Predicate logic - Natural deduction

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Equality

- The rule for introducing equality is fairly simple: it simply says that any term t is equal to itself, and that no premises are needed to conclude this.
- The "elimination" rule for equality is more interesting; it describes how substitution may be used in formulas.
- If t_1 and t_2 are free for z in ϕ, then we may substitute the term t_2 for t_1 in ϕ.

$$\frac{t_1 = t_2 \quad \phi[t_1/z]}{\phi[t_2/z]} = \mathrm{e}$$

- Conversely, we may substitute the term t_1 for t_2 in ϕ.

$$\frac{t_1 = t_2 \quad \phi[t_2/z]}{\phi[t_1/z]} = \mathrm{e}_R$$

Universal quantification

- To eliminate the $\forall$ quantifier, we choose one of the many "values" quantified over, namely any term *t* (which is free for *x* in ϕ):

$$\frac{\forall x \phi}{\phi[t/x]} \; \forall x \text{ e}$$

- $\forall$-introduction is not quite so simple.
- To introduce a quantifier $\forall x$, we must prove the formula being quantified for all possible "values". This seems impossible.

$\forall$-introduction

- Let's take a look at a somewhat informal proof below:

Theorem

Every even natural number is the sum of two odd natural numbers whose difference is at most 2.

Proof.

Let n be an even natural number. Then n is of the form $2k$, for some $k \geq 1$. If k is odd, then we can write $n = k + k$, and the two k's satisfy the theorem. If k is even, we can write $n = (k - 1) + (k + 1)$, and the numbers $k - 1$ and $k + 1$ satisfy the theorem. $\square$

$\forall$-introduction

- If we look closely at this proof, we are taking an arbitrary even natural number, which we are calling n, and expressing it as the sum of two odd natural numbers whose difference is at most 2. That allows us to conclude that the property holds for all natural numbers n, since there is nothing special about n (that is, it doesn't appear in the statement of the theorem or anywhere else outside the proof).
- This suggests that to prove a formula of the form $\forall x \phi$, we can prove ϕ with some arbitrary but fresh variable x_0 substituted for x. That is, we want to prove the formula $\phi[x_0/x]$.

$\forall$-introduction

- The word “fresh” means that the variable has never been used before in the proof. Furthermore, it will not be used once $\phi[x_0/x]$ has been proved.
- It is “local” to this part of the proof. As we did with assumptions, we enforce this locality by surrounding this part of the proof with a proof box, putting the fresh variable as a label in the top left corner.
- We label the top of the proof box with the fresh variable, but so far we only know what the last line inside the proof box should look like: $\phi[x_0/x]$.
- The first line can be anything we consider useful. In certain cases, we might even leave it blank (for example, if we are immediately going to open another proof box inside it).

∀-introduction

- The template looks like

$$\dfrac{\boxed{\begin{array}{l} x_0 \\ \quad\quad \vdots \\ \quad\quad \phi[x_0/x] \end{array}}}{\forall x \phi} \ \forall x\mathrm{i}$$

- Example: Prove

$$\forall x \big(P(x) \vee Q(x)\big),\ \forall x \big(\neg P(x)\big) \vdash \forall x Q(x).$$

$\forall$-introduction

	1	$\forall x(P(x) \vee Q(x))$	premise
	2	$\forall x(\neg P(x))$	premise
x_0	3	$P(x_0) \vee Q(x_0)$	$\forall x$e 1
	4	$P(x_0)$	assumption
	5	$\neg P(x_0)$	$\forall x$e 2
	6	$\perp$	$\neg$e $4, 5$
	7	$Q(x_0)$	$\perp$e 6
	8	$Q(x_0)$	assumption
	9	$Q(x_0)$	$\vee$e $3, 4-7, 8$
	10	$\forall x Q(x)$	$\forall x$i $3, 9$

Existential quantification

- The first line needs to be ϕ with the fresh variable substituted for the quantified variable to be eliminated.
- This line is labeled as an assumption.
- The last line cannot involve the fresh variable, as it is not permitted to appear outside the proof box.
- Prove:

$$\forall x (P(x) \vee Q(x)),\ \exists x (\neg P(x)) \vdash \exists x Q(x)$$

Existential quantification

	1	$\forall x(P(x) \lor Q(x))$	premise
	2	$\exists x(\neg P(x))$	premise
x_0	3	$\neg P(x_0)$	assumption
	4	$P(x_0) \lor Q(x_0)$	$\forall x$e 1
	5	$P(x_0)$	assumption
	6	$\bot$	$\neg$e $3, 5$
	7	$Q(x_0)$	$\bot$e 6
	8	$Q(x_0)$	assumption
	9	$Q(x_0)$	$\lor$e $4, 5-7, 8$
	10	$\exists x Q(x)$	$\exists x$i 9
	11	$\exists x Q(x)$	$\exists x$e $2, 3-10$

Equivalences

Prove that $\exists x(\neg\phi) \vdash \neg\forall x\phi$.

	1	$\exists x(\neg\phi)$	premise
x_0	2	$(\neg\phi)[x_0/x]$	assumption
	3	$\neg(\phi[x_0/x])$	identical
	4	$\forall x\phi$	assumption
	5	$\phi[x_0/x]$	$\forall x$e
	6	$\perp$	$\neg$e $3, 5$
	7	$\neg\forall x\phi$	$\neg$i $4-6$
	8	$\neg\forall x\phi$	$\exists x$e $1, 2-7$

Equivalences

Prove that $\neg\forall x\phi \vdash \exists x(\neg\phi)$.

	1	$\neg\forall x\phi$	premise
	2	$\neg\exists x\neg\phi$	assumption
x_0	3		
	4	$\neg(\phi[x_0/x])$	assumption
	5	$(\neg\phi)[x_0/x]$	identical
	6	$\exists x\neg\phi$	$\exists$i 5
	7	$\perp$	$\neg$e 2, 6
	8	$\phi[x_0/x]$	PBC 4–7
	9	$\forall x\phi$	$\forall$i 4–8
	10	$\perp$	$\neg$e $9, 1$
	11	$\exists x\neg\phi$	PBC 2–10