Computational Packages Concerning General Relativity

Working Paper · January 2016

DOI: 10.13140/RG.2.1.1507.8484

Ceren Guzelgun
Istanbul Technical University

İSTANBUL TECHNICAL UNIVERSITY
FACULTY OF SCIENCE AND LETTERS
ADVANCED PHYSICS PROJECT REPORT

COMPUTATIONAL PACKAGES CONCERNING GENERAL RELATIVITY

Ceren GÜZELGÜN

Branch : Engineering Physics

Advisor: : Yrd. Doç. Dr. Tolga BİRKANDAN

GÜZ 2015

To family and friends who had been supportive.
Also, special thanks to Elif, Mustafa,
and Tolga Birkandan.

Contents

1. Introduction

General relativity (GR) is one of the most important progresses in science that had begun in 1907, just after Einstein formulated his special theory of relativity. This theory, now involving gravitational interactions, was presented in November 1915 to the Prussian Academy of Sciences. Even though, Albert Einstein was the first developer of the subject, later on the field kept growing and improving with the contributions of many others.

Working on general relativity requires a sophisticated way of thinking, and to be skilled in advanced calculations that can be significantly challenging at times to do on paper.

In this work, we tried to utilize some programs with special packages in order to efficiently perform GR calculations with computers, while decreasing the time required and increasing the accuracy with no calculation errors.

The chosen programs are Maple, Maxima, Sage and Python while the packages working on them are GrTensor II, ctensor, SageManifolds and GraviPy. Only one of these programs, namely Maple is commercial and all other programs and packages are freely available.

After a brief introduction to GR, we will study the aforementioned computational tools in detail.

2. Introduction to General Relativity

In this section, we will give some basic concepts of general relativity and study the black hole solutions.

2.1. Basic Concepts of General Relativity

General theory of relativity is the geometric theory of gravitation that was first presented by Albert Einstein in 1915 [1]. The theory generalizes special relativity and Newton's law of universal gravitation, while providing a new gravity description in terms of space and time, or as often called; spacetime.

According to Newton's law of universal gravitation, two bodies in the Universe attract each other with a force. This force is proportional to the product of the bodies' masses, and inversely proportional to the square of the distance in between. This relation defines the mass as the gravitational mass, m_G. According to another law of Newton that is the second law of motion, the velocity of an object can be increased or decreased through an external force. This force can also change the direction of that velocity. With this relation, the second law defines the mass as the resistance to the change of velocity; namely, the inertial mass, m_I. [2]

General Relativity starts with the principle

$$m_I = m_G \tag{1}$$

which states that the inertial mass and the gravitational mass are identical to each other. The explanation of this was first discovered by Albert Einstein, and he called it the Principle of Equivalence. [3] From this principle, Einstein was able to further continue studying General Relativity.

According to GR, the time coordinate (t) is not different from the spatial coordinates (x, y, z) and they are all regarded as “spacetime coordinates”.

The coordinates of the flat spacetime (also known as the Minkowski spacetime) are

$$x^4 = (x^0, x^1, x^2, x^3) = (t, x, y, z) \tag{2}$$

From this, the Minkowski line element (the distance from one point in this spacetime to another) is

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 \tag{3}$$

where we used the convention in which the speed of light in vacuum c is unity (c = 1). If we write the line element in tensorial form, it is

$$ds^2 = \eta_{ab} dx^a dx^b \quad \text{where} \quad \eta_{ab} = diag(1, -1, -1, -1) \tag{4}$$

In Minkowski spacetime, η_{ab} is constant and the curvature dependent quantitiy Γ^a_{bc} vanishes. From here, it can be seen that the Riemann curvature tensor, R^a_{bcd}, which will be given shortly, also vanishes. This explains why the Minkowski metric is flat, for the condition of being flat is if the Riemann tensor vanishes [4].

This was the definition of a flat spacetime. However, spacetime in general is much more complicated than being flat as a whole. When one is working with a *curved* spacetime, a different geometry is needed. For example, Einstein used the geodesics (the shortest path between two points) of a curved geometry while identifying the trajectories of a falling body. The trajectories are locally straight, however, when looked globally they do not remain parallel [5]. From this starting point, Einstein illustrated the concept of the curved (or warped) spacetime through the curvature tensors.

To express the extent to which the Euclidean parallelism fails, one can use Riemann tensor [6]:

$$R^{\alpha}_{\beta\nu\mu}=\Gamma^{\alpha}_{\beta\nu,\mu}-\Gamma^{\alpha}_{\beta\mu,\nu}+\Gamma^{\alpha}_{\sigma\mu}\Gamma^{\sigma}_{\beta\nu}-\Gamma^{\alpha}_{\sigma\nu}\Gamma^{\sigma}_{\beta\mu} \tag{5}$$

Ricci tensor [7] can be calculated by contraction accordingly

$$R_{\alpha\beta}=R^{\mu}_{\sigma\mu\beta}=R_{\beta\alpha} \tag{6}$$

and Ricci scalar [8] is given as

$$R=g^{\mu\nu}R_{\mu\nu}=g^{\mu\nu}g^{\alpha\beta}R_{\alpha\mu\beta\nu} \tag{7}$$

Finally, the Einstein tensor [9] is formulated using these curvature tensors. Using the Ricci tensor, Ricci scalar and the spacetime metric, the Einstein tensor can be given by

$$G_{\mu\nu}=R_{\mu\nu}-\frac{1}{2}R\,g_{\mu\nu} \tag{8}$$

Using this tensor, one can write the Einstein's equation which gives a relation of the geometry of spacetime and its matter content. In tensorial form, the Einstein's equation is given by

$$G_{\mu\nu}=8\pi G\,T_{\mu\nu} \tag{9}$$

Here, G is the Newton's constant, $G_{\mu\nu}$ (Einstein tensor) carries the geometric information of the spacetime and $T_{\mu\nu}$, namely the “energy-momentum tensor” carries the information on the matter present in the spacetime. If the energy-momentum tensor is zero, this means that we have a “vacuum”, where there is no matter (or radiation) in the spacetime.

At the first glance, Einstein's equation seems to contain 16 equations as the Einstein tensor and the energy-momentum tensor can both be written as 4x4 matrices (as we have 4 spacetime dimensions) and each element is regarded as a single equation. However, as these tensors are symmetric (e.g. $G_{tr}=G_{rt}$ and $T_{tr}=T_{rt}$) the number of independent equations drop to 10. With the help of some identities (Bianchi identities, which are beyond the scope of our work), the number of independent equations is 6.

Therefore one needs to solve 6 coupled partial differential equations to find the geometric structure of the spacetime. The solution of the Einstein's equation is the metric tensor $g_{\mu\nu}$ which describes the line element of the spacetime and it contains the curvature information.

Surprisingly, this set of difficult equations have exact solutions. Probably the most intriguing exact solutions are the black holes. We will study the most general black hole solutions in classical general relativity in the next part.

2.2. Black Holes

Even though the expression “hole” can be deceiving, a black hole could not be more different than *empty space*. Instead, a black hole can be defined as a gigantic amount of mass packed into a very small volume. It can be imagined as a star that is so much more massive than the Sun, squeezed into a volume with a diameter of a mid-sized state on Earth [10].

A black hole causes a gravitational field so strong that nothing, not even photons can escape from the inside. The region where escaping is not a possibility anymore is called the *event horizon.*

Because of their stupendous mass, black holes bend space-time so much that the center of it becomes a one dimensional point. This point is called the *singularity*, where we cannot perform physical calculations. In mathematical language, the curvature of spacetime is singular at this point.

The black holes, when they were firstly introduced, were believed to be some “mathematical” structures that Einstein's equation yields, and they were not seriously considered as physical objects. After the observation of the first black hole candidates in the Universe, they gained more importance. Today, they are especially indispensable for the research of quantum gravity which tries to unify quantum mechanics and gravitation.

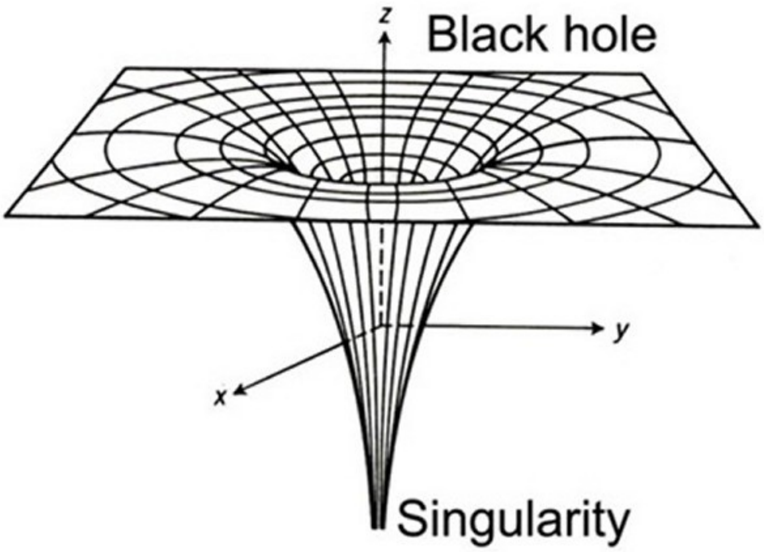

Figure 1: Spacetime Geometry of a Black Hole

A black hole is characterized by its mass, electric charge and angular momentum. According to these qualities, classical black holes are studied under these four categories:

- Non-rotating, uncharged (*Schwarzschild*)
- Non-rotating, charged (*Reissner – Nordström*)
- Rotating, uncharged (*Kerr*)
- Rotating, charged (*Kerr-Newman*)

These are all exact solutions of Einstein's equation. [11]

2.2.1 Schwarzschild Metric

The Schwarzschild solution defines black holes that have no charge, and are not rotating. It was discovered by Karl Schwarzschild in 1915 as the first exact solution of Einstein's equation [12].

Schwarzschild metric is the geometry of the spacetime around the black hole of this certain type. However, it can also be used to understand the behavior of planets and stars which are rotating slowly like the planets in our system and our sun. It was used as the theoretical background for the classical tests of general relativity such as the deflection of light by the sun and the perihelion precession of Mercury.

In spherical coordinates, the metric can be written as; [13]

$$ds^2 = -(1-\frac{2GM}{r})dt^2 + (1-\frac{2GM}{r})^{-1} dr^2 + r^2 d\Omega^2 \tag{10}$$

Where $d\Omega^2$ is the metric of a unit sphere:

$$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2 \tag{11}$$

M is the mass of the black hole that is causing the gravitational force and G is Newton's gravitational constant.

As one can easily see, the coefficient of the radial part, namely $(1-\frac{2GM}{r})^{-1}$ is singular at $r=2GM$. This radius defines the “event horizon”. This is not a real singularity as we can change its location via coordinate transformations. The real and unavoidable singularity is the curvature singularity. We see that $r = 0$ is the real singularity when we calculate the curvature scalars. This point is singular in all coordinate choices.

2.2.2 Reissner-Nordström Metric

This is the solution for a non-rotating charged black hole. It was discovered by Hans Reissner and Gunnar Nordström in 1918. The difference between Reissner-Nordström and Schwarzschild types is one of them is charged. A charged black hole possesses an electric charge.

Compressing an electrically charged mass contains a great electromagnetic repulsion. Approximately 40 orders of magnitude times greater than the gravitational attraction. Hence; it is not likely for a black hole with a significant electric charged to be formed.

The Reissner-Nordström metric is the solution of Einstein-Maxwell field equations for a non-rotating charged black hole. This means solving Einstein's equation with Maxwell equations as a system of equations.

In spherical coordinates, the metric can be written as; [14]

$$ds^2 = (1-\frac{r_s}{r}+\frac{r_Q^2}{r^2})c^2 dt^2 - \frac{1}{1-\frac{r_s}{r}+\frac{r_Q^2}{r^2}} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta\, d\phi^2 \tag{12}$$

Here, r is the radial component, c is the speed of light, t is the time coordinate.

r_s is the Schwarzschild radius of the mass:

$$r_s = \frac{2GM}{c^2} \tag{13}$$

r_Q Is the characteristic length scale:

$$r_Q = \frac{Q^2 G}{4\pi\epsilon_0 c^4} \tag{14}$$

Where c is the speed of light and $1/4\pi\epsilon_0$ is the Coulomb force constant.

2.2.3 Kerr Metric

Kerr metric, also known as Kerr vacuum, is discovered by the mathematician Roy Kerr in 1963. It is a generalized form of the Schwarzschild metric.

The Schwarzschild metric is the geometry of spacetime around a spherically-symmetric, “non-rotating”, uncharged body, where the Kerr metric works with spherically-symmetric, “rotating”, uncharged black-holes. Therefore the Kerr solution describes black holes that are astrophysical. The black holes that we are trying to observe in the Universe are all Kerr black holes.

The metric can be written for a mass M that is rotating with the angular momentum J, as follows; [15]

$$\begin{aligned} c^2 d\tau^2 = & \left(1 - \frac{r_s r}{\rho^2}\right) c^2 dt^2 - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2 \\ & - \left(r^2 + \alpha^2 + \frac{r_s r \alpha^2}{\rho^2} \sin^2\theta\right) \sin^2\theta \, d\phi^2 + \frac{2 r_s r \alpha \sin^2\theta}{\rho^2} c\,dt\,d\phi \end{aligned} \tag{15}$$

Where r_s is the Schwarzschild radius.

$$r_s = 2GM/c^2 \tag{16}$$

$$\alpha=\frac{J}{Mc} \tag{17}$$

$$\rho^2=r^2+\alpha^2\cos^2\theta \tag{18}$$

$$\Delta=r^2-r_s r+\alpha^2 \tag{19}$$

Einstein's equation has a very complicated form when we seek for a rotating solution. The equations which are needed to find the Kerr's solution are said to be the most complicated set of partial differential equations in science that have a real physical meaning.

2.2.4 Kerr-Newman Metric

Kerr-Newman metric is a solution of Einstein-Maxwell equations in general relativity that describes the geometry of spacetime around a rotating & charged black hole. It was first discovered by Ezra Ted Newman in 1965. This metric is the generalized version of Kerr metric with an electric charge.

The metric can be written as; [16]

$$\begin{gathered} c^2 d\tau^2=-\left(\frac{dr^2}{\Delta}+d\theta^2\right)\rho^2 \\ +\left(cdt-\alpha\sin^2\theta\, d\phi\right)^2\frac{\Delta}{\rho^2}-\left(\left(r^2+\alpha^2\right)d\phi-\alpha\, cdt\right)^2\sin^2\frac{\theta}{\rho^2} \end{gathered} \tag{20}$$

Where r_s is the Schwarzschild radius.

$$r_s=2GM/c^2 \quad , \quad r_Q^2=\frac{Q^2 G}{4\pi\epsilon_0 c^4} \tag{21}$$

$$\alpha=\frac{J}{Mc} \tag{22}$$

$$\rho^2=r^2+\alpha^2\cos^2\theta \tag{23}$$

$$\Delta=r^2-r_s r+\alpha^2+r_Q^2 \tag{24}$$

3. MAPLE & GRTensor II

3. 1 General Information

Maple is a computer algebra system firstly developed by the Symbolic Computation group at the University of Waterloo. Then Maplesoft was founded in order to commercialize and sell the technology. They were known as Waterloo Maple Inc. back then.

The program functions with traditional mathematical notation, and supports numeric computations, symbolic computation and visualization. There are interfaces in order to use C, C#, Fortran, Java, MATLAB and Visual Basic with Maple.

Maple's journey began in 1980 at the University of Waterloo. The researches aimed to buy a computer that was resourceful enough to run Macsyma (MAC's Symbolic Manipulator, now known as Maxima). But the plans had changed when they decided to develop their own computer algebra system, which was mapped out to run even on the computers with lower features. They named their program as Maple, to make a reference to Canadian culture. In three years, they managed to distribute Maple over 50 universities.

In 1984, Watcom Products Inc was put in charge of licensing and distributing Maple, whereas in 1988 Waterloo Maple Inc was founded. At first the only goal was to present and grant global access to Maple, however later on the company has evolved enough to accommodate a Research and Development section that is doing most of the job to enhance Maple products today.

The first graphical UI for Maple was developed in 1989, and it started giving support for Macintosh. In 1990 the Maple V was released with an additional support for X11 and Windows.

1999 was the year that Maple 6 was released, and this time it included NAG Numerical Libraries. [17] NAG stands for the Numerical Algorithms Group, and it is a software library of numerical analysis routines. The contemporary interface that we still use today was first introduced in 1999 with Maple 9.

During time, Maple lost its popularity and a remarkable amount of market share due to rough competition in the business. One of the main reasons were the user interface that remained incapable compared to the opponents'. To prevent further devolution, Maple introduced Maple 12 and Maple 13 respectively in 2008 and 2009. New features were added to the program such as MATLAB toolbox, style sheets, bracket matching, command completion templates, syntax checking and 3D plotting.

Maple and Maplesoft were purchased by Cybernet Systems in 2009, September.

3. 2 Basic Computations

Basic mathematical operations are computed in Maple as the following examples.

3. 2. 1 Addition – Subtraction – Multiplication – Division

This basic operations can be computed in Maple by typing '+', '-', '*' and '/' respectively.

Example:

```
> 334+557;
                    891
> 556-330;
                    226
> 34*45;
                    1530
> 56/5;
```

$$\frac{56}{5}$$

3. 2. 2 Root – Exponential

To compute the real odd root of a negative real number, the function **surd** is used. It's basic form is surd(x,n) where x is a complex number and n is an integer. Surd computes the n^{th} root of x.

Example:

```
> surd(-8, 3);
                         -2
> surd( 8, 3);
                                 2
```

The function **root** also computes the n^{th} root of x, where x can be a real or complex floating constant, an integer or complex rational constant, or a general symbolic expression.

If x is a real or complex floating-point constant, Maple computes the result in floating-point arithmetic. Else, the function will try to simplify **x^(1/n)**.

Example:

```
> root( 8, 3);
```

$$2$$

```
> root(-8, 3);
```

$$2(-1)^{1/3}$$

To compute an exponential, simply type '^' between the variables.

Example:

```
> 3^4;
```

$$81$$

3. 2. 3 Derivative

The function **diff** is used to take derivations. It can be used in multiple ways.

$\mathrm{diff}(f, x1, \ldots, xj)$:

$$\frac{d^j}{dx_j \ldots dx_1} f$$

$\mathrm{diff}(f, [x1\$n])$:

$$\frac{d^n}{dx_1^n} f$$

Where f is an algebraic expression or an equation, x1, x2, … , xj are the names representing differentiation values and n is the n^{th} order of the derivative.

Example:

```
>diff(sin(x),x$2)
```

$-\sin(x)$

```
>diff(x^3,x$2)
```

$6x$

3. 2. 4 Integration

Integrals can be computed with the function **int**.

- To compute a single, indefinite integral;

$$\text{int}(expression, x, options) \rightarrow \int \text{expression dx}$$

Example:

```
> f := 2*x;
```

$2\,x$

```
> int(f, x);
```

x^2

- To compute a single, definite integral;

$$\text{int}(expression, x=a..b) \rightarrow \int_a^b \text{expression dx}$$

Example:

```
> f:= 2*x;
```

$2\,x$

```
> int(f, x = 1 .. 2);
```

3

- To compute a double integral;

$$\text{int}(expression, [x, y, ...]) \rightarrow \int\int \text{expression dx dy}$$

Example:

```
> f:= 2*x^2 + y^3 ;
```

$$2x^2+y^3$$

```
> int(f, [x,y]);
```

$$\frac{2}{3}x^3y+\frac{1}{4}y^4x$$

```
> int(f, [x=1..2,y=2..3]);
```

$$\frac{251}{12}$$

3.3 GRTensor II

3.3.1 General Information

GRTensor II is a computer algebra package that can be used to do calculations on general differential geometry and general relativity. It is developed by Peter Musgrave, Denis Pollney and Kayll Lake.

The main purpose of the package is to do tensor calculations on curved spacetimes, that is been specified as a metric, or a set of basis vectors. It can:

- Calculate standard objects such as Riemann and Ricci tensors for a given metric;
- Calculate scalar and differential invariants of a metric;
- Do these calculations in any dimension;
- Detect junction conditions between to spaces;
- Work simultaneously with more than one metric
- Define new tensors

The package does not work on its own, it requires an algebraic engine. Even though it is initially designed for MapleV, it can also be used with other versions of Maple. Also there is a special version called GRTensorM that was imported into Mathematica.

GRTensor II supports Windows, Unix and Macintosh. It is free of charge and accessible from http://grtensor.phy.queensu.ca/. [18]

The Newman-Penrose calculator of the GRTensor II package is problematic for Euclidean metrics. For those metrics one can use the "NPInstanton" package written on Maple and GRTensor II by T. Birkandan. [19]

3.3.2 Calculations of the Kerr Metric

3.3.2.1 Defining the Metric

In this project, Kerr metric is chosen to be studied. By using the GRTensor II package, we will now do the calculations of the metric. The following calculations are done in Maple, with the help of GRTensor II.

Before moving any further, the first thing that needs to be done is to manually describe the Kerr metric in Maple.

$$dssq := \left(1 - \frac{2 \cdot M \cdot r}{r^2 + a^2 \cdot \cos(theta)^2}\right) \cdot d[t]^2 - \left(\frac{r^2 + a^2 \cdot \cos(theta)^2}{r^2 - 2 \cdot M \cdot r + a^2}\right) d[r]^2$$
$$-\left(r^2 + a^2 \cdot \cos(theta)^2\right) d[theta]^2 - \left(r^2 + a^2 + \frac{2 \cdot M \cdot a^2 \cdot r}{r^2 + a^2 \cdot \cos(theta)^2} \cdot \sin(theta)^2\right) \cdot \sin(theta)^2 \cdot d[phi]^2$$
$$\frac{+2 \cdot 2 \cdot M \cdot r \cdot a \cdot \sin(theta)^2}{r^2 + a^2 \cdot \cos(theta)^2} \cdot d[t]^2 \cdot d[phi]$$

Here, we define the metric as “dssq” to use later. Then call the GRTensor II by the command

>***grtw( )***

To define the metric *dssq* as the Kerr metric from scratch , the following command must be used.

>makeg(kerrgrt)

Makeg 2.0: GRTensor metric/basis entry utility

To quit makeg, type 'exit' at any prompt.

Do you wish to enter a 1) metric [g(dn,dn)],
2) line element [ds],
3) non-holonomic basis [e(1)...e(n)], or
4) NP tetrad [l,n,m,mbar]?

Option [2] must be chosen.

Enter coordinates as a LIST (eg. [t,r,theta,phi]):

The coordinates for the Kerr metric must be entered, which is [t,r,theta,phi]

Enter the line element using d[coord] to indicate differentials.
(for example, r^2(d[theta]^2 + sin(theta)^2*d[phi]^2)*
[Type 'exit' to quit makeg]
ds^2 =

Here, ds^2=dssq must be typed. One can write the whole metric here manually but defining it beforehand and using its name ("dssq" in this case) is more convenient.

If there are any complex valued coordinates, constants or functions for this spacetime, please enter them as a SET (eg. { z, psi }).

Complex quantities [default={}]:

{} must be typed in the dialogue box, then proceed.

Finally, the elements for the Kerr metric can be seen.

The values you have entered are:
Coordinates = [t, r, theta, phi]

$$g_{tt}=1-\frac{2Mr}{r^2+a^2\cos(\theta)^2}$$

$$g_{tr}=0$$

$$g_{t\theta}=0$$

$$g_{t\phi}=\frac{2Mra\sin(\theta)^2}{r^2+a^2\cos(\theta)^2}$$

$$g_{rt}=0$$

$$g_{rr}=\frac{-r^2}{r^2-2Mr+a^2}-\frac{a^2\cos(\theta)^2}{r^2-2Mr+a^2}$$

$$g_{r\theta}=0$$

$$g_{r\phi}=0$$

$$g_{\theta t}=0$$

$$g_{\theta r}=0$$

$$g_{\theta\theta}=-r^2-a^2\cos(\theta)^2$$

$$g_{\theta\phi}=0$$

$$g_{\phi t}=\frac{2 M r a \sin(\theta)^2}{r^2+a^2\cos(\theta)^2}$$

$$g_{\phi r}=0$$

$$g_{\phi\theta}=0$$

$$g_{\phi\phi}=-\sin(\theta)^2 r^2-\frac{2\sin(\theta)^4 M a^2 r}{r^2+a^2\cos(\theta)^2}$$

3.3.2.2 Calculating the Riemann Tensor

To calculate the Riemann covariant tensor of a Kerr metric; type:

>grcalc(R(dn,dn,dn,dn))

Before displaying, simplify the solution with

>gralter(R(dn,dn,dn,dn))

Then display.

>grdisplay(R(dn,dn,dn,dn))

For the kerrgrt spacetime:
Covariant Riemann

$$R_{trtr}=-\frac{M\left(12 M r a^2\cos(\theta)^2-4 M r^3+3 a^4\cos(\theta)^4-9 a^4\cos(\theta)^2-7 r^2 a^2\cos(\theta)^2+3 r^2 a^2+2 r^4\right) r}{\left(-r^2+2 M r-a^2\right)\left(3 r^2 a^4\cos(\theta)^4+a^6\cos(\theta)^6+3 r^4 a^2\cos(\theta)^2+r^6\right)}$$

$$R_{trt\theta}=\frac{3 M a^2\left(a^2\cos(\theta)^2-3 r^2\right)\cos(\theta)\sin(\theta)}{3 r^2 a^4\cos(\theta)^4+a^6\cos(\theta)^6+3 r^4 a^2\cos(\theta)^2+r^6}$$

$$R_{trr\phi}=-\frac{M a\left(3 a^2\cos(\theta)^2-r^2\right) r\left(4 M r-3 r^2-3 a^2\right)\sin(\theta)^2}{\left(-r^2+2 M r-a^2\right)\left(3 r^2 a^4\cos(\theta)^4+a^6\cos(\theta)^6+3 r^4 a^2\cos(\theta)^2+r^6\right)}$$

$$R_{tr\theta\phi}=-\frac{a M\sin(\theta)\cos(\theta)\left(a^4\cos(\theta)^4-3 a^4\cos(\theta)^2-5 r^2 a^2\cos(\theta)^2+9 r^2 a^2+6 r^4\right)}{3 r^2 a^4\cos(\theta)^4+a^6\cos(\theta)^6+3 r^4 a^2\cos(\theta)^2+r^6}$$

$$R_{t\theta t\theta}=-\frac{M r\left(6 M r a^2\cos(\theta)^2-2 M r^3+6 a^4\cos(\theta)^4-9 a^4\cos(\theta)^2-5 r^2 a^2\cos(\theta)^2+3 r^2 a^2+r^4\right)}{3 r^2 a^4\cos(\theta)^4+a^6\cos(\theta)^6+3 r^4 a^2\cos(\theta)^2+r^6}$$

$$R_{t\theta r\phi} = -\frac{a\,M\sin(\theta)\cos(\theta)\left(2\,a^4\cos(\theta)^4 - 3\,a^4\cos(\theta)^2 - 7\,r^2\,a^2\cos(\theta)^2 + 9\,r^2\,a^2 + 3\,r^4\right)}{3\,r^2\,a^4\cos(\theta)^4 + a^6\cos(\theta)^6 + 3\,r^4\,a^2\cos(\theta)^2 + r^6}$$

$$R_{t\theta\theta\phi} = -\frac{M\,a\,r\left(3\,a^2\cos(\theta)^2 - r^2\right)\left(2\,M\,r - 3\,r^2 - 3\,a^2\right)\sin(\theta)^2}{3\,r^2\,a^4\cos(\theta)^4 + a^6\cos(\theta)^6 + 3\,r^4\,a^2\cos(\theta)^2 + r^6}$$

$$R_{t\phi t\phi} = -\frac{M\,r\sin(\theta)^2\left(-r^2 + 2\,M\,r - a^2\right)\left(3\,a^2\cos(\theta)^2 - r^2\right)}{3\,r^2\,a^4\cos(\theta)^4 + a^6\cos(\theta)^6 + 3\,r^4\,a^2\cos(\theta)^2 + r^6}$$

$$R_{t\phi r\theta} = -\frac{a\,M\left(a^2\cos(\theta)^2 - 3\,r^2\right)\cos(\theta)\sin(\theta)}{2\,r^2\,a^2\cos(\theta)^2 + a^4\cos(\theta)^4 + r^4}$$

$$R_{r\theta r\theta} = \frac{M\,r\left(3\,a^2\cos(\theta)^2 - r^2\right)}{\left(-r^2 + 2\,M\,r - a^2\right)\left(r^2 + a^2\cos(\theta)^2\right)}$$

$$R_{r\phi r\phi} = \Big(M\big(-6\,a^6\cos(\theta)^4 - 6\,r^2\,a^4\cos(\theta)^4 + 12\,a^4\cos(\theta)^4\,M\,r + 9\cos(\theta)^2\,a^6 - 12\,M\,r\,a^4\cos(\theta)^2$$
$$+ 14\,a^4\,r^2\cos(\theta)^2 - 4\,M\,r^3\,a^2\cos(\theta)^2 + 5\,r^4\,a^2\cos(\theta)^2 - 3\,r^2\,a^4 + 4\,M\,r^3\,a^2 - 4\,a^2\,r^4 - r^6\big)\sin(\theta)^2\,r\Big)\Big/$$
$$\Big(\left(-r^2 + 2\,M\,r - a^2\right)\left(3\,r^2\,a^4\cos(\theta)^4 + a^6\cos(\theta)^6 + 3\,r^4\,a^2\cos(\theta)^2 + r^6\right)\Big)$$

$$R_{r\phi\theta\phi} = \frac{3\,a^2\left(a^2\cos(\theta)^2 - 3\,r^2\right)\left(r^2 + a^2\right)M\cos(\theta)\sin(\theta)^3}{3\,r^2\,a^4\cos(\theta)^4 + a^6\cos(\theta)^6 + 3\,r^4\,a^2\cos(\theta)^2 + r^6}$$

$$R_{\theta\phi\theta\phi} = \frac{1}{3\,r^2\,a^4\cos(\theta)^4 + a^6\cos(\theta)^6 + 3\,r^4\,a^2\cos(\theta)^2 + r^6}\Big(M\,r\big(-3\,a^6\cos(\theta)^4 + 6\,a^4\cos(\theta)^4\,M\,r$$
$$- 3\,r^2\,a^4\cos(\theta)^4 + 9\cos(\theta)^2\,a^6 - 6\,M\,r\,a^4\cos(\theta)^2 + 16\,a^4\,r^2\cos(\theta)^2 - 2\,M\,r^3\,a^2\cos(\theta)^2$$
$$+ 7\,r^4\,a^2\cos(\theta)^2 - 3\,r^2\,a^4 + 2\,M\,r^3\,a^2 - 5\,a^2\,r^4 - 2\,r^6\big)\sin(\theta)^2\Big)$$

-

3.3.2.3 Calculating the Ricci Tensor

To calculate the Ricci covariant tensor of a Kerr metric; type:

>grcalc(R(dn,dn))

Before displaying, simplify the solution with

>gralter(R(dn,dn))

Then display.

>grdisplay(R(dn,dn))

For the kerrgrt spacetime:
Covariant Ricci
R(dn,dn)
R[a][b]=All components are zero.

3.3.2.4 Calculating the Ricci Scalar

To calculate the Ricci scalar of a Kerr metric; type:

>grcalc(Ricciscalar)

Before displaying, simplify the solution with

>gralter(Ricciscalar)

Then display.

>grdisplay(Ricciscalar)

For the kerrgrt spacetime:
Ricci Scalar
R=0

3.3.2.5 Calculating the Einstein Tensor

To calculate the Ricci scalar of a Kerr metric; type:

>grcalc(eins(dn,dn))

Before displaying, simplify the solution with

>gralter(eins(dn,dn))

Then display.

>grdisplay(eins(dn,dn))

For the kerrgrt spacetime:
eins(dn,dn)
eins_[a][b]=All components are zero.

3.3.2.6 Calculating the Christoffel Symbol of the First Kind

To calculate the Christoffel symbols of a Kerr metric; type:

>grcalc(Chr(dn,dn,dn))

Before displaying, simplify the solution with

>gralter(Chr(dn,dn,dn))

Then display.

>grdisplay(Chr(dn,dn,dn))

For the kerrgrt spacetime:

Christoffel symbol of the first kind (symmetric in first two indices)

$$\Gamma_{t\,t\,r} = \frac{M\left(-r^2 + a^2 \cos(\theta)^2\right)}{2\,r^2\,a^2 \cos(\theta)^2 + a^4 \cos(\theta)^4 + r^4}$$

$$\Gamma_{t\,t\,\theta} = \frac{2\,M\,a^2\,r \cos(\theta)\,\sin(\theta)}{2\,r^2\,a^2 \cos(\theta)^2 + a^4 \cos(\theta)^4 + r^4}$$

$$\Gamma_{t\,r\,t} = -\frac{M\left(-r^2 + a^2 \cos(\theta)^2\right)}{2\,r^2\,a^2 \cos(\theta)^2 + a^4 \cos(\theta)^4 + r^4}$$

$$\Gamma_{t\,r\,\phi} = \frac{M\,a \sin(\theta)^2 \left(-r^2 + a^2 \cos(\theta)^2\right)}{2\,r^2\,a^2 \cos(\theta)^2 + a^4 \cos(\theta)^4 + r^4}$$

$$\Gamma_{t\,\theta\,t} = -\frac{2\,M\,a^2\,r \cos(\theta)\,\sin(\theta)}{2\,r^2\,a^2 \cos(\theta)^2 + a^4 \cos(\theta)^4 + r^4}$$

$$\Gamma_{t\,\theta\,\phi} = \frac{2\,M\,r\,a \sin(\theta) \cos(\theta)\left(r^2 + a^2\right)}{2\,r^2\,a^2 \cos(\theta)^2 + a^4 \cos(\theta)^4 + r^4}$$

$$\Gamma_{t\,\phi\,r} = -\frac{M\,a \sin(\theta)^2 \left(-r^2 + a^2 \cos(\theta)^2\right)}{2\,r^2\,a^2 \cos(\theta)^2 + a^4 \cos(\theta)^4 + r^4}$$

$$\Gamma_{t\,\phi\,\theta} = -\frac{2\,M\,r\,a \sin(\theta) \cos(\theta)\left(r^2 + a^2\right)}{2\,r^2\,a^2 \cos(\theta)^2 + a^4 \cos(\theta)^4 + r^4}$$

$$\Gamma_{r\,r\,r} = -\frac{-M\,r^2 + r\,a^2 - a^2 \cos(\theta)^2\,r + a^2 \cos(\theta)^2\,M}{\left(-r^2 + 2\,M\,r - a^2\right)^2}$$

$$\Gamma_{r r \theta} = \frac{a^2 \cos(\theta) \sin(\theta)}{-r^2 + 2 M r - a^2}$$

$$\Gamma_{r \theta r} = -\frac{a^2 \cos(\theta) \sin(\theta)}{-r^2 + 2 M r - a^2}$$

$$\Gamma_{r \theta \theta} = -r$$

$$\Gamma_{r \phi t} = \frac{M a \sin(\theta)^2 \left(-r^2 + a^2 \cos(\theta)^2\right)}{2 r^2 a^2 \cos(\theta)^2 + a^4 \cos(\theta)^4 + r^4}$$

$$\Gamma_{r \phi \phi} = \frac{1}{2 r^2 a^2 \cos(\theta)^2 + a^4 \cos(\theta)^4 + r^4} \Big(\sin(\theta)^2 \big(-r a^4 \cos(\theta)^4 + \cos(\theta)^4 M a^4 - M a^4 \cos(\theta)^2 - 2 r^3 a^2 \cos(\theta)^2 - \cos(\theta)^2 r^2 a^2 M - r^5 + M a^2 r^2\big)\Big)$$

$$\Gamma_{\theta \theta r} = r$$

$$\Gamma_{\theta \theta \theta} = a^2 \cos(\theta) \sin(\theta)$$

$$\Gamma_{\theta \phi t} = \frac{2 M r a \sin(\theta) \cos(\theta) \left(r^2 + a^2\right)}{2 r^2 a^2 \cos(\theta)^2 + a^4 \cos(\theta)^4 + r^4}$$

$$\Gamma_{\theta \phi \phi} = \frac{1}{2 r^2 a^2 \cos(\theta)^2 + a^4 \cos(\theta)^4 + r^4} \Big(\sin(\theta) \cos(\theta) \big(-r^2 a^4 \cos(\theta)^4 - a^6 \cos(\theta)^4 - 2 r^4 a^2 \cos(\theta)^2 - 2 a^4 r^2 \cos(\theta)^2 - r^6 - a^2 r^4 - 4 M r^3 a^2 - 2 M a^4 r + 2 a^4 \cos(\theta)^4 M r + 4 M r^3 a^2 \cos(\theta)^2\big)\Big)$$

$$\Gamma_{\phi \phi r} = -\frac{1}{2 r^2 a^2 \cos(\theta)^2 + a^4 \cos(\theta)^4 + r^4} \Big(\sin(\theta)^2 \big(-r a^4 \cos(\theta)^4 + \cos(\theta)^4 M a^4 - M a^4 \cos(\theta)^2 - 2 r^3 a^2 \cos(\theta)^2 - \cos(\theta)^2 r^2 a^2 M - r^5 + M a^2 r^2\big)\Big)$$

$$\Gamma_{\phi \phi \theta} = -\frac{1}{2 r^2 a^2 \cos(\theta)^2 + a^4 \cos(\theta)^4 + r^4} \Big(\sin(\theta) \cos(\theta) \big(-r^2 a^4 \cos(\theta)^4 - a^6 \cos(\theta)^4 - 2 r^4 a^2 \cos(\theta)^2 - 2 a^4 r^2 \cos(\theta)^2 - r^6 - a^2 r^4 - 4 M r^3 a^2 - 2 M a^4 r + 2 a^4 \cos(\theta)^4 M r + 4 M r^3 a^2 \cos(\theta)^2\big)\Big)$$

4. Sage & SageManifolds

4.1 General Information

Sage stands for *System for Algebra and Geometry Experimentation.* It's a mathematical software that has been used in mathematics, especially for calculus, algebra, numerical mathematics and combinatorics.

It was first released nearly ten years ago, on 24th of February, 2005. William Stein was the leader of the project. The goal in making Sage, was to be the open source alternative to Magma, Maple, Mathematica and MATLAB [20].

Sage is written in Python & Cython and can be said to use a syntax that is Python like [21]. Its symbolic calculator is based on another free package Maxima and SymPy which is the symbolic module of Python.

One can install Sage on Linux and Mac but Sage is also installed on a powerful cloud server with many of its freely available packages: https://cloud.sagemath.com/.

We will be using SageManifolds package for our relativistic calculations. It is installed on the cloud server.

4.2 Basic Computations

4.2.1 Addition – Subtraction – Multiplication – Division

The four operations are easy to compute in Sage.

Example:

```
3+4
    7
```

```
55-42
    13
```

```
23*25
    574
```

```
556/4
    139
```

4.2.2 Root – Power

To compute with roots and exponentials,

Example:

```
sqrt(144)
```

```
    12
```

Example:

```
2**2
```

```
    4
```

```
2^2
    4
```

4.2.3 Derivative

With Sage, derivatives of many functions can be computed. For example, to differentiate *cos(x)* once with respect to *x*:

Example:

```
x = var('x')
diff (cos(x),x)
```

```
    -sin (x)
```

4.2.4 Integration

Integrals can also be computed using Sage. To compute an indefinite integral,

Example:

$\int x e^{-x^2} dx$

```
    integral (x*e^(-x^2), x)
```

```
        -1/2*e^(-x^2)
```

To compute a definite integral,

Example:

$$\int_0^\infty x e^{-x^2} dx$$

```
    integral (x*e^(-x^2), x, 0, infinity)
```

```
        1/2
```

4.3 Calculations With SageManifolds

4.3.1 Writing the Spacetime Manifold

First, the Kerr Spacetime is defined as a 4-dimensional manifold.

```
    M=Manifold (4, 'M', r'\mathcal {M}')
```

```
    M0 = M.open_subset('M0', r'\mathcal{M}_0')
    # BL = Boyer-Lindquist
    BL.<t,r,th,ph> = M0.chart(r't r:(0,+oo) th:
(0,pi):\theta ph:(0,2*pi):\phi')
    print BL ; BL
```

$$Chart\left(M0,\left(t,r,th,ph\right)\right)$$
$$\left(M_0,\left(t,r,\theta,\phi\right)\right)$$

```
    BL[0], BL [1]
```

$$\left(t,r\right)$$

4.3.2 Writing the Metric

The parameters m and a of the Kerr spacetime must be declared as symbolic variables, and the spacetime metric is introduced.

```
var('m,a')
g= M.lorentzian_metric('g')
```

The components of the metric is then defined.

```
rho2 = r^2 + (a*cos(th))^2
Delta = r^2 -2*m*r + a^2
g[0,0] = -(1-2*m*r/rho2)
g[0,3] = -2*a*m*r*sin(th)^2/rho2
g[1,1], g[2,2] = rho2/Delta, rho2
g[3,3] = (r^2+a^2+2*m*r*(a*sin(th))^2/rho2)*sin(th)^2
```

Now the metric can be displayed.

```
g.display()
```

$$g=\left(\frac{-a^2\cos(\theta)^2-2mr+r^2}{a^2\cos(\theta)^2+r^2}\right)dt\otimes dt+\left(\frac{-2amr\sin(\theta)^2}{a^2\cos(\theta)^2+r^2}\right)dt\otimes d\phi+\left(\frac{a^2\cos(\theta)^2+r^2}{a^2-2mr+r^2}\right)dr\otimes dr$$
$$+\left(a^2\cos(\theta)^2+r^2\right)d\theta\otimes d\theta+\left(\frac{-2amr\sin(\theta)^2}{a^2\cos(\theta)^2+r^2}\right)d\phi\otimes dt$$
$$\left(\frac{2a^2mr\sin(\theta)^4+\left(a^2r^2+r^4+\left(a^4+a^2r^2\right)\cos(\theta)^2\right)\sin(\theta)^2}{a^2\cos(\theta)^2+r^2}\right)d\phi\otimes d\phi$$

To see the components in the matrix form, type:

```
g[:]
```

$$\begin{pmatrix} -\frac{a^2\cos(\theta)^2-2mr+r^2}{a^2\cos(\theta)^2+r^2} & 0 & 0 & -\frac{2amr\sin(\theta)^2}{a^2\cos(\theta)^2+r^2} \\ 0 & \frac{a^2\cos(\theta)^2+r^2}{a^2-2mr+r^2} & 0 & 0 \\ 0 & 0 & a^2\cos(\theta)^2+r^2 & 0 \\ -\frac{2amr\sin(\theta)^2}{a^2\cos(\theta)^2+r^2} & 0 & 0 & \frac{2a^2mr\sin(\theta)^4+\left(a^2r^2+r^4+\left(a^4+a^2r^2\right)\cos(\theta)^2\right)\sin(\theta)^2}{a^2\cos(\theta)^2+r^2} \end{pmatrix}$$

To see the list of the non-zero components, type:

```
g.display_comp()
```

$$g_{t\,t} = -\frac{a^{2}\cos(\theta)^{2}-2\,mr+r^{2}}{a^{2}\cos(\theta)^{2}+r^{2}}$$

$$g_{t\,\phi} = -\frac{2\,amr\sin(\theta)^{2}}{a^{2}\cos(\theta)^{2}+r^{2}}$$

$$g_{r\,r} = \frac{a^{2}\cos(\theta)^{2}+r^{2}}{a^{2}-2\,mr+r^{2}}$$

$$g_{\theta\,\theta} = a^{2}\cos(\theta)^{2}+r^{2}$$

$$g_{\phi\,t} = -\frac{2\,amr\sin(\theta)^{2}}{a^{2}\cos(\theta)^{2}+r^{2}}$$

$$g_{\phi\,\phi} = \frac{2\,a^{2}mr\sin(\theta)^{4}+\left(a^{2}r^{2}+r^{4}+(a^{4}+a^{2}r^{2})\cos(\theta)^{2}\right)\sin(\theta)^{2}}{a^{2}\cos(\theta)^{2}+r^{2}}$$

4.3.3 Calculating the Christoffel Symbols

To display the Christoffel symbols, type:

```
g.christoffel_symbols_display()
```

$$\Gamma^{t}{}_{t\,r} = -\frac{a^{4}m-mr^{4}-(a^{4}m+a^{2}mr^{2})\sin(\theta)^{2}}{a^{2}r^{4}-2\,mr^{5}+r^{6}+(a^{6}-2\,a^{4}mr+a^{4}r^{2})\cos(\theta)^{4}+2\,(a^{4}r^{2}-2\,a^{2}mr^{3}+a^{2}r^{4})\cos(\theta)^{2}}$$

$$\Gamma^{t}{}_{t\,\theta} = -\frac{2\,a^{2}mr\cos(\theta)\sin(\theta)}{a^{4}\cos(\theta)^{4}+2\,a^{2}r^{2}\cos(\theta)^{2}+r^{4}}$$

$$\Gamma^{t}{}_{r\,\phi} = -\frac{\left(a^{3}mr^{2}+3\,amr^{4}-(a^{5}m-a^{3}mr^{2})\cos(\theta)^{2}\right)\sin(\theta)^{2}}{a^{2}r^{4}-2\,mr^{5}+r^{6}+(a^{6}-2\,a^{4}mr+a^{4}r^{2})\cos(\theta)^{4}+2\,(a^{4}r^{2}-2\,a^{2}mr^{3}+a^{2}r^{4})\cos(\theta)^{2}}$$

$$\Gamma^{t}{}_{\theta\,\phi} = -\frac{2\,\left(a^{5}mr\cos(\theta)\sin(\theta)^{5}-(a^{5}mr+a^{3}mr^{3})\cos(\theta)\sin(\theta)^{3}\right)}{a^{6}\cos(\theta)^{6}+3\,a^{4}r^{2}\cos(\theta)^{4}+3\,a^{2}r^{4}\cos(\theta)^{2}+r^{6}}$$

$$\Gamma^{r}{}_{t\,t} = \frac{a^{2}mr^{2}-2\,m^{2}r^{3}+mr^{4}-(a^{4}m-2\,a^{2}m^{2}r+a^{2}mr^{2})\cos(\theta)^{2}}{a^{6}\cos(\theta)^{6}+3\,a^{4}r^{2}\cos(\theta)^{4}+3\,a^{2}r^{4}\cos(\theta)^{2}+r^{6}}$$

$$\Gamma^{r}{}_{t\,\phi} = -\frac{\left(a^{3}mr^{2}-2\,am^{2}r^{3}+amr^{4}-(a^{5}m-2\,a^{3}m^{2}r+a^{3}mr^{2})\cos(\theta)^{2}\right)\sin(\theta)^{2}}{a^{6}\cos(\theta)^{6}+3\,a^{4}r^{2}\cos(\theta)^{4}+3\,a^{2}r^{4}\cos(\theta)^{2}+r^{6}}$$

$$\Gamma^{r}{}_{r\,r} = \frac{a^{2}m-mr^{2}-(a^{2}m-a^{2}r)\sin(\theta)^{2}}{a^{2}r^{2}-2\,mr^{3}+r^{4}+(a^{4}-2\,a^{2}mr+a^{2}r^{2})\cos(\theta)^{2}}$$

$$\Gamma^{r}{}_{\text{[illegible]}} = -\frac{a^{2}\cos(\theta)\sin(\theta)}{\text{[illegible]}}$$

$$\Gamma^{\theta}{}_{t\,t} = -\frac{2\,a^{2}mr\cos(\theta)\sin(\theta)}{a^{6}\cos(\theta)^{6}+3\,a^{4}r^{2}\cos(\theta)^{4}+3\,a^{2}r^{4}\cos(\theta)^{2}+r^{6}}$$

$$\Gamma^{\theta}{}_{t\,\phi} = \frac{2\,(a^{3}mr+amr^{3})\cos(\theta)\sin(\theta)}{a^{6}\cos(\theta)^{6}+3\,a^{4}r^{2}\cos(\theta)^{4}+3\,a^{2}r^{4}\cos(\theta)^{2}+r^{6}}$$

$\sin(\theta)^{2}$

$$\Gamma^{\theta}{}_{r\,r} = \frac{a^{2}\cos(\theta)\sin(\theta)}{a^{2}r^{2}-2\,mr^{3}+r^{4}+(a^{4}-2\,a^{2}mr+a^{2}r^{2})\cos(\theta)^{2}}$$

$$\Gamma^{\theta}{}_{r\,\theta} = \frac{r}{a^{2}\cos(\theta)^{2}+r^{2}}$$

$$\Gamma^{\theta}{}_{\theta\,\theta} = -\frac{a^{2}\cos(\theta)\sin(\theta)}{a^{2}\cos(\theta)^{2}+r^{2}}$$

$$\Gamma^{\theta}{}_{\phi\,\phi} = -\frac{\left((a^{6}-2\,a^{4}mr+a^{4}r^{2})\cos(\theta)^{5}+2\,(a^{4}r^{2}-2\,a^{2}mr^{3}+a^{2}r^{4})\cos(\theta)^{3}+(2\,a^{4}mr+4\,a^{2}mr^{3}+a^{2}r^{4}+r^{6})\cos(\theta)\right)\sin(\theta)}{a^{6}\cos(\theta)^{6}+3\,a^{4}r^{2}\cos(\theta)^{4}+3\,a^{2}r^{4}\cos(\theta)^{2}+r^{6}}$$

$$\Gamma^{\phi}{}_{t\,r} = -\frac{a^{3}m\cos(\theta)^{2}-amr^{2}}{a^{2}r^{4}-2\,mr^{5}+r^{6}+(a^{6}-2\,a^{4}mr+a^{4}r^{2})\cos(\theta)^{4}+2\,(a^{4}r^{2}-2\,a^{2}mr^{3}+a^{2}r^{4})\cos(\theta)^{2}}$$

$$\Gamma^{\phi}{}_{t\,\theta} = -\frac{2\,amr\cos(\theta)}{\left(a^{4}\cos(\theta)^{4}+2\,a^{2}r^{2}\cos(\theta)^{2}+r^{4}\right)\sin(\theta)}$$

$$\Gamma^{\phi}{}_{r\,\phi} = -\frac{a^{2}mr^{2}+2\,mr^{4}-r^{5}+(a^{4}m-a^{4}r)\cos(\theta)^{4}-(a^{4}m-a^{2}mr^{2}+2\,a^{2}r^{3})\cos(\theta)^{2}}{a^{2}r^{4}-2\,mr^{5}+r^{6}+(a^{6}-2\,a^{4}mr+a^{4}r^{2})\cos(\theta)^{4}+2\,(a^{4}r^{2}-2\,a^{2}mr^{3}+a^{2}r^{4})\cos(\theta)^{2}}$$

$$\Gamma^{\phi}{}_{\theta\,\phi} = \frac{a^{4}\cos(\theta)^{5}-2\,(a^{2}mr-a^{2}r^{2})\cos(\theta)^{3}+(2\,a^{2}mr+r^{4})\cos(\theta)}{\left(a^{4}\cos(\theta)^{4}+2\,a^{2}r^{2}\cos(\theta)^{2}+r^{4}\right)\sin(\theta)}$$

4.3.4 Calculating the Ricci Tensor

There are two ways to check the Ricci Tensor.

1.

```
Ric == 0
```

True

The feedback “True” means that the all elements of the Ricci tensor are zero.

2.

```
Ric.display()
```

Ric (g)=0

4.3.5 Calculating the Ricci Scalar

To calculate the Ricci scalar, type:

```
g.ricci_scalar().display()
```

$$\begin{array}{llcl} \mathrm{r}\left(g\right): & \mathcal{M} & \longrightarrow & \mathbb{R} \\ \text{on } \mathcal{M}_0: & (t, r, \theta, \phi) & \longmapsto & 0 \end{array}$$

4.3.6 Calculating the Riemann Tensor

To display the Riemann tensor, type:

```
R = g.riemann() ; print R
```

Then type the indices of to see the component you would like to be displayed. For example, to display the component R^0_{123} , type:

```
R[0,1,2,3]
```

$$-\frac{\left(a^{7} m - 2 \, a^{5} m^{2} r + a^{5} m r^{2}\right) \cos\left(\theta\right) \sin\left(\theta\right)^{5} + \left(a^{7} m + 2 \, a^{5} m^{2} r + 6 \, a^{5} m r^{2} - 6 \, a^{3} m^{2} r^{3} + 5 \, a^{3} m r^{4}\right) \cos\left(\theta\right) \sin\left(\theta\right)^{3} - 2 \, \left(a^{7} m - a^{5} m r^{2} - 5 \, a^{3} m r^{4} - 3 \, a m r^{6}\right) \cos\left(\theta\right) \sin\left(\theta\right)}{a^{2} r^{6} - 2 \, m r^{7} + r^{8} + \left(a^{8} - 2 \, a^{6} m r + a^{6} r^{2}\right) \cos\left(\theta\right)^{6} + 3 \, \left(a^{6} r^{2} - 2 \, a^{4} m r^{3} + a^{4} r^{4}\right) \cos\left(\theta\right)^{4} + 3 \, \left(a^{4} r^{4} - 2 \, a^{2} m r^{5} + a^{2} r^{6}\right) \cos\left(\theta\right)^{2}}$$

4.3.7 The Einstein Tensor

The Einstein tensor can be defined as;

```
G= Ric – 1/2*g.ricci_scalar()*g ; print G
```

Field of symmetric bilinear forms +Ric(g) on the 4-dimensional differentiable manifold M

In section 4.2.6 we have seen that the Ricci scalar of the Kerr metric is zero. So, here, the Einstein tensor will reduce to the Ricci tensor.

```
G == Ric
```

True.

We can display the elements of the Einstein tensor when needed.

5. Maxima & ctensor

5.1 General Information

Maxima, is a free computer algebra system that has been based on the MIT developed general purpose program Macsyma.

It is written in Common LISP and supports Windows, OS X, Unix, BSD, Linux and Android. Although Maxima is specialized in symbolic operations, it also offers numerical support [22] such as arbitrary-precision arithmetic.

5.2 Basic Computations

5.2.1 Addition – Subtraction – Multiplication – Division

The four basic operations in Maxima are computed as follows;

Example:

```
    5+4;
        9
```

```
    443-23;
        420
```

```
    45*56;
        2520
```

```
    1250/24;
        50
```

5.2.2 Root – Power

To compute with roots and exponentials,

Example:

```
sqrt(1250);
```

$25\sqrt{2}$

Example:

```
2**3;
```

8

```
2^3;
```

8

5.2.3 Derivative

With Maxima, derivatives of functions can be computed. For example, to differentiate *cos(x)* once with respect to *x, we write*:

Example:

```
diff(cos(x),x);
```

-sin (x)

5.2.4 Integration

Integrals can also be computed using Maxima. To compute an indefinite integral,

Example:

$$\int x^3 dx$$

```
integrate(x^3,x);
```

(x^4)/4

To compute a definite integral,

Example:

$$\int_0^1 x^2 dx$$

```
defint(x^2,x,0,1);
```

```
        1/3
```

5.3 Calculations

The tensor calculations in Maxima are done with the help of ctensor, and it is a component tensor manipulation package. It must be loaded in Maxima each time before doing any tensor calculations. The necessary steps are down below:

```
load(ctensor);
```

5.3.1 Writing the Metric

5.3.1.1 With csetup

```
csetup();
```

After this command, Maxima will ask a series of questions to define the metric accordingly. The first question is about the dimension of the coordinate system.

```
Enter the dimension of the coordinate system:
```

For Kerr metric, four dimensions are needed.

```
4;
```

Next question is about the names of the coordinates. Since changing them is unnecessary in this case, the question must be answered with

```
n;
```

Some options on building the metric will then appear.

```
Do you want to
1. Enter a new metric?
2. Enter a metric from a file?
3. Approximate a metric with a Taylor series?
```

To enter the metric from scratch, continue with

```
    1;
```

Now it has come to the form of the metric.

```
    Is the matrix
        1. Diagonal
        2. Symmetric
        3. Antisymmetric
        4. General
```

There are two possible paths here. One can choose the general form, and type in all the components. Or, since Kerr metric is symmetric; one can choose

```
        2;
```

and type in only the components Maxima asked. Which makes the process remarkably easy in this case.

```
    Row 1 Column 1:  -(a^2*cos(theta)^2-2*m*r+r^2)/
(a^2*cos(theta)^2);
    Row 1 Column 2: 0;
    Row 1 Column 3: 0;
    Row 1 Column 4: -(2*a*m*r*sin(theta)^2)/
(a^2*cos(theta)^2*r^2);
    Row 2 Column 2: (a^2*cos(theta)^2*r^2)/(a^2-
2*m*2+r^2);
    Row 2 Column 3: 0;
    Row 2 Column 4: 0;
    Row 3 Column 3: a^2*cos(theta)^2+r^2;
    Row 3 Column 4: 0;
    Row 4 Column 4: (2*a^2*m*r*sin(theta)^4+(a^2*r^2+r^4+
(a^4+a^2*r^2)*cos(theta)^2)*sin(theta)^2)/
(a^2*cos(theta)^2+r^2);

Matrix entered.
```

To display the metric in matrix form, proceed as:

```
    Enter functional dependencies with DEPENDS or 'N' if 
none
    N;
    Do you wish to see the metric?
```

```
    Yes;
```

$$\begin{bmatrix} \frac{-a^2\cos(\theta)^2-r^2+2\,m\,r}{a^2\cos(\theta)^2} & 0 & 0 & -\frac{2\,m\sin(\theta)^2}{a\,r\cos(\theta)^2} \\ 0 & \frac{a^2\,r^2\cos(\theta)^2}{r^2-4\,m+a^2} & 0 & 0 \\ 0 & 0 & a^2\cos(\theta)^2+r^2 & 0 \\ -\frac{2\,m\sin(\theta)^2}{a\,r\cos(\theta)^2} & 0 & 0 & \frac{2\,a^2\,m\,r\sin(\theta)^4+\left(\left(a^2\,r^2+a^4\right)\cos(\theta)^2+r^4+a^2\,r^2\right)\sin(\theta)^2}{a^2\cos(\theta)^2+r^2} \end{bmatrix}$$

5.3.1.2 With lg:ident

```
    lg:ident(4);
```

Defines a 4x4 identity matrix. To define our metric, we will modify the corresponding elements of this matrix one by one.

```
    lg[1,1]:-(a^2*cos(theta)^2-2*m*r+r^2)/
(a^2*cos(theta)^2);
    lg[1,4]:-(2*a*m*r*sin(theta)^2)/
(a^2*cos(theta)^2*r^2);
    lg[2,2]:(a^2*cos(theta)^2*r^2)/(a^2-2*m*2+r^2);
    lg[3,3]:a^2*cos(theta)^2+r^2;
    lg[4,1]:-(2*a*m*r*sin(theta)^2)/
(a^2*cos(theta)^2*r^2);
    lg[4,4]:(2*a^2*m*r*sin(theta)^4+(a^2*r^2+r^4+
(a^4+a^2*r^2)*cos(theta)^2)*sin(theta)^2)/
(a^2*cos(theta)^2+r^2);
```

Now the new matrix, which is Kerr metric in matrix form, can be displayed by:

```
    display(lg);
```

$$lg = \begin{bmatrix} \frac{-a^2\cos(\theta)^2 - r^2 + 2\,m\,r}{a^2\cos(\theta)^2} & 0 & 0 & -\frac{2\,m\sin(\theta)^2}{a\,r\cos(\theta)^2} \\ 0 & \frac{a^2\,r^2\cos(\theta)^2}{r^2 - 4\,m + a^2} & 0 & 0 \\ 0 & 0 & a^2\cos(\theta)^2 + r^2 & 0 \\ -\frac{2\,m\sin(\theta)^2}{a\,r\cos(\theta)^2} & 0 & 0 & \frac{2\,a^2\,m\,r\sin(\theta)^4 + \left(\left(a^2\,r^2 + a^4\right)\cos(\theta)^2 + r^4 + a^2\,r^2\right)\sin(\theta)^2}{a^2\cos(\theta)^2 + r^2} \end{bmatrix}$$

5.3.2 Calculating The Christoffel Symbols

To calculate and display the Christoffel symbols, type

```
    christoff(mcs);
```

Here, It is important to emphasize that ctensor results are not simplified completely like our other example systems. Hence, it is not practical to include all the output here as they are inconveniently long. The rest of the calculations will be displayed as little portions of the complete results, just to demonstrate that the calculations can indeed be done on Maxima.

$$(\%t19)\quad mcs_{1,4,2} = -\frac{\left(m\,r^2 - 4\,m^2 + a^2\,m\right)\sin(\theta)^2}{a^3\,r^4\cos(\theta)^4}$$

$$(\%t20)\quad mcs_{1,4,3} = \frac{2\,m\sin(\theta)^3 + 2\,m\cos(\theta)^2\sin(\theta)}{a^3\,r\cos(\theta)^5 + a\,r^3\cos(\theta)^3}$$

$$(\%t21)\quad mcs_{2,2,2} = -\frac{4\,m - a^2}{r^3 + \left(a^2 - 4\,m\right)r}$$

$$(\%t22)\quad mcs_{2,2,3} = \frac{a^2\,r^2\cos(\theta)\sin(\theta)}{\left(a^2\,r^2 - 4\,a^2\,m + a^4\right)\cos(\theta)^2 + r^4 + \left(a^2 - 4\,m\right)r^2}$$

$$(\%t23)\quad mcs_{2,3,2} = -\frac{\sin(\theta)}{\cos(\theta)}$$

$$(\%t24)\quad mcs_{2,3,3} = \frac{r}{a^2\cos(\theta)^2 + r^2}$$

5.3.3 Calculating The Ricci Tensor

To calculate and display the Ricci tensor of the Kerr metric, type

```
    ricci(dis);
```

$$\begin{gathered}
(\%t79)\ ric_{1,1} = (2\,((2\,a^4\,m\,r\cos(\theta)^3+2\,a^2\,m\,r^3\cos(\theta))\sin(\theta)^2+2\,a^4\,m\,r\cos(\theta)^5+(4\,a^2\,m\,r^3-4\,a^2\,m^2\,r^2)\cos(\theta)^3+(2\,m\,r^5-4\,m^2\,r^4)\cos(\theta))\,(2\,m\sin(\theta)^3+2\,m\cos(\theta)^2\sin(\theta)))/(\\
(a^3\,r\cos(\theta)^5+a\,r^3\cos(\theta)^3)\,((2\,a^5\,m\,r^3\cos(\theta)^4+(2\,a^3\,m\,r^5-4\,a^3\,m^2\,r^4+4\,a^3\,m^2)\cos(\theta)^2+4\,a\,m^2\,r^2)\sin(\theta)^3+ \\
((a^5\,r^4+a^7\,r^2)\cos(\theta)^6+(2\,a^3\,r^6-2\,a^3\,m\,r^5+2\,a^5\,r^4-2\,a^5\,m\,r^3)\cos(\theta)^4+(a\,r^8-2\,a\,m\,r^7+a^3\,r^6-2\,a^3\,m\,r^5)\cos(\theta)^2)\sin(\theta)))+((r^3-m\,r^2+(a^2-4\,m)\,r+4\,m^2-a^2\,m)\,(\\
(a^6\,m\,r^3\cos(\theta)^6+(-2\,a^4\,m^2\,r^4-2\,a^4\,m^2)\cos(\theta)^4+(-a^2\,m\,r^7+2\,a^2\,m^2\,r^6-4\,a^2\,m^2\,r^2)\cos(\theta)^2-2\,m^2\,r^4)\sin(\theta)^2+a^6\,r^4\cos(\theta)^8+(3\,a^4\,r^6-2\,a^4\,m\,r^5)\cos(\theta)^6+(3\,a^2\,r^8-4\,a^2\,m\,r^7) \\
\cos(\theta)^4+(r^{10}-2\,m\,r^9)\cos(\theta)^2))/(a^4\,r^2\cos(\theta)^4\,((2\,a^6\,m\,r^4\cos(\theta)^6+(4\,a^4\,m\,r^6-4\,a^4\,m^2\,r^5+4\,a^4\,m^2\,r)\cos(\theta)^4+(2\,a^2\,m\,r^8-4\,a^2\,m^2\,r^7+8\,a^2\,m^2\,r^3)\cos(\theta)^2+4\,m^2\,r^5)\sin(\theta)^2+ \\
(a^6\,r^5+a^8\,r^3)\cos(\theta)^8+(3\,a^4\,r^7-2\,a^4\,m\,r^6+3\,a^6\,r^5-2\,a^6\,m\,r^4)\cos(\theta)^6+(3\,a^2\,r^9-4\,a^2\,m\,r^8+3\,a^4\,r^7-4\,a^4\,m\,r^6)\cos(\theta)^4+(r^{11}-2\,m\,r^{10}+a^2\,r^9-2\,a^2\,m\,r^8)\cos(\theta)^2))-(\\
(r^3-m\,r^2+(a^2-4\,m)\,r+4\,m^2-a^2\,m)\,((((2\,a^2\,m\,r^5-2\,a^2\,m^2\,r^4-2\,a^2\,m^2)\cos(\theta)^2-2\,m^2\,r^2)\sin(\theta)^2+(a^2\,r^6-a^2\,m\,r^5+a^4\,r^4-a^4\,m\,r^3)\cos(\theta)^4+(r^8-m\,r^7+a^2\,r^6-a^2\,m\,r^5)\cos(\theta)^2))/(a^4 \\
r^2\cos(\theta)^4\,((2\,a^4\,m\,r^4\cos(\theta)^4+(2\,a^2\,m\,r^6-4\,a^2\,m^2\,r^5+4\,a^2\,m^2\,r)\cos(\theta)^2+4\,m^2\,r^3)\sin(\theta)^2+(a^4\,r^5+a^6\,r^3)\cos(\theta)^6+(2\,a^2\,r^7-2\,a^2\,m\,r^6+2\,a^4\,r^5-2\,a^4\,m\,r^4)\cos(\theta)^4+ \\
(r^9-2\,m\,r^8+a^2\,r^7-2\,a^2\,m\,r^6)\cos(\theta)^2))+(2\,(m\,r^2-4\,m^2+a^2\,m)\,(a^4\,m\cos(\theta)^4+(4\,a^2\,m\,r^2-4\,a^2\,m^2\,r)\cos(\theta)^2+3\,m\,r^4-4\,m^2\,r^3)\sin(\theta)^2)/(a^3\,r^4\cos(\theta)^4\,(\\
(2\,a^5\,m\,r^3\cos(\theta)^4+(2\,a^3\,m\,r^5-4\,a^3\,m^2\,r^4+4\,a^3\,m^2)\cos(\theta)^2+4\,a\,m^2\,r^2)\sin(\theta)^2+(a^5\,r^4+a^7\,r^2)\cos(\theta)^6+(2\,a^3\,r^6-2\,a^3\,m\,r^5+2\,a^5\,r^4-2\,a^5\,m\,r^3)\cos(\theta)^4+ \\
(a\,r^8-2\,a\,m\,r^7+a^3\,r^6-2\,a^3\,m\,r^5)\cos(\theta)^2))+((r^2-2\,m\,r)\sin(\theta)\,((2\,a^6\,m\,r^3\cos(\theta)^6+(2\,a^4\,m\,r^5-4\,a^4\,m^2\,r^4+4\,a^4\,m^2)\cos(\theta)^4+8\,a^2\,m^2\,r^2\cos(\theta)^2+4\,m^2\,r^4)\sin(\theta)^4+ \\
(4\,a^6\,m\,r^3\cos(\theta)^8+(8\,a^4\,m\,r^5-8\,a^4\,m^2\,r^4+4\,a^4\,m^2)\cos(\theta)^6+(4\,a^2\,m\,r^7-8\,a^2\,m^2\,r^6+8\,a^2\,m^2\,r^2)\cos(\theta)^4+4\,m^2\,r^4\cos(\theta)^2)\sin(\theta)^2+(a^6\,r^4+a^8\,r^2)\cos(\theta)^{10}+ \\
(3\,a^4\,r^6-2\,a^4\,m\,r^5+3\,a^6\,r^4-2\,a^6\,m\,r^3)\cos(\theta)^8+(3\,a^2\,r^8-4\,a^2\,m\,r^7+3\,a^4\,r^6-4\,a^4\,m\,r^5)\cos(\theta)^6+(r^{10}-2\,m\,r^9+a^2\,r^8-2\,a^2\,m\,r^7)\cos(\theta)^4))/((a^4\cos(\theta)^5+a^2\,r^2\cos(\theta)^3)\,(\\
(2\,a^6\,m\,r^3\cos(\theta)^7+(4\,a^4\,m\,r^5-4\,a^4\,m^2\,r^4+4\,a^4\,m^2)\cos(\theta)^5+(2\,a^2\,m\,r^7-4\,a^2\,m^2\,r^6+8\,a^2\,m^2\,r^2)\cos(\theta)^3+4\,m^2\,r^4\cos(\theta))\sin(\theta)^3+ \\
((a^6\,r^4+a^8\,r^2)\cos(\theta)^9+(3\,a^4\,r^6-2\,a^4\,m\,r^5+3\,a^6\,r^4-2\,a^6\,m\,r^3)\cos(\theta)^7+(3\,a^2\,r^8-4\,a^2\,m\,r^7+3\,a^4\,r^6-4\,a^4\,m\,r^5)\cos(\theta)^5+(r^{10}-2\,m\,r^9+a^2\,r^8-2\,a^2\,m\,r^7)\cos(\theta)^3)\sin(\theta)))-(\\
(r^2-2\,m\,r)\sin(\theta) \\
(((2\,a^2\,m\,r^5-4\,a^2\,m^2\,r^4+4\,a^2\,m^2)\cos(\theta)^2+4\,m^2\,r^2)\sin(\theta)^3+((a^2\,r^6-2\,a^2\,m\,r^5+a^4\,r^4-2\,a^4\,m\,r^3+4\,a^2\,m^2)\cos(\theta)^4+(r^8-2\,m\,r^7+a^2\,r^6-2\,a^2\,m\,r^5+4\,m^2\,r^2)\cos(\theta)^2)\sin(\theta)))/(\\
(a^4\cos(\theta)^5+a^2\,r^2\cos(\theta)^3)\,((2\,a^4\,m\,r^3\cos(\theta)^5+(2\,a^2\,m\,r^5-4\,a^2\,m^2\,r^4+4\,a^2\,m^2)\cos(\theta)^3+4\,m^2\,r^2\cos(\theta))\sin(\theta)^2+(a^4\,r^4+a^6\,r^2)\cos(\theta)^7+(2\,a^2\,r^6-2\,a^2\,m\,r^5+2\,a^4\,r^4-2\,a^4\,m\,r^3) \\
\cos(\theta)^5+(r^8-2\,m\,r^7+a^2\,r^6-2\,a^2\,m\,r^5)\cos(\theta)^3))-\frac{a^2\,(r^2-2\,m\,r)\cos(\theta)\sin(\theta)^2}{(a^2\cos(\theta)^2+r^2)\,(a^4\cos(\theta)^5+a^2\,r^2\cos(\theta)^3)}-\frac{(r^2-2\,m\,r)\sin(\theta)^2}{\cos(\theta)\,(a^4\cos(\theta)^5+a^2\,r^2\cos(\theta)^3)}- \\
\frac{(r^2-2\,m\,r)\sin(\theta)\,(-5\,a^4\cos(\theta)^4\sin(\theta)-3\,a^2\,r^2\cos(\theta)^2\sin(\theta))}{(a^4\cos(\theta)^5+a^2\,r^2\cos(\theta)^3)^2}+\frac{(r^2-2\,m\,r)\cos(\theta)}{a^4\cos(\theta)^5+a^2\,r^2\cos(\theta)^3}+\frac{r^3-m\,r^2+(a^2-4\,m)\,r+4\,m^2-a^2\,m}{a^4\,r\cos(\theta)^4\,(a^2\cos(\theta)^2+r^2)}-\frac{(4\,m-a^2)\,(r^3-m\,r^2+(a^2-4\,m)\,r+4\,m^2-a^2\,m)}{a^4\,r^2\,(r^3+(a^2-4\,m)\,r)\cos(\theta)^4}- \\
\frac{2\,(r^3-m\,r^2+(a^2-4\,m)\,r+4\,m^2-a^2\,m)}{a^4\,r^3\cos(\theta)^4}+\frac{3\,r^2-2\,m\,r-4\,m+a^2}{a^4\,r^2\cos(\theta)^4}
\end{gathered}$$

5.3.4 Calculating The Ricci Scalar

To calculate and display the Ricci scalar of the Kerr metric, type

```
scurvature();
```

```
<< Expression too long to display! >>
<< Expression too long to display! >>
<< Expression too long to display! >>
<< Expression too long to display! >>
(%o84) done
```

Maxima couldn't display the result.

5.3.5 Calculating The Riemann Tensor

To calculate and display the Riemann tensor of the Kerr metric, type

```
riemann(dis);
```

(%t75) $riem_{4,4,2,2} = ((4\,a^{10}\,m^2\,r^8\cos(\theta)^6+(4\,a^8\,m^2\,r^{10}-8\,a^8\,m^3\,r^9+8\,a^8\,m^3\,r^5)\cos(\theta)^4+8\,a^6\,m^3\,r^7\cos(\theta)^2)\sin(\theta)^8+($

$$(2\,a^{10}\,m\,r^9+8\,a^{10}\,m^2\,r^8+2\,a^{12}\,m\,r^7+a^{10}\,m^2\,r^6+(3\,a^{12}\,m^2-12\,a^{10}\,m^3)\,r^4)\cos(\theta)^8+(4\,a^8\,m\,r^{11}+12\,a^8\,m^2\,r^{10}+(4\,a^{10}\,m-16\,a^8\,m^3)\,r^9+(11\,a^8-4\,a^{10})\,m^2\,r^8-2\,a^8\,m^3$$

$$r^7+(13\,a^{10}\,m^2-52\,a^8\,m^3)\,r^6+(24\,a^8\,m^4+(16\,a^8-6\,a^{10})\,m^3)\,r^5-6\,a^8\,m^3\,r^3+(8\,a^8\,m^4-2\,a^{10}\,m^3)\,r)\cos(\theta)^6+(2\,a^6\,m\,r^{13}+4\,a^6\,m^2\,r^{12}+(2\,a^8\,m-16\,a^6\,m^3)\,r^{11}+$$

$$(7\,a^6-4\,a^8)\,m^2\,r^{10}-20\,a^6\,m^3\,r^9+(5\,a^8\,m^2-20\,a^6\,m^3)\,r^8+(80\,a^6\,m^4+(32\,a^6-20\,a^8)\,m^3)\,r^7+14\,a^6\,m^3\,r^5+(18\,a^8\,m^3-72\,a^6\,m^4)\,r^3)\cos(\theta)^4+$$

$$(-3\,a^4\,m^2\,r^{12}+6\,a^4\,m^3\,r^{11}+(20\,a^4\,m^3-5\,a^6\,m^2)\,r^{10}+((10\,a^6+16\,a^4)\,m^3-40\,a^4\,m^4)\,r^9+14\,a^4\,m^3\,r^7+(10\,a^6\,m^3-40\,a^4\,m^4)\,r^5)\cos(\theta)^2-6\,a^2\,m^3\,r^9+$$

$$(40\,a^2\,m^4-10\,a^4\,m^3)\,r^7)\sin(\theta)^6+((6\,a^{10}\,m\,r^9+6\,a^{12}\,m\,r^7+(3\,a^{12}\,m-12\,a^{10}\,m^2)\,r^5+(a^{14}\,m-4\,a^{12}\,m^2)\,r^3)\cos(\theta)^{10}+(18\,a^8\,m\,r^{11}-12\,a^8\,m^2\,r^{10}+(18\,a^{10}+2\,a^8)\,m$$

$$r^9-12\,a^{10}\,m^2\,r^8+(18\,a^{10}\,m-48\,a^8\,m^2)\,r^7+(24\,a^8\,m^3+(4\,a^8-6\,a^{10})\,m^2)\,r^6+(8\,a^{12}\,m-32\,a^{10}\,m^2)\,r^5+(8\,a^{10}\,m^3+(-2\,a^{12}+4\,a^{10}-9\,a^8)\,m^2)\,r^4+(20\,a^8\,m^3-6\,a^{10}\,m^2)$$

$$r^2+4\,a^{10}\,m^3-a^{12}\,m^2)\cos(\theta)^8+(18\,a^6\,m\,r^{13}-24\,a^6\,m^2\,r^{12}+(18\,a^8-2\,a^6)\,m\,r^{11}+(-24\,a^8-4\,a^6)\,m^2\,r^{10}+(20\,a^8\,m-40\,a^6\,m^2)\,r^9+(72\,a^6\,m^3+(12\,a^6-30\,a^8)\,m^2)\,r^8+$$

$$(10\,a^{10}\,m-40\,a^8\,m^2)\,r^7+(56\,a^8\,m^3+(-14\,a^{10}+12\,a^8-36\,a^6)\,m^2)\,r^6+(80\,a^6\,m^3-24\,a^8\,m^2)\,r^4+(16\,a^8\,m^3-4\,a^{10}\,m^2)\,r^2)\cos(\theta)^6+(6\,a^4\,m\,r^{15}-12\,a^4\,m^2\,r^{14}+$$

$$(6\,a^6-10\,a^4)\,m\,r^{13}+(8\,a^4-12\,a^6)\,m^2\,r^{12}+(16\,a^4\,m^2-2\,a^6\,m)\,r^{11}+(8\,a^4\,m^3+(12\,a^4-10\,a^6)\,m^2)\,r^{10}+(24\,a^6\,m^3+(-6\,a^8+12\,a^6-54\,a^4)\,m^2)\,r^8+$$

$$(120\,a^4\,m^3-36\,a^6\,m^2)\,r^6+(24\,a^6\,m^3-6\,a^8\,m^2)\,r^4)\cos(\theta)^4+(-6\,a^2\,m\,r^{15}+12\,a^2\,m^2\,r^{14}+(20\,a^2\,m^2-7\,a^4\,m)\,r^{13}+((14\,a^4+4\,a^2)\,m^2-40\,a^2\,m^3)\,r^{12}+$$

$$(12\,a^4\,m^2-3\,a^6\,m)\,r^{11}+((6\,a^6+4\,a^4-36\,a^2)\,m^2-24\,a^4\,m^3)\,r^{10}+(80\,a^2\,m^3-24\,a^4\,m^2)\,r^8+(16\,a^4\,m^3-4\,a^6\,m^2)\,r^6)\cos(\theta)^2-9\,m^2\,r^{12}+(20\,m^3-6\,a^2\,m^2)\,r^{10}+$$

$$(4\,a^2\,m^3-a^4\,m^2)\,r^8)\sin(\theta)^4+((a^{10}\,r^{10}+2\,a^{12}\,r^8+(a^{14}-4\,a^{10}\,m)\,r^6)\cos(\theta)^{12}+$$

$$(4\,a^8\,r^{12}-2\,a^8\,m\,r^{11}+8\,a^{10}\,r^{10}-4\,a^{10}\,m\,r^9+(4\,a^{12}-20\,a^8\,m)\,r^8+(8\,a^8\,m^2-2\,a^{12}\,m)\,r^7)\cos(\theta)^{10}+$$

$$(6\,a^6\,r^{14}-6\,a^6\,m\,r^{13}+12\,a^8\,r^{12}-12\,a^8\,m\,r^{11}+(6\,a^{10}-40\,a^6\,m)\,r^{10}+(32\,a^6\,m^2-6\,a^{10}\,m)\,r^9)\cos(\theta)^8+$$

$$(4\,a^4\,r^{16}-6\,a^4\,m\,r^{15}+8\,a^6\,r^{14}-12\,a^6\,m\,r^{13}+(4\,a^8-40\,a^4\,m)\,r^{12}+(48\,a^4\,m^2-6\,a^8\,m)\,r^{11})\cos(\theta)^6+$$

$$(a^2\,r^{18}-2\,a^2\,m\,r^{17}+2\,a^4\,r^{16}-4\,a^4\,m\,r^{15}+(a^6-20\,a^2\,m)\,r^{14}+(32\,a^2\,m^2-2\,a^6\,m)\,r^{13})\cos(\theta)^4+(8\,m^2\,r^{15}-4\,m\,r^{16})\cos(\theta)^2)\sin(\theta)^2)/((2\,a^{12}\,m\,r^7\cos(\theta)^{12}+$$

$$(8\,a^{10}\,m\,r^9-4\,a^{10}\,m^2\,r^8+4\,a^{10}\,m^2\,r^4)\cos(\theta)^{10}+(12\,a^8\,m\,r^{11}-12\,a^8\,m^2\,r^{10}+16\,a^8\,m^2\,r^6)\cos(\theta)^8+(8\,a^6\,m\,r^{13}-12\,a^6\,m^2\,r^{12}+24\,a^6\,m^2\,r^8)\cos(\theta)^6+$$

$$(2\,a^4\,m\,r^{15}-4\,a^4\,m^2\,r^{14}+16\,a^4\,m^2\,r^{10})\cos(\theta)^4+4\,a^2\,m^2\,r^{12}\cos(\theta)^2)\sin(\theta)^2+(a^{12}\,r^8+a^{14}\,r^6)\cos(\theta)^{14}+(5\,a^{10}\,r^{10}-2\,a^{10}\,m\,r^9+5\,a^{12}\,r^8-2\,a^{12}\,m\,r^7)\cos(\theta)^{12}+$$

$$(10\,a^8\,r^{12}-8\,a^8\,m\,r^{11}+10\,a^{10}\,r^{10}-8\,a^{10}\,m\,r^9)\cos(\theta)^{10}+(10\,a^6\,r^{14}-12\,a^6\,m\,r^{13}+10\,a^8\,r^{12}-12\,a^8\,m\,r^{11})\cos(\theta)^8+(5\,a^4\,r^{16}-8\,a^4\,m\,r^{15}+5\,a^6\,r^{14}-8\,a^6\,m\,r^{13})$$

$$\cos(\theta)^6+(a^2\,r^{18}-2\,a^2\,m\,r^{17}+a^4\,r^{16}-2\,a^4\,m\,r^{15})\cos(\theta)^4)$$

5.3.6 Calculating The Einstein Tensor

After calculating the Christoffel symbols and Ricci tensor, now Maxima can compute the Einstein tensor.

```
    einstein(dis);
```

(%t86) $ein_{1,4} = ((8\,a^8\,m^2\,r^7\cos(\theta)^6+(4\,a^6\,m^2\,r^9-8\,a^6\,m^3\,r^8+8\,a^6\,m^3\,r^4)\cos(\theta)^4)\sin(\theta)^4+((6\,a^8\,m\,r^8+4\,a^8\,m^2\,r^7+6\,a^{10}\,m\,r^6+5\,a^8\,m^2\,r^5+(7\,a^{10}\,m^2-28\,a^8\,m^3)\,r^3)\cos(\theta)^8+$

$$(8\,a^6\,m\,r^{10}+(8\,a^8\,m-16\,a^6\,m^3)\,r^8+(11\,a^6-8\,a^8)\,m^2\,r^7-24\,a^6\,m^3\,r^6+(21\,a^8\,m^2-84\,a^6\,m^3)\,r^5+(144\,a^6\,m^4-36\,a^8\,m^3)\,r^4+4\,a^6\,m^3\,r^2-32\,a^6\,m^4+8\,a^8\,m^3)\cos(\theta)^6+(2\,a^4\,m\,r^{12}+$$

$$(2\,a^6\,m-16\,a^4\,m^3)\,r^{10}+(16\,a^4\,m^4+(3\,a^4-4\,a^6)\,m^2)\,r^9-32\,a^4\,m^3\,r^8+(32\,a^4\,m^4-68\,a^4\,m^3+17\,a^6\,m^2)\,r^7+(256\,a^4\,m^4-64\,a^6\,m^3)\,r^6+((48\,a^6-16\,a^4)\,m^4-192\,a^4\,m^5)\,r^5-8\,a^4\,m^3\,r^4+$$

$$(12\,a^6\,m^3-48\,a^4\,m^4)\,r^2+(64\,a^4\,m^5-16\,a^6\,m^4)\,r)\cos(\theta)^4+$$

$$(-3\,a^2\,m^2\,r^{11}+8\,a^2\,m^3\,r^{10}+(3\,a^4\,m^2-12\,a^2\,m^3)\,r^9+(48\,a^2\,m^4-12\,a^4\,m^3)\,r^8+(16\,a^4\,m^4-64\,a^2\,m^5)\,r^7-36\,a^2\,m^3\,r^6+16\,a^2\,m^4\,r^5+(32\,a^2\,m^4-8\,a^4\,m^3)\,r^4+(64\,a^2\,m^5-16\,a^4\,m^4)\,r^3)\cos(\theta)^2$$

$$-24\,m^3\,r^8+16\,m^4\,r^7+(48\,m^4-12\,a^2\,m^3)\,r^6)\sin(\theta)^2+(6\,a^8\,m\,r^8-4\,a^8\,m^2\,r^7+(6\,a^{10}+3\,a^8)\,m\,r^6+(6\,a^{10}\,m-16\,a^8\,m^2)\,r^4+(3\,a^{12}\,m-12\,a^{10}\,m^2)\,r^2)\cos(\theta)^{10}+(14\,a^6\,m\,r^{10}-28\,a^6\,m^2\,r^9+$$

$$(16\,a^6\,m^3+(14\,a^8+12\,a^6)\,m)\,r^8+(-20\,a^8-15\,a^6)\,m^2\,r^7+(24\,a^8\,m-72\,a^6\,m^2)\,r^6+(84\,a^6\,m^3-30\,a^8\,m^2)\,r^5+(-8\,a^6\,m^3-48\,a^8\,m^2+12\,a^{10}\,m)\,r^4+(60\,a^8\,m^3-15\,a^{10}\,m^2)\,r^3)\cos(\theta)^8+(10\,a^4$$

$$m\,r^{12}-32\,a^4\,m^2\,r^{11}+(32\,a^4\,m^3+(10\,a^6+18\,a^4)\,m)\,r^{10}+((-28\,a^6-45\,a^4)\,m^2-16\,a^4\,m^4)\,r^9+((16\,a^6+20\,a^4)\,m^3-120\,a^4\,m^2+36\,a^6\,m)\,r^8+(268\,a^4\,m^3-90\,a^6\,m^2)\,r^7+$$

$$(-112\,a^4\,m^4+(40\,a^6-16\,a^4)\,m^3-72\,a^6\,m^2+18\,a^8\,m)\,r^6+(16\,a^4\,m^4+180\,a^6\,m^3-45\,a^8\,m^2)\,r^5+(20\,a^8\,m^3-80\,a^6\,m^4)\,r^4)\cos(\theta)^6+(2\,a^2\,m\,r^{14}-8\,a^2\,m^2\,r^{13}+(8\,a^2\,m^3+(2\,a^4+12\,a^2)\,m)\,r^{12}$$

$$+(-8\,a^4-41\,a^2)\,m^2\,r^{11}+((8\,a^4+32\,a^2)\,m^3-88\,a^2\,m^2+24\,a^4\,m)\,r^{10}+(268\,a^2\,m^3-82\,a^4\,m^2)\,r^9+(-192\,a^2\,m^4+(64\,a^4-8\,a^2)\,m^3-48\,a^4\,m^2+12\,a^6\,m)\,r^8+(16\,a^2\,m^4+164\,a^4\,m^3-41\,a^6\,m^2)\,r^7$$

$$+(32\,a^6\,m^3-128\,a^4\,m^4)\,r^6)\cos(\theta)^4+$$

$$(3\,m\,r^{14}-11\,m^2\,r^{13}+(12\,m^3-24\,m^2+6\,a^2\,m)\,r^{12}+(84\,m^3-22\,a^2\,m^2)\,r^{11}+(-80\,m^4+24\,a^2\,m^3-12\,a^2\,m^2+3\,a^4\,m)\,r^{10}+(44\,a^2\,m^3-11\,a^4\,m^2)\,r^9+(12\,a^4\,m^3-48\,a^2\,m^4)\,r^8)\cos(\theta)^2)/((4\,a^{11}$$

$$m^2\,r^9\cos(\theta)^{10}+(8\,a^9\,m^2\,r^{11}-16\,a^9\,m^3\,r^{10}+16\,a^9\,m^3\,r^6)\cos(\theta)^8+(4\,a^7\,m^2\,r^{13}-16\,a^7\,m^3\,r^{12}+16\,a^7\,m^4\,r^{11}+32\,a^7\,m^3\,r^8-32\,a^7\,m^4\,r^7+16\,a^7\,m^4\,r^3)\cos(\theta)^6+$$

$$(16\,a^5\,m^3\,r^{10}-32\,a^5\,m^4\,r^9+32\,a^5\,m^4\,r^5)\cos(\theta)^4+16\,a^3\,m^4\,r^7\cos(\theta)^2)\sin(\theta)^4+((4\,a^{11}\,m\,r^{10}+4\,a^{13}\,m\,r^8)\cos(\theta)^{12}+$$

$$(12\,a^9\,m\,r^{12}-16\,a^9\,m^2\,r^{11}+12\,a^{11}\,m\,r^{10}-16\,a^{11}\,m^2\,r^9+8\,a^9\,m^2\,r^7+8\,a^{11}\,m^2\,r^5)\cos(\theta)^{10}+$$

$$(12\,a^7\,m\,r^{14}-32\,a^7\,m^2\,r^{13}+(16\,a^7\,m^3+12\,a^9\,m)\,r^{12}-32\,a^9\,m^2\,r^{11}+16\,a^9\,m^3\,r^{10}+24\,a^7\,m^2\,r^9-16\,a^7\,m^3\,r^8+24\,a^9\,m^2\,r^7-16\,a^9\,m^3\,r^6)\cos(\theta)^8+$$

$$(4\,a^5\,m\,r^{16}-16\,a^5\,m^2\,r^{15}+(16\,a^5\,m^3+4\,a^7\,m)\,r^{14}-16\,a^7\,m^2\,r^{13}+16\,a^7\,m^3\,r^{12}+24\,a^5\,m^2\,r^{11}-32\,a^5\,m^3\,r^{10}+24\,a^7\,m^2\,r^9-32\,a^7\,m^3\,r^8)\cos(\theta)^6+$$

$$(8\,a^3\,m^2\,r^{13}-16\,a^3\,m^3\,r^{12}+8\,a^5\,m^2\,r^{11}-16\,a^5\,m^3\,r^{10})\cos(\theta)^4)\sin(\theta)^2+(a^{11}\,r^{11}+2\,a^{13}\,r^9+a^{15}\,r^7)\cos(\theta)^{14}+(4\,a^9\,r^{13}-4\,a^9\,m\,r^{12}+8\,a^{11}\,r^{11}-8\,a^{11}\,m\,r^{10}+4\,a^{13}\,r^9-4\,a^{13}\,m\,r^8)$$

$$\cos(\theta)^{12}+(6\,a^7\,r^{15}-12\,a^7\,m\,r^{14}+(4\,a^7\,m^2+12\,a^9)\,r^{13}-24\,a^9\,m\,r^{12}+(8\,a^9\,m^2+6\,a^{11})\,r^{11}-12\,a^{11}\,m\,r^{10}+4\,a^{11}\,m^2\,r^9)\cos(\theta)^{10}+$$

$$(4\,a^5\,r^{17}-12\,a^5\,m\,r^{16}+(8\,a^5\,m^2+8\,a^7)\,r^{15}-24\,a^7\,m\,r^{14}+(16\,a^7\,m^2+4\,a^9)\,r^{13}-12\,a^9\,m\,r^{12}+8\,a^9\,m^2\,r^{11})\cos(\theta)^8+$$

$$(a^3\,r^{19}-4\,a^3\,m\,r^{18}+(4\,a^3\,m^2+2\,a^5)\,r^{17}-8\,a^5\,m\,r^{16}+(8\,a^5\,m^2+a^7)\,r^{15}-4\,a^7\,m\,r^{14}+4\,a^7\,m^2\,r^{13})\cos(\theta)^6)$$

<< Expression too long to display! >>

As can be seen in the last line of the cropped computation, Maxima has trouble displaying longer expressions, at least in tensor calculations. This feature can be changed using Maxima's properties but these long expressions are not helpful in physical calculations.

6. Python & GraviPy

6.1 General Information

Python is a high level programming language that has been first developed by Guido van Rossum in the 1990s [23], and to this day it continues to evolve by millions of developers across the world. It has been designed to make the codes more readable, and to express the concepts in fewer lines.

6.2 Basic Computations

Basic computations can easily be done with Python, as the other programs and packages covered here. Before starting any calculations, import the sympy library.

```
from sympy import *
X, y, z = symbols('x y z')
init_printing(use_unicode=True)
```

6.2.1 Addition – Subtraction – Multiplication – Division

The four operators are computed as follows, without applying any special notation.

Example:

```
20+45
        65
```

```
233-21
        212
```

```
34*54
        1386
```

```
44/11
        4
```

6.2.2 Root – Power

The root and powered numbers are computed as follows, without applying any special notation.

Example:

```
sqrt(144)
         12
```

```
12**2
         144
```

6.2.3 Derivative

To take derivatives;

Example:

```
diff(cos(x), x)
    -sin(x)
```

6.2.4 Integration

To take definite and indefinite integrals;

Example:

$$\int_0^{\infty} e^{-x} dx$$

```
integrate(exp(-x),(x,0,oo))
    1
```

$$\int \cos(x) dx$$

```
integrate(cox(x),x)
    -sin(x)
```

6.3 Calculations

Before starting the calculations, import the GraviPy library and initialize the pretty-printing mode by;

```
    from gravipy import *
    init_printing()
```

6.3.1 Writing The Metric

Choose coordinates, define the metric as a SymPy matrix.

```
    a, t, r, theta, phi, M = symbols('a, t, r, 
\\theta, \\phi, M')
    x = Coordinates('\chi', [t, r, theta, phi])
    Metric = Matrix([[-(a**2*(cos(theta))**2-
2*m*r+r**2)/(a**2*(cos(theta))**2), 0, 0, -
(2*a*m*r*(sin(theta))**2)/
(a**2*(cos(theta))**2*r**2)],
   [0, (a**2*(cos(theta))**2*r**2)/(a**2-
2*m*2+r**2), 0, 0],
   [0, 0, a**2*(cos(theta))**2+r**2, 0],
   [-(2*a*m*r*(sin(theta))**2)/
(a**2*(cos(theta))**2*r**2), 0, 0,
(2*a**2*m*r*(sin(theta))**4+(a**2*r**2+r**4+
(a**4+a**2*r**2)*(cos(theta))**2)*(sin(theta))**2)
/(a**2*(cos(theta))**2+r**2)]])
```

To display the metric, type

```
    Metric
```

$$\begin{bmatrix} \frac{1}{a^2\cos^2(\theta)}\left(-a^2\cos^2(\theta)+2mr-r^2\right) & 0 & 0 & -\frac{2m\sin^2(\theta)}{ar\cos^2(\theta)} \\ 0 & \frac{a^2r^2\cos^2(\theta)}{a^2-4m+r^2} & 0 & 0 \\ 0 & 0 & a^2\cos^2(\theta)+r^2 & 0 \\ -\frac{2m\sin^2(\theta)}{ar\cos^2(\theta)} & 0 & 0 & \frac{1}{a^2\cos^2(\theta)+r^2}\left(2a^2mr\sin^4(\theta)+\left(a^2r^2+r^4+\left(a^4+a^2r^2\right)\cos^2(\theta)\right)\sin^2(\theta)\right) \end{bmatrix}$$

Now, the metric can be defined.

```
    g = MetricTensor('g', x, Metric)
```

6.3.2 Calculating The Christoffel Symbols

To calculate the Christoffel symbols of the first kind, type

```
    Ga = Christoffel('Ga', g)
```

To display all the components;

```
    Ga(All,All,All)
```

To display the components individually;

Example

```
    Ga(1,2,1)
```

$$\frac{m-r}{a^2\cos^2(\theta)}$$

```
    Ga(2,2,3)
```

$$-\frac{a^2r^2\sin(2\theta)}{2a^2-8m+2r^2}$$

6.3.3 Calculating the Ricci Tensor

To calculate the Ricci tensor, type

```
    Ri = Ricci('Ri', g)
```

To display all the components;

```
    Ri(All,All)
```

To display the components individually;

```
    Ri(1,1)
```

$$
\begin{gathered}
\frac{1}{8a^4 r^4 \left(a^2 \cos^2(\theta) + r^2\right)^2 \left(4m^2 \left(a^2 \cos^2(\theta) + r^2\right) \sin^2(\theta)\right.} (\\
+ r^2 \left(a^2 \cos^2(\theta) - 2mr + r^2\right) \left(2a^2 mr \sin^2(\theta) + a^2 r^2 + a^2 \left(a^2 + r^2\right) \cos^2(\theta) + r^4\right) \cos^2(\theta)) \left(a^2 \cos^2(\theta) - 2mr + r^2\right) \cos^6(\theta) \tan^2(\theta) \\
-16a^4 r^5 (2m - r) (4m^2 \left(a^2 \cos^2(\theta) + r^2\right) \sin^2(\theta) + r^2 \left(a^2 \cos^2(\theta) - 2mr + r^2\right) \left(2a^2 mr \sin^2(\theta) + a^2 r^2 + a^2 \left(a^2 + r^2\right) \cos^2(\theta) + r^4\right) \cos^2(\theta \\
\left(a^2 \cos^2(\theta) - 2mr + r^2\right) \sin^4(\theta) \cos^2(\theta) \\
+ 4a^4 r^5 (2m - r) (4m^2 \left(a^2 \cos^2(\theta) + r^2\right) \sin^2(\theta) + r^2 \left(a^2 \cos^2(\theta) - 2mr + r^2\right) \left(2a^2 mr \sin^2(\theta) + a^2 r^2 + a^2 \left(a^2 + r^2\right) \cos^2(\theta) + r^4\right) \cos^2(\theta) \\
\left(a^2 \cos^2(\theta) - 2mr + r^2\right) \sin^3(\theta) \sin(2\theta) \cos(\theta) + 64a^2 m^2 r^4 \left(a^2 \cos^2(\theta) + r^2\right)^2 \left(a^2 - 2mr + r^2\right) \left(a^2 \cos^2(\theta) - 2mr + r^2\right) \sin^2(\theta) \cos^2(\theta) \\
- 24a^2 r^5 (2m - r) \left(a^2 \cos^2(\theta) + r^2\right) (4m^2 \left(a^2 \cos^2(\theta) + r^2\right) \sin^2(\theta) \\
+ r^2 \left(a^2 \cos^2(\theta) - 2mr + r^2\right) \left(2a^2 mr \sin^2(\theta) + a^2 r^2 + a^2 \left(a^2 + r^2\right) \cos^2(\theta) + r^4\right) \cos^2(\theta)) \left(a^2 \cos^2(\theta) - 2mr + r^2\right) \sin^4(\theta) \\
+ \frac{4a^2 r^5 \sin^3(\theta)}{\cos(\theta)} (2m - r) \left(a^2 \cos^2(\theta) + r^2\right) (4m^2 \left(a^2 \cos^2(\theta) + r^2\right) \sin^2(\theta) \\
+ r^2 \left(a^2 \cos^2(\theta) - 2mr + r^2\right) \left(2a^2 mr \sin^2(\theta) + a^2 r^2 + a^2 \left(a^2 + r^2\right) \cos^2(\theta) + r^4\right) \cos^2(\theta)) \left(a^2 \cos^2(\theta) - 2mr + r^2\right) \sin(2\theta) \\
- 8a^2 r^5 (2m - r) \left(a^2 \cos^2(\theta) + r^2\right) (4m^2 \left(a^2 \cos^2(\theta) + r^2\right) \sin^2(\theta) \\
+ r^2 \left(a^2 \cos^2(\theta) - 2mr + r^2\right) \left(2a^2 mr \sin^2(\theta) + a^2 r^2 + a^2 \left(a^2 + r^2\right) \cos^2(\theta) + r^4\right) \cos^2(\theta)) \left(a^2 \cos^2(\theta) - 2mr + r^2\right) \cos^4(\theta) \tan^2(\theta) \\
+ 8a^2 r^5 (2m - r) \left(a^2 \cos^2(\theta) + r^2\right) \left(4m^2 \left(a^2 \cos^2(\theta) + r^2\right) \left(a^2 \cos^2(\theta) - 2mr + r^2\right) \sin^2(\theta)\right.
\end{gathered}
$$

The result is too long to display here completely.

6.3.4 Calculating the Ricci Scalar

To calculate the Ricci scalar, type

```
    Ri.scalar()
```

6.3.5 Calculating the Riemann Tensor

To calculate the Riemann tensor, type

```
    Rm = Riemann('Rm', g)
```

To display all the components;

```
    Rm(All,All,All,All)
```

To display the components individually;

```
    Ri(1,1,2,1)
```

6.3.6 Calculating the Einstein Tensor

To calculate the Einstein tensor, type

```
    G = Einstein('G', Ri)
```

To display all the components;

```
    G(All,All)
```

To display the components individually;

```
    G(1,2)
```

The most important feature of the GraviPy module is that it is the slowest one among the packages we have studied. It is also not easy to use simplfication commands of SymPy which is the symbolic manipulation module of Python.

7. Conclusions

We picked the Kerr metric and calculated its curvature tensors, Ricci scalar and Christoffel symbols with various programs and packages. Our tools were

- Maple & GR Tensor II
- Maxima & ctensor
- SageManifolds & Sage
- Python & GraviPy

Among these, Maple is a commercial software, while rest of the tools are freely distributed. Sage has a cloud option for those who use an operating system that is not supported.

Even though Maple can be inconvenient in financial aspects, it is a stable program that successfully cooperates with GrTensor II package which is freely available. This combination is very useful for the tensor calculations, for there is a defined altering (simplification) function. When the altering function is used for simplifying the solutions, the results are summed up and displayed neatly.

Maxima was harder to work with, and we could not get healthy results, for ctensor do not have a developed simplification function defined in its library. For this reason, the expressions were often too long for Maxima to display. However, ctensor has a well detailed method to manually enter metrics.

Sage has a clean interface and is very user friendly. It lets us do the calculations quickly, as well as accurate. The cloud option is powerful, even though it had problems launching fast enough at times.

Python & GraviPy seems the less favorable choice among others as it is very slow and the simplification routines are not sufficient. Thus, the results are not usable for further computations.

We can conclude that Maple & GRTensor II is the best option when one considers both calculation speed and simplification of the results. The second best option would be using Sage & SageManifolds as it is the best free, yet powerful choice. The user should choose the most eligible program and package according to the needs of the calculation.

8. References

[1] O'Connor, J.J. and Robertson, E.F. (1996), *General relativity.Mathematical Physics index*, School of Mathematics and Statistics,University of St. Andrews, Scotland. Retrieved 2015-02-04.

[2] Rindler, W. (2006). *Relativity special, general, and cosmological* (2nd ed., p. 11). Oxford: Oxford University Press.

[3] Rindler, W. (2006). *Relativity special, general, and cosmological* (2nd ed., p. 18). Oxford: Oxford University Press.

[4] D'Inverno, R. (1992). *Introducing Einstein's Relativity* (pp. 107-108). Oxford: Clarendon Press.

[5] Schutz, B. (2009). *A First Course in General Relativity* (p. 117). Cambridge: Cambridge University Press.

[6] Schutz, B. (2009). *A First Course in General Relativity* (p. 159). Cambridge: Cambridge University Press.

[7] Schutz, B. (2009). *A First Course in General Relativity* (p. 164). Cambridge: Cambridge University Press.

[8] Schutz, B. (2009). *A First Course in General Relativity* (p. 164). Cambridge: Cambridge University Press.

[9] Schutz, B. (2009). *A First Course in General Relativity* (p. 165). Cambridge: Cambridge University Press.

[10] Black Holes - NASA Science. (n.d.). Retrieved December 26, 2015, from http://science.nasa.gov/astrophysics/focus-areas/black-holes/

[11] Stephani, H.; Kramer, D.; MacCallum, M.; Hoenselaers, C.; & Herlt, E. (2003). *Exact Solutions of Einstein's Field Equations* (2nd edn.). Cambridge: Cambridge University Press.

[12] Frolov, V., & Zelnikov, A. (2011). *Introduction to black hole physics* (p. 168). Oxford: Oxford University Press.

[13] L.D. Landau, E.M. Lifshitz, "The Classical Theory of Fields" (Fourth Revised English Edition), Pergamon Press, New York (1975) pp. 321–330

[14] Misner, C., & Thorne, K. (1973). *Gravitation* (p. 841). San Francisco: W.H. Freeman and Company.

[15] Landau, L. D.; Lifshitz, E. M. (1975). *The Classical Theory of Fields (Course of Theoretical Physics, Vol. 2)* (revised 4th English ed.). New York: Pergamon Press. pp. 321–330.

[16] Misner, C., & Thorne, K. (1973). *Gravitation* (p. 877). San Francisco: W.H. Freeman and Company.

[17] Maple 6.0. (n.d.). Retrieved December 26, 2015, from http://www.macworld.com/article/1001870/21reviewsmaple.html

[18] Background. (n.d.). Retrieved December 28, 2015, from *http://grtensor.phy.queensu.ca/*

[19] Birkandan, T. "A Newman-Penrose Calculator for Instanton Metrics",

International Journal of Modern Physics C 19 (8), 1277 (2008). arXiv: 0711.0613 [gr-qc]
[20] Stein, William (2007-06-12). *"Sage Days 4"* (PDF). Archived from http://www.sagemath.org/why/stein-sd4.pdf on 2007-06-27. Retrieved 2007-08-02.
[21] Anastassiou, George A. and Razvan A. Mezei (2015). *Numerical Analysis Using Sage*. New York: Springer. pp. xi,1. *ISBN* 9783319167381.
[22] Barnes, David J. & Chu, Dominique (2010). "Chapter 5". *Introduction to Modeling for Biosciences*. Springer. *ISBN* *978-1-84996-325-1*.
[23] History and License. (n.d.). Retrieved December 29, 2015, from https://docs.python.org/2/license.html