DISCRETE STRUCTURES
Lecture 4. Partial Orderings

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Content

1. Maximal and Minimal Elements
2. Topological Sorting
3. Algorithms

Minimal and Maximal Elements

Definition 1.1

An element a is maximal in the poset $(S, \preccurlyeq)$ if there is no element $b \in S$ such that $a \preccurlyeq b$.

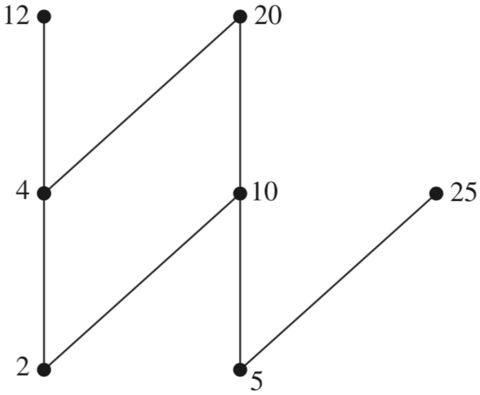
In other words, an element of a poset is called maximal if it is not less than any *comparable* element of the poset.

Definition 1.2

An element a is minimal in the poset $(S, \preccurlyeq)$ if there is no element $b \in S$ such that $b \preccurlyeq a$.

In other words, an element of a poset is called minimal if it is not greater than any *comparable* element of the poset.

Minimal and Maximal Elements



1. 2 and 5 are minimal elements.
2. 12, 20 and 25 are maximal elements.
3. The minimal and the maximal elements may not be unique.

Greatest and Least Elements

Definition 1.3

An element a is greatest element in the poset $(S, \preccurlyeq)$ if $b \preccurlyeq a$ for all $b \in S$. The greatest element is unique when it exists.

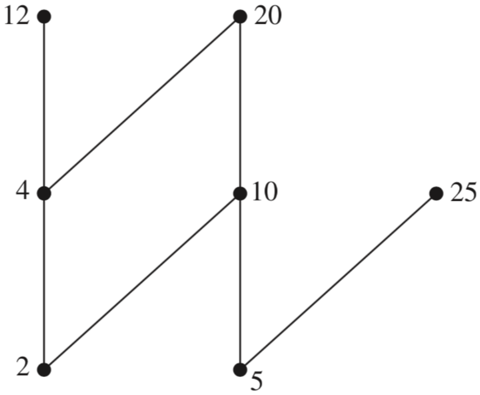
In other words, an element of a poset is called greatest element if it is greater than every other elements of S.

Definition 1.4

An element a is least element in the poset $(S, \preccurlyeq)$ if $a \preccurlyeq b$ for all $b \in S$. The least element is unique when it exists.

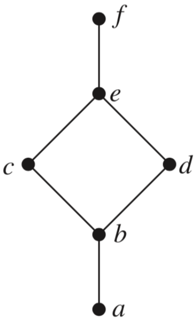
In other words, an element of a poset is called least element if it is less than every other elements of S.

Greatest and Least Elements



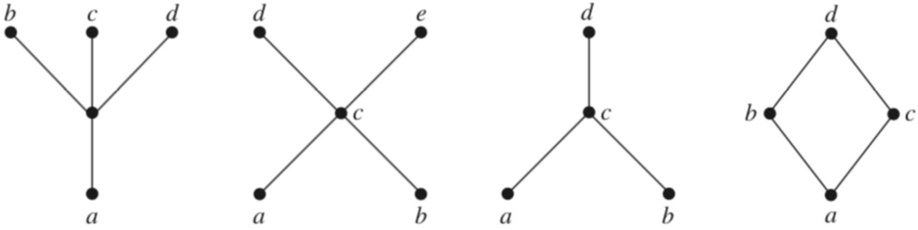
1. There is no greatest element.
2. There is no least element.

Greatest and Least Elements



1. f is greatest element.
2. a least element.
3. greatest element and least element are unique.

Examples

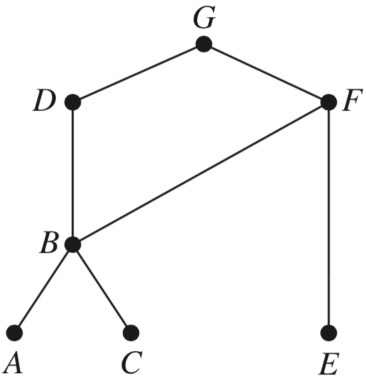


(a) (b) (c) (d)

Suppose that a project is made up of 20 different tasks. Some tasks can be completed only after others have been finished. How can an order be found for these tasks? To model this problem we set up a partial order on the set of tasks so that $a \prec b$ if and only if a and b are tasks where b cannot be started until a has been completed. To produce a schedule for the project, we need to produce an order for all 20 tasks that is compatible with this partial order. We will show how this can be done.

Example

A development project at a computer company requires the completion of seven tasks. Some of these tasks can be started only after other tasks are finished. A partial ordering on tasks is set up by considering task X $\prec$ task Y if task Y cannot be started until task X has been completed. The Hasse diagram for the seven tasks, with respect to this partial ordering, is shown in Figure. Find an order in which these tasks can be carried out to complete the project.



Definition 1.5

A total ordering $\preccurlyeq$ is said to be compatible with the partial ordering R if $a \preccurlyeq b$ whenever aRb.
Constructing a compatible total ordering from a partial ordering is called topological sorting.

Lemma 1.6

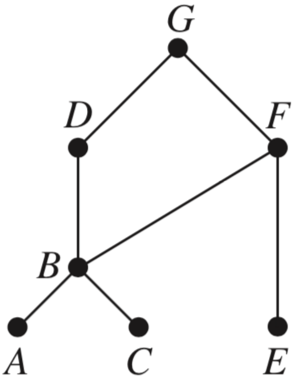
Every finite nonempty poset $(S, \preccurlyeq)$ has at least one minimal element.

Kahn's Algorithm

ALGORITHM 1 Topological Sorting.

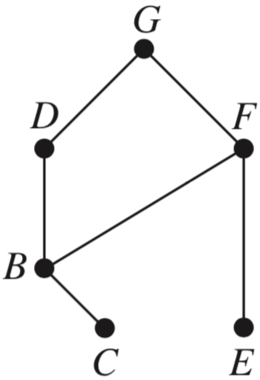
procedure *topological sort* $((S, \preccurlyeq)$: finite poset)
$k := 1$
while $S \neq \emptyset$
    $a_k := $ a minimal element of S {such an element exists by Lemma 1}
    $S := S - \{a_k\}$
    $k := k + 1$
return $a_1, a_2, \ldots, a_n$ $\{a_1, a_2, \ldots, a_n$ is a compatible total ordering of $S\}$

Example



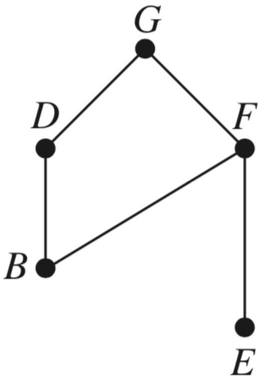
- Minimal element chosen: A
- Total Order Set: A

Example



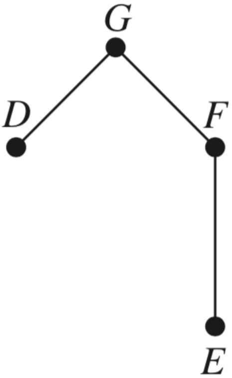
- Minimal element chosen: C
- Total Order Set: $A \prec C$

Example



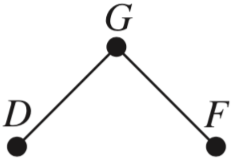
- Minimal element chosen: B
- Total Order Set: $A \prec C \prec B$

Example



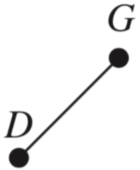
- Minimal element chosen: E
- Total Order Set: $A \prec C \prec B \prec E$

Example



- Minimal element chosen: F
- Total Order Set: $A \prec C \prec B \prec E \prec F$

Example



- Minimal element chosen: D
- Total Order Set: $A \prec C \prec B \prec E \prec F \prec D$

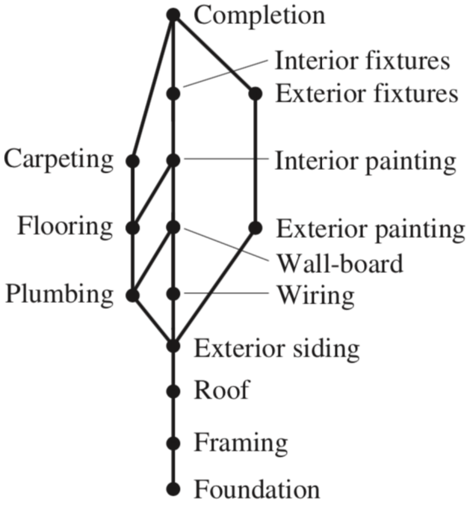
Example

- Minimal element chosen: G
- Total Order Set: $A \prec C \prec B \prec E \prec F \prec D \prec G$

Topological Sorting Using Indegree Method

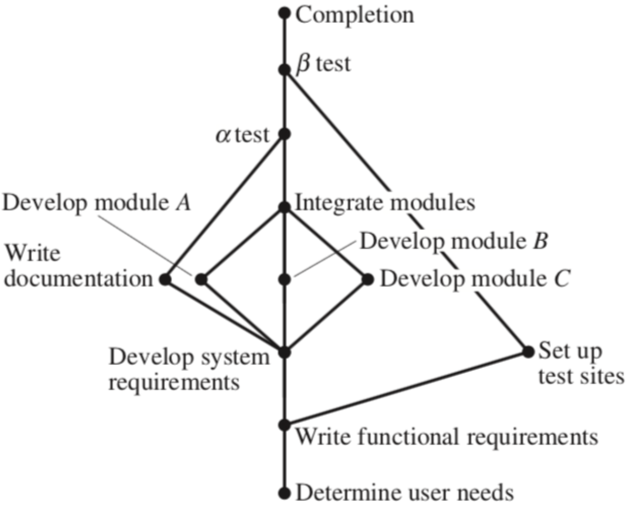
https://www.cs.usfca.edu/ galles/visualization/TopoSortIndegree.html

Q1. Schedule the tasks needed to build a house, by specifying their order, if the Hasse diagram representing these tasks is as shown in the figure.



Exercises

Q2. Find an ordering of the tasks of a software project if the Hasse diagram for the tasks of the project is as shown.



Topological Sorting Using DFS

https://www.cs.usfca.edu/ galles/visualization/TopoSortDFS.html

Implementations

1. Kahn's Algorithm
2. DFS