

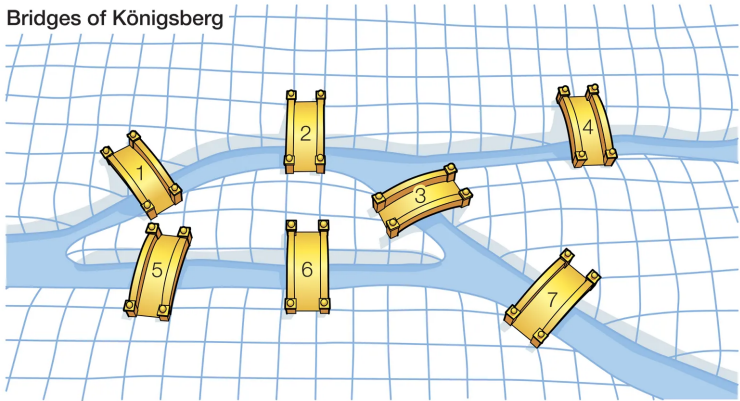
Proof For Closed Eulerian Trails

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September 13, 2022
Math 479

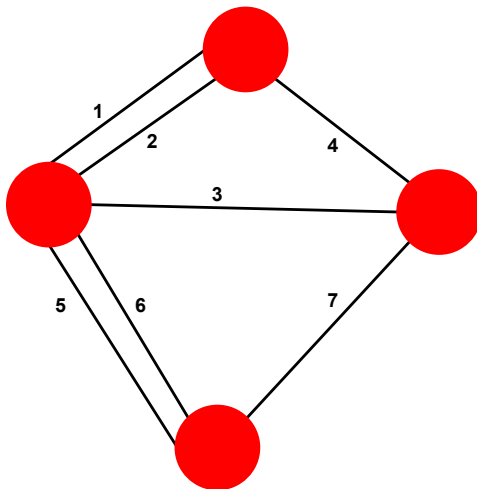
Seven Bridges of Königsberg

Bridges of Königsberg



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Seven Bridges of Königsberg



Definition

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- A **closed Eulerian trail** is when an Eulerian trail begins and ends on the same vertex.

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The **degree** of a vertex, v , is the number of edges incident with v .

Theorem

A connected graph has a closed Eulerian trail if and only if all of its vertices have even degree.

Proof.

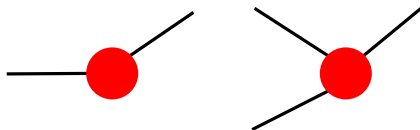
⇒ Assume a connected graph G has a closed Eulerian trail, denoted as C . Prove that every vertex has an even degree.

Take an arbitrary vertex v . As we traverse C , each time we enter v we must be able to exit on a distinct edge. Thus every vertex must be adjacent to $2k$ edges, where k is the number of times a vertex is visited. Therefore every vertex has even degree.

Proof.

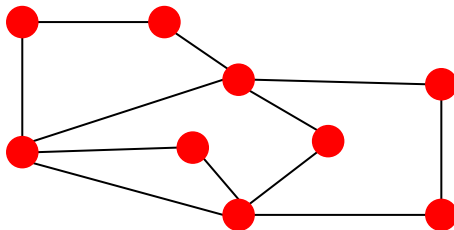
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Proof (cont).

⇐ Assume G is a connected graph with every vertex having even degree. Prove that it has a closed Eulerian trail.

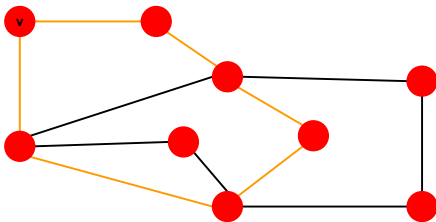


Proof (cont).

Start at an arbitrary vertex v , begin constructing a trail. Continue until we reach a vertex we cannot leave. If this vertex is not v , that means we entered on one edge and do not have a corresponding exit, meaning this vertex has an odd degree, which is a contradiction. So we know that we are at v , meaning we have found a closed trail, which we will call C_1 .

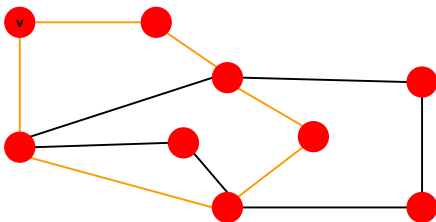
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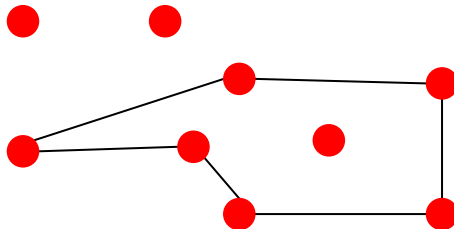
Proof (cont).

If C_1 contains all the edges of G , we have found an Eulerian trail and are done. If not remove all the edges of C_1 from G , let's call the resulting graph G' . G' is not necessarily connected.



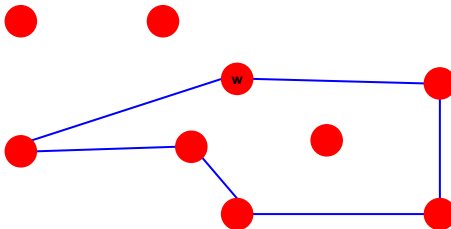
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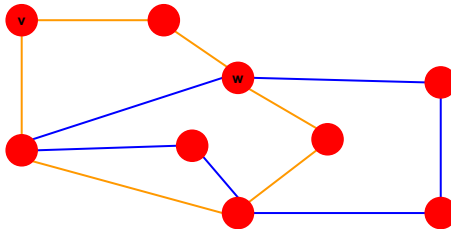
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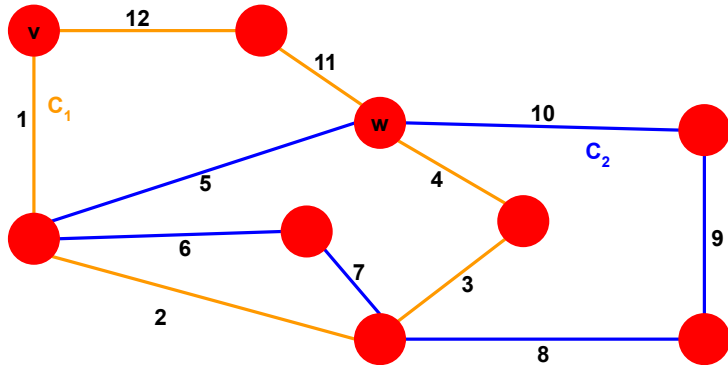
Select a vertex w that is in both C_1 and G . Then begin tracing a trail in G' starting at w until we arrive back at w . Name the resulting closed trail C_2 .



Proof (cont).

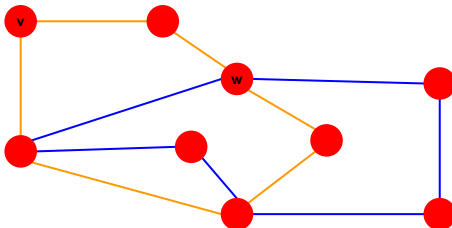
Combine C_1 and C_2 to form another closed trail denoted as C' . If C' contains all the edges of G , we have found a closed Eulerian trail and are done. If not, remove all the edges of C' from G to get G'' . Choose a vertex in both C' and G'' and repeat the steps above until we have used all the edges of G . □



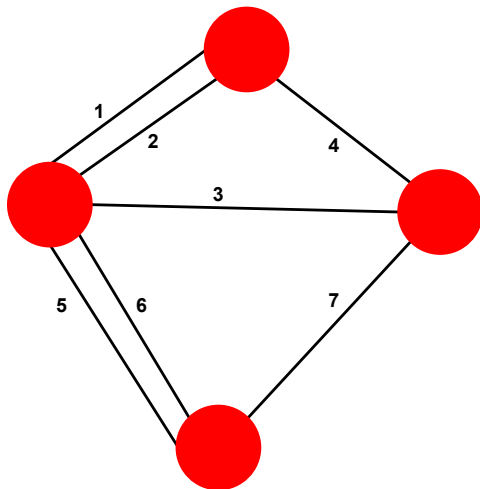


Proof cont.

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Seven Bridges of Königsberg Revisited



Applications

- Computer Science

Applications

- Computer Science
- Networking

Applications

- Computer Science
- Networking
- Routing

References

- [1] Belcastro, *Discrete Mathematics with Ducks*, CRC Press, 2012.
- [2] Epp, *Discrete Mathematics with Applications*, Brooks Cole, 1996.