# DISCRETE STRUCTURES
# Lecture 6. Mathematical Induction

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# Content

# Examples

Many mathematical statements assert that a property is true for all positive integers.

Examples of such statements are that for every positive integer $n$:

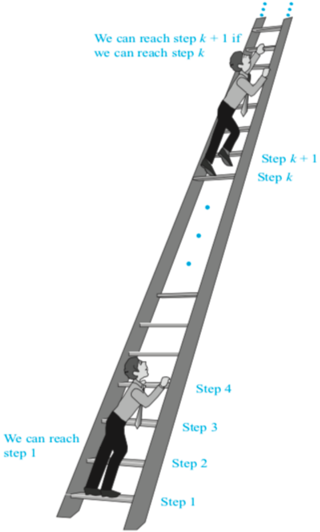
1. $n! \leq n^n$
2. $n^3 - n$ is divisible by 3
3. a set with $n$ elements has $2^n$ subsets
4. sum of the first $n$ positive integers is $n(n+1)/2$

One method of proving those statements is Mathematical Induction.

# Induction

Suppose that we have an infinite ladder and we want to know whether we can reach every step on this ladder. The idea for mathematical induction are:

1. We can reach the first rung of the ladder
2. If we can reach a particular rung of the ladder, then we can reach the next rung



# Induction

## Principle of mathematical induction

To prove that $P(n)$ is true for all positive integers $n$, where $P(n)$ is a propositional function, we complete two steps:

- **Basis step:** We verify that $P(1)$ true.
- **Inductive step:** We show that the conditional statement $P(k) \to P(k+1)$ is true for all positive integers $k$.

Expressed as a rule of inference, this proof technique can be stated as

$$(P(1) \wedge \forall k(P(k) \to P(k+1))) \to \forall n P(n)$$

where the domain is the set of all positive integers.

**Note:** The assumption that $P(k)$ is true is called the inductive hypothesis.

# Why Mathematical Induction is Valid?

## Well-ordering property

Every nonempty subset of the set of positive integers has a least element.

- Suppose that $P(1)$ and the implication $P(k) \rightarrow P(k+1)$ is true for all positive $k$.
- Let $S$ be the set of all positive integers $n$ for which $P(n)$ is false. Suppose that $S$ is non-empty.
- $S$ has a least element, say $m$, by the well-ordering property. And hence $m-1 \notin S$. Means $P(m-1)$ is true. And then by the assumption, $P(m)$ is true. This is a contradiction.

# Induction: proving summation formulae

## Example 1.1

Show that if $n$ is a positive integer, then

$$1+2+\cdots+n=\frac{n(n+1)}{2}$$

Let $P(n)$ be the proposition that the sum of the first $n$ positive integers, $1+2+\cdots+n=\frac{n(n+1)}{2}$.

We must do:

1. Prove the basis step, $P(1)$ is true.
2. Prove the inductive step, conditional statement $P(k)$ implies $P(k+1)$ for any arbitrary $k$.

# What is bad about Mathematical Induction?

1. It can not find a new theorem.
2. Do not provide insights of the problem.

# Conjecture and Proof

**Question:** Formula for the sum of first $n$ positive odd integers?

## Conjecture

$$1 + 3 + \cdots + (2n - 1) = n^2.$$

## Proof.

By mathematical induction. $\square$

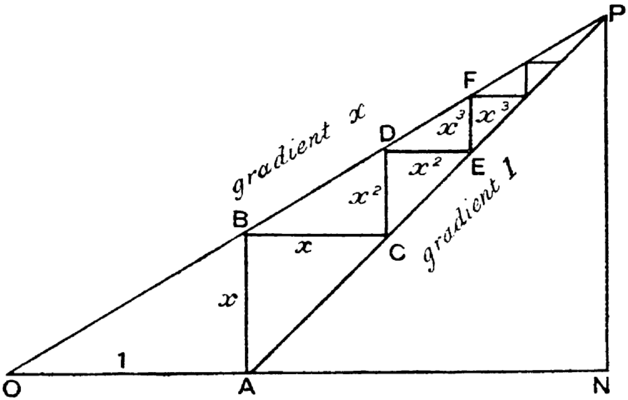
# Sums of Geometric Progressions

Use mathematical induction to prove this formula for the sum of a finite number of terms of a geometric progression with initial term a and common ratio $r$:

$$\sum_{j=0}^{n} ar^j = a + ar + ar^2 + \cdots + ar^n = \frac{ar^{n+1} - a}{r - 1}, \quad \text{when } r \neq 1,$$

where $n$ is a nonnegative integer.

# Why "geometric"?



# Induction: proving inequalities

## Example 1.2

Use mathematical induction to prove the inequality

$$n < 2^n$$

for all positive integers $n$.

1. Let $P(n)$ be the proposition that $n < 2^n$
2. Basis step ...
3. Inductive step ...

# Inequality for Harmonic Numbers

The harmonic numbers $H_j$ defined by

$$H_j = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{j}$$

Use mathematical induction to show that

$$H_{2^n} \geq 1 + \frac{n}{2}$$

whenever $n$ is nonnegative integer.

**Note:** This result shows that the harmonic series is a divergent infinite series.

# Exercises

**Q1.** Prove that if $h > -1$, then $1 + nh \leq (1+h)^n$ for all nonnegative integers $n$. This is called Bernoulli's inequality.

**Q2.** Suppose that $a$ and $b$ are real numbers with $0 < b < a$. Prove that if $n$ is a positive integer, then $a^n - b^n \leq na^{n-1}(a-b)$.

**Q3.** Prove that $H_{2^n} \leq 1 + n$ whenever $n$ is a nonnegative integer.

**Q4.** Prove that $H_1 + H_2 + \cdots + H_n = (n+1)H_n - n$.