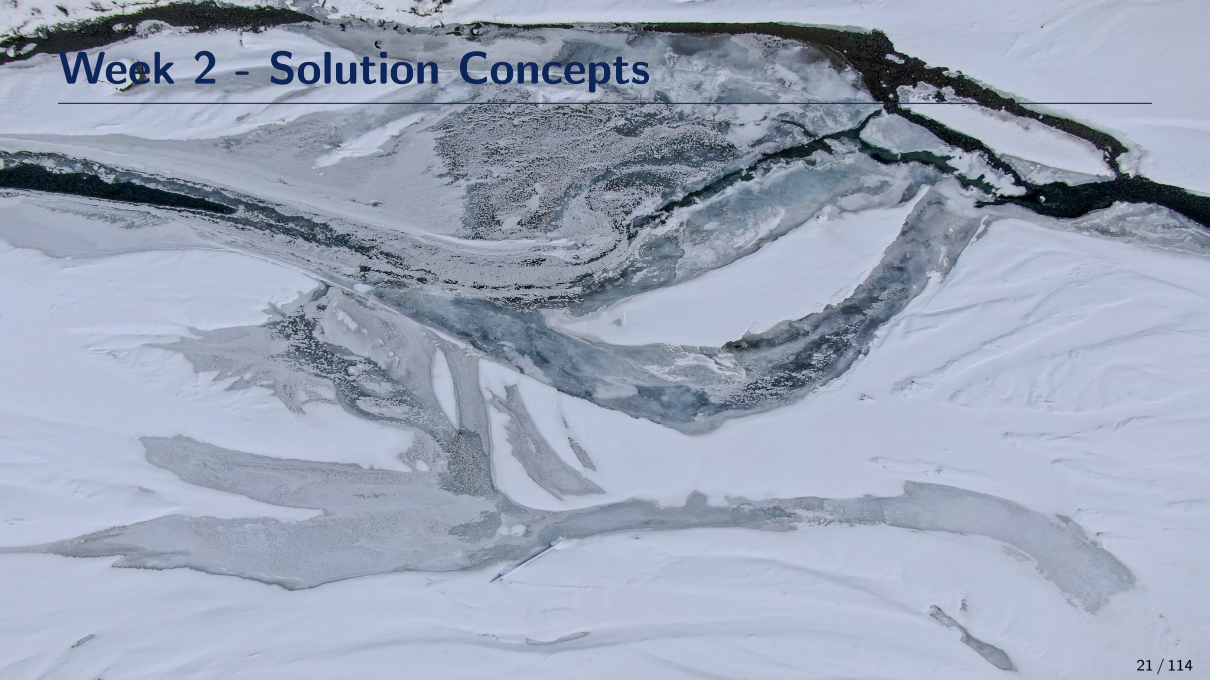
Week 2 - Solution Concepts



Normal Form Games

- Unlike RL, the return/reward is a function of all the agents in the game
- Fixing the the the policies of other players results in a single agent environment
- Minmax vs Nash solution concepts

Fixed Strategies

- We assume the agents follow a fixed strategy
  - Does not allow for opponent adaptation or online learning
- The analysis is for tabular, explicit representations
  - But works for implicit and online algorithms as long as they allow for tabularization [1]
- Training and evaluation paradigm - as used in all of the major games AI milestones

[1] Sustr M, Schmid M, Moravcik M, Burch N, Lanctot M, Bowling M. Sound search in imperfect information games.

Train/Eval Paradigm

(a) Train

(b) Eval

Search

- Online algorithm
- Common idea used in many perfect information games (chess, go, ...)
- Many appealing properties
- We still assume that it is consistent with a fixed, tabular strategy
- Makes only sense in sequential decision making - we will come back to this later

Solution Concepts

(a) Maximin

(b) Nash equilibrium

Maximin

- Maximizing the worst-case scenario
- Assumes everyone else is "out there to get you"

Definition: Maximin Policy

Maximin policy of a player i is:

$$\arg\max_{\pi_i \in \Pi_i} \min_{\pi_{-i} \in \Pi_{-i}} R_i(\pi_i, \pi_{-i}) = \arg\max_{\pi_i \in \Pi_i} BRV_i(\pi_i) \tag{1}$$

Nash equilibrium

- Everyone is happy

Definition: Nash Equilibrium

Strategy profile (π_i, π_{-i}) forms a Nash equilibrium if none of the players benefit by deviating from their policy.

$$\forall i \in N, \forall \pi_i' \, : \, R_i(\pi_i, \pi_{-i}) \geq R_i(\pi_i', \pi_{-i})$$

Maximin vs Nash

- One is defined for strategy, the other for strategy profile
- We will see some interesting differences
- But we will also see that they are sometimes the same!

Maximin

Maximin in Pure Strategies

	Cooperate	**Defect**
Cooperate	(-6, -6)	(0, -10)
Defect	(-10, 0)	(-1, -1)

Table: Prisoner's dillema

	Stop	**Go**
Stop	(0, 0)	(0, 1)
Go	(1, 0)	(-10, -10)

Table: Chicken's game

Maximin in Pure Strategies

	Rock	**Paper**	**Scissors**
Rock	0	1	-1
Paper	-1	0	1
Scissors	1	-1	0

Table: Rock paper scissors

Maximin

- Let's consider pure strategies
- When we mix, we can do better!
- Opponent does not care about their reward at all!
- How can we find the best mixed strategy?

Optimizing Against Best Response

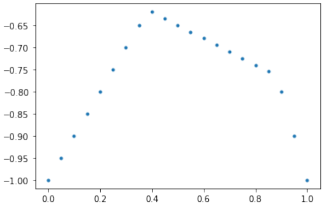
- For two-player zero sum games, we have

$$\arg\max_{\pi_i \in \Pi_i} \min_{\pi_{-i} \in \Pi_{-i}} R_i(\pi_i, \pi_{-i}) = \arg\max_{\pi_i \in \Pi_i} R_i(\pi_i, br_{-i}(\pi_i))$$

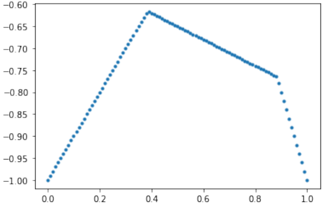
- We are thus optimizing against a best-responding player
- Let's visualise the best-response value function $f(\pi_i) = R_i(\pi_i, br_{-i}(\pi_i))$

Best Response Value Function

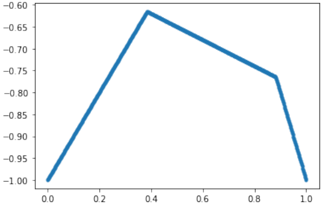
	A	B	C
X	-1	0	-0.8
1-X	1	-1	-0.5



(a) Step size 0.05



(b) Step size 0.01



(c) Step size 0.001

Nash

Nash equilibrium

Nash equlibrium

Strategy profile (π_i, π_{-i}) forms a Nash equilibrium if none of the players benefit by deviating from their policy.

$$\forall i \in N, \forall \pi_i' : R_i(\pi_i, \pi_{-i}) \geq R_i(\pi_i', \pi_{-i})$$

- Easy to verify - all strategies are best response
- Brute-force - enumerate all possible pairs and then verify

Nash in Pure Strategies

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Table: Chicken's game

Nash in Pure Strategies

	Rock	**Paper**	**Scissors**
Rock	0	1	-1
Paper	-1	0	1
Scissors	1	-1	0

Table: Rock paper scissors

Nash equilibrium

- Everyone is happy
- Pure Nash - enumerate, but might not exist!

Mixed Nash Equilibrium

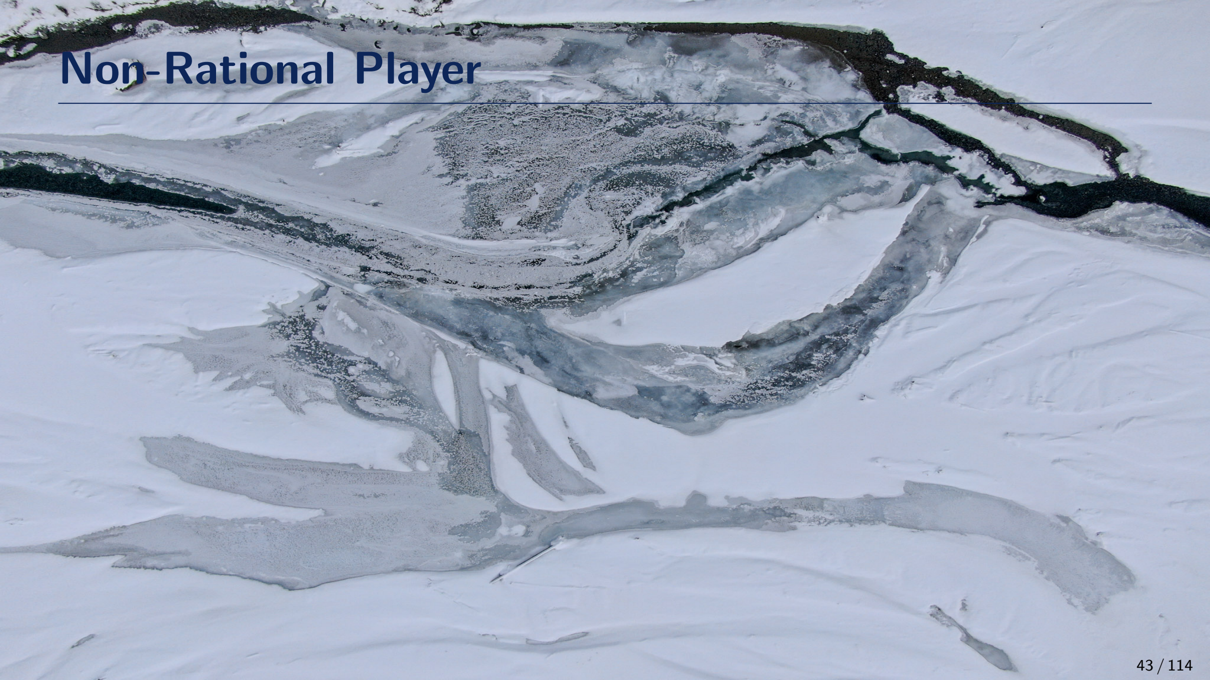
$$\forall i \in N, \forall \pi'_i : R_i(\pi_i, \pi_{-i}) \geq R_i(\pi'_i, \pi_{-i})$$

- Recall that the opponents are best-responding
- We also know that for best-response strategy, all the actions in the support have the same value
- For sparse x, y consider the corresponding elements of xA_1, A_2y^T

Enumerating Support

- For sparse x, y consider the corresponding elements of xA_1, A_2y^T
- All the elements must correspond to the best-response value from the perspective of the **other** player

Non-Rational Player



Non-Rational Players - Deviating from Nash

Let's elaborate on the properties of this solution concept!

- What are the implications for the players?
- What are the situations we would/wouldn't play Nash?

Non-Rational Players

Suppose we play versus a stupid opponent

- Non-rational player does not maximize his utility, he can play arbitrarily
- Given Nash equilibrium $\pi = (\pi_0, \pi_1)$, we decided to play π_0, what do we know?
- Even though π_1 maximizes the utility for the opponent, he can make mistakes and select different (non-equilibristic) strategy π_1'
- Choosing different strategy than π_1' is no better for the opponent
- But it can be much worse for us! It can be the case that $u_0(\pi_0, \pi_1) \gg u_0(\pi_0, \pi_1')$

Moral of the story: opponent mistakes can hurt us!

Deviating from Nash

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Table: Prisoner's dillema

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Table: Chicken's game

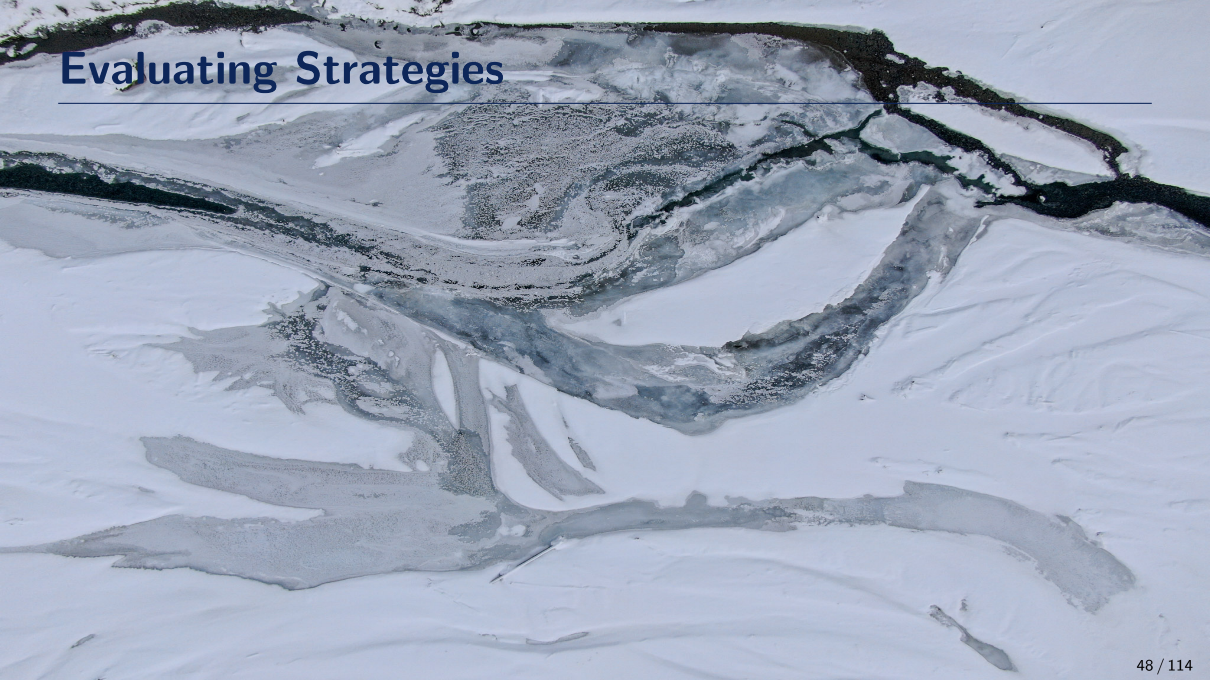
Rational Players, Multiple Equilibria (I)

Suppose there are two optimal strategy profiles in the game
(π_0, π_1^a) and (π_0, π_1^b)

- The opponent is indifferent between his two strategies, he does not care which strategy he chooses (given our strategy π_0), since $u_1(\pi_0, \pi_1^a) = u_1(\pi_0, \pi_1^b)$
- We care!
- $u_i(\pi_0, \pi_1^a) \neq u_0(\pi_0, \pi_1^b)$

Moral of the story: even thought both players play optimally, different optimal strategies can lead to different utilities!

Evaluating Strategies



Loosening The Definitions

- Exact solution might be hard (large games, numerical issues, ...)
- Given a strategy profile, we still need to know how "good" it is, even if it's not exactly optimal
- The standard measures tell us how "close" to an optimal policy we are - terms of performance rather then i.e. KL divergence. [2]

[2] Timbers F, Lockhart E, Lanctot M, Schmid M, Schrittwieser J, Hubert T, Bowling M. Approximate exploitability: Learning a best response in large games. arXiv preprint arXiv:2004.09677. 2020 Apr 20.

ϵ-Nash Equilibrium

ϵ-Nash Equilibrium

A strategy profile π is said to be a ϵ-**Nash equilibrium** if for all players i and each his alternate strategy π_i', we have that:

$$u_i(\pi_i, \pi_{-i}) \geq u_i(\pi_i', \pi_{-i}) - \epsilon$$

Standard Metrics

- Player's incentive to deviate is:

$$\delta_i(\pi) = u_i(br_i(\pi_{-i}), \pi_{-i}) - u_i(\pi)$$

NashConv

$$NASHCONV(\pi) = \sum_i \delta_i(\pi)$$

Exploitability

$$EXPLOITABILITY(\pi) = NASHCONV(\pi)/n$$

ϵ-Nash Equilibrium

Policy π for which:

$$max_i \delta_i(\pi) \leq \epsilon$$

Week 2 Homework

1. Evaluating policy pair
   1.1 Compute Δ_i
   1.2 Compute ϵ
2. Draw best-response value function for a $(2 \times N)$ matrix game
3. For a two-player matrix game (does not have to be a zero-sum), enumerate all possible supports and try to find a Nash Equilibrium for each support