2.1.11, 19, 2.2.15, 2.3.5, 16, 2.4.14, 2.5.5

2.1.11 Given: Newton-Raphson method for $\sqrt{a}$
① Initial guess $= r_0$
② $r_{n+1} = \frac{1}{2}(r_n + a/r_n)$

Find: a) If $r_n < \sqrt{a}$, then $r_{n+1} > \sqrt{a}$
b) If $r_n > \sqrt{a}$, then $\sqrt{a} < r_{n+1} < r_n$
c) Let $f(x) = \frac{1}{2}(x + a/x) \Rightarrow r_{n+1} = f(r_n)$.
Show $0 < f'(x) < \frac{1}{2}$ if $x > \sqrt{a}$.
d) The Law of the mean $f(x) - \sqrt{a} = f(x) - f(\sqrt{a}) = f'(b)(x - \sqrt{a})$ for some b between x and $\sqrt{a}$. Show for $n \geq 1$, $0 < r_{n+1} - \sqrt{a} < \frac{1}{2}(r_n - \sqrt{a})$.
e) Show $0 < r_{n+1} - \sqrt{a} < (\frac{1}{2})^n (r_1 - \sqrt{a})$, proving $r_n \to \sqrt{a}$

Assumptions: none

Solution: a) let $r_n = \varepsilon\sqrt{a}$ where $0 < \varepsilon < 1$.
$\Rightarrow r_{n+1} = \frac{1}{2}(\varepsilon\sqrt{a} + a/\varepsilon\sqrt{a}) = \sqrt{a}\cdot\frac{1}{2}(\varepsilon + 1/\varepsilon)$
$\frac{1}{2}(\varepsilon + 1/\varepsilon) > 1 \quad \forall \varepsilon \Rightarrow r_{n+1} > \sqrt{a}$

b) let $r_n = c\sqrt{a}$ where $c > 1$.
$r_{n+1} = \frac{1}{2}(c\sqrt{a} + a/c\sqrt{a}) = \sqrt{a}\cdot\frac{1}{2}(c + 1/c)$

$\frac{1}{2}(c + \frac{1}{c})$ is monotonically increasing and $> 1 \ \forall c$.
$\Rightarrow r_{n+1} > \sqrt{a}$

$\frac{1}{2}(c + \frac{1}{c}) < c \quad \forall c \Rightarrow r_{n+1} < r_n$

$\therefore \sqrt{a} < r_{n+1} < r_n$

c) $f'(x) = \frac{1}{2}(1 - a/x^2)$

For $x > \sqrt{a}$, $a/x^2 < 1 \Rightarrow 1 - a/x^2 > 0$ and

$a/x^2 > 0 \Rightarrow 1 - a/x^2 < 1 \Rightarrow 0 < f'(x) < \frac{1}{2}$.

d) By parts (a) and (b), for any $n > 1$

$\Rightarrow \sqrt{a} < r_{n+1} < r_n \Rightarrow r_{n+1} - \sqrt{a} > 0$

By the Law of the mean,
$f(r_n) - \sqrt{a} = f(r_n) - f(\sqrt{a}) = f'(b)(r_n - \sqrt{a})$

$< \frac{1}{2}(r_n - \sqrt{a})$ for some $\sqrt{a} < b < r_n$

$\Rightarrow 0 < r_{n+1} - \sqrt{a} = f(r_n) - \sqrt{a} < \frac{1}{2}(r_n - \sqrt{a})$

e) $0 < r_2 - \sqrt{a} < \frac{1}{2}(r_1 - \sqrt{a})$
$0 < r_3 - \sqrt{a} < \frac{1}{2}(r_2 - \sqrt{a}) < \frac{1}{2}(\frac{1}{2}(r_1 - \sqrt{a}))$
$\Rightarrow 0 < r_{n+1} - \sqrt{a} < (\frac{1}{2})^n (r_1 - \sqrt{a}) \to 0$ as $n \to \infty$
$\therefore r_{n+1} \to \sqrt{a}$

2.1.19 | Given: Newton-Raphson method

Find: $\sqrt{5}$ using $r_0 = 5$ and as many iterations as needed to get 3, 4, 5, ... decimal places of accuracy.
Find pattern for # decimal places of accuracy based on # of iterations.
Check pattern with $\sqrt{50}$, $\sqrt{500}$

Assumptions: none

Solution: $r_0 = 5$

Using R: $\sqrt{5} \approx 2.236067977499789805051$

see HW3.R

$r_1 = 3$

options(digits = 22) max is 22 but macheps?

$r_2 = 2.\bar{3}$

getOption("digits")

$r_3 = 2.238$ (2 digits)

$r_4 = 2.236068$ (5)

$r_5 = 2.236067977499$ (12)

$r_6 = 2.236067977499789805051$ (>21, macheps?)

for $\sqrt{50}$

5 iterations → 2
6 iterations → 6
7 iterations → 14
8 → >21

for $\sqrt{500}$

7 → 2
8 → 7
9 → >21
10

2.2.15 | Given: $N(\mu, \sigma)$: $y = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$

Find: Verify area between $x = \mu - \sigma$ and $x = \mu + \sigma$ is 0.68, with $\mu = 0$ and $\sigma = 5$.
Use enough approximating rectangles to get close.

Assumptions: none

Solution: See HW3.R Midpoint rule w/ n rectangles

$$\Delta x = \frac{b-a}{n}, \quad \bar{x}_k = a + (k - \tfrac{1}{2})\Delta x$$

$$\text{Area} \approx \sum_{k=1}^{n} f(\bar{x}_k)\Delta x \quad \Rightarrow 6 \text{ rectangles} \Rightarrow \sim 0.68$$

2.3.5| Given: Data on population of 15-24 year olds in 1930. Suggestion that figures are *higher* for 21 y.o. males and 18 y.o. females.

Find: a) Cite evidence for or against overreporting.

b) Choose best method to modify data and why:
1. moving average w/ central window of size 3
2. Grouping in 2 yr age groups
3. " " 5 yr

c) Is there a better method?

Assumptions: none

Solution: a) There appears to be significant change in the trend of the pops through these 2 age groups. This alone may not be enough evidence but the ratios of males to females bolsters the claim.

b) The moving average seems best. The 2 year age groups would put 19 and 20 y.o. males together and still likely underreport. The 5 year age groups would eliminate most of the resolution.

c) A better method might be to use interpolation as in example 4, fitting a quadratic using 19 and 22 y.o. males with the sum of 20 and 21 y.o. males and 16 and 19 y.o. females with the sum of 17 and 18 y.o. females.

2.3.16| Given: Data in Table 3: and model $d = \frac{c}{p}$

p	d
9	1200
10	1000
11	975

Least squares method $c = \frac{\sum d_i / p_i}{\sum 1/p_i^2}$

Find: Write computer algorithm to compute c.

Assumptions: none

Solution: See HW3.R

$\Rightarrow$ $c = 10518$

<u>2.4.14</u> <u>Given</u>: Weighted voting game with
n number of players
w the weight for each player
l the line up for a vote

<u>Find</u>: Write code to determine all crucial and semicrucial voters.

<u>Assumptions</u>: none

<u>Solution</u>: See HW3.R

<u>2.5.5</u> <u>Given</u>: Data of Figure 1 which plots atomic volume against atom weight of elements.

<u>Find</u>: Would it make sense to adjust data? Justify.

<u>Assumptions</u>: none

<u>Solution</u>: It would not make sense to adjust the data here since we would be looking for specific trends in the characteristics at a very detailed level. The measurements are not likely caused by error that is not biased (systematic error). The math methods here deal with random error.