3

Beta and Gamma Functions

3.1 BETA FUNCTION

Definition. If $m > 0$, $n > 0$, then the integral $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$, which is obviously a function of m and n, is called a **Beta function** and is denoted by $B(m, n)$.

Thus $B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \;\forall\; m > 0, n > 0$. Beta function is also called the **First Eulerian Integral.**

For example,

(i) $\int_0^1 x^3(1-x)^5\,dx = B(3+1, 5+1) = B(4, 6)$

(ii) $\int_0^1 \sqrt{x}(1-x)^3\,dx = B\left(\frac{1}{2}+1, 3+1\right) = B\left(\frac{3}{2}, 4\right)$

(iii) $\int_0^1 x^{-2/3}(1-x)^{-\frac{1}{2}}\,dx = B\left(-\frac{2}{3}+1, -\frac{1}{2}+1\right) = B\left(\frac{1}{3}, \frac{1}{2}\right)$

(iv) $\int_0^1 x^{-3}(1-x)^5\,dx$ is not a Beta function since $m = -3+1 = -2 < 0$.

3.2 PROPERTIES OF BETA FUNCTION

Property I. *Symmetry of Beta function i.e., $B(m, n) = B(n, m)$.*

Proof. By definition, $B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx,\; m > 0, n > 0$

Changing x to $1-x$ $\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

$$B(m, n) = \int_0^1 (1-x)^{m-1}[1-(1-x)]^{n-1}\,dx$$

$$= \int_0^1 (1-x)^{m-1}x^{n-1}\,dx = \int_0^1 x^{n-1}(1-x)^{m-1}\,dx = B(n, m)$$

Hence, $B(m, n) = B(n, m)$.

Property II. *If m, n are positive integers, then* $B(m, n) = \dfrac{(m-1)!(n-1)!}{(m+n-1)!}$.

Proof. $B(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1}\, dx$

Integrating by parts $= \left[x^{m-1} \cdot \dfrac{(1-x)^n}{n(-1)}\right]_0^1 - \int_0^1 (m-1)\, x^{m-2} \cdot \dfrac{(1-x)^n}{n(-1)}\, dx$

$$= \frac{m-1}{n} \int_0^1 x^{m-2} (1-x)^n\, dx$$

$$= \frac{m-1}{n} \int_0^1 x^{m-2} (1-x)^{n-1} (1-x)\, dx$$

$$= \frac{m-1}{n} \int_0^1 [x^{m-2} (1-x)^{n-1} - x^{m-1} (1-x)^{n-1}]\, dx$$

$$= \frac{m-1}{n} \int_0^1 x^{m-2} (1-x)^{n-1}\, dx - \frac{m-1}{n} \int_0^1 x^{m-1} (1-x)^{n-1}\, dx$$

$$= \frac{m-1}{n} B(m-1, n) - \frac{m-1}{n} B(m, n)$$

$$\Rightarrow \left(1 + \frac{m-1}{n}\right) B(m, n) = \frac{m-1}{n} B(m-1, n)$$

$$\Rightarrow \quad B(m, n) = \frac{m-1}{m+n-1} B(m-1, n) \qquad \ldots(1)$$

Changing m to $(m-1)$, we have $B(m-1, n) = \dfrac{m-2}{m+n-2} B(m-2, n)$

Putting this value of $B(m-1, n)$ in (1), we have

$$B(m, n) = \frac{(m-1)(m-2)}{(m+n-1)(m+n-2)} B(m-2, n) \qquad \ldots(2)$$

Generalising from (1) and (2)

$$B(m, n) = \frac{(m-1)(m-2)\ldots\ldots 1}{(m+n-1)(m+n-2)\ldots\ldots(n+1)} B(1, n) \qquad \ldots(3)$$

But $B(1, n) = \int_0^1 x^0 (1-x)^{n-1}\, dx = \left[\dfrac{(1-x)^n}{n(-1)}\right]_0^1 = \dfrac{1}{n}$

$\therefore$ From (3), we get

$$B(m, n) = \frac{(m-1)(m-2)\ldots\ldots 1}{(m+n-1)(m+n-2)\ldots\ldots(n+1)n} = \frac{(m-1)!}{(m+n-1)(m+n-2)\ldots\ldots(n+1)\,n}$$

Multiplying the num. and denom. by $(n-1)!$, we have

$$B(m, n) = \frac{(m-1)!(n-1)!}{(m+n-1)(m+n-2)\ldots\ldots(n+1)\,n \cdot (n-1)!} = \frac{(m-1)!(n-1)!}{(m+n-1)!}$$

Property III. $B(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx \, , \, m > 0, n > 0.$

Proof. $B(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$

Put $x = \frac{1}{1+y}$ then $dx = \frac{-1}{(1+y)^2} dy$

and $1 - x = 1 - \frac{1}{1+y} = \frac{y}{1+y}$

Also $y \to \infty$ when $x \to 0$ and $y = 0$ when $x = 1$

$$\therefore \quad B(m, n) = \int_\infty^0 \frac{1}{(1+y)^{m-1}} \left(\frac{y}{1+y}\right)^{n-1} \cdot \frac{-1}{(1+y)^2} dy$$

$$= -\int_\infty^0 \frac{y^{n-1}}{(1+y)^{m-1+n-1+2}} dy = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}} dy$$

Changing y to x,

$$B(m, n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

Interchanging m and n,

$$B(n, m) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{n+m}} dx$$

But $B(m, n) = B(n, m)$

Hence, $B(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx .$

3.3 GAMMA FUNCTION

If n is positive, then the definite integral $\int_0^\infty e^{-x} x^{n-1} dx$, which is a function of n, is called the Gamma function (or Eulerian integral of second kind) and is denoted by $\Gamma(n)$. Thus,

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx, n > 0.$$

In particular, $\Gamma(1) = \int_0^\infty e^{-x} dx = \left[-e^{-x} \right]_0^\infty = 1.$

$$\therefore \quad \boxed{\Gamma(1) = 1}$$

3.4 REDUCTION FORMULA FOR $\Gamma(n)$

$$\Gamma(n+1) = \int_0^\infty e^{-x} x^n \, dx = \int_0^\infty x^n e^{-x} \, dx$$

Integrating by parts, we have

$$\Gamma(n+1) = \left| -x^n e^{-x} \right|_0^\infty + n\int_0^\infty x^{n-1} e^{-x} \, dx$$

$$= n\int_0^\infty e^{-x} x^{n-1} \, dx \qquad \left[\because \lim_{x \to \infty} \frac{x^n}{e^x} = 0\right]$$

$$= n\Gamma(n)$$

$\therefore$ $\mathbf{\Gamma(n+1) = n\Gamma(n)}$, which is the reduction formula for $\Gamma(n)$.

Note 1. If n is a positive integer, then by repeated application of above formula, we get

$$\begin{aligned}\Gamma(n+1) &= n\Gamma(n)\\ &= n(n-1)\,\Gamma(n-1)\\ &= n(n-1)(n-2)\,\Gamma(n-2)\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &= n(n-1)(n-2)\ldots 3\cdot 2\cdot 1\,\Gamma(1)\\ &= n!, \quad \text{since } \Gamma(1) = 1\end{aligned}$$

Hence $\Gamma(n+1) = n!$ when n is a positive integer.

Note 2. If n is a positive fraction, then by repeated application of above formula, we get

$$\Gamma(n) = (n-1)(n-2) \times \text{go on decreasing by 1} \ldots$$

the series of factors being continued so long as the factors remain positive, multiplied by Γ (last factor).

Thus, $$\Gamma\left(\frac{11}{4}\right) = \frac{7}{4}\Gamma\left(\frac{7}{4}\right) = \frac{7}{4}\cdot\frac{3}{4}\Gamma\left(\frac{3}{4}\right)$$

The value of $\Gamma\left(\frac{3}{4}\right)$ can be obtained from the table of gamma functions.

Note 3. $\Gamma(n+1) = n\,\Gamma(n)$

$$\Rightarrow \qquad \Gamma(n) = \frac{\Gamma(n+1)}{n}, \; n \neq 0$$

$$= \frac{(n+1)\Gamma(n+1)}{n(n+1)} = \frac{\Gamma(n+2)}{n(n+1)}, \; n \neq 0, -1$$

$$= \frac{(n+2)\Gamma(n+2)}{n(n+1)(n+2)} = \frac{\Gamma(n+3)}{n(n+1)(n+2)}, \; n \neq 0, -1, -2$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$= \frac{\Gamma(n+k+1)}{n(n+1)(n+2)\ldots(n+k)}, \; n \neq 0, -1, -2, \ldots, -k$$

This result defines $\Gamma(n)$ for $n < 0$, k being the least positive integer such that $n + k + 1 > 0$.

For example, to evaluate $\Gamma(-3.4)$

$$n + k + 1 > 0 \;\Rightarrow\; -3.4 + k + 1 > 0 \;\Rightarrow\; k > 2.4$$

We choose $k = 3$

$$\therefore \quad \Gamma(-3.4) = \frac{\Gamma(-3.4+3+1)}{(-3.4)(-2.4)(-1.4)(-.4)} = \frac{\Gamma(.6)}{(3.4)(2.4)(1.4)(.4)}$$

The value of $\Gamma(.6)$ can be obtained from the table of gamma functions.

Also we observe that $\Gamma(n)$ is infinite when $n = 0$ or a negative integer.

3.5(a) RELATION BETWEEN BETA AND GAMMA FUNCTIONS

$$\mathbf{B}(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

We know that $\Gamma(m) = \int_0^\infty e^{-t}\, t^{m-1}\, dt$

Putting $t = x^2$ so that $dt = 2x\, dx$

$$\Gamma(m) = 2\int_0^\infty e^{-x^2} x^{2m-1}\, dx \qquad \ldots(1)$$

Similarly, $\Gamma(n) = 2\int_0^\infty e^{-y^2}\, x^{2n-1}\, dy$

Now we use the following result from double integrals:

If $f(x)$ and $g(y)$ are functions of x and y only, and the limits of integration are constants, then the double integral can be represented as a product of two integrals. Thus,

$$\Gamma(m)\,\Gamma(n) = 4\int_0^\infty e^{-x^2} x^{2m-1}\, dx. \int_0^\infty e^{-y^2}\, y^{2n-1}\, dy$$

$$= 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)} x^{2m-1}\, y^{2n-1}\, dx\, dy$$

Changing to polar co-ordinates, we have

$$\Gamma(m)\,\Gamma(n) = 4\int_0^{\frac{\pi}{2}}\int_0^\infty e^{-r^2}\cdot(r\cos\theta)^{2m-1}\cdot(r\sin\theta)^{2n-1}\cdot r dr d\theta$$

$$= 4\int_0^{\pi/2}\int_0^\infty e^{-r^2}\, r^{2(m+n)-1}\, \cos^{2m-1}\theta \sin^{2n-1}\theta\, dr\, d\theta$$

$$= 4\int_0^\infty e^{-r^2}\, r^{2(m+n)-1}\, dr \cdot \int_0^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta\, d\theta \qquad \ldots(2)$$

Now, $2\int_0^\infty e^{-r^2}\, r^{2(m+n)-1}\, dr = \Gamma\,(m+n)$, from (1)

and putting $\sin^2\theta = z$ so that $2\sin\theta\cos\theta\, d\theta = dz$

$$2\int_0^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta\, d\theta = \int_0^1 (1-z)^{m-1}\, z^{n-1}\, dz = \int_0^1 z^{n-1}\,(1-z)^{m-1}\, dz$$

$= \mathrm{B}\,(n, m) = \mathrm{B}\,(m, n)$ by symmetry of Beta Function

$\therefore$ From (2), we have $\Gamma(m)\,\Gamma(n) = \Gamma(m+n)\,\mathrm{B}\,(m, n)$

Hence, $$\mathrm{B}\,(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

3.5(b) VALUE OF $\Gamma\left(\frac{1}{2}\right)$

Using $B(m, n) = \dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$, we have

$$B\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)} \Rightarrow B\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{\left[\Gamma\left(\frac{1}{2}\right)\right]^2}{\Gamma(1)}$$

$$\Rightarrow \qquad \left[\Gamma\left(\frac{1}{2}\right)\right]^2 = B\left(\frac{1}{2}, \frac{1}{2}\right) \qquad [\because \Gamma(1) = 1]$$

$$= \int_0^1 x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}\,dx = \int_0^1 \frac{dx}{\sqrt{x(1-x)}}$$

[Put $x = \sin^2\theta$ so that $dx = 2\sin\theta\cos\theta\,d\theta$]

$$= \int_0^{\frac{\pi}{2}} \frac{2\sin\theta\cos\theta\,d\theta}{\sin\theta\cos\theta} = 2\int_0^{\frac{\pi}{2}} d\theta = 2[\theta]_0^{\frac{\pi}{2}} = 2\left(\frac{\pi}{2}\right) = \pi$$

$$\therefore \qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

For Example:

1. $\Gamma(3.5) = \Gamma\left(\dfrac{7}{2}\right) = \dfrac{5}{2}\cdot\dfrac{3}{2}\cdot\dfrac{1}{2}\,\Gamma\left(\dfrac{1}{2}\right) = \dfrac{15}{8}\sqrt{\pi}$

2. $B(2.5, 1.5) = B\left(\dfrac{5}{2}, \dfrac{3}{2}\right) = \dfrac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{5}{2}+\frac{3}{2}\right)} = \dfrac{\left[\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\right]\left[\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\right]}{\Gamma(4)}$

$$= \frac{\frac{3}{8}\left[\Gamma\left(\frac{1}{2}\right)\right]^2}{3!} = \frac{\frac{3}{8}(\sqrt{\pi})^2}{3\cdot 2\cdot 1} = \frac{\pi}{16}$$

3. $B\left(\dfrac{9}{2}, \dfrac{7}{2}\right) = \dfrac{\Gamma\left(\frac{9}{2}\right)\Gamma\left(\frac{7}{2}\right)}{\Gamma\left(\frac{9}{2}+\frac{7}{2}\right)} = \dfrac{\left[\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\right]\left[\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\right]}{\Gamma(8)}$

$$= \frac{7\cdot 5\cdot 3\cdot 5\cdot 3\left[\Gamma\left(\frac{1}{2}\right)\right]^2}{16\times 8\times 7!} = \frac{7\cdot 5\cdot 3\cdot 5\cdot 3\,(\sqrt{\pi})^2}{128\times 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} = \frac{5\pi}{2048}.$$

3.6 TO EVALUATE $\int_0^{\pi/2} \sin^p x \cos^q x\, dx$

Putting $\sin^2 x = t$ so that $2 \sin x \cos x\, dx = dt$

$$\int_0^{\pi/2} \sin^p x \cos^q x\, dx = \int_0^{\pi/2} (\sin^{p-1} x \cos^{q-1} x) \sin x \cos x\, dx$$

$$= \int_0^{\pi/2} (\sin^2 x)^{(p-1)/2} \cdot (1 - \sin^2 x)^{(q-1)/2} \sin x \cos x\, dx$$

$$= \frac{1}{2}\int_0^1 t^{(p-1)/2}(1-t)^{(q-1)/2}\, dt$$

$$= \frac{1}{2} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}.$$

For example: $\int_0^{\frac{\pi}{2}} \sin^2 \theta \cos^4 \theta\, d\theta$

(Here $p = 2$, $q = 4$)

$$= \frac{\Gamma\left(\frac{2+1}{2}\right)\Gamma\left(\frac{4+1}{2}\right)}{2\Gamma\left(\frac{2+4+2}{2}\right)} = \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{5}{2}\right)}{2\Gamma(4)}$$

$$= \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{2[(4-1)!]} = \frac{\frac{3}{8}\left[\Gamma\left(\frac{1}{2}\right)\right]^2}{2\times 6} = \frac{\left(\sqrt{\pi}\right)^2}{32} = \frac{\pi}{32}$$

Cor. 1. Putting $p = q = 0$, we have

$$\frac{[\Gamma(\frac{1}{2})]^2}{2\Gamma(1)} = \int_0^{\pi/2} dx = \frac{\pi}{2} \quad \text{or} \quad [\Gamma(\tfrac{1}{2})]^2 = \pi, \text{ since } \Gamma(1) = 1$$

$\therefore$ $$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

Cor. 2. Putting $p = n$ and $q = 0$, we have

$$\int_0^{\pi/2} \sin^n x\, dx = \frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{n+2}{2}\right)} = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}\cdot\frac{\sqrt{\pi}}{2}$$

Similarly, $$\int_0^{\pi/2} \cos^n x\, dx = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}\cdot\frac{\sqrt{\pi}}{2}.$$

ILLUSTRATIVE EXAMPLES

Example 1. *Evaluate the following:*

(i) $\int_0^\infty x^{1/4} e^{-\sqrt{x}}\, dx$ (ii) $\int_0^\infty x^{n-1} e^{-h^2x^2}\, dx$

(iii) $\int_0^\infty \frac{x^a}{a^x}\, dx.$ (iv) $\int_0^1 x^m (\log x)^n\, dx$; n is a +ve integer $m > -1$.

Sol. (i) Let $$I = \int_0^\infty x^{1/4} e^{-\sqrt{x}}\, dx$$

Let $\sqrt{x} = t \quad \therefore \quad \frac{1}{2\sqrt{x}}\, dx = dt \quad \therefore \quad dx = 2t\, dt$

$$I = \int_0^\infty t^{1/2} e^{-t} \cdot 2t\, dt = 2\int_0^\infty t^{3/2} e^{-t}\, dt$$

$$= 2\int_0^\infty e^{-t}\, t^{3/2}\, dt \text{ which is Gamma of } \frac{3}{2} + 1 = \frac{5}{2}$$

$$\therefore \quad I = 2.\Gamma\left(\frac{5}{2}\right) = 2\cdot\frac{3}{2}\cdot\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right) = \frac{3}{2}\sqrt{\pi}$$

(ii) Let $$I = \int_0^\infty x^{n-1} e^{-h^2x^2}\, dx$$

Let $h^2x^2 = t \quad \therefore \quad 2h^2x\, dx = dt$

$$\therefore \quad I = \int_0^\infty x^{n-1} e^{-t}\, \frac{dt}{2h^2x} = \frac{1}{2h^2}\int_0^\infty x^{n-2}\, e^{-t}\, dt$$

$$= \frac{1}{2h^2}\int_0^\infty \left(\frac{t}{h^2}\right)^{\frac{n-2}{2}} e^{-t}\, dt = \frac{1}{2h^2}\int_0^\infty \frac{1}{h^{n-2}}\, t^{\frac{n}{2}-1}\, e^{-t}\, dt$$

$$= \frac{1}{2h^n}\int_0^\infty e^{-t}\, t^{\frac{n}{2}-1}\, dt = \frac{1}{2h^n}\,\Gamma\left(\frac{n}{2}\right)$$

(iii) Let $$I = \int_0^\infty \frac{x^a}{a^x}\, dx$$

Let $$a^x = e^t$$

Taking logs of both sides,

$$x \log a = t, \quad \therefore \quad x = \frac{t}{\log a}$$

$$\therefore \quad dx = \frac{1}{\log a}\, dt$$

$$I = \int_0^\infty \left(\frac{t}{\log a}\right)^a \frac{1}{e^t}\cdot\frac{dt}{\log a} = \int_0^\infty \frac{1}{(\log a)^{a+1}}\cdot t^a\, e^{-t}\, dt$$

$$= \frac{1}{(\log a)^{a+1}}\int_0^\infty e^{-t}\, t^a\, dt = \frac{1}{(\log a)^{a+1}}\,\Gamma(a+1).$$

(*iv*) Let $$I = \int_0^1 x^m (\log x)^n \, dx$$

Let $\log \frac{1}{x} = t \quad \therefore$ when $x = 0,\ t = \infty$

When $x = 1,\ t = 0$

$$\therefore \quad \frac{1}{x} = e^t \quad \therefore \quad x = e^{-t} \Rightarrow dx = -e^{-t}\, dt$$

$$= \int_\infty^0 e^{-mt} (-t)^n \cdot (-e^{-t})\, dt = \int_\infty^0 (-1)^{n+1} t^n \, e^{-(m+1)t}\, dt$$

$$= (-1)^n \int_0^\infty e^{-(m+1)t}\, t^n \, dt$$

Put $(m+1)\, t = z \quad \therefore \quad dt = \dfrac{dz}{m+1}$

$$= (-1)^n \int_0^\infty e^{-z} \left(\frac{z}{m+1}\right)^n \cdot \frac{dz}{m+1} = \frac{(-1)^n}{(m+1)^{n+1}} \int_0^\infty e^{-z} z^n \, dz$$

$$= \frac{(-1)^n}{(m+1)^{n+1}} \int_0^\infty e^{-z} z^{(n+1)-1} \, dz$$

$$= \frac{(-1)^n}{(m+1)^{n+1}} \Gamma(n+1) \quad \because \quad \Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx$$

$$= \frac{(-1)^n}{(m+1)^{n+1}}\, n! \quad \because \quad \Gamma(n+1) = n! \text{ where } n \text{ is a +ve integer.}$$

Example 2. *Prove the following:*

(*i*) $1 \cdot 3 \cdot 5 \ldots (2n-1) = \dfrac{2^n\, \Gamma\left(n + \frac{1}{2}\right)}{\sqrt{\pi}}$ (*ii*) $\displaystyle\int_0^\infty e^{-y^{\frac{1}{m}}}\, dy = m\,\Gamma(m)$

Sol. (*i*) $1 \cdot 3 \cdot 5 \ldots (2n-1)$

$$= 2^n \cdot \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} \ldots \frac{2n-3}{2} \cdot \frac{2n-1}{2}$$

$$= 2^n \left\{ \frac{2n-1}{2} \cdot \left(\frac{2n-1}{2} - 1\right)\left(\frac{2n-1}{2} - 2\right) \ldots \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \right\}$$

Multiply and divide by $\Gamma\left(\frac{1}{2}\right)$

$$= \frac{2^n \left\{\left(\frac{2n-1}{2}\right)\left(\frac{2n-3}{2}\right)\left(\frac{2n-5}{2}\right) \ldots \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2}\, \Gamma\left(\frac{1}{2}\right)\right\}}{\Gamma\left(\frac{1}{2}\right)}$$

$$= 2^n \frac{\Gamma\left(\frac{2n-1}{2}+1\right)}{\Gamma\left(\frac{1}{2}\right)} = 2^n \frac{\Gamma\left(\frac{2n+1}{2}\right)}{\sqrt{\pi}} = 2^n \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}}$$

(*ii*) $\int_0^\infty e^{-y^{\frac{1}{m}}}\, dy$ Let $y^{\frac{1}{m}} = t \quad \therefore \quad y = t^m$

$$dy = mt^{m-1}\, dt$$

$$= \int_0^\infty e^{-t} \cdot mt^{m-1}\, dt = m\int_0^\infty e^{-t}\, t^{m-1}\, dt = m\,\Gamma(m)$$

Example 3. *Prove the following:*

(*i*) $\int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}\, dx \times \int_0^\infty x^2\, e^{-x^4}\, dx = \frac{\pi}{4\sqrt{2}}$

(*ii*) $\int_0^\infty x^n e^{-a^2x^2}\, dx = \frac{1}{2a^{n+1}}\,\Gamma\left(\frac{n+1}{2}\right)$; $n > -1$

Deduce that $\int_{-\infty}^\infty e^{-a^2x^2}\, dx = \frac{\sqrt{\pi}}{a}$.

Sol. (*i*) Let $\int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}\, dx \times \int_0^\infty x^2 e^{-x^4}\, dx$

$$= I_1 \times I_2 \qquad \ldots(1)$$

where $\quad I_1 = \int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}\, dx$ and $I_2 = \int_0^\infty x^2\, e^{-x^4}\, dx$

Find I_1 : Put $x^2 = t \quad \therefore \quad 2x\, dx = dt$

$$I_1 = \int_0^\infty \frac{e^{-t}}{t^{\frac{1}{4}}} \frac{dt}{2\,t^{1/2}} = \frac{1}{2}\int_0^\infty t^{-3/4}\, e^{-t}\, dt = \frac{1}{2}\int_0^\infty e^{-t}\, t^{-3/4}\, dt$$

$$= \frac{1}{2}\Gamma\left(-\frac{3}{4}+1\right) = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)$$

$$I_2 = \int_0^\infty x^2 e^{-x^4}\, dx$$

Put $x^4 = t \quad \therefore \quad 4x^3 dx = dt$

$$I_2 = \int_0^\infty x^2 e^{-t} \frac{dt}{4x^3} = \frac{1}{4}\int_0^\infty t^{-\frac{1}{4}}\, e^{-t}\, dt$$

$$= \frac{1}{4}\int_0^\infty e^{-t} t^{-\frac{1}{4}}\, dt$$

From (1), $\int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}\,dx \times \int_0^\infty x^2 e^{-x^4}\,dx = \frac{1}{2}\Gamma\left(\frac{1}{4}\right) \times \frac{1}{4}\Gamma\left(\frac{3}{4}\right) = \frac{1}{8}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$

$$= \frac{1}{8}\sqrt{2}\,\pi \qquad \text{[See example 14 (next)]}$$

$$= \frac{\pi}{4\sqrt{2}}.$$

(*ii*) $\int_0^\infty x^n e^{-a^2x^2}\,dx$

Put $a^2x^2 = t \quad \therefore \quad 2a^2x\,dx = dt$

$$= \int_0^\infty x^n e^{-t}\frac{dt}{2a^2x} = \frac{1}{2a^2}\int_0^\infty e^{-t}\,x^{n-1}\,dt = \frac{1}{2a^2}\int_0^\infty e^{-t}\left(\frac{t^{1/2}}{a}\right)^{n-1} dt$$

$$= \frac{1}{2a^2\,a^{n-1}}\int_0^\infty e^{-t}\,.\,t^{\frac{n-1}{2}}\,dt = \frac{1}{2a^{n+1}}\Gamma\left(\frac{n-1}{2}+1\right) = \frac{1}{2a^{n+1}}\Gamma\left(\frac{n+1}{2}\right)$$

For deduction Put $n = 0$, $\int_0^\infty e^{-a^2x^2}\,dx = \frac{1}{2a}\Gamma\left(\frac{1}{2}\right)$

Now, $\int_{-\infty}^\infty e^{-a^2x^2}\,dx = 2\int_0^\infty e^{-a^2x^2}\,dx \quad \because$ integrand is even

$$= 2\frac{1}{a}\sqrt{\pi} \qquad \Big|\because\ \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Example 4. *Prove that* $\int_0^\infty e^{-ax}\,x^{n-1}\,dx = \frac{\Gamma(n)}{a^n}$, *where a, n are positive. Deduce that*

(*i*) $\int_0^\infty e^{-ax}\,x^{n-1}\cos bx\,dx = \frac{\Gamma(n)}{r^n}\cos n\theta$

(*ii*) $\int_0^\infty e^{-ax}\,x^{n-1}\sin bx\,dx = \frac{\Gamma(n)}{r^n}\sin n\theta$

where $r^2 = a^2 + b^2$ *and* $\theta = \tan^{-1}\frac{b}{a}$.

Sol. Put $ax = z$ in $\int_0^\infty e^{-ax}x^{n-1}dx$ so that $dx = \frac{dz}{a}$

$$\therefore \quad \int_0^\infty e^{-ax}x^{n-1}\,dx = \int_0^\infty e^{-z}\left(\frac{z}{a}\right)^{n-1}\cdot\frac{dz}{a} = \frac{1}{a^n}\int_0^\infty e^{-z}z^{n-1}\,dz = \frac{\Gamma(n)}{a^n}.$$

Deduction. Replacing a by $(a + ib)$, we have

$$\int_0^\infty e^{-(a+ib)x}\,x^{n-1}\,dx = \frac{\Gamma(n)}{(a+ib)^n} \qquad \ldots(1)$$

Now, $\qquad e^{-(a+ib)x} = e^{-ax} \cdot e^{-ibx} = e^{-ax}(\cos bx - i \sin bx)$

Also, putting $a = r\cos\theta$ and $b = r\sin\theta$

so that $\qquad r^2 = a^2 + b^2$ and $\theta = \tan^{-1}\dfrac{b}{a}$

$$(a+ib)^n = (r\cos\theta + ir\sin\theta)^n = r^n(\cos\theta + i\sin\theta)^n$$
$$= r^n(\cos n\theta + i\sin n\theta) \qquad \text{[De Moivre's Theorem]}$$

$\therefore$ From (1), we have

$$\int_0^\infty e^{-ax}(\cos bx - i\sin bx)\, x^{n-1}\, dx$$

$$= \frac{\Gamma(n)}{r^n(\cos n\theta + i\sin n\theta)} = \frac{\Gamma(n)}{r^n}(\cos n\theta + i\sin n\theta)^{-1}$$

$$= \frac{\Gamma(n)}{r^n}(\cos n\theta - i\sin n\theta)$$

Now equate real and imaginary parts on the two sides. We get

(i) $$\int_0^\infty e^{-ax}\cos bx \cdot x^{n-1}\, dx = \frac{\Gamma(n)}{r^n}\cos n\theta$$

and (ii) $$\int_0^\infty e^{-ax}\sin bx\, x^{n-1}\, dx = \frac{\Gamma(n)}{r^n}\sin n\theta$$

Example 5. *Show that* $\displaystyle\int_0^1 y^{q-1}\left(\log\frac{1}{y}\right)^{p-1} dy = \frac{\Gamma(p)}{q^p}$ *where* $p > 0,\ q > 0.$

Sol. Put $\log\dfrac{1}{y} = x$ so that $\dfrac{1}{y} = e^x$ or $y = e^{-x}$ and $dy = -e^{-x}\,dx$

$$\therefore \quad \int_0^1 y^{q-1}\left(\log\frac{1}{y}\right)^{p-1} dy = \int_\infty^0 e^{-(q-1)x}\cdot x^{p-1}(-e^{-x})\,dx = \int_0^\infty e^{-qx}\, x^{p-1}\, dx$$

$$= \int_0^\infty e^{-t}\cdot\left(\frac{t}{q}\right)^{p-1}\cdot\frac{dt}{q}, \quad \text{where } qx = t$$

$$= \frac{1}{q^p}\int_0^\infty e^{-t}t^{p-1}\,dt = \frac{\Gamma(p)}{q^p}.$$

Example 6. *Prove that:*

$$\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}}\,dx = \frac{B(m,n)}{a^n(a+b)^m}.$$

Sol. Put $$\frac{x}{a+bx} = \frac{t}{a+b} \qquad \ldots(1)$$

Differentiating both sides:

$$\frac{(a+bx)\cdot 1 - x\cdot b}{(a+bx)^2}\, dx = \frac{dt}{a+b}$$

$$\therefore \qquad \frac{dx}{(a+bx)^2} = \frac{dt}{a(a+b)}$$

From (1) $x(a+b) = at + btx \quad \therefore \quad x = \dfrac{at}{a+b-bt}$

When $\quad x = 0, \quad t = 0 \quad$ when $x = 1\,;\, t = 1$

$$\therefore \quad \int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}}\, dx = \int_0^1 \frac{x^{m-1}\cdot x^{n-1}}{(a+bx)^{m+n-2}} \cdot \frac{dx}{(a+bx)^2}$$

$$= \int_0^1 \frac{x^{m+n-2}\left(\dfrac{1}{x}-1\right)^{n-1}}{(a+bx)^{m+n-2}} \cdot \frac{dt}{a(a+b)}$$

$$= \int_0^1 \left(\frac{x}{a+bx}\right)^{m+n-2} \left(\frac{1}{x}-1\right)^{n-1} \cdot \frac{dt}{a(a+b)}$$

$$\frac{1}{x} - 1 = \frac{a+b-bt}{at} - 1 = \frac{a+b-bt-at}{at} = \frac{(a+b)(1-t)}{at}$$

$$= \int_0^1 \frac{t^{m+n-2}}{(a+b)^{m+n-2}} \cdot \frac{(a+b)^{n-1}}{a^{n-1}} \cdot \frac{(1-t)^{n-1}}{t^{n-1}} \cdot \frac{dt}{a(a+b)}$$

$$= \frac{1}{a^n (a+b)^m} \int_0^1 t^{m-1}(1-t)^{n-1}\, dt = \frac{1}{a^n (a+b)^m}\, \mathrm{B}(m, n)$$

Hence $\displaystyle \int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}}\, dx = \frac{\mathrm{B}(m,n)}{a^n (a+b)^m}$.

Example 7. *Show that* $B(p, q) = \displaystyle\int_0^\infty \frac{y^{q-1}}{(1+y)^{p+q}}\, dy = \int_0^1 \frac{x^{p-1} + x^{q-1}}{(1+x)^{p+q}}\, dx$.

Sol. $\qquad \mathrm{B}(p, q) = \displaystyle\int_0^1 x^{p-1}(1-x)^{q-1}\, dx$

Putting $x = \dfrac{1}{1+y}$ so that $dx = -\dfrac{1}{(1+y)^2}\, dy$

$$\mathrm{B}(p, q) = \int_\infty^0 \left(\frac{1}{1+y}\right)^{p-1} \left(\frac{y}{1+y}\right)^{q-1} \cdot \frac{-1}{(1+y)^2}\, dy = \int_0^\infty \frac{y^{q-1}}{(1+y)^{p+q}}\, dy$$

$$= \int_0^1 \frac{y^{q-1}}{(1+y)^{p+q}}\, dy + \int_1^\infty \frac{y^{q-1}}{(1+y)^{p+q}}\, dy \qquad \ldots(1)$$

Now, putting $y = \frac{1}{z}$ in the second integral, we have

$$\int_1^\infty \frac{y^{q-1}}{(1+y)^{p+q}} dy = \int_1^0 \frac{\left(\frac{1}{z}\right)^{q-1}}{\left(1+\frac{1}{z}\right)^{p+q}}\left(-\frac{1}{z^2}\right) dz = \int_0^1 \frac{z^{p-1}}{(1+z)^{p+q}} dz$$

$\therefore$ From (1), we have

$$B(p, q) = \int_0^1 \frac{y^{q-1}}{(1+y)^{p+q}} dy + \int_0^1 \frac{z^{p-1}}{(1+z)^{p+q}} dz = \int_0^1 \frac{x^{p-1}+x^{q-1}}{(1+x)^{p+q}} dx.$$

Example 8. *Prove that:* $\int_0^1 x^3 (1-x)^{\frac{4}{3}} dx = \frac{243}{7280}$.

Sol. $\int_0^1 x^3 (1-x)^{\frac{4}{3}} dx = \int_0^1 x^{4-1} (1-x)^{\frac{7}{3}-1} dx$

Compare it with $\int_0^1 x^{m-1} (1-x)^{n-1} dx = B(m, n)$

$$\therefore \quad \int_0^1 x^3 (1-x)^{\frac{4}{3}} dx = B\left(4, \frac{7}{3}\right)$$

We know that $\quad B(m, n) = B(n, m)$

$$\therefore \quad \int_0^1 x^3 (1-x)^{\frac{4}{3}} dx = B\left(\frac{7}{3}, 4\right) = \int_0^1 x^{\frac{4}{3}} (1-x)^3 dx.$$

$$= \int_0^1 x^{\frac{4}{3}} [1 - 3x + 3x^2 - x^3] dx$$

$$= \int_0^1 \left[x^{\frac{4}{3}} - 3x^{\frac{7}{3}} + 3x^{\frac{10}{3}} - x^{\frac{13}{3}}\right] dx$$

$$= \frac{3}{7} x^{\frac{7}{3}} - 3 \cdot \frac{3x^{\frac{10}{3}}}{10} + 3 \cdot \frac{3}{13} x^{\frac{13}{3}} - \frac{3}{16} \cdot x^{\frac{16}{3}} \Bigg|_0^1$$

$$= \frac{3}{7} - \frac{9}{10} + \frac{9}{13} - \frac{3}{16} = \frac{243}{7280}.$$

Example 9. *Prove the following:*

(*i*) $\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{\sqrt{\pi}}{4} \cdot \frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$ (*ii*) $\int_0^1 \frac{x dx}{\sqrt{1-x^5}}$

Sol. (*i*) Put $x^4 = t$ *i.e.*, $x = t^{1/4}$ so that $dx = \frac{1}{4} t^{-3/4}\, dt$

$$\therefore \quad \int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{1}{4}\int_0^1 t^{-3/4}(1-t)^{-1/2}\, dt = \frac{1}{4}\int_0^1 t^{1/4-1}(1-t)^{1/2-1}\, dt$$

$$= \frac{1}{4} \mathrm{B}\left(\frac{1}{4}, \frac{1}{2}\right) = \frac{1}{4} \cdot \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{4}+\frac{1}{2}\right)} = \frac{\sqrt{\pi}}{4} \cdot \frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}. \qquad \Big|\, \because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

(*ii*) Put $x^5 = t \quad \therefore \quad 5x^4\, dx = dt \quad \therefore \quad x\, dx = \frac{1}{5x^3}\, dt = \frac{dt}{5t^{3/5}}$

$$\int_0^1 \frac{x\, dx}{\sqrt{1-x^5}} = \int_0^1 \frac{1}{\sqrt{1-t}} \frac{dt}{5t^{3/5}}$$

$$= \frac{1}{5}\int_0^1 t^{-3/5}(1-t)^{-1/2}\, dt = \frac{1}{5}\int_0^1 t^{\frac{2}{5}-1}(1-t)^{\frac{1}{2}-1}\, dt$$

$$= \frac{1}{5}\mathrm{B}\left(\frac{2}{5}, \frac{1}{2}\right) \qquad \because \quad \mathrm{B}(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx.$$

Example 10. *Prove that* $\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}\, dx = 0.$

Sol.
$$\int_0^\infty \frac{x^8 - x^{14}}{(1+x)^{24}}\, dx = \int_0^\infty \frac{x^8}{(1+x)^{24}}\, dx - \int_0^\infty \frac{x^{14}}{(1+x)^{24}}\, dx \qquad \ldots(1)$$

$$\mathrm{B}(m, n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}$$

$$\therefore \quad \int_0^\infty \frac{x^8}{(1+x)^{24}}\, dx = \int_0^\infty \frac{x^{9-1}}{(1+x)^{9+15}}\, dx = \mathrm{B}(15, 9)$$

and
$$\int_0^\infty \frac{x^{14}}{(1+x)^{24}}\, dx = \int_0^\infty \frac{x^{15-1}}{(1+x)^{15+9}}\, dx = \mathrm{B}(9, 15) = \mathrm{B}(15, 9)$$

$$\Big|\, \because \quad \mathrm{B}(m, n) = \mathrm{B}(n, m)$$

Substituting (1), $\int_0^\infty \frac{x^8 - x^{14}}{(1+x)^{24}}\, dx = 0.$

Example 11. *Prove that* $\int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}}\, dx = \frac{\mathrm{B}(m, n)}{a^n b^m}$, *where m, n, a, b, are positive.*

Sol. Put $bx = at$ *i.e.*, $x = \frac{at}{b}$ so that $dx = \frac{a}{b}\, dt$

$$\therefore \quad \int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}}\, dx = \int_0^\infty \frac{\left(\frac{at}{b}\right)^{m-1}}{(a+at)^{m+n}} \cdot \frac{a}{b}\, dt = \frac{1}{a^n b^m}\int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}}\, dt = \frac{\mathrm{B}(m, n)}{a^n b^m}.$$

Example 12. *Prove that $B(m, n) = B(m + 1, n) + B(m, n + 1)$.*

Sol. $B(m + 1, n) + B(m, n + 1)$

$$= \int_0^1 x^m (1-x)^{n-1}\, dx + \int_0^1 x^{m-1}(1-x)^n\, dx$$

$$= \int_0^1 x^{m-1}(1-x)^{n-1}[x + 1 - x]\, dx = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx = B(m, n).$$

Example 13. *Prove the following:*

(*i*) $B\left(m, \frac{1}{2}\right) = 2^{2m-1} B(m, m)$ (*ii*) $B(m, m) \cdot B\left(m + \frac{1}{2}, m + \frac{1}{2}\right) = \frac{\pi m^{-1}}{2^{4m-1}}$.

Sol. (*i*) We know that

$$B(m, n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$$

Put $n = \frac{1}{2}$

$$B\left(m, \frac{1}{2}\right) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\, d\theta \quad \ldots(1) \quad \Big| \because \ 2n - 1 = 0$$

Now for RHS put $n = m$

$$B(m, m) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2m-1}\theta\, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} (\sin\theta\cos\theta)^{2m-1}\, d\theta$$

$$= \frac{2}{2^{2m-1}}\int_0^{\frac{\pi}{2}} (2\sin\theta\cos\theta)^{2m-1}\, d\theta = \frac{2}{2^{2m-1}}\int_0^{\frac{\pi}{2}} (\sin 2\theta)^{2m-1}\, d\theta$$

Put $2\theta = \phi \quad \therefore \quad 2d\theta = d\phi$

$$\therefore \quad B(m, n) = \frac{1}{2^{2m-2}}\int_0^{\pi} (\sin\phi)^{2m-1} \cdot \frac{d\phi}{2}$$

$$= \frac{1}{2^{2m-2}}\, 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\phi\, d\phi \quad \Big| \begin{array}{l} \because f(a-x) = f(x) \\ \therefore \int_0^{2a} f(a-x)\,dx = 2\int_0^a f(x)\,dx \end{array}$$

$$= \frac{1}{2^{2m-2}}\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\, d\theta \text{ changing } \phi \text{ to } \theta$$

From (1), $$B\left(m, \frac{1}{2}\right) = 2 \cdot 2^{2m-2}\, B(m, m) = 2^{2m-1}\, B(m, m)$$

(*ii*) From part (*i*) $$B\left(m, \frac{1}{2}\right) = 2^{2m-1}\, B(m, m)$$

$$\therefore \quad B(m, m) = \frac{1}{2^{2m-1}} B\left(m, \frac{1}{2}\right) = \frac{1}{2^{2m-1}} \frac{\Gamma(m)\,\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(m + \frac{1}{2}\right)}$$

Change m to $m + \frac{1}{2}$

$$B\left(m+\frac{1}{2}, m+\frac{1}{2}\right) = \frac{1}{2^{2\left(m+\frac{1}{2}\right)-1}} \cdot \frac{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(m+\frac{1}{2}+\frac{1}{2}\right)}$$

$$= \frac{1}{2^{2m}} \cdot \frac{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(m+1)}$$

$$\therefore \quad B(m, m)\, B\left(m+\frac{1}{2}, m+\frac{1}{2}\right) = \frac{1}{2^{2m-1}} \cdot \frac{1}{2^{2m}} \frac{\Gamma(m)\,\Gamma\frac{1}{2}}{\Gamma\left(m+\frac{1}{2}\right)} \times \frac{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(m+1)}$$

$$= \frac{1}{2^{4m-1}} \frac{\Gamma(m)}{m\,\Gamma(m)} \pi$$

$$\Big| \because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \text{ and } \Gamma(m+1) = m\Gamma(m)$$

$$\therefore \quad B(m, m)\, B\left(m+\frac{1}{2}, m+\frac{1}{2}\right) = \frac{\pi m^{-1}}{2^{4m-1}}.$$

Example 14. *Prove that* $\Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}\cdot\Gamma(2n+1)}{2^{2n}\,\Gamma(n+1)}$.

Sol. Since, $$B(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

$$\therefore \quad B\left(n+\frac{1}{2}, n+\frac{1}{2}\right) = \frac{\left[\Gamma\left(n+\frac{1}{2}\right)\right]^2}{\Gamma(2n+1)} \quad \ldots(1)$$

Now, $$B\left(n+\frac{1}{2}, n+\frac{1}{2}\right) = \int_0^1 x^{n-1/2}(1-x)^{n-1/2}\, dx$$

Putting $x = \sin^2\theta$ so that $dx = 2\sin\theta\cos\theta\, d\theta$

$$B\left(n+\frac{1}{2}, n+\frac{1}{2}\right) = \int_0^{\pi/2} \sin^{2n-1}\theta\cos^{2n-1}\theta \cdot 2\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2n}\theta\cos^{2n}\theta\, d\theta = 2\int_0^{\pi/2} \left(\frac{\sin 2\theta}{2}\right)^{2n} d\theta$$

$$= \frac{1}{2^{2n-1}} \int_0^{\pi/2} \sin^{2n} 2\theta\, d\theta$$

$$= \frac{1}{2^{2n}} \int_0^{\pi} \sin^{2n} \phi \, d\phi, \quad \text{where } \phi = 2\theta$$

$$= \frac{1}{2^{2n}} \cdot 2\int_0^{\pi/2} \sin^{2n} \phi \, d\phi$$

$$\left[\because \int_0^{2a} f(x)\, dx = 2\int_0^{a} f(x)\, dx, \text{ if } f(2a-x) = f(x)\right]$$

$$= \frac{1}{2^{2n-1}} \cdot \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma\left(\dfrac{2n+1}{2}\right)}{\Gamma\left(\dfrac{2n+2}{2}\right)} = \frac{\sqrt{\pi}\,\Gamma\left(n+\dfrac{1}{2}\right)}{2^{2n}\,\Gamma(n+1)}$$

$\therefore$ From (1), we have $\dfrac{\sqrt{\pi}\,\Gamma\left(n+\dfrac{1}{2}\right)}{2^{2n}\,\Gamma(n+1)} = \dfrac{\left[\Gamma\left(n+\dfrac{1}{2}\right)\right]^2}{\Gamma(2n+1)}$

$$\Rightarrow \qquad \Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}\,\Gamma(2n+1)}{2^{2n}\,\Gamma(n+1)}.$$

Cor. Putting $n = \dfrac{1}{4}$, we have

$$\Gamma\left(\frac{3}{4}\right) = \frac{\sqrt{\pi}\,\Gamma(3/2)}{\sqrt{2}\,\Gamma(5/4)} = \frac{\sqrt{\pi}\cdot\dfrac{1}{2}\,\Gamma\left(\dfrac{1}{2}\right)}{\sqrt{2}\cdot\dfrac{1}{4}\,\Gamma\left(\dfrac{1}{4}\right)}$$

$$\Rightarrow \qquad \Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \sqrt{2}\,\pi.$$

Example 15. *Prove that* $\displaystyle\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta = \int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta = \frac{1}{2}\,\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{\pi}{\sqrt{2}}.$

Sol. $\displaystyle\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta = \int_0^{\pi/2} \sqrt{\frac{\sin\theta}{\cos\theta}}\, d\theta = \int_0^{\pi/2} \sin^{1/2}\theta \cos^{-1/2}\theta \, d\theta$

$$= \frac{\Gamma\left(\dfrac{\frac{1}{2}+1}{2}\right)\Gamma\left(\dfrac{-\frac{1}{2}+1}{2}\right)}{2\,\Gamma\left(\dfrac{\frac{1}{2}-\frac{1}{2}+2}{2}\right)} = \frac{\Gamma\left(\dfrac{3}{4}\right)\Gamma\left(\dfrac{1}{4}\right)}{2\,\Gamma(1)} = \frac{\Gamma\left(\dfrac{1}{4}\right)\Gamma\left(\dfrac{3}{4}\right)}{2} \qquad \ldots(1)$$

$$\int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta = \int_0^{\pi/2} (\sin\theta)^{-1/2} (\cos\theta)^{1/2}\, d\theta = \frac{\Gamma\left(\dfrac{-\frac{1}{2}+1}{2}\right)\Gamma\left(\dfrac{\frac{1}{2}+1}{2}\right)}{2\,\Gamma\left(\dfrac{-\frac{1}{2}+\frac{1}{2}+2}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{2\Gamma(1)} = \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{2} \qquad \ldots(2)$$

$\therefore$ Combining (1) and (2),

$$\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta = \int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta = \frac{1}{2}\,\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{1}{2}\sqrt{2}\,\pi = \frac{\pi}{\sqrt{2}}.$$

Example 16. *Prove that:*

(*a*) $\iint_D x^{l-1} y^{m-1}\, dxdy = \dfrac{\Gamma(l)\,\Gamma(m)}{\Gamma(l+m+1)}\, a^{l+m}$, *where D is the domain* $x \geq 0$, $y \geq 0$ *and* $x + y \leq a$.

(*b*) *Establish Dirichlet's integral:*

$$\iiint_V x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz = \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n+1)}$$

where V is the region $x \geq 0$, $y \geq 0$, $z \geq 0$ *and* $x + y + z \leq 1$.

Sol. (*a*) Putting $x = a\text{X}$ and $y = a\text{Y}$, the given integral reduces to

$$\text{I} = \iint_{\text{D}'} (a\text{X})^{l-1} (a\text{Y})^{m-1}\, a^2\, d\text{X}d\text{Y}$$

where D′ is the domain $\text{X} \geq 0$, $\text{Y} \geq 0$ and $\text{X} + \text{Y} \leq 1$

$$= a^{l+m} \int_0^1 \int_0^{1-\text{X}} \text{X}^{l-1}\, \text{Y}^{m-1}\, d\text{Y}d\text{X}$$

$$= a^{l+m} \int_0^1 \text{X}^{l-1} \left| \frac{\text{Y}^m}{m} \right|_0^{1-\text{X}} d\text{X} = \frac{a^{l+m}}{m} \int_0^1 \text{X}^{l-1} (1-\text{X})^m\, d\text{X}$$

$$= \frac{a^{l+m}}{m}\, \beta(l, m+1) = \frac{a^{l+m}}{m} \cdot \frac{\Gamma(l)\,\Gamma(m+1)}{\Gamma(l+m+1)}$$

$$= a^{l+m} \frac{\Gamma(l)\,\Gamma(m)}{\Gamma(l+m+1)}, \quad \text{since } \frac{\Gamma(m+1)}{m} = \Gamma(m).$$

(*b*) Taking $y + z \leq 1 - x = a$ (say), the given integral

$$\text{I} = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x^{l-1}\, y^{m-1}\, z^{n-1}\, dzdydx$$

$$= \int_0^1 x^{l-1}\left[\int_0^a\int_0^{a-y} y^{m-1} z^{n-1}\, dzdy\right] dx$$

$$= \int_0^1 x^{l-1} \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n+1)} a^{m+n} dx \qquad \text{[by part } (a)]$$

$$= \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n+1)} \int_0^1 x^{l-1}(1-x)^{m+n}\, dx, \quad \text{since } a = 1-x$$

$$= \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n+1)} \mathrm{B}(l, m+n+1) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n+1)} \cdot \frac{\Gamma(l)\,\Gamma(m+n+1)}{\Gamma(l+m+n+1)}$$

$$= \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n+1)}.$$

Example 17. *Prove that:*

$$\int_0^\infty \int_0^\infty e^{-(ax^2+by^2)}\, x^{2m-1}\, y^{2n-1}\, dxdy = \frac{\Gamma(m)\,\Gamma(n)}{4a^m b^m}\;; \quad a, b, m, n \text{ are +ve.}$$

Sol. $\displaystyle\int_0^\infty \int_0^\infty e^{-ax^2} \cdot e^{-by^2}\, x^{2m-1} y^{2n-1}\, dxdy = \int_0^\infty e^{-ax^2}\, x^{2m-1}\, dx \int_0^\infty e^{-by^2}\, y^{2n-1}\, dy$

Put $ax^2 = t$ Put $by^2 = t$

$$= \int_0^\infty e^{-t}\left(\frac{t}{a}\right)^{\frac{2m-1}{2}} \frac{dt}{2ax} \times \int_0^\infty e^{-t}\left(\frac{t}{b}\right)^{\frac{2n-1}{2}} \frac{dt}{2by}$$

$$= \int_0^\infty e^{-t} \frac{t^{\frac{2m-1}{2}}\, dt}{2a^{\frac{2m+1}{2}} \dfrac{t^{1/2}}{a^{1/2}}} \times \int_0^\infty e^{-t} \frac{t^{\frac{2n-1}{2}}\, dt}{2b^{\frac{2n+1}{2}} \dfrac{t^{1/2}}{b^{1/2}}}$$

$$= \frac{1}{2a^m} \int_0^\infty e^{-t} t^{\frac{2m-2}{2}}\, dt \times \frac{1}{2b^n} \int_0^\infty e^{-t}, t^{\frac{2n-2}{2}}\, dt$$

$$= \frac{1}{4a^n b^n} \int_0^\infty e^{-t}\, t^{m-1}\, dt \times \int_0^\infty e^{-t} \cdot t^{n-1}\, dt$$

$$= \frac{1}{4a^m b^n}\, \Gamma(m)\, \Gamma(n).$$

Example 18. *The plane* $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$ *meets the axes in A, B and C. Apply Dirichlet's integral to find the volume of the tetrahedron OABC. Also find its mass if the density at any point is kxyz.*

Sol. Put $\dfrac{x}{a} = u$, $\dfrac{y}{b} = v$, $\dfrac{z}{c} = w$, then $u \ge 0$, $v \ge 0$, $w \ge 0$ and $u + v + w \le 1$.

Also, $dx = a\, du$, $dy = b\, dv$, $dz = c\, dw$.

Volume OABC $= \displaystyle\iiint_D dx\, dy\, dz = \iiint_{D'} abc\, du\, dv\, dw$, where $u + v + w \le 1$

$$= abc \iiint_{D'} u^{1-1} v^{1-1} w^{1-1} \, du \, dv \, dw$$

$$= abc \frac{\Gamma(1)\Gamma(1)\Gamma(1)}{\Gamma(1+1+1+1)} = \frac{abc}{3!} = \frac{abc}{6}$$

$$\text{Mass} = \iiint_D kxyz \, dx \, dy \, dz = \iiint_D k(au)(bv)(cw) \, abc \, du \, dv \, dw$$

$$= ka^2b^2c^2 \iiint_{D'} u^{2-1} v^{2-1} w^{2-1} \, du \, dv \, dw$$

$$= ka^2b^2c^2 \frac{\Gamma(2)\Gamma(2)\Gamma(2)}{\Gamma(2+2+2+1)} = ka^2b^2c^2 \frac{1!\,1!\,1!}{6!} = \frac{ka^2b^2c^2}{720}.$$

TEST YOUR KNOWLEDGE

Prove that:

1. $\int_{-\infty}^{\infty} e^{-k^2x^2} dx = \frac{\sqrt{\pi}}{k}$.

2. $\int_0^{\infty} x^3 e^{-x^3} dx = \frac{1}{9}\Gamma\left(\frac{1}{3}\right)$.

3. $\int_0^{\infty} \sqrt{x}\, e^{-x^3} dx = \frac{\sqrt{\pi}}{3}$.

4. $\int_0^{\infty} e^{-x^2} x^{2n-1} dx = \frac{1}{2}\Gamma(n)$.

5. $\int_0^{\infty} x^{2n-1} e^{-ax^2} dx = \frac{\Gamma(n)}{2a^n}$.

6. $\int_0^1 \left(\log \frac{1}{x}\right)^{n-1} dx = \Gamma(n),\ n > 0.$ $\left[\textbf{Hint.} \text{ Put } \log \frac{1}{x} = t\right]$

7. $\int_0^{\infty} \sqrt{x} e^{-x^2} dx \times \int_0^{\infty} \frac{e^{-x^2}}{\sqrt{x}} dx = \frac{\pi}{2\sqrt{2}}$.

8. $\int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta \times \int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} = \pi$.

9. $\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}$, given $\int_0^{\infty} \frac{x^{n-1}}{1+x} dx = \frac{\pi}{\sin n\pi}$.

10. $y\mathrm{B}(x+1, y) = x\mathrm{B}(x, y+1)$.

11. $\int_0^2 (8-x^3)^{-1/3} dx = \frac{1}{3}\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)$. [**Hint.** Put $x^3 = at$]

12. $\int_0^a x^{n-1}(a-x)^{m-1} dx = a^{m+n-1}\mathrm{B}(m, n)$. [**Hint.** Put $x = at$]

13. $\int_a^b (x-a)^m (b-x)^n dx = (b-a)^{m+n+1}\mathrm{B}(m+1, n+1)$.

    [**Hint.** Put $x = a + (b-a)z$]

14. $\int_0^1 x^5(1-x^3)^3 dx = \frac{1}{60}$.

    [**Hint.** Put $x^3 = t$]

15. $\frac{\mathrm{B}(p, q+1)}{q} = \frac{\mathrm{B}(p+1, q)}{p} = \frac{\mathrm{B}(p, q)}{p+q}$.

16. $\frac{\mathrm{B}(m+1, n)}{\mathrm{B}(m, n)} = \frac{m}{m+n}$

17. $\frac{\mathrm{B}(m+2, n-2)}{\mathrm{B}(m, n)} = \frac{m(m+1)}{(n-1)(n-2)}$.

18. $\int_0^{\frac{\pi}{2}} \sin^3 x \cos^{5/2} x \, dx = \frac{8}{77}$.

19. $B(n, n) = \dfrac{\sqrt{\pi}\,\Gamma(n)}{2^{2n-1}\,\Gamma\left(n + \frac{1}{2}\right)}$.

20. $2^n \Gamma\left(n + \frac{1}{2}\right) = 1 \cdot 3 \cdot 5 \,......\, (2n - 1)\sqrt{\pi}$.

21. $\displaystyle\int_0^1 \frac{dx}{\sqrt{1 - x^n}} = \frac{\sqrt{\pi}}{n} \frac{\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{1}{n} + \frac{1}{2}\right)}$.

[Hint. Put $x^2 = \tan\theta$]

22. (i) $\displaystyle\int_0^1 \frac{x^2\,dx}{\sqrt{1 - x^4}} = \sqrt{\pi}\, \frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}$.

[Hint. Put $x^4 = t$ or $x^2 = \sin\theta$]

(ii) $\displaystyle\int_0^1 \frac{dx}{\sqrt{1 + x^4}} = \frac{1}{4\sqrt{2}}\, B\left(\frac{1}{4}, \frac{1}{2}\right)$

[Hint. Put $x^2 = \tan\theta$]

23. Prove that:

(i) $\displaystyle\int_0^\infty \frac{x^4 (1 + x^5)}{(1 + x)^{15}}\, dx = \frac{1}{5005}$

(ii) $\displaystyle\int_0^{\pi/2} \sin^n \theta \, d\theta = \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}, \; n > -1$

24. Show that $\iint x^{m-1} y^{n-1}\, dx\, dy$ over the positive octant of the ellipse

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \text{ is } \frac{a^m b^n}{2n}\, B\left(\frac{m}{2}, \frac{n}{2} + 1\right).$$

25. Show that the volume of the solid bounded by the coordinate planes and the surface

$$\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} + \sqrt{\frac{z}{c}} = 1 \text{ is } \frac{abc}{90}.$$

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