Math 207 Section A: Test 1

- Calculators are not allowed
- Show all work and enclose final answer in a box

1. (15 points) Find the parametric solution to the following system of equations

$$\begin{aligned} -x_1 + 3x_2 - 2x_3 + 4x_4 &= 4 \\ 3x_1 - 9x_2 + 6x_3 - 10x_4 &= -2 \end{aligned}$$

$$\left[\begin{array}{cccc|c} -1 & 3 & -2 & 4 & 4 \\ 3 & -9 & 6 & -10 & -2 \end{array}\right] \xrightarrow{3R_1+R_2} \left[\begin{array}{cccc|c} -1 & 3 & -2 & 4 & 4 \\ 0 & 0 & 0 & 2 & 10 \end{array}\right]$$

$$\xrightarrow[-R_1]{\frac{1}{2}R_2} \left[\begin{array}{cccc|c} 1 & -3 & 2 & -4 & -4 \\ 0 & 0 & 0 & 1 & 5 \end{array}\right] \xrightarrow{4R_2+R_1} \left[\begin{array}{cccc|c} 1 & -3 & 2 & 0 & 16 \\ 0 & 0 & 0 & 1 & 5 \end{array}\right]$$

x_2, x_3 free

Let $x_2 = s$

$x_3 = t$

$$\boxed{\begin{aligned} x_1 &= 16 + 3s - 2t \\ x_2 &= s \\ x_3 &= t \\ x_4 &= 5 \end{aligned}}$$

2. (6 points) Find a constant k so that the solution set will include infinitely many solutions

$$A = \left[\begin{array}{cc|c} 2 & -3 & k \\ -4 & 6 & 3 \end{array}\right]$$

$$\xrightarrow{2R_1+R_2} \left[\begin{array}{cc|c} 2 & -3 & k \\ 0 & 0 & 2k+3 \end{array}\right]$$

Need $2k+3=0$

So $\boxed{k = -\frac{3}{2}}$

3. (8 points) Let

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -2 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix}.$$

Compute the following (if possible). If the operation is not possible, explain why.

(a) $B^T A$

$$\underset{3\times 2}{\begin{bmatrix} 2 & 1 \\ 0 & 0 \\ -1 & -1 \end{bmatrix}} \underset{2\times 3}{\begin{bmatrix} -1 & 0 & 1 \\ 0 & -2 & 2 \end{bmatrix}} = \boxed{\begin{bmatrix} -2 & -2 & 4 \\ 0 & 0 & 0 \\ 1 & 2 & -3 \end{bmatrix}}$$

(b) AB^T

$$\underset{2\times 3}{\begin{bmatrix} -1 & 0 & 1 \\ 0 & -2 & 2 \end{bmatrix}} \underset{3\times 2}{\begin{bmatrix} 2 & 1 \\ 0 & 0 \\ -1 & -1 \end{bmatrix}} = \boxed{\begin{bmatrix} -3 & -2 \\ -2 & -2 \end{bmatrix}}$$

4. (15 points) Show all work. Find the LU factorization for

$$A = \begin{bmatrix} 3 & 2 & 3 \\ 3 & 1 & -2 \\ -6 & -4 & -4 \end{bmatrix}$$

$$\xrightarrow{-R_1+R_2} \begin{bmatrix} 3 & 2 & 3 \\ 0 & -1 & -5 \\ -6 & -4 & -4 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{2R_1+R_3} U = \begin{bmatrix} 3 & 2 & 3 \\ 0 & -1 & -5 \\ 0 & 0 & 2 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$E_2 E_1 A = U$, so $A = E_1^{-1} E_2^{-1} U = LU$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}} = L$$

$$\boxed{U = \begin{bmatrix} 3 & 2 & 3 \\ 0 & -1 & -5 \\ 0 & 0 & 2 \end{bmatrix}}$$

5. Let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ -2 & 2 & -2 \end{bmatrix}.$$

(a) (15 points) Compute the inverse of A.

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ -2 & 2 & -2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ -2 & 2 & -2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{2R_1+R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 2 & 1 \end{array}\right] \xrightarrow{-2R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & -2 & 2 & 1 \end{array}\right]$$

$$\xrightarrow{-R_3+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & -1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & -2 & 2 & 1 \end{array}\right] \xrightarrow{\frac{1}{2}R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & -1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1/2 \end{array}\right]$$

$$\boxed{A^{-1} = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & 0 \\ -1 & 1 & 1/2 \end{bmatrix}}$$

(b) (5 points) Use your answer from (a) to solve for $\mathbf{x}$ for $A\mathbf{x} = \mathbf{b}$ where

$$\mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

No credit will be given if another method is used.

$$\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & 0 \\ -1 & 1 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \boxed{= \begin{bmatrix} 2 \\ 1 \\ -3/2 \end{bmatrix}}$$

6. (16 points) For the following matrix of transition probabilities, use linear algebra (do not guess and check) to find the steady state matrix if the combined popluation of all states is 100.

$$P = \begin{bmatrix} 0.7 & 0.1 & 0.1 \\ 0.2 & 0.8 & 0.3 \\ 0.1 & 0.1 & 0.6 \end{bmatrix}.$$

$$P\bar{X} = \bar{X}$$

$$(P - I)\bar{X} = \bar{0}$$

$$\left[\begin{array}{ccc|c} -.3 & .1 & .1 & 0 \\ .2 & -.2 & .3 & 0 \\ .1 & .1 & -.4 & 0 \end{array}\right]$$

$$\xrightarrow[10R_3]{10R_1,\,10R_2} \left[\begin{array}{ccc|c} -3 & 1 & 1 & 0 \\ 2 & -2 & 3 & 0 \\ 1 & 1 & -4 & 0 \end{array}\right] \xrightarrow[-2R_3+R_2]{3R_3+R_1} \begin{bmatrix} 0 & 4 & -11 \\ 0 & -4 & 11 \\ 1 & 1 & -4 \end{bmatrix}$$

$$\xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & -4 & 0 \\ 0 & -4 & 11 & 0 \\ 0 & 4 & -11 & 0 \end{array}\right] \xrightarrow{R_2+R_3} \left[\begin{array}{ccc|c} 1 & 1 & -4 & 0 \\ 0 & -4 & 11 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{4}R_2} \left[\begin{array}{ccc|c} 1 & 1 & -4 & 0 \\ 0 & 1 & -11/4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-R_2+R_1} \left[\begin{array}{ccc|c} 1 & 0 & -5/4 & 0 \\ 0 & 1 & -11/4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = 5/4\, x_3$

$x_2 = 11/4\, x_3$

$x_3 = x_3$

$x_1 + x_2 + x_3 = 100$

$20/4\, x_3 = 100,\ x_3 = 20$

$$\boxed{\begin{aligned} x_1 &= 25 \\ x_2 &= 55 \\ x_3 &= 20 \end{aligned}}$$

7. (12 points) Find the determinant of

$$A = \begin{bmatrix} -2 & 1 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 & 3 \\ 4 & 1 & 1 & 2 & 1 \\ \boxed{2} & 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & -4 & 1 \end{bmatrix}$$

$$= -2 \begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & \boxed{3} \\ 1 & 1 & 2 & 1 \\ 3 & 1 & -4 & 1 \end{vmatrix} = -2(3) \begin{vmatrix} \boxed{1} & 0 & 0 \\ 1 & 1 & 2 \\ 3 & 1 & -4 \end{vmatrix} = -2(3) \begin{vmatrix} 1 & 2 \\ 1 & -4 \end{vmatrix}$$

$$= -2(3)(-4-2) \boxed{= 36}$$

8. (8 points) Find all values of λ that will make the following determinant equal to zero

$$A = \begin{bmatrix} 3-\lambda & 2 \\ 1 & 4-\lambda \end{bmatrix}.$$

$$|A| = (3-\lambda)(4-\lambda) - 2 = \lambda^2 - 7\lambda + 10 = 0$$

$$(\lambda-2)(\lambda-5) = 0$$

$$\boxed{\lambda = 2, 5}$$