3

Linear Differential Equations

This chapter of "Linear Differential Equations" is divided into three parts.

Part I : It deals with '**Linear differential equations of first order**'.

Part II : It deals with "**linear differential equations of higher order with constant coefficients**".

Part III : It deals with "**Linear Differential Equations of higher order with variable coefficients** and **Simultaneous Linear Differential Equations with constant coefficients**".

PART-I

(Linear Differential Equations of First Order)

3.1. DEFINITION

A differential equation is said to be **linear** *if the dependent variable and its derivatives occur only in the first degree and are not multiplied together.* **(P.T.U., May 2005, June 2003)**

The general form of a linear differential equation of the first order is $\frac{dy}{dx} + Py = Q$ ...(1)

where P and Q are functions of x only or may be constants.

Equation (1) is also known as ***Leibnitz's linear equation***. **(P.T.U., May 2007)**

3.1(a). SOLVE THE LINEAR DIFFERENTIAL EQUATION $\frac{dy}{dx} + Py = Q$

(P.T.U., Dec. 2006)

To solve it, we multiply both sides by $e^{\int P\,dx}$, we get

$$\frac{dy}{dx}e^{\int P\,dx} + y\left(e^{\int P\,dx}\,P\right) = Qe^{\int P\,dx}$$

CM. $\frac{d}{dx}$ IPy = ?

or $$\frac{d}{dx}\left(y\,e^{\int P\,dx}\right) = Qe^{\int P\,dx}$$

Integrating both sides, we have $y\,e^{\int P\,dx} = \int Qe^{\int P\,dx}dx + c$

which is the required solution.

Note 1. In the general form of a linear differential equation, the coefficient of $\frac{dy}{dx}$ is unity.

The equation $R\frac{dy}{dx} + Sy = T$, where R, S and T are functions of x only or constants, must be divided by R to bring it to the general linear form.

Note 2. The factor $e^{\int P\,dx}$ on multiplying by which the L.H.S. of (1) becomes the differential coefficient of a single function is called the integrating factor (briefly written as I.F.) of (1).

Thus $\text{I.F.} = e^{\int P\,dx}$ and the solution is $y\,(\text{I.F.}) = \int Q(\text{I.F.})\,dx + c$.

Note 3. Sometimes a differential equation takes linear form if we regard x as dependent variable and y as independent variable. The equation can then be put as $\frac{dx}{dy} + Px = Q$, where P, Q are functions of y only or constants.

The integrating factor in this case is $e^{\int P\,dy}$ and the solution is $x\,(\text{I.F.}) = \int Q\,(\text{I.F.})\,dy + c$.

Note 4. $e^{\log f(x)} = f(x)$.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve* : $x\left(1-x^2\right)\frac{dy}{dx} + \left(2x^2-1\right)\, y = x^3$.

Sol. Dividing by $x\,(1-x^2)$ to make the coefficient of $\frac{dy}{dx}$ unity, the given equation becomes

$$\frac{dy}{dx} + \frac{2x^2-1}{x\left(1-x^2\right)}\, y = \frac{x^2}{1-x^2}$$

Comparing it with $\frac{dy}{dx} + Py = Q$, we have $P = \frac{2x^2-1}{x\left(1-x^2\right)}$, $Q = \frac{x^2}{1-x^2}$

Now, $$P = \frac{2x^2-1}{x(1-x)\,(1+x)} = -\frac{1}{x} + \frac{1}{2(1-x)} - \frac{1}{2(1+x)} \text{ by partial fractions}$$

$$\int P\,dx = -\log x - \frac{1}{2}\log(1-x) - \frac{1}{2}\log(1+x) = -\log\left[x(1-x)^{\frac{1}{2}}\,(1+x)^{\frac{1}{2}}\right]$$

$$= -\log\left[x\left(1-x^2\right)^{\frac{1}{2}}\right] = \log\frac{1}{x\sqrt{1-x^2}}$$

$$\text{I.F.} = e^{\int P\,dx} = e^{\log\frac{1}{x\sqrt{1-x^2}}} = \frac{1}{x\sqrt{1-x^2}}$$

Thus the solution is

$$y\,(\text{I.F.}) = \int Q(\text{I.F.})\,dx + c \quad \text{or} \quad y.\frac{1}{x\sqrt{1-x^2}} = \int\frac{x^2}{1-x^2}\times\frac{1}{x\sqrt{1-x^2}}\,dx + c = \int\frac{x}{\left(1-x^2\right)^{\frac{3}{2}}}\,dx + c$$

$$= -\frac{1}{2}\int\left(1-x^2\right)^{-\frac{3}{2}}(-2x)\,dx + c = \left(1-x^2\right)^{-\frac{1}{2}} + c \quad \because \int[f(x)]^n\, f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1}\,(n \neq -1)$$

or $$y = x + cx\sqrt{1-x^2}.$$

Example 2. *Solve* : $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)\frac{dx}{dy} = 1$.

Sol. The given equation can be written as $\frac{dy}{dx} + \frac{y}{\sqrt{x}} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$

Comparing it with $\frac{dy}{dx}+Py=Q$, we have $P=\frac{1}{\sqrt{x}}$, $Q=\frac{e^{-2\sqrt{x}}}{\sqrt{x}}$

$$\text{I.F.}= e^{\int P\,dx}=e^{\int\frac{1}{\sqrt{x}}dx}=e^{\frac{x^{1/2}}{1/2}}=e^{2\sqrt{x}}$$

$\therefore$ The solution is $\quad y(\text{I.F.})=\int Q(\text{I.F.})\,dx+c$ or $\quad ye^{2\sqrt{x}}=\int\frac{e^{-2\sqrt{x}}}{\sqrt{x}}\cdot e^{2\sqrt{x}}dx+c$

or
$$ye^{2\sqrt{x}}=\int x^{-\frac{1}{2}}\,dx+c=2\sqrt{x}+c$$

Example 3. *Solve :* $\left(1+y^2\right)dx=\left(tan^{-1}\,y-x\right)dy$.

Sol. The given equation can be written as $\left(1+y^2\right)\frac{dx}{dy}+x=\tan^{-1}y$

Dividing by $(1+y^2)$, we get $\quad \frac{dx}{dy}+\frac{x}{1+y^2}=\frac{\tan^{-1}y}{1+y^2}$

which is of the form $\quad \frac{dx}{dy}+Px=Q$

Here $\quad P=\frac{1}{1+y^2},\ Q=\frac{\tan^{-1}y}{1+y^2}$

$$\text{I.F.}= e^{\int P\,dy}=e^{\int\frac{1}{1+y^2}dy}=e^{\tan^{-1}y}$$

$\therefore$ The solution is $x\,(\text{I.F.})=\int Q(\text{I.F.})\,dy+c$

or
$$xe^{\tan^{-1}y}=\int\frac{\tan^{-1}y}{1+y^2}\,e^{\tan^{-1}y}\,dy+c=\int te^t\,dt+c,\ \text{where } t=\tan^{-1}y$$

$$=te^t-\int 1\,.\,e^t\,dt+c=te^t-e^t+c=\left(\tan^{-1}y-1\right)e^{\tan^{-1}y}+c$$

or
$$x=\tan^{-1}y-1+ce^{-\tan^{-1}y}.$$

TEST YOUR KNOWLEDGE

Solve the following differential equations :

1. $\frac{dy}{dx}+\frac{y}{x}=x^3-3$
2. $x\log x\,\frac{dy}{dx}+y=2\log x$
3. $(x+1)\frac{dy}{dx}-y=e^x(x+1)^2$
4. $\left(x^2+1\right)\frac{dy}{dx}+2xy=x^2$
5. $\cos^2x\,\frac{dy}{dx}+y=\tan x$
6. $\left(1+x^3\right)\frac{dy}{dx}+6x^2y=1+x^2$

7. $\frac{dy}{dx} + y\cot x = \cos x$

8. $\left(1+x^2\right)\frac{dy}{dx} + y = e^{\tan^{-1} x}$

9. $\frac{dy}{dx} + y\cot x = 4x\,\text{cosec}\,x$, if $y = 0$ when $x = \frac{\pi}{2}$

10. $\frac{dy}{dx} + y\cot x = 5e^{\cos x}$, if $y = -4$ when $x = \frac{\pi}{2}$

11. $\frac{dy}{dx} - y\tan x = 3e^{-\sin x}$, if $y = 4$ when $x = 0$

12. $\left(1-x^2\right)\frac{dy}{dx} + 2xy = x\sqrt{1-x^2}$

13. $x\frac{dy}{dx} + y = e^x - xy$

14. $\left(1+y^2\right) + \left(x - e^{-\tan^{-1} y}\right)\frac{dy}{dx} = 0$

15. $e^{-y}\sec^2 y\,dy = dx + x\,dy$

16. $\left(x+2y^3\right)\frac{dy}{dx} = y$

17. $ye^y\,dx = \left(y^3 + 2xe^y\right)dy$

18. $\frac{dy}{dx} = \frac{y}{2y\log y + y - x}$

19. $x^2\frac{dy}{dx} = 3x^2 - 2xy + 1$

20. $\sqrt{1-y^2}\,dx = \left(\sin^{-1} y - x\right)dy$

ANSWERS

1. $10xy = 2x^5 - 15x^2 + c$

2. $y\log x = (\log x)^2 + c$

3. $y = (x+1)(e^x + c)$

4. $y(x^2+1) = \frac{x^3}{3} + c$

5. $y = \tan x - 1 + c\,e^{-\tan x}$

6. $y\left(1+x^3\right)^2 = y + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^6}{6} + c$

7. $y\sin x = \frac{1}{2}\sin^2 x + c$

8. $2ye^{\tan^{-1} x} = e^{2\tan^{-1} x} + c$

9. $y\sin x = 2x^2 - \frac{\pi^2}{2}$

10. $y\sin x = 5e^{\cos x} - 9$

11. $y\cos x = 7 - 3e^{-\sin x}$

12. $y = \sqrt{1-x^2} + c\left(1-x^2\right)$

13. $xy = \frac{1}{2}e^x + ce^{-x}$

14. $x = e^{-\tan^{-1} y}\left(\tan^{-1} y + c\right)$

15. $xe^y = \tan y + c$

16. $x = y^3 + cy$

17. $x = y^2\left(c - e^{-y}\right)$

18. $x = \frac{c}{y} + y\log y$

19. $y = x + x^{-1} + cx^{-2}$

20. $x = \sin^{-1} y - 1 + ce^{-\sin^{-1} y}$

.2. EQUATIONS REDUCIBLE TO THE LINEAR FORM (Bernoulli's Equation)

(P.T.U., May 2007, Jan 2009)

(i) An equation of the form $\frac{dy}{dx} + Py = Qy^n$

where P and Q are functions of x only or constants is known as *Bernoulli's equation*. Though not linear, it can be made linear.

Dividing both sides of (1) by y^n, we have $y^{-n}\frac{dy}{dx}+Py^{1-n}=Q$ ...(2)

Putting $y^{1-n}=z$ so that $(1-n)y^{-n}\frac{dy}{dx}=\frac{dz}{dx}$

or $y^{-n}\frac{dy}{dx}=\frac{1}{1-n}\frac{dz}{dx}$

Equation (2) becomes $\frac{1}{1-n}\frac{dz}{dx}+Pz=Q$ or $\frac{dz}{dx}+(1-n)Pz=(1-n)Q$

which is a linear differential equation with z as the dependent variable.

(*ii*) **General equation reducible to linear form is $f'(y)\frac{dy}{dx}+Pf(y)=Q$** ...(1)

where P and Q are functions of x only or constants.

Putting $f(y)=z$ so that $f'(y)\frac{dy}{dx}=\frac{dz}{dx}$

Equation (1) becomes $\frac{dz}{dx}+Pz=Q$

which is linear.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve the following differential equations.*

(*i*) $x\frac{dy}{dx}+y=x^3y^6$ **(P.T.U., May 2007)**

(*ii*) $y'+y=y^2$ **(P.T.U., May 2008)**

(*iii*) $\left(xy^2-e^{1/x^3}\right)dx-x^2y\,dy=0$ **(P.T.U., Dec. 2001)**

Sol. (*i*) $x\frac{dy}{dx}+y=x^3y^6$

Dividing by xy^6, we get

$$y^{-6}\frac{dy}{dx}+\frac{1}{x}y^{-5}=x^2 \quad ...(1)$$

Put $y^{-5}=z$ so that

$$-5y^{-6}\frac{dy}{dx}=\frac{dz}{dx}$$

Equation(1) becomes

$$-\frac{1}{5}\frac{dz}{dx}+\frac{1}{x}z=x^2$$

or

$$\frac{dz}{dx}-\frac{5}{x}z=-5x^2$$

which is linear in z where $P=-\frac{5}{x}, Q=-5x^2$

$$\text{I.F.}=e^{\int-\frac{5}{x}dx}=e^{-5\log x}=e^{\log x^{-5}}=x^{-5}=\frac{1}{x^5}$$

∴ Solution in z is

$$z(\text{I.F.}) = \int Q \cdot \text{I.F.}\, dx + c$$

$$z \cdot \frac{1}{x^5} = \int -5x^2 \cdot \frac{1}{x^5}\, dx + c$$

$$= -5\int x^{-3}\, dx + c$$

$$= -5\,\frac{x^{-2}}{-2} + c = \frac{5}{2x^2} + c$$

Substituting the value of z, we get

$$\frac{1}{y^5} \cdot \frac{1}{x^5} = \frac{5}{2x^2} + c$$

or $$\frac{1}{y^5} = \frac{5x^3}{2} + cx^5$$

(ii) $$y' + y = y^2$$

or $$\frac{dy}{dx} + y = y^2$$

Divide by y^2;

$$y^{-2}\frac{dy}{dx} + \frac{1}{y} = 1 \quad ...(1)$$

Put $$\frac{1}{y} = z \quad \therefore \quad -\frac{1}{y^2}\frac{dy}{dx} = \frac{dz}{dx}$$

Substituting in (1);

$$-\frac{dz}{dx} + z = 1 \quad \text{or} \quad \frac{dz}{dx} - z = -1$$

which is linear differential equation in z

where $$P = -1, Q = -1$$

$$\text{I.F.} = e^{\int -1\, dx} = e^{-x}$$

Solution in z is

$$z(\text{I.F.}) = \int Q \cdot (\text{I.F.})\, dx + c$$

or $$z\, e^{-x} = \int (-1)\, e^{-x}\, dx + c = e^{-x} + c$$

or $$z = 1 + ce^{x}$$

Substituting the value of z ;

$$\frac{1}{y} = 1 + ce^{x}$$

or $$y = \frac{1}{1 + ce^{x}}$$

(iii) $\left(xy^2 - e^{\frac{1}{x^3}}\right) dx - x^2 y\, dy = 0$

$$x^2y\frac{dy}{dx}-xy^2+e^{\frac{1}{x^3}}=0$$

or

$$x^2y\frac{dy}{dx}-xy^2=-e^{\frac{1}{x^3}}$$

or

$$y\frac{dy}{dx}-\frac{1}{x}y^2=-\frac{1}{x^2}e^{\frac{1}{x^3}}$$

Put $y^2=z$;

$$2y\frac{dy}{dx}=\frac{dz}{dx}$$

$$\frac{1}{2}\frac{dz}{dx}-\frac{1}{x}z=-\frac{1}{x^2}e^{\frac{1}{x^3}}$$

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or

$$\frac{dz}{dx}-\frac{2}{x}z=-\frac{2}{x^2}e^{\frac{1}{x^3}}$$

which is linear differential equation in z, where $P=-\frac{2}{x}$, $Q=-\frac{2}{x^2}e^{\frac{1}{x^3}}$

$$\text{I.F.}=e^{\int\frac{-2}{x}dx}=e^{-2\log x}=e^{\log x^{-2}}=\frac{1}{x^2}$$

Solution in z is

$$z(\text{I.F.})=\int Q\cdot(\text{I.F.})\,dx+c$$

or

$$z\left(\frac{1}{x^2}\right)=\int\frac{1}{x^2}\left(-\frac{2}{x^2}e^{\frac{1}{x^3}}\right)dx+c=-2\int\frac{1}{x^4}e^{\frac{1}{x^3}}\,dx+c$$

Put $\frac{1}{x^3}=t$ $\therefore$

$$\frac{-3}{x^4}dx=dt$$

$\therefore$

$$z\cdot\frac{1}{x^2}=(-2)\int e^t\frac{dt}{-3}+c=\frac{2}{3}e^t+c=\frac{2}{3}e^{\frac{1}{x^3}}+c$$

or

$$\frac{y^2}{x^2}=\frac{2}{3}e^{\frac{1}{x^3}}+c \quad\text{or}\quad 3y^2=2x^2e^{\frac{1}{x^3}}+cx^2$$

Example 2. *Solve :* $xy\left(1+xy^2\right)\frac{dy}{dx}=1.$ **(P.T.U., May 2009)**

Sol. The given equation can be written as $\frac{dx}{dy}-yx=y^3x^2$

Dividing by x^2, we have $x^{-2}\dfrac{dx}{dy} - y\,x^{-1} = y^3$

Putting $x^{-1} = z$ so that $-x^{-2}\dfrac{dx}{dy} = \dfrac{dz}{dy}$ or $x^{-2}\dfrac{dx}{dy} = -\dfrac{dz}{dy}$

Equation (1) becomes $-\dfrac{dz}{dy} - yz = y^3$

or $\dfrac{dz}{dy} + yz = -y^3$ which is linear in z

$$\text{I.F.} = e^{\int y\,dy} = e^{\frac{1}{2}y^2}$$

∴ The solution is $z\,(\text{I.F.}) = \int -y^3\,(\text{I.F.})\,dy + c$

or $$z\,e^{\frac{1}{2}y^2} = \int -y^3 e^{\frac{1}{2}y^2}\,dy + c$$

$$z\,e^{\frac{1}{2}y^2} = -\int y^2\,e^{\frac{1}{2}y^2}\,y\,dy + c = -\int 2t\,e^t\,dt + c \quad \text{where } t = \frac{1}{2}y^2$$

or $$z\,e^{\frac{1}{2}y^2} = -2\left[t\,e^t - \int 1 \cdot e^t\,dt\right] + c = -2\left(t e^t - e^t\right) + c = -2e^{\frac{1}{2}y^2}\left(\frac{1}{2}y^2 - 1\right) + c$$

or $$z = -2\left(\frac{1}{2}y^2 - 1\right) + c\,e^{-\frac{1}{2}y^2}$$

or $$\frac{1}{x} = 2 - y^2 + c\,e^{-\frac{1}{2}y^2} \qquad \left(\because\ z = \frac{1}{x}\right)$$

Example 3. *Solve* $\dfrac{dy}{dx} + x\sin 2y = x^3\cos^2 y$ **(P.T.U., May 2002)**

Sol. Dividing by $\cos^2 y$, we have $\sec^2 y\dfrac{dy}{dx} + x\,\dfrac{2\sin y\cos y}{\cos^2 y} = x^3$

or $$\sec^2 y\,\frac{dy}{dx} + 2x\tan y = x^3 \qquad \ldots(1)$$

Putting $\tan y = z$ so that $\sec^2 y\,\dfrac{dy}{dx} = \dfrac{dz}{dx}$

Equation (1) becomes $\dfrac{dz}{dx} + 2xz = x^3$ which is linear in z.

$$\text{I.F.} = e^{\int 2x\,dx} = e^{x^2}$$

∴ The solution is $z\,.\,e^{x^2} = \int x^3\,.\,e^{x^2}\,dx + c = \int x^2\,e^{x^2}\,.\,x\,dx + c$

$$= \frac{1}{2}\int t\,e^t\,dt + c \quad \text{where } t = x^2$$

$$= \frac{1}{2}(t-1)e^t + c = \frac{1}{2}\left(x^2 - 1\right)e^{x^2} + c$$

$$z = \frac{1}{2}\left(x^2 - 1\right) + c\,e^{-x^2}$$

$$\tan y = \frac{1}{2}\left(x^2 - 1\right) + c\,e^{-x^2}. \qquad (\because \; z = \tan y)$$

Example 4. *Solve :* $e^y y' = e^x (e^x - e^y)$. **(P.T.U., May 2004)**

Sol.
$$e^y y' = e^x (e^x - e^y) \qquad \ldots(1)$$

Put
$$e^y = z$$

Differentiating w.r.t. x
$$e^y \frac{dy}{dx} = \frac{dz}{dx}$$

$$e^y y' = \frac{dz}{dx} \qquad \ldots(2)$$

Substituting in (1)

$$\frac{dz}{dx} = e^{2x} - e^x \,.\, z$$

$$\frac{dz}{dx} + e^x \,.\, z = e^{2x}$$ which is a linear differential equation in z.

$$\text{I.F.} = e^{\int e^x dx} = e^{e^x}$$

Solution is
$$z \,.\, e^{e^x} = \int e^{2x} \,.\, e^{e^x} dx + c$$

Put
$$e^x = t \quad \therefore \quad e^x dx = dt$$

$\therefore$
$$z \,.\, e^{e^x} = \int t\, e^t\, dt + c \quad \text{Integrate by parts}$$
$$= (t - 1)\, e^t + c$$

$\therefore$
$$e^y \,.\, e^{e^x} = \left(e^x - 1\right) e^{e^x} + c \quad \text{or} \quad e^{e^x}\left(1 - e^x + e^y\right) = c.$$

Example 5. *Solve : (2x log x – xy) dy = – 2y dx.* **(P.T.U., Dec. 2004)**

Sol.
$$2x \log x - xy = -2y \frac{dx}{dy}$$

$$2y \frac{dx}{dy} - y \,.\, x + 2x \log x = 0$$

$$\frac{dx}{dy} - \frac{1}{2}x + \frac{x}{y} \log x = 0$$

Divide by x;
$$\frac{1}{x}\frac{dy}{dx} + \frac{1}{y} \log x = \frac{1}{2}$$

Put $\log x = z$ $\therefore$
$$\frac{1}{x}\frac{dx}{dy} = \frac{dz}{dy}$$

$$\frac{dz}{dy} + \frac{1}{y} z = \frac{1}{2}$$, which is linear differential equation in z.

$$\text{I.F.} = e^{\int \frac{1}{y} dy} = e^{\log y} = y$$

$\therefore$ Solution is $$z \cdot y = \int y \cdot \frac{1}{2} dy + c = \frac{y^2}{4} + c$$

or $$y \log x = \frac{y^2}{4} + c.$$

Example 6. *Solve* : $\dfrac{dy}{dx} - \tan xy = -y^2 \sec^2 x.$ **(P.T.U., Dec. 2004)**

Sol. $\dfrac{dy}{dx} - \tan x \cdot y = -y^2 \sec^2 x$

Divide by y^2 : $\dfrac{1}{y^2} \dfrac{dy}{dx} - \tan x \dfrac{1}{y} = -\sec^2 x$

Put $$\frac{1}{y} = z \quad \therefore \quad -\frac{1}{y^2}\frac{dy}{dx} = \frac{dz}{dx}$$

$\therefore$ $$-\frac{dz}{dx} - \tan x \cdot z = -\sec^2 x$$

or $$\frac{dz}{dx} + \tan x \cdot z = \sec^2 x$$

which is linear differential equation in z

$$\text{I.F.} = e^{\int \tan x\, dx} = e^{-\log \cos x} = e^{\log \sec x} = \sec x$$

$\therefore$ Its solution is

$$z \sec x = \int \sec^2 x \cdot \sec x\, dx + c = \int \sec^3 x\, dx + c \quad \ldots(1)$$

Let $$\text{I} = \int \sec^3 x\, dx = \int \sec x \sec^2 x\, dx \text{ Integrate by parts}$$

$$= (\sec x)(\tan x) - \int \sec x \tan x \tan x\, dx$$

$$= \sec x \tan x - \int \sec x (\sec^2 x - 1)\, dx$$

$$= \sec x \tan x - \text{I} + \int \sec x\, dx$$

$\therefore$ $$2\text{I} = \sec x \tan x + \log(\sec x + \tan x)$$

$\therefore$ $$\text{I} = \frac{1}{2}\left[\sec x \tan x + \log(\sec x + \tan x)\right]$$

Substituting in Equation (1),

$$z \,.\, \sec x = \frac{1}{2}\,[\sec x \tan x + \log(\sec x + \tan x)] + c$$

Substitute the value of z,

$$\frac{1}{y}\sec x = \frac{1}{2}\,[\sec x \tan x + \log(\sec x + \tan x)] + c.$$

TEST YOUR KNOWLEDGE

Solve the following differential equations :

1. $2\frac{dy}{dx} = \frac{y}{x} + \frac{y^2}{x^2}$

2. $\frac{dy}{dx} - x^2 y = y^2 e^{-\frac{1}{3}x^3}$

3. $(x^3 y^2 + xy)\, dx = dy$

4. (a) $e^y\left(\frac{dy}{dx}+1\right) = e^x$ **(P.T.U., May 2002)**

   [**Hint:** Put $e^y = z$]

   (b) $(x+1)\frac{dy}{dx} + 1 = 2e^{-y}$

   [**Hint:** Divide by $(x + 1)e^{-y}$ and Put $e^y = z$]

   (P.T.U., May 2001)

5. $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)\, e^x \sec y$

6. $\frac{dy}{dx} + y\tan x = y^3 \cos x$

7. $\frac{dy}{dx} + \frac{x}{1-x^2}\, y = x\sqrt{y}$

8. $\frac{dy}{dx} + \frac{y\log y}{x} = \frac{y(\log y)^2}{x^2}$

9. $y - \cos x\frac{dy}{dx} = y^2(1-\sin x)\cos x$, given that $y = 2$ when $x = 0$.

10. $y\left(2xy + e^x\right) dx = e^x\, dy$

11. $\tan y\frac{dy}{dx} + \tan x = \cos y \cos^2 x$

ANSWERS

1. $\frac{x}{y} = 1 + c\sqrt{x}$

2. $y(c-x) = e^{\frac{1}{3}x^3}$

3. $\frac{1}{y} = -\,x^2 + 2 + ce^{-\frac{1}{2}x^2}$

4. (a) $e^{x+y} = \frac{1}{2}e^{2x} + c$ (b) $(x+1)\,e^y = 2x + c$

5. $\sin y = (1+x)\,(e^x + c)$

6. $\cos^2 x = y^2\left(c - 2\sin x + \frac{2}{3}\sin^3 x\right)$

7. $\sqrt{y} = -\frac{1}{3}\left(1-x^2\right) + c\left(1-x^2\right)^{\frac{1}{4}}$

8. $\frac{1}{x\log y} = \frac{1}{2x^2} + c$

9. $2\,(\tan x + \sec x) = y\,(2\sin x + 1)$

10. $e^x = y\left(c - x^2\right)$

11. $\sec y = (c + \sin x)\cos x.$

PART-II

(Linear differential equations of higher order with constant coefficients)

3.3. DEFINITIONS

A **linear differential equation** is that in which the dependent variable and its derivatives occur only in the first degree and are not multiplied together. Thus, the general linear differential equation of the nth order is of the form $\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + P_{n-1} \frac{dy}{dx} + P_n y = X$ where $P_1, P_2, \dots, P_{n-1}, P_n$ and X are functions of x only.

A **linear differential equation with constant coefficients** is of the form

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + a_{n-1} \frac{dy}{dx} + a_n y = X \quad \dots(1)$$

where $a_1, a_2, \dots, a_{n-1}, a_n$ are constants and X is either a constant or a function of x only.

3.4. THE OPERATOR D

The part $\frac{d}{dx}$ of the symbol $\frac{dy}{dx}$ may be regarded as an operator such that when it operates on y, the result is the derivative of y.

Similarly, $\frac{d^2}{dx^2}, \frac{d^3}{dx^3}, \dots, \frac{d^n}{dx^n}$ may be regarded as operators.

For brevity, we write $\frac{d}{dx} \equiv D, \frac{d^2}{dx^2} \equiv D^2, \dots, \frac{d^n}{dx^n} \equiv D^n$

Thus, the symbol D is a **differential operator or simply an operator.**

Written in symbolic form, equation (1) becomes $(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_{n-1} D + a_n) y = X$

or $\quad f(D)\, y = X$

where $\quad f(D) = D^n + a_1 D^{n-1} + a_1 D^{n-2} + \dots + a_{n-1} D + a_n$

i.e., $f(D)$ is a polynomial in D.

The operator D can be treated as an algebraic quantity.

i.e.,
$$D(u + v) = Du + Dv$$
$$D(\lambda u) = \lambda Du$$
$$D^p D^q u = D^{p+q} u$$
$$D^p D^q u = D^q D^p u$$

The polynomial $f(D)$ can be factorised by ordinary rules of algebra and the factors may be written in any order.

3.5. THEOREMS

Theorem 1. If $y = y_1, y = y_2, \dots, y = y_n$ *are n linearly independent solutions of the differential equation*

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_n) y = 0 \quad \dots(1)$$

then $u = c_1 y_1 + c_2 y_2 + \dots + c_n y_n$ *is also its solution, where* $c_1, c_2, \dots, c_n$ *are arbitrary constants.*

Proof. Since $y=y_1$, $y=y_2$, ..., $y=y_n$ are solution of equation (*i*).

$$\therefore \left.\begin{array}{l} D^n y_1 + a_1 D^{n-1} y_1 + a_2 D^{n-2} y_1 + \cdots + a_n y_1 = 0 \\ D^n y_2 + a_1 D^{n-1} y_2 + a_2 D^{n-2} y_2 + \cdots + a_n y_2 = 0 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ D^n y_n + a_1 D^{n-1} y_n + a_2 D^{n-2} y_n + \cdots + a_n y_n = 0 \end{array}\right\} \quad \ldots(ii)$$

Now, $D^n u + a_1 D^{n-1} u + a_2 D^{n-2} u + \ldots\ldots + a_n u$

$$= D^n (c_1 y_1 + c_2 y_2 + \ldots\ldots + c_n y_n) + a_1 D^{n-1} (c_1 y_1 + c_2 y_2 + \ldots\ldots + c_n y_n)$$
$$+ a_2 D^{n-2} (c_1 y_1 + c_2 y_2 + \ldots\ldots + c_n y_n) + \ldots \quad \ldots \quad \ldots \quad \ldots \quad + a_n (c_1 y_1 + c_2 y_2 + \ldots + c_n y_n)$$
$$= c_1 (D^n y_1 + a_1 D^{n-1} y_1 + a_2 D^{n-2} y_1 + \ldots + a_n y_1) + c_2 (D^n y_2 + a_1 D^{n-1} y_2 + a_2 D^{n-2} y_n + \ldots + a_n y_2)$$
$$+ \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad + c_n (D^n y_n + a_1 D^{n-1} y_n + a_2 D^{n-2} y_n + \ldots + a_n y_n)$$
$$= c_1 (0) + c_2 (0) + \ldots + c_n (0) \qquad [\because \text{ of } (ii)]$$
$$= 0$$

which shows that $u = c_1 y_1 + c_2 y_2 + \ldots\ldots + c_n y_n$ is also the solution of equation (*i*).

Since this solution contains n arbitrary constants, it is the general or complete solution of equation (*i*).

Theorem 2. *If $y = u$ is the complete solution of the equation $f(D)\, y = 0$ and $y = v$ is a particular solution (containing no arbitrary constants) of the equation $f(D)\, y = X$, then the complete solution of the equation*

$$f(D)\, y = X \ \text{ is } \ y = u + v.$$

Proof. Since $y = u$ is the complete solution of the equation $f(D)\, y = 0$ ...(*i*)

$$\therefore \qquad f(D)\, u = 0 \qquad \ldots(ii)$$

Also, $y = v$ is a particular solution of the equation $f(D)\, y = X$ ...(

$$\therefore \qquad f(D)\, v = X \qquad \ldots($$

Adding (*ii*) and (*iv*), we have $f(D)\,(u + v) = X$

Thus $y = u + v$ satisfies the equation (*iii*), hence it is the **complete solution (C.S.)** because it contains n arbitrary constants.

The part $y = u$ is called the **complementary function (C.F.)** and the part $y = v$ is called the **particular integral (P.I.)** of the equation (*iii*). **(P.T.U., Jan. 2010)**

$\therefore$ The complete solution of equation (*iii*), is **y = C.F. + P.I.**

Thus in order to solve the equation (iii), we first find the C.F. i.e., the C.S. of equation (i) and then the P.I. i.e., a particular solution of equation (iii).

3.6. AUXILIARY EQUATION (A.E.)

Consider the differential equation $(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots\ldots\ldots + a_n)\, y = 0$ ...(*i*)

Let $y = e^{mx}$ be a solution of (*i*), then $Dy = m\, e^{mx}$, $D^2 y = m^2 e^{mx}$, ..., $D^{n-2} y = m^{n-2} e^{mx}$

$$D^{n-1} y = m^{n-1} e^{mx}, \ D^n y = m^n e^{mx}$$

Substituting the values of y, Dy, $D^2 y$, ..., $D^n y$ in (*i*), we get

$$(m^n + a_1 m^{n-1} + a_2 m^{n-2} + \ldots\ldots + a_n)\, e^{mx} = 0$$

or
$$m^n + a_1 m^{n-1} + a_2 m^{n-2} + \ldots + a_n = 0, \text{ since } e^{mx} \neq 0 \qquad \ldots(ii)$$

Thus $y = e^{mx}$ will be a solution of equation (*i*) if m satisfies equation (*ii*).

Equation (*ii*) is called the auxiliary equation for the differential equation (*i*).

Replacing m by D in (*ii*), we get $D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_n = 0$

Equation (*ii*) gives the same values of m as equation (*iii*) gives of D. In practice, we take equation (*iii*) as the auxiliary equation which is obtained by equating to zero the symbolic co-efficient of y in equation (*i*).

Definition. The equation obtained by equating to zero the symbolic coefficient of y is called the **auxiliary equation**, briefly written as A.E.

3.7. RULES FOR FINDING THE COMPLEMENTARY FUNCTION

Consider the equation $(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_n) y = 0$ ...(*i*)

where all the a_i's are constant

Its auxiliary equation is $D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_n = 0$ ...(*ii*)

Let $D = m_1, m_2, m_3, \ldots, m_n$ be the roots of the A.E. The solution of equation (*i*) depends upon the nature of roots of the A.E. The following cases arise

Case I. If all the roots of the A.E. are real and distinct, then equation (*ii*) is equivalent is

$$(D - m_1)(D - m_2) \ldots (D - m_n) = 0 \qquad \ldots(iii)$$

Equation (*iii*) will be satisfied by the solutions of the equations

$$(D - m_1) y = 0, (D - m_2) y = 0, \ldots, (D - m_n) y = 0$$

Now, consider the equation $(D - m_1) y = 0$, *i.e.*, $\frac{dy}{dx} - m_1 y = 0$

dy/dx + Py

It is a linear equation and I.F. $= e^{\int -m_1 dx} = e^{-m_1 x}$

Its solution is $y \cdot e^{-m_1 x} = \int 0 \cdot e^{-m_1 x} dx + c_1$ or $y = c_1 e^{m_1 x}$

Similarly, the solution of $(D - m_2) y = 0$ is $y = c_2 e^{m_2 x}$

....

The solution of $(D - m_n) y = 0$ is $y = c_n e^{m_n x}$

Hence the complete solution of equation (*i*) is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x} \qquad \ldots(iv)$$

Case II. If two roots of the A.E. are equal, let $m_1 = m_2$

The solution obtained in equation (*iv*) becomes

$$y = (c_1 + c_2) e^{m_1 x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x} = c e^{m_1 x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$$

It contains $(n - 1)$ arbitrary constants and is, therefore, not the complete solution of equation (*i*).

The part of the complete solution corresponding to the repeated root is the complete solution of

$$(D - m_1)(D - m_1) y = 0$$

Putting $(D - m_1) y = v$, it becomes $(D - m_1) v = 0$ *i.e.*, $\frac{dv}{dx} - m_1 v = 0$

As in case I, its solution is $v = c_1 e^{m_1 x}$

$\therefore$ $(D - m_1) y = c_1 e^{m_1 x}$ or $\frac{dy}{dx} - m_1 y = c_1 e^{m_1 x}$

which is a linear equation and I.F. $= e^{-m_1 x}$

Its solution is $y \cdot e^{-m_1 x} = \int c_1 e^{m_1 x} \cdot e^{-m_1 x} dx + c_2 = c_1 x + c_2$

or $$y = (c_1 x + c_2) e^{m_1 x}$$

Thus, the complete solution of equation (*i*) is

$$y=(c_1 x+c_2)\, e^{m_1 x}+c_3 e^{m_3 x}+\cdots+c_n e^{m_n x}$$

If, however, three roots of the A.E. are equal, say $m_1=m_2=m_3$, then proceeding as above, the solution becomes

$$y=\left(c_1 x^2+c_2 x+c_3\right)e^{m_1 x}+c_4 e^{m_4 x}+\cdots+c_n e^{m_n x}$$

Case III. If two roots of the A.E. are imaginary,

Let $m_1=\alpha+i\beta$ and $m_2=\alpha-i\beta$ ($\because$ in a real equation imaginary roots occur in conjugate pair)

The solution obtained in equation (*iv*) becomes

$$y=c_1 e^{(\alpha+i\beta)x}+c_2 e^{(\alpha-i\beta)x}+c_3 e^{m_3 x}+\cdots+c_n e^{m_n x}$$

$$=e^{\alpha x}\left(c_1 e^{i\beta x}+c_2 e^{-i\beta x}\right)+c_3 e^{m_3 x}+\cdots+c_n e^{m_n x}$$

$$=e^{\alpha x}\left[c_1(\cos\beta x+i\sin\beta x)+c_2(\cos\beta x-i\sin\beta x)\right]+c_3 e^{m_3 x}+\cdots+c_n e^{m_n x}$$

$$\left[\because \text{ by Euler's Theorem, } e^{i\theta}=\cos\theta+i\sin\theta\right]$$

$$=e^{\alpha x}\left[(c_1+c_2)\cos\beta x+i(c_1-c_2)\sin\beta x\right]+c_3 e^{m_3 x}+\cdots+c_n e^{m_n x}$$

$$=e^{\alpha x}\left(C_1\cos\beta x+C_2\sin\beta x\right)+c_3 e^{m_3 x}+\cdots+c_n e^{m_n x}$$

[taking $c_1+c_2=C_1$, $i(c_1-c_2)=C_2$]

Case IV. If two pairs of imaginary roots be equal

Let $m_1=m_2=\alpha+i\beta$ and $m_3=m_4=\alpha-i\beta$

Then by case II, the complete solution is

$$y=e^{\alpha x}\left[(c_1 x+c_2)\cos\beta x+(c_3 x+c_4)\sin\beta x\right]+c_5 e^{m_5 x}+\cdots+c_n e^{m_n x}.$$

ILLUSTRATIVE EXAMPLES

Example 1. *Solve* $9y'''+3y''-5y'+y=0.$ **(P.T.U., May 2008)**

Sol. Symbolic form of given equation is

$$(9D^3+3D^2-5D+1)\,y=0$$

A.E. is $9D^3+3D^2-5D+1=0$

or $\quad (D+1)(3D-1)^2=0$

or $\quad D=-1,\ \frac{1}{3},\ \frac{1}{3}$

$\therefore$ C.S is $y=c_1 e^{-x}+(c_2+c_3 x)\,e^{\frac{1}{3}x}$.

Example 2. *Solve :* $\dfrac{d^4 x}{dt^4}+4x=0.$

Sol. Given equation in symbolic form is $(D^4+4)\,x=0$, where $D=\dfrac{d}{dt}$

Its A.E. is $\quad D^4+4=0 \quad$ or $\quad (D^4+4D^2+4)-4D^2=0$

or $\quad (D^2+2)^2-(2D)^2=0 \quad$ or $\quad (D^2+2D+2)(D^2-2D+2)=0$

whence $\quad D=\dfrac{-2\pm\sqrt{-4}}{2}$ and $\dfrac{2\pm\sqrt{-4}}{2}$ *i.e.*, $D=-1\pm i$ and $1\pm i$

Hence the C.S. is $\quad x=e^{-t}(c_1\cos t+c_2\sin t)+e^{t}(c_3\cos t+c_4\sin t)$.

Example 3. *If* $\frac{d^2x}{dt^2} + \frac{g}{b}(x - a) = 0$; $(a > 0,\ b > 0,\ g > 0)$ *and* $x = a',\ \frac{dx}{dt} = 0$ *when* $t = 0$, *show that*

$x = a + (a' - a)\cos\sqrt{\frac{g}{b}}\, t$ (P.T.U., May 200[illegible])

Sol. $$\frac{d^2x}{dt^2} + \frac{g}{b}(x - a) = 0$$

Put $x - a = y$ ∴ $\frac{d^2x}{dt^2} = \frac{d^2y}{dt^2}$

∴ $$\frac{d^2y}{dt^2} + \frac{g}{b}y = 0 \quad \text{A.E. is } m^2 + \frac{g}{b} = 0 \quad \therefore m^2 = -\frac{g}{b} \ (\text{–ve})$$

∴ $$m = \pm i\sqrt{\frac{g}{b}} \quad \therefore y = c_1\cos\sqrt{\frac{g}{b}}\, t + c_2\sin\sqrt{\frac{g}{b}}\, t$$

$$x - a = c_1\cos\sqrt{\frac{g}{b}}\, t + c_2\sin\sqrt{\frac{g}{b}}\, t$$

When $$x = a',\ t = 0;\quad a' - a = c_1$$

∴ $$x - a = (a' - a)\cos\sqrt{\frac{g}{b}}\, t + c_2\sin\sqrt{\frac{g}{b}}\, t$$

$$\frac{dx}{dt} = -(a' - a)\sqrt{\frac{g}{b}}\sin\sqrt{\frac{g}{b}}\, t + c_2\sqrt{\frac{g}{b}}\cos\sqrt{\frac{g}{b}}\, t$$

$$t = 0,\ \frac{dx}{dt} = 0 \quad \therefore 0 = c_2\sqrt{\frac{g}{b}} \quad \therefore c_2 = 0$$

∴ $$x - a = (a' - a)\cos\sqrt{\frac{g}{b}}\, t$$

Hence $$x = a + (a' - a)\cos\sqrt{\frac{g}{b}}\, t$$

TEST YOUR KNOWLEDGE

Solve the following differential equations :

1. $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 4y = 0$
2. $\frac{d^2y}{dx^2} + (a + b)\frac{dy}{dx} + aby = 0$
3. $\frac{d^3y}{dx^3} - 4\frac{dy}{dx} + y = 0$
4. $\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 9x = 0$
5. $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = 0$
6. $\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} + 6y = 0$
7. $\frac{d^4y}{dx^4} - 5\frac{d^2y}{dx^2} + 4y = 0$
8. $\frac{d^4y}{dx^4} + 8\frac{d^2y}{dx^2} + 16y = 0$

$D^3 - 3D^2 + 3D - 1$

9. $(D^2+1)^3(D^2+D+1)^2 y = 0$ $(D^2+1)^3 = 0$ $(D^2+D+1)^2 = 0$

10. $\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 0$, given that when $t = 0$, $x = 0$ and $\frac{dx}{dt} = 0$ $D^2 - 1$

11. $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 29y = 0$, given that when $x = 0$, $y = 0$ and $\frac{dy}{dx} = 15$

12. If $\frac{d^4x}{dt^4} = m^4x$, show that $x = c_1 \cos mt + c_2 \sin mt + c_3 \cosh mt + c_4 \sinh mt$. $y = C_1 + C_2 + C_3$

ANSWERS

1. $y = c_1 e^{4x} + c_2 e^{-x}$
2. $y = c_1 e^{-ax} + c_2 e^{-bx}$
3. $y = c_1 e^{(2+\sqrt{3})x} + c_2 e^{(2-\sqrt{3})x}$
4. $x = (c_1 + c_2 t) e^{-3t}$
5. $y = (c_1 + c_2 x + c_3 x^2) e^x$
6. $y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{-3x}$
7. $y = c_1 e^x + c_2 e^{-x} + c_3 e^{2x} + c_4 e^{-2x}$
8. $y = (c_1 + c_2 x) \cos 2x + (c_3 + c_4 x) \sin 2x$
9. $y = \left(c_1 + c_2 x + c_3 x^2\right)\cos x + \left(c_4 + c_5 x + c_6 x^2\right)\sin x + e^{-\frac{1}{2}x}\left[\left(c_7 + c_8 x\right) \cos\frac{\sqrt{3}}{2}x + \left(c_9 + c_{10} x\right) \sin\frac{\sqrt{3}}{2}x\right]$
10. $x = 0$
11. $y = 3e^{-2x} \sin 5x$.

3.8. THE INVERSE OPERATOR $\frac{1}{f(D)}$

Definition: $\frac{1}{f(D)}$ X is that function of x, *free from arbitrary constants,* which when operated upon by $f(D)$ gives X.

Thus $f(D)\left\{\frac{1}{f(D)} X\right\} = X$

$\therefore$ $f(D)$ and $\frac{1}{f(D)}$ are inverse operators.

Theorem 1. $\frac{1}{f(D)} X$ *is the particular integral of* $f(D)\, y = X$.

Proof. The given equation is $f(D)\, y = X$ ...(1)

Putting $y = \frac{1}{f(D)}X$ in (1), we have $f(D)\left\{\frac{1}{f(D)} X\right\} = X$ or $X = X$

which is true.

$$y = \frac{1}{f(D)}X \text{ is a solution of (1).}$$

Since it contains no arbitrary constants, it is the particular integral of $f(D)\, y = X$.

Case V. When X is any other function of x.

Resolve $f(D)$ into linear factors.

Let $f(D) = (D - m_1)(D - m_2) \ldots\ldots\ldots (D - m_n) X$

Then P.I. $= \dfrac{1}{f(D)} X = \dfrac{1}{(D - m_1)(D - m_2)\cdots(D - m_n)} X$

$$= \left(\frac{A_1}{D - m_1} + \frac{A_2}{D - m_2} + \cdots + \frac{A_n}{D - m_2}\right) X \qquad \text{(Partial Fractions)}$$

$$= A_1 \frac{1}{D - m_1} X + A_2 \frac{1}{D - m_2} X + \cdots + A_n \frac{1}{D - m_n} X$$

$$= A_1 e^{m_1 x} \int X e^{-m_1 x}\, dx + A_2 e^{m_2 x} \int X e^{-m_2 x}\, dx + \cdots + A_n e^{m_n x} \int X e^{-m_n x}\, dx$$

See solved example 11 (art. 3.10) $\left[\because \dfrac{1}{D - m} X = e^{mx} \int X e^{-mx}\, dx \right].$

3.10. WORKING RULE TO SOLVE THE EQUATION

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} \cdots a_{n-1} \frac{dy}{dx} + a_n y = X$$

Step 1. Write the equation in symbolic form

$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots\ldots\ldots\ldots\ldots + a_{n-1} D + a_n) y = X$

Step 2. Solve the auxiliary equation

$D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots\ldots\ldots\ldots\ldots + a_{n-1} D + a_n = 0$

Step 3. Write the complementary function with the help of following table.

Roots of the A.E.	C.F.
1. If roots are real and distinct say m_1, m_2, m_3	$c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \ldots\ldots$
2. If two real roots are equal say $m_1 = m_2 = m$	C.F. $= (c_1 + c_2 x) e^{mx} + c_3 e^{m_3 x} + \ldots\ldots$
3. If three roots are equal $m_1 = m_2 = m_3 = m$	C.F. $= (c_1 + c_2 x + c_3 x^2) e^{mx} + c_4 e^{m_4 x} + \ldots\ldots$
4. If roots are a pair of imaginary (non-repeated) numbers (say) $\alpha \pm i\beta$.	C.F. $= e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$
5. If pair of imaginary roots is repeated twice, *i.e.*, $\alpha \pm i\beta,\ \alpha \pm i\beta$.	C.F. $= e^{\alpha x}\{(c_1 + c_2 x)\cos \beta x + (c_3 + c_4 x)\sin \beta x\}$

Step 4. Find the particular integral *i.e.*, P.I. $= \dfrac{1}{D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n} X$ with the help of following rules.

Functions	Particular Integrals
(1) When $X = e^{ax}$ then	P.I. $= \dfrac{e^{ax}}{f(D)}$ Put $D = a$ $= \dfrac{e^{ax}}{f(a)}$ provided $f(a) \neq 0$.

In case $f(a) = 0$ then multiply by x and differentiate the denominator w.r.t. D and continue this process untill denominator ceases to be zero on putting $D = a$.

(2) When $X = \sin(ax+b)$ or $\cos(ax+b)$	P.I. $= \dfrac{1}{\phi(D^2)} \sin(ax+b)$ or $\cos(ax+b)$ Put $D^2 = -a^2$ $= \dfrac{1}{\phi(-a^2)} \sin(ax+b)$ or $\cos(ax+b)$ provided $\phi(-a^2) \neq 0$

In case of failure apply to above mentioned rule of (1) case.

(3) When $X = x^m$ then	P.I. $= [f(D)]^{-1} x^m$ expand $f(D)$ by binomial theorem up to D^m and then operate on x^m.
(4) When $X = e^{ax} V$,	P.I. $= e^{ax} \dfrac{1}{f(D+a)} V$

(5) If X is any other function of x then P.I. $= \dfrac{1}{f(D)} X$. Resolve $\dfrac{1}{f(D)}$ into partial fractions and operate each partial fraction on X.

(6) Remember $\dfrac{1}{D} X = \int X\, dx$ and $\dfrac{1}{D-a} X = e^{ax} \int X e^{-ax}\, dx$

Step 5. Then write the C.S. which is C.S. = C.F. + P.I.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve :* $(D^2 + D + 1)y = (1 + \sin x)^2$.

(P.T.U., May 2007)

Sol. $(D^2 + D + 1)\, y = (1 + \sin x)^2$

$$\text{A.E is } D^2 + D + 1 = 0 \therefore D = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm i \frac{\sqrt{3}}{2}$$

$$\text{C.F.} = e^{\frac{-x}{2}} \left[c_1 \cos \frac{\sqrt{3}}{2} x + c_2 \sin \frac{\sqrt{3}}{2} x \right]$$

$$\text{P.I.} = \frac{1}{D^2 + D + 1} (1 + \sin x)^2$$

$$= \frac{1}{D^2 + D + 1} \{1 + 2\sin x + \sin^2 x\}$$

$$= \frac{1}{D^2 + D + 1} \left\{1 + 2\sin x + \frac{1 - \cos 2x}{2}\right\} = \frac{1}{D^2 + D + 1} \left\{\frac{3}{2} + 2\sin x - \frac{1}{2} \cos 2x\right\}$$

$$= \frac{3}{2} \cdot \frac{1}{D^2 + D + 1} \cdot e^{0.x} + 2 \frac{1}{D^2 + D + 1} \sin x - \frac{1}{2} \frac{1}{D^2 + D + 1} \cos 2x$$

(Put D = 0) (Put $D^2 = -1$) (Put $D^2 = -4$)

$$= \frac{3}{2} \cdot 1 + 2 \cdot \frac{1}{D} \sin x - \frac{1}{2} \frac{1}{D-3} \cos 2x$$

$$= \frac{3}{2} + 2(-\cos x) - \frac{1}{2} \frac{D+3}{D^2 - 9} \cos 2x$$

$$= \frac{3}{2} - 2\cos x - \frac{1}{2} \frac{D+3}{-13} \cos 2x$$

$$= \frac{3}{2} - 2\cos x + \frac{1}{26} [-2\sin 2x + 3\cos 2x]$$

$$= \frac{3}{2} - 2\cos x - \frac{1}{13} \sin 2x + \frac{3}{26} \cos 2x$$

C.S. is $y = e^{\frac{-x}{2}} \left[c_1 \cos \frac{\sqrt{3}}{2} x + c_2 \sin \frac{\sqrt{3}}{2} x \right] + \frac{3}{2} - 2\cos x - \frac{1}{13} \sin 2x + \frac{3}{26} \cos 2x.$

Example 2. *Solve :* $(D-2)^2 y = 8(e^{2x} + \sin 2x + x^2)$. **(P.T.U., May 2009)**

Sol. A.E. is $(D-2)^2 = 0 \quad \therefore \quad D = 2, 2$

C.F. $= (c_1 + c_2 x) e^{2x}$

$$\text{P.I.} = \frac{1}{(D-2)^2} \left[8\left(e^{2x} + \sin 2x + x^2\right) \right] = 8 \left[\frac{1}{(D-2)^2} e^{2x} + \frac{1}{(D-2)^2} \sin 2x + \frac{1}{(D-2)^2} x^2 \right]$$

Now, $\frac{1}{(D-2)^2} e^{2x}$ Put D = 2; case of failure

$= x \cdot \frac{1}{2(D-2)} e^{2x}$ | Put D = 2. Case of failure

$= x^2 \cdot \frac{1}{2} e^{2x} = \frac{x^2}{2} e^{2x}$

$$\frac{1}{(D-2)^2} \sin 2x = \frac{1}{D^2 - 4D + 4} \sin 2x = \frac{1}{-2^2 - 4D + 4} \sin 2x \qquad [\text{Putting } D^2 = -2^2]$$

$$= -\frac{1}{4D} \sin 2x = -\frac{1}{4} \int \sin 2x \, dx = -\frac{1}{4}\left(-\frac{\cos 2x}{2}\right) = \frac{1}{8} \cos 2x$$

$$\frac{1}{(D-2)^2} x^2 = \frac{1}{(2-D)^2} x^2 = \frac{1}{4\left(1 - \frac{D}{2}\right)^2} x^2 = \frac{1}{4}\left(1 - \frac{D}{2}\right)^{-2} x^2$$

$$= \frac{1}{4}\left[1 - 2\left(-\frac{D}{2}\right) + \frac{(-2)(-3)}{2}\left(\frac{D}{2}\right)^2 \cdots\right] x^2$$

$$= \frac{1}{4}\left[1 + D + \frac{3}{4} D^2 + \cdots\right] x^2 = \frac{1}{4}\left(x^2 + 2x + \frac{3}{2}\right)$$

$$\therefore \qquad \text{P.I.} = 8\left[\frac{x^2}{2} e^{2x} + \frac{1}{8}\cos 2x + \frac{1}{4}\left(x^2 + 2x + \frac{3}{2}\right)\right] = 4x^2 e^{2x} + \cos 2x + 2x^2 + 4x + 3$$

Hence the C.S. is $\qquad y = (c_1 + c_2 x) e^{2x} + 4x^2 e^{2x} + \cos 2x + 2x^2 + 4x + 3.$

Example 3. *Solve :* $(D + 2)(D - 1)^2 y = e^{-2x} + 2 \sinh x.$

Sol. A.E. is $(D + 2)(D - 1)^2 = 0 \quad \therefore \quad D = -2, 1, 1$

$$\text{C.F.} = c_1 e^{-2x} + (c_2 + c_3 x) e^x$$

$$\text{P.I.} = \frac{1}{(D + 2)(D - 1)^2}\left(e^{-2x} + 2\sinh x\right)$$

$$= \frac{1}{(D + 2)(D - 1)^2}\left(e^{-2x} + e^x - e^{-x}\right) \qquad \left[\because \sinh x = \frac{e^x - e^{-x}}{2}\right]$$

Now, $$\frac{1}{(D + 2)(D - 1)^2} e^{-2x} = \frac{1}{D + 2}\left[\frac{1}{(D - 1)^2} e^{-2x}\right] = \frac{1}{D + 2}\left[\frac{1}{(-2 - 1)^2} e^{-2x}\right]$$

$$= \frac{1}{9} \cdot \frac{1}{D + 2} e^{-2x} \qquad \left| \begin{array}{l} \text{Put } D = -2 \\ \text{Case of failure} \end{array} \right.$$

$$= \frac{1}{9} x \cdot \frac{1}{1} e^{-2x} = \frac{x}{9} e^{-2x}$$

$$\frac{1}{(D + 2)(D - 1)^2} e^x = \frac{1}{(D - 1)^2}\left[\frac{1}{D + 2} e^x\right] = \frac{1}{(D - 1)^2}\left[\frac{1}{1 + 2} e^x\right]$$

$$= \frac{1}{3} \cdot \frac{1}{(D - 1)^2} e^x \qquad \left| \begin{array}{l} \text{Put } D = 1 \\ \text{Case of failure} \end{array} \right.$$

$$= \frac{1}{3} \cdot x \frac{1}{2(D - 1)} e^x \qquad \left| \begin{array}{l} \text{Put } D = 1 \\ \text{Case of failure} \end{array} \right.$$

$$= \frac{1}{3} \cdot x^2 \cdot \frac{1}{2} e^x = \frac{1}{6} x^2 e^x$$

$$\frac{1}{(D + 2)(D - 1)^2} e^{-x} = \frac{1}{(-1 + 2)(-1 - 1)^2} e^{-x} = \frac{1}{4} e^{-x}$$

$$\therefore \qquad \text{P.I.} = \frac{x}{9} e^{-2x} + \frac{x^2}{6} e^x + \frac{1}{4} e^{-x}$$

Hence the C.S. is $\qquad y = c_1 e^{-2x} + (c_2 + c_3 x) e^x + \frac{x}{9} e^{-2x} + \frac{x^2}{6} e^x + \frac{1}{4} e^{-x}.$

Example 4. *Solve :* $\frac{d^4y}{dx^4} - y = \cos x \cosh x.$ **(P.T.U., May 2001, 2007)**

Sol. Given equation in symbolic form is $(D^4 - 1)\,y = \cos x \cosh x$

A.E. is $D^4 - 1 = 0$ or $(D^2 - 1)(D^2 + 1) = 0 \quad \therefore \quad D = \pm 1, \pm i$

$$\text{C.F.} = c_1 e^x + c_2 e^{-x} + e^{0x}(c_3 \cos x + c_4 \sin x)$$
$$= c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$$

$$\text{P.I.} = \frac{1}{D^4 - 1} \cos x \cosh x = \frac{1}{D^4 - 1} \cos x \left(\frac{e^x + e^{-x}}{2}\right)$$

$$= \frac{1}{2}\left[\frac{1}{D^4 - 1} e^x \cos x + \frac{1}{D^4 - 1} e^{-x} \cos x\right]$$

$$= \frac{1}{2}\left[e^x \frac{1}{(D+1)^4 - 1} \cos x + e^{-x} \frac{1}{(D-1)^4 - 1} \cos x\right]$$

$$= \frac{1}{2}\left[e^x \frac{1}{D^4 + 4D^3 + 6D^2 + 4D} \cos x + e^{-x} \frac{1}{D^4 - 4D^3 + 6D^2 - 4D} \cos x\right]$$

Put $D^2 = -1$

$$= \frac{1}{2}\left[e^x \frac{1}{(-1)^2 + 4D(-1) + 6(-1) + 4D} \cos x + e^{-x} \frac{1}{(-1)^2 - 4D(-1) + 6(-1) - 4D} \cos x\right]$$

$$= \frac{1}{2}\left[e^x \frac{1}{-5} \cos x + e^{-x} \frac{1}{-5} \cos x\right] = -\frac{1}{5}\left(\frac{e^x + e^{-x}}{2}\right) \cos x = -\frac{1}{5} \cosh x \cos x$$

Hence the C.S. is $y = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x - \frac{1}{5} \cos x \cosh x$.

Example 5. *Solve* $\frac{d^2y}{dx^2} + 4y = x \sin 2x.$ **(P.T.U., Dec. 2002)**

Sol. S.F. of given equation is

$$(D^2 + 4)\,y = x \sin 2x$$

$$\text{A.E. is } D^2 + 4 = 0 \quad \therefore \quad D = \pm 2i$$

C.F. is $e^{0x}(\cos 2x + i \sin 2x) = \cos 2x + i \sin 2x$

$$\text{P.I.} = \frac{1}{D^2 + 4} x \sin 2x = \text{Imaginary part of } \frac{1}{D^2 + 4} x e^{i2x}$$

$$= \text{Imaginary part of } e^{i2x} \frac{1}{(D + 2i)^2 + 4} x$$

$$= \text{Imaginary part of } e^{i2x} \frac{1}{D^2 + 4iD - 4 + 4} x$$

$$= \text{Imaginary part of } e^{i2x} \frac{1}{4iD\left[1 + \frac{D^2}{4iD}\right]} x$$

$$= \text{Imaginary part of } \frac{e^{i2x}}{4iD}\left[1 - \frac{iD}{4}\right]^{-1} x$$

$$= \text{Imaginary part of } \frac{-ie^{i2x}}{4} \cdot \frac{1}{D}\left[1 + \frac{iD}{4}\right] x$$

$$= \text{Imaginary part of } \frac{-ie^{i2x}}{4} \cdot \frac{1}{D} \cdot \left[x + \frac{i}{4}\right]$$

$$= \text{Imaginary part of } \frac{-i(\cos 2x + i\sin 2x)}{4} \cdot \left(\frac{x^2}{2} + \frac{ix}{4}\right)$$

$$= -\frac{x^2}{8}\cos 2x + \frac{x}{16}\sin 2x$$

C.S. is $y = \text{C.F.} + \text{P.I.}$

$$= c_1 \cos 2x + c_2 \sin 2x - \frac{x^2}{8}\cos 2x + \frac{x}{16}\sin 2x.$$

Example 6. *Solve :* $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x \sin x$. **(P.T.U., Dec. 2001, 2003; Jan. 2010)**

Sol. Given equation in symbolic form is $(D^2 - 2D + 1)y = xe^x \sin x$

A.E. is $D^2 - 2D + 1 = 0$ or $(D-1)^2 = 0 \quad \therefore \quad D = 1, 1$

C.F. $= (c_1 + c_2 x)e^x$

$$\text{P.I.} = \frac{1}{(D-1)^2} e^x . x\sin x = e^x . \frac{1}{(D+1-1)^2} x \sin x$$

$$= e^x \frac{1}{D^2} x \sin x = e^x \frac{1}{D}\int x \sin x \, dx$$

Integrating by parts $= e^x \frac{1}{D}\left[x(\cos x)\int 1(\cos x)\,dx\right] = e^x \frac{1}{D}(x\cos x \;\sin x)$

$$= e^x \int(-x\cos x + \sin x)\,dx = e^x\left[-\left\{x\sin x - \int 1.\sin x\,dx\right\} - \cos x\right]$$

$$= e^x[-x\sin x - \cos x - \cos x] = -e^x(x\sin x + 2\cos x)$$

Hence the C.S. is $y = (c_1 + c_2 x)e^x - e^x(x\sin x + 2\cos x)$.

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Example 7. *Solve :* $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x \sin x.$ **(P.T.U., May 2006)**

Sol. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x \sin x$

S.F. is $(D^2 - 2D + 1)\, y = e^x \sin x$

A.E. is $m^2 - 2m + 1 = 0$ *i.e.*, $m = 1, 1.$

C.F. is $(c_1 + c_2 x)\, e^x$

$$\text{P.I.} = \frac{1}{D^2 - 2D + 1} e^x \sin x = \frac{1}{(D-1)^2} e^x \sin x$$

$$= e^x \frac{1}{(D+1-1)^2} \sin x = e^x \cdot \frac{1}{D^2} \sin x$$

Put $D^2 = -1$

$$= e^x \frac{\sin x}{-1} = -e^x \sin x$$

$\therefore$ C.S. is $y =$ C.F. + P.I.

$$y = (c_1 + c_2 x)\, e^x - e^x \sin x$$
$$= e^x [c_1 + c_2 x - \sin x].$$

Example 8. *Solve the differential equation* $(D^4 + D^2 + 1)\, y = e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}}{2} x\right).$ **(P.T.U., May 2004)**

Sol. $(D^4 + D^2 + 1)\, y = e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}}{2} x\right)$

A.E. is $m^4 + m^2 + 1 = 0$

$$m^2 = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$$m^2 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} \text{ and } \cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}$$

i.e., $m^2 = \text{cis}\,\frac{2\pi}{3}$ and $\text{cis}\left(-\frac{2\pi}{3}\right)$

when $m^2 = \text{cis}\,\frac{2\pi}{3}$

then $m = \left[\text{cis}\left(2k\pi + \frac{2\pi}{3}\right)\right]^{\frac{1}{2}}$

$$m = \text{cis}\,\frac{2k\pi + \frac{2\pi}{3}}{2},\ k = 0, 1$$

$$m = \text{cis}\,\frac{2\pi}{6},\ \text{cis}\,\frac{8\pi}{6} = \text{cis}\,\frac{\pi}{3},\ \text{cis}\,\frac{4\pi}{3}$$

i.e., two values of m corresponding to $\text{cis}\,\frac{\pi}{3}, \text{cis}\,\frac{4\pi}{3}$ are $\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

Other two values are obtained by changing i to $-i$.

$\therefore$ Four values of m are $\frac{1}{2} \pm i\frac{\sqrt{3}}{2},\ -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

$$\text{C.F.} = e^{\frac{1}{2}x}\left(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right) + e^{-\frac{1}{2}x}\left(c_3 \cos\frac{\sqrt{3}}{2}x + c_4 \sin\frac{\sqrt{3}}{2}x\right)$$

$$\text{P.I.} = \frac{1}{D^4 + D^2 + 1} e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}}{2}x\right)$$

$$= e^{-\frac{x}{2}} \frac{1}{\left(D - \frac{1}{2}\right)^4 + \left(D - \frac{1}{2}\right)^2 + 1} \cos\left(\frac{\sqrt{3}}{2}x\right) \quad \textbf{(Using 3.9 case IV)}$$

$$= e^{-\frac{x}{2}} \frac{1}{D^4 - 2D^3 + \frac{5}{2}D^2 - \frac{3}{2}D + \frac{21}{16}} \cos\left(\frac{\sqrt{3}}{2}x\right)$$

Put $D^2 = -\frac{3}{4}$

$$= e^{-\frac{x}{2}} \frac{1}{\frac{9}{16} + \frac{3}{2}D - \frac{15}{8} - \frac{3}{2}D + \frac{21}{16}} \cos\left(\frac{\sqrt{3}}{2}x\right)$$

$$= e^{-\frac{x}{2}} \frac{1}{0} \cos\left(\frac{\sqrt{3}}{2}x\right) \quad \textit{i.e., case of failure}$$

$$= e^{-\frac{x}{2}} \frac{x}{4D^3 - 6D^2 + 5D - \frac{3}{2}} \cos\left(\frac{\sqrt{3}}{2}\right)x$$

Put $D^2 = -\frac{3}{4}$

$$= e^{-\frac{x}{2}} \frac{x}{-3D + \frac{9}{2} + 5D - \frac{3}{2}} \cos\left(\frac{\sqrt{3}}{2}x\right)$$

$$= xe^{-\frac{x}{2}} \frac{1}{2D+3} \cos\left(\frac{\sqrt{3}}{2}x\right) = xe^{-\frac{x}{2}} \frac{2D-3}{4D^2-9} \cos\frac{\sqrt{3}}{2}x$$

$$= xe^{-\frac{x}{2}} \frac{(2D-3)}{-12} \cos\frac{\sqrt{3}}{2}x = \frac{-xe^{-\frac{x}{2}}}{12} \cdot \left\{-2.\frac{\sqrt{3}}{2}\sin\frac{\sqrt{3}}{2}x - 3\cos\frac{\sqrt{3}}{2}x\right\}$$

$$= \frac{xe^{-\frac{x}{2}}}{12}\left[\sqrt{3}\sin\frac{\sqrt{3}}{2}x + 3\cos\frac{\sqrt{3}}{2}x\right]$$

C.S. is $y =$ C.F. + P.I.

$$y = e^{\frac{x}{2}}\left(c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x\right) + e^{-\frac{x}{2}}\left(c_3\cos\frac{\sqrt{3}}{2}x + c_4\sin\frac{\sqrt{3}}{2}x\right)$$

$$+ \frac{xe^{-\frac{x}{2}}}{12} \cdot \left[\sqrt{3}\sin\frac{\sqrt{3}}{2}x + 3\cos\frac{\sqrt{3}}{2}x\right].$$

Example 9. *Solve :* $(D^2 - 6D + 13)\, y = 8e^{3x}\sin 4x + 2^x$. **(P.T.U., Dec. 2005)**

Sol. A.E. is $D^2 - 6D + 13 = 0$

$$D = \frac{6 \pm \sqrt{36-12}}{2} = 3 \pm 2i$$

$$\text{C.F.} = e^{3x}(c_1\cos 2x + c_2\sin 2x)$$

$$\text{P.I.} = \frac{1}{D^2 - 6D + 13}(8e^{3x}\sin 4x + 2^x)$$

$$= 8\frac{1}{D^2-6D+13}e^{3x}\sin 4x + \frac{1}{D^2-6D+13}2^x$$

$$= 8e^{3x}\frac{1}{(D+3)^2 - 6(D+3) + 13}\sin 4x + \frac{1}{D^2-6D+13}e^{\log 2^x}$$

$$= 8e^{3x}\frac{1}{D^2+4}\sin 4x + \frac{1}{D^2-6D+13}e^{x\log 2}$$

(Put $D^2 = -16$) (Put $D = \log 2$)

$$= 8e^{3x} \cdot \frac{1}{-16+4}\sin 4x + \frac{1}{(\log 2)^2 - 6\log 2 + 13}e^{x\log 2}$$

$$= \frac{8e^{3x}}{-12}\sin 4x + \frac{1}{(\log 2)^2 - 6\log 2 + 13}2^x$$

$$\text{C.S. is } y = e^{3x}(c_1\cos 2x + c_2\sin 2x) - \frac{2}{3}e^{3x}\sin 4x + \frac{1}{(\log 2)^2 - 6\log 2 + 13}2^x.$$

Example 10. *Solve* : $(D^2 + 2D + 2)\,y = e^{-x}\sec x$. (P.T.U., Dec. 2002)

Sol. A.E. is $D^2 + 2D + 2 = 0$

8

$$D = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

$$\text{C.F.} = e^{-x}[c_1\cos x + c_2\sin x)$$

$$\text{P.I.} = \frac{1}{D^2 + 2D + 2}e^{-x}\sec x = e^{-x}\frac{1}{(D-1)^2 + 2(D-1) + 2}\sec x$$

$$= e^{-x}\frac{1}{D^2 + 1}\sec x = e^{-x}\frac{1}{(D+i)(D-i)}\sec x$$

$$= e^{-x}\left[\frac{\frac{1}{2i}}{D-i} - \frac{\frac{1}{2i}}{D+i}\right]\sec x \qquad \text{[By partial fractions]}$$

$$= \frac{e^{-x}}{2i}\left[\frac{1}{D-i} - \frac{1}{D+i}\right]\sec x$$

Now, $$\frac{1}{D-i}\sec x = e^{ix}\int e^{-ix}\sec x\,dx = e^{ix}\int(\cos x - i\sin x)\frac{1}{\cos x}dx$$

$$= e^{ix}\int(1 - i\tan x)\,dx = e^{ix}[x + i\log\cos x]$$

Similarly, $$\frac{1}{D+i}\sec x = e^{-ix}[x - i\log\cos x]$$

$\therefore$ $$\text{P.I.} = \frac{e^{-x}}{2i}[e^{ix}(x + i\log\cos x) - e^{-ix}(x - i\log\cos x)]$$

$$= \frac{e^{-x}}{2i}[x(e^{ix} - e^{-ix}) + i\log\cos x\,(e^{ix} + e^{-ix})]$$

$$= \frac{e^{-x}}{2i}[x\,.\,2i\sin x + i\log\cos x\,.\,2\cos x] = e^{-x}(x\sin x + \cos x\log\cos x)$$

$$\text{C.S. is } y = e^{-x}[c_1\cos x + c_2\sin x + x\sin x + \cos x\log\cos x].$$

Example 11. *Solve :* $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = e^{e^x}$. **(P.T.U., Dec. 2003)**

Sol. $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = e^{e^x}$

$e^{x}\int e^{-e^x} dx$

S.F. is $(D^2 + 3D + 2)\, y = e^{e^x}$

A.E. is $m^2 + 3m + 2 = 0 \quad \therefore \quad m = -1, -2$

C.F. is $c_1 e^{-x} + c_2 e^{-2x}$

$$\text{P.I.} = \frac{1}{D^2 + 3D + 2} e^{e^x} = \frac{1}{(D+1)(D+2)} e^{e^x}$$

$$= \left[\frac{1}{D+1} - \frac{1}{D+2}\right] e^{e^x} \text{ by partial fractions}$$

$$= \frac{1}{D+1} e^{e^x} - \frac{1}{D+2} e^{e^x}$$

$$= e^{-x} \int e^x . e^{e^x} dx - e^{-2x} \int e^{2x} e^{e^x} dx \quad \because \frac{1}{D-a} X = e^{ax} \int e^{-ax} X\, dx$$

Put $\qquad e^x = t \quad \therefore \quad e^x\, dx = dt$

$$= e^{-x} \int e^t\, dt - e^{-2x} \int t\, e^t\, dt = e^{-x} e^t - e^{-2x} (t-1)\, e^t$$

$$= e^{-x} e^{e^x} - e^{-2x} (e^x - 1)\, e^{e^x}$$

$$= e^{e^x} [e^{-x} - e^{-x} + e^{-2x}] = e^{-2x} e^{e^x}.$$

Example 12. *Solve : $(D^2 - 4D + 4)\, y = 8x^2 e^{2x} \sin 2x$.* **(P.T.U., May 2002)**

Sol. $(D^2 - 4D + 4)\, y = 8x^2 e^{2x} \sin 2x$

A.E. is $\qquad m^2 - 4m + 4 = 0 \quad \text{or} \quad (m-2)^2 = 0 \quad \therefore \quad m = 2, 2$

$\therefore \qquad$ C.F. $= (c_1 + c_2 x)\, e^{2x}$

$$\text{P.I.} = \frac{1}{D^2 - 4D + 4} 8x^2 e^{2x} \sin 2x = \frac{1}{(D-2)^2} 8x^2 e^{2x} \sin 2x$$

$$= 8e^{2x} . \left(\frac{1}{(D+2-2)^2}\right) x^2 \sin 2x = 8e^{2x} . \frac{1}{D^2} x^2 \sin 2x$$

$$= 8e^{2x} \cdot \frac{1}{D}\int x^2 \sin 2x \, dx \text{ Integrate by parts}$$

$$= 8e^{2x} \cdot \frac{1}{D}\left\{(x^2)\left(-\frac{\cos 2x}{2}\right)-(2x)\left(-\frac{\sin 2x}{4}\right)+2\left(\frac{\cos 2x}{8}\right)\right\}$$

$$= 8e^{2x} \cdot \frac{1}{D}\left\{-\frac{x^2\cos 2x}{2}+\frac{x\sin 2x}{2}+\frac{\cos 2x}{4}\right\}$$

$$= 4e^{2x}\int\left(-x^2\cos 2x + x\sin 2x + \frac{\cos 2x}{2}\right)dx \text{ Integrate by parts}$$

$$= 4e^{2x}\left[-\left\{x^2\left(\frac{\sin 2x}{2}\right)-(2x)\left(-\frac{\cos 2x}{4}\right)+(2)\left(-\frac{\sin 2x}{8}\right)+x\left(-\frac{\cos 2x}{2}\right)-(1)\left(-\frac{\sin 2x}{4}\right)+\frac{\sin 2x}{4}\right\}\right]$$

$$= 2e^{2x}\left[-x^2\sin 2x - x\cos 2x + \frac{\sin 2x}{2} - x\cos 2x + \frac{\sin 2x}{2} + \frac{\sin 2x}{2}\right]$$

$$= 2e^{2x}\left[-x^2\sin 2x - 2x\cos 2x + \frac{3}{2}\sin 2x\right]$$

$= -e^{2x}[(2x^2-3)\sin 2x + 4x\cos 2x]$

C.S. is $y = (c_1 + c_2 x)e^{2x} - e^{2x}[(2x^2-3)\sin 2x + 4x\cos 2x]$.

Example 13. *Solve :* $(D^3 + 2D^2 + D)y = x^2 e^x + \sin^2 x$. **(P.T.U., June 2003)**

Sol. A.E. is $m^3 + 2m^2 + m = 0$

$m(m^2+2m+1) = 0$ or $m(m+1)^2 = 0$ or $m = 0, -1, -1$

C.F. $= c_1 e^{0x} + (c_2 + c_3 x)e^{-x} = c_1 + (c_2 + c_3 x)e^{-x}$

$$\text{P.I.} = \frac{1}{D^3+2D^2+D}\left(x^2 e^x + \sin^2 x\right) = \frac{1}{D^3+2D^2+D}\left(x^2 e^x + \frac{1-\cos 2x}{2}\right)$$

$$= \frac{1}{D^3+2D^2+D}x^2 e^x + \frac{1}{D^3+2D^2+D}\left(\frac{1}{2}\right) - \frac{1}{D^3+2D^2+D}\left(\frac{1}{2}\cos 2x\right)$$

$$= e^x \frac{1}{(D+1)^3+2(D+1)^2+(D+1)}x^2 + \frac{1}{2}\cdot\frac{1}{D^3+2D^2+D}e^{0.x} - \frac{1}{2}\cdot\frac{1}{D^3-2D^2+D}\cos 2x$$

(Put D = 0; Case of failure) (Put $D^2 = -4$)

$$= e^x \frac{1}{D^3+5D^2+8D+4}x^2 + \frac{1}{2}\frac{x}{3D^2+4D+1}e^{0.x} - \frac{1}{2}\cdot\frac{1}{-4D-8+D}\cos 2x$$

(Put D = 0)

$$= \frac{e^x}{4}\left[1+\frac{8D+5D^2+D^3}{4}\right]^{-1} x^2+\frac{x}{2}+\frac{1}{2}\cdot\frac{1}{3D+8}\cos 2x$$

$$= \frac{e^x}{4}\left[1-\frac{8D+5D^2+D^3}{4}+\left(\frac{8D+5D^2+D^3}{4}\right)^2\right]x^2+\frac{x}{2}+\frac{3D-8}{2\left(9D^2-64\right)}\cos 2x$$

$$= \frac{e^x}{4}\left[1-\frac{8D}{4}-\frac{5D^2}{4}+4D^2\right]x^2+\frac{x}{2}+\frac{3D-8}{2(-36-64)}\cos 2x$$

$$= \frac{e^x}{4}\left[x^2-\frac{8}{4}(2x)+\frac{11}{4}(2)\right]+\frac{x}{2}-\frac{1}{200}\left[3(-2\sin 2x)-8\cos 2x\right]$$

$$= \frac{e^x}{4}\left[x^2-4x+\frac{11}{2}\right]+\frac{x}{2}+\frac{3}{100}\sin 2x+\frac{\cos 2x}{25}$$

$$\text{C.S. } y = c_1+(c_2+c_3 x)e^{-x}+\frac{e^x}{4}\left[x^2-4x+\frac{11}{2}\right]+\frac{x}{2}+\frac{3}{100}\sin 2x+\frac{\cos 2x}{25}$$

which is the required solution.

TEST YOUR KNOWLEDGE

Solve the following differential equations :

1. $\frac{d^3y}{dx^3}+y=3+5e^x$

2. $\frac{d^2y}{dx^2}-4y=\left(1+e^x\right)^2$

3. $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+5y=-2\cosh x$

4. (*a*) $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+5y=\sin 3x$

   (*b*) $(D^2+a^2)\,y=\sin ax$ **(P.T.U., May 2009)**

5. $\frac{d^3y}{dx^3}+\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=\sin 2x$

6. $\frac{d^3y}{dx^3}+y=\sin 3x-\cos^2\frac{x}{2}$

7. $\left(D^2-4D+3\right)y=\sin 3x\cos 2x$ **(P.T.U., Jan. 2009)**

8. $(D^2-3D+2)\,y=6e^{-3x}+\sin 2x$

9. $\frac{d^2y}{dx^2}+4y=e^x+\sin 2x$

10. $\frac{d^3y}{dx^3}-2\frac{d^2y}{dx^2}+4\frac{dy}{dx}=e^{2x}+\sin 2x$

11. $\frac{d^2y}{dx^2}-4y=x^2$

12. $\frac{d^3y}{dx^3}-\frac{d^2y}{dx^2}-6\frac{dy}{dx}=1+x^2$

13. $\frac{d^2y}{dx^2}+\frac{dy}{dx}=x^2+2x+4$

14. $\frac{d^2y}{dx^2}+y=e^{2x}+\cosh 2x+x^3$

15. $\left(D^2-3D+2\right)y=2e^x\cos\frac{x}{2}$ **(P.T.U., Dec. 2003)**

$$\left[\textbf{Hint: } \text{P.I.}=\frac{1}{D^2-3D+2}2e^x\cos\frac{x}{2}=2e^x\frac{1}{(D+1)^2-3(D+1)+2}=2e^x\frac{1}{D^2-D}\cos\frac{x}{2}\right]$$

16. $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = xe^{3x} + \sin 2x$

17. $\dfrac{d^4y}{dx^4} - y = e^x \cos x$

18. $(D^2 - 2D)\, y = e^x \sin x$

19. $(D^2 + 4D + 8)\, y = 12\, e^{-2x} \sin x \sin 3x$

20. $\dfrac{d^2y}{dx^2} + 2y = x^2 e^{3x} + e^x \cos 2x$

21. $\dfrac{d^2y}{dx^2} + y = \operatorname{cosec} x$

22. $(D-1)^2 (D+1)^2\, y = \sin^2 \dfrac{x}{2} + e^x + x$

23. $\dfrac{d^2y}{dx^2} - 4y = x \sinh x$

24. $(D^2 - 1)\, y = x \sin x + (1 + x^2)\, e^x$

25. $\dfrac{d^2y}{dx^2} + a^2 y = \sec ax$ (P.T.U., Dec. 200?)

$\left[\textbf{Hint : } \text{P.I} = \dfrac{1}{D^2 + a^2} \sec ax = \dfrac{1}{2ia}\left[\dfrac{1}{D - ia} - \dfrac{1}{D + ia}\right] \sec ax\right.$

$\left. \text{now see S.E. 10} \right]$

26. $\dfrac{d^2y}{dx^2} + 4y = 4 \tan 2x$

ANSWERS

1. $y = c_1 e^{-x} + e^{\frac{1}{2}x}\left(c_2 \cos\dfrac{\sqrt{3}}{2}x + c_3 \sin\dfrac{\sqrt{3}}{2}x\right) + 3 + \dfrac{5}{2}e^x$

2. $y = c_1 e^{2x} + c_2 e^{-2x} - \dfrac{1}{4} - \dfrac{2}{3}e^x + \dfrac{1}{4}xe^{2x}$

3. $y = e^{-2x}(c_1 \cos x + c_2 \sin x) - \dfrac{1}{10}e^x - \dfrac{1}{2}e^{-x}$

4. (a) $y = e^x(c_1 \cos 2x + c_2 \sin 2x) + \dfrac{1}{26}(3\cos 3x - 2\sin 3x)$

(b) $y = c_1 \cos ax + c_2 \sin ax - \dfrac{x \cos ax}{2a}$

5. $y = c_1 e^{-x} + c_2 \cos x + c_3 \sin x + \dfrac{1}{15}(2\cos 2x - \sin 2x)$

6. $y = c_1 e^{-x} + e^{\frac{1}{2}x}\left(c_2 \cos\dfrac{\sqrt{3}}{2}x + c_3 \sin\dfrac{\sqrt{3}}{2}x\right) + \dfrac{1}{730}(\sin 3x + 27\cos 3x) - \dfrac{1}{2} - \dfrac{1}{4}(\cos x - \sin x)$

7. $y = c_1 e^x + c_2 e^{3x} + \dfrac{1}{884}(10\cos 5x - 11\sin 5x) + \dfrac{1}{20}(\sin x + 2\cos x)$

8. $y = c_1 e^x + c_2 e^{2x} + \dfrac{3}{10}e^{-3x} + \dfrac{1}{20}(3\cos 2x - \sin 2x)$

9. $y = c_1 \cos 2x + c_2 \sin 2x + \dfrac{1}{5}e^x - \dfrac{x}{4}\cos 2x$

10. $y = c_1 + e^x\left(c_2 \cos\sqrt{3}x + c_3 \sin\sqrt{3}x\right) + \dfrac{1}{8}\left(e^{2x} + \sin 2x\right)$

11. $y = c_1 e^{2x} + c_2 e^{-2x} - \dfrac{1}{4}\left(x^2 + \dfrac{1}{2}\right)$

12. $y = c_1 + c_2 e^{3x} + c_3 e^{-2x} - \frac{1}{18}\left(x^3 - \frac{x^2}{2} + \frac{25}{6}x\right)$

13. $y = c_1 + c_2 e^{-x} + \frac{x^3}{3} + 4x$

14. $y = c_1 \cos x + c_2 \sin x + \frac{1}{5}e^{2x} + \frac{1}{5}\cosh 2x + x^3 - 6x$

15. $y = c_1 e^{x} + c_2 e^{2x} - \frac{8}{5}e^{x}\left(2\sin\frac{x}{2} + \cos\frac{x}{2}\right)$

16. $y = c_1 e^{x} + c_2 e^{2x} + \frac{1}{4}e^{3x}(2x-3) + \frac{1}{20}(3\cos 2x - \sin 2x)$

17. $y = c_1 e^{x} + c_2 e^{-x} + c_3 \cos x + c_4 \sin x - \frac{1}{5}e^{x}\cos x$

18. $y = c_1 + c_2 e^{2x} - \frac{1}{2}e^{x}\sin x$

19. $y = e^{-2x}(c_1 \cos 2x + c_2 \sin 2x) + \frac{1}{2}e^{-2x}(3x\sin 2x + \cos 4x)$

20. $y = c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}\,x + \frac{e^{3x}}{11}\left(x^2 - \frac{12}{11}x + \frac{50}{121}\right) + \frac{e^{x}}{17}(4\sin 2x - \cos 2x)$

21. $y = c_1 \cos x + c_2 \sin x + \sin x \log \sin x - x\cos x$

22. $y = (c_1 + c_2 x)e^{x} + (c_3 + c_4 x)\, e^{-x} + \frac{1}{2} - \frac{1}{8}\cos x + \frac{x^2}{8}e^{x} + x$

23. $c_1 e^{2x} + c_2 e^{-2x} - \frac{x}{3}\sinh x - \frac{2}{9}\cosh x$

24. $y = c_1 e^{x} + c_2 e^{-x} - \frac{1}{2}(x\sin x + \cos x) + \frac{1}{12}xe^{x}(2x^2 - 3x + 9)$

25. $y = c_1 \cos ax + c_2 \sin ax + \frac{1}{a}\left(x\sin ax + \cos ax\,\frac{\log\cos ax}{a}\right)$

26. $y = c_1 \cos 2x + c_2 \sin 2x - \cos 2x \log(\sec 2x + \tan 2x)$.

3.11. METHOD OF VARIATION OF PARAMETERS TO FIND P.I.

(P.T.U., May 2004)

Consider the linear equation of **second order** with constant coefficients

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2 y = X \qquad \ldots(1)$$

Let its C.F. be $y = c_1 y_1 + c_2 y_2$ so that y_1 and y_2 satisfy the equation

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2 y = 0 \qquad \ldots(2)$$

Now, let us assume that the P.I. of (1) is $y = uy_1 + vy_2$ where u and v are unknown functions of x. ...(3)

Differentiating (3) w.r.t. x, we have $y' = uy_1' + vy_2' + u'y_1 + v'y_2 = uy_1' + vy_2'$ ...(4)

assuming that u, v satisfy the equation $u'y_1 + v'y_2 = 0$ ...(5)

Differentiating (4) w.r.t. x, we have $y'' = uy_1'' + u'y_1' + vy_2'' + v'y_2'$

Substituting the values of y, y' and y'' in (1), we get

$$(uy_1'' + u'y_1' + vy_2'' + v'y_2') + a_1(uy_1' + vy_2') + a_2(uy_1 + vy_1) = X$$

or $$u(y_1'' + a_1y_1' + a_2y_1) + v(y_2'' + a_1y_2' + a_2y_2) + u'y_1' + v'y_2' = X$$

or $$u'y_1' + v'y_2' = X \text{ Since } y_1 \text{ and } y_2 \text{ satisfy (2).} \quad ...(6)$$

Solving (5) and (6), we get $$u' = \begin{vmatrix} 0 & y_2 \\ X & y_2' \end{vmatrix} \div \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = -\frac{y_2 X}{W}$$

and $$v' = \begin{vmatrix} y_1 & 0 \\ y_1' & X \end{vmatrix} \div \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \frac{y_1 X}{W}$$

where $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ is called the Wronskian of y_1, y_2.

Integrating $$u = -\int \frac{y_2 X}{W}\,dx, \quad v = \int \frac{y_1 X}{W}\,dx$$

Substituting in (3), the P.I. is known.

Note 1. As the solution is obtained by varying the arbitrary constants c_1, c_2 of the C.F., the method is known as *variation of parameters*.

Note 2. Method of variation of parameters is to be used if instructed to do so.

ILLUSTRATIVE EXAMPLES

Example 1. *Find the general solution of the equation $y'' + 16y = 32 \sec 2x$; using method of variation of parameters.* **(P.T.U., May 2008; May 2010)**

Sol. Given equation in symbolic form is $(D^2 + 16)y = 32 \sec 2x$

A.E. is $D^2 + 16 = 0 \quad \therefore \quad D = \pm 4i$

C.F. is $y = c_1 \cos 4x + c_2 \sin 4x$

Here $y_1 = \cos 4x, y_2 = \sin 4x, X = 32 \sec 2x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 4x & \sin 4x \\ -4\sin 4x & 4\cos 4x \end{vmatrix} = 4$$

$$\text{P.I.} = uy_1 + vy_2 \text{ where } u = -\int \frac{y_2 X}{W}\,dx \text{ and } v = \int \frac{y_1 X}{W}$$

$\therefore$ $$\text{P.I.} = -\cos 4x \int \frac{\sin 4x \cdot 32 \sec 2x}{4}\,dx + \sin 4x \int \frac{\cos 4x \cdot 32 \sec 2x}{4}\,dx$$

$$= -8\cos 4x \int 2\sin 2x \cos 2x \cdot \frac{1}{\cos 2x}\,dx + 8\sin 4x \int \frac{2\cos^2 2x - 1}{\cos 2x}\,dx$$

$$= -16\cos 4x \int \sin 2x\,dx + 8\sin 4x \int (2\cos 2x - \sec 2x)\,dx$$

$$= -16\cos 4x\left[-\frac{\cos 2x}{2}\right] + 8\sin 4x\left[\frac{2\sin 2x}{2} - \frac{\log(\sec 2x + \tan 2x)}{2}\right]$$

$$= 8\cos 4x\cos 2x + 8\sin 4x\sin 2x - 4\sin 4x\log(\sec 2x + \tan 2x)$$

$$= 8\cos(4x - 2x) - 4\sin 4x\log(\sec 2x + \tan 2x)$$

$$= 8\cos 2x - 4\sin 4x\log(\sec 2x + \tan 2x)$$

C.S. is $y = \text{C.F.} + \text{P.I.}$

$$= c_1\cos 4x + c_2\sin 4x + 8\cos 2x - 4\sin 4x\log(\sec 2x + \tan 2x).$$

Example 2. *Solve :* $y'' - 6y' + 9y = \dfrac{e^{3x}}{x^2}$ *by variation of parameter method.* **(P.T.U., May 2010)**

Sol. Equation in the symbolic form is

$$(D^2 - 6D + 9)\,y = \frac{e^{3x}}{x^2}$$

A.E. is $D^2 - 6D + 9 = 0$ *i.e.*, $(D-3)^2 = 0$ *i.e.*, $D = 3, 3$

C.F. $= (c_1 + c_2x)\,e^{3x} = c_1 e^{3x} + c_2 x e^{3x} = c_1y_1 + c_2y_2$, where $y_1 = e^{3x}$, $y_2 = x e^{3x}$

and

$$X = \frac{e^{3x}}{x^2}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & (1+3x)\,e^{3x} \end{vmatrix} = e^{6x}$$

P.I. $= uy_1 + vy_2$, where $u = -\displaystyle\int \frac{y_2 X}{W}\,dx$ and $v = \displaystyle\int \frac{y_1 X}{W}\,dx$

∴

$$\text{P.I.} = -e^{3x}\int \frac{xe^{3x}\,.\,e^{3x}}{e^{6x}\,.\,x^2}\,dx + xe^{3x}\int \frac{e^{3x}\,.\,e^{3x}}{e^{6x}\,.\,x^2}\,dx = -e^{3x}\int \frac{1}{x}\,dx + xe^{3x}\int \frac{1}{x^2}\,dx$$

$$= -e^{3x}\log x + xe^{3x}\left(-\frac{1}{x}\right) = -e^{3x}\,(1 + \log x)$$

C.S. is $y = (c_1 + c_2x)\,e^{3x} - e^{3x}(1 + \log x)$

$$= e^{3x}\,[c_1 + c_2x - 1 - \log x] = e^{3x}\,[(c_1 - 1) + c_2x - \log x]$$

$$= e^{3x}\,[c_1' + c_2x - \log x], \text{ where } c_1' = c_1 - 1 \text{ is the required solution.}$$

Example 3. *Solve by method of variation of parameters the differential equation* $\dfrac{d^2y}{dx^2} + y = \sec x$.

(P.T.U., Dec. 2003, Dec. 2005)

Sol. S.F. of the equation is

$$(D^2+1)y = \sec x$$

A.E. is $D^2 + 1 = 0 \quad \therefore \quad D = \pm i$

$$\text{C.F.} = c_1 \cos x + c_2 \sin x = c_1 y_1 + c_2 y_2$$

where $\quad y_1 = \cos x,\ y_2 = \sin x,\ X = \sec x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

$u = -\int$ [illegible]

$$\text{P.I.} = uy_1 + vy_2, \text{ where } u = -\int \frac{y_2 X}{W}\,dx,\ v = \int \frac{y_1 X}{W}\,dx$$

$$\therefore \quad \text{P.I.} = -\cos x \int \frac{\sin x \sec x}{1}\,dx + \sin x \int \frac{\cos x . \sec x}{1}\,dx$$

$$= -\cos x \int \tan x\,dx + \sin x \int 1 . dx$$

$$= \cos x \log \cos x + x \sin x$$

C.S. is $y = c_1 \cos x + c_2 \sin x + \cos x \log \cos x + x \sin x$.

TEST YOUR KNOWLEDGE

Solve by the method of variation of parameters :

1. $\dfrac{d^2y}{dx^2} + y = \operatorname{cosec} x$

2. $\dfrac{d^2y}{dx^2} + 4y = \tan 2x$

3. $\dfrac{d^2y}{dx^2} + 4y = \sec 2x$ **(P.T.U., May 2004)**

   [Hint: Consult S.E. 1**]**

4. $\dfrac{d^2y}{dx^2} + 4y = 4\sec^2 2x$ **(P.T.U., Jan. 2009)**

5. $\dfrac{d^2y}{dx^2} + y = x \sin x$

6. $y'' - 2y' + 2y = e^x \tan x$.

ANSWERS

1. $y = c_1 \cos x + c_2 \sin x - x\cos x + \sin x \log \sin x$

2. $y = c_1 \cos 2x + c_2 \sin 2x - \dfrac{1}{4} \cos 2x \log(\sec 2x + \tan 2x)$

3. $y = c_1 \cos 2x + c_2 \sin 2x + \dfrac{1}{4} \cos 2x \log \cos 2x + \dfrac{x}{2} \sin 2x$

4. $y = c_1 \cos 2x + c_2 \sin 2x - 1 + \sin 2x \log(\sec 2x + \tan 2x)$

5. $y = c_1 \cos x + c_2 \sin x + \dfrac{x}{2} \sin x - \dfrac{x^2}{4} \cos x$

6. $y = e^x(c_1 \cos x + c_2 \sin x) - e^x \cos x \log(\sec x + \tan x)$.

3.12. METHOD OF UNDETERMINED COEFFICIENTS FOR FINDING PARTICULAR INTEGRAL OF A LINEAR DIFFERENTIAL EQUATION $f(D)y = X$

This method is useful only when X contains terms in some special forms. To find P.I., we assume a trial solution containing unknown constants which are determined by substitution in the given equation.

The following table suggests the trial solution to be used corresponding to a special form of X.

Table

S.No.	Special form of X	Trial solution for P.I.
1.	x^n or $a_n x^n$ or $a_0 + a_1 x + a_2 x^2 + \ldots\ldots + a_n x^n$	$A_0 + A_1 x + A_2 x^2 + \ldots\ldots + A_n x^n$
2.	e^{ax} or $p\, e^{ax}$	$A e^{ax}$
3.	$a_n x^n e^{ax}$ or $e^{ax}(a_0 + a_1 x + a_2 x^2 + \ldots\ldots + a_n x^n)$	$e^{ax}(A_0 + A_1 x + A_2 x^2 + \ldots\ldots + A_n x^n)$
4.	$p \sin ax$ or $q \cos ax$ or $p \sin ax + q \cos ax$	$A \sin ax + B \cos ax$
5.	$p\, e^{bx} \sin ax$ or $q\, e^{bx} \cos ax$ or $e^{bx}(p \sin ax + q \cos ax)$	$e^{bx}(A \sin ax + B \cos ax)$
6.	$x^n \sin ax$ or $a_n x^n \sin ax$ or $x^n \cos ax$ or $a_n x^n \cos ax$ or $(a_0 + a_1 x + \ldots + a_n x^n) \sin ax$ or $(a_0 + a_1 x + \ldots\ldots + a_n x^n) \cos ax$	$(A_0 + A_1 x + \ldots + A_n x^n) \sin ax + (A_0' + A_1' x + \ldots\ldots + A_n' x^n) \cos ax$

If X is a linear combination of more than one special forms of the above table, then the trial solution must be the sum of corresponding trial solutions with appropriate constant coefficients to be evaluated later on.

Particular Cases : *When a term of X is also a term of C.F. of the given equation then the procedure to write trial solution in as follows :*

(1) If a term of X, (say u) is also a term of the C.F. corresponding to r fold root m then in the trail solution we introduce a term $x^r u$ + terms arising from it by differentiation of x^r.

e.g., Consider $(D-2)^2 (D+3)\, y = e^{2x} + x^2$ ...(1)

Its C.F. $= (c_1 + c_2 x)\, e^{2x} + c_3 e^{-3x}$

$\because x^2$ is not occuring in the C.F. $\therefore$ the corresponding contribution to trial solution of PI is $A_1 + A_2 x + A_3 x^2$.

But e^{2x} is present in C.F. and of double root $m = 2$.

$\therefore$ Trial solution corresponding to e^{2x} will be

$$A_4 x^2 e^{2x} + A_5 x e^{2x} + A_6 e^{2x}$$

$\therefore$ Total trial solution of P.I. of (1) is

$$(A_1 + A_2 x + A_3 x^2) + (A_4 x^2 e^{2x} + A_5 x e^{2x} + A_6 e^{2x})$$

Note. We can easily omit $A_5 x e^{2x}$ and $A_6 e^{2x}$ from trial solution as xe^{2x} and e^{2x} are already present in C.F.

$\therefore$ Trial solution $= (A_1 + A_2 x + A_3 x^2) + A_4 x^2 e^{2x}$

(2) If a term of X is $x^s u$ where u is also a term of C.F. and is of r fold root m then in the trial solution we introduce $x^{r+s} u$ + terms arising from it by differentiation of x^{r+s}

e.g., Consider $(D-2)^3 (D+3)\, y = x^2 e^{2x} + x^2$ ...(1)

Here x^2 is not occurring in C.F.

$$\text{C.F.} = (c_1 + c_2 x + c_3 x^2)\, e^{2x} + c_4 e^{-3x}$$

$\therefore$ Its contribution to trial solution is $A_1 + A_2 x + A_3 x^2$. But e^{2x} is present in C.F. and of multiplicity three. Also it is multiplied by x^2.

$\therefore$ Trial solution corresponding to $x^2 e^{2x}$ will be

$$A_4 x^5 e^{2x} + A_5 x^4 e^{2x} + A_6 x^3 e^{2x} + A_7 x^2 e^{2x} + A_8 x e^{2x} + A_9 e^{2x}$$

$\therefore$ Total trial solution of (1) is

$$A_1 + A_2 x + A_3 x^2 + A_4 x^5 e^{2x} + A_5 x^4 e^{2x} + A_6 x^3 e^{2x} + A_7 x^2 e^{2x} + A_8 x e^{2x} + A_9 e^{2x}.$$

Note. Omit $A_7 x^2 e^{2x} + A_8 x e^{2x} + A_9 e^{2x}$ as they are already present in C.F.

$\therefore$ Trial solution for P.I. $= A_1 + A_2 x + A_3 x^2 + e^{2x}(A_4 x^5 + A_5 x^4 + A_6 x^3)$

$$= A_1 + A_2 x + A_3 x^2 + x^3 e^{2x}(A_4 x^2 + A_5 x + A_6).$$

ILLUSTRATIVE EXAMPLES

Example 1. *Solve* $\dfrac{d^2 y}{dx^2} + 2\dfrac{dy}{dx} + 4y = 2x^2 + 3e^{-x}$ *by method of undermined coefficients.*

(P.T.U., May 2007)

Sol. $$\frac{d^2 y}{dx^2} + 2\frac{dy}{dx} + 4y = 2x^2 + 3e^{-x} \quad \text{...(1)}$$

S.F. is $(D^2 + 2D + 4)y = 2x^2 + 3e^{-x}$

A.E. is $D^2 + 2D + 4 = 0$

$$D = \frac{-2 \pm \sqrt{4-16}}{2} = \frac{-2 \pm i2\sqrt{3}}{2} = -1 \pm i\sqrt{3}$$

$$\text{C.F.} = e^{-x}\left(c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x\right)$$

None of the terms in $X = 2x^2 + 3e^{-x}$ is present in C.F. $\therefore$ We write the trail solutions corresponding to x^2 and (1) e^{-x} and sum them up

$\therefore$ Trial solution for P.I.

$$y = \left(A_0 + A_1 x + A_2 x^2\right) + A_3 e^{-x}$$

$$\frac{dy}{dx} = A_1 + 2A_2 x - A_3 e^{-x}$$

$$\frac{d^2 y}{dx^2} = 2A_2 + A_3 e^{-x}$$

Substituting $\dfrac{d^2 y}{dx^2}, \dfrac{dy}{dx}, y$ in (1)

$$2A_2 + A_3 e^{-x} + 2A_1 + 4A_2 x - 2A_3 e^{-x} + 4A_0 + 4A_1 x + 4A_2 x^2 + 4A_3 e^{-x} = 2x^2 + 3e^{-x}$$

Equating coefficient of like terms on both sides

Coefficient of e^{-x}; $A_3 - 2A_3 + 4A_3 = 3 \quad \therefore \quad A_3 = 1$

Coefficient of x^2; $\quad 4A_2 = 2 \quad \therefore \quad A_2 = \dfrac{1}{2}$

Coefficient of x; $\quad 4A_2 + 4A_1 = 0 \quad \therefore \quad A_1 = -\dfrac{1}{2}$

Constant terms; $2A_2 + 2A_1 + 4A_0 = 0$; $A_0 = 0$

$$\text{P.I.} = 0 - \frac{1}{2}x + \frac{1}{2}x^2 + e^{-x} = -\frac{1}{2}x + \frac{1}{2}x^2 + e^{-x}$$

$\therefore$ C.S. is $y = \text{C.F.} + \text{P.I.}$

$\therefore$

$$= e^{-x}\left(c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x\right) - \frac{1}{2}x + \frac{1}{2}x^2 + e^{-x}$$

Example 2. *Solve : $(D^2 + 1)\, y = \sin x$.*

Sol. $(D^2 + 1)\, y = \sin x$ ...(1)

A.E. is $D^2 + 1 = 0 \quad \therefore \quad D = \pm i$

$$\text{C.F.} = c_1 \cos x + c_2 \sin x$$

Now, $X = \sin x$; but $\sin x$ is also present in C.F. but of multiplicity one only

$\therefore$ Trial solution corresponding to $\sin x$ will be

$$y = x\,(A_1 \sin x + A_2 \cos x)$$

Diff. w.r.t. x; $\dfrac{dy}{dx} = x\left(A_1 \cos x - A_2 \sin x\right) + \left(A_1 \sin x + A_2 \cos x\right)$

$$\frac{d^2y}{dx^2} = x\left(-A_1 \sin x - A_2 \cos x\right) + \left(A_1 \cos x - A_2 \sin x\right) + A_1 \cos x - A_2 \sin x$$

$$= \left(-A_1 x - 2A_2\right)\sin x + \left(-A_2 x + 2A_1\right)\cos x$$

Substituting the values of y and $\dfrac{d^2y}{dx^2}$ in (1)

$$\left(-A_1 x - 2A_2\right)\sin x + \left(-A_2 x + 2A_1\right)\cos x + xA_1 \sin x + xA_2 \cos x = \sin x$$

Comparing coefficients of like terms on both sides

$$-2A_2 = 1,\ \ 2A_1 = 0 \ \ \therefore \ \ A_1 = 0,\ \ A_2 = -\frac{1}{2}$$

$$\text{P.I.} = x\left(-\frac{1}{2}\cos x\right) = -\frac{x\cos x}{2}$$

$\therefore$ C.S. is $y = c_1 \cos x + c_2 \sin x - \dfrac{x}{2}\cos x$.

Example 3. *Solve : $\left(D^2 - 4D + 4\right)y = x^3 e^{2x} + xe^{2x}$.*

Sol. $(D^2 - 4D + 4)\, y = x^3 e^{2x} + x\, e^{2x}$ ...(1)

A.E. is $D^2 - 4D + 4 = 0$ or $(D-2)^2 = 0$ or $D = 2, 2$

$\therefore$ C.F. $= (c_1 + c_2 x)\, e^{2x}$

Now, $X = x^3 e^{2x} + x\, e^{2x}$

Here $x^3 e^{2x}$ is of the type $x^r u$ where $u = e^{2x}$ is present in the C.F. and is repeated twice ($\therefore$ by art. 3.12 particular case 2) trial solution corresponding to $x^3 e^{2x}$ is

$$A_1 x^5 e^{2x} + A_2 x^4 e^{2x} + A_3 x^3 e^{2x} + A_4 x^2 e^{2x} + A_5 x\, e^{2x} + A_6 e^{2x}$$

Now, trial solution corresponding to $x\, e^{2x}$ is included in the trial solution corresponding to $x^3 e^{2x}$. Also from the above trial solution we can easily omit $A_5 x\, e^{2x} + A_6 e^{2x}$ $\because$ e^{2x} and $x\, e^{2x}$ are already present in C.F.

$\therefore$ Final trial solution for the P.I.

$y = (A_1 x^5 + A_2 x^4 + A_3 x^3 + A_4 x^2) e^{2x}$ ...(2)

Differentiating twice w.r.t. x

$$\frac{dy}{dx} = (A_1 x^5 + A_2 x^4 + A_3 x^3 + A_4 x^2) 2e^{2x} + (5A_1 x^4 + 4A_2 x^3 + 3A_3 x^2 + 2A_4 x) e^{2x}$$

$$= e^{2x} [2A_1 x^5 + (2A_2 + 5A_1) x^4 + (2A_3 + 4A_2) x^3 + (2A_4 + 3A_3) x^2 + 2A_4 x]$$

$$\frac{d^2y}{dx^2} = 2e^{2x} [2A_1 x^5 + (2A_2 + 5A_1) x^4 + (2A_3 + 4A_2) x^3 + (2A_4 + 3A_3) x^2 + 2A_4 x]$$

$$+ e^{2x} [10A_1 x^4 + (8A_2 + 20A_1) x^3 + (6A_3 + 12A_2) x^2 + (4A_4 + 6A_3) x + 2A_4]$$

$$= e^{2x} [4A_1 x^5 + (4A_2 + 20A_1) x^4 + (4A_3 + 16A_2 + 20A_1) x^3$$

$$+ (4A_4 + 12A_3 + 12A_2) x^2 + (8A_4 + 6A_3) x + 2A_4]$$

Putting the values of $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$ in (1)

$$e^{2x} \{4A_1 x^5 + (4A_2 + 20A_1) x^4 + (4A_3 + 16A_2 + 20A_1) x^3 + (4A_4 + 12A_3 + 12A_2) x^2$$

$$+ (8A_4 + 6A_3) x + 2A_4\} - 4e^{2x} [2A_1 x^5 + (2A_2 + 5A_1) x^4 + (2A_3 + 4A_2) x^3 + (2A_4 + 3A_3) x^2 +$$

$$2A_4 x] + 4e^{2x} [A_1 x^5 + A_2 x^4 + A_3 x^3 + A_4 x^2] = e^{2x} (x^3 + x)$$

$$(4A_1 - 8A_1 + 4A) x^5 + (4A_2 + 20A_1 - 8A_2 - 20A_1 + 4A_2) x^4$$

$$+ (4A_3 + 16A_2 + 20A_1 - 8A_3 - 32A_2) x^3 + 4A_3 + (4A_4 + 12A_3 + 12A_2 - 8A_4 - 12A_3 + 4A_4) x^2$$

$$+ (8A_4 + 6A_3 - 8A_4) x + (2A_4) = x^3 + x$$

Comparing coeffs. of like terms

Coeff. of $x^5 \to 0 = 0$

Coeff of $x^4 \to \quad 0 = 0$

Coeff of $x^3 \to +20A_1 = 1 \quad \therefore \quad A_1 = \frac{1}{20}$

Coeff of $x^2 \to \quad 12A_2 = 0 \quad \therefore \quad A_2 = 0$

Coeff of $x \to \quad 6A_3 = 1 \quad A_3 = \frac{1}{6}$

Constant term $\quad 2A_4 = 0 \quad A_4 = 0$

$\therefore$ From (2), P.I. $= \left(\frac{1}{20} x^5 + \frac{1}{6} x^3\right) e^{2x}$

$\therefore$ C.S. is $y = (c_1 + c_2 x) e^{2x} + \left(\frac{1}{20} x^5 + \frac{1}{6} x^3\right) e^{2x} = \left(c_1 + c_2 x + \frac{1}{6} x^3 + \frac{1}{20} x^5\right) e^{2x}$.

Example 4. *Solve* : $(D^2 + 4) y = x^2 \sin 2x$.

Sol. $(D^2 + 4) y = x^2 \sin 2x$ ...(1)

A.E. is $D^2 + 4 = 0 \quad \therefore \quad D = \pm 2i$

C.F. $= c_1 \cos 2x + c_2 \sin 2x$

Now, $X = x^2 \sin 2x$ it is of the type $x^r u$ where $u = \sin 2x$ is present in the C.F. of multiplicity one.

$\therefore$ Its trial solution (by art 3.12 particular case 2)

$$= x^3 (A_1 \sin 2x + A_2 \cos 2x) + x^2 (A_3 \sin 2x + A_4 \cos 2x) + x (A_5 \sin 2x + A_6 \cos 2x)$$

$$+ (A_7 \sin 2x + A_8 \cos 2x)$$

$A_7 \sin 2x + A_8 \cos 2x$ may be omitted as it is present in C.F.

$\therefore$ Trial solution $y = x^3 (A_1 \sin 2x + A_2 \cos 2x) + x^2 (A_3 \sin 2x + A_4 \cos 2x) + x (A_5 \sin 2x + A_6 \cos 2x)$

$$y = (A_1 x^3 + A_3 x^2 + A_5 x) \sin 2x + (A_2 x^3 + A_4 x^2 + A_6 x) \cos 2x \quad \ldots(2)$$

$$\frac{dy}{dx} = (A_1 x^3 + A_3 x^2 + A_5 x)\, 2 \cos 2x + (3A_1 x^2 + 2A_3 x + A_5) \sin 2x + (3A_2 x^2 + 2A_4 x + A_6) \cos 2x + (A_2 x^3 + A_4 x^2 + A_6 x)(-2 \sin 2x)$$

$$\frac{dy}{dx} = \left[2A_1 x^3 (2A_3 + 3A_2) x^2 + (2A_5 + 2A_4) x + A_6\right] \cos 2x + \left[-2A_2 x^3 + (3A_1 - 2A_4) x^2 + (2A_3 - 2A_6) x + A_5\right] \sin 2x$$

$$\frac{d^2y}{dx^2} = \left[2A_1 x^3 + (2A_3 + 3A_2) x^2 + (2A_5 + 2A_4) x + A_6\right][-2 \sin 2x]$$
$$+\left[6A_1 x_2 + (4A_3 + 6A_2) x + (2A_5 + 2A_4)\right] \cos 2x$$
$$+\left[-2A_2 x^3 + (3A_1 - 2A_4) x^2 + (2A_3 - 2A_6) x + A_5\right][2 \cos 2x]$$
$$+\left[-6A_2 x^2 + (6A_1 - 4A_4) x + (2A_3 - 2A_6)\right] \sin 2x$$

$$= \left[-4A_1 x^3 + (-4A_3 - 6A_2 - 6A_2) x^2 + (-4A_5 - 4A_4 + 6A_1 - 4A_4)\right] x$$
$$+ (-2A_6 + 2A_3 - 2A_6) \sin 2x \Big] x$$
$$+\left[-4A_2 x^3 + (6A_1 + 6A_1 - 4A_4) + (4A_3 + 6A_2 + 4A_3 - 4A_6) x\right.$$
$$\left. + (2A_5 + 2A_4 + 2A_5)\right] \cos 2x$$

$$= \left[-4A_1 x^3 + (-4A_3 - 12A_2) x^2 + (-4A_5 - 8A_4 + 6A_1) x\right.$$
$$\left. + (2A_3 - 4A_6)\right] \sin 2x$$
$$+\left[-4A_2 x^3 + (12A_1 - 4A_4) x^2 + (8A_3 + 6A_2 - 4A_6) x + (4A_5 + 2A_4)\right] \cos 2x$$

Substituting $\frac{d^2y}{dx^2}$ and y in (1),

$$[-4A_1 x^3 + (-4A_3 - 12A_2) x^2 + (-4A_5 - 8A_4 + 6A_1) x + (2A_3 - 4A_6)] \sin 2x$$
$$+ [-4A_2 x^3 + (12A_1 - 4A_4) x^2 + (8A_3 + 6A_2 - 4A_6) x + (4A_5 + 2A_4)] \cos 2x$$
$$+ [4A_1 x^3 + 4A_3 x^2 + 4A_5 x] \sin 2x + 4A_2 x^3 + 4A_4 x^2 + 4xA_6] \cos 2x = x^2 \sin 2x$$
$$(-4A_1 + 4A_1) x^3 \sin 2x + (-4A_3 - 12A_2 + 4A_3) x^2 \sin 2x$$
$$+ (-4A_5 - 8A_4 + 6A_1 + 4A_5) x \sin 2x + (2A_3 - 4A_6) \sin 2x$$
$$+ (-4A_2 + 4A_2) x^2 \cos 2x + (12A_1 - 4A_4 + 4A_4) x^2 \cos 2x$$
$$\ldots 4A_6) x \cos 2x + (4A_5 + 2A_4) \cos 2x = x^2 \sin 2x$$

Comparing coefficients of like terms :

$$-12A_2 = 1 \qquad A_2 = -\frac{1}{12}$$

$$-8A_4 + 6A_1 = 0 \qquad 8A_4 = 6A_1$$

$$2A_3 - 4A_6 = 0 \qquad A_3 = 2A_6$$

$$12A_1 = 0 \qquad A_1 = 0$$

$$8A_3 + 6A_2 = 0 \qquad 4A_3 = -3A_2$$

$$4A_5 + 2A_4 = 0 \qquad 2A_5 = -A_4$$

$$\therefore \quad A_1 = 0 \Rightarrow A_4 = 0, A_5 = 0$$

$$A_2 = -\frac{1}{12} \Rightarrow A_3 = \frac{1}{16}, A_6 = \frac{1}{32}.$$

$\therefore$ From (2),

$$\text{P.I.} = x^3\left(-\frac{1}{12}\right)\cos 2x + x^2\left(\frac{1}{16}\right)\sin 2x + x.\frac{1}{32}\cos 2x$$

C.S. is $y = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{12}x^3 \cos 2x + \frac{1}{16}x^2 \sin 2x + \frac{1}{32}x\cos 2x$.

TEST YOUR KNOWLEDGE

Solve the following differential equations by method of undetermined coefficients :

1. $\frac{d^2y}{dx^2} - 9y = x + e^{2x} - \sin 2x$
2. $\frac{d^2y}{dx^2} - y = e^{3x}\cos 2x - e^{2x}\sin 3x$ **(P.T.U., May 2009)**
3. $(D^2 - 2D)\, y = e^x \sin x$ **(P.T.U., 2000)**
4. $\frac{d^2y}{dx^2} + y = 2\cos x$ **(P.T.U., 2000)**
5. $(D^2 - 3D + 2) = x^2 + e^x$
6. $(D^2 - 2D + 3)\, y = x^3 + \sin x$
7. $(D^3 - 2D^2 - D - 2)\, y = e^x + x^2$.

ANSWERS

1. $y = c_1 e^{3x} + c_2 e^{-3x} - \frac{1}{9}x - \frac{1}{5}e^{2x} + \frac{1}{13}\sin 2x$
2. $y = c_1 e^x + c_2 e^{-x} + \frac{1}{30}e^{2x}(2\cos 3x + \sin 3x) + \frac{1}{40}e^{3x}(\cos 2x + 3\sin 2x)$
3. $y = c_1 + c_2 e^{2x} - \frac{1}{2}e^x \sin x$
4. $y = c_1 \cos x + c_2 \sin x - x\sin x$
5. $y = c_1 e^x + c_2 e^{2x} + \frac{1}{2}\left(x^2 + 3x + 3.5 - 2xe^x\right)$
6. $y = e^x\left(c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x\right) + \frac{1}{27}\left(9x^3 + 18x^2 + 6x - 8\right) + \frac{1}{4}(\sin x + \cos x)$
7. $y = c_1 e^x + c_2 e^{-x} + c_3 e^{-2x} - \frac{5}{4} + \frac{1}{2}x - \frac{1}{2}x^2 + \frac{1}{6}xe^x$.

PART-III

[(a) Linear Differential Equations of Higher Order with Variable Coefficients]
[(b) Simultaneous Linear Differential Equations with constant Coefficients.]

3.13. CAUCHY'S HOMOGENEOUS LINEAR EQUATION

(P.T.U., Dec. 2004)

An equation of the form $x^n \dfrac{d^n y}{dx^n} + a_1 x^{n-1} \dfrac{d^{n-1}}{dx^{n-1}} + a_2 x^{n-2} \dfrac{d^{n-2} y}{dx^{n-2}} + \ldots\ldots\ldots + a_{n-1} x \dfrac{dy}{dx} + a_n y = X$ ...(1)

where a_i's are constant and X is a function of x, is called Cauchy 's homogeneous linear equation.

Such equation can be reduced to linear differential equations with constant coefficients by the substitution $x = e^z$ *i.e.*, $z = \log x$.

so that
$$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = \frac{dy}{dz}\cdot\frac{1}{x} \text{ or } x\frac{dy}{dx} = \frac{dy}{dz} = \text{D}y \text{ where D} = \frac{d}{dz}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\cdot\frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\cdot\frac{d^2y}{dz^2}\cdot\frac{dz}{dx}$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2} \qquad \left(\because \frac{dz}{dx} = \frac{1}{x}\right)$$

or
$$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz} = \text{D}^2y - \text{D}y = \text{D}(\text{D}-1)y$$

Similarly,
$$x^3\frac{d^3y}{dx^3} = \text{D}\,(\text{D}-1)\,(\text{D}-2)y \text{ and so on.}$$

Substituting these values in equation (*i*), we get a linear differential equation with constant coefficients, which can be solved by the methods already discussed.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve* : $x^2 y^2 + 4xy'' + 2y = 0.$ **(P.T.U., May 2006)**

Sol. $x^2\dfrac{d^2y}{dx^2} + 4x\dfrac{dy}{dx} + 2y = 0$ which is Cauchy's homogeneous linear equation.

Put
$$x = e^z \quad \therefore \quad z = \log x$$

D(D-1)

$$x\frac{dy}{dx} = \text{D}y, x^2\frac{d^2y}{dx^2} = \text{D}\,(\text{D}-1)\,y, \text{ we get}$$

D(D-1)
$D^2 - D$

or
$$\text{D}\,(\text{D}-1)\,y + 4\text{D}y + 2y = 0$$
$$(\text{D}^2 + 3\text{D} + 2)\,y = 0$$

$\therefore$ A.E. is $\text{D}^2 + 3\text{D} + 2 = 0$

$C_1 e$

$$\text{D} = -1, -2$$

Solution is
$$y = c_1 e^{-z} + c_2 e^{-2z} = c_1 \cdot \frac{1}{x} + c_2 \cdot \frac{1}{x^2}.$$

Example 2. *Solve :* $x^3 \dfrac{d^3y}{dx^3} + 2x^2 \dfrac{d^2y}{dx^2} + 2y = 10\left(x + \dfrac{1}{x}\right).$ **(P.T.U., June 2003, May 2009)**

Sol. Given equation is Cauchy's homogeneous linear equation

Put $$x = e^z \quad i.e.\ z = \log x$$

so that $$x\frac{dy}{dx} = \mathrm{D}y,\ x^2\frac{d^2y}{dx^2} = \mathrm{D(D-1)}y$$

$$x^3\frac{d^3y}{dx^3} = \mathrm{D\,(D-1)(D-2)}y, \text{ where } \mathrm{D} = \frac{d}{dz}$$

Dy

D(D-1)

Substituting these values in the given equation, it reduces to

$$[\mathrm{D\,(D-1)\,(D-2) + 2D\,(D-1) + 2}]y = 10(e^z + e^{-z})$$

or $$(\mathrm{D^3 - D^2 + 2})\, y = 10(e^z + e^{-z})$$

which is a linear equation with constant coefficients.

e^z, e^{-z}

Its A.E. is $\mathrm{D^3 - D^2 + 2 = 0}$ or $\mathrm{(D+1)\,(D^2 - 2D + 2) = 0}$

$1, e^{-z}$

$$\therefore \quad \mathrm{D} = -1, \frac{2 \pm \sqrt{4-8}}{2} = -1, 1 \pm i$$

$$\therefore \quad \text{C.F.} = c_1 e^{-z} + e^z(c_2 \cos z + c_3 \sin z) = \frac{c_1}{x} + x[c_2 \cos(\log x) + c_3 \sin(\log x)]$$

$$\text{P.I} = 10\frac{1}{\mathrm{D^3 - D^2 + 2}}(e^z + e^{-z}) = 10\left(\frac{1}{\mathrm{D^3 - D^2 + 2}}e^z + \frac{1}{\mathrm{D^3 - D^2 + 2}}e^{-z}\right)$$

$$= 10\left(\frac{1}{1^3 - 1^2 + 2}e^z + z.\frac{1}{3\mathrm{D}^2 - 2\mathrm{D}}e^{-z}\right) = 10\left(\frac{1}{2}e^z + z.\frac{1}{3(-1)^2 - 2(-1)}e^{-z}\right)$$

$$= 5e^z + 2ze^{-z} = 5x + \frac{2}{x}\log x$$

Hence the C.S. is $y = \dfrac{c_1}{x} + x\big[c_2 \cos(\log x) + c_3 \sin(\log x)\big] + 5x + \dfrac{2}{x}\log x.$

Example 3. *Solve the differential equation*

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} = \frac{12 \log x}{x^2}.$$ **(P.T.U., Dec. 2005)**

Sol. $$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} = \frac{12 \log x}{x^2}$$

Multiply by x^2;

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 12 \log x \text{ which is Cauchy's homogeneous linear equation}$$

Put $$x = e^z \quad \therefore \quad z = \log x$$

$$x\frac{dy}{dx} = \mathrm{D}y;\ x^2\frac{d^2y}{dx^2} = \mathrm{D\,(D-1)}\,y, \text{ we get}$$

$$D(D-1)y + Dy = 12z$$

$$D^2 y = 12z$$

A.E. is $D^2 = 0 \quad \therefore \quad D = 0, 0$

$$\text{C.F.} = c_1 + c_2 z$$

$$\text{P.I.} = \frac{1}{D^2}(12z) = 12 \cdot \frac{1}{D}\left(\frac{z^2}{2}\right) = 12 \cdot \frac{z^3}{6} = 2z^3$$

C.S. is $y = \text{C.F.} + \text{P.I.}$

$$= c_1 + c_2 z + 2z^3$$

$$y = c_1 + c_2 \log x + 2(\log x)^3.$$

Example 4. *Solve* $x^2 \dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + y = \log x \dfrac{\sin(\log x) + 1}{x}.$

Sol. Given equation is Cauchy's homogeneous linear equation

$\therefore$ Put $\quad x = e^z \quad \therefore \quad z = \log x;\ x\dfrac{dy}{dx} = Dy,\ x^2\dfrac{d^2y}{dx^2} = D^2y$, where $D = \dfrac{d}{dz}$

$$\therefore \quad [D(D-1) - 3D + 1]\,y = z\,\frac{\sin(z) + 1}{e^z}$$

or

$$[D^2 - 4D + 1]\,y = e^{-z} z(\sin z + 1)$$

A.E. is $D^2 - 4D + 1 = 0$ *i.e.*, $D = \dfrac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$

$$\therefore \quad \text{C.F.} = c_1 e^{(2+\sqrt{3})z} + c_2 e^{(2-\sqrt{3})z} = c_1 x^{2+\sqrt{3}} + c_2 x^{2-\sqrt{3}}$$

$$\text{P.I.} = \frac{1}{D^2 - 4D + 1} e^{-z} z(\sin z + 1)$$

$$= e^{-z} \frac{1}{(D-1)^2 - 4(D-1) + 1} z(1 + \sin z)$$

$$= e^{-z}\left\{\frac{1}{D^2 - 6D + 6} z + \frac{1}{D^2 - 6D + 6} z \sin z\right\}$$

$$= e^{-z}[I_1 + I_2] \qquad \ldots(1)$$

where

$$I_1 = \frac{1}{D^2 - 6D + 6} z = \frac{1}{6}\left[1 - D + \frac{D^2}{6}\right]^{-1} z = \frac{1}{6}\left[1 + D + \frac{D^2}{6}\right] z$$

$$= \frac{1}{6}[z + 1]$$

$$I_2 = \frac{1}{D^2 - 6D + 6} z \sin z$$

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We know that $\frac{1}{f(D)}(xV) = x\frac{1}{f(D)}V + \left[\frac{d}{dD}f(D)\right]V$ [Note this P.I.]

$$= z\frac{1}{D^2-6D+6}\sin z + \frac{d}{dD}\left(\frac{1}{D^2-6D+6}\right)\sin z$$

Put $D^2 = -1$

$$= z\frac{1}{5-6D}\sin z - \frac{2D-6}{\left(D^2-6D+6\right)^2}\sin z$$

$$= z\frac{5+6D}{25-36D^2}\sin z - \frac{2D-6}{\left(D^2-6D+6\right)^2}\sin z$$

Put $D^2 = -1$ ✓

$$= z\frac{(5+6D)}{61}\sin z - \frac{(2D-6)}{(5-6D)^2}\sin z$$

$$= z\frac{5\sin z + 6\cos z}{61} - \frac{1}{25+36D^2-60D}(2\cos z - 6\sin z)$$

Put $D^2 = -1$,

$$= \frac{z}{61}(5\sin z + 6\cos z) + \frac{1}{11+60D}(2\cos z - 6\sin z)$$

$$= \frac{z}{61}(5\sin z + 6\cos z) + \frac{11-60D}{121-3600D^2}(2\cos z - 6\sin z)$$

Put $D^2 = -1$

$$= \frac{z}{61}(5\sin z + 6\cos z) + \frac{11-60D}{3721}(2\cos z - 6\sin z)$$

$$= \frac{z}{61}(5\sin z + 6\cos z) + \frac{1}{3721}[22\cos z - 66\sin z + 120\sin z + 360\cos z]$$

$$= \frac{z}{61}(5\sin z + 6\cos z) + \frac{1}{3721}(54\sin z + 382\cos z)$$

$\therefore$ From (1),

$$\text{P.I.} = e^{-z}\left[\frac{z+1}{6} + \frac{z}{61}(5\sin z + 6\cos z) + \frac{2}{3721}(27\sin z + 191\cos z)\right]$$

$$= \frac{1}{x}\left[\frac{1}{6}(\log x + 1) + \frac{\log x}{61}\{5\sin(\log x) + 6\cos(\log x)\} + \frac{2}{(61)^2}\{27\sin(\log x) + 191\cos(\log x)\}\right]$$

C.S. is $y = c_1 x^{2+\sqrt{3}} + c_2 x^{2-\sqrt{3}} + \frac{1}{6x}(1+\log x) + \frac{1}{61x}\Big[\log x\,\{5\sin(\log x) + 6\cos(\log x)\}$

$+\frac{2}{61}\{27\sin(\log x) + 191\cos(\log x)\}\Big]$

Example 5. *Solve* : $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = \log x \sin(\log x)$. **(P.T.U., Dec. 2003; Jan 2010)**

Sol. Given equation is a Cauchy's homogeneous linear equation.

Put $x = e^z$ *i.e.* $z = \log x$ so that $x\frac{dy}{dx} = \text{D}y,\ x^2\frac{d^2y}{dx^2} = \text{D}(\text{D}-1)y$

where $\text{D} = \frac{d}{dz}$.

Substituting these values in the given equation, it reduces to $[\text{D}(\text{D}-1) + \text{D} + 1]\, y = z\sin z$

or $(\text{D}^2 + 1)\, y = z\sin z$

Its A.E. is $\text{D}^2 + 1 = 0$ so that $\text{D} = \pm i$

$$\text{C.F.} = c_1\cos z + c_2\sin z = c_1\cos(\log x) + c_2\sin(\log x)$$

$$\text{P.I.} = \frac{1}{\text{D}^2+1} z\sin z = \text{Imaginary part of } \frac{1}{\text{D}^2+1} ze^{iz}$$

$$= \text{I.P. of } e^{iz}\frac{1}{(\text{D}+i)^2+1}z = \text{I.P. of } e^{iz}\frac{1}{\text{D}^2+2i\text{D}}z$$

$$= \text{I.P. of } e^{iz}\frac{1}{2i\text{D}\left(1+\frac{\text{D}}{2i}\right)}z = \text{I.P. of } e^{iz}\frac{1}{2i\text{D}\left(1-\frac{i\text{D}}{2}\right)}z$$

$$= \text{I.P. of } \frac{1}{2i}e^{iz}\frac{1}{\text{D}}\left(1-\frac{i\text{D}}{2}\right)^{-1}z = \text{I.P. of } \frac{1}{2i}e^{iz}\frac{1}{\text{D}}\left(1+\frac{i\text{D}}{2}+.....\right)z$$

$$= \text{I.P. of } \frac{1}{2i}e^{iz}\frac{1}{\text{D}}\left(z+\frac{i}{2}\right) = \text{I.P. of } \frac{1}{2i}e^{iz}\int\left(z+\frac{i}{2}\right)dz$$

$$= \text{I.P. of } -\frac{i}{2}e^{iz}\left(\frac{z^2}{2}+\frac{i}{2}z\right) = \text{I.P. of } e^{iz}\left(-\frac{i}{4}z^2+\frac{z}{4}\right)$$

$$= \text{I.P. of } (\cos z + i\sin z)\left(-\frac{i}{4}z^2+\frac{z}{4}\right) = -\frac{z^2}{4}\cos z + \frac{z}{4}\sin z$$

$$= -\frac{1}{4}(\log x)^2\cos(\log x) + \frac{1}{4}\log x\sin(\log x).$$

Hence the C.S. is $y = c_1\cos(\log x) + c_2\sin(\log x) - \frac{1}{4}(\log x)^2\cos(\log x) + \frac{1}{4}\log x\sin(\log x)$.

Example 6. *Solve :* $x^2 \dfrac{d^2 y}{dx^2} + 3x \dfrac{dy}{dx} + y = \dfrac{1}{(1-x)^2}$. (P.T.U., Dec. 2002)

Sol. Given equation is Cauchy's homogeneous linear equation

Put $x = e^z$ $\therefore$ $z = \log x, x \dfrac{dy}{dx} = \dfrac{dy}{dz} = \text{D}y$

$$x^2 \frac{d^2 y}{dx^2} = \text{D}(\text{D}-1)y, \text{ where D} = \frac{d}{dz}$$

Substituting the values in given equation

$$[\text{D}(\text{D}-1) + 3\text{D} + 1]y = \frac{1}{(1-e^z)^2}$$

$$(\text{D}^2 + 2\text{D} + 1)y = \frac{1}{(1-e^z)^2}$$

A.E. is $\text{D}^2 + 2\text{D} + 1 = 0$, *i.e.*, $(\text{D}+1)^2 = 0 \therefore \text{D} = -1, -1$

$$\text{C.F.} = (c_1 + c_2 z)e^{-z}$$

$$\text{P.I.} = \frac{1}{(\text{D}+1)^2} \cdot \frac{1}{(1-e^z)^2} = \frac{1}{\text{D}+1}\left\{\frac{1}{\text{D}+1} \cdot \frac{1}{(1-e^z)^2}\right\}$$

$$= \frac{1}{\text{D}+1}\left[e^{-z}\int e^z \cdot \frac{1}{(1-e^z)^2}\,dz\right] \quad \text{By using } \frac{1}{\text{D}-a}\text{X} = e^{ax}\int e^{-ax}\text{X}\,dx$$

$$= \frac{1}{\text{D}+1}\left[e^{-z}\int (1-e^z)^{-2} e^z\,dz\right] = \frac{1}{\text{D}+1}\left[e^{-z}\int -(1-e^z)^{-2}(-e^z)\,dz\right]$$

$$= \frac{1}{\text{D}+1}\left[e^{-z}(-1)\frac{(1-e^z)^{-1}}{-1}\right] \quad \text{By using } \int [f(z)]^n f'(z)\,dz = \frac{[f(z)]^{n+1}}{n+1}; n \neq -1$$

$$= \frac{1}{\text{D}+1}\left(\frac{e^{-z}}{1-e^z}\right)$$

$$= e^{-z}\int e^z \cdot \frac{e^{-z}}{1-e^z}\,dz = e^{-z}\int \frac{dz}{1-e^z}$$

Put $e^z = t \therefore e^z\,dz = dt \therefore dz = \dfrac{1}{t}\,dt$

$$\text{P.I.} = e^{-z}\int \frac{1}{t(1-t)}\,dt$$

$$= e^{-z} \int \left(\frac{1}{t} + \frac{1}{1-t} \right) dt$$ By Partial fractions

$$= e^{-z} [\log t - \log (1-t)] = e^{-z} \log \frac{t}{1-t} = e^{-z} \log \frac{e^z}{1-e^z}$$

C.S. is $$y = (c_1 + c_2 z) e^{-z} + e^{-z} \log \frac{e^z}{1-e^z} = (c_1 + c_2 \log x) \frac{1}{x} + \frac{1}{x} \log \frac{x}{1-x}$$

$$= \frac{1}{x} \left[c_1 + c_2 \log x + \log \frac{x}{1-x} \right]$$

Example 7. *Solve* : $u = r \dfrac{d}{dr} \left(r \dfrac{du}{dr} \right) + ar^3$. **(P.T.U., May 2001)**

Sol. $$u = r \left[r \frac{d^2u}{dr^2} + \frac{du}{dr} \right] + ar^3$$

$$r^2 \frac{d^2u}{dr^2} + r \frac{du}{dr} - u = -ar^3$$ which is Cauchy's homogeneous linear equation in u and r

Put $r = e^z$ and Let $D = \dfrac{d}{dz}$

$\therefore$ $$(D(D-1) + D - 1) u = -a e^{3z}$$
$$(D^2 - 1) u = -a e^{3z}$$

A.E. is $$D^2 - 1 = 0$$ $\therefore D = -1, 1$

$$\text{C.F.} = c_1 e^z + c_2 e^{-z}$$

$$\text{P.I.} = \frac{1}{D^2 - 1} (-a e^{3z}) = \frac{-a e^{3z}}{8}$$ by putting $D = 3$

$$\text{C.S. is } u = c_1 e^z + c_2 e^{-z} - \frac{a e^{3z}}{8} = c_1 r + \frac{c_2}{r} - \frac{a}{8} r^3.$$

Example 8. *Solve* $x^2y'' - 4xy' + 8y = 4x^3 + 2 \sin (\log x)$. **(P.T.U., May 2006)**

Sol. $$x^2 \frac{d^2y}{dx^2} - 4x \frac{dy}{dx} + 8y = 4x^3 + 2 \sin (\log x)$$

which is Cauchy's homogeneous linear = x

Put $$x = e^z \text{ i.e., } z = \log x$$

$$x \frac{dy}{dx} = Dy, \; x^2 \frac{d^2y}{dx^2} = D(D-1) y$$

$\therefore$ $$D(D-1) y - 4Dy + 8y = 4e^{3z} + 2 \sin z$$
$$(D^2 - 5D + 8) y = 4e^{3z} + 2 \sin z$$

A.E. is $D^2 - 5D + 8 = 0 \quad \therefore \; D = \frac{5}{2} \pm i\frac{\sqrt{7}}{2}$

$$\text{C.F.} = e^{\frac{5}{2}z}\left[c_1 \cos\frac{\sqrt{7}}{2}z + c_2 \sin\frac{\sqrt{7}}{2}z\right]$$

$$\text{P.I.} = \frac{1}{D^2 - 5D + 8}\left(4e^{3z} + 2\sin z\right)$$

$$= 4\underset{(\text{Put } D = 3)}{\frac{1}{D^2 - 5D + 8}}e^{3z} + 2\underset{(\text{Put } D^2 = -1)}{\frac{1}{D^2 - 5D + 8}}\sin z$$

$$= 4.\frac{1}{2}e^{3z} + 2.\frac{1}{-5D + 7}\sin z = 2e^{3z} - 2\frac{5D + 7}{25D^2 - 49}\sin z$$

$$= 2e^{3z} - 2\frac{5D + 7}{-74}\sin z = 2e^{3z} + \frac{1}{37}[5\cos z + 7\sin z]$$

$$\text{C.S. } y = e^{\frac{5}{2}z}\left[c_1 \cos\frac{\sqrt{7}}{2}z + c_2 \sin\frac{\sqrt{7}}{2}z\right] + 2e^{3z} + \frac{5}{37}\cos z + \frac{7}{37}\sin z$$

$$\therefore \quad y = x^{\frac{5}{2}}\left[c_1 \cos\left(\frac{\sqrt{7}}{2}\log x\right) + c_2 \sin\left(\frac{\sqrt{7}}{2}\log x\right)\right] + 2x^3 + \frac{5}{37}\cos(\log x) + \frac{7}{37}\sin(\log x).$$

3.14. LEGENDRE'S LINEAR EQUATION

(P.T.U., May 2007, Dec. 2005)

An equation of the form

$$(a + bx)^n \frac{d^n y}{dx^n} + a_1 (a + bx)^{n-1} \frac{d^{n-1}}{dx^{n-1}} + \ldots.. + a_{n-1}(a + bx)\frac{dy}{dx} + a_n y = X \qquad \ldots(i)$$

where a_i's are constants and X is a function x, is called Legendre's linear equation.

Such equations can be reduced to linear differential equations with constant coefficients, by the substitutions

$$a + bx = e^z \quad i.e., \; z = \log(a + bx) \quad \text{so that } \frac{dy}{dx} = \frac{dy}{dz}.\frac{dz}{dx} = \frac{b}{a + bx}\frac{dy}{dz}$$

or $$(a + bx)\frac{dy}{dx} = b\frac{dy}{dz} = bDy, \text{ where } D = \frac{d}{dz}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{b}{a + bx}\frac{dy}{dz}\right) = -\frac{b^2}{(a + bx)^2}\frac{dy}{dz} + \frac{b}{a + bx}.\frac{d^2y}{dz^2}.\frac{dy}{dx}$$

$$= -\frac{b^2}{(a + bx)^2}\frac{dy}{dz} + \frac{b}{a + bx}\frac{d^2y}{dz^2}.\frac{b}{a + bx} = \frac{b^2}{(a + bx)^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$$

or $$(a+bx)^2\frac{d^2y}{dx^2} = b^2(D^2y - Dy) = b^2 D(D-1)y$$

Similarly, $(a+bx)^3\dfrac{d^3y}{dx^3} = b^3 D(D-1)(D-2)y.$

Substituting these values in equation (i), we get a linear differential equation with constant coefficient which can be solved by the methods already discussed.

Example 9. *Solve :* $(3x+2)^2\dfrac{d^2y}{dx^2} + 3(3x+2)\dfrac{dy}{dx} - 36y = 3x^2 + 4x + 1$.

Sol. Given equation is a Legendre's linear equation.

Put $\quad 3x+2 = e^z \quad$ *i.e.*, $\quad z = \log(3x+2)$ so that $(3x+2)\dfrac{dy}{dx} = 3Dy$.

$$(3x+2)^2\frac{d^2y}{dx^2} = 3^2 D(D-1)y, \text{ where } D = \frac{d}{dz}.$$

Substituting these values in the given equation, it reduces to

$$[3^2 D(D-1) + 3.3D - 36]y = 3\left(\frac{e^z-2}{3}\right)^2 + 4\left(\frac{e^z-2}{3}\right) + 1$$

or $$9(D^2-4)y = \frac{1}{3}e^{2z} - \frac{1}{3} \text{ or } (D^2-4)y = \frac{1}{27}(e^{2z}-1)$$

which is a linear equation with constant coefficients.

Its A.E. is $D^2 - 4 = 0 \therefore D = \pm 2$

$$\text{C.F.} = c_1e^{2z} + c_2e^{-z} = c_1(3x+2)^2 + c_2(3x+2)^{-2}$$

$$\text{P.I.} = \frac{1}{27}\cdot\frac{1}{D^2-4}\left(e^{2z}-1\right) = \frac{1}{27}\left[\frac{1}{D^2-4}e^{2z} - \frac{1}{D^2-4}e^{0z}\right].$$

$$= \frac{1}{27}\left[z.\frac{1}{2D}e^{2z} - \frac{1}{0-4}e^{0z}\right] = \frac{1}{27}\left[\frac{z}{2}\int e^{2z}dz + \frac{1}{4}\right]$$

$$= \frac{1}{27}\left[\frac{z}{4}e^{2z} + \frac{1}{4}\right] = \frac{1}{108}\left(ze^{2z}+1\right) = \frac{1}{108}[(3x+2)^2\log(3x+2)+1]$$

Hence the C.S. is $y = c_1(3x+2)^2 + c_2(3x+2)^{-2} + \dfrac{1}{108}[(3x+2)^2\log(3x+2)+1]$.

Example 10. *Solve :* $(1+x)^2\dfrac{d^2y}{dx^2} + (1+x)\dfrac{dy}{dx} + y = \sin[2\log(1+x)]$. **(P.T.U., Dec. 2006)**

Sol. Given equation is Legendre's linear equation

$\therefore$ Put $\quad 1+x = e^z \quad \therefore \quad z = \log(1+x)$

$$(1+x)\frac{dy}{dx} = Dy, \quad (1+x)^2\frac{d^2y}{dx^2} = D(D-1)y, \text{ where } D = \frac{d}{dz}$$

$\therefore \quad D(D-1)y + Dy + y = \sin(2z)$

or $\quad (D^2+1)y = \sin 2z$

which is linear differential equation with constant coefficients

A.E. is $D^2 + 1 = 0 \quad \therefore \; D = \pm i$

$$\text{C.F.} = c_1 \cos z + c_2 \sin z$$

$$\text{P.I.} = \frac{1}{D^2+1} \sin 2z$$

Put $D^2 = -4$

$$\therefore \quad \text{P.I.} = -\frac{1}{3}\sin 2z$$

$$\text{C.S. is } y = c_1 \cos z + c_2 \sin z - \frac{1}{3}\sin 2z$$

Put $z = \log(1+x)$

$$\therefore \quad y = c_1 \cos[\log(1+x)] + c_2 \sin[\log(1+x)] - \frac{1}{3}\sin[2\log(1+x)].$$

TEST YOUR KNOWLEDGE

Solve :

1. $x^2\dfrac{d^2y}{dx^2} + 9x\dfrac{dy}{dx} + 25y = 50$

2. $x^2\dfrac{d^2y}{dx^2} - 2y = x^2 + \dfrac{1}{x}$

3. $x^2\dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} - 20y = (x+1)^2$

4. $x^2\dfrac{d^3y}{dx^3} - 4x\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} = 4$

   [Hint : Multiply throughout by x]

5. (*i*) $x^4\dfrac{d^3y}{dx^3} + 2x^3\dfrac{d^2y}{dx^2} - x^2\dfrac{dy}{dx} + xy = 1$ (*ii*) $x^2\dfrac{d^2y}{dx^2} - 4x\dfrac{dy}{dx} + 6y = x^2$

6. The radial displacement u in a rotating disc at a distance r from the axis is given by $r^2\dfrac{d^2u}{dr^2} + r\dfrac{du}{dr} - u + kr^3 = 0$, where k is a constant. Solve the equation under the conditions $u = 0$ when $r = 0$, $u = 0$ when $r = a$.

7. $x^2\dfrac{d^2y}{dx} - x\dfrac{dy}{dx} + y = \log x$ **(P.T.U., May 2009)**

8. $x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + 2y = x\log x$ **(P.T.U., May 2010)**

9. $x^2\dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} - 12y = x^3\log x$

10. $x^2\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} - 4y = x^2 + 2\log x$

11. $x^2\dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + 5y = \sin(\log x)$

12. $x^3\dfrac{d^3y}{dx^3} + 3x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + 8y = 65\cos(\log x)$

13. $x^2\dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + 5y = x^2\sin(\log x)$

14. $x^2\dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + y = \log x\dfrac{\sin(\log x) + 1}{x}$

15. $x^2\frac{d^2y}{dx^2}-x\frac{dy}{dx}-3y=x^2\log x$

16. $(1+x)^2\frac{d^2y}{dx^2}+(1+x)\frac{dy}{dx}+y=4\cos\log(1+x)$

17. $(1+2x)^2\frac{d^2y}{dx^2}-6(1+2x)\frac{dy}{dx}+16y=8(1+2x)^2$

18. $(2x+3)^2\frac{d^2y}{dx}-2(2x+3)\frac{dy}{dx}-12y=6x$.

ANSWERS

1. $y=x^{-4}[c_1\cos(3\log x)+c_2\sin(3\log x)]+2$

2. $y=c_1x^2+\frac{c_2}{x}+\frac{1}{3}\left(x^2-\frac{1}{x}\right)\log x$

3. $y=c_1x^4+c_2x^{-5}-\frac{x^2}{14}-\frac{x}{9}-\frac{1}{20}$

4. $y=c_1+c_2x^3+c_3x^4+\frac{2}{3}x$

5. (*i*) $y=(c_1+c_2\log x)x+c_3x^{-1}+\frac{1}{4x}\log x$

(*ii*) $y=c_1x^2+c_2x^2-x^2\log x$

6. $u=\frac{kr}{8}(a^2-r^2)$

7. $y=(c_1+c_2\log x)x+\log x+2$

8. $y=x[c_1\cos(\log x)+c_2\sin(\log x)]+x\log x$

9. $y=c_1x^3+c_2x^{-4}+\frac{x^2}{98}\log x\,(7\log x-2)$

10. $y=c_1x^{-1}+c_2x^4-\frac{x^2}{6}-\frac{1}{2}\log x+\frac{3}{8}$

11. $y=x^2[c_1\cos(\log x)+c_2\sin(\log x)]+\frac{1}{8}[\sin(\log x)+\cos(\log x)]$

12. $y=c_1x^{-2}+x(c_2\cos\sqrt{3}\log x)+c_3\sin(\sqrt{3}\log x)+8\cos(\log x)-\sin(\log x)$

13. $y=x^2[c_1\cos(\log x)+c_2\sin(\log x)]-\frac{1}{2}x^2\log x\cos(\log x)$

14. $y=c_1x^{2+\sqrt{3}}+c_2x^{2-\sqrt{3}}+\frac{1}{61x}$

$$\left[\log x\{5\sin(\log x)+6\cos(\log x)\}+\frac{2}{61}\{21\sin(\log x)+191\cos(\log x)\}\right]+\frac{1}{6x}(1+\log x)$$

15. $y=c_1x^3+\frac{c_2}{x}-\frac{x^2}{3}\left(\log x+\frac{2}{3}\right)$

16. $y=c_1\cos[\log(1+x)]+c_2\sin[\log(1+x)]+2\log(1+x)+\sin[\log(1+x)]$

17. $y=(1+2x)^2[c_1+c_2\log(1+2x)+\{\log(1+2x)\}^2]$

18. $y=c_1(2x+3)^{-1}+c^2(2x+3)^3-\frac{3}{4}(2x+3)+3$.

3.15. SIMULTANEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

Now, we discuss differential equations in which there is one independent variable and two or more than two dependent variables. Such equations are called *simultaneous linear equations.* To solve such equations completely, we must have as many simultaneous equations as the number of dependent variables. Here, we shall consider simultaneous linear equations with constant coefficients only.

Let x, y be the two dependent variables and t the independent variable. Consider the simultaneous equations

$$f_1(D)x + f_2(D)y = T_1 \quad \ldots(1) \qquad \phi_1(D)x + \phi_2(D)y = T_2 \quad \ldots(2)$$

where $D = \frac{d}{dt}$ and T_1, T_2 are functions of t.

To eliminate y, operating on both sides of (1) by $\phi_2(D)$ and on both sides of (2) by $f_2(D)$ and subtracting,

we get $$[f_1(D)\phi_2(D) - \phi_1(D)f_2(D)]x = \phi_2(D)T_1 - f_2(D)T_2 \text{ or } f(D)x = T$$

which is a linear equation in x and t and can be solved by the methods already discussed. Substituting the value of x in either (1) or (2), we get the value of y.

Note. We can also eliminate x to get a linear equation in y and t.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve :* $\frac{dx}{dt} + 4x + 3y = t$

$$\frac{dy}{dt} + 2x + 5y = e^t.$$

Sol. Writing D for $\frac{d}{dt}$, the given equations become $(D+4)x + 3y = t$ ...(1)

and $$2x + (D+5)y = e^t$$

To eliminate y, operating on both sides of (1) by $(D+5)$ and on both sides of (2) by 3 and subtracting,

we get $$[(D+4)(D+5) - 6]x = (D+5)t - 3e^t$$

or $$(D^2 + 9D + 14)x = 1 + 5t - 3e^t$$

Its A.E. is $$D^2 + 9D + 14 = 0$$

$$(D+2)(D+7) = 0 \qquad \therefore \quad D = -2, -7$$

$$\text{C.F.} = c_1e^{-2t} + c_2e^{-7t}$$

$$\text{P.I.} = \frac{1}{D^2 + 9D + 14}(1 + 5t - 3e^t)$$

$$= \frac{1}{D^2 + 9D + 14}e^{0t} + 5\frac{1}{D^2 + 9D + 14}t - 3\frac{1}{D^2 + 9D + 14}e^t$$

$$= \frac{1}{0^2 + 9(0) + 14}e^{0t} + 5.\frac{1}{14\left(1 + \frac{9D}{14} + \frac{D^2}{14}\right)}t - 3\frac{1}{1^2 + 9(1) + 14}e^t$$

$$= \frac{1}{14} + \frac{5}{14}\left[1 + \left(\frac{9D}{14} + \frac{D^2}{14}\right)\right]^{-1}t - \frac{1}{8}e^t = \frac{1}{14} + \frac{5}{14}\left[1 - \left(\frac{9D}{14} + \frac{D^2}{14}\right) + \ldots\right]t - \frac{1}{8}e^t$$

$$= \frac{1}{14} + \frac{5}{14}\left(t - \frac{9}{14}\right) - \frac{1}{8}e^t = \frac{1}{14} + \frac{5}{14}t - \frac{45}{196} - \frac{1}{8}e^t = \frac{5}{14}t - \frac{31}{196} - \frac{1}{8}e^t$$

Now, $\frac{dx}{dt} = -2c_1e^{-2t} - 7c_2e^{-7t} + \frac{5}{14} - \frac{1}{8}e^t$

Substituting the values of x and $\frac{dx}{dt}$ in (1), we have $3y = t - \frac{dx}{dt} - 4x$

$$= t + 2c_1e^{-2t} + 7c_2e^{-7t} - \frac{5}{14} + \frac{1}{8}e^t - 4c_1e^{-2t} - 4c_2e^{-7t} - \frac{10}{7}t + \frac{31}{49} + \frac{1}{2}e^t$$

$$\therefore \quad y = \frac{1}{3}\left[-2c_1e^{-2t} + 3c_2e^{-7t} - \frac{3}{7}t + \frac{27}{98} + \frac{5}{8}e^t\right]$$

Hence $\quad x = c_1e^{-2t} + c_2e^{-7t} + \frac{5}{14}t - \frac{31}{196} - \frac{1}{8}e^t$

$$y = -\frac{2}{3}c_1e^{-2t} + c_2e^{-7t} - \frac{1}{7}t + \frac{9}{98} + \frac{5}{24}e^t.$$

Example 2. *Solve :* $\frac{dx}{dt} + 2y = e^t$

$$\frac{dy}{dt} - 2x = e^{-t}.$$

(P.T.U., Dec. 2001)

Sol. $$\frac{dx}{dt} + 2y = e^t \qquad \text{...(1)}$$

$$\frac{dy}{dt} - 2x = e^{-t} \qquad \text{...(2)}$$

To eliminate y, differentiate (1) w.r.t. t

$$\frac{d^2x}{dt^2} + 2\frac{dy}{dt} = e^t$$

From (2), $\quad \frac{dy}{dt} = 2x + e^{-t}$

$$\therefore \quad \frac{d^2x}{dt^2} + 4x + 2e^{-t} = e^t$$

$$\frac{d^2x}{dt^2} + 4x = e^t - 2e^{-t}$$

which is linear differential equation with constant coefficients.

Its S.F. is $(D^2 + 4)x = e^t - 2e^{-t}$ where $D = \frac{d}{dt}$

A.E. is $D^2 + 4 = 0$ $\quad \therefore \quad D = \pm 2i$

C.F. $= c_1 \cos 2t + c_2 \sin 2t$

P.I. $= \frac{1}{D^2+4}\left(e^t - 2e^{-t}\right) = \frac{1}{D^2+4}e^t - 2\frac{1}{D^2+4}e^{-t}$

(Put D = 1) (Put D = – 1)

$$= \frac{1}{5}e^t - \frac{2}{5}e^{-t}$$

$\therefore$ C.S. is $x = c_1 \cos 2t + c_2 \sin 2t + \frac{1}{5}e^t - \frac{2}{5}e^{-t}$

$$\frac{dx}{dt} = -2c_1 \sin 2t + 2c_2 \cos 2t + \frac{1}{5}e^t + \frac{2}{5}e^{-t}$$

From (1),

$$2y = 2c_1 \sin 2t - 2c_2 \cos 2t - \frac{1}{5}e^t - \frac{2}{5}e^{-t} + e^t$$

$\therefore$ $$y = c_1 \sin 2t - c_2 \cos 2t + \frac{2}{5}e^t - \frac{1}{5}e^{-t}$$

Hence $$x = c_1 \cos 2t + c_2 \sin 2t + \frac{1}{5}e^t - \frac{2}{5}e^{-t}$$

and $y = c_1 \sin 2t - c_2 \cos 2t + \frac{2}{5}e^t - \frac{1}{5}e^{-t}$ is the required solution.

Example 3. *Solve the system equations*

$$(2D-4)y_1 + (3D+5)y_2 = 3t+2$$
$$(D-2)y_1 + (D+1)y_2 = t$$

(P.T.U., Jan 2008)

Sol. $$(2D-4)y_1 + (3D+5)y_2 = 3t+2 \quad ...(1)$$
$$(D-2)y_1 + (D+1)y_2 = t \quad ...(2)$$

Multiply (2) by 2 and Subtract from (1)

$$(D+3)y_2 = t+2$$

or $\frac{dy_2}{dt} + 3y_2 = t+2$ which is linear differential equation in t.

Its $$\text{I.F.} = e^{\int 3dt} = e^{3t}$$

Its solution is $$y_2 e^{3t} = \int (t+2)e^{3t}\,dt + c_1$$

Integrating by parts

$$= (t+2)\frac{e^{3t}}{3} - (1)\cdot\frac{e^{3t}}{9} + c_1$$

$\therefore$ $$y_2 = \frac{t+2}{3} - \frac{1}{9} + c_1 e^{-3t} = \frac{3t+5}{9} + c_1 e^{-3t}$$

Substituting the value of y_2 in (2), we get

$$(D-2)y_1 + (D+1)\left[\frac{3t+5}{9} + c_1 e^{-3t}\right] = t$$

or $$(D-2)y_1 + \left[\frac{1}{3} - 3c_1 e^{-3t} + \frac{3t+5}{9} + c_1 e^{-3t}\right] = t$$

or $$(D-2)y_1 + \left[\frac{8+3t}{9} - 2c_1 e^{-3t}\right] = t$$

or $$(D-2)y_1 = t - \frac{8}{9} - \frac{1}{3}t + 2c_1 e^{-3t}$$

or $$\frac{dy_1}{dt} - 2y_1 = \frac{2}{3}t - \frac{8}{9} + 2c_1 e^{-3t}$$

which is linear differential equation in t

Its I.F. $= e^{-2t}$

Its solution is
$$y_1 e^{-2t} = \int e^{-2t}\left(\frac{2}{3}t - \frac{8}{9} + 2c_1 e^{-3t}\right)dt + c_2$$
$$= \frac{2}{3}\int te^{-2t}\,dt - \frac{8}{9}\int e^{-2t}\,dt + 2c_1\int e^{-5t}\,dt + c_2$$
$$= \frac{2}{3}\left[t\frac{e^{-2t}}{-2} - 1\left(\frac{e^{-2t}}{4}\right)\right] - \frac{8}{9}\frac{e^{-2t}}{-2} + 2c_1\frac{e^{-5t}}{-5} + c_2$$
$$= -\frac{t}{3}e^{-2t} - \frac{1}{6}e^{-2t} + \frac{4}{9}e^{-2t} - \frac{2}{5}c_1 e^{-5t} + c_2$$
$$= -\frac{t}{3}e^{-2t} + \frac{5}{18}e^{-2t} - \frac{2}{5}c_1 e^{-5t} + c_2$$

$\therefore$
$$y_1 = -\frac{t}{3} + \frac{5}{18} - \frac{2}{5}c_1 e^{-3t} + c_2 e^{+2t}$$

Hence
$$y_1 = -\frac{1}{3}t + \frac{5}{18} - \frac{2}{5}c_1 e^{-3t} + c_2 e^{2t}$$
$$y_2 = \frac{3t+5}{9} + c_1 e^{-3t}.$$

Example 4. *Solve :* $\dfrac{d^2x}{dt^2} + 4x + 5y = t^2$

$$\frac{d^2y}{dt^2} + 5x + 4y = t + 1.$$

Sol. Writing D for $\dfrac{d}{dt}$, the given equations becomes $(D^2+4)x + 5y = t^2$ ...(1)

and $5x + (D^2+4)y = t + 1$ ...(2)

To eliminate y, operating on both sides of (1) by $(D^2 + 4)$ and on both sides of (2) by 5 and subtracting, we get
$$[(D^2+4)^2 - 25]\,x = (D^2+4)\,t^2 - 5(t+1)$$
or
$$(D^4 + 8D^2 - 9)x = 2 + 4t^2 - 5t - 5$$
or
$$(D^4 + 8D^2 - 9)\,x = 4t^2 - 5t - 3$$
Its A.E. is
$$D^4 + 8D^2 - 9 = 0$$
or
$$(D^2+9)(D^2-1) = 0 \quad \therefore \quad D = \pm 1. \pm 3i$$
$$\text{C.F.} = c_1e^t + c_2e^{-t} + c_3\cos 3t + c_4 \sin 3t$$
$$\text{P.I.} = \frac{1}{D^4 + 8D^2 - 9}(4t^2 - 5t - 3) = \frac{1}{-9\left(1 - \dfrac{8D^2}{9} - \dfrac{D^4}{9}\right)}(4t^2 - 5t - 3)$$
$$= -\frac{1}{9}\left[1 - \left(\frac{8D^2}{9} + \frac{D^4}{9}\right)\right]^{-1}(4t^2 - 5t - 3)$$

$$= -\frac{1}{9}\left[1+\left(\frac{8D^2}{9}+\frac{D^4}{9}\right)+.....\right](4t^2-5t-3)$$

$$= -\frac{1}{9}\left[4t^2-5t-3+\frac{8}{9}(8)\right] = -\frac{1}{9}\left(4t^2-5t+\frac{37}{9}\right)$$

$$\therefore \quad x = c_1e^t + c_2e^{-t} + c_3\cos 3t + c_4\sin 3t - \frac{4}{9}t^2 + \frac{5}{9}t - \frac{37}{81}$$

Now, $$\frac{dx}{dt} = c_1e^t - c_2e^{-t} - 3c_3\sin 3t + 3c_4\cos 3t - \frac{8}{9}t + \frac{5}{9}$$

$$\frac{d^2x}{dt^2} = c_1e^t + c_2e^{-t} - 9c_3\cos 3t - 9c_4\sin 3t - \frac{8}{9}$$

Substituting the values of x and $\frac{d^2x}{dt^2}$ in (1), we have from (1) $5y = t^2 - 4x - \frac{d^2x}{dt^2}$

$$\therefore \quad 5y = t^2 - 4c_1e^t - 4c_2e^{-t} - 4c_3\cos 3t - 4c_4\sin 3t$$

$$+\frac{169}{9}t^2 - \frac{20}{9}t + \frac{148}{81} - c_1e^t - c_2e^{-t} + 9c_3\cos 3t + 9c_4\sin 3t + \frac{8}{9}$$

$$\therefore \quad y = \frac{1}{5}\left[-5c_1e^t - 5c_2e^{-t} + 5c_3\cos 3t + 5c_4\sin 3t + \frac{25}{9}t^2 - \frac{20}{9}t + \frac{220}{81}\right]$$

Hence $$x = c_1e^t + c_2e^{-t} + c_3\cos 3t + c_4\sin 3t - \frac{1}{9}\left(4t^2 - 5t + \frac{37}{9}\right)$$

$$y = -c_1e^t - c_2e^{-t} + c_3\cos 3t + c_4\sin 3t - \frac{1}{2}\left(5t^2 - 4t + \frac{44}{9}\right).$$

Example 5. *Solve the simultaneous equations :*

$$t\frac{dx}{dt} + y = 0, \; t\frac{dy}{dt} + x = 0 \text{ given } x(1) = 1, \; y(-1) = 0.$$

Sol. The given equations are $$t\frac{dx}{dt} + y = 0 \qquad ...(1)$$

$$t\frac{dy}{dt} + x = 0 \qquad ...(2)$$

Differentiating (1) w.r.t. t, we have

$$t\frac{d^2x}{dt^2} + \frac{dx}{dt} + \frac{dy}{dt} = 0$$

Multiplying throughout by t

$$t^2\frac{d^2x}{dt^2} + t\frac{dx}{dt} + t\frac{dy}{dt} = 0$$

or $$t^2\frac{d^2x}{dt^2} + t\frac{dx}{dt} - x = 0 \qquad \text{[Using (2)]} \; ...(3)$$

which is Cauchy's homogeneous linear equation.

Putting $t = e^u$ *i.e.*, $u = \log t$, so that $t\dfrac{d}{dt} = \dfrac{d}{du} = D$, equation (3) becomes

$$[D(D-1)+D-1]\,x = 0 \quad \text{or} \quad (D^2-1)\,x = 0$$

Its A.E. is $D^2 - 1 = 0$ whence $D = \pm 1$

$$\therefore \quad x = c_1e^u + c_2e^{-u} = c_1t + \frac{c_2}{t} \qquad \ldots(4)$$

From (1), $$y = -t\frac{dx}{dt} = -t\left(c_1 - \frac{c_2}{t^2}\right) = -c_1t + \frac{c_2}{t} \qquad \ldots(5)$$

Since $x(1) = 1$, $\quad\therefore$ from (4), we have $\quad 1 = c_1 + c_2$

Also, $y(-1) = 0 \quad\therefore$ from (5), we have $\quad 0 = c_1 - c_2$

Solving $$c_1 = c_2 = \frac{1}{2}$$

Hence $$x = \frac{1}{2}\left(t + \frac{1}{t}\right), y = \frac{1}{2}\left(-t + \frac{1}{t}\right).$$

Example 6. *Solve the following simultaneous equations :*

$$\frac{dx}{dt} = 2y, \frac{dy}{dt} = 2z, \frac{dz}{dt} = 2x.$$

Sol. The given equations are

$$\frac{dx}{dt} = 2y \quad \ldots(1) \qquad \frac{dy}{dt} = 2z \quad \ldots(2) \qquad \frac{dz}{dt} = 2x \qquad \ldots(3)$$

Differentiating (1) w.r.t. t, $$\frac{d^2x}{dt^2} = 2\frac{dy}{dt} = 2(2z) \qquad \text{[Using (2)]}$$

Differentiating again w.r.t. t, $$\frac{d^3x}{dt^3} = 4\frac{dz}{dt} = 4(2x) \quad \text{or} \quad (D^3-8)x = 0$$

where $$D = \frac{d}{dt}$$

Its A.E. is $\quad D^3 - 8 = 0 \quad$ or $(D-2)(D^2+2D+4) = 0$

whence $$D = 2, \frac{-2 \pm 2i\sqrt{3}}{2} \quad \text{or} \quad D = 2, -1 \pm i\sqrt{3}$$

(See note at the end of the questions)

$$\therefore \quad x = c_1e^{2t} + c_2e^{-t}\cos\left(\sqrt{3}t - c_3\right)$$

From (1), $$y = \frac{1}{2}\frac{dx}{dt}$$

$$= \frac{1}{2}\left[2c_1e^{2t} - c_2e^{-t}\cos\left(\sqrt{3}t - c_3\right) - c_2\sqrt{3}e^{-t}\sin\left(\sqrt{3}t - c_3\right)\right]$$

$$= c_1e^{2t} + c_2e^{-t}\left[-\frac{1}{2}\cos\left(\sqrt{3}t - c_3\right) - \frac{\sqrt{3}}{2}\sin\left(\sqrt{3}t - c_3\right)\right]$$

$$= c_1e^{2t} + c_2e^{-t}\left[\cos\frac{2\pi}{3}\cos\left(\sqrt{3}t - c_3\right) - \sin\frac{2\pi}{3}\sin\left(\sqrt{3}t - c_3\right)\right]$$

$$\because \cos\frac{2\pi}{3} = -\frac{1}{2} \text{ and } \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$$

$$= c_1e^{2t} + c_2e^{-t}\cos\left(\sqrt{3}t - c_3 + \frac{2\pi}{3}\right)$$

From (2), $$z = \frac{1}{2}\frac{dy}{dt}$$

$$= \frac{1}{2}\left[2c_1e^{2t} - c_2e^{-t}\cos\left(\sqrt{3}t - c_3 + \frac{2\pi}{3}\right) - c_2\sqrt{3}\, e^{-t}\sin\left(\sqrt{3}t - c_3 + \frac{2\pi}{3}\right)\right]$$

$$= c_1e^{2t} + c_2e^{-t}\left[\cos\frac{2\pi}{3}\cos\left(\sqrt{3}t - c_3 + \frac{2\pi}{3}\right) - \sin\frac{2\pi}{3}\sin\left(\sqrt{3}t - c_3 + \frac{2\pi}{3}\right)\right]$$

$$= -c_3e^{2t} + c_2e^{-t}\cos\left(\sqrt{3}t - c_3 + \frac{4\pi}{3}\right).$$

Note. $c_1\cos\beta x + c_2\sin\beta x$ can be replaced by $c_1\cos(\beta x - c_2)$ or $c_1\cos(\beta x + c_2)$ or $c_1\sin(\beta x - c_2)$ or $c_1\sin(\beta x + c_2)$.

TEST YOUR KNOWLEDGE

Solve the following simultaneous equations :

1. $\frac{dx}{dt} = 7x - y, \frac{dy}{dt} = 2x + 5y$
2. $\frac{dx}{dt} + y = \sin t, \frac{dy}{dt} + x = \cot t$; given that $x = 2$ and $y = 0$ when $t = 0$.
3. $\frac{dx}{dt} + 5x - 2y = t, \frac{dy}{dt} + 2x + y = 0$; given that $x = y = 0$ when $t = 0$. **(P.T.U., May 2010)**
4. $(D + 1)x + (2D + 1)y = e^t$, $(D - 1)x + (D + 1)y = 1$.
5. $\frac{dx}{dt} + 2x + 3y = 0, 3x + \frac{dy}{dt} + 2y = 2e^{2t}$.
6. $(D - 1)x + Dy = 2t + 1$, $(2D + 1)x + 2Dy = t$.
7. $\frac{d^2x}{dt^2} - 3x - 4y = 0, \frac{d^2y}{dt^2} + x + y = 0$.
8. $\frac{d^2x}{dt^2} - \frac{dy}{dt} = 2x + 2t, \frac{dx}{dt} + 4\frac{dx}{dt} = 3y$.
9. $\frac{d^2y}{dt^2} + \frac{dy}{dt} - 2y = \sin t, \frac{dx}{dt} + x - 3y = 0$.
10. A mechanical system with two degrees of freedom satisfies the equations $2\frac{d^2x}{dt^2} + 3\frac{dy}{dt} = 4$, $2\frac{d^2y}{dt^2} - 3\frac{dx}{dt} = 0$.

    Obtained expressions for x and y is terms of t, given $x, y, \frac{dx}{dt}, \frac{dy}{dt}$ all vanish at $t = 0$.
11. $\frac{d^2x}{dt^2} + y = \sin t, \frac{d^2y}{dt^2} + x = \cos t$.

ANSWERS

1. $x = e^{6t}(c_1\cos t + c_2\sin t), y = e^{6t}[(c_1 - c_2)\cos t + (c_1 + c_2)\sin t]$
2. $x = e^t + e^{-t}, y = e^{-1} - e^t + \sin t$

3. $x = -\frac{1}{27}(1+6t)e^{-3t} + \frac{1}{27}(1+3t)$, $y = -\frac{2}{27}(2+3t)e^{-3t} + \frac{2}{27}(2-3t)$

4. $x = \frac{1}{2}\left[e^t - 1 - ac_1e^{at} + bc_2e^{bt}\right]$, $y = c_1e^{at} + c_2e^{bt} + \frac{1}{2}$ where $a = \frac{1}{2}(3+\sqrt{17}), b = \frac{1}{2}(3-\sqrt{17})$

5. $x = c_1e^t + c_2e^{-5t} - \frac{6}{7}e^{2t}$, $y = c_2e^{-5t} - c_1e^t + \frac{8}{7}e^{2t}$

6. $x = -t - \frac{2}{3}$, $y = \frac{1}{2}t^2 + \frac{4}{3}t + c$

7. $x = (c_1 + c_2t)e^{-t} + (c_3 + c_4t)e^t$, $y = -\frac{1}{2}\left[c_1 + c_2(1+t)\right]e^{-t} + \frac{1}{2}\left[c_4(1-t) - c_3\right]e^t$

8. $x = (c_1 + c_2t)\, e^t + c_3e^{-3/2t} - t$, $y = [c_2(3-t) - c_1]e^t - \frac{1}{6}c_3e^{-32t}$

9. $x = \frac{3}{2}c_1e^t - 3c_2e^{-2t} + c_3e^{-t} + \frac{3}{10}e^t(\cos t - 2\sin t)$; $y = c_1e^t + c_2e^{-2t} - \frac{1}{10}(\cos t + 3\sin t)$

10. $x = \frac{8}{9}\left(1 - \cos\frac{3t}{2}\right)$, $y = \frac{4}{3}t - \frac{8}{9}\sin\frac{3t}{3}$

11. $x = c_1e^t + c_2e^{-t} + c_3\cos t + c_4\sin t + \frac{t}{4}(\sin t - \cos t)$

$y = -c_1e^t + c_2e^{-t} - c_3\cos t + c_4\sin t + \frac{1}{4}(2+t)(\sin t - \cos t).$

SHORT ANSWER TYPE QUESTIONS

1. Define Leibnitz's linear differential equation of first order. Also give an example. **(P.T.U., May 2005, 2007)**

2. Solve $\frac{dy}{dx} + Py = Q$, where P, Q are functions of x or constants. **(P.T.U., June 2003, Dec. 2006)**

3. Define Bernoulli's linear differential equation and write its standard form. **(P.T.U., May 2007, Jan. 2009)**

4. How will you reduce $f'(y)\frac{dy}{dx} + Pf(y) = Q$ to linear differential equation (where, P, Q are functions of x or constants).

5. Solve the following differential equations :

(i) $(x+1)\frac{dy}{dx} - y = e^x(x+1)^2$ (ii) $\frac{dy}{dx} + y\cot x = \cos x$ (iii) $xy(1+xy^2)\frac{dy}{dx} = 1$

[Hint : S.E. 2 art. 3.2]

(iv) $2\frac{dy}{dx} = \frac{y}{x} + \frac{y^2}{x^2}$. (v) $x\frac{dy}{dx} + y = x^3y^6$ **(P.T.U., May 2007)**

[Hint : S.E. 1(*i*) art. 3.2]

(vi) $(x+1)\frac{dy}{dx} + 1 = 2e^{-y}$. **(P.T.U., May 2001)**

[Hint: Divide by $(x+1)e^{-y}$; $e^y\frac{dy}{dx} + \frac{1}{x+1}e^y = \frac{2}{x+1}$. Put $e^y = t$, we get $\frac{dt}{dx} + \frac{1}{x+1}t = \frac{2}{x+1}$ I.F. $= x+1$]

(vii) $y' + y = y^2$ **[Hint:** S.E. 1(*ii*) art. 3.2] **(P.T.U., May 2008)**

6. (*a*) What do you understand by complementary function? Explain. (P.T.U., Jan 2010)

(*b*) If $y = u$ is the complete solution of the equation $f(D)\, y = 0$ and $y = v$ is a particular solution of the equation $f(D)\, y = X$, then the complete solution of the equation $f(D)\, y = X$ is $y = u + v$.

7. Define Auxiliary Equation of a linear differential equation.

8. What is the solution of the differential equation corresponding to roots of the A.E. if

(*i*) roots are all real and distinct.

(*ii*) roots are imaginary and distinct.

9. Solve the following differential equations :

(*i*) $\dfrac{d^2y}{dx^2} + (a+b)\dfrac{dy}{dx} + aby = 0$

[**Hint :** A.E. is $D^2 + (a+b)D + ab = 0 \therefore D = -a, D = -b \therefore y = c_1 e^{-ax} + c_2 e^{-bx}$]

(*ii*) $9y''' + 3y'' - 5y' + y = 0$ [**Hint :** S.E. 1 art. 3.7] (P.T.U., May 2008)

(*iii*) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = e^x \sin x$. [**Hint :** S.E. 7 art. 3.10] (P.T.U., June 2003, May 2006)

10. Find particular solutions of the following differential equations :

(*i*) $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = e^{e^x}$ [**Hint :** S.E. 11 art. 3.10] (P.T.U., Dec. 2003)

(*ii*) $(D^2 - 2D + 4)\, y = e^x \sin x$

(*iii*) $(D^2 - 3D + 2)\, y = 2e^x \cos \dfrac{x}{2}$. [**Hint :** Consult S.E. 4 art. 3.9] (P.T.U., Dec. 2003)

(*iv*) $(D^3 - 3D^2 + 4)\, y = e^{2x}$. [**Hint :** S.E. 1 art. 3.9]

(*v*) $y''' - y'' + 4y' - 4y = \sin 3x$ [**Hint :** S.E. 2 art. 3.9] (P.T.U., May 2008)

(*vi*) $(D^2 + a^2)y = \sin ax$ (P.T.U., May 2009)

11. Explain method of variation of parameters to find P.I. of a differential equation. (P.T.U., May 2004)

12. Explain briefly the method of undetermined coefficients for finding P.I. of a linear differential equation.

13. Define Cauchy's homogeneous linear differential equation and give one example. (P.T.U., Dec. 2004)

14. Solve the following differential equations :

(*i*) $x^2\dfrac{d^2y}{dx^2} + 4x\dfrac{dy}{dx} + 2y = 0$ [**Hint :** S.E. 1 art. 3.13] (P.T.U., May 2006)

(*ii*) $x^2\dfrac{d^2y}{dx^2} - 2y = x^2 + \dfrac{1}{x}$.

15. Define Legendre's linear equation and give one example. (P.T.U., Dec. 2005, May 2007)

16. Solve the following simultaneous linear differential equations :

(*i*) $\dfrac{dx}{dt} + 2y = e^t$; $\dfrac{dy}{dt} - 2x = e^{-t}$ [**Hint :** S.E. 2 art. 3.15]

(*ii*) $\dfrac{dx}{dt} = -2x + y$; $\dfrac{dy}{dt} = -4x + 3y + 10\cos t$. (P.T.U., Dec. 2002)

ANSWERS

4. By putting $f(y) = z \therefore f'(y)\dfrac{dy}{dx} = \dfrac{dz}{dx}$ and given equation reduces to $\dfrac{dz}{dx} + Pz = Q$.

5. (i) $y = (x+1)(e^x + c)$ (ii) $y \sin x = \frac{1}{2}\sin^2 x + c$

(iii) $\frac{1}{x} = 2 - y^2 + c\,e^{-\frac{1}{2}y^2}$ (iv) $\frac{x}{y} = 1 + c\sqrt{x}$

(v) $\frac{1}{y^5} = \frac{5}{2}x^3 + cx^5$ (vi) $(x+1)e^y = 2x + c$

(vii) $y = \frac{1}{1 + ce^x}$

9. (i) $y = c_1 e^{-ax} + c_2 e^{-bx}$ (ii) $y = c_1 e^{(2+\sqrt{3})x} + c_2 e^{(2-\sqrt{3})x}$

(iii) $y = (c_1 + c_2 x)\,e^x - e^x \sin x$

10. (i) $e^{-2x} e^{e^x}$ (ii) $\frac{1}{2}e^x \sin x$

(iii) $-\frac{8}{5}e^x\left[\cos\frac{x}{2} + 2\sin\frac{x}{2}\right]$ (iv) $\frac{x^2}{6}e^{2x}$

(v) $\frac{1}{50}(\sin 3x + 3\cos 3x)$ (vi) $\frac{-x}{2a}\cos ax$

14. (i) $y = \frac{c_1}{x} + \frac{c_2}{x^2}$ (ii) $y = c_1 x^2 + \frac{c_2}{x} + \frac{1}{3}\left(x^2 - \frac{1}{x}\right)\log x$

16. (i) $x = c_1 \cos 2t + c_2 \sin 2t + \frac{1}{5}e^t - \frac{2}{5}e^{-t}$

$y = c_1 \sin 2t - c_2 \cos 2t + \frac{2}{5}e^t - \frac{1}{5}e^{-t}$

(ii) $x = c_1 e^{2t} + c_2 e^{-2t} - \sin t - 3\cos t$

$y = 4c_1 e^{2t} - 3c_2 e^{-t} - 7\cos t + \sin t.$