DISCRETE STRUCTURES
Lecture 3. Relations

Bui Anh Tuan

Advanced Program in Computer Science

Fall, 2018

Content

1. Relations, their properties and representations
2. Equivalence relations

Definitions

Definition 2.1 (Cartesian product)

Let A and B be sets. The Cartesian product of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. ie.,

$$A \times B = \{(a, b) | a \in A \wedge b \in B\}.$$

Example 2.2

Let $A = \{0, 1, 2\}$ and $B = \{a, b\}$. Then

$$A \times B = \{(0, a), (0, b), (1, a), (1, b), (2, a), (2, b)\}.$$

Note: $(a, b), (c, d) \in A \times B,\ (a, b) = (c, d) \Leftrightarrow a = c \wedge b = d.$

Binary Relation

Definition 2.3

Let A and B be sets. A binary relation from A to B is a subset R of $A \times B$. A relation from a set A to itself is called a relation on A.

Example 2.4

Let $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$.
Consider $R = \{(a,3), (c,1), (c,2), (c,3)\} \subset A \times B$. Then R is a binary relation from A to B. Since $(a,3) \in R$, we say that
"a is related to 3" or "$aR3$"

Examples

Example 2.5

Let $A = \{1, 2, 3, 4\}$ and define

$$R = \{(a, b) | a \text{ divides } b\}.$$

Example 2.6

List the ordered pairs in the relation R from $A = \{0, 1, 2, 3, 4\}$ to $B = \{0, 1, 2, 3\}$, where $(a, b) \in R$ if and only if

a) a = b.
b) a + b = 4.
c) a > b.
d) gcd(a, b) = 1.

Number of Relations

Question

If A is a finite set with $|A| = n$, how many different relations are there on A?

1. How many subset of a set of n elements?
2. How many elements in $A \times A$?

Representing a Relation

There are several ways to represent a relation

1. Using table
2. Using matrix
3. Using directed graph

Representing a Relation Using Table

Example 2.7

Let $A = \{0, 1, 2\}$ and $B = \{a, b\}$. Then $R = \{(0, a), (0, b), (1, a), (2, b)\}$ is a relation from A to B.

R	a	b
0	×	×
1	×	
2		×

Representing a Relation Using Matrix

Example 2.8

Let $A = \{0, 1, 2\}$ and $B = \{a, b\}$. Then $R = \{(0, a), (0, b), (1, a), (2, b)\}$ is a relation from A to B.

Representing a Relation Using Matrix

Example 2.8

Let $A = \{0, 1, 2\}$ and $B = \{a, b\}$. Then $R = \{(0, a), (0, b), (1, a), (2, b)\}$ is a relation from A to B.

- Choose arbitrary order for A and B
- Elements of A is the index for row and elements of B is the index of column
- Define the matrix as:

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{otherwise} \end{cases}$$

Representing a Relation Using Matrix

Example 2.9

Let $A = \{0, 1, 2\}$ and $B = \{a, b\}$. Then $R = \{(0, a), (0, b), (1, a), (2, b)\}$ is a relation from A to B.

Representing a Relation Using Matrix

Example 2.9

Let $A = \{0, 1, 2\}$ and $B = \{a, b\}$. Then $R = \{(0, a), (0, b), (1, a), (2, b)\}$ is a relation from A to B.

$$M_R = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Note: Such a representation depends on the ordering used for A and B.

Representing a Relation Using Directed Graph

Definition 2.10 (Directed Graph)

A directed graph $G = (V, E)$, or digraph, consists of a set V of vertices (or nodes) together with a set E of edges (or arcs). The vertex a is called the initial vertex of the edge (a,b), and the vertex b is called the terminal vertex of this edge.

Representing a Relation Using Directed Graph

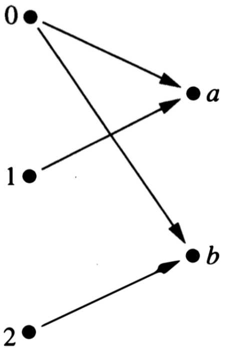
Example 2.11

Let $A = \{0, 1, 2\}$ and $B = \{a, b\}$. Then $R = \{(0, a), (0, b), (1, a), (2, b)\}$ is a relation from A to B.

Representing a Relation Using Directed Graph

Example 2.11

Let $A = \{0, 1, 2\}$ and $B = \{a, b\}$. Then $R = \{(0, a), (0, b), (1, a), (2, b)\}$ is a relation from A to B.



Representing a Relation Using Table

Example 2.12

Let $A = \{1,2,3,4\}$ and define $R = \{(a,b)|a \text{ divides } b\}$.

$$R = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(4,4)\}.$$

Representing a Relation Using Table

Example 2.12

Let $A = \{1,2,3,4\}$ and define $R = \{(a,b) | a \text{ divides } b\}$.

$$R = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(4,4)\}.$$

Using table:

R	1	2	3	4
1	×	×	×	×
2		×		×
3			×	
4				×

Representing a Relation Using Matrix

Example 2.13

Let $A = \{1,2,3,4\}$ and define $R = \{(a,b) | a \text{ divides } b\}$.

$$R = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(4,4)\}.$$

Using matrix:

Representing a Relation Using Matrix

Example 2.13

Let $A = \{1,2,3,4\}$ and define $R = \{(a,b) | a \text{ divides } b\}$.

$$R = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(4,4)\}.$$

Using matrix:

$$M_R = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Representing a Relation Using Directed Graph

Example 2.14

Let $A = \{1, 2, 3, 4\}$ and define $R = \{(a, b) | a \text{ divides } b\}$.

$$R = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(4,4)\}.$$

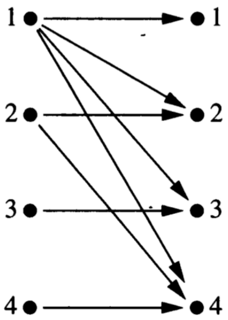
Representing a Relation Using Directed Graph

Example 2.14

Let $A = \{1,2,3,4\}$ and define $R = \{(a,b) | a \text{ divides } b\}$.

$$R = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(4,4)\}.$$

Using graph:



Properties

1. Reflexive
2. Symmetric
3. Anti-symmetric
4. Transitive

Properties

1. Reflexive: if $(a, a) \in R$ for all element $a \in A$.

Properties

1. Reflexive: if $(a, a) \in R$ for all element $a \in A$.
2. Symmetric: if $(a, b) \in R$ implies that $(b, a) \in R$ for all $a, b \in A$

Properties

1. Reflexive: if $(a,a) \in R$ for all element $a \in A$.
2. Symmetric: if $(a,b) \in R$ implies that $(b,a) \in R$ for all $a,b \in A$.
3. Anti-symmetric: if, for all $a,b \in A, (a,b) \in R$ and $(b,a) \in R$ then $a = b$.

Properties

1. Reflexive: if $(a,a) \in R$ for all element $a \in A$.
2. Symmetric: if $(a,b) \in R$ implies that $(b,a) \in R$ for all $a,b \in A$.
3. Anti-symmetric: if, for all $a,b \in A, (a,b) \in R$ and $(b,a) \in R$ then $a = b$.
4. Transitive: if, whenever $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$, for all $a,b,c \in A$.

Reflexive Relation

Definition 2.15

A relation R on a set A is called reflexive if $(a, a) \in R$ for all element $a \in A$.

Example 2.16

Let A be the set $\{1, 2, 3, 4\}$ and R be the relation $R = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\}$. Is this relation reflexive? If it is not, can we add more pairs to R in order to get a transitive relation?

Symmetric Relation

Definition 2.17

A relation R on a set A is called symmetric if $(a,b) \in R$ implies that $(b,a) \in R$ for all $a,b \in A$.

Example 2.18

Let A be the set $\{1,2,3,4\}$ and R be the relation $R = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\}$. Is this relation symmetric? If it is not, can we add more pairs to R in order to get a symmetric relation?

Anti-Symmetric Relation

Definition 2.19

A relation R on a set A is called anti-symmetric if, for all $a, b \in A, (a, b) \in R$ and $(b, a) \in R$ then $a = b$.

Example 2.20

Let A be the set $\{1, 2, 3, 4\}$ and R be the relation $R = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\}$. Is this relation anti-symmetric? If it is not, can we add more pairs to R in order to get a anti-symmetric relation?

Transitive Relation

Definition 2.21

A relation R on a set A is called transitive if, whenever $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$, for all $a,b,c \in A$.

Example 2.22

Let A be the set $\{1,2,3,4\}$ and R be the relation $R = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\}$. Is this relation transitive? If it is not, can we add more pairs to R in order to get a transitive relation?

Exercises

1. Determine whether the relations represented by the matrices below are reflexive, irreflexive, symmetric, antisymmetric, and/or transitive.

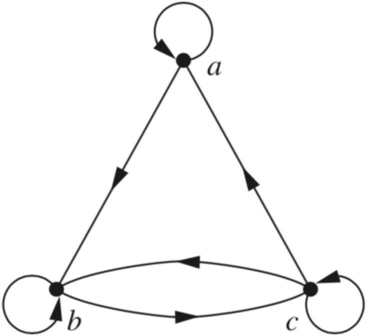
a) $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

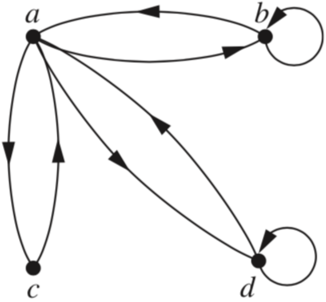
c) $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

Exercises

2. Determine whether the relations for the directed graphs shown in Figures are reflexive, sym- metric, antisymmetric, and/or transitive.



(a) Directed graph of R



(b) Directed graph of S