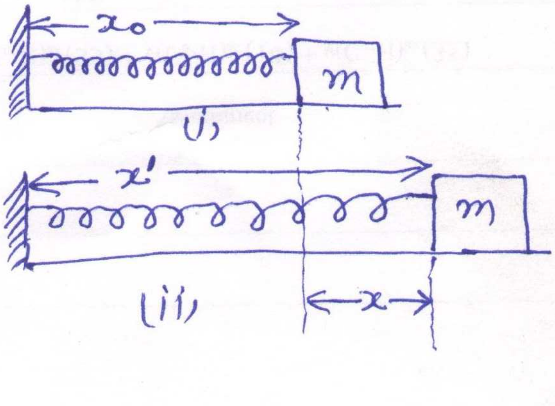
Wave Mechanics & Solid Mechanics (WMSM)

Periodic Motion:- A motion, that repeats itself after a regular interval of time is called Periodic motion.

Oscillatory Motion and Vibratory Motion:- When a physical quantity fluctuates periodically about its, equilibrium value, the process is called 'oscillation'. However when a physical object moves periodically about its equilibrium or mean position, then the process is called vibratory motion. One example of oscillation is fluctuation of charge across plates of a capacitor in LC circuit (called LC oscillations) and one example of vibratory motion is that of a simple pendulum. It should be noted that every vibration is a special case of oscillation.

Simple Harmonic Motion:- (SHM) Any motion that repeats itself after a fixed interval of time such that acceleration is always directly proportional to displacement from mean position and is directed toward mean position is called SHM.

Equation of SHM:- Consider an object of mass 'm' connected to a spring of equilibrium length 'x_0' as shown in fig (i). Let the object is pulled away from



equilibrium position and released. After releasing the object starts vibrating about its equilibrium position under the influence of restoring force (F). The direction of restoring force is always toward equilibrium position of object as it always tries to restore original length of spring. However the inertia of attached mass results in overshooting when object reaches equilibrium position. Let at any instant of time, extension in the length of spring (instantaneous length (x') − equilibrium length (x_0)) is x. Then according to Hooke's law, the variable restoring Force (F) is directly proportional to x. That is

$$F \propto x$$

$$\text{or } F = -kx \quad \text{——— (1)}$$

Where k is constant of proportionality, called Stiffness constant. It is also denoted by symbol S. Its unit is N/m. The negative sign in (1) indicates that F is always directed toward mean position.

Let a = acceleration of object of mass m

$$\therefore \; a = \frac{dv}{dt} = \frac{d^2x'}{dt^2}$$

$$= \frac{d^2}{dt^2}(x+x_0) \qquad (\because x' = x + x_0)$$

$$= \frac{d^2x}{dt^2} \quad \text{——— (2)} \quad (\because x_0 = \text{const}^t.)$$

$$\therefore \; F = ma = m\frac{d^2x}{dt^2} \quad \text{——— (3)} \quad (\text{Put in (1)})$$

WMSM

$\therefore \quad m\dfrac{d^2x}{dt^2} = -kx$

or $\quad \dfrac{d^2x}{dt^2} + \dfrac{k}{m}x = 0$

or $\quad \dfrac{d^2x}{dt^2} + \omega^2 x = 0$ ——— (4)

where $\quad \omega^2 = \dfrac{k}{m}$ ——— (5)

Equation (4) is standard equation of SHM. ω is a constant, called angular frequency of SHM.

To find the solution of equation (4), we put

$D = \dfrac{d}{dt}$ so that $D^2 = \dfrac{d^2}{dt^2}$

equation (4) can be re-written as follows:-

$D^2x + \omega^2 x = 0$

$\Rightarrow (D^2 + \omega^2)x = 0$

The corresponding auxiliary equation is given by

$D^2 + \omega^2 = 0$

$\therefore D^2 = -\omega^2$

$= i^2\omega^2$

$\therefore D = \pm i\omega$

Thus roots of auxiliary equation are $i\omega$ and $-i\omega$.

Hence solution of equation (4) is

$x = c_1 e^{i\omega t} + c_2 e^{-i\omega t}$ ——— (6)

where c_1, c_2 are constants of integration

we know that $e^{i\theta} = \cos\theta + i\sin\theta$

∴ Equation 6 can be written as

$$x = (C_1 + C_2)\cos\omega t + (C_1 + i C_2)\sin\omega t$$

or $x = A\cos\omega t + B\sin\omega t$ ——— (7)

where $A = C_1 + C_2$ and $B = C_1 + iC_2$ are new constants.

If we put $A = R\sin\theta$ ——— (8)
and $B = R\cos\theta$ ——— (9)

Then equation (8) simplifies as follows:-

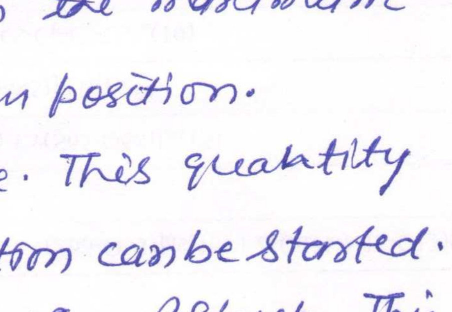
$$x = R[\sin\theta\cos\omega t + \cos\theta\sin\omega t]$$

$$\Rightarrow x = R\sin(\omega t + \theta) \quad\text{——— (10)}$$

Equation (10) is general solution of SHM. From (10), we see that $x_{max} = R = \sqrt{A^2+B^2}$ (using (8) & (9))

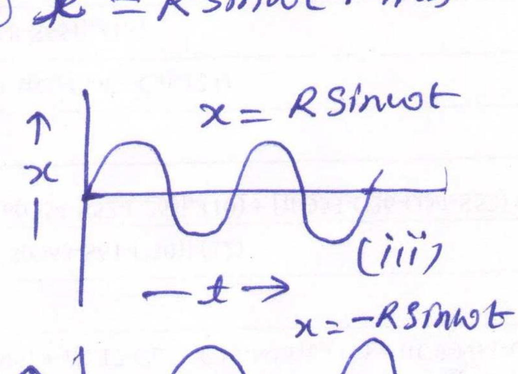
and $\theta = \tan^{-1}\left(\frac{A}{B}\right)$ (using (8) & (9))

R is called amplitude of SHM. It is the maximum displacement of object from its mean position. θ is called Epoch or initial phase. This quantity controlls the manner in which motion can be started. For example if θ = 0, then from (10) $x = R\sin\omega t$. This equation represents a SHM, which starts from mean position toward +X axis as shown in fig (iii)



(iii)

If $x = \pi$, then $x = R\sin(\omega t + \pi)$
$= -R\sin\omega t$



(iv)

In this case SHM starts from mean position toward −ve X axis as shown in fig (iv)

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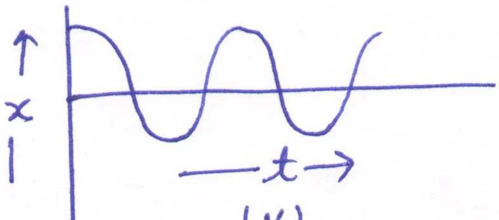
For $x = \pi/2$ we get $x = R\cos\omega t$



(V)

It is the case of in which SHM starts from extreme position toward +X axis. The reader can verify different other ways of starting SHM by taking other values of θ.

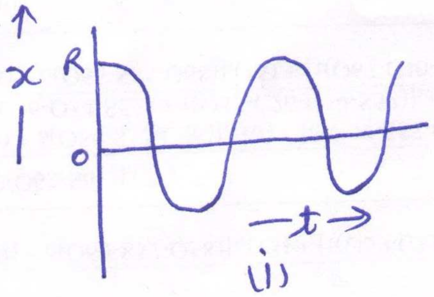
From (10), we can give a new definition of SHM as follows:- "It is a motion, that repeats itself periodically in sinusoidal manner".

Velocity and Acceleration of SHM:- The standard equation of SHM is

$$x = R\sin(\omega t + \theta) \quad \text{——} (1)$$

Let v is velocity of object executing SHM at any time.

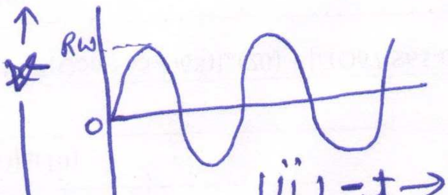
Thus
$$v = \frac{dx}{dt}$$
$$= \frac{d}{dt}(R\sin(\omega t+\theta))$$
$$= R\omega\cos(\omega t+\theta) \quad \text{——} (2)$$



(i)

Let a = acceleration of SHM

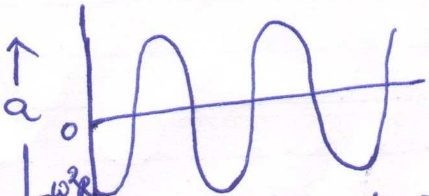
$$\therefore a = \frac{dv}{dt}$$
$$= \frac{d}{dt}(R\omega\cos(\omega t+\theta))$$
$$= -\omega^2 R\sin(\omega t+\theta)$$



(ii) —t→

or $a = -\omega^2 x$ —— (3)

(using (1))



(iii) —t→

If we assume that $\theta = \pi/2$, then, equations (1), (2), (3) will become

$x = R\cos\omega t$, $v = R\omega\sin\omega t$, $a = -\omega^2 x$
$= -\omega^2 R\sin\omega t$

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Thus we see that when an object executes SHM, then its velocity and accelaration execute SHO (Simple Harmonic Oscillations) with exactly same frequency as that of SHM. However velocity is $\pi/2$ out of phase from displacement and acceleration is out of phase from displacement by angle (π).

The ~~time perio~~ frequency of SHM (ν) is given by

$$\nu = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \text{ —— } ④ \quad \left(\because \omega^2 = \frac{k}{m}\right)$$

Time period of SHM is given by

$$T = \frac{1}{\nu} = 2\pi\sqrt{\frac{m}{k}} \text{ —— } ⑤$$

$$T = 2\pi\sqrt{\frac{\text{displacement}}{\text{acceleration}}}$$

Maximum displacement (amplitude) is

$$x_{max} = R \quad \text{(at extreme position)}$$

Maximum velocity (also called velocity amplitude) is

$$v_{max} = R\omega \quad \text{(at mean position)}$$

and Maximum acceleration (called acceleration amplitude) is

$$a_{max} = R\omega^2 \quad \text{(at extreme position)}$$

Kinetic Energy, Potential Energy and Total Energy of SHM:-

Let K = kinetic Energy of SHM

$$\therefore K = \frac{1}{2}mv^2$$

$$= \frac{1}{2}m\omega^2R^2\cos^2(\omega t+\theta) \text{ —— } ①$$

$$(\because x = R\sin(\omega t+\theta) \therefore v = R\omega\cos(\omega t+\theta))$$

Equation ① can also be re-written as follows

$$K = \frac{1}{2} m\omega^2 R^2 \left[1 - \sin^2(\omega t + \theta)\right]$$

$$= \frac{1}{2} m\omega^2 \left[R^2 - R^2 \sin^2(\omega t + \theta)\right]$$

$$K = \frac{1}{2} m\omega^2 \left[R^2 - x^2\right] \quad \text{——} ②$$

Equation ② shows variation of Kinetic Energy (K) with respect to displacement of SHM.

From ②, we see that when $x = 0$ then $K = \max = \frac{1}{2} m\omega^2 R^2$

and when $x = \pm R$ then $K = 0 = \text{minimum}$.

Thus KE of SHM is maximum at mean position and minimum at extreme positions.

Let U = potential energy of SHM at any time t, when displacement from mean position is x.

The restoring force on the object at this instant is

$$F = -kx$$

Small amount of work done by the applied force, F' ($F' = -F$) to extend the spring quasistatically by small distance dx is given by

$$dW = F' dx \cos 0° \qquad (\because \text{both } F' \,\&\, dx \text{ are in same direction})$$

$$= -F dx$$

$$= kx dx$$

$\therefore$ U = pot. Energy = work done to produce extension x

$$= \int_0^x dW$$

$$= k\int_0^x x\,dx$$

$$= \frac{1}{2}kx^2$$

$$\therefore \quad U = \frac{1}{2}kx^2 \text{ —— } (3) \qquad \text{OR } U = \frac{1}{2}m\omega^2x^2 \text{ —— } (3) \quad \left(\because \omega^2 = \frac{k}{m}\right)$$

Equation (3) gives potential energy of SHM w.r.t displacement from mean position.

at $x = 0$, $U = min = 0$

at $x = \pm R$, $U = max = \frac{1}{2}kR^2$

Let E = total mechanical energy of SHM

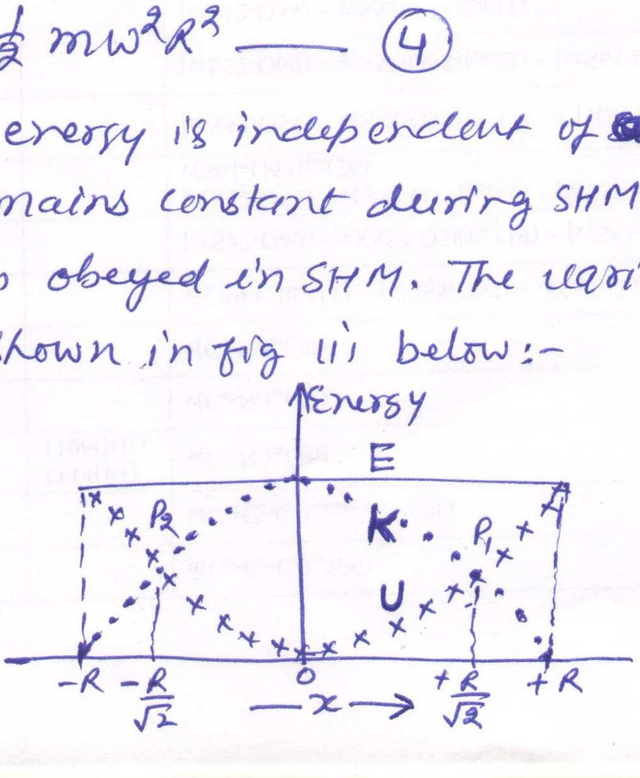
$$\therefore E = K + U$$

$$= \frac{1}{2}m\omega^2(R^2 - x^2) + \frac{1}{2}kx^2$$

$$= \frac{1}{2}m\omega^2(R^2 - x^2) + \frac{1}{2}m\omega^2x^2 \qquad \left(\because \omega^2 = \frac{k}{m}\right)$$

$$E = \frac{1}{2}m\omega^2R^2 \text{ —— } (4)$$

Thus Total energy is independent of displacement and always remains constant during SHM. Thus law of conservation of energy is obeyed in SHM. The variation of K, U & E with x is shown in fig (ii) below:-



(ii)

There are two points P_1 & P_2 on the ~~curves~~ graph where curves of KE & PE are intersecting. These points can be determined by putting $K = U$

$$\therefore \frac{1}{2}m\omega^2(R^2 - x^2) = \frac{1}{2}m\omega^2 x^2$$

$$\therefore R^2 - x^2 = x^2$$

$$\therefore 2x^2 = R^2$$

$$\therefore x = \pm \frac{R}{\sqrt{2}}$$

To understand how energies of SHM vary w.r.t. time, we rewrite equations (2), (3) & (4) as follows:-

$$K = \frac{1}{2}m\omega^2\left[R^2 - x^2\right]$$

$$= \frac{1}{2}m\omega^2\left[R^2 - R^2\sin^2(\omega t + \theta)\right]$$

$$= \frac{1}{2}m\omega^2R^2\left[\cos^2(\omega t + \theta)\right]$$

$$K = \frac{1}{4}m\omega^2R^2\left[1 + \cos(2\omega t + 2\theta)\right] \quad \text{--- (5)}$$

$$U = \frac{1}{2}m\omega^2x^2$$

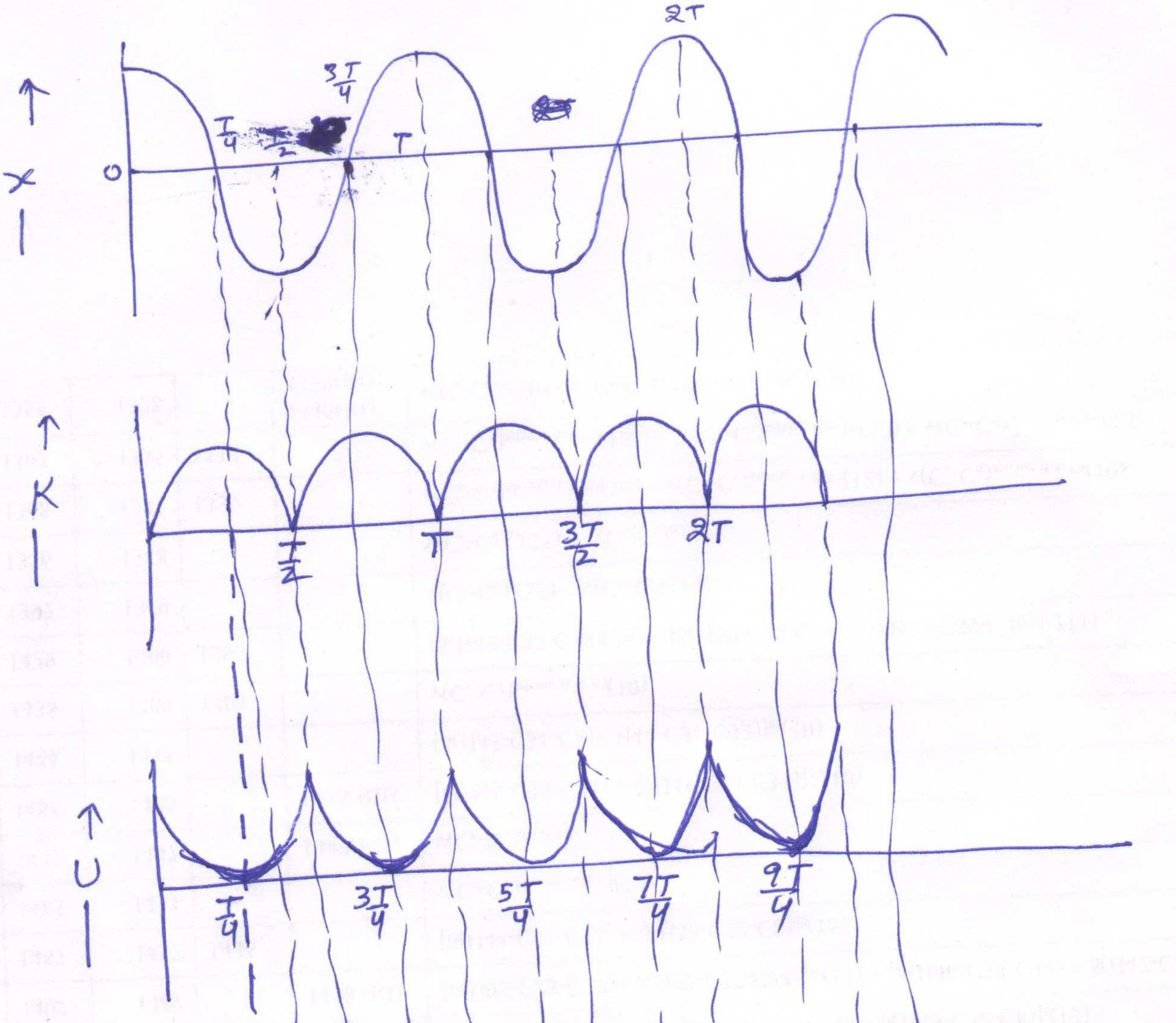
$$= \frac{1}{2}m\omega^2R^2\sin^2(\omega t + \theta)$$

$$= \frac{1}{4}m\omega^2R^2\left[1 - \cos(2\omega t + 2\theta)\right] \quad \text{--- (6)}$$

$$E = \frac{1}{2}m\omega^2R^2 = \text{constant.} \quad \text{--- (7)}$$

Thus we see that when a particle executes SHM with angular frequency ω, then its KE & PE execute SHO with twice frequency 2ω. Moreover KE & PE

are out of phase by a phase angle of π. While total energy E always remains constant during SHM.
The variation of KE, PE & TE, w.r.t. time is shown in figure (ii) below:- (assuming $\theta = \pi/2$)



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Free or Natural Oscillations:- When a system is oscillating without ~~the~~ any external aid, then the oscillations are called free or natural oscillations. For exemple when a pendulum is slightly disturbed from its mean position & then allowed to oscillate due to its own elasticity and inertia, then its oscillations are called natural or free oscillations. In this context, the vibrations discussed so far in this chapter are natural or free oscillations.

Damped Harmonic oscillations:- The mechanism, that results in dissipating the energy of an oscillator is called damping. Damping may result due to (i) viscous drag (ii) Friction and (iii) structure/shape of the system. It is observed that damping force F_d is directly proportional to the velocity of the oscillator.

Let x is extension from mean position at time t

F_d = damping force at time t

then $F_d \propto v$

or $F_d \propto \frac{dx}{dt}$

$$\Rightarrow F_d = -r\frac{dx}{dt} \quad \text{--- (1)}$$

Here constant of proportionality r is called damping constant and negative sign indicates that damping force opposes the motion.

Thus the basic equation of damped oscillations can be written by using Newton's second law of motion as follows:-

mass x acceleration = damping force + resistive force

or $$m\frac{d^2x}{dt^2} = -r\frac{dx}{dt} - kx$$

or $$\frac{d^2x}{dt^2} + \frac{r}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

or $$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega_0^2 x = 0 \quad \text{--- (2)}$$

Where $b = \frac{r}{2m}$ and $\omega_0^2 = \frac{k}{m}$ --- (3)

Equation (2) can also be written as

$$(D^2 + 2bD + \omega_0^2)x = 0 \quad \text{--- (4)}$$

where $D = \frac{d}{dt}$

The auxiliary equation is

$$D^2 + 2bD + \omega_0^2 = 0$$

Roots of this equation are

$$D = \frac{-2b \pm \sqrt{4b^2 - 4\omega_0^2}}{2}$$

$$= -b \pm \sqrt{b^2 - \omega_0^2} \quad \text{--- (5)}$$

Hence general solution of the equation is

$$x = A_1 e^{(-b + \sqrt{b^2-\omega_0^2})t} + A_2 e^{(-b - \sqrt{b^2-\omega_0^2})t}$$

$$= e^{-bt}\left[A_1 e^{(\sqrt{b^2-\omega_0^2})t} + A_2 e^{-(\sqrt{b^2-\omega_0^2})t}\right] \quad \text{--- (6)}$$

Where A_1, A_2 are integration constants.

WMSM

Depending upon relative values of b and ω_0, three different kinds of dampings can arise:-

(i) Over damped or heavily damped oscillations:- The oscillator is said to be overdamped if

$$b^2 > \omega_0^2$$

$$\Rightarrow \frac{r^2}{4m^2} > \frac{k}{m} \quad \text{—— (7)}$$

In this case $b^2 - \omega_0^2$ is real and positive. Moreover damping term (containing b) is dominating over stiffness term (containing ω_0^2 or k).

Equation (6) can be modified as follows:-

$$x = e^{-bt}\left[A_1\left(\sinh\sqrt{b^2-\omega_0^2}\,t + \cosh\sqrt{b^2-\omega_0^2}\,t\right) + A_2\left(\cosh\sqrt{b^2-\omega_0^2}\,t - \sinh\sqrt{b^2-\omega_0^2}\,t\right)\right]$$

$$= e^{-bt}\left[(A_1+A_2)\cosh\sqrt{b^2-\omega_0^2}\,t + (A_1-A_2)\sinh\sqrt{b^2-\omega_0^2}\,t\right] \quad \text{—— (8)}$$

Kindly note that hyperbolic functions are non-periodic in nature. Thus equation (8) shows that when oscillator is heavily damped, then motion is

Note:-

$$\sinh x = \frac{e^{x} - e^{-x}}{2}$$

$$\cosh x = \frac{e^{x} + e^{-x}}{2}$$

$$\tanh x = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$$

By adding ~~sth~~ $\sinh x$ and $\cosh x$, we get

$$\sinh x + \cosh x = e^{x}$$

Subtracting $\sinh x$ from $\cosh x$, we get

$$\cosh x - \sinh x = e^{-x}$$

not oscillatory at all. Rather displacement x decreases exponentially and oscillator energy is dissipated at very fast rate so that motion stops in a short time (practically) after it is started. The variation of displacement is shown in fig. 1.

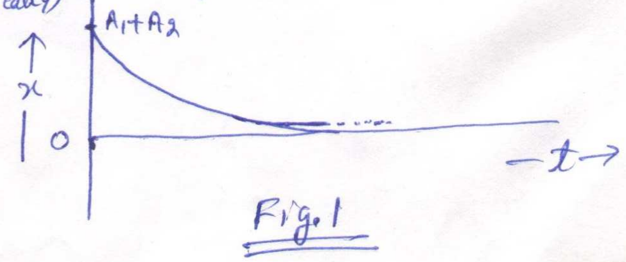

Fig. 1

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(ii) Lightly damped oscillations:- The oscillations are said to be lightly damped if stiffness term (ω_0^2) is greater than damping term (b^2). i.e. $b^2 < \omega_0^2$

$$\text{or} \quad \frac{r^2}{4m^2} < \frac{k}{m} \quad \text{—— (9)}$$

In this case $b^2-\omega_0^2$ is negative, so that $\sqrt{b^2-\omega_0^2}$ becomes imaginary. We can therefore write $\sqrt{b^2-\omega_0^2}$ as $i\sqrt{\omega_0^2-b^2}$. Thus equation (6) reduces to :-

$$x = e^{-bt}\left[A_1 e^{i\sqrt{\omega_0^2-b^2}\,t} + A_2 e^{-i\sqrt{\omega_0^2-b^2}\,t}\right]$$

$$= e^{-bt}\left[A_1\left(\cos\sqrt{\omega_0^2-b^2}\,t + i\sin\sqrt{\omega_0^2-b^2}\,t\right)\right.$$

$$\left. + A_2\left(\cos\sqrt{\omega_0^2-b^2}\,t - i\sin\sqrt{\omega_0^2-b^2}\,t\right)\right]$$

$$= e^{-bt}\left[(A_1+A_2)\cos\sqrt{\omega_0^2-b^2}\,t + i(A_1-A_2)\sin\sqrt{\omega_0^2-b^2}\,t\right] \quad \text{—— (10)}$$

Note

$e^{ix} = \cos x + i\sin x$

&

$e^{-ix} = \cos x - i\sin x$

$\therefore e^{ix} + e^{-ix} = 2\cos x$

or $\cos x = \dfrac{e^{ix}+e^{-ix}}{2}$

Similarly

$\sin x = \dfrac{e^{ix}-e^{-ix}}{2i}$

Not needed here

If we substitute

$$A_1 + A_2 = A_0\cos\phi \quad \text{—— (11)}$$

and

$$i(A_1-A_2) = A_0\sin\phi \quad \text{—— (12)}$$

then equation (10) becomes $x = A_0 e^{-bt}\left[\cos\left(\sqrt{\omega_0^2-b^2}\,t-\phi\right)\right]$ —— (13)

$$\cancel{x = A_0 e^{-bt}\left[\cos\right.} \quad \cancel{x = A_0 e^{-bt}\sin\sqrt{\omega_0^2-b^2}\,t \quad (13)}$$

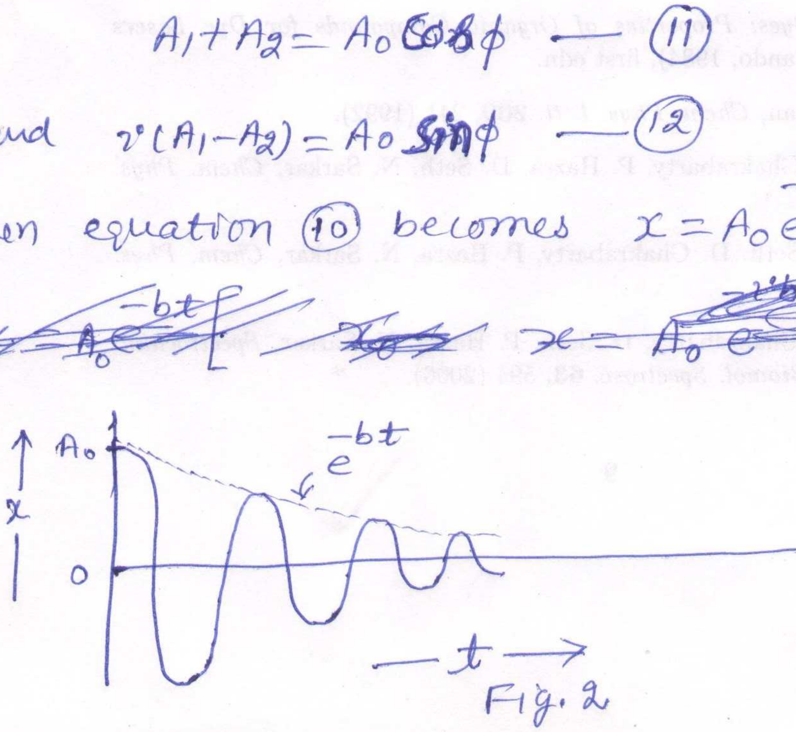

Fig. 2

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From (11) & (12), we can write

$$A_0^2(\cos^2\phi + \sin^2\phi) = (A_1+A_2)^2 + i^2(A_1-A_2)^2$$

$$\Rightarrow A_0^2 = A_1^2 + A_2^2 + 2A_1A_2 - (A_1^2 + A_2^2 - 2A_1A_2)$$

$$= 4A_1A_2$$

$$\therefore A_0 = 2\sqrt{A_1A_2} \quad \text{—— (14)}$$

and $\tan\phi = \dfrac{i(A_1-A_2)}{A_1+A_2}$ —— (15)

$\therefore$ If A_1 & A_2 are known, then A_0 & ϕ can be calculated using equations (14) & (15).

If we assume $R_0 = A_0 e^{-bt} = A_0 e^{-\frac{r}{2m}t}$ —— (16)

and $w_0' = \sqrt{w_0^2 - b^2} = \sqrt{\left(\frac{k}{m} - \frac{r^2}{4m^2}\right)}$ —— (17)

then equation (13) simplifies to

$$x = R_0 \cos(w_0't - \phi) \quad \text{—— (18)}$$

Equation (18) represents oscillations of a ~~critically~~ lightly damped oscillator. Clearly R_0 is amplitude of oscillations at time t and w_0' is frequency of oscillations. ϕ represents initial phase. Since at $t=0$, $R_0 = A_0 e^{-b(0)} = A_0$. Thus A_0 represent initial amplitude of lightly damped oscillator.

The variation of x versus t of lightly damped oscillator is shown in Fig. 2 by assuming $\phi = 0$. Clearly the oscillations are decreasing in amplitude after each cycle. The amplitude decays exponentially from initial value A_0 to value R_0 at time t according to the relation $R_0 = A_0 e^{-bt}$.

WMSM

The angular frequency of oscillation of a critically damped oscillator is $w_0' = \sqrt{\frac{k}{m} - \frac{r^2}{4m^2}}$ — (A) which is lower than angular frequency of corresponding undamped oscillator (i.e. $\sqrt{\frac{k}{m}}$).

Larger the value of r, smaller the value of frequency of lightly damped oscillator.

Note that from equation (18), the expression of velocity and acceleration of lightly damped oscillations are given as follows:-

$$v = \frac{dx}{dt} = -R_0 w_0' \sin(w_0't - \phi) + \frac{dR_0}{dt}\cos(w_0't - \phi)$$

$$= -R_0 w_0' \sin(w_0't - \phi) - bR_0\cos(w_0't - \phi)$$

$$= -R_0[w_0' \sin(w_0't - \phi) + b\cos(w_0't - \phi)]$$

$\because \frac{dR_0}{dt} = \frac{d}{dt}(A_0e^{-bt}) = -bA_0e^{-bt} = -bR_0$

and $a = \frac{dv}{dt}$

$$= -\frac{dR_0}{dt}[w_0' \sin(w_0't - \phi) + b\cos(w_0't - \phi)]$$

$$- R_0[w_0'^2 \cos(w_0't - \phi) - bw_0' \sin(w_0't - \phi)]$$

$$= bR_0[w_0' \sin(w_0't - \phi) + b\cos(w_0't - \phi)] - R_0[w_0'^2 \cos(w_0't - \phi)$$

$$- bw_0' \sin(w_0't - \phi)]$$

$$= R_0[2bw_0' \sin(w_0't - \phi) + (b^2 - w_0'^2)\cos(w_0't - \phi)]$$

$$= R_0[2bw_0' \sin(w_0't - \phi) + (2b^2 - w_0^2)\cos(w_0't - \phi)] \quad (19)$$

$\because w_0'^2 = w_0^2 - b^2$

$\therefore b^2 - w_0'^2 = 2b^2 - w_0^2$

Equation (19) shows that ~~displacement~~ acceleration is not directly proportional to displacement (this is also visible from equation (2), but that is for all cases of damped oscillations). Therefore the oscillations are not simple harmonic.

(iii) <u>Critically damped oscillations</u>:- An oscillator is said to be critically damped if $b^2 = \omega_0^2$ i.e. damping and stiffness terms are balancing each other. In this state, equation (6) reduces to

$$x = e^{-bt}\left[A_1 e^0 + A_2 e^0\right]$$

$$= (A_1 + A_2)\, e^{-bt}$$

or

$$x = G e^{-bt} \qquad (20)$$

where $G = A_1 + A_2$ = some constant.

equation (20) is not periodic in nature ($\because$ G is constant and e^{-bt} is non-periodic). Thus system will not execute ~~only~~ any oscillation in this stage just as for heavily damped oscillator. Thus graph between x & t should be similar to that of heavily damped oscillator. However it should be noted that equation (20) contains only one exponential function, while equation for ~~heavy~~ heavy damping (equation ~~8 or~~ (8) or better refer to equation (6) & keep in mind that $b > \omega_0$) contains ~~product of two exponentials term~~ sum of two terms $\left[e^{-bt} \cdot A_1 e^{\sqrt{b^2-\omega_0^2}\,t} \text{ and } e^{-bt} \cdot A_2 e^{-\sqrt{b^2-\omega_0^2}\,t}\right]$.

Initially when $t=0$ $x = G = A_1 + A_2$ for heavy as well as critical damping. As the time passes, displacement decays with constant exponential e^{-bt} for critical damping. While it decays with exponential term $e^{(-b+\sqrt{b^2-\omega_0^2})t}$ after some time ($\because$ 2nd term quickly becomes zero). Since $-b+\sqrt{b^2-\omega_0^2}$ is smaller than b. Thus decay is slower in heavy damping than in critical damping as shown in Fig. 3.

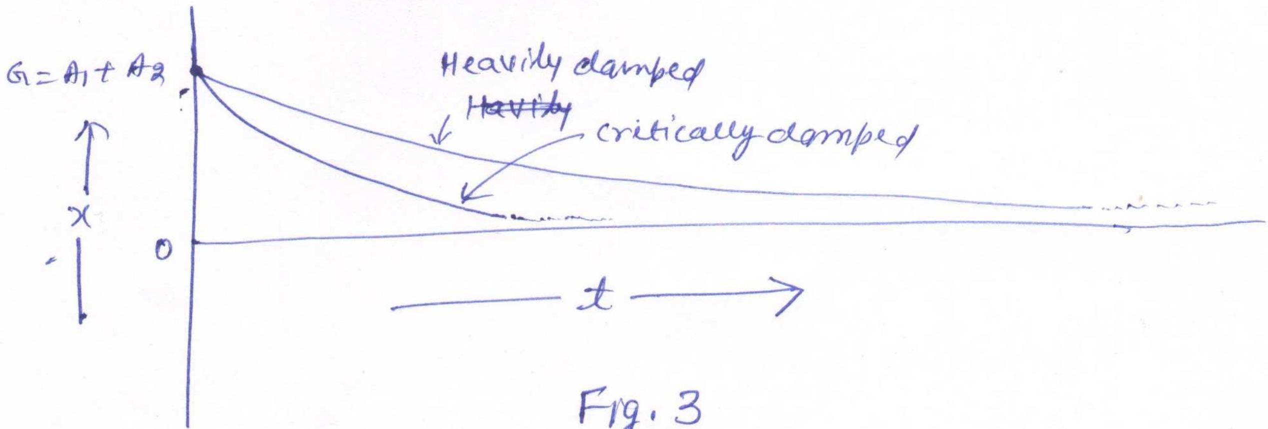

Fig. 3

Problem Show that $x = (c_1 + c_2 t) e^{-\frac{r}{2m}t}$ is a solution of the differential equation $\frac{d^2x}{dt^2} + \frac{r}{m}\frac{dx}{dt} + \frac{k}{m} = 0$ for the critically damped oscillations.

Hint:- Use $b^2 = \omega_0^2$ i.e. $\frac{r^2}{4m^2} = \frac{k}{m}$

Problem:- Show that damping force F_d cannot be directly proportional to displacement (x) or acceleration (a). It can only be directly proportional to velocity of oscillator.

Hint:- A force is said to be dissipative or damping or non-conservative, if it wastes energy while doing work. Therefore work done in one complete cycle of oscillation by that force must be non zero. Now assume any oscillator $x = A\cos(\omega t + \phi)$. Find its velocity $v = \frac{dx}{dt}$ & acceleration $a = \frac{d^2x}{dt}$. Now assume $F_d \propto x$, then $F_d \propto v$ then $F_d \propto a$ and write $F_d = -cx$ or $F_d = -cv$ or $F_d = -ca$ where c = some constant. Now find work done by F_d over one full cycle. It will be seen that work done will be non zero only in case (2), while it will be zero

WMSM

in case (1) & (3). This will give desired result.

Logarithmic decrement (δ) :- Logarithmic decrement of a damped oscillator is defined as natural logarithm of the ratio of amplitudes of two consecutive oscillations.

Note Since damped oscillator shows oscillatory behaviour only in case of light damping, therefore this quantity is relevant only for that case.

The angular frequency of lightly damped oscillation is given by $\omega_0' = \sqrt{\frac{k}{m} - \frac{r^2}{4m^2}}$ —— (1)

$\therefore$ If T' is time period of damped oscillator then

$$T' = \frac{2\pi}{\omega_0'} = \frac{2\pi}{\sqrt{\frac{k}{m} - \frac{r^2}{4m^2}}} \quad \text{—— (2)}$$

We know that amplitude of lightly damped oscillator at any time t is given as

$$R_0 = A_0 e^{-bt} \quad \text{—— (3)}$$

Thus amplitude of oscillator after exactly one cycle (when time will become $t + T'$) will be given by

$$R_0' = A_0 e^{-b(t+T')} \quad \text{—— (4)}$$

$\therefore$ δ = Logarithmic decrement is given as follows:-

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$$\delta = \ln\left(\frac{R_0}{R_0'}\right) = \ln\left(\frac{A_0 e^{-bt}}{A_0 e^{-b(t+T')}}\right)$$

$$= \ln\left(e^{bT'}\right)$$

$$\therefore \cancel{\boxed{\delta = bT'}} \qquad \therefore \boxed{\delta = bT' = \frac{rT'}{2m}} \quad -(5)$$

$$\text{or } \delta = \frac{r}{2m} \times \frac{2\pi}{\sqrt{\frac{k}{m} - \frac{r^2}{4m^2}}} \quad (\text{using } (2))$$

$$\therefore \boxed{\delta = \frac{2\pi r}{\sqrt{4mk - r^2}}} \quad -(6)$$

Note that δ has no dimensions.

Relaxation time (τ) :- It is defined as the time during which amplitude of a damped oscillator becomes $\frac{1}{e}$ of initial value.

Note that again this term is relevant only in lightly damped oscillator.

Now amplitude decays as $R_0 = A_0 e^{-bt}$ — (7)

$\therefore$ When $t = \tau$ = relaxation time

then $R_0 = \frac{A_0}{e}$

Putin in (7), we get $\therefore \frac{A_0}{e} = A_0 e^{-b\tau}$

or $e^{-1} = e^{-b\tau}$

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$\therefore \quad b\tau = 1$

$$\text{or} \quad \boxed{\tau = \frac{1}{b} = \frac{2m}{r}} \quad \text{——} \ (8) \qquad \left(\because b = \frac{r}{2m}\right)$$

From (5) & (8), we see that δ and τ are related as follows:-

$$\cancel{\boxed{\delta = \tau T'}} \quad \cancel{\text{——} \ (9)}$$

$$\boxed{\delta = \frac{T'}{\tau}} \quad \text{——} \ (9)$$

Problem:- If A_0 is initial amplitude and A_n is amplitude at the end of nth cycle, then show that logarethmic decrement δ can also be written as $\delta = \frac{1}{n} \ln\left(\frac{A_0}{A_n}\right)$

Quality Factor (or Q-factor) :- This parameter ~~represents the rate~~ defined as 2π times the instantaneous energy of an oscillator to the energy lost in one cycle.

$$Q = 2\pi \times \left(\frac{\text{Energy at the start of cycle}}{\text{loss of energy in one cycle}}\right) \quad \text{——} \ (10)$$

Obviously if denominator of equation (10) is small then less energy is dissipated per cycle and oscillations ~~will~~ can be sustained for larger time. This will increase the value of Q and quality of oscillator will will improve. Hence the name quality factor.

We know that in any oscillator the total energy stored at any time t is directly proportional to square of amplitude at that time.

WMSM

i.e. $E \propto R_0^2$

$$\Rightarrow E = C R_0^2 = C A_0 e^{-2bt} \quad \text{——} (11) \qquad (\because R_0 = A_0 e^{-bt})$$

where C is some constant.

$E' =$ energy after one cycle

$$= C A_0 e^{-2b(t+T')} \quad \text{——} (12)$$

$$\Delta E = \text{Energy lost per cycle} = E - E'$$

$$= C A_0 \left[e^{-2bt} - e^{-2b(t+T')} \right]$$

$$= C A_0 e^{-2bt} \left[1 - e^{-2bT'} \right]$$

$$Q = 2\pi \times \frac{E}{\Delta E}$$

$$= 2\pi \times \frac{C A_0 e^{-2bt}}{C A_0 e^{-2bt} \left[1 - e^{-2bT'} \right]}$$

$$= \frac{2\pi}{1 - e^{-2bT'}}$$

$$\approx \frac{2\pi}{1 - (1 - 2bT')}$$

(Expanding by binomial and neglecting higher powers because oscillator is lightly damped)

$$\therefore \boxed{Q = \frac{\pi}{bT'}} \quad \text{——} (13)$$

or

$$\boxed{Q = \frac{2\pi m}{r T'}} \quad \text{——} (14) \qquad \left(\because b = \frac{r}{2m}\right)$$

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From (5) and (14), we can write

$$\cancel{\frac{\delta}{Q} = \frac{rT'}{2m} \times} \qquad \delta \times Q = \frac{rT'}{2m} \times \frac{2\pi m}{rT'}$$

$$= \pi$$

$$\therefore \boxed{\delta = \frac{\pi}{Q}} \quad \text{--- (15)}$$

Forced Oscillator :- An oscillator, to which a continuous excitation is provided by some external agency is called Forced oscillator and vibrations/oscillations thus executed are called forced oscillations.

The ~~oscillator~~ agency which can provide energy to an oscillator must itself be executing oscillations (that is only an oscillator can give energy to another oscillator).

The oscillator giving energy (external agency) is called driver oscillator and the oscillator receiving energy is called driven oscillator.

Let m is mass of driven oscillator (also called forced oscillator)

x = instantaneous displacement of forced oscillator

There are three kinds of forces acting on a forced oscillator namely (i) restoring force $-kx$

(ii) damping force $F_d = -r\frac{dx}{dt}$

(iii) Driving force $\tilde{F} = F_0 \, e^{i\omega t}$

Note that $\tilde{F}$ denotes that $\tilde{F}$ is a complex quantity, while F_0 is a real quantity. F_0 is maximum force that can be applied by driver oscillator. Note that ω is angular frequency of the driver oscillator. It may or may not be equal to the frequency of driven oscillator. Moreover actual force acting on driven oscillator is always given by real part of $\tilde{F}$ (i.e. $F_0 \cos\omega t$).

The equation of motion of forced oscillator is obtained by using Newton's second law as follows:-

mass × acceleration = Net force

$$m\frac{d^2\tilde{x}}{dt^2} = -k\tilde{x} - r\frac{d\tilde{x}}{dt} + F_0 e^{i\omega t} \quad \text{—— (1)}$$

Since $\tilde{F} = F_0 e^{i\omega t}$ is a complex number. Therefore x can also be complex. That is the reason to use $\tilde{x}$ notation instead of x.

Initially at $t=0$, the driven oscillator will try to oscillate with its own angular frequency ω_0' (see damped oscillator) and driver will try to oscillate it with its own frequency ω. Therefore the behaviour of oscillator will be uncertain (transient). However as time progresses, the driver oscillator will lose its natural frequency of vibration (ω_0') and start vibrating with the frequency of driver oscillator (ω). When this stage is achieved, the oscillator is said to be in steady state.

(i) Transient Behaviour:- Initially system will vibrate with its natural frequency so that equation (1) becomes

$$m\frac{d^2x}{dt^2} = -kx - r\frac{dx}{dt}$$

(∵ $F_0 e^{i\omega t}$ is absent, so x will be real)

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or $m\frac{d^2x}{dt} + kx + r\frac{dx}{dt} = 0$ —— (2)

This is basic equation of damped oscillator ~~undergoing~~ showing/giving natural oscillations. Therefore displacement is given as

$$x = A_0 e^{-bt} \cos(\sqrt{\omega_0^2 - b^2}\, t - \phi) \quad \text{—— (3)}$$

(See previous discussion on lightly damped oscillator and solve it here if it is asked in question paper).

We can rewrite equation (3) in complex notation as follows:-

$$\tilde{x} = A_0 e^{-bt} e^{i[\sqrt{\omega_0^2 - b^2}\, t - \phi]} \quad \text{—— (4)}$$

Where actual solution is given by real part of equation (4).

(ii) Steady state behaviour :- Due to damping, soon the natural vibrations of damped oscillator will die out and it will start vibrating with frequency ω and equation of motion will be

$$m\frac{d^2\tilde{x}}{dt^2} = -k\tilde{x} - r\frac{d\tilde{x}}{dt} + F_0 e^{i\omega t}$$

$$\text{or } m\frac{d^2\tilde{x}}{dt^2} + k\tilde{x} + r\frac{d\tilde{x}}{dt} - F_0 e^{i\omega t} = 0 \quad \text{—— (5)}$$

Let the solution of equation (5) is of the form

$$\tilde{x} = \tilde{A} e^{i\omega t} \quad \text{—— (6)}$$

From now onward we drop complex notation, but we will keep in mind that x and A are complex quantities.

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$$\therefore \frac{dx}{dt} = A(i\omega)e^{i\omega t} \quad \text{--- (7)}$$

and
$$\frac{d^2x}{dt^2} = A(i\omega)(i\omega)e^{i\omega t}$$
$$= -A\omega^2 e^{i\omega t} \quad \text{--- (8)}$$

For equation (6) to be true solution of (5), the values of $x, \frac{dx}{dt}$ & $\frac{d^2x}{dt^2}$ obtained in (6)–(8) must satisfy equation (5). Therefore putting these values in (5), we get

$$\cancel{-mA\omega^2 + i\omega r + k}$$

$$e^{i\omega t}\left[-mA\omega^2 + kA + rA(i\omega)\right] - F_0 e^{i\omega t} = 0$$

or
$$\cancel{e^{i\omega t}} \times A\left[-m\omega^2 + k + i\omega r\right] = F_0 \cancel{e^{i\omega t}}$$

$$\Rightarrow A = \frac{F_0}{k - m\omega^2 + i\omega r}$$

$$= \frac{-iF_0}{-i(k - m\omega^2 + i\omega r)}$$

$$= \frac{-iF_0}{-i(k - m\omega^2) + \omega r}$$

$$= \frac{-iF_0}{\omega\left[r + i\left(m\omega - \frac{k}{m}\right)\right]} \quad \text{--- (9)}$$

or
$$\tilde{A} = \frac{-iF_0}{\omega \tilde{Z}_m} \quad \text{--- (10)}$$

Here $\tilde{Z}_m = r + i\left(m\omega - \frac{k}{m}\right)$ —— (11)

$\tilde{Z}_m$ is called complex mechanical impedance of the forced oscillator. It consist of two parts

(a) r :- This term represents the resistance to the oscillatory motion due to friction, viscosity or any other dissipating property.

(b) $\left(m\omega - \frac{k}{m}\right)$:- This term is the reactance due to combined effect of elasticity & inertia. ωm is called inertial reactance while $\frac{k}{m}$ is called elastic reactance.

Phase difference between driving force and displacement in steady state :- Put $r = |Z_m|\cos\phi$ and $m\omega - \frac{k}{m} = |Z_m|\sin\phi$,

we get $\tilde{Z}_m = |Z_m|(\cos\phi + i\sin\phi)$

or $\tilde{Z}_m = |Z_m| e^{i\phi}$ —— (12)

$\therefore |Z_m| = \sqrt{r^2 + \left(m\omega - \frac{k}{m}\right)^2}$ —— (13)

$|Z_m|$ is amplitude of $\tilde{Z}_m$

Moreover $\tan\phi = \frac{|Z_m|\sin\phi}{|Z_m|\cos\phi}$

$= \frac{m\omega - \frac{k}{m}}{r} = \frac{X_m}{r}$ —— (14)

where $X_m = m\omega - \frac{k}{m}$ —— (15)

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Equation (14) can be used to plot Phasor diagram as shown in Fig. 1 below :-

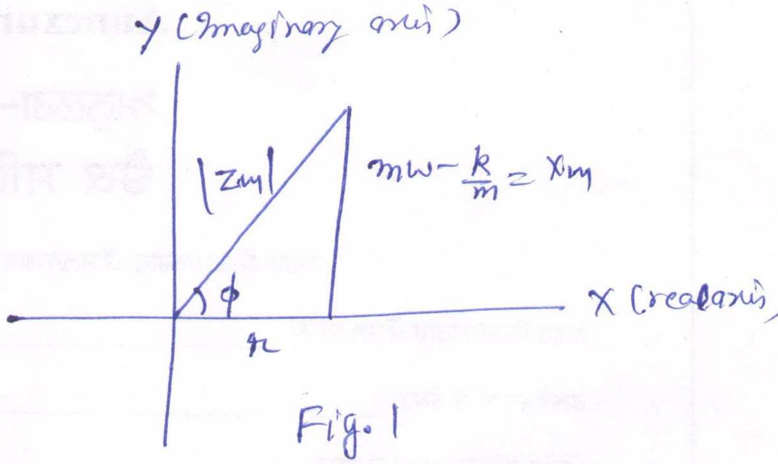

Fig. 1

~~Putting values~~

Now displacement x is given by :- $x = Ae^{i\omega t}$

where

$$A = \frac{-iF_0}{\omega Z_m}$$

$$= \frac{-iF_0}{\omega |Z_m| e^{i\phi}}$$

$$\therefore \quad x = Ae^{i\omega t}$$

$$= \frac{-iF_0 e^{i\omega t}}{\omega |Z_m| e^{i\phi}}$$

$$\text{or} \quad \boxed{\tilde{x} = \frac{-iF_0}{\omega |Z_m|} e^{i(\omega t - \phi)}} \quad \text{—— (15)}$$

~~This equation shows that phase difference between force and displacement~~

Equation (15) can also be written as

$$\tilde{x} = \frac{e^{-i\pi/2} F_0 e^{i(\omega t - \phi)}}{\omega |Z_m|} \qquad \left(\because e^{-i\frac{\pi}{2}} = -i\right)$$

$$= \frac{F_0}{\omega |Z_m|} e^{i(\omega t - \phi - \frac{\pi}{2})} \quad \text{—— (16)}$$

Since driving force is given by the expression

$$\tilde{F} = F_0 e^{i\omega t} \quad \text{—— (17)}$$

Therefore comparison of (16) & (17) shows that phase of displacement

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lags than phase of driving force by a value of $\phi + \pi/2$.
The reactive part of impedance that is $X_m = \omega m - k/\omega$ introduces a phase ϕ and presence of term $-i$ in equation (15) introduces additional phase difference of $-\pi/2$. Thus even when $\phi = 0$, ~~force and~~ driving force and displacement will still differ in phase by $\pi/2$.

~~Velocity of Forced oscillations:- It is given as follows:-~~

Since only real quantities can be measured, therefore we are interested in real displacement of oscillator caused by real driving force. To accomplish this task, we do following steps:-

$$\tilde{F} = F_0 e^{i\omega t} = F_0 \cos\omega t + i F_0 \sin\omega t$$

$$\therefore F_{real} = F_0 \cos\omega t \quad \text{---} \quad (18)$$

$$\tilde{x} = \frac{-i F_0}{\omega |Z_m|} e^{i(\omega t - \phi)}$$

$$= \frac{-i F_0}{\omega |Z_m|} \left[\cos(\omega t - \phi) + i \sin(\omega t - \phi)\right]$$

$$= \frac{F_0}{\omega |Z_m|} \left[\sin(\omega t - \phi) - i \cos(\omega t - \phi)\right] \quad \text{---} \quad (19)$$

$$\therefore x_{real} = \frac{F_0 \sin(\omega t - \phi)}{\omega |Z_m|} \quad \text{---} \quad (20)$$

Equation (20) can also be written as

$$x_{real} = \frac{F_0}{\omega |Z_m|} \cos\left(\omega t - \phi - \frac{\pi}{2}\right) \quad \text{---} \quad (21)$$

The comparison of (18) and (21) ~~sho~~ ~~als~~ again shows that phase difference ~~bto~~ between real driving force and real displacement is also $\phi + \pi/2$ and that displacement lags force by $\phi + \pi/2$.

Velocity and acceleration of Forced oscillations:- It is given as follows:-

$$\tilde{v} = \frac{d\tilde{x}}{dt}$$

$$= \frac{d}{dt}\left\{ \frac{-iF_0}{\omega|Z_m|} e^{i(\omega t-\phi)} \right\}$$

$$= \frac{-iF_0}{\omega|Z_m|} \times (i\omega) \times e^{i(\omega t-\phi)}$$

$$\therefore \boxed{\tilde{v} = \frac{F_0}{|Z_m|} e^{i(\omega t-\phi)}} \quad \text{—— (22)}$$

Similarly the acceleration is given by

$$\tilde{a} = \frac{d\tilde{v}}{dt}$$

$$\therefore \boxed{\tilde{a} = \frac{i\omega F_0}{|Z_m|} e^{i(\omega t-\phi)}} \quad \text{—— (23)}$$

equation (23) can also be written as follows :-

$$\tilde{a} = \frac{\omega F_0}{|Z_m|} e^{i(\omega t-(\phi-\pi/2))} \quad \text{—— (24)}$$

$$(\because i = e^{i\pi/2})$$

comparison of (17) and (22) shows that velocity lags force by a phase angle of ϕ. Similarly comparison of (17) and (24) shows that driving force and acceleration of driven oscillator have a phase difference of $\phi - \pi/2$. If $\phi > \pi/2$ then acceleration will ~~lago~~ lag driving force. While for $\phi < \pi/2$ acceleration of driven oscillator lead driving force.

From equations 16, 22 and 24, we can conclude that frequency of oscillation of displacement, velocity and acceleration of forced oscillator in steady state is equal to the frequency of driver oscillator.

From 16, the amplitude of displacement is

$$x_0 = A_0 = \frac{F_0}{\omega |Z_m|} \quad \text{——— (25)}$$

From 22, the amplitude (peak value) of velocity is

$$V_0 = \frac{F_0}{|Z_m|} \quad \text{——— (26)}$$

and from 24, we see that amplitude of acceleration is

$$a_0 = \frac{\omega F_0}{|Z_m|} \quad \text{——— (27)}$$

Equation 22 can also be written as

$$\tilde{v} = \frac{F_0 e^{i(\omega t - \phi)}}{|Z_m|}$$

$$= \frac{F_0 e^{i\omega t}}{|Z_m| e^{i\phi}}$$

$$= \frac{\tilde{F}}{\tilde{Z}}$$

$$\therefore \boxed{\tilde{Z} = \frac{\tilde{F}}{\tilde{v}}} \quad \text{——— (28)}$$

Thus complex impedance is the ratio of complex driving force to the complex velocity in steady state.

Resonance:— Equation (25) shows that amplitude of a forced oscillator in steady state is a function of frequency.

$$\therefore A_0 = \frac{F_0}{\omega|Z_m|}$$

$$= \frac{F_0}{\omega\left[r^2+\left(m\omega-\frac{k}{m}\right)^2\right]^{1/2}}$$

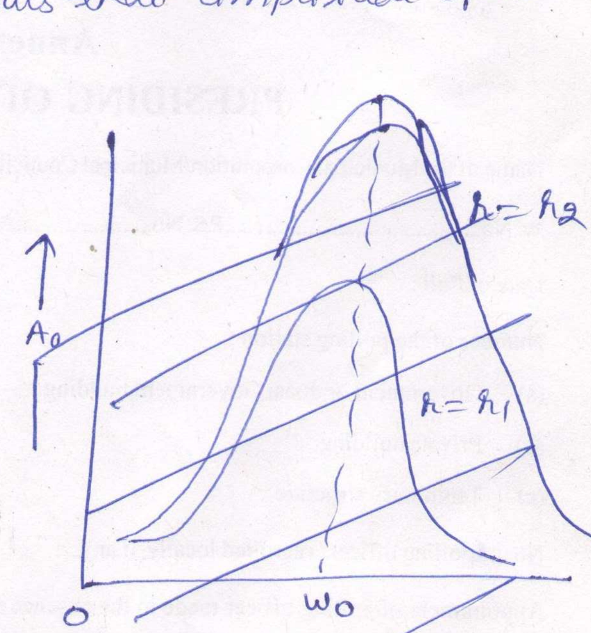

Fig. 2

The variation of A_0 with ω is shown in fig 2. ~~Three cases of damp~~. To understand this curve which is also called frequency response curve of a forced oscillator, we discuss three cases.

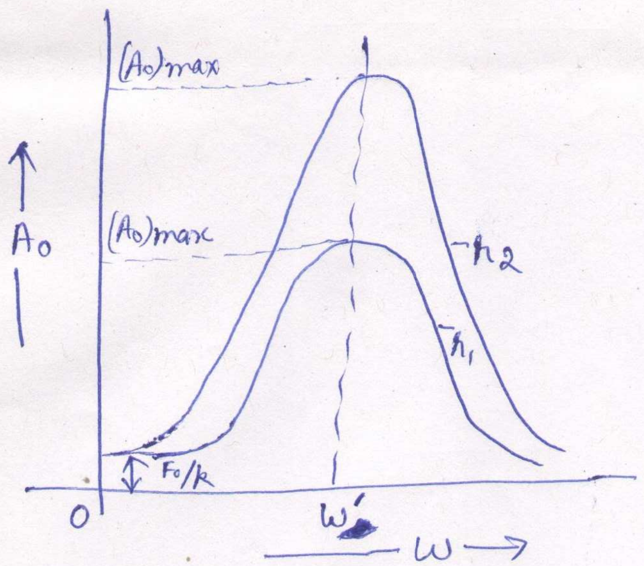

Fig. 2

(a) when $\omega \to 0$

At very low frequencies $m\omega \to 0$, $\frac{k}{\omega} \to \infty$, so that r^2 and $m\omega$ can be neglected compared with $\frac{k}{\omega}$.

Thus at low frequencies A_0 can be approximated as

$$A_0 \simeq \frac{F_0}{\omega\left[0+\left(0-\frac{k}{\omega}\right)^2\right]^{\frac{1}{2}}}$$

$$= \frac{F_0}{k}$$

This amplitude is independent of frequency.

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(b) <u>When $w \to \infty$</u> In this case $mw \to \infty$ and $\frac{k}{\omega} \to 0$ Thus $\frac{k}{\omega}$ and r^2 can be neglected in comparison to $m\omega$. Hence A_0 becomes

$$A_0 \approx \frac{F_0}{\omega[0^2 + (m\omega - 0)^2]^{\frac{1}{2}}}$$

$$= \frac{F_0}{m\omega^2}$$

Thus amplitude decreases fastly with frequency ~~of~~ at higher values.

(c) <u>Maximum value of amplitude</u> :- Since $A_0 = \frac{F_0}{w|Z_m|}$ and F_0 is constant. Thus A_0 will be maximum when the denominator $w|Z_m|$ is minimum. To obtain this stationary point, we put

$$\frac{d}{dw}(w|Z_m|) = 0$$

$$\text{or} \quad \frac{d}{dw}\left[w\sqrt{r^2 + \left(mw - \frac{k}{\omega}\right)^2}\right] = 0$$

$$\Rightarrow \quad \frac{d}{dw}\left[\sqrt{\omega^2 r^2 + (m\omega^2 - k)^2}\right] = 0$$

$$\Rightarrow \quad \frac{1}{2}\left(\omega^2 r^2 + (m\omega^2 - k)^2\right)^{-\frac{1}{2}} \times \left[2wr^2 + 2(\omega^2 m - k)(2wm)\right] = 0$$

$$\Rightarrow \quad \frac{wr^2 + (w^2 m - k)(2wm)}{\sqrt{\omega^2 r^2 + (m\omega^2 - k^2)^2}} = 0$$

$$\Rightarrow \quad wr^2 + (w^2 m - k)(2wm) = 0$$

$$\Rightarrow \quad w\left[r^2 + 2m(w^2 m - k)\right] = 0$$

$$\Rightarrow \quad r^2 + 2m(w^2 m - k) = 0 \qquad (\because w = 0 \text{ is trivial solution})$$

WMSM

$$\Rightarrow \quad 2m^2\omega^2 = 2mk - r^2$$

$$\Rightarrow \omega^2 = \frac{k}{m} - \frac{r^2}{2m^2}$$

$$\Rightarrow \quad \omega = \sqrt{\frac{k}{m} - \frac{r^2}{2m^2}} = \omega' \text{ (say)} \quad \text{——} \quad (29)$$

Since $\omega_0^2 = \frac{k}{m}$ and $b = \frac{r}{2m} \Rightarrow \frac{r^2}{2m^2} = 2b^2$

$$\therefore \omega' = \sqrt{\frac{k}{m} - \frac{r^2}{2m^2}} = \sqrt{\omega_0^2 - \frac{r^2}{2m^2}} = \sqrt{\omega^2 - 2b^2} \quad \text{—} (30)$$

$\therefore \quad \omega' < \omega_0$

Thus peak value of amplitude occurs at a frequency less than natural frequency of vibration of an undamped oscillator. The frequency ω' is called resonance frequency and oscillations occuring at this frequency, which are of maximum amplitude are called resonant oscillations. The maximum value of amplitude can be obtained as follows:-

$$A_0 = \frac{F_0}{\omega|Z_m|} = \frac{F_0}{\omega\sqrt{r^2 + (m\omega - \frac{k}{\omega})^2}} = \frac{F_0}{\sqrt{\omega^2 r^2 + (m\omega^2 - k)^2}}$$

A_0 is maximum $((A_0)_{max})$ when $\omega = \omega' = \sqrt{\frac{k}{m} - \frac{r^2}{2m^2}}$

$$\therefore (A_0)_{max} = \frac{F_0}{\sqrt{\omega'^2 r^2 + (m\omega'^2 - k^2)^2}}$$ (Put value of ω')

$$\therefore \quad (A_0)_{max} = \frac{F_0}{\sqrt{\left(\frac{k}{m} - \frac{r^2}{2m^2}\right) r^2 + \left[m\left(\frac{k}{m} - \frac{r^2}{2m^2}\right) - k\right]^2}}$$

$$= \frac{F_0}{\sqrt{\frac{k}{m} r^2 - \frac{r^4}{2m^2} + \frac{r^4}{4m^2}}}$$

WMSM

$$= \frac{F_0}{\sqrt{\frac{k}{m} r^2 - \frac{r^4}{4m^2}}}$$

$$= \frac{F_0}{r\sqrt{\frac{k}{m} - \frac{r^2}{2m^2}}}$$

$$\Rightarrow \boxed{(A_0)_{max} = \frac{F_0}{r\omega_0'}} \quad \text{—— (31)}$$

where $\omega_0' = \sqrt{\frac{k}{m} - \frac{r^2}{2m^2}}$ is natural frequency of vibration of corresponding damped oscillator (see page (16), equation (*)).

Impedance Matching :- When two or more media are in contact with each other and vibrational energy due to waves/oscillations is incident from one medium, then at the common boundary of two different media in contact, some amount of energy is reflected back into first medium and remaining energy flows into the second medium. The amount of energy/power that can transfer from one medium to other depends upon elastic and inertial properties of two media in contact. Thus sometimes it may not be possible to transfer vibrational energy from one medium (say medium (1)) to another medium (say medium (2)), when these are in direct contact with each other. However, it is possible to increase the transfer efficiency by introducing/ sandwiching an intervening medium between ~~above~~ two given media and this process is called Impedance Matching.

Fig. 1 shows a string which ~~is~~ consists of two diffrent media of impedences Z_1 and Z_2 and these are joined to each other with the help of a small matching medium of length l and impedence Z. The impedence of matching medium is chosen in such a way that maximum power is transferred from medium Z_1 to medium Z_2.

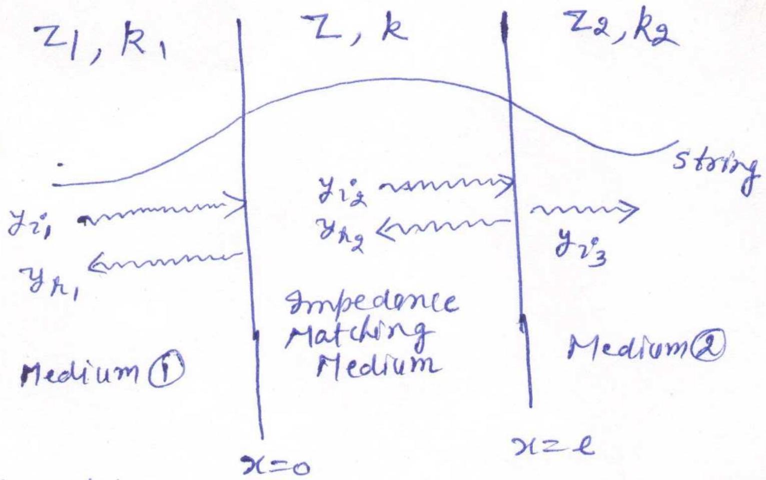

Fig. 1

The part ① of string as input and part ② (medium Z_2) acts as output. When vibrational energy is incident from medium ① to matching medium at $x=0$, then some part of energy is reflected back and remaing energy/power enters in matching medium. This energy is then incident on output medium (Z_2) at $x=l$, where again some power is reflected back into matching medium and remaining power is transferred into output medium.

Let incident wave at $x=0$ is given as

$$y_{i_1} = A_1 e^{i(\omega t - k_1 x)}$$

where A_1, ω, k_1 represents amplitude, angular frequency and propagation constant.

It should be noted that when a wave enters from one medium to other, then its velocity and wavelength are changed while frequency remains same. However when a wave is reflected back into same medium, then only direction of propagation is changed while frequency, wavelength and speed remains same (although velocity gets changed due to change in direction).

Morever in addition, amplitude of wave may remain same if there is pure reflection or pure transmission (and no absorption). But if a wave is partially transmitted and partially reflected, ~~then~~ or when energy dissipation is taking place at the boundary, then amplitudes of reflected and transmitted waves also get changed because square of amplitude represents power or intensity (and division of power came come only through change in amplitude). Thus equation of reflected wave y_{r_1} and transmitted wave y_{i_2} at the first boundary $x=0$ can be written as follows:-

$$y_{r_1} = B_1 e^{i(\omega t + k_1 x)}$$

and $$y_{i_2} = A_2 e^{i(\omega t - k_2 x)}$$

The transmitted wave is written as y_{i_2}, because it acts as incident wave at the second boundary ($x=l$). There again a part of wave (y_{r_2}) is reflected and remaining part (y_{i_3}) gets transmitted. The equations of two waves are given as follows:-

$$y_{r_2} = B_2 e^{i(\omega t + k_2 x)}$$

and $$y_{i_3} = A_3 e^{i(\omega t - k_3(x-l))}$$

The power transferred is maximum when (i) wave function y is continuous across all the boundaries

that is $$(y_{i_1} + y_{r_1})_{x=0} = (y_{i_2} + y_{r_2})_{x=0}$$

and $$(y_{i_2} + y_{r_2})_{x=l} = (y_{i_3})_{x=l}$$

It should be noted that all y, A, B are complex while ω, k, ~~and~~ x are real. However for simplicity complex notation is not followed.

(ii) Transvers component of force must be continuous at across the boundaries. That is

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$$\cancel{T\left(\frac{\partial y}{\partial x}\right)} \quad \left(T\left(\frac{\partial y_{i_1}}{\partial x}\right) + T\left(\frac{\partial y_{r_1}}{\partial x}\right)\right)_{x=0} = T\left(\left(\frac{\partial y_{i_2}}{\partial x}\right) + T\left(\frac{\partial y_{r_2}}{\partial x}\right)\right)_{x=0}$$

and
$$\left[T\left(\frac{\partial y_{i_2}}{\partial x}\right) + T\left(\frac{\partial y_{r_2}}{\partial x}\right)\right]_{x=l} = T\left(\frac{\partial y_{i_3}}{\partial x}\right)_{x=l}$$

Here T is tension in the string and it gets cancelled out.

Let λ_1, λ and λ_2 be the wavelength of wave in medium Z_1, Z and Z_2 respectively. Then

$$\lambda_1 = \frac{2\pi}{k_1}, \quad \lambda = \frac{2\pi}{k} \text{ and } \lambda_2 = \frac{2\pi}{k_2}$$

For maximum power transfer third condition to be satisfied is that length of matching medium must be equal to quarter of wavelength λ in that medium. That is

$$l = \frac{\lambda}{4} = \frac{\pi}{2k}$$

The energy of incident wave at $x=0$ is given by

$$E_i = Z_1 (A_1)^2$$

While energy of ~~[illegible]~~ final transmitted wave (y_{i_3}) is given by

$$E_t = Z_2 (A_2)^2$$

The fourth and final condition for maximum power transfer is that
$$E_t = E_i$$

or
$$\frac{E_t}{E_i} = 1$$

or
$$\frac{Z_2 (A_2)^2}{Z_1 (A_1)^2} = 1$$

When all these conditions are simultaneously satisfied (the process involves big mathematics, which is avoided here in this qualitative discussion), then we get following relation between Z_1, Z and Z_2:

$$\boxed{Z = \sqrt{Z_1 Z_2}} \longrightarrow \text{condition for perfect matching.} \quad \circledast$$

Therefore for perfect matching, the impedance of coupling or matching medium must be equal to harmonic mean of impedances of two given media and ~~to~~ length of coupling medium must be equal to $\lambda/4$.

Matching of impedances plays very important role in transmission ~~of~~ of electric power across lines, where two lines are connected by inserting quarter wavelength long stubs between them so that condition $\circledast$ is satisfied.

Similarly to reduce reflecting properties ~~of~~ of lenses, these are coated with thin films of thickness equal to quarter wave-length of light in that medium and choosing the refractive index of the film material (n) as

$$n = \sqrt{n_1 n_2}$$

where n_1, n_2 represent refractive index of incident medium (usually air) and lens medium (usually crown glass) respectively. With such coating light is transmitted efficiently from air into lens medium and almost no reflection of light takes place.

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