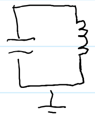
Class 4

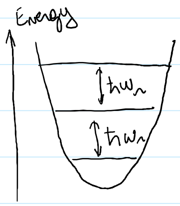
quinta-feira, 28 de agosto de 2025 16:30

Last Lecture

1) Hamiltonian of the LC oscillator



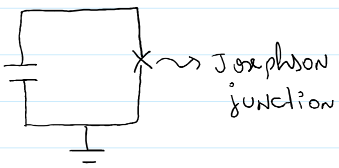
- Lumped-element approach
- Lagrange-Hamilton formulation
- Canonical quantization



$$\hat{H} = \frac{\hat{Q}}{2C} + \frac{\hat{\Phi}}{2L} \quad , \quad [\hat{\Phi}, \hat{Q}] = i\hbar$$

2) Artificial atom
- Non-linear element

$$\hat{H} = 4E_c(\hat{n} - n_g)^2 - E_J \cos\hat{\varphi}$$ Cooper-pair box (CPB) Hamiltonian



CPB Hamiltonian

Goal: Understand the energy levels of this Hamiltonian

For that, we will work at the charge basis $\{|n\rangle\}$ (important: not a Fock basis!)

$$\hat{n}|n\rangle = n|n\rangle \quad , \quad \text{with} \quad n = -N, -N+1, \ldots, 0, \ldots, N-1, N$$

And $\cos\hat{\varphi} = \dfrac{e^{i\hat{\varphi}} + e^{-i\hat{\varphi}}}{2}$. But what is $e^{i\hat{\varphi}}|n\rangle$?

Remember that

$$[\hat{\Phi}, \hat{Q}] = i\hbar \longrightarrow \left[\frac{\Phi_0}{2\pi}\hat{\varphi}, 2e\hat{n}\right] = i\hbar \longrightarrow \frac{2e\Phi_0}{2\pi}[\hat{\varphi}, n] = \frac{h}{2\pi}[\hat{\varphi}, \hat{n}] = i\hbar$$

So $[\hat{\varphi}, \hat{n}] = i$

From Q.M. class, we see that this is analogous to $[\hat{x}, -i\frac{\partial}{\partial \hat{x}}] = i$, so they are conjugated variables with a Fourier transform between them

$$\langle \varphi | n \rangle = \frac{1}{\sqrt{2\pi}} e^{in\varphi}$$

Now, we can make

$$e^{i\hat{\varphi}}|n\rangle \longrightarrow \langle \varphi | e^{i\hat{\varphi}} | n \rangle = e^{i\varphi}\langle \varphi | n \rangle = \frac{1}{\sqrt{2\pi}} e^{i(n+1)\varphi} = \langle \varphi | n+1 \rangle$$

Therefore

$$e^{\pm i\hat{\varphi}}|n\rangle = |n \pm 1\rangle$$

And

$$\cos\hat{\varphi} = \frac{1}{2}\sum_n |n+1\rangle\langle n| + |n\rangle\langle n+1|$$

Plugging both together, we obtain the Hamiltonian in the charge basis

$$\hat{H} = 4E_C \sum_n (n - n_g)^2 |n\rangle\langle n| - \frac{E_J}{2}\sum_n (|n+1\rangle\langle n| + |n\rangle\langle n+1|)$$

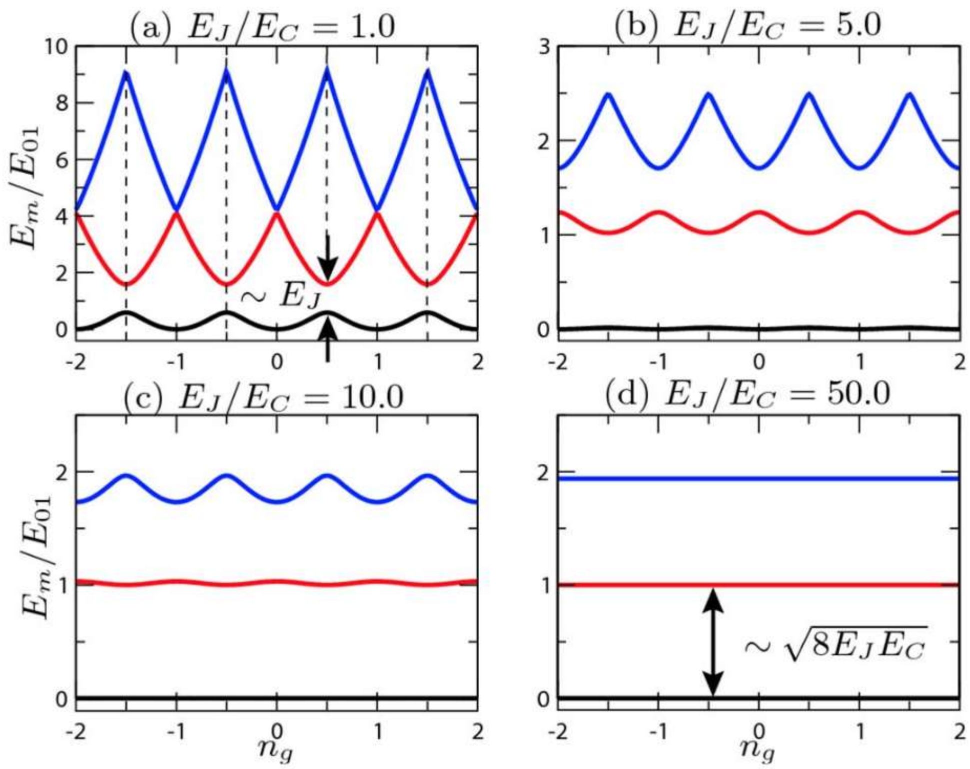
In matrix form, it has three diagonals

$$\hat{H} = \begin{pmatrix} \ddots & \ddots & & 0 \\ \ddots & 4E_C(n-n_g)^2 & -E_J/2 & \\ & -E_J/2 & \ddots & \ddots \\ 0 & & \ddots & \ddots \end{pmatrix}$$

$n = -N, \ldots, N$

This is a tight-binding model matrix

For instance, we can diagonalize this matrix and find the eigenenergies as a function of n_g. We have:



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So, we have two main regimes

- when $E_C \gtrsim E_J$, we have an extremely non-linear system, at cost of an extreme sensitivity to charge fluctuations. These are called charge qubits

- when $E_J \gg E_C$, we have an almost linear system, but insensitive to charge fluctuations. These are the transmon qubits. The transmons are the most widespread type of superconducting qubits today.

Again, let's analyze the Hamiltonian

$$\hat{H} = 4E_C(\hat{n} - n_g)^2 - E_J \cos\hat{\varphi}$$

If E_C is big, it favors charge localization

If E_J is big, it favors phase localization

So, when $E_J \gg E_C$, we can expand

$$\cos\hat{\varphi} \cong 1 - \frac{\hat{\varphi}^2}{2} + \frac{\hat{\varphi}^4}{24} + \dots$$

And we have, ignoring constants

$$\underbrace{\hat{H} = 4E_c\hat{n}^2 + \frac{E_J}{2}\hat{\varphi}^2}_{\text{Harmonic}} \underbrace{- \frac{E_J}{24}\hat{\varphi}^4}_{\text{weakly anharmonic contribution}} + \dots$$

If we make, as in a previous lecture,

$$\hat{\varphi} = \left(\frac{2E_c}{E_J}\right)^{1/4}(\hat{b}^+ + \hat{b})$$

$$\hat{n} = \frac{i}{2}\left(\frac{E_J}{2E_c}\right)^{1/4}(\hat{b}^+ - \hat{b})$$

$\hat{b}^+, \hat{b}$ are creation/annihilation operators

With some algebra, we get

$$\hat{H} = \sqrt{8E_cE_J}\,\hat{b}^+\hat{b} - \frac{E_c}{12}(\hat{b}^+ + \hat{b})^4$$

$$\boxed{\hat{H} \cong \underbrace{\hbar\omega_q\hat{b}^+\hat{b}}_{\text{harmonic}} - \underbrace{\frac{E_c}{2}\hat{b}^+\hat{b}^+\hat{b}\hat{b}}_{\text{Kerr term}}}$$

(Keeping only terms with the same number of $\hat{b}, \hat{b}^+$)

where

$$\hbar\omega_q = \sqrt{8E_cE_J} - E_c$$

Also important: the anharmonicity

$$\alpha \equiv \omega_{12} - \omega_{10} \cong -\frac{E_c}{\hbar}$$

Let's think about it: if we have

- $E_J/E_c \gg 1$ (transmon regime)
- The linewidth of the transition $\gamma \ll \alpha$

then we can consider (and control) only the two lowest levels

$\Rightarrow$ we have a superconducting qubit!

In that case, we can approximate (to facilitate calculations)

$$\hat{b} = \begin{pmatrix} 0 & 1 & & & \\ 0 & 0 & \sqrt{2} & & 0 \\ 0 & & 0 & \sqrt{3} & \\ \vdots & & & 0 & \\ & & & & \ddots \end{pmatrix} \longrightarrow \hat{\sigma}_- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

And the qubit Hamiltonian becomes

$$\hat{H} \cong \hbar \omega_q \hat{\sigma}_+ \hat{\sigma}_- \longrightarrow \boxed{\hat{H} = \frac{\hbar \omega_q}{2} \hat{\sigma}_z}$$

(here, we assumed that average of the eigenenergies is 0)

$$\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$