Mixed-Integer Linear Programming for a PDE-Constrained Dynamic Network Flow

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To Be looked Up

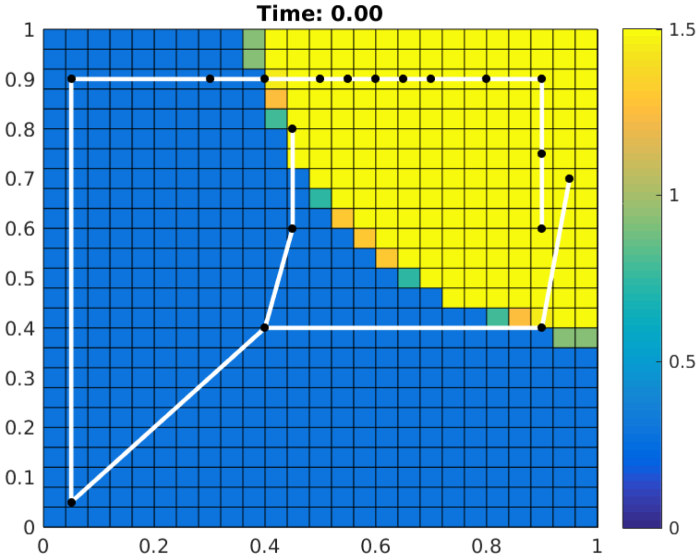
Introduction

- Coupling PDE with a network flow
- Interaction goes both ways, flow controls the PDE, network constrained by PDE
- Linear PDE, Objective and Constraints lead to an MILP

Example - Firefighting Mission

- The spread of fire can be modeled using Convection Diffusion Equations, a linear approximation can give insight to the diffusive and convective behavior
- Firefighters are often constrained as they can only maneuver on the roads in a given area
- Firefighters influence the spread of the fire (PDE)
- The fire constraints the possibilities of the paths that are available for the firefighters

Embedded Animation



The PDE

$$u_t(x,t) - c \cdot \nabla u(x,t) - D\Delta u(x,t) = y(x,t,w) \qquad \forall\ (x,t) \in \Omega \times (0,T) \tag{1}$$

$$\frac{\partial}{\partial n} u(x,t) = h_R(u_R - u(x,t)) \qquad \forall\ (x,t) \in \partial\Omega \times (0,T) \tag{2}$$

$$u(x,0) = f(x) \qquad \forall x \in \Omega \tag{3}$$

$$u(x,t) \geq 0, \qquad \forall (x,t) \in \Omega \times (0,T) \tag{4}$$

The Control Function

$$y(x, t, w) = \sum_{i \in D} -\lambda w_{i,t} \chi_{[t, t+T_E)}(t) \exp\left(-\frac{(x - x_i)^2 + (y - y_i)^2}{\sigma_i^2}\right) \tag{5}$$

The Network Flow

$$x_{i,j,0} = 0 \qquad \forall (i,j) \in A : i \notin S \tag{6}$$

$$\sum_{s \in Q : s \leq \min(\delta_{i,j}, T-t)} w_{i,j,t+s} \leq c_{i,j} z_{j,z} \qquad \forall (i,j) \in A, t \in Q \tag{7}$$

$$\sum_{i \in V : (i,k) \in A} w_{i,k,t-1} = \sum_{j \in V : (k,j) \in A} w_{i,j,t} + w_{k,t} \quad \forall k \in V, t \in Q \setminus 0 \tag{8}$$

$$u((x_i, y_i), t) - (1 - z_{i,t})M \leq U_B \qquad \forall i \in V \tag{9}$$

$$w_{i,t} \leq M z_{i,t} \qquad \forall i \in V, t \in Q \tag{10}$$

$$z_{i,t} \in \{0, 1\} \tag{11}$$

Eliminating The PDE from The Constraints

If the PDE was independent of the network variables, one could solve the PDE first and only use the temperatures to formulate the MILP, effectively eliminating the PDE from the MILP. In the case of linear PDEs it is possible to use two properties to calculate a 'basis' for the PDE and substitute the temperature by a linear combination of this basis, again eliminating the PDE from the equation.

The Discretization

Different methods exist:
Finite Elements: Find a solution for the weak formulation of the PDE restricted to a finite dimensional space of test functions.
Finite Differences: Formulate Difference equations for a finite set of points

A possible Callback

for $t \leq T$ **do**
  $u \leftarrow \min_{y \in \Omega} u_{y,t}$
  $i \leftarrow \arg\min_{y \in \Omega} u_{y,t}$
  if $u \leq 0$ **then**
    add Constraint $u_{i,t} \geq 0$
  end if
end for

Computational Results

Nice graphs, table, yay