DISCRETE STRUCTURES
Lecture 3. Relations

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Advanced Program in Computer Science

Fall, 2018

Content

1. Combining relations
2. Equivalence relations

Combining Relations

Let R and R be relations from A to B, then the following set operations are defined as usual:

1. $R \cup S = \{x | x \in R \vee x \in S\}$
2. $R \cap S = \{x | x \in R \wedge x \in S\}$
3. $R - S = \{x | x \in R \wedge x \notin S\}$
4. $R \oplus S = R \cup S - R \cap S$

Combining Relations

Example 2.1

Let A = {students}   B = {courses}

- R = {(a,b): student a takes the course b}
- S={(a,b): student a requires course b to graduate}

Then

- $R \cap S$ ={(a,b): student a takes course b and require course b to graduate}
- $R - S$ ={(a,b): student a takes course b but does not need course b to graduate (b is an elective course)}

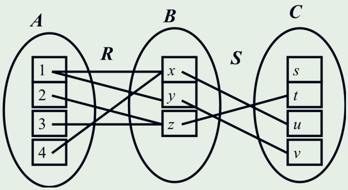
Composite

Definition 2.2

Let R be a relation from A to B and S a relation from B to C. Their composite is defined as :

$$S \circ R = \{(a,c) \in A \times C, \exists b \in B : (a,b) \in R \wedge (b,c) \in S\}.$$

Example 2.3



$$S \circ R = \{(1,u),(1,v),(2,t),(3,t),(4,u)\}$$

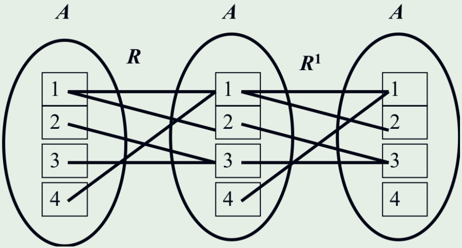
Power

Definition 2.4

Let R be a relation on A. The powers R^n are defined recursively by:

$$R^1 = R \qquad \text{and} \qquad R^{n+1} = R^n \circ R$$

Example 2.5



$R^2 = R \circ R = \{(1,1),(1,2),(1,3),(2,3),(3,3),(4,1),(4,2)\}$

Definitions

Definition 3.1

A relation R on a set A is called an equivalence relation if it is reflexive, symmetric, and transitive.

Definition 3.2

Two elements a and b that are related by an equivalence relation are called equivalent. The notation $a \equiv b$ is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation.

Example 3.3

Let R be the relation on the set of real numbers such that aRb if and only if $a-b$ is an integer. Is R an equivalence relation?

Example: Modulo n

Let a an integer and m a positive integer with $m > 1$. The notation a mod m is the remainder when a is divided by m.

Definition 3.4

If a and b are integers and m is a positive integer, then a is congruent to b modulo m if m divides $a - b$. Notation: $a \equiv b$ (mod m).

Theorem 3.5

Let a and b be integers, and let m be a positive integer. Then $a \equiv b$ (mod m) if and only if a and b have the same remainder when divided by m, i.e., a mod m = b mod m.

Example: Modulo n

Theorem 3.6

Let m be a positive integer with $m > 1$. *The relation*

$$R = \{(a, b) | a \equiv b (mod \ m)\}$$

is an equivalence relation.

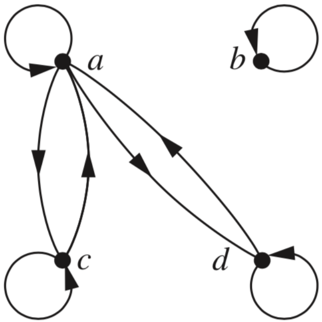
Example

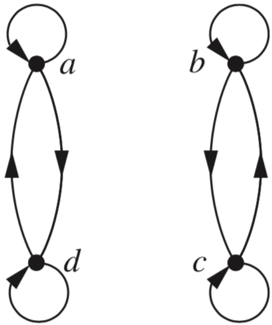
Example 3.7

Let $A = \mathbb{Z}$ and $R = \{(a, b) | a \text{ divides } b\}$.
R is not an equivalence relation because it is not symmetric.

Exercises

1. Determine whether the relation with the directed graph shown is an equivalence relation.





Exercises

2. Determine whether the relation with the representing matrix shown is an equivalence relation.

a) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ **b)** $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ **c)** $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Equivalence classes

Definition 3.8

Let R be an equivalence relation over a set A. The set of all elements that are related to an element x of A is called the equivalence class of x.
The equivalence class of x with respect to R is denoted as $[x]_R$. When only one relation is under consideration, we will use just [x] to denote the equivalence class of a with respect to R.

Note:

- $[x]_R = \{y \in A | x \equiv y\}$
- if $b \in [x]_R$ then b is called a representative of class $[x]_R$.

Equivalence classes

Exercise

Let A = 1; 2; 3; 5; 6; 10; 11; 12 and R be a relation defined on A by

$$R = \{(a; b) | a - b \text{ is divisible by} 4\}$$

a) Show that R is an equivalence relation.
b) Write down the equivalence class [2], [6]

Equivalence classes

Theorem 3.9

Let R be an equivalence relation on a set A. These statements for elements a and b of A are equivalent:

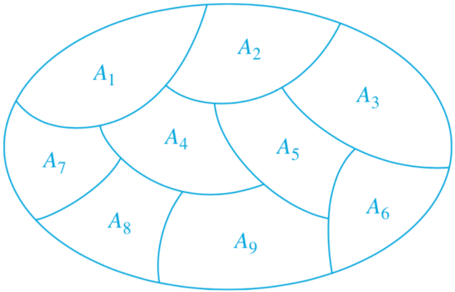
1. aRb
2. $[a] = [b]$
3. $[a] \cap [b] = \emptyset$

Partition

Definition 3.10

A partition of a set S is a collection of disjoint nonempty subsets of S that have S as their union. In other words, the collection of subsets $A_i,\ i \in I$ (where I is an index set) forms a partition of S if and only if

1. $A_i \neq \emptyset \quad \forall i \in I$
2. $A_i \cap A_j = \emptyset$, for $i \neq j$
3. $\bigcup_{i \in I} A_i = S$



Partition

Theorem 3.11

Let R be an equivalence relation on a set S. Then the equivalence classes of R form a partition of S. Conversely, given a partition $\{A_i | i \in I\}$ *of the set S , there is an equivalence relation R that has the sets* A_i, $i \in I$ *, as its equivalence classes.*

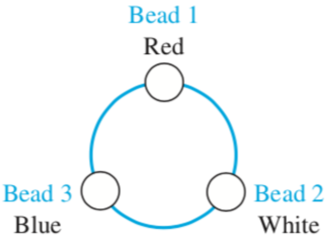
Exercises

Exercise 1

What are the sets in the partition of the integers arising from congruence modulo 2?

Exercise 2

Each bead on a bracelet with three beads is either red, white, or blue, as illustrated in the figure shown.



Define the relation R between bracelets as: (B_1, B_2), where B_1 and B_2 are bracelets, belongs to R if and only if B_2 can be obtained from B_1 by rotating it or rotating it and then reflecting it.

a) Show that R is an equivalence relation.

b) What are the equivalence classes of R?

Exercises

Exercise 3

Let R be the relation on the set of all colorings of the 2×2 checkerboard where each of the four squares is colored either red or blue so that (C1 , C2), where C1 and C2 are 2×2 checkerboards with each of their four squares colored blue or red, belongs to R if and only if C2 can be obtained from C1 either by rotating the checkerboard or by rotating it and then reflecting it.
a) Show that R is an equivalence relation.
b) What are the equivalence classes of R?