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Differential Calculus

The Chapter of Differential Calculus is divided into two Sections—Section A : Curve Tracing and Section B : Curvature.

SECTION A: CURVE TRACING

1.1. INTRODUCTION

Students are already familiar with the idea of tracing a curve by giving different values to x and finding corresponding values of y and joining these points by free hand curves. But this type of tracing gives us only a rough idea of what the graph looks like. But to have more accuracy in sketching the curves we will now use the techniques of domains, intervals of increase and decrease of the curve, maxima, minima, concepts of concavity and convexity, asymptotes etc. Many of these techniques are already studied in details by the students in lower classes but some of these like concavity convexity, asymptotes, multiple points are new to them. So first of all we will have discussion about these new topics.

1.2. CONCAVITY, CONVEXITY, POINT OF INFLEXION

Def. (a) Concave upward or convex downward

Let $y = f(x)$ be any continuous curve in (a, b).

A curve is said to concave upward (or convex downward) on (a, b) if all the points of the curve lie above any tangent to it on that interval [See Fig. (i)].

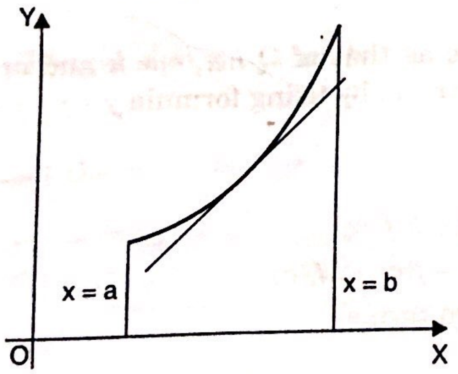


Fig. (i)

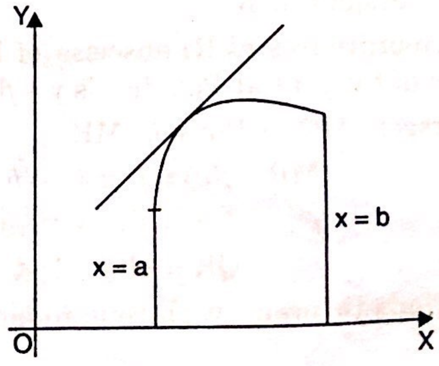


Fig. (ii)

(b) Concave downward or Convex upward

A curve is said to be concave downward (or convex upward) on (a, b) if all the points of the curve lies below any tangent to it on that interval [See Fig. (ii)].

(c) Point of Inflexion (*P.T.U., Dec. 2005, Dec. 2007*)

A point on the curve is said to be a point of inflexion if the curve crosses the tangent at that point *i.e.*, two portions of the curve on the two sides of P lie on different sides of the tangent at that point [See Fig. (*iii*)].

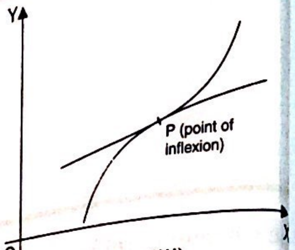


Fig. (*iii*)

1.3. CRITERIA FOR CONCAVITY, CONVEXITY AND INFLEXION

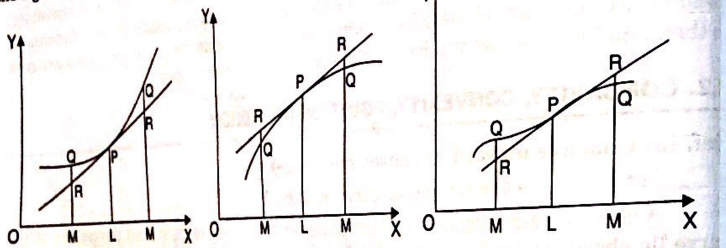
To determine whether a curve $y = f(x)$ is concave upward, concave downward or has a point of inflexion; we take P(c, $f(c)$) any point on the curve such that $f''(c)$ is finite then

(*i*) Curve is concave upward or convex downward at P if $f''(c) > 0$

(*ii*) Curve is concave downward or convex upward at P if $f''(c) < 0$

(*iii*) Curve has a point of inflexion at P if $f''(c) = 0$ but $f'''(c) \neq 0$.

Proof. Let Q($c + h$, $f(c + h)$) be another point on the curve very near to P. Q will lie on the right hand side of P if $h > 0$ and will be on the left hand side of P if $h < 0$ (See Figs.).



Draw tangent at P and perpendiculars from P and Q on OX ; Let QM ; the perpendicular from Q meets tangent at R.

(Find co-ordinates of R) abscissa of R is same as that of Q *i.e.*, $c + h$ and ordinate is MR. Equation of tangent at P(c, $f(c)$) is $y - f(c) = f'(c)(x - c)$ by using formula $y - y_1 = m(x - x_1)$.

It intersects MQ at R($c + h$, MR)

$\therefore$ $\quad$ $MR - f(c) = f'(c)(c + h - c)$

$MR = f(c) + f'(c) \cdot h = f(c) + h f'(c)$

Now $\quad$ $QR = MQ - MR = f(c + h) - f(c) - hf'(c)$ $\quad$ ...(1)

By Taylor's theorem with remainder after two terms

$$f(c + h) = f(c) + hf'(c) + \frac{h^2}{2!} f''(c + \theta h) \quad \text{where } 0 < \theta < 1$$

$\therefore$ *From (1),*

$$QR = \frac{h^2}{2!} f''(c + \theta h) \qquad 0 < \theta < 1 \quad ...(2)$$

Case I. Let $\quad f''(c) > 0 \quad \because \quad f''(x)$ is continuous at c

$\therefore \quad f''(x) > 0 \quad \forall \; x$ in the neighbourhood of c

$c + \theta h \in$ neighbourhood of c

$\therefore \quad f''(c + \theta h) > 0, \quad \frac{h^2}{2!}$ is +ve

$\therefore$ From (2) QR is +ve

$\therefore$ Hence curve is concave upward at P if $f''(c) > 0$.

Case II. $\quad$ If $f''(c) < 0$

Then $\quad f''(c + \theta h) < 0, \; \frac{h^2}{2!}$ is positive

$\therefore$ QR is negative

Hence curve is concave downward at P.

Case III. Let $\; f''(c) = 0 \;$ but $\; f'''(c) \neq 0$

Then again by Taylor's theorem with remainder after three terms.

$$f(c + h) = f(c) + h f'(c) + \frac{h^2}{2!} f''(c) + \frac{h^3}{3!} f'''(a + \theta_1 h) \; ; \; 0 < \theta_1 < 1$$

$\because \quad f''(c) = 0$

$$\therefore \quad f(c + h) = f(c) + h f'(c) + \frac{h^3}{3!} f'''(a + \theta_1 h)$$

From (1) $\quad QR = \frac{h^3}{3!} f'''(a + \theta_1 h)$ where $0 < \theta_1 < 1$

Suppose $f'''(x)$ is also continuous at c

$\therefore \quad f'''(x)$ and $f'''(c)$ have the same sign

$\therefore \quad f'''(c + \theta_1 h)$ has the same sign as that of $f'''(c)$.

But $\frac{h^3}{3!}$ is +ve when h is +ve and is –ve when h is –ve

i.e., $\frac{h^3}{3!}$ changes sign with the change in the sign of h *i.e.,* QR changes sign with the change in sign of h.

Hence the curve has point of inflexion at P if $f''(c) = 0$ and $f'''(c) \neq 0$.

1.4. (a) WORKING RULE TO FIND CONCAVITY AND CONVEXITY OF A CURVE IN AN INTERVAL (a, b)

(1) Evaluate $\frac{d^2y}{dx^2}$.

(2) Find the interval (a, b) for which $\frac{d^2y}{dx^2} > 0$; then (a, b) is the interval for the curve being concave upward or convex downward.

(3) Find the interval (a, b) for which $\frac{d^2y}{dx^2} < 0$; then (a, b) is the interval for the curve being concave downward or convex upward.

1.4. (b) WORKING RULE TO FIND THE POINTS OF INFLEXION

(1) Evaluate $\dfrac{d^2y}{dx^2}$.

(2) Find the values of x for which $\dfrac{d^2y}{dx^2} = 0$ and also the values of x (if any) where $\dfrac{d^2y}{dx^2}$ does not exist.

(3) All these values of x obtained in step (2) will be expected points of inflexion.

(4) Let $x = a$ be any one of the values.

It will be the point of inflexion if

$$\frac{d^3y}{dx^3} \neq 0 \quad \text{at} \quad x = a.$$

or

$$\frac{d^2y}{dx^2} \text{ changes sign at } x = a.$$

Note. For point of inflexion $\dfrac{d^2y}{dx^2} = 0$ but converse is not true *i.e.*, if $\dfrac{d^2y}{dx^2} = 0$ at $x = a$ then $x = a$ may not be the point of inflexion.

1.5. INCREASING AND DECREASING FUNCTION

A function $y = f(x)$ is said to increasing in an interval (a, b) if $f'(x) > 0 \ \forall \ x \in (a, b)$ and decreasing in an interval (a, b) if $f'(x) < 0 \ \forall \ x \in (a, b)$

e.g., if

$$y = x^3 - 3x + 2$$

then

$$\frac{dy}{dx} = 3x^2 - 3 = 3(x^2 - 1)$$

$y = f(x)$ will be increasing in the interval where

$$f'(x) = \frac{dy}{dx} > 0 \text{ i. e., } x^2 - 1 > 0 \quad \text{or} \quad x^2 > 1 \text{ or } |x| > 1$$

i.e., either $x > 1$ or $x < -1$

i.e., function increases in the interval $(-\infty, -1) \cup (1, \infty)$ and function decreases when $x^2 - 1 < 0$

i.e., $x^2 < 1$ or $|x| < 1$ or $-1 < x < 1$

i.e., function decreases in the interval $(-1, 1)$.

ILLUSTRATIVE EXAMPLES

Example 1. *Find the values of x for which $y = x^4 - 6x^3 + 12x^2 + 5x + 7$ is concave upwards or downwards. Also determine the points of inflexion.*

Sol.

$$y = x^4 - 6x^3 + 12x^2 + 5x + 7$$

$$\frac{dy}{dx} = 4x^3 - 18x^2 + 24x + 5$$

$$\frac{d^2y}{dx^2} = 12x^2 - 36x + 24$$

Curve will concave upward in the interval for which

i.e., $\dfrac{d^2y}{dx^2} > 0$

or $12x^2 - 36x + 24 > 0$

or $x^2 - 3x + 2 > 0$

or $\left(x - \dfrac{3}{2}\right)^2 > \dfrac{9}{4} - 2$

i.e., $\left(x - \dfrac{3}{2}\right)^2 > \dfrac{1}{4}$ or $\left|x - \dfrac{3}{2}\right| > \dfrac{1}{2}$

or

i.e., either $x - \dfrac{3}{2} > \dfrac{1}{2}$ or $x - \dfrac{3}{2} < -\dfrac{1}{2}$

$x > 2$ or $x < 1$

$x \in (-\infty, 1) \cup (2, \infty)$

$\therefore$ Curve is concave upward in $(-\infty, 1) \cup (2, \infty)$

Curve will concave down if $\dfrac{d^2y}{dx^2} < 0$

i.e., $x^2 - 3x + 2 < 0$

or $\left|x - \dfrac{3}{2}\right| < \dfrac{1}{2}$

or $\dfrac{3}{2} - \dfrac{1}{2} < x < \dfrac{3}{2} + \dfrac{1}{2}$ or $1 < x < 2$

$\therefore$ $x \in (1, 2)$

$\therefore$ Curve is concave downward in $(1, 2)$.

For points of inflexion ;

$$\frac{d^2y}{dx^2} = 0 \quad \text{or} \quad \frac{d^2y}{dx^2} \text{ does not exit.}$$

i.e., $x^2 - 3x + 2 = 0$

or $x = 1, 2$

$$\frac{d^3y}{dx^3} = 24x - 36$$

At $x = 1$, $\dfrac{d^3y}{dx^3} = -12 \neq 0$; for $x = 1, y = 19$ | from (1)

$\therefore$ (1, 19) is the point of inflexion.

At $x = 2$, $\dfrac{d^3y}{dx^3} = 48 - 36 = 12 \neq 0$ for $x = 2, y = 23$

$\therefore$ (2, 23) is also the point of inflexion.

Hence (1, 19) and (2, 23) are two points of inflexion.

Example 2. *Find the points of inflexion of the following curves:* **(P.T.U., Dec. 2005)**

(i) $y = x^3 + 8x^2 - 270x$

(ii) $x = 3y^4 - 4y^3 + 5$.

Sol. (*i*)

$$y = x^3 + 8x^2 - 270x$$

$$\frac{dy}{dx} = 3x^2 + 16x - 270$$

$$\frac{d^2y}{dx^2} = 6x + 16$$

$$\frac{d^3y}{dx^3} = 6$$

$$\frac{d^2y}{dx^2} = 0 \text{ gives } x = -\frac{16}{6} = -\frac{8}{3}$$

and at $x = -\frac{8}{3}$, $\frac{d^3y}{dx^3} \neq 0$

$\therefore$ $x = -\frac{8}{3}$ gives the point of inflexion.

From (1), $y = -\frac{512}{27} + \frac{512}{9} + \frac{2160}{3} = \frac{20464}{27}$

$\therefore$ Point of inflexion is $\left(-\frac{8}{3}, \frac{20464}{27}\right)$.

(*ii*) $x = 3y^4 - 4y^3 + 5$

$$\frac{dx}{dy} = 12y^3 - 12y^2 = 12(y^3 - y^2)$$

$$\frac{d^2x}{dy^2} = 12(3y^2 - 2y)$$

$$\frac{d^3x}{dy^3} = 12(6y - 2) = 24(3y - 1)$$

for points of inflexion $\frac{d^2x}{dy^2} = 0 \Rightarrow 3y^2 - 2y = 0$ which gives $y = 0, y = \frac{2}{3}$

for both the values of y, $\frac{d^3x}{dy^3} \neq 0$ $\therefore$ $y = 0, y = \frac{2}{3}$ gives points of inflexion.

$\therefore$ Points of inflexion are $(5, 0)$ and $\left(\frac{119}{27}, \frac{2}{3}\right)$.

Example 3. *Find points of inflexion of the curve*
$x = a(2\theta - \sin\theta)$, $y = a(2 - \cos\theta)$.

Sol. $x = a(2\theta - \sin\theta)$, $y = a(2 - \cos\theta)$

$$\frac{dx}{d\theta} = a(2 - \cos\theta), \quad \frac{dy}{d\theta} = a\sin\theta$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\sin\theta}{a(2-\cos\theta)} = \frac{\sin\theta}{2-\cos\theta}$$

$$\frac{d^2y}{dx^2} = \frac{(2-\cos\theta)\cos\theta - \sin\theta(\sin\theta)}{(2-\cos\theta)^2}\frac{d\theta}{dx}$$

$$= \frac{2\cos\theta - \cos^2\theta - \sin^2\theta}{(2-\cos\theta)^2} \cdot \frac{1}{a(2-\cos\theta)} = \frac{2\cos\theta - 1}{a(2-\cos\theta)^3}$$

$\frac{d^2y}{dx^2} = 0$ when $2\cos\theta - 1 = 0$ $\therefore$ $\cos\theta = \frac{1}{2} = \cos\frac{\pi}{3}$

$\therefore$ $\theta = 2n\pi \pm \frac{\pi}{3}$; $n \in N$

Now $\frac{d^2y}{dx^2} < 0$ if $\cos\theta < \frac{1}{2}$ $\quad | \because$ $2 - \cos\theta$ is +ve as $\cos\theta < 1$

and $\frac{d^2y}{dx^2} > 0$ if $\cos\theta > \frac{1}{2}$

$\therefore$ $\frac{d^2y}{dx^2}$ changes sign at $\theta = 2n\pi \pm \frac{\pi}{3}$

$\therefore$ $x = a\left[4n\pi \pm \frac{2\pi}{3} \mp \frac{\sqrt{3}}{2}\right], y = \frac{3a}{2}$

$\therefore$ $\left\{a\left(4n\pi \pm \frac{2\pi}{3} \mp \frac{\sqrt{3}}{2}\right), \frac{3a}{2}\right\}$ are points of inflexion.

Example 4. *Determine a, b so that the curve $y = ax^3 + 3bx^2$ has a point of inflexion at (–1, 2).*

Sol.

$$y = ax^3 + 3bx^2 \quad \text{...(1)}$$

$$\frac{dy}{dx} = 3ax^2 + 6bx$$

$$\frac{d^2y}{dx^2} = 6ax + 6b \quad \text{...(2)}$$

$$\frac{d^3y}{dx^3} = 6a \quad \text{...(3)}$$

$\because$ P(–1, 2) is the point of inflexion.

$\therefore$ (*i*) P(–1, 2) lies on the curve (1) (*ii*) at (–1, 2) $\frac{d^2y}{dx^2} = 0$ (*iii*) $\frac{d^3y}{dx^3}$ at $(-1, 2) \neq 0$

From (1) $2 = -a + 3b$

From (2) $-6a + 6b = 0$ $\therefore$ $a = b$

$\therefore$ $2 = -a + 3a$ $\therefore$ $2a = 2$ $\therefore$ $a = 1$

$\therefore$ $a = 1, b = 1$

$$\frac{d^3y}{dx^3} = 6a = 6 \neq 0$$

$\therefore$ We have $a = 1, b = 1$

Example 5. *Find the points of inflexion of the curve* $y = \dfrac{x^2+1}{x^2-1}$. *Also find the value x for which function is concave upward and concave downward.*

Sol.
$$y = \frac{x^2+1}{x^2-1}$$
$$\frac{dy}{dx} = \frac{(x^2-1)2x-(x^2+1)2x}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2}$$
$$\frac{d^2y}{dx^2} = -4\frac{(x^2-1)^2 - x\cdot 2(x^2-1)2x}{(x^2-1)^4}$$
$$= \frac{-4[x^4-2x^2+1-4x^4+4x^2]}{(x^2-1)^4} = \frac{4(3x^4-2x^2-1)}{(x^2-1)^4}$$
$$= \frac{4(x^2-1)(3x^2+1)}{(x^2-1)^4} = \frac{4(3x^2+1)}{(x^2-1)^3}$$

The expected points of inflexion can be the values of x for which $\dfrac{d^2y}{dx^2} = 0$ but $\dfrac{d^3y}{dx^3}$ and also the values of x for which $\dfrac{d^2y}{dx^2}$ does not exist. $\dfrac{d^2y}{dx^2} = 0$ gives $3x^2+1=0$ which does give any real value of x. Secondly the points at which $\dfrac{d^2y}{dx^2}$ does not exist are given by $x =$ but these two points do not belong to the domain of the given function ∴ these points can be considered for points of inflexion.

∴ $f(x)$ curve has no points of inflexion.

Now curve will concave upward if $\dfrac{d^2y}{dx^2} > 0$

i.e., $\dfrac{4(3x^2+1)}{(x^2-1)^3} > 0$ i.e., $x^2-1>0$

or $x^2 > 1$ or $|x| > 1$ i.e., either $x > 1$ or $x <$

∴ Curve concave upwards in the interval $(-\infty, -1) \cup (1, \infty)$.

Curve will concave downward if $\dfrac{d^2y}{dx^2} < 0$

i.e., $x^2 - 1 < 0$ ∴ $x^2 < 1$ or $|x| < 1$

i.e., $-1 < x < 1$

∴ Curve concaves downward is the interval $(-1, 1)$.

Example 6. *Find the points of inflexion (if any) of the curve* $y = (x^3 - 6x^2 + 9x + 6)/6$. **(P.T.U., May 200**

Sol. Equation of the curve is
$$y = \frac{1}{6}(x^3 - 6x^2 + 9x + 6)$$
$$\frac{dy}{dx} = \frac{1}{6}(3x^2 - 12x + 9)$$
$$\frac{d^2y}{dx^2} = \frac{1}{6}(6x - 12) = x - 2$$

$$\frac{d^3y}{dx^3} = 1$$

For points of inflexion $\dfrac{d^2y}{dx^2} = 0$ but $\dfrac{d^3y}{dx^3} \neq 0$ at those points.

$\dfrac{d^2y}{dx^2} = 0$ gives $x - 2 = 0$ i.e., $x = 2$ and $\dfrac{d^3y}{dx^3} \neq 0$

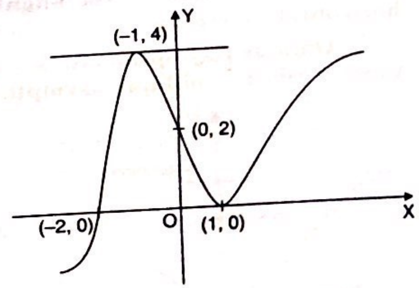
∴ $x = 2$ is the point of inflexion and for $x = 2, y = \dfrac{1}{6}(8 - 24 + 18 + 6) = \dfrac{4}{3}$

∴ Coordinates of the point of inflexion are $\left(2, \dfrac{4}{3}\right)$.

Example 7. *As x moves from left to right through the point c = 2, is the graph of* $f(x) = x^3 - 3x + 2$ *rising or is it falling ? Give reason for your answer.* **(P.T.U., Dec. 2004)**

Sol. Given $f(x) = x^3 - 3x + 2$
$$f'(x) = 3x^2 - 3$$

For graph of the curve find its intersection with x-axis and y-axis
$$y = f(x) = x^3 - 3x + 2 = (x-1)^2(x+2)$$



Curve intersects x-axis at $(1, 0)$ and $(-2, 0)$ and y-axis at $(0, 2)$

Tangent to the curve is parallel to x-axis where

$\dfrac{dy}{dx} = 0$ i.e., $f'(x) = 0$ i.e., $3x^2 - 3 = 0$

i.e., $x^2 = 1$ or $x = \pm 1$.

∴ Tangent is parallel to x-axis at $(1, 0)$ and at $(-1, 4)$.

Curve increases when $f'(x) > 0$ i.e. $x^2 - 1 > 0$

or $x^2 > 1$ or $|x| > 1$

i.e., either $x > 1$ or $x < -1$ and decreases when $|x| < 1$ i.e., $-1 < x < 1$. Graph of the curve is shown in the figure.

∴ As x moves from left to right $f(x)$ is rising in the interval $(-\infty, -1) \cup (1, \infty)$ and falling in the interval $(-1, 1)$.

i.e., graph of $f(x)$ is falling at $c = 2$.

TEST YOUR KNOWLEDGE

1. Examine the curve $y = x^3 - 18x^2 + 5x + 7$ for concavity upward and concavity downward.
2. Determine the concavity and convexity of the curve $y = x + \dfrac{4}{x}$.
3. Find the points of inflexion of the following curves

(i) $y = \dfrac{x(x^2-1)}{3x^2+1}$ (ii) $a^2y^2 = x^2(a^2 - x^2)$

(iii) $54y = (x+5)^2(x^3 - 10)$

4. Show that the line joining the two points of inflexion of the curve $y^2(x - a) = x^2(x + a); x \neq \pm$ subtends an angle $\frac{\pi}{3}$ at the origin.

5. Show that abscissa of the points of inflexion on the curve $x = a - b \cos\theta,\ y = a\theta - b \sin\theta$ is $\frac{a^2 - b^2}{a}$.

Answers

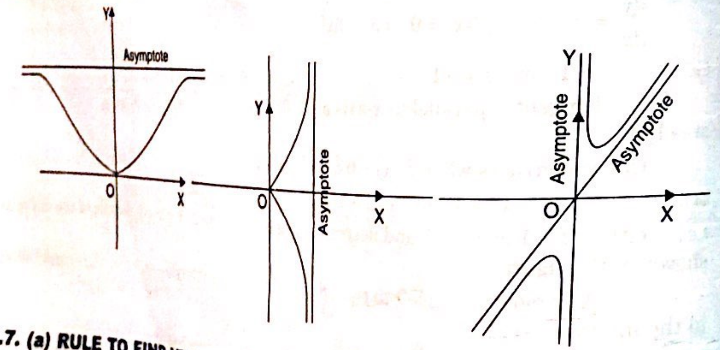
1. Concave upward in $(6, \infty)$, concave downward in $(-\infty, 6)$
2. Concave upward in $(0, \infty)$, concave downward in $(-\infty, 0)$
3. (i) $(0, 0), (1, 0), (-1, 0)$ (ii) $(0, 0)$ (iii) $x = -2, -4 \pm \sqrt{18}$.

1.6. ASYMPTOTES

Definition. *A straight line 'l' is called an asymptote of an infinite branch of a curve if the line lies on one side of branch of curve and the perpendicular distance of a point on it from any point P on that branch of the curve tends to zero as P approaches infinity along that branch.*

Rectangular Asymptotes. If an asymptote to a curve is either parallel to x-axis or y-axis then it is called a **rectangular asymptote**. Asymptote parallel to x-axis is called **horizontal asymptote** and asymptote parallel to y-axis is called **vertical asymptote**.

Oblique Asymptote. An asymptote which is neither parallel to x-axis nor parallel to y-axis is called an **oblique asymptote** (See Figs. below).



1.7. (a) RULE TO FIND VERTICAL ASYMPTOTES

Case I. *If the equation of the curve can be put into the form* $y = \frac{N(x)}{D(x)}$ *when* $N(x)$, $D(x)$ *are polynomials without any common factor then vertical asymptotes are obtained by equating to zero the real linear factors of* $D(x)$.

Case II. *If the equation of the curve cannot be put into the form* $y = \frac{N(x)}{D(x)}$ *then vertical Asymptotes are obtained by equating to zero the real linear factors in the coefficient of the highest power of* y *in the equation of the given curve.*

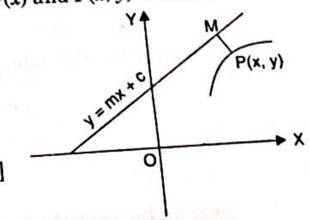
1.7. (b) RULE TO FIND HORIZONTAL ASYMPTOTES

Case I. If the equation of the curve can be put into the form $x = \frac{N(y)}{D(y)}$ where $N(y)$ and $D(y)$ are two polynomials in y without any common factor then horizontal asymptotes are obtained by equating to zero the real linear factors of $D(y)$.

Case II. If the equation of the curve cannot be put into the form $x = \frac{N(y)}{D(y)}$ then horizontal asymptotes are obtained by equating to zero the real linear factors in the coefficient of highest power of x in the equation of the given curve.

1.8. CONDITION THAT A LINE $y = mx + c$ MAY BE AN ASYMPTOTE OF THE CURVE $y = f(x)$; m AND c BOTH BEING REAL

Proof. Let $y = mx + c$ be any oblique asymptote of the curve $y = f(x)$ and $P(x, y)$ be any point on the curve. Draw $PM \perp$ on $y = mx + c$ then



$$PM = \frac{|y - mx - c|}{\sqrt{1 + m^2}} \quad \text{...(1)}$$

As $x \to \infty$ or $-\infty$

$|PM| \to 0$ $\quad [\because\ y = mx + c$ is an asymptote$]$

$$\therefore \quad \underset{x \to \infty}{\text{Lt}} \frac{|y - mx - c|}{\sqrt{1 + m^2}} = 0$$

or $$\underset{x \to \infty}{\text{Lt}} |y - mx - c| = 0 \quad \text{or} \quad \underset{x \to \infty}{\text{Lt}} (y - mx - c) = 0$$

or $$\underset{x \to \infty}{\text{Lt}} (y - mx) = c$$

Also $$\underset{x \to \infty}{\text{Lt}} (y - mx - c) = 0 \text{ gives us } \underset{x \to \infty}{\text{Lt}} \left(\frac{y}{x} - m - \frac{c}{x}\right) = 0$$

i.e., $$m = \underset{x \to \infty}{\text{Lt}} \left(\frac{y}{x} - \frac{c}{x}\right) = \underset{x \to \infty}{\text{Lt}} \frac{y}{x}$$

Hence $y = mx + c$ will be an asymptote of the curve if

$$m = \underset{x \to \infty}{\text{Lt}} \frac{y}{x} \quad \text{and} \quad c = \underset{\substack{x \to \infty \\ \text{or } -\infty}}{\text{Lt}} (y - mx).$$

1.9. OBLIQUE ASYMPTOTES OF THE CURVE IN GENERAL FORM i.e., $f(x, y) = 0$

Proof. Given curve is $f(x, y) = 0$

Let this equation be of nth order

Given equation can be put in the form

$$x^n \phi_n\left(\frac{y}{x}\right) + x^{n-1}\phi_{n-1}\left(\frac{y}{x}\right) + x^{n-2}\phi_{n-2}\left(\frac{y}{x}\right) + \dots + x\phi_1\left(\frac{y}{x}\right) + \phi_0\left(\frac{y}{x}\right) = 0 \quad \text{...(1)}$$

Divide by x^n

$$\phi_n\left(\frac{y}{x}\right)+\frac{1}{x}\phi_{n-1}\left(\frac{y}{x}\right)+\frac{1}{x^2}\phi_{n-2}\left(\frac{y}{x}\right)-\dots-\frac{1}{x^{n-1}}\phi_1\left(\frac{y}{x}\right)+\frac{1}{x^n}\phi_0\left(\frac{y}{x}\right)=0$$

Let $y = mx + c$ be any one of its asymptotes then $\underset{x\to\infty}{\text{Lt}}\ \frac{y}{x} = m$

$\therefore$ Let $x \to \infty$ in (1), we get

$$\phi_n(m) = 0$$

$\therefore$ Roots of the equation (3) will give us the values of m and the real values of m represent the slopes of the different asymptotes.

Let m_1 be any one of the real roots of (3) $\therefore$ $\phi_n(m_1) = 0$

and the asymptote becomes

$$y = m_1x + c_1$$

$$\frac{y}{x} = m_1 + \frac{c_1}{x}$$

Substituting the value of $\frac{y}{x}$ in (1)

$$x^n\phi_n\left(m_1+\frac{c_1}{x}\right)+x^{n-1}\phi_{n-1}\left(m_1+\frac{c_1}{x}\right)+x^{n-2}\phi_{n-2}\left(m_1+\frac{c_1}{x}\right)$$
$$+\dots+x\phi_1\left(m_1+\frac{c_1}{x}\right)+\phi_0\left(m_1+\frac{c_1}{x}\right)$$

Expand by Taylor's theorem, we get

$$x^n\left\{\phi_n(m_1)+\frac{c_1}{x}\phi_n'(m_1)+\frac{c_1^2}{2x^2}\phi_n''(m_1)+\dots\right\}$$
$$+x^{n-1}\left\{\phi_{n-1}(m_1)+\frac{c_1}{x}\phi'_{n-1}(m_1)+\frac{c_1^2}{2x^2}\phi''_{n-1}(m_1)\right.$$
$$+x^{n-2}\left\{\phi_{n-2}(m_1)+\frac{c_1}{x}\phi'_{n-2}(m_1)+\frac{c_1^2}{2x^2}\phi''_{n-2}(m_1)\right\}+\dots=$$

or

$$x^n[\phi_n(m_1)]+x^{n-1}[c_1\phi_n'(m_1)+\phi_{n-1}(m_1)]+x^{n-2}\left\{\frac{c_1^2}{2}\phi''_n(m_1)+c_1\phi'_{n-1}(m_1)+\phi_{n-2}(m)\right\}$$
$$+\dots=$$

Put $\phi_n(m_1) = 0$ and divide by x^{n-1}

$$[c_1\phi_n'(m_1)+\phi_{n-1}(m_1)]+\frac{1}{x}\left[\frac{c_1^2}{2}\phi_n''(m_1)+c_1\phi_{n-1}'(m_1)+\phi_{n-2}(m_1)\right]+\dots=0 \quad \dots($$

Let $x \to \infty$, we get

$$c_1\phi_n'(m_1)+\phi_{n-1}(m_1)=0$$

$\therefore$

$$c_1 = -\frac{\phi_{n-1}(m_1)}{\phi_n'(m_1)} \text{ provided } \phi_n'(m_1) \neq 0$$

$\therefore$ Asymptote corresponding to the slope m_1 is

$$y = m_1x - \frac{\phi_{n-1}(m_1)}{\phi_n'(m_1)} \text{ when } \phi_n'(m_1) \neq 0$$

Similarly, if $m_2, m_3, \dots$ are the roots of (3)

Then we have corresponding asymptotes

$$y = m_2x - \frac{\phi_{n-1}(m_2)}{\phi_n'(m_2)},\ y = m_3x - \frac{\phi_{n-1}(m_3)}{\phi_n'(m_3)}$$

...ovided $\phi_n'(m_2) \neq 0,\ \phi_n'(m_3) \neq 0\dots$

Cor 1. If $\phi_n'(m_1) = 0$ suppose $\phi_{n-1}(m_1) \neq 0$, then

$$c_1\phi_n'(m_1) + \phi_{n-1}(m_1) = 0$$

...oes not determine any finite value of c_1 $\therefore$ There is no asymptote corresponding to m_1.

Cor 2. If $\phi_n'(m_1) = 0$ and $\phi_{n-1}(m_1) = 0$. Then from (5) we have when $x \to \infty$

$$\frac{c_1^2}{2}\phi_n''(m_1) + c_1\phi_{n-1}(m_1) + \phi_{n-2}(m_1) = 0$$

...hich is quadratic in c_1 provided $\phi_n''(m_1) \neq 0$

$\therefore$ When $\phi_n'(m_1) = 0,\ \ \phi_{n-1}(m_1) = 0$ but $\phi_n''(m_1) \neq 0$

...e., m_1 is a repeated root of $\phi_n(m) = 0$ twice then there are two paralled asymptotes to the ...urve corresponding to the slope m_1, *i.e.*, $y = m_1x + c'$, and $y = m_1x + c''$, are two asymptotes.

.10. WORKING RULE TO FIND OBLIQUE ASYMPTOTES

Step 1. Put $x = 1$, $y = m$ in nth degree terms and $(n-1)$th degree terms in $f(x, y) = 0$

Step 2. Obtain $\phi_n(m)$, $\phi_{n-1}(m)$

Step 3. Find all the real roots of $\phi_n(m) = 0$

Step 4. For every non repeated real root of $\phi_n(m) = 0$, corresponding value of c is given by

$$c\phi_n'(m) + \phi_{n-1}(m) = 0 \quad \text{provided } \phi_n'(m) \neq 0$$

The ansymptote $y = mx + c$ is obtained by substituing the values of m and c

$\phi_n'(m) = 0 \Rightarrow$ there is no asymptote corresponding to this value of m.

Step 5. For every repeated real root of $\phi_n(m) = 0$ occuring twice, corresponding values of c are given by

$$\frac{c^2}{2!}\phi_n''(m) + c\phi'_{n-1}(m) + \phi_{n-2}(m) = 0$$

provided $\phi_n''(m) \neq 0$. In this case we get two parallel asymptotes to the curve.

$\because$ for one value of m we get two values of c.

Note 1. For every root of $\phi_n(m) = 0$ repeated thrice or more we can proceed similarly.

Note 2. A rational algebraic curve of degree n cannot have more than n *asymptotes*.

ILLUSTRATIVE EXAMPLES

Example 1. *Find the asymptotes (if any) parallel to the co-ordinate axes of the follow curves*

(i) $x^2y^2 - xy^2 - x^2y + x + y + 1 = 0$ (P.T.U., Dec. 2012) (ii) $(x + y)^2 = x^2y^2 - 7$

(iii) $x^2y^3 + x^2 + 3y^2 - 9xy + 8x - 25 = 0$ (P.T.U., Dec. 20

Sol. (i) Equation of the curve is $x^2y^2 - xy^2 - x^2y + x + y + 1 = 0$

For asymptotes parallel to x-axis equate coefficient of highest power of x to zero i coefficient of x^2 to zero which gives $y^2 - y = 0$ or $y(y - 1) = 0$ or $y = 0, y = 1$

$\therefore$ Asymptotes parallel to x-axis are $y = 0, y = 1$

For asymptotes parallel to y-axis equate coefficient of highest power of y to zero i coefficient of y^2 to zero which gives $x^2 - x = 0$ i.e., $x(x - 1) = 0$ or $x = 0, x = 1$

$\therefore$ Asymptotes parallel to y-axis are $x = 0, x = 1$

(ii) $(x + y)^2 = x^2y^2 - 7$

$x^2 + 2xy + y^2 = x^2y^2 - 7$

or $x^2 + 2xy + y^2 - x^2y^2 + 7 = 0$

Highest power of x is 2

$\therefore$ Asymptotes parallel to x-axis is

$1 - y^2 = 0 \quad \therefore \quad y = \pm 1$

Highest power of y is 2

Asymptote parallel to y-axis is

$1 - x^2 = 0 \quad \therefore \quad x = \pm 1$

$\therefore$ Asymptotes parallel to co-ordinate axes are

$y - 1 = 0, y + 1 = 0, x - 1 = 0, x + 1 = 0$

(iii) Equation of the curve is $x^2y^3 + x^2 + 3y^2 - 9xy + 8x - 25 = 0$

For asymptotes parallel to x-axis equate coefficient of highest power of x to zero i.e. equate coefficient of x^2 to zero i.e., $y^3 + 1 = 0$ or $(y + 1)(y^2 - y + 1) = 0$ which gives $y = -1$, the only real asymptote parallel to x-axis. For asymptote parallel to y-axis equate coefficient of highest power of y to zero i.e., equate coefficient of y^3 to zero i.e., $x^2 = 0$ which gives $x = 0$ as asymptote parallel to y-axis.

Hence the asymptotes parallel to coordinate axes are $x = 0$ and $y = -1$.

Example 2. *Find the asymptotes of the curve*

$$x^3 - x^2y - xy^2 + y^3 + 2x^2 - 4y^2 + 2xy + x + y + 1 = 0.$$

Sol. Coefficient of highest power of $x = 1$

$\therefore$ no asymptote parallel to x-axis.

Coefficient of highest power of $y = 1$

$\therefore$ no asymptote parallel to y-axis.

For oblique asymptotes.

Put $x = 1$, $y = m$ in 3rd degree terms.

We get $\phi_3(m) = 1 - m - m^2 + m^3$

Put $x = 1$, $y = m$ in 2nd degree terms, we get

$\phi_2(m) = 2 - 4m^2 + 2m$

Put $x = 1$, $y = m$ in the 1st degree terms

$$\phi_1(m) = 1 + m$$
$$\phi_0(m) = 1$$

$\therefore$

$$\phi_3(m) = 1 - m - m^2 + m^3$$
$$\phi_3'(m) = -1 - 2m + 3m^2$$
$$\phi_3''(m) = -2 + 6m$$
$$\phi_3'''(m) = 6$$
$$\phi_2(m) = 2 - 4m^2 + 2m \; ; \; \phi_2'(m) = -8m + 2$$
$$\phi_1(m) = 1 + m \; ; \; \phi_1'(m) = 1$$
$$\phi_0(m) = 1$$

$\phi_3(m) = 0$ gives $1 - m - m^2 + m^3 = 0$

$$(1 - m)(1 - m^2) = 0$$
$$(1 - m)^2(1 + m) = 0$$

$\therefore$ $m = -1, m = 1, 1$

For $m = -1$, c is given by

$$\phi_3'(m)c + \phi_2(m) = 0$$
$$(-1 + 2 + 3)c + (2 - 4 - 2) = 0$$
$$4c - 4 = 0 \quad \therefore \quad c = 1$$

$\therefore$ Corresponding asymptote is

$$y = -x + 1 \quad \text{or} \quad x + y - 1 = 0$$

when $m = 1, 1$ *i.e.*, 1 is the repeated root of $\phi_3(m) = 0$

$\therefore$ The two values of c are given by

$$\frac{c^2}{2!}\phi_3''(m) + c \cdot \phi_2'(m) + \phi_1(m) = 0$$

or

$$\frac{c^2}{2}(-2 + 6) + c(-8 + 2) + 2 = 0$$
$$2c^2 - 6c + 2 = 0 \quad \text{or} \quad c^2 - 3c + 1 = 0$$
$$c = \frac{3 \pm \sqrt{9 - 4}}{2} = \frac{3 \pm \sqrt{5}}{2}$$

$\therefore$ Asymptotes corresponding to $m = 1$ are

$$y = x + \frac{3 \pm \sqrt{5}}{2}$$

$\therefore$ $x - y + \dfrac{3 + \sqrt{5}}{2} = 0$

and $x - y + \dfrac{3 - \sqrt{5}}{2} = 0$

Hence Asymptotes of (1) are

$$x + y = 1, \quad x - y + \frac{3 \pm \sqrt{5}}{2} = 0.$$

1.11. ASYMPTOTES FOR POLAR CURVES

Prove that the asymptotes of the polar curve $\frac{1}{r} = f(\theta)$ are given by $r \sin(\theta - \alpha) = \frac{1}{f'(\alpha)}$ where α is a root of the equation $f(\theta) = 0$.

Proof. Equation of the curve is $\frac{1}{r} = f(\theta)$ ...(1)

Let $P(r, \theta)$ be any point on the curve we know that in cartesian system we have

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right] \quad ...(2)$$

Substituting the value of r from (1) in (2)

$$x = \frac{\cos\theta}{f(\theta)}, \quad y = \frac{\sin\theta}{f(\theta)}$$

$\therefore$ $$\frac{y}{x} = \tan\theta$$

If $P(r, \theta) \to \infty$ along the curve then $r \to \infty$ and hence $\frac{1}{r} \to 0$ and from (1) : $f(\theta) \to 0$

Let α be the root of $f(\theta) = 0 \;\therefore\; f(\alpha) = 0$

$\therefore$ $$\theta \to \alpha \text{ as } r \to \infty$$

In cartesian system, the equation of the oblique asymptote to the curve $y = f(x)$ is

$y = mx + c$

where $$m = \underset{P \to \infty}{\text{Lt}} \frac{y}{x} \text{ and } c = \underset{P \to \infty}{\text{Lt}} (y - mx)$$ (art. 1.8)

In polar form $m = \underset{\theta \to \alpha}{\text{Lt}} \tan\theta = \tan\alpha$

$$c = \underset{\theta \to \alpha}{\text{Lt}} [y - mx] = \underset{\theta \to \alpha}{\text{Lt}} [r\sin\theta - \tan\alpha . r\cos\theta]$$

$$c = \underset{\theta \to \alpha}{\text{Lt}} \left[\frac{1}{f(\theta)}\sin\theta - \frac{\tan\alpha . \cos\theta}{f(\theta)}\right]$$

$$= \underset{\theta \to \alpha}{\text{Lt}} \frac{1}{f(\theta)}\left[\sin\theta - \frac{\sin\alpha . \cos\theta}{\cos\alpha}\right] = \underset{\theta \to \alpha}{\text{Lt}} \frac{\sin(\theta - \alpha)}{f(\theta)\cos\alpha} \text{ which is } \left(\frac{0}{0}\right) \text{ form}$$

Apply L' Hospital Rule (differentiate numerator and denominator separately w.r.t. θ)

$$= \underset{\theta \to \alpha}{\text{Lt}} \frac{\cos(\theta - \alpha)}{f'(\theta)\cos\alpha} = \frac{1}{f'(\alpha)\cos\alpha}$$

$\therefore$ Equation of the asymptote is

$$y = mx + c$$

i.e.,

$$r\sin\theta = \tan\alpha . r\cos\theta + \frac{1}{f'(\alpha)\cos\alpha}$$

$$r[\sin\theta - \tan\alpha\cos\theta] = \frac{1}{f'(\alpha)\cos\alpha}$$

$$r\left[\sin\theta - \frac{\sin\alpha\cos\theta}{\cos\alpha}\right] = \frac{1}{f'(\alpha)\cos\alpha}$$

$$r\sin(\theta - \alpha) = \frac{1}{f'(\alpha)}$$

$$r\sin(\theta - \alpha) = \frac{1}{f'(\alpha)}$$

$$r = \frac{1}{f'(\alpha)\sin(\theta - \alpha)}$$

Note 1. Only those polar curves will have asymptotes which possess an infinite branch or in other words the polar curve for which r remains finite has no asymptote.

Note 2. If $r \to \infty$ when $\theta \to \alpha$ it is not necessary that $\theta = \alpha$ is an asymptote of the curve. Asymptote might not exist and even if it exists it may be parallel to the radius vector along $\theta = \alpha$.

Suggestion. For finding asymptotes of the polar curves it is always better that, if convenient, convert polar curve to cartesian by the transformation $x = r\cos\theta$, $y = r\sin\theta$.

Example 3. *Find asymptote of the curve $r\sin\theta = a\cos 2\theta$.*

Sol. Equation of the curve is

$$r\sin\theta = a\cos 2\theta \quad \text{or} \quad \frac{1}{r} = \frac{\sin\theta}{a\cos 2\theta}$$

Compare it with $\frac{1}{r} = f(\theta) \;\therefore\; f(\theta) = \frac{\sin\theta}{a\cos 2\theta}$

$f(\theta) = 0$ gives $\frac{\sin\theta}{a\cos 2\theta} = 0$ or $\sin\theta = 0$ or $\theta = n\pi$

$$f'(\theta) = \frac{1}{a}\,\frac{\cos 2\theta\cos\theta - \sin\theta . 2(-\sin 2\theta)}{\cos^2 2\theta} = \frac{1}{a}\,\frac{\cos 2\theta\cos\theta + 2\sin\theta\sin 2\theta}{\cos^2 2\theta}$$

Put $\theta = n\pi$

$$f'(n\pi) = \frac{1}{a\cos^2 2n\pi}[\cos 2n\pi\cos n\pi + 2\sin 2n\pi\sin n\pi]$$

$$= \frac{(-1)^n}{a}$$

Hence the asymptote to (1) is $\left[r\sin(\theta - \alpha) = \frac{1}{f'(\alpha)}\right]$

$$r\sin(\theta - n\pi) = \frac{1}{(-1)^n/a}$$

$$-r\sin(n\pi - \theta) = \frac{a}{(-1)^n}$$

$$-r(-1)^{n-1}\sin\theta = \frac{a}{(-1)^n} \qquad \left|\because \sin(n\pi-\theta) = (-1)^{n-1}\sin\theta\right.$$

or $r(-1)^{2n}\sin\theta = a$ or $r\sin\theta = a$.

Example 4. *Find the asymptotes of $r = a\tan\theta + b\sec\theta$.*

Sol. Equation of the curve is $r = a\tan\theta + b\sec\theta$ ...(1)

Change it to cartesian form by putting $x = r\cos\theta$, $y = r\sin\theta$

(1) is $r = \dfrac{a\sin\theta}{\cos\theta} + \dfrac{b}{\cos\theta}$ or $r\cos\theta = a\sin\theta + b$

or $x = a\sin\theta + b$ or $(x-b)^2 = a^2\sin^2\theta$

or $(x-b)^2 = a^2\cdot\dfrac{y^2}{r^2} = \dfrac{a^2y^2}{x^2+y^2}$

or $(x-b)^2(x^2+y^2) = a^2y^2$

or $(x^2-2bx+b^2)(x^2+y^2) = a^2y^2$

or $x^4 + x^2y^2 - 2bx^3 - 2bxy^2 + b^2x^2 + b^2y^2 - a^2y^2 = 0$ ...(2)

highest power of x is 4 and coeff. of $x^4 = 1$

$\therefore$ There is no asymptotes parallel to x-axis

Highest power of y is 2 and coeff. of y^2 is

$x^2 - 2bx + b^2 - a^2 = 0$ or $(x-b)^2 = a^2$

or $x - b = \pm a$

$\therefore$ There are two asymptotes parallel to y-axis which are $x = b + a$ and $x = b - a$.

To find oblique asymptotes put $x = 1, y = m$ in highest degree terms is (2).

$1 + m^2 = 0$ it does not given any real value of m $\therefore$ there is no oblique asymptote

$\therefore$ The only asymptotes to the curve are $r\cos\theta = b + a$ and $r\cos\theta = b - a$.

TEST YOUR KNOWLEDGE

1. (*i*) Show that the asymptotes parallel to the coordinate axes of the curve $a^2y^2 + b^2x^2 = x^2y^2$ form a rectangle of area $4\,|ab|$ square units.
   (*ii*) Find the asymptote parallel to co-ordinate axes of the curve $x^2y^2(x^2+y^2) = 4x^4 + 9y^4$

Find asymptotes of the following curves from (2 to 10)

2. $y^3 - 3x^2y + xy^2 - 3x^3 + 2y^2 + 2xy + 4x + 5y + 6 = 0$
3. $x^3 - 2y^3 + xy(2x-y) + y(x-y) + 1 = 0$
4. $xy(x^2-y^2) + 2x^2 + 2xy + 2y^2 - 1 = 0$
5. $r = a\tan\theta$
6. $r\log\theta = a$
7. $r = 2a/1 - 2\cos\theta$
8. $r^2 = \dfrac{a^2}{\theta}$
9. $r = \dfrac{a}{\theta}$
10. $r\sin 2\theta = a\cos 3\theta$ [**Hint.** Change to cartesian system]

Answers

1. (*ii*) $y = \pm 2, x = \pm 3$
2. $x + y = 0, \sqrt{3}x - y - 1 = 0, \sqrt{3}x + y + 1 = 0$
3. $x - y = 0, x + y + 1 = 0, x + 2y - 1 = 0$
4. $x = 0, y = 0, y + x = 0, y - x = 0$
5. $r\cos\theta + a = 0, r\cos\theta - a = 0$
6. $r\,\cancel{\cos}(\theta - 1) = a$ sin θ.
7. $r\sin\left(\dfrac{\pi}{3} \pm \theta\right) = -\dfrac{2a}{\sqrt{3}}$
8. $\theta = 0$
9. $r\sin\theta = a$
10. $2r\sin\theta = a, \theta = \dfrac{\pi}{2}$.

1.12. MULTIPLE POINTS

A point through which two or more than two branches of a curve pass, is called a multiple point. In particular, if only two branches of a curve pass through one point, then the point is known as **double points.**

Double points are of three types:

(1) Node. A double point is called a node if the two tangents at the double point are real and distinct (See Fig. (*i*)).

(2) Cusp. A double point is called a cusp if the two tangents at the double point are real and coincident (See Fig. (*ii*))

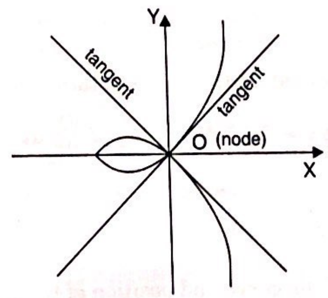


Fig. (*i*)

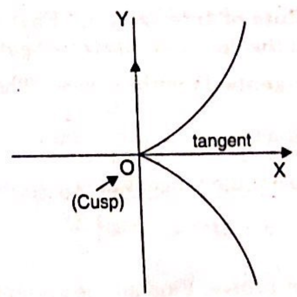


Fig. (*ii*)

(3) Conjugate or Isolated Point. A double point is called a conjugate or isolated point if the two tangents at the double point are not real or in other words there is no real point on the curve in the neighbourhood of the double point.

1.13. SOME SALIENT FEATURES OF THE CURVE TRACING (*P.T.U., Dec. 2004*)

Tracing of a curve depends upon the following points:

Let the equation of the curve be $f(x, y) = 0$

(1) Symmetry. Curve $f(x, y) = 0$ will be

(*a*) Symmetric about x-axis if it is unchanged on changing y to $-y$

i.e., $$f(x, -y) = f(x, y)$$

(*b*) Symmetric about y-axis if it is unchanged on changing x to $-x$

i.e.,
$$f(-x, y) = f(x, y)$$

(*c*) Symmetric about origin if it is unchanged on changing x to $-x$ and y to $-y$

If
$$f(-x, -y) = f(x, y)$$

(*d*) Symmetric about $y = x$

If
$$f(y, x) = f(x, y) \quad \textit{i.e.}, \text{ on interchanging } x \text{ and } y,\ f(x, y) \text{ is unchanged}$$

(*e*) Symmetric about $y = -x$ if it is unchanged on changing x to $-y$ and y to $-x$

i.e.,
$$f(-y, -x) = f(x, y)$$

(2) Origin. Check whether origin lies on the curve. If curve passes through the origin find tangents at the origin and also check whether origin is node, cusp or conjugate point.

[Rule to find tangent at the origin (if it exists) ; Tangent at the origin to any curve is obtained by equating to zero the lowest degree terms of $f(x, y)$ *e.g.*, if the curve is $x^2 + 3axy + y^2 + x + y = 0$ then tangent at the origin is $x + y = 0$].

(3) Domain and Range. Find domain and range of the curve

[Rule to find domain and range ; If curve is put in the form $y = f(x)$ then $D_f = \{x ; f(x)$ is defined$\}$ and $R_f = \{y = f(x); x \in D_f\}$ *i.e.*, Find all those possible values of x for which $f(x)$ is real and defined. Also find values of y for all the values of $x \in D_f$ then values of y will form R_f.] Knowledge of D_f, R_f gives us an idea of the extent of the curve in the xy-plane.

(4) Points of Intersection. Find the points of intersection of the curve with co-ordinate axes and the line of symmetry by putting $x = 0$, $y = 0$ and $y = x$.

(5) Tangents. Find the slopes of the tangents at all the points obtained in step 4. (note that slope of the tangent at any point of the curve $y = f(x)$ is the value of $\frac{dy}{dx}$ at that point)

Equation of the tangent at (x_1, y_1) is
$$y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} (x - x_1).$$

(6) Asymptotes. Find all the asymptotes of the curve and position of the curve relative to its asymptotes [for asymptotes consult articles 1.6 to 1.9].

(7) Maxima and Minima. Find the points where the function has maximum or minimum values .

[Rule to find maxima, minima ; put the curve in form $y = f(x)$ find $\frac{dy}{dx}, \frac{d^2y}{dx^2}$; for extreme values $\frac{dy}{dx} = 0$; for maximum value $\frac{dy}{dx} = 0, \frac{d^2y}{dx^2} < 0$; for minimum value $\frac{dy}{dx} = 0, \frac{d^2y}{dx^2} > 0$].

(8) Intervals of Increase and Decrease. Find the intervals in which y increases or decreases. Consult art 1.5.

(9) Intervals of concavity convexity. Consult art. 1.4(*a*).

(10) Points of inflexion. Consult art. 1.4(*b*).

(11) Additional Points. Find the points on the curve at which $\frac{dy}{dx} = 0$ or $\frac{dy}{dx} = \infty$

Because where $\frac{dy}{dx} = 0$, at that point tangent to the curve is parallel to x-axis and where $\frac{dy}{dx} = \infty$; tangent is parallel to y-axis.

Note 1. It is obvious that it is not possible for the students to verify all the above properties of a curve before tracing the curve as there is limitation of time in examination. So while tracing, students can easily omit inconvenient steps but they can do so only if they have sufficient practice of curve tracing before the examination. So students should have as much practice as they can have.

Note 2. In paper, students must mention the headings of all the salient features of curve tracing mentioned above.

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ILLUSTRATIVE EXAMPLES

Example 1. *Trace the curve* $x^3 + y^3 = 3axy, \quad a \geq 0.$ **(P.T.U., May 2003, Dec. 2012)**

Sol. The equation of the curve is

$$x^3 + y^3 - 3axy = 0 \quad \text{...(1)}$$

1. Symmetry. The curve is neither symmetric about x-axis, nor y-axis but about $y = x$.

2. Origin. The curve passes through the origin (0, 0) and the tangents at the origin are given by

$$3axy = 0$$

i.e.,

$$x = 0, y = 0 \quad \textit{i.e.,} \quad x\text{-axis and } y\text{-axis.}$$

3. Domain and Range. From (1) it is clear that x and y both cannot be negative $\because$ then L.H.S. will be negative but R.H.S. will be positive which is impossible $\therefore$ no portion of the curve will lie in 3rd quadrant.

4. Points of Intersection. Curve meets x-axis (put $y = 0$ in (1) we get $x = 0$) *i.e.*, at (0, 0) Curve meets y-axis (put $x = 0$ we get $y = 0$,) *i.e.*, at (0, 0) $\therefore$ curve only passes through (0, 0); Curve intersects $y = x$ where $x^3 + x^3 = 3ax^2$ or $2x^3 = 3ax^2$ or $x = \frac{3a}{2}$ $\therefore$ Point of intersection with $y = x$ is $\left(\frac{3a}{2}, \frac{3a}{2}\right)$

5. Tangents. To find slope of the tangent at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ differentiate (1) w.r.t. x

$$3x^2 + 3y^2\frac{dy}{dx} = 3ax\frac{dy}{dx} + 3ay$$

$$(y^2 - ax)\frac{dy}{dx} = ay - x^2$$

$$\frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}; \quad \left(\frac{dy}{dx}\right)_{\left(\frac{3a}{2}, \frac{3a}{2}\right)} = \frac{\frac{3a^2}{2} - \frac{9a^2}{4}}{\frac{9a^2}{4} - \frac{3a^2}{2}} = -1$$

$\therefore$ At $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ slope of the tangent $= -1$

$\therefore$ Tangent makes an angle of 135° with x-axis at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$

6. Asymptotes. Asymptote of $x^3 + y^3 = 3axy$ are given by putting $x = 1, y = m$

$\phi_3(m) = 1 + m^3, \phi_3'(m) = 3m^2, \phi_3''(m) = 6m$

$\phi_2(m) = -3am, \ \phi_2'(m) = -3a$

$\phi_3(m) = 0$ given $1 + m^3 = 0$

or $(1 + m)(1 - m + m^2) = 0$ only real value of m is -1.

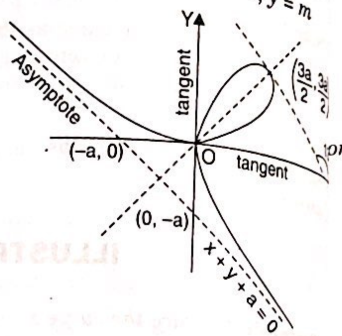
$\therefore$ Asymptote with slope $m = -1$ is $y = -x + c$ where c is given by

$$c\phi_3'(m) + \phi_2(m) = 0 \text{ at } m = -1$$

$$c(3) + (-3a)(-1) = 0$$

or $c + a = 0 \ \therefore \ c = -a$

$\therefore$ Asymptote is $y = -x - a$ or $x + y + a = 0$



Sketch of the curve is shown in the figure.

Example 2. *Trace the curve* $a^2y^2 = x^2(a^2 - x^2)$. **(P.T.U., May**

Sol. Equation of the curve is

$$a^2y^2 = x^2(a^2 - x^2)$$

1. Symmetry. Curve is symmetrical about x-axis, y-axis and origin.

2. Domain. $y^2 = \frac{x^2(a^2 - x^2)}{a^2}$

y is defined when $x^2 < a^2$ *i.e.,* $|x| < a$ *i.e.,* $-a < x < a$

$\therefore$ $D_f = (-a, a)$ $\therefore$ curve lies between $x = a$ and $x = -a$.

3. Origin. The curve passes through the origin and tangent at the origin

$$a^2x^2 - a^2y^2 = 0 \quad \text{or} \quad y = \pm x.$$

4. Points of Intersection. The curve intersects x-axis where $y = 0$

i.e., at $x = 0, x = \pm a$ *i.e.,* at $(0, 0), (a, 0)$; $(-a, 0)$.

It intersects y-axis where $x = 0$ $\therefore$ $y = 0$ $\therefore$ it intersects y-axis at $(0, 0)$.

5. Tangents. Now slope of the tangents at $(a, 0)$ and $(-a, 0)$ are given by

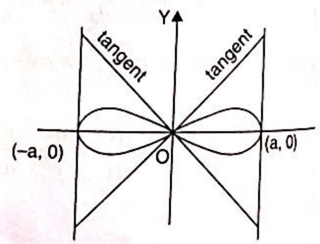
Differentiate (1) w.r.t. x

$$2a^2y\frac{dy}{dx} = x^2(-2x) + (a^2 - x^2) \cdot 2x$$

$$= 2x(-x^2 + a^2 - x^2)$$

$$\therefore \quad \frac{dy}{dx} = \frac{x(a^2 - 2x^2)}{a^2y}$$

$$\left(\frac{dy}{dx}\right)_{(a, 0)} = \frac{a(-a^2)}{2} = \infty$$



$\therefore$ Tangent at $(a, 0)$ is parallel to y-axis.

Similarly, $\left(\frac{dy}{dx}\right)_{(-a, 0)} = \infty$

$\therefore$ Tangent at $(-a, 0)$ is also parallel to y-axis.

6. Asymptotes. Asymptotes to $x^4 - a^2y^2 - a^2x^2 = 0$; **curve has no asymptote.**

Sketch of the curve is shown in figure.

Example 3. *Sketch the curve* $x(x^2 + y^2) = a(x^2 - y^2)$ **(P.T.U., May 2006)**

$$y^2(a + x) = x^2(a - x) \qquad \text{...(1)}$$

Sol. Equation of the curve is $x(x^2 + y^2) = a(x^2 - y^2)$

1. Symmetry. Curve is symmetric about x-axis (changing y to $-y$ does not alter the curve)

2. Domain. From (1), $y^2 = \frac{x^2(a - x)}{a + x} \quad x \neq -a$

or $$y = \pm x\sqrt{\frac{a - x}{a + x}}$$

y will exist when $a - x > 0$ and $a + x > 0$

i.e., $a > x$ and $x > -a$ *i.e.,* when $-a < x < a$

$a - x$ and $a + x$ both cannot be negative. $\because$ $a < -a$ is impossible.

$\therefore$ Curve lies within $x = -a$ and $x = a$

3. Origin. Curve passes through the origin and tangents at the origin are given by $x^2 - y^2 = 0$ *i.e.,* tangents are $y = x$ and $y = -x$

4. Intersection with Axes. Curve intersects x-axis at (put $y = 0$) $x = 0, x = a$, *i.e.,* at $(0, 0)$ and $(a, 0)$

It intersects y-axis (put $x = 0$) at $(0, 0)$ only

5. Tangents. From (1), we have

$$y^2(a + x) = x^2(a - x)$$

Differentiate w.r.t. x

$$2y\frac{dy}{dx}(a + x) + y^2 = x^2(-1) + (a - x)2x$$

or $$2y(a + x)\frac{dy}{dx} = -x^2 + 2ax - 2x^2 - y^2$$

or $$\frac{dy}{dx} = \frac{-3x^2 + 2ax - \frac{x^2(a - x)}{a + x}}{2y(a + x)}$$

$$= \frac{-3ax^2 - 3x^3 + 2a^2x + 2ax^2 - ax^2 + x^3}{2y(a + x)^2}$$

$$\frac{dy}{dx} = \pm\frac{-2x^3 - 2ax^2 + 2a^2x}{2x\sqrt{\frac{a - x}{a + x}}(a + x)^2} = \pm\frac{-2x(x^2 + ax - a^2)}{2x\sqrt{a - x}(a + x)^{3/2}}$$

$$= \mp \frac{x^2 + ax - a^2}{\sqrt{a-x}\,(a+x)^{3/2}}$$

At $(a, 0)$; $\frac{dy}{dx} = \infty$ $\therefore$ Tangent is parallel to y-axis

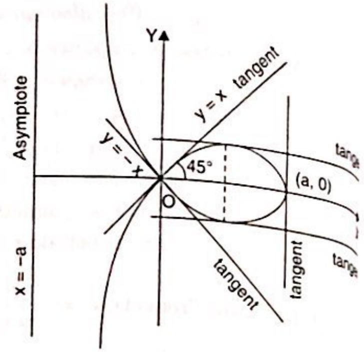
Tangent is parallel to x-axis at $\left(\text{where } \frac{dy}{dx} = 0\right)$

i.e., at points given by $x^2 + ax - a^2 = 0$

i.e., $\quad x = \frac{-a \pm \sqrt{a^2 + 4a^2}}{2} = \frac{-1 \pm \sqrt{5}}{2}\, a.$

i.e., at $\quad x = \frac{-1+\sqrt{5}}{2}\, a$ only

$\because \quad \frac{-1-\sqrt{5}}{2}\, a \notin Df$



6. Asymptote. Asymptote parallel to y-axis is (equate coefficient of highest power of y to zero)

$$a + x = 0 \quad i.e., \quad x = -a$$

Sketch of the curve is shown in the figure :

Example 4. *Sketch the curve* $y = x + \frac{1}{x}$.

Sol. Equation of the curve is

$$y = x + \frac{1}{x} \qquad ...(i$$

1. Symmetry. Curve is only symmetric about origin *i.e.,* x changed to $-x$; y change to $-y$ does not change the curve.

Also when x is positive, y is +ve

When x is negative ; y is –ve

$\therefore$ Curve lies in Ist and 3rd quadrants only.

2. Domain and Range. y is not defined at $x = 0$ $\therefore$ D_f = R except 0

$\therefore$ Curve is discontinuous at $x = 0$

For range $\quad xy = x^2 + 1$ or $x^2 - xy + 1 = 0$

$\therefore \quad x = \frac{y \pm \sqrt{y^2 - 4}}{2}$; x is defined when

$$y^2 - 4 \geq 0 \quad i.e., \quad y^2 \geq 4 \quad \text{or} \quad |y| \geq 2$$

$\therefore$ either $\quad y \geq 2$ or $y \leq -2$

$\therefore \quad R_f = (-\infty, -2] \cup [2, \infty)$

3. Intersection with Co-ordinate Axes. Curve intersects x-axis (put $y = 0$); $x^2 + 1 =$ which does not give any real value of x $\therefore$ curve does not intersect x-axis.

It intersects y-axis where $x = 0$ but $x = 0 \notin D_f$ $\therefore$ Curve does not intersect any of the axes.

4. Asymptotes. Equation of the curve is

$$xy = x^2 + 1 \quad \text{or} \quad xy - x^2 - 1 = 0$$

Asymptote parallel to y-axis is $x = 0$

There is no asymptote parallel to x-axis

For oblique asymptotes (put $x = 1, y = m$ in second degree terms)

$$\phi_2(m) = m - 1$$
$$\phi_1(m) = 0 \quad \phi_2'(m) = 1$$
$$\phi_0(m) = -1$$
$$\phi_2(m) = 0 \quad \text{gives } m = 1$$

c is given by $\quad c\phi_2'(m) + \phi_1(m) = 0$

$\therefore \quad c \cdot 1 + 0 = 0 \quad \therefore \quad c = 0$

$\therefore$ Asymptote is $\quad y = mx + c \quad i.e., \; y = x$

5. Tangents. $\quad \frac{dy}{dx} = 1 - \frac{1}{x^2}$

Tangent is parallel to x-axis where $\frac{dy}{dx} = 0$

i.e., $\quad 1 - \frac{1}{x^2} = 0$ or $x^2 = 1$ or $x = \pm 1$

when $x = 1, y = 2$; when $x = -1, y = -2$

$\therefore$ Tangent is $\parallel$ to x-axis at (1, 2) and (–1, –2)

6. Intervals of Increase and Decrease

$$\frac{dy}{dx} > 0 \quad \text{or} \quad 1 - \frac{1}{x^2} > 0 \text{ or } x^2 > 1$$

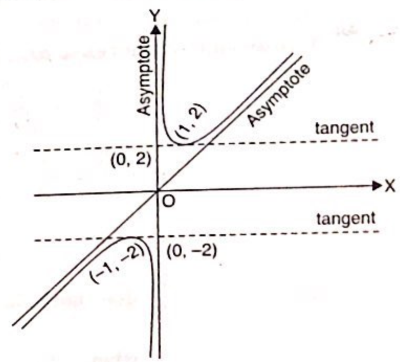
or $\quad |x| > 1 \quad \Rightarrow$ either $x > 1$ or $x < -1$

$\therefore$ Curve increases in $(-\infty, -1) \cup (1, \infty)$ and decreases in $(-1, 1)$ when $\frac{dy}{dx} < 0$

7. Intervals of Concavity and Convexity

$$\frac{d^2y}{dx^2} = \frac{2}{x^3}$$
$$\frac{d^3y}{dx^3} = -\frac{6}{x^4}$$

Curve concaves upward when $\frac{2}{x^3} > 0$ or $x^3 > 0$ *i.e.,* $x > 0$ *i.e.,* $x \in (0, \infty)$ concaves downward when $x < 0$ *i.e.,* $x \in (-\infty, 0)$. There is no point of inflexion.



Sketch of the curve is shown in the figure.

Example 5. *Trace the curve* $y^2 (a - x) = x^3$, $a > 0$.

Sol. Equation of the curve is $y^2(a - x) = x^3$

1. Symmetry. The curve is symmetric about x-axis $\because$ If we change y to $-y$ equation remains unchanged.

2. Domain and Range. $y^2 = \dfrac{x^3}{a-x}$ $\quad \therefore \quad y = \pm\sqrt{\dfrac{x^3}{a-x}}, \quad x \neq a$

$\therefore$ y is defined when $\dfrac{x^3}{a-x} \geq 0$

When $x > 0$, then $\dfrac{x^3}{a-x} \geq 0$ iff $a - x > 0$ or $x < a$ *i.e.*, $0 < x < a$

When $x < 0$, then $\dfrac{x^3}{a-x} \geq 0$ iff $a - x < 0$ or $x > a$ *i.e.*, $a < x < 0$ which

impossible $\because$ $a > 0$

$\therefore$ $D_f = [0, a)$ $\therefore$ curve lies between 0 and a.

As y can have both +ve as well as –ve values $\therefore$ Range is the set of all real numb

3. Origin. Curve passes through the origin and the tangent at the origin is ay^2

i.e., $y = 0$ [obtained by equating the lowest degree terms in the equation to zero.]

4. Points of Intersection with Axes. Curve intersects x-axis put $(y = 0)$ at $x^3 = 0$

at (0, 0). It intersects y-axis (put $x = 0$) at (0, 0) origin.

5. Asymptotes. In the equation of the curve

coeff. of $x^3 = 1$ $\therefore$ no horizontal asymptotes

coeff. of y^2 is $a - x$ $\therefore$ vertical asymptote is $x = a$

For oblique asymptote write the equation of the curve in the form $x^3 + xy^2 - ay^2 = 0$

Put $x = 1, y = m$ in the highest degree terms, we get $1 + m^2 = 0$ it gives no real value of

$\therefore$ There is no oblique asymptote.

6. Intervals of Increase and Decrease

$$y = \pm\frac{x^{3/2}}{\sqrt{a-x}} \text{ take only +ve sign, we get } y = \frac{x^{3/2}}{\sqrt{a-x}}$$

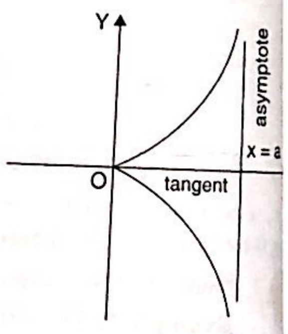
$$\frac{dy}{dx} = \frac{\sqrt{x}\,(3a-2x)}{2(a-x)^{3/2}}$$

$$\frac{dy}{dx} > 0 \text{ when } 3a - 2x > 0 \quad i.e., \quad x < \frac{3a}{2}$$

As $\quad x = \dfrac{3a}{2}$ does not belong to the domain

$\therefore \quad \dfrac{dy}{dx} > 0$ when $0 < x < a$

$\therefore$ y increases in $0 < x < a$.



7. Additional Points

Tangents parallel to x-axis is given by $\dfrac{dy}{dx} = 0$ *i.e.*, at $x = \dfrac{3a}{2}$ which is outside th

n.

Within the domain there is no tangent parallel to x-axis.

Tangent parallel to y-axis is given by $\dfrac{dy}{dx} = \infty$, *i.e.*, at $x = a$. Which again does not belong to the domain.

Now the sketch of the curve is given in the figure

Example 6. *Trace the curve* $y = \dfrac{x^2+1}{x+1}$

Sol. Equation of the curve is

$$y = \frac{x^2+1}{x+1} \quad ...(1)$$

1. Symmetry. Curve is not symmetrical about any line.

2. Domain. y exists for all real values of x except $x = -1$

$\therefore$ Curve is discontinuous at $x = -1$

$\therefore \quad D_f = R$ except -1

for Range $\quad yx + y = x^2 + 1$

or $\quad x^2 - y \cdot x + (1 - y) = 0$

or $\quad x = \dfrac{y \pm \sqrt{y^2 - 4(1-y)}}{2} = \dfrac{y \pm \sqrt{y^2 + 4y - y}}{2}$

$$x = \frac{y \pm \sqrt{(y+2)^2 - 8}}{2}$$

x exists if $(y + 2)^2 - 8 \geq 0$ or $(y + 2)^2 \geq 8$

$$|y + 2| \geq 2\sqrt{2} \quad \therefore \quad \text{either } y + 2 \geq 2\sqrt{2} \text{ or } y + 2 \leq -2\sqrt{2}$$

$\therefore \quad y \geq 2\sqrt{2} - 2$ or $y \leq -2\sqrt{2} - 2$

$\therefore \quad y \in (-\infty, -2\sqrt{2} - 2] \cup [2\sqrt{2} - 2, \infty)$

$\therefore$ Curve lies outside $y = 2(\sqrt{2} - 1)$ and $y = -2(\sqrt{2} + 1)$

3. Origin. Curve does not pass through the origin.

4. Intersection with Co-ordinate Axes

Curve intersects x-axis (put $y = 0$) ; $x^2 + 1 = 0$ no real point

Curve intersect y-axis (put $x = 0$) ; $y = 1$

$\therefore$ Point of intersection with y-axis is (0, 1)

5. Tangents. Slope of the tangent at (0, 1) is obtained from $\dfrac{dy}{dx}$

$$\frac{dy}{dx} = \frac{(x+1)2x - (x^2+1)}{(x+1)^2} = \frac{2x^2 + 2x - x^2 - 1}{(x+1)^2}$$

$$\frac{dy}{dx} = \frac{x^2 + 2x - 1}{(x+1)^2} = \frac{(x+1)^2 - 2}{(x+1)^2}$$

$$\left(\frac{dy}{dx}\right)_{(0,1)} = \frac{1-2}{1} = -1$$

$\therefore$ Tangent to the curve makes an angle of 135° at (0, 1) with +ve direction of x-axis

6. Asymptotes. Equation of the curve is

$$y(x+1) = x^2 + 1$$

Asymptote parallel to y-axis is $x + 1 = 0$; *i.e.*, $x = -1$

Now equation of the curve can be rewritten as

$$(xy - x^2) + y - 1 = 0$$

[Put $x = 1, y = m$ in 2nd degree term

For oblique asymptotes $\phi_2(m) = m - 1$

$$\phi_2'(m) = 1$$

$$\phi_1(m) = m$$

$$\phi_2(m) = 0 \text{ gives } m = 1$$

c is given by $\quad c\phi_2'(m) + \phi_1(m) = 0$

$$c \cdot 1 + 1 = 0 \quad \therefore \quad c = -1$$

$\therefore$ Asymptote is $\quad y = mx + c \quad$ *i.e.,* $\quad y = x - 1$

7. Maxima, Minima

For maxima, minima $\quad \frac{dy}{dx} = 0 \quad$ *i.e.,* $\quad (x+1)^2 - 2 = 0$

i.e., $\quad x + 1 = \pm\sqrt{2} \quad \therefore \quad x = \sqrt{2} - 1, -\sqrt{2} - 1$

$\therefore$ Tangent is $\parallel$ to x-axis at $x = \sqrt{2} - 1$ and at $x = -\sqrt{2} - 1$

When $x = \sqrt{2} - 1$, $\quad y = \frac{(\sqrt{2}-1)^2 + 1}{\sqrt{2} - 1 + 1} = \frac{2 + 1 - 2\sqrt{2} + 1}{\sqrt{2}} = \frac{4 - 2\sqrt{2}}{\sqrt{2}} = 2\sqrt{2} - 2 = 2(\sqrt{2} - 1)$

When $x = -\sqrt{2} - 1$, $\quad y = \frac{(\sqrt{2}+1)^2 + 1}{-\sqrt{2} - 1 + 1} = \frac{2 + 2\sqrt{2} + 1 + 1}{-\sqrt{2}} = \frac{4 + 2\sqrt{2}}{-\sqrt{2}}$

$$= -2\sqrt{2} - 2 = -2(\sqrt{2} + 1)$$

$$\frac{d^2y}{dx^2} = \frac{(x+1)^2 \cdot 2(x+1) - [(x+1)^2 - 2][2(x+1)]}{(x+1)^4}$$

$$= 2\,\frac{(x+1)^2 - (x+1)^2 + 2}{(x+1)^3} = \frac{4}{(x+1)^3}$$

At $\quad x = \sqrt{2} - 1, \quad \frac{d^2y}{dx^2} > 0$

$\therefore$ at $(\sqrt{2} - 1, 2\sqrt{2} - 2)$ curve has minimum value

At $\quad x = -\sqrt{2} - 1, \quad \frac{d^2y}{dx^2} < 0$

$\therefore$ at $(-\sqrt{2} - 1, -2\sqrt{2} - 2)$ curve has maximum value

8. Intervals of Increase and Decrease.

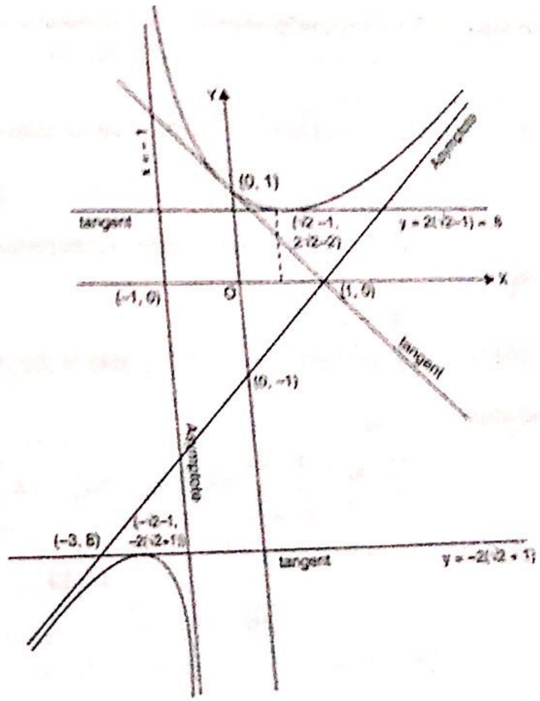
$$\frac{dy}{dx} > 0 \quad \text{when} \quad (x+1)^2 > 2$$

$|x + 1| > \sqrt{2}$ $\quad$ *i.e.*, either $\quad x + 1 > \sqrt{2}$; *i.e.*, $x > \sqrt{2} - 1$

or $\quad x + 1 < -\sqrt{2}$; $x < -\sqrt{2} - 1$

$\therefore$ Curve increases in $(-\infty, -\sqrt{2} - 1) \cup (\sqrt{2} - 1, \infty)$

and $\quad$ decreases in $(-\sqrt{2} - 1, \sqrt{2} - 1)$



9. Concavity, Convexity

$$\frac{d^2y}{dx^2} = \frac{(x+1)^2 \cdot 2(x+1) - [(x+1)^2 - 2]\,2(x+1)}{(x+1)^4}$$

$$= 2 \cdot \frac{(x+1)^2 - (x+1)^2 + 2}{(x+1)^3} = \frac{4}{(x+1)^3}$$

Curve concaves upward if $x+1>0$ *i.e.*, $x>-1$ and concaves downward when $x<-1$

$\frac{d^2y}{dx^2} \neq 0$ ∴ *There is no point of inflexion.*

Sketch of the curve is shown in figure.

Example 7. *Trace the curve* $y = \frac{x^3+1}{x}$. **(P.T.U., Dec. 2004)**

Sol. Equation of the curve is

$$y = \frac{x^3+1}{x} \qquad \ldots(1)$$

1. Symmetry. Curve is not symmetric about any line.

2. Domain. y does not exist at $x = 0$

$$D_f = \text{all real values of } x \text{ except } 0.$$

∴ Curve is discontinuous at $x = 0$

3. Intersection with Co-ordinate Axes. Curve intersects x-axis at (put $y = 0$)

$$x^3 + 1 = 0 \quad \therefore \quad x = -1 \text{ i.e., at } (-1, 0)$$

Curve does not intersect y-axis

4. Tangent. Slope of the tangent at $(-1, 0)$ is obtained by differentiating (1) w.r.t. x

$$\frac{dy}{dx} = 2x - \frac{1}{x^2}; \left(\frac{dy}{dx}\right)_{(-1,0)} = -2-1 = -3$$

∴ Tangent at $(-1, 0)$ makes as an $\pi - \tan^{-1} 3$ with +ve direction of x-axis.

5. Asymptote

$$xy = x^3 + 1$$

Curve has asymptote $x = 0$, parallel to y-axis *i.e.*, y-axis is the only asymptote to the curve.

6. Maxima, Minima

$$\frac{dy}{dx} = 2x - \frac{1}{x^2}; \frac{dy}{dx} = 0 \text{ gives } x^3 = \frac{1}{2}. \quad i.e., \quad x = \frac{1}{2^{1/3}} = 0.79$$

$$x = \frac{1}{2^{1/3}}, y = \frac{\frac{1}{2}+1}{\frac{1}{2^{1/3}}} = \frac{3}{2} \times 2^{1/3} = 1.19$$

$$\frac{d^2y}{dx^2} = 2 + \frac{2}{x^3}$$

At $x = \frac{1}{2^{1/3}}$; $\frac{d^2y}{dx^2}$ is positive

∴ Curve has minimum value at $\left(\frac{1}{2^{1/3}}, \frac{3}{2}2^{1/3}\right)$ and tangent at this point is parallel to x-axis.

7. Concavity, Convexity, Point of Inflexion

$$\frac{d^2y}{dx^2} > 0 \text{ for } 2 + \frac{2}{x^3} > 0$$

For $(x > 0)$ $\quad x^3 + 1 > 0 \quad \therefore \quad x + 1 > 0, x > -1$; But $x > 0 \quad \therefore \quad x \in (0, \infty)$

for $x < 0$, $x^3 + 1 < 0 \quad \therefore \quad x < -1 \Rightarrow x \in (-\infty, -1)$

∴ Curve is concave upward in $(-\infty, -1) \cup (0, \infty)$

Concave downward for $\frac{d^2y}{dx^2} < 0$

$$2 + \frac{2}{x^3} < 0 \quad \text{or} \quad 1 + \frac{1}{x^3} < 0$$

When $x > 0$ $\quad x^3 + 1 < 0$ or $x + 1 < 0$ *i.e.*, $x < -1$ impossible

When $x < 0$ $\quad x^3 + 1 > 0$ or $x + 1 > 0$ or $x > -1$

∴ $-1 < x < 0$ ∴ Concave downward in the interval $(-1, 0)$

$$\frac{d^2y}{dx^2} = 0, \quad x^3 + 1 = 0 \Rightarrow x + 1 = 0 \Rightarrow x = -1;$$

$$\frac{d^3y}{dx^3} = \frac{-6}{x^4} \neq 0 \text{ at } x = -1$$

When $x = -1, y = 0$

∴ $(-1, 0)$ is the point of Inflexion of the curve.

8. Intervals of Increase and Decrease

$$\frac{dy}{dx} > 0; 2x - \frac{1}{x^2} > 0 \quad \text{or} \quad 2x^3 > 1 \quad \text{or} \quad x > \frac{1}{2^{1/3}}$$

∴ Curve increases (rises up) in the interval $\left(\frac{1}{2^{1/3}}, \infty\right)$ *i.e.*, $(0.79, \infty)$. Curve decreases when $\frac{dy}{dx} < 0$ *i.e.*, $\frac{2x^3-1}{x^2} < 0$ *i.e.*, $2x^3 - 1 < 0$ ∵ x^2 is always +ve *i.e.*, $x < \frac{1}{2^{1/3}} = 0.79$

But at $x = 0$, curve is discontinuous

∴ Curve decreases in the interval $(0, 0.79)$

Also when $x < 0$, $\frac{dy}{dx} < 0$.

∴ Curve also decreases in the interval $(-\infty, 0)$

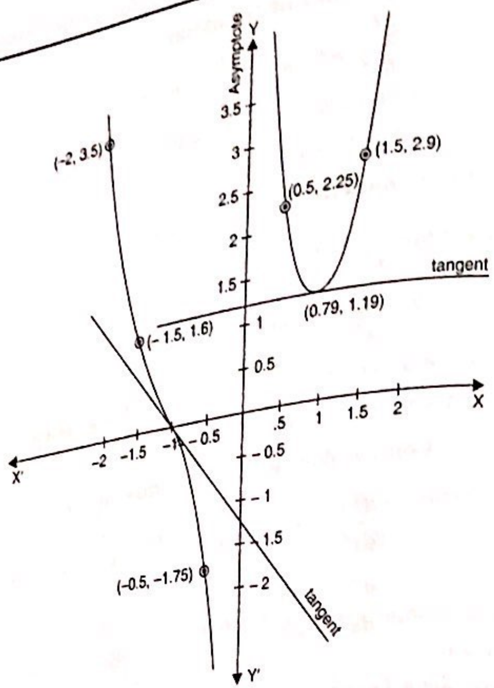
∴ Curve decreases in the interval $(-\infty, 0) \cup (0, 0.79)$

9. Additional Points

As the above conditions are not sufficient to trace the accurate curve.

We will find some additional points on the curve

x	−3	−2	−1.5	−1	−.5	.5	1	2
y	8.6	3.5	1.6	0	−1.75	2.25	2	4.5



Example 8. *Trace the curve* $y = \frac{x^3}{1+x^2}$.

Sol. Equation of the curve is

$$y = \frac{x^3}{1+x^2} \quad ...(1)$$

1. Symmetry. Curve is not symmetric about any line.

When x is negative y is negative when x is positive y is +ve

$\therefore$ Curve lies in first and third quadrants.

2. Origin. Curve passes through the origin and the equation of the tangent at the origin is (obtained by equating the lowest degree terms to zero) *i.e.*, $y = 0$ *i.e.*, x-axis is tangent at the origin

3. Domain. y exists for all real values of x $\therefore$ $D_f = R$.

4. Intersection with Co-ordinate Axes. For intersection with x-axis put $(y = 0)$, we get $x = 0$. For intersection with y-axis put $x = 0$ we get $y = 0$ $\therefore$ Curve only passes through origin.

5. Asymptotes. Curve is $(1 + x^2)y = x^3$ or $x^2y - x^3 + y = 0$

There is no real rectangular asymptote.

For oblique asymptote

$$\phi_3(m) = m - 1 \qquad (\text{Put } x = 1, y = m \text{ in } x^2y - x^3)$$

$$\phi_2(m) = 0 \qquad \phi_3'(m) = 1$$

$$\phi_3(m) = 0 \Rightarrow m = 1$$

$$\therefore \quad c\phi_3'(m) + \phi_2(m) = 0$$

$$c \cdot 1 + 0 = 0 \quad \therefore \quad c = 0$$

$\therefore$ Asymptote is $y = x$

6. Maxima, Minima.

$$y = \frac{x^3}{1+x^2}$$

$$\frac{dy}{dx} = \frac{(1+x^2)3x^2 - x^3 \cdot 2x}{(1+x^2)^2} = \frac{3x^2+x^4}{(1+x^2)^2} \quad ...(2)$$

For maximum, minimum values

$$\frac{dy}{dx} = 0 \Rightarrow \frac{3x^2+x^4}{(1+x^2)^2} = 0$$

$$x^2(3 + x^2) = 0 \Rightarrow x = 0$$

$$\frac{d^2y}{dx^2} = \frac{(1+x^2)^2 . (6x+4x^3) - (3x^2+x^4)2(1+x^2)2x}{(1+x^2)^4}$$

$$= \frac{(1+x^2)(6x+4x^3) - 4x(3x^2+x^4)}{(1+x^2)^3}$$

$$= \frac{6x+6x^3+4x^3+4x^5-12x^3-4x^5}{(1+x^2)^3} = \frac{6x-2x^3}{(1+x^2)^3} = \frac{2x(3-x^2)}{(1+x^2)^3}$$

At $x = 0$; $\frac{d^2y}{dx^2} = 0$

So (0, 0) is a saddle point. There is no point of maxima, minima on the curve.

7. Concavity, Convexity, Point of Inflexion.

$$\frac{d^2y}{dx^2} > 0 \Rightarrow x(3 - x^2) > 0$$

i.e., when $x > 0$, $3 > x^2$ or $x^2 < 3$ or $|x| < \sqrt{3}$ or $-\sqrt{3} < x < \sqrt{3}$

But $x > 0$ $\therefore$ Curve concaves upward in the interval $(0, \sqrt{3})$

When $x < 0$; $3 - x^2 < 0$ $\therefore$ $3 < x^2$ or $x^2 > 3$

or $|x| > \sqrt{3}$ $\therefore$ either $x > \sqrt{3}$ or $x < -\sqrt{3}$

But $x \not> \sqrt{3}$ $\because$ $x < 0$

$\therefore$ Curve concaves upward in $(-\infty, -\sqrt{3})$

$\therefore$ Combining the two, the curve concaves upward in $(-\infty, -\sqrt{3}) \cup (0, \sqrt{3})$

If $\frac{d^2y}{dx^2} < 0$ then $x(3 - x^2) < 0$

$\therefore$ when $x > 0$, $3 - x^2 < 0$ or $x^2 > 3$

$\therefore$ either $x > \sqrt{3}$ or $x < -\sqrt{3}$

But $x \not< -\sqrt{3}$ $\because$ $x > 0$ $\therefore$ $x > \sqrt{3}$ $\therefore$ $x \in (\sqrt{3}, \infty)$

$\therefore$ Curve concaves downward in $(\sqrt{3}, \infty)$

when $x < 0$, $3 - x^2 > 0$, *i.e.*, $-\sqrt{3} < x < \sqrt{3}$

But $x < 0$ $\therefore$ Curve concaves downward in $(-\sqrt{3}, 0)$

$\therefore$ Combining the two curve concaves downward in $(-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)$

Points of Inflexion.

$$\frac{d^2y}{dx^2} = 0 \text{ for } x = 0, x = \sqrt{3}, x = -\sqrt{3}$$

$$\frac{d^3y}{dx^3} = 2\,\frac{(1+x^2)^3(3-3x^2) - (3x - x^3)\cdot 3(1+x^2)^2 \cdot 2x}{(1+x^2)^6}$$

$$= 2\,\frac{3(1+x^2)(1-x^2) - 6x(3x - x^3)}{(1+x^2)^4}$$

$$= 2\cdot\frac{3 - 3x^4 - 18x^2 + 6x^4}{(1+x^2)^4} = 2\cdot\frac{3 - 18x^2 + 3x^4}{(1+x^2)^4} = \frac{6(1-6x^2+x^4)}{(1+x^2)^4}$$

At $x = 0$, $\frac{d^3y}{dx^3} \neq 0$ $\therefore$ $x = 0$ is a point of inflexion

At $x = \pm\sqrt{3}$, $\frac{d^3y}{dx^3} = \frac{6(1-6.3+9)}{(1+3)^4} = \frac{6(-8)}{4^4} \neq 0$

$\therefore$ $x = \sqrt{3}$, $y = \frac{3\sqrt{3}}{4}$; $x = -\sqrt{3}$, $y = \frac{-3\sqrt{3}}{4}$

$\therefore$ Points of inflexion are $(0, 0)$, $\left(\sqrt{3}, \frac{3\sqrt{3}}{4}\right)$, $\left(-\sqrt{3}, \frac{-3\sqrt{3}}{4}\right)$

Slope of the tangents at $\left(\sqrt{3}, \frac{3\sqrt{3}}{4}\right)$ and $\left(-\sqrt{3}, \frac{-3\sqrt{3}}{4}\right)$ are given by

$$\tan\theta = \frac{9+9}{4^2}, \frac{9+9}{4^2} \quad i.e., \tan\theta = \frac{9}{8} \text{ at each point.} \qquad \text{(Using}$$

$\therefore$ Tangent at both the points of inflexion make $\tan^{-1}\frac{9}{8}$ angle with +ve direction x-axis.

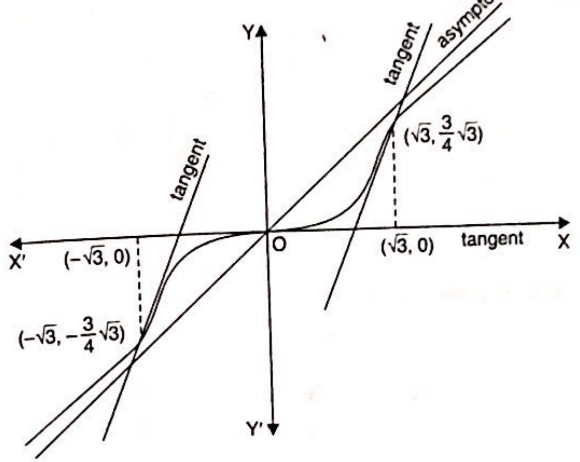
8. Intervals of Increase and Decrease. $\frac{dy}{dx} > 0$ for all values of x.

$\therefore$ Curve is increasing and never decreases.

9. Additional Points

We can choose some additional points also

x	-3	-2	-1	0	1	2	3
y	-2.7	-1.6	-.5	0	.5	1.6	2.7



Example 9. *Trace the curve* $y = \frac{x^3 - 3}{2x - 4}$. **(P.T.U., May 2005)**

Sol. Equation of the curve is $y = \frac{x^3 - 3}{2x - 4} = \frac{x^3 - 3}{2(x - 2)}$

1. Symmetry. The curve is not symmetrical about any line.

2. Domain. y exists $\forall$ x except $x = 2$, but when x is negative y is positive and when x is positive y may be positive or negative except at $x = 2$ $\therefore$ curve does not lie in 3rd quadrant.

3. Origin. Curve does not pass through the origin.

4. Intersection with Axes. Curve intersects x-axis at $(3^{1/3}, 0) = (1.4, 0)$ and y-axis at $\left(0, \frac{3}{4}\right) = (0, 0.75)$.

5. Asymptotes. Equation of the curve can be put into the form $2y(x-2) = x^3 - 3$.

It has only one asymptote parallel to y-axis which is $x - 2 = 0$ *i.e.*, $x = 2$.

6. Tangents. $\frac{dy}{dx} = \frac{1}{2}\,\frac{(x-2).3x^2 - (x^3 - 3)}{(x-2)^2} = \frac{1}{2}\,\frac{2x^3 - 6x^2 + 3}{(x-2)^2}$

At $(1.4, 0)$; $\frac{dy}{dx}$ is –ve $\therefore$ tangent at $(1.4, 0)$ makes obtuse angle with the +ve direction of x-axis.

At $(0, 0.75)$; $\frac{dy}{dx}$ is +ve $\therefore$ tangent at $(0, 0.75)$ makes acute angle with the +ve direction of x-axis.

7. Additional points. We can choose some additional points on the curve

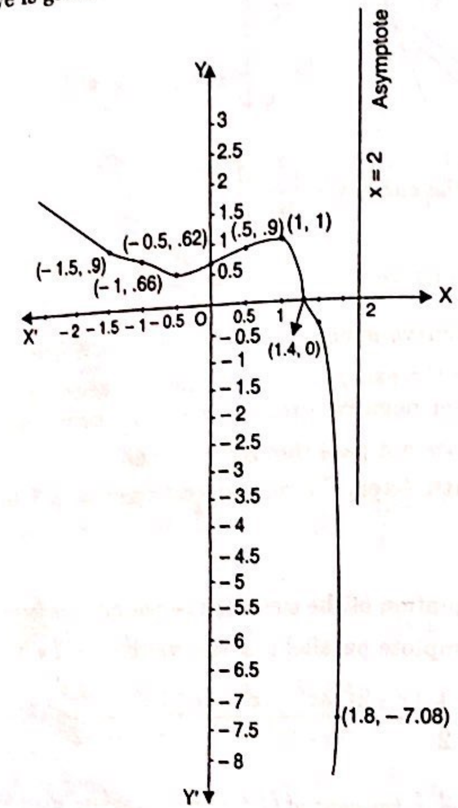
We have $$y = \frac{x^3 - 3}{2(x-2)} = \frac{1 - \frac{3}{x^3}}{2\left(\frac{1}{x^2} - \frac{2}{x^3}\right)}$$

[∵ when x is –ve, y i

When $x \to -\infty$ $y \to +\infty$

x	–1.5	–1	–0.5	0	0.5	1	1.5
y	$\frac{51}{56} = .9$	$\frac{2}{3} = .66$	$\frac{5}{8} = .62$	$\frac{3}{4} = .75$	$\frac{23}{24} = .9$	1	$-\frac{3}{8} = -.37$
$\frac{dy}{dx}$	–ve	–ve	+ve	+ve	+ve	–ve	–ve

Sketch of the curve is given below:



Example 10. *Trace the curve* $y = x^3 - 12x - 16$. **(P.T.U., May 20**

Sol. Equation of the curve is

$$y = x^3 - 12x - 16$$

1. Symmetry. Curve is not symmetric about any line

2. Origin. Curve does not pass through the origin

3. Domain. Curve exists ∀ real x.

4. Intersection with Coordinate Axes. Curve intersects x-axis (put $y = 0$)

i.e., $x^3 - 12x - 16 = 0$

or $(x + 2)^2 (x - 4) = 0$ or $x = -2, x = 4$ *i.e.,* at $(-2, 0), (4, 0)$

It intersects y-axis (put $x = 0$) at $(0, -16)$

5. Asymptote. Curve has no asymptote

6. Tangents. $\frac{dy}{dx} = 3x^2 - 12$

At $(-2, 0)$; $\frac{dy}{dx} = 0$ ∴ Tangent is parallel to x-axis at $(-2, 0)$

At $(4, 0)$; $\frac{dy}{dx} = 36$ *i.e.,* $\tan\theta = 36$

∴ at $(4, 0)$ tangent makes an acute angle $\tan^{-1} 36$ with x-axis

Also $\frac{dy}{dx} = 0$ at $3x^2 - 12 = 0$ *i.e.,* $x^2 = 4$ *i.e.,* $x = \pm 2$

∴ Tangent is also parallel to x-axis at $(2, -32)$

7. Concavity. $y = x^3 - 12x - 16$

$$\frac{dy}{dx} = 3x^2 - 12$$

$$\frac{d^2y}{dx^2} = 6x$$

∴ Curve concaves upward for $x > 0$

and downward for $x < 0$

$x = 0$ is the point of inflexion

8. Intervals of Increase and Decrease . $\frac{dy}{dx} = 3(x^2 - 4)$

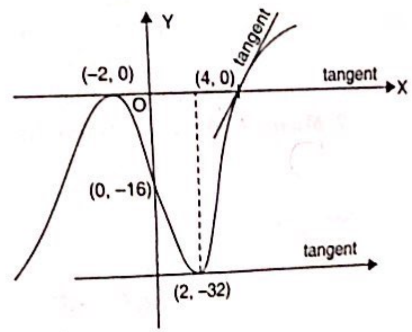
Curve increases for $\frac{dy}{dx} > 0$ *i.e.,* $x^2 > 4$ or $|x| > 2$ *i.e.,* increases in the interval $(-\infty, -2) \cup (2, \infty)$ and decreases in the interval $(-2, 2)$

9. Additional Points. $y \to \infty$ as $x \to \infty$

and $y \to -\infty$ as $x \to -\infty$

y is +ve for $x > 4$ and –ve for $x < 4$

Sketch of the curve is shown in the figure:



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Example 11. *Sketch the curve* $y = \frac{1}{6}(x^3 - 6x^2 + 9x + 6)$.

Sol. 1. Symmetry. No symmetry

2. Domain. $D_f = R$

3. Origin. Does not pass through the origin

4. Intersection with Co-ordinate Axes. Curve intersects x-axis (put $y = 0$)

$$x^3 - 6x^2 + 9x + 6 = 0$$

It is not convenient to find points of intersection. $\therefore$ we can only guess the location of points of intersection by taking

$$f(0) = 6 = +ve$$
$$f(1) = 1 - 6 + 9 + 6 = 10 = +ve$$
$$f(2) = 8 - 24 + 18 + 6 = 8 = +ve$$

which shows that on R.H.S. of origin curve does not meet x-axis.

Now $\quad f(-1) = -1 - 6 - 9 + 6 = -10 = -ve, f(0) = 1 = +ve$

$$f(-1) < 0, f(0) > 0$$

$\therefore$ One point of intersection lies between $(-1, 0)$.

5. Asymptote. No asymptote of the curve.

6. Intervals of Increase and Decrease.

$$\frac{dy}{dx} = \frac{1}{6}(3x^2 - 12x + 9) = \frac{1}{2}(x^2 - 4x + 3)$$

$$\frac{dy}{dx} > 0 \text{ when } x^2 - 4x + 3 > 0$$

or $\quad (x-2)^2 > 4 - 3 = 1$

or $\quad |x - 2| > 1$

i.e., either $x - 2 > 1$ or $x - 2 < -1$.

or $\quad x > 3$ or $x < 1$

$\therefore$ Curve increases in $(3, \infty)$ or $(-\infty, 1)$ *i.e.*, $(-\infty, 1) \cup (3, \infty)$

$$\frac{dy}{dx} < 0 \text{ when } (x-2)^2 < 1$$

or $\quad |x - 2| < 1$ or $2 - 1 < x < 2 + 1$

$$1 < x < 3$$

$\therefore$ Curve decreases in $(1, 3)$.

7. Maxima, Minima. Curve has extreme value where

$$\frac{dy}{dx} = 0 \quad i.e., \quad x^2 - 4x + 3 = 0$$

$\therefore \quad x = 1, 3$

$$\frac{d^2y}{dx^2} = \frac{1}{2}(2x - 4) = x - 2$$

At $x = 1$, $\quad \frac{d^2y}{dx^2} = -ve$

$\therefore$ Curve has maximum value at $x = 1$ *i.e.*, at $\left(1, \frac{5}{3}\right)$

At $x = 3$; $\quad \frac{d^2y}{dx^2} = 1 = +ve$

$\therefore$ Curve has minimum value at $(3, 1)$.

$\because$ Tangents to the curve are parallel to x-axis at $(1, 5/3)$ and $(3, 1)$

8. Concavity, Convexity, Point of Inflexion

$$\frac{d^2y}{dx^2} = x - 2; \quad \frac{d^3y}{dx^3} = 1$$

Curve concaves upward for $x > 2$ *i.e*, in $(2, \infty)$

Curve concaves downward of $x < 2$ *i.e.*, $(-\infty, 2)$

Point of inflexion at $x = 2$ $\because$ at $x = 2$ $\frac{d^2y}{dx^2} = 0$ but $\frac{d^3y}{dx^3} \neq 0$ $\therefore$ $\left(2, \frac{4}{3}\right)$ is point of inflexion

Slope of tangent at $x = 2$ is given by

$$\left(\frac{dy}{dx}\right)_{x=2} = \frac{1}{2}(4 - 8 + 3) = -\frac{1}{2}.$$

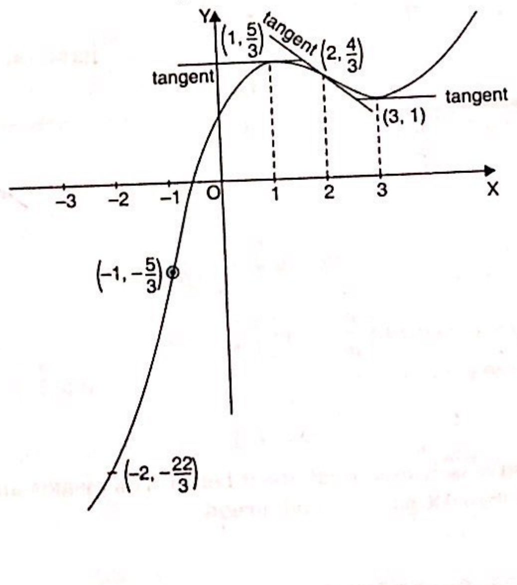
$\therefore$ Tangent at $(2, 4/3)$ makes an angle of $\pi - \frac{1}{2}$ with +ve direction of x-axis.

9. Additional Points. We can find some more points on the curve as given below:

x	-3	-2	-1	0	1	2	3
y	-17	$-\frac{22}{3}$	$-\frac{5}{3}$	1	$\frac{5}{3}$	$\frac{4}{3}$	1

When $x \to \infty, y \to \infty$

When $x \to -\infty, y \to -\infty$



Example 12. *Trace the curve* $x = (y-1)(y-2)(y-3)$. **(P.T.U., May 20**

Sol. 1. Symmetry. Nil or $x = y^3 - 6y^2 + 11y - 6$

2. Origin. Does not pass through the origin

3. Domain Range. All real values of y

4. Intersection with Co-ordinate Axes. Curve intersects y-axis (put $x = 0$) at $y =$ 1, $y = 2, y = 3$ *i.e.,* at (0, 1), (0, 2), (0, 3) curve intersects x-axis (put $y = 0$) ; $x = -6$ *i.e.,* (– 6,

5. Asymptotes. Nil

6. Tangents. $\dfrac{dx}{dy} = (y-1)(y-2) + (y-2)(y-3) + (y-3)(y-1) = 3y^2 - 12y + 1$

$$\frac{dx}{dy} = 0 \text{ when } y = \frac{12 \pm \sqrt{144-132}}{6} = \frac{12 \pm 2\sqrt{3}}{6} = 2 \pm \frac{1}{\sqrt{3}}$$

$\therefore$ $\dfrac{dx}{dy} = 0$ at $y = 2.6, 1.4$

At $y = 2.6$; $x = -3.84$

and at $y = 1.4$; $x = 3.84$

$\therefore$ Tangents to the curve are parallel to y-axis at (– 3.84, 2.6) and (3.84, 1.4)

7. Additional Points. when $y < 0$ then $x < 0$

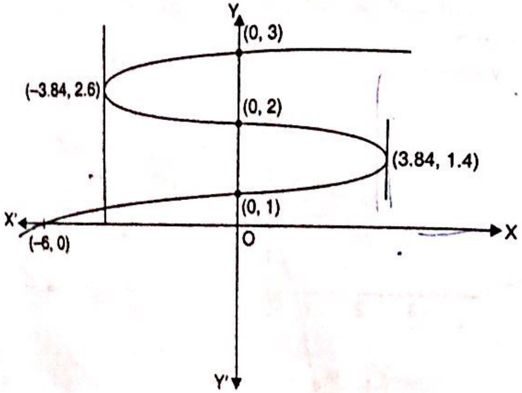
when y lies between 0 and 1; $x < 0$

when y lies between 1 and 2; $x > 0$

when y lies between 2 and 3; $x < 0$

when y is greater than 3 $x > 0$

when $x \to \infty, y \to \infty$



Example 13. *Trace the curve* $\left(\dfrac{x}{a}\right)^{2/3} + \left(\dfrac{y}{b}\right)^{2/3} = 1$. **(P.T.U., Jan 2010**

Sol. Equation of the curve is

$$\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1 \quad ...(1$$

1. Symmetry. Curve is symmetrical about both x-axis , y-axis and origin.

2. Origin. Curve does not pass through origin.

3. Domain and Range.

$$\left(\frac{y}{b}\right)^{2/3} = 1 - \left(\frac{x}{a}\right)^{2/3}$$

or
$$y = b\left\{1 - \left(\frac{x}{a}\right)^{2/3}\right\}^{3/2}$$

y exists if $1 - \left(\dfrac{x}{a}\right)^{2/3} > 0$ or $1 > \left(\dfrac{x}{a}\right)^{2/3}$

or
$$\left(\frac{x}{a}\right)^{2/3} < 1 \text{ or } \left(\frac{x}{a}\right)^2 < 1 \text{ or } \left|\frac{x}{a}\right| < 1$$

or
$$-1 < \frac{x}{a} < 1 \text{ or } -a < x < a$$

$\therefore$ Domain of the curve (– a, a) and similarly range of the curve (– b, b).

4. Intersection with Co-ordinate Axes. Curve intersects x-axis (put $y = 0$)

$$\left(\frac{x}{a}\right)^{2/3} = 1 \text{ or } \left(\frac{x}{a}\right)^2 = 1 \text{ or } x = \pm a$$

$\therefore$ Points of intersection with x-axis (a, 0) ; (–a, 0)

Points of intersection with y-axis (0, b) ; (0, –b)

5. Tangents. Slopes of the Tangents at ($\pm a$, 0) and (0, $\pm b$) are obtained from the value of $\dfrac{dy}{dx}$

$$\frac{2}{3}\left(\frac{x}{a}\right)^{-\frac{1}{3}}\left(\frac{1}{a}\right) + \frac{2}{3}\left(\frac{y}{b}\right)^{-\frac{1}{3}}\left(\frac{1}{b}\right)\frac{dy}{dx} = 0$$

$\therefore$
$$\frac{dy}{dx} = -\frac{\left(\frac{x}{a}\right)^{-1/3}\frac{1}{a}}{\left(\frac{y}{b}\right)^{-1/3}\left(\frac{1}{b}\right)} = -\frac{y^{1/3}b^{2/3}}{x^{1/3}a^{2/3}}$$

$$\left(\frac{dy}{dx}\right)_{(a,0)} = 0 \,;\, \left(\frac{dy}{dx}\right)_{(-a,0)} = 0$$

$\therefore$ At (a, 0), (– a, 0) tangents are parallel to x-axis

$$\left(\frac{dy}{dx}\right)_{(0,b)} = -\infty, \left(\frac{dy}{dx}\right)_{(0,-b)} = -\infty$$

$\therefore$ Tangents are parallel to y-axis at (0, b) and (0, –b).

6. Asymptote. Nil.

7. Maxima, Minima. Curve has extreme value where

$$\frac{dy}{dx} = 0 \quad i.e., \quad y = 0 \quad \therefore \quad x = \pm a$$

extreme values at ($\pm a$, 0)

$$\frac{d^2y}{dx^2} = -\left(\frac{b}{a}\right)^{2/3} \frac{x^{1/3}\cdot\frac{1}{3}y^{-2/3}\frac{d}{dx}y - y^{1/3}\frac{1}{3}x^{-2/3}}{x^{2/3}}$$

$$= -\left(\frac{b}{a}\right)^{2/3} \frac{1}{3x^{2/3}}\left\{-x^{1/3}y^{-2/3}\frac{y^{1/3}}{x^{1/3}}\left(\frac{b}{a}\right)^{2/3} - y^{1/3}x^{-2/3}\right\}$$

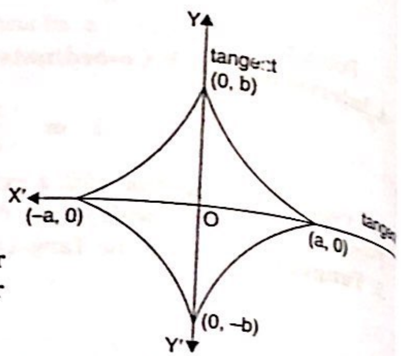
$$= +\left(\frac{b}{a}\right)^{2/3} \frac{1}{3x^{2/3}}\cdot b^{2/3}x^{-2/3}y^{-1/3}\left[\frac{x^{2/3}}{a^{2/3}} + \frac{y^{2/3}}{b^{2/3}}\right]$$

$$= \frac{1}{3}\left(\frac{b^2}{a}\right)^{2/3}\cdot\frac{1}{x^{4/3}y^{1/3}} \qquad | \because \text{ of (1)}$$

$$= \frac{1}{3}\left[\frac{b^2}{ax^2}\right]^{2/3}\cdot\frac{1}{y^{1/3}}$$

$$\frac{d^2y}{dx^2} > 0 \text{ if } y > 0$$

And $$\frac{d^2y}{dx^2} < 0 \text{ if } y < 0$$



$\therefore$ Curve has maximum values at $(\pm a, 0)$ for $y < 0$ and curve has minimum value at $(\pm a, 0)$ for $y > 0$

Sketch of the curve is shown in the figure.

Example 14. *Trace the curve* $y^2 = \dfrac{x-3}{x^2-6x-7}$.

Sol. Equation of the curve is

$$y^2 = \frac{x-3}{x^2-6x-7} = \frac{x-3}{(x+1)(x-7)}$$

1. Symmetry. Curve is symmetric about x-axis (changing y to $-y$ does not change equation)

2. Origin. Curve does not pass through the origin

3. Domain and Range. y is not defined at $x = -1$, $x = 7$ also when $x > 3, x < 7$

Curve is not defined in $(3, 7)$ also when $x < -1$, y is not defined

$\therefore$ Curve is not defined in $(-\infty, -1) \cup (3, 7)$

For range; solve the equation for x

$$x^2y^2 - 6xy^2 - 7y^2 = x - 3$$

or $$(y^2)x^2 - (6y^2+1)x + (-7y^2+3) = 0$$

$$x = \frac{6y^2 + 1 \pm \sqrt{(6y^2+1)^2 - 4y^2(-7y^2+3)}}{2y^2}$$

x exists when

$$(6y^2+1)^2 - 4y^2(-7y^2+3) \geq 0$$

$$36y^4 + 12y^2 + 1 + 28y^4 - 12y^2 \geq 0$$

$$64y^4 + 1 \geq 0$$

$\therefore$ x exists for all values of y $\therefore$ Range = R

4. Intersection with Co-ordinate Axes.

Curve intersects x-axis where $y = 0$ *i.e.*, at $x = 3$ *i.e.*, at $(3, 0)$

and y-axis (put $x = 0$) $\qquad y^2 = \dfrac{-3}{-7} = \dfrac{3}{7}$

$\therefore$ At $\left(0, \sqrt{\frac{3}{7}}\right)$ and $\left(0, -\sqrt{\frac{3}{7}}\right)$

i.e., At $(0, 0.65)$ and $(0, -0.65)$

5. Tangents. Slopes of the tangent at these points are obtained by evaluating $\dfrac{dy}{dx}$ at these points

$$2y\frac{dy}{dx} = \frac{(x^2-6x-7)(1) - (x-3)(2x-6)}{(x^2-6x-7)^2}$$

$$= \frac{x^2-6x-7-2x^2+12x-18}{(x^2-6x-7)^2} = \frac{-x^2+6x-25}{(x^2-6x-7)^2}$$

$$\frac{dy}{dx} = \frac{\pm 1}{2\cdot\dfrac{\sqrt{x-3}}{\sqrt{x^2-6x-7}}} \times \frac{-(x^2-6x+25)}{(x^2-6x-7)^2}$$

$$\frac{dy}{dx} = \pm\frac{x^2-6x+25}{2\sqrt{x-3}\,(x^2-6x-7)^{3/2}}$$

$$\left(\frac{dy}{dx}\right)_{(3,0)} = \infty \quad \therefore \text{ At } (3, 0) \text{ tangent is parallel to } y\text{-axis.}$$

At $\left(0, \pm\sqrt{\frac{3}{7}}\right)$, $\qquad \dfrac{dy}{dx} = \pm\dfrac{25}{2\sqrt{-3}(-7)^{3/2}}$ imaginary

$\therefore$ No tangent at $\left(0, \pm\sqrt{\frac{3}{7}}\right)$

6. Asymptotes. Equation of the curve is $(x^2 - 6x - 7)\,y^2 = x - 3$

Asymptotes ‖ to x-axis are $y^2 = 0$ *i.e.*, $y = 0$, *i.e.*, x-axis itself.

[Equating coefficient of highest power of x to zero]

Asymptotes ∥ to y-axis are

$$x^2 - 6x - 2 = 0 \quad i.e., \quad x = -1, x = 7$$

No oblique asymptotes.

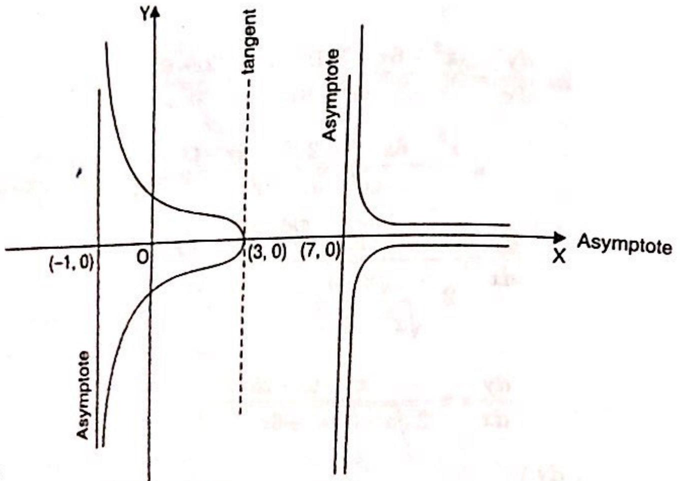
7. Maxima, Minima. Curve attains the extreme value at $\frac{dy}{dx} = 0$

i.e., at
$$x^2 - 6x + 25 = 0$$
$$x = \frac{6 \pm \sqrt{36 - 100}}{2} = 3 \pm i4 \quad \text{(imaginary)}$$

no extreme point.

8. Additional Points. When $x > 7$ or $-1 < x < 3$ we get real values of y

x	7	8	9	10
y	∞	$\frac{5}{9}$	$\frac{3}{10}$	$\frac{7}{33}$



Example 15. *Trace the curve* $y = c \cosh \frac{x}{c}$. **(P.T.U., Dec. 2003)**

Sol. 1. Symmetry. The curve is symmetric about y-axis

[∵ Changing of x to $-x$ does not change the equation]

2. Origin. Curve does not pass through the origin.

3. Domain and Range. y is defined for all real values of x is $D_f = R$. But y always has +ve values $\forall x$ ∴ Range is the set of +ve values of y only ∴ curve lies in the upper half plane.

4. Intersection with Axes. Curve intersects x-axis (put $y = 0$) *i.e.,* $c \cosh \frac{x}{c} = 0$ but $c \neq 0$ ∴ $\cosh \frac{x}{c} = 0$ *i.e.,* $\frac{e^{x/c} + e^{-x/c}}{2} = 0$ *i.e.,* $e^{\frac{2x}{c}} = -1$ which is impossible ∴ curve does not intersect x-axis.

5. Tangents. $\frac{dy}{dx} = \sinh \frac{x}{c}$

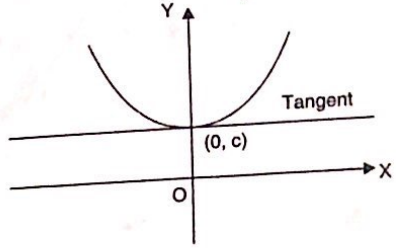
Tangent is parallel to x-axis when $\frac{dy}{dx} = 0$ *i.e.,* $\sinh \frac{x}{c} = 0$ *i.e.,* $x = 0$ *i.e.,* at $(0, c)$; $\frac{dy}{dx} = \infty$ at no real point. ∴ no where tangent is parallel to y-axis.

6. Asymptotes. Curve does not have any asymptote.

7. Maxima, Minima. The curve has maximum or minimum values where ; $\frac{dy}{dx} = 0$ *i.e.,* at $x = 0$ *i.e.,* at $(0, c)$ and at $(0, c)$ $\frac{d^2y}{dx^2} = \frac{1}{c} \cosh \frac{x}{c} > 0$

∴ curve has minimum value at $(0, c)$

Now sketch of the curve is given in the figure:



8. Additional Points. We see that when $x \to \infty$ y also tends to $+\infty$ and when $x \to -\infty$ again y tends to $+\infty$.

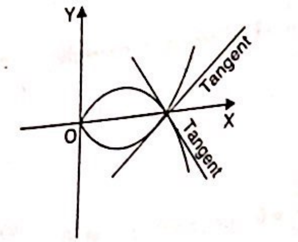
TEST YOUR KNOWLEDGE

Trace the following curves from 1 to 19:

1. $y^2 = x(x - a)^2$; $a > 0$
2. $(a^2 + x^2)\, y^2 = x^2(a^2 - x^2)$
3. $y^2 = (x - 2)(x - 5)^2$
4. $ay^2 = x^2(a - x)$
5. $y = \frac{x^2}{1 + x^2}$
6. $y = \frac{x}{x^2 - 1}$
7. $y = \frac{x^2}{x^2 - 1}$
8. $y = x^3 - 3x^2 + 3$
9. $y = x(x + 1)(x - 2)$ or $y = x^3 - x^2 - 2x$
10. $y^2(a + x) = x^2(3a - x)$
    [**Hint.** Similar to solved examples 3]
11. $y^2(2a - x) = x^3$
12. $xy^2 = 4a^2(2a - x)$
13. $y^2 = x^3$ ***(P.T.U., Jan. 2009)***
14. $y^2 = (x + 1)^3$
15. $y^2(x^2 - 9) = -9x^2$
16. $a^2 x^2 = y^3(2a - y)$
17. $x^3 + y^3 = ax^2$
18. $x^4 - y^4 + xy = 0$
19. $y = \frac{(x - 1)(x - 3)}{x^2}$. ***(P.T.U., Dec. 2007)***

Answers

1.



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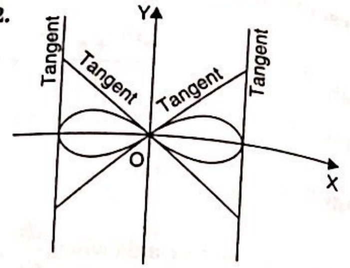

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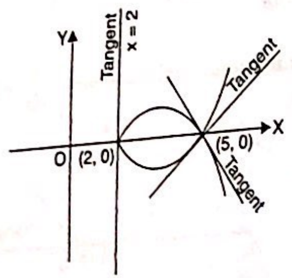

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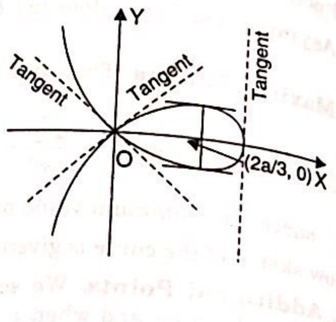

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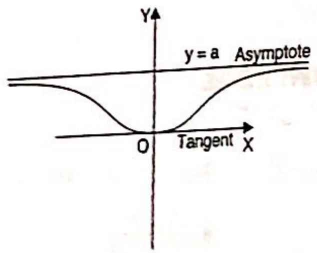

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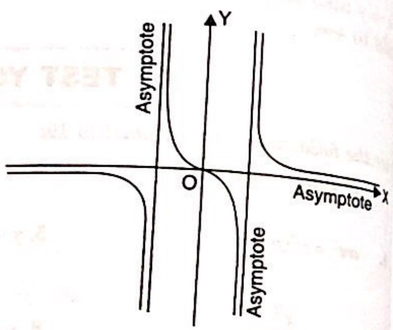

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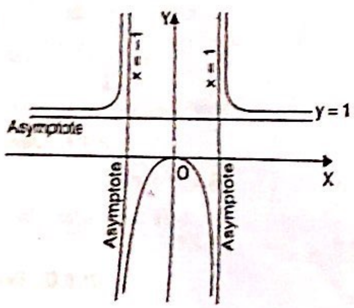

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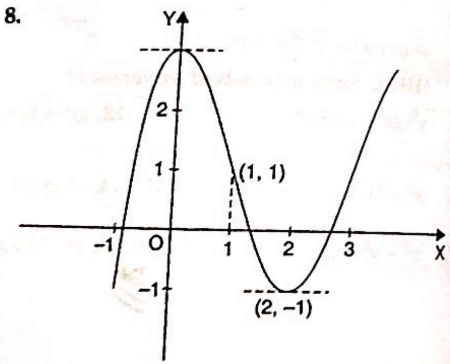

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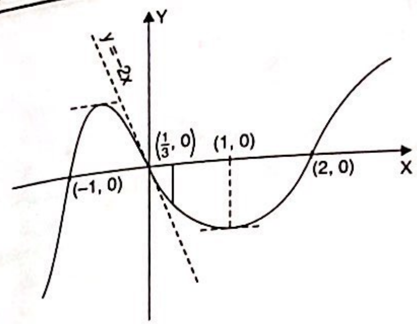

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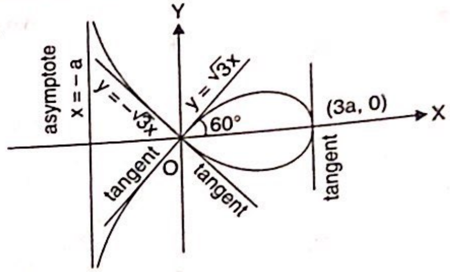

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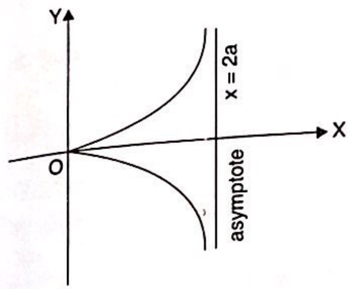

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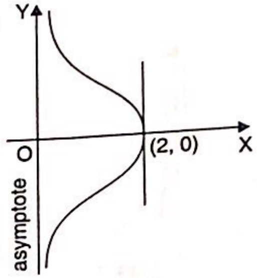

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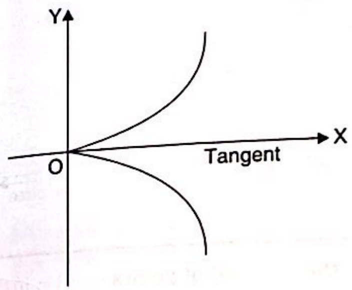

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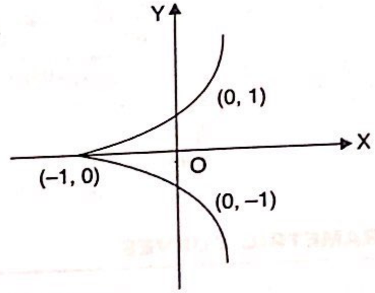

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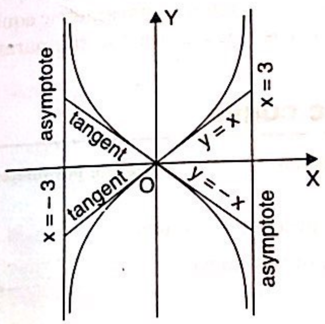

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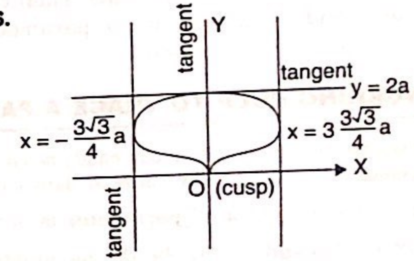

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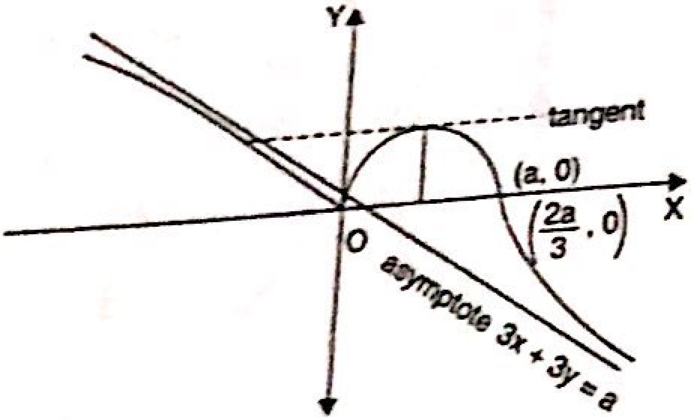
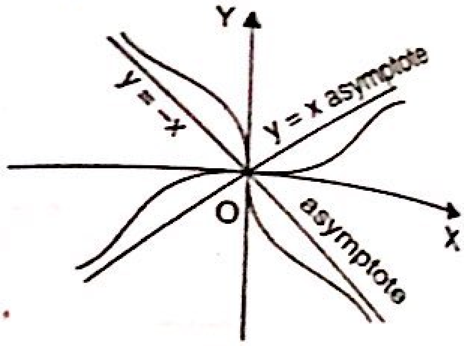
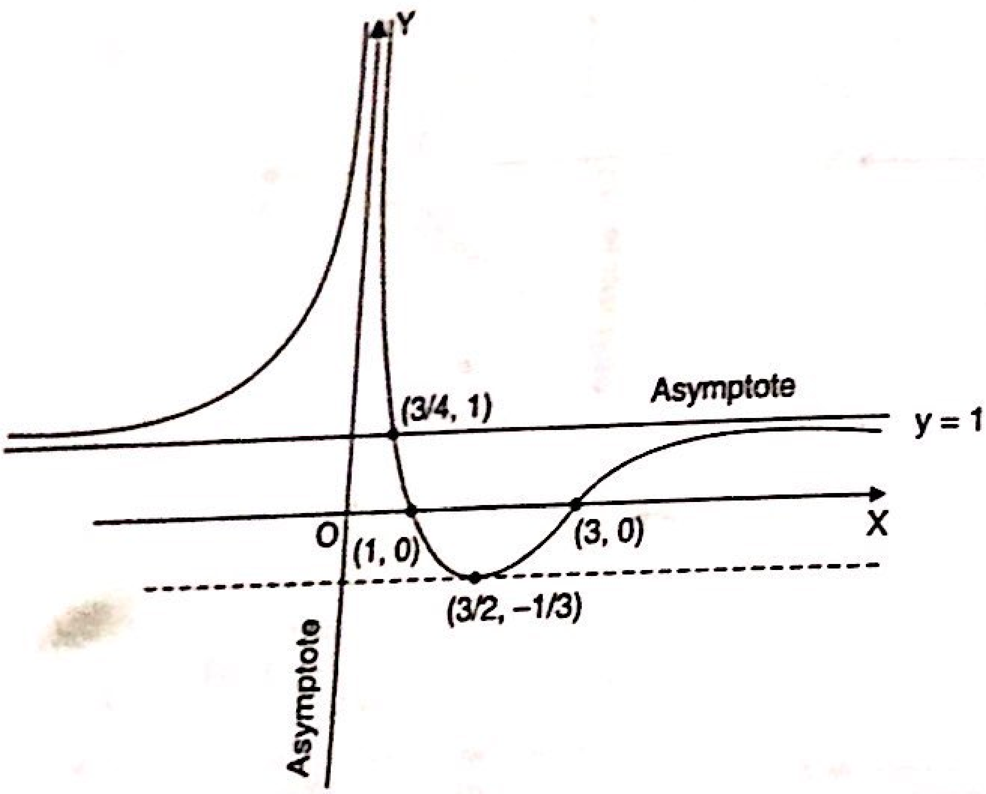

Y
tangent
(a, 0)
$\left(\frac{2a}{3}, 0\right)$
X
O
asymptote 3x + 3y = a

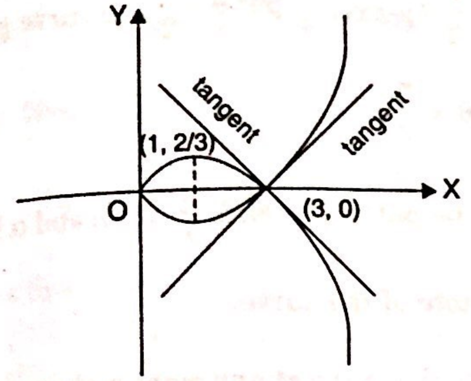
18.


Y
y = −x
y = x asymptote
O
asymptote
X

19.


Y
(3/4, 1)
Asymptote
y = 1
O
(1, 0)
(3, 0)
X
(3/2, −1/3)
Asymptote

9.



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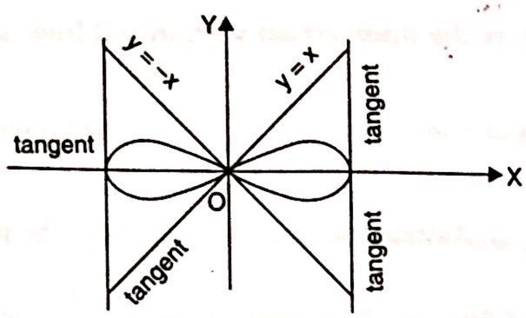


1.16. WORKING RULE TO TRACE A CURVE IN POLAR CO-ORDINATES

Salient features to trace a curve in polar co-ordinates are same as in the case of cartesian co-ordinates but first of all we will discuss how to apply these features when curve is given in polar co-ordinates.

In polar co-ordinates we have one point called the pole (generally taken as origin) and one line called the initial line (generally taken as x-axis).

Let the equation of the curve in polar co-ordinates be $f(r, \theta) = 0$

1. **Symmetry.** Curve is symmetric

(*a*) w.r.t. initial line (*i.e.*, x-axis) if equation remains unchanged when θ is changed to $-\theta$.

(*b*) w.r.t. line $\theta = \frac{\pi}{2}$ (*i.e.*, y-axis) if equation remains unchanged when θ is changed to $\pi - \theta$ or when θ is changed to $-\theta$ and r is changed to $-r$.

(*c*) w.r.t. line $\theta = \frac{\pi}{4}$ (*i.e.*, $y = x$) if equation remains unchanged when θ is changed to $\frac{\pi}{2} - \theta$.

(*d*) w.r.t. line $\theta = \frac{3\pi}{4}$ (*i.e.*, $y = -x$) if equation remains unchanged when θ is changed to $\frac{3\pi}{2} - \theta$.

(*e*) w.r.t. pole (origin) if equation remains unchanged when θ is changed to $\pi + \theta$ and when r is changed to $-r$.

2. **Pole or Origin.** Curve passes through the pole if $r = 0$ for some real values of θ and if $r = 0$ gives no real value of θ then curve does not pass through the pole.

3. **Tangent at the Pole (Origin).** The real value of θ for which $r = 0$ gives tangent at the pole.

4. **Points of Intersection with Initial Line and the Line $\theta = \frac{\pi}{2}$.** Find the points where the curve meets the initial line $\theta = 0$ (put $\theta = 0$ in curve get the value of r say r_1, then

(r_1, 0) is the intersection with initial line) and for $\theta = \frac{\pi}{2}$ (y-axis) $\left(\text{put } \theta = \frac{\pi}{2}\right)$ in curve value of r say r_2 then $\left(r_2, \frac{\pi}{2}\right)$ is the intersection with $\theta = \frac{\pi}{2}$.

5. Asymptotes. Let equation of the polar curve be put in the form $\frac{1}{r} = f(\theta)$ and α root of $f(\theta) = 0$. Then $r \sin (\theta - r) = \frac{1}{f'(\alpha)}$ is an asymptote of the curve.

6. The value of ϕ. ϕ is the angle which tangent to the curve at any point makes with radius vector at that point. We know from differential calculus

$$\tan \phi = \frac{r\,d\theta}{dr} \quad \text{(discussed in the chapter of curvature)}$$

Find the values of θ for which $\phi = 0$ or $\frac{\pi}{2}$.

Note that the values of θ for which $\phi = 0$ are tangents to the curve at the pole and values of θ for which $\phi = \frac{\pi}{2}$ cut the curve at right angles.

7. Region. Solve the polar curve for r or θ. Now the following steps follow:

(a) If for $\alpha < \theta < \beta$; r is imaginary, then no part of the curve lies between $\theta = \alpha$, $\theta = \beta$.

(b) If the greatest numerical value of r is a, then curve entirely lies within the circle $r = a$.

(c) If the least numerical value of r be b, then the curve lies outside the circle $r = b$.

(d) Trace the variations of r when θ varies in the interval $(-\infty, 0)$ and $(0, \infty)$.

(e) Find the values of θ for which r is zero or maximum or minimum.

(f) If only periodic functions sin θ or cos θ occur in the equation. Then consider values of θ from 0 to 2π $\because$ after 2π, the same values of r will be repeated.

Note 1. If convenient, convert polar curve to cartesian by means of the transformation $x = r\cos\theta$, $y = r \sin \theta$.

2. While tracing a curve, inconvenient steps may be avoided.

ILLUSTRATIVE EXAMPLES

Example 1. *Trace the curve (known as cardioid) $r = a(1 - \cos \theta)$.*

(P.T.U. Dec. 2005, Dec. 2010; May 2012)

Sol. 1. Symmetry. If we change θ to $-\theta$ equation remains unchanged $\therefore$ curve is symmetrical about initial line.

2. Pole or origin. $r = 0$ when $\cos \theta = 1$ *i.e.,* $\theta = 0$ (real)

$\therefore$ curve passes through pole and tangent at pole is $\theta = 0$

3. Points of intersection with initial line and $\theta = \frac{\pi}{2}$

Curve intersects initial line where $\theta = 0$

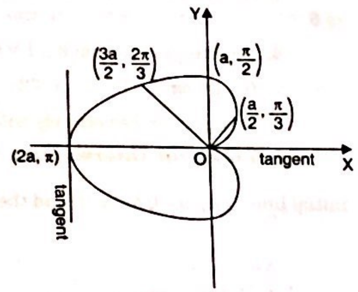
$\therefore$ $r = 0$ *i.e.,* at (0, 0)

and the line $\theta = \frac{\pi}{2}$

at $r = a$ *i.e.,* at $\left(a, \frac{\pi}{2}\right)$

and the line $\theta = \pi$

at $r = 2a$, *i.e.,* at $(2a, \pi)$



4. Asymptote. Curve has no asymptote

5. Region. We know that $-1 \le \cos \theta \le 1$

$\therefore$ $r \le 2a$ $\because$ $-1 \le \cos \theta$

$1 \ge -\cos \theta$ or $-\cos \theta \le 1$

$1 - \cos \theta \le 2$

$\therefore$ curve lies within the circle $r = 2a$.

6. Additional Points.

θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	0	$\frac{a}{2}$	a	$\frac{3a}{2}$	$2a$
ϕ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$

$$\tan \phi = r\frac{d\theta}{dr}$$

$$\tan \phi = \frac{a(1-\cos\theta)}{a \sin \theta} = \tan\frac{\theta}{2}$$

$\therefore$ $\phi = \frac{\theta}{2}$. (See the Fig.)

Example 2. *Trace the curve $r = a(1 + \sin \theta)$.* **(P.T.U., Dec. 2011)**

Sol. 1. Symmetry. The curve is symmetric about $\theta = \frac{\pi}{2}$, $\because$ on changing θ to $\pi - \theta$ equation remains unchanged.

2. Pole or Origin. $r = 0$ gives $1 + \sin \theta = 0$ *i.e.,* $\sin \theta = -1$ *i.e.,* $\theta = \frac{3\pi}{2}$ $\therefore$ curve passes through the pole and $\theta = \frac{3\pi}{2}$ is the tangent at pole.

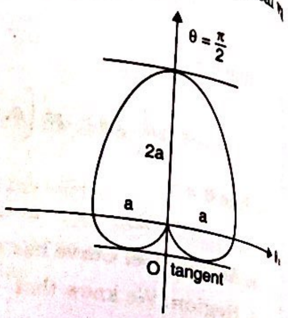
3. Asymptote. The curve has no asymptote $\because$ r does not tend to ∞ for any real value of θ.

4. Region. As $-1 \leq \sin\theta \leq 1 \ \forall \ \theta$

$\therefore \ 0 \leq 1 + \sin\theta \leq 2 \quad \therefore \quad r \leq 2a$

$\therefore$ given curve lies entirely within the circle $r = 2a$.

5. Points of Intersection. Curve intersects the initial line *i.e.,* $\theta = 0$ at $(a, 0)$ and the line $\theta = \frac{\pi}{2}$ at $\left(2a, \frac{\pi}{2}\right)$.



6. Additional Points. $\tan\phi = r\dfrac{d\theta}{dr}$

$$= a(1+\sin\theta)\cdot\frac{1}{a\cos\theta}$$

$$\tan\phi = \frac{1+\sin\theta}{\cos\theta} = \frac{(\cos\theta/2+\sin\theta/2)^2}{\cos^2\theta/2-\sin^2\theta/2} = \frac{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}} = \frac{1+\tan\theta/2}{1-\tan\theta/2}$$

$$= \tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$$

$$\therefore \qquad \phi = \frac{\pi}{4}+\frac{\theta}{2}$$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	2π
r	a	$1.5a$	$1.87a$	$2a$	a	a
ϕ	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5}{12}\pi$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	$\frac{5\pi}{4}$

Example 3. *Trace the curve* $r = a \sin 3\theta$, $a > 0$. (Called three leaved rose)

Sol. 1. Symmetry. The curve is symmetrical about the line $\theta = \frac{\pi}{2}$

2. Pole (origin). $r = 0$ for $\sin 3\theta = 0 = \sin 0, \pi, 2\pi, 3\pi,$

i.e., when $\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \ldots$

$\therefore$ Curve passes through the pole and tangents at the pole are $\theta = 0$, $\theta = \frac{\pi}{3}$, $\theta = \frac{2\pi}{3}$, $\theta = \pi$.

3. Asymptote. Curve has no asymptote.

4. Region. We know that $-1 \leq \sin 3\theta \leq 1 \ \therefore -a \leq r \leq a \ \therefore$ curve lies within the circle $r = a$.

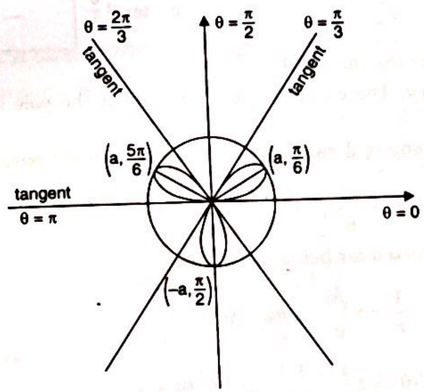
5. Additional Points.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
r	0	a	0	$-a$	0	a	0

$$\tan\phi = r\frac{d\theta}{dr} = \frac{a\sin 3\theta}{3a\cos 3\theta}$$

$$\tan\phi = \frac{1}{3}\tan 3\theta$$

Plot all these points and trace the curve.



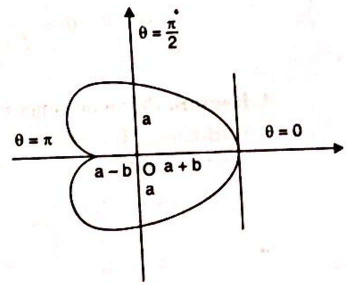
Example 4. *Trace the curve* $r = a + b\cos\theta$ $(a > b)$.

Sol. 1. Symmetry. The curve is symmetric about the initial line $\because$ on changing θ to $-\theta$ curve remains unchanged.

2. Pole or Origin. Put $r = 0$ we get $a + b\cos\theta = 0$

i.e., $\cos\theta = -\frac{a}{b}$ which is impossible

$\because$ $\frac{a}{b} > 1$ and $\cos\theta < -1$ $\therefore$ curve does not pass through the pole.



3. Asymptote. The given curve does not have any asymptote $\because$ r remains always finite.

4. Region. as $|\cos\theta| \leq 1$ and $a > b$ $\therefore$ r is always positive.

5. Points of Intersection. Curve intersects initial line where $\theta = 0$ at $(a + b, 0)$ and line $\theta = \frac{\pi}{2}$ at $\left(a, \frac{\pi}{2}\right)$

Intersection of the curve with $\theta = \pi$ is $(a - b, \pi)$.

6. Additional Points. $\tan\phi = r\dfrac{d\theta}{dr} = (a + b\cos\theta)\dfrac{1}{-b\sin\theta} = -\dfrac{a + b\cos\theta}{b\sin\theta}$

θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	$a+b$	$a+\frac{b}{2}$	a	$a-\frac{b}{2}$	$a-b$
ϕ	$-\frac{\pi}{2}$	—	$\pi - \tan^{-1}\frac{a}{b}$	—	$-\frac{\pi}{2}$

Example 5. *Trace the curve* $r^2\theta = a^2$.

Sol. 1. Symmetry. The curve is symmetric about the pole (r changed to $-r$ does not change the equation)

2. Pole. $r \neq 0$ for any real value of θ. $\therefore$ Curve does not pass through the pole.

3. Asymptotes. $r^2 = \dfrac{a^2}{\theta}$; $r \to \infty$ as $\theta \to 0$

$\therefore$ $\theta = 0$ may be tested for being asymptote

$$\frac{1}{r} = \pm\frac{\sqrt{\theta}}{a} = f(\theta) \text{ say}$$

$$f'(\theta) = \pm\frac{1}{a}\cdot\frac{1}{2\sqrt{\theta}} \Rightarrow f'(0) = \infty$$

$\therefore$ Asymptote is given by

$$r\sin(\theta - 0) = \frac{1}{f'(0)} \Rightarrow r\sin\theta = 0$$

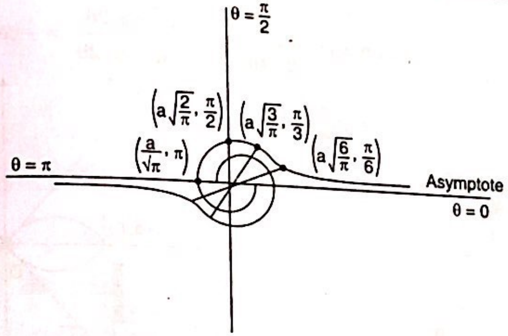
$\Rightarrow$ $\theta = 0$ is the asymptote $\because r \neq 0$

4. Region. Curve does not exist for –ve values of θ.

5. Additional Points.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π
r	∞	$\pm a\sqrt{\frac{6}{\pi}}$	$\pm a\sqrt{\frac{3}{\pi}}$	$\pm a\sqrt{\frac{2}{\pi}}$	$\pm a\sqrt{\frac{1}{\pi}}$

Note that as θ increases from 0 to π, r decreases from ∞ to $\dfrac{a}{\sqrt{\pi}}$; r has equal and opposite values.



Example 6. *Trace the curve* $r^2 = a^2\cos 2\theta$ *(known as lamniscate).* **(P.T.U., Dec. 2013)**

Sol. Equation of the curve is $r^2 = a^2\cos 2\theta$ ...(1)

1. Symmetry. On changing r to $-r$, curve remains unchanged $\therefore$ Curve is symmetric about initial line $\theta = 0$.

2. Pole or Origin. Put $r = 0$ in (1), we get $\cos 2\theta = 0$ *i.e.,* $\cos 2\theta = \cos\dfrac{\pi}{2} = \cos\left(2n\pi \pm \dfrac{\pi}{2}\right)$

$\therefore$ $\theta = n\pi \pm \dfrac{\pi}{4}$ for $n = 0, 1, 2, 3, \ldots$

$\therefore$ θ has the values $-\dfrac{\pi}{4}, \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}, \ldots$

$\therefore$ Tangents at the pole are; $\theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}, \ldots$

3. Region. $\cos 2\theta$ is –ve in 2nd and 3rd quadrants.

i.e., $\dfrac{\pi}{2} < 2\theta < \pi$ and $\pi < 2\theta < \dfrac{3\pi}{2}$

i.e., $\dfrac{\pi}{4} < \theta < \dfrac{\pi}{2}$ and $\dfrac{\pi}{2} < \theta < \dfrac{3\pi}{4}$

$\therefore$ Curve does not lie in the portion between $\left(\dfrac{\pi}{4} \text{ and } \dfrac{\pi}{2}\right)$ and $\left(\dfrac{\pi}{2}, \dfrac{3\pi}{4}\right)$.

4. Points of Intersection. Curve intersects $\theta = 0$ at $r^2 = a^2$ *i.e.,* $r = \pm a$ *i.e.,* at $(-9, 0)$ and $(9, 0)$.

5. Asymptotes. Since r is finite $\therefore$ There is no asymptote.

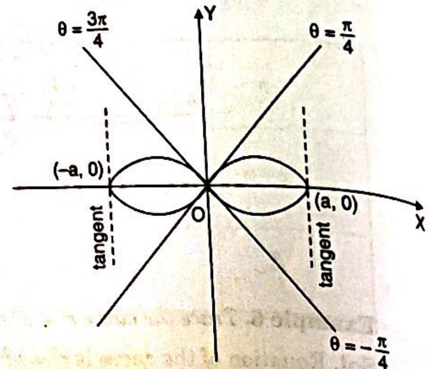
6. Additional Points. $\tan\phi = r\dfrac{d\theta}{dr}$

from (1); $2r\dfrac{dr}{d\theta} = a^2(-\sin 2\theta)\cdot 2$

$$\therefore \quad \tan\phi = r\,\frac{r}{-a^2\sin 2\theta} = \frac{a^2\cos 2\theta}{-a^2\sin 2\theta} = -\cot 2\theta$$

$$\therefore \quad \phi = \frac{\pi}{2} + 2\theta$$

θ	0	$\frac{\pi}{4}$	$\frac{3\pi}{4}$	π
r	a	0	0	$-a$
ϕ	$\frac{\pi}{2}$	π	2π	$\frac{5\pi}{2}$



Sketch of the curve is shown in adjoining figure.

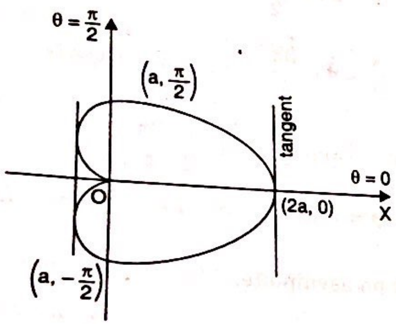
TEST YOUR KNOWLEDGE

Trace the following curves:

1. $r = a(1 + \cos\theta)$ **(P.T.U., Dec. 2011; Dec. 2012)**
2. $r = 2a\cos\theta$
3. $r = a + b\cos\theta\ (a < b)$
4. $r = a\cos 3\theta$
5. $r = a\sin 2\theta$
6. $r^2 = a^2\sin 2\theta$
7. $r = ae^{m\theta}\ (a > 0, m > 0)$
8. $r^{-1/2} = a^{-1/2}\cos\frac{\theta}{2}$ or $a = r\cos^2\frac{\theta}{2}$
9. $r\cos\theta = a\sin^2\theta$ [**Hint.** Change to cartesian coordinates $(x^2 + y^2)\,x = ay^2$]
10. $x^5 + y^5 = 5ax^2y^2$.

Answers

1.



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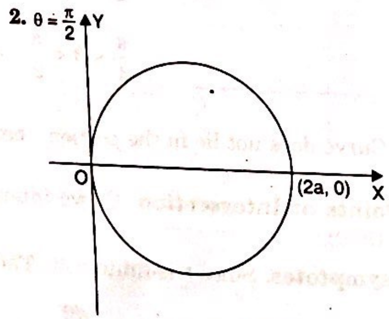

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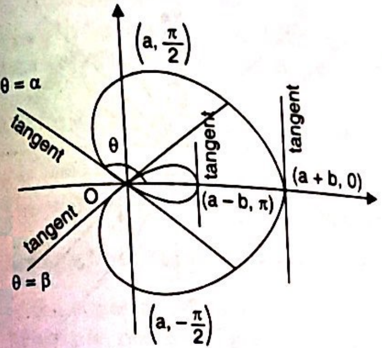
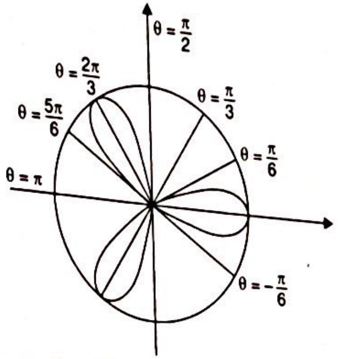


$\theta = \cos^{-1}\left(-\frac{a}{b}\right)$ has two values in the interval $(0, 2\pi)$ which lie in 2nd and 3rd quadrant (say α, β)

4.



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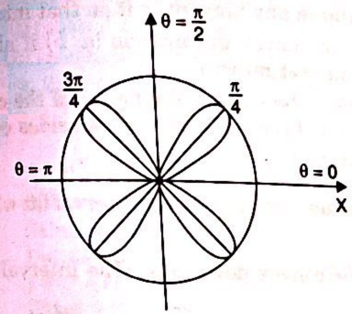

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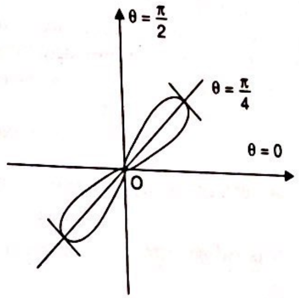

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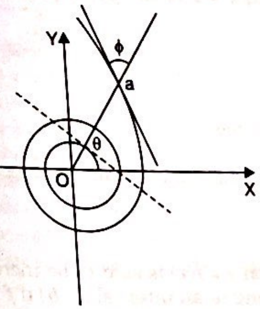

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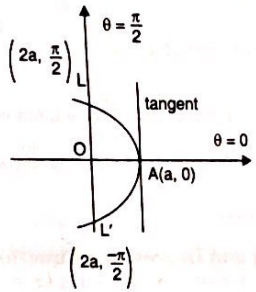
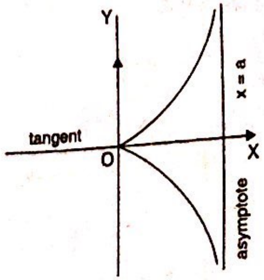
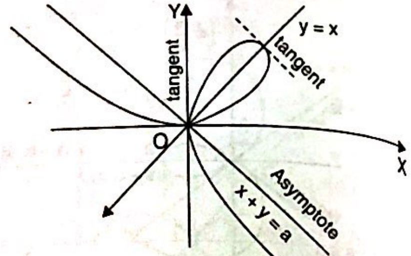


9.



10.



REVIEW OF THE CHAPTER

1. **Concavity, Convexity**—A curve $y = f(x)$ is said to concave upward (or convex downward) on (a, b) if all the points of the curve lie **above** any tangent to it on that interval. A curve $y = f(x)$ is said to concave downward (or convex upward) on (a, b) if all the points of the curve lie **below** any tangent to it on that interval.
2. **Point of Inflexion**—A point on the curve is said to be a point of inflexion if the curve crosses the tangent at that point *i.e.,* two portions of the curve on the two sides of the point lie on different sides of the tangent at that point.
3. **Criteria for Concavity, Convexity and Inflexion**—Find $\frac{d^2y}{dx^2}$, the interval for which $\frac{d^2y}{dx^2} > 0$ is called interval of concave upward or convex downward. The interval for which $\frac{d^2y}{dx^2} < 0$ is called interval of concave downward or convex upward. If $\frac{d^2y}{dx^2} = 0$ for certain value of x (say a) but $\frac{d^3y}{dx^3} \neq 0$ at $x = a$ then $x = a$ is the point of inflexion (or if $\frac{d^2y}{dx^2}$ changes sign at $x = a$).

   Note: For point of inflexion $\frac{d^2y}{dx^2} = 0$ but converse is not true.

   For points of inflexion we consider those points also (belonging to the domain of $y = f(x)$) for which $\frac{d^2y}{dx^2}$ does not exist.
4. **Increasing and Decreasing Function**—A function $y = f(x)$ is said to be increasing in an interval (a, b) if $f'(x) > 0 \ \forall \ x \in (a, b)$ and decreasing in an interval (a, b) if $f'(x) < 0 \ \forall \ x \in (a, b)$.
5. **Asymptote**—A straight line is called an asymptote of an infinite branch of a curve iff the line lies on one side of the branch of the curve and ⊥ distance from any point on curve on it tends to zero as the point approaches infinity along that branch.
6. **How to Find Asymptotes**—Asymptotes parallel to x-axis and y-axis are called rectangular asymptotes.

   Coefficient of the highest power of x equated to zero gives horizontal asymptote and coefficient of the highest power of y equated to zero gives the vertical asymptote.

   To find oblique asymptote put $x = 1$, $y = m$ in the n^{th} degree (highest degree) and $(n-1)$th degree terms and obtain $\phi_n(m), \phi_{n-1}(m)$. Find the real values of m from $\phi_n(m) = 0$ then substitute in $\phi_n'(m) + \phi_{n-1}(m) = 0$; $\phi_n'(m) \neq 0$ and get the value of c. Then asymptote will be $y = mx + c$. If $\phi_n'(m) = 0$ then c will have two values given by $\frac{c^2}{2!}\phi_n''(m) + c\,\phi_{n-1}'(m) + \phi_{n-2}(m) = 0$ where $\phi_n''(m) \neq 0$, we will get two parallel asymptotes given by $y = mx + c$. The process can be similarly repeated.
7. Asymptote of the polar curve $\frac{1}{r} = f(\theta)$ is given by $r \sin(\theta - \alpha) = \frac{1}{f'(\alpha)}$, where α is a root of the equation $f(\theta) = 0$.
8. **Multiple point**—A point through which two or more than two branches of a curve pass, is called a multiple point.
9. **Double point**—A point through which only two branches of a curve pass is called a double point.
10. **Node, cusp, conjugate point**—A double point is said to be a
    (*i*) node if two tangents at the double point are real and distinct
    (*ii*) cusp if two tangents at that point are real and coincident
    (*iii*) conjugate if two tangents at that point are not real.
11. Salient features of the curve tracing (see article 1.13).

SHORT ANSWER TYPE QUESTIONS

1. Define concavity and convexity of a curve $y = f(x)$.
   [Hint. See art 1.2]
2. Define point of inflexion of a curve and find the point(s) of inflexion for the curve $y = x^3 + 8x^2 - 270x$. ***(P.T.U., Dec. 2005)***
   [Hint. S.E. 2(*i*) art. 1.4]
3. Find the points of inflexion (if any) of the following curves. ***(P.T.U., May 2005)***
   (*i*) $y = (x^3 - 6x^2 + 9x + 6)/$ **[Hint.** S.E. 6 art. 1.4]
   (*ii*) $y = \frac{x^2+1}{x^2-1}$
   [Hint. S.E. 5 art. 1.4]
   (*iii*) $x = 3y^4 - 4y^3 + 5$ **[Hint.** S.E. 2(*ii*) art. 1.4]
   (*iv*) $x = a(2\theta - \sin\theta), y = a(2 - \cos\theta)$ **[Hint.** S.E. 3 art. 1.4]
   (*v*) $x = a - b\cos\theta, y = a\theta - b\sin\theta$
4. Find the values of x for which the curve $y = x^4 - 6x^3 + 12x^2 + 5x + 7$ is concave upward or downward:
   [Hint. S.E. 1 art. 1.4]
5. Examine the curve $y = x^3 - 9x^2 + 10x + 5$ for concavity and convexity upward.
6. Define a decreasing function and state the derivative test for decreasing function. ***(P.T.U., May 2005)***
   [Hint. Consult art 1.5]
7. Find the rectangular asymptotes (if any) of the following curves:
   (*i*) $x^2y^2(x^2 + y^2) = 4x^4 + 9y^4$ (*ii*) $(x + y)^2 = x^2y^2 - 7$
   (*iii*) $x^2y^2 - xy^2 - x^2y + x + y + 1 = 0$ ***(P.T.U., Dec. 2012)***
   (*iv*) $y^2x - a^2(x + a) = 0$ (*v*) $y = x(x - 1)(x - 2)$ ***(P.T.U., Dec. 2013)***
   (*vi*) $x^2y^3 + x^2 + 3y^2 - 9xy + 8x - 25 = 0$
   [Hint. S.E. 1 (*iii*) art. 1.10]

8. As x moves from left to right through the point $c = 2$, is the graph $y = x^3 - 3x + 2$ rising or is falling? Give reason for your answer.
[**Hint.** S.E. 7 art. 1.4(*a*)] **(P.T.U., Dec. 2004)**

9. What do you understand by a parametric curve ? Give an example of the parametric curve involving the parameters. **(P.T.U., Dec. 2005)**

10. State salient features of the curve tracing.
[**Hint.** Consult art 1.13]

11. Trace the curve $x^3 + y^3 = 3axy$.
[**Hint.** S.E. 1 art. 1.13]. **(P.T.U., May 2003)**

12. Draw rough sketch of the curve $y^2 = x + 5$. **(P.T.U., May 2008)**

13. Trace the curves **(P.T.U., Dec. 2005, 2010; May 2012)**
(*i*) $r = a(1 - \cos\theta)$
(*ii*) $r = a(1 + \cos\theta)$ **(P.T.U., Dec. 2011, Des. 2012)**

Answers

2. $\left(-\dfrac{8}{3}, \dfrac{20464}{27}\right)$

3. (*i*) $\left(2, \dfrac{4}{3}\right)$

(*ii*) Nil

(*iii*) $(5, 0), \left(\dfrac{119}{27}, \dfrac{2}{3}\right)$

(*iv*) $\left\{a\left(4n\pi \pm \dfrac{2\pi}{3} \mp \dfrac{\sqrt{3}}{2}\right), \dfrac{3a}{2}\right\}$

(*v*) $x = \dfrac{a^2 - b^2}{a}, y = a\cos^{-1}\dfrac{b}{a} - \dfrac{b}{a}\sqrt{a^2 - b^2}$.

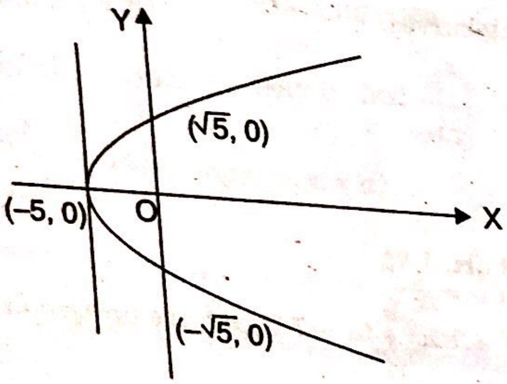
4. Concave upward in $(-\infty, 1) \cup (2, \infty)$ concave downward in $(1, 2)$

5. Concave upward for $(3, \infty)$; convex upward for $(-\infty, 3)$.

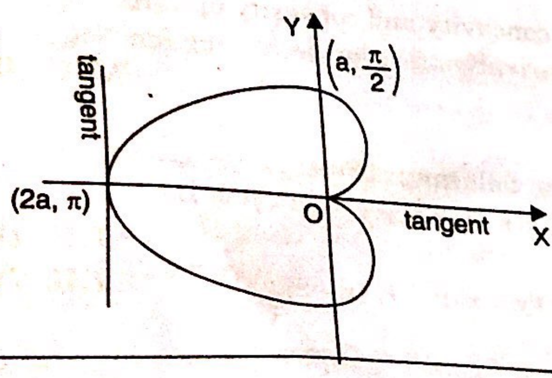
7. (*i*) $y = \pm 2, x = \pm 3$ (*ii*) $y = \pm 1, x = \pm 1$ (*iii*) $x = 0, x = 1, y = 0, y = 1$ (*iv*) $x = 0, y = \pm a$.
(*v*) no asymptote (*vi*) $x = 0, y = -1$

8. $f(x)$ is rising in $(-\infty, -1) \cup (1, \infty)$ and falling in $(-1, 1)$.

12.



13. (*i*)



(*ii*)