BYZANTINE GENERALS PROBLEM

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Abstract. In this paper we explore how many guards are needed to guard a museum. We construct a lemma showing how to triangulate a convex polygon. We then use that and vertex coloring to develop a proof by induction to find the amount of guards that could sufficiently guard a museum.

1. Introduction

The following problem was coined by Victor Klee in 1973. Suppose the manager of a museum wants to place guards to watch every point of the building at all times. Guards are stationed at fixed points, and they have the ability to turn around. How many guards are needed to achieve this? Before we dive into the problem, let's introduce some preliminary definitions and corresponding examples.

Definition. *A polygon is called **convex** when all of its interior angles are less than* $180°$*.*

figures/convex-polygon.pdf figures/non-convex-polygon.pdf

Figure 1. Convex polygon on the left, non-convex polygon on the right.

Definition. *A graph is called **planar** when all of its vertices and edges are contained in a single plane.*

figures/planar-graph.pdf

Figure 2. An example of a planar graph.

Definition. *A planar graph G is said to be* ***triangulated*** *when adding another edge to G results in a nonplanar graph.*

figures/triangulation.pdf

FIGURE 3. An example of a triangulated graph.

Note that this definition is equivalent to **maximal planar**. Some sources will state a graph is triangulated when no straight edges can be added, an example of that would be a square with a single diagonal; you could draw a new edge connecting the two non-connected vertices, but the edge would not be straight. For this paper, we are only concerned about the interior of the polygons. Therefore, we will consider a polygon triangulated when adding an edge that is contained within the polygon will make it nonplanar.

Definition. *A proper* ***vertex coloring*** *of a graph G is an assignment of colors to the vertices of G such that no two adjacent vertices receive the same color.*

figures/coloring.pdf

FIGURE 4. This is a 3-coloring of G.

Coming back to the question we originally introduced, we will imagine the walls of the museum are a polygon consisting of n sides. If the polygon is convex, then we only need one guard to oversee the entire museum.

figures/convex-museum.pdf

However, the walls of the museum can take on the shape of **any** closed polygon.

figures/floor-plan.png

FIGURE 5. Floor layout of the second floor of the Met. (From NY Times, 2017)

Let's consider a comb-shaped museum with $n = 3m$ walls.

figures/comb-shape.pdf

Notice that point 1 can only be seen by a guard stationed anywhere in the shaded triangle containing the point. This applies to the other points 2, 3,..., m. We can see that this requires **at least** $m = \frac{n}{3}$ guards, one for each shaded triangle.

However, m guards are also sufficient. Since the guards can be placed at the bottom of the triangles, and each guard can watch their own triangle as well as down the hall. Thus, the floor of $\frac{n}{3}$ guards are needed for an n-walled museum.

2. Triangulation

Before we get into the main proof of our paper, we'd like to go over a proof of a lemma we will use. This lemma proves that there exists a triangulation for any non-convex polygon.

We will look at the triangulation of a pentagon to show the basic process of triangulating a polygon. Draw a diagonal between any two vertices and then split the pentagon at the diagonal. The result will be a triangle and a quadrilateral. We then want to draw a diagonal between any two points of the quadrilateral, and then split it on the diagonal. The result will be two more triangles and a triangulated pentagon.

figures/triangulate-pentagon-split-fb-dos.pdf

Figure 6. This shows the process of triangulating a pentagon.

Proof. Let P be a non-convex polygon with n sides. We will be using induction for this proof. Our base case is $n = 3$, which is a triangle. For our inductive hypothesis, assume P can be triangulated if n is less than k, k being any integer greater than 3. We will show that P can be triangulated when $n = k$.

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To prove this, we will show that there exists a diagonal that can split P into two smaller polygons that can be triangulated. A vertex A is convex if its interior angle is less than 180°. We know that the sum of the interior angles of a polygon is $(n-2)180$. If we imagine that there are n vertices being placed into $n-2$ boxes of 180°, then by the pigeonhole principle, there must be a convex vertex A. Looking at the two neighbors B and C of A. If the segment BC is entirely in P, then we have our diagonal. If the segment is not entirely in P, then the triangle ABC contains other vertices.

figures/triangulation-proof-dotted.pdf

Slide BC towards A until you hit the last vertex Z in ABC. Since AZ is within P, we have our diagonal.

figures/triangulation-proof-diagonal.pdf figures/triangulation-proof-split.pdf

Now that we have our diagonal we can split P into smaller polygons, both with side AZ, with less than k sides. By our inductive hypothesis, both of these polygons can be triangulated. Thus P can be triangulated. □

3. Art Gallery Theorem

Now that we know that any polygon can be triangulated we can continue on to our main proof.

Theorem 1 (Art Gallery Theorem)**.** *For any museum with n walls, the floor of $\frac{n}{3}$ guards suffice.*

Proof. Let P be a polygon with n walls. Because of our lemma, we know we can triangulate P. You can think of P as a planar graph with the corners as vertices and the walls and diagonals as edges.

figures/art-gallery-triangulated.pdf

We will now show that P is 3-colorable. For $n = 3$, the coloring trivial, as it would be a triangle and every vertex would be a different color. For $n > 3$, pick any two vertices u and v connected by a diagonal and split P along uv.

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This will give us two smaller triangulated graphs that both contain the edge uv. Continue in this manner until only triangles remain.

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We can color each triangle using three colors. Paste these colorings together to form a 3-coloring of P.

figures/art-gallery-triangulated-fb-10.pdf figures/art-gallery-colored.pdf

We choose one of the three colors. For every vertex of that color, we assign a guard. Since every triangle contains that color, we know that each triangle is guarded. So the whole museum can be guarded by the floor of $\frac{n}{3}$ guards. □

figures/art-gallery-guarded.pdf

In our example we placed our guards at the yellow vertices, which required three guards to guard the museum. If we had instead place guards at the green vertices then we would have only needed to use two guards. This helps show that the theorem is only stating how many guards will be sufficient to guard the museum, it does not say that $\frac{n}{3}$ will be the minimum number of guards that could guard the museum.

4. Conclusion

There are several variants to the art gallery theorem. For example, we may only want to guard the walls, since that's where the actual paintings will be, or guards may only be stationed at edges. An unsolved variant states: suppose each guard may patrol one wall of the museum, so they walk along their wall and see anything that can be seen from any point along that wall. How many "wall guards" do we then need to keep control?

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Godfried Toussaint constructed the example of a museum which shows that the floor of $\frac{n}{4}$ guards might be necessary. This polygon has 28 sides, and only requires 7 wall-guards to guard it. The way this is accomplished is by placing a guard at the foot of each leg of the polygon. It is conjectured that, besides some small values of n, that the floor of $\frac{n}{4}$ is also sufficient, but there is still no proof for it.

References

1. M. Aigner and G. Ziegler, "Proofs From THE BOOK," 3^{rd} edition, Springer-Verlag, 2004. 203-205.