DISCRETE STRUCTURES
Lecture 1. Propositional logic and equivalence

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    - Proposition
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Declarative Sentence

Definition 1.1

Declarative sentence is a sentence that declares a fact.

Example 1.2

Declarative sentences

1. Paris is the capital of France.
2. The sun rises in the east.
3. $1 + 1 = 2$.
4. $x + 1 = 2$.

Example 1.3

Not declarative sentences

1. What time is it?
2. No smoking!
3. What a beautiful day!

Propositions

Definition 1.4

A *proposition* is a declarative sentence that is either TRUE or FALSE, but not both.

Example 1.5

Propositions

1. Paris is the capital of France.
2. The sun rises in the east.
3. $1 + 1 = 2$.

Example 1.6

Not propositions

1. $x + 1 = 2$.

Propositions

Notations

- We use letters to denote propositions such as p, q, r, s,...
- We denote the truth value of a proposition as T (true) or F (false).

Example 1.7

Truth value

1. Paris is the capital of France: T
2. The sun rises in the east: F
3. $1 + 1 = 2$: T
4. $1 + 1 = 3$: F

Compound Propositions

Question:

If it rains, I will stay at home.

Is it a proposition?

Definition 1.8

Compound propositions, are formed from existing propositions using logical operators.

Example 1.9

1. $3 > 4$ and Paris is the capital of France
2. $3 > 4$ or Paris is the capital of France
3. If $2 = 1$ then Newton and Pascal are one person.
4. It is cloudy if and only if it is raining.

Negation

Example 1.10

Let

$$p := \text{"It is raining."}$$
$$q := \text{"It is not raining"}.$$

Is there any relationship between p and q?

Definition 1.11

Negation Let p be a proposition. The compound proposition

"it is not the case that p"

is an other proposition, called the negation of p, and denoted $\neg p$. The truth value of the negation of p is the opposite of the truth value of p. The proposition $\neg p$ is read "not p".

Negation

Example 1.12

Find the negation of the following:

"At least 1 0 inches of rain fell today in Miami."

Negation: "It is not the case that at least 1 0 inches of rain fell today in Miami."

Simple English: "Less than 1 0 inches of rain fell today in Miami."

Negation of comparative operators

p	$\neg p$
$>$	$\leq$
$\geq$	$<$
$<$	$\geq$
$\leq$	$>$

Truth table

Definition 1.13

A truth table presents the relations between the truth value of many propositions involved in a compound proposition. This table has a row for each possible truth value of the propositions.

Truth table for the negation $\neg p$ of the proposition p:

p	$\neg p$
T	F
F	T

Conjunction

Definition 1.14

Let p and q be propositions. The compound proposition

"p and q"

denoted $p \wedge q$, is true when both p and q are true and false otherwise. This compound proposition $p \wedge q$ is called the conjunction of p and q.

Truth table for the conjunction $p \wedge q$ of the propositions p and q:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Conjunction

Example 1.15

Determine whether these statements are true or false.

a) $3 > 4$ and Paris is the capital of France

b) 2 is an even and prime number

c) An is drinking water and singing a song

c) $3^2 > 9$ and the sun rises in the west

Disjunction

Definition 1.16

Let p and q be propositions. The compound proposition

"p or q"

denoted $p \vee q$, is false when both p and q are false and true otherwise. This compound proposition $p \vee q$ is called the disjunction of p and q.

Truth table for the disjunction $p \vee q$ of the propositions p and q:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Conjunction

Example 1.17

Determine whether these statements are true or false.

a) $3 > 4$ or Paris is the capital of France

b) 4 is an even number or 5 is a prime number

c) $\pi > 4$ or the sun rises in the west

Exclusive Disjunction

Definition 1.18

Let p and q be propositions. The compound proposition

"p exclusive or q"

denoted $p \oplus q$, is true when exactly one of p and q is true and false otherwise. This compound proposition $p \oplus q$ is called the exclusive disjunction of p and q.

Truth table for the exclusive disjunction $p \oplus q$ of the propositions p and q:

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Implication

Example 1.19

"If $2 = 1$ then Newton and Pascal are one person"

Definition 1.20

Let p and q be propositions. The compound proposition

"if p, then q"

denoted $p \to q$, is false when p is true and q is false, and true otherwise. This compound proposition $p \to q$ is called the implication (conditional statement) of p and q. In this implication, p is called the hypothesis and q is called the conclusion.

Implication

Truth table for the implication $p \to q$ of the propositions p and q:

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Example 1.21

"If $2 = 1$ then Newton and Pascal are one person": T

Implication

Variety of terminology is used to express the implication $p \to q$:

- if p, then q;
- p implies q;
- q if p;
- p only if q;
- q when p;
- p is sufficient for q;
- a sufficient condition for q is p;
- q follows from p;
- q whenever p.

Converse, Inverse and Contrapositive

We can form some new conditional statements starting with the implication $p \to q$. Here are some popular ones:

1. The converse of $p \to q$ is the proposition $q \to p$.
2. The inverse of $p \to q$ is the proposition $\neg p \to \neg q$.
3. The contrapositive of $p \to q$ is the proposition $\neg q \to \neg p$.

Example 1.22

"The home team is wining whenever it is raining."

Equivalent sentence: "If it is raining, the home team is wining."

- Converse: If the home team wins, then it is raining.
- Inverse: If it is not raining, then the home team does not win
- Contrapositive: If the home team does not win, then it is not raining.

Bi-implication

Definition 1.23

Let p and q be propositions. The compound proposition

"p if and only if q"

denoted $p \leftrightarrow q$, is true when p and q have the same truth value, and false otherwise. This compound proposition $p \leftrightarrow q$ is called the bi-implication (biconditional statement) of p and q.

Some common equivalent saying:

- p is necessary and sufficient for q;
- if p then q, and conversely;
- p iff q.

Bi-implication

Truth table for the bi-implication $p \to q$ of the propositions p and q:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Exercises

Construct the truth table of the compound propositions:

$$(p \lor \neg q) \to (p \land q)$$

Precedence of Logical Operators

Operators	Precedence
$\neg$	1
$\land$	2
$\lor$	3
$\to$	4
$\leftrightarrow$	5