DISCRETE STRUCTURES
Lecture 7. Recurrences

Bui Anh Tuan

Advanced Program in Computer Science

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Content

Examples

Example 1.1

Given a number $x \in \mathbb{R}$, how to define x^n?

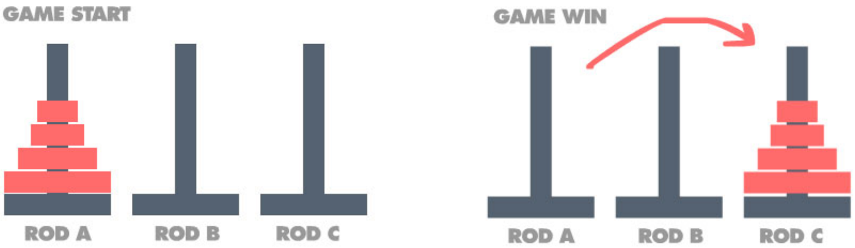
Example 1.2

A sum of P_0 is deposited in the saving account with the interest rate of r % compounded annually. How much will be in the account after n years?

$$P_n = (1 + r)P_{n-1}$$

Geometric progression, solution: $P_n = (1 + r)^n P_0$.

Tower of Hanoi

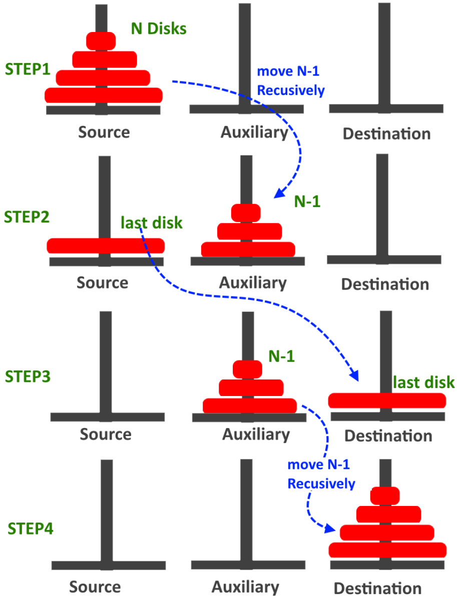


How can we move the disks to the 3rd rod, one in a time, following the rule: larger disks are never placed on top of smaller ones?

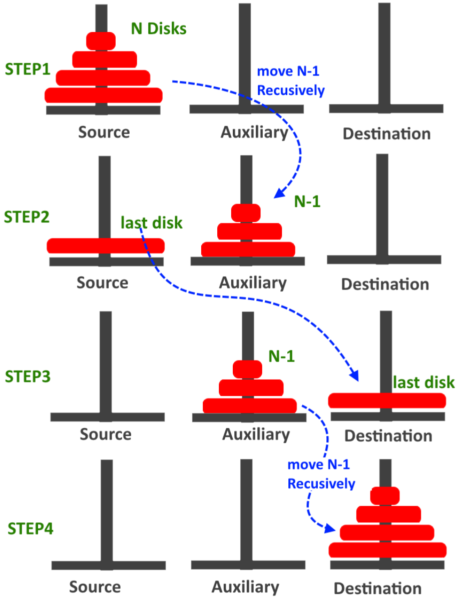
Tower of Hanoi

https://www.mathsisfun.com/games/towerofhanoi.html

What is the minimum number of moves to win the game?



Let H_n be the minimum number of moves to complete the puzzle.



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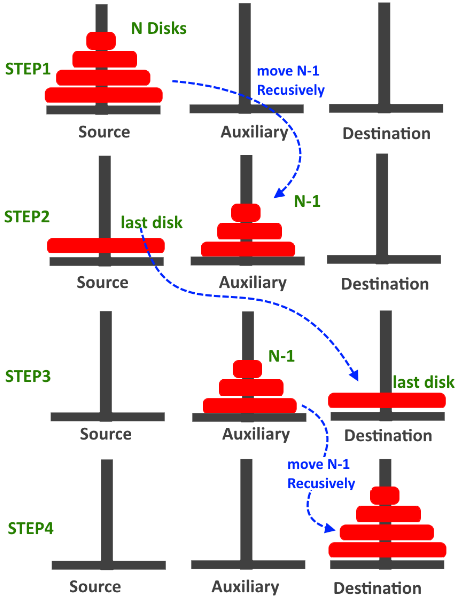
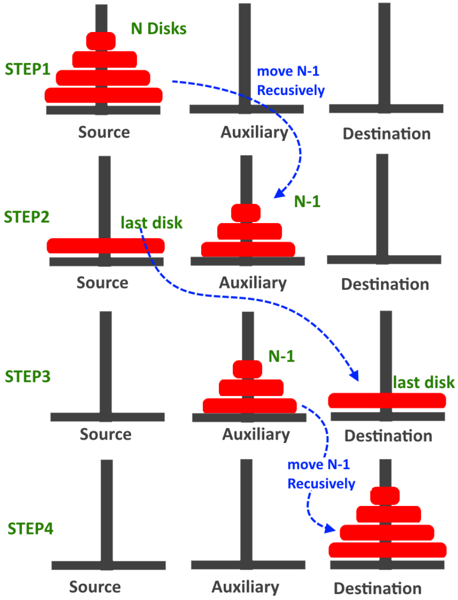
Step 1: 0 moves.



Let H_n be the minimum number of moves to complete the puzzle.

Step 1: 0 moves.

Step 2: H_{n-1} moves.



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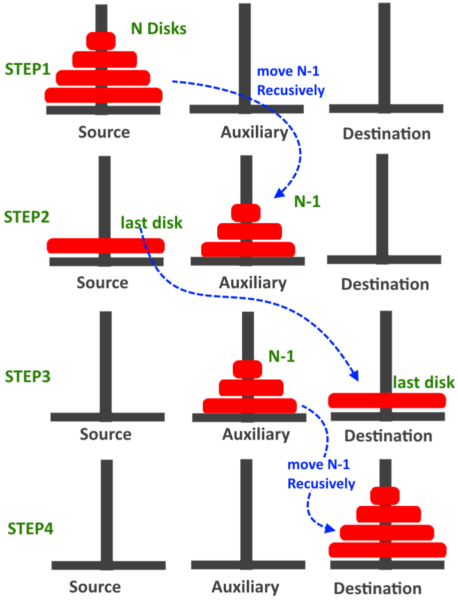
Step 3: 1 move.



Let H_n be the minimum number of moves to complete the puzzle.

Step 1: 0 moves.

Step 2: H_{n-1} moves.

Step 3: 1 move.

Step 4: H_{n-1} moves.

$$H_n = 2H_{n-1} + 1$$

Solve the recursion

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2. $H_n + 1 = 2^{n-1} H_1$

Solve the recursion

1. $H_n = 2H_{n-1} + 1 \Leftrightarrow H_n + 1 = 2(H_{n-1} + 1)$: geometric progression.
2. $H_n + 1 = 2^{n-1}H_1 = 2^n$ (since $H_1 = 1$ obviously).
3. Therefore, $H_n = 2^n - 1$ moves.

Example 1.3

$$H_{64} = 18,446,744,073,709,551,615$$

Suppose that it takes one second for one move, then the time spend for completing the game is:

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Example 1.3

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Suppose that it takes one second for one move, then the time spend for completing the game is: 584,942,417,355 years ≈ 600 billion years.

Compare to:

- the age of the earth: 0.05 billion years
- the age of this universe: 13.8 billion years

Linear Recurrences

Example 1.4 (Fibonacci)

http://setosa.io/ev/eigenvectors-and-eigenvalues/

$$F_n = F_{n-1} + F_{n-2}$$

given that $F_1 = F_2 = 1$.

Solution: $F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$.

Linear Recurrence

Definition 1.5

A linear homogeneous recurrence relation of degree k with constant coefficients is a relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

where $c_1, c_2, \cdots, c_k$ are real numbers and $c_k \neq 0$.

We try to find solution of the form r^n:

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k}$$

This is equivalent to $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k = 0$.
This is called the characteristic equation.

Linear Recurrence

Theorem 1.6

Consider the recurrence $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$ *and its characteristic equation* $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k = 0$.

- $\{r_1, r_2, \ldots, r_m\}$ *are solutions of the characteristic equation;*
- $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ *are constants.*

Then $\alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_m r_m^n$ *is solution of the recurrence.*

Example

Example 1.7

Consider the Fibonacci sequence satisfies: $F_n = F_{n-1} + F_{n-2}$, where $F_1 = F_2 = 1$.

Characteristic equation: $r^2 - r - 1 = 0$.

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Consider the Fibonacci sequence satisfies: $F_n = F_{n-1} + F_{n-2}$, where $F_1 = F_2 = 1$.

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One solution of the recurrence $F_n = \alpha_1 r_1^n + \alpha_2 r_2^n$

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Characteristic equation: $r^2 - r - 1 = 0$.
This equation has 2 solutions: $r_1 = \frac{1+\sqrt{5}}{2}$ and $r_2 = \frac{1-\sqrt{5}}{2}$.
One solution of the recurrence $F_n = \alpha_1 r_1^n + \alpha_2 r_2^n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$
Initial conditions: $F_1 = F_2 = 1$ then

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Linear Recurrence

Example 1.8

What is the solution of the following recurrence

$$a_n = 6a_{n-1} - 9a_{n-2}$$

with $a_0 = 1$ and $a_1 = 6$.

Linear Recurrence

Proposition 1.1

Assume r_0 *is a solution of the characteristic equation with multiplicity* m*. Then* $\alpha n^j r_0^n$ *is solution of the recurrence, for all* $0 \le j \le m-1$*.*

Linear Recurrence

Proposition 1.1

Assume r_0 is a solution of the characteristic equation with multiplicity m. Then $\alpha n^j r_0^n$ is solution of the recurrence, for all $0 \le j \le m-1$.

Theorem 1.9

Consider the recurrence $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$ and its characteristic equation $r^k - c_1 a^{k-1} - c_2 a^{k-2} - \cdots - c_k = 0$.

- *$\{r_1, r_2, \ldots, r_t\}$ are solutions with multiplicity $m_1, m_2, \ldots, m_t$ of the characteristic equation;*
- *$\{\alpha_{ij}\}$ are constants.*

A solution:

$$\begin{aligned} a_n = &(\alpha_{01} + \alpha_{11} n + \alpha_{21} n^2 + \cdots + \alpha_{m_1-1} n^{m_1-1,1}) r_1^n \\ &+ (\alpha_{02} + \alpha_{12} n + \alpha_{22} n^2 + \cdots + \alpha_{m_2-1} n^{m_2-1,2}) r_2^n \\ &\ldots \\ &+ (\alpha_{0t} + \alpha_{1t} n + \alpha_{2t} n^2 + \cdots + \alpha_{m_t-1} n^{m_t-1,t}) r_t^n \end{aligned}$$

Linear Recurrence

Example 1.10

What is the solution of the following recurrence

$$a_n = 6a_{n-1} - 9a_{n-2}$$

with $a_0 = 1$ and $a_1 = 6$.

Characteristic Equation: $r^2 - 6t + 9 = 0$ has solution r=3 multiplicity 2.

Linear Recurrence

Example 1.10

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with $a_0 = 1$ and $a_1 = 6$.

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A solution: $a_n = (\alpha_0 + \alpha_1 n)3^n$

Linear Recurrence

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A solution: $a_n = (\alpha_0 + \alpha_1 n)3^n$
With initial conditions: $a_0 = 1$ and $a_1 = 6$,
We have $a_n = 3^n + n3^n$.