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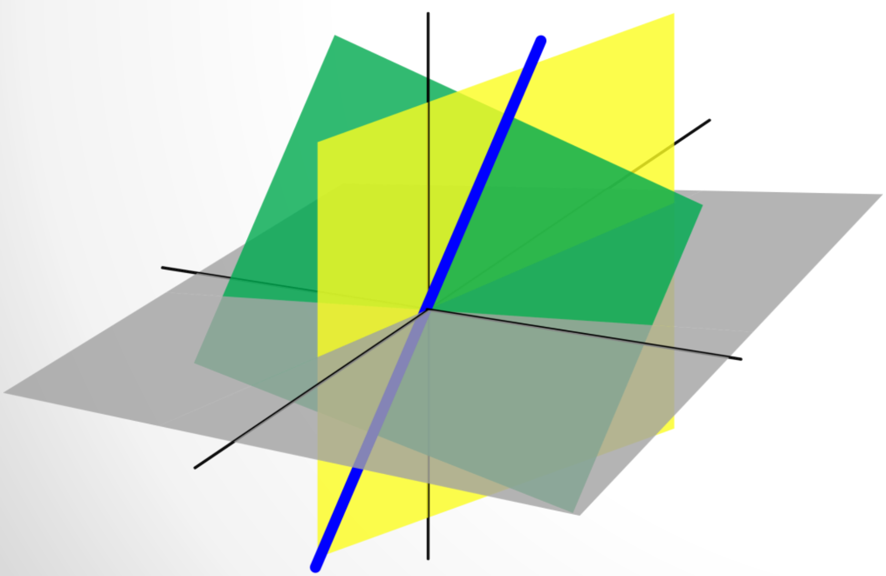
Linear Algebra



In this section, you will look at another classic problem in linear algebra called the diagonalization problem.

Expressed in terms of matrices, the problem is this:

For a square matrix A, does there exist an invertible matrix P such that $P^{-1}AP$ is diagonal?

Diagonalization

Diagonalization

Definition

Two square matrices A and B are called similar if there exists an invertible matrix P such that

$$B = P^{-1}AP$$

Matrices that are similar to diagonal matrices are called diagonalizable.

Diagonalization

Definition (Diagonalizable Matrix)

An $n \times n$ matrix A is **diagonalizable** if it is similar to a diagonal matrix.

That is, A is diagonalizable if there exists an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

Diagonalization

Provided with this definition, the diagonalization problem can be stated as follows:

Which square matrices are diagonalizable?

Clearly, every diagonal matrix D is diagonalizable, because the identity matrix I can play the role of P to yield

$$D = I^{-1}DI$$

Diagonalization

Example 1. A Diagonalizable Matrix

The matrix

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

is diagonalizable because

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

has the property

$$P^{-1}AP = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Diagonalization

The eigenvalue problem is related closely to the diagonalization problem.

The next two theorems shed more light on this relationship.

Theorem 1 (Similar Matrices Have the Same Eigenvalues)

If A and B are similar $n \times n$ matrices, then they have the same eigenvalues.

Diagonalization

Proof

Because A and B are similar, there exists an invertible matrix P such that $P^{-1}AP = B$.

By the properties of determinants, it follows that

$$|\lambda I - B| = \left|\lambda I - P^{-1}AP\right| = \left|P^{-1}\lambda IP - P^{-1}AP\right| =$$

$$= \left|P^{-1}(\lambda I - A)P\right| =$$

$$= \left|P^{-1}\right| \cdot |\lambda I - A| \cdot |P| =$$

$$= \left|P^{-1}\right| \cdot |P| \cdot |\lambda I - A| =$$

$$= |\lambda I - A|$$

Diagonalization

Proof (continued)

But this means that A and B have the same characteristic polynomial.

So, they must have the same eigenvalues.

Diagonalization

Example 2. Finding Eigenvalues of Similar Matrices

The matrices *A* and *D* are similar:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & -2 & 4 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Use the last Theorem to find their eigenvalues.

Solution

Because *D* is a diagonal matrix, its eigenvalues are simply the entries on its main diagonal — that is,

$$\lambda_1 = 1, \qquad \lambda_2 = 2, \qquad \lambda_3 = 3$$

Diagonalization

Solution (continued)

Moreover, because A is said to be similar to D, you know from Theorem 1 that A has the same eigenvalues.

Task

Find the eigenvalues of the matrix A and show that they are the same as of matrix D.

Diagonalization

Remark

Example 2 simply states that matrices A and D are similar.

Try checking that $D = P^{-1}AP$ using the matrices

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad \text{and} \quad P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Diagonalization

In fact, the columns of P are precisely the eigenvectors of A corresponding to the eigenvalues

$$\lambda_1 = 1, \qquad \lambda_2 = 2, \qquad \lambda_3 = 3$$

The two diagonalizable matrices in Examples 1 and 2 provide a clue to the diagonalization problem.

Diagonalization

Each of these matrices has a set of three linearly independent eigenvectors.

This is characteristic of diagonalizable matrices, as stated in the following Theorem.

Theorem 2. Condition for Diagonalization

An $n \times n$ matrix A is diagonalizable if and only if it has linearly independent eigenvectors.

Linear Dependence and Independence

The following notion illustrates an important type of problem in linear algebra — writing one vector (column matrix) x as the sum of scalar multiples of other vectors (column matrices) $v_1, v_2, \dots, v_n$.

$$x = c_1v_1 + c_2v_2 + \cdots + c_nv_n$$

The vector x is called a linear combination of the vectors $v_1, v_2, \dots, v_n$ ($c_1, c_2, \dots, c_n$ are scalars).

Linear Dependence and Independence

It is often useful to represent a vector $u = (u_1, u_2, \dots, u_n)$ in R^n as either a $1 \times n$ row matrix (row vector),

$$u = [u_1, u_2, \dots, u_n]$$

or an $n \times 1$ column matrix (column vector)

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \dots \\ u_n \end{bmatrix}$$

Linear Dependence and Independence

For a given set of vectors $S = \{v_1, v_2, \dots, v_k\}$ in R^n, the vector equation

$$c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0 \qquad (*)$$

always has the trivial solution

$$c_1 = 0, c_2 = 0, \dots, c_k = 0 \qquad (**)$$

Often, however, there are also nontrivial solutions.

Linear Dependence and Independence

Definition (Linear Dependence and Independence)

A set of vectors $S = \{v_1, v_2, \dots, v_k\}$ in R^n is called **linearly independent** if the vector equation (*) has only the trivial solution (**).

If there are also nontrivial solutions, then S is called **linearly dependent**.

Linear Dependence and Independence

Definition (Testing for Linear Independence and Dependence)

Let $S = \{v_1, v_2, \dots, v_k\}$ be a set of vectors in R^n. To determine whether S is linearly independent or linearly dependent, perform the following steps.

1. From the vector equation $c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0$ write a homogeneous system of linear equations in the variables $c_1, c_2, \dots, c_k$.

Linear Dependence and Independence

Definition (Testing for Linear Independence and Dependence)

2. Use Gaussian elimination to determine whether the system has a unique solution.
3. If the system has only the trivial solution, $c_1 = 0, c_2 = 0, \dots, c_k = 0$, then the set S is linearly independent. If the system also has nontrivial solutions, then S is linearly dependent.

Linear Dependence and Independence

A Property of Linearly Dependent Sets

A set $S = \{v_1, v_2, \dots, v_k\}$, $k \geq 2$, is linearly dependent if and only if at least one of the vectors in S can be written as a linear combination of the other vectors in S.

This property has a practical corollary that provides a simple test for determining whether two vectors are linearly dependent.

Linear Dependence and Independence

Corollary

Two vectors u and v in R^n are linearly dependent if and only if one is a scalar multiple of the other, that is

$$u = \alpha v$$

or

$$v = \beta u$$

Diagonalization

Proof of Theorem 2

- First, assume that A is diagonalizable. Then there exists an invertible matrix P such that $P^{-1}AP = D$ is diagonal.

Leting the main entries of D be $\lambda_1, \lambda_2, \dots, \lambda_n$ and the column vectors of P be $p_1, p_2, \dots, p_n$ yields

$$PD = [p_1 \vdots p_2 \vdots \cdots \vdots p_n] \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} = \cdots$$

Diagonalization

Proof

$$\ldots = [\lambda_1 p_1 \vdots \lambda_2 p_2 \vdots \cdots \vdots \lambda_n p_n]$$

Because $P^{-1}AP = D$, then $AP = PD$, which implies that

$$[Ap_1 \vdots Ap_2 \vdots \cdots \vdots Ap_n] = [\lambda_1 p_1 \vdots \lambda_2 p_2 \vdots \cdots \vdots \lambda_n p_n]$$

In other words, $Ap_i = \lambda_i p_i$ for each column vector p_i – this means that the column vectors p_i of P are eigenvectors of the matrix A.

Diagonalization

Proof

Moreover, because P is invertible, its column vectors are linearly independent. Thus A has n linearly independent eigenvectors.

- Conversely, assume that A has n linearly independent eigenvectors $p_1, p_2, \dots, p_n$ with corresponding eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$.

Diagonalization

Proof

Let P be the matrix whose columns are these n eigenvectors, that is

$$P = [p_1 \vdots p_2 \vdots \cdots \vdots p_n]$$

Because each p_i is an eigenvector of A, you have $Ap_i = \lambda_i p_i$ and

$$AP = A[p_1 \vdots p_2 \vdots \cdots \vdots p_n] =$$

$$= [\lambda_1 p_1 \vdots \lambda_2 p_2 \vdots \cdots \vdots \lambda_n p_n]$$

Diagonalization

Proof

The right-hand matrix in this equation can be written as the matrix product below

$$AP = [p_1 \vdots p_2 \vdots \cdots \vdots p_n]\begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} = PD$$

Finally, because the vectors $p_1, p_2, \dots, p_n$ are linearly independent, P is invertible and you can write the equation $AP = PD$ as $P^{-1}AP = D$.

Diagonalization

A key result of this proof is the fact that for diagonalizable matrices, the columns of P consist of the n linearly independent eigenvectors.

Example 3 verifies this important property for the matrices in Examples 1 and 2.

Diagonalization

Example 3. Diagonalizable Matrices

(a) **The matrix in Example 1 has the eigenvalues and corresponding eigenvectors listed below:**

$$\lambda_1 = 4, p_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix};\ \lambda_2 = -2, p_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}; \lambda_3 = -2, p_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The matrix P whose columns correspond to these eigenvectors is

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Diagonalization

Example 3 (continued)

Moreover, because P is row-equivalent to the identity matrix, the eigenvectors p_1, p_2, and p_3 are linearly independent.

(b) The matrix in Example 2 has the eigenvalues and corresponding eigenvectors listed below:

$$\lambda_1 = 1, p_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}; \ \lambda_2 = 2, p_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}; \lambda_3 = 3, p_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

Diagonalization

Example 3 (continued)

The matrix P whose columns correspond to these eigenvectors is

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Again, because P is row-equivalent to the identity matrix, the eigenvectors p_1, p_2, and p_3 are linearly independent.

Diagonalization

Guidelines for Diagonalizing an $n \times n$ Square Matrix

Let A be an $n \times n$ matrix.

1. Find n linearly independent eigenvectors

$$p_1, p_2, \ldots, p_n$$

for A with corresponding eigenvalues

$$\lambda_1, \lambda_2, \ldots, \lambda_n$$

Diagonalization

Guidelines for Diagonalizing an $n \times n$ Square Matrix

If n linearly independent eigenvectors do not exist, then A is not diagonalizable.

2. If A has n linearly independent eigenvectors, let P be the $n \times n$ matrix whose columns consist of these eigenvectors. That is,

$$P = [p_1 \vdots p_2 \vdots \cdots \vdots p_n]$$

Diagonalization

Guidelines for Diagonalizing an $n \times n$ Square Matrix

3. The diagonal matrix $D = P^{-1}AP$ will have the eigenvalues

$$\lambda_1, \lambda_2, \ldots, \lambda_n$$

on its main diagonal (and zeros elsewhere).

Note that the order of the eigenvectors used to form P will determine the order in which the eigenvalues appear on the main diagonal of D.

Diagonalization

Example 4. A Matrix That Is Not Diagonalizable

Show that the matrix A is not diagonalizable:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

Solution

Because A is triangular, the eigenvalues are simply the entries on the main diagonal. So, the only eigenvalue is $\lambda_1 = 1$.

Diagonalization

Solution (continued)

The matrix $(I - A)$ has the reduced row-echelon form shown below:

$$I - A = \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

This implies that $x_2 = 0$, and letting $x_1 = t$, you can find that every eigenvector of A has the form

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Diagonalization

Solution (continued)

So, A does not have two linearly independent eigenvectors, and you can conclude that A is not diagonalizable.

Diagonalization

Example 5. Diagonalizing a Matrix

Show that the matrix A is diagonalizable:

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix}$$

Then find a matrix P such that $P^{-1}AP$ is diagonal.

Task: Solve the problem

Diagonalization

Example 6. Diagonalizing a Matrix

Show that the matrix A is diagonalizable:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & -10 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}$$

Then find a matrix P such that $P^{-1}AP$ is diagonal.

Task: Solve the problem

Diagonalization

Theorem 3. Sufficient Condition for Diagonalization

If an $n \times n$ matrix A has distinct eigenvalues, then the corresponding eigenvectors are linearly independent and A is diagonalizable.

Proof

Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be n distinct eigenvalues of A corresponding to the eigenvectors $x_1, x_2, \dots, x_n$.

Diagonalization

Proof (continued)

To begin, assume that the set of eigenvectors is linearly dependent.

Moreover, consider the eigenvectors to be ordered so that the first m eigenvectors are linearly independent, but the first $m+1$ are dependent, where $m < n$.

Diagonalization

Proof (continued)

Then x_{m+1} can be written as a linear combination of the first m eigenvectors:

$$x_{m+1} = c_1 x_1 + c_2 x_2 + \cdots + c_m x_m \quad \text{Equation 1}$$

where the c_i's are not all zero.

Multiply both sides of Equation 1 by A:

$$Ax_{m+1} = Ac_1 x_1 + Ac_2 x_2 + \cdots + Ac_m x_m$$

Diagonalization

Proof (continued)

The last expression yields

$$\lambda_{m+1}x_{m+1}$$
$$= c_1\lambda_1x_1 + c_2\lambda_2x_2 + \cdots + c_m\lambda_mx_m \qquad \text{Equation 2}$$

Multiply both sides of Equation 1 by λ_{m+1}:

$$\lambda_{m+1}x_{m+1}$$
$$= c_1\lambda_{m+1}x_1 + c_2\lambda_{m+1}x_2 + \cdots + c_m\lambda_{m+1}x_m$$

Equation 3

Diagonalization

Proof (continued)

Now, subtract Equation 2 from Equation 3:

$$c_1(\lambda_{m+1} - \lambda_1)x_1 + c_2(\lambda_{m+1} - \lambda_2)x_2 + \cdots +$$
$$+c_m(\lambda_{m+1} - \lambda_m)x_m = 0$$

Using the fact that the first m eigenvectors are linearly independent, you can conclude that all coefficients of this equation must be zero.

Diagonalization

Proof (continued)

That is

$$c_1(\lambda_{m+1} - \lambda_1) = c_2(\lambda_{m+1} - \lambda_2) = \cdots$$
$$= c_m(\lambda_{m+1} - \lambda_m) = 0$$

Because all the eigenvalues are distinct, it follows that $c_i = 0, \mathrm{i} = 1, 2, \dots, m$.

But this result contradicts our assumption that x_{m+1} can be written as a linear combination of the first m eigenvectors.

Diagonalization

Proof (continued)

From Theorem 2 it follows that the matrix A is diagonalizable.

This completes the proof of the theorem.

Diagonalization

Example 7. Determining Whether a Matrix is Diagonalizable

Determine whether the matrix A is diagonalizable:

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -3 \end{bmatrix}$$

Task: Solve the problem

Diagonalization

Remark

Remember that the condition in the last Theorem is sufficient but not necessary for diagonalization, as demonstrated in Example 6.

In other words, a diagonalizable matrix need not have distinct eigenvalues.

Thank You for Attention