DISCRETE STRUCTURES
Lecture 5. Counting

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Content

The Pigeonhole Principle

The pigeonhole principle

If $k+1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

Example 1.1

1. If there are 11 players in a soccer team that wins 12-0, there must be at least one player in the team who scored at least twice.
2. If you have 6 classes from Monday to Friday, there must be at least one day on which you have at least two classes.

The Generalized Pigeonhole Principle

Question: If we draw 10 cards, how many cards of a same suit are guaranteed?

The Generalized Pigeonhole Principle

If N objects are placed into k boxes, then there is some box containing at least $\lceil N/k \rceil$ of the objects.

Example 1.2

1. In a 60-student class, at least 12 students will get the same letter grade (A, B, C, D or F).
2. In a 61-student class, at least 13 students will get the same letter grade.

The Generalized Pigeonhole Principle

Example 1.3

Assume you have a drawer containing a random distribution of a dozen brown socks and a dozen black socks. It is dark, so how many socks do you have to pick to be sure that among them there is a matching pair?

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Solution: There are two types of socks, so if you pick at least 3 socks, there must be either at least two brown socks or at least two black socks.

Applications of the Pigeonhole Principle

Example 1.4

How many cards must be selected from a standards deck of 52 cards to ensure that we get at least 3 cards of the same suit?

Applications of the Pigeonhole Principle

Example 1.5

Assume in a group of 6 people, any pair consists of either 2 friends or 2 enemies. Then there are either 3 mutual friends or 3 mutual enemies.

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Let's consider person A, then A either has at least 3 friends or 3 enemies.

Assume that A has 3 friends B,C,D.

- If B,C,D are mutual enemies then the problems is proved.
- Otherwise, 2 of them are friends, say B,C. Then A,B,C are mutual friends.

The case A has 3 enemies is treated similarly.

Exercises

Q1. How many numbers must be selected from the set 1; 3; 5; 7; 9; 11; 13; 15 to guarantee that at least one pair of these numbers add up to 16?

Q2. A computer network consists of six computers. Each computer is directly connected to at least one of the other computers. Show that there are at least two computers in the network that are directly connected to the same number of other computers.

Q3. During a month with 30 days, a baseball team plays at least one game a day, but no more than 45 games in total. Show that there must be a period of some number of consecutive days during which the team must play exactly 14 games.

Q4. Show that among any $n+1$ positive integers not exceeding $2n$ there must be an integer that divides one of the other integers.

Ramsey number

Definition 1.6

The Ramsey number $R(m,n)$, where m and n are positive integers greater than or equal to 2, denotes the minimum number of people at a party such that there are either m mutual friends or n mutual enemies, assuming that every pair of people at the party are friends or enemies.

Example 1.7

We know that $R(3,3) \leq 6$. But in a group of five people, there may not be there friends or three enemies. Hence $R(3,3) = 6$.

1. $R(m,n) = R(n,m)$
2. $R(2,n) = n$
3. $R(4,4) = 18$

Exercises

Q1. Show that in a group of 10 people (when any two people are either friends or enemies), there are either three mutual friends or four mutual enemies, and there are either three mutual enemies or four mutual friends.

Q2. Show that among any group of 20 people (when any two people are either friends or enemies), there are either 4 mutual friends or 4 mutual enemies.

Permutations

Some questions

1. In how many ways can we select two students from a group of four students to stand in line for a picture?
2. In how many ways can we select two students from a group of four students to stand in line for a picture?

Let $S = \{1, 2, 3\}$.

1. The arrangement 3,1,2 is a permutation of S.
2. The arrangement 3,2 is a 2-permutation of S.

The number of r-permutations of a set with n distinct elements is denoted by $P(n, r)$.

Permutations

Question: $P(n,r) =$??? (number of r-permutations of a set with n distinct elements)

1. There are n choices for the first element
2. There are $n-1$ choices for the second element
   $\dots$
3. There are $n-r+1$ choices for the rth element.

Hence $P(n,r) = n \cdot (n-1) \cdot \dots \cdot (n-r+1)$.

Example 2.1

Suppose that a saleswoman has to visit eight different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order she wishes. How many possible orders can the saleswoman use when visiting these cities?

Permutations

Definition 2.2

The factorial of a non-negative integer n, denoted by $n!$, is the product of all positive integers less than or equal to n,

$$n! = n.(n-1).(n-2)\ldots 2.1$$

General Formula

$$P(n,r) = \frac{n!}{(n-r)!}$$

Especially, $P(n,n) = n!$.

Exercises

Q1. How many permutations of the letters ABCDEFGH contain the string ABC ?

Q2. How many permutations of the letters ABCDEFG contain

1. the string BCD?
2. the string CFGA?
3. the strings BA and GF?
4. the strings ABC and DE?
5. the strings ABC and CDE?
6. the strings CBA and BED?

Combinations

Definition 2.3

An r-combination of elements of a set is an unordered selection of r elements from the set.
In other words, an r-combination is simply a subset of the set with r elements.
The number of r-combinations of a set with n distinct elements is denoted by $C(n, r)$.

Example 2.4

Let $S = \{1, 2, 3, 4\}$

1. $\{1, 2, 3\}$ is a 3-combination from S.
2. The set of all 2-combinations from S

$$\{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}$$

Combinations

Question: Calculate $C(n, r)$ number of r-combinations of a set with n distinct elements.

Solution:

1. form all the r-combinations of the set (there are $C(n, r)$ such r-combinations).
2. for each r-combination B of A we select an arbitrary ordering $(b_1, b_2, \ldots, b_r)$ of B.

Hence, $P(n, r) = C(n, r) \cdot P(r, r)$. Therefore,

$$C(n, r) = \frac{n!}{r!(n-r)!}$$

Note:

$$C(n, r) = C(n, n-r)$$

Combinations

Example 2.5

A soccer club has 2 goal keepers, 7 defenders, 6 midfielders and 3 strikers. For today's match, the coach wants to adopt the 4 - 4 - 2 formation. How many possible ways to pick a team?

1. one goal keeper out of two:

Combinations

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1. one goal keeper out of two: $C(2,1) = 2$ ways
2. 4 defenders out of seven:

Combinations

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1. one goal keeper out of two: $C(2,1) = 2$ ways
2. 4 defenders out of seven:

$$C(7,4) = \frac{7!}{4!3!} = 35$$

3. 4 midfielders out of six:

Combinations

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$$C(7,4) = \frac{7!}{4!3!} = 35$$

3. 4 midfielders out of six: $C(6,4) = 15$
4. 2 strikers out of three:

Combinations

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1. one goal keeper out of two: $C(2,1) = 2$ ways
2. 4 defenders out of seven:
$$C(7,4) = \frac{7!}{4!3!} = 35$$
3. 4 midfielders out of six:$C(6,4) = 15$
4. 2 strikers out of three: $C(3,2) = 3$

By the Product Rule, the number of possible ways to pick a team is:

$$2 \times 35 \times 15 \times 3 = 3150$$

Exercises

Q1. How many bit strings of length 10 contain

a) exactly four 1s?

b) at most four 1s?

c) at least four 1s?

d) an equal number of 0s and 1s?

Q2. Suppose that a department contains 10 men and 15 women. How many ways are there to form a committee with six members if it must have more women than men?

Q3. How many bit strings contain exactly eight 0s and 10 1s if every 0 must be immediately followed by a 1?

Binomial Coefficients

Consider

$$(x+y)^3 = (x+y)(x+y)(x+y)$$

What is the coefficient of x^2y?

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What is the coefficient of x^2y?Choose 2 x from 3 factors $\Longrightarrow C(3,2)=3$ ways $\Longrightarrow$ coefficient of x^2y is 3. Hence,

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

Binomial Theorem

$$(x+y)^n = \sum_{i=0}^{n} C(n,i)x^{n-i}y^i$$

In this case, $C(n,i)$ is called binomial coefficient and denoted by $\binom{n}{i}$.

Binomial Coefficients

Example 3.1

$$(x+y)^4 = \sum_{i=0}^{4} \binom{4}{i} x^{n-i} y^i$$

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$$(x+y)^4 = \sum_{i=0}^{4} \binom{4}{i} x^{n-i} y^i = \binom{4}{0} x^4 + \binom{4}{1} x^3 y + \binom{4}{2} x^2 y^2 + \binom{4}{3} x y^3 + \binom{4}{4} y^4$$

Thus,

$$(x+y)^4 = x^4 + 4x^3 y + 6x^2 y^2 + 4xy^3 + y^4$$

Example 3.2

What is the coefficient of $x^{12}y^{13}$ in the expansion of $(x+y)^{25}$?

Binomial Coefficients

Example 3.1

$$(x+y)^4 = \sum_{i=0}^{4} \binom{4}{i} x^{n-i} y^i = \binom{4}{0} x^4 + \binom{4}{1} x^3 y + \binom{4}{2} x^2 y^2 + \binom{4}{3} x y^3 + \binom{4}{4} y^4$$

Thus,

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Example 3.2

What is the coefficient of $x^{12} y^{13}$ in the expansion of $(x+y)^{25}$?

$$\binom{25}{12} = \frac{25}{12!13!} = 5,200,300.$$

Binomial Coefficients

Consider

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i$$

By putting $x = y = 1$, we obtain

$$\sum_{i=0} \binom{n}{i} = 2^n.$$

What is special about this number?

Binomial Coefficients

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$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i$$

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What is special about this number?

Proposition 3.1

The total number of subsets of an n-element set is 2^n.

Permutations with Repetition

Example 3.3

How many strings of length r can be formed from the English alphabet?

Solution. 26^r strings.

Theorem 3.4

The number of r-permutations of a set of n objects with repetition allowed is n^r.

Combinations with Repetition

Example 3.5

How many ways are there to select 4 pieces of fruits from a bowl containing plenty of apples, oranges and pears?

Each selection is called a $4-$combinations from a three-element set with repetition allowed. And in this example, there are **15** of them.

Question: How to find this number?

Combinations with Repetition

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How many ways are there to select 4 pieces of fruits from a bowl containing plenty of apples, oranges and pears?

Each selection is called a $4-$combinations from a three-element set with repetition allowed. And in this example, there are **15** of them.

Question: How to find this number?

Theorem 3.6

The number of $r-$combinations from a set of n elements with repetition allowed is

$$C(n+r-1, r)$$

Exercises

Q1. How many solutions does the equation

$$x_1 + x_2 + x_3 = 11$$

have, where x_1, x_2, x_3 are non-negative integers?

Q2. How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 = 14$$

where x_i are non-negative integers for all i such that

a) $x_1 \geq 1$

b) $x_i \geq 2$ for all i

c) $0 \leq x_1 \leq 10$

d) $0 \leq x_1 \leq 3,\ 1 \leq x_2 < 4$, and $x_3 \geq 15$.