A class of warped filter bank frames tailored to non-linear frequency scales

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Abstract

A method for constructing non-uniform filter banks is presented. Starting from a uniform system of translates, generated by a prototype filter, a non-uniform covering of the frequency axis is obtained by composition with a warping function. The warping function is a $\mathcal{C}^1$-diffeomorphism that determines the frequency progression and can be chosen freely, apart from minor technical restrictions. The resulting functions are interpreted as filter frequency responses and, combined with appropriately chosen downsampling factors, give rise to a non-uniform analysis filter bank. Classical Gabor and wavelet filter banks are obtained as special cases. Beyond the state-of-the-art, we construct a filter bank adapted to the auditory ERB frequency scale and families of filter banks that can be interpreted as an interpolation between linear (Gabor) and logarithmic (wavelet, ERBlet) frequency scales, closely related to the α-transform. For any arbitrary warping function, we derive straightforward decay conditions on the prototype filter and bounds for the downsampling factors, such that the resulting warped filter bank forms a frame. In particular, we obtain a simple and constructive method for obtaining tight frames with bandlimited filters by invoking previous results on generalized shift-invariant systems.

Keywords: time-frequency; adaptive systems; frames; generalized shift-invariant systems; non-uniform filter banks; warping

1. Introduction

In this contribution, we introduce a class of non-uniform time-frequency systems optimally adapted to non-linear frequency scales. The central paradigm of our construction, and what differentiates it from previous approaches, is to provide uniform frequency resolution *on the target frequency scale*. Invertible time-frequency systems are of particular importance, since they allow for stable recovery of signals from the time-frequency representation coefficients. Therefore, we also derive necessary and sufficient conditions for the resulting systems to form a frame.

To demonstrate the flexibility and importance of our construction, illustrative examples recreating (or imitating) classical time-frequency representations such as Gabor [1, 2, 3, 4], wavelet [5, 6]

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or α-transforms [7, 8, 9] are provided. While this paper considers the setting of (discrete) Hilbert space frames, the properties of continuous warped time-frequency systems are investigated in the related contribution [10]. Whenever a time-frequency filter bank adapted to a given frequency progression and with linear time-progression in each channel is desired, we believe that the proposed *warped filter banks* provide an optimal framework for its design.

In the proposed method, generalized shift-invariant (GSI) systems [11, 12, 13, 14] over $\mathbf{L}^2(\mathbb{R})$ are constructed from a prototype frequency response via composition with a *warping function* that specifies the desired frequency scale/progression. To highlight the relation of the resulting warped time-frequency systems to non-uniform filter banks, we use terminology from filter bank theory and refer to GSI systems as (analysis) filter banks, despite operating on $\mathbf{L}^2(\mathbb{R})$ instead of the sequence space $\ell^2(\mathbb{Z})$.

It will be shown that warped time-frequency systems provide a very natural and intuitive framework for time-frequency analysis on non-linear frequency scales. Most importantly, invertible systems are constructed with ease, in particular tight filter bank frames with bandlimited filters can be obtained through a very simple procedure. Moreover, the selection of appropriate downsampling factors (sampling steps) is simplified by the filter bandwidths' direct link to the derivative of the warping function, leading to a canonical choice of *natural downsampling factors*. These downsampling factors are determined for all frequency channels simultaneously by the selection of a single parameter. The later sections of this contribution are concerned with providing some examples for the given abstract framework, as well as its adaptation to digital signals in $\ell^2(\mathbb{Z})$, see also [15], where warped filter banks for $\ell^2(\mathbb{Z})$ where first presented.

Adapted time-frequency systems. Time-frequency (or time-scale) representations are an indispensable tool for signal analysis and processing. The most widely used and most thoroughly explored such representations are certainly Gabor and wavelet transforms and their variations, e.g. windowed modified cosine [16, 17] or wavelet packet [18] transforms. The aforementioned transforms unite two very important properties: There are various, well-known necessary and/or sufficient conditions for stable inversion from the transform coefficients, i.e. for the generating function system to form a frame. In addition to the perfect reconstruction property, the frame property ensures stability of the synthesis operation after coefficient modification, enabling controlled time-frequency processing. Furthermore, efficient algorithms for the computation of the transform coefficients and the synthesis operation exist for each of the mentioned transforms [19, 5].

While providing a sound and well-understood mathematical foundation, Gabor and wavelet transforms are designed to follow two specified frequency scales, i.e. linear and logarithmic, respectively. A wealth of approaches exists to soften this restriction, e.g. decompositions using filter banks [20], in particular based on perceptive frequency scales [21]. Adaptation over time is considered in approaches such as modulated lapped transforms [22], adapted local trigonometric transforms [23] or (time-varying) wavelet packets [24]. Techniques that jointly offer flexible time-frequency resolution and redundancy, the perfect reconstruction property and efficient computation are scarce however. The setting of so-called nonstationary Gabor transforms [14], a recent generalization of classical Gabor transforms, provides the latter 2 properties while allowing for freely chosen time progression and varying resolution. In this construction, the frequency scale is still linear, but the sampling density may be changed over time. The properties of nonstationary Gabor systems are a matter of ongoing investigation, but a number of results already exist [25, 26, 27]. When desiring increased flexibility along frequency, generalized shift-invariant systems [11, 12, 28, 29, 13], or equivalently (non-uniform) *filter banks* [30], provide the analogous concept. They offer full flexibility in frequency, with a linear time progression in each filter, but flexible sampling density across the filters. Analogous, continuously indexed systems are considered in [31, 32]. Indeed, nonstationary Gabor systems are equivalent to filter banks via an application of the (inverse) Fourier transform to the generating functions. Note that all the widely used transforms mentioned in the previous paragraph can be interpreted as filter banks.

Adaptation to non-linear frequency scales through warping. There have been previous attempts to construct adapted filter banks by frequency warping. All previous methods have in common, however, that they focus on unitary warping operators that cannot provide the shape-preserving property that is central to our approach. Therefore, the properties of the resulting systems and the challenges faced in their construction are quite different.

For example, Braccini and Oppenheim [33], as well as Twaroch and Hlawatsch [34], propose a unitary warping of a collection system of translates, interpreted as filter frequency responses. In [33] only spectral analysis is desired, while time-frequency distributions are constructed in [34], without considering signal reconstruction.

The application of unitary warping to an entire Gabor or wavelet system has also been investigated [35, 36, 37, 38]. Although unitary transformation bequeaths basis (or frame) properties to the warped atoms, the resulting system is not anymore a filter bank. Instead, the warped system produces undesirable, dispersive time-shifts and the resulting representation is not easily interpreted, see [37]. Only for the continuous short-time Fourier transform, or under quite strict assumptions on a Gabor system, a *redressing* procedure can be applied to recover a GSI system [39]. In all other cases, the combination of unitary warping with redressing complicates the efficient, exact computation of redressed warped Gabor frames, such that approximate implementations are considered [40].

Finally, it should be noted that the idea of a (non-unitary) logarithmic warping of the frequency axis to obtain wavelet systems from a system of translates was already used in the proof of the so called *painless conditions* for wavelets systems [41]. However, the idea has never been relaxed to other frequency scales so far. While the parallel work by Christensen and Goh [42] focuses on exposing the duality between Gabor and wavelet systems via the mentioned logarithmic warping, we will allow for more general warping functions to generate time-frequency transformations beyond wavelet and Gabor systems. The proposed warping procedure has already proven useful in the area of graph signal processing [43].

2. Preliminaries

We use the following normalization of the Fourier transform

$$\hat{f}(\xi) := \mathcal{F}f = \int_{\mathbb{R}} f(t)e^{-2\pi i t\xi}\, dt, \text{for all } f \in \mathbf{L}^1(\mathbb{R})$$

and its unitary extension to $\mathbf{L}^2(\mathbb{R})$. The inverse Fourier transform is denoted by $\check{f} = \mathcal{F}^{-1}f$. Further, we require the *modulation operator* and the *translation operator* defined by $\mathbf{M}_\omega f = f \cdot e^{2\pi i\omega(\cdot)}$ and $\mathbf{T}_x f = f(\cdot - x)$ respectively for all $f \in \mathbf{L}^2(\mathbb{R})$. The composition $f(g(\cdot))$ of two functions f and g is denoted by $f \circ g$ and the standard Lebesgue measure by μ.

When discussing the properties of the constructed function systems in the following sections, we will repeatedly use the notions of weight functions and weighted $\mathbf{L}^2$-spaces. Weighted $\mathbf{L}^2$-spaces are defined as

$$\mathbf{L}^2_w(D) := \left\{ f : D \mapsto \mathbb{C} \ : \ \int_D w^2(t)|f(t)|^2\, dt < \infty \right\},$$

with a continuous, nonnegative function $w : D \mapsto \mathbb{R}$ called *weight function.* The associated norm is defined as expected. In the following, when the term weight function is used, continuity and non-negativity are always implied. Two special classes of weight functions are of particular interest: *Continuous, positive weight functions* $v : \mathbb{R} \to \mathbb{R}^+$ and $w : \mathbb{R} \to \mathbb{R}^+$ are called *submultiplicative* and *v-moderate* respectively if they satisfy

$$v(x+y) \leq v(x)v(y), \text{ and } w(x+y) \leq Cv(x)w(y), \tag{1}$$

for all $x, y \in \mathbb{R}$ and some positive constant C. In particular, we can (and will) always choose the constant to equal 1 in the latter inequality, using that $\max\{C, 1\}v$ is submultiplicative, whenever v is. Submultiplicative and moderate weight functions play an important role in the theory of function spaces, as they are closely related to the translation-invariance of the corresponding weighted spaces [44, 2], see also [45] for an in-depth analysis of weight functions and their role in harmonic analysis.

A generalized shift-invariant (GSI) system on $\mathbf{L}^2(\mathbb{R})$ is a union of shift-invariant systems $\{\mathbf{T}_{na_m}\widecheck{g_m} \in \mathbf{L}^2(\mathbb{R}) \ : \ n \in \mathbb{Z}\}$, with $\widecheck{g_m} \in \mathbf{L}^2(\mathbb{R})$ and $a_m \in \mathbb{R}^+$. The representation coefficients of a function $f \in \mathbf{L}^2(\mathbb{R})$ with respect to the GSI system are given by the inner products

$$c_{n,m} := c_f(n,m) := \langle f, \mathbf{T}_{na_m}\widecheck{g_m}\rangle = \left(f * \overline{\widecheck{g_m}(-\cdot)}\right)(na_m),$$

for all n, m. The above representation of the coefficients in terms of a convolution alludes to the fact that $c_f(\cdot, m)$ is a filtered, and sampled, version of f. This relation justifies our use of filter bank terminology when discussing GSI systems. Instead of arbitrary functions in $\mathbf{L}^2(\mathbb{R})$, we consider here functions whose Fourier spectrum is restricted to an interval D contained in $\mathbb{R}$, i.e. elements of

$$\mathbf{L}^{2,\mathcal{F}}(D) := \mathcal{F}^{-1}\left(\mathbf{L}^2(D)\right) \subseteq \mathbf{L}^2(\mathbb{R}).$$

Prototypical cases are $D = \mathbb{R}$ and $D = \mathbb{R}^+$, the latter leading to analytic functions.

Definition 1. Let $\mathbf{g} = (g_m)_{m\in\mathbb{Z}} \subset \mathbf{L}^2(D)$ and $\mathbf{a} = (a_m)_{m\in\mathbb{Z}}$, with $a_m \in \mathbb{R}^+$ for all $m \in \mathbb{Z}$. We call the system

$$\mathcal{G}(\mathbf{g}, \mathbf{a}) = (g_{m,n})_{m,n\in\mathbb{Z}}, \quad g_{m,n} := T_{na_m}\widecheck{g_m} = T_{na_m}\mathcal{F}^{-1}(g_m), \text{ for all } n, m \in \mathbb{Z}, \tag{2}$$

a *(non-uniform) filter bank* for $\mathbf{L}^{2,\mathcal{F}}(D)$. The elements of $\mathbf{g}$ are called *frequency responses* and $\mathbf{a}$ are the corresponding *downsampling factors.*

Such filter banks can be used to analyze signals in $\mathbf{L}^{2,\mathcal{F}}(D)$ and for a given signal $f \in \mathbf{L}^{2,\mathcal{F}}(D)$, we refer to the sequence $c_f := (c_f(n,m))_{n,m\in\mathbb{Z}} = (\langle f, g_{m,n}\rangle)_{m,n\in\mathbb{Z}}$ as the *filter bank (analysis) coefficients.*

For many applications it is of great importance that all the considered signals can be reconstructed from these coefficients, in a stable fashion. It is a central observation of frame theory that this is equivalent to the existence of constants $0 < A \leq B < \infty$, such that

$$A\|f\|_2^2 \leq \|c_f\|^2_{\ell^2(\mathbb{Z}^2)} \leq B\|f\|_2^2, \text{ for all } f \in \mathbf{L}^{2,\mathcal{F}}(D). \tag{3}$$

A system $\mathcal{G}(\mathbf{g}, \mathbf{a})$ that satisfies this condition is called *filter bank frame* [46, 29], and *tight (filter bank) frame* if equality can be achieved in (3). If $\mathcal{G}(\mathbf{g}, \mathbf{a})$ is a frame, then the frame operator given by

$$\mathbf{S} : \mathbf{L}^{2,\mathcal{F}}(D) \to \mathbf{L}^{2,\mathcal{F}}(D), \quad \mathbf{S}f = \sum_{m,n\in\mathbb{Z}} c_f(m,n) g_{m,n}, \text{ for all } f \in \mathbf{L}^{2,\mathcal{F}}(D), \tag{4}$$

is invertible. The frame operator is tremendously important and the key component for an appropriate synthesis system that maps the coefficient space $\ell^2(\mathbb{Z}^2)$ to the signal space $\mathbf{L}^{2,\mathcal{F}}(D)$. Namely, the *canonical dual frame* $(\widetilde{g_{m,n}})_{m,n\in\mathbb{Z}}$, obtained by applying the inverse of the frame operator to the frame elements, i.e. $\widetilde{g_{m,n}} := \mathbf{S}^{-1}(g_{m,n})$, for all $m, n \in \mathbb{Z}$, facilitates *perfect reconstruction* from the analysis coefficients:

$$f = \sum_{m,n\in\mathbb{Z}} c_f(m,n) \widetilde{g_{m,n}}, \text{ for all } f \in \mathbf{L}^{2,\mathcal{F}}(D). \tag{5}$$

If at least the upper inequality in (4) is satisfied, then $\mathcal{G}(\mathbf{g}, \mathbf{a})$ is a *Bessel sequence.*

Abstract filter bank frames have received considerable attention, e.g. in [20], as (generalized) shift-invariant systems in [47, 12, 11, 13, 32] and as (frequency-side) nonstationary Gabor systems in [14, 26, 27, 25]. In contrast, this contribution is concerned with a concrete, structured family of filter bank systems and how the superimposed structure can be used to construct filter bank frames.

3. Warped filter banks

In signal analysis, the usage of different frequency scales has a long history. Linear and logarithmic scales arise naturally when constructing a filter bank through modulation or dilation of a single prototype filter, respectively. In this way, the classical Gabor and wavelet transforms are obtained. The consideration of alternative frequency scales can be motivated, for example, from (a) theoretical interest in a family of time-frequency representations that serve as an interpolation between the two extremes, as is the case for the α-transform, which can in fact be related to polynomial scales, or (b) specific applications and/or signal classes. A prime example for the second case is audio signal processing with respect to an auditory frequency scale, e.g. in gammatone filter banks [48, 49] adapted to the ERB scale [50]. All the mentioned methods have several things in common: They are based on a single prototype filter and possess the structure of a GSI (or filter bank) system. Moreover, the bandwidth of the filters g_k is directly linked to their center frequency through the frequency progression rate.

The filter banks we propose in this section have the property that they are designed as a system of translates on a given frequency scale. This scale determines a conversion from frequency to a new unit (e.g. ERB) with respect to which the designed filters provide a uniform resolution. In the next sections we will show that this construction admits a special class of non-uniform filter banks with a simplified structure compared to general filter banks.

Formally, a frequency scale is specified by a continuous, bijective function $\Phi : D \to \mathbb{R}$ and the transition between the non-linear scale Φ and the unit linear scale is achieved by Φ and Φ^{-1}. Hence, we construct filter frequency responses from a prototype function $\theta : \mathbb{R} \mapsto \mathbb{C}$ by

$$(\theta_{\Phi,m})_{m\in\mathbb{Z}}, \text{ where } \theta_{\Phi,m} = (\mathbf{T}_m\theta) \circ \Phi. \tag{6}$$

This general formulation provides tremendous flexibility for frequency scale design. Furthermore, it comes as no surprise that choosing Φ as $\Phi(\xi) \mapsto a\xi$ or $\Phi(\xi) \mapsto \log_a(\xi)$, for a finite constant

$a > 0$, yields systems of translates $\mathbf{T}_{m/a}(\theta(a\cdot))$ and dilates $(\theta \circ \log_a)(\cdot/a^m)$, respectively. Such Φ will provide the starting point for recovering Gabor and wavelet filter banks in our framework.

Definition 2. Let $D \subseteq \mathbb{R}$ be any open interval. A diffeomorphism $\Phi : D \to \mathbb{R}$, i.e. $\Phi, \Phi^{-1} \in \mathcal{C}^1(D)$, is called *warping function*, if

(i) The derivative Φ' of Φ is positive, i.e. $\Phi' > 0$, and

(ii) There is a submultiplicative weight v, such that the weight function $w := \left(\Phi^{-1}\right)' = \frac{1}{\Phi'(\Phi^{-1}(\cdot))}$ is v-moderate, i.e. $w(\tau_0 + \tau_1) \leq v(\tau_0)w(\tau_1)$, for all $\tau_0, \tau_1 \in \mathbb{R}$.

Proposition 1. *If $\Phi : D \to \mathbb{R}$ is a warping function as per Definition 2, then $\tilde{\Phi} := c\Phi(\cdot/d)$ is a warping function with domain dD, for all positive, finite constants $c, d \in \mathbb{R}^+$.*

Proof. Item (i) of Definition 2 follows from positivity of c, d after noting that $\tilde{\Phi}' = \frac{c}{d}\Phi'(\cdot/d)$. For item (ii), elementary calculations show $\tilde{\Phi}^{-1} = d\Phi^{-1}(\cdot/c)$ and thus

$$(\tilde{\Phi}^{-1})' = \frac{d}{c\Phi'\left(\Phi^{-1}(\cdot/c)\right)} = \frac{d}{c}(\Phi^{-1})'(\cdot/c).$$

We obtain, for all $\tau_0, \tau_1 \in \mathbb{R}$,

$$(\tilde{\Phi}^{-1})'(\tau_0 + \tau_1) = \frac{d}{c}(\Phi^{-1})'\left(\frac{\tau_0 + \tau_1}{c}\right) \leq v(\tau_0/c) \cdot \frac{d}{c}(\Phi^{-1})'\left(\frac{\tau_1}{c}\right) = v(\tau_0/c)(\tilde{\Phi}^{-1})'(\tau_1).$$

Consequently, $(\tilde{\Phi}^{-1})'$ is $v(\cdot/c)$-moderate and submultiplicativity of $v(\cdot/c)$ is easily seen. □

Several things should be noted when considering the definition and proposition above.

- Proposition 1 shows that it really is sufficient to consider integer translates of the prototype θ when constructing the frequency responses $\theta_{\Phi,m}$. If $a > 0$ is arbitrary, then with $\theta_a := \theta(\cdot/a)$, we have
$$(\mathbf{T}_m\theta_a) \circ (a\Phi) = \theta_a\left(a\Phi(\cdot) - m\right) = \theta\left(\Phi(\cdot) - m/a\right) = (\mathbf{T}_{m/a}\theta) \circ \Phi.$$

- Moderateness of $w = \left(\Phi^{-1}\right)'$ ensures translation invariance of the associated weighted $\mathbf{L}^p$-spaces. In particular,
$$\|(T_m\theta) \circ \Phi\|^2_{\mathbf{L}^2(D)} = \|T_m\theta\|^2_{\mathbf{L}^2_{\sqrt{w}}(\mathbb{R})} \leq \begin{cases} v(m)\|\theta\|^2_{\mathbf{L}^2_{\sqrt{w}}(\mathbb{R})} & \text{, if } \theta \in \mathbf{L}^2_{\sqrt{w}}(\mathbb{R}) \\ w(m)\|\theta\|^2_{\mathbf{L}^2_{\sqrt{v}}(\mathbb{R})} & \text{, if } \theta \in \mathbf{L}^2_{\sqrt{v}}(\mathbb{R}). \end{cases} \tag{7}$$

  Note that moderateness of w provides $w(\tau) \leq w(0)v(\tau)$ for all $\tau \in \mathbb{R}$. Therefore $\|\theta\|^2_{\mathbf{L}^2_{\sqrt{w}}(\mathbb{R})} \leq w(0)\|\theta\|^2_{\mathbf{L}^2_{\sqrt{v}}(\mathbb{R})}$ and $L^2_{\sqrt{v}}(\mathbb{R}) \subseteq L^2_{\sqrt{w}}(\mathbb{R})$ follows.

A warped filter bank can now be constructed easily. To do so, after selecting the warping function Φ, one simply chooses an appropriate prototype frequency response θ and positive, real downsampling factors $(a_m)_{m\in\mathbb{Z}}$. From here on, $w = (\Phi^{-1})'$ and v will always be weights as specified in Definition 2. Although, in theory, the choice of downsampling factors is arbitrary, the warping function Φ induces a canonical choice, which relates a_m^{-1} to the essential support of the frequency responses and is particularly suited for the creation of warped filter bank frames, see Section 4.

Definition 3. Let $\Phi : D \to \mathbb{R}$ be a warping function and $\theta \in \mathbf{L}^2_{\sqrt{w}}(\mathbb{R})$. Furthermore, let $\mathbf{a} = (a_m)_{m\in\mathbb{Z}}$ be a set of downsampling factors. Then the *warped filter bank* with respect to the triple $(\Phi, \theta, \mathbf{a})$ is given by

$$\mathcal{G}(\Phi, \theta, \mathbf{a}) := (T_{na_m}\widecheck{g_m})_{m,n\in\mathbb{Z}} = \left(T_{na_m}\mathcal{F}^{-1}(g_m)\right)_{m,n\in\mathbb{Z}}, \tag{8}$$

with

$$g_m := \sqrt{a_m}\theta_{\Phi,m} = \sqrt{a_m}(\mathbf{T}_m\theta) \circ \Phi. \tag{9}$$

If either $a_m = \tilde{a}/v(m)$ or both $\theta \in \mathbf{L}^2_{\sqrt{v}}(\mathbb{R})$ and $a_m = \tilde{a}/w(m)$, for all $m \in \mathbb{Z}$ and some $\tilde{a} > 0$, then we say that $\mathbf{a}$ is a set of *natural downsampling factors (for (Φ, θ)).*

Natural downsampling factors are very important, as they guarantee uniform L^2-boundedness of the g_m, recall (7), which is in turn easily seen to be necessary for the Bessel property. Depending on how much the submultiplicative weight v deviates from $w = (\Phi^{-1})'$, the two sets of natural downsampling factors may differ significantly. Similarly, there can be a large gap between the spaces $\mathbf{L}^2_{\sqrt{w}}(\mathbb{R})$ and $\mathbf{L}^2_{\sqrt{v}}(\mathbb{R})$ of prototype functions. Therefore, the results in this contribution will be presented as to cover both settings.

To conclude, we give some examples of warping functions that are of particular interest, e.g. because they encompass important frequency scales. In Proposition 2 at the end of this section, we show that the presented examples indeed define warping functions in the sense of Definition 2. Some instances of the warping functions in the following examples can be seen in Figure 1.

Example 1 (Wavelets). Choosing $\Phi = \log$, with $D = \mathbb{R}^+$ leads to a system of the form

$$\theta_{\Phi,m}(\xi) = \theta(\log(\xi) - m) = \theta_{\Phi,0}(\log(\xi e^{-m})).$$

This warping function therefore leads to $\theta_{\Phi,m}$ being a dilated version of $\theta_{\Phi,0}$ up to normalization. The natural downsampling factors are given by $a_m = \tilde{a}/w(m) = \tilde{a}e^{-m}$. This shows that $\mathcal{G}(\theta, \log, \tilde{a}e^{-m})$ is indeed a wavelet system, with the minor modification that our scales are reciprocal to the usual definition of wavelets.

Example 2. The family of warping functions $\Phi_l(\xi) = c\left((\xi/d)^l - (\xi/d)^{-l}\right)$, for some $c, d > 0$ and $l \in (0, 1]$, is an alternative to the logarithmic warping for the domain $D = \mathbb{R}^+$. The logarithmic warping in the previous example can be interpreted as the limit of this family for $l \to 0$ in the sense that for any fixed $\xi \in \mathbb{R}^+$,

$$\Phi_l'(\xi) = \frac{lc}{d}\left((\xi/d)^{-1+l} + (\xi/d)^{-1-l}\right) \overset{l\to 0}{\to} \frac{2lc}{\xi} = \frac{2lc}{d}\log'(\xi/d).$$

This type of warping provides a frequency scale that approaches the limits 0 and ∞ of the frequency range D in a slower fashion than the wavelet warping. In other words, $\theta_{\Phi_l,m}(\xi)$ is less deformed for $m > 0$, but more deformed for $m < 0$ than in the case $\Phi = \log$. Furthermore, the property that $\theta_{\Phi,m}$ can be expressed as a dilated version of $\theta_{\Phi,0}$ is lost.

Example 3 (ERBlets). In psychoacoustics, the investigation of filter banks adapted to the spectral resolution of the human ear has been subject to a wealth of research, see [51] for an overview. We mention here the Equivalent Rectangular Bandwidth scale (ERB-scale) described in [50], which introduces a set of bandpass filters modeling human perception. In [52] the authors construct a filter bank that is designed to be adapted to the ERB-scale. In our terminology the ERB warping function is given by

$$\Phi_{\mathrm{ERB}}(\xi) = \operatorname{sgn}(\xi)\; c \log\left(1 + \frac{|\xi|}{d}\right),$$

where the constants are given by $c = 9.265$ and $d = 228.8$. Using this function, we can construct a similar filter bank, but with better control over the frame properties. The transform based on an ERB filter bank has potential applications in audio signal processing, as it provides a perfectly invertible transform adapted to the human perception of sound.

Example 4. The family of warping functions $\Phi_l(\xi) = \operatorname{sgn}(\xi)\,\big((|\xi|+1)^l - 1\big)$, for some $l \in (0,1]$ leads to filter banks that are structurally very similar to the *α-transform*, see [7, 8, 9, 53]. The α-transform provides a family of time-frequency transforms with varying time-frequency resolution. The time-frequency atoms are constructed from a single prototype by a combination of translation, modulation and dilation:

$$g_{\xi,x} = \eta_\alpha(\xi)^{1/2}\mathbf{M}_\xi g\left(\eta_\alpha(\xi)(\cdot - x)\right),$$

for $\eta_\alpha(\xi) = (1+|\xi|)^\alpha$, with $\alpha \in [0,1]$. If $\mathcal{F}g$ is a symmetric bump function centered at frequency 0, with a bandwidth[1] of 1, then $\mathcal{F}g_{\xi,0}$ is a symmetric bump function centered at frequency ξ, with a bandwidth of $\eta_\alpha(\xi)$. Up to a phase factor, $g_{\xi,x} = \mathbf{T}_x g_{\xi,0}$. Varying α, one can *interpolate* between the Gabor transform ($\alpha = 0$, constant time-frequency resolution) and a wavelet-like (or more precisely ERB-like) transform with the dilation depending linearly on the center frequency ($\alpha = 1$).

Through our construction, we can obtain a transform with similar properties by using the warping functions $\Phi_l(t) = l^{-1}\operatorname{sgn}(t)\big((1+|t|)^l - 1\big)$, for $l \in (0,1]$, and $\Phi_0(t) = \operatorname{sgn}(t)\log(1+|t|)$, introduced here and in Example 3. Take θ a symmetric bump function centered at frequency 0, with a bandwidth of 1. Then $\theta_{\Phi_l,m}$ is still a bump function with peak frequency $\Phi_l^{-1}(m)$, but only symmetric if $l = 1$ or $m = 0$. Moreover, the bandwidth of $\theta_{\Phi_l,m}$ equals

$$\Phi_l^{-1}(m+1/2) - \Phi_l^{-1}(m-1/2) = \int_{-1/2}^{1/2} (\Phi_l^{-1})'(m+s)\, ds \approx 1/(\Phi_l)'(\Phi_l^{-1}(m)).$$

Note that $(\Phi_l)'(\xi) = (1+|\xi|)^{l-1} = 1/\eta_{1-l}(\xi)$, for $l \in]0,1]$, and for $\Phi_0(\xi) = \operatorname{sgn}(\xi)\log(1+|\xi|)$ we obtain $(\Phi_l)'(\xi) = (1+|\xi|)^{-1} = 1/\eta_1(\xi)$. Finally, the time-frequency atoms are again obtained by translation of $\mathcal{F}^{-1}(\theta_{\Phi_l,m})$. All in all, it can be expected that the obtained warped filter banks provide a time-frequency representation very similar to sampled α-transforms with the corresponding choice of α.

Example 5. Finally, we propose a warping function for representing functions band-limited to the interval $D = (-\pi,\pi)$. For this purpose set $\Phi(\xi) = \tan(\xi)$. Necessarily, the frequency responses $g_m = \sqrt{a_m}(\mathbf{T}_m\theta)\circ\Phi$ are all compactly supported on D and increasingly peaky and concentrated at the upper and lower borders of D, as m tends to ∞ and $-\infty$, respectively. By using the equivalence of GSI systems and nonstationary Gabor systems [14] through application of the Fourier transform, we can thus construct time-frequency systems on arbitrary open intervals. Frames for intervals have been proposed previously by Abreu et al. [54].

Proposition 2. *The following functions are warping functions:*

[1] The exact definition of bandwidth, e.g. frequency support or -3db bandwidth, is not important for this example.

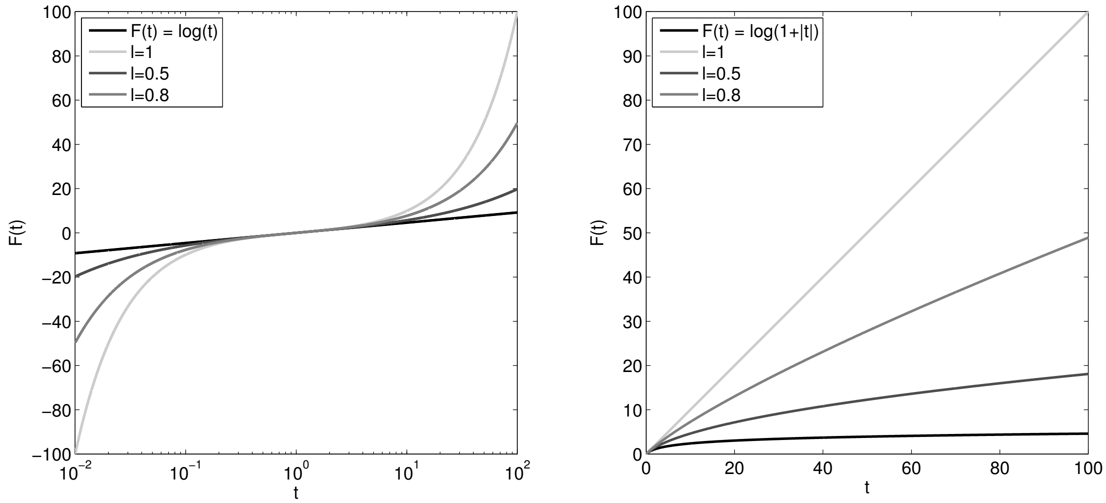

Figure 1: (left) Warping functions from Examples 1 and 2: This plot shows logarithmic (wavelet) warping function (black) and $\Phi_l = l^{-1}(\xi^l - \xi^{-l})$, for $l = 1$ (light gray), $l = 0.5$ (dark gray) and $l = 0.8$ (medium gray). Note that the horizontal axis is logarithmic. (right) Warping functions from Examples 3 and 4: This plot shows the ERBlet warping function (black) and $\Phi_l = l^{-1}\,\mathrm{sgn}(\xi)((1+|\xi|)^l - 1)$, for $l = 1$ (light gray), $l = 0.5$ (dark gray) and $l = 0.8$ (medium gray). The horizontal axis is linear.

(a) $\Phi : \mathbb{R}^+ \to \mathbb{R},\ \xi \mapsto c\log(\xi/d)$ *for some* $c, d > 0$.
(b) $\Phi : \mathbb{R}^+ \to \mathbb{R},\ \xi \mapsto c\left((\xi/d)^l - (\xi/d)^{-l}\right)$ *for some* $c, d > 0$ *and* $l \in (0, 1]$.
(c) $\Phi : \mathbb{R} \to \mathbb{R},\ \xi \mapsto \mathrm{sgn}(\xi)\, c\log(1 + |\xi|/d)$ *for some* $c, d > 0$.
(d) $\Phi : \mathbb{R} \to \mathbb{R},\ \xi \mapsto \mathrm{sgn}(\xi)\, c\left((|\xi|/d + 1)^l - 1\right)$ *for some* $l \in (0, 1]$.
(e) $\Phi : (-\pi, \pi) \to \mathbb{R},\ \xi \mapsto c\tan(\xi/d)$.

Proof. We only consider for each example the case $c = d = 1$, the result for arbitrary $c, d > 0$ then follows from Proposition 1.

(a): In the first case we find that

$$w(\tau_0) = \left(\Phi^{-1}\right)'(\tau_0) = e^{\tau_0}.$$

For this weight function we find $w(\tau_0 + \tau_1) = e^{\tau_0} w(\tau_1)$. Therefore w is even submultiplicative.

(b): Φ is in $\mathcal{C}^\infty(\mathbb{R}^+)$ and

$$w(\tau_0) = \left(\Phi^{-1}\right)'(\tau_0) = \frac{1}{\Phi'(\Phi^{-1}(\tau_0))} = l^{-1}\frac{\Phi^{-1}(\tau_0)}{\Phi^{-1}(\tau_0)^l + \Phi^{-1}(\tau_0)^{-l}}.$$

The inverse of Φ is given by $\Phi^{-1}(\tau_0) = 2^{-1/l}(\tau_0 + \sqrt{\tau_0^2 + 4})^{1/l}$. Assume that Φ^{-1} is v-moderate, then

$$w(\tau_0 + \tau_1) = l^{-1}\frac{\Phi^{-1}(\tau_0 + \tau_1)}{\Phi^{-1}(\tau_0 + \tau_1)^l + \Phi^{-1}(\tau_0 + \tau_1)^{-l}} \le w(\tau_0)\max\{v(\tau_1), v(-\tau_1)v(0)^{-1}\}^{1+l},$$

where we used that submultiplicativity of v implies $v(\tau_1)^{-1} \leq v(-\tau_1)v(0)^{-1}$. Therefore, $w = (\Phi^{-1})'$ is $\tilde{v}$-moderate with $\tilde{v} := \max\{v, v(-\cdot)v(0)^{-1}\}^{1+l}$. Note that

$$\frac{\Phi^{-1}(\tau_1+\tau_0)}{\Phi^{-1}(\tau_0)} = \frac{\tau_1+\tau_0+\sqrt{(\tau_1+\tau_0)^2+4}}{\tau_0+\sqrt{\tau_0^2+4}} = 1 + \frac{\tau_1+\sqrt{(\tau_1+\tau_0)^2+4}-\sqrt{\tau_0^2+4}}{\tau_0+\sqrt{\tau_0^2+4}} \leq v_0(\tau_1), \quad (10)$$

for some submultiplicative function v_0, implies that Φ^{-1} is v-moderate with $v = v_0^{1/l}$. Hence, it is sufficient to show (10).

Observe $\tau_0 + \sqrt{\tau_0^2+4} \geq 0$ and $|\sqrt{(\tau_0+\tau_1)^2+4} - \sqrt{\tau_0^2+4}| \leq |\tau_1|$ to see that

$$\frac{\tau_1+\sqrt{(\tau_1+\tau_0)^2+4}-\sqrt{\tau_0^2+4}}{\tau_0+\sqrt{\tau_0^2+4}} \leq 0 \text{ for all } \tau_1 \leq 0.$$

For any fixed $\tau_1 > 0$, the global maximum over $\tau_0 \in \mathbb{R}$ of $\frac{\tau_1+\sqrt{(\tau_1+\tau_0)^2+4}-\sqrt{\tau_0^2+4}}{\tau_0+\sqrt{\tau_0^2+4}}$ is attained for $\tau_0 = -\tau_1/2$, i.e.

$$\frac{\tau_1+\sqrt{(\tau_1+\tau_0)^2+4}-\sqrt{\tau_0^2+4}}{\tau_0+\sqrt{\tau_0^2+4}} \leq \frac{\tau_1}{\sqrt{\tau_1^2/4+4}-\tau_1/2}.$$

Wit $f = \sqrt{\tau_1^2/4+\cdot}$, the fundamental theorem of calculus yields

$$\sqrt{\tau_1^2/4+4} - y/2 = \int_0^4 \frac{1}{2\sqrt{\tau_1^2/4+\tau}}\, d\tau \geq \frac{2}{\sqrt{\tau_1^2/4+4}}.$$

Therefore,

$$\frac{\tau_1}{\sqrt{\tau_1^2/4+4}-\tau_1/2} \leq \frac{\tau_1^2+16}{4} \leq (2+|\tau_1|/2)^2$$

holds for all $\tau_1 > 0$. Consequently, (10) holds with $v_0 = (2+|\cdot|/2)^2$ and w is $\tilde{v}$-moderate with $\tilde{v} := v_0^{\frac{l+1}{l}} = (2+|\cdot|/2)^{\frac{2+2l}{l}}$.

(c): The third function is easily identified as being in $\mathcal{C}^1$. The corresponding weight function is given by

$$w(\tau_0) = \left(\Phi^{-1}\right)'(\tau_0) = e^{|\tau_0|}.$$

Similar to Example 1, we find that $w(\tau_0+\tau_1) = e^{|\tau_0|}w(\tau_1)$, which shows that w is indeed submultiplicative.

(d): Note that Φ is in $\mathcal{C}^1(\mathbb{R})$. Further, we find

$$w(\tau_0) = \left(\Phi^{-1}\right)'(\tau_0) = \frac{1}{l}(1+|\tau_0|)^{1/l-1}.$$

Moreover, $v := (1+|\cdot|)^{1/l-1}$ is submultiplicative since

$$(1+|\tau_0+\tau_1|)^{1/l-1} \leq (1+|\tau_0|+|\tau_1|+|\tau_0\tau_1|)^{1/l-1} = (|\tau_0|+1)^{1/l-1}(|\tau_1|+1)^{1/l-1}, \text{ for all } \tau_0, \tau_1 \in \mathbb{R}.$$

Since $w = v/l$ this also shows that w is v-moderate.

(e): Let $I_\pi := (-\pi, \pi)$. It is well-known that $\tan \in \mathcal{C}^\infty(I_\pi)$ and $\tan' = \cos^{-2}$. By the inverse function theorem, $\arctan' = \cos^2 \circ \arctan$ and, with $\cos \circ \arctan(\tau) = (1+\tau^2)^{-1/2}$, we obtain $(\Phi^{-1})'(\tau) = w(\tau) = \arctan'(\tau) = (1+\tau^2)^{-1}$. Now, note that

$$\frac{w(\tau_0)}{w(\tau_0+\tau_1)} = 1 + \frac{\tau_1(\tau_0+\tau_1)}{1+\tau_0^2}, \text{ for all } \tau_0, \tau_1 \in \mathbb{R}.$$

If $\operatorname{sgn}(\tau_0) = \operatorname{sgn}(\tau_1)$ or $|\tau_1| \geq |\tau_0|$, then the latter fraction is nonnegative. If $\operatorname{sgn}(\tau_0) \neq \operatorname{sgn}(\tau_1)$ and $|\tau_1| < |\tau_0|$, then

$$\left| \frac{\tau_1(\tau_0+\tau_1)}{1+\tau_0^2} \right| \leq \frac{\tau_0^2}{4(1+\tau_0^2)} \leq \frac{1}{4},$$

and thus

$$w(\tau_0+\tau_1) \leq 4/3 w(\tau_0), \text{ for all } \tau_0, \tau_1 \in \mathbb{R}.$$

Therefore, $w(\tau) = (1+\tau^2)^{-1}$ is v-moderate with $v(\tau) = 4/3$ and $\Phi = \tan$ is a warping function as per Definition 2. $\square$

4. Warped filter bank frames

This contribution is concerned only with filter bank frames that possess additional structure, namely warped filter bank frames. Nonetheless, our results are derived from structural properties and results obtained in the general, abstract filter bank (or GSI) setting. As such, the structure imposed on warped filter banks can be seen as a constructive means to satisfy, or simplify, the conditions of these these abstract results. Therefore, before delving into the frame theory of warped filter banks proper, we recall some previous results on filter banks.

In order to exclude pathologic cases from the study of filter bank frames, it has proven useful to assume that $\mathcal{G}(\mathbf{g}, \mathbf{a})$ satisfies a local integrability condition [12, 32, 13]. This enables the generalization of numerous important results, e.g. a characterization of dual frames, from the frame theory of Gabor systems [2] and uniform filter banks [47], i.e. $a_m = a$ for all $m \in \mathbb{Z}$.

Definition 4. Denote by $\mathcal{D}$ the set of all functions $f \in \mathbf{L}^2(D)$, such that $f \in \mathbf{L}^\infty(D)$ with compact support. We say that $\mathcal{G}(\mathbf{g}, \mathbf{a})$ satisfies the α-local integrability condition (α-LIC), if

$$L_\alpha(f) := \sum_{m\in\mathbb{Z}} \sum_{l\in\mathbb{Z}} a_m^{-1} \int_{\mathbb{R}} \left| f(\xi) f(\xi + l a_m^{-1}) \right| \cdot \left| g_m(\xi) g_m(\xi + l a_m^{-1}) \right| \, d\xi < \infty, \tag{11}$$

for all $f \in \mathcal{D}$. If even

$$L(f) := \sum_{m\in\mathbb{Z}} \sum_{l\in\mathbb{Z}} a_m^{-1} \int_{\operatorname{supp}(f)} \left| f(\xi + l a_m^{-1}) g_m(\xi) \right|^2 \, d\xi < \infty, \tag{12}$$

for all $f \in \mathcal{D}$, then we say that $\mathcal{G}(\mathbf{g}, \mathbf{a})$ satisfies the local integrability condition (LIC).

These local integrability conditions might seem intimidating and opaque at first, but once we impose some structure on $\mathcal{G}(\mathbf{g}, \mathbf{a})$, they can often be substituted by mild conditions on the frequency responses g_m and downsampling factors a_m. The (stronger) LIC implies a straightforward condition on a filter bank to satisfy a lower frame bound, as has been shown in [13]. Complementing that result with statements from [14] and [25], we arrive at the following proposition.

Proposition 3. *Let $\mathcal{G}(\mathbf{g}, \mathbf{a})$ be a filter bank for $\mathbf{L}^{2,\mathcal{F}}(D)$.*

(i) If $\mathcal{G}(\mathbf{g}, \mathbf{a})$ is a Bessel sequence with bound $B < \infty$, then

$$\sum_{m\in\mathbb{Z}} \frac{1}{a_m} |g_m(\xi)|^2 \leq B, \text{ for almost all } \xi \in D. \tag{13}$$

If $\mathcal{G}(\mathbf{g}, \mathbf{a})$ satisfies the LIC (12) and it is a frame with lower frame bound $A > 0$, then

$$A \leq \sum_{m\in\mathbb{Z}} \frac{1}{a_m} |g_m(\xi)|^2, \text{ for almost all } \xi \in D. \tag{14}$$

(ii) Assume that there are some constants $c_m, d_m \in \mathbb{R}$, such that $\operatorname{supp}(g_m) \subseteq [c_m, d_m]$ and a_m satisfies $a_m^{-1} \geq d_m - c_m$, for all $m \in \mathbb{Z}$. Then $\mathcal{G}(\mathbf{g}, \mathbf{a})$ forms a frame, with frame bounds A, B, for $\mathbf{L}^{2,\mathcal{F}}(D)$ if and only if

$$0 < A \leq \sum_{m\in\mathbb{Z}} \frac{1}{a_m} |g_m(\xi)|^2 \leq B < \infty, \text{ for almost all } \xi \in D. \tag{15}$$

Furthermore, $\mathcal{G}(\widetilde{\mathbf{g}}, \mathbf{a}) = ((\widetilde{g_m})_m, (a_m)_m)$, where

$$\widetilde{g_m} = \frac{g_m}{\sum_{l\in\mathbb{Z}} \frac{1}{a_l} |g_l|^2}, \text{ for all } m \in \mathbb{Z}, \tag{16}$$

is the canonical dual frame for $\mathcal{G}(\mathbf{g}, \mathbf{a})$.

We refer to the setup in Proposition 3(ii) as the *painless case*, a generalization of the classical painless nonorthogonal expansions introduced in [41]. The whole of Proposition 3 serves as a strong indicator that for any *snug* frame, i.e. with $B/A \approx 1$, the sum $\sum_m a_m^{-1}|g_m|^2$, which equals the *frequency response* of the filter bank, must necessarily be close to constant. Hence, there is an intimate relationship between stability of the filter bank $\mathcal{G}(\widetilde{\mathbf{g}}, \mathbf{a})$ and the shape of its frequency response. The next result alludes to the fact that the g_m should be small outside an interval of size a_m^{-1}. This generalization of a well-known sufficient frame condition for Gabor and wavelet systems, originally by Daubechies [55], holds for abstract generalized-translation invariant systems on locally compact Abelian groups, see [32], which filter banks on $\mathbf{L}^{2,\mathcal{F}}(D)$ are a special case of.

Proposition 4. *Let $\mathcal{G}(\mathbf{g}, \mathbf{a})$ be a filter bank for $\mathbf{L}^{2,\mathcal{F}}(D)$. If*

$$A := \operatorname*{ess\,inf}_{\xi\in D} \left[\sum_{m\in\mathbb{Z}} \frac{1}{a_m} \left(|g_m(\xi)|^2 - \sum_{l\neq 0} |g_m(\xi)\overline{g_m}(\xi - l/a_m)| \right) \right] > 0 \tag{17}$$

$$B := \operatorname*{ess\,sup}_{\xi\in D} \left[\sum_{m\in\mathbb{Z}} \sum_{l\in\mathbb{Z}} \frac{1}{a_m} |g_m(\xi)\overline{g_m}(\xi - l/a_m)| \right] < \infty, \tag{18}$$

almost everywhere, then $\mathcal{G}(\mathbf{g}, \mathbf{a})$ constitutes a frame for $\mathbf{L}^{2,\mathcal{F}}(D)$. Furthermore, we have $A \leq A_0 \leq B_0 \leq B$, where A_0, B_0 are the optimal frame bounds for $\mathcal{G}(\mathbf{g}, \mathbf{a})$. If at least (18) is satisfied, then $\mathcal{G}(\mathbf{g}, \mathbf{a})$ is a Bessel sequence.

The sufficient Bessel condition (18) has additional significance, since it implies the α-LIC (11), see [32, Lemma 3.8]. After some straightforward adaptations, a direct proof of the Proposition is easily derived from the proofs of the corresponding variant results in [6] or [56].

It should be noted that, although Proposition 4 can be used to verify the frame property, it is not particularly useful for *constructing* filter bank frames without any further assumptions on the frequency responses g_m. As a mild generalization of the painless case condition in Proposition 3(ii) it only suggests that the frame property can be achieved through sufficient overlap and decay of the g_m, if the downsampling factors a_m are small enough, i.e. there exist c_m and $d_m = c_m + a_m^{-1}$ such that $\|g_m|_{\mathbb{R}\setminus[c_m,d_m]}\|_2 \ll \|g_m\|_2$ for all $m \in \mathbb{Z}$. However, the real value of Proposition 4 can be seen when combined with structural assumptions on $\mathcal{G}(\mathbf{g}, \mathbf{a})$, similar to the results on nonstationary Gabor systems in [26]. Its implications for warped filter banks are explored in the following.

4.1. Constructing warped filter bank frames

The next step now is to apply Propositions 3 and 4 to warped filter banks. This will provide important insight into the appropriate choice of the prototype filter θ and the downsampling factors a_m, with the aim to construct warped filter bank frames. A number of the following observations are consequences of the simple, but important identity

$$\sum_{m\in\mathbb{Z}} a_m^{-1}|g_m(\xi)|^2 = \sum_{m\in\mathbb{Z}} |(\mathbf{T}_m\theta)\circ\Phi(\xi)|^2, \text{ for all } \xi \in D, \tag{19}$$

which holds for every warped filter bank $\mathcal{G}(\Phi, \theta, \mathbf{a})$. A straightforward application of Proposition 3 leads to the following corollary.

Corollary 1. *Let $\mathcal{G}(\Phi, \theta, \mathbf{a})$ be a warped filter bank for $\mathbf{L}^{2,\mathcal{F}}(D)$.*

(i) If $\mathcal{G}(\Phi, \theta, \mathbf{a})$ is a Bessel sequence with bound $B < \infty$, then

$$\sum_m |\mathbf{T}_m\theta(\tau)|^2 \leq B < \infty, \textit{ for almost all } \tau \in \mathbb{R}. \tag{20}$$

If $\mathcal{G}(\Phi, \theta, \mathbf{a})$ is a frame with lower bound $0 < A$ satisfying the LIC (12), then

$$0 < A \leq \sum_{m\in\mathbb{Z}} |\mathbf{T}_m\theta(\tau)|^2, \textit{ for almost all } \tau \in \mathbb{R}. \tag{21}$$

(ii) Assume that, there are constants $c < d$, such that $\operatorname{supp}(\theta) \subseteq [c, d]$. If $a_m^{-1} \geq \Phi^{-1}(d + m) - \Phi^{-1}(c + m)$, for all $m \in \mathbb{Z}$, then $\mathcal{G}(\Phi, \theta, \mathbf{a})$ forms a frame for $\mathbf{L}^{2,\mathcal{F}}(D)$, if and only if $0 < A \leq \sum_{m\in\mathbb{Z}} |\mathbf{T}_m\theta(\tau)|^2 \leq B < \infty$ almost everywhere. Furthermore, $\mathcal{G}(\Phi, \tilde{\theta}, \mathbf{a})$, with

$$\tilde{\theta}(\tau) = \frac{\theta(\tau)}{\sum_{l\in\mathbb{Z}} |\mathbf{T}_l\theta(\tau)|^2} \tag{22}$$

is the canonical dual frame for $\mathcal{G}(\Phi, \theta, \mathbf{a})$.

Proof. Both (i) and (ii) are immediate consequences of Equation (19) and Proposition 3. For technical correctness, just note that the diffeomorphism Φ^{-1} preserves zero sets, i.e. if $E \subset \mathbb{R}$ is any measurable set with Lebesgue measure $\mu(E) = 0$, then $\Phi^{-1}(E) \subset D$ is a measurable set with $\mu(\Phi^{-1}(E)) = 0$. $\square$

Note that the canonical dual frame in (ii) only differs from $\mathcal{G}(\Phi, \theta, \mathbf{a})$ by the prototype filter. The necessary frame condition (21) still depends on the LIC, the direct verification of which is nontrivial. However, it is easy to show that any warped filter bank with reasonable downsampling factors a_m satisfies the LIC, without additional assumptions on the prototype filter θ.

Theorem 1. *Let $\Phi : D \to \mathbb{R}$ be a warping function and $\theta \in \mathbf{L}^2_{\sqrt{w}}(\mathbb{R})$, with $w = (\Phi^{-1})'$ being v-moderate. If either of the following holds,*

$$\sup_{m\in\mathbb{Z}} a_m w(m) < \infty \text{ and } \theta \in \mathbf{L}^2_{\sqrt{v}}(\mathbb{R}) \qquad \text{or} \qquad \sup_{m\in\mathbb{Z}} a_m v(m) < \infty,$$

and the warped filter bank $\mathcal{G}(\Phi, \theta, \mathbf{a})$ is a Bessel sequence, then the local integrability condition (12) *is satisfied. In particular, a Bessel sequence $\mathcal{G}(\Phi, \theta, \mathbf{a})$ satisfies the LIC, if $\mathbf{a}$ are natural downsampling factors.*

Proof. First note that, instead of considering all compactly supported and essentially bounded functions $f \in \mathbf{L}^2(D)$, it is sufficient to verify the LIC (12) only for the characteristic functions $\mathbb{1}_I$ on compact intervals $I \subset D$. These functions are clearly contained in $\mathbf{L}^2(D)$. It is easy to see that

$$\operatorname{supp}(f) \subseteq I \quad \Longrightarrow \quad L(f) \leq \|f\|_\infty^2 L(\mathbb{1}_I).$$

For $\mathbb{1}_I$, the LIC reads

$$L(\mathbb{1}_I) = \sum_{m\in\mathbb{Z}} a_m^{-1} \sum_{l\in\mathbb{Z}} \int_I \mathbb{1}_{I+la_m^{-1}}(\xi) \left|g_m(\xi)\right|^2 \, d\xi. \tag{23}$$

If the right hand side of (23) is finite, then it converges absolutely and we can interchange sums and integrals freely. Hence,

$$\begin{aligned} L(\mathbb{1}_I) &= \sum_{m\in\mathbb{Z}} a_m^{-1} \int_I \left|g_m(\xi)\right|^2 \sum_{l\in\mathbb{Z}} \mathbb{1}_{I+la_m^{-1}}(\xi) \, d\xi \\ &< \sum_{m\in\mathbb{Z}} \frac{a_m \mu(I) + 1}{a_m} \int_I \left|g_m(\xi)\right|^2 \, d\xi \\ &= \sum_{m\in\mathbb{Z}} (a_m \mu(I) + 1) \int_I \left|(\mathbf{T}_m \theta) \circ \Phi(\xi)\right|^2 \, d\xi, \end{aligned}$$

where we used that $\sum_{l\in\mathbb{Z}} \mathbb{1}_{I+la_m^{-1}}(\xi) \leq \lceil a_m \mu(I) \rceil < a_m \mu(I) + 1$ for arbitrary $\xi \in D$. We split the upper estimate into two terms and interchange integration and summation once more to obtain

$$L(\mathbb{1}_I) < \int_I \sum_{m\in\mathbb{Z}} \left|(\mathbf{T}_m \theta) \circ \Phi(\xi)\right|^2 \, d\xi + \sum_{m\in\mathbb{Z}} a_m \mu(I) \cdot \int_I \left|(\mathbf{T}_m \theta) \circ \Phi(\xi)\right|^2 \, d\xi.$$

Since $\mathcal{G}(\Phi, \theta, \mathbf{a})$ is a Bessel sequence, (20) holds for some positive constant B and we can conclude that

$$\int_I \sum_{m\in\mathbb{Z}} \left|(\mathbf{T}_m \theta) \circ \Phi(\xi)\right|^2 \, d\xi \leq \mu(I) B.$$

The change of variable $\tau = \Phi(\xi) - m$ yields

$$\sum_{m\in\mathbb{Z}} a_m \cdot \int_I \left|(\mathbf{T}_m\theta)\circ\Phi(\xi)\right|^2 \, d\xi = \sum_{m\in\mathbb{Z}} a_m \cdot \int_{\Phi^{-1}(I)+m} w(\tau+m)\left|\theta(\tau)\right|^2 \, d\tau = (*).$$

Now assume $\sup_{m\in\mathbb{Z}} a_m w(m) < \infty$ and $\theta \in \mathbf{L}^2_{\sqrt{v}}(\mathbb{R})$. To estimate the right hand side of the above equation, use v-moderateness of w:

$$\begin{aligned}
(*) &\leq \sum_{m\in\mathbb{Z}} a_m w(m) \cdot \int_{\Phi^{-1}(I)+m} v(\tau)\left|\theta(\tau)\right|^2 \, d\tau \\
&= \sum_{m\in\mathbb{Z}} a_m w(m) \cdot \int_D \mathbb{1}_{\Phi^{-1}(I)+m} v(\tau)\left|\theta(\tau)\right|^2 \, d\tau \\
&= \int_D a_m w(m)\cdot v(\tau)\left|\theta(\tau)\right|^2 \sum_{m\in\mathbb{Z}} \mathbb{1}_{\Phi^{-1}(I)+m} \, d\tau \\
&< \left(1+\mu(\Phi^{-1}(I))\right) \int_D a_m w(m)\cdot v(\tau)\left|\theta(\tau)\right|^2 \, d\tau \\
&\leq \left(1+\mu(\Phi^{-1}(I))\right) \cdot \sup_{m\in\mathbb{Z}} a_m w(m) \cdot \|\theta\|^2_{\mathbf{L}^2_{\sqrt{v}}}.
\end{aligned}$$

Together, we obtain

$$L(\mathbb{1}_I) < \mu(I)\cdot\left(B + \left(1+\mu(\Phi^{-1}(I))\right)\cdot \sup_{m\in\mathbb{Z}} a_m w(m)\cdot\|\theta\|^2_{\mathbf{L}^2_{\sqrt{v}}}\right) < \infty,$$

which both establishes the desired result and justifies all previous interchanges between sums and integrals a posteriori. In particular, if $\mathbf{a}$ is a set of natural downsampling factors, then $a_m w(m) = \tilde{a}$ for all $m \in \mathbb{Z}$. By simply exchanging the roles of w and v, the result $L(\mathbb{1}_I) < \infty$ is obtained for the variant assumption $\sup_{m\in\mathbb{Z}} a_m v(m) < \infty$ and $\theta \in \mathbf{L}^2_{\sqrt{w}}(\mathbb{R})$. □

Corollary 2. *If $\mathcal{G}(\Phi,\theta,\mathbf{a})$ is a warped filter bank frame, with frame bounds A, B, that satisfies*

$$\sup_{m\in\mathbb{Z}} a_m w(m) < \infty \text{ and } \theta \in \mathbf{L}^2_{\sqrt{v}}(\mathbb{R}) \qquad \text{or} \qquad \sup_{m\in\mathbb{Z}} a_m v(m) < \infty,$$

then

$$0 < A \leq \sum_{m\in\mathbb{Z}} \left|\mathbf{T}_m\theta(\tau)\right|^2 \leq B < \infty, \text{ a.e.} \tag{24}$$

The corollary above shows that in order to construct snug frames, it is imperative that the translates of the original window θ have good summation properties in the sense of (24) (with $B/A \approx 1$). On the other hand, the painless case result in Corollary 1 is sharp in the sense that the conditions on $\mathrm{supp}(\theta)$ and a_m cannot be relaxed without losing the equivalence of the frame property to Equation (24). However, close-to-optimal values for the a_m are once more provided by a set of natural downsampling factors.

Proposition 5. *Let $\mathcal{G}(\Phi,\theta,\mathbf{a})$ be a warped filter bank with compactly supported prototype $\theta \in$*

$\mathbf{L}^2_{\sqrt{w}}(\mathbb{R})$. *Furthermore, define*

$$c_0 := \inf \operatorname{supp}(\theta) \quad \textit{and} \quad d_0 := \sup \operatorname{supp}(\theta)$$

and let $\widetilde{a_w} := V(c_0, d_0)^{-1}$ *and* $\widetilde{a_v} := \left(\Phi^{-1}(d_0) - \Phi^{-1}(c_0)\right)^{-1}$, *where* $V(\tau_0, \tau_1) := \int_{\tau_0}^{\tau_1} v(\tau) d\tau$, *for all* $\tau_0, \tau_1 \in \mathbb{R}$. *If*

$$a_m \leq \max\{\widetilde{a_w}/w(m), \widetilde{a_v}/v(m)\},$$

then $\mathcal{G}(\Phi, \theta, \mathbf{a})$ *is a frame with frame bounds* A, B, *if and only if Equation* (24) *holds. In that case, the prototype* $\tilde{\theta} = \frac{\theta}{\sum_{l\in\mathbb{Z}} |\mathbf{T}_l \theta|^2}$ *gives rise to the canonical dual frame* $\mathcal{G}(\Phi, \tilde{\theta}, \mathbf{a})$.

Proof. All we need to do is to invoke the fundamental theorem of calculus to show that $\Phi^{-1}(d_0 + m) - \Phi^{-1}(c_0 + m) \leq a_m^{-1}$ for all $m \in \mathbb{Z}$.

$$\begin{aligned}\Phi^{-1}(d_0 + m) - \Phi^{-1}(c_0 + m) &= \int_{c_0+m}^{d_0+m} w(\tau)\ d\tau \\ &\leq \begin{cases} w(m) \cdot \int_{c_0}^{d_0} v(\tau)\ d\tau = V(c_0, d_0) w(m) \\ v(m) \cdot \int_{c_0}^{d_0} w(\tau)\ d\tau = (\Phi^{-1}(d_0) - \Phi^{-1}(c_0)) v(m). \end{cases}\end{aligned}$$

Therefore,

$$\Phi^{-1}(d_0 + m) - \Phi^{-1}(c_0 + m) \leq \min\{w(m)/\widetilde{a_w}, v(m)/\widetilde{a_v}\} \leq a_m^{-1},$$

as per the assumption. Apply Corollary 1(ii) to finish the proof. □

In fact, without additional assumptions on the warping function Φ, the conditions of Proposition 5 cannot be improved. To see this we have to find a warping function, such that any choice $a_m > \max\{\widetilde{a_w}/w(m), \widetilde{a_v}/v(m)\}$ yields $a_m^{-1} < \Phi^{-1}(d) - \Phi^{-1}(c)$, for all such intervals $[c, d]$. Hence, the conditions of Corollary 1(ii) are violated and equivalence of the frame property to Equation (24) is lost. Choose now $\Phi(\xi) = \log(\xi)$. Then $\Phi^{-1}(\tau) = e^\tau = w(\tau)$ for all $\tau \in \mathbb{R}$. We can choose $v = w$ and obtain

$$e^{d_0+m} - e^{c_0+m} = e^m \int_{c_0}^{d_0} e^\tau d\tau = V(c_0, d_0) w(m) = (\Phi^{-1}(d_0) - \Phi^{-1}(c_0)) v(m),$$

to show that for a logarithmic warping function the natural downsampling factors are indeed the coarsest possible downsampling factors to satisfy the painless case conditions. Note that $\int_{c_0}^{d_0} v(\tau)\ d\tau$ can be estimated from above by $(d_0 - c_0) \max_{\tau\in[c_0,d_0]} w(\tau)$ and similarly for v instead of w. This coarser estimate can be used for an even simpler computation of downsampling factors appropriate for Proposition 5, e.g. if w, or v, is nondecreasing away from zero.

Whenever the prototype frequency response θ is not compactly supported, or larger downsampling factors, not permitted by Corollary 1(ii), are desired, the verification of the frame property becomes substantially harder. However, if at least the conditions of Proposition 4 are satisfied, we can still guarantee that $\mathcal{G}(\Phi, \theta, \mathbf{a})$ is a frame. The construction of such warped filter banks $\mathcal{G}(\Phi, \theta, \mathbf{a})$ is our next goal. One would be tempted to just refer to [26, Theorem 3.4], which provides such a result. However, the conditions on the filter bank $\mathcal{G}(\mathbf{g}, \mathbf{a})$ given there are far too restrictive for the treatment of warped filter banks. In particular, the frequency responses are required to decay polynomially around a δ-separated set $(b_m)_{m\in\mathbb{Z}}$, i.e. $\inf_{k,m\in\mathbb{Z}} |b_m - b_k| \geq \delta > 0$, in the following way:

$$|g_m| \leq C_m(1+|\cdot - b_m|)^{p_m}, \text{ with } p_m \in I_p,\ C_m \in I_C, \text{ for all } m \in \mathbb{Z} \text{ and compact intervals } I_p, I_C$$

For warped filter banks $\mathcal{G}(\Phi,\theta,\mathbf{a})$, the points b_m are always given by $\Phi^{-1}(m)$. Whenever D is a true subset of $\mathbb{R}$, the set $(\Phi^{-1}(m))_{m\in\mathbb{Z}}$ can never be δ-separated for any $\delta > 0$, however. More importantly, the condition above implies that there are some fixed $p_L, C_U > 0$ such that the condition holds uniformly for $p_m = p_L$ and $C_m = C_U$, for all $m \in \mathbb{Z}$. While the first condition alone does not pose a severe restriction, the combination of both implies for $\mathcal{G}(\Phi,\theta,\mathbf{a})$ that $w = (\Phi^{-1})'$ is bounded above and thus Φ must have at least linear asymptotic growth. Such a limitation is clearly not desired, see Examples 1-4.

Instead, we establish that a mild decay condition on θ ensures the Bessel property and, when complemented by sufficiently small downsampling factors, even the frame property. This is of central interest, as it shows that warped filter banks also admit the construction of frames for many prototype filters θ with full support. For functions $f_0 \in \mathbf{L}^1_{\text{loc}}(\mathbb{R})$, let the Landau symbol $\mathcal{O}(f_0)$ denote the set of functions that do not have faster asymptotic growth than f_0, i.e.

$$\mathcal{O}(f_0) = \{f \in \mathbf{L}^1_{\text{loc}}(\mathbb{R}) \ : \ \exists\, C > 0 \text{ s.t. } f \leq C f_0 \text{ a.e.}\}.$$

Furthermore, if $w = (\Phi^{-1})'$ is v-moderate, then we denote, as in Proposition 5, by V the function

$$V(\tau_0,\tau_1) = \int_{\tau_0}^{\tau_1} v(\tau)\ d\tau, \text{ for all } \tau_0,\tau_1 \in \mathbb{R}.$$

Theorem 2. *Let $\Phi : D \to \mathbb{R}$ be a warping function with $\theta \in \mathbf{L}^2_{\sqrt{w}}(\mathbb{R})$. If either*

(i) $\theta \in \mathcal{O}\left((1+\max\{|\cdot|, |\Phi^{-1}(\cdot)-\Phi^{-1}(0)|\})^{-1-\epsilon}\right)$ for some $\epsilon > 0$ and $a_m \leq \tilde{a}/v(m)$, for all $m \in \mathbb{Z}$ and some $\tilde{a} > 0$, or

(ii) $\theta \in \mathcal{O}\left((1+\max\{|\cdot|, |V(0,\cdot)|\})^{-1-\epsilon}\right)$ and $a_m \leq \tilde{a}/w(m)$, for all $m \in \mathbb{Z}$ and some $\tilde{a} > 0$,

then $\mathcal{G}(\Phi,\theta,\mathbf{a})$ is a Bessel sequence. If additionally, there is a constant $A_1 > 0$ such that

$$0 < A_1 \leq \sum_{m\in\mathbb{Z}} |\mathbf{T}_m\theta|^2 \textit{ almost everywhere,}$$

then the following hold:

(iii) If $\theta \in \mathcal{O}\left((1+\max\{|\cdot|, |\Phi^{-1}(\cdot)-\Phi^{-1}(0)|\})^{-1-\epsilon}\right)$ for some $\epsilon > 0$, there is a constant $\tilde{a}_{v,0} > 0$ such that $\mathcal{G}(\Phi,\theta,\mathbf{a})$ is a frame, whenever $a_m \leq \tilde{a}_{v,0}/v(m)$, for all $m \in \mathbb{Z}$.

(iv) If $\theta \in \mathcal{O}\left((1+\max\{|\cdot|, |V(0,\cdot)|\})^{-1-\epsilon}\right)$, there is a constant $a_{w,0} > 0$ such that $\mathcal{G}(\Phi,\theta,\mathbf{a})$ is a frame, whenever $a_m \leq \tilde{a}_{w,0}/w(m)$, for all $m \in \mathbb{Z}$.

Before we proceed to prove Theorem 2 above, we require the following auxiliary results. Also note that $(1+|V(0,\cdot)|)^{-1-\epsilon} \in \mathbf{L}^2_{\sqrt{v}}(\mathbb{R})$, such that (i),(ii) imply the LIC (12) by Theorem 1 and uniform boundedness of all the frame elements $g_{m,n} \in \mathbf{L}^{2,\mathcal{F}}(D)$, see (7) in Section 3.

Lemma 1. *Let $\Phi : D \to \mathbb{R}$ be a warping function such that w is v-moderate. There are bijective, increasing functions $A_w : \mathbb{R} \to \mathbb{R}$ and $A_v : \mathbb{R} \to \mathbb{R}$, such that $A_w(0) = A_v(0) = 0$ and the following hold:*

(i) For all $c \in \mathbb{R}^+$,

$$\left|\Phi^{-1}(\tau_1) - \Phi^{-1}(\tau_0)\right| \geq cv(\tau_0) \quad \Longrightarrow |\tau_1 - \tau_0| \geq \begin{cases} |A_w^{-1}(c)| & \text{if } \tau_1 \geq \tau_0 \\ |A_w^{-1}(-c)| & \text{else} \end{cases}, \text{ for all } \tau_0, \tau_1 \in \mathbb{R}.$$

(ii) For all $c \in \mathbb{R}^+$,

$$\left|\Phi^{-1}(\tau_1) - \Phi^{-1}(\tau_0)\right| \geq cw(\tau_0) \quad \Longrightarrow |\tau_1 - \tau_0| \geq \begin{cases} |A_v^{-1}(c)| & \text{if } \tau_1 \geq \tau_0 \\ |A_v^{-1}(-c)| & \text{else} \end{cases}, \text{ for all } \tau_0, \tau_1 \in \mathbb{R}.$$

Proof. Observe that for all $\tau_0, \tau_1 \in \mathbb{R}$,

$$\left|\Phi^{-1}(\tau_1) - \Phi^{-1}(\tau_0)\right| = \left|\int_0^{\tau_1-\tau_0} w(\tau + \tau_0)\, d\tau\right| \leq |\tau_1 - \tau_0| \cdot \begin{cases} v(\tau_0) \max_{\tau \in [0,\tau_1-\tau_0] \cup [\tau_1-\tau_0,0]} w(\tau) \\ w(\tau_0) \max_{\tau \in [0,\tau_1-\tau_0] \cup [\tau_1-\tau_0,0]} v(\tau). \end{cases} \tag{25}$$

For item (i), assume $\left|\Phi^{-1}(\tau_1) - \Phi^{-1}(\tau_0)\right| \leq cv(\tau_0)$. Then,

$$v(\tau_0)\, |\tau_1 - \tau_0| \max_{\tilde{\tau} \in [0,\tau_1-\tau_0] \cup [\tau_1-\tau_0,0]} w(\tilde{\tau}) \leq cv(\tau_0)$$

or equivalently

$$c \geq |\tau_1 - \tau_0| \max_{\tilde{\tau} \in [0,\tau_1-\tau_0] \cup [\tau_1-\tau_0,0]} w(\tilde{\tau}) = |A_w(\tau_1 - \tau_0)|, \tag{26}$$

with $A_w : \mathbb{R} \to \mathbb{R}$ given by

$$A_w(\tau) := \tau \max_{\tilde{\tau} \in [0,\tau] \cup [\tau,0]} w(\tilde{\tau}), \text{ for all } \tau \in \mathbb{R}.$$

It is easy to see that A_w is increasing, a bijection and $A_w(0) = 0$, owing to the fact that w is continuous and positive. Since $c \geq |A_w(\tau_1 - \tau_0)|$ implies $\left|\Phi^{-1}(\tau_1) - \Phi^{-1}(\tau_0)\right| \leq cv(\tau_0)$ by assumption,

$$\left|\Phi^{-1}(\tau_1) - \Phi^{-1}(\tau_0)\right| \geq cv(\tau_0) \Longrightarrow c \leq |A_w(\tau_1 - \tau_0)|\,.$$

Using the inverse A_w^{-1} of A_w, we obtain

$$|\tau_1 - \tau_0| \geq \begin{cases} |A_w^{-1}(c))| & \text{if } \tau_1 - \tau_0 \geq 0, \\ |A_w^{-1}(-c))| & \text{else.} \end{cases}$$

This proves item (i). For item (ii), assume $\left|\Phi^{-1}(\tau_1) - \Phi^{-1}(\tau_0)\right| \leq cv(\tau_0)$. By the second estimate in (25), this is equivalent to

$$c \geq |\tau_1 - \tau_0| \max_{\tilde{\tau} \in [0,\tau_1-\tau_0] \cup [\tau_1-\tau_0,0]} v(\tilde{\tau}) = |A_v(\tau_1 - \tau_0)|, \tag{27}$$

with $A_v : \mathbb{R} \to \mathbb{R}$ given by

$$A_v(\tau) := \tau \max_{\tilde{\tau} \in [0,\tau] \cup [\tau,0]} v(\tilde{\tau}), \text{ for all } \tau \in \mathbb{R}.$$

Similar to A_w, A_v is is increasing (since $0 < w \leq v/w(0)$ by v-moderateness of w) and a bijection with $A_v(0) = 0$ and we obtain

$$|\tau_1 - \tau_0| \geq \begin{cases} |A_v^{-1}(c))| & \text{if } \tau_1 - \tau_0 \geq 0, \\ |A_v^{-1}(-c))| & \text{else} \end{cases},$$

as desired. □

Lemma 1 allows us to derive the following result.

Lemma 2. *For a given warped filter bank $\mathcal{G}(\Phi, \theta, \mathbf{a})$, with $\mathbf{a} = (a_m)_{m \in \mathbb{Z}}$ as before, define*

$$\mathbf{P}(\xi) := \mathbf{P}_{\Phi,\theta,\mathbf{a}}(\xi) := \sum_{m \in \mathbb{Z}} \left(|\theta(\Phi(\xi) - m)| \cdot \sum_{k \in \mathbb{Z} \setminus \{0\}} \left|\theta(\Phi(\xi + k a_m^{-1}) - m)\right| \right), \text{ for all } \xi \in D. \tag{28}$$

The following hold:

(i) If $\theta \in \mathcal{O}\left((1 + \max\{|\cdot|, |\Phi^{-1}(\cdot) - \Phi^{-1}(0)|\})^{-1-\epsilon}\right)$, for some $\epsilon > 0$, and $a_m \leq \tilde{a}/v(m)$, for all $m \in \mathbb{Z}$ and some $\tilde{a} > 0$, then

$$\operatorname*{ess\,sup}_{\xi \in D} \mathbf{P}_{\Phi,\theta,\mathbf{a}}(\xi) < \infty \quad \text{and} \quad \operatorname*{ess\,sup}_{\xi \in D} \mathbf{P}_{\Phi,\theta,\mathbf{a}}(\xi) \overset{\tilde{a} \to 0}{\longrightarrow} 0.$$

(ii) If $\theta \in \mathcal{O}\left((1 + \max\{|\cdot|, |V(0, \cdot)|\})^{-1-\epsilon}\right)$, for some $\epsilon > 0$, and $a_m \leq \tilde{a}/w(m)$, for all $m \in \mathbb{Z}$ and some $\tilde{a} > 0$, then

$$\operatorname*{ess\,sup}_{\xi \in D} \mathbf{P}_{\Phi,\theta,\mathbf{a}}(\xi) < \infty \quad \text{and} \quad \operatorname*{ess\,sup}_{\xi \in D} \mathbf{P}_{\Phi,\theta,\mathbf{a}}(\xi) \overset{\tilde{a} \to 0}{\longrightarrow} 0.$$

Proof. In our estimation, we use *Hurwitz' zeta function* [57] repeatedly; it is defined as

$$\zeta(s, q) = \sum_{k \in \mathbb{N}_0} \frac{1}{(q + k)^s}. \tag{29}$$

The function $\zeta(s, q)$ is finite for all $q > 0, s > 1$ and tends towards zero for s fixed and $q \to \infty$ or vice versa. Note that v-moderateness of w implies

$$\begin{aligned} \left|s - \Phi^{-1}(m)\right| = \left|\Phi^{-1}(\Phi(s)) - \Phi^{-1}(m)\right| &= \left|\int_m^{\Phi(s)} w(\tau)\ d\tau\right| \\ &= \left|\int_0^{\Phi(s)-m} w(\tau + m)\ d\tau\right| \\ &\leq \begin{cases} v(m) |\int_0^{\Phi(s)-m} w(\tau)\ d\tau| = v(m)|\Phi^{-1}(\Phi(s) - m) - \Phi^{-1}(0)| \\ w(m) |\int_0^{\Phi(s)-m} v(\tau)\ d\tau| = w(m)|V(0, \Phi(s) - m)|, \end{cases} \end{aligned} \tag{30}$$

Moreover, $\theta \in \mathcal{O}\left((1+|\cdot|)^{-1-\epsilon}\right)$ yields

$$\operatorname*{ess\,sup}_{\tau\in\mathbb{R}} \sum_{m\in\mathbb{Z}} \mathbf{T}_m\theta(\tau) \le 2C_0 \sum_{m\in\mathbb{N}_0} \frac{1}{(1+m)^{1+\epsilon}} = 2C_0\zeta(1,1+\epsilon) =: \tilde{B} < \infty \tag{31}$$

We begin by estimating the inner sum in (28). To that end, define

$$P_m := \sum_{k\in\mathbb{Z}\setminus\{0\}} \left|\theta(\Phi(\cdot + ka_m^{-1}) - m)\right|, \text{ for all } m \in \mathbb{Z}.$$

Now, assume that $\theta \in \mathcal{O}\left((1+|\Phi^{-1}(\cdot)-\Phi^{-1}(0)|)^{-1-\epsilon}\right)$ with constant $C_w > 0$. Insert into the definition of P_m to see that

$$\begin{aligned} P_m(\xi) &\le C_w \sum_{k\in\mathbb{Z}\setminus\{0\}} \left|(1+|\Phi^{-1}(\Phi(\xi+ka_m^{-1})-m)-\Phi^{-1}(0)|)^{-1-\epsilon}\right| \\ &\le C_w \sum_{k\in\mathbb{Z}\setminus\{0\}} \left(1+\left|\frac{\xi+ka_m^{-1}-\Phi^{-1}(m)}{v(m)}\right|\right)^{-1-\epsilon}, \end{aligned}$$

for almost every $\xi \in D$. Here, we used (30) with $s = \xi + ka_m^{-1}$ to obtain the second inequality. For any pair (ξ, m), there is a unique $k_{(\xi,m)} \in \mathbb{Z}$ such that $|\xi + k_{(\xi,m)}a_m^{-1} - \Phi^{-1}(m)| \in [-(2a_m)^{-1}, (2a_m)^{-1})$. Let

$$M_\xi := \{m \in \mathbb{Z} \ : \ k_{(\xi,m)} = 0\} \quad \text{ and } \quad M_\xi^\dagger := \mathbb{Z} \setminus M_\xi.$$

First assume that $m \in M_\xi$. Clearly,

$$\begin{aligned} &\sum_{k\in\mathbb{Z}\setminus\{0\}} \left(1+\left|\frac{\xi+ka_m^{-1}-\Phi^{-1}(m)}{v(m)}\right|\right)^{-1-\epsilon} \\ &\le \sum_{k\in\mathbb{N}_0} \left(1+\left|\frac{(2a_m)^{-1}+ka_m^{-1}}{v(m)}\right|\right)^{-1-\epsilon} + \sum_{k\in\mathbb{N}_0} \left(1+\left|-\frac{(2a_m)^{-1}+ka_m^{-1}}{v(m)}\right|\right)^{-1-\epsilon} \\ &= 2\sum_{k\in\mathbb{N}_0} \left(\left|\frac{v(m)+(2a_m)^{-1}+ka_m^{-1}}{v(m)}\right|\right)^{-1-\epsilon} = (*). \end{aligned}$$

If $a_m \le \tilde{a}/v(m)$, then $|(2a_m)^{-1} + ka_m^{-1}| \ge |(1/2+k)v(m)/\tilde{a}|$ and

$$(*) \le 2\sum_{k\in\mathbb{N}_0} \left(\frac{\tilde{a}}{\tilde{a}+1/2+k}\right)^{1+\epsilon} = 2\tilde{a}^{1+\epsilon} \sum_{k\in\mathbb{N}_0} \frac{1}{(\tilde{a}+1/2+k)^{1+\epsilon}} = 2\tilde{a}^{1+\epsilon}\zeta(\tilde{a}+1/2, 1+\epsilon).$$

Consequently, we obtain that

$$P_m(\xi) \le 2C_w\tilde{a}^{1+\epsilon}\zeta(\tilde{a}+1/2, 1+\epsilon), \text{ almost everywhere.}$$

Now, if $m \in M_\xi^\dagger$, then a similar estimation yields

$$\begin{aligned} P_m(\xi) &\leq \left|\theta(\Phi(\cdot + k_{\xi,m}a_m^{-1}) - m)\right| + C_w \sum_{k\in\mathbb{Z}\setminus\{0,k_{\xi,m}\}} \left(1 + \left|\frac{\xi + ka_m^{-1} - \Phi^{-1}(m)}{v(m)}\right|\right)^{-1-\epsilon} \\ &\leq \left|\theta(\Phi(\cdot + k_{\xi,m}a_m^{-1}) - m)\right| + 2C_w\tilde{a}^{1+\epsilon}\zeta(\tilde{a}+1/2, 1+\epsilon), \end{aligned}$$

almost everywhere. These estimates can now be inserted into the expression (28) for $\mathbf{P}$:

$$\begin{aligned} \mathbf{P}(\xi) &= \sum_{m\in\mathbb{Z}} |\theta(\Phi(\xi) - m)| \cdot P_m(\xi) \\ &\leq \sum_{m\in M_\xi^\dagger} |\theta(\Phi(\xi) - m)| \left|\theta(\Phi(\xi + k_{\xi,m}a_m^{-1}) - m)\right| + C_{w,\tilde{a},\epsilon} \sum_{m\in\mathbb{Z}} |\theta(\Phi(\xi) - m)|, \end{aligned}$$

almost everywhere, with $C_{w,\tilde{a},\epsilon} := 2C_w\tilde{a}^{1+\epsilon}\zeta(\tilde{a}+1/2, 1+\epsilon)$. By (31),

$$C_{w,\tilde{a},\epsilon} \sum_{m\in\mathbb{Z}} |\theta(\Phi(\xi) - m)| \leq \tilde{B}C_{w,\tilde{a},\epsilon} < \infty, \text{ a.e.}$$

It remains to estimate $\sum_{m\in M_\xi^\dagger} |\theta(\Phi(\xi) - m)| \left|\theta(\Phi(\cdot + k_{\xi,m}a_m^{-1}) - m)\right|$. Using (31), we easily obtain that

$$\begin{aligned} &\sum_{m\in M_\xi^\dagger} |\theta(\Phi(\xi) - m)| \left|\theta(\Phi(\cdot + k_{\xi,m}a_m^{-1}) - m)\right| \\ &\leq C_w \sum_{m\in\mathbb{Z}} |\theta(\Phi(\xi) - m)| \leq \tilde{B}C_w < \infty, \end{aligned}$$

showing that

$$\operatorname*{ess\,sup}_{\xi\in D} \mathbf{P}(\xi) \leq \tilde{B}C_w \cdot \left(1 + \tilde{a}^{1+\epsilon}\zeta(\tilde{a}+1/2, 1+\epsilon)\right) < \infty.$$

For the second assertion of item (i), we additionally require convergence of the essential supremum to 0 for $\tilde{a} \to 0$. To this end, we first need to show that there is a function $A : \mathbb{R}^+ \to \mathbb{R}^+$ with

$$A(\tilde{a}) \overset{\tilde{a}\to 0}{\longrightarrow} \infty \quad \text{ and } \quad m \in M_\xi^\dagger \Longrightarrow |\Phi(\xi) - m| \geq A(\tilde{a}).$$

By definition, $m \in M_\xi^\dagger$ implies $|\xi - \Phi^{-1}(m)| \geq (2a_m)^{-1} \geq v(m)(2\tilde{a})^{-1}$. Hence, we can apply Lemma 1, item (i), with $\tau_0 = m,\ \tau_1 = \Phi(\xi)$ and $c = (2\tilde{a})^{-1}$ to obtain

$$|\Phi(\xi) - m| \geq \begin{cases} \left|A_w^{-1}\left(1/(2\tilde{a})\right)\right| & \text{if } \Phi(\xi) - m \geq 0, \\ \left|A_w^{-1}\left(-1/(2\tilde{a})\right)\right| & \text{else.} \end{cases}$$

In other words, we can rewrite

$$
\begin{aligned}
\sum_{m\in M_\xi^\dagger} |\theta(\Phi(\xi)-m)| &\le \sum_{k\in\mathbb{N}_0} |\theta(|A_w^{-1}\left(1/(2\tilde a)\right)|+k)| + \sum_{k\in\mathbb{N}_0} |\theta(|A_w^{-1}\left(-1/(2\tilde a)\right)|+k)| \\
&\le C_0\left(\sum_{k\in\mathbb{N}_0} \frac{1}{\left(1+|A_w^{-1}\left(1/(2\tilde a)\right)|+k\right)^{1+\epsilon}} + \sum_{k\in\mathbb{N}_0} \frac{1}{\left(1+|A_w^{-1}\left(-1/(2\tilde a)\right)|+k\right)^{1+\epsilon}}\right) \\
&= C_0\left(\zeta(1+|A_w^{-1}\left(1/(2\tilde a)\right)|, 1+\epsilon) + \zeta(1+|A_w^{-1}\left(-1/(2\tilde a)\right)|, 1+\epsilon)\right).
\end{aligned}
$$

All in all, we obtain for $\mathbf{P}$ the estimate

$$
\begin{aligned}
\mathbf{P}_{\Phi,\theta,\mathbf{a}}(\xi) \le C_w C_0 &\left(\zeta(1+|A_w^{-1}\left(1/(2\tilde a)\right)|, 1+\epsilon) + \zeta(1+|A_w^{-1}\left(-1/(2\tilde a)\right)|, 1+\epsilon)\right) \\
&+ 2\tilde B C_w \tilde a^{1+\epsilon}\zeta(\tilde a + 1/2, 1+\epsilon), \text{ a.e.}
\end{aligned}
\tag{32}
$$

Since $\zeta(\tilde a+1/2,1+\epsilon) \le \zeta(1/2,1+\epsilon)$ and $A_w^{-1}(\tau) \overset{\tau\to\pm\infty}{\longrightarrow} \infty$, we finally obtain

$$
\operatorname*{ess\,sup}_{\xi\in D} \mathbf{P}_{\Phi,\theta,\mathbf{a}}(\xi) \overset{\tilde a\to 0}{\longrightarrow} 0, \text{ if } a_m \le \tilde a/v(m) \text{ for all } m\in\mathbb{Z},
$$

as desired, finishing the proof of (i).

Item (ii) is essentially proven with the same steps. The main difference is using now the second estimate derived in Equation (30) and Lemma 1, item (ii). Together with $\theta \in \mathcal{O}\left((1+V(0,\cdot))^{-1-\epsilon}\right)$ (with the constant $C_v > 0$), we obtain

$$
P_m(\xi) \le C_v \sum_{k\in\mathbb{Z}\setminus\{0\}} \left(1+\left|\frac{\xi + k a_m^{-1} - \Phi^{-1}(m)}{w(m)}\right|\right)^{-1-\epsilon} \le 2C_v\tilde a^{1+\epsilon}\zeta(\tilde a+1/2, 1+\epsilon)
$$

and

$$
m\in M_\xi^\dagger \Longrightarrow (2\tilde a)^{-1} \le |A_v(\Phi(\xi)-m)|,
$$

where $A_v(\tau_0) := \tau_0 \max_{\tau\in[0,\tau_0]\cup[\tau_0,0]} v(\tau)$, for all $\tau_0\in\mathbb{R}$ is again a bijection. The final estimate for $\mathbf{P}$ in the setting (ii) reads

$$
\begin{aligned}
\mathbf{P}_{\Phi,\theta,\mathbf{a}}(\xi) \le C_v C_0 &\left(\zeta(1+|A_v^{-1}\left(1/(2\tilde a)\right)|, 1+\epsilon) + \zeta(1+|A_v^{-1}\left(-1/(2\tilde a)\right)|, 1+\epsilon)\right) \\
&+ 2\tilde B C_v \tilde a^{1+\epsilon}\zeta(\tilde a + 1/2, 1+\epsilon), \text{ a.e.,}
\end{aligned}
\tag{33}
$$

which, by the same reasoning as before, is finite for all $\mathbf{a}$ with $a_m \le \tilde a/w(m)$ for all $m\in\mathbb{Z}$ and converges to 0 if $\tilde a \to 0$. ☐

With Lemma 2 in place, proving Theorem 2 only requires a few simple steps.

Proof of Theorem 2. The main observation to consider is the following. For any given warped filter bank $\mathcal{G}(\Phi,\theta,\mathbf{a})$, we have

$$\begin{aligned}
&\sum_{m\in\mathbb{Z}} a_m^{-1}|g_m(\xi)|^2 \pm \sum_{m\in\mathbb{Z}}\sum_{l\in\mathbb{Z}\setminus\{0\}} a_m^{-1}|g_m(\xi)\overline{g_m(\xi+ka_m^{-1})}| \\
&= \sum_{m\in\mathbb{Z}} |\theta(\Phi(\xi))|^2 \pm \sum_{m\in\mathbb{Z}}\left(|\theta(\Phi(\xi))|\cdot \sum_{l\in\mathbb{Z}\setminus\{0\}} |\theta(\Phi(\xi+ka_m^{-1})-m)|\right) \\
&= \sum_{m\in\mathbb{Z}} |\theta(\Phi(\xi))|^2 \pm \mathbf{P}_{\Phi,\theta,\mathbf{a}}(\xi), \text{ for almost every } \xi\in D.
\end{aligned}$$

First, assume that $\theta \in \mathcal{O}\left((1+\max\{|\cdot|, |\Phi^{-1}(\cdot)-\Phi^{-1}(0)|\})^{-1-\epsilon}\right)$ for some $\epsilon>0$ and $a_m \leq \tilde{a}/v(m)$, for all $m\in\mathbb{Z}$ and some $\tilde{a}>0$. By Lemma 2, item (i), $\operatorname{ess\,sup}_{\xi\in D}\mathbf{P}_{\Phi,\theta,\mathbf{a}}(\xi)<\infty$. Moreover, since $\theta\in\mathcal{O}\left((1+|\cdot|)^{-1-\epsilon}\right)$, we obtain the estimate

$$\operatorname*{ess\,sup}_{\tau\in\mathbb{R}} \sum_{m\in\mathbb{Z}} \mathbf{T}_m\theta(\tau) \leq= 2C_0\zeta(1,1+\epsilon) =: \tilde{B} < \infty$$

as per Equation (31). In total, with B as in Proposition 4,

$$B = \operatorname*{ess\,sup}_{\xi\in D}\left(\sum_{m\in\mathbb{Z}} |\theta(\Phi(\xi))|^2 + \mathbf{P}_{\Phi,\theta,\mathbf{a}}(\xi)\right) \leq \operatorname*{ess\,sup}_{\tau\in\mathbb{R}}\sum_{m\in\mathbb{Z}}\mathbf{T}_m\theta(\tau) + \operatorname*{ess\,sup}_{\xi\in D}\mathbf{P}_{\Phi,\theta,\mathbf{a}}(\xi) < \infty,$$

and $\mathcal{G}(\Phi,\theta,\mathbf{a})$ is a Bessel sequence by Proposition 4, proving item (i). Similarly, with A as in Proposition 4,

$$\begin{aligned}
A &= \operatorname*{ess\,inf}_{\xi\in D}\left(\sum_{m\in\mathbb{Z}} |\theta(\Phi(\xi))|^2 - \mathbf{P}_{\Phi,\theta,\mathbf{a}}(\xi)\right) \\
&\geq \operatorname*{ess\,inf}_{\tau\in\mathbb{R}}\sum_{m\in\mathbb{Z}}\mathbf{T}_m\theta(\tau) - \operatorname*{ess\,sup}_{\xi\in D}\mathbf{P}_{\Phi,\theta,\mathbf{a}}(\xi) = \tilde{A} - \operatorname*{ess\,sup}_{\xi\in D}\mathbf{P}_{\Phi,\theta,\mathbf{a}}(\xi).
\end{aligned}$$

By Lemma 2, item (i), there is a constant $\tilde{a}_{v,0}>0$, such that $a_m \leq \tilde{a}_{v,0}/v(m)$, for all $m\in\mathbb{Z}$, implies

$$\operatorname*{ess\,sup}_{\xi\in D}\mathbf{P}_{\Phi,\theta,\mathbf{a}}(\xi) < \tilde{A}.$$

Thus, Proposition 4 is applicable and $\mathcal{G}(\Phi,\theta,\mathbf{a})$ constitutes a frame. This proves item (iii). In order to prove items (ii) and (iv), follow the same steps, but apply Lemma 2, item (ii) instead of item (i). □

Corollary 3. *If the conditions of Theorem 2, item (i) or (ii), are satisfied, then $\mathcal{G}(\Phi,\theta,\mathbf{a})$ satisfies the local integrability condition* (12)*. In particular, $\mathcal{G}(\Phi,\theta,\mathbf{a})$ also satisfies the α-local integrability condition* (11)*.*

Proof. The conditions of Theorem 2, item (i) or (ii), imply the conditions of Theorem 1. Hence, Theorem 1 can be applied and $\mathcal{G}(\Phi,\theta,\mathbf{a})$ satisfies the LIC. Moreover, since the LIC implies the α-LIC, the α-LIC is also satisfied. □

Note that, by proving that the warped system $\mathcal{G}(\Phi,\theta,\mathbf{a})$ satisfies the LIC for all choices of natural downsampling factors and under a mild decay condition on θ, the results presented in [12, 32, 13] are available to a large class of warped filter banks.

4.2. Tight warped filter bank frames

The remainder of this section is devoted to the construction of tight frames following a given frequency scale. Using the framework of warped filter banks this is easily realized via the painless case result presented in Corollary 1. This is one of the major assets of the presented construction. Tight frames are important for various reasons, the most important surely being that they provide a perfect reconstruction system in which the synthesis frame equals the analysis system up to a constant. Hence, there is no need for computing and/or storing a dual frame, which might be highly inefficient. Furthermore, the usage of tight frames guarantees that the synthesis shares the properties of the analysis, e.g. in terms of time-frequency localization.

Under the conditions of Corollary 1, we can formulate very simple necessary and sufficient conditions for warped filter banks to form tight frames.

Corollary 4. *Let $\mathcal{G}(\Phi,\theta,\mathbf{a})$ be a warped filter bank for $\mathbf{L}^{2,\mathcal{F}}(D)$.*

(i) If $\mathcal{G}(\Phi,\theta,\mathbf{a})$ forms a tight frame with frame bound $C>0$ and satisfies the LIC (12), then

$$\sum_{m\in\mathbb{Z}} |T_m\theta|^2 = C, \text{ a.e.} \tag{34}$$

In particular, if $\theta \in \mathcal{O}((1+\max\{|\cdot|,|\Phi^{-1}(\cdot)-\Phi^{-1}(0)|\})^{-1-\epsilon})$ and $a_m \leq \tilde{a}/v(m)$, for all $m\in\mathbb{Z}$, or $\theta\in\mathcal{O}((1+\max\{|\cdot|,|V(0,\cdot)|\})^{-1-\epsilon})$ and $a_m\leq \tilde{a}/w(m)$, for all $m\in\mathbb{Z}$, for some constant $\tilde{a}>0$, then (34) is a necessary condition for the tight frame property.

(ii) If $\operatorname{supp}(\theta)\subseteq[c,d]$ and $a_m^{-1}\geq \Phi^{-1}(d+m)-\Phi^{-1}(c+m)$, for all $m\in\mathbb{Z}$, then $\mathcal{G}(\Phi,\theta,\mathbf{a})$ is a tight frame with frame bound $C>0$ if and only if Eq. (34) is satisfied. In particular, if $a_m\leq\max\{\tilde{a}_w/w(m),\tilde{a}_v/v(m)\}$, for all $m\in\mathbb{Z}$, with $\tilde{a}_w,\tilde{a}_v$ as in Proposition 5, then Eq. (34) is equivalent to $\mathcal{G}(\Phi,\theta,\mathbf{a})$ forming a tight frame with frame bound $C>0$

Proof. The first parts of both items are immediate consequences of Corollary 1. Note that $\theta\in\mathcal{O}((1+\max\{|\cdot|,|\Phi^{-1}(\cdot)-\Phi^{-1}(0)|\})^{-1-\epsilon})$ and $a_m\leq\tilde{a}/v(m)$, for all $m\in\mathbb{Z}$, for some constant $\tilde{a}>0$, implies the LIC (12). To see that, note that Theorem 2(i) is applicable implying the Bessel property and thus the conditions of Theorem 1 are satisfied. If instead $\theta\in\mathcal{O}((1+\max\{|\cdot|,|V(0,\cdot)|\})^{-1-\epsilon})$ and $a_m\leq\tilde{a}/w(m)$, for all $m\in\mathbb{Z}$, for some constant $\tilde{a}>0$, then Theorem 2(ii) is applicable and again the conditions of Theorem 1 are satisfied. Thus, (34) holds, if $\mathcal{G}(\Phi,\theta,\mathbf{a})$ is a tight frame with frame bound $C>0$. Moreover, by Proposition 5, Eq. (34) is equivalent to $\mathcal{G}(\Phi,\theta,\mathbf{a})$ being a tight frame with frame bound $C>0$, if $a_m\leq\max\{\tilde{a}_w/w(m),\tilde{a}_v/v(m)\}$. This finishes the proof. □

We see that, even if $a_m^{-1}<\Phi^{-1}(d_0+m)-\Phi^{-1}(c_0+m)$, with c_0,d_0 as in Proposition 5, Eq. (34) is still a necessary condition for tightness of the warped filter bank $\mathcal{G}(\Phi,\theta,\mathbf{a})$. Therefore, θ that satisfy Eq. (34) are the optimal starting point when aiming to construct warped filter bank frames with small frame bound ratio, i.e. $B/A\approx 1$.

Although surely not the only methods for obtaining functions satisfying Eq. (34), we highlight here two classical methods that provide both compact support, which is important to satisfy Corollary 4, and a prescribed smoothness: B-splines [58] and windows constructed as a superposition of truncated cosine waves of different frequency [59]. The second class contains classical window functions such as the Hann, Hamming and Blackman windows. In the following we recall a procedure to construct such functions that also satisfy Eq. (34). The method has previously been reported in [43].

Proposition 6. *Let $K \in \mathbb{N}$ and $c_k \in \mathbb{R}$ for $k \in \{0, 1, \dots, K\}$, and define*

$$\vartheta(\tau) := \sum_{k=0}^{K} c_k \cos(2\pi k\tau)\mathbf{1}_{[-1/2,1/2[}. \tag{35}$$

Then for any $R > 2K$

$$\sum_{m\in\mathbb{Z}} \left|\vartheta\left(\frac{\tau - m}{R}\right)\right|^2 = Rc_0^2 + \frac{R}{2}\sum_{k=1}^{K} c_k^2, \quad \forall \tau \in \mathbb{R};$$

i.e. the sum of squares of a system of regular translates $(T_m\theta)_{m\in\mathbb{Z}}$, with $\theta = \vartheta(\cdot/R)$, is constant.

The proposition above can be combined with Corollary 4 to easily construct tight frames by choosing the downsampling factors a_m to satisfy

$$a_m^{-1} \geq \Phi^{-1}(m + R/2) - \Phi^{-1}(m - R/2). \tag{36}$$

In the following, we will demonstrate this for the examples given in Section 3. The well-studied wavelet case behaves similar to Example 6 below and is therefore omitted.

For the purpose of all the following examples, we choose

$$\vartheta(\tau) = \begin{cases} 1/2 + 1/2\cos(2\pi\tau) & \text{for } \tau \in [-1/2, 1/2] \\ 0 & \text{else.} \end{cases}$$

The function above is often called the *Hann window.* The Hann window is among the most popular finitely supported Gabor windows or filters for time-frequency signal analysis and satisfies Proposition 6 for any $R \geq 3$.

Example 6 ($\Phi(\xi) = \operatorname{sgn}(\xi)\log(1 + |\xi|)$). For this choice of Φ, (36) takes the form

$$a_m^{-1} \geq \operatorname{sgn}(m + R/2)(e^{|m+R/2|} - 1) - \operatorname{sgn}(m - R/2)(e^{|m-R/2|} - 1),$$

or equivalently

$$a_m^{-1} \geq \begin{cases} (e^{|m|+R/2} - 1) - (e^{|m|-R/2} - 1) = e^{|m|}(e^{R/2} - e^{-R/2}) & \text{for } |m| \geq R/2, \\ (e^{m+R/2} - 1) + (e^{-m+R/2} - 1) = e^{R/2}(e^{|m|} + e^{-|m|}) - 2 & \text{else,} \end{cases}$$

where the latter case concerns the filters were $\operatorname{supp}(\mathbf{T}_m\theta)$ contains both positive and negative numbers. We see that in both cases, a_m^{-1} equals $e^{|m|}$, up to a constant depending solely on R. If we set $R = 3$, then a tight frame is obtained by choosing

$$a_m = \begin{cases} e^{-|m|}(e^{3/2} - e^{-3/2})^{-1} \geq \frac{1}{4.26e^{|m|}} & \text{for } |m| \geq 2, \\ e^{3/2}(e^1 + e^{-1}) - 2 > \frac{1}{11.84} & \text{for } |m| = 1, \\ (2e^{3/2} - 2)^{-1} > \frac{1}{6.97} & \text{for } m = 0. \end{cases}$$

On the other hand, Proposition 5 yields $\widetilde{a_w} = \widetilde{a_v} = (e^{3/2} - e^{-3/2})^{-1}$ and $w(m) = v(m) = e^{|m|}$, i.e. $a_m = e^{-|m|}(e^{3/2} - e^{-3/2})^{-1}$ for all $m \in \mathbb{Z}$, which is slightly more conservative.

Example 7 ($\Phi_l(\xi) = \operatorname{sgn}(\xi)((1 + |\xi|)^l - 1)$). Assume that $p := 1/l \in \mathbb{N}$. Then Eq. (36) can be rewritten as

$$a_m^{-1} \geq \begin{cases} (1 + |m| + R/2)^p - (1 + |m| - R/2|)^p & \text{for } |m| \geq R/2, \\ (1 + R/2 + m)^p + (1 + R/2 - m)^p - 2 & \text{else.} \end{cases}$$

If $l = 1/2$, i.e. $p = 2$, evaluation of the above conditions yields a tight warped filter bank with

$$a_m = \begin{cases} \frac{1}{6+6|m|} & \text{for } |m| \geq 2, \\ \frac{2}{25} & \text{for } |m| = 1, \\ \frac{2}{21} & \text{for } m = 0. \end{cases}$$

In this setting, Proposition 5 yields $\widetilde{a_v} = \frac{2}{21} = \widetilde{a_w}/2$ and $v(m) = 1 + |m| = w(m)/2$, see also Example 4.

Example 8 ($\Phi_l(\xi) = \xi^l - \xi^{-l}$). Recall that we have shown in the proof of Proposition 2, item (ii), that $w = (\Phi^{-1})'$, given by

$$w(\tau) = l^{-1}\frac{(\tau + \sqrt{\tau^2 + 4})^{1/l}}{(\tau + \sqrt{\tau^2 + 4}) + (\tau + \sqrt{\tau^2 + 4})^{-1}} \leq l^{-1}(\tau + \sqrt{\tau^2 + 4})^{1/l - 1}, \text{ for all } \tau \in \mathbb{R},$$

is v-moderate for $v = (2 + |\cdot|/2)^{\frac{2+2l}{l}}$. The application of Proposition 5 provides us with

$$a_m \leq \max\{l\widetilde{a_w}(m + \sqrt{m^2 + 4})^{1/l-1}, \widetilde{a_v}(2 + |\cdot|/2)^{-\frac{2+2l}{l}},$$

for some constants $\widetilde{a_w}, \widetilde{a_v} > 0$. It is easy to see that, asymptotically, the second estimate is much stricter than the first. For example, if $l = 1$, then we have $a_m \leq \max\{\widetilde{a_w}, \widetilde{a_v}(2 + |\cdot|/2)^{-4}\}$. For $l = 1/2$, we obtain $a_m \leq \max\{\frac{\widetilde{a_w}}{2}(m + \sqrt{m^2 + 4})^{-1}, \widetilde{a_v}(2 + |\cdot|/2)^{-6}\}$ and so on for other choices of $l \in (0, 1]$.

Inserting $\Phi_{1/2}^{-1} = (\cdot + \sqrt{(\cdot)^2 + 4})^2$ into Eq. (36) directly does not provide very instructive estimates. However, we can easily see that

$$\begin{aligned} \Phi^{-1}(m + R/2) - \Phi^{-1}(m - R/2) &= \int_{-R/2}^{R/2} w(\tau + m)\, d\tau \\ &\leq 2\int_{-R/2}^{R/2} \tau + m + \sqrt{(\tau + m)^2 + 4})\, d\tau \\ &\leq R(m + R/2 + \sqrt{(m + R/2)^2 + 4}), \end{aligned}$$

where we used that w is nondecreasing. Set $R = 4$ to obtain

$$\Phi^{-1}(m+R/2)-\Phi^{-1}(m-R/2) \leq 4(m+2+\sqrt{m^2+4m+8}) \leq 4(m+2+\sqrt{(m+4)^2}) \leq 4(m+|m+4|)$$

and if we choose $\mathbf{a} = (a_m)_{m\in\mathbb{Z}}$ with

$$a_m = \frac{1}{4m+4|m+4|}, \text{ for all } m \in \mathbb{Z},$$

then $\mathcal{G}(\Phi_l, \vartheta(\cdot/4), \mathbf{a})$ is a tight frame.

Example 9 ($\Phi(\xi) = \tan(\xi)$). For the tangent warping function, we have shown in Proposition 2 that $w(\tau) = (1+\tau^2)^{-1}$ and the moderating weight v can be chosen as constant 4/3. We obtain from Proposition 5 the following sets of almost optimal natural downsampling factors

$$\widetilde{a_w}/w(m) = \frac{3(1+m^2)}{4R} \quad \text{and} \quad \widetilde{a_v}/v(m) = \frac{3}{8\arctan(R/2)}.$$

With $R = 3$, a tight frame is obtained for $a_0 = 3/8$ and $a_m = (1+m^2)/4$ for $m \neq 0$.

5. Warped filter banks for digital signals

In the following, we consider sequences in $x \in \ell^2(\mathbb{Z})$, interpreted as the samples of signals sampled at frequency ξ_s Hz. The discrete time Fourier transform (DTFT) and its inverse are denoted in the same fashion as the continuous Fourier transform before, i.e. $\hat{x}(\xi) := \mathcal{F}x(\xi) = \sum_{\mathbb{Z}} x(l) = e^{-2\pi i l\xi}$ and $\check{x} := \mathcal{F}^{-1}x$. Discrete translation and modulation operators are given as usual. Note that $x \in \ell^2(\mathbb{Z})$ implies $\hat{x} \in \mathbf{L}^2(\mathbb{T})$, with the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Finally, let $\underline{M} := \{0,\ldots,M-1\}$. A slightly more detailed account of the construction of warped filter banks for signals in $\ell^2(\mathbb{Z})$ can be found in [15].

A *(M channel, non-uniform analysis) filter bank* $(g_m, a_m)_{m\in\underline{M}}$, with $g_m \in \mathbf{L}^2(\mathbb{T})$ and $a_m \in \mathbb{N}$ is the set of finite energy sequences

$$g_{m,n} := \mathbf{T}_{na_m}\widecheck{g_m}, \text{ for all } m \in \underline{M}, n \in \mathbb{Z}, \tag{37}$$

Filter bank frames on $\ell^2(\mathbb{Z})$ are defined analogous to those on $\mathbf{L}^{2,\mathcal{F}}(D)$.

We will now construct filters on $g_m \in \mathbf{L}^2(\mathbb{T})$ by restricting the warping function Φ to $I_{\xi_s,D} := D \cap (-\xi_s/2, \xi_s/2]$, i.e. we construct filter banks for sequences $x \in \ell^2(\mathbb{Z})$ with $\operatorname{supp}(\hat{x}) \subset I_{\xi_s,D}/\xi_s$. Here, the interval $(-1/2, 1/2]$ is interpreted as one period of the torus $\mathbb{T}$.

Although it is not strictly necessary, we will assume from now on, that $\operatorname{supp}(\theta) \subseteq [c,d] \subset \Phi(I_{\xi_s,D})$. Let

$$M_{\max} = \max\{m \in \mathbb{Z} \ : \ \Phi^{-1}(m+d) \leq \sup(I_{\xi_s,D})\}$$
$$M_{\min} = \min\{m \in \mathbb{Z} \ : \ \Phi^{-1}(m+c) > \inf(I_{\xi_s,D})\},$$

and define the frequency responses

$$g_m(\xi) := \sqrt{a_m}(\mathbf{T}_m\theta) \circ \Phi(\xi \cdot \xi_s), \quad \text{for all } \xi \in \mathbb{T},\ m \in \{M_{\min},\ldots,M_{\max}\}, \tag{38}$$

where the constants $a_m \in \mathbb{N}$ can be selected freely. The support restriction ensures that $\operatorname{supp}(g_m) \subseteq I_{\xi_s,D}/\xi_s$. Clearly, if $I_{\xi_s,D}$ is a strict subset of $(-\xi_s/2, \xi_s/2]$, then either $M_{\min}$ or $M_{\max}$ is not finite. Hence, for the construction of a filter bank with a finite number of channels, we only consider the filters g_m for $m \in \{m_{\min}, \dots, m_{\max}\}$, where $-\infty < m_{\min}, m_{\max} < \infty$ satisfy $m_{\min} \geq M_{\min}$ and $m_{\max} \leq M_{\max}$.

In order to cover the full frequency range $I_{\xi_s,D}/\xi_s$, we need to design additional band-pass filters. We distinguish two particular cases:

(i) If $I_{\xi_s,D} = (-\xi_s/2, \xi_s/2]$, then

$$g_{m_{\max}+1}(\xi) := \left(a_{m_{\max}+1} \sum_{m \in \mathbb{Z} \setminus [m_{\min}, m_{\max}]} |(\mathbf{T}_m \theta) \circ \Phi(\xi \cdot \xi_s)|^2 \right)^{1/2}, \text{ for all } \xi \in \mathbb{T}. \tag{39}$$

(ii) If $I_{\xi_s,D} \subsetneq (-\xi_s/2, \xi_s/2]$, then

$$\begin{aligned} g_{m_{\min}-1}(\xi/\xi_s) &:= \left(a_{m_{\min}-1} \sum_{m < m_{\min}} |(\mathbf{T}_m \theta) \circ \Phi(\xi \cdot \xi_s)|^2 \right)^{1/2} \quad \text{and} \\ g_{m_{\max}+1}(\xi/\xi_s) &:= \left(a_{m_{\max}+1} \sum_{m > m_{\max}} |(\mathbf{T}_m \theta) \circ \Phi(\xi \cdot \xi_s)|^2 \right)^{1/2}, \text{ for all } \xi \in I_{\xi_s,D}/\xi_s, \end{aligned} \tag{40}$$

and 0 elsewhere. Once again $a_{m_{\min}-1}, a_{m_{\max}+1} \in \mathbb{N}$ can be selected freely.

The final filter bank contains $M := m_{\max} - m_{\min} + 1$ filters (or $M := m_{\max} - m_{\min} + 2$ if $I_{\xi_s,D} \subsetneq (-\xi_s/2, \xi_s/2]$) and after shifting the index set by $m_s := -m_{\min}$ (or $m_s := -m_{\min} - 1$), we obtain the M-channel, discrete warped filter bank $(g_m, a_m)_{m \in \underline{M}}$. Since the downsampling factors $a_m \in \mathbb{N}$ only act as a normalization factor in the definition of the g_m, they can easily be chosen (and varied) a posteriori. For some exemplar frequency responses derived from the warping functions introduced in Example 1-4, see Figure 2. Note that the square-root warping corresponds to Example 4 with $l = 1/2$. In Figure 3, we show the time-frequency plots of a test signal with respect to the same warping functions.

Necessary and sufficient frame conditions for discrete warped filter banks are analogous to the continuous case. In particular,

$$0 < A \leq \sum_{m \in \mathbb{Z}} |(\mathbf{T}_m \theta) \circ \Phi(\xi \cdot \xi_s)|^2 = \sum_{m \in \underline{M}} a_m^{-1} |g_m(\xi)|^2 \leq B < \infty, \text{ for all } \xi \in I_{\xi_s,D}/\xi_s, \tag{41}$$

is a necessary condition for $(g_m, a_m)_{m \in \underline{M}}$ to constitute a frame with frame bounds A, B.

Recall that we assume $\operatorname{supp}(\theta) \subseteq [c, d]$ and define $M_{\text{bp}} = \{M - 1\}$, if $I_{\xi_s,D} = (-\xi_s/2, \xi_s/2]$ and $M_{\text{bp}} = \{0, M - 1\}$. For all $m \in \underline{M} \setminus M_{\text{bp}}$ choose the downsampling factors a_m such that

$$\xi_s/a_m \geq \Phi^{-1}(d + m + m_s) - \Phi^{-1}(c + m + m_s). \tag{42}$$

If additionally

$$\begin{gathered} \xi_s/a_{M-1} \geq \Phi^{-1}(d + m_s) + \xi_s - \Phi^{-1}(c + M + m_s - 1), \text{ if } I_{\xi_s,D} = (-\xi_s/2, \xi_s/2], \text{ or} \\ \xi_s/a \geq \Phi^{-1}(d + m_s) - \inf(I_{\xi_s,D}) \text{ and } \xi_s/a_{M-1} \geq \sup(I_{\xi_s,D}) - \Phi^{-1}(c + M + m_s - 1), \end{gathered} \tag{43}$$

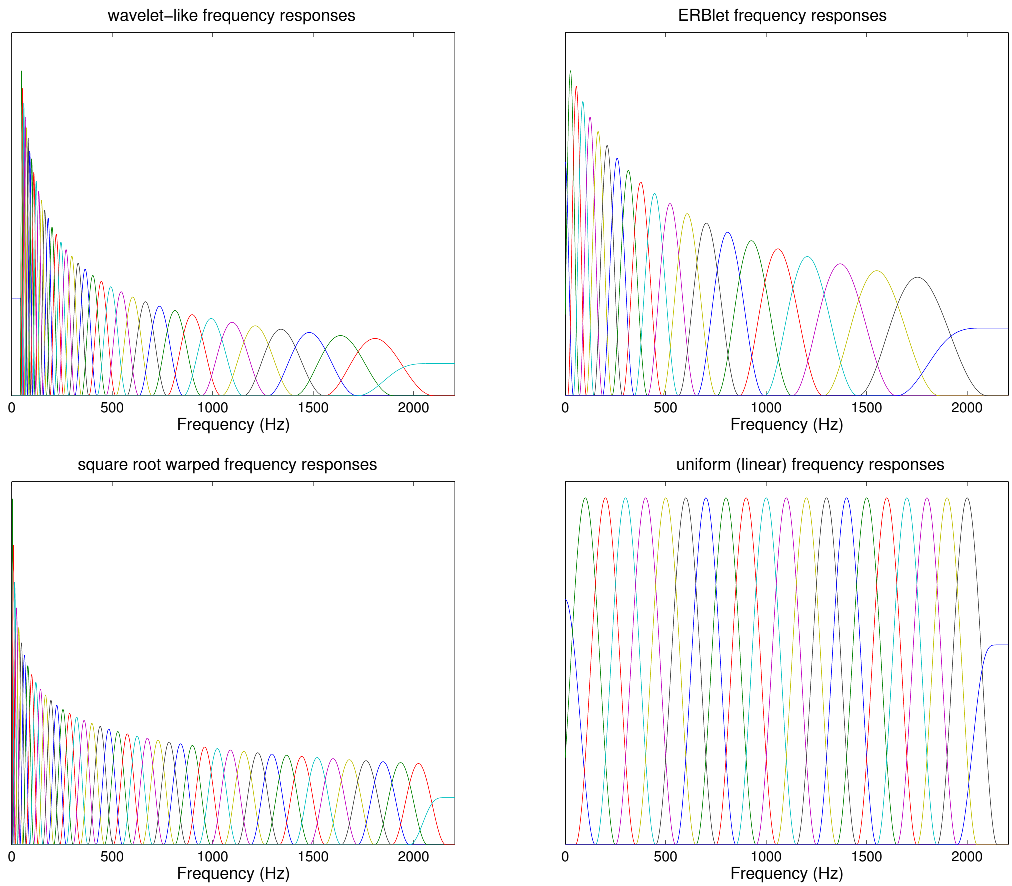

Figure 2: Frequency responses of warped filters using a Hann window prototype: (top-left) logarithmic warping $F(x) = 10\log(x)$, (top-right) ERBlet warping $F(x) = 21.4\,\mathrm{sgn}(x)\log_{10}(1+|x|/229)$, (bottom-left) square root warping $F(x) = \mathrm{sgn}(x)(\sqrt{1+|x|}-1)$ and (bottom-right) linear warping $F(x) = x/100$. The systems use 1 bin/unit and were restricted to the frequency range 0 Hz-8.82 kHz for visualization. Note the low-pass and high-pass filters defined in (39) and (40).

then (41) is equivalent to the frame property (with bounds A, B) and a dual filter bank is obtained analogous to Corollary 1(ii). The proof for this equivalence is a straightforward application of the discrete variant of an appropriate, discrete variant of Proposition 3(ii) and left to the reader.

Similarly, a discrete variant of Proposition 4 can be used to show that

$$\infty > \sum_{m\in\underline{M}} a_m^{-1}|g_m|^2(\xi) > \sum_{m\in\underline{M}} \left(a_m^{-1}|g_m| \sum_{k=1}^{a_m-1} |\mathbf{T}_{na_m^{-1}}\overline{g_m}|(\xi) \right) =: \mathcal{A}(\xi), \text{ almost everywhere,} \quad (44)$$

implies that $(g_m, a_m)_{m\in\underline{M}}$ is a frame. Since $\mathcal{A}$ depends continuously on the downsampling factors a_m (at least if $\theta \in \mathcal{C}$), this shows that discrete warped filter bank frames can be obtained, even if the downsampling regime discussed in the previous paragraph is not strictly satisfied. This is also

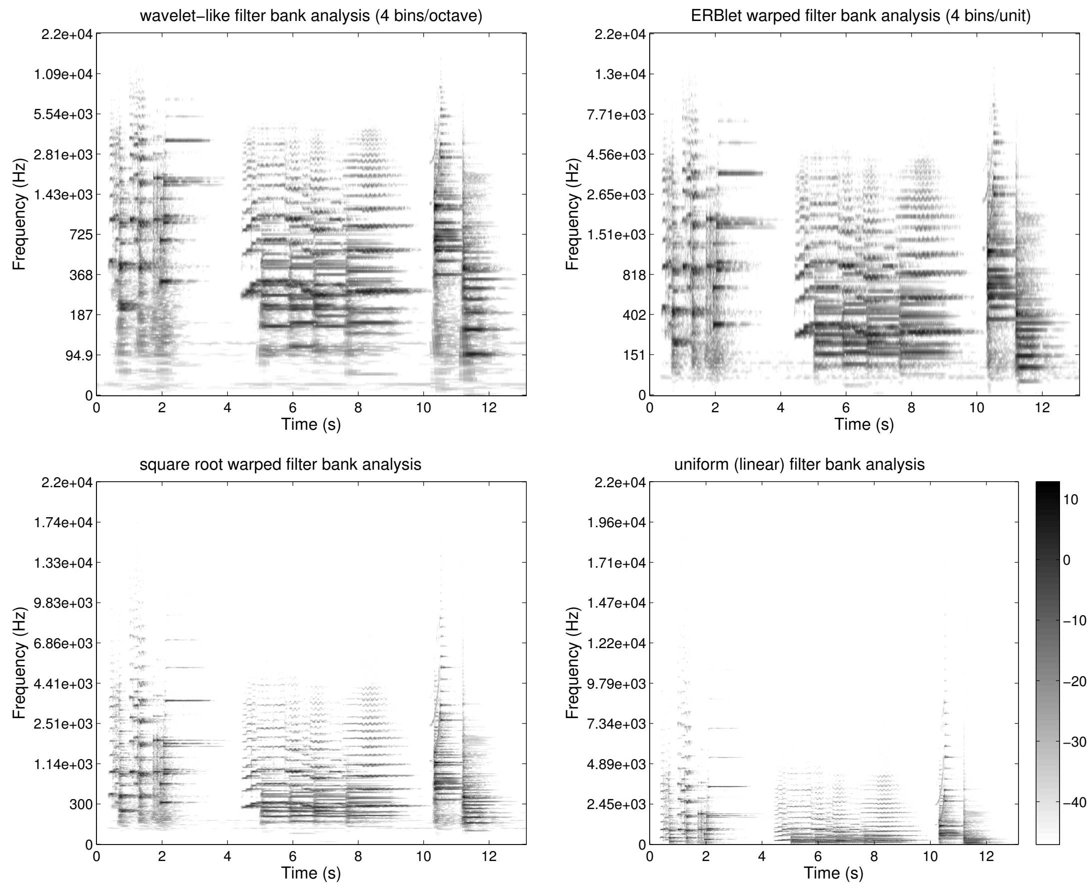

Figure 3: Time-frequency plots of a test signal comprised of three short piano and violin excerpts. The following warping functions were selected, using 4 bins/unit each: (top-left) logarithmic warping $F(x) = 10\log(x)$, (top-right) ERBlet warping $F(x) = 21.4\,\mathrm{sgn}(x)\log_{10}(1+|x|/229)$, (bottom-left) square root warping $F(x) = \mathrm{sgn}(x)(\sqrt{1+|x|}-1)$ and (bottom-right) linear warping $F(x) = x/100$. Intensity is in dB, the colorbar on the bottom-right is valid for all plots.

illustrated by Table 1 that provides numerically computed frame bounds of different warped filter banks with varying redundancy. All the downsampling factors were chosen to be anti-proportional to the filter frequency support, but only the first column represents filter banks that satisfy Eq.s (42) and (43), cf. [15] for more details[2]. The values presented in the table (a) indicate that, for fixed redundancy, all the warping functions provide systems with similarly small frame bound ratio and (b) serve as numerical verification of Theorem 2.

[2]In fact, due to a bug in older versions of the LTFAT Toolbox (`ltfat.github.io`) used for the frame bound calculations, the frame bound ratios reported in [15] are too large. The ratios in Table 1 have been corrected and updated with new, lower redundancies. Updated code reproducing the table and the figures in this section are available at `http://ltfat.github.io/notes/039/`

Table 1: Frame bound ratios of various warped filter banks. From top to bottom: linear warping $F(x) = x/100$, square root warping $F(x) = \operatorname{sgn}(x)(\sqrt{1+|x|} - 1)$, ERBlet warping $F(x) = 21.4\,\operatorname{sgn}(x)\log_{10}(1+|x|/229)$ and logarithmic warping $F(x) = 10\log(x)$. The columns correspond to warped filter banks with *approximately* the same redundancy $C_{red} = \sum_m 1/a_m$.

$C_{red}(\approx)$	3	2	1.5	1.25	1.125
linear	1.000	1.220	1.961	3.880	6.868
square root	1.003	1.237	1.980	3.938	7.315
ERB	1.000	1.240	1.970	3.860	7.122
logarithmic	1.014	1.240	1.973	3.876	7.159

6. Conclusion and Outlook

In this contribution, we have introduced a novel, flexible family of structured time-frequency filter banks. These warped filter banks are able to recreate or imitate important classical time-frequency representations, while providing additional design freedom. Warped filter banks allow for intuitive handling and the application of important results from the theory of generalized-shift invariant frames. In particular, the construction of tight frames of bandlimited filters reduces to the selection of a compactly supported prototype function whose integer translates satisfy a simple summation condition and sufficiently small downsampling factors a_m. Moreover, the warping construction induces a natural choice of downsampling factors that further simplifies the design of warped filter bank frames. With several examples, we have illustrated not only the flexibility of our method when selecting a non-linear frequency scale, but also the ease with which tight frames or snug frames can be constructed.

The complementary manuscript [10] discusses warped time-frequency representations in the context of continuous frames, determines the associated (generalized) coorbit spaces and the warped time-frequency representations' sampling properties in the context of atomic decompositions and Banach frames [60, 61, 62]. Future work will continue to explore practical applications of warped time-frequency representations and their finite dimensional equivalents on $\mathbb{C}^L$, as well as extending the warping method to multidimensional signals.

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