Algorithmic Game Theory

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Overview

About the Course

Class

- Simultaneous and sequential decision making
- Solution concepts and optimal policies
- Practical algorithm for finding the optimal policies

Homeworks

- You will get to implement the games and algorithms!

Understand These!

(a) AlphaZero

(b) AlphaStar

(c) DeepStack

Game Theory - Reinforcement Learning

Reinforcement Learning

- Single agent settings
- Maximize reward
- Scalable practical algorithms

Game Theory

- Multi agent settings
- Analyzes agent interaction, incentives
- Optimal solution concepts
- Algorithms (historically) tabular and not scalable

Terminology

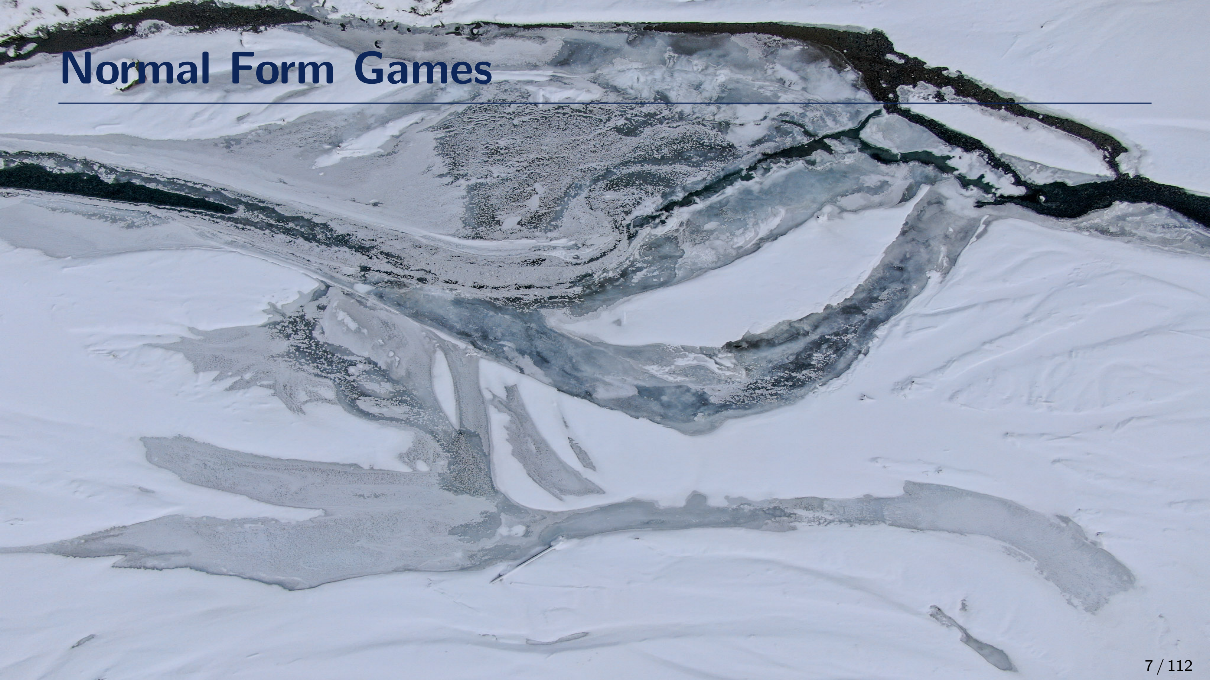
Reinforcement Learning

1. Environment
2. Agent
3. Policy
4. Reward

Game Theory

1. Game
2. Player
3. Strategy
4. Utility

Normal Form Games



Normal Form Games

The normal form games is a model in which each player chooses his strategy, and then all players play simultaneously. The outcome depends on the actions chosen by the players.

Definition: Normal Form Game

is a tuple $\langle N, (A_i), (u_i) \rangle$, where

- N is the **finite** set of players
- A_i is the nonempty set of actions available to the player i
- u_i is a **payoff/utility** function for the player i. Let $A = \times_{i \in N} A_i$.
$$u_i : A \to \mathbb{R}$$

Normal Form Games

- If there are only two players ($|N| = 2$) , we can conveniently described the game using a table

	Rock	Paper	Scissors
Rock	(0, 0)	(-1, 1)	(1, -1)
Paper	(1, -1)	(0, 0)	(-1, 1)
Scissors	(-1, 1)	(1, -1)	(0, 0)

(a) Rock-Paper-Scissors

	Confess	Be quiet
Confess	(8, 8)	(0, 10)
Be quiet	(10, 0)	(2, 2)

(c) Prisoner's dilemma

- Rows/columns correspond to actions of player one/two
- The cell (i, j) contains the players' payoffs $u_1(i, j)$ and $u_2(i, j)$

Constant Sum Games

- Constant-sum game is a game for which $u_1 + u_2 = c$
- Zero-sum game is a constant-sum game for $c = 0$, so $u_1 = -u_2$
- Critical implications!

Zero Sum Games

	Rock	Paper	Scissors
Rock	(0, 0)	(-1, 1)	(1, -1)
Paper	(1, -1)	(0, 0)	(-1, 1)
Scissors	(-1, 1)	(1, -1)	(0, 0)

	Rock	Paper	Scissors
Rock	0	-1	1
Paper	1	0	-1
Scissors	-1	1	0

Normal Form Game Strategies

Definition: Pure Strategy

$a_i \in A_i$ is player i's pure strategy. This strategy is referred to as pure, because there's no probability involved. For example, the player can always play Scissors.

Definition: Mixed Strategy

is a probability measure over the player's pure strategies. The set of player i's mixed strategies is denoted as Π_i. Given $\pi_i \in \Pi_i$, we denote the probability that the player chooses the action $a_j \in A_i$ as $\pi^{\pi_i}(a_j)$ Mixed strategies allow a player to probabilistically choose actions.

Definition: Strategy profile

Is the set of all players' strategies (one for every player), denoted as $\pi = (\pi_0, \pi_1 \ldots \pi_n)$. Finally, π_{-i} refers to all the strategies in π except π_i.

Normal Form Game Strategies II

- **Pure Strategy** $a_i \in A_i$ is player i's pure strategy.
- **Mixed Strategy** is a probability measure over the player's pure strategies. The set of player i's mixed strategies is denoted as Π_i. Given $\pi_i \in \Pi_i$, we denote the probability that the player chooses the action $a_j \in A_i$ as $\pi_i(a_j)$
- **Strategy profile** Is a tuple of all players' strategies, denoted as $\pi = (\pi_0, \pi_1 \ldots \pi_n)$. Finally, π_{-i} refers to all the strategies in π except for π_i.

Outcome

- Given a pure strategies of all players, we can easily compute the utilities. Player i's utility $= u_i(a)$
- How to compute the outcome if the players use mixed strategy (they randomize among the pure strategies)? We simply compute the expected value given the probability measure.
- Since the players choose the actions simultaneously, the events are independent and consequently $\pi^{\pi}((a_0, a_1, \ldots, a_n)) = \pi^{\pi_0}(a_0)\pi^{\pi_1}(a_1)\ldots\pi^{\pi_n}(a_n)$
- Using this fact, computing the expected value is easy
$$u_i(\pi) = \sum_{a \in A} \pi^{\pi}(a) u_i(a)$$

Best Response

- One of the key concepts, that you will see throughout the class
- Given the strategies π_{-i} of the opponents, the **best response** is the strategy that maximizes the utility for the player.

Definition: Best Response

Best response against a policy π_i is:

$$\arg\max_{\pi_{-i} \in \Pi_{-i}} R_{-i}(\pi_i, \pi_{-i})$$

We use $\mathbb{BR}(\pi_i)$ to denote the set of best response policies against the policy π_i.

Best Response

Note that for zero-sum games, opponent maximizing their reward is equivalent to opponent minimizing our reward.

$$\arg\max_{\pi_{-i}} R_{-i}(\pi_i, \pi_{-i}) = \arg\min_{\pi_{-i}} R_i(\pi_i, \pi_{-i})$$

As this means the player's value against any best-response strategy is unique, we denote this unique value as $BRV_i(\pi_i)$.

$$BRV_i(\pi_i) = \min_{\pi_{-i}} R_i(\pi_i, \pi_{-i}) = -\max_{\pi_{-i}} R_{-i}(\pi_i, \pi_{-i})$$

Best Response

Lemma

For any best response strategy $\pi_i \in \mathbb{BR}_i(\pi_{-i})$, all the actions in the support have the same expected value.

Lemma

The best response set $\mathbb{BR}(\pi_{-i})$ is convex.

Dominated Strategies

- Some actions can be clearly poor choises, and it makes no sense for a rational player to take.
- Strategy π_i^a **strictly dominates** π_i^b iff for any π_{-i}
$$u_i(\pi_i^a, \pi_{-i}) > u_i(\pi_i^b, \pi_{-i})$$
- Strategy π_i^a **weakly dominates** π_i^b iff for any π_{-i}
$$u_i(\pi_i^a, \pi_{-i}) \geq u_i(\pi_i^b, \pi_{-i})$$
- Strategy is **strictly/weakly** dominated if there's a strategy that strictly/weakly dominates it.
- Strategies π_i^a, π_i^b are **intransitive** iff one neither dominates nor is dominated by the other.

Examples

Can a weakly/strictly dominated strategy be a best response?

Iterated elimination of dominated strategies

- A rational player does not play dominated strategy
- Iterated elimination of dominated strategies

	Left	**Center**	**Right**
Top	(13, 3)	(1, 4)	(7, 3)
Middle	(4, 1)	(3, 3)	(6, 2)
Up	(-1, 9)	(2, 8)	(8, -1)

Examples

Can a weakly/strictly dominated strategy that we found during the iterated elimination be a best response in the original game?

Week 1 Homework

1. Python and notebooks
2. Strategy pair evaluation for a matrix game
3. Best response calculation
4. Strategy evaluation against a best response
5. Iterated removal of dominated strategies
6. OpenSpiel