9. The position vector of a moving particle at a time t is $\vec{R}(t) = t^2\hat{i} + -t^3\hat{j} + t^4\hat{k}$. Find the tangential and normal components of its acceleration at time $t = 1$. **(P.T.U., May 2010)**
[**Hint.** See S.E. 11]

10. A particle moves along the curve $x = e^{-t}$, $y = 2\cos 3t$, $z = 2\sin 3t$, where t is the time. Determine its velocity and acceleration vectors and also the magnitudes of velocity and acceleration at $t = 0$.
[**Hint.** See solved example 9] **(P.T.U., Dec. 2002)**

11. A particle moves along the curve $x = 2t^2$, $y = t^2 - 4t$, $z = 3t - 5$ where t is the time. Find the components of its velocity and acceleration at time $t = 1$, in the direction $\hat{i} - 3\hat{j} + 2\hat{k}$.

12. A particle moves so that its position vector is given by $\vec{r} = \cos \omega t\,\hat{i} + \sin \omega t\,\hat{j}$. Show that the velocity $\vec{v}$ of the particle is perpendicular to $\vec{r}$ and $\vec{r} \times \vec{v}$ is a constant vector.

13. The position vector of a point at any time t is given by $\vec{r} = e^t\left(\cos t\,\hat{i} + \sin t\,\hat{j}\right)$. Show that $\vec{a} = 2(\vec{v} - \vec{r})$ where $\vec{a}$ and $\vec{v}$ are acceleration and velocity of a particle.

14. A particle having position vector $\vec{r}$ is moving in a circle with constant angular velocity ω. Show by vector methods, that the acceleration is equal to $-\omega^2\vec{r}$.

Answers

4. $-4\left(\hat{i} + 2\hat{j}\right)$

5. (i) $\dfrac{d\vec{r}}{dt} \cdot \vec{a}$ (ii) $\dfrac{d\vec{r}}{dt} \times \vec{a}$ (iii) $\vec{r} \times \dfrac{d^2\vec{r}}{dt^2}$ (iv) $\left(\dfrac{d\vec{r}}{dt}\right)^2 + \vec{r} \cdot \dfrac{d^2\vec{r}}{dt^2}$

7. (i) $\dfrac{t\hat{i} + 2\hat{j} + (2t-3)\hat{k}}{\sqrt{5t^2 - 12t + 13}}$, $\dfrac{1}{3}\left(2\hat{i} + 2\hat{j} + \hat{k}\right)$ (ii) $\dfrac{(-a\sin t)\hat{i} + (a\cos t)\hat{j} + b\hat{k}}{\sqrt{a^2 + b^2}}$

8. $\dfrac{1}{3}\left(2\hat{i} + 2\hat{j} + \hat{k}\right)$, $\dfrac{1}{3\sqrt{5}}\left(2\hat{i} + 2\hat{k}\right)$

9. $\dfrac{70}{\sqrt{29}}$, $2\sqrt{46}$

10. $\sqrt{37}$, $5\sqrt{13}$

11. $\dfrac{8\sqrt{14}}{7}$, $\dfrac{-\sqrt{14}}{7}$

7.13. SCALAR AND VECTOR POINT FUNCTIONS

Point Function. A variable quantity whose value at any point in a region of space depends upon the position of the point, is called a point function.

Point functions are of two types:

(i) Scalar Point Function (ii) Vector Point Function.

(i) **Scalar Point Function.** A function $\phi(x, y, z)$ is called a scalar point function if it associates a scalar with every point in region R of a space. Region R is called scalar field. The temperature distribution in a heated body, density of a body and potential due to gravity are examples of scalar point functions.

(*ii*) **Vector Point Function.** If a function $\vec{V}\ (x, y, z)$ defines a vector at every point of the region R of a space then $\vec{V}\ (x, y, z)$ is called a vector point function and R is called a vector field. Every vector $\vec{v}$ of the field is regarded as a localized vector attached to the corresponding point (x, y, z).

The velocity of a moving fluid at any instant, gravitational forces are examples of vector point function.

7.14. GRADIENT OF A SCALAR FIELD

(*P.T.U., Jan. 2009*)

Let $\phi\ (x, y, z)$ be a function defining a scalar field, then the vector $\hat{i}\dfrac{\partial\phi}{\partial x} + \hat{j}\dfrac{\partial\phi}{\partial y} + \hat{k}\dfrac{\partial\phi}{\partial z}$ is called the gradient of the scalar field ϕ and is denoted by grad ϕ.

Thus $$\text{grad}\ \phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

Grad ϕ is a vector quantity.

The gradient of scalar field ϕ is obtained by operating on ϕ the vector operator.

$$\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

This operator is denoted by symbol ∇ (read as del) (also called nable)

Thus $$\text{grad}\ \phi = \nabla\ \phi.$$

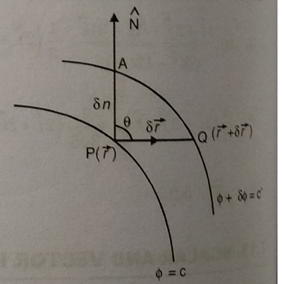
7.15. GEOMETRICAL INTERPRETATION OF GRADIENT

(*P.T.U., May 2010, May 2012*)

If a surface $\phi\ (x, y, z) = c$ is drawn through any point P such that at each point on the surface, the function has the same value as at P, then such a surface is called a **level surface** through P.

Through any point passes one and only one level surface. Also no two level surfaces can intersect.

Consider the level surface through the point P at which the function has value ϕ and let $\phi + \delta\phi$ be another level surface through the neighbouring point Q.



Let $\vec{r}$ and $\vec{r} + \delta\vec{r}$ be the position vectors of P and Q respectively then $\overrightarrow{PQ} = \delta\vec{r}$

Now,
$$\nabla\phi \cdot \delta\vec{r} = \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right) \cdot \left(\hat{i}\,\delta x + \hat{j}\,\delta y + \hat{k}\,\delta z\right)$$
$$= \frac{\partial\phi}{\partial x}\delta x + \frac{\partial\phi}{\partial y}\delta y + \frac{\partial\phi}{\partial z}\delta z = \delta\phi \qquad \text{...(1)}$$

If Q lies on the same surface as P, then $\delta\,\phi = 0$

$\therefore$ (1) reduces to $\nabla\phi \cdot \delta\vec{r} = 0$

$\therefore$ $\nabla\phi$ is perpendicular to $\delta\vec{r}$ which is true for all values of r.

Hence $\nabla\phi$ is normal to the surface $\phi(x, y, z) = c$.

Let $\nabla\phi = |\nabla\phi|\,\hat{N}$, where $\hat{N}$ is a unit normal to $\phi = c$ at P. Let PA $= \delta n$ be the perpendicular distance between the two level surfaces $\phi = c$ and $\phi + \delta\phi = c'$. Then rate of change of ϕ in the direction of normal to the surface through P is $\frac{\partial\phi}{\partial n} = \underset{\delta n \to 0}{\text{Lt}} \frac{\delta\phi}{\delta n} = \underset{\delta n \to 0}{\text{Lt}} \frac{\nabla\phi \cdot \delta\vec{r}}{\delta n}$ [$\because$ of (1)]

$$= \underset{\delta n \to 0}{\text{Lt}} \frac{|\nabla\phi|\,\hat{N} \cdot \delta\vec{r}}{\delta n}$$

Since, $$\hat{N} \cdot \delta\vec{r} = |\hat{N}|\,|\delta\vec{r}| \cos\theta = 1 \cdot \text{PQ} \cos\theta = \delta n$$

$$\therefore \quad \frac{\partial\phi}{\partial n} = \underset{\delta n \to 0}{\text{Lt}} \frac{|\nabla\phi|\,\delta n}{\delta n} = |\nabla\phi|$$

$$\therefore \quad |\nabla\phi| = \frac{\partial\phi}{\partial n}.$$

Hence the gradient of a scalar field ϕ is a vector normal to the surface $\phi = c$ and has a magnitude equal to the rate of change of ϕ along the normal.

Cor 1. Equation of the tangent plane to a surface $\phi(x, y, z) = c$ at a point A (x_1, y_1, z_1) can be derived from the gradient vector at that point.

Since gradient vector at a point A (x_1, y_1, z_1) on the surface $\phi(x, y, z) = c$ represents normal to the surface at that point $\therefore$ if we take P(x, y, z) be any point on the tangent plane at (x_1, y_1, z_1) then the vector $(x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k}$ will be perpendicular to the normal vector at (x_1, y_1, z_1)

$$\therefore \quad (\nabla\phi)_{(x_1, y_1, z_1)} \cdot \left[(x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k}\right] = 0$$

i.e., $$\left(\frac{\partial f}{\partial x}\right)_{(x_1, y_1, z_1)} (x - x_1) + \left(\frac{\partial\phi}{\partial z}\right)_{(x_1, y_1, z_1)} (y - y_1) + \left(\frac{\partial\phi}{\partial z}\right)_{(x_1, y_1, z_1)} (z - z_1) = 0$$

which is the equation of the tangent plane at (x_1, y_1, z_1) to $\phi(x, y, z) = c$.

Cor 2. Equation of the normal at A (x_1, y_1, z_1) to the surface $\phi(x, y, z) = c$.

Let P (x, y, z) be any variable point on the normal to the surface $\phi(x, y, z) = c$. Then $\overrightarrow{\text{AP}}$ is parallel to normal vector $\nabla\phi$ at (x_1, y_1, z_1)

$$\therefore \quad \overrightarrow{\text{AP}} \times (\nabla\phi)_{(x_1, y_1, z_1)} = \vec{0}$$

i.e., $$\left[(x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k}\right] \times \left[\left(\frac{\partial\phi}{\partial x}\right)_{(x_1, y_1, z_1)} \hat{i} + \left(\frac{\partial\phi}{\partial y}\right)_{(x_1, y_1, z_1)} \hat{j} + \left(\frac{\partial\phi}{\partial z}\right)_{(x_1, y_1, z_1)} \hat{k}\right] = \vec{0}$$

$$\therefore \quad \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x - x_1 & y - y_1 & z - z_1 \\ \left(\frac{\partial \phi}{\partial x}\right)_{(x_1, y_1, z_1)} & \left(\frac{\partial \phi}{\partial y}\right)_{(x_1, y_1, z_1)} & \left(\frac{\partial \phi}{\partial z}\right)_{(x_1, y_1, z_1)} \end{vmatrix} = \vec{0}$$

$$\therefore \quad \sum \left[(y - y_1) \left(\frac{\partial \phi}{\partial z}\right)_{(x_1, y_1, z_1)} - (z - z_1) \left(\frac{\partial \phi}{\partial y}\right)_{(x_1, y_1, z_1)} \right] \hat{i} = \vec{0}$$

$$\therefore \quad (y - y_1) \left(\frac{\partial \phi}{\partial z}\right)_{(x_1, y_1, z_1)} - (z - z_1) \left(\frac{\partial \phi}{\partial y}\right)_{(x_1, y_1, z_1)} = 0; \quad \therefore \quad \frac{y - y_1}{\left(\frac{\partial \phi}{\partial y}\right)_{(x_1, y_1, z_1)}} = \frac{z - z_1}{\left(\frac{\partial \phi}{\partial z}\right)_{(x_1, y_1, z_1)}}$$

Similarly, by equating components of $\hat{j}$ and $\hat{k}$ to zero, we get

$$\frac{z - z_1}{\left(\frac{\partial \phi}{\partial z}\right)_{(x_1, y_1, z_1)}} = \frac{x - x_1}{\left(\frac{\partial \phi}{\partial x}\right)_{(x_1, y_1, z_1)}} \quad \text{and} \quad \frac{x - x_1}{\left(\frac{\partial \phi}{\partial x}\right)_{(x_1, y_1, z_1)}} = \frac{y - y_1}{\left(\frac{\partial \phi}{\partial y}\right)_{(x_1, y_1, z_1)}}$$

Combining all three, the equation of the normal at A (x_1, y_1, z_1) is

$$\frac{x - x_1}{\left(\frac{\partial \phi}{\partial x}\right)_{(x_1, y_1, z_1)}} = \frac{y - y_1}{\left(\frac{\partial \phi}{\partial y}\right)_{(x_1, y_1, z_1)}} = \frac{z - z_1}{\left(\frac{\partial \phi}{\partial z}\right)_{(x_1, y_1, z_1)}}.$$

7.16. DIRECTIONAL DERIVATIVE

(P.T.U., Jan. 2010)

Let PQ = δr then $\underset{\delta r \to 0}{\text{Lt}} \frac{\delta \phi}{\delta r} = \frac{\partial \phi}{\partial r}$ is called directional derivative of ϕ at P in the direction of PQ.

Let $\hat{N}'$ be a unit vector in the direction of PQ then $\hat{N} . \hat{N}' = \cos \theta$

$$\delta r = \frac{\delta n}{\cos \theta} = \frac{\delta n}{\hat{N} \cdot \hat{N}'}$$

$$\therefore \quad \frac{\partial \phi}{\partial r} = \underset{\delta r \to 0}{\text{Lt}} \frac{\delta \phi}{\delta n} \hat{N} \cdot \hat{N}' = \hat{N} \cdot \hat{N}' \frac{\partial \phi}{\partial n}$$

$$\therefore \quad \frac{\partial \phi}{\partial r} = \hat{N} \cdot \hat{N}' |\nabla \phi| \qquad \therefore \quad |\nabla \phi| = \frac{\partial \phi}{\partial n} \quad \text{from art. 7·15}$$

$$= \hat{N}' \cdot |\nabla \phi| \hat{N} = \hat{N}' \cdot \nabla \phi \quad \because \quad \hat{N} |\nabla \phi| = \nabla \phi .$$

Thus the directional derivative $\frac{\partial \phi}{\partial r}$ *is the resolved part of* $\nabla\phi$ *in the direction of* $\hat{N}'$

i.e., $\vec{PQ}$

Since $$\frac{\partial \theta}{\partial r} = \hat{N}' \cdot \nabla\phi = |\nabla\phi| \cos\theta \le |\nabla\phi|$$

$\therefore$ **$\nabla\phi$ gives the maximum rate of change of ϕ and the magnitude of this maximum rate of change is $|\nabla\phi|$.**

7.17. PROPERTIES OF GRADIENT

(1) If ϕ is a constant scalar point function, then $\nabla\phi = \vec{0}$.

(2) If ϕ_1 and ϕ_2 are two scalar point functions, then

(*a*) $\nabla(\phi_1 \pm \phi_2) = \nabla\phi_1 \pm \nabla\phi_2$

(*b*) $\nabla(c_1\phi_1 + c_2\phi_2) = c_1\nabla\phi_1 + c_2\nabla\phi_2$, *where* c_1 *and* c_2 *are constants.*

(*c*) $\nabla(\phi_1\phi_2) = \phi_1\nabla\phi_2 + \phi_2\nabla\phi_1$

(*d*) $\nabla\left(\frac{f_1}{f_2}\right) = \frac{f_2\nabla f_1 - \phi_1\nabla f_2}{f_2^2}, \phi_2 \ne 0.$

Proof. (1) $$\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z} = 0$$

$$\left[\because \ \phi \text{ is constant } \therefore \frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial y} = \frac{\partial\phi}{\partial z} = 0\right]$$

(2) (*a*) $$\nabla(\phi_1 \pm \phi_2) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(\phi_1 \pm \phi_2)$$

$$= \hat{i}\frac{\partial}{\partial x}(\phi_1 \pm \phi_2) + \hat{j}\frac{\partial}{\partial y}(\phi_1 \pm \phi_2) + \hat{k}\frac{\partial}{\partial z}(\phi_1 \pm \phi_2)$$

$$= \hat{i}\left[\frac{\partial\phi_1}{\partial x} \pm \frac{\partial\phi_2}{\partial x}\right] + \hat{j}\left[\frac{\partial\phi_1}{\partial y} \pm \frac{\partial\phi_2}{\partial y}\right] + \hat{k}\left[\frac{\partial\phi_1}{\partial z} \pm \frac{\partial\phi_2}{\partial z}\right]$$

$$\nabla(\phi_1 \pm \phi_2) = \left(\hat{i}\frac{\partial\phi_1}{\partial x} + \hat{j}\frac{\partial\phi_1}{\partial y} + \hat{k}\frac{\partial\phi_1}{\partial z}\right) \pm \left(\hat{i}\frac{\partial\phi_2}{\partial x} + \hat{j}\frac{\partial\phi_2}{\partial y} + \hat{k}\frac{\partial\phi_2}{\partial z}\right)$$

$$= \nabla\phi_1 \pm \nabla\phi_2$$

(*b*) Students can easily prove it.

(*c*) $$\nabla(\phi_1\phi_2) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi_1\phi_2$$

$$= \hat{i}\frac{\partial}{\partial x}(\phi_1\phi_2) + \hat{j}\frac{\partial}{\partial y}(\phi_1\phi_2) + \hat{k}\frac{\partial}{\partial z}(\phi_1\phi_2)$$

$$= \hat{i}\left[\phi_1 \frac{\partial \phi_2}{\partial x} + \phi_2 \frac{\partial \phi_1}{\partial x}\right] + \hat{j}\left[\phi_1 \frac{\partial \phi_2}{\partial y} + \phi_2 \frac{\partial \phi_1}{\partial y}\right] + \hat{k}\left[\phi_1 \frac{\partial \phi_2}{\partial z} + \phi_2 \frac{\partial \phi_1}{\partial z}\right]$$

$$= \left(\hat{i}\,\phi_1 \frac{\partial \phi_2}{\partial x} + \hat{j}\,\phi_1 \frac{\partial \phi_2}{\partial y} + \hat{k}\,\phi_1 \frac{\partial \phi_2}{\partial z}\right) + \left(\hat{i}\,\phi_2 \frac{\partial \phi_1}{\partial x} + \hat{j}\,\phi_2 \frac{\partial \phi_1}{\partial y} + \hat{k}\,\phi_2 \frac{\partial \phi_1}{\partial z}\right)$$

$$= \phi_1 \nabla \phi_2 + \phi_2 \nabla \phi_1$$

(d) $$\nabla\left(\frac{\phi_1}{\phi_2}\right) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\frac{\phi_1}{\phi_2}\right) = \hat{i}\frac{\partial}{\partial x}\left(\frac{\phi_1}{\phi_2}\right) + \hat{j}\frac{\partial}{\partial y}\left(\frac{\phi_1}{\phi_2}\right) + \hat{k}\frac{\partial}{\partial z}\left(\frac{\phi_1}{\phi_2}\right)$$

$$= \hat{i}\,\frac{\phi_2 \frac{\partial}{\partial x}\phi_1 - \phi_1 \frac{\partial}{\partial x}\phi_2}{\phi_2^2} + \hat{j}\,\frac{\phi_2 \frac{\partial}{\partial y}\phi_1 - \phi_1 \frac{\partial}{\partial y}\phi_2}{\phi_2^2} + \hat{k}\,\frac{\phi_2 \frac{\partial}{\partial z}\phi_1 - \phi_1 \frac{\partial}{\partial z}\phi_2}{\phi_2^2}$$

$$= \frac{1}{\phi_2^2}\left[\left(\hat{i}\,\phi_2 \frac{\partial \phi_1}{\partial x} + \hat{j}\,\phi_2 \frac{\partial \phi_1}{\partial y} + \hat{k}\,\phi_2 \frac{\partial \phi_1}{\partial z}\right) - \left(\hat{i}\,\phi_1 \frac{\partial \phi_2}{\partial x} + \hat{j}\,\phi_1 \frac{\partial \phi_2}{\partial y} + \hat{k}\,\phi_1 \frac{\partial \phi_2}{\partial z}\right)\right]$$

$$= \frac{1}{\phi_2^2}\left[\phi_2 \nabla \phi_1 - \phi_1 \nabla \phi_2\right]$$

$$\therefore \quad \nabla\left(\frac{\phi_1}{\phi_2}\right) = \frac{\phi_2 \nabla \phi_1 - \phi_1 \nabla \phi_2}{\phi_2^2}; \quad \phi_2 \neq 0.$$

ILLUSTRATIVE EXAMPLES

Example 1. *Find gradient of the following functions*

(i) $\phi = y^2 - 4xy$ *at (1, 2)*

(ii) $\phi = x^3 + y^3 + 3xyz$ *at (1, –2, –1).* **(P.T.U., May 2008)**

Sol. *(i)* $\phi = y^2 - 4xy$

$$\text{grad. } \phi = \nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(y^2 - 4xy)$$

$$= \hat{i}(-4y) + \hat{j}(2y - 4x)$$

At (1, 2); $\quad \text{grad. } \phi = -8\hat{i} + 0\hat{j} = -8\hat{i}$

(ii) $\quad \phi = x^3 + y^3 + 3xyz$

$$\text{grad. } \phi = \nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^3 + y^3 + 3xyz)$$

$$= \hat{i}(3x^2 + 3yz) + \hat{j}(3y^2 + 3xz) + \hat{k}(3xy)$$

At (1, –2, –1) ; $\text{grad. } \phi = 9\hat{i} + 9\hat{j} - 6\hat{k} = 3(3\hat{i} + 3\hat{j} - 2\hat{k})$.

Example 2. *Find a unit vector normal to the surface $x^3 + y^3 + 3xyz = 3$ at the point $(1, 2, -1)$.* **(P.T.U., Dec. 2012)**

Sol. Let $\phi(x, y, z) = x^3 + y^3 + 3xyz - 3$

We know that $\nabla\phi$ is a vector normal to the surface $\phi = c$

$$\therefore \quad \nabla\phi = \text{grad}\,\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^3 + y^3 + 3xyz - 3)$$

$$= \hat{i}(3x^2 + 3yz) + \hat{j}(3y^2 + 3xz) + \hat{k}(3xy).$$

Normal vector at $(1, 2, -1)$ is $-3\hat{i} + 9\hat{j} + 6\hat{k}$

Unit normal vector at $(1, 2, -1)$

$$= \frac{3(-\hat{i} + 3\hat{j} + 2\hat{k})}{3\sqrt{1+9+4}} = \frac{-\hat{i} + 3\hat{j} + 2\hat{k}}{\sqrt{14}}$$

Example 3. *Find the normal vector and the equation of the tangent plane to the surface $z = \sqrt{x^2 + y^2}$ at the point $(1, 1, \sqrt{2})$.* **(P.T.U., Jan. 2008)**

Sol. Let $\phi(x, y, z) = \sqrt{x^2 + y^2} - z$

$$\frac{\partial\phi}{\partial x} = \frac{1}{2\sqrt{x^2+y^2}} \cdot 2x = \frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{\partial\phi}{\partial y} = \frac{1}{2\sqrt{x^2+y^2}} \cdot 2y = \frac{y}{\sqrt{x^2+y^2}}$$

$$\frac{\partial\phi}{\partial z} = -1$$

We know that $\nabla\phi$ is a vector normal to the surface $\phi = C$

$$\therefore \quad \nabla\phi = \text{grad}\,\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi$$

$$= \hat{i}\frac{x}{\sqrt{x^2+y^2}} + \hat{j}\frac{y}{\sqrt{x^2+y^2}} + \hat{k}(-1)$$

$$= \frac{x\hat{i} + y\hat{j} - \hat{k}\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}$$

Normal vector at $\quad (1, 1, \sqrt{2}) = \dfrac{\hat{i} + \hat{j} - \sqrt{2}\,\hat{k}}{\sqrt{2}}$

Unit normal vector at $(1, 1, \sqrt{2}) = \dfrac{\hat{i} + \hat{j} - \sqrt{2}\hat{k}}{\sqrt{2}\sqrt{\frac{1}{2} + \frac{1}{2} + 1}} = \dfrac{\hat{i} + \hat{j} - \sqrt{2}\hat{k}}{2}$

Now the equation of the tangent plane at $(1, 1, \sqrt{2})$ is

$$(x-1)\left(\frac{\partial\phi}{\partial x}\right)_{(1,1,\sqrt{2})} + (y-1)\left(\frac{\partial\phi}{\partial y}\right)_{(1,1,\sqrt{2})} + (z-\sqrt{2})\left(\frac{\partial\phi}{\partial z}\right)_{(1,1,\sqrt{2})} = 0$$

i.e.,
$$(x-1)\frac{1}{\sqrt{2}}+(y-1)\left(\frac{1}{\sqrt{2}}\right)+(z-\sqrt{2})(-1)=0$$

or
$$x-1+y-1-\sqrt{2}\,(z-\sqrt{2})=0 \text{ or } x+y-\sqrt{2}z=0.$$

Example 4. *If* $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$, *show that*

(i) *grad* $r=\frac{\vec{r}}{r}$ (ii) *grad* $\left(\frac{1}{r}\right)=-\frac{\vec{r}}{r^3}$ **(P.T.U., Dec. 2003)**

(iii) $\nabla r^n=n\,r^{n-2}\vec{r}$

(iv) $\nabla(\vec{a}\cdot\vec{r})=\vec{a}$, *where* $\vec{a}$ *is a constant vector.* **(P.T.U., Dec. 2002)**

Sol. (i) $r=|\vec{r}|=\sqrt{x^2+y^2+z^2}$ or $r^2=x^2+y^2+z^2$

Differentiate partially w.r.t. x, y and z respectively, we get $2r\frac{\partial r}{\partial x}=2x$, $\frac{\partial r}{\partial x}=\frac{x}{r}$.

Similarly, $\frac{\partial r}{\partial y}=\frac{y}{r}$, $\frac{\partial r}{\partial z}=\frac{z}{r}$

Now, $$\text{grad } r=\nabla r=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(r)$$
$$=\hat{i}\frac{\partial r}{\partial x}+\hat{j}\frac{\partial r}{\partial y}+\hat{k}\frac{\partial r}{\partial z}=\hat{i}\frac{x}{r}+\hat{j}\frac{y}{r}+\hat{k}\frac{z}{r}=\frac{1}{r}\left(x\hat{i}+y\hat{j}+z\hat{k}\right)=\frac{\vec{r}}{r}$$

Hence $$\text{grad } r=\frac{\vec{r}}{r}$$

(ii) $$\text{grad }\frac{1}{r}=\nabla\left(\frac{1}{r}\right)=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\left(\frac{1}{r}\right)$$
$$=\hat{i}\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial x}+\hat{j}\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial y}+\hat{k}\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial z}$$
$$=-\frac{1}{r^2}\left(\hat{i}\frac{x}{r}+\hat{j}\frac{y}{r}+\hat{k}\frac{z}{r}\right)=-\frac{1}{r^3}\left(x\hat{i}+y\hat{j}+z\hat{k}\right)=-\frac{\vec{r}}{r^3}$$

(iii) $$\nabla r^n=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)r^n$$
$$=\hat{i}\,nr^{n-1}\frac{\partial r}{\partial x}+\hat{j}\,nr^{n-1}\frac{\partial r}{\partial y}+\hat{k}\,nr^{n-1}\frac{\partial r}{\partial z}$$
$$=nr^{n-1}\left[\hat{i}\frac{x}{r}+\hat{j}\frac{y}{r}+\hat{k}\frac{z}{r}\right]=nr^{n-2}\left(x\hat{i}+y\hat{j}+z\hat{k}\right)=nr^{n-2}\vec{r}$$

(iv) $\nabla(\vec{a}\cdot\vec{r})$

Let $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, where a_1, a_2, a_3 are constants.

$$\vec{a}\cdot\vec{r} = a_1x + a_2y + a_3z \qquad \text{Since } \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\nabla(\vec{a}\cdot\vec{r}) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(a_1x + a_2y + a_3z)$$

$$= \hat{i}\frac{\partial}{\partial x}(a_1x + a_2y + a_3z) + \hat{j}\frac{\partial}{\partial y}(a_1x + a_2y + a_3z) + \hat{k}\frac{\partial}{\partial z}(a_1x + a_2y + a_3z)$$

$$= \hat{i}\,a_1 + \hat{j}\,a_2 + \hat{k}\,a_3 = a_1\hat{i} + a_2\hat{j} + a_3\hat{k} = \vec{a}$$

Hence $\nabla(\vec{a}\cdot\vec{r}) = \vec{a}$

Example 5. *What is the directional derivative of the function $xy^2 + yz^3$ at the point $(2, -1, 1)$ in the direction of the vector $\hat{i} + 2\hat{j} + 2\hat{k}$?* **(P.T.U., Dec. 2011)**

Sol. Let $\phi(x, y, z) = xy^2 + yz^3$

Gradiert of ϕ $= \nabla\phi = \hat{i}\dfrac{\partial\phi}{\partial x} + \hat{j}\dfrac{\partial\phi}{\partial y} + \hat{k}\dfrac{\partial\phi}{\partial z}$

$$= \hat{i}\,y^2 + \hat{j}(2xy + z^3) + \hat{k}(3yz^2)$$

$$\nabla\phi \text{ at } (2, -1, 1) = \hat{i} - 3\hat{j} - 3\hat{k}$$

If $\hat{n}$ is a unit vector in the direction of $\hat{i} + 2\hat{j} + 2\hat{k}$, then $\hat{n} = \dfrac{\hat{i} + 2\hat{j} + 2\hat{k}}{\sqrt{1+4+4}} = \dfrac{1}{3}(\hat{i} + 2\hat{j} + 2\hat{k})$

$\therefore$ Directional derivative of the given function ϕ at $(2, -1, 1)$ in the direction of

$$\hat{i} + 2\hat{j} + 2\hat{k} = [\nabla\phi \text{ at } (2, -1, 1)]\cdot\hat{n}$$

$$= (\hat{i} - 3\hat{j} - 3\hat{k})\cdot\frac{1}{3}(\hat{i} + 2\hat{j} + 2\hat{k}) = \frac{1 - 6 - 6}{3} = -\frac{11}{3}$$

Example 6. *Find all the directional derivative of the function $f = x^2 - y^2 + 2z^2$ at the point $P(1, 2, 3)$ in the direction of the line PQ, where Q is the point $(5, 0, 4)$.*

In what direction it will be maximum? Find also the magnitude of this maximum.

Sol. Gradient of $f = \nabla f$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 - y^2 + 2z^2) = \hat{i}(2x) + \hat{j}\big((-2y) + \hat{k}(4z)\big)$$

$$\nabla f \text{ at } (1, 2, 3) = 2\hat{i} - 4\hat{j} + 12\hat{k}.$$

Now $\overrightarrow{PQ}$ = P.V of Q − P.V. of P $= 5\hat{i} + 4\hat{k} - (\hat{i} + 2\hat{j} + 3\hat{k}) = 4\hat{i} - 2\hat{j} + \hat{k}$

If $\hat{n}$ is a unit vector in the direction $\overrightarrow{PQ}$, then $\hat{n} = \dfrac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{16+4+1}} = \dfrac{1}{\sqrt{21}}(4\hat{i} - 2\hat{j} + \hat{k})$

Direction derivative of f at (1, 2, 3) in the direction of $\overrightarrow{PQ} = [\nabla f \text{ at } (1,2,3)] \cdot \hat{n}$

$$= (2\hat{i}+4\hat{j}+12\hat{k}) \cdot \frac{1}{\sqrt{21}}(4\hat{i}+2\hat{j}+\hat{k}) = \frac{1}{\sqrt{21}}(8+8+12) = \frac{28}{\sqrt{21}} = \frac{28}{21}\sqrt{21} = \frac{4}{3}\sqrt{21}$$

The directional derivative of f is maximum in the direction of the normal to the given surface *i.e.*, in the direction of $\nabla f = (2\hat{i}-4\hat{j}+12\hat{k})$.

The maximum value of this directional derivative $= |\nabla f| = \sqrt{4+16+144}$

$$= \sqrt{164} = 2\sqrt{41}.$$

Example 7. *Find the directional derivative of* $\phi = e^{2x} \cos yz$ *at the origin in the direction of the tangent to the curve* $x = a \sin t$, $y = a \cos t$, $z = at$ *at* $t = \frac{\pi}{4}$.

Sol. Gradient of $\phi = \nabla \phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(e^{2x} \cos yz)$

$$= \hat{i}(2e^{2x}\cos yz) + \hat{j}(-e^{2x} z \sin yz) + \hat{k}\{e^{2x}(-\sin yz)y\}$$

At the origin *i.e.*, when $x = 0, y = 0, z = 0$.

$$\nabla\phi = \hat{i}(2) = 2\hat{i}$$

Equation of the curve is $x = a \sin t, y = a \cos t, z = at$

Any point on the curve is $\vec{r} = \hat{i}(a \sin t) + \hat{j}(a \cos t) + \hat{k}(at)$

Direction of the tangent is given by $= \frac{d\vec{r}}{dt} = (a \cos t)\hat{i} - (a \sin t)\hat{j} + a\hat{k}$

At $t = \frac{\pi}{4}$, direction of tangent $= \frac{a}{\sqrt{2}}\hat{i} - \frac{a}{\sqrt{2}}\hat{j} + a\hat{k}$

$\hat{n}$ = unit direction of the tangent

$$= \frac{\frac{a}{\sqrt{2}}\hat{i} - \frac{a}{\sqrt{2}}\hat{j} + a\hat{k}}{\sqrt{\frac{a^2}{2} + \frac{a^2}{2} + a^2}} = \frac{\frac{a}{\sqrt{2}}(\hat{i} - \hat{j} + \sqrt{2}\,\hat{k})}{\sqrt{2}\,a} = \frac{1}{2}(\hat{i} - \hat{j} + \sqrt{2}\,\hat{k})$$

Directional derivative of ϕ at (0, 0, 0) in the direction of tangent at $t = \frac{\pi}{4}$ is $= \nabla\phi \cdot \hat{n}$ at (0, 0, 0).

$$= 2\hat{i} \cdot \frac{1}{2}(\hat{i} - \hat{j} + \sqrt{2}\,\hat{k}) = 1.$$

Example 8. *Find the directional derivative of $\vec{V}^2$, where $\vec{V} = xy^2\hat{i} + zy^2\hat{j} + xz^2\hat{k}$ at the point (2, 0, 3) in the direction of outward normal to the sphere $x^2 + y^2 + z^2 = 14$ at the point (3, 2, 1).*

Sol.
$$\vec{V} = xy^2\hat{i} + zy^2\hat{j} + xz^2\hat{k}$$

$$\vec{V}^2 = \vec{V}\cdot\vec{V} = x^2y^4 + z^2y^4 + x^2z^4$$

$$\text{Gradient of } \vec{V}^2 = \nabla(V^2) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(x^2y^4 + z^2y^4 + x^2z^4\right)$$

$$= \hat{i}\left(2xy^4 + 2xz^4\right) + \hat{j}\left(4x^2y^3 + 4z^2y^3\right) + \hat{k}\left(2zy^4 + 4x^2z^3\right)$$

Gradient of $\vec{V}^2$ at $(2, 0, 3) = \hat{i}(324) + \hat{j}(0) + \hat{k}(432) = 108(3\hat{i} + 4\hat{k})$

Normal to the sphere $x^2 + y^2 + z^2 = 14$ is ∇f

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(x^2 + y^2 + z^2 - 14\right) = \hat{i}(2x) + \hat{j}(2y) + \hat{k}(2z)$$

Normal vector at $(3, 2, 1) = 6\hat{i} + 4\hat{j} + 2\hat{k}$

$$\hat{n} = \text{Unit Normal vector at } (3, 2, 1) = \frac{6\hat{i} + 4\hat{j} + 2\hat{k}}{\sqrt{36+16+4}} = \frac{6\hat{i} + 4\hat{j} + 2\hat{k}}{2\sqrt{14}} = \frac{3\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{14}}$$

Directional derivative of $\vec{V}^2$ at (2, 0, 3) along the normal at (3, 2, 1)

$$= 108\left(3\hat{i} + 4\hat{k}\right)\cdot\frac{3\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{14}}$$

$$= \frac{108}{\sqrt{14}}(9+4) = \frac{(108)(13)}{\sqrt{14}} = \frac{1404}{\sqrt{14}}.$$

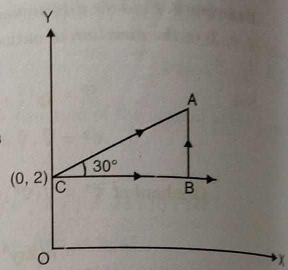
Example 9. *For the function $\phi(x, y) = \dfrac{x}{x^2 + y^2}$, find the magnitude of the directional derivative along a line making an angle 30° with the positive axis at (0, 2).*

Sol. Gradient of $\phi = \nabla\phi$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\frac{x}{x^2 + y^2}\right) = \hat{i}\left[\frac{\left(x^2 + y^2\right)\cdot 1 - x\cdot 2x}{\left(x^2 + y^2\right)^2}\right] + \hat{j}\left(\frac{-x\cdot 2y}{\left(x^2 + y^2\right)^2}\right)$$

$$= \frac{y^2 - x^2}{\left(x^2 + y^2\right)^2}\hat{i} - \frac{2xy}{\left(x^2 + y^2\right)^2}\hat{j}$$

Gradient of ϕ at $(0, 2) = \frac{4}{16}\hat{i} - 0 = \frac{\hat{i}}{4}$



Now, $\overrightarrow{CA} = \overrightarrow{CB} + \overrightarrow{BA}$

$\overrightarrow{CB} = CA\cos 30^\circ\,\hat{i}$ $\because$ CB is $\parallel$ to X-axis

$\overrightarrow{BA} = CA\sin 30^\circ\,\hat{j}$ $\because$ BA is $\parallel$ to Y-axis

$\therefore$ $\overrightarrow{CA} = CA\left\{\cos 30^\circ\,\hat{i} + \sin 30^\circ\,\hat{j}\right\}$

$$\frac{\overrightarrow{CA}}{CA} = \frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j} = \frac{\sqrt{3}\hat{i} + \hat{j}}{2}$$

$$\widehat{CA} = \frac{1}{2}\left(\sqrt{3}\,\hat{i} + \hat{j}\right)$$

Directional derivative of ϕ at (0, 2) in the direction of $\overrightarrow{CA} = \frac{\hat{i}}{4}\cdot\widehat{CA} = \frac{\hat{i}}{4}\cdot\frac{1}{2}\left(\sqrt{3}\hat{i} + \hat{j}\right) = \frac{\sqrt{3}}{8}$.

Example 10. *The temperature at any point in space is given by $T = xy + yz + zx$. Determine the directional derivative of T in the direction of the vector $3\hat{i} - 4\hat{k}$ at the point (1, 1, 1).*

Sol. $T = xy + yz + zx$

Gradient of $T = \nabla T = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(xy + yz + zx) = \hat{i}(y+z) + \hat{j}(z+x) + \hat{k}(x+y)$

Gradient of T at $(1, 1, 1) = 2\hat{i} + 2\hat{j} + 2\hat{k}$

Directional derivative of T at (1, 1, 1) in the direction of $\left(3\hat{i} - 4\hat{k}\right)$

$$= \left(2\hat{i} + 2\hat{j} + 2\hat{k}\right)\cdot\left(\frac{3\hat{i} - 4\hat{k}}{\sqrt{9+16}}\right) = \frac{1}{5}(2\cdot 3 - 2\cdot 4) = \frac{-2}{5}.$$

Example 11. *(i) In what direction from (3, 1, – 2) is the directional derivative of $f = x^2y^2z^4$ maximum? Find also the magnitude of this maximum.*

(ii) Find the maximum value of directional derivative of $f = x^2 - 2y^2 + 4z^2$ at the point (1, 1, –1). **(P.T.U., May 2009)**

Sol. (*i*) Gradient of $\phi = \nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(x^2y^2z^4\right)$

$$= \hat{i}\left(2xy^2z^4\right) + \hat{j}\left(2x^2yz^4\right) + \hat{k}\left(4x^2y^2z^3\right)$$

Gradient of ϕ at $(3, 1, -2) = 96\hat{i} + 288\hat{j} - 288\hat{k}$

Directional derivative is maximum in the direction given by $\Delta\phi$ at $(3, 1, -2)$ $= 96\hat{i} + 288\hat{j} - 288\hat{k}$

In any other direction the magnitude of the directional derivative will be less than its maximum value which is

$$= \sqrt{(96)^2 + (288)^2 + (288)^2} = 96\sqrt{1+9+9} = 96\sqrt{19}.$$

(ii) We know that the maximum value of directional derivative of $f = x^2 - 2y^2 + 4z^2$ at $(1, 1, -1)$ is obtained from the value of gradient of f

$$\therefore \quad \text{grad } f = \nabla f$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 - 2y^2 + 4z^2)$$

$$= 2x\hat{i} - 4y\hat{j} + 8z\hat{k}$$

Value of ∇f at $(1, 1, -1)$

$$= 2\hat{i} - 4\hat{j} - 8\hat{k}$$

$\therefore$ Maximum value of directional derivative of f at $(1, 1, -1)$

$$= \sqrt{4+16+64} = \sqrt{84} = 2\sqrt{21}$$

Example 12. *Let $f(x, y, z)$ and $\phi(x, y, z)$ be two scalar functions. Find an expression for $\nabla^2(fg)$ in terms of $\nabla^2 f$, $\nabla^2 g$, ∇f and ∇g.*

Sol.

$$\nabla(fg) = f\nabla(g) + g(\nabla f)$$

$$\nabla^2(fg) = \nabla[\nabla(fg)] = \nabla\{f(\nabla g)\} + \nabla\{g(\nabla f)\}$$

$$= f(\nabla^2 g) + (\nabla f) \cdot (\nabla g) + (\nabla g) \cdot (\nabla f) + g(\nabla^2 f)$$

$$= f(\nabla^2 g) + 2(\nabla f) \cdot (\nabla g) + g(\nabla^2 f).$$

Example 13. *Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point $(2, -1, 2)$.*

Sol. Let $\quad \phi_1 = x^2 + y^2 + z^2 - 9$

and $\quad \phi_2 = x^2 + y^2 - z - 3$

$$\text{grad } \phi_1 = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2 - 9) = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

Similarly, $\quad \text{grad } \phi_2 = 2x\hat{i} + 2y\hat{j} - \hat{k}$

Now, angle between the two surfaces at $(2, -1, 2)$ is the angle between their normals at the point.

$\therefore$ Let $\vec{n}_1$ and $\vec{n}_2$ be the normal vectors to ϕ_1 and ϕ_2 respectively at $(2, -1, 2)$.

Now,

$$\vec{n}_1 = \text{grad } \phi_1 = \nabla\phi_1 \text{ at } (2, -1, 2) = 4\hat{i} - 2\hat{j} + 4\hat{k}.$$

$$\vec{n}_2 = \text{grad } \phi_2 = \nabla\phi_2 \text{ at } (2, -1, 2) = 4\hat{i} - 2\hat{j} - \hat{k}$$

If θ be the angle between $\vec{n}_1$ and $\vec{n}_2$ then

$$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|} = \frac{4(4) - 2(-2) + 4(-1)}{\sqrt{16+4+16}\sqrt{16+4+1}}$$

$$\cos\theta = \frac{16+4-4}{6\sqrt{21}} = \frac{16}{6\sqrt{21}} = \frac{8}{3\sqrt{21}}$$

$$\therefore \quad \theta = \cos^{-1}\frac{8}{3\sqrt{21}}.$$

TEST YOUR KNOWLEDGE

1. Find grad ϕ at $(1, -2, -1)$ if
   (i) $\phi = 3x^2y - y^3z^2$ (ii) $\phi = x^2 + yz$
   (iii) $\phi = \log(x^2 + y^2 + z^2)$
2. Find a unit vector normal to the following surfaces
   (i) $z^2 = x^2 + y^2$ at the point $(1, 0, -1)$
   (ii) $xy^3z^2 = 4$ at the point $(-1, -1, 2)$
   (iii) $x^2y + 2xz = 4$ at the point $(2, -2, 3)$
   (iv) $z = x^3 + y^2$ at $(1, -2, 5)$ **(P.T.U. May 2002)**
   (v) $x^2 + 3y^2 + 2z^2 = 6$ at $(1, 0, 1)$
   (vi) $z = \sqrt{x^2 + y^2}$ at $(3, 4, 5)$ **(P.T.U. Jan. 2008)**
   [**Hint.** Consult S.E. 3]
3. If $r = |\vec{r}|$ where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, prove that
   (i) $\nabla f(r) = f'(r)\nabla(r)$ (ii) $\nabla(\log r) = \frac{\vec{r}}{r^2}$
   [**Hint.** S.E. 10 art 7.23]
   (iii) $\nabla\left(e^{r^2}\right) = 2e^{r^2}\vec{r}$ (iv) grad $|\vec{r}|^2 = 2\vec{r}$
4. (i) Find the directional derivative of $\phi = x^2 + y^2 + z^2$ at the point $(2, 2, 1)$ in the direction of $2\hat{i} + 2\hat{j} + \hat{k}$.
   (ii) What is the directional derivative of $2xy + z^2$ at the point $(1, -1, 3)$ in the direction of the vector $\hat{i} + 2\hat{j} + 2\hat{k}$? **(P.T.U. May 2004)**
5. Find the directional derivative of $\phi = 4xz^3 - 3x^2yz^2$ at the point $(2, -1, 2)$ along Z-axis.
6. Find the directional derivative of $f = 3e^{2x-y+z}$ at the point A $(1, 1, -1)$ in the direction $\overline{AB}$ where B is the point $(-3, 5, 6)$.
7. (i) Calculate the directional derivative of the function $\phi(x, y, z) = xy^2 + yz^3$ at the point $(1, -1, 1)$ in the direction of $(3, 1, -1)$.
   (ii) Find the directional derivative of $f(x, y, z) = x^2y^2z^2$ at the point $(1, 1, -1)$ in the direction of tangent to the curve $x = e^t$, $y = 2\sin t + 1$, $z = t - \cos t$ at $t = 0$.
   [**Hint.** See solved example 7]
8. Find the direction in which the directional derivative of $f(x, y) = (x^2 - y^2)xy$ at $(1, 1)$ is zero.

9. Find the directional derivative of the function $\phi = xy^2 + yz^3$
   (*i*) In the direction of $\hat{i} + 2\hat{j} + 2\hat{k}$ at the point (2, –1, 1)
   [**Hint.** See solved example 5] **(P.T.U., Dec. 2011)**
   (*ii*) In the direction of outward normal to the surface $x \log z - y^2 + 4 = 0$ at (–1, 2, 1)
   [**Hint.** See solved example 8]
10. Find the directional derivative of the scalar function $f(x, y, z) = xyz$ in the direction of the outer normal to the surface $z = xy$ at the point (3, 1, 3).
11. What is the greatest rate of increase of $u = x^2 + y z^2$ at the point (1, –1, 3) ?
12. If θ is the acute angle between the surfaces $xy^2z = 3x + z^2$ and $3x^2 - y^2 + 2z = 1$ at the point (1, –2, 1), show that $\cos q = \dfrac{3}{7\sqrt{6}}$.
13. Calculate the angle between the normals to the surface $xy = z^2$ at the point (4, 1, 2) and (3, 3, –3).
14. Find the angle between tangent planes to the surfaces $x \log z = y^2 - 1$ and $x^2 y = 2 - z$ at the point (1, 1, 1).
15. Find the values of constants a and b so that the surfaces $ax^2 - byz = (a + 2)x$ and $4x^2 y + z^3 = 4$ may intersect orthogonally at the point (1, –1, 2).
   [**Hint.** The point P (1, –1, 2) lies on both the surfaces and $(\nabla\phi_1 \text{ at P}) \cdot (\nabla\phi_2 \text{ at P}) = 0$]
16. The temperature at any point (x, y, z) in space is given by $T(x, y, z) = x^2 + y^2 - z$. A mosquito located at (1, 1, 2) desires to fly in such a direction that it will get warm as soon as possible. In which direction should it fly ?
17. If the directional derivative of $f(x, y, z) = axy + byz + czx$ at (1, 1, 1) has the maximum magnitude 4 in a direction parallel to x-axis then find the values of a, b, c.
18. Find the equation of the tangent plane to the surface $2xz^2 - 3xy - 4x = 7$ at the point (1, –1, 2). Also find the equation of the normal at (1, –1, 2).

Answers

1. (*i*) $-12\hat{i} - 9\hat{j} - 16\hat{k}$ (*ii*) $2x\hat{i} + 2\hat{j} + y\hat{k}$ (*iii*) $\frac{1}{3}(\hat{i} - 2\hat{j} - \hat{k})$

2. (*i*) $\frac{1}{\sqrt{2}}(\hat{i} + \hat{k})$ (*ii*) $-\frac{1}{\sqrt{11}}(\hat{i} + 3\hat{j} - \hat{k})$ (*iii*) $\frac{1}{3}(-\hat{i} + 2\hat{j} + 2\hat{k})$

   (*iv*) $\frac{2\hat{i} - 4\hat{j} - \hat{k}}{\sqrt{21}}$ (*v*) $\frac{\hat{i} + 2\hat{k}}{\sqrt{5}}$ (*vi*) $\frac{3\hat{i} + 4\hat{j} - 5\hat{k}}{5\sqrt{2}}$

4. (*i*) 6 (*ii*) $\frac{14}{3}$ 5. 144

6. $-\frac{5}{3}$ 7. (*i*) $\frac{5}{\sqrt{11}}$ (*ii*) $\frac{2\sqrt{6}}{3}$

8. $\frac{\hat{i} + \hat{j}}{\sqrt{2}}$. 9. (*i*) $-\frac{11}{3}$ (*ii*) $\frac{15}{\sqrt{17}}$

10. $\frac{27}{\sqrt{11}}$ 11. $2\hat{i} + 9\hat{j} - 6\hat{k}$ 13. $\cos^{-1}\sqrt{\frac{3}{62}}$

14. $\cos^{-1}\frac{1}{\sqrt{30}}$ 15. $a = 2.5, b = 1$ 16. $\frac{1}{3}(2\hat{i} + 2\hat{j} - \hat{k})$

17. $a = 2, b = -2, c = 2$ 18. $7x - 3y + 8z - 26 = 0; \frac{x-1}{7} = \frac{y+1}{-3} = \frac{z-2}{8}$.

7.18. DIVERGENCE OF A VECTOR FUNCTION

The divergence of a continuously differentiable vector point function $\vec{V}$ is denoted by div $\vec{V}$ and is defined as

$$\text{div } \vec{V} = \nabla \cdot \vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot \vec{V} = \hat{i}\cdot\frac{\partial \vec{V}}{\partial x} + \hat{j}\cdot\frac{\partial \vec{V}}{\partial y} + \hat{k}\cdot\frac{\partial \vec{V}}{\partial z}$$

Clearly, divergence of a vector point function is a scalar point function.

If $\quad \vec{V} = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$

then $\quad \text{div } \vec{V} = \nabla \cdot \vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(V_1\hat{i} + V_2\hat{j} + V_3\hat{k}\right) = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}.$

For example, if $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then $\text{div } \vec{r} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1 = 3.$

7.19. PHYSICAL INTERPRETATION OF DIVERGENCE

Let us consider the case of a fluid flow. Consider a small rectangular parallelopiped of dimensions dx, dy, dz parallel to X-axis, Y-axis and Z-axis respectively.

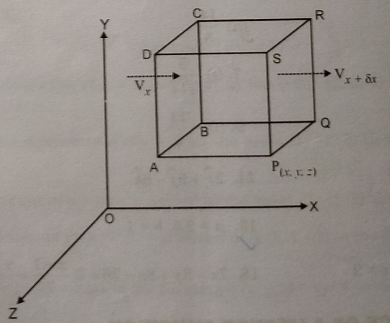
Let $\vec{V} = V_x\hat{i} + V_y\hat{j} + V_z\hat{k}$ be the velocity of fluid at P (x, y, z) where (V_x, V_y, V_z) are components of $\vec{V}$ parallel to X-axis, Y-axis, Z-axis respectively.

Mass of the fluid flowing in through the face ABCD per unit time = Velocity × area of the face = $V_x\,(dy\,dz)$

∴ Mass of the fluid flowing out across the face PQRS per unit time = $V_{x+\delta x}\,(dy\,dz)$.

$$V_{x+\delta x} = V_x + \delta x\,\frac{\partial V_x}{\partial x} + \cdots \text{ by Taylor's theorem.}$$

$$\therefore \quad V_{x+\delta x}\,(dydz) = \left(V_x + \frac{\partial V_x}{\partial x}\,dx\right)(dydz)$$



Decrease in mass of fluid in the parallelopiped corresponding to the flow along X-axis per unit time

$$= V_x\,dy\,dz - \left(V_x + \frac{\partial V_x}{\partial x}dx\right)dy\,dz = -\frac{\partial V_x}{\partial x}dx\,dy\,dz \qquad \text{(–ve sign shows decrease)}$$

Similarly, the decrease in mass of fluid to the flow along Y-axis $= \frac{\partial V_y}{\partial y}dx\,dy\,dz$ and decrease in mass along Z-axis $= \frac{\partial V_z}{\partial z}dx\,dy\,dz$.

Total decrease in mass of fluid per unit time $= \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right)dx\,dy\,dz$

The rate of loss of fluid per unit volume $= \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\hat{i}V_x + \hat{j}V_y + \hat{k}V_z\right) = \nabla\cdot\vec{V} = \text{div}\,\vec{V} \qquad \ldots(1)$$

$\therefore$ div $\vec{V}$ is the rate at which the fluid is flowing at a point per unit volume. If the flux entering any element of the space is the same as that leaving it *i.e.,* div $\vec{V} = 0$ everywhere then such a point function is called a **Solenoidal Vector Function.**

Equation (1) is also called the equation of continuity or conservation of mass.

Note. **For details of the solenoidal vector function consult chapter 8 art 8.2.**

7.20. CURL OF A VECTOR POINT FUNCTION *(P.T.U., May 2007)*

The curl of a continuously differentiable vector point function $\vec{V}$ is defined by the equation

$$\text{Curl}\,\vec{V} = \nabla\times\vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times\left(\vec{V}\right)$$

$$\text{Curl}\,\vec{V} = \hat{i}\times\frac{\partial\vec{V}}{\partial x} + \hat{j}\times\frac{\partial\vec{V}}{\partial y} + \hat{k}\times\frac{\partial\vec{V}}{\partial z}$$

Clearly the curl of a vector point function is a vector point function.

If $\qquad \vec{V} = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$

Then $\qquad \text{curl}\,\vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times\left(V_1\hat{i} + V_2\hat{j} + V_3\hat{k}\right)$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix} = \hat{i}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) + \hat{j}\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right) + \hat{k}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)$$

Note. Curl of a vector point function is also called rotation of a vector point function.

7.21. PHYSICAL INTERPRETATION OF CURL OF A VECTOR POINT FUNCTION

(P.T.U., Dec. 2003, May 2010, Dec. 2011)

Consider a rigid body rotating about a fixed axis through the point O with uniform angular velocity ω. If $\vec{V}$ be the Linear Velocity and $\vec{r}$ be the position vector of any point on the rotating body.

then $\vec{V} = \vec{\omega} \times \vec{r}$

then $\text{curl } \vec{V} = \nabla \times \vec{V} = \nabla \times (\vec{\omega} \times \vec{r})$

Let $\vec{\omega} = \omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k}$ and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

then $\text{curl } \vec{V} = \nabla \times \left\{ \left(\omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k} \right) \times \left(x\hat{i} + y\hat{j} + z\hat{k} \right) \right\}$

$$= \nabla \times \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix} = \nabla \times \left\{ \hat{i}\,(\omega_2 z - \omega_3 y) + \hat{j}\,(\omega_3 x - \omega_1 z) + \hat{k}\,(\omega_1 y - \omega_2 x) \right\}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right) \times \left\{ (\omega_2 z - \omega_3 y)\,\hat{i} + (\omega_3 x - \omega_1 z)\,\hat{j} + (\omega_1 y - \omega_2 x)\,\hat{k} \right\}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \omega_2 z - \omega_3 y & \omega_3 x - \omega_1 z & \omega_1 y - \omega_2 x \end{vmatrix}$$

$$= \hat{i} \left\{ \frac{\partial}{\partial y}(\omega_1 y - \omega_2 x) - \frac{\partial}{\partial z}(\omega_3 x - \omega_1 z) \right\} + \hat{j} \left\{ \frac{\partial}{\partial z}(\omega_2 z - \omega_3 y) - \frac{\partial}{\partial x}(\omega_1 y - \omega_2 x) \right\}$$
$$+ \hat{k} \left\{ \frac{\partial}{\partial x}(\omega_3 x - \omega_1 z) - \frac{\partial}{\partial y}(\omega_2 z - \omega_3 y) \right\}$$

$$= \hat{i}\,(\omega_1 + \omega_1) + \hat{j}\,(\omega_2 + \omega_2) + \hat{k}\,(\omega_3 + \omega_3) = \hat{i}\,2\omega_1 + \hat{j}\,2\omega_2 + \hat{k}\,2\omega_3$$

$$= 2\left(\omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k} \right) = 2\vec{\omega}$$

Curl $\vec{V} = 2\vec{\omega}$ shows that curl of a vector field is connected with rotational properties of the vector field and justifies the name *rotation* used for *curl*.

Hence the angular velocity at any point is equal to half the curl of linear velocity at that point of the body.

Cor. ***If curl $\vec{V} = \vec{0}$, then $\vec{V}$ is called irrotational vector and the field V is termed irrotational.***

Note. For details of the irrotational vectors consult chapter 8 art 8.3 and 8.4.

7.22. PROPERTIES OF DIVERGENCE AND CURL

1. For a constant vector $\vec{a}$, **div** $\vec{a} = \mathbf{0}$, **curl** $\vec{a} = \vec{0}$

2. div $(\vec{A}+\vec{B}) = \text{div}\,\vec{A} + \text{div}\,\vec{B}$ **or** $\nabla\cdot(\vec{A}+\vec{B}) = \nabla\cdot\vec{A} + \nabla\cdot\vec{B}$

Proof.
$$\text{div}(\vec{A}+\vec{B}) = \nabla\cdot(\vec{A}+\vec{B}) = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot(\vec{A}+\vec{B})$$
$$= \hat{i}\cdot\frac{\partial}{\partial x}(\vec{A}+\vec{B})+\hat{j}\cdot\frac{\partial}{\partial y}(\vec{A}+\vec{B})+\hat{k}\cdot\frac{\partial}{\partial z}(\vec{A}+\vec{B})$$
$$= \hat{i}\cdot\left(\frac{\partial\vec{A}}{\partial x}+\frac{\partial\vec{B}}{\partial x}\right)+\hat{j}\cdot\left(\frac{\partial\vec{A}}{\partial y}+\frac{\partial\vec{B}}{\partial y}\right)+\hat{k}\cdot\left(\frac{\partial\vec{A}}{\partial z}+\frac{\partial\vec{B}}{\partial z}\right)$$
$$= \left(\hat{i}\cdot\frac{\partial\vec{A}}{\partial x}+\hat{j}\cdot\frac{\partial\vec{A}}{\partial y}+\hat{k}\cdot\frac{\partial\vec{A}}{\partial z}\right)+\left(\hat{i}\cdot\frac{\partial\vec{B}}{\partial x}+\hat{j}\cdot\frac{\partial\vec{B}}{\partial y}+\hat{k}\cdot\frac{\partial\vec{B}}{\partial z}\right)$$
$$= \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot\vec{A}+\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot\vec{B}$$
$$= \nabla\cdot\vec{A}+\nabla\cdot\vec{B}$$
$$= \text{div}\,\vec{A}+\text{div}\,\vec{B}$$

3. Curl $(\vec{A}+\vec{B}) = \text{curl}\,\vec{A}+\text{curl}\,\vec{B}$ or $\nabla\times(\vec{A}+\vec{B}) = \nabla\times\vec{A}+\nabla\times\vec{B}$

Proof. Curl $(\vec{A}+\vec{B}) = \nabla\times(\vec{A}+\vec{B})$
$$= \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\times(\vec{A}+\vec{B})$$
$$= \hat{i}\times\frac{\partial}{\partial x}(\vec{A}+\vec{B})+\hat{j}\times\frac{\partial}{\partial y}(\vec{A}+\vec{B})+\hat{k}\times\frac{\partial}{\partial z}(\vec{A}+\vec{B})$$
$$= \hat{i}\times\left(\frac{\partial\vec{A}}{\partial x}+\frac{\partial\vec{B}}{\partial x}\right)+\hat{j}\times\left(\frac{\partial\vec{A}}{\partial y}+\frac{\partial\vec{B}}{\partial y}\right)+\hat{k}\times\left(\frac{\partial\vec{A}}{\partial z}+\frac{\partial\vec{B}}{\partial z}\right)$$
$$= \left(\hat{i}\times\frac{\partial\vec{A}}{\partial x}+\hat{j}\frac{\partial\vec{A}}{\partial y}+\hat{k}\frac{\partial\vec{A}}{\partial z}\right)+\left(\hat{i}\times\frac{\partial\vec{B}}{\partial x}+\hat{j}\frac{\partial\vec{B}}{\partial y}+\hat{k}\frac{\partial\vec{B}}{\partial z}\right)$$
$$\text{Curl}(\vec{A}+\vec{B}) = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\times\vec{A}+\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\times\vec{B}$$
$$= \nabla\times\vec{A}+\nabla\times\vec{B} = \text{curl}\,\vec{A}+\text{curl}\,\vec{B}$$

Hence curl $(\vec{A}+\vec{B}) = \text{curl}\,\vec{A}+\text{curl}\,\vec{B}$

4. If $\vec{A}$ is a vector function and ϕ is a scalar function, then

$$\text{div}\left(\phi\vec{A}\right) = \phi\left(\text{div}\,\vec{A}\right) + (\text{grad}\,\phi)\cdot\vec{A}$$

or

$$\nabla\cdot\left(\phi\vec{A}\right) = \phi\left(\nabla\cdot\vec{A}\right) + (\nabla\phi)\cdot\vec{A}$$

Proof. $\text{div}\left(\phi\vec{A}\right) = \nabla\cdot\left(\phi\vec{A}\right) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\phi\vec{A}\right)$

$$= \hat{i}\cdot\frac{\partial}{\partial x}\left(\phi\vec{A}\right) + \hat{j}\cdot\frac{\partial}{\partial y}\left(\phi\vec{A}\right) + \hat{k}\cdot\frac{\partial}{\partial z}\left(\phi\vec{A}\right) = \Sigma\hat{i}\cdot\left(\phi\frac{\partial}{\partial x}\vec{A} + \frac{\partial\phi}{\partial x}\vec{A}\right)$$

$$= \phi\Sigma\hat{i}\cdot\frac{\partial}{\partial x}\vec{A} + \Sigma\left(\hat{i}\frac{\partial\phi}{\partial x}\right)\cdot\vec{A}$$

$$= \phi\left\{\hat{i}\cdot\frac{\partial}{\partial x}\vec{A} + \hat{j}\cdot\frac{\partial}{\partial y}\vec{A} + \hat{k}\cdot\frac{\partial}{\partial z}\vec{A}\right\} + \left\{\left(\hat{i}\frac{\partial\phi}{\partial x}\right)\cdot\vec{A} + \left(\hat{j}\frac{\partial\phi}{\partial y}\right)\cdot\vec{A} + \left(\hat{k}\frac{\partial\phi}{\partial z}\right)\cdot\vec{A}\right\}$$

$$= \phi\left\{\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right\}\cdot\vec{A} + \left\{\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right\}\cdot\vec{A}$$

$$= \phi\left(\nabla\cdot\vec{A}\right) + (\nabla\phi)\cdot\vec{A}$$

$$\therefore\ \text{div}\left(\phi\vec{A}\right) = \phi\left(\text{div}\,\vec{A}\right) + (\text{grad}\,\phi)\cdot\vec{A}$$

5. If $\vec{A}$ is a vector function and ϕ is a scalar function then ***(P.T.U., Jan. 2010)***

$$\text{Curl}\left(\phi\vec{A}\right) = (\text{grad}\,\phi)\times\vec{A} + \phi\,\text{curl}\,\vec{A}$$

or

$$\nabla\times\left(\phi\vec{A}\right) = (\nabla\phi)\times\vec{A} + \phi\left(\nabla\times\vec{A}\right).$$

Proof. $\text{Curl}\left(\phi\vec{A}\right) = \nabla\times\left(\phi\vec{A}\right) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times\left(\phi\vec{A}\right)$

$$= \hat{i}\times\frac{\partial}{\partial x}\left(\phi\vec{A}\right) + \hat{j}\times\frac{\partial}{\partial y}\left(\phi\vec{A}\right) + \hat{k}\times\frac{\partial}{\partial z}\left(\phi\vec{A}\right)$$

$$= \hat{i}\times\left\{\frac{\partial\phi}{\partial x}\vec{A} + \phi\frac{\partial\vec{A}}{\partial x}\right\} + \hat{j}\times\left\{\frac{\partial\phi}{\partial y}\vec{A} + \phi\frac{\partial\vec{A}}{\partial y}\right\} + \hat{k}\times\left\{\frac{\partial\phi}{\partial z}\vec{A} + \phi\frac{\partial\vec{A}}{\partial z}\right\}$$

$$= \left\{\hat{i}\times\left(\frac{\partial\phi}{\partial x}\vec{A}\right) + \hat{j}\times\left(\frac{\partial\phi}{\partial y}\vec{A}\right) + \hat{k}\times\left(\frac{\partial\phi}{\partial z}\vec{A}\right)\right\} + \phi\left\{\hat{i}\times\frac{\partial\vec{A}}{\partial x} + \hat{j}\times\frac{\partial\vec{A}}{\partial y} + \hat{k}\times\frac{\partial\vec{A}}{\partial z}\right\}$$

$$= \left\{\frac{\partial\phi}{\partial x}\hat{i}\times\vec{A} + \frac{\partial\phi}{\partial y}\hat{j}\times\vec{A} + \frac{\partial\phi}{\partial z}\hat{k}\times\vec{A}\right\} + \phi\left\{\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right\}\times\vec{A}$$

$$= \left\{\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\phi\right\}\times\vec{A}+\phi\left\{\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right\}\times\vec{A}$$

$$= (\nabla\phi)\times\vec{A}+\phi(\nabla\times\vec{A}) = (\text{grad }\phi)\times\vec{A}+\phi\,(\text{curl }\vec{A})$$

6. $\nabla(\vec{A}\cdot\vec{B}) = (\vec{A}\cdot\nabla)\vec{B}+(\vec{B}\cdot\nabla)\vec{A}+\vec{A}\times(\nabla\times\vec{B})+\vec{B}\times(\nabla\times\vec{A})$

Proof. $\nabla(\vec{A}\cdot\vec{B}) = \Sigma\,\hat{i}\,\frac{\partial}{\partial x}(\vec{A}\cdot\vec{B}) = \Sigma\,\hat{i}\left\{\vec{A}\cdot\frac{\partial\vec{B}}{\partial x}+\frac{\partial\vec{A}}{\partial x}\cdot\vec{B}\right\} = \Sigma\left(\vec{A}\cdot\frac{\partial\vec{B}}{\partial x}\right)\hat{i}+\Sigma\left(\vec{B}\cdot\frac{\partial\vec{A}}{\partial x}\right)\hat{i}$

Now, we know that $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$

$\therefore \quad (\vec{a}\cdot\vec{b})\vec{c} = (\vec{a}\cdot\vec{c})\vec{b}-\vec{a}\times(\vec{b}\times\vec{c})$

$\therefore \quad \left(\vec{A}\cdot\frac{\partial\vec{B}}{\partial x}\right)\hat{i} = (\vec{A}\cdot\hat{i})\frac{\partial\vec{B}}{\partial x}-\vec{A}\times\left(\frac{\partial\vec{B}}{\partial x}\times\hat{i}\right) = (\vec{A}\cdot\hat{i})\frac{\partial\vec{B}}{\partial x}+\vec{A}\times\left(\hat{i}\times\frac{\partial\vec{B}}{\partial x}\right)$

$\therefore \quad \Sigma\left(\vec{A}\cdot\frac{\partial\vec{B}}{\partial x}\right)\hat{i} = \left(\vec{A}\cdot\Sigma\,\hat{i}\,\frac{\partial}{\partial x}\right)\vec{B}+\vec{A}\times\Sigma\left(\hat{i}\times\frac{\partial\vec{B}}{\partial x}\right) = (\vec{A}\cdot\nabla)\vec{B}+\vec{A}\times(\nabla\times\vec{B})$

Similarly, $\quad \Sigma\left(\vec{B}\cdot\frac{\partial\vec{A}}{\partial x}\right)\hat{i} = (\vec{B}\cdot\nabla)\vec{A}+\vec{B}\times(\nabla\times\vec{A})$ (By interchanging A and B)

$\therefore \quad \nabla(\vec{A}\cdot\vec{B}) = (\vec{A}\cdot\nabla)\vec{B}+\vec{A}\times(\nabla\times\vec{B})+(\vec{B}\cdot\nabla)\vec{A}+\vec{B}\times(\nabla\times\vec{A})$

$$= (\vec{A}\cdot\nabla)\vec{B}+(\vec{B}\cdot\nabla)\vec{A}+\vec{A}\times(\nabla\times\vec{B})+\vec{B}\times(\nabla\times\vec{A})$$

7. $\nabla\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\nabla\times\vec{A})-\vec{A}\cdot(\nabla\times\vec{B})$ ***(P.T.U., Jan. 2010)***

Proof. $\nabla\cdot(\vec{A}\times\vec{B}) = \Sigma\,\hat{i}\cdot\frac{\partial}{\partial x}(\vec{A}\times\vec{B}) = \Sigma\,\hat{i}\cdot\left\{\vec{A}\times\frac{\partial\vec{B}}{\partial x}+\frac{\partial\vec{A}}{\partial x}\times\vec{B}\right\}$

$$= \Sigma\left\{\hat{i}\cdot\left(\vec{A}\times\frac{\partial\vec{B}}{\partial x}\right)\right\}+\Sigma\left\{\hat{i}\cdot\left(\frac{\partial\vec{A}}{\partial x}\times\vec{B}\right)\right\} = \Sigma\hat{i}\cdot\left(\frac{\partial\vec{A}}{\partial x}\times\vec{B}\right)-\Sigma\hat{i}\cdot\left(\frac{\partial\vec{B}}{\partial x}\times\vec{A}\right)$$

$$= \Sigma\left(\hat{i}\times\frac{\partial\vec{A}}{\partial x}\right)\cdot\vec{B}-\Sigma\left(\hat{i}\times\frac{\partial\vec{B}}{\partial x}\right)\cdot\vec{A} = (\nabla\times\vec{A})\cdot\vec{B}-(\nabla\times\vec{B})\cdot\vec{A}$$

$$= \vec{B}\cdot(\nabla\times\vec{A})-\vec{A}\cdot(\nabla\times\vec{B})$$

8. $\nabla \times (\vec{A} \times \vec{B}) = (\nabla \cdot \vec{B})\vec{A} - (\nabla \cdot \vec{A})\vec{B} + (\vec{B} \cdot \nabla)\vec{A} - (\vec{A} \cdot \nabla)\vec{B}$ ***(P.T.U., May 2010, May 2011)***

Proof. $\nabla \times (\vec{A} \times \vec{B}) = \Sigma \hat{i} \times \frac{\partial}{\partial x}(\vec{A} \times \vec{B}) = \Sigma \hat{i} \times \left\{\frac{\partial \vec{A}}{\partial x} \times \vec{B} + \vec{A} \times \frac{\partial \vec{B}}{\partial x}\right\}$

$$= \Sigma \hat{i} \times \left(\frac{\partial \vec{A}}{\partial x} \times \vec{B}\right) + \Sigma \hat{i} \times \left(\vec{A} \times \frac{\partial \vec{B}}{\partial x}\right)$$

$$= \Sigma\left\{(\hat{i} \cdot \vec{B})\frac{\partial \vec{A}}{\partial x} - \left(\hat{i} \cdot \frac{\partial \vec{A}}{\partial x}\right)\vec{B}\right\} + \Sigma\left\{\left(\hat{i} \cdot \frac{\partial \vec{B}}{\partial x}\right)\vec{A} - (\hat{i} \cdot \vec{A})\frac{\partial \vec{B}}{\partial x}\right\}$$

$$= \Sigma(\vec{B} \cdot \hat{i})\frac{\partial \vec{A}}{\partial x} - \left\{\Sigma \hat{i} \cdot \frac{\partial \vec{A}}{\partial x}\right\}\vec{B} + \left\{\Sigma \hat{i} \cdot \frac{\partial \vec{B}}{\partial x}\right\}\vec{A} - \Sigma(\vec{A} \cdot \hat{i})\frac{\partial \vec{B}}{\partial x}$$

$$= \left(\vec{B} \cdot \Sigma \hat{i} \frac{\partial}{\partial x}\right)\vec{A} - (\nabla \cdot \vec{A})\vec{B} + (\nabla \cdot \vec{B})\vec{A} - \left(\vec{A} \cdot \Sigma \hat{i} \frac{\partial}{\partial x}\right)\vec{B}$$

$$= (\vec{B} \cdot \nabla)\vec{A} - (\nabla \cdot \vec{A})\vec{B} + (\nabla \cdot \vec{B})\vec{A} - (\vec{A} \cdot \nabla)\vec{B}$$

$$= (\nabla \cdot \vec{B})\vec{A} - (\nabla \cdot \vec{A})\vec{B} + (\vec{B} \cdot \nabla)\vec{A} - (\vec{A} \cdot \nabla)\vec{B}$$

7.23. REPEATED OPERATIONS BY ∇

Before starting with the repeated operations by ∇, students are advised to note the following:

If $\phi(x, y, z)$ and $\vec{V}(x, y, z)$ be scalar and vector point functions respectively, then

(*i*) Since ϕ is scalar we can take its gradient only.

(*ii*) Since grad ϕ and $\vec{V}$ are both vector functions we can take their divergence as well as curl.

(*iii*) Since div $\vec{V}$ is a scalar function we can take its gradient only.

1. **Div (grad ϕ) = $\nabla^2 \phi$ where** $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

Proof. Div (grad ϕ) $= \nabla \cdot (\nabla \phi)$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial z}\right) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi = \nabla^2 \phi$$

Note 1. ∇^2 is called Laplacian Operator and $\nabla^2 \phi = 0$ is called **Laplace Equation.**

Note 2. A function satisfying Laplace Equation is called **Harmonic Function** *i.e.,* ϕ is Harmonic Function.

2. **Curl (grad ϕ) = $\nabla \times (\nabla \phi) = \vec{0}$**

Proof. Curl (grad ϕ) = $\nabla \times (\nabla \phi)$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix} = \Sigma \hat{i} \left\{\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial y}\right)\right\} = \Sigma \hat{i} \left\{\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y}\right\} = \vec{0} \qquad \left(\because \frac{\partial^2 \phi}{\partial y \partial z} = \frac{\partial^2 \phi}{\partial z \partial y}\right)$$

Hence curl (grad ϕ) $= \vec{0}$

Note. Curl (grad ϕ) $= \vec{0}$ implies that gradient field describes an irrotational motion.

3. Div. $\left(\textbf{Curl } \vec{V}\right) = \nabla \cdot \left(\nabla \times \vec{V}\right) = 0$

Proof. Div $\left(\textbf{Curl } \vec{V}\right) = \nabla \cdot \left(\nabla \times \vec{V}\right)$

Let $\qquad \vec{V} = V_1 \hat{i} + V_2 \hat{j} + V_3 \hat{k}$

$$\therefore \qquad \nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix} = \Sigma \hat{i} \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right)$$

$$\therefore \quad \text{Div}\left(\text{Curl } \vec{V}\right) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(\Sigma \hat{i}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right)\right)$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left\{\hat{i}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) + \hat{j}\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right) + \hat{k}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)\right\}$$

$$= \frac{\partial}{\partial x}\left\{\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right\} + \frac{\partial}{\partial y}\left\{\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right\} + \frac{\partial}{\partial z}\left\{\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right\}$$

$$= \frac{\partial^2 V_3}{\partial x \partial y} - \frac{\partial^2 V_2}{\partial x \partial z} + \frac{\partial^2 V_1}{\partial y \partial z} - \frac{\partial^2 V_3}{\partial y \partial x} + \frac{\partial^2 V_2}{\partial z \partial x} - \frac{\partial^2 V_1}{\partial z \partial y} = 0$$

Hence Div $\cdot \left(\text{Curl } \vec{V}\right) = 0$

Note. Div $\left(\text{curl } \vec{V}\right) = 0$ implies that curl $\vec{V}$ is a solenoidal vector point function.

4. Curl $\left(\textbf{Curl } \vec{V}\right) = \textbf{grad div } \vec{V} - \nabla^2 \vec{V}$ ***(P.T.U., June 2003, Dec. 2005)***

or Curl $\left(\textbf{Curl } \vec{V}\right) = \nabla\left(\nabla \cdot \vec{V}\right) - (\nabla \cdot \nabla)\vec{V}$

Proof. Let $\vec{V} = V_1 \hat{i} + V_2 \hat{j} + V_3 \hat{k}$

$$\text{Curl } \vec{V} = \hat{i}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) + \hat{j}\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right) + \hat{k}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)$$

$$\text{Curl Curl } \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z} & \frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x} & \frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y} \end{vmatrix}$$

$$= \Sigma \hat{i}\left\{\frac{\partial}{\partial y}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right)\right\} = \Sigma \hat{i}\left\{\frac{\partial^2 V_2}{\partial y\,\partial x} - \frac{\partial^2 V_1}{\partial y^2} - \frac{\partial^2 V_1}{\partial z^2} + \frac{\partial^2 V_3}{\partial z\,\partial x}\right\}$$

$$= \Sigma \hat{i}\left\{\frac{\partial^2 V_2}{\partial y\,\partial x} + \frac{\partial^2 V_3}{\partial z\,\partial x} - \frac{\partial^2 V_1}{\partial y^2} - \frac{\partial^2 V_1}{\partial z^2}\right\}$$

Add and subtract $\frac{\partial^2 V_1}{\partial x^2}$

$$= \Sigma \hat{i}\left\{\left(\frac{\partial^2 V_1}{\partial x^2} + \frac{\partial^2 V_2}{\partial x\,\partial y} + \frac{\partial^2 V_3}{\partial x\,\partial z}\right) - \left(\frac{\partial^2 V_1}{\partial x^2} + \frac{\partial^2 V_1}{\partial y^2} + \frac{\partial^2 V_1}{\partial z^2}\right)\right\}$$

$$= \Sigma \hat{i}\left[\frac{\partial}{\partial x}\left(\frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}\right) - \left(\frac{\partial^2 V_1}{\partial x^2} + \frac{\partial^2 V_1}{\partial y^2} + \frac{\partial^2 V_1}{\partial z^2}\right)\right]$$

$$= \Sigma \hat{i}\left\{\frac{\partial}{\partial x}\left(\nabla \cdot \vec{V}\right) - \nabla^2 V_1\right\} = \Sigma \hat{i}\,\frac{\partial}{\partial x}\left(\nabla \cdot \vec{V}\right) - \nabla^2\, \Sigma \hat{i}\, V_1$$

$$= \nabla\left(\nabla \cdot \vec{V}\right) - \nabla^2 \vec{V} = \text{grad }(\text{div } \vec{V}) - \nabla^2 \vec{V}$$

Cor. From above result we can also deduce

$$\textbf{grad }\left(\textbf{div } \vec{V}\right) = \text{Curl}\left(\text{Curl } \vec{V}\right) + \nabla^2 \vec{V}$$ **(P.T.U. Dec. 2011)**

or $$\nabla\left(\nabla \cdot \vec{V}\right) = \nabla \times \left(\nabla \times \vec{V}\right) + \nabla^2 \vec{V}$$

Note. For application in questions, the results of repeated application of ∇ can easily be written down (treating ∇ as a vector)

(i) $\nabla \cdot \nabla \phi = \nabla^2 \phi$ $\quad \because \vec{a} \cdot \vec{a} = a^2$

(ii) $\nabla \times \nabla \phi = \vec{0}$ $\quad \because \vec{a} \times \vec{a} = \vec{0}$

(iii) $\nabla \cdot \left(\nabla \times \vec{V}\right) = 0$ $\quad \because$ in scalar triple product $\vec{a} \cdot \left(\vec{a} \times \vec{b}\right) = 0$

(iv) $\nabla \times (\nabla \times \vec{V}) = (\nabla \cdot \vec{V})\,\nabla - \nabla^2 V$ $\quad \because \vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\,\vec{b} - (\vec{a} \cdot \vec{b})\,\vec{c}$.

ILLUSTRATIVE EXAMPLES

Example 1. *Evaluate the following:*

(i) *Div* $\left[(xy\sin z)\hat{i} + (y^2\sin x)\hat{j} + (z^2\sin xy)\hat{k}\right]$ *at the point* $\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right)$.

(ii) *Curl Curl of* $\vec{V} = (2xz^2)\hat{i} - yz\,\hat{j} + (3xz^3)\hat{k}$ *at (1, 1, 1)* **(P.T.U., May 2006)**

Sol. (i) Div $\left[(xy\sin z)\hat{i} + (y^2\sin x)\hat{j} + (z^2\sin xy)\hat{k}\right]$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left[(xy\sin z)\hat{i} + (y^2\sin x)\hat{j} + (z^2\sin xy)\hat{k}\right]$$

$$= \frac{\partial}{\partial x}(xy\sin z) + \frac{\partial}{\partial y}(y^2\sin x) + \frac{\partial}{\partial z}(z^2\sin xy)$$

$$= y\sin z + 2y\sin x + 2z\sin xy$$

At the point $\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\text{div}\left\{(xy\sin z)\hat{i} + (y^2\sin x)\hat{j} + (z^2\sin xy)\hat{k}\right\} = \frac{\pi}{2} + 0 + 0 = \frac{\pi}{2}$$

(ii) $$\vec{V} = (2xz^2)\hat{i} - yz\,\hat{j} + (3xz^3)\hat{k}$$

$$\text{Curl }\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xz^2 & -yz & 3xz^3 \end{vmatrix}$$

$$= \hat{i}\left\{\frac{\partial}{\partial y}(3xz^3) - \frac{\partial}{\partial z}(-yz)\right\} - \hat{j}\left\{\frac{\partial}{\partial x}(3xz^3) - \frac{\partial}{\partial z}(2xz^2)\right\} + \hat{k}\left\{\frac{\partial}{\partial x}(-yz) - \frac{\partial}{\partial y}(2xz^2)\right\}$$

$$= \hat{i}\{y\} - \hat{j}\{3z^3 - 4xz\} + \hat{k}\{0\} = y\hat{i} + (-3z^3 + 4xz)\hat{j}$$

$$\text{Curl Curl }\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -3z^3 + 4xz & 0 \end{vmatrix}$$

$$= \hat{i}\left\{\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(-3z^3 + 4xz)\right\} - \hat{j}\left\{\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(y)\right\} + \hat{k}\left\{\frac{\partial}{\partial x}(-3z^3 + 4xz) - \frac{\partial}{\partial y}(y)\right\}$$

$$= -\hat{i}\left(-9z^2 + 4x\right) + \hat{j}(0) + \hat{k}\{4z - 1\} = \left(9z^2 - 4x\right)\hat{i} + (4z - 1)\hat{k}$$

At (1, 1, 1) curl curl $\vec{V} = 5\hat{i} + 3\hat{k}$.

Example 2. *(i) Find div* $\vec{F}$ *and curl* $\vec{F}$*, where* $\vec{F} = grad\,(x^3 + y^3 + z^3 - 3xyz)$.

(ii) If $\phi(x, y, z) = x - y + 2z^2$ *compute the value of* $\nabla \times (\nabla\phi)$. **(P.T.U., Dec. 2013)**

Sol. (i) $\vec{F} = \text{grad}\,(x^3 + y^3 + z^3 - 3xyz) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^3 + y^3 + z^3 - 3xyz)$

$$= \hat{i}\left(3x^2 - 3yz\right) + \hat{j}\left(3y^2 - 3zx\right) + \hat{k}\left(3z^2 - 3xy\right)$$

Now, $\text{div}\,\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left\{3\left(x^2 - yz\right)\hat{i} + 3\left(y^2 - zx\right)\hat{j} + 3\left(z^2 - xy\right)\hat{k}\right\}$

$$= 3\left\{\frac{\partial}{\partial x}\left(x^2 - yz\right) + \frac{\partial}{\partial y}\left(y^2 - zx\right) + \frac{\partial}{\partial z}\left(z^2 - xy\right)\right\}$$

$$= 3\,\{2x + 2y + 2z\} = 6\,(x + y + z)$$

$\text{Curl}\,\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left\{3\left(x^2 - yz\right)\hat{i} + 3\left(y^2 - zx\right)\hat{j} + 3\left(z^2 - xy\right)\hat{k}\right\}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - yz & y^2 - zx & z^2 - xy \end{vmatrix} = 3\Sigma\hat{i}\left\{\frac{\partial}{\partial y}\left(z^2 - xy\right) - \frac{\partial}{\partial z}\left(y^2 - zx\right)\right\} = 3\Sigma\hat{i}\,\{-x + x\}$$

$$= 3\,\Sigma\,\hat{i}\,.\,0 = \vec{0}.$$

(ii) $\phi(x, y, z) = x - y + 2z^2$

$$\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x - y + 2z^2)$$

$$= \hat{i}\,(1) + \hat{j}(-1) + \hat{k}(4z) = \hat{i} - \hat{j} + 4z\hat{k}$$

$$\nabla \times \nabla\phi = \text{curl}\,(\hat{i} - \hat{j} + 4z\hat{k}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 1 & -1 & 4z \end{vmatrix}$$

$$= \hat{i}\,(0 - 0) - \hat{j}\,(0 - 0) + \hat{k}\,(0 - 0) = \vec{0}$$

$\therefore$ $\nabla \times \nabla\phi = \vec{0}$.

Example 3. *If* $u = x^2 + y^2 + z^2$, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$*, then find div* $(u\vec{r})$ *in terms of u.* **(P.T.U., Dec. 2005)**

Sol. $\text{div}\,(u\vec{r}) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot (u\vec{r})$

Now, $(u\vec{r}) = (x^2 + y^2 + z^2)\left(x\hat{i} + y\hat{j} + z\hat{k}\right)$

$$= x\left(x^2 + y^2 + z^2\right)\hat{i} + y\left(x^2 + y^2 + z^2\right)\hat{j} + z\left(x^2 + y^2 + z^2\right)\hat{k}$$

$$\frac{\partial}{\partial x}(u\vec{r}) = \left(3x^2 + y^2 + z^2\right)\hat{i} + (2xy)\,\hat{j} + (2xz)\,\hat{k}$$

$$\frac{\partial}{\partial y}(u\vec{r}) = (2xy)\,\hat{i} + \left(x^2 + z^2 + 3y^2\right)\hat{j} + (2yz)\,\hat{k}$$

$$\frac{\partial}{\partial z}(u\vec{r}) = (2xz)\,\hat{i} + (2yz)\,\hat{j} + \left(x^2 + y^2 + 3z^2\right)\hat{k}$$

$$\therefore \quad \operatorname{div}(u\vec{r}) = \hat{i}\cdot\frac{\partial}{\partial x}(u\vec{r}) + \hat{j}\cdot\frac{\partial}{\partial y}(u\vec{r}) + \hat{k}\cdot\frac{\partial}{\partial z}(u\vec{r})$$

$$= (3x^2 + y^2 + z^2) + (x^2 + z^2 + 3y^2) + (x^2 + y^2 + 3z^2)$$

$$= 5x^2 + 5y^2 + 5z^2 = 5(x^2 + y^2 + z^2) = 5u.$$

Example 4. *Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\vec{a}$ be a constant vector, find the value of $\operatorname{div}\dfrac{\vec{a}\times\vec{r}}{r^n}$.*

Sol. Let $\quad \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$$\vec{a}\times\vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} = \hat{i}\,(a_2z - a_3y) + \hat{j}\,(a_3x - a_1z) + \hat{k}\,(a_1y - a_2x)$$

$$\frac{\vec{a}\times\vec{r}}{r^n} = \frac{\Sigma\hat{i}\,(a_2z - a_3y)}{\left(x^2 + y^2 + z^2\right)^{\frac{n}{2}}}$$

$$\operatorname{div}\left(\frac{\vec{a}\times\vec{r}}{r^n}\right) = \nabla\cdot\frac{\vec{a}\times\vec{r}}{r^n}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\frac{(a_2z - a_3y)\hat{i} + (a_3x - a_1z)\hat{j} + (a_1y - a_2x)\hat{k}}{\left(x^2 + y^2 + z^2\right)^{\frac{n}{2}}}$$

$$= \frac{\partial}{\partial x}\frac{(a_2z - a_3y)}{\left(x^2 + y^2 + z^2\right)^{\frac{n}{2}}} + \frac{\partial}{\partial y}\frac{(a_3x - a_1z)}{\left(x^2 + y^2 + z^2\right)^{\frac{n}{2}}} + \frac{\partial}{\partial z}\frac{(a_1y - a_2x)}{\left(x^2 + y^2 + z^2\right)^{\frac{n}{2}}}$$

$$= \frac{(a_2z - a_3y)\left(-\frac{n}{2}\right)(2x)}{\left(x^2 + y^2 + z^2\right)^{\frac{n}{2}+1}} + \frac{(a_3x - a_1z)\left(-\frac{n}{2}\right)(2y)}{\left(x^2 + y^2 + z^2\right)^{\frac{n}{2}+1}} + \frac{(a_1y - a_2x)\left(-\frac{n}{2}\right)(2z)}{\left(x^2 + y^2 + z^2\right)^{\frac{n}{2}+1}}$$

$$= \frac{-n}{\left(x^2 + y^2 + z^2\right)^{\frac{n+2}{2}}}\left[x\,(a_2z - a_3y) + y\,(a_3x - a_1z) + z\,(a_1y - a_2x)\right]$$

$$= \frac{-n}{\left(x^2 + y^2 + z^2\right)^{\frac{n+2}{2}}}\cdot 0 = 0$$

Hence $\operatorname{div}\left(\dfrac{\vec{a}\times\vec{r}}{r^n}\right) = 0.$

Example 5. *Given the vector field* $\vec{V} = (x^2 - y^2 + 2xz)\hat{i} + (xz - xy + yz)\hat{j} + (z^2 + x^2)\hat{k}$ *find Curl* $\vec{V}$. *Show that the vectors given by Curl* $\vec{V}$ *at P (1, 2, – 3) and Q (2, 3, 12) are orthogonal.*

Sol. $\vec{V} = (x^2 - y^2 + 2xz)\hat{i} + (xz - xy + yz)\hat{j} + (z^2 + x^2)\hat{k}$

Curl $\vec{V} = \nabla \times \vec{V}$.

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 - y^2 + 2xz & xz - xy + yz & z^2 + x^2 \end{vmatrix}$$

$$= \hat{i}\left\{\frac{\partial}{\partial y}(z^2 + x^2) - \frac{\partial}{\partial z}(xz - xy + yz)\right\} + \hat{j}\left\{\frac{\partial}{\partial z}(x^2 - y^2 + 2xz) - \frac{\partial}{\partial x}(z^2 + x^2)\right\}$$

$$+ \hat{k}\left\{\frac{\partial}{\partial x}(xz - xy + yz) - \frac{\partial}{\partial y}(x^2 - y^2 + 2xz)\right\}$$

$$= \hat{i}\{-(x + y)\} + \hat{j}\{2x - 2x\} + \hat{k}\{z - y + 2y\} = -(x + y)\hat{i} + (y + z)\hat{k}$$

Curl $\vec{V}$ at P (1, 2, – 3) $= -3\hat{i} - \hat{k}$

Curl $\vec{V}$ at Q (2, 3, 12) $= -5\hat{i} + 15\hat{k}$.

Curl $\vec{V}$ at P, Q will be orthogonal if their dot product is zero.

i.e., $\quad (-3\hat{i} - \hat{k}) \cdot (-5\hat{i} + 15\hat{k}) = 15 - 15 = 0.$

Hence curl vectors at P and Q are orthogonal.

Example 6. *If* $u\vec{F} = \nabla v$, *where u, v are scalars fields and* $\vec{F}$ *is a vector field, show that* $\vec{F} \cdot \text{curl}\, \vec{F} = 0$.

Sol. $\quad \text{Curl}\, \vec{F} = \text{Curl}\left(\frac{1}{u}\nabla v\right) = \nabla \times \left(\frac{1}{u}\nabla v\right)$

We know that $\nabla \times (\phi\vec{A}) = \nabla\phi \times \vec{A} + \phi\nabla \times \vec{A}$

$$\therefore \quad \text{Curl}\, \vec{F} = \left(\nabla\frac{1}{u}\right) \times (\nabla v) + \frac{1}{u}\nabla \times (\nabla v) = \left(\nabla\frac{1}{u}\right) \times (\nabla v) \qquad (\because \nabla \times (\nabla v) = \vec{0})$$

$$\vec{F} \cdot \text{Curl}\, \vec{F} = \left(\frac{1}{u}\nabla v\right) \cdot \left(\nabla\frac{1}{u}\right) \times (\nabla v) = \frac{1}{u}\left\{\nabla v \cdot \nabla\frac{1}{u} \times \nabla v\right\}$$

$$= \frac{1}{u}\left[\nabla v, \nabla \frac{1}{u}, \nabla v\right] = \frac{1}{u} \cdot 0 = 0$$

[$\because$ In scalar triple product two vectors are equal]

Hence $\vec{F} \cdot \text{Curl } \vec{F} = 0$

Example 7. *If $\vec{A} = \nabla \times (\phi \hat{i})$ where $\nabla^2 \phi = 0$, show that $\vec{A} \cdot \nabla \times \vec{A} = \frac{\partial \phi}{\partial z} \cdot \frac{\partial^2 \phi}{\partial y\, \partial x} - \frac{\partial \phi}{\partial y} \cdot \frac{\partial^2 \phi}{\partial z\, \partial x}$.*

Sol. $\vec{A} = \nabla \times (\phi \hat{i}) = \nabla \phi \times \hat{i}$

$$\nabla \times \vec{A} = \nabla \times (\nabla \phi \times \hat{i}) = (\nabla \cdot \hat{i}) \nabla \phi - (\nabla \cdot \nabla \phi)\hat{i} \quad \text{using } \vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{b})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$$

$$\nabla \cdot \hat{i} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \hat{i} = \frac{\partial}{\partial x}$$

$$\therefore \nabla \times \vec{A} = \frac{\partial}{\partial x}(\nabla \phi) - (\nabla^2 \phi)\hat{i} = \frac{\partial}{\partial x}\left[\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}\right] - \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}\right)\hat{i}$$

$$= \hat{i}\frac{\partial^2 \phi}{\partial x^2} + \hat{j}\frac{\partial^2 \phi}{\partial x\, \partial y} + \hat{k}\frac{\partial^2 \phi}{\partial x\, \partial z} - \frac{\partial^2 \phi}{\partial x^2}\hat{i} - \frac{\partial^2 \phi}{\partial y^2}\hat{i} - \frac{\partial^2 \phi}{\partial z^2}\hat{i}$$

$$= -\left(\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}\right)\hat{i} + \frac{\partial^2 \phi}{\partial x\, \partial y}\hat{j} + \frac{\partial^2 \phi}{\partial x\, \partial z}\hat{k}$$

$$\vec{A} \cdot \nabla \times \vec{A} = (\nabla \times \phi \hat{i}) \cdot \left[-\left(\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}\right)\hat{i} + \frac{\partial^2 \phi}{\partial x\, \partial y}\hat{j} + \frac{\partial^2 \phi}{\partial x\, \partial z}\hat{k}\right]$$

$$= (\nabla \phi \times \hat{i}) \cdot \left[-\left(\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}\right)\hat{i} + \frac{\partial^2 \phi}{\partial x\, \partial y}\hat{j} + \frac{\partial^2 \phi}{\partial x\, \partial z}\hat{k}\right]$$

$$= \left[\left(\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}\right) \times \hat{i}\right] \cdot \left[-\left(\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}\right)\hat{i} + \frac{\partial^2 \phi}{\partial x\, \partial y}\hat{j} + \frac{\partial^2 \phi}{\partial x\, \partial z}\hat{k}\right]$$

$$= \left[-\frac{\partial \phi}{\partial y}\hat{k} + \frac{\partial \phi}{\partial z}\hat{j}\right] \cdot \left[-\left(\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}\right)\hat{i} + \frac{\partial^2 \phi}{\partial x\, \partial y}\hat{j} + \frac{\partial^2 \phi}{\partial x\, \partial z}\hat{k}\right]$$

$$\left(\because \hat{i} \times \hat{i} = \vec{0},\ \hat{j} \times \hat{i} = -\hat{k},\ \hat{k} \times \hat{i} = \hat{j}\right)$$

$$= -\frac{\partial \phi}{\partial y} \cdot \frac{\partial^2 \phi}{\partial x\, \partial z} + \frac{\partial \phi}{\partial z} \cdot \frac{\partial^2 \phi}{\partial x\, \partial y} = \frac{\partial \phi}{\partial z} \cdot \frac{\partial^2 \phi}{\partial y\, \partial x} - \frac{\partial \phi}{\partial y}\frac{\partial^2 \phi}{\partial z\, \partial x}$$

Example 8. *By taking* $\vec{F} = u \nabla v$, *where u and v are scalars, prove that*

$\nabla \bullet \vec{F} = u \nabla^2 v + \nabla u \bullet \nabla v$.

Sol.
$$\vec{F} = u \nabla v = u\left(\hat{i}\frac{\partial v}{\partial x} + \hat{j}\frac{\partial v}{\partial y} + \hat{k}\frac{\partial v}{\partial z}\right)$$

$$\nabla \bullet \vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \bullet u\left(\hat{i}\frac{\partial v}{\partial x} + \hat{j}\frac{\partial v}{\partial y} + \hat{k}\frac{\partial v}{\partial z}\right)$$

Differentiate by product rule.

$$= \left(\hat{i}\frac{\partial u}{\partial x} + \hat{j}\frac{\partial u}{\partial y} + \hat{k}\frac{\partial u}{\partial z}\right) \bullet \left(\hat{i}\frac{\partial v}{\partial x} + \hat{j}\frac{\partial v}{\partial y} + \hat{k}\frac{\partial v}{\partial z}\right)$$

$$+ u\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \bullet \left(\hat{i}\frac{\partial v}{\partial x} + \hat{j}\frac{\partial v}{\partial y} + \hat{k}\frac{\partial v}{\partial z}\right)$$

$$= \nabla u \bullet \nabla v + u\left[\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial v}{\partial z}\right)\right]$$

$$= \nabla u \bullet \nabla v + u\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right] = \nabla u \bullet \nabla v + u\nabla^2 v$$

$$= u \nabla^2 v + \nabla u \bullet \nabla v$$

Example 9. *If r is the distance of a point (x, y, z) from the origin, prove that* $curl\left(\hat{k} \times grad\frac{1}{r}\right) + grad\left(\hat{k} \cdot grad\frac{1}{r}\right) = 0$, *where* $\hat{k}$ *is a unit vector in the direction of z.*

Sol.
$$r = \sqrt{x^2 + y^2 + z^2}$$

$\therefore$
$$\frac{1}{r} = \left(x^2 + y^2 + z^2\right)^{-\frac{1}{2}}$$

$$\text{grad}\,\frac{1}{r} = \nabla\left(x^2 + y^2 + z^2\right)^{-\frac{1}{2}} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(x^2 + y^2 + z^2\right)^{-\frac{1}{2}}$$

$$= -\frac{1}{2}\left(x^2 + y^2 + z^2\right)^{-\frac{3}{2}}\left[2x\hat{i} + 2y\hat{j} + 2z\hat{k}\right]$$

$$= \frac{-1}{\left(x^2 + y^2 + z^2\right)^{\frac{3}{2}}}\left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

Now $\hat{k} \times \text{grad}\, \dfrac{1}{r} = \hat{k} \times \left\{ \dfrac{-1}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}} \right\} \left\{ x\hat{i} + y\hat{j} + z\hat{k} \right\} = \dfrac{-1}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}} \left\{ x\hat{j} - y\hat{i} \right\}$

$$\text{Curl}\left(\hat{k} \times \text{grad}\, \frac{1}{r}\right) = \nabla \times \left[\frac{-1}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}} \right] \left[x\hat{j} - y\hat{i} \right]$$

$$= \nabla \times \left\{ \frac{-x}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}} \hat{j} + \frac{y}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}} \hat{i} \right\}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{y}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}} & \dfrac{-x}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}} & 0 \end{vmatrix}$$

$$= \hat{i} \left\{ 0 - \frac{\partial}{\partial z} \frac{-x}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}} \right\} + \hat{j} \left\{ \frac{\partial}{\partial z} \left(\frac{y}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}} \right) - 0 \right\}$$

$$+ \hat{k} \left\{ -\frac{\partial}{\partial x} \frac{x}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}} - \frac{\partial}{\partial y} \frac{y}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}} \right\}$$

$$= \hat{i} \left\{ \frac{-3xz}{\left(x^2+y^2+z^2\right)^{\frac{5}{2}}} \right\} + \hat{j} \left\{ \frac{-3yz}{\left(x^2+y^2+z^2\right)^{\frac{5}{2}}} \right\}$$

$$+ \hat{k} \left\{ \frac{3x^2 - x^2 - y^2 - z^2 + 3y^2 - x^2 - y^2 - z^2}{\left(x^2+y^2+z^2\right)^{\frac{5}{2}}} \right\}$$

$$\therefore \text{Curl}\left(\hat{k} \times \text{grad}\, \frac{1}{r}\right) = \frac{\hat{i}(-3xz) + \hat{j}(-3yz) + \hat{k}\left(x^2+y^2-2z^2\right)}{\left(x^2+y^2+z^2\right)^{\frac{5}{2}}} \qquad \ldots(1)$$

Now, $\hat{k} \bullet \text{grad}\, \dfrac{1}{r} = \hat{k} \bullet \dfrac{-1}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}} \left(x\hat{i} + y\hat{j} + z\hat{k}\right) = -\dfrac{1}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}} z$

$$\text{grad}\left(\hat{k}\cdot\text{grad}\,\frac{1}{r}\right)=\nabla\left\{-\frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right\}=\left\{\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right\}\left(\frac{-z}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right)$$

$$=\left[\frac{\hat{i}(-z)\left(-\frac{3}{2}\right)(2x)}{(x^2+y^2+z^2)^{\frac{5}{2}}}+\hat{j}\frac{(-z)\left(-\frac{3}{2}\right)(2y)}{(x^2+y^2+z^2)^{\frac{5}{2}}}+\hat{k}\left\{\frac{(-z)\left(-\frac{3}{2}\right)(2z)}{(x^2+y^2+z^2)^{\frac{5}{2}}}-\frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right\}\right]$$

$$=\frac{\hat{i}\,(3xz)+\hat{j}\,(3yz)+\hat{k}\,(3z^2-x^2-y^2-z^2)}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$

$$=\frac{\hat{i}\,(3xz)+\hat{j}\,(3yz)+\hat{k}\,(-x^2-y^2+2z^2)}{(x^2+y^2+z^2)^{\frac{5}{2}}} \qquad ...(2)$$

Adding (1) and (2), we get $Curl\left(\hat{k}\times grad\,\frac{1}{r}\right)+grad\left(\hat{k}\cdot grad\,\frac{1}{r}\right)=0.$

Example 10. *Prove that (i)* $\nabla^2 f(r)=f''(r)+\frac{2}{r}\,f'(r)$.

(ii) $\nabla^2(r^n)=n\,(n+1)\,r^{n-2}$ **(P.T.U., May 2007, May 2008)**

Sol. (*i*) $\nabla^2 f(r)=\nabla\cdot\{\nabla f(r)\}=\text{div}\,\{\text{grad}\,f(r)\}$

$$\text{grad}\,f(r)=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)f(r)=\hat{i}\frac{\partial}{\partial x}f(r)+\hat{j}\frac{\partial}{\partial y}f(r)+\hat{k}\frac{\partial}{\partial z}f(r)$$

$$=\hat{i}\,f'(r)\frac{\partial r}{\partial x}+\hat{j}\,f'(r)\frac{\partial r}{\partial y}+\hat{k}\,f'(r)\frac{\partial r}{\partial z}=f'(r)\left[\hat{i}\frac{\partial r}{\partial x}+\hat{j}\frac{\partial r}{\partial y}+\hat{k}\frac{\partial r}{\partial z}\right]$$

$$=f'(r)\,\text{grad}\,r \qquad ...(1)$$

$\because \qquad \nabla^2 f(r)=\text{div}\,[f'(r)\,\text{grad}\,r]$

We know that $\text{grad}\,r=\frac{\vec{r}}{r}$ (See S.E. 4 art 7.17)

$$=\text{div}\left[f'(r)\,\frac{\vec{r}}{r}\right]$$

$$=\text{div}\left[\frac{f'(r)}{r}\,\vec{r}\right]$$

which is of the type $\text{div}\,(\phi\vec{a})$ where $\phi=\frac{f'(r)}{r}$ and $\vec{a}=\vec{r}$

$$\therefore \quad \operatorname{div}\left[\frac{f'(r)}{r}\vec{r}\right] = \frac{f'(r)}{r}\operatorname{div}\vec{r} + \vec{r}\cdot\operatorname{grad}\frac{f'(r)}{r}$$

Since $\operatorname{div}\vec{r} = 3$ and from (1) grad $f(r) = f'(r)$ grad r; Replace $f(r)$ by $\frac{f'(r)}{r}$, we get

$$\operatorname{grad}\frac{f'(r)}{r} = \frac{d}{dr}\left[\frac{f'(r)}{r}\right]\operatorname{grad} r$$

$$= \frac{f'(r)}{r}3 + \vec{r}\cdot\left[\frac{d}{dr}\left(\frac{f'(r)}{r}\right)\operatorname{grad} r\right]$$

$$= \frac{3f'(r)}{r} + \vec{r}\cdot\left\{\frac{rf''(r)-f'(r)}{r^2}\right\}\cdot\frac{\vec{r}}{r}$$

$$= \frac{3f'(r)}{r} + \left\{\frac{f''(r)}{r^2} - \frac{f'(r)}{r^3}\right\}\vec{r}\cdot\vec{r} = \frac{3f'(r)}{r} + \left\{f''(r) - \frac{f'(r)}{r}\right\}\frac{r^2}{r^2}$$

$$= \frac{3f'(r)}{r} + f''(r) - \frac{f'(r)}{r} = f''(r) + \frac{2}{r}f'(r)$$

(ii) $\nabla^2 r^n = \nabla\cdot\nabla r^n = \operatorname{div}\operatorname{grad} r^n$

$$\operatorname{grad} r^n = \hat{i}\frac{\partial}{\partial x}r^n + \hat{j}\frac{\partial}{\partial y}r^n + \hat{k}\frac{\partial}{\partial z}r^n$$

$$= \hat{i}\cdot nr^{n-1}\frac{\partial r}{\partial x} + \hat{j}\,nr^{n-1}\frac{\partial r}{\partial y} + \hat{k}\,nr^{n-1}\frac{\partial r}{\partial z}$$

$$= nr^{n-1}\left\{\hat{i}\frac{x}{r} + \hat{j}\frac{y}{r} + \hat{k}\frac{z}{r}\right\} = nr^{n-1}\frac{\vec{r}}{r} = nr^{n-2}\vec{r}$$

Now $\quad \nabla^2 r^n = \operatorname{div}\left[\left(nr^{n-2}\right)\vec{r}\right]$ which is of the type $\operatorname{div}(\phi\vec{a})$

$$= \left(nr^{n-2}\right)\operatorname{div}\vec{r} + \vec{r}\cdot\operatorname{grad}\left(nr^{n-2}\right)$$

$$= \left(nr^{n-2}\right)3 + \vec{r}\;n\operatorname{grad} r^{n-2}$$

$$= 3nr^{n-2} + n\vec{r}\cdot(n-2)r^{n-4}\vec{r}$$

$\because$ grad $r^n = nr^{n-2}\vec{r}$; change n to $n-2$

grad $r^{n-2} = (n-2)r^{n-4}\vec{r}$

$$= 3n\, r^{n-2} + n\,(n-2)\, r^{n-4}\,(\vec{r} \cdot \vec{r})$$
$$= 3n\, r^{n-2} + n\,(n-2)\, r^{n-4} \cdot r^2$$
$$= 3n\, r^{n-2} + n\,(n-2)\, r^{n-2}$$
$$= (3n + n^2 - 2n)\, r^{n-2}$$
$$= (n + n^2)\, r^{n-2} = n\,(n+1)\, r^{n-2}.$$

Example 11. *Find directional derivative of div $(\vec{u})$ at the point (1, 2, 2) in the direction of the outer normal of the sphere $x^2 + y^2 + z^2 = 9$ for $\vec{u} = x^4\,\hat{i} + y^4\,\hat{j} + z^4\,\hat{k}$.*

Sol. div $(\vec{u}) = \nabla \cdot \vec{u}$

$$= \left(\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}\right) \cdot \left(x^4\,\hat{i} + y^4\,\hat{j} + z^4\,\hat{k}\right) = 4\,(x^3 + y^3 + z^3)$$

Outer normal to the sphere $= \nabla(x^2 + y^2 + z^2 - 9)$

$$= \left(\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}\right) \cdot \left(x^2 + y^2 + z^2 - 9\right)$$
$$= \hat{i}(2x) + \hat{j}(2y) + \hat{k}\,(2z) = 2\left(x\,\hat{i} + y\,\hat{j} + z\,\hat{k}\right)$$

Outer normal at the point (1, 2, 2) $= 2\left(\hat{i} + 2\hat{j} + 2\hat{k}\right)$

Gradient of div $\vec{u} = \nabla\,(4\,x^3 + 4\,y^3 + 4\,z^3)$

$$= \left(\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}\right)\left(4x^3 + 4y^3 + 4z^3\right) = 12\left(x^2\,\hat{i} + y^2\,\hat{j} + z^2\hat{k}\right)$$

Gradient of div $\vec{u}$ at (1, 2, 2) $= 12\,\left(\hat{i} + 4\hat{j} + 4\hat{k}\right)$

Directional derivative of div $\vec{u}$ in the direction of outer normal

$$= 12\left(\hat{i} + 4\hat{j} + 4\hat{k}\right) \cdot \frac{\left(2\hat{i} + 4\hat{j} + 4\hat{k}\right)}{\sqrt{4 + 16 + 16}}$$
$$= \frac{12}{6}\,(1.2 + 4.4 + 4.4) = 2\,(2 + 16 + 16) = 68.$$

TEST YOUR KNOWLEDGE

1. If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, show that div $\vec{r} = 3$, curl $\vec{r} = \vec{0}$. **(P.T.U., Dec. 2006)**
2. (*i*) Find divergence and Curl of the vector $\vec{V} = (xyz)\,\hat{i} + \left(3x^2 y\right)\hat{j} + \left(xz^2 - y^2 z\right)\hat{k}$ at the point (2, –1, 1).

   (*ii*) Show that for the vector field $\vec{F} = 2xy\hat{i} + xe^y\hat{j} + 2z\hat{k}$, $\nabla \cdot (\nabla \times \vec{F}) = 0$. **(P.T.U. Dec. 2013)**
3. (*i*) If $\vec{F} = (x + y + 1)\,\hat{i} + \hat{j} - (x + y)\,\hat{k}$, show that $\vec{F} \cdot \text{curl}\,\vec{F} = 0$ **(P.T.U., Dec. 2012)**

   (*ii*) Find $\nabla \times (\nabla\phi)$ where $\phi\,(x, y, z) = -\,2x^3\,yz^2$. **(P.T.U. Dec. 2013)**

4. If $\vec{A} = (3xz^2)\hat{i} - (yz)\hat{j} + (x + 2z)\hat{k}$, find Curl (Curl $\vec{A}$).

5. If $\vec{V} = \dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}}$, show that $\nabla \bullet \vec{V} = \dfrac{2}{\sqrt{x^2 + y^2 + z^2}}$ and $\nabla \times \vec{V} = \vec{0}$ or If $\vec{V} = \dfrac{\vec{r}}{r}$, show that divergence of $\vec{V} = \dfrac{2}{r}$ and Curl $\vec{V} = \vec{0}$.

6. If $\vec{V}_1$ and $\vec{V}_2$ be the vectors joining the fixed points (x_1, y_1, z_1) and (x_2, y_2, z_2) to a variable point (x, y, z) prove that

(i) $\text{div}\left(\vec{V}_1 \times \vec{V}_2\right) = 0$ (ii) $\text{grad}\left(\vec{V}_1 \bullet \vec{V}_2\right) = \vec{V}_1 + \vec{V}_2$

(iii) $\text{Curl}\left(\vec{V}_1 \times \vec{V}_2\right) = 2\left(\vec{V}_1 - \vec{V}_2\right)$.

7. If $\vec{a}$ is a constant vector and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ prove that

(i) $\text{div}(\vec{a} \times \vec{r}) = 0$ (ii) $\text{Curl}\left[(\vec{a} \bullet \vec{r})\vec{r}\right] = \vec{a} \times \vec{r}$

(iii) $\nabla(\vec{a} \bullet \vec{a}) = 2(\vec{a} \bullet \nabla)\vec{a} + 2\vec{a} \times (\nabla \times \vec{a})$.

(iv) $\text{Curl}(\vec{a} \times \vec{r}) = 2\vec{a}$ **(P.T.U., May 2010)**

(v) $\text{grad}(\vec{a} \bullet \vec{r}) = \vec{a}$

8. If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, prove that

(i) $\nabla^2\left(\dfrac{1}{r}\right) = 0$ ***(P.T.U., June 2003)*** (ii) $\nabla^2\left(r^n\right)\vec{r} = n(n+1)r^{n-2}\,\vec{r}$

(P.T.U., Dec. 2003)

(iii) $\nabla \bullet \left\{r \nabla\left(\dfrac{1}{r^3}\right)\right\} = \dfrac{3}{r^4}$ (iv) $\nabla^2\left\{\nabla \bullet \left(\dfrac{\vec{r}}{r^2}\right)\right\} = 2r^{-4}$.

9. Find the directional derivative of $\nabla \bullet (\nabla\phi)$ at the point $(1, -2, 1)$ in the direction of outer normal to the surface $xy^2z = 3x + z^2$ where, $\phi = 2x^3 y^2 z^4$.

Answers

2. (i) $14, 2\hat{i} - 3\hat{j} - 14\hat{k}$ (ii) -3 3. (ii) $\vec{0}$ 4. $-6x\,\hat{i} + (6z - 1)\,\hat{k}$

9. $\dfrac{1724}{\sqrt{21}}$.

7.24. INTEGRATION OF VECTORS

Definition. Let $\overrightarrow{f(t)}$ and $\overrightarrow{g(t)}$ be two vectors functions of a scalar variable t such that $\dfrac{d}{dt}\overrightarrow{g(t)} = \overrightarrow{f(t)}$ then $\overrightarrow{g(t)}$ is called an integral of $\overrightarrow{f(t)}$ with respect to t and we write

$$\int \overrightarrow{f(t)}\,dt = \overrightarrow{g(t)}$$

If $\vec{c}$ is any arbitrary constant vector then we know that $\frac{d}{dt}\left[\overrightarrow{g(t)}+\vec{c}\right]=\overrightarrow{f(t)}$

$$\therefore \qquad \int \overrightarrow{f(t)}\,dt = \overrightarrow{g(t)}+\vec{c}$$

$\overrightarrow{g(t)}$ is called the indefinite integral of $\overrightarrow{f(t)}$. The constant vector $\vec{c}$ is called constant of integration and can be determined if some initial conditions are given.

The definite integral of $\overrightarrow{f(t)}$ between the limits $t = a$ and $t = b$ is written as

$$\int_a^b \overrightarrow{f(t)}\,dt = \overrightarrow{g(t)}\Big|_a^b = g(b) - g(a)$$

Note 1. If $\overrightarrow{f(t)} = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$

then
$$\int \overrightarrow{f(t)}\,dt = \hat{i}\int f_1(t)dt + \hat{j}\int f_2(t)dt + \hat{k}\int f_3(t)dt$$

Thus in order to integrate a vector function integrate its components.

Note 2. We can obtain some standard results for integration of vector functions by considering the derivatives of suitable vector functions. For example:

(i) $\frac{d}{dt}(\vec{r}\cdot\vec{s}) = \frac{d\vec{r}}{dt}\cdot\vec{s} + \vec{r}\cdot\frac{d\vec{s}}{dt}$

$\Rightarrow \int\left(\frac{d\vec{r}}{dt}\cdot\vec{s}+\vec{r}\cdot\frac{d\vec{s}}{dt}\right)dt = \vec{r}\cdot\vec{s}+c$, where c is a scalar quantity since integrand is scalar.

(ii) $\frac{d\vec{r}}{dt}(\vec{r}^2) = 2\vec{r}\cdot\frac{d\vec{r}}{dt}$

$\Rightarrow \int\left(2\vec{r}\cdot\frac{d\vec{r}}{dt}\right)dt = \vec{r}^2 + c$, where c is scalar quantity since integrand is scalar.

(iii) $\frac{d}{dt}\left(\vec{r}\times\frac{d\vec{r}}{dt}\right) = \frac{d\vec{r}}{dt}\times\frac{d\vec{r}}{dt} + \vec{r}\times\frac{d^2\vec{r}}{dt^2} = \vec{r}\times\frac{d^2\vec{r}}{dt^2} \quad \because \quad \frac{d\vec{r}}{dt}\times\frac{d\vec{r}}{dt}=\vec{0}$

$\Rightarrow \int\left(\vec{r}\times\frac{d^2\vec{r}}{dt^2}\right)dt = \vec{r}\times\frac{d\vec{r}}{dt}+\vec{c}$, where $\vec{c}$ is a vector quantity since integrand is vector.

(iv) If $\vec{a}$ is a constant vector then, $\frac{d}{dt}(\vec{a}\times\vec{r}) = \vec{a}\times\frac{d\vec{r}}{dt}$

$\Rightarrow \int\left(\vec{a}\times\frac{d\vec{r}}{dt}\right)dt = \vec{a}\times\vec{r}+\vec{c}$, where $\vec{c}$ is a vector quantity since integrand is vector.

ILLUSTRATIVE EXAMPLES

Example 1. *If* $\overrightarrow{f(t)} = (5t^2-3t)\hat{i}+6t^3\hat{j}-7t\hat{k}$ *evaluate* $\int_{t=2}^{t=4} f(\vec{t})\,dt$.

Sol.
$$\int_{t=2}^{t=4} f(\vec{t})dt = \int_2^4 (5t^2-3t)\hat{i}+(6t^3)\hat{j}-(7t)\hat{k}\,dt$$
$$= \left(\frac{5t^3}{3}-\frac{3t^2}{2}\right)\hat{i}+\frac{6t^4}{4}\hat{j}-\frac{7t^2}{2}\hat{k}\Big|_2^4$$
$$= \left(\frac{320}{3}-24\right)\hat{i}+384\hat{j}-56\hat{k}-\left(\frac{40}{3}-6\right)\hat{i}-24\hat{j}+14\hat{k}$$
$$= \frac{248}{3}\hat{i}+384\hat{j}-56\hat{k}-\frac{22}{3}\hat{i}-24\hat{j}+14\hat{k} = \frac{226}{3}\hat{i}+360\hat{j}-42\hat{k}.$$

Example 2. *The acceleration of a particle at any time t is given by* $\vec{a} = 18\cos 3t\hat{i} - 8\sin 2t\hat{j}+6t\hat{k}$. *If the velocity* $\vec{v}$ *and displacement* $\vec{r}$ *be zero at* $t = 0$. *Find* $\vec{v}$ *and* $\vec{r}$ *at any time t.*

Sol. $\vec{a} = \frac{d^2\vec{r}}{dt^2} = 18\cos 3t\hat{i} - 8\sin 2t\hat{j}+6t\hat{k}$

Integrating w.r.t. t, we have
$$\vec{v} = \frac{d\vec{r}}{dt} = \int\left\{(18\cos 3t)\hat{i}-(8\sin 2t)\hat{j}+(6t)\hat{k}\right\}dt+\vec{c}$$
$$= 6\sin 3t\hat{i}+4\cos 2t\hat{j}+3t^2\hat{k}+\vec{c}$$

At $t = 0$, $\quad \vec{v} = \vec{0}$

$\therefore \quad \vec{0} = 4\hat{j}+\vec{c} \quad \therefore \quad \vec{c} = -4\hat{j}$

$\therefore \quad \vec{v} = \frac{d\vec{r}}{dt} = 6\sin 3t\hat{i}+4(\cos 2t-1)\hat{j}+3t^2\hat{k}.$

Integrating again w.r.t. t
$$\vec{r} = -2\cos 3t\hat{i}+4\left(\frac{\sin 2t}{2}-t\right)\hat{j}+t^3\hat{k}+\vec{c}_1$$

At $t = 0$, $\quad \vec{r} = \vec{0} \therefore \quad \vec{0} = -2\hat{i}+\vec{c}_1 \quad \therefore \quad \vec{c}_1 = 2\hat{i}$

$\therefore \quad \vec{r} = -(2\cos 3t)\hat{i}+(2\sin 2t-4t)\hat{j}+t^3\hat{k}+2\hat{i}$

$$\vec{r} = 2(1-\cos 3t)\hat{i}+2(\sin 2t-2t)\hat{j}+t^3\hat{k}.$$

Example 3. *If* $\overrightarrow{r(t)} = 5t^2\hat{i} + t\hat{j} - t^3\hat{k}$, *prove that* $\int_1^2 \vec{r} \times \frac{d^2\vec{r}}{dt^2}\, dt = -14\hat{i} + 75\hat{j} - 15\hat{k}$.

Sol. $\frac{d}{dt}\left(\vec{r} \times \frac{d\vec{r}}{dt}\right) = \frac{d\vec{r}}{dt} \times \frac{d\vec{r}}{dt} + \vec{r} \times \frac{d^2\vec{r}}{dt^2} = \vec{r} \times \frac{d^2\vec{r}}{dt^2} \quad \because \quad \frac{d\vec{r}}{dt} \times \frac{d\vec{r}}{dt} = \vec{0}$

Integrating both sides w.r.t. t between the limits $t = 1$ and $t = 2$, we get

$$\int_1^2 \left(\vec{r} \times \frac{d^2\vec{r}}{dt^2}\right) dt = \vec{r} \times \frac{d\vec{r}}{dt}\Bigg|_1^2 \qquad \text{...(1)}$$

[Now find $\vec{r} \times \frac{d\vec{r}}{dt}$]

$$\vec{r} \times \frac{d\vec{r}}{dt} = \left\{(5t^2)\hat{i} + (t)\hat{j} - (t^3)\hat{k}\right\} \times \left\{(10t)\hat{i} + \hat{j} - 3t^2\hat{k}\right\} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5t^2 & t & -t^3 \\ 10t & 1 & -3t^2 \end{vmatrix}$$

$$= -2t^3\hat{i} + 5t^4\hat{j} - 5t^2\hat{k}$$

$$\therefore \quad \int_1^2 \vec{r} \times \frac{d^2\vec{r}}{dt^2}\, dt = -2t^3\hat{i} + 5t^4\hat{j} - 5t^2\hat{k}\Big|_1^2 = -16\hat{i} + 80\hat{j} - 20\hat{k} + 2\hat{i} - 5\hat{j} + 5\hat{k}$$

$$= -14\hat{i} + 75\hat{j} - 15\hat{k}.$$

Example 4. *Given* $\vec{r}(t) = 2\hat{i} - \hat{j} + 2\hat{k}$ *when* $t = 2$ *and* $\vec{r}(t) = 4\hat{i} - 2\hat{j} + 3\hat{k}$ *when* $t = 3$.

Show that $\int_2^3 \left(\vec{r} \cdot \frac{d\vec{r}}{dt}\right) dt = 10$.

Sol. $\frac{d}{dt}\vec{r}^2 = 2\vec{r} \cdot \frac{d\vec{r}}{dt}$

Integrate both sides w.r.t. t, we have $\int_2^3 2\vec{r} \cdot \frac{d\vec{r}}{dt}\, dt = \vec{r}^2 \Big|_2^3$

$$\therefore \quad \int_2^3 \vec{r} \cdot \frac{d\vec{r}}{dt}\, dt = \frac{1}{2}\vec{r}^2 \Big|_2^3 \qquad \text{...(1)}$$

Now, At $t = 3$, $\quad \vec{r} = 4\hat{i} - 2\hat{j} + 3\hat{k}$

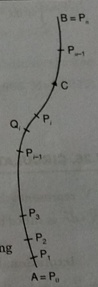
$\therefore \quad \vec{r}^2 = \vec{r} \cdot \vec{r} = 16 + 4 + 9 = 29$

At $t = 2$, $\quad \vec{r} = 2\hat{i} - \hat{j} + 2\hat{k}$

$\vec{r}^2 = 4 + 1 + 4 = 9$

$\therefore$ From (1), $\int_2^3 \vec{r} \cdot \frac{d\vec{r}}{dt}\, dt = \frac{1}{2} \cdot [29 - 9] = 10.$

7.25. LINE INTEGRAL



Definition. Any integral which is evaluated along a curve is called a line integral. Consider a continuous vector point function $\overrightarrow{F(P)}$ which is defined at each point of the curve C in space.

Divide the curve C into n parts by the points.

$A = P_0, P_1, P_2 \cdots P_{i-1}, P_i \cdots P_{n-1}, P_n = B$ and let $\vec{r}_0, \vec{r}_1, \vec{r}_2, \cdots, \vec{r}_{i-1}, \vec{r}_i, \cdots, \vec{r}_{n-1}, \vec{r}_n$ be the position vectors of these points.

Let Q_i be any point on the arc $P_{i-1}P_i$. Then the limit of the sum

$$\sum_{i=1}^{n} \overrightarrow{F(Q_i)} \cdot \delta\vec{r}_i, \text{ where } \delta\vec{r}_i = \vec{r}_i - \vec{r}_{i-1} \qquad \text{...(1)}$$

as $n \to \infty$ and every $|\delta r_i| \to 0$ (if it exists) is called a line integral of $\vec{F}$ along C and is denoted by

$$\int_C \vec{F} \cdot \overrightarrow{dR} \quad \text{or} \quad \int_C \vec{F} \cdot \frac{\overrightarrow{dR}}{dt}\, dt$$

Clearly, it is a scalar quantity.

It is called **tangential line integral of $\vec{F}$ along C.**

Note 1. If the scalar products in (1) are replaced by vector products then the corresponding line integral is $\int_C \vec{F} \times d\vec{r}$ which is a vector.

Note 2. If the vector function $\vec{F}$ is replaced by a scalar function ϕ, then the corresponding line integral is defined as $\int_C \phi\, d\vec{r}$ which is a vector.

Note 3. If $\quad \vec{F}(x, y, z) = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ then

$$\int_C \vec{F} \cdot d\vec{r} = \int_{t_1}^{t_2} \left(f_1\frac{dx}{dt} + f_2\frac{dy}{dt} + f_3\frac{dz}{dt}\right) dt$$

Note 4. If C is a closed curve, then the integral sign $\int_C$ is replaced by $\oint_C$

Example 5. *Evaluate* $\int_C x\,dx - yz\,dy + e^z dz$ *where C is the curve defined by* $x = t^3$, $y = -t$, $z = t^2$ *for t lying in the interval* $0 \le t \le 2$. **(P.T.U. Dec. 2013)**

Sol. Given $x = t^3, y = -t, z = t^2$ and $0 \le t \le 2$.

$$dx = 3t^2\,dt,\ dy = -\,dt,\ dz = 2t\,dt$$

$$\int_C x\,dx - yz\,dy + e^z dz = \int_0^2 t^3(3t^2\,dt) - (-t)(t^2)(-dt) + e^{t^2}(2t\,dt)$$

$$= \int_0^2 (3t^5 - t^3 + 2t\,e^{t^2})\,dt$$

$$= \left.\frac{3t^6}{6} - \frac{t^4}{4} + e^{t^2}\right|_0^2 \qquad \left[\because \int e^{t^2}\cdot 2t\,dt = e^{t^2}\right]$$

$$= (32 - 4 + e^4) - (1) = 27 + e^4.$$

7.26. CIRCULATION

If $\vec{V}$ represents the velocity of a fluid particle and C is a closed curve, then the line integral $\int_C \vec{V}\cdot\vec{dr}$ is called circulation of $\vec{V}$ round the curve C.

Irrotational. If the circulation of $\vec{V}$ round every closed curve in a region D vanishes, then $\vec{V}$ is said to be irrotational in D.

7.27. WORK DONE BY A FORCE

If $\vec{F}$ represents a force acting on a particle moving along an arc AB of a curve C, then the work done during a small displacement $\delta\vec{r}$ is $\vec{F}\cdot\delta\vec{r}$

$\therefore$ Total work done by $\vec{F}$ during the displacement from A to B is given by $\int_A^B \vec{F}\cdot\vec{dr}$.

If the force is conservative, then there exists a scalar function ϕ such that

$$\vec{F} = \nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$\therefore$ The work done by $\vec{F}$ during displacement from A to B $= \int_A^B \vec{F}\cdot\vec{dr}$

$$= \int_A^B \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)\cdot\left(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz\right)$$

$$= \int_A^B \left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right) = \int_A^B d\phi = \phi\Big|_A^B = \phi_B - \phi_A.$$

Thus in a conservative field work done depends upon the value of ϕ at the end points A and B **and not on the path joining A and B.**

Note. If $\int_A^B \vec{F}\cdot\vec{dr}$ is to be proved to be independent of path, then prove $\vec{F} = \nabla\phi$

Here $\vec{F}$ is called conservative (irrotational) vector field and ϕ is called scalar potential.

And $\nabla\times\vec{F} = \nabla\times\nabla\phi = \vec{0}$.

Example 6. *A vector field is given by* $\vec{F} = (\sin y)\hat{i} + x(1 + \cos y)\hat{j}$. *Evaluate the line integral over a circular path given by* $x^2 + y^2 = a^2, z = 0$. **(P.T.U., June, 2003, May 2012)**

Sol. Line integral of $\vec{F}$ is $\int_C \vec{F}\cdot\vec{dr}$

$$= \oint_C \left[(\sin y)\hat{i} + x(1 + \cos y)\hat{j}\right]\cdot\left[dx\,\hat{i} + dy\,\hat{j}\right]$$

$$= \oint_C \sin y\,dx + (x + x\cos y)\,dy$$

$$= \oint_C (\sin y\,dx + x\cos y\,dy) + x\,dy$$

$$= \oint_C d(x\sin y) + \oint_C x\,dy$$

The parametric equation of C is $x = a\cos\theta, y = a\sin\theta$, where $0 \le \theta \le 2\pi$.

$$\therefore \quad \text{Line integral} = \int_0^{2\pi} d\left[a\cos\theta\sin(a\sin\theta)\right] + \int_0^{2\pi}(a\cos\theta)(a\cos\theta\,d\theta)$$

$$= a\cos\theta\sin(a\sin\theta)\Big|_0^{2\pi} + a^2\int_0^{2\pi}\cos^2\theta\,d\theta$$

$$= 0 + a^2\,4\int_0^{\frac{\pi}{2}}\cos^2\theta\,d\theta = 4a^2\,\frac{1}{2}\,\frac{\pi}{2} = \pi a^2$$

Example 7. *Compute* $\int_c \vec{F}\cdot\vec{dr}$, *where* $\vec{F} = \dfrac{\hat{i}\,y - \hat{j}\,x}{x^2 + y^2}$ *and c is the circle* $x^2 + y^2 = 1$ *transversed counter clockwise.* **(P.T.U., Dec. 2004)**

Sol. $$\int_c \vec{F}\cdot d\vec{r} = \int_c \left(\frac{\hat{i}\,y-\hat{j}\,x}{x^2+y^2}\right)\cdot\left(dx\,\hat{i}+dy\,\hat{j}+dz\,\hat{k}\right)$$

$$= \int_c \frac{y}{x^2+y^2}\,dx - \frac{x\,dy}{x^2+y^2} = \int_c \frac{y\,dx - x\,dy}{1} \qquad (\because x^2+y^2=1)$$

Parametric equation of the circle (with radius 1) is $x = 1\,.\cos\theta$, $y = 1\,.\sin\theta$

$\therefore$ $dx = -\sin\theta\,d\theta$, $dy = \cos\theta\,d\theta$

$$\therefore \quad \int_c \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \sin\theta\,(-\sin\theta)\,d\theta - \cos\theta\,(\cos\theta\,d\theta)$$

$$= -\int_0^{2\pi}\left(\sin^2\theta+\cos^2\theta\right)d\theta = -\int_0^{2\pi} d\theta = -\theta\Big|_0^{2\pi} = -2\pi$$

Hence $\int_c \vec{F}\cdot d\vec{r} = -2\pi$.

Example 8. *If $\vec{F} = 3xy\,\hat{i} - y^2\,\hat{j}$. Evaluate $\int_c \vec{F}\cdot d\vec{r}$, where c is the curve in xy-plane $y = 2x^2$ from (0, 0) to (1, 2).* **(P.T.U., May 2004)**

Sol. $$\int_c \vec{F}\cdot d\vec{r} = \int_c \left(3xy\,\hat{i} - y^2\,\hat{j}\right)\cdot\left(dx\,\hat{i}+dy\,\hat{j}\right) = \int_c 3xy\,dx - y^2 dy$$

$$= \int_c 3xy\,dx - \int_c y^2\,dy$$

To integrate; change 1st integrand to function of x only with the help of $y = 2x^2$.

$$\therefore \quad \int_c \vec{F}\cdot d\vec{r} = \int_0^1 3x\,.\,2x^2\,dx - \int_0^2 y^2\,dy \quad (x \text{ varies from 0 to 1 and } y \text{ varies from 0 to 2}).$$

$$= 6\,\frac{x^4}{4}\Big|_0^1 - \frac{y^3}{3}\Big|_0^2 = \frac{6}{4}-\frac{8}{3} = \frac{3}{2}-\frac{8}{3} = \frac{-7}{6}.$$

Second Method. Take parametric equation of the parabola $y = 2x^2$ which is $x = t$, $y = 2t^2$

$\therefore$ when $x = t$; $dx = dt$, when $y = 2t^2$; $dy = 4t\,dt$

At the point (0, 0), $x = 0$ $\therefore$ $t = 0$

At the point (1, 2); $x = 1$ $\therefore$ $t = 1$

$$\therefore \quad \int_c \vec{F}\cdot d\vec{r} = \int_c 3xy\,dx - y^2\,dy = \int_0^1 3\,.\,t\,.\,2t^2 - 4t^4\,.\,4t\,dt = \int_0^1\left(6t^3-16t^5\right)dt$$

$$= 6\,\frac{t^4}{4} - 16\,\frac{t^6}{6}\Big|_0^1 = \frac{6}{4}-\frac{8}{3} = -\frac{7}{6}.$$

Example 9. *Find the work done in moving a particle in the force field $\vec{F} = 3x^2\,\hat{i} + (2xz-y)\,\hat{j} + z\,\hat{k}$ along*

(i) the straight line from (0, 0, 0) to (2, 1, 3). **(P.T.U., May 2005, May 2010)**

(ii) the curve defined by $x^2 = 4y$, $3x^3 = 8z$ from $x = 0$ to $x = 2$.

Sol. Work done $$= \int \vec{F}\cdot d\vec{r} = \int_c \left(3x^2\,\hat{i} + (2xz-y)\,\hat{j} + z\,\hat{k}\right)\cdot\left(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz\right)$$

$$= \int_c 3x^2\,dx + (2xz-y)\,dy + z\,dz$$

(*i*) Equation of the line through (0, 0, 0) to (2, 1, 3) is

$$\frac{x-0}{2-0} = \frac{y-0}{1-0} = \frac{z-0}{3-0} \quad \text{or} \quad \frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t \text{ (say)}$$

$\therefore$ $x = 2t$, $y = t$, $z = 3t$

$dx = 2dt$, $dy = dt$, $dz = 3\,dt$

When $x = 0$, $t = 0$

When $x = 2$, $t = 1$

$\therefore$ t varies from 0 to 1.

$$\text{Work done} = \int_{t=0}^{1} 3\left(4t^2\right)(2dt) + \left(12t^2 - t\right)dt + 7t\,dt$$

$$= \int_0^1\left(24t^2+12t^2-t+9t\right)dt = \int_0^1\left(36t^2+8t\right)dt = 36\,\frac{t^3}{3} + 8\,\frac{t^2}{2}\Big|_0^1 = 12+4 = 16.$$

(*ii*) Along the curve $x^2 = 4y$, $3x^3 = 8z$

For parametric equation of the curve, we take $x = t$ $\therefore$ $y = \frac{t^2}{4}$, $z = \frac{3t^3}{8}$

$\therefore$ $dx = dt$, $dy = \frac{t\,dt}{2}$, $dz = \frac{9t^2}{8}\,dt$

When $x = 0$, $t = 0$

When $x = 2$, $t = 2$ $\therefore$ t varies from 0 to 2.

$$\therefore \quad \text{Work done} = \int_0^2 \left(3t^2\right)dt + \left(\frac{3}{4}t^4 - \frac{t^2}{4}\right)\frac{t}{2}\,dt + \left(\frac{3}{8}t^3\right)\left(\frac{9}{8}t^2\right)dt$$

$$= \int_0^2\left(3t^2 + \frac{3}{8}t^5 - \frac{t^3}{8} + \frac{27}{64}t^5\right)dt = \int_0^2\left(3t^2 - \frac{t^3}{8} + \frac{51}{64}t^5\right)dt$$

$$= t^3 - \frac{t^4}{32} + \frac{51}{64}\cdot\frac{t^6}{6}\Big|_0^2 = 8 - \frac{1}{2} + \frac{51}{6} = 8 - \frac{1}{2} + \frac{17}{2} = 8 + \frac{16}{2} = 16.$$

Example 10. *Find the work done by the force* $\vec{F} = x\hat{i} - z\hat{j} + 2y\hat{k}$ *in the displacement along the closed path c consisting of the segments* c_1, c_2, c_3 *where*

On c_1 $\quad 0 \le x \le 1 \quad y = x, \quad z = 0$

On c_2 $\quad 0 \le z \le 1 \quad x = 1, \quad y = 1$

On c_3 $\quad 1 \ge x \ge 0 \quad y = z = x$

Sol. Total work done $= \int_c \vec{F} \cdot d\vec{r} = \int_c \left(x\hat{i} - z\hat{j} + 2y\hat{k}\right) \cdot \left(\hat{i}\, dx + \hat{j}\, dy + \hat{k}\, dz\right)$

$$= \int_c x\,dx - z\,dy + 2y\,dz$$

$$= \int_{c_1} x\,dx - z\,dy + 2y\,dz + \int_{c_2} x\,dx - z\,dy + 2y\,dz + \int_{c_3} x\,dx - z\,dy + 2y\,dz$$

$$= W_1 + W_2 + W_3$$

$$W_1 = \int_{c_1} x\,dx - z\,dy + 2y\,dz$$

On c_1 $\quad y = x, \quad z = 0$ and x varies from 0 to 1

$\therefore$ $\quad dy = dx,\ dz = 0$

$$W_1 = \int_0^1 x\,dx - 0 + 0 = \left.\frac{x^2}{2}\right|_0^1 = \frac{1}{2}$$

$$W_2 = \int_{c_2} x\,dx - z\,dy + 2y\,dz$$

On c_2 $\quad x = 1, \quad y = 1$ and z varies from 0 to 1

$$dx = 0, \quad dy = 0$$

$$W_2 = \int_0^1 0 - 0 + 2 \cdot 1\,dz = 2z\,\Big|_0^1 = 2 \text{ and}$$

$$W_3 = \int_{c_3} x\,dx - z\,dy + 2y\,dz$$

On c_3 $\quad y = z = x \quad \therefore \left.\begin{matrix} y = x \\ z = x \end{matrix}\right\}$ and x varies from 1 to 0

$$dy = dx, \quad dz = dx$$

$$W_3 = \int_1^0 x\,dx - x\,dx + 2x\,dx = x^2\Big|_1^0 = -1$$

$\therefore$ Total work done $= \frac{1}{2} + 2 - 1 = \frac{1}{2} + 1 = \frac{3}{2}$.

Example 11. *Show that the integral* $\int_{(1,2)}^{(3,4)} \left(xy^2 + y^3\right)dx + \left(x^2y + 3xy^2\right)dy$ *is independent of the path joining the points (1, 2) and (3, 4). Hence evaluate the integral.*

Sol. $\int_{(1,2)}^{(3,4)} \left(xy^2 + y^3\right)dx + \left(x^2y + 3xy^2\right)dy$

$$= \int_{(1,2)}^{(3,4)} \left\{\left(xy^2 + y^3\right)\hat{i} + \left(x^2y + 3xy^2\right)\hat{j}\right\} \cdot \left\{dx\,\hat{i} + dy\,\hat{j}\right\}$$

$$= \int_{(1,2)}^{(3,4)} \vec{F} \cdot d\vec{r} \quad \therefore \quad \vec{F} = \left(xy^2 + y^3\right)\hat{i} + \left(x^2y + 3xy^2\right)\hat{j}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 + y^3 & x^2 + 3xy^2 & 0 \end{vmatrix}$$

$$= \hat{i}\left[-\frac{\partial}{\partial z}\left(x^2y + 3xy^2\right)\right] + \hat{j}\left[\frac{\partial}{\partial z}\left(xy^2 + y^3\right) - \frac{\partial}{\partial x}(0)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}\left(x^2y + 3xy^2\right) - \frac{\partial}{\partial y}\left(xy^2 + y^3\right)\right]$$

$$= \hat{i} \cdot 0 + \hat{j} \cdot 0 + \hat{k}\left[2xy + 3y^2 - 2xy - 3y^2\right] = \vec{0}$$

$\therefore \quad \nabla \times \vec{F} = \vec{0} \quad \therefore$ Integral is independent of the path of integration.

Also, if $\nabla \times \vec{F} = \vec{0}$, then $\vec{F} = \nabla\phi$

$$\left(xy^2 + y^3\right)\hat{i} + \left(x^2y + 3xy^2\right)\hat{j} = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$\therefore \quad \frac{\partial\phi}{\partial x} = xy^2 + y^3; \quad \frac{\partial\phi}{\partial y} = x^2y + 3xy^2, \quad \frac{\partial\phi}{\partial z} = 0$

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = (x\,y^2 + y^3)\,dx + (x^2\,y + 3x\,y^2)\,dy = \vec{F} \cdot d\vec{r}$$

$$= (xy^2\,dx + x^2\,y\,dy) + (y^3\,dx + 3x\,y^2\,dy)$$

$$= \frac{1}{2}d\left(x^2y^2\right) + d\left(xy^3\right) = d\left[\frac{1}{2}x^2y^2 + xy^3\right]$$

Integrate

$$\therefore \quad \int_{(1,2)}^{(3,4)} d\phi = \int_{(1,2)}^{(3,4)} d\left[\frac{1}{2}x^2y^2 + xy^3\right] = \frac{1}{2}x^2y^2 + xy^3\Bigg]_{(1,2)}^{(3,4)}$$

$$= \left(\frac{1}{2}9 \cdot 16 + 3 \cdot 64\right) - (2+8) = 254$$

Hence $\displaystyle\int_{(1,2)}^{(3,4)} \left(xy^2 + y^3\right) dx + \left(x^2 y + 3xy^2\right) dy = 254.$

Example 12. *Find circulation of $\vec{F}$ round the curve C, where $\vec{F} = e^x \sin y \hat{\imath} + e^x \cos y \hat{\jmath}$ and C is the rectangle whose vertices are (0, 0), (1, 0), $\left(1, \frac{\pi}{2}\right)$, $\left(0, \frac{\pi}{2}\right)$.*

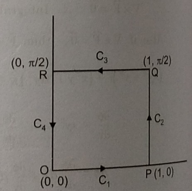
Sol. Circulation of $\vec{F}$ round the curve C

$$= \oint_C \vec{F} \cdot d\vec{r} = \oint_C \left\{\left(e^x \sin y\right)\hat{\imath} + e^x \cos y \hat{\jmath}\right\} \cdot \left(dx\hat{\imath} + \hat{\jmath}\, dy\right)$$

$$= \oint_C e^x \sin y\, dx + e^x \cos y\, dy$$

$$= \int_{C_1} e^x \sin y\, dx + e^x \cos y\, dy + \int_{C_2} e^x \sin y\, dx + e^x \cos y\, dy$$

$$+ \int_{C_3} e^x \sin y\, dx + e^x \cos y\, dy + \int_{C_4} e^x \sin y\, dx + e^x \cos y\, dy$$



Along C_1, $y = 0$; x varies from 0 to 1

$\therefore \quad dy = 0$

$$\int_{C_1} e^x \sin y\, dx + e^x \cos y\, dy = \int_0^1 0 + e^x \cdot 0 = 0$$

$\therefore$ Integral along $\quad C_1 = 0$

Along C_2, $x = 1$; y varies from 0 to $\frac{\pi}{2}$

$$dx = 0$$

$$\int_{C_2} e^x \sin y dx + e^x \cos y dy = \int_0^{\pi/2} e \sin y \cdot 0 + e \cos y dy = e \sin y \Big|_0^{\pi/2} = e$$

$\therefore$ Integral along $\quad C_2 = e$

Along C_3; $y = \frac{\pi}{2}$; x varies from 1 to 0

$$dy = 0$$

$$\int_{C_3} e \sin y\, dx + e^x \cos y\, dy = \int_1^0 e^x \cdot 1\, dx + e^x \cdot 0 = e^x \Big|_1^0 = 1 - e$$

$\therefore$ Integral along $\quad C_3 = 1 - e$

Along C_4; $x = 0$; y varies from $\frac{\pi}{2}$ to 0

$$dx = 0$$

$\therefore$

$$\int_{C_4} e^x \sin y\, dx + e^x \cos y\, dy = \int_{\frac{\pi}{2}}^0 \sin y \cdot 0 + 1 \cdot \cos y\, dy = \sin y \Big|_{\frac{\pi}{2}}^0 = 0 - 1 = -1$$

$\therefore$ Integral along $\quad C_4 = -1$

$\therefore$ Circulation of $\quad \vec{F} = 0 + e + 1 - e - 1 = 0.$

Example 13. *Show that $\int x^2 y^2 ds = \frac{\pi a^5}{4}$ around the circle $x^2 + y^2 = a^2$.* **(P.T.U., 2004)**

Sol. Parametric equation of the circle is $x = a \cos \theta$, $y = a \sin \theta$.

$$\therefore \quad \frac{ds}{d\theta} = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = \sqrt{(-a \sin\theta)^2 + (a\cos\theta)^2} = a$$

$$\therefore \quad \int x^2 y^2 ds = 4\int_0^{\frac{\pi}{2}} \left(a^2 \cos^2\theta\right)\left(a^2 \sin^2\theta\right) \bullet a d\theta$$

$$= 4a^5 \int_0^{\frac{\pi}{2}} \sin^2\theta \cos^2\theta\, d\theta = 4a^5 \int_0^{\frac{\pi}{2}} \frac{1}{4}(\sin 2\theta)^2 d\theta$$

$$= a^5 \int_0^{\frac{\pi}{2}} \frac{1 - \cos 4\theta}{2} d\theta = \frac{a^5}{2}\left[\theta - \frac{\sin 4\theta}{8}\right]_0^{\frac{\pi}{2}} = \frac{a^5}{2}\frac{\pi}{2} = \frac{\pi a^5}{4}$$

TEST YOUR KNOWLEDGE

1. Evaluate $\int_C (x^2 + yz)\, dz$ where C is the curve defined by $x = t$, $y = t^2$, $z = 3t$ for t lying in the interval $1 \le t \le 2$. *(P.T.U. May 2008)*
[**Hint:** Consult S.E. 5]

2. If $\frac{d^2\vec{p}}{dt^2} = 6t\hat{\imath} - 12t^2\hat{\jmath} + 4\cos t \hat{k}$, find $\vec{p}$ given that $\frac{d\vec{p}}{dt} = -\hat{\imath} - 3\hat{k}$ and $\vec{p} = 2\hat{\imath} + \hat{\jmath}$ at $t = 0$.

3. The acceleration of a particle at any point is given by $\vec{a} = 12\cos 2t\,\hat{i} - 8\sin 2t\,\hat{j} + 16t\,\hat{k}$. If the velocity $\vec{v}$ and displacement $\vec{r}$ are zero at $t = 0$. Find $\vec{v}$ and $\vec{r}$ at any time t.

4. If $\vec{r} = t\hat{i} - t^2\hat{j} + (t-1)\hat{k}$ and $\vec{s} = 2t^2\hat{i} + 6t\,\hat{k}$, evaluate

   (i) $\int_0^2 \vec{r} \cdot \vec{s}\, dt$ (ii) $\int_0^2 (\vec{r} \times \vec{s})\, dt$

5. Find the value of $\vec{r}$ satisfying the equation $\dfrac{d^2\vec{r}}{dt^2} = 6t\,\hat{i} - 24t^2\,\hat{j} + 4\sin t\,\hat{k}$ given that $\vec{r} = 2\hat{i} + \hat{j}$ and $\dfrac{d\vec{r}}{dt} = -\hat{i} - 3\hat{k}$ at $t = 0$.

6. If $\phi = 2xyz^2$, $\vec{F} = xy\,\hat{i} - z\,\hat{j} + x^2\,\hat{k}$ and c is the curve $x = t^2$, $y = 2t$, $z = t^3$ from $t = 0$ to $t = 1$.
   Evaluate the

   (i) $\int_c \phi\, d\vec{r}$ (ii) $\int_c \vec{F} \times d\vec{r}$

7. Evaluate $\int_c \left(x^2 + xy\right)dx + \left(x^2 + y^2\right)dy$, where c is the square formed by the lines $y = \pm 1$, $x = \pm 1$.

8. If $\vec{F} = \left(5xy - 6x^2\right)\hat{i} + (2y - 4x)\hat{j}$; evaluate $\int_c \vec{F} \cdot d\vec{r}$ along the curve c in the xy-plane $y = x^3$ from the point (1, 1) to (2, 8).

9. If $\vec{F} = \left(3x^2 + 6y\right)\hat{i} - 14\,yz\,\hat{j} + 20\,xz^2\,\hat{k}$ evaluate the integral $\int_c \vec{F} \cdot d\vec{r}$ from (0, 0, 0) to (1, 1, 1) along the following paths c :
   (i) $x = t$, $y = t^2$, $z = t^3$
   (ii) the straight line joining (0, 0, 0) to (1, 1, 1)
   (iii) the straight line from (0, 0, 0) to (1, 0, 0) then to (1, 1, 0) and then to (1, 1, 1).

10. Compute the line integral $\int_c \left(y^2\, dx - x^2\, dy\right)$ along the triangle whose vertices are (1, 0); (0, 1) and (–1, 0). **(P.T.U Dec. 2012)**

11. Evaluate $\int_c (y + 3z)\,dx + (2z + x)\,dy + (3x + 2y)\,dz$ where c is the arc of helix $x = a\cos\theta$, $y = a\sin\theta$, $z = \dfrac{2a\theta}{\pi}$ between the points $(a, 0, 0)$ and $(0, a, a)$.

12. Evaluate $\int_c \left\{\left(2x^2y + y + z^2\right)\hat{i} + 2\left(1 + yz^3\right)\hat{j} + \left(2z + 3y^2z^2\right)\hat{k}\right\} \cdot d\vec{r}$ along the curve c; $y^2 + z^2 = a^2$, $x = 0$.

13. If a force $\vec{F} = 2x^2y\hat{i} + 3xy\,\hat{j}$ displaces a particle in xy-plane from (0, 0) to (1, 4) along the curve $y = 4x^2$. Find the work done.

14. (i) Find the work done by a force $y\hat{i} + x\hat{j}$ which displaces a particle from origin to a point $\left(\hat{i} + \hat{j}\right)$.

   (ii) If $\vec{F} = \dfrac{-y\hat{i} + x\hat{j}}{x^2 + y^2}$, find work done by $\vec{F}$ along the upper half of the circle passing through the points (–1, 0) and (1, 0).

15. Find the work done when a force $\vec{F} = \left(x^2 - y^2 + x\right)\hat{i} - (2xy + y)\hat{j}$ moves a particle in xy-plane from (0, 0) to (1, 1) along the parabola $y^2 = x$. Is the work done different when the path is the straight line $y = x$?

16. Compute the work done by the force $\vec{F} = (2y + 3)\,\hat{i} + xz\,\hat{j} + (yz - x)\,\hat{k}$ when it moves a particle from the point (0, 0, 0) to the point (2, 1, 1) along the curve $x = 2t^2$, $y = t$, $z = t^3$.

17. (i) Find the work done in moving a particle in the force field $\vec{F} = 3x^2\hat{i} + (2xz - y)\hat{j} + z\hat{k}$ along the curve defined by $x^2 = 4y$, $3x^3 = 8z$, from $x = 0$ to $x = 2$.

   (ii) Find the work done by the force $\vec{F} = -xy\hat{i} + y^2\hat{j} + z\hat{k}$ in moving a particle over the circular path $x^2 + y^2 = 4$, $z = 0$ from (2, 0, 0) at (0, 2, 0). **(P.T.U., Jan. 2008, May 2010)**

18. Show that $\vec{V} = \left(2xy + z^3\right)\hat{i} + x^2\,\hat{j} + 3xz^2\,\hat{k}$ is a conservative field. Find its scalar potential ϕ such that $\vec{V}$ = grad ϕ. Find the work done by the force $\vec{V}$ in moving a particle from (1, –2, 1) to (3, 1, 4).

19. Show that the integral $\int_c (2xy + 3)\,dx + \left(x^2 - 4z\right)dy - 4y\,dz$, where c is any path joining (0, 0, 0) to (1, –1, 3) does not depend on the path c and evaluate the integral.

20. Find the circulation of $\vec{F}$ round the curve c where $\vec{F} = y\hat{i} + z\hat{j} + x\hat{k}$ and c is the circle $x^2 + y^2 = 1$, $z = 0$. **(P.T.U., Dec. 2005)**

21. Evaluate (i) $\int_C x^2y\, dS$, where C is curve defined by $x = 3\cos t$, $y = 3\sin t$; $0 \le t \le \dfrac{\pi}{2}$.

   (ii) $\int_C \left(x^2 + yz\right)dS$, where C is the curve defined by $x = 4y$, $z = 3$ from $\left(2, \dfrac{1}{2}, 3\right)$ to (4, 1, 3).

   [Hint. Let $x = t$ $\therefore$ $y = \dfrac{t}{4}$, $z = 3$ and $2 \le t \le 4$; $\dfrac{dS}{dt} = \sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2}$].

Answers

1. $\dfrac{163}{4}$

2. $\left(t^3 - t + 2\right)\hat{i} + \left(1 - t^4\right)\hat{j} + (4 - 4\cos t - 3t)\hat{k}$

3. $\vec{v} = 6\sin 2t\,\hat{i} + 4(\cos 2t - 1)\hat{j} + 8t^2\,\hat{k}$; $\vec{r} = 3(1 - \cos 2t)\hat{i} + 2(\sin 2t - 2t)\hat{j} + \dfrac{8}{3}t^3\hat{k}$