DISCRETE STRUCTURES
Lecture 2. Rules of Inference

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Contents

Predicates

Definition 2.1

A predicate, or propositional function, is a function defined on a set U and returns a proposition as its value.
The set U is called the universe of discourse.

- We often denote a predicate by $P(X)$.
- Note that $P(x)$ is not a proposition, but $P(a)$ where a is some fixed element of U is a proposition with well determined truth value.

Example 2.2

Let $Q(x,y) = \text{“}x > y\text{''}$. Which statements are propositions?

1. $Q(x,y)$: NO
2. $Q(3,4)$: YES
3. $Q(x,9)$: NO

Quantifiers - The Universal Quantifier

Let $P(x)$ be a predicate on some universe of discourse U. Methods to obtain a proposition from $P(x)$:

1. Substitute x by a fixed element of U
2. Use the universal quantifier.

Consider the statement:

"$P(x)$ is true for all x in the universe of discourse."

- We write $\forall x, P(x)$ and read "for all $x, P(x)$.

- The symbol $\forall$ is the universal quantifier.

Quantifiers - The Universal Quantifier

Let $P(x)$ be a predicate on some universe of discourse U.

Consider the statement:

"$P(x)$ is true for all x in the universe of discourse."

The proposition $\forall x, P(x)$ is called the universal quantification of the predicate $P(x)$. And it is:

- TRUE if $P(a)$ is true for all a in U
- FALSE if there is an a in U for which $P(a)$ is false.

Quantifiers - The Existential Quantifier

Let $P(x)$ be a predicate on some universe of discourse U. Methods to obtain a proposition from $P(x)$:

1. Substitute x by a fixed element of U
2. Use the universal quantifier.
3. Use the existential quantifier.

Consider the statement:

"There exists an element x in the universe of discourse such that $P(x)$ is true."

- We write $\exists x, P(x)$ and read "for some $x, P(x)$.

- The symbol $\exists$ is the existential quantifier.

Quantifiers - The Existential Quantifier

Let $P(x)$ be a predicate on some universe of discourse U.

Consider the statement:

"There exists an element x in the universe of discourse such that $P(x)$ is true."

The proposition $\exists x, P(x)$ is called the existential quantification of the predicate $P(x)$. And it is:

- TRUE if there is an a for which $P(a)$ is true.
- FALSE if $P(x)$ is false for every single x in U.

Quantifiers - Examples

Example 2.3

Let $P(x)$ be the statement "$x = x^2$." If the domain consists of the integers, what are these truth values?

- $P(0)$, $P(1)$, $P(2)$
- $P(-1)$, $\exists x P(x)$, $\forall x P(x)$

Example 2.4

Determine the truth value of each of these statements if the domain consists of all integers.

- $\forall n, (n+1 > n)$, $\exists n, (2n = 3n)$
- $\exists n, (n = -n)$, $\forall n, (3n \leq 4n)$

Quantifiers - Examples

Example 2.5

Translate these statements into English, where
$C(x)$ is "x is a comedian" and
$F(x)$ is "x is funny"
and the domain consists of all people.

1. $\forall x, (C(x) \rightarrow F(x))$
2. $\forall x, (C(x) \land F(x))$
3. $\exists x, (C(x) \rightarrow F(x))$
4. $\exists x, (C(x) \land F(x))$

Quantifiers - Examples

Example 2.6

Let $P(x) =$ "x can speak Russian" and
$Q(x) =$ "x knows the computer language C++."
Express each of these sentences in term of $P(x)$ and $Q(x)$ quantifiers, and logical connectives. The domain for quantifiers consists of all students at your school.

1. There is a student at your school who can speak Russian and who knows C++.
2. There is a student at your school who can speak Russian but who does not know C++.
3. Every student at your school either can speak Russian or knows C++.
4. No student at your school can speak Russian or knows C++.

Universal Instantiation

$$\frac{\forall x P(x)}{\therefore P(c)}$$

If a propositional function is true for all element x of the universe of discourse, then it is true for a particular element c of the universe of discourse.

Example 2.7

All women are wise. Then Lisa is wise.

Universal Instantiation and modus ponens

The universal instantiation and the modus ponens are used together to form the universal modus ponens.

Example 2.8

Show that the premises "Everyone in this discrete mathematics class has taken a course in computer science" and "Marla is a student in this class" imply the conclusion "Marla has taken a course in computer science."

Let $D(x)$ denote "x is in this discrete mathematics class," and $C(x)$ denote "x has taken a course in computer science."

Step	**Reason**
1. $\forall x(D(x) \to C(x))$	Premise
2. $D(Marla) \to C(Marla)$	Universal instantiation from (1)
3. $D(Marla)$	Premise
4. $C(Marla)$	Modus ponens from (2) and (3)

Universal Generalization

$$\frac{P(c) \text{ for an arbitrary } c}{\therefore \forall x P(x)}$$

We must first define the universe of discourse. Then, we must show that $P(c)$ is true for an arbitrary, and not a specific, element c of the universe of discourse. We have no control over c and we can not make any other assumptions about c other than it comes from the domain of discourse. The error of adding unwarranted assumptions about the arbitrary element c is common and is an incorrect reasoning.

Existential Instantiation

$$\frac{\exists x P(x)}{\therefore P(c) \text{ for some } c}$$

The existential instantiation is the rule that allow us to conclude that there is an element c in the universe of discourse for which $P(c)$ is true if we know that $\exists x P(x)$ is true. We can not select an arbitrary value of c here, but rather it must be a c for which $P(c)$ is true.

Existential Generalization

$$\frac{P(c) \text{ for some } c}{\therefore \exists x P(x)}$$

If we know one element c in the universe of discourse for which $P(c)$ is true, therefore we know that $\exists x P(x)$ is true.

Examples

Example 2.9

Show that the premises "A student in this class has not read the book," and "Everyone in this class passed the first exam" imply the conclusion "Someone who passed the first exam has not read the book."