2

INDETERMINATE FORMS

2.1 If a function is such that for a certain assigned value of the variable involved, its value cannot be found by simply substituting that value of the variable, the function is said to take an indeterminate form.

The forms $\frac{0}{0}, \frac{\infty}{\infty}, 0 \times \infty, \infty - \infty, 0^\circ, 1^\infty, \infty^\circ$ are called indeterminate forms. The limiting value of an indeterminate form is called its **true value.**

We know that, in general, $\underset{x \to a}{\text{Lt}} \frac{f(x)}{g(x)} = \frac{\underset{x \to a}{\text{Lt}} f(x)}{\underset{x \to a}{\text{Lt}} g(x)}$

But, if $\underset{x \to a}{\text{Lt}} f(x) = 0 = \underset{x \to a}{\text{Lt}} g(x)$, then $\underset{x \to a}{\text{Lt}} \frac{f(x)}{g(x)}$ reduces to the indeterminate form $\frac{0}{0}$.

This does not mean that $\underset{x \to a}{\text{Lt}} \frac{f(x)}{g(x)}$ does not exist. The only conclusion is that the method adopted is not suitable.

For example, if $f(x) = x^2 - 1$ and $g(x) = x - 1$ with $a = 1$

We have
$$\underset{x \to a}{\text{Lt}} \frac{f(x)}{g(x)} = \underset{x \to 1}{\text{Lt}} \frac{x^2 - 1}{x - 1} \qquad \left| \frac{0}{0} \text{ form} \right.$$
$$= \underset{x \to 1}{\text{Lt}} (x + 1) = 2$$

The forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$ are regarded as fundamental forms and all other forms are dealt with by converting them to one of these two forms. The methods of effecting such conversions will be explained at their proper places.

2.2 L'HOSPITAL'S RULE

State and prove L'Hospital's rule for determining the true value of the indeterminate form 0/0

Statement. *If $f(x)$ and $\phi(x)$ be two functions such that*

(i) they are continuous in the neighbourhood of a.

(ii) they are derivable in the deleted neighbourhood of a,

(iii) $f(a) = 0 = \phi(a)$, i.e., $\underset{x \to a}{Lt} f(x) = 0 = \underset{x \to a}{Lt} \phi(x)$,

and (iv) $\phi'(x) \neq 0$ *in the deleted neighbourhood of a, then* $\underset{x \to a}{Lt} \dfrac{f(x)}{\phi(x)} = \underset{x \to a}{Lt} \dfrac{f'(x)}{\phi'(x)} = \dfrac{f'(a)}{\phi'(a)}$ *provided the latter limit exists, whether finite* (*zero or non-zero*) *or infinite.*

Proof. Let $x = a + h$ be any point in the neighbourhood of $x = a$ whether h is positive or negative.

Now $$\frac{f(x)}{\phi(x)} = \frac{f(a+h)}{\phi(a+h)} = \frac{f(a) + hf'(a+\theta_1 h)}{\phi(a) + h\phi'(a+\theta_2 h)} \text{ where } 0 < \theta_1 < 1,\ 0 < \theta_2 < 1$$

[on using Lagrange's mean-value theorem]

$\because$ $f(a) = 0 = \phi(a)$ given

$$\therefore \quad \frac{f(x)}{\phi(x)} = \frac{hf'(a+\theta_1 h)}{h\phi'(a+\theta_2 h)} = \frac{f'(a+\theta_1 h)}{\phi'(a+\theta_2 h)}$$

$$\therefore \quad \underset{x \to a}{\text{Lt}} \frac{f(x)}{\phi(x)} = \underset{h \to 0}{\text{Lt}} \frac{f'(a+\theta_1 h)}{\phi'(a+\theta_2 h)} \quad [\because \ x = a + h, \ \therefore \ x \to a, h \to 0]$$

$$= \frac{f'(a)}{\phi'(a)} = \underset{x \to a}{\text{Lt}} \frac{f'(x)}{\phi'(x)}.$$

This proves the result.

Note. Generalisation of L'Hospital Rule

If $f(x)$ and $\phi(x)$ vanish at $x = a$ and have their first $(n-1)$ derivatives all zero at $x = a$ while their nth derivatives are finite and non-zero, then

$$\underset{x \to a}{\text{Lt}} \frac{f(x)}{\phi(x)} = \underset{x \to a}{\text{Lt}} \frac{f^n(x)}{\phi^n(x)} = \frac{f^n(a)}{\phi^n(a)},$$ provided this limit exists finitely or infinitely.

Working Rule for finding the value of $\underset{x \to a}{\text{Lt}} \dfrac{f(x)}{\phi x}$ where $f(a) = 0 = \phi(a)$

1. Differentiate the numerator and denominator **separately.**
2. Put $x = a$ and remove the word limit.
3. If the indeterminate form $\dfrac{0}{0}$ still persists, repeat the above process.

Caution. *It should be carefully noted that* $\dfrac{f(x)}{\phi(x)}$ *should not be differentiated as a fraction. The numerator and denominator should be differentiated* **separately.**

ILLUSTRATIVE EXAMPLES

Example 1. *Evaluate the following limits:*

(a) $\underset{x \to 0}{Lt} \dfrac{(1+x)^n - 1}{x}$

(b) $\underset{x \to 0}{Lt} \dfrac{xe^x - \log(1+x)}{x^2}$

(c) $\underset{x \to 0}{Lt} \dfrac{x - \tan x}{x^3}$

(d) $\underset{x \to 0}{Lt} \dfrac{\log(1-x^2)}{\log \cos x}$

(e) $\underset{x \to 1}{Lt} \dfrac{x^x - x}{1 - x + \log x}$.

Sol. (*a*) $\underset{x \to 0}{\text{Lt}} \dfrac{(1+x)^n - 1}{x}$ $\left[\text{form } \dfrac{0}{0}\right]$

[*Differentiate num. and denom. separately*]

$$= \underset{x \to 0}{\text{Lt}} \frac{n(1+x)^{n-1}}{1} = n.$$

(*b*) $\underset{x \to 0}{\text{Lt}} \dfrac{xe^x - \log(1+x)}{x^2}$

When we put $x = 0$, the given expression takes the form $\dfrac{0}{0}$.

So, we differentiate the num. and denom. separately.

$$\therefore \quad \underset{x \to 0}{\text{Lt}} \frac{xe^x - \log(1+x)}{x^2} \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= \underset{x \to 0}{\text{Lt}} \frac{xe^x + e^x \,.\, 1 - \dfrac{1}{1+x}}{2x} \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= \underset{x \to 0}{\text{Lt}} \frac{x \,.\, e^x + e^x \,.\, 1 + e^x + \dfrac{1}{(1+x)^2}}{2} = \frac{0+1+1+1}{2} = \frac{3}{2}.$$

(*c*) $\underset{x \to 0}{\text{Lt}} \dfrac{x - \tan x}{x^2}$ $\left[\text{form } \dfrac{0}{0}\right]$

$$= \underset{x \to 0}{\text{Lt}} \frac{1 - \sec^2 x}{3x^2} \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= \underset{x \to 0}{\text{Lt}} \frac{-2 \sec x \,.\, \sec x \tan x}{6x} = \underset{x \to 0}{\text{Lt}} \frac{-\sec^2 x \tan x}{3x} \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= \underset{x \to 0}{\text{Lt}} \left(-\frac{1}{3} \sec^2 x\right) \times \underset{x \to 0}{\text{Lt}} \frac{\tan x}{x} = -\frac{1}{3} \times 1 = -\frac{1}{3}.$$

(*d*) $\underset{x \to 0}{\text{Lt}} \dfrac{\log(1-x^2)}{\log \cos x}$ $\left[\text{form } \dfrac{0}{0}\right]$

$$= \underset{x \to 0}{\text{Lt}} \frac{\dfrac{-2x}{1-x^2}}{\dfrac{1}{\cos x} \,.\, (-\sin x)} = \underset{x \to 0}{\text{Lt}} \frac{2x \cos x}{(1-x^2) \sin x} \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= \underset{x \to 0}{\text{Lt}} \frac{2[-x \sin x + \cos x \,.\, 1]}{(1-x^2) \cos x - 2x \sin x} = \frac{2}{1} = 2.$$

(e) $\underset{x \to 1}{\text{Lt}} \dfrac{x^x - x}{1 - x + \log x}$ $\left[\text{form } \dfrac{0}{0}\right]$

$$= \underset{x \to 1}{\text{Lt}} \frac{x^x(1 + \log x) - 1}{-1 + \dfrac{1}{x}} \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= \underset{x \to 1}{\text{Lt}} \frac{x^x\left(\dfrac{1}{x}\right) + (1 + \log x) \,.\, x^x(1 + \log x)}{-\dfrac{1}{x^2}}$$

$$= \frac{1 + (1 + 0) \times 1 \times (1 + 0)}{-1} = -2.$$

$\because$ If $u = x^x$ then $\log u = x \log x$

Differentiating w.r.t. x

$$\frac{1}{u}\frac{du}{dx} = x \,.\, \frac{1}{x} + \log x \,.\, 1$$

$$\frac{du}{dx} = u(1 + \log x)$$

$$= x^x (1 + \log x)$$

Example 2. *Evaluate the following limits :*

(a) $\underset{x \to 0}{Lt} \dfrac{e^x - 2\cos x + e^{-x}}{x \sin x}$ (b) $\underset{x \to 0}{Lt} \dfrac{\tan x - x}{x^2 \tan x}$

(c) $\underset{x \to a}{Lt} \dfrac{x^a - a^x}{x^x - a^a}$ (d) $\underset{x \to 0}{Lt} \dfrac{e^x \sin x - x - x^2}{x^2 + x \log(1 - x)}$.

Sol. (a) $\underset{x \to 0}{\text{Lt}} \dfrac{e^x - 2\cos x + e^{-x}}{x \sin x}$ $\left[\text{form } \dfrac{0}{0}\right]$

[Differentiate num. and denom. separately]

$$= \underset{x \to 0}{\text{Lt}} \frac{e^x + 2\sin x - e^{-x}}{x \cos x + \sin x} \qquad \left[\text{form } \frac{0}{0}\right]$$

[Differentiate the num. and denom. separately again]

$$= \underset{x \to 0}{\text{Lt}} \frac{e^x + 2\cos x + e^{-x}}{-x \sin x + \cos x + \cos x} = \frac{1 + 2 + 1}{0 + 1 + 1} = \frac{4}{2} = 2.$$

(b) $\underset{x \to 0}{\text{Lt}} \dfrac{\tan x - x}{x^2 \tan x}$ $\left|\text{form } \dfrac{0}{0}\right.$

$$= \underset{x \to 0}{\text{Lt}} \frac{\tan x - x}{x^3} \cdot \frac{x}{\tan x}$$ | **Note this step**

$$= \underset{x \to 0}{\text{Lt}} \frac{\tan x - x}{x^3} \cdot \underset{x \to 0}{\text{Lt}} \frac{x}{\tan x} = \underset{x \to 0}{\text{Lt}} \frac{\tan x - x}{x^3}$$ $\left|\because \underset{x \to 0}{\text{Lt}} \dfrac{x}{\tan x} = 1\right.$

$\left|\text{form } \dfrac{0}{0}\right.$

$$= \underset{x \to 0}{\text{Lt}} \frac{\sec^2 x - 1}{3x^2}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{2\sec x \,.\, \sec x \tan x}{6x} = \underset{x \to 0}{\text{Lt}} \frac{1}{3} \sec^2 x \,.\, \frac{\tan x}{x} = \frac{1}{3} \times 1 \times 1 = \frac{1}{3}.$$

(c) $\underset{x \to a}{\text{Lt}} \dfrac{x^a - a^x}{x^x - a^a}$ | form $\dfrac{0}{0}$

$$= \underset{x \to 0}{\text{Lt}} \frac{ax^{a-1} - a^x \log a}{x^x(1+\log x) - 0}$$

$$= \frac{a \cdot a^{a-1} - a^a \log a}{a^a(1+\log a)}$$

$$= \frac{a^a - a^a \log a}{a^a(1+\log a)} = \frac{a^a(1-\log a)}{a^a(1+\log a)}$$

$$= \frac{1-\log a}{1+\log a}$$

$\because$ if $y = x^x$

$\log y = \log x^x = x \log x$

$\therefore \dfrac{1}{y} \cdot \dfrac{dy}{dx} = x \cdot \dfrac{1}{x} + \log x \cdot 1 = 1 + \log x$

or $\dfrac{dy}{dx} = y(1 + \log x) = x^x(1 + \log x)$

(d) $\underset{x \to 0}{\text{Lt}} \dfrac{e^x \sin x - x - x^2}{x^2 + x \log(1-x)}$ | form $\dfrac{0}{0}$

$$= \underset{x \to 0}{\text{Lt}} \frac{e^x \cos x + e^x \sin x - 1 - 2x}{2x + x\left(\dfrac{-1}{1-x}\right) + \log(1-x)}$$ | form $\dfrac{0}{0}$

$$= \underset{x \to 0}{\text{Lt}} \frac{-e^x \sin x + e^x \cos x + e^x \cos x + e^x \sin x - 2}{2 - \dfrac{(1-x) \cdot 1 - x(-1)}{(1-x)^2} - \dfrac{1}{1-x}}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{2e^x \cos x - 2}{2 - \dfrac{1}{(1-x)^2} - \dfrac{1}{1-x}}$$ | form $\dfrac{0}{0}$

$$= \underset{x \to 0}{\text{Lt}} \frac{-2e^x \sin x + 2e^x \cos x}{\dfrac{-2}{(1-x)^3} - \dfrac{1}{(1-x)^2}} = \frac{0+2}{-2-1} = -\frac{2}{3}.$$

Example 3. *Evaluate the following limits :*

(a) $\underset{x \to 0}{Lt} \dfrac{e^{ax} - e^{-ax}}{\log(1+bx)}$ (b) $\underset{x \to 0}{Lt} \dfrac{\sin^2 x - x^2}{x^2 \sin^2 x}$

Sol. (a) $\underset{x \to 0}{\text{Lt}} \dfrac{e^{ax} - e^{-ax}}{\log(1+bx)}$ | form $\dfrac{0}{0}$

$$= \underset{x \to 0}{\text{Lt}} \frac{ae^{ax} + ae^{-ax}}{\dfrac{1}{1+bx} \cdot b} = \frac{a+a}{b} = \frac{2a}{b}$$

(*b*) $\underset{x \to 0}{\text{Lt}} \dfrac{\sin^2 x - x^2}{x^2 \sin^2 x}$ | form $\dfrac{0}{0}$

$$= \underset{x \to 0}{\text{Lt}} \frac{\sin^2 x - x^2}{x^4} \cdot \left(\frac{x}{\sin x}\right)^2$$ | **Note carefully**

$$= \underset{x \to 0}{\text{Lt}} \frac{\sin^2 x - x^2}{x^4} \cdot \underset{x \to 0}{\text{Lt}} \left(\frac{x}{\sin x}\right)^2$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\sin^2 x - x^2}{x^4} \qquad \left[\because \underset{x \to 0}{\text{Lt}} \frac{x}{\sin x} = 1\right]$$

$$= \underset{x \to 0}{\text{Lt}} \frac{2 \sin x \cos x - 2x}{4x^3} = \underset{x \to 0}{\text{Lt}} \frac{\sin 2x - 2x}{4x^3}$$ | form $\dfrac{0}{0}$

$$= \underset{x \to 0}{\text{Lt}} \frac{2 \cos 2x - 2}{12x^2}$$ | form $\dfrac{0}{0}$

$$= \underset{x \to 0}{\text{Lt}} \frac{-4 \sin 2x}{24x}$$ | form $\dfrac{0}{0}$

$$= \underset{x \to 0}{\text{Lt}} \frac{-8 \cos 2x}{24} = \frac{-8}{24} = -\frac{1}{3}.$$

Example 4. *What is wrong with the following application of L'Hospital's rule?*

$$\underset{x \to 1}{Lt} \frac{x^3 + 3x - 4}{2x^2 + x - 3} = \underset{x \to 1}{Lt} \frac{3x^2 + 3}{4x + 1} = \underset{x \to 1}{Lt} \frac{6x}{4} = \frac{3}{2}.$$

Sol. $\underset{x \to 1}{\text{Lt}} \dfrac{x^3 + 3x - 4}{2x^2 + x - 3}$ | form $\dfrac{0}{0}$

$$= \underset{x \to 1}{\text{Lt}} \frac{3x^2 + 3}{4x + 1}$$

Now the expression $\dfrac{3x^2 + 3}{4x + 1}$ is not of the form $\dfrac{0}{0}$ as $x \to 1$. Therefore it is not correct to apply L'Hospital's rule to evaluate $\underset{x \to 1}{\text{Lt}} \dfrac{3x^2 + 3}{4x + 1}$.

In fact, $\underset{x \to 1}{\text{Lt}} \dfrac{3x^2 + 3}{4x + 1} = \dfrac{3 + 3}{4 + 1} = \dfrac{6}{5}$.

Example 5. (*a*) *For what value of a does* $\dfrac{\sin 2x + a \sin x}{x^3}$ *tend to a finite limit l as* $x \to 0$? *When a has this value, what is the value of l ?*

(*b*) *Find the values of a and b in order that* $\underset{x \to 0}{Lt} \dfrac{x(1 - a \cos x) + b \sin x}{x^3}$ *may be equal to* $\dfrac{1}{3}$.

Sol. (*a*) $\underset{x \to 0}{\text{Lt}} \dfrac{\sin 2x + a \sin x}{x^3}$ $\left| \text{form } \dfrac{0}{0} \right.$

$$= \underset{x \to 0}{\text{Lt}} \frac{2 \cos 2x + a \cos x}{3x^2} \qquad ...(1)$$

The denominator of (1) $\to 0$ as $x \to 0$ but (1) $\to a$ finite limit l. (given)

$\therefore$ The numerator $(2 \cos 2x + a \cos x)$ must $\to 0$ as $x \to 0$

$$\therefore \qquad 2 + a = 0 \quad \text{or} \quad a = -2$$

With this value of a,

$$(1) \qquad = \underset{x \to 0}{\text{Lt}} \frac{2 \cos 2x - 2 \cos x}{3x^2} \qquad \left| \text{form } \frac{0}{0} \right.$$

$$= \underset{x \to 0}{\text{Lt}} \frac{-4 \sin 2x + 2 \sin x}{6x} \qquad \left| \text{form } \frac{0}{0} \right.$$

$$= \underset{x \to 0}{\text{Lt}} \frac{-8 \cos 2x + 2 \cos x}{6} = \frac{-8+2}{6} = -1$$

$$\therefore \qquad l = -1.$$

(*b*) $\underset{x \to 0}{\text{Lt}} \dfrac{x(1 - a \cos x) + b \sin x}{x^3}$ $\left| \text{form } \dfrac{0}{0} \right.$

$$= \underset{x \to 0}{\text{Lt}} \frac{1.(1 - a \cos x) + x(a \sin x) + b \cos x}{3x^2} \qquad ...(1)$$

The denominator of (1) $\to 0$ as $x \to 0$ but (1) $\to a$ finite limit $\frac{1}{3}$. (given)

$\therefore$ The numerator $(1 - a \cos x + ax \sin x + b \cos x)$ must $\to 0$ as $x \to 0$.

$$\therefore \qquad 1 - a + b = 0 \qquad ...(2)$$

Also, if the relation (2) holds, then

$$(1) \qquad = \underset{x \to 0}{\text{Lt}} \frac{1 - a \cos x + ax \sin x + b \cos x}{3x^2} \text{ is of the form } \frac{0}{0}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{a \sin x + a \sin x + ax \cos x - b \sin x}{6x} \qquad \left| \text{form } \frac{0}{0} \right.$$

$$= \underset{x \to 0}{\text{Lt}} \frac{a \cos x + a \cos x + a \cos x - ax \sin x - b \cos x}{6}$$

$$= \frac{3a - b}{6} = \frac{1}{3} \qquad \text{(given)}$$

$$\therefore \qquad 3a - b = 2 \qquad ...(3)$$

From (2) and (3), $\quad a = \dfrac{1}{2},\ b = -\dfrac{1}{2}.$

TEST YOUR KNOWLEDGE

Evaluate the following limits (1 – 26):

1. $\lim\limits_{x \to 3} \dfrac{x^2 - 9}{x - 3}$.

2. $\lim\limits_{x \to -2} \dfrac{x + 2}{x^2 - 4}$

3. $\lim\limits_{x \to 0} \dfrac{\sin ax}{\tan bx}$

4. $\lim\limits_{x \to 1} \dfrac{x^2 - 2x + 1}{2x^3 - 3x^2 + 1}$

5. $\lim\limits_{x \to 0} \dfrac{x - \sin x}{\tan^3 x}$

6. $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{3x^2}$

7. $\lim\limits_{x \to 0} \dfrac{\sin x^2}{x}$

8. $\lim\limits_{x \to 1} \dfrac{x - 1}{\log x - \sin \pi x}$

9. $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{x \log (1 + x)}$

10. $\lim\limits_{x \to 1} \dfrac{\log x}{x - 1}$

11. $\lim\limits_{x \to 0} \dfrac{\cosh x - \cos x}{x \sin x}$

12. $\lim\limits_{x \to 0} \dfrac{x \cos x - \log (1 + x)}{x^2}$

13. $\lim\limits_{x \to 0} \dfrac{a \sin x - \sin ax}{x(\cos x - \cos ax)}$

14. $\lim\limits_{x \to 0} \dfrac{a^x - 1}{b^x - 1}$, $a > 0$, $b > 0$

15. $\lim\limits_{x \to 0} \dfrac{1 - \sqrt{1 - x^2}}{x^2}$

16. $\lim\limits_{x \to 0} \dfrac{\log (1 + x^3)}{\sin^3 x}$

17. $\lim\limits_{x \to 0} \dfrac{\log (1 + kx^2)}{1 - \cos x}$

18. $\lim\limits_{x \to 0} \dfrac{e^x - e^{-x} - 2x}{x^2 \sin x}$

19. $\lim\limits_{x \to 0} \dfrac{e^x \sin x - x - x^2}{x^3}$

20. $\lim\limits_{\theta \to 0} \dfrac{\sin \theta - \theta \cos \theta}{\sin \theta - \theta}$

21. $\lim\limits_{x \to \frac{1}{2}} \dfrac{\cos^2 \pi x}{e^{2x} - 2ex}$

22. $\lim\limits_{x \to 0} \dfrac{\sin 2x + 2 \sin^2 x - 2 \sin x}{\cos x - \cos^2 x}$

23. $\lim\limits_{x \to 0} \dfrac{e^x - e^{-x} - 2 \log (1 + x)}{x \sin x}$

24. $\lim\limits_{x \to 0^+} \dfrac{3^x - 2^x}{\sqrt{x}}$

25. $\lim\limits_{x \to 0} \dfrac{1 + \sin x - \cos x + \log (1 - x)}{x \tan^2 x}$

26. $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\left(\frac{\pi}{2} - x\right)^2 \cdot \sin x}{\cos^2 x}$

27. What is wrong with the following application of L'Hospital's rule?

(*i*) $\lim\limits_{x \to 1} \dfrac{x^4 - 4x^3 + 3}{3x^2 - x - 2} = \lim\limits_{x \to 1} \dfrac{4x^3 - 12x^2}{6x - 1} = \lim\limits_{x \to 1} \dfrac{12x^2 - 24x}{6} = \dfrac{12 - 24}{6} = -2$

(*ii*) $\lim\limits_{x \to 0} \dfrac{x^2 + 2x}{x^2 - \sin x} = \lim\limits_{x \to 0} \dfrac{2x + 2}{2x - \cos x} = \lim\limits_{x \to 0} \dfrac{2}{2 + \sin x} = \dfrac{2}{2 + 0} = 1$

28. Find the values of a and b in order that

$$\lim_{x \to 0} \frac{x(1 + a \cos x) - b \sin x}{x^3} \text{ may be equal to 1.}$$

29. Find the values of a, b and c so that

$$\lim_{x \to 0} \frac{{}^{x} - b \cos x + ce^{-x}}{x \sin x} \text{ may be equal to 2.}$$

Answers

1. 6	**2.** $-\frac{1}{4}$	**3.** $\frac{a}{b}$	**4.** $\frac{1}{3}$	**5.** $\frac{1}{6}$
6. $\frac{1}{6}$	**7.** 0	**8.** $\frac{1}{1+\pi}$	**9.** $\frac{1}{2}$	**10.** 1
11. 1	**12.** $\frac{1}{2}$	**13.** $\frac{a}{3}$	**14.** $\log_b a$	**15.** $\frac{1}{2}$
16. 1	**17.** $2k$	**18.** $\frac{1}{3}$	**19.** $\frac{1}{3}$	**20.** -2
21. $\frac{\pi^2}{2e}$	**22.** 4	**23.** 1	**24.** 0	**25.** $-\frac{1}{2}$
26. 1				

27. (*i*) $\lim_{x \to 1} \frac{4x^3 - 12x^2}{6x - 1}$ is not $\frac{0}{0}$ form (*ii*) $\lim_{x \to 0} \frac{2x + 2}{2x - \cos x}$ is not $\frac{0}{0}$ form.

28. $a = -\frac{5}{2}$, $b = -\frac{3}{2}$ **29.** $a = 1$, $b = 2$, $c = 1$.

2.3 TRUE VALUE OF THE INDETERMINATE FORM $\frac{\infty}{\infty}$

If $f(x)$ and $\phi(x)$ be two functions such that

$$\underset{x \to a}{Lt}\, f(a) = \infty \text{ and } \underset{x \to a}{Lt}\, \phi(x) = \infty, \text{ then } \underset{x \to a}{Lt} \frac{f(x)}{\phi(x)} = \underset{x \to a}{Lt} \frac{f'(x)}{\phi'(x)}.$$

Note 1. The above result is also true when $x \to \infty$.

Note 2. In most of the problems of the form $\frac{\infty}{\infty}$, it is necessary to change it into the form $\frac{0}{0}$ at the proper stage, otherwise the process will never terminate. **[Remember]**

ILLUSTRATIVE EXAMPLES

Example. *Evaluate the following limits :*

(*a*) $\underset{x \to 0}{Lt} \frac{\log x^2}{\cot x^2}$ (*b*) $\underset{\theta \to \pi/2}{Lt} \frac{\log(\theta - \pi/2)}{\tan \theta}$

(*c*) $\underset{x \to 0}{Lt} \frac{\operatorname{cosec} x}{\log x}$ (*d*) $\underset{x \to 0+}{Lt} \frac{\log \tan x}{\log x}$.

Sol. (*a*) $\displaystyle \mathop{\text{Lt}}_{x \to 0} \frac{\log x^2}{\cot x^2} = \mathop{\text{Lt}}_{x \to 0} \frac{2 \log x}{\cot x^2}$ | form $\dfrac{\infty}{\infty}$ $\because$ $\log 0 = -\infty, \cot 0 = \infty$

[Differentiate the num. and denom. separately]

$$= \mathop{\text{Lt}}_{x \to 0} \frac{2 \cdot \frac{1}{x}}{-\operatorname{cosec}^2 x^2 \cdot 2x} = \mathop{\text{Lt}}_{x \to 0} -\frac{1}{x^2 \operatorname{cosec}^2 x^2}$$

$$= \mathop{\text{Lt}}_{x \to 0} \frac{-\sin^2 x^2}{x^2}$$ | form $\dfrac{0}{0}$ **[Note this step]**

$$= \mathop{\text{Lt}}_{x \to 0} \frac{-2 \sin x^2 \cdot \cos x^2 \cdot (2x)}{2x} = \mathop{\text{Lt}}_{x \to 0} -\sin(2x^2) = 0.$$

(*b*) $\displaystyle \mathop{\text{Lt}}_{\theta \to \pi/2} \frac{\log(\theta - \pi/2)}{\tan \theta}$ | form $\dfrac{\infty}{\infty}$ $\because$ $\log 0 = -\infty, \tan \dfrac{\pi}{2} = \infty$

$$= \mathop{\text{Lt}}_{\theta \to \pi/2} \frac{\frac{1}{\theta - \pi/2}}{\sec^2 \theta}$$ | form $\dfrac{0}{0}$

$$= \mathop{\text{Lt}}_{\theta \to \pi/2} \frac{\cos^2 \theta}{\theta - \pi/2}$$ | form $\dfrac{0}{0}$ **[Note this step]**

$$= \mathop{\text{Lt}}_{\theta \to \pi/2} \frac{-2 \cos \theta \sin \theta}{1} = \mathop{\text{Lt}}_{\theta \to \pi/2} -\sin 2\theta = -\sin \pi = 0$$

(*c*) $\displaystyle \mathop{\text{Lt}}_{x \to 0} \frac{\operatorname{cosec} x}{\log x}$ | form $\dfrac{\infty}{\infty}$ $\because$ $\operatorname{cosec} 0 = -\infty, \log 0 = -\infty$

$$= \mathop{\text{Lt}}_{x \to 0} \frac{-\operatorname{cosec} x \cot x}{\frac{1}{x}}$$ | form $\dfrac{\infty}{\infty}$

$$= \mathop{\text{Lt}}_{x \to 0} \frac{-x \cos x}{\sin^2 x}$$ | form $\dfrac{0}{0}$ **[Note this step]**

$$= \mathop{\text{Lt}}_{x \to 0} \frac{-\cos x + x \sin x}{2 \sin x \cos x} = \frac{-1}{0} = -\infty.$$

(*d*) $\displaystyle \mathop{\text{Lt}}_{x \to 0+} \frac{\log \tan x}{\log x}$ | form $\dfrac{\infty}{\infty}$

$$= \mathop{\text{Lt}}_{x \to 0+} \frac{\frac{1}{\tan x} \cdot \sec^2 x}{\frac{1}{x}} = \mathop{\text{Lt}}_{x \to 0+} \frac{x}{\sin x \cos x}$$

$$= \mathop{\text{Lt}}_{x \to 0+} \frac{2x}{\sin 2x}$$ | form $\dfrac{\infty}{\infty}$

$$= \mathop{\text{Lt}}_{x \to 0+} \frac{2}{2 \cos 2x} = \frac{2}{2} = 1.$$

TEST YOUR KNOWLEDGE

Evaluate the following limits (1 – 12):

1. $\lim_{x \to \infty} \dfrac{\log x}{x}$

2. $\lim_{x \to 0^+} \dfrac{\log \sin x}{\cot x}$

3. $\lim_{x \to 0^+} \dfrac{\log \sin x}{\log x}$

4. $\lim_{x \to 0^+} \dfrac{\log x}{\cot x}$

5. $\lim_{x \to 0} \dfrac{\log \sin ax}{\log \sin bx}$, $(a, b > 0)$

6. $\lim_{x \to \infty} \dfrac{\log x}{x^m}$ $(m > 0)$

7. $\lim_{x \to a} \dfrac{\log (x - a)}{\log (e^x - e^a)}$

8. $\lim_{x \to \frac{\pi}{2}} \dfrac{\tan x}{\tan 3x}$

9. $\lim_{x \to 1^-} \dfrac{\log (1 - x)}{\cot (\pi x)}$

10. $\lim_{x \to \left(\frac{\pi}{2}\right)^-} \dfrac{\sec x}{1 + \tan x}$

11. $\lim_{x \to 0^+} \dfrac{\log (x^2 + 2x)}{\log x}$

12. $\lim_{x \to 0^+} \dfrac{\log (e^x - 1)}{\log x}$

Answers

1. 0	2. 0	3. 1	4. 0	5. 1
6. 0	7. 1	8. 3	9. 0	10. 1
11. 1	12. 1			

2.4 OTHER INDETERMINATE FORMS

The indeterminate forms $\infty - \infty$, $0 \times \infty$, 0^0, ∞^0 and 1^∞ can be reduced to any one of the two indeterminate forms $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$ by a suitable simplification and hence can be evaluated by L'Hospital's rule. This is illustrated by the following examples:

ILLUSTRATIVE EXAMPLES

[Form $0 \times \infty$]

Example 1. *Evaluate the following limits:*

(a) $\underset{x \to 1}{Lt}\ (1 - x) \cdot \tan \dfrac{\pi x}{2}$

(b) $\underset{x \to 0}{Lt}\ \sin x \cdot \log x$

(c) $\underset{x \to 0}{Lt}\ x^m (\log x)^n$, *where m and n are + ve integers*

(d) $\underset{x \to \infty}{Lt}\ x^n e^{-x}$, *where n is a + ve integer.*

Sol. (*a*) $\underset{x \to 1}{\text{Lt}} (1-x) \cdot \tan \dfrac{\pi x}{2}$ | form $0 \times \infty$

$$= \underset{x \to 1}{\text{Lt}} \frac{1-x}{\cot \dfrac{\pi x}{2}} \qquad \left| \text{form } \frac{\infty}{\infty} \quad \because \quad \cot \frac{\pi}{2} = 0 \right.$$

$$= \underset{x \to 1}{\text{Lt}} \frac{-1}{-\operatorname{cosec}^2 \dfrac{\pi x}{2} \cdot \dfrac{\pi}{2}} = \frac{1}{\dfrac{\pi}{2}} = \frac{2}{\pi}.$$

(*b*) $\underset{x \to 0}{\text{Lt}} \sin x \log x$ | form $0 \times \infty$

$$= \underset{x \to 0}{\text{Lt}} \frac{\log x}{\operatorname{cosec} x}$$

form $\dfrac{\infty}{\infty}$ [**Note this step,** *when one of the two factors in the form* $0 \times \infty$ *is* $\log x$, **do not take** $\log x$ *in the denominator*.]

$$= \underset{x \to 0}{\text{Lt}} \frac{\dfrac{1}{x}}{-\operatorname{cosec} x \cot x} \qquad \left| \text{form } \frac{\infty}{\infty} \right.$$

$$= \underset{x \to 0}{\text{Lt}} \frac{-\sin^2 x}{x \cos x} \qquad \left| \text{form } \frac{\infty}{\infty} \right. \text{ [Note this step]}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{-2 \sin x \cos x}{\cos x \cdot 1 - x \sin x} = \frac{0}{1-0} = 0.$$

(*c*) $\underset{x \to 0}{\text{Lt}} x^m (\log x)^n$ | form $0 \times \infty$

$$= \underset{x \to 0}{\text{Lt}} \frac{(\log x)^n}{x^{-m}} \qquad \left| \text{form } \frac{\infty}{\infty} \right.$$

$$= \underset{x \to 0}{\text{Lt}} \frac{n(\log x)^{n-1} \cdot \dfrac{1}{x}}{-m \cdot x^{-m-1}} = \underset{x \to 0}{\text{Lt}} \frac{n(\log x)^{n-1}}{(-m)\, x^{-m}} \qquad \left| \text{form } \frac{\infty}{\infty} \right.$$

$$= \underset{x \to 0}{\text{Lt}} \frac{n(n-1)(\log x)^{n-2} \cdot \dfrac{1}{x}}{(-m)^2\, x^{-m-1}} = \underset{x \to 0}{\text{Lt}} \frac{n(n-1) \cdot (\log x)^{n-2}}{(-m)^2 \cdot x^{-m}}$$

$$= \ldots\ldots = \underset{x \to 0}{\text{Lt}} \frac{n(n-1) \cdot 3.2.1}{(-m)^n \cdot x^{-m}} = \underset{x \to 0}{\text{Lt}} \frac{n!\, x^m}{(-m)^n} = 0$$

(*d*) $\underset{x \to \infty}{\text{Lt}} x^n e^{-x}$ | form $\infty \times 0$

$$= \underset{x \to \infty}{\text{Lt}} \frac{x^n}{e^x} \qquad \left| \text{form } \frac{\infty}{\infty} \right.$$

[Differentiate the num. and denom. separately n times]

$$= \underset{x \to \infty}{\text{Lt}} \frac{n!}{e^x} = \frac{n!}{\infty} = 0.$$

[Form $\infty - \infty$]

Example 2. *Evaluate the following limits :*

(a) $\underset{x \to 0}{Lt} \left(\dfrac{1}{x^2} - \dfrac{1}{x \tan x} \right)$ (b) $\underset{x \to 0}{Lt} \left(\dfrac{1}{x} - \cot x \right)$

(c) $\underset{x \to 0}{Lt} \left(\cot^2 x - \dfrac{1}{x^2} \right)$ (d) $\underset{x \to \pi/2}{Lt} (\sec x - \tan x)$

(e) $\underset{x \to 0}{Lt} \left[\dfrac{1}{x} - \dfrac{1}{x^2} \log (1+x) \right]$ (f) $\underset{x \to 0}{Lt} \left[\dfrac{\pi}{4x} - \dfrac{\pi}{2x(e^{\pi x} + 1)} \right]$

(g) $\underset{x \to 4}{Lt} \left[\dfrac{1}{\log (x-3)} - \dfrac{1}{x-4} \right]$ (h) $\underset{x \to 0}{Lt} \left(\dfrac{1}{e^x - 1} - \dfrac{1}{x} \right)$.

Sol. (a) $\underset{x \to 0}{\text{Lt}} \left(\dfrac{1}{x^2} - \dfrac{1}{x \tan x} \right)$ | form $\infty - \infty$

$$= \underset{x \to 0}{\text{Lt}} \frac{\tan x - x}{x^2 \tan x} \qquad \left| \text{form } \frac{0}{0} \right.$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\tan x - x}{x^2} \cdot \frac{x}{\tan x}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\tan x - x}{x^3} \times \underset{x \to 0}{\text{Lt}} \frac{x}{\tan x}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\tan x - x}{x^3} \qquad \left[\because \underset{x \to 0}{\text{Lt}} \frac{x}{\tan x} = 1 \right] \qquad \left| \text{form } \frac{0}{0} \right.$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\sec^2 x - 1}{3x^2} \qquad \left| \text{form } \frac{0}{0} \right.$$

$$= \underset{x \to 0}{\text{Lt}} \frac{2 \sec x . \sec x \tan x}{6x} = \underset{x \to 0}{\text{Lt}} \frac{\sec^2 x}{3} . \underset{x \to 0}{\text{Lt}} \frac{\tan x}{x} = \frac{1}{3} \times 1 = \frac{1}{3}.$$

(b) $\underset{x \to 0}{\text{Lt}} \left(\dfrac{1}{x} - \cot x \right)$ | form $\infty - \infty$

$$= \underset{x \to 0}{\text{Lt}} \left(\frac{1}{x} - \frac{\cos x}{\sin x} \right) = \underset{x \to 0}{\text{Lt}} \frac{\sin x - x \cos x}{x \sin x} \qquad \left| \text{form } \frac{0}{0} \right.$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\sin x - x \cos x}{x^2} . \frac{x}{\sin x}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\sin x - x \cos x}{x^2} . \underset{x \to 0}{\text{Lt}} \frac{x}{\sin x}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\sin x - x \cos x}{x^2} \qquad \left[\because \underset{x \to 0}{\text{Lt}} \frac{x}{\sin x} = 1 \right] \qquad \left| \text{form } \frac{0}{0} \right.$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\cos x - \cos x + x \sin x}{2x} = \underset{x \to 0}{\text{Lt}} \frac{\sin x}{2} = 0.$$

(c) $\underset{x \to 0}{\text{Lt}} \left(\cot^2 x - \frac{1}{x^2} \right)$

$$= \underset{x \to 0}{\text{Lt}} \left(\frac{\cos^2 x}{\sin^2 x} - \frac{1}{x^2} \right) \qquad | \text{ form } \infty - \infty$$

$$= \underset{x \to 0}{\text{Lt}} \frac{x^2 \cos^2 x - \sin^2 x}{x^2 \sin^2 x}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{x^2 \cos^2 x - \sin^2 x}{x^4} \cdot \frac{x^2}{\sin^2 x} \qquad \left| \text{ form } \frac{0}{0} \right.$$

$$= \underset{x \to 0}{\text{Lt}} \frac{x^2 \cos^2 x - \sin^2 x}{x^4} \cdot \underset{x \to 0}{\text{Lt}} \left(\frac{x}{\sin x} \right)^2$$

$$= \underset{x \to 0}{\text{Lt}} \frac{x^2 \left(\frac{1 + \cos 2x}{2} \right) - \left(\frac{1 - \cos 2x}{2} \right)}{x^4} \times 1$$

$$= \underset{x \to 0}{\text{Lt}} \frac{x^2 (1 + \cos 2x) - (1 - \cos 2x)}{2x^4}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{x^2 - 1 + (x^2 + 1) \cos 2x}{2x^4}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{2x + 2x \cos 2x - 2(x^2 + 1) \sin 2x}{8x^3} \qquad \left| \text{ form } \frac{0}{0} \right.$$

$$= \underset{x \to 0}{\text{Lt}} \frac{2 + 2 \cos 2x - 4x \sin 2x - 4x \sin 2x - 4(x^2 + 1) \cos 2x}{24x^2} \qquad \left| \text{ form } \frac{0}{0} \right.$$

$$= \underset{x \to 0}{\text{Lt}} \frac{2 - 8x \sin 2x - (4x^2 + 2) \cos 2x}{24x^2}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{-8 \sin 2x - 16x \cos 2x - 8x \cos 2x + 2(4x^2 + 2) \sin 2x}{48x} \qquad \left| \text{ form } \frac{0}{0} \right.$$

$$= \underset{x \to 0}{\text{Lt}} \frac{-24x \cos 2x + (8x^2 - 4) \sin 2x}{48x}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{-24 \cos 2x + 48x \sin 2x + 16x \sin 2x + 2(8x^2 - 4) \cos 2x}{48} \qquad \left| \text{ form } \frac{0}{0} \right.$$

$$= \frac{-24 + 0 + 0 - 8}{48} = \frac{-32}{48} = -\frac{2}{3}.$$

(d) $\underset{x \to \frac{\pi}{2}}{\text{Lt}} (\sec x - \tan x)$ $\qquad | \text{ form } \infty - \infty$

$$= \underset{x \to \frac{\pi}{2}}{\text{Lt}} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x} \right) = \underset{x \to \frac{\pi}{2}}{\text{Lt}} \frac{1 - \sin x}{\cos x} \qquad \left| \text{ form } \frac{0}{0} \right.$$

$$= \underset{x \to \frac{\pi}{2}}{\text{Lt}} \frac{-\cos x}{-\sin x} = \underset{x \to \frac{\pi}{2}}{\text{Lt}} \cot x = 0.$$

(e) $$\underset{x\to 0}{\text{Lt}}\left[\frac{1}{x}-\frac{\log(1+x)}{x^2}\right]=\underset{x\to 0}{\text{Lt}}\frac{x-\log(1+x)}{x^2} \qquad \left|\ \text{form } \frac{0}{0}\right.$$

$$=\underset{x\to 0}{\text{Lt}}\frac{1-\dfrac{1}{1+x}}{2x}=\underset{x\to 0}{\text{Lt}}\frac{1}{2(1+x)}=\frac{1}{2}.$$

(f) $$\underset{x\to 0}{\text{Lt}}\left[\frac{\pi}{4x}-\frac{\pi}{2x(e^{\pi x}+1)}\right]=\underset{x\to 0}{\text{Lt}}\ \pi\left[\frac{e^{\pi x}+1-2}{4x(e^{\pi x}+1)}\right] \qquad \left|\ \text{form } \frac{0}{0}\right.$$

$$=\underset{x\to 0}{\text{Lt}}\ \pi\left[\frac{\pi e^{\pi x}}{4x(e^{\pi x}\,.\,\pi)+4\,.\,(e^{\pi x}+1)}\right]=\frac{\pi^2}{0+4(1+1)}=\frac{\pi^2}{8}.$$

(g) $$\underset{x\to 4}{\text{Lt}}\left[\frac{1}{\log(x-3)}-\frac{1}{x-4}\right] \qquad |\ \text{form } \infty-\infty$$

$$=\underset{x\to 4}{\text{Lt}}\frac{(x-4)-\log(x-3)}{(x-4)\log(x-3)} \qquad \left|\ \text{form } \frac{0}{0}\right.$$

$$=\underset{x\to 4}{\text{Lt}}\frac{1-\dfrac{1}{x-3}}{(x-4)\,.\,\dfrac{1}{x-3}+\log(x-3}=\underset{x\to 4}{\text{Lt}}\frac{1-\dfrac{1}{x-3}}{1-\dfrac{1}{x-3}+\log(x-3)} \qquad \left|\ \text{form } \frac{0}{0}\right.$$

$$=\underset{x\to 4}{\text{Lt}}\frac{\dfrac{1}{(x-3)^2}}{\dfrac{1}{(x-3)^2}+\dfrac{1}{x-3}}=\frac{1}{1+1}=\frac{1}{2}.$$

(h) $$\underset{x\to 0}{\text{Lt}}\left(\frac{1}{e^x-1}-\frac{1}{x}\right) \qquad |\ \text{form } \infty-\infty$$

$$=\underset{x\to 0}{\text{Lt}}\frac{x-e^x+1}{x(e^x-1)} \qquad \left|\ \text{form } \frac{0}{0}\right.$$

$$=\underset{x\to 0}{\text{Lt}}\frac{1-e^x}{(e^x-1)+xe^x} \qquad \left|\ \text{form } \frac{0}{0}\right.$$

$$=\underset{x\to 0}{\text{Lt}}\frac{-e^x}{e^x+e^x+xe^x}=\frac{-1}{1+1+0}=-\frac{1}{2}.$$

[Forms 0^0, 1^∞, ∞^0]

Example 3. *Evaluate the following limits :*

(a) $\underset{x\to\infty}{Lt}\left(1+\dfrac{a}{x}\right)^x$

(b) $\underset{x\to 1}{Lt}\,(x)^{\frac{1}{1-x}}$

(c) $\underset{x\to 0}{Lt}\left(\dfrac{\tan x}{x}\right)^{\frac{1}{x^2}}$

(d) $\underset{x\to 0}{Lt}\left(\dfrac{\sinh x}{x}\right)^{\frac{1}{x^2}}$

(e) $\underset{x\to a}{Lt}\left(2-\dfrac{x}{a}\right)^{\tan\frac{\pi x}{2a}}$

(f) $\underset{x\to \pi/2}{Lt}(\sin x)^{\tan x}$

Sol. (*a*) Let $$l = \underset{x \to \infty}{\text{Lt}} \left(1+\frac{a}{x}\right)^x = l \text{ (say)}$$ | form 1^∞

$\therefore$ $$\log l = \underset{x \to \infty}{\text{Lt}} \log\left(1+\frac{a}{x}\right)^x$$ | Taking logs on both sides

$$= \underset{x \to \infty}{\text{Lt}} x \log\left(1+\frac{a}{x}\right)$$ | form $0 \times \infty$

$$= \underset{x \to \infty}{\text{Lt}} \frac{\log\left(1+\frac{a}{x}\right)}{\frac{1}{x}}$$ | form $\frac{0}{0}$

$$= \underset{x \to \infty}{\text{Lt}} \frac{\frac{1}{\left(1+\frac{a}{x}\right)}\left(-\frac{a}{x^2}\right)}{-\frac{1}{x^2}} = \underset{x \to \infty}{\text{Lt}} \frac{a}{1+\frac{a}{x}} = \frac{a}{1+0} = a$$

$\therefore$ $$l = e^a \quad \text{or} \quad \underset{x \to \infty}{\text{Lt}} \left(1+\frac{a}{x}\right)^x = e^a.$$

(*b*) Let $$l = \underset{x \to 1}{\text{Lt}} (x)^{\frac{1}{1-x}}$$ | form 1^∞

$\therefore$ $$\log l = \underset{x \to 1}{\text{Lt}} \log (x)^{\frac{1}{1-x}} = \underset{x \to 1}{\text{Lt}} \frac{1}{1-x} \log x = \underset{x \to 1}{\text{Lt}} \frac{\log x}{1-x}$$ | form $\frac{0}{0}$

$$= \underset{x \to 1}{\text{Lt}} \frac{\frac{1}{x}}{-1} = \underset{x \to 1}{\text{Lt}} \frac{-1}{x} = -1$$

$\therefore$ $$l = e^{-1} \quad \text{or} \quad \underset{x \to 1}{\text{Lt}} (x)^{\frac{1}{1-x}} = e^{-1}.$$

(*c*) $$\underset{x \to 0}{\text{Lt}} \left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}} = l \text{ (say)}$$ | form 1^∞

$\therefore$ $$\log l = \underset{x \to 0}{\text{Lt}} \log\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}} = \underset{x \to 0}{\text{Lt}} \frac{\log\left(\frac{\tan x}{x}\right)}{x^2}$$ | form $\frac{0}{0}$

$$= \underset{x \to 0}{\text{Lt}} \frac{\left[\frac{x}{\tan x} \cdot \frac{x \sec^2 x - \tan x}{x^2}\right]}{2x}$$ | L'Hospital's Rule

$$= \underset{x \to 0}{\text{Lt}} \frac{x \sec^2 x - \tan x}{2x^2 \tan x}$$ | form $\frac{0}{0}$

$$= \underset{x \to 0}{\text{Lt}} \frac{\sec^2 x + 2x \sec^2 x \tan x - \sec^2 x}{2x^2 \sec^2 x + 4x \tan x}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\sec^2 x \tan x}{x \sec^2 x + 2 \tan x} \qquad \left| \text{form } \frac{0}{0} \right.$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\sec^4 x + 2 \sec^2 x \tan^2 x}{2x \sec^2 x \tan x + \sec^2 x + 2 \sec^2 x} = \frac{1}{3}$$

$$\therefore \qquad l = e^{\frac{1}{3}} \quad \text{or} \quad \underset{x \to 0}{\text{Lt}} \left(\frac{\tan x}{x} \right)^{\frac{1}{x^2}} = e^{\frac{1}{3}}.$$

(*d*) Let $l = \underset{x \to 0}{\text{Lt}} \left(\frac{\sinh x}{x} \right)^{\frac{1}{x^2}}$ | form 1^∞

$$\therefore \quad \log l = \underset{x \to 0}{\text{Lt}} \frac{1}{x^2} \log \left(\frac{\sinh x}{x} \right) = \underset{x \to 0}{\text{Lt}} \frac{\log \left(\frac{\sinh x}{x} \right)}{x^2} \qquad \left| \text{form } \frac{0}{0} \right.$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\left[\frac{x}{\sinh x} \cdot \frac{x \cosh x - \sinh x}{x^2} \right]}{2x} = \underset{x \to 0}{\text{Lt}} \frac{x \cosh x - \sinh x}{2x^2 \sinh x} \qquad \left| \text{form } \frac{0}{0} \right.$$

$$= \underset{x \to 0}{\text{Lt}} \frac{x \sinh x + \cosh x - \cosh x}{4x \sinh x + 2x^2 \cosh x} = \underset{x \to 0}{\text{Lt}} \frac{\sinh x}{4 \sinh x + 2x \cosh x} \qquad \left| \text{form } \frac{0}{0} \right.$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\cosh x}{4 \cosh x + 2x \sinh x + 2 \cosh x} = \frac{1}{6}$$

$$\therefore \qquad l = e^{\frac{1}{6}}.$$

(*e*) Let $l = \underset{x \to a}{\text{Lt}} \left(2 - \frac{x}{a} \right)^{\tan \frac{\pi x}{2a}}$ | form 1^∞

$$\therefore \qquad \log l = \underset{x \to a}{\text{Lt}} \tan \frac{\pi x}{2a} \log \left(2 - \frac{x}{a} \right) \qquad | \text{ form } \infty \times 0$$

$$= \underset{x \to a}{\text{Lt}} \frac{\log \left(2 - \frac{x}{a} \right)}{\cot \frac{\pi x}{2a}} \qquad \left| \text{form } \frac{0}{0} \right.$$

$$= \underset{x \to a}{\text{Lt}} \frac{\frac{1}{2 - \frac{x}{a}} \cdot \frac{-1}{a}}{-\frac{\pi}{2a} \operatorname{cosec}^2 \frac{\pi x}{2a}} = \frac{2}{\pi}$$

$$\therefore \qquad l = e^{\frac{2}{\pi}} \quad \text{or} \quad \underset{x \to a}{\text{Lt}} \left(2 - \frac{x}{a} \right)^{\tan \frac{\pi x}{2a}} = e^{\frac{2}{\pi}}.$$

(*f*) Let $$l = \underset{x \to \pi/2}{\text{Lt}} (\sin x)^{\tan x}$$ | form 1^{∞}

$\therefore$ $$\log l = \underset{x \to \pi/2}{\text{Lt}} \tan x \log \sin x = \underset{x \to \pi/2}{\text{Lt}} \frac{\log \sin x}{\cot x}$$ | form $\frac{0}{0}$

$$= \underset{x \to \pi/2}{\text{Lt}} \frac{\frac{1}{\sin x} . \cos x}{-\operatorname{cosec}^2 x} = \frac{0}{-1} = 0$$

$\therefore$ $$l = e^0 = 1 \quad \text{or} \quad \underset{x \to \pi/2}{\text{Lt}} (\sin x)^{\tan x} = 1$$

Example 4. *Evaluate the following limits :*

(*a*) $\underset{x \to \infty}{Lt} (1+x)^{\frac{1}{x}}$

(*b*) $\underset{x \to 0}{Lt} (\operatorname{cosec} x)^{\frac{1}{\log x}}$

(*c*) $\underset{x \to 0}{Lt} \left(\frac{1}{x}\right)^{\tan x}$

(*d*) $\underset{x \to 0+}{Lt} (\cot x)^x$

(*e*) $\underset{x \to \pi/2}{Lt} (\tan x)^{\sin 2x}$

(*f*) $\underset{x \to 0}{Lt} \frac{\pi}{2} (\sin x)^{\tan^2 x}$.

Sol. (*a*) Let $$l = \underset{x \to \infty}{\text{Lt}} (1+x)^{\frac{1}{x}}$$ | form ∞^{0}

$\therefore$ $$\log l = \underset{x \to \infty}{\text{Lt}} \frac{1}{x} \log (1+x) = \underset{x \to \infty}{\text{Lt}} \frac{\log (1+x)}{x}$$ | form $\frac{\infty}{\infty}$

$$= \underset{x \to \infty}{\text{Lt}} \frac{\frac{1}{1+x}}{1} = \frac{1}{\infty} = 0.$$

$\therefore$ $$l = 0 = 1.$$

(*b*) Let $$l = \underset{x \to \infty}{\text{Lt}} (\operatorname{cosec} x)^{\frac{1}{\log x}}$$ | form ∞^{0}

$\therefore$ $$\log l = \underset{x \to 0}{\text{Lt}} \frac{1}{\log x} . \log \operatorname{cosec} x$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\log \operatorname{cosec} x}{\log x}$$ | form $\frac{\infty}{\infty}$

$$= \underset{x \to 0}{\text{Lt}} \frac{\frac{1}{\operatorname{cosec} x}}{\frac{1}{x}} (-\operatorname{cosec} x \cot x)$$

$$= \underset{x \to 0}{\text{Lt}} (-x \cot x) = \underset{x \to 0}{\text{Lt}} \frac{-x}{\tan x}$$ | form $\frac{\infty}{\infty}$

$$= \underset{x \to 0}{\text{Lt}} \frac{-1}{\sec^2 x} = -1$$

$\therefore$ $$l = e^{-1} = \frac{1}{e}.$$

(c) Let $$l = \underset{x \to 0}{\text{Lt}} \left(\frac{1}{x}\right)^{\tan x}$$ | form ∞^0

$$\log l = \underset{x \to 0}{\text{Lt}} \tan x \,.\, \log \frac{1}{x} = \underset{x \to 0}{\text{Lt}} \frac{-\log x}{\cot x}$$ | form $\frac{\infty}{\infty}$ **[Note this step]**

$$= \underset{x \to 0}{\text{Lt}} \frac{-\frac{1}{x}}{\operatorname{cosec}^2 x} = \underset{x \to 0}{\text{Lt}} \frac{\sin^2 x}{x}$$ | form $\frac{0}{0}$

$$= \underset{x \to 0}{\text{Lt}} \frac{2 \sin x \cos x}{1} = 0$$

$\therefore$ $$l = e^0 = 1.$$

(d) Let $$l = \underset{x \to 0+}{\text{Lt}} (\cot x)^x$$ | form ∞^0

$\therefore$ $$\log l = \underset{x \to 0}{\text{Lt}} x \log (\cot x)$$ | form $0 \times \infty$

$$= \underset{x \to 0+}{\text{Lt}} \frac{\log (\cot x)}{\frac{1}{x}}$$ | form $\frac{0}{0}$

$$= \underset{x \to 0+}{\text{Lt}} \frac{\frac{1}{\cot x} . (-\operatorname{cosec}^2 x)}{-\frac{1}{x^2}} = \underset{x \to 0+}{\text{Lt}} \frac{x^2 \tan x}{\sin^2 x} = \underset{x \to 0+}{\text{Lt}} \left(\frac{x}{\sin x}\right)^2 . \underset{x \to 0+}{\text{Lt}} \tan x$$

$$= (1)^2 \times 0 = 0$$

$\therefore$ $$l = e^0 = 1.$$

(e) Let $$l = \underset{x \to \pi/2}{\text{Lt}} (\tan x)^{\sin 2x}$$ | form ∞^0

$\therefore$ $$\log l = \underset{x \to \pi/2}{\text{Lt}} \sin 2x \log (\tan x)$$ | form $0 \times \infty$

$$= \underset{x \to \pi/2}{\text{Lt}} \frac{\log (\tan x)}{\operatorname{cosec} 2x}$$ | form $\frac{\infty}{\infty}$

$$= \underset{x \to \pi/2}{\text{Lt}} - \frac{\frac{1}{\tan x} . \sec^2 x}{2 \operatorname{cosec} 2x \cot 2x} = \underset{x \to \pi/2}{\text{Lt}} \frac{\frac{\cos x}{\sin x} . \frac{1}{\cos^2 x}}{-2 . \frac{1}{2 \sin x \cos x} . \cot 2x}$$

$$= \underset{x \to \pi/2}{\text{Lt}} - \tan 2x = 0$$

$\therefore$ $$l = e^0 = 1.$$

(f) $\underset{x \to 0}{\text{Lt}} \frac{\pi}{2} (\sin x)^{\tan^2 x} = \frac{\pi}{2} \underset{x \to 0}{\text{Lt}} (\sin x)^{\tan^2 x}$ ...(1)

Let $l = \underset{x \to 0}{\text{Lt}} (\sin x)^{\tan^2 x}$ | form 0^0

$\therefore$ $\log l = \underset{x \to 0}{\text{Lt}} \tan^2 x \log (\sin x)$ | form $0 \times \infty$

$$= \underset{x \to 0}{\text{Lt}} \frac{\log \sin x}{\cot^2 x} \qquad \Big| \text{ form } \frac{\infty}{\infty}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\frac{1}{\sin x} . \cos x}{2 \cot x . (-\operatorname{cosec}^2 x)} = \underset{x \to 0}{\text{Lt}} \frac{\cot x}{-2 \cot x \operatorname{cosec}^2 x}$$

$$= \underset{x \to 0}{\text{Lt}} \left(-\frac{1}{2} \sin^2 x \right) = 0 \quad \Rightarrow \quad l = e^0 = 1$$

$\therefore$ Form (1), $\underset{x \to 0}{\text{Lt}} \frac{\pi}{2} (\sin x)^{\tan^2 x} = \frac{\pi}{2} \times 1 = \frac{\pi}{2}$.

Example 5. *Evaluate the following limits :*

(a) $\underset{x \to 0}{Lt}\ x^x$ (b) $\underset{x \to 1}{Lt} (1 - x^2)^{\frac{1}{\log (1 - x)}}$

(c) $\underset{x \to 0+}{Lt} (\cot x)^{\sin x}$

Sol. (a) Let $l = \underset{x \to 0}{\text{Lt}}\ x^x$ | form 0^0

$\therefore$ $\log l = \underset{x \to 0}{\text{Lt}}\ x \log x = \underset{x \to 0}{\text{Lt}} \frac{\log x}{\frac{1}{x}}$ $\Big|$ form $\frac{\infty}{\infty}$

$$= \underset{x \to 0}{\text{Lt}} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \underset{x \to 0}{\text{Lt}} (-x) = 0$$

$\therefore$ $l = e^0 = 1.$

(b) Let $l = \underset{x \to 1}{\text{Lt}} (1 - x^2)^{\frac{1}{\log (1 - x)}}$ | form 0^0

$\therefore$ $\log l = \underset{x \to 1}{\text{Lt}} \frac{1}{\log (1 - x)} . \log (1 - x^2) = \underset{x \to 1}{\text{Lt}} \frac{\log (1 - x^2)}{\log (1 - x)}$ $\Big|$ form $\frac{\infty}{\infty}$

$$= \underset{x \to 1}{\text{Lt}} \frac{\frac{-2x}{1 - x^2}}{\frac{-1}{1 - x}} = \underset{x \to 1}{\text{Lt}} \frac{2x}{1 + x} = \frac{2}{1 + 1} = 1$$

$\therefore$ $l = e^1 = e.$

(c) Let

$$l = \underset{x \to 0+}{\text{Lt}} (\cot x)^{\sin x} \qquad | \text{ form } \infty^0$$

$$\log l = \underset{x \to 0+}{\text{Lt}} \sin x \log (\cot x) \qquad | \text{ form } 0 \times \infty$$

$$= \underset{x \to 0+}{\text{Lt}} \frac{\log (\cot x)}{\operatorname{cosec} x} \qquad \left| \text{ form } \frac{\infty}{\infty} \right.$$

$$= \underset{x \to 0+}{\text{Lt}} \frac{\frac{1}{\cot x} . (-\operatorname{cosec}^2 x)}{-\operatorname{cosec} x \cot x} = \underset{x \to 0+}{\text{Lt}} \frac{\operatorname{cosec} x}{\cot^2 x}$$

$$= \underset{x \to 0+}{\text{Lt}} \frac{\tan^2 x}{\sin x} \qquad \left| \text{ form } \frac{0}{0} \right.$$

$$= \underset{x \to 0+}{\text{Lt}} \frac{2 \tan x . \sec^2 x}{\cos x} = \frac{2 \times 0 \times 1}{1} = 0$$

$$\Rightarrow \qquad l = e^0 = 1.$$

TEST YOUR KNOWLEDGE

Evaluate the following limits:

1. $\lim\limits_{x \to 0^+} x \log x$
2. $\lim\limits_{x \to \infty} x \tan \frac{1}{x}$
3. $\lim\limits_{x \to 1} \sec \frac{\pi}{2x} \log x$
4. $\lim\limits_{x \to a} (a - x) \tan \frac{\pi x}{2a}$
5. $\lim\limits_{x \to \infty} 2^x \sin \frac{a}{2^x}$
6. $\lim\limits_{x \to 0} x \log \tan x$
7. $\lim\limits_{x \to \frac{\pi}{2}} \left(x - \frac{\pi}{2} \right) \tan x$
8. $\lim\limits_{x \to 0} (\operatorname{cosec} x - \cot x)$
9. $\lim\limits_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right)$
10. $\lim\limits_{x \to 0} \left(\frac{1}{x^2} - \operatorname{cosec}^2 x \right)$
11. $\lim\limits_{x \to 0} \left(\frac{a}{x} - \cot \frac{x}{a} \right)$
12. $\lim\limits_{x \to 1} \left(\frac{x}{x-1} - \frac{1}{\log x} \right)$
13. $\lim\limits_{x \to 1} \left(\frac{2}{x^2 - 1} - \frac{1}{x-1} \right)$
14. $\lim\limits_{x \to \frac{\pi}{2}} (2x \tan x - \pi \sec x)$
15. $\lim\limits_{x \to 0^+} x^x$
16. $\lim\limits_{x \to \frac{\pi}{2}} (\cos x)^{\cos x}$
17. $\lim\limits_{x \to 0^+} (\sin x)^{\tan x}$
18. $\lim\limits_{x \to 0} \left(\frac{1}{x} \right)^{2 \sin x}$

19. $\lim_{x \to 1} (1-x^2)^{\frac{1}{\log(1-x)}}$

20. $\lim_{x \to \infty} \left(\frac{\pi}{2} - \tan^{-1} x\right)^{1/x}$

21. $\lim_{x \to \frac{\pi}{2}} (\sec x)^{\cot x}$

22. $\lim_{x \to 0} \left(\frac{\sin x}{x}\right)^{\frac{1}{x}}$

23. $\lim_{x \to 0} \left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}$

24. $\lim_{x \to 0} (\cos x)^{\cot x}$

25. $\lim_{x \to 0} (\cos x)^{1/x^2}$

26. $\lim_{x \to 0} \left(\frac{a^x + b^x}{2}\right)^{1/x}$

27. $\lim_{x \to 0} (a^x + x)^{1/x}$

28. $\lim_{x \to 0} \left(\frac{1}{x}\right)^{1-\cos x}$

Answers

1. 0	**2.** 1	**3.** $\frac{2}{\pi}$	**4.** $\frac{2a}{\pi}$	**5.** a
6. 0	**7.** -1	**8.** 0	**9.** 0	**10.** $-\frac{1}{3}$
11. 0	**12.** $\frac{1}{2}$	**13.** $-\frac{1}{2}$	**14.** -2	**15.** 1
16. 1	**17.** 1	**18.** 1	**19.** e	**20.** 1
21. 1	**22.** 1	**23.** $e^{-1/6}$	**24.** 1	**25.** $\frac{1}{\sqrt{e}}$
26. $\sqrt{ab}$	**27.** ae	**28.** 1		

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