STFT and DGT phase conventions and phase derivatives interpretation

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About this document

The purpose of this document is to sumarize common STFT and DGT phase conventions and their effect on the phase derivatives.

Consequently, the phase derivatives are linked to terms like instantaneous time $\widehat{t}(\omega, t)$, instantaneous frequency $\widehat{\omega}(\omega, t)$, chanelized instantaneous frequency $\mathrm{CIF}(\omega, t)$, local group delay $\mathrm{LGD}(\omega, t)$, and to derivatives of $\widehat{t}(\omega, t)$ and $\widehat{\omega}(\omega, t)$ over time or frequency $\frac{\partial \widehat{t}}{\partial \omega}$, $\frac{\partial \widehat{\omega}}{\partial t}$, $\frac{\partial \widehat{t}}{\partial t}$ and $\frac{\partial \widehat{\omega}}{\partial \omega}$.

Numerical approximation of the phase derivatives using DGT is discussed also.

1 STFT phase conventions

STFT of signal $f(t)$ using window $g(t)$ and phase convention pc will be denoted as

$$V_{\mathrm{pc}}(\omega, t) = M(\omega, t)\mathrm{e}^{i\phi_{\mathrm{pc}}(\omega, t)}. \tag{1}$$

The translation and modulation operators are defined as

$$T_\tau f(t) = f(t-\tau) \qquad \text{and} \qquad M_\omega f(t) = f(t)\mathrm{e}^{j\omega t}. \tag{2}$$

STFT with *frequency-invariant* phase is obtained as

$$\begin{align} V_{\mathrm{fi}}(\omega, t) &= \langle f, M_\omega T_t g\rangle \tag{3}\\ &= \int_{-\infty}^{\infty} f(\tau)\mathrm{e}^{-j\omega\tau}\overline{g(\tau - t)}\,\mathrm{d}\tau. \tag{4}\end{align}$$

STFT with *time-invariant* phase is obtained as

$$\begin{align} V_{\mathrm{ti}}(\omega, t) &= \langle f, T_t M_\omega g\rangle \tag{5}\\ &= \int_{-\infty}^{\infty} f(\tau)\mathrm{e}^{-j\omega(\tau - t)}\overline{g(\tau - t)}\,\mathrm{d}\tau \tag{6}\\ &= \int_{-\infty}^{\infty} f(\tau + t)\mathrm{e}^{-j\omega\tau}\overline{g(\tau)}\,\mathrm{d}\tau \tag{7}\\ &= \mathrm{e}^{j\omega t}V_{\mathrm{fi}}(\omega, t). \tag{8}\end{align}$$

STFT with *symmetric* phase is obtained as

$$\begin{align} V_{\mathrm{sym}}(\omega, t) &= \langle f, M_{\omega/2} T_t M_{\omega/2} g\rangle \tag{9}\\ &= \int_{-\infty}^{\infty} f(\tau)\mathrm{e}^{-j\omega\tau/2}\mathrm{e}^{-j\omega(\tau - t)/2}\overline{g(\tau - t)}\,\mathrm{d}\tau \tag{10}\\ &= \int_{-\infty}^{\infty} f(\tau)\mathrm{e}^{-j\omega(\tau - t/2)}\overline{g(\tau - t)}\,\mathrm{d}\tau \tag{11}\\ &= \int_{-\infty}^{\infty} f(\tau + t/2)\mathrm{e}^{-j\omega\tau}\overline{g(\tau - t/2)}\,\mathrm{d}\tau \tag{12}\\ &= \mathrm{e}^{j\omega t/2}V_{\mathrm{fi}}(\omega, t) \tag{13}\end{align}$$

The overline $\overline{g(t)}$ denotes complex conjugation.

The relations between the phases follow directly

$$\phi_{\mathrm{fi}}(\omega, t) = \phi_{\mathrm{ti}}(\omega, t) - \omega t = \phi_{\mathrm{sym}}(\omega, t) - \omega t/2. \tag{14}$$

1.1 Phase derivatives

The phase derivatives along *frequency* are linked in the following way:

$$\widehat{t}(\omega,t) = t - \frac{\partial \phi_{\mathrm{ti}}}{\partial \omega} = -\frac{\partial \phi_{\mathrm{fi}}}{\partial \omega} = \frac{t}{2} - \frac{\partial \phi_{\mathrm{sym}}}{\partial \omega}, \tag{15}$$

where $\widehat{t}(\omega,t)$ denotes the absolute instantaneous time.

Similarly, phase derivatives along *time* are linked in the following way:

$$\widehat{\omega}(\omega,t) = \frac{\partial \phi_{\mathrm{ti}}}{\partial t} = \omega + \frac{\partial \phi_{\mathrm{fi}}}{\partial t} = \frac{\omega}{2} + \frac{\partial \phi_{\mathrm{sym}}}{\partial t}, \tag{16}$$

where $\widehat{\omega}(\omega,t)$ denotes the instantaneous frequency.

Second derivatives do not depend on the phase convention anymore i.e.

$$\frac{\partial \widehat{t}(\omega,t)}{\partial \omega} = -\frac{\partial^2 \phi_{\mathrm{ti}}}{\partial \omega^2} = -\frac{\partial^2 \phi_{\mathrm{fi}}}{\partial \omega^2} = -\frac{\partial^2 \phi_{\mathrm{sym}}}{\partial \omega^2} \tag{17}$$

and

$$\frac{\partial \widehat{\omega}(\omega,t)}{\partial t} = \frac{\partial^2 \phi_{\mathrm{ti}}}{\partial t^2} = \frac{\partial^2 \phi_{\mathrm{fi}}}{\partial t^2} = \frac{\partial^2 \phi_{\mathrm{sym}}}{\partial t^2}. \tag{18}$$

Mixed phase derivatives are linked as follows:

$$\frac{\partial \widehat{t}(\omega,t)}{\partial t} = 1 - \frac{\partial^2 \phi_{\mathrm{ti}}}{\partial t \partial \omega} = -\frac{\partial^2 \phi_{\mathrm{fi}}}{\partial t \partial \omega} = \frac{1}{2} - \frac{\partial^2 \phi_{\mathrm{sym}}}{\partial t \partial \omega} \tag{19}$$

and

$$\frac{\partial \widehat{\omega}(\omega,t)}{\partial \omega} = \frac{\partial^2 \phi_{\mathrm{ti}}}{\partial \omega \partial t} = 1 + \frac{\partial^2 \phi_{\mathrm{fi}}}{\partial \omega \partial t} = \frac{1}{2} + \frac{\partial^2 \phi_{\mathrm{sym}}}{\partial \omega \partial t}. \tag{20}$$

and they do not depend on the order of the directions.

In [1], the following quantities were defined: the *chanelized instantaneous frequency* as

$$\mathrm{CIF}(\omega,t) = \frac{\partial \phi_{\mathrm{ti}}}{\partial t}(\omega,t) \tag{21}$$

and *local group delay* as

$$\mathrm{LGD}(\omega,t) = -\frac{\partial \phi_{\mathrm{ti}}}{\partial \omega}(\omega,t). \tag{22}$$

We define *relative instantaneous frequency* as

$$\mathrm{RIF}(\omega,t) = \frac{\partial \phi_{\mathrm{fi}}}{\partial t}(\omega,t) \tag{23}$$

Moreover, *instantaneous slope* was defined as a directional derivative in the direction of the unit vector $\boldsymbol{\alpha} = [\alpha_\omega, \alpha_t]$

$$\mathrm{IS}_{\boldsymbol{\alpha}}(\omega,t) = \frac{\frac{\partial \widehat{\omega}}{\partial \alpha}}{\frac{\partial \widehat{t}}{\partial \alpha}} \tag{24}$$

which for $\boldsymbol{\alpha} = [0,1]$ becomes

$$\mathrm{IS}_{[0,1]}(\omega,t) = \frac{\frac{\partial^2 \phi_{\mathrm{ti}}}{\partial t^2}}{-\frac{\partial^2 \phi_{\mathrm{fi}}}{\partial t \partial \omega}} \tag{25}$$

and for $\boldsymbol{\alpha} = [1,0]$ becomes

$$\mathrm{IS}_{[1,0]}(\omega,t) = \frac{\frac{\partial^2 \phi_{\mathrm{ti}}}{\partial \omega \partial t}}{-\frac{\partial^2 \phi_{\mathrm{fi}}}{\partial t^2}} \tag{26}$$

The authors of [2] describe the physical meaning of the mixed derivatives such that

$$\frac{\partial \widehat{\omega}(\omega,t)}{\partial \omega} \approx 0 \tag{27}$$

for constant sinusoidal components and similarly

$$\frac{\partial \widehat{t}(\omega,t)}{\partial t} \approx 0 \tag{28}$$

for impulsive components.

2 DGT

DGT of signal $f \in \mathbb{C}^L$ using window g and phase convention pc will be denoted as

$$c_{\mathrm{pc}}(m,n) = s(m,n)\mathrm{e}^{i\varphi_{\mathrm{pc}}(m,n)}. \tag{29}$$

where m and n are frequency and time indices respectivelly.
Frequency-invariant phase is obtained as

$$\begin{aligned} c_{\mathrm{fi}}(m,n) &= \langle f, M_{mb}T_{na}g\rangle & (30)\\ &= \sum_{l=0}^{L-1} f(l)e^{-i2\pi mbl/L}\overline{g(l-an)} & (31)\end{aligned}$$

for $m = 0,\dots,M-1$ and $n = 0,\dots,N-1$ where $M = L/b$ is the number of channels, $N = L/a$ numer of time shifts, a is a hop factor in time and b hop factor in frequency. T_{na} is a circular translation operator $(T_{na}h)(l) = h(l-an)$ and M_{mb} is a modulation operator $(M_{mb}h)(l) = h(l)e^{i2\pi mbl/L}$. $l-an$ is assumed to be evaluated modulo L.
Time-invariant phase is obtained as

$$\begin{aligned} c_{\mathrm{ti}}(m,n) &= \langle f, T_{na}M_{mb}g\rangle & (32)\\ &= \sum_{l=0}^{L-1} f(l)e^{-i2\pi mb(l-an)/L}\overline{g(l-an)} & (33)\\ &= \sum_{l=0}^{L-1} f(l+an)e^{-i2\pi mbl/L}\overline{g(l)} & (34)\\ &= e^{i2\pi mbna/L}c_{\mathrm{fi}}(m,n) & (35)\end{aligned}$$

$l+am$ is assumed to be evaluated modulo L.
Symmetric phase is obtained as

$$\begin{aligned} c_{\mathrm{sym}}(m,n) &= \langle f, M_{mb/2}T_{na}M_{mb/2}g\rangle & (36)\\ &= \sum_{l=0}^{L-1} f(l)e^{-i2\pi mb(l-an/2)/L}\overline{g(l-an)} & (37)\\ &= e^{i\pi mbna/L}c_{\mathrm{fi}}(m,n). & (38)\end{aligned}$$

Note that the symmetric case introduces a phase discontinuity at the borders if b is odd.

The relations between the phases follow directly

$$\varphi_{\mathrm{fi}}(m,n) = \varphi_{\mathrm{ti}}(m,n) - 2\pi mbna/L = \varphi_{\mathrm{sym}}(m,n) - \pi mbna/L \tag{39}$$

for m,n as before.

2.1 Phase derivatives

DGT amounts to STFT defined on L-periodic signals sampled uniformly with step 1. In the discrete case, we can only approximate the derivatives numerically. In this document, we abuse the notation slightly and we denote the numerical approximation of a phase derivatives as $\frac{\partial\varphi(m,n)}{\partial\omega}$ and $\frac{\partial\varphi(m,n)}{\partial t}$ respectivelly. We further assume that the derivatives are scaled such that the values are in samples.

In the similar spirit as in sec. **??**, we list the relationships between the approximate phase derivatives. The phase derivatives along *frequency* are linked in the following way:

$$\widehat{n}(m,n)a = na - \frac{\partial\varphi_{\mathrm{ti}}}{\partial\omega} = -\frac{\partial\varphi_{\mathrm{fi}}}{\partial\omega} = \frac{an}{2} - \frac{\partial\varphi_{\mathrm{sym}}}{\partial\omega}, \tag{40}$$

where $\widehat{n}(m,n)$ denotes the possibly noninteger instantaneous time index.

Similarly, phase derivatives along *time* are linked in the following way:

$$\widehat{m}(m,n)b = \frac{\partial \varphi_{\mathrm{ti}}}{\partial t} = bm + \frac{\partial \varphi_{\mathrm{fi}}}{\partial t} = \frac{bm}{2} + \frac{\partial \varphi_{\mathrm{sym}}}{\partial t}, \tag{41}$$

where $\widehat{m}(m,n)$ denotes the possibly noninteger instantaneous frequency index.

Second derivatives do not depend on the phase convention anymore i.e.

$$\frac{\partial \widehat{t}(\omega,t)}{\partial \omega} = -\frac{\partial^2 \varphi_{\mathrm{ti}}}{\partial \omega^2} = -\frac{\partial^2 \varphi_{\mathrm{fi}}}{\partial \omega^2} = -\frac{\partial^2 \varphi_{\mathrm{sym}}}{\partial \omega^2} \tag{42}$$

and

$$\frac{\partial \widehat{\omega}(\omega,t)}{\partial t} = \frac{\partial^2 \varphi_{\mathrm{ti}}}{\partial t^2} = \frac{\partial^2 \varphi_{\mathrm{fi}}}{\partial t^2} = \frac{\partial^2 \varphi_{\mathrm{sym}}}{\partial t^2}. \tag{43}$$

Mixed phase derivatives are linked as follows:

$$\frac{\partial \widehat{n}(m,n)}{\partial t} = 1 - \frac{\partial^2 \varphi_{\mathrm{ti}}}{\partial t \partial \omega} = -\frac{\partial^2 \varphi_{\mathrm{fi}}}{\partial t \partial \omega} = \frac{1}{2} - \frac{\partial^2 \varphi_{\mathrm{sym}}}{\partial t \partial \omega} \tag{44}$$

and

$$\frac{\partial \widehat{m}(m,n)}{\partial \omega} = \frac{\partial^2 \varphi_{\mathrm{ti}}}{\partial \omega \partial t} = 1 + \frac{\partial^2 \varphi_{\mathrm{fi}}}{\partial \omega \partial t} = \frac{1}{2} + \frac{\partial^2 \varphi_{\mathrm{sym}}}{\partial \omega \partial t}. \tag{45}$$

and they do not depend on the order of the directions.

2.2 Matlab functions

Function `[tgrad,fgrad]=gabphasegrad(...)` returns `tgrad` and `fgrad` which approximate the following:

$$\texttt{tgrad} \rightarrow \frac{\partial \varphi_{\mathrm{fi}}}{\partial t} = \mathrm{RIF}(m,n) \tag{46}$$

$$\texttt{fgrad} \rightarrow -\frac{\partial \varphi_{\mathrm{ti}}}{\partial \omega} = \mathrm{LGD}(m,n) \tag{47}$$

We define this convention as *relative* as the time-frequency center of gravity functions contain only additions:

$$\widehat{n}(m,n)a = na + \mathrm{LGD}(m,n) \tag{48}$$

$$\widehat{m}(m,n)b = mb + \mathrm{RIF}(m,n). \tag{49}$$

`gabphasederiv` computes approximates of the derivatives directly. It accepts flags `'freqinv'` (default), `'freqinv'`, `'symphase'` and `'relative'`.

2.3 Algorithms

The numerical differentiation is a well established part of numerical analysis. It however cannot be applied directly to phases as they live on a torus $[-\pi, \pi[$. For dense enough sampling, the *phase unwrapping* can fix that. The simplest algorithm for phase unwrapping adds or substracts 2π when an abs. value of a phase difference of two neighboring coefficients is greater than π.

This however only works for dense enough sampling since it relies on the fact that the abs. difference of phase of two neighboring coefficients is less than π.

In practise, a prior information is used to help the phase unwrapping task such as a constant factor phase increase is assumed in each channel m along time index n see chapter 7.3.5 in [**?**] (note that φ and $\widetilde{\varphi}$ denote frequency and time invariant phases respectively).

The phase wrapping problem arises in other fields such as like radar interferometry and more advanced algorithms has been developed there.

TBD: Auger Flandrin algorithm

TBD: Nelson algorithm

3 Reassignment

Equations (40),(41) already give the formulas for the T-F reassignment [3, 4]. Recently, an adjustable version of the reassignment was introduced in [5]. The new time-frequency center of gravity depends on parameter $\mu \in \mathbb{R}^+$ and is given by

$$\begin{pmatrix} \widetilde{\omega}(\omega, t) \\ \widetilde{t}(\omega, t) \end{pmatrix} = \begin{pmatrix} \omega \\ t \end{pmatrix} - \left(\nabla^t R(\omega, t) + \mu I \right)^{-1} R(\omega, t) \tag{50}$$

where I is two-dimensional identity matrix, $R(\omega, t)$ is a negative of a relative displacement vector

$$R(\omega, t) = \begin{pmatrix} \omega \\ t \end{pmatrix} - \begin{pmatrix} \widehat{\omega}(\omega, t) \\ \widehat{t}(\omega, t) \end{pmatrix} \tag{51}$$

and $\nabla^t R(\omega, t)$ is a Jacobian matrix such that

$$\nabla^t R(\omega, t) = \left(\frac{\partial R(\omega,t)}{\partial \omega}, \; \frac{\partial R(\omega,t)}{\partial t} \right) \tag{52}$$

Equation (50) can be rewritten such that

$$\left(\nabla^t R(\omega, t) + \mu I \right) \left[\begin{pmatrix} \widetilde{\omega}(\omega, t) \\ \widetilde{t}(\omega, t) \end{pmatrix} - \begin{pmatrix} \omega \\ t \end{pmatrix} \right] = \begin{pmatrix} \widehat{\omega}(\omega, t) \\ \widehat{t}(\omega, t) \end{pmatrix} - \begin{pmatrix} \omega \\ t \end{pmatrix} \tag{53}$$

4 Alternative conventions

In the literature, two deviations from the conventions presented in (4),(8) and (13) can be found. Namely the window g is sometimes time-reversed, then for fixed ω the formulas become convolutions and the window is sometimes not conjugated. Both facts have no effect when using odd-symmetric and real windows.

References

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