Taylor Theorem (notes)

I: If a function $f(x)$ defined on $[a, a+h]$ is such that

(i) $f^{n-1}(x)$ is continuous in $[a, a+h]$

(ii) $f^{n}(x)$ exist in $(a, a+h)$

then there exists at least one real number θ between 0 and 1 such that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{n!} f^n(a+\theta h)$$

where $R_n = \frac{h^n}{n!} f^n(a+\theta h)$ is called Lagrange's form of remainder.

II: If we put $a+h = x$ then Taylor's Theorem reads,

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{(x-a)^n}{n!} f^n(a+\theta(x-a))$$

where $R_n = \frac{(x-a)^n}{n!} f^n(a+\theta(x-a))$ is called Lagrange's form of remainder.

Maclaurin's Theorem

If a function $f(x)$ defined on $[0,x]$ is such that
(i) $f^{n-1}(x)$ is continuous in $[0,x]$ and
(ii) $f^n(x)$ exists in $(0,x)$

then there exists at least one real number θ between 0 and 1 such that

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0)$$

$$+ \frac{x^n}{n!} f^n(\theta x),$$

where

$R_n = \frac{x^n}{n!} f^n(\theta x)$ is called lagrange form of remainder.

Q1. Expand $f(x) = \tan^{-1}x$ in powers of $\left(x - \frac{\pi}{4}\right)$ using Taylor Theorem.

Solution $f(x) = \tan^{-1}x$	$f\left(\frac{\pi}{4}\right) = \tan^{-1}\frac{\pi}{4} = 1$
$f'(x) = \frac{1}{(1+x^2)} = (1+x^2)^{-1}$	$f'\left(\frac{\pi}{4}\right) = \frac{1}{\left(1+\frac{\pi^2}{16}\right)}$
$f''(x) = (-1)(1+x^2)^{-2}\cdot(2x)$ $= \frac{-2x}{(1+x^2)^2}$	$f''\left(\frac{\pi}{4}\right) = \frac{-\pi}{2\left(1+\frac{\pi^2}{16}\right)^2}$

$\therefore$ By Taylor Theorem $\left(a = \frac{\pi}{4}\right)$

$$f(x) = f\left(\frac{\pi}{4}\right) + \left(x - \frac{\pi}{4}\right) f'\left(\frac{\pi}{4}\right) + \frac{\left(x - \frac{\pi}{4}\right)^2}{2!} f''\left(\frac{\pi}{4}\right) + \cdots$$

$$= 1 + \left(x - \frac{\pi}{4}\right) \cdot \frac{1}{\left(1 + \frac{\pi^2}{16}\right)} + \frac{\left(x - \frac{\pi}{4}\right)^2}{(2)} \cdot \left(\frac{-\pi}{2\left(1 + \frac{\pi^2}{16}\right)^2}\right) + \cdots$$

$$\tan^{-1} x = \left\{ 1 + \frac{\left(x - \frac{\pi}{4}\right)}{\left(1 + \frac{\pi^2}{16}\right)} - \frac{\pi}{4} \cdot \frac{\left(x - \frac{\pi}{4}\right)^2}{\left(1 + \frac{\pi^2}{16}\right)^2} + \cdots \right\}$$

Q2. Expand $\log(1+x)$ using Maclaurin Theorem

Solution.

$f(x) = \log(1+x)$	$f(0) = \log 1 = 0$
$f'(x) = \frac{1}{(1+x)} = (1+x)^{-1}$	$f'(0) = 1$
$f''(x) = (-1)(1+x)^{-2}$ $= \frac{-1}{(1+x)^2}$	$f''(0) = -1$
$f'''(x) = (-1)(-2)(1+x)^{-3}$ $= \frac{2}{(1+x)^3}$	$f'''(0) = 2$

$\therefore$ By Maclaurin Theorem

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

$$= 0 + \frac{x}{(1)}(1) + \frac{x^2}{(2)}(-1) + \frac{x^3}{(6)}(2) + \cdots$$

$$\therefore \log(1+x) = \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\right)$$

Q7. Expand $f(x) = (2x^3 + 7x^2 + x - 1)$ in powers of $(x-2)$.

Solution Here $a = 2$

$f(x) = (2x^3 + 7x^2 + x - 1)$, $f(2) = 45$

$f'(x) = (6x^2 + 14x + 1)$, $f'(2) = 53$

$f''(x) = (12x + 14)$, $f''(2) = 38$

$f'''(x) = 12$, $f'''(2) = 12$

and all other derivatives are zero.

$\therefore$ By Taylor Theorem

$$f(x) = f(2) + \frac{(x-2)}{1!} f'(2) + \frac{(x-2)^2}{2!} f''(2) + \frac{(x-2)^3}{3!} f'''(2)$$

$$= 45 + \frac{(x-2)}{(1)}(53) + \frac{(x-2)^2}{(2)}(38) + \frac{(x-2)^3}{6}(12)$$

$$\therefore f(x) = \left[45 + 53(x-2) + 19(x-2)^2 + 2(x-2)^3\right]$$

Q4. If $f(x) = (x^3 + 8x^2 + 15x - 24)$, calculate the value of $f\left(\frac{11}{10}\right)$ using Taylor Theorem.

Solution. $f\left(\frac{11}{10}\right) = f(1.1) = f(1 + 0.1)$
$= f(a+h)$

Here $a = 1,\ h = 0.1$

$f(x) = x^3 + 8x^2 + 15x - 24$, $f(1) = 0$

$f'(x) = 3x^2 + 16x + 15$, $f'(1) = 34$

$f''(x) = 6x + 16$, $f''(1) = 22$

$f'''(x) = 6$, $f'''(1) = 6$

$f^{IV}(x) = 0$

By Taylor Theorem

$$f(a+h) = f(a) + \frac{h}{1!} f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a)$$

put $a = 1,\ h = 0.1$

$$f(1.1) = f(1) + \frac{(0.1)}{1!} f'(1) + \frac{(0.1)^2}{2!} f''(1) + \frac{(0.1)^3}{3!} f'''(1)$$

$$= 0 + \frac{(0.1)}{(1)}(34) + \frac{(0.1)^2}{(2)}(22) + \frac{(0.1)^3}{(6)}(6)$$

$$= 0 + 3.4 + 0.11 + 0.001$$

$$= 3.511.$$

practice problems

① Expand the following functions by Maclaurin's Theorem:

(a) $e^{\sin x}$ (b) $\log(1-x)$.

② Expand

(i) $\log x$ in powers of $(x-1)$ using Taylor Theorem.

(ii) $\sin x$ in powers of $\left(x-\frac{\pi}{2}\right)$ using Taylor Theorem

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Indeterminate forms

Def: Let f and g be real valued functions. We know

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}, \quad \lim_{x\to a} g(x) \neq 0$$

But if both $\lim_{x\to a} f(x)=0$, $\lim_{x\to a} g(x)=0$ then we get $\frac{0}{0}$ which is meaningless. Thus $\frac{0}{0}$ is one of the indeterminate forms. Similarly if $\lim_{x\to a} f(x)=\infty$, $\lim_{x\to a} g(x)=\infty$ then we get $\frac{\infty}{\infty}$, which is also indeterminate. The other indeterminate forms are $\infty-\infty$, $0\times\infty$, 0^0, 1^∞ and ∞^0.

L'HOSPITAL RULE: Let f and g be functions such that

(i) $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$

(ii) $f'(a)$, $g'(a)$ exist and $g'(a)\neq 0$

Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

Q1 Evaluate $\lim_{x\to 0}\left(\frac{e^x - e^{-x} - 2x}{x - \sin x}\right)$

Solution: $\lim_{x\to 0}\left(\frac{e^x - e^{-x} - 2x}{x - \sin x}\right)$ $\Big/ \frac{0}{0}$

$= \lim_{x\to 0}\left(\frac{e^x + e^{-x} - 2}{1 - \cos x}\right)$ using L'HOSPITAL RULE $\Big/ \frac{0}{0}$

$$= \lim_{x \to 0} \frac{e^x - e^{-x}}{\sin x} \quad \left| \frac{0}{0} \right.$$

$$= \lim_{x \to 0} \frac{e^x + e^{-x}}{\cos x} = \frac{1+1}{1} = \boxed{2}$$

Q2. Evaluate $\lim_{x \to 0} \left(\frac{x - \sin x}{\tan^2 x} \right)$

Solution $\lim_{x \to 0} \left(\frac{x - \sin x}{\tan^2 x} \right) \quad \left| \frac{0}{0} \right.$

$$= \lim_{x \to 0} \left(\frac{1 - \cos x}{2 \tan x \cdot \sec^2 x} \right) \quad \left| \frac{0}{0} \right.$$

$$= \lim_{x \to 0} \frac{\sin x}{2 [\tan x \cdot 2 \sec x (\sec x \tan x) + \sec^2 x \cdot \sec^2 x]}$$

$$= \lim_{x \to 0} \frac{\sin x}{2 [2 \sec^2 x \tan^2 x + \sec^4 x]} = \frac{0}{2(0+1)}$$

$$= 0.$$

Indeterminate form $\left(\frac{\infty}{\infty} \right)$

Q3 Evaluate $\lim_{x \to \frac{\pi}{2}} \frac{\log (x - \frac{\pi}{2})}{\tan x}.$

Solution $\lim_{x\to\frac{\pi}{2}} \frac{\log\left(x-\frac{\pi}{2}\right)}{\tan x} \quad \Big| \frac{\infty}{\infty}$

$= \lim_{x\to\frac{\pi}{2}} \frac{\frac{1}{\left(x-\frac{\pi}{2}\right)}}{\sec^2 x} = \lim_{x\to\frac{\pi}{2}} \frac{\cos^2 x}{\left(x-\frac{\pi}{2}\right)} \quad \Big| \frac{0}{0}$

$= \lim_{x\to\frac{\pi}{2}} \frac{2\cos x \cdot (-\sin x)}{1}$

$= \frac{2\cos\frac{\pi}{2}\left(-\sin\frac{\pi}{2}\right)}{1} = 0$

Indeterminate form $(\infty - \infty)$

Q4 Evaluate $\lim_{x\to 0} \left(\frac{1}{x} - \cot x\right)$

Solution $\lim_{x\to 0} \left(\frac{1}{x} - \cot x\right) \quad \Big| \infty - \infty$

$= \lim_{x\to 0} \left(\frac{1}{x} - \frac{\cos x}{\sin x}\right)$

$= \lim_{x\to 0} \left(\frac{\sin x - x\cos x}{x\sin x}\right) \quad \Big| \frac{0}{0}$

$= \lim_{x\to 0} \frac{\cancel{\cos x} - (-x\sin x + \cancel{\cos x})}{x\cos x + \sin x}$

$= \lim_{x\to 0} \frac{x\sin x}{(x\cos x + \sin x)} \quad \Big| \frac{0}{0}$

$$= \lim_{x \to 0} \frac{x\cos x + \sin x}{(-x\sin x + \cos x + \cos x)}$$

$$= \frac{0}{2} = 0.$$

Indeterminate form $(0 \times \infty)$

Q5. Evaluate $\lim_{x \to 0} (x \log x)$

Solution $\lim_{x \to 0} (x \log x)$ $\left[0 \times \infty \right.$

$$= \lim_{x \to 0} \frac{\log x}{\left(\frac{1}{x}\right)} \quad \left[\frac{\infty}{\infty} \right.$$

$$= \lim_{x \to 0} \frac{\left(\frac{1}{x}\right)}{\left(\frac{-1}{x^2}\right)} = \lim_{x \to 0} (-x) = 0.$$

Indeterminate forms $0^0, 1^\infty, \infty^0$

Q6. Evaluate $\lim_{x \to a} (x-a)^{x-a}$

Solution $\lim_{x \to a} (x-a)^{x-a}$ $\left[0^0 \right.$

let $y = (x-a)^{x-a}$

$$\log y = (x-a)\log(x-a) = \frac{\log(x-a)}{\frac{1}{(x-a)}}$$

Therefore

$$\lim_{x\to a} \log y = \lim_{x\to a} \frac{\log(x-a)}{\frac{1}{(x-a)}} \quad \Big| \frac{\infty}{\infty}$$

$$= \lim_{x\to a} \frac{\frac{1}{(x-a)}}{\frac{-1}{(x-a)^2}} = \lim_{x\to a} -(x-a) = 0$$

Thus $\lim_{x\to a} \log y = 0$

$$\Rightarrow \lim_{x\to a} y = \lim_{x\to a} (x-a)^{x-a} = e^0 = 1.$$

Q7 Evaluate $\lim_{x\to 0} \left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$

Solution

$$\lim_{x\to 0} \left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}} \quad \Big| 1^\infty$$

Let $y = \left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$

$$\log y = \log\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}} = \frac{1}{x^2} \log\left(\frac{\tan x}{x}\right)$$

Thus $\lim_{x\to 0} \log y = \lim_{x\to 0} \frac{\log\left(\frac{\tan x}{x}\right)}{x^2} \quad \Big| \frac{0}{0}$

$$= \lim_{x\to 0} \frac{\frac{1}{\left(\frac{\tan x}{x}\right)}\left(\frac{x\sec^2 x - \tan x}{x^2}\right)}{2x}$$

$$= \lim_{x\to 0} \left(\frac{x\sec^2 x - \tan x}{2x^2 \tan x}\right) \quad \Big| \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{x(2\sec x \cdot \sec x \tan x) + \cancel{\sec^2 x} - \cancel{\sec^2 x}}{2[x^2 \cdot \sec^2 x + 2x \tan x]}$$

$$= \lim_{x\to 0} \frac{2\cancel{x} \sec^2 x \tan x}{2x^{\cancel{2}} \sec^2 x + 4\cancel{x} \tan x} \quad \Big| \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{2\sec^2 x \tan x}{2x \sec^2 x + 4\tan x} \quad \Big| \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{2[\sec^2 x \cdot \sec^2 x + (2\sec x \cdot \sec x \tan x)\tan x]}{2x(2\sec x) \cdot \sec x \tan x + 2\sec^2 x + 4\sec^2 x}$$

$$= \frac{2(1+0)}{0+6} = \frac{2}{6} = \frac{1}{3}$$

$$\lim_{x\to 0} \log y = \lim_{x\to 0} \log\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}} = \frac{1}{3}$$

$$\Rightarrow \lim_{x\to 0} y = e^{1/3} \quad \text{or} \quad \lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}} = e^{1/3}.$$

practice problems

Evaluate the following limits:

(1) $\lim\limits_{x \to 0} \dfrac{x\cos x - \log(1+x)}{x^2}$

(2) $\lim\limits_{x \to 0} \dfrac{e^x \sin x - x - x^2}{x^2 + x\log(1-x)}$

(3) $\lim\limits_{x \to \frac{\pi}{2}} \left(\dfrac{\tan x}{\tan 3x}\right)$

(4) $\lim\limits_{x \to 0} \left(\dfrac{\operatorname{cosec} x}{\log x}\right)$

(5) $\lim\limits_{x \to \frac{\pi}{2}} (\sec x - \tan x)$

(6) $\lim\limits_{x \to 0} \left(\dfrac{1}{x^2} - \dfrac{\operatorname{cosec} x}{x}\right)$

(7) $\lim\limits_{x \to 0} x\cot x$

(8) $\lim\limits_{x \to 0} \left(\dfrac{\sin x}{x}\right)^{\frac{1}{x}}$

(9) $\lim\limits_{x \to \infty} (1+x)^{1/x}$

(10) $\lim\limits_{x \to 0} (\cot x)^{\frac{1}{\log x}}$

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