Homework 2

Due: 1:00 pm, 09/18/2023

Question 1

Consider an economy with a representative, infinitely lived consumer who has the utility function

$$U = \sum_{t=0}^{\infty} \beta^t \log(c_t)$$

where $\beta \in (0,1)$. The consumer owns one unit of labor in each period and $\overline{k}_0$ units of capital in period 0. The depreciation rate on capital is δ.

Feasible plans need to satisfy

$$c_t + k_{t+1} - (1-\delta)k_t \leq F(k_t, \ell_t)$$

where $F(k_t, \ell_t) = \theta k_t^{\alpha} \ell_t^{1-\alpha}$ and $\ell_t \in [0,1]$.

(1) Suppose that the consumer borrows b_{t+1} bonds in period t to be paid off in period $t+1$. The initial endowment of bonds is $\overline{b}_0 = 0$, the wage rate in period t is w_t, the capital rental rate is r_t^k, and the interest rate on bonds is r_t^b.

Write down the consumer's problem in the sequential markets economy. Explain why you need to include a non-Ponzi condition. Write down the Euler condition and transversality condition for this problem.

(2) Define a sequential markets equilibrium with borrowing and lending for this economy. What is the relationship between $r_t^k - \delta$ and r_t^b in equilibrium.

(3) Suppose that the consumers sells his endowment of capital to the firm in period 0. Thereafter, firms buy and sell capital from each other. Describe the production set for the Arrow-Debreu economy, the set of feasible $k_0, k_1, \ldots, \ell_0, \ell_1, \ldots, c_0, c_1, \ldots$

(4) Define the Arrow-Debreu equilibrium for this economy.

(5) Suppose that the consumer can buy new capital in each period and rent capital services to the firm. Define the Arrow-Debreu equilibrium for this economy.

(6) State theorems that relate the equilibrium allocations in parts (2) (4) and (5).

Question 2

Consider the following social planner's problem

$$\max_{\{k_t\},\{c_t\}} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma} - 1}{1-\sigma}$$

subject to

$$\begin{aligned} c_t + k_{t+1} - (1-\delta)k_t &\leq \theta k_t^{\alpha} \\ k_t, c_t &\geq 0 \\ k_0 &\text{ given} \end{aligned}$$

where $\beta, \delta, \alpha \in (0,1)$ and $\sigma > 0$.

(1) Rewrite the problem in the following form

$$\max_{\{k_t\}} \sum_{t=0}^{\infty} \beta^t F(k_t, k_{t+1})$$

subject to

$$\begin{gathered} k_{t+1} \in \Gamma(k_t) \\ k_0 \text{ given} \end{gathered}$$

Write down the Euler condition and transversality condition for this problem.

(2) Prove that if there exists a sequence $\{k_t\}$ satisfies the Euler condition and transversality condition, then it solves the planner's problem in part (1).

(3) Show that the sequence of capital $\{k_t\}$ that solves the planner's problem needs to satisfy a second-order difference equation

$$S(k_t, k_{t+1}, k_{t+2}) = 0$$

(4) Show that the sequence of capital in equilibrium in Question 1 needs to satisfy the same second-order difference equation and transversality condition as in the planner's problem.

Question 3

Consider the following social planner's problem

$$\max_{\{k_t\},\{c_t\}} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma} - 1}{1-\sigma}$$

subject to

$$\begin{aligned} c_t + k_{t+1} - (1-\delta)k_t &\leq \theta k_t^{\alpha} \\ k_t, c_t &\geq 0 \\ k_0 &\text{ given} \end{aligned}$$

where $\beta, \delta, \alpha \in (0,1)$ and $\sigma > 0$.

(1) Derive a formula for the non-zero steady state level of capital $\overline{k}$ and consumption $\overline{c}$.

(2) Write down a code which uses the shooting algorithm to solve for the sequences of capital and consumption that converge to the steady state.

(3) Assume $\sigma = 2$. Choose θ, δ, α, and β such that in the steady state, output equals to 1, the labor income share is 64%, the capital-to-output ratio is 3, and the consumption-to-output ratio is 0.8.

(4) Use your code from part (2) and the parameters in part (3) to plot the sequences of capital when $k_0 = 0.5\overline{k}$ and $k_0 = 1.5\overline{k}$. Also plot the associated sequences of $\{r_t^b\}$, $\{r_t^k\}$, and $\{w_t\}$ in a sequential markets equilibrium. Note that the length of the sequences you plot should be sufficient to show the convergence.