DISCRETE STRUCTURES
Lecture 4. Partial Orderings

Bui Anh Tuan

Advanced Program in Computer Science

Fall, 2018

Content

Examples

Determine whether these relations are reflexive, symmetric, anti-symmetric, transitive

1. Let $S = \{1, 2, 3, 4, 5, 6\}$ and the relation aRb if and only if $a|b$
2. Let $S = \{1, 2, 3, 4\}$ and relation $R = \{(a, b) \in S \times S : a \leq b\}$
3. Let $S = \{a, b, c\}$ and the relation
   $R = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$

Question: What are the common properties of these relations?

Definitions

Definition 2.1

A relation R on a set A is a partial order if it is reflexive, anti-symmetric and transitive.
A set S together with a partial ordering R is called a partially ordered set, or poset, and is denoted by (S, R). Members of S are called elements of the poset.
We often denote a partial order by the pair $(A, \preccurlyeq)$.

Example 2.2

The divisibility relation $|$ is a partial ordering on the set of positive integers. Hence $(\mathbb{Z}^+, |)$ is a poset.

Exercises

Inclusion relation

Show that the inclusion relation $\subseteq$ is a partial ordering on the power set of a set S.

Age

Let R be the relation on the set of people such that xRy if x and y are people and x is older than y. Show that R is not a partial ordering.

Comparable and Incomparable

Definition 2.3

The elements a and b of a poset $(S, \preccurlyeq)$ are called comparable if either $a \preccurlyeq b$ or $b \preccurlyeq a$.
When a and b are elements of S such that neither $a \preccurlyeq b$ nor $b \preccurlyeq a$, then a and b are called incomparable.

Example 2.4

In the poset $(\mathbb{Z}^+, |)$, are the integers 3 and 9 comparable? Are 5 and 7 comparable?

Total Order

Definition 2.5

If $(S, \preccurlyeq)$ is a poset and every two elements of S are comparable, then S is called a totally ordered set or linearly ordered set, and $\preccurlyeq$ is called a total order or a linear order. A totally ordered set is also called a chain.

Example 2.6

The poset $(\mathbb{Z}, \leq)$ is totally ordered.

Example 2.7

The poset $(\mathbb{Z}+, |)$ is not totally ordered.

Lexicographic Order

The words in the dictionary are listed in alphabetic, or lexicographic, order, which is based on the ordering of the letters in the alphabet. This is a special case of an ordering of strings on a set constructed from a partial ordering on the set.

Lexicographic Order

Definition 2.8

Consider two posets, $(A_1, \preccurlyeq_1)$ and $(A_2, \preccurlyeq_2)$. The lexicographic ordering $\preccurlyeq$ on $A_1 \times A_2$ is defined by specifying that one pair is less than a second pair if the first entry of the first pair is less than (in A_1) the first entry of the second pair, or if the first entries are equal, but the second entry of this pair is less than (in A_2) the second entry of the second pair. In other words, (a_1, a_2) is less than (b_1, b_2), that is,

$$(a_1, a_2) \prec (b_1, b_2),$$

either if $a_1 \prec_1 b_1$ or if both $a_1 = b_1$ and $a_2 \prec_2 b_2$. We obtain a partial ordering $\preccurlyeq$ by adding equality to the ordering $\prec$ on $A_1 \times A_2$.

Lexicographic Order

Example 2.9

Determine whether $(3,5) \prec (4,8)$, whether $(3,8) \prec (4,5)$, and whether $(4,9) \prec (4,11)$ in the poset $(\mathbb{Z} \times \mathbb{Z}, \preccurlyeq)$, where $\preccurlyeq$ is the lexicographic ordering constructed from the usual $\leq$ relation on $\mathbb{Z}$.

Lexicographic Order

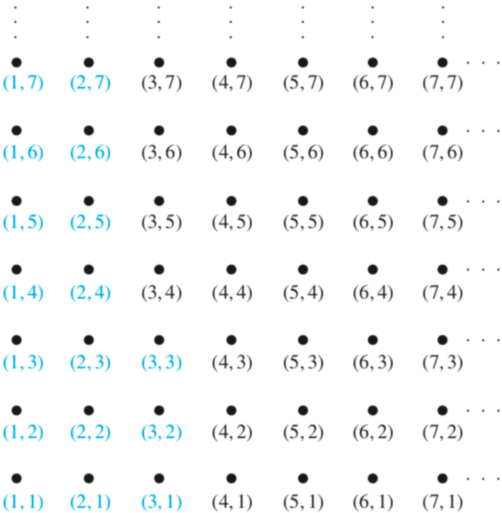
A lexicographic ordering can be defined on the Cartesian product of n posets $(A_1, \preccurlyeq_1), (A_2, \preccurlyeq_2), ..., (A_n, \preccurlyeq_n)$. Define the partial ordering $\preccurlyeq$ on $A_1 \times A_2 \times \cdots \times A_n$ by

$$(a_1, a_2, ..., a_n) \prec (b_1, b_2, ..., b_n)$$

if $a_1 \prec_1 b_1$,or if there is an integer $i > 0$ such that $a_1 = b_1, ..., a_i = b_i$, and $a_{i+1} \prec_{i+1} b_{i+1}$.

Lexicographic Order

The Ordered Pairs Less Than (3, 4) in Lexicographic Order.



Lexicographic Order (String)

Definition 2.10

Consider the strings $a_1a_2\cdots a_m$ and $b_1b_2\cdots b_n$ on a partially ordered set S. Suppose these strings are not equal. Let t be the minimum of m and n. The definition of lexicographic ordering is that the string $a_1a_2\cdots a_m$ is less than the string $b_1b_2\cdots b_n$ if and only if

$$(a_1, a_2, .., a_t) \prec (b_1, b_2, ..., b_t)$$

or

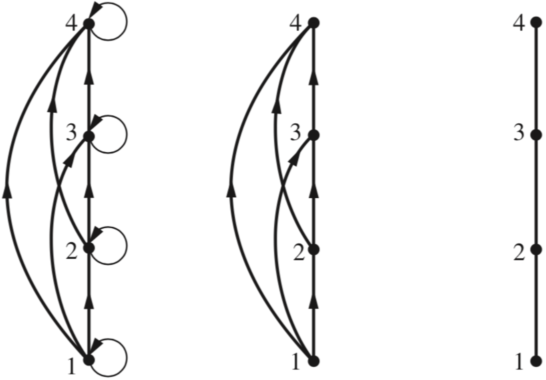
$$(a_1, a_2, .., a_t) = (b_1, b_2, ..., b_t)$$

and $m < n$, where $\prec$ in this inequality represents the lexicographic ordering of S^t.

Example 2.11

1. discreet $\prec$ discrete
2. discreet $\prec$ discreetness
3. discrete $\prec$ discretion

Hasse Diagram



(a) (b) (c)

1. Many edges do not have to be shown because they must be present.
2. If we assume that all edges are pointed "upward", we do not have to show the directions of the edges.

Construction of Hasse Diagram

Construction

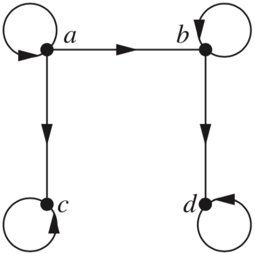
Step 1 of 4: We remove all loops caused by reflexivity.
Step 2 of 4: We remove all edges implied by the transitivity property.
Step 3 of 4: We redraw edges and vertices such that the initial vertex of each edge is below its terminal vertex.
Step 4 of 4: Remove all arrows from the directed edges, since they are all upward. The diagram at right is the Hasse diagram.

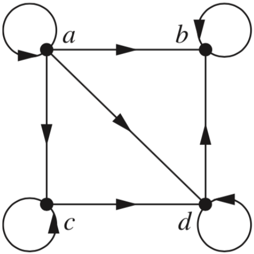
Example 2.12

Construct Hasse Diagram for the poset $(\{2, 4, 5, 10, 12, 20, 25\}, |)$

Exercises

Q1. Determine whether the relation with the directed graph shown is a partial order.





Exercises

Q2. Find the lexicographic ordering of these n-tuples:

a) (1,1,2), (1,2,1) b) (0,1,2,3), (0,1,3,2) c) (1,0,1,0,1), (0,1,1,1,0)

Q3. Find the lexicographic ordering of these strings of lower- case English letters:

a) quack,quick,quicksilver,quicksand,quacking

b) open, opener, opera, operand, opened

c) zoo, zero, zoom, zoology, zoological

Exercises

Q4. Draw the Hasse diagram for divisibility on the set

a) $\{1,2,3,4,5,6\}$.

b) $\{3,5,7,11,13,16,17\}$.

c) $\{2,3,5,10,11,15,25\}$.

d) $\{1,3,9,27,81,243\}$.

Q5. Draw the Hasse diagram for divisibility on the set

a) $\{1,2,3,4,5,6,7,8\}$.

b) $\{1,2,3,5,7,11,13\}$.

c) $\{1,2,3,6,12,24,36,48\}$.

d) $\{1,2,4,8,16,32,64\}$.

Q6. Draw the Hasse diagram for inclusion on the set $P(S)$, where $S = \{a,b,c,d\}$.