Last Lecture

- CPB Hamiltonian in the charge basis

$$\hat{H} = 4E_c \sum_n (n-n_g)^2 |n\rangle\langle n| - \frac{E_J}{2} \sum_n \left(|n+1\rangle\langle n| + |n\rangle\langle n+1| \right)$$

- $E_c \gtrsim E_J \longrightarrow$ charge qubits
- $E_J \gg E_c \longrightarrow$ transmon qubits

- It can be approximated to

$$\hat{H} \simeq \hbar \omega_q \hat{b}^\dagger \hat{b} - \frac{E_c}{2} \hat{b}^\dagger \hat{b}^\dagger \hat{b} \hat{b} \qquad \hbar\omega_q = \sqrt{8E_cE_J} - E_c$$

$$\alpha \cong -\frac{E_c}{\hbar}$$

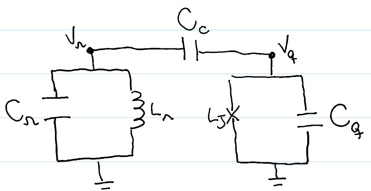
- Or to a TLS

$$\hat{H} \cong \frac{\hbar \omega_q}{2} \hat{\sigma}_z$$

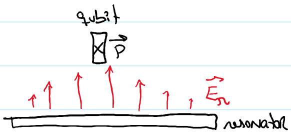
Today's lecture: how do they interact?

Two perspectives

1) Circuit picture



2) Dipolar interaction picture



We will follow 1) today

We look for the interaction Hamiltonian

$$\frac{Q_c}{C_c} = V_q - V_r = \frac{Q_q}{C_q} - \frac{Q_r}{C_r}$$

So, the energy term of the coupling capacitor is

$$\frac{Q_c^2}{2C_c} = \frac{C_c}{2}(V_q - V_r)^2 = \frac{C_c}{2}\left(\frac{Q_q^2}{C_q^2} - \frac{2Q_qQ_r}{C_qC_r} + \frac{Q_r^2}{C_r^2} \right)$$

$$= \frac{C_cQ_q^2}{2C_q^2} + \frac{C_cQ_r^2}{2C_r^2} - \frac{C_c}{C_qC_r} Q_q Q_r$$

Plugging into a total circuit Hamiltonian

$$H = H_r + H_q + H_c = \frac{Q_r^2}{2C_r} + \frac{\Phi_r^2}{2L} + \frac{Q_q^2}{2C_q} - E_J\cos\varphi + \frac{Q_c^2}{2C_c}$$

$$= \frac{Q_r^2}{2C_r}\left(1 + \frac{C_c}{C_r}\right) + \frac{\Phi_r^2}{2L} + \frac{Q_q^2}{2C_q}\left(1 + \frac{C_c}{C_q}\right) - E_J\cos\varphi - \frac{C_c}{C_q C_r} Q_q Q_r \quad *$$

Therefore, we have two terms that we have already explored, but now with rescaled capacitances C_k^{eff} (that lead to rescaled ω_k) plus an interaction term

$$* \rightarrow \hat{H} = \underbrace{\hbar\omega_r \hat{a}_r^+ \hat{a}_r}_{\text{from class 3}} + \underbrace{\hbar\omega_q \hat{a}_q^+ \hat{a}_q}_{\text{from class 4}} - \frac{C_c}{C_q C_r}\hat{Q}_q \hat{Q}_r$$

where we used

$$\hat{\Phi}_k = \Phi_{ZPF,k}(\hat{a}_k^+ + \hat{a}_k) \qquad \Phi_{ZPF,k} = \sqrt{\frac{\hbar}{2C_k^{eff}\omega_k}}$$

$$** \quad \hat{Q}_k = i Q_{ZPF,k}(\hat{a}_k^+ - \hat{a}_k) \qquad Q_{ZPF,k} = \sqrt{\frac{\hbar C_k^{eff}\omega_k}{2}}$$

$$\omega_k = \frac{1}{\sqrt{L_k C_k^{eff}}}$$

We can also use ** into the interaction term

$$\hat{H}_{int} = -\frac{C_c}{C_q C_r}\hat{Q}_q\hat{Q}_r = -\frac{C_c}{C_q C_r}\left[i Q_{ZPF,q}(\hat{a}_q^+ - \hat{a}_q)\right]\left[i Q_{ZPF,r}(\hat{a}_r^+ - \hat{a}_r)\right]$$

$$= \frac{C_c}{C_q C_r} Q_{ZPF,q} Q_{ZPF,r}\left[\hat{a}_q^+\hat{a}_r^+ - \hat{a}_q\hat{a}_r^+ - \hat{a}_q^+\hat{a}_r + \hat{a}_q\hat{a}_r\right]$$

We define a coupling constant

$$\hbar g \equiv \frac{C_c}{C_q C_r} Q_{ZPF,q} Q_{ZPF,r}$$

If $g \ll \omega_k$, the interaction is a small perturbation and terms that do not conserve energy in the bare Hamiltonian (no interaction) can be ignored. This is called the rotating wave approximation. So

$$\hat{H}_{int} = \hbar g(\hat{a}_r^+\hat{a}_q + \hat{a}_r\hat{a}_q^+)$$

Beam-splitter Hamiltonian

Finally, if we truncate the qubit to the two first levels, the total $\hat{H}$ is

$$\boxed{\hat{H}_{JC} = \hbar\omega_n \hat{a}_n^{+}\hat{a}_n + \frac{\hbar\omega_q}{2}\hat{\sigma}_z + \hbar g\left(\hat{a}_n\hat{\sigma}_+ + \hat{a}_n^{+}\hat{\sigma}_-\right)}$$

Jaynes - Cummings Hamiltonian

Now, we can look for the energy levels of $\hat{H}_{JC}$ $\rightarrow$ diagonalization.
We use the composite basis $\{|g,n\rangle, |e,n\rangle\}$, for $n=0,1,2\ldots$

Defining the total excitation number operator $\hat{N} = \hat{a}^{+}\hat{a} + \hat{\sigma}_+\hat{\sigma}_-$,
we see that

$$[\hat{H}, \hat{N}] = 0$$

Therefore, $\hat{N}$ is conserved under a unitary evolution $\hat{U}(t) = e^{-i\hat{H}t/\hbar}$.
This means, that states are always related by pairs

$$|g,1\rangle \longleftrightarrow |e,0\rangle \qquad \text{and} \qquad |g,0\rangle \text{ static}$$
$$|g,2\rangle \longleftrightarrow |e,1\rangle$$

Our Hamiltonian will then be block diagonal

$$\hat{H} = \begin{pmatrix} H_0 & & & 0 \\ & \boxed{H_1} & & \\ & & \boxed{H_2} & \\ 0 & & & \ddots \end{pmatrix}$$

Let's check the terms of each block

- Diagonal terms

$$\langle e,n|\hat{H}|e,n\rangle \qquad\qquad \langle g,n+1|\hat{H}|g,n+1\rangle$$

$$\begin{aligned} \hat{a}^{+}\hat{a}|e,n\rangle &= n|e,n\rangle \\ \hat{\sigma}_z|e,n\rangle &= |e,n\rangle \\ \hat{a}\hat{\sigma}_+|e,n\rangle &= 0 \\ \hat{a}^{+}\hat{\sigma}_-|e,n\rangle &= \sqrt{n+1}\,|g,n+1\rangle \end{aligned} \qquad\qquad \begin{aligned} \hat{a}^{+}\hat{a}|g,n+1\rangle &= (n+1)|g,n+1\rangle \\ \hat{\sigma}_z|g,n+1\rangle &= -|g,n+1\rangle \\ \hat{a}\hat{\sigma}_+|g,n+1\rangle &= |e,n\rangle \\ \hat{a}^{+}\hat{\sigma}_-|g,n+1\rangle &= 0 \end{aligned}$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\langle e,n|\hat{H}|e,n\rangle = \hbar\left(\omega_n n + \frac{\omega_q}{2}\right) \qquad \langle g,n+1|\hat{H}|g,n+1\rangle = \hbar\left[\omega_n(n+1) - \frac{\omega_q}{2}\right]$$

- Off-diagonal terms (of the block matrix)

$$\langle e,n|\hat{H}|g,n+1\rangle$$
only $\hat{a}\hat{\sigma}_+$ contributes

$$\hbar g \hat{a}\hat{\sigma}_+ |g, n+1\rangle = \hbar g \hat{a} |e, n+1\rangle = \hbar g \sqrt{n+1}\, |e, n\rangle$$

and

$$\langle g, n+1| \hat{H} |e, n\rangle$$

only $\hat{a}^\dagger \hat{\sigma}_-$

$$\hbar g \hat{a}^\dagger \hat{\sigma}_- |e, n\rangle = \hbar g \sqrt{n+1}\, |g, n+1\rangle$$

Then, each block matrix will be

$$\hat{H}_n = \begin{pmatrix} \hbar \omega_r n + \frac{\hbar \omega_q}{2} & \hbar g \sqrt{n+1} \\ \hbar g \sqrt{n+1} & \hbar \omega_r (n+1) - \frac{\hbar \omega_q}{2} \end{pmatrix}, \quad \text{for } n = 1, 2, \ldots$$

We can subtract a constant term $\hbar \omega_r (n + 1/2)$ from the diagonal and make $\Delta \equiv \omega_q - \omega_r$, so we have

$$\hat{H}_n' = \begin{pmatrix} \frac{\hbar \Delta}{2} & \hbar g \sqrt{n+1} \\ \hbar g \sqrt{n+1} & -\frac{\hbar \Delta}{2} \end{pmatrix}$$

So, If we diagonalize the block matrices

$$\det \begin{pmatrix} \Delta/2 - \lambda & g\sqrt{n+1} \\ g\sqrt{n+1} & -\Delta/2 - \lambda \end{pmatrix} = 0$$

$$\lambda^2 - \frac{\Delta^2}{4} - g^2(n+1) = 0 \quad \rightarrow \quad \lambda_\pm = \pm \frac{\sqrt{4g^2(n+1) + \Delta^2}}{2} = \pm \frac{\Omega_n}{2}$$

Then, the unshifted energies are

$$E_{n,\pm} = \hbar \omega_r \left(n + \frac{1}{2} \right) \pm \frac{\hbar \Omega_n}{2}$$

Finally, the eigenvectors are

$$\begin{pmatrix} \Delta/2 - \Omega_n/2 & g\sqrt{n+1} \\ g\sqrt{n+1} & -\Delta/2 - \Omega_n/2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0$$

Since $|\alpha|^2 + |\beta|^2 = 1$, it is common to define an angle θ_n such that $\alpha = \cos\theta_n$ and $\beta = \sin\theta_n$. So, the eigenvectors will be

$$|n,+\rangle = \cos\theta_n |e,n\rangle + \sin\theta_n |g,n+1\rangle$$

$$|n,-\rangle = -\sin\theta_n |e,n\rangle + \cos\theta_n |g,n+1\rangle$$

with $\tan(2\theta_n) = \dfrac{2g\sqrt{n+1}}{\Delta}$

Therefore, the states oscillate between $|e,n\rangle \longleftrightarrow |g,n+1\rangle$. These are called Rabi oscillations. Their frequencies are Ω_n, called Rabi frequencies.

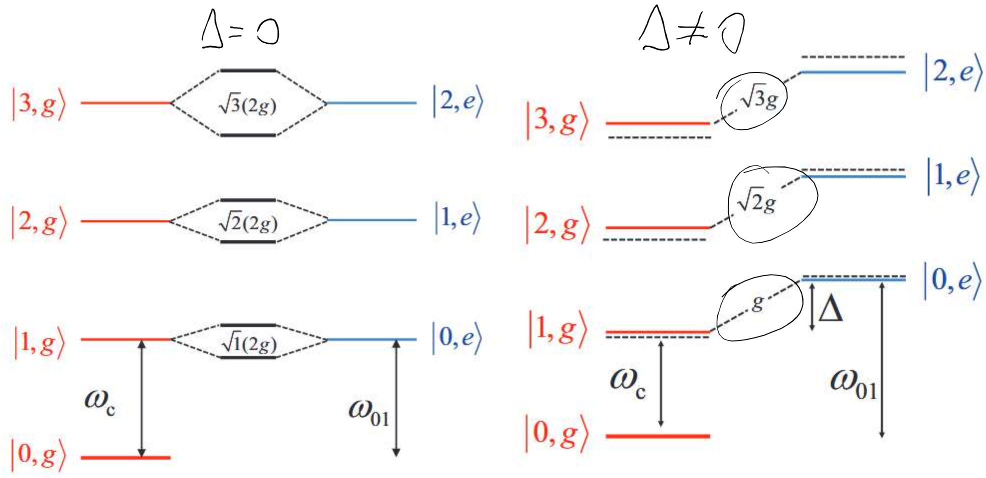


Fig. 6.2 Jaynes-Cummings ladder or 'dressed-atom' level structure. (left panel) Degenerate case $\omega_{01} = \omega_c$. The degenerate levels mix and split by an amount proportional to the vacuum Rabi splitting g. (right panel) Dispersive case $\omega_{01} = \omega_c + \Delta$. For $\Delta > 0$ the level repulsion causes the cavity frequency to decrease when the qubit is in the ground state and increase when the qubit is in the excited state.