Quantum aspects of the classical Maxwell's equations in free space from the perspective of the correspondence principle

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Due to the advent of Quantum Mechanics' 100th anniversary, we wrote this review paper in order to present a discussion that addresses the foundations of this theory. And since the creation of this Mechanics and other quantum theories was guided by correspondence principles that needed to be identified between them and other well-established physical theories, this paper will be devoted to discussing the correspondence between these quantum theories and Maxwell's theory of electromagnetism. More precisely, what we will do throughout this paper is discuss how the Maxwell's electromagnetic theory in free space and the stronger formulation of the correspondence principle already pointed, together, to the basis of a Quantum Mechanics that was only be formulated half a century later. And, in order to make this very clear, we will show that the quantum-mechanical description of a photon can already be identified, simply, by manipulating Maxwell's equations in free space with mathematical resources that, for instance, were already well known before the advent of Quantum Mechanics.

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I. INTRODUCTION

If, for some reason, the scientific community needed to choose the greatest climaxes in the history of Physics, most of them would certainly date from the beginning of the 19th Century until the first half of the 20th Century. After all, it was precisely in this period that some theories, which have a crucial importance in Physics, emerged and broke some paradigms. And one of these theories was the dynamical theory of the electromagnetic field developed by James Clerk Maxwell (1831 – 1879) that, among other things, demonstrated to everyone that light could actually be interpreted as an electromagnetic wave [1].

By the way, the brilliance and adequacy of the electromagnetic theory developed by Maxwell were so great that it not only unleashed a great technological revolution [2], but also to give rise to new theories shortly thereafter. And one of these new theories, which emerged in the same century, was proposed by Hendrik Lorentz (1853 – 1928), who tried to describe the propagation of light through a luminiferous aether [3–10]. Incidentally, Maxwell himself was also believed in the existence of this aether since, for instance, all the non-luminous waves known at that time needed a material medium to propagate [11–13]. However, in spite of the mathematical foundation of Lorentz's theory being flawless, the only (and big!) problem that did not allow it to be interpreted as a physical theory was precisely the impossibility of detecting (experimentally) the existence of this aether [14–18].

Given this impossibility of detecting such aether, the entire mathematical foundation of Lorentz's theory was adapted to a new (and more practical) proposal that Albert Einstein (1879 – 1955) made in 1905 [19, 20]. In this new proposal, Einstein abolished the idea of this aether in order to base the mathematical foundation of Lorentz's theory only on two physical principles: (i) the laws of physics should be independent of the uniform motion of an inertial frame of reference; and (ii) the speed of light should have the same constant value in any such frame [21]. The implications of this new Lorentz-Einstein theory [22] (which became popularly known as special theory of relativity) were extensively investigated and, due to the agreement of its predictions with the various theoretical and experimental tests, it not only was accepted by the scientific community, but also ended up being used as a first step towards a more general theory of relativity, where gravitational effects need to be considered [23–25].

However, although classical electromagnetism and special relativity have revealed pro-

perties of light that look perfect and non-conflicting, Einstein also published, in that same year of 1905, another paper that blew the minds of those who were already accustomed to the idea that light was a wave [26]. After all, by reviewing a paper that Max Planck (1858 – 1947) had published, in 1901, in order to explain the black body radiation [27], Einstein came to a curious conclusion: while Planck assumed (even without knowing how to explain why) that the walls of a black body emitted, when heated, light with an energy

$$E = hf \tag{1}$$

proportional to the frequency (f) of the electromagnetic radiation that characterized it, Einstein concluded that it was light itself that has this quantized nature. By according to this curious conclusion, not only monochromatic lights, but all the electromagnetic waves could be interpreted as a swarm of discrete energy packets (i.e., particles that have this energy (1)) since it was exactly this interpretation that could explain, for instance, the photoelectric effect. And as this Einstein's conclusion was so successful that it won him the Nobel Prize in Physics, in 1921, precisely for his discovery of the law of the photoelectric effect, the new question that needed to be answered by the scientific community was: how to conciliate the interpretation of light as an electromagnetic wave with this new Einstein's proposal, where light needs to be described as a stream of particles?

In order to understand the answer to this question, it is important to note that, since the second half of the 19th Century, several researches have been done in pursuit of some new theory that could explain some microscopic phenomena [28–37]. And as these early researches sought to describe the atomic and molecular behaviour of various materials, it is not wrong to say that all these early researches did not deal with relativistic considerations because, for instance, atoms are massive. Nevertheless, by showing that light could be interpreted in this quantum way, Einstein ended up making it quite clear that everything, which was already being developed to describe the microscopic phenomena, also needed to be mixed in some way not only with this corpuscular interpretation of light, but also with electromagnetism and special relativity because, among other things,

- it was already known that an atom could be unequivocally characterized by its light spectrum [38], and

- all the particles that compose the atoms/molecules (i.e., protons, neutrons and elec-

trons) interact with the electromagnetic fields that are described by Maxwell's equations [39].

Given this new scenario, it was pretty clear that any theory, which was able to describe light in this quantum way revealed by Einstein, needed not only to respect the Maxwell's formulation of electromagnetism, but also to be necessarily relativistic from its inception. However, as unexpected as it was for the scientific community to revive the wave-particle debate on the nature of light, the fact is that, in addition to Maxwell's equations already agreed with special relativity [40], it is possible to prove that these equations also already agreed with what was yet to be formulated by Werner Heisenberg (1901 – 1976), Erwin Schrödinger (1887 – 1961) and the other physicists more than half a century later. In particular, it is possible to show that Maxwell's electromagnetic theory already supports some quantum description of light since, among other things, part of Maxwell's equations in free space takes the same form as the Schrödinger equation [41, 42]. In this way, as this fact is a little overshadowed in the most references that introduce quantum electrodynamics and, in most cases, it is not even mentioned in Quantum Mechanics textbooks, we can affirm that the basic purpose of this paper is to present a critical and pedagogical review on this subject in order to clarify, for instance, how the Maxwell's electromagnetic theory in free space and the correspondence principle [43] already pointed, together, to the basis of a Quantum Mechanics that was only be formulated half a century later.

II.   SOME WAVE CONSIDERATIONS

The first thing we need to do here, in order to begin to understand why Maxwell's equations in free space[1] [44]

$$\boldsymbol{\nabla}_{\boldsymbol{r}} \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0} \ , \tag{2a}$$

$$\boldsymbol{\nabla}_{\boldsymbol{r}} \cdot \boldsymbol{B} = 0 \ , \tag{2b}$$

$$\boldsymbol{\nabla}_{\boldsymbol{r}} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \quad \text{and} \tag{2c}$$

$$\boldsymbol{\nabla}_{\boldsymbol{r}} \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \frac{1}{c^2} \frac{\partial \boldsymbol{E}}{\partial t} \tag{2d}$$

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[1] Here, the variable $\boldsymbol{r}$ that indexes the operator $\boldsymbol{\nabla}$ represents the differentiation variable. An analogous comment applies when this same operator is indexed by other variables.

already support some quantum description, is to recognize that all the waves that actually exist in Nature are not really flat and monochromatic, and that some of their general simultaneous solutions can be written as[2]

$$\boldsymbol{E}\left(\boldsymbol{r}, t\right) = \int \boldsymbol{\mathcal{E}}_0\left(\boldsymbol{k}\right) e^{i\left[\boldsymbol{k}\cdot\boldsymbol{r} - \omega(\boldsymbol{k})t\right]} d\boldsymbol{k} \quad \text{and} \quad \boldsymbol{B}\left(\boldsymbol{r}, t\right) = \int \boldsymbol{\mathcal{B}}_0\left(\boldsymbol{k}\right) e^{i\left[\boldsymbol{k}\cdot\boldsymbol{r} - \omega(\boldsymbol{k})t\right]} d\boldsymbol{k}, \quad (3)$$

among which are the so-called *wave packets*: i.e., the electromagnetic fields $\left(\boldsymbol{E}\left(\boldsymbol{r}, t\right), \boldsymbol{B}\left(\boldsymbol{r}, t\right)\right)$ that can be modelled by the same superpositions (3), but where the functions $\boldsymbol{\mathcal{E}}_0\left(\boldsymbol{k}\right)$ and $\boldsymbol{\mathcal{B}}_0\left(\boldsymbol{k}\right)$ are proportional to

$$\exp\left[-\frac{\left|\boldsymbol{k} - \boldsymbol{k}_0\right|^2}{2\left(\Delta k\right)^2}\right] .$$

In other words, $\boldsymbol{\mathcal{E}}_0\left(\boldsymbol{k}\right)$ and $\boldsymbol{\mathcal{B}}_0\left(\boldsymbol{k}\right)$ are four *Gaussian functions* [45] in this wave packet case, which have the same maximum point $\boldsymbol{k} = \boldsymbol{k_0}$, where Δk represents their peak width as illustrated in Figure 1.

Regarding the function $\omega\left(\boldsymbol{k}\right)$ that appears in (3), it is correct to say that, in addition to it being interpreted as the angular frequency of the components

$$\boldsymbol{\mathcal{E}}\left(\boldsymbol{k}, t\right) = \boldsymbol{\mathcal{E}}_0\left(\boldsymbol{k}\right) e^{-i\omega(\boldsymbol{k})\cdot t} \quad \text{and} \quad \boldsymbol{\mathcal{B}}\left(\boldsymbol{k}, t\right) = \boldsymbol{\mathcal{B}}_0\left(\boldsymbol{k}\right) e^{-i\omega(\boldsymbol{k})\cdot t} , \quad (4)$$

it can also be interpreted as the function that models the *dispersion* of an electromagnetic wave packet. And as this last interpretation allows us to recognize that the group velocity of this wave packet is

$$\boldsymbol{v}_g = \boldsymbol{\nabla}_{\boldsymbol{k}} \, \omega\left(\boldsymbol{k}\right) ,$$

this allows us to conclude, for instance, that such dispersion can be approximated by

$$\omega\left(\boldsymbol{k}\right) \approx \omega_0 + \left(\boldsymbol{k} - \boldsymbol{k}_0\right)^T \cdot \boldsymbol{v}_g + \frac{1}{2}\left(\boldsymbol{k} - \boldsymbol{k}_0\right)^T \cdot \mathsf{H}_\omega \cdot \left(\boldsymbol{k} - \boldsymbol{k}_0\right) ,$$

where H_ω is the Hessian of $\omega\left(\boldsymbol{k}\right)$ [47]. Therefore, by noting that, under a change of variable $\boldsymbol{k} \rightarrow \boldsymbol{\kappa} = \boldsymbol{k} - \boldsymbol{k}_0$, the expressions of these wave packets can be rewritten as

$$\boldsymbol{E}\left(\boldsymbol{r}, t\right) = e^{i\boldsymbol{k}_0\cdot\boldsymbol{r}} \int \boldsymbol{\mathcal{E}}_0\left(\boldsymbol{k}_0 + \boldsymbol{\kappa}\right) e^{i\left[\boldsymbol{\kappa}\cdot\boldsymbol{r} - \omega(\boldsymbol{\kappa})t\right]} d\boldsymbol{\kappa} \quad \text{and}$$

$$\boldsymbol{B}\left(\boldsymbol{r}, t\right) = e^{i\boldsymbol{k}_0\cdot\boldsymbol{r}} \int \boldsymbol{\mathcal{B}}_0\left(\boldsymbol{k}_0 + \boldsymbol{\kappa}\right) e^{i\left[\boldsymbol{\kappa}\cdot\boldsymbol{r} - \omega(\boldsymbol{\kappa})t\right]} d\boldsymbol{\kappa} ,$$

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[2] Although we are not concerned with writing the domain of integration in (3), it is worth observing that such a domain is the entire $\boldsymbol{k}$-space. A similar comment will also apply to other integrals that are written without detailing their domains of integration.
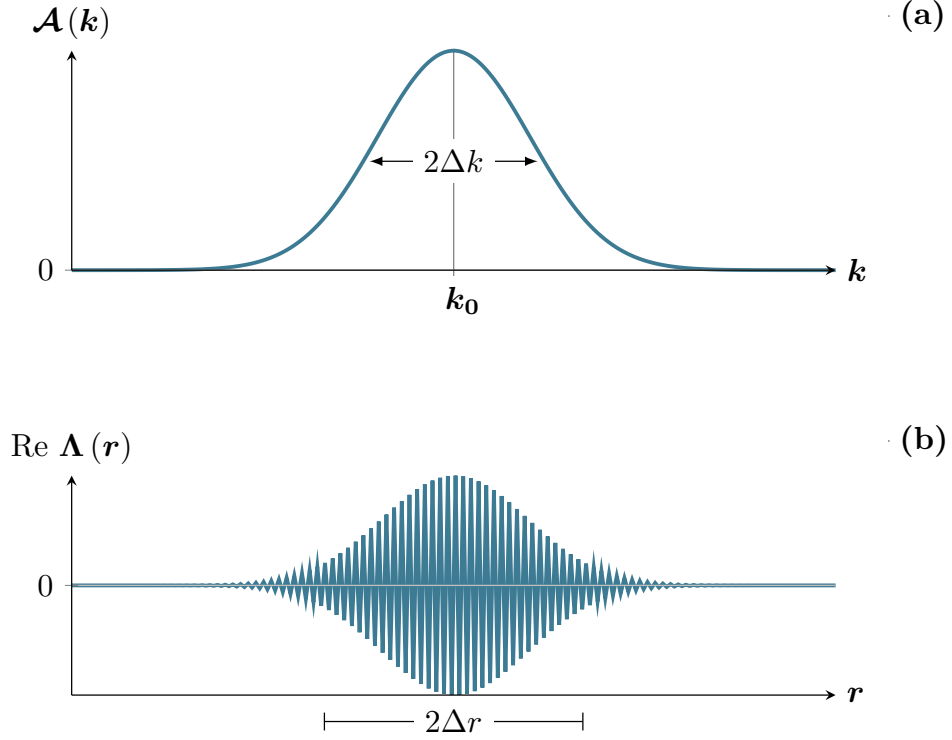
Figura 1: In figure $(\mathbf{a})$, above, we see the graph of a function $\mathcal{A}(\mathbf{k})$ that has the same properties as $\mathcal{E}_0(\mathbf{k})$ and $\mathcal{B}_0(\mathbf{k})$: i.e., $\mathcal{A}(\mathbf{k})$ is, for instance, a non-negative real function, which is symmetric around the line $\mathbf{k} = \mathbf{k_0}$. Now, in figure $(\mathbf{b})$, we see the graph of only the real part of a function $\mathbf{\Lambda}(\mathbf{r})$ that assumes the same form as the fields in (3) at a fixed time instant. Here, the widths Δk and Δr are related by the expression (5), which is justified by the fact that $\mathcal{A}(\mathbf{k})$ and $\mathbf{\Lambda}(\mathbf{r})$ are related by a *Fourier transform* [46]: in plain English, it means that the graph of $\mathcal{A}(\mathbf{k})$ is only that way because this is the only way to justify the shape of the wave packet $\mathbf{\Lambda}(\mathbf{r})$. It is worth noting that, although the imaginary part of $\mathbf{\Lambda}(\mathbf{r})$ has not been graphed here, its shape is analogous to that shown in Figure $(\mathbf{b})$.

we can conclude that they are all are, for instance, proportional to

$$e^{i(\mathbf{k_0}\cdot\mathbf{r}-\omega_0 t)} \int e^{i\boldsymbol{\kappa}^T\cdot(\mathbf{r}-\mathbf{r_0}-\mathbf{v}_g t)} \exp\left[\frac{\boldsymbol{\kappa}^T}{2}\cdot\left(\frac{1}{(\Delta k)^2}+i\,\mathbf{H}_\omega t\right)\cdot\boldsymbol{\kappa}\right]\,d\boldsymbol{\kappa}\ .$$

In this fashion, since this last result makes clear that the modules of this electromagnetic are proportional to

$$\exp\left[-\frac{|\mathbf{r}-\mathbf{r_0}-\mathbf{v}_g t|^2}{2(\Delta r)^2}\right]\ ,$$

it also becomes clear that Δr and Δk satisfy a kind of uncertainty principle that is very

similar to Heisenberg's because

$$\Delta r = \frac{1}{2\,\Delta k}\ \sqrt{1+\left(\boldsymbol{\kappa}^T\cdot\mathbf{H}_\omega\cdot\boldsymbol{\kappa}\right)^2 t^2}\quad\Leftrightarrow\quad \Delta r\cdot\Delta k = \frac{1}{2}\ \sqrt{1+\left(\boldsymbol{\kappa}^T\cdot\mathbf{H}_\omega\cdot\boldsymbol{\kappa}\right)^2 t^2}\geqslant\frac{1}{2}\ . \quad (5)$$

III. MAXWELL'S EQUATIONS WRITTEN IN k-SPACE

Given the result we have just obtained, it is quite tempting to conclude that (5) has some relationship with the Heisenberg's uncertainty principle [48]

$$\Delta\boldsymbol{r}\cdot\Delta\boldsymbol{p}\geqslant\frac{\hbar}{2}\qquad\qquad(6)$$

in the electromagnetic context. After all, if this relationship actually exists in order to, for example, allows us to recognize (5) and (6) as equivalent, this could be of utmost importance in view of the proposal of this paper. However, as all the algebraic manipulations we have done so far still do not endorse this equivalence in the electromagnetic context, it is important to make new developments in order to see if they show whether this equivalence between (5) and (6) is actually valid or not. And since the last Section showed us that the expressions (3) already satisfy the inequality (5), it is essential to evaluate whether the replacement of these expressions in Maxwell's equations (2) allows us to recognize such an equivalence.

But, although this is exactly what we will do throughout this Section, it is convenient to make a small caveat because you, the reader, will see that, instead of replacing the expressions (3) in the equations (2), we will replace

$$\boldsymbol{E}\left(\boldsymbol{r},t\right) = \int\boldsymbol{\mathcal{E}}\left(\boldsymbol{k},t\right)e^{i\boldsymbol{k}\cdot\boldsymbol{r}}d\boldsymbol{k}\quad\text{and}\qquad(7\mathrm{a})$$

$$\boldsymbol{B}\left(\boldsymbol{r},t\right) = \int\boldsymbol{\mathcal{B}}\left(\boldsymbol{k},t\right)e^{i\boldsymbol{k}\cdot\boldsymbol{r}}d\boldsymbol{k}\ . \qquad(7\mathrm{b})$$

And why will we do this? We will do this because, in addition to these expressions (3) and (7) being exactly the same under the identification made in (4), what we want to do here, with the replacement of (7) in Maxwell's equations (2), is to rewrite these equations in $\boldsymbol{k}$-space. And since the electromagnetic fields in the equations (2) are expressed as a function of $(\boldsymbol{r},t)$, it is essential that this replacement uses functions of $(\boldsymbol{k},t)$. Observe that, in order for us to be able to rewrite Maxwell's equations in $\boldsymbol{k}$-space, it will also be necessary to take

$$\rho\left(\boldsymbol{r},t\right) = \int\varrho\left(\boldsymbol{k},t\right)e^{i\boldsymbol{k}\cdot\boldsymbol{r}}d\boldsymbol{k}\quad\text{and}\qquad(8\mathrm{a})$$

$$\boldsymbol{J}\left(\boldsymbol{r},t\right) = \int\boldsymbol{\mathcal{J}}\left(\boldsymbol{k},t\right)e^{i\boldsymbol{k}\cdot\boldsymbol{r}}d\boldsymbol{k}\ . \qquad(8\mathrm{b})$$

By the way, one thing that is quite clear is that, when we substitute the expression of the fields (7a) and (8a) in the equation (2a), we obtain

$$\boldsymbol{\nabla_r} \cdot \int \boldsymbol{\mathcal{E}}\left(\boldsymbol{k}, t\right) e^{i\boldsymbol{k}\cdot\boldsymbol{r}} d\boldsymbol{k} = \int \boldsymbol{\nabla_r} \cdot \left[\boldsymbol{\mathcal{E}}\left(\boldsymbol{k}, t\right) e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\right] d\boldsymbol{k} = \frac{1}{\varepsilon_0} \int \varrho\left(\boldsymbol{k}, t\right) e^{i\boldsymbol{k}\cdot\boldsymbol{r}} d\boldsymbol{k} \ .$$

Thus, by noting that

$$\boldsymbol{\nabla_r} \cdot \left[\boldsymbol{\mathcal{E}}\left(\boldsymbol{k}, t\right) e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\right] = e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \underbrace{\left[\boldsymbol{\nabla_r} \cdot \boldsymbol{\mathcal{E}}\left(\boldsymbol{k}, t\right)\right]}_{=0} + \boldsymbol{\mathcal{E}}\left(\boldsymbol{k}, t\right) \cdot \boldsymbol{\nabla_r} \ e^{i\boldsymbol{k}\cdot\boldsymbol{r}} = \left[i\boldsymbol{k} \cdot \boldsymbol{\mathcal{E}}\left(\boldsymbol{k}, t\right)\right] e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \ ,$$

this development ends up showing that

$$i\boldsymbol{k} \cdot \boldsymbol{\mathcal{E}}\left(\boldsymbol{k}, t\right) = \frac{1}{\varepsilon_0} \ \varrho\left(\boldsymbol{k}, t\right) \ .$$

Note that, as equation (2b) is analogous to (2a), this last result also allows to conclude that

$$i\boldsymbol{k} \cdot \boldsymbol{\mathcal{B}}\left(\boldsymbol{k}, t\right) = 0 \ .$$

Now, in relation to what follows from the replacement of (7a) and (7b) in equation (2c), it is immediate to see that

$$\begin{aligned}\boldsymbol{\nabla_r} \times \int \boldsymbol{\mathcal{E}}\left(\boldsymbol{k}, t\right) e^{i\boldsymbol{k}\cdot\boldsymbol{r}} d\boldsymbol{k} &= -\frac{\partial}{\partial t} \int \boldsymbol{\mathcal{B}}\left(\boldsymbol{k}, t\right) e^{i\boldsymbol{k}\cdot\boldsymbol{r}} d\boldsymbol{k} \\ &\Rightarrow \int \boldsymbol{\nabla_r} \times \left[\boldsymbol{\mathcal{E}}\left(\boldsymbol{k}, t\right) e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\right] d\boldsymbol{k} = -\int \frac{\partial}{\partial t} \left[\boldsymbol{\mathcal{B}}\left(\boldsymbol{k}, t\right) e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\right] d\boldsymbol{k} \ .\end{aligned} \tag{9}$$

Thus, by noting that

$$\boldsymbol{\nabla_r} \times \left[\boldsymbol{\mathcal{E}}\left(\boldsymbol{k}, t\right) e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\right] = \underbrace{e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \cdot \boldsymbol{\nabla_r} \times \boldsymbol{\mathcal{E}}\left(\boldsymbol{k}, t\right)}_{=0} + \left(\boldsymbol{\nabla_r} \ e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\right) \times \boldsymbol{\mathcal{E}}\left(\boldsymbol{k}, t\right) = i\boldsymbol{k} \ e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \times \boldsymbol{\mathcal{E}}\left(\boldsymbol{k}, t\right) \ ,$$

it is not difficult to conclude that (9) becomes

$$\int \left[i\boldsymbol{k} e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \times \boldsymbol{\mathcal{E}}\left(\boldsymbol{k}, t\right)\right] d\boldsymbol{k} = -\int \frac{\partial}{\partial t} \left[\boldsymbol{\mathcal{B}}\left(\boldsymbol{k}, t\right) e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\right] d\boldsymbol{k} = -\int \left(\frac{\partial \boldsymbol{\mathcal{B}}}{\partial t}\right) e^{i\boldsymbol{k}\cdot\boldsymbol{r}} d\boldsymbol{k} \ ,$$

which makes clear that the sufficient condition for this equality to hold is

$$\boldsymbol{k} \times \boldsymbol{\mathcal{E}}\left(\boldsymbol{k}, t\right) = -i\frac{\partial \boldsymbol{\mathcal{B}}}{\partial t} \ .$$

So, since the replacement of (7a), (7b) and (8b) in the equation (2d) also shows us that

$$i\boldsymbol{k} \times \boldsymbol{\mathcal{B}}\left(\boldsymbol{k}, t\right) = \mu_0 \boldsymbol{\mathcal{J}}\left(\boldsymbol{k}, t\right) + \frac{1}{c^2}\frac{\partial \boldsymbol{\mathcal{E}}}{\partial t} \ ,$$

it is valid to say that all the equations

$$i\boldsymbol{k} \cdot \boldsymbol{\mathcal{E}}\left(\boldsymbol{k}, t\right) = \frac{1}{\varepsilon_0} \varrho\left(\boldsymbol{k}, t\right) \ , \tag{10a}$$

$$i\boldsymbol{k} \cdot \boldsymbol{\mathcal{B}}\left(\boldsymbol{k}, t\right) = 0 \ , \tag{10b}$$

$$i\boldsymbol{k} \times \boldsymbol{\mathcal{E}}\left(\boldsymbol{k}, t\right) = -\frac{\partial \boldsymbol{\mathcal{B}}}{\partial t} \quad \text{and} \tag{10c}$$

$$i\boldsymbol{k} \times \boldsymbol{\mathcal{B}}\left(\boldsymbol{k}, t\right) = \mu_0 \boldsymbol{\mathcal{J}}\left(\boldsymbol{k}, t\right) + \frac{1}{c^2} \frac{\partial \boldsymbol{\mathcal{E}}}{\partial t} \tag{10d}$$

are equivalent to the same Maxwell's equations (2). Note that, by replacing (8a) and (8b) into the continuity equation

$$\boldsymbol{\nabla}_{\boldsymbol{r}} \cdot \boldsymbol{J} + \frac{\partial \rho}{\partial t} = 0 \ ,$$

manipulations analogous to the ones we did above lead us to

$$i\boldsymbol{k} \cdot \boldsymbol{\mathcal{J}}\left(\boldsymbol{k}, t\right) + \frac{\partial \varrho}{\partial t} = 0 \ . \tag{11}$$

A. Electromagnetic wave equations written in k-space

From the results we have just derived in the last Section, it is not difficult to see that, by manipulating these equations (10), we obtain two differential equations that, for example, show us that the electromagnetic field behaves like a harmonic oscillator. And, in order to obtain the first of them, it is enough to notice that equation (10c) shows us that

$$\begin{aligned} i\boldsymbol{k} \times \left[\boldsymbol{k} \times \boldsymbol{\mathcal{E}}\left(\boldsymbol{k}, t\right)\right] &= \left[i\boldsymbol{k} \cdot \boldsymbol{\mathcal{E}}\left(\boldsymbol{k}, t\right)\right] \boldsymbol{k} - i\left(\boldsymbol{k} \cdot \boldsymbol{k}\right) \boldsymbol{\mathcal{E}}\left(\boldsymbol{k}, t\right) \\ &= \left[\frac{1}{\varepsilon_0} \varrho\left(\boldsymbol{k}, t\right)\right] \boldsymbol{k} - ik^2 \boldsymbol{\mathcal{E}}\left(\boldsymbol{k}, t\right) = -\boldsymbol{k} \times \frac{\partial \boldsymbol{\mathcal{B}}}{\partial t} \ , \end{aligned} \tag{12}$$

where $k = |\boldsymbol{k}|$. After all, since equation (10d) also shows that

$$\boldsymbol{k} \times \frac{\partial \boldsymbol{\mathcal{B}}}{\partial t} = \frac{\partial}{\partial t}\left[\boldsymbol{k} \times \boldsymbol{\mathcal{B}}\left(\boldsymbol{k}, t\right)\right] = -i\mu_0 \frac{\partial \boldsymbol{\mathcal{J}}}{\partial t} - \frac{i}{c^2} \frac{\partial^2 \boldsymbol{\mathcal{E}}}{\partial t^2} \ ,$$

the substitution of this last result into (12) makes it quite clear that

$$\left[\frac{\partial^2}{\partial t^2} + \left(ck\right)^2\right] \boldsymbol{\mathcal{E}}\left(\boldsymbol{k}, t\right) = -\frac{1}{\varepsilon_0} \left\{\left[ic^2 \varrho\left(\boldsymbol{k}, t\right)\right] \boldsymbol{k} + \frac{\partial \boldsymbol{\mathcal{J}}}{\partial t}\right\} \ . \tag{13}$$

Now, in order to obtain the second differential equation, it is enough to observe that the manipulation of the equation (10d) shows us that

$$\begin{aligned} i\boldsymbol{k} \times \left[\boldsymbol{k} \times \boldsymbol{\mathcal{B}}\left(\boldsymbol{k}, t\right)\right] &= \underbrace{\left[\boldsymbol{k} \cdot \boldsymbol{\mathcal{B}}\left(\boldsymbol{k}, t\right)\right]}_{=0} \boldsymbol{k} - \left(\boldsymbol{k} \cdot \boldsymbol{k}\right) \boldsymbol{\mathcal{B}}\left(\boldsymbol{k}, t\right) \\ &= -ik^2 \boldsymbol{\mathcal{B}}\left(\boldsymbol{k}, t\right) = \mu_0 \left[\boldsymbol{k} \times \boldsymbol{\mathcal{J}}\left(\boldsymbol{k}, t\right)\right] + \frac{1}{c^2}\left(\boldsymbol{k} \times \frac{\partial \boldsymbol{\mathcal{E}}}{\partial t}\right) \ . \end{aligned} \tag{14}$$

In this way, since the equation (10c) also shows us that

$$\boldsymbol{k} \times \frac{\partial \boldsymbol{\mathcal{E}}}{\partial t} = \frac{\partial}{\partial t}\left[\boldsymbol{k} \times \boldsymbol{\mathcal{E}}\left(\boldsymbol{k}, t\right)\right] = i\frac{\partial^2 \boldsymbol{\mathcal{B}}}{\partial t^2} \ ,$$

it is precisely the substitution of this last result into (14) that leads us to

$$\left[\frac{\partial^2}{\partial t^2} + (ck)^2\right]\boldsymbol{\mathcal{B}}\left(\boldsymbol{k}, t\right) = -i\mu_0\left[\boldsymbol{k} \times \boldsymbol{\mathcal{J}}\left(\boldsymbol{k}, t\right)\right] \ . \tag{15}$$

Clearly, differential equations (13) and (15) are telling us that the electromagnetic field, which satisfies that Maxwell's equations (10), can be recognized as an electromagnetic wave whose natural frequency of oscillation is $\omega = ck$. But, as much as the right-hand sides of these same differential equations characterize this electromagnetic field as a forced harmonic oscillator, it is interesting to highlight that there is a special case, which is of extreme interest, where the relationship between $\boldsymbol{\mathcal{J}}\left(\boldsymbol{k}, t\right)$ and $\boldsymbol{\mathcal{E}}\left(\boldsymbol{k}, t\right)$ is ohmic: i.e., where this relationship is such that

$$\boldsymbol{\mathcal{J}}\left(\boldsymbol{k}, t\right) = \sigma \boldsymbol{\mathcal{E}}\left(\boldsymbol{k}, t\right) \ , \tag{16}$$

where the electrical conductivity σ can be considered constant. And since this ohmic relationship shows that

$$\frac{\partial \boldsymbol{\mathcal{J}}}{\partial t} = \sigma \frac{\partial \boldsymbol{\mathcal{E}}}{\partial t} \quad \text{and} \quad \boldsymbol{k} \times \boldsymbol{\mathcal{J}}\left(\boldsymbol{k}, t\right) = \sigma\left[\boldsymbol{k} \times \boldsymbol{\mathcal{E}}\left(\boldsymbol{k}, t\right)\right] = i\sigma \frac{\partial \boldsymbol{\mathcal{B}}}{\partial t} \ ,$$

the substitution of these last results into (13) and (15) respectively leads us to

$$\left[\frac{\partial^2}{\partial t^2} + \frac{\sigma}{\varepsilon_0}\frac{\partial}{\partial t} + (ck)^2\right]\boldsymbol{\mathcal{E}}\left(\boldsymbol{k}, t\right) = -\left[\frac{ic^2}{\varepsilon_0}\varrho\left(\boldsymbol{k}, t\right)\right]\boldsymbol{k} \quad \text{and} \tag{17a}$$

$$\left[\frac{\partial^2}{\partial t^2} + \frac{\sigma}{\varepsilon_0}\frac{\partial}{\partial t} + (ck)^2\right]\boldsymbol{\mathcal{B}}\left(\boldsymbol{k}, t\right) = \boldsymbol{0} \ , \tag{17b}$$

which makes it quite clear that, in this ohmic special case, the electromagnetic field describes a damped harmonic oscillation, whose damping factor is $\gamma = \sigma/\varepsilon_0$.

B. A remark on the components of the electric and magnetic fields

Note that, in accordance with these equations (17), one of the cases where this damping factor disappears is in free space: i.e., a case where, because there are also no electric charges and currents in the medium where the electromagnetic field is defined, these equations (17b)

reduce to

$$\left[\frac{\partial^2}{\partial t^2} + (ck)^2\right] \boldsymbol{\mathcal{E}}\left(\boldsymbol{k}, t\right) = \boldsymbol{0} \quad \text{and} \tag{18a}$$

$$\left[\frac{\partial^2}{\partial t^2} + (ck)^2\right] \boldsymbol{\mathcal{B}}\left(\boldsymbol{k}, t\right) = \boldsymbol{0} \tag{18b}$$

But, before we pay attention to this free space case, whose analysis is crucial in view of the proposal of this paper, it is important to remember that, as $\boldsymbol{E}\left(\boldsymbol{r}, t\right)$ and $\boldsymbol{B}\left(\boldsymbol{r}, t\right)$ describe an electromagnetic field, these last two functions must be real. And remembering this is very significant because this reality condition, being such that

$$\boldsymbol{E}\left(\boldsymbol{r}, t\right) = \boldsymbol{E}^*\left(\boldsymbol{r}, t\right) \quad \text{and} \quad \boldsymbol{B}\left(\boldsymbol{r}, t\right) = \boldsymbol{B}^*\left(\boldsymbol{r}, t\right) , \tag{19}$$

makes it clear that $\boldsymbol{\mathcal{E}}\left(\boldsymbol{k}, t\right)$ and $\boldsymbol{\mathcal{B}}\left(\boldsymbol{k}, t\right)$ need to satisfy some constraints. Note that such constraints can be obtained as long as we observe not only that

$$\boldsymbol{E}^*\left(\boldsymbol{r}, t\right) = \int \boldsymbol{\mathcal{E}}^*\left(\boldsymbol{k}, t\right) e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} d\boldsymbol{k} \quad \text{and} \quad \boldsymbol{B}^*\left(\boldsymbol{r}, t\right) = \int \boldsymbol{\mathcal{B}}^*\left(\boldsymbol{k}, t\right) e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} d\boldsymbol{k} , \tag{20}$$

but also that a change of variable $\boldsymbol{k} \to -\boldsymbol{k}$ leads us to

$$\boldsymbol{E}\left(\boldsymbol{r}, t\right) = -\int \boldsymbol{\mathcal{E}}\left(-\boldsymbol{k}, t\right) e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} d\boldsymbol{k} = \int \boldsymbol{\mathcal{E}}\left(-\boldsymbol{k}, t\right) e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} d\boldsymbol{k} \quad \text{and}$$

$$\boldsymbol{B}\left(\boldsymbol{r}, t\right) = -\int \boldsymbol{\mathcal{B}}\left(-\boldsymbol{k}, t\right) e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} d\boldsymbol{k} = \int \boldsymbol{\mathcal{B}}\left(-\boldsymbol{k}, t\right) e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} d\boldsymbol{k} . \tag{21}$$

After all, by replacing (20) and (21) in the reality condition (19), it is not difficult to conclude that these constraints are

$$\boldsymbol{\mathcal{E}}^*\left(\boldsymbol{k}, t\right) = \boldsymbol{\mathcal{E}}\left(-\boldsymbol{k}, t\right) \quad \text{and} \quad \boldsymbol{\mathcal{B}}^*\left(\boldsymbol{k}, t\right) = \boldsymbol{\mathcal{B}}\left(-\boldsymbol{k}, t\right) . \tag{22}$$

IV.   GETTING A SCHRÖDINGER EQUATION IN FREE SPACE

Given this result (22), it is not difficult to see, for instance, that, with the help of another function $\boldsymbol{\varphi}\left(\boldsymbol{k}, t\right)$, the function $\boldsymbol{\mathcal{E}}\left(\boldsymbol{k}, t\right)$ can be expressed as

$$\boldsymbol{\mathcal{E}}\left(\boldsymbol{k}, t\right) = n\left(k\right)\left[\boldsymbol{\varphi}\left(\boldsymbol{k}, t\right) + \boldsymbol{\varphi}^*\left(-\boldsymbol{k}, t\right)\right] , \tag{23}$$

where $n\left(k\right)$ is a function that only depends on the modulus $k = |\boldsymbol{k}|$. And in order to get a similar expression for $\boldsymbol{\mathcal{B}}\left(\boldsymbol{k}, t\right)$, the first thing we can do is notice, for instance, that the equation (14) leads us to

$$\boldsymbol{\mathcal{B}}\left(\boldsymbol{k}, t\right) = i\boldsymbol{k} \times \left[\frac{\sigma\mu_0}{k^2} + \frac{1}{(ck)^2}\frac{\partial}{\partial t}\right] \boldsymbol{\mathcal{E}}\left(\boldsymbol{k}, t\right) .$$

After all, by noticing that, when $\sigma = 0$, this last result reduces to

$$\boldsymbol{\mathcal{B}}\left(\boldsymbol{k}, t\right) = \frac{i}{\left(ck\right)^2} \, \boldsymbol{k} \times \dot{\boldsymbol{\mathcal{E}}}\left(\boldsymbol{k}, t\right) \ , \tag{24}$$

this is exactly what allows us to conclude that, in free space, $\boldsymbol{\mathcal{B}}\left(\boldsymbol{k}, t\right)$ can be expressed, at least, as a function of the (first order) time derivative of $\boldsymbol{\varphi}\left(\boldsymbol{k}, t\right)$ because (23) shows us that

$$\dot{\boldsymbol{\mathcal{E}}}\left(\boldsymbol{k}, t\right) = n\left(k\right) \left[\dot{\boldsymbol{\varphi}}\left(\boldsymbol{k}, t\right) + \dot{\boldsymbol{\varphi}}^*\left(-\boldsymbol{k}, t\right)\right] \ . \tag{25}$$

Therefore, as the development of (18a) shows us that

$$\left(\frac{\partial}{\partial t} + ick\right) \boldsymbol{\mathcal{E}}\left(\boldsymbol{k}, t\right) = \boldsymbol{0} \quad \text{or} \quad \left(\frac{\partial}{\partial t} - ick\right) \boldsymbol{\mathcal{E}}\left(\boldsymbol{k}, t\right) = \boldsymbol{0} \ ,$$

the expressions (23) and (25) make it clear that solving this equation (18a) means making explicit the relationship that exists between $\boldsymbol{\varphi}\left(\boldsymbol{k}, t\right)$ and $\dot{\boldsymbol{\varphi}}\left(\boldsymbol{k}, t\right)$ in free space: more specifically, a relationship that, in free space, can be chosen, for example, as

$$ck\,\boldsymbol{\varphi}\left(\boldsymbol{k}, t\right) = i\frac{\partial \boldsymbol{\varphi}}{\partial t} \tag{26}$$

which, when substituted in (24), allows us to conclude that

$$\boldsymbol{\mathcal{B}}\left(\boldsymbol{k}, t\right) = \frac{n\left(k\right)}{ck} \, \boldsymbol{k} \times \left[\boldsymbol{\varphi}\left(\boldsymbol{k}, t\right) - \boldsymbol{\varphi}^*\left(-\boldsymbol{k}, t\right)\right] \ . \tag{27}$$

Since the Maxwell's equations in free space reduce to

$$i\boldsymbol{k} \cdot \boldsymbol{\mathcal{E}}\left(\boldsymbol{k}, t\right) = 0 \ , \tag{28a}$$

$$i\boldsymbol{k} \cdot \boldsymbol{\mathcal{B}}\left(\boldsymbol{k}, t\right) = 0 \ , \tag{28b}$$

$$i\boldsymbol{k} \times \boldsymbol{\mathcal{E}}\left(\boldsymbol{k}, t\right) = -\frac{\partial \boldsymbol{\mathcal{B}}}{\partial t} \quad \text{and} \tag{28c}$$

$$i\boldsymbol{k} \times \boldsymbol{\mathcal{B}}\left(\boldsymbol{k}, t\right) = \frac{1}{c^2}\frac{\partial \boldsymbol{\mathcal{E}}}{\partial t} \ , \tag{28d}$$

it is not wrong to say that (26) was obtained by manipulating only the (28c) and (28d), and not the (28a) and (28b). However, by substituting either of these expressions (23) or (27) into Maxwell's equations (28a) and (28b), what we get is

$$\boldsymbol{k} \cdot \boldsymbol{\varphi}\left(\boldsymbol{k}, t\right) = \sum_{\beta=1}^{3} k_\beta \cdot \varphi_\beta\left(\boldsymbol{k}, t\right) = 0 \ , \tag{29}$$

which makes clear that another way to rewrite the Maxwell's equations (2) in free space is through

$$ck\,\boldsymbol{\varphi}\left(\boldsymbol{k}, t\right) = i\frac{\partial \boldsymbol{\varphi}}{\partial t} \quad \text{and} \quad \boldsymbol{k} \cdot \boldsymbol{\varphi}\left(\boldsymbol{k}, t\right) = 0 \ . \tag{30}$$

After all, note that solving this new system (30) means determining the function $\boldsymbol{\varphi}(\boldsymbol{k}, t)$ that defines the functions (23) and (27), and consequently determining the electric $\boldsymbol{E}(\boldsymbol{r}, t)$ and magnetic $\boldsymbol{B}(\boldsymbol{r}, t)$ fields that satisfy Maxwell's equations (2) in free space.

Yet, another thing we need to note here is that, due to the linear independence of the equations in the system (30), this is not the only way to express the Maxwell's equations in free space in terms of $\boldsymbol{\varphi}(\boldsymbol{k}, t)$: another way is by

(i) multiplying the equation (26) by a real constant a,

$$ack\,\boldsymbol{\varphi}(\boldsymbol{k}, t) = ack\left[\sum_{\beta=1}^{3} \varphi_\beta(\boldsymbol{k}, t)\right] = ack\left[\sum_{\alpha=1}^{3}\sum_{\beta=1}^{3} \delta_{\alpha\beta}\varphi_\beta(\boldsymbol{k}, t)\right] = ia\frac{\partial\boldsymbol{\varphi}}{\partial t} \ ,$$

(ii) multiplying the equation (29) by the vector $c\boldsymbol{k}/k$ previously multiplied by the same constant a mentioned in (i),

$$\frac{ac\boldsymbol{k}}{k}\left[\boldsymbol{k}\cdot\boldsymbol{\varphi}(\boldsymbol{k}, t)\right] = \frac{ac}{k}\left(\sum_{\alpha=1}^{3} k_\alpha\right)\left[\sum_{\beta=1}^{3} k_\beta\cdot\varphi_\beta(\boldsymbol{k}, t)\right] = 0 \ , \quad \text{and}$$

(iii) subtracting the result (ii) from the (i),

$$\sum_{\alpha=1}^{3}\sum_{\beta=1}^{3} ack\left(\delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}\right)\varphi_\beta(\boldsymbol{k}, t) = ia\frac{\partial\boldsymbol{\varphi}}{\partial t} \ , \tag{31}$$

And, by taking a as $\hbar$ in the last result, this brings us to a new equation

$$\mathbb{H}\boldsymbol{\varphi}(\boldsymbol{k}, t) = i\hbar\frac{\partial\boldsymbol{\varphi}}{\partial t} \tag{32}$$

that is very peculiar, since $\mathbb{H}\boldsymbol{\varphi}(\boldsymbol{k}, t)$ is a vector whose components are

$$\mathbb{H}\varphi_\alpha(\boldsymbol{k}, t) = \sum_{\beta=1}^{3} \mathbb{H}_{\alpha\beta}\cdot\varphi_\beta(\boldsymbol{k}, t) \ , \quad \text{where} \quad \mathbb{H}_{\alpha\beta} = \hbar ck\left(\delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}\right) \ . \tag{33}$$

In other words, this new development, which has just been described by items (i), (ii) and (iii), is leading us to a new system

$$\mathbb{H}\boldsymbol{\varphi}(\boldsymbol{k}, t) = i\hbar\frac{\partial\boldsymbol{\varphi}}{\partial t} \quad \text{and} \quad \boldsymbol{k}\cdot\boldsymbol{\varphi}(\boldsymbol{k}, t) = 0 \tag{34}$$

that covers the same content as Maxwell's equations in free space, but with one of its equations taking the same form as the Schrödinger equation.

A. Some considerations on the energy

Although it is already quite tempting to claim that (32) is a Schrödinger equation, we still cannot do this without first checking whether, for example, $\mathbb{H}$ can be interpreted as a Hamiltonian operator. And since every Hamiltonian operator measures the energy of the system associated with it, a good way to assess whether $\mathbb{H}$ really can be interpreted as such is by evaluating not only whether there is any relationship between it and the energy

$$\mathbb{E} = \frac{\varepsilon_0}{2} \int \left[\boldsymbol{E}^2 \left(\boldsymbol{r}, t \right) + c^2 \boldsymbol{B}^2 \left(\boldsymbol{r}, t \right) \right] d\boldsymbol{r} \tag{35}$$

of electromagnetic waves that propagate through free space, but also whether this energy (35) leads to a known result, which is the energy of a photon.

One of the things we can do to make this assessment is, for example, to deal with the result

$$\int e^{i(\boldsymbol{k}+\boldsymbol{k}')\boldsymbol{r}} \, d\boldsymbol{r} = (2\pi)^3 \, \delta \left(\boldsymbol{k} + \boldsymbol{k}' \right) \ . \tag{36}$$

After all, when we substitute (7a) and (7b into equation (35), it allows us to observe that

$$\mathbb{E} = \frac{\varepsilon_0}{2} \int \int \left[\boldsymbol{\mathcal{E}} \left(\boldsymbol{k}, t \right) \cdot \boldsymbol{\mathcal{E}} \left(\boldsymbol{k}', t \right) + c^2 \boldsymbol{\mathcal{B}} \left(\boldsymbol{k}, t \right) \cdot \boldsymbol{\mathcal{B}} \left(\boldsymbol{k}', t \right) \right] \int e^{i(\boldsymbol{k}+\boldsymbol{k}')\boldsymbol{r}} \, d\boldsymbol{r} \, d\boldsymbol{k} \, d\boldsymbol{k}'$$

$$= \frac{(2\pi)^3 \, \varepsilon_0}{2} \int \left[\boldsymbol{\mathcal{E}} \left(\boldsymbol{k}, t \right) \cdot \boldsymbol{\mathcal{E}} \left(-\boldsymbol{k}, t \right) + c^2 \boldsymbol{\mathcal{B}} \left(\boldsymbol{k}, t \right) \cdot \boldsymbol{\mathcal{B}} \left(-\boldsymbol{k}, t \right) \right] d\boldsymbol{k} \ . \tag{37}$$

Observe that, with the help of (23), the first term of this integral (37) can be developed as

$$\boldsymbol{\mathcal{E}} \left(\boldsymbol{k}, t \right) \cdot \boldsymbol{\mathcal{E}} \left(-\boldsymbol{k}, t \right) = \left[n \left(k \right) \right]^2 \left[\boldsymbol{\varphi} \left(\boldsymbol{k}, t \right) \cdot \boldsymbol{\varphi} \left(-\boldsymbol{k}, t \right) + \boldsymbol{\varphi} \left(\boldsymbol{k}, t \right) \cdot \boldsymbol{\varphi}^* \left(\boldsymbol{k}, t \right) \right]$$

$$+ \left[n \left(k \right) \right]^2 \left[\boldsymbol{\varphi}^* \left(-\boldsymbol{k}, t \right) \cdot \boldsymbol{\varphi} \left(-\boldsymbol{k}, t \right) + \boldsymbol{\varphi}^* \left(-\boldsymbol{k}, t \right) \cdot \boldsymbol{\varphi}^* \left(\boldsymbol{k}, t \right) \right] \tag{38}$$

and that, with the help of (27), it is not difficult to see that

$$\boldsymbol{\mathcal{B}} \left(\boldsymbol{k}, t \right) \cdot \boldsymbol{\mathcal{B}} \left(-\boldsymbol{k}, t \right) = \frac{1}{\left(ck \right)^4} \left(\boldsymbol{k} \cdot \boldsymbol{k} \right) \left[\dot{\boldsymbol{\mathcal{E}}} \left(\boldsymbol{k}, t \right) \cdot \dot{\boldsymbol{\mathcal{E}}} \left(-\boldsymbol{k}, t \right) \right] + \frac{1}{\left(ck \right)^4} \left[\dot{\boldsymbol{\mathcal{E}}} \left(\boldsymbol{k}, t \right) \cdot \boldsymbol{k} \right] \left[\boldsymbol{k} \cdot \dot{\boldsymbol{\mathcal{E}}} \left(-\boldsymbol{k}, t \right) \right] \ .$$

Thus, given that the equation (28a) is such that

$$\boldsymbol{k} \cdot \boldsymbol{\mathcal{E}} \left(\boldsymbol{k}, t \right) = 0 \ \Rightarrow \ \frac{\partial}{\partial t} \left[\boldsymbol{k} \cdot \boldsymbol{\mathcal{E}} \left(\boldsymbol{k}, t \right) \right] = \boldsymbol{k} \cdot \frac{\partial \boldsymbol{\mathcal{E}}}{\partial t} = 0$$

and, therefore, leads to

$$\boldsymbol{\mathcal{B}} \left(\boldsymbol{k}, t \right) \cdot \boldsymbol{\mathcal{B}} \left(-\boldsymbol{k}, t \right) = \frac{1}{\left(ck \right)^2} \left[\dot{\boldsymbol{\mathcal{E}}} \left(\boldsymbol{k}, t \right) \cdot \dot{\boldsymbol{\mathcal{E}}} \left(-\boldsymbol{k}, t \right) \right]$$

$$= -\frac{\left[n \left(k \right) \right]^2}{c^2} \left[\boldsymbol{\varphi} \left(\boldsymbol{k}, t \right) \cdot \boldsymbol{\varphi} \left(-\boldsymbol{k}, t \right) - \boldsymbol{\varphi} \left(\boldsymbol{k}, t \right) \cdot \boldsymbol{\varphi}^* \left(\boldsymbol{k}, t \right) \right]$$

$$+ \frac{\left[n \left(k \right) \right]^2}{c^2} \left[\boldsymbol{\varphi}^* \left(-\boldsymbol{k}, t \right) \cdot \boldsymbol{\varphi} \left(-\boldsymbol{k}, t \right) - \boldsymbol{\varphi}^* \left(-\boldsymbol{k}, t \right) \cdot \boldsymbol{\varphi}^* \left(\boldsymbol{k}, t \right) \right] \ , \tag{39}$$

it is exactly this last result that, together with (38), shows us that this energy (37) can be rewritten as

$$\mathbb{E} = (2\pi)^3 \, \varepsilon_0 \left[\int [n\,(k)]^2 \, \boldsymbol{\varphi}\,(\boldsymbol{k}, t) \cdot \boldsymbol{\varphi}^*\,(\boldsymbol{k}, t) \;\; d\boldsymbol{k} \; + \int [n\,(k)]^2 \, \boldsymbol{\varphi}^*\,(-\boldsymbol{k}, t) \cdot \boldsymbol{\varphi}\,(-\boldsymbol{k}, t) \;\; d\boldsymbol{k} \right] \; .$$

And why is interesting to obtain such a result? Because, when we perform a change $\boldsymbol{k} \rightarrow -\boldsymbol{k}$ in the last integral, this expression, which we got for $\mathbb{E}$, can be further reduced to

$$\mathbb{E} = 2\,(2\pi)^3 \, \varepsilon_0 \int [n\,(k)]^2 \, \boldsymbol{\varphi}^*\,(\boldsymbol{k}, t) \cdot \boldsymbol{\varphi}\,(\boldsymbol{k}, t) \;\; d\boldsymbol{k} \; , \tag{40}$$

which makes this task of evaluating whether (32) actually corresponds to a Schrödinger equation become easier. And as the equation (26) also helps us to recognize, for instance, that

$$\frac{\partial^2 \boldsymbol{\varphi}}{\partial t^2} = -ck\left(i\frac{\partial \boldsymbol{\varphi}}{\partial t} \right) = -\,(ck)^2 \, \boldsymbol{\varphi}\,(\boldsymbol{k}, t) \;\; \Rightarrow \;\; \frac{\partial^2 \boldsymbol{\varphi}}{\partial t^2} + (ck)^2 \, \boldsymbol{\varphi}\,(\boldsymbol{k}, t) = 0 \; ,$$

this makes it very clear that, in this region, where there are no electric charges or currents, such an evaluation can be performed by considering

$$\boldsymbol{\varphi}\,(\boldsymbol{k}, t) = \boldsymbol{\varphi}_0\,(\boldsymbol{k})\, e^{-i\omega t} \tag{41}$$

as solution of the equation (32), where $\boldsymbol{\varphi}_0\,(\boldsymbol{k}) = \boldsymbol{\varphi}\,(\boldsymbol{k}, 0)$ and $\omega = ck$ is its angular frequency of oscillation[3]. After all, since the substitution of this monochromatic solution (41) in the equation (32) shows us that

$$\mathbb{H}\boldsymbol{\varphi}\,(\boldsymbol{k}, t) = \hbar\omega \; \boldsymbol{\varphi}\,(\boldsymbol{k}, t) \;\; , \tag{42}$$

it is precisely this that makes it quite clear that

$$\int \boldsymbol{\varphi}^*\,(\boldsymbol{k}, t) \cdot [\mathbb{H}\boldsymbol{\varphi}\,(\boldsymbol{k}, t)] \;\; d\boldsymbol{k} = \hbar\omega \int \boldsymbol{\varphi}^*\,(\boldsymbol{k}, t) \cdot \boldsymbol{\varphi}\,(\boldsymbol{k}, t) \;\; d\boldsymbol{k} \; .$$

In plain English, it is precisely this that makes it quite clear that, in order for the right-hand side of this last equality to be identified, for instance, as the energy of a photon, it must be valid that[4]

$$\int \boldsymbol{\varphi}^*\,(\boldsymbol{k}, t) \cdot \boldsymbol{\varphi}\,(\boldsymbol{k}, t) \;\; d\boldsymbol{k} = 1 \; . \tag{43}$$

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[3] Note that dealing with this solution (41) makes perfect sense since, for example, it is fully consistent with the fact that, by taking $\omega\,(\boldsymbol{k}) = \omega$, the expressions (3) and (7) are exactly the same under the identification made in (4).

[4] Note also that, due to (41), this integral has the same value for all values of t. In other words, analogously to what Quantum Mechanics argues, the probability density that the wave function $\boldsymbol{\varphi}\,(\boldsymbol{k}, t)$ determines is independent of time. We will discuss this point, in more detail, in the next Section because it is there that the interpretation of $\boldsymbol{\varphi}^*\,(\boldsymbol{k}, t) \cdot \boldsymbol{\varphi}\,(\boldsymbol{k}, t)$ as a probability density will become more explicit.

In this way, by taking into account the need for this normalization condition (43), it is imperative to conclude that (40) can actually be recognized as the energy of a photon as long as it is valid that

$$n\left(k\right) = \sqrt{\frac{\hbar c k}{2\left(2\pi\right)^3 \varepsilon_0}} \;\; \text{or, equivalently,} \;\; n\left(\omega\right) = \sqrt{\frac{\hbar \omega}{2\left(2\pi\right)^3 \varepsilon_0}} \;.$$

B.   Some considerations on the linear momentum

Although the results obtained in the last Section seem quite satisfactory for the purpose of recognizing (32) as a Schrödinger equation, we still need to evaluate other things to validate such recognition and, therefore, claiming that Maxwell's equations in free space already provide some support for the quantum description of light. And one of those other things that still needs to be evaluated is, for example, the electromagnetic linear momentum

$$\boldsymbol{p} = \varepsilon_0 \int \boldsymbol{E}\left(\boldsymbol{r},t\right) \times \boldsymbol{B}\left(\boldsymbol{r},t\right) \; d\boldsymbol{r} \tag{44}$$

that is associated with this light. After all, if these equations (2) already provide some support for the quantum description of light, it is expected that the development of (44) reveals to us that the relation (5) can actually be interpreted as the same Heisenberg's uncertainty principle (6).

In fact, given all the calculations that we have developed so far, it is valid to say that making such an evaluation is not difficult since, with the help of (36), the development of (44) shows us that

$$\boldsymbol{p} = \varepsilon_0 \int \int \boldsymbol{\mathcal{E}}\left(\boldsymbol{k},t\right) \times \boldsymbol{\mathcal{B}}\left(\boldsymbol{k}',t\right) \int e^{i\left(\boldsymbol{k}+\boldsymbol{k}'\right)\boldsymbol{r}} \; d\boldsymbol{r}\,d\boldsymbol{k}\,d\boldsymbol{k}'$$
$$= \left(2\pi\right)^3 \varepsilon_0 \int \boldsymbol{\mathcal{E}}\left(\boldsymbol{k},t\right) \times \boldsymbol{\mathcal{B}}\left(-\boldsymbol{k},t\right) \; d\boldsymbol{k} \;. \tag{45}$$

Thus, by considering that

$$\boldsymbol{\mathcal{E}}\left(\boldsymbol{k},t\right) \times \boldsymbol{\mathcal{B}}\left(-\boldsymbol{k},t\right) = \boldsymbol{\mathcal{E}}\left(\boldsymbol{k},t\right) \times \left[-\frac{i}{\left(ck\right)^2}\,\boldsymbol{k} \times \dot{\boldsymbol{\mathcal{E}}}\left(-\boldsymbol{k},t\right)\right]$$
$$= -\frac{i}{\left(ck\right)^2} \left\{ \left[\boldsymbol{\mathcal{E}}\left(\boldsymbol{k},t\right) \cdot \dot{\boldsymbol{\mathcal{E}}}\left(-\boldsymbol{k},t\right)\right] \boldsymbol{k} - \underbrace{\left[\boldsymbol{\mathcal{E}}\left(\boldsymbol{k},t\right) \cdot \boldsymbol{k}\right]}_{=0} \dot{\boldsymbol{\mathcal{E}}}\left(-\boldsymbol{k},t\right) \right\}$$
$$= -\frac{i}{\left(ck\right)^2}\,\boldsymbol{k} \left[\boldsymbol{\mathcal{E}}\left(\boldsymbol{k},t\right) \cdot \dot{\boldsymbol{\mathcal{E}}}\left(-\boldsymbol{k},t\right)\right] \;,$$

and that the relations (23), (25) and (26), together, lead us to

$$\boldsymbol{\mathcal{E}}\left(\boldsymbol{k},t\right)\cdot\dot{\boldsymbol{\mathcal{E}}}\left(-\boldsymbol{k},t\right)=-\frac{i}{2}\frac{\hbar\left(ck\right)^{2}}{\left(2\pi\right)^{3}\varepsilon_{0}}\left[\boldsymbol{\varphi}\left(\boldsymbol{k},t\right)\cdot\boldsymbol{\varphi}\left(-\boldsymbol{k},t\right)-\boldsymbol{\varphi}\left(\boldsymbol{k},t\right)\cdot\boldsymbol{\varphi}^{*}\left(\boldsymbol{k},t\right)\right]$$
$$-\frac{i}{2}\frac{\hbar\left(ck\right)^{2}}{\left(2\pi\right)^{3}\varepsilon_{0}}\left[\boldsymbol{\varphi}^{*}\left(-\boldsymbol{k},t\right)\cdot\boldsymbol{\varphi}\left(-\boldsymbol{k},t\right)-\boldsymbol{\varphi}^{*}\left(-\boldsymbol{k},t\right)\cdot\boldsymbol{\varphi}^{*}\left(\boldsymbol{k},t\right)\right]\ ,\ \ (46)$$

it is immediate to observe that (45) reduces to

$$\boldsymbol{p}=-\frac{\hbar}{2}\int\boldsymbol{k}\left[\boldsymbol{\varphi}\left(\boldsymbol{k},t\right)\cdot\boldsymbol{\varphi}\left(-\boldsymbol{k},t\right)-\boldsymbol{\varphi}\left(\boldsymbol{k},t\right)\cdot\boldsymbol{\varphi}^{*}\left(\boldsymbol{k},t\right)\right]\ d\boldsymbol{k}$$
$$-\frac{\hbar}{2}\int\boldsymbol{k}\left[\boldsymbol{\varphi}^{*}\left(-\boldsymbol{k},t\right)\cdot\boldsymbol{\varphi}\left(-\boldsymbol{k},t\right)-\boldsymbol{\varphi}^{*}\left(-\boldsymbol{k},t\right)\cdot\boldsymbol{\varphi}^{*}\left(\boldsymbol{k},t\right)\right]\ d\boldsymbol{k}\ ,\ \ \ \ \ \ (47)$$

and, therefore, to

$$\boldsymbol{p}=\hbar\int\boldsymbol{k}\left[\boldsymbol{\varphi}^{*}\left(\boldsymbol{k},t\right)\cdot\boldsymbol{\varphi}\left(\boldsymbol{k},t\right)\right]\ d\boldsymbol{k}\ \ \ \ \ \ \ (48)$$

after a change $\boldsymbol{k}\rightarrow-\boldsymbol{k}$ in the last integral appearing in (47).

Note that this result (48) that we have just obtained is exactly what we needed to be able to interpret (5) as the Heisenberg's uncertainty principle (6) in this electrodynamic context. After all, as the normalization condition (43) allows us to interpret $\boldsymbol{k}\left[\boldsymbol{\varphi}^{*}\left(\boldsymbol{k},t\right)\cdot\boldsymbol{\varphi}\left(\boldsymbol{k},t\right)\right]$ as the probability density of finding a photon with a wave vector $\boldsymbol{k}$, it is precisely this result (48) that shows us, for example, that $\boldsymbol{p}$ can be interpreted as the expected value for the linear momentum of a photon. In this fashion, and by noting that this proportionality relationship (48) that exists between $\boldsymbol{p}$ and $\boldsymbol{k}$ leads us to

$$\Delta\boldsymbol{p}=\hbar\,\Delta\boldsymbol{k}\ ,\ \ \ \ \ \ \ (49)$$

there is no way not to recognize that there is actually a correspondence, between the Maxwell's equations in free space and this new formulation (34), that is leading us to a quantum interpretation of these same equations.

C.   Some considerations on the angular momentum

It is obvious that everything that we have presented so far raises some new questions, which can also be answered with additional calculations. And, among these new questions, we can point out those that, for example, are related to the spin that is associated with light. After all, as much as Schrodinger obtained his wave equation by exploring a non-relativistic context [41, 42], it is important to note that Maxwell's equations in free space,

and consequently (34), are describing the behaviour of electromagnetic radiation that is relativistic. In this sense, and in order to understand how (34) supports the description of the spin of a photon, the first thing that we need to do is remember that the spin of a particle can always be interpreted as and intrinsic angular momentum. And, as the manipulation of (35) has already shown us that the energy of an electromagnetic light wave, when written in terms of $\varphi\left(\boldsymbol{k}, t\right)$, can be interpreted as the eigenvalue of an operator $\mathbb{H}$ that measures the energy of a photon, it is interesting to evaluate what an analogous manipulation, now of the expression

$$\boldsymbol{M} = \varepsilon_0 \int \boldsymbol{r} \times \left[\boldsymbol{E}\left(\boldsymbol{r}, t\right) \times \boldsymbol{B}\left(\boldsymbol{r}, t\right)\right] d\boldsymbol{r} \tag{50}$$

of the total angular momentum of this same wave, reveals to us.

In fact, by proceeding analogously to what was done in Section IV, it is not difficult to see that the replacement of (7a) and (7b) into this (50) shows us that

$$\boldsymbol{M} = \varepsilon_0 \int \int \int \left[\boldsymbol{r} e^{i(\boldsymbol{k}+\boldsymbol{k}')\boldsymbol{r}}\right] \times \left[\boldsymbol{\mathcal{E}}\left(\boldsymbol{k}, t\right) \times \boldsymbol{\mathcal{B}}\left(\boldsymbol{k}', t\right)\right] \, d\boldsymbol{r} \, d\boldsymbol{k} \, d\boldsymbol{k}' \, . \tag{51}$$

In this way, and by observing that

$$\int \boldsymbol{r} e^{i(\boldsymbol{k}+\boldsymbol{k}')\boldsymbol{r}} d\boldsymbol{r} = -i\boldsymbol{\nabla}_{\boldsymbol{k}'} \int e^{i(\boldsymbol{k}+\boldsymbol{k}')\boldsymbol{r}} d\boldsymbol{r} \, ,$$

we can conclude that

$$\boldsymbol{M} = i\left(2\pi\right)^3 \varepsilon_0 \int \int \delta\left(\boldsymbol{k}+\boldsymbol{k}'\right) \boldsymbol{\nabla}_{\boldsymbol{k}'} \times \left[\boldsymbol{\mathcal{E}}\left(\boldsymbol{k}, t\right) \times \boldsymbol{\mathcal{B}}\left(\boldsymbol{k}', t\right)\right] \, d\boldsymbol{k} \, d\boldsymbol{k}' \tag{52}$$

because, according to the Stoke's [49],

$$\int \int \boldsymbol{\nabla}_{\boldsymbol{k}'} \times \left[\delta\left(\boldsymbol{k}+\boldsymbol{k}'\right) \, \boldsymbol{\mathcal{E}}\left(\boldsymbol{k}, t\right) \times \boldsymbol{\mathcal{B}}\left(\boldsymbol{k}', t\right)\right] \, d\boldsymbol{k} \, d\boldsymbol{k}' = 0 \, .$$

1. Some considerations on the integrand of (52)

By the way, it is interesting to note that the expression (24) allows us to see that

$$\boldsymbol{\mathcal{E}}\left(\boldsymbol{k}, t\right) \times \boldsymbol{\mathcal{B}}\left(\boldsymbol{k}', t\right) = \boldsymbol{\mathcal{E}}\left(\boldsymbol{k}, t\right) \times \left[\frac{i}{\left(ck'\right)^2} \, \boldsymbol{k}' \times \dot{\boldsymbol{\mathcal{E}}}\left(\boldsymbol{k}', t\right)\right]$$

$$= \frac{i}{\left(ck'\right)^2} \left\{\left[\boldsymbol{\mathcal{E}}\left(\boldsymbol{k}, t\right) \cdot \dot{\boldsymbol{\mathcal{E}}}\left(\boldsymbol{k}', t\right)\right] \boldsymbol{k}' - \left[\boldsymbol{\mathcal{E}}\left(\boldsymbol{k}, t\right) \cdot \boldsymbol{k}'\right] \dot{\boldsymbol{\mathcal{E}}}\left(\boldsymbol{k}', t\right)\right\} \, ,$$

from which it is clear, for instance, that

$$\boldsymbol{\nabla}_{\boldsymbol{k}'} \times [\boldsymbol{\mathcal{E}}(\boldsymbol{k},t) \times \boldsymbol{\mathcal{B}}(\boldsymbol{k}',t)]$$
$$= \frac{i}{(ck')^2} \left\{ \left[\boldsymbol{\mathcal{E}}(\boldsymbol{k},t) \cdot \dot{\boldsymbol{\mathcal{E}}}(\boldsymbol{k}',t) \right] \boldsymbol{\nabla}_{\boldsymbol{k}'} \times \boldsymbol{k}' + \boldsymbol{\nabla}_{\boldsymbol{k}'} \left[\boldsymbol{\mathcal{E}}(\boldsymbol{k},t) \cdot \dot{\boldsymbol{\mathcal{E}}}(\boldsymbol{k}',t) \right] \times \boldsymbol{k}' \right\}$$
$$- \frac{i}{(ck')^2} \left\{ [\boldsymbol{\mathcal{E}}(\boldsymbol{k},t) \cdot \boldsymbol{k}'] \boldsymbol{\nabla}_{\boldsymbol{k}'} \times \dot{\boldsymbol{\mathcal{E}}}(\boldsymbol{k}',t) + \boldsymbol{\nabla}_{\boldsymbol{k}'} [\boldsymbol{\mathcal{E}}(\boldsymbol{k},t) \cdot \boldsymbol{k}'] \times \dot{\boldsymbol{\mathcal{E}}}(\boldsymbol{k}',t) \right\} \ .$$

In this way, and by taking into account

$$\boldsymbol{\nabla}_{\boldsymbol{k}'} \times \boldsymbol{k}' = 0 \quad \text{and} \quad \boldsymbol{\nabla}_{\boldsymbol{k}'} [\boldsymbol{\mathcal{E}}(\boldsymbol{k},t) \cdot \boldsymbol{k}'] = \boldsymbol{\mathcal{E}}(\boldsymbol{k},t) \ ,$$

it is not difficult to observe that

$$\boldsymbol{\nabla}_{\boldsymbol{k}'} \times [\boldsymbol{\mathcal{E}}(\boldsymbol{k},t) \times \boldsymbol{\mathcal{B}}(\boldsymbol{k}',t)]$$
$$= -\frac{i}{(ck')^2} \boldsymbol{k}' \times \boldsymbol{\nabla}_{\boldsymbol{k}'} \left[\boldsymbol{\mathcal{E}}(\boldsymbol{k},t) \cdot \dot{\boldsymbol{\mathcal{E}}}(\boldsymbol{k}',t) \right]$$
$$- \frac{i}{(ck')^2} [\boldsymbol{\mathcal{E}}(\boldsymbol{k},t) \cdot \boldsymbol{k}'] \boldsymbol{\nabla}_{\boldsymbol{k}'} \times \dot{\boldsymbol{\mathcal{E}}}(\boldsymbol{k}',t) - \frac{i}{(ck')^2} \boldsymbol{\mathcal{E}}(\boldsymbol{k},t) \times \dot{\boldsymbol{\mathcal{E}}}(\boldsymbol{k}',t)$$

and, wherefore, that

$$\boldsymbol{M} = \frac{(2\pi)^3 \varepsilon_0}{c^2} \int \int \delta(\boldsymbol{k}+\boldsymbol{k}') \frac{\boldsymbol{k}'}{(k')^2} \times \boldsymbol{\nabla}_{\boldsymbol{k}'} \left[\boldsymbol{\mathcal{E}}(\boldsymbol{k},t) \cdot \dot{\boldsymbol{\mathcal{E}}}(\boldsymbol{k}',t) \right] \ d\boldsymbol{k} \ d\boldsymbol{k}'$$
$$+ \frac{(2\pi)^3 \varepsilon_0}{c^2} \int \int \delta(\boldsymbol{k}+\boldsymbol{k}') \frac{\boldsymbol{\mathcal{E}}(\boldsymbol{k},t) \cdot \boldsymbol{k}'}{(k')^2} \boldsymbol{\nabla}_{\boldsymbol{k}'} \times \dot{\boldsymbol{\mathcal{E}}}(\boldsymbol{k}',t) \ d\boldsymbol{k} \ d\boldsymbol{k}'$$
$$+ \frac{(2\pi)^3 \varepsilon_0}{c^2} \int \int \delta(\boldsymbol{k}+\boldsymbol{k}') \frac{\boldsymbol{\mathcal{E}}(\boldsymbol{k},t) \times \dot{\boldsymbol{\mathcal{E}}}(\boldsymbol{k}',t)}{(k')^2} \ d\boldsymbol{k} \ d\boldsymbol{k}' \ .$$

For this reason, and by noting that equation (28a) also allows us to conclude that

$$\boldsymbol{\mathcal{E}}(\boldsymbol{k},t) \cdot \boldsymbol{k}'|_{\boldsymbol{k}'=-\boldsymbol{k}} = -\boldsymbol{\mathcal{E}}(\boldsymbol{k},t) \cdot \boldsymbol{k} = 0 \ ,$$

it becomes clear that such total angular momentum can be reduced as

$$\boldsymbol{M} = \boldsymbol{M}' + \boldsymbol{M}'' \ , \tag{53}$$

where

$$\boldsymbol{M}' = \frac{(2\pi)^3 \varepsilon_0}{(ck)^2} \int \boldsymbol{k} \times \overline{\boldsymbol{\nabla}}_{\boldsymbol{k}} \left[\boldsymbol{\mathcal{E}}(\boldsymbol{k},t) \cdot \underline{\dot{\boldsymbol{\mathcal{E}}}}(-\boldsymbol{k},t) \right] \ d\boldsymbol{k}$$

and

$$\boldsymbol{M}'' = \frac{(2\pi)^3 \varepsilon_0}{(ck)^2} \int \boldsymbol{\mathcal{E}}(\boldsymbol{k},t) \times \dot{\boldsymbol{\mathcal{E}}}(-\boldsymbol{k},t) \ d\boldsymbol{k} \ .$$

Here, the bars in $\overline{\boldsymbol{\nabla}}_{\boldsymbol{k}}$ and $\underline{\dot{\boldsymbol{\mathcal{E}}}}(-\boldsymbol{k},t)$ only serve to highlight the fact that $\boldsymbol{\nabla}_{\boldsymbol{k}}$ acts only on the function $\dot{\boldsymbol{\mathcal{E}}}(-\boldsymbol{k},t)$. That is, these bars indicate that, in carrying out the differentiation, $\boldsymbol{\mathcal{E}}(\boldsymbol{k},t)$ should be regarded constant.

2. Some considerations on $\boldsymbol{M}'$ and $\boldsymbol{M}''$

In view of (46) allows us to get

$$
\boldsymbol{k} \times \overline{\boldsymbol{\nabla}}_{\boldsymbol{k}} \left[\boldsymbol{\mathcal{E}} \left(\boldsymbol{k}, t \right) \cdot \underline{\dot{\boldsymbol{\mathcal{E}}}} \left(-\boldsymbol{k}, t \right) \right]
$$

$$
= -i \, \frac{\hbar \left(ck \right)^2}{2 \left(2\pi \right)^3 \varepsilon_0} \, \boldsymbol{k} \times \overline{\boldsymbol{\nabla}}_{\boldsymbol{k}} \left[\boldsymbol{\varphi} \left(\boldsymbol{k}, t \right) \cdot \underline{\boldsymbol{\varphi}} \left(-\boldsymbol{k}, t \right) - \boldsymbol{\varphi} \left(\boldsymbol{k}, t \right) \cdot \underline{\boldsymbol{\varphi}}^* \left(\boldsymbol{k}, t \right) \right]
$$

$$
- i \, \frac{\hbar \left(ck \right)^2}{2 \left(2\pi \right)^3 \varepsilon_0} \, \boldsymbol{k} \times \overline{\boldsymbol{\nabla}}_{\boldsymbol{k}} \left[\boldsymbol{\varphi}^* \left(-\boldsymbol{k}, t \right) \cdot \underline{\boldsymbol{\varphi}} \left(-\boldsymbol{k}, t \right) - \boldsymbol{\varphi}^* \left(-\boldsymbol{k}, t \right) \cdot \underline{\boldsymbol{\varphi}}^* \left(\boldsymbol{k}, t \right) \right] ,
$$

the development of $\boldsymbol{M}'$ is not difficult. After all, since

$$
\int \boldsymbol{k} \times \overline{\boldsymbol{\nabla}}_{\boldsymbol{k}} \left[\boldsymbol{\varphi} \left(\boldsymbol{k}, t \right) \cdot \underline{\boldsymbol{\varphi}}^* \left(\boldsymbol{k}, t \right) \right] \, d\boldsymbol{k} = \int \boldsymbol{k} \times \overline{\boldsymbol{\nabla}}_{\boldsymbol{k}} \left[\boldsymbol{\varphi} \left(-\boldsymbol{k}, t \right) \cdot \underline{\boldsymbol{\varphi}}^* \left(-\boldsymbol{k}, t \right) \right] \, d\boldsymbol{k}
$$

$$
= - \int \boldsymbol{k} \times \overline{\boldsymbol{\nabla}}_{\boldsymbol{k}} \left[\underline{\boldsymbol{\varphi}}^* \left(\boldsymbol{k}, t \right) \cdot \boldsymbol{\varphi} \left(\boldsymbol{k}, t \right) \right] \, d\boldsymbol{k} , \quad \text{and}
$$

$$
\int \boldsymbol{k} \times \overline{\boldsymbol{\nabla}}_{\boldsymbol{k}} \left[\boldsymbol{\varphi} \left(\boldsymbol{k}, t \right) \cdot \underline{\boldsymbol{\varphi}}^* \left(-\boldsymbol{k}, t \right) \right] \, d\boldsymbol{k} = \int \boldsymbol{k} \times \overline{\boldsymbol{\nabla}}_{\boldsymbol{k}} \left[\boldsymbol{\varphi} \left(-\boldsymbol{k}, t \right) \cdot \underline{\boldsymbol{\varphi}}^* \left(\boldsymbol{k}, t \right) \right] \, d\boldsymbol{k}
$$

$$
= - \int \boldsymbol{k} \times \overline{\boldsymbol{\nabla}}_{\boldsymbol{k}} \left[\underline{\boldsymbol{\varphi}}^* \left(-\boldsymbol{k}, t \right) \cdot \boldsymbol{\varphi} \left(\boldsymbol{k}, t \right) \right] \, d\boldsymbol{k}
$$

are valid relations, the first therm of (53) can be reduced as

$$
\boldsymbol{M}' = -i\hbar \int \boldsymbol{k} \times \overline{\boldsymbol{\nabla}}_{\boldsymbol{k}} \left[\underline{\boldsymbol{\varphi}}^* \left(\boldsymbol{k}, t \right) \cdot \boldsymbol{\varphi} \left(\boldsymbol{k}, t \right) \right] \, d\boldsymbol{k} . \tag{54}
$$

Now, in order to develop the second therm of (53), we can note, for instance, that the same expressions (23), (25) and (26), taken together, leads us to

$$
\boldsymbol{\mathcal{E}} \left(\boldsymbol{k}, t \right) \times \dot{\boldsymbol{\mathcal{E}}} \left(-\boldsymbol{k}, t \right) = -\frac{i}{2} \, \frac{\hbar \left(ck \right)^2}{\left(2\pi \right)^3 \varepsilon_0} \, \left[\boldsymbol{\varphi} \left(\boldsymbol{k}, t \right) \times \boldsymbol{\varphi} \left(-\boldsymbol{k}, t \right) - \boldsymbol{\varphi} \left(\boldsymbol{k}, t \right) \times \boldsymbol{\varphi}^* \left(\boldsymbol{k}, t \right) \right]
$$

$$
- \frac{i}{2} \, \frac{\hbar \left(ck \right)^2}{\left(2\pi \right)^3 \varepsilon_0} \, \left[\boldsymbol{\varphi}^* \left(-\boldsymbol{k}, t \right) \times \boldsymbol{\varphi} \left(-\boldsymbol{k}, t \right) - \boldsymbol{\varphi}^* \left(-\boldsymbol{k}, t \right) \times \boldsymbol{\varphi}^* \left(\boldsymbol{k}, t \right) \right] .
$$

After all, by taking a change of variable $\boldsymbol{k} \to -\boldsymbol{k}$, we get

$$
\int \boldsymbol{\mathcal{E}} \left(\boldsymbol{k}, t \right) \times \dot{\boldsymbol{\mathcal{E}}} \left(-\boldsymbol{k}, t \right) \, d\boldsymbol{k} = \frac{i}{2} \, \frac{\hbar \left(ck \right)^2}{\left(2\pi \right)^3 \varepsilon_0} \int \left[\boldsymbol{\varphi} \left(\boldsymbol{k}, t \right) \times \boldsymbol{\varphi}^* \left(\boldsymbol{k}, t \right) - \boldsymbol{\varphi}^* \left(\boldsymbol{k}, t \right) \times \boldsymbol{\varphi} \left(\boldsymbol{k}, t \right) \right] \, d\boldsymbol{k}
$$

$$
= -i \, \frac{\hbar \left(ck \right)^2}{\left(2\pi \right)^3 \varepsilon_0} \int \boldsymbol{\varphi}^* \left(\boldsymbol{k}, t \right) \times \boldsymbol{\varphi} \left(\boldsymbol{k}, t \right) \, d\boldsymbol{k} ,
$$

which shows us that this second term of (53) can be rewritten as

$$
\boldsymbol{M}'' = -i\hbar \int \boldsymbol{\varphi}^* \left(\boldsymbol{k}, t \right) \times \boldsymbol{\varphi} \left(\boldsymbol{k}, t \right) \, d\boldsymbol{k} . \tag{55}
$$

3. Getting the expression for the total angular momentum operator

Here, it is worth noting that, since $\boldsymbol{\varphi}\left(\boldsymbol{k},t\right) = \left(\varphi_1\left(\boldsymbol{k},t\right), \varphi_2\left(\boldsymbol{k},t\right), \varphi_3\left(\boldsymbol{k},t\right)\right)$, the fact of $\boldsymbol{\varphi}^*\left(\boldsymbol{k},t\right)\cdot\boldsymbol{\varphi}\left(\boldsymbol{k},t\right)$ is a dot product allows us to recognize that the components of the first term of (53) are

$$\mathbb{M}'_\alpha = -i\hbar \int \left(\boldsymbol{k}\times\overline{\boldsymbol{\nabla}}_{\boldsymbol{k}}\right)_\alpha \left[\sum_{\beta=1}^{3}\underline{\varphi}^*_\beta\left(\boldsymbol{k},t\right)\cdot\varphi_\beta\left(\boldsymbol{k},t\right)\right]\,d\boldsymbol{k}$$

$$= -i\hbar \int \sum_{\beta=1}^{3}\varphi^*_\beta\left(\boldsymbol{k},t\right)\cdot\left[\left(\boldsymbol{k}\times\boldsymbol{\nabla}_{\boldsymbol{k}}\right)_\alpha\varphi_\beta\left(\boldsymbol{k},t\right)\right]\,d\boldsymbol{k}\,. \tag{56}$$

In this fashion, as $\boldsymbol{\varphi}^*\left(\boldsymbol{k},t\right)\times\boldsymbol{\varphi}\left(\boldsymbol{k},t\right)$ is a cross product, whose components are given by

$$\left[\boldsymbol{\varphi}^*\left(\boldsymbol{k},t\right)\times\boldsymbol{\varphi}\left(\boldsymbol{k},t\right)\right]_\alpha = \sum_{\beta=1}^{3}\sum_{\gamma=1}^{3}\epsilon_{\alpha\beta\gamma}\,\varphi^*_\beta\left(\boldsymbol{k},t\right)\cdot\varphi_\gamma\left(\boldsymbol{k},t\right) \tag{57}$$

with $\epsilon_{\alpha\beta\gamma}$ being the Levi-Civita symbol, it is immediate to conclude that the components of (53) can be perfectly recognized as

$$\mathbb{M}_\alpha = \int \sum_{\beta=1}^{3}\varphi^*_\beta\left(\boldsymbol{k},t\right)\cdot\left[-i\hbar\,\left(\boldsymbol{k}\times\boldsymbol{\nabla}_{\boldsymbol{k}}\right)_\alpha\varphi_\beta\left(\boldsymbol{k},t\right) - i\hbar\sum_{\gamma=1}^{3}\epsilon_{\alpha\beta\gamma}\,\varphi_\gamma\left(\boldsymbol{k},t\right)\right]\,d\boldsymbol{k}\,. \tag{58}$$

However, it is also interesting to observe that, due to the antisymmetry of $\epsilon_{\alpha\beta\gamma}$, it becomes perfectly feasible to obtain the same result (57) with the help of an operator $\boldsymbol{S} = \left(\mathbb{S}_1, \mathbb{S}_2, \mathbb{S}_3\right)$ that, for instance, is such that

$$\mathbb{S}_\alpha\varphi_\beta\left(\boldsymbol{k},t\right) = -\mathbb{S}_\beta\varphi_\alpha\left(\boldsymbol{k},t\right) = -i\hbar\sum_{\gamma=1}^{3}\epsilon_{\alpha\beta\gamma}\,\varphi_\gamma\left(\boldsymbol{k},t\right)\,. \tag{59}$$

After all, since replacing this last result in (57) allows us to recognize that

$$-i\hbar\,\left[\boldsymbol{\varphi}^*\left(\boldsymbol{k},t\right)\times\boldsymbol{\varphi}\left(\boldsymbol{k},t\right)\right]_\alpha = \sum_{\beta=1}^{3}\varphi^*_\beta\left(\boldsymbol{k},t\right)\cdot\left[\mathbb{S}_\alpha\varphi_\beta\left(\boldsymbol{k},t\right)\right]\,, \tag{60}$$

it is precisely this that, when replaced in (58), leads us to

$$\mathbb{M}_\alpha = \int \sum_{\beta=1}^{3}\varphi^*_\beta\left(\boldsymbol{k},t\right)\cdot\left[\mathbb{J}_\alpha\varphi_\beta\left(\boldsymbol{k},t\right)\right]\,d\boldsymbol{k}\,,$$

where $\mathbb{J}_\alpha = \left[-i\hbar\,\left(\boldsymbol{k}\times\boldsymbol{\nabla}_{\boldsymbol{k}}\right) + \boldsymbol{S}\right]_\alpha = -i\hbar\,\left(\boldsymbol{k}\times\boldsymbol{\nabla}_{\boldsymbol{k}}\right)_\alpha + \mathbb{S}_\alpha$.

Something that is quite clear about this last result is that it assumes a form quite similar to the one we obtained in (48) for the linear momentum $\boldsymbol{p}$. That is, once again, after we do

some algebraic manipulations, we are faced with a classical result hat is also very familiar from the point of view of Quantum Mechanics. After all, just as (48) is equivalent to

$$\boldsymbol{p} = \int \sum_{\alpha=1}^{3} \varphi_{\alpha}^{*}\left(\boldsymbol{k}, t\right) \cdot \left[\boldsymbol{P}\,\varphi_{\alpha}\left(\boldsymbol{k}, t\right)\right]\, d\boldsymbol{k}$$

as long as relation

$$\boldsymbol{P}\,\boldsymbol{\varphi}\left(\boldsymbol{k}, t\right) = \hbar\boldsymbol{k}\,\boldsymbol{\varphi}\left(\boldsymbol{k}, t\right) \tag{61}$$

holds, the same can also be said about $\boldsymbol{J}$ since the result

$$\boldsymbol{M} = \int \sum_{\alpha=1}^{3} \varphi_{\alpha}^{*}\left(\boldsymbol{k}, t\right) \cdot \left[\boldsymbol{J}\,\varphi_{\alpha}\left(\boldsymbol{k}, t\right)\right]\, d\boldsymbol{k} \tag{62}$$

can also be seen as the average value of the action of operator $\boldsymbol{J}$ on the wave function.

However, in order to better understand this connection between what is classical and what "seems" to be quantum in this electrodynamic context, it is important to observe, for example, that the result of the action of $\boldsymbol{J}$ on the function $\boldsymbol{\varphi}\left(\boldsymbol{k}, t\right)$ is identical to the result of the action of the total angular momentum operator. By the way, note that its first term

$$\boldsymbol{L} = -i\hbar\ \left(\boldsymbol{k} \times \boldsymbol{\nabla_{k}}\right) \tag{63}$$

has exactly the same expression of the orbital angular momentum of the Quantum Mechanics when expressed in momentum space [50]. And since the action of $\boldsymbol{L} = \left(\mathbb{L}_1, \mathbb{L}_2, \mathbb{L}_3\right)$ is, for instance, such that

$$\begin{aligned}
\boldsymbol{L}^2\ Y_{\ell m}\left(\boldsymbol{\theta}\right) &= \ell\left(\ell+1\right)\hbar^2\ Y_{\ell m_\ell}\left(\boldsymbol{\theta}\right) \quad \text{and} \\
\mathbb{L}_3\ Y_{\ell m_\ell}\left(\boldsymbol{\theta}\right) &= m_\ell\hbar\ Y_{\ell m_\ell}\left(\boldsymbol{\theta}\right)\ ,
\end{aligned} \tag{64}$$

where $Y_{\ell m_\ell}\left(\boldsymbol{\theta}\right)$ are spherical harmonic functions, it is not wrong to state that the action of $\boldsymbol{L}$ is restricted, here, only to the parameter $\boldsymbol{\theta} = \left(\theta_1, \theta_2\right)$ whose unit vector is perpendicular to $\boldsymbol{k}$ [5]. After all, when we notice that the components of $\boldsymbol{k}$ take the form

$$\begin{aligned}
k_1 &= \kappa\left(\sin\theta_1\right)\left(\cos\theta_2\right)\ , & (65) \\
k_2 &= \kappa\left(\sin\theta_1\right)\left(\sin\theta_2\right) \quad \text{and} & (66) \\
k_3 &= \kappa\left(\cos\theta_1\right) & (67)
\end{aligned}$$

---

[5] Although it is usual to denote the spherical harmonic functions as $Y_{\ell m_\ell}\left(\theta, \phi\right)$ (since the polar and azimuthal angles are popularly represented by the Greek letters θ and ϕ respectively), we preferred to write them as $Y_{\ell m_\ell}\left(\theta_1, \theta_2\right)$ for the sake of a better understanding. After all, since we have already denoted an arbitrary infinitesimal rotation as $\delta\boldsymbol{\theta} = \left(\delta\theta_1, \delta\theta_2\right)$, we needed to make it clear to you, the reader, that such a rotation is performed over the same subspace where such functions are defined.

when expressed by using spherical coordinates $(\kappa, \boldsymbol{\theta})$, it is precisely this that, when analysed from the perspective of the relations contained in (64), allows us to recognize, for example, that [51]

$$\varphi_\alpha\left(\boldsymbol{k}, t\right) = \sum_{\ell=0}^{\infty} \sum_{m_\ell=-\ell}^{m_\ell=\ell} f_{\alpha\ell}\left(\kappa, t\right) Y_{\ell m_\ell}\left(\boldsymbol{\theta}\right)$$

because the two indices ℓ and m_ℓ, which appear in (64), are two integer numbers such that $-\ell \leqslant m_\ell \leqslant \ell$.

Of course, in order to ratify the interpretation of $\boldsymbol{J}$ as the operator that measures the total angular momentum of a photon, we still need to demonstrate that $\boldsymbol{S}$ is the operator that measures the spin of a photon. And, for us to be able to do this, we need to observe that, unlike what happens with $\boldsymbol{L}$, the result (59) is showing us that each component of $\mathbb{S}_\alpha$ acts, in a specific way, on the components of $\boldsymbol{\varphi}\left(\boldsymbol{k}, t\right)$. In fact, since the general expression of $\boldsymbol{J}$ reinforces that $\boldsymbol{S}$ also needs to be interpreted vectorially, it is precisely this result (59) that allows us to conclude, among other things, that the action of $\boldsymbol{S}$ on $\boldsymbol{\varphi}\left(\boldsymbol{k}, t\right)$ can be represented with the help of matrices

$$\mathbb{S}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i\hbar \\ 0 & i\hbar & 0 \end{pmatrix}, \quad \mathbb{S}_2 = \begin{pmatrix} 0 & 0 & i\hbar \\ 0 & 0 & 0 \\ -i\hbar & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbb{S}_3 = \begin{pmatrix} 0 & -i\hbar & 0 \\ i\hbar & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{68}$$

as long as the matrix representation

$$\boldsymbol{\varphi}\left(\boldsymbol{k}, t\right) = \begin{pmatrix} \varphi_1\left(\boldsymbol{k}, t\right) \\ \varphi_2\left(\boldsymbol{k}, t\right) \\ \varphi_3\left(\boldsymbol{k}, t\right) \end{pmatrix} \tag{69}$$

is also valid. Note that one of the useful consequences of dealing with this representation (68) is that, with it, it is not difficult to see that all components $\mathbb{S}_\alpha$ have three distinct eigenvalues, which are equal to $-\hbar$, 0 and $+\hbar$. As a consequence, another thing that is not difficult to notice is that, in the case of component $\mathbb{S}_3$, its three normalized eigenstates, and respectively associated with these three eigenvalues, can be expressed as

$$\boldsymbol{s}_{-\hbar}\left(\boldsymbol{k}, t\right) = \frac{1}{\sqrt{2}}\left[\varphi_1\left(\boldsymbol{k}, t\right)\right]\boldsymbol{e}_1 - \frac{i}{\sqrt{2}}\left[\varphi_2\left(\boldsymbol{k}, t\right)\right]\boldsymbol{e}_2, \tag{70}$$

$$\boldsymbol{s}_0\left(\boldsymbol{k}, t\right) = \left[\varphi_1\left(\boldsymbol{k}, t\right)\right]\boldsymbol{e}_3 \quad \text{and} \tag{71}$$

$$\boldsymbol{s}_{+\hbar}\left(\boldsymbol{k}, t\right) = -\frac{1}{\sqrt{2}}\left[\varphi_1\left(\boldsymbol{k}, t\right)\right]\boldsymbol{e}_1 - \frac{i}{\sqrt{2}}\left[\varphi_2\left(\boldsymbol{k}, t\right)\right]\boldsymbol{e}_2. \tag{72}$$

Here,

$$\boldsymbol{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \;,\quad \boldsymbol{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \;,\quad \boldsymbol{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{73}$$

are the matrix representations of the unit vectors that serve as the basis not only for (69), but also for, for example,

$$\boldsymbol{k} = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} \;. \tag{74}$$

Because of not only the expressions of these eigenstates, but also the orthonormality in (73), another thing that becomes clear here is that the two eigenstates $\boldsymbol{s}_{-\hbar}(\boldsymbol{k}, t)$ and $\boldsymbol{s}_{+\hbar}(\boldsymbol{k}, t)$ are actually describing two independent polarizations for a photon: more specifically, while $\boldsymbol{s}_{-\hbar}(\boldsymbol{k}, t)$ describes a left-handed circular polarization, $\boldsymbol{s}_{+\hbar}(\boldsymbol{k}, t)$ is describing a right-handed one. And when we also remember that, for such a wave function $\boldsymbol{\varphi}(\boldsymbol{k}, t)$, the relation (29) must also be valid, this is what explains why, for instance, nothing similar can be associated with the eigenstate $\boldsymbol{s}_0(\boldsymbol{k}, t)$. After all, as the relation (29) only exists because $\boldsymbol{k}$, by defining the direction of linear momentum, is always orthogonal to the components of the electromagnetic field that satisfy (10), the matrix representation (74) makes it quite clear that

- the only components that describe such a field in this basis (73) (and, therefore, lead to two independent circular polarizations) are $\varphi_1(\boldsymbol{k}, t)$ and $\varphi_2(\boldsymbol{k}, t)$, and

- the $\varphi_3(\boldsymbol{k}, t)$ is not circularly polarizable in this basis (73) because, in this basis, this component needs to be zero since, due to the condition (29), it has the same direction as $\boldsymbol{k} = (0, 0, k_3) = (0, 0, k)$.

Note that, given this last comment, it becomes valid to identify the unit vector $\boldsymbol{e}_3$ as $\boldsymbol{k}/k$. And one of the direct consequences of such identification is that, as it leads us to

$$\boldsymbol{e}_{1,\alpha} \cdot \boldsymbol{e}_{1,\beta} + \boldsymbol{e}_{2,\alpha} \cdot \boldsymbol{e}_{2,\beta} = \delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2} \;,$$

the comparison of this new result with (33) makes clear that

$$\mathbb{H}_{\alpha\beta} = \hbar c k \sum_{\lambda=1,2} \boldsymbol{e}_{\lambda,\alpha} \cdot \boldsymbol{e}_{\lambda,\beta} \;.$$

In other words, this last result is showing us, in a very clear way, that the energy of a photon, whose movement is in the $\boldsymbol{k}$ direction, is due solely and exclusively to the $\varphi_1\left(\boldsymbol{k}, t\right)$ and $\varphi_2\left(\boldsymbol{k}, t\right)$ components of the wave function that describes it.

By the way, and by exploring this same context, it is also quite curious to note that the eigenvalues $-\hbar$ and $+\hbar$, which are respectively associated with $s_{-\hbar}\left(\boldsymbol{k}, t\right)$ and $s_{+\hbar}\left(\boldsymbol{k}, t\right)$, are exactly the same as those associated with the helicity (i.e., the projection of the spin) of a photon (on the direction of its linear momentum $\boldsymbol{p}$). And why is it quite curious to note this? Because it is precisely this that reinforces, even more, the consequent interpretation of $\boldsymbol{S}$ as the operator that measures the spin of a photon since, for instance, these three eigenvalues of $\mathbb{S}_3$ also satisfy a relation that is quite analogous to (64). After all, as the result

$$\boldsymbol{S}^2 = \left(\mathbb{S}_1\right)^2 + \left(\mathbb{S}_2\right)^2 + \left(\mathbb{S}_3\right)^2 = \begin{pmatrix} 2\hbar^2 & 0 & 0 \\ 0 & 2\hbar^2 & 0 \\ 0 & 0 & 2\hbar^2 \end{pmatrix}$$

also makes clear that the three eigenvalues of $\boldsymbol{S}^2$ are all equal to $2\hbar^2$, it is not at all difficult to recognize that this complete analogy, between

$$\boldsymbol{S}^2\, \boldsymbol{\varphi}\left(\boldsymbol{k}, t\right) = s\left(s+1\right)\hbar^2\, \boldsymbol{\varphi}\left(\boldsymbol{k}, t\right) \quad \text{and}$$

$$\mathbb{S}_3\, \boldsymbol{\varphi}\left(\boldsymbol{k}, t\right) = m_s\hbar\, \boldsymbol{\varphi}\left(\boldsymbol{k}, t\right)\ , \tag{75}$$

and all the results contained in (64), as long as it is valid that $s = 1$ since this guarantees that all possible integer values for m_s are actually such that $-s \leqslant m_s \leqslant s$. In other words, even though we have made purely algebraic manipulations by taking, as a starting point, the expression for angular momentum (50) that comes from Maxwell's electromagnetic theory, it is quite clear that we have obtained results that allow us to recognize

$$\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$$

as the operator that measures the total angular momentum of a photon. And a very simple way to prove this is by noting that, since the operators $\boldsymbol{L}$ and $\boldsymbol{S}$ act on parameters of disjoint subspaces, these operators are necessarily such that

$$\left[\boldsymbol{L}^2, \boldsymbol{S}^2\right] = \left[\mathbb{L}_\alpha, \boldsymbol{S}^2\right] = \left[\boldsymbol{L}^2, \mathbb{S}_\beta\right] = \left[\mathbb{L}_\alpha, \mathbb{S}_\beta\right] = 0\ .$$

After all, since the expressions in (63) and (68) allows to conclude that

$$\left[\mathbb{L}_\alpha, \mathbb{L}_\beta\right] = i\hbar\epsilon_{\alpha\beta\gamma}\mathbb{L}_\gamma \quad \text{and} \quad \left[\mathbb{S}_\alpha, \mathbb{S}_\beta\right] = i\hbar\epsilon_{\alpha\beta\gamma}\mathbb{S}_\gamma\ ,$$

it is precisely this that allows us to demonstrate the analogous result

$$[\mathbb{J}_\alpha, \mathbb{J}_\beta] = i\hbar\epsilon_{\alpha\beta\gamma}\mathbb{J}_\gamma$$

that, for example, leads us to

$$\boldsymbol{J}^2 \, \boldsymbol{\varphi}(\boldsymbol{k}, t) \;=\; j\,(j+1)\,\hbar^2 \, \boldsymbol{\varphi}(\boldsymbol{k}, t) \quad \text{and}$$
$$\mathbb{J}_3 \, \boldsymbol{\varphi}(\boldsymbol{k}, t) \;=\; m_j\,\hbar\,\boldsymbol{\varphi}(\boldsymbol{k}, t) \;, \tag{76}$$

where $-j \leqslant m_j \leqslant j$.

V.   REMARKS ON THE FOUNDATIONS OF QUANTUM THEORIES

As interesting as it is to evaluate what the more general Maxwell's equations (2) (i.e., Maxwell's equations where the four-current density $(c\rho(\boldsymbol{r}, t), \boldsymbol{J}(\boldsymbol{r}, t))$ is not necessarily (a) zero (vector)) have to tell us when we perform algebraic manipulations analogous to those in the last Section, it is more important to pay attention, first, to a crucial fact. And what is this crucial fact? That, even though Maxwell formulated his electromagnetic field theory without any quantum concerns, it is impossible not to recognize that all the results we have just presented are endorsing that this electromagnetic field theory in free space is defining an example of quantum theory: more specifically, all the results we have just presented, by analysing the electromagnetic field theory in free space, are defining the quantum mechanics of the photon.

Of course, due to the algebraic manipulations we made in items **(i)** and **(ii)**, a more attentive reader could counter-argue what we just said in the last paragraph. After all, since these algebraic manipulations depended on multiplications by the same real constant a, which was later taken as $\hbar$, it gives the impression that the quantum nature of Maxwell's equations in free space only becomes apparent due to a dirty trick. And, at first glance, we cannot even disagree with this counter-argument because, by performing these multiplications, it really does seem that we are performing a procedure analogous to what we do when, for example, we want to tune a radio receiver in to station that plays our favourite songs. In other words, it gives the impression that there is a very specific and artificial way of "tuning" the quantum mechanics of the photon through Maxwell's equations in free space and, as this way is by choosing the "radio station" $a = \hbar$, this seems to be a big dirty trick

because, at the time Maxwell formulated his electromagnetic field theory, no one had the slightest knowledge of what this constant $\hbar$.

However, as relevant as this counter-argument may be, it is also very important to note that, although $\hbar$ has a very specific value in the International System of Units, this system of units is not the only one that exists. And one of the other systems of units that exist is, for example, the Lorentz-Heaviside system where, among other things, both the speed of light c and this same Planck constant $\hbar$ are reduced to the same dimensionless unitary constant for theoretical purposes [52]. Therefore, even though it may seem that the multiplications, described in items **(i)** and **(ii)**, were made to force the "tuning" of this Quantum Mechanics underlying Maxwell's equations in free space, what this new observation reveals to us is that, in fact, such multiplications were not even necessary for this purpose. In plain English, all these mathematical developments are showing us that the electromagnetic field theory, which was developed by Maxwell in the 19th Century, was already "tuned", by definition, to a Quantum Mechanics that would still be developed in the 20th Century. After all, note that all the mathematical developments presented above, which led us to an equation whose form is, today, credited to Schrödinger, were made using only resources that were already known at the beginning of the 19th Century.

Given what we have just said, you, the reader, may be feeling a little uncomfortable. After all, how to justify, in a more fundamental way, the fact that we arrived, for instance, at this equation (32), which has the same form and content as the wave equation that Schrödinger would get about twenty years after Einstein's quantum interpretation of light? That is, why do Maxwell's equations in free space, when rewritten in the form (32), have features that are so familiar to a Quantum Mechanics that would only be formulated almost half a century after these equations? And the best answer that we can give to these questions can be summed up in a single term: **correspondence principle**.

A. Some words on the correspondence principle

Although this principle became popularly known only due to the works of Niels Bohr (1885 – 1962) in the advent of Quantum Mechanics, it is not wrong to say that it was always respected by all those who helped to build the most diverse physical theories. By the way, it is also not wrong to say that this principle only received its name, in 1920,

because, contrary to what happened before the 17th Century, the language of Physics in the 20th Century was already extremely mathematical. After all, within this mathematical scenario where Quantum Mechanics was being born, Paul Dirac (1902 – 1984) also noted that this predicate "correspondence" also had to be necessarily related to the mathematical correspondence

$$\mathcal{F} : U_A \to V_B \subset U_B \tag{77}$$

that had to exist between two physical theories: in the case, between a theory A, which was already well established due to several experiments and (theoretical) considerations (as, for example, Classical Mechanics was), and another B that was beginning to be proposed/constructed (as, for example, Quantum Mechanics was). By the way, due to the work not only of Bohr, but mainly of Dirac (who, for instance, improved the definition of this correspondence principle a little further, by modelling quantum mechanical systems in correspondence with the classical Hamiltonian systems) [53], it became clear that this correspondence $\mathcal{F}$ had to be established so that [54]

- its domain U_A contained all the functions that already describe a physical system in the context of the theory A, and

- its codomain U_B contained all the new functions that describe this same physical system in the context of the theory B.

Just as a matter of curiosity, it is interesting to note that, since $\mathcal{F}$ is a mathematical correspondence that relates two sets containing functions, some current research has been dedicated to investigating whether this correspondence principle can be, for example, interpreted from the point of view of category theory [54]. But what is most important to note here is that, as every new theory B always seeks to answer the questions that A does not yet provide, the image V_B does not necessarily identify with U_B. However, as the new theory B cannot contradict the correct predictions of A, this correspondence needs to be established so that these theories give the same results when some limits are considered [55].

Note that this definition, which is the most basic and general that we can give to this correspondence principle, is the one that manages to justify, for example, why:

- the results of Special Relativity are reduced to those of non-relativistic Mechanics (Newtonian Physics) when the speed of light is taken as infinity [56]; and

- Statistical Mechanics is able to reproduce all the results of Thermodynamics when the number of particles in a system is supposed to be infinite [57].

However, as much as these two examples seem to show that these correspondences exist between two physical theories in different contexts, it is worth noting that this difference is not always necessary. And a good example where this difference is not necessary is evident in the three laws that Isaac Newton (1643 – 1727) chose to base his Mechanics. After all, if Newton had not thought it reasonable that his Mechanics should respect this correspondence principle, he would never have, for instance, incorporated the Galileo Galilei's (1564 – 1642) inertia principle as the first of these laws [58–60]. Therefore, it is valid to say that this correspondence, which predicates the name of this principle, may be identified as an inclusion map in some cases: i.e., it may be identified as a correspondence

$$\mathcal{F} : U_A \hookrightarrow V_B \ , \tag{78}$$

where domain and image are exactly the same set. Note that another example, where this inclusion relation (in fact, where an equivalence relation) is very clear, is the one that justifies why the functions of the Hamiltonian Mechanics can be obtained by means of a Legendre transformation on the functions of the Lagrangian Mechanics [61].

1. When classical and quantum interact

Given what we have just said about the concept of the correspondence principle, it is not wrong to say that this is precisely what helps to justify one of the reasons behind the quantum aspects of Maxwell's equations in free space. How? By noting, for instance, that the foundation of the photoelectric effect lies in the interaction (of photons) of light with (the electrons that compose) matter. After all, in addition to requiring that the corpuscular description of light needed to not clash with Maxwell's electromagnetic theory, this photoelectric effect also makes it clear that this quantum theory (which would only be fully formulated about 20 years after the Einstein's proposal of the photon concept) needed to be able to describe the interactions between classical and quantum systems. And although this last requirement makes a lot of sense from the experimental point of view (because all the experimental devices are composed of matter), this same requirement also shows that this

correspondence, when established between classical and quantum physical theories, needs to make the first theories can be interpreted as "special cases" of the second ones [62].

Of course, by analysing the whole process of creating a physical theory, it is very unfair to say that any physical theory A can be interpreted as a special case of another B, especially when they describe physical systems in different contexts. After all, when someone says something like that, it sounds like someone is arguing that B is better than A. And how is it possible to argue that B is better than A in a situation where, for instance, B is Quantum Mechanics and A is Classical Mechanics formulated by Newton? Could, for instance, Einstein, Planck, Dirac, Schrödinger and Heisenberg have been better scientists than Newton and, therefore, helped to formulate a better Quantum Mechanics than Classical Mechanics? In other words, it is not fair to say such a thing, especially knowing that any new theory is only formulated due to the needs of the occasion.

Note that another thing that reinforces that it is not fair to say that Quantum Mechanics is better than Classical Mechanics is, for instance, the simple fact that it is precisely the latter that manages to describe, quite accurately, most of the movements that happen in our daily lives. By the way, if it were not for the concepts that were created in the classical context (such as position, momentum, energy etc.), it would not even make sense to talk about these same concepts in the quantum context. However, the fact is that, due to this need for classical and quantum systems to interact with each other, there is this "gap" (which is not wrong, but that is unfair) that allows some researches to claim, with the support of this correspondence principle, that Quantum Mechanics "brings" all of Classical Mechanics as a special case. And since all matter is composed of electrons, protons and neutrons, this is precisely what also allows us to assert that any physical theory, which is able to describe electromagnetic fields quantum-mechanically, needs to "bring" all of Maxwell's electromagnetic theory as a special case.

In fact, and in view of this conclusion (that, in order to carry out any measurement in Quantum Mechanics, it is necessary that there are interactions between classical and quantum systems), each of these interactions, including those that are never observed, was raised to the post of **measurement** in all the quantum theories [62]. And, as strange as this (new) concept of measurement may seem, its reasonableness is already quite evident in the photoelectric phenomenon. After all, as different sources of light are always illuminating some places of the world/Universe, there is nothing that, for instance, prevents the most

diverse photoelectric effects that may be occurring in places where there may not be a nearby observer. In this fashion, as this same comment also extends to all neutrinos that are always passing through us, to the cosmic rays that are "tearing" the Universe apart and interacting with the most diverse objects etc. [63], it is not wrong to say that all measurements in quantum theories are analogous to those in a crazy bingo game where, although there is someone drawing (i.e., measuring) all the numbers (which can be repeated) and reading them aloud, no player (i.e., observer) is participating in it. In other words, since we cannot control all the quantum interactions that are always taking place in the Universe, all these measurements are always being released by nature regardless of whether there is any observer. And just like in this crazy bingo game, writing down the values (of part) of these quantum measurements is just a matter of luck, opportunity and (experimental) preparation for it.

B.   Some words on the postulates of Quantum Mechanics

In fact, since we have just talked about the fact that interactions were raised to the post of measurements with the advent of Quantum Mechanics (i.e., since we have just talked about something that was of utmost importance for the foundation of Quantum Mechanics), it is also important to draw attention to the fact that all the results, which we obtained in the previous Section, already satisfy all the postulates that were chosen to structure Quantum Mechanics [62]. After all, observe that, besides expression (32) already making it clear that $\varphi(\boldsymbol{k}, t)$ is a solution of a Schrödinger equation whose Hamiltonian operator takes us to the eigenenergies of a photon, result (43) also points to the fact that this function is actually capable of describing the behavior of this photon as long as it is, for example, square integrable: i.e., the function $\varphi(\boldsymbol{k}, t)$ actually describes the physical state of a photon as long as it belongs to an example of Hilbert space [64, 65]. Furthermore, observe that (42), (61), (64), (75) and (76) also make it quite clear that [51]

- each of the energy and (linear, orbital, spin, and total angular) momentum observables is associated with a Hermitian operator, and

- $\varphi(\boldsymbol{k}, t)$ can be interpreted as an eigenfunction of the operators in (42), (61), (64), (75) and (76).

In this way, by noting that result (43) allows us to interpret $\boldsymbol{\varphi}^*\left(\boldsymbol{k}, t\right) \cdot \boldsymbol{\varphi}\left(\boldsymbol{k}, t\right)$ as a probability density [66] that, for example, is also in expressions (40), (48) and (62), it is not difficult to recognize these same expressions as averages and that, therefore, the entire Quantum Mechanics of a photon is already, in fact, contained in Maxwell's equations in free space.

1.   A few more words on the correspondence principle

Given what we have just said, it is impossible not to recognize that all the results of the last Section make the interpretation of the correspondence principle even clearer along the lines we have already given above: i.e., as the mathematical correspondence (77) that needs to be established between two physical theories A and B. In fact, when we recognize theory A as Maxwell's electromagnetic theory in free space and theory B as the Quantum Mechanics of a photon, there is no way not to recognize that all the results above are, in fact, telling us that this mathematical correspondence can be identified as an inclusion map (78). In other words, this is something that reinforces, even further, the comment we made above about the fact that it is not wrong to say that theory B brings theory A as a special case.

By the way, since this identification could be made with the help of Fourier transformations, it is interesting to highlight that it was precisely the work with Fourier series (i.e., with an approximation of Fourier transforms for periodic functions [67]) that led Heisenberg, Max Born (1881 – 1970) and Pascual Jordan (1902 – 1980) to obtain the matrix mechanics that, for example, was able to justify the emission spectrum of the hydrogen atom [68–70]. However, since all the results we have obtained for the quantum mechanics of the photon in free space belong to a wave context, it seems much more interesting to pay attention to the considerations that led Schrödinger to his wave mechanics. After all, since Schrödinger obtained his wave equation by exploiting the Hamilton-Jacobi formalism [41, 42, 71], under

- the assumption that Hamilton's principal function S was identical to the phase of this wave, and

- the requirement that the linear momentum, of the non-relativistic particle he wanted to treat as a wave, had to be perpendicular to S at all instants of time,

it is interesting to note that, since Hamilton's principal function associated with the elec-

tromagnetic field theory in free space is, indeed, identical to the phase

$$S\left(\boldsymbol{r}, \boldsymbol{k}, t\right) = \exp\left(\boldsymbol{k} \cdot \boldsymbol{r} \pm \omega t + \phi\right) = \exp\left(\frac{\left(\hbar \boldsymbol{k}\right) \cdot \boldsymbol{r}}{\hbar} \pm \frac{\left(\hbar \omega\right) t}{\hbar} + \phi\right)$$

of the electromagnetic field in free space, it is not at all difficult to see that $\boldsymbol{p}$ is already perpendicular to this S because

$$\boldsymbol{\nabla}_r S = S\boldsymbol{p}/\hbar \ .$$

That is, while Schrödinger needed to impose that the linear momentum, of the non-relativistic particle he wanted to treat as a wave, was equal to the gradient of Hamilton's principal function so that the wave-particle duality could be established by his wave equation, in Maxwell's electromagnetic theory in free space this is not even necessary.

VI. FINAL REMARKS

As important as it is to make further algebraic developments in order to reveal the quantum aspects that underlie Maxwell's equations where $\left(c\rho\left(\boldsymbol{r}, t\right), \boldsymbol{J}\left(\boldsymbol{r}, t\right)\right) \neq \left(0, \boldsymbol{0}\right)$, we will stop here and delegate such discussion to a future paper. After all, besides not wanting to make this paper even longer than it already is, what we have presented above makes it quite clear how Maxwell's electromagnetic theory in free space and the correspondence principle already pointed, together, to the basis of a Quantum Mechanics that was only formulated more than half a century after Maxwell's work.

However, it is interesting to note that, according to the differential equations (17), there already exists a situation that does not seem to be difficult to analyse: the one where the four-current density is defined by $\rho\left(\boldsymbol{r}, t\right) = 0$, but where $\boldsymbol{J}\left(\boldsymbol{r}, t\right)$ satisfies the ohmic condition (16). And in order to understand why this is, it is enough to see that, in addition to equations (10a) and (10b) continuing to guarantee that $\boldsymbol{\mathcal{E}}\left(\boldsymbol{k}, t\right)$, $\boldsymbol{\mathcal{B}}\left(\boldsymbol{k}, t\right)$ and $\boldsymbol{k}$ maintain the same perpendicularity relations as in free space, it also continues to be perfectly possible to deal with an expression

$$\boldsymbol{\mathcal{E}}\left(\boldsymbol{k}, t\right) = n\left(k\right)\left[\boldsymbol{\varphi}\left(\boldsymbol{k}, t\right) + \boldsymbol{\varphi}^{*}\left(-\boldsymbol{k}, t\right)\right]$$

that, when substituted into (17a), makes it very clear that

$$\left[\frac{\partial^2}{\partial t^2} + \gamma \frac{\partial}{\partial t} + \omega^2\right] \boldsymbol{\varphi}\left(\boldsymbol{k}, t\right) = 0 \ . \tag{79}$$

Regardless of what the other results in Section III can show us about the expression that defines $\boldsymbol{\mathcal{B}}(\boldsymbol{k}, t)$, it is interesting to highlight what follows, for example, from the use of slowly varying envelope approximation [72] on this last equation. After all, when we analyse a situation where the function that solves (79) can be expressed as

$$\boldsymbol{\varphi}(\boldsymbol{k}, t) = \boldsymbol{\Upsilon}(\boldsymbol{k}, t) e^{-\overline{\omega}t},$$

with the help of an envelope $\boldsymbol{\Upsilon}(\boldsymbol{k}, t)$ whose average angular frequency $\overline{\omega}$ that characterizes it is such that

$$\left| \frac{\partial^2 \boldsymbol{\Upsilon}}{\partial t^2} \right| \ll \overline{\omega} \left| \frac{\partial \boldsymbol{\Upsilon}}{\partial t} \right|,$$

it is not difficult to conclude that this leads us to

$$(-2i\overline{\omega} + \gamma) \frac{\partial \boldsymbol{\Upsilon}}{\partial t} + \left[\left(\omega^2 - \overline{\omega}^2 \right) - i\gamma\overline{\omega} \right] \boldsymbol{\Upsilon}(\boldsymbol{k}, t) \approx \boldsymbol{0}$$

and, therefore, to

$$\frac{1}{2} \left(\frac{\omega^2 - \overline{\omega}^2}{\overline{\omega}} - i\gamma \right) \boldsymbol{\Upsilon}(\boldsymbol{k}, t) \approx i \frac{\partial \boldsymbol{\Upsilon}}{\partial t} \tag{80}$$

in the special cases where $\gamma \ll \overline{\omega}$. In this fashion, and even though we have already said that we will only analyse the quantum aspects that underlie Maxwell's equations where $(c\rho(\boldsymbol{r}, t), \boldsymbol{J}(\boldsymbol{r}, t)) \neq (0, \boldsymbol{0})$ in a future paper, there is no way we can close this present paper without recognizing that, when we multiply both sides of equation (80) by the same constant $\hbar$, this leads us to an equation

$$\frac{\hbar}{2} \left(\frac{\omega^2 - \overline{\omega}^2}{\overline{\omega}} \right) \boldsymbol{\Upsilon}(\boldsymbol{k}, t) - \frac{i\hbar\gamma}{2} \boldsymbol{\Upsilon}(\boldsymbol{k}, t) \approx i\hbar \frac{\partial \boldsymbol{\Upsilon}}{\partial t}$$

that, again, is very similar to Schrödinger's, but which contains a damping factor.

VII.   ACKNOWLEDGEMENTS

And, among them, the one we find most interesting at the moment is Ref. [73], which we actually used as a guide to write some parts of this paper, especially its Section IV C.

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