# DISCRETE STRUCTURES
## Lecture 3. Relations

Bui Anh Tuan

Advanced Program in Computer Science

Fall, 2018

# Content

1. Relations, their properties and representations
2. Equivalence relations

# Definitions

## Definition 2.1 (Cartesian product)

Let A and B be sets. The Cartesian product of A and B, denoted by $A \times B$, is the set of all ordered pairs $(a, b)$, where $a \in A$ and $b \in B$. ie.,

$$A \times B = \{(a, b) | a \in A \wedge b \in B\}.$$

## Example 2.2

Let $A = \{0, 1, 2\}$ and $B = \{a, b\}$. Then

$$A \times B = \{(0, a), (0, b), (1, a), (1, b), (2, a), (2, b)\}.$$

**Note:** $(a, b), (c, d) \in A \times B,\ (a, b) = (c, d) \Leftrightarrow a = c \wedge b = d.$

# Binary Relation

## Definition 2.3

Let $A$ and $B$ be sets. A binary relation from A to B is a subset R of $A \times B$. A relation from a set A to itself is called a relation on A.

## Example 2.4

Let $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$.
Consider $R = \{(a, 3), (c, 1), (c, 2), (c, 3)\} \subset A \times B$. Then $R$ is a binary relation from A to B. Since $(a, 3) \in R$, we say that
"a is related to 3" or "$aR3$"

# Examples

## Example 2.5

Let $A = \{1, 2, 3, 4\}$ and define

$$R = \{(a, b) | a \text{ divides } b\}.$$

## Example 2.6

List the ordered pairs in the relation R from $A = \{0, 1, 2, 3, 4\}$ to $B = \{0, 1, 2, 3\}$, where $(a, b) \in R$ if and only if

a) a = b.
b) a + b = 4.
c) a > b.
d) gcd(a, b) = 1.

# Number of Relations

**Question**

If A is a finite set with $|A| = n$, how many different relations are there on A?

1. How many subset of a set of $n$ elements?
2. How many elements in $A \times A$?

# Representing a Relation

There are several ways to represent a relation

1. Using table
2. Using matrix
3. Using directed graph

# Representing a Relation Using Table

## Example 2.7

Let $A = \{0, 1, 2\}$ and $B = \{a, b\}$. Then $R = \{(0, a), (0, b), (1, a), (2, b)\}$ is a relation from A to B.

| $R$ | $a$ | $b$ |
|---|---|---|
| 0 | × | × |
| 1 | × | |
| 2 | | × |

# Representing a Relation Using Matrix

## Example 2.8

Let $A = \{0,1,2\}$ and $B = \{a,b\}$. Then $R = \{(0,a),(0,b),(1,a),(2,b)\}$ is a relation from A to B.

# Representing a Relation Using Matrix

## Example 2.8

Let $A = \{0, 1, 2\}$ and $B = \{a, b\}$. Then $R = \{(0, a), (0, b), (1, a), (2, b)\}$ is a relation from A to B.

- Choose arbitrary order for A and B
- Elements of A is the index for row and elements of B is the index of column
- Define the matrix as:

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{otherwise} \end{cases}$$

# Representing a Relation Using Matrix

## Example 2.9

Let $A = \{0, 1, 2\}$ and $B = \{a, b\}$. Then $R = \{(0, a), (0, b), (1, a), (2, b)\}$ is a relation from A to B.

# Representing a Relation Using Matrix

## Example 2.9

Let $A = \{0, 1, 2\}$ and $B = \{a, b\}$. Then $R = \{(0, a), (0, b), (1, a), (2, b)\}$ is a relation from A to B.

$$M_R = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

**Note:** Such a representation depends on the ordering used for A and B.

# Representing a Relation Using Directed Graph

**Definition 2.10 (Directed Graph)**

A directed graph $G = (V, E)$, or digraph, consists of a set V of vertices (or nodes) together with a set E of edges (or arcs). The vertex a is called the initial vertex of the edge (a,b), and the vertex b is called the terminal vertex of this edge.

# Representing a Relation Using Directed Graph

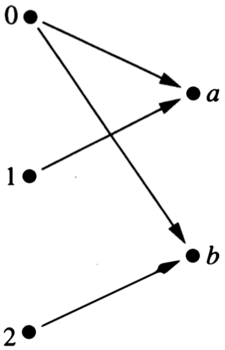
## Example 2.11

Let $A = \{0, 1, 2\}$ and $B = \{a, b\}$. Then $R = \{(0, a), (0, b), (1, a), (2, b)\}$ is a relation from A to B.

# Representing a Relation Using Directed Graph

## Example 2.11

Let $A = \{0, 1, 2\}$ and $B = \{a, b\}$. Then $R = \{(0, a), (0, b), (1, a), (2, b)\}$ is a relation from A to B.



# Representing a Relation Using Table

## Example 2.12

Let $A = \{1,2,3,4\}$ and define $R = \{(a,b) | a \text{ divides } b\}$.

$$R = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(4,4)\}.$$

# Representing a Relation Using Table

## Example 2.12

Let $A = \{1,2,3,4\}$ and define $R = \{(a,b) | a \text{ divides } b\}$.

$$R = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(4,4)\}.$$

Using table:

| $R$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | × | × | × | × |
| 2 | | × | | × |
| 3 | | | × | |
| 4 | | | | × |

# Representing a Relation Using Matrix

## Example 2.13

Let $A = \{1,2,3,4\}$ and define $R = \{(a,b) | a \text{ divides } b\}$.

$$R = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(4,4)\}.$$

Using matrix:

# Representing a Relation Using Matrix

## Example 2.13

Let $A = \{1, 2, 3, 4\}$ and define $R = \{(a, b) | a \text{ divides } b\}$.

$$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}.$$

Using matrix:

$$M_R = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

# Representing a Relation Using Directed Graph

## Example 2.14

Let $A = \{1,2,3,4\}$ and define $R = \{(a,b) | a$ divides $b\}$.

$$R = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(4,4)\}.$$

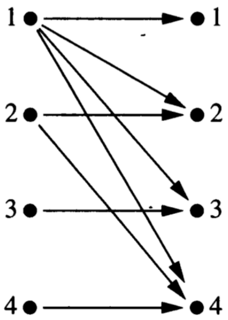
# Representing a Relation Using Directed Graph

## Example 2.14

Let $A = \{1,2,3,4\}$ and define $R = \{(a,b) | a \text{ divides } b\}$.

$$R = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(4,4)\}.$$

Using graph:



# Properties

1. Reflexive
2. Symmetric
3. Anti-symmetric
4. Transitive

# Properties

1. Reflexive: if $(a, a) \in R$ for all element $a \in A$.

# Properties

1. Reflexive: if $(a,a) \in R$ for all element $a \in A$.
2. Symmetric: if $(a,b) \in R$ implies that $(b,a) \in R$ for all $a,b \in A$

# Properties

1. Reflexive: if $(a, a) \in R$ for all element $a \in A$.
2. Symmetric: if $(a, b) \in R$ implies that $(b, a) \in R$ for all $a, b \in A$ .
3. Anti-symmetric: if, for all $a, b \in A, (a, b) \in R$ and $(b, a) \in R$ then $a = b$.

## Properties

1. Reflexive: if $(a, a) \in R$ for all element $a \in A$.
2. Symmetric: if $(a, b) \in R$ implies that $(b, a) \in R$ for all $a, b \in A$ .
3. Anti-symmetric: if, for all $a, b \in A, (a, b) \in R$ and $(b, a) \in R$ then $a = b$.
4. Transitive: if, whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, for all $a, b, c \in A$.

# Reflexive Relation

## Definition 2.15

A relation R on a set A is called reflexive if $(a, a) \in R$ for all element $a \in A$.

## Example 2.16

Let A be the set $\{1,2,3,4\}$ and R be the relation $R = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\}$. Is this relation transitive? If it is not, can we add more pairs to $R$ in order to get a transitive relation?

# Symmetric Relation

## Definition 2.17

A relation R on a set A is called symmetric if $(a, b) \in R$ implies that $(b, a) \in R$ for all $a, b \in A$.

## Example 2.18

Let A be the set $\{1, 2, 3, 4\}$ and R be the relation $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$. Is this relation symmetric? If it is not, can we add more pairs to $R$ in order to get a symmetric relation?

# Anti-Symmetric Relation

## Definition 2.19

A relation R on a set A is called anti-symmetric if, for all $a, b \in A, (a, b) \in R$ and $(b, a) \in R$ then $a = b$.

## Example 2.20

Let A be the set $\{1, 2, 3, 4\}$ and R be the relation $R = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\}$. Is this relation anti-symmetric? If it is not, can we add more pairs to $R$ in order to get a anti-symmetric relation?

# Transitive Relation

## Definition 2.21

A relation R on a set A is called transitive if, whenever $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$, for all $a,b,c \in A$.

## Example 2.22

Let A be the set $\{1,2,3,4\}$ and R be the relation $R = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\}$. Is this relation transitive? If it is not, can we add more pairs to $R$ in order to get a transitive relation?