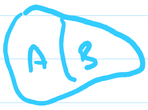
Class 2

quinta-feira, 21 de agosto de 2025 16:00

Quantum State Representation

Vector State

Ex: $|\Psi\rangle = a|0\rangle + b|1\rangle$



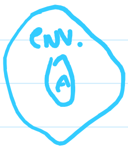
$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

Ex: $|\Psi\rangle = |\Psi_A\rangle \otimes |\Psi_B\rangle$

If not, $|\Psi\rangle$ is entangled

$p_i \rightarrow |\Psi_i\rangle$ — $tr(\rho^2) = 1$ pure states

— $tr(\rho^2) < 1$ mixed states



$\rho_A = tr_{ENV}\{\rho_{LOCAL}\}$

How do they transform?

Quantum State Transformation

For closed systems, we want to understand what type of U leads to

$$|\Psi_1\rangle = U|\Psi_0\rangle$$

$$\rho_1 = |\Psi_1\rangle\langle\Psi_1| = U|\Psi_0\rangle\langle\Psi_0|U^\dagger = U\rho_0 U^\dagger$$

Ψ_0 — Ψ_1

- Reversible
- Norm preserving
- linear

$\Big\}$ U is unitary $UU^\dagger = U^\dagger U = \mathbb{1}$

$$U^\dagger|\Psi_1\rangle = U^\dagger U|\Psi_0\rangle = |\Psi_0\rangle$$

If we want the time evolution, we use the Schrödinger

$$i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle = \hat{H}|\Psi(t)\rangle \qquad |\Psi(t_0)\rangle = |\Psi_0\rangle$$

$\hat{H} \rightarrow$ Hamiltonian

If we want to solve this equation

$$|\Psi(t)\rangle = \exp\left[-\frac{i}{\hbar}H(t-t_0)\right]|\Psi_0\rangle = U(t,t_0)|\Psi_0\rangle$$

$\hookrightarrow$ propagator

$$\frac{d\rho}{dt} = \frac{d}{dt}\left(U \rho_0 U^+\right) = \frac{dU}{dt}\rho_0 U^+ + U \rho_0 \frac{dU^+}{dt} \qquad \frac{dU}{dt} = \frac{-i\hat{H}}{\hbar} U$$

$$\frac{d\rho}{dt} = \frac{-iH}{\hbar} U \rho U^+ + \frac{i}{\hbar} U \rho U^+ H = -\frac{i}{\hbar}\left[\hat{H}, \rho(t)\right]$$

$$\frac{d\rho}{dt} = \frac{-i}{\hbar}\left[\hat{H}, \rho\right] \qquad \text{VON - NEUMANN equation}$$

Ex 1: A two-level system completely isolated. The Hamiltonian is

$$\hat{H} = \frac{\hbar\omega}{2}\hat{\sigma}_z \qquad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{(Pauli matrix)}$$

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

At what time t' $|\psi(t')\rangle = |\psi(0)\rangle$?

Unitary evolution

$$|\psi(t)\rangle = \exp\left[\frac{-i}{\hbar}\hat{H}t\right]|\psi(0)\rangle$$

$$= \begin{pmatrix} e^{-i\omega t/2} & 0 \\ 0 & e^{i\omega t/2} \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \qquad e^A = \sum_{n=0}^{\infty}\frac{1}{n!}A^n$$

$$= \frac{1}{\sqrt{2}}\left(e^{-i\omega t/2}|0\rangle + e^{i\omega t/2}|1\rangle\right)$$

$$e^{-i\omega t/2} = e^{i\omega t/2} = 1 \longrightarrow \frac{\omega t}{2} = 2\pi \longrightarrow t = \frac{4\pi}{\omega}\cdot n \longrightarrow t' = \frac{4\pi}{\omega}$$

Non-unitary evolution

It is needed when we lose information about the state. In other words, the operation is not reversible

$$\rho_A = \mathrm{tr}_{ENV}\{\rho_{tot}\}$$

To model the transformation, we use a trace-preserving completely positive maps, usually called quantum map

$$\rho_1 = \mathcal{E}(\rho_0) \qquad \mathcal{E}: \text{quantum map}$$

To be a valid operation, $\mathcal{E}$ must satisfy

To be a valid operation, $\mathcal{E}$ must satisfy

① Linearity $\quad \mathcal{E}(\alpha\rho_A+\beta\rho_B)=\alpha\,\mathcal{E}(\rho_A)+\beta\,\mathcal{E}(\rho_B)$

② Trace-preserving $\quad \mathrm{tr}(\mathcal{E}(\rho))=1$

③ Complete positivity

$$\langle\phi_A|\,\mathcal{E}(\rho)\,|\phi_A\rangle\geq 0 \qquad \forall\ \phi_A\in\mathcal{H}_A$$

$$\langle\phi_{AB}|\,\mathcal{E}_A\otimes I\,(\rho_{AB})\,|\phi_{AB}\rangle\geq 0 \qquad \forall\ \phi_{AB}\in\mathcal{H}_{AB}=\mathcal{H}_A\otimes\mathcal{H}_B$$

④ Hermiticity-preserving

$$\text{If}\quad \rho=\rho^\dagger \longrightarrow \mathcal{E}(\rho)=[\mathcal{E}(\rho)]^\dagger$$

This set of properties indicate that $\mathcal{E}$ can be represented as

$$\boxed{\begin{array}{ll}\mathcal{E}(\rho)=\sum_j K_j\,\rho\,K_j^\dagger & \text{Kraus-sum representation}\\ \text{with}\ \sum_j K_j^\dagger K_j=\mathbb{1} & \end{array}}$$

Ex:

① If we keep track of the environment, we know U

$$U:\begin{cases}U|0\rangle_A|0\rangle_E\longrightarrow|0\rangle_A|0\rangle_E\\ U|1\rangle_A|0\rangle_E\longrightarrow\sqrt{p}\,|0\rangle_A|1\rangle_E+\sqrt{1-p}\,|1\rangle_A|0\rangle_E\end{cases}$$

② If we don't keep track of env. $\rightarrow$ it must trace it out

Suppose: $\rho_{AE}=\rho_A\otimes|0\rangle\langle 0|_E \longrightarrow \rho_{AE}(p)=U(\rho_A\otimes|0\rangle\langle 0|_E)U^\dagger$

$$\mathrm{tr}_{\mathrm{ENV.}}(\rho_{AE}(p))=\sum_{j=1}^{N_E}(\mathbb{1}_A\otimes\langle j|_E)\,U\rho_A\otimes|0\rangle\langle 0|_E\,U^\dagger\,(I_A\otimes|j\rangle_E)$$

$$=\sum_{j=1}^{N_E}\mathbb{1}_A\underbrace{(\langle j|U|0\rangle_E)}_{K_j}\rho_A\underbrace{(\langle 0|U^\dagger|j\rangle_E)\,\mathbb{1}_A}_{K_j^\dagger}=\sum_{j=1}^{N_E}K_j\,\rho_A\,K_j^\dagger$$

- $K_0|0\rangle_A=\langle 0|U|0\rangle_E|0\rangle_A=\langle 0|0\rangle_E^{=1}|0\rangle_A=|0\rangle_A$

- $K_0 |0\rangle_A = \langle 0|U|0\rangle_E |0\rangle_A = \langle 0|0\rangle_E |0\rangle_A = |0\rangle_A$
- $K_0 |1\rangle_A = \langle 0|U|0\rangle_E |1\rangle_A = \sqrt{p}\, \overset{=0}{\langle 0|1\rangle_E} |0\rangle_A + \sqrt{1-p}\, \overset{=1}{\langle 0|0\rangle_E} |1\rangle_A = \sqrt{1-p}\, |1\rangle_A$

- $K_1 |0\rangle_A = \langle 1|U|0\rangle_E |0\rangle_A = \overset{=0}{\langle 1|0\rangle_E} |0\rangle_A = 0$
- $K_1 |1\rangle_A = \langle 1|U|0\rangle_E |1\rangle_A = \sqrt{p}\, |0\rangle_A$

Using the basis $\{|0\rangle, |1\rangle\}$

$$K_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix} \qquad K_1 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}$$

Sanity check

$$K_0^\dagger K_0 + K_1^\dagger K_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1-p \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \sqrt{p} & 0 \end{pmatrix}\begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1-p \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & p \end{pmatrix} = \mathbb{1}$$

This particular $\mathcal{E}(p)$ is called amplitude-damping channel

Exercise: Start with a TLS in the state

$$|\psi\rangle_A = |1\rangle_A$$

Apply the quantum map $\mathcal{E}(p)$ amplitude-damping channel over this state. Under what condition $\mathcal{E}(\rho_A)$ is a pure state? Use the Kraus state decomposition.

$$\rho_A = |1\rangle\langle 1| = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathcal{E}(\rho) = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix} + \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ \sqrt{p} & 0 \end{pmatrix} = \begin{pmatrix} p & 0 \\ 0 & 1-p \end{pmatrix}$$

$$\mathrm{tr}\left([\mathcal{E}(\rho_A)]^2\right) = \mathrm{tr}\left[\begin{pmatrix} p^2 & 0 \\ 0 & (1-p)^2 \end{pmatrix}\right] = \underbrace{p^2 + (1-p)^2 = 1}_{\text{solutions } p=0 \text{ or } p=1}$$

Lindblad Master Equation

Now we are ready to look at the quantum equation of motion for a state ρ_A. We would like to express this incremental evolution in the Kraus form

$$\mathcal{E}_{\Delta t}(\rho_A) = \sum_j K_j(\Delta t)\,\rho_A(t)\,K_j^\dagger(\Delta t) = \rho_A(t+\Delta t)$$

* Disclaimer: we have assumed $\rho_{A,E} = \rho_A \otimes |0\rangle\langle 0|_E$, i.e., they are not entangled

This approximation is valid as long as the env. has a number of DoF very large, so that the correlation with the system τ_c is very short

$$\underbrace{\tau_c \ll \Delta t}_{\text{Markov approximation}} \ll T$$

↳ timescale we want to describe *

So we want to find

$$\frac{d\rho_A}{dt} = \lim_{\Delta t\to 0} \frac{\rho_A(t+\Delta t) - \rho_A(t)}{\Delta t}$$

We can assume

$$K_0 = \mathbb{1} - i\Delta t\, K + O(\Delta t^2) \qquad \text{(close to the identity)}$$

$$K_j = \sqrt{\Delta t}\, L_j \qquad j \geq 1 \qquad \text{(very small)}$$

Let's split K in a Hermitian/Anti-Hermitian

$$K = \frac{H}{\hbar} - iJ \qquad H = \hbar\frac{K+K^\dagger}{2} = H^\dagger \qquad J = i\frac{K-K^\dagger}{2} = J^\dagger$$

For the map $\mathcal{E}(\rho_A)$, using the first order in Δt

$$K_0\rho_A K_0^\dagger = (\mathbb{1} - i\Delta t K)\rho_A(\mathbb{1} + i\Delta t K^\dagger)$$

$$= \left[\mathbb{1} - i\Delta t\left(\frac{H}{\hbar} - iJ\right)\right]\rho_A\left[\mathbb{1} + i\Delta t\left(\frac{H}{\hbar} + iJ\right)\right]$$

$$= \rho_A - \frac{i\Delta t}{\hbar}H\rho_A - \Delta t J\rho_A + \frac{i\Delta t}{\hbar}\rho_A H - \Delta t\rho_A J$$

$$= \rho_A - \frac{i\Delta t}{\hbar}[H,\rho_A] - \Delta t(J\rho_A + \rho_A J) \;*$$

The other elements $(j \neq 0)$

$$K_j\rho_A K_j^\dagger = \Delta t\, L_j\rho_A L_j^\dagger \;*$$

Also

$$\sum_j K_j^\dagger K_j = \mathbb{1} \longrightarrow K_0^\dagger K_0 + \sum_{j\neq 0} \Delta t\, L_j^\dagger L_j = \mathbb{1}$$

$$K_0^\dagger K_0 = (\mathbb{1} + i\Delta t\, K^\dagger)(\mathbb{1} - i\Delta t\, K) = \mathbb{1} + \Delta t\,(i\,(K^\dagger - K)) = \mathbb{1} - 2\Delta t J$$

$$\mathbb{1} - 2\Delta t J + \sum_{j\neq 0} \Delta t\, L_j^\dagger L_j = \mathbb{1} \longrightarrow J = \frac{1}{2}\sum_{j\neq 0} L_j^\dagger L_j \quad *$$

Plugging equations * into ■, we have

$$\frac{d\rho_A}{dt} = \frac{\rho_A - \frac{i\Delta t}{\hbar}[H,\rho_A] - \Delta t\, \frac{1}{2}\sum_{j\neq 0}(L_j^\dagger L_j \rho_A + \rho_A L_j^\dagger L_j) + \Delta t \sum_{j\neq 0} L_j \rho_A L_j^\dagger - \rho_A(t)}{\Delta t}$$

$$\boxed{\frac{d\rho_A}{dt} = -\frac{i}{\hbar}[H,\rho_A] + \sum_{j\neq 0} L_j \rho_A L_j^\dagger - \tfrac{1}{2}(L_j^\dagger L_j \rho_A + \rho_A L_j^\dagger L_j)}$$

Master equation in the Lindblad form

$$\frac{d\rho}{dt} = \mathcal{L}[\rho]$$

$\mathcal{L}$ → Lindbladian superoperator

The L_j are called jump or collapse operator

- with probability $p_j = \Delta t\, \mathrm{tr}[L_j^\dagger \rho L_j]$ a quantum jump would occur

- If not, we have an evolution with the effective Hamiltonian

$$H_{eff} = H - \frac{i\hbar}{2}\sum_{j\neq 0} L_j^\dagger L_j$$

$$\frac{d\rho}{dt} = \frac{-i}{\hbar}(H_{eff}\,\rho - \rho H_{eff}^\dagger)$$