Class 1

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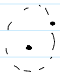
Quantum Systems

A quantum system is a physical system for which at least one of the degrees of freedom (DOF) can only be described using the laws of QM

Ex: Hydrogen atom



All DOF of the system need of QM to be described
- energy levels
- spin
- probability of finding the electron at position $\vec{r}$

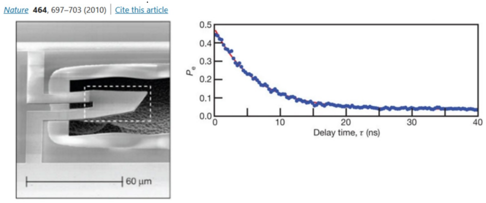
Ex 2:

DOF: mechanical motion

$f \sim 10\,Hz$ $\quad T \sim 300K$

$\bar{n} = \dfrac{1}{e^{hf/k_BT}-1}$ $\quad \bar{n}(10Hz, 300K) \sim 6.25 \times 10^{11}$

Ex 3:



Quantum ground state and single-phonon control of a mechanical resonator

A. D. O'Connell, M. Hofheinz, M. Ansmann, Radoslaw C. Bialczak, M. Lenander, Erik Lucero, M. Neeley, D. Sank, H. Wang, M. Weides, J. Wenner, John M. Martinis & A. N. Cleland

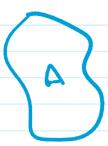
Nature **464**, 697–703 (2010) | Cite this article

$f \sim 6\,GHz$ $\quad T \sim 25\,mK$

$\bar{n} \cong 10^{-5}$ (ground state)

QM must be used

Quantum state representations



We can describe the state of an isolated quantum system by using a complex vector space with an inner product. This space is called a Hilbert space

A state of the system will be fully described by a vector in this space.

state of quantum system $\longrightarrow \psi, |\psi\rangle$
↳(ket)

Ex: If we have a binary $\{|0\rangle, \{1\rangle\}$

$|\psi\rangle = a|0\rangle + b|1\rangle$ $\quad$ a, b are complex numbers

A conventional way to operate these state is describing them by column vectors

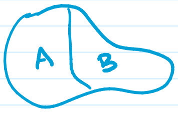
$$|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix} \quad ; \quad |0\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |1\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and we can look at the complex conjugate as a row vector

$\langle\psi| = (a^* \ b^*)$ $\implies$ inner product $\langle\psi|\psi\rangle = |a|^2 + |b|^2 \equiv 1$

(bra)↵

Composite states



Suppose we have a Hilbert space such that $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, with A and B subspaces.

Some states of $\mathcal{H}$ can be represented as

$$|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle = |\psi_A, \psi_B\rangle = |\psi_A \psi_B\rangle$$

Ex 1: $|\psi_A\rangle = a|0\rangle + b|1\rangle$

$|\psi_B\rangle = c|0\rangle + d|1\rangle$

$$\longrightarrow |\psi\rangle = ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle = \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix}$$

Ex2: $|\psi_A\rangle = a|g\rangle + b|e\rangle$

$|\psi_B\rangle = c|0\rangle + d|1\rangle + f|2\rangle$

$$\longrightarrow |\psi\rangle = ac|g0\rangle + ad|g1\rangle + af|g2\rangle + bc|e0\rangle + bd|e1\rangle + bf|e2\rangle$$

$$\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \otimes \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} = \begin{bmatrix} a_{1,1}\begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} & a_{1,2}\begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} \\ a_{2,1}\begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} & a_{2,2}\begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} a_{1,1}b_{1,1} & a_{1,1}b_{1,2} & a_{1,2}b_{1,1} & a_{1,2}b_{1,2} \\ a_{1,1}b_{2,1} & a_{1,1}b_{2,2} & a_{1,2}b_{2,1} & a_{1,2}b_{2,2} \\ a_{2,1}b_{1,1} & a_{2,1}b_{1,2} & a_{2,2}b_{1,1} & a_{2,2}b_{1,2} \\ a_{2,1}b_{2,1} & a_{2,1}b_{2,2} & a_{2,2}b_{2,1} & a_{2,2}b_{2,2} \end{bmatrix}$$

However, there exist states that can not be decomposed into tensor products of state of the individual subsystems.

Ex: $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ (Bell state)

These states are called <u>entangled states</u>.

Density operators

There are situations in we don't have full knowledge of our quantum states. In these cases, we can describe the system statistically, assigning probabilities to different states

$$p_i \longrightarrow |\psi_i\rangle$$

For these situations, we use a different formalism called <u>density operator</u>.

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

Properties

1) ρ is a complex square matrix

2) ρ is Hermitian $\rho^{+} = \rho$

3) $Tr(\rho) = 1$

Ex: binary state

• $p_0 = 1$

$$\rho = |0\rangle\langle 0| \longrightarrow \rho = \begin{array}{cc} |0\rangle \;\; |1\rangle & \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{array}{c} \langle 0| \\ \langle 1| \end{array} \end{array}$$

$$tr(\rho) = 1 \rightarrow tr(\rho^2) = tr\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right] = 1 \longrightarrow \underline{\text{pure state}}$$

• $p_0 = p_1 = 1/2$

$$\rho = 1/2\,|0\rangle\langle 0| + 1/2\,|1\rangle\langle 1| \longrightarrow \rho = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

$\mathrm{tr}(\rho)=1 \longrightarrow \mathrm{tr}(\rho^2)=\mathrm{tr}\left[\begin{pmatrix}1/2 & 0\\ 0 & 1/2\end{pmatrix}\begin{pmatrix}1/2 & 0\\ 0 & 1/2\end{pmatrix}\right]=1/2 \longrightarrow$ mixed state

- Pure states are states that can be written as vector states. They have $\mathrm{tr}(\rho^2)=1$

- Mixed states are states that can not be represented as projection onto a single vector in the Hilbert space. They have $\mathrm{tr}(\rho^2)<1$

When using the formalism of density operators, it is straightforward to write a composite state. If $\mathcal{H}=\mathcal{H}_A\otimes\mathcal{H}_B$ and $\rho_A\in\mathcal{H}_A$ and $\rho_B\in\mathcal{H}_B$, then we have $\rho=\rho_A\otimes\rho_B\in\mathcal{H}$.

Analogously, we have entangled states when ρ can not be decomposed onto density matrices of pure states of subsystems.

Partial trace

We can always associate density matrices to the subsystems taking the partial trace

$$\rho_A=\mathrm{tr}_B[\rho]=\mathrm{tr}_B[\rho_A\otimes\rho_B]=\sum_i (I_A\otimes\langle i|_B)\,\rho\,(I_A\otimes|i\rangle)$$
$$\rho_B=\mathrm{tr}_A[\rho]=\mathrm{tr}_A[\rho_A\otimes\rho_B]=\sum_i (\langle i|_A\otimes I_B)\,\rho\,(|i\rangle\otimes I_B)$$

Ex:

$$\rho_A=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \qquad \rho_B=\begin{pmatrix}1/2 & 0\\ 0 & 1/2\end{pmatrix}$$

1) Construct $\rho=\rho_A\otimes\rho_B$

$$\rho=\begin{array}{c}\begin{array}{cccc}|00\rangle & |10\rangle & |01\rangle & |11\rangle\end{array}\\ \begin{pmatrix}1/2 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1/2 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\end{array}=1/2\,|00\rangle\langle 00|+1/2\,|01\rangle\langle 01|$$

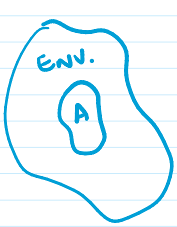
2) $\rho_A=\mathrm{tr}_B(\rho)= 1/2\,|0\rangle_A\left(\langle 0|0\rangle_B\langle 0|0\rangle_B+\langle 1|0\rangle_B\langle 0|1\rangle_B\right)\langle 0|_A$ $\qquad \langle 0|0\rangle=1$

$\qquad +1/2\,|0\rangle_A\left(\langle 0|1\rangle_B\langle 1|0\rangle_B+\langle 1|1\rangle_B\langle 1|1\rangle_B\right)\langle 0|_A$ $\qquad \langle 0|1\rangle=0$

$\qquad =1/2\,|0\rangle_A\langle 0|_A+1/2\,|0\rangle_A\langle 0|_A=|0\rangle_A\langle 0|_A=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$ $\qquad \langle 1|1\rangle=1$

3) Check that

$$\mathrm{tr}_A(\rho)=\rho_B$$



Open Systems

These are systems that interact with the environment

The environment can be seen as an "observer" from which we don't have full knowledge about the outcomes.

Lack of information about the state $\longrightarrow$ density matrices

To be more precise, system and environment are

Lack of information about the state $\longrightarrow$ density matrices

To be more precise, system and environment are entangled, so tracing out the environment we end up with a mixed state

$$\text{If} \quad \rho_A = tr_{ENV}(\rho_{total}) \longrightarrow tr(\rho_A^2) < 1$$