2.8 TAYLOR'S SERIES

Statement: If $f(x+h)$ be a given function of h which can be expanded into a convergent series of positive ascending integral powers of h, then

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots\ldots + \frac{h^n}{n!} f^n(x) + \ldots\ldots$$

Proof: Let $f(x+h)$ be a function of h which can be expanded into positive ascending integral powers of h, then

$$f(x+h) = a_0 + a_1 h + a_2 h^2 + a_3 h^3 + a_4 h^4 + \ldots\ldots \quad \ldots (1)$$

Differentiating w.r.t. h successively,

$$f'(x+h) = a_1 + a_2 \cdot 2h + a_3 \cdot 3h^2 + a_4 \cdot 4h^3 + \ldots\ldots \quad \ldots (2)$$

$$f''(x+h) = a_2 \cdot 2 + a_3 \cdot 6h + a_4 \cdot 12h^2 + \ldots\ldots \quad \ldots (3)$$

$$f'''(x+h) = a_3 \cdot 6 + a_4 \cdot 24h + \ldots\ldots \quad \ldots (4)$$

and so on

Putting $h = 0$ in Eq. (1), (2), (3) and (4),

$$a_0 = f(x)$$
$$a_1 = f'(x)$$
$$a_2 = \frac{1}{2!} f''(x)$$
$$a_3 = \frac{1}{3!} f'''(x) \text{ and so on}$$

Substituting a_0, a_1, a_2 and a_3 in Eq. (1),

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots\ldots + \frac{h^n}{n!} f^n x + \ldots\ldots$$

This is known as **Taylor's Series.**

Putting $x = a$ and $h = x - a$ in above series, we get Taylor's Series in powers of $(x-a)$ as

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \ldots\ldots\ldots$$
$$+ \frac{(x-a)^n}{n!} f^n(a) + \ldots\ldots$$

Example 1: Prove that $f(mx) = f(x) + (m-1)x f'(x) + \frac{(m-1)^2}{2!} x^2 f''(x) + \ldots.$

Solution: $f(mx) = f(mx - x + x) = f[x + (m-1)x]$

By Taylor's series,

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots\ldots\ldots\ldots\ldots$$

Putting $h = (m-1)x$,

$$f[x+(m-1)x] = f(mx) = f(x) + (m-1)x f'(x) + \frac{(m-1)^2}{2!} x^2 f''(x) + \ldots\ldots\ldots$$

Example 2: Prove that

$$f\left(\frac{x^2}{1+x}\right) = f(x) - \frac{x}{1+x} f'(x) + \frac{x^2}{2!(1+x)^2} f''(x) - \frac{x^3}{3!(1+x)^3} f'''(x) + \ldots.$$

Solution:

$$\frac{x^2}{1+x} = x - \frac{x}{1+x},$$

By Taylor's series,

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots$$

Putting $h = -\dfrac{x}{1+x}$,

$$f\left(x - \frac{x}{1+x}\right) = f\left(\frac{x^2}{1+x}\right)$$

$$= f(x) - \frac{x}{1+x} f'(x) + \frac{x^2}{2!\,(1+x)^2} f''(x) - \frac{x^3}{3!\,(1+x)^3} f'''(x) + \dots$$

Example 3: Expand $f(x) = x^5 - x^4 + x^3 - x^2 + x - 1$ in powers of $(x-1)$ and find $f(0.99)$.

Solution: $f(x) = x^5 - x^4 + x^3 - x^2 + x - 1$

By Taylor's series,

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \dots\dots$$

Putting $a = 1$,

$$f(x) = x^5 - x^4 + x^3 - x^2 + x - 1$$

$$= f(1) + (x-1) f'(1) + \frac{(x-1)^2}{2!} f''(1) + \frac{(x-1)^3}{3!} f'''(1)$$

$$+ \frac{(x-1)^4}{4!} f^{iv}(1) + \frac{(x-1)^5}{5!} f^{v}(1) + \dots\dots \quad \dots(1)$$

$$f(1) = 1 - 1 + 1 - 1 + 1 - 1 = 0$$

Differentiating $f(x)$ w.r.t. x successively,

$f'(x) = 5x^4 - 4x^3 + 3x^2 - 2x + 1$, $\quad f'(1) = 5 - 4 + 3 - 2 + 1 = 3$

$f''(x) = 20x^3 - 12x^2 + 6x - 2$, $\quad f''(1) = 20 - 12 + 6 - 2 = 12$

$f'''(x) = 60x^2 - 24x + 6$, $\quad f'''(1) = 60 - 24 + 6 = 42$

$f^{iv}(x) = 120x - 24$, $\quad f^{iv}(1) = 120 - 24 = 96$

$f^{v}(x) = 120$, $\quad f^{v}(1) = 120$

Substituting in Eq. (1),

$$f(x) = 0 + (x-1)3 + \frac{(x-1)^2}{2!}(12) + \frac{(x-1)^3}{3!}(42) + \frac{(x-1)^4}{4!}(96) + \frac{(x-1)^5}{5!}(120)$$

$$= 3(x-1) + 6(x-1)^2 + 7(x-1)^3 + 4(x-1)^4 + (x-1)^5$$

Putting $x = 0.99$,

$$f(0.99) = 3(0.99-1) + 6(0.99-1)^2 + 7(0.99-1)^3 + 4(0.99-1)^4 + (0.99-1)^5$$

$$= 3(-0.01) + 6(-0.01)^2 + 7(-0.01)^3 + 4(-0.01)^4 + (-0.01)^5$$

$$= -0.02939$$

Example 4: Prove that $\dfrac{1}{1-x}=\dfrac{1}{3}+\dfrac{(x+2)}{3^2}+\dfrac{(x+2)^2}{3^3}+\dfrac{(x+2)^3}{3^4}+\ldots$.

Solution: Let $f(x)=\dfrac{1}{1-x}$

By Taylor's series,

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\frac{(x-a)^3}{3!}f'''(a)+\ldots\ldots\ldots$$

Putting $a=-2$,

$$f(x)=\frac{1}{1-x}=f(-2)+(x+2)f'(-2)+\frac{(x+2)^2}{2!}f''(-2)+\frac{(x+2)^3}{3!}f'''(-2)+\ldots\ldots \quad \ldots(1)$$

$$f(-2)=\frac{1}{3}$$

Differentiating $f(x)$ w.r.t. x successively,

$$f'(x)=\frac{1}{(1-x)^2},\quad f'(-2)=\frac{1}{3^2}$$

$$f''(x)=\frac{2}{(1-x)^3},\quad f''(-2)=\frac{2!}{3^3}$$

$$f'''(x)=\frac{2.3}{(1-x)^4},\quad f'''(-2)=\frac{3!}{3^4}\ \text{and so on}$$

Substituting in Eq. (1),

$$f(x)=\frac{1}{1-x}=\frac{1}{3}+\frac{(x+2)}{3^2}+\frac{(x+2)^2}{3^3}+\frac{(x+2)^3}{3^4}+\ldots\ldots\ldots$$

Example 5: Expand log (cos x) about $\dfrac{\pi}{3}$.

Solution: Let $f(x)=\log(\cos x)$

By Taylor's series,

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\frac{(x-a)^3}{3!}f'''(a)+\ldots\ldots$$

Putting $a=\dfrac{\pi}{3}$,

$$f(x)=\log(\cos x)$$

$$=f\left(\frac{\pi}{3}\right)+\left(x-\frac{\pi}{3}\right)f'\left(\frac{\pi}{3}\right)+\frac{1}{2!}\left(x-\frac{\pi}{3}\right)^2 f''\left(\frac{\pi}{3}\right)+\frac{1}{3!}\left(x-\frac{\pi}{3}\right)^3 f'''\left(\frac{\pi}{3}\right)+\ldots \quad \ldots(1)$$

$$f\left(\frac{\pi}{3}\right)=\log\left(\cos\frac{\pi}{3}\right)=\log\left(\frac{1}{2}\right)=-\log 2$$

Differentiating $f(x)$ w.r.t. x successively,

$$f'(x)=\frac{1}{\cos x}(-\sin x)=-\tan x, \qquad f'\left(\frac{\pi}{3}\right)=-\tan\frac{\pi}{3}=-\sqrt{3}$$

$$f''(x)=-\sec^2 x, \qquad f''\left(\frac{\pi}{3}\right)=-\sec^2\frac{\pi}{3}=-4$$

$$f'''(x)=-2\sec^2 x\tan x, \qquad f'''\left(\frac{\pi}{3}\right)=-2\sec^2\frac{\pi}{3}\tan\frac{\pi}{3}=-2(4)\sqrt{3}=-8\sqrt{3} \text{ and so on}$$

Substituting in Eq. (1),

$$f(x)=\log(\cos x)=-\log 2+\left(x-\frac{\pi}{3}\right)\left(-\sqrt{3}\right)+\frac{1}{2!}\left(x-\frac{\pi}{3}\right)^2(-4)$$
$$+\frac{1}{3!}\left(x-\frac{\pi}{3}\right)^3\left(-8\sqrt{3}\right)+\ldots$$
$$=-\log 2-\sqrt{3}\left(x-\frac{\pi}{3}\right)-2\left(x-\frac{\pi}{3}\right)^2-\frac{4\sqrt{3}}{3}\left(x-\frac{\pi}{3}\right)^3-\ldots$$

Example 6: Obtain $\tan^{-1}x$ in powers of $(x-1)$.

Solution: Let $f(x)=\tan^{-1}x$

By Taylor's series,

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\frac{(x-a)^3}{3!}f'''(a)+\ldots\ldots$$

Putting $a=1$,

$$f(x)=\tan^{-1}x=f(1)+(x-1)f'(1)+\frac{(x-1)^2}{2!}f''(1)+\frac{(x-1)^3}{3!}f'''(1)+\ldots$$

$$f(1)=\tan^{-1}1=\frac{\pi}{4}$$

Differentiating $f(x)$ w.r.t. x successively,

$$f'(x)=\frac{1}{1+x^2}, \qquad f'(1)=\frac{1}{2}$$

$$f''(x)=-\frac{2x}{(1+x^2)^2}, \qquad f''(1)=-\frac{2}{4}=-\frac{1}{2} \text{ and so on}$$

$$f'''(x)=-\frac{2}{(1+x^2)^2}+\frac{8x^2}{(1+x^2)^3}, \qquad f'''(1)=\frac{1}{2}$$

Substituting in Eq. (1),

$$f(x) = \tan^{-1} x = \frac{\pi}{4} + (x-1)\left(\frac{1}{2}\right) + \frac{(x-1)^2}{2!}\left(-\frac{1}{2}\right) + \frac{(x-1)^3}{3!}\left(\frac{1}{2}\right) + \ldots\ldots$$

$$= \frac{\pi}{4} + \frac{1}{2}(x-1) - \frac{1}{4}(x-1)^2 + \frac{1}{12}(x-1)^3 \ldots$$

Example 7: Prove that

$$\log[\sin(x+h)] = \log\sin x + h\cot x - \frac{h^2}{2}\operatorname{cosec}^2 x + \frac{h^3}{3}\frac{\cos x}{\sin^3 x} + \ldots\ldots\ldots$$

Solution: Let $f(x) = \log(\sin x)$, $f(x+h) = \log[\sin(x+h)]$

By Taylor's series,

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots\ldots\ldots \quad \ldots (1)$$

Differentiating $f(x)$ w.r.t. x successively,

$$f'(x) = \frac{1}{\sin x}\cos x = \cot x$$

$$f''(x) = -\operatorname{cosec}^2 x$$

$$f'''(x) = 2\operatorname{cosec}^2 x \cot x = \frac{2\cos x}{\sin^3 x} \text{ and so on}$$

Substituting in Eq. (1),

$$f(x+h) = \log[\sin(x+h)]$$

$$= \log\sin x + h\cot x - \frac{h^2}{2!}\operatorname{cosec}^2 x + \frac{h^3}{3!}\frac{2\cos x}{\sin^3 x} + \ldots\ldots\ldots$$

$$= \log\sin x + h\cot x - \frac{h^2}{2}\operatorname{cosec}^2 x + \frac{h^3}{3}\frac{\cos x}{\sin^3 x} + \ldots\ldots\ldots$$

Example 8: Expand $\tan^{-1}(x+h)$ in powers of h and hence, find the value of $\tan^{-1}(1.003)$ up to 5 places of decimal.

Solution: Let $f(x) = \tan^{-1} x$, $f(x+h) = \tan^{-1}(x+h)$

By Taylor's series,

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots\ldots\ldots \quad \ldots (1)$$

Differentiating $f(x)$ w.r.t. x successively,

$$f'(x) = \frac{1}{1+x^2}, \quad f''(x) = -\frac{2x}{(1+x^2)^2}$$

Substituting in Eq. (1),

$$\sqrt{1+x+2x^2} = 1+\frac{1}{2}(x+2x^2)-\frac{1}{4}\frac{(x^2+4x^3+4x^4)}{2}+\frac{3}{8}\frac{(x^3+\ldots)}{6}+\ldots\ldots\ldots$$

$$= 1+\frac{x}{2}+\frac{7x^2}{8}-\frac{7x^3}{16}+\ldots$$

Example 10: Expand $\sqrt{1+x+2x^2}$ in powers of $(x-1)$.

Solution: $\sqrt{1+x+2x^2} = \sqrt{4+2(x-1)^2+5(x-1)}$ [Expressing in terms of $(x-1)$]

Let $f(x) = \sqrt{x}$, $f(x+h) = \sqrt{x+h}$

By Taylor's series,

$$f(x+h) = f(x)+h\,f'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+\ldots\ldots\ldots$$

Putting $x = 4$, $h = 2(x-1)^2+5(x-1)$,

$$f(x+h) = \sqrt{x+h} = \sqrt{4+2(x-1)^2+5(x-1)}$$

$$= f(4)+[2(x-1)^2+5(x-1)]\,f'(4)+\frac{[2(x-1)^2+5(x-1)]^2}{2!}f''(4)+\ldots \quad \ldots (1)$$

$f(x) = \sqrt{x}$, $\quad f(4) = 2$

Differentiating $f(x)$ w.r.t. x successively,

$$f'(x) = \frac{1}{2\sqrt{x}}, \qquad f'(4) = \frac{1}{4}$$

$$f''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\frac{1}{x^{\frac{3}{2}}}, \qquad f''(4) = -\frac{1}{32} \text{ and so on}$$

Substituting in Eq. (1),

$$\sqrt{4+2(x-1)^2+5(x-1)} = 2+[2(x-1)^2+5(x-1)]\frac{1}{4}$$

$$+\frac{[2(x-1)^2+5(x-1)]^2}{2!}\left(-\frac{1}{32}\right)+\ldots\ldots\ldots$$

$$\sqrt{1+x+2x^2} = 2+\frac{5}{4}(x-1)+\frac{7}{64}(x-1)^2+\ldots\ldots\ldots\ldots$$

Example 11: Using Taylor's theorem, evaluate up to 4 places of decimals:

(i) $\sqrt{1.02}$

(ii) $\sqrt{25.15}$

(iii) $\sqrt{9.12}$

(iv) $\sqrt{10}$

Solution: Let $f(x)=\sqrt{x}$, $f(x+h)=\sqrt{x+h}$

By Taylor's series,

$$f(x+h)=f(x)+h\,f'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+\ldots\ldots\ldots\ldots$$

(i) Putting $x=1,\ h=0.02$,

$$f(x+h)=\sqrt{x+h}=\sqrt{1+0.02}$$

$$=f(1)+(0.02)\,f'(1)+\frac{(0.02)^2}{2!}f''(1)+\ldots\ldots\ldots$$

$$f(x)=\sqrt{x},\qquad f(1)=1$$

Differentiating $f(x)$ w.r.t. x successively,

$$f'(x)=\frac{1}{2\sqrt{x}},\qquad f'(1)=\frac{1}{2}$$

$$f''(x)=-\frac{1}{4x^{\frac{3}{2}}},\qquad f''(1)=-\frac{1}{4}\ \text{and so on}$$

Substituting in Eq. (2) and considering only first 3 terms,

$$\sqrt{1.02}=1+(0.02)\frac{1}{2}+\frac{(0.02)^2}{2!}\left(-\frac{1}{4}\right)$$

$$=1.0099\ \text{approx.}$$

(ii) Putting $x=25,\ h=0.15$ in Eq. (1),

$$f(x+h)=\sqrt{x+h}=\sqrt{25+0.15}$$

$$=f(25)+(0.15)\,f'(25)+\frac{(0.15)^2}{2!}f''(25)+\ldots$$

$$f(x)=\sqrt{x},\qquad f(25)=5$$

Differentiating $f(x)$ w.r.t. x successively,

$$f'(x)=\frac{1}{2\sqrt{x}},\qquad f'(25)=\frac{1}{10}=0.1$$

$$f''(x)=-\frac{1}{4x^{\frac{3}{2}}},\qquad f''(25)=-\frac{1}{500}=-0.002\ \text{and so on}$$

Substituting in Eq. (2) and considering only first 3 terms,

$$\sqrt{25.15}=5+(0.15)(0.1)+\frac{(0.15)^2}{2}(-0.002)$$

$$=5.0150\ \text{approx.}$$

(iii) Putting $x = 9, h = 0.12$ in Eq. (1),

$$f(x+h) = \sqrt{x+h} = \sqrt{9+0.12}$$

$$= f(9) + (0.12)\, f'(9) + \frac{(0.12)^2}{2!} f''(9) + \ldots \quad \ldots (3)$$

$$f(x) = \sqrt{x}, \qquad f(9) = 3$$

Differentiating $f(x)$ w.r.t. x successively,

$$f'(x) = \frac{1}{2\sqrt{x}}, \qquad f'(9) = \frac{1}{6}$$

$$f''(x) = -\frac{1}{4x^{\frac{3}{2}}}, \qquad f''(9) = -\frac{1}{108} \text{ and so on}$$

Substituting in Eq. (2) and considering only first 3 terms,

$$\sqrt{9.12} = 3 + (0.12)\left(\frac{1}{6}\right) + \frac{(0.12)^2}{2}\left(-\frac{1}{108}\right)$$

$$= 3 + 0.02 - (0.12)(0.06)(0.0093)$$

$$= 3.0199 \text{ approx.}$$

(iv) Putting $x = 9, h = 1$ in Eq. (1),

$$f(x+h) = \sqrt{x+h} = \sqrt{9+1} = f(9) + f'(9) + \frac{1}{2!} f''(9) + \ldots \quad \ldots (4)$$

$$\sqrt{10} = 3 + \frac{1}{6} - \frac{1}{216} \qquad \text{[refer (iii)]}$$

$$= 3.1620 \text{ approx.}$$

Example 12: Find the value of tan (43°).

Solution: Let $f(x) = \tan x, \quad f(x+h) = \tan(x+h)$

By Taylor's series,

$$f(x+h) = f(x) + h\, f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots\ldots\ldots\ldots$$

Putting $x = 45^\circ, h = -2^\circ = -\frac{2\pi}{180} = -\frac{\pi}{90} = -0.0349$,

$$\tan(x+h) = \tan(45^\circ - 2^\circ) = \tan 43^\circ$$

$$= f(45^\circ) + (-0.0349) f'(45^\circ) + \frac{(-0.0349)^2}{2!} f''(45^\circ) + \ldots\ldots\ldots \quad \ldots (1)$$

$$f(x) = \tan x, \qquad f(45^\circ) = \tan(45^\circ) = 1$$

Differentiating $f(x)$ w.r.t. x successively,

$$f'(x) = \sec^2 x, \quad f'(45^\circ) = \sec^2 45^\circ = 2$$

$$f''(x) = 2\sec^2 x \tan x, \quad f''(45^\circ) = 2\sec^2 45^\circ \tan 45^\circ = 4$$

and so on

Substituting in Eq. (1) and considering only first 3 terms,

$$\tan 43^\circ = 1 + (-0.0349)(2) + \frac{(-0.0349)^2}{2!}(4)$$

$$= 0.9326 \text{ approx.}$$

Example 13: Find cosh (1.505) given sinh (1.5) = 2.1293 and cosh (1.5) = 2.3524.

Solution: Let $f(x) = \cosh x$

By Taylor's series,

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots\ldots\ldots$$

Putting $x = 1.5,\ h = 0.005$,

$$f(x+h) = \cosh(x+h) = \cosh(1.5 + 0.005)$$

$$= f(1.5) + (0.005) f'(1.5) + \frac{(0.005)^2}{2!} f''(1.5) + \frac{(0.005)^3}{3!} f'''(1.5) + \cdots \quad \ldots (1)$$

$$f(x) = \cosh x,\ f(1.5) = \cosh(1.5) = 2.3524$$

Differentiating $f(x)$ w.r.t. x successively,

$$f'(x) = \sinh x,\ f'(1.5) = \sinh(1.5) = 2.1293$$

$$f''(x) = \cosh x,\ f''(1.5) = \cosh(1.5) = 2.3524$$

and so on

Substituting in Eq. (1) and considering only first 3 terms,

$$\cosh(1.505) = \cosh(1.5) + (0.005)\sinh(1.5) + \frac{(0.005)^2}{2!}\cosh(1.5) + \ldots$$

$$= 2.3524 + (0.005)(2.1293) + (12.5)(10^{-6})(2.3524)$$

$$= 2.3631 \text{ approx.}$$

Exercise 2.6

1. Expand e^x in powers of $(x-1)$.

$$\left[\textbf{Ans.: } e\left(1 + (x-1) + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \ldots\right)\right]$$

2. Expand $2x^3 + 7x^2 + x - 1$ in powers of $x - 2$.

[**Ans.:** $45 + 53(x-2) + 19(x-2)^2 + 2(x-2)^3$]

3. Expand $x^5 - 5x^4 + 6x$ ... $+ 8x - 9$ in powers of $(x - 1)$

[**Ans.:** $-6 - 3($ ... $x-1)^2$ ... $-4(x$ ... $1)^5$]

4. Expand $x^4 - 3x^3 +$ ... powers of $(x - 3)$.

[**Ans.:** $16 + 38(x$ ... $-3)^3$ ... $+9(x -$ ...]

20. Arrange in powers of x, by Taylor's theorem, $17 + 6(x+2) + 3(x+2)^3 + (x+2)^4 - (x+2)^5$.

[**Ans.:** $37 - 6x - 38x^2 - 29x^3 - 9x^4 - x^5$]

21. Arrange in powers of $(x+1)$, by Taylor's theorem, $(x+2)^4 + 5(x+2)^3 + 6(x+2)^2 + 7(x+2) + 8$.

Hint: $f(x) = x^4 + 5x^3 + 6x^2 + 7x + 8$,

$$f[(x+1)+1] = f(1) + (x+1)f'(1) + \frac{(x+1)^2}{2!}f''(1) + \ldots$$

[**Ans.:** $27 + 38(x+1) + 27(x+1)^2 + 9(x+1)^3 + (x+1)^4$]

22. Prove that $\sinh(x+a) = \sinh a + x\cosh a + \dfrac{x^2}{2!}\sinh a + \ldots$

Given $\sinh(1.5) = 2.1293$, $\cosh(1.5) = 2.3524$, find the value of $\sinh(1.505)$.

[**Ans.:** 2.141...]

2.9 MACLAURIN'S SERIES

Statement: If $f(x)$ be a given function of x which can be expanded in positive ascending integral powers of x, then

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots\ldots + \frac{x^n}{n!}f^n(0) + \ldots\ldots$$

Proof: Let $f(x)$ be a function of x which can be expanded into positive ascending integral powers of x, then

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots\ldots \quad \ldots (1)$$

Differentiating w.r.t. x successively,

$$f'(x) = a_1 + a_2 \cdot 2x + a_3 \cdot 3x^2 + a_4 \cdot 4x^3 + \ldots\ldots \quad \ldots (2)$$

$$f''(x) = a_2 \cdot 2 + a_3 \cdot 6x + a_4 \cdot 12x^2 + \ldots\ldots \quad \ldots (3)$$

$$f'''(x) = a_3 \cdot 6 + a_4 \cdot 24x + \ldots\ldots \quad \ldots (4)$$

and so on

Putting $x = 0$ in Eq. (1), (2), (3) and (4),

$$a_0 = f(0)$$

$$a_1 = f'(0)$$

$$a_2 = \frac{1}{2!}f''(0)$$

$$a_3 = \frac{1}{3!}f'''(0) \quad \text{and so on.}$$

Substituting a_0, a_1, a_2 and a_3 in Eq. (1),

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots\ldots + \frac{x^n}{n!}f^n(0) + \ldots\ldots$$

This is known as **Maclaurin's Series.**

This series can also be written as,

$$y = y(0) + xy_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \ldots\ldots + \frac{x^n}{n!} y_n(0) + \ldots\ldots\ldots$$

2.9.1 Standard Expansions

Using Maclaurin's series, expansion of some standard functions can be obtained. These expansions can be directly used while solving the examples.

(1) Expansion of e^x (Exponential series)

Proof: Let $y = e^x$, $y(0) = e^0 = 1$

Now $y_n = \frac{d^n}{dx^n}(e^x) = e^x$, $y_n(0) = e^0 = 1$ for all values of n.

Substituting in Maclaurin's series,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\ldots$$

This series is known as the exponential series.

Note: In the above series

(i) Replacing x by $-x$,

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots\ldots\ldots$$

(ii) Replacing x by ax,

$$e^{ax} = 1 + ax + \frac{a^2x^2}{2!} + \frac{a^3x^3}{3!} + \ldots\ldots\ldots$$

(2) Expansion of sin x (Sine series)

Proof: Let $y = \sin x$, $y(0) = \sin 0 = 0$

Now

$$y_n = \frac{d^n}{dx^n}(\sin x) = \sin\left(x + \frac{n\pi}{2}\right)$$

$$y_n(0) = \sin\left(\frac{n\pi}{2}\right)$$

Putting $n = 1, 2, 3, 4, 5, \ldots\ldots$

$y_1(0) = 1$, $y_2(0) = 0$, $y_3(0) = -1$, $y_4(0) = 0$, $y_5(0) = 1$, and so on.

Substituting in Maclaurin's series,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\ldots\ldots$$

This series is known as the sine series.

(3) Expansion of cos x (Cosine series)

Proof: Let $y = \cos x$, $y(0) = \cos 0 = 1$

$$y_n = \frac{d^n}{dx^n}(\cos x) = \cos\left(x + \frac{n\pi}{2}\right)$$

Now

$$y_n(0) = \cos\left(\frac{n\pi}{2}\right)$$

Putting $n = 1, 2, 3, 4, \ldots$

$y_1(0) = 0,\ y_2(0) = -1,\ y_3(0) = 0,\ y_4(0) = 1,$ and so on.

Substituting in Maclaurin's series,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\ldots\ldots$$

This series is known as the cosine series.

(4) Expansion of tan x (Tangent series)

Proof: Let $y = \tan x$, $\quad y(0) = 0$

$y_1 = \sec^2 x = 1 + \tan^2 x = 1 + y^2, \quad y_1(0) = 1$

$y_2 = 2yy_1, \quad y_2(0) = 2y(0)y_1(0) = 2(0)(1) = 0$

$y_3 = 2y_1^2 + 2yy_2, \quad y_3(0) = 2(1)^2 + 2(0)(0) = 2$

$y_4 = 4y_1y_2 + 2y_1y_2 + 2yy_3 \quad y_4(0) = 6(1)(0) + 2(0)(2)$

$\quad = 6y_1y_2 + 2yy_3, \quad = 0$

$y_5 = 6y_2^2 + 6y_1y_3 + 2y_1y_3 + 2yy_4 \quad y_5(0) = 0 + 8(1)(2) + 0$

$\quad = 6y_2^2 + 8y_1y_3 + 2yy_4, \quad = 16$

Substituting in Maclaurin's series,

$$\tan x = x + \frac{x^3}{3!}(2) + \frac{x^5}{5!}(16) + \ldots\ldots\ldots$$

$$= x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots\ldots\ldots$$

This series is known as the tangent series.

Note: This series can also be obtained by dividing the sine and cosine series since $\tan x = \dfrac{\sin x}{\cos x}$.

(5) Expansion of sinh x

Proof: We have $\sinh x = \dfrac{e^x - e^{-x}}{2}$

Substituting e^x and e^{-x} from above exponential series,

$$\sinh x = \frac{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right) - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots\right)}{2}$$

$$= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\ldots\ldots$$

(6) **Expansion of cosh x**

Proof: We have $\sinh x = \dfrac{e^x + e^{-x}}{2}$

Substituting exponential series e^x and e^{-x},

$$\cosh x = \frac{\left(1+x+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\ldots\right)+\left(1-x+\dfrac{x^2}{2!}-\dfrac{x^3}{3!}+\ldots\right)}{2}$$

$$= 1+\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots\ldots\ldots$$

(7) **Expansion of tanh x**

Proof: Expansion of tanh x can be obtained by dividing the series of sinh x and cosh x.

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{x+\dfrac{x^3}{3!}+\dfrac{x^5}{5!}+\dfrac{x^7}{7!}+\ldots}{1+\dfrac{x^2}{2!}+\dfrac{x^4}{4!}+\dfrac{x^6}{6!}+\ldots}$$

$$= x-\frac{x^2}{3}+\frac{2}{15}x^5-\ldots\ldots\ldots$$

Note: This series can also be obtained by using Maclaurin's series (refer tangent series)

(8) **Expansion of log $(1+x)$ (Logarithmic series)**

Proof: Let $y = \log(1+x)$, $y(0) = \log 1 = 0$

Now $$y_n = \frac{d^n}{dx^n}[\log(1+x)] = (-1)^{n-1}\cdot\frac{(n-1)!}{(x+1)^n}$$

$$y_n(0) = (-1)^{n-1}\cdot(n-1)!$$

Putting $n = 1, 2, 3, 4, \ldots\ldots$

$$y_1(0) = 1,\ y_2(0) = -1,\ y_3(0) = 2! \text{ and so on}$$

Substituting in Maclaurin's series,

$$\log(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots\ldots\ldots$$

This series is known as the Logarithmic series and is valid for $-1 < x < 1$.

Note: In above series replacing x by $-x$, we get expansion of log $(1-x)$

$$\log(1-x) = -x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\frac{x^5}{5}-\ldots$$

(9) **Expansion of $(1+x)^m$ (Binomial series)**

Proof: Let $y = (1+x)^m$, $y(0) = (1+0)^m = 1$

Now
$$y_n = m(m-1)(m-2)\ldots\ldots\ldots(m-n+1)(1+x)^{m-n}$$
$$y_n(0) = m(m-1)(m-2)\ldots\ldots\ldots(m-n+1)$$

Putting $n = 1, 2, 3, 4, \ldots\ldots$

$y_1(0) = m,\ y_2(0) = m(m-1),\ y_3(0) = m(m-1)(m-2)$ and so on

Substituting in Maclaurin's series,

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \ldots\ldots\ldots$$

This series is known as the Binomial series and is valid for $-1 < x < 1$.

By Definition

Example 1: Expand 5^x up to the first three non-zero terms of the series.

Solution: Let
$$f(x) = 5^x, f(0) = 5^0 = 1$$
$$f'(x) = 5^x \log 5,\ f'(0) = 5^0 \log 5 = \log 5$$
$$f''(x) = 5^x (\log 5)^2, f''(0) = 5^0 (\log 5)^2 = (\log 5)^2$$

Substituting in Maclaurin's series,

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots\ldots\ldots$$

$$5^x = 1 + x \log 5 + \frac{x^2}{2!}(\log 5)^2 + \ldots\ldots\ldots$$

Aliter: $f(x) = 5^x = e^{\log 5^x} = e^{x \log 5}$

$$= 1 + x\log 5 + \frac{(x \log 5)^2}{2!} + \ldots\ldots\ldots$$ [Using Exponential series]

Example 2: Obtain the series log $(1 + x)$ and find the series $\log\left(\frac{1+x}{1-x}\right)$ and hence, find the value of $\log_e\left(\frac{11}{9}\right)$.

Solution: Let $y = \log(1+x)$

$$y_1 = \frac{1}{1+x}, y_2 = -\frac{1}{(1+x)^2}, y_3 = \frac{(2!)}{(1+x)^3}, y_4 = -\frac{(3!)}{(1+x)^4} \text{ etc.}$$

At $x = 0, y = 0,\ y_1 = 1,\ y_2 = -1,\ y_3 = 2!,\ y_4 = -(3!)$ etc.

Substituting in Maclaurin's series,

$$y = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \frac{x^4}{4!}y_4(0) + \ldots\ldots\ldots$$

$$= 0 + x - \frac{x^2}{2!} + \frac{x^3}{3!}(2!) - \frac{x^4}{4!}(3!) + \ldots\ldots\ldots$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\ldots$$

Replacing x by $-x$,

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots\dots\dots$$

Now:

$$\log\left(\frac{1+x}{1-x}\right) = \log(1+x) - \log(1-x)$$

$$= 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \dots\right)$$

Putting $x = \frac{1}{10}$, and considering first three terms,

$$\log_e\left(\frac{11}{9}\right) = 2\left[\frac{1}{10} + \frac{1}{3}\cdot\frac{1}{(10)^3} + \frac{1}{5}\cdot\frac{1}{(10)^5}\right] = 0.20067$$

Example 3: If $x^3 + y^3 + xy - 1 = 0$, prove that $y = 1 - \frac{x}{3} - \frac{26}{81}x^3 - \dots$

Solution: $x^3 + y^3 + xy - 1 = 0,$

Putting $x = 0$, $\quad y(0) = 1$

Differentiating w.r.t. x,

$$3x^2 + 3y^2y_1 + xy_1 + y = 0 \qquad \dots(1)$$

Putting $x = 0$, $y_1(0) = \frac{-1}{3}$

Differentiating Eq. (1) w.r.t. x,

$$6x + 6yy_1^2 + 3y^2y_2 + 2y_1 + xy_2 = 0 \qquad \dots(2)$$

Putting $x = 0$, $6\left(-\frac{1}{3}\right)^2 + 3y_2(0) + 2\left(-\frac{1}{3}\right) = 0$

$y_2(0) = 0$

Differentiating Eq. (2) w.r.t. x,

$$6 + 6y_1^3 + 12yy_1y_2 + 3y^2y_3 + 6yy_1y_2 + 3y_2 + xy_3 = 0$$

Putting $x = 0$,

$$6 + 6\left(\frac{-1}{27}\right) + 0 + 3y_3(0) = 0$$

$$y_3(0) = \frac{-52}{27} \quad \text{and so on.}$$

Substituting in Maclaurin's series,

$$y = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \dots\dots$$

$$y = 1 - \frac{x}{3} + \frac{x^2}{2!}(0) + \frac{x^3}{3!}\left(\frac{-52}{27}\right) + \dots\dots$$

$$= 1 - \frac{x}{3} - \frac{26}{81}x^3 - \dots\dots$$

Example 4: If $x^3 + 2xy^2 - y^3 + x - 1 = 0$, expand y in ascending powers of x.

Solution:
$$x^3 + 2xy^2 - y^3 + x - 1 = 0$$

Putting $x = 0$, $y(0) = -1$

Differentiating w.r.t. x,
$$3x^2 + 2y^2 + 4xyy_1 - 3y^2y_1 + 1 = 0 \quad \ldots(1)$$

Putting $x = 0$,
$$2 - 3y_1(0) + 1 = 0$$
$$y_1(0) = 1$$

Differentiating Eq. (1) w.r.t. x,
$$6x + 4yy_1 + 4yy_1 + 4xy_1^2 + 4xyy_2 - 6yy_1^2 - 3y^2y_2 = 0$$

Putting $x = 0$,
$$-8 + 6 - 3y_2(0) = 0$$
$$y_2(0) = -\frac{2}{3} \text{ and so on.}$$

Substituting in Maclaurin's series,
$$y = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \cdots\cdots$$
$$y = -1 + x + \frac{x^2}{2!}\left(-\frac{2}{3}\right) + \cdots\cdots$$
$$= -1 + x - \frac{x^2}{3} + \cdots\cdots$$

Example 5: If $x = y(1 + y^2)$, prove that $y = x - x^3 + 3x^5 + \ldots$.

Solution:
$$x = y(1 + y^2)$$

Putting $x = 0$, $y(0) = 0$

Differentiating w.r.t. x,
$$1 = y_1 + 3y^2y_1 \quad \ldots(1)$$

Putting $x = 0$,
$$1 = y_1(0)$$
$$y_1(0) = 1$$

Differentiating Eq. (1) w.r.t. x,
$$0 = y_2 + 6yy_1^2 + 3y^2y_2 \quad \ldots(2)$$

Putting $x = 0$, $y_2(0) = 0$,

Differentiating Eq. (2) w.r.t. x,
$$0 = y_3 + 12yy_1y_2 + 6y_1^3 + 6yy_1y_2 + 3y^2y_3 \quad \ldots(3)$$
$$0 = y_3(1 + 3y^2) + 18yy_1y_2 + 6y_1^3$$

Putting $x = 0$,
$$0 = y_3(0) + 6$$
$$y_3(0) = -6$$

2.10 INDETERMINATE FORMS

We have studied certain rules to evaluate the limits. But some limits cannot be evaluated by using these rules. These limits are known as indeterminate forms. There are seven types of indeterminate forms given as:

(i) $\frac{0}{0}$ (ii) $\frac{\infty}{\infty}$ (iii) $0 \times \infty$ (iv) $\infty - \infty$

(v) 1^{∞} (vi) 0^{0} (vii) ∞^{0}

These limits can be evaluated by using L'Hospital's Rule.

2.10.1 L'Hospital's Rule

Statement: If $f(x)$ and $g(x)$ are two functions of x which can be expanded by Taylor's series in the neighbourhood of $x = a$ and

$\lim\limits_{x \to a} f(x) = f(a) = 0, \quad \lim\limits_{x \to a} g(x) = g(a) = 0,$ then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

Proof: Let $x = a + h$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{h \to 0} \frac{f(a+h)}{g(a+h)}$$

$$= \lim_{h \to 0} \frac{f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \dots\dots}{g(a) + hg'(a) + \frac{h^2}{2!} g''(a) + \dots\dots} \qquad \text{[By Taylor's theorem]}$$

$$= \lim_{h \to 0} \frac{hf'(a) + \frac{h^2}{2!} f''(a) + \dots\dots}{hg'(a) + \frac{h^2}{2!} g''(a) + \dots\dots} \qquad [\because f(a) = 0,\ g(a) = 0]$$

$$= \lim_{h \to 0} \frac{f'(a) + \frac{h}{2!} f''(a) + \dots\dots}{g'(a) + \frac{h}{2!} g''(a) + \dots\dots}$$

$$= \frac{f'(a)}{g'(a)}$$

$$= \lim_{x \to a} \frac{f'(x)}{g'(x)}, \text{ provided } g'(a) \neq 0.$$

2.10.2 Standard Limits

Following standard limits can be used to solve the problems:

(1) $\lim_{x\to 0}\frac{\sin x}{x}=1$

(2) $\lim_{x\to 0}\frac{\tan x}{x}=1$

(3) $\lim_{x\to 0}\frac{a^x-1}{x}=\log a$

(4) $\lim_{x\to 0}\frac{e^x-1}{x}=1$

(5) $\lim_{x\to 0}(1+x)^{\frac{1}{x}}=e$

(6) $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e$

(7) $\lim_{x\to 0}\frac{\sin^{-1}x}{x}=1$

(8) $\lim_{x\to 0}\frac{\sinh x}{x}=1$

(9) $\lim_{x\to 0}\frac{\tan^{-1}x}{x}=1$

2.10.3 Type 1 : $\left(\frac{0}{0}\right)$

Problems under this type are solved by using L'Hospital's rule considering the fact that

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)} \text{ if } \lim_{x\to a}f(x)=0 \text{ and } \lim_{x\to a}g(x)=0.$$

Example 1: Evaluate $\lim_{x\to 0}\frac{xe^x-\log(1+x)}{x^2}$.

Solution: Let

$$l=\lim_{x\to 0}\frac{xe^x-\log(1+x)}{x^2}\quad\left[\frac{0}{0}\right]$$

$$=\lim_{x\to 0}\frac{e^x+xe^x-\frac{1}{1+x}}{2x}\quad\left[\frac{0}{0}\right]\quad\text{[Applying L'Hospital's rule]}$$

$$=\lim_{x\to 0}\frac{e^x+e^x+xe^x+\frac{1}{(1+x)^2}}{2}\quad\text{[Applying L'Hospital's rule]}$$

$$=\frac{3}{2}.$$

Example 2: Evaluate $\lim_{x\to 0}\frac{\log(1+x^3)}{\sin^3 x}$.

Solution: Let

$$l=\lim_{x\to 0}\frac{\log(1+x^3)}{\sin^3 x}\quad\left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{\frac{1}{1+x^3}\cdot 3x^2}{3\sin^2 x \cos x}$$ [Applying L'Hospital's rule]

$$= \lim_{x\to 0} \left(\frac{x}{\sin x}\right)^2 \frac{1}{(1+x^3)\cos x}$$

$$= 1$$ $\left[\lim_{x\to 0} \frac{x}{\sin x} = 1\right]$

Example 3: Evaluate $\lim_{x\to \frac{1}{2}} \frac{\cos^2 \pi x}{e^{2x} - 2xe}$.

Solution: Let $l = \lim_{x\to \frac{1}{2}} \frac{\cos^2 \pi x}{e^{2x} - 2xe}$ $\left[\frac{0}{0}\right]$

$$= \lim_{x\to \frac{1}{2}} \frac{2\cos \pi x(-\pi \sin \pi x)}{2e^{2x} - 2e}$$ [Applying L'Hospital's rule]

$$= \lim_{x\to \frac{1}{2}} \frac{-\pi \sin 2\pi x}{2(e^{2x} - e)} \quad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to \frac{1}{2}} \frac{-2\pi^2 \cos 2\pi x}{2\cdot 2e^{2x}}$$ [Applying L'Hospital's rule]

$$= \frac{\pi^2}{2e}.$$

Example 4: Evaluate $\lim_{x\to y} \frac{x^y - y^x}{x^x - y^y}$.

Solution: Let $l = \lim_{x\to y} \frac{x^y - y^x}{x^x - y^y}$ $\left[\frac{0}{0}\right]$

$$= \lim_{x\to y} \frac{yx^{y-1} - y^x \log y}{x^x(1+\log x) - 0}$$ [Applying L'Hospital's rule]

$$= \frac{y^y - y^y \log y}{y^y(1+\log y)} = \frac{(1-\log y)}{(1+\log y)}$$

Example 5: Evaluate $\lim_{x\to 0} \frac{2^x - 1}{(1+x)^{\frac{1}{2}} - 1}$.

Solution: Let $$l = \lim_{x\to 0}\frac{2^x - 1}{(1+x)^{\frac{1}{2}} - 1} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0}\frac{2^x \log 2}{\frac{1}{2}(1+x)^{-\frac{1}{2}}} = 2\log 2.$$

Example 6: Evaluate $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\sin(x\cos x)}{\cos(x\sin x)}$.

Solution: Let $$l = \lim_{x\to\frac{\pi}{2}}\frac{\sin(x\cos x)}{\cos(x\sin x)} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to\frac{\pi}{2}}\frac{\cos(x\cos x)(\cos x - x\sin x)}{-\sin(x\sin x)(\sin x + x\cos x)} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \frac{\pi}{2}.$$

Example 7: Prove that $\lim\limits_{\theta\to\alpha}\dfrac{1-\cos(\theta-\alpha)}{(\sin\theta-\sin\alpha)^2} = \dfrac{1}{2}\sec^2\alpha$.

Solution: Let $$l = \lim_{\theta\to\alpha}\frac{1-\cos(\theta-\alpha)}{(\sin\theta-\sin\alpha)^2} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{\theta\to\alpha}\frac{\sin(\theta-\alpha)}{2(\sin\theta-\sin\alpha)\cos\theta} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{\theta\to\alpha}\frac{\sin(\theta-\alpha)}{(\sin 2\theta - 2\sin\alpha\cos\theta)} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{\theta\to\alpha}\frac{\cos(\theta-\alpha)}{2\cos 2\theta + 2\sin\alpha\sin\theta} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \frac{\cos 0}{2\cos 2\alpha + 2\sin\alpha\sin\alpha}$$

$$= \frac{1}{2(1-2\sin^2\alpha)+2\sin^2\alpha} = \frac{1}{2-2\sin^2\alpha}$$

$$= \frac{1}{2\cos^2\alpha} = \frac{1}{2}\sec^2\alpha.$$

Example 8: Evaluate $\lim\limits_{x\to 0}\dfrac{2x^2 - 2e^{x^2} + 2\cos(x^{\frac{3}{2}}) + \sin^3 x}{x^2}$.

Solution: Let $l = \lim_{x\to 0} \dfrac{2x^2 - 2e^{x^2} + 2\cos x^{\frac{3}{2}} + \sin^3 x}{x^2}$ $\left[\dfrac{0}{0}\right]$

$$= \lim_{x\to 0} \frac{4x - 2e^{x^2}(2x) - 2\sin x^{\frac{3}{2}}\left(\frac{3}{2}x^{\frac{1}{2}}\right) + 3\sin^2 x\cos x}{2x} \quad \left[\frac{0}{0}\right]$$

[Applying L'Hospital's rule]

$$= \lim_{x\to 0} \frac{4 - 4(e^{x^2} + xe^{x^2}\cdot 2x) - 3\left(\sqrt{x}\cos x^{\frac{3}{2}}\cdot\frac{3}{2}x^{\frac{1}{2}} + \frac{1}{2\sqrt{x}}\sin x^{\frac{3}{2}}\right) + 6\sin x\cos^2 x - 3\sin^3 x}{2}$$

[Applying L'Hospital's rule]

$$= \frac{4 - 4 - \lim_{x\to 0}\frac{\sin x^{\frac{3}{2}}}{2\sqrt{x}}\cdot\frac{x}{x}}{2}$$

$$= \frac{-1}{2}\cdot\frac{1}{2}\lim_{x\to 0}\frac{\sin x^{\frac{3}{2}}}{x^{\frac{3}{2}}}\cdot x \qquad \left[\because \lim_{x\to 0}\frac{\sin x^{\frac{3}{2}}}{x^{\frac{3}{2}}} = 1\right]$$

$= 0$

Example 9: Evaluate $\lim_{x\to 0} \dfrac{x^{\frac{1}{2}}\tan x}{(e^x - 1)^{\frac{3}{2}}}$.

Solution: Let

$$l = \lim_{x\to 0}\frac{\sqrt{x}\tan x}{(e^x - 1)^{\frac{3}{2}}} \quad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0}\frac{x\sqrt{x}}{(e^x - 1)^{\frac{3}{2}}}\cdot\frac{\tan x}{x}$$

$$= \lim_{x\to 0}\frac{x\sqrt{x}}{(e^x - 1)^{\frac{3}{2}}}\cdot\lim_{x\to 0}\left(\frac{\tan x}{x}\right)$$

$$= \lim_{x\to 0}\left(\frac{x}{e^x - 1}\right)^{\frac{3}{2}} \qquad \left[\because \lim_{x\to 0}\frac{\tan x}{x} = 1\right]$$

Now,

$$\lim_{x\to 0}\frac{x}{e^x - 1} = \lim_{x\to 0}\frac{1}{e^x} = 1 \qquad \text{[Applying L'Hospital's rule]}$$

Hence,

$$\lim_{x\to 0}\left(\frac{x}{e^x - 1}\right)^{\frac{3}{2}} = (1)^{\frac{3}{2}} = 1$$

Example 10: Evaluate $\lim_{x\to 0}\dfrac{\log_{\sec x}\cos\frac{x}{2}}{\log_{\sec\frac{x}{2}}\cos x}$.

Solution: Let
$$l=\lim_{x\to 0}\frac{\log_{\sec x}\cos\frac{x}{2}}{\log_{\sec\frac{x}{2}}\cos x}$$
$$=\lim_{x\to 0}\frac{\log\cos\frac{x}{2}}{\log\sec x}\cdot\frac{\log\sec\frac{x}{2}}{\log\cos x}$$
$$=\lim_{x\to 0}\frac{\log\cos\frac{x}{2}}{(-\log\cos x)}\cdot\frac{\left(-\log\cos\frac{x}{2}\right)}{\log\cos x}$$
$$=\lim_{x\to 0}\left(\frac{\log\cos\frac{x}{2}}{\log\cos x}\right)^2 \qquad \left[\frac{0}{0}\right]$$

Now,
$$\lim_{x\to 0}\frac{\log\cos\frac{x}{2}}{\log\cos x}=\lim_{x\to 0}\frac{\frac{1}{\cos\frac{x}{2}}\cdot\left(-\frac{1}{2}\sin\frac{x}{2}\right)}{\frac{1}{\cos x}(-\sin x)} \qquad \text{[Applying L'Hospital's rule]}$$
$$=\lim_{x\to 0}\frac{\tan\frac{x}{2}}{2\tan x}$$
$$=\lim_{x\to 0}\frac{1}{4}\left(\frac{\tan\frac{x}{2}}{\frac{x}{2}}\right)\cdot\left(\frac{x}{\tan x}\right)$$
$$=\frac{1}{4} \qquad \left[\because \lim_{x\to 0}\frac{\tan x}{x}=1\right]$$
$$\lim_{x\to 0}\left(\frac{\log\cos\frac{x}{2}}{\log\cos x}\right)^2=\left(\frac{1}{4}\right)^2=\frac{1}{16}.$$

Example 11: Prove that $\lim_{x\to 0}\dfrac{(1+x)^{\frac{1}{x}}-e}{x}=-\dfrac{e}{2}$.

Solution: Let
$$l=\lim_{x\to 0}\frac{(1+x)^{\frac{1}{x}}-e}{x}\qquad\left[\frac{0}{0}\right]$$

$$=\lim_{x\to 0}\frac{e^{\frac{1}{x}\log(1+x)}-e}{x}$$

[Applying L'Hospital's rule]

$$=\lim_{x\to 0}\frac{e^{\frac{1}{x}\log(1+x)}\left[-\frac{1}{x^2}\log(1+x)+\frac{1}{x(1+x)}\right]}{1}$$

[Applying L'Hospital's rule]

$$=\lim_{x\to 0}(1+x)^{\frac{1}{x}}\lim_{x\to 0}\frac{[-\{\log(1+x)\}(1+x)+x]}{x^2(1+x)}\qquad\left[\frac{0}{0}\right]$$

$$=e\lim_{x\to 0}\left[\frac{-\log(1+x)-1+1}{2x+3x^2}\right]\qquad\left[\frac{0}{0}\right]$$

[Applying L'Hospital's rule]

$$=e\lim_{x\to 0}\left(\frac{-\frac{1}{1+x}}{2+6x}\right)=-\frac{e}{2}.$$

Example 12: Prove that $\lim\limits_{x\to 0}\dfrac{\left(\sqrt{1-x}-1\right)^{2n}}{(1-\cos x)^n}=2^{-n}$.

Solution: Let
$$l=\lim_{x\to 0}\frac{\left(\sqrt{1-x}-1\right)^{2n}}{(1-\cos x)^n}\cdot\frac{\left(\sqrt{1-x}+1\right)^{2n}}{\left(\sqrt{1-x}+1\right)^{2n}}$$

$$=\lim_{x\to 0}\frac{(1-x-1)^{2n}}{\left(2\sin^2\frac{x}{2}\right)^n\left(\sqrt{1-x}+1\right)^{2n}}$$

$$=\lim_{x\to 0}\frac{(-x)^{2n}}{2^n\left(\sin\frac{x}{2}\right)^{2n}\left(\sqrt{1-x}+1\right)^{2n}}\cdot\frac{2^n}{2^n}$$

$$=\lim_{x\to 0}\left(\frac{\frac{x}{2}}{\sin\frac{x}{2}}\right)^{2n}\frac{2^n}{\left(\sqrt{1-x}+1\right)^{2n}}\qquad\left[\because(-x)^{2n}=\{(-x)^2\}^n=x^{2n}\right]$$

$$=\frac{1}{2^n}.$$

Example 13: If $\lim\limits_{x\to 0}\dfrac{\sin 2x + p\sin x}{x^3}$ is finite, find the value of p and hence, the limit.

Solution: Let $l = \lim\limits_{x\to 0}\dfrac{\sin 2x + p\sin x}{x^3}$, where l is finite

$$l = \lim_{x\to 0}\frac{\sin 2x + p\sin x}{x^3} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0}\frac{2\cos 2x + p\cos x}{3x^2} = \frac{2+p}{0} \qquad \text{[Applying L'Hospital's rule]}$$

But limit is finite, therefore, numerator must be zero.

$$2 + p = 0,\ p = -2$$

Thus,

$$l = \lim_{x\to 0}\frac{2\cos 2x - 2\cos x}{3x^2} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0}\frac{-4\sin 2x + 2\sin x}{6x} \qquad \left[\frac{0}{0}\right] \quad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 0}\frac{-8\cos 2x + 2\cos x}{6}$$

$$= -1$$

Hence, $p = -2$ and $l = -1$

Example 14: Find the values of a and b such that $\lim\limits_{x\to 0}\dfrac{a\sin^2 x + b\log\cos x}{x^4} = \dfrac{1}{2}$.

Solution: $\dfrac{1}{2} = \lim\limits_{x\to 0}\dfrac{a\sin^2 x + b\log\cos x}{x^4}$

$$= \lim_{x\to 0}\frac{a\sin^2 x + b\log\cos x}{x^4}$$

$$= \lim_{x\to 0}\frac{a\cdot 2\sin x\cos x + b\cdot\dfrac{1}{\cos x}(-\sin x)}{4x^3} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 0}\frac{a\sin 2x - b\tan x}{4x^3} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0}\frac{2a\cos 2x - b\sec^2 x}{12x^2} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \frac{2a-b}{0}$$

But limit is finite, therefore, numerator must be zero.

$$2a - b = 0$$

$$b = 2a$$

Thus, $\frac{1}{2} = \lim_{x\to 0} \frac{2a\cos 2x - 2a\sec^2 x}{12x^2}$ $\left[\frac{0}{0}\right]$

$$= \lim_{x\to 0} \frac{-4a\sin 2x - 4a\sec^2 x\tan x}{24x}$$ [Applying L'Hospital's rule]

$$= \lim_{x\to 0}\left(\frac{-a}{3}\frac{\sin 2x}{2x} - \frac{a}{6}\sec^2 x\cdot\frac{\tan x}{x}\right)$$

$$\frac{1}{2} = -\frac{a}{3} - \frac{a}{6} = -\frac{a}{2}$$ $\left[\because \lim_{x\to 0}\frac{\sin 2x}{2x} = 1 \text{ and } \lim_{x\to 0}\frac{\tan x}{x} = 1\right]$

Hence, $a = -1,\ b = -2$

Example 15: Find a and b if $\lim_{x\to 0}\frac{a\sin x - \sin 2x}{\tan^3 x} = b$.

Solution: $$b = \lim_{x\to 0}\frac{a\sin x - \sin 2x}{\tan^3 x} = \lim_{x\to 0}\frac{a\sin x - \sin 2x}{x^3\left(\frac{\tan x}{x}\right)^3}$$

$$= \lim_{x\to 0}\frac{a\sin x - \sin 2x}{x^3}$$ $\left[\frac{0}{0}\right]$ $\left[\because \lim_{x\to 0}\frac{\tan x}{x} = 1\right]$

$$= \lim_{x\to 0}\frac{a\cos x - 2\cos 2x}{3x^2}$$ [Applying L'Hospital's rule]

$$= \frac{a-2}{0}$$

But limit is finite, therefore, numerator must be zero.

$$a - 2 = 0,\ a = 2$$

Thus, $$b = \lim_{x\to 0}\frac{2\cos x - 2\cos 2x}{3x^2}$$ $\left[\frac{0}{0}\right]$

$$= \lim_{x\to 0}\frac{-2\sin x + 4\sin 2x}{6x}$$ [Applying L'Hospital's rule]

$$= \lim_{x\to 0}\left[-\frac{2}{6}\left(\frac{\sin x}{x}\right) + \frac{4}{3}\left(\frac{\sin 2x}{2x}\right)\right]$$

$$= -\frac{2}{6} + \frac{4}{3} = 1$$ $\left[\because \lim_{x\to 0}\frac{\sin x}{x} = 1\right]$

$$a = 2, \ b = 1$$

Hence,

Example 16: Find a, b, c if $\lim\limits_{x\to 0} \dfrac{ae^x - b\cos x + ce^{-x}}{x\sin x} = 2.$

Solution:

$$2 = \lim_{x\to 0} \frac{ae^x - b\cos x + ce^{-x}}{x\sin x}$$

$$= \lim_{x\to 0} \frac{ae^x - b\cos x + ce^{-x}}{x \cdot x\left(\dfrac{\sin x}{x}\right)}$$

$$= \lim_{x\to 0} \frac{ae^x - b\cos x + ce^{-x}}{x^2} \qquad \left[\because \lim_{x\to 0}\frac{\sin x}{x} = 1\right]$$

$$= \frac{a-b+c}{0}$$

But limit is finite, therefore, numerator must be zero.

$$a - b + c = 0 \qquad \text{...(1)}$$

Thus,

$$2 = \lim_{x\to 0} \frac{ae^x - b\cos x + ce^{-x}}{x^2} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{ae^x + b\sin x - ce^{-x}}{2x} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \frac{a-c}{0}$$

But limit is finite, therefore, numerator must be zero.

$$a - c = 0, \ a = c \qquad \text{...(2)}$$

$$2 = \lim_{x\to 0} \frac{ae^x + b\sin x - ae^{-x}}{2x} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{ae^x + b\cos x + ae^{-x}}{2} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \frac{a+b+a}{2}$$

$$2a + b = 4 \qquad \text{...(3)}$$

From Eqs (1) and (2), we have

$$2a - b = 0 \qquad \text{...(4)}$$

Solving Eqs (3) and (4),

$$a = 1, \quad b = 2, \text{ and } c = 1$$

2.10.4 Type 2: $\left(\frac{\infty}{\infty}\right)$

Problems under this type are also solved by using L'Hospital's rule considering the fact that

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)} \text{ if } \lim_{x\to a} f(x)=\infty \text{ and } \lim_{x\to a} g(x)=\infty.$$

Example 1: Prove that $\lim_{x\to\frac{\pi}{2}}\dfrac{\log\left(x-\frac{\pi}{2}\right)}{\tan x}=0.$

Solution: Let $l=\lim_{x\to\frac{\pi}{2}}\dfrac{\log\left(x-\frac{\pi}{2}\right)}{\tan x}$ $\quad\left[\frac{\infty}{\infty}\right]$

$$=\lim_{x\to\frac{\pi}{2}}\frac{\dfrac{1}{x-\frac{\pi}{2}}}{\sec^2 x} \quad \text{[Applying L'Hospital's rule]}$$

$$=\lim_{x\to\frac{\pi}{2}}\frac{\cos^2 x}{x-\frac{\pi}{2}} \quad \left[\frac{0}{0}\right]$$

$$=\lim_{x\to\frac{\pi}{2}}\frac{2\cos x(-\sin x)}{1}=0 \quad \text{[Applying L'Hospital's rule]}$$

Example 2: Prove that $\lim_{x\to a}\dfrac{\log(x-a)}{\log(a^x-a^a)}=1.$

Solution: Let $l=\lim_{x\to a}\dfrac{\log(x-a)}{\log(a^x-a^a)}$ $\quad\left[\frac{\infty}{\infty}\right]$

$$=\lim_{x\to a}\frac{\dfrac{1}{(x-a)}}{\dfrac{1}{a^x-a^a}\cdot a^x\log a} \quad \text{[Applying L'Hospital's rule]}$$

$$=\lim_{x\to a}\left(\frac{a^x-a^a}{x-a}\right)\cdot\lim_{x\to a}\frac{1}{a^x\log a}$$

$$=\lim_{x\to a}\frac{a^x\log a}{1}\cdot\frac{1}{a^a\log a} \quad \text{[Applying L'Hospital's rule for first term]}$$

$$=a^a\log a\cdot\frac{1}{a^a\log a}=1$$

Example 3: Prove that $\lim\limits_{x\to\infty}\dfrac{\sinh^{-1}x}{\cosh^{-1}x}=1$.

Solution: Let $l=\lim\limits_{x\to\infty}\dfrac{\sinh^{-1}x}{\cosh^{-1}x}$

$$=\lim_{x\to\infty}\frac{\log\left(x+\sqrt{x^2+1}\right)}{\log\left(x+\sqrt{x^2-1}\right)}\qquad\left[\frac{\infty}{\infty}\right]$$

$$l=\lim_{x\to\infty}\frac{\dfrac{1}{\left(x+\sqrt{x^2+1}\right)}\cdot\left(1+\dfrac{1}{2\sqrt{x^2+1}}\cdot 2x\right)}{\dfrac{1}{\left(x+\sqrt{x^2-1}\right)}\cdot\left(1+\dfrac{1}{2\sqrt{x^2-1}}\cdot 2x\right)}\qquad\text{[Applying L'Hospital's rule]}$$

$$=\lim_{x\to\infty}\frac{\dfrac{\sqrt{x^2+1}+x}{\left(x+\sqrt{x^2+1}\right)\sqrt{x^2+1}}}{\dfrac{\sqrt{x^2-1}+x}{\left(x+\sqrt{x^2-1}\right)\sqrt{x^2-1}}}$$

$$=\lim_{x\to\infty}\frac{\sqrt{x^2-1}}{\sqrt{x^2+1}}=\lim_{x\to\infty}\frac{\sqrt{1-\dfrac{1}{x^2}}}{\sqrt{1+\dfrac{1}{x^2}}}=1$$

Example 4: Prove that $\lim\limits_{x\to 0}\log_x\sin x=1$.

Solution: Let $l=\lim\limits_{x\to 0}\log_x\sin x$

$$=\lim_{x\to 0}\frac{\log\sin x}{\log x}\qquad\left[\frac{\infty}{\infty}\right]\qquad\text{[Change of base property]}$$

$$=\lim_{x\to 0}\frac{\dfrac{1}{\sin x}\cdot\cos x}{\dfrac{1}{x}}\qquad\text{[Applying L'Hospital's rule]}$$

$$=\lim_{x\to 0}\frac{x}{\sin x}\cdot\lim_{x\to 0}\cos x\qquad\left[\because \lim_{x\to 0}\frac{\sin x}{x}=1\right]$$

$$=1$$

Example 5: Prove that $\lim\limits_{x\to\infty}\dfrac{e^{\frac{1}{x}}+e^{\frac{2}{x}}+e^{\frac{3}{x}}+\ldots+e^{\frac{x}{x}}}{x}=e-1.$

Solution: Let $l=\lim\limits_{x\to\infty}\dfrac{e^{\frac{1}{x}}+e^{\frac{2}{x}}+e^{\frac{3}{x}}+\ldots+e^{\frac{x}{x}}}{x}$

$$=\lim_{x\to\infty}\frac{e^{\frac{1}{x}}\left[1-\left(e^{\frac{1}{x}}\right)^{x}\right]}{1-e^{\frac{1}{x}}}\cdot\frac{1}{x} \qquad \text{[Sum of G.P.]}$$

$$=\lim_{x\to\infty}\frac{e^{\frac{1}{x}}(e-1)}{e^{\frac{1}{x}}-1}\cdot\frac{1}{x}$$

Putting $\dfrac{1}{x}=y$, when $x\to\infty, y\to 0$

$$l=\lim_{y\to 0}\frac{(e-1)e^{y}y}{e^{y}-1} \qquad \left[\frac{0}{0}\right]$$

$$=\lim_{y\to 0}\frac{(e-1)(ye^{y}+e^{y})}{e^{y}} \qquad \text{[Applying L'Hospital's rule]}$$

$$=e-1$$

Example 6: Prove that $\lim\limits_{x\to\infty}\dfrac{x^{n}}{e^{kx}}=0.$

Solution: Let $l=\lim\limits_{x\to\infty}\dfrac{x^{n}}{e^{kx}}$ $\left[\dfrac{\infty}{\infty}\right]$

$$=\lim_{x\to\infty}\frac{nx^{n-1}}{ke^{kx}} \qquad \left[\frac{\infty}{\infty}\right] \qquad \text{[Applying L'Hospital's rule]}$$

$$=\lim_{x\to\infty}\frac{n(n-1)x^{n-2}}{k^{2}e^{kx}} \qquad \left[\frac{\infty}{\infty}\right] \qquad \text{[Applying L'Hospital's rule]}$$

Applying L'Hospital's rule n times,

$$l=\lim_{x\to\infty}\frac{n(n-1)(n-2)\ldots 2.1}{k^{n}e^{kx}}=\lim_{x\to\infty}\frac{n!}{k^{n}e^{kx}}=0 \qquad [\because \lim_{x\to\infty} e^{kx} \ldots$$

Example 7: **Prove that** $\lim\limits_{x\to\infty}\dfrac{1^2+2^2+3^2+\dots+x^2}{x^3}=\dfrac{1}{3}$.

Solution: Let $l=\lim\limits_{x\to\infty}\dfrac{1^2+2^2+3^2+\dots+x^2}{x^3}$

$$=\lim_{x\to\infty}\frac{x(x+1)(2x+1)}{6x^3}\qquad\left[\because \sum n^2=\frac{n(n+1)(2n+1)}{6}\right]$$

$$=\lim_{x\to\infty}\frac{2x^3+3x^2+x}{6x^3}=\lim_{x\to\infty}\frac{2+\frac{3}{x}+\frac{1}{x^2}}{6}$$

$$=\frac{2}{6}=\frac{1}{3}.$$

Example 8: **Prove that** $\lim\limits_{x\to\infty}\dfrac{e^x}{\left[\left(1+\frac{1}{x}\right)^x\right]^x}=e^{\frac{1}{2}}$.

Solution: Let $l=\lim\limits_{x\to\infty}\dfrac{e^x}{\left[\left(1+\frac{1}{x}\right)^x\right]^x}\qquad\left[\dfrac{\infty}{\infty}\right]$

$$=\lim_{x\to\infty}\frac{e^x}{\left(1+\frac{1}{x}\right)^{x^2}}$$

Taking logarithm on both the sides,

$$\log l=\lim_{x\to\infty}\left[\log e^x-\log\left(1+\frac{1}{x}\right)^{x^2}\right]=\lim_{x\to\infty}\left[x-x^2\log\left(1+\frac{1}{x}\right)\right]$$

$$=\lim_{x\to\infty}x^2\left[\frac{1}{x}-\log\left(1+\frac{1}{x}\right)\right]=\lim_{x\to\infty}\frac{\frac{1}{x}-\log\left(1+\frac{1}{x}\right)}{\frac{1}{x^2}}\qquad\left[\frac{0}{0}\right]$$

$$=\lim_{x\to\infty}\frac{-\frac{1}{x^2}-\frac{1}{1+\frac{1}{x}}\left(-\frac{1}{x^2}\right)}{-\frac{2}{x^3}}\qquad\text{[Applying L'Hospital's rule]}$$

$$=\lim_{x\to\infty}\frac{1-\frac{1}{1+\frac{1}{x}}}{\frac{2}{x}}$$

$$= \frac{1}{2}\lim_{x\to\infty}\frac{1}{1+\frac{1}{x}} = \frac{1}{2}$$

Hence,

$$l = e^{\frac{1}{2}}$$

Exercise 2.11

1. Prove that $\lim_{x\to\infty}\frac{\log x}{x^n} = 0\,(n>0)$.
2. Prove that $\lim_{x\to0}\frac{\log x}{\cot x} = 0$.
3. Prove that $\lim_{x\to1}\frac{\log(1-x)}{\cot \pi x} = 0$.
4. Prove that $\lim_{x\to0}\frac{\log_{\sin x}\cos x}{\log_{\sin\frac{x}{2}}\cos\frac{x}{2}} = 4$.
5. Prove that $\lim_{x\to0}\log_{\tan x}\tan 2x = 1$.
6. Prove that $\lim_{x\to0}\log_{\sin x}\sin 2x = 1$.
7. Prove that $\lim_{x\to\infty}\frac{\log(1+e^{3x})}{x} = 3$.
8. Prove that $\lim_{x\to0}\frac{\log x^2}{\cot x^2} = 0$.

   [Hint : Put $x^2 = y$]
9. Prove that $\lim_{x\to\infty}\frac{x^m}{e^x} = 0\,(m>0)$.
10. Prove that

$$\lim_{n\to\infty}\frac{\left(\frac{1}{e}\right)+\left(\frac{1}{e}\right)^2+\left(\frac{1}{e}\right)^3+\ldots\ldots+\left(\frac{1}{e}\right)^n}{n} = 0.$$

2.10.5 Type 3 : $(0\times\infty)$

To solve the problems of the type

$$\lim_{x\to a}[f(x)\cdot g(x)], \text{ when } \lim_{x\to a}f(x)=0,\ \lim_{x\to a}g(x)=\infty \text{ (i.e. } 0\times\infty \text{ form)}$$

We write $\lim_{x\to a}f(x)\cdot g(x) = \lim_{x\to a}\frac{f(x)}{\frac{1}{g(x)}}$ or $\lim_{x\to a}\frac{g(x)}{\frac{1}{f(x)}}$.

These new forms are of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ respectively, which can be solved using L'Hospital's rule.

Example 1: Prove that $\lim_{x\to0}\sin x\log x = 0$.

Solution: Let $l = \lim_{x\to0}\sin x\log x$ $\qquad [0\times\infty]$

$$= \lim_{x \to 0} \frac{\log x}{\operatorname{cosec} x} \qquad \left[\frac{\infty}{\infty}\right]$$

$$= \lim_{x \to 0} \frac{\frac{1}{x}}{-\operatorname{cosec} x \cot x} \qquad \text{[Applying L'Hospital's rule]}$$

$$= -\lim_{x \to 0} \sin x \cdot \frac{\tan x}{x}$$

$$= -\lim_{x \to 0} \sin x \cdot \lim_{x \to 0} \frac{\tan x}{x}$$

$$= 0 \qquad \left[\because \lim_{x \to 0} \frac{\tan x}{x} = 1\right]$$

Example 2: $\lim_{x \to \infty} 2^x \cdot \sin\left(\frac{a}{2^x}\right) = a.$

Solution: Let $l = \lim_{x \to \infty} 2^x \cdot \sin\left(\frac{a}{2^x}\right)$

Taking $2^x = \frac{1}{t}$, $t = \frac{1}{2^x}$,

when $x \to \infty$, $2^x \to \infty$, $\quad t \to 0$

$$l = \lim_{t \to 0} \frac{\sin at}{t} = \lim_{t \to 0} \frac{a \sin at}{at} = a \cdot 1 = a \qquad \left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1\right]$$

Example 3: Prove that $\lim_{x \to \infty}\left(a^{\frac{1}{x}} - 1\right)x = \log a$.

Solution: Let $l = \lim_{x \to \infty}\left(a^{\frac{1}{x}} - 1\right) \cdot x \qquad [0 \times \infty]$

$$= \lim_{x \to \infty} \frac{\left(a^{\frac{1}{x}} - 1\right)}{\frac{1}{x}} \qquad \left[\frac{0}{0}\right]$$

Taking $\frac{1}{x} = t$, when $x \to \infty$, $t \to 0$

$$l = \lim_{t \to 0} \frac{a^t - 1}{t} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{t \to 0} \frac{a^t \log a}{1} \qquad \text{[Applying L'Hospital's rule]}$$

$$= a^0 \log a = \log a$$

Example 4: $\lim\limits_{x\to 1} \tan^2\left(\frac{\pi x}{2}\right)(1+\sec \pi x) = -2.$

Solution: Let $l = \lim\limits_{x\to 1} \tan^2\left(\frac{\pi x}{2}\right)(1+\sec \pi x)$ $\quad [\infty \times 0]$

$$= \lim_{x\to 1} \frac{1+\sec \pi x}{\cot^2\left(\frac{\pi x}{2}\right)} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 1} \frac{\pi \sec \pi x \tan \pi x}{2\cot\left(\frac{\pi x}{2}\right)\left(-\text{cosec}^2 \frac{\pi x}{2}\right)\frac{\pi}{2}} \qquad \text{[Applying L'Hospital's rule]}$$

$$= -\left(\lim_{x\to 1} \frac{\sec \pi x}{\text{cosec}^2 \frac{\pi x}{2}}\right)\left(\lim_{x\to 1} \frac{\tan \pi x}{\cot \frac{\pi x}{2}}\right)$$

$$= -\left(\frac{\sec \pi}{\text{cosec}^2 \frac{\pi}{2}}\right) \lim_{x\to 1} \frac{\tan \pi x}{\cot \frac{\pi x}{2}} \qquad \left[\frac{0}{0}\right]$$

$$= -(-1) \lim_{x\to 1} \frac{\pi \sec^2 \pi x}{\left(-\text{cosec}^2 \frac{\pi x}{2}\right)\frac{\pi}{2}} \qquad \text{[Applying L'Hospital's rule]}$$

$$= -2 \frac{\sec^2 \pi}{\text{cosec}^2 \frac{\pi}{2}} = -2$$

Example 5: $\lim\limits_{x\to a} \sin^{-1}\sqrt{\frac{a-x}{a+x}} \,\text{cosec}\sqrt{a^2-x^2} = \frac{1}{2a}.$

Solution: Let $l = \lim\limits_{x\to a} \sin^{-1}\sqrt{\frac{a-x}{a+x}} \,\text{cosec}\sqrt{a^2-x^2}$ $\quad [0 \times \infty]$

$$= \lim_{x\to a} \frac{\sin^{-1}\sqrt{\frac{a-x}{a+x}}}{\sin\sqrt{a^2-x^2}}$$

Here applying L'Hospital's rule will make the expression complicated, so we rearrange the terms to apply the limits directly.

Let $\sqrt{\dfrac{a-x}{a+x}}=\alpha,\ \sqrt{a^2-x^2}=\beta$

when $x\to a,\ \alpha\to 0$ and $\beta\to 0$

Hence,
$$l=\lim_{\alpha\to 0}\sin^{-1}\alpha\lim_{\beta\to 0}\frac{1}{\sin\beta}$$

$$=\left[\lim_{\alpha\to 0}\left(\frac{\sin^{-1}\alpha}{\alpha}\right)\cdot\alpha\right]\left[\lim_{\beta\to 0}\left(\frac{\beta}{\sin\beta}\right)\cdot\frac{1}{\beta}\right]$$

$$=\lim_{\alpha\to 0}\alpha\cdot\lim_{\beta\to 0}\frac{1}{\beta}\qquad\left[\because\lim_{x\to 0}\left(\frac{\sin^{-1}x}{x}\right)=1 \text{ and } \lim_{x\to 0}\left(\frac{\sin x}{x}\right)=1\right]$$

$$=\lim_{x\to a}\sqrt{\frac{a-x}{a+x}}\cdot\frac{1}{\sqrt{a^2-x^2}}\qquad[\text{Resubstituting } \alpha \text{ and } \beta]$$

$$=\lim_{x\to a}\sqrt{\frac{a-x}{a+x}}\cdot\frac{1}{\sqrt{a+x}\sqrt{a-x}}$$

$$=\lim_{x\to a}\frac{1}{a+x}=\frac{1}{2a}$$

Example 6: Evaluate $\lim\limits_{x\to 0}x^m(\log x)^n$, where m and n are positive integers.

Solution: Let $l=\lim\limits_{x\to 0}x^m(\log x)^n \qquad [0\times\infty]$

$$=\lim_{x\to 0}\frac{(\log x)^n}{\frac{1}{x^m}}\qquad\left[\frac{\infty}{\infty}\right]$$

$$=\lim_{x\to 0}\frac{n(\log x)^{n-1}\frac{1}{x}}{-m(x)^{-m-1}}\qquad[\text{Applying L'Hospital's rule}]$$

$$=\lim_{x\to 0}\frac{(-1)^1 n(\log x)^{n-1}}{m(x)^{-m}}\qquad\left[\frac{\infty}{\infty}\right]$$

$$=\lim_{x\to 0}\frac{(-1)^1 n(n-1)(\log x)^{n-2}\cdot\frac{1}{x}}{m(-m)^1(x)^{-m-1}}\qquad[\text{Applying L'Hospital's rule}]$$

$$=\lim_{x\to 0}\frac{(-1)^2 n(n-1)(\log x)^{n-2}}{m^2(x)^{-m}}\qquad\left[\frac{\infty}{\infty}\right]$$

Applying L'Hospital's rule $(n-2)$ times in the above expression,

$$l = \lim_{x\to 0} \frac{(-1)^n n!(\log x)^0}{m^n (x)^{-m}}$$

$$= \lim_{x\to 0} \frac{(-1)^n n!}{m^n} \cdot x^m = 0$$

Exercise 2.12

1. Prove that $\lim\limits_{x\to 0} x \log x = 0$.

2. Prove that $\lim\limits_{x\to\infty} x^2 e^{-x} = 0$.

3. Prove that $\lim\limits_{x\to\infty} x^2 \left(1 - e^{-\frac{2gy}{x^2}}\right) = 2gy$.

4. Prove that $\lim\limits_{x\to 0} \tan x \log x = 0$.

5. Prove that

$$\lim_{x\to 1} (x^2 - 1) \tan\left(\frac{\pi x}{2}\right) = -\frac{4}{\pi}.$$

6. Prove that

$$\lim_{x\to 1} (1 + \sec \pi x) \tan \frac{\pi x}{2} = 0.$$

7. Prove that

$$\log\left(2 - \frac{x}{a}\right) \cot(x-a) = -\frac{1}{a}.$$

8. Prove that

$$\lim_{x\to 1} \log(1-x) \cot\left(\frac{\pi x}{2}\right) = 0.$$

9. Prove that $\lim\limits_{x\to 0} \log\left(\frac{1+x}{1-x}\right) \cot x = 2$

10. Prove that

$$\lim_{x\to a} \sqrt{\frac{a+x}{a-x}} \tan^{-1} \sqrt{a^2 - x^2} = 2a.$$

11. Prove that

$$\lim_{x\to 2} \sqrt{\frac{2+x}{2-x}} \tan^{-1} \sqrt{4 - x^2} = 4.$$

2.10.6 Type 4 : $(\infty - \infty)$

To evaluate the limits of the type $\lim\limits_{x\to a} [f(x) - g(x)]$, when $\lim\limits_{x\to a} f(x) = \infty$ and $\lim\limits_{x\to a} g(x) = \infty$ [i.e., $(\infty - \infty)$ form], we reduce the expression in the form of $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by taking LCM or by rearranging the terms and then appling L'Hospital's rule.

Example 1: Prove that $\lim\limits_{x\to\infty} (\cosh^{-1} x - \log x) = \log 2$.

Solution: Let $l = \lim\limits_{x\to\infty} (\cosh^{-1} x - \log x) \quad [\infty - \infty]$

$$= \lim_{x\to\infty} \left[\log\left(x + \sqrt{x^2 - 1}\right) - \log x\right]$$

$$= \lim_{x\to\infty} \log\left(\frac{x + \sqrt{x^2 - 1}}{x}\right)$$

$$= \lim_{x\to\infty} \log\left(1 + \sqrt{1 - \frac{1}{x^2}}\right)$$

$$= \log\left(1 + \sqrt{1 - 0}\right) = \log 2$$

Example 2: Prove that $\lim\limits_{x\to 1}\left(\dfrac{1}{\log x}-\dfrac{x}{x-1}\right)=-\dfrac{1}{2}$.

Solution: Let $l=\lim\limits_{x\to 1}\left(\dfrac{1}{\log x}-\dfrac{x}{x-1}\right)$ $[\infty-\infty]$

$$=\lim_{x\to 1}\left[\frac{x-1-x\log x}{(x-1)\log x}\right] \qquad \left[\frac{0}{0}\right]$$

$$=\lim_{x\to 1}\frac{1-x\cdot\frac{1}{x}-\log x}{(x-1)\cdot\frac{1}{x}+\log x} \qquad \text{[Applying L'Hospital's rule]}$$

$$=\lim_{x\to 1}\frac{-\log x}{1-\frac{1}{x}+\log x} \qquad \left[\frac{0}{0}\right]$$

$$=\lim_{x\to 1}\frac{-\frac{1}{x}}{\frac{1}{x^2}+\frac{1}{x}}=-\frac{1}{2} \qquad \text{[Applying L'Hospital's rule]}$$

Example 3: Prove that $\lim\limits_{x\to 0}\left(\dfrac{a}{x}-\cot\dfrac{x}{a}\right)=0$.

Solution: Let $l=\lim\limits_{x\to 0}\left(\dfrac{a}{x}-\cot\dfrac{x}{a}\right)$ $[\infty-\infty]$

Taking $\dfrac{x}{a}=y$, when $x\to 0,\ y\to 0$

$$l=\lim_{y\to 0}\left(\frac{1}{y}-\cot y\right)$$

$$=\lim_{y\to 0}\left(\frac{1}{y}-\frac{1}{\tan y}\right) \qquad [\infty-\infty]$$

$$=\lim_{y\to 0}\left(\frac{\tan y-y}{y\tan y}\right) \qquad \left[\frac{0}{0}\right]$$

$$=\lim_{y\to 0}\left(\frac{\tan y-y}{y^2}\right)\cdot\lim_{y\to 0}\frac{1}{\left(\frac{\tan y}{y}\right)}$$

$$=\lim_{y\to 0}\frac{\tan y-y}{y^2}\cdot 1 \qquad \left[\frac{0}{0}\right]\left[\because \lim_{y\to 0}\frac{\tan y}{y}=1\right]$$

$$= \lim_{y \to 0} \frac{\sec^2 y - 1}{2y} \qquad \left[\frac{0}{0}\right] \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{y \to 0} \frac{2\sec y \cdot \sec y \tan y}{2} \qquad \text{[Applying L'Hospital's rule]}$$

$$= 0$$

Example 4: Prove that $\lim\limits_{x \to 0} \left[\frac{1}{2x} - \frac{1}{x(e^{\pi x} + 1)}\right] = \frac{\pi}{4}$.

Solution: Let $l = \lim\limits_{x \to 0} \left[\frac{1}{2x} - \frac{1}{x(e^{\pi x} + 1)}\right] \qquad [\infty - \infty]$

$$= \lim_{x \to 0} \frac{e^{\pi x} + 1 - 2}{2x(e^{\pi x} + 1)} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x \to 0} \frac{\pi e^{\pi x}}{2\left[(e^{\pi x} + 1) + x(\pi e^{\pi x})\right]}$$

$$= \frac{\pi}{2} \frac{e^0}{(e^0 + 1)} = \frac{\pi}{4}$$

Example 5: Prove that $\lim\limits_{x \to \infty} \left(x + \frac{1}{2}\right)\left[\log\left(x + \frac{1}{2}\right) - \log x\right] = \frac{1}{2}$.

Solution: Let $l = \lim\limits_{x \to \infty} \left(x + \frac{1}{2}\right)\left[\log\left(x + \frac{1}{2}\right) - \log x\right] \qquad [\infty - \infty]$

$$= \lim_{x \to \infty} \left(x + \frac{1}{2}\right) \log\left(\frac{x + \frac{1}{2}}{x}\right)$$

$$= \lim_{x \to \infty} \left[x \log\left(1 + \frac{1}{2x}\right) + \frac{1}{2}\log\left(1 + \frac{1}{2x}\right)\right]$$

$$= \lim_{x \to \infty} \frac{1}{2} \log\left(1 + \frac{1}{2x}\right)^{2x} + \frac{1}{2}\lim_{x \to \infty} \log\left(1 + \frac{1}{2x}\right)$$

$$= \frac{1}{2}\log e + \frac{1}{2}\log 1 \qquad \left[\because \lim_{x \to \infty}\left(1 + \frac{1}{ax}\right)^{ax} = e\right]$$

$$= \frac{1}{2}.$$

Example 6: If $\lim_{x\to 0}\left(\frac{a\cot x}{x}+\frac{b}{x^2}\right)=\frac{1}{3}$, find a and b.

Solution:

$$\frac{1}{3}=\lim_{x\to 0}\left(\frac{a\cot x}{x}+\frac{b}{x^2}\right)$$

$$=\lim_{x\to 0}\left(\frac{a}{x\tan x}+\frac{b}{x^2}\right)$$

$$=\lim_{x\to 0}\left(\frac{ax+b\tan x}{x^2\tan x}\right) \qquad \left[\frac{0}{0}\right]$$

$$=\lim_{x\to 0}\frac{(ax+b\tan x)}{(x^2\cdot x)\left(\frac{\tan x}{x}\right)}$$

$$=\lim_{x\to 0}\frac{ax+b\tan x}{x^3}\cdot\lim_{x\to 0}\left(\frac{x}{\tan x}\right)$$

$$=\lim_{x\to 0}\frac{ax+b\tan x}{x^3}\cdot 1 \qquad \left[\frac{0}{0}\right]\left[\because \lim_{x\to 0}\frac{x}{\tan x}=1\right]$$

$$=\lim_{x\to 0}\left(\frac{a+b\sec^2 x}{3x^2}\right) \qquad \text{[Applying L'Hospital's rule]}$$

$$=\frac{a+b\sec 0}{0}=\frac{a+b}{0}$$

But limit is finite, therefore, numerator must be zero.

$$a+b=0,\ a=-b \qquad \ldots(1)$$

Thus,

$$\frac{1}{3}=\lim_{x\to 0}\frac{-b+b\sec^2 x}{3x^2} \qquad \left[\frac{0}{0}\right]$$

$$=\lim_{x\to 0}\frac{b\cdot 2\sec x\sec x\tan x}{6x} \qquad \text{[Applying L'Hospital's rule]}$$

$$=\left(\lim_{x\to 0}\frac{b}{3}\sec^2 x\right)\cdot\left(\lim_{x\to 0}\frac{\tan x}{x}\right)$$

$$=\frac{b}{3}\sec 0\cdot 1 \qquad \left[\because \lim_{x\to 0}\frac{\tan x}{x}=1\right]$$

$$\frac{1}{3}=\frac{b}{3},\ b=1$$

From Eq. (1),

$$a=-b=-1$$

Hence,

$$a=-1,\ b=1.$$

Exercise 2.12

1. Prove that $\lim\limits_{x \to 0}\left(\cot x - \dfrac{1}{x}\right) = 0.$

2. Prove that

$$\lim_{x \to a}\left[\frac{1}{x-a} - \cot(x-a)\right] = 0.$$

3. Prove that

$$\lim_{x \to \frac{\pi}{2}} (\sec x - \tan x) = 0.$$

4. Prove that

$$\lim_{x \to \frac{\pi}{2}}\left(\tan x - \frac{2x \sec x}{\pi}\right) = \frac{2}{\pi}.$$

5. Prove that

$$\lim_{x \to 0}\left[\frac{1}{x-a} - \frac{1}{\log(x+1-a)}\right] =$$

6. Prove that

$$\lim_{x \to 3}\left[\frac{1}{x-3} - \frac{1}{\log(x-2)}\right] = -\frac{1}{2}.$$

7. Prove that $\lim\limits_{x \to 0}\left[\dfrac{1}{x} - \dfrac{\log(1+x)}{x^2}\right] = \dfrac{1}{2}.$

8. Prove that $\lim\limits_{x \to 0}\left(\dfrac{1}{x} - \dfrac{1}{e^x - 1}\right) = \dfrac{1}{2}.$

2.10.7 Type 5 : $1^\infty, \infty^0, 0^0$

To evaluate the limits of the type $\lim\limits_{x \to a}[f(x)]^{g(x)}$ which takes any one of the above form, we proceed as follows:

Let $$l = \lim_{x \to a} [f(x)]^{g(x)}$$

$$\log l = \lim_{x \to a} [g(x) \cdot \log f(x)] \quad [\text{if } f(x) > 0]$$

which takes the form $\infty \times 0$, i.e., type 3 form.

Example 1: Prove that $\lim\limits_{x \to 0} (a^x + x)^{\frac{1}{x}} = ae.$

Solution: Let $l = \lim\limits_{x \to 0} (a^x + x)^{\frac{1}{x}}$ $\quad [1^\infty]$

$$\log l = \lim_{x \to 0} \frac{1}{x} \cdot \log(a^x + x)$$

$$= \lim_{x \to 0} \frac{\log(a^x + x)}{x} \quad \left[\frac{0}{0}\right]$$

$$= \lim_{x \to 0} \frac{\frac{1}{a^x + x}(a^x \log a + 1)}{1} \quad \text{[Applying L'Hospital's rule]}$$

$$= \frac{a^0 \log a + 1}{a^0 + 0} = \frac{\log_e a + \log_e e}{1}$$

$$\log l = \log ae$$

Hence,

$$l = ae$$

Example 2: Prove that $\lim_{x\to 0}\left(\frac{a^x+b^x}{2}\right)^{\frac{1}{x}} = \sqrt{ab}$.

Solution: Let $l = \lim_{x\to 0}\left(\frac{a^x+b^x}{2}\right)^{\frac{1}{x}}$ $[1^\infty]$

$$\log l = \lim_{x\to 0}\frac{1}{x}\log\left(\frac{a^x+b^x}{2}\right)$$

$$= \lim_{x\to 0}\frac{\log\left(\frac{a^x+b^x}{2}\right)}{x} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0}\left(\frac{2}{a^x+b^x}\right)\cdot\frac{(a^x\log a+b^x\log b)}{2} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \left(\frac{2}{a^0+b^0}\right)\frac{(a^0\log a+\mathrm{b}^0\log b)}{2}$$

$$= \frac{1}{2}\cdot\log ab$$

Hence,

$$\log l = \log(ab)^{\frac{1}{2}}$$

$$l = \sqrt{ab}$$

Example 3: Prove that $\lim_{x\to\infty}\left(\frac{a^{\frac{1}{x}}+b^{\frac{1}{x}}+c^{\frac{1}{x}}+d^{\frac{1}{x}}}{4}\right)^x = (abcd)^{\frac{1}{4}}$.

Solution: Let $l = \lim_{x\to\infty}\left(\frac{a^{\frac{1}{x}}+b^{\frac{1}{x}}+c^{\frac{1}{x}}+d^{\frac{1}{x}}}{4}\right)^x$

Taking $\frac{1}{x} = y$, when $x\to\infty$, $y\to 0$

$$l = \lim_{y\to 0}\left(\frac{a^y+b^y+c^y+d^y}{4}\right)^{\frac{1}{y}} \qquad [1^\infty]$$

$$\log l = \lim_{y\to 0}\frac{1}{y}\log\left(\frac{a^y+b^y+c^y+d^y}{4}\right)$$

$$= \lim_{y\to 0}\frac{\log\left(\dfrac{a^y+b^y+c^y+d^y}{4}\right)}{y} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{y\to 0}\left(\frac{4}{a^y+b^y+c^y+d^y}\right)\left(\frac{a^y\log a+b^y\log b+c^y\log c+d^y\log d}{4}\right)$$

[Applying L'Hospital's rule]

$$= \frac{\log a+\log b+\log c+\log d}{4}$$

$$= \frac{1}{4}\log(abcd)$$

Hence, $\log l = \log(abcd)^{\frac{1}{4}}$

$$l = (abcd)^{\frac{1}{4}}$$

Example 4: Prove that $\lim_{x\to\infty}\left(\dfrac{ax+1}{ax-1}\right)^x = e^{\frac{2}{a}}$.

Solution: Let $l = \lim_{x\to\infty}\left(\dfrac{ax+1}{ax-1}\right)^x$

$$= \lim_{x\to\infty}\left(\frac{1+\dfrac{1}{ax}}{1-\dfrac{1}{ax}}\right)^x \qquad [1^\infty]$$

$$\log l = \lim_{x\to\infty} x\log\left(\frac{1+\dfrac{1}{ax}}{1-\dfrac{1}{ax}}\right)$$

$$= \lim_{x\to\infty}\frac{ax}{a}\left[\log\left(1+\frac{1}{ax}\right)-\log\left(1-\frac{1}{ax}\right)\right]$$

$$= \lim_{x\to\infty}\frac{1}{a}\left[\log\left(1+\frac{1}{ax}\right)^{ax}+\log\left(1-\frac{1}{ax}\right)^{-ax}\right]$$

$$= \frac{1}{a}(\log e+\log e) = \frac{1}{a}(1+1) = \frac{2}{a} \qquad \left[\because \lim_{x\to\infty}\left(1+\frac{1}{ax}\right)^{ax} = \right.$$

$$\log l = \frac{2}{a}$$

Hence,

$$l = e^{\frac{2}{a}}$$

Example 5: Prove that $\lim\limits_{x \to a}\left(2-\frac{x}{a}\right)^{\tan\frac{\pi x}{2a}} = e^{\frac{2}{\pi}}$.

Solution: Let $l = \lim\limits_{x \to a}\left(2-\frac{x}{a}\right)^{\tan\frac{\pi x}{2a}}$ $\qquad [1^\infty]$

$$\log l = \lim_{x \to a}\tan\left(\frac{\pi x}{2a}\right)\log\left(2-\frac{x}{a}\right)$$

$$= \lim_{x \to a}\frac{\log\left(2-\frac{x}{a}\right)}{\cot\left(\frac{\pi x}{2a}\right)} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x \to a}\frac{1}{\left(2-\frac{x}{a}\right)}\left(-\frac{1}{a}\right)\frac{1}{\left(-\operatorname{cosec}^2\frac{\pi x}{2a}\right)\left(\frac{\pi}{2a}\right)} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \frac{2}{\pi}$$

Hence,

$$\log l = \frac{2}{\pi}$$

$$l = e^{\frac{2}{\pi}}$$

Example 6: Prove that $\lim\limits_{x \to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}} = e^{\frac{1}{3}}$.

Solution: Let $l = \lim\limits_{x \to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$ $\qquad [1^\infty] \quad \left[\because \lim\limits_{x \to 0}\frac{\tan x}{x} = 1\right]$

$$\log l = \lim_{x \to 0}\frac{1}{x^2}\log\left(\frac{\tan x}{x}\right)$$

$$= \lim_{x \to 0}\frac{\log\left(\frac{\tan x}{x}\right)}{x^2} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x \to 0}\frac{x}{\tan x}\left(\frac{x\sec^2 x - \tan x}{x^2}\right)\cdot\frac{1}{2x} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 0} \frac{x\sec^2 x - \tan x}{2x^3} \qquad \left[\frac{0}{0}\right] \qquad \left[\because \lim_{x\to 0} \frac{x}{\tan x} = 1\right]$$

$$= \lim_{x\to 0} \frac{\sec^2 x + x\cdot 2\sec^2 x \tan x - \sec^2 x}{6x^2} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 0} \frac{\sec^2 x}{3} \cdot \frac{\tan x}{x} = \frac{1}{3}$$

Hence,

$$\log l = \frac{1}{3}$$

$$l = e^{\frac{1}{3}}.$$

Example 7: Prove that $\lim\limits_{x\to 0} \left(\dfrac{\sinh x}{x}\right)^{\frac{1}{x^2}} = e^{\frac{1}{6}}$.

Solution: Let $l = \lim\limits_{x\to 0} \left(\dfrac{\sinh x}{x}\right)^{\frac{1}{x^2}}$ $\qquad [1^\infty] \qquad \left[\because \lim\limits_{x\to 0} \dfrac{\sinh x}{x} = 1\right]$

$$\log l = \lim_{x\to 0} \frac{1}{x^2} \log\left(\frac{\sinh x}{x}\right)$$

$$= \lim_{x\to 0} \frac{\log\left(\frac{\sinh x}{x}\right)}{x^2} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{x}{\sinh x}\left(\frac{x\cosh x - \sinh x}{x^2}\right)\cdot \frac{1}{2x} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 0} \frac{x\cosh x - \sinh x}{2x^3} \quad \left[\frac{0}{0}\right] \qquad \left[\because \lim_{x\to 0} \frac{x}{\sinh x} = 1\right]$$

$$= \lim_{x\to 0} \frac{x\sinh x + \cosh x - \cosh x}{6x^2} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 0} \frac{1}{6} \cdot \frac{\sinh x}{x} = \frac{1}{6} \qquad \left[\because \lim_{x\to 0} \frac{\sinh x}{x} = 1\right]$$

Hence, $\log l = \dfrac{1}{6}$

$$l = e^{\frac{1}{6}}$$

Example 8: Prove that $\lim\limits_{x\to 0} \left(\dfrac{1}{x}\right)^{1-\cos x} = 1$.

Solution: Let $l = \lim\limits_{x\to 0} \left(\dfrac{1}{x}\right)^{1-\cos x}$ $\qquad [\infty^0]$

$$\log l = \lim_{x\to 0}(1-\cos x)\log\left(\frac{1}{x}\right)$$

$$= \lim_{x\to 0}\left(2\sin^2\frac{x}{2}\right)(-\log x)$$

$$= \lim_{x\to 0}\frac{2\left(\sin\frac{x}{2}\right)^2\left(\frac{x}{2}\right)^2}{\left(\frac{x}{2}\right)^2}(-\log x)$$

$$= \lim_{x\to 0}\frac{x^2(-\log x)}{2} \qquad \left[\because \lim_{x\to 0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)=1\right]$$

$$= \lim_{x\to 0}\frac{1}{2}\frac{(-\log x)}{\left(\frac{1}{x^2}\right)} \qquad \left[\frac{\infty}{\infty}\right]$$

$$= \frac{1}{2}\lim_{x\to 0}\left(\frac{\frac{-1}{x}}{\frac{-2}{x^3}}\right) \qquad \text{[Applying L'Hospital's rule]}$$

$$= \frac{1}{2}\lim_{x\to 0}\left(\frac{x^2}{2}\right) = 0$$

Hence, $\log l = 0$

$$l = e^0 = 1$$

Example 9: Prove that $\lim\limits_{x\to\infty} e^{\frac{\sinh^{-1}x}{\cosh^{-1}x}} = e.$

Solution: Let $l = \lim\limits_{x\to\infty} e^{\frac{\sinh^{-1}x}{\cosh^{-1}x}}$

$$= \lim_{x\to\infty}\left(e^{\sinh^{-1}x}\right)^{\frac{1}{\cosh^{-1}x}} \qquad [\infty^0]$$

$$\log l = \lim_{x\to\infty}\frac{\sinh^{-1}x}{\cosh^{-1}x}\cdot\log e$$

$$= \lim_{x\to\infty}\frac{\log(x+\sqrt{x^2+1})}{\log(x+\sqrt{x^2-1})} \qquad \left[\frac{\infty}{\infty}\right]$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x+\sqrt{x^2+1}}\left(1+\frac{1}{2\sqrt{x^2+1}}\cdot 2x\right)}{\frac{1}{x+\sqrt{x^2-1}}\left(1+\frac{1}{2\sqrt{x^2-1}}\cdot 2x\right)}$$ [Applying L'Hospital's rule]

$$= \lim_{x\to\infty} \frac{\sqrt{x^2-1}}{\sqrt{x^2+1}} = \lim_{x\to\infty} \sqrt{\frac{1-\frac{1}{x^2}}{1+\frac{1}{x^2}}} = 1$$

Hence,
$$\log l = 1$$
$$l = e^1 = e.$$

Example 10: Prove that $\lim_{x\to\infty}\left(\frac{1}{x}\right)^{\frac{1}{x}} = 1.$

Solution: Let $l = \lim_{x\to\infty}\left(\frac{1}{x}\right)^{\frac{1}{x}}$ $[0^0]$

$$\log l = \lim_{x\to\infty}\frac{1}{x}\log\left(\frac{1}{x}\right) = \lim_{x\to\infty}\frac{-\log x}{x} \qquad \left[\frac{\infty}{\infty}\right]$$

$$= -\lim_{x\to\infty}\frac{\frac{1}{x}}{1}$$ [Applying L'Hospital's rule]

$$= 0$$

Hence, $\log l = 0$

$$l = e^0 = 1$$

Example 11: Prove that $\lim_{x\to 1}(1-x^2)^{\frac{1}{\log(1-x)}} = e.$

Solution: Let $l = \lim_{x\to 1}(1-x^2)^{\frac{1}{\log(1-x)}}$ $[0^0]$

$$\log l = \lim_{x\to 1}\frac{1}{\log(1-x)}\log(1-x^2) \qquad \left[\frac{\infty}{\infty}\right]$$

$$= \lim_{x\to 1}\frac{\frac{-2x}{(1-x^2)}}{\frac{1}{(1-x)}(-1)}$$ [Applying L'Hospital's rule]

$$= \lim_{x\to 1}\frac{2x(1-x)}{(1-x)(1+x)} = \lim_{x\to 1}\frac{2x}{1+x} = 1$$

$$\log l = 1$$

Hence, $$l = e$$

Example 12: Prove that $\lim\limits_{x\to 0} \dfrac{e^x}{\left[\left(1+\dfrac{1}{x}\right)^x\right]^x} = 1$.

Solution: Let $l = \lim\limits_{x\to 0}\left[\left(1+\dfrac{1}{x}\right)^x\right]^x$

$$= \lim_{x\to 0}\left(1+\frac{1}{x}\right)^{x^2} \qquad (\infty^0)$$

$$\log l = \lim_{x\to 0} x^2 \log\left(1+\frac{1}{x}\right)$$

$$= \lim_{x\to 0}\frac{\log\left(1+\frac{1}{x}\right)}{\frac{1}{x^2}} \qquad \left[\frac{\infty}{\infty}\right]$$

$$= \lim_{x\to 0}\frac{\frac{1}{1+\frac{1}{x}}\left(-\frac{1}{x^2}\right)}{-\frac{2}{x^3}} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 0}\frac{x}{2\left(1+\frac{1}{x}\right)} = \lim_{x\to 0}\frac{x^2}{2(x+1)} = 0$$

Hence,

$$\log l = 0$$

$$l = e^0 = 1$$

$$\lim_{x\to 0}\left[\left(1+\frac{1}{x}\right)^x\right]^x = 1$$

Hence,

$$\lim_{x\to 0}\frac{e^x}{\left[\left(1+\frac{1}{x}\right)^x\right]^x} = \frac{e^0}{1} = \frac{1}{1} = 1.$$

Example 13: Prove that $\lim_{x\to 0}\frac{1-x^{\sin x}}{x\log x}=-1$.

Solution: Let $l_1=\lim_{x\to 0}x^{\sin x}$ $\quad [0^0]$

$$\log l_1=\lim_{x\to 0}\sin x\cdot\log x=\lim_{x\to 0}\frac{\log x}{\operatorname{cosec} x}\qquad\left[\frac{\infty}{\infty}\right]$$

$$=\lim_{x\to 0}\frac{\frac{1}{x}}{-\operatorname{cosec} x\cot x}\qquad\text{[Applying L'Hospital's rule]}$$

$$=\lim_{x\to 0}-\frac{\sin^2 x}{x\cos x}=-\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2\cdot\frac{x}{\cos x}=0\qquad\left[\because \lim_{x\to 0}\frac{\sin x}{x}=1\right]$$

$$\log l_1=0,\ l_1=e^0=1,\ \therefore\ \lim_{x\to 0}x^{\sin x}=1\qquad\ldots(1)$$

Let $l_2=\lim_{x\to 0}x\log x$ $\quad [0\times\infty]$

$$=\lim_{x\to 0}\frac{\log x}{\frac{1}{x}}\qquad\left[\frac{\infty}{\infty}\right]$$

$$=\lim_{x\to 0}\frac{\frac{1}{x}}{-\frac{1}{x^2}}\qquad\text{[Applying L'Hospital's rule]}$$

$$=\lim_{x\to 0}(-x)=0\qquad\ldots(2)$$

Let $l=\lim_{x\to 0}\frac{1-x^{\sin x}}{x\log x}$ $\quad \left[\frac{0}{0}\right]$ $\qquad$ [Using Eqs (1) and (2)]

$$=\lim_{x\to 0}\frac{1-e^{\sin x\log x}}{x\log x}$$

$$\lim_{x\to 0}\frac{1-x^{\sin x}}{x\log x}=\lim_{x\to 0}\frac{-e^{\sin x\log x}\left(\frac{\sin x}{x}+\cos x\cdot\log x\right)}{1+\log x}\qquad\text{[Applying L'Hospital's rule]}$$

$$=\lim_{x\to 0}\frac{-x^{\sin x}\left[\left(\frac{\sin x}{x}\right)\cdot\frac{1}{\log x}+\cos x\right]}{\frac{1}{\log x}+1}\qquad\ldots(3)$$

[Dividing numerator and denominator by $\log x$]

$$=-\frac{1\left(1\cdot\frac{1}{\infty}+\cos 0\right)}{\frac{1}{\infty}+1}=-1\qquad\left[\text{Using Eq. (1) and }\lim_{x\to 0}\frac{\sin x}{x}=1\right]$$

2.10.8 Type 6 : Using Expansion

In some cases, it is difficult to differentiate the numerator or denominator, or in some cases, power of x in the denominator is very large. In such cases, we use expansion of the function to find the limit.

Example 1: Prove that $\lim\limits_{x\to 0}\dfrac{\sin^{-1}x-x}{x^3}=\dfrac{1}{6}$.

Solution: Let $l=\lim\limits_{x\to 0}\dfrac{\sin^{-1}x-x}{x^3}$

$$=\lim_{x\to 0}\frac{\left(x+\frac{x^3}{6}+\frac{3}{40}x^5+\ldots\right)-x}{x^3}$$

$$=\lim_{x\to 0}\left(\frac{1}{6}+\frac{3}{40}x^2+\ldots\right)=\frac{1}{6}.$$

Example 2: Evaluate $\lim\limits_{x\to 0}\dfrac{\sin x-\tan^{-1}x}{x^2\log(1+x)}$.

Solution: Let $l=\lim\limits_{x\to 0}\dfrac{\sin x-\tan^{-1}x}{x^2\log(1+x)}$

$$=\lim_{x\to 0}\frac{\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots\right)-\left(x-\frac{x^3}{3}+\frac{x^5}{5}-\ldots\right)}{x^2\left(1-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots\right)}$$

$$=\lim_{x\to 0}\frac{x^3\left(\frac{1}{6}-\frac{23}{120}x^2+\ldots\right)}{x^3\left(1-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\ldots\right)}=\frac{1}{6}.$$

Example 3: Evaluate $\lim\limits_{x\to 0}\dfrac{\sin x\sin^{-1}x-x^2}{x^6}$.

Solution: Let $$l=\lim_{x\to 0}\frac{\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots\right)\left(x+\frac{x^3}{6}+\frac{3}{40}x^5+\ldots\right)-x^2}{x^6}$$

$$=\lim_{x\to 0}\frac{\left(x^2+\frac{x^4}{6}+\frac{3}{40}x^6-\frac{x^4}{6}-\frac{x^6}{36}-\frac{x^8}{80}+\frac{x^6}{120}+\frac{x^8}{720}+\ldots\right)-x^2}{x^6}$$

$$= \lim_{x\to 0} \frac{\frac{x^6}{18} + \text{Higher powers of } x}{x^6}$$

$$= \frac{1}{18}.$$

Example 4: Evaluate $\lim_{x\to 0} \dfrac{\tan x \tan^{-1} x - x^2}{x^6}$.

Solution: Let $l = \lim_{x\to 0} \dfrac{\left(x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots\right)\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\right) - x^2}{x^6}$

$$= \lim_{x\to 0} \frac{\left(x^2 - \frac{x^4}{3} + \frac{x^6}{5} - \frac{x^8}{7} + \frac{x^4}{3} - \frac{x^6}{9} + \frac{x^8}{15} + \frac{2}{15}x^6 - \frac{2}{45}x^8 + \ldots\right) - x^2}{x^6}$$

$$= \lim_{x\to 0} \frac{x^6\left(\frac{1}{5} - \frac{1}{9} + \frac{2}{15}\right) + \text{Higher powers of } x \text{ more than } 6}{x^6}$$

$$= \frac{2}{9}.$$

Example 5: Prove that $\lim_{x\to 0} \dfrac{e^x \sin x - x - x^2}{x^2 + x\log(1-x)} = -\dfrac{2}{3}$.

Solution: Let

$$l = \lim_{x\to 0} \frac{e^x \sin x - x - x^2}{x^2 + x\log(1-x)}$$

$$= \lim_{x\to 0} \frac{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right)\left(x - \frac{x^3}{3!} + \ldots\right) - x - x^2}{x^2 + x\left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots\right)}$$

$$= \lim_{x\to 0} \frac{\left(x - \frac{x^3}{3!} + x^2 - \frac{x^4}{3!} + \frac{x^3}{2!} - \frac{x^5}{2!3!} + \frac{x^4}{3} - \frac{x^6}{3!3!} + \ldots\right) - x - x^2}{x^2 - x^2 - \frac{x^3}{2} - \frac{x^4}{3} - \ldots}$$

$$= \lim_{x\to 0} \frac{\frac{x^3}{3} - \frac{x^5}{12} + \ldots}{-\frac{x^3}{2} - \frac{x^4}{3} - \ldots}$$

$$= \lim_{x\to 0} \frac{\frac{1}{3} - \frac{x^2}{12} + \ldots}{-\frac{1}{2} - \frac{x}{3} - \ldots}$$

$$= \frac{\frac{1}{3}}{-\frac{1}{2}} = -\frac{2}{3}.$$

Example 6: Prove that $\lim_{x\to 0} \frac{(1+x)^{\frac{1}{x}} - e}{x} = -\frac{e}{2}$.

Solution: Let
$$l = \lim_{x\to 0} \frac{(1+x)^{\frac{1}{x}} - e}{x}$$

$$= \lim_{x\to 0} \frac{e^{\frac{1}{x}\log(1+x)} - e}{x}$$

$$= \lim_{x\to 0} \frac{e^{\frac{1}{x}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\right)} - e}{x}$$

$$= \lim_{x\to 0} \frac{e^{\left(1 - \frac{x}{2} + \frac{x^2}{3} - \ldots\right)} - e}{x}$$

$$= \lim_{x\to 0} \frac{ee^{\left(-\frac{x}{2} + \frac{x^2}{3} - \ldots\right)} - e}{x}$$

$$= \lim_{x\to 0} \frac{e\left[1 + \left(-\frac{x}{2} + \frac{x^2}{3} - \ldots\right) + \frac{1}{2}\left(-\frac{x}{2} + \ldots\right)^2 + \ldots\right] - e}{x}$$

$$= \lim_{x\to 0} e\left(-\frac{1}{2} + \frac{x}{3} - \ldots\right)$$

$$= -\frac{e}{2}.$$

Example 7: Prove that $\lim_{x\to 0} \frac{(1+x)^{\frac{1}{x}} - e + \frac{ex}{2}}{x^2} = \frac{11e}{24}$.

Solution: Let $l = \lim_{x\to 0} \frac{(1+x)^{\frac{1}{x}} - e + \frac{ex}{2}}{x^2}$

$$= \lim_{x\to 0} \frac{e^{\frac{1}{x}\log(1+x)} - e + \frac{ex}{2}}{x^2}$$

$$= \lim_{x\to 0} \frac{e^{\frac{1}{x}\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\dots\right)} - e + \frac{ex}{2}}{x^2}$$

$$= \lim_{x\to 0} \frac{e^{\left(1-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\dots\right)} - e + \frac{ex}{2}}{x^2}$$

$$= \lim_{x\to 0} \frac{e e^{\left(-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\dots\right)} - e + \frac{ex}{2}}{x^2}$$

$$= \lim_{x\to 0} \frac{\left[1+\left(-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\dots\right)+\frac{1}{2!}\left(-\frac{x}{2}+\frac{x^2}{3}-\dots\right)^2+\dots\right] - e + \frac{ex}{2}}{x^2}$$

$$= \lim_{x\to 0} \frac{\left(e-\frac{ex}{2}+\frac{ex^2}{3}+\frac{ex^2}{8}-\frac{ex^3}{4}-\frac{ex^3}{6}+\dots\right) - e + \frac{ex}{2}}{x^2}$$

$$= \lim_{x\to 0} \left(\frac{11e}{24}-\frac{5e}{12}x+\dots\right) = \frac{11e}{24}$$

Example 8: Prove that $\lim_{x\to 0}\left[2\left(\frac{\cosh x-1}{x^2}\right)\right]^{\frac{1}{x^2}} = e^{\frac{1}{12}}$**.**

Solution: Let $l = \lim_{x\to 0}\left[2\left(\frac{\cosh x-1}{x^2}\right)\right]^{\frac{1}{x^2}}$

$$\log l = \lim_{x\to 0} \frac{1}{x^2}\log\left[2\left(\frac{\cosh x-1}{x^2}\right)\right]$$

$$= \lim_{x\to 0} \frac{1}{x^2}\log\left[2\left\{\frac{\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\dots\right)-1}{x^2}\right\}\right]$$

$$= \lim_{x\to 0} \frac{1}{x^2}\cdot\log\left[2\left\{\frac{\frac{x^2}{2}+\frac{x^4}{24}+\frac{x^6}{720}+\dots}{x^2}\right\}\right]$$

$$= x^2 + \frac{x^6}{36} - 2x\frac{x^3}{6} + 2x\cdot\frac{x^5}{120} + \ldots$$

$$= x^2 - \frac{x^4}{3} + \frac{2}{45}x^6 - \ldots$$

$$x\sin(\sin x) - \sin^2 x = \frac{x^6}{10} - \frac{2}{45}x^6 - \ldots$$

$$= \frac{1}{18}x^6 - \ldots$$

Hence,
$$l = \lim_{x\to 0}\frac{\frac{1}{18}x^6 - \text{Higher powers of } x}{x^6} = \frac{1}{18}.$$

Example 12: Find a, b, c if $\lim\limits_{x\to 0}\frac{x(a+b\cos x)-c\sin x}{x^5} = 1$.

Solution: $1 = \lim\limits_{x\to 0}\frac{x(a+b\cos x)-c\sin x}{x^5}$

$$= \lim_{x\to 0}\frac{x\left[a+b\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots\right)\right]-c\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots\right)}{x^5}$$

$$= \lim_{x\to 0}\frac{(a+b-c)x+x^3\left(-\frac{b}{2}+\frac{c}{6}\right)+x^5\left(\frac{b}{4!}-\frac{c}{5!}\right)+x^7\left(-\frac{b}{6!}+\frac{c}{7!}\right)+\ldots}{x^5}$$

$$= \lim_{x\to 0}\frac{(a+b-c)+x^2\left(-\frac{b}{2}+\frac{c}{6}\right)+x^4\left(\frac{b}{4!}-\frac{c}{5!}\right)+x^6\left(-\frac{b}{6!}+\frac{c}{7!}\right)+\ldots}{x^4}$$

But limit is given as 1.

$$a+b-c=0,\ -\frac{b}{2}+\frac{c}{6}=0,\ \frac{b}{24}-\frac{c}{120}=1,$$

$$a+b-c=0,\ -3b+c=0,\ 5b-c=120\cdot$$

Solving all the equations, we get $a = 120$, $b = 60$, $c = 180$.

Exercise 2.13

1. Prove that $\lim\limits_{x\to 0}\frac{x-\sin x}{\tan^3 x} = \frac{1}{6}$.

2. Prove that $\lim\limits_{x\to 0}\frac{\tan x - x}{x^2\tan x} = \frac{1}{3}$.

3. Prove that $\lim\limits_{x\to 0}\frac{\tan^2 x - x^2}{x^2\tan^2 x} = \frac{2}{3}$.

4. Prove that
$$\lim_{x\to 0}\frac{\sin x - x + \frac{x^3}{6}}{x^5} = \frac{1}{120}.$$

5. Prove that $\lim_{x\to 0}\dfrac{2\sinh x-2x}{x^2\sin x}=\dfrac{1}{3}$.

6. Prove that $\lim_{x\to 0}\dfrac{\log(1+x^3)}{\sin^3 x}=1$.

7. Prove that

$$\lim_{x\to 0}\frac{\tanh x-2\sin x+x}{x^5}=\frac{7}{60}.$$

8. Prove that $\lim_{x\to 0}\dfrac{e^x-e^{x\cos x}}{x-\sin x}=3$.

9. Prove that $\lim_{x\to 0}\dfrac{\sin x-\tan^{-1}x}{x^2\log(1+x)}=\dfrac{1}{3}$.

10. Prove that

$$\lim_{x\to 0}\frac{e^{x\sin x}-\cosh\left(x\sqrt{2}\right)}{x^4}=\frac{1}{6}.$$

11. Prove that

$$\lim_{x\to 0}\frac{2x^2-2e^{x^2}+2\cos\left(x^{\frac{3}{2}}\right)+\sin^3 x}{x^4}=-1.$$

12. Prove that $\lim_{x\to 0}\dfrac{\sinh x-x}{\sin x-x\cos x}=\dfrac{1}{2}$.

FORMULAE

n^{th} Order Derivative of Some Standard Functions

(i) $\dfrac{d^n}{dx^n}(ax+b)^m$

$$=\frac{a^n m!(ax+b)^{m-n}}{(m-n)!},\ \text{if } n<m$$

$= n!\,a^n$, if $n=m$

$= 0$, if $n>m$

(ii) $\dfrac{d^n}{dx^n}(ax+b)^{-m}$

$$=(-1)^n\frac{(m+n-1)!}{(m-1)!}\frac{a^n}{(ax+b)^{m+n}}$$

$\dfrac{d^n}{dx^n}(ax+b)^{-1}$

$$=(-1)^n n!\frac{a^n}{(ax+b)^{1+n}}$$

(iii) $\dfrac{d^n}{dx^n}\log(ax+b)$

$$=\frac{(-1)^{n-1}(n-1)!\,a^n}{(ax+b)^n}$$

(iv) $\dfrac{d^n}{dx^n}e^{ax}=a^n e^{ax}$

(v) $\dfrac{d^n}{dx^n}a^{mx}=m^n a^{mx}(\log a)^n$

(vi) $\dfrac{d^n}{dx^n}[\sin(ax+b)]$

$$=a^n\sin\left(\frac{n\pi}{2}+ax+b\right)$$

(vii) $\dfrac{d^n}{dx^n}[\cos(ax+b)]$

$$=a^n\cos\left(\frac{n\pi}{2}+ax+b\right)$$

(viii) $\dfrac{d^n}{dx^n}[e^{ax}\sin(bx+c)]$

$$=r^n e^{ax}\sin(bx+c+n\theta),$$

where $r=\sqrt{a^2+b^2}$, $\theta=\tan^{-1}\dfrac{b}{a}$.

(ix) $\dfrac{d^n}{dx^n}[e^{ax}\cos(bx+c)]$

$$=r^n e^{ax}\cos(bx+c+n\theta),$$

where $r=\sqrt{a^2+b^2}$, $\theta=\tan^{-1}\dfrac{b}{a}$.

Leibnitz's Theorem

$$y_n=(uv)_n$$
$$=u_n v+{}^nC_1 u_{n-1}v_1+{}^nC_2 u_{n-2}v_2+{}^nC_3 u_{n-3}v_3+\ldots+{}^nC_n uv_n.$$