9 Special Theory of Relativity

(1) INTRODUCTION

One of the most important problems in physics for over a century was the problem of absolute motion *i.e.* motion relative to a fixed frame of reference and the explanation of various experimentally observed facts in terms of absolute motion : Attempts were made to find a material body, which is at absolute rest, however, it was observed that everybody in this universe is moving with respect to some other body *e.g.* earth and other planets revolve around sun, which itself is moving towards other stars and these stars are themselves scattering randomly. Thus there is no object at absolute rest.

However, in the progress of wave theory of light, concept of a new medium namely 'Ether' was introduced, because at that time, it was supposed that light requires medium for propagation. Whole universe was supposed to be filled with ether and all bodies, heavy or light including earth were assumed to be moving freely through this hypothetical medium. Since elasticity and inertia are properties required by a medium to allow the propagation of waves, so it was assumed that density of ether is very small & it is highly elastic because no damping is observed as light travels through it. It was thought that this medium might serve the purpose of absolute frame of refrence.

But what happens to ether when material bodies move through it. Two cases arise, either ether may be at rest or it may be in motion itself. If the material bodies moving in space carried along with them the Ether (*i.e.* ether is dragged with bodies), then there would be no relative motion between ether and the bodies moving in Ether System. Thus velocity of light in ether frame and body frame would be same. As a result, the absolute motion of moving bodies could not be detected by optical phenomenon. If, on the other hand, ether remains perfectly at rest, while bodies move through it, there will be relative motion between the two, so that the velocity of light in the body system will be different from its value in ether system and as a consequence, the absolute motion of moving bodies can be detected using some optical phenomenon *e.g.* Let C is speed of light in ether and V is velocity of earth in ether in the same direction as C. Then velocity of light relative to earth (according to Galilean Transformation) will be $C - V$ and it will be $C + V$, if earth is moving in the opposite direction. If we are at rest or moving in any other direction, then the velocity of light, relative to us will lie between $C - V$ to $C + V$. Thus, if we measure the velocity of light coming to us from various directions and choose the maximum and minimum values, then half the difference between them will give us value of V

$\left(\because \frac{(C+V)-(C-V)}{2}=V\right)$, which is the absolute velocity of earth. But problem received different solutions from different scientists.

According to Hertz, the ether is dragged along with bodies moving through it. If this view is correct, then there would be no relative motion between ether and any other body and velocity of body relative to ether could not be measured or detected by optical experiments. But it follows from Hertz explanation that velocity of light must depend upon the velocity of source emitting light. But experimentally, it has been found that the velocity of light has the same value in all systems of reference, independent of whether or not emitting source moves or how it moves. Hence Hertz view could not be accepted. Hence if ether exists, then it is not dragged alongwith the bodies.

Lorentz put forth a stationary ether theory. According to him ether is absolutely at rest, even that part of it, through which other bodies are moving. This means space is sea of ether with all systems resting in it or moving relative to it.

But if ether is at rest, then one could think of bodies, which move relative to it and there must be some means of discovering such movements. Thus a number of experiments were conducted all over the world. But no experiment could prove existence of ether. The most famous among them being that conducted by Michelson and Moreley during 1881 and 1889.

(2) MICHELSON MORELEY EXPERIMENT

Principle & Objective

The main objective of this experiment was to measure the velocity of earth relative to ether frame. Earth is propgating through stationary ether with a uniform velocity. If a beam of light is sent from source to observer in the direction of motion of earth through ether, then it should take more time for journery than if sent in opposite direction. If one could measure the difference in time of journery of these two beams, then velocity of earth relative to ether frame can be found.

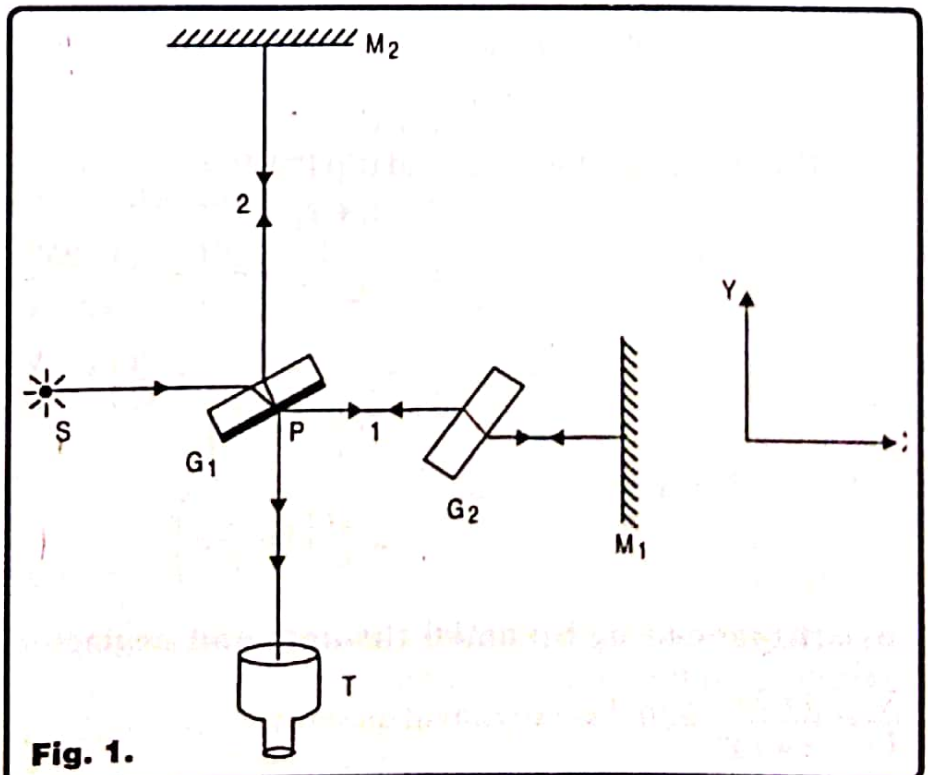

Fig. 1.

Experimental Arrangement

A schematic diagram of experimental arrangement is shown in figure (1). S is a monochromatic light source. A light beam from S is incident on a glass slab G_1 at angle of incidence 45°. The outer surface of G_1 is partially silvered (shown shaded). The beam of light is divided into two parts at point P on this face. Both beams are of equal intensity. Beam 1 (transmitted by G_1) goes straight towards another slab G_2 (which is not silvered) & then falls normally on plane mirror

M_1. It is then relfected back by M_1 & reaches again at point P after passing through slab G_2 again. At point P the beam suffers 50% reflection. The reflected part of beam enters in telescope T.

Beam 2, which is reflected at P by glass slab G_1 moves toward plane mirror M_2 and falls normally on it. After reflection from M_2, it also reaches back to point P on G_1, where it suffers 50% transmission. The transmitted part enters in the telescope T. Thus an interference pattern will be seen in the telescope due to the two beams, which enter in it.

The function of plate G_2 is to make path of beams 1 and 2 identical. If G_2 is not there then it can be clearly seen that beam 1 will travel all distance in ether in going from point P to M_1 and back to P, while beam 2 travels through Glass plate G_1, two times in going from point P to M_2 and back to P.

Both mirrors M_1 and M_2 are placed at equal distance l from Glass plate G_1. Thus with introduction of plate G_2, both geometrical and optical paths of beam 1 become identical.

Working and Theory

(*i*) Time for round trip of light beam 1 travelling along X-axis : We assume direction of motion of earth is towards X-axis. Thus w.r.t. earth, the effective velocity of light becomes $V_1 = C - V$.

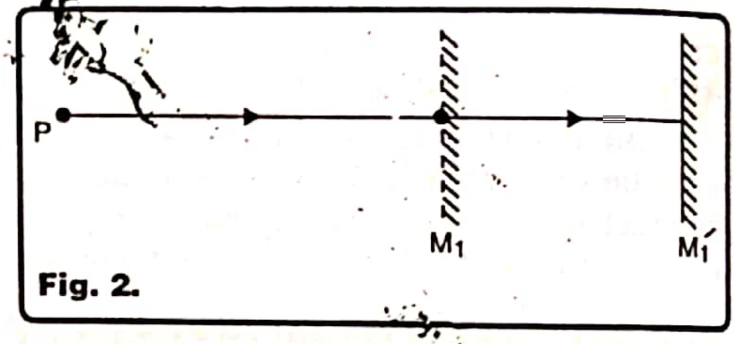

Fig. 2.

where V = Velocity of earth in Ether Frame

& C = Velocity of light in ether frame

Thus time taken by light from P to strike mirror M_1 is

$$t_1 = \frac{l}{V_1} = \frac{l}{C-V} \quad ...(1)$$

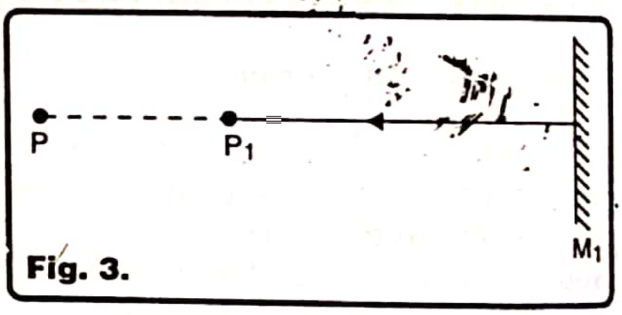

Fig. 3.

In the return journey from M_1 to point P, effective velocity of light w.r.t. Earth would become

$$V_2 = C + V$$

Thus time taken by light for its return journey form M_1 to P is

$$t_2 = \frac{l}{C+V} \quad ...(2)$$

Thus total time for the round trip PM_1P is

$$T_1 = t_1 + t_2$$

$$T_1 = \frac{l}{C-V} + \frac{l}{C+V} = \frac{2lC}{C^2 - V^2}$$

$$= \frac{2l}{C\left(1 - \frac{V^2}{C^2}\right)} = \frac{2l}{C}\left(1 - \frac{V^2}{C^2}\right)^{-1}$$

$$\approx \frac{2l}{C}\left(1 + \frac{V^2}{C^2}\right) \quad ...(3)$$

(Expanded by binomial theorem and neglected higher powers of $\frac{V}{C}$ because $\frac{V}{C} \approx \frac{3 \times 10^4}{3 \times 10^8} = 10^{-4}$ is very small quantity).

(ii) **Time for round trip of beam 2 travelling in Y-direction :** Since earth is moving with velocity $\vec{V}$ along X-axis and beam 2 goes along Y-axis from point P to mirror M_2 w.r.t. earth. It means the velocity $\vec{C}$ of light in Ether frame should be directed along PA (See fig. (4)) so that in earth frame the relative velocity $\vec{u} = \vec{C} - \vec{V}$ becomes exactly along Y-axis.

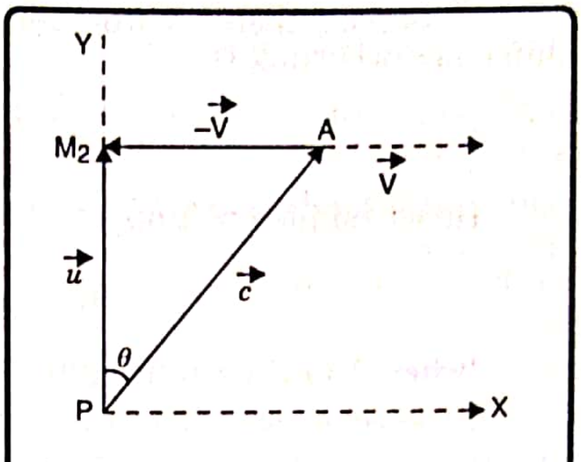

Fig. 4.

Thus relation $\vec{u} = \vec{C} - \vec{V}$ gives us

$$\vec{C} = \vec{u} + \vec{V}$$

Now $$C^2 = \vec{C}.\vec{C}$$

$$= (\vec{u} + \vec{V}).(\vec{u} + \vec{V}) = u^2 + V^2 + 2\vec{u}.\vec{V}$$

$$= u^2 + V^2 \qquad (\because \vec{u} \perp \vec{V} \text{ see fig.})$$

Thus $$u = \sqrt{C^2 - V^2} \qquad ...(4)$$

The time taken by beam 2 to move from P to M_2 is

$$t_3 = \frac{l}{u} = \frac{l}{\sqrt{C^2 - V^2}}$$

Similarly, time taken by beam 2 to move from M_2 to P back will be $t_4 = \dfrac{l}{\sqrt{C^2 - V^2}}$.

Thus total time taken by beam 2 for complete journey from P to M_2 to P is

$$T_2 = t_3 + t_4$$

$$= \frac{2l}{\sqrt{C^2 - V^2}} = \frac{2l}{C}\left(1 - \frac{V^2}{C^2}\right)^{\frac{-1}{2}}$$

$$T_2 \approx \frac{2l}{C}\left(1 + \frac{V^2}{2C^2}\right) \qquad ...(5)$$

(Expanded by Binomial Theorem and neglected higher powers of $\frac{V}{C}$).

(iii) **Expected Fringe Shift :** The time difference between two beams for their round trips is

$$T = T_1 - T_2$$

$$= \frac{2l}{C}\left(1 + \frac{V^2}{C^2}\right) - \frac{2l}{C}\left(1 + \frac{V^2}{2C^2}\right) = \frac{lV^2}{C^3} \qquad ...(6)$$

Thus optical path difference between the beams corresponding to a time difference T is

$$\Delta = CT = \frac{lV^2}{C^2} \qquad ...(7)$$

If interferometer (this experimental arrangement is called Michelson's Interferometer) is suddenly brought to rest (V is made zero), then the path difference Δ would become zero (See eq. (7)). We know that if the path difference between two beams changes by λ, there is a shift of one fringe across the cross wire of telescope in the field of view.

Thus if we change V from initial value V to final value zero, then the change in path difference occurring is

$$\delta = \frac{lV^2}{C^2} - \frac{l\,(0)^2}{C^2} = \frac{l\,V^2}{C^2}$$

Hence number of fringes that will shift across cross wire in this case will be

$$\Delta N = \frac{\delta}{\lambda} = \frac{l}{\lambda}\frac{V^2}{C^2} \qquad \text{...(8)}$$

(where λ = light wavelength)

However as every body can understand, we cannot make V equal to zero. Thus in actual practice, we have to resort to some alternative. For this the whole apparatus was placed on a block of stone floating in mercury. This apparatus was rotated by 90°. With this the new path difference between two beams was found to be $\Delta' = -\frac{lV^2}{C^2}$. The change in path difference that occurred due to rotation of apparatus was

$$\delta = \Delta - \Delta'$$

$$= \frac{lV^2}{C^2} - \left(-\frac{lV^2}{C^2}\right) = 2l\frac{V^2}{C^2} \qquad \text{...(9)}$$

Thus fringe shift expected in this case was

$$\Delta N = \frac{\delta}{\lambda} = \frac{2l}{\lambda}\frac{V^2}{C^2} \qquad \text{...(10)}$$

In the original experiment, wavelength of light used was λ = 5000 Å and distance l was 11 *m*. Also V $\approx 3 \times 10^4$ ms^{-1}.

Thus expected fringe shift is

$$\Delta N = \frac{2 \times 11}{5500 \times 10^{-10}} \times \left(\frac{3 \times 10^4}{3 \times 10^8}\right)^2 = 0.4$$

Thus a shift of 40% of fringe width was expected.

Result

The result of the experiment was quite surprising. Experimentally no fringe shift was observed, when apparatus was rotated by 90°. The experiment was repeated during various times of day, and various seasons of the year, but no shift was observed. Thus motion of earth through ether could not be experimentally detected.

Explanation of the Negative Result

Three separate explanations were given to negative result of Michelson Morely Experiment.

(i) **Ether Drag Hypothesis.** According to this hypothesis the ether is dragged along with the moving earth so that there is no relative motion between earth and ether. However, Fizeau had shown experimentally that a moving body can drag the light waves only partially. Also this hypothesis goes against the aberration of light from stars. Hence this hypothesis was rejected.

(ii) **Fitzgerald-Lorentz Contraction Hypothesis.** According to this hypothesis, the length of an object in the direction of motion decreases by a factor of $\sqrt{1 - V^2/C^2}$, whereas it remains

unchangted in a direction perpendicular to the direction of motion. If this hypothesis is used then it can be seen that T_1 & T_2 become equal and there will be no fringe shift. However, according to Rayleigh, such a contraction would produce double refraction, which was never observed.

(iii) **Light Velocity Hypothesis.** According to this hypothesis, the velocity of light is the vector sum of the natural velocity of light and the velocity of source. If this were true then negative result of the experiment can be explained. However this hypothesis was also rejected, because it goes against wave theory of light.

Einstein's Idea

According to Einstein no motion is absolute. Only the relative motion with respect to some frame of reference has physical significance. If we are isolated in universe, then there is no method, by which we can tell whether we are in motion or not. This idea was ultimately developed in the theory of relativity.

(3) INERTIAL FRAME OF REFERENCE

The absolute space is an imaginary frame work, in which bodies move, but it itself is immovable and is same everywhere.

"A frame of reference, which is at absolute rest w.r.t. the absolute space is called Absolute Inertial Frame of Reference."

Since experiments can reveal only relative motion, there is no absolute motion. Accordingly, there is no absolute space and hence no absolute inertial frame of reference.

In practice "An inertial frame of reference is that, in which law of inertia (*i.e.* Newton's 1st law of motion) holds well" *i.e.* if no external force acting on a body in an inertial frame, it will continue to be in state of rest or uniform motion relative to this inertial frame. This means that no Pseudo or Fictitious force can arise in inertial frame. We know that in any frame of reference, which is either at rest or moving with uniform velocity, no Pseudo force is produced. Hence a frame of reference moving with uniform velocity can be taken as Inertial frame of reference and all other frames, which are moving with uniform velocity w.r.t. an inertial frame of reference are also inertial frames.

A frame of reference moving under accelerated motion is called non inertial frame of reference.

Since motion of earth in space around sun is accelerated motion, thus ideally speaking, Earth is Non Inertial Frame of reference. However, acceleration is very small and for experiments which can be conducted on earth in small intervals of time, the effect of acceleration of earth can be ignored and earth behaves as inertial frame of reference to fairly good approximation.

(4) GALILEAN TRANSFORMATION

The equations which connect the position co-ordinates of a particle in two inertial frames of reference (at low speeds as compared to speed of light) are called Galilean Transformations.

Consider a frame of reference S, which is inertial. Let S′ is another frame, which is moving with uniform velocity $\vec{V}$ w.r.t. S (See fig. 5). The respective axes of S and S′ are assumed to be parallel to each other.

Suppose time is counted from the instant, when the two origins O and O′ coincide. At this time the two clocks (one in S and other in S′) are synchronized.

Since the velocity of the frame S′ is uniform, the displacement $\overrightarrow{OO'}$ of the moving origin O′ at any time t is

$$\overrightarrow{OO'} = \vec{V}\,t \qquad ...(11)$$

One of the basic assumption of Newtonian Mechanics is that motion has no effect on time. This means that if we synchronize two clocks of S and S′ at any instant, then they will agree at all later times. If a particular instant is recorded as t in S and t' in S′ then we must have

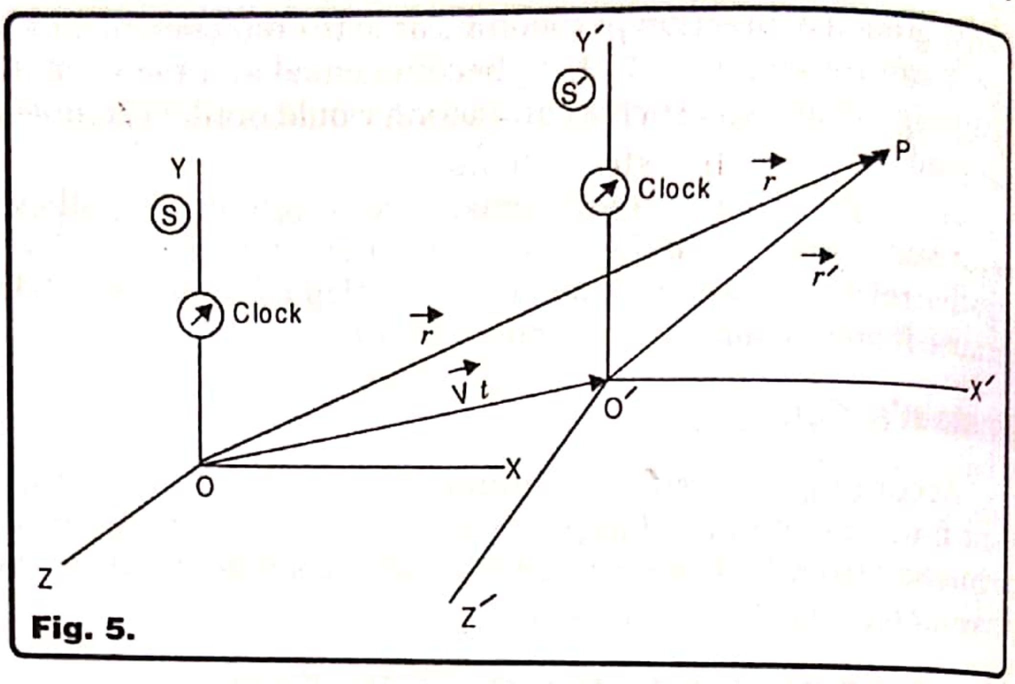

Fig. 5.

$$t = t' \qquad ...(12)$$

This equation simply says that the time measurements are same in the two frames. In this sense, time is considered as absolute quantity in classical mechanics.

Let a particle P is moving in space. Let at any instant, the position vector of P from origin O and O′ be $\vec{r}$ and $\vec{r}'$ respectively. Then

$$\vec{r} = \overrightarrow{OP} \qquad \text{and} \qquad \vec{r}' = \overrightarrow{O'P}$$

By Δ law of vectors to the triangle OO′P, we have

$$\overrightarrow{OP} = \overrightarrow{OO'} + \overrightarrow{O'P}$$

$$\Rightarrow \qquad \vec{r} = \vec{r}' + \vec{V}\,t$$

$$\text{or} \qquad \vec{r}' = \vec{r} - Vt \qquad ...(13)$$

The set of equations (12) and (13) are called Galilean Transformation.

Special Case

(*i*) **Suppose the velocity of the frame S′ with respect to frame S is along X-direction.** Then

$$\vec{V} = V\,\hat{i} \qquad ...(14)$$

Let (x, y, z) are co-ordinates of P w.r.t. O at any instant and (x', y', z') are co-ordinates of P w.r.t. O′ at same instant.

$$\left.\begin{aligned} &\text{Then} \quad \vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k} \\ &\text{and} \quad \vec{r}' = x'\,\hat{i} + y'\,\hat{j} + z'\,\hat{k} \end{aligned}\right\} \qquad ...(15)$$

Put values from (15) & (14) in (13), we get

$$x'\,\hat{i} + y'\,\hat{j} + z'\,\hat{k} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k} - V\,\hat{i}\,t$$

Equate coefficients of respective terms, we get

$$\left.\begin{aligned} x' &= x - Vt \\ y' &= y \\ z' &= z \end{aligned}\right\} \qquad ...(16)$$

Also time is absolute in Newtonian Mechanics.

$\Rightarrow$ $$t' = t \qquad ...(17)$$

Set of equations (16) and (17) are Galilean Transformation in this case.

(ii) **Velocity Transformations.**

Let $\vec{u}' = u'_x \hat{i} + u'_y \hat{j} + u'_z \hat{k}$ is velocity of P at any instant w.r.t. origin O'.

Let $\vec{u} = u_x \hat{i} + u_y \hat{j} + u_z \hat{k}$ is velocity of P w.r.t. O at the same instant. Then we have

$$\frac{d\vec{r}}{dt} = \vec{u} \quad \text{and} \quad \frac{d\vec{r}'}{dt'} = \vec{u}' \qquad ...(18)$$

Also $$u'_x = \frac{dx'}{dt'},\ u'_y = \frac{dy'}{dt'},\ u'_z = \frac{dz'}{dt'}$$

and $$u_x = \frac{dx}{dt},\ u_y = \frac{dy}{dt},\ u_z = \frac{dz}{dt} \qquad ...(19)$$

Differentiate both sides of equation (13) w.r.t. t, we get

$$\frac{d\vec{r}'}{dt} = \frac{d\vec{r}}{dt} - \vec{V} \qquad (\because \vec{V} \text{ is constant})$$

or $$\frac{d\vec{r}'}{dt'}\frac{dt'}{dt} = \frac{d\vec{r}'}{dt'} - \vec{V}$$

or $$\vec{u}'(1) = \vec{u} - \vec{V} \qquad (\because t' = t)$$

or $$\vec{u}' = \vec{u} - \vec{V} \qquad ...(20)$$

Equation (20) is called Galilean Velocity Transformation. Again if we have S' moving along X-axis only then $\vec{V} = V\hat{i}$.

So that equation (20) becomes

$$u'_x \hat{i} + u'_y \hat{j} + u'_z \hat{k} = u_x \hat{i} + u_y \hat{j} + u_z \hat{k} - V\hat{i}$$

Compare coefficients of respective terms on both sides of equation, we get

$$\left.\begin{aligned} u'_x &= u_x - V \\ u'_y &= u_y \\ u'_z &= u_z \end{aligned}\right\} \qquad ...(21)$$

Equations (21) are Galilean Velocity Transformation in this case.

(5) INVARIANCE OF AN EQUATION

Whenever a physical quantity or a law or an equation does not change its form due to change in the frame of reference, then that quantity or law or equation is called Invariant.

(6) BASIC INVARIANT QUANTITIES

There are two fundamental assumptions of Newtonian mechanics :

(i) The mass of a body is same for all the observers *i.e.* it is independent of motion of observer. Thus mass of a body is an invariant quantity in Newtonian mechanics.

(ii) The motion has no effect on time. This means that if the clocks in two frames, which are in motion agree at one instant, they will agree at all later times. This implies

$$t = t'$$

where t and t' are times of a particular event recorded in S and S' respectively. In other words, we may say that time is invariant in Newtonian mechanics.

Example 1. *Show that Newton's second law of motion is invariant under Galilean Transformations.*

Solution. Let m is mass of a particle P. Let S is an inertial frame and S′ is another inertial frame moving with uniform velocity $\vec{V}$ w.r.t. S.

Let $\vec{u}\,(u_x, u_y, u_z)$ be the velocity of particle at any instant t w.r.t. frame S.

Let $\vec{u}'\,(u'_x, u'_y, u'_z)$ be the velocity of same particle P at same instant t' as noted by the clock of S′.

Then by Galilean Transformation

$$\vec{u}' = \vec{u} - \vec{V} \qquad \text{...(22)}$$

Let $\vec{a}$ is acceleration of particle P at time t, then

$$\vec{a} = \frac{d\vec{u}}{dt} \qquad \text{...(23)}$$

Let $\vec{a}'$ = acceleration of particle at same instant w.r.t. S′

$$\therefore \quad \vec{a}' = \frac{d\vec{u}'}{dt'} = \frac{d\vec{u}'}{dt} \times \frac{dt}{dt'}$$

$$= \frac{d\vec{u}'}{dt} \qquad (\because t' = t) \text{...(24)}$$

Take derivative of equation (22) w.r.t. time, we get

$$\frac{d\vec{u}'}{dt} = \frac{d\vec{u}}{dt} - 0 \qquad (\because \vec{V} = \text{constant})$$

or $$\vec{a}' = \vec{a} \qquad \text{...(25) (using (23) and (24))}$$

In Newtonian mechanics mass of a body is assumed to be invariant *i.e.* $m' = m$ ..(26)

Multiply equations (25) and (26) we get

$$m'\vec{a}' = m\vec{a} \qquad \text{...(27)}$$

Thus we see that Newton's 2nd law of motion retains its form in both of the frames S and S′ under Galilean Transformations. Hence it is Galilean invariant.

Example 2. *According to Maxwell, any electromagnetic wave propagates in space with velocity 'C' according to the relation*

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} - \frac{1}{C^2}\frac{\partial^2\psi}{\partial t^2} = 0 \qquad \text{...(28)}$$

where ψ is wave function and $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial t}$ represent the partial derivatives w.r.t. x, y, z, t respectively. This equation is called Maxwell's wave equation. Show that this equation is not invariant under Galilean Transformation.

Solution. Let S and S′ are two inertial frames of reference such that S′ is moving with uniform velocity $\vec{V} = V\hat{i}$ along X-axis of frame S. The respective axes of S and S′ remain parallel to each other. The Galilean transformations thus can be written as

$$\left.\begin{aligned} x' &= x - Vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned}\right\} \qquad \text{...(29)}$$

The wave function ψ depends on variables x, y, z, t in frame S and on variables x', y', z', t' in frame S'. Let us transform equation (28) from frame S to frame S'. By chain formula of calculus, the partial derivative is given as

$$\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial \psi}{\partial y'}\frac{\partial y'}{\partial x} + \frac{\partial \psi}{\partial z'}\frac{\partial z'}{\partial x} + \frac{\partial \psi}{\partial t'}\frac{\partial t'}{\partial x} \quad ...(30)$$

From (29), we can observe that :

$$\left.\frac{\partial x'}{\partial x} = 1, \frac{\partial y'}{\partial x} = 0, \frac{\partial z'}{\partial x} = 0, \frac{\partial t'}{\partial x} = 0\right\} \quad ...(31)$$

Put these values in (30), we get

$$\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial x'} \quad ...(32)$$

Again apply chain formula of calculus to equation (32), we get

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x}\right)$$

$$= \frac{\partial}{\partial x'}\left(\frac{\partial \psi}{\partial x}\right)\frac{\partial x'}{\partial x} + \frac{\partial}{\partial y'}\left(\frac{\partial \psi}{\partial x}\right)\frac{\partial y'}{\partial x} + \frac{\partial}{\partial z'}\left(\frac{\partial \psi}{\partial x}\right)\frac{\partial z'}{\partial x} + \frac{\partial}{\partial t'}\left(\frac{\partial \psi}{\partial x}\right)\frac{\partial t'}{\partial x}$$

$$= \frac{\partial}{\partial x'}\left(\frac{\partial \psi}{\partial x}\right)(1) + 0 + 0 + 0 \qquad \text{(using 31)}$$

$$= \frac{\partial}{\partial x'}\left(\frac{\partial \psi}{\partial x'}\right) \qquad \text{(using 32)}$$

$$= \frac{\partial^2 \psi}{\partial x'^2} \quad ...(33)$$

Similarly one may show that

$$\frac{\partial^2 \psi}{\partial y^2} = \frac{\partial^2 \psi}{\partial y'^2} \quad ...(34)$$

and

$$\frac{\partial^2 \psi}{\partial z^2} = \frac{\partial^2 \psi}{\partial z'^2} \quad ...(35)$$

But

$$\frac{\partial \psi}{\partial t} = \frac{\partial \psi}{\partial x'}\frac{\partial x'}{\partial t} + \frac{\partial \psi}{\partial y'}\frac{\partial y'}{\partial t} + \frac{\partial \psi}{\partial z'}\frac{\partial z'}{\partial t} + \frac{\partial \psi}{\partial t'}\frac{\partial t'}{\partial t} \quad ...(36)$$

From equations (29) we can find that

$$\frac{\partial x'}{\partial t} = -\mathrm{V}, \frac{\partial y'}{\partial t} = 0, \frac{\partial z'}{\partial t} = 0, \frac{\partial t'}{\partial t} = 1 \quad ...(37)$$

Put these values in (36), we get

$$\frac{\partial \psi}{\partial t} = -\mathrm{V}\frac{\partial \psi}{\partial x'} + 0 + 0 + \frac{\partial \psi}{\partial t'}$$

$$\therefore \quad \frac{\partial \psi}{\partial t} = \frac{\partial \psi}{\partial t'} - V\frac{\partial \psi}{\partial x'} \quad ...(38)$$

To find 2nd derivative w.r.t. t, we resort to short cut method. If we ignore ψ from both sides of equation (38), we get

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t'} - \mathrm{V}\frac{\partial}{\partial x'} \quad ...(39)$$

Now 2nd derivative w.r.t. t is given by

$$\frac{\partial^2 \psi}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial \psi}{\partial t}\right)$$

$$= \left(\frac{\partial}{\partial t'} - V\frac{\partial}{\partial x'}\right)\left(\frac{\partial \psi}{\partial t'} - V\frac{\partial \psi}{\partial x'}\right) \quad \text{(using (38) and (39))}$$

$$= \frac{\partial^2 \psi}{\partial t'^2} - V\frac{\partial^2 \psi}{\partial t'\partial x'} - V\frac{\partial^2 \psi}{\partial x'\partial t'} + V^2\frac{\partial^2 \psi}{\partial x'^2} \quad (\because V = \text{constant})$$

$$\Rightarrow \quad \frac{\partial^2 \psi}{\partial t} = \frac{\partial^2 \psi}{\partial t'^2} - 2V\frac{\partial^2 \psi}{\partial x'\partial t'} + V^2\frac{\partial^2 \psi}{\partial x'^2} \quad \ldots(40) \left(\because \frac{\partial^2 \psi}{\partial x'\partial t'} = \frac{\partial^2 \psi}{\partial t'\partial x'}\right)$$

Put values from (33), (34), (35) and (40) in (28), we get

$$\frac{\partial^2 \psi}{\partial x'^2} + \frac{\partial^2 \psi}{\partial y'^2} + \frac{\partial^2 \psi}{\partial z'^2} - \frac{1}{C^2}\frac{\partial^2 \psi}{\partial t'^2} + \frac{2V}{C^2}\frac{\partial^2 \psi}{\partial x'\partial t'} - \frac{V^2}{C^2}\frac{\partial^2 \psi}{\partial x'^2} = 0 \quad \ldots(41)$$

Conclusion

It is evident that equation (41) does not resemble in form with equation (28) [equation (41) contains last two terms extra as compared to equation (28)]. Hence Maxwell's electromagnetic wave equation is not invariant under Galilean Transformation.

Discussion

(*i*) Non invariance of Maxwell's equation under Galilean transformation indicates that either Maxwell's equations are wrong or Galilean Transformations are wrong. However, the propagation of e.m. waves through space was completely explained using Maxwell's equation. Further a no. of other laws related with e.m. waves [Faraday's laws of e.m. induction, Gauss Law, Radiation by accelerated charged particle etc.] can be derived using Maxwell's equations. This suggests that Galilean Transformations are not correct.

(*ii*) If $V << C$ then $\frac{V}{C^2}$ and $\frac{V^2}{C^2}$ can be neglected. Hence two extra terms in equation (41) will disappear. Then it will become identical to equation (28). Thus we may conclude that the Galilean Transformations are valid only at low speeds.

(7) NEED FOR A NEW SET OF TRANSFORMATION EQUATIONS AND TO GUESS THEIR BASIC NATURE

It is very necessary for us that all laws of physics must be invariant in all inertial frames of reference, otherwise we have to remember each different form of same law depending upon the frame of reference and hence solve the problem accordingly. Since Maxwell's equation is not invariant under Galilean Transformation, thus there is need to replace these transformation with a new one.

Galilean transformation was linear (See eq. (16)). The question arises whether new transformation is linear or not. To check this we assume the new transformation connecting x and x' is non linear. Then graph between x and x' can be of the form shown in figure (6).

It follows that two rods AB and CD of equal lengths in frame S will appear as A'B' and C'D' in the frame S' after transformation *i.e.* they will no longer appear equal in length.

If on the other hand new transformation is also linear, then graph between x and x' will be a straight line as shown in figure (7). In this case equal rods AB and CD in frame S will appear as A'B' and C'D' in S' but still these will be equal in length *i.e.* if AB = CD then A'B' = C'D'.

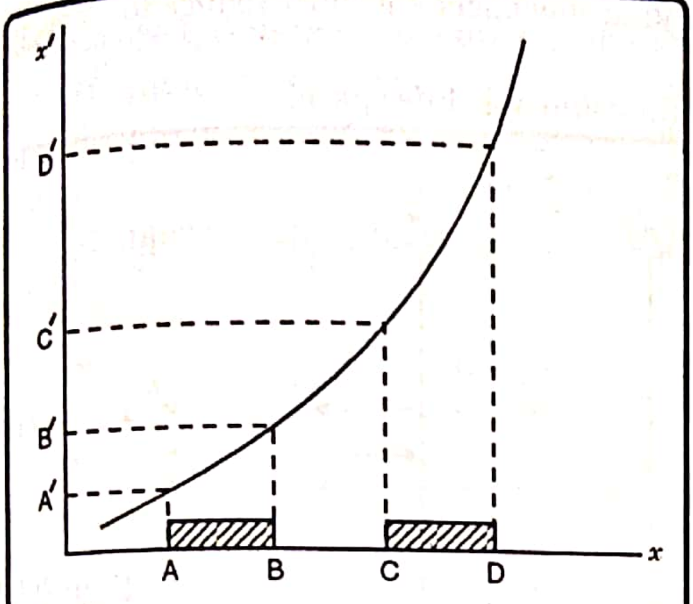

Fig. 6.

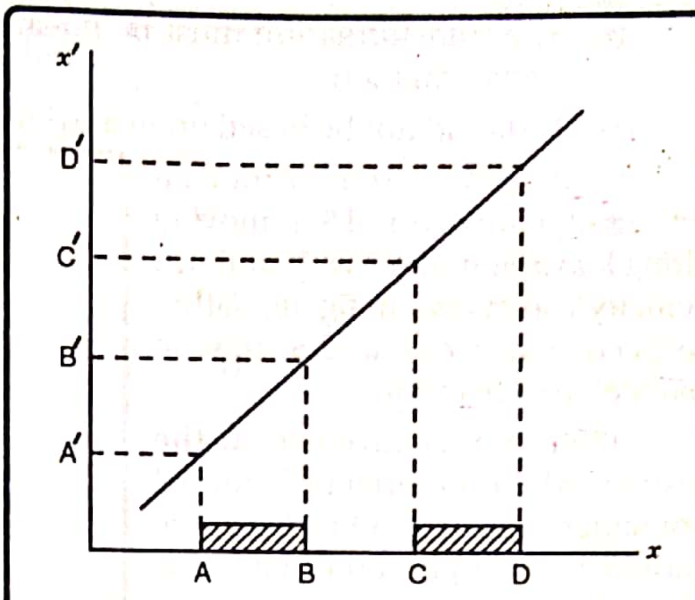

Fig. 7.

The second case is acceptable to us because it is in accordance with principle of Homogeneity of space, which says that if two objects at rest have equal lengths in one frame, these will have equal lengths at rest in all other frames.

Also we have discussed earlier that Galilean transformation hold good at low speeds. Thus we can now guess nature of new transformation, which should replace Galilean transformation. The conditions on new transformation to be found are :

(*i*) These must be linear just like Galilean transformation.

(*ii*) These must reduce to Galilean transformation under the limit of low velocities.

(8) EINSTEIN'S SPECIAL THEORY OF RELATIVITY

The failure of Michelson Moreley experiment to detect the absolute motion of earth relative to ether and consequent invariance of speed of light lead Albert Einstein to put forth his special theory of relativity in 1905. The two postulates of this theory are :

(*i*) "All laws of physics have the same form in all inertial frames of reference moving with a constant velocity relative to one another." This postulate is called principle of relativity.

This postulate expresses the absence of a universal frame of reference. If laws of Physics were different for observers in different frames in relative motion, it could be determined from these differences, which objects are stationary in space and which are moving. But there is no universal frame of reference, this distinction between objects cannot be made.

(*ii*) "The speed of light in free space is the same in all frames of reference." This postulate is called principle of constancy of speed of light. This postulate follows directly from the result of Michelson-Moreley experiment.

It should be noted that the second postulate is more important and more basic to the theory than the first, for the theory departs from the classical ideas not through the first postulate but through the second—the constancy of speed of light.

(9) LORENTZ TRANSFORMATION EQUATIONS

The Lorentz transformation equations suitable for special theory of relativity must fulfill the following requirements :

(*i*) The speed of light C must have same value in every frame of reference.

(*ii*) The transformation must be linear and at low speeds it must reduce to Galilean transformation.

(*iii*) It should not be based upon absolute time and absolute space.

Let S and S′ be two inertial frames of reference and S′ is moving along X axis of frame S with uniform velocity V as shown in fig. (8). Other respective axes of S and S′ remain parallel to each other.

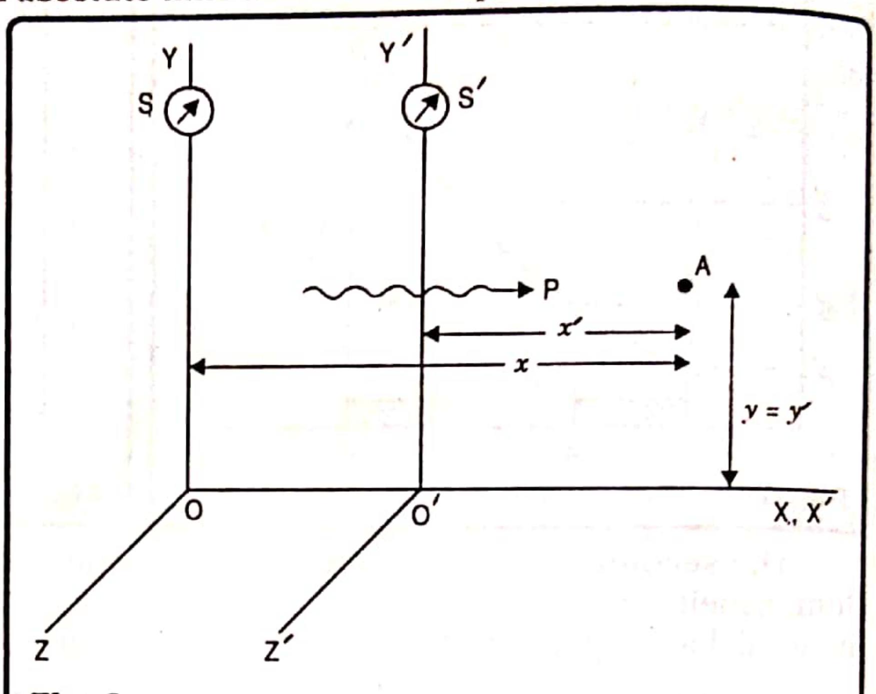

Fig. 8.

Time is measured from the instant, when the origins O and O′ coincided with each other. Let at the same instant a photon of light P is sent along X-axis in space. This photon travels with constant speed C w.r.t. both O and O′ (2nd postulate of Special Theory of Relativity).

Consider any point A in space. Let the co-ordinates of A at any instant are (x, y, z) *w.r.t.* origin o and (x', y', z') w.r.t origin o′. Suppose readings of clocks of S and S′ at that instant are t and t' respectively.

Since frame of refrence S′ is travelling only along X-axis with uniform velocity V, so Y co-ordinate of point A will be same in S and S′. Similarly Z co-dordinate of point A will be same in S and S′.

Thus

and
$$\left.\begin{aligned} y' &= y \\ z' &= z \end{aligned}\right\} \qquad ...(42)$$

Also photon is travelling along X-axis with speed c. When photon will reach the point A, then values of x and x' must be given by ct and ct' respectively *i.e.*

When
$$x = ct, \text{ then } x' = ct' \qquad ...(43)$$

Now, we want to relate x and x' (just like equations (42)). We know that relation between x and x' must be linear and it should reduce to Galilean transformations at low speed of S′. Thus we can anticipate that relation between x and x' should be of the form

$$x' = K(x - Vt) \qquad ...(44)$$

where K is some parameter to be found.

According to first postulate, the equations of physics must have same form in both S and S′. Therefore corresponding equation of x in terms of x' and t' will be similar to (44) except that V should be replaced by – V. Thus

$$x = K(x' + Vt') \qquad ...(45)$$

Substitute value of x' from (44), we get

$$x = K[K(x - Vt) + Vt']$$

or
$$t' = \left(\frac{1 - K^2}{KV}\right)x + Kt \qquad ...(46)$$

equation (46) shows that $t' \neq t$ i.e. time of two clocks will not agree though both were initially synchronized. Thus time no more remains absolute.

Substitute condition (43) in equation (44) and (45), we get

$$Ct' = Kt\,(C - V) \quad \text{...(47)}$$

and

$$Ct = Kt'\,(C + V) \quad \text{...(48)}$$

Multiply (47) and (48), we get

$$C^2\, tt' = K^2\, tt'\,(C^2 - V^2)$$

$\Rightarrow$

$$K^2 = \frac{C^2}{C^2 - V^2}$$

or

$$K = \frac{1}{\sqrt{1 - \dfrac{V^2}{C^2}}} \quad \text{...(49)}$$

Put value of K in (44) and (46) and arrange the results along with equations (42), we get

$$\left.\begin{aligned} x' &= \frac{x - Vt}{\sqrt{1 - \dfrac{V^2}{C^2}}} \\ y' &= y \\ z' &= z \\ t' &= \frac{t - \dfrac{Vx}{C^2}}{\sqrt{1 - \dfrac{V^2}{C^2}}} \end{aligned}\right\} \quad \text{...(50)}$$

Set of equations (50) is Lorentz Transformation Equations. It is clear that if S′ is moving at low speed then V << C.

$\therefore$

$$\frac{V}{C^2} \approx 0 \text{ and } \frac{V^2}{C^2} \approx 0 \text{ and } K \approx 1$$

$\therefore$ (50) $\Rightarrow$

$$x' = x - Vt$$

$$y' = y$$

$$z' = z$$

$$t' = t$$

which is old Galilean transformation.

Inverse Lorentz Transformation can be obtained directly from (50) by interchanging primed and unprimed co-ordinates and changing V to – V. Thus, inverse Lorentz transformations are

$$\left.\begin{aligned} x &= \frac{x' + Vt'}{\sqrt{1 - V^2/C^2}} \\ y &= y' \\ z &= z' \\ t &= \frac{t' + Vx'/C^2}{\sqrt{1 - V^2/C^2}} \end{aligned}\right\} \quad \text{...(51)}$$

(10) ADDITION OF VELOCITIES OR LORENTZ VELOCITY TRANSFORMATION

Let S and S′ are two inertial frames of reference with their respective axis parallel to each other. S′ is moving with uniform velocity V along X-axis of S. Time is measured from the instant, when origins O and O′ coincided with each other.

A particle P is moving in space. Let at any instant when clocks of S and S′ show time t and t' respectively, the position co-ordinates of P are (x, y, z) w.r.t. O and (x', y', z') w.r.t. O′. Let velocity of P at this instant w.r.t. frame S is $\vec{u}$ where

$$\vec{u} = u_x \hat{i} + u_y \hat{j} + u_z \hat{k} \quad ...(52)$$

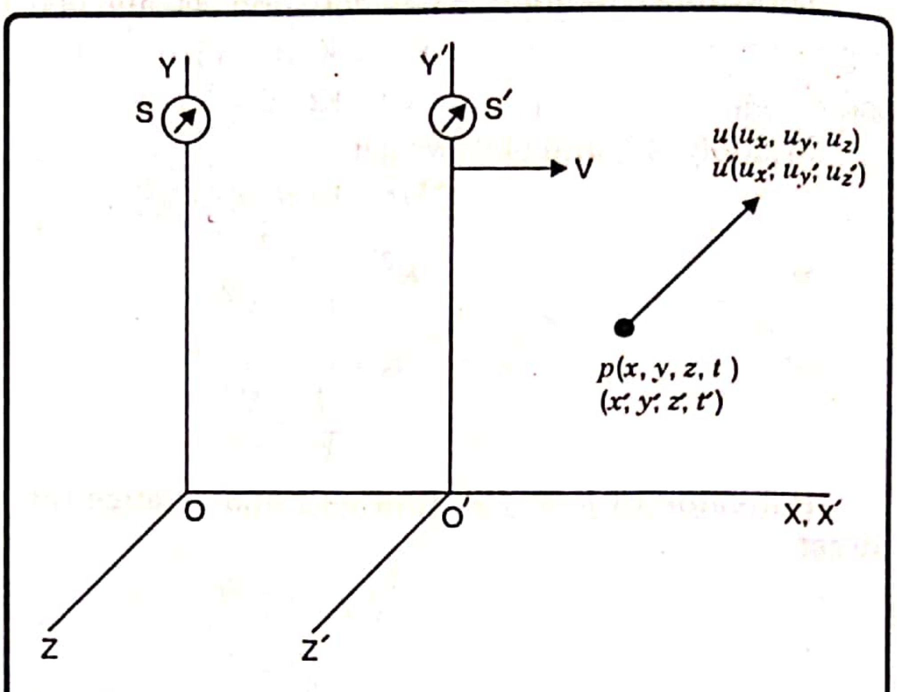

Fig. 9.

and w.r.t. S′, its velocity was found at same instant to be $\vec{u'}$ where

$$\vec{u'} = u'_x \hat{i} + u'_y \hat{j} + u'_z \hat{k} \quad ...(53)$$

The components of velocity are defined in S and S′ as

$$\left.\begin{aligned} u'_x = \frac{dx'}{dt'},\ u'_y = \frac{dy'}{dt'},\ u'_z = \frac{dz'}{dt'} \\ \text{and} \quad u_x = \frac{dx}{dt},\ u_y = \frac{dy}{dt},\ u_z = \frac{dz}{dt} \end{aligned}\right\} \quad ...(54)$$

According to Lorentz transformations, we have

$$\left.\begin{aligned} x' &= \frac{x - Vt}{\sqrt{1 - V^2/C^2}} \\ y' &= y \\ z' &= z \\ t' &= \frac{t - Vx/C^2}{\sqrt{1 - V^2/C^2}} \end{aligned}\right\} \quad ...(55)$$

Differentiate these equations (V, C are constants), we get

$$dx' = \frac{dx - Vdt}{\sqrt{1 - V^2/C^2}} \quad ...(56)$$

$$dy' = dy \quad ...(57)$$

$$dz' = dz \quad ...(58)$$

$$dt' = \frac{dt - (V/C^2)\,dx}{\sqrt{1 - \frac{V^2}{C^2}}} \quad ...(59)$$

Divide (56) by (59) we get

$$\frac{dx'}{dt'}=\frac{\frac{dx}{dt}-V}{1-\frac{V}{C^2}\frac{dx}{dt}} \quad \text{or} \quad u'_x=\frac{u_x-V}{1-\frac{V}{C^2}u_x} \qquad ...(60)$$

Divide (57) by (59) we get

$$u'_y=\frac{u_y\sqrt{1-V^2/C^2}}{1-\frac{V}{C^2}u_x} \qquad ...(61) \text{ (using (54))}$$

Divide (58) by (59) and use (54), we get

$$u'_z=\frac{u_z\sqrt{1-V^2/C^2}}{1-\frac{V}{C^2}u_x} \qquad ...(62)$$

The set of equations (60) – (62) is called Lorentz Velocity Transformation.

Inverse Lorentz velocity transformation can be written directly from (60) – (62) by interchanging primed and unprimed variables and changing V to – V.

Thus these are given as

$$\left.\begin{aligned} u_x &= \frac{u'_x+V}{1+\frac{V}{C^2}u'_x} \\ u_y &= \frac{u'_y\sqrt{1-V^2/C^2}}{1+\frac{V}{C^2}u'_x} \\ u_z &= \frac{u'_z\sqrt{1-V^2/C^2}}{1+\frac{V}{C^2}u'_x} \end{aligned}\right\} \qquad ...(63)$$

(11) VARIATION OF MASS WITH VELOCITY

In the light of relativistic ideas of time and space and the way in which velocities add together, the concept of mass as an invariable property of body remaining constant under all circumstances has to be abandoned. In fact, mass is a function of velocity of body.

Fig. 10 shows inertial frames S and S′ such that S′ is moving with velocity V along X-axis.

Consider two exactly identical particles A and B having equal mass m in frame S′, which are approaching each other with velocity u' in frame S′.

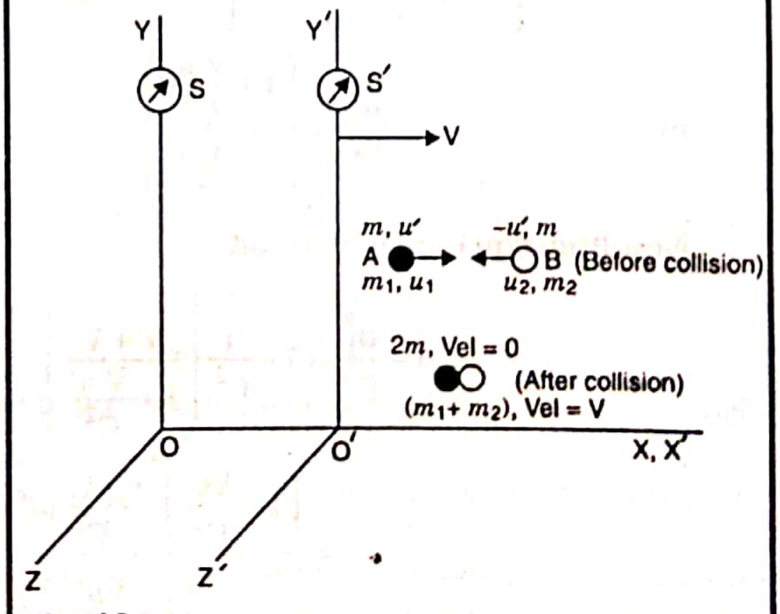

Fig. 10.

Let in frame S the bodies appear to have velocities u_1 and u_2 before collision. Since their velocities are different in S, so their masses will also be different in S because mass is function of velocity. Let their masses in frame S are m_1, m_2 respectively.

These bodies collide each other perfectly inelastically and stick together. Thus in frame S' the combined mass will be $2m$ and it will be at rest in S'. In frame S the two bodies will have combined mass $m_1 + m_2$ and it will move with the velocity V *i.e.* velocity of S'.

According to inverse Lorentz velocity transformation

$$u_x = \frac{u'_x + V}{1 + \frac{V}{C^2} u'_x} \quad \text{...(64)}$$

For body A, we put $u_x = u_1$, $u'_x = u'$, thus (64) gives

$$u_1 = \frac{u' + V}{1 + \frac{Vu'}{C^2}} \quad \text{...(65)}$$

For body B, we put $u_x = u_2$, $u'_x = -u'$. Thus (64) gives

$$u_2 = \frac{-u' + V}{1 - \frac{Vu'}{C^2}} \quad \text{...(66)}$$

Apply law of conservation of linear momentum in frame S, we get

$$m_1 u_1 + m_2 u_2 = (m_1 + m_2) V$$

or
$$m_1 \left(\frac{u' + V}{1 + \frac{V u'}{C^2}} \right) + m_2 \left(\frac{-u' + V}{1 - \frac{V u'}{C^2}} \right) = (m_1 + m_2) V \quad \text{(using (65) and (66))}$$

or
$$m_1 \left[\frac{u' + V}{1 + \frac{Vu'}{C^2}} - V \right] = m_2 \left[V - \left(\frac{-u' + V}{1 - \frac{Vu'}{C^2}} \right) \right]$$

or
$$\frac{m_1}{m_2} = \left(\frac{1 + \frac{V u'}{C^2}}{1 - \frac{V u'}{C^2}} \right) \quad \text{...(67)}$$

Now from equation (65) we get

$$1 - \frac{u_1^2}{C^2} = 1 - \frac{1}{C^2} \left(\frac{u' + V}{1 + \frac{V u'}{C^2}} \right)^2$$

$$= \frac{\left(1 + \frac{Vu'}{C^2}\right)^2 - \frac{1}{C^2}(u' + V)^2}{\left(1 + \frac{Vu'}{C^2}\right)^2} = \frac{\left(1 - \frac{u'^2}{C^2}\right)\left(1 - \frac{V^2}{C^2}\right)}{\left(1 + \frac{Vu'}{C^2}\right)^2} \quad \text{...(68)}$$

Similarly from (66), we get

$$1-\frac{u_2^2}{C^2}=\frac{\left(1-\frac{u'^2}{C^2}\right)\left(1-\frac{V^2}{C^2}\right)}{\left(1-\frac{Vu'}{C^2}\right)^2} \quad \text{...(69)}$$

Divide (69) by (68), we get

$$\frac{1-\frac{u_2^2}{C^2}}{1-\frac{u_1^2}{C^2}}=\left(\frac{1+\frac{Vu'}{C^2}}{1-\frac{Vu'}{C^2}}\right)^2=\left(\frac{m_1}{m_2}\right)^2 \quad \text{(using 67)}$$

or

$$\frac{m_1}{m_2}=\frac{\sqrt{1-\frac{u_2^2}{C^2}}}{\sqrt{1-\frac{u_1^2}{C^2}}} \quad \text{...(70)}$$

The relation is valid for all values of u_1 and u_2. If we choose u' in such a way that particle B has zero velocity in frame S before collision (This is possible if we choose $u'=V$), then $u_2=0$ and $m_2=m_0$ = rest mass of object B. Then (70) reduces to

$$\frac{m_1}{m_0}=\frac{1}{\sqrt{1-\frac{u_1^2}{C^2}}}$$

or

$$m_1=\frac{m_0}{\sqrt{1-\frac{u_1^2}{C^2}}} \quad \text{...(71)}$$

Since two particles are exactly identical, the rest mass of the other body is also m_0. Hence the above equation can be considered as applying to one and the same particle. Then replacing m_1 by m and u_1 by u, equation (71) becomes

$$m=\frac{m_0}{\sqrt{1-\frac{u^2}{C^2}}} \quad \text{...(72)}$$

This is mass variation formula. Here m_0 is mass of particle at rest and m is mass when it moves with velocity u.

Conclusions

(i) As the velocity of particle relative to an observer increases, the mass of the particle also increases.

(ii) as $u \to C$, $m \to \infty$. This means that no material particle can travel with velocity equal to greater than speed of light.

(iii) When $u << C$, then u^2/C^2 can be neglected then $m \approx m_0$. This means that at ordinary velocities, difference between m and m_0 is too small to be detectable.

Experimental Verification

In 1909, Bucherer determined the ratio of e/m as a function of velocity for fast electrons emitted from radioactive nuclei and showed that the mass of electron varies with its velocity according to above relation.

(12) LIMITING VELOCITY OF A MATERIAL PARTICLE

When a particle is accelerated its mass goes on increasing and approaches ∞ as its velocity approaches velocity of light. Thus ∞ force is required for a particle to move with speed of light in vacuum. But no external agency can provide ∞ force. Hence no material particle (whose rest mass is not zero is called material particle) can ever reach speed of light in vacuum.

However this fact is not true in a medium. Light travels slowly in different media compared to its speed in vacuum (*e.g.* in glass speed of light is $2 \times 10^8\ ms^{-1}$). But very energetic atomic particles are however capable of moving faster than speed of light in medium (but never faster than speed of light in vacuum). This phenomenon gives rise to light radiation known as 'Cerenkov Radiation'. Hence the motion of high energy electron beam across the screen faster than light does not violate the theory of relativity.

(13) EINSTEIN'S MASS ENERGY EQUIVALENCE RELATION

The variation of mass with velocity has modified our ideas. Consider an object of rest mass m_0. The object is totally free to move in any direction. Thus if force is applied to the object, it will immediately start moving in the direction of force and because of increase in velocity its mass would also increase. Let at any time when object has velocity V, then its mass has become m. Thus

$$m = \frac{m_0}{\sqrt{1 - V^2/C^2}} \qquad \text{...(73)}$$

Let force is applied along X-axis, so motion of object is also along X-axis. Small amount of work done by the force to displace object through small distance dx is given by

$$\begin{aligned} dW &= F dx \\ &= \frac{d}{dt}(mV)\, dx && \left(\because F = \frac{dp}{dt} = \frac{d}{dt}(mV)\right) \\ &= \left(V\frac{dm}{dt} + m\frac{dV}{dt}\right) dx && (\because m \text{ changes with } V) \\ &= \left(V\frac{dx}{dt}dm + m\frac{dx}{dt}dV\right) \\ &= V^2 dm + mVdV && \text{...(74)} \left(\because V = \frac{dx}{dt}\right) \end{aligned}$$

Square equation (73), we get

$$m^2\left(1 - \frac{V^2}{C^2}\right) = m_0^2$$

or $$m^2C^2 - m^2V^2 = m_0^2C^2$$

Differentiate, we get

$$C^2(2m\,dm) - m^2(2V\,dV) - V^2(2m\,dm) = 0$$

or $$V^2 dm + mV\,dV = C^2 dm \quad ...(75)$$

From (75) and (74), we get

$$dW = C^2 dm$$

Total amount of work done by the applied force in order to change its velocity from 0 to V (or equivalently to change its mass from m_0 to m) is given as

$$W = C^2 \int_{m_0}^{m} dm$$

or $$W = C^2 (m - m_0) \quad ...(76)$$

This work is done for giving motion to the object. Hence it must be equal to kinetic energy (T) acquired by body.

Thus $$KE = T = C^2 (m - m_0) \quad ...(77)$$

We know that potential energy of a system (object) is defined as energy because of its shape, size or position. The initial mass of object was m_0 (before applying force). But this must also be possessing energy because of its shape and size. This energy is obviously potential energy of object when it rests and we call it as Rest Mass Energy of object.

By definition of potential energy, the rest mass energy of object must be equal to work done to bring all particles (which make the object of mass m_0) from ∞ to a place. Thus from (76) expression for rest mass energy is

Rest Mass Energy = work done to form an object of mass m_0

$$= C^2 \int_0^{m_0} dm$$

$\Rightarrow$ Rest Mass Energy $= m_0C^2 \quad ...(78)$

The total energy of object is given as

$$E = \text{Kinetic Energy} + \text{Rest Mass Energy}$$
$$= (m - m_0)C^2 + m_0C^2$$

$\Rightarrow$ $$E = mC^2 \quad ...(79)$$

Equation (79) is famous mass energy relation. It is valid at all speeds.

Discussion

(*i*) This relation shows that with a mass m, an energy of mC^2 is associated or conversely with energy E, a mass $m = E/C^2$ is associated.

(*ii*) Since mass and energy are inter convertible. Thus in any process (physical or chemical), mass and energy are not conserved separately, but it is the mass plus energy of an isolated system, that is conserved during any process.

Verification

Equation (79) has been verified in a number of phenomenon.

(*i*) **Compton Effect.** In this effect, X-ray scattering was considered as elastic collision between a photon and an electron, after which the scattered photon moves in a new direction and electron recoils with a velocity. The wavelength of scattered photon was found to be more than that of incident photon and the change in wavelength of scattered photon was independent of nature of target. These results were successfully explained when law of conservation of energy & momentum were applied according to mass energy equivalence.

(*ii*) **Fine Structure of Spectral Lines.** Sommerfield explained the fine structure of spectral lines on the basis of relativistic variation of mass. This agreement of his theory with experiment provides another verification of mass energy relation.

(*iii*) **Nuclear Phenomena.** The explanation of mass defect and release of tremendous amount of energy in nuclear fission, which is entirely based upon mass energy relation, gives a strong support to the relation.

(14) NON RELATIVISTIC KINETIC ENERGY OF A BODY

The relativistic kinetic energy of an object is given as

$$T = (m - m_0)\,C^2 \qquad \text{...(80)}$$

But

$$m = \frac{m_0}{\sqrt{1 - V^2/C^2}}$$

Thus equation (80) becomes

$$T = m_0C^2\left[\frac{1}{\sqrt{1 - V^2/C^2}} - 1\right]$$

If particle is moving non-relativistically then V << C. Hence $\frac{V^4}{C^4}$ and higher powers can be neglected after expanding above equation by Binomial Theorem. Thus at low speeds

$$T = m_0C^2\left[\left(1 - \frac{V^2}{C^2}\right)^{\frac{-1}{2}} - 1\right]$$

$$\approx m_0C^2\left[1 + \frac{V^2}{2C^2} - 1\right] = \frac{1}{2}m_0V^2 \qquad \text{...(81)}$$

This relation is classical expression for kinetic energy. The classical kinetic energy is less than rest energy m_0C^2. This gives us a rule to make a distinction between relativistic and non relativistic case. "If kinetic energy of a body in less than its rest mass energy, then non relativistic formulae are used to solve the problem. For Kinetic energy more than rest mass energy, use relativistic formulae to solve the problem."

(15) RELATIVISTIC ENERGY MOMENTUM RELATION

The total energy of an object is given as

$$E = mC^2 = \frac{m_0C^2}{\sqrt{1 - V^2/C^2}} \qquad \text{...(82)}$$

The relativistic momentum of a particle is given by

$$p = mV = \frac{m_0V}{\sqrt{1 - V^2/C^2}} \qquad \text{...(83)}$$

Using (82), (83) we find value of $E^2 - p^2C^2$. We have

$$E^2 - p^2C^2 = \frac{m_0^2C^4}{\left(1 - \frac{V^2}{C^2}\right)} - \frac{m_0^2V^2C^2}{\left(1 - \frac{V^2}{C^2}\right)} = \frac{m_0^2C^2}{\left(1 - \frac{V^2}{C^2}\right)}[C^2 - V^2]$$

$$= m_0^2C^4$$

or
$$E^2 = m_0^2C^4 + p^2C^2 \text{ or } E = \sqrt{p^2C^2 + m_0^2C^4} \qquad ...(84)$$

Equation (84) is relativistic relation between total energy and momentum of particle.

Discussion

(*i*) The kinetic energy of the particle in non relativistic case is given as

$$T = E - m_0C^2 = \sqrt{m_0^2C^4 + p^2C^2} - m_0C^2$$
$$= m_0C^2\left[\left(1 + \frac{p^2}{m_0^2C^2}\right)^{\frac{1}{2}} - 1\right] \approx m_0C^2\left[1 + \frac{1}{2}\frac{p^2}{m_0^2C^2} - 1\right]$$
$$= \frac{p^2}{2m_0}$$

($\because$ particle is moving non-relativistically. So $\frac{p^2}{m_0^2C^2}$ is small)

This is classical/non-relativistic relation between kinetic energy and momentum of particle.

(*ii*) **Concept of Massless Particles.** A massless particle is one, which has zero rest mass. Classically, such a particle cannot exist. In relativistic mechanics, however, a particle having zero rest mass can exist and show particle like properties as energy and momentum. Let us now find conditions under which such a particle can exist.

For a massless particle $m_0 = 0$. Thus relation (84) gives us

$$E = pC \qquad ...(85)$$

This relation gives the condition that total energy of a massless particle is always equal to product of its linear momentum and speed of light. Further if $p = 0$ then $E = 0$ *i.e.* a massless particle cannot exist at rest.

Since massless particle should always be in motion. Let us find what velocities it can acquire.

If $V < C$ then (82) and (84) give us

$$E = 0, \quad p = 0 \quad \text{(Note for massless particle } m_0 = 0)$$

Thus a massless particle can never have velocity smaller than velocity of light.

For $V > C$, these equations again imply $E = 0$, $p = 0$. Thus a massless particle cannot travel faster than speed of light.

For $V = C$, these equations give us

$$E = \frac{0}{0}$$
$$p = \frac{0}{0}$$

Both of these equations have ∞ many solutions. Thus a massless particle can exist only if it is travelling with speed of light and it can possess ∞ many different values of energy and ∞ many different values of momentum and E and p of massless particle are always related to each other by the relation $E = pC$.

(16) RELATIVITY OF LENGTH (LORENTZ FITZ GERALD LENGTH CONTRACTION)

Relative motion affects measurement of length. A body moving with a velocity V relative to an observer appears to be contracted in length in the direction of motion by a factor $\sqrt{1-V^2/C^2}$.

Proof

Figure (11) shows a rod AB lying at rest along X-axis of frame S', which itself is moving with uniform speed V along X-axis of another inertial frame S such that their respective axes remain parallel to each other. Time is measured from the instant, when origins O and O' coincided with each other.

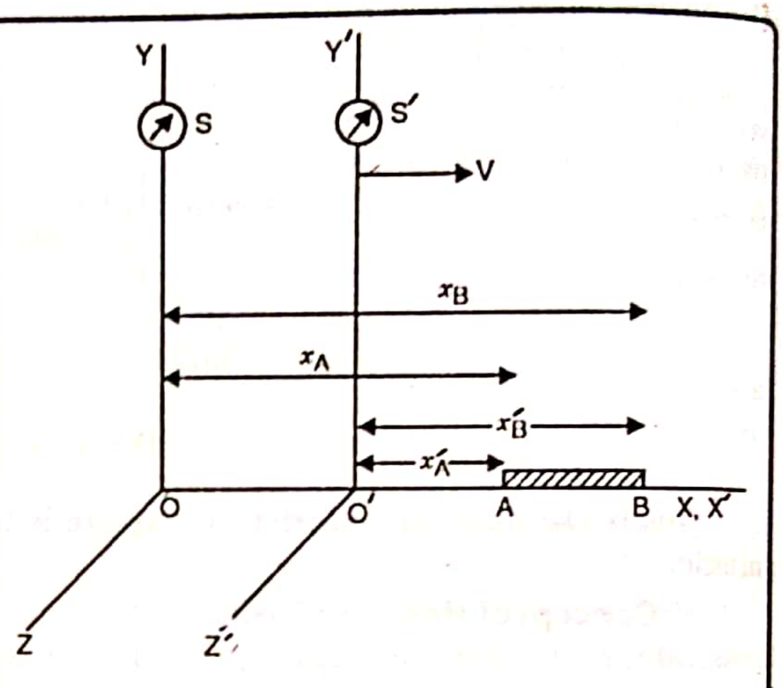

Fig. 11.

Let x'_A and x'_B be the positions of end points of the rod from origin O' measured at any instant t' w.r.t. clock of S'.

Let x_A and x_B be the positions of end points of rod from origin O measured at same instant. However clock of S would give reading t (say) at same instant.

Since rod is at rest in S' thus positions x'_A and x'_B are independent of time t'. Hence length of the rod in S' is given by

$$L_0 = x'_B - x'_A \quad \text{...(86)}$$

This length of rod is called Proper Length (but it is not absolute length of rod) because it is measured in a frame in which rod always remains at rest (*i.e.* frame S').

W.r.t. frame S, rod appears to be moving with velocity V along X-axis. Hence positions x_A and x_B must be function of time and hence both x_A and x_B should be measured simultaneously. The length of rod in frame S is given by

$$L = x_B - x_A \quad \text{...(87)}$$

This length is called improper length as it is measured in a frame S w.r.t. which rod appears to be moving.

According to direct Lorentz transformation*

$$x'_B = \frac{x_B - Vt}{\sqrt{1-V^2/C^2}} \text{ and } x'_A = \frac{x_A - Vt}{\sqrt{1-V^2/C^2}} \quad \text{...(88)}$$

Put values in (86), we get

$$L_0 = \frac{x_B - x_A}{\sqrt{1-V^2/C^2}}$$

*We must apply direct Lorentz Transformation here because we have already stated that x_A and x_B are function of time t and x'_A and x'_B do not depend on variable t'. Thus we must choose a relation involving x_A, x_B, t as variables. This is possible if we apply direct Lorentz Transformation only.

or
$$L_0 = \frac{L}{\sqrt{1 - V^2/C^2}}$$

or
$$L = L_0 \sqrt{1 - V^2/C^2} \quad \text{...(89)}$$

Discussion

(i) This equation shows that to the stationary observer in S, the rod placed in moving frame S′ appears to be contracted ($\because L < L_0$) along the direction of its motion by amount $\sqrt{1 - V^2/C^2}$.

(ii) If $V = C$, then $L = 0$. It means that a rod moving with speed of light will appear as reduced to a point to a stationary observer.

(iii) If rod is lying at rest in frame S′ along Y-axis of frame S′ then the length of rod w.r.t. S′ and S is given as

$$L' = y'_B - y'_A$$
$$L = y_B - y_A$$

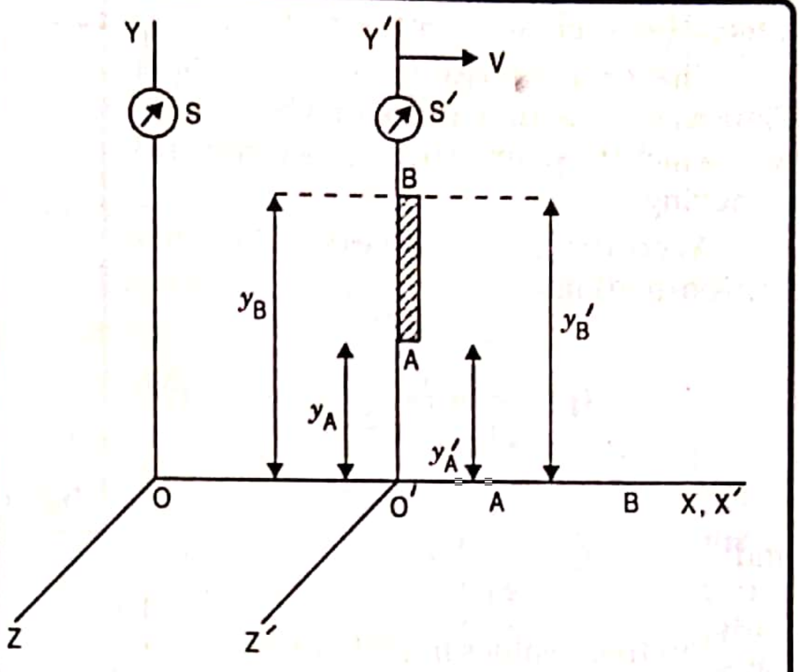

Fig. 12.

But according to Lorentz transformation $y = y'$

Thus $y'_A = y_A, y'_B = y_B$

Hence $\qquad L' = L$

i.e. length contraction does not take place in a direction perpendicular to the motion of rod.

(iv) If Rod is lying at rest in frame S at rest along X-axis then proper length will be $L_0 = x_B - x_A$

and improper length will be $\qquad L = x'_B - x'_A$

Now apply Inverse Lorentz transformations, we shall again get same result i.e. $L = L_0\sqrt{1 - V^2/C^2}$

i.e. $L < L_0$ i.e. length will contract by a factor $\sqrt{1 - V^2/C^2}$.

(17) RELATIVITY OF TIME (TIME DILATION)

Suppose a clock is placed at point x' in frame S′. Let an event takes place in frame S′ (may be a light flash is produced or a cracker bursts) at point A (whose position is x') when clock of S′ records its time as t'_1, while clock of S records time of occurrence of the same event as t_1.

Let at a later stage, another event takes place at same point A, when clock of S′ records the time of 2nd event t'_2, while clock of S records time of 2nd event as t_2.

According to clock of S′, the time interval between two events is

$$\tau_0 = t'_2 - t'_1 \quad \text{...(90)}$$

This time interval is called Proper Time because it is the time shown by a clock, w.r.t. which the point A at which both events occurred remains at rest all the times.

According to clock of S, the time interval between two events is $\tau = t_2 - t_1$ ...(91)

This time interval is called Improper Time as it is the time measured by a clock, w.r.t. which the point of occurrence of events is moving.

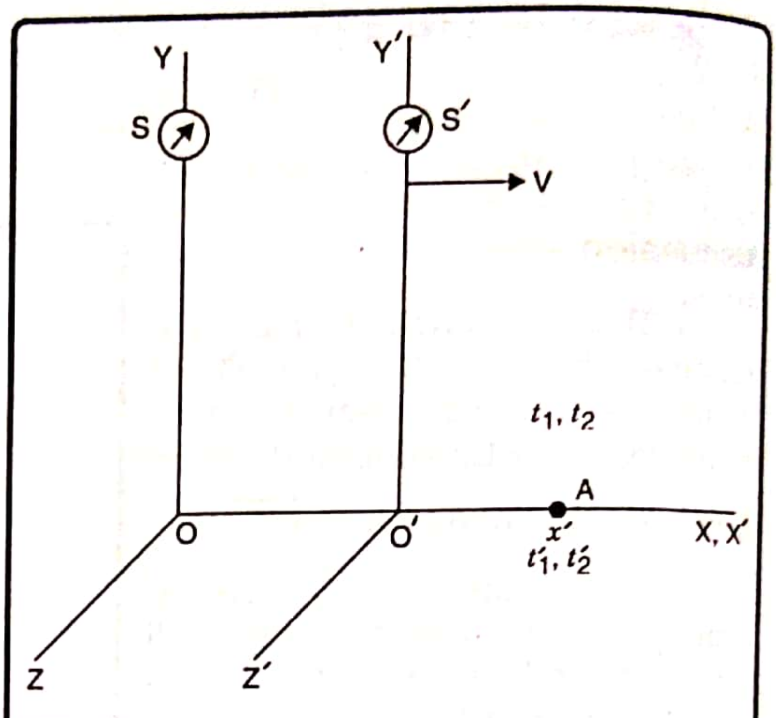

Fig. 13.

According to inverse Lorentz Transformation

$$t_1 = \frac{t_1' + \frac{Vx'}{C^2}}{\sqrt{1 - V^2/C^2}} \quad ...(92)$$

and

$$t_2 = \frac{t_2' + \frac{Vx'}{C^2}}{\sqrt{1 - V^2/C^2}} \quad ...(93)$$

Put these values in (91), we get

$$\tau = \frac{t_2' - t_1'}{\sqrt{1 - \frac{V^2}{C^2}}}$$

or

$$\tau = \frac{\tau_0}{\sqrt{1 - \frac{V^2}{C^2}}} \quad ...(94)$$

Thus $\tau > \tau_0$. This equation shows that to the stationary observer in S, the time interval appears to be lengthened by a factor $\frac{1}{\sqrt{1 - V^2/C^2}}$. Thus a stationary clock (of S) measures a longer time interval between events occurring in a moving frame of reference, than does a clock in the moving frame (S′). In other words, moving clock (of S′) appears to be slowed down. This phenomenon is called time dilation.

When $V \to C$ then $\tau \to \infty$ or $\tau_0 \to 0$

In other words if S′ starts moving with speed of light, then there will be no passage of time in frame S′. But this can never happen practically as no material particle can travel with speed of light.

(18) ILLUSTRATION OF TIME DILATION

(i) Decay of μ-mesons. μ-mesons are elementary particles formed in the atmosphere by the action of Cosmic Ray Showers. The half life time of μ-mesons (time in which no. of μ-mesons become half of initial number) is 3.1 μs in a frame where these are at rest *i.e.* their proper half time is 3.1 μs. The speed of these particles is estimated to be 0.9C.

Thus distance travelled by these particles in one half life is

$$d = V\tau_0 = 0.90 \times 3 \times 10^8 \times 3.1 \times 10^{-6} \approx 840 \ m.$$

If we place two meson counters at a distance of 840 m apart, we expect the number of counts in the upper one should be double that of lower. Experimentally, however, the number of counts in the lower counter are found to be much more than half of the counts of the upper counter.

Let us try to answer this observation. A μ-meson can be assumed as frame S′ and earth as frame S and line joining μ-meson and centre of Earth as X-axis. Then the time $\tau_0 = 3.1\,\mu s$ is w.r.t. rest frame of μ-meson *i.e.* frame S′. However w.r.t. Earth frame (S), the half life time of μ-mesons would be

$$\tau = \frac{\tau_0}{\sqrt{1 - V^2/C^2}} = \frac{3.1\,\mu s}{\sqrt{1 - \left(\frac{0.9\,C}{C}\right)^2}} = 7.2\,\mu s$$

Thus no. of μ-mesons should reduce to half after a distance

$$d = V\tau = 0.9 \times 3 \times 10^8 \times 7.2 \times 10^{-6}$$
$$= 1920\ m$$

This is indeed in agreement with experimental observations.

(*ii*) **Twin Paradox.** A direct consequence of time dilation is what is usually called as differential aging. To understand this term, let us consider following hypothetical experiment.

Consider two twin brothers. One of the twins stays at home and other goes to a long space journey. According to brother on journey time will be delayed and when he returns back then he will find himself younger than the brother, who stayed at home. However we can treat the problem in reverse way also. Since every motion is relative so it can be said that the brother, who stayed home is in motion with respect to the brother in spaceship. Thus at the end of journey, the brother staying at home should find himself younger. The symmetry of the problem tells that both brothers will find themselves equally young at the end of journey.

However there will be some difference in their age. It is because of the fact that the brother, who stayed home did not suffer any acceleration. While brother who went in the ship will definitely experience acceleration (*e.g.* at the time of returning back). Hence differential aging will exist. However practically it will be negligible because no space ship can travel with speed comparable to speed of light. But differential aging has been observed in unstable elementary particles, which travel with high speeds and show accelerated motion very frequently.

(19) RELATIVITY OF SIMULTANEITY

Suppose two events take place at two different points A and B in frame S′ but at same time t'. Let positions of points A and B from O′ are x_1' and x_2' respectively.

Let an observer in frame S notes times of occurrence of these events as t_1 and t_2 respectively. Then by inverse Lorentz velocity transformations, we have

$$t_1 = \frac{t' + \frac{Vx_1'}{C^2}}{\sqrt{1 - V^2/C^2}} \qquad ...(95)$$

and

$$t_2 = \frac{t' + \frac{Vx_2'}{C^2}}{\sqrt{1 - V^2/C^2}} \qquad ...(96)$$

According to observer of S, the time difference between two events in S is

$$t_2 - t_1 = \frac{\frac{V}{C^2}(x_2' - x_1')}{\sqrt{1 - V^2/C^2}}$$

Clearly $t_2 \neq t_1$ *i.e.* events which are simultaneous in frame S′ are not simultaneous in frame S. The question arises who is right? Actually both are correct because each one measures, what he sees. Hence, we conclude that there is no such thing as absolute time, which is same for all observers. Time is relative and different for different observers in motion. However if $x_1' = x_2'$, then above equation gives $t_2 = t_1$ *i.e.* two events can be simultaneous to all observers, if they occur at same time and at same place.

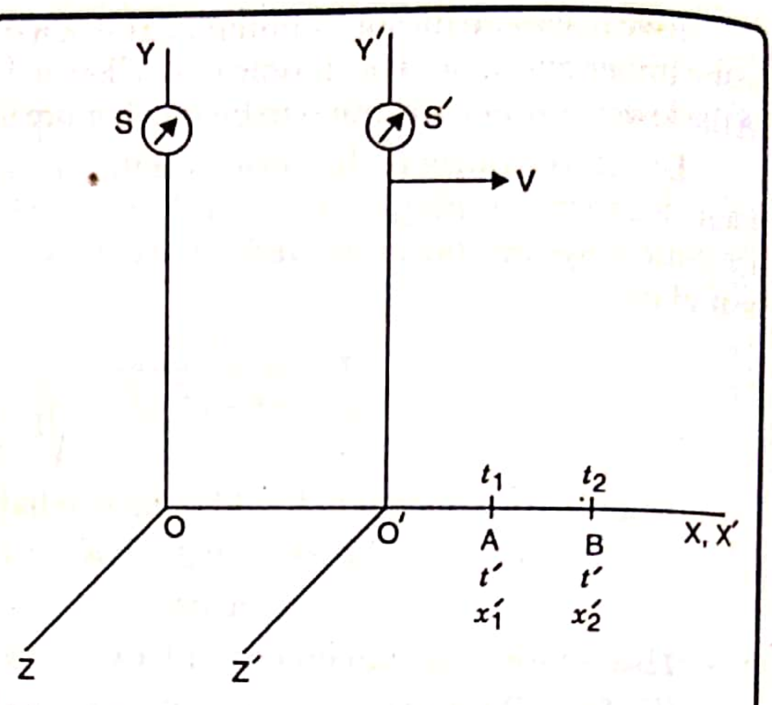

Fig. 14.

(20) PAST, PRESENT & FUTURE

We are now convinced about relative character of time. But still there is a sense of time distinguishing past, present and future. No observer can see an event before it actually occurs. It is because the light signal takes finite time to reach the observer. Further, we can order a sequence of events. Let an event A occurs at earlier time t_1 and a second event 'B' occur at some later time t_2. Then to every observer in any frame the event A will occur earlier than event B. The only difference will be in the time interval $t_2 - t_1$ for various observers.

However if an event occurs at a point P and S_1 and S_2 are two observers, such that S_1 is close to P than S_2, then light signal will reach S_1 earlier than S_2. Thus for S_1 event is future of S_2 & for S_2, event is past of S_1. Then S_1 can think of telling S_2 about his future by sending a signal to S_2. But that signal must travel faster than speed of light, only then S_2 can know about his future. Since no signal can travel faster than speed of light, so practically it is not possible to look into future.

Example 3. ***Show that Maxwell's wave equation is invariant under Lorentz Transformation.***

Solution. For a wave function ψ, Maxwell's e.m. wave equation is

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} - \frac{1}{C^2}\frac{\partial^2\psi}{\partial t^2} = 0 \qquad \text{...(97)}$$

According to Lorentz Transformation

$$\left.\begin{aligned} x' &= K(x - Vt) \\ y' &= y \\ z' &= z \\ t' &= K\left(t - \frac{Vx}{C^2}\right) \end{aligned}\right\} \qquad \text{...(98)}$$

where
$$K = \frac{1}{\sqrt{1 - V^2/C^2}}$$

By calculus we have

$$\frac{\partial\psi}{\partial x}=\frac{\partial\psi}{\partial x'}\frac{\partial x'}{\partial x}+\frac{\partial\psi}{\partial y'}\frac{\partial y'}{\partial x}+\frac{\partial\psi}{\partial z'}\frac{\partial z'}{\partial x}+\frac{\partial\psi}{\partial t'}\frac{\partial t'}{\partial x}$$

$$=\frac{\partial\psi}{\partial x'}(K)+0+0+\frac{\partial\psi}{\partial t'}\left(-\frac{KV}{C^2}\right)\qquad\left[\begin{array}{l}\because \text{ from } (1.98)\\ \frac{\partial x'}{\partial x}=K,\frac{\partial y'}{\partial x}=0,\frac{\partial z'}{\partial x}=0\\ \frac{\partial t'}{\partial x}=-\frac{KV}{C^2}\end{array}\right]$$

$$\Rightarrow\qquad \frac{\partial\psi}{\partial x}=K\frac{\partial\psi}{\partial x'}-\frac{KV}{C^2}\frac{\partial\psi}{\partial t'}\qquad ...(99)$$

Hence $$\frac{\partial}{\partial x}=K\frac{\partial}{\partial x'}-\frac{KV}{C^2}\frac{\partial}{\partial t'}\qquad ...(100)$$

Now $$\frac{\partial^2\psi}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial x}\right)$$

$$=\left(K\frac{\partial}{\partial x'}-\frac{KV}{C^2}\frac{\partial}{\partial t'}\right)\left(K\frac{\partial\psi}{\partial x'}-\frac{KV}{C^2}\frac{\partial\psi}{\partial t'}\right)$$

$$=K^2\frac{\partial^2\psi}{\partial x'^2}-2\frac{KV}{C^2}\frac{\partial^2\psi}{\partial t'\partial x'}+\frac{K^2V^2}{C^4}\frac{\partial^2\psi}{\partial t'^2}\qquad ...(101)$$

Similarly $$\left.\begin{array}{l}\frac{\partial^2\psi}{\partial y^2}=\frac{\partial^2\psi}{\partial y'^2}\\ \frac{\partial^2\psi}{\partial z^2}=\frac{\partial^2\psi}{\partial z'^2}\end{array}\right\}\qquad ...(102)$$

And $$\frac{\partial\psi}{\partial t}=\frac{\partial\psi}{\partial x'}\frac{\partial x'}{\partial t}+\frac{\partial\psi}{\partial y'}\frac{\partial y'}{\partial t}+\frac{\partial\psi}{\partial z'}\frac{\partial z'}{\partial t}+\frac{\partial\psi}{\partial t'}\frac{\partial t'}{\partial t}$$

$$=\frac{\partial\psi}{\partial x'}(-KV)+0+0+\frac{\partial\psi}{\partial t'}(K)\qquad \text{(using (98))}$$

$$=K\frac{\partial\psi}{\partial t'}-KV\frac{\partial\psi}{\partial x'}\qquad ...(103)$$

Hence $$\frac{\partial}{\partial t}=K\frac{\partial}{\partial t'}-KV\frac{\partial}{\partial x'}\qquad ...(104)$$

$$\therefore\qquad \frac{\partial^2\psi}{\partial t^2}=\frac{\partial}{\partial t}\left(\frac{\partial\psi}{\partial t}\right)$$

$$=\left(K\frac{\partial}{\partial t'}-KV\frac{\partial}{\partial x'}\right)\left(K\frac{\partial\psi}{\partial t'}-KV\frac{\partial\psi}{\partial x'}\right)$$

$$=K^2\frac{\partial^2\psi}{\partial t'^2}-2KV\frac{\partial^2\psi}{\partial t'\partial x'}+K^2V^2\frac{\partial^2\psi}{\partial x'^2}\qquad ...(105)$$

Put values from (101), (102) and (105) in (97), we get

$$K^2\left(1-\frac{V^2}{C^2}\right)\frac{\partial^2\psi}{\partial x'^2}+\frac{\partial^2\psi}{\partial y'^2}+\frac{\partial^2\psi}{\partial z'^2}-\frac{1}{C^2}K^2\left(1-\frac{V^2}{C^2}\right)\frac{\partial^2\psi}{\partial t'^2}=0$$

$$\frac{\partial^2\psi}{\partial x'^2}+\frac{\partial^2\psi}{\partial y'^2}+\frac{\partial^2\psi}{\partial z'^2}-\frac{1}{C^2}\frac{\partial^2\psi}{\partial t'^2}=0 \qquad \left(\because K^2\left(1-\frac{V^2}{C^2}\right)=1\right)$$

This equation is similar to equation (97). Hence under Lorentz Transformation, Maxwell's wave equation is invariant.

Example 4. *Show that $x^2+y^2+z^2-C^2t^2=0$ is invariant under Lorentz Transformation.*

Solution. Given equation is

$$x^2+y^2+z^2-C^2t^2=0 \qquad ...(106)$$

According to Inverse Lorentz Transformations

$$x=K(x'+Vt')$$

$$y=y' \qquad z=z'$$

$$t=K(t'+Vx'/C^2) \qquad \text{where} \qquad K=\frac{1}{\sqrt{1-V^2/C^2}}$$

Put these values in (106) we get

$$K^2(x'^2+V^2t'^2+2x'Vt')+y'^2+z'^2-C^2K^2\left[t'^2+\frac{V^2x'^2}{C^4}+\frac{2Vt'x'}{C^2}\right]=0$$

or
$$K^2\left(1-\frac{V^2}{C^2}\right)x'^2+y'^2+z'^2-C^2K^2\left(1-\frac{V^2}{C^2}\right)t'^2=0$$

or
$$x'^2+y'^2+z'^2-C^2t'^2=0$$

This equation is similar to equation (106). Hence the law $x^2+y^2+z^2-C^2t^2=0$ is invariant under Lorentz Transformations.

Exercise 1. The distance travelled by light along X-axis is given by equation $x^2=C^2t^2$. Show that this equation is invariant under Lorentz Transformations.

Example 5. *In a laboratory experiment, the speeds of two electrons moving in opposite directions are each found to be 0.8 C. What is their relative speed ?* (*P.T.U. Exam.*)

Solution. We assume laboratory as frame S. One electron A is assumed to be moving frame S′ and its direction of motion as X-axis.

Thus

V = Speed of S′ (Electron A) w.r.t. lab (Frame S)
= 0.8 C

$u_x=-0.8$ C
= Speed of 2nd electron (B) w.r.t. lab
(– ve sign shows that 2nd e^- is moving toward – X axis)

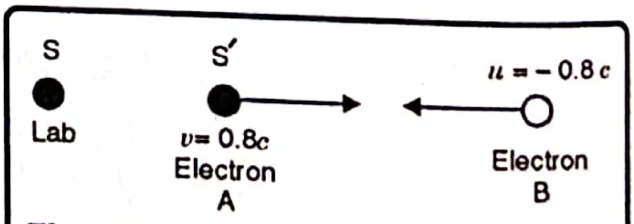

Fig. 15.

u'_x = Vol. of 2nd elution w.r.t. first

$$=\frac{u_x-V}{1-\frac{V}{C^2}u_x}=\frac{-0.8\,C-0.8\,C}{1-\frac{(0.8\,C)(-0.8\,C)}{C^2}}=\frac{-1.6\,C}{1.64}$$

$$u'_x=-0.9756\,C$$

– ve sign means 2nd electron is moving along – X axis.

Example 6. ***Show that velocity of light is invariant under Lorentz Transformation.***

Solution. Consider a photon P moving along X-axis of frame S with velocity $u_x = C$. Let S′ is another frame moving with velocity V along X-axis of S. u'_x = Velocity of photon w.r.t. frame S′

$$\therefore \quad u_x' = \frac{u_x - V}{1 - \frac{V}{C^2} u_x}$$

$$= \frac{C - V}{1 - \frac{V}{C^2} \cdot C} \qquad (\because u_x = C)$$

$$= C$$

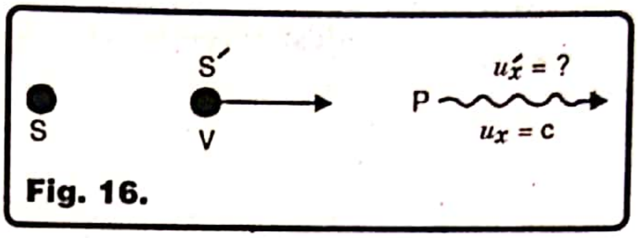

Fig. 16.

Thus velocity of light is invariant under Lorentz Transformations.

Example 7. ***Two photons are approaching each other with velocity C in space. Find relative speed of one photon w.r.t. other.***

Solution. Let S is an inertial frame. We designate two photons as A and B and their motion is considered along X-axis.

Attach a frame S′ to photon A. Thus V = C. The velocity of photon B w.r.t. S is $u_x = -C$. The velocity of this photon w.r.t. frame S′ (*i.e.* w.r.t. photon A) is

$$u'_x = \frac{u_x - V}{1 - \frac{V}{C^2} u_x} = \frac{-C - C}{1 - \frac{(C)(-C)}{C^2}}$$

$$= \frac{-2C}{2} = -C$$

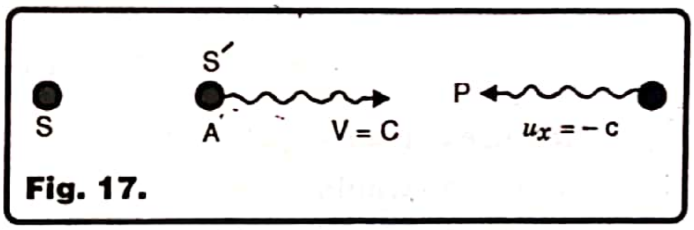

Fig. 17.

Thus w.r.t. photon A, the photon B appears to be approaching with velocity C.

Example 8. ***Prove that the four dimensional volume element dx, dy, dz, dt is invariant under Lorentz Transformation.***

Solution. Consider a frame S′ moving with a constant velocity w.r.t. another frame S. According to length contraction effect, the length intervals dx, dy, dz in the frame S will appear to the observer of S′ as

$$dx' = dx\sqrt{1 - V^2/C^2}$$

$$dy' = dy$$

$$dz' = dz$$

Also time interval dt in S will appear in frame S′ as

$$dt' = \frac{dt}{\sqrt{1 - V^2/C^2}}$$

Thus four dimensional volume element $dx'\, dy'\, dz'\, dt'$ in S′ will appear as

$$dx'\, dy'\, dz'\, dt' = dx\sqrt{1 - V^2/C^2}\; dy\, dz \frac{dt}{\sqrt{1 - \frac{V^2}{C^2}}}$$

$$= dx\, dy\, dz\, dt$$

Hence four dimensional volume element $dx\, dy\, dz\, dt$ is invariant under Lorentz Transformations.

Example 9. *What is the total energy E of a 250 MeV electron ?* *(Mass of electron = 9.109×10^{-31} kg).*

Solution. Given kinetic energy $= 250$ MeV

Rest mass energy in MeV $= \dfrac{m_0 C^2}{1.6 \times 10^{-13}}$ MeV

$$= \frac{9.109 \times 10^{-3} \times 9 \times 10^{16}}{1.0 \times 10^{-13}} = 0.512 \text{ MeV}$$

Thus total energy = KE + Rest mass energy

Total energy $= 250 + 0.512 = 250.512$ MeV

Example 10. *A woman leaves the earth in a space craft, that makes a round trip to the nearest star 4 light years distant, at a speed of 0.9 C. How many days younger is she, upon her return than her twin sister, who remained on earth ?*

Solution. Total distance covered w.r.t. earth $L = 2 \times 4 \times 9.46 \times 10^{15} = 7.57 \times 10^{16}\, m$

Time spent for journey w.r.t. earth $= \tau = \dfrac{L}{V} = \dfrac{7.57 \times 10^{16}}{0.9 \times 3 \times 10^{8}} = 2.86 \times 10^{6}\, s$

Time spent according to clock of woman on journey $\tau_0 = \tau\sqrt{1 - V^2/C^2}$

or $$\tau_0 = 2.86 \times 10^{6} \times \sqrt{1 - \left(\frac{0.9\,C}{C}\right)^2} = 1.25 \times 10^{6} s$$

Difference $= \Delta\tau = \tau - \tau_0 = 2.86 \times 10^{6} - 1.25 \times 10^{6} = 1.6 \times 10^{6}\, s = 18.83$ days

Thus woman would appear younger by 18.63 days than her twin sister.

Example 11. *Show that phase difference between two light beams in Michelson Moreley experiment can be written as* $\Delta\phi = \dfrac{2\pi LV^2}{\lambda C^2}$ *where symbols have their usual meanings.*

Solution. We know that path difference between the waves in Michelson Moreley Experiment is given as

$$\Delta x = \frac{LV^2}{C^2}$$

Phase difference is given by

$$\Delta\phi = \frac{2\pi}{\lambda} \times \Delta x = \frac{2\pi}{\lambda} \cdot \frac{LV^2}{C^2}$$

Example 12. *A rod of proper length 60 cm is placed in a stationary frame with its one end at the origin. Its direction cosines are* $\left(\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right)$. *What will be its length and direction cosines, when viewed from a frame moving with a velocity 0.6 C along X-direction relative to a stationary frame ?*

Solution. Let rod makes angles α, β, γ with X, Y, Z axes in frame S.

$\therefore$ $\cos\alpha = 2/3$, $\cos\beta = -1/3$, $\cos\gamma = 2/3$.

The components of length along X, Y, Z direction in frame S are

$$L_x = L_0 \cos\alpha = 60 \times \frac{2}{3} = 40 \text{ cm}$$

$$L_y = L_0 \cos\beta = 60 \times \left(-\frac{1}{3}\right) = -20 \text{ cm (– ve sign only indicates orientation)}$$

$$L_z = L_0 \cos\alpha = 60 \times \frac{2}{3} = 40 \text{ cm}$$

The rod is at rest in frame S. So L_x, L_y, L_z represent proper lengths.

The frame S′ is moving along X-axis with velocity V = 0.6 C. Thus length components in S′ are improper and are given as

$$L'_x = L_x \sqrt{1 - V^2/C^2} = 40 \times \sqrt{1 - \left(\frac{0.6\,C}{C}\right)^2} = 40 \times 0.8 = 32 \text{ cm}$$

$$L'_y = L_y = -20 \text{ cm}$$

$$L'_z = L_z = 40 \text{ cm}$$

∴ Length of rod appears to be L in S′, where L is given as

$$L = \sqrt{L'^2_x + L'^2_y + L'^2_z} = \sqrt{(32)^2 + (-20)^2 + (40)^2} \approx 55 \text{ cm}$$

Let $\cos\alpha'$, $\cos\beta'$, $\cos\gamma'$ are direction cosines of rod in frame S′

∴
$$\cos\alpha' = \frac{L'_x}{L} = \frac{32}{55}$$

$$\cos\beta' = \frac{L'_y}{L} = \frac{-20}{55}$$

$$\cos\gamma' = \frac{L'_z}{L} = \frac{40}{55}$$

∴ Direction cosines of rod in S′ are $\left(\frac{32}{55}, \frac{-20}{55}, \frac{40}{55}\right)$.

Example 13. *What will be fringe shift according to Michelson Moreley experiment, if the effective path length of light is 7 metres and wavelength of light has 7000 Å wavelength ? The velocity of earth is 3×10^4 m/s.*

Solution. For observing fringe shift the apparatus has to be rotated through 90°. Thus fringe shift is given as

$$\Delta N = \frac{2l\,V^2}{C^2 \lambda} = \frac{2 \times 7 \times (3 \times 10^4)^2}{(3 \times 10^8)^2 \times (7 \times 10^{-7})^2} = 0.2 \text{ Ans.}$$

Example 14. *An event occurs at x = 100 km, y = 10 km and z = 1.0 km at t = 2.0×10^{-4} s in a reference frame S. Another frame S′ is moving with speed 0.95 C relative to S along common X-axis, the origins coinciding at t = t′ = 0. Compute the co-ordinates x′, y′, z′, t′ of the event in S′.*

Solution.
$$x' = \frac{x - Vt}{\sqrt{1 - V^2/C^2}} = \frac{100 \text{ km} - (0.95 \times 3 \times 10^5 \text{ km/s})\,(2 \times 10^{-4} \text{ s})}{\sqrt{1 - (0.95)^2}}$$

$$= 137.8 \text{ km}$$

$$y' = y = 10 \text{ km}$$

$$z' = z = 1.0 \text{ km}$$

$$t' = \frac{t - \left(\frac{Vx}{C^2}\right)}{\sqrt{1 - V^2/C^2}} = \frac{(2 \times 10^{-4} \text{ s}) - \frac{100 \text{ km} \times 0.95}{3 \times 10^5 \text{ km/s}}}{\sqrt{1 - (0.95)^2}} = -3.74 \times 10^{-4} \text{ s}$$

Example 15. *Calculate the length and orientation of a meter rod in a frame of reference, which is moving with a velocity equal to 0.6 C, in a direction making an angle 30° with the rod.*

Solution. Component of length of metre rod in the direction of motion is $L_x = 1 \cos 30° = 0.8660\ m$

Component of length ⊥ to direction of motion is

$$L_y = 1 \sin 30° = 0.5\ m$$

Component of length along X-axis in moving frame is

$$L'_x = L_x \sqrt{1 - V^2/C^2} = 0.8660 \sqrt{1-(0.6)^2} = 0.693\ m$$

Component of length ⊥ to motion in moving frame is

$$L'_y = L_y = 0.5\ m$$

Length of rod in moving frame is

$$L = \sqrt{L'^2_x + L'^2_y} = \sqrt{(0.5)^2 + (0.693)^2} = 0.854\ m$$

Let rod appears to make angle θ with the direction of motion in moving frame then

$$\tan\theta = \frac{L'_y}{L'_x} = \frac{0.5}{0.693} = 0.72$$

$$\therefore \qquad \theta = \tan^{-1}(0.72) = 35°\ 49'$$

Example 16. *A cube has side L_0 at rest. Find the apparent volume of cube when it moves with velocity βC normal to one of its faces.*

Solution. Volume of cube at rest $= L_0^3$

Velocity of moving cube $= V = \beta C \Rightarrow \beta = \dfrac{V}{C}$

When cube moves normal to one of its faces then the length of edge of cube parallel to direction of motion will contract to $L = L_0\sqrt{1 - V^2/C^2} = L_0\sqrt{1-\beta^2}$.

While length of other two edges which are normal to direction of motion remains same as L_0.

Thus in moving frame it will look like a cuboid having sides L_0, L_0, $L_0\sqrt{1-\beta^2}$ and its volume will be $V = L_0^3\sqrt{1-\beta^2}$.

Exercise 2. Calculate the percentage contraction in a rod moving with velocity 0.6 C in a direction making an angle 60° to its own length. [Ans. 4.6%]

Exercise 3. A rod of length 6*m* lies at rest in XY plane of frames. It is inclined at an angle 30° to the X-axis of frame S. Find the length and orientation of the rod as observed from a frame S′ moving with velocity 0.5 C along X-axis. [Ans. 5.42 *m*, 33°41′]

Example 17. *A spaceship moving away from earth with velocity 0.6 C fires a rocket, whose velocity relative to spaceship is 0.7 C (a) away from earth (b) towards earth. What will be the velocity of rocket as observed from earth in two cases ?*

Solution. (*a*) We assume earth as frame S, spaceship as frame S′ and rocket as a particle. The direction of motion is taken X-axis.

Here $\qquad V = 0.6C$

u'_x = Vol. of rocket w.r.t. spaceship = 0.7 C
u_x = Vol. of rocket w.r.t. earth

By Inverse Lorentz Transformations

$$u_x = \frac{u'_x + V}{1 + \frac{V}{C^2} u_x} = \frac{0.7\,C + 0.6\,C}{1 + (0.7)\,(0.6)} = 0.915\,C \text{ (away from earth)}$$

(b) In this case $u'_x = -0.7\,C$

$V = 0.6\,C$

$$u_x = \frac{u'_x + V}{1 + \frac{V}{C^2} u'_x} = \frac{-0.7\,C + 0.6\,C}{1 + (-0.7)\,(0.6)} = -0.172\,C \text{ (towards earth)}$$

Example 18. *A particle moves with velocity u in XY plane. The direction of motion of the particle makes an angle θ with X-axis. If the particle appears to move at an angle φ w.r.t. X-direction in frame S′, show that* $\tan\phi = \dfrac{\sin\theta\sqrt{1 - V^2/C^2}}{\cos\theta - V/u}$.

Solution. The components of velocity in frames S are

$$u_x = u\cos\theta,\ u_y = u\sin\theta$$

The component of velocity of particle in S′ frame are

$$u'_x = u\cos\phi,\ u'_y = u'\sin\phi$$

where u' = velocity of particle in frame S′.

We have
$$u'_x = \frac{u_x - V}{1 - \frac{V}{C^2} u_x}$$

or
$$u'\cos\phi = \frac{u\cos\theta - V}{1 - \frac{V}{C^2} u\cos\theta} \quad ...(1)$$

Also
$$u'_y = \frac{u_y\sqrt{1 - V^2/C^2}}{1 - \frac{V}{C^2} u_x}$$

or
$$u'\sin\phi = \frac{u\sin\theta\sqrt{1 - V^2/C^2}}{1 - \frac{V}{C^2}\cos\phi} \quad ...(2)$$

Divide (2) by (1)

⇒
$$\tan\phi = \frac{u\sin\theta\sqrt{1 - V^2/C^2}}{u\cos\theta - V} = \frac{\sin\theta\sqrt{1 - V^2/C^2}}{\cos\theta - \frac{V}{u}}$$

Example 19. *A particle in a stationary frame S is moving in XY plane with velocity 0.8 C inclined to 60° along X-axis. What will be the velocity of particle observed by a person in frame S′ moving relative to S with a velocity 0.4 C ?*

Solution.
$$u_x = u\cos 60° = 0.8\,C \times \frac{1}{2} = 0.4\,C,\ V = 0.4\,C$$

$$u_y = u \sin 60° = 0.8\,C \times 0.8660 = 0.693\,C$$

$$u'_x = \frac{u_x - V}{1-(V/C^2)\,u_x} = \frac{0.4\,C - 0.4\,C}{1-(0.4)(0.4)} = 0$$

$$u'_y = \frac{u_y\sqrt{1-V^2/C^2}}{1-\dfrac{V}{C^2}u_x} = \frac{0.693\,C\sqrt{1-(0.4)^2}}{1-(0.4)(0.4)} = 0.756\,C$$

Since $u'_x = 0$. Thus total velocity $= \sqrt{u'^2_x + u'^2_y} = 0.756\,C$ along Y-axis.

Exercise 4. Rockets A and B are observed from earth to be travelling with velocities 0.8 C and 0.7 C in the same line in the same direction. What is the velocity of B as seen by observer of A ? [Ans. 0.23 C]

Exercise 5. A radioactive atom moves with a velocity 0.1 C along X-axis of a system S. It emits a β-particle of velocity 0.95 C relative to the system S′, in which, the radioactive atom is at rest. If β-particle is emitted along X-axis in S′, find its speed relative to S. [Ans. 0.959 C]

Example 20. *At what speed is a particle moving, if its mass is equal to 4 times its rest mass.*

Solution. $m = 4\,m_0$

$$\Rightarrow \quad \frac{m_0}{\sqrt{1-V^2/C^2}} = 4m_0$$

$$\Rightarrow \quad \frac{1}{16} = 1 - \frac{V^2}{C^2}$$

$$\Rightarrow \quad \frac{V}{C} = \sqrt{\frac{15}{16}} = 0.968$$

$$\Rightarrow \quad V = 0.968\,C$$

Example 21. *Electrons in a betatron machine have been accelerated to a speed of 90% of velocity of light. What is their relativistic mass ?*

Solution. $V = 0.9\,C$

$$\therefore \quad m = \frac{m_0}{\sqrt{1-V^2/C^2}} = \frac{m_0}{\sqrt{1-(0.9)^2}} = 2.29\,m_0$$

$$= 2.29 \times 9.1 \times 10^{-31} = 2.08 \times 10^{-30}\ \text{kg.}$$

Example 22. *How much mass is lost when 1 kg of water at 0°C converts into ice at 0°C?*

Solution. According to Einstein's mass energy relation

$$E = \Delta m C^2$$

or $\quad mL = \Delta m C^2 \quad$ where L = Latent heat of fusion of water

$$\Delta m = \frac{mL}{C^2} = \frac{1\,\text{kg} \times 4.186 \times 80 \times 10^3}{(3\times 10^8)^2} \qquad \left[\begin{aligned} L &= 80\ \text{cal g}^{-1} \\ &= 80 \times 4.186 \times 10^3\ \text{J kg}^{-1}\end{aligned}\right]$$

$$= 3.72 \times 10^{-12}\ \text{kg}$$

Example 23. *Sun radiates continuously and the solar energy is reaching the top of earth's atmosphere at a rate of 1.35 kW m^{-2}. Calculate the decrease in mass of sun per second, if the average distance between sun and earth is 1.5×10^{11} m.*

Solution. Total energy emitted per second by sun is

$$E = 4\pi r^2 \times \text{Solar constant}$$
$$= 4 \times 3.142 \times (1.5 \times 10^{11})^2 \times (1350)$$
$$= 3.82 \times 10^{26}\ J/s$$

mass lost per second by sun is

$$\Delta m = \frac{E}{C^2} = \frac{3.82 \times 10^{26}}{9 \times 10^{16}} = 4.24 \times 10^9\ kg/s$$

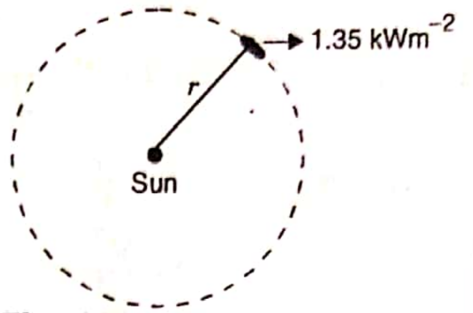

Fig. 18.

Example 24. *Calculate the speed of electron accelerated through a potential difference of 10^6 volt.*

Solution. Before accelerating, the energy of electron is m_0C^2. After acceleration energy of electron becomes mC^2.

Obviously the increase in energy must be the energy provided by accelerating voltage.

Thus
$$eV = mC^2 - m_0C^2$$
$$= m_0C^2\left(\frac{1}{\sqrt{1 - V^2/C^2}} - 1\right)$$

Note. Energy given to electron is $qV = 10^6 eV = 1$ MeV

This energy is much more than rest mass energy (0.51 MeV) of electron. Hence it is relativistic problem.

$$\Rightarrow \quad \frac{eV}{m_0C^2} + 1 = \frac{1}{\sqrt{1 - V^2/C^2}}$$

$$\Rightarrow \quad 1 - V^2/C^2 = \frac{1}{\left(\frac{eV}{m_0C^2} + 1\right)^2} = \frac{1}{\left(\frac{1.6 \times 10^{-19} \times 10^6}{9.1 \times 10^{-31} \times 9 \times 10^{16}} + 1\right)^2}$$

$$= 0.115$$

$$\Rightarrow \quad \frac{V^2}{C^2} = 1 - 0.115 = 0.885$$

$$\Rightarrow \quad V = \sqrt{0.885}\, C = 0.94\, C$$

Exercise 6. Calculate the velocity, which a particle of mass 1 a.m.u. will have, when its kinetic energy is equal to three times the rest mass energy. [Ans. 0.968 C]

Exercise 7. What is the error involved in calculating the kinetic energy of the particle according to classical physics and according to relativistic physics, when it is moving with a speed of 0.5 C ? [Ans. 19.4%]

Example 25. *Show that Lorentz transformation can be put in the form*

$$x' = x \cos h\,\theta - Ct \sin h\,\theta$$
$$Ct' = -x \sin h\,\theta + Ct \cos h\,\theta$$

where θ is a parameter defined by $\frac{V}{C} = \tan h\,\theta$.

Solution. According to Lorentz Transformations

$$x' = \frac{x - Vt}{\sqrt{1 - V^2/C^2}} \quad ...(1)$$

and
$$t' = \frac{t - \frac{Vx}{C^2}}{\sqrt{1 - V^2/C^2}} \quad ...(2)$$

Given
$$\frac{V}{C} = \tan h\,\theta$$

$\therefore$
$$1 - \frac{V^2}{C^2} = 1 - \tan h^2\,\theta = \sec h^2\,\theta \qquad (\because \sec h^2\,\theta + \tan h^2\,\theta = 1)$$

$\therefore$ (1) $\Rightarrow$
$$x' = \frac{x - C(\tan h\,\theta)\,t}{\sqrt{\sec h^2\,\theta}} \qquad (\because V = \tan h\theta)$$

$$= \cos h\theta\left[x - \frac{Ct\sin h\theta}{\cos h\theta}\right] = \cos h\theta\left[\frac{x\cos h\theta - Ct\sin h\theta}{\cos h\theta}\right]$$

$\Rightarrow$
$$x' = x\cos h\theta - Ct\sin h\theta$$

Also (2) $\Rightarrow$
$$t' = \frac{t - \frac{(C\tan h\theta)\,x}{C^2}}{\sqrt{\sec h^2\,\theta}} = \cos h\theta\left[t - \frac{x}{C}\frac{\sin h\theta}{\cos h\theta}\right]$$

$$= \frac{Ct\cos h\theta - x\sin h\theta}{C}$$

$\Rightarrow$
$$Ct' = -x\sin h\theta + Ct\cos h\theta$$

Example 26. *A particle is subjected to a constant force for time t. Show that the speed of particle cannot exceed speed of light irrespective of value of t.*

Solution. The force acting on particle is given as

$$F = \frac{dp}{dt}$$

Thus momentum gained by particle in time t is

$$p = \int_0^t dp = \int_0^t F\,dt = Ft \qquad ...(1)\ (\because F = \text{constant})$$

But
$$p = mV = \frac{m_0 V}{\sqrt{1 - V^2/C^2}} \quad ...(2)$$

Equate (1) and (2)

$\Rightarrow$
$$\frac{m_0 V}{\sqrt{1 - V^2/C^2}} = Ft$$

$\Rightarrow$
$$m_0^2 V^2 = F^2 t^2 \left(1 - \frac{V^2}{C^2}\right)$$

$\Rightarrow$
$$\left(m_0^2 + \frac{F^2 t^2}{C^2}\right) V^2 = F^2 t^2$$

$\Rightarrow$
$$V^2 = \frac{F^2 t^2}{m_0^2 + \frac{F^2 t^2}{C^2}} = C^2\left[\frac{F^2 t^2}{F^2 t^2 + m_0^2 C^2}\right]$$

$$\Rightarrow \quad V = \frac{C}{\sqrt{1 + \frac{m_0^2 C^2}{F^2 t^2}}}$$

This equation shows that speed of particle will always be less than C, however finite larger t may be.

Example 27. *Show that rest mass of a particle is given by*

$$m_0 = \frac{p^2 C^2 - T^2}{2TC^2}$$

where p and T denote momentum and kinetic energy of particle.

Solution. We know total energy is equal to sum of rest mass energy and kinetic energy

i.e. $$E = m_0 C^2 + T \qquad ...(1)$$

Also $$E = \sqrt{p^2 C^2 + m_0^2 C^4} \qquad ...(2)$$

Equate (1) and (2)

$$\Rightarrow \quad p^2 C^2 + m_0^2 C^4 = m_0^2 C^4 + T^2 + 2T\, m_0 C^2$$

$$\Rightarrow \quad p^2 C^2 - T^2 = 2T\, m_0 C^2$$

$$\Rightarrow \quad m_0 = \frac{p^2 C^2 - T^2}{2TC^2}$$

Example 28. *On her 20th birthday, a young lady decides that she will like to remain 20 years atleast for 10 years. She decides to go on a journey into outer space with uniform velocity. What is the minimum speed, with which she should travel w.r.t. earth, so that when she returns after 10 years, she can still say, quite truthfully, that she is only 20 ?*

Solution. This is possible if she returns to the earth on the same day on calender in the spaceship.

Thus time spent on spaceship should be 1 day and it should appear as 10 years on earth.

Thus $$\tau_0 = 1 \text{ day}$$

and $$\tau = 10 \text{ years} = 3650 \text{ days}$$

Now $$\tau = \frac{\tau_0}{\sqrt{1 - V^2/C^2}}$$

$$\Rightarrow \quad 3650 = \frac{1}{\sqrt{1 - V^2/C^2}}$$

$$\Rightarrow \quad \frac{V^2}{C^2} = 1 - \frac{1}{3650} = 0.9997$$

$$\Rightarrow \quad V = \sqrt{0.9997}\, C$$

$$V = 0.9998\, C$$

Example 29. *A clock keeps correct time. With what speed should it be moved so that it may loose 4 minutes in one day ?*

Solution. Here τ_0 = Proper time = 24 hrs = 1440 min

τ = time measured by a stationary clock on earth

= 1444 min.

Now $$\tau = \frac{\tau_0}{\sqrt{1-V^2/C^2}}$$

$$\Rightarrow \quad 1444 = \frac{1440}{\sqrt{1-V^2/C^2}}$$

$$\Rightarrow \quad V = 0.0745\,C$$

Example 30. *A particle of mass M disintegrates while at rest into two parts having masses $\frac{M}{2}$ and $\frac{M}{4}$. Show that their relativistic kinetic energies are $\frac{3Mc^2}{32}$ and $\frac{5Mc^2}{32}$ respectively.*

Solution. Let T_1 and T_2 be relativistic kinetic energy of mass $\frac{M}{2}$ and $\frac{M}{4}$ respectively. Since initially the particle of mass M was at rest so linear momentum before distintegration was zero. So in order to conserve linear momentum, the two particles must posses equal and opposite momenta *i.e.*

$$\vec{p}_1 = -\vec{p}_2 = \vec{p} \text{ (say)}$$

or $$p_1 = p_2 = p \quad \text{(in magnitude)} \qquad ...(i)$$

Further the kinetic energy of two fragments is due to decrease in mass of body at the time of desintegration *i.e.*

$$T_1 + T_2 = \Delta Mc^2$$

$$= \left(M - \left(\frac{M}{2} + \frac{M}{4}\right)\right)c^2 = \frac{M}{4}c^2. \qquad ...(ii)$$

Let E_1, E_2 are total energies of $\frac{M}{2}$ and $\frac{M}{4}$ respectively.

Thus according to energy momentum relation, we have

$$E_1 = T_1 + \frac{M}{2}c^2$$

or $$\sqrt{p_1^2c^2 + \left(\frac{M}{2}\right)^2 c^4} = T_1 + \frac{M}{2}c^2$$

or $$p^2c^2 + \frac{M^2c^4}{4} = \left(T_1 + \frac{M}{2}c^2\right)^2$$

or $$p^2c^2 + \frac{M^2c^4}{4} = T_1^2 + \frac{M^2c^4}{4} + Mc^2T_1$$

$$p^2c^2 = T_1^2 + Mc^2T_1 \qquad ...(iii)$$

Similarly for particle $\frac{M}{4}$, we have

$$E_2 = T_2 + \frac{M}{4}c^2$$

$$\Rightarrow \quad \sqrt{p_2^2c^2 + \left(\frac{M}{4}\right)^2 c^4} = T_2 + \frac{M}{4}c^2$$

$$\Rightarrow \quad \sqrt{p^2c^2 + \frac{M^2c^4}{16}} = \frac{Mc^2}{4} - T_1 + \frac{M}{4}c^2 \qquad \text{[Using (i) and (ii)]}$$

$$\Rightarrow \quad \sqrt{p^2c^2+\frac{M^2c^4}{16}} = \frac{Mc^2}{2}-T_1$$

$$\Rightarrow \quad p^2c^2+\frac{M^2c^4}{16} = \frac{M^2c^4}{4}+T_1^2-Mc^2T_1$$

$$\Rightarrow \quad T_1^2+Mc^2T_1+\frac{M^2c^4}{16} = \frac{M^2c^4}{4}+T_1^2-Mc^2T_1 \qquad \text{[Using (iii)]}$$

$$\Rightarrow \quad 2Mc^2T_1 = M^2c^4\left(\frac{1}{4}-\frac{1}{16}\right)$$

$$\Rightarrow \quad 2T_1 = \frac{3}{16}Mc^2$$

$$\Rightarrow \quad T_1 = \frac{3}{32}Mc^2$$

put in (*ii*), we get

$$T_2 = \frac{M}{4}c^2-T_1 = \frac{M}{4}c^2-\frac{3}{32}Mc^2$$

$$= \left(\frac{1}{4}-\frac{3}{32}\right)Mc^2$$

$$= \frac{5}{32}Mc^2$$

Example 31. *An event occurs in the frame S at t = 1 ms at x = 5 km. The position of the point of occurrence of event in frame S' appears to be x' = 35 km. Find the time of occurrence of the event (t') in the frame S'.*

Solution. Given

$$t = 1\text{ ms} = 10^{-3}\ s$$

$$x = 5\text{ km} = 5\times10^3\ m$$

$$x' = 35\text{ km} = 35\times10^3\ m$$

$$t' = ?$$

According to Lorentz Transformation

$$x' = \frac{x-Vt}{\sqrt{1-\frac{V^2}{C^2}}}$$

$$\Rightarrow \quad 35\times10^3 = \frac{5\times10^3-V\times10^{-3}}{\sqrt{1-\frac{V^2}{9\times10^{16}}}}$$

$$\Rightarrow 1.225\times10^9\left(1-\frac{V^2}{9\times10^{16}}\right) = (5\times10^3-V\times10^{-3})^2$$

$$\Rightarrow 1.225\times10^9-1.36\times10^{-8}V^2 = 25\times10^6+V^2\times10^{-6}-10V$$

$$\Rightarrow \quad (10^{-6}+1.36\times10^{-8}) - 10V + (25\times10^6-1.225\times10^9) = 0$$

$$\Rightarrow \quad 1.0136\times10^{-6}V^2-10V-1.2\times10^9 = 0$$

$$\Rightarrow \quad V = \frac{10\pm\sqrt{100-4(1.0136\times10^{-6})(-1.2\times10^9)}}{2\times1.0136\times10^{-6}}$$

$$= \frac{10 \pm 70.46}{2 \times 1.0136 \times 10^{-6}}$$

$$= \frac{10 + 70.46}{2 \times 1.0136 \times 10^{-6}} = 3.97 \times 10^{7}\ ms^{-1}$$

Again from Lorentz Transformation

$$t' = \frac{t - \frac{Vx}{c^2}}{\sqrt{1 - \frac{V^2}{c^2}}} = \frac{10^{-3} - \frac{3.97 \times 10^7 \times 5 \times 10^3}{9 \times 10^{16}}}{\sqrt{1 - \left(\frac{3.97 \times 10^7}{3 \times 10^8}\right)^2}}$$

$$= (10^{-3} - 2.21 \times 10^{-1.6}) \times 1.009$$

$$= 1.0068 \times 10^{-3}\ s = 1.0068\ ms$$

Exercise 8. What is the velocity of π-mesons, whose proper mean life is $2.5 \times 10^{-8} s$ and observed mean life is $2.5 \times 10^{-7} s$. [Ans. 0.995 C]

Exercise 9. A person in a train moving at $3 \times 10^7\ m/s$ sleeps at 10 : 00 p.m. by his watch and gets up at 5 a.m. How long did he sleep according to the clocks at the station ? [Ans. 7.04 h]

Exercise 10. Two events separated by a distance of 15 m and time interval of $3 \times 10^{-8} s$ and occur in frame S. Find the distance and time separations between two events as observed from frame S′ moving with speed $\frac{C}{3}$ along the direction of 15m line.

$$\left[\begin{array}{l}\text{Ans. } \Delta x' = 12.63\ m, \\ \qquad \Delta t' = 3.18 \times 10^{-8}\ s\end{array}\right]$$

SHORT ANSWER TYPE QUESTIONS

Q. 1. What is the meaning of optical path of ray of light ?

Ans. Let a ray of light travels for time t in a medium of refractive index μ. The speed of light in this medium is given by $V = \frac{C}{\mu}$. The distance travelled by light in medium in time t is given by $x = Vt = \frac{Ct}{\mu}$

Also in same time t, light can travel a distance $x' = Ct$ in vacuum. Thus $x = \frac{x'}{\mu}$

or $$x' = \mu x$$

Hence distance x travelled by a ray of light in a medium is analogous to distance μx travelled by light in vacuum. This distance μx is called optical path. During the interference, it is the optical path difference that is considered than geometrical path difference.

Q. 2. Why the apparatus of Michelson Moreley experiment was rotated by 90° ?

Ans. For observing fring shift, the change in the path difference is must. Since path difference $S = \frac{lV^2}{C^2}$ depends on velocity V of earth and we cannot alter the velocity of earth. Thus we can change δ only by rotating the apparatus. Because with this the direction of light beam is altered as compared to direction of motion of earth, so that δ becomes negative, and we may expect to observe fringe shift.

Q. 3. What are limitations of Galilean Transformation ?

Ans. (*i*) Velocity of light is not is invariant in it.

(*ii*) Maxwell's equations are not invariant.

(*iii*) Transformation is valid only at low speeds.

(*iv*) Time is considered to be an absolute quantity in these transformations.

Q. 4. What are absolute quantities ? Name three absolute quantities in (*a*) Classical Newtonian Mechanics (*b*) Relativistic Mechanics.

Ans. The quantities which have same value in all frames of reference are called absolute quantities. (*a*) In newtonian mechanics, the absolute quantities are mass length and time. (*b*) In relativistic mechanics, the absolute quantities are velocity of light in free space, phase of wave and charge.

Q. 5. Does the postulate of constancy of speed of light hold for inertial frames only ?

Ans. No, the speed of light is an absolute quantity and it remains same in all frames of refrence whether inertial or non inertial.

Q. 6. What will happen to special theory of relativity, if speed of light becomes infinite?

Ans. According to Lorentz Transformations

$$x' = \frac{x - vt}{\sqrt{1 - V^2/C^2}}$$

$$y' = y$$

$$z' = z$$

$$t' = \frac{t - \dfrac{Vx}{C^2}}{\sqrt{1 - V^2/C^2}}$$

If $C \to \infty$ then above equations transform to

$$x' = x - Vt$$

$$y' = y$$

$$z' = z$$

$$t' = t$$

These are Galilean Transformation equations. Hence special theory of relativity and old Newtonian mechanics (Galilean Transformations) will become identical.

Q. 7. If an observer sits on photon, will the photon appear to be at rest to him ?

Ans. No, because speed of light is an absolute quantity independent of state of motion of observer. Thus to the observer, photon will appear to be moving with velocity C.

Q. 8. A circular ring is moving in a direction along (*i*) one of its diameter (*ii*) Normal of its plane (*i.e.* along its axial line). What will be the shape of the ring in both cases as observed in laboratory frames.

Ans. (*i*) In this case the length contraction will take place along the direction of motion. Thus the diameter of ring along the motion will be smaller than any other diameter, and diameter normal to the direction of motion will remain unchanged. Hence shape will appear to be ellipse.

(*ii*) In this case the circumferencial length of the ring is perpendicular to the direction of motion. Hence there will be no length contraction. Thus shape remains uncharged.

Q. 9. Who is observer in relativity ?

Ans. In relativity, an observer is an infinite set of recording clocks distributed throghout the space at rest and synchronized w.r.t each other.

Q. 10. Name the transformation under which Maxwell's equations are invariant.

Ans. Maxwell's equations are invariant under Lorentz Transformations.

Q. 11. Why electrons cannot be accelerated using cyclotron ?

Ans. The mass of a particle moving with velocity V is given as

$$m = \frac{m_0}{\sqrt{1 - V^2/C^2}} \qquad ...(1)$$

The change in mass with respect to velocity V is given as

$$\frac{dm}{dV} = m_0 \left(\frac{-1}{2}\right)\left[1 - \frac{V^2}{C^2}\right]^{-3/2} \times \left(\frac{-2V}{C^2}\right)$$

$$= \frac{m_0 V}{C^2 \left[1 - V^2/C^2\right]^{3/2}}$$

We see that rate of increase of mass increases with velocity of particle $\left(\because \frac{dm}{dV} \propto V\right)$. Thus if an electron is accelerated using cyclotron, then with velocity the rate of increase of mass will also increase and soon the increase in mass will be comparable to the rest mass of electron. Due to this the electron will take more time to cover circular path in a dee. But the polarity of dees will reverse after constant time interval. Hence soon resonance between circular motion of electron & the oscillator is broken and electron will get deaccelerated by the dees because of this.

Q. 12. Why is compensating glass plate used in Michelson Moreley Experiment ?

Ans. It is introduced, so that optical paths of both the reflected and transmitted beam are same.

Q. 13. How relativistic motion can affect colour perception of human eye ?

Ans. We know that Doppler Effect is shown by light, when there is relative motion between source of light and observer. If source and observer are approaching each other, then due to Doppler Effect, the observed frequency of light increases *e.g.* if a vehicle is approaching (with speed comparable to speed of lgiht) towards a red light, then frequency of light observed by driver will be more than actual frequency and to him, the red signal may appear green.

Q. 14. Why cannot a material body move with speed of light ?

Ans. Let a particle has rest mass m_0, then its mass in motion is given by $m = \frac{m_0}{\sqrt{1 - V^2/C^2}}$.

If $m_0 \neq 0$ and $V = C$ we see that $m = \infty$.

Thus mass of particle tends to be infinite. Hence no material particle can travel with speed of light.

On the other hand if $m_0 = 0$ then if we put V = C, thus we get $m = \frac{0}{0}$, which has infinite solutions. Hence a photon can travel with speed of light.

Q. 15. Why are the consequences of special theory of relativity not apparent in daily observations ?

Ans. Because the objects in daily life are moving at a speed much small as compared to speed of light.

QUESTIONS

1. Define an inertial frame of reference. Does an inertial frame exist ?
2. Derive an expression for variation of mass of a body with its speed.
3. Describe in detail the Michelson Moreley Experiment.
4. What are Einstein's postulates of relativity ?
5. Derive the equations of Lorentz Transformation.
6. Explain why Galilean Transformation is inadequate ?
7. Explain length contraction and obtain an expression for contracted length.
8. Show that two simultaneous events at different positions in a frame of reference are not in general simultaneous in another inertial frame in relative motion.
9. Obtain the relativistic formula for addition of velocities. Hence show that the velocity of light is absolute constant independent of the frame of reference and that the addition of any velocity to the velocity of light merely reproduces the velocity of light.
10. Derive an expression for mass energy equivalence. Discuss its implications. Give some evidence, showing its validity.
11. What are massless particles ? Show that massless particles can exist only if they move with the speed of light and that their energy E and momentum p must have the relation $E = pC$.
12. Show that simultaneity is relative.
13. What do you understand by time dilation ? Define proper time interval and obtain an expression for it.
14. Briefly state the significance of Michelson Moreley experiment.
15. What is Twin Paradox ? How it is resolved ?