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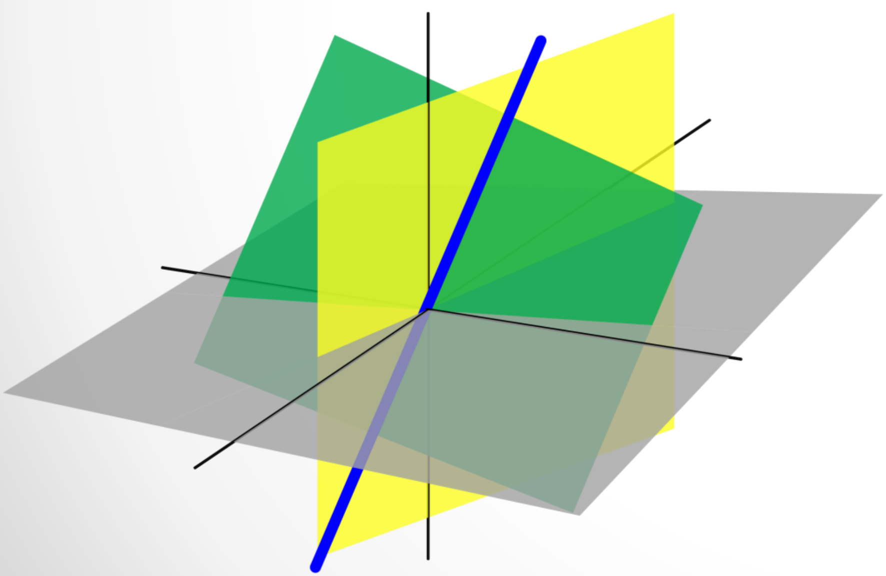
Linear Algebra



In this lecture, you are introduced to the concept of the **rank of a matrix**.

Rank enables one to relate matrices to vectors, and vice versa.

Rank is a unifying tool that enables us to bring together many of the concepts discussed so far.

Rank of a Matrix

Rank of a Matrix

Solutions to certain systems of linear equations, singularity of a matrix, and invertibility of a matrix all come together under the umbrella of rank.

Definition

Let A be an $m \times n$ matrix.

The $n-$tuples corresponding to the rows of A are called the **row vectors** of A:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Row Vectors of A

$$(a_{11}, a_{12}, \ldots, a_{1n})$$
$$(a_{21}, a_{22}, \ldots, a_{2n})$$
$$\vdots$$
$$(a_{m1}, a_{m2}, \ldots, a_{mn})$$

The rows of A may be viewed as row vectors

$$r_1, r_2, \ldots, r_m$$

Each row vector will have n components.

Rank of a Matrix

Definition

Similarly, the columns of A are called the column vectors of A

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Column Vectors of A

$$\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} \cdots \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

The columns of A may be viewed as column vectors

$$\boldsymbol{c}_1, \boldsymbol{c}_2, \dots, \boldsymbol{c}_n$$

Each column vector will have m components.

Example 1. Row Vectors and Column Vectors

For the matrix

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -2 & 3 & 4 \end{bmatrix}$$

the row vectors are

$$r_1 = (0 \quad 1 \quad -1) \qquad \text{and} \qquad r_2 = (-2 \quad 3 \quad 4)$$

The column vectors are

$$c_1 = \begin{pmatrix} 0 \\ -2 \end{pmatrix}, \qquad c_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad \text{and} \quad c_3 = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$$

Note that for an $m \times n$ matrix A, the row vectors are vectors in R^n and the column vectors are vectors in R^m.

Vectors

Definition

In physics and engineering, a **vector** is characterized by two quantities –

- length
- direction

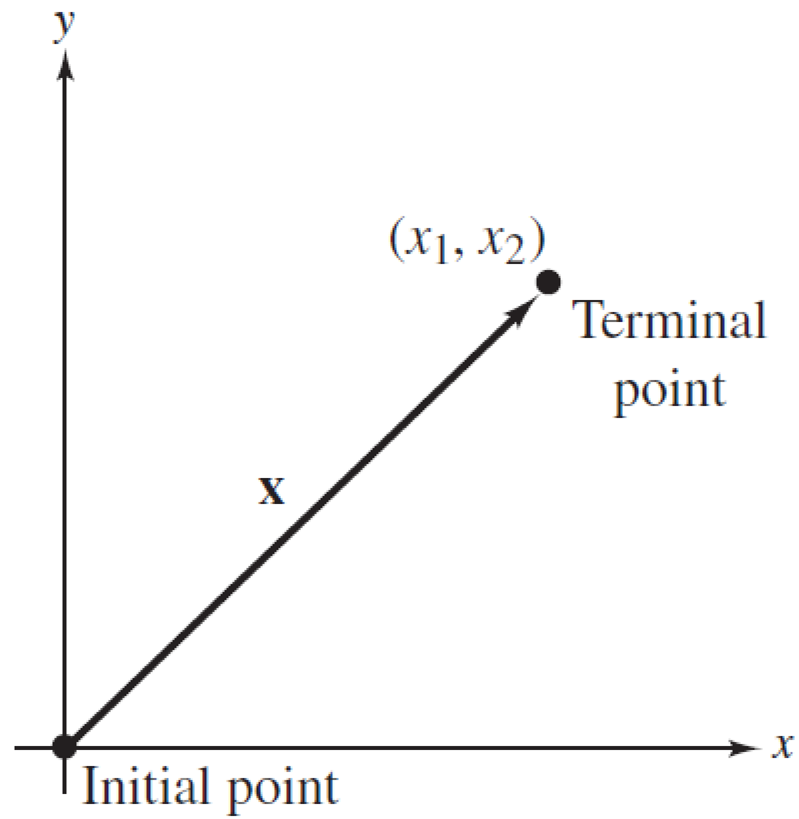


Figure 1

Vectors

A vector in the plane is represented geometrically by a directed line segment whose initial point is the origin and whose terminal point is the point (x_1, x_2) as shown in Figure 1.

Vectors

This vector is represented by the same ordered pair used to represent its terminal point, that is,

$$x = (x_1, x_2)$$

The coordinates x_1 and x_2 are called the components of the vector x.

Two vectors in the plane $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are equal if and only if $u_1 = v_1$ and $u_2 = v_2$.

Definition

A vector in $n-$space is represented by an ordered n $-$tuple.

For instance, an ordered **triple** has the form

$$(x_1, x_2, x_3)$$

an ordered **quadruple** has the form

$$(x_1, x_2, x_3, x_4)$$

and a general ordered n $-$**tuple** has the form

$$(x_1, x_2, \ldots, x_n)$$

Operations on Vectors

Definition (Vector Addition)

The first basic vector operation is **vector addition**.

To add two vectors in the plane, add their corresponding components –

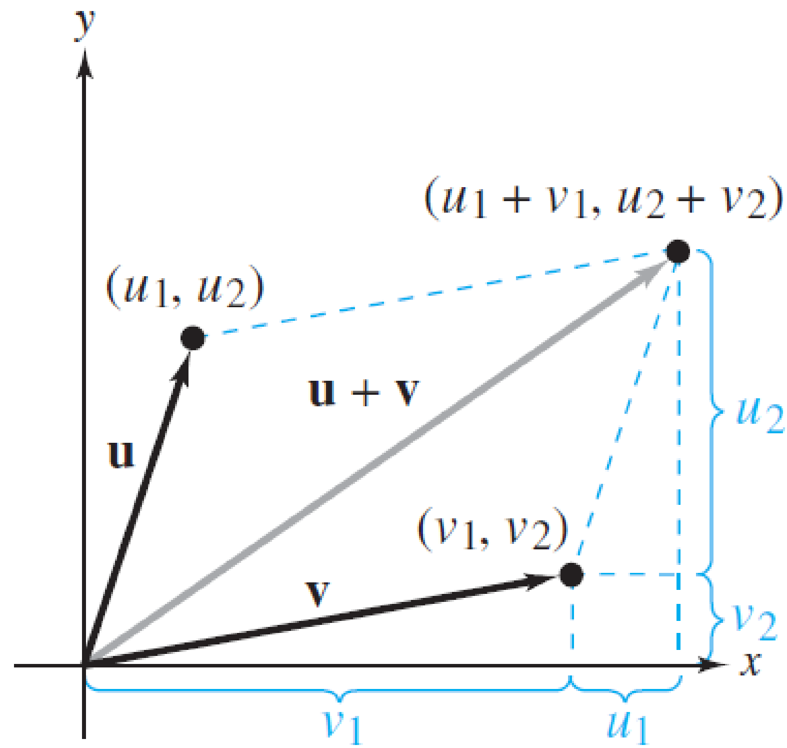
that is, the sum of $\boldsymbol{u}$ and $\boldsymbol{v}$ is the vector

$$\boldsymbol{u} + \boldsymbol{v} = (u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2)$$

Operations on Vectors

Geometrical Interpretation of Addition

Geometrically, the sum of two vectors in the plane is represented as the diagonal of a parallelogram having $\boldsymbol{u}$ and $\boldsymbol{v}$ as its adjacent sides, as shown in Figure 2.



Vector Addition

Figure 2

Operations on Vectors

Definition (Zero Vector)

The vector $(0,0)$ is called the **zero vector**. The zero vector is denoted by $\mathbf{0}$.

Definition (Scalar Multiplication)

The second basic vector operation is called **scalar multiplication**. To multiply a vector v by a scalar c, multiply each of the components of v by c: $cv = c(v_1, v_2) = (cv_1, cv_2)$

Operations on Vectors

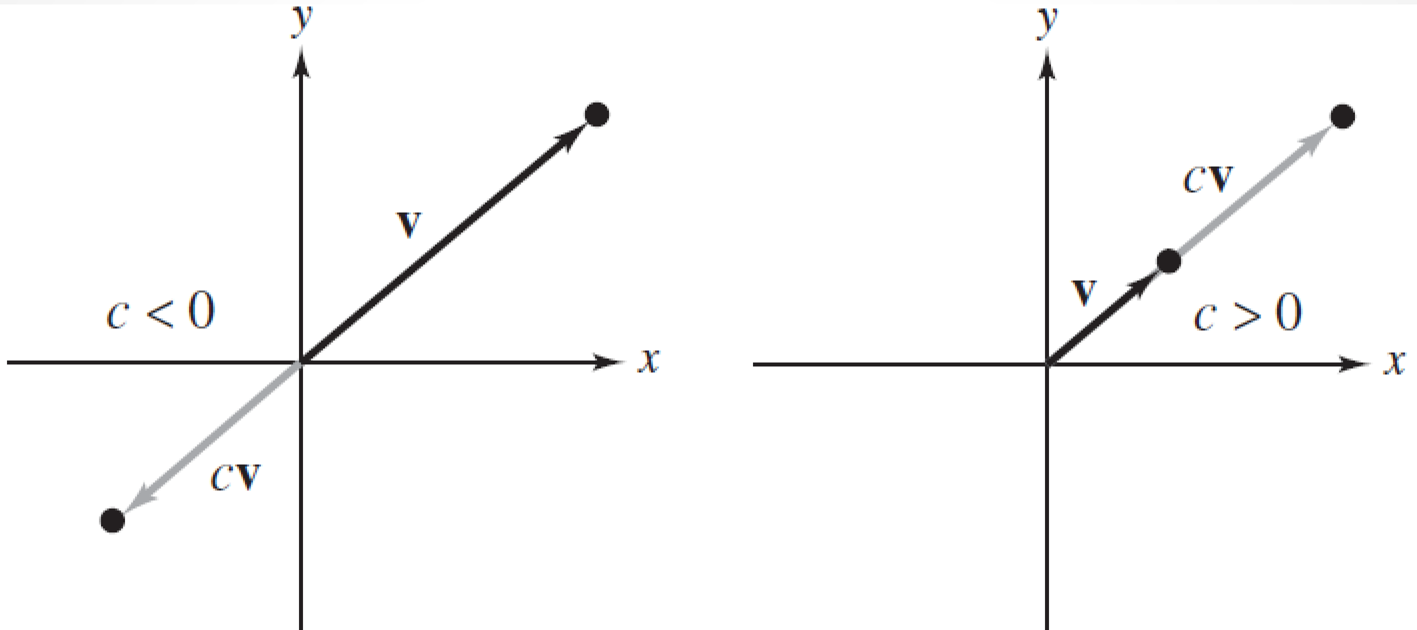


Figure 3

Definition

The vector $-v$ is called the negative of v.

Operations on Vectors

Definition (Difference of Vectors)

The difference of $\boldsymbol{u}$ and $\boldsymbol{v}$ is defined as

$$\boldsymbol{u} - \boldsymbol{v} = \boldsymbol{u} + (-\boldsymbol{v})$$

and you can say $\boldsymbol{v}$ is subtracted from $\boldsymbol{u}$.

Definition (Real Vector Space)

The set of all n −tuples is called n −space and is denoted by R^n.

- R^1 is 1−space = set of all real numbers;
- R^2 is 2−space = set of all ordered pairs of real numbers;
- R^3 is 3−space = set of all ordered triples of real numbers;

Vector Spaces

- R^4 is 4−space = set of all ordered quadruples of real numbers;
- R^n is n −space = set of all ordered n −tuples of real numbers.

The sum of two vectors in R^n and the scalar multiple of a vector in R^n are called the standard operations in R^n and are defined as follows.

Definition (Vector Addition and Scalar Multiplication)

Let $u = (u_1, u_2, \dots, u_n)$ and $v = (v_1, v_2, \dots, v_n)$ be vectors in R^n and let c be a real number.

Then the sum of u and v is defined as the vector

$$u + v = (u_1 + v_1, u_2 + v_2, \dots, u_n + v_n)$$

and the scalar multiple of u by c is defined as the vector

$$cu = (cu_1, cu_2, \dots, cu_n)$$

Theorem 1. Properties of Vector Addition and Scalar Multiplication in the Plane

Let u, v, and w be vectors in the plane, and let c and d be scalars.

1. $\mathbf{u} + \mathbf{v}$ is a vector in the plane.	Closure under addition
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	Commutative property of addition
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$	Associative property of addition
4. $\mathbf{u} + \mathbf{0} = \mathbf{u}$	Additive identity property
5. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$	Additive inverse property
6. $c\mathbf{u}$ is a vector in the plane.	Closure under scalar multiplication
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$	Distributive property
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$	Distributive property
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$	Associative property of multiplication
10. $1(\mathbf{u}) = \mathbf{u}$	Multiplicative identity property

Definition

The zero vector $\mathbf{0}$ in R^n is called the additive identity in R^n.

Similarly, the vector $-v$ is called the additive inverse of v.

The theorem below summarizes several important properties of the additive identity and additive inverse in R^n.

Theorem 2. Properties of Additive Identity and Additive Inverse

Let v be a vector in R^n and let c be a scalar. Then the following properties are true:

1. The additive identity of v is unique.
2. The additive inverse of v is unique.
3. $0v = 0$.
4. $c0 = 0$.
5. If $cv = 0$, then $c = 0$ or $v = 0$.
6. $-(-v) = v$.

Example 1

Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 \\ 3 & 4 & 1 & 6 \\ 5 & 4 & 1 & 0 \end{bmatrix}$$

The row vectors of the matrix are

$$r_1 = (1, 2, -1, -2), r_2 = (3, 4, 1, 6), r_3 = (5, 4, 1, 0)$$

The column vectors of the matrix are

$$c_1 = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, c_2 = \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}, c_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, c_4 = \begin{bmatrix} 2 \\ 6 \\ 0 \end{bmatrix}$$

Rank of a Matrix

Definition

Each matrix can be associated with two ranks: row rank (rank of rows) and column rank (rank of columns).

- The rank of rows is the maximum number of linearly independent rows of the matrix.
- The rank of columns is the maximum number of linearly independent columns of the matrix.

Rank of a Matrix

Theorem 3. Property of Row and Column Ranks of a Matrix

The row rank of a matrix is equal to its column rank.

Example 2

Determine the rank of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 5 & 8 \end{bmatrix}$$

Solution

We can see by inspection that the third row of the matrix is a linear combination of the first two rows:

$$(2, 5, 8) = 2 \cdot (1, 2, 3) + 1 \cdot (0, 1, 2)$$

Hence the three rows of the matrix are linearly dependent.

Rank of a Matrix

Solution (continued)

Therefore, the rank of the matrix must be less than 3.

Since $(1, 2, 3)$ is not a scalar multiple of $(0, 1, 2)$, these two vectors are linearly independent.

Thus, the rank of the matrix, denoted by $\text{rank}(A)$, is definitely 2.

Remark

This method, based on the definition, is not practical for determining the ranks of larger matrices. (We shall give a more systematic method for finding the rank of a matrix.)

The following theorem, which paves the way for the method, tells us that the rank of a matrix that is in reduced echelon form is immediately known.

Rank of a Matrix

Theorem 4. The Rank of a Matrix

The nonzero row vectors of a matrix that is in reduced row-echelon form are a basis for the row space of this matrix.

The rank of such a matrix is the number of nonzero row vectors.

Example 3

Determine the rank of a matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution

This matrix is in reduced row-echelon form.

There are three nonzero row vectors, namely the first three rows of the matrix. According to Theorem 4, the rank of the matrix equals 3.

Rank of a Matrix

Theorem 5. Ranks of Equivalent Matrices

Let A and B be row equivalent matrices.

Then A and B have the same ranks:

$$\text{rank}\,(A) = \text{rank}\,(B)$$

Theorem 6. Finding the Rank of a Given Matrix

Let E be the reduced row-echelon form of a matrix A. The rank of A is the number of nonzero row vectors in E.

Example 4

Determine the rank of a matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 1 & 1 & 5 \end{bmatrix}$$

Solution

Use elementary row operations to find the reduced row-echelon form of the given matrix. We obtain

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 1 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank of a Matrix

Solution (continued)

The last matrix is in reduced row-echelon form.

We have two non-zero rows in the last matrix.

Therefore the rank of the initial matrix is 2.

Example 5

Determine the rank of a matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & -1 & -4 & -2 \\ 3 & 4 & 11 & 8 \end{bmatrix}$$

Solution

Determine the reduced row-echelon form of the given matrix. We obtain

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & -1 & -4 & -2 \\ 3 & 4 & 11 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank of a Matrix

Solution (continued)

The last matrix is in reduced row-echelon form.

We have two non-zero rows in the last matrix.

Therefore the rank of the initial matrix is 2.

The next theorem brings together a number of results and concepts that have appeared so far in a convenient manner.

Rank of a Matrix

Theorem 7

Let A be an $n \times n$ matrix.

The following statements are equivalent:

(a) A is invertible

(b) A is non-singular, i.e. $\det(A) \neq 0$

(c) The system of equations $Ax = b$ has a unique solution

(d) $Ax = 0$ has only a trivial solution

(e) A is row equivalent to I_n

(f) $\text{rank}\,(A) = n$

Remark

The last theorem tells us how rank gives information about the uniqueness of the solution to a system of n linear equations in n unknowns (variables).

Rank of a Matrix

The concept of rank plays an important role in understanding the behavior of systems of linear equations of all sizes (both overdetermined and underdetermined systems).

We have seen how systems of linear equations can have a unique solution, many solutions, or no solution at all.

Rank of a Matrix

These situations can be categorized in terms of the ranks of the augmented matrix and the matrix of coefficients.

Theorem 8

Consider a system $Ax = b$ of m linear equations in n variables.

a) If the augmented matrix and the matrix of coefficients have the same rank r and $r = n$, the solution of the system is unique.

Theorem 8 (continued)

b) If the augmented matrix and the matrix of coefficients have the same rank r and $r < n$, there are an infinite number of solutions of the system.

c) If the augmented matrix and the matrix of coefficients do not have the same rank, a solution of the system does not exist.

Example 6

Consider the following system of linear equations

$$x + y + z = 2$$
$$2x + 3y + z = 3$$
$$x - y - 2z = -6$$

Solution

The augmented matrix of this system of equations is as follows:

$$\bar{A} = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 3 & 1 & 3 \\ 1 & -1 & -2 & -6 \end{bmatrix}$$

Rank of a Matrix

Solution (continued)

The augmented matrix contains the matrix of coefficients as a submatrix (shown in dark crimson).

Its reduced row-echelon form is as follows:

$$\bar{E} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Solution (continued)

We see that ranks of the augmented matrix and the matrix of coefficients are equal, both being 3.

The system thus has a unique solution and the reduced row-echelon form suggests the solution

$$x_1 = -1, \qquad x_2 = 1, \qquad x_3 = 2$$

Thank You for Attention