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Differential Equations

1. Definitions. 2. Variables separable. 3. Homogeneous equations. 4. Equations reducible to homogeneous form. 5. Leibnitz's linear equation. 6. Bernoulli's equation. 7. Exact differential equations. 8. Equations reducible to exact equations. 9. Linear equations with constant coefficients. 10. Rules for finding the complementary function. 11. Rules for finding the particular integral. 12. Working procedure. 13. Method of variation of parameters. 14. Equations reducible to linear equations with constant coefficients. 15. Simultaneous linear equations with constant coefficients.

20·1. (1) *A differential equation is an equation which involves independent and dependent variables and the derivatives of the dependent variables. Thus*

(*i*) $\frac{dy}{dx} + y \sin x = \cos x,$ (*ii*) $\frac{d^2x}{dt^2} + n^2x = 0$

(*iii*) $y = x\frac{dy}{dx} + 1/dy/dx$ (*iv*) $\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{3/2} = \frac{d^2y}{dx^2}$

are examples of differential equations.

(2) *The* **order** *of a differential equation is the order of highest derivative appearing in it.*

The **degree** *of a differential equation is the degree of the highest derivative occuring in it, after the equation has been expressed in a form free from radicals and fractions so far as the derivatives are concerned.*

Thus from the above differential equations

(*i*) is of the first order and first degree ;

(*ii*) is of the second order and first degree ;

(*iii*) written as $y\frac{dy}{dx} = x\left(\frac{dy}{dx}\right)^2 + 1$, is of the first order and second degree ;

and (*iv*) written as $\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^3 = \left(\frac{d^2y}{dx^2}\right)^2$, is of the second order and second degree.

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(3) *A* **solution** *of a differential equation is a relation between the variables which satisfies the given differential equation.*

For instance, if $y = A\cos x + B\sin x$ ...(1)

then $$\frac{dy}{dx} = -A\sin x + B\cos x$$

and $$\frac{d^2y}{dx^2} = -A\cos x - B\sin x = -y$$

i.e. $$\frac{d^2y}{dx^2} + y = 0 \qquad ...(2)$$

Thus (1) is a solution of (2).

The **general** (*or* **complete**) **solution** *of a differential equation is that in which the number of arbitrary constants is equal to the order of the differential equation.*

Thus (1) a general solution of (2) as the number of arbitrary constants (A, B) in (1) is the same as the order of (2).

A **particular solution** *is that which can be obtained from the general solution by giving particular values to the arbitrary constants.*

For example $y = 2\cos x + 3\sin x$

is a particular solution of (2) as it can be derived from the general solution (1) by putting $A = 2, B = 3$.

Example 20·1. *Form the differential equation of all circles of radius a.*

Such a circle is $(x-h)^2 + (y-k)^2 = a^2$ ...(*i*)

where h and k, the co-ordinates of the centre, are the arbitrary constants.

Differentiating it twice, we have

$$x - h + (y-k)\frac{dy}{dx} = 0$$

and $$1 + (y-k)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$$

Then $$y - k = -\frac{1 + (dy/dx)^2}{d^2y/dx^2}$$

and $$x - h = -(y-k)\,dy/dx = \frac{\dfrac{dy}{dx}\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]}{d^2y/dx^2}$$

To eliminate h and k, substituting these in (*i*) and simplifying, we get

$$[1 + (dy/dx)^2]^3 = a^2\,(d^2y/dx^2)^2 \qquad ...(ii)$$

as the required differential equation.

Obs. We have seen that a differential equation is formed by eliminating the arbitrary constants from a relation in the variables and constants.

Problems

1. Show that $x = a \sin(\omega t + b)$ is a solution of the differential equation $d^2x/dt^2 + \omega^2 x = 0$.

Form the differential equations from the following equations (2 and 3) :

2. $y = ax^3 + bx^2$.

3. $xy = Ae^x + Be^{-x}$ *(Nagarjuna, 1998 S)*

4. $y = e^x (A \cos x + B \sin x)$. *(Andhra, 1998)*

Find the differential equations of :

5. A family of circles passing through the origin and having centres on the x-axis. *(Karnataka, 1993)*

6. All circles of radius 5, with their centres on the y-axis.

Equations of the first order and first degree

20·2. Variables separable. If in an equation it is possible to collect all functions of x and dx on one side and all the functions of y and dy on the other side, then the *variables are said to be separable.* Thus the general form of such an equation is

$$f(y)\, dy = \phi(x)\, dx$$

Integrating both sides, we get

$$\int f(y)\, dy = \int \phi(x)\, dx + c$$

as its solution.

Example 20·2. *Solve* $(xy^2 + x)\, dx + (yx^2 + y)\, dy = 0.$

We have $\quad x(y^2 + 1)\, dx + y(x^2 + 1)\, dy = 0$

or $$\frac{xdx}{x^2 + 1} + \frac{ydy}{y^2 + 1} = 0$$

Integrating $$\int \frac{xdx}{x^2 + 1} - \int \frac{ydy}{y^2 + 1} = c$$

or $$\tfrac{1}{2} \log(x^2 + 1) + \tfrac{1}{2} \log(y^2 + 1) = c$$

$$\log(x^2 + 1)(y^2 + 1) = 2c$$

or $$(x^2 + 1)(y^2 + 1) = e^{2c} = c' \quad \text{(say)}$$

which is the required solution.

Example 20·3. *Solve* $\dfrac{dy}{dx} = e^{3x - 2y} + x^2 e^{-2y}$.

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We have $$\frac{dy}{dx} = e^{-2y}(e^{3x} + x^2)$$

or $$e^{2y}\,dy = (e^{3x} + x^2)\,dx$$

Integrating both sides,

$$\int e^{2y}\,dy = \int (e^{3x} + x^2)\,dx + c$$

or $$\frac{e^{2y}}{2} = \frac{e^{3x}}{3} + \frac{x^3}{3} + c$$

or $$3e^{2y} = 2(e^{3x} + x^3) + c' \qquad [c' = 6c]$$

which is the desired solution.

Problems

Solve the following differential equations :

1. $y\sqrt{(1-x^2)}\,dy + x\sqrt{(1-y^2)}\,dx = 0.$
2. $(x^2 - yx^2)\dfrac{dy}{dx} + y^2 + xy^2 = 0.$
3. $xy\dfrac{dy}{dx} = 1 + x + y + xy.$ (*Andhra, 1998*)
4. $(x+1)\dfrac{dy}{dx} + 1 = e^{-y}$ (*Madras, 1992*)
5. $3e^x \tan y\,dx + (1 - e^x)\sec^2 y\,dy = 0.$ (*Mangalore, 1999*)
6. $\dfrac{dy}{dx} = xe^{y-x^2}$, if $y = 0$ when $x = 0$.
7. $\dfrac{dy}{dx} = e^{2x-3y} + 4x^2 e^{-3y}.$
8. $y = 3x + \log p$ where $p = \dfrac{dy}{dx}.$ (*Pondicherry, 1998*)
9. $\dfrac{dy}{dx} = \dfrac{x(2\log x + 1)}{\sin y + y\cos y}.$
10. $y - x\dfrac{dy}{dx} = a\left(y^2 + \dfrac{dy}{dx}\right).$ (*Bhopal, 1991*)
11. $(x-y)^2\dfrac{dy}{dx} = a^2.$ (*A.M.I.E., 1997*)
12. $\cos(x+y)\,dy = dx.$ (*Patna, 1997 S*)

20·3. Homogeneous equations are of the form

$$\frac{dy}{dx} = \frac{f(x, y)}{\phi(x, y)}$$

where $f(x, y)$ and $\phi(x, y)$ are homogeneous functions of the same degree in x and y (see page 380).

To solve a homogeneous equation

(*i*) Put $y = vx$, then $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$,

(*ii*) Separate the variables v and x, and integrate.

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Example 20·4. *Solve* $(x^2 - y^2)\,dx = 2xydy$. (*Mangalore, 1997*)

Given equation is $$\frac{dy}{dx} = \frac{x^2 - y^2}{2xy} \quad ...(i)$$

which is homogeneous in x and y.

Put $y = vx$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

$\therefore$ (i) becomes $v + x\frac{dv}{dx} = \frac{1 - v^2}{2v}$

or $$x\frac{dv}{dx} = \frac{1 - v^2}{2v} - v = \frac{1 - 3v^2}{2v}.$$

Separating the variables,

$$\frac{2v}{1 - 3v^2}\,dv = \frac{dx}{x}$$

Integrating both sides,

$$\int \frac{2v\,dv}{1 - 3v^2} = \int \frac{dx}{x} + c \qquad \text{or} \quad -\frac{1}{3}\int \frac{-6v}{1 - 3v^2}\,dv = \int \frac{dx}{x} + c$$

or $-\frac{1}{3}\log(1 - 3v^2) = \log x + c$ or $3\log x + \log(1 - 3v^2) = -3c$

or $\log x^3(1 - 3v^2) = -3c$ [Put $v = y/x$]

or $x^3(1 - 3y^2/x^2) = e^{-3c} = c'$

Hence the required solution is $x(x^2 - 3y^2) = c'$.

Problems

Solve the following differential equations :

1. $2xy\frac{dy}{dx} = 3y^2 + x^2$. (*Andhra, 1998*)
2. $(y^2 - 2xy)\,dx = (x^2 - 2xy)\,dy$. (*Hamirpur, 1996 S*)
3. $(x^2 + 2y^2)\,dx - xydy = 0$, given $y = 0$ when $x = 1$.
4. $x\,dy - y\,dx = \sqrt{x^2 + y^2}\,dx$. (*Kanpur, 1998*)
5. $y^2 + x^2\frac{dy}{dx} = xy\frac{dy}{dx}$. (*Triputi, 1998 S*)
6. $x^3\,dx - y^3\,dy = 3xy\,(ydx - xdy)$.

[*Equations solvable like homogeneous equations* : When a differential equation contains y/x a number of times, solve it like a homogeneous equation by putting $y/x = v$].

7. $\frac{dy}{dx} = \frac{y}{x} + \sin\frac{y}{x}$. (*Kuvempu, 1998 S*)
8. $(1 + e^{x/y})\,dx + e^{x/y}(1 - x/y)\,dy = 0$. (*Triputi, 1998*)

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20·4. Equations reducible to homogeneous form

The equations of the form $\frac{dy}{dx} = \frac{ax + by + c}{a'x + b'y + c'}$ ...(1)

can be reduced to the homogeneous form as follows :

Case I. When $\frac{\mathbf{a}}{\mathbf{a'}} \neq \frac{\mathbf{b}}{\mathbf{b'}}$

Putting $x = X + h, y = Y + k$, (h, k being constants)

so that $dx = dX, dy = dY$, (1) becomes

$$\frac{dY}{dX} = \frac{aX + bY + (ah + bk + c)}{a'X + b'Y + (a'h + b'k + c')} \qquad ...(2)$$

Choose h, k so that (2) may become homogeneous.

Put $ah + bk + c = 0$, and $a'h + b'k + c' = 0$

so that $$\frac{h}{bc' - b'c} = \frac{k}{ca' - c'a} = \frac{1}{ab' - ba'}$$

or $$h = \frac{bc' - b'c}{ab' - b'a}, \quad k = \frac{ca' - c'a}{ab' - ba'} \qquad ...(3)$$

Thus when $ab' - ba' \neq 0$, (2) becomes

$$\frac{dY}{dX} = \frac{aX + bY}{a'X + b'Y}$$

which is homogeneous in X, Y and can be solved by putting $Y = vX$.

Case II. When $\frac{\mathbf{a}}{\mathbf{a'}} = \frac{\mathbf{b}}{\mathbf{b'}}$,

i.e. $ab' - ba' = 0$, the above method fails as h and k become infinite or indeterminate.

Now $$\frac{a}{a'} = \frac{b}{b'} = \frac{1}{m} \text{ (say)}$$

$\therefore$ $a' = am, b' = bm$ and (1) becomes

$$\frac{dy}{dx} = \frac{(ax + by) + c}{m(ax + by) + c'} \qquad ...(4)$$

Put $ax + by = t$, so that $a + b\frac{dy}{dx} = \frac{dt}{dx}$ or $\frac{dy}{dx} = \frac{1}{b}\left(\frac{dt}{dx} - a\right)$

$\therefore$ (4) becomes $\frac{1}{b}\left(\frac{dt}{dx} - a\right) = \frac{t + c}{mt + c'}$

or $$\frac{dt}{dx} = a + \frac{bt + bc}{mt + c'} = \frac{(am + b)\,t + ac' + bc}{mt + c'}$$

so that the variables are separable. In this solution, putting $t = ax + by$, we get the required solution of (1).

Example 20·5. *Solve* $\frac{dy}{dx} = \frac{y + x - 2}{y - x - 4}$. *(Mangalore, 1999)*

Given equation is $\frac{dy}{dx} = \frac{y + x - 2}{y - x - 4}$ $\left[\text{Case } \frac{a}{a'} \neq \frac{b}{b'}\right]$ ...(*i*)

Putting $x = X + h, y = Y + k$, (h, k being constants)

so that $dx = dX, dy = dY$, (*i*) becomes

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$$\frac{dY}{dX}=\frac{Y+X+(k+h-2)}{Y-X+(k-h-4)} \qquad ...(ii)$$

Put $k+h-2=0$ and $k-h-4=0$ so that $h=-1$, $k=3$.

$\therefore$ (ii) becomes $\dfrac{dY}{dX}=\dfrac{Y+X}{Y-X}$ ...(iii)

which is homogeneous in X and Y.

$\therefore$ Put $Y=vX$, then $\qquad \dfrac{dY}{dX}=v+X\dfrac{dv}{dX}$

$\therefore$ (iii) becomes $\qquad v+X\dfrac{dv}{dX}=\dfrac{v+1}{v-1}$

or $$X\frac{dv}{dX}=\frac{v+1}{v-1}-v=\frac{1+2v-v^2}{v-1}$$

or $$\frac{v-1}{1+2v-v^2}\,dv=\frac{dX}{X}.$$

Integrating both sides,

$$-\frac{1}{2}\int\frac{2-2v}{1+2v-v^2}\,dv=\int\frac{dX}{X}+c$$

or $$-\frac{1}{2}\log(1+2v-v^2)=\log X+c$$

or $$\log\left(1+\frac{2Y}{X}-\frac{Y^2}{X^2}\right)+\log X^2=-2c$$

or $$\log(X^2+2XY-Y^2)=-2c$$

or $$X^2+2XY-Y^2=e^{-2c}=c' \qquad ...(iv)$$

Putting $X=x-h=x+1$, $Y=y-k=y-3$, (iv) becomes

$$(x+1)^2+2(x+1)(y-3)-(y-3)^2=c'$$

or $$x^2+2xy-y^2-4x+8y-14=c'$$

which is the required solution.

Example 20·6. *Solve* $(3y+2x+4)\,dx-(4x+6y+5)\,dy=0.$

(*Karnataka, 1993*)

Given equation is $\qquad \dfrac{dy}{dx}=\dfrac{(2x+3y)+4}{2(2x+3y)+5}$ ...(i)

Putting $2x+3y=t$ so that $2+3\dfrac{dy}{dx}=\dfrac{dt}{dx}$

$\therefore$ (i) becomes $\dfrac{1}{3}\left(\dfrac{dt}{dx}-2\right)=\dfrac{t+4}{2t+5}$

or $$\frac{dt}{dx}=2+\frac{3t+12}{2t+5}=\frac{7t+22}{2t+5} \quad \text{or} \quad \frac{2t+5}{7t+22}\,dt=dx$$

Integrating both sides,

$$\int\frac{2t+5}{7t+22}\,dt=\int dx+c$$

or $$\int\left(\frac{2}{7}-\frac{9}{7}\cdot\frac{1}{7t+22}\right)dt=x+c$$

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or $$\frac{2}{7}t - \frac{9}{49}\log(7t+22) = x + c$$

Putting $t = 2x + 3y$, we have

$$14(2x+3y) - 9\log(14x+21y+22) = 49x + 49c$$

or $$21x - 42y + 9\log(14x+21y+22) = c'$$

which is the required solution.

Problems

Solve the following differential equations :

1. $\dfrac{dy}{dx} = \dfrac{2y-x-4}{y-3x+3}$. *(Kerala, 1990 S)*
2. $(2x+y-3)\,dy = (x+2y-3)\,dx$ *(Andhra, 1998)*
3. $(2x+5y+1)\,dx - (5x+2y-1)\,dy = 0$. *(Madurai, 1990 S)*
4. $\dfrac{dy}{dx} = \dfrac{x+y-1}{2x+2y+2}$. *(Madras, 1992)*
5. $(4x-6y-1)\,dx + (3y-2x-2)\,dy = 0$. *(Kanpur, 1998)*
6. $(x+2y)(dx-dy) = dx+dy$. *(Bangalore, 1990)*

20·5. Leibnitz's linear equation. *A differential equation is said to be linear if the dependent variable and its differential coefficients occur only in the first degree and are not multiplied together. The standard form of a linear equation of the first order, commonly known as* **Leibnitz's linear equation,** *is*

$$\frac{dy}{dx} + Py = Q \qquad ...(1)$$

where P, Q are any functions of x.

To solve this equation, multiply both sides by $e^{\int P dx}$ so that we get

$$\frac{dy}{dx}\cdot e^{\int P dx} + y\,(e^{\int P dx}P) = Qe^{\int P dx}$$

i.e. $$\frac{d}{dx}(ye^{\int P dx}) = Qe^{\int P dx}$$

Integrating both sides, we get

$$ye^{\int P dx} = \int Qe^{\int P dx}\,dx + c$$

as the required solution.

Obs. The factor $e^{\int P dx}$ on multiplying by which the left-hand side of (1) becomes the differential coefficient of a single function, is called the **integrating factor (I.F.)** of the linear equation (1).

It is important to remember that

$$\text{I.F.} = e^{\int P dx}$$

and the solution is $$y\,(\text{I.F.}) = \int Q\,(\text{I.F.})\,dx + c.$$

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Example 20·7. *Solve* $(x+1)\dfrac{dy}{dx} - y = e^{3x}(x+1)^2$. *(Andhra, 1998)*

Dividing throughout by $(x + 1)$, given equation becomes

$$\frac{dy}{dx} - \frac{y}{x+1} = e^{3x}(x+1) \qquad ...(i)$$

which is Leibnitz's equation.

Here $$P = -\frac{1}{x+1}$$

and $$\int P dx = -\int \frac{dx}{x+1} = -\log(x+1) = \log(x+1)^{-1}$$

$$\therefore \quad \text{I.F.} = e^{\int P dx} = e^{\log(x+1)^{-1}} = \frac{1}{x+1}$$

Thus the solution of (i) is

$$y\,(\text{I.F.}) = \int [e^{3x}(x+1)](\text{I.F.})\,dx + c$$

or $$\frac{y}{x+1} = \int e^{3x}\,dx + c = \frac{1}{3}e^{3x} + c$$

or $$y = (\tfrac{1}{3}e^{3x} + c)(x+1).$$

Example 20·8. *Solve* $(1+y^2)\,dx = (\tan^{-1} y - x)\,dy$.

(Mangalore, 1995)

This equation contains y^2 and $\tan^{-1} y$ and is, therefore, not a linear in y ; but since only x occurs, it can be written as

$$(1+y^2)\frac{dx}{dy} = \tan^{-1} y - x$$

or $$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1} y}{1+y^2}$$

which is a Leibnitz's equation in x.

$$\therefore \quad \text{I.F.} = e^{\int P dy} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$$

Thus the solution is

$$x\,(\text{I.F.}) = \int \frac{\tan^{-1} y}{1+y^2}(\text{I.F.})\,dy + c$$

or $$x e^{\tan^{-1} y} = \int \frac{e^{\tan^{-1} y}}{1+y^2} \cdot e^{\tan^{-1} y}\,dy + c \qquad \left\{ \begin{array}{l} \text{Put } \tan^{-1} y = t \\ \therefore \quad \dfrac{dy}{1+y^2} = dt \end{array} \right.$$

$$= \int t e^t\,dt + c$$

$$= t \cdot e^t - \int 1 \cdot e^t\,dt + c \qquad \text{(Integrating by parts)}$$

$$= t \cdot e^t - e^t + c$$

$$= (\tan^{-1} y - 1)\,e^{\tan^{-1} y} + c$$

or $$x = \tan^{-1} y - 1 + c e^{-\tan^{-1} y}.$$

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Problems

Solve the following differential equations :

1. $\dfrac{dy}{dx} + y \tan x = \cos x.$ (*Madras, 1992*)

2. $x \log x \dfrac{dy}{dx} + y = \log x^2.$ (*Nagarjuna, 1998 S*)

3. $\dfrac{dy}{dx} = x^3 - 2xy$, if $y = 2$ when $x = 1$.

4. $(1 + x^3) \dfrac{dy}{dx} + 6x^2 y = e^x.$

5. $\dfrac{dy}{dx} - \dfrac{2y}{x+1} = (x + 1)^3.$

6. $\cosh x \dfrac{dy}{dx} + y \sinh x = 2 \cosh^2 x \sinh x.$ (*Marathwada, 1998*)

7. $\dfrac{dy}{dx} = -\dfrac{x + y \cos x}{1 + \sin x}.$

8. $dr + (2r \cot \theta + \sin 2\theta)\, d\theta = 0.$

9. $(3e^{3x} y - 2x)\, dx + e^{3x}\, dy = 0$

10. $(x + 2y^3) \dfrac{dy}{dx} = y.$ (*Andhra, 1994*)

20·6. Bernoulli's equation. *The equation*

$$\frac{dy}{dx} + Py = Qy^n \qquad ...(1)$$

where P, Q are functions of x, is reducible to the Leibnitz's linear equation and is usually called the Bernoulli's equation.

To solve (1), divide both sides by y^n, so that

$$y^{-n} \frac{dy}{dx} + Py^{1-n} = Q \qquad ...(2)$$

Put $y^{1-n} = z$ so that $(1 - n)\, y^{-n} \dfrac{dy}{dx} = \dfrac{dz}{dx}$.

$\therefore$ (2) becomes $\dfrac{1}{1-n} \dfrac{dz}{dx} + Pz = Q$

or
$$\frac{dz}{dx} + P(1 - n)z = Q(1 - n),$$

which is Leibnitz's linear in z and can be solved easily.

Example 20·9. *Solve* $x \dfrac{dy}{dx} + y = x^3 y^6.$ (*Andhra, 1998*)

Dividing throughout by xy^6,

$$y^{-6} \frac{dy}{dx} + \frac{y^{-5}}{x} = x^2 \qquad ...(i)$$

Put $y^{-5} = z$, so that $-5y^{-6} \dfrac{dy}{dx} = \dfrac{dz}{dx}$

$\therefore$ (*i*) becomes $-\dfrac{1}{5} \dfrac{dz}{dx} + \dfrac{z}{x} = x^2$

or
$$\frac{dz}{dx} - \frac{5}{x} z = -5x^2 \qquad ...(ii)$$

which is Leibnitz's linear in z.

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$$\text{I.F.} = e^{-\int (5/x)dx} = e^{-5 \log x} = e^{\log x^{-5}} = x^{-5}$$

$\therefore$ the solution of (*ii*) is

$$z\ (\text{I.F.}) = \int (-5x^2)\ (\text{I.F.})\ dx + c$$

or
$$zx^{-5} = \int (-5x^2)\ x^{-5}\ dx + c$$

or
$$y^{-5}\ x^{-5} = -5 \cdot \frac{x^{-2}}{-2} + c \qquad [\because z = y^{-5}]$$

Dividing throughout by $y^{-5}\ x^{-5}$,

$$1 = (2{\cdot}5 + cx^2)x^3y^5$$

which is the required solution.

Problems

Solve the following equations :

1. $\dfrac{dy}{dx} + y \tan x = y^2 \sec x.$ (*Kurukshetra, 1998*)
2. $\dfrac{dy}{dx} + y \cos x = y^3 \sin 2x.$ (*Pondicherry, 1998 S*)
3. $r \sin \theta - \dfrac{dr}{d\theta} \cos \theta = r^2.$ **4.** $(x^3y^2 + xy)\ dx = dy.$ (*Delhi, 1994*)
5. $\dfrac{dy}{dx} = \dfrac{x^2 + y^2 + 1}{2xy}$
6. $x(x - y)\ dy + y^2\ dx = 0.$
7. $xy(1 + xy^2)\ \dfrac{dy}{dx} = 1.$
8. $y(2xy + e^x)\ dx - e^x dy = 0.$ (*Bangalore, 1990*)

20·7. Exact differential equations

(1) **Def.** *A differential equation of the form*

$$\mathbf{M}(x, y)\ \mathbf{dx} + \mathbf{N}\ (x, y)\ \mathbf{dy} = \mathbf{0} \qquad ...(1)$$

is said to be **exact** *if its left hand member is the exact differential of some function $u(x, y)$.*

i.e.
$$du \equiv Mdx + Ndy = 0$$

Its solution, therefore, is $u(x, y) = c$.

(2) *If* $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, *the equation* (*i*) is exact and its solution is

$$\int_{(y\ \text{constant})} \mathbf{Mdx} + \int (\textbf{terms of N not containing x})\ \mathbf{dy} = \mathbf{c}$$

Example 20·10. *Solve* $\dfrac{dy}{dx} + \dfrac{y \cos x + \sin y + y}{\sin x + x \cos y + x} = 0.$

(*Mangalore, 1999*)

Given equation can be written as

$$(y \cos x + \sin y + y)\ dx + (\sin x + x \cos y + x)\ dy = 0.$$

Here $M = y \cos x + \sin y + y$ and $N = \sin x + x \cos y + x$.

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$$\therefore \qquad \frac{\partial M}{\partial y} = \cos x + \cos y + 1 = \frac{\partial N}{\partial x}.$$

Thus the equation is exact and its solution is

$$\int_{(y \text{ const.})} M dx + \int (\text{terms of } N \text{ not containing } x)\, dy = c$$

i.e. $$\int (y \cos x + \sin y + y)\, dx + \int (0)\, dx = c$$

or $$y \sin x + (\sin y + y)\, x = c.$$

Problems

Solve the following equations :

1. $(x^2 - ay)\, dx = (ax - y^2)\, dy.$ (*Kurukshetra, 1998*)
2. $(x^2 + y^2 - a^2)\, xdx + (x^2 - y^2 - b^2)\, ydy = 0.$ (*Kanpur, 1996*)
3. $(x^2 - 4xy - 2y^2)\, dx + (y^2 - 4xy - 2x^2)\, dy = 0.$ (*Madras, 1998S*)
4. $(x^2 - 2xy + 3y^2)\, dx + (4y^2 + 6xy - x^2)\, dy = 0.$ (*Pondicherry, 1998S*)
5. $(2x^3 - xy^2 - 2y + 3)dx - (x^2y + 2x)\, dy = 0.$
6. $ye^{xy}dx + (xe^{xy} + 2y)\, dy = 0.$
7. $(3x^2 + 6xy^2)\, dx + (6x^2 y + 4y^3)\, dy = 0$ (*Mysore, 1995*)
8. $\dfrac{2x}{y^3}\, dx + \dfrac{y^2 - 3x^2}{y^4}\, dy = 0.$
9. $(ye^{xy} - 2y^3)\, dx + (xe^{xy} - 6xy^2 - 2y)\, dy = 0.$ (*Madras, 1992*)

20·8. Equations reducible to exact equations. Sometimes a differential equation which is not exact, can be made so on multiplication by a suitable factor called an *integrating factor*. In a number of cases, the integrating factor can be found after regrouping the terms of the equation and recognizing each group as being a part of an exact differential. In this connection the following integrable combinations prove quite useful :

$$xdy + ydx = d\,(xy) \quad ; \quad \frac{xdy - ydx}{x^2} = d\left(\frac{y}{x}\right)$$

$$\frac{xdy - ydx}{xy} = d\left[\log\left(\frac{y}{x}\right)\right] \quad ; \quad \frac{xdy - ydx}{y^2} = -\, d\left(\frac{x}{y}\right)$$

$$\frac{xdy - ydx}{x^2 + y^2} = d\left(\tan^{-1}\frac{y}{x}\right) \quad ; \quad \frac{xdy - ydx}{x^2 - y^2} = d\left(\frac{1}{2}\log\frac{x+y}{x-y}\right).$$

Example 20·11. *Solve* $y\,(2x^2y + e^x)\, dx = (e^x + y^3)\, dy.$

It is easy to note that the terms $ye^x dx$ and $e^x dy$ should be put together.

$$\therefore \qquad (ye^x dx - e^x dy) + 2x^2y^2dx - y^3dy = 0$$

Now we observe that the term $2x^2y^2\, dx$ should not involve y^2. This suggest that $1/y^2$ may be the I.F. Multiplying throughout by $1/y^2$, it becomes

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$$\frac{ye^x\,dx - e^x\,dy}{y^2} + 2x^2\,dx - y\,dy = 0$$

or
$$d\left(\frac{e^x}{y}\right) + 2x^2\,dx - y\,dy = 0$$

Integrating, we get
$$\frac{e^x}{y} + \frac{2x^3}{3} - \frac{y^2}{2} = c$$

which is the required solution.

Example 20·12. *Solve* $(1 + xy)\,ydx + (1 - xy)\,xdy = 0.$

(*Madurai, 1990 S*)

Given equation can be written as

$$(ydx + xdy) + (xy^2\,dx - x^2y\,dy) = 0 \qquad ...(i)$$

If we divide by x^2y^2, each of the groups will become integrable.

$\therefore$ Taking I.F. $= 1/x^2y^2$, (i) becomes

$$\frac{ydx + xdy}{x^2y^2} + \left(\frac{dx}{x} - \frac{dy}{y}\right) = 0$$

or
$$\frac{d(xy)}{(xy)^2} + d\,(\log x - \log y) = 0$$

or
$$\frac{dt}{t^2} + d\left(\log\frac{x}{y}\right) = 0, \quad \text{where } t = xy$$

Integrating,
$$-\frac{1}{t} + \log\frac{x}{y} = c \quad \text{or} \quad \log\frac{x}{y} - \frac{1}{xy} = c$$

which is the required solution.

Problems

Solve the following equations :

1. $xdy - ydx + a(x^2 + y^2)\,dx = 0.$ (*Andhra, 1993*)
2. $xdx + ydy = a^2\,\dfrac{xdy - ydx}{x^2 + y^2}$
3. $ydx - xdy + \log x\,dx = 0$ (*Marathwada, 1994*)
4. $x\dfrac{dy}{dx} - y = x\sqrt{(x^2 + y^2)}.$
5. $(2xy^2 - y)\,dx + xdy = 0.$
6. $ydx - xdy - 3x^2y^2e^{x^2}dx = 0.$ (*Kottayam, 1996*)
7. $(x^4e^x - 2mxy^2)\,dx + 2mx^2ydy = 0$ (*J.N.T.U., 1998*)

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20·9. Linear equations with constant coefficients

An equations of the form

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + k_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots\ldots + k_n y = X \qquad \ldots(1)$$

where $k_1, k_2, \ldots, k_n$ are constants and X is a function of x only, is called a **linear equation with constant coefficients.**

(1) can be written in the *symbolic form*

$$D^n y + k_1 D^{n-1} y + k_2 D^{n-2} y + \ldots\ldots + k_n y = X$$

or $$(D^n + k_1 D^{n-1} + k_2 D^{n-2} + \ldots\ldots k_n)\, y = X$$

or $$f(D)\, y = X \qquad \ldots(2)$$

where $f(D) = D^n + k_1 D^{n-1} + k_2 D^{n-2} + \ldots\ldots + k_n$.

To solve (2), we first solve the equation $f(D)\, y = 0$. ...(3)

Its solution is called the **complementary function** (*C.F.*) and contains n arbitrary constants.

Next, we observe that

$$f(D) \left\{ \frac{1}{f(D)} X \right\} = X$$

i.e. $\frac{1}{f(D)} X$ *satisfies the equation (2) and is therefore, its* **particular integral** (*P.I.*) which does not contain any arbitrary constant.

Thus $$\text{P.I.} = \frac{1}{f(D)} X$$

The sum of C.F. and P.I. also contains n arbitrary constants and is therefore, the **complete solution** (*C.S.*) *of (2).* Thus *the C.S. of (2) is*

$$\mathbf{y = C.F. + P.I.}$$

Obs. $f(D)$ and $1/f(D)$ are *inverse operators.* In particular, D and $1/D$ being inverse operators, we have $\frac{1}{D} X = \int X dx$.

$$\left[\because\ DX = \frac{d}{dx}(X) \right.$$

20·10. Rules for finding the complementary function

To solve the equation

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + k_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots\ldots + k_n y = 0 \qquad \ldots(1)$$

where k's are constants.

The equation (1) in *symbolic form* is

$$(D^n + k_1 D^{n-1} + k_2 D^{n-2} + \ldots\ldots + k_n) y = 0 \qquad \ldots(2)$$

Its symbolic co-efficient equated to zero *i.e.*

$$D^n + k_1 D^{n-1} + k_2 D^{n-2} + \ldots\ldots + k_n = 0$$

is called the *auxiliary equation* (A.E.). Let $m_1, m_2, \ldots\ldots, m_n$ be its roots.

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Case I. *If all the roots be real and different*, then (2) is equivalent to

$$(D - m_1)(D - m_2)\ldots(D - m_n)y = 0 \qquad \ldots(3)$$

Now (3) will be satisfied by the solution of

$$(D - m_n)y = 0, \quad i.e. \text{ by } \frac{dy}{dx} - m_n y = 0.$$

This is a Leibnitz's linear and I.F. $= e^{-m_n x}$

$\therefore$ Its solution is $\qquad ye^{-m_n x} = c_n$, *i.e.* $y = c_n e^{m_n x}$

Similarly, since the factors in (3) can be taken in any order, it will be satisfied by the solutions of

$$(D - m_1)y = 0, (D - m_2)y = 0 \text{ etc.}$$

i.e. by $\qquad y = c_1 e^{m_1 x}, y = c_2 e^{m_2 x}$ etc.

Thus the complete solution of (1) is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots\ldots + c_n e^{m_n x} \qquad \ldots(4)$$

Case II. *If two roots are equal (i.e.* $m_1 = m_2$), then the complete solution of (1) is

$$y = (c_1 x + c_2)\, e^{m_1 x} + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}$$

For its proof refer to author's *Higher Engineering Mathematics.*

Case III. *If one pair of roots be imaginary, i.e.* $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$, then the complete solution is

$$\begin{aligned} y &= c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x} + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x} \\ &= e^{\alpha x}(c_1 e^{i\beta x} + c_2 e^{-i\beta x}) + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x} \\ &= e^{\alpha x}[c_1(\cos\beta x + i\sin\beta x) + c_2(\cos\beta x - i\sin\beta x] \\ &\qquad + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x} \end{aligned}$$

[$\because$ by *Euler's Theorem*, $e^{i\theta} = \cos\theta + i\sin\theta$

$$= e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x) + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}$$

where $C_1 = c_1 + c_2$ and $C_2 = i(c_1 - c_2)$.

Example 20·13. *Solve (i)* $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 2y = 0.$

(ii) $\dfrac{d^2x}{dt^2} + 6\dfrac{dx}{dt} + 9x = 0$

(*i*) Given equation in *symbolic form* is

$$(D^2 + D - 2)y = 0.$$

Its A.E. is $D^2 + D - 2 = 0$, *i.e.* $\quad (D + 2)(D - 1) = 0$

whence $\qquad D = -2, 1.$

Hence the C.S. is $y = c_1 e^{-2x} + c_2 e^x$.

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(*ii*) Given equation in *symbolic form is*

$$(D^2 + 6D + 9)\,x = 0$$

$\therefore$ A.E. is $D^2 + 6D + 9 = 0$, *i.e.* $(D + 3)^2 = 0$

whence $D = -3, -3$.

Hence the C.S. is $x = (c_1 + c_2 t)\,e^{-3t}$.

Example 20·14. *Solve (i)* $(D^3 + D^2 + 4D + 4) = 0.$

(*Mangalore, 1997*)

$$(ii)\ \frac{d^4x}{dt^4} + 4x = 0.$$

(*i*) Here the A.E. is $D^3 + D^2 + 4D + 4 = 0$

i.e. $(D^2 + 4)(D + 1) = 0$

$\therefore$ $D = -1, \pm 2i.$

Hence the C.S. is $y = c_1 e^{-x} + e^{0x}(c_2 \cos 2x + c_3 \sin 2x)$

i.e. $y = c_1 e^{-x} + c_2 \cos 2x + c_3 \sin 2x.$

(*ii*) Given equation in *symbolic form is* $(D^4 + 4)x = 0$

$\therefore$ A.E. is $D^4 + 4 = 0$

or $(D^4 + 4D^2 + 4) - 4D^2 = 0$

or $(D^2 + 2)^2 - (2D)^2 = 0$ or $(D^2 + 2D + 2)(D^2 - 2D + 2) = 0$

$\therefore$ either $D^2 + 2D + 2 = 0$ or $D^2 - 2D + 2 = 0$

i.e. $D = \dfrac{-2 \pm \sqrt{(-4)}}{2}$ and $\dfrac{2 \pm \sqrt{(-4)}}{2}$

or $D = -1 \pm i$ and $1 \pm i$.

Hence the required solution is

$$x = e^{-t}(c_1 \cos t + c_2 \sin t) + e^t (c_3 \cos t + c_4 \sin t).$$

Problems

Solve :

1. $(D^2 + 1)\,y = 0$ (*Madras, 1996*) 2. $\dfrac{d^2x}{dt^2} + 3a\dfrac{dx}{dt} - 4a^2x = 0.$

3. $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = 0.$ 4. $\dfrac{d^3y}{dx^3} + y = 0.$

5. $(D - 3)^3\,y = 0$ (*Nagarjuna, 1998 S*)

6. $\dfrac{d^4y}{dx^4} + 8\dfrac{d^2y}{dx^2} + 16y = 0.$

7. If $\dfrac{d^4x}{dt^4} = m^4x$, show that

$$x = c_1 \cos mt + c_2 \sin mt + c_3 \cosh mt + c_4 \sinh mt.$$

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20·11. Rules for finding the particular integral

Consider the equation

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + k_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots\ldots + k_n y = X$$

which in *symbolic form is*

$$(D^n + k_1 D^{n-1} + k_2 D^{n-2} + \ldots\ldots + k_n) y = X.$$

$$\therefore \text{ P.I.} = \frac{1}{D^n + k_1 D^{n-1} + k_2 D^{n-2} + \ldots\ldots + k_n} X.$$

Case I. When $\mathbf{X = e^{ax}}$

Since
$$De^{ax} = ae^{ax}$$
$$D^2 e^{ax} = a^2 e^{ax}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$D^n e^{ax} = a^n e^{ax}$$

$$\therefore \quad (D^n + k_1 D^{n-1} + \ldots\ldots + k_n)\, e^{ax} = (a^n + k_1 a^{n-1} + \ldots\ldots + k_n) e^{ax}$$

i.e.
$$f(D) e^{ax} = f(a) e^{ax}$$

Operating on both sides by $\frac{1}{f(D)}$

$$\frac{1}{f(D)} f(D) e^{ax} = \frac{1}{f(D)} f(a) e^{ax} \quad \text{or} \quad e^{ax} = f(a) \frac{1}{f(D)} e^{ax}$$

$\therefore$ dividing by $f(a)$,

$$\frac{1}{\mathbf{f(D)}} \mathbf{e}^{ax} = \frac{1}{\mathbf{f(a)}} \mathbf{e}^{ax}, \textit{ provided } f(a) \neq 0 \qquad \ldots(1)$$

If f(a) = 0, *the above rule fails and we have*

$$\frac{1}{\mathbf{f(D)}} \mathbf{e}^{ax} = \mathbf{x} \frac{1}{\mathbf{f'(a)}} \mathbf{e}^{ax} \qquad \ldots(2)$$

Example 20·15. *Find the P.I. of* (*i*) $(D^2 + 5D + 6)\, y = e^x$.
(*ii*) $(D + 2)(D - 1)^2\, y = e^{-2x}$.

(*i*) $$P.I. = \frac{1}{D^2 + 5D + 6} e^x \qquad [\text{Put } D = 1.$$

$$= \frac{1}{1^2 + 5 \cdot 1 + 6} e^x = \frac{e^x}{12}.$$

(*ii*) $$P.I. = \frac{1}{(D+2)(D-1)^2} e^{-2x} = \frac{1}{D+2} \cdot \left[\frac{1}{(D-1)^2} e^{-2x}\right]$$

$$= \frac{1}{D+2} \cdot \frac{1}{(-2-1)^2} e^{-2x} = \frac{1}{9} \cdot \frac{1}{D+2} e^{-2x}$$

$$= \frac{1}{9} \cdot x \cdot \frac{1}{1} e^{-2x} \qquad \left[\because \frac{d}{dD}(D+2) = 1\right.$$

$$= \frac{x}{9} e^{-2x}$$

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Case II. When X = sin (ax + b) or cos (ax + b).

Since
$$D \sin(ax+b) = a \cos(ax+b)$$
$$D^2 \sin(ax+b) = -a^2 \sin(ax+b)$$
$$D^3 \sin(ax+b) = -a^3 \cos(ax+b)$$
$$D^4 \sin(ax+b) = a^4 \sin(ax+b)$$

i.e.
$$D^2 \sin(ax+b) = (-a^2) \sin(ax+b)$$
$$(D^2)^2 \sin(ax+b) = (-a^2)^2 \sin(ax+b).$$

In general
$$(D^2)^r \sin(ax+b) = (-a^2)^r \sin(ax+b).$$

$\therefore$
$$f(D^2) \sin(ax+b) = f(-a^2) \sin(ax+b).$$

Operating on both sides $1/f(D^2)$,

$$\frac{1}{f(D^2)} \cdot f(D^2) \sin(ax+b) = \frac{1}{f(D^2)} f(-a^2) \sin(ax+b)$$

or
$$\sin(ax+b) = f(-a^2) \frac{1}{f(D^2)} \sin(ax+b).$$

$\therefore$ dividing by $f(-a^2)$,

$$\frac{1}{\mathbf{f(D^2)}} \mathbf{\sin(ax+b)} = \frac{1}{\mathbf{f(-a^2)}} \mathbf{\sin(ax+b)}$$

provided $f(-a^2) \neq 0$ ...(4)

If $f(-a^2) = 0$ *the above rule fails and we have*

$\therefore$
$$\frac{1}{\mathbf{f(D^2)}} \mathbf{\sin(ax+b)} = \mathbf{x} \frac{1}{\mathbf{f'(-a^2)}} \mathbf{\sin(ax+b)}$$

provided $f'(-a)^2) \neq 0$ ...(4)

Similarly, $\dfrac{1}{\mathbf{f(D^2)}} \mathbf{\cos(ax+b)} = \dfrac{1}{\mathbf{f(-a^2)}} \mathbf{\cos(ax+b)},$

provided $f(-a)^2 \neq 0,$

If $f(-a^2) = 0, \dfrac{1}{\mathbf{f(D^2)}} \mathbf{\cos(ax+b)} = \mathbf{x} \cdot \dfrac{1}{\mathbf{f'(-a^2)}} \mathbf{\cos(ax+b)},$

provided $f'(-a^2) \neq 0.$

Example 20·16. *Find the P.I. of* $(D^3 + 4D)\, y = \sin 2x$

$$P.I. = \frac{1}{D(D^2+4)} \sin 2x$$

[$\because D^2 + 4 = 0$ for $D^2 = -2^2$, $\therefore$ Apply (4) above

$$= x \frac{1}{3D^2+4} \sin 2x \qquad \left[\because \frac{d}{dD}[D^3+4D] = 3D^2+4 \right.$$

[Put $D^2 = -2^2 = -4$

$$= x \frac{1}{3(-4)+4} \sin 2x = -\frac{x}{8} \sin 2x.$$

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Case III. When $X = x^m$.

Here $P.I. = \frac{1}{f(D)} x^m = [f(D)]^{-1} x^m$.

Expand $[f(D)]^{-1}$ in ascending powers of D as far as the term in D^m and operate on x^m term by term. Since the $(m+1)$th and higher derivatives of x^m are zero, we need not consider terms beyond D^m.

Example 20·17. *Find the P.I. of* $\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2 + 2x + 4$ (Kanpur, 1998)

Given equation in *symbolic form* is

$$(D^2 + D)\,y = x^2 + 2x + 4.$$

$$\therefore \quad P.I. = \frac{1}{D(D+1)}(x^2 + 2x + 4) = \frac{1}{D}(1+D)^{-1}(x^2 + 2x + 4)$$

$$= \frac{1}{D}(1 - D + D^2 - \ldots\ldots)(x^2 + 2x + 4)$$

$$= \frac{1}{D}[x^2 + 2x + 4 - (2x + 2) + 2]$$

$$= \int (x^2 + 4)\,dx = \frac{x^3}{3} + 4x.$$

Case IV. When $X = e^{ax} V$, *V being a function of x.*

$$\frac{1}{f(D)}(e^{ax}V) = e^{ax}\frac{1}{f(D+a)}V \qquad \ldots(5)$$

For proofs of (2), (4) and (5) refer to author's '*Higher Engineering Mathematics*'.

Example 20·18. *Find P.I. of* $(D^2 - 2D + 4)\,y = e^x \sin x$.

$$P.I. = \frac{1}{D^2 - 2D + 4} e^x \sin x \qquad \text{[Replace } D \text{ by } D+1]$$

$$= e^x \frac{1}{(D+1)^2 - 2(D+1) + 4} \sin x$$

$$= e^x \frac{1}{D^2 + 3} \sin x \qquad [\text{Put } D^2 = -1^2 = -1$$

$$= e^x \frac{1}{-1+3} \cos x = \frac{1}{2} e^x \sin x.$$

20·12. Working procedure to solve the equation

$$\frac{d^ny}{dx^n} + k_1 \frac{d^{n-1}y}{dx^{n-1}} + \ldots\ldots + k_{n-1}\frac{dy}{dx} + k_n y = X$$

of which the *symbolic form* is

$$(D^n + k_1 D^{n-1} + \ldots\ldots + k_{n-1} D + k_n)\,y = X.$$

Step I. To find the complementary function

(*i*) *Write the A.E.*

i.e. $$D^n + k_1 D^{n-1} + \ldots\ldots + k_{n-1} D + k_n = 0$$

and solve it for D.

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(*ii*) *Write the C.F.* as follows :

	Roots of A.E.	C.F.
1.	m_1, m_2, m_3 ...... (real and different roots)	$c_1e^{m_1x} + c_2e^{m_2x} + c_3e^{m_3x} + \ldots$
2.	m_1, m_1, m_3 ...... (two real and equal roots)	$(c_1 + c_2 x)\, e^{m_1x} + c_3e^{m_3x} + \ldots$
3.	$\alpha + i\beta, \alpha - i\beta, m_3$ ...... (a pair of imaginary roots)	$e^{\alpha x}\,(c_1 \cos \beta x + c_2 \sin \beta x) + c_3e^{m_3x} + \ldots$
4.	$\alpha \pm i\beta, \alpha \pm i\beta, m_5$ ...... (2 pairs of equal imaginary roots)	$e^{\alpha x}\,[(c_1 + c_2x) \cos \beta x + (c_3 + c_4x) \sin \beta x] + c_5e^{m_5x} + \ldots$

Step II. To find the particular integral

From symbolic form

$$P.I. = \frac{1}{D^n + k_1D^{n-1} + \ldots + k_{n-1}D + k_n} X$$

$$= \frac{1}{f(D)} \text{ or } \frac{1}{\phi(D^2)} X.$$

(*i*) *When* $X = e^{ax}$

$$P.I. = \frac{1}{f(D)} e^{ax}, \text{ put } D = a, \qquad [f(a) \neq 0$$

$$= x \frac{1}{f'(D)} e^{ax}, \text{ put } D = a, \qquad [f(a) = 0, f'(a) \neq 0$$

$$= x^2 \frac{1}{f''(D)} e^{ax}, \text{ put } D = a, \qquad [f'(a) = 0, f''(a) \neq 0$$

and so on.

where $f'(D)$ = diff. coeff. of $f(D)$ w.r.t. D

$f''(D)$ = diff. coeff. of $f'(D)$ w.r.t. D, etc.

(*ii*) *When* $X = \sin(ax + b)$ or $\cos(ax + b)$.

$$P.I. = \frac{1}{\phi(D^2)} \sin(ax + b) \text{ [or } \cos(ax + b)], \text{ put } D^2 = -a^2$$

$$[\phi(-a^2) \neq 0.$$

$$= x \frac{1}{\phi'(D^2)} \sin(ax + b) \text{ [or } \cos(ax + b)], \text{ put } D^2 = -a^2$$

$$[\phi(-a^2) = 0, \phi'(-a^2) \neq 0.$$

$$= x^2 \frac{1}{\phi''(D^2)} \sin(ax + b) \text{ [or } \cos(ax + b)], \text{ put } D^2 = -a^2$$

$$[\phi'(-a^2) = 0, \phi''(-a^2) \neq 0.$$

and so on.

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where $\phi'(D^2)$ = diff. coeff. of $\phi(D^2)$ w.r.t. D,

$\phi''(D^2)$ = diff. coeff. of $\phi'(D^2)$ w.r.t. D, etc.

(*iii*) *When* $X = x^m$, m being a positive integer.

$$P.I. = \frac{1}{f(D)} x^m = [f(D)]^{-1} x^m.$$

To evaluate it, expand $[f(D)]^{-1}$ in ascending powers of D by Binomial theorem as far as D^m and operate on x^m term by term.

(*iv*) *When* $X = e^{ax}V$, where V is a function of x

$$P.I. = \frac{1}{f(D)} e^{ax} V = e^{ax} \frac{1}{f(D+a)} V$$

and then evaluate $\dfrac{1}{f(D+a)} V$ as in (*i*), (*ii*) and (*iii*).

Step III. To find the complete solution

Then the C.S. is y = C.F. + P.I.

Example 20·19. *Solve* $(D-2)^2 y = 8(e^{2x} + \sin 2x + x^2)$.

(*i*) *To find C.F.*

Its A.E. is $(D-2)^2 = 0$, $\therefore D = 2, 2.$

Thus C.F. $= (c_1 + c_2 x)e^{2x}$.

(*ii*) *To find P.I.*

$$P.I. = 8\left[\frac{1}{(D-2)^2} e^{2x} + \frac{1}{(D-2)^2} \sin 2x + \frac{1}{(D-2)^2} x^2\right]$$

Now $\dfrac{1}{(D-2)^2} e^{2x} = x^2 \dfrac{1}{2(1)} e^{2x}$

[$\because$ by putting $D = 2$, $(D-2)^2 = 0$, $2(D-2) = 0$

$$= \frac{x^2 e^{2x}}{2}.$$

$$\frac{1}{(D-2)^2} \sin 2x = \frac{1}{D^2 - 4D + 4} \sin 2x = \frac{1}{(-2^2) - 4D + 4} \sin 2x$$

$$= -\frac{1}{4}\int \sin 2x \, dx = -\frac{1}{4}\left(\frac{-\cos 2x}{2}\right)$$

$$= \tfrac{1}{8} \cos 2x$$

and

$$\frac{1}{(D-2)^2} x^2 = \frac{1}{4}\left(1 - \frac{D}{2}\right)^{-2} x^2 = \frac{1}{4}\left[1 + (-2)\left(\frac{D}{2}\right) + \frac{(-2)(-3)}{2!}\left(-\frac{D}{2}\right)^2 + \ldots\ldots\right] x^2$$

$$= \frac{1}{4}\left(1 + D + \frac{3D^2}{4} + \ldots\ldots\right) x^2 = \frac{1}{4}\left(x^2 + 2x + \frac{3}{2}\right)$$

Thus P.I. $= 4x^2 e^{2x} + \cos 2x + 2x^2 + 4x + 3.$

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(*iii*) Hence the C.S. is

$$y = (c_1 + c_2 x)e^{2x} + 4x^2 e^{2x} + \cos 2x + 2x^2 + 4x + 3.$$

Example 20·20. *Solve* $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 13y = e^{2x}\cos 3x.$

(i) To find C.F.

Its A.E. is $D^2 - 4D + 13 = 0$

$$D = \frac{4 \pm \sqrt{(16 - 52)}}{2} = 2 \pm 3i$$

$$\therefore \qquad \text{C.F.} = e^{2x}(c_1 \cos 3x + c_2 \sin 3x)$$

(ii) To find P.I.

$$\text{P.I.} = \frac{1}{D^2 - 4D + 13} e^{2x} \cos 3x$$

$$= e^{2x} \frac{1}{(D+2)^2 - 4(D+2) + 13} \cos 3x$$

$$= e^{2x} \frac{1}{D^2 + 9} \cos 3x = e^{2x} \cdot x \frac{1}{2D} \cos 3x$$

$$= \frac{1}{2} x e^{2x} \int \cos 3x dx = \frac{1}{6} x e^{2x} \sin 3x$$

(*iii*) Hence the *C.S.* is

$$y = e^{2x}(c_1 \cos 3x + c_2 \sin 3x) + \frac{1}{6} x e^{2x} \sin 3x.$$

Problems

Solve :

1. $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 5y = -2\cosh x.$
Also find y when $y = 0$, $dy/dx = 1$ at $x = 0$.
2. $(D^2 + D - 2)y = 3e^x.$ (*J.N.T.U., 1995*)
3. $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = \cos^2 x.$ (*Triputi, 1998 S*)
4. $(D^2 - 4D + 3)y = \sin 3x \cos 2s.$ (*Karnataka, 1995*)
5. $(D^2 + 9)y = 2\cos 3x + e^{-3x}.$ (*Pondicherry, 1998 S*)
6. $\dfrac{d^2y}{dx^2} - 4y = x^2$ (*Bhopal, 1991*)
7. $\dfrac{d^2y}{dx^2} + 4y = x^2 + \cos 2x.$ (*Kurukshetra, 1998*)
8. $(D^2 + 5D + 6)y = \cos^2 2x + x.$ (*Kottayam, 1996*)
9. $(D^2 - 2D)y = e^x \sin x.$ (*Andhra, 1991*)

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10. $\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} + 6y = e^{-2x} \sin 2x.$ (*Panjab, 1997*)

11. $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = xe^{3x} + \sin 2x$ (*Bangalore, 1990*)

12. $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 3y = e^{-3x} \cdot x^2$ (*Madras, 1993*)

13. $(D^3 + 3D^2 + D)y = x^2e^{2x} + \sin^2 x.$ (*Nagarjuna, 1998*)

20·13. Method of variation of parameters. This method applies to equations of the form

$$y'' + py' + qy = X \qquad \text{....(1)}$$

where p, q and X are functions of x.

If y_1, y_2 are the solutions of $y'' + py' + qy = 0$

and $$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, \text{ then}$$

$$\text{P.I. of (1)} = -y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X}{W} dx.$$

Example 20·21. *Solve* $\dfrac{d^2y}{dx^2} + 4y = \tan 2x$ (*Mangalore, 1997*)

Given equation is $(D^2 + 4)\, y = \tan 2x$

(*i*) *To find C.F.*

Its A.E. is $D^2 + 4 = 0$, $\therefore$ $D = \pm 2i$.

Thus C.F. is $y = c_1 \cos 2x + c_2 \sin 2x$.

(*ii*) *To find P.I.*

Here $y_1 = \cos 2x$, $y_2 = \sin 2x$, $X = \tan 2x$.

$$\therefore \quad W = \begin{vmatrix} \cos 2x & \sin 2x \\ -2 \sin 2x & 2 \cos 2x \end{vmatrix} = 2.$$

Thus
$$P.I. = -y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X}{W} dx$$

$$= -\cos 2x \int \frac{\sin 2x \tan 2x}{2} dx + \sin 2x \int \frac{\cos 2x \tan 2x}{2} dx$$

$$= -\tfrac{1}{2} \cos 2x \int (\sec 2x - \cos 2x)\, dx + \frac{1}{2} \sin 2x \int \sin 2x\, dx$$

$$= -\tfrac{1}{4} \cos 2x\, [\log (\sec 2x + \tan 2x) - \sin 2x] - \tfrac{1}{4} \sin 2x \cos 2x$$

$$= -\tfrac{1}{4} \cos 2x \log (\sec 2x + \tan 2x).$$

Hence the C.S. is

$$y = c_1 \cos 2x + c_2 \sin 2x - \tfrac{1}{4} \cos 2x \log (\sec 2x + \tan 2x)$$

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Problems

Solve by the method of variation of parameters :

1. $\frac{d^2y}{dx^2} + y = \sec x.$ (*J.N.T.U., 1998*)

2. $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$ (*Madras, 1998*)

3. $\frac{d^2y}{dx^2} + y = \tan x$ (*Pondicherry, 1998 S*)

4. $\frac{d^2y}{dx^2} + y = x \sin x$ (*Andhra, 1998*)

5. $y'' - 6y' + 9y = e^{3x}/x^2$ (*Raipur, 1998*)

6. $y'' - 2y' + 2y = e^x \tan x.$ (*Nagarjuna, 1998*)

20·14. Equations reducible to linear equations with constant coefficients. Equations of the form

$$x^3 \frac{d^3y}{dx^3} + k_1 x^2 \frac{d^2y}{dx^2} + k_2 x \frac{dy}{dx} + y = X$$

where X is a function of x, can be reduced to linear equations with constant coefficients by putting $x = e^t$ or $t = \log x$. Then if $D = d/dt$,

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \cdot \frac{1}{x} \; i.e. \; x\frac{dy}{dx} = Dy.$$

Similarly, $x^2 \frac{d^2y}{dx^2} = D(D-1)\,y,\; x^3 \frac{d^3y}{dx^3} = D(D-1)(D-2)\,y$ etc.

Example 20·22. *Solve* $x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = \log x$

(*Pondicherry, 1998 S*)

Put $x = e^t$ *i.e.* $t = \log x$, so that

$$x\frac{dy}{dx} = Dy,\; x^2 \frac{d^2y}{dx^2} = D(D-1)\,y \qquad \text{where } D = \frac{d}{dt}.$$

Then the given equation becomes

$$[D(D-1) - D + 1]\,y = t \quad \text{or} \quad (D-1)^2\,y = t \qquad ...(i)$$

which is a linear equation with constant coefficients.

Its A.E. is $(D-1)^2 = 0$ which gives $D = 1, 1$.

$\therefore$ $\quad$ C.F. $= (c_1 + c_2 t)\,e^t$

and $\quad$ P.I. $= \frac{1}{(D-1)^2}\,t = (1-D)^{-2}\,t$

$= (1 + 2D + ...)\,t = t + 2.$

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Hence the solution of (*i*) is

$$y = (c_1 + c_2 t)\, e^t + t + 2$$

or putting $t = \log x$ and $e^t = x$, we get

$$y = (c_1 + c_2 \log x)\, x + \log x + 2.$$

as the required solution.

Problems

Solve :

1. $x^2 \dfrac{d^2y}{dx^2} - 4x \dfrac{dy}{dx} + 6y = x^2$ (*Madras, 1997*)

2. $x^2 \dfrac{d^2y}{dx^2} - 2x \dfrac{dy}{dx} - 4y = x^4$ (*Andhra, 1998*)

3. $\dfrac{d^2y}{dx^2} + \dfrac{1}{x}\dfrac{dy}{dx} = \dfrac{12 \log x}{x^2}$ (*A.M.I.E., 1997*)

4. $x^2 \dfrac{d^2y}{dx^2} - x \dfrac{dy}{dx} + 2y = x \sin(\log x)$

5. $x^2 \dfrac{d^2y}{dx^2} + 2x \dfrac{dy}{dx} - 12y = x^3 \log x$ (*Triputi, 1998*)

6. $x^3 \dfrac{d^3y}{dx^3} + 2x^2 \dfrac{d^2y}{dx^2} + 2y = 10\,(x + 1/x).$ (*Sambalpur, 1998*)

20·15. Simultaneous linear equations with constant coefficients. Such equations are solved by eliminating all but *one* of the dependent variables and solving the resulting equation as before. Each dependent variable is obtained in a similar manner.

Example 20·23. *Solve the simultaneous equations :*

$$\frac{dx}{dt} + 5x - 2y = t, \quad \frac{dy}{dt} + 2x + y = 0$$

being given $x = y = 0$ when $t = 0$. (*Mangalore, 1999*)

Taking $d/dt = D$, the given equations become

$$(D + 5)x - 2y = t \qquad \ldots(i)$$

$$2x + (D + 1)\, y = 0 \qquad \ldots(ii)$$

Eliminate x as if D were an ordinary algebraic multiplier. Multiplying (*i*) by 2 and operating on (*ii*) by $D + 5$ and then subtracting, we get

$$[-4 - (D + 5)(D + 1)]\, y = 2t$$

or

$$(D^2 + 6D + 9)\, y = -2t.$$

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Its auxiliary equation is $D^2 + 6D + 9 = 0$, *i.e.* $(D + 3)^2 = 0$ which gives $D = -3, -3$.

$$\therefore \qquad C.F. = (c_1 + c_2 t)\, e^{-3t}$$

and
$$P.I. = \frac{1}{(D+3)^2}(-2t) = -\frac{2}{9}\left(1 + \frac{D}{3}\right)^{-2} t$$
$$= -\frac{2}{9}\left(1 - \frac{2D}{3} + \ldots\right) t = -\frac{2t}{9} + \frac{4}{27}$$

Hence
$$y = (c_1 + c_2 t)\, e^{-3t} - \frac{2t}{9} + \frac{4}{27} \qquad \ldots(iii)$$

Now to find x, either eliminate y from (i) and (ii) and solve the resulting equation or substitute the value of y in (ii). Here, it is more convenient to adopt the latter method.

From (iii), $Dy = c_2 e^{-3t} + (c_1 + c_2 t)(-3)\, e^{-3t} - \frac{2}{9}$

$\therefore$ Substituting for y and Dy in (ii), we get

$$x = -\tfrac{1}{2}\,[Dy + y]$$
$$= [(c_1 - \tfrac{1}{2} c_2) + c_2 t]\, e^{-3t} + \frac{t}{9} + \frac{1}{27} \qquad \ldots(iv)$$

Hence (iii) and (iv) constitute the solutions of the given equations.

Since $x = y = 0$ when $t = 0$, the equations (iii) and (iv) give

$$0 = c_1 + \frac{4}{27} \text{ and } c_1 - \frac{1}{2} c_2 + \frac{1}{27} = 0$$

whence
$$c_1 = -\frac{4}{27},\; c_2 = -\frac{2}{9}.$$

Hence the desired solutions are

$$x = -\frac{1}{27}(1 + 6t)\, e^{-3t} + \frac{1}{27}(1 + 3t).$$
$$y = -\frac{2}{27}(2 + 3t)\, e^{-3t} + \frac{1}{27}(2 - 3t).$$

Problems

Solve the following simultaneous equations :

1. $\dfrac{dx}{dt} - y = t,\; \dfrac{dy}{dt} + x = t$ given $x(0) = 1, y(0) = 2$. (*A.M.I.E., 1997*)

2. $\dfrac{dx}{dt} + y = \sin t,\; \dfrac{dy}{dt} + x = \cos t$; given that $x = 2$ and $y = 0$ when $t = 0$. (*Andhra, 1998*)

3. $\dfrac{dx}{dt} + 2x + 3y = 0,\; 3x + \dfrac{dy}{dt} + 2y = 2e^{2t}$. (*Nagarjuna, 1998*)

4. $\dfrac{dx}{dt} + 2y = e^t,\; \dfrac{dy}{dt} - 2x = e^{-t}$. (*Pondicherry, 1998 S*)

5. $t\dfrac{dx}{dt} + y = 0,\; t\dfrac{dy}{dt} + x = 0$ given $x(1) = 1, y(-1) = 0$.

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