Class 3

segunda-feira, 25 de agosto de 2025 17:18

Last Lecture

Unitary Transformations

$|\psi_1\rangle = U|\psi_0\rangle \qquad \rho_1 = U\rho_0 U^\dagger \qquad U$ unitary

$i\hbar \dfrac{\partial}{\partial t}|\psi\rangle = \hat{H}|\psi\rangle \qquad U(t,t_0) = \exp\left[-\dfrac{i}{\hbar}\hat{H}(t-t_0)\right]$

$$\frac{\partial \rho}{\partial t} = -\frac{i}{\hbar}[\hat{H},\rho]$$

Non-unitary transformations

$\rho_1 = \mathcal{E}(\rho_0)$ quantum maps

$\mathcal{E}(\rho) = \sum_j K_j \rho K_j^\dagger$ Kraus-sum

$$\sum K_j^\dagger K_j = \mathbb{1}$$

$$\frac{\partial \rho}{\partial t} = -\frac{i}{\hbar}[H,\rho] + \sum_{j\neq 0} L_j \rho L_j^\dagger - \tfrac{1}{2}\left(L_j^\dagger L_j \rho + \rho L_j^\dagger L_j\right)$$

Master equation in the Lindblad form

H – hamiltonian

L_j – jump or collapse operators

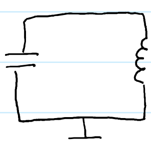
So we want to control Hamiltonians and decay channels

1) What systems provide freedom of control over H?
2) How to describe the quantum Hamiltonian of a system?

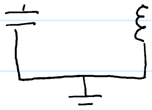
} today

LC oscillator

We are going to study circuit elements in the "lumped element" approach, assuming $\lambda \gg$ size of the devices (mm vs. µm)



Let's consider a parallel LC circuit



Let's consider a parallel LC circuit

→ Energy oscillates between electrical energy in the capacitor and magnetic energy in the inductor

Recalling that the instantaneous stored power in a circuit element is

$$P(t) = V(t)\, I(t)$$

So the instantaneous energy is

$$E(t) = \int_{-\infty}^{t} V(t')\, I(t')\, dt' \qquad (1)$$

We don't know how to directly derive the quantum Hamiltonian, but we have a procedure to derive the classical Hamiltonian

Lagrange - Hamilton formulation (classical mechanics)

1) Lagrangian $\mathcal{L} = T - U$ $\quad$ T = Kinetic energy, U = potential energy

2) Define a generalized position "q"
a generalized momentum "p" $= \dfrac{\partial \mathcal{L}}{\partial \dot{q}}$

3) Then we have the Hamiltonian

$$H = p\dot{q} - \mathcal{L}$$

So, using (1), we have

• In the capacitor

$$\mathcal{E} = -\frac{d\Phi}{dt} \longrightarrow V = \dot{\Phi}$$

$$I = \frac{dQ}{dt} = \frac{d}{dt}(CV) = C\frac{dV}{dt} = C\ddot{\Phi}$$

$$\Rightarrow E(t) = \int_{-\infty}^{t} \dot{\Phi} \cdot C\ddot{\Phi}\, dt'$$

(integrate by parts)
$u = \dot{\Phi}$ $\quad$ $dv = \ddot{\Phi}\, dt'$
$du = d\dot{\Phi}$ $\quad$ $v = \dot{\Phi}$

$$= C\left[\dot{\Phi}^2 - \int \dot{\Phi}\, d\dot{\Phi}\right]\Big|^{t}$$

$$= C\left[\dot{\Phi}^2 - \int \dot{\Phi}\, d\dot{\Phi}\right]\Big|_{-\infty}^{t} \qquad u = \dot{\Phi} \quad dv = \ddot{\Phi}\, dt' \quad du = d\dot{\Phi} \quad v = \dot{\Phi}$$

$$\int u\, dv = uv - \int v\, du$$

$$= C\left[\dot{\Phi}^2 - \frac{\dot{\Phi}^2}{2}\right]\Big|_{-\infty}^{t}$$

$$= \frac{C\dot{\Phi}^2(t)}{2} \quad (\text{assuming } \dot{\Phi}(-\infty) \to 0)$$

- In the inductor

$$\Phi = LI \qquad (L = \text{inductance})$$

$$I = \frac{dQ}{dt} = \dot{Q}$$

$$V = \frac{d\Phi}{dt} = L\frac{dI}{dt} = L\frac{d^2Q}{dt^2} = L\ddot{Q}$$

$$E = L\int_{-\infty}^{t} \ddot{Q} \cdot \dot{Q}\, dt' \qquad u = \dot{Q} \quad du = d\dot{Q} \quad dv = \ddot{Q}\, dt' \quad v = \dot{Q}$$

$$= L\left[\dot{Q}^2 - \int \dot{Q}\, d\dot{Q}\right]\Big|_{-\infty}^{t}$$

$$= \frac{L\dot{Q}^2}{2} = \frac{LI^2}{2} = \frac{\Phi^2}{2L}$$

So

$$E_C = \frac{C\dot{\Phi}^2}{2} \longrightarrow \text{Kinetic energy}$$

$$E_L = \frac{\Phi^2}{2L} \longrightarrow \text{potential energy}$$

in that case

$$\text{"}q\text{"} \longrightarrow \Phi \ (\text{generalized position})$$

So, we get

$$\mathcal{L} = \frac{C\dot{\Phi}^2}{2} - \frac{\Phi^2}{2L}$$

and

$$\text{"}p\text{"} = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} = C\dot{\Phi} = CV = Q \longrightarrow \text{"}p\text{"} = Q$$

Finally

$$H = p\dot{q} - \mathcal{L} = C\dot{\Phi}^2 - C\dot{\Phi}^2 + \Phi^2$$

Finally

$$H = p\dot{q} - \mathcal{L} = C\dot{\Phi}^2 - \frac{C\dot{\Phi}^2}{2} + \frac{\Phi^2}{2L}$$

So the classical Hamiltonian

$$H = \frac{C\dot{\Phi}^2}{2} + \frac{\Phi^2}{2L}$$

Canonical Quantization

1) Find a pair of variables p, q such that $\{A, B\} = 1$

Poisson Bracket $\quad \{A, B\} = \frac{\partial A}{\partial q}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial q}$

2) Replace $\quad \{A, B\} = \frac{1}{i\hbar}[\hat{A}, \hat{B}] \equiv \frac{1}{i\hbar}(\hat{A}\hat{B} - \hat{B}\hat{A}),$

promoting variables A, B to quantum operators $\hat{A}$ and $\hat{B}$

And our choice of variables was convenient

$$\{\Phi, Q\} = \frac{\partial \Phi}{\partial \Phi}\frac{\partial Q}{\partial Q} - \frac{\partial \Phi}{\partial Q}\frac{\partial Q}{\partial \Phi} = 1$$

So

$$\{\Phi, Q\} \Rightarrow [\hat{\Phi}, \hat{Q}] = i\hbar$$

Such that we get the quantum Hamiltonian

$$\boxed{H = \frac{C\hat{\dot{\Phi}}^2}{2} + \frac{\hat{\Phi}^2}{2L} = \frac{\hat{Q}^2}{2C} + \frac{\hat{\Phi}^2}{2L}}$$

Exercise: Replace in the Hamiltonian

$$\hat{\Phi} = \sqrt{\frac{\hbar}{2\omega_r C}}(\hat{a}^+ + \hat{a}) \quad , \text{ where } \omega_r = \frac{1}{\sqrt{LC}} \text{ and } [\hat{a}, \hat{a}^+] = 1$$

$$\hat{Q} = i\sqrt{\frac{\hbar\omega_r C}{2}}(\hat{a}^+ - \hat{a})$$

What Hamiltonian do we obtain?

Answer:

$$\frac{\hat{Q}^2}{2C} = -\frac{\hbar\omega_r}{4}\left(\hat{a}^\dagger\hat{a}^\dagger - \hat{a}^\dagger\hat{a} - \hat{a}\hat{a}^\dagger + \hat{a}\hat{a}\right)$$

$$\frac{\hat{\Phi}^2}{2L} = \frac{\hbar\omega_r}{4}\left(\hat{a}^\dagger\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger + \hat{a}\hat{a}\right)$$

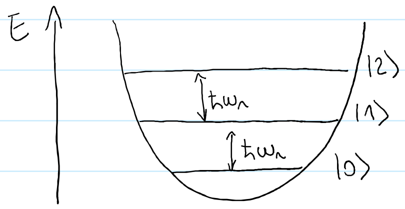
$$\Rightarrow \quad \hat{H} = \frac{\hbar\omega_r}{4}\left(2\hat{a}^\dagger\hat{a} + 2\hat{a}\hat{a}^\dagger\right)$$

but $\hat{a}\hat{a}^\dagger = 1 + \hat{a}^\dagger\hat{a}$

so $$\hat{H} = \frac{\hbar\omega_r}{2}\left(\hat{a}^\dagger\hat{a} + 1 + \hat{a}^\dagger\hat{a}\right)$$

$$\boxed{\hat{H} = \hbar\omega_r\left(\hat{a}^\dagger\hat{a} + 1/2\right)}$$ Quantum harmonic oscillator

A harmonic system has all energy levels equally spaced



It's possible to show that by using Hamiltonians that are at most quadratic in their operators, the Gaussianity of states is preserved.

If we want to build more complex states, we need some type of nonlinearity / anharmonicity.

Building an artificial atom

Assume now that I can replace my linear inductor by a non-linear element

$$I = \frac{1}{L}\Phi \longrightarrow I(t) = I_c \sin\left(\frac{2\pi}{\Phi_0}\Phi(t)\right) \qquad \Phi_0 = \frac{h}{2e} \text{ - flux quantum}$$

$$E(t) = \int_{-\infty}^{t} V(t')\, I(t')\, dt'$$

$$= I_c \int_{-\infty}^{t'} \frac{d\Phi}{dt'} \sin\left[\frac{2\pi}{\Phi_0} \Phi(t')\right] dt'$$

$$= -\frac{I_c \Phi_0}{2\pi} \cos\left[\frac{2\pi \Phi}{\Phi_0}\right]$$ (ignoring irrelevant constant term)

We follow the same quantization process to get

$$\hat{H} = \frac{\hat{Q}^2}{2C} - \frac{I_c \Phi_0}{2\pi} \cos\left[\frac{2\pi}{\Phi_0} \Phi\right]$$

Let's rename some variables

$\hat{n} = \frac{\hat{Q}}{2e}$ — charge number operator

↳ electron charge

$E_c = \frac{e^2}{2C}$ — charging energy

$\hat{\varphi} = \frac{2\pi}{\Phi_0} \hat{\Phi}$ — phase operator

$E_J = \frac{I_c \Phi_0}{2\pi}$ — inductive or Josephson energy

We can also include the possibility of a charge offset, observed in experiments, due to fluctuations in the electric field

$$n_g = \frac{Q_g}{2e}$$

So, we end up with

$$\boxed{\hat{H} = 4E_c(\hat{n} - n_g)^2 - E_J \cos\hat{\varphi}}$$ Cooper-pair box Hamiltonian

This Hamiltonian was made possible by the invention of the Josephson junction, a superconductor-based device that introduces the desired non-linear current.

Exercise: Assume typical values for single Josephson junction devices

$$C = 70\,fF \qquad I_c = 30\,nA$$

Compute E_c and E_J. What frequency scale are we working with? Remember to convert energy to frequency.

$$E_c = \frac{(1.6\times10^{-19})^2}{2\times70\times10^{-15}} \cong 1.83\times10^{-25}\,J \qquad \frac{E_c}{h} \cong 0.277\,GHz$$

$$E_J = \frac{30\times10^{-9}\cdot2\times10^{-15}}{2\pi} = 9.9\times10^{-24}\,J \qquad \frac{E_J}{h} \cong 14.9\,GHz$$

Note that $\frac{E_J}{E_c} \sim 54 \longrightarrow$ transmon regime