DISCRETE STRUCTURES
Lecture 1. Propositional logic and equivalence

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Contents

Review

Construct the truth table of the compound propositions:

$$(p \oplus q) \wedge \neg r \to p$$

Precedence of Logical Operators

Operators	Precedence
$\neg$	1
$\wedge$	2
$\vee$	3
$\to$	4
$\leftrightarrow$	5

Logical Equivalence

Definition 2.1

The propositions p and q are logically equivalent if they have the same truth tables. We also write $p \equiv q$.

Example 2.2

Using truth table, prove that the below propositions are logically equivalence.

$$\neg p \vee q \text{ and } p \to q$$

Tautology and Contradiction

Definition 2.3

1. A tautology (denoted by T) is a compound proposition that is always true.
2. A contradiction (denoted by F) is a compound proposition that is always false.

Thus p and q are logical equivalent if and only if $p \leftrightarrow q$ is a tautology.

Table: Truth tables for $p \wedge \neg p$ and $p \vee \neg p$

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

Then, $p \wedge \neg p$ is a contradiction and $p \vee \neg p$ is tautology.

Tautology and Contradiction

Example 2.4

Check if the following is tautology or contradiction:

1. $(p \wedge (p \rightarrow q)) \rightarrow q$
2. $p \wedge (\neg p \rightarrow q) \rightarrow r$

De Morgan's Laws

De Morgan's Law 1

The compound propositions $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent.

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

De Morgan's Law 2

The compound propositions $\neg(p \wedge q)$ and $\neg p \vee \neg q$ are logically equivalent.

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

Example 2.5

Using De Morgan's laws to find the negation of the following sentences:

- a) Jan is rich and happy.
- b) Mei walks or takes the bus to class.

Logical Equivalences

Equivalence	Name
$p \wedge T \equiv p$ $p \vee F \equiv p$	Identity laws
$p \vee T \equiv T$ $p \wedge F \equiv F$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double Negation law
$p \vee \neg p \equiv T$ $p \wedge \neg p \equiv F$	Negation laws

Logical Equivalences (continued)

Equivalence	Name
$p \vee q \equiv q \vee q$ $p \wedge q \equiv q \wedge q$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws

Logical Equivalences Involving Implications

$$p \to q \equiv \neg p \lor q$$

$$p \to q \equiv \neg q \to \neg p \text{ (contrapositive)}$$

$$p \lor q \equiv \neg p \to q$$

$$p \land q \equiv \neg(p \to \neg q)$$

$$\neg(p \to q) \equiv p \land \neg q$$

$$(p \to q) \land (p \to r) \equiv p \to (q \land r)$$

$$(p \to r) \land (q \to r) \equiv (p \lor q) \to r$$

$$(p \to q) \lor (p \to r) \equiv p \to (q \lor r)$$

$$(p \to r) \lor (q \to r) \equiv (p \land q) \to r$$

Logical Equivalences Involving Bi-Implications

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

$$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$$

$$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$$

$$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$$

Disjunctive Normal Form

Definition 2.6

A compound proposition is said to be in disjunctive normal form if it is a disjunction of conjunctions of the variables or their negations.

Example 2.7

The compound proposition

$$(p \wedge q \wedge r) \vee (\neg p \wedge q \neg r) \vee (\neg p \wedge \neg q \wedge r)$$

Logical Equivalences

Question: How to prove two propositions are equivalent?

Two methods:

1. Using truth tables:
   - Not good for long formula
2. Using the logical equivalences
   - The preferred method.

Logically Equivalences

Example 2.8

Show that $(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

1. Using the truth table
2. Using the logical equivalences

$$
\begin{aligned}
(p \to r) \vee (q \to r) &\equiv (\neg p \vee r) \vee (\neg q \vee r) && \text{Implication} \\
&\equiv \neg p \vee r \neg q \vee r && \text{Associative law} \\
&\equiv \neg p \vee \neg q \vee r \vee r && \text{Commutative law} \\
&\equiv (\neg p \vee \neg q) \vee (r \vee r) && \text{Associative law} \\
&\equiv \neg (p \wedge q) \vee r && \text{De Morgan and Idempotent} \\
&\equiv (p \wedge q) \to r
\end{aligned}
$$

Exercises

1. Show that $(\neg p \rightarrow r) \wedge (q \rightarrow r)$ and $(p \rightarrow q) \rightarrow r$ are logically equivalent.
2. Show that $(p \wedge q) \rightarrow r$ and $(p \rightarrow r) \wedge (q \rightarrow r)$ are not logically equivalent.