DISCRETE STRUCTURES
Lecture 6. Mathematical Induction

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Advanced Program in Computer Science

Fall, 2018

Content

Examples

Example 1.1

Show that if n is an integer greater than 1, then n can be written as the product of primes.

Strong Induction

Principle of Strong Induction

To prove that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function, we complete two steps:

1. **Basis step:** Verify that the proposition $P(1)$ is true.
2. **Inductive step:** We show that the conditional statement $[P(1) \land P(2) \land \cdots \land P(k)] \to P(k+1)$ is true for all positive integers k.

Strong induction is sometimes called the second principle of mathematical induction or complete induction.

Examples

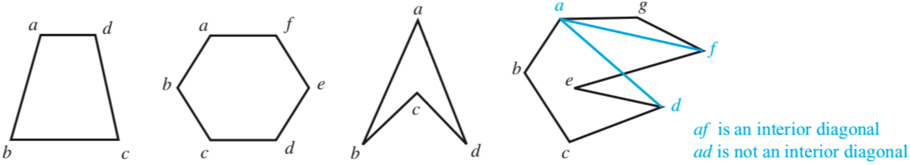
Example 1.2

Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

Method 1: Using Mathematical Induction

Method 2: Using Strong Induction

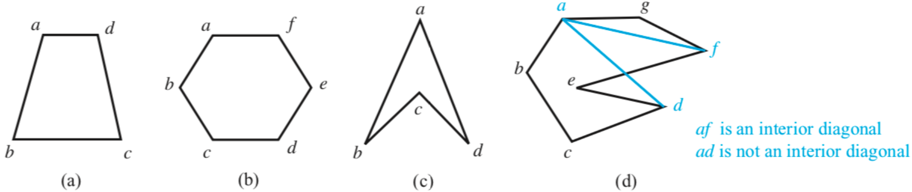
Strong Induction in Computational Geometry



(a) (b) (c) (d)

- A **polygon** consisting of a sequence of line segments $s_1, s_2, ..., s_n$, called **sides**. Each pair (s_i, s_{i+1}), $i = 1, 2, \ldots, n-1$, and (s_n, s_1), of the polygon meet at a common endpoint, called a **vertex**.

Strong Induction in Computational Geometry



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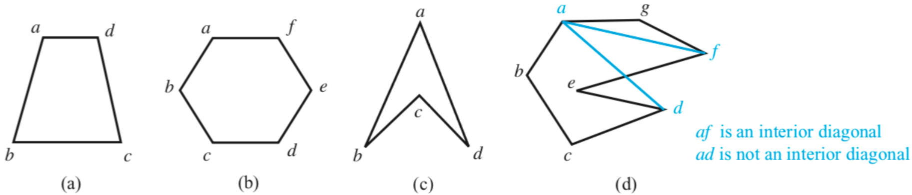

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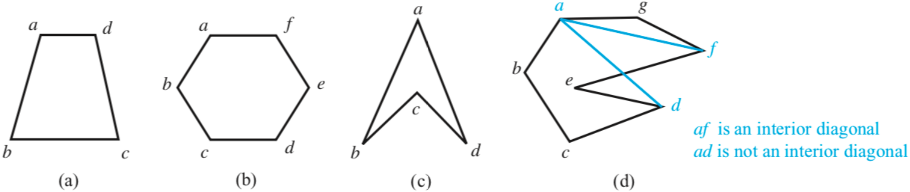

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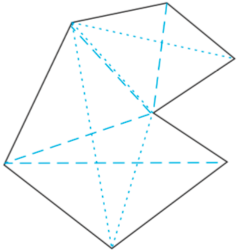

Two different triangulations of a simple polygon with seven sides into five triangles, shown with dotted lines and with dashed lines, respectively

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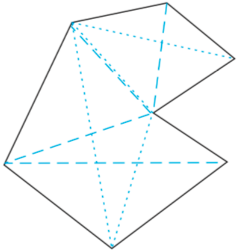

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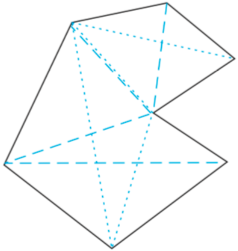

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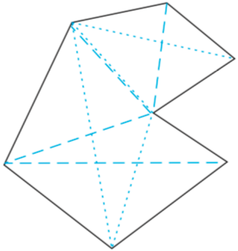

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Strong Induction in Computational Geometry

Theorem 1.3

A simple polygon with n sides, where n is an integer with $n \geq 3$, can be triangulated into $n-2$ triangles.

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Lemma 1.4

Every simple polygon with at least four sides has an interior diagonal.

Strong Induction in Computational Geometry

Lemma 1.5

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- Suppose that P is a simple polygon drawn in the plane.

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- Let b be the point of P or in the interior of P with the least y-coordinate among the vertices with the smallest x-coordinate. Then b must be a vertex of P.

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- Let b be the point of P or in the interior of P with the least y-coordinate among the vertices with the smallest x-coordinate. Then b must be a vertex of P.
- Let a and C be adjacent vertices of b. Then the angle abc is less than 180 degree. Let T be the triangle abc.

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- *Case 1:* There is no vertex inside T then ac is a interior diagonal.

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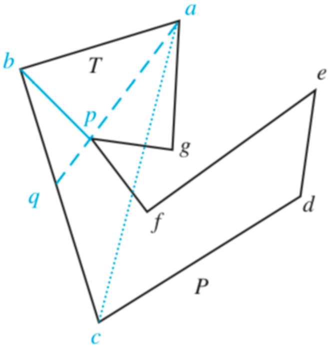
- Let b be the point of P or in the interior of P with the least y-coordinate among the vertices with the smallest x-coordinate. Then b must be a vertex of P.
- Let a and C be adjacent vertices of b. Then the angle abc is less than 180 degree. Let T be the triangle abc.
- *Case 1:* There is no vertex inside T then ac is a interior diagonal.
- *Case 2:* Select a vertex p such that the angle bap is smallest. the triangle baq cannot contain any vertices of P in its interior. Hence, we can connect b and p to produce an interior diagonal of P.

Strong Induction in Computational Geometry



T is the triangle abc

p is the vertex of P inside T such that the $\angle bap$ is smallest

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Strong Induction in Computational Geometry

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1. **Basis step:** $T(3)$ is true because a simple polygon with three sides is a triangle.

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1. **Basis step:** $T(3)$ is true because a simple polygon with three sides is a triangle.
2. **Inductive step:** Assume that $T(j)$ is true for all integers j with $3 \leq j \leq k$.

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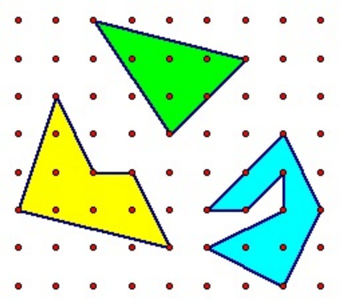
Exercise

Question: How many lattice points (points with integer coordinates) inside a simple polygon?



Exercise

Pick's Theorem

Pick's theorem says that the area of a simple polygon P in the plane with vertices that are all lattice points (that is, points with integer coordinates) equals $I(P)+B(P)/2-1$, where $I(P)$ and $B(P)$ are the number of lattice points in the interior of P and on the boundary of P , respectively. Use strong induction on the number of vertices of P to prove Pick's theorem.

- Prove the theorem for rectangles.
- Prove the theorem for right triangles.
- Prove the theorem for all triangles.
- Use induction to prove the general case.