Predicate logic

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Table of Contents

Limitations of Propositional Logic

- Propositional logic is good for dealing with statements built from basic propositions using *and*, *or*, *not*, *if … then*

> Example:
>
> *“**If** a student hands in his or her assignment late **and** there is a good reason for the late submission, **then** there will be no penalty.”*

- More complex statements cannot be expressed naturally in propositional logic

> Example:
>
> *“Every student is younger than some instructor.”*

Predicates

- Recall the statement:

  "Every student is younger than some instructor."

- What is it about?
  - Being a student
  - Being younger than somebody else
  - Being an instructor

  these are properties

- Idea: use predicates to express such properties

Example

Predicates:

- Student: is a student
- Younger: is younger than
- Instructor: is an instructor

statement	expressed using predicates
Alice is a student	$Student(alice)$
Alice is younger than Bob	$Younger(alice, bob)$
Bob is an instructor	$Instructor(bob)$

Predicates are Not Enough

- Predicates are not yet enough to express our statement:

> "Every student is younger than some instructor."

- We need a mechanism to:
    - Express "***every …***" and "***some …***".
    - Talk about students and instructors without mentioning each one by name.

      (The above statement doesn't really care about the names of students and instructors, just the connection between students and instructors.)

Variables to the Rescue

Idea: use variables

- as place holders for concrete values (like students, instructors, account numbers, etc.)
- Notation: x, y, z, … (possibly with decorations like subscripts)

Examples:

$\mathit{Student}(x)$	… "x is a student"
$\mathit{Younger}(\mathit{alice}, y)$	… "Alice is younger than y"
$\mathit{Younger}(z_1, x)$	… "z_1 is younger than x"
$\mathit{Instructor}(z)$	… "z is an instructor"

Quantifiers

Our original sentence:

> "Every student is younger than some instructor."

can be rephrased using predicates and variables as:

> "**For every** individual x with $\mathit{Student}(x)$, **there is** an individual y such that $\mathit{Younger}(x, y)$ and $\mathit{Instructor}(y)$."

To express "*for all*" and "*exists*", we use quantifiers $\forall$ and $\exists$:

- $\forall$ … for "for all"
- $\exists$ … for "there exists an"

The Statement in Predicate Logic

The original statement can now be written symbolically using predicates, variables, and quantifiers:

$$\forall x \Big(\mathit{Student}(x) \rightarrow \exists y \, (\mathit{Younger}(x,y) \wedge \mathit{Instructor}(y)) \Big)$$

- This is a typical formula in predicate logic!
- Translating the formula back yields:

  "For every individual x, if x is a student, then there is an individual y such that x is younger than y and y is an instructor."

Translation into Predicate Logic II

"Not all birds can fly."

1. Pick suitable predicates: *Bird* (being a bird), *CanFly* (can fly)
2. Encode the sentence in predicate logic:

$$\neg\forall x\big(\mathit{Bird}(x) \rightarrow \mathit{CanFly}(x)\big)$$

"It is not the case that for all individuals x, if x is a bird, then x can fly."

Equivalently:

$$\exists x\big(\mathit{Bird}(x) \wedge \neg\mathit{CanFly}(x)\big)$$

"There is an individual that is a bird and that cannot fly."

This equivalence can be made precise (later lecture).

Exercises

P. 157, 158 Exercises 1,2,3,4.

Equality and Function Symbols

Predicate logic has two additional features:

1. Equality: allows to express that individuals are equal or not

   Example: *"At least two students are registered for COMP118."*

   $$\exists x\, \exists y\, \big(Registered(x, comp118) \land Registered(y, comp118) \land \neg x = y \big)$$

2. Function symbols: allows to express functional dependencies between individuals

   Example: *"Alice and Bob have the same mother."*

   $$Mother(alice) = Mother(bob)$$

Simulating Function Symbols by Predicates

- Function symbols can often be "simulated" by predicates, so they are not really necessary

Example: $\mathit{Mother}(\mathit{alice}) = \mathit{Mother}(\mathit{bob})$ can be expressed as

$$\forall x \forall y \, (\mathit{IsMotherOf}(x, \mathit{alice}) \wedge \mathit{IsMotherOf}(y, \mathit{bob}) \rightarrow x = y)$$

- But function symbols lead to more natural descriptions

Example: compare the two formulae above

The Elements of Predicate Logic
Summary

- Predicate logic allows us to express statements about
  - objects (humans, animals, account numbers, …)
  - and their properties and relations to each other (being a student, being younger, being able to fly, …).
- New mechanisms:
  - Predicates (to express properties of objects and their relationship to each other),
  - Variables and Quantifiers (to express "*for all*", "*there exists*")
  - Equality and Function Symbols

Syntax of Predicate Logic

Two parts:

1. Terms: These are names for objects (not to be confused with the objects themselves).

2. Formulae: These correspond to statements about objects and their properties.

Table of Contents

Signatures

- A signature defines the "vocabulary" relative to which formulae can be expressed.
- It is the set of all the symbols that can be used in a formula:
    - Predicate symbols: *Student*, *Younger*, *Bird*, …
    - Constant symbols: *alice*, *comp118*, *42*, …
    - Function symbols: *Mother*, *Plus*, *Minus*, …
- It also fixes an arity (number of parameters) for each predicate and function symbol in it.

  Example: In the formulae of the last lecture, we assumed that *Student* and *Younger* have arity 1 and 2, respectively.

Signatures

Definition

A signature S is a set consisting of:

- predicate symbols, each with an associated arity
- function symbols, each with an associated arity
- constant symbols

Notation:

- $P, Q, R, \ldots$, possibly with subscripts/superscripts, denote predicate symbols (unless otherwise stated)
- $F, G, \ldots$ denote function symbols
- $a, b, c, \ldots$ denote constant symbols

Example: Kinship Relations

A signature S_K for expressing kinship relations:

$$S_K = \{Male, Female, Parent, Sibling, alice, Mother, Father\},$$

where

- *Male* and *Female* are predicate symbols of arity 1
- *Parent* and *Sibling* are predicate symbols of arity 2
- *alice* is a constant symbol,
- *Mother* and *Father* are function symbols of arity 1

Note: We'll see later how to assign meaning to the symbols. For the moment, keep in mind that symbols do not have any meaning per se. The descriptive names above merely *hint at the intended meaning*.

Notation

We call a predicate or function symbol

- unary if its arity is 1
- binary if its arity is 2
- k-ary if its arity is k

Example: Arithmetic

A signature S_A for arithmetic:

$$S_A = \{Smaller, 0, 1, Plus, Times\},$$

where

- *Smaller* is a binary predicate symbol
- *0* and *1* are constant symbols,
- *Plus* and *Times* are binary function symbols

Table of Contents

Variables

In the remaining lectures, $x, y, z, \ldots$ (possibly with decorations like subscripts/superscripts) always denote variables, unless specified otherwise.

Terms

Recall: Terms are *names for objects*.

Definition

Let S be a signature. The set of all S-terms (or just *terms* if S is understood) is defined inductively as follows:

- Each variable is an S-term.
- Each constant in S is an S-term.
- If F is a k-ary function symbol in S, and $t_1, \ldots, t_k$ are S-terms, then $F(t_1, \ldots, t_k)$ is an S-term.

Important: The set of terms depends on the signature S.

Example

Consider the signature $S = \{P, c, F, G\}$, where

- P is a binary predicate symbol,
- c is a constant symbol,
- F is a unary and G is a binary function symbol.

S-terms	*no* S-terms
c	$F(x, y)$ (wrong arity)
x	$G(c)$ (wrong arity)
$G(x, c)$	$P(c, z)$ (P is no function symbol)
$G\big(F(y), G(c, z)\big)$	

Example: Kinship Relations (cont'd)

Signature $S = \{alice, Mother, Father\}$, where *alice* is a constant symbol, and *Mother*, *Father* are unary function symbols

- *alice* can be used to refer to Alice.
- $Mother(alice)$ can be used to refer to Alice's mother.
- $Father(Mother(x))$ can be used to refer to x's maternal grandfather.
- etc.

Question

$S = \{P, c\}$, where

- P is a binary predicate symbol,
- c is a constant symbol.

What are the S-terms?

Formulae

Definition

Let S be a signature. The set of all S-formulae is defined by induction as follows:

1. If P is a predicate in S of arity k, and if $t_1, \ldots, t_k$ are S-terms, then $P(t_1, \ldots, t_k)$ is an S-formula.
2. If t_1, t_2 are S-terms, then $t_1 = t_2$ is an S-formula.
3. If φ and ψ are S-formulae, then $\neg\varphi$, $(\varphi \wedge \psi)$, and $(\varphi \vee \psi)$ are S-formulae.
4. If φ is an S-formula and x is a variable, then $\exists x\, \varphi$ and $\forall x\, \varphi$ are S-formulae.

Atomic S-formulae are those formulae formed by rules 1. and 2.

Example

Consider again the signature $S = \{P, c, F, G\}$, where

- P is a binary predicate symbol,
- c is a constant symbol,
- F is a unary and G is a binary function symbol.

S-formulae that are atomic	S-formulae that are *not* atomic
$P(x, F(y))$	$\neg P(x, F(y))$
$P(G(F(x), c), F(z))$	$\big(P(x, F(y)) \wedge x = F(z)\big)$
$x = c$	$\exists x\, x = c$
$G(F(x), c) = F(y)$	$\forall x\big(\neg P(x, F(x)) \vee \exists y\, G(c, y) = F(x)\big)$

Notation

- As with propositional logic, we omit outer parentheses, e.g.:

$$\exists x\, \varphi \vee \psi \quad \text{instead of} \quad (\exists x\, \varphi \vee \psi)$$

- We use $\varphi \rightarrow \psi$ as an abbreviation for $\neg\varphi \vee \psi$.
- We use $\varphi \leftrightarrow \psi$ as an abbreviation for $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$.

Example: Arithmetic

$S_A = \{Smaller, 0, 1, Plus, Times\}$

- x is an even number:

$$\exists y\, x = Plus(y, y)$$

- x is prime:

$$Smaller(1, x) \wedge \forall y \forall z \left(Times(y, z) = x \rightarrow (y = 1 \vee y = x) \right)$$

Important: This assumes that the universe of discourse is the set of natural numbers.

Example: Kinship Relation (cont'd)

Signature $S = \{\mathit{Male}, \mathit{Female}, \mathit{ParentOf}, \mathit{Sibling}, \mathit{alice}, \mathit{Mother}\}$

What are S-formulae that correspond to:

- Alice is female.
- Alice has a sister.
- Alice's mother has a brother.

Exercises

p.158-160: 2.2: 1,2,3,4,5

Table of Contents

Free Variables – Informally

Variables may occur bound (in the scope of a quantifier) or free:

bound occurrences of y free occurrence of y

$$\exists y \, \underbrace{\big(P(F(x), y) \land \forall z \, \underbrace{G(x, F(z)) = F(y)}_{\text{scope of } \forall z} \big)}_{\text{scope of } \exists y} \land Q(y)$$

Observation: A variable may occur both free and bound.

Free Variables

The set free(φ) is the set of all the variables that occur free in φ.

Definition

Let φ be an S-formula. The set free(φ) of the free variables of φ is defined inductively:

- If φ is $R(t_1, \ldots, t_k)$, then free(φ) is the set of all the variables that occur in the terms $t_1, \ldots, t_k$.
- If φ is $t_1 = t_2$, then free(φ) is the set of all the variables that occur in the terms t_1, t_2.
- If φ is $\neg\psi$, then $\text{free}(\varphi) = \text{free}(\psi)$.
- If φ is $\psi \wedge \chi$ or $\psi \vee \chi$, then $\text{free}(\varphi) = \text{free}(\psi) \cup \text{free}(\chi)$.
- If φ is $\exists x\, \psi$ or $\forall x\, \psi$, then $\text{free}(\varphi) = \text{free}(\psi) \setminus \{x\}$.

Example

- free($P(x,\ G(F(z), x))$) $= \{x, z\}$
- free($\exists x\, P(x, z)$) $= \{z\}$
- free($\exists x\, P(x, z) \vee \neg \forall z\, Q(x, z)$) $= \{z, x\}$
- free$\Big(\forall y\, \exists x\, \big(P(x, y, z) \vee \exists z\, Q(z, y)\big) \wedge \exists z\, R(x, z) \Big) = ?$

Semantics of Formulae

What does this formula mean ?

$$\forall x \Big(\mathit{Student}(x) \rightarrow \exists y \big(\mathit{Younger}(x, y) \wedge \mathit{Instructor}(y) \big) \Big)$$

- Every student is younger than some instructor at our school ?
- Every student on earth is younger than some instructor at our school ?
- Something else ?

It has no meaning per se !

Semantics of Formulae

What does this formula mean?

$$\exists x\, F(\underline{2}, x) = y$$

- y is two more than x? (if F is addition)
- y is even? (if F is integer multiplication)
- Something else?

As before: It has no meaning per se!

Semantics of Formulae

How do we assign meaning to formulae like:

$$\forall x \Big(Student(x) \rightarrow \exists y \left(Younger(x, y) \wedge Instructor(y) \right) \Big)$$

$$\exists x \, F(\underline{2}, x) = y$$

We have to specify:

1. The domain (of discourse).

   This is the set of all objects we are talking about (people, birds, integers, reals, etc.)

2. The meaning of all the symbols mentioned in the formula (predicate/function/constant symbols).

Example: Students and Instructors

$$\forall x\Big(Student(x) \to \exists y\,(Younger(x,y) \wedge Instructor(y))\Big)$$

- A possible domain: $D = \{\text{Alice}, \text{Bob}, \text{Carol}\}$
- To say that Alice and Bob are students, we could interpret the predicate symbol *Student* with the set:

$$\{\text{Alice}, \text{Bob}\}$$

- To say that Alice is younger than Bob, and Bob is younger than Carol, we could interpret *Younger* with the set:

$$\{(\text{Alice}, \text{Bob}),\ (\text{Bob}, \text{Carol}),\ (\text{Alice}, \text{Carol})\}$$

Example: Arithmetic

$$\exists x\, F(\underline{2}, x) = y$$

- A possible domain: $D = \{0, 1, 2, 3, 4, \dots\}$
- To say that $F(x, y)$ should mean "$x + y$", we could interpret the function symbol F with the function $F \colon D^2 \to D$:

$$F(d, d') = d + d' \qquad \text{for all } (d, d') \in D^2$$

- To say that $\underline{2}$ should correspond to the number 2, we interpret it with the number 2 (which is in D).