2

IMPROPER INTEGRALS

2.1 FINITE AND INFINITE INTERVALS

An interval is said to be finite or infinite according as its length is finite or infinite. Thus the intervals $[a, b]$, $[a, b)$, $(a, b]$, (a, b), each with length $(b - a)$, are finite (or bounded) if both a and b are finite. The intervals $[a, \infty)$, (a, ∞), $(-\infty, b]$, $(-\infty, b)$ and $(-\infty, \infty)$ are infinite (or unbounded) intervals.

2.2 BOUNDED FUNCTION

A function f is said to be bounded if its range is bounded. Thus, $f : [a, b] \to \mathrm{R}$ is bounded if there exist two real numbers m and M, $(m \leq \mathrm{M})$ such that

$$m \leq f(x) \leq \mathrm{M} \quad \forall\, x \in [a, b]$$

f is also bounded if there exists a positive real number K such that

$$|f(x)| \leq \mathrm{K} \quad \forall\, x \in [a, b].$$

2.3 PROPER INTEGRAL

The definite integral $\int_a^b f(x)\, dx$ is called a proper integral if

(*i*) the interval of integration $[a, b]$ is finite (or bounded)

(*ii*) the integrand f is bounded on $[a, b]$

If $\mathrm{F}(x)$ is an indefinite integral of $f(x)$, then $\int_a^b f(x)\, dx = \mathrm{F}(b) - \mathrm{F}(a)$.

2.4 IMPROPER INTEGRAL

The definite integral $\int_a^b f(x)\, dx$ is called an improper integral if either or both the above conditions are not satisfied. Thus $\int_a^b f(x)\, dx$ is an improper integral if either the interval of integration $[a, b]$ is not finite or f is not bounded on $[a, b]$ or neither the interval $[a, b]$ is finite nor f is bounded over it.

(*i*) In the definite integral $\int_a^b f(x)\,dx$, if either a or b or both a and b are infinite so that the interval of integration is unbounded but f is bounded, then $\int_a^b f(x)\,dx$ is called an **improper integral of the first kind.**

For example, $\int_1^\infty \frac{dx}{\sqrt{x}}, \int_{-\infty}^0 e^{2x}dx, \int_{-\infty}^\infty \frac{dx}{x^2+2x+2}$ are improper integrals of the first kind.

(*ii*) In the definite integral $\int_a^b f(x)\,dx$, if both a and b are finite so that the interval of integration is finite but f has one or more points of infinite discontinuity *i.e.*, f is not bounded on $[a, b]$, then $\int_a^b f(x)\,dx$ is called an **improper integral of the second kind.**

For example, $\int_0^1 \frac{dx}{x^2}, \int_1^2 \frac{dx}{2-x}, \int_1^4 \frac{dx}{(x-1)(4-x)}$ are improper integrals of the second kind.

(*iii*) In the definite integral $\int_a^b f(x)dx$, if the interval of integration is unbounded (so that a or b or both are infinite) and f is also unbounded, then $\int_a^b f(x)dx$ is called an **improper integral of the third kind or mixed kind.**

For example, $\int_0^\infty \frac{e^{-x}}{\sqrt{x}}dx$ is an improper integral of the third kind.

2.5 IMPROPER INTEGRAL AS THE LIMIT OF A PROPER INTEGRAL

(*a*) *When the improper integral is of the first kind, either a or b or both a and b are infinite but f is bounded. We define*

(*i*) $\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$, $(t > a)$

The improper integral $\int_a^\infty f(x)\,dx$ is said to be **convergent** if the limit on the right hand side exists finitely and the integral is said to be **divergent** if the limit is $+\infty$ or $-\infty$.

If the integral is neither convergent nor divergent, then it is said to be **oscillating.**

(*ii*) $\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$, $(t < b)$

The improper integral $\int_{-\infty}^b f(x)\,dx$ is said to be convergent if the limit on the right hand side exists finitely and the integral is said to be divergent if the limit is $+\infty$ or $-\infty$.

(*iii*) $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx$, where c is any real number.

$$= \lim_{t_1 \to -\infty} \int_{t_1}^{c} f(x)\,dx + \lim_{t_2 \to \infty} \int_{c}^{t_2} f(x)\,dx$$

The improper integral $\int_{-\infty}^{\infty} f(x)\,dx$ is said to be convergent if both the limits on the right hand side exist finitely and independent of each other, otherwise it is said to be divergent.

Note. $\int_{-\infty}^{\infty} f(x)\,dx \neq \lim_{t \to \infty} \left[\int_{-t}^{c} f(x)\,dx + \int_{c}^{t} f(x)\,dx\right]$.

(*b*) *When the improper integral is of the second kind, both a and b are finite but f has one (or more) points of infinite discontinuity on $[a, b]$.*

(*i*) If $f(x)$ becomes infinite at $x = b$ only, we define $\int_{a}^{b} f(x)\,dx = \lim_{\varepsilon \to 0+} \int_{a}^{b-\varepsilon} f(x)\,dx$.

The improper integral $\int_{a}^{b} f(x)\,dx$ is said to be convergent if the limit on the right hand side exists finitely and the integral is said to be divergent if the limit is $+\infty$ or $-\infty$.

(*ii*) If $f(x)$ becomes infinite at $x = a$ only, we define $\int_{a}^{b} f(x)\,dx = \lim_{\varepsilon \to 0+} \int_{a+\varepsilon}^{c} f(x)\,dx$.

The improper integral $\int_{a}^{b} f(x)\,dx$ converges if the limit on the right hand side exists finitely, otherwise it is said to be divergent.

(*iii*) If $f(x)$ becomes infinite at $x = c$ only where $a < c < b$, we define

$$\int_{a}^{b} f(x)\,dx = \int_{a}^{c} f(x)\,dx + \int_{c}^{b} f(x)\,dx = \lim_{\varepsilon_1 \to 0+} \int_{a}^{c-\varepsilon_1} f(x)\,dx + \lim_{\varepsilon_2 \to 0+} \int_{c+\varepsilon_2}^{b} f(x)\,dx.$$

The improper integral $\int_{a}^{b} f(x)\,dx$ is said to be convergent if both the limits on the right hand side exist finitely and independent of each other, otherwise it is said to be divergent.

Note 1. If f has infinite discontinuity at an end point of the interval of integration, then the point of discontinuity is approached from within the interval.

Thus if the interval of integration is $[a, b]$ and

(*i*) f has infinite discontinuity at 'a', we consider $[a + \varepsilon, b]$ as $\varepsilon \to 0+$.

(*ii*) f has infinite discontinuity at 'b', we consider $[a, b - \varepsilon]$ as $\varepsilon \to 0+$.

Note 2. A proper integral is always convergent.

Note 3. If $\int_{a}^{b} f(x)\,dx$ is convergent, then

(*i*) $\int_{a}^{b} kf(x)\,dx$ is convergent, $k \in \mathbb{R}$,

(*ii*) $\int_{a}^{b} f(x)\,dx = \int_{a}^{c} f(x)\,dx + \int_{c}^{b} f(x)\,dx$ where $a < c < b$ and each integral or right hand side is convergent.

ILLUSTRATIVE EXAMPLES

Example 1. *Examine the convergence of the improper integrals:*

(i) $\int_1^\infty \frac{1}{x}\, dx$ (ii) $\int_1^\infty \frac{dx}{x^{3/2}}$

Sol. (i) By definition, $\int_0^\infty \frac{dx}{x} = \lim_{t\to\infty} \int_0^t \frac{dx}{x} = \lim_{t\to\infty} \left[\log x\right]_1^t = \lim_{t\to\infty} \log t = \infty$

$\Rightarrow$ $\int_0^\infty \frac{dx}{x}$ is divergent.

(ii) By definition, $\int_1^\infty \frac{dx}{x^{3/2}} = \lim_{t\to\infty} \int_1^t x^{-3/2}\, dx = \lim_{t\to\infty} \left[\frac{x^{-1/2}}{-\frac{1}{2}}\right]_1^t$

$$= \lim_{t\to\infty} \left[\frac{-2}{\sqrt{x}}\right]_1^t = \lim_{t\to\infty} \left(\frac{-2}{\sqrt{t}} + 2\right) = 0 + 2 = 2 \text{ which is finite.}$$

$\Rightarrow$ $\int_1^\infty \frac{dx}{x^{3/2}}$ is convergent and its value is 2.

Example 2. *Examine for convergence the improper integrals:*

(i) $\int_0^\infty e^{-mx}\, dx\ (m > 0)$ (ii) $\int_a^\infty \frac{x}{1+x^2}\, dx$

(iii) $\int_0^\infty \sin x\, dx$ (iv) $\int_0^\infty \frac{dx}{(1+x)^3}$

Sol. (i) By definition, $\int_0^\infty e^{-mx}\, dx = \lim_{t\to\infty} \int_0^t e^{-mx}\, dx = \lim_{t\to\infty} \left[\frac{e^{-mx}}{-m}\right]_0^t$

$$= \lim_{t\to\infty} -\frac{1}{m}\,(e^{-mt} - 1)$$

$$= -\frac{1}{m}(0 - 1) = \frac{1}{m} \text{ which is finite.}$$

$\Rightarrow$ $\int_0^\infty e^{-mx}\, dx$ is convergent and its value is $\frac{1}{m}$.

(ii) By definition,

$$\int_a^\infty \frac{x}{1+x^2}\, dx = \lim_{t\to\infty} \int_a^t \frac{x}{1+x^2}\, dx = \lim_{t\to\infty} \int_a^t \frac{1}{2}\left(\frac{2x}{1+x^2}\right) dx$$

$$= \lim_{t\to\infty} \left[\frac{1}{2}\log(1+x^2)\right]_a^t = \lim_{t\to\infty} \tfrac{1}{2}[\log(1+t^2) - \log(1+a^2)] = \infty.$$

$\Rightarrow$ $\int_a^\infty \frac{x}{1+x^2}\, dx$ is divergent.

(iii) $\int_0^{\infty} \sin x \, dx = \lim_{t \to \infty} \int_0^t \sin x \, dx = \lim_{t \to \infty} [-\cos x]_0^t = \lim_{t \to \infty} (1 - \cos t)$

which does not exist uniquely since cos t oscillates between -1 and $+1$ when $t \to \infty$.

$\Rightarrow$ $\int_0^{\infty} \sin x \, dx$ oscillates.

(iv) $$\int_0^{\infty} \frac{dx}{(1+x)^3} = \lim_{t \to \infty} \int_0^t (1+x)^{-3} \, dx = \lim_{t \to \infty} \left[\frac{(1+x)^{-2}}{-2} \right]_0^t$$

$$= \lim_{t \to \infty} -\frac{1}{2}\left[\frac{1}{(1+t)^2} - 1 \right] = -\frac{1}{2}(0-1) = \frac{1}{2} \text{ which is finite.}$$

$\Rightarrow$ $\int_0^{\infty} \frac{dx}{(1+x)^3}$ is convergent and its value is $\frac{1}{2}$.

Example 3. *Examine for convergence the improper integrals:*

(i) $\int_2^{\infty} \frac{2x^2}{x^4 - 1} dx$ (ii) $\int_1^{\infty} \frac{x}{(1+2x)^3} dx$

Sol. (i) $$\int_2^{\infty} \frac{2x^2}{x^4 - 1} dx = \lim_{t \to \infty} \int_2^t \frac{(x^2+1)+(x^2-1)}{(x^2+1)(x^2-1)} dx$$

$$= \lim_{t \to \infty} \int_2^t \left(\frac{1}{x^2-1} + \frac{1}{x^2+1} \right) dx = \lim_{t \to \infty} \left[\frac{1}{2} \log \frac{x-1}{x+1} + \tan^{-1} x \right]_2^t$$

$$= \lim_{t \to \infty} \left[\frac{1}{2} \log \frac{t-1}{t+1} + \tan^{-1} t - \frac{1}{2} \log \frac{1}{3} - \tan^{-1} 2 \right]$$

$$= \frac{1}{2} \lim_{t \to \infty} \log \frac{1 - \frac{1}{t}}{1 + \frac{1}{t}} + \frac{\pi}{2} + \frac{1}{2} \log 3 - \tan^{-1} 2$$

$$= \frac{1}{2} \log 1 + \frac{\pi}{2} + \frac{1}{2} \log 3 - \tan^{-1} 2$$

$$= \frac{\pi}{2} + \frac{1}{2} \log 3 - \tan^{-1} 2 \text{ which is finite.}$$

$\Rightarrow$ $\int_2^{\infty} \frac{2x^2}{x^4 - 1} dx$ is convergent and its value is $\frac{\pi}{2} + \frac{1}{2} \log 3 - \tan^{-1} 2$.

(ii) $\int_1^{\infty} \frac{x}{(1+2x)^3}\,dx = \lim_{t\to\infty}\int_1^t \frac{x}{(1+2x)^3}\,dx = \lim_{t\to\infty}\int_1^t \frac{\frac{1}{2}(1+2x)-\frac{1}{2}}{(1+2x)^3}\,dx$

$$= \lim_{t\to\infty}\int_1^t \left[\frac{1}{2}(1+2x)^{-2} - \frac{1}{2}(1+2x)^{-3}\right] dx$$

$$= \lim_{t\to\infty}\left[\frac{1}{2}\cdot\frac{(1+2x)^{-1}}{-1\times 2} - \frac{1}{2}\cdot\frac{(1+2x)^{-2}}{-2\times 2}\right]_1^t$$

$$= \lim_{t\to\infty}\left[\frac{-1}{4(1+2x)} + \frac{1}{8(1+2x)^2}\right]_1^t$$

$$= \lim_{t\to\infty}\left[\frac{-1}{4(1+2t)} + \frac{1}{8(1+2t)^2} + \frac{1}{12} - \frac{1}{72}\right]$$

$$= 0 + 0 + \frac{1}{12} - \frac{1}{72} = \frac{5}{72} \text{ which is finite.}$$

$\Rightarrow$ $\int_1^{\infty} \frac{x}{(1+2x)^3}\,dx$ is convergent and its value is $\frac{5}{72}$.

Example 4. *Examine for convergence the integrals:*

(i) $\int_1^{\infty} \frac{dx}{(1+x)\sqrt{x}}$ (ii) $\int_0^{\infty} e^{-x} \sin x\, dx$

Sol. (i) $\int_1^{\infty} \frac{dx}{(1+x)\sqrt{x}} = \lim_{t\to\infty}\int_1^t \frac{dx}{(1+x)\sqrt{x}}$

Put $\sqrt{x} = z$ so that $\frac{1}{2\sqrt{x}}\,dx = dz$

When $x = 1$, $z = 1$; when $x = t$, $z = \sqrt{t}$

$\therefore$ $\int_1^{\infty} \frac{dx}{(1+x)\sqrt{x}} = \lim_{t\to\infty}\int_1^{\sqrt{t}} \frac{2dz}{1+z^2} = \lim_{t\to\infty}\left[2\tan^{-1} z\right]_1^{\sqrt{t}}$

$$= \lim_{t\to\infty} 2\left[\tan^{-1}\sqrt{t} - \tan^{-1} 1\right] = 2\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{\pi}{2} \text{ which is finite.}$$

$\Rightarrow$ $\int_1^{\infty} \frac{dx}{(1+x)\sqrt{x}}$ is convergent and its value is $\frac{\pi}{2}$.

(ii) $\int_0^{\infty} e^{-x} \sin x\, dx = \lim_{t\to\infty}\int_0^t e^{-x} \sin x\, dx$

$$= \lim_{t\to\infty}\left[\frac{e^{-x}}{(-1)^2 + 1^2}(-1\sin x - 1\cos x)\right]_0^t$$

$$\left[\because \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\right]$$

$$= \lim_{t \to \infty} \left[-\frac{1}{2} e^{-x} (\sin x + \cos x) \right]_0^t = \lim_{t \to \infty} -\frac{1}{2} [e^{-t} (\sin t + \cos t) - 1]$$

$$= -\frac{1}{2} [(0 \times \text{a finite quantity}) - 1] = \frac{1}{2} \text{ which is finite.}$$

$\Rightarrow \int_0^\infty e^{-x} \sin x \, dx$ is convergent and its value is $\frac{1}{2}$.

Example 5. *Examine the convergence of the integrals:*

(i) $\int_1^\infty \frac{dx}{x(x+1)}$ (ii) $\int_1^\infty \frac{\tan^{-1} x}{x^2} dx$

Sol. (i) $\int_1^\infty \frac{dx}{x(x+1)} = \lim_{t \to \infty} \int_1^t \frac{dx}{x(x+1)} = \lim_{t \to \infty} \int_1^t \left(\frac{1}{x} - \frac{1}{x+1} \right) dx$ [Partial Fractions]

$$= \lim_{t \to \infty} [\log x - \log (x+1)]_1^t = \lim_{t \to \infty} \left[\log \frac{x}{x+1} \right]_1^t$$

$$= \lim_{t \to \infty} \left[\log \frac{t}{t+1} - \log \frac{1}{2} \right] = \lim_{t \to \infty} \left(\log \frac{1}{1+\frac{1}{t}} \right) + \log 2$$

$$= \log 1 + \log 2 = \log 2 \text{ which is finite.}$$

$\Rightarrow \int_1^\infty \frac{dx}{x(x+1)}$ is convergent and its value is log 2.

(ii) $\int_1^\infty \frac{\tan^{-1} x}{x^2} dx = \lim_{t \to \infty} \int_1^t \frac{\tan^{-1} x}{x^2} dx$

Put $x = \tan \theta$, so that $dx = \sec^2 \theta \, d\theta$

$$\int \frac{\tan^{-1} x}{x^2} dx = \int \frac{\theta}{\tan^2 \theta} \sec^2 \theta \, d\theta = \int \theta \operatorname{cosec}^2 \theta \, d\theta$$

$$= \theta(-\cot \theta) - \int 1(-\cot \theta) \, d\theta$$

$$= -\theta \cot \theta + \log \sin \theta = -\frac{\tan^{-1} x}{x} + \log \frac{x}{\sqrt{1+x^2}}$$

$$\therefore \int_1^\infty \frac{\tan^{-1} x}{x^2} dx = \lim_{t \to \infty} \left[-\frac{\tan^{-1} x}{x} + \log \frac{x}{\sqrt{1+x^2}} \right]_1^t$$

$$= \lim_{t \to \infty} \left[-\frac{\tan^{-1} t}{t} + \log \frac{t}{\sqrt{1+t^2}} + \tan^{-1} 1 - \log \frac{1}{\sqrt{2}} \right]$$

$$= 0 + \lim_{t \to \infty} \log \frac{1}{\sqrt{\frac{1}{t^2} + 1}} + \frac{\pi}{4} + \frac{1}{2} \log 2$$

$$= \log 1 + \frac{\pi}{4} + \frac{1}{2} \log 2 = \frac{\pi}{4} + \frac{1}{2} \log 2 \text{ which is finite.}$$

$\Rightarrow \quad \int_0^{\infty} \frac{\tan^{-1} x}{x^2} dx$ is convergent and its value is $\frac{\pi}{4} + \frac{1}{2} \log 2$.

Example 6. *Examine the convergence of the integrals:*

(i) $\int_{-\infty}^{0} \frac{dx}{p^2 + q^2 x^2}$ (ii) $\int_{-\infty}^{0} e^{-x} dx$

Sol. (i) $\int_{-\infty}^{0} \frac{dx}{p^2 + q^2x^2} = \lim_{t \to -\infty} \int_t^0 \frac{dx}{q^2\left(\frac{p^2}{q^2} + x^2\right)} = \lim_{t \to -\infty} \left[\frac{1}{q^2} \cdot \frac{1}{p/q} \tan^{-1} \frac{x}{p/q}\right]_t^0$

$$= \lim_{t \to -\infty} \frac{1}{pq}\left[0 - \tan^{-1} \frac{qt}{p}\right]$$

$$= -\frac{1}{pq}\left(-\frac{\pi}{2}\right) = \frac{\pi}{2pq} \text{ which is finite.}$$

$\Rightarrow \quad \int_{-\infty}^{0} \frac{dx}{p^2 + q^2x^2}$ is convergent and its value is $\frac{\pi}{2pq}$.

(ii) $\int_{-\infty}^{0} e^{-x} dx = \lim_{t \to -\infty} \int_t^0 e^{-x}\, dx = \lim_{t \to -\infty} \left[-e^{-x}\right]_t^0 = \lim_{t \to -\infty} (-1 + e^{-t}) = -1 + \infty = \infty$

$\Rightarrow \quad \int_{-\infty}^{0} e^{-x}\, dx$ is divergent and diverges to $+\infty$.

Example 7. *Examine the convergence of the integrals:*

(i) $\int_{-\infty}^{\infty} e^{-x}\, dx$ (ii) $\int_{-\infty}^{\infty} \frac{dx}{1 + x^2}$ (iii) $\int_{-\infty}^{\infty} \frac{dx}{e^x + e^{-x}}$

Sol. (i) $\int_{-\infty}^{\infty} e^{-x}\, dx = \int_{-\infty}^{0} e^{-x}\, dx + \int_0^{\infty} e^{-x}\, dx = \lim_{t_1 \to -\infty} \int_{t_1}^{0} e^{-x}\, dx + \lim_{t_2 \to \infty} \int_0^{t_2} e^{-x}\, dx$

$$= \lim_{t_1 \to -\infty} \left[-e^{-x}\right]_{t_1}^{0} + \lim_{t_2 \to \infty} \left[-e^{-x}\right]_0^{t_2}$$

$$= \lim_{t_1 \to -\infty} (-1 + e^{-t_1}) + \lim_{t_2 \to \infty} (-e^{-t_2} + 1) = (-1 + \infty) + (0 + 1) = \infty$$

$\Rightarrow \quad \int_{-\infty}^{\infty} e^{-x}\, dx$ is divergent and diverges to ∞.

(ii) $\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2}$

$$= \lim_{t_1 \to -\infty} \int_{t_1}^{0} \frac{dx}{1+x^2} + \lim_{t_2 \to \infty} \int_{0}^{t_2} \frac{dx}{1+x^2}$$

$$= \lim_{t_1 \to -\infty} \left[\tan^{-1} x\right]_{t_1}^{0} + \lim_{t_2 \to \infty} \left[\tan^{-1} x\right]_{0}^{t_2}$$

$$= \lim_{t_1 \to -\infty} \left[-\tan^{-1} t_1\right] + \lim_{t_2 \to \infty} \left[\tan^{-1} t_2\right] = -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2} = \pi \text{ which is finite.}$$

$\Rightarrow$ $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$ is convergent and its value is π.

(iii) $\int_{-\infty}^{\infty} \frac{dx}{e^x + e^{-x}} = \int_{-\infty}^{0} \frac{dx}{e^x + e^{-x}} + \int_{0}^{\infty} \frac{dx}{e^x + e^{-x}}$

$$= \lim_{t_1 \to -\infty} \int_{t_1}^{0} \frac{e^x}{e^{2x}+1} dx + \lim_{t_2 \to \infty} \int_{0}^{t_2} \frac{e^x}{e^{2x}+1} dx$$

Now, $\int \frac{e^x}{e^{2x}+1} dx = \int \frac{dz}{z^2+1}$ where $z = e^x$

$$= \tan^{-1} z = \tan^{-1} e^x$$

$\therefore$ $\int_{-\infty}^{\infty} \frac{dx}{e^x + e^{-x}} = \lim_{t_1 \to -\infty} \left[\tan^{-1} e^x\right]_{t_1}^{0} + \lim_{t_2 \to \infty} \left[\tan^{-1} e^x\right]_{0}^{t_2}$

$$= \lim_{t_1 \to -\infty} \left[\tan^{-1} 1 - \tan^{-1} e^{t_1}\right] + \lim_{t_2 \to \infty} \left[\tan^{-1} e^{t_2} - \tan^{-1} 1\right]$$

$$= \left(\frac{\pi}{4} - \tan^{-1} 0\right) + \left(\tan^{-1} \infty - \frac{\pi}{4}\right) = \frac{\pi}{2} \text{ which is finite.}$$

$\Rightarrow$ $\int_{-\infty}^{\infty} \frac{dx}{e^x + e^{-x}}$ is convergent and its value is $\frac{\pi}{2}$.

Example 8. *Test the convergence of the integrals:*

(i) $\int_0^1 \frac{dx}{\sqrt{x}}$ (ii) $\int_0^1 \frac{dx}{x^2}$ (iii) $\int_1^2 \frac{x}{\sqrt{x-1}} dx$

Sol. (i) 0 is the only point of infinite discontinuity of the integrand on [0, 1].

$\therefore$ $\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{\varepsilon \to 0+} \int_{0+\varepsilon}^{1} x^{-1/2} dx$

$$= \lim_{\varepsilon \to 0+} \left[2\sqrt{x}\right]_{\varepsilon}^{1} = \lim_{\varepsilon \to 0+} 2\left(1 - \sqrt{\varepsilon}\right) = 2 \text{ which is finite.}$$

$\Rightarrow$ $\int_0^1 \frac{dx}{\sqrt{x}}$ is convergent and its value is 2.

(*ii*) 0 is the only point of infinite discontinuity of the integrand on [0, 1].

$$\therefore \int_0^1 \frac{dx}{x^2} = \lim_{\varepsilon \to 0+} \int_{0+\varepsilon}^1 x^{-2}\, dx = \lim_{\varepsilon \to 0+} \left[-\frac{1}{x}\right]_\varepsilon^1 = \lim_{\varepsilon \to 0+} \left(-1 + \frac{1}{\varepsilon}\right) = \infty$$

$\Rightarrow \quad \int_0^1 \frac{dx}{x^2}$ diverges to ∞.

(*iii*) 1 is the only point of infinite discontinuity of the integrand on [1, 2].

$$\therefore \int_1^2 \frac{x}{\sqrt{x-1}}\, dx = \lim_{\varepsilon \to 0+} \int_{1+\varepsilon}^2 \frac{(x-1)+1}{\sqrt{x-1}}\, dx$$

$$= \lim_{\varepsilon \to 0+} \int_{1+\varepsilon}^2 \left(\sqrt{x-1} + \frac{1}{\sqrt{x-1}}\right) dx = \lim_{\varepsilon \to 0+} \left[\frac{2}{3}(x-1)^{3/2} + 2\sqrt{x-1}\right]_{1+\varepsilon}^2$$

$$= \lim_{\varepsilon \to 0+} \left[\frac{2}{3} + 2 - \frac{2}{3}\varepsilon^{3/2} - 2\sqrt{\varepsilon}\right] = \frac{8}{3} \text{ which is finite.}$$

$\Rightarrow \quad \int_1^2 \frac{x}{\sqrt{x-1}}\, dx$ is convergent and its value is $\frac{8}{3}$.

Example 9. *Examine the convergence of the integrals:*

(*i*) $\int_0^2 \frac{dx}{\sqrt{4-x^2}}$ (*ii*) $\int_0^{\pi/2} \frac{\cos x}{\sqrt{1-\sin x}}\, dx$ (*iii*) $\int_0^1 \frac{dx}{x^2 - 3x + 2}$.

Sol. (*i*) 2 is the only point of infinite discontinuity of the integrand on [0, 2].

$$\therefore \int_0^2 \frac{dx}{\sqrt{4-x^2}} = \lim_{\varepsilon \to 0+} \int_0^{2-\varepsilon} \frac{dx}{\sqrt{4-x^2}} = \lim_{\varepsilon \to 0+} \left[\sin^{-1}\frac{x}{2}\right]_0^{2-\varepsilon}$$

$$= \lim_{\varepsilon \to 0+} \left[\sin^{-1}\frac{2-\varepsilon}{2} - \sin^{-1} 0\right] = \sin^{-1} 1 - 0 = \frac{\pi}{2} \text{ which is finite.}$$

$\Rightarrow \quad \int_0^2 \frac{dx}{\sqrt{4-x^2}}$ converges to $\frac{\pi}{2}$.

(*ii*) $\frac{\pi}{2}$ is the only point of infinite discontinuity of the integrand on $\left[0, \frac{\pi}{2}\right]$.

$$\therefore \int_0^{\pi/2} \frac{\cos x}{\sqrt{1-\sin x}}\, dx = \lim_{\varepsilon \to 0+} \int_0^{\frac{\pi}{2}-\varepsilon} -(1-\sin x)^{-1/2}(-\cos x)\, dx$$

$$= \lim_{\varepsilon \to 0+} \left[-2\sqrt{1-\sin x}\right]_0^{\frac{\pi}{2}-\varepsilon}$$

$$= \lim_{\varepsilon \to 0+} -2\left[\sqrt{1-\sin\left(\frac{\pi}{2}-\varepsilon\right)} - 1\right] = -2\left[\sqrt{1-\sin\frac{\pi}{2}} - 1\right] = 2$$

$\Rightarrow \quad \int_0^{\pi/2} \frac{\cos x}{\sqrt{1-\sin x}}\, dx$ converges to 2.

(*iii*) 1 is the only point of infinite discontinuity of the integrand on [0, 1].

$$\therefore \int_0^1 \frac{dx}{x^2 - 3x + 2} = \lim_{\varepsilon \to 0+} \int_0^{1-\varepsilon} \frac{dx}{(1-x)(2-x)} = \lim_{\varepsilon \to 0+} \int_0^{1-\varepsilon} \left(\frac{1}{1-x} - \frac{1}{2-x} \right) dx$$

$$= \lim_{\varepsilon \to 0+} \left[-\log(1-x) + \log(2-x) \right]_0^{1-\varepsilon}$$

$$= \lim_{\varepsilon \to 0+} \left[\log \frac{2-x}{1-x} \right]_0^{1-\varepsilon} = \lim_{\varepsilon \to 0+} \left[\log \frac{1+\varepsilon}{\varepsilon} - \log 2 \right]$$

$$= \lim_{\varepsilon \to 0+} \log\left(1 + \frac{1}{\varepsilon}\right) - \log 2 = \log \infty - \log 2 = \infty.$$

$\Rightarrow \quad \int_0^1 \frac{dx}{x^2 - 3x + 2}$ diverges to ∞.

Example 10. *Examine the convergence of the integrals:*

(*i*) $\int_{-1}^{1} \frac{dx}{x^2}$ (*ii*) $\int_a^{3a} \frac{dx}{(x-2a)^2}$

Sol. (*i*) The integrand becomes infinite at $x = 0$ and $-1 < 0 < 1$

$$\therefore \int_{-1}^{1} \frac{dx}{x^2} = \int_{-1}^{0} \frac{dx}{x^2} + \int_0^1 \frac{dx}{x^2} = \lim_{\varepsilon_1 \to 0+} \int_1^{0-\varepsilon_1} \frac{dx}{x^2} + \lim_{\varepsilon_2 \to 0+} \int_{0+\varepsilon_2}^{1} \frac{dx}{x^2}$$

so that 0 enclosed within $(-\varepsilon_1, \varepsilon_2)$ is excluded.

$$= \lim_{\varepsilon_1 \to 0+} \left[-\frac{1}{x} \right]_{-1}^{-\varepsilon_1} + \lim_{\varepsilon_2 \to 0+} \left[-\frac{1}{x} \right]_{\varepsilon_2}^{1}$$

$$= \lim_{\varepsilon_2 \to 0+} \left(\frac{1}{\varepsilon_1} - 1 \right) + \lim_{\varepsilon_2 \to 0+} \left(-1 + \frac{1}{\varepsilon_2} \right) = (\infty - 1) + (-1 + \infty) = \infty$$

$\Rightarrow \quad \int_{-1}^{1} \frac{dx}{x^2}$ diverges to $+\infty$.

(*ii*) The integrand becomes infinite at $x = 2a$ and $a < 2a < 3a$.

$$\therefore \int_a^{3a} \frac{dx}{(x-2a)^2} = \int_a^{2a} \frac{dx}{(x-2a)^2} + \int_{2a}^{3a} \frac{dx}{(x-2a)^2}$$

$$= \lim_{\varepsilon_1 \to 0+} \int_a^{2a-\varepsilon_1} \frac{dx}{(x-2a)^2} + \lim_{\varepsilon_2 \to 0+} \int_{2a+\varepsilon_2}^{3a} \frac{dx}{(x-2a)^2}$$

$$= \lim_{\varepsilon_1 \to 0+} \left[\frac{-1}{x-2a} \right]_a^{2a-\varepsilon_1} + \lim_{\varepsilon_1 \to 0+} \left[\frac{-1}{x-2a} \right]_{2a+\varepsilon_2}^{3a}$$

$$= \lim_{\varepsilon_1 \to 0+} \left(\frac{1}{\varepsilon_1} - \frac{1}{a} \right) + \lim_{\varepsilon_2 \to 0+} \left(-\frac{1}{a} + \frac{1}{\varepsilon_2} \right) = \left(\infty - \frac{1}{a} \right) + \left(-\frac{1}{a} + \infty \right) = \infty$$

$\Rightarrow \quad \int_a^{3a} \frac{dx}{(x-2a)^2}$ diverges to ∞.

Example 11. *Examine the convergence of the integrals:*

(*i*) $\int_{-a}^{a} \frac{x}{\sqrt{a^2 - x^2}}\, dx$ (*ii*) $\int_{0}^{\pi} \frac{dx}{\sin x}$.

Sol. (*i*) Both the end points $-a$ and a are points of infinite discontinuity of the integrand on $[-a, a]$.

$$\therefore \int_{-a}^{a} \frac{x}{\sqrt{a^2 - x^2}}\, dx = \int_{-a}^{0} \frac{x}{\sqrt{a^2 - x^2}}\, dx + \int_{0}^{a} \frac{x}{\sqrt{a^2 - x^2}}\, dx$$

$$= \lim_{\varepsilon_1 \to 0+} \int_{-a+\varepsilon_1}^{0} -\frac{1}{2}(a^2 - x^2)^{-1/2}(-2x)\, dx$$

$$+ \lim_{\varepsilon_2 \to 0+} \int_{0}^{a-\varepsilon_2} -\frac{1}{2}(a^2 - x^2)^{-1/2}(-2x)\, dx$$

$$= \lim_{\varepsilon_1 \to 0+} \left[-\sqrt{a^2 - x^2}\right]_{-a+\varepsilon_1}^{0} + \lim_{\varepsilon_2 \to 0+} \left[-\sqrt{a^2 - x^2}\right]_{0}^{a-\varepsilon_2}$$

$$= \lim_{\varepsilon_1 \to 0+} \left[-a + \sqrt{\varepsilon_1(2a - \varepsilon_1)}\right] + \lim_{\varepsilon_2 \to 0+} \left[-\sqrt{\varepsilon_2(2a - \varepsilon_2)} + a\right]$$

$$= -a + a = 0.$$

$$\therefore \quad \int_{-a}^{a} \frac{x}{\sqrt{a^2 - x^2}}\, dx \text{ converges to } 0.$$

(*ii*) Both the end points 0 and π are points of infinite discontinuity of the integrand on $[0, \pi]$.

$$\therefore \int_{0}^{\pi} \frac{dx}{\sin x} = \int_{0}^{\pi/2} \operatorname{cosec} x\, dx + \int_{\pi/2}^{\pi} \operatorname{cosec} x\, dx$$

$$= \lim_{\varepsilon_1 \to 0+} \int_{0+\varepsilon_1}^{\pi/2} \operatorname{cosec} x\, dx + \lim_{\varepsilon_2 \to 0+} \int_{\pi/2}^{\pi-\varepsilon_2} \operatorname{cosec} x\, dx$$

$$= \lim_{\varepsilon_1 \to 0+} \left[\log \tan \frac{x}{2}\right]_{\varepsilon_1}^{\pi/2} + \lim_{\varepsilon_2 \to 0+} \left[\log \tan \frac{x}{2}\right]_{\pi/2}^{\pi-\varepsilon_2}$$

$$= \lim_{\varepsilon_1 \to 0+} \left[\log \tan \frac{\pi}{4} - \log \tan \frac{\varepsilon_1}{2}\right] + \lim_{\varepsilon_2 \to 0+} \left[\log \tan \left(\frac{\pi}{2} - \frac{\varepsilon_2}{2}\right) - \log \tan \frac{\pi}{4}\right]$$

$$= 0 - (-\infty) + \infty - 0 = \infty$$

$$\Rightarrow \quad \int_{0}^{\pi} \frac{dx}{\sin x} \text{ diverges to } \infty.$$

TEST YOUR KNOWLEDGE

Examine the convergence of the following improper integrals:

1. $\int_0^\infty \frac{dx}{1+x^2}$
2. $\int_1^\infty \frac{dx}{\sqrt{x}}$
3. $\int_0^\infty \frac{dx}{x^2+4a^2}$
4. $\int_0^\infty e^{2x}\,dx$
5. $\int_3^\infty \frac{dx}{(x-2)^2}$
6. $\int_0^\infty \frac{dx}{(1+x)^{2/3}}$
7. $\int_{\sqrt{2}}^\infty \frac{dx}{x\sqrt{x^2-1}}$
8. $\int_1^\infty \frac{x}{(1+x)^3}\,dx$
9. $\int_1^\infty xe^{-x}\,dx$
10. $\int_0^\infty xe^{-x^2}\,dx$
11. $\int_2^\infty \frac{dx}{x\log x}$
12. $\int_0^\infty e^{-ax}\cos bx\,dx$
13. $\int_0^\infty e^{-\sqrt{x}}\,dx$
14. $\int_{-\infty}^0 e^{2x}\,dx$
15. $\int_{-\infty}^0 \frac{x}{1+x^2}\,dx$
16. $\int_{-\infty}^\infty \frac{dx}{(1+x^2)^2}$
17. $\int_{-\infty}^\infty \frac{x}{1+x^2}\,dx$
18. $\int_{-\infty}^\infty \frac{dx}{x^2+2x+2}$
19. $\int_1^2 \frac{dx}{\sqrt{x-1}}$
20. $\int_1^2 \frac{dx}{x\sqrt{x^2-1}}$
21. $\int_2^3 \frac{x-1}{\sqrt{x-2}}\,dx$
22. $\int_0^1 \log x\,dx$
23. $\int_1^2 \frac{dx}{x\log x}$
24. $\int_0^e \frac{dx}{x(\log x)^3}$
25. $\int_0^a \frac{dx}{\sqrt{a-x}}$
26. $\int_1^2 \frac{dx}{2-x}$
27. $\int_0^1 \frac{dx}{x^2-1}$
28. $\int_0^{\pi/2} \tan\theta\,d\theta$
29. $\int_0^{2a} \frac{dx}{(x-a)^2}$
30. $\int_0^4 \frac{dx}{x(4-x)}$
31. $\int_0^2 \frac{dx}{2x-x^2}$
32. $\int_0^\pi \frac{dx}{1+\cos x}$.

Answers

1. Converges to $\frac{\pi}{2}$
2. Divergent
3. Converges to $\frac{\pi}{4a}$
4. Divergent
5. Converges to 1
6. Divergent
7. Converges to $\frac{\pi}{4}$
8. Converges to $\frac{3}{8}$
9. Converges to $\frac{2}{e}$

10. Converges to $\frac{1}{2}$

11. Divergent

12. Converges to $\frac{a}{a^2+b^2}$

13. Converges to 2

14. Converges to $\frac{1}{2}$

15. Divergent

16. Converges to $\frac{\pi}{2}$

17. Divergent

18. Converges to π

19. Converges to 2

20. Converges to $\frac{\pi}{3}$

21. Converges to $\frac{8}{3}$

22. Converges to -1

23. Divergent

24. Converges to $-\frac{1}{2}$

25. Converges to $2\sqrt{a}$

26. Divergent

27. Divergent

28. Divergent

29. Divergent

30. Divergent

31. Divergent

32. Divergent.

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