.T.U., 2006)

Solution. Here

$$\begin{aligned}
f(x, y) &= e^x \log (1 + y) & \therefore\quad f(0, 0) &= 0\\
f_x(x, y) &= e^x \log (1 + y) & f_x(0, 0) &= 0\\
f_y(x, y) &= e^x \frac{1}{1+y} & f_y(0, 0) &= 1\\
f_{xx}(x, y) &= e^x \log (1 + y) & f_{xx}(0, 0) &= 0\\
f_{xy}(x, y) &= e^x \frac{1}{1+y} & f_{xy}(0, 0) &= 1\\
f_{yy}(x, y) &= -e^x (1 + y)^{-2} & f_{yy}(0, 0) &= -1\\
f_{xxx}(x, y) &= e^x \log (1 + y) & f_{xxx}(0, 0) &= 0\\
f_{xxy}(x, y) &= e^x \frac{1}{1+y} & f_{xxy}(0, 0) &= 1\\
f_{xyy}(x, y) &= -e^x (1 + y)^{-2} & f_{xyy}(0, 0) &= -1\\
f_{yyy}(x, y) &= 2e^x (1 + y)^{-3} & f_{yyy}(0, 0) &= 2
\end{aligned}$$

Now Maclaurin's expansion of $f(x, y)$ gives

$$f(x, y) = f(0, 0) + xf_x(0, 0) + yf_y(0, 0) + \frac{1}{2!}\{x^2 f_{xx}(0, 0) + 2xy\, f_{xy}(0, 0) + y^2 f_{yy}(0, 0)\}$$
$$+ \frac{1}{3!}\{x^3 f_{xxx}(0, 0) + 3x^2 y f_{xxy}(0, 0) + 3xy^2 f_{xyy}(0, 0) + y^3 f_{yyy}(0, 0)\} + ...$$

$$\therefore \quad e^x \log (1 + y) = 0 + x(0) + y(1) + \frac{1}{2!}\{x^2 (0) + 2xy(1) + y^2 (-1)\}$$
$$+ \frac{1}{3!}\{x^3(0) + 3x^2y(1) + 3xy^2(-1) + y^3(2)\} + ...$$
$$= y + xy - \frac{1}{2}y^2 + \frac{1}{2}(x^2y - xy^2) + \frac{1}{3}y^3 + ...$$

Example 5.33. *Expand $x^2y + 3y - 2$ in powers of $(x - 1)$ and $(y + 2)$ using Taylor's theorem.*

(P.T.U., 2010 ; V.T.U., 2008 ; U.P.T.U., 2006 ; Anna, 2005)

Solution. Taylor's expansion of $f(x, y)$ in powers of $(x - a)$ and $(x - b)$ is given by

$$f(x, y) = f(a, b) + [(x - a) f_x(a, b) + (y - b) f_y(a, b)] + \frac{1}{2!}[(x - a)^2 f_{xx}(a, b)$$
$$+ 2(x - a)(y - b) f_{xy}(a, b) + (y - b)^2 f_{yy}(a, b)] + \frac{1}{3!}[(x - a)^3 f_{xxx}(a, b)$$
$$+ 3(x - a)^2 (y - b) f_{xxy}(a, b) + 3(x - a)(y - b)^2 f_{xyy}(a, b)$$
$$+ (y - b)^3 f_{yyy}(a, b)] + ... \quad ...(i)$$

Hence $a = 1$, $b = -2$ and $f(x, y) = x^2y + 3y - 2$

$\therefore$ $f(1, -2) = -10, f_x = 2xy, f_x(1, -2) = -4; f_y = x^2 + 3, f_y(1, -2) = 4; f_{xx} = 2y,$
$f_{xx}(1, -2) = -4; f_{xy} = 2x, f_{xy}(1, -2) = 2; f_{yy} = 0, f_{yy}(1, -2) = 0; f_{xxx} = 0, f_{xxx} = (1, -2) = 0;$
$f_{xxy}(1, -2) = 2, f_{xyy}(1, -2) = 0, f_{yyy}(1, -2) = 0$

All partial derivatives of higher order vanish.

Substituting these in (*i*), we get

$$x^2y + 3y - 2 = -10 + [(x-1)(-4) + (y+2)4] + \frac{1}{2}[(x-1)^2(-4) + 2(x-1)(y+2)(2)$$
$$+ (y+2)^2(0)] + \frac{1}{6}[(x-1)^3(0) + 3(x-1)^2(y+2)(2) + 3(x-1)(y+2)^2(0) + (y+2)^3(0)]$$
$$= -10 - 4(x-1) + 4(y+2) - 2(x-1)^2 + 2(x-1)(y+2) + (x-1)^2(y+2).$$

Example 5.34. *Expand $f(x, y) = \tan^{-1}(y/x)$ in powers of $(x-1)$ and $(y-1)$ upto third-degree terms. Hence compute $f(1.1, 0.9)$ approximately.* *(V.T.U., 2010 ; J.N.T.U., 2006 ; U.P.T.U., 2006)*

Solution. Here $a = 1, b = 1$ and $f(1, 1) = \tan^{-1}(1) = \pi/4$.

$$f_x = \frac{-y}{x^2+y^2} \qquad f_x(1,1) = -\frac{1}{2}; \qquad f_y = \frac{x}{x^2+y^2}, \qquad f_y(1,1) = \frac{1}{2}$$

$$f_{xx} = \frac{2xy}{(x^2+y^2)^2}, \qquad f_{xx}(1,1) = \frac{1}{2}; \qquad f_{xy} = \frac{y^2-x^2}{(x^2+y^2)^2}, \qquad f_{xy}(1,1) = 0$$

$$f_{yy} = \frac{-2xy}{(x^2+y^2)^2}, \qquad f_{yy}(1,1) = -\frac{1}{2};$$

$$f_{xxx} = \frac{2y^3 - 6x^2y}{(x^2+y^2)^3}, \qquad f_{xxx}(1,1) = -\frac{1}{2}; \qquad f_{xxy} = \frac{2x^3 - 6xy^2}{(x^3+y^2)^3}, \qquad f_{xxy}(1,1) = -\frac{1}{2}$$

$$f_{xyy} = \frac{6x^2y - 2y^3}{(x^2+y^2)^3}, \qquad f_{xyy}(1,1) = \frac{1}{2}; \qquad f_{yyy} = \frac{6xy^2 - 2x^3}{(x^2+y^2)^3}, \qquad f_{yyy}(1,1) = \frac{1}{2}$$

Taylor's expansion of $f(x, y)$ in powers of $(x-1)$ and $(y-1)$ is given by

$$f(x,y) = f(1,1) + \frac{1}{1!}\{(x-1)f_x(1,1) + (y-1)f_y(1,1)\} + \frac{1}{2!}\{(x-1)^2 f_{xx}(1,1) + 2(x-1)(y-1)$$
$$f_{xy}(1,1) + (y-1)^2 f_{yy}(1,1) + \frac{1}{3!}\{(x-1)^3 f_{xxx}(1,1) + 3(x-1)^2(y-1)f_{xxy}(1,1)$$
$$+ 3(x-1)(y-1)^2 f_{xyy}(1,1) + (y-1)^3 f_{yyy}(1,1)\} + \dots$$

$$\therefore \quad \tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{4} + \left\{(x-1)\left(-\frac{1}{2}\right) + (y-1)\frac{1}{2}\right\} + \frac{1}{2!}\left\{(x-1)^2\frac{1}{2} + 2(x-1)(y-1)(0) + (y-1)^2\left(-\frac{1}{2}\right)\right\}$$
$$+ \frac{1}{3!}\left\{(x-1)^3\left(-\frac{1}{2}\right) + 3(x-1)^2(y-1)\left(-\frac{1}{2}\right) + 3(x-1)(y-1)^2\frac{1}{2} + (y-1)^3\frac{1}{2}\right\} + \dots$$
$$= \frac{\pi}{4} - \frac{1}{2}\{(x-1)-(y-1)\} + \frac{1}{4}\{(x-1)^2-(y-1)^2\} - \frac{1}{12}\{(x-1)^3 + 3(x-1)^2(y-1)$$
$$- 3(x-1)(y-1)^2 - (y-1)^3\} + \dots$$

Putting $x = 1.1$ and $y = 0.9$, we get

$$f(1.1, 0.9) = \frac{\pi}{4} - \frac{1}{2}(0.2) + \frac{1}{4}(0) - \frac{1}{12}\{(0.1)^3 - 3(0.1)^3 - 3(0.1)^3 - (-0.1)^3\}$$
$$= 0.7854 - 0.1000 + 0.0003 = 0.6857.$$

5.10 (1) ERRORS AND APPROXIMATIONS

Let $f(x, y)$ be a continuous function of x and y. If δx and δy be the increments of x and y, then the new value of $f(x, y)$ will be $f(x + \delta x, y + \delta y)$. Hence

$$\delta f = f(x + \delta x, y + \delta y) - f(x, y).$$

Expanding $f(x + \delta x, y + \delta y)$ by Taylor's theorem and supposing δx, δy to be so small that their products, squares and higher powers can be neglected, we get

$$\delta \mathbf{f} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \delta \mathbf{x} + \frac{\partial \mathbf{f}}{\partial \mathbf{y}} \delta \mathbf{y}, \text{ approximately.}$$

Similarly if f be a function of several variables $x, y, z, t, \ldots$, then

$$\delta f = \frac{\partial f}{\partial x} \delta x + \frac{\partial f}{\partial y} \delta y + \frac{\partial f}{\partial z} \delta z + \frac{\partial f}{\partial t} \delta t + \ldots \text{ approximately.}$$

These formulae are very useful in correcting the effect of small errors in measured quantities.

(2) Total Differential

If u is a function of two variables x and y, the *total differential* of u is defined as

$$\mathbf{du} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \mathbf{dx} + \frac{\partial \mathbf{u}}{\partial \mathbf{y}} \mathbf{dy} \qquad \ldots(1)$$

The differentials dx and dy are respectively the increments δx and δy in x and y. If x and y are not independent variables but functions of another variable t even then the formula (1) holds and we write $dx = \frac{dx}{dt} dt$ and $dy = \frac{dy}{dt} dt$. Similar definition can be given for a function of three or more variables.

Example 5.35. *The diameter and altitude of a can in the shape of a right circular cylinder are measured as 4 cm and 6 cm respectively. The possible error in each measurement is 0.1 cm. Find approximately the maximum possible error in the values computed for the volume and the lateral surface.*

Solution. Let x be the diameter and y the height of the can. Then its volume $V = \frac{\pi}{4} x^2 y$

$$\therefore \qquad \delta V = \frac{\partial V}{\partial x} \delta x + \frac{\partial V}{\partial y} \delta y = \frac{\pi}{4} (2xy\delta x + x^2 \delta y)$$

When $x = 4$ cm., $y = 6$ cm. and $\delta x = \delta y = 0.1$ cm.

$$\therefore \qquad \delta V = \frac{\pi}{4}(2 \times 4 \times 6 \times 0.1 + 4^2 \times 0.1) = 1.6\pi \text{ cm}^3$$

Also its lateral surface $S = \pi xy$

$$\therefore \qquad \delta S = \pi(y\, \delta x + x\, \delta y)$$

When $x = 4$ cm., $y = 6$ cm. and $\delta x = \delta y = 0.1$ cm., we have $\delta S = \pi(6 \times 0.1 + 4 \times 0.1) = \pi$ cm^2.

Example 5.36. *The period of a simple pendulum is $T = 2\pi\sqrt{(l/g)}$, find the maximum error in T due to the possible error upto 1% in l and 2.5% in g.* (U.P.T.U., 2004)

Solution. We have $T = 2\pi\sqrt{(l/g)}$

or
$$\log T = \log 2\pi + \frac{1}{2} \log l - \frac{1}{2} \log g$$

$$\therefore \qquad \frac{1}{T} \delta \mathrm{T} = 0 + \frac{1}{2} \frac{1}{l} \delta l - \frac{1}{2} \frac{1}{g} \delta g$$

$$\frac{\delta T}{T} 100 = \frac{1}{2}\left(\frac{\delta l}{l} 100 - \frac{\delta g}{g} 100\right) = \frac{1}{2}(1 \pm 2.5) = 1.75 \text{ or } -0.75$$

Thus the maximum error in $T = 1.75\%$

Example 5.37. *A balloon is in the form of right circular cylinder of radius 1.5 m and length 4 m and is surmounted by hemispherical ends. If the radius is increased by 0.01 m and length by 0.05 m, find the percentage change in the volume of balloon.* (U.P.T.U., 2005)

Solution. Let the volume of the balloon (Fig. 5.3) be V, so that

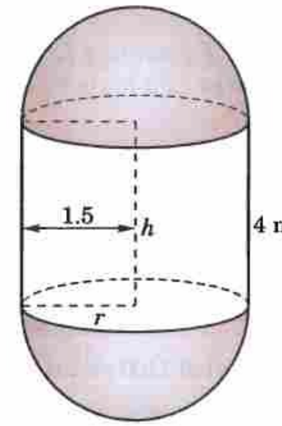

Fig. 5.3

$$V = \pi r^2 h + \frac{2}{3}\pi r^3 + \frac{2}{3}\pi r^3 = \pi r^2 h + \frac{4}{3}\pi r^3$$

$$\therefore \quad \delta V = 2\pi r \delta r h + \pi r^2 \delta h + \frac{4}{3}\pi 3r^2 \delta r$$

or
$$\frac{\delta V}{V} = \frac{\pi r[2h\delta r + r\delta h + 4r\delta r]}{\pi r^2 h + \frac{4}{3}\pi r^3}$$

$$= \frac{2(h+2r)\,\delta r + r\delta h}{rh + \frac{4}{3}r^2} = \frac{2(4+3)(.01) + 1.5(.05)}{1.5 \times 4 + \frac{4}{3}(1.5)^2}$$

$$= \frac{0.14 + 0.075}{6+3} = \frac{0.215}{9}$$

Hence, the percentage change in $V = 100\,\dfrac{\delta V}{V} = \dfrac{21.5}{9} = 2.39\%$

Example 5.38. *In estimating the cost of a pile of bricks measured as 2 m × 15 m × 1.2 m, the tape is stretched 1% beyond the standard length. If the count is 450 bricks to 1 cu. m. and bricks cost ₹ 530 per 1000, find the approximate error in the cost.* (V.T.U., 2001)

Solution. Let x, y and z m be the length, breadth and height of the pile so that its volume $V = xyz$

or
$$\log V = \log x + \log y + \log z \quad \therefore \quad \frac{\delta V}{V} = \frac{\delta x}{x} + \frac{\delta y}{y} + \frac{\delta z}{z}$$

Since
$$V = 2 \times 15 \times 1.2 = 36 \text{ m}^3, \text{ and } \frac{\delta x}{x} = \frac{\delta y}{y} = \frac{\delta z}{z} = \frac{1}{100}$$

$$\therefore \quad \delta V = 36\left(\frac{3}{100}\right) = 1.08 \text{ m}^3.$$

Number of bricks in $\delta V = 1.08 \times 450 = 486$

Thus error in the cost $= 486 \times \dfrac{530}{1000} =$ ₹ 257.58 which is a loss to the brick seller.

Example 5.39. *The height h and semi-vertical angle α of a cone are measured and from them A, the total area of the surface of the cone including the base is calculated. If h and α are in error by small quantities δh and δα respectively, find the corresponding error in the area. Show further that if α = π/6, an error of + 1% in h will be approximately compensated by an error of – 0.33 degrees in α.*

Solution. If r be the base radius and l the slant height of the cone, (Fig. 5.4), then total area

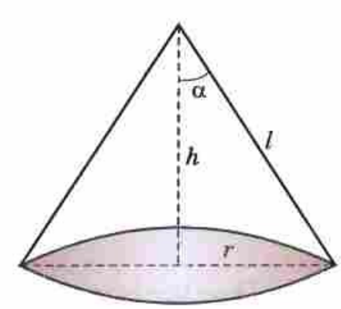

Fig. 5.4

$$A = \text{area of base} + \text{area of curved surface}$$
$$= \pi r^2 + \pi r l = \pi r(r + l)$$
$$= \pi h \tan\alpha\,(h\tan\alpha + h\sec\alpha)$$
$$= \pi h^2(\tan^2\alpha + \tan\alpha\sec\alpha)$$

$$\therefore \quad \delta A = \frac{\delta A}{\delta h}\delta h + \frac{\delta A}{\delta\alpha}\delta\alpha$$
$$= 2\pi h(\tan^2\alpha + \tan\alpha\sec\alpha)\delta h$$
$$+ \pi h^2(2\tan\alpha\sec^2\alpha + \sec^3\alpha + \tan\alpha\sec\alpha\tan\alpha)\delta\alpha$$

which gives the error in the area A.

Putting $\delta h = h/100$ and $\alpha = \pi/6$, we get

$$\delta A = 2\pi h\left[\left(\frac{1}{\sqrt{3}}\right)^2 + \frac{1}{\sqrt{3}}\cdot\frac{2}{\sqrt{3}}\right]\frac{h}{100} + \pi h^2\left[2.\frac{1}{\sqrt{3}}\cdot\frac{4}{3} + \frac{8}{3\sqrt{3}} + \frac{1}{\sqrt{3}}\cdot\frac{2}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}}\right]\delta\alpha$$

$$= \frac{2\pi h^2}{100} + 2\sqrt{3}\pi h^2 \ \delta\alpha$$

The error in h will be compensated by the error in α, when

$$\delta A = 0 \text{ i.e., } \frac{2\pi h^2}{100} + 2\sqrt{3}\pi h^2 \delta\alpha = 0$$

or $$\delta\alpha = -\frac{1}{100\sqrt{3}} \text{ radians} = -\frac{.01}{1.732} \times 57.3° = -0.33°.$$

Example 5.40. *Show that the approximate change in the angle A of a triangle ABC due to small changes δa, δb, δc in the sides a, b, c respectively, is given by*

$$\delta A = \frac{a}{2\Delta} (\delta a - \delta b \cos C - \delta c \cos B)$$

where Δ is the area of the triangle. Verify that $\delta A + \delta B + \delta C = 0$.

Solution. We know that $a^2 = b^2 + c^2 - 2bc \cos A$

so that $$2a\delta a = 2b\delta b + 2c\delta c - 2(c\delta b \cos A - b\delta c \cos A + bc \sin A \ \delta A)$$

$\therefore$ $$bc \sin A \ \delta A = a\delta a - (b - c \cos A)\ \delta b - (c - b \cos A)\ \delta c$$

or $$2\ \Delta\delta A = a\delta a - (c \cos A + a \cos C - c \cos A)\ \delta b - (a \cos B + b \cos A - b \cos A)\ \delta c$$

[$\because$ $b = c \cos A + a \cos C$ etc. ...(*i*)]

$$= a\delta a - a \cos C \ \delta b - a \cos B \ \delta c$$

or $$\delta A = \frac{a}{2\Delta} (\delta a - \delta b \cos C - \delta c \cos B)$$

By symmetry, we have

$$\delta B = \frac{b}{2\Delta} (\delta b - \delta c \cos A - \delta a \cos C)$$

$$\delta C = \frac{c}{2\Delta} (\delta c - \delta a \cos B - \delta b \cos A)$$

$\therefore$ $$\delta A + \delta B + \delta C = \frac{1}{2\Delta} (a - b \cos C - c \cos B)\ \delta a + (b - c \cos A - a \cos C)\ \delta b + (c - a \cos B - b \cos A)]$$

$$= \frac{1}{2\Delta} [(a - a)\ \delta a + (b - b)\ \delta b + (c - c)\ \delta c] = 0$$ [By (*i*)]

Example 5.41. *If the sides of a plane triangle ABC vary in such a way that its circumradius remains constant, prove that* $\dfrac{da}{\cos A} + \dfrac{db}{\cos B} + \dfrac{dc}{\cos C} = 0.$

Solution. The circumradius R of ΔABC is given by

$$R = \frac{a}{2 \sin A} = \frac{b}{2 \sin B} = \frac{c}{2 \sin C}$$

$\therefore$ $$a = 2R \sin A$$ [$\because$ R is constant

Taking differentials, $da = 2R \cos A \ dA$ or $\dfrac{da}{\cos A} = 2RdA$

Similarly, $$\frac{db}{\cos B} = 2RdB, \ \frac{dc}{\cos C} = 2RdC$$

$\therefore$ $$\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 2R\ (dA + dB + dC)$$

Now $A + B + C = \pi$, gives $dA + dB + dC = 0$ ...(*i*)

Thus $$\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 0$$ [By (*i*)]

PROBLEMS 5.9

1. Expand the following functions as far as terms of third degree :
   (i) $\sin x \cos y$ (V.T.U., 2009) (ii) $e^x \sin y$ at $(-1, \pi/4)$ (Anna, 2009)
   (iii) $xy^2 + \cos xy$ about $(1, \pi/2)$. (Hissar, 2005 S ; V.T.U., 2003)
2. Expand $f(x, y) = x^y$ in powers of $(x-1)$ and $(y-1)$. (U.T.U., 2009)
3. If $f(x, y) = \tan^{-1} xy$, compute $f(0.9, -1.2)$ approximately.
4. If the kinetic energy $k = wv^2/2g$, find approximately the change in the kinetic energy as w changes from 49 to 49.5 and v changes from 1600 to 1590. (V.T.U., 2006)
5. Find the possible percentage error in computing the resistance r from the formula $1/r = 1/r_1 + 1/r_2$, if r_1, r_2 are both in error by 2%.
6. The voltage V across a resistor is measured with an error h, and the resistance R is measured with an error k. Show that the error in calculating the power $W(V, R) = V^2/R$ generated in the resistor, is $VR^{-2}(2Rh - Vk)$. (V.T.U., 2009)
7. Find the percentage error in the area of an ellipse if one per cent error is made in measuring the major and minor axes. (V.T.U., 2011)
8. The time of oscillation of a simple pendulum is given by the equation $T = 2\pi\sqrt{(l/g)}$. In an experiment carried out to find the value of g, errors of 1.5% and 0.5% are possible in the values of l and T respectively. Show that the error in the calculated value of g is 0.5%. (Cochin, 2005)
9. If $pv^2 = k$ and the relative errors in p and v are respectively 0.05 and 0.025, show that the error in k is 10%. (Mysore, 1999)
10. If the H.P. required to propel a steamer varies as the cube of the velocity and square of the length. Prove that a 3% increase in velocity and 4% increase in length will require an increase of about 17% in H.P.
11. The range R of a projectile which starts with a velocity v at an elevation α is given by $R = (v^2 \sin 2\alpha)/g$. Find the percentage error in R due to an error of 1% in v and an error of $\frac{1}{2}$% in α. (Kurukshetra, 2009)
12. In estimating the cost of a pile of bricks measured as 6 m × 50 m × 4 m, the tape is stretched 1% beyond the standard length. If the count is 12 bricks in 1 m³ and bricks cost ₹ 100 per 1000, find the approximate error in the cost. (U.T.U., 2010 ; U.P.T.U., 2005)
13. The deflection at the centre of a rod of length l and diameter d supported at its ends, loaded at the centre with a weight w varies at wl^3d^{-4}. What is the increase in the deflection corresponding to p% increase in w, q% decrease in l and r% increase in d ?
14. The work that must be done to propel a ship of displacement D for a distance s in time t is proportional to $(s^2D^{2/3}/t^2)$. Find approximately the increase of work necessary when the displacement is increased by 1%, the time is diminished by 1% and the distance diminished by 2%.
15. The indicated horse power I of an engine is calculated from the formula $I = PLAN/33{,}000$, where $A = \pi d^2/4$. Assuming that error of r per cent may have been made in measuring P, L, N and d, find the greatest possible error in I.
16. The torsional rigidity of a length of wire is obtained from the formula $N = 8\pi Il/t^2r^4$. If l is decreased by 2%, r is increased by 2%, t is increased by 1.5%, show that the value of N is diminished by 13% approximately. (V.T.U., 2003)
17. If $x^2 + y^2 + z^2 - 2xyz = 1$, show that $\dfrac{dx}{\sqrt{(1-x^2)}} + \dfrac{dy}{\sqrt{(1-y^2)}} + \dfrac{dz}{\sqrt{(1-z^2)}} = 0$.
    [**Hint.** $2(x - yz)\,dx + 2(y - zx)\,dy + 2(z - xy)\,dz = 0$. Also $(x - yz)^2 = (1 - y^2)(1 - z^2)$, ...]

5.11 (1) MAXIMA AND MINIMA OF FUNCTIONS OF TWO VARIABLES

Def. *A function $f(x, y)$ is said to have a* **maximum** *or* **minimum** *at $x = a$, $y = b$, according as*

$$f(a + h, b + k) < \text{ or } > f(a, b),$$

for all positive or negative small values of h and k.

In other words, if $\Delta = f(a + h, b + k) - f(a, b)$, *is of the same sign for all small values of h, k, and if this sign is negative, then $f(a, b)$ is a maximum. If this sign is positive, $f(a, b)$ is a minimum.*

Considering $z = f(x, y)$ as a surface, maximum value of z occurs at the top of an elevation (*e.g.*, a dome) from which the surface descends in every direction and a minimum value occurs at the bottom of a depression (*e.g.*, a bowl) from which the surface ascends in every direction. Sometimes the maximum or minimum value may form a *ridge* such that the surface descends or ascends in all directions except that of the ridge. Besides these, we have such a point of the surface, where the tangent plane is horizontal and the surface looks like leather seat on the horse's back [Fig. 5.5 (*c*)] which falls for displacement in certain directions and rises for displacements in other directions. Such a point is called a **saddle point.**

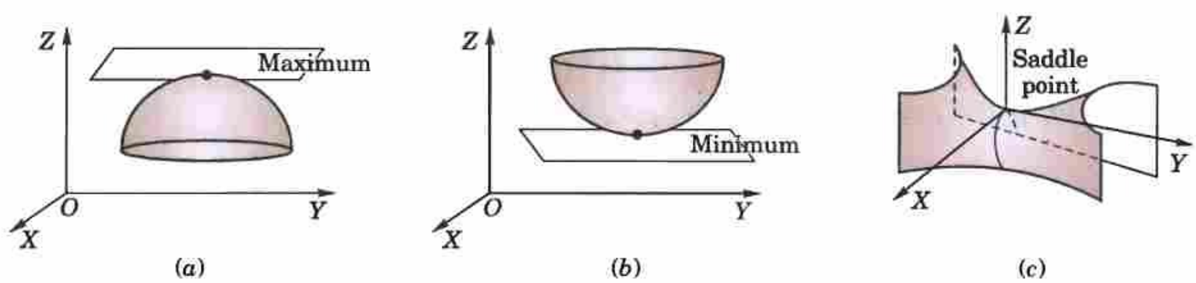

Fig. 5.5

Note. *A maximum or minimum value of a function is called its* **extreme value.**

(2) Conditions for f(x, y) to be maximum or minimum

Using Taylor's theorem page 235, we have $\Delta = f(a + h, b + k) - f(a, b)$

$$= \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right)_{a,b} + \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right) + \ldots \quad \ldots(i)$$

For small values of h amd k, the second and higher order terms are still smaller and hence may be neglected. Thus

$$\text{sign of } \Delta = \text{sign of } [hf_x(a, b) + kf_y(a, b)].$$

Taking $h = 0$ we see that the right hand side changes sign when k changes sign. Hence $f(x, y)$ cannot have a maximum or a minimum at (a, b) unless $f_y(a, b) = 0$.

Similarly taking $k = 0$, we find that $f(x, y)$ cannot have a maximum or minimum at (a, b) unless $f_x(a, b) = 0$.

Hence *the necessary conditions for $f(x, y)$ to have a maximum or minimum at (a, b) are that*

$$\mathbf{f_x(a, b) = 0,\ f_y(a, b) = 0.}$$

If these conditions are satisfied, then for small value of h and k, (i) gives

$$\text{sign of } \Delta = \text{sign of } \left[\frac{1}{2!}(h^2r + 2hks + k^2t)\right] \text{ where } r = f_{xx}(a, b),\ s = f_{xy}(a, b) \text{ and } t = f_{yy}(a, b).$$

Now $\quad h^2r + 2hks + k^2t = \frac{1}{r}\left[(h^2r^2 + 2hkrs + k^2rt)\right] = \frac{1}{r}\left[(hr + ks)^2 + k^2(rt - s^2)\right]$

Thus sign of Δ = sign of $\frac{1}{2r}\left\{(hr + ks)^2 + k^2(rt - s^2)\right\} \quad \ldots(ii)$

In (ii), $(hr + ks)^2$ is always positive and $k^2(rt - s^2)$ will be positive if $rt - s^2 > 0$. In this case, Δ will have the same sign as that of r for all values of h and k.

Hence if $rt - s^2 > 0$, then $f(x, y)$ *has a maximum or a minimum at (a, b) according as $r <$ or > 0.*

If $rt - s^2 < 0$, then Δ will change with h and k and hence there is no maximum or minimum at (a, b) *i.e.*, it is a *saddle point*.

If $rt - s^2 = 0$, further investigation is required to find whether there is a maximum or minimum at (a, b) or not.

Note. Stationary value. *$f(a, b)$ is said to be a stationary value of $f(x, y)$, if $f_x(a, b) = 0$* and *$f_y(a, b) = 0$ i.e.* the function is stationary at (a, b).

Thus every extreme value is a stationary value but the converse may not be true.

(3) Working rule to find the maximum and minimum values of f(x, y)

1. Find $\partial f/\partial x$ and $\partial f/\partial y$ *and equate each to zero. Solve these as simultaneous equations in x and y.* Let (a, b), *(c, d), ... be the pairs of values.*
2. *Calculate the value of $r = \partial^2 f/\partial x^2$, $s = \partial^2 f/\partial x \partial y$, $t = \partial^2 f/\partial y^2$ for each pair of values.*

3. *(i) If $rt - s^2 > 0$ and $r < 0$ at (a, b), $f(a, b)$ is a max. value.*
(ii) If $rt - s^2 > 0$ and $r > 0$ at (a, b), $f(a, b)$ is a min. value.
(iii) If $rt - s^2 < 0$ at (a, b), $f(a, b)$ is not an extreme value, i.e., (a, b) is a saddle point.
(iv) If $rt - s^2 = 0$ at (a, b), the case is doubtful and needs further investigation.

Similarly examine the other pairs of values one by one.

Example 5.42. *Examine the following function for extreme values :*

$$f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2.$$ *(J.N.T.U., 2003)*

Solution. We have $f_x = 4x^3 - 4x + 4y$; $f_y = 4y^3 + 4x - 4y$

and $r = f_{xx} = 12x^2 - 4$, $s = f_{xy} = 4$, $t = f_{yy} = 12y^2 - 4$ ...(i)

Now $f_x = 0, f_y = 0$ give $x^3 - x + y = 0$, ...(i) $y^3 + x - y = 0$ ...(ii)

Adding these, we get $4(x^3 + y^3) = 0$ or $y = -x$.

Putting $y = -x$ in (i), we obtain $x^3 - 2x = 0$, *i.e.* $x = \sqrt{2}, -\sqrt{2}, 0$.

$\therefore$ Corresponding values of y are $-\sqrt{2}, \sqrt{2}, 0$.

At $(\sqrt{2}, -\sqrt{2})$, $rt - s^2 = 20 \times 20 - 4^2 = +$ ve and r is also + ve. Hence $f(\sqrt{2}, -\sqrt{2})$ is a minimum value.

At $(-\sqrt{2}, \sqrt{2})$ also both $rt - s^2$ and r are + ve.

Hence $f(-\sqrt{2}, \sqrt{2})$, is also a minimum value.

At (0, 0), $rt - s^2 = 0$ and, therefore, further investigation is needed.

Now $f(0, 0) = 0$ and for points along the x-axis, where $y = 0$, $f(x, y) = x^4 - 2x^2 = x^2(x^2 - 2)$, which is negative for points in the neighbourhood of the origin.

Again for points along the line $y = x$, $f(x, y) = 2x^4$ which is positive.

Thus in the neighbourhood of (0, 0) there are points where $f(x, y) < f(0, 0)$ and there are points where $f(x, y) > f(0, 0)$.

Hence $f(0, 0)$ is not an extreme value *i.e.*, it is a *saddle point*.

Example 5.43. *Discuss the maxima and minima of $f(x, y) = x^3y^2(1 - x - y)$.*

(Anna, 2009 ; J.N.T.U., 2006 ; Bhopal, 2002)

Solution. We have $f_x = 3x^2y^2 - 4x^3y^2 - 3x^2y^3$; $f_y = 2x^3y - 2x^4y - 3x^3y^2$

and $r = f_{xx} = 6xy^2 - 12x^2y^2 - 6xy^3$; $s = f_{xy} = 6x^2y - 8x^3y - 9x^2y^2$; $t = f_{yy} = 2x^3 - 2x^4 - 6x^3y$.

When $f_x = 0, f_y = 0$, we have $x^2y^2(3 - 4x - 3y) = 0$, $x^3y(2 - 2x - 3y) = 0$

Solving these, the stationary points are (1/2, 1/3), (0, 0).

Now $rt - s^2 = x^4y^2[12(1 - 2x - y)(1 - x - 3y) - (6 - 8x - 9y)^2]$

At (1/2, 1/3), $rt - s^2 = \frac{1}{16} \cdot \frac{1}{9}\left[12\left(1 - 1 - \frac{1}{3}\right)\left(1 - \frac{1}{2} - 1\right) - (6 - 4 - 3)^2\right] = \frac{1}{14} > 0$

Also $r = 6\left(\frac{1}{2} \cdot \frac{1}{9} - \frac{2}{4} \cdot \frac{1}{9} - \frac{1}{2} \cdot \frac{1}{27}\right) = -\frac{1}{9} < 0$

Hence $f(x, y)$ has a maximum at (1/2, 1/3) and maximum value $= \frac{1}{8} \cdot \frac{1}{9}\left(1 - \frac{1}{2} - \frac{1}{3}\right) = \frac{1}{432}$.

At (0, 0), $rt - s^2 = 0$ and therefore further investigation is needed.

For points along the line $y = x$, $f(x, y) = x^5(1 - 2x)$ which is positive for $x = 0.1$ and negative for $x = -0.1$ *i.e.*, in the neighbourhood of (0, 0) there are points where $f(x, y) > f(0, 0)$ and there are points where $f(x, y) < f(0, 0)$. Hence $f(0, 0)$ is not an extreme value.

Example 5.44. *In a plane triangle, find the maximum value of cos A cos B cos C.*

(V.T.U., 2010 ; Nagpur, 2009 ; Anna, 2005 S)

Solution. We have $A + B + C = \pi$ so that $C = \pi - (A + B)$.

$$\cos A \cos B \cos C = \cos A \cos B \cos [\pi - (A + B)]$$
$$= -\cos A \cos B \cos (A + B) = f(A, B), \text{ say.}$$

We get $$\frac{\partial f}{\partial A} = \cos B\,[\sin A \cos(A+B) + \cos A \sin(A+B)]$$
$$= \cos B \sin(2A+B)$$

and $$\frac{\partial f}{\partial B} = \cos A \sin(A+2B)$$

$$\frac{\partial f}{\partial A} = 0,\ \frac{\partial f}{\partial B} = 0 \quad \text{only when } A = B = \pi/3.$$

Also $$r = \frac{\partial^2 f}{\partial A^2} = 2\cos B\cos(2A+B),\ t = \frac{\partial^2 f}{\partial B^2} = 2\cos A\cos(A+2B)$$

$$s = \frac{\partial^2 f}{\partial A \partial B} = -\sin B \sin(2A+B) + \cos B\cos(2A+B) = \cos(2A+2B)$$

When $A = B = \pi/3$, $r = -1$, $s = -1/2$, $t = -1$ so that $rt - s^2 = 3/4$.

These show that $f(A, B)$ is maximum for $A = B = \pi/3$.

Then $C = \pi - (A+B) = \pi/3$.

Hence $\cos A \cos B \cos C$ is maximum when each of the angles is $\pi/3$ *i.e.*, triangle is equilateral and its maximum value = 1/8.

5.12 LAGRANGE'S METHOD OF UNDERTERMINED MULTIPLIERS

Sometimes it is required to find the stationary values of a function of several variables which are not all independent but are connected by some given relations. Ordinarily, we try to convert the given function to the one, having least number of independent variables with the help of given relations. Then solve it by the above method. When such a procedure becomes impracticable, Lagrange's method* proves very convenient. Now we explain this method.

Let $$u = f(x, y, z) \qquad ...(1)$$

be a function of three variables x, y, z which are connected by the relation.

$$\phi(x, y, z) = 0 \qquad ...(2)$$

For u to have stationary values, it is necessary that

$$\partial u/\partial x = 0,\ \partial u/\partial y = 0,\ \partial u/\partial z = 0.$$

$$\therefore \quad \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = du = 0 \qquad ...(3)$$

Also differentiating (2), we get $\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = d\phi = 0$ ...(4)

Multiply (4) by a parameter λ and add to (3). Then

$$\left(\frac{\partial u}{\partial x} + \lambda\frac{\partial \phi}{\partial x}\right)dx + \left(\frac{\partial u}{\partial y} + \lambda\frac{\partial \phi}{\partial y}\right)dy + \left(\frac{\partial u}{\partial z} + \lambda\frac{\partial \phi}{\partial z}\right)dz = 0$$

This equation will be satisfied if $\frac{\partial u}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0,\ \frac{\partial u}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0,\ \frac{\partial u}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 0.$

These three equations together with (2) will determine the values of x, y, z and λ for which u is stationary.

Working rule : 1. *Write* $F = f(x, y, z) + \lambda\phi(x, y, z)$

2. *Obtain the equations* $\frac{\partial F}{\partial x} = 0,\ \frac{\partial F}{\partial y} = 0,\ \frac{\partial F}{\partial z} = 0.$

3. *Solve the above equations together with* $\phi(x, y, z) = 0$.

The values of x, y, z so obtained will give the stationary value of $f(x, y, z)$.

Obs. Although the Lagrange's method is often very useful in application yet *the drawback is that we cannot determine the nature of the stationary point.* This can sometimes, be determined from physical considerations of the problem.

*See footnote page 142.

Example 5.45. *A rectangular box open at the top is to have volume of 32 cubic ft. Find the dimensions of the box requiring least material for its construction.* (Kurukshetra, 2006 ; P.T.U., 2006 ; U.P.T.U., 2005)

Solution. Let x, y and z ft. be the edges of the box and S be its surface.

Then $$S = xy + 2yz + 2zx \quad ...(i)$$

and $$xyz = 32 \quad ...(ii)$$

Eliminating z from (*i*) with the help of (*ii*), we get $S = xy + 2(y + x)\dfrac{32}{xy} = xy + 64\left(\dfrac{1}{x} + \dfrac{1}{y}\right)$

$\therefore$ $$\partial S/\partial x = y - 64/x^2 = 0 \quad \text{and} \quad \partial S/\partial y = x - 64/y^2 = 0.$$

Solving these, we get $x = y = 4$.

Now $r = \partial^2 S/\partial x^2 = 128/x^3$, $s = \partial^2 S/\partial x \partial y = 1$, $t = \partial^2 S/\partial y^2 = 128/y^3$.

At $x = y = 4$, $rt - s^2 = 2 \times 2 - 1 = +$ ve and r is also + ve.

Hence S is minimum for $x = y = 4$. Then from (*ii*), $z = 2$.

Otherwise (*by Lagrange's method*) :

Write $$F = xy + 2yz + 2zx + \lambda(xyz - 32)$$

Then $$\frac{\partial F}{\partial x} = y + 2z + \lambda yz = 0 \quad ...(iii)$$

$$\frac{\partial F}{\partial y} = x + 2z + \lambda zx = 0 \quad ...(iv)$$

$$\frac{\partial F}{\partial z} = 2y + 2x + \lambda xy = 0 \quad ...(v)$$

Multiplying (*iii*) by x and (*iv*) by y and subtracting, we get $2zx - 2zy = 0$ or $x = y$.

[The value $z = 0$ is neglected, as it will not satisfy (*ii*)]

Again multiplying (*iv*) by y and (*v*) by z and subtracting, we get $y = 2z$.

Hence the dimensions of the box are $x = y = 2z = 4$ ...(*vi*)

Now let us see what happens as z increases from a small value to a large one. When z is small, the box is flat with a large base showing that S is large. As z increases, the base of the box decreases rapidly and S also decreases. After a certain stage, S again starts increasing as z increases. Thus S must be a minimum at some intermediate stage which is given by (*vi*). Hence S is minimum when $x = y = 4$ ft and $z = 2$ ft.

Example 5.46. *Given $x + y + z = a$, find the maximum value of $x^m y^n z^p$.* (Anna, 2009)

Solution. Let $f(x, y, z) = x^m y^n z^p$ and $\phi(x, y, z) = x + y + z - a$.

Then $$F(x, y, z) = f(x, y, z) + \lambda\, \phi(x, y, z)$$
$$= x^m y^n z^p + \lambda(x + y + z - a).$$

For stationary values of F, $\dfrac{\partial F}{\partial x} = 0, \dfrac{\partial F}{\partial y} = 0, \dfrac{\partial F}{\partial z} = 0$

$\therefore$ $$mx^{m-1}y^n z^p + \lambda = 0,\ nx^m y^{n-1} z^p + \lambda = 0,\ px^m y^n z^{p-1} + \lambda = 0$$

or $$-\lambda = mx^{n-1}y^n z^p = nx^m y^{n-1} z^p = px^m y^n z^{p-1}$$

i.e. $$\frac{m}{x} = \frac{n}{y} = \frac{p}{z} = \frac{m+n+p}{x+y+z} = \frac{m+n+p}{a} \qquad [\because \ x + y + z = a$$

$\therefore$ The maximum value of f occurs when

$$x = am/(m + n + p),\ y = an/(m + n + p),\ z = ap/(m + n + p)$$

Hence the maximum value of $f(x, y, z) = \dfrac{a^{m+n+p} \cdot m^m \cdot n^n \cdot p^p}{(m+n+p)^{m+n+p}}$.

Example 5.47. *Find the maximum and minimum distances of the point (3, 4, 12) from the sphere $x^2 + y^2 + z^2 = 4$.* (U.T.U., 2010)

Solution. Let $P(x, y, z)$ be any point on the sphere and $A(3, 4, 12)$ the given point so that

$$AP^2 = (x - 3)^2 + (y - 4)^2 + (z - 12)^2 = f(x, y, z), \text{ say} \quad ...(i)$$

We have to find the maximum and minimum values of $f(x, y, z)$ subject to the condition

$$\phi(x, y, z) = x^2 + y^2 + z^2 - 4 = 0 \quad ...(ii)$$

Let $F(x, y, z) = f(x, y, z) + \lambda\phi(x, y, z)$

$$= (x-3)^2 + (y-4)^2 + (z-12)^2 + \lambda(x^2 + y^2 + z^2 - 4)$$

Then $$\frac{\partial F}{\partial x} = 2(x-3) + 2\lambda x, \ \frac{\partial F}{\partial y} = 2(y-4) + 2\lambda y, \ \frac{\partial F}{\partial z} = 2(z-12) + 2\lambda z$$

$\therefore$ $\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0$ and $\frac{\partial F}{\partial z} = 0$ give

$$x - 3 + \lambda x = 0, y - 4 + \lambda y = 0, z - 12 + \lambda z = 0 \quad ...(iii)$$

which give $$\lambda = -\frac{x-3}{x} = -\frac{y-4}{y} = -\frac{z-12}{z}$$

$$= \pm \frac{\sqrt{[(x-3)^2 + (y-4)^2 + (z-12)^2]}}{\sqrt{(x^2 + y^2 + z^2)}} = \pm \frac{\sqrt{f}}{1}$$

Substituting for λ in (iii), we get

$$x = \frac{3}{1+\lambda} = \frac{3}{1 \pm \sqrt{f}}, y = \frac{4}{1 \pm \sqrt{f}}, z = \frac{12}{1 \pm \sqrt{f}}$$

$\therefore$ $$x^2 + y^2 + z^2 = \frac{9 + 16 + 144}{(1 \pm \sqrt{f})^2} = \frac{169}{(1 \pm \sqrt{f})^2}$$

Using (ii), $$1 = \frac{169}{(1 \pm \sqrt{f})^2} \quad \text{or} \quad 1 \pm \sqrt{f} = \pm 13, \sqrt{f} = 12, 14.$$

[We have left out the negative values of $\sqrt{f}$, because $\sqrt{f} = AP$ is + ve by (i)]

Hence maximum $AP = 14$ and minimum $AP = 12$.

Example 5.48. *Show that the rectangular solid of maximum volume that can be inscribed in a sphere is a cube.* (*Kurukshetra, 2006 ; U.P.T.U., 2004*)

Solution. Let $2x$, $2y$, $2z$ be the length, breadth and height of the rectangular solid so that its volume

$$V = 8xyz \quad ...(i)$$

Let R be the radius of the sphere so that $x^2 + y^2 + z^2 = R^2$ ...(ii)

Then $F(x, y, z) = 8xyz + \lambda(x^2 + y^2 + z^2 - R^2)$

and $\partial F/\partial x = 0, \partial F/\partial y = 0$ and $\partial F/\partial z = 0$ give

$$8yz + 2x\lambda = 0, 8zx + 2y\lambda = 0, 8xy + 2z\lambda = 0$$

or $$2x^2\lambda = -8xyz = 2y^2\lambda = 2z^2\lambda$$

Thus for a maximum volume $x = y = z$.

i.e., the rectangular solid is a cube.

Example 5.49. *A tent on a square base of side x, has its sides vertical of height y and the top is a regular pyramid of height h. Find x and y in terms of h, if the canvas required for its construction is to be minimum for the tent to have a given capacity.*

Solution. Let V be the volume enclosed by the tent and S be its surface area (Fig. 5.6).

Then V = cuboid $(ABCD, A'B'C'D')$ + pyramid $(K, A'B'C'D')$

$$= x^2y + \frac{1}{3}x^2h = x^2(y + h/3)$$

$$S = 4(ABGF) + 4\Delta KGH = 4xy + 4\frac{1}{2}(x \cdot KM)$$

$$= 4xy + x\sqrt{(x^2 + 4h^2)} \quad [\because KM = \sqrt{(KL^2 + LM^2)} = \sqrt{[h^2 + (x/2)^2]}]$$

For constant V, we have

$$\delta V = 2x(y + h/3)\,\delta x + x^2(\delta y) + \frac{x^2}{3}\,\delta h = 0$$

For minimum S, we have

$$\delta S = [4y + \sqrt{(x^2 + 4h^2)} + x \cdot \frac{1}{2}(x^2 + 4h^2)^{-1/2} \cdot 2x]\,\delta x$$
$$+ 4x\delta y + x \cdot \frac{1}{2}(x^2 + 4h^2)^{-1/2} \cdot 8h\delta h = 0$$

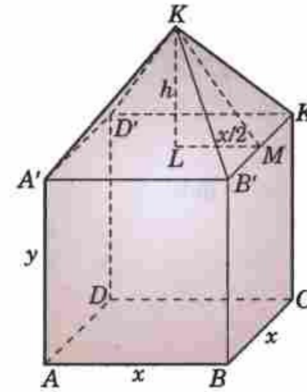

Fig. 5.6

By Lagrange's method,

$$[4y + \sqrt{(x^2 + 4h^2)} + x^2 (x^2 + 4h^2)^{-1/2}] + \lambda \cdot 2x(y + h/3) = 0 \quad ...(i)$$
$$4x + \lambda \cdot x^2 = 0 \quad ...(ii)$$
$$4hx(x^2 + 4h^2)^{-1/2} + \lambda \cdot x^2/3 = 0 \quad ...(iii)$$

(ii) gives $\lambda = -4/x$. Then (iii) becomes

$$4hx(x^2 + 4h^2)^{-1/2} - 4x/3 = 0 \quad \text{or} \quad x = \sqrt{5}\,h$$

Now putting $x = \sqrt{5}\,h$, $\lambda = -4/x$ in (i), we get

$$4y + 3h + \frac{5}{3}h - \frac{4}{x} \cdot 2x(y + h/3) = 0 \quad \text{or} \quad 4y + \frac{14}{3}h - 8y - \frac{8h}{3} = 0, \quad i.e., \quad y = h/2.$$

Example 5.50. *If $u = a^3x^2 + b^3y^2 + c^3z^2$ where $x^{-1} + y^{-1} + z^{-1} = 1$, show that the stationary value of u is given by $x = \Sigma a / a$, $y = \Sigma a / b$, $z = \Sigma a / c$.* (Kerala, 2005)

Solution. Let $u = f(x, y, z) = a^3x^2 + b^3y^2 + c^3z^2$

and $\phi(x, y, z) = x^{-1} + y^{-1} + z^{-1} - 1$

Let $F(x, y, z) = f(x, y, z) + \lambda\,\phi(x, y, z)$
$= a^3x^2 + b^3y^2 + c^3z^2 + \lambda(x^{-1} + y^{-1} + z^{-1} - 1)$

Then $\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0$ and $\frac{\partial F}{\partial z} = 0$ give

$$2a^3x^2 - \lambda/x^2 = 0, \quad 2b^3y - \lambda/y^2 = 0, \quad 2c^3z - \lambda/z^2 = 0$$

or

$$2a^3x^3 = \lambda,\ 2b^3y^3 = \lambda,\ 2c^3z^3 = \lambda$$

which give $ax = by = cz = k$ (say) *i.e.*, $x = k/a$, $y = k/b$, $z = k/c$.

Substituting these in $x^{-1} + y^{-1} + z^{-1} = 1$, we get $k = a + b + c$

Hence the stationary value of u is given by

$$x = \Sigma a/a, \quad y = \Sigma a/b \quad \text{and} \quad z = \Sigma a/c.$$

Example 5.51. *Find the volume of the greatest rectangular parallelopiped that can be inscribed in the ellipsoid*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$ (U.T.U., 2010 ; Anna, 2009 ; Madras, 2006)

Solution. Let the edges of the parallelopiped be $2x$, $2y$ and $2z$ which are parallel to the axes. Then its volume $V = 8xyz$.

Now we have to find the maximum value of V subject to the condition that

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \quad ...(i)$$

Write $$F = 8xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$$

Then $$\frac{\partial F}{\partial x} = 8yz + \lambda\left(\frac{2x}{a^2}\right) = 0 \quad ...(ii)$$

$$\frac{\partial F}{\partial y} = 8zx + \lambda\left(\frac{2y}{b^2}\right) = 0 \quad ...(iii) \qquad \frac{\partial F}{\partial z} = 8xy + \lambda\left(\frac{2z}{c^2}\right) = 0 \quad ...(iv)$$

Equating the values of λ from (*ii*) and (*iii*), we get $x^2/a^2 = y^2/b^2$

Similarly from (*iii*) and (*iv*), we obtain $y^2/b^2 = z^2/c^2 \quad \therefore \quad x^2/a^2 = y^2/b^2 = z^2/c^2$

Substituting these in (*i*), we get $x^2/a^2 = \frac{1}{3}$ *i.e.* $\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2} = \frac{1}{3}$

These give $x = a/\sqrt{3}$, $y = b/\sqrt{3}$, $z = c/\sqrt{3}$ ...(*v*)

When $x = 0$, the parallelopiped is just a rectangular sheet and as such its volume $V = 0$.

As x increases, V also increases continuously.

Thus V must be greatest at the stage given by (*v*).

Hence the greatest volume $= \dfrac{8abc}{3\sqrt{3}}$.

PROBLEMS 5.10

1. Find the maximum and minimum values of
   (*i*) $x^3 + y^3 - 3axy$ (*U.P.T.U., 2005*) (*ii*) $xy + a^3/x + a^3/y$.
   (*iii*) $x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$ (*Mumbai, 2007*) (*iv*) $2(x^2 - y^2) - x^4 + y^4$ (*Osmania, 2003*)
   (*v*) $\sin x \sin y \sin(x + y)$.
2. If $xyz = 8$, find the values of x, y for which $u = 5xyz/(x + 2y + 4z)$ is a maximum. (*S.V.T.U., 2007 ; Kurukshetra, 2005*)
3. Find the minimum value of $x^2 + y^2 + z^2$, given that
   (*i*) $xyz = a^3$ (*P.T.U., 2009 ; Osmania, 2003*) (*ii*) $ax + by + cz = p$. (*V.T.U., 2010 ; U.P.T.U., 2006*)
   (*iii*) $xy + yz + zx = 3a^2$ (*Anna, 2009*)
4. Find the dimensions of the rectangular box, open at the top, of maximum capacity whose surface is 432 sq. cm. (*Madras, 2000 S*)
5. The sum of three numbers is constant. Prove that their product is maximum when they are equal.
6. Find the points on the surface $z^2 = xy + 1$ nearest to the origin. (*Burdwan, 2003 ; Andhra, 2000*)
7. Show that, if the perimeter of a triangle is constant, the triangle has maximum area when it is equilateral.
8. Find the maximum and minimum distances from the origin to the curve $5x^2 + 6xy + 5y^2 - 8 = 0$.
9. The temperature T at any point (x, y, z) in space is $T = 400\,xyz^2$. Find the highest temperature on the surface of the unit sphere $x^2 + y^2 + z^2 = 1$. (*V.T.U., 2009 ; Hissar, 2005 S*)
10. Divide 24 into three parts such that the continued product of the first, square of the second and the cube of the third may be maximum. (*Bhillai, 2005*)
11. Find the stationary values of $u = x^2 + y^2 + z^2$ subject to $ax^2 + by^2 + cz^2 = 1$ and $lx + my + nz = 0$. (*S.V.T.U., 2008*)

5.13 DIFFERENTIATION UNDER THE INTEGRAL SIGN

If a function $f(x, \alpha)$ of two variables x and α (called a parameter), be integrated with respect to x between the limits a and b, then $\int_a^b f(x, \alpha)\,dx$ is a function of $\alpha : F(\alpha)$, say. To find the derivative of $F(\alpha)$, when it exists, it is not always possible to first evaluate this integral and then to find the derivative. Such problems are solved by the following rules :

(1) Leibnitz's rule*

If $f(x, \alpha)$ and $\dfrac{\partial f(x, \alpha)}{\partial \alpha}$ *be continuous functions of* x *and* α, *then*

$$\frac{\mathbf{d}}{\mathbf{d}\alpha}\left[\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{f(x, \alpha)\,dx}\right] = \int_{\mathbf{a}}^{\mathbf{b}} \frac{\partial \mathbf{f(x, \alpha)}}{\partial \alpha}\,\mathbf{dx} \quad \textit{where, } a, b \textit{ are constants independent of } \alpha.$$

*See foot note on p. 139.

Let $\int_a^b f(x, \alpha)dx = F(\alpha)$,

then $F(\alpha + \delta\alpha) - F(\alpha) = \int_a^b f(x, \alpha + \delta\alpha)dx - \int_a^b f(x, \alpha)dx = \int_a^b [f(x, \alpha + \delta\alpha) - f(x, \alpha)]dx$

$= \delta\alpha \int_a^b \frac{\partial f(x, \alpha + \theta\delta\alpha)}{\partial\alpha}dx, \ (0 < \theta < 1)$ $\left\{\begin{array}{l}\because \ f(x, \alpha + h) - f(x, \alpha) = hf'(x, \alpha + \theta h) \\ \text{where } 0 < \theta < 1, \text{ by Mean Value Theorem}\end{array}\right.$

Proceeding to limits as $\delta\alpha \to 0$, $\underset{\delta\alpha \to 0}{\text{Lt}} \frac{F(\alpha + \delta\alpha) - F(\alpha)}{\delta\alpha} = \int_a^b \frac{\partial f(x, \alpha + \theta \cdot 0)}{\partial\alpha}dx$

or $\frac{dF}{d\alpha} = \int_a^b \frac{\partial f(x, \alpha)}{\partial\alpha}dx$ which is the desired result.

Obs. 1. Leibnitz's rule enables us to derive from the value of a simple definite integral, the value of another definite integral which it may otherwise be difficult, or even impossible, to evaluate.

Obs. 2. The rule for differentiation under the integral sign of an infinite integral is the same as for a definite integral.

Example 5.52. *Evaluate* $\int_0^1 \frac{x^\alpha - 1}{\log x}dx$, $\alpha \geq 0$. *(V.T.U., 2010)*

Solution. Let $F(\alpha) = \int_0^1 \frac{x^\alpha - 1}{\log x}dx$ ...(*i*)

then $F(\alpha) = \int_0^1 \frac{\partial}{\partial\alpha}\left(\frac{x^\alpha - 1}{\log x}\right)dx = \int_0^1 \frac{x^\alpha \log x}{\log x}dx$

$= \int_0^1 x^\alpha \, dx = \left| \frac{x^{\alpha+1}}{\alpha + 1} \right|_0^1 = \frac{1}{1 + \alpha}$ $\left[\because \ \frac{d}{dt}(n^t) = n^t \log n\right]$

Now integrating both sides w.r.t. α, $F(\alpha) = \log(1 + \alpha) + c$ ...(*ii*)

From (*i*), when $\alpha = 0$, $F(0) = 0$

$\therefore$ From (*ii*), $F(0) = \log(1) + c$, *i.e.*, $c = 0$. Hence (*ii*) gives, $F(\alpha) = \log(1 + \alpha)$.

Example 5.53. *Given* $\int_0^\pi \frac{dx}{a + b\cos x} = \frac{\pi}{\sqrt{(a^2 - b^2)}}$ $(a > b)$,

evaluate $\int_0^\pi \frac{dx}{(a + b\cos x)^2}$ *and* $\int_0^\pi \frac{\cos x}{(a + b\cos x)^2}dx$ *(Madras, 2006)*

Solution. We have $\int_0^\pi \frac{dx}{a + b\cos x} = \frac{\pi}{\sqrt{(a^2 - b^2)}}$ ...(*i*)

Differentiating both sides of (*i*) w.r.t. a,

$$\int_0^\pi \frac{\partial}{\partial a}\left(\frac{1}{a + b\cos x}\right)dx = \frac{\partial}{\partial a}\left\{\frac{\pi}{\sqrt{(a^2 - b^2)}}\right\}$$

i.e.
$$\int_0^\pi \frac{-dx}{(a + b\cos x)^2} = \pi \cdot \left(-\frac{1}{2}\right)(a^2 - b^2)^{-3/2} \cdot 2a$$

$\therefore$
$$\int_0^\pi \frac{dx}{(a + b\cos x)^2} = \frac{\pi a}{(a^2 - b^2)^{3/2}}$$

Now differentiating both sides of (*i*) w.r.t. b,

$$\int_0^\pi -(a + b\cos x)^{-2} \cdot \cos x \, dx = \pi\left(-\frac{1}{2}\right)(a^2 - b^2)^{-3/2} \cdot (-2b)$$

$$\therefore \qquad \int_0^{\pi} \frac{\cos x}{(a + b\cos x)^2}\,dx = \frac{\pi b}{(a^2 - b^2)^{3/2}}.$$

(2) Leibnitz's rule *for variable limits of integration*

If $f(x, \alpha)$, $\dfrac{\partial f(x, \alpha)}{\partial \alpha}$ *be continuous functions of x and* α, *then*

$$\frac{\mathbf{d}}{\mathbf{d}\alpha}\left\{\int_{\phi(\alpha)}^{\psi(\alpha)} \mathbf{f}(\mathbf{x}, \alpha)\,\mathbf{dx}\right\} = \int_{\phi(\alpha)}^{\psi(\alpha)} \frac{\partial \mathbf{f}(\mathbf{x}, \alpha)}{\partial \alpha}\,\mathbf{dx} + \frac{\mathbf{d}\psi}{\mathbf{d}\alpha}\,\mathbf{f}[\psi(\alpha), \alpha] - \frac{\mathbf{d}\phi}{\mathbf{d}\alpha}\,\mathbf{f}[\phi(\alpha), \alpha]$$

provided $\phi(\alpha)$ *and* $\psi(\alpha)$ *possesses continuous first order derivatives w.r.t.* α.

Its proof is beyond the scope of this book.

Example 5.54. *Evaluate* $\displaystyle\int_0^{\alpha} \frac{\log(1 + \alpha x)}{1 + x^2}\,dx$ *and hence show that*

$$\int_0^{1} \frac{\log(1 + x)}{1 + x^2}\,dx = \frac{\pi}{8}\log_e 2$$

(*Hissar, 2005 S*)

Solution. Let $\qquad F(\alpha) = \displaystyle\int_0^{\alpha} \frac{\log(1 + \alpha x)}{1 + x^2}\,dx \qquad$ ...(*i*)

Then by the above rule, $F'(\alpha) = \displaystyle\int_0^{\alpha} \frac{\partial}{\partial \alpha}\left(\frac{\log(1 + \alpha x)}{1 + x^2}\right)dx + \frac{d(\alpha)}{d\alpha}\cdot\frac{\log(1 + \alpha^2)}{1 + \alpha^2} - 0$

$$= \int_0^{\alpha} \frac{x}{(1 + \alpha x)(1 + x^2)}\,dx + \frac{\log(1 + \alpha^2)}{1 + \alpha^2} \qquad \text{...(ii)}$$

Breaking the integrand into partial fractions,

$$\int_0^{\alpha} \frac{x\,dx}{(1 + \alpha x)(1 + x^2)} = -\frac{\alpha}{1 + \alpha^2}\int_0^{\alpha} \frac{dx}{1 + \alpha x} + \frac{1}{2(1 + \alpha^2)}\int_0^{\alpha} \frac{2x}{1 + x^2}\,dx + \frac{\alpha}{1 + \alpha^2}\int_0^{\alpha} \frac{dx}{1 + x^2}$$

$$= -\frac{1}{1 + \alpha^2}\Big|\log(1 + \alpha x)\Big|_0^{\alpha} + \frac{1}{2(1 + \alpha^2)} \times \Big|\log(1 + x^2)\Big|_0^{\alpha} + \frac{\alpha}{1 + \alpha^2}\Big|\tan^{-1} x\Big|_0^{\alpha}$$

$$= -\frac{\log(1 + \alpha^2)}{1 + \alpha^2} + \frac{\log(1 + \alpha^2)}{2(1 + \alpha^2)} + \frac{\alpha \tan^{-1}\alpha}{1 + \alpha^2}$$

Substituting this value in (*ii*), $F'(\alpha) = \dfrac{\log(1 + \alpha^2)}{2(1 + \alpha^2)} + \dfrac{\alpha \tan^{-1}\alpha}{1 + \alpha^2}$

Now integrating both sides w.r.t. α,

$$F(\alpha) = \frac{1}{2}\int \log(1 + \alpha^2)\cdot\frac{1}{1 + \alpha^2}\,d\alpha + \int \frac{\alpha \tan^{-1}\alpha}{1 + \alpha^2}\,d\alpha \qquad \text{[Integrating by parts]}$$

$$= \frac{1}{2}\left[\log(1 + \alpha^2)\cdot\tan^{-1}\alpha - \int \frac{2\alpha}{1 + \alpha^2}\cdot\tan^{-1}\alpha\,d\alpha\right] + \int \frac{\alpha \tan^{-1}\alpha}{1 + \alpha^2}\,d\alpha + c$$

$$= \frac{1}{2}\log(1 + \alpha^2)\cdot\tan^{-1}\alpha + c \qquad \text{...(iii)}$$

But from (*i*), when $\alpha = 0$, $F(0) = 0$.

$\therefore$ From (*iii*), $F(0) = 0 + c$, *i.e.*, $c = 0$. Hence (*iii*) gives, $F(\alpha) = \dfrac{1}{2}\log(1 + \alpha^2)\tan^{-1}\alpha$

Putting $\alpha = 1$, we get $\displaystyle\int_0^{1} \frac{\log(1 + x)}{1 + x^2}\,dx = F(1) = \frac{\pi}{8}\log_e 2.$

PROBLEMS 5.11

1. Differentiating $\int_0^{\infty} \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$ under the integral sign, find the value of $\int_0^{\infty} \frac{dx}{(x^2 + a^2)^2}$.

2. By successive differentiation of $\int_0^1 x^m \, dx = \frac{1}{m+1}$ w.r.t. m, evaluate $\int_0^1 x^m (\log x)^n \, dx$.

3. Evaluate $\int_0^{\pi} \log(1 + a \cos x) \, dx$, using the method of differentiation under the sign of integration.

4. Given that $\int_0^{\pi} \frac{dx}{a - \cos x} = \frac{\pi}{\sqrt{(a^2 - 1)}}$, evaluate $\int_0^{\pi} \frac{dx}{(a - \cos x)^2}$. *(V.T.U., 2009)*

Prove that :

5. $\int_0^{\infty} e^{-x} \cdot \frac{\sin ax}{x} dx = \tan^{-1} a.$ $\left[\textbf{Hint.} \text{ Use } \int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx)\right]$

6. $\int_0^{\infty} e^{-ax} \frac{\sin x}{x} dx = \tan^{-1} \frac{1}{a}$. Hence show that $\int_0^{\infty} \frac{\sin x}{x} = \frac{\pi}{2}$. *(Rohtak, 2003)*

7. $\int_0^{\infty} \frac{\tan^{-1} ax}{x(1 + x^2)} dx = \frac{\pi}{2} \log(1 + a)$ where $a \geq 0$. *(V.T.U., 2010 ; S.V.T.U., 2009 ; Rohtak, 2006 S ; Anna, 2005 S)*

8. $\int_0^{\infty} \frac{e^{-x}}{x} (1 - e^{-ax}) \, dx = \log (1 + a), (a > - 1).$

9. $\int_0^{\pi/2} \log(\alpha^2 \cos^2 \theta + \beta^2 \sin^2 \theta) \, d\theta = \pi \log \frac{\alpha + \beta}{2}$ *(S.V.T.U., 2008)*

10. $\int_0^{\pi/2} \frac{\log (1 + y \sin^2 x)}{\sin^2 x} dx = \pi [\sqrt{(1 + y)} - 1]$ *(S.V.T.U., 2008)*

11. $\int_0^{\pi} \frac{\log (1 + \alpha \cos x)}{\cos x} dx = \pi \sin^{-1} \alpha.$ *(V.T.U., 2007)*

12. $\int_0^{\infty} e^{-x^2} \cos \alpha x \, dx = \frac{\sqrt{\pi}}{2} e^{-\alpha^2/4}$ *(Mumbai, 2009 S)*

13. $\frac{d}{da} \int_0^{a^2} \tan^{-1} \frac{x}{a} dx = 2a \tan^{-1} a - \frac{1}{2} \log (a^2 + 1)$. Verify your result by direct integration.

14. $\int_{\pi/2 - \alpha}^{\pi/2} \sin \theta \cos^{-1} (\cos \alpha \cos \theta) \, d\theta = \frac{\pi}{2} (1 - \cos \alpha).$ *(Burdwan, 2003)*

15. If $y = \int_0^x f(t) \sin[k(x - t)] dt$, prove that y satisfies the differential equation $\frac{d^2y}{dx^2} + k^2 y = k f(x)$.

5.14 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 5.12

Select the correct answer or fill up the blanks in each of the following problems :

1. If $u = e^x (x \cos y - y \sin y)$, then $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = ...$

2. If $x = uv$, $y = u/v$, then $\frac{\partial (x, y)}{\partial (u, v)}$ is

(a) $-2u/v$ (b) $-2v/u$ (c) 0 (d) 1. *(V.T.U., 2010)*

3. If $J_1 = \frac{\partial(u, v)}{\partial(x, y)}$ and $J_2 = \frac{\partial(x, y)}{\partial(u, v)}$, then $J_1 J_2 = ...$

4. If $u = f(y/x)$, then

(a) $x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = 0$ (b) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$ (c) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u$ (d) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1$.

5. If $u = x^y$, then $\partial u/\partial x$ is

(a) 0 (b) yx^{y-1} (c) $x^y \log x$.

6. If $x = r\cos\theta$, $y = r\sin\theta$, then

(a) $\frac{\partial x}{\partial r} = 1/\frac{\partial r}{\partial x}$ (b) $\frac{\partial x}{\partial r} = \frac{\partial r}{\partial x}$ (c) $\frac{\partial x}{\partial r} = 0$.

7. If $u = x^y$, then $\partial u/\partial y$ is

(a) yx^{y-1} (b) 0 (c) $x^y \log x$.

8. If $u = x^3 + y^3$, then $\frac{\partial^2 u}{\partial x \partial y}$ is equal to

(a) -3 (b) 3 (c) 0 (d) $3x + 3y$ *(V.T.U., 2010 S)*

9. If $u = x^2 + 2xy + y^2 + x + y$, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ is equal to

(a) $2u$ (b) u (c) 0 (d) none of these.

10. If $u = \log\frac{x^2}{y}$, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ is equal to

(a) $2u$ (b) $3u$ (c) u (d) 1. *(V.T.U., 2010 S)*

11. If $x = r\cos\theta$, $y = r\sin\theta$, then $\frac{\partial(x, y)}{\partial(r, \theta)}$ is equal to

(a) 1 (b) r (c) $1/r$ (d) 0. *(V.T.U., 2010 S)*

12. If $A = f_{xx}(a, b)$, $B = f_{xy}(a, b)$, $C = f_{yy}(a, b)$, then $f(x, y)$ will have a maximum at (a, b) if

(a) $f_x = 0, f_y = 0$, $AC < B^2$ and $A < 0$ (b) $f_x = 0, f_y = 0$, $AC = B^2$ and $A > 0$
(c) $f_x = 0, f_y = 0$, $AC > B^2$ and $A > 0$ (d) $f_x = 0, f_y = 0$, $AC > B^2$ and $A < 0$.

13. If $z = \sin^{-1}\frac{\sqrt{x^2 + y^2}}{x + y}$, then $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$ is

(a) 0 (b) 1/2 (c) 1 (d) 2. *(Bhopal, 2008)*

14. If $u = \sin^{-1}(x/y) + \tan^{-1}(y/x)$, then $x\,\partial u/\partial x + y\,\partial u/\partial y$ equals

(a) $\sin^{-1}(x/y) + \tan^{-1}(y/x)$ (b) $2[\sin^{-1}(x/y) + \tan^{-1}(y/x)]$
(c) $3[\sin^{-1}(x/y) + \tan^{-1}(y/x)]$ (d) zero.

15. If an error of 1% is made in measuring its length and breadth, the percentage error in the area of a rectangle is

(a) 0.2% (b) 0.02% (c) 2% (d) 1%. *(V.T.U., 2010)*

16. $\frac{\partial(u, v)}{\partial(x, y)} \times \frac{\partial(x, y)}{\partial(u, v)}$ equals

(a) -1 (b) 1 (c) zero (d) none of these.

17. $\frac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}$ is a homogeneous function of degree ......

18. If $z = \log(x^3 + y^3 - x^2y - xy^2)$, then $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}$ is equal to ......

19. If $r = \partial^2 f/\partial x^2$, $s = \partial^2 f/\partial x \partial y$ and $t = \partial^2 f/\partial y^2$, then the condition for the saddle point is ......

20. If $f(x, y) = \frac{1}{x^3} + \frac{1}{y^3} + \frac{1}{x^3 + y^3}$, then $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}$ is

(a) 0 (b) $3f$ (c) 9 (d) $-3f$. *(V.T.U., 2009 S)*

21. If $u = x^4 + y^4 + 3x^2y^2$, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = ...$

22. If u and v are functions of r, s where r, s are functions of x, y, then $\dfrac{\partial(u, v)}{\partial(r, s)} \cdot \dfrac{\partial(r, s)}{\partial(x, y)} = \ldots\ldots$

23. The necessary conditions for a function $f(x, y)$ to have an extreme at (a, b) are ......

24. If $u = (x - y)^4 + (y - z)^4 + (z - x)^4$, then $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z}$ is

(*a*) 1 (*b*) u (*c*) $4u$ (*d*) 0. (*V.T.U., 2010*)

25. If u is a composite function of t, defined by the relations $u = f(x, y)$; $x = \phi(t)$, $y = \psi(t)$, then total derivative $\dfrac{du}{dt} = \ldots\ldots$

26. If $u = \cos^{-1}(x/y) + \tan^{-1}(y/x)$, then $x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy}$ is

(*a*) u (*b*) $2u$ (*c*) 0 (*d*) 1. (*V.T.U., 2010*)

27. If $f(x, y, z) = 0$, then $\dfrac{\partial x}{\partial y} \cdot \dfrac{\partial y}{\partial z} \cdot \dfrac{\partial z}{\partial x} = \ldots\ldots$

28. If $u = f(x + ay) + g(x - ay)$, then $\dfrac{\partial^2 u}{\partial y^2}$ equals

(*a*) $\dfrac{\partial^2 u}{\partial x^2}$ (*b*) $a\dfrac{\partial^2 u}{\partial x^2}$ (*c*) $a^2\dfrac{\partial^2 u}{\partial x^2}$ (*d*) $\dfrac{\partial^2 u}{\partial x \partial y}$. (*V.T.U., 2010*)

29. If sum of three numbers is constant, then their product is a maximum when the numbers are ......

30. $y = \cosh(\lambda x)\cosh(-\lambda at)$ is a solution of $\dfrac{\partial^2 y}{\partial t^2} = a^2\dfrac{\partial^2 y}{\partial x^2}$. (*True* or *False*)

CHAPTER 6

Integral Calculus and Its Applications

1. Reduction formulae. 2. Reduction formulae for $\int \sin^n x\,dx$, $\int \cos^n x\,dx$ and evaluation of $\int_0^{\pi/2} \sin^n x\,dx$, $\int_0^{\pi/2} \cos^n x\,dx$. 3. Reduction formula for $\int \sin^m x \cos^n x\,dx$ and evaluation of $\int_0^{\pi/2} \sin^m x \cos^n x\,dx$. 4. Reduction formulae for $\int \tan^n x\,dx$ $\int \cot^n x\,dx$. 5. Reduction formulae for $\int \sec^n x\,dx$, $\int \operatorname{cosec}^n x\,dx$. 6. Reduction formulae for $\int x^n e^{ax}\,dx$, $\int x^m (\log x)^n\,dx$. 7. Reduction formulae for $\int x^n \sin mx\,dx$, $\int x^n \cos nx\,dx$ and $\int \cos^m x \sin nx\,dx$. 8. Definite integrals. 9. Integral as the limit of a sum. 10. Areas of curves. 11. Lengths of curves. 12. Volumes of revolution. 13. Surface areas of revolution. 14. Objective Type of Questions.

6.1 REDUCTION FORMULAE

The reader is already familiar with some standard methods of integrating functions of a single variable. However, there are some integrals which cannot be evaluated by the afore-said methods. In such cases, the method of reduction formulae proves useful. A reduction formula connects an integral with another of the same type but of lower order. The successive application of the reduction formula enables us to evaluate the given integral. Now we shall derive some standard reduction formulae.

6.2 (1) REDUCTION FORMULAE for

(a) $\int \sin^n x\,dx$ **(b)** $\int \cos^n x\,dx.$

(a) $\int \sin^n x\,dx = \int \sin^{n-1} x \cdot \sin x\,dx$ [Integrated by parts]

$$= \sin^{n-1} x \cdot (-\cos x) - \int (n-1) \sin^{n-2} x \cos x (-\cos x)\,dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x (1 - \sin^2 x)\,dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x\,dx - (n-1) \int \sin^n x\,dx$$

Transposing

$$n \int \sin^n x\,dx = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x\,dx$$

or

$$\int \sin^n x\,dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x\,dx \qquad ...(i)$$

(*b*) Similarly, $$\int \cos^n x\,dx = \frac{\sin x \cos^{n-1} x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,dx$$

Thus we have the required reduction formulae.

Obs. To integrate $\int \sin^n x\,dx$ or $\int \cos^n x\,dx$,

(a) when the index of sin x is odd put cos x = t

when the index of cos x is odd, put sin x = t

(b) when the index is an even positive integer, express the integrand as a series of cosines of multiple angles and integrate term by term if n is small, *otherwise use the method of reduction formulae.*

(2) To show that $\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx$

$$= \frac{(n-1)(n-3)(n-5)\ldots}{n(n-2)(n-4)\ldots} \times \left(\frac{\pi}{2}, \textit{only if } n \textit{ is even}\right)$$

From (*i*), we have

$$I_n = \int_0^{\pi/2} \sin^n x\,dx = -\left|\frac{\sin^{n-1} x \cos x}{n}\right|_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx$$

i.e. $$I_n = \frac{n-1}{n} I_{n-2}$$

Case I. *When n is odd*

Similarly $$I_{n-2} = \frac{n-3}{n-2} I_{n-4},\quad I_{n-4} = \frac{n-5}{n-4} I_{n-6}$$

....

$$I_5 = \frac{4}{5} I_3,\ I_3 = \frac{2}{3} I_1 = \frac{2}{3}\int_0^{\pi/2} \sin x\,dx = \frac{2}{3}\left|-\cos x\right|_0^{\pi/2} = \frac{2}{3}.$$

Form these, we get $I_n = \dfrac{(n-1)(n-3)(n-5)\ldots 2}{n(n-2)(n-4)\ldots 3}$ ...(*ii*)

Case II. *When n is even*

We have $$I_{n-2} = \frac{n-3}{n-2} I_{n-4},\ I_{n-4} = \frac{n-5}{n-4} I_{n-6}$$

....

$$I_4 = \frac{3}{4} I_2,\ I_2 = \frac{1}{2} I_0 = \int_0^{\pi/2} \sin^0 x\,dx = \frac{1}{2}\int_0^{\pi/2} dx = \frac{1}{2}\cdot\frac{\pi}{2}.$$

Form these, we obtain $I_n = \dfrac{(n-1)(n-3)(n-5)\ldots 3\cdot 1}{n(n-2)(n-4)\ldots 4\cdot 2}\cdot\dfrac{\pi}{2}$ ...(*iii*)

Combining (*ii*) and (*iii*), we get the required result for $\int_0^{\pi/2} \sin^n x\,dx$.

Proceeding exactly as above, we get the result for $\int_0^{\pi/2} \cos^n x\,dx$.

Example 6.1. *Integrate (i) $\int \sin^4 x\,dx$ (ii) $\int_0^{\pi/2} \cos^6 x\,dx$.*

Solution. (*i*) We have the reduction formula

$$\int \sin^n x\,dx = \frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$

Putting $n = 4, 2$ successively,

$$\int \sin^4 x\,dx = -\frac{\sin^3 x \cos x}{4} + \frac{3}{4}\int \sin^2 x\,dx \qquad \ldots(\alpha)$$

$$\int \sin^2 x\, dx = -\frac{\sin x \cos x}{2} + \frac{1}{2}\int (\sin x)^0\, dx$$

But $\int (\sin x)^0\, dx = \int dx = x.\quad \therefore\quad \int \sin^2 x\, dx = -\frac{\sin x \cos x}{2} + \frac{x}{2}$

Substituting this in (α), we get

$$\int \sin^4 x\, dx = -\frac{\sin^3 x \cos x}{4} + \frac{3}{4}\left(-\frac{\sin x \cos x}{2} + \frac{x}{2}\right)$$

(*ii*) We know that $\int_0^{\pi/2} \cos^n x\, dx = \frac{(n-1)(n-3)(n-5)\ldots}{n(n-2)(n-4)\ldots}\left(\frac{\pi}{2} \text{ if } n \textit{ is even}\right)$

Putting $n = 6$, we get

$$\int_0^{\pi/2} \cos^6 x\, dx = \frac{5.3.1}{6.4.2}\frac{\pi}{2} = \frac{5\pi}{16}.$$

Example 6.2. *Evaluate*

(*i*) $\int_0^a \frac{x^7\, dx}{\sqrt{(a^2 - x^2)}}$ (*V.T.U., 2006*) (*ii*) $\int_0^\pi \frac{\sqrt{(1-\cos x)}}{1+\cos x} \sin^2 x\, dx$ (*iii*) $\int_0^\infty \frac{dx}{(a^2 + x^2)^n}$.

Solution. (*i*) $\int_0^a \frac{x^7}{\sqrt{(a^2 - x^2)}} dx$ | Put $x = a \sin\theta$, so that $dx = a\cos\theta\, d\theta$
Also when $x = 0$, $\theta = 0$, when $x = a$, $\theta = \pi/2$

$$= \int_0^{\pi/2} \frac{a^7 \sin^7\theta}{a\cos\theta} . a\cos\theta\, d\theta = a^7 \int_0^{\pi/2} \sin^7\theta\, d\theta = a^7 . \frac{6.4.2}{7.5.3.1} = \frac{16}{35} a^7$$

(*ii*) Putting $x = 2\theta$, we get

$$\int_0^\pi \frac{\sqrt{(1-\cos x)}}{1+\cos x}\sin^2 x\, dx = \int_0^{\pi/2} \frac{\sqrt{(1-\cos 2\theta)}}{1+\cos 2\theta}\sin^2 2\theta \cdot 2d\theta$$

$$= 2\int_0^{\pi/2} \frac{\sqrt{2}\sin\theta}{2\cos^2\theta}\cdot(2\sin\theta\cos\theta)^2\, d\theta = 4\sqrt{2}\int_0^{\pi/2}\sin^3\theta\, d\theta = 4\sqrt{2}\cdot\frac{2}{3} = \frac{8\sqrt{2}}{3}.$$

(*iii*) $\int_0^\infty \frac{dx}{(a^2 + x^2)^n}$ | Put $x = a\tan\theta$, so that $dx = a\sec^2\theta\, d\theta$
Also when $x = 0$, $\theta = 0$, when $x = \infty$, $\theta = \pi/2$

$$= \int_0^{\pi/2} \frac{a\sec^2\theta\, d\theta}{a^{2n}\sec^{2n}\theta} = \frac{1}{a^{2n-1}}\int_0^{\pi/2}\cos^{2n-2}\theta\, d\theta = \frac{1}{a^{2n-1}}\cdot\frac{(2n-3)(2n-5)\ldots 3.1}{(2n-2)(2n-4)\ldots 4.2}\cdot\frac{\pi}{2}.$$

Example 6.3. *Evaluate* $\int_0^a \frac{x^n}{\sqrt{(a^2 - x^2)}} dx$. *Hence find the value of* $\int_0^1 x^n \sin^{-1} x\, dx$.

Solution. Putting $x = a\sin\theta$, we get

$$\int_0^a \frac{x^n}{\sqrt{(a^2 - x^2)}} dx = \int_0^{\pi/2} \frac{(a\sin\theta)^n}{a\cos\theta}(a\cos\theta)\, d\theta = a^n \int_0^{\pi/2}\sin^n\theta\, d\theta$$

$$\left.\begin{aligned} &= \frac{(n-1)(n-3)\ldots.2}{n(n-2)\ldots.3}a^n, \text{ if } n \text{ is odd} \\ &= \frac{(n-1)(n-3)\ldots 1}{n(n-2)\ldots 2}\cdot\frac{\pi}{2}a^n, \text{ if } n \text{ is even}\end{aligned}\right\} \quad \ldots(i)$$

Now integrating by parts, we have

$$\int_0^1 x^n \sin^{-1} x\, dx = \left|(\sin^{-1} x)\cdot\frac{x^{n+1}}{n+1}\right|_0^1 - \int_0^1 \frac{x^{n+1}}{n+1}\frac{1}{\sqrt{(1-x^2)}} dx$$

$$= \frac{1}{(n+1)}\left[\frac{\pi}{2} - \int_0^1 \frac{x^{n+1}}{(1-x^2)}\right] \quad \text{[Using (i) p. 241]}$$

$$= \frac{1}{n+1}\left\{\frac{\pi}{2} - \frac{n(n-2)(n-4)\ldots 1}{(n+1)(n-1)(n-3)\ldots 2}\frac{\pi}{2}\right\} \quad \text{when } n \text{ is odd}$$

$$= \frac{1}{n+1}\left\{\frac{\pi}{2} - \frac{n(n-2)(n-4)\ldots 2}{(n+1)(n-1)(n-3)\ldots 3}\right\} \quad \text{when } n \text{ is even}$$

Evaluate 6.4. *Evaluate* $I_n = \int_0^a (a^2 - x^2)^n \, dx$ *where n is a positive integer. Hence show that*

$$I_n = \frac{2n}{2n+1} a^2 I_{n-1}.$$

Solution. Putting $n = a \sin \theta$, we get

$$I_n = \int_0^a (a^2 - x^2)^n \, dx = \int_0^{\pi/2} (a^2 - a^2 \sin^2 \theta)^n \, a \cos \theta \, d\theta = a^{2n+1} \int_0^{\pi/2} \cos^{2n+1} \theta \, d\theta$$

$$= a^{2n+1} \cdot \frac{(2n)(2n-2)(2n-4)\ldots 4.2}{(2n+1)(2n-1)(2n-3)\ldots 5.3} \quad [\because \ (2n+1) \text{ is always odd}]$$

Now replacing n by $n-1$, we get

$$I_{n-1} = a^{2n-1} \frac{(2n-2)(2n-4)\ldots 4.2}{(2n-1)(2n-3)\ldots 5.3} \quad \therefore \quad \frac{I_n}{I_{n-1}} = a^2 \cdot \frac{2n}{2n+1} \quad \text{or} \quad I_n = \frac{2n}{2n+1} a^2 I_{n-1}.$$

which is the second desired result.

6.3 (1) REDUCTION FORMULAE for $\int \sin^m x \cos^n x \, dx$

$$\int \sin^m x \cos^n x \, dx = \int \sin^{m-1} x \cdot \cos^n x \cdot \sin x \, dx \quad \text{[Integrate by parts]}$$

$$= \sin^{m-1} x \cdot \left(\frac{-\cos^{n+1} x}{n+1}\right) - \int (m-1) \sin^{m-2} x \cos x \cdot \left(-\frac{\cos^{n+1} x}{n+1}\right) dx.$$

$$= -\frac{\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{m+1} \int \sin^{m-2} x (1 - \sin^2 x) \cos^n x \, dx$$

$$= -\frac{\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{n+1} \int \sin^{m-2} x \cos^n x \, dx - \frac{m-1}{n+1} \int \sin^m x \cos^n x \, dx$$

Transposing the last term to the left and dividing by $1 + (m-1)/(n+1)$, *i.e.*, $(m+n)/(n+1)$, we obtain the reduction formula

$$\int \sin^m x \cos^n x \, dx = -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} \int \sin^{m-2} x \cos^n x \, dx \quad \ldots(1)$$

Obs. To integrate $\int \sin^m x \cos^n x \, dx$,

(*a*) *when m is odd*, put $\cos x = t$

when n is odd, put $\sin x = t$

(*b*) *when m and n both are even integers*, express the integrand as a series of cosines of multiple angles and integrate term by term if m and n are small, *otherwise* use the method of reduction formulae.

(2) To show that

$$\int_0^{\pi/2} \sin^m x \cos^n x \, dx = \frac{(m-1)(m-3)\ldots \times (n-1)(n-3)\ldots}{(m+n)(m+n-2)(m+n-4)\ldots} \times \left(\frac{\pi}{2}, \textit{ only if both m and n are even}\right)$$

From (*i*), we have

$$I_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x\,dx = \left| -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} \right|_0^{\pi/2} + \frac{m-1}{m+n}\int_0^{\pi/2} \sin^{m-2} x \cos^n x\,dx$$

i.e.,
$$I_{m,n} = \frac{m-1}{m+n} I_{m-2,n}.$$

Case I. *When m is odd*

Similarly, $$I_{m-2,n} = \frac{m-3}{m+n-2} I_{m-4,n}, I_{m-4,n} = \frac{m-5}{m+n-4} I_{m-6,n}$$

....

$$I_{5,n} = \frac{4}{n+5} I_{3,n}$$

Finally $$I_{3,n} = \frac{2}{n+3} I_{1,n} = \frac{2}{n+3}\int_0^{\pi/2} \sin x \cos^n x\,dx$$

$$= \frac{2}{n+3}\left| -\frac{\cos^{n+1} x}{n+1} \right|_0^{\pi/2} = \frac{2}{(n+3)(n+1)} \qquad \ldots(ii)$$

From these, we obtain

$$I_{m,n} = \frac{(m-1)(m-3)(m-5)\ldots 4.2}{(m+n)(m+n-2)(m+n-4)\ldots(n+3)(n+1)}$$

Case II. *When m is even*

We have, $$I_{m-2,n} = \frac{m-3}{m+n-2} I_{m-4,n}, \quad I_{m-4,n} = \frac{m-5}{m+n-4} I_{m-6,n}$$

....

$$I_{4,n} = \frac{3}{n+4} I_{2,n}, I_{2,n} = \frac{1}{n+2} I_{0,n} = \frac{1}{n+2}\int_0^{\pi/2} \cos^n x\,dx$$

From these, we have $$I_{m,n} = \frac{(m-1)(m-3)(m-5)\ldots 1}{(m+n)(m+n-2)(m+n-4)\ldots(n+2)}\int_0^{\pi/2} \cos^n x\,dx$$

$$= \frac{(m-1)(m-3)\ldots 1}{(m+n)(m+n-2)\ldots(n+2)} \cdot \frac{(n-1)(n-3)\ldots}{n(n-2)\ldots} \times (\pi/2 \text{ only if } n \text{ is even}) \qquad \ldots(iii)$$

Combining (*ii*) and (*iii*), we get the desired result.

Example 6.5. *Integrate* (*i*) $\int \sin^4 x \cos^2 x\,dx$ *(Raipur, 2005)*

(*ii*) $\int_0^{\infty} \frac{t^6}{(1+t^2)^7}\,dt$ (*iii*) $\int_0^{\infty} \frac{x^2}{(1+x^2)^{7/2}}\,dx$ *(V.T.U., 2010 S)*

Solution. (*i*) Taking $n = 2$, in (*i*) of page 241, we have the reduction formula :

$$\int \sin^m x \cos^2 x\,dx = \frac{\sin^{m-1} x \cos^3 x}{m+2} + \frac{m-1}{m+2}\int \sin^{m-2} x \cos^2 x\,dx$$

Putting $m = 4, 2$ successively,

$$\int \sin^4 x \cos^2 x\,dx = -\frac{\sin^3 x \cos^3 x}{6} + \frac{3}{6}\int \sin^2 x \cos^2 x\,dx \qquad \ldots(1)$$

$$\int \sin^2 x \cos^2 x\,dx = -\frac{\sin x \cos^3 x}{4} + \frac{1}{4}\int \cos^2 x\,dx$$

But $$\int \cos^2 x\,dx = \frac{1}{2}\int (1+\cos 2x)\,dx = \frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right)$$

$\therefore \qquad \int \sin^2 x \cos^2 x \, dx = -\dfrac{\sin x \cos^3 x}{4} + \dfrac{1}{16}(2x + \sin 2x)$

Substituting this in (1), we get

$$\int \sin^4 x \cos^2 x \, dx = -\frac{\sin^3 x \cos^3 x}{6} + \frac{1}{2}\left\{-\frac{\sin x \cos^3 x}{4} + \frac{1}{16}(2x + \sin 2x)\right\}$$

(*ii*) Putting $t = \tan \theta$, so that

$$\int_0^\infty \frac{t^6}{(1+t^2)^7} dt = \int_0^{\pi/2} \frac{\tan^6 \theta}{\sec^{14} \theta} \sec^2 \theta \, d\theta = \int_0^{\pi/2} \sin^6 \theta \, \cos^6 \theta \, d\theta = \frac{5 \cdot 3 \cdot 1 \times 5 \cdot 3 \cdot 1}{12 \cdot 10 \cdot 8 \cdot 6 \cdot 4 \cdot 2} \times \frac{\pi}{2} = \frac{5\pi}{2048}.$$

(*iii*) Putting $x = \tan \theta$, so that

$$\int_0^\infty \frac{x^2}{(1+x^2)^{7/2}} dx = \int_0^{\pi/2} \frac{\tan^2 \theta}{\sec^7 \theta} \sec^2 \theta \, d\theta = \int_0^{\pi/2} \sin^2 \theta \cos^3 \theta \, d\theta = \frac{1.2}{53.1} = \frac{2}{15}.$$

Example 6.6. *Evaluate* : (*i*) $\int_0^{\pi/6} \cos^4 3\theta \sin^3 6\theta \, d\theta$ *(V.T.U., 2003 S)*

(*ii*) $\int_0^1 x^4 (1-x^2)^{3/2} dx$ (*iii*) $\int_0^{2a} x^2 \sqrt{(2ax - x^2)} \, dx.$ *(V.T.U., 2010)*

Solution. (*i*) $\int_0^{\pi/6} \cos^4 3\theta \sin^3 6\theta \, d\theta = \int_0^{\pi/6} \cos^4 3\theta \, (2 \sin 3\theta \cos 3\theta)^3 \, d\theta$

$$= 8 \int_0^{\pi/6} \sin^3 3\theta \cos^7 3\theta \, d\theta$$

$$= \frac{8}{3} \int_0^{\pi/2} \sin^3 x \cos^7 x \, dx$$

$$= \frac{8}{3} \cdot \frac{2 \times 6 \cdot 4 \cdot 2}{10 \cdot 8 \cdot 6 \cdot 4 \cdot 2} = \frac{1}{15}.$$

Put $3\theta = x$ so that $3d\theta = dx$

Also when $\theta = 0, x = 0$; when $\theta = \pi/6, x = \pi/2$.

(*ii*) $\int_0^1 x^4 (1-x^2)^{3/2} dx$

Put $x = \sin t$ so that $dx = \cos t \, dt$

When $x = 0, t = 0$; when $x = 1, t = \pi/2$

$$= \int_0^{\pi/2} \sin^4 t \, (\cos^2 t)^{3/2} \cdot \cos t \, dt = \int_0^{\pi/2} \sin^4 t \cos^4 t \, dt$$

$$= \frac{3 \cdot 1 \times 3 \cdot 1}{8 \cdot 6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2} = \frac{3\pi}{256}.$$

(*iii*) $\int_0^{2a} x^2 \sqrt{(2ax - x^2)} \, dx$

$$= \int_0^{\pi/2} x^{5/2} \sqrt{(2a - x)} \, dx$$

Put $x = 2a \sin^2 \theta$

$\therefore \; dx = 4a \sin \theta \cos \theta \, d\theta$

$$= \int_0^{\pi/2} (2a \sin^2 \theta)^{5/2} \sqrt{(2a)} \cos \theta \cdot 4a \sin \theta \cos \theta \, d\theta$$

$$= 2^5 a^4 \int_0^{\pi/2} \sin^6 \theta \cos^2 \theta \, d\theta = 32\, a^4 \cdot \frac{5 \cdot 3 \cdot 1 \times 1}{8 \cdot 6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2} = \frac{5\pi a^4}{8}.$$

PROBLEMS 6.1

Evaluate :

1. (i) $\int_0^{\frac{\pi}{2}} \cos^9 x \, dx$ (ii) $\int_0^{\pi/6} \sin^5 3\theta \, d\theta$
2. (i) $\int_0^1 \frac{x^9}{\sqrt{(1-x^2)}} dx$ (ii) $\int_0^1 x^5 \sin^{-1} x \, dx$
3. (i) $\int_0^\infty \frac{dx}{(1+x^2)^n} \, (n > 1)$ *(V.T.U., 2008 S)* (ii) $\int_0^{\pi/4} \sin^2 x \cos^4 x \, dx.$ *(J.N.T.U., 2003)*

4. If $I_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x \, dx$ $(m > 0, n > 0)$, show that $I_{m,n} = \frac{m-1}{m+n} I_{m-2,n}$.

Hence evaluate $\int_0^{\pi/2} \sin^4 x \cos^8 x \, dx$

Evaluate :

5. (*i*) $\int_0^{\pi/2} \sin^4 x \cos^6 x \, dx$ *(Cochin, 2005)* (*ii*) $\int_0^{\pi/2} \sin^{15} x \cos^3 x \, dx$

6. (*i*) $\int_0^1 x^6 \sqrt{(1-x^2)} \, dx$ (*ii*) $\int_0^{\pi/2} \cos^4 3\theta \sin^3 6\theta \, d\theta$

7. (*i*) $\int_0^{2a} x^{7/2} (2a-x)^{-1/2} \, dx$ (*ii*) $\int_0^{2a} \frac{x^3 \, dx}{\sqrt{(2ax-x^2)}}$ *(Madras, 2000 S)*

8. (*i*) $\int_0^2 x^{5/2} \sqrt{(2-x)} \, dx$ (*ii*) $\int_0^4 x^3 \sqrt{(4x-x^2)} \, dx$ *(V.T.U., 2004)*

9. If $I_n = \int x^n \sqrt{(a-x)} \, dx$, prove that $(2n+3) I_n = 2an\, I_{n-1} - 2x^n (a-x)^{3/2}$ *(Marathwada, 2008)*

10. If n is a positive integer, show that $\int_0^{2a} x^n \sqrt{(2ax-x^2)} \, dx = \frac{2n+1}{(n+2)\, n!} \cdot \frac{a^{n+2}}{2n} \pi$ *(V.T.U., 2007)*

6.4 REDUCTION FORMULAE for (a) $\int \tan^n x \, dx$ (b) $\int \cot^n x \, dx$

(*a*) Let $I_n = \int \tan^n x \, dx = \int \tan^{n-2} x \cdot \tan^2 x \, dx = \int \tan^{n-2} x \cdot (\sec^2 x - 1) \, dx$

$$= \int \tan^{n-2} x \sec^2 x \, dx - \int \tan^{n-2} x \, dx$$

Thus, $I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$ which is the required reduction formula.

(*b*) Let $I_n = \int \cot^n x \, dx = \int \cot^{n-2} x \cot^2 x \, dx = \int \cot^{n-2} x (\operatorname{cosec}^2 x - 1) \, dx$

$$= \int \cot^{n-2} x \operatorname{cosec}^2 x \, dx - \int \cot^{n-2} x \, dx$$

Thus $I_n = -\frac{\cot^{n-1} x}{n-1} - I_{n-2}$

which is the required reduction formula.

Example 6.7. *Evaluate (i) $\tan^5 x \, dx$ (ii) $\int \cot^6 x \, dx$.*

Solution. (*i*) Putting $n = 5, 3$ successively in the reduction formula for $\int \tan^n x \, dx$, we get

$$I_5 = \frac{1}{4} \tan^4 x - I_3 ; \quad I_3 = \frac{1}{2} \tan^2 x - I_1$$

Thus $I_5 = \frac{1}{4} \tan^4 x - \frac{1}{2} \tan^2 x + I_1$

i.e., $\int \tan^5 x \, dx = \frac{1}{4} \tan^4 x - \frac{1}{2} \tan^2 x + \int \tan x \, dx = \frac{1}{4} \tan^4 x - \frac{1}{2} \tan^2 x - \log \cos x.$

(*ii*) Putting $n = 6, 4, 2$ successively in the reduction formula for $\int \cot^n x \, dx$, we get

$$I_6 = -\frac{1}{5} \cot^5 x - I_4 ; I_4 = -\frac{1}{3} \cot^3 x - I_2 ; I_2 = -\cot x - I_0$$

Thus $I_6 = \frac{-1}{5} \cot^5 x + \frac{1}{3} \cot^3 x - \cot x - \int dx$

i.e., $$\int \cot^6 x\, dx = -\frac{1}{5}\cot^5 x + \frac{1}{3}\cot^3 x - \cot x - x.$$

Example 6.8. *If* $I_n = \int_0^{\pi/4} \tan^n \theta\, d\theta$, *prove that* $n(I_{n-1} + I_{n+1}) = 1$. (V.T.U., 2003)

Solution. The reduction formula for $\int_0^{\pi/4} \tan^n \theta\, d\theta$ is

$$I_n = \frac{1}{n-1}\left| \tan^n x \right|_0^{\pi/4} - I_{n-2} = \frac{1}{n-1} - I_{n-2} \quad \text{or} \quad I_n + I_{n-2} = \frac{1}{n-1}$$

Changing n to $n + 1$, we obtain

$$I_{n+1} + I_{n-1} = \frac{1}{(n+1)} \quad \text{or} \quad (n+1)(I_{n+1} + I_{n-1}) = 1.$$

6.5 REDUCTION FORMULAE for (a) $\int \sec^n x\, dx$ (b) $\int \operatorname{cosec}^n x\, dx$

(*a*) Let $$I_n = \int \sec^n x\, dx = \int \sec^{n-2} x \,.\, \sec^2 x\, dx$$

Integrating by parts, we have

$$I_n = \sec^{n-2} x.\, \tan x - \int \{(n-2)\sec^{n-3} x \cdot \sec x \tan x\} \tan x\, dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x \cdot \tan^2 x\, dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x \cdot (\sec^2 x - 1)\, dx$$

$$= \sec^{n-2} x \tan x - (n-2) I_n + (n-2) I_{n-2}$$

Transposing, we have

$$(n-1)I_n = \sec^{n-2} x \tan x + (n-2)I_{n-2}$$

Thus $$I_n = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}$$ which is the desired reduction formula.

(*b*) Let $$I_n = \int \operatorname{cosec}^n x\, dx = \int \operatorname{cosec}^{n-2} x \cdot \operatorname{cosec}^2 x\, dx$$

Integrating by parts, we have

$$I_n = \operatorname{cosec}^{n-2} x.\, (-\cot x) - \int [(n-2)\operatorname{cosec}^{n-3} x \cdot (-\operatorname{cosec} x \cot x) \cdot (-\cot x)\, dx$$

$$= -\cot x \operatorname{cosec}^{n-2} x - (n-2) \int \operatorname{cosec}^{n-2} x (\operatorname{cosec}^2 x - 1)\, dx$$

$$= -\cot x \operatorname{cosec}^{n-2} x - (n-2)I_n + (n-2)I_{n-2}$$

or $$[1 + (n-2)]I_n = -\cot x \operatorname{cosec}^{n-2} x + (n-2)I_{n-2}$$

Thus $$I_n = -\frac{\cot x \operatorname{cosec}^{n-2} x}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

which is the required reduction formula.

Example 6.9. *Evaluate* (*i*) $\int_0^{\pi/4} \sec^4 x\, dx$ (*ii*) $\int_{\pi/3}^{\pi/2} \operatorname{cosec}^3 \theta\, d\theta$. (V.T.U., 2008)

Solution. (*i*) Putting $n = 4$ in the reduction formula for $\int \sec^n x\, dx$, we get $I_4 = \frac{\sec^2 x \tan x}{3} + \frac{2}{3} I_2$

$$\therefore \quad \int_0^{\pi/4} \sec^4 x\, dx = \left| \frac{\sec^2 x \tan x}{3} \right|_0^{\pi/4} + \frac{2}{3} \int_0^{\pi/4} \sec^2 x\, dx$$

$$= \frac{2}{3} + \frac{2}{3} \left| \tan x \right|_0^{\pi/4} = \frac{2}{3}(1+1) = 4/3.$$

(ii) Putting $n = 3$ in the reduction formula for $\int \operatorname{cosec}^n x\,dx$, we get

$$I_3 = -\frac{1}{2}\cot x \operatorname{cosec} x + \frac{1}{2} I_1$$

$$\therefore \quad \int_{\pi/3}^{\pi/2} \operatorname{cosec}^3 x\,dx = -\frac{1}{2}\left| \cot x \operatorname{cosec} x \right|_{\pi/3}^{\pi/2} + \frac{1}{2}\int_{\pi/3}^{\pi/2} \operatorname{cosec} x\,dx$$

$$= -\frac{1}{2}\left(0 - \frac{2}{3}\right) + \frac{1}{2}\left| \log(\operatorname{cosec} x - \cot x) \right|_{\pi/3}^{\pi/2}$$

$$= \frac{1}{3} + \frac{1}{2}\left[\log 1 - \log\left(\frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right)\right] = \frac{1}{3} + \frac{1}{4}\log 3.$$

PROBLEMS 6.2

1. Evaluate (i) $\int \tan^6 x\,dx$ (V.T.U., 2007) (ii) $\int \cot^5 x\,dx$.
2. Show that $\int_0^{\pi/4} \tan^7 x\,dx = \frac{1}{12}(5 - 6\log 2)$
3. If $I_n = \int_0^{\pi/4} \tan^n x\,dx$, prove that $(n-1)(I_n + I_{n-2}) = 1$. (V.T.U., 2009)
   Hence evaluate I_5. (Madras, 2000)
4. If $I_n = \int_{\pi/4}^{\pi/2} \cot^n \theta\,d\theta\ (n > 2)$, prove that $I_n = \frac{1}{n-1} - I_{n-1}$. Hence evaluate I_4. (Marathwada, 2008)
5. Obtain the reduction formula for $\int_0^{\pi/4} \sec^n \theta\,d\theta$. (V.T.U., 2010 S)
6. Evaluate (i) $\int \sec^6 \theta\,d\theta$ (ii) $\int_{\pi/6}^{\pi/2} \operatorname{cosec}^5 d\theta$.
7. Evaluate $\int_0^a (a^2 + x^2)^{5/2}\,dx$.
8. If $I_n = \int \frac{t^n}{1+t^2}\,dt$, show that $I_{n+2} = \frac{t^{n+1}}{n+1} - I_n$. Hence evaluate I_6.

6.6 REDUCTION FORMULAE for

(a) $\int x^n e^{ax}\,dx$ **(b)** $\int x^m (\log x)^n\,dx$.

(a) Let $\quad I_n = \int x^n e^{ax}\,dx$

Integrating by parts, we have

$$I_n = x^n \cdot \frac{e^{ax}}{a} - \int n x^{n-1} \cdot \frac{e^{ax}}{a}\,dx$$

or $\quad I_n = \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-1}$ which is the required reduction formula. (Madras, 2006)

(b) Let $\quad I_{m,n} = \int x^m (\log x)^n\,dx = \int (\log x)^n \cdot x^m\,dx$

Integrating by parts, we have

$$I_{m,n} = (\log x)^n \cdot \frac{x^{m+1}}{m+1} - \int n(\log x)^{n-1} \cdot \frac{1}{x} \cdot \frac{x^{m+1}}{m+1}\,dx$$

$$= \frac{x^{m+1}}{m+1}(\log x)^n - \frac{n}{m+1}\int x^m (\log x)^{n-1}\,dx \quad \text{or} \quad I_{m,n} = \frac{x^{m+1}}{m+1}(\log x)^n - \frac{n}{m+1} I_{m,n-1}$$

which is the desired reduction formula.

6.7 REDUCTION FORMULAE for

(a) $\int \mathbf{x^n \sin mx\, dx}$ (b) $\int \mathbf{x^n \cos mx\, dx}$ (c) $\int \mathbf{\cos^m x \sin nx\, dx}$

(a) Let $I_n = \int x^n \sin mx\, dx$

Integrating by parts, we get

$$I_n = x^n\left(\frac{-\cos mx}{m}\right) - \int n\, x^{n-1}\left(\frac{-\cos mx}{m}\right) dx$$

$$= -\frac{x^n \cos mx}{m} + \frac{n}{m}\int x^{n-1} \cos mx\, dx \qquad \text{[Again integrate by parts]}$$

$$= -\frac{x^n \cos mx}{m} + \frac{n}{m}\left\{x^{n-1} \cdot \frac{\sin mx}{m} - \left\{\int (n-1) x^{n-2} \cdot \frac{\sin mx}{m}\, dx\right\}\right\}$$

or
$$I_n = -\frac{x^n \cos mx}{m} + \frac{n}{m^2}\, x^{n-1} \sin mx - \frac{n(n-1)}{m^2} I_{n-2}$$

which is the desired reduction formula. (*Madras, 2003*)

(b) Let $I_n = \int x^n \cos mx\, dx$

Integrating twice by parts as above, we get

$$I_n = \frac{x^n \sin mx}{m} + \frac{n}{m^2}\, x^{n-1} \cos mx - \frac{n(n-1)}{m^2}\, I_{n-2}$$

(c) Let $I_{m,n} = \int \cos^m x \sin nx\, dx$

Integrating by parts,

$$I_{m,n} = -\cos^m x \cdot \frac{\cos nx}{n} - \int m \cos^{m-1} x\,(-\sin x) \cdot \left(\frac{-\cos nx}{n}\right) dx$$

$$= -\frac{1}{n}\cos^m x \cos nx - \frac{m}{n}\int \cos^{m-1} x \cdot \cos nx \sin x\, dx$$

$$\left[\begin{array}{l} \because\ \sin(n-1)x = \sin nx \cos x - \cos nx \sin x \\ \text{or } \cos nx \sin x = \sin nx \cos x - \sin(n-1)x \end{array}\right.$$

$$= -\frac{1}{n}\cos^m x \cos nx - \frac{m}{n}\int \cos^{m-1} x\,\{\sin nx \cos x - \sin(n-1)x\}\, dx$$

$$= -\frac{1}{n}\cos^m x \cos nx - \frac{m}{n}(I_{m,n} - I_{m-1,n-1})$$

Transposing, we get

$$\left(1 + \frac{m}{n}\right) I_{m,n} = -\frac{1}{n}\cos^m x \cos nx + \frac{m}{n}\, I_{m-1,n-1}$$

or
$$I_{m,n} = -\frac{\cos^m x \cos nx}{m+n} + \frac{m}{m+n}\, I_{m-1,n-1}$$

which is the desired reduction formula.

Example 6.10. *Show that* $\int_0^{\pi/2} \cos^m x \cos nx\, dx = \frac{m}{m+n}\int_0^{\pi/2} \cos^{m-1} x \cos(n-1)x\, dx$

Hence deduce that $\int_0^{\pi/2} \cos^n x \cos nx\, dx = \frac{\pi}{2^{n+1}}$. (*S.V.T.U., 2008*)

Solution. Let $I_{m,n} = \int_0^{\pi/2} \cos^m x \cos nx\, dx$

Integrating by parts

$$I_{m,n} = \left|\cos^m x \cdot \frac{\sin nx}{n}\right|_0^{\pi/2} - \int_0^{\pi/2} m \cos^{m-1} x\,(-\sin x) \times \frac{\sin nx}{n}\, dx$$

$$= \frac{m}{n}\int_0^{\pi/2} \cos^{m-1} x \sin nx \sin x\, dx \qquad \left[\begin{array}{l}\because \cos(n-1)x = \cos nx \cos x + \sin nx \sin x \\ \text{or } \sin nx \sin x = \cos(n-1)x - \cos nx \cos x\end{array}\right.$$

$$= \frac{m}{n}\int_0^{\pi/2} \cos^{m-1} x\,[\cos(n-1)x - \cos nx \cos x]\, dx = \frac{m}{n}(I_{m-1,\,n-1} - I_{m,\,n})$$

Transposing and dividing by $(1 + m/n)$, we get

$$I_{m,\,n} = \frac{m}{m+n} I_{m-1,\,n-1}$$

which is the required result.

Putting $\quad m = n, I_n \left(= \int_0^{\pi/2} \cos^n x \cos nx\, dx\right) = \frac{1}{2} I_{n-1}$

Changing n to $n - 1$,

$$I_{n-1} = \frac{1}{2} I_{n-2}$$

$$\therefore \quad I_n = \frac{1}{2}\left(\frac{1}{2} I_{n-2}\right) = \frac{1}{2^2} I_{n-2} = \frac{1}{2^3} I_{n-3} \ldots\ldots = \frac{1}{2^n} I_{n-n} = \frac{1}{2^n}\cdot\int_0^{\pi/2} (\cos x)^0\, dx$$

Hence $\quad I_n = \frac{1}{2^n}\cdot\frac{\pi}{2} = \frac{\pi}{2^{n+1}}.$

Example 6.11. *Find a reduction formula for* $\int e^{ax} \sin x\, dx$. *Hence evaluate* $\int e^x \sin^3 x\, dx$.

Solution. Let $I_n = \int e^{ax} \sin^n x\, dx = \int \underset{I}{\sin^n x} \cdot \underset{II}{e^{ax}}\, dx$

Integrating by parts,

$$I_n = \sin^n x \cdot \frac{e^{ax}}{a} - \int (n \sin^{n-1} x \cos x) \cdot \frac{e^{ax}}{a}\, dx$$

$$= \frac{e^{ax} \sin^n x}{a} - \frac{n}{a}\int (\sin^{n-1} x \cos x) \cdot e^{ax}\, dx \qquad \text{[Again integrating by parts]}$$

$$= \frac{e^{ax} \sin^n x}{a} - \frac{n}{a}\left[\sin^{n-1} x \cos x \cdot \frac{e^{ax}}{a} - \int \{(n-1)\sin^{n-2} x \times \cos x \cdot \cos x + \sin^{n-1} x(-\sin x)\} \frac{e^{ax}}{a}\, dx\right]$$

$$= \frac{e^{ax} \sin^{n-1} x}{a^2}(a \sin x - n \cos x) + \frac{n}{a^2}\int [(n-1)\sin^{n-2} x \times (1 - \sin^2 x) - \sin^n x]\, e^{ax}\, dx$$

$$= \frac{e^{ax} \sin^{n-1} x}{a}(a \sin x - n \cos x) + \frac{n(n-1)}{a^2} I_{n-2} - \frac{n^2}{a^2} I_n$$

Transposing and dividing by $(1 + n^2/a^2)$, we get

$$I_n = \frac{e^{ax} \sin^{n-1} x\,(a \sin x - n \cos x)}{a^2 + n^2} + \frac{n(n-1)}{a^2 + n^2} I_{n-2}$$

which is the required reduction formula.

Putting $a = 1$ and $n = 3$, we get

$$I_3 = \frac{e^x \sin^2 x\,(\sin x - 3\cos x)}{1^2 + 9} + \frac{3 \cdot 2}{1^2 + 9} I_1$$

But $\quad I_1 = \int e^x \sin x\, dx = \frac{e^{ax}}{\sqrt{2}} \sin(x - \tan^{-1} 1).$

$$\therefore \quad I_3 = \frac{e^x \sin^2 x\,(\sin x - 3\cos x)}{10} + \frac{3}{5}\cdot\frac{e^{ax}}{\sqrt{2}} \sin(x - \pi/4).$$

PROBLEMS 6.3

1. If $I_n = \int x^n e^x \, dx$, show that $I_n + n I_{n-1} = x^n e^x$. Hence find I_4. *(Madras, 2000)*

2. If $u_n = \int_0^a x^n e^{-x} \, dx$, prove that $u_n - (n+a) u_{n-1} + a(n-1) u_{n-2} = 0$. *(Madras, 2003)*

3. Obtain a reduction formula for $\int x^m (\log x)^n \, dx$. Hence evaluate $\int_0^1 x^5 (\log x)^3 \, dx$. *(S.V.T.U., 2009 ; Bhillai, 2005)*

4. If n is a positive integer, show that $\int_0^1 x^m (\log x)^n \, dx = \dfrac{(-1)^n n!}{(m+1)^{n+1}}$, $m > -1$.

5. If $I_n = \int_0^{\pi/2} x \sin^n x \, dx$ $(n > 1)$, prove that $n^2 I_n = n(n-1) I_{n-2} + 1$. Hence evaluate I_5.

6. If $I_n = \int_0^{\pi/2} x \cos^n x \, dx$ $(n > 1)$, prove that $I_n = \dfrac{n-1}{n} I_{n-2} - \dfrac{1}{n^2}$. Hence evaluate I_4.

7. If $u_n = \int_0^{\pi/2} x^n \sin x \, dx$, $(n > 1)$, prove that $u_n + n(n-1) u_{n-2} = n(\pi/2)^{n-1}$. Hence evaluate u_2. *(Madras, 2000 S)*

8. If $I_n = \int x^n \sin ax \, dx$, show that $a^2 I_n = -ax^n \cos ax + nx^{n-1} \sin ax - n(n-1) I_{n-2}$. *(Marathwada, 2008)*

9. Prove that $\int_0^{\pi/2} \cos^{n-2} x \sin nx \, dx = \dfrac{1}{n-1}$, $n > 1$.

10. If $I_{m,n} = \int_0^{\pi/2} \cos^m x \cos nx \, dx$, prove that $I_{m,n} = \dfrac{m(m-1)}{m^2 - n^2} I_{m-2,n}$

11. Find a reduction formula for $\int e^{ax} \cos^n x \, dx$. Hence evaluate $\int_0^{\pi/2} e^{2x} \cos^3 x \, dx$.

12. Obtain a reduction formula for $I_m = \int_0^\infty e^{-x} \sin^m x \, dx$ where $m \geq 2$ in the form

    $(1 + m^2) I_m = m(m-1) I_{m-2}$. Hence evaluate I_4. *(Gorakhpur, 1999)*

6.8 DEFINITE INTEGRALS

Property I. $\int_a^b f(x) \, dx = \int_a^b f(t) \, dt$

(i.e., the value of a definite integral depends on the limits and not on the variable of integration).

Let $\int f(x) \, dx = \phi(x)$; $\quad \therefore \int_a^b f(x) \, dx = \phi(b) - \phi(a)$.

Then $\int f(t) \, dt = \phi(t)$; $\quad \therefore \int_a^b f(t) \, dt = \phi(b) - \phi(a)$.

Hence the result.

Property II. $\int_a^b f(x) \, dx = -\int_b^a f(x) \, dx$

(i.e., the interchange of limits changes the sign of the integral).

Let $\int f(x) \, dx = \phi(x)$; $\quad \therefore \int_a^b f(x) \, dx = \phi(b) - \phi(a)$

and
$$-\int_b^a f(x) \, dx = -\left| \phi(x) \right|_b^a = -[\phi(a) - \phi(b)] = \phi(b) - \phi(a).$$

Hence the result.

Property III. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

Let $\int f(x)\,dx = \phi(x)$, so that $\int_a^b f(x)\,dx = \phi(b) - \phi(a)$ ...(1)

Also $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = |\,\phi(x)\,|_a^c + |\,\phi(x)\,|_c^b$

$$= [\phi(c) - \phi(a)] + [\phi(b) - \phi(c)] = \phi(b) - \phi(a) \quad ...(2)$$

Hence the result follows from (1) and (2).

Property IV. $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

Put $x = a - t$, so that $dx = -dt$. Also when $x = 0$, $t = a$; when $x = a$, $t = 0$.

$$\therefore \quad \int_0^a f(x)\,dx = -\int_a^0 f(a-t)\,dt = \int_0^a f(a-t)\,dt = \int_0^a f(a-x)\,dx \quad \text{[Prop. II]}$$

Example 6.12. *Evaluate* $\int_0^{\pi/2} \frac{\sqrt{(\sin x)}}{\sqrt{(\sin x)} + \sqrt{(\cos x)}}\,dx.$

Solution. Let
$$I = \int_0^{\pi/2} \frac{\sqrt{(\sin x)}}{\sqrt{(\sin x)} + \sqrt{(\cos x)}}\,dx$$

Then
$$I = \int_0^{\pi/2} \frac{\sqrt{\left[\sin\left(\frac{1}{2}\pi - x\right)\right]}}{\sqrt{\left[\sin\left(\frac{1}{2}\pi - x\right)\right]} + \sqrt{\left[\cos\left(\frac{1}{2}\pi - x\right)\right]}}\,dx \quad \text{[Prop. IV]}$$

$$= \int_0^{\pi/2} \frac{\sqrt{(\cos x)}}{\sqrt{(\cos x)} + \sqrt{(\sin x)}}\,dx$$

Adding
$$2I = \int_0^{\pi/2} \frac{\sqrt{(\sin x)} + \sqrt{(\cos x)}}{\sqrt{(\sin x)} + \sqrt{(\cos x)}}\,dx = \int_0^{\pi/2} dx = |\,x\,|_0^{\pi/2} = \frac{\pi}{2}.$$

Hence $I = \pi/4$.

Example 6.13. *Evaluate* $\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx.$ *(Cochin, 2005)*

Solution. Let
$$I = \int_0^1 \frac{\log(1+x)}{1+x^2}\,dx \qquad \left[\begin{array}{l}\text{Put } x = \tan\theta \text{ so that } dx = \sec^2\theta\,d\theta \\ \text{When } x = 0, \theta = 0 \text{ ; when } x = 1, \theta = \pi/4\end{array}\right.$$

$$= \int_0^{\pi/4} \frac{\log(1+\tan\theta)}{1+\tan^2\theta}\,.\sec^2\theta\,d\theta = \int_0^{\pi/4} \log(1+\tan\theta)\,d\theta$$

$$= \int_0^{\pi/4} \log\left[1 + \tan\left(\frac{\pi}{4} - \theta\right)\right]d\theta = \int_0^{\pi/4} \log\left(1 + \frac{1-\tan\theta}{1+\tan\theta}\right)d\theta \quad \text{[Prop. IV]}$$

$$= \int_0^{\pi/4} \log\left(\frac{2}{1+\tan\theta}\right)d\theta = \log 2 \int_0^{\pi/4} d\theta - I$$

Transposing, $2I = \log 2 \cdot |\,\theta\,|_0^{\pi/4} = \frac{\pi}{4}\log 2$. Hence $I = \frac{\pi}{8}\log 2$.

Example 6.14. *Evaluate* $\int_0^{\pi} \frac{x\sin^3 x}{1+\cos^2 x}\,dx.$ *(Madras, 2006)*

Solution. Let
$$I = \int_0^{\pi} \frac{x\sin^3 x}{1+\cos^2 x}\,dx$$

Then $$I = \int_0^\pi \frac{(\pi - x)\sin^3(\pi - x)}{1 + \cos^2(\pi - x)}\, dx$$ [Prop. IV]

$$= \int_0^\pi \frac{(\pi - x)\sin^3 x}{1 + \cos^2 x}\, dx = \pi \int_0^\pi \frac{\sin^3 x}{1 + \cos^2 x}\, dx - I$$

Transposing, $$2I = \pi \int_0^\pi \frac{\sin^3 x}{1 + \cos^2 x}\, dx$$

$$= -\pi \int_1^{-1} (1 - t^2) \frac{dt}{1 + t^2}$$

Put $\cos x = t$ so that $-\sin x\, dx = dt$
When $x = 0, t = 1$; When $x = \pi, t = -1$;

$$= \pi \int_1^{-1} \frac{-2 + (1 + t^2)}{1 + t^2}\, dt = -2\pi \int_1^{-1} \frac{dt}{1 + t^2} + \pi \int_1^{-1} dt$$

$$= -2\pi \left| \tan^{-1} t \right|_1^{-1} + \pi \left| t \right|_1^{-1} = -2\pi \left(-\frac{\pi}{4} - \frac{\pi}{4}\right) - 2\pi.$$ Hence, $I = \pi^2/2 - \pi$.

Property V. $\int_{-a}^{a} \mathbf{f(x)\, dx = 2 \int_0^a f(x)\, dx}$ **, if f(x) is an even function,**

= 0 if f(x) is an odd function. *(Bhopal, 2008)*

$$\int_{-a}^{a} f(x)\, dx = \int_{-a}^{0} f(x)\, dx + \int_0^a f(x)\, dx \quad \text{...(1) [Prop. I]}$$

In $\int_{-a}^{0} f(x)\, dx$, put $x = -t$, so that $dx = -dt$

$$\therefore \quad \int_{-a}^{0} f(x)\, dx = -\int_a^0 f(-t)\, dt = \int_0^a f(-t)\, dt = \int_0^a f(-x)\, dx \quad \text{[Prop. II]}$$

Substituting in (1), we get

$$\int_{-a}^{a} f(x)\, dx = \int_0^a f(-x)\, dx + \int_0^a f(x)\, dx \quad \text{...(2)}$$

(*i*) *If* $f(x)$ *is an even function,* $f(-x) = f(x)$.

$\therefore$ from (2), $\int_{-a}^{a} f(x)\, dx = \int_0^a f(x)\, dx + \int_0^a f(x)\, dx = 2\int_0^a f(x)\, dx$

(*ii*) *If* $f(x)$ *is an odd function,* $f(-x) = -f(x)$.

$\therefore$ from (2), $\int_{-a}^{a} f(x)\, dx = -\int_0^a f(x)\, dx + \int_0^a f(x)\, dx = 0.$

Property VI. $\int_0^{2a} \mathbf{f(x)\, dx = 2 \int_0^a f(x)\, dx}$ **, if f (2a – x) = f(x)**

= 0, if f(2a – x) = – f(x)

$$\int_0^{2a} f(x)\, dx = \int_0^a f(x)\, dx + \int_a^{2a} f(x)\, dx \quad \text{...(1) [Prop. III]}$$

In $\int_0^{2a} f(x)\, dx$, put $x = 2a - t$, so that $dx = -dt$

Also when $x = a, t = a$; when $x = 2a, t = 0$.

$$\therefore \quad \int_0^{2a} f(x)\, dx = -\int_a^0 f(2a - t)\, dt = \int_0^a f(2a - t)\, dt = \int_0^a f(2a - x)\, dx \quad \text{[Prop. II]}$$

Subsituting in (1), we get

$$\int_0^{2a} f(x)\, dx = \int_0^a f(x)\, dx + \int_0^a f(2a - x)\, dx \quad \text{...(2)}$$

(*i*) *If* $f(2a - x) = f(x)$, then from (2)

$$\int_0^{2a} f(x)\, dx = \int_0^a f(x)\, dx + \int_0^a f(x)\, dx = 2\int_0^a f(x)\, dx$$

(*ii*) If $f(2a - x) = -f(x)$, then from (2)

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx - \int_0^a f(x)\,dx = 0.$$

Cor. 1. *If n is even,* $\int_0^{\pi} \sin^m x \cos^n x\,dx = 2\int_0^{\pi/2} \sin^m x \cos^n x\,dx$ *and if n is odd,* $\int_0^{\pi} \sin^m x \cos^n x\,dx = 0.$

Cor. 2. *If m is odd,* $\int_0^{2\pi} \sin^m x \cos^n x\,dx = 0$

and *if m is even,* $\int_0^{2\pi} \sin^m x \cos^n x\,dx = 2\int_0^{\pi} \sin^m x \cos^n x\,dx$

$$= 4\int_0^{\pi/2} \sin^m x \cos^n x\,dx, \text{ if } n \text{ is even} = 0, \text{ if } n \text{ is odd.}$$

Example 6.15. *Evaluate* $\int_0^{\pi} \theta \sin^2\theta \cos^4\theta\,d\theta$. *(V.T.U., 2009 S)*

Solution. Let $I = \int_0^{\pi} \theta \sin^2\theta \cos^4\theta\,d\theta$

Then $I = \int_0^{\pi} (\pi - \theta)\sin^2(\pi - \theta)\cos^4(\pi - \theta)\,d\theta = \pi\int_0^{\pi} \sin^2\theta\cos^4\theta\,d\theta - I$ [Prop. IV]

or $2I = \pi\int_0^{\pi} \sin^2\theta\cos^4\theta\,d\theta = 2\pi\int_0^{\pi/2} \sin^2\theta\cos^4\theta\,d\theta$ [Prop. VI Cor. 2]

$$= 2\pi \cdot \frac{1\cdot3\cdot1}{6\cdot4\cdot2} = \frac{\pi}{2} = \frac{\pi^2}{16}$$

Hence $I = \dfrac{\pi^2}{32}$

Example 6.16. *Evaluate* $\int_0^{\pi/2} \log \sin x\,dx$. *(Anna, 2005 S)*

Solution. Let $I = \int_0^{\pi/2} \log \sin x\,dx$ ...(*i*)

then $I = \int_0^{\pi/2} \log \sin(\pi/2 - x)\,dx = \int_0^{\pi/2} \log \cos x\,dx$ ...(*ii*)

Adding (*i*) and (*ii*)

$$2I = \int_0^{\pi/2} (\log \sin x + \log \cos x)\,dx$$

$$= \int_0^{\pi/2} \log(\sin x + \cos x)\,dx = \int_0^{\pi/2} \log\left(\frac{\sin 2x}{2}\right)dx$$

$$= \int_0^{\pi/2} \log \sin 2x\,dx - \int_0^{\pi/2} \log 2\,dx = \int_0^{\pi/2} \log \sin 2x\,dx - \log 2\int_0^{\pi/2} dx$$

$$= \int_0^{\pi/2} \log \sin 2x\,dx - \log 2\,|x|_0^{\pi/2} = I' - \frac{\pi}{2}\log 2 \qquad ...(iii)$$

where $I' = \int_0^{\pi/2} \log \sin 2x\,dx$ [Put, $2x = t$, so that $2dx = dt$ When $x = 0, t = 0$; when $x = \pi/2, t = \pi$

$$= \frac{1}{2}\int_0^{\pi} \log \sin t\,dt = \frac{1}{2}\int_0^{\pi} \log \sin x\,dx$$ [$\because$ $\log\sin(\pi - x) = \log \sin x$, Prop. IV]

$$= \frac{1}{2}\cdot 2\int_0^{\pi/2} \log \sin x\,dx = I.$$

Thus from (*iii*), $2I = I - (\pi/2)\log 2$, *i.e.*, $I = -(\pi/2)\log 2$.

Obs. The following are its immediate deductions :

$$\int_0^{\pi/2} \log \sin x\, dx = \int_0^{\pi/2} \log \cos x\, dx = -\frac{\pi}{2} \log 2$$

and $$\int_0^{\pi} \log \sin x\, dx = -\pi \log 2.$$

Example 6.17. *Evaluate* $\int_0^1 \frac{\sin^{-1} x}{x} dx$.

Solution. Put $\sin^{-1} x = \theta$ or $x = \sin\theta$ so that $dx = \cos\theta\, d\theta$

Also when $x = 0, \theta = 0$; when $x = 1, \theta = \pi/2$.

$$\therefore \quad \int_0^1 \frac{\sin^{-1} x}{x} dx = \int_0^{\pi/2} \theta \,.\, \frac{\cos\theta}{\sin\theta} d\theta \qquad \text{[Integrate by parts]}$$

$$= [\theta \,.\, \log \sin\theta]_0^{\pi/2} - \int_0^{\pi/2} 1 \,.\, \log \sin\theta\, d\theta$$

$$= -\int_0^{\pi/2} \log \sin\theta\, d\theta = -\left(-\frac{\pi}{2} \log 2\right) = \frac{\pi}{2} \log 2 \qquad \left[\because \operatorname*{Lt}_{x \to 0} (x \log x) = 0\right]$$

PROBLEMS 6.4

Prove that :

1. (i) $\int_0^{\pi/2} \log \tan x \, dx = 0$ (ii) $\int_0^{\pi/2} \sin 2x \log \tan x \, dx = 0$

2. (i) $\int_0^{\infty} \frac{x^7 (1 - x^{12})}{(1+x)^{28}} dx = 0$ (ii) $\int_0^{\pi/4} \log (1 + \tan\theta)\, d\theta = \frac{\pi}{8} \log_e 2$ *(Madras, 2000)*

3. (i) $\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx = \frac{\pi}{4}$ (ii) $\int_0^{a} \frac{dx}{x + \sqrt{(a^2 + x^2)}} = \frac{\pi}{4}$

4. (i) $\int_0^{\pi/2} \frac{dx}{1 + \sqrt{\cot x}} = \frac{\pi}{4}$. (ii) $\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x} dx = \frac{\pi}{4}$.

5. (i) $\int_0^{\pi/2} \frac{x \tan x}{\sec x + \cos x} dx = \frac{\pi^2}{4}$. (ii) $\int_0^{\pi} \frac{x}{1 + \sin x} dx = \pi$ *(Anna, 2002 S)*

6. (i) $\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} = \frac{1}{2} \pi (\pi - 2)$. (ii) $\int_0^{\pi/2} \frac{x\, dx}{\sin x + \cos x} = \frac{\pi}{2\sqrt{2}} \log (\sqrt{2} + 1)$.

Evaluate :

7. (i) $\int_0^{\pi} \sin^4 x \, dx$ (ii) $\int_0^{2\pi} \cos^6 x \, dx$
   (iii) $\int_0^{\pi} \sin^6 x \cos^4 x \, dx$ *(V.T.U., 2001)* (iv) $\int_0^{2\pi} \sin^4 x \cos^6 x \, dx$

8. (i) $\int_0^{\pi} x \sin^7 x \, dx$ *(V.T.U., 2009)* (ii) $\int_0^{\pi} x \cos^4 x \sin^5 x \, dx$ *(Marathwada, 2008)*

Prove that :

9. (i) $\int_0^{\pi} \frac{x\, dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{\pi^2}{2ab}$ (ii) $\int_0^{\pi/2} \frac{x\, dx}{2 \sin^2 x + \cos^2 x} = \frac{\pi^2}{2\sqrt{2}}$

10. (i) $\int_0^{\pi} \frac{x\, dx}{a^2 - \cos^2 x} = \frac{\pi^2}{2a\sqrt{(a^2 - 1)}}$ $(a > 1)$ (ii) $\int_0^{\pi} \frac{x\, dx}{1 + \sin^2 x} = \frac{\pi^2}{2\sqrt{2}}$.

11. $\int_0^{\pi} \log(1+\cos\theta)\, d\theta = -\pi \log_e 2$ (*Madras, 2003*)

12. (*i*) $\int_0^{\infty} \frac{\log(1+x^2)}{1+x^2}\, dx = \pi \log_e 2$ (*ii*) $\int_0^{\infty} \frac{\log(x+1/x)}{1+x^2}\, dx = \pi \log_e 2.$

6.9 (1) INTEGRAL AS THE LIMIT OF A SUM

We have so far considered integration as inverse of differentiation. We shall now define the definite integral as the limit of a sum :

Def. *If $f(x)$ is continuous and single valued in the interval $[a, b]$, then the definite integral of $f(x)$ between the limits a and b is defined by the equation*

$$\int_a^b \mathbf{f(x)\,dx} = \underset{h \to 0}{\mathbf{Lt}}\ \mathbf{h\,[f(a) + f(a+h) + f(a+2h) + \ldots + f(a+\overline{n-1}\,h)\,]},$$

where $nh = b - a$. ...(1)

(2) Evaluation of limits of series

The summation definition of a definite integral enables us to express the limits of sums of certain types of series as definite integrals which can be easily evaluated. We rewrite (1) as follows :

$$\int_a^b f(x)\,dx = \underset{n \to \infty}{\text{Lt}}\ h \sum_{r=0}^{n-1} f(a + rh), \text{ where } nh = b - a.$$

Putting $a = 0$ and $b = 1$, so that $h = 1/n$, we get

$$\underset{n \to \infty}{\mathbf{Lt}}\ \frac{\mathbf{1}}{\mathbf{n}} \sum_{r=0}^{n-1} \mathbf{f}\left(\frac{\mathbf{r}}{\mathbf{n}}\right) = \int_0^1 \mathbf{f(x)\ dx}$$

Thus to express a given series as definite integral:

(*i*) Write the general term (T_r or T_{r+1} whichever involves r)

i.e., $f(r/n) \,.\, 1/n$

(*ii*) Replace r/n by x and $1/n$ by dx,

(*iii*) Integrate the resulting expression, taking

the lower limit = $\underset{n \to \infty}{\text{Lt}}\ (r/n)$ where r is as in the first term,

and the upper limit = $\underset{n \to \infty}{\text{Lt}}\ (r/n)$ where r is as in the last term.

Example 6.18. *Find the limit, when $n \to \infty$, of the series*

$$\frac{n}{n^2} + \frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \ldots + \frac{n}{n^2+(n-1)^2}$$

Solution. Here the general term $(= T_{r+1}) = \dfrac{n}{n^2+r^2} = \dfrac{n}{1+(r/n)^2} \cdot \dfrac{1}{n}$

$$= \frac{1}{1+x^2}\, dx \qquad \text{[Putting } r/n = x \text{ and } 1/n = dx]$$

Now for the first term $r = 0$ and for the last term $r = n - 1$

$\therefore$ the lower limit of integration $= \underset{n \to \infty}{\text{Lt}} \left(\dfrac{0}{n}\right) = 0$

and the upper limit of integration $= \underset{n \to \infty}{\text{Lt}} \left(\dfrac{n-1}{n}\right) = \underset{n \to \infty}{\text{Lt}} \left(1 - \dfrac{1}{n}\right) = 1.$

Hence, the required limit $= \displaystyle\int_0^1 \frac{dx}{1+x^2} = \Big|\tan^{-1} x\Big|_0^1 = \tan^{-1}(1) - \tan^{-1}(0) = \pi/4.$

[To find limit of a product by integration :

Let $P = \underset{n \to \infty}{\text{Lt}}$ (given product)

Take logs of both sides, so that

$$\log P = \underset{n \to \infty}{\text{Lt}} \text{ (a series) } = k \text{ (say). Then } P = e^k.]$$

Example 6.19. *Evaluate* $\underset{n \to \infty}{Lt} \left\{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\ldots\left(1+\frac{n}{n}\right)\right\}^{1/n}$. *(Bhopal, 2008)*

Solution. Let $P = \underset{n \to \infty}{\text{Lt}} \left\{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\ldots\left(1+\frac{n}{n}\right)\right\}^{1/n}$.

Taking logs of both sides,

$$\log P = \underset{n \to \infty}{\text{Lt}} \frac{1}{n}\left\{\log\left(1+\frac{1}{n}\right) + \log\left(1+\frac{2}{n}\right) + \ldots + \log\left(1+\frac{n}{n}\right)\right\}$$

Its general term $= \log\left(1+\frac{r}{n}\right) \cdot \frac{1}{n} = \log(1+x) \cdot dx$ [Putting $r/n = x$ and $1/n = dx$]

Also for first term $r = 1$ and for the last term $r = n$.

$\therefore$ The lower limit of integration $= \underset{n \to \infty}{\text{Lt}} (1/n) = 0$ and the upper limit $= \underset{n \to \infty}{\text{Lt}} (n/n) = 1$

Hence $\log P = \int_0^1 \log(1+x)\,dx = \int_0^1 \log(1+x) \cdot 1\,dx$ [Integrate by parts]

$$= \left| \log(1+x) \cdot x \right|_0^1 - \int_0^1 \frac{1}{1+x} \cdot x\,dx$$

$$= \log 2 - \int_0^1 \frac{1+x-1}{1+x}\,dx = \log 2 - \int_0^1 dx + \int_0^1 \frac{dx}{1+x}$$

$$= \log 2 - \left| x \right|_0^1 + \left| \log(1+x) \right|_0^1 = \log 2 - 1 + \log 2$$

$$= \log 2^2 - \log_e e = \log(4/e). \text{ Hence, } P = 4/e.$$

PROBLEMS 6.5

Find the limit, as $n \to \infty$, of the series :

1. $\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \ldots + \frac{1}{2n}$. *(Bhopal, 2009)*

2. $\frac{1}{n^3+1} + \frac{4}{n^3+8} + \frac{9}{n^3+27} + \ldots + \frac{r^2}{n^3+r^3} + \ldots + \frac{1}{2n}$.

3. $\underset{n \to \infty}{\text{Lt}} \frac{\sqrt{n}}{\sqrt{n^3}} + \frac{\sqrt{n}}{\sqrt{(n+3)^3}} + \frac{\sqrt{n}}{\sqrt{(n+6)^3}} + \ldots + \frac{\sqrt{n}}{\sqrt{[n+3(n-1)]^3}}$.

Evaluate :

4. $\underset{n \to \infty}{\text{Lt}} \sum_{r=1}^{n-1} \frac{1}{\sqrt{(n^2-r^2)}}$ *(Bhopal, 2008)*

5. $\underset{n \to \infty}{\text{Lt}} \frac{[(n+1)(n+2)\ldots(n+n)]^{1/n}}{n}$.

6. $\underset{n \to \infty}{\text{Lt}} \left\{\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\left(1+\frac{3^2}{n^2}\right)\ldots\left(1+\frac{n^2}{n^2}\right)\right\}^{1/n}$ *(Bhopal, 2008)*

6.10 AREAS OF CARTESIAN CURVES

(1) *Area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$, $x = b$ is* $\int_a^b$ **y dx.**

Let AB be the curve $y = f(x)$ between the ordinates LA $(x = a)$ and MB $(x = b)$. (Fig. 6.1)

Let $P(x, y)$, $P'(x + \delta x, y + \delta y)$ be two neighbouring points on the curve and NP, $N'P'$ be their respective ordinates.

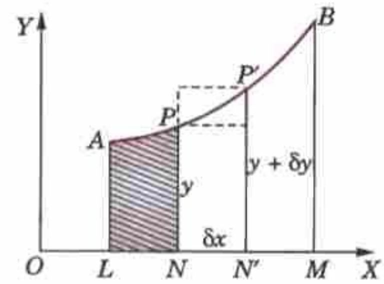

Fig. 6.1

Let the area $ALNP$ be A, which depends on the position of P whose abscissa is x. Then the area $PNN'P' = \delta A$.

Complete the rectangles PN' and $P'N'$.

Then the area $PNN'P'$ lies between the areas of the rectangles PN' and $P'N$.

i.e., δA lies between $y\delta x$ and $(y + \delta y)\, \delta x$

$\therefore$ $\dfrac{\delta A}{\delta x}$ lies between y and $y + \delta y$.

Now taking limits as $P' \to P$ *i.e.*, $\delta x \to 0$ (and $\therefore$ $\delta y \to 0$),

$dA/dx = y$

Integrating both sides between the limits $x = a$ and $x = b$, we have

$$| A |_a^b = \int_a^b y\, dx$$

or (value of A for $x = b$) – (value of A for $x = a$) = $\int_a^b y\, dx$

Thus area $ALMB = \int_a^b y\, dx.$

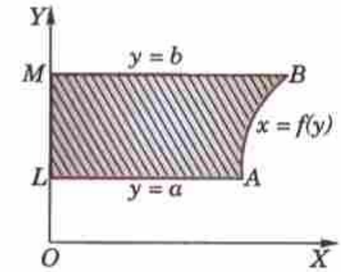

Fig. 6.2

(2) Interchanging x and y in the above formula, we see that *the area bounded by the curve $x = f(y)$, the y-axis and the abscissae $y = a$, $y = b$ is* $\int_a^b$ **x dy.** (Fig. 6.2)

Obs. 1. The area bounded by a curve, the x-axis and two ordinates is called the **area under the curve.** The process of finding the area of plane curves is often called **quadrature.**

Obs. 2. **Sign of an area.** *An area whose boundary is described in the anti-clockwise direction is considered positive and an area whose boundary is described in the clockwise direction is taken as negative.*

In Fig. 6.3, the area $ALMB \left(= \int_a^b y\, dx\right)$ which is described in the anti-clockwise direction and lies above the x-axis, will give a positive result.

In Fig. 6.4, the area $ALMB \left(= \int_a^b y\, dx\right)$ which is described in the clockwise direction and lies below the x-axis, will give a negative result.

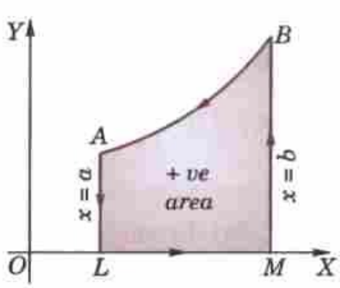

Fig. 6.3

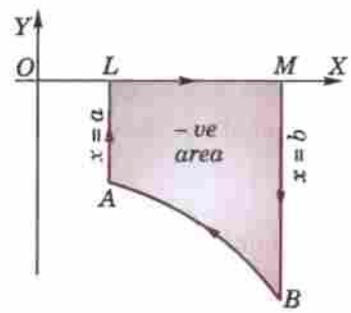

Fig. 6.4

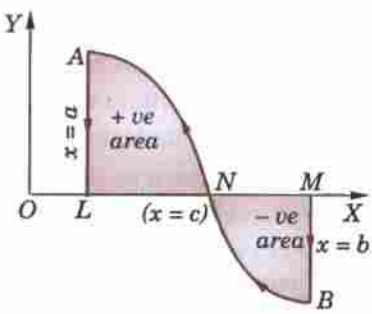

Fig. 6.5

In Fig. 6.5, the area $ALMB \left(= \int_a^b y\, dx\right)$ will not consist of the sum of the area $ALN \left(= \int_a^c y\, dx\right)$ and the area NMB $\left(= \int_c^b y\, dx\right)$, but their difference.

Thus to find the total area in such cases the numerical value of the area of each portion must be evaluated separately and their results added afterwards.

Example 6.20. *Find the area of the loop of the curve $ay^2 = x^2(a - x)$.* (*S.V.T.U., 2009 ; Osmania, 2000*)

Solution. Let us trace the curve roughly to get the limits of integration.

(*i*) The curve is symmetrical about x-axis.

(*ii*) It passes through the origin. The tangents at the origin are $ay^2 = ax^2$ or $y = \pm x$. $\therefore$ Origin is a node.

(*iii*) The curve has no asymptotes.

(*iv*) The curve meets the x-axis at (0, 0) and (a, 0). It meets the y-axis at (0, 0) only.

From the equation of the curve, we have $y = \dfrac{x}{\sqrt{a}}\sqrt{(a-x)}$

For $x > a$, y is imaginary. Thus no portion of the curve lies to the right of the line $x = a$. Also $x \to -\infty, y \to \infty$.

Thus the curve is as shown in Fig. 6.6.

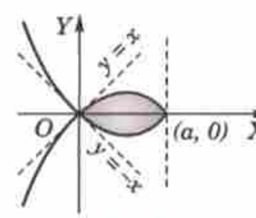

Fig. 6.6

$\therefore$ Area of the loop = 2 (area of upper half of the loop)

$$= 2\int_0^a y\,dx = 2\int_0^a x\sqrt{\left(\frac{a-x}{a}\right)}\,dx = \frac{2}{\sqrt{a}}\int_0^a [a-(a-x)]\sqrt{(a-x)}\,dx$$

$$= \frac{2}{\sqrt{a}}\int_0^a [a(a-x)^{1/2} - (a-x)^{3/2}]\,dx = 2\sqrt{a}\left|\frac{(a-x)^{3/2}}{-3/2}\right|_0^a - \frac{2}{\sqrt{a}}\left|\frac{(a-x)^{5/2}}{-5/2}\right|_0^a$$

$$= -\frac{4}{3}\sqrt{a}(0-a^{3/2}) + \frac{4}{5\sqrt{a}}(0-a^{5/2}) = \frac{4}{3}a^2 - \frac{4}{5}a^2 = \frac{8}{15}a^2.$$

Example 6.21. *Find the area included between the curve $y^2(2a-x) = x^3$ and its asymptote.* (V.T.U., 2003)

Solution. The curve is as shown in Fig. 4.23.

Area between the curve and the asymptote

$$= 2\int_0^{2a} y\,dx = 2\int_0^{2a}\sqrt{\left(\frac{x^3}{2a-x}\right)}\,dx$$

Put $x = 2a\sin^2\theta$ so that $dx = 4a\sin\theta\cos\theta\,d\theta$

$$= 2\int_0^{\pi/2}\sqrt{\left\{\frac{(2a\sin^2\theta)^3}{2a\cos^2\theta}\right\}}\cdot 4a\sin\theta\cos\theta\,d\theta$$

$$= 16a^2\int_0^{\pi/2}\sin^4\theta\,d\theta = 16a^2\cdot\frac{3\cdot1}{4\cdot2}\cdot\frac{\pi}{2} = 3\pi a^2.$$

Example 6.22. *Find the area enclosed by the curve $a^2x^2 = y^3(2a-y)$.*

Solution. Let us first find the limits of integration.

(*i*) The curve is symmetrical about y-axis.

(*ii*) It passes through the origin and the tangents at the origin are $x^2 = 0$ or $x = 0, x = 0$.

$\therefore$ There is a cusp at the origin.

(*iii*) The curve has no asymptote.

(*iv*) The curve meets the x-axis at the origin only and meets the y-axis at (0, 2a). From the equation of the curve, we have

$$x = \frac{y}{a}\sqrt{\{y(2a-y)\}}$$

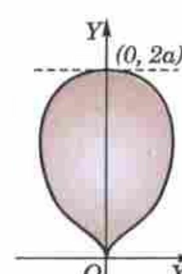

Fig. 6.7

For $y < 0$ or $y > 2a$, x is imaginary. Thus the curve entirely lies between $y = 0$ (x-axis) and $y = 2a$, which is shown in Fig. 6.7.

$$\therefore \text{ Area of the curve } = 2\int_0^{2a} x\,dy = \frac{2}{a}\int_0^{2a} y\sqrt{[y(2a-y)]}\,dy$$

Put $y = 2a\sin^2\theta$ $\therefore$ $dy = 4a\sin\theta\cos\theta\,d\theta$

$$= \frac{2}{a}\int_0^{\pi/2} 2a\sin^2\theta\sqrt{[2a\sin^2\theta(2a-2a\sin^2\theta)]}\times 4a\sin\theta\cos\theta\,d\theta$$

$$= 32a^2\int_0^{\pi/2}\sin^4\theta\cos^2\theta\,d\theta = 32a^2\,\frac{3\cdot1\times1}{6\cdot4\cdot2}\cdot\frac{\pi}{2} = \pi a^2.$$

Example 6.23. *Find the area enclosed between one arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$; and its base.* (V.T.U., 2000)

Solution. To describe its first arch, θ varies from 0 to 2π *i.e.*, x varies from 0 to $2a\pi$ (Fig. 6.8).

$\therefore$ Required area $= \int_{x=0}^{2\pi a} y\,dx$

where $y = a(1 - \cos\theta)$, $dx = a(1 - \cos\theta)\,d\theta$.

$$= \int_{\theta=0}^{\pi/2} a(1 - \cos\theta)\cdot a(1 - \cos\theta)\,d\theta$$

$$= 2a^2\int_0^{\pi}(1 - \cos\theta)^2\,d\theta = 8a^2\int_0^{\pi}\sin^4\frac{\theta}{2}\,d\theta$$

$$= 16a^2\int_0^{\pi/2}\sin^4\phi\,d\phi, \text{ putting } \theta/2 = \phi \text{ so that } d\theta = 2d\phi.$$

$$= 16a^2\cdot\frac{3\cdot 1}{4\cdot 2}\cdot\frac{\pi}{2} = 3\pi a^2.$$

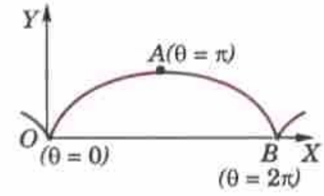

Fig. 6.8

Example 6.24. *Find the area of the tangent cut off from the parabola $x^2 = 8y$ by the line $x - 2y + 8 = 0$.*

Solution. Given parabola is $x^2 = 8y$ ...(*i*)

and the straight line is $x - 2y + 8 = 0$ ...(*ii*)

Substituting the value of y from (*ii*) in (*i*), we get

$$x^2 = 4(x + 8) \text{ or } x^2 - 4x - 32 = 0$$

or $(x - 8)(x + 4) = 0 \quad \therefore \quad x = 8, -4.$

Thus (*i*) and (*ii*) intersect at P and Q where $x = 8$ and $x = -4$. (Fig. 6.9)

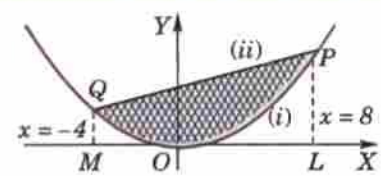

Fig. 6.9

$\therefore$ Required area POQ (*i.e.*, dotted area) = area bounded by straight line (*ii*) and x-axis from $x = -4$ to $x = 8$ – area bounded by parabola (*i*) and x-axis from $x = -4$ to $x = 8$.

$$= \int_{-4}^{8} y\,dx, \text{ from } (ii) - \int_{-4}^{8} y\,dx, \text{ from } (i)$$

$$= \int_{-4}^{8}\frac{x+8}{2}\,dx - \int_{-4}^{8}\frac{x^2}{8}\,dx = \frac{1}{2}\left|\frac{x^2}{2} + 8x\right|_{-4}^{8} - \frac{1}{8}\left|\frac{x^3}{3}\right|_{-4}^{8}$$

$$= \frac{1}{2}\{(32 + 64) - (-24)\} - \frac{1}{24}(512 + 64) = 36.$$

Example 6.25. *Find the area common to the parabola $y^2 = ax$ and the circle $x^2 + y^2 = 4ax$.*

Solution. Given parabola is $y^2 = ax$ ...(*i*)

and the circle is $x^2 + y^2 = 4ax$. ...(*ii*)

Both these curves are symmetrical about x-axis. Solving (*i*) and (*ii*) for x, we have

$$x^2 + ax = 4ax \text{ or } x(x - 3a) = 0$$

or $x = 0, 3a.$

Thus the two curves intersect at the points where $x = 0$ and $x = 3a$. (Fig. 6.10).

Also (*ii*) meets the x-axis at $A(4a, 0)$.

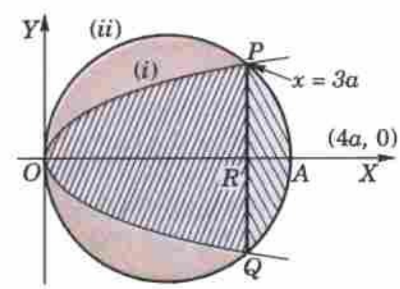

Fig. 6.10

Area common to (*i*) and (*ii*) *i.e.*, the shaded area

$= 2$[Area ORP + Area PRA] (By symmetry)

$$= 2\left[\int_0^{3a} y\,dx, \text{ from } (i) + \int_{3a}^{4a} y\,dx, \text{ from } (ii)\right]$$

$$= 2\left[\int_0^{3a}\sqrt{(ax)}\,dx + \int_{3a}^{4a}\sqrt{(4ax - x^2)}\,dx\right]$$

$$= 2\sqrt{a}\left|\frac{x^{3/2}}{3/2}\right|_0^{3a} + 2\int_{3a}^{4a}\sqrt{[4a^2-(x-2a)^2]}\,dx$$

$$= \frac{4\sqrt{a}}{3}(3a)^{3/2} + 2\left[\frac{1}{2}(x-2a)\sqrt{[4a^2-(x-2a)^2]} + \frac{4a^2}{2}\sin^{-1}\frac{x-2a}{2a}\right]_{3a}^{4a}$$

$$= 4\sqrt{3}\,a^2 + 2[\{0 - \frac{1}{2}a\sqrt{3}a\} + 2a^2\{\pi/2 - \pi/6\}]$$

$$= 4\sqrt{3}\,a^2 - \sqrt{3}a^2 + \frac{4}{3}\pi a^2 = \left(3\sqrt{3} + \frac{4}{3}\pi\right)a^2.$$

PROBLEMS 6.6

1. (i) Find the area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. (Kerala, 2005 ; V.T.U., 2003 S)

   (ii) Find the area bounded by the parabola $y^2 = 4ax$ and its latus-rectum.
2. Find the area bounded by the curve $y = x(x-3)(x-5)$ and the x-axis.
3. Find the area included between the curve $ay^2 = x^3$, the x-axis and the ordinates $x = a$.
4. Find the area of the loop of the curve :

   (i) $3ay^2 = x(x-a)^2$ (Rajasthan, 2005) (ii) $x(x^2+y^2) = a(x^2-y^2)$ (P.T.U., 2010)
5. Find the whole area of the curve :

   (i) $a^2x^2 = y^3(2a-y)$ (Nagpur, 2009) (ii) $8a^2y^2 = x^2(a^2-x^2)$ (V.T.U., 2006)
6. Find the area included between the curve and its asymptotes in each case :

   (i) $xy^2 = a^2(a-x)$. (V.T.U., 2003) (ii) $x^2y^2 = a^2(y^2-x^2)$. (V.T.U., 2007)
7. Show that the area of the loop of the curve $y^2(a+x) = x^2(3a-x)$ is equal to the area between the curve and its asymptote.
8. Find the whole area of the *astroid* $x^{2/3}+y^{2/3}=a^{2/3}$ or $x = a\cos^3\theta$, $y = a\sin^3\theta$. (V.T.U., 2005)
9. Find the area bounded by the curve $\sqrt{x}+\sqrt{y}=\sqrt{a}$ and the coordinate axes.
10. Find the area included between the *cycloid* $x = a(\theta+\sin\theta)$, $y = a(1-\cos\theta)$ *and its base*. Also find the area between the curve and the x-axis. (Gorakhpur, 1999)
11. Find the area common to the circles $x^2+y^2=4$ and $x^2+y^2=4x$.
12. Prove that the area common to the parabolas $x^2 = 4ay$ and $y^2 = 4ax$ is $16a^2/3$. (S.V.T.U., 2008 ; Kurukshetra, 2005)
13. Find the area included between the circle $x^2+y^2=2ax$ and the parabola $y^2 = ax$.
14. Find the area bounded by the parabola $y^2 = 4ax$ and the line $x + y = 3a$.
15. Find the area of the segment cut off from the parabola $y = 4x - x^2$ by the straight line $y = x$.

    (V.T.U., 2010 ; S.V.T.U., 2008)

(2) Areas of polar curves. *Area bounded by the curve $r = f(\theta)$ and the radii vectors*

$$\theta = \alpha, \theta = \beta \textit{ is } \frac{1}{2}\int_\alpha^\beta \mathbf{r^2}\,\mathbf{d\theta}.$$

Let AB be the curve $r = f(\theta)$ between the radii vectors $OA\,(\theta=\alpha)$ and $OB\,(\theta=\beta)$. Let $P(r,\theta)$, $P'(r+\delta r, \theta+\delta\theta)$ be any two neighbouring points on the curve. (Fig. 6.11)

Let the area $OAP = A$ which is a function of θ. Then the area $OPP' = \delta A$. Mark circular arcs PQ and $P'Q'$ with centre O and radii OP and OP'.

Evidently area OPP' lies between the sectors OPQ and $OP'Q'$ *i.e.*, δA lies between $\frac{1}{2}r^2\,\delta\theta$ and $\frac{1}{2}(r+\delta r)^2\,\delta\theta$.

$$\therefore \quad \frac{\delta A}{\delta\theta} \text{ lies between } \tfrac{1}{2}r^2 \text{ and } \tfrac{1}{2}(r+\delta r)^2.$$

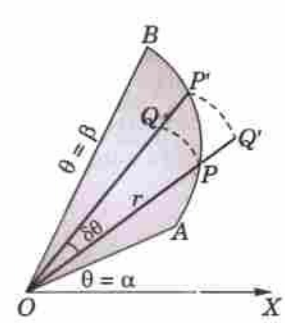

Fig. 6.11

Now taking limits as $\delta\theta \to 0$ ($\therefore \quad \delta r \to 0$), $\dfrac{dA}{d\theta} = \dfrac{1}{2} r^2$

Integrating both sides from $\theta = \alpha$ to $\theta = \beta$, we get $| A |_\alpha^\beta = \int_\alpha^\beta \frac{1}{2} r^2 \, d\theta$

or (value of A for $\theta = \beta$) – (value of A for $\theta = \alpha$) $= \dfrac{1}{2} \displaystyle\int_\alpha^\beta r^2 \, d\theta$

Hence the required area $OAB = \dfrac{1}{2} \displaystyle\int_\alpha^\beta r^2 \, d\theta$.

Example 6.26. *Find the area of the cardioid $r = a(1 - \cos\theta)$.* (V.T.U., 2004)

Solution. The curve is as shown in Fig. 6.12. Its upper half is traced from $\theta = 0$ to $\theta = \pi$.

$\therefore$ Area of the curve

$$= 2 \cdot \frac{1}{2} \int_0^\pi r^2 \, d\theta = a^2 \int_0^\pi (1 - \cos\theta)^2 \, d\theta$$

$$= a^2 \int_0^\pi (2 \sin^2 \theta/2)^2 \, d\theta = 4a^2 \int_0^\pi \sin^4 \theta/2 \cdot d\theta$$

$$= 8a^2 \int_0^{\pi/2} \sin^4 \phi \, d\phi, \text{ putting } \theta/2 = \phi \text{ and } d\theta = 2d\phi.$$

$$= 8a^2 \cdot \frac{3 \cdot 1}{4 \cdot 2} \cdot \frac{\pi}{2} = \frac{3\pi a^2}{2}.$$

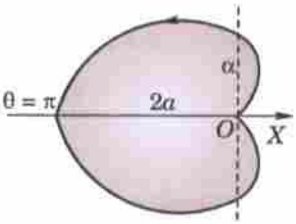

Fig. 6.12

Example 6.27. *Find the area of a loop of the curve $r = a \sin 3\theta$.*

Solution. The curve is as shown in Fig. 4.35. It consists of three loops.
Putting $r = 0$, $\sin 3\theta = 0 \quad \therefore \quad 3\theta = 0$ or π *i.e.*, $\theta = 0$ or $\pi/3$ which are the limits for the first loop.

$\therefore$ Area of a loop

$$= \frac{1}{2} \int_0^{\pi/3} r^2 \, d\theta = \frac{1}{2} a^2 \int_0^{\pi/3} \sin^2 3\theta \, d\theta = \frac{a^2}{4} \int_0^{\pi/3} (1 - \cos 6\theta) \, d\theta$$

$$= \frac{a^2}{4} \left| \theta - \frac{\sin 6\theta}{6} \right|_0^{\pi/3} = \frac{a^2}{4} \left(\frac{\pi}{3} - 0 \right) = \frac{\pi a^2}{12}.$$

Obs. The limits of integration for a loop of $r = a \sin n\theta$ or $r = a \cos n\theta$ are the two consecutive values of θ when $r = 0$.

Example 6.28. *Prove that the area of a loop of the curve $x^3 + y^3 = 3axy$ is $3a^2/2$.*

Solution. Changing to polar form (by putting $x = r\cos\theta$, $y = r \sin\theta$), $r = \dfrac{3a \sin\theta \cos\theta}{\cos^3\theta + \sin^3\theta}$

Putting $r = 0$, $\sin\theta \cos\theta = 0$.

$\therefore \quad \theta = 0, \pi/2$, which are the limits of integration for its loop.

$\therefore$ Area of the loop

$$= \frac{1}{2} \int_0^{\pi/2} r^2 \, d\theta = \frac{1}{2} \int_0^{\pi/2} \frac{9a^2 \sin^2\theta \cos^2\theta}{(\cos^3\theta + \sin^3\theta)^2} \, d\theta$$

$$= \frac{9a^2}{2} \int_0^{\pi/2} \frac{\tan^2\theta \sec^2\theta}{(1 + \tan^3\theta)^2} \, d\theta \qquad \text{[Dividing num. and denom. by } \cos^6\theta]$$

$$= \frac{3a^2}{2} \int_1^\infty \frac{dt}{t^2}, \qquad \text{putting } 1 + \tan^3\theta = t \text{ and } 3\tan^2\theta \sec^2\theta \, d\theta = dt.$$

$$= \frac{3a^2}{2} \left| \frac{t^{-1}}{-1} \right|_1^\infty = \frac{3a^2}{2}(-0 + 1) = \frac{3a^2}{2}.$$

Example 6.29. *Find the area common to the circles*

$$r = a\sqrt{2} \text{ and } r = 2a\cos\theta$$

Solution. The equations of the circles are $r = a\sqrt{2}$ ...(*i*) and $r = 2a\cos\theta$ ...(*ii*)

(*i*) represents a circle with centre at (0, 0) and radius $a\sqrt{2}$. (*ii*) represents a circle symmetrical about *OX*, with centre at $(a, 0)$ and radius *a*.

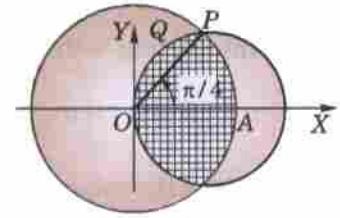

Fig. 6.13

The circles are shown in Fig. 6.13. At their point of intersection *P*, eliminating *r* from (*i*) and (*ii*),

$$a\sqrt{2} = 2a\cos\theta \text{ i.e., } \cos\theta = 1\sqrt{2}$$

or
$$\theta = \pi/4$$

$\therefore$ Required area = 2 × area *OAPQ* (By symmetry)

$$= 2(\text{area } OAP + \text{area } OPQ)$$

$$= 2\left[\frac{1}{2}\int_0^{\pi/4} r^2\, d\theta, \text{ for } (i) + \frac{1}{2}\int_{\pi/4}^{\pi/2} r^2\, d\theta, \text{ for } (ii)\right]$$

$$= \int_0^{\pi/4} (a\sqrt{2})^2\, d\theta + \int_{\pi/4}^{\pi/2} (2a\cos\theta)^2\, d\theta = 2a^2\,|\,\theta\,|_0^{\pi/4} + 4a^2\int_{\pi/4}^{\pi/2} \frac{1+\cos 2\theta}{2}\, d\theta$$

$$= 2a^2(\pi/4 - 0) + 2a^2\left|\,\theta + \frac{\sin 2\theta}{2}\right|_{\pi/4}^{\pi/2} = \frac{\pi a^2}{2} + 2a^2\left(\frac{\pi}{2} - \frac{\pi}{4} - \frac{1}{2}\right) = a^2(\pi - 1).$$

Example 6.30. *Find the area common to the cardioids $r = a(1 + \cos\theta)$ and $r = a(1 - \cos\theta)$.*

(*Kurukshetra, 2006 ; V.T.U., 2006*)

Solution. The cardioid $r = a(1 + \cos\theta)$ is *ABCOB′A* and the cardioid $r = a(1 - \cos\theta)$ is *OC′BA′B′O*. Both the cardioids are symmetrical about the initial line *OX* and intersect at *B* and *B′* (Fig. 6.14)

$\therefore$ Required area (shaded) = 2 area *OC′BCO*

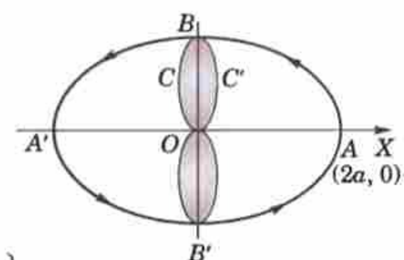

Fig. 6.14

$$= 2\,[\text{area } OC'BO + \text{area } OBCO]$$

$$= 2\left[\left\{\int_0^{\pi/2} \frac{1}{2}r^2\, d\theta\right\}_{r = a(1-\cos\theta)} + \left\{\int_{\pi/2}^{\pi} \frac{1}{2}r^2\, d\theta\right\}_{r = a(1+\cos\theta)}\right]$$

$$= a^2\int_0^{\pi/2} (1 - \cos\theta)^2\, d\theta + a^2\int_{\pi/2}^{\pi} (1 + \cos\theta)^2\, d\theta$$

$$= a^2\left\{\int_0^{\pi/2} (1 - 2\cos\theta + \cos^2\theta)\, d\theta + \int_{\pi/2}^{\pi} [1 + 2\cos\theta + \cos^2\theta]\, d\theta\right\}$$

$$= a^2\left\{\int_0^{\pi} (1 + \cos^2\theta)\, d\theta - 2\int_0^{\pi/2} \cos\theta\, d\theta + 2\int_{\pi/2}^{\pi} \cos\theta\, d\theta\right\}$$

$$= a^2\left\{\int_0^{\pi}\left(1 + \frac{1+\cos 2\theta}{2}\right) d\theta - 2\,|\sin\theta\,|_0^{\pi/2} + 2\,|\sin\theta\,|_{\pi/2}^{\pi}\right\}$$

$$= a^2\left\{\left|\frac{3}{2}\theta + \frac{\sin 2\theta}{4}\right|_0^{\pi} - 2(1-0) + 2(0-1)\right\} = \left(\frac{3\pi}{2} - 4\right)a^2.$$

PROBLEMS 6.7

1. Find the whole area of
   (*i*) the *cardioid* $r = a(1 + \cos\theta)$ (*V.T.U., 2008*) (*ii*) the *lemniscate* $r^2 = a^2\cos 2\theta$; (*V.T.U., 2006*)
2. Find the area of one loop of the curve
   (*i*) $r = a\sin 2\theta$. (*ii*) $r = a\cos 3\theta$.
3. Show that the area included between the folium $x^3 + y^3 = 3axy$ and its asymptote is equal to the area of loop.
4. Prove that the area of the loop of the curve $x^3 + y^3 = 3axy$ is three times the area of the loop of the curve $r^2 = a^2\cos 2\theta$.
5. Find the area inside the circle $r = a\sin\theta$ and lying outside the *cardioid* $r = a(1 - \cos\theta)$. (*Anna, 2009*)
6. Find the area outside the *circle* $r = 2a\cos\theta$ and inside the *cardioid* $r = a(1 + \cos\theta)$. (*Kurukshetra, 2006*)

6.11 LENGTHS OF CURVES

(1) *The length of the arc of the curve $y = f(x)$ between the points where $x = a$ and $x = b$ is*

$$\int_a^b \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}\,dx$$

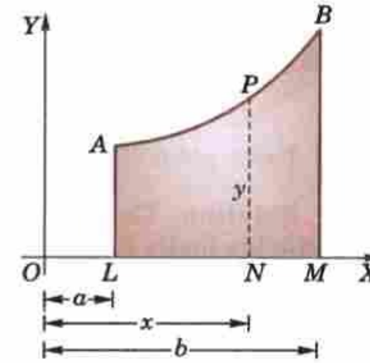

Fig. 6.15

Let AB be the curve $y = f(x)$ between the points A and B where $x = a$ and $x = b$ (Fig. 6.15)

Let $P(x, y)$ be any point on the curve and arc $AP = x$ so that it is a function of x.

Then $$\frac{ds}{dx} = \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}$$ [(1) of p. 164]

$$\therefore \quad \int_a^b \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}\,dx = \int_a^b \frac{ds}{dx}\cdot dx = |\,s\,|_{x=a}^{x=b}$$

= (value of s for $x = b$) – (value of s for $x = a$) = arc AB – 0

Hence, the arc $AB = \int_a^b \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}\,dx.$

(2) *The length of the arc of the curve $x = f(y)$ between the points where $y = a$ and $y = b$, is*

$$\int_a^b \sqrt{\left[1+\left(\frac{dx}{dy}\right)^2\right]}\,dy$$ [Use (2) of p. 165]

(3) *The length of the arc of the curve $x = f(t)$, $y = \phi(t)$ between the points where $t = a$ and $t = b$, is*

$$\int_a^b \sqrt{\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right]}\,dt.$$ [Use (3) p. 165]

(4) *The length of the arc of the curve $r = f(\theta)$ between the points where $\theta = \alpha$ and $\theta = \beta$, is*

$$\int_\alpha^\beta \sqrt{\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right]}\,d\theta.$$ [Use (1) of p. 165]

(5) *The length of the arc of the curve $\theta = f(r)$ between the points where $r = a$ and $r = b$, is*

$$\int_a^b \sqrt{\left[1+\left(r\frac{d\theta}{dr}\right)^2\right]}\,dr$$ [Use (2) of p. 166]

Example 6.31. *Find the length of the arc of the parabola $x^2 = 4ay$ measured from the vertex to one extremity of the latus-rectum.* *(Delhi, 2002)*

Solution. Let A be the vertex and L an extremity of the latus-rectum so that at A, $x = 0$ and at L, $x = 2a$. (Fig. 6.16).

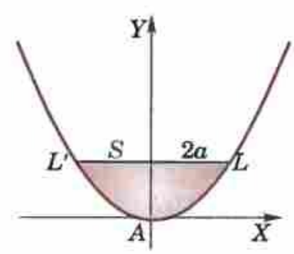

Fig. 6.16

Now $$y = x^2/4a \text{ so that } \frac{dy}{dx} = \frac{1}{4a}\cdot 2x = \frac{x}{2a}$$

$$\therefore \quad \text{arc } AL = \int_0^{2a} \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}\,dx$$

$$= \int_0^{2a} \sqrt{\left[1+\left(\frac{x}{2a}\right)^2\right]}\,dx = \frac{1}{2a}\int_0^{2a} \sqrt{[(2a)^2+x^2]}\,dx$$

$$= \frac{1}{2a}\left[\frac{x\sqrt{[(2a)^2 + x^2]}}{2} + \frac{(2a)^2}{2}\sinh^{-1}\frac{x}{2a}\right]_0^{2a} = \frac{1}{2a}\left[\frac{2a\sqrt{(8a)^2}}{2} + 2a^2\sinh^{-1}1\right]$$

$$= a[\sqrt{2} + \sinh^{-1}1] = a[\sqrt{2} + \log(1+\sqrt{2})] \qquad [\because \sinh^{-1}x = \log[x + \sqrt{(1+x^2)}]]$$

Example 6.32. *Find the perimeter of the loop of the curve $3ay^2 = x(x-a)^2$.*

Solution. The curve is symmetrical about the x-axis and the loop lies between the limits $x = 0$ and $x = a$. (Fig. 6.17).

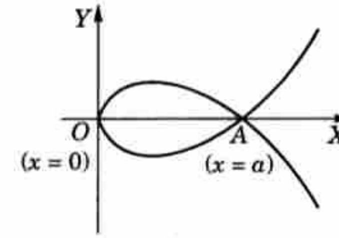

Fig. 6.17

We have $\quad y = \dfrac{\sqrt{x}(x-a)}{\sqrt{(3a)}}$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\sqrt{(3a)}}\left[\frac{3}{2}x^{1/2} - \frac{a}{2}\cdot x^{-1/2}\right] = \frac{1}{2\sqrt{(3a)}}\,\frac{3x-a}{\sqrt{x}}$$

$$\therefore \quad \text{Perimeter of the loop} = 2\int_0^a \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}\,dx \qquad \text{(By symmetry)}$$

$$= 2\int_0^a \sqrt{\left[1 + \frac{(3x-a)^2}{12ax}\right]}\,dx = 2\int_0^a \frac{\sqrt{(9x^2 + 6ax + a^2)}}{\sqrt{(12ax)}}\,dx$$

$$= \frac{1}{\sqrt{(3a)}}\int_0^a \frac{3x+a}{\sqrt{x}}\,dx = \frac{1}{\sqrt{(3a)}}\int_0^a (3x^{1/2} + ax^{-1/2})\,dx$$

$$= \frac{1}{\sqrt{(3a)}}\left|\frac{3x^{3/2}}{3/2} + a\frac{x^{1/2}}{1/2}\right|_0^a = \frac{1}{\sqrt{(3a)}}(4a^{3/2}) = \frac{4a}{\sqrt{3}}.$$

Example 6.33. *Find the length of one arch of the cycloid*

$x = a(t - \sin t)$, $y = a(1 - \cos t)$. *(P.T.U., 2009 ; V.T.U., 2004)*

Solution. As a point moves from one end O to the other end of its first arch, the parameter t increases from 0 to 2π. [see Fig. 6.8]

Also $\quad \dfrac{dx}{dt} = a(1-\cos t),\ \dfrac{dy}{dt} = a\sin t.$

$$\therefore \quad \text{Length of an arch} = \int_0^{2\pi}\sqrt{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]}\,dx$$

$$= \int_0^{2\pi}\sqrt{\{[a(1-\cos t)]^2 + (a\sin t)^2\}}\,dt = a\int_0^{2\pi}\sqrt{[2(1-\cos t)]}\,dt$$

$$= 2a\int_0^{2\pi}\sin t/2\,dt = 2a\left|-\frac{\cos t/2}{1/2}\right|_0^{2\pi} = 4a[(-\cos\pi) - (-\cos 0)] = 8a.$$

Example 6.34. *Find the entire length of the cardioid $r = a(1+\cos\theta)$.*

(P.T.U., 2010 ; Bhopal, 2008 ; Kurukshetra, 2005)

Also show that the upper half is bisected by $\theta = \pi/3$. *(Bhillai, 2005)*

Solution. The cardioid is symmetrical about the initial line and for its upper half, θ increases from 0 to π (Fig. 6.18)

Also $\quad \dfrac{dr}{d\theta} = -a\sin\theta.$

$$\therefore \quad \text{Length of the curve} = 2\int_0^{\pi}\sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]}\,d\theta$$

$$= 2\int_0^{\pi} \sqrt{\{[a(1+\cos\theta)]^2 + (-a\sin\theta)^2\}}\, d\theta = 2a\int_0^{\pi} \sqrt{[2(1+\cos\theta)]}\, d\theta$$

$$= 4a\int_0^{\pi} \cos\theta/2\, d\theta = 4a\left|\frac{\sin\theta/2}{1/2}\right|_0^{\pi} = 8a(\sin\pi/2 - \sin 0) = 8a.$$

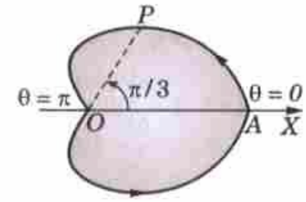

Fig. 6.18

$\therefore$ Length of upper half of the curve is $4a$. Also length of the arc AP from 0 to $\pi/3$.

$$= a\int_0^{\pi/3} \sqrt{[2(1+\cos\theta)]}\, d\theta = 2a\int_0^{\pi/3} \cos\theta/2 \cdot d\theta$$

$$= 4a\left|\sin\theta/2\right|_0^{\pi/3} = 2a = \text{half the length of upper half of the cardioid.}$$

PROBLEMS 6.8

1. Find the length of the arc of the *semi-cubical parabola* $ay^2 = x^3$ from the vertex to the ordinate $x = 5a$.
2. Find the length of the curve (*i*) $y = \log \sec x$ from $x = 0$ to $x = \pi/3$. (*V.T.U., 2010 S ; P.T.U., 2007*)
   (*ii*) $y = \log [(e^x - 1)/(e^x + 1)]$ from $x = 1$ to $x = 2$.
3. Find the length of the arc of the parabola $y^2 = 4ax$ (*i*) from the vertex to one end of the latus-rectum.
   (*ii*) cut off by the line $3y = 8x$. (*V.T.U., 2008 S ; Mumbai, 2006*)
4. Find the perimeter of the loop of the following curves :
   (*i*) $ay^2 = x^2(a - x)$ (*ii*) $9y^2 = (x - 2)(x - 5)^2$.
5. Find the length of the curve $y^2 = (2x - 1)^2$ cut off by the line $x = 4$. (*V.T.U., 2000 S*)
6. Show that the whole length of the curve $x^2(a^2 - x^2) = 8a^2y^2$ is $\pi a\sqrt{2}$.
7. (*a*) Find the length of an arch of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$.
   (*b*) By finding the length of the curve show that the curve $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, is divided in the ratio 1 : 3 at $\theta = 2\pi/3$. (*S.V.T.U., 2009*)
8. Find the whole length of the curve $x = a\cos^3 t$, $y = a\sin^3 t$ *i.e.*, $x^{2/3} + y^{2/3} = a^{2/3}$.
   (*V.T.U., 2010 ; Marathwada, 2008 ; Rajasthan, 2006*)
   Also show that the line $\theta = \pi/3$ divides the length of this *astroid* in the first quadrant in the ratio 1 : 3. (*Mumbai, 2001*)
9. Find the length of the loop of the curve $x = t^2$, $y = t - t^3/3$. (*Mumbai, 2001*)
10. For the curve $r = ae^{\theta}\cot\alpha$, prove that s/r = constant, s being measured from the origin.
11. Find the length of the curve $\theta = \dfrac{1}{2}\left(r + \dfrac{1}{r}\right)$ from $r = 1$ to $r = 3$. (*Marathwada, 2008*)
12. Find the perimeter of the *cardioid* $r = a(1 - \cos\theta)$. Also show that the upper half of the curve is bisected by the line $\theta = 2\pi/3$.
13. Find the whole length of the *lemniscate* $r^2 = a^2\cos 2\theta$.
14. Find the length of the parabola $r(1 + \cos\theta) = 2a$ as cut off by the latus-rectum. (*J.N.T.U., 2003*)

6.12 (1) VOLUMES OF REVOLUTION

(*a*) **Revolution about x-axis.** *The volume of the solid generated by the revolution about the x-axis, of the area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$, $x = b$ is*

$$\int_a^b \pi \mathbf{y^2\, dx}.$$

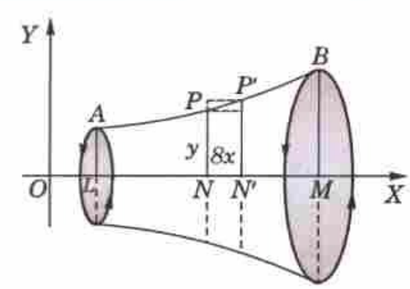

Fig. 6.19

Let AB be the curve $y = f(x)$ between the ordinates $LA(x = a)$ and $MB(x = b)$.

Let $P(x, y)$, $P'(x + \delta x, y + \delta y)$ be two neighbouring points on the curve and NP, $N'P'$ be their respective ordinates (Fig. 6.19).

Let the volume of the solid generated by the revolution about x-axis of the area $ALNP$ be V, which is clearly a function of x. Then the volume of the solid generated by the revolution of the area $PNN'P'$ is δV. Complete the rectangles PN' and $P'N$.

The δV lies between the volumes of the right circular cylinders generated by the revolution of rectangles PN' and $P'N$,

i.e., δV lies between $\pi y^2\,\delta x$ and $\pi(y+\delta y)^2\,\delta x$.

$\therefore \quad \dfrac{\delta V}{\delta x}$ lies between πy^2 and $\pi(y+\delta y)^2$.

Now taking limits as $P' \to P$, *i.e.*, $\delta x \to 0$ (and $\therefore \; \delta y \to 0$), $\dfrac{dV}{dx} = \pi y^2$

$$\therefore \quad \int_a^b \frac{dV}{dx}\,dx = \int_a^b \pi y^2\,dx \quad \text{or} \quad |\,V\,|_{x=a}^{b} = \int_a^b \pi y^2\,dx$$

or (value of V for $x = b$) – (value of V for $x = a$)

i.e., volume of the solid obtained by the revolution of the area $ALMB = \int_a^b \pi y^2\,dx$.

Example 6.35. *Find the volume of a sphere of radius a.* (S.V.T.U., 2007)

Solution. Let the sphere be generated by the revolution of the semi-circle ABC, of radius a about its diameter CA (Fig. 6.20)

Taking CA as the x-axis and its mid-point O as the origin, the equation of the circle ABC is $x^2 + y^2 = a^2$.

$\therefore$ Volume of the sphere = 2 (volume of the solid generated by the revolution about x-axis of the quadrant OAB)

$$= 2\int_0^a \pi y^2\,dx = 2\pi \int_0^b (a^2 - x^2)\,dx$$

$$= 2\pi \left| a^2x - \frac{x^3}{3} \right|_0^a = 2\pi\left[a^3 - \frac{a^3}{3} - (0-0)\right] = \frac{4}{3}\pi a^3.$$

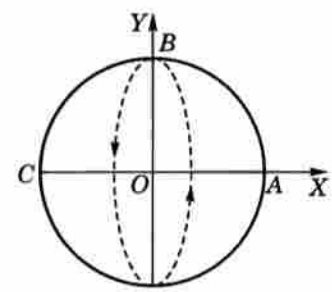

Fig. 6.20

Example 6.36. *Find the volume formed by the revolution of loop of the curve $y^2(a+x) = x^2(3a-x)$, about the x-axis.* (Marathwada, 2008)

Solution. The curve is symmetrical about the x-axis, and for the upper half of its loop x varies from 0 to $3a$ (Fig. 6.21)

$$\therefore \quad \text{Volume of the loop} = \int_0^{3a} \pi y^2\,dx = \pi \int_0^{3a} \frac{x^2(3a-x)}{a+x}\,dx$$

$$= \pi \int_0^{3a} \frac{-x^3 + 3ax^2}{x+a}\,dx$$

[Divide the numerator by the denominator]

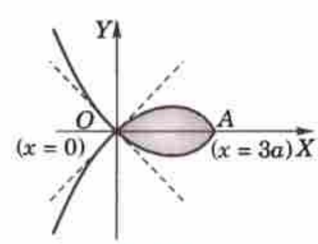

Fig. 6.21

$$= \pi \int_0^{3a} \left[-x^2 + 4ax - 4a^2 + \frac{4a^3}{x+a}\right] dx = \pi \left| -\frac{x^3}{3} + 4a\cdot\frac{x^2}{2} - 4a^2x + 4a^3 \log(x+a) \right|_0^{3a}$$

$$= \pi\left[\frac{-27a^3}{3} + 2a\cdot 9a^2 - 4a^2\cdot 3a + 4a^3 \log 4a - (4a^3 \log a)\right]$$

$$= \pi a^3(-3 + 4\log 4) = \pi a^3(8\log 2 - 3).$$

Example 6.37. *Prove that the volume of the reel formed by the revolution of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1-\cos\theta)$ about the tangent at the vertex is $\pi^2 a^3$.* (V.T.U., 2003)

Solution. The arch AOB of the cycloid is symmetrical about the y-axis and the tangent at the vertex is the x-axis. For half the cycloid OA, θ varies from 0 to π. (Fig. 4.31).

Hence the required volume

$$= 2\int_{\theta=0}^{\theta=\pi} \pi y^2\,dx = 2\pi \int_0^{\pi} a^2(1-\cos\theta)^2 \,.\, a(1+\cos\theta)\,d\theta$$

$$= 2\pi a^3 \int_0^{\pi} (2 \sin^2 \theta/2)^2 \cdot (2 \cos^2 \theta/2)\, d\theta$$

$$= 16\pi a^3 \int_0^{\pi} \sin^4 \theta/2 \cdot \cos^2 \theta/2 \cdot d\theta \qquad [\text{Put } \theta/2 = \phi,\ d\theta = 2d\phi]$$

$$= 32\pi a^3 \int_0^{\pi/2} \sin^4 \phi \cos^2 \phi\, d\phi = 32\pi a^3 \cdot \frac{3\,.\,1 \times 1}{6\,.\,4\,.\,2} \cdot \frac{\pi}{2} = \pi^2 a^3.$$

Example 6.38. *Find the volume of the solid formed by revolving about x-axis, the area enclosed by the parabola $y^2 = 4ax$, its evolute $27ay^2 = 4(x - 2a)^3$ and the x-axis.*

Solution. The curve $27ay^2 = 4(x - 2a)^3$ ...(*i*)
is symmetrical about x-axis and is a semi-cubical parabola with vertex at A $(2a, 0)$. The parabola $y^2 = 4ax$ and (*i*) intersect at B and C where $27a\,(4ax) = 4(x - 2a)^3$ or $x^3 - 6ax^2 - 15a^2x - 8a^3 = 0$ which gives $x = -a, -a, 8a$. Since x is not negative, therefore we have $x = 8a$ (Fig. 6.22).

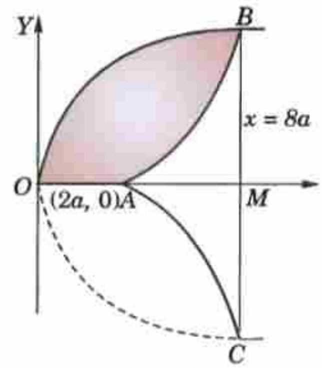

Fig. 6.22

$\therefore$ Required volume = Volume obtained by revolving the shaded area OAB about x-axis = Vol. obtained by revolving area $OMBO$ – Vol. obtained by revolving area $ADBA$

$$= \int_0^{8a} \pi y^2\,(= 4ax)\, dx - \int_{2a}^{8a} \pi y^2\,[\text{for } (i)]\, dx$$

$$= 4a\pi \left| \frac{x^2}{2} \right|_0^{8a} - \frac{4\pi}{27a} \int_{2a}^{8a} (x - 2a)^3\, dx$$

$$= 128\pi a^3 - \frac{4\pi}{27a} \left| \frac{(x - 2a)^4}{4} \right|_{2a}^{8a}$$

$$= 128\pi a^3 - \frac{\pi}{27a} (6a)^4 = 80\pi a^3.$$

(*b*) **Revolution about the y-axis.** Interchanging x and y in the above formula, we see that the *volume of the solid generated by the revolution about y-axis, of the area, bounded by the curve $x = f(y)$, the y-axis and the abscissae $y = a$, $y = b$ is*

$$\int_a^b \pi\, \mathbf{x}^2\, \mathbf{dy}.$$

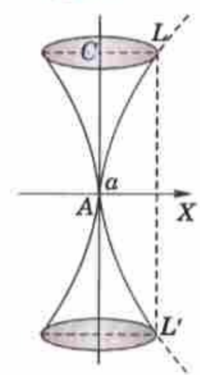

Fig. 6.23

Example 6.39. *Find the volume of the reel-shaped solid formed by the revolution about the y-axis, of the part of the parabola $y^2 = 4ax$ cut off by the latus-rectum.* (Rohtak, 2003)

Solution. Given parabola is $x = y^2/4a$.

Let A be the vertex and L one extremity of the latus-rectum. For the arc AL, y varies from 0 to $2a$ (Fig. 6.23).

$\therefore$ required volume = 2 (volume generated by the revolution about the y-axis of the area ALC)

$$= 2\int_0^{2a} \pi x^2\, dy = 2\pi \int_0^{2a} \frac{y^4}{16a^2} \cdot dy = \frac{\pi}{8a^2} \left| \frac{y^5}{5} \right|_0^{2a} = \frac{\pi}{40a^2} (32a^5 - 0) = \frac{4\pi a^3}{5}.$$

(*c*) **Revolution about any axis.** *The volume of the solid generated by the revolution about any axis LM of the area bounded by the curve AB, the axis LM and the perpendiculars AL, BM on the axis, is*

$$\int_{OL}^{OM} \pi (\mathbf{PN})^2\, \mathbf{d(ON)}$$

where O is a fixed point in LM and PN is perpendicular from any point P of the curve AB on LM.

With O as origin, take OLM as the x-axis and OY, perpendicular to it as the y-axis (Fig. 6.24).

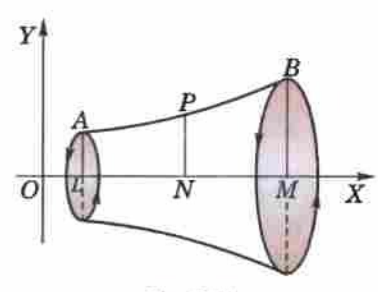

Fig. 6.24

Let the coordinates of P be (x, y) so that $x = ON$, $y = NP$

If $OL = a$, $OM = b$, then required volume $= \int_a^b \pi y^2 \, dx = \int_{OL}^{OM} \pi (PN)^2 \, d(ON)$.

Example 6.40. *Find the volume of the solid obtained by revolving the cissoid $y^2 (2a - x) = x^3$ about its asymptote.* (V.T.U., 2000)

Solution. Given curve is $y = \dfrac{x^3}{2a - x}$ ...(i)

It is symmetrical about x-axis and the asymptote is $x = 2a$. (See Fig. 4.23). If $P(x, y)$ be any point on it and PN is perpendicular on the asymptote AN then $PN = 2a - x$ and

$$AN = y = \frac{x^{3/2}}{\sqrt{(2a - x)}} \qquad \text{[From (i)]}$$

$$\therefore \qquad d(AN) = dy = \frac{\sqrt{(2a-x)}\,(3/2)\sqrt{x} - x^{3/2} \cdot \frac{1}{2}(2a-x)^{-1/2}(-1)}{2a - x}\, dx$$

$$= \frac{3\sqrt{x}(2a - x) + x^{3/2}}{2(2a - x)^{3/2}}\, dx = \frac{3ax^{1/2} - x^{3/2}}{(2a - x)^{3/2}}\, dx$$

$$\therefore \text{ Required volume } = 2\int_{x=0}^{x=2a} \pi (PN)^2 \, d(AN) = 2\pi \int_0^{2a} (2a - x)^2 \cdot \frac{3ax^{1/2} - x^{3/2}}{(2a - x)^{3/2}} \cdot dx$$

$$= 2\pi \int_0^{2a} \sqrt{(2a - x)}\,(3a - x)\sqrt{x}\; dx \qquad \left[\begin{array}{l}\text{Put } x = 2a \sin^2\theta \\ \text{then } dx = 4a \sin\theta\cos\theta\, d\theta\end{array}\right.$$

$$= 2\pi \int_0^{\pi/2} \sqrt{(2a)}\, \cos\theta\, (3a - 2a\sin^2\theta)\, x\, \sqrt{(2a)}\, \sin\theta \,.\, 4a \sin\theta\cos\theta\, d\theta$$

$$= 16\pi a^3 \left[3\int_0^{\pi/2} \sin^2\theta\cos^2\theta\, d\theta - 2\int_0^{\pi/2} \sin^4\theta\cos^2\theta\, d\theta\right]$$

$$= 16\pi a^3 \left[3 \cdot \frac{1 \times 1}{4 \cdot 2} \cdot \frac{\pi}{2} - 2 \cdot \frac{3 \cdot 1 \times 1}{6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2}\right] = 2\pi^2 a^3.$$

(2) Volumes of revolution (polar curves). *The volume of the solid generated by the revolution of the area bounded by the curve $r = f(\theta)$ and the radii vectors $\theta = \alpha$, $\theta = \beta$ (Fig. 6.25)*

(a) about the initial line OX $(\theta = 0) = \displaystyle\int_\alpha^\beta \frac{2\pi}{3} \mathbf{r}^3 \sin\theta\, \mathbf{d}\theta$

(b) about the line OY $(\theta = \pi/2) = \displaystyle\int_\alpha^\beta \frac{2\pi}{3} \mathbf{r}^3 \cos\theta\, \mathbf{d}\theta.$

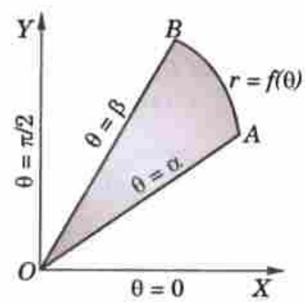

Fig. 6.25

Example 6.41. *Find the volume of the solid generated by the revolution of the cardioid $r = a(1 + \cos\theta)$ about the initial line.* (V.T.U., 2010 ; Kurukshetra, 2009 S)

Solution. The cardioid is symmetrical about the initial line and for its upper half θ varies from 0 to π. [Fig. 6.18].

$$\therefore \text{ required volume} = \int_0^\pi \frac{2}{3}\pi r^3 \sin\theta\, d\theta = \frac{2\pi}{3}\int_0^\pi a^3 (1 + \cos\theta)^3 \sin\theta\, d\theta$$

$$= -\frac{2\pi a^3}{3}\int_0^\pi (1 + \cos\theta)^3 \cdot (-\sin\theta)\, d\theta = -\frac{2\pi a^3}{3}\left|\frac{(1 + \cos\theta)^4}{4}\right|_0^\pi = -\frac{\pi a^3}{6}[0 - 16] = \frac{8}{3}\pi a^3.$$

Example 6.42. *Find the volume of the solid generated by revolving the lemniscate $r^2 = a^2\cos 2\theta$ about the line $\theta = \pi/2$.* (V.T.U., 2006)

Solution. The curve is symmetrical about the pole. For the upper half of the R.H.S. loop, θ varies from 0 to $\pi/4$. (Fig. 4.34).

$\therefore$ required volume = 2(volume generated by the half loop in the first quadrant)

$$= 2\int_0^{\pi/4} \frac{2}{3}\pi r^3 \cos\theta\, d\theta = \frac{4\pi}{3}\cdot\int_0^{\pi/4} a^3(\cos 2\theta)^{3/2}\cos\theta\, d\theta \qquad [\because\ r = a(\cos 2\theta)^{1/2}]$$

$$= \frac{4\pi a^3}{3}\int_0^{\pi/4}(1-2\sin^2\theta)^{3/2}\cos\theta\, d\theta \qquad \left[\begin{array}{l}\text{Put } \sqrt{2}\sin\theta = \sin\phi \\ \therefore\ \sqrt{2}\cos\theta\, d\theta = \cos\phi\, d\phi\end{array}\right.$$

$$= \frac{4\pi a^3}{3}\int_0^{\pi/2}(1-\sin^2\phi)^{3/2}\cdot\frac{1}{\sqrt{2}}\cos\phi\, d\phi = \frac{4\pi a^3}{3\sqrt{2}}\int_0^{\pi/2}\cos^4\phi\, d\phi = \frac{4\pi}{3\sqrt{2}}a^3\,\frac{3.1}{4.2}\cdot\frac{\pi}{2} = \frac{\pi a^3}{4\sqrt{2}}.$$

PROBLEMS 6.9

1. Find the volume generated by the revolution of the area bounded by x-axis, the *catenary* $y = c\cosh x/c$ and the ordinates $x = \pm c$, about the axis of x.
2. Find the volume of a spherical segment of height h cut off from a sphere of radius a.
3. Find the volume generated by revolving the portion of the parabola $y^2 = 4ax$ cut off by its latus-rectum about the axis of the parabola. *(V.T.U., 2009)*
4. Find the volume generated by revolving the area bounded by the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$, $x = 0$, $y = 0$ about the x-axis.
5. Find the volume of the solid generated by revolving the ellipse $x^2/a^2 + y^2/b^2 = 1$.
   (*i*) about the major axis. *(Bhopal, 2002 S)* (*ii*) about the minor axis. *(Bhillai, 2005)*
6. Obtain the volume of the frustrum of a right circular cone whose lower base has radius R, upper base is of radius r and altitude is h.
7. Find the volume generated by the revolution of the curve $27ay^2 = 4(x - 2a)^3$ about the x-axis.
8. Find the volume of the solid formed by the revolution, about the x-axis, of the loop of the curve :
   (*i*) $y^2(a - x) = x^2(a + x)$ (*ii*) $2ay^2 = x(x - a)^2$ (*iii*) $y^2 = x(2x - 1)^2$.
9. Find the volume obtained by revolving one arch of the cycloid
   (*i*) $x = a(t - \sin t)$, $y = a(1 - \cos t)$, about its base. *(Kurukshetra, 2006 ; V.T.U., 2005)*
   (*ii*) $x = a(\theta + \sin\theta)$, $y = a(1 + \cos\theta)$, about the x-axis.
10. Find the volume of the spindle-shaped solid generated by the revolution of the *astroid* $x^{2/3} + y^{2/3} = a^{2/3}$ about the x-axis. *(P.T.U., 2010 ; S.V.T.U., 2008)*
11. Find the volume of the solid formed by the revolution, about the y-axis, of the area enclosed by the curve $xy^2 = 4a^2(2a - x)$ and its asymptote. *(V.T.U., 2006)*
12. Prove that the volume of the solid formed by the revolution of the curve $(a^2 + x^2) = a^3$, about its asymptote is $\frac{1}{2}\pi^2 a^3$.
13. Find the volume generated by the revolution about the initial line of
    (*i*) $r = 2a\cos\theta$ (*ii*) $r = a(1 - \cos\theta)$. *(P.T.U., 2006)*
14. Determine the volume of the solid obtained by revolving the *lemicon* $r = a + b\cos\theta$ $(a > b)$ about the initial line. *(Gorakhpur, 1999)*
15. Find the volume of the solid formed by revolving a loop of the lemniscate $r^2 = a^2\cos 2\theta$ about the initial line. *(J.N.T.U., 2003 ; Delhi, 2002)*

6.13 SURFACE AREAS OF REVOLUTION

(*a*) **Revolution about x-axis.** *The surface area of the solid generated by the revolution about x-axis, of the arc of the curve $y = f(x)$ from $x = a$ to $x = b$, is*

$$\int_{x=a}^{x=b} 2\pi y\, ds.$$

Let AB be the curve $y = f(x)$ between the ordinates LA $(x = a)$ and MB $(x = b)$. Let $P(x, y)$, $P'(x + \delta x y + \delta y)$ be two neighbouring points on the curve and NP, $N'P'$ be their respective ordinates (Fig. 6.19).

Let the arc $AP = s$ so that arc $PP' = \delta s$. Let the surface-area generated by the revolution about x-axis of the arc AP be S and that generated by the revolution of the arc PP' be δS.

Since δs is small, the surface area δS may be regarded as lying between the curved surfaces of the right cylinders of radii PN and $P'N'$ and of same thickness δs.

Thus δS lies between $2\pi y\ \delta s$ and $2\pi (y + \delta y)\ \delta s$

$\therefore \quad \dfrac{\delta S}{\delta s}$ lies between $2\pi y$ and $2\pi (y + \delta y)$

Taking limits as $P' \to P$, *i.e.*, $\delta s \to 0$ and $\delta y \to 0$, $dS/dx = 2\pi y$

$$\therefore \qquad \int_{x=a}^{x=b} \frac{dS}{ds}\, ds = \int_{x=a}^{x=b} 2\pi y\, ds \quad \text{or} \quad \Big| S \Big|_{x=a}^{x=b} = \int_{x=a}^{x=b} 2\pi y\, ds$$

or (value of S for $x = b$) – (value of S for $x = a$) = $\displaystyle\int_{x=a}^{x=b} 2\pi y\ dx$

or surface area generated by the revolution of the arc $AB - 0 = \displaystyle\int_{x=a}^{x=b} 2\pi y\ ds$.

Hence, the required surface area = $\displaystyle\int_{x=a}^{x=b} 2\pi y\ ds$.

Obs. Practical forms of the formula $S = \int 2\pi y\, ds$.

(*i*) *Cartesian form* [*for the curve* $y = f(x)$]

$$S = \int 2\pi y \frac{ds}{dx}\, dx, \text{ where } \frac{ds}{dx} = \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}$$

(*ii*) *Parametric form* [*for the curve* $x = f(t), y = \phi(t)$]

$$S = \int 2\pi y \frac{ds}{dt}\, dt, \text{ where } \frac{ds}{dt} = \sqrt{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]}$$

(*iii*) *Polar form* [*for the curve* $r = f(\theta)$]

$$S = \int 2\pi y \frac{ds}{d\theta}\, d\theta, \text{ where } y = r\sin\theta,\ \frac{ds}{d\theta} = \sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]}.$$

Example 6.43. *Find the surface of the solid formed by revolving the cardioid $r = a\,(1 + \cos\theta)$ about the initial line.* (V.T.U., 2009 ; Rajasthan, 2006 ; J.N.T.U., 2003)

Solution. The cardioid is symmetrical about the initial line and for its upper half, θ varies from 0 to π (Fig. 6.18).

Also
$$\frac{ds}{d\theta} = \sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]} = \sqrt{[a^2(1+\cos\theta)^2 + a^2\sin^2\theta]}$$

$$= a\sqrt{[2(1+\cos\theta)]} = a\sqrt{[2.2\cos^2\theta/2]} = 2a\cos\theta/2$$

$$\therefore \quad \text{required surface} = \int_0^{\pi} 2\pi y \frac{ds}{d\theta}\, d\theta = 2\pi \int_0^{\pi} r\sin\theta \cdot 2a\cos\theta/2\ d\theta$$

$$= 4\pi a \int_0^{\pi} a(1+\cos\theta)\ \sin\theta \cdot \cos\theta/2\ d\theta = 4\pi a^2 \int_0^{\pi} 2\cos^2\frac{\theta}{2} \,.\, 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \,.\, \cos\frac{\theta}{2}\, d\theta$$

$$= 16\pi a^2 \int_0^{\pi} \cos^4\frac{\theta}{2}\sin\frac{\theta}{2}\, d\theta = 16\pi a^2\,(-2) \int_0^{\pi} \cos^4\frac{\theta}{2}\left(-\sin\frac{\theta}{2}\cdot\frac{1}{2}\right) d\theta$$

$$= -32\pi a^2 \left| \frac{\cos^5\theta/2}{5} \right|_0^{\pi} = \frac{-32\pi a^2}{5}(0-1) = \frac{32\pi a^2}{5}.$$

(*b*) **Revolution about y-axis.** Interchanging x and y in the above formula, we see that *the surface of the solid generated by the revolution about y-axis, of the arc of the curve $x = f(y)$ from $y = a$ to $y = b$ is*

$$\int_{y=a}^{y=b} 2\pi x\ ds.$$

Example 6.44. *Find the surface area of the solid generated by the revolution of the astroid $x = a\cos^3 t$, $y = a\sin^3 t$, about the y-axis.*

Solution. The astroid is symmetrical about the x-axis, and for its portion in the first quadrant t varies from 0 to $\pi/2$. (Fig. 4.29).

Also $$\frac{dx}{dt} = -3a\cos^2 t \sin t,\ \frac{dy}{dt} = 3a \sin^2 t \cos t.$$

$$\frac{ds}{dt} = \sqrt{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]} = \sqrt{[9a^2 \cos^4 t \sin^2 t + 9a^2 \sin^4 t \cos^2 t]}$$

$$= 3a \sin t \cos t \sqrt{(\cos^2 t + \sin^2 t)} = 3a \sin t \cos t$$

$\therefore$ $$\text{required surface} = 2\int_0^{\pi/2} 2\pi x \frac{ds}{dt} \cdot dt = 4\pi \int_0^{\pi/2} a \cos^3 t \cdot 3a \sin t \cos t \, dt$$

$$= 12\pi a^2 \int_0^{\pi/2} \sin t \cos^4 t \, dt = 12\pi a^2 \frac{3.1}{5.3.1} = \frac{12\pi a^2}{5}.$$

PROBLEMS 6.10

1. Find the area of the surface generated by revolving the arc of the catenary $y = c \cosh x/c$ from $x = 0$ to $x = c$ about the x-axis.
2. Find the area of the surface formed by the revolution of $y^2 = 4ax$ about its axis, by the arc from the vertex to one end of the latus-rectum.
3. Find the surface of the solid generated by the revolution of the *ellipse* $x^2/a^2 + y^2/b^2 = 1$ about the x-axis. *(Raipur, 2005 ; Bhopal, 2002 S)*
4. Find the volume and surface of the *right circular cone* formed by the revolution of a right-angled triangle about a side which contains the right angle.
5. Obtain the surface area of a *sphere* of radius a.
6. Show that the surface area of the solid generated by the revolution of the curve $x = a \cos^3 t$, $y = a \sin^3 t$ about the x-axis, is $12\pi^2/5$.
7. The arc of the curve $x^{2/3} + y^{2/3} = a^{2/3}$ in the first quadrant revolves about x-axis. Show that the area of the surface generated is $6\pi a^2/5$.
8. Find the surface area of the solid generated by revolving the *cycloid* $x = a(t - \sin t)$, $y = a(1 - \cos t)$ about the base. *(Marathwada, 2008 ; Cochin, 2005 ; Kurukshetra, 2005)*
9. Find the surface area of the solid got by revolving the arch of the *cycloid*
$x = a(\theta + \sin\theta)$, $y = a(1 + \cos\theta)$ about the base. *(V.T.U., 2010 S)*
10. Prove that the surface and volume of the solid generated by the revolution about the x-axis, of the loop of the curve
$x = t^2$, $y = t - t^3/3$, [or $9y^2 = x(x-3)^2$],
are respectively 3π and $3\pi/4$.
11. Prove that the surface of the solid generated by the revolution of the *tractrix* $x = a \cos t + \frac{a}{2} \log \tan^2 t/2$, $y = a \sin t$, about x-axis is $4\pi a^2$.
12. Find the surface area of the solid of revolution of the curve $r = 2a \cos\theta$ about the initial line. *(V.T.U., 2009)*
13. Find the surface of the solid generated by the revolution of the *cardioid* $r = a(1 - \cos\theta)$ about the initial line.
14. Find the surface of the solid generated by the revolution of the *lemniscate* $r^2 = a^2 \cos 2\theta$ about the initial line. *(V.T.U., 2005)*
15. The part of *parabola* $y^2 = 4ax$ cut off by the latus-rectum revolves about the tangent at the vertex. Find the curved surface of the reel thus formed.

6.14 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 6.11

Choose the correct answer or fill up the blanks in the following problems :

1. If $f(x) = f(2a - x)$, then $\int_0^{2a} f(x)\, dx$ is equal to

(a) $\int_a^0 f(2a-x)\,dx$ (b) $2\int_0^a f(x)\,dx$ (c) $-2\int_0^a f(x)\,dx$ (d) 0.

2. $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$ is equal to

(a) 0 (b) 1 (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{2}$.

3. The value of definite integral $\int_{-a}^{a} |x|\,dx$ is equal to

(a) a (b) a^2 (c) 0 (d) $2a$.

4. $\lim_{n\to\infty}\left[\frac{n}{n^2}+\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\ldots+\frac{n}{n^2+(n-1)^2}\right]$ is equal to

(a) $-\frac{\pi}{4}$ (b) 0 (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{3}$.

5. $\int_0^{\pi/2} \frac{\cos 2x}{\cos x+\sin x}\,dx$ equals

(a) -1 (b) 0 (c) 1 (d) 2.

6. $\lim_{n\to\infty}\left(\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{3n}\right)$ equals

(a) $\log_e 2$ (b) $2\log_e 2$ (c) $\log_e 3$ (d) $2\log_e 3$.

7. $\int_0^{\pi} \sin^5\left(\frac{x}{2}\right)$ is equal to

(a) $\frac{16}{15}$ (b) $\frac{15}{16}\pi$ (c) $\frac{16}{15}\pi^2$ (d) $\frac{15}{16}$.

8. $\int_0^{\pi/2} \sin^{99} x\,\cos x\,dx$ is equal to

(a) $\frac{1}{99}$ (b) $\frac{\pi}{100}$ (c) $\frac{99}{100}$ (d) None of these. (*V.T.U., 2009*)

9. The value of $\int_{-\pi/2}^{\pi/2} \cos^7 x\,dx$ is

(a) $\frac{32\pi}{35}$ (b) $\frac{32}{35}$ (c) zero.

10. The length of the arc of the equiangular spiral $r = ae^{\theta\cot\alpha}$ between the points for which the radii vectors are r_1 and r_2 is

(a) (r_2-r_1) cosec α (b) $(r_2-r_1)\cos\alpha$ (c) $(r_2-r_1)\sin\alpha$ (d) $(r_2-r_1)\sec\alpha$.

11. The area of the region in the first quadrant bounded by the y-axis and the curves $y=\sin x$ and $y=\cos x$ is

(a) $\sqrt{2}$ (b) $\sqrt{2}+1$ (c) $\sqrt{2}-1$ (d) $2\sqrt{2}-1$.

12. The value of $\int_0^1 x^{3/2}(1-x)^{3/2}\,dx$ is

(a) $\pi/32$ (b) $-\pi/32$ (c) $3\pi/128$ (d) $-3\pi/128$. (*V.T.U., 2010*)

13. The entire length of the cardioid $r = 5(1+\cos\theta)$ is

(a) 40 (b) 30 (c) 20 (d) 5. (*V.T.U., 2009*)

14. The area of the cardioid $r = a(1-\cos\theta)$ is ....

15. If S_1 and S_2 are surface areas of the solids generated by revolving the ellipses $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$ about the y-axis, then

(a) $S_1 > S_2$ (b) $S_1 < S_2$ (c) $S_1 = S_2$ (d) can't predict.

16. The area of the loop of the curve $r = a\sin 3\theta$ is ....

17. If $I_n = \int_0^{\pi/4} \tan^n\theta\,d\theta$, then $n(I_{n-1}+I_{n+1}) =$ ....

18. $\int_0^2 x^3\sqrt{(2x-x^2)}\,dx =$ ....

19. $\int_0^{\pi/2} \sin 2\theta \log \tan \theta \, d\theta$ is equal to

(a) 1 (b) -1 (c) 0 (d) $\pi/2$.

20. The area of the loop of the folium of Descartes $x^3 + y^3 - 3xy = 0$ is

(a) π (b) $\pi/2$ (c) 1.5 (d) 3.

21. The volume of the frustrum of a right circular cone whose lower base has radius r_1 and upper base has radius r_2 and altitude is h = ...

22. The length of the arc of the curve $y = \log \sec x$ from $x = 0$ to $x = \pi/4$ is

(a) $\log_e 2$ (b) $\log_e 3$ (c) $\log_e (1 + \sqrt{2})$ (d) $\log_e (1 + \sqrt{3})$. (*Bhopal, 2008*)

23. If v_1 = volume of the solid generated by revolving the area included between x-axis and $x^2 + y^2 = a^2$ about x-axis ; v_2 = volume of the solid generated by revolving the entire area of the circle $x^2 + y^2 = a^2$ about x-axis, then

(a) $v_1 = v_2$ (b) $v_2 = 2v_1$ (c) $v_2 = 4v_1$ (d) $v_2 = 16v_1$.

24. If $f(r, \theta) = f(-r, \theta)$, then the curve is symmetrical about the ....

(a) initial line (b) pole (c) origin (d) tangential line. (*V.T.U., 2010*)

25. The volume generated by the revolution of the curve $y = a^3 (a^2 + x^2)^{-1}$ about its asymptote is

(a) $\pi^2 a^3/2$ (b) $\pi a^3/2$ (c) $\pi a^2/2$ (d) $\pi a/2$. (*V.T.U., 2010*)

CHAPTER 7

Multiple Integrals and Beta, Gamma Functions

1. Double integrals. 2. Change of order of integration. 3. Double integrals in Polar coordinates. 4. Areas enclosed by plane curves. 5. Triple integrals. 6. Volume of solids. 7. Change of variables. 8. Area of a curved surface. 9. Calculation of mass. 10. Centre of gravity. 11. Centre of pressure. 12. Moment of inertia. 13. Product of inertia; Principal axes. 14. Beta function. 15. Gamma function. 16. Relation between beta and gamma functions. 17. Elliptic integrals. 18. Error function or Probability integral. 19. Objective Type of Questions.

7.1 DOUBLE INTEGRALS

The definite integral $\int_a^b f(x)\,dx$ is defined as the limit of the sum

$$f(x_1)\,\delta x_1 + f(x_2)\,\delta x_2 + \ldots + f(x_n)\,\delta x_n,$$

where $n \to \infty$ and each of the lengths $\delta x_1, \delta x_2, \ldots$ tends to zero. A double integral is its counterpart in two dimensions.

Consider a function $f(x, y)$ of the independent variables x, y defined at each point in the finite region R of the xy-plane. Divide R into n elementary areas $\delta A_1, \delta A_2, \ldots, \delta A_n$. Let (x_r, y_r) be any point within the rth elementary area δA_r. Consider the sum

$$f(x_1, y_1)\,\delta A_1 + f(x_2, y_2)\,\delta A_2 + \ldots + f(x_n, y_n)\,\delta A_n, \text{ i.e., } \sum_{r=1}^{n} f(x_r, y_r)\,\delta A_r$$

The limit of this sum, if it exists, as the number of sub-divisions increases indefinitely and area of each sub-division decreases to zero, is defined as the *double integral of f(x, y) over the region R* and is written as $\iint_R f(x, y)\,dA$.

Thus $$\iint_R f(x, y)\,dA = \underset{\substack{n \to \infty \\ \delta A \to 0}}{\text{Lt}} \sum_{r=1}^{n} f(x_r, y_r)\,\delta A_r \qquad \ldots(1)$$

The utility of double integrals would be limited if it were required to take limit of sums to evaluate them. However, there is another method of evaluating double integrals by successive single integrations.

For purpose of evaluation, (1) is expressed as the repeated integral $\int_{x_1}^{x_2} \int_{y_1}^{y_2} f(x, y)\,dxdy$.

Its value is found as follows :

(*i*) *When y_1, y_2 are functions of x and x_1, x_2 are constants,* $f(x, y)$ is first integrated w.r.t. y keeping x fixed between limits y_1, y_2 and then resulting expression is integrated w.r.t. x within the limits x_1, x_2 *i.e.,*

$$\boxed{I_1 = \int_{x_1}^{x_2} \boxed{\int_{y_1}^{y_2} f(x, y)\,dy}\,dx}$$

where integration is carried from the inner to the outer rectangle.

Figure 7.1 illustrates this process. Here *AB* and *CD* are the two curves whose equations are $y_1 = f_1(x)$ and $y_2 = f_2(x)$. *PQ* is a vertical strip of width dx.

Then the inner rectangle integral means that the integration is along one edge of the strip *PQ* from *P* to *Q* (x remaining constant), while the outer rectangle integral corresponds to the sliding of the edge from *AC* to *BD*.

Thus the whole region of integration is the area *ABDC*.

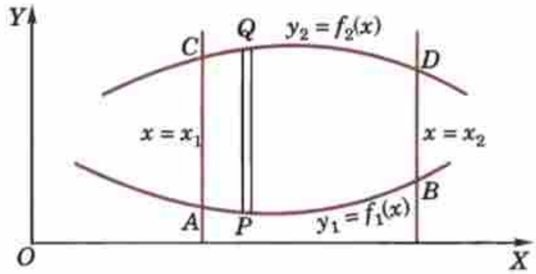

Fig. 7.1

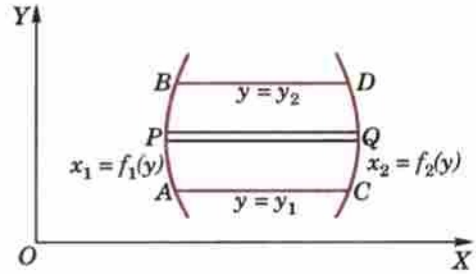

Fig. 7.2

(*ii*) *When x_1, x_2 are functions of y and y_1, y_2 are constants,* $f(x, y)$ is first integrated w.r.t. x keeping y fixed, within the limits x_1, x_2 and the resulting expression is integrated w.r.t. y between the limits y_1, y_2, *i.e.*,

$$I_2 = \boxed{\int_{y_1}^{y_2} \boxed{\int_{x_1}^{x_2} f(x, y)\, dx}\, dy}$$ which is geometrically illustrated by Fig. 7.2.

Here *AB* and *CD* are the curves $x_1 = f_1(y)$ and $x_2 = f_2(y)$. *PQ* is a horizontal strip of width dy.

Then inner rectangle indicates that the integration is along one edge of this strip from *P* to *Q* while the outer rectangle corresponds to the sliding of this edge from *AC* to *BD*.

Thus the whole region of integration is the area *ABDC*.

(*iii*) *When both pairs of limits are constants,* the region of integration is the rectangle *ABDC* (Fig. 7.3).

In I_1, we integrate along the vertical strip *PQ* and then slide it from *AC* to *BD*.

In I_2, we integrate along the horizontal strip *P′Q′* and then slide it from *AB* to *CD*.

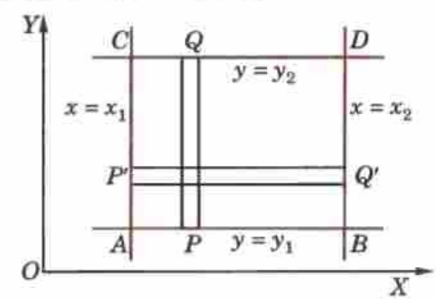

Fig. 7.3

Here obviously $I_1 = I_2$.

Thus *for constant limits, it hardly matters whether we first integrate w.r.t. x and then w.r.t. y or vice versa.*

Example 7.1. *Evaluate* $\int_0^5 \int_0^{x^2} x(x^2 + y^2)\, dxdy$.

Solution. $$I = \int_0^5 dx \int_0^{x^2} (x^3 + xy^3)\, dy = \int_0^5 \left[x^3 y + x \cdot \frac{y^3}{3}\right]_0^{x^2} dx = \int_0^5 \left[x^3 \cdot x^2 + x \cdot \frac{y^6}{3}\right] dx$$

$$= \int_0^5 \left(x^5 + \frac{x^7}{3}\right) dx = \left|\frac{x^6}{6} + \frac{x^8}{24}\right|_0^5 = 5^6\left[\frac{1}{6} + \frac{5^2}{24}\right] = 18880.2 \text{ nearly.}$$

Example 7.2. *Evaluate* $\iint_A xy\, dx\, dy$, *where A is the domain bounded by x-axis, ordinate $x = 2a$ and the curve $x^2 = 4ay$.*

Solution. The line $x = 2a$ and the parabola $x^2 = 4ay$ intersect at $L(2a, a)$. Figure 7.4 shows the domain *A* which is the area *OML*.

Integrating first over a vertical strip *PQ*, *i.e.*, w.r.t. y from $P(y = 0)$ to $Q(y = x^2/4a)$ on the parabola and then w.r.t. x from $x = 0$ to $x = 2a$, we have

$$\iint_A xy\, dx\, dy = \int_0^{2a} dx \int_4^{x^2/4a} xy\, dy = \int_0^{2a} x\left[\frac{y^2}{2}\right]_0^{x^2/4a} dx$$

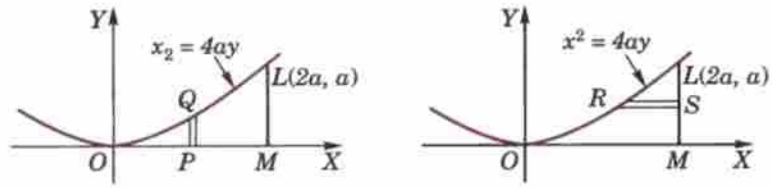

Fig. 7.4

$$= \frac{1}{32a^2} \int_0^{2a} x^5 \, dx = \frac{1}{32a^2} \left| \frac{x^6}{6} \right|_0^{2a} = \frac{a^4}{3}.$$

Otherwise integrating first over a horizontal strip RS, *i.e.*, w.r.t. x from, R ($x = 2\sqrt{ay}$) on the parabola to $S(x = 2a)$ and then w.r.t. y from $y = 0$ to $y = a$, we get

$$\iint_A xy \, dx \, dy = \int_0^a dx \int_{2\sqrt{(ay)}}^{2a} xy \, dx = \int_0^a y \left[\frac{x^2}{2} \right]_{2\sqrt{(ay)}}^{2a} dy$$

$$= 2a \int_0^a (ay - y^2) \, dy = 2a \left[\frac{ay^2}{2} - \frac{y^3}{3} \right]_0^a = \frac{a^4}{3}.$$

Example 7.3. *Evaluate* $\iint_R x^2 \, dx \, dy$ *where R is the region in the first quadrant bounded by the lines $x = y$, $y = 0$, $x = 8$ and the curve $xy = 16$.*

Solution. The line AL ($x = 8$) intersects the hyperbola $xy = 16$ at A (8, 2) while the line $y = x$ intersects this hyperbola at B (4, 4). Figure 7.5 shows the region R of integration which is the area $OLAB$. To evaluate the given integral, we divide this area into two parts OMB and $MLAB$.

$$\therefore \quad \iint_R x^2 \, dxdy = \int_{x \text{ at } 0}^{x \text{ at } M} \int_{y \text{ at } P}^{y \text{ at } Q} x^2 \, dxdy + \int_{x \text{ at } M}^{x \text{ at } L} \int_{y \text{ at } P'}^{y \text{ at } Q'} x^2 \, dxdy$$

$$= \int_0^4 \int_0^x x^2 \, dxdy + \int_4^8 \int_0^{16/x} x^2 \, dxdy$$

$$= \int_0^4 x^2 \, dx \left| y \right|_0^x + \int_4^8 x^2 \, dx \left| y \right|_0^{16/x}$$

$$= \int_0^4 x^3 \, dx + \int_4^8 16x \, dx = \left| \frac{x^4}{4} \right|_0^4 + 16 \left| \frac{x^2}{2} \right|_4^8 = 448$$

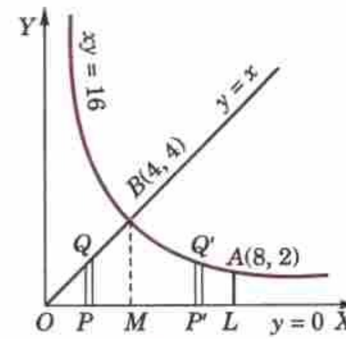

Fig. 7.5

7.2 CHANGE OF ORDER OF INTEGRATION

In a double integral with variable limits, the change of order of integration changes the limit of integration. While doing so, sometimes it is required to split up the region of integration and the given integral is expressed as the sum of a number of double integrals with changed limits. To fix up the new limits, it is always advisable to draw a rough sketch of the region of integration.

The change of order of integration quite often facilitates the evaluation of a double integral. The following examples will make these ideas clear.

Example 7.4. *By changing the order of integration of* $\int_0^\infty \int_0^\infty e^{-xy} \sin px \, dxdy$, *show that*

$$\int_0^\infty \frac{\sin px}{x} \, dx = \frac{\pi}{2}. \qquad \textit{(U.P.T.U., 2004)}$$

Solution. $\int_0^\infty \int_0^\infty e^{-xy} \sin px \, dxdy = \int_0^\infty \left(\int_0^\infty e^{-xy} \sin px \, dx \right) dy$

$$= \int_0^\infty \left| -\frac{e^{-xy}}{p^2+y^2}(p\cos px + y\sin px)\right|_0^\infty dy$$

$$= \int_0^\infty \frac{p}{p^2+y^2}\,dy = \left|\tan^{-1}\left(\frac{y}{p}\right)\right|_0^\infty = \frac{\pi}{2} \qquad ...(i)$$

On changing the order of integration, we have

$$\int_0^\infty \int_0^\infty e^{-xy}\sin px\,dx\,dy = \int_0^\infty \sin px\left\{\int_0^\infty e^{-xy}\,dy\right\}dx$$

$$= \int_0^\infty \sin px\left|\frac{e^{-xy}}{-x}\right|_0^\infty dx = \int_0^\infty \frac{\sin px}{x}\,dx \qquad ...(ii)$$

Thus from (*i*) and (*ii*), we have $\int_0^\infty \frac{\sin px}{x}\,dx = \frac{\pi}{2}$.

Example 7.5. *Change the order of integration in the integral*

$$I = \int_{-a}^{a}\int_0^{\sqrt{(a^2-y^2)}} f(x,y)\,dx\,dy.$$

Solution. Here the elementary strip is parallel to x-axis (such as PQ) and extends from $x = 0$ to $x = \sqrt{(a^2-y^2)}$ (*i.e.*, to the circle $x^2 + y^2 = a^2$) and this strip slides from $y = -a$ to $y = a$. This shaded semi-circular area is, therefore, the region of integration (Fig. 7.6).

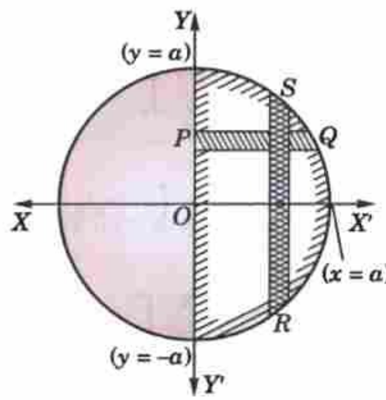

Fig. 7.6

On changing the order of integration, we first integrate w.r.t. y along a vertical strip RS which extends from R [$y = -\sqrt{(a^2-y^2)}$] to S [$y = \sqrt{(a^2-y^2)}$]. To cover the given region, we then integrate w.r.t. x from $x = 0$ to $x = a$.

Thus $$I = \int_0^a dx\int_{-\sqrt{(a^2-x^2)}}^{\sqrt{(a^2-x^2)}} f(x,y)\,dy$$

or $$= \int_0^a \int_{-\sqrt{(a^2-x^2)}}^{\sqrt{(a^2-x^2)}} f(x,y)\,dydx.$$

Example 7.6. *Evaluate* $\int_0^1\int_{e^x}^{e} dydx/\log y$ *by changing the order of integration.*

Solution. Here the integration is first w.r.t. y from P on $y = e^x$ to Q on the line $y = e$. Then the integration is w.r.t. x from $x = 0$ to $x = 1$, giving the shaded region ABC (Fig. 7.7).

On changing the order of integration, we first integrate w.r.t. x from R on $x = 0$ to S on $x = \log y$ and then w.r.t. y from $y = 1$ to $y = e$.

Thus $$\int_0^1\int_{e^x}^{e}\frac{dydx}{\log y} = \int_1^e\int_0^{\log y}\frac{dxdy}{\log y}$$

$$= \int_1^e \frac{dy}{\log y}\left|x\right|_0^{\log y} = \int_1^e dy = \left|y\right|_1^e = e - 1.$$

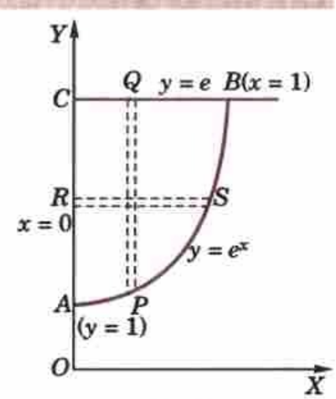

Fig. 7.7

Example 7.7. *Change the order of integration in* $I = \int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dydx$ *and hence evaluate.*

(Nagpur, 2009 ; P.T.U., 2009 S)

Solution. Here integration is first w.r.t. y and P on the parabola $x^2 = 4ay$ to Q on the parabola $y^2 = 4ax$ and then w.r.t. x from $x = 0$ to $x = 4a$ giving the shaded region of integration (Fig. 7.8).

On changing the order of integration, we first integrate w.r.t. x from R to S, then w.r.t. y from $y = 0$ to $y = 4a$

$$\therefore \quad I = \int_0^{4a} \int_{y^2/4a}^{2\sqrt{ay}} dxdy = \int_0^{4a} dy \left| x \right|_{y^2/4a}^{2\sqrt{ay}} = \int_0^{4a} (2\sqrt{ay} - y^2/4a)\, dy$$

$$= \left| 2\sqrt{a}\, \frac{y^{3/2}}{3/2} - \frac{y^3}{12a} \right|_0^{4a} = \frac{32a^2}{3} - \frac{16a^2}{3} = \frac{16a^2}{3}.$$

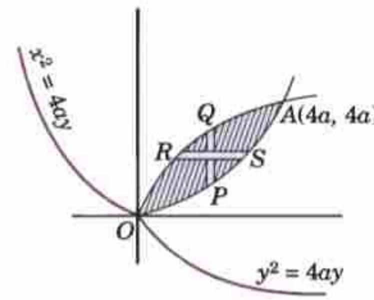

Fig. 7.8

Example 7.8. *Change the order of integration and hence evaluate*

$$I = \int_0^a \int_{\sqrt{ax}}^a \frac{y^2\, dxdy}{\sqrt{(y^4 - a^2x^2)}} \qquad \text{(S.V.T.U., 2006 S)}$$

Solution. Here integration is first w.r.t. y from P on the parabola $y^2 = ax$ to Q on the line $y = a$, then w.r.t. x from $x = 0$ to $x = a$, giving the shaded region OAB of integration (Fig. 7.9).

On changing the order of integration, we first integrate w.r.t. x from R to S, then w.r.t. y from $y = 0$ to $y = a$.

$$\therefore \quad I = \int_0^a \int_0^{y^2/a} \frac{y^2\, dy}{\sqrt{(y^4 - a^2x^2)}}\, dx = \frac{1}{a} \int_0^a \int_0^{y^2/a} y^2\, dy \frac{dx}{\sqrt{[(y^2/a)^2 - x^2]}}\, dx$$

$$= \frac{1}{a} \int_0^a y^2 dy \left| \sin^{-1}\left(\frac{xa}{y^2}\right) \right|_0^{y^2/a} = \frac{1}{a} \int_0^a y^2 dy\ [\sin^{-1}(1) - \sin^{-1}(0)]$$

$$= \frac{\pi}{2a} \int_0^a y^2 dy = \frac{\pi}{2a} \left| \frac{y^3}{3} \right|_0^a = \frac{\pi a^2}{6}.$$

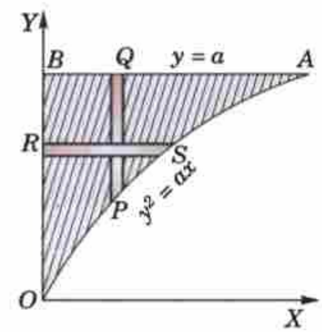

Fig. 7.9

Example 7.9. *Change the order of integration in* $I = \int_0^1 \int_{x^2}^{2-x} xy\, dxdy$ *and hence evaluate the same.*

(Bhopal, 2008 ; V.T.U., 2008 ; S.V.T.U., 2007 ; P.T.U., 2005 ; U.P.T.U., 2005)

Solution. Here the integration is first w.r.t. y along a vertical strip PQ which extends from P on the parabola $y = x^2$ to Q on the line $y = 2 - x$. Such a strip slides from $x = 0$ to $x = 1$, giving the region of integration as the curvilinear triangle OAB (shaded) in Fig. 7.10.

On changing the order of integration, we first integrate w.r.t. x along a horizontal strip $P'Q'$ and that requires the splitting up of the region OAB into two parts by the line AC ($y = 1$), *i.e.*, the curvilinear triangle OAC and the triangle ABC.

For the region OAC, the limits of integration for x are from $x = 0$ to $x = \sqrt{y}$ and those for y are from $y = 0$ to $y = 1$. So the contribution to I from the region OAC is

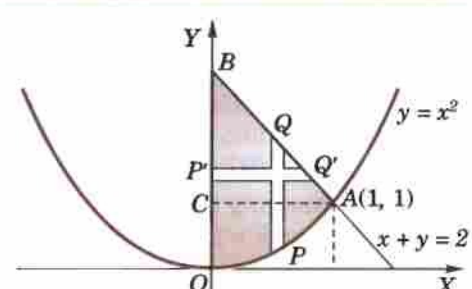

Fig. 7.10

$$I_1 = \int_0^1 dy \int_0^{\sqrt{y}} xy\, dx$$

For the region ABC, the limits of integration for x are from $x = 0$ to $x = 2 - y$ and those for y are from $y = 1$ to $y = 2$. So the contribution to I from the region ABC is

$$I_2 = \int_1^2 dy \int_0^{2-y} xy\, dx.$$

Hence, on reversing the order of integration,

$$I = \int_0^1 dy \int_0^{\sqrt{y}} xy\,dx + \int_1^2 dy \int_0^{2-y} xy\,dx$$

$$= \int_0^1 dy \left| \frac{x^2}{2} \cdot y \right|_0^{\sqrt{y}} + \int_1^2 dy \left| \frac{x^2}{2} \cdot y \right|_0^{2-y} = \frac{1}{2}\int_0^1 y^2 dy + \frac{1}{2}\int_1^2 y(2-y)^2\,dy = \frac{1}{6} + \frac{5}{24} = \frac{3}{8}.$$

Example 7.10. *Change the order of integration in* $I = \int_0^1 \int_x^{\sqrt{(2-x^2)}} \frac{x}{\sqrt{(x^2+y^2)}}\,dx$ *and hence evaluate it.*

(J.N.T.U., 2005 ; Rohtak, 2003)

Solution. Here the integration is first w.r.t. y along PQ which extends from P on the line $y = x$ to Q on the circle $y = \sqrt{(2 \cdot x^2)}$. Then PQ slides from $y = 0$ to $y = 1$, giving the region of integration OAB as in Fig. 7.11.

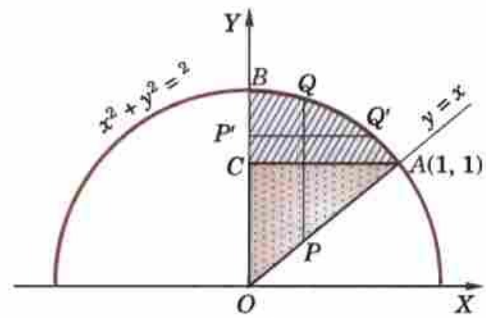

Fig. 7.11

On changing the order of integration, we first integrate w.r.t. x from P' to Q' and that requires splitting the region OAB into two parts OAC and ABC.

For the region OAC, the limits of integration for x are from $x = 0$ to $x = 1$ and those for y are from $y = 0$ to $y = 1$. So the contribution to I from the region OAC is

$$I_1 = \int_0^1 dy \int_0^y \frac{x}{\sqrt{(x^2+y^2)}}\,dx.$$

For the region ABC, the limits of integration for x are 0 to $\sqrt{(2-y^2)}$ and these for y are from 1 to $\sqrt{2}$. So the contribution to I from the region ABC is

$$I_2 = \int_1^{\sqrt{2}} dy \int_0^{\sqrt{(2-y^2)}} \frac{x}{\sqrt{(x^2+y^2)}}\,dx$$

Hence
$$I = \int_0^1 \left| (x^2+y^2)^{1/2} \right|_0^y dy + \int_1^{\sqrt{2}} \left| (x^2+y^2)^{1/2} \right|_0^{\sqrt{(2-y^2)}} dy$$

$$= \int_0^1 (\sqrt{2}-1)\,y\,dy + \int_1^{\sqrt{2}} \sqrt{(2-y)}\,dy = \frac{1}{2}(\sqrt{2}-1) + \sqrt{2}\sqrt{(2-1)} - \frac{1}{2} = 1 - 1/\sqrt{2}.$$

7.3 DOUBLE INTEGRALS IN POLAR COORDINATES

To evaluate $\int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} f(r, \theta)\,dr\,d\theta$, we first integrate w.r.t. r between limits $r = r_1$ and $r = r_2$ keeping θ fixed and the resulting expression is integrated w.r.t. θ from θ_1 to θ_2. In this integral, r_1, r_2 are functions of θ and θ_1, θ_2 are constants.

Figure 7.12 illustrates the process geometrically.

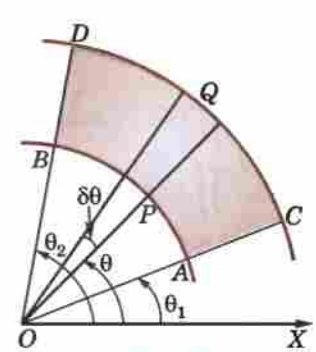

Fig. 7.12

Here AB and CD are the curves $r_1 = f_1(\theta)$ and $r_2 = f_2(\theta)$ bounded by the lines $\theta = \theta_1$ and $\theta = \theta_2$. PQ is a wedge of angular thickness $\delta\theta$.

Then $\int_{r_1}^{r_2} f(r, \theta)\,dr$ indicates that the integration is along PQ from P to Q while the integration w.r.t. θ corresponds to the turning of PQ from AC to BD.

Thus the whole region of integration is the area $ACDB$. The order of integration may be changed with appropriate changes in the limits.

Example 7.11. *Evaluate* $\iint r \sin\theta \, dr \, d\theta$ *over the cardioid* $r = a(1 - \cos\theta)$ *above the initial line.* *(Kerala, 2005)*

Solution. To integrate first w.r.t. r, the limits are from 0 ($r = 0$) to P [$r = a(1 - \cos\theta)$] and to cover the region of integration R, θ varies from 0 to π (Fig. 7.13).

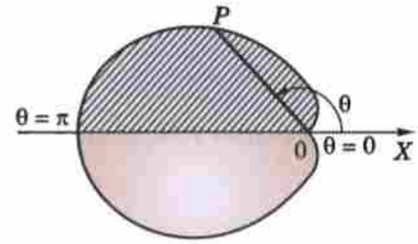

Fig. 7.13

$$\therefore \quad \iint_R r \sin\theta \, dr d\theta = \int_0^\pi \sin\theta \left[\int_0^{r = a(1-\cos\theta)} r dr\right] d\theta$$

$$= \int_0^\pi \sin\theta \, d\theta \left| \frac{r^2}{2} \right|_0^{a(1-\cos\theta)} = \frac{a^2}{2} \int_0^\pi (1 - \cos\theta)^2 \, . \sin\theta \, d\theta$$

$$= \frac{a^2}{2} \left| \frac{(1-\cos\theta)^3}{3} \right|_0^\pi = \frac{a^2}{2} \cdot \frac{8}{3} = \frac{4a^2}{3}.$$

Example 7.12. *Calculate* $\iint r^3 \, dr \, d\theta$ *over the area included between the circles* $r = 2\sin\theta$ *and* $r = 4\sin\theta$.

Solution. Given circles $r = 2\sin\theta$ ...(*i*)

and $r = 4\sin\theta$ ...(*ii*)

are shown in Fig. 7.14. The shaded area between these circles is the region of integration.

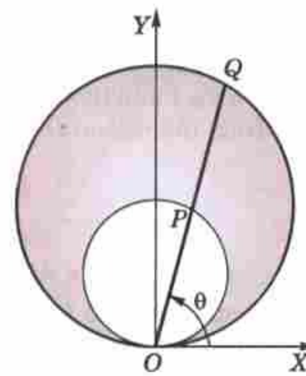

Fig. 7.14

If we integrate first w.r.t. r, then its limits are from $P(r = 2\sin\theta)$ to $Q(r = 4\sin\theta)$ and to cover the whole region θ varies from 0 to π. Thus the required integral is

$$I = \int_0^\pi d\theta \int_{2\sin\theta}^{4\sin\theta} r^3 \, dr = \int_0^\pi d\theta \left[\frac{r^4}{4}\right]_{2\sin\theta}^{4\sin\theta}$$

$$= 60 \int_0^\pi \sin^4\theta \, d\theta = 60 \times 2 \int_0^{\pi/2} \sin^4\theta \, d\theta = 120 \times \frac{3.1}{4.2} \cdot \frac{\pi}{2} = 22.5\,\pi.$$

PROBLEMS 7.1

Evaluate the following integrals (1–7) :

1. $\int_1^2 \int_1^3 xy^2 \, dxdy$.

2. $\int_0^1 \int_x^{\sqrt{x}} (x^2 + y^2) \, dx \, dy$. *(V.T.U., 2000)*

3. $\int_0^1 \int_0^x e^{x/y} \, dxdy$. *(P.T.U., 2005)*

4. $\int_0^1 \int_0^{\sqrt{(1+x^2)}} \frac{dydx}{1 + x^2 + y^2}$ *(Rajasthan, 2005)*

5. $\iint xy \, dxdy$ over the positive quadrant of the circle $x^2 + y^2 = a^2$. *(Rajasthan, 2006)*

6. $\iint (x + y)^2 dxdy$ over the area bounded by the ellipse $x^2/a^2 + y^2/b^2 = 1$. *(Kurukshetra, 2009 S ; U.P.T.U., 2004 S)*

7. $\iint xy(x + y) \, dxdy$ over the area between $y = x^2$ and $y = x$. *(V.T.U., 2010)*

Evaluate the following integrals by changing the order of integration (8–15) :

8. $\int_0^a \int_y^a \frac{xdxdy}{x^2 + y^2}$. *(Bhopal, 2008)*

9. $\int_0^3 \int_1^{\sqrt{(4-y)}} (x + y) \, dxdy$. *(V.T.U., 2005 ; Anna, 2003 S ; Delhi, 2002)*

10. $\int_0^1 \int_x^{\sqrt{(2-x^2)}} \frac{x\,dydx}{\sqrt{(x^2+y^2)}}$. (P.T.U., 2010 ; Marathwada, 2008 ; U.P.T.U., 2006)

11. $\int_0^{a/\sqrt{2}} \int_y^{\sqrt{(a^2-y^2)}} \log(x^2+y^2)\,dxdy \ (a>0)$.

12. $\int_0^1 \int_x^{\sqrt{x}} xy\,dydx$. (V.T.U., 2010)

13. $\int_0^a \int_{a-\sqrt{(a^2-y^2)}}^{a+\sqrt{(a^2-y^2)}} xy\,dxdy$. (Anna, 2009)

14. $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dydx$. (Bhopal, 2009 ; S.V.T.U., 2009 ; V.T.U., 2007)

15. $\int_0^\infty \int_0^x xe^{-x^2/y}\,dydx$. (S.V.T.U., 2006 ; U.P.T.U., 2005 ; V.T.U., 2004)

16. Sketch the region of integration of the following integrals and change the order of integrations,

(i) $\int_0^{2a} \int_{\sqrt{(2ax-x^2)}}^{\sqrt{(2ax)}} f(x)\,dxdy$ (Rajasthan, 2006) (ii) $\int_0^{ae^{\pi/4}} \int_{2\log(r/a)}^{\pi/2} f(r,\theta)\,r\,drd\theta$.

17. Show that $\iint_R r^2 \sin\theta\,drd\theta = 2a^3/3$, where R is the semi-circle $r = 2a\cos\theta$ above the initial line.

18. Evaluate $\iint \frac{r\,drd\theta}{\sqrt{a^2+r^2}}$ over one loop of the lemniscate $r^2 = a^2\cos 2\theta$. (Rohtak, 2006 S ; P.T.U., 2005)

19. Evaluate $\iint r^3\,drd\theta$ over the area bounded between the circles $r = 2\cos\theta$ and $r = 4\cos\theta$.

(Anna, 2009 ; Madras, 2006)

7.4 AREA ENCLOSED BY PLANE CURVES

(1) *Cartesian coordinates.*

Consider the area enclosed by the curves $y = f_1(x)$ and $y = f_2(x)$ and the ordinates $x = x_1$, $x = x_2$ [Fig. 7.15 (a)].

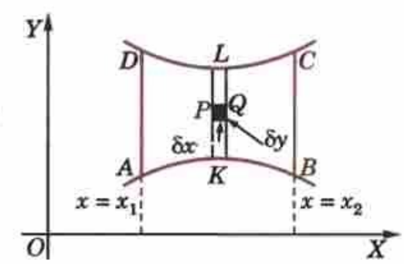

Fig. 7.15(a)

Divide this area into vertical strips of width δx. If $P(x, y)$, $Q(x+\delta x, y+\delta y)$ be two neighbouring points, then the area of the small rectangle $PQ = \delta x\delta y$.

$\therefore$ area of strip $KL = \underset{\delta y \to 0}{\text{Lt}} \Sigma\, \delta x\, \delta y$.

Since for all rectangles in this strip δx is the same and y varies from $y = f_1(x)$ to $y = f_2(x)$.

$$\therefore \quad \text{area of the strip } KL = \delta x \underset{\delta y \to 0}{\text{Lt}} \sum_{f_1(x)}^{f_2(x)} dy = \delta x \int_{f_1(x)}^{f_2(x)} dy.$$

Now adding up all such strips from $x = x_1$ to $x = x_2$, we get the area $ABCD$

$$= \underset{\delta x \to 0}{\text{Lt}} \sum_{x_1}^{x_2} \delta x \,.\int_{f_1(x)}^{f_2(x)} dy = \int_{x_1}^{x_2} dx \int_{f_1(x)}^{f_2(x)} dy = \int_{x_1}^{x_2} \int_{f_1(x)}^{f_2(x)} dxdy$$

Similarly, dividing the area $A'B'C'D$ [Fig. 7.15(b)] into horizontal strips of width δy, we get the area $A'B'C'D'$.

$$= \int_{y_1}^{y_2} \int_{f_1(y)}^{f_2(y)} dxdy$$

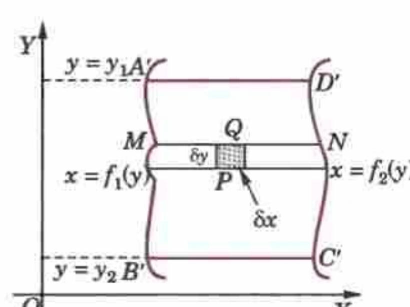

Fig. 7.15 (b)

(2) *Polar coordinates.*

Consider an area A enclosed by a curve whose equation is in polar coordinates.

Let $P(r, \theta)$, $Q(r+\delta r, \theta+\delta\theta)$ be two neighbouring points. Mark circular areas of radii r and $r+\delta r$ meeting OQ in R and OP (produced) in S (Fig. 7.16).

Since arc $PR = r\delta\theta$ and $PS = \delta r$.

$\therefore$ area of the curvilinear rectangle $PRQS$ is approximately $= PR \,.\, PS = r\delta\theta \,.\, \delta r$.

If the whole area is divided into such curvilinear rectangles, the sum $\Sigma\Sigma r\delta\theta\delta r$ taken for all these rectangles, gives in the limit the area A.

Hence $$A = \underset{\substack{\delta r \to 0 \\ \delta\theta \to 0}}{\text{Lt}} \Sigma\Sigma r\delta\theta\delta r = \iint r d\theta dr$$

where the limits are to be so chosen as to cover the entire area.

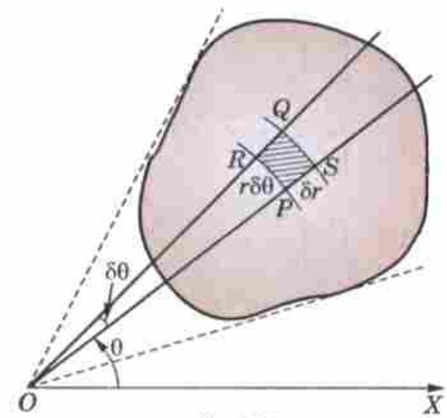

Fig. 7.16

Example 7.13. *Find the area of a plate in the form of a quadrant of the ellipse*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$ *(V.T.U., 2001 ; Osmania, 2000 S)*

Solution. Dividing the area into vertical strips of width δx, y varies from $K(y = 0)$ to $L[y = b\sqrt{(1 - x^2/b^2)}]$ and then x varies from 0 to a (Fig. 7.17).

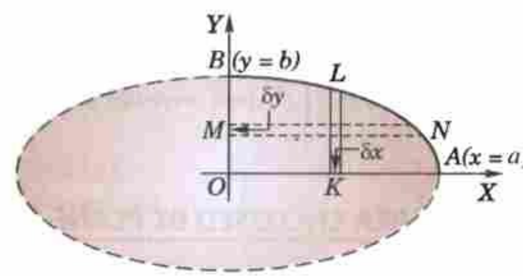

Fig. 7.17

$\therefore$ required area

$$= \int_0^a dx \int_0^{b\sqrt{(1-x^2/a^2)}} dy = \int_0^a dx\, [y]_0^{b\sqrt{(1-x^2/a^2)}}$$

$$= \frac{b}{a}\int_0^a \sqrt{(a^2 - x^2)}\, dx = \pi ab/4.$$

Otherwise, dividing this area into horizontal strips of width δy, x varies from $M(x = 0)$ to $N[x = a\sqrt{(1 - y^2/b^2)}\,]$ and then y varies from 0 to b.

$\therefore$ required area $$= \int_0^b dy \int_0^{a\sqrt{(1-y^2/b^2)}} dx = \int_0^b dy\, [x]_0^{a\sqrt{(1-y^2/b^2)}}$$

$$= \frac{a}{b}\int_0^b \sqrt{(b^2 - y^2)}\, dy = \pi ab/4.$$

Obs. The change of the order of integration does not in any way affect the value of the area.

Example 7.14. *Show that the area between the parabolas* $y^2 = 4ax$ *and* $x^2 = 4ay$ *is* $\frac{16}{3}a^2$.

(Kerala, 2005 ; Rohtak, 2003)

Solution. Solving the equations $y^2 = 4ax$ and $x^2 = 4ay$, it is seen that the parabolas intersect at O (0, 0) and A ($4a$, $4a$). As such for the shaded area between these parabolas (Fig. 7.18) x varies from 0 to $4a$ and y varies from P to Q *i.e.*, from $y = x^2/4a$ to $y = 2\sqrt{(ax)}$. Hence the *required area*

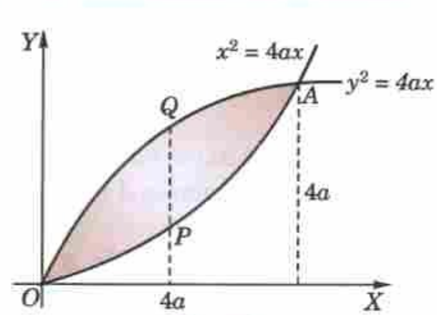

Fig. 7.18

$$= \int_0^{4a}\int_{x^2/4a}^{2\sqrt{(ax)}} dydx = \int_0^{4a} (2\sqrt{(ax)} - x^2/4a)\, dx$$

$$= \left| 2\sqrt{a}\,.\,\frac{2}{3}x^{3/2} - \frac{1}{4a}\,.\,\frac{x^3}{3}\right|_0^{4a} = \frac{32}{3}a^2 - \frac{16}{3}a^2 = \frac{16}{3}a^2.$$

Example 7.15. *Calculate the area included between the curve $r = a(\sec\theta + \cos\theta)$ and its asymptote.*

Solution. The curve is symmetrical about the initial line and has an asymptote $r = a\sec\theta$ (Fig. 7.19).

Draw any line OP cutting the curve at P and its asymptote at P'. Along this line, θ is constant and r varies from $a\sec\theta$ at P' to $a(\sec\theta + \cos\theta)$ at P. Then to get the upper half of the area, θ varies from 0 to $\pi/2$.

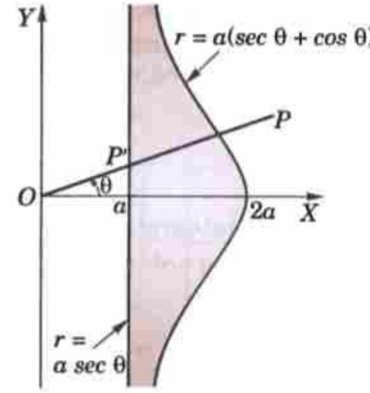

Fig. 7.19

$$\therefore \quad \text{required area} = 2\int_0^{\pi/2}\int_{a\sec\theta}^{a(\sec\theta+\cos\theta)} r\,dr d\theta$$

$$= 2\int_0^{\pi/2}\left[\frac{r^2}{2}\right]_{a\sec\theta}^{a(\sec\theta+\cos\theta)} d\theta$$

$$= a^2\int_0^{\pi/2}(2+\cos^2\theta)\,d\theta = 5\pi a^2/4.$$

Example 7.16. *Find the area lying inside the cardioid $r = a(1 + \cos\theta)$ and outside the circle $r = a$.*

Solution. In Fig. 7.20, $ABODA$ represents the cardioid $r = a(1 + \cos\theta)$ and $CBA'DC$ is the circle $r = a$.

Required area (shaded) = 2 (area $ABCA$)

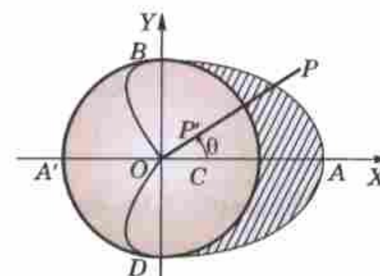

Fig. 7.20

$$= 2\int_0^{\pi/2}\int_{r=OP'}^{r=OP} r\,d\theta\,dr = 2\int_0^{\pi/2}\int_a^{a(1+\cos\theta)}(rdr)\,d\theta$$

$$= 2\int_0^{\pi/2}\left|\frac{r^2}{2}\right|_a^{a(1+\cos\theta)} d\theta = a^2\int_0^{\pi/2}[(1+\cos\theta)^2 - 1]\,d\theta$$

$$= a^2\int_0^{\pi/2}(\cos^2\theta + 2\cos\theta)\,d\theta = a^2\left(\frac{1}{2}\cdot\frac{\pi}{2}+2\right) = \frac{a^2}{4}(\pi+8).$$

PROBLEMS 7.2

1. Find, by double integration, the area lying between the parabola $y = 4x - x^2$ and the line $y = x$.
2. Find the area lying between the parabola $y = x^2$ and the line $x + y - z = 0$. *(Anna, 2009)*
3. By double integration, find the whole area of the curve $a^2x^2 = y^3(2a - y)$. *(U.P.T.U., 2001)*
4. Find, by double integration, the area enclosed by the curves $y = 3x/(x^2 + 2)$ and $4y = x^2$. *(J.N.T.U., 2005)*
5. Find, by double integration, the area of the lemniscate $r^2 = a^2\cos 2\theta$. *(Madras, 2000 S)*
6. Find, by double integration, the area lying inside the circle $r = a\sin\theta$ and outside the cardioid $r = a(1 - \cos\theta)$. *(Anna 2009 ; Mumbai, 2006)*
7. Find the area lying inside the cardioid $r = 1 + \cos\theta$ and outside the parabola $r(1 + \cos\theta) = 1$.
8. Find the area common to the circles $r = a\cos\theta$, $r = a\sin\theta$ by double integration. *(Mumbai, 2007)*

7.5 TRIPLE INTEGRALS

Consider a function $f(x, y, z)$ defined at every point of the 3-dimensional finite region V. Divide V into n elementary volumes $\delta V_1, \delta V_2, \ldots, \delta V_n$. Let (x_r, y_r, z_r) be any point within the rth sub-division δV_r. Consider the sum

$$\sum_{r=1}^{\infty} f(x_r, y_r, z_r)\delta V_r.$$

The limit of this sum, if it exists, as $n \to \infty$ and $\delta V_r \to 0$ is called the *triple integral of $f(x, y, z)$ over the region V* and is denoted by

$$\iiint f(x, y, z)\,dV.$$

For purposes of evaluation, it can also be expressed as the repeated integral

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x, y, z)\,dxdydz.$$

If x_1, x_2 are constants ; y_1, y_2 are either constants or functions of x and z_1, z_2 are either constants or functions of x and y, then this integral is evaluated as follows :

First $f(x, y, z)$ is integrated w.r.t. z between the limits z_1 and z_2 keeping x and y fixed. The resulting expression is integrated w.r.t. y between the limits y_1 and y_2 keeping x constant. The result just obtained is finally integrated w.r.t. x from x_1 to x_2.

Thus
$$I = \boxed{\int_{x_1}^{x_2} \boxed{\int_{y_1(x)}^{y_2(x)} \boxed{\int_{z_1(x,y)}^{z_2(x,y)} f(x, y, z) dz}\, dy}\, dx}$$

where the integration is carried out from the innermost rectangle to the outermost rectangle.

The order of integration may be different for different types of limits.

Example 7.17. *Evaluate* $\int_{-1}^{1} \int_{0}^{z} \int_{x-z}^{x+z} (x + y + z)\, dx dy dz$. *(J.N.T.U., 2006 ; Cochin, 2005)*

Solution. Integrating first w.r.t. y keeping x and z constant, we have

$$I = \int_{-1}^{1} \int_{0}^{z} \left| xy + \frac{y^2}{2} + yz \right|_{x-z}^{x+z} dx\, dz = \int_{-1}^{1} \int_{0}^{z} \left[(x + z)(2z) + \frac{1}{2} 4xz \right] dx\, dz$$

$$= 2 \int_{-1}^{1} \left| \frac{x^2 z}{2} + z^2 x + \frac{x^2}{2} z \right|_0^z dz = 2 \int_{-1}^{1} \left(\frac{z^3}{2} + z^3 + \frac{z^3}{2} \right) dz = 4 \left| \frac{z^4}{4} \right|_{-1}^{1} = 0.$$

Example 7.18. *Evaluate* $\int_{0}^{1} \int_{0}^{\sqrt{(1-x^2)}} \int_{0}^{\sqrt{(1-x^2-y^2)}} xyz\, dxdydz$. *(V.T.U., 2003 S)*

Solution. We have

$$I = \int_0^1 x \left[\int_0^{\sqrt{(1-x^2)}} y \left\{ \int_0^{\sqrt{(1-x^2-y^2)}} z\, dz \right\} dy \right] dx = \int_0^1 x \left\{ \int_0^{\sqrt{(1-x^2)}} y \cdot \left| \frac{z^2}{2} \right|_0^{\sqrt{(1-x^2-y^2)}} dy \right\} dx$$

$$= \int_0^1 x \left\{ \int_0^{\sqrt{(1-x^2)}} y \cdot \frac{1}{2}(1 - x^2 - y^2)\, dy \right\} dx = \frac{1}{2} \int_0^1 x \left| (1 - x^2)\frac{y^2}{2} - \frac{y^4}{4} \right|_0^{\sqrt{(1-x^2)}} dx$$

$$= \frac{1}{8} \int_0^1 [(1 - x^2)^2 \cdot 2x - (1 - x^2)^4 \cdot x]\, dx = \frac{1}{8} \int_0^1 (x - 2x^3 + x^5)\, dx$$

$$= \frac{1}{8} \left| \frac{x^2}{2} - \frac{2x^4}{4} + \frac{x^6}{6} \right|_0^1 = \frac{1}{8}\left(\frac{1}{2} - \frac{1}{2} + \frac{1}{6} \right) = \frac{1}{48}.$$

PROBLEMS 7.3

Evaluate the following integrals :

1. $\int_0^a \int_0^b \int_0^c (x^2 + y^2 + z^2)\, dx\, dy\, dz$. *(Anna, 2009)*
2. $\int_{-c}^{c} \int_{-b}^{b} \int_{-a}^{a} (x^2 + y^2 + z^2)\, dx\, dy\, dz$ *(S.V.T.U., 2009 ; V.T.U. 2000)*
3. $\int_0^1 \int_{y^2}^1 \int_0^{1-x} x\, dz\, dx\, dy$ *(Nagpur, 2009)*
4. $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}\, dz\, dy\, dx$. *(V.T.U., 2010 ; Kurukshetra, 2009 S ; J.N.T.U., 2005)*
5. $\int_0^{\log 2} \int_0^x \int_0^{x+\log y} e^{x+y+z}\, dx\, dy\, dz$. *(Bhopal, 2008)*
6. $\int_1^e \int_1^{\log y} \int_1^{e^x} \log z\, dz\, dx\, dy$. *(S.V.T.U., 2008 ; Rohtak, 2005)*
7. $\int_0^{\pi/2} \int_0^{a \sin\theta} \int_0^{\frac{a^2 - r^2}{a}} r dz\, dr\, d\theta$. *(V.T.U., 2009)*

7.6 VOLUMES OF SOLIDS

(1) Volumes as double integrals. Consider a surface $z = f(x, y)$. Let the orthogonal projection on XY-plane of its portion S' be the area S (Fig. 7.21).

Divide S into elementary rectangles of area $\delta x\, \delta y$ by drawing lines parallel to X and Y-axes. With each of these rectangles as base, erect a prism having its length parallel to OZ.

$\therefore$ volume of this prism between S and the given surface $z = f(x, y)$ is $z\delta x\delta y$.

Hence the volume of the solid cylinder on S as base, bounded by the given surface with generators parallel to the Z-axis.

$$= \underset{\substack{\delta x \to 0 \\ \delta y \to 0}}{\text{Lt}} \Sigma\Sigma\, z\delta x\delta y$$

$$= \iint z\, dx\, dy \quad \text{or} \quad \iint f(x, y)\, dx\, dy$$

where the integration is carried over the area S.

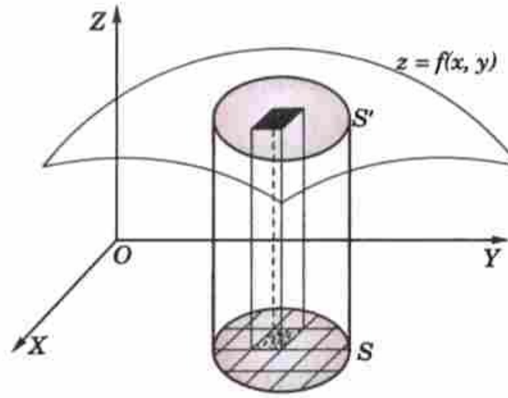

Fig. 7.21

Obs. While using polar coordinates, divide S into elements of area $r\delta\theta\, \delta r$.

$\therefore$ replacing $dxdy$ by $r\delta\theta\delta r$, we get the required volume $= \iint zr\, d\theta\, dr$.

Example 7.19. *Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ and $z = 0$.* (S.V.T.U., 2007 ; Cochin, 2005 ; Madras, 2000 S)

Solution. From Fig. 7.22, it is self-evident that $z = 4 - y$ is to be integrated over the circle $x^2 + y^2 = 4$ in the XY-plane. To cover the shaded half of this circle, x varies from 0 to $\sqrt{(4 - y^2)}$ and y varies from -2 to 2.

$\therefore$ Required volume

$$= 2\int_{-2}^{2} \int_{0}^{\sqrt{(4-y^2)}} z\, dx\, dy = 2\int_{-2}^{2} \int_{0}^{\sqrt{(4-y^2)}} (4 - y)\, dx\, dy$$

$$= 2\int_{-2}^{2} (4 - y)\, [x]_0^{\sqrt{(4-y^2)}}\, dy = 2\int_{-2}^{2} (4 - y)\sqrt{(4 - y^2)}\, dy$$

$$= 2\int_{-2}^{2} 4\sqrt{(4 - y^2)}\, dy - 2\int_{-2}^{2} y\sqrt{(4 - y^2)}\, dy$$

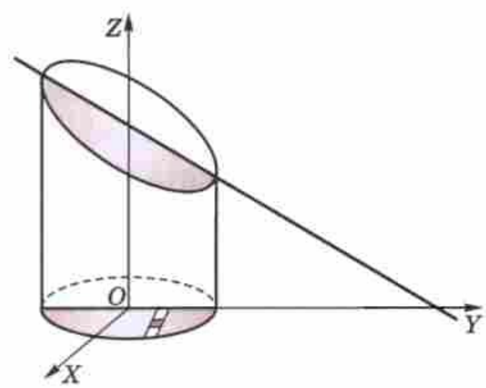

Fig. 7.22

$$= 8\int_{-2}^{2} \sqrt{(4 - y^2)}\, dy$$ [The second term vanishes as the integrand is an odd function.]

$$= 8\left|\frac{y\sqrt{(4 - y)^2}}{2} + \frac{4}{2}\sin^{-1}\frac{y}{2}\right|_{-2}^{2} = 16\pi.$$

(2) Volume as triple integral

Divide the given solid by planes parallel to the coordinate planes into rectangular parallelopipeds of volume $\delta x\, \delta y\, \delta z$ (Fig. 7.23).

$$\therefore \text{ the total volume} = \underset{\substack{\delta x \to 0 \\ \delta y \to 0 \\ \delta z \to 0}}{\text{Lt}} \Sigma\Sigma\Sigma\, \delta x\, \delta y\, \delta z$$

$$= \iiint dx\, dy\, dz$$

with appropriate limits of integration.

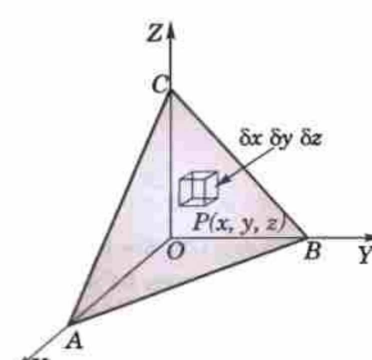

Fig. 7.23

Example 7.20. *Calculate the volume of the solid bounded by the planes $x = 0, y = 0, x + y + z = a$ and $z = 0$.* (P.T.U., 2009)

Solution. Volume required $= \int_0^a \int_0^{a-x} \int_0^{a-x-y} dz\, dy\, dx$

$$= \int_0^a \int_0^{a-x} (a - x - y)\, dy\, dx = \int_0^a \left| (a-x)y - \frac{y^2}{2} \right|_0^{a-x} dx$$

$$= \int_0^a \left\{ (a-x)^2 - \frac{(a-x)^2}{2} \right\} dx = \frac{1}{2}\int_0^a (a-x)^2\, dx = \frac{1}{2}\left| -\frac{(a-x)^3}{3} \right|_0^a = \frac{a^3}{6}.$$

Example 7.21. *Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.* (Anna, 2009 ; P.T.U., 2006 ; Kottayam, 2005)

Solution. Let $OABC$ be the positive octant of the given ellipsoid which is bounded by the planes OAB $(z = 0)$, OBC $(x = 0)$, OCA $(y = 0)$ and the surface ABC, *i.e.*,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Divide this region R into rectangular parallellopipeds of volume $\delta x \delta y \delta z$. Consider such an element at $P(x, y, z)$. (Fig. 7.24)

$\therefore$ the required volume $= 8 \iiint_R dx\, dy\, dz$.

In this region R,

(*i*) z varies from 0 to MN where

$$MN = c\sqrt{(1 - x^2/a^2 - y^2/b^2)}.$$

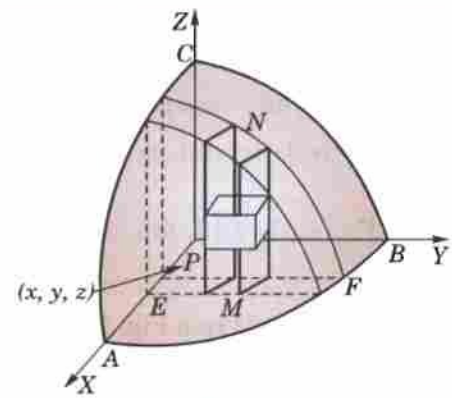

Fig. 7.24

(*ii*) y varies from 0 to EF, where $EF = b\sqrt{(1 - x^2/a^2)}$ from the equation of the ellipse OAB, *i.e.*,

$$x^2/a^2 + y^2/b^2 = 1.$$

(*iii*) x varies from 0 to $OA = a$.

Hence the volume of the whole ellipsoid

$$= 8\int_0^a \int_0^{b\sqrt{(1-x^2/a^2)}} \int_0^{c\sqrt{(1-x^2/a^2-y^2/b^2)}} dx\,dy\,dz = 8\int_0^a dx \int_0^{b\sqrt{(1-x^2/a^2)}} dy \mid z \mid_0^{c\sqrt{(1-x^2/a^2-y^2/b^2)}}$$

$$= 8c\int_0^a dx \int_0^{b\sqrt{(1-x^2/a^2)}} \sqrt{(1 - x^2/a^2 - y^2/b^2)}\, dy$$

$$= \frac{8c}{b}\int_0^a dx \int_0^{\rho} \sqrt{(\rho^2 - y^2)}\, dy \quad \text{when } \rho = b\sqrt{1 - x^2/a^2}.$$

$$= \frac{8c}{b}\int_0^a dx \left[\frac{y\sqrt{(\rho^2 - y^2)}}{2} + \frac{\rho^2}{2}\sin^{-1}\frac{y}{\rho} \right]_0^{\rho} = \frac{8c}{b}\int_0^a \frac{b^2}{2}\left(1 - \frac{x^2}{a^2}\right)\frac{\pi}{2}\, dx$$

$$= 2\pi\, bc \int_0^a \left(1 - \frac{x^2}{a^2}\right) dx = 2\pi\, bc \left| x - \frac{x^3}{3a^2} \right|_0^a = \frac{4\pi abc}{3}.$$

Otherwise. See Problem 27 page 292.

(3) Volumes of solids of revolution

Consider an elementary area $\delta x \delta y$ at the point $P(x, y)$ of a plane area A. (Fig. 7.25)

As this elementary area revolves about x-axis, we get a ring of volume

$$= \pi[(y + \delta y)^2 - y^2]\,\delta x = 2\pi y\, \delta x\, \delta y,$$

nearly to the first powers of δy.

Hence the total volume of the solid formed by the revolution of the area A about x-axis.

$$= \iint_A 2\pi y \, dxdy.$$

In polar coordinates, the above formula for the volume becomes

$$\iint_A 2\pi r \sin\theta \,.\, r d\theta dr, \text{ i.e. } \iint_A 2\pi r^2 \sin\theta \, d\theta dr$$

Similarly, the volume of the solid formed by the revolution of the area A about y-axis $= \iint_A 2\pi x \, dx \, dy$.

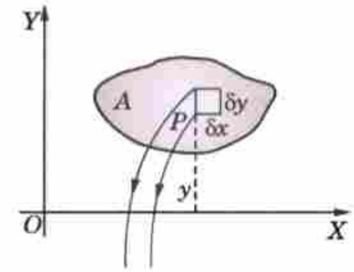

Fig. 7.25

Example 7.22. *Calculate by double integration, the volume generated by the revolution of the cardioid $r = a(1 - \cos\theta)$ about its axis.*

Solution. Required volume

$$= \int_0^{\pi} \int_0^{a(1-\cos\theta)} 2\pi r^2 \sin\theta \, dr \, d\theta$$

$$= 2\pi \int_0^{\pi} \left| \frac{r^3}{3} \right|_0^{a(1-\cos\theta)} \sin\theta \, d\theta$$

$$= \frac{2\pi a^3}{3} \int_0^{\pi} (1-\cos\theta)^3 \,.\, \sin\theta \, d\theta = \frac{2\pi a^3}{3} \left| \frac{(1-\cos\theta)^4}{4} \right|_0^{\pi} = \frac{8\pi a^3}{3}.$$

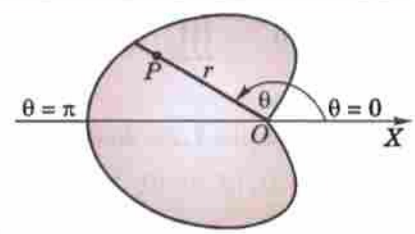

Fig. 7.26

7.7 CHANGE OF VARIABLES

An appropriate choice of co-ordinates quite often facilitates the evaluation of a double or a triple integral. By changing the variables, a given integral can be transformed into a simpler integral involving the new variables.

(1) **In a double integral,** let the variables x, y be changed to the new variables u, v by the transformation.

$$x = \phi(u, v), y = \psi(u, v)$$

where $\phi(u, v)$ and $\psi(u, v)$ are continuous and have continuous first order derivatives in some region R'_{uv} in the uv-plane which corresponds to the region R_{xy} in the xy-plane. Then

$$\iint_{R_{xy}} f(x, y) \, dxdy = \iint_{R'_{uv}} f[\phi(u, v), \psi(u, v)] \, |J| \, dudv \quad ...(1)$$

where $$J = \frac{\partial(x, y)}{\partial(u, v)} \; (\neq 0)$$

is the *Jacobian of transformation** from (x, y) to (u, v) coordinates.

(2) **For triple integrals,** the formula corresponding to (1) is

$$\iiint_{R_{xyz}} f(x, y, z) \, dxdydz = \iiint_{R'_{uvw}} f[x(u, v, w), y(u, v, w), z(u, v, w)] \, |J| \, dudvdw$$

where $$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} \; (\neq 0)$$

is the *Jacobian of transformation* from (x, y, z) to (u, v, w) coordinates.

Particular cases :

(*i*) ***To change cartesian coordinates (x, y) to polar coordinates (r, θ),*** *we have* $x = r\cos\theta, y = r\sin\theta$ *and*

$$J = \frac{\partial(x, y)}{\partial(r, \theta)} = r$$ [Ex. 5.25, p. 216]

$$\therefore \quad \iint_{R_{xy}} f(x, y) \, dx \, dy = \iint_{R'_{r\theta}} f(r\cos\theta, r\sin\theta) \,.\, r \, dr \, d\theta.$$

* See footnote page 215.

(*ii*) ***To change rectangular coordinates (x, y, z) to cylindrical coordinates*** (ρ, ϕ, z) — *Fig. 8.27, we have*

$$x = \rho \cos \phi, y = \rho \sin \phi, z = z$$

and
$$J = \frac{\partial(x, y, z)}{\partial(\rho, \phi, z)} = \rho \qquad \text{[Ex. 5.25]}$$

Then
$$\iiint_{R_{xyz}} f(x, y, z)\, dxdydz = \iiint_{R'_{\rho\phi z}} f(\rho \cos \phi, \rho \sin \phi, z)\,.\,\rho d\rho d\phi dz.$$

(*iii*) ***To change rectangular coordinates (x, y, z) to spherical polar coordinates*** (r, θ, ϕ)—*Fig. 8.28, we have*

$$x = r \sin \theta \cos \phi, y = r \sin \theta \sin \phi, z = r \cos \theta$$

and
$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2 \sin \theta \qquad \text{[Ex. 5.25]}$$

Then
$$\iiint_{R_{xyz}} f(x,y,z)\, dxdydz = \iiint_{R'_{r\theta\phi}} f(r \sin \theta \cos \phi, r \sin \theta, \sin \phi, r \cos \theta)\,.\,r^2 \sin \theta\, drd\theta d\phi$$

Example 7.23. *Evaluate* $\iint_R (x + y)^2\, dxdy$, *where R is the parallelogram in the xy-plane with vertices (1, 0), (3, 1), (2, 2), (0, 1) using the transformation* $u = x + y$ *and* $v = x - 2y$. *(U.P.T.U., 2004)*

Solution. The region R, *i.e.*, parallelogram $ABCD$ in the xy-plane becomes the region R', *i.e.*, rectangle $A'B'C'D'$ in the uv-plane as shown in Fig. 7.27, by taking

$$u = x + y \quad \text{and} \quad v = x - 2y \qquad ...(i)$$

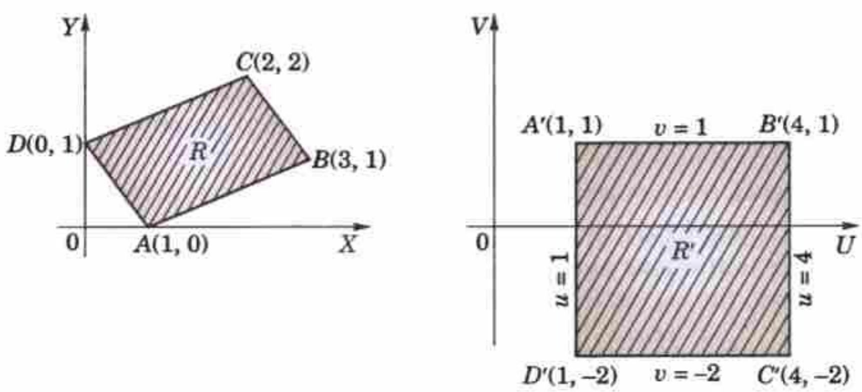

Fig. 7.27

From (*i*), we have
$$x = \frac{1}{3}(2u + v), y = \frac{1}{3}(u - v)$$

$\therefore$
$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{vmatrix} = -\frac{1}{3}$$

Hence, the given integral

$$= \iint_{R'} u^2\, |J|\, dudv = \int_1^4 \int_{-2}^1 u^2 \,.\, \frac{1}{3}\,.\, dudv = \frac{1}{3} \left| \frac{u^3}{3} \right|_1^4 .\, |v|_{-2}^1 = 21.$$

Example 7.24. *Evaluate* $\iint_D xy\sqrt{(1 - x - y)}\, dx\, dy$ *where D is the region bounded by* $x = 0$, $y = 0$ *and* $x + y = 1$ *using the transformation* $x + y = u$, $y = uv$. *(Marathwada, 2008)*

Solution. We have $x = u - uv$, $y = uv$

$$\therefore \qquad J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \partial x/\partial u & \partial y/\partial u \\ \partial x/\partial v & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} 1-v & v \\ -u & u \end{vmatrix} = u.$$

Also when $x = 0$, $u = 0$, $v = 1$; when $y = 0$, $u = 0$, $v = 0$ and when $x + y = 1$, $u = 1$

$\therefore$ the limits of u are from 0 to 1 and limits of v are from 0 to 1.

Thus $$\iint_D xy\sqrt{(1-x-y)}\, dxdy = \int_0^1 \int_0^1 u(1-v)\, uv\, (1-u)^{1/2}\, |J|\, dudv$$

$$= \int_0^1 \int_0^1 u^3 (1-u)^{1/2}\, v(1-v)\, du\, dv$$

where $u = \sin^2\theta$
$du = 2\sin\theta\cos\theta\, d\theta$
$u = 0, \theta = 0$
$u = 1, \theta = \pi/2$

$$= \int_0^1 u^3(1-u)^{1/2}\, du \times \int_0^1 v(1-v)\, dv$$

$$= \int_0^{\pi/2} \sin^6\theta\cos\theta\,.\,2\sin\theta\cos d\theta \times \left| \frac{v^2}{2} - \frac{v^3}{3} \right|_0^1$$

$$= 2\int_0^{\pi/2} \sin^7\theta\cos^2\theta\, d\theta \left(\frac{1}{6}\right) = \frac{1}{3}\cdot\frac{6\,.\,1}{9\,.\,7\,.\,5\,.\,3} = \frac{2}{945}.$$

Example 7.25. *Evaluate* $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dxdy$ *by changing to polar coordinates.* (Anna, 2003)

Hence show that $\int_0^\infty e^{-x^2}\, dx = \sqrt{\pi/2}$. (Madras, 2003 ; U.P.T.U., 2003 ; J.N.T.U., 2000)

Solution. The region of integration being the first quadrant of the xy-plane, r varies from 0 to ∞ and θ varies from 0 to $\pi/2$. Hence,

$$I = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dxdy = \int_{\theta=0}^{\pi/2} \int_{r=0}^{\infty} e^{-r^2}\, rdrd\theta$$

$$= -\frac{1}{2}\int_0^{\pi/2} \left\{\int_0^\infty e^{-r^2}(-2r)\, dr\right\} d\theta = -\frac{1}{2}\int_0^{\pi/2} \left| e^{-r^2} \right|_0^\infty d\theta = \frac{1}{2}\int_0^{\pi/2} d\theta = \frac{\pi}{4}. \qquad ...(i)$$

Also $$I = \int_0^\infty e^{-x^2}\, dx \times \int_0^\infty e^{-y^2}\, dy = \left\{\int_0^\infty e^{-x^2}\, dx\right\}^2 \qquad ...(ii)$$

Thus, from (*i*) and (*ii*), we have $\int_0^\infty e^{-x^2} = \sqrt{\pi/2}$. ...(*iii*)

Example 7.26. *Find the volume bounded by the paraboloid $x^2 + y^2 = az$, the cylinder $x^2 + y^2 = 2ay$ and the plane $z = 0$.*

Solution. The required volume is found by integrating $z = (x^2 + y^2)/a$ over the circle $x^2 + y^2 = 2ay$.

Changing to polar coordinates in the xy-plane, we have $x = r\cos\theta$, $y = r\sin\theta$ so that $z = r^2/a$ and the polar equation of the circle is $r = 2a\sin\theta$.

To cover this circle, r varies from 0 to $2a\sin\theta$ and θ varies from 0 to π. (Fig. 7.28)

Hence the required volume

$$= \int_0^\pi \int_0^{2a\sin\theta} z\,.\,r\, d\theta\, dr = \frac{1}{a}\int_0^\pi d\theta \int_0^{2a\sin\theta} r^3\, dr$$

$$= \frac{1}{a}\int_0^\pi d\theta \left| \frac{r^4}{4} \right|_0^{2a\sin\theta} = 4a^3 \int_0^\pi \sin^4\theta\, d\theta = \frac{3\pi a^3}{2}.$$

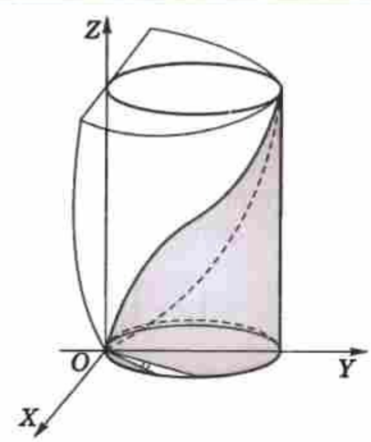

Fig. 7.28

Example 7.27. *Find, by triple integration, the volume of the sphere* $x^2 + y^2 + z^2 = a^2$. (*Bhopal, 2009 ; Madras, 2006 ; V.T.U., 2003 S*)

Solution. Changing to polar spherical coordinates by putting

$$x = r \sin\theta \cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$$

we have $dx\,dy\,dz = r^2 \sin\theta\, dr\, d\theta\, d\phi$.

Also the volume of the sphere is 8 times the volume of its portion in the positive octant for which r varies from 0 to a, θ varies from 0 to $\pi/2$ and ϕ varies from 0 to $\pi/2$.

$\therefore$ volume of the sphere

$$= 8\int_0^a \int_0^{\pi/2}\int_0^{\pi/2} r^2 \sin\theta\, dr\, d\theta\, d\phi = 8\int_0^a r^2\, dr \,.\int_0^{\pi/2} .\sin\theta\, d\theta \,.\int_0^{\pi/2} d\phi$$

$$= 8\,.\left|\frac{r^3}{3}\right|_0^a .\left| -\cos\theta \right|_0^{\pi/2} .\frac{\pi}{2} = 4\pi\,.\frac{a^3}{3}\,.(-0+1) = \frac{4}{3}\pi a^3.$$

Example 7.28. *Find the volume of the portion of the sphere* $x^2 + y^2 + z^2 = a^2$ *lying inside the cylinder* $x^2 + y^2 = ay$. (*Rohtak, 2003*)

Solution. The required volume is easily found by changing to cylindrical coordinates (ρ, ϕ, z). We therefore, have

$$x = \rho\cos\phi,\ y = \rho\sin\phi,\ z = z$$

and
$$J = \frac{\partial(x, y, z)}{\partial(\rho, \phi, z)} = \rho.$$

Then the equation of the sphere becomes $\rho^2 + z^2 = a^2$ and that of cylinder becomes $\rho = a\sin\phi$.

The volume inside the cylinder bounded by the sphere is twice the volume shown shaded in the Fig. 7.29 for which z varies from 0 to $\sqrt{(a^2-\rho^2)}$, ρ varies from 0 to $a\sin\phi$ and ϕ varies from 0 to π.

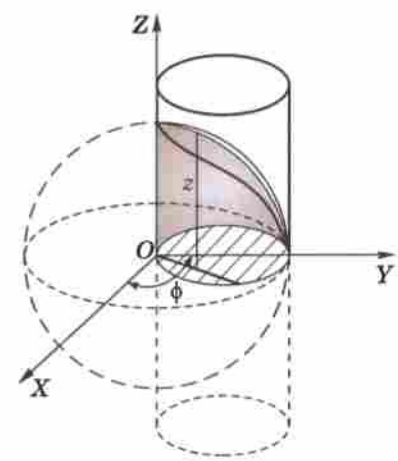

Fig. 7.29

Hence the required volume $= 2\int_0^{\pi}\int_0^{a\sin\phi}\int_0^{\sqrt{(a^2-\rho^2)}} \rho\, dz\, d\rho\, d\phi$

$$= 2\int_0^{\pi}\int_0^{a\sin\phi} \rho\sqrt{(a^2-\rho^2)}\, d\rho\, d\phi = 2\int_0^{\pi}\left| -\frac{1}{3}(a^2-\rho^2)^{3/2}\right|_0^{a\sin\phi} d\phi$$

$$= \frac{2a^3}{3}\int_0^{\pi}(1-\cos^3\phi)\, d\phi = \frac{2a^3}{9}(3\pi - 4).$$

Example 7.29. *Evaluate* $\int_0^1 \int_0^{\sqrt{(1-x^2)}} \int_{\sqrt{(x^2+y^2)}}^1 \frac{dz\, dy\, dx}{\sqrt{(x^2+y^2+z^2)}}$. (*V.T.U., 2008*)

Solution. We change to spherical polar coordinates (r, θ, ϕ), so that

$$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$$

and
$$J = r^2\sin\theta,\ x^2+y^2+z^2 = r^2.$$

The region of integration is common to the cone $z^2 = x^2 + y^2$ and the cylinder $x^2 + y^2 = 1$ bounded by the plane $z = 1$ in the positive octant (Fig. 7.30). Hence θ varies from 0 to $\pi/4$, r varies from 0 to $\sec\theta$ and ϕ varies from 0 to $\pi/2$.

$\therefore$ given integral becomes

$$\int_0^{\pi/2}\int_0^{\pi/4}\int_0^{\sec\theta}\frac{1}{r}.r^2\sin\theta\, dr\, d\theta\, d\phi = \int_0^{\pi/2} d\phi \int_0^{\pi/4}\left|\frac{r^2}{2}\right|_0^{\sec\theta}\sin\theta\, d\theta$$

$$= \frac{\pi}{2}\int_0^{\pi/4}\frac{\sec^2\theta}{2}\sin\theta\, d\theta = \frac{\pi}{4}\int_0^{\pi/4}\sec\theta\tan\theta\, d\theta = \frac{\pi}{4}\left|\sec\theta\right|_0^{\pi/4} = \frac{(\sqrt{2}-1)\pi}{4}.$$

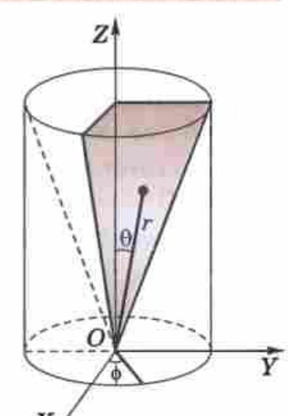

Fig. 7.30

Example 7.30. *Find the volume of the solid surrounded by the surface*

$$(x/a)^{2/3} + (y/b)^{2/3} + (z/c)^{2/3} = 1.$$ (Hissar, 2005 S)

Solution. Changing the variables, x, y, z to X, Y, Z where, $(x/a)^{1/3} = X$, $(y/b)^{1/3} = Y$, $(z/c)^{1/3} = Z$

i.e., $x = aX^3, y = bY^3, z = cZ^3$ so that $J = \partial(x, y, z)/\partial(X, Y, Z) = 27\, abc\, X^2Y^2Z^2$.

$$\therefore \quad \text{required volume} = \iiint dx\, dy\, dz = 27\, abc \iiint X^2Y^2Z^2\, dX\, dY dZ$$

taken throughout the sphere $X^2 + Y^2 + Z^2 = 1$. ...(i)

Now change X, Y, Z to spherical polar coordinates r, θ, ϕ so that $X = r \sin\theta \cos\phi$, $Y = r\sin\theta \sin\phi$, $Z = r\cos\theta$, and $\partial(X, Y, Z)/\partial(r, \theta, \phi) = r^2 \sin\theta$. To describe the positive octant of the sphere (i), r varies from 0 to 1, θ from 0 to $\pi/2$ and ϕ from 0 to $\pi/2$.

$$\therefore \quad \text{required volume} = 27abc \times 8 \int_0^1 \int_0^{\pi/2} \int_0^{\pi/2} r^2 \sin^2\theta \cos^2\phi \times r^2 \sin^2\theta \sin^2\phi \,.\, r^2\cos^2\theta \,.\, r^2 \sin\theta\, dr d\theta d\phi$$

$$= 216\, abc \int_0^1 r^8\, dr \int_0^{\pi/2} \sin^4\theta \cos^2\theta\, d\theta \int_0^{\pi/2} \sin^2\phi \cos^2\phi\, d\phi = 4\pi\, abc/35.$$

PROBLEMS 7.4

Evaluate the following integrals by changing to polar co-ordinates :

1. $\int_0^1 \int_0^{\sqrt{(1-y^2)}} (x^2 + y^2)\, dy\, dx$. (P.T.U., 2010)

2. $\int_0^2 \int_0^{\sqrt{(2x-x^2)}} \frac{x\, dx\, dy}{x^2+y^2}$ (Anna, 2009)

3. $\int_0^{4a} \int_{y^2/4a}^{y} \frac{x^2-y^2}{x^2+y^2} dxdy$ (Mumbai, 2006)

4. $\iint xy(x^2+y^2)^{n/2}\, dxdy$ over the positive quadrant of $x^2 + y^2 = 4$, supposing $n + 3 > 0$. (S.V.T.U., 2007)

5. $\iint \frac{dxdy}{(1+x^2+y^2)^2}$ over one loop of the lemniscate $(x^2 + y^2) = x^2 - y^2$. (Mumbai, 2007)

6. Transform the following to cartesian form and hence evaluate $\int_0^{\pi} \int_0^{a} r^3 \sin\theta \cos\theta\, dr d\theta$. (P.T.U., 2005)

7. $\iint y^2\, dx\, dy$ over the area outside $x^2 + y^2 - ax = 0$ and inside $x^2 + y^2 - 2ax = 0$. (Mumbai, 2006)

8. By using the transformation $x + y = u, y = uv$, show that $\int_0^1 \int_0^{1-x} e^{y/(x+y)}\, dydx = \frac{1}{2}(e-1)$. (P.T.U., 2003)

9. Transform $\int_0^{\pi/2} \int_0^{\pi/2} \sqrt{\frac{\sin\phi}{\sin\theta}}\, d\phi\, d\theta$ by the substitution $x = \sin\phi\cos\theta$, $y = \sin\phi \sin\theta$ and show that its value is π. (U.P.T.U., 2001)

Evaluate the following integrals by changing to spherical coordinates :

10. $\int_0^1 \int_0^{\sqrt{(1-x^2)}} \int_0^{\sqrt{(1-x^2-y^2)}} \frac{dx\, dy\, dz}{\sqrt{(1-x^2-y^2-z^2)}}$. (V.T.U., 2006 ; Kottayam, 2005)

11. $\iiint_V \frac{dx\, dy\, dz}{x^2+y^2+z^2}$ where V is the volume of the sphere $x^2 + y^2 + z^2 = a^2$. (Anna, 2009)

12. Evaluate $\iiint \frac{dx\, dy\, dz}{(1+x+y+z)^3}$ over the volume of the tetrahedron $x = 0, y = 0, z = 0, x + y + z = 1$. (Mumbai, 2007)

13. Show that $\iiint \frac{dx\, dy\, dz}{\sqrt{(a^2-x^2-y^2-z^2)}} = \frac{\pi^2 a^3}{8}$, the integral being extended for all the values of the variables for which the expression is real. (U.T.U., 2010)

14. $\iiint z^2\, dx\, dy\, dz$, taken over the volume bounded by the surfaces $x^2 + y^2 = a^2$, $x^2 + y^2 = z$ and $z = 0$.

15. Find the volume bounded by the xy-plane, the cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 3$. (*I.S.M., 2001*)
16. Find the volume bounded by the xy-plane, the paraboloid $2z = x^2 + y^2$ and the cylinder $x^2 + y^2 = 4$. (*Raipur, 2005*)
17. Find the volume cut from the sphere $x^2 + y^2 + z^2 = a^2$ by the cone $x^2 + y^2 = z^2$.
18. Find the volume common to the cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$. (*S.V.T.U., 2006*)
19. Find the volume cut off from the cylinder $x^2 + y^2 = ax$ by the planes $z = 0$ and $z = x$. (*U.P.T.U., 2006*)
20. Find the volume enclosed by the cylinders $x^2 + y^2 = 2ax$ and $z^2 = 2ax$. (*Marathwada, 2008*)
21. Find the volume of the cylinder $x^2 + y^2 - 2ax = 0$, intercepted between the paraboloid $x^2 + y^2 = 2az$ and the xy-plane.
22. Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the hyperboloid $x^2 + y^2 - z^2 = 1$.
23. Find the volume of the region bounded by $z = x^2 + y^2$, $z = 0$, $x = -a$, $x = a$ and $y = -a$, $y = a$.
24. Prove, by using a double integral that the volume generated by the revolution of the cardioid $r = a(1 + \cos\theta)$ about its axis is $8\pi a^3/3$. (*V.T.U., 2000*)
25. Evaluate $\iiint (x + y + z)\, dx\, dy\, dz$ over the tetrahedron bounded by the planes $x = 0, y = 0, z = 0$ and $x + y + z = 1$. [See Fig. 7.34]
26. Find the volume of the tetrahedron bounded by the coordinate planes and the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$. (*Burdwan, 2003*)
27. Work out example 7.21 by changing the variables.

7.8 AREA OF A CURVED SURFACE

Consider a point P of the surface $S : z = f(x, y)$. Let its projection on the xy-plane be the region A. Divide it into area elements by drawing lines parallel to the axes of X and Y. (Fig. 7.31).

On the element $\delta x \delta y$ as base, erect a cylinder having generators parallel to OZ and meeting the surface S in an element of area δS.

As $\delta x \delta y$ is the projection of δS on the xy-plane,

$\therefore$ $\delta x \delta y = \delta S \,.\, \cos\gamma$, where γ is the angle between the xy-plane and the tangent plane to S at P, *i.e.*, it is the angle between the Z-axis and the normal to S at $P (= \angle Z'PN)$.

Now since the direction cosines of the normal to the surface $F(x, y, z) = 0$ proportional to

$$\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}.$$

$\therefore$ the direction cosines of the normal to $S [F = f(x, y) - z]$ are proportional to $-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}$, 1 and those of the z-axis are 0, 0, 1.

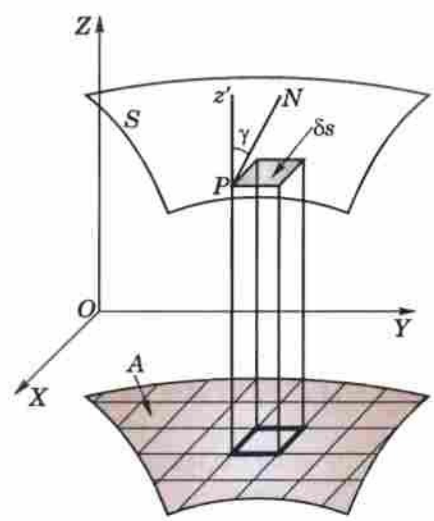

Fig. 7.31

Hence $$\cos\gamma = \frac{1}{\sqrt{\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right]}} \quad \therefore \quad \delta S = \frac{\delta x\, \delta y}{\cos\gamma} = \sqrt{\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right]}\, \delta x\, \delta y$$

Hence $$S = \underset{\delta S \to 0}{\text{Lt}} \sum \delta S = \iint_A \sqrt{\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right]}\, dx dy$$

Similarly, if B and C be the projections of S on the yz-and zx-planes respectively, then

$$S = \iint_B \sqrt{\left[\left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2 + 1\right]}\, dy\, dz$$

and $$S = \iint_C \sqrt{\left[\left(\frac{\partial y}{\partial z}\right)^2 + \left(\frac{\partial y}{\partial x}\right)^2 + 1\right]}\, dz\, dx.$$

Example 7.31. *Find the area of the portion of the cylinder $x^2 + z^2 = 4$ lying inside the cylinder $x^2 + y^2 = 4$.*

Solution. Figure 7.32 shows one-eighth of the required area. Its projection on the xy-plane is a quadrant circle $x^2 + y^2 = 4$.

For the cylinder $\quad x^2 + z^2 = 4, \quad$ ...(*i*)

we have $$\frac{\partial z}{\partial x} = -\frac{x}{z}, \frac{\partial z}{\partial y} = 0.$$

so that $$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1 = \frac{x^2+z^2}{z^2} = \frac{4}{4-x^2}.$$

Hence the required surface area = 8 (surface area of the upper portion of (*i*) lying within the cylinder $x^2 + y^2 = 4$ in the positive octant)

$$= 8\int_0^2 \int_0^{\sqrt{(4-x^2)}} \frac{2}{\sqrt{(4-x^2)}}\,dxdy = 16\int_0^2 dx = 32 \text{ sq. units.}$$

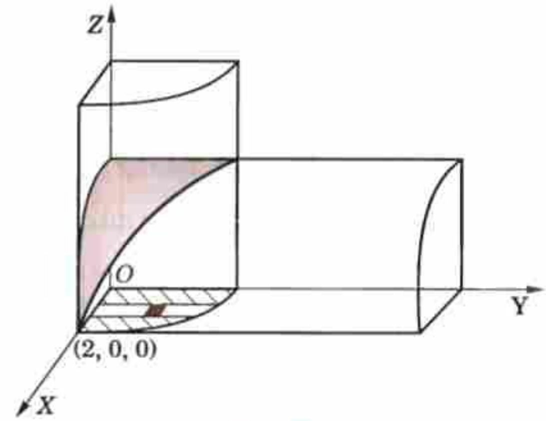

Fig. 7.32

Example 7.32. *Find the area of the portion of the sphere $x^2 + y^2 + z^2 = 9$ lying inside the cylinder $x^2 + y^2 = 3y$.*

Solution. Figure 7.33 shows one-fourth of the required area. Its projection on the xy-plane is the semi-circle $x^2 + y^2 = 3y$ bounded by the Y-axis.

For the sphere

$$x^2 + y^2 + z^2 = 9, \frac{\partial z}{\partial x} = -\frac{x}{z} \text{ and } \frac{\partial z}{\partial y} = -\frac{y}{z}$$

$$\therefore \quad \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1 = (x^2 + y^2 + z^2)/z^2$$

$$= \frac{9}{9 - x^2 - y^2} = \frac{9}{9-r^2} \quad \text{when} \quad x = r\cos\theta, y = r\sin\theta.$$

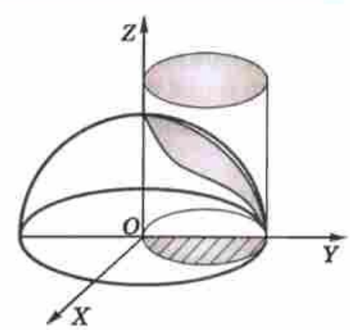

Fig. 7.33

Using polar coordinates, the required area is found by integrating $3/\sqrt{(9-r^2)}$ over the semi-circle $r = 3\sin\theta$, for which r varies from 0 to $3\sin\theta$ and θ varies from 0 to $\pi/2$.

Hence the required surface area

$$= 4\int_0^{\pi/2}\int_0^{3\sin\theta} \frac{3}{\sqrt{(9-r^2)}}\,r d\theta\, dr = -6\int_0^{\pi/2} \left| \frac{\sqrt{(9-r^2)}}{1/2} \right|_0^{3\sin\theta} d\theta$$

$$= 36\int_0^{\pi/2} (1-\cos\theta)\,d\theta = 36\,|\,\theta - \sin\theta\,|_0^{\pi/2} = 18(\pi - 2) \text{ sq. units.}$$

PROBLEMS 7.5

1. Show that the surface area of the sphere $x^2 + y^2 + z^2 = a^2$ is $4\pi a^2$.
2. Find the area of the portion of the cylinder $x^2 + y^2 = 4y$ lying inside the sphere $x^2 + y^2 + z^2 = 16$.
3. Find the area of the portion of the sphere $x^2 + y^2 + z^2 = a^2$ lying inside the cylinder $x^2 + y^2 = ax$.
4. Find the area of the surface of the cone $x^2 + y^2 = z^2$ cut off by the surface of the cylinder $x^2 + y^2 = a^2$ above the xy-plane.
5. Compute the area of that part of the plane $x + y + z = 2a$ which lies in the first octant and is bounded by the cylinder $x^2 + y^2 = a^2$. (*Burdwan, 2003*)

7.9 CALCULATION OF MASS

(*a*) **For a plane lamina,** if the surface density at the point $P(x, y)$ be $\rho = f(x, y)$ then the elementary mass at $P = \rho\delta x\delta y$.

$\therefore$ *total mass of the lamina* $= \iint \rho\, dx\, dy$ ...(*i*)

with integrals embracing the whole area of the lamina.

In polar coordinates, taking $\rho = \phi(r, \theta)$ at the point $P(r, \theta)$,

total mass of the lamina $= \iint \rho r\, d\theta\, dr$ ...(*ii*)

(*b*) **For a solid,** if the density at the point $P(x, y, z)$ be $\rho = f(x, y, z)$, then

total mass of the solid $= \iiint \rho\, dx\, dy\, dz$ with appropriate limits of integration.

Example 7.33. *Find the mass of the tetrahedron bounded by the coordinates planes and the plane* $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, *the variable density* $\rho = \mu xyz$. (*Rohtak, 2003 ; U.P.T.U., 2003*)

Solution. Elementary mass at $P = \mu xyz \,.\, \delta x\, \delta y\, \delta z$.

$\therefore$ the whole mass $= \iiint \mu xyz\, dx\, dy\, dz$,

the integrals embracing the whole volume $OABC$ (Fig. 7.34). The limits for z are from 0 to $z = c(1 - x/a - y/b)$.

The limits for y are from 0 to $y = b(1 - x/a)$ and limits for x are from 0 to a.

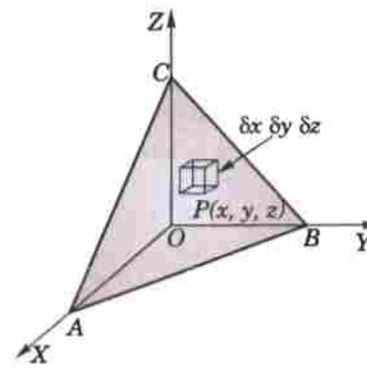

Fig. 7.34

Hence the required mass

$$= \int_0^a \int_0^{b(1-x/a)} \int_0^{c(1-x/a-y/b)} \mu xyz\, dz\, dy\, dx$$

$$= \mu \int_0^a \int_0^{b(1-x/a)} xy \left| z^2/2 \right|_0^{c(1-x/a-y/b)} dydz$$

$$= \mu \int_0^a \int_0^{b(1-x/a)} xy \,.\, \frac{c^2}{2}\left(1 - \frac{x}{a} - \frac{y}{b}\right)^2 dydx$$

$$= \frac{\mu c^2}{2} \int_0^a \int_0^{b(1-x/a)} x \,.\left[\left(1-\frac{x}{a}\right)^2 y - 2\left(1-\frac{x}{a}\right)\frac{y^2}{b} + \frac{y^3}{b^2}\right] dy\, dx$$

$$= \frac{\mu c^2}{2} \int_0^a x \left| \left(1-\frac{x}{a}\right)^2 \frac{y^2}{2} - 2\left(1-\frac{x}{a}\right)\frac{y^3}{3b} + \frac{y^4}{4b^2} \right|_0^{b(1-x/a)} dx$$

$$= \frac{\mu c^2}{2} \int_0^a x \left[\frac{b^2}{2}\left(1-\frac{x}{a}\right)^4 - \frac{2b^2}{3}\left(1-\frac{x}{a}\right)^4 + \frac{b^2}{4}\left(1-\frac{x}{a}\right)^4\right] dx = \frac{\mu b^2 c^2}{24} \int_0^a x\,(1 - x/a)^4\, dx = \frac{\mu a^2 b^2 c^2}{720}.$$

7.10 CENTRE OF GRAVITY

(*a*) **To find the C.G. $(\bar{x}, \bar{y})$ of a plane lamina,** take the element of mass $\rho\delta x\delta y$ at the point $P(x, y)$. Then

$$\bar{x} = \frac{\iint x\rho\, dxdy}{\iint \rho\, dxdy}, \quad \bar{y} = \frac{\iint y\, \rho dxdy}{\iint \rho\, dxdy}$$ with integrals embracing the whole lamina.

While using polar coordinates, take the elementary mass as $\rho r\delta\theta\delta r$ at the point $P(r, \theta)$ so that $x = r\cos\theta$, $y = r\sin\theta$.

$$\therefore \quad \bar{x} = \frac{\iint r\cos\theta\, \rho r\, d\theta dr}{\iint \rho\, rd\theta dr}, \quad \bar{y} = \frac{\iint r\sin\theta\, \rho\, rd\theta dr}{\iint \rho\, rd\theta dr}$$

(*b*) **To find the C.G. $(\bar{x}, \bar{y}, \bar{z})$ of a solid,** take an element of mass $\rho\delta x\delta y\delta z$ enclosing the point $P(x, y, z)$. *Then*

$$\bar{x} = \frac{\iiint x\rho\, dxdydz}{\iiint \rho\, dxdydz},\ \bar{y} = \frac{\iiint y\rho\, dxdydz}{\iiint \rho\, dxdydz} \text{ and } \bar{z} = \frac{\iiint z\rho\, dxdydz}{\iiint \rho\, dxdydz}.$$

Example 7.34. *Find by double integration, the centre of gravity of the area of the cardioid $r = a(1 + \cos\theta)$.*

Solution. The cardioid being symmetrical about the initial line, its C.G. lies on OX, *i.e.*, $\bar{y} = 0$ (Fig. 7.35).

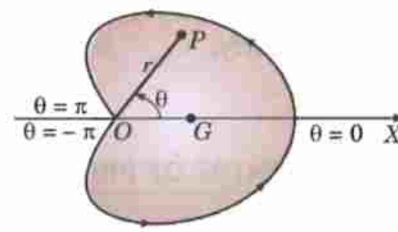

Fig. 7.35

$$\therefore \quad \bar{x} = \frac{\iint \rho\, r\cos\theta \,.\, rd\theta dr}{\iint \rho\, rd\theta dr} = \frac{\int_{-\pi}^{\pi}\int_0^{a(1+\cos\theta)} \cos\theta.\, r^2 dr\,.\, d\theta}{\int_{-\pi}^{\pi}\int_0^{a(1+\cos\theta)} rdr\,.\, d\theta}$$

$$= \frac{\int_{-\pi}^{\pi} \cos\theta \left| \frac{r^3}{3} \right|_0^{a(1+\cos\theta)} d\theta}{\int_{-\pi}^{\pi} \left| \frac{r^2}{2} \right|_0^{a(1+\cos\theta)} d\theta} = \frac{2a}{3} \cdot \frac{\int_{-\pi}^{\pi} \cos\theta\,(1+\cos\theta)^3\, d\theta}{\int_{-\pi}^{\pi} (1+\cos\theta)^2\, d\theta}$$

$$= \frac{2a}{3} \cdot \frac{2 \,.\, \int_0^{\pi} (3\cos^2\theta + \cos^4\theta)\, d\theta}{2 \,.\, \int_0^{\pi} (1+\cos^2\theta)\, d\theta} \qquad \left\{\because \int_{-\pi}^{\pi} \cos^n\theta\, d\theta = 2\int_0^{\pi} \cos^n\theta\, d\theta \text{ or } 0 \text{ according as } n \text{ in even or odd.}\right\}$$

$$= \frac{2a}{3} \cdot \frac{2 \,.\, \int_0^{\pi/2} (3\cos^2\theta + \cos^4\theta)\, d\theta}{2 \,.\, \int_0^{\pi/2} (1+\cos^2\theta)\, d\theta} \text{ (as the powers of } \cos\theta \text{ are even)} = \frac{2a}{3} \cdot \frac{3 \,.\, \frac{1}{2}\cdot\frac{\pi}{2} + \frac{3\,.\,1}{4\,.\,2}\cdot\frac{\pi}{2}}{\frac{\pi}{2} + \frac{1}{2}\cdot\frac{\pi}{2}} = \frac{5a}{6}$$

Hence the C.G. of the cardioid is at $G(5a/6, 0)$.

Example 7.35. *Using double integration, find the C.G. of a lamina in the shape of a quadrant of the curve $(x/a)^{2/3} + (y/b)^{2/3} = 1$, the density being $\rho = kxy$, where k is a constant.*

Solution. Let $G\ (\bar{x}, \bar{y})$ be the C.G. of the lamina OAB (Fig. 7.36), so that

$$\bar{x} = \frac{\iint kxy\,.\,xdxdy}{\iint kxy\,.\,dxdy} = \frac{\iint x^2y\, dxdy}{\iint xy\, dxdy}$$

where the integrals are taken over the area OAB so that y varies from 0 to y (to be found from the equation of the curve in terms of x) and then x varies from 0 to a.

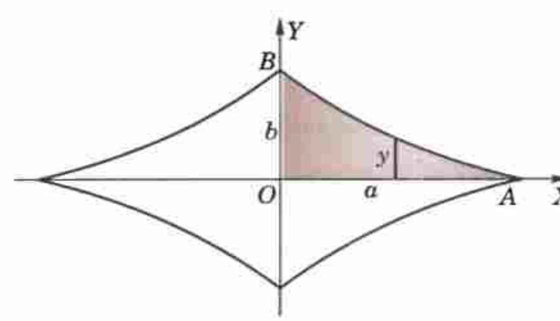

Fig. 7.36

Thus

$$\bar{x} = \frac{\int_0^a \int_0^y x^2y\, dydx}{\int_0^a \int_0^y xy\, dydx} = \frac{\int_0^a x^2 \,.\, \left| y^2/2 \right|_0^y dx}{\int_0^a x \,.\, \left| y^2/2 \right|_0^y dx} = \frac{\int_0^a x^2y^2\, dx}{\int_0^a xy^2\, dx}$$

For any point on the curve, we have

$$x = a\cos^3\theta,\ y = b\sin^3\theta \text{ so that}$$
$$dx = -3a\cos^2\theta\sin\theta\, d\theta.$$

Also when $\quad x = 0,\ \theta = \pi/2$; when $x = a,\ \theta = 0$.

Hence $$\bar{x} = \frac{\int_{\pi/2}^{0} a^2 \cos^6 \theta . b^2 \sin^6 \theta . (-3a \cos^2 \theta \sin \theta)\, d\theta}{\int_{\pi/2}^{0} a \cos^3 \theta . b^2 \sin^6 \theta . (-3a \cos^2 \theta \sin \theta)\, d\theta}$$

$$= a \frac{\int_0^{\pi/2} \sin^7 \theta \cos^8 \theta\, d\theta}{\int_0^{\pi/2} \sin^7 \theta \cos^5 \theta\, d\theta} = \frac{128}{429} a$$

Similarly, $$\bar{y} = \frac{\int_0^a \int_0^y kxy . y\, dxdy}{\int_0^a \int_0^y kxy . dxdy} = \frac{128}{429} b.$$ Hence the required C.G. is $G\left(\frac{128}{429}a, \frac{128}{429}b\right)$.

7.11 CENTRE OF PRESSURE

Consider plane area A immersed vertically in a homogeneous liquid. Take the line of intersection of the given plane with the free surface of the liquid as the x-axis and any line lying in this plane and perpendicular to it downwards as the y-axis (Fig. 7.37).

If p be the pressure at the point $P(x, y)$ of the area A, then the pressure on an elementary area $\delta x \delta y$ at P is $p\delta x \delta y$ which is normal to the plane.

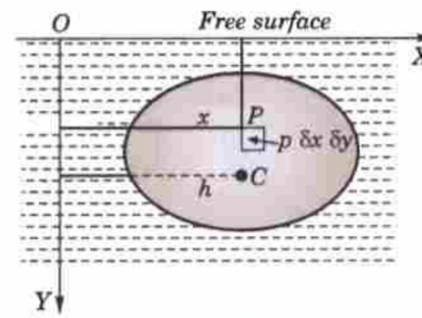

Fig. 7.37

$\therefore$ the resultant pressure on $A = \iint p dxdy$.

If this resultant pressure acting at $C(h, k)$ is equivalent to pressure at various points such as $p\delta x\delta y$ distributed over the whole area A, then C is called the *centre of pressure*.

$\therefore$ taking the moment of the resultant pressure at C and the sum of the moments of the individual pressures such as $p\delta x\delta y$ at $P(x, y)$ about the y-axis, we get

$$h \iint p\, dxdy = \iint x . p dxdy, \quad i.e., h = \iint x . dxdy \Big/ \iint p dxdy$$

Similarly, taking moments about x-axis, we have

$$k = \iint y . pdxdy \Big/ \iint p dxdy$$ with integrals embracing the whole of the area A.

While using polar coordinates, replace x by $r \cos \theta$, y by $r \sin \theta$ and $dxdy$ by $r\, d\theta dr$ in the above formulae.

Example 7.36. *A horizontal boiler has a flat bottom and its ends are plane and semi-circular. If it is just full of water, show that the depth of the centre of pressure of either end is 0.7 × total depth approximately.*

Solution. Let the semi-circle $x^2 + y^2 = a^2$ represent an end of the given boiler (Fig. 7.38). By symmetry, its centre of pressure lies on OY.

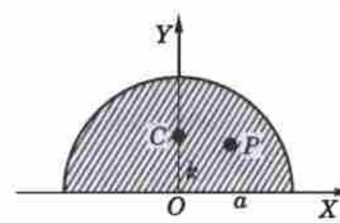

Fig. 7.38

If w be the weight of water per unit volume, then the pressure p at the point $P(x, y) = w(a - y)$.

$\therefore$ the height k of the C.P. above OX, is given by

$$k = \frac{\iint y \cdot pdxdy}{\iint pdxdy} = \frac{\int_{-a}^{a} \int_0^{\sqrt{(a^2-x^2)}} w(a-y) y\, dy \cdot dx}{\int_{-a}^{a} \int_0^{\sqrt{(a^2-x^2)}} w(a-y)\, dy \cdot dx}$$

$$= \frac{\int_{-a}^{a} \left| ay^2/2 - y^3/3 \right|_0^{\sqrt{(a^2-x^2)}} dx}{\int_{-a}^{a} \left| ay - y^2/2 \right|_0^{\sqrt{(a^2-x^2)}} dx} = \frac{\int_{-a}^{a} \left[\frac{a}{2}(a^2-x^2) - \frac{1}{3}(a^2-x^2)^{3/2}\right] dx}{\int_{-a}^{a} \left[a(a^2-x^2)^{1/2} - \frac{1}{2}(a^2-x^2)\right] dx}$$

Now put $x = a \sin\theta$, so that $dx = a \cos\theta \, d\theta$.

Also when $x = -a$, $\theta = -\pi/2$, and when $x = a$, $\theta = \pi/2$.

$$\therefore \qquad k = \frac{\int_{-\pi/2}^{\pi/2} \left[\frac{a^3}{2}\cos^2\theta - \frac{a^3}{3}\cos^3\theta\right] a\cos\theta \, d\theta}{\int_{-\pi/2}^{\pi/2} \left[a^2\cos\theta - \frac{a^2}{2}\cos^2\theta\right] a\cos\theta \, d\theta}$$

$$= \frac{a}{3} \cdot \frac{2\int_0^{\pi/2} (3\cos^3\theta - 2\cos^4\theta)\, d\theta}{2\int_0^{\pi/2} (2\cos^2\theta - \cos^3\theta)\, d\theta} = \frac{a}{4}\left(\frac{16 - 3\pi}{3\pi - 4}\right) = 0.3a \text{ nearly.}$$

Hence, the depth of the C.P. $= a - k = 0.7a$ approximately.

PROBLEMS 7.6

1. A lamina is bounded by the curves $y = x^2 - 3x$ and $y = 2x$. If the density at any point is given by λxy, find by double integration, the mass of the lamina.
2. Find the mass of a lamina in the form of cardioid $r = a(1 + \cos\theta)$ whose density at any point varies as the square of its distance from the initial line.
3. Find the mass of a solid in the form of the positive octant of the sphere $x^2 + y^2 + z^2 = 9$, if the density at any point is $2xyz$.
4. Find the centroid of the area enclosed by the parabola $y^2 = 4ax$, the axis of x and its latus-rectum.
5. The density at any point (x, y) of a lamina is $\sigma(x + y)/a$ where σ and a are constants. The lamina is bounded by the lines $x = 0, y = 0, x = a, y = b$. Find the position of its centre of gravity.
6. Find the centroid of a loop of the lemniscate $r^2 = a^2 \cos 2\theta$.
7. A plane in the form of a quadrant of the ellipse $(x/a)^2 + (y/b)^2 = 1$ is of small but varying thickness, the thickness at any point being proportional to the product of the distances of that point from the axes ; show that the coordinates of the centroid are $(8a/15, 8b/15)$. *(Nagpur, 2009)*
8. In a semi-circular disc bounded by a diameter OA, the density at any point varies as the distance from O ; find the position of the centre of gravity.
9. Find the centroid of the tetrahedron bounded by the coordinate planes and the plane $x + y + z = 1$, the density at any point varying as its distance from the face $z = 0$.
10. Find $\bar{x}$ where $(\bar{x}, \bar{y}, \bar{z})$ is the centroid of the region R bounded by the parabolic cylinder $z = 4 - x^2$ and the planes $x = 0, y = 0, y = 6, z = 0$. (Assume that the density is constant).
11. If the density at any point of the solid octant of the ellipsoid $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$ varies as xyz, find the coordinates of the C.G. of the solid. *(P.T.U., 2005)*
12. A horizontal boiler has a flat bottom and its ends consist of a square 1 metre wide surmounted by an isosceles triangle of height 0.5 metre. Determine the depth of the centre of pressure of either end when the boiler is just full.
13. A quadrant of a circle is just, immersed vertically in a heavy homogeneous liquid with one edge in the surface. Find the centre of pressure.
14. Find the depth of the centre of pressure of a square lamina immersed in the liquid, with one vertex in the surface and the diagonal vertical.
15. Find the centre of pressure of a triangular lamina immersed in a homogeneous liquid with one side in the free surface. *(P.T.U., 2003)*
16. A uniform semi-circular is lamina immersed in a fluid with its plane vertical and its boundary diameter on the free surface. If the density at any point of the fluid varies as the depth of the point below the free surface, find the position of the centre of pressure of the lamina.

7.12 (1) MOMENT OF INERTIA

If a particle of mass m of a body be at a distance r from a given line, then mr^2 is called the *moment of inertia of the particle about the given line* and the sum of similar expressions taken for all the particles of the body, *i.e.*, Σmr^2 is called the *moment of inertia of the body about the given line* (Fig. 7.39).

If M be the total mass of the body and we write its moment of inertia $= Mk^2$, then k is called the *radius of gyration* of the body about the axis.

(2) M.I. of plane lamina. Consider the elementary mass $\rho\delta x\delta y$ at the point $P(x, y)$ of a plane area A so that its M.I. about x-axis $= \rho\delta x\delta y\, y^2$.

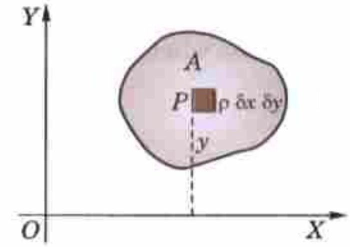

Fig. 7.39

$\therefore$ M.I. of the lamina about x-axis, *i.e.* $I_x = \iint_A \rho\, y^2\, dxdy.$

Similarly, M.I. of the lamina about y-axis' *i.e.*, $I_y = \iint_A \rho\, x^2\, dxdy.$

Also M.I. of the lamina about an axis perpendicular to the xy-plane, *i.e.*,

$$I_z = \iint_A \rho(x^2 + y^2)\, dxdy.$$

(3) M.I. of a solid. Consider an elementary mass $\rho\delta x\delta y\delta z$ enclosing a point $P(x, y, z)$ of a solid of volume V.

Distance of P from the x-axis $= \sqrt{(y^2 + z^2)}$.

$\therefore$ M.I. of this element about the x-axis $= \rho\ \delta x\delta y\delta z\ (y^2 + z^2)$.

Thus *M.I. of this solid about x-axis, i.e.,* $I_x = \iiint_V \rho(y^2 + z^2)\, dxdydz.$

Similarly, *its M.I. about y-axis, i.e.,* $I_y = \iiint_V \rho(z^2 + x^2)\, dxdydz$

and M.I. *about z-axis, i.e.,* $I_z = \iiint_V \rho(x^2 + y^2)\, dxdydz.$

(4) Sometimes we require the moment of inertia of a body about axes other than the principal axes. The following theorems prove useful for this purpose :

I. Theorem of perpendicular axis. *If the moment of inertia of a lamina about two perpendicular axes OX, OY in its plane are I_x and I_y, then the moment of inertia of the lamina about an axis OZ, perpendicular to it is given by $I_z = I_x + I_y$.*

Its proof follows from the relations giving I_x, I_y and I_z for a plane lamina [(2) above].

II. Steiner's theorem*. *If the moment of inertia of a body of mass M about an axis through its centre of gravity is I, then I′, moment of inertia about a parallel axis at a distance d from the first axis, is given by $I' = I + Md^2$.*

Its proof will be found in any text book on Dynamics of a Rigid Body.

Example 7.37. *Find the M.I. of the area bounded by the curve $r^2 = a^2 \cos 2\theta$ about its axis.*

Solution. Given curve is symmetrical about the pole and for half of the loop in the first quadrant θ varies from 0 to $\pi/4$ (Fig. 7.40).

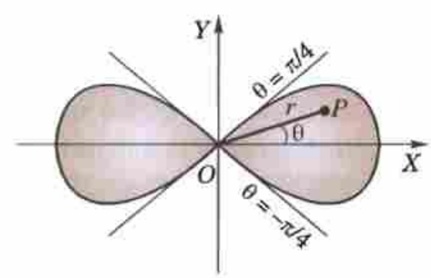

Fig. 7.40

Elementary area at $P(r, \theta) = r\, d\theta dr$.

If ρ be the surface density, then elementary mass

$$= \rho r\, d\theta dr \qquad ...(i)$$

$$\therefore \text{ its total mass } M = 4\int_0^{\pi/4}\int_0^{a\sqrt{(\cos 2\theta)}} \rho r\, dr d\theta$$

$$= 2\rho a^2 \int_0^{\pi/4} \cos 2\theta\, d\theta = \rho a^2 \qquad ...(ii)$$

Now M.I. of the elementary mass (i) about the x-axis.

$$= \rho r\, d\theta dr \cdot y^2 = \rho r d\theta\, dr\, (r \sin\theta)^2 = \rho r^3 \sin^2\theta\, drd\theta$$

Hence the M.I. of the whole area

$$= 4\int_0^{\pi/4}\int_0^{a\sqrt{(\cos 2\theta)}} \rho r^3 \sin^2\theta\, drd\theta = 4\rho \int_0^{\pi/4} \sin^2\theta \cdot \Big| r^4/4 \Big|_0^{a\sqrt{(\cos 2\theta)}} d\theta$$

$$= \rho a^2 \int_0^{\pi/4} \cos^2 2\theta \cdot \sin^2\theta\, d\theta = \rho a^4 \int_0^{\pi/2} \cos^2\phi \cdot \sin^2\frac{\phi}{2}\cdot\frac{d\phi}{2} \qquad [\text{Put } 2\theta = \phi,\ d\theta = d\phi/2]$$

$$= \frac{\rho a^4}{4}\int_0^{\pi/2} (\cos^2\phi - \cos^3\phi)\, d\phi = \frac{\rho a^4}{48}(3\pi - 8) = \frac{Ma^2}{48}(3\pi - 8). \qquad [\text{By } (ii)]$$

*Named after a Swiss geometrer *Jacob Steiner* (1796–1863) who was a professor at Berlin University.

Example 7.38. *Find the moment of inertia of a hollow sphere about a diameter, its external and internal radii being 5 metres and 4 metres.*

Solution. Let ρ be the density of the given hollow sphere. Then the M.I. about the diameter, *i.e.*, x-axis is

$$I_x = \iiint_V \rho(y^2 + z^2)\, dx\, dy\, dz$$

Changing to polar spherical coordinates, we get

$$I_x = \int_0^{2\pi} \int_0^{\pi} \int_4^5 \rho\{(r \sin\theta \sin\phi)^2 + (r\cos\theta)^2\}\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= \rho\left\{\int_0^{2\pi} \sin^2\phi\, d\phi \cdot \int_0^{\pi} \sin^3\theta\, d\theta \left|\frac{r^5}{5}\right|_4^5 + \int_0^{2\pi} d\phi \int_0^{\pi} \cos^2\theta \sin\theta\, d\theta \cdot \left|\frac{r^5}{5}\right|_4^5\right\}$$

$$= \frac{8\pi\rho}{15}(5^5 - 4^5) = 1120.5 \text{ m}.$$

Example 7.39. *A solid body of density ρ is in the shape of the solid formed by revolution of the centroid $r = a(1 + \cos\theta)$ about the initial line. Show that its moment of inertia about a straight line through the pole perpendicular to the initial line is $\frac{352}{105}\pi\rho a^5$.* *(U.P.T.U., 2001)*

Solution. An elementary area $rd\theta dr$, when revolved about OX generates a circular ring of radius $LP = r\sin\theta$ (Fig. 7.41).

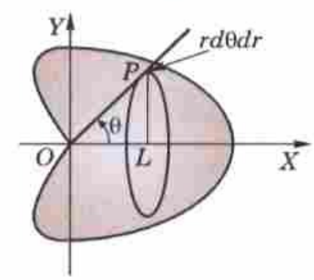

Fig. 7.41

M.I. of this ring about a diameter parallel to OY

$$= (2\pi r \sin\theta)(rd\theta dr)\rho \cdot \frac{(r\sin\theta)^2}{2}.$$

[∵ M.I. of a ring about a diameter $= Ma^2/2$.]

Now using Steiner's theorem, we have M.I. of the ring about OY = M.I. of the ring about a diameter LP parallel to OY + Mass of the ring $(OL)^2$ $(r\cos\theta)^2$

$$= 2\pi\rho r^4 \sin^3\theta\, d\theta dr + 2\pi r \sin\theta\, (rd\theta dr)\, (r\cos\theta)^2$$

Hence M.I. of the solid generated by revolution about OY

$$= \pi\rho \int_0^{\pi} \int_0^{r=a(1+\cos\theta)} (r^4 \sin^3\theta + 2r^4 \sin\theta \cos^2\theta)\, d\theta dr$$

$$= \pi\rho \int_0^{\pi} (\sin^3\theta + 2\sin\theta\cos^2\theta)\, d\theta \int_0^{a(1+\cos\theta)} r^4\, dr$$

$$= \frac{\pi\rho a^5}{5} \int_0^{\pi} \sin\theta\, (1 + \cos^2\theta)(1 + \cos\theta)^5\, d\theta \qquad \text{[Put } \theta = 2\phi]$$

$$= \frac{\pi\rho a^5}{5} \int_0^{\pi/2} \sin 2\phi\, (1 + \cos^2 2\phi)(1 + \cos 2\phi)^5\, 2d\phi$$

$$= \frac{\pi\rho a^5}{5} \int_0^{\pi/2} 2\sin\phi\cos\phi\, \{1 + (2\cos^2\phi - 1)^2\}\, (2\cos^2\phi)^5\, 2d\phi$$

$$= \frac{256\,\pi\rho a^5}{5} \int_0^{\pi/2} (\cos^{11}\phi - 2\cos^{13}\phi + 2\cos^{15}\phi)\sin\phi\, d\phi$$

$$= \frac{256\,\pi\rho a^5}{5}\left| -\frac{\cos^{12}\phi}{12} + \frac{2\cos^{14}\phi}{14} - \frac{2\cos^{16}\phi}{16}\right|_0^{\pi/2} = \frac{352\,\pi\rho a^5}{105}.$$

Example 7.40. *A hemisphere of radius R has a cylindrical hole of radius a drilled through it, the axis of the hole being along the radius normal to the plane face of the hemisphere. Find its radius of gyration about a diameter of this face.*

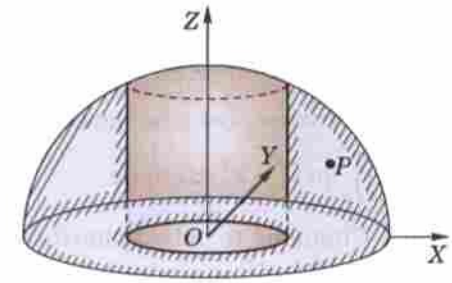

Fig. 7.42

Solution. M.I. of the given solid about x-axis

$$= \iiint \rho(y^2 + z^2)\, dx\, dy\, dz$$

The limits of integration for z are from 0 to $z = \sqrt{(R^2 - x^2 - y^2)}$ found from the equation of the sphere $x^2 + y^2 + z^2 = R^2$. The limits for x and y are to be such as to cover the shaded area A in the xy-plane between the concentric circles of radii a and R (Fig. 7.42).

Thus the required M.I. about x-axis

$$= \rho \iint_A \int_0^{\sqrt{(R^2 - x^2 - y^2)}} (y^2 + z^2)\, dz\, dx\, dy$$

$$= \rho \iint_A \left| y^2 z + z^3/3 \right|_0^{\sqrt{(R^2 - x^2 - y^2)}} dx\, dy = \rho \iint_A \left[y^2 (R^2 - x^2 - y^2)^{1/2} + \frac{1}{3}(R^2 - x^2 - y^2)^{3/2} \right] dx\, dy.$$

Now changing to polar coordinates, we have $x = r\cos\theta$, $y = r\sin\theta$ and $dxdy = rd\theta dr$.

Also to cover the area A, r varies from a to R and θ varies from 0 to 2π.

Hence the required M.I. about x-axis

$$= \rho \int_a^R \int_0^{2\pi} \left[r^2 \sin^2\theta \cdot (R^2 - r^2)^{1/2} + \frac{1}{3}(R^2 - r^2)^{3/2} \right] r d\theta dr$$

$$= \rho \int_a^R \int_0^{2\pi} \left[\frac{1}{2} r^2 (1 - \cos 2\theta) + \frac{1}{3}(R^2 - r^2) \right] d\theta \cdot r(R^2 - r^2)^{1/2}\, dr$$

$$= \rho \int_a^R \left| \frac{r^2}{2}\left(\theta - \frac{\sin 2\theta}{2}\right) + \frac{1}{3}(R^2 - r^2)\,\theta \right|_0^{2\pi} \cdot r(R^2 - r^2)^{1/2}\, dr$$

$$= \rho \int_a^R 2\pi \left(\frac{r^2}{2} + \frac{R^2 - r^2}{3} \right) . \, r(R^2 - r^2)^{1/2}\, dr$$

$$= \frac{\pi\rho}{3} \int_a^R (2R^2 + r^2)(R^2 - r^2)^{1/2} \cdot r dr \qquad \text{[Put } r^2 = t \text{ and } rdr = dt/2]$$

$$= \frac{\pi\rho}{6} \int_{a^2}^{R^2} (2R^2 + t)(R^2 - t)^{1/2}\, dt \qquad \text{[Integrate by parts]}$$

$$= \frac{\pi\rho}{9} \left[(2R^2 + a^2)(R^2 - a^2)^{3/2} + \frac{2}{5}(R^2 - a^2)^{5/2} \right] = \frac{2\pi\rho}{3}(R^2 - a^2)^{3/2} \times \frac{1}{10}(4R^2 + a^2)$$

$$\left[\because \ \text{Mass} = \rho \int_0^{2\pi} \int_a^R \int_0^{\sqrt{(R^2 - r^2)}} dz \cdot rdr \cdot d\theta = \frac{2\pi\rho}{3}(R^2 - a^2)^{3/2} \right]$$

Hence, the radius of gyration $= [(4R^2 + a^2)/10]^{1/2}$.

7.13 (1) PRODUCT OF INERTIA

If a particle of mass m of a body be at distances x and y from two given perpendicular lines, then $\sum mxy$ is called the *product of inertia* of the body about the given lines.

Consider an elementary mass $\delta x \delta y \delta z$ enclosing the point $P(x, y, z)$ of solid of volume V. Then the product of inertia (P.I.) of this element about the axes of x and $y = \rho \delta x \delta y \delta z\, xy$.

$\therefore$ P.I. of the solid about x and y-axes, *i.e.*, $P_{xy} = \iiint_V \rho\, xy\, dxdydz$

Similarly, $P_{yz} = \iiint_V \rho\, yz\, dxdydz$ and $P_{zx} = \iiint_V \rho\, zx\, dxdydz$.

In particular, for a plane lamina of surface density ρ and covering a region A in the xy-plane,

$$P_{xy} = \iint_A \rho\, xy\, dxdy \ \text{ whereas } P_{yz} = P_{zx} = 0. \qquad [\because \ z = 0]$$

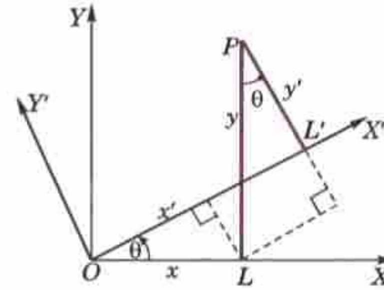

Fig. 7.43

(2) Principal axes. *The principal axes* of a lamina at a given point are that pair of axes in its plane through the given point, about which the product of inertia of the lamina vanishes.

Let $P(x, y)$ be a point of the plane area A referred to rectangular axes OX, OY. Let (x', y') be the coordinates of P referred to another pair of rectangular axes OX', OY' in the same plane and inclined at an angle θ to the first pair (Fig. 7.43).

Then $$x' = x\cos\theta + y\sin\theta$$
$$y' = y\cos\theta - x\sin\theta$$

If I_x, I_y be the moments of inetria of the area A about OX and OY and P_{xy} be its product of inertia about these axes, then

$$I_x = \iint_A \rho y^2\, dA,\ I_y = \iint_A \rho x^2\, dA,\ P_{xy} = \iint \rho\, xy\, dA.$$

$\therefore$ the product of inertia P'_{xy} about OX' and OY' is given by

$$P'_{xy} = \int_A \rho x'y'\, dA = \iint_A \rho(x\cos\theta + y\sin\theta)(y\cos\theta - x\sin\theta)\, dA$$

$$= \sin\theta\cos\theta \iint_A \rho(y^2 - x^2)\, dA + (\cos^2\theta - \sin^2\theta)\iint_A \rho xy\, dA$$

$$= 1/2 \sin 2\theta \cdot (I_x - I_y) + \cos 2\theta\, P_{xy}.$$

Now OX', OY' will be the principal axes of the area A if P'_{xy} vanishes.

i.e., If $\quad 1/2 \sin 2\theta\, (I_x - I_y) + \cos 2\theta\, P_{xy} = 0$

i.e., If $\quad \mathbf{\tan 2\theta = 2P_{xy}/(I_y - I_x)}$.

This gives two values of θ differing by $\pi/2$.

Example 7.41. *Show that the principal axes at the node of a half-loop of the lemniscate $r^2 = a^2\cos 2\theta$ are inclined to the initial line at angles*

$$\frac{1}{2}\tan^{-1}\frac{1}{2} \text{ and } \frac{\pi}{2} + \frac{1}{2}\tan^{-1}\frac{1}{2}.$$

Solution. Let the element of mass at $P(r, \theta)$ be $\rho r d\theta dr$.

Then $$I_x = \rho\int_0^{\pi/4}\int_0^{a\sqrt{(\cos 2\theta)}} r^2\sin^2\theta \cdot r d\theta dr$$ [See Fig. 7.40]

$$= \frac{\rho a^4}{4}\int_0^{\pi/4}\sin^2\theta\cos^2 2\theta\, d\theta = \frac{\rho a^4}{16}\left(\frac{\pi}{4} - \frac{2}{3}\right)$$

$$I_y = \rho\int_0^{\pi/4}\int_0^{a\sqrt{(\cos 2\theta)}} r^2\cos^2\theta \cdot r\, d\theta dr = \frac{\rho a^4}{16}\left(\frac{\pi}{4} + \frac{2}{3}\right)$$

and $$P_{xy} = \rho\int_0^{\pi/4}\int_0^{a\sqrt{(\cos 2\theta)}} r^2\sin\theta\cos\theta \cdot r d\theta dr = \frac{\rho a^4}{48}.$$

Hence the required direction of the principal axes at O are given by

$$\tan 2\theta = \frac{2P_{xy}}{I_y - I_x} = \frac{\rho a^4/24}{(\rho a^4/16)\times(4/3)} = \frac{1}{2}$$

or by $$\theta = \frac{1}{2}\tan^{-1}\frac{1}{2} \text{ and } \frac{\pi}{2} + \frac{1}{2}\tan^{-1}\frac{1}{2}.$$

PROBLEMS 7.7

1. Using double integrals, find the moment of inertia about the x-axis of the area enclosed by the lines $x = 0, y = 0, (x/a) + (y/b) = 1$. (*P.T.U., 2005*)
2. Find the moment of inertia of a circular plate about a tangent.
3. Find the moment of inertia of the area $y = \sin x$ from $x = 0$ to $x = 2\pi$ about OX.

4. Find the moment of inertia of a quadrant of the ellipse $(x/a)^2 + (y/b)^2 = 1$ of mass M about the x-axis, if the density at a point is proportional to xy.
5. Find the moment of inertia about the initial line of the cardioid $r = a(1 + \cos\theta)$.
6. Find the moment of inertia of a uniform spherical ball of mass M and radius R about a diameter.
7. Find the moment of inertia of a solid right circular cylinder about (*i*) its axis (*P.T.U., 2006*)
(*ii*) a diameter of the base.
8. Find the M.I. of a solid right circular cone having base-radius r and height h, about (*i*) its axis, (*ii*) an axis through the vertex and perpendicular to its axis, (*iii*) a diameter of its base.
9. Find the moment of inertia of a hollow sphere about a diameter, its external and internal radii being 51 metres and 49 metres.
10. Find the moment of inertia about z-axis of a homogeneous tetrahedron bounded by the planes $x = 0, y = 0, z = x + y$ and $z = 1$.
11. Find the moment of inertia of an octant of the ellipsoid $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$, about the x-axis.
12. Find the product of inertia of a quadrant of the ellipse $(x/a)^2 + (y/b)^2 = 1$, about the coordinate axes.
13. Show that the principal axes at the origin of the triangle enclosed by $x = 0, y = 0, (x/a) + (y/b) = 1$ are inclined to the x-axis at angles α and $\alpha + \pi/2$, where $\alpha = \frac{1}{2}\tan^{-1}[ab/(a^2 - b^2)]$ (*U.P.T.U., 2002*)
14. The lengths AB and AD of the sides of a rectangle $ABCD$ are $2a$ and $2b$. Show that the inclination to AB of one of the principal axes at A is $\frac{1}{2}\tan^{-1}\left\{\frac{3ab}{2(a^2 - b^2)}\right\}$.

7.14 BETA FUNCTION

The beta function is defined as

$$\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad \begin{cases} m > 0 \\ n > 0 \end{cases} \qquad \text{...(1)}$$

Putting $x = 1 - y$ in (1), we get $\beta(m, n) = -\int_1^0 (1-y)^{m-1}\, y^{n-1}\,dy$

$$= \int_0^1 y^{n-1}(1-y)^{m-1}\,dy = \beta(n, m)$$

Thus $\beta(m, n) = \beta(n, m)$ ...(2)

Putting $x = \sin^2\theta$ so that $dx = 2\sin\theta\cos\theta\,d\theta$, (1) becomes

$$\beta(m, n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1}(\cos^2\theta)^{n-1}\,2\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta \qquad \text{...(3)}$$

which is another form of $\beta(m, n)$.

This function is also *Euler's integral of the first kind**.

7.15 (1) GAMMA FUNCTION

The gamma function is defined as

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\,dx \ (n > 0) \qquad \text{...(i)}$$

This integral is also known as *Euler's integral of the second kinds.* It defines a function of n for positive values of n.

*After an enormously creative Swiss mathematician *Leonhard Euler (1707–1783).* He studied under *John Bernoulli* and became a professor of mathematics in St. Petersburg, Russia. Even after becoming totally blind in 1771, he contributed to almost all branches of mathematics.

In particular, $\Gamma(1) = \int_0^\infty e^{-x}\,dx = \left| -e^{-x} \right|_0^\infty = 1.$ ...(ii)

(2) Reduction formula for Γ(n).

Since $\Gamma(n+1) = \int_0^\infty e^{-x} x^n\,dx$ [Integrating by parts] $= \left| -x^n e^{-x} \right|_0^\infty + n\int_0^\infty e^{-x} x^{n-1} dx$

$\therefore \quad \Gamma(n+1) = n\Gamma(n)$ ...(iii)

which is the reduction formula for $\Gamma(n)$. From this formula, it is clear that if $\Gamma(n)$ is known throughout a unit interval say : $1 < n \le 2$, then the values of $\Gamma(n)$ throughout the next unit interval $2 < n \le 3$ are found, from which the values of $\Gamma(n)$ for $3 < n \le 4$ are determined and so on. In this way, *the values of Γ(n) for all positive values of n > 1 may be found by successive application of (iii).*

Also using *(iii)* in the form

$$\Gamma(n) = \frac{\Gamma(n+1)}{n} \quad \text{...(iv)}$$

We can define $\Gamma(n)$ for values of n for which the definition (1) fails. It gives the value of $\Gamma(n)$ for $0 < n \le 1$ in terms of the values of $\Gamma(n)$ for $1 < n \le 2$. Thus we can define $\Gamma(n)$ for all $n < 0$ provided its value for $1 < n \le 2$ is known. Also if $-1 < n < 0$, (4) gives $\Gamma(n)$ in terms of its values for $0 < n < 1$. Then we may find, $\Gamma(n)$ for $-2 < n < -1$ and so on.

Thus (i) and (iv) together give a complete definition of Γ(n) for all values of n except when n is zero or a negative integer and its graph is as shown in Fig. 7.44. The values of *Γ(n)* for *1 < n ≤ 2* are given in (Table I-Appendix 2) from which the values of *Γ(n)* for values of n outside the interval $1 < n \le 2$ $(n \ne 0, -1, -2, -3 \ldots..)$ may be found.

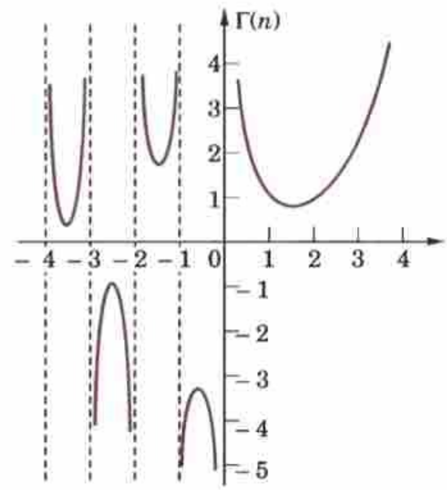

Fig. 7.44

(3) Value of Γ(n) in terms of factorial.

Using $\Gamma(n+1) = n\Gamma(n)$ successively, we get

$$\Gamma(2) = 1 \times \Gamma(1) = 1\,!$$
$$\Gamma(3) = 2 \times \Gamma(2) = 2 \times 1 = 2\,!$$
$$\Gamma(4) = 3 \times \Gamma(3) = 3 \times 2\,! = 3\,!$$

....

In general $\Gamma(n+1) = n\,!$ provided n is a positive integer ...(v)

Taking $n = 0$, it defines $0\,! = \Gamma(1) = 1$.

(4) Value of $\Gamma\left(\frac{1}{2}\right)$. We have

$$\Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty e^{-x} x^{-1/2}\,dx \qquad [\text{Put } x = y^2 \text{ so that } dx = 2y\,dy]$$

$$= 2\int_0^\infty e^{-y^2}\,dy \text{ which is also } = 2\int_0^\infty e^{-x^2}\,dx$$

$$\therefore \quad \left[\Gamma\left(\tfrac{1}{2}\right)\right]^2 = 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy = 4\int_0^{\pi/2}\int_0^\infty e^{-r^2}\,r\,dr\,d\theta$$

$$= 4 \cdot \frac{\pi}{2}\int_0^\infty e^{-r^2}\,r dr = 2\pi\left[\left(-\frac{1}{2}\right)e^{-r^2}\right]_0^\infty = \pi$$

whence $\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi} = 1.772$ ...(vi) *(V.T.U., 2006)*

7.16 RELATION BETWEEN BETA AND GAMMA FUNCTIONS

$$\beta(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

We have $\Gamma(m) = \int_0^\infty e^{-t}\,t^{m-1}$ [Put $t = x^2$ so that $dt = 2x\,dx$

$$= 2\int_0^\infty e^{-x^2} x^{2m-1}\, dx \qquad ...(2)$$

Similarly, $\Gamma(n) = 2\int_0^\infty e^{-y^2} y^{2n-1}\, dy$

$\therefore$ $\Gamma(m)\Gamma(n) = 4\int_0^\infty e^{-x^2} x^{2m-1}\, dx \int_0^\infty e^{-y^2} y^{2n-1}\, dy$

$$= 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} x^{2m-1} y^{2n-1}\, dx\, dy \qquad ...(3) \quad [\because \text{ the limits of integration are constant.}]$$

Now change to polar coordinates by writing $x = r\cos\theta$, $y = r\sin\theta$ and $dx\,dy = rd\theta dr$. To cover the region in (3) which is the entire first quadrant, r varies from 0 to ∞ and θ from 0 to $\pi/2$. Thus (3) becomes

$$\Gamma(m)\Gamma(n) = 4\int_0^\infty \int_0^{\pi/2} e^{-r^2} r^{2(m+n)-1} \cos^{2m-1}\theta \sin^{2n-1}\theta\, d\theta dr$$

$$= \left[2\int_0^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta\, d\theta\right] \times \left[2\int_0^\infty e^{-r^2} r^{2(m+n)-1}\, dr\right] \qquad ...(4)$$

But by (2), $2\int_0^\infty e^{-r^2} r^{2(m+n)-1}\, dr = \Gamma(m+n)$

and by (3) of § 7.14, $2\int_0^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta\, d\theta = \beta(m, n)$.

Thus (4) gives $\Gamma(m)\Gamma(n) = \beta(m, n)\,\Gamma(m+n)$ *(U.T.U., 2010 ; Bhopal, 2009 ; V.T.U., 2008 S)*
whence follows (1) which is extremely useful for evaluating definite integrals in terms of gamma functions.

Cor. *Rule to evaluate* $\int_0^{\pi/2} \sin^p x \cos^q x\, dx$.

$$\int_0^{\pi/2} \sin^p x \cos^q x\, dx = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) \qquad \text{[By (3) of § 7.14]}$$

$$= \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)} \qquad ...(5)$$

In particular, when $q = 0$, and $p = n$, we have

$$\left.\begin{aligned} \int_0^{\pi/2} \sin^n x\, dx &= \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} \cdot \frac{\sqrt{\pi}}{2} \\ \text{Similarly,} \quad \int_0^{\pi/2} \cos^n x\, dx &= \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} \cdot \frac{\sqrt{\pi}}{2} \end{aligned}\right\} \qquad ...(6)$$

Example 7.42. *Show that*

(a) $\Gamma(n) = \int_0^1 \left(\log \frac{1}{y}\right)^{n-1} dy \; (n > 0)$. *(J.N.T.U., 2003 ; Madras, 2003 S)*

(b) $\beta(p, q) = \int_0^\infty \frac{y^{q-1}}{(1+y)^{p+q}}\, dy$ *(V.T.U., 2003 ; Gauhati, 1999)*

$= \int_0^1 \frac{x^{p-1} + x^{q-1}}{(1+x)^{p+q}}\, dx.$ *(V.T.U., 2008 ; Osmania, 2003 ; Rohtak, 2003)*

Solution. (*a*) $\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\, dx \; (n > 0)$

$$= \int_1^0 \left(\log \frac{1}{y}\right)^{n-1} y\left(-\frac{1}{y}dy\right) = \int_0^1 \left(\log \frac{1}{y}\right)^{n-1} dy.$$

Put $y = e^{-x}$
i.e., $x = \log(1/y)$
so that $dx = -(1/y)\,dy$

(*b*) $$\beta(p, q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx$$

$$= \int_\infty^0 \frac{1}{(1+y)^{p-1}}\left(\frac{y}{1+y}\right)^{q-1} \frac{-1}{(1+y)^2}\,dy$$

Put $x = \dfrac{1}{1+y}$ *i.e.*, $y = \dfrac{1}{x} - 1$
so that $dx = \dfrac{-1}{(1+y)^2}\,dy$

$$= \int_0^\infty \frac{y^{q-1}}{(1+y)^{p+q}}\,dy = \int_0^1 \frac{y^{q-1}}{(1+y)^{p+q}}\,dy + \int_1^\infty \frac{y^{q-1}}{(1+y)^{p+q}}\,dy$$

Now substituting $y = 1/z$ in the second integral, we get

$$\int_1^\infty \frac{y^{q-1}}{(1+y)^{p+q}}\,dy = \int_1^0 \frac{1}{z^{q-1}} \cdot \frac{1}{(1+1/z)^{p+q}}\left(-\frac{1}{z^2}\right)dz = \int_0^1 \frac{z^{p-1}}{(1+z)^{p+q}}\,dz.$$

Hence, $$\beta(p, q) = \int_0^1 \frac{y^{q-1}}{(1+y)^{p+q}}\,dy + \int_0^1 \frac{z^{p-1}}{(1+z)^{p+q}}\,dz = \int_0^1 \frac{x^{p-1} + x^{q-1}}{(1+x)^{p+q}}\,dx.$$

Example 7.43. *Express the following integrals in terms of gamma functions :*

(*a*) $\displaystyle\int_0^1 \frac{dx}{\sqrt{(1-x^4)}}$ (*b*) $\displaystyle\int_0^{\pi/2} \sqrt{(\tan\theta)}\,d\theta$. (*Madras, 2006*)

(*c*) $\displaystyle\int_0^\infty \frac{x^c}{c^x}\,dx$ (*U.P.T.U., 2006*) (*d*) $\displaystyle\int_0^\infty a^{-bx^2}\,dx$.

(*e*) $\displaystyle\int_0^1 x^5\,[\log(1/x)]^3\,dx$ (*Madras, 2000*)

Solution. (*a*) $\displaystyle\int_0^1 \frac{dx}{\sqrt{(1-x^4)}}$

Put $x^2 = \sin\theta$, *i.e.*, $x = \sin^{1/2}\theta$
so that $dx = 1/2\sin^{-1/2}\theta\cos\theta\,d\theta$

$$= \int_0^{\pi/2} \frac{1}{2} \cdot \frac{\sin^{-1/2}\theta \,.\, \cos\theta\,d\theta}{\sqrt{(1-\sin^2\theta)}} = \frac{1}{2}\int_0^{\pi/2} \sin^{-1/2}\theta\,d\theta = \frac{1}{2}\cdot\frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma\left(\dfrac{-\frac{1}{2}+1}{2}\right)}{\Gamma\left(\dfrac{-\frac{1}{2}+2}{2}\right)} = \frac{\sqrt{\pi}}{4}\,\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$$

(*b*) $$\int_0^{\pi/2}\sqrt{(\tan\theta)}\,d\theta = \int_0^{\pi/2} \sin^{1/2}\theta\cos^{-1/2}\theta\,d\theta$$

$$= \frac{\Gamma\left(\dfrac{\frac{1}{2}+1}{2}\right)\Gamma\left(\dfrac{-\frac{1}{2}+1}{2}\right)}{2\Gamma\left(\dfrac{\frac{1}{2}-\frac{1}{2}+2}{2}\right)} = \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{2\Gamma(1)} = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$$

(*c*) $$\int_0^\infty \frac{x^c}{c^x}\,dx = \int_0^\infty \frac{x^c}{e^{x\log c}}\,dx$$ $[\because\ c^x = e^{\log c^x} = e^{x\log c}]$

$$= \int_0^\infty e^{-x\log c}\,x^c\,dx$$ [Put $x\log c = t$ so that $dx = dt/\log c$]

$$= \int_0^\infty e^{-t}\left(\frac{t}{\log c}\right)^c \frac{dt}{\log c} = \frac{1}{(\log c)^{c+1}} \int_0^\infty t^c\, e^{-t}\, dt = \Gamma(c+1)/(\log c)^{c+1}$$

(*d*) $\int_0^\infty a^{-bx^2}\, dx = \int_0^\infty e^{-bx^2 \log a}\, dx$ $\left[\begin{array}{l}\text{Put } (b \log a)\, x^2 = t \\ \text{so that } dx = dt/2\sqrt{(b \log a)}\end{array}\right.$

$$= \frac{1}{2\sqrt{(b\log a)}} \int_0^\infty e^{-t}\, t^{-1/2}\, dt = \frac{\Gamma\left(\frac{1}{2}\right)}{2\sqrt{(b\log a)}} = \frac{\sqrt{\pi}}{2\sqrt{(b\log a)}}$$

(*e*) $\int_0^1 x^4\, [\log(1/x)]^3\, dx = \frac{1}{625} \int_0^\infty e^{-t}\,.\, t^3\, dt$ $\left[\begin{array}{l}\text{Put } x = e^{-t/5} \text{ so that } \log(1/x) = t/5 \\ dx = -\frac{1}{5} e^{-t/5}\, dt\end{array}\right]$

$$= \frac{\Gamma(4)}{625} = \frac{6}{625}.$$

Example 7.44. *Evaluate* $\int_0^\infty e^{-ax}\, x^{m-1}\, \sin bx\, dx$ *in terms of Gamma function.* (*U.P.T.U., 2003*)

Solution. We have $\Gamma(m) = \int_0^\infty e^{-x}\, x^{m-1}\, dx$ [Put $x = ay$, $dx = a\,dy$]

$$= \int_0^\infty e^{-ay}\, a^m\, y^{m-1}\, dy \quad \text{or} \quad \int_0^\infty e^{-ay}\, y^{m-1}\, dy = \Gamma(m)/a^m \qquad \ldots(i)$$

Then $I = \int_0^\infty e^{-ax}\, x^{m-1} \sin bx\, dx = \int_0^\infty e^{-ax}\, x^{m-1}\, (\text{Imaginary part of } e^{ibx})\, dx$

$= \text{I.P. of } \int_0^\infty e^{-(a-ib)x}\, x^{m-1}\, dx$

$= \text{I.P. of } \{\Gamma(m)/(a-ib)^m$ [By (*i*)]

$= \text{I.P. of } \{\Gamma(m)/(r^m (\cos\theta - i \sin\theta)^m]$ where $a = r\cos\theta$, $b = r\sin\theta$

$= \text{I.P. of } \Gamma(m)/(r^m (\cos m\theta - i\sin m\theta)]$ (Using Demoivere's theorem §19.5)

$$= \text{I.P. of } \left\{\frac{\Gamma(m)\,.\,(\cos m\theta + i\sin m\theta)}{r^m (\cos m\theta + i \sin m\theta)(\cos m\theta - i\sin m\theta)}\right\}$$

$= \frac{\Gamma(m)}{r^m} \sin m\theta$ where $r\ \sqrt{(a^2+b^2)}$, $\theta = \tan^{-1} b/a$.

Example 7.45. *Prove that* $\int_0^1 \frac{x^2\,dx}{\sqrt{(1-x^4)}} \times \int_0^1 \frac{dx}{\sqrt{(1+x^4)}} = \frac{\pi}{4\sqrt{2}}$

Solution. $\int_0^1 \frac{x^2 dx}{\sqrt{(1-x^4)}} \times \int_0^{\pi/2} \frac{\sin\theta}{\cos\theta} \cdot \frac{\cos\theta}{2\sqrt{(\sin\theta)}}\, d\theta$ $\left[\text{Putting } x^2 = \sin\theta,\ dx = \frac{\cos\theta\, d\theta}{2\sqrt{(\sin\theta)}}\right.$

$$= \frac{1}{2}\int_0^{\pi/2} \sqrt{(\sin\theta)}\, d\theta = \frac{1}{4}\beta\left(\frac{3}{4}, \frac{1}{2}\right) = \frac{1}{4}\frac{\Gamma(3/4)\,\Gamma(1/2)}{\Gamma(5/4)} = \frac{\Gamma(3/4)\,\Gamma(1/2)}{\Gamma(1/4)}$$

$\int_0^1 \frac{dx}{\sqrt{(1+x^4)}} = \int_0^{\pi/4} \frac{\sec^2\theta\, d\theta}{2\sqrt{(\tan\theta)}\sec\theta}$ $\left[\text{Putting } x^2 = \tan\theta,\ dx = \frac{\sec^2\theta\, d\theta}{2\sqrt{(\tan\theta)}}\right.$

$= \frac{1}{\sqrt{2}} \int_0^{\pi/4} \frac{d\theta}{\sqrt{(\sin 2\theta)}} = \frac{1}{2\sqrt{2}} \int_0^{\pi/2} \sin^{-1/2}\phi\, d\phi$ $\left[\text{Putting } 2\theta = \phi,\ d\theta = \frac{1}{2} d\phi\right.$

$$= \frac{1}{4\sqrt{2}}\beta\left(\frac{1}{4}, \frac{1}{2}\right) = \frac{1}{4\sqrt{2}} \frac{\Gamma(1/4)\,\Gamma(1/2)}{\Gamma(3/4)}$$

$$\therefore \qquad \int_0^1 \frac{x^2\,dx}{\sqrt{(1-x^4)}} \times \int_0^1 \frac{dx}{\sqrt{(1+x^4)}} = \frac{1}{4\sqrt{2}}\left\{\Gamma\left(\frac{1}{2}\right)\right\}^2 = \frac{\pi}{4\sqrt{2}}.$$

Example 7.46. *Prove that (i)* $\beta\,(m, 1/2) = 2^{2m-1}\,\beta\,(m, m)$ *(V.T.U., 2004)*

$$(ii)\ \Gamma\,(m)\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\,\Gamma\,(2m) \qquad \textit{(Duplication Formula)}$$

(V.T.U., 2010 ; Kerala, M.E., 2005 ; Madras, 2003 S)

Solution. (*i*) We know that $\beta\,(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta$ ...(1)

Putting $n = \frac{1}{2}$, we have $\beta\left(m, \frac{1}{2}\right) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\,d\theta$ ...(2)

Again putting $n = m$ in (*i*), we get $\beta(m, m) = 2\int_0^{\pi/2} (\sin\theta\cos\theta)^{2m-1}\,d\theta$

$$= \frac{1}{2^{2m-2}}\int_0^{\pi/2} \sin^{2m-1} 2\theta\,d\theta$$

$$= \frac{1}{2^{2m-1}}\int_0^{\pi} \sin^{2m-1}\phi\,d\phi, \text{ putting } 2\theta = \phi$$

$$= \frac{1}{2^{2m-1}}\cdot 2\int_0^{\pi/2} \sin^{2m-1}\phi\,d\phi$$

or $\qquad 2^{2m-1}\,\beta(m, m) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\,d\theta = \beta\left(m, \frac{1}{2}\right)$ [by (2)

(*ii*) Rewriting the above result in terms of Γ functions, we get

$$2^{2m-1}\,\frac{\Gamma\,(m)\,\Gamma\,(m)}{\Gamma\,(m+m)} = \frac{\Gamma\,(m)\,\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(m+\frac{1}{2}\right)} \qquad \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

or $\qquad \Gamma\,(m)\,\Gamma\left(m+\frac{1}{2}\right) = \dfrac{\sqrt{\pi}\Gamma\,(2m)}{2^{2m-1}}.$

Example 7.47. *Prove that*

(*a*) $\iint_D x^{l-1}\,y^{m-1}\,dx\,dy = \dfrac{\Gamma\,(l)\,\Gamma\,(m)}{\Gamma\,(l+m+1)}\,h^{l+m}$ *where D is the domain* $x \ge 0$, $y \ge 0$ *and* $x + y \le h$.

(U.P.T.U., 2005)

(*b*) $\iiint_V x^{l-1}\,y^{n-1}\,z^{n-1}\,dxdydz = \dfrac{\Gamma\,(l)\,\Gamma\,(m)\,\Gamma\,(n)}{\Gamma\,(l+m+n+1)}$

where V is the region $x \ge 0$, $y \ge 0$, $z \ge 0$ *and* $x + y + z \le 1$. *This important result is known as* **Dirichlet's integral*.**

Solution. (*a*) Putting $x/h = X$ and $y/h = Y$, we see that the given integral

$$= \iint_{D'} (hX)^{l-1}\,(hY)^{m-1}\,h^2\;\;dXdY \text{ where } D' \text{ is the domain } X \ge 0, Y \ge 0 \text{ and } X + Y \le 1.$$

$$= h^{l+m}\int_0^1\int_0^{1-X} X^{l-1}\,Y^{m-1}\,dY\,dX = h^{l+m}\int_0^1 X^{l-1}\left|\frac{Y^m}{m}\right|_0^{1-X} dX$$

$$= \frac{h^{l+m}}{m}\int_0^1 X^{l-1}\,(1-X)^m\,dX = \frac{h^{l+m}}{m}\,\beta\,(l, m+1) = \frac{h^{l+m}}{m}\cdot\frac{\Gamma\,(l)\,\Gamma(m+1)}{\Gamma\,(l+m+1)}$$

*Named after a German mathematician *Peter Gustav Lejeune Dirichlet* (1805–1859) who studied under Cauchy and succeeded Gauss at Gottingen. He is known for his contributions to Fourier series and number theory.

$$= h^{l+m}\frac{\Gamma(l)\Gamma(m)}{\Gamma(l+m+1)} \qquad \ldots(i)\ [\because\ \Gamma(m+1)/m = \Gamma(m)]$$

(*b*) Taking $y + z \le 1 - x (= h\text{: say})$, the triple integral

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x^{l-1} y^{m-1} z^{n-1}\, dzdydx$$

$$= \int_0^1 x^{l-1}\left[\int_0^h \int_0^{h-y} y^{m-1} z^{n-1}\, dzdy\right] dx = \int_0^1 x^{l-1} \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n+1)} h^{m+n}\, dx \qquad \ldots \text{[By } (i)]$$

$$= \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n+1)} \int_0^1 x^{l-1}(1-x)^{m+n}\, dx = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n+1)}\, \beta(l, m+n+1)$$

$$= \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n+1)} \cdot \frac{\Gamma(l)\Gamma(m+n+1)}{\Gamma(l+m+n+1)} = \frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n+1)}.$$

Example 7.48. *Evaluate the integral* $\iiint x^{l-1} y^{m-1} z^{n-1}\, dxdydz$ *where x, y, z are all positive with condition,* $(x/a)^p + (y/b)^q + (z/c)^r \le 1$. *(U.P.T.U., 2005 S)*

Solution. Put $(x/a)^p = u$, *i.e.*, $x = au^{1/p}$ so that $dx = \frac{a}{p} u^{1/p-1}\, du$

$(y/b)^q = v$, *i.e.*, $y = bv^{1/q}$ so that $dy = \frac{b}{q} v^{1/q-1}\, dv$

and $(z/c)^r = w$, *i.e.*, $z = cw^{1/r}$ so that $dz = \frac{c}{r} w^{1/r-1}\, dw$

Then $\iiint x^{l-1} y^{m-1} z^{n-1}\, dxdydz$

$$= \iiint (au^{1/p})^{l-1} (bv^{1/q})^{m-1} (cw^{1/r})^{n-1} \left(\frac{a}{p}\right) u^{1/p-1} \left(\frac{b}{q}\right) v^{1/q-1} \left(\frac{c}{r}\right) w^{1/r-1}\, dudvdw$$

$$= \frac{a^l b^m c^n}{pqr} \iiint u^{l/p-1} v^{m/q-1} w^{n/r-1}\, du\, dv\, dw \text{ where } u + v + w \le 1.$$

$$= \frac{a^l b^m c^n}{pqr} \frac{\Gamma(l/p)\Gamma(m/q)\Gamma(n/r)}{\Gamma(l/p + m/q + n/r + 1)} \qquad \text{[By Dirichlet's integral]}$$

Example 7.49. *The plane* $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ *meets the axes in A, B and C. Apply Dirichlet's integral to find the volume of the tetrahedron OABC. Also find its mass if the density at any point is kxyz.* *(U.P.T.U., 2004)*

Solution. Put $x/a = u$, $y/b = v$, $z/c = w$ then the tetrahedron *OABC* has $u \ge 0, v \ge 0, w \ge 0$ and $u + v + w \le 1$.

$\therefore$ volume of this tetrahedron $= \iiint_D dx\, dy\, dz$

$$= \iiint_{D'} abc\, du\, dv\, dw \qquad \left[\begin{array}{l} a\ dx = adu,\ dy = bdv,\ dz = cdw \\ \text{for } D' = u \ge 0, v \ge 0, w \ge 0\ \&\ u+v+w \le 1. \end{array}\right.$$

$$= abc \iiint_{D'} u^{1-1} v^{1-1} w^{1-1}\, du\, dv\, dw$$

$$= abc \frac{\Gamma(1)\Gamma(1)\Gamma(1)}{\Gamma(1+1+1+1)} = \frac{abc}{6} \qquad \text{[By Dirichlet's integral]}$$

$$\text{Mass} = \iiint kxyz dx\, dy\, dz = \iiint k(au)(bv)(cw) abc\, du\, dv\, dw$$

$$= ka^2b^2c^2 \iiint u^{2-1} v^{2-1} w^{2-1}\, du\, dv\, dw$$

$$= ka^2b^2c^2 \frac{\Gamma(2)\Gamma(2)\Gamma(2)}{\Gamma(2+2+2+1)} ka^2b^2c^2 \cdot \frac{1}{6!} = \frac{k}{720} a^2b^2c^2.$$

PROBLEMS 7.8

1. Compute :
   (*i*) $\Gamma(3.5)$ (*Assam, 1998*) (*ii*) $\Gamma(4.5)$
   (*iii*) $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$ (*S.V.T.U., 2009*) (*iv*) $\beta(2.5, 1.5)$ (*v*) $\beta\left(\frac{9}{2}, \frac{7}{2}\right)$. (*Andhra, 2000*)
2. Express the following integrals in terms of gamma functions :
   (*i*) $\int_0^\infty e^{-x^2}\, dx$ (*ii*) $\int_0^\infty x^{p-1} e^{-kx}\, dx (k > 0)$ (*Delhi, 2002 ; V.T.U., 2000*)
   (*iii*) $\int_0^\infty \sqrt{x}\, e^{-x^3}\, dx$ (*J.N.T.U., 2003*) (*iv*) $\int_0^\infty \frac{dx}{x^{p+1}.(x-1)^q} (-p < q < 1)$
3. Show that :
   (*i*) $\int_0^\infty \frac{x^4}{4^x}\, dx = \frac{\Gamma(5)}{(\log 4)^5}$ (*Marathwada, 2008*)
   (*ii*) $\int_0^{\pi/2} \sqrt{(\cot\theta)}\, d\theta = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$ (*Osmania, 2003 S ; V.T.U., 2001*)
   (*iii*) $\int_0^{\pi/2} [\sqrt{(\tan\theta)} + \sqrt{(\sec\theta)}]\, d\theta = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\left\{\Gamma\left(\frac{3}{4}\right) + \sqrt{\pi}/\Gamma\left(\frac{3}{4}\right)\right\}$ (*J.N.T.U., 2000*)
   (*iv*) $\int_0^{\pi/2} \sqrt{(\sin\theta)}\, d\theta \times \int_0^{\pi/2} \frac{d\theta}{\sqrt{(\sin\theta)}} = \pi.$ (*V.T.U., 2007*)
4. Given $\int_0^\infty \frac{x^{n-1}}{1+x}\, dx = \frac{\pi}{\sin n\pi}$, show that $\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}$. (*S.V.T.U., 2008*)
   Hence evaluate $\int_0^\infty \frac{dy}{1+y^4}$. (*V.T.U., 2006 ; J.N.T.U., 2005*)
5. Prove that :
   (*i*) $\int_0^1 \frac{x\,dx}{\sqrt{(1-x^5)}} = \frac{1}{5}\beta\left(\frac{2}{5}, \frac{1}{2}\right)$ (*Raipur, 2006*) (*ii*) $\int_0^1 \frac{dx}{\sqrt{(1+x^4)}} = \frac{1}{4\sqrt{2}}\beta\left(\frac{1}{4}, \frac{1}{2}\right)$. (*V.T.U., 2003*)
   (*iii*) $\int_0^1 x^3 (1-\sqrt{x})^5\, dx = 2\beta(8, 6)$. (*Marathwada, 2008 ; J.N.T.U., 2006*)
6. Show that (*i*) $\int_a^b (x-a)^{m-1}(b-x)^{n-1}\, dx = (b-a)^{m+n-1}\beta(m, n)$ (*P.T.U., 2010 ; Mumbai, 2005*)
   (*ii*) $\int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}}\, dx = \frac{1}{a^n . b^m}\beta(m, n)$ (*Nagpur, 2009*) (*iii*) $\int_0^\infty \frac{x^{10} - x^{18}}{(1+x)^{30}}\, dx = 0$ (*Mumbai, 2005*)
   (*iv*) $\int_0^1 \frac{(1-x^4)^{3/4}}{(1+x^4)^2}\, dx = \frac{1}{2^{9/2}}\beta\left(\frac{7}{4}, \frac{1}{4}\right)$ (*Mumbai, 2007*)
7. Prove that $\int_0^1 x^m (\log x)^n\, dx = \frac{(-1)^n\, n!}{(m+1)^{n+1}}$, where n is a positive integer and $m > -1$. (*S.V.T.U., 2006*)
   Hence evaluate $\int_0^1 x(\log x)^3\, dx$. (*Nagpur, 2009*)
8. Show that $\int_0^1 y^{q-1}\left(\log\frac{1}{y}\right)^{p-1} dy = \frac{\Gamma(p)}{q^p}$, where $p > 0, q > 0$. (*Rohtak, 2006 S*)
9. Express $\int_0^1 x^m (1-x^n)^p\, dx$ in terms of gamma functions (*Marathwada, 2008*)
   Hence evaluate : (*i*) $\int_0^1 x(1-x^3)^{10}\, dx$. (*Bhopal, 2008*) (*ii*) $\int_0^1 \frac{dx}{\sqrt{(1-x^n)}}$ (*Anna, 2005*)

10. Prove that $\int_0^\infty \frac{dx}{(e^x + e^{-x})^n} = \frac{1}{4}\beta\left(\frac{n}{2}, \frac{n}{2}\right)$ and hence evaluate $\int_0^\infty \text{sech}^8 x\, dx$.

11. Prove that $\beta\left(n + \frac{1}{2}, n + \frac{1}{2}\right) = \frac{\Gamma(n + 1/2)\sqrt{\pi}}{2^{2n}\,\Gamma(n+1)}$. Hence show that $2^n\,\Gamma(n + 1/2) = 1 \cdot 3 \cdot 5 \ldots (2n-1)\sqrt{\pi}$

(Mumbai, 2007)

12. Prove that :

(i) $\frac{\beta(m+1, n)}{m} = \frac{\beta(m, n+1)}{n} = \frac{\beta(m, n)}{m+n}$

(ii) $\beta(n, n) = \frac{\sqrt{\pi}\,\Gamma(n)}{2^{2n-1}\,\Gamma\left(n + \frac{1}{2}\right)}$

(iii) $\Gamma\left(n + \frac{1}{2}\right) = \frac{\Gamma(2n+1)\sqrt{\pi}}{2^{2n}\,.\,\Gamma(n+1)}$

(iv) $\beta(m+1) + \beta(m, n+1) = \beta(m, n)$

(Bhopal, 2008 ; J.N.T.U., 2006 ; Madras, 2003)

13. Show that $\iint x^{m-1}\, y^{n-1}\, dx\, dy$ over the positive quadrant of the ellipse

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \text{ is } \frac{a^m b^n}{2n}\,\beta\left(\frac{m}{2}, \frac{n}{2} + 1\right).$$

14. Show that the area in the first quadrant enclosed by the curve $(x/a)^\alpha + (y/b)^\beta = 1$, $\alpha > 0$, $\beta > 0$, is given by

$$\frac{ab}{\alpha + \beta}\,\frac{\Gamma(1/\alpha)\,\Gamma(1/\beta)}{\Gamma(1/\alpha + 1/\beta)}.$$

15. Find the mass of an octant of the ellipsoid $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$, the density at any point being $\rho = kxyz$.

(U.P.T.U., 2002)

7.17 (1) ELLIPTIC INTEGRALS

In Applied Mathematics, we often come across integrals of the form $\int_0^1 e^{-x^2}\, dx$ or $\int_0^1 \sin x^2\, dx$ which cannot be evaluated by any of the standard methods of integration. In such cases, we may find the value to any desired degree of accuracy by expanding their integrands as power series. An important class of such integrals is the *elliptic integrals.*

Def. The integral $F(k, \phi) = \int_0^\phi \frac{dx}{\sqrt{(1 - k^2 \sin^2 x)}}\ (k^2 < 1)$ ...(i)

which is a function of the two variables k and ϕ, is called the *elliptic integral of the first kind with modulus k* and *amplitude* ϕ.

The integral $E(k, \phi) = \int_0^\phi \sqrt{(1 - k^2 \sin^2 x)}\, dx\ (k^2 < 1)$ ...(ii)

is called the *elliptic integral of the second kind* with *modulus k and amplitude* ϕ.

The name *elliptic integral* arose from its original application in finding the length of an elliptic arc (Fig.7.45), For instance, consider the ellipse

$$x = a\cos\phi, \qquad y = b\sin\phi, \qquad (a < b)$$

Then length of its arc

$$AP = \int_0^\phi \sqrt{\left[\left(\frac{dx}{d\phi}\right)^2 + \left(\frac{dy}{d\phi}\right)^2\right]}\, d\phi = \int_0^\phi \sqrt{[(-a\sin\phi)^2 + (b\cos\phi)^2]}\, d\phi$$

$$= \int_0^\phi \sqrt{[(b^2 + (a^2 - b^2)\sin^2\phi]}\, d\phi = b\int_0^\phi \sqrt{\left[1 - \left(1 - \frac{a^2}{b^2}\right)\sin^2\phi\right]}\, d\phi$$

$= bE(k, \phi)$ for $k^2 = 1 - a^2/b^2 \le 1$.

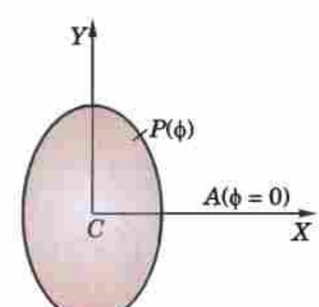

Fig. 7.45

Also the perimeter of the ellipse

$$= 4b \int_0^{\pi/2} \sqrt{(1 - k^2 \sin^2 \phi)}\ d\phi = 4bE(k, \pi/2).$$

This particular integral with upper limit $\phi = \pi/2$ is called the *complete elliptic integral of the second kind and is denoted by* $E(k)$.

Thus $$E(k) = \int_0^{\pi/2} (1 - k^2 \sin^2 \phi)\, d\phi\ (k^2 < 1) \qquad ...(iii)$$

Similarly, the *complete elliptic integral of first kind is*

$$K(k) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{(1 - k^2 \sin^2 \phi)}} (k^2 < 1) \qquad ...(iv)$$

To evaluate it, we expand the integral in the form

$$(1 - k^2 \sin^2 \phi)^{-1/2} = 1 + \frac{k^2}{2} \sin^2 \phi + \frac{3k^4}{4} \sin^4 \phi + ...$$

This series can be shown to be uniformly convergent for all k, and may, therefore, be integrated term by term [See § 9.19-II]. Then we have

$$K(k) = \int_0^{\pi/2} \left(1 + \frac{k^2}{2} \sin^2 \phi + \frac{3k^4}{8} \sin^4 \phi + \frac{5k^6}{16} \sin^6 \phi + ...\right) d\phi$$

$$= \frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1.3}{2.4}\right)^2 k^4 + \left(\frac{1.3.5}{2.4.6.}\right)^2 k^6 + ...\right] \qquad ...(v)$$

This series may be used to compute K for various values of k. In particular, if $k = \sin 10°$; we have

$$K = \frac{\pi}{2}(1 + 0.00754 + 0.00012 + ...) = 1.5828 \qquad ...(vi)$$

In this way tables of the elliptic integrals are constructed. Values of $F(k, \phi)$ and $E(k, \phi)$ are readily available for $0 \le \phi \le \pi/2$, $0 < k < 1$. *(See Peirce's short tables).*

Example 7.50. *Express* $\int_0^{\pi/6} \frac{dx}{\sqrt{(\sin x)}}$ *in terms of elliptic integral.*

Solution. Put $\cos x = \cos^2 \phi$ and $dx = \dfrac{2 \cos \phi\, d\phi}{\sqrt{(1 + \cos^2 \phi)}}$

Then $$I = \int_0^{\pi/2} \frac{2 \cos^2 \phi}{\sqrt{(1 + \cos^2 \phi)}} d\phi = 2 \int_0^{\pi/2} \frac{(1 + \cos^2 \phi) - 1}{\sqrt{(1 + \cos^2 \phi)}} d\phi$$

$$= 2\left\{\int_0^{\pi/2} \sqrt{(1 + \cos^2 \phi)}\, d\phi - \int_0^{\pi/2} \frac{d\phi}{\sqrt{(1 + \cos^2 \phi)}}\right\} = 2\left\{\int_0^{\pi/2} \sqrt{(2 - \sin^2 \phi)}\, d\phi - \int_0^{\pi/2} \frac{d\phi}{\sqrt{(2 - \sin^2 \phi)}}\right\}$$

$$= 2\sqrt{2} \int_0^{\pi/2} \sqrt{(1 - 1/2 \sin^2 \phi)}\, d\phi - \sqrt{2} \int_0^{\pi/2} \frac{d\phi}{\sqrt{(1 - 1/2 \sin^2 \phi)}} = 2\sqrt{2}\, E\left(\frac{1}{\sqrt{2}}\right) - \sqrt{2}\, K\left(\frac{1}{\sqrt{2}}\right)$$

(2) **Elliptic functions.** By putting $\sin x = t$ and $\sin \phi = z$, (i) becomes

$$u = \int_0^{z} \frac{dt}{\sqrt{[(1 - t^2)(1 - k^2 t^2)]}} (k^2 < 1) \qquad ...(vii)$$

This is known as *Jacobi's form of the elliptic integral of first kind** whereas (i) is the *Legendre's form*†.

If $k = 0$, (vii) gives $u = \sin^{-1} z$. By analogy, we denote (vii) $sn^{-1} z$ for a fixed non-zero value of k. This leads to the functions $sn\ u = z = \sin \phi$ and $cn\ u = \cos \phi$
which are called the *Jacobi's elliptic functions.*

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* See footnote p. 215.

† A French mathematician *Adrien Marie Legendre* (1752–1833) who made important contributions to number theory, special functions, calculus of variations and elliptic integrals.

The elliptic functions $sn\, u$ and $cn\, u$ are periodic with a period depending on k and an amplitude equal to unity. These behave somewhat like $\sin u$ and $\cos u$. For instance

$$sn(0) = 0, cn(1) = 1 \quad \text{and} \quad sn(-u) = -sn(u), cn(-u) = cn(u).$$

Example 7.51. *Show that* $\int_0^{a/2} \frac{dx}{\sqrt{(2ax - x^2)}\sqrt{(a^2 - x^2)}} = \frac{2}{3a} K\left(\frac{1}{3}\right)$.

Solution. Putting $x = \frac{a}{2}(1 - \sin\theta), dx = -\frac{a}{2}\cos\theta\, d\theta$,

$$2ax - x^2 = \frac{a^2}{4}(1 - \sin\theta)(3 + \sin\theta) \text{ and } a^2 - x^2 = \frac{a^2}{4}(1 + \sin\theta)(3 - \sin\theta)$$

Also when $x = 0, \theta = \pi/2$; when $x = a/2, \theta = 0$.

Thus the given integral

$$= \frac{4}{a^2}\int_{\pi/2}^{0} \frac{-(a/2)\cos\theta\, d\theta}{\sqrt{[(1-\sin^2\theta)(2-\sin^2\theta)]}} = \frac{2}{3a}\int_0^{\pi/2}\frac{d\theta}{\sqrt{[(1-(1/3)^2\sin^2\theta]}} = \frac{2}{3a}K\left(\frac{1}{3}\right).$$

7.18 (1) ERROR FUNCTION OR PROBABILITY INTEGRAL

The error function or the probability integral is defined as

$$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\, dt.$$

This integral arises in the solution of certain partial differential equations of applied mathematics and occupies an important position in the probability theory.

The complementary error function $erfc(x)$ is defined as $erfc(x) = 1 - erf(x)$.

(2) Properties : (*i*) $erf(-x) = -erf(x)$; (*ii*) $erf(0) = 0$

$$(iii)\ erf(\infty) = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-t^2} dt = \frac{2}{\sqrt{\pi}}\frac{\sqrt{\pi}}{2} = 1$$ [By (*iii*), p. 289]

This proves that the total area under the *Normal* or *Gaussian error function curve* is unity – § 26.16.

PROBLEMS 7.9

1. By means of the substitution $k \sin x = \sin z$, show that

   (*i*) $\int_0^\pi \frac{dx}{\sqrt{(1-k^2\sin^2 x)}} = \frac{1}{k}F\left(\frac{1}{k}, \phi'\right)$;

   (*ii*) $\int_0^\phi \sqrt{(1-k^2\sin^2 x)}\, dx = \left(\frac{1}{k} - k\right)F\left(\frac{1}{k}, \phi'\right) + kE\left(\frac{1}{k}, \phi'\right)$

   where $k > 1$ and $\phi' = \sin^{-1}(k \sin\phi)$.

   Express the following integrals in terms of elliptic integrals :

2. $\int_0^{\pi/2} \frac{dx}{\sqrt{(1+3\sin^2 x)}}$. (*Kerala, M.E., 2005*)
3. $\int_0^{\pi/2} \frac{dx}{\sqrt{(2-\cos x)}}$.
4. $\int_0^{\pi/2} \sqrt{(\cos x)}\, dx$.
5. Expand $erf(x)$ in ascending powers of x. Hence evaluate $erf(0)$. (*P.T.U., 2009 S*)
6. Compute (*i*) $erf(0.3)$, (*ii*) $erf(0.5)$, correct to three decimal places.
7. Show that (*i*) $erf(x) + erf(-x) = 0$ (*ii*) $erfc(x) + erfc(-x) = 2$
8. Prove that

   (*i*) $\frac{d}{dx}[erf(ax)] = \frac{2a}{\sqrt{\pi}}e^{-a^2x^2}$ (*Osmania, 2003*) (*ii*) $\frac{d}{dx}[erfc(ax)] = -\frac{2a}{\sqrt{\pi}}e^{-a^2x^2}$.
9. Prove that $\int_0^\infty e^{-x^2-2ax}\, dx = \frac{\sqrt{\pi}}{2}e^{a^2}[1 - erf(0)]$

7.19 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 7.10

Fill up the blanks or choose the correct answer from the following problems :

1. $\int_0^2 \int_0^x (x+y)\,dx\,dy =$ ....
2. $\int_0^1 \int_0^{1-x} dx\,dy$ ....
3. $\int_0^1 e^{-x^2}\,dx =$ ....
4. $\beta\left(\frac{1}{2}, \frac{1}{2}\right) =$ .... (*V.T.U., 2010*)
5. $\Gamma(3.5) =$ ....
6. The surface area of the sphere $x^2 + y^2 + z^2 + 2x - 4y + 8z - 2 = 0$ is ....
7. $\int_0^2 \int_1^3 \int_1^2 xy^2 z\,dzdydx =$ ....
8. If $u = x + y$ and $v = x - 2y$, then the area-element $dxdy$ is replaced by .... $dudv$.
9. In terms of Beta function $\int_0^{\pi/2} \sin^7\theta\sqrt{\cos\theta}\,d\theta =$ ....
10. The value of $\beta(2, 1) + \beta(1, 2)$ is ....
11. $\int_0^1 \int_1^2 xy\,dy\,dx =$ ....
12. Volume bounded by $x \ge 0, y \ge 0, z \ge 0$ and $x^2 + y^2 + z^2 = 1$ as a triple integral integral.
13. Value of $\int_0^1 \int_0^{x^2} xe^y\,dy\,dx$ is equal to

    (*a*) $e/2$ (*b*) $e - 1$ (*c*) $1 - e$ (*d*) $e/2 - 1$. (*Bhopal, 2008*)
14. $\iint x^2y^3\,dxdy$ over the rectangle $0 \le x \le 1$ and $0 \le y \le 3$ is ....
15. $\int_0^{\pi} \int_0^{a\sin\theta} r\,dr\,d\theta =$ ....
16. $\int_{x=0}^{x=3} \int_{y=0}^{y=1/x} ye^{xy}\,dxdy =$ ....
17. $\int_0^{\pi/2} \int_0^{\infty} \frac{r\,dr\,d\theta}{\left(r^2 + a^2\right)} =$ ....
18. $\int_0^1 \int_x^{\sqrt{x}} (x^2 + y^2)\,dx\,dy =$ ....
19. To change cartesian coordinates (x, y, z) to spherical polar coordinate (r, θ, ϕ); $dx\,dy\,dz$ is replaced by ....
20. $\int_0^2 \int_0^{x^2} e^{y/x}\,dydx =$ ....
21. $\iint (x + y)^2\,dxdy$ over the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, is ....
22. $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dx\,dy}{1 + x^2 + y^2} =$ ....
23. $\iint xy(x + y)\,dx\,dy$ over the area between $y + x^2$ and $y = x$, is ....
24. Value of $\int_0^1 \int_x^{\sqrt{x}} xy\,dx\,dy$ is

    (*a*) zero (*b*) $-1/24$ (*c*) $1/24$ (*d*) 24. (*V.T.U., 2010*)
25. $\iint dx\,dy$ over the area bounded by $x = 0, y = 0, x^2 + y^2 = 1$ and $5y = 3$, is ....
26. $\iint_R y\,dx\,dy$ where R is the region bounded by the parabolas $y^2 = 4x$ and $x^2 = 4y$, is ....
27. $\iint (x^2 + y^2)\,dx\,dy$ in the positive quadrant for which $x + y \le 1$, is ....
28. Area between the parabolas $y^2 = 4x$ and $x^2 = 4y$ is ....
29. Changing the order of integration in $\int_{-a}^{a} \int_0^{\sqrt{a^2-y^2}} f(x, y)\,dx\,dy =$ ....
30. $\Gamma(1/4)\Gamma(3/4) =$ .... (*V.T.U., 2011*)
31. $\beta(5/2, 7/2) =$ ....
32. $\int_0^{\infty} \int_0^{x} xe^{-x^2/y}\,dydx =$ ....
33. On changing to polar coordinates $\int_0^{2a} \int_0^{\sqrt{(2ax-x^2)}} dx\,dy$ becomes ....

34. A square lamina is immersed in the liquid with one vertex in the surface and the diagonal of length vertical. Its centre of pressure is at a depth ....
35. The centroid of the area enclosed by the parabola $y^2 = 4x$, x-axis and its latus-rectum is ....
36. The moment of inertia of a uniform spherical ball of mass 10 gm and radius 2 cm about a diameter is ....
37. M.I. of a solid right circular cone (base-radius r and height h) about its axis is ....

38. $erf_c(-x) - erf(x) =$ ....
39. $\int_0^1 \frac{x-1}{\log x}\, dx =$ ....
40. $\Gamma\left(\frac{3}{2}\right) =$ ....

41. Value of $\int_0^a \int_0^b \int_0^c x^2y^2z^2\, dx\, dy\, dz$ is

(a) $\frac{abc}{3}$ (b) $\frac{a^2b^2c^2}{27}$ (c) $\frac{a^3b^3c^3}{27}$ (d) $\frac{a^2b^2c^2}{9}$.

42. The integral $\int_0^1 \int_0^{\sqrt{(1-x^2)}} (x+y)\, dy\, dx$ after changing the order of integration.

(a) $\int_0^2 \int_0^{\sqrt{(1-y^2)}} (x+y)\, dx\, dy$ (b) $\int_0^1 \int_0^{\sqrt{(1-y^2)}} (x+y)\, dx\, dy$

(c) $\int_0^1 \int_0^{\sqrt{(1+y^2)}} (x+y)\, dx\, dy$ (d) $\int_0^{-1} \int_0^{\sqrt{(1-y^2)}} (x+y)\, dx\, dy$. (V.T.U., 2011)