Wavepacket

Whenever a material particle is in motion, it is associated with a wave, and a wave is described with the help of some wavefunction.
Now if we consider $\psi(x,t) = A\sin(\omega t - kx)$ then this is the eqn. of a plane monochromatic wave which extends from $-\infty$ to $+\infty$. But a particle is finite and can be localized.
So a single infinite monochromatic wave cannot be associated with finite material & moving particle. Instead, an infinite number of such monochromatic waves which differ slightly in freq. and k are associated with particle. These waves undergo superimposition/interference to generate a finite wave, known as wavepacket and hence amplitude becomes variable, which will lead to different probabilities to locate the particle at different points in a region.
If only single monochromatic wave is associated with a particle, then probability to locate particle at different points in space at a given time will come out to be same, which is not possible for a material particle

A wave packet is formed when infinite no. of monochromatic waves differing slightly in w & k superimpose each other. (20)

Velocity of individual/component wave is called the phase/wave velocity (u or v_p)

$$\text{i.e. } v_p = \nu\lambda = \frac{w}{k} = \frac{E}{p} = \frac{c^2}{v}$$

Velocity of the wave packet i.e. average velocity of all the component waves in a group of waves (i.e wave packet) is known as group velocity (v_g) & $v_g = \frac{dw}{dk} = \frac{dE}{dp}$.

Formation of a wave packet :- Let us consider the following two monochromatic waves are associated with the motion of a material particle (Actually the no. is infinite).

$$\psi_1(x) = A\sin(\omega t - kx) \;\&\; \psi_2(x) = A\left[\sin(\omega+d\omega)t - (k+dk)x\right]$$

After superimposition of waves,

$$\psi(x) = \psi_1(x) + \psi_2(x)$$

$$= A\left[\{\sin(\omega t - kx)\} + \{\sin(\omega+d\omega)t - (k+dk)x\}\right]$$

$$\underbrace{\sin A + \sin B}_{\text{formula}}$$

$$= 2A\left(\sin\frac{2\omega+d\omega}{2}t - \frac{2k+dk}{2}x\right)\cos\left(-\frac{d\omega}{2}t + \frac{dk}{2}x\right)$$

$$= 2A\sin(\omega t - kx)\cos\left(\frac{d\omega}{2}t - \frac{dk}{2}x\right)$$

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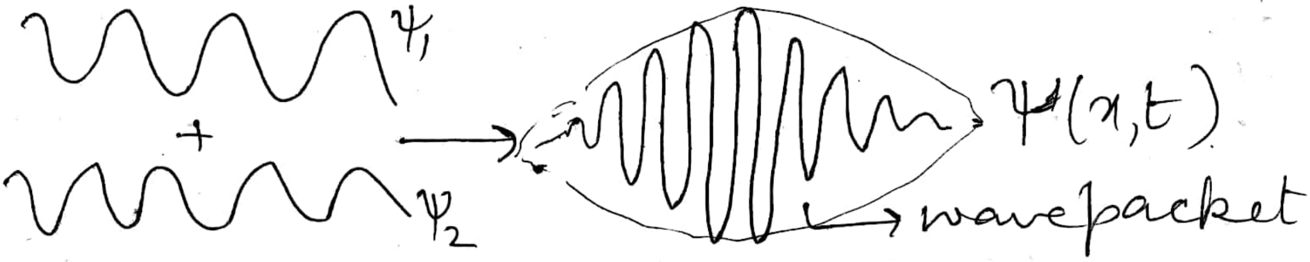
Note that small changes $d\omega$ & dk can be ignored in comparison with 2ω & $2k$, and

$$\cos\left(-\frac{d\omega}{2}t+\frac{dk}{2}x\right)=\cos\left(\frac{d\omega}{2}t-\frac{dk}{2}x\right)$$

$$\therefore \psi(x,t)=2A\sin(\omega t-kx)\cos\left(\frac{d\omega}{2}t-\frac{dk}{2}x\right)$$

$$\Rightarrow \psi(x,t)=R\sin(\omega t-kx),\quad \text{where } R=2A\cos\left(\frac{d\omega}{2}t-\frac{dk}{2}x\right)$$

Note now amplitude A is not constant, rather it has become function of x & t, i.e. variable.



It is just like modulation. $\psi_2(x,t)$ has been superimposed (data) on the carrier wave $\psi_1(x,t)$. Now $\psi(x,t)$ is not infinite, rather it has become finite in nature with variable amplitude, leading to the concept of wave packet. Where the amplitude will be maximum, probability to locate particle will maximum there at. It is important to note that it is group velocity which is equal to particle velocity and not the phase (wave) velocity is particle velocity.

Q. Prove that particle velocity (v) is equal to group velocity (v_g) and not to v_p or u, i.e phase velocity. (29)

Ans. (i) For non relativistic motions

$$E = \frac{p^2}{2m} + U \qquad v_g = \frac{d\omega}{dk} = \frac{dE}{dp}$$

$$\Rightarrow dE = \frac{\cancel{2}p\,dp}{\cancel{2}m} + 0$$

$$\Rightarrow \frac{dE}{dp} = \frac{mv}{m} = v$$

$$\Rightarrow v_g = v$$

(ii) For Relativistic motions

$$E^2 = p^2c^2 + m_0^2c^4$$

$$\Rightarrow \cancel{2}E\,dE = \cancel{2}p\,dp\,c^2 + 0$$

$$\Rightarrow \frac{dE}{dp} = \frac{mv}{mc^2}c^2 = v$$

$$\Rightarrow v_g = v$$

Hence proved.

Q. Relationship b/w v_p & v_g, also known as dispersion relation.

Ans. $v_g = \frac{d\omega}{dk}$; $v_p = \frac{\omega}{k} \Rightarrow \omega = v_p k$

$$\therefore v_g = \frac{d(v_p k)}{dk}$$

$$\Rightarrow v_g = v_p + k\frac{dv_p}{dk}$$

$$\Rightarrow v_g = v_p + \frac{2\pi}{\lambda}\frac{dv_p}{d\lambda}\frac{d\lambda}{dk}$$

Now $k = \frac{2\pi}{\lambda} \Rightarrow \frac{dk}{d\lambda} = -\frac{2\pi}{\lambda^2}$

$$\therefore v_g = v_p - \lambda\frac{dv_p}{d\lambda}$$

Case (i) when v_p either increases or decreases monotonically with λ, $\frac{dv_p}{d\lambda}$ is +ve

$\therefore v_g < v_p \rightarrow$ Normal dispersion.

(ii) When v_p does not depend on λ, $\frac{dv_p}{d\lambda} = 0$ then $v_g = v_p$ [For light in free space, $v_g = v_p = c$]

No dispersion

(iii) When $\frac{dv_p}{d\lambda}$ is −ve, $v_g > v_p \rightarrow$ Anomalous dispersion, not possible

* Write short note on de-Broglie hypothesis and prepare numericals based on it.

* Write short note on Wave-particle duality.

* Write failures of classical physics/ mechanics.