3

Partial Differentiation

3.1. FUNCTIONS OF TWO VARIABLES

If three variables x, y, z are so related that the value of z depends upon the values of x and y, then z is called a function of two variables x and y, and this is denoted by $z = f(x, y)$.

z is called the dependent variable while x and y are called independent variables.

For example, the area of a triangle is determined when its base and altitude are known. Thus, area of a triangle is a function of two variables, base and altitude.

(In a similar way, a function of more than two variables can be defined).

Geometrically. Let $z = f(x, y)$ be a function of two independent variables x and y defined for all pairs of values of x and y which belong to an area A of the xy-plane. Then to each point (x, y) of this area corresponds a value of z given by the relation $z = f(x, y)$. Representing all these values (x, y, z) by points in space, we get a surface.

Hence the function z = f(x, y) represents a surface.

3.2. (a) LIMIT

A function $f(x, y)$ is said to tend to the limit l iff corresponding to a positive number ε, however small, there exists another positive number δ such that $| f(x, y) - l | < \varepsilon$ for every point (x, y) within the circle with its centre at (a, b) and radius δ *i.e.*, whenever $0 < (x - a)^2 + (y - b)^2 < \delta^2$.

Alternatively. Function $f(x, y)$ is said to tend to the limit l as $x \to a$ and $y \to b$ if and only if l is independent of the path followed by the point (x, y) as $x \to a$ and $y \to b$ and it is written as $\underset{\substack{x \to a \\ y \to b}}{\text{Lt}} f(x, y) = l$.

Properties of the limits

If $\quad \underset{\substack{x \to a \\ y \to b}}{\text{Lt}} f(x, y) = l \quad$ and $\quad \underset{\substack{x \to a \\ y \to b}}{\text{Lt}} g(x, y) = m$

Then (*i*) $\underset{\substack{x \to a \\ y \to b}}{\text{Lt}} \{f(x, y) \pm g(x, y)\} = l \pm m$

(*ii*) $\underset{\substack{x \to a \\ y \to b}}{\text{Lt}} \{f(x, y)\, g(x, y)\} = lm$

(*iii*) $\underset{\substack{x \to a \\ y \to b}}{\text{Lt}} \left\{\dfrac{f(x, y)}{g(x, y)}\right\} = \dfrac{l}{m} \quad (m \neq 0).$

3.2. (b) CONTINUITY

A function $f(x, y)$ is said to be continuous at a point (a, b) if, for any arbitrarily chosen positive number ε, however small, we can find a corresponding number δ such that $|f(x, y) - f(a, b)| < \varepsilon$ for every point (x, y) within the circle with its centre at (a, b) and radius δ *i.e.*, whenever

$$0 < (x - a)^2 + (y - b)^2 < \delta^2.$$

Alternatively, $f(x, y)$ is said to be continuous at (a, b) if $\underset{\substack{x \to a \\ y \to b}}{\text{Lt}} f(x, y) = f(a, b)$ irrespective of the path along which $x \to a, y \to b$.

It should not be assumed that the path along which the point (x, y) tends to (a, b) is immaterial, because $\underset{x \to a}{\text{Lt}} \{\underset{y \to b}{\text{Lt}} f(x, y)\}$ is not always equal to $\underset{y \to b}{\text{Lt}} \{\underset{x \to a}{\text{Lt}} f(x, y)\}$.

For example consider, $\underset{\substack{x \to 0 \\ y \to 0}}{\text{Lt}} \dfrac{x + y}{x + 2y}$

$$\underset{x \to 0}{\text{Lt}} \left\{ \underset{y \to 0}{\text{Lt}} \frac{x + y}{x + 2y} \right\} = \underset{x \to 0}{\text{Lt}} \frac{x}{x} = \underset{x \to 0}{\text{Lt}}\, 1 = 1$$

$$\underset{y \to 0}{\text{Lt}} \left\{ \underset{x \to 0}{\text{Lt}} \frac{x + y}{x + 2y} \right\} = \underset{y \to 0}{\text{Lt}} \left\{ \frac{y}{2y} \right\} = \underset{y \to 0}{\text{Lt}} \frac{1}{2} = \frac{1}{2}$$

$$\therefore \quad \underset{x \to 0}{\text{Lt}}\ \underset{y \to 0}{\text{Lt}} \frac{x + y}{x + 2y} \neq \underset{y \to 0}{\text{Lt}}\ \underset{x \to 0}{\text{Lt}} \frac{x + y}{x + 2y}$$

Otherwise also we see that

$$\underset{\substack{x \to 0 \\ y \to 0}}{\text{Lt}} \frac{x + y}{x + 2y} \quad \text{let } x \to 0, y \to 0 \text{ along the line } y = mx$$

$$= \underset{x \to 0}{\text{Lt}} \frac{x + mx}{x + 2mx} = \underset{x \to 0}{\text{Lt}} \frac{(1 + m)x}{(1 + 2m)x} = \frac{1 + m}{1 + 2m}$$

Which is different for different values of m

$$\therefore \quad \underset{x \to 0}{\text{Lt}}\ \underset{y \to 0}{\text{Lt}} \frac{x + y}{x + 2y} \neq \underset{y \to 0}{\text{Lt}}\ \underset{x \to 0}{\text{Lt}} \frac{x + y}{x + 2y}$$

However, usually, the limit is the same irrespective of the path of approach.

In what follows, we shall assume that the functions considered are continuous and their partial differential coefficients as defined in the next section, exist.

Properties of continuity of functions : If $f(x, y)$, $g(x, y)$ are continuous at (a, b) then so are the functions

(*i*) $f(x, y) \pm g(x, y)$ (*ii*) $f(x, y)\, g(x, y)$

(*iii*) $\dfrac{f(x, y)}{g(x, y)}$ $g(x, y) \neq 0$.

3.3. (a) PARTIAL DERIVATIVES OF FIRST ORDER

Let $z = f(x, y)$ be a function of two independent variables x and y. If y is kept constant and x alone is allowed to vary, then z becomes a function of x only. The derivative of z, with respect to x, treating y as constant, is called partial derivative of z w.r.t. x and is denoted by

$$\frac{\partial z}{\partial x} \quad \text{or} \quad \frac{\partial f}{\partial x} \quad \text{or} \quad f_x.$$

Thus, $$\frac{\partial z}{\partial x} = \underset{h \to 0}{\text{Lt}} \frac{f(x+h, y) - f(x, y)}{h}$$

Similarly, the derivative of z, with respect to y, treating x as constant, is called partial derivative of z w.r.t. y and is denoted by $\frac{\partial z}{\partial y}$ or $\frac{\partial f}{\partial y}$ or f_y.

Thus, $$\frac{\partial z}{\partial y} = \underset{k \to 0}{\text{Lt}} \frac{f(x, y+k) - f(x, y)}{k}$$

$\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are called **first order partial derivatives of z.**

[**In general,** if z is a function of two or more independent variables, then the **partial derivative of z** w.r.t. **any one** of the independent variables is the **ordinary derivative of z** w.r.t. that variable, **treating all other variables as constant.**]

3.3. (b) GEOMETRICAL MEANING OF PARTIAL DERIVATIVES OF FIRST ORDER

(*P.T.U. May 2011*)

Let $z = f(x, y)$ be a function of two variables x and y. Then by Art. 3.1, it represents a surface S. If $y = k$, a constant, then $y = k$ represents a plane parallel to the zx-plane.

$\therefore$ $z = f(x, y)$ and $y = k$ represent a plane curve C which is the section of S by $y = k$.

$\frac{\partial z}{\partial x}$ represents the slope of tangent to C at (x, k, z).

Thus, $\frac{\partial z}{\partial x}$ gives the slope of the tangent drawn to the curve of intersection of the surface $z = f(x, y)$ and a plane parallel to zx-plane.

Similarly, $\frac{\partial z}{\partial y}$ gives the slope of the tangent drawn to the curve of intersection of the surface $z = f(x, y)$ and a plane parallel to yz-plane.

3.4. PARTIAL DERIVATIVES OF HIGHER ORDER

Since the first order partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are themselves functions of x and y, they can be further differentiated partially w.r.t. x as well as y. These are called second order partial derivatives of z. The usual notations for these second order partial derivatives are:

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} \text{ or } f_{xx}\ ; \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} \text{ or } f_{yy}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} \text{ or } f_{xy}\ ; \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} \text{ or } f_{yx}$$

In general, $$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} \text{ or } f_{xy} = f_{yx}.$$

Note 1. If $z = f(x)$, a function of single independent variable x, we get $\frac{dz}{dx}$.

If $z = f(x, y)$, a function of two independent variables x and y, we get $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Similarly, for a function of more than two independent variables $x_1, x_2, \ldots, x_n$, we get $\frac{\partial z}{\partial x_1}, \frac{\partial z}{\partial x_2}, \ldots, \frac{\partial z}{\partial x_n}$.

Note 2. (*i*) If $z = u + v$, where $u = f(x, y)$, $v = \phi(x, y)$ then z is a function of x and y.

$$\therefore \quad \frac{\partial z}{\partial x} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}; \quad \frac{\partial z}{\partial y} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$$

(*ii*) If $z = uv$, where $u = f(x, y)$, $v = \phi(x, y)$ then $\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(uv) = u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x}$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(uv) = u\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y}$$

(*iii*) If $z = \frac{u}{v}$, where $u = f(x, y)$, $v = \phi(x, y)$ then $\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(\frac{u}{v}\right) = \frac{v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}}{v^2}$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(\frac{u}{v}\right) = \frac{v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}}{v^2}$$

(*iv*) If $z = f(u)$, where $u = \phi(x, y)$ then $\frac{\partial z}{\partial x} = \frac{dz}{du}\cdot\frac{\partial u}{\partial x}$; $\frac{\partial z}{\partial y} = \frac{dz}{du}\cdot\frac{\partial u}{\partial y}$.

ILLUSTRATIVE EXAMPLES

Example 1. *Evaluate the following limits:*

(*i*) $\underset{\substack{x \to 1 \\ y \to 2}}{Lt} \frac{2x^2y}{x^2 + y^2 + 1}$ (*ii*) $\underset{\substack{x \to \infty \\ y \to 2}}{Lt} \frac{xy + 4}{x^2 + 2y^2}$

(*iii*) $f(x, y) = \frac{2x - y}{2x + y}$ *when* $x \to 0, y \to 0$.

Sol. (*i*) $\underset{\substack{x \to 1 \\ y \to 2}}{\text{Lt}} \frac{2x^2y}{x^2 + y^2 + 1} = \underset{x \to 1}{\text{Lt}} \left\{ \underset{y \to 2}{\text{Lt}} \frac{2x^2y}{x^2 + y^2 + 1} \right\}$

$$= \underset{x \to 1}{\text{Lt}} \frac{4x^2}{x^2 + 5} = \frac{4}{6} = \frac{2}{3}$$

$$\underset{\substack{x \to 1 \\ y \to 2}}{\text{Lt}} \frac{2x^2y}{x^2 + y^2 + 1} = \underset{y \to 2}{\text{Lt}} \left\{ \underset{x \to 1}{\text{Lt}} \frac{2x^2y}{x^2 + y^2 + 1} \right\} = \underset{y \to 2}{\text{Lt}} \frac{2y}{y^2 + 2} = \frac{4}{4 + 2} = \frac{2}{3}$$

Hence $\underset{\substack{x \to 1 \\ y \to 2}}{\text{Lt}} \frac{2x^2y}{x^2 + y^2 + 1} = \frac{2}{3}$

(*ii*) $$\underset{\substack{x \to \infty \\ y \to 2}}{\text{Lt}} \frac{xy + 4}{x^2 + 2y^2} = \underset{x \to \infty}{\text{Lt}} \left\{ \underset{y \to 2}{\text{Lt}} \frac{xy + 4}{x^2 + 2y^2} \right\}$$

$$= \underset{x \to \infty}{\text{Lt}} \frac{2x + 4}{x^2 + 8} = \underset{x \to 0}{\text{Lt}} \frac{2 + \frac{4}{x}}{x\left(1 + \frac{8}{x^2}\right)} = \frac{2}{\infty} = 0$$

$$\underset{\substack{x \to \infty \\ y \to 2}}{\text{Lt}} \frac{xy + 4}{x^2 + 2y^2} = \underset{y \to 2}{\text{Lt}} \left\{ \underset{x \to \infty}{\text{Lt}} \frac{xy + 4}{x^2 + 2y^2} \right\}$$

$$= \underset{y \to 2}{\text{Lt}} \left\{ \underset{x \to \infty}{\text{Lt}} \frac{y + \frac{4}{x}}{x\left(1 + \frac{2y^2}{x^2}\right)} \right\} = \underset{y \to 2}{\text{Lt}} \frac{y}{\infty} = \underset{y \to 2}{\text{Lt}} = \frac{2}{\infty} = 0$$

$$\therefore \qquad \underset{\substack{x \to \infty \\ y \to 2}}{\text{Lt}} \frac{xy + 4}{x^2 + 2y^2} = 0.$$

(*iii*)
$$\underset{\substack{x \to 0 \\ y \to 0}}{\text{Lt}} \frac{2x - y}{2x + y} = \underset{x \to 0}{\text{Lt}} \left\{ \underset{y \to 0}{\text{Lt}} \frac{2x - y}{2x + y} \right\} = \underset{x \to 0}{\text{Lt}} \left(\frac{2x}{2x} \right) = 1$$

$$\underset{\substack{x \to 0 \\ y \to 0}}{\text{Lt}} \frac{2x - y}{2x + y} = \underset{y \to 0}{\text{Lt}} \left\{ \underset{x \to 0}{\text{Lt}} \frac{2x - y}{2x + y} \right\} = \underset{y \to 0}{\text{Lt}} \frac{-y}{y} = -1$$

$$\underset{x \to 0}{\text{Lt}} \left\{ \underset{y \to 0}{\text{Lt}} f(x, y) \right\} \neq \underset{y \to 0}{\text{Lt}} \left\{ \underset{x \to 0}{\text{Lt}} f(x, y) \right\}$$

$\therefore$ limit does not exist.

Example 2. *Show that the function $f(x, y)$*

$$f(x, y) = \begin{cases} 2x^2 + y; & (x, y) \neq (1, 2) \\ 0 \quad ; & (x, y) = (1, 2) \end{cases}$$

is discontinuous at (1, 2).

Sol.
$$\underset{\substack{x \to 1 \\ y \to 2}}{\text{Lt}} f(x, y) = \underset{\substack{x \to 1 \\ y \to 2}}{\text{Lt}} (2x^2 + y) = 3$$

But $f(1, 2) = 0$ (given)

$$\therefore \qquad \underset{\substack{x \to 1 \\ y \to 2}}{\text{Lt}} f(x, y) \neq f(1, 2)$$

$\therefore$ function is discontinuous at (1, 2).

Example 3. *Show that the function*

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + 2y^2}; & (x, y) \neq (0, 0) \\ 0 \quad ; & (x, y) = (0, 0) \end{cases}$$

is not continuous at (0, 0) but its partial derivatives f_x, f_y exist at (0, 0).

(P.T.U., Dec. 2004

Sol.
$$\underset{\substack{x \to 0 \\ y \to 0}}{\text{Lt}} f(x, y) = \underset{\substack{x \to 0 \\ y \to 0}}{\text{Lt}} \frac{xy}{x^2 + 2y^2}$$

$\therefore$ Let $x \to 0, y \to 0$ along $y = mx$

$$\therefore \qquad \underset{\substack{x \to 0 \\ y \to 0}}{\text{Lt}} \frac{xy}{x^2 + 2y^2} = \underset{x \to 0}{\text{Lt}} \frac{x(mx)}{x^2 + 2m^2x^2} = \frac{m}{1 + 2m^2}$$

which is different for different values of m

$\therefore$ limit of $f(x, y)$ does not exist at (0, 0)

$\therefore$ function is discontinuous at (0, 0).

For partial derivative at (0, 0)

$$f_x(0, 0) = \left(\frac{\partial f}{\partial x}\right)_{\text{at }(0,0)} = \underset{h \to 0}{\text{Lt}} \frac{f(0+h, 0) - f(0,0)}{h}$$

$$= \underset{h \to 0}{\text{Lt}} \frac{f(h, 0) - f(0,0)}{h} = \underset{h \to 0}{\text{Lt}} \frac{0-0}{h} = 0 \qquad [\because \; h \neq 0]$$

$$f_y(0, 0) = \left(\frac{\partial f}{\partial y}\right)_{\text{at }(0,0)} = \underset{k \to 0}{\text{Lt}} \frac{f(0, 0+k) - f(0,0)}{k}$$

$$= \underset{k \to 0}{\text{Lt}} \frac{f(0, k) - f(0,0)}{k} = \underset{k \to 0}{\text{Lt}} \frac{0-0}{k} = 0 \qquad [\because \; k \neq 0]$$

$\therefore$ Partial derivatives exist at (0, 0).

Example 4. *Compute $f_{xy}(0, 0)$ and $f_{yx}(0, 0)$ for the function*

$$f(x, y) = \begin{cases} \dfrac{xy^3}{x+y^2}; & (x, y) \neq (0,0) \\ 0 \quad ; & (x, y) = (0,0) \end{cases}$$

Also discuss the continuity of f_{yx} and f_{xy} at (0, 0).

Sol.
$$f(x, y) = \begin{cases} \dfrac{xy^3}{x+y^2}; & (x, y) \neq (0,0) \\ 0 \quad ; & (x, y) = (0,0) \end{cases}$$

$$f_x(0, 0) = \underset{h \to 0}{\text{Lt}} \frac{f(h, 0) - f(0,0)}{h} = \underset{h \to 0}{\text{Lt}} \frac{0-0}{h} = 0 \qquad [\because \; h \neq 0]$$

$$f_y(0, 0) = \underset{k \to 0}{\text{Lt}} \frac{f(0, k) - f(0,0)}{k} = \underset{k \to 0}{\text{Lt}} \frac{0-0}{k} = 0 \qquad [\because \; k \neq 0]$$

Now
$$\left.\begin{aligned} f_{xy}(0, 0) &= \underset{h \to 0}{\text{Lt}} \frac{f_y(h, 0) - f_y(0,0)}{h} \\ \text{and} \quad f_{yx}(0, 0) &= \underset{k \to 0}{\text{Lt}} \frac{f_x(0, k) - f_x(0,0)}{k} \end{aligned}\right\} \qquad \ldots(1)$$

To find f_{xy} and f_{yx} at (0, 0), we have to find $f_y(h, 0)$ and $f_x(0, k)$

$$f_y(h, 0) = \underset{k \to 0}{\text{Lt}} \frac{f(h, k) - f(h,0)}{k} = \underset{k \to 0}{\text{Lt}} \frac{\dfrac{hk^3}{h+k^2} - 0}{k} = \underset{k \to 0}{\text{Lt}} \frac{hk^2}{h+k^2} = 0$$

$$f_x(0, k) = \underset{h \to 0}{\text{Lt}} \frac{f(h, k) - f(0,k)}{h} = \underset{h \to 0}{\text{Lt}} \frac{\dfrac{hk^3}{h+k^2} - 0}{h} = \underset{h \to 0}{\text{Lt}} \frac{k^3}{h+k^2} = k \; (\neq 0)$$

From (1), $f_{xy}(0, 0) = \underset{h \to 0}{\text{Lt}} \dfrac{0-0}{h} = 0$

$$f_{yx}(0, 0) = \underset{k \to 0}{\text{Lt}} \frac{k-0}{k} = 1$$

$\therefore$ $f_{xy}(0, 0) = 0$ but $f_{yx}(0, 0) = 1$

$f_{xy}(0, 0) \neq f_{yx}(0, 0)$ $\therefore$ f_{xy} and f_{yx} are not continuous at (0, 0).

Example 5. *(i) If $f(x, y) = x^3y - xy^3$, find* $\left\{\dfrac{1}{\frac{\partial f}{\partial x}} + \dfrac{1}{\frac{\partial f}{\partial y}}\right\}_{\substack{x=1\\y=2}}$.

(ii) If $f(x, y, z) = \begin{vmatrix} x^2 & y^2 & z^2 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$, *prove that* $f_x + f_y + f_z = 0$.

Sol. (i)
$$f(x, y) = x^3y - xy^3$$
$$\frac{\partial f}{\partial x} = 3x^2y - y^3;\ \frac{\partial f}{\partial y} = x^3 - 3xy^2$$
$$\therefore \quad \frac{1}{\frac{\partial f}{\partial x}} + \frac{1}{\frac{\partial f}{\partial y}} = \frac{1}{3x^2y - y^3} + \frac{1}{x^3 - 3xy^2}$$
$$\left\{\frac{1}{\frac{\partial f}{\partial x}} + \frac{1}{\frac{\partial f}{\partial y}}\right\}_{\substack{x=1\\y=2}} = \frac{1}{6-8} + \frac{1}{1-12} = -\frac{1}{2} - \frac{1}{11} = \frac{-13}{22}.$$

(ii)
$$f(x, y, z) = \begin{vmatrix} x^2 & y^2 & z^2 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$$

First of all evaluate the determinant operate $C_2 - C_1$, $C_3 - C_2$

$$f(x, y, z) = \begin{vmatrix} x^2 & y^2 - x^2 & z^2 - y^2 \\ x & y - x & z - y \\ 1 & 0 & 0 \end{vmatrix} = (y - x)(z - y)\begin{vmatrix} x^2 & y + x & z + y \\ x & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix}$$
$$= (y - x)(z - y)\,[y + x - z - y]$$
$$= (x - y)(y - z)(z - x)$$

Now,
$$f_x = \frac{\partial f}{\partial x} = (y - z)\,[(x - y)(-1) + 1\,(z - x)] = (y - z)(y + z - 2x)$$
$$\therefore \quad f_x = (y^2 - z^2) - 2(xy - xz)$$

Similarly,
$$f_y = (z^2 - x^2) - 2(yz - yx)$$
$$f_z = (x^2 - y^2) - 2\,(zx - zy)$$
$$\therefore \quad f_x + f_y + f_z = 0 - 2 \cdot 0 = 0$$

Example 6. *(i) If $z(x + y) = x^2 + y^2$, show that* $\left(\dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y}\right)^2 = 4\left(1 - \dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y}\right)$.

(ii) If $z = e^{ax + by} f(ax - by)$ find $b\dfrac{\partial z}{\partial x} + a\dfrac{\partial z}{\partial y}$ **(P.T.U., May 2006)**

Sol. (i)
$$z = \frac{x^2 + y^2}{x + y} \qquad [z \text{ is symmetrical w.r.t. } x \text{ and } y]$$

$$\frac{\partial z}{\partial x} = \frac{(x+y)\frac{\partial}{\partial x}(x^2+y^2) - (x^2+y^2)\frac{\partial}{\partial x}(x+y)}{(x+y)^2}$$

$$= \frac{(x+y)\cdot 2x - (x^2+y^2)\cdot 1}{(x+y)^2} = \frac{x^2+2xy-y^2}{(x+y)^2}$$

Similarly, $$\frac{\partial z}{\partial y} = \frac{y^2+2xy-x^2}{(x+y)^2}$$

Now, $$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = \left[\frac{2x^2-2y^2}{(x+y)^2}\right]^2 = \frac{4(x+y)^2(x-y)^2}{(x+y)^4} = \frac{4(x-y)^2}{(x+y)^2}$$

$$4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right) = 4\left[1 - \frac{x^2+2xy-y^2}{(x+y)^2} - \frac{y^2+2xy-x^2}{(x+y)^2}\right]$$

$$= 4\left[\frac{x^2+2xy+y^2-x^2-2xy+y^2-y^2-2xy+x^2}{(x+y)^2}\right]$$

$$= \frac{4(x^2-2xy+y^2)}{(x+y)^2} = \frac{4(x-y)^2}{(x+y)^2}$$

$\therefore$ $$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right).$$

(*ii*) $$z = e^{ax+by} f(ax-by)$$

$$\frac{\partial z}{\partial x} = e^{ax+by} f'(ax-by)\, a + ae^{ax+by} f(ax-by)$$

$$\frac{\partial z}{\partial y} = e^{ax+by} f'(ax-by)(-b) + be^{ax+by} f(ax-by)$$

$$b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = abe^{ax+by} f'(ax-by) + abe^{ax+by} f(ax-by)$$

$$- ab\, e^{ax+by} f'(ax-by) + ab\, e^{ax+by} f(ax-by)$$

$$= abe^{ax+by} [f'(ax-by) + f(ax-by) - f'(ax-by) + f(ax-by)]$$

$$= ab\, e^{ax+by} \cdot 2f(ax-by) = 2ab\, e^{ax+by} f(ax-by) = 2ab.\, z.$$

Example 7. (*i*) *If* $z = \log(x^2+xy+y^2)$ *prove that* $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2$ **(P.T.U., Dec. 2013)**

(*ii*) *If* $u = e^{x^2+y^2}$ *prove that* $x\frac{\partial u}{\partial x} + y\frac{\partial v}{\partial y} = 2u \log u$ **(P.T.U., May 2014)**

(*iii*) *If* $z = x\, f(x+y) + y\, g(x+y)$, *show that* $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0.$

Sol. (*i*) $$z = \log(x^2+xy+y^2)$$

$$\frac{\partial z}{\partial x} = \frac{1}{x^2+xy+y^2}(2x+y), \quad \frac{\partial z}{\partial y} = \frac{1}{x^2+xy+y^2}(x+2y)$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{2x^2+xy}{x^2+xy+y^2} + \frac{xy+2y^2}{x^2+xy+y^2} = \frac{2(x^2+xy+y^2)}{x^2+xy+y^2}$$

$$\therefore \qquad x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2.$$

(ii)

$$u = e^{x^2+y^2}$$

$$\frac{\partial u}{\partial x} = e^{x^2+y^2} \,.\, 2x$$

$$\frac{\partial u}{\partial y} = e^{x^2+y^2} \,.\, 2y$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2e^{x^2+y^2}(x^2+y^2)$$

$$= 2u \,.\, \log u \qquad \left|\begin{array}{l} \because \quad e^{x^2+y^2} = u \\ \therefore \quad x^2+y^2 = \log u \end{array}\right.$$

(iii)

$$z = x\, f(x+y) + y\, g(x+y)$$

$$\frac{\partial z}{\partial x} = x\, f'(x+y) + f(x+y) + y\, g'(x+y)$$

$$\frac{\partial^2 z}{\partial x^2} = x\, f''(x+y) + f'(x+y) + f'(x+y) + y\, g''(x+y)$$

$$= x\, f''(x+y) + 2f'(x+y) + y\, g''(x+y)$$

$$\frac{\partial z}{\partial y} = x\, f'(x+y) + y\, g'(x+y) + g(x+y)$$

$$\frac{\partial^2 z}{\partial y^2} = x\, f''(x+y) + y\, g''(x+y) + 1.g'(x+y) + g'(x+y)$$

$$= x\, f''(x+y) + 2g'(x+y) + y\, g''(x+y)$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left[x\, f'(x+y) + y\, g'(x+y) + g(x+y)\right]$$

$$= x\, f''(x+y) + f'(x+y) + y\, g''(x+y) + g'(x+y)$$

Now $\dfrac{\partial^2 z}{\partial x^2} - 2\dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{\partial^2 z}{\partial y^2} = x\, f''(x+y) + 2f'(x+y) + yg''(x+y)$

$$- 2x\, f''(x+y) - 2f'(x+y) - 2yg''(x+y) - 2g'(x+y)$$
$$+ x\, f''(x+y) + 2g'(x+y) + y\, g''(x+y) = 0$$

Hence $\dfrac{\partial^2 z}{\partial x^2} - 2\dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{\partial^2 z}{\partial y^2} = 0$

Example 8. *Prove that if* $f(x, y) = \dfrac{1}{\sqrt{y}} \cdot e^{-\frac{(x-a)^2}{4y}}$, *then* $f_{xy} = f_{yx}$.

Sol. $f(x, y) = \dfrac{1}{\sqrt{y}} \cdot e^{-\frac{(x-a)^2}{4y}} = y^{-\frac{1}{2}}\, e^{-\frac{(x-a)^2}{4y}}$

$$f_x = \frac{\partial f}{\partial x} = y^{-\frac{1}{2}} \cdot e^{-\frac{(x-a)^2}{4y}} \cdot \frac{\partial}{\partial x}\left[-\frac{(x-a)^2}{4y}\right]$$

$$= y^{-\frac{1}{2}} \cdot e^{-\frac{(x-a)^2}{4y}} \cdot \left[-\frac{2(x-a)}{4y}\right] = -\frac{1}{2}\, y^{-\frac{3}{2}}\,(x-a)\, e^{-\frac{(x-a)^2}{4y}}$$

$$f_y = \frac{\partial f}{\partial y} = -\frac{1}{2} y^{-\frac{3}{2}} \cdot e^{-\frac{(x-a)^2}{4y}} + y^{-\frac{1}{2}} \cdot e^{-\frac{(x-a)^2}{4y}} \cdot \frac{\partial}{\partial y}\left[-\frac{(x-a)^2}{4y}\right]$$

$$= e^{-\frac{(x-a)^2}{4y}} \left[-\frac{1}{2} y^{-\frac{3}{2}} + y^{-\frac{1}{2}} \cdot \frac{(x-a)^2}{4y^2}\right] = \frac{1}{4} y^{-\frac{3}{2}} e^{-\frac{(x-a)^2}{4y}} \quad [-2 + y^{-1}(x-a)^2]$$

$$f_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{1}{4} y^{-\frac{3}{2}} \left\{ e^{-\frac{(x-a)^2}{4y}} \cdot \frac{\partial}{\partial x}\left[-\frac{(x-a)^2}{4y}\right] \cdot [-2 + y^{-1}(x-a)^2] + e^{-\frac{(x-a)^2}{4y}} \cdot 2y^{-1}(x-a) \right\}$$

$$= \frac{1}{4} y^{-\frac{3}{2}} \cdot e^{-\frac{(x-a)^2}{4y}} \left\{ -\frac{2(x-a)}{4y} [-2 + y^{-1}(x-a)^2] + 2y^{-1}(x-a) \right\}$$

$$= \frac{1}{4} y^{-\frac{3}{2}} \cdot e^{-\frac{(x-a)^2}{4y}} \cdot \frac{x-a}{y} \left\{ -\frac{1}{2} [-2 + y^{-1}(x-a)^2] + 2 \right\}$$

$$= \frac{1}{4} y^{-\frac{5}{2}} (x-a) e^{-\frac{(x-a)^2}{4y}} \left[3 - \frac{(x-a)^2}{2y}\right]$$

$$f_{yx} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = -\frac{1}{2}(x-a)\left[-\frac{3}{2} y^{-\frac{5}{2}} \cdot e^{-\frac{(x-a)^2}{4y}} + y^{-\frac{3}{2}} \cdot e^{-\frac{(x-a)^2}{4y}} \cdot \frac{(x-a)^2}{4y^2}\right]$$

$$= -\frac{1}{4}(x-a) y^{-\frac{5}{2}} \cdot e^{-\frac{(x-a)^2}{4y}} \left[-3 + \frac{(x-a)^2}{2y}\right]$$

$$= \frac{1}{4} y^{-\frac{5}{2}} (x-a) e^{-\frac{(x-a)^2}{4y}} \left[3 - \frac{(x-a)^2}{2y}\right]$$

$\therefore \quad f_{xy} = f_{yx}.$

Example 9. *If* $u = x^y$, *show that* $\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial^3 u}{\partial x \partial y \partial x}$.

Sol.
$$u = x^y$$

Take logs on both sides

$$\log u = y \log x \qquad \ldots(1)$$

Differentiate (1) partially w.r.t. y,

$$\therefore \quad \frac{1}{u}\frac{\partial u}{\partial y} = \log x \qquad \therefore \quad \frac{\partial u}{\partial y} = x^y \log x = u \log x$$

Differentiate (1) partially w.r.t. x,

$$\frac{1}{u}\frac{\partial u}{\partial x} = \frac{y}{x} \qquad \therefore \quad \frac{\partial u}{\partial x} = x^y \frac{y}{x} = yx^{y-1} \quad \text{or} \quad u\frac{y}{x}$$

Now,
$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}[u \log x] = u \cdot \frac{1}{x} + \log x \cdot \frac{\partial u}{\partial x} = \frac{x^y}{x} + yx^{y-1} \log x$$

$$= yx^{y-1}\log x + x^y \cdot \frac{1}{x} = x^{y-1}(y\log x + 1)$$

$$\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x \partial y}\right) = \frac{\partial}{\partial x}\,[x^{y-1}(y\log x + 1)] \qquad \ldots(2)$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{uy}{x}\right) = \frac{1}{x}\left\{u \cdot 1 + y\frac{\partial u}{\partial y}\right\}$$

$$= \frac{1}{x}\,[x^y + yx^y \log x] = x^{y-1} + yx^{y-1}\log x = x^{y-1}(y\log x + 1)$$

$$\frac{\partial^3 u}{\partial x \partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial y \partial x}\right) = \frac{\partial}{\partial x}\,[x^{y-1}(y\log x + 1)] \qquad \ldots(3)$$

From (2) and (3), $\dfrac{\partial^3 u}{\partial x^2 \partial y} = \dfrac{\partial^3 u}{\partial x \partial y \partial x}$.

Example 10. *Find p, q if $x = \sqrt{a}\,(\sin u + \cos v)$; $y = \sqrt{a}\,(\cos u - \sin v)$, $z = 1 + \sin(u - v)$ where p, q means $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$ respectively.*

Sol. Given

$$x = \sqrt{a}\,(\sin u + \cos v)$$

$$y = \sqrt{a}\,(\cos u - \sin v)$$

$$z = 1 + \sin(u - v)$$

$$x^2 + y^2 = a\{\sin^2 u + \cos^2 v + 2\sin u\cos v + \cos^2 u + \sin^2 v - 2\cos u \sin v\}$$

$$= a\,[2 + 2\sin(u - v)]$$

$$= 2a \cdot z$$

$$\therefore \quad z = \frac{1}{2a}(x^2 + y^2)$$

$$p = \frac{\partial z}{\partial x} = \frac{2x}{2a} = \frac{x}{a}$$

$$q = \frac{\partial z}{\partial y} = \frac{2y}{2a} = \frac{y}{a}.$$

Example 11. *If $\theta = t^n e^{-\frac{r^2}{4t}}$, find the value of n which will make $\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial \theta}{\partial r}\right) = \dfrac{\partial \theta}{\partial t}$.*

(P.T.U., May 2002, Dec. 2011)

Sol.

$$\theta = t^n e^{-\frac{r^2}{4t}}$$

$$\frac{\partial \theta}{\partial r} = t^n \cdot e^{-\frac{r^2}{4t}} \cdot \left(-\frac{2r}{4t}\right) = -\frac{1}{2} r t^{n-1} e^{-\frac{r^2}{4t}}$$

$$\therefore \quad r^2 \frac{\partial \theta}{\partial r} = -\frac{1}{2} r^3 \cdot t^{n-1} e^{-\frac{r^2}{4t}}$$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = -\frac{1}{2} t^{n-1}\left[3r^2 e^{-\frac{r^2}{4t}} + r^3 e^{-\frac{r^2}{4t}}\left(-\frac{2r}{4t}\right)\right] = -\frac{1}{2} t^{n-1} r^2 e^{-\frac{r^2}{4t}}\left[3 - \frac{r^2}{2t}\right]$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = \frac{1}{2} t^{n-1} e^{-\frac{r^2}{4t}}\left(\frac{r^2}{} - 3\right)$$

Also, $$\frac{\partial\theta}{\partial t} = nt^{n-1}e^{-\frac{r^2}{4t}} + t^n e^{-\frac{r^2}{4t}} \cdot \left(\frac{r^2}{4t^2}\right) = t^{n-1} e^{-\frac{r^2}{4t}}\left(n + \frac{r^2}{4t}\right)$$

Since, $$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial\theta}{\partial r}\right) = \frac{\partial\theta}{\partial t}$$ [given]

$$\therefore \quad \frac{1}{2}t^{n-1}e^{-\frac{r^2}{4t}}\left(\frac{r^2}{2t} - 3\right) = t^{n-1}e^{-\frac{r^2}{4t}}\left(n + \frac{r^2}{4t}\right)$$

$$\Rightarrow \quad \frac{r^2}{4t} - \frac{3}{2} = n + \frac{r^2}{4t} \quad \therefore \quad n = -\frac{3}{2}.$$

Example 12. *If* $u = (1 - 2xy + y^2)^{-1/2}$, *prove that* $\frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial u}{\partial x}\right\} + \frac{\partial}{\partial y}\left\{y^2 \frac{\partial u}{\partial y}\right\} = 0.$

Sol. $u = (1 - 2xy + y^2)^{-1/2} = V^{-1/2}$, where $V = 1 - 2xy + y^2$

$$\frac{\partial u}{\partial x} = -\frac{1}{2}V^{-3/2} \cdot \frac{\partial V}{\partial x} = -\frac{1}{2}V^{-3/2}(-2y) = yV^{-3/2}$$

$$\frac{\partial^2 u}{\partial x^2} = y \cdot \frac{\partial}{\partial x}(V^{-3/2}) = y \cdot \left(-\frac{3}{2}\right)V^{-5/2} \cdot \frac{\partial V}{\partial x} = -\frac{3}{2}yV^{-5/2}(-2y) = 3y^2 V^{-5/2}$$

$$\therefore \quad \frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial u}{\partial x}\right\} = (1-x^2) \cdot \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} \cdot \frac{\partial}{\partial x}(1-x^2)$$

$$= (1-x^2) \cdot 3y^2V^{-5/2} + yV^{-3/2}(-2x) = yV^{-3/2}[3yV^{-1}(1-x^2) - 2x] \qquad ...(1)$$

Also, $$\frac{\partial u}{\partial y} = -\frac{1}{2}V^{-3/2}\frac{\partial V}{\partial y} = -\frac{1}{2}V^{-3/2} \cdot (-2x + 2y) = V^{-3/2} \cdot (x - y)$$

$$\frac{\partial^2 u}{\partial y^2} = V^{-3/2} \cdot \frac{\partial}{\partial y}(x-y) + (x-y) \cdot \frac{y}{\partial y}(V^{-3/2})$$

$$= V^{-3/2} \cdot (-1) + (x-y) \cdot \left(-\frac{3}{2}V^{-5/2}\right) \cdot \frac{\partial V}{\partial y}$$

$$= -V^{-3/2} - \frac{3}{2}(x-y)V^{-5/2} \cdot (-2x + 2y) = -V^{-3/2} + 3(x-y)^2 V^{-5/2}$$

$$\therefore \quad \frac{\partial}{\partial y}\left\{y^2 \frac{\partial u}{\partial y}\right\} = y^2 \cdot \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} \cdot \frac{\partial}{\partial y}(y^2)$$

$$= y^2[-V^{-3/2} + 3(x-y)^2 V^{-5/2}] + V^{-3/2}(x-y) \cdot 2y$$

$$= yV^{-3/2}[-y + 3y(x-y)^2 V^{-1} + 2(x-y)]$$

$$= yV^{-3/2}[3y(x-y)^2 V^{-1} + (2x - 3y)] \qquad ...(2)$$

Adding (1) and (2), we have

$$\frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial u}{\partial x}\right\} + \frac{\partial}{\partial y}\left\{y^2\frac{\partial u}{\partial y}\right\} = yV^{-3/2}[3yV^{-1}(1-x^2) - 2x + 3y(x-y)^2V^{-1} + 2x - 3y]$$

$$= yV^{-3/2}[3yV^{-1}(1 - x^2 + x^2 - 2xy + y^2) - 3y]$$

$$= yV^{-3/2}[3yV^{-1}(1 - 2xy + y^2) - 3y]$$

$$= yV^{-3/2}[3y - 3y] \qquad | \because \quad V = 1 - 2xy + y^2$$

Example 13. *If* $u = \log(x^3 + y^3 + z^3 - 3xyz)$, *show that*

(*i*) $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$

(P.T.U., May 2003, Dec. 2004, 2005, May 2007, Jan. 2009, May 2012)

(*ii*) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + 2\frac{\partial^2 u}{\partial y \partial z} + 2\frac{\partial^2 u}{\partial z \partial x} + 2\frac{\partial^2 u}{\partial x \partial y} = \frac{-9}{(x+y+z)^2}$.

Sol. (*i*) $u = \log(x^3 + y^3 + z^3 - 3xyz)$

$$\frac{\partial u}{\partial x} = \frac{3x^2 - 3yz}{x^3 + y^3 + z^3 - 3xyz}; \quad \frac{\partial u}{\partial y} = \frac{3y^2 - 3zx}{x^3 + y^3 + z^3 - 3xyz}$$

$$\frac{\partial u}{\partial z} = \frac{3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz}$$

Adding, $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3(x^2 + y^2 + z^2 - xy - yz - zx)}{x^3 + y^3 + z^3 - 3xyz} = \frac{3}{x+y+z}$

[$\because$ $x^3 + y^3 + z^3 - 3xyz = (x+y+z)(x^2 + y^2 + z^2 - xy - yz - zx)$]

Now $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) u$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right) = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$= -\frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} = -\frac{9}{(x+y+z)^2} \quad ...(1)$$

(*ii*) $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)$

$$= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial x \partial z} + \frac{\partial^2 u}{\partial y \partial x} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial y \partial z} + \frac{\partial^2 u}{\partial z \partial x} + \frac{\partial^2 u}{\partial z \partial y} + \frac{\partial^2 u}{\partial z^2}$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + 2\frac{\partial^2 u}{\partial y \partial z} + 2\frac{\partial^2 u}{\partial z \partial x} + 2\frac{\partial^2 u}{\partial x \partial y}$$

$$\left[\because \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^2 u}{\partial z \partial y} = \frac{\partial^2 u}{\partial y \partial z}, \frac{\partial^2 u}{\partial x \partial z} = \frac{\partial^2 u}{\partial z \partial x}\right]$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + 2\frac{\partial^2 u}{\partial y \partial z} + 2\frac{\partial^2 u}{\partial z \partial x} + 2\frac{\partial^2 u}{\partial x \partial y} = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$$

[from (1)]

Example 14. *If* $x^x y^y z^z = c$, *show that at* $x = y = z$, $\frac{\partial^2 z}{\partial x \partial y} = -(x \log ex)^{-1}$.

Sol. $x^x y^y z^z = c$ defines z as a function of x and y.

Taking logs on both sides $x \log x + y \log y + z \log z = \log c$

Differentiating partially w.r.t. y, we have

$$y \cdot \frac{1}{y} + 1 \cdot \log y + z \cdot \frac{1}{z} \cdot \frac{\partial z}{\partial y} + 1 \cdot \log z \cdot \frac{\partial z}{\partial y} = 0$$

or

$$1 + \log y + (1 + \log z)\frac{\partial z}{\partial y} = 0 \qquad ...(1)$$

or

$$\left.\begin{aligned} \frac{\partial z}{\partial y} &= -\frac{1+\log y}{1+\log z} \\ \frac{\partial z}{\partial x} &= -\frac{1+\log x}{1+\log z} \end{aligned}\right\} \qquad ...(2)$$

Similarly,

Differentiating (1) partially w.r.t. x, we have

$$\left(\frac{1}{z}\frac{\partial z}{\partial x}\right)\frac{\partial z}{\partial y} + (1+\log z)\frac{\partial^2 z}{\partial x \partial y} = 0 \quad \text{or} \quad \frac{\partial^2 z}{\partial x \partial y} = -\frac{1}{z(1+\log z)}\frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y} \qquad ...(3)$$

When $x = y = z$

From (2), $\qquad \dfrac{\partial z}{\partial y} = -1, \ \dfrac{\partial z}{\partial x} = -1$

From (3), $\dfrac{\partial^2 z}{\partial x \partial y} = -\dfrac{1}{x(1+\log x)}(-1)(-1) = -\dfrac{1}{x(\log e + \log x)} = -\dfrac{1}{x(\log ex)} = -(x \log ex)^{-1}$.

Example 15. (*i*) *If* $V = f(r)$ *and* $r^2 = x^2 + y^2 + z^2$ *prove that* $V_{xx} + V_{yy} + V_{zz} = f''(r) + \frac{2}{r} f'(r)$.

(*ii*) *If* $V = r^m$ *where* $r^2 = x^2 + y^2 + z^2$, *show that* $V_{xx} + V_{yy} + V_{zz} = m(m+1)\, r^{m-2}$.

(P.T.U., Dec. 2006, 2010)

Sol. (*i*) $\qquad V = f(r), \ r^2 = x^2 + y^2 + z^2$

$\therefore \qquad 2r \dfrac{\partial r}{\partial x} = 2x \quad \therefore \quad \dfrac{\partial r}{\partial x} = \dfrac{x}{r}$. Similarly, $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}, \ \dfrac{\partial r}{\partial z} = \dfrac{z}{r}$

$$V_x = \frac{\partial V}{\partial x} = f'(r) \cdot \frac{\partial r}{\partial x} = \frac{x}{r} f'(r)$$

$$V_{xx} = x \cdot \frac{1}{r} \cdot f''(r)\frac{\partial r}{\partial x} + x f'(r)\left(\frac{-1}{r^2}\right)\frac{\partial r}{\partial x} + \frac{f'(r)}{r}$$

$$= \frac{x}{r} f''(r) \cdot \frac{x}{r} - f'(r)\frac{x}{r^2} \cdot \frac{x}{r} + \frac{f'(r)}{r}$$

$$= \frac{x^2}{r^2} f''(r) + f'(r)\left[\frac{1}{r} - \frac{x^2}{r^3}\right]$$

$$= \frac{x^2}{r^2} f''(r) + \frac{1}{r^3}(r^2 - x^2) f'(r)$$

Similarly,

$$V_{yy} = \frac{y^2}{r^2} f''(r) + \frac{r^2 - y^2}{r^3} f'(r)$$

$$V_{zz} = \frac{z^2}{r^2} f''(r) + \frac{r^2 - z^2}{r^3} f'(r)$$

$$\therefore \quad V = \frac{x^2 + y^2 + z^2}{r^2} f''(r) + \frac{3r^2 - x^2 - y^2 - z^2}{r^3} f'(r)$$

$$= \frac{r^2}{r^2} f''(r) + \frac{3r^2 - r^2}{r^3} f'(r)$$

$$= f''(r) + \frac{2}{r} f'(r)$$

(ii) $V = r^m$, $\quad r^2 = x^2 + y^2 + z^2 \quad \therefore \quad \frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial y} = \frac{z}{r}$

$$V_x = m\, r^{m-1} \frac{\partial r}{\partial x} = m\, r^{m-1} \cdot \frac{x}{r} = m\, r^{m-2}\, x$$

$$V_{xx} = m \left\{ r^{m-2} \cdot 1 + x \cdot (m-2)\, r^{m-3} \cdot \frac{\partial r}{\partial x} \right\}$$

$$= m\, r^{m-2} + m(m-2)\, r^{m-3} \cdot \frac{x^2}{r}$$

$$V_{xx} = m\, r^{m-2} + m(m-2)\, r^{m-4} \cdot x^2$$

Similarly, $\quad V_{yy} = m\, r^{m-2} + m(m-2)\, r^{m-4}\, y^2$

$$V_{zz} = m\, r^{m-2} + m\,(m-2)\, r^{m-4}\, z^2$$

$$V_{xx} + V_{yy} + V_{zz} = 3m\, r^{m-2} + m(m-2)\, r^{m-4}\, (x^2 + y^2 + z^2)$$

$$= 3m\, r^{m-2} + m(m-2)\, r^{m-4}\, r^2$$

$$= 3m\, r^{m-2} + m\,(m-2)\, r^{m-2}$$

$$= m\,(m+1)\, r^{m-2}$$

Hence $\quad V_{xx} + V_{yy} + V_{zz} = m\,(m+1)\, r^{m-2}$.

Example 16. *If $x = r\cos\theta$, $y = r\sin\theta$, prove that*

(i) $\frac{\partial r}{\partial x} = \frac{\partial x}{\partial r}$ $\quad$ (ii) $\frac{1}{r} \cdot \frac{\partial x}{\partial \theta} = r \frac{\partial \theta}{\partial x}$ $\quad$ (iii) $\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0.$

(iv) $\frac{\partial^2 r}{\partial x^2} \cdot \frac{\partial^2 r}{\partial y^2} = \left(\frac{\partial^2 r}{\partial x \partial y} \right)^2$ $\quad$ (v) $\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r} \left[\left(\frac{\partial r}{\partial x} \right)^2 + \left(\frac{\partial r}{\partial y} \right)^2 \right]$.

Sol. (i) $\frac{\partial r}{\partial x}$ means $\left(\frac{\partial r}{\partial x} \right)_y$ = the partial derivative of r w.r.t. x, treating y as constant.

$\therefore$ We express r in terms of x and y.

Squaring and adding the given relations, $r^2 = x^2 + y^2$

Differentiating partially w.r.t. x, we get $2r \frac{\partial r}{\partial x} = 2x$ or $\frac{\partial r}{\partial x} = \frac{x}{r}$

$\frac{\partial x}{\partial r}$ means $\left(\frac{\partial x}{\partial r} \right)_\theta$ = the partial derivative of x w.r.t. r treating θ as constant.

$\therefore$ we express x in terms of r and θ.

Thus, $\quad x = r\cos\theta \quad$ (given)

$$\frac{\partial x}{\partial r} = \cos\theta = \frac{x}{r} \qquad \left(\because \cos\theta = \frac{x}{r} \right)$$

$$\therefore \quad \frac{\partial r}{\partial x} = \frac{\partial x}{\partial r}$$

(ii) Expressing x in terms of r and θ, we have $x = r\cos\theta$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta = -y \quad \Rightarrow \quad \frac{1}{r}\frac{\partial x}{\partial \theta} = -\frac{y}{r}$$

Expressing θ in terms of x and y, we have $\tan\theta = \frac{y}{x}$ or $\theta = \tan^{-1}\frac{y}{x}$

$$\frac{\partial\theta}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}\cdot\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2} = \frac{-y}{r^2(\cos^2\theta+\sin^2\theta)} = -\frac{y}{r^2}$$

$$\Rightarrow \qquad r\frac{\partial\theta}{\partial x} = -\frac{y}{r} \qquad \therefore \quad \frac{1}{r}\cdot\frac{\partial x}{\partial\theta} = r\frac{\partial\theta}{\partial x}.$$

(*iii*) Expressing θ in terms of x and y, we have $\tan\theta = \frac{y}{x}$ or $\theta = \tan^{-1}\frac{y}{x}$

$$\frac{\partial\theta}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}\cdot\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2} = -y(x^2+y^2)^{-1}$$

$$\frac{\partial^2\theta}{\partial x^2} = y(x^2+y^2)^{-2}\cdot 2x = \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial\theta}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}}\cdot\frac{1}{x} = \frac{x}{x^2+y^2} = x(x^2+y^2)^{-1}$$

$$\frac{\partial^2\theta}{\partial y^2} = -x(x^2+y^2)^{-2}\cdot 2y = -\frac{2xy}{(x^2+y^2)^2}$$

$$\therefore \qquad \frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0.$$

(*iv*) from (*i*) part $r^2 = x^2 + y^2 \quad \therefore \quad r = (x^2+y^2)^{1/2}$.

$$\frac{\partial r}{\partial x} = \frac{1}{2}(x^2+y^2)^{-1/2}\cdot 2x = \frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{\partial r}{\partial y} = \frac{1}{2}(x^2+y^2)^{-1/2}\cdot 2y = \frac{y}{\sqrt{x^2+y^2}}$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{\sqrt{x^2+y^2}\cdot 1 - x\cdot\frac{1}{2\sqrt{x^2+y^2}}\cdot 2x}{x^2+y^2} = \frac{x^2+y^2-x^2}{(x^2+y^2)^{3/2}} = \frac{y^2}{(x^2+y^2)^{3/2}}$$

$$\frac{\partial^2 r}{\partial y^2} = \frac{\sqrt{x^2+y^2}\cdot 1 - y\cdot\frac{1}{2\sqrt{x^2+y^2}}\cdot 2y}{x^2+y^2} = \frac{x^2+y^2-y^2}{(x^2+y^2)^{3/2}} = \frac{x^2}{(x^2+y^2)^{3/2}}$$

$$\frac{\partial^2 r}{\partial x\partial y} = y\cdot\frac{-1}{2}(x^2+y^2)^{-3/2}\cdot 2x = \frac{-xy}{(x^2+y^2)^{3/2}}.$$

Now $$\frac{\partial^2 r}{\partial x^2}\cdot\frac{\partial^2 r}{\partial y^2} = \frac{y^2}{(x^2+y^2)^{3/2}}\cdot\frac{x^2}{(x^2+y^2)^{3/2}} = \frac{x^2y^2}{(x^2+y^2)^{3/2}} = \left(\frac{\partial^2 r}{\partial x\partial y}\right)^2$$

(*v*) And $$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{y^2}{(x^2+y^2)^{3/2}} + \frac{x^2}{(x^2+y^2)^{3/2}} = \frac{x^2+y^2}{(x^2+y^2)^{3/2}} = \frac{1}{(x^2+y^2)^{1/2}} = \frac{1}{r}$$

$$\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 = \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} = \frac{x^2+y^2}{x^2+y^2} = 1$$

$$\therefore \quad \frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r} \cdot 1 = \frac{1}{r}\left\{\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right\}.$$

Example 17. *If $x = e^{r\cos\theta}\cos(r\sin\theta)$ and $y = e^{r\cos\theta}\sin(r\sin\theta)$, prove that*

$$\frac{\partial x}{\partial r} = \frac{1}{r}\cdot\frac{\partial y}{\partial \theta} \cdot \frac{\partial y}{\partial r} = -\frac{1}{r}\cdot\frac{\partial x}{\partial \theta}.$$

Hence deduce that $\dfrac{\partial^2 x}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial x}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 x}{\partial \theta^2} = 0.$

Sol. $x = e^{r\cos\theta}\cos(r\sin\theta)$

$$\therefore \quad \frac{\partial x}{\partial r} = e^{r\cos\theta}\cdot\cos\theta\cdot\cos(r\sin\theta) - e^{r\cos\theta}\sin(r\sin\theta)\cdot\sin\theta$$

$$= e^{r\cos\theta}[\cos\theta\cos(r\sin\theta) - \sin\theta\sin(r\sin\theta)]$$

$$= e^{r\cos\theta}\cos(\theta + r\sin\theta) \quad ...(1)$$

$$\frac{\partial x}{\partial \theta} = e^{r\cos\theta}\cdot(-r\sin\theta)\cdot\cos(r\sin\theta) - e^{r\cos\theta}\sin(r\sin\theta)\cdot r\cos\theta$$

$$= -re^{r\cos\theta}[\sin\theta\cos(r\sin\theta) + \cos\theta\sin(r\sin\theta)]$$

$$= -re^{r\cos\theta}\sin(\theta + r\sin\theta) \quad ...(2)$$

Also, $y = e^{r\cos\theta}\sin(r\sin\theta)$

$$\therefore \quad \frac{\partial y}{\partial r} = e^{r\cos\theta}\cdot\cos\theta\cdot\sin(r\sin\theta) + e^{r\cos\theta}\cdot\cos(r\sin\theta)\sin\theta$$

$$= e^{r\cos\theta}[\sin\theta\cos(r\sin\theta) + \cos\theta\sin(r\sin\theta)]$$

$$= e^{r\cos\theta}\sin(\theta + r\sin\theta) \quad ...(3)$$

$$\frac{\partial y}{\partial \theta} = e^{r\cos\theta}(-r\sin\theta)\sin(r\sin\theta) + e^{r\cos\theta}\cos(r\sin\theta)\times r\cos\theta$$

$$= re^{r\cos\theta}[\cos\theta\cos(r\sin\theta) - \sin\theta\sin(r\sin\theta)]$$

$$= re^{r\cos\theta}\cos(\theta + r\sin\theta) \quad ...(4)$$

From (1) and (4), $\dfrac{\partial x}{\partial r} = \dfrac{1}{r}\cdot\dfrac{\partial y}{\partial \theta}$ ...(5)

From (2) and (3), $\dfrac{\partial y}{\partial r} = -\dfrac{1}{r}\cdot\dfrac{\partial x}{\partial \theta}$ ...(6)

From (5), $\dfrac{\partial^2 x}{\partial r^2} = -\dfrac{1}{r^2}\cdot\dfrac{\partial y}{\partial \theta} + \dfrac{1}{r}\cdot\dfrac{\partial^2 y}{\partial r\partial\theta}$ ...(7)

From (6), $\dfrac{\partial x}{\partial \theta} = -r\dfrac{\partial y}{\partial r}$

$$\therefore \quad \frac{\partial^2 x}{\partial \theta^2} = -r\frac{\partial^2 y}{\partial\theta\partial r} = -r\frac{\partial^2 y}{\partial r\partial\theta} \quad ...(8)$$

$$\therefore \quad \frac{\partial^2 x}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial x}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2 x}{\partial\theta^2} = -\frac{1}{r^2}\cdot\frac{\partial y}{\partial\theta} + \frac{1}{r}\cdot\frac{\partial^2 y}{\partial r\partial\theta} + \frac{1}{r^2}\cdot\frac{\partial y}{\partial\theta} - \frac{1}{r}\cdot\frac{\partial^2 y}{\partial r\partial\theta} = 0.$$

[Using (5), (7), (8)]

Example 18. *If* $\dfrac{x^2}{a^2+u}+\dfrac{y^2}{b^2+u}+\dfrac{z^2}{c^2+u}=1$, *prove that*

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=2\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right).$$ **(P.T.U., Dec. 2003)**

Sol. Given $$\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}=1 \quad ...(1)$$

or $$x^2(a^2+u)^{-1}+y^2(b^2+u)^{-1}+z^2(c^2+u)^{-1}=1$$

Differentiating partially w.r.t. x, we have

$$2x(a^2+u)^{-1}-x^2(a^2+u)^{-2}\cdot\frac{\partial u}{\partial x}-y^2(b^2+u)^{-2}\cdot\frac{\partial u}{\partial x}-z^2(c^2+u)^{-2}\cdot\frac{\partial u}{\partial x}=0$$

or $$\frac{2x}{a^2+u}=\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]\frac{\partial u}{\partial x}$$

or $$\frac{2x}{a^2+u}=V\frac{\partial u}{\partial x} \text{ where } V=\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}$$

or $$\frac{\partial u}{\partial x}=\frac{2x}{V(a^2+u)}$$

Similarly, $$\frac{\partial u}{\partial y}=\frac{2y}{V(b^2+u)} \text{ and } \frac{\partial u}{\partial z}=\frac{2z}{V(c^2+u)}$$

$$\therefore \left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=\frac{4}{V^2}\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]$$

$$=\frac{4}{V^2}(V)=\frac{4}{V} \quad ...(2)$$

Now, $$2\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right)=2\left[\frac{2x^2}{V(a^2+u)}+\frac{2y^2}{V(b^2+u)}+\frac{2z^2}{V(c^2+u)}\right]$$

$$=\frac{4}{V}\left[\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}\right]$$

$$=\frac{4}{V}(1) \quad \text{[Using (1)]}$$

$$=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2 \quad \text{[Using (2)]}$$

Example 19. *If* $x^2=au+bv$, $y^2=au-bv$, *prove that* $\left(\dfrac{\partial u}{\partial x}\right)_y\left(\dfrac{\partial x}{\partial u}\right)_v=\dfrac{1}{2}=\left(\dfrac{\partial v}{\partial y}\right)_x\left(\dfrac{\partial y}{\partial v}\right)_u$.

Sol. As $$x^2=au+bv \quad ...(1)$$
$$y^2=au-bv \quad ...(2)$$

Add and subtract, we get $u=\dfrac{x^2+y^2}{2a}$, $v=\dfrac{x^2-y^2}{2b}$

$$\left(\frac{\partial u}{\partial x}\right)_y = \frac{x}{a}, \left(\frac{\partial v}{\partial y}\right)_x = -\frac{y}{b}$$

From (1), $2x\left(\frac{\partial x}{\partial u}\right)_v = a \therefore \left(\frac{\partial x}{\partial u}\right)_v = \frac{a}{2x}$

From (2), $2y\left(\frac{\partial y}{\partial v}\right)_u = -b \quad \therefore \left(\frac{\partial y}{\partial v}\right)_u = -\frac{b}{2y}$

$$\text{L.H.S.} = \left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v = \frac{x}{a}\cdot\frac{a}{2x} = \frac{1}{2}$$

$$\text{R.H.S.} = \left(\frac{\partial v}{\partial y}\right)_x \left(\frac{\partial y}{\partial v}\right)_u = \left(-\frac{y}{b}\right)\left(-\frac{b}{2y}\right) = \frac{1}{2}$$

Hence, $\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v = \frac{1}{2} = \left(\frac{\partial v}{\partial y}\right)_x \left(\frac{\partial y}{\partial v}\right)_u$.

Example 20. *If $u = lx + my$, $v = mx - ly$, show that*

$$\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v = \frac{l^2}{l^2+m^2}, \left(\frac{\partial y}{\partial v}\right)_x \left(\frac{\partial v}{\partial y}\right)_u = \frac{l^2+m^2}{l^2}.$$

Sol. Given

$$u = lx + my \qquad ...(1)$$

$$v = mx - ly \qquad ...(2)$$

(*i*) $\left(\frac{\partial u}{\partial x}\right)_y$ = The partial derivative of u w.r.t. x keeping y constant.

$\therefore$ We need a relation expressing u as a function of x and y.

From (1), $\left(\frac{\partial u}{\partial x}\right)_y = l$

$\left(\frac{\partial x}{\partial u}\right)_v$ = The partial derivative of x w.r.t. u keeping v constant.

$\therefore$ We need a relation expressing x as a function of u and v.

Eliminating y between (1) and (2) by multiplying (1) by l, (2) by m and adding the products, we have

$$lu + mv = (l^2 + m^2)x \quad \text{or} \quad x = \frac{lu+mv}{l^2+m^2}$$

$$\therefore \quad \left(\frac{\partial x}{\partial u}\right)_v = \frac{l}{l^2+m^2}$$

Hence $\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v = \frac{l^2}{l^2+m^2}$

(*ii*) $\left(\frac{\partial y}{\partial v}\right)_x$ = the partial derivative of y w.r.t. v keeping x constant.

$\therefore$ We need a relation expressing y as a function of v and x.

From (2), $y = \dfrac{mx - v}{l}$ $\quad\therefore\quad \left(\dfrac{\partial y}{\partial v}\right)_x = -\dfrac{1}{l}$

Also $\left(\dfrac{\partial v}{\partial y}\right)_u$ = partial derivative of v w.r.t. y keeping u constant.

$\therefore$ We need a relation expressing v as a function of y and u.

Eliminating x between (1) and (2), we have $v = \dfrac{mu - (l^2 + m^2)y}{l}$

$$\therefore \quad \left(\frac{\partial v}{\partial y}\right)_u = -\frac{l^2 + m^2}{l}$$

Hence $\left(\dfrac{\partial y}{\partial v}\right)_x \left(\dfrac{\partial v}{\partial y}\right)_u = \left(-\dfrac{1}{l}\right)\left(-\dfrac{l^2 + m^2}{l}\right) = \dfrac{l^2 + m^2}{l^2}$.

Example 21. *If* $u = f(ax^2 + 2hxy + by^2)$, $v = \phi(ax^2 + 2hxy + by^2)$, *prove that*

$$\frac{\partial}{\partial y}\left(u\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial y}\right).$$

Sol. Let
$$z = ax^2 + 2hxy + by^2 \qquad ...(1)$$
$\therefore$
$$u = f(z),\ v = \phi(z) \qquad ...(2)$$

We have to prove $\dfrac{\partial}{\partial y}\left(u\dfrac{\partial v}{\partial x}\right) = \dfrac{\partial}{\partial x}\left(u\dfrac{\partial v}{\partial y}\right)$

i.e., to prove $u\dfrac{\partial^2 v}{\partial y\,\partial x} + \dfrac{\partial u}{\partial y}\cdot\dfrac{\partial v}{\partial x} = u\cdot\dfrac{\partial^2 v}{\partial x\,\partial y} + \dfrac{\partial u}{\partial x}\cdot\dfrac{\partial v}{\partial y}$

As
$$\frac{\partial^2 v}{\partial y\,\partial x} = \frac{\partial^2 v}{\partial x\,\partial y}$$

$\therefore$ We have to prove only $\dfrac{\partial u}{\partial y}\cdot\dfrac{\partial v}{\partial x} = \dfrac{\partial u}{\partial x}\cdot\dfrac{\partial v}{\partial y}$

From (2),
$$\frac{\partial u}{\partial x} = f'(z)\frac{\partial z}{\partial x};\ \frac{\partial u}{\partial y} = f'(z)\frac{\partial z}{\partial y}$$

and
$$\frac{\partial v}{\partial x} = \phi'(z)\frac{\partial z}{\partial x};\ \frac{\partial v}{\partial y} = \phi'(z)\frac{\partial z}{\partial y}$$

From (1),
$$\frac{\partial z}{\partial x} = 2ax + 2hy$$
$$\frac{\partial z}{\partial y} = 2hx + 2by$$

$\therefore$
$$\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x} = f'(z)\,(2hx + 2by)\,\phi'(z)\,(2ax + 2hy)$$
$$= 4\,f'(z)\,\phi'(z)\,(hx + by)\,(ax + hy)$$

And
$$\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y} = f'(z)\,(2ax + 2hy).\ \phi'(z)\,(2hx + 2by)$$
$$= 4\,f'(z)\,\phi'(z)\,(hx + by)\,(ax + hy)$$

Hence $$\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}=\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y}$$

$$\therefore \quad \frac{\partial}{\partial y}\left(u\frac{\partial v}{\partial x}\right)=\frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial y}\right).$$

TEST YOUR KNOWLEDGE

1. Show that the function $f(x, y) = \begin{cases} \dfrac{x^2+y^2}{|x|+|y|}; & (x, y) \neq 0 \\ 0 \quad ; & (x, y) = 0 \end{cases}$ is continuous at (0, 0) but its partial derivatives do not exist at (0, 0).

2. For the function $f(x, y) = \begin{cases} \dfrac{xy(2x^2-3y^2)}{x^2+y^2}; & (x, y) \neq (0, 0) \\ 0 \quad ; & (x, y) = (0, 0) \end{cases}$

   find $f_{xy}(0, 0)$ and $f_{yx}(0, 0)$ and prove that f_{xy}, f_{yx} are discontinuous at (0, 0).

3. Find the first order partial derivatives of the following functions:

   (*i*) $u = y^x$ (*ii*) $u = \log(x^2 + y^2)$

   (*iii*) $u = x^2 \sin\dfrac{y}{x}$ (*iv*) $u = \dfrac{x}{y}\tan^{-1}\left(\dfrac{y}{x}\right)$.

4. If $u = x^2 + y^2 + z^2$, prove that $xu_x + yu_y + zu_z = 2u$.

5. If $z = \log(x^2 + xy + y^2)$, prove that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 2$. **(P.T.U., Dec. 2013)**

6. If $u = x^2y + y^2z + z^2x$, prove that $u_x + u_y + u_z = (x + y + z)^2$.

7. If $u = \dfrac{y}{z} + \dfrac{z}{x} + \dfrac{x}{y}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 0$. **(P.T.U., Dec. 2012)**

8. If $u = \log(\tan x + \tan y)$, prove that $\sin 2x\dfrac{\partial u}{\partial x} + \sin 2y\dfrac{\partial u}{\partial y} = 2$.

9. Verify that $\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial y \partial x}$ for the following functions:

   (*i*) $u = ax^2 + 2hxy + by^2$ (*ii*) $u = \tan^{-1}\left(\dfrac{x}{y}\right)$ (*iii*) $u = \log\left(\dfrac{x^2+y^2}{xy}\right)$

   (*iv*) $u = e^{ax}\sin by$ (*v*) $u = \log(x \sin y + y \sin x)$

   (*vi*) $\sin\left(\dfrac{y}{x}\right)$ (*vii*) $\log x \tan^{-1}(x^2 + y^2)$.

10. If $z = \log(e^x + e^y)$, show that $rt - s^2 = 0$; where $r = \dfrac{\partial^2 z}{\partial x^2}$, $t = \dfrac{\partial^2 z}{\partial y^2}$, $s = \dfrac{\partial^2 z}{\partial x \partial y}$.

11. If $u = \tan^{-1}\dfrac{xy}{\sqrt{1+x^2+y^2}}$, show that $\dfrac{\partial^2 u}{\partial x \partial y} = (1 + x^2 + y^2)^{-3/2}$.

12. If $u = e^{xyz}$, prove that $\dfrac{\partial^3 u}{\partial x \partial y \partial z} = (1 + 3xyz + x^2y^2z^2)e^{xyz}$.

13. If $u = \log (x^2 + y^2) + \tan^{-1} \dfrac{y}{x}$, prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

14. If $u = \tan^{-1}\left(\dfrac{2xy}{x^2 - y^2}\right)$, prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

15. Find the value of n so that the equation $V = r^n (3 \cos^2 \theta - 1)$ satisfies the relation

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial V}{\partial\theta}\right) = 0.$$

16. If $z = \tan (y + ax) - (y - ax)^{3/2}$, show that $\dfrac{\partial^2 z}{\partial x^2} = a^2 \dfrac{\partial^2 z}{\partial y^2}$.

17. If $u = \log (x^2 + y^2 + z^2)$, prove that $x \dfrac{\partial^2 u}{\partial y \partial z} = y \dfrac{\partial^2 u}{\partial z \partial x} = z \dfrac{\partial^2 u}{\partial x \partial y}$.

18. If $u = \log \sqrt{x^2 + y^2 + z^2}$, prove that $(x^2 + y^2 + z^2)\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}\right) = 1$.

19. If $x^2 + y^2 + z^2 = \dfrac{1}{u^2}$, prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$.

[**Hint.** Consult S.E. 15(*ii*) replace m by -1]

20. If $u = \sqrt{x^2 + y^2 + z^2}$, show that

(*i*) $\left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial u}{\partial y}\right)^2 + \left(\dfrac{\partial u}{\partial z}\right)^2 = 1$ (*ii*) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = \dfrac{2}{u}$.

21. (*i*) If $u = e^{x - at} \cos (x - at)$, show that $\dfrac{\partial^2 u}{\partial t^2} = a^2 \dfrac{\partial^2 u}{\partial x^2}$.

(*ii*) If $v = \dfrac{1}{\sqrt{t}} e^{-\frac{x^2}{4a^2 t}}$, prove that $\dfrac{\partial v}{\partial t} = a^2 \dfrac{\partial^2 v}{\partial x^2}$

22. (*i*) If $u = e^x(x \cos y - y \sin y)$, show that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

(*ii*) If $z = f(x + ay) + \phi(x - ay)$ prove that $\dfrac{\partial^2 z}{\partial y^2} = a^2 \dfrac{\partial^2 z}{\partial x^2}$.

23. If $u = \log (x^3 + y^3 - x^2 y - xy^2)$, show that $\dfrac{\partial^2 u}{\partial x^2} + 2\dfrac{\partial^2 u}{\partial x \partial y} + \dfrac{\partial^2 u}{\partial y^2} = -4(x + y)^{-2}$.

[**Hint.** $u = \log \{x^2(x - y) - y^2(x - y)\} = \log (x - y)(x^2 - y^2) = \log (x - y)^2(x + y)$
$= 2 \log (x - y) + \log (x + y)$]

Answers

2. $f_{xy}(0, 0) = 0$, $f_{yx}(0, 0) = -3$.

3. (*i*) $y^x \log y,\ xy^{x-1}$; (*ii*) $\dfrac{2x}{x^2 + y^2}, \dfrac{2y}{x^2 + y^2}$ (*iii*) $2x \sin \dfrac{y}{x} - y \cos \dfrac{y}{x},\ x \cos \dfrac{y}{x}$

(*iv*) $\dfrac{-x}{x^2 + y^2} + \dfrac{1}{y} \tan^{-1} \dfrac{y}{x},\ \dfrac{x^2}{y(x^2 + y^2)} - \dfrac{x}{y^2} \tan^{-1} \dfrac{y}{x}$ **15.** $n = -3, 2$.

3.5. HOMOGENEOUS FUNCTIONS

(*P.T.U., May 2005, May 2009*)

A function $f(x, y)$ is said to be homogeneous of degree (or order) n in the variables x and y if it can be expressed in the form $x^n \phi\left(\frac{y}{x}\right)$ or $y^n \phi\left(\frac{x}{y}\right)$.

An alternative test for a function $f(x, y)$ to be homogeneous of degree (or order) n is that

$$f(tx, ty) = t^n f(x, y).$$

For example, if $\quad f(x, y) = \dfrac{x + y}{\sqrt{x} + \sqrt{y}}$, then

$$(i)\ f(x, y) = \frac{x\left(1 + \frac{y}{x}\right)}{\sqrt{x}\left(1 + \sqrt{\frac{y}{x}}\right)} = x^{1/2}\ \phi\left(\frac{y}{x}\right)$$

$\Rightarrow$ $f(x, y)$ is a homogeneous function of degree $\frac{1}{2}$ in x and y.

$$(ii)\ f(x, y) = \frac{y\left(\frac{x}{y} + 1\right)}{\sqrt{y}\left(\sqrt{\frac{x}{y}} + 1\right)} = y^{1/2}\ \phi\left(\frac{x}{y}\right)$$

$\Rightarrow$ $f(x, y)$ is a homogeneous function of degree $\frac{1}{2}$ in x and y.

$$(iii)\ f(tx, ty) = \frac{tx + ty}{\sqrt{tx} + \sqrt{ty}} = \frac{t(x + y)}{\sqrt{t}\,(\sqrt{x} + \sqrt{y})} = t^{1/2} f(x, y)$$

$\Rightarrow$ $f(x, y)$ is a homogeneous function of degree $\frac{1}{2}$ in x and y.

Similarly, a function $f(x, y, z)$ is said to be homogeneous of degree (or order) n in the variables x, y, z if

$$f(x, y, z) = x^n \phi\left(\frac{y}{x}, \frac{z}{x}\right) \quad \text{or} \quad y^n \phi\left(\frac{x}{y}, \frac{z}{y}\right) \quad \text{or} \quad z^n \phi\left(\frac{x}{z}, \frac{y}{z}\right).$$

Alternative test is $f(tx, ty, tz) = t^n f(x, y, z)$.

3.6. EULER'S THEOREM ON HOMOGENEOUS FUNCTIONS

(*P.T.U. May 2003, Dec. 2004, May 2005, May 2006, Jan. 2009, May 2009, Dec. 2010*)

If u is a homogeneous function of degree n in x and y, then $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = nu.$

Proof. Since u is a homogeneous function of degree n in x and y, it can be expressed as

$$u = x^n f\left(\frac{y}{x}\right)$$

$$\therefore \quad \frac{\partial u}{\partial x} = nx^{n-1} f\left(\frac{y}{x}\right) + x^n f'\left(\frac{y}{x}\right)\cdot\left(-\frac{y}{x^2}\right)$$

$$\Rightarrow \qquad x\frac{\partial u}{\partial x} = nx^n f\left(\frac{y}{x}\right) - x^{n-1} yf'\left(\frac{y}{x}\right) \qquad ...(1)$$

Also, $$\frac{\partial u}{\partial y} = x^n f'\left(\frac{y}{x}\right).\frac{1}{x} = x^{n-1} f'\left(\frac{y}{x}\right)$$

$$\Rightarrow \qquad y\frac{\partial u}{\partial y} = x^{n-1} yf'\left(\frac{y}{x}\right) \qquad ...(2)$$

Adding (1) and (2), we get $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^n f\left(\frac{y}{x}\right) = nu.$

Note. Euler's theorem can be extended to a homogeneous function of any number of variables. Thus, if u is a homogeneous function of degree n in x, y and z, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu.$

3.7. IF u IS A HOMOGENEOUS FUNCTION OF DEGREE n IN x AND y, THEN $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$

Proof. Since u is a homogeneous function of degree n in x and y

$\therefore$ By Euler's Theorem, we have $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu \qquad ...(1)$$

Differentiating (1) partially w.r.t. x, we have $$1\cdot\frac{\partial u}{\partial x} + x\frac{\partial^2 u}{\partial x^2} + y\cdot\frac{\partial^2 u}{\partial x\partial y} = n\frac{\partial u}{\partial x} \qquad ...(2)$$

Differentiating (1) partially, w.r.t. y, we have $$x\frac{\partial^2 u}{\partial y\partial x} + 1\cdot\frac{\partial u}{\partial y} + y\cdot\frac{\partial^2 u}{\partial y^2} = n\cdot\frac{\partial u}{\partial y}$$

But $$\frac{\partial^2 u}{\partial y\partial x} = \frac{\partial^2 u}{\partial x\partial y}$$

$$\therefore \qquad x\frac{\partial^2 u}{\partial x\partial y} + \frac{\partial u}{\partial y} + y\frac{\partial^2 u}{\partial y^2} = n\frac{\partial u}{\partial y} \qquad ...(3)$$

Multiplying (2) by x, (3) by y and adding

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = n\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

or $$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + nu = n\cdot nu \qquad \text{[Using (1)]}$$

$$\Rightarrow \qquad x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n^2u - nu = n(n-1)u.$$

ILLUSTRATIVE EXAMPLES

Example 1. *Verify Euler's theorem for the functions:*

(i) $u = (x^{1/2} + y^{1/2})(x^n + y^n)$ (ii) $u = \sin^{-1}\dfrac{x}{y} + \tan^{-1}\dfrac{y}{x}$

(iii) $f(x, y, z) = 3x^2yz + 5xy^2z + 4z^4$ **(P.T.U., Dec., 2005)**

Sol. (i) $u = (x^{1/2} + y^{1/2})(x^n + y^n)$ ...(1)

$$= x^{1/2}\left(1+\frac{y^{1/2}}{x^{1/2}}\right)x^n\left(1+\frac{y^n}{x^n}\right) = x^{n+1/2}\left[1+\left(\frac{y}{x}\right)^{1/2}\right]\left[1+\left(\frac{y}{x}\right)^n\right] = x^{n+1/2} f\left(\frac{y}{x}\right)$$

[or $f(tx, ty) = t^{n+1/2} f(x, y)$]

$\Rightarrow$ u is a homogeneous function of degree $\left(n+\dfrac{1}{2}\right)$ in x and y

$\therefore$ By Euler's theorem, we should have $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \left(n+\dfrac{1}{2}\right)u$ ...(2)

From (1),
$$\frac{\partial u}{\partial x} = \frac{1}{2}x^{-1/2}(x^n + y^n) + nx^{n-1}(x^{1/2} + y^{1/2})$$
$$x\frac{\partial u}{\partial x} = \frac{1}{2}x^{1/2}(x^n + y^n) + nx^n(x^{1/2} + y^{1/2})$$
$$\frac{\partial u}{\partial y} = \frac{1}{2}y^{-1/2}(x^n + y^n) + ny^{n-1}(x^{1/2} + y^{1/2})$$
$$y\frac{\partial u}{\partial y} = \frac{1}{2}y^{1/2}(x^n + y^n) + ny^n(x^{1/2} + y^{1/2})$$

$\therefore$
$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}(x^{1/2} + y^{1/2})(x^n + y^n) + n(x^n + y^n)(x^{1/2} + y^{1/2})$$
$= \frac{1}{2}u + nu = (n + \frac{1}{2})u$ which is the same as (2). Hence the verification.

(ii) $$u = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x}$$ ...(1)
$$= \operatorname{cosec}^{-1}\frac{y}{x} + \tan^{-1}\frac{y}{x} = x^0 f\left(\frac{y}{x}\right)$$

[OR $f(tx, ty) = f(x, y) = t^0 f(x, y)$]

$\Rightarrow$ u is a homogeneous function of degree 0 in x and y.

$\therefore$ By Euler's theorem, we should have $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0 \times u = 0$ ...(2)

From (1),
$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\cdot\frac{1}{y} + \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) = \frac{1}{\sqrt{y^2-x^2}} - \frac{y}{x^2+y^2}$$
$$x\frac{\partial u}{\partial x} = \frac{x}{\sqrt{y^2-x^2}} - \frac{xy}{x^2+y^2}$$

$$\frac{\partial u}{\partial y}=\frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\left(-\frac{x}{y^2}\right)+\frac{1}{1+\frac{y^2}{x^2}}\cdot\frac{1}{x}=-\frac{x}{y\sqrt{y^2-x^2}}+\frac{x}{x^2+y^2}$$

$$y\frac{\partial u}{\partial y}=-\frac{x}{\sqrt{y^2-x^2}}+\frac{xy}{x^2+y^2}$$

$\therefore\quad x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=0$ which is the same as (2). Hence the verification.

(*iii*) $f(x, y, z) = 3x^2yz + 5xy^2z + 4z^4$ ...(1)

$f(x, y, z)$ is a homogeneous function of x, y, z of degree 4.

By Euler's theorem $x\dfrac{\partial f}{\partial x}+y\dfrac{\partial f}{\partial y}+z\dfrac{\partial f}{\partial z}=nf=4f$

Differentiable (1) partially w.r.t. x, y, z successively, we get

$$\frac{\partial f}{\partial x}=6xyz+5y^2z$$

$$\frac{\partial f}{\partial y}=3x^2z+10xyz$$

$$\frac{\partial f}{\partial z}=3x^2y+5xy^2+16z^3$$

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=6x^2yz+5xy^2z+3x^2yz+10xy^2z+3x^2yz+5xy^2z+16z^4$$

$$=12x^2yz+20xy^2z+16z^4=4(3x^2yz+5xy^2z+4z^4)=4f(x, y, z).$$

Example 2. *If* $u=xf\left(\dfrac{y}{x}\right)+g\left(\dfrac{y}{x}\right)$, *prove that* $x^2\dfrac{\partial^2 u}{\partial x^2}+2xy\dfrac{\partial^2 u}{\partial x\partial y}+y^2\dfrac{\partial^2 u}{\partial y^2}=0.$

(P.T.U., May 2012)

Sol. $u=xf\left(\dfrac{y}{x}\right)+g\left(\dfrac{y}{x}\right)=u_1+u_2$ where

$$u_1=xf\left(\frac{y}{x}\right),\ u_2=g\left(\frac{y}{x}\right)$$

u_1 is a homogeneous function of x, y of degree 1

$\therefore$ By Euler's theorem

$$x\frac{\partial u_1}{\partial x}+y\frac{\partial u_1}{\partial y}=1\cdot u_1 \qquad \text{...(1)}$$

u_2 is a homogeneous function of x, y of degree zero

$$\therefore\quad x\frac{\partial u_2}{\partial x}+y\frac{\partial u_2}{\partial y}=0 \qquad \text{...(2)}$$

Differentiate (1) and (2) partially w.r.t. x and y

$$x\frac{\partial^2 u_1}{\partial x^2}+\frac{\partial u_1}{\partial x}+y\frac{\partial^2 u_1}{\partial x\partial y} = \frac{\partial u_1}{\partial x}$$

or
$$x\frac{\partial^2 u_1}{\partial x^2}+y\frac{\partial^2 u_1}{\partial x\partial y} = 0 \qquad ...(3)$$

$$x\frac{\partial^2 u_1}{\partial x\partial y}+y\frac{\partial^2 u_1}{\partial y^2}+\frac{\partial u_1}{\partial y} = \frac{\partial u_1}{\partial y}$$

or
$$x\frac{\partial^2 u_1}{\partial x\partial y}+y\frac{\partial^2 u_1}{\partial y^2} = 0 \qquad ...(4)$$

$$x\frac{\partial^2 u_2}{\partial x^2}+\frac{\partial u_2}{\partial x}+y\frac{\partial^2 u_2}{\partial x\partial y} = 0 \qquad ...(5)$$

$$x\frac{\partial^2 u_2}{\partial x\partial y}+y\frac{\partial^2 u_2}{\partial y^2}+\frac{\partial u_2}{\partial y} = 0 \qquad ...(6)$$

Multiple (3), (5) each by x and (4), (6) each by y and add

$$x^2\frac{\partial^2 u_1}{\partial x^2}+xy\frac{\partial^2 u_1}{\partial x\partial y}+xy\frac{\partial^2 u_1}{\partial x\partial y}+y^2\frac{\partial^2 u_1}{\partial y^2}+x^2\frac{\partial^2 u_2}{\partial x^2}+x\frac{\partial u_2}{\partial x}+xy\frac{\partial^2 u_2}{\partial x\partial y}$$
$$+xy\frac{\partial^2 u_2}{\partial x\partial y}+y^2\frac{\partial^2 u_2}{\partial y^2}+y\frac{\partial u_2}{\partial y} = 0$$

or
$$x^2\left(\frac{\partial^2 u_1}{\partial x^2}+\frac{\partial^2 u_2}{\partial x^2}\right)+2xy\left(\frac{\partial^2 u_1}{\partial x\partial y}+\frac{\partial^2 u_2}{\partial x\partial y}\right)+y^2\left(\frac{\partial^2 u_1}{\partial y^2}+\frac{\partial^2 u_2}{\partial y^2}\right)+\left(x\frac{\partial u_2}{dx}+y\frac{\partial u_2}{\partial y}\right) = 0$$

or
$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+0 = 0 \qquad (\because \text{ of } 2)$$

or
$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2} = 0$$

Example 3. (*i*) *If* $u = \sin^{-1}\dfrac{x^3+y^3+z^3}{ax+by+cz}$, *prove that* $xu_x+yu_y+zu_z = 2\tan u$.

(P.T.U., May 2008)

(*ii*) *If* $u = \tan^{-1}\dfrac{x^3+y^3}{x-y}$ *prove that* $x^2\dfrac{\partial^2 u}{\partial x^2}+2xy\dfrac{\partial^2 u}{\partial x\partial y}+y^2\dfrac{\partial^2 u}{\partial y^2} = \sin 4u-\sin 2u = 2\cos 3u$ *sin u.*

(P.T.U., Jan. 2010, Dec. 2011)

Sol. (*i*)
$$u = \sin^{-1}\frac{x^3+y^3+z^3}{ax+by+cz}$$

u is not a homogeneous function

$$\therefore \qquad \sin u = \frac{x^3+y^3+z^3}{ax+by+cz} = \frac{x^3\left[1+\left(\frac{y}{x}\right)^3+\left(\frac{z}{x}\right)^3\right]}{x\left[a+b\frac{y}{x}+c\frac{z}{x}\right]} = x^2 f\left(\frac{y}{x},\frac{z}{x}\right)$$

$\therefore$ sin u is a homogeneous function of order 2

$\therefore$ By Euler's theorem

$$x \frac{\partial}{\partial x}(\sin u) + y \frac{\partial}{\partial y}(\sin u) + z \frac{\partial}{\partial z}(\sin u) = 2 \sin u$$

or
$$x \cos u \frac{\partial u}{\partial x} + y \cos u \frac{\partial u}{\partial y} + z \cos u \frac{\partial u}{\partial z} = 2 \sin u$$

or
$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = 2 \tan u$$

(*ii*) $u = \tan^{-1} \dfrac{x^3 + y^3}{x - y}$

u is not a homogeneous function of x and y

$\therefore$
$$\tan u = \frac{x^3 + y^3}{x - y} = x^2 \frac{1 + \left(\frac{y}{x}\right)^3}{1 - \left(\frac{y}{x}\right)} = x^2 f\left(\frac{y}{x}\right)$$

$\tan u$ is a homogeneous function of degree 2

$\therefore$ By Euler's theorem

$$x \frac{\partial}{\partial x}(\tan u) + y \frac{\partial}{\partial y}(\tan u) = 2 \tan u$$

or
$$x \sec^2 u \frac{\partial u}{\partial x} + y \sec^2 u \frac{\partial u}{\partial y} = 2 \tan u$$

or
$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2 \frac{\tan u}{\sec^2 u} = 2 \sin u \cos u = \sin 2u \qquad \text{...(1)}$$

Differentiate (1) partially w.r.t. x and y

$$x \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y \frac{\partial^2 u}{\partial x \partial y} = \cos 2u \cdot 2 \frac{\partial u}{\partial x} \qquad \text{...(2)}$$

$$x \frac{\partial^2 u}{\partial y \partial x} + y \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = \cos 2u \cdot 2 \frac{\partial u}{\partial y} \qquad \text{...(3)}$$

Multiply (2) by x and (3) by y and add

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}\right) + y^2 \frac{\partial^2 u}{\partial y^2} = 2 \cos 2u \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}\right)$$

or
$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = (2 \cos 2u - 1)\left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}\right)$$

$$= (2 \cos 2u - 1)(\sin 2u) \qquad | \text{ Using (1)}$$

$$= \sin 4u - \sin 2u$$

$$= 2 \cos 3u \sin u.$$

Example 4. (*i*) *If* $u = \cos\left(\dfrac{xy + yz + zx}{x^2 + y^2 + z^2}\right)$, *prove that* $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 0.$

(*ii*) *If* $u = \log\dfrac{x^5 + y^5 + z^5}{x^2 + y^2 + z^2}$, *prove that* $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 3.$

Sol. (*i*)
$$u = \cos\frac{xy + yz + zx}{x^2 + y^2 + z^2}$$
$$= \cos\frac{x^2\left(\dfrac{y}{x} + \dfrac{y}{x}\cdot\dfrac{z}{x} + \dfrac{z}{x}\right)}{x^2\left[1 + \left(\dfrac{y}{x}\right)^2 + \left(\dfrac{z}{x}\right)^2\right]}$$
$$u = x^0 \cos\frac{\dfrac{y}{x} + \dfrac{y}{x}\cdot\dfrac{z}{x} + \dfrac{z}{x}}{1 + \left(\dfrac{y}{x}\right)^2 + \left(\dfrac{z}{x}\right)^2} = x^0 f\left(\frac{y}{x}, \frac{z}{x}\right)$$

which is a homogeneous function of x, y, z of degree 0 $\therefore$ by Euler's theorem

$\therefore$ By Euler's theorem $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 0 \cdot u = 0$

Hence $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 0.$

Aliter: Let $u = f(x, y, z)$

$\therefore$
$$f(x, y, z) = \cos\frac{xy + yz + zx}{x^2 + y^2 + z^2}$$
$$f(tx, ty, tz) = \cos\frac{t^2xy + t^2yz + t^2zx}{t^2x^2 + t^2y^2 + t^2z^2} = t^0 \cos\frac{xy + yz + zx}{x^2 + y^2 + z^2} = t^0 f(x, y, z)$$

$\therefore$ $f(x, y, z)$ is homogeneous function of x, y, z of degree 0.

$\therefore$ By Euler's theorem $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 0$

(*ii*) $u = \log\dfrac{x^5 + y^5 + z^5}{x^2 + y^2 + z^2} = \log x^3 \cdot \dfrac{1 + \left(\dfrac{y}{x}\right)^5 + \left(\dfrac{z}{x}\right)^5}{1 + \left(\dfrac{y}{x}\right)^2 + \left(\dfrac{z}{x}\right)^2}$ it is not homogeneous function of x, y, z $\because$ it cannot be expressed as $x^n f\left(\dfrac{y}{x}, \dfrac{z}{x}\right)$

Now
$$e^u = x^3 \frac{1 + \left(\dfrac{y}{x}\right)^5 + \left(\dfrac{z}{x}\right)^5}{1 + \left(\dfrac{y}{x}\right)^2 + \left(\dfrac{z}{x}\right)^2}, \text{ let } \phi(u) = e^u \quad \therefore \quad \phi(u) = x^3 f\left(\frac{y}{x}, \frac{z}{x}\right);$$

$\phi(u)$ is homogeneous function of x, y, z of degree 3 $\quad\therefore\quad$ by Euler's theorem

$$x\frac{\partial\phi}{\partial x}+y\frac{\partial\phi}{\partial y}+z\frac{\partial\phi}{\partial z}=3\phi \quad\text{or}\quad x\frac{\partial}{\partial x}(e^u)+y\frac{\partial}{\partial y}(e^u)+z\frac{\partial}{\partial z}(e^u)=3e^u$$

or
$$x\,.\,e^u\frac{\partial u}{\partial x}+ye^u\frac{\partial u}{\partial y}+ze^u\frac{\partial u}{\partial z}=3e^u$$

or
$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=3.$$

Example 5. *Given* $z=x^n f_1\left(\frac{y}{x}\right)+y^{-n}f_2\left(\frac{x}{y}\right)$, *prove that*

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}+x\frac{\partial z}{\partial y}+y\frac{\partial z}{\partial y}=n^2 z.$$

Sol. Let
$$u=x^n f_1\left(\frac{y}{x}\right),\ v=y^{-n}f_2\left(\frac{x}{y}\right)$$

$$\therefore\qquad z=u+v$$

u is a homogeneous function of x, y of degree n.

$$\therefore\qquad x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu \qquad ...(1)$$

v is a homogeneous function of x, y of degree $-n$.

$$\therefore\qquad x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=-nv \qquad ...(2)$$

Diff. both (1) and (2) partially w.r.t. x and y

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\partial y}=n\frac{\partial u}{\partial x} \qquad ...(3)$$

$$x\frac{\partial^2 u}{\partial y\partial x}+y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}=n\frac{\partial u}{\partial y} \qquad ...(4)$$

$$x\frac{\partial^2 v}{\partial x^2}+\frac{\partial v}{\partial x}+y\frac{\partial^2 v}{\partial x\partial y}=-n\frac{\partial v}{\partial x} \qquad ...(5)$$

$$x\frac{\partial^2 v}{\partial y\partial x}+y\frac{\partial^2 v}{\partial y^2}+\frac{\partial v}{\partial y}=-n\frac{\partial v}{\partial y} \qquad ...(6)$$

Multiply (3) and (5) by x and (4), (6) by y and adding,

$$x^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x^2}\right)+x\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x}\right)+xy\left(\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 v}{\partial x\partial y}\right)+xy\left(\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial^2 v}{\partial y\partial x}\right)$$
$$+y^2\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 v}{\partial y^2}\right)+y\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}\right)$$
$$=n\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)-n\left(x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}\right)$$

$$x^2 \frac{\partial^2}{\partial x^2}(u+v) + y^2 \frac{\partial^2}{\partial y^2}(u+v) + 2xy \frac{\partial^2}{\partial x \partial y}(u+v)$$

$$+ x\frac{\partial}{\partial x}(u+v) + y\frac{\partial}{\partial y}(u+v) = n \cdot nu - n(-nv)$$

or $$x^2 \frac{\partial^2 z}{\partial x^2} + y^2 \frac{\partial^2 z}{\partial y^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = n^2 z.$$

Example 6. *If* $u = \tan^{-1} \frac{y^2}{x}$, *show that* $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = -\sin 2u \sin^2 u.$

(P.T.U., May 2006, May 2011)

Sol. $$u = \tan^{-1} \frac{y^2}{x}$$

$$\tan u = \frac{y^2}{x}$$

Let $f(x, y) = \tan u = \frac{y^2}{x} = x\frac{y^2}{x^2} = x^1 \left(\frac{y}{x}\right)^2$ which is a homogeneous function in x, y of degree 1.

$\therefore$ By Euler's theorem $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 1 \cdot f$

or $$x\frac{\partial}{\partial x}(\tan u) + y\frac{\partial}{\partial y}(\tan u) = \tan u$$

or $$x \sec^2 u \cdot \frac{\partial u}{\partial x} + y \sec^2 u \frac{\partial u}{\partial y} = \tan u$$

or $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u \cos^2 u = \sin u \cos u = \frac{1}{2} \sin 2u \quad ...(1)$$

Differentiating it partially w.r.t. x and y

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x \partial y} = \cos 2u \cdot \frac{\partial u}{\partial x} \quad ...(2)$$

$$x\frac{\partial^2 u}{\partial y \partial x} + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = \cos 2u \cdot \frac{\partial u}{\partial y} \quad ...(3)$$

Multiply (2) by x and (3) by y and add,

$$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = \cos 2u \left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right]$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (\cos 2u - 1)\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = -2\sin^2 u \cdot \frac{1}{2} \sin 2u$$

[Using (1)]

Hence $$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\sin 2u \sin^2 u.$$

Example 7. *If* $u = \text{cosec}^{-1}\left(\dfrac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}\right)^{1/2}$, *prove that*

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{13+\tan^2 u}{144},$$

Sol. $$u = \text{cosec}^{-1}\left(\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}\right)^{1/2}$$

$$f(x, y) = \text{cosec}\, u = \left(\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}\right)^{1/2} = \frac{x^{1/4}\left[1+\left(\dfrac{y}{x}\right)^{1/2}\right]^{1/2}}{x^{1/6}\left[1+\left(\dfrac{y}{x}\right)^{1/3}\right]^{1/2}} = x^{\frac{1}{12}}\phi\left(\frac{y}{x}\right)$$

$\therefore$ $f(x, y)$ is a homogeneous function of degree $\dfrac{1}{12}$

$\therefore$ By Euler's theorem

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=\frac{1}{12}f$$

or $$x\frac{\partial}{\partial x}(\text{cosec}\, u)+y\frac{\partial}{\partial y}(\text{cosec}\, u)=\frac{1}{12}\text{cosec}\, u$$

$$-x\,\text{cosec}\, u\cot u\frac{\partial u}{\partial x}-y\,\text{cosec}\, u\cot u\frac{\partial u}{\partial y}=\frac{1}{12}\text{cosec}\, u$$

or $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{-1}{12}\text{cosec}\, u\cdot\frac{1}{\text{cosec}\, u\cot u}=-\frac{1}{12}\tan u$$

$\therefore$ $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-\frac{1}{12}\tan u \quad ...(1)$$

Differentiate (1) partially w.r.t. x and y

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial y}+y\frac{\partial^2 u}{\partial x\partial y}=-\frac{1}{12}\sec^2 u\frac{\partial u}{\partial x} \quad ...(2)$$

$$x\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial u}{\partial y}+y\frac{\partial^2 u}{\partial y^2}=-\frac{1}{12}\sec^2 u\frac{\partial u}{\partial y} \quad ...(3)$$

Multiply (2) by x and (3) by y and add

$$x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)=-\frac{1}{12}\sec^2 u\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

or $$x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+2xy\frac{\partial^2 u}{\partial x\partial y}=-\left[1+\frac{1}{12}\sec^2 u\right]\left[u\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right]=-\left[\frac{12+1+\tan^2 u}{12}\right]\left[-\frac{1}{12}\right.$$

$$=\frac{13+\tan^2 u}{144}$$

Hence $$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{13+\tan^2 u}{144}.$$

TEST YOUR KNOWLEDGE

1. Verify Euler's theorem for the functions:

(i) $f(x, y) = ax^2 + 2hxy + by^2$

(ii) $u = \dfrac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}$

[**Hint.** (ii) Homogeneous function of degree $\frac{1}{20}$]

(iii) $f(x, y) = \dfrac{x^2 (x^2 - y^2)^3}{(x^2 + y^2)^3}$.

[**Hint.** Homogeneous function of degree 2]

2. (i) If $V = \dfrac{x^3y^3}{x^3 + y^3}$, show that $x \dfrac{\partial V}{\partial x} + y \dfrac{\partial V}{\partial y} = 3V$.

(ii) If $V = \log \dfrac{x^4 + y^4}{x + y}$, show that $x \dfrac{\partial V}{\partial x} + y \dfrac{\partial V}{\partial y} = 3$.

[**Hint.** $e^V = \dfrac{x^4 + y^4}{x + y}$ is a homogeneous function of degree 3]

(iii) If $V = f\left(\dfrac{y}{x}\right)$ show that $x \dfrac{\partial V}{\partial x} + y \dfrac{\partial V}{\partial y} = 0$.

(iv) If $z = \log (x^2 + xy + y^2)$ show that $x \dfrac{\partial z}{\partial x} + y \dfrac{\partial z}{\partial y} = 2$. **(P.T.U., Dec. 2013)**

[**Hint.** $e^z = x^2 + xy + y^2$ is a homogeneous function of degree 2]

(v) If $u = e^{x^2 + y^2}$ prove that $x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y} = 2u \log u$. **(P.T.U., Dec. 2014)**

[**Hint.** $\log u = x^2 + y^2$ is a homogeneous function of degree 2]

3. If $u = \cos^{-1} \dfrac{x + y}{\sqrt{x} + \sqrt{y}}$, show that $x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y} + \dfrac{1}{2} \cot u = 0$.

4. (i) If $u = \sin^{-1} \left(\dfrac{x^2 + y^2}{x + y}\right)$, prove that $x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y} = \tan u$ **(P.T.U., Dec. 2013)**

(ii) If $u = \sin^{-1} \dfrac{x}{y} + \tan^{-1} \dfrac{y}{x}$, prove that $x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y} = 0$ **(P.T.U., Dec. 2010)**

[**Hint.** $u = \text{cosec}^{-1} \dfrac{y}{x} + \tan^{-1} \dfrac{y}{x}$ which is homogeneous function of degree 0]

5. (i) If $u = \sin^{-1} \dfrac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}$, show that $\dfrac{\partial u}{\partial x} = -\dfrac{y}{x} \dfrac{\partial u}{\partial y}$.

(ii) If $\sin u = \dfrac{x^2y^2}{x + y}$, show that $x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y} = 3 \tan u$. **(P.T.U., Dec. 2012)**

(iii) If $f(x, y) = \dfrac{1}{x^2} + \dfrac{1}{xy} + \dfrac{\log x - \log y}{x^2 + y^2}$, show that $x \dfrac{\partial f}{\partial x} + y \dfrac{\partial f}{\partial y} = -2f$. **(P.T.U., May 2004)**

[**Hint.** $f(x, y) = x^{-2} \left[1 + \dfrac{x}{y} + \dfrac{\log \frac{x}{y}}{1 + \left(\frac{y}{x}\right)^2} \right] = x^{-2} \left[1 + \dfrac{1}{\frac{y}{x}} - \dfrac{\log \frac{y}{x}}{1 + \left(\frac{y}{x}\right)^2} \right]$

which is homogeneous function of degree -2]

6. If $u = \dfrac{x^2y^2}{x+y}$, show that

(i) $x \dfrac{\partial^2 u}{\partial x^2} + y \dfrac{\partial^2 u}{\partial x \partial y} = 2 \dfrac{\partial u}{\partial x}$ (ii) $x \dfrac{\partial^2 u}{\partial x \partial y} + y \dfrac{\partial^2 u}{\partial y^2} = 2 \dfrac{\partial u}{\partial y}$.

7. (i) If $u = (x^2 + y^2)^{1/3}$, show that $x^2 \dfrac{\partial^2 u}{\partial x^2} + 2xy \dfrac{\partial^2 u}{\partial x \partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2} = -\dfrac{2u}{9}$.

[**Hint.** Homogeneous function of degree 2/3]

(ii) If $u = x f\left(\dfrac{y}{x}\right)$ show that $x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y} = u$ and $x^2 \dfrac{\partial^2 u}{\partial x^2} + 2xy \dfrac{\partial^2 u}{\partial x \partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2} = 0$.

(P.T.U., May 2008)

8. If $u = \sin^{-1} \dfrac{x+y}{\sqrt{x} + \sqrt{y}}$, prove that $x^2 \dfrac{\partial^2 u}{\partial x^2} + 2xy \dfrac{\partial^2 u}{\partial x \partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2} = -\dfrac{\sin u \cos 2u}{4 \cos^3 u}$.

(P.T.U., May 2008)

9. If $u = x^2 \tan^{-1}\left(\dfrac{y}{x}\right) - y^2 \tan^{-1}\left(\dfrac{x}{y}\right)$, prove that $x^2 \dfrac{\partial^2 u}{\partial x^2} + 2xy \dfrac{\partial^2 u}{\partial x \partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2} = 2u$.

10. (i) If $f(x, y) = \sqrt{x^2 - y^2} \sin^{-1} \dfrac{y}{x}$, show that $x \dfrac{\partial f}{\partial x} + y \dfrac{\partial f}{\partial y} = f(x, y)$

(ii) If $f(x, y) = \sqrt{y^2 - x^2} \sin^{-1} \dfrac{x}{y} + \dfrac{x^2 - y^2}{\sqrt{x^2 + y^2}}$, show that $x \dfrac{\partial f}{\partial x} + y \dfrac{\partial f}{\partial y} = f(x, y)$.

3.8. COMPOSITE FUNCTIONS; TOTAL DERIVATIVE ***(P.T.U., May 2007)***

(i) If $u = f(x, y)$ where $x = \phi(t)$, $y = \psi(t)$

then u is called a composite function of the **single variable** t and we can find $\dfrac{du}{dt}$ which is called the **Total Derivative** of u.

(ii) If $z = f(x, y)$ where $x = \phi(u, v)$, $y = \psi(u, v)$

then z is called a composite function of **two variables** u and v so that we can find $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$.

3.9. (a) DIFFERENTIATION OF COMPOSITE FUNCTIONS

If u is composite function of t, defined by the relations $u = f(x, y)$; $x = \phi(t)$, $y = \psi(t)$, then find the total derivative of u w.r.t. t without actually substituting the values of x, y in f(x, y).

OR

To prove

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}.$$

Proof. Here $u = f(x, y)$ ...(1)

Let δt be an increment in t and δx, δy, δu the corresponding increments in x, y and u respectively. Then, we have

$$u + \delta u = f(x + \delta x, y + \delta y) \quad \text{...(2)}$$

Subtracting (1) from (2), we get

$$\delta u = f(x + \delta x, y + \delta y) - f(x, y)$$
$$= f(x + \delta x, y + \delta y) - f(x, y + \delta y) + f(x, y + \delta y) - f(x, y)$$
$$\frac{\delta u}{\delta t} = \frac{f(x + \delta x, y + \delta y) - f(x, y + \delta y)}{\delta t} + \frac{f(x, y + \delta y) - f(x, y)}{\delta t}$$
$$= \frac{f(x + \delta x, y + \delta y) - f(x, y + \delta y)}{\delta x} \cdot \frac{\delta x}{\delta t} + \frac{f(x, y + \delta y) - f(x, y)}{\delta y} \cdot \frac{\delta y}{\delta t} \quad ...(3)$$

As $\delta t \to 0$, δx and δy both $\to 0$, so that

$$\underset{\delta t \to 0}{\text{Lt}} \frac{\delta u}{\delta t} = \frac{du}{dt}, \quad \underset{\delta t \to 0}{\text{Lt}} \frac{\delta x}{\delta t} = \frac{dx}{dt}, \quad \underset{\delta t \to 0}{\text{Lt}} \frac{\delta y}{\delta t} = \frac{dy}{dt}$$
$$\underset{\delta x \to 0}{\text{Lt}} \frac{f(x + \delta x, y + \delta y) - f(x, y + \delta y)}{\delta y} = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x}$$
$$\underset{\delta y \to 0}{\text{Lt}} \frac{f(x, y + \delta y) - f(x, y)}{\delta y} = \frac{\partial f}{\partial y} = \frac{\partial u}{\partial y}$$

$\therefore$ From (1), $$\mathbf{\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}}$$

Cor. 1. If $u = f(x, y, z)$ and x, y, z are function of t, then u is a composite function of t and

$$\mathbf{\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt}}.$$

Cor. 2. If $z = f(x, y)$ and x, y are functions of u and v, then

$$\mathbf{\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}; \quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}}.$$

Cor. 3. If $u = f(x, y)$ where $y = \phi(x)$ then since $u = \psi(x)$, u is a composite function of x.

$$\therefore \quad \frac{du}{dx} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} \Rightarrow \mathbf{\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}}.$$

3.9. (b) DIFFERENTIATION OF IMPLICIT FUNCTIONS (*P.T.U., Dec. 2004*)

If we are given an implicit function $f(x, y) = c$, then $u = f(x, y)$ where $u = c$

Using Cor. 3, we have $$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}$$

But $\frac{du}{dx} = 0 \quad \therefore$ $$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} = 0 \quad \text{or} \quad \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} = 0$$

$$\therefore \quad \frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{f_x}{f_y}$$

Hence $\mathbf{\frac{dy}{dx} = -\frac{f_x}{f_y}}$.

which gives, the important formula for the first differential coefficient of an implicit equation.

Cor. 1. If $f(x, y) = c$ then, we have $\frac{dy}{dx} = -\frac{f_x}{f_y}$ (proved above)

Differentiating again w.r.t. x, we get

$$\frac{d^2y}{dx^2} = -\frac{f_y \frac{d}{dx}(f_x) - f_x \frac{d}{dx}(f_y)}{f_y^2} = -\frac{f_y\left[\frac{\partial f_x}{\partial x} + \frac{\partial f_x}{\partial y}\cdot\frac{dy}{dx}\right] - f_x\left[\frac{\partial f_y}{\partial x} + \frac{\partial f_y}{\partial y}\cdot\frac{dy}{dx}\right]}{f_y^2}$$

$$= -\frac{f_y\left[f_{xx} - f_{yx}\cdot\frac{f_x}{f_y}\right] - f_x\left[f_{xy} - f_{yy}\cdot\frac{f_x}{f_y}\right]}{f_y^2} = -\frac{f_{xx}f_y^2 - f_xf_yf_{xy} - f_xf_yf_{xy} - f_{yy}f_x^2}{f_y^3}$$

Hence $$\frac{d^2y}{dx^2} = -\frac{f_{xx}f_y^2 - 2f_xf_yf_{xy} + f_{yy}f_x^2}{f_y^3}.$$

which is the important formula for the second differential of an implicit function.

3.9. (c) CHANGE OF VARIABLES

If $z = f(x, y)$ and $x = \phi(u, v)$, $y = \psi(u, v)$, it is often necessary to change the expressions involving $z, x, y, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ etc to expressions involving $z, u, v, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}$ etc.

The necessary formulae for change of variables are easily obtained. If v is regarded constant then x, y, z all will be functions of u only.

$\therefore$ From Cor. 2 (3.9 a)

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} \quad \text{...(1)}$$

Similarly if u is regarded constant then x, y, z all will be functions of v only

$$\therefore \quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v} \quad \text{...(2)}$$

On solving (1) and (2) as simultaneous functions in $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, we get their values in terms of $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$, z, u, v.

If instead of x, y being functions of u, v it is given that u, v are functions of x, y it is easier to use the formula.

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} \quad \text{...(3)}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} \quad \text{...(4)}$$

The higher derivatives of u can be found by repeated application of the formula (1) and (2).

Cor. 1. If $f(x, y) = c$ then, we have $\dfrac{dy}{dx} = -\dfrac{f_x}{f_y}$ (proved above)

Differentiating again w.r.t. x, we get

$$\frac{d^2y}{dx^2} = -\frac{f_y \dfrac{d}{dx}(f_x) - f_x \dfrac{d}{dx}(f_y)}{f_y^2} = -\frac{f_y\left[\dfrac{\partial f_x}{\partial x} + \dfrac{\partial f_x}{\partial y}\cdot\dfrac{dy}{dx}\right] - f_x\left[\dfrac{\partial f_y}{\partial x} + \dfrac{\partial f_y}{\partial y}\cdot\dfrac{dy}{dx}\right]}{f_y^2}$$

$$= -\frac{f_y\left[f_{xx} - f_{yx}\cdot\dfrac{f_x}{f_y}\right] - f_x\left[f_{xy} - f_{yy}\cdot\dfrac{f_x}{f_y}\right]}{f_y^2} = -\frac{f_{xx}f_y^2 - f_xf_yf_{xy} - f_xf_yf_{xy} - f_{yy}f_x^2}{f_y^3}$$

Hence $\mathbf{\dfrac{d^2y}{dx^2} = -\dfrac{f_{xx}f_y^2 - 2f_xf_yf_{xy} + f_{yy}f_x^2}{f_y^3}}$.

which is the important formula for the second differential of an implicit function.

3.9. (c) CHANGE OF VARIABLES

If $z = f(x, y)$ and $x = \phi(u, v)$, $y = \psi(u, v)$, it is often necessary to change the expressions involving $z, x, y, \dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$ etc to expressions involving $z, u, v, \dfrac{\partial z}{\partial u}, \dfrac{\partial z}{\partial v}$ etc.

The necessary formulae for change of variables are easily obtained. If v is regarded constant then x, y, z all will be functions of u only.

$\therefore$ From Cor. 2 (3.9 *a*)

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} \quad ...(1)$$

Similarly if u is regarded constant then x, y, z all will be functions of v only

$$\therefore \quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v} \quad ...(2)$$

On solving (1) and (2) as simultaneous functions in $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$, we get their values in terms of $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$, z, u, v.

If instead of x, y being functions of u, v it is given that u, v are functions of x, y it is easier to use the formula.

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} \quad ...(3)$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} \quad ...(4)$$

The higher derivatives of u can be found by repeated application of the formula (1) and (2).

ILLUSTRATIVE EXAMPLES

Example 1. *Find* $\frac{du}{dt}$ *when* $u = x^2 + y^2, x = at^2, y = 2at$. *Also verify by direct substitution.*

Sol. $u = x^2 + y^2, \; x = at^2, y = 2at$

u is a function of x, y which are further functions of t

$\therefore$ u is a composite function of t

$$\therefore \quad \frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$= (2x)(2at) + (2y) \cdot 2a$$

$$= 2at^2 \cdot 2at + 2 \cdot 2at \cdot 2a$$

$$= 4a^2 t^3 + 8a^2 t = 4a^2 t (t^2 + 2) \quad \ldots(1)$$

Direct substitution gives

$$u = (at^2)^2 + (2at)^2 = a^2 t^4 + 4a^2 t^2$$

$$\frac{du}{dt} = a^2 . 4t^3 + 4a^2 . 2t = 4 a^2 t^3 + 8a^2 t$$

$$= 4a^2 t + (t^2 + 2) \text{ which is same as (1)}$$

Example 2. (*i*) *If* $u = \sin^{-1}(x - y), x = 3t, y = 4t^3$, *show that* $\frac{du}{dt} = \frac{3}{\sqrt{1-t^2}}$.

(P.T.U., Dec. 2014, May 2014)

(*ii*) *If* $u = \sin \frac{x}{y}, x = e^t, y = t^2$, *find* $\frac{du}{dt}$. **(P.T.U., May 2006)**

Sol. (*i*) The given equations define u as a composite function of t.

$$\therefore \quad \frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} \text{ where } u = \sin^{-1}(x - y), x = 3t, y = 4t^3$$

$$= \frac{1}{\sqrt{1-(x-y)^2}} \cdot 3 + \frac{1}{\sqrt{1-(x-y)^2}} (-1) \cdot 12t^2$$

$$= \frac{3(1-4t^2)}{\sqrt{1-(x-y)^2}} = \frac{3(1-4t^2)}{\sqrt{1-(3t-4t^3)^2}} = \frac{3(1-4t^2)}{\sqrt{1-9t^2+24t^4-16t^6}}$$

$$= \frac{3(1-4t^2)}{\sqrt{(1-t^2)(1-8t^2+16t^4)}} = \frac{3(1-4t^2)}{\sqrt{(1-t^2)(1-4t^2)^2}} = \frac{3}{\sqrt{1-t^2}}.$$

(*ii*) $u = \sin \frac{x}{y}\; ; x = e^t, y = t^2$

We know that $$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

$$\therefore \quad \frac{du}{dt} = \frac{1}{y} \cos \frac{x}{y} \cdot e^t + \cos \frac{x}{y} \left(-\frac{x}{y^2}\right) 2t$$

$$= \frac{e^t}{t^2} \cos \frac{e^t}{t^2} - \frac{2t\,e^t}{t^4} \cos \frac{e^t}{t^2}$$

$$= e^t \cos \frac{e^t}{t^2} \left\{ \frac{1}{t^2} - \frac{2}{t^3} \right\} = e^t \cos \frac{e^t}{t^2} \left\{ \frac{t-2}{t^3} \right\}$$

Hence $$\frac{du}{dt} = e^t \, \frac{t-2}{t^3} \cos \frac{e^t}{t^2}.$$

Example 3. (*i*) *If $w = x^2 + y^2$; $x = r - s$, $y = r + s$ find $\frac{\partial w}{\partial r}, \frac{\partial w}{\partial s}$ in terms of r, s.*

(P.T.U., May 2012)

(*ii*) *If z is a function of x and y, where $x = e^u + e^{-v}$ and $y = e^{-u} - e^v$, show that*

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x \frac{\partial z}{\partial x} - y \frac{\partial z}{\partial y}.$$

Sol. (*i*) w is a composite function of r and s.

$$\therefore \quad \frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial r} \qquad \ldots(1)$$

and $$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial s} \qquad \ldots(2)$$

We have $w = x^2 + y^2$, $x = r - s$, $y = r + s$

$$\therefore \quad \frac{\partial w}{\partial x} = 2x; \quad \frac{\partial w}{\partial y} = 2y$$

$$\frac{\partial x}{\partial r} = 1, \quad \frac{\partial x}{\partial s} = -1, \quad \frac{\partial y}{\partial r} = 1, \quad \frac{\partial y}{\partial s} = 1$$

$\therefore$ From (1) and (2)

$$\frac{\partial w}{\partial r} = 2x \cdot 1 + 2y \cdot 1 = 2(x + y) = 2(2r) = 4r$$

$$\frac{\partial w}{\partial s} = 2x \cdot (-1) + 2y \cdot 1 = -2(x - y) = -2(-2s) = 4s$$

$$\therefore \quad \frac{\partial w}{\partial r} = 4r, \quad \frac{\partial w}{\partial s} = 4s.$$

(*ii*) Given $x = e^u + e^{-v}$, $y = e^{-u} - e^v$.

Here z is a composite function of u and v.

$$\therefore \quad \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u} = \frac{\partial z}{\partial x} \cdot e^u + \frac{\partial z}{\partial y} (-e^{-u})$$

and $$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v} = \frac{\partial z}{\partial x} (-e^{-v}) + \frac{\partial z}{\partial y} (-e^v)$$

Subtracting, $$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = (e^u + e^{-v}) \frac{\partial z}{\partial x} - (e^{-u} - e^v) \frac{\partial z}{\partial y} = x \frac{\partial z}{\partial x} - y \frac{\partial z}{\partial y}.$$

Example 4. (*i*) *If $u = f(y - z, z - x, x - y)$, prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$*

(P.T.U., Dec. 2003, Dec. 2011)

(*ii*) *If* $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$, *then prove that* $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0.$

(P.T.U., May 2003, 2014, Dec. 2010)

Sol. (*i*) Here $u = f(X, Y, Z)$, where $X = y - z, Y = z - x, Z = x - y$

$\therefore$ u is a composite function of x, y and z.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial X}\cdot\frac{\partial X}{\partial x} + \frac{\partial u}{\partial Y}\cdot\frac{\partial Y}{\partial x} + \frac{\partial u}{\partial Z}\cdot\frac{\partial Z}{\partial x} = \frac{\partial u}{\partial X}(0) + \frac{\partial u}{\partial Y}(-1) + \frac{\partial u}{\partial Z}(1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial X}\cdot\frac{\partial X}{\partial y} + \frac{\partial u}{\partial Y}\cdot\frac{\partial Y}{\partial y} + \frac{\partial u}{\partial Z}\cdot\frac{\partial Z}{\partial y} = \frac{\partial u}{\partial X}(1) + \frac{\partial u}{\partial Y}(0) + \frac{\partial u}{\partial Z}(-1)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial X}\cdot\frac{\partial X}{\partial z} + \frac{\partial u}{\partial Y}\cdot\frac{\partial Y}{\partial z} + \frac{\partial u}{\partial Z}\cdot\frac{\partial Z}{\partial z} = \frac{\partial u}{\partial X}(-1) + \frac{\partial u}{\partial Y}(1) + \frac{\partial u}{\partial Z}(0)$$

Adding, $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$

(*ii*) $$u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$$

Let $X = \frac{x}{y}$, $Y = \frac{y}{z}$, $Z = \frac{z}{x}$

$\therefore$ $u = f(X, Y, Z)$

$\therefore$ u is a composite function of x, y, z

(Here u is a function of X, Y, Z and further X, Y, Z are functions of x, y, z)

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial X}\cdot\frac{\partial X}{\partial x} + \frac{\partial u}{\partial Y}\frac{\partial Y}{\partial x} + \frac{\partial u}{\partial Z}\frac{\partial Z}{\partial x} = \frac{\partial u}{\partial X}\left(\frac{1}{y}\right) + \frac{\partial u}{\partial Y}(0) + \frac{\partial u}{\partial Z}\left(-\frac{z}{x^2}\right)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial X}\cdot\frac{\partial X}{\partial y} + \frac{\partial u}{\partial Y}\cdot\frac{\partial Y}{\partial y} + \frac{\partial u}{\partial Z}\cdot\frac{\partial Z}{\partial y} = \frac{\partial u}{\partial X}\left(-\frac{x}{y^2}\right) + \frac{\partial u}{\partial Y}\left(\frac{1}{z}\right) + \frac{\partial u}{\partial Z}(0)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial X}\cdot\frac{\partial X}{\partial z} + \frac{\partial u}{\partial Y}\frac{\partial Y}{\partial z} + \frac{\partial u}{\partial Z}\frac{\partial Z}{\partial z} = \frac{\partial u}{\partial X}(0) + \frac{\partial u}{\partial Y}\left(-\frac{y}{z^2}\right) + \frac{\partial u}{\partial Z}\left(\frac{1}{x}\right)$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \frac{\partial u}{\partial X}\left(\frac{x}{y} - \frac{x}{y} + 0\right) + \frac{\partial u}{\partial Y}\left(0 + \frac{y}{z} - \frac{y}{z}\right) + \frac{\partial u}{\partial Z}\left(\frac{-z}{x} + 0 + \frac{z}{x}\right)$$

$$= \frac{\partial u}{\partial X}\cdot 0 + \frac{\partial u}{\partial Y}\cdot 0 + \frac{\partial u}{\partial Z}\cdot 0 = 0$$

Aliter:

Given $$u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$$

i.e., $$u(x, y, z) = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$$

$$u(tx, ty, tz) = f\left(\frac{tx}{ty}, \frac{ty}{tz}, \frac{tz}{tx}\right) = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right) = u(x, y, z)$$

$\therefore$ $u(tx, ty, tz) = t^0\, u(x, y, z)$

$\because$ u is a homogeneous functions of x, y, z of degree zero

$\therefore$ By Euler's theorem, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu = 0\cdot u = 0.$

Example 5. *(i) If $x = u + v + w$, $y = vw + wu + uv$, $z = uvw$ and F is a function of x, y, z, show that* $u\frac{\partial F}{\partial u} + v\frac{\partial F}{\partial v} + w\frac{\partial F}{\partial w} = x\frac{\partial F}{\partial x} + 2y\frac{\partial F}{\partial y} + 3z\frac{\partial F}{\partial z}$.

(ii) If $u = f(x^2 + 2yz, y^2 + 2zx)$, prove that: $(y^2 - zx)\frac{\partial u}{\partial x} + (x^2 - yz)\frac{\partial u}{\partial y} + (z^2 - xy)\frac{\partial u}{\partial z} = 0$.

Sol. (*a*) F is a function of x, y, z and x, y, z are further functions of u, v, w

$\therefore$ F is a composite function of u, v, w.

$$x = u + v + w, \quad y = vw + wu + uv, \; z = uvw$$

$\therefore$

$$\frac{\partial F}{\partial u} = \frac{\partial F}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial F}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial u} = \frac{\partial F}{\partial x}\cdot 1 + \frac{\partial F}{\partial y}(w + v) + \frac{\partial F}{\partial z}\cdot vw \quad ...(1)$$

$$\frac{\partial F}{\partial v} = \frac{\partial F}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial F}{\partial y}\cdot\frac{\partial y}{\partial v} + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial v} = \frac{\partial F}{\partial x}\cdot 1 + \frac{\partial F}{\partial y}\cdot(w + u) + \frac{\partial F}{\partial z}\cdot uw \quad ...(2)$$

$$\frac{\partial F}{\partial w} = \frac{\partial F}{\partial x}\cdot\frac{\partial x}{\partial w} + \frac{\partial F}{\partial y}\cdot\frac{\partial y}{\partial w} + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial w} = \frac{\partial F}{\partial x}\cdot 1 + \frac{\partial F}{\partial y}\cdot(v + u) + \frac{\partial F}{\partial z}\cdot uv \quad ...(3)$$

Multiply (1) by u, (2) by v, (3) by w and add

$$u\frac{\partial F}{\partial u} + v\frac{\partial F}{\partial v} + w\frac{\partial F}{\partial w} = (u + v + w)\frac{\partial F}{\partial x} + (uw + uv + vw + vu + wv + wu)\frac{\partial F}{\partial y} + (uvw + vuw + wuv)\frac{\partial F}{\partial z}$$

$$= x\frac{\partial F}{\partial x} + 2y\frac{\partial F}{\partial y} + 3z\frac{\partial F}{\partial z}$$

Hence $\quad u\frac{\partial F}{\partial u} + v\frac{\partial F}{\partial v} + w\frac{\partial F}{\partial w} = x\frac{\partial F}{\partial x} + 2y\frac{\partial F}{\partial y} + 3z\frac{\partial F}{\partial z}$.

(*ii*) Let $\quad X = x^2 + 2yz, \; Y = y^2 + 2zx \quad \therefore \quad u = f(X, Y)$

u is a function of X, Y ; X, Y are functions of x, y, z

$\therefore$ u is a composite function of x, y, z

$\therefore$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial X}\cdot\frac{\partial X}{\partial x} + \frac{\partial u}{\partial Y}\cdot\frac{\partial Y}{\partial x} = \frac{\partial u}{\partial X}\cdot 2x + \frac{\partial u}{\partial Y}\cdot 2z \quad ...(1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial X}\cdot\frac{\partial X}{\partial y} + \frac{\partial u}{\partial Y}\cdot\frac{\partial Y}{\partial y} = \frac{\partial u}{\partial X}\,.\,2z + \frac{\partial u}{\partial Y}\,.\,2y \quad ...(2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial X}\cdot\frac{\partial X}{\partial z} + \frac{\partial u}{\partial Y}\cdot\frac{\partial Y}{\partial z} = \frac{\partial u}{\partial X}\,.\,2y + \frac{\partial u}{\partial Y}\,.\,2x \quad ...(3)$$

Multiply (1) by $y^2 - zx$; (2) by $(x^2 - yz)$, (3) by $z^2 - xy$ and add

$$(y^2 - zx)\frac{\partial u}{\partial x} + (x^2 - yz)\frac{\partial u}{\partial y} + (z^2 - xy)\frac{\partial u}{\partial z}$$

$$= \frac{\partial u}{\partial X}(2xy^2 - 2zx^2 + 2x^2z - 2yz^2 + 2yz^2 - 2xy^2)$$

$$+ \frac{\partial u}{\partial Y}[2y^2z - 2z^2x + 2x^2y - 2y^2z + 2z^2x - 2x^2y]$$

$$= \frac{\partial u}{\partial X}\cdot 0 + \frac{\partial u}{\partial Y}\cdot 0 = 0.$$

Example 6. *If $w = f(x, y)$, $x = r\cos\theta$, $y = r\sin\theta$ show that*

$$\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2. \qquad \text{(P.T.U., Dec. 2007)}$$

Sol. The given equations define w as a composite function of r and θ.

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r} = \frac{\partial w}{\partial x}\cdot\cos\theta + \frac{\partial w}{\partial y}\cdot\sin\theta$$

or $$\frac{\partial w}{\partial r} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta \qquad ...(1) \quad [\because \; w = f(x, y)]$$

Also $$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial \theta} = \frac{\partial w}{\partial x}(-r\sin\theta) + \frac{\partial w}{\partial y}(r\cos\theta)$$

or $$\frac{1}{r}\frac{\partial w}{\partial \theta} = -\frac{\partial f}{\partial x}\sin\theta + \frac{\partial f}{\partial y}\cos\theta \qquad ...(2)$$

Squaring and adding (1) and (2), we get

$$\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2.$$

Example 7. *If $u = xe^y z$, where $y = \sqrt{a^2 - x^2}$, $z = \sin^2 x$, find $\frac{du}{dx}$.*

Sol. Here u is a function of x, y and z while y and z are functions of x.

$\therefore$ u is a composite function of x.

$$\therefore \quad \frac{du}{dx} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dx}$$

$$= e^y z \cdot 1 + xe^y z \cdot \tfrac{1}{2}(a^2 - x^2)^{-1/2}(-2x) + xe^y \cdot 2\sin x\cos x$$

$$= e^y\left[z - \frac{x^2 z}{\sqrt{a^2 - x^2}} + x\sin 2x\right].$$

Example 8. *Find $\frac{du}{dx}$ if $u = \sin(x^2 + y^2)$, where $a^2x^2 + b^2y^2 = c^2$.*

Sol. The given equations are the form $u = f(x, y)$ and $\phi(x, y) = k$

$\therefore$ u is a composite function of x.

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx} \qquad ...(1)$$

Now, $$\frac{\partial u}{\partial x} = 2x\cos(x^2 + y^2), \quad \frac{\partial u}{\partial y} = 2y\cos(x^2 + y^2)$$

Also, differentiating $a^2x^2 + b^2y^2 = c$ w.r.t. x, we have

$$2a^2x + 2b^2y \cdot \frac{dy}{dx} = 0 \quad \text{or} \quad \frac{dy}{dx} = -\frac{a^2x}{b^2y}$$

$\therefore$ From (1), $$\frac{du}{dx} = 2x \cos(x^2 + y^2) + 2y \cos(x^2 + y^2) \cdot \left[-\frac{a^2x}{b^2y}\right]$$

$$= 2\left[x - \frac{a^2x}{b^2}\right] \cos(x^2 + y^2) = \frac{2(b^2 - a^2)x}{b^2} \cdot \cos(x^2 + y^2).$$

Example 9. *Find* $\frac{dy}{dx}$, *when*

(*i*) $x^y + y^x = c$ (*ii*) $(\cos x)^y = (\sin y)^x$.

Sol. (*i*) Let $f(x, y) = x^y + y^x$, then $f(x, y) = c$

[Using art. 3.9*b*] $$\frac{dy}{dx} = -\frac{f_x}{f_y}$$

(Find f_x and f_y)

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^y) + \frac{\partial}{\partial x}(y^x) = yx^{y-1} + y^x \log y$$

$$f_y = \frac{\partial}{\partial y}(x^y) + \frac{\partial}{\partial y}(y^x) = x^y \log x + xy^{x-1}$$

$\Rightarrow$ $$\frac{dy}{dx} = -\frac{yx^{y-1} + y^x \log y}{x^y \log x + xy^{x-1}}.$$

(*ii*) Let $f(x, y) = (\cos x)^y - (\sin y)^x = 0$

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{y(\cos x)^{y-1} \cdot (-\sin x) - (\sin y)^x \log(\sin y)}{(\cos x)^y \log(\cos x) - x(\sin y)^{x-1} \cdot \cos y}$$

$$= \frac{y(\cos x)^{y-1} \sin x + (\cos x)^y \log(\sin y)}{(\cos x)^y \log(\cos x) - x(\cos x)^y (\sin y)^{-1} \cos y}$$

$[\because \quad (\sin y)^x = (\cos x)^y]$

$$= \frac{(\cos x)^y \left[y \cdot \frac{\sin x}{\cos x} + \log \sin y\right]}{(\cos x)^y [\log \cos x - x \cot y]} = \frac{y \tan x + \log \sin y}{\log \cos x - x \cot y}.$$

Example 10. *If* $z = xy\, f\left(\frac{y}{x}\right)$ *and z is constant show that* $\dfrac{f'\left(\frac{y}{x}\right)}{f\left(\frac{y}{x}\right)} = \dfrac{x\left[y + x\frac{dy}{dx}\right]}{y\left[y - x\frac{dy}{dx}\right]}$.

Sol. $z = xy\, f\left(\frac{y}{x}\right)$; z is constant

$\therefore$ Let $\phi(x, y) = xy\, f\left(\frac{y}{x}\right)$, $\phi(x, y) = c$

[Using art. 3.9(*b*)] $$\frac{dy}{dx} = -\frac{\phi_x}{\phi_y} \qquad \text{...(1)}$$

$$\phi_x = \frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x}\left[xy\, f\left(\frac{y}{x}\right)\right]$$

$$= y\left\{x\, f'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right) + f\left(\frac{y}{x}\right)\right\}$$

$$= y\left[-\frac{y}{x} f'\left(\frac{y}{x}\right) + f\left(\frac{y}{x}\right)\right]$$

$$\phi_y = \frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y}\left[xy\, f\left(\frac{y}{x}\right)\right] = x\left[y\, f'\left(\frac{y}{x}\right)\cdot\frac{1}{x} + f\left(\frac{y}{x}\right)\right]$$

$$= x\left[\frac{y}{x} f'\left(\frac{y}{x}\right) + f\left(\frac{y}{x}\right)\right]$$

From (1),
$$\frac{dy}{dx} = -\frac{y\left[-\frac{y}{x} f'\left(\frac{y}{x}\right) + f\left(\frac{y}{x}\right)\right]}{x\left[\frac{y}{x} f'\left(\frac{y}{x}\right) + f\left(\frac{y}{x}\right)\right]}$$

or
$$y\, f'\left(\frac{y}{x}\right)\frac{dy}{dx} + xf\left(\frac{y}{x}\right)\frac{dy}{dx} = \frac{y^2}{x} f'\left(\frac{y}{x}\right) - yf\left(\frac{y}{x}\right)$$

or
$$f'\left(\frac{y}{x}\right)\left\{y\frac{dy}{dx} - \frac{y^2}{x}\right\} = f\left(\frac{y}{x}\right)\left[-x\frac{dy}{dx} - y\right]$$

or
$$\frac{f'\left(\frac{y}{x}\right)}{f\left(\frac{y}{x}\right)} = \frac{-\left(y + x\frac{dy}{dx}\right)}{-\frac{y}{x}\left(y - x\frac{dy}{dx}\right)} = \frac{x}{y}\cdot\frac{y + x\frac{dy}{dx}}{y - x\frac{dy}{dx}}.$$

Example 11. *If* $\phi(x, y, z) = 0$, *show that* $\left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial x}{\partial y}\right)_z = -1.$

(P.T.U., Dec. 2011)

Sol. The given relation defines y as a function of x and z. Treating x as constant

$$\left(\frac{\partial y}{\partial z}\right)_x = -\frac{\frac{\partial \phi}{\partial z}}{\frac{\partial \phi}{\partial y}}$$

The given relation defines z as a function of x and y. Treating y as constant

$$\left(\frac{\partial z}{\partial x}\right)_y = -\frac{\frac{\partial \phi}{\partial x}}{\frac{\partial \phi}{\partial z}}$$

Similarly,
$$\left(\frac{\partial x}{\partial y}\right)_z = -\frac{\frac{\partial \phi}{\partial y}}{\frac{\partial \phi}{\partial x}}$$

Multiplying, we get

$$\left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial x}{\partial y}\right)_z = -1.$$

Example 12. *If z is a function of x and y and u, v be two other variables such that $u = lx + my$, $v = ly - mx$, show that* $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = (l^2 + m^2)\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right)$. **(P.T.U., May 2004)**

Sol. z is a composite function of x, y

$$\therefore \quad \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} = l\frac{\partial z}{\partial u} - m\frac{\partial z}{\partial v}$$

$$\Rightarrow \quad \frac{\partial}{\partial x} = l\frac{\partial}{\partial u} - m\frac{\partial}{\partial v}$$

Now
$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \left(l\frac{\partial}{\partial u} - m\frac{\partial}{\partial v}\right)\left(l\frac{\partial z}{\partial u} - m\frac{\partial z}{\partial v}\right)$$

$$= l^2\frac{\partial^2 z}{\partial u^2} - lm\frac{\partial^2 z}{\partial u \partial v} - lm\frac{\partial^2 z}{\partial v \partial u} + m^2\frac{\partial^2 z}{\partial v^2}$$

$$\therefore \quad \frac{\partial^2 z}{\partial x^2} = l^2\frac{\partial^2 z}{\partial u^2} - 2lm\frac{\partial^2 z}{\partial u \partial v} + m^2\frac{\partial^2 z}{\partial v^2} \quad \text{...(1)}$$

Now
$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} = m\frac{\partial z}{\partial u} + l\frac{\partial z}{\partial v}$$

$$\frac{\partial}{\partial y} = m\frac{\partial}{\partial u} + l\frac{\partial}{\partial v}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \left(m\frac{\partial}{\partial u} + l\frac{\partial}{\partial v}\right)\left(m\frac{\partial z}{\partial u} + l\frac{\partial z}{\partial v}\right)$$

$$= m^2\frac{\partial^2 z}{\partial u^2} + 2lm\frac{\partial^2 u}{\partial u \partial v} + l^2\frac{\partial^2 z}{\partial v^2} \quad \text{...(2)}$$

Adding (1) and (2),

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = (l^2 + m^2)\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right).$$

Example 13. *Prove that* $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial y^2}$

where $x = u\cos\alpha - v\sin\alpha$, $y = u\sin\alpha + v\cos\alpha$ **(P.T.U., Dec. 2011)**

Or

By changing the independent variables u and v to x and y by means of the relations $x = u\cos\alpha - v\sin\alpha$, $y = u\sin\alpha + v\cos\alpha$, *show that* $\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}$ *transforms into* $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}$.

Sol. Here z is a composite function of u and v

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} = \cos\alpha\frac{\partial z}{\partial x} + \sin\alpha\frac{\partial z}{\partial y}$$

or $$\frac{\partial}{\partial u}(z) = \left(\cos\alpha \frac{\partial}{\partial x} + \sin\alpha \frac{\partial}{\partial y}\right) z \Rightarrow \frac{\partial}{\partial u} \equiv \cos\alpha \frac{\partial}{\partial x} + \sin\alpha \frac{\partial}{\partial y} \quad \ldots(1)$$

Also $$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v} = -\sin\alpha \frac{\partial z}{\partial x} + \cos\alpha \frac{\partial z}{\partial y}$$

or $$\frac{\partial}{\partial v}(z) = \left(-\sin\alpha \frac{\partial}{\partial x} + \cos\alpha \frac{\partial}{\partial y}\right) z$$

$\Rightarrow$ $$\frac{\partial}{\partial v} \equiv -\sin\alpha \frac{\partial}{\partial x} + \cos\alpha \frac{\partial}{\partial y} \quad \ldots(2)$$

Now we shall make use of the equivalence of operators as given by (1) and (2).

$$\frac{\partial^2 z}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right) = \left(\cos\alpha \frac{\partial}{\partial x} + \sin\alpha \frac{\partial}{\partial y}\right)\left(\cos\alpha \frac{\partial z}{\partial x} + \sin\alpha \frac{\partial z}{\partial y}\right)$$

$$= \cos^2\alpha \frac{\partial^2 z}{\partial x^2} + \cos\alpha \sin\alpha \frac{\partial^2 z}{\partial x \partial y} + \sin\alpha\cos\alpha \frac{\partial^2 z}{\partial y \partial x} + \sin^2\alpha \frac{\partial^2 z}{\partial y^2}$$

$$= \cos^2\alpha \frac{\partial^2 z}{\partial x^2} + 2\cos\alpha\sin\alpha \frac{\partial^2 z}{\partial x \partial y} + \sin^2\alpha \frac{\partial^2 z}{\partial y^2} \quad \ldots(3)$$

$$\frac{\partial^2 z}{\partial v^2} = \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right) = \left(-\sin\alpha \frac{\partial}{\partial x} + \cos\alpha \frac{\partial}{\partial y}\right)\left(-\sin\alpha \frac{\partial z}{\partial x} + \cos\alpha \frac{\partial z}{\partial y}\right)$$

$$= \sin^2\alpha \frac{\partial^2 z}{\partial x^2} - \sin\alpha\cos\alpha \frac{\partial^2 z}{\partial x \partial y} - \cos\alpha\sin\alpha \frac{\partial^2 z}{\partial y \partial x} + \cos^2\alpha \frac{\partial^2 z}{\partial y^2}$$

$$= \sin^2\alpha \frac{\partial^2 z}{\partial x^2} - 2\cos\alpha\sin\alpha \frac{\partial^2 z}{\partial x \partial y} + \cos^2\alpha \frac{\partial^2 z}{\partial y^2} \quad \ldots(4)$$

Adding (3) and (4), $\dfrac{\partial^2 z}{\partial u^2} + \dfrac{\partial^2 z}{\partial v^2} = \dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2}$.

Example 14. *Transform the equation* $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ *into polar co-ordinates.*

Sol. The relations connecting cartesian co-ordinates, (x, y) with polar co-ordinates (r, θ) are

$$x = r\cos\theta,\ y = r\sin\theta$$

Squaring and adding, $r^2 = x^2 + y^2$

Dividing, $\tan\theta = \dfrac{y}{x}$

$\therefore$ $$r = \sqrt{x^2 + y^2},\ \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \frac{r\cos\theta}{r} = \cos\theta$$

$$\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} = \frac{r\sin\theta}{r} = \sin\theta$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}\cdot\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2} = -\frac{r\sin\theta}{r^2} = -\frac{\sin\theta}{r}$$

$$\frac{\partial\theta}{\partial y}=\frac{1}{1+\frac{y^2}{x^2}}\cdot\frac{1}{x}=\frac{x}{x^2+y^2}=\frac{r\cos\theta}{r^2}=\frac{\cos\theta}{r}$$

Here u is a composite function of x and y

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x}+\frac{\partial u}{\partial\theta}\cdot\frac{\partial\theta}{\partial x}=\cos\theta\frac{\partial u}{\partial r}-\frac{\sin\theta}{r}\frac{\partial u}{\partial\theta}$$

or $$\frac{\partial}{\partial x}(u)=\left(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)u$$

$\Rightarrow$ $$\frac{\partial}{\partial x}\equiv\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta} \quad ...(1)$$

Also $$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y}+\frac{\partial u}{\partial\theta}\cdot\frac{\partial\theta}{\partial y}=\sin\theta\frac{\partial u}{\partial r}+\frac{\cos\theta}{r}\frac{\partial u}{\partial\theta}$$

or $$\frac{\partial}{\partial y}(u)=\left(\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)u \;\Rightarrow\; \frac{\partial}{\partial y}\equiv\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta} \quad ...(2)$$

Now we shall make use of the equivalence of cartesian and polar operators as given by (1) and (2),

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=\left(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\cos\theta\frac{\partial u}{\partial r}-\frac{\sin\theta}{r}\frac{\partial u}{\partial\theta}\right)$$

$$=\cos\theta\frac{\partial}{\partial r}\left(\cos\theta\frac{\partial u}{\partial r}-\frac{\sin\theta}{r}\frac{\partial u}{\partial\theta}\right)-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial u}{\partial r}-\frac{\sin\theta}{r}\frac{\partial u}{\partial\theta}\right)$$

$$=\cos\theta\left[\cos\theta\frac{\partial^2 u}{\partial r^2}-\sin\theta\frac{\partial u}{\partial\theta}\left(-\frac{1}{r^2}\right)-\frac{\sin\theta}{r}\cdot\frac{\partial^2 u}{\partial r\partial\theta}\right]$$

$$-\frac{\sin\theta}{r}\left[-\sin\theta\frac{\partial u}{\partial r}+\cos\theta\frac{\partial^2 u}{\partial\theta\partial r}-\frac{\cos\theta}{r}\frac{\partial u}{\partial\theta}-\frac{\sin\theta}{r}\frac{\partial^2 u}{\partial\theta^2}\right]$$

$$=\cos^2\theta\frac{\partial^2 u}{\partial r^2}+\frac{2\cos\theta\sin\theta}{r^2}\frac{\partial u}{\partial\theta}+\frac{\sin^2\theta}{r}\frac{\partial u}{\partial r}-\frac{2\cos\theta\sin\theta}{r^2}\frac{\partial^2 u}{\partial r\partial\theta}+\frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2} \quad ...(3)$$

$$\frac{\partial^2 u}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)=\left(\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\sin\theta\frac{\partial u}{\partial r}+\frac{\cos\theta}{r}\frac{\partial u}{\partial\theta}\right)$$

$$=\sin\theta\frac{\partial}{\partial r}\left(\sin\theta\frac{\partial u}{\partial r}+\frac{\cos\theta}{r}\frac{\partial u}{\partial\theta}\right)+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial r}+\frac{\cos\theta}{r}\frac{\partial u}{\partial\theta}\right)$$

$$=\sin\theta\left[\sin\theta\frac{\partial^2 u}{\partial r^2}-\frac{\cos\theta}{r^2}\frac{\partial u}{\partial\theta}+\frac{\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta}\right]$$

$$+\frac{\cos\theta}{r}\left[\cos\theta\frac{\partial u}{\partial r}+\sin\theta\frac{\partial^2 u}{\partial\theta\partial r}-\frac{\sin\theta}{r}\frac{\partial u}{\partial\theta}+\frac{\cos\theta}{r}\frac{\partial^2 u}{\partial\theta^2}\right]$$

$$=\sin^2\theta\frac{\partial^2 u}{\partial r^2}-\frac{2\cos\theta\sin\theta}{r^2}\frac{\partial u}{\partial\theta}+\frac{\cos^2\theta}{r}\frac{\partial u}{\partial r}+\frac{2\cos\theta\sin\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta}+\frac{\cos^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2} \quad ...(4)$$

Adding (3) and (4), $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=\dfrac{\partial^2 u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2}$

$\therefore$ $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$ transforms into $\dfrac{\partial^2 u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2}=0.$

Example 15. *If* $x = e^r \cos\theta,\ y = e^r \sin\theta$, *show that* $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=e^{-2r}\left[\dfrac{\partial^2 u}{\partial r^2}+\dfrac{\partial^2 u}{\partial \theta^2}\right]$ *where* $u = f(x, y)$.

Sol. $u = f(x, y)\ ;\ x = e^r \cos\theta,\ y = e^r\sin\theta$

u is a function of x, y; x, y are functions of r, θ

$\therefore$ u is a composite function of r, θ

$$\therefore\quad \frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}=(e^r\cos\theta)\frac{\partial u}{\partial x}+(e^r\sin\theta)\frac{\partial u}{\partial y}=x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}$$

and
$$\frac{\partial u}{\partial \theta}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \theta}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \theta}=(-e^r\sin\theta)\frac{\partial u}{\partial x}+(e^r\cos\theta)\frac{\partial u}{\partial y}$$

$$=-y\frac{\partial u}{\partial x}+x\frac{\partial u}{\partial y}$$

$$\therefore\quad \frac{\partial u}{\partial r}=\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)u \quad\text{or}\quad \frac{\partial}{\partial r}=x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}$$

$$\frac{\partial u}{\partial \theta}=\left(-y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}\right)u \quad\text{or}\quad \frac{\partial}{\partial \theta}=-y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}$$

Now
$$\frac{\partial^2 u}{\partial r^2}=\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\right)=\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

$$=x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2} \qquad ...(1)$$

$$\frac{\partial^2 u}{\partial \theta^2}=\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial \theta}\right)=\left(-y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}\right)\left(-y\frac{\partial u}{\partial x}+x\frac{\partial u}{\partial y}\right)$$

$$=y^2\frac{\partial^2 u}{\partial x^2}-2xy\frac{\partial^2 u}{\partial x\partial y}+x^2\frac{\partial^2 u}{\partial y^2} \qquad ...(2)$$

Adding (1) and (2),

$$\frac{\partial^2 u}{\partial r^2}+\frac{\partial^2 u}{\partial \theta^2}=(x^2+y^2)\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right]$$

$$=[e^{2r}\cos^2\theta+e^{2r}\sin^2\theta]\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right]=e^{2r}\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right]$$

Hence
$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=e^{-2r}\left[\frac{\partial^2 u}{\partial r^2}+\frac{\partial^2 u}{\partial \theta^2}\right].$$

Example 16. *If $u = f(x, y, z)$ and $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r\cos\theta$, then show that* $\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2 = \left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2 + \frac{1}{r^2 \sin^2\theta}\left(\frac{\partial f}{\partial \phi}\right)^2$. **(P.T.U., Dec. 2004)**

Sol.
$$u = f(x, y, z)$$
$$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$$
$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial r}$$
$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \theta} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial \theta}$$
$$\frac{\partial f}{\partial \phi} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial \phi} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \phi} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial \phi}$$

or
$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}(\sin\theta\cos\phi) + \frac{\partial f}{\partial y}(\sin\theta\sin\phi) + \frac{\partial f}{\partial z}(\cos\theta)$$
$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}(r\cos\theta\cos\phi) + \frac{\partial f}{\partial y}(r\cos\theta\sin\phi) + \frac{\partial f}{\partial z}(-r\sin\theta)$$
$$\frac{\partial f}{\partial \phi} = \frac{\partial f}{\partial x}(-r\sin\theta\sin\phi) + \frac{\partial f}{\partial y}(r\sin\theta\cos\phi) + 0$$
$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}(\sin\theta\cos\phi) + \frac{\partial f}{\partial y}(\sin\theta\sin\phi) + \frac{\partial f}{\partial z}(\cos\theta)$$
$$\frac{1}{r}\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}(\cos\theta\cos\phi) + \frac{\partial f}{\partial y}(\cos\theta\sin\phi) + \frac{\partial f}{\partial z}(-\sin\theta)$$
$$\frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi} = \frac{\partial f}{\partial x}(-\sin\phi) + \frac{\partial f}{\partial y}(\cos\phi)$$

Squaring and adding all the above three functions, we get
$$\left(\frac{\partial f}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial f}{\partial \theta}\right)^2 + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial f}{\partial \phi}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2.$$
all other terms cancel with each other proved.

Example 17. *Show that* $a\frac{\partial w}{\partial y} = b\frac{\partial w}{\partial x}$ *if a and b are constants and $w = f(ax + by)$ is a differentiable function of $u = ax + by$.* **(P.T.U., May 2005)**

Sol. Given $w = f(ax + by)$

and $u = ax + by$ ...(1)

$\therefore$ $w = f(u)$ where u is a function of x and y

Now
$$\frac{\partial w}{\partial x} = \frac{\partial}{\partial x} f(u) = \frac{\partial f}{\partial u}\times\frac{\partial u}{\partial x} = \frac{\partial f}{\partial u}\cdot a \qquad \left[\text{from (1), } \frac{\partial u}{\partial x} = a\right]$$
$$\frac{\partial w}{\partial y} = \frac{\partial}{\partial y} f(u) = \frac{\partial f}{\partial u}\times\frac{\partial u}{\partial y} = \frac{\partial f}{\partial u}\times b \qquad \left[\text{from (1), } \frac{\partial u}{\partial y} = b\right]$$

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$$\therefore \qquad a\frac{\partial w}{\partial y} = ab\frac{\partial f}{\partial u};$$

$$b\frac{\partial w}{\partial x} = ab\frac{\partial f}{\partial u}$$

Hence $$a\frac{\partial w}{\partial y} = b\frac{\partial w}{\partial x}.$$

TEST YOUR KNOWLEDGE

1. (*i*) If $u = x^2 + y^2 + z^2$ and $x = e^{2t}, y = e^{2t}\cos 3t, z = e^{2t}\sin 3t$; find $\frac{du}{dt}$ as a total derivative and verify the result by direct substitution.

   (*ii*) If $u = x^3 + y^3$ where $x = a\cos t, y = b\sin t$, find $\frac{du}{dt}$ and verify result by direct substitution.

2. (*i*) If $z = u^2 + v^2, u = r\cos\theta, v = r\sin\theta$, find $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$.

   (*ii*) If $z = \log(u^2 + v); u = e^{x^2+y^2}, v = x^2 + y$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

3. If $u = f(r, s), r = x + y, s = x - y$, show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2\frac{\partial u}{\partial r}$.

4. If $x = u + v, y = uv$ and z is a function of x, y; show that $u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} + 2y\frac{\partial z}{\partial y}$.

5. If $u = f(r, s), r = x + at, s = y + bt$, show that $\frac{\partial u}{\partial t} = a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y}$.

6. If $u = x\log(xy)$, where $x^3 + y^3 + 3xy = 1$, find $\frac{du}{dx}$.

7. (*i*) If $u = f(r, s, t)$ and $r = \frac{x}{y}, s = \frac{y}{z}, t = \frac{z}{x}$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$.

   [**Hint.** S.E. 4(*b*)] ***(P.T.U., May 2003)***

   (*ii*) If $u = f(2x - 3y, 3y - 4z, 4z - 2x)$, prove that $\frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z} = 0$. ***(P.T.U., Dec. 2011)***

8. (*i*) If $z = x^2y$ and $x^2 + xy + y^2 = 1$, show that $\frac{dz}{dx} = 2xy - \frac{x^2(2x + y)}{x + 2y}$.

   (*ii*) If $z = \sqrt{x^2 + y^2}$ and $x^3 + y^3 + 3axy = 5a^2$, find the value of $\frac{dz}{dx}$ when $x = y = a$.

   (P.T.U., May 2009)

9. If $x^3 + 3x^2y + 6xy^2 + y^3 = 1$, find $\frac{dy}{dx}$.

10. If $x^y = y^x$, show that $\frac{dy}{dx} = \frac{y(y - x\log y)}{x(x - y\log x)}$ using partial derivative method.

    [**Hint.** Consult S.E. 9(*a*)]

11. If $x^3 + y^3 - 3axy = 0$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ using partial derivative method.

12. Prove that if $y^3 - 3ax^2 + x^3 = 0$, then $\frac{d^2y}{dx^2} + \frac{2a^2x^2}{y^5} = 0$.

13. If $f(x, y) = 0$, $\phi(y, z) = 0$, show that $\frac{\partial f}{\partial y} \cdot \frac{\partial \phi}{\partial z} \cdot \frac{dz}{dx} = \frac{\partial f}{\partial x} \cdot \frac{\partial \phi}{\partial y}$

14. If $x = \rho \cos \phi$, $y = \rho \sin \phi$, and $v = f(x, y)$ show that $\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 = \left(\frac{\partial v}{\partial \rho}\right)^2 + \frac{1}{\rho^2}\left(\frac{\partial v}{\partial \phi}\right)^2$

[**Hint.** See S.E. 6]

15. By changing the independent variables x and y to u and v by means of the relations $u = x - ay$, $v = x + ay$, show that $a^2 \frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2}$ transforms into $4a^2 \frac{\partial^2 z}{\partial u \partial v}$.

[**Hint.** Consult S.E. 13]

16. Prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial t^2}$, where $x = s \cos \alpha - t \sin \alpha$ and $y = s \sin \alpha + t \cos \alpha$.

[**Hint.** Consult S.E. 13]

17. If by substitution $u = x^2 - y^2$, $v = 2xy$, $f = \phi(u, v)$, show that $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = u(x^2 + y^2)\left[\frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2}\right]$.

Answers

1. (*i*) $8e^{4t}$ (*ii*) $3 \sin t \cos t\,(b^3 \sin t - a^3 \cos t)$

2. (*i*) $2r, 0$ (*ii*) $\frac{2x}{u^2 + v}(2u^2 + 1), \frac{1}{u^2 + v}(4yu^2 + 1)$

6. $1 + \log(xy) - \frac{x(x^2 + y)}{y(y^2 + x)}$

8. (*ii*) $= 0$ 9. $-\frac{x^2 + 2xy + 2y^2}{x^2 + 4xy + y^2}$ 11. $\frac{x^2 - ay}{ax - y^2}, \frac{2a^3xy}{(ax - y^2)^3}$.

3.10. JACOBIANS

If u and v are functions of two independent variables x and y, then the determinant $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ is called Jacobian of u, v with respect to x, y and is denoted by the symbol $J\left(\frac{u, v}{x, y}\right)$ or $\frac{\partial(u, v)}{\partial(x, y)}$.

Similarly, if u, v, w be functions of x, y, z, then the Jacobian of u, v, w with respect to x, y, z is

$$J\left(\frac{u, v, w}{x, y, z}\right) \text{ or } \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}.$$

3.11. PROPERTIES OF JACOBIANS

I. *If u, v are functions of r, s where r, s are functions of x, y, then*

$$J\left(\frac{u, v}{x, y}\right) = \frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial(u, v)}{\partial(r, s)} \times \frac{\partial(r, s)}{\partial(x, y)}.$$

Proof. Since u, v are composite functions of x, y

$$\therefore \quad \left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} = u_r r_x + u_s s_x \\ \frac{\partial u}{\partial y} &= \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y} = u_r r_y + u_s s_y \\ \frac{\partial v}{\partial x} &= \frac{\partial v}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial v}{\partial s}\cdot\frac{\partial s}{\partial x} = v_r r_x + v_s s_x \\ \frac{\partial v}{\partial y} &= \frac{\partial v}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial v}{\partial s}\cdot\frac{\partial s}{\partial y} = v_r r_y + v_s s_y \end{aligned}\right] \quad \text{...(A)}$$

Now $$\frac{\partial(u,v)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)} = \begin{vmatrix} u_r & u_s \\ v_r & v_s \end{vmatrix} \cdot \begin{vmatrix} r_x & r_y \\ s_x & s_y \end{vmatrix}$$

Interchanging rows and columns in the second determinant

$$= \begin{vmatrix} u_r & u_s \\ v_r & v_s \end{vmatrix} \cdot \begin{vmatrix} r_x & s_x \\ r_y & s_y \end{vmatrix} = \begin{vmatrix} u_r r_x + u_s s_x & u_r r_y + u_s s_y \\ v_r r_x + v_s s_x & v_r r_y + v_s s_y \end{vmatrix} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \quad \text{[Using (A)]}$$

$$= \frac{\partial(u, v)}{\partial(x, y)}.$$

II. *If J_1 is the Jacobian of u, v, with respect to x, y and J_2 is the Jacobian of x, y, with respect to u, v, then $J_1 J_2 = 1$ i.e.,* $\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(u,v)} = 1.$ **(P.T.U., Dec. 2012)**

Proof. Let $u = u(x, y)$ and $v = v(x, y)$, so that u and v are functions of x, y.
Differentiating partially w.r.t. u and v, we get

$$\left.\begin{aligned} 1 &= \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial u} = u_x x_u + u_y y_u \\ 0 &= \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial v} = u_x x_v + u_y y_v \\ 0 &= \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial u} = v_x x_u + v_y y_u \\ 1 &= \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial v} = v_x x_v + v_y y_v \end{aligned}\right] \quad \text{...(B)}$$

Now, $\dfrac{\partial(u,v)}{\partial(x,y)}\cdot\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$

Interchanging rows and columns in the second determinant

$$= \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}\begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} = \begin{vmatrix} u_x x_u + u_y y_u & u_x x_v + u_y y_v \\ v_x x_u + v_y y_u & v_x x_v + v_y y_v \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1. \qquad \text{[Using (B)]}$$

ILLUSTRATIVE EXAMPLES

Example 1. *(i) If $x = r\cos\theta$, $y = r\sin\theta$, verify* $\dfrac{\partial(x,y)}{\partial(r,\theta)}\cdot\dfrac{\partial(r,\theta)}{\partial(x,y)} = 1$. **(P.T.U., May 2005, Dec. 2011)**

(ii) If $x = r\cos\theta$, $y = r\sin\theta$, $z = z$ evaluate $\dfrac{\partial(x,y,z)}{\partial(r,\theta,z)}$. **(P.T.U., Dec. 2013)**

(iii) If $x = u(1+v)$, $y = v(1+u)$, find the value of $\dfrac{\partial(x,y)}{\partial(u,v)}$. **(P.T.U., May 2014)**

Sol. (*i*) $x = r\cos\theta,\ y = r\sin\theta$

Squaring and adding $r^2 = x^2 + y^2 \quad \therefore \quad r = \sqrt{x^2+y^2}$

Dividing $\tan\theta = \dfrac{y}{x} \quad \therefore \quad \theta = \tan^{-1}\dfrac{y}{x}$

Now, $$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

$$\frac{\partial r}{\partial x} = \frac{1}{2\sqrt{x^2+y^2}}\cdot 2x = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r}$$

$$\frac{\partial r}{\partial y} = \frac{1}{2\sqrt{x^2+y^2}}\cdot 2y = \frac{y}{\sqrt{x^2+y^2}} = \frac{y}{r}$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+\dfrac{y^2}{x^2}}(-y/x^2) = \frac{y}{x^2+y^2} = -\frac{y}{r^2}$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{1+\dfrac{y^2}{x^2}}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2} = \frac{x}{r^2}$$

$$\therefore \quad \frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial r}{\partial x} & \dfrac{\partial r}{\partial y} \\ \dfrac{\partial \theta}{\partial x} & \dfrac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \dfrac{x}{r} & \dfrac{y}{r} \\ -\dfrac{y}{r^2} & \dfrac{x}{r^2} \end{vmatrix} = \frac{x^2+y^2}{r^3} = \frac{r^2}{r^3} = \frac{1}{r}$$

$$\therefore \quad \frac{\partial(x,y)}{\partial(r,\theta)} \cdot \frac{\partial(r,\theta)}{\partial(x,y)} = r \cdot \frac{1}{r} = 1.$$

(ii) $x = r\cos\theta,\ y = r\sin\theta,\ z = z$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

(iii) $x = u(1+v) \qquad y = v(1+u)$

$$\frac{\partial x}{\partial u} = 1 + v \qquad \frac{\partial y}{\partial u} = v$$

$$\frac{\partial x}{\partial v} = u \qquad \frac{\partial y}{\partial v} = 1 + u$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1+v & u \\ v & 1+u \end{vmatrix}$$

$$= (1+v)(1+u) - uv$$

$$= 1 + v + u.$$

Example 2. *If* $u = \dfrac{x+y}{1-xy}$, $v = \tan^{-1} x + \tan^{-1} y$, *find* $\dfrac{\partial(u,v)}{\partial(x,y)}$. **(P.T.U., May 2012)**

Sol. $$u = \frac{x+y}{1-xy} \quad \therefore \quad \frac{\partial u}{\partial x} = \frac{(1-xy)\cdot 1 - (x+y)(-y)}{(1-xy)^2} = \frac{1+y^2}{(1-xy)^2}$$

$$\frac{\partial u}{\partial y} = \frac{(1-xy)\cdot 1 - (x+y)(-x)}{(1-xy)^2} = \frac{1+x^2}{(1-xy)^2}$$

$$v = \tan^{-1} x + \tan^{-1} y; \ \frac{\partial v}{\partial x} = \frac{1}{1+x^2}, \ \frac{\partial v}{\partial y} = \frac{1}{1+y^2}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix} = 0.$$

Example 3. *If* $x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$, *show that* $\dfrac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$.

Sol. $$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\cos\phi & r\sin\theta\sin\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

Taking out common factors (r from second column and $r \sin\theta$ from third column)

$$= r^2 \sin\theta \begin{vmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{vmatrix}$$

Expanding by third row

$$= r^2 \sin\theta \left\{ \cos\theta \begin{vmatrix} \cos\theta\cos\phi & -\sin\phi \\ \cos\theta\sin\phi & \cos\phi \end{vmatrix} + \sin\theta \begin{vmatrix} \sin\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\phi \end{vmatrix} \right\}$$

$$= r^2 \sin\theta\,[\cos\theta\,(\cos\theta\cos^2\phi + \cos\theta\sin^2\phi) + \sin\theta\,(\sin\theta\cos^2\phi + \sin\theta\sin^2\phi)]$$

$$= r^2 \sin\theta\,(\cos^2\theta + \sin^2\theta) = r^2 \sin\theta.$$

Example 4. *If* $u = x \sin y$, $v = y \sin x$ *find* $\dfrac{\partial(u,v)}{\partial(x,y)}$ **(P.T.U., May 2006)**

Sol. $u = x \sin y \qquad v = y \sin x$

$$\therefore \quad \frac{\partial u}{\partial x} = \sin y \qquad \frac{\partial v}{\partial x} = y \cos x$$

$$\frac{\partial u}{\partial y} = x \cos y \qquad \frac{\partial v}{\partial y} = \sin x$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \sin y & x\cos y \\ y\cos x & \sin x \end{vmatrix}$$

$$= \sin x \sin y - xy \cos x \cos y.$$

Example 5. *If* $u = \dfrac{x}{y-z}$, $v = \dfrac{y}{z-x}$, $w = \dfrac{z}{x-y}$, *show that* $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = 0$.

Sol.

$$u = \frac{x}{y-z};\ \frac{\partial u}{\partial x} = \frac{1}{y-z},\ \frac{\partial u}{\partial y} = \frac{-x}{(y-z)^2},\ \frac{\partial u}{\partial z} = \frac{x}{(y-z)^2}$$

$$v = \frac{y}{z-x};\ \frac{\partial v}{\partial x} = \frac{y}{(z-x)^2},\ \frac{\partial v}{\partial y} = \frac{1}{z-x},\ \frac{\partial v}{\partial z} = \frac{-y}{(z-x)^2}$$

$$w = \frac{z}{x-y};\ \frac{\partial w}{\partial x} = \frac{-z}{(x-y)^2},\ \frac{\partial w}{\partial y} = \frac{z}{(x-y)^2},\ \frac{\partial w}{\partial z} = \frac{1}{x-y}$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \dfrac{1}{y-z} & \dfrac{-x}{(y-z)^2} & \dfrac{x}{(y-z)^2} \\ \dfrac{y}{(z-x)^2} & \dfrac{1}{z-x} & \dfrac{-y}{(z-x)^2} \\ \dfrac{-z}{(x-y)^2} & \dfrac{z}{(x-y)^2} & \dfrac{1}{x-y} \end{vmatrix}$$

$$= \frac{1}{(x-y)^2(y-z)^2(z-x)^2} \begin{vmatrix} y-z & -x & x \\ y & z-x & -y \\ -z & z & x-y \end{vmatrix}$$

Operate $R_1 \to R_1 - R_2 - R_3$

$$= \frac{1}{(x-y)^2(y-z)^2(z-x)^2} \begin{vmatrix} 0 & -2z & 2y \\ y & z-x & -y \\ -z & z & x-y \end{vmatrix}$$

Expand w.r.t. R_1

$$= \frac{1}{(x-y)^2(y-z)^2(z-x)^2}\begin{bmatrix} 2z(xy-y^2-yz) \\ +2y(yz+z^2-zx) \end{bmatrix}$$

$$= \frac{2}{(x-y)^2(y-z)^2(z-x)^2}\begin{bmatrix} xyz-y^2z-yz^2 \\ +y^2z+yz^2-xyz \end{bmatrix}$$

$$= \frac{2}{(x-y)^2(y-z)^2(z-x)^2}\cdot 0$$

$$= 0$$

Hence $\quad \dfrac{\partial(u,v,w)}{\partial(x,y,z)} = 0.$

Example 6. *If $u = x + y + z$, $uv = y + z$, $uvw = z$, show that* $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = u^2 v.$

(P.T.U., Jan. 2010, Dec. 2011)

Sol. $\quad u = x + y + z,\ uv = y + z \cdot uvw = z$

$\therefore \quad z = uvw,\ y = uv - uvw,\ x = u - (y+z) = u - uv$

$\therefore \quad x = u - uv \quad y = uv - uvw,\ z = uvw$

$$\frac{\partial x}{\partial u} = 1 - v,\ \frac{\partial x}{\partial v} = -u,\ \frac{\partial x}{\partial w} = 0$$

$$\frac{\partial y}{\partial u} = v - vw,\ \frac{\partial y}{\partial v} = u - uw,\ \frac{\partial y}{\partial w} = -uv$$

$$\frac{\partial z}{\partial u} = vw,\ \frac{\partial z}{\partial v} = uw,\ \frac{\partial z}{\partial w} = uv$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1-v & -u & 0 \\ v(1-w) & u(1-w) & -uv \\ vw & uw & uv \end{vmatrix}$$

Operate $R_2 \to R_2 + R_3$

$$= \begin{vmatrix} 1-v & -u & 0 \\ v & u & 0 \\ vw & uw & uv \end{vmatrix} = u\begin{vmatrix} 1-v & -1 & 0 \\ v & 1 & 0 \\ vw & w & uv \end{vmatrix}$$

Expand with $C_3 = u \cdot uv\ (1 - v + v) = u \cdot uv \cdot 1 = u^2 v.$

Example 7. *If $u = xyz$, $v = xy + yz + zx$, $w = x + y + z$, show that*

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{1}{(x-y)(y-z)(z-x)}.$$

Sol. $\quad u = xyz,\ v = xy + yz + zx,\ w = x + y + z$

As x, y, z cannot be easily expressed in terms of u, v, w $\therefore$ we apply the Jacobian property

that $\quad \dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \dfrac{1}{\dfrac{\partial(u,v,w)}{\partial(x,y,z)}}$

$\therefore \quad \dfrac{\partial u}{\partial x} = yz,\ \dfrac{\partial u}{\partial y} = zx,\ \dfrac{\partial u}{\partial z} = xy$

$$\frac{\partial v}{\partial x} = y + z, \ \frac{\partial v}{\partial y} = z + x, \ \frac{\partial v}{\partial z} = x + y$$

$$\frac{\partial w}{\partial x} = 1, \ \frac{\partial w}{\partial y} = 1, \ \frac{\partial w}{\partial z} = 1$$

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} yz & zx & xy \\ y+z & z+x & x+y \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} yz & z(x-y) & x(y-z) \\ y+z & x-y & y-z \\ 1 & 0 & 0 \end{vmatrix}, \text{ operate } C_2 - C_1, C_3 - C_2$$

$$= (x-y)(y-z) \begin{vmatrix} yz & z & x \\ y+z & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix} = (x-y)(y-z)(z-x)$$

$$\therefore \quad \frac{\partial(x, y, z)}{\partial(u, v, w)} = \frac{1}{(x-y)(y-z)(z-x)}.$$

TEST YOUR KNOWLEDGE

1. If $u = x(1-y)$, $v = xy$, prove that $JJ' = 1$.
2. If $u = \frac{yz}{x}, v = \frac{zx}{y}$ and $w = \frac{xy}{z}$, show that $\frac{\partial(u, v, w)}{\partial(x, y, z)} = 4$.
3. (i) If $u = \frac{y^2}{2x}, v = \frac{x^2 + y^2}{2x}$, find $\frac{\partial(u, v)}{\partial(x, y)}$.

   (ii) If $u = x^3 + xy$, $v = xy$, find $\frac{\partial(u, v)}{\partial(x, y)}$. ***(P.T.U., May 2009)***
4. If $x = a \cos \xi \cosh \eta$, $y = a \sinh \xi \sin \eta$, show that $\frac{\partial(x, y)}{\partial(\xi, \eta)} = \frac{a^2}{2} (\cosh 2\xi - \cos 2\eta)$.

   $\left[\textbf{Hint. } \cosh^2 x = \frac{\cosh 2x + 1}{2}, \sinh^2 x = \frac{\cosh 2x - 1}{2}\right]$
5. If $F = xu + v - y$, $G = u^2 + vy + w$, $H = zu - v + vw$, compute $\frac{\partial(F, G, H)}{\partial(u, w, v)}$.
6. If $u = x^2 - 2y$, $v = x + y + z$, $w = x - 2y + 3z$, find $\frac{\partial(u, v, w)}{\partial(x, y, z)}$.

Answers

3. (i) $-\frac{y}{2x}$ (ii) $3x^3$ **5.** $xw - x - xyv + 2uv - z$ **6.** $10x + 4$.

$$\frac{\partial v}{\partial x} = y + z, \quad \frac{\partial v}{\partial y} = z + x, \quad \frac{\partial v}{\partial z} = x + y$$

$$\frac{\partial w}{\partial x} = 1, \quad \frac{\partial w}{\partial y} = 1, \quad \frac{\partial w}{\partial z} = 1$$

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} yz & zx & xy \\ y+z & z+x & x+y \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} yz & z(x-y) & x(y-z) \\ y+z & x-y & y-z \\ 1 & 0 & 0 \end{vmatrix}, \text{ operate } C_2 - C_1,\ C_3 - C_2$$

$$= (x-y)(y-z) \begin{vmatrix} yz & z & x \\ y+z & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix} = (x-y)(y-z)(z-x)$$

$$\therefore \quad \frac{\partial(x, y, z)}{\partial(u, v, w)} = \frac{1}{(x-y)(y-z)(z-x)}.$$

TEST YOUR KNOWLEDGE

1. If $u = x(1-y)$, $v = xy$, prove that $JJ' = 1$.
2. If $u = \frac{yz}{x}$, $v = \frac{zx}{y}$ and $w = \frac{xy}{z}$, show that $\frac{\partial(u, v, w)}{\partial(x, y, z)} = 4$.
3. (*i*) If $u = \frac{y^2}{2x}$, $v = \frac{x^2 + y^2}{2x}$, find $\frac{\partial(u, v)}{\partial(x, y)}$.

   (*ii*) If $u = x^3 + xy$, $v = xy$, find $\frac{\partial(u, v)}{\partial(x, y)}$. ***(P.T.U., May 2009)***
4. If $x = a \cos \xi \cosh \eta$, $y = a \sinh \xi \sin \eta$, show that $\frac{\partial(x, y)}{\partial(\xi, \eta)} = \frac{a^2}{2}(\cosh 2\xi - \cos 2\eta)$.

   $\left[\textbf{Hint.} \cosh^2 x = \frac{\cosh 2x + 1}{2}, \sinh^2 x = \frac{\cosh 2x - 1}{2}\right]$
5. If $F = xu + v - y$, $G = u^2 + vy + w$, $H = zu - v + vw$, compute $\frac{\partial(F, G, H)}{\partial(u, w, v)}$.
6. If $u = x^2 - 2y$, $v = x + y + z$, $w = x - 2y + 3z$, find $\frac{\partial(u, v, w)}{\partial(x, y, z)}$.

Answers

3. (*i*) $-\frac{y}{2x}$ (*ii*) $3x^3$ **5.** $xw - x - xyv + 2uv - z$ **6.** $10x + 4$.

3.12. JACOBIAN OF IMPLICIT FUNCTIONS

If x, y, u, v are connected by implicit functions $f_1(x, y, u, v) = 0$, $f_2(x, y, u, v) = 0$ where u, v are implicit functions of x and y then

$$\frac{\partial(u, v)}{\partial(x, y)} = (-1)^2 \frac{\dfrac{\partial(f_1, f_2)}{\partial(x, y)}}{\dfrac{\partial(f_1, f_2)}{\partial(u, v)}}.$$

Proof. Given

$$f_1(x, y, u, v) = 0 \quad ...(1)$$

$$f_2(x, y, u, v) = 0 \quad ...(2)$$

Differentiate (1), (2) w.r.t. x, y

$$\frac{\partial f_1}{\partial x} + \frac{\partial f_1}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f_1}{\partial v} \cdot \frac{\partial v}{\partial x} = 0 \quad ...(3)$$

$$\frac{\partial f_1}{\partial y} + \frac{\partial f_1}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f_1}{\partial v} \cdot \frac{\partial v}{\partial y} = 0 \quad ...(4)$$

$$\frac{\partial f_2}{\partial x} + \frac{\partial f_2}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f_2}{\partial v} \cdot \frac{\partial v}{\partial x} = 0 \quad ...(5)$$

$$\frac{\partial f_2}{\partial y} + \frac{\partial f_2}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f_2}{\partial v} \cdot \frac{\partial v}{\partial y} = 0 \quad ...(6)$$

Now
$$\frac{\partial(f_1, f_2)}{\partial(u, v)} \times \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \dfrac{\partial f_1}{\partial u} & \dfrac{\partial f_1}{\partial v} \\ \dfrac{\partial f_2}{\partial u} & \dfrac{\partial f_2}{\partial v} \end{vmatrix} \times \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{\partial f_1}{\partial u} \cdot \dfrac{\partial u}{\partial x} + \dfrac{\partial f_1}{\partial v} \cdot \dfrac{\partial v}{\partial x} & \dfrac{\partial f_1}{\partial u} \cdot \dfrac{\partial u}{\partial y} + \dfrac{\partial f_1}{\partial v} \cdot \dfrac{\partial v}{\partial y} \\ \dfrac{\partial f_2}{\partial u} \cdot \dfrac{\partial u}{\partial x} + \dfrac{\partial f_2}{\partial v} \cdot \dfrac{\partial v}{\partial x} & \dfrac{\partial f_2}{\partial u} \cdot \dfrac{\partial u}{\partial y} + \dfrac{\partial f_2}{\partial v} \cdot \dfrac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} -\dfrac{\partial f_1}{\partial x} & -\dfrac{\partial f_1}{\partial y} \\ -\dfrac{\partial f_2}{\partial x} & -\dfrac{\partial f_2}{\partial y} \end{vmatrix} = (-1)^2 \frac{\partial(f_1, f_2)}{\partial(x, y)} \quad \text{[using (3), (4), (5), (6)]}$$

$$\therefore \quad \frac{\partial(f_1, f_2)}{\partial(u, v)} \times \frac{\partial(u, v)}{\partial(x, y)} = (-1)^2 \frac{\partial(f_1, f_2)}{\partial(x, y)}$$

$$\therefore \quad \frac{\partial(u, v)}{\partial(x, y)} = (-1)^2 \frac{\partial(f_1, f_2)}{\partial(x, y)} \Bigg/ \frac{\partial(f_1, f_2)}{\partial(u, v)}$$

In general the variables $x_1, x_2, \cdots, x_n$ are connected with $u_1, u_2, \cdots, u_n$ implicity as

$$f_1(x_1, x_2, \cdots, x_n, u_1, u_2, \cdots, u_n) = 0$$

$$f_2(x_1, x_2, \cdots, x_n, u_1, u_2, \cdots, u_n) = 0$$

....

....

$$f_n(x_1, x_2, \cdots, x_n, u_1, u_2, \cdots, u_n) = 0$$

then we have $\dfrac{\partial(u_1,u_2,\cdots,u_n)}{\partial(x_1,x_2,\cdots,x_n)} = (-1)^n \dfrac{\partial(f_1,f_2,\cdots,f_n)}{\partial(x_1,x_2,\cdots,x_n)} \bigg/ \dfrac{\partial(f_1,f_2,\cdots,f_n)}{\partial(u_1,u_2,\cdots,u_n)}$.

ILLUSTRATIVE EXAMPLES

Example 1. *If $u^3 + v + w = x + y^2 + z^2$; $u + v^3 + w = x^2 + y + z^2$, $u + v + w^3 = x^2 + y^2 + z$. Prove that*

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \frac{1 - 4(xy + yz + zx) + 16xyz}{2 - 3(u^2 + v^2 + w^2) + 27u^2v^2w^2}.$$

Sol. Let

$$f_1 = u^3 + v + w - x - y^2 - z^2$$
$$f_2 = u + v^3 + w - x^2 - y - z^2$$
$$f_3 = u + v + w^3 - x^2 - y^2 - z$$

$$\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)} = \begin{vmatrix} -1 & -2y & -2z \\ -2x & -1 & -2z \\ -2x & -2y & -1 \end{vmatrix} = -1 + 4(yz + zx + xy) - 16xyz$$

and

$$\frac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)} = \begin{vmatrix} 3u^2 & 1 & 1 \\ 1 & 3v^2 & 1 \\ 1 & 1 & 3w^2 \end{vmatrix} = 2 - 3(u^2 + v^2 + w^2) + 27u^2v^2w^2$$

Now

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = (-1)^3 \frac{\dfrac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)}}{\dfrac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)}} = (-1)^3 \frac{-1 + 4(yz + zx + xy) - 16xyz}{2 - 3(u^2 + v^2 + w^2) + 27u^2v^2w^2}$$

$$= \frac{1 - 4(yz + zx + xy) + 16xyz}{2 - 3(u^2 + v^2 + w^2) + 27u^2v^2w^2}.$$

Example 2. *If u, v, w are the roots of the equation $(\lambda - x)^3 + (\lambda - y)^3 + (\lambda - z)^3 = 0$. Find $\dfrac{\partial(u, v, w)}{\partial(x, y, z)}$.*

Sol. $(\lambda - x)^3 + (\lambda - y)^3 + (\lambda - z)^3 = 0$

or $\quad 3\lambda^3 - 3(x + y + z)\lambda^2 + 3(x^2 + y^2 + z^2)\lambda - (x^3 + y^3 + z^3) = 0$

Sum of the roots $u + v + w = x + y + z$

Sum of the roots taken two at a time, $uv + vw + wu = x^2 + y^2 + z^2$

Product of the roots $uvw = \dfrac{1}{3}(x^3 + y^3 + z^3)$

Let

$$f_1 = u + v + w - x - y - z$$
$$f_2 = uv + vw + wu - x^2 - y^2 - z^2$$
$$f_3 = uvw - \frac{1}{3}(x^3 + y^3 + z^3)$$

$$\frac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)} = \begin{vmatrix} -1 & -1 & -1 \\ -2x & -2y & -2z \\ -x^2 & -y^2 & -z^2 \end{vmatrix} = -2(x-y)(y-z)(z-x)$$

$$\frac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)} = \begin{vmatrix} 1 & 1 & 1 \\ v+w & w+u & u+v \\ vw & wu & uv \end{vmatrix} = -(u-v)(v-w)(w-u)$$

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = (-1)^3 \frac{\dfrac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)}}{\dfrac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)}} = (-1)^3 \frac{-2(x-y)(y-z)(z-x)}{-(u-v)(v-w)(w-u)}$$

$$= -2\frac{(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)}.$$

Example 3. *If* $u = \dfrac{x}{\sqrt{1-r^2}}$, $v = \dfrac{y}{\sqrt{1-r^2}}$, $w = \dfrac{z}{\sqrt{1-r^2}}$, *show that* $\dfrac{\partial(u, v, w)}{\partial(x, y, z)} = \dfrac{1}{(1-r^2)^{5/2}}$

where $r^2 = x^2 + y^2 + z^2$.

Sol. $$u = \frac{x}{\sqrt{1-r^2}}, \quad \frac{\partial u}{\partial x} = \frac{\sqrt{1-r^2} \cdot 1 - x \cdot \dfrac{1}{2\sqrt{1-r^2}}(-2r)\dfrac{\partial r}{\partial x}}{1-r^2} = \frac{1-r^2 + xr\left(\dfrac{x}{r}\right)}{(1-r^2)^{3/2}}$$

$$= \frac{1-r^2+x^2}{(1-r^2)^{3/2}} = \frac{1+x^2-x^2-y^2-z^2}{(1-r^2)^{3/2}} = \frac{1-y^2-z^2}{(1-r^2)^{3/2}}$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{1-y^2-z^2}{(1-r^2)^{3/2}};$$

$$\frac{\partial u}{\partial y} = x\left(-\frac{1}{2}\right)(1-r^2)^{-3/2}(-2r)\frac{\partial r}{\partial y} = \frac{xr}{(1-r^2)^{3/2}} \cdot \frac{y}{r} = \frac{xy}{(1-r^2)^{3/2}}$$

$$\frac{\partial u}{\partial z} = x\left(-\frac{1}{2}\right)(1-r^2)^{-3/2}(-2r)\frac{\partial r}{\partial z} = \frac{xz}{(1-r^2)^{3/2}}$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{1-y^2-z^2}{(1-r^2)^{3/2}}, \frac{\partial u}{\partial y} = \frac{xy}{(1-r^2)^{3/2}}, \frac{\partial u}{\partial z} = \frac{xz}{(1-r^2)^{3/2}}$$

Similarly, $$\frac{\partial v}{\partial x} = \frac{xy}{(1-r^2)^{3/2}}; \frac{\partial v}{\partial y} = \frac{1-z^2-x^2}{(1-r^2)^{3/2}}; \frac{\partial v}{\partial z} = \frac{yz}{(1-r^2)^{3/2}}$$

$$\frac{\partial w}{\partial x} = \frac{xz}{(1-r^2)^{3/2}}; \frac{\partial w}{\partial y} = \frac{yz}{(1-r^2)^{3/2}}; \frac{\partial w}{\partial z} = \frac{1-x^2-y^2}{(1-r^2)^{3/2}}$$

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} \dfrac{1-y^2-z^2}{(1-r^2)^{3/2}} & \dfrac{xy}{(1-r^2)^{3/2}} & \dfrac{xz}{(1-r^2)^{3/2}} \\ \dfrac{xy}{(1-r^2)^{3/2}} & \dfrac{1-z^2-x^2}{(1-r^2)^{3/2}} & \dfrac{yz}{(1-r^2)^{3/2}} \\ \dfrac{xz}{(1-r^2)^{3/2}} & \dfrac{yz}{(1-r^2)^{3/2}} & \dfrac{1-x^2-y^2}{(1-r^2)^{3/2}} \end{vmatrix}$$

$$= \frac{1}{(1-r^2)^{9/2}} \begin{vmatrix} 1-y^2-z^2 & xy & xz \\ xy & 1-z^2-x^2 & yz \\ xz & yz & 1-x^2-y^2 \end{vmatrix}$$

Operate $R_1(x)$, $R_2(y)$, $R_3(z)$

$$= \frac{1}{(1-r^2)^{9/2}} \frac{1}{xyz} \begin{vmatrix} x(1-y^2-z^2) & x^2y & x^2z \\ xy^2 & y(1-z^2-x^2) & y^2z \\ xz^2 & yz^2 & z(1-x^2-y^2) \end{vmatrix}$$

Operate $C_1\left(\frac{1}{x}\right)$, $C_2\left(\frac{1}{y}\right)$, $C_3\left(\frac{1}{z}\right)$

$$= \frac{1}{(1-r^2)^{9/2}} \frac{xyz}{xyz} \begin{vmatrix} 1-y^2-z^2 & x^2 & x^2 \\ y^2 & 1-z^2-x^2 & y^2 \\ z^2 & z^2 & 1-x^2-y^2 \end{vmatrix}$$

Operate $R_1 + R_2 + R_3$

$$= \frac{1}{(1-r^2)^{9/2}} \begin{vmatrix} 1 & 1 & 1 \\ y^2 & 1-z^2-x^2 & y^2 \\ z^2 & z^2 & 1-x^2-y^2 \end{vmatrix}$$

Operate $C_2 - C_1$, $C_3 - C_1$

$$= \frac{1}{(1-r^2)^{9/2}} \begin{vmatrix} 1 & 0 & 0 \\ y^2 & 1-x^2-y^2-z^2 & 0 \\ z^2 & 0 & 1-x^2-y^2-z^2 \end{vmatrix}$$

$$= \frac{1}{(1-r^2)^{9/2}} \begin{vmatrix} 1 & 0 & 0 \\ y^2 & 1-r^2 & 0 \\ z^2 & 0 & 1-r^2 \end{vmatrix}$$

Expand with first row

$$= \frac{1}{(1-r^2)^{9/2}} (1-r^2)^2 = \frac{1}{(1-r^2)^{5/2}}$$

Hence $$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \frac{1}{(1-r^2)^{5/2}}.$$

3.13. PARTIAL DERIVATIVE OF IMPLICIT FUNCTIONS BY JACOBIAN

Statement. *Given* $f_1(x, y, u, v) = 0$, $f_2(x, y, u, v) = 0$

to prove $$\frac{\partial u}{\partial x} = -\frac{\partial(f_1, f_2)}{\partial(x, v)} \Big/ \frac{\partial(f_1, f_2)}{\partial(u, v)}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial(f_1, f_2)}{\partial(u, x)} \Big/ \frac{\partial(f_1, f_2)}{\partial(v, u)}.$$

Proof. $f_1(x, y, u, v) = 0$; $f_2(x, y, u, v) = 0$ ...(1)

$$\therefore \quad \frac{\partial f_1}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial f_1}{\partial v}\cdot\frac{\partial v}{\partial x}+\frac{\partial f_1}{\partial x}\cdot 1=0$$

$$\frac{\partial f_2}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial f_2}{\partial v}\cdot\frac{\partial v}{\partial x}+\frac{\partial f_2}{\partial x}.1=0$$

Solve for $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}$

$$\frac{\frac{\partial u}{\partial x}}{\begin{vmatrix}\frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial x}\\ \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial x}\end{vmatrix}}=\frac{\frac{\partial v}{\partial x}}{-\begin{vmatrix}\frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial x}\\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial x}\end{vmatrix}}=\frac{1}{\begin{vmatrix}\frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v}\\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v}\end{vmatrix}}$$

$$\frac{\frac{\partial u}{\partial x}}{\frac{\partial(f_1,f_2)}{\partial(v,x)}}=-\frac{\frac{\partial v}{\partial x}}{\frac{\partial(f_1,f_2)}{\partial(u,x)}}=\frac{1}{\frac{\partial(f_1,f_2)}{\partial(u,v)}}$$

$$\therefore \quad \frac{\partial u}{\partial x}=\frac{\frac{\partial(f_1,f_2)}{\partial(v,x)}}{\frac{\partial(f_1,f_2)}{\partial(u,v)}}=-\frac{\frac{\partial(f_1,f_2)}{\partial(x,v)}}{\frac{\partial(f_1,f_2)}{\partial(u,v)}}$$

$$\frac{\partial v}{\partial x}=-\frac{\partial(f_1,f_2)}{\partial(u,x)}\Big/\frac{\partial(f_1,f_2)}{\partial(u,v)}$$

Note: To remember this result note that numerator of $\frac{\partial u}{\partial x}$ is obtained from the denominator by replacing u by x in the Jacobian $\frac{\partial(f_1,f_2)}{\partial(u,v)}$. And the numerator of $\frac{\partial v}{\partial x}$ is obtained from the denominator by replacing v by x in the Jacobian.

Similarly on differentiating (1) partially w.r.t. y, we get

$$\frac{\partial u}{\partial y}=-\frac{\frac{\partial(f_1,f_2)}{\partial(y,v)}}{\frac{\partial(f_1,f_2)}{\partial(u,v)}};\quad \frac{\partial v}{\partial y}=-\frac{\frac{\partial(f_1,f_2)}{\partial(u,y)}}{\frac{\partial(f_1,f_2)}{\partial(u,v)}}$$

To get $\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}$ and $\frac{\partial y}{\partial u}, \frac{\partial y}{\partial v}$ differentiate partially w.r.t. u and v respectively, we get

$$\frac{\partial x}{\partial u}=-\frac{\frac{\partial(f_1,f_2)}{\partial(u,y)}}{\frac{\partial(f_1,f_2)}{\partial(x,y)}};\quad \frac{\partial y}{\partial u}=-\frac{\frac{\partial(f_1,f_2)}{\partial(x,u)}}{\frac{\partial(f_1,f_2)}{\partial(x,y)}}$$

and

$$\frac{\partial x}{\partial v}=-\frac{\frac{\partial(f_1,f_2)}{\partial(v,y)}}{\frac{\partial(f_1,f_2)}{\partial(x,y)}};\quad \frac{\partial y}{\partial v}=-\frac{\frac{\partial(f_1,f_2)}{\partial(x,v)}}{\frac{\partial(f_1,f_2)}{\partial(x,y)}}$$

Example 4. *If* $x = u^2 - v^2$, $y = 2uv$, *find* $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}, \dfrac{\partial(u,v)}{\partial(x,y)}$.

Sol. Let

$$f_1 = x - u^2 + v^2$$
$$f_2 = y - 2uv$$

$$\frac{\partial u}{\partial x} = -\frac{\dfrac{\partial(f_1,f_2)}{\partial(x,v)}}{\dfrac{\partial(f_1,f_2)}{\partial(u,v)}} = -\frac{\begin{vmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial v} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial v} \end{vmatrix}}{\begin{vmatrix} \dfrac{\partial f_1}{\partial u} & \dfrac{\partial f_1}{\partial v} \\ \dfrac{\partial f_2}{\partial u} & \dfrac{\partial f_2}{\partial v} \end{vmatrix}} = -\frac{\begin{vmatrix} 1 & 2v \\ 0 & -2u \end{vmatrix}}{\begin{vmatrix} -2u & 2v \\ -2v & -2u \end{vmatrix}}$$

$$= \frac{2u}{4(u^2+v^2)} = \frac{u}{2(u^2+v^2)}$$

$$\frac{\partial u}{\partial y} = -\frac{\dfrac{\partial(f_1,f_2)}{\partial(y,v)}}{\dfrac{\partial(f_1,f_2)}{\partial(u,v)}} = -\frac{\begin{vmatrix} \dfrac{\partial f_1}{\partial y} & \dfrac{\partial f_1}{\partial v} \\ \dfrac{\partial f_2}{\partial y} & \dfrac{\partial f_2}{\partial v} \end{vmatrix}}{\begin{vmatrix} \dfrac{\partial f_1}{\partial u} & \dfrac{\partial f_1}{\partial v} \\ \dfrac{\partial f_2}{\partial u} & \dfrac{\partial f_2}{\partial v} \end{vmatrix}} = -\frac{\begin{vmatrix} 0 & 2v \\ 1 & -2u \end{vmatrix}}{4(u^2+v^2)}$$

$$\therefore \quad \frac{\partial u}{\partial y} = \frac{2v}{4(u^2+v^2)} = \frac{v}{2(u^2+v^2)}$$

$$\frac{\partial v}{\partial x} = -\frac{\dfrac{\partial(f_1,f_2)}{\partial(u,x)}}{\dfrac{\partial(f_1,f_2)}{\partial(u,v)}} = -\frac{\begin{vmatrix} \dfrac{\partial f_1}{\partial u} & \dfrac{\partial f_1}{\partial x} \\ \dfrac{\partial f_2}{\partial u} & \dfrac{\partial f_2}{\partial x} \end{vmatrix}}{\begin{vmatrix} \dfrac{\partial f_1}{\partial u} & \dfrac{\partial f_1}{\partial v} \\ \dfrac{\partial f_2}{\partial u} & \dfrac{\partial f_2}{\partial v} \end{vmatrix}} = -\frac{\begin{vmatrix} -2u & 1 \\ -2v & 0 \end{vmatrix}}{4(u^2+v^2)}$$

$$\therefore \quad \frac{\partial v}{\partial x} = \frac{-v}{2(u^2+v^2)}$$

$$\frac{\partial v}{\partial y} = -\frac{\dfrac{\partial(f_1,f_2)}{\partial(u,y)}}{\dfrac{\partial(f_1,f_2)}{\partial(u,v)}} = -\frac{\begin{vmatrix} \dfrac{\partial f_1}{\partial u} & \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial f_2}{\partial u} & \dfrac{\partial f_2}{\partial y} \end{vmatrix}}{\begin{vmatrix} \dfrac{\partial f_1}{\partial u} & \dfrac{\partial f_1}{\partial v} \\ \dfrac{\partial f_2}{\partial u} & \dfrac{\partial f_2}{\partial v} \end{vmatrix}} = -\frac{\begin{vmatrix} -2u & 0 \\ -2x & 1 \end{vmatrix}}{4(u^2+v^2)}$$

$$\frac{\partial v}{\partial y}=\frac{2u}{4(u^2+v^2)}=\frac{u}{2(u^2+v^2)}$$

$$\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}$$ (As u, v cannot be expressed in terms of x, y easily)

$\therefore$ We can write

$$\frac{\partial(u,v)}{\partial(x,y)}=\frac{1}{\frac{\partial(x,y)}{\partial(u,v)}}=\frac{1}{\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}}$$

$$=\frac{1}{\begin{vmatrix}2u & -2v\\ 2v & 2u\end{vmatrix}}=\frac{1}{4(u^2+v^2)}.$$

Example 5. *Use Jacobian to find* $\frac{\partial u}{\partial x}$ *if* $u^2+xv^2-xy=0$ *and* $u^2+xyv+v^2=0.$

Sol. Let
$$f_1=u^2+xv^2-xy$$
$$f_2=u^2+xyv+v^2$$

$$\frac{\partial u}{\partial x}=-\frac{\frac{\partial(f_1,f_2)}{\partial(x,v)}}{\frac{\partial(f_1,f_2)}{\partial(u,v)}}=-\frac{\begin{vmatrix}\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial v}\\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial v}\end{vmatrix}}{\begin{vmatrix}\frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v}\\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v}\end{vmatrix}}=-\frac{\begin{vmatrix}v^2-y & 2xv\\ yv & xy+2v\end{vmatrix}}{\begin{vmatrix}2u & 2xv\\ 2u & xy+2v\end{vmatrix}}$$

$$=-\frac{(v^2-y)(xy+2v)-2xyv^2}{2\{xyu+2uv-2xuv\}}=-\frac{xyv^2-xy^2+2v^3-2yv-2xyv^2}{2xyu+4uv-4xuv}$$

$$=\frac{xyv^2+xy^2-2v^3+2yv}{2xyu+4uv-4xuv}.$$

Example 6. *If* $x=u+v+w,\ y=u^2+v^2+w^2,\ z=u^3+v^3+w^3$, *show that* $\frac{\partial u}{\partial x}=\frac{vw}{(u-v)(u-w)}.$

Sol. Let
$$f_1=u+v+w-x$$
$$f_2=u^2+v^2+w^2-y$$
$$f_3=u^3+v^3+w^3-z$$

$$\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}=\begin{vmatrix}1 & 1 & 1\\ 2u & 2v & 2w\\ 3u^2 & 3v^2 & 3w^2\end{vmatrix}=6(u-v)(v-w)(w-u)$$

Replace u by x

$$\frac{\partial(f_1,f_2,f_3)}{\partial(x,v,w)} = \begin{vmatrix} -1 & 1 & 1 \\ 0 & 2v & 2w \\ 0 & 3v^2 & 3w^2 \end{vmatrix} = 6vw\,(v-w)$$

Now
$$\frac{\partial u}{\partial x} = -\frac{\dfrac{\partial(f_1,f_2,f_3)}{\partial(x,v,w)}}{\dfrac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}} = -\frac{6vw(v-w)}{6(u-v)(v-w)(w-u)}$$

$$= \frac{-6vw}{(u-v)(w-u)} = \frac{6vw}{(u-v)(u-w)}.$$

Example 7. *If* $u = x + y^2$, $v = y + z^2$, $w = z + x^2$, *prove that* $\dfrac{\partial x}{\partial u} = \dfrac{1}{1+8xyz}$. *Also find* $\dfrac{\partial^2 x}{\partial u^2}$.

Sol. Let
$$f_1 = u - x - y^2$$
$$f_2 = v - y - z^2$$
$$f_3 = w - z - x^2$$

$$\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)} = \begin{vmatrix} -1 & -2y & 0 \\ 0 & -1 & -2z \\ -2x & 0 & -1 \end{vmatrix} = -1 - 8\,xyz$$

Replace x by u.

$$\frac{\partial(f_1,f_2,f_3)}{\partial(u,y,z)} = \begin{vmatrix} 1 & -2y & 0 \\ 0 & -1 & -2z \\ 0 & 0 & -1 \end{vmatrix} = 1$$

$\therefore$
$$\frac{\partial x}{\partial u} = -\frac{\dfrac{\partial(f_1,f_2,f_3)}{\partial(u,y,z)}}{\dfrac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}} = -\frac{1}{-1-8xyz} = \frac{1}{1+8xyz}.$$

Also
$$\frac{\partial^2 x}{\partial u^2} = \frac{\partial}{\partial u}\left[\frac{1}{1+8xyz}\right] = -\frac{1}{(1+8xyz)^2}\left[\frac{\partial}{\partial u}(1+8xyz)\right]$$

$$= -\frac{1}{(1+8xyz)^2}\,8\left\{xy\frac{\partial z}{\partial u} + yz\frac{\partial x}{\partial u} + zx\frac{\partial y}{\partial u}\right\} \qquad \text{...(1)}$$

Find $\dfrac{\partial y}{\partial u}, \dfrac{\partial z}{\partial u}$

$$\frac{\partial y}{\partial u} = -\frac{\dfrac{\partial(f_1,f_2,f_3)}{\partial(x,u,z)}}{\dfrac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}}, \quad \frac{\partial z}{\partial u} = -\frac{\dfrac{\partial(f_1,f_2,f_3)}{\partial(x,y,u)}}{\dfrac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}}$$

$$\frac{\partial(f_1,f_2,f_3)}{\partial(x,u,z)} = \begin{vmatrix} -1 & 1 & 0 \\ 0 & 0 & -2z \\ -2x & 0 & -1 \end{vmatrix} = 4xz$$

$$\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,u)}=\begin{vmatrix}-1 & -2y & 1\\ 0 & -1 & 0\\ -2x & 0 & 0\end{vmatrix}=-2x$$

$$\therefore \quad \frac{\partial y}{\partial u}=-\frac{4xz}{-1-8xyz}=\frac{4xz}{1+8xyz}$$

$$\frac{\partial z}{\partial u}=-\frac{-2x}{-1-8xyz}=-\frac{2x}{1+8xyz}$$

$\therefore$ From (1),

$$\frac{\partial^2 x}{\partial u^2}=\frac{-8}{(1+8xyz)^2}\left[xy\frac{-2x}{1+8xyz}+yz\frac{1}{1+8xyz}+zx\cdot\frac{4xz}{1+8xyz}\right]$$

$$=\frac{-8}{(1+8xyz)^3}[-2x^2y+yz+4x^2z^2].$$

TEST YOUR KNOWLEDGE

1. If $x^2+y^2+u^2-v^2=0$ and $uv+xy=0$, prove that $\dfrac{\partial(u,v)}{\partial(x,y)}=\dfrac{x^2-y^2}{u^2+v^2}$.

2. If $x+y+z=u$, $y+z=uv$, $z=uvw$, show that $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}=u^2v$.

3. If u, v, w are the roots of the equation in λ and $\dfrac{x}{a+\lambda}+\dfrac{y}{b+\lambda}+\dfrac{z}{c+\lambda}=1$, find $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$.

[**Hint.** Consult Ex. 2]

Answer

3. $\dfrac{(u-v)(v-w)(w-u)}{(a-b)(b-c)(c-a)}$.

REVIEW OF THE CHAPTER

1. **Limit of a Function of Two Variables *i.e.*, $f(x, y)$.** A function $f(x, y)$ is said to tend to the limit l iff corresponding to a positive number ε, $\exists$s another positive number δ s.t. $|f(x,y)-l|<\varepsilon\ \forall\ (x,y)$ within the circle with its centre at (a, b) and radius δ, *i.e.* $0<(x-a)^2+(y-b)^2<\delta^2$

or $f(x, y)$ is said to tend limit l as $x \to a$, $y \to b$ iff l is independent of the path followed by the point (x, y) as $x \to a$ and $y \to b$ and is written as $\underset{\substack{x\to a\\ y\to b}}{\text{Lt}} f(x,y)=l$

Properties: If $\underset{\substack{x\to a\\ y\to b}}{\text{Lt}} f(x,y)=l$ and $\underset{\substack{x\to a\\ y\to b}}{\text{Lt}} g(x,y)=m$

then (*i*) $\underset{\substack{x\to a\\ y\to b}}{\text{Lt}} [f(x,y)\pm g(x,y)]=l\pm m$

(*ii*) $\underset{\substack{x\to a\\ y\to b}}{\text{Lt}} f(x,y), g(x,y)=lm.$

(*iii*) $\underset{\substack{x\to a\\ y\to b}}{\text{Lt}} \dfrac{f(x,y)}{g(x,y)}=\dfrac{l}{m}\quad (m\neq 0).$

2. **Continuity of f(x, y).** A function is said to be continuous at a point (a, b) iff corresponding to a positive number ε, however small, $\exists$s a +ve number δ s.t.

$$| f(x, y) - f(a, b) | < \varepsilon \ \forall \ (x, y)$$

within the circle with its centre at (a, b) and radius δ *i.e.*,

$$\underset{\substack{x \to a \\ y \to b}}{\text{Lt}} f(x, y) = f(a, b).$$

Properties. If $f(x, y)$ and $g(x, y)$ are continuous functions of x, y at (a, b) then $f(x) \pm g(x)$; $f(x)\, g(x)$; $f(x, y)/g(x, y)$ $[g(x, y) \neq 0]$ are also continuous functions of x, y.

3. **Partial Derivatives.** If $z = f(x, y)$ then

$$\frac{\partial z}{\partial x} = \underset{h \to 0}{\text{Lt}} \frac{f(x+h, y) - f(x, y)}{h} \quad \text{and} \quad \frac{\partial z}{\partial y} = \underset{k \to 0}{\text{Lt}} \frac{f(x, y+k) - f(x, y)}{k}$$

$\frac{\partial z}{\partial x}$ is represented by f_x and $\frac{\partial z}{\partial y}$ is represented by f_y

$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$ and is represented by f_{xx}

$\frac{\partial^2 z}{\partial y^2}$ is represented by f_{yy} and $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$ is represented by f_{xy} or f_{yx}.

4. **Homogeneous Function.** A function $f(x, y)$ is said to be homogeneous function of degree n if it can be expressed as $x^n \phi\left(\frac{y}{x}\right)$ or $y^n \phi\left(\frac{x}{y}\right)$ or alternatively if

$$f(tx, ty) = t^n f(x, y).$$

5. **Euler's Theorem on Homogeneous Functions.** If u is a homogeneous function of degree n in x and y then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu \text{ and } x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)\,u.$$

6. **Composite Functions.** If $u = f(x, y)$ and $x = \phi(t)$, $y = \psi(t)$, then u is called composite function of single variable t.

7. **Total Derivative.** If $u = f(x, y)$, $x = \phi(t)$, $y = \psi(t)$ *i.e.*, u is a composite funciton of t then $\frac{du}{dt}$ is called total derivative and can be obtained without actually substituting the values of x, y in $f(x, y)$. Total derivative is $\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$.

8. Differentiation of Implicit Functions. If given implicit function $f(x, y) = c$; we can write $u = f(x, y)$ where $u = c$

then $$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{f_x}{f_y}$$

and $$\frac{d^2y}{dx^2} = -\frac{f_{xx}f_y^{\,2} - 2f_xf_yf_{xy} + f_{yy}f_x^2}{f_y^3}$$

9. Change of Variables. If $z = f(x, y)$ and $x = \phi(u, v)$, $y = \psi(u, v)$, then

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

or we can express

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}.$$

10. Jacobians. If u, v are functions of two variables x and y then Jacobian of u, v w.r.t. x, y is defined as $$J\left(\frac{u, v}{x, y}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial(u, v)}{\partial(x, y)}$$

and $$J\left(\frac{u, v, w}{x, y, z}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \frac{\partial(u, v, w)}{\partial(x, y, z)}.$$

Properties of Jacobians:

(*i*) If u, v are functions of r, s where r, s are functions of x, y then

$$J\left(\frac{u, v}{x, y}\right) = \frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial(u, v)}{\partial(r, s)} \times \frac{\partial(r, s)}{\partial(x, y)}.$$

(*ii*) If J_1 is Jacobian of u, v w.r.t. x, y and J_2 is the Jacobian of x, y w.r.t. u, v then $J_1 J_2 = 1$

i.e., $$\frac{\partial(u, v)}{\partial(x, y)} \times \frac{\partial(x, y)}{\partial(u, v)} = 1.$$

11. **Jacobian of Implicit Functions.** If x, y, u, v are connected by implicit functions $f_1(x, y, u, v) = 0$ and $f_2(x, y, u, v) = 0$ where u, v are implicit functions of x, y, then

$$\frac{\partial(u, v)}{\partial(x, y)} = (-1)^2 \frac{\dfrac{\partial(f_1, f_2)}{\partial(x, y)}}{\dfrac{\partial(f_1, f_2)}{\partial(u, v)}}$$

Generalised form is $\dfrac{\partial(u_1, u_2, \cdots, u_n)}{\partial(x_1, x_2, \cdots, x_n)} = (-1)^n \dfrac{\partial(f_1, f_2, \cdots f_n)}{\partial(x_1, x_2, \cdots, x_n)} \Big/ \dfrac{\partial(f_1, f_2, \cdots, f_n)}{\partial(u_1, u_2, \cdots, u_n)}$.

12. **Partial Derivatives of Implicit Functions by Jacobian.** If $f_1(x, y, u, v) = 0$ and $f_2(x, y, u, v) = 0$.

Then $\dfrac{\partial u}{\partial x} = -\dfrac{\partial(f_1, f_2)}{\partial(x, v)} \Big/ \dfrac{\partial(f_1, f_2)}{\partial(u, v)}$

(numerator is obtained by replacing u by x in denominator)

$$\frac{\partial v}{\partial x} = -\frac{\partial(f_1, f_2)}{\partial(u, x)} \Big/ \frac{\partial(f_1, f_2)}{\partial(u, v)}$$

$$\frac{\partial u}{\partial y} = -\frac{\dfrac{\partial(f_1, f_2)}{\partial(y, v)}}{\dfrac{\partial(f_1, f_2)}{\partial(u, v)}}, \quad \frac{\partial v}{\partial y} = -\frac{\dfrac{\partial(f_1, f_2)}{\partial(u, y)}}{\dfrac{\partial(f_1, f_2)}{\partial(u, v)}}.$$

SHORT ANSWER TYPE QUESTIONS

1. If $z = f(x, y)$ be a surface then what is the geometrical meaning of $\dfrac{\partial z}{\partial x}$? **(P.T.U., May 2011)**

[**Hint.** Art. 3.3(*i*)]

2. (*i*) If $z = e^{ax + by} f(ax - by)$, find $b\dfrac{\partial z}{\partial x} + a\dfrac{\partial z}{\partial y}$. **(P.T.U., May 2006)**

[**Hint.** S.E. 6(*ii*) art. 3.4]

(*ii*) If $u = (1 - 2x + y^2)^{-1/2}$, prove that $x\dfrac{\partial u}{\partial x} - y\dfrac{\partial u}{\partial y} = y^2 u^3$.

[**Hint.** S.E. 12(*i*) art. 3.4]

3. If $x = r\cos\theta$, $y = r\sin\theta$, prove that

(*i*) $\dfrac{\partial r}{\partial x} = \dfrac{\partial x}{\partial r}$ (*ii*) $\dfrac{1}{r}\dfrac{\partial x}{\partial \theta} = r\dfrac{\partial \theta}{\partial x}$.

[**Hint.** S.E. 16 art. 3.4]

4. Find the first order derivatives of the following functions:

(*i*) $u = x^2 \sin\dfrac{y}{x}$ (*ii*) $u = \log(x^2 + y^2)$ (*iii*) $u = y^x$.

5. (*i*) If $u = x^2y + y^2z + z^2x$ prove that $u_x + u_y + u_z = (x + y + z)^2$

(*ii*) If $u = \dfrac{y}{z} + \dfrac{z}{x} + \dfrac{x}{y}$ prove that $xu_x + yu_y + zu_z = 0$.

6. If $u = \log(\tan x + \tan y)$; prove that $2x\frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} = 2$.

7. Verify $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ for the following functions:

(i) $u = \tan^{-1}\frac{x}{y}$

(ii) $u = ax^2 + 2hxy + by^2$

(iii) $u = e^{ax} \sin by$

(iv) $u = \log(x \sin y + y \sin x)$

(v) $u = \sin^{-1}\frac{x}{y}$

(vi) $u = \log \frac{x^2 + y^2}{xy}$.

$\left[\textbf{Hint.}\ \log \frac{x^2+y^2}{xy} = \log(x^2+y^2) - \log x - \log y\right]$

(P.T.U., May 2011)

8. If $u = x^y$, find $\frac{\partial^2 u}{\partial x \partial y}$ at (1, 2)

[Hint: S.E. 9 art 3.4]

9. If $z = \log(e^x + e^y)$ show that $rt - s^2 = 0$, where $r = \frac{\partial^2 z}{\partial x^2}, s = \frac{\partial^2 z}{\partial x \partial y}, t = \frac{\partial^2 z}{\partial y^2}$.

10. If $u = \log(x^2 + y^2) + \tan^{-1}\frac{y}{x}$ prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

11. If $u = (x^2 + y^2 + z^2)^{-1/2}$ prove that $u_{xx} + u_{yy} + u_{zz} = 0$. [Hint. S.E.6(*i*) art. 3.4]

12. (i) If $z(x + y) = x^2 + y^2$ show that $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$. (P.T.U., Dec. 2013)

(ii) If $z = \log(x^2 + xy + y^2)$ prove that $x\frac{\partial z}{\partial y} + y\frac{\partial z}{\partial y} = 2$.

[Hint. S.E.7(*ii*) art. 3.4]

13. Define homogeneous function with the help of an example. (P.T.U., 2005, May 2009, May 2010)

[Hint. Art. 3.5]

14. (i) State and prove, Euler's theorem on homogeneous functions (P.T.U., 2003, 2006, Dec. 2004, Jan. 2009, May 2009)

[Hint. Art. 3.6]

(ii) State Euler's theorem and use it to prove $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$ where $u = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x}$. (P.T.U., Dec. 2010)

[Hint. S.E. I (*ii*) art. 3.7]

15. Verify Euler's theorem for $f(x, y, z) = 3x^2yz + 5xy^2z + 4z^4$ (P.T.U., Dec. 2005)

[Hint. S.E. I (*iii*) art 3.7]

16. (i) If $u = f\left(\frac{y}{x}\right)$ show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$

[Hint. S.E. 2 art. 3.7]

(*ii*) If $u = xf\left(\frac{y}{x}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u$ and $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0$

(*P.T.U., May 2008*)

17. Verify Euler's theorem for the function $z = ax^2 + 2hxy + by^2$.

18. State Euler's theorem and use it to prove $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u \log u$, where $u = e^{x^2+y^2}$.

(*P.T.U., May 2014*)

19. If $f(x, y) = \sqrt{x^2 - y^2} \sin^{-1}\left(\frac{y}{x}\right)$ prove that $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = f(x, y)$.

20. (*i*) If $u = \cos^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\cot u = 0$.

(*ii*) If $u = \sin^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u$. **(*P.T.U., May 2008*)**

[**Hint.** In each case u is not homogeneous function but $\cos u$ and $\sin u$ both are homogeneous function of degree $\frac{1}{2}$ apply Euler's theorem to each]

(*iii*) If $u = \sin^{-1}\frac{x^2+y^2}{x+y}$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u$. **(*P.T.U., Dec. 2013*)**

21. If $u = \log\frac{x^4+y^4}{x+y}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3$.

22. Define composite function of single and double variables. **(*P.T.U., May 2007*)**

[**Hint.** Art. 3.8]

23. State Euler's theorem and establish the same for any suitable function.

[**Hint.** Art. 3.6] **(*P.T.U., May 2005*)**

24. If given $f(x, y) = c$, an implicit function prove that $\frac{dy}{dx} = -\frac{f_x}{f_y}$.

[**Hint.** Art. 3.9(*b*)]

25. (*i*) If $u = f(y-z, z-x, x-y)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$ **(*P.T.U., Dec 2003, Dec. 2011*)**

[**Hint.** S.E.4 (*a*) art 3.9 (*c*)]

(*ii*) If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$ prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$.

(*P.T.U., May 2003, Dec. 2010, May 2014*)

[**Hint.** S.E. 4(*ii*) art. 3.9 (*c*)]

26. Find $\frac{du}{dt}$ when

(*i*) $u = x^2 + y^2, x = at^2, y = 2at$.

[**Hint.** S.E. 1 art. 3.9(*c*)]

(*ii*) $u = x^2 + y^2 + z^2$; $x = e^{2t}, y = e^{2t}\cos 3t, z = e^{2t}\sin 3t$.

(iii) $u = \sin \dfrac{x}{y}$; $x = e^t, y = t^2$. (P.T.U., May 2006)

[**Hint.** S.E. 2 art. 3.9(c)]

(iv) $u = x^3 + y^3, x = a \cos t, y = b \sin t$.

27. If $w = x^2 + y^2, x = r - s, y = r + s$ find $\dfrac{\partial w}{\partial r}, \dfrac{\partial w}{\partial s}$ in terms of r, s. (P.T.U., May 2012)

[**Hint.** S.E. 3(i) art. 3.9]

28. If $x = u + v, y = uv$ and z is a function of x, y show that $u\dfrac{\partial z}{\partial u} + v\dfrac{\partial z}{\partial v} = x\dfrac{\partial z}{\partial x} + 2y\dfrac{\partial z}{\partial y}$.

29. If $z = x^2 y$ and $x^2 + xy + y^2 = 1$ show that $\dfrac{dz}{dx} = 2xy - \dfrac{x^2 (2x + y)}{x + 2y}$.

30. If $z = \sqrt{x^2 + y^2}$ and $x^3 + y^3 + z\,axy = 5a^2$, find the value of $\dfrac{dz}{dx}$ at $x = y = a$. (P.T.U., May 2009)

31. When $x^y + y^x = C$, show that $\dfrac{dy}{dx} = \dfrac{yx^{y-1} + y^x \log y}{x^y \log x + xy^{x-1}}$.

[**Hint.** S.E. 9(i) art. 3.9(c)]

32. When $(\cos x)^y = (\sin y)^x$, prove that $\dfrac{dy}{dx} = \dfrac{x \tan x + \log \sin y}{\log \cos x - x \cot y}$.

[**Hint.** S.E. 9(ii) art. 3.9(c)]

33. If $F(x, y, z) = 0$, prove that $\left(\dfrac{\partial y}{\partial x}\right)_z \left(\dfrac{\partial z}{\partial y}\right)_x \left(\dfrac{\partial x}{\partial z}\right)_y = -1$. (P.T.U., Dec. 2011)

[**Hint.** S.E. 11 art 3.9 (i)]

34. Find total derivative of $z = \tan^{-1}\left(\dfrac{x}{y}\right)$ when $(x, y) \neq (0, 0)$. (P.T.U., Dec. 2007)

$$\left[\textbf{Hint. } dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = \frac{y\,dx - x\,dy}{x^2 + y^2}\right]$$

35. If $x = r \cos \theta, y = r \sin \theta$ verify $\dfrac{\partial(x, y)}{\partial(r, \theta)} \times \dfrac{\partial(r, \theta)}{\partial(x, y)} = 1$. (P.T.U., Dec. 2005, May 2005, Dec. 2011)

[**Hint.** S.E. 1(i) art. 3.11]

36. If $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1} \dfrac{y}{x}$ evaluate $\dfrac{\partial(r, \theta)}{\partial(x, y)}$.

37. $x = r \cos \theta, y = r \sin \theta, z = z$ prove that $\dfrac{\partial(x, y, z)}{\partial(r, \theta, z)} = r$. (P.T.U., Dec. 2013)

[**Hint.** S.E. 1 (ii) art. 3.11]

38. (i) If $u = x - y, v = xy$, prove that $JJ' = 1$.

(ii) If $x = u (1 + v), y = v (1 + u)$, find the value of $\dfrac{\partial(x, y)}{\partial(u, v)}$. (P.T.U., May 2014)

[**Hint.** S.E. 1 (iii) art. 3.11]

39. (i) If $u = x^2 - 2y, v = x + y$, prove that $\dfrac{\partial(u, v)}{\partial(x, y)} = 2x + 2$. (P.T.U., May 2006)

(ii) If $u = \dfrac{y^2}{2x}, v = \dfrac{x^2 + y^2}{2x}$, prove that $\dfrac{\partial(u, v)}{\partial(x, y)} = -\dfrac{y}{2x}$.

(iii) If $u = \frac{yz}{x}$, $v = \frac{zx}{y}$, $w = \frac{xy}{z}$, prove that $\frac{\partial(u, v, w)}{\partial(x, y, z)} = 4$.

(iv) If $u = x \sin y$, $v = y \sin x$, prove that $\frac{\partial(u, v)}{\partial(x, y)} = \sin x \sin y - xy \cos x \cos y$. **(P.T.U., May 2006)**

[**Hint.** S.E. 4 art. 3.11]

(v) If $u = x^3 + xy$, $v = xy$, prove that $\frac{\partial(u, v)}{\partial(x, y)} = 3x^3$. **(P.T.U., May 2009)**

(vi) If $u = \frac{x+y}{1-xy}$; $v = \tan^{-1} x + \tan^{-1} y$, prove that $\frac{\partial(u,v)}{\partial(x,y)} = 0$. **(P.T.U., May 2012)**

Answers

2. (i) $2abz$

4. (i) $2x \sin \frac{y}{x} - y \cos \frac{y}{x}, x \cos \frac{y}{x}$ (ii) $\frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2}$ (iii) $y^x \log y, xy^{x-1}$

8. 1

26. (i) $4a^2 t (t^2 + 2)$ (ii) $8e^{4t}$

(iii) $\cos \frac{e^t}{t^2}$ (iv) $-3a^3 \cos^2 t \sin t$.

27. $4r, 4s$ **30.** 0 **34.** $\frac{ydx - xdy}{x^2 + y^2}$ **36.** $\frac{1}{r}$

38. (ii) $1 + u + v$.