1 Model Problem

$$
\begin{aligned}
\Delta u &= 0 \quad \text{in} \quad \Omega = \Omega_1 \cup \Omega_2, \\
\mathsf{T}_D^2 u &= \mathfrak{g} \quad \text{on} \quad \Gamma_D, \\
\mathsf{T}_N^2 u &= \eta \quad \text{on} \quad \Gamma_N,
\end{aligned}
\tag{1}
$$

where T_D^2 and T_N^2 are the trace operators from within Ω_2. The dielectrics are homogeneous and isotropic in each subdomain.

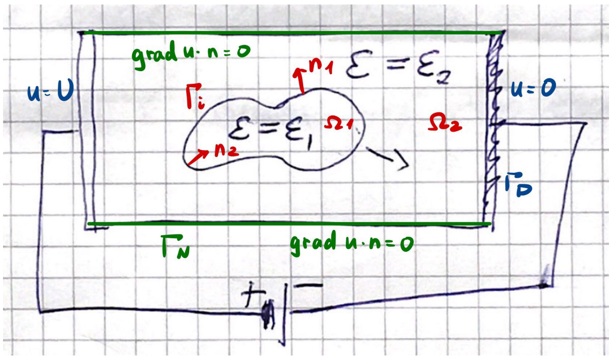

Figure 1: Geometric setting

2 BIE

2.1 Transmission conditions

On the interface Γ_i,

$$
\begin{aligned}
\mathsf{T}_D^1 u &= \mathsf{T}_D^2 u, \\
\varepsilon_1 \mathsf{T}_N^1 u = \varepsilon_1 \nabla u \cdot \boldsymbol{n}_1 &= -\varepsilon_2 \nabla u \cdot \boldsymbol{n}_2 = -\varepsilon_2 \mathsf{T}_N^2 u.
\end{aligned}
\tag{2}
$$

2.2 Unknown quantities

$$
\begin{aligned}
\mathfrak{u}_i &:= \mathsf{T}_D^2 u\big|_{\Gamma_i}, \\
\psi_i &:= \mathsf{T}_N^2 u\big|_{\Gamma_i}, \\
\mathfrak{u} &:= \mathsf{T}_D^2 u\big|_{\Gamma_N}, \\
\psi &:= \mathsf{T}_N^2 u\big|_{\Gamma_D}.
\end{aligned}
\tag{3}
$$

2.3 BIEs

In the following equations, Id may denote identity operators acting on different domains.

For subdomain Ω_1,

$$
(\frac{1}{2}\mathsf{Id} + \mathsf{K}_1)(\mathsf{T}_D^1 u) - \mathsf{V}_1(\mathsf{T}_N^1 u) = 0 \quad \text{in} \quad H^{\frac{1}{2}}(\partial\Omega_1), \tag{4}
$$

$$
-\mathsf{W}_1(\mathsf{T}_D^1 u) + (\frac{1}{2}\mathsf{Id} - \mathsf{K}_1')(\mathsf{T}_N^1 u) = 0 \quad \text{in} \quad H^{-\frac{1}{2}}(\partial\Omega_1). \tag{5}
$$

For subdomain Ω_2,

$$
(\frac{1}{2}\mathsf{Id} + \mathsf{K}_2)(\mathsf{T}_D^2 u) - \mathsf{V}_2(\mathsf{T}_N^2 u) = 0 \quad \text{in} \quad H^{\frac{1}{2}}(\partial\Omega_2), \tag{6}
$$

$$
-\mathsf{W}_2(\mathsf{T}_D^2 u) + (\frac{1}{2}\mathsf{Id} - \mathsf{K}_2')(\mathsf{T}_N^2 u) = 0 \quad \text{in} \quad H^{-\frac{1}{2}}(\partial\Omega_2). \tag{7}
$$

Using transmission conditions (2) and replacing traces by symbols defined in (1) and (3) ($\tilde{f}$ denotes the extension by zero to $\partial\Omega_2$ of f), the above equations can be rewritten as

$$(\frac{1}{2}\mathsf{Id} + \mathsf{K}_1)(\mathfrak{u}_i) + \frac{\varepsilon_2}{\varepsilon_1}\mathsf{V}_1(\psi_i) = 0 \quad \text{in} \quad H^{\frac{1}{2}}(\partial\Omega_1), \tag{8}$$

$$\mathsf{W}_1(\mathfrak{u}_i) + \frac{\varepsilon_2}{\varepsilon_1}(\frac{1}{2}\mathsf{Id} - \mathsf{K}_1')(\psi_i) = 0 \quad \text{in} \quad H^{-\frac{1}{2}}(\partial\Omega_1), \tag{9}$$

$$(\frac{1}{2}\mathsf{Id} + \mathsf{K}_2)(\tilde{\mathfrak{u}} + \tilde{\mathfrak{u}}_i + \tilde{\mathfrak{g}}) - \mathsf{V}_2(\tilde{\psi} + \tilde{\psi}_i + \tilde{\eta}) = 0 \quad \text{in} \quad H^{\frac{1}{2}}(\partial\Omega_2), \tag{10}$$

$$-\mathsf{W}_2(\tilde{\mathfrak{u}} + \tilde{\mathfrak{u}}_i + \tilde{\mathfrak{g}}) + (\frac{1}{2}\mathsf{Id} - \mathsf{K}_2')(\tilde{\psi} + \tilde{\psi}_i + \tilde{\eta}) = 0 \quad \text{in} \quad H^{-\frac{1}{2}}(\partial\Omega_2). \tag{11}$$

2.4 Variational formulation

1. Test (8) with $\phi_i \in H^{-\frac{1}{2}}(\partial\Omega_1)$.
2. Test (9) with $\mathfrak{v}_i \in H^{\frac{1}{2}}(\partial\Omega_1)$.
3. Test (10) with $\tilde{\phi}_i \in H^{-\frac{1}{2}}_{\Gamma_D\cup\Gamma_N}(\partial\Omega_2) = \{\varphi_i \in H^{-\frac{1}{2}}(\partial\Omega_2) : \varphi_i|_{\Gamma_D\cup\Gamma_N} = 0\}$.
4. Test (10) with $\tilde{\phi} \in H^{-\frac{1}{2}}_{\Gamma_i\cup\Gamma_N}(\partial\Omega_2) = \{\varphi \in H^{-\frac{1}{2}}(\partial\Omega_2) : \varphi|_{\Gamma_i\cup\Gamma_N} = 0\}$.
5. Test (11) with $\tilde{\mathfrak{v}}_i \in H^{\frac{1}{2}}_{\Gamma_D\cup\Gamma_N}(\partial\Omega_2) = \{\mathfrak{f}_i \in H^{\frac{1}{2}}(\partial\Omega_2) : \mathfrak{f}_i|_{\Gamma_D\cup\Gamma_N} = 0\}$.
6. Test (11) with $\tilde{\mathfrak{v}} \in H^{\frac{1}{2}}_{\Gamma_i\cup\Gamma_D}(\partial\Omega_2) = \{\mathfrak{f} \in H^{\frac{1}{2}}(\partial\Omega_2) : \mathfrak{f}|_{\Gamma_i\cup\Gamma_D} = 0\}$.
7. Replace operators in equations of step 1-2 using
$$\mathsf{V}_1 = \mathsf{V}_2|_{H^{-\frac{1}{2}}(\partial\Omega_1)}, \quad \mathsf{K}_1 = -\,\mathsf{K}_2|_{H^{\frac{1}{2}}(\partial\Omega_1)}, \quad \mathsf{K}_1' = -\,\mathsf{K}_2'|_{H^{-\frac{1}{2}}(\partial\Omega_1)}, \quad \mathsf{W}_1 = \mathsf{W}_2|_{H^{\frac{1}{2}}(\partial\Omega_1)}. \tag{12}$$
8. Identify $H^{-\frac{1}{2}}_{\Gamma_D\cup\Gamma_N}(\partial\Omega_2)$ with $H^{-\frac{1}{2}}(\partial\Omega_1)$ and subtract the equation of step 1 from that of step 3.
9. Identify $H^{\frac{1}{2}}_{\Gamma_D\cup\Gamma_N}(\partial\Omega_2)$ with $H^{\frac{1}{2}}(\partial\Omega_1)$ and subtract the equation of step 2 from that of step 5.

The variational formulation of the first-kind BIEs of (1) is

$$\mathfrak{u}_i \in H^{\frac{1}{2}}(\Gamma_i),\ \psi_i \in H^{-\frac{1}{2}}(\Gamma_i),\ \mathfrak{u} \in H^{\frac{1}{2}}_{\Gamma_D}(\Gamma_D\cup\Gamma_N),\ \psi \in H^{-\frac{1}{2}}_{\Gamma_N}(\Gamma_D\cup\Gamma_N):$$

$$\left(\frac{\varepsilon_1}{\varepsilon_2}+1\right)\mathsf{a}_{W,ii}(\mathfrak{u}_i,\mathfrak{v}_i) + 2\mathsf{a}_{K,ii}(\mathfrak{v}_i,\psi_i) + \mathsf{a}_{W,Ni}(\mathfrak{u},\mathfrak{v}_i) + \mathsf{a}_{K,iD}(\mathfrak{v}_i,\psi) = -\mathsf{b}_{W,Di}(\mathfrak{g},\mathfrak{v}_i) - \mathsf{b}_{K,iN}(\mathfrak{v}_i,\eta) \qquad \forall\mathfrak{v}_i \in H^{\frac{1}{2}}(\Gamma_i), \tag{13}$$

$$2\,\mathsf{a}_{K,ii}(\mathfrak{u}_i,\phi_i) - \left(\frac{\varepsilon_2}{\varepsilon_1}+1\right)\mathsf{a}_{V,ii}(\psi_i,\phi_i) + \mathsf{a}_{K,Ni}(\mathfrak{u},\phi_i) - \mathsf{a}_{V,Di}(\psi,\phi_i) = -\mathsf{b}_{K,Di}(\mathfrak{g},\phi_i) + \mathsf{b}_{V,Ni}(\eta,\phi_i) \qquad \forall\phi_i \in H^{-\frac{1}{2}}(\Gamma_i), \tag{14}$$

$$\mathsf{a}_{W,iN}(\mathfrak{u}_i,\mathfrak{v}) + \mathsf{a}_{K,Ni}(\mathfrak{v},\psi_i) + \mathsf{a}_{W,NN}(\mathfrak{u},\mathfrak{v}) + \mathsf{a}_{K,ND}(\mathfrak{v},\psi) = -\mathsf{b}_{W,DN}(\mathfrak{g},\mathfrak{v}) + \frac{1}{2}\ell_\eta(\mathfrak{v}) - \mathsf{b}_{K,NN}(\mathfrak{v},\eta) \qquad \forall\mathfrak{v} \in H^{\frac{1}{2}}_{\Gamma_D}(\Gamma_D\cup\Gamma_N), \tag{15}$$

$$\mathsf{a}_{K,iD}(\mathfrak{u}_i,\phi) - \mathsf{a}_{V,iD}(\psi_i,\phi) + \mathsf{a}_{K,ND}(\mathfrak{u},\phi) - \mathsf{a}_{V,DD}(\psi,\phi) = -\frac{1}{2}\ell_{\mathfrak{g}}(\phi) - \mathsf{b}_{K,DD}(\mathfrak{g},\phi) + \mathsf{b}_{V,ND}(\eta,\phi) \qquad \forall\phi \in H^{-\frac{1}{2}}_{\Gamma_N}(\Gamma_D\cup\Gamma_N), \tag{16}$$

with

$$
\begin{aligned}
\mathsf{a}_{V,mn}(\varphi,\phi) &:= \int_{\Gamma_n}\int_{\Gamma_m} G(\boldsymbol{x},\boldsymbol{y})\,\varphi(\boldsymbol{y})\,\phi(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{y})\mathrm{d}S(\boldsymbol{x}) && m,n\in\{i,D\},\\
\mathsf{a}_{K,mn}(\mathfrak{v},\phi) &:= \int_{\Gamma_n}\int_{\Gamma_m} \nabla_{\boldsymbol{y}} G(\boldsymbol{x},\boldsymbol{y})\cdot\boldsymbol{n}(\boldsymbol{y})\,\mathfrak{v}(\boldsymbol{y})\,\phi(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{y})\mathrm{d}S(\boldsymbol{x}) && m\in\{i,N\},\ n\in\{i,D\},\\
\mathsf{a}_{W,mn}(\mathfrak{f},\mathfrak{v}) &:= \int_{\Gamma_n}\int_{\Gamma_m} G(\boldsymbol{x},\boldsymbol{y})\,\frac{d\mathfrak{f}}{ds}(\boldsymbol{y})\,\frac{d\mathfrak{v}}{ds}(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{y})\mathrm{d}S(\boldsymbol{x}) && m,n\in\{i,N\},\\
\mathsf{b}_{V,Nm}(\eta,\phi) &:= \int_{\Gamma_m}\int_{\Gamma_N} G(\boldsymbol{x},\boldsymbol{y})\,\eta(\boldsymbol{y})\,\phi(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{y})\mathrm{d}S(\boldsymbol{x}) && m\in\{i,D\},\\
\mathsf{b}_{K,Dm}(\mathfrak{g},\phi) &:= \int_{\Gamma_m}\int_{\Gamma_D} \nabla_{\boldsymbol{y}} G(\boldsymbol{x},\boldsymbol{y})\cdot\boldsymbol{n}(\boldsymbol{y})\,\mathfrak{g}(\boldsymbol{y})\,\phi(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{y})\mathrm{d}S(\boldsymbol{x}) && m\in\{i,D\},\\
\mathsf{b}_{K,mN}(\mathfrak{v},\eta) &:= \int_{\Gamma_N}\int_{\Gamma_m} \nabla_{\boldsymbol{y}} G(\boldsymbol{x},\boldsymbol{y})\cdot\boldsymbol{n}(\boldsymbol{y})\,\mathfrak{v}(\boldsymbol{y})\,\eta(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{y})\mathrm{d}S(\boldsymbol{x}) && m\in\{i,N\},\\
\mathsf{b}_{W,Dm}(\mathfrak{g},\mathfrak{v}) &:= \int_{\Gamma_m}\int_{\Gamma_D} G(\boldsymbol{x},\boldsymbol{y})\,\frac{d\mathfrak{g}}{ds}(\boldsymbol{y})\,\frac{d\mathfrak{v}}{ds}(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{y})\mathrm{d}S(\boldsymbol{x}) && m\in\{i,N\},\\
\ell_{\mathfrak{g}}(\phi) &:= \int_{\Gamma_D} \mathfrak{g}(\boldsymbol{x})\,\phi(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{x}),\\
\ell_{\eta}(\mathfrak{v}) &:= \int_{\Gamma_N} \eta(\boldsymbol{x})\,\mathfrak{v}(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{x}),
\end{aligned}
\tag{17}
$$

and the fundamental solution $G : \{(\boldsymbol{x},\boldsymbol{y})\in\mathbb{R}^2\times\mathbb{R}^2 : \boldsymbol{x}\neq\boldsymbol{y}\}\to\mathbb{R}$

$$
G(\boldsymbol{x},\boldsymbol{y}) := -\frac{1}{2\pi}\log\|\boldsymbol{x}-\boldsymbol{y}\|. \tag{18}
$$

3 Shape Calculus

3.1 Virtual work principle

The domain $\Omega = \Omega_1\cup\Omega_2$ is fixed and the total energy is a function of the shape of subdomain Ω_1. $u = u(\Omega_1)$ is the solution of (1).

Energy of the electric field

$$
\begin{aligned}
\mathcal{E}_F(\Omega_1) &:= \frac{1}{2}\int_{\Omega_1}\varepsilon_1\nabla u(\boldsymbol{x})\cdot\nabla u(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} + \frac{1}{2}\int_{\Omega_2}\varepsilon_2\nabla u(\boldsymbol{x})\cdot\nabla u(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\\
&= \frac{\varepsilon_1}{2}\int_{\partial\Omega_1} u(\boldsymbol{x})\,\nabla u(\boldsymbol{x})\cdot\boldsymbol{n}_1(x)\,\mathrm{d}S(\boldsymbol{x}) + \frac{\varepsilon_2}{2}\int_{\partial\Omega_2} u(\boldsymbol{x})\,\nabla u(\boldsymbol{x})\cdot\boldsymbol{n}_2(x)\,\mathrm{d}S(\boldsymbol{x})\\
&= \frac{\varepsilon_2}{2}\int_{\Gamma_D}\mathfrak{g}(\boldsymbol{x})\,\psi(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{x}) + \frac{\varepsilon_2}{2}\int_{\Gamma_N}\mathfrak{u}(\boldsymbol{x})\,\eta(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{x}).
\end{aligned}
\tag{19}
$$

Energy stored in the battery

$$
\Delta\mathcal{E}_B(\Omega_1) = -2\Delta\mathcal{E}_F(\Omega_1). \tag{20}
$$

Total energy

$$
\mathcal{E}(\Omega_1) := \mathcal{E}_F(\Omega_1) + \mathcal{E}_B(\Omega_1) = -\frac{\varepsilon_2}{2}\int_{\Gamma_D}\mathfrak{g}(\boldsymbol{x})\,\psi(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{x}) - \frac{\varepsilon_2}{2}\int_{\Gamma_N}\mathfrak{u}(\boldsymbol{x})\,\eta(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{x}). \tag{21}
$$

Deformation vector field $\mathcal{V}\in\left(C_0^\infty(\Omega)\right)^2$ spawns the one-parameter family of perturbation maps

$$
\mathsf{T}^t_{\mathcal{V}} : \Omega\to\mathbb{R}^2,\quad \mathsf{T}^t_{\mathcal{V}}(\boldsymbol{x}) := \boldsymbol{x} + t\mathcal{V}(\boldsymbol{x}),\quad t\in\mathbb{R}. \tag{22}
$$

Deformed subdomains and interfaces

$$
\Omega_t := \mathsf{T}^t_{\mathcal{V}}(\Omega_1),\quad \Gamma_t := \mathsf{T}^t_{\mathcal{V}}(\Gamma_i), \tag{23}
$$

where $|t| < \delta(\mathcal{V})$ guarantees certain properties of deformed geometries.

3.2 Pullback of BIEs

3.2.1 t-dependent version of BIEs

The t-dependent version of (13)(14)(15)(16) is

$$\mathfrak{u}_i(t) \in H^{\frac{1}{2}}(\Gamma_t),\ \psi_i(t) \in H^{-\frac{1}{2}}(\Gamma_t),\ \mathfrak{u}(t) \in H^{\frac{1}{2}}_{\Gamma_D}(\Gamma_D \cup \Gamma_N),\ \psi(t) \in H^{-\frac{1}{2}}_{\Gamma_N}(\Gamma_D \cup \Gamma_N):$$

$$\begin{aligned}\left(\frac{\varepsilon_1}{\varepsilon_2}+1\right)\mathsf{a}_{W,ii}(t;\mathfrak{u}_i(t),\mathfrak{v}_i) + 2\,\mathsf{a}_{K,ii}(t;\mathfrak{v}_i,\psi_i(t)) + \mathsf{a}_{W,Ni}(t;\mathfrak{u}(t),\mathfrak{v}_i) + \mathsf{a}_{K,iD}(t;\mathfrak{v}_i,\psi(t))\\ = -\mathsf{b}_{W,Di}(t;\mathfrak{g},\mathfrak{v}_i) - \mathsf{b}_{K,iN}(t;\mathfrak{v}_i,\eta) \quad \forall \mathfrak{v}_i \in H^{\frac{1}{2}}(\Gamma_t),\end{aligned} \tag{24}$$

$$\begin{aligned}2\,\mathsf{a}_{K,ii}(t;\mathfrak{u}_i(t),\phi_i) - \left(\frac{\varepsilon_2}{\varepsilon_1}+1\right)\mathsf{a}_{V,ii}(t;\psi_i(t),\phi_i) + \mathsf{a}_{K,Ni}(t;\mathfrak{u}(t),\phi_i) - \mathsf{a}_{V,Di}(t;\psi(t),\phi_i)\\ = -\mathsf{b}_{K,Di}(t;\mathfrak{g},\phi_i) + \mathsf{b}_{V,Ni}(t;\eta,\phi_i) \quad \forall \phi_i \in H^{-\frac{1}{2}}(\Gamma_t),\end{aligned} \tag{25}$$

$$\begin{aligned}\mathsf{a}_{W,iN}(t;\mathfrak{u}_i(t),\mathfrak{v}) + \mathsf{a}_{K,Ni}(t;\mathfrak{v},\psi_i(t)) + \mathsf{a}_{W,NN}(\mathfrak{u}(t),\mathfrak{v}) + \mathsf{a}_{K,ND}(\mathfrak{v},\psi(t))\\ = -\mathsf{b}_{W,DN}(\mathfrak{g},\mathfrak{v}) + \frac{1}{2}\ell_\eta(\mathfrak{v}) - \mathsf{b}_{K,NN}(\mathfrak{v},\eta) \quad \forall \mathfrak{v} \in H^{\frac{1}{2}}_{\Gamma_D}(\Gamma_D \cup \Gamma_N).\end{aligned} \tag{26}$$

$$\begin{aligned}\mathsf{a}_{K,iD}(t;\mathfrak{u}_i(t),\phi) - \mathsf{a}_{V,iD}(t;\psi_i(t),\phi) + \mathsf{a}_{K,ND}(\mathfrak{u}(t),\phi) - \mathsf{a}_{V,DD}(\psi(t),\phi)\\ = -\frac{1}{2}\ell_{\mathfrak{g}}(\phi) - \mathsf{b}_{K,DD}(\mathfrak{g},\phi) + \mathsf{b}_{V,ND}(\eta,\phi) \quad \forall \phi \in H^{-\frac{1}{2}}_{\Gamma_N}(\Gamma_D \cup \Gamma_N),\end{aligned} \tag{27}$$

Note that Γ_D and Γ_N, as well as the variational forms defined on these boundaries, remain unchanged. The total energy (21) also becomes a function of t

$$\begin{aligned}&\mathcal{E}(\mathcal{V};t) = J(\mathfrak{u}(t),\psi(t)),\\ &J(\mathfrak{f},\varphi) := -\frac{\varepsilon_2}{2}\int_{\Gamma_D}\mathfrak{g}(\boldsymbol{x})\,\varphi(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{x}) - \frac{\varepsilon_2}{2}\int_{\Gamma_N}\mathfrak{f}(\boldsymbol{x})\,\eta(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{x}),\ \mathfrak{f} \in H^{\frac{1}{2}}_{\Gamma_D}(\Gamma_D \cup \Gamma_N),\ \varphi \in H^{-\frac{1}{2}}_{\Gamma_N}(\Gamma_D \cup \Gamma_N).\end{aligned} \tag{28}$$

3.2.2 Transformation rules

Let $\gamma_t : [0,1] \to \mathbb{R}^2$ be a C^2-parameterization of the deformed curve $\Sigma_t \in \Gamma_t$. Given a surface density $f : \Sigma_t \to \mathbb{R}$, intrinsically there exists a curve $\Sigma \in \Gamma_i$, a C^2-parameterization $\gamma : [0,1] \to \mathbb{R}^2$ and a density $\widehat{f} : \Sigma \to \mathbb{R}$ that describe the same quantity. To derive the explicit expression of $\widehat{f}$, consider the surface integral

$$\begin{aligned}\int_{\Sigma_t} f(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{x}) &= \int_0^1 f(\gamma_t(\tau))\|\dot{\gamma}_t(\tau)\|\mathrm{d}\tau\\ &= \int_0^1 f(\gamma_t(\gamma^{-1}(\gamma(\tau))))\frac{\|\dot{\gamma}_t(\tau)\|}{\|\dot{\gamma}(\tau)\|}\|\dot{\gamma}(\tau)\|\,\mathrm{d}\tau\\ &= \int_\Sigma f(\gamma_t(\gamma^{-1}(\widehat{\boldsymbol{x}})))\,\frac{\|\dot{\gamma}_t(\gamma^{-1}(\widehat{\boldsymbol{x}}))\|}{\|\dot{\gamma}(\gamma^{-1}(\widehat{\boldsymbol{x}}))\|}\,\mathrm{d}S(\widehat{\boldsymbol{x}})\\ &= \int_\Sigma \widehat{f}(\widehat{\boldsymbol{x}})\,\mathrm{d}S(\widehat{\boldsymbol{x}}).\end{aligned} \tag{29}$$

Then $\widehat{f}$, the pullback of f, can be written as

$$\widehat{f} = \left(\frac{\|\dot{\gamma}_t\|}{\|\dot{\gamma}\|}\,f \circ \gamma_t\right) \circ \gamma^{-1}. \tag{30}$$

The pullback is generally defined as $\widehat{f} = \omega_t\, f \circ \mathsf{T}^t_{\mathcal{V}}$. It's easy to check that the two definitions are equivalent with the following relationship

$$\gamma_t = \mathsf{T}^t_{\mathcal{V}} \circ \gamma = \gamma + t\mathcal{V} \circ \gamma, \quad \omega_t = \frac{\|\dot{\gamma}_t\|}{\|\dot{\gamma}\|} \circ \gamma^{-1}. \tag{31}$$

In this way, surface integral of the arclength derivative is transformed as follows

$$\begin{aligned}\int_{\Sigma_t} \frac{df}{ds_t}(\boldsymbol{x})\, \mathrm{d}S(\boldsymbol{x}) &= \int_0^1 \frac{1}{\|\dot{\gamma}_t(\tau)\|} \frac{df(\gamma_t(\tau))}{d\tau} \|\dot{\gamma}_t(\tau)\|\, \mathrm{d}\tau \\ &= \int_0^1 \frac{\|\dot{\gamma}(\tau)\|}{\|\dot{\gamma}_t(\tau)\|} \frac{d\widehat{f}(\gamma(\tau))}{d\tau} + \frac{d}{d\tau}\left(\frac{\|\dot{\gamma}(\tau)\|}{\|\dot{\gamma}_t(\tau)\|}\right) \widehat{f}(\gamma(\tau))\, \mathrm{d}\tau \\ &= \int_\Sigma \frac{1}{\omega_t(\widehat{\boldsymbol{x}})} \frac{d\widehat{f}}{ds}(\widehat{\boldsymbol{x}}) - \frac{1}{\omega_t^2(\widehat{\boldsymbol{x}})} \frac{d\omega_t}{ds}(\widehat{\boldsymbol{x}})\, \widehat{f}(\widehat{\boldsymbol{x}})\, \mathrm{d}S(\widehat{\boldsymbol{x}}).\end{aligned} \tag{32}$$

Generally, the surface integral of function $g : \Omega \to \mathbb{R}$ is transformed to Γ_i using

$$\int_{\Gamma_t} g(\boldsymbol{x})\, \mathrm{d}S(\boldsymbol{x}) = \int_{\Gamma_i} g(\mathsf{T}^t_{\mathcal{V}}(\widehat{\boldsymbol{x}}))\, \omega_t(\widehat{\boldsymbol{x}})\, \mathrm{d}S(\widehat{\boldsymbol{x}}), \quad \omega_t(\widehat{\boldsymbol{x}}) = \|\mathbf{C}(\mathsf{DT}^t_{\mathcal{V}}(\widehat{\boldsymbol{x}}))\, \boldsymbol{n}(\widehat{\boldsymbol{x}})\|, \tag{33}$$

where $\mathbf{C}(\mathbf{M})$ denotes the co-factor matrix for $\mathbf{M} \in \mathbb{R}^{2,2}$. The unit normal vector field $\boldsymbol{n}_t$ on Γ_t is transformed according to

$$\boldsymbol{n}_t(\boldsymbol{x}) = \frac{\mathbf{C}(\mathsf{DT}^t_{\mathcal{V}}(\widehat{\boldsymbol{x}}))\, \boldsymbol{n}(\widehat{\boldsymbol{x}})}{\omega_t(\widehat{\boldsymbol{x}})}, \quad \boldsymbol{x} := \mathsf{T}^t_{\mathcal{V}}(\widehat{\boldsymbol{x}}),\ \widehat{\boldsymbol{x}} \in \Gamma_i. \tag{34}$$

3.2.3 Transformed BIEs

Transformed variational BIEs

$$\widehat{\mathfrak{u}}_i(t) \in H^{\frac{1}{2}}(\Gamma_i),\ \widehat{\psi}_i(t) \in H^{-\frac{1}{2}}(\Gamma_i),\ \mathfrak{u}(t) \in H^{\frac{1}{2}}_{\Gamma_D}(\Gamma_D \cup \Gamma_N),\ \psi(t) \in H^{-\frac{1}{2}}_{\Gamma_N}(\Gamma_D \cup \Gamma_N):$$

$$\begin{aligned}\left(\frac{\varepsilon_1}{\varepsilon_2} + 1\right) \widehat{\mathsf{a}}_{W,ii}(t; \widehat{\mathfrak{u}}_i(t), \widehat{\mathfrak{v}}_i) + 2\, \widehat{\mathsf{a}}_{K,ii}(t; \widehat{\mathfrak{v}}_i, \widehat{\psi}_i(t)) + \widehat{\mathsf{a}}_{W,Ni}(t; \mathfrak{u}(t), \widehat{\mathfrak{v}}_i) + \widehat{\mathsf{a}}_{K,iD}(t; \widehat{\mathfrak{v}}_i, \psi(t)) \\ = -\widehat{\mathsf{b}}_{W,Di}(t; \mathfrak{g}, \widehat{\mathfrak{v}}_i) - \widehat{\mathsf{b}}_{K,iN}(t; \widehat{\mathfrak{v}}_i, \eta) \quad \forall \widehat{\mathfrak{v}}_i \in H^{\frac{1}{2}}(\Gamma_i),\end{aligned} \tag{35}$$

$$\begin{aligned}2\, \widehat{\mathsf{a}}_{K,ii}(t; \widehat{\mathfrak{u}}_i(t), \widehat{\phi}_i) - \left(\frac{\varepsilon_2}{\varepsilon_1} + 1\right) \widehat{\mathsf{a}}_{V,ii}(t; \widehat{\psi}_i(t), \widehat{\phi}_i) + \widehat{\mathsf{a}}_{K,Ni}(t; \mathfrak{u}(t), \widehat{\phi}_i) - \widehat{\mathsf{a}}_{V,Di}(t; \psi(t), \widehat{\phi}_i) \\ = -\widehat{\mathsf{b}}_{K,Di}(t; \mathfrak{g}, \widehat{\phi}_i) + \widehat{\mathsf{b}}_{V,Ni}(t; \eta, \widehat{\phi}_i) \quad \forall \widehat{\phi}_i \in H^{-\frac{1}{2}}(\Gamma_i),\end{aligned} \tag{36}$$

$$\begin{aligned}\widehat{\mathsf{a}}_{W,iN}(t; \widehat{\mathfrak{u}}_i(t), \mathfrak{v}) + \widehat{\mathsf{a}}_{K,Ni}(t; \mathfrak{v}, \widehat{\psi}_i(t)) + \mathsf{a}_{W,NN}(\mathfrak{u}(t), \mathfrak{v}) + \mathsf{a}_{K,ND}(\mathfrak{v}, \psi(t)) \\ = -\mathsf{b}_{W,DN}(\mathfrak{g}, \mathfrak{v}) + \frac{1}{2}\ell_\eta(\mathfrak{v}) - \mathsf{b}_{K,NN}(\mathfrak{v}, \eta) \quad \forall \mathfrak{v} \in H^{\frac{1}{2}}_{\Gamma_D}(\Gamma_D \cup \Gamma_N).\end{aligned} \tag{37}$$

$$\begin{aligned}\widehat{\mathsf{a}}_{K,iD}(t; \widehat{\mathfrak{u}}_i(t), \phi) - \widehat{\mathsf{a}}_{V,iD}(t; \widehat{\psi}_i(t), \phi) + \mathsf{a}_{K,ND}(\mathfrak{u}(t), \phi) - \mathsf{a}_{V,DD}(\psi(t), \phi) \\ = -\frac{1}{2}\ell_{\mathfrak{g}}(\phi) - \mathsf{b}_{K,DD}(\mathfrak{g}, \phi) + \mathsf{b}_{V,ND}(\eta, \phi) \quad \forall \phi \in H^{-\frac{1}{2}}_{\Gamma_N}(\Gamma_D \cup \Gamma_N),\end{aligned} \tag{38}$$

with

$$
\begin{aligned}
\widehat{\mathsf{a}}_{V,ii}(t;\widehat{\varphi}_i,\widehat{\phi}_i) &= \int_{\Gamma_i}\int_{\Gamma_i} G(\mathsf{T}^t_{\mathcal{V}}(\widehat{\boldsymbol{x}}),\mathsf{T}^t_{\mathcal{V}}(\widehat{\boldsymbol{y}}))\,\widehat{\varphi}_i(\widehat{\boldsymbol{y}})\,\widehat{\phi}_i(\widehat{\boldsymbol{x}})\,\mathrm{d}S(\widehat{\boldsymbol{y}})\mathrm{d}S(\widehat{\boldsymbol{x}}),\\
\widehat{\mathsf{a}}_{V,iD}(t;\widehat{\varphi}_i,\phi) &= \int_{\Gamma_D}\int_{\Gamma_i} G(\boldsymbol{x},\mathsf{T}^t_{\mathcal{V}}(\widehat{\boldsymbol{y}}))\,\widehat{\varphi}_i(\widehat{\boldsymbol{y}})\,\phi(\boldsymbol{x})\,\mathrm{d}S(\widehat{\boldsymbol{y}})\mathrm{d}S(\boldsymbol{x}),\\
\widehat{\mathsf{a}}_{V,Di}(t;\varphi,\widehat{\phi}_i) &= \int_{\Gamma_i}\int_{\Gamma_D} G(\mathsf{T}^t_{\mathcal{V}}(\widehat{\boldsymbol{x}}),\boldsymbol{y})\,\varphi(\boldsymbol{y})\,\widehat{\phi}_i(\widehat{\boldsymbol{x}})\,\mathrm{d}S(\boldsymbol{y})\mathrm{d}S(\widehat{\boldsymbol{x}}),\\
\widehat{\mathsf{a}}_{K,ii}(t;\widehat{\mathfrak{v}}_i,\widehat{\phi}_i) &= \int_{\Gamma_i}\int_{\Gamma_i} \nabla_{\boldsymbol{y}} G(\mathsf{T}^t_{\mathcal{V}}(\widehat{\boldsymbol{x}}),\mathsf{T}^t_{\mathcal{V}}(\widehat{\boldsymbol{y}}))\cdot\mathbf{C}(\mathsf{DT}^t_{\mathcal{V}}(\widehat{\boldsymbol{y}}))\,\boldsymbol{n}(\widehat{\boldsymbol{y}})\,\omega_t^{-1}(\widehat{\boldsymbol{y}})\,\widehat{\mathfrak{v}}_i(\widehat{\boldsymbol{y}})\,\widehat{\phi}_i(\widehat{\boldsymbol{x}})\,\mathrm{d}S(\widehat{\boldsymbol{y}})\mathrm{d}S(\widehat{\boldsymbol{x}}),\\
\widehat{\mathsf{a}}_{K,iD}(t;\widehat{\mathfrak{v}}_i,\phi) &= \int_{\Gamma_D}\int_{\Gamma_i} \nabla_{\boldsymbol{y}} G(\boldsymbol{x},\mathsf{T}^t_{\mathcal{V}}(\widehat{\boldsymbol{y}}))\cdot\mathbf{C}(\mathsf{DT}^t_{\mathcal{V}}(\widehat{\boldsymbol{y}}))\,\boldsymbol{n}(\widehat{\boldsymbol{y}})\,\omega_t^{-1}(\widehat{\boldsymbol{y}})\,\widehat{\mathfrak{v}}_i(\widehat{\boldsymbol{y}})\,\phi(\boldsymbol{x})\,\mathrm{d}S(\widehat{\boldsymbol{y}})\mathrm{d}S(\boldsymbol{x}),\\
\widehat{\mathsf{a}}_{K,Ni}(t;\mathfrak{v},\widehat{\phi}_i) &= \int_{\Gamma_i}\int_{\Gamma_N} \nabla_{\boldsymbol{y}} G(\mathsf{T}^t_{\mathcal{V}}(\widehat{\boldsymbol{x}}),\boldsymbol{y})\cdot\boldsymbol{n}(\boldsymbol{y})\,\mathfrak{v}(\boldsymbol{y})\,\widehat{\phi}_i(\widehat{\boldsymbol{x}})\,\mathrm{d}S(\boldsymbol{y})\mathrm{d}S(\widehat{\boldsymbol{x}}),\\
\widehat{\mathsf{a}}_{W,ii}(t;\widehat{\mathfrak{f}}_i,\widehat{\mathfrak{v}}_i) &= \int_{\Gamma_i}\int_{\Gamma_i} G(\mathsf{T}^t_{\mathcal{V}}(\widehat{\boldsymbol{x}}),\mathsf{T}^t_{\mathcal{V}}(\widehat{\boldsymbol{y}}))\left(\omega_t^{-1}(\widehat{\boldsymbol{y}})\frac{d\widehat{\mathfrak{f}}_i}{ds}(\widehat{\boldsymbol{y}})-\omega_t^{-2}(\widehat{\boldsymbol{y}})\frac{d\omega_t}{ds}(\widehat{\boldsymbol{y}})\,\widehat{\mathfrak{f}}_i(\widehat{\boldsymbol{y}})\right)\\
&\qquad\left(\omega_t^{-1}(\widehat{\boldsymbol{x}})\frac{d\widehat{\mathfrak{v}}_i}{ds}(\widehat{\boldsymbol{x}})-\omega_t^{-2}(\widehat{\boldsymbol{x}})\frac{d\omega_t}{ds}(\widehat{\boldsymbol{x}})\,\widehat{\mathfrak{v}}_i(\widehat{\boldsymbol{x}})\right)\,\mathrm{d}S(\widehat{\boldsymbol{y}})\mathrm{d}S(\widehat{\boldsymbol{x}}),\\
\widehat{\mathsf{a}}_{W,iN}(t;\widehat{\mathfrak{f}}_i,\mathfrak{v}) &= \int_{\Gamma_N}\int_{\Gamma_i} G(\boldsymbol{x},\mathsf{T}^t_{\mathcal{V}}(\widehat{\boldsymbol{y}}))\left(\omega_t^{-1}(\widehat{\boldsymbol{y}})\frac{d\widehat{\mathfrak{f}}_i}{ds}(\widehat{\boldsymbol{y}})-\omega_t^{-2}(\widehat{\boldsymbol{y}})\frac{d\omega_t}{ds}(\widehat{\boldsymbol{y}})\,\widehat{\mathfrak{f}}_i(\widehat{\boldsymbol{y}})\right)\frac{d\mathfrak{v}}{ds}(\boldsymbol{x})\,\mathrm{d}S(\widehat{\boldsymbol{y}})\mathrm{d}S(\boldsymbol{x}),\\
\widehat{\mathsf{a}}_{W,Ni}(t;\mathfrak{f},\widehat{\mathfrak{v}}_i) &= \int_{\Gamma_i}\int_{\Gamma_N} G(\mathsf{T}^t_{\mathcal{V}}(\widehat{\boldsymbol{x}}),\boldsymbol{y})\,\frac{d\mathfrak{f}}{ds}(\boldsymbol{y})\left(\omega_t^{-1}(\widehat{\boldsymbol{x}})\frac{d\widehat{\mathfrak{v}}_i}{ds}(\widehat{\boldsymbol{x}})-\omega_t^{-2}(\widehat{\boldsymbol{x}})\frac{d\omega_t}{ds}(\widehat{\boldsymbol{x}})\,\widehat{\mathfrak{v}}_i(\widehat{\boldsymbol{x}})\right)\,\mathrm{d}S(\boldsymbol{y})\mathrm{d}S(\widehat{\boldsymbol{x}}),\\
\widehat{\mathsf{b}}_{V,Ni}(t;\eta,\widehat{\phi}_i) &= \int_{\Gamma_i}\int_{\Gamma_N} G(\mathsf{T}^t_{\mathcal{V}}(\widehat{\boldsymbol{x}}),\boldsymbol{y})\,\eta(\boldsymbol{y})\,\widehat{\phi}_i(\widehat{\boldsymbol{x}})\,\mathrm{d}S(\boldsymbol{y})\mathrm{d}S(\widehat{\boldsymbol{x}}),\\
\widehat{\mathsf{b}}_{K,Di}(t;\mathfrak{g},\widehat{\phi}_i) &= \int_{\Gamma_i}\int_{\Gamma_D} \nabla_{\boldsymbol{y}} G(\mathsf{T}^t_{\mathcal{V}}(\widehat{\boldsymbol{x}}),\boldsymbol{y})\cdot\boldsymbol{n}(\boldsymbol{y})\,\mathfrak{g}(\boldsymbol{y})\,\widehat{\phi}_i(\widehat{\boldsymbol{x}})\,\mathrm{d}S(\boldsymbol{y})\mathrm{d}S(\widehat{\boldsymbol{x}}),\\
\widehat{\mathsf{b}}_{K,iN}(t;\widehat{\mathfrak{v}}_i,\eta) &= \int_{\Gamma_N}\int_{\Gamma_i} \nabla_{\boldsymbol{y}} G(\boldsymbol{x},\mathsf{T}^t_{\mathcal{V}}(\widehat{\boldsymbol{y}}))\cdot\mathbf{C}(\mathsf{DT}^t_{\mathcal{V}}(\widehat{\boldsymbol{y}}))\,\boldsymbol{n}(\widehat{\boldsymbol{y}})\,\omega_t^{-1}(\widehat{\boldsymbol{y}})\,\widehat{\mathfrak{v}}_i(\widehat{\boldsymbol{y}})\,\eta(\boldsymbol{x})\,\mathrm{d}S(\widehat{\boldsymbol{y}})\mathrm{d}S(\boldsymbol{x}),\\
\widehat{\mathsf{b}}_{W,Di}(t;\mathfrak{g},\widehat{\mathfrak{v}}_i) &= \int_{\Gamma_i}\int_{\Gamma_D} G(\mathsf{T}^t_{\mathcal{V}}(\widehat{\boldsymbol{x}}),\boldsymbol{y})\,\frac{d\mathfrak{g}}{ds}(\boldsymbol{y})\left(\omega_t^{-1}(\widehat{\boldsymbol{x}})\frac{d\widehat{\mathfrak{v}}_i}{ds}(\widehat{\boldsymbol{x}})-\omega_t^{-2}(\widehat{\boldsymbol{x}})\frac{d\omega_t}{ds}(\widehat{\boldsymbol{x}})\,\widehat{\mathfrak{v}}_i(\widehat{\boldsymbol{x}})\right)\,\mathrm{d}S(\boldsymbol{y})\mathrm{d}S(\widehat{\boldsymbol{x}}).
\end{aligned}
\tag{39}
$$

3.3 BIE-Constrained shape derivative

3.3.1 Lagrangian function

Define the Lagrangian function

$$\begin{aligned}
L(t;(\widehat{\mathfrak{f}}_i,\widehat{\varphi}_i,\mathfrak{f},\varphi),&(\widehat{\mathfrak{v}}_i,\widehat{\phi}_i,\mathfrak{v},\phi)) \\
:=&J(\mathfrak{f},\varphi) \\
&+\left(\frac{\varepsilon_1}{\varepsilon_2}+1\right)\widehat{\mathsf{a}}_{W,ii}(t;\widehat{\mathfrak{f}}_i,\widehat{\mathfrak{v}}_i)+2\,\widehat{\mathsf{a}}_{K,ii}(t;\widehat{\mathfrak{v}}_i,\widehat{\varphi}_i)+\widehat{\mathsf{a}}_{W,Ni}(t;\mathfrak{f},\widehat{\mathfrak{v}}_i)+\widehat{\mathsf{a}}_{K,iD}(t;\widehat{\mathfrak{v}}_i,\varphi) \\
&+\widehat{\mathsf{b}}_{W,Di}(t;\mathfrak{g},\widehat{\mathfrak{v}}_i)+\widehat{\mathsf{b}}_{K,iN}(t;\widehat{\mathfrak{v}}_i,\eta) \\
&+2\,\widehat{\mathsf{a}}_{K,ii}(t;\widehat{\mathfrak{f}}_i,\widehat{\phi}_i)-\left(\frac{\varepsilon_2}{\varepsilon_1}+1\right)\widehat{\mathsf{a}}_{V,ii}(t;\widehat{\varphi}_i,\widehat{\phi}_i)+\widehat{\mathsf{a}}_{K,Ni}(t;\mathfrak{f},\widehat{\phi}_i)-\widehat{\mathsf{a}}_{V,Di}(t;\varphi,\widehat{\phi}_i) \\
&+\widehat{\mathsf{b}}_{K,Di}(t;\mathfrak{g},\widehat{\phi}_i)-\widehat{\mathsf{b}}_{V,Ni}(t;\eta,\widehat{\phi}_i) \\
&+\widehat{\mathsf{a}}_{W,iN}(t;\widehat{\mathfrak{f}}_i,\mathfrak{v})+\widehat{\mathsf{a}}_{K,Ni}(t;\mathfrak{v},\widehat{\varphi}_i)+\mathsf{a}_{W,NN}(\mathfrak{f},\mathfrak{v})+\mathsf{a}_{K,ND}(\mathfrak{v},\varphi) \\
&+\mathsf{b}_{W,DN}(\mathfrak{g},\mathfrak{v})-\frac{1}{2}\ell_\eta(\mathfrak{v})+\mathsf{b}_{K,NN}(\mathfrak{v},\eta) \\
&+\widehat{\mathsf{a}}_{K,iD}(t;\widehat{\mathfrak{f}}_i,\phi)-\widehat{\mathsf{a}}_{V,iD}(t;\widehat{\varphi}_i,\phi)+\mathsf{a}_{K,ND}(\mathfrak{f},\phi)-\mathsf{a}_{V,DD}(\varphi,\phi) \\
&+\frac{1}{2}\ell_{\mathfrak{g}}(\phi)+\mathsf{b}_{K,DD}(\mathfrak{g},\phi)-\mathsf{b}_{V,ND}(\eta,\phi).
\end{aligned} \tag{40}$$

Then $\widehat{\mathcal{E}}(\mathcal{V};t)$ can be expressed as

$$\begin{aligned}
\mathcal{E}(\mathcal{V};t)=J(\mathfrak{u}(t),\psi(t))=L(t;(\widehat{\mathfrak{u}}_i(t),\widehat{\psi}_i(t),\mathfrak{u}(t),\psi(t)),(\widehat{\mathfrak{v}}_i,\widehat{\phi}_i,\mathfrak{v},\phi)) \\
\forall(\widehat{\mathfrak{v}}_i,\widehat{\phi}_i,\mathfrak{v},\phi)\in H^{\frac{1}{2}}(\Gamma_i)\times H^{-\frac{1}{2}}(\Gamma_i)\times H^{\frac{1}{2}}_{\Gamma_D}(\Gamma_D\cup\Gamma_N)\times H^{-\frac{1}{2}}_{\Gamma_N}(\Gamma_D\cup\Gamma_N).
\end{aligned} \tag{41}$$

3.3.2 Adjoint problem

The shape derivative $\frac{d\mathcal{E}}{d\Omega_1}(\Omega_1;\mathcal{V})=\frac{d\widehat{\mathcal{E}}}{dt}(\mathcal{V};0)$ can be computed as the derivative of the Lagrangian function with respect to t. Since the dependence of the state solution $(\widehat{\mathfrak{u}}_i(t),\widehat{\psi}_i(t),\mathfrak{u}(t),\psi(t))$ on t is complicated, solve the adjoint variational problem to eliminate the partial derivative of L with respect to it: seek $(\rho_i,\pi_t,\rho,\pi)\in H^{\frac{1}{2}}(\Gamma_i)\times H^{-\frac{1}{2}}(\Gamma_i)\times H^{\frac{1}{2}}_{\Gamma_D}(\Gamma_D\cup\Gamma_N)\times H^{-\frac{1}{2}}_{\Gamma_N}(\Gamma_D\cup\Gamma_N)$ such that

$$\begin{aligned}
\left\langle\frac{\partial L}{\partial(\widehat{\mathfrak{f}}_i,\widehat{\varphi}_i,\mathfrak{f},\varphi)}(0;(\widehat{\mathfrak{u}}_i(0),\widehat{\psi}_i(0),\mathfrak{u}(0),\psi(0)),(\rho_i,\pi_i,\rho,\pi)),(\widehat{\mathfrak{v}}_i,\widehat{\phi},\mathfrak{v},\phi)\right\rangle=0 \\
\forall(\widehat{\mathfrak{v}}_i,\widehat{\phi}_i,\mathfrak{v},\phi)\in H^{\frac{1}{2}}(\Gamma_i)\times H^{-\frac{1}{2}}(\Gamma_i)\times H^{\frac{1}{2}}_{\Gamma_D}(\Gamma_D\cup\Gamma_N)\times H^{-\frac{1}{2}}_{\Gamma_N}(\Gamma_D\cup\Gamma_N).
\end{aligned} \tag{42}$$

To be specific,

$$\rho_i\in H^{\frac{1}{2}}(\Gamma_i),\ \pi_i\in H^{-\frac{1}{2}}(\Gamma_i),\ \rho\in H^{\frac{1}{2}}_{\Gamma_D}(\Gamma_D\cup\Gamma_N),\ \pi\in H^{-\frac{1}{2}}_{\Gamma_N}(\Gamma_D\cup\Gamma_N):$$

$$\left(\frac{\varepsilon_1}{\varepsilon_2}+1\right)\widehat{\mathsf{a}}_{W,ii}(0;\widehat{\mathfrak{v}}_i,\rho_i)+2\,\widehat{\mathsf{a}}_{K,ii}(0;\widehat{\mathfrak{v}}_i,\pi_i)+\widehat{\mathsf{a}}_{W,iN}(0;\widehat{\mathfrak{v}}_i,\rho)+\widehat{\mathsf{a}}_{K,iD}(0;\widehat{\mathfrak{v}}_i,\pi)=0\quad\forall\widehat{\mathfrak{v}}_i\in H^{\frac{1}{2}}(\Gamma_i), \tag{43}$$

$$2\,\widehat{\mathsf{a}}_{K,ii}(0;\rho_i,\widehat{\phi}_i)-\left(\frac{\varepsilon_2}{\varepsilon_1}+1\right)\widehat{\mathsf{a}}_{V,ii}(0;\widehat{\phi}_i,\pi_i)+\widehat{\mathsf{a}}_{K,Ni}(0;\rho,\widehat{\phi}_i)-\widehat{\mathsf{a}}_{V,iD}(0;\widehat{\phi}_i,\pi)=0\quad\forall\widehat{\phi}_i\in H^{-\frac{1}{2}}(\Gamma_i), \tag{44}$$

$$\begin{aligned}
\widehat{\mathsf{a}}_{W,Ni}(0;\mathfrak{v},\rho_i)+\widehat{\mathsf{a}}_{K,Ni}(0;\mathfrak{v},\pi_i)+\mathsf{a}_{W,NN}(\mathfrak{v},\rho)+\mathsf{a}_{K,ND}(\mathfrak{v},\pi)=-\left\langle\frac{\partial J}{\partial\mathfrak{f}}(\mathfrak{u}(0),\psi(0)),\mathfrak{v}\right\rangle \\
\forall\mathfrak{v}\in H^{\frac{1}{2}}_{\Gamma_D}(\Gamma_D\cup\Gamma_N),
\end{aligned} \tag{45}$$

$$\widehat{\mathsf{a}}_{K,iD}(0;\rho_i,\phi) - \widehat{\mathsf{a}}_{V,Di}(0;\phi,\pi_i) + \mathsf{a}_{K,ND}(\rho,\phi) - \mathsf{a}_{V,DD}(\phi,\pi) = -\left\langle \frac{\partial J}{\partial \varphi}(\mathfrak{u}(0),\psi(0)),\phi \right\rangle \quad \forall \phi \in H^{-\frac{1}{2}}_{\Gamma_N}(\Gamma_D\cup\Gamma_N), \tag{46}$$

which is equivalent to

$$\left(\frac{\varepsilon_1}{\varepsilon_2}+1\right)\mathsf{a}_{W,ii}(0;\mathfrak{v}_i,\rho_i) + 2\,\mathsf{a}_{K,ii}(0;\mathfrak{v}_i,\pi_i) + \mathsf{a}_{W,iN}(0;\mathfrak{v}_i,\rho) + \mathsf{a}_{K,iD}(0;\mathfrak{v}_i,\pi) = 0 \quad \forall \mathfrak{v}_i \in H^{\frac{1}{2}}(\Gamma_i), \tag{47}$$

$$2\,\mathsf{a}_{K,ii}(0;\rho_i,\phi_i) - \left(\frac{\varepsilon_2}{\varepsilon_1}+1\right)\mathsf{a}_{V,ii}(0;\phi_i,\pi_i) + \mathsf{a}_{K,Ni}(0;\rho,\phi_i) - \mathsf{a}_{V,iD}(0;\phi_i,\pi) = 0 \quad \forall \phi_i \in H^{-\frac{1}{2}}(\Gamma_i), \tag{48}$$

$$\mathsf{a}_{W,Ni}(0;\mathfrak{v},\rho_i) + \mathsf{a}_{K,Ni}(0;\mathfrak{v},\pi_i) + \mathsf{a}_{W,NN}(\mathfrak{v},\rho) + \mathsf{a}_{K,ND}(\mathfrak{v},\pi) = \frac{\varepsilon_2}{2}\int_{\Gamma_N} \mathfrak{v}(\boldsymbol{x})\,\eta(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{x}) \quad \forall \mathfrak{v} \in H^{\frac{1}{2}}_{\Gamma_D}(\Gamma_D\cup\Gamma_N), \tag{49}$$

$$\mathsf{a}_{K,iD}(0;\rho_i,\phi) - \mathsf{a}_{V,Di}(0;\phi,\pi_i) + \mathsf{a}_{K,ND}(\rho,\phi) - \mathsf{a}_{V,DD}(\phi,\pi) = \frac{\varepsilon_2}{2}\int_{\Gamma_D} \mathfrak{g}(\boldsymbol{x})\,\phi(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{x}) \quad \forall \phi \in H^{-\frac{1}{2}}_{\Gamma_N}(\Gamma_D\cup\Gamma_N). \tag{50}$$

3.3.3 Shape derivative

$$\begin{aligned}
\frac{d\widehat{\mathcal{E}}}{dt}(\mathcal{V};0) &= \frac{\partial L}{\partial t}(0;(\widehat{\mathfrak{u}}_i(0),\widehat{\psi}_i(0),\mathfrak{u}(0),\psi(0)),(\rho_i,\pi_i,\rho,\pi)) \\
&= \left(\frac{\varepsilon_1}{\varepsilon_2}+1\right)\frac{\partial\widehat{\mathsf{a}}_{W,ii}}{\partial t}(0;\mathfrak{u}_i,\rho_i) + 2\frac{\partial\widehat{\mathsf{a}}_{K,ii}}{\partial t}(0;\rho_i,\psi_i) + \frac{\partial\widehat{\mathsf{a}}_{W,Ni}}{\partial t}(0;\mathfrak{u},\rho_i) + \frac{\partial\widehat{\mathsf{a}}_{K,iD}}{\partial t}(0;\rho_i,\psi) \\
&\quad + \frac{\partial\widehat{\mathsf{b}}_{W,Di}}{\partial t}(0;\mathfrak{g},\rho_i) + \frac{\partial\widehat{\mathsf{b}}_{K,iN}}{\partial t}(0;\rho_i,\eta) \\
&\quad + 2\frac{\partial\widehat{\mathsf{a}}_{K,ii}}{\partial t}(0;\mathfrak{u}_i,\pi_i) - \left(\frac{\varepsilon_2}{\varepsilon_1}+1\right)\frac{\partial\widehat{\mathsf{a}}_{V,ii}}{\partial t}(0;\psi_i,\pi_i) + \frac{\partial\widehat{\mathsf{a}}_{K,Ni}}{\partial t}(0;\mathfrak{u},\pi_i) - \frac{\partial\widehat{\mathsf{a}}_{V,Di}}{\partial t}(0;\psi,\pi_i) \\
&\quad + \frac{\partial\widehat{\mathsf{b}}_{K,Di}}{\partial t}(0;\mathfrak{g},\pi_i) - \frac{\partial\widehat{\mathsf{b}}_{V,Ni}}{\partial t}(0;\eta,\pi_i) \\
&\quad + \frac{\partial\widehat{\mathsf{a}}_{W,iN}}{\partial t}(0;\mathfrak{u}_i,\rho) + \frac{\partial\widehat{\mathsf{a}}_{K,Ni}}{\partial t}(0;\rho,\psi_i) \\
&\quad + \frac{\partial\widehat{\mathsf{a}}_{K,iD}}{\partial t}(0;\mathfrak{u}_i,\pi) - \frac{\partial\widehat{\mathsf{a}}_{V,iD}}{\partial t}(0;\psi_i,\pi),
\end{aligned} \tag{51}$$

with building blocks

$$\begin{aligned}
\frac{\partial\widehat{\mathsf{a}}_{V,ii}}{\partial t}(0;\psi_i,\pi_i) &= \int_{\Gamma_i}\int_{\Gamma_i} \left.\frac{dG(\mathsf{T}^t_{\mathcal{V}}(\widehat{\boldsymbol{x}}),\mathsf{T}^t_{\mathcal{V}}(\widehat{\boldsymbol{y}}))}{dt}\right|_{t=0} \psi_i(\widehat{\boldsymbol{y}})\,\pi_i(\widehat{\boldsymbol{x}})\,\mathrm{d}S(\widehat{\boldsymbol{y}})\mathrm{d}S(\widehat{\boldsymbol{x}}) \\
&= \int_{\Gamma_i}\int_{\Gamma_i} \left(\nabla_{\boldsymbol{x}}G(\boldsymbol{x},\boldsymbol{y})\cdot\mathcal{V}(\boldsymbol{x}) + \nabla_{\boldsymbol{y}}G(\boldsymbol{x},\boldsymbol{y})\cdot\mathcal{V}(\boldsymbol{y})\right)\psi_i(\boldsymbol{y})\,\pi_i(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{y})\mathrm{d}S(\boldsymbol{x}) \\
&= -\frac{1}{2\pi}\int_{\Gamma_i}\int_{\Gamma_i} \frac{\boldsymbol{x}-\boldsymbol{y}}{\|\boldsymbol{x}-\boldsymbol{y}\|^2}\cdot(\mathcal{V}(\boldsymbol{x})-\mathcal{V}(\boldsymbol{y}))\,\psi_i(\boldsymbol{y})\,\pi_i(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{y})\mathrm{d}S(\boldsymbol{x}),
\end{aligned} \tag{52}$$

$$
\begin{aligned}
\frac{\partial\widehat{\mathsf{a}}_{K,ii}}{\partial t}(0;\mathfrak{u}_i,\pi_i) &= \int_{\Gamma_i}\int_{\Gamma_i} \frac{d\left(\nabla_{\boldsymbol{y}} G(\mathsf{T}^t_{\mathcal{V}}(\widehat{\boldsymbol{x}}),\mathsf{T}^t_{\mathcal{V}}(\widehat{\boldsymbol{y}}))\cdot\mathbf{C}(\mathsf{DT}^t_{\mathcal{V}}(\widehat{\boldsymbol{y}}))\,\boldsymbol{n}(\widehat{\boldsymbol{y}})\,\omega_t^{-1}(\widehat{\boldsymbol{y}})\right)}{dt}\Bigg|_{t=0} \mathfrak{u}_i(\widehat{\boldsymbol{y}})\,\pi_i(\widehat{\boldsymbol{x}})\,\mathrm{d}S(\widehat{\boldsymbol{y}})\mathrm{d}S(\widehat{\boldsymbol{x}})\\
&= \frac{1}{2\pi}\int_{\Gamma_i}\int_{\Gamma_i} \frac{d}{dt}\left(\frac{\mathsf{T}^t_{\mathcal{V}}(\boldsymbol{x})-\mathsf{T}^t_{\mathcal{V}}(\boldsymbol{y})}{\|\mathsf{T}^t_{\mathcal{V}}(\boldsymbol{x})-\mathsf{T}^t_{\mathcal{V}}(\boldsymbol{y})\|^2}\right)\Bigg|_{t=0}\cdot\boldsymbol{n}(\boldsymbol{y})\,\mathfrak{u}_i(\boldsymbol{y})\,\pi_i(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{y})\mathrm{d}S(\boldsymbol{x})\\
&+\int_{\Gamma_i}\int_{\Gamma_i}\nabla_{\boldsymbol{y}}G(\boldsymbol{x},\boldsymbol{y})\cdot\left(\nabla\cdot\mathcal{V}(\boldsymbol{y})\,\boldsymbol{n}(\boldsymbol{y})-\mathsf{D}\mathcal{V}^\top(\boldsymbol{y})\,\boldsymbol{n}(\boldsymbol{y})\right)\mathfrak{u}_i(\boldsymbol{y})\,\pi_i(\boldsymbol{y})\,\mathrm{d}S(\boldsymbol{y})\mathrm{d}S(\boldsymbol{x})\\
&-\int_{\Gamma_i}\int_{\Gamma_i}\nabla_{\boldsymbol{y}}G(\boldsymbol{x},\boldsymbol{y})\cdot\boldsymbol{n}(\boldsymbol{y})\left(\nabla\cdot\mathcal{V}(\boldsymbol{y})-\boldsymbol{n}(\boldsymbol{y})\cdot\mathsf{D}\mathcal{V}^\top(\boldsymbol{y})\,\boldsymbol{n}(\boldsymbol{y})\right)\mathfrak{u}_i(\boldsymbol{y})\,\pi_i(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{y})\mathrm{d}S(\boldsymbol{x})\\
&=-\frac{1}{2\pi}\int_{\Gamma_i}\int_{\Gamma_i}2\,\frac{\boldsymbol{x}-\boldsymbol{y}}{\|\boldsymbol{x}-\boldsymbol{y}\|^4}\cdot\boldsymbol{n}(\boldsymbol{y})\left((\boldsymbol{x}-\boldsymbol{y})\cdot(\mathcal{V}(\boldsymbol{x})-\mathcal{V}(\boldsymbol{y}))\right)\mathfrak{u}_i(\boldsymbol{y})\,\pi_i(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{y})\mathrm{d}S(\boldsymbol{x})\\
&+\frac{1}{2\pi}\int_{\Gamma_i}\int_{\Gamma_i}\frac{\boldsymbol{n}(\boldsymbol{y})\cdot(\mathcal{V}(\boldsymbol{x})-\mathcal{V}(\boldsymbol{y}))}{\|\boldsymbol{x}-\boldsymbol{y}\|^2}\,\mathfrak{u}_i(\boldsymbol{y})\,\pi_i(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{y})\mathrm{d}S(\boldsymbol{x})\\
&-\frac{1}{2\pi}\int_{\Gamma_i}\int_{\Gamma_i}\frac{\boldsymbol{x}-\boldsymbol{y}}{\|\boldsymbol{x}-\boldsymbol{y}\|^2}\cdot\mathsf{D}\mathcal{V}^\top(\boldsymbol{y})\,\boldsymbol{n}(\boldsymbol{y})\,\mathfrak{u}_i(\boldsymbol{y})\,\pi_i(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{y})\mathrm{d}S(\boldsymbol{x})\\
&+\frac{1}{2\pi}\int_{\Gamma_i}\int_{\Gamma_i}\frac{\boldsymbol{x}-\boldsymbol{y}}{\|\boldsymbol{x}-\boldsymbol{y}\|^2}\cdot\boldsymbol{n}(\boldsymbol{y})\left(\boldsymbol{n}(\boldsymbol{y})\cdot\mathsf{D}\mathcal{V}^\top(\boldsymbol{y})\,\boldsymbol{n}(\boldsymbol{y})\right)\mathfrak{u}_i(\boldsymbol{y})\,\pi_i(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{y})\mathrm{d}S(\boldsymbol{x}),
\end{aligned}
\tag{53}
$$

$$
\begin{aligned}
\frac{\partial\widehat{\mathsf{a}}_{W,ii}}{\partial t}(0;\mathfrak{u}_i,\rho_i) &= \int_{\Gamma_i}\int_{\Gamma_i}\frac{dG(\mathsf{T}^t_{\mathcal{V}}(\widehat{\boldsymbol{x}}),\mathsf{T}^t_{\mathcal{V}}(\widehat{\boldsymbol{y}}))}{dt}\Bigg|_{t=0}\frac{d\mathfrak{u}_i}{ds}(\widehat{\boldsymbol{y}})\,\frac{d\rho_i}{ds}(\widehat{\boldsymbol{x}})\,\mathrm{d}S(\widehat{\boldsymbol{y}})\mathrm{d}S(\widehat{\boldsymbol{x}})\\
&+\int_{\Gamma_i}\int_{\Gamma_i}G(\widehat{\boldsymbol{x}},\widehat{\boldsymbol{y}})\,\frac{d}{dt}\left(\omega_t^{-1}(\widehat{\boldsymbol{y}})\frac{d\mathfrak{u}_i}{ds}(\widehat{\boldsymbol{y}})-\omega_t^{-2}(\widehat{\boldsymbol{y}})\frac{d\omega_t}{ds}\,\mathfrak{u}_i(\widehat{\boldsymbol{y}})\right)\Bigg|_{t=0}\frac{d\rho_i}{ds}(\widehat{\boldsymbol{x}})\,\mathrm{d}S(\widehat{\boldsymbol{y}})\mathrm{d}S(\widehat{\boldsymbol{x}})\\
&+\int_{\Gamma_i}\int_{\Gamma_i}G(\widehat{\boldsymbol{x}},\widehat{\boldsymbol{y}})\,\frac{d\mathfrak{u}_i}{ds}(\widehat{\boldsymbol{y}})\,\frac{d}{dt}\left(\omega_t^{-1}(\widehat{\boldsymbol{x}})\frac{d\rho_i}{ds}(\widehat{\boldsymbol{x}})-\omega_t^{-2}(\widehat{\boldsymbol{x}})\frac{d\omega_t}{ds}\,\rho_i(\widehat{\boldsymbol{x}})\right)\Bigg|_{t=0}\mathrm{d}S(\widehat{\boldsymbol{y}})\mathrm{d}S(\widehat{\boldsymbol{x}})\\
&=\int_{\Gamma_i}\int_{\Gamma_i}\left(\nabla_{\boldsymbol{x}}G(\boldsymbol{x},\boldsymbol{y})\cdot\mathcal{V}(\boldsymbol{x})+\nabla_{\boldsymbol{y}}G(\boldsymbol{x},\boldsymbol{y})\cdot\mathcal{V}(\boldsymbol{y})\right)\frac{d\mathfrak{u}_i}{ds}(\boldsymbol{y})\,\frac{d\rho_i}{ds}(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{y})\mathrm{d}S(\boldsymbol{x})\\
&-\int_{\Gamma_i}\int_{\Gamma_i}G(\boldsymbol{x},\boldsymbol{y})\left(\nabla\cdot\mathcal{V}(\boldsymbol{y})-\boldsymbol{n}(\boldsymbol{y})\cdot\mathsf{D}\mathcal{V}^\top(\boldsymbol{y})\,\boldsymbol{n}(\boldsymbol{y})\right)\frac{d\mathfrak{u}_i}{ds}(\boldsymbol{y})\,\frac{d\rho_i}{ds}(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{y})\mathrm{d}S(\boldsymbol{x})\\
&-\int_{\Gamma_i}\int_{\Gamma_i}G(\boldsymbol{x},\boldsymbol{y})\,\frac{d}{dt}\left(\frac{d\omega_t}{ds}(\boldsymbol{y})\right)\Bigg|_{t=0}\mathfrak{u}_i(\boldsymbol{y})\,\frac{d\rho_i}{ds}(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{y})\mathrm{d}S(\boldsymbol{x})\\
&-\int_{\Gamma_i}\int_{\Gamma_i}G(\boldsymbol{x},\boldsymbol{y})\left(\nabla\cdot\mathcal{V}(\boldsymbol{x})-\boldsymbol{n}(\boldsymbol{x})\cdot\mathsf{D}\mathcal{V}^\top(\boldsymbol{x})\,\boldsymbol{n}(\boldsymbol{x})\right)\frac{d\mathfrak{u}_i}{ds}(\boldsymbol{y})\,\frac{d\rho_i}{ds}(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{y})\mathrm{d}S(\boldsymbol{x})\\
&-\int_{\Gamma_i}\int_{\Gamma_i}G(\boldsymbol{x},\boldsymbol{y})\,\frac{d}{dt}\left(\frac{d\omega_t}{ds}(\boldsymbol{x})\right)\Bigg|_{t=0}\frac{d\mathfrak{u}_i}{ds}(\boldsymbol{x})\,\rho_i(\boldsymbol{y})\,\mathrm{d}S(\boldsymbol{y})\mathrm{d}S(\boldsymbol{x})\\
&=-\frac{1}{2\pi}\int_{\Gamma_i}\int_{\Gamma_i}\frac{\boldsymbol{x}-\boldsymbol{y}}{\|\boldsymbol{x}-\boldsymbol{y}\|^2}\cdot(\mathcal{V}(\boldsymbol{x})-\mathcal{V}(\boldsymbol{y}))\,\frac{d\mathfrak{u}_i}{ds}(\boldsymbol{y})\,\frac{d\rho_i}{ds}(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{y})\mathrm{d}S(\boldsymbol{x})\\
&+\frac{1}{2\pi}\int_{\Gamma_i}\int_{\Gamma_i}\log\|\boldsymbol{x}-\boldsymbol{y}\|\left(\nabla\cdot\mathcal{V}(\boldsymbol{y})-\boldsymbol{n}(\boldsymbol{y})\cdot\mathsf{D}\mathcal{V}^\top(\boldsymbol{y})\,\boldsymbol{n}(\boldsymbol{y})\right)\frac{d\mathfrak{u}_i}{ds}(\boldsymbol{y})\,\frac{d\rho_i}{ds}(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{y})\mathrm{d}S(\boldsymbol{x})\\
&+\frac{1}{2\pi}\int_{\Gamma_i}\int_{\Gamma_i}\log\|\boldsymbol{x}-\boldsymbol{y}\|\,\frac{d}{dt}\left(\frac{d\omega_t}{ds}(\boldsymbol{y})\right)\Bigg|_{t=0}\mathfrak{u}_i(\boldsymbol{y})\,\frac{d\rho_i}{ds}(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{y})\mathrm{d}S(\boldsymbol{x})\\
&+\frac{1}{2\pi}\int_{\Gamma_i}\int_{\Gamma_i}\log\|\boldsymbol{x}-\boldsymbol{y}\|\left(\nabla\cdot\mathcal{V}(\boldsymbol{x})-\boldsymbol{n}(\boldsymbol{x})\cdot\mathsf{D}\mathcal{V}^\top(\boldsymbol{x})\,\boldsymbol{n}(\boldsymbol{x})\right)\frac{d\mathfrak{u}_i}{ds}(\boldsymbol{y})\,\frac{d\rho_i}{ds}(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{y})\mathrm{d}S(\boldsymbol{x})\\
&+\frac{1}{2\pi}\int_{\Gamma_i}\int_{\Gamma_i}\log\|\boldsymbol{x}-\boldsymbol{y}\|\,\frac{d}{dt}\left(\frac{d\omega_t}{ds}(\boldsymbol{x})\right)\Bigg|_{t=0}\frac{d\mathfrak{u}_i}{ds}(\boldsymbol{y})\,\rho_i(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{y})\mathrm{d}S(\boldsymbol{x}),
\end{aligned}
\tag{54}
$$

The expressions of other terms $\partial\widehat{\mathsf{a}}_{T,mn}/\partial t$ and $\partial\widehat{\mathsf{b}}_{T,mn}/\partial t$ with $T\in\{V,K,W\},\ m,n\in\{i,D,N\}$ can be written similarly by setting the velocity field on the boudary to zero.

The shape derivative of $d\omega_t/ds$ can be computed in the parameter domain as defined in Section 3.2.2

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{d\omega_t}{ds}\circ\gamma\right)\Big|_{t=0} &= \frac{d}{dt}\left(\frac{1}{\|\dot{\gamma}\|}\frac{d}{d\tau}\left(\frac{\|\dot{\gamma}_t\|}{\|\dot{\gamma}\|}\right)\right)\Big|_{t=0} \\
&= \frac{d}{dt}\left(\frac{1}{\|\dot{\gamma}\|^2\|\dot{\gamma}_t\|}\dot{\gamma}_t\cdot\ddot{\gamma}_t - \frac{\|\dot{\gamma}_t\|}{\|\dot{\gamma}\|^4}\dot{\gamma}\cdot\ddot{\gamma}\right)\Big|_{t=0} \\
&= -\frac{2}{\|\dot{\gamma}\|^4}\frac{d\|\dot{\gamma}_t\|}{dt}\Big|_{t=0}\dot{\gamma}\cdot\ddot{\gamma} + \frac{1}{\|\dot{\gamma}\|^3}\frac{d\dot{\gamma}_t}{dt}\Big|_{t=0}\cdot\ddot{\gamma} + \frac{1}{\|\dot{\gamma}\|^3}\dot{\gamma}\cdot\frac{d\ddot{\gamma}_t}{dt}\Big|_{t=0} \\
&= -\frac{2\dot{\gamma}\cdot(\mathsf{D}\mathcal{V}\circ\gamma)\dot{\gamma}}{\|\dot{\gamma}\|^5}\dot{\gamma}\cdot\ddot{\gamma} + \frac{1}{\|\dot{\gamma}\|^3}((\mathsf{D}\mathcal{V}\circ\gamma)\dot{\gamma})\cdot\ddot{\gamma} + \frac{1}{\|\dot{\gamma}\|^3}\dot{\gamma}\cdot\left(\left(\mathsf{D}^2\mathcal{V}\circ\gamma\right)(\dot{\gamma},\dot{\gamma}) + (\mathsf{D}\mathcal{V}\circ\gamma)\ddot{\gamma}\right).
\end{aligned}
\tag{55}
$$

4 BEM

4.1 Link

Code is available at https://github.com/gninr/FCSCD.git

4.2 Notations

- $\vec{\boldsymbol{\mu}}_i, \vec{\boldsymbol{\varphi}}_i, \vec{\boldsymbol{\mu}}, \vec{\boldsymbol{\varphi}}$ denote the coefficient vectors of state solution $\mathfrak{u}_i, \psi_i, \mathfrak{u}, \psi$.
- $\vec{\boldsymbol{\rho}}_i, \vec{\boldsymbol{\pi}}_i, \vec{\boldsymbol{\rho}}, \vec{\boldsymbol{\pi}}$ denote the coefficient vectors of adjoint solution ρ_i, π_i, ρ, π.
- $\vec{\boldsymbol{\gamma}}, \vec{\boldsymbol{\eta}}$ denote the coefficient vectors of interpolants of $\mathfrak{g}, \eta$.
- $\mathbf{A}_{mn}$ denotes the block of Galerkin matrix $\mathbf{A}$ corresponding to basis functions associated with entities $\in \Gamma_m, \Gamma_n$, where $m, n \in \{i, D, N\}$. Note that the order of m and n is reversed compared to that in (17).

4.3 Linear system of equations

The discrete version of state problem is

$$
\begin{bmatrix}
\left(\frac{\varepsilon_1}{\varepsilon_2}+1\right)\mathbf{W}_{ii} & 2\left(\mathbf{K}^\top\right)_{ii} & \mathbf{W}_{iN} & \left(\mathbf{K}^\top\right)_{iD} \\
2\mathbf{K}_{ii} & -\left(\frac{\varepsilon_2}{\varepsilon_1}+1\right)\mathbf{V}_{ii} & \mathbf{K}_{iN} & -\mathbf{V}_{iD} \\
\mathbf{W}_{Ni} & \left(\mathbf{K}^\top\right)_{Ni} & \mathbf{W}_{NN} & \left(\mathbf{K}^\top\right)_{ND} \\
\mathbf{K}_{Di} & -\mathbf{V}_{Di} & \mathbf{K}_{DN} & -\mathbf{V}_{DD}
\end{bmatrix}
\begin{bmatrix} \vec{\boldsymbol{\mu}}_i \\ \vec{\boldsymbol{\varphi}}_i \\ \vec{\boldsymbol{\mu}} \\ \vec{\boldsymbol{\varphi}} \end{bmatrix}
= \begin{bmatrix}
-\mathbf{W}_{iD} & -\left(\mathbf{K}^\top\right)_{iN} \\
-\mathbf{K}_{iD} & \mathbf{V}_{iN} \\
-\mathbf{W}_{ND} & \frac{1}{2}\left(\mathbf{M}^\top\right)_{NN} - \left(\mathbf{K}^\top\right)_{NN} \\
-\frac{1}{2}\mathbf{M}_{DD} - \mathbf{K}_{DD} & \mathbf{V}_{DN}
\end{bmatrix}
\begin{bmatrix} \vec{\boldsymbol{\gamma}} \\ \vec{\boldsymbol{\eta}} \end{bmatrix}
\tag{56}
$$

The discrete version of adjoint problem is

$$
\begin{bmatrix}
\left(\frac{\varepsilon_1}{\varepsilon_2}+1\right)\mathbf{W}_{ii} & 2\left(\mathbf{K}^\top\right)_{ii} & \mathbf{W}_{iN} & \left(\mathbf{K}^\top\right)_{iD} \\
2\mathbf{K}_{ii} & -\left(\frac{\varepsilon_2}{\varepsilon_1}+1\right)\mathbf{V}_{ii} & \mathbf{K}_{iN} & -\mathbf{V}_{iD} \\
\mathbf{W}_{Ni} & \left(\mathbf{K}^\top\right)_{Ni} & \mathbf{W}_{NN} & \left(\mathbf{K}^\top\right)_{ND} \\
\mathbf{K}_{Di} & -\mathbf{V}_{Di} & \mathbf{K}_{DN} & -\mathbf{V}_{DD}
\end{bmatrix}
\begin{bmatrix} \vec{\boldsymbol{\rho}}_i \\ \vec{\boldsymbol{\pi}}_i \\ \vec{\boldsymbol{\rho}} \\ \vec{\boldsymbol{\pi}} \end{bmatrix}
= \begin{bmatrix} \vec{\mathbf{0}} \\ \vec{\mathbf{0}} \\ \frac{\varepsilon_2}{2}\left(\mathbf{M}^\top\right)_{NN}\vec{\boldsymbol{\eta}} \\ \frac{\varepsilon_2}{2}\mathbf{M}_{DD}\vec{\boldsymbol{\gamma}} \end{bmatrix}
\tag{57}
$$

4.4 Building blocks

Defined in `factors.hpp`.

4.4.1 Kernels

- `Kernel1`

$$-\frac{1}{2\pi}\frac{\boldsymbol{x}-\boldsymbol{y}}{\|\boldsymbol{x}-\boldsymbol{y}\|^2}\cdot(\mathcal{V}(\boldsymbol{x})-\mathcal{V}(\boldsymbol{y})) \tag{58}$$

- `Kernel2`

$$-\frac{1}{2\pi}\left(2\frac{\boldsymbol{x}-\boldsymbol{y}}{\|\boldsymbol{x}-\boldsymbol{y}\|^4}\cdot\boldsymbol{n}(\boldsymbol{y})\left((\boldsymbol{x}-\boldsymbol{y})\cdot(\mathcal{V}(\boldsymbol{x})-\mathcal{V}(\boldsymbol{y}))\right)-\frac{\boldsymbol{n}(\boldsymbol{y})\cdot(\mathcal{V}(\boldsymbol{x})-\mathcal{V}(\boldsymbol{y}))}{\|\boldsymbol{x}-\boldsymbol{y}\|^2}\right.$$
$$\left.+\frac{\boldsymbol{x}-\boldsymbol{y}}{\|\boldsymbol{x}-\boldsymbol{y}\|^2}\cdot\mathsf{D}\mathcal{V}^\top(\boldsymbol{y})\,\boldsymbol{n}(\boldsymbol{y})-\frac{\boldsymbol{x}-\boldsymbol{y}}{\|\boldsymbol{x}-\boldsymbol{y}\|^2}\cdot\boldsymbol{n}(\boldsymbol{y})\left(\boldsymbol{n}(\boldsymbol{y})\cdot\mathsf{D}\mathcal{V}^\top(\boldsymbol{y})\,\boldsymbol{n}(\boldsymbol{y})\right)\right) \tag{59}$$

- `LogKernel`

$$-\frac{1}{2\pi}\log\|\boldsymbol{x}-\boldsymbol{y}\| \tag{60}$$

4.4.2 Factors

Let b denote a shape function.

- `Factor1`

$$b(\boldsymbol{x}) \tag{61}$$

- `Factor2`

$$\frac{db}{ds}(\boldsymbol{x}) \tag{62}$$

- `Factor3`

$$\left(\nabla\cdot\mathcal{V}(\boldsymbol{x})-\boldsymbol{n}(\boldsymbol{x})\cdot\mathsf{D}\mathcal{V}^\top(\boldsymbol{x})\,\boldsymbol{n}(\boldsymbol{x})\right)\frac{db}{ds}(\boldsymbol{x}) \tag{63}$$

- `Factor4`

$$\frac{d}{dt}\left(\frac{d\omega_t}{ds}(\boldsymbol{x})\right)\Bigg|_{t=0} b(\boldsymbol{x}) \tag{64}$$

4.5 Force computation

Total force is computed using the following expression

$$F=\begin{bmatrix}\vec{\boldsymbol{\rho}}_i & \vec{\boldsymbol{\pi}}_i & \vec{\boldsymbol{\rho}} & \vec{\boldsymbol{\pi}}\end{bmatrix}\begin{bmatrix}\left(\frac{\varepsilon_1}{\varepsilon_2}+1\right)\mathbf{DW}_{ii} & 2\left(\mathbf{DK}^\top\right)_{ii} & \mathbf{DW}_{iN} & \left(\mathbf{DK}^\top\right)_{iD}\\ 2\mathbf{DK}_{ii} & -\left(\frac{\varepsilon_2}{\varepsilon_1}+1\right)\mathbf{DV}_{ii} & \mathbf{DK}_{iN} & -\mathbf{DV}_{iD}\\ \mathbf{DW}_{Ni} & \left(\mathbf{DK}^\top\right)_{Ni} & \mathbf{O}_{NN} & \mathbf{O}_{ND}\\ \mathbf{DK}_{Di} & -\mathbf{DV}_{Di} & \mathbf{O}_{DN} & \mathbf{O}_{DD}\end{bmatrix}\begin{bmatrix}\vec{\boldsymbol{\mu}}_i\\ \vec{\boldsymbol{\varphi}}_i\\ \vec{\boldsymbol{\mu}}\\ \vec{\boldsymbol{\varphi}}\end{bmatrix}$$
$$+\begin{bmatrix}\vec{\boldsymbol{\rho}}_i & \vec{\boldsymbol{\pi}}_i & \vec{\boldsymbol{\rho}} & \vec{\boldsymbol{\pi}}\end{bmatrix}\begin{bmatrix}\mathbf{W}_{iD} & \left(\mathbf{K}^\top\right)_{iN}\\ \mathbf{K}_{iD} & -\mathbf{V}_{iN}\\ \mathbf{O}_{ND} & \mathbf{O}_{NN}\\ \mathbf{O}_{DD} & \mathbf{O}_{DN}\end{bmatrix}\begin{bmatrix}\vec{\boldsymbol{\gamma}}\\ \vec{\boldsymbol{\eta}}\end{bmatrix} \tag{65}$$

where $\mathbf{DT},\ \mathbf{T}\in\{\mathbf{V},\mathbf{K},\mathbf{W}\}$ denotes shape derivatives of corresponding variational forms and $\mathbf{O}$ is the zero matrix.

4.6 Validation

4.6.1 State problem

An FEM-implementation is used for validation of the BEM-implementation. The simple test case used is shown in Figure 2. An informal qualitative validation by comparing the Dirichlet trace of solution on top and bottom sides of the outer dielectric, i.e., $\vec{\boldsymbol{\mu}}$, is given in Figure 3.

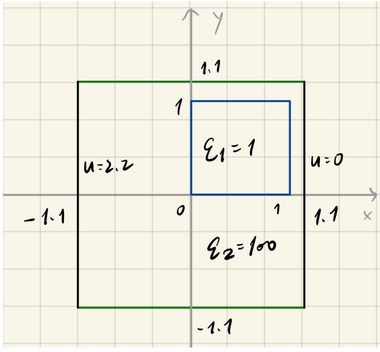

Figure 2: Square inner dielectric for validation

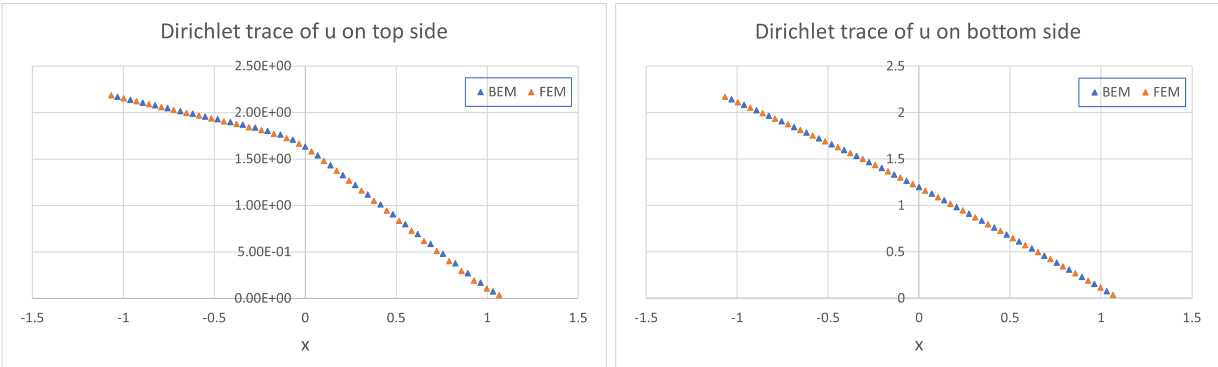

Figure 3: Result

4.6.2 Force computation

Test cases:

- $\varepsilon_1 = \varepsilon_2 \longrightarrow F = 0$: Pass
- $U = 0 \longrightarrow F = 0$: Pass
- Dielectrics are symmetric about the origin $\longrightarrow F = 0$: Pass
- $F_{\text{BEM}} = F_{\text{FEM}}$: Fail