Chvátal's Theorem for Tree-Complete Ramsey Numbers

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Sample Presentation for Math 479

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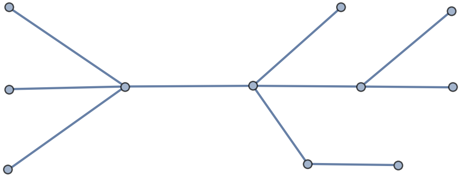
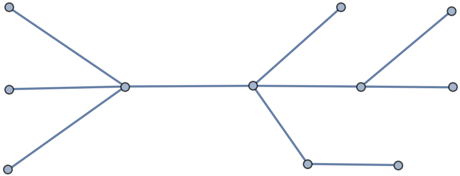
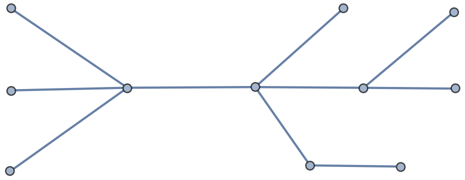


- Recall that a **tree** can be defined by one of several equivalent definitions. In particular,
  - A tree is a connected graph that does not contain any cycles.



- Recall that a **tree** can be defined by one of several equivalent definitions. In particular,
  - A tree is a connected graph that does not contain any cycles.
  - A tree is a graph that can be constructed edge-by-edge, with each new edge including exactly one vertex from the previous graph (ie., the addition of each new edge requires the addition of one new vertex).



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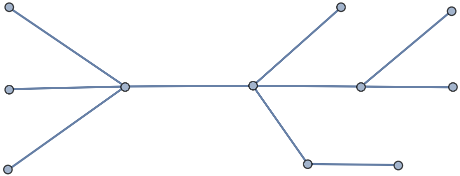
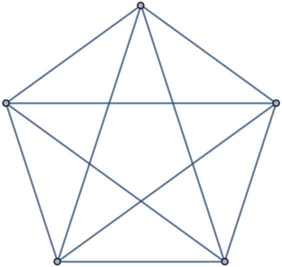
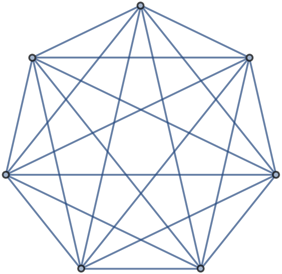


- From this second definition, it follows that every tree contains at least one **leaf** (a vertex of degree one).

- The **complete graph** K_n is a graph of order n in which every pair of distinct vertices are adjacent.



K_5



K_7

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- If T is any tree containing at least one edge, then $\chi(T) = 2$.
- For all $n \in \mathbb{N}$, $\chi(K_n) = n$.

- For any graphs G_1 and G_2, the **Ramsey number** $R(G_1, G_2)$ is the least natural number p such that every red/blue coloring of the edges of K_p contains a red subgraph isomorphic to G_1 or a blue subgraph isomorphic to G_2.

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- The existence of Ramsey numbers follows from Frank Ramsey's foundational work [4].

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    3. $R(G_1, G_2) = R(G_2, G_1)$, for all graphs G_1 and G_2.

In 1972, Chvátal and Harary [2] proved the following theorem.

Theorem (Chvátal and Harary)

For all graphs G_1 and G_2,

$$R(G_1, G_2) \geq (c(G_1) - 1)(\chi(G_2) - 1) + 1,$$

where $c(G_1)$ is the order of the largest connected component in G_1.

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- From this result, it follows that if T_m is any tree of order m, then
$$R(T_m, K_n) \geq (m-1)(n-1) + 1.$$

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- In fact, in 1977, Chvátal [1] proved that this is always the case.

Theorem (Chvátal)

For every tree T_m of order m,

$$R(T_m, K_n) = (m-1)(n-1) + 1.$$

The Proof of Chvátal's Theorem

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  are the base cases.
- Inductive hypothesis: suppose that

$$R(T_{m'}, K_{n'}) \leq (m'-1)(n'-1)+1$$

  for all $m'+n' < m+n$.

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- Let T' be the tree formed by removing a single leaf from T_m and denote by x the vertex in T' that was adjacent to the removed leaf.
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- It follows that there exists a red T' or a blue K_n. Suppose the former case.

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- So, our coloring contains a red T'. Other than the T', there are

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- Applying the inductive hypothesis again, we find that

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- Hence the two coloring of the remaining vertices contains either a red T_m or a blue K_{n-1}. Assume the latter case.

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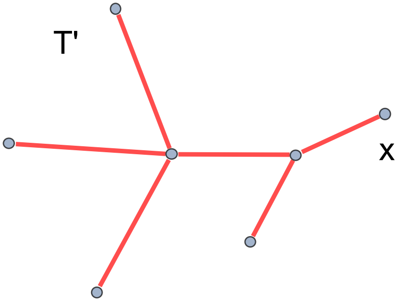
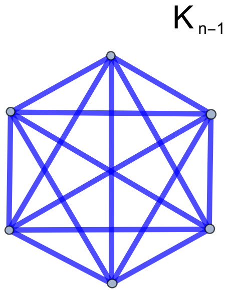
- Thus, the original red/blue coloring of $K_{(m-1)(n-1)+1}$ contains a red T' and a blue K_{n-1} that are disjoint.





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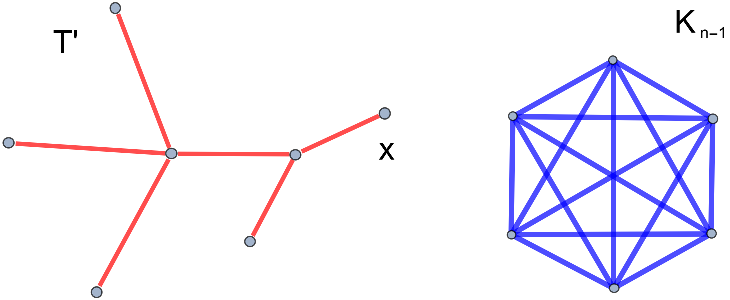

- Consider the edges connecting x to the vertices in the blue K_{n-1}.

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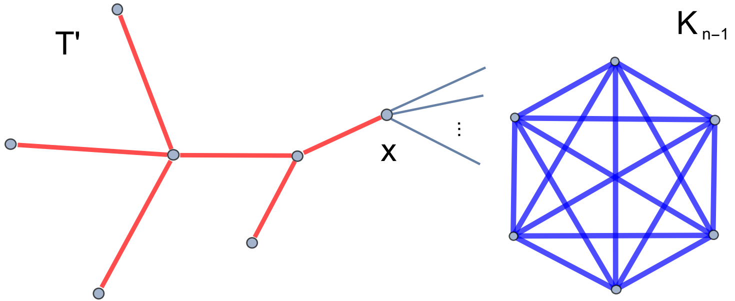

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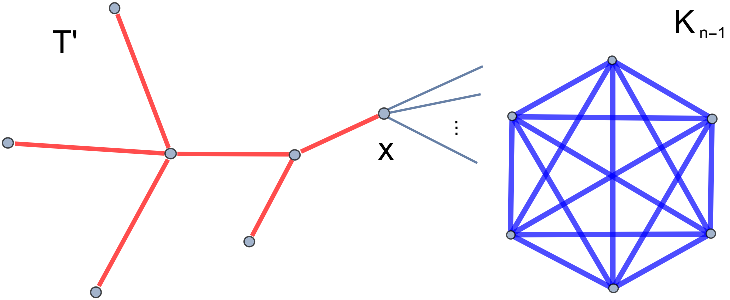

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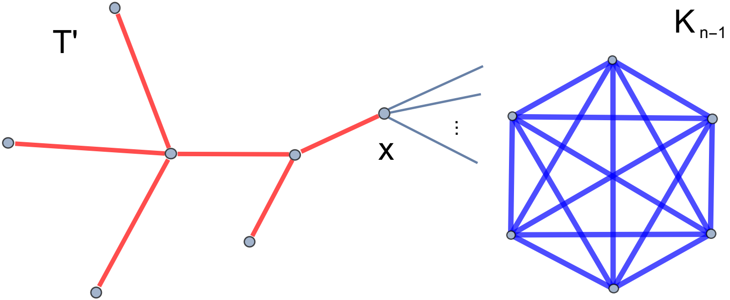

- If any such edge is red, then we obtain a red subgraph isomorphic to T_m.
- Otherwise, all such edges are blue, and a blue K_n is formed.

The Proof of Chvátal's Theorem

- We have proved that every red/blue coloring of $K_{(m-1)(n-1)+1}$ contains a red T_m or a blue K_n.

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- We have proved that every red/blue coloring of $K_{(m-1)(n-1)+1}$ contains a red T_m or a blue K_n.
- Hence,

$$R(T_m, K_n) \leq (m-1)(n-1)+1,$$

completing the proof of the theorem. $\square$

Concluding Remarks:

1. A similar proof using induction can be used to show that

$$R(T_m, K_{1,n}) \leq m + n - 1,$$

where $K_{1,n}$ is a star (a tree with n leaves and one vertex of order n).

Concluding Remarks:

1. A similar proof using induction can be used to show that

$$R(T_m, K_{1,n}) \leq m + n - 1,$$

where $K_{1,n}$ is a star (a tree with n leaves and one vertex of order n).

2. When a connected graph G of order m satisfies

$$R(G, K_n) \leq (m-1)(n-1) + 1,$$

it is called *n-good*. Ongoing research focuses on classifying all n-good graphs.

References

[1] Chvátal, *Tree-complete Graph Ramsey Numbers,* J. Graph Theory **1** (1977), 93.

[2] Chvátal and Harary, *Generalized Ramsey Theory for Graphs III, Small Off-diagonal Numbers,* Pacific J. Math. **41** (1972), 335-345.

[3] Radziszowski, *Small Ramsey Numbers,* Elec. Journ. of Combin., Dynamic Survey **1**, last updated 2017.

[4] Ramsey, *On a Problem of Formal Logic,* Proc. London Math. Soc. **30** (1930), 264-286.