Differential ideals and homomorphisms of Lie groups

Supporting material of the talk

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1 Left invariant forms

Proposition 1.1. *Left invariant forms are smooth.*

Proof. Let $\omega \in \Omega^p_{l\,\mathrm{inv}}(G)$. It suffices to check that for $X_1, \ldots, X_p \in \mathfrak{X}(G)$, $\omega(X_1, \ldots, X_p)$ is smooth. This is equivalent to

$$\begin{aligned} \omega(X_1, \ldots, X_p)(\sigma) &= \omega|_\sigma(X_1|_\sigma, \ldots, X_p|_\sigma) \\ &= (l_{\sigma^{-1}}{}^* \omega)|_\sigma(X_1|_\sigma, \ldots, X_p|_\sigma) \\ &= \omega|_e(\mathrm{d}_\sigma l_{\sigma^{-1}} X_1|_\sigma, \ldots, \mathrm{d}_\sigma l_{\sigma^{-1}} X_p|_\sigma) \end{aligned} \tag{1}$$

for all $\sigma \in G$. Let $\tilde{\omega}$ be a smooth form such that $\tilde{\omega}|_e = \omega|_e$. Then for $\varphi : G \times G \to G, (a, b) \mapsto ab$, $\varphi^* \tilde{\omega}$ is smooth. Also $(0, X_i)$ is a smooth vector field on $G \times G$, and $i : G \to G \times G, \tau \mapsto (\tau^{-1}, \tau)$ is smooth. Therefore, $\varphi^* \tilde{\omega}((0, X_1), \ldots, (0, X_p))(i(\sigma))$ is smooth with respect to σ. But,

$$\begin{aligned} &\varphi^* \tilde{\omega}\left((0, X_1), \ldots, (0, X_r)\right)(i(\sigma)) \\ &= \varphi^* \tilde{\omega}|_{(\sigma^{-1},\sigma)} \left((0, X_1)|_{(\sigma^{-1},\sigma)}, \ldots, (0, X_r)|_{(\sigma^{-1},\sigma)}\right) \\ &= \tilde{\omega}|_e \left(\mathrm{d}\varphi(0, X_1)|_{(\sigma^{-1},\sigma)}, \ldots, \mathrm{d}\varphi(0, X_r)|_{(\sigma^{-1},\sigma)}\right). \end{aligned} \tag{2}$$

Furthermore,

$$\mathrm{d}\varphi(0, X_i)|_{(a,b)}(f) = (0, X_i)|_{(a,b)}(f \circ \varphi) = X_i|_b(f \circ l_a) = \mathrm{d}l_a X_i|_b(f). \tag{3}$$

Therefore

$$\begin{aligned} &\tilde{\omega}|_e \left(\mathrm{d}\varphi(0, X_1)|_{(\sigma^{-1},\sigma)}, \ldots, \mathrm{d}\varphi(0, X_r)|_{(\sigma^{-1},\sigma)}\right) \\ &= \tilde{\omega}|_e \left(\mathrm{d}l_{\sigma^{-1}} X_i|_\sigma, \ldots, \mathrm{d}l_{\sigma^{-1}} X_r|_\sigma\right), \end{aligned} \tag{4}$$

and the result follows with equation (1). ☐

2 Lie derivative

Proposition 2.1. *Let $X \in \mathfrak{X}(M)$ a smooth vector field on a differentiable manifold M. For each $m \in M$ there exists $a(m), b(m) \in \mathbb{R} \cup \{\pm\infty\}$, and a smooth curve*

$$\gamma_m : (a(m), b(m)) \to M \tag{5}$$

with

(a) $0 \in (a(m), b(m))$ *and* $\gamma_m(0) = m$.

(b) γ_m *is an integral curve of* X.

(c) If $\mu : (c, d) \to M$ *is a smooth curve satisfying conditions (a) and (b), then* $(c, d) \subset (a(m), b(m))$ *and* $\mu = \gamma_m|_{(c,d)}$.

Proof. See [Warner, 1.48]. □

Proposition 2.2. *Let $X \in \mathfrak{X}(M)$, $t \in \mathbb{R}$ and $\mathfrak{D}_t \equiv \{m \in M | t \in (a(m), b(m))\}$. Then $X_t : \mathfrak{D}_t \to \mathfrak{D}_{-t}, m \mapsto \gamma_m(t)$ is a diffeomorphism with inverse X_{-t}. Furthermore:*

(a) For each $m \in M$, there exist an open neighborhood V of m and $\epsilon > 0$, such that the map

$$(-\epsilon, \epsilon) \times V \to M, (t, p) \mapsto X_t(p) \tag{6}$$

is $C^\infty((-\epsilon, \epsilon) \times V, M)$.

(b) $\mathfrak{D}_t$ *is open for each* $t \in \mathbb{R}$.

(c) $\cup_{t>0} \mathfrak{D}_t = M$.

(d) Let $s, t \in \mathbb{R}$. Then the domain of $X_s \circ X_t$ is contained but generally not equal to $\mathfrak{D}_{s+t}$. However, the domain of $X_s \circ X_t$ is $\mathfrak{D}_{s+t}$ in the cas in which s and t both have the same sing. Moreover, on the domain of $X_s \circ X_t$ we have

$$X_s \circ X_t = X_{s+t}. \tag{7}$$

Proof. See [Warner, 1.48]. □

Definition 2.3. Let $X \in \mathfrak{X}(M)$ a smooth vector field on M with transformation X_t, let $Y \in \mathfrak{X}(M)$ another smooth vector field on M and let $m \in M$.
The Lie derivative of Y with respect to X at m is defined by

$$(\mathrm{L}_X Y)|_m = \lim_{t \to 0} \frac{\mathrm{d}_m X_{-t}\left(Y|_{X_t(m)}\right) - Y|_m}{t} = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \Big(\mathrm{d}_m X_{-t}\left(Y|_{X_t(m)}\right)\Big). \tag{8}$$

In a similar way we define the Lie derivative of a differential form $\omega \in \Omega^*(M)$ with respect to X at m by

$$(\mathrm{L}_X \omega)|_m = \lim_{t \to 0} \frac{\left({X_t}^* \omega\right)\big|_m - \omega\big|_m}{t} = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \left({X_t}^* \omega\right)\big|_m. \tag{9}$$

Remark. The Lie derivative L_X can be extended to arbitrary tensor fields in the obvious way. See [Warner, 2.24].

Proposition 2.4. *$\mathrm{L}_X f = X(f)$ whenever $f \in C^\infty(M)$ and $X \in \mathfrak{X}(M)$.*

Proof. Let $m \in M$, $f \in C^\infty(M) = \Omega^0(M)$ and $X \in \mathfrak{X}(M)$. Then

$$\begin{aligned}(\mathrm{L}_X f)\,|_m &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\big({X_t}^* f\big)\big|_m = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\big(f \circ X_t\big)\big|_m \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\big(f \circ \gamma_m(t)\big) = X|_m(f).\end{aligned} \tag{10}$$

□

Proposition 2.5. *Let $X, Y \in \mathfrak{X}(M)$. Then*

$$\mathrm{L}_X Y = [X, Y]. \tag{11}$$

Proof. Let $X, Y \in \mathfrak{X}(M)$ with transformations X_t and Y_t respectively. We need to show that $(\mathrm{L}_X Y)(f) = [X, Y](f)$ for each $f \in C^\infty(M)$. Let $m \in M$. Then

$$\begin{aligned}(\mathrm{L}_X Y)|_m(f) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\Big(\mathrm{d}_m X_{-t}\left(Y|_{X_t(m)}\right)(f)\Big) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\Big(Y|_{X_t(m)}\,(f \circ X_{-t})\,|_m\Big).\end{aligned} \tag{12}$$

Define a real-valued function H on a neighborhood of $(0,0)$ in $\mathbb{R}^2$ by setting

$$H(t,u) = f\big(X_{-t}(Y_u(X_t(m)))\big). \tag{13}$$

Then by proposition 2.4 and

$$Y|_{X_t(m)}\left((f \circ X_{-t})|_m\right) = \frac{\partial}{\partial r_2}\Big|_{(t,0)} H(t,u), \tag{14}$$

equation (12) becomes

$$(\mathrm{L}_X Y)|_m(f) = \frac{\partial^2 H}{\partial r_1 \partial r_2}\Big|_{(0,0)}. \tag{15}$$

To evaluate this derivative, we set

$$K(t,u,s) = f\big(X_s(Y_u(X_t(m)))\big). \tag{16}$$

Then $H(t,u) = K(t,u,-t)$. It follows from the chain rule that

$$\frac{\partial^2 H}{\partial r_1 \partial r_2}\Big|_{(0,0)} = \frac{\partial^2 K}{\partial r_1 \partial r_2}\Big|_{(0,0,0)} - \frac{\partial^2 K}{\partial r_3 \partial r_2}\Big|_{(0,0,0)}. \tag{17}$$

Now, $K(t,u,0) = f\big(Y_u(X_t(m))\big)$. Hence

$$\frac{\partial K}{\partial r_2}\Big|_{(0,0,0)} = Y|_{X_t(m)}(f), \tag{18}$$

$$\frac{\partial^2 K}{\partial r_1 \partial r_2}\Big|_{(0,0,0)} = X|_m(Y|_m f). \tag{19}$$

Also $K(0,u,s) = f\big(X_s(Y_u(m))\big)$. Hence

$$\frac{\partial K}{\partial r_3}\Big|_{(0,0,0)} = X|_m f\big(Y_u(m)\big), \tag{20}$$

$$\frac{\partial^2 K}{\partial r_2 \partial r_3}\Big|_{(0,0,0)} = Y|_m(X|_m f). \tag{21}$$

The proposition now follows from equation (15), (17), (19) and (21). □

Lemma 2.6. $\mathrm{L}_X : \Omega^*(M) \to \Omega^*(M)$ *is a derivation of degree* 0 *which commutes with* d *for every* $X \in \mathfrak{X}(M)$.

Proof. Let $X \in \mathfrak{X}(M)$, $\omega \in \Omega^u(M)$ and $\eta \in \Omega^s(M)$. Then

$$\begin{aligned}
\mathrm{L}_X(\omega \wedge \eta) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\big({X_t}^*(\omega\wedge\eta)\big) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\big(({X_t}^*\omega)\wedge({X_t}^*\eta)\big) \\
&= \left(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}({X_t}^*\omega)\right)\wedge({X_t}^*\eta)\big|_{t=0} \\
&+ ({X_t}^*\omega)\big|_{t=0}\wedge\left(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}({X_t}^*\eta)\right) \\
&= \left(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}({X_t}^*\omega)\right)\wedge\eta + \omega\wedge\left(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}({X_t}^*\eta)\right) \\
&= \mathrm{L}_X(\omega)\wedge\eta + \omega\wedge\mathrm{L}_X(\eta)
\end{aligned} \tag{22}$$

That shows the first part of the lemma. Hence we need only show that $\mathrm{L}_X \circ \mathrm{d} = \mathrm{d} \circ \mathrm{L}_X$ for all functions $f \in C^\infty(M)$ and 1-forms $\mathrm{d}f$:

$$\begin{aligned}
(\mathrm{L}_X \circ \mathrm{d})(f) &= (\mathrm{L}_X)(\mathrm{d}f) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}({X_t}^*\,\mathrm{d}f) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\big(\mathrm{d}({X_t}^* f)\big) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\big(\mathrm{d}(f\circ X_t)\big) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\left(\sum_i \frac{\partial}{\partial x^i}(f\circ X_t)\,\mathrm{d}x^i\right) \\
&= \sum_i \frac{\partial}{\partial x^i}\left(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(f\circ X_t)\,\mathrm{d}x^i\right) = \sum_i \frac{\partial}{\partial x^i}\mathrm{L}_X\,(f)\mathrm{d}x^i \\
&= \mathrm{d}\big(\mathrm{L}_X(f)\big) = (\mathrm{d}\circ\mathrm{L}_X)(f),
\end{aligned} \tag{23}$$

$$(\mathrm{L}_X \circ \mathrm{d})(\mathrm{d}f) = \mathrm{L}_X(\mathrm{d}^2 f) = 0 = \mathrm{d}^2(\mathrm{L}_X(f)) = \mathrm{d}(\mathrm{L}_X(\mathrm{d}f)) = (\mathrm{d}\circ\mathrm{L}_X)(\mathrm{d}f). \tag{24}$$

□

Lemma 2.7. *Every derivation D of $\Omega^*(M)$ of degree 0, which commutes with the exterior derivativ* d*, is equal to the Lie derivation* L_X *of a smooth vector field $X \in \mathfrak{X}(M)$ on M.*

Proof. Let D be a derivation of $\Omega^*(M)$ of degree 0, which commutes with d. Then D is a derivation of $C^\infty(M) = \Omega^0(M)$. Hence, there exist $X \in \mathfrak{X}(M)$ with $D(f) = X(f) = \mathrm{L}_X(f)$ for all $f \in C^\infty(M)$. Let $D' = D - \mathrm{L}_X$ and $\sum_i f_i \mathrm{d}x^i \in \Omega^1(M)$. Then

$$
\begin{aligned}
D'\left(\sum_i f_i \mathrm{d}x^i\right) &= \sum_i D'\left(f_i \mathrm{d}x^i\right) = \sum_i D'\left(f_i\right)\, \mathrm{d}x^i + f_i\, D'\left(\mathrm{d}x^i\right) \\
&= \sum_i \left(D(f_i) - \mathrm{L}_X(f_i)\right) \mathrm{d}x^i + f_i\, \mathrm{d}\left(D(x^i) - \mathrm{L}_X(x^i)\right) \\
&= \sum_i \left(X(f_i) - X(f_i)\right) \mathrm{d}x^i + f_i\, \mathrm{d}\left(X(x^i) - X(x^i)\right) = 0.
\end{aligned}
\tag{25}
$$

$\square$

Proposition 2.8. *Let $X \in \mathfrak{X}(M)$. Then*

$$\mathrm{L}_X = \mathrm{d} \circ i_X + i_X \circ \mathrm{d}. \tag{26}$$

Proof. Let $X \in \mathfrak{X}(M)$. $\mathrm{d} \circ i_X + i_X \circ \mathrm{d}$ is a derivation of degree 0, which commutes with d. With lemma 2.7, there exists $Y \in \mathfrak{X}(M)$ with

$$\mathrm{d} \circ i_X + i_X \circ \mathrm{d} = \mathrm{L}_Y. \tag{27}$$

Let $f \in C^\infty(M)$. Then

$$Y(f) = (\mathrm{d} \circ i_X + i_X \circ \mathrm{d})(f) = i_X \circ \mathrm{d}f = \mathrm{d}f(X) = X(f), \tag{28}$$

and therefore $X = Y$. $\square$

Lemma 2.9. *Let $\omega \in \Omega^p(M)$ and $Y_0, \ldots, Y_p \in \mathfrak{X}(M)$. Then*

$$
\begin{aligned}
\mathrm{L}_{Y_0}\left(\omega\left(Y_1, \ldots, Y_p\right)\right) &= \left(\mathrm{L}_{Y_0}\omega\right)\left(Y_1, \ldots, Y_p\right) \\
&\quad + \sum_{i=1}^{p} \omega\left(Y_1, \ldots, Y_{i-1}, \mathrm{L}_{Y_0} Y_i, Y_{i+1}, \ldots, Y_p\right).
\end{aligned}
\tag{29}
$$

Proof. Let $Y_0, \ldots, Y_p \in \mathfrak{X}(M)$. Since the equation (29) is linear with respect to ω, it is enoug to consider the case $\omega = g\, \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_p} \in \Omega^p(M)$. Then

$$
\begin{aligned}
\mathrm{L}_{Y_0}\big(\omega(Y_1, \ldots, Y_p)\big) &= Y_0\left(g \det\big(Y_1(x^{i_j}), \ldots, Y_p(x^{i_j})\big)_{1 \le j \le p}\right) \\
&= Y_0(g) \det\big(Y_1(x^{i_j}), \ldots, Y_p(x^{i_j})\big)_{1 \le j \le p} \\
&\quad + g\, Y_0\left(\det\big(Y_1(x^{i_j}), \ldots, Y_p(x^{i_j})\big)_{1 \le j \le p}\right)
\end{aligned}
\tag{30}
$$

$$
\begin{aligned}
\mathrm{L}_{Y_0}(\omega)(Y_1,\dots,Y_p) = Y_0(g)\det\bigl(Y_1(x^{i_j}),\dots,Y_p(x^{i_j})\bigr)_{1\le j\le p} \\
+\sum_{k=1}^{p} g\det\bigl(Y_1(x^{i_j}),\dots,Y_k(Y_0(x^{i_j})),\dots,Y_p(x^{i_j})\bigr)_{1\le j\le p}
\end{aligned} \tag{31}
$$

$$
\begin{aligned}
&\omega(Y_1,\dots,L_{Y_0}Y_k,\dots,Y_p) \\
&= g\det\bigl(Y_1(x^{i_j}),\dots,Y_0(Y_k(x^{i_j})),\dots,Y_p(x^{i_j})\bigr)_{1\le j\le p} \\
&-g\det\bigl(Y_1(x^{i_j}),\dots,Y_k(Y_0(x^{i_j})),\dots,Y_p(x^{i_j})\bigr)_{1\le j\le p}
\end{aligned} \tag{32}
$$

The lemma follows by combining equation (30), (31) and (32). □

Proposition 2.10. *Let $\omega \in \Omega^p(M)$ and $Y_0,\dots,Y_p \in \mathfrak{X}(M)$. Then*

$$
\begin{aligned}
\mathrm{d}\omega\,(Y_0,\dots,Y_p) = \sum_{i=0}^{p}(-1)^i\cdot Y_i\cdot\omega\left(Y_0,\dots,\hat{Y}_i,\dots,Y_p\right) \\
+\sum_{i<j}(-1)^{i+j}\cdot\omega\left([Y_i,Y_j]\,,Y_0,\dots,\hat{Y}_i,\hat{Y}_j,\dots,Y_p\right).
\end{aligned} \tag{33}
$$

Proof. We will proof this proposition by induction on p. For the case $p = 1$, we have from lemma 2.9

$$
\mathrm{L}_{Y_0}\left(\omega(Y_1)\right) = \left(\mathrm{L}_{Y_0}\omega\right)(Y_1) + \omega\left([Y_0,Y_1]\right). \tag{34}
$$

Applying lemma 2.4 and proposition 2.8 to (34), one obtains

$$
\begin{aligned}
Y_0\omega(Y_1) &= \bigl((i_{Y_0}\circ \mathrm{d} + \mathrm{d}\circ i_{Y_0})(\omega)\bigr)(Y_1) + \omega([Y_0,Y_1]) \\
&= \mathrm{d}\omega(Y_0,Y_1) + Y_1(\omega(Y_1)) + \omega([Y_0,Y_1]),
\end{aligned} \tag{35}
$$

which is result (33) for the case $p = 1$. Now assume that (33) holds for $p-1$. Then (33) is obtained for p by again starting with 2.9 and applying 2.8 and the induction hypothesis. □

3 Differential ideals

Definition 3.1. Let $\mathscr{D}$ be a p-dimensional smooth distribution on M.
A q-form ω is said to annihilate $\mathscr{D}$ if for each $m \in M$ and $v_1, \ldots, v_q \in \mathscr{D}(m)$

$$\omega|_m (v_1, \ldots, v_q) = 0. \tag{36}$$

A form $\omega \in \Omega^*(M)$ is said to annihilate $\mathscr{D}$ if each of the homogenous parts of ω annihilates $\mathfrak{D}$. We let

$$\mathscr{I}(\mathscr{D}) = \{\omega \in \Omega^*(M) |\, \omega \,\text{annihilates}\, \mathscr{D}\}. \tag{37}$$

Definition 3.2. A collection $\omega_1, \ldots, \omega_n$ of 1-forms on M is called independent if they form an independent set in ${T_m}^*M$ for each $m \in M$.

Proposition 3.3. *Let $\mathscr{D}$ be a smooth p-dimensional distribution on M. $\mathscr{I}(\mathscr{D})$ is an ideal in $\Omega^*(M)$.*

Proof. The proposition follows from the definiton of $\mathscr{I}(\mathscr{D})$ and the definition of multiplication in $\Omega^*(M)$. □

Lemma 3.4. *Let $\mathscr{D}$ be a smooth p-dimensional distribution on M. $\mathscr{I}(\mathscr{D})$ is locally generated by $d-p$ independent 1-forms.*
That is, to each $m \in M$ there corresponds a neighborhood U of m and a set of independent 1-forms $\omega_1, \ldots, w_{d-p}$ on U such that:

(i) If $\omega \in \mathscr{I}(\mathscr{D})$, then $\omega|_U$ belongs to the ideal in $\Omega^*(U)$ generated by $\omega_1, \ldots, w_{d-p}$.

(ii) If $\omega \in \Omega^*(M)$, and if there is a cover of M by sets U (as above) such that for each U in the cover, $\omega|_U$ belongs to the ideal generated by $\omega_1, \ldots, w_{d-p}$, then $\omega \in \mathscr{I}(\mathscr{D})$.

Proof. Let $m \in M$. Since $\mathscr{D}$ is smooth and p-dimensional, there exist smooth vector fields $X_{d-p+1}, \ldots, X_d$ defined and spanning $\mathscr{D}$ at each point of a neighborhood of m. This collection can be completed to acollection $X_1, \ldots, X_d$ of smooth vector fields forming a masis of T_mM for each n in a neighborhood U of m. Let $\omega_1, \ldots, \omega_d$ be the dual 1-forms; that is,

$$\omega_i(X_j)(n) = \delta_{ij} \tag{38}$$

for each n in U. Then $\omega_1, \ldots, \omega_{d-p}$ are the desierd 1-forms on U. They are independent smooth 1-forms on U. If $\omega \in \mathscr{I}(\mathscr{D})$, $\omega|_U = \sum a_\Phi \, \omega_{i_1} \wedge \cdots \wedge \omega_{i_r}$, wher Φ runs over nonempty subsets $\{i_1, \ldots, i_r\} \subset \{1, \ldots, d\}$, and where the a_Ψ must be identically zero unless $\{i_1, \ldots, i_r\} \cap \{1, \ldots, d-p\} = \emptyset$. Thus $\omega|_U$ belongs to the ideal in $\Omega^*(U)$ generated by $\omega_1, \ldots, \omega_{d-p}$, then clearly $\omega \in \mathscr{I}(\mathscr{D})$. This proves the lemma. □

Lemma 3.5. *If $\mathscr{I} \subset \Omega^*(M)$ is an ideal locally generated by $d-p$ independent 1-forms, then there exists a unique smooth distribution $\mathscr{D}$ of dimension p on M for which $\mathscr{I} = \mathscr{I}(\mathscr{D})$.*

Proof. Let $m \in M$, and let the independent 1-forms $\omega_1, \ldots, \omega_{d-p}$ generate $\mathscr{I}$ on a neighborhood U. Define $\mathscr{D}(m)$ to be the subspace of $T_m M$ whose annihilator is the subspace of $T_m{}^* M$ spanned by the collection $\{\omega_i(m) \mid i \in \{1, \ldots, d-p\}\}$. It follows that $\mathscr{D}$ is a smooth p-dimensional distribution on M and that $\mathscr{I} = \mathscr{I}(\mathscr{D})$. Uniqueness of $\mathscr{D}$ follows from the fact that $\mathscr{D} \neq \mathscr{D}_1$ implies $\mathscr{I}(\mathscr{D}) \neq \mathscr{I}(\mathscr{D}_1)$. $\square$

Definition 3.6. An ideal $\mathscr{I} \subset \Omega^*(M)$ is called a differential ideal if it is closed under exterior differentiation d; that is,

$$\mathrm{d}\,(\mathscr{I}) \subset \mathscr{I}. \tag{39}$$

Lemma 3.7. *A smooth distribution $\mathscr{D}$ on M is involutive if and only if the ideal $\mathscr{I}(\mathscr{D})$ is a differential ideal.*

Proof. Let ω be a q-form in $\mathscr{I}(\mathscr{D})$, and let $X_0, \ldots, X_q$ be smooth vector fields lying in $\mathscr{D}$. Then proposition 2.10 together with the involutiveness of $\mathscr{D}$ implies $\mathrm{d}\omega(X_0, \ldots, X_q) \equiv 0$. Hence $\mathrm{d}\omega \in \mathscr{I}(\mathscr{D})$, and $\mathscr{I}(\mathscr{D})$ is a differential ideal. Conversely, suppose that $\mathscr{I}(\mathscr{D})$ is a differential ideal. Let Y_0 and Y_1 be vector fields lying in $\mathscr{D}$, and let $m \in M$. By proposition 3.4, there are independent 1-forms $\omega_1, \ldots, \omega_{d-p}$ generating $\mathscr{I}(\mathscr{D})$ on a neighborhood U of m. Extend these forms to M by multiplying by a $C^\infty(M)$ function which is 1 on a neighborhood of m and has support in U. We shall denote the extended forms similarly by $\omega_1, \ldots, \omega_{d-p}$. By 2.10,

$$\omega([Y_0, Y_1]) = -\mathrm{d}\omega_i(Y_0, Y_1) + Y_0\, \omega_i(Y_0) - Y_1\, \omega_i(Y_0) \tag{40}$$

for $i \in \{1, \ldots, d-p\}$. The right-hand side of (40) is identically zero on M since $\mathscr{I}(\mathscr{D})$ is a differential ideal and since $\omega_i \in \mathscr{I}(\mathscr{D})$. Thus $\omega_i([Y_0, Y_1])(m) = 0$ for $i \in \{1, \ldots, d-p\}$. Now $\mathscr{D}(m)$ is the subspace of $T_m M$ whose annihilator is the subspace of $T_m{}^* M$ spanned by the collection $\{\omega_i(m) \mid i \in \{1, \ldots, d-p\}\}$. Thus $[Y_0, Y_1](m) \in \mathscr{D}(m)$, and $\mathscr{D}$ is involutive. $\square$

Definition 3.8. A submanifold (N, ψ) of M is an integral manifold of an ideal $\mathscr{I} \subset \Omega^*(M)$ if for every $\omega \in \mathscr{I}$, $\psi^*\, \omega \equiv 0$. A connected integral manifold of an ideal $\mathscr{I}$ is maximal if its image is not a proper subset of the image of any other connected integral manifold of the ideal.

Theorem 3.9. *Let $\mathscr{I} \subset \Omega^*(M)$ be a differential ideal locally generated by $d-p$ independent 1-forms. Let $m \in M$. Then there exists a unique maximal, connected, integral manifold of $\mathscr{I}$ through m, and this integral manifold has dimension p.*

Proof. Let $\mathscr{I} \subset \Omega^*(M)$ be a differential ideal locally generated by $d-p$ independent 1-forms. By lemma 3.5, there exists a unique smooth distribution $\mathscr{D}$ of dimension p on M for which $\mathscr{I} = \mathscr{I}(\mathscr{D})$. Hence, by lemma 3.7, the distribution $\mathscr{D}$ is involutive. Let $m \in M$. By theorem [Warner, 1.64] there exist a unique maximal connected integral manifold (N, ψ) of $\mathscr{D}$ passing through m. Then

$$\mathrm{d}\psi(T_n N) = \mathscr{D}(\psi(n)) \quad \text{for each}\, n \in N. \tag{41}$$

This implies

$$0 = \omega(d_n\psi(v)) = (\psi^*\,\omega)|_n(v) \tag{42}$$

for every $n \in N$, $\omega \in \mathscr{I} = \mathscr{I}(\mathscr{D})$ and $v \in T_nN$. Therefore, (N, ψ) is a unique maximal, connected, integral manifold of $\mathscr{I}$ through m with dimension p. $\square$

Proposition 3.10. *Let N^c and M^d be differentiable manifolds, and let π_1 and π_2 be the canonical projections of $N \times M$ onto N and M respectively. Suppose that there exists a basis $\left\{w_i \in \Omega^1(M) \,|\, i \in \{1,\ldots,d\}\right\}$ for the 1-forms on M.*

(a) If $f : N \to M$ is $C^\infty(M,N)$, then the graph of f is an integral maniflod of the ideal of forms on $N \times M$ generated by

$$\left\{{\pi_1}^*\, f^*\,\omega_i - {\pi_2}^*\,\omega_i \,|\, i \in \{1,\ldots,d\}\right\}. \tag{43}$$

(b) If $\left\{\alpha_i \in \Omega^1(M) \,|\, i \in \{1,\ldots,d\}\right\}$ are 1-forms on N, and if the ideal of forms on $N \times M$ generated by

$$\left\{{\pi_1}^*\,\alpha_i - {\pi_2}^*\,\omega_i \,|\, i \in \{1,\ldots,d\}\right\} \tag{44}$$

is a differential ideal, then given $n_0 \in N$ and $m_0 \in M$ there exists a neighborhood U of n_0 and a $C^\infty(U,M)$ map $f : U \to M$ sucht that $f(n_0) = m_0$ and such that

$$f^*\,\omega_i = \alpha_i|_U \tag{45}$$

for all $i \in \{1,\ldots,d\}$. Moreover, if U is any connected open set containing n_0 for which there exist a $C^\infty(U,M)$ map $f : U \to M$ satisfying both $f(n_0) = m_0$ and equation (45), then there exists a unique such map on U.

Proof. Part (a): For each i, we define a form μ_i on $N \times M$ by setting

$$\mu_i = {\pi_1}^*\, f^*\,\omega_i - {\pi_2}^*\,\omega_i. \tag{46}$$

Let $\mathscr{I}$ the ideal in $\Omega^*(N \times M)$ generated by the μ_i. Now the graph of f is the submanifold (N, g) of $N \times M$ where

$$g(n) = (n, f(n)). \tag{47}$$

We claim that the graph is an integral manifold of the ideal $\mathscr{I}$. For this, it is suffices to show that $g^*\,\mu_i = 0$ for each i. Now, $\pi_1 \circ g = \mathrm{id}$, and $\pi_2 \circ g = f$, and thus it follows that

$$g^*\,\mu_i = (\pi_1 \circ g)^*\, f^*\,\omega_i - (\pi_2 \circ g)^*\,\omega_i = f^*\,\omega_i - f^*\,\omega_i = 0. \tag{48}$$

Part (b): Let us define forms μ_i on $N \times M$ by setting

$$\mu_i = {\pi_1}^*\,\alpha_i - {\pi_2}^*\,\omega_i, \tag{49}$$

and we let $\mathscr{I}$ the ideal in $\Omega^*(N \times M)$ which they generate. Let $(n_0, m_0) \in N \times M$. Then since $\mathscr{I}$ is locally generated by d independent 1-forms, the Frobenius theorem 3.9 guarantees that there is a maximal, connected, integral manifold I of $\mathscr{I}$ of dimension c through (n_0, m_0). Let $q \in I$ and $v \in T_p I$ with $\mathrm{d}|_p \pi_1(v) = 0$. Then since $\mu_i = 0$, it follows from equation (49) that $\omega_i(\mathrm{d}|_p(v)) = 0$ for $i \in \{1, \ldots, d\}$; and this implies that $\mathrm{d}|_p \pi_2(v) = 0$ and therefore $v = 0$. Hence $\mathrm{d}|_p \pi_1|_{I_q}$ is one-to-one. Thus $\pi_1|_I : I \to N$ is locally a diffeomorphism. So there exist neighborhoods V of (n_0, m_0) in I and U of n_0 such that $\pi_1|_V : V \to U$ is a diffeomorphism. We define $f : U \to M$ by setting

$$f = \pi_2 \circ (\pi_1|_V)^{-1}. \tag{50}$$

Then $f(n_0) = m_0$ and the graph of f is an open submanifold of I. Moreover,

$$\begin{aligned} 0 &= \mu_i\big(\mathrm{d}|_q (\pi_1|_V)^{-1}(v)\big) \\ &= \alpha_i(v) - \omega_i(\mathrm{d}|_{(\pi_1|_V)^{-1}(q)} \pi_2 \circ \mathrm{d}|_q (\pi_1|_V)^{-1}(v)) \\ &= \alpha_i(v) - f^* \omega_i(v), \end{aligned} \tag{51}$$

which shows $f^* \omega_i = \alpha_i$. If U is a connected open neighborhood of n_0 in N for which ther exist such f, then there is a unique such map on U. For let $\tilde{f}$ be any other such map. Let $(U, \tilde{g})$ and (U, g) be the graphs of $\tilde{f}$ and f over U respectively. Thus

$$\tilde{g}(n) = (n, \tilde{f}(n)) \quad \text{and} \quad g(n) = (n, f(n)) \tag{52}$$

for $n \in U$. By (a), not only is (U, g) an integral manifold of $\mathscr{I}$ through (n_0, m_0), but so is $(U, \tilde{g})$. Now, the subset of U on which g and $\tilde{g}$ agree is non-empty since it contains n_0, and is closed by continuitty, and is moreover open. Then it follows from the uniqueness of integral manifolds that there exist sufficiently small neighborhoods W and $\tilde{W}$ of n so that

$$g(W) = g(\tilde{W}). \tag{53}$$

It follows from equation (52) that $W = \tilde{W}$. Hence, since U is connected, $g = \tilde{g}$ on U, which implies that $f = \tilde{f}$ on U. This proves uniqueness. □