Byzantine Generals Problem

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Math 479

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- Generals can only communicate by sending messengers.

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B. A small number of traitors cannot cause the loyal generals to adopt a bad plan.

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- Condition B is met by having the generals use a robust decision making method.

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Condition 1 can be rewritten as:

1′. For every i, any two loyal generals use the same value of v_i.

- Loyal generals cannot take a value v_i at face value.

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- Condition $1'$ and Condition 2 are both contingent on a single v_i sent by the i^{th} general.

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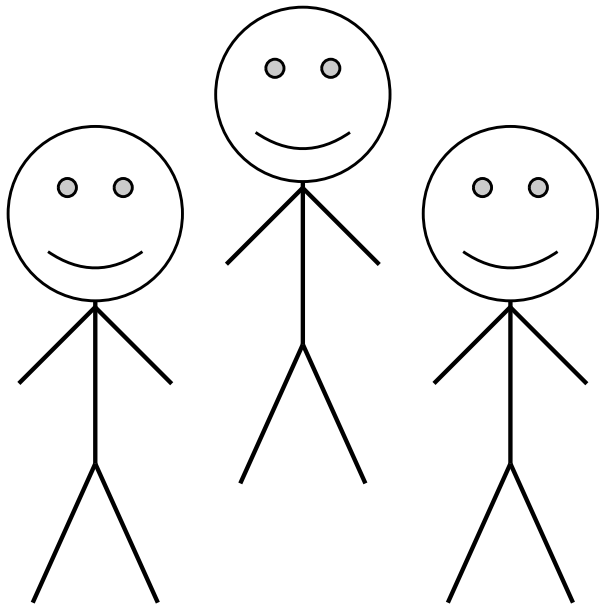
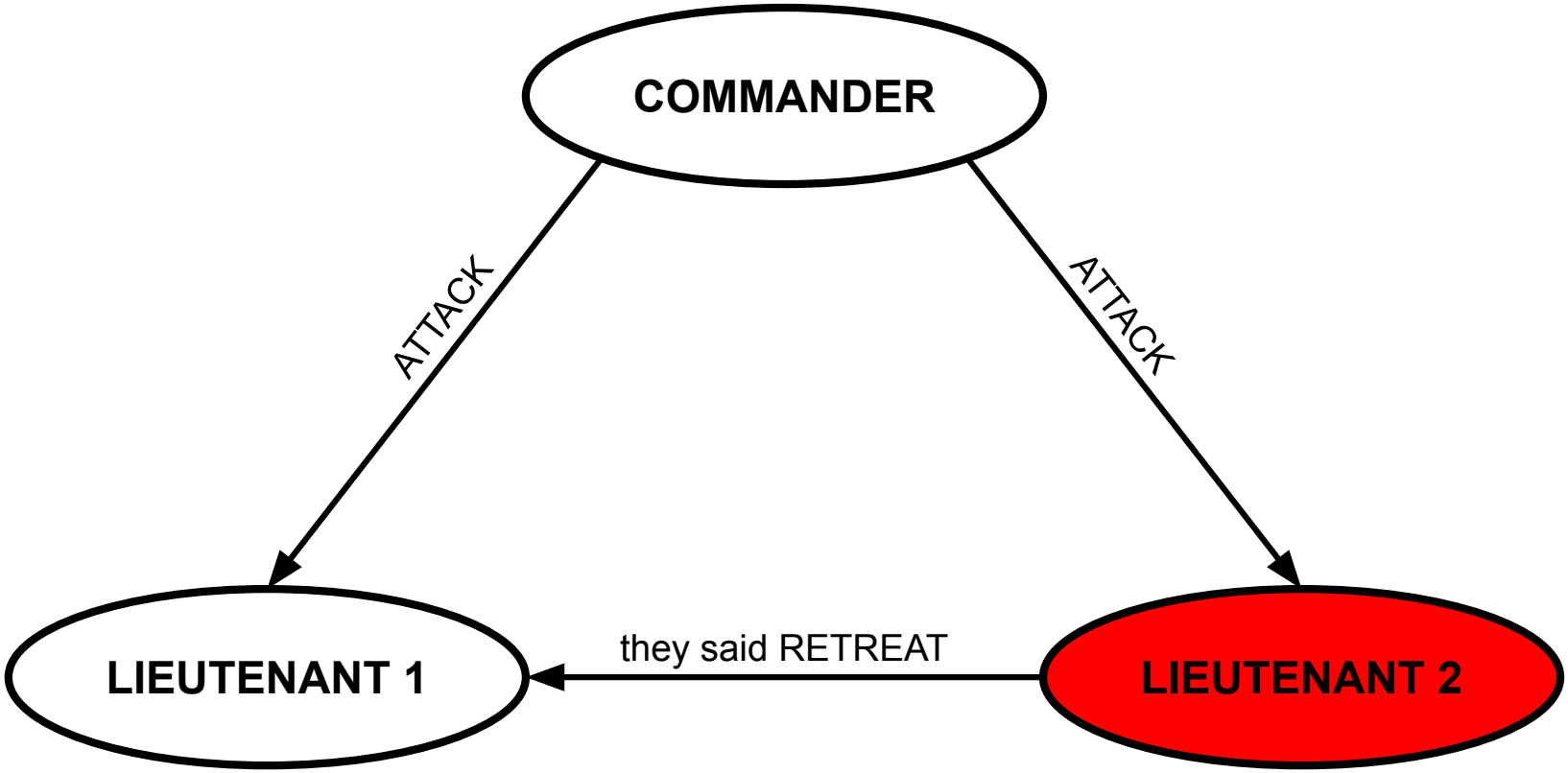
IC1. All loyal lieutenants obey the same order.

IC2. If the commander is loyal, then every loyal lieutenant obeys the order they send.

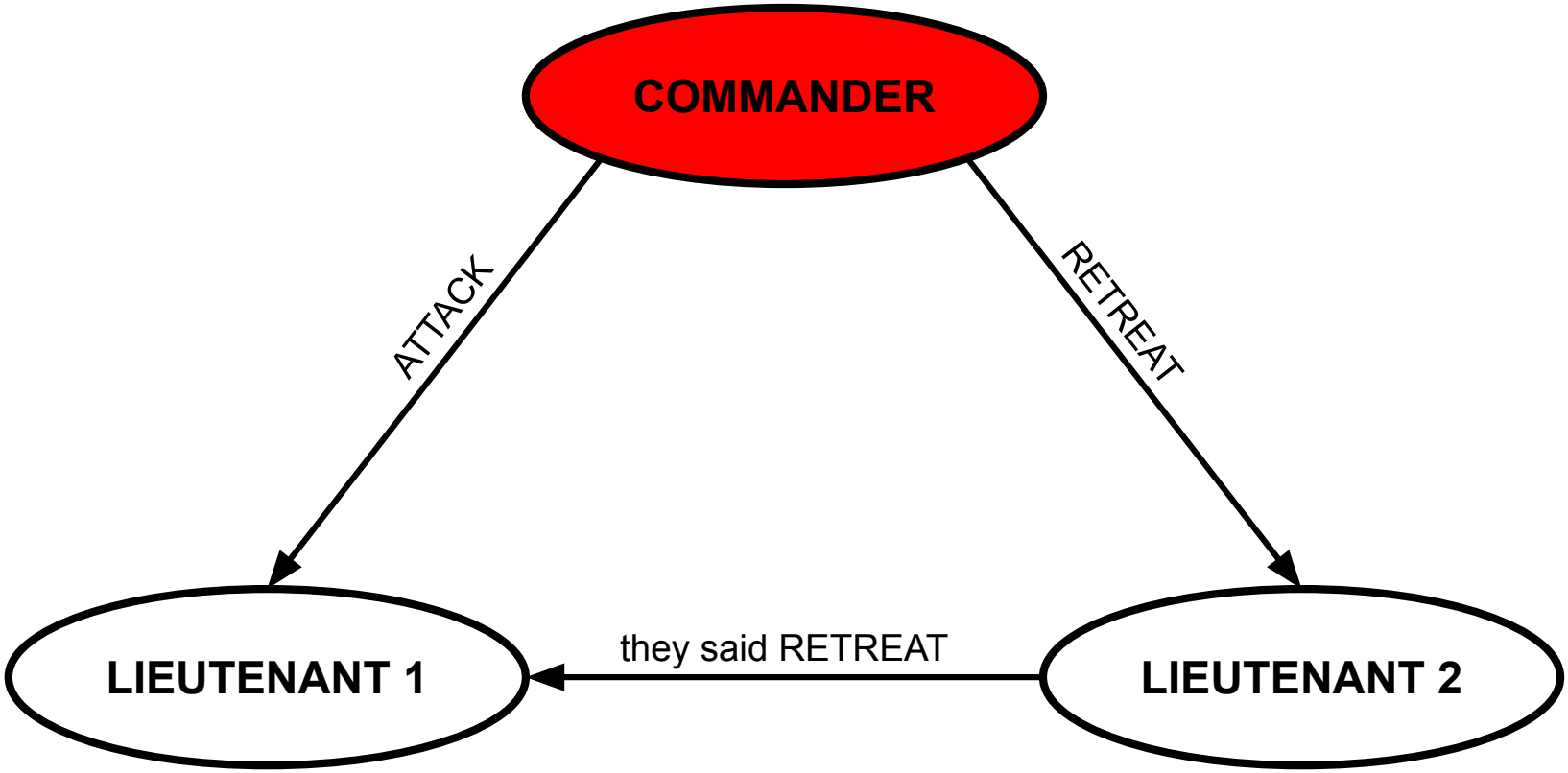
Three Generals Problem

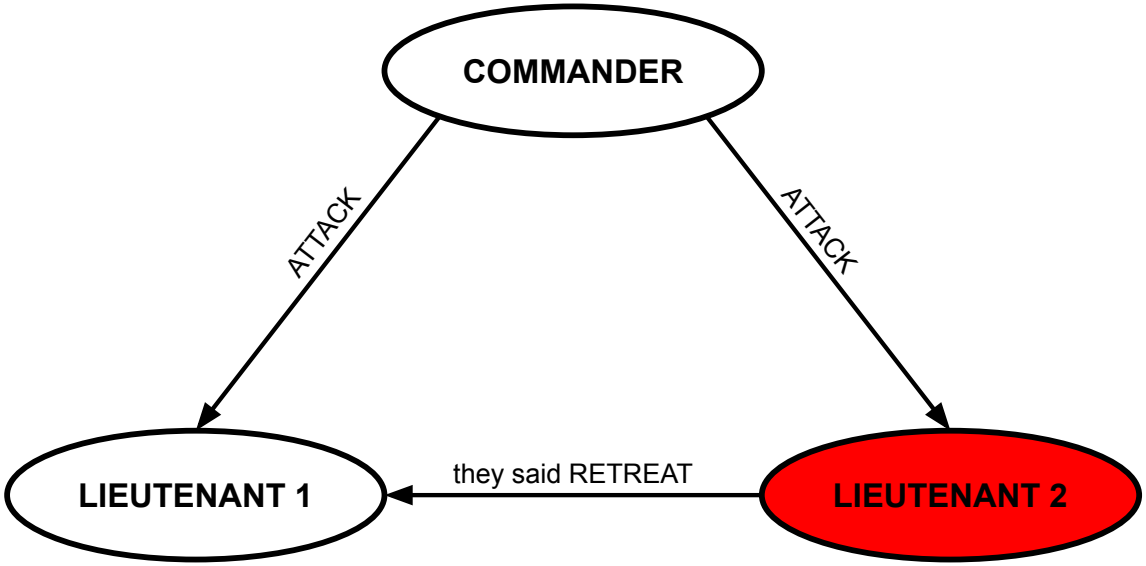


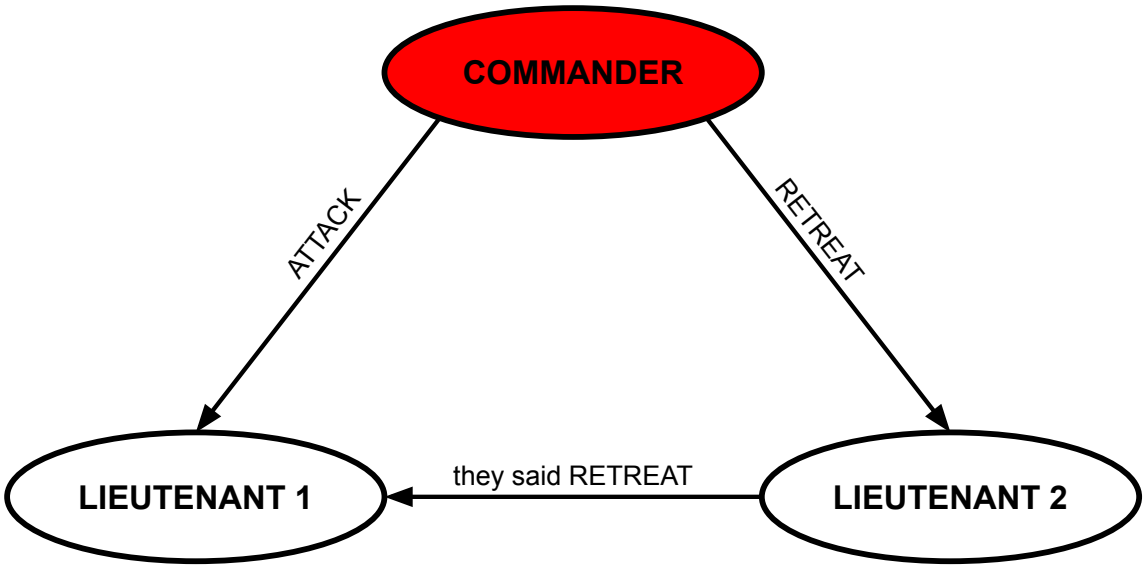
Situation 1: Commander is Loyal



Situation 1: Commander is a Traitor




COMMANDER
ATTACK
ATTACK
LIEUTENANT 1
they said RETREAT
LIEUTENANT 2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Theorem (Three Generals)

No solution exists for $n < 3m + 1$ generals with m traitors and $n > 3$.

Proof.

Assume that a solution exists for $3m$ or less Albanian generals.
We will show a solution exists for three Byzantine generals with a single traitor.

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We know that there is a single Byzantine traitor.
Since each Byzantine general represents at most m Albanian generals, we know there is at most m Albanian traitors.

Proof.

The assumed solution means that IC1 and IC2 is true for the Albanian generals.

Since up to m Albanian generals are represented by a Byzantine general, then IC1 and IC2 must also be true for the Byzantine generals, which we know is impossible, forming a contradiction. □

We know we need $n \geq 3m + 1$ generals if we have m traitors.

Oral Solution

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A3. The absence of a message can be detected.

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2. Each lieutenant uses the value they received from the commander. If they received no value, default to RETREAT.

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2. For each i,
   a. Lieutenant i receives a value v_i from the commander. Default to RETREAT if they receive no value.
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3. For each i, and each j not equal to i,
   a. let v_j be the value Lieutenant i received from Lieutenant j in step (2b). Default to RETREAT if Lieutenant i received no value from Lieutenant j.

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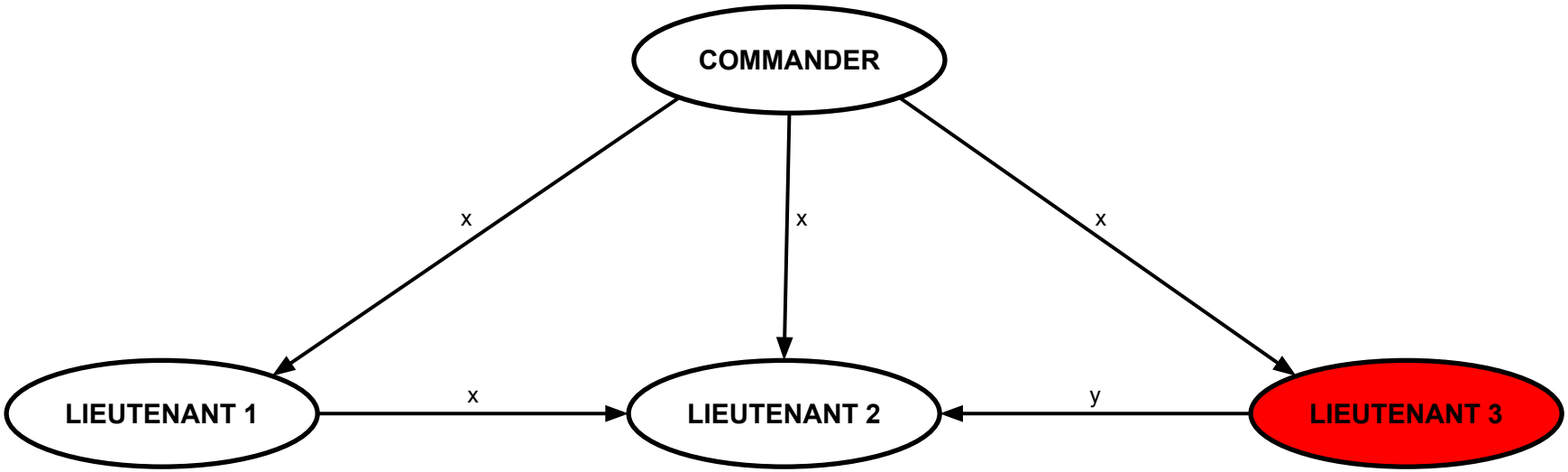
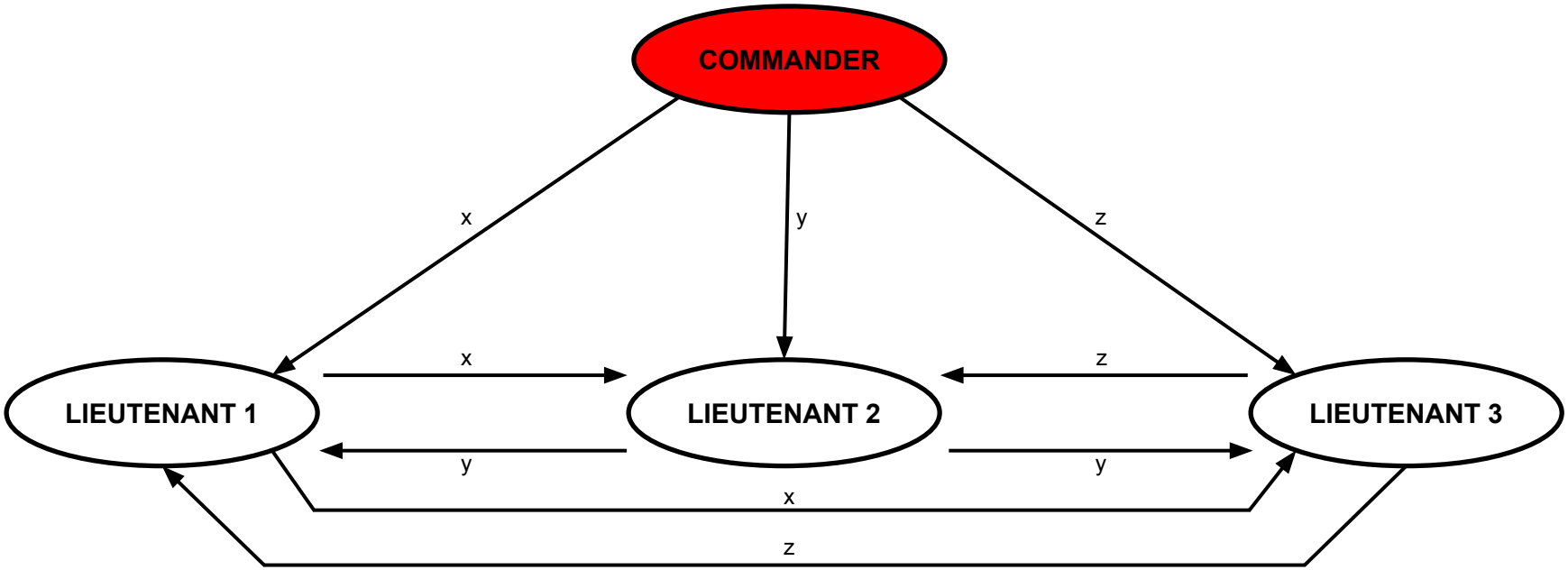
   a. let v_j be the value Lieutenant i received from Lieutenant j in step (2b). Default to RETREAT if Lieutenant i received no value from Lieutenant j.
   b. Lieutenant i uses the value $majority(v_1, ..., v_{n-1})$.

OM(m) Commander is Loyal



OM(m) Commander is a Traitor



Signed Messages

A4. (a) A loyal general's signature cannot be forged, and any alterations of the contents of their signed messages can be detected.

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(b) Anyone can verify the authenticity of a general's signature.

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2. $\mathit{choice}(\emptyset) =$ RETREAT, where $\emptyset$ is the empty set.

- The value x signed by General i is denoted as $x : i$.

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- That means $x : i : j$ is the value x signed by General i and then General j.

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      i. they add v to V_i.

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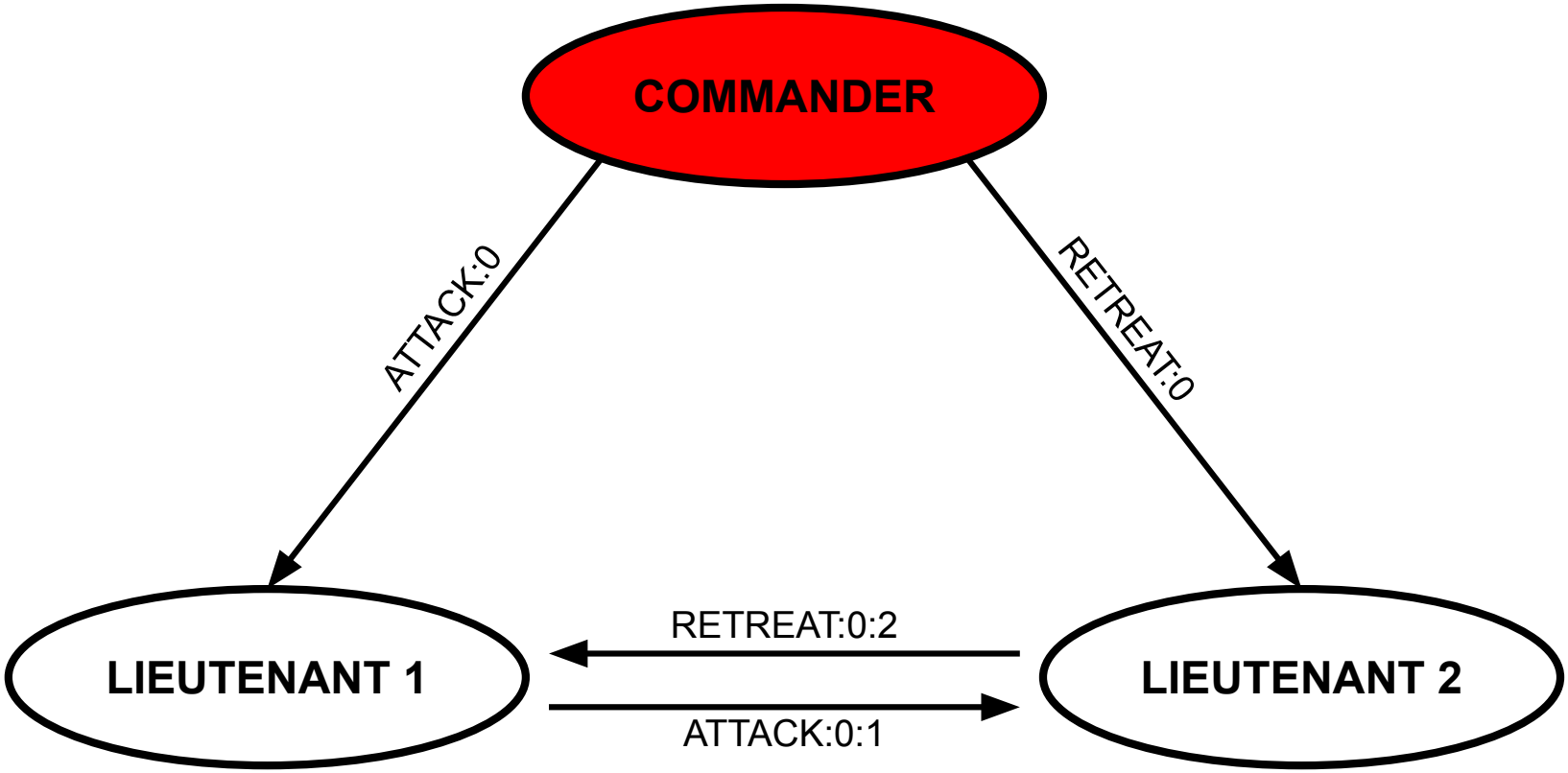
   c. For each i, when Lieutenant i receives no more messages, they follow the result from $choice(V_i)$.

OM(m) Commander is a Traitor



Applications

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- Nodes on a network.

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- Blockchain

References

[1] L. Lamport, R. Shostak, and M. Pease, "The Byzantine Generals Problem", ACM Transactions on Programming Languages and Systems, Vol. 4, No. 3, 1982, 382-401.

[2] M. Pease, R. Shostak, and L. Lamport, "Reaching agreement in the presence of faults.", J. ACM Transactions 27, 2, 1980, 228-234