K-SAMPLE ANALOGUES OF THE KOLMOGOROV-SMIRNOV AND CRAMÉR-V. MISES TESTS

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0. Summary. The main purpose of this paper is to obtain the limiting distribution of certain statistics described in the title. It was suggested by the author in [1] that these statistics might be useful for testing the homogeneity hypothesis H_1 that k random samples of real random variables have the same continuous probability law, or the goodness-of-fit hypothesis H_2 that all of them have some specified continuous probability law. Most tests of H_1 discussed in the existing literature, or at least all such tests known to the author before [1] in the case $k > 2$, have only been shown to have desirable consistency or power properties against limited classes of alternatives (see e.g., [2], [3], [4] for lists of references on these tests), while those suggested here are shown to be consistent against all alternatives and to have good power properties. Some test statistics whose distributions can be computed from known results are also listed.

1. Introduction. Let X_{ji} be independent random variables ($1 \leqq i \leqq n_j$, $1 \leqq j \leqq k$), X_{ji} having unknown continuous distribution function (d.f.)F_j. We are going to consider tests of two hypotheses, the homogeneity hypothesis

$$H_1: F_1 = F_2 = \cdots = F_k \tag{1.1}$$

and the goodness-of-fit hypothesis

$$H_2: F_1 = F_2 = \cdots = F_k = G, \tag{1.2}$$

where G is some specified continuous d.f. In the case of H_1, the hypothesis allows the common unknown d.f. to be any continuous d.f. The class of alternatives to H_1 or H_2 can be considered to be all sets $(F_1, \cdots, F_k)$ which violate (1.1) or (1.2), respectively; in discussing power under alternatives, continuity of the F_i is irrelevant.

Let

$$S_{n_j}^{(j)}(x) = n_j^{-1} \quad (\text{number of } X_{ji} \leqq x,\ 1 \leqq i \leqq n_j)$$

be the sample d.f. of the n_j observations in the jth set. We shall omit the subscript n_j whenever this causes no confusion. For $k = 1$ the Kolmogorov test [5] and Cramér-v. Mises ω^2 test [6] of H_2, and for $k = 2$ the Smirnov test [7] and the 2-sample analogue of the ω^2 test of H_1 considered by Lehmann [8] and Rosenblatt [9], may be thought of as test criteria based on simple measurements of distance between $S^{(1)}$ and G or between $S^{(1)}$ and $S^{(2)}$, respectively. (In this

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paper, the word "distance" is not used in the technical sense; see [23], following (5.1).) In [1], several analogous measurements of distance (dispersion) among the $S^{(j)}$ were suggested for testing H_1 or H_2 when k is larger than 2. For example, for testing H_1, some of the most obvious analogues are

$$U = \sum_{q,r} \sup_x C_{q,r} \mid S^{(q)}(x) - S^{(r)}(x)\mid,$$

$$V = \sup_{q,r,x} C_{q,r} \mid S^{(q)}(x) - S^{(r)}(x)\mid,$$

$$T = \sup_x \sum_j C_j[S^{(j)}(x) - \bar{S}(x)]^2,$$

$$W = \int_{-\infty}^{\infty} \sum_j C_j[S^{(j)}(x) - \bar{S}(x)]^2 \, d\bar{S}(x),$$

$$Z = \max_j \int_{-\infty}^{\infty} C_j[S^{(j)}(x) - \bar{S}(x)]^2 \, d\bar{S}(x),$$

where $C_{q,r}$ and C_j are positive constants (see, however, the next paragraph) and $\bar{S}(x) = \sum_j n_j S_{n_j}^{(j)}(x)/\sum_j n_j$ is the sample d.f. of the pooled k samples. Similarly, for testing H_2, one might use corresponding statistics U', V', T', W' or Z', obtained from the above by writing G for $S^{(r)}$ or $\bar{S}$. Each of this last collection of statistics has a distribution which does not depend on G in the case that H_2 is true, and each of the first collection has a distribution which does not depend on what the common d.f. is when H_1 is true. In all cases, large values of the statistic lead to rejection of the hypothesis. It is clear that an appropriate choice of the C_j and $C_{q,r}$ in the case $k = 1$ of H_2 or the case $k = 2$ of H_1, reduces each of these tests to one of those previously mentioned for those cases in [5], [6], [7], [8], [9] (in the case of [8] and [9], the integrating measure is altered slightly, as discussed in connection with (2.8) below).

Many tests may be constructed along similar lines by allowing the C_j and $C_{q,r}$ to be functions (of the $S^{(j)}$ for H_1 and of $G(x)$ for H_2) as in the treatments of Kac [11] and Anderson and Darling [12] when $k = 1$, by using other measures of distance or dispersion, etc. In Section 5 we shall mention a few statistics whose limiting distributions are easy to obtain from those of the usual Kolmogorov-Smirnov and ω^2 statistics, but which are intuitively less appealing than those we have mentioned, especially from a practical point of view. In fact, the limiting distribution of V' or Z' (suitably normalized) is that of the maximum of multiples of k independent random variables with limiting Kolmogorov or ω^2 distributions, and is thus trivial to obtain from these latter distributions. From a practical point of view, the problem of testing H_2 may thus seem to be satisfactorily answered by these statistics.

Thus, our main goal is to obtain the limiting distribution under H_1 of appropriate statistics for testing that hypothesis, and the corresponding results we shall obtain for tests of H_2 are less important by-products of the investigation. Specifically, in Section 3 we shall obtain the limiting distribution of T (and T') for $C_j = n_j$, as the $n_j \to \infty$, while in Section 4 we obtain the limiting distribution of W (and W') under the same conditions. The limiting distributions of U, V, and Z seem more difficult to obtain, and the methods of this paper do not apply at all to those statistics.

Many different proofs of the Kolmogorov-Smirnov results [5] and [7] now exist. Combinatorial proofs such as those of Feller [10] and of several papers by Russian authors (such as Smirnov, Gnedenko, Korolyuk) seem inapplicable to the problem of obtaining the limiting distribution of the generalizations T and T' of the Kolmogorov-Smirnov statistics. The geometric aspects of Doob's proof [13] clearly cannot be directly generalized. However, the approach used by Kac in several papers since 1949, e.g., in [11], to obtain various results such as that of Kolmogorov, can be generalized with some slight technical modifications to give results on the Wiener process in dimensions >1 which can be used with an analogue of Donsker's result [14] to obtain the limiting distribution of T; such results for closely related problems have in fact been studied by Rosenblatt [17]. The method of Anderson and Darling [12] could also be used, but perhaps guessing the solution to the appropriate diffusion equation is more difficult than the approach used here.

In Section 2, therefore, we reduce the problem of finding the limiting distribution of T or T' to a calculation regarding a multidimensional Wiener process, and outline the steps to be carried out in performing this calculation. The solution is then obtained in Section 3. A similar method will work for the limiting distributions of W and W', but these may be obtained more easily by convolving the usual ω^2 distribution with itself an appropriate number of times (Section 4). In Section 5 the statistics mentioned three paragraphs above and whose distributions may be obtained from existing tables, are discussed. The power of the tests considered in this paper is discussed briefly in Section 6, where several other remarks are made. Finally, Section 7 contains tables of some of the limiting distributions obtained in the paper.

2. Reduction of the problem. We hereafter write N for the vector $(n_1, \cdots, n_k)$ and consider (now exhibiting the dependence on N)

$$\begin{aligned} T_N &= \sup_x \textstyle\sum_j n_j \,[S_{n_j}^{(j)}(x) - \bar{S}_N(x)]^2, \\ T'_N &= \sup_x \textstyle\sum_j n_j \,[S_{n_j}^{(j)}(x) - G(x)]^2, \\ W_N &= \int_{-\infty}^{\infty} \textstyle\sum_j n_j \,[S_{n_j}^{(j)}(x) - \bar{S}_N(x)]^2 \, d\bar{S}_N(x), \\ W'_N &= \int_{-\infty}^{\infty} \textstyle\sum_j n_j \,[S_{n_j}^{(j)}(x) - G(x)]^2 \, dG(x). \end{aligned}$$

(We shall also consider extensions of W_N; see equation (2.8).) Since the distribution of each of these statistics does not depend on G (resp., on the common d.f.) if H_2 (resp., H_1) is true, we shall as usual perform our calculations under the assumption that G and all F_i are the uniform d.f. on the unit interval.

Let $Y_1, Y_2, \cdots, Y_h$ be h independent separable Gaussian processes whose sample functions are functions of the same "time" parameter t, $0 \leqq t \leqq 1$, and such that $EY_i(t) = 0$ and $EY_i(t)Y_i(s) = \min(s, t) - st$ for each i. Thus, the Y_i are independent "tied-down Wiener processes" which may be represented as $Y_i(t) = (1-t)^{-1}w_i(t/(1-t))$, where the w_i are independent Wiener processes of the usual variety; i.e., w_i is a separable Gaussian process of independent increments with $Ew_i(\tau) = 0$ and $Ew_i(\tau)w_i(\sigma) = \min(\tau, \sigma)$ for $0 \leqq \tau, \sigma < \infty$. The use of such processes in [11], [12], [13] to obtain the Kolmogorov-Smirnov results is well known. Let

$$A_h(a) = P\{\max_{0 \leqq t \leqq 1} \textstyle\sum_{i=1}^h [Y_i(t)]^2 \leqq a\}. \tag{2.1}$$

and

$$B_h(a) = P\left\{\int_0^1 \textstyle\sum_{i=1}^h [Y_i(t)]^2\, dt \leqq a\right\}. \tag{2.2}$$

When G is the uniform d.f., the k random functions

$$\nu_{n_j}^{(j)}(t) = \sqrt{n_j}(S_{n_j}^{(j)}(t) - t), \qquad 0 \leqq t \leqq 1$$

are independent of each other and as $n_j \to \infty$ their behavior approaches that of the processes $Y_1, \cdots, Y_h$ with $h = k$. More precisely, an obvious extension of the argument of Donsker [14] or Theorem 2 of Kiefer and Wolfowitz [15] to the present case shows at once that, at all continuity points of the limit (which, we shall see, means for all a),

$$\lim_{\text{all } n_j \to \infty} P\{T_N' \leqq a\} = A_k(a) \tag{2.3}$$

and

$$\lim_{\text{all } n_j \to \infty} P\{W_N' \leqq a\} = B_k(a). \tag{2.4}$$

Similarly, let H be a $k \times k$ orthogonal matrix such that the jth element of the first row of H is $(n_j/\sum n_j)^{\frac{1}{2}}$ for $1 \leqq j \leqq k$, and write ν_N for the k-vector whose jth component is the random function $\nu_{n_j}^{(j)}$. We have already discussed the asymptotic behavior of ν_N as the $n_j \to \infty$. The extension of the results of Donsker [14] or Kiefer and Wolfowitz [15] to the present case shows, on considering the sum of squares of the last $k-1$ components of $H\nu_N$, which sum is equal to $\sum_j n_j[S_{n_j}^{(j)}(t) - \bar{S}_N(t)]^2$, that

$$\lim_{\text{all } n_j \to \infty} P\{T_N \leqq a\} = A_{k-1}(a). \tag{2.5}$$

We remark that, as in the case $h = 1$, if F_1 is not continuous, the statistics T_N and T_N' are equivalent to statistics obtained for the case of continuous F_1 by taking the supremum over a restricted range; thus, the d.f. of T_N or T_N' in such a case is not larger than what it is for continuous F_1.

Next, we consider W_N. Since we need to prove statements which differ slightly from those of Rosenblatt [9], and since the partial integrations in [9] require some alterations, we shall carry out the required demonstration in full here rather than to refer elsewhere.[2] We shall actually prove without extra difficulty a more general result than that needed here, but one which is useful in reducing the calculation of the limiting distribution of other integral criteria in the same way that we reduce that of W_n. Our result is (roughly) that an integral criterion formed by integrating with respect to a consistent estimator of the common F_i has the same limiting distribution if the consistent estimator is replaced by F_1. The following statement of it is thus easily generalized:

LEMMA. *Let $D \geqq 0$ be a continuous function of $k - 1$ real variables which is bounded on bounded sets and such that*

$$(2.6) \quad \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} D(t_1, \cdots, t_{k-1}) \,|\, t_1 \cdots t_{k-1} \,|\, e^{-(t_1^2 + \cdots + t_{k-1}^2)/2} dt_1 \cdots dt_{k-1} < \infty .$$

Then, for each j, when all F_i are uniform on [0, 1],

$$(2.7) \quad \int_0^1 D(\nu_{n_1}^{(1)}(t) - \nu_{n_k}^{(k)}(t), \cdots, \nu_{n_{k-1}}^{(k-1)}(t) - \nu_{n_k}^{(k)}(t))\, d(S_{n_j}^{(j)}(t) - t)$$

converges to 0 *in probability as all $n_i \to \infty$.*

PROOF: It was proved by Dvoretzky, Kiefer, and Wolfowitz [16] that $P\{\sup_t \nu_{n_j}^{(j)}(t) > r\} < ce^{-2r^2}$ for all n_j and r, where c is a positive constant. Hence, (2.6) implies that if in (2.7) we replace the function D by max (D, L), where L is a constant, (2.7) is altered by a quantity which goes to 0 in probability as the constant $L \to \infty$, uniformly in the n_j. Hence, it suffices to prove (2.7) assuming D is bounded and uniformly continuous, which we now assume. The proof of Theorem 2 of Kiefer and Wolfowitz [15] shows that for any $\epsilon > 0$ there is a value m such that the probability that

$$\sup_{i/m \leqq t \leqq (i+1)/m} |\, \nu_{n_j}^{(j)}(t) - \nu_{n_j}^{(j)}(i/m) \,| < \epsilon$$

for all $i(0 \leqq i \leqq m - 1)$ is at least $1 - \epsilon$ for all sufficiently large n_j. Thus, given any $\epsilon' > 0$, we can choose ϵ (and thus m) with regard to the modulus of continuity of D, so that for all n_j sufficiently large the probability will be $>1 - \epsilon'$ that the value of the integrand of (2.7) varies over a range of length $< \epsilon'$ as t varies from i/m to $(i + 1)/m$, simultaneously for all i. On the other hand, when the n_j are sufficiently large, $S_{n_j}^{(j)}$ assigns measure arbitrarily close to $1/m$ to each of the intervals $i/m \leqq t \leqq (i + 1)/m$, with probability arbitrarily close to 1. Since we have seen that D may be assumed bounded, the assertion of the lemma now follows easily.

We conclude at once from the lemma and the use of the orthogonal transformation H discussed in connection with T_N that if $a_1, \cdots, a_k$ are real numbers with $\sum a_i = 1$, then

$$(2.8) \quad \lim_{\text{all } n_j \to \infty} P\left\{\int_{-\infty}^{\infty} \sum_j n_j\, [S_{n_j}^{(j)}(x) - \bar{S}_N(x)]^2\, d[\sum_i a_i S_{n_i}^{(i)}(x)] \leqq a\right\} = B_{k-1}(a);$$

[2] Professor Rosenblatt has informed the author that he has constructed another correct proof of the result of [9], and has indicated that some corrections to [17] will appear shortly.

in particular,

$$\lim_{\text{all } n_j \to \infty} P\{W_N \leqq a\} = B_{k-1}(a). \tag{2.9}$$

The extension (2.8) of (2.9) includes, for example, integration with respect to $k^{-1}\sum_j S_{n_j}^{(j)}$, which is what is done in the case $k = 2$ by Rosenblatt [9]. It is easy to extend (2.8) to allow the a_i to vary slightly with N, etc.

We note that we nowhere require the ratios n_i/n_j to approach positive finite limits. This requirement, which is made in [7], [9], [10], and [13] in the case $k = 2$ of H_1, is inessential, and our remarks show that the results there hold without this restriction.

3. The limiting distribution of T_N and T_N'. In [17] Rosenblatt studies the distribution of a class of suitably regular functionals of the h-dimensional process $Y = (Y_1, \cdots, Y_h)$ on $0 \leqq t \leqq 1$. We shall only state briefly the results we need from [17] and Kac's paper [11]. In fact, writing

$$\Lambda_c(t) = [(Y_1(t) + ct)^2 + \sum_2^h (Y_i(t))^2]^{\frac{1}{2}}$$

for $c \geqq 0$, if one considers only nonnegative functions v of Λ_c which satisfy the regularity conditions of [17], then the analysis there may be shortened somewhat, and we now summarize the results we need in that briefer form; the reader may consult [11] or [17] for details.

For any h-vector x and $t > 0$, with primes denoting transposes, write

$$Q_0(x, t) = (2\pi t)^{-h/2} e^{-x'x/2t} \tag{3.1}$$

and, for $n > 0$, with E^h denoting Euclidean h-space and $d\xi = d\xi_1 d\xi_2 \cdots d\xi_h$,

$$Q_{n+1}(x, t) = \int_0^t \int_{E^h} Q_0(x - \xi, t - \tau) v([\xi'\xi]^{\frac{1}{2}}) Q_n(\xi, \tau)\, d\xi\, d\tau. \tag{3.2}$$

It is easy to see that Q_n depends on x only through $x'x = r^2$ (say), so that we can write $Q_n(x, t) = \bar{Q}_n(r, t)$. Define the generating function (in $u \geqq 0$)

$$Q(r, t, u) = \sum_{n=0}^{\infty} (-u)^n \bar{Q}_n(r, t) \tag{3.3}$$

and, for $r > 0$, its transform (in $s \geqq 0$)

$$\psi(r) \equiv \psi_{s,u}(r) = \int_0^\infty Q(r, t, u) e^{-st}\, dt. \tag{3.4}$$

Write

$$\phi(r) \equiv \phi_{s,u}(r) = r^{(h-1)/2} \psi_{s,u}(r). \tag{3.5}$$

One proves easily that ψ is the unique solution of the ordinary differential equation (for $r > 0$)

$$\psi''(r) + \frac{h-1}{r} \psi'(r) - [2s + 2uv(r)]\psi(r) = 0 \tag{3.6}$$

which satisfies

$$(3.7)\qquad \begin{aligned} &\text{(a) } \psi(r) \to 0 \text{ as } r \to \infty;\\ &\text{(b) } \psi'(r) \text{ is continuous for } r > 0;\\ &\text{(c) as } r \to 0,\ \psi'(r) \sim -\Gamma(h/2)\pi^{-h/2}r^{1-h}. \end{aligned}$$

It is sometimes convenient to rewrite (3.6) and (3.7) in other terms. For example, for $h > 1$ and suitably regular v, we can obtain ϕ as the unique solution (for $r > 0$) of

$$(3.6a)\qquad \phi''(r) - \left[2s + \frac{(h-1)(h-3)}{4r^2} + 2uv(r)\right]\phi(r) = 0$$

which satisfies

$$(3.7a)\qquad \begin{aligned} &\text{(a) } \phi(r) \to 0 \text{ as } r \to \infty;\\ &\text{(b) } \phi'(r) \text{ is continuous for } r > 0;\\ &\text{(c) as } \quad r \to 0, \qquad \phi(r) \sim \begin{cases} -\pi^{-1}r^{\frac{1}{2}} \log r & \text{if} \quad h = 2,\\ \Gamma(h/2)\pi^{-h/2}r^{(3-h)/2}/(h-2) & \text{if} \quad h > 2. \end{cases} \end{aligned}$$

(Equation (3.6) is merely the reduction to an ordinary differential equation of the partial differential equation of [17, equation (1.14)] when v depends only on $x'x$; (3.7) for the case $h \geqq 1$ is the analogue of [11], equation (3.14), for the case $h = 1$.)

Let $(w_1, \cdots, w_h)$ be the h-dimensional Wiener process described just above (2.1). Let

$$(3.8)\qquad \begin{aligned} \zeta(t) &= \int_0^t v\big(\big(\textstyle\sum_i [w_i(\tau)]^2\big)^{\frac{1}{2}}\big)\, d\tau,\\ \sigma(q; t) &= P\{\zeta(t) < q\}. \end{aligned}$$

The function Q in the case of more general v is studied by Rosenblatt [17] because, as in the case $h = 1$ of Kac [11], it is desired to compute σ, and

$$(3.9)\qquad \int_0^\infty e^{-uq}\, d_q\, \sigma(q, t) = \int_{E^h} Q([x'x]^{\frac{1}{2}}, t, u)\, dx.$$

But it can also be seen, as it was in [11], equation (6.16), when $h = 1$, that if

$$(3.10)\qquad \begin{aligned} \eta_c &= \int_0^1 v(\Lambda_c(t))\, dt,\\ p_c(q) &= P\{\eta_c < q\}, \end{aligned}$$

then

$$(3.11)\qquad \int_0^\infty e^{-uq}\, d_q\, p_c(q) = (2\pi)^{k/2}e^{c^2/2}Q(c, 1, u).$$

This is the use of Q which concerns us in obtaining distributions like those of (2.1) and (2.2).

In Kac's paper [11] it is only necessary to consider η_0, since p_0 is what we actually want to determine. However, $\psi_{s,u}(0)$ is infinite when $h > 1$, so that we are forced to consider η_c, determine $\psi_{s,u}(c)$ for c near 0, invert this to obtain $Q(c, 1, u)$, and the let $c \to 0$ to obtain $Q(0, 1, u)$. This continuity in c of $Q(c, 1, u)$ is proved by Rosenblatt [17] (it is also evident from the probabilistic meaning of η_c); the particular case of interest to us here involves another limit operation and will be discussed in the next paragraph.

In order to obtain the function A_h of (2.1), we consider, as did Kac [11] for the case $h = 1$, the function

$$v(r) = \begin{cases} 0 & \text{if } r < a, \\ 1 & \text{if } r \geqq a, \end{cases} \tag{3.12}$$

where $a > 0$. From (3.10) and (3.11) we then have

$$P\{\max_{0 \leqq t \leqq 1} \Lambda_c(t) < a\} = (2\pi)^{h/2} e^{c^2/2} \lim_{u \to \infty} Q(c, 1, u). \tag{3.13}$$

It is convenient to interchange the order of inverting with respect to s and letting $u \to \infty$; i.e., by bounded convergence we have

$$\psi_{s,\infty}(c) \equiv \lim_{u \to \infty} \psi_{s,u}(c) = \int_0^\infty \lim_{u \to \infty} Q(c, t, u) e^{-st}\, dt, \tag{3.14}$$

so that we can invert $(2\pi)^{h/2} e^{c^2/2} \psi_{s,\infty}(c)$ with respect to s and set $t = 1$ to obtain the left side of (3.13) and then, from the probabilistic meaning of Λ_c, let $c \to 0$ and obtain, for $a \geqq 0$,

$$A_h(a^2) = \lim_{c \to 0} P\{\max_{0 \leqq t \leqq 1} \Lambda_c(t) < a\}. \tag{3.15}$$

For the v of (3.12), the solution of (3.6) satisfying the conditions (3.7) is easily obtained in terms of modified Bessel functions of the first and third kind ([18], Vol. 2, [20]). The solution is of the form $\phi(r) = r^{(h-1)/2}\psi(r) = C_1 r^{\frac{1}{2}} K_{(h-2)/2}(r(2s)^{\frac{1}{2}}) + C_2 r^{\frac{1}{2}} I_{(h-2)/2}(r(2s)^{\frac{1}{2}})$ for $0 < r < a$, and of the same form with s replaced by $s + u$ and with C_1 and C_2 replaced by C_1' and C_2' (say) for $r \geqq a$, where the C_i and C_i' depend on s and u. From (3.7)(a) or (3.7a)(a) we obtain $C_2' = 0$, and from (3.7)(c) or (3.7a)(c) we obtain

$$C_1 = 2(2s)^{(h-2)/4}(2\pi)^{-h/2}$$

The other two constants are obtained from the continuity of ϕ and ϕ' at $r = a$. In particular, we obtain, writing $a(2s)^{\frac{1}{2}} = \alpha$ and $a(2s + 2u)^{\frac{1}{2}} = \beta$,

$$\frac{C_2}{C_1} = \frac{K_{h/2}(\alpha)K_{(h-2)/2}(\beta) - (\beta/\alpha)K_{(h-2)/2}(\alpha)K_{h/2}(\beta)}{I_{h/2}(\alpha)K_{(h-2)/2}(\beta) + (\beta/\alpha)I_{(h-2)/2}(\alpha)K_{h/2}(\beta)}. \tag{3.16}$$

When we let u (i.e., β) go to ∞, this ratio approaches the limit

$$-K_{(h-2)/2}(\alpha)/I_{(h-2)/2}(\alpha).$$

Thus, we have, for $0 < r < a$ and $h \geqq 1$,

$$(2\pi)^{h/2}\psi_{s,\infty}(r) \tag{3.17}$$
$$= \frac{2(2s)^{(h-2)/4}}{r^{(h-2)/2}}\left\{K_{(h-2)/2}(r(2s)^{\frac{1}{2}}) - \frac{K_{(h-2)/2}(a(2s)^{\frac{1}{2}})I_{(h-2)/2}(r(2s)^{\frac{1}{2}})}{I_{(h-2)/2}(a(2s)^{\frac{1}{2}})}\right\}.$$

(The corresponding formula and subsequent inversion in [17] is incorrect,[2] due to a mistake in evaluating C_1).

To invert (3.17), we consider the Fourier-Bessel expansion of [18], Vol. 2, p. 104, equation (58):

$$\frac{\pi J_\nu(xz)}{zJ_\nu(z)}[J_\nu(z)Y_\nu(Xz) - Y_\nu(z)J_\nu(Xz)] = \sum_{n=1}^{\infty}\frac{J_\nu(\gamma_{\nu,n}\, x)J_\nu(\gamma_{\nu,n}\, X)}{(z^2 - \gamma_{\nu,n}^2)[J_{\nu+1}(\gamma_{\nu,n})]^2}, \tag{3.18}$$

where $\gamma_{\nu,n}(n = 1, 2, \cdots)$ are the positive zeros of J_ν, ν and z are arbitrary, and $0 < x \leqq X \leqq 1$. (A similar formula of Watson ([20], p. 499) seems incorrect, as can be seen in the case $\nu = \frac{1}{2}$, $z \to 0$ there.) Divide both sides of (3.8) by $J_\nu(xz)$ and let $x \to 0$, noting that $J_\nu(\gamma_{\nu,n}x)/J_\nu(xz) \to (\gamma_{\nu,n}/z)^\nu$; it is easy to justify taking the limit inside the sum. Put $z = ia(2s)^{\frac{1}{2}}$ and $X = r/a$. We then obtain, from (3.17), (3.18), and the relation of I and K to J and Y, where $\nu = (h-2)/2$,

$$(2\pi)^{h/2}\psi_{s,\infty}(r) = 4\sum_{n=1}^{\infty}\left(\frac{\gamma_{\nu,n}}{ar}\right)^\nu \frac{J_\nu(r\gamma_{\nu,n}/a)}{[J_{\nu+1}(\gamma_{\nu,n})]^2(2a^2s + \gamma_{\nu,n}^2)}. \tag{3.19}$$

It is easy to see that this series can be inverted term-by-term with respect to s; inverting and setting $t = 1$, we have from (3.14),

$$P\{\max_{0\leqq t\leqq 1}\Lambda_r(t) < a\} = 2e^{r^2/2}\sum_{n=1}^{\infty}\left(\frac{\gamma_{\nu,n}}{ar}\right)^\nu \frac{J_\nu(r\gamma_{\nu,n}/a)e^{-\gamma_{\nu,n}^2/2a^2}}{[J_{\nu+1}(\gamma_{\nu,n})]^2a^2}. \tag{3.20}$$

Finally, letting $r \to 0$, we have, from (3.15) and (3.20),

THEOREM. For $h \geqq 1$ (*see also* (3.27) *and* (3.31)),

$$A_h(a^2) = \frac{4}{\Gamma\left(\frac{h}{2}\right)2^{h/2}a^h}\sum_{n=1}^{\infty}\frac{(\gamma_{(h-2)/2,n})^{h-2}\exp[-(\gamma_{(h-2)/2,n})^2/2a^2]}{[J_{h/2}(\gamma_{(h-2)/2,n})]^2}. \tag{3.21}$$

Thus, writing $\Phi_k(x) = A_k(x^2)$ for $x > 0$ and $\Phi_k(x) = 0$ otherwise, Φ_{k-1} and Φ_k are the limiting d.f.'s of $\sqrt{T_N}$ and $\sqrt{T_N'}$, respectively.

The series converges rapidly (see also the discussion of the two succeeding paragraphs for large a), but reduces to an expression in terms of elementary functions only when $h = 1$ or $h = 3$. When $h = 1$, we have $\gamma_{-\frac{1}{2},n} = (2n-1)\pi/2$ and thus $[J_{\frac{1}{2}}(\gamma_{-\frac{1}{2},n})]^2 = 4/(2n-1)\pi^2$. Thus, for $a > 0$,

$$A_1(a^2) = \frac{(2\pi)^{\frac{1}{2}}}{a}\sum_{n=1}^{\infty}e^{-(2n-1)^2\pi^2/8a^2}, \tag{3.22}$$

which is Smirnov's result, since T_N is the square of the usual Smirnov statistic when $k = 2$. Similarly, for $h = 3$ we obtain, for $a > 0$,

$$(3.23) \qquad A_3(a^2) = \frac{2^{\frac{1}{2}}\pi^{\frac{5}{2}}}{a^3} \sum_{n=1}^{\infty} n^2 e^{-n^2\pi^2/2a^2}$$

In these cases we can obtain alternative expressions which are more useful for computations when a is large. These may be obtained directly by using an appropriate transformation on a theta function, or by noting that (3.17) reduces to

$$\pi^{\frac{1}{2}} \sinh [(a - r)(2s)^{\frac{1}{2}}]/s^{\frac{1}{2}} \cosh [a(2s)^{\frac{1}{2}}]$$

when $h = 1$ and to

$$(2\pi)^{\frac{1}{2}} \sinh [(a - r)(2s)^{\frac{1}{2}}]/r \sinh [a(2s)^{\frac{1}{2}}]$$

when $h = 3$, and these are tabled as theta function transforms in [19], Vol. 1, p. 258, equations (34) and (31), the first of which is wrong in sign. For $h = 1$ we obtain, letting $r \to 0$, the more familiar form of A_1 for $a > 0$,

$$(3.24) \qquad A_1(a^2) = 1 + 2\sum_{n=1}^{\infty} (-1)^n e^{-2n^2a^2}$$

(For $h = 1$, but not for $h = 3$, we could have let $r \to 0$ before inverting, and used [19], Vol. 1, p. 257, equation (24).) For $h = 3$, the inverse Laplace transform is given in terms of a derivative of the theta function θ_4; letting $r \to 0$ yields

$$(3.25) \qquad A_3(a^2) = 1 + 4\sum_{n=1}^{\infty} [\tfrac{1}{2} - 2n^2a^2]e^{-2n^2a^2}$$

The existence of the two forms for A_1 and A_3 suggests that a form more useful than (3.21) for large a might be found. There seems to be no simple analogue of the theta function transformation for the series of (3.21), but in this and the next two paragraphs we mention other computational approaches which may prove useful. There are other Fourier-Bessel expansions which can be employed in inverting (3.17). For example, one series for $J_\nu(xz)/J_\nu(z)$ ([18], Vol. 2, p. 104, equation (59)) gives (writing ν for $(h - 2)/2$)

$$(3.26) \qquad \begin{aligned} &(2\pi)^{h/2}\psi_{s,\infty}(r) \\ &= 2\,\frac{(2s)^{\nu/2}}{r^\nu}\left\{K_\nu(r(2s)^{\frac{1}{2}}) - 2\sum_{n=1}^{\infty} \frac{J_\nu(\gamma_{\nu,n}\, r/a)\gamma_{\nu,n}\, K_\nu(a(2s)^{\frac{1}{2}})}{J_{\nu+1}(\gamma_{\nu,n})(2a^2s + \gamma_{\nu,n}^2)}\right\}. \end{aligned}$$

Now, by [19], Vol. 1, p. 283, equation (40), $2(2s)^{\nu/2}K_\nu(r(2s)^{\frac{1}{2}})/r^\nu$ is the transform of $t^{-\nu-1}e^{-r^2/2t}$, which becomes 1 at $t = 1$, $r \to 0$. Since $(2a^2s + \gamma^2)^{-1}$ is the transform of $e^{-\gamma^2 t/2a^2}/2a^2$ and $(a/r)^\nu J_\nu(\gamma r/a) \to \gamma^\nu/2^\nu\Gamma(\nu + 1)$ as $r \to 0$, we obtain

$$(3.27) \qquad A_h(a^2) = 1 - \frac{1}{2^\nu a^2\Gamma(\nu+1)} \sum_{n=1}^{\infty} \frac{(\gamma_{\nu,n})^{\nu+1}}{J_{\nu+1}(\gamma_{\nu,n})} \int_0^1 t^{-\nu-1}e^{-a^2/2t}e^{-(\gamma_{\nu,n})^2(1-t)/2a^2}\, dt.$$

For computational purposes, this formula has the disadvantage of involving a numerical quadrature, but it has the advantage that the series converges rapidly for a large.

Another way of trying to obtain a more useful formula for large a is to try to use the theta function transformation on a function close to that of (3.21). The following is such an approach *when h is odd*. We again write $\nu = (h - 2)/2$. Now, for large n we have $\gamma_{\nu,n} \sim \pi(4n + 2\nu - 1)/4$ (see, e.g., [18], Vol. 2, pp. 60 and 85) and $[J_{\nu+1}(\gamma_{\nu,n})]^2 \sim 2/\pi\gamma_{\nu,n}$. Thus, an approximation to the summand of (3.21) is

$$(3.28) \qquad f(\nu, n, a) = \frac{\pi^{2\nu+2}}{2}\left[n + \frac{2\nu - 1}{4}\right]^{2\nu+1} e^{-\pi^2[n+(2\nu-1)/4]^2/2a^2}$$

How good an approximation this is of course depends on the exponential term; but the form of (3.28) is suggestive of theta functions. In fact, the transformation $\theta_3(t^{-1}v \mid -t^{-1}) = (-it)^{\frac{1}{2}}e^{i\pi v^2/t}\theta_3(v / t)$ ([18], Vol. 2, p. 370), on putting $t = -1/i\pi x$, becomes

$$(3.29) \qquad (\pi x)^{\frac{1}{2}}\sum_{n=-\infty}^{\infty} e^{-\pi^2[n+v]^2x} = e^{-2\pi^2v^2x}\sum_{n=-\infty}^{\infty} e^{i\pi vn}e^{-n^2/x},$$

so that, for $2v$ a nonnegative integer,

$$(3.30) \qquad \begin{aligned}\sum_{n=1}^{\infty} e^{-\pi^2[n+v]^2x} &= \frac{e^{-2\pi^2v^2x}}{2(\pi x)^{\frac{1}{2}}}\sum_{n=-\infty}^{\infty} e^{i\pi vn-n^2/x} - \frac{1}{2}\sum_{n=-2v}^{0} e^{-\pi^2[n+v]^2x} \\ &= q_1(x) - q_2(x)(\text{say}).\end{aligned}$$

Putting $v = (2\nu - 1)/4$, differentiating $(2\nu + 1)/2$ times with respect to x, and denoting the summand of (3.17) by $g(\nu, n, a)$, we thus obtain for odd $h \geqq 3$,

$$(3.31) \qquad \begin{aligned}\frac{\Gamma\left(\frac{h}{2}\right)2^{h/2}a^h}{4} A_h(a^2) &= \sum_{n=1}^{\infty} [g(\nu, n, a) - f(\nu, n, a)] \\ &\quad + \frac{(-1)^{(2\nu+1)/2}\pi}{2}\left(\frac{d}{dx}\right)^{(2\nu+1)/2} [q_1(x) - q_2(x)]\,\big|_{x=\frac{1}{2}a^2}.\end{aligned}$$

When f is close to g, this will be a convenient formula, since q_1 converges rapidly as $a \to \infty$ and q_2 will contain only $2\nu + 1$ terms.

Another approach to obtain different expressions from (3.17) to invert, and which allows us to let $r \to 0$ before inverting with respect to s, is to note that although the Laplace transform ψ of $Q(r, t, u)$ is infinite for $r = 0$, the transform of $t^mQ(r, t, u)$ is finite there for m an integer $> h/2$. But this is just $d^m\psi_{s,u}(r)/ds^m$. Thus, performing such a differentiation and letting $u \to \infty$ and $r \to 0$, we obtain an expression whose inverse transform with respect to s at $t = 1$ give $(2\pi)^{-h/2}A_h(a^2)$.

Tables of the functions A_h will be found in Section 7. Even when h is even, the computation is not very difficult. For example, when $h = 2$ the denominator of the summand of (3.21) is approximately $2/\pi\gamma_{0,n}$, as we have seen, and the series is easy to work with. For the next odd h above those we have considered in detail, $h = 5$, the $\gamma_{\nu,n}$ are solutions of $\tan x = x$ and the summand of (3.21) is $\pi\gamma_{\nu,n}^2(\gamma_{\nu,n}^2 + 1)e^{-(\gamma_{\nu,n})^2/2a^2}/2$.

4. The limiting distribution of W_N and W_N'. The differential equation of (3.6) and (3.7) can be solved, when $v(r) = r^2$, in terms of a confluent hypergeometric function (specifically, by (3.7)(a), in terms of the Whittaker function $W_{\kappa,\mu}$); but a more direct approach is to note, on reversing the order of integration and summation in (2.2), that the distribution B_h is merely the h-fold convolution of B_1 with itself. In the case $h = 1$, it is well known that $(2\pi)^{\frac{1}{2}}Q(0, 1, u) = [(2u)^{\frac{1}{2}}/\sinh(2u)^{\frac{1}{2}}]^{\frac{1}{2}}$. Raising this to the hth power, we obtain $(2\pi)^{h/2}Q(0, 1, u)$ for general h. We can now follow a procedure like that of Anderson and Darling ([12], p. 201): we obtain, on integrating by parts,

$$\int_0^1 e^{-ua}B_h(a)\,da = u^{-1}[(2u)^{\frac{1}{2}}/\sinh(2u)^{\frac{1}{2}}]^{h/2} \tag{4.1}$$

Using the binomial expansion on $[1 - e^{-2(2u)^{\frac{1}{2}}}]^{-h/2}$, (4.1) becomes

$$2^{3h/4}\sum_{j=0}^{\infty}\frac{\Gamma(j + h/2)}{j!\Gamma(h/2)}u^{-1+h/4}e^{-(8u)^{\frac{1}{2}}(j+h/4)} \tag{4.2}$$

This series can be inverted term-by-term in terms of tabled transforms, without computations like those of [12]: from [19], Vol. 1, p. 246, equation (9), we find that $u^{-1+h/4}e^{-(8u)^{\frac{1}{2}}(j+h/4)}$ is the Laplace transform of

$$2^{(2-h)/4}\pi^{-\frac{1}{2}}t^{-h/4}e^{-(j+h/4)^2/t}D_{(h-2)/2}(2(j + h/4)t^{-\frac{1}{2}}),$$

where D is the parabolic cylinder function. Thus, inverting (4.2) with respect to u, we obtain, for $a > 0$,

$$B_h(a) = \frac{2^{(h+1)/2}}{\pi^{\frac{1}{2}}a^{h/4}}\sum_{j=0}^{\infty}\frac{\Gamma(j + h/2)}{j!\Gamma(h/2)}e^{-(j+h/4)^2/a}D_{(h-2)/2}((2j + h/2)/a^{\frac{1}{2}}). \tag{4.3}$$

Thus, B_k and B_{k-1} are the limiting d.f.'s of W_n and W_n', respectively.

B_h can be written in a more convenient form if h is even. In that case if we write H_n for the nth Hermite polynomial, i.e., $H_n(x) = (-1)^n e^{x^2}d^n e^{-x^2}/dx^n$, we obtain from the relation between D_n and H_n ([18], Vol. 2, p. 117), for $a > 0$ and h even,

$$B_h(a) = \frac{2^{(h+1)/2}}{\pi^{\frac{1}{2}}a^{h/4}}\sum_{j=0}^{\infty}\frac{\Gamma(j + h/2)}{j!\Gamma(h/2)}e^{-2(j+h/4)^2/a}H_{(h-2)/a}((2j + h/2)/(2a)^{\frac{1}{2}}). \tag{4.4}$$

When h is odd, (4.3) can be written in terms of the Bessel functions $K_{\frac{1}{4}}$ and $K_{\frac{3}{4}}$, as follows: Since ([18], Vol. 2, p. 119) $D_{-\frac{1}{2}}(z) = (z/2\pi)^{\frac{1}{2}}K_{\frac{1}{4}}(z^2/4)$ and $D_{\frac{1}{2}}(z) = -e^{z^2/4}d[e^{-z^2/4}D_{-\frac{1}{2}}(z)]/dz = \pi^{-\frac{1}{2}}(z/2)^{\frac{3}{2}}[K_{\frac{1}{4}}(z^2/4) + K_{\frac{3}{4}}(z^2/4)]$, successive use of the recursion relation $D_{\nu+1}(z) = zD_\nu(z) - \nu D_{\nu-1}(z)$ and the fact that $K_\nu = K_{-\nu}$ yields $D_{m-\frac{1}{2}}$, for m a positive integer, in terms of $K_{\frac{1}{4}}$ and $K_{\frac{3}{4}}$.

In the case $h = 1$, substitution of the formula for $D_{-\frac{1}{2}}$ in terms of $K_{\frac{1}{2}}$ gives the formula of [12], equation (4.35).

Tables of B_h will be found in Section 7.

5. Criteria whose distributions may be obtained from previously known results. We limit our discussion to criteria for testing H_1 ; analogues for testing H_2 are obvious, and some criteria have been mentioned in Section 1. We shall also limit our discussion to criteria of the Kolmogorov-Smirnov type, ones of the integral (ω^2-) type being obtained similarly. Symbols newly defined in this section need not have their earlier meaning.

One of the simplest tests whose size may be computed from previously known results is that based on the maximum of the $k - 1$ random variables

$$Y_j = C_j'' \sup_x \mid S_j(x) - \sum_{i<j} n_i S_i(x) / \sum_{i<j} n_i \mid, \qquad (2 \leqq j \leqq k)'$$

which are obviously independent under H_1 (since, for example, the conditional distribution of $\sup_x \mid S_1(x) - S_2(x) \mid$ given the value of the function $n_1 S_1 + n_2 S_2$ does not depend on the latter). Y_j is distributed like a multiple of the Smirnov 2-sample criterion for sample sizes n_j and $\sum_{i<j} n_i$; thus, the tables of Massey [21] may be used in an obvious way to compute the d.f. of $\max_j Y_j$. Of course, asymptotically one may use the Kolmogorov-Smirnov distribution $A_1(a^2)$.

This test may be made more symmetrical by choosing at random the indexing j of the k sets. Another method of symmetrizing is to subdivide each of the k original sets of observations into $k!$ subsets, form $k!$ collections each of which contains one subset of each original set, index the subsets in each collection in a different one of the $k!$ possible ways, compute the maximum of the Y_j for each collection, and take the maximum of these over all collections.

A test based upon the Y_j of the previous paragraph is a special case of the class of tests based on the $k - 1$ quantities $Z_j = \sup_x \mid R_j(x) \mid$ $(j = 2, \cdots, k)$ where the R_j are any $k - 1$ orthogonal linear combinations of the S_j which are orthogonal to $\bar{S}$; however, the Z_j will in general be independently distributed only in the limit, not for finite n_j as with the Y_j .

For $k = 3$, the asymptotic behavior of max (Y_2, Y_3) was also noted by Fisz [22]. For $k > 3$ Fisz suggests dividing the k samples into approximately $k/3$ collections of 3 or 2 samples each, computing the above or the Smirnov statistic from each collection, and then computing the maximum of these. The resulting test is clearly inferior to those we have considered: it is not even consistent, since it tests effectively only differences *within* the various collections.

Another simple test whose size may be computed from previously known results is the following: Let the n_j observations in the jth sample be divided at random into $k - 1$ subsets, each subset containing approximately the same number of observations, and call the sample d.f.'s of the observations in the $k - 1$ subsets of the jth samples $S_{jr}(x)$ $(1 \leqq r \leqq k, r \neq j)$; for any j_1, j_2 with $j_1 \neq j_2$, the distribution of $Z_{j_1 j_2} = C'_{j_1 j_2} \sup_x |S_{j_1 j_2}(x) - S_{j_2 j_1}(x)|$ (where $C'_{j_1 j_2}$ is a suitable normalizing constant) may again be obtained from Massey's tables [21], and the size of a test of H_1 based on such a statistic as $\max_{j_1,j_2} Z_{j_1,j_2}$ is again easily computed, since the $Z_{j_1 j_2}$ are independent.

Tests based on statistics like $\sum Y_j$ are less convenient to use, since the computation of size entails the convolution of the Kolmogorov-Smirnov d.f. $\Phi_1(x) = A_1(x^2)$ with itself. For example, a single convolution of Φ_1 with itself using term-by-term integration of (3.24) yields the d.f. G_2 given for $z > 0$ by a slowly converging double sum of terms involving the normal d.f., and this is extremely poor for computational purposes. It is in fact easier to obtain G_2 by numerical integration of the convolution formula, and this has been done to obtain a table of G_2 in Section 7.

6. Power; miscellaneous remarks. We again limit the discussion to tests of H_1, similar remarks applying for H_2. We use the notation of Section 1.

It is easily seen that, for the test of size (approximately) $\alpha > 0$ based on T, U, V, or any of the procedures listed in the previous section (excluding that of Fisz [22] for $k > 3$), for any $\beta < 1$ there is a value $\delta(\alpha, \beta)$ such that any of these tests has power $>\beta$ against all alternatives for which

$$\sup_{q,r,x}\{|F_q(x) - F_r(x)| \min(n_q^{\frac{1}{2}}, n_r^{\frac{1}{2}})\} > \delta(\alpha, \beta).$$

However, tests based on criteria such as Z or W cannot be guaranteed to have the property just cited; this may be demonstrated exactly as it was for ω^2-type tests in another problem in the paper by Kac, Kiefer, and Wolfowitz [23]. Similar results may be proved relative to other measures of distance of alternatives from H_1, as in [23]. Thus, distance tests of the Kolmogorov-Smirnov type seem preferable in applications to those of the ω^2-type.

We note that the distribution of Λ_r obtained in Section 3 gives an asymptotic computation of power for certain alternatives when T is used.

We remark that the methods of this paper may be modified along the lines of the papers by Darling [24] and Kac, Kiefer, and Wolfowitz [23] in parametric cases, e.g., to test the hypothesis H_1 under the assumption that the F_j are all normal, or to test that the F_j are equal *and* normal.

In the case $k = 3$ of H_1, when all n_j are equal, David [25] has used a clever device to compute the distribution of $\max_{j,x}[S_j(x) - S_{j+1}(x)]$, where the subscripts are taken mod 3. The method does not seem to generalize.

The use of "distance" criteria in various nonparametric multi-decision problems, e.g., problems of ranking or of classification, is to be recommended, but the appropriate distribution theory is more complicated.

The author plans to return in another paper to consideration of some of the limiting distributions discussed here using a method somewhat similar to that of Doob [13].

7. Tables. The functions A_h of Section 3 and B_h of Section 4 ($1 \leqq h \leqq 5$), and the function G_2 defined in Section 5, have been tabled by the Cornell Computing Center's 650. I am indebted to Miss Susan Litt, Miss Virginia Walbran, Mrs. Jane Wiegand, Professor R. J. Walker, and Mr. R. C. Lesser, for carrying out this work.

(*Continued at the foot of p. 438*)

TABLE 1

Tables of $\Phi_i(x) = A_i(x^2)$ for $i = 1, 2, 3, 4, 5$

x	$\Phi_1(x)$	$\Phi_2(x)$	$\Phi_3(x)$	$\Phi_4(x)$	$\Phi_5(x)$
0.37	.000826				
0.38	.001285				
0.39	.001929				
0.40	.002808				
0.41	.003972				
0.42	.005476				
0.43	.007377				
0.44	.009730				
0.45	.012589				
0.46	.016005				
0.47	.020022				
0.48	.024682				
0.49	.030017				
0.50	.036055				
0.51	.042814				
0.52	.050306				
0.53	.058534	.000894			
0.54	.067497	.001256			
0.55	.077183	.001731			
0.56	.087577	.002342			
0.57	.098656	.003115			
0.58	.110394	.004079			
0.59	.122760	.005262			
0.60	.135717	.006696			
0.61	.149229	.008412			
0.62	.163255	.010441			
0.63	.177752	.012816			
0.64	.192677	.015566			
0.65	.207987	.018720	.000762		
0.66	.223637	.022307	.001035		
0.67	.239582	.026350	.001383		
0.68	.255780	.030874	.001824		
0.69	.272188	.035897	.002373		
0.70	.288765	.041437	.003050		
0.71	.305470	.047507	.003874		
0.72	.322265	.054116	.004866		
0.73	.339114	.061271	.006050		
0.74	.355981	.068976	.007447		
0.75	.372833	.077230	.009081		
0.76	.389640	.086029	.010977	.000820	
0.77	.406372	.095367	.013159	.001080	
0.78	.423002	.105233	.015649	.001406	
0.79	.439505	.115614	.018472	.001810	
0.80	.455858	.126496	.021649	.002306	
0.81	.472039	.137859	.025201	.002907	
0.82	.488028	.149685	.029149	.003631	
0.83	.503809	.161950	.033510	.004493	
0.84	.519365	.174632	.038300	.005511	

TABLE 1—*Continued*

x	$\Phi_1(x)$	$\Phi_2(x)$	$\Phi_3(x)$	$\Phi_4(x)$	$\Phi_5(x)$
0.85	.534681	.187705	.043534	.006704	
0.86	.549745	.201142	.049223	.008092	.000897
0.87	.564545	.214917	.055378	.009694	.001157
0.88	.579071	.229001	.062006	.011530	.001476
0.89	.593315	.243366	.069112	.013621	.001867
0.90	.607269	.257982	.076699	.015986	.002340
0.91	.620928	.272822	.084766	.018645	.002908
0.92	.634285	.287855	.093313	.021618	.003584
0.93	.647337	.303054	.102333	.024924	.004382
0.94	.660081	.318390	.111821	.028579	.005317
0.95	.672514	.333834	.121767	.032600	.006407
0.96	.684636	.349361	.132160	.037004	.007666
0.97	.696445	.364942	.142988	.041802	.009113
0.98	.707941	.380554	.154236	.047009	.010765
0.99	.719126	.396169	.165887	.052634	.012639
1.00	.730000	.411765	.177923	.058687	.014754
1.01	.740566	.427319	.190326	.065174	.017127
1.02	.750825	.442809	.203074	.072101	.019777
1.03	.760781	.458214	.216146	.079471	.022720
1.04	.770436	.473514	.229521	.087284	.025972
1.05	.779794	.488690	.243174	.095541	.029551
1.06	.788860	.503725	.257083	.104239	.033471
1.07	.797637	.518603	.271223	.113372	.037747
1.08	.806130	.533308	.285569	.122935	.042390
1.09	.814343	.547826	.300099	.132919	.047414
1.10	.822282	.562143	.314786	.143314	.052828
1.11	.829951	.576248	.329607	.154110	.058642
1.12	.837356	.590130	.344538	.165291	.064862
1.13	.844502	.603779	.359554	.176846	.071495
1.14	.851395	.617184	.374632	.188756	.078545
1.15	.858040	.630340	.389749	.201006	.086015
1.16	.864443	.643237	.404883	.213577	.093904
1.17	.870610	.655871	.420012	.226450	.102213
1.18	.876546	.668235	.435114	.239605	.110938
1.19	.882258	.680325	.450170	.253023	.120075
1.20	.887750	.692137	.465159	.266681	.129619
1.21	.893030	.703668	.480064	.280558	.139562
1.22	.898102	.714916	.494865	.294632	.149895
1.23	.902973	.725879	.509546	.308881	.160607
1.24	.907648	.736555	.524090	.323283	.171687
1.25	.912134	.746946	.538483	.337815	.183121
1.26	.916435	.757050	.552710	.352455	.194895
1.27	.920557	.766869	.566758	.367181	.206993
1.28	.924506	.776403	.580613	.381971	.219400
1.29	.928288	.785655	.594266	.396804	.232097
1.30	.931908	.794626	.607703	.411658	.245067
1.31	.935371	.803319	.620917	.426513	.258290
1.32	.938682	.811737	.633898	.441348	.271746
1.33	.941847	.819883	.646638	.456145	.285417
1.34	.944871	.827761	.659129	.470884	.299281

TABLE 1—*Continued*

x	$\Phi_1(x)$	$\Phi_2(x)$	$\Phi_3(x)$	$\Phi_4(x)$	$\Phi_5(x)$
1.35	.947758	.835374	.671366	.485547	.313318
1.36	.950514	.842727	.683343	.500117	.327506
1.37	.953143	.849824	.695055	.514577	.341825
1.38	.955651	.856670	.706498	.528911	.356254
1.39	.958041	.863269	.717669	.543104	.370771
1.40	.960318	.869627	.728564	.557141	.385356
1.41	.962487	.875748	.739183	.571009	.399989
1.42	.964551	.881638	.749523	.584696	.414648
1.43	.966515	.887302	.759585	.598190	.429314
1.44	.968383	.892745	.769367	.611479	.443968
1.45	.970158	.897973	.778871	.624554	.458590
1.46	.971846	.902992	.788096	.637405	.473163
1.47	.973448	.907808	.797046	.650025	.487667
1.48	.974969	.912425	.805720	.662404	.502087
1.49	.976413	.916849	.814122	.674537	.516406
1.50	.977782	.921086	.822255	.686418	.530607
1.51	.979080	.925142	.830121	.698041	.544676
1.52	.980310	.929023	.837724	.709401	.558598
1.53	.981475	.932733	.845067	.720496	.572360
1.54	.982579	.936278	.852154	.731321	.585948
1.55	.983623	.939664	.858990	.741874	.599352
1.56	.984610	.942897	.865579	.752155	.612560
1.57	.985544	.945980	.871926	.762160	.625561
1.58	.986427	.948921	.878036	.771890	.638346
1.59	.987261	.951723	.883913	.781345	.650906
1.60	.988048	.954393	.889563	.790525	.663233
1.61	.988791	.956934	.894991	.799432	.675320
1.62	.989492	.959352	.900203	.808066	.687161
1.63	.990154	.961651	.905203	.816430	.698749
1.64	.990777	.963837	.909998	.824526	.710081
1.65	.991364	.965913	.914593	.832356	.721151
1.66	.991917	.967885	.918994	.839925	.731957
1.67	.992438	.969756	.923206	.847235	.742495
1.68	.992928	.971530	.927235	.854290	.752763
1.69	.993389	.973213	.931087	.861094	.762760
1.70	.993823	.974807	.934766	.867651	.772485
1.71	.994230	.976317	.938280	.873967	.781936
1.72	.994612	.977746	.941633	.880045	.791116
1.73	.994972	.979099	.944830	.885891	.800024
1.74	.995309	.980378	.947878	.891509	.808660
1.75	.995625	.981586	.950781	.896905	.817028
1.76	.995922	.982728	.953546	.902084	.825130
1.77	.996200	.983807	.956176	.907052	.832966
1.78	.996460	.984824	.958676	.911813	.840542
1.79	.996704	.985784	.961053	.916375	.847859
1.80	.996932	.986689	.963311	.920741	.854921
1.81	.997146	.987542	.965455	.924919	.861732
1.82	.997346	.988345	.967488	.928913	.868296
1.83	.997533	.989102	.969417	.932729	.874618
1.84	.997707	.989813	.971245	.936373	.880703

TABLE 1—*Continued*

x	$\Phi_1(x)$	$\Phi_2(x)$	$\Phi_3(x)$	$\Phi_4(x)$	$\Phi_5(x)$
1.85	.997870	.990483	.972976	.939851	.886554
1.86	.998023	.991112	.974615	.943167	.892177
1.87	.998165	.991703	.976166	.946328	.897578
1.88	.998297	.992259	.977633	.949338	.902760
1.89	.998421	.992780	.979019	.952204	.907731
1.90	.998536	.993269	.980329	.954931	.912494
1.91	.998644	.993728	.981566	.957524	.917056
1.92	.998744	.994158	.982733	.959987	.921423
1.93	.998837	.994560	.983833	.962326	.925599
1.94	.998924	.994938	.984871	.964547	.929591
1.95	.999004	.995291	.985848	.966653	.933404
1.96	.999079	.995621	.986769	.968649	.937044
1.97		.995930	.987635	.970541	.940517
1.98		.996219	.988450	.972332	.943827
1.99		.996489	.989216	.974027	.946981
2.00		.996741	.989936	.975631	.949984
2.01		.996976	.990612	.977146	.952842
2.02		.997195	.991247	.978578	.955560
2.03		.997400	.991843	.979930	.958142
2.04		.997591	.992402	.981206	.960595
2.05		.997768	.992925	.982409	.962924
2.06		.997934	.993416	.983543	.965133
2.07		.998088	.993875	.984612	.967227
2.08		.998231	.994305	.985618	.969211
2.09		.998364	.994707	.986565	.971090
2.10		.998488	.995083	.987455	.972868
2.11		.998603	.995434	.988292	.974549
2.12		.998710	.995762	.989079	.976139
2.13		.998809	.996069	.989817	.977640
2.14		.998901	.996355	.990511	.979058
2.15		.998987	.996621	.991161	.980396
2.16		.999066	.996870	.991770	.981657
2.17		.999139	.997101	.992342	.982846
2.18			.997317	.992877	.983966
2.19			.997518	.993377	.985020
2.20			.997704	.993846	.986012
2.21			.997878	.994284	.986945
2.22			.998039	.994693	.987821
2.23			.998189	.995075	.988645
2.24			.998328	.995432	.989418
2.25			.998458	.995765	.990143
2.26			.998577	.996076	.990823
2.27			.998688	.996366	.991460
2.28			.998791	.996635	.992057
2.29			.998887	.996887	.992616
2.30			.998975	.997120	.993139
2.31			.999057	.997338	.993628
2.32			.999132	.997540	.994085
2.33				.997728	.994512
2.34				.997902	.994910
2.35				.998064	.995282

TABLE 1—*Continued*

x	$\Phi_1(x)$	$\Phi_2(x)$	$\Phi_3(x)$	$\Phi_4(x)$	$\Phi_5(x)$
2.36				.998215	.995629
2.37				.998354	.995952
2.38				.998483	.996253
2.39				.998603	.996534
2.40				.998714	.996795
2.41				.998817	.997038
2.42				.998911	.997263
2.43				.998999	.997473
2.44				.999080	.997668
2.45				.999155	.997849
2.46					.998016
2.47					.998172
2.48					.998316
2.49					.998449
2.50					.998573
2.51					.998687
2.52					.998793
2.53					.998891
2.54					.998981
2.55					.999065
2.56					.999142

TABLE 2

Table of the inverses $\Phi_i^{-1}(p)$

p	$\Phi_1^{-1}(p)$	$\Phi_2^{-1}(p)$	$\Phi_3^{-1}(p)$	$\Phi_4^{-1}(p)$	$\Phi_5^{-1}(p)$
.25	0.67645	0.89456	1.05493	1.18776	1.30375
.50	0.82757	1.05751	1.22349	1.35992	1.47855
.75	1.01918	1.25299	1.42047	1.55788	1.67728
.80	1.07275	1.30614	1.47337	1.61065	1.72997
.85	1.13795	1.37025	1.53692	1.67388	1.79299
.90	1.22385	1.45399	1.61960	1.75593	1.87462
.95	1.35810	1.58379	1.74726	1.88226	2.00005
.98	1.51743	1.73699	1.89743	2.03053	2.14698
.99	1.62762	1.84273	2.00092	2.13257	2.24798
.995	1.73082	1.94172	2.09773	2.22797	2.34235
.999	1.94948	2.15162	2.30296	2.43009	2.54217
.9999	2.22530	2.41695	2.56244	2.68565	2.79481

$\Phi_h(x) = A_h(x^2)$ is tabled in Table 1 for $1 \leqq h \leqq 5$ and for x in steps of .01 from $\Phi_h^{-1}(.001)$ to $\Phi_h^{-1}(.999)$. Tables of $\Phi_h^{-1}(p)$ for various often used values of p are given in Table 2. Thus, in using the statistic T (resp., T') to test H_1 (resp., H_2) when the n_j are large, with a test of size α, one should reject the hypothesis when $\sqrt{T} > \Phi_{k-1}^{-1}(1 - \alpha)$ (resp., $\sqrt{T'} > \Phi_k^{-1}(1 - \alpha)$).

(*Continued on p. 444*)

TABLE 3

Tables of B_i (x) for i = 1, 2, 3, 4, 5

x	$B_1(x)$	$B_2(x)$	$B_3(x)$	$B_4(x)$	$B_5(x)$
0.01	.000006				
0.02	.002892				
0.03	.023832				
0.04	.066851				
0.05	.123719	.000324			
0.06	.186020	.001566			
0.07	.248436	.004768			
0.08	.308145	.010891			
0.09	.363856	.020564			
0.10	.415127	.034001			
0.11	.461959	.051075	.000914		
0.12	.504575	.071420	.001966		
0.13	.543293	.094544	.003735		
0.14	.578461	.119910	.006438		
0.15	.610424	.146986	.010272		
0.16	.639507	.175283	.015396		
0.17	.666005	.204366	.021924	.000708	
0.18	.690186	.233862	.029920	.001249	
0.19	.712291	.263459	.039405	.002067	
0.20	.732530	.292900	.050357	.003240	
0.21	.751092	.321978	.062721	.004848	
0.22	.768144	.350530	.076413	.006971	
0.23	.783833	.378432	.091332	.009682	
0.24	.798290	.405587	.107364	.013049	.000675
0.25	.811630	.431928	.124383	.017130	.001043
0.26	.823958	.457406	.142264	.021971	.001566
0.27	.835364	.481991	.160881	.027605	.002274
0.28	.845930	.505668	.180110	.034056	.003184
0.29	.855730	.528431	.199832	.041333	.004359
0.30	.864829	.550283	.219937	.049437	.005830
0.31	.873285	.571236	.240320	.058356	.007632
0.32	.881153	.591305	.260885	.068071	.009813
0.33	.888478	.610511	.281544	.078555	.012394
0.34	.895305	.628877	.302218	.089771	.015414
0.35	.901673	.646428	.322835	.101682	.018906
0.36	.907617	.663191	.343331	.114243	.022887
0.37	.913168	.679193	.363651	.127406	.027378
0.38	.918358	.694464	.383745	.141122	.032397
0.39	.923211	.709031	.403570	.155340	.037951
0.40	.927753	.722922	.423088	.170007	.044054
0.41	.932006	.736166	.442268	.185074	.050702
0.42	.935990	.748790	.461084	.200488	.057898
0.43	.939724	.760820	.479514	.216199	.065629
0.44	.943226	.772283	.497538	.232160	.073892
0.45	.946512	.783203	.515144	.248323	.082674
0.46	.949595	.793605	.532320	.264643	.091955
0.47	.952490	.803513	.549056	.281078	.101720
0.48	.955210	.812950	.565349	.297587	.111948
0.49	.957765	.821936	.581193	.314133	.122617

TABLE 3—*Continued*

x	$B_1(x)$	$B_2(x)$	$B_3(x)$	$B_4(x)$	$B_5(x)$
0.50	.960167	.830494	.596590	.330680	.133701
0.51	.962425	.838642	.611537	.347194	.145177
0.52	.964549	.846400	.626039	.363646	.157017
0.53	.966547	.853787	.640097	.380006	.169195
0.54	.968427	.860819	.653717	.396248	.181679
0.55	.970197	.867515	.666904	.412349	.194449
0.56	.971864	.873889	.679663	.428287	.207471
0.57	.973433	.879957	.692004	.444042	.220721
0.58	.974912	.885734	.703933	.459597	.234170
0.59	.976305	.891233	.715458	.474935	.247790
0.60	.977618	.896468	.726589	.490043	.261557
0.61	.978855	.901451	.737333	.504908	.275444
0.62	.980022	.906195	.747701	.519519	.289426
0.63	.981122	.910710	.757702	.533868	.303480
0.64	.982159	.915008	.767344	.547945	.317582
0.65	.983138	.919100	.776639	.561745	.331712
0.66	.984061	.922995	.785596	.575262	.345847
0.67	.984932	.926702	.794224	.588492	.359967
0.68	.985754	.930231	.802533	.601431	.374053
0.69	.986530	.933590	.810532	.614076	.388088
0.70	.987262	.936787	.818232	.626427	.402054
0.71	.987954	.939830	.825641	.638482	.415937
0.72	.988607	.942727	.832769	.650242	.429721
0.73	.989224	.945485	.839624	.661707	.443394
0.74	.989806	.948110	.846217	.672878	.456943
0.75	.990356	.950608	.852555	.683757	.470349
0.76	.990876	.952986	.858647	.694347	.483607
0.77	.991367	.955250	.864502	.704649	.496713
0.78	.991831	.957405	.870127	.714668	.509646
0.79	.992270	.959455	.875532	.724407	.522402
0.80	.992684	.961408	.880723	.733869	.534981
0.81	.993076	.963266	.885707	.743059	.547361
0.82	.993447	.965035	.890494	.751980	.559556
0.83	.993797	.966718	.895090	.760639	.571546
0.84	.994128	.968321	.899501	.769038	.583319
0.85	.994441	.969846	.903735	.777183	.594903
0.86	.994737	.971298	.907797	.785079	.606259
0.87	.995017	.972680	.911696	.792732	.617411
0.88	.995282	.973995	.915436	.800145	.628332
0.89	.995532	.975248	.919024	.807326	.639045
0.90	.995769	.976439	.922465	.814278	.649538
0.91	.995993	.977574	.925765	.821007	.659801
0.92	.996205	.978654	.928930	.827519	.669848
0.93	.996406	.979681	.931964	.833819	.679675
0.94	.996596	.980660	.934874	.839912	.689284
0.95	.996776	.981591	.937663	.845803	.698668
0.96	.996946	.982477	.940336	.851499	.707832
0.97	.997107	.983321	.942898	.857003	.716780

TABLE 3—*Continued*

x	$B_1(x)$	$B_2(x)$	$B_3(x)$	$B_4(x)$	$B_5(x)$
0.98	.997259	.984124	.945353	.862321	.725508
0.99	.997403	.984889	.947706	.867459	.734026
1.00	.997540	.985616	.949960	.872421	.742332
1.01	.997669	.986309	.952120	.877213	.750424
1.02	.997791	.986968	.954190	.881839	.758311
1.03	.997907	.987596	.956172	.886304	.765992
1.04	.998017	.988193	.958070	.890614	.773472
1.05	.998121	.988761	.959889	.894771	.780754
1.06	.998219	.989302	.961630	.898782	.787834
1.07	.998312	.989817	.963298	.902651	.794727
1.08	.998400	.990308	.964895	.906382	.801427
1.09	.998484	.990775	.966425	.909979	.807943
1.10	.998563	.991219	.967888	.913447	.814272
1.11	.998638	.991642	.969291	.916790	.820424
1.12	.998709	.992044	.970632	.920011	.826397
1.13	.998776	.992427	.971916	.923115	.832199
1.14	.998840	.992792	.973146	.926106	.837833
1.15	.998900	.993139	.974322	.928986	.843298
1.16	.998957	.993469	.975448	.931761	.848602
1.17	.999011	.993784	.976525	.934433	.853750
1.18	.999063	.994083	.977557	.937006	.858742
1.19		.994368	.978544	.939484	.863580
1.20		.994639	.979488	.941868	.868274
1.21		.994897	.980391	.944164	.872821
1.22		.995143	.981256	.946373	.877227
1.23		.995377	.982082	.948499	.881497
1.24		.995599	.982873	.950544	.885630
1.25		.995811	.983630	.952512	.889635
1.26		.996013	.984354	.954405	.893515
1.27		.996205	.985047	.956226	.897268
1.28		.996388	.985708	.957977	.900902
1.29		.996562	.986341	.959661	.904419
1.30		.996727	.986947	.961281	.907818
1.31		.996885	.987526	.962837	.911110
1.32		.997035	.988080	.964334	.914292
1.33		.997178	.988610	.965773	.917370
1.34		.997313	.989116	.967156	.920346
1.35		.997443	.989600	.968485	.923223
1.36		.997566	.990063	.969762	.926004
1.37		.997683	.990506	.970989	.928692
1.38		.997795	.990929	.972169	.931287
1.39		.997901	.991334	.973302	.933797
1.40		.998002	.991721	.974390	.936220
1.41		.998098	.992091	.975435	.938560
1.42		.998190	.992444	.976439	.940821
1.43		.998277	.992782	.977404	.943003
1.44		.998360	.993104	.978330	.945110
1.45		.998439	.993413	.979219	.947145
1.46		.998514	.993708	.980073	.949108
1.47		.998586	.993990	.980893	.951002

TABLE 3—*Continued*

x	$B_1(x)$	$B_2(x)$	$B_3(x)$	$B_4(x)$	$B_5(x)$
1.48		.998654	.994259	.981680	.952831
1.49		.998718	.994517	.982436	.954595
1.50		.998780	.994763	.983161	.956298
1.51		.998839	.994998	.983857	.957937
1.52		.998895	.995223	.984526	.959519
1.53		.998948	.995437	.985167	.961044
1.54		.998999	.995643	.985782	.962520
1.55		.999047	.995839	.986373	.963941
1.56		.999093	.996026	.986939	.965311
1.57			.996205	.987483	.966629
1.58			.996376	.988005	.967897
1.59			.996539	.988505	.969129
1.60			.996695	.988985	.970307
1.61			.996844	.989445	.971452
1.62			.966987	.989887	.972538
1.63			.997123	.990311	.973602
1.64			.997253	.990717	.974615
1.65			.997377	.991106	.975598
1.66			.997495	.991480	.976544
1.67			.997608	.991838	.977450
1.68			.997717	.992182	.978329
1.69			.997820	.992511	.979165
1.70			.997919	.992827	.979979
1.71			.998013	.993129	.980765
1.72			.998103	.993420	.981511
1.73			.998189	.993698	.982239
1.74			.998271	.993964	.982932
1.75			.998349	.994220	.983606
1.76			.998424	.994465	.984252
1.77			.998496	.994700	.984865
1.78			.998564	.994925	.985462
1.79			.998629	.995140	.986040
1.80			.998692	.995347	.986590
1.81			.998751	.995545	.987123
1.82			.998808	.995734	.987635
1.83			.998862	.995916	.988124
1.84			.998914	.996090	.988597
1.85			.998963	.996257	.989056
1.86			.999011	.996417	.989493
1.87			.999056	.996570	.989915
1.88				.996717	.990315
1.89				.996857	.990709
1.90				.996992	.991077
1.91				.997121	.991439
1.92				.997244	.991781
1.93				.997363	.992111
1.94				.997476	.992431
1.95				.997584	.992742
1.96				.997688	.993039
1.97				.997788	.993321

TABLE 3—*Continued*

x	$B_1(x)$	$B_2(x)$	$B_3(x)$	$B_4(x)$	$B_5(x)$
1.98				.997883	.993593
1.99				.997974	.993853
2.00				.998061	.994107
2.01				.998145	.994346
2.02				.998225	.994577
2.03				.998302	.994802
2.04				.998375	.995014
2.05				.998445	.995219
2.06				.998513	.995417
2.07				.998577	.995605
2.08				.998639	.995787
2.09				.998698	.995963
2.10				.998754	.996132
2.11				.998808	.996290
2.12				.998860	.996445
2.13				.998909	.996596
2.14				.998957	.996737
2.15				.999002	.996873
2.16				.999046	.997004
2.17					.997131
2.18					.997252
2.19					.997367
2.20					.997479
2.21					.997584
2.22					.997687
2.23					.997787
2.24					.997882
2.25					.997971
2.26					.998059
2.27					.998143
2.28					.998224
2.29					.998298
2.30					.998373
2.31					.998446
2.32					.998512
2.33					.998578
2.34					.998637
2.35					.998699
2.36					.998756
2.37					.998812
2.38					.998866
2.39					.998916
2.40					.998962
2.41					.999012
2.42					.999055

The corresponding tables of B_h, the limiting d.f. of W (with $h = k - 1$) and of W' (with $h = k$), and of B_h^{-1}, are Tables 3 and 4.

Tables 1 and 2 were computed from equation (3.21), while Tables 3 and 4 were computed using the form of (4.3) given in (4.4) and the paragraph following (4.4). A program developed at Cornell was used to obtain the Bessel functions by power series or asymptotic series in appropriate regions.

As a check, the tables for $h = 1$ were compared with that of Φ_1 of Smirnov [26] and that of B_1 of Anderson and Darling [12]. In the case of Φ_1, the last tabled figure often differed slightly; wherever a discrepency was noted in the last *two* places, the tables were checked by differencing, and Smirnov's appeared to be in error. The table of [12] checked with that of B_1 here.

As mentioned in Section 5, the easiest way to compute tables of the convolution G_2 of Φ_1 with itself appeared to be by numerical integration, and Table 5 was computed in this way. Thus, for example, to test H_1 with size α when $k = 3$, one can use the statistic $Y_2 + Y_3$ of Section 5 with $C_2'' = [n_1n_2/(n_1 + n_2)]^{\frac{1}{2}}$ and $C_3'' = [n_3(n_1 + n_2)/(n_1 + n_2 + n_3)]^{\frac{1}{2}}$, rejecting the hypothesis for large n_i when $Y_2 + Y_3 > G_2^{-1}(1 - \alpha)$.

Added in proof: The author has recently learned that the following independently obtained results, which overlap some of those of this paper, appeared somewhat after [1] and the submission of earlier versions of the present paper: the limiting d.f. of T_N has been considered by J. J. Gichman in *Teorya Veryotnostei i yeyau primenyenya*, vol. 2 (1957), pp. 380–384, using an approach like that of [12], and two papers by L. C. Chang and M. Fisz in *Science Record*, vol. 1 (1957), pp. 335–346, consider tests like those discussed in the second and fourth paragraphs of Section 5.

TABLE 4

Table of the inverses $B_i^{-1}(p)$

p	$B_1^{-1}(p)$	$B_2^{-1}(p)$	$B_3^{-1}(p)$	$B_4^{-1}(p)$	$B_5^{-1}(p)$
.25	0.07026	0.18545	0.31472	0.45103	0.59161
.50	0.11888	0.27757	0.44138	0.60668	0.77253
.75	0.20939	0.42098	0.62227	0.81775	1.00947
.80	0.24124	0.46640	0.67691	0.87980	1.07785
.85	0.28406	0.52481	0.74592	0.95734	1.16268
.90	0.34730	0.60704	0.84116	1.06311	1.27748
.95	0.46136	0.74752	1.00018	1.23730	1.46466
.98	0.61981	0.93320	1.20561	1.45913	1.70028
.99	0.74346	1.07366	1.35861	1.62263	1.87215
.995	0.86939	1.21412	1.51010	1.78345	2.03935
.999	1.16786	1.54027	1.85773	2.14949	2.40774
.9999	1.60443	2.00691	2.3495	2.66130	2.825

TABLE 5

Table of $G_2(x)$

x	$G_2(x)$	x	$G_2(x)$	x	$G_2(x)$	x	$G_2(x)$
.92	.0008	1.42	.2005	1.92	.7157	2.42	.9531
.93	.0011	1.43	.2100	1.93	.7238	2.43	.9549
.94	.0013	1.44	.2197	1.94	.7319	2.44	.9569
.95	.0016	1.45	.2295	1.95	.7396	2.45	.9586
.96	.0020	1.46	.2396	1.96	.7474	2.46	.9605
.97	.0024	1.47	.2497	1.97	.7549	2.47	.9621
.98	.0028	1.48	.2601	1.98	.7624	2.48	.9638
.99	.0034	1.49	.2705	1.99	.7695	2.49	.9653
1.00	.0040	1.50	.2811	2.00	.7767	2.50	.9669
1.01	.0048	1.51	.2917	2.01	.7835	2.51	.9682
1.02	.0056	1.52	.3025	2.02	.7904	2.52	.9697
1.03	.0065	1.53	.3133	2.03	.7969	2.53	.9709
1.04	.0076	1.54	.3242	2.04	.8035	2.54	.9723
1.05	.0087	1.55	.3352	2.05	.8097	2.55	.9734
1.06	.0100	1.56	.3463	2.06	.8160	2.56	.9747
1.07	.0115	1.57	.3573	2.07	.8219	2.57	.9757
1.08	.0131	1.58	.3685	2.08	.8278	2.58	.9769
1.09	.0149	1.59	.3796	2.09	.8335	2.59	.9779
1.10	.0168	1.60	.3909	2.10	.8391	2.60	.9790
1.11	.0189	1.61	.4020	2.11	.8445	2.61	.9798
1.12	.0212	1.62	.4133	2.12	.8499	2.62	.9808
1.13	.0238	1.63	.4244	2.13	.8549	2.63	.9816
1.14	.0265	1.64	.4356	2.14	.8600	2.64	.9826
1.15	.0294	1.65	.4467	2.15	.8648	2.65	.9833
1.16	.0326	1.66	.4579	2.16	.8697	2.66	.9841
1.17	.0359	1.67	.4689	2.17	.8742	2.67	.9848
1.18	.0395	1.68	.4801	2.18	.8788	2.68	.9856
1.19	.0434	1.69	.4910	2.19	.8830	2.69	.9862
1.20	.0475	1.70	.5020	2.20	.8873	2.70	.9869
1.21	.0528	1.71	.5127	2.21	.8914	2.71	.9874
1.22	.0564	1.72	.5236	2.22	.8954	2.72	.9881
1.23	.0612	1.73	.5342	2.23	.8992	2.73	.9886
1.24	.0663	1.74	.5449	2.24	.9030	2.74	.9892
1.25	.0717	1.75	.5554	2.25	.9066	2.75	.9896
1.26	.0773	1.76	.5658	2.26	.9102	2.76	.9902
1.27	.0832	1.77	.5761	2.27	.9135	2.77	.9906
1.28	.0893	1.78	.5864	2.28	.9169	2.78	.9912
1.29	.0957	1.79	.5964	2.29	.9200	2.79	.9915
1.30	.1023	1.80	.6064	2.30	.9232	2.80	.9920
1.31	.1092	1.81	.6162	2.31	.9261	2.81	.9923
1.32	.1164	1.82	.6260	2.32	.9291	2.82	.9928
1.33	.1237	1.83	.6355	2.33	.9318	2.83	.9930
1.34	.1314	1.84	.6451	2.34	.9346	2.84	.9934
1.35	.1392	1.85	.6543	2.35	.9371	2.85	.9937
1.36	.1474	1.86	.6636	2.36	.9397	2.86	.9941
1.37	.1557	1.87	.6726	2.37	.9420	2.87	.9943
1.38	.1642	1.88	.6816	2.38	.9445	2.88	.9946
1.39	.1730	1.89	.6902	2.39	.9466	2.89	.9948
1.40	.1820	1.90	.6989	2.40	.9489	2.90	.9952
1.41	.1911	1.91	.7073	2.41	.9509	2.91	.9953

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