DISCRETE STRUCTURES
Lecture 5. Counting

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Advanced Program in Computer Science

Fall, 2018

Content

Basic Counting Principles

Example 1.1

Counting problems are of the following kind:

- How many different 8-letter passwords are there?

Basic Counting Principles

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Counting problems are of the following kind:

- How many different 8-letter passwords are there?
  1. lower case letters = 26
  2. upper case letters = 26
  3. digits = 10
  4. punctuations & special characters = 33 (period, comma, question mark,...)

Basic Counting Principles

Example 1.1

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- How many possible ways are there to pick 11 soccer players out of a 20-player team?

Basic Counting Principles

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- Counting steps in average/worst case of an algorithm.

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  2. upper case letters = 26
  3. digits = 10
  4. punctuations & special characters = 33 (period, comma, question mark,...)
- How many possible ways are there to pick 11 soccer players out of a 20-player team?
- Counting steps in average/worst case of an algorithm.
- Counting is the basis for computing probabilities of discrete events. What is the probability of winning Vietlot?

Basic Counting Principles

Example 1.2

The department will award a free computer to either a CS student or a CS professor. How many different choices are there, if there are 530 students and 15 professors?

Solution. There are $530 + 15 = 545$ choices.

The Sum Rule

If a first task can be done in m ways and a second task in n ways, and if these two tasks cannot be done at the same time, then there are $m + n$ ways to do either task.

Basic Counting Principles

Generalized Sum Rule

If we have tasks $T_1, T_2, \ldots, T_m$ that can be done in $n_1, n_2, \ldots, n_m$ ways, respectively, and no two of these tasks can be done at the same time, then there are

$$n_1 + n_2 + \cdots + n_m$$

ways to do one of these tasks.

Basic Counting Principles

The Product Rule

Suppose that a procedure can be broken down into two successive tasks. If there are n_1 ways to do the first task and n_2 ways to do the second task after the first task has been done, then there are $n_1 \times n_2$ ways to do the procedure.

Generalized Product Rule

If we have a procedure consisting of sequential tasks $T_1, T_2, \ldots, T_m$ that can be done in $n_1, n_2, \ldots, n_m$ ways, respectively, then there are

$$n_1 \times n_2 \times \cdots \times n_m$$

ways to carry out the procedure.

Examples

Example 1.3

How many possible different license plates can be issued in Ho Chi Minh city? Note that a license plate is of the form: XY LD - DDD.DD, where XY is a two-digit number from 50-59, L stands for Letter and D stands for digit.

Examples

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Example 1.4

How many functions are there from an m-element set A to an n-element set B?

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Example 1.4

How many functions are there from an m-element set A to an n-element set B?

Solution.

Let $A = \{a_1, a_2, \ldots, a_m\}$. Then a function $f : A \to B$ is completely determined by the selection of the images $f(a_1), f(a_2), \ldots, f(a_m)$.

There are n choices (among the elements of B) for each of these images.

Examples

Example 1.5

How many one-to-one functions are there from an m-element set A to an n-element set B?

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Solution. The images $f(a_1), f(a_2), \ldots, f(a_m)$ are different.

Examples

Example 1.6

Each user on a computer system has a password, which is six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

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Solution.

Let P_n be the number of possible passwords of length n.

We now that

$$P_n = 36^n - 26^n$$

where 36^n is the total number of n-length passwords and 26^n is the number of n-length passwords do not contain any digits.

Basic Counting Principles

Set Theory: Sum Rule

Let $A_1, A_2, \ldots, A_m$ be disjoint sets. Then the number of ways to choose any element from one of these sets is

$$|A_1 \cup A_2 \cup \cdots \cup A_m| = |A_1| + |A_2| + \cdots + |A_3|$$

Set Theory: Product Rule

Let $A_1, A_2, \ldots, A_m$ be finite sets. Then the number of ways to choose one element from each set in the order $A_1, A_2, \ldots, A_m$ is

$$|A_1 \times A_2 \times \cdots \times A_m| = |A_1| \cdot |A_2| \cdot \cdots \cdot |A_3|$$

Exercises

Q1. Suppose that a password for a computer system must have at least 8, but no more than 12, characters, where each character in the password is a lowercase English letter, an uppercase English letter, a digit, or one of the six special characters ., $>$, $<$, !, $+$, and $=$

a) How many different passwords are available for this computer system?

b) How many of these passwords contain at least one of the six special characters?

c) Using your answer to part (a), determine how long it takes a hacker to try every possible password, assuming that it takes one nanosecond for a hacker to check each possible password.

Inclusion-Exclusion

Example 1.7

How many bit strings of length 8 either start with a 1 or end with 00?

Task 1: Construct a string of length 8 that starts with a 1

- There is one way to pick the first bit (1),
- two ways to pick the second bit (0 or 1),
- two ways to pick the third bit (0 or 1),

  ....

- two ways to pick the eighth bit (0 or 1).

Product Rule: Task 1 can be done in $1 \times 2^7 = 128$ ways

Inclusion-Exclusion

Example 1.8

How many bit strings of length 8 either start with a 1 or end with 00?

Task 2: Construct a string of length 8 that ends with 00.

- There are two ways to pick the first bit (0 or 1),
- two ways to pick the second bit (0 or 1),

  ....
- two ways to pick the sixth bit (0 or 1),
- one way to pick the seventh bit (0), and
- one way to pick the eighth bit (0).

Product Rule: Task 2 can be done in $2^6 \times 1 \times 1 = 64$ ways

Inclusion-Exclusion

Example 1.9

How many bit strings of length 8 either start with a 1 or end with 00?

Task 1: 128 ways; **Task 2:** 64 ways $\Leftrightarrow$ 128+64 = 192 ways?

Inclusion-Exclusion

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Task 1: 128 ways; **Task 2:** 64 ways $\Leftrightarrow$ $128+64 = 192$ ways?

- No!, because here Task 1 and Task 2 can be done at the same time.
- When we carry out Task 1 and create strings starting with 1, some of these strings end with 00.

Inclusion-Exclusion

Example 1.9

How many bit strings of length 8 either start with a 1 or end with 00?

Task 1: 128 ways; **Task 2:** 64 ways $\Leftrightarrow$ 128+64 = 192 ways?

- No!, because here Task 1 and Task 2 can be done at the same time.
- When we carry out Task 1 and create strings starting with 1, some of these strings end with 00.

Solution. Subtract the cases that Task 1 and Task 2 are carried out at the same time.

This is called the Principle of Inclusion-Exclusion.

Inclusion-Exclusion

Principle of Inclusion-Exclusion

In set theory, for two sets, we have

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Inclusion-Exclusion

Principle of Inclusion-Exclusion

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Principle of Inclusion-Exclusion

In set theory, for two sets, we have

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$$

Generalized Principle of Inclusion-Exclusion

In set theory, for two sets, we have

$$|A_1 \cup A_2 \cup \cdots \cup A_m| = \sum_{i=1}^{m} |A_i| - \sum_{1 \le i_1 < i_2 \le m} |A_{i_1} \cap A_{i_2}| + \cdots + (-1)^{m-1} |A_1 \cap A_2 \cap \cdots \cap A_m|$$

Example

Example 1.10

Use the principle of Inclusion-Exclusion to find the number of positive integers less than 1,000,000 that are not divisible by either 4 or by 6.

Tree Diagram

Example 1.11

How many bit strings of length four do not have two consecutive 1s?

Tree Diagram

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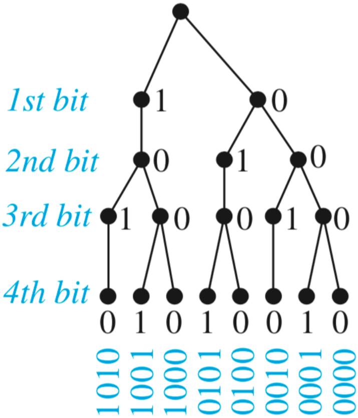

Exercises

Q1. Let S be the set of points whose coordinates x,y and z are integers that satisfy $0 \le x \le 2$, $0 \le y \le 3$ and $0 \le z \le 4$. Two distinct points are randomly chosen from S. What is the possibility that the midpoint of the segment they determine also belongs to S?

Exercises

Q2. Let n be a positive integer with $n \geq 2$. Fix $2n$ points in space in such a way that no four of them are in the same plane, and select any n^2+1 segments determined by the given points. Prove that these segments form at least one triangles.

Q3. Let n be a positive integer with $n \geq 2$. Fix $2n$ points in space in such a way that no four of them are in the same plane, and select any n^2+1 segments determined by the given points. Prove that these segments form at least n triangles.