1 Projective geometry

1. Projective geometry is an extension of standard, affine geometry that facilitates the analysis of geometric objects at infinity. Two important examples of objects at infinity are intersection points of parallel lines and asymptotic points of unbounded curves. Affine geometry does not model these concepts directly, so they generate special cases: two lines intersect in a unique point *except* when they are parallel; a polynomial $f(x, y)$ defines a curve that consists of closed loops of points *except* where it is unbounded. Projective geometry models the special cases in the same way as the regular cases, so that no exceptions arise.

2. There is a projective space of every dimension. The projective space of dimension n extends the affine space $\Re^n$ with an $n-1$ dimensional space of points at infinity. We will start with the projective line ($n = 1$) where the concepts are simplest. We will focus on the projective plane $(n = 2)$, which is the most important case for computer graphics. We will briefly discuss $n = 3$ and the general case.

3. The projective line is the space of rays through the origin in the affine plane. A ray is represented by any of its points. The representation is called homogeneous coordinates. Homogeneous coordinates are not unique, since (a, b) and (ka, kb) represent the same point for any non-zero k.

4. The space of rays is hard to visualize. There are three ways to transform rays into points in a manner that reveals their structure: the line model, the circle model, and the semicircle model.

5. In the line model, the ray (a, b) is intersected with the horizontal line $y = 1$ to obtain the point $(a/b, 1)$. A one-to-one mapping, $(a, b) \to a/b$, is established between the rays with $b \neq 0$ and the points of the affine line. The only missing ray is $(1, 0)$, which is parallel to $y = 1$, hence does not intersect it. Thus, the line model represents the projective line with an affine line plus an extra ray, called the point at infinity.

6. In the circle model, the ray (a, b) is intersected with the unit circle $x^2 + y^2 = 1$ to obtain the points $\pm(a/k, b/k)$ with $k = \sqrt{a^2 + b^2}$. A one-to-two mapping is thus established between the rays and the points of the circle. The circle model is simpler than the line model in that there is no point at infinity. It is more complicated in that each ray maps to a pair of points, which must be treated as a single entity.

7. In the semicircle model, the ray (a, b) is intersected with the upper half of the unit circle $x^2 + y^2 = 1, y \geq 0$. A one-to-one mapping is established between the rays with $b \neq 0$ and the points of the open semicircle with $y > 0$. The ray $(1, 0)$ maps to the points $(\pm 1, 0)$ that bound the open semicircle. It is called the point at infinity. This model resembles the

line model in that every ray is represented by a point, except for one ray called the point at infinity. It resembles the circle model in that all the rays are mapped to a single curve.

8. The projective plane is the space of rays through the origin in affine (x, y, z) space. Homogeneous coordinates are defined as before. There are three ways to transform rays into points in a manner that reveals their structure: the plane model, the sphere model, and the hemisphere model.

9. In the plane model, the ray (a, b, c) is intersected with the plane $z = 1$ to obtain the point $(a/c, b/c, 1)$. A one-to-one mapping, $(a, b, c) \to (a/c, b/c)$, is established between the rays with $c \neq 0$ and the points of the affine plane. The missing rays are parallel to the $z = 1$ plane, hence lie in the $z = 0$ plane and have the form $(a, b, 0)$. These rays form a projective line, called the line at infinity, with $(a, b, 0)$ representing the ray (a, b). Thus, the plane model represents the projective plane with an affine plane plus a projective line at infinity.

10. In the sphere model, the ray (a, b, c) is intersected with the unit sphere $x^2 + y^2 + z^2 = 1$ to obtain the points $\pm(a/k, b/k, c/k)$ with $k = \sqrt{a^2 + b^2 + c^2}$. A one-to-two mapping is established between the rays and the points of the sphere. The sphere model is simpler than the plane model in that there is no point at infinity. It is more complicated in that each ray maps to a pair of points, which must be treated as a single entity.

11. In the hemisphere model, the ray (a, b, c) is intersected with the upper half of the unit sphere $x^2 + y^2 + z^2 = 1, z \geq 0$. A one-to-one mapping is established between the rays with $c \neq 0$ and the points of the open hemisphere $z > 0$. The rays $(a, b, 0)$ map to the circle $x^2 + y^2 = 1, z = 0$ that bounds the open hemisphere. They form a projective line at infinity, as in the plane model. The hemisphere model resembles the plane model in that every ray is represented by a point, except for points at infinity. It resembles the sphere model in that all the rays are mapped to a single surface.

12. Projective space contains analogs of lines called projective lines. A projective line is defined as the set of rays that lie on a plane through the origin of (x, y, z) space. The plane with normal $n = (u, v, w)$ is writen as $\langle u, v, w \rangle$. The plane equation, $ux + vy + wz = 0$, is linear in x, y, z, so it is true/false for all the homogeneous coordinates of a ray.

13. Although the projective line is a plane in (x, y, z), the points cluster into rays in the projective plane. The plane, sphere, and hemisphere models help us see their projective geometric structure. In the $z = 1$ plane, the projective line $ux + vy + wz = 0$ maps to the affine line $ux + vy + w = 0$ plus the point at infinity $(-v, u, 0)$. Each point (x, y) on the affine line maps to the ray $(x, y, 1)$ on the projective line. In the sphere model, a projective line maps to a great circle. In the hemisphere model, it maps to a great semi-circle.

14. The intersection of two projective lines is a projective point because the intersection of two planes through the origin is a line through the origin. In the $z = 1$ plane model, the two projective lines normally map to two affine lines plus two points at infinity. If the affine lines intersect, the projective lines intersect at the same point. If the affine line are parallel, the projective lines intersect at their common point at infinity. One of the projective lines can be $z = 0$, which consists of all the points at infinity for this plane model. This line is called the line at infinity. It intersects every other projective line at its point at infinity. In the sphere model, two projective lines map to two great circles that intersect at two antipodal (opposite) points that model a single projective point. In the hemisphere model, two projective lines map to two semicircles that intersect at one affine point on the open hemisphere or at two antipodal points on the boundary circle that model a point at infinity.

15. There is a unique projective line through every pair of projective points because there is a unique plane through the origin that is orthogonal to two lines through the origin. The normal of the line through a and b is $a \times b$. In the plane model, the projective line through two affine points is the projective line corresponding to the affine line through the two points. In the sphere/hemisphere models, it is the great circle/semicircle through the two points.

16. Examples: the affine lines $x + y - 1 = 0$ and $x - y - 1 = 0$ are perpendicular and intersect at $(1, 0)$. Their projective lines, $x + y - z = 0$ and $x - y - z = 0$, intersect at the projective point $(1, 0, 1)$. The affine lines $x - y - 1 = 0$ and $x - y - 2 = 0$ are parallel, but their projective lines, $x - y - z = 0$ and $x - y - 2z = 0$, intersect at $(1, 1, 0)$. The projective lines are obtained by homogenizing the affine lines, substituting x/z for x and y/z for y and clearing the denominator, as discussed below. The affine points $(1, 1)$ and $(2, 3)$ define the line $-2x + y + 1 = 0$. Their projective points, $(1, 1, 1)$ and $(2, 3, 1)$, have the projective line $-2x + y + z = 0$, since $(1, 1, 1) \times (2, 3, 1) = (-2, 1, 1)$. The affine line through (a, b) in direction (c, d) is the projective line through $(a, b, 1)$ and $(c, d, 0)$. The line at infinity, $z = 0$, goes through any two projective points at infinity, $(a, b, 0)$ and $(c, d, 0)$, because $(a, b, 0) \times (c, d, 0) = (0, 0, ad - bc)$.

17. There is a natural duality between the point $p = (a, b, c)$ and the line $\hat{p} = \langle a, b, c \rangle$. Unlike the affine case, every line has a dual. If a point, p, is on a line, l, then $\hat{l}$ is on $\hat{p}$, since the original equation is $p \cdot l = 0$ and the dual equation is $\hat{l} \cdot \hat{p} = 0$. If a line, l, passes through two points, p and q, then $\hat{p}$ and $\hat{q}$ intersect at $\hat{l}$, since $l = p \times q$ implies $l \times p = 0$ and $l \times q = 0$.

18. A projective variety is the zero set of a homogeneous polynomial $p(x, y, z)$. Homogeneous means that every term of the polynomial has the same degree d. For example, a projective line is homogeneous with $d = 1$ and $xy - z^2$ is homogeneous with $d = 2$. The reason that the polynomial has to be homogeneous is that a homogeneous polynomial is zero or nonzero for all the homogeneous coordinates of a ray: the coordinates have the form (ka, kb, kc) and $p(ka, kb, kc) = k^d p(a, b, c)$.

19. We have seen that a projective line consists of an affine line plus a point at infinity in the plane model. A similar analysis applies to projective varieties of higher degree. In the $z = 1$ plane model, the projective variety $p(x, y, z) = 0$ consists of the affine variety $p(x, y, 1) = 0$, which is the intersection of the projective variety with the plane $z = 1$, plus the points at infinity $p(x, y, 0) = 0$, which are the intersection of the projective variety with the plane $z = 0$. For example, $xy - z^2 = 0$ consists of the hyperbola $xy = 1$ plus the points at infinity $(1, 0, 0)$ and $(0, 1, 0)$, which represents the asympotes $x = 0, y = \pm\infty$ and $x = \pm\infty, y = 0$.

20. A polynomial $p(x, y) = 0$ of degree d can be converted to a homogeneous polynomial of degree d in x, y, z by multiplying each term of degree e by z^{d-e}. This process is called homogenization. For example, the affine line $ux+vy+w = 0$ homogenizes to the projective line $ux + vy + wz = 0$ and the affine hyperbola $xy - 1 = 0$ homogenizes to $xy - z^2 = 0$. Another way of obtaining the same result is to substitute x/z for x and y/z for y in the polynomial then to clear the denominator. The original polynomial is recovered by setting $z = 1$. This is called dehomogenization.

21. Let $q(x, y, z)$ be the homogenization of $p(x, y)$. The affine variety of p equals the finite part of the projective variety of q, that is the points with $z = 1$. The points at infinity of q are the zeroes of the leading (highest degree) terms of p, since the other terms of q are zero for $z = 0$.

22. The line $y = 2x + 2$ homogenizes to $2x - y + 2z = 0$ with point at infinity $(1, 2, 0)$ that equals $(0.447, 0.894, 0)$ in the hemisphere model. This point converts the affine line into a loop. (input5l)

23. The parabola $y = x^2$ homogenizes to $yz - x^2 = 0$ with point at infinity $(0, 1, 0)$ that converts the affine parabola into a loop. (input5p)

24. The ellipse $x^2 + 4y^2 = 4$ homogenizes to $x^2 + 4y^2 - 4z^2 = 0$ with no points at infinity, since the affine ellipse is already closed. (input5e)

25. The hyperbola $xy = 1$ homogenizes to $xy - z^2 = 0$ with points at infinity $(1, 0, 0)$ and $(0, 1, 0)$. These points convert the two components of the affine hyperbola into a single loop. (input5h)

26. The cubic $y = x^3$ homogenizes to $yz^2 - x^3 = 0$ with point at infinity $(0, 1, 0)$ that converts the affine variety to a loop. (input5c)

27. The ultimate setting for algebraic geometry is complex projective space. For example, the circle $x^2 + y^2 = 1$ homogenizes to $x^2 + y^2 = z^2$ with points at infinity $(\pm 1, i)$.

28. Bezout's theorem: If polynomials p and q of degrees m and n do not have a common component, they have mn complex projective roots counting multiplicity. Likewise in higher dimensions.

29. Since every circle has the same two points at infinity, the intersection of two circles consists of these two points plus two real or complex affine points.

30. Although the projective plane eliminates the special cases of the affine plane, it also has disadvantages. The projective plane is not orientable. Lines have only one side: removing a line leaves a connected set. Segments are ambiguous: two points split their line into two connected parts that cannot be distinguished. Likewise, the direction from a to b is ambiguous, e.g. each point at infinity lies in two directions from every affine point. Convexity is undefined. Stolfi defines an oriented version of projective geometry that solves these problems at the cost of increased complexity. Each projective point is split into two oriented points: the line ka is split into the rays ka and $-ka$ with $k > 0$. Each projective line is split into two oriented lines likewise. In the sphere model, opposite points are no longer identified and great circles are oriented. In this model, the convex hull of a set of points is the dual of the envelope of the dual lines.

31. Three-dimensional projective space is the space of rays in (x, y, z, w) affine space. We encountered this space when we used homogeneous coordinates, transformation matrices, projection, and clipping. The planar concepts extend to three dimensions and to any dimension.