New ideas in reassignment: General time-frequency filter banks, sampling and processing

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***Abstract*—Reassignment is a popular technique for deconvolution of quadratic time-frequency representations, in particular spectrogram and scalogram signal representations. In differential reassignment, the time and frequency phase derivatives of a complex-valued time-frequency representation are used to determine the instantaneous frequency and group delay associated to the individual representation coefficients. Subsequently, the coefficient energy is *reassigned* to the determined position. In this contribution, we show that classical and efficient reassignment methods can be used with arbitrary time-frequency localized filters and subsampling rates, after simple modification. The main drawback of reassignment is the non-invertibility of the resulting representation. However, we propose a processing scheme that benefits both from the improved localization of the reassigned representation and the frame properties of the underlying complex-valued representation.**

I. Introduction

Time-frequency representations such as the short-time Fourier transform (STFT) [12], [13] are ubiquitous in signal analysis. In particular their squared magnitude, the *spectrogram* is frequently used to determine the local frequency content of an analyzed signal. The spectrogram however provides a biased, smoothed representation highly dependent on the chosen STFT window function. The smoothing effect in the spectrogram is subject to Heisenberg's uncertainty inequality and thus cannot be arbitrarily reduced.

Therefore, various alternative time-frequency representations have been proposed. The quadratic time-frequency representations in Cohen's class [7], [8] are given by the Wigner-Ville distribution (WVD) [28], [27] convolved with a smoothing kernel. While the spectrogram suffers from a large amount of smoothing, the WVD produces undesirable interference terms. Cohen's class representations often sit between these two extrema and are designed with a certain trade-off between smoothing and interference attenuation in mind, e.g. the smoothed pseudo WVD [19] and Born-Jordan distributions.

Another approach is the construction of time-frequency representations with varying window functions, either according to a fixed rule or in a signal-dependent fashion. Varying the window along frequency leads to (nonuniform) filter banks [25], [1]. In particular, wavelet filter banks [26], [18] (often referred to as time-scale representations) can be constructed through a dilation rule. Variation along time leads to nonstationary Gabor transforms [4], [17]. Joint time-frequency adaptation has been studied e.g. in [14], [11]. All these methods share the property that they do not reduce the smoothing effect overall, but instead try to select locally a good trade-off between time and frequency smoothing with respect to the signal characteristics.

Here, we recall the (differential) reassignment method originally proposed by Kodera et al. [16] to obtain a sharper time-frequency picture. Reassignment attempts the deconvolution of the spectrogram by means of the partial derivatives of the phase of the underlying STFT. The resulting *reassigned spectrogram* achieves a sharp representation similar to the WVD, but with little interference. Reassignment was made popular by Auger and Flandrin [2], [6], who generalize the method to general Cohen's class time-frequency representations and an equivalent class of time-scale representations. Most importantly, they discovered an efficient means of computation and the application of reassignment. In contrast to the previously mentioned methods, the reassigned spectrogram is not bilinear and even approximate reconstruction is a nontrivial task. Therefore, although a valuable analysis tool, reassigned time-frequency representations are deemed unsuitable for signal processing so far.

A variant of the reassignment method dealing with the reconstruction problem, dubbed *synchrosqueezing*, has been proposed by Daubechies et al. [10], [9], [3]. The main differences between reassignment and the synchrosqueezing transform are that the latter acts on a complex valued time-frequency representation, hence preserving phase information, and only performs *frequency reassignment*, i.e. the time position of a representation coefficient is never modified. Clever use of the independence of the continuous wavelet (short-time Fourier) transform reconstruction formula from the synthesis wavelet (window) yields an inversion procedure and thus a means of synthesizing processed signals. Moreover, the synchrosqueezing transform works for representations considering only a discrete subset of all frequency bands, i.e. it is compatible with frequency sampling.

In this contribution, we propose application of the reassignment method to Gabor transforms (sampled STFTs) [12], [13] and from there continue to show that, with the obvious adjustments, the reassignment method can be applied to any filter bank representation. Furthermore, we demonstrate that the reassigned spectrogram, with respect to a *(over-)complete system*, can be used as an interface for time-frequency processing. Hence, the benefits of a sharpened representation can be combined with the invertibility of the underlying filter bank representation.

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Organization of the paper: After recalling essential concepts and the reassignment method applied to the STFT in Section II and III, we demonstrate in Section IV how to apply reassignment to nonuniform filter bank coefficients. Finally, Section V discusses some issues in implementation and Section VI presents a method for using reassigned representations in signal processing.

II. Preliminaries

Although the results presented in this contribution are applicable for discrete signals, see Section V, we will present the theoretical material for continuous time signals $f \in \mathbf{L}^2(\mathbb{R})$ to stay consistent with previous publications on reassignment.

We denote *translation* and *modulation* operators

$$\mathbf{T}_x f(t) = f(t-x) \text{ and } \mathbf{M}_\omega f(t) = f(t)e^{2\pi i \omega t},$$

for $f \in \mathbf{L}^2(\mathbb{R})$, $t, x, \omega \in \mathbb{R}$.

For a signal $f \in \mathbf{L}^2(\mathbb{R})$, its STFT with respect to the window $g \in \mathbf{L}^2(\mathbb{R})$ is given by

$$\begin{aligned} V_g f(x,\omega) &= \langle f, \mathbf{T}_x \mathbf{M}_\omega g \rangle \\ &= \sqrt{\mathcal{S}_g f(x,\omega)}\, e^{2\pi i \phi(x,\omega)}, \end{aligned} \tag{1}$$

where $\mathcal{S}_g f = |\mathcal{V}_g f|^2$ is the spectrogram and $\phi : \mathbb{R}^2 \to \mathbb{R}$ is the phase of the STFT. The collection $\mathcal{G}(g) := \{g_{x,\omega}\}_{x,\omega \in \mathbb{R}}$, with $g_{x,\omega} = \mathbf{T}_x \mathbf{M}_\omega g$ is the (continuous) Gabor system with respect to g. Without loss of generality, we assume that $g, \hat{g}$ are well-localized around 0.

Let $\mathbf{g} = \{g_m \in \mathbf{L}^2(\mathbb{R})\}_{m \in \mathbb{Z}}$, $\mathbf{a} = \{a_m \in \mathbb{R}^+\}_{m \in \mathbb{Z}}$ a sequence of functions and time steps, respectively. We call the system $\mathcal{G}(\mathbf{g}, \mathbf{a}) := \{\mathbf{T}_{na_m} g_m \ : \ a_m \in \mathbb{R}^+\}_{n,m \in \mathbb{Z}}$ a *time-frequency filter bank*, if g_m is well-localized around time 0 and $\widehat{g_m}$ is well-localized around frequency ω_m with $\omega_j < \omega_m$ if $j < m$ and $\lim_{m \to -\infty} \omega_m = -\infty$, $\lim_{m \to \infty} \omega_m = \infty$. The associated *filter bank analysis* is given by

$$c_{n,m} := c_f[n,m] := \langle f, \mathbf{T}_{na_m} g_m \rangle. \tag{2}$$

If $g_m = \mathbf{M}_{mb} g$ for some $g \in \mathbf{L}^2(\mathbb{R})$ and $b \in \mathbb{R}^+$, then $\mathcal{G}(\mathbf{g}, \mathbf{a})$ is a regular (discrete) Gabor system.

A filter bank forms a frame, if there are constants $0 < A \leqslant B < \infty$, such that

$$A\|f\|_2^2 \leqslant \sum_{n,m} |c_{n,m}|^2 \leqslant B\|f\|_2^2, \text{ for all } f \in \mathbf{L}^2(\mathbb{R}). \tag{3}$$

The frame property guarantees the stable invertibility of the coefficient mapping by means of a dual frame $\{\widetilde{g_{n,m}}\}_{n,m \in \mathbb{Z}}$, i.e.

$$f = \sum_{n,m} c_{n,m} \widetilde{g_{n,m}}, \text{ for all } f \in \mathbf{L}^2(\mathbb{R}). \tag{4}$$

III. Reassignment for the STFT

In differential reassignment, STFT coefficients are assigned new positions according to the local *group delay*

$$x_0(x,\omega) = -\frac{\partial}{\partial \omega} \phi(x,\omega) \tag{5}$$

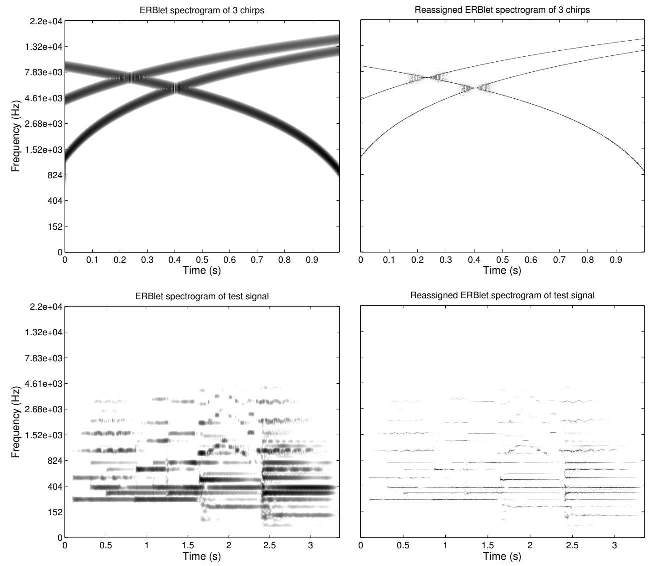

Fig. 1. Reassignment of ERBlet filter bank representations: (top) Reassignment perfectly localizes linear chirps that are well separated in the time-frequency plane. Separation quality decreases in the vicinity of crossover points. (bottom) Reassignment of a violin and piano test signal. Both examples use the auditory ERBlet filter bank proposed in [21], [20].

and *instantaneous frequency*

$$\omega_0(x,\omega) = \omega + \frac{\partial}{\partial t} \phi(x,\omega). \tag{6}$$

The reassigned spectrogram of f with respect to the window g is defined as

$$\mathcal{R}_g f(x,\omega) = \int_{T_{x,\omega}} \mathcal{S}_g f(y,\xi) \ d\xi \ dy, \tag{7}$$

where $T_{x,\omega} := \{(y,\xi) \in \mathbb{R}^2 \ : \ x_0(y,\xi) = t \text{ and } \omega_0(y,\xi) = \omega\}$.

Auger and Flandrin [2] have shown that the reassignment operators can be expressed as the pointwise product of the STFTs with respect to 3 window functions, depending on the original window g:

$$\begin{aligned} x_0(x,\omega) &= t - \mathbf{Re}\left(\mathcal{V}_{g_T} f(x,\omega) / \mathcal{V}_g f(x,\omega)\right) \\ &= t - \mathbf{Re}\left(\frac{\mathcal{V}_{g_T} f(x,\omega)\overline{\mathcal{V}_g f(x,\omega)}}{\mathcal{S}_g f(x,\omega)}\right) \end{aligned} \tag{8}$$

and

$$\begin{aligned} \omega_0(x,\omega) &= \omega + \mathbf{Im}\left(\mathcal{V}_{g'} f(x,\omega) / \mathcal{V}_g f(x,\omega)\right) \\ &= \omega + \mathbf{Im}\left(\frac{\mathcal{V}_{g'} f(x,\omega)\overline{\mathcal{V}_g f(x,\omega)}}{\mathcal{S}_g f(x,\omega)}\right), \end{aligned} \tag{9}$$

whenever $\mathcal{S}_g f(x,\omega) \neq 0$ and 0 else. Here, $g'(t) = \frac{\partial}{\partial t} g(t)$ is the derivative and $g_T(t) := t g(t)$ is a time weighted version of g.

IV. Reassignment for filter banks

Due to the fact that derivation inherently is an infinitesimal operation, reassignment has only been proposed for continuously indexed time-frequency representations so far, see [2]. However, such representations are highly redundant and more often than not, a representation with reduced redundancy is desired. For practical applications in particular, it is often unfeasible to compute the full representation for any signal of considerable length. Therefore, time-frequency representations are usually sampled, i.e. only evaluated at a discrete set of points. Consequently, the application of reassignment to such sampled representations is of great interest.

Although it would be possible to apply a finite difference scheme to a regularly sampled STFT in order to obtain an approximation of the operators (5) and (6), the approximation quality would be highly dependent on the sampling density and the dynamics of the analyzed signal. It seems to have gone unnoticed that Eqs. (8) and (9) immediately enable the reassignment of subsampled representations without any additional work. Before we discuss the general case, let us illustrate the principal idea at the simple example of a regular Gabor system, comprised of $g_{n,m} = \mathbf{T}_{na}\mathbf{M}_{mb}g$, $n, m \in \mathbb{Z}$. Then

$$c_f[n,m] = \langle f, g_{n,m}\rangle = V_g f(na, mb). \tag{10}$$

Let g^T be a time-weighted version of g as before and denote $g^F := g'$ and define

$$c_f^T[n,m] := \langle f, g^T_{n,m}\rangle = V_{g^T} f(na, mb) \tag{11}$$

and

$$c_f^F[n,m] := \langle f, g^F_{n,m}\rangle = V_{g'} f(na, mb). \tag{12}$$

Then obviously

$$x_0(na, mb) = na - \mathbf{Re}\left(c_f^T[n,m]/c_f[n,m]\right), \tag{13}$$

whenever $c_f[n,m] \neq 0$ and similarly

$$\omega_0(na, mb) = mb + \mathbf{Im}\left(c_f^F[n,m]/c_f[n,m]\right). \tag{14}$$

Clearly, $(x_0(na, mb), \omega_0(na, mb))$ rarely coincides with a sampling point $(\tilde{n}a, \tilde{m}b)$. For some applications, the precise values might be useful for signal analysis, but more often we will want to reassign energy to the nearest sampling point to obtain a reassigned spectrogram on the same sampling grid. Denoting by $\lfloor x \rceil$ rounding of $x \in \mathbb{R}$ to the next integer, this is easily achieved by defining

$$c_f^{\mathcal{R}}[n,m] := \sum_{(l,k)\in L_{n,m}} |c_f[l,k]|^2, \tag{15}$$

where

$$L_{n,m} := \{(l,k) \in \mathbb{Z}^2 \;:\; n_0[l,k] = n,\; m_0[l,k] = m\}, \tag{16}$$

$$n_0[n,m] := \lfloor x_0(na, mb)/a \rceil,\;\; m_0[n,m] = \lfloor \omega_0(na, mb)/b \rceil, \tag{17}$$

for all $n, m \in \mathbb{Z}$.

The same principle can be applied to time-frequency FBs[1], only the correct formulation of the reassignment operators is slightly more complicated. Define

$$g_m^0 := \mathbf{M}_{-\omega_m} g_m, \tag{18}$$

$$g_m^T(t) := t g_m(t),\;\; g_m^F := \mathbf{M}_{\omega_m}(\mathbf{M}_{-\omega_m} g_m)', \tag{19}$$

for all $t \in \mathbb{R}$, $m \in \mathbb{Z}$. Then g_m^0 is centered at 0 in time and frequency,

$$c_f[n,m] = \langle f, g_{n,m}\rangle = V_{g_m^0} f(na_m, \omega_m) \tag{20}$$

and c_f^T, c_f^F can be defined analogously. Moreover, we obtain

$$x_0(na_m, \omega_m) = na_m - \mathbf{Re}\left(c_f^T[n,m]/c_f[n,m]\right), \tag{21}$$

whenever $c_f[n,m] \neq 0$ and similarly

$$\omega_0(na_m, \omega_m) = \omega_m + \mathbf{Im}\left(c_f^F[n,m]/c_f[n,m]\right). \tag{22}$$

Once more, we might prefer the reassigned spectrogram to be defined on the sampling points (na_m, ω_m). This is achieved by

$$c_f^{\mathcal{R}}[n,m] := \sum_{(l,k)\in L_{n,m}} |c_f[l,k]|^2, \tag{23}$$

where

$$L_{n,m} := \{(l,k) \in \mathbb{Z}^2 \;:\; (m_0[l,k], n_0[l,k]) = (n,m)\}, \tag{24}$$

where

$$m_0[l,k] := \arg\min_{m\in\mathbb{Z}} |\omega_m - \omega_0(la_k, \omega_k)| \text{ and} \tag{25}$$

$$n_0[l,k] := \left\lfloor \frac{x_0(la_k, x_k)}{a_{m_0[l,k]}} \right\rceil. \tag{26}$$

Note that in this setup, frequency reassignment has priority over time reassignment.

Remark 1. Clearly, an analogous procedure can be applied for any function system $\{g_j\}_{j\in I}$, with a countable index set I, if there is a sequence $(x_j, \omega_j)_{j\in I}$, such that g_j $\widehat{g_j}$ are well-localized around time x_j and frequency ω_j, respectively. In particular, we can use the reassignment method for time-frequency frames.

V. Notes on implementation

In applications, we are mostly concerned with discrete signals in $\ell^2(\mathbb{Z})$ (or $\mathbb{C}^L$). Of course, the reassignment operators (5) and (6) can be discretized by substitution of the partial derivatives with a suitable discrete derivation, e.g. a finite difference scheme or the spectral derivative $f' := \mathcal{F}^{-1}\left((\cdot)\hat{f}(\cdot)\right)$. However, it is again preferable to use Eqs. (8) and (9). Then, the discrete derivation is applied to g_i's to obtain discrete variants of g_m^F's. In many cases, g_m is even a sampled version of a function in $\mathbf{L}^2(\mathbb{R})$ and we might be able to find an explicit formula for $(\mathbf{M}_{-\omega_i} g_m)'$ possibly providing even higher precision. On the other hand, g_m^T is obtained simply as $(l - x_i)g_m[l]$.

[1] For the reassignment procedure to make sense, the points ω_m, $m \in \mathbb{Z}$, must be well-distributed on the frequency axis.

We provide implementations for Gabor and filter bank reassignment in the LTFAT Toolbox [22], available at ltfat.sourceforge.net. These implementations use the spectral derivative to compute g_m^F's. That way, we use the same type of discretization for obtaining g_m^F's as for g_m^T's, since $\mathbf{M}_{\omega_m}(\mathbf{M}_{-\omega_m} g_m)' = \mathcal{F}^{-1}\left((\cdot - \omega_m)\hat{g_m}(\cdot)\right)$. The phase derivatives are provided by the routines `gabphasegrad` and `filterbankphasegrad`, respectively. The reassignment procedure is performed by `gabreassign` and `filterbankreassign`. For examples, please visit our website http://ltfat.sourceforge.net/notes/041, where MATLAB scripts reproducing Figure 1 and Examples 1 and 2 can be found.

VI. Processing with reassigned filter banks

So far, the lack of an efficient inversion procedure has hindered the spread of reassignment as a tool for signal processing, therefore limiting its application to pure analysis purposes. Here, we propose a method to use the reassigned spectrogram as an interface for interaction with the underlying filter bank representation.

For that purpose, we define the *inverse reassignment map*

$$\begin{aligned}&\mathbf{iR}[n,m]\\&= \{(l,k) \in \mathbb{Z}^2 \ : \ (n_0[l,k], m_0[l,k]) = (n,m)\},\end{aligned} \tag{27}$$

which serves as a lookup table during processing. In other words, the user selects a time-frequency region $\Omega \subset \mathbb{Z}^2$ in the reassigned spectrogram to be processed and decides on a processing operation. That processing operation is then applied automatically to the filter bank coefficients $c_f[\mathbf{iR}[\Omega]]$. If the filter bank forms a frame, then the modified coefficients can be synthesized to obtain a processed signal.

Such a scheme is easily implemented for a wide variety of processing methods, in particular pointwise coefficient modifications of any kind, e.g. isolation and amplification of signal components, denoising, time-frequency displacement (with respect to the filter bank specific frequency scale) or multiplicative/additive changes to the phase derivative.

We present here 2 toy examples using a constant-Q filter bank [5], [15], [23]: (a) Denoising by hard thresholding and (b) transposition of a harmonic signal component.

Example 1. We create a noisy version of the test signal already shown in Figure 1(bottom), with uniformly distributed noise with the same energy as the signal, resulting in a signal-to-noise ratio (SNR) of 0 dB, where

$$\mathrm{SNR}_{\mathrm{dB}}(f_{\text{noisy}}) = 20 \log_{10}(\|f_{\text{clean}}\| / \|\text{noise}\|).$$

A denoising mask is determined by hard thresholding in the reassigned domain; a good threshold value is determined experimentally at 23 dB below the maximum energy coefficient. Through the inverse reassignment map $\mathbf{iR}[\Omega]$, where Ω is the area where the denoising mask is larger than 0, we obtain a denoised version of the test signal with a SNR of approximately 11.5 dB. The intermediate representations can be seen in Figure 2 and the experimental setup can be reproduced using the MATLAB script `exp1_denoise.m` provided on the webpage.

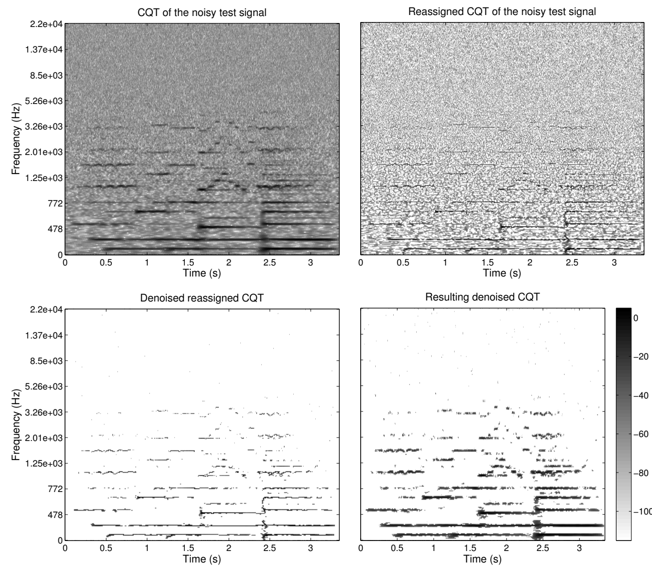

Fig. 2. **Denoising experiment:** (top) Spectrogram of the test signal contaminated with noise (0 dB SNR). (bottom) Denoised reassigned coefficients and resulting denoised constant-Q representation. SNR after denoising 11.5 dB.

Example 2. Due to the geometric spacing of frequency bands in a constant-Q filter bank, harmonic structures are invariant under translation along the frequency bands. This has been used in [15], [24] to provide a straightforward method for component transposition in audio. We use the same technique on a single note of a glockenspiel test signal. The harmonics of the note are isolated using a binary mask created in the reassigned domain and translated to the filter bank representation using $\mathbf{iR}[\Omega]$ similar to the previous example. The isolated note is shifted downward by 48 bins which amounts to 1 octave in the chosen constant-Q representation. After a phase adjustment, see e.g. [24], both the remainder signal and the isolated, transposed note are synthesized and added. Some intermediate representations are shown in Figure 3. In particular, a constant-Q representation of the resulting synthesized signal is shown in the lower right. The experimental setup can be reproduced using the MATLAB script `exp2_transpose.m` provided on the webpage.

VII. Conclusion

We have motivated and demonstrated the application of the reassignment method to general time-frequency filter banks with arbitrary downsampling factors, using the connection of the individual filter bank coefficients to certain (possibly different) short-time Fourier coefficients. The proposed implementation is based on a well-known representation of the reassignment operators introduced by Auger and Flandrin [2] and provides good precision independent of the number of frequency bands and/or decimation.

Furthermore, we have shown that the reassigned time-frequency representation can be used as an interface for signal processing, providing a possibility to harness the improved time-frequency localization for various processing purposes,

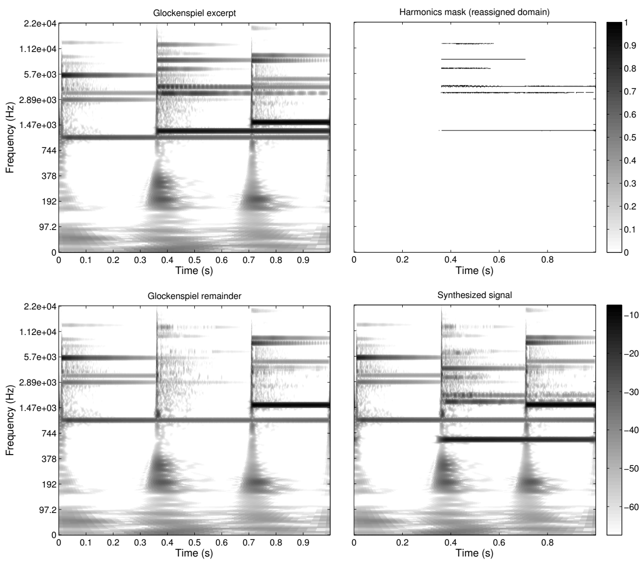

Fig. 3. **Transposition experiment (-1 octave):** Separate and transpose the 2nd glockenspiel note. Using the reassigned representation as an interface, the harmonics of a note are separated. The isolated coefficients are shifted downwards by 48 bins and the phases are adjusted. The remainder (bottom left) and the transposed component are synthesized are added. The lower right plot shows the re-analyzed output signal.

while preserving the perfect reconstruction properties of the underlying filter bank representation. Thus, we overcome a major perceived weakness of the reassignment method over other procedures that attempt to improve the localization properties of time-frequency representations, e.g. synchrosqueezing.

Efficient implementations of the presented methods have been included in the open-source MATLAB/octave toolbox LTFAT, and are thus freely available to the community. The figures and examples in this contribution can be easily reproduced using the script files we provide, see http://ltfat.sourceforge.net/notes/041.

The methods provided here are easily extended to general time-frequency frames, not necessarily possessing a filter bank structure, see Remark 1.

Acknowledgement: This work was supported by the Austrian Science Fund (FWF) START-project FLAME ("Frames and Linear Operators for Acoustical Modeling and Parameter Estimation"; Y 551-N13).

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