April 10, 2017

1 determine the solution set of the following systems of equations

a1

solve for real numbers:

$$\begin{bmatrix} 2 & 3 & 1 & 2 \\ 4 & 3 & 1 & 1 \\ 5 & 11 & 3 & 2 \\ 2 & 5 & 1 & 1 \\ 1 & -7 & -1 & 2 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 2 \\ 1 \\ 7 \end{bmatrix}$$

solution:
since there are 5 equations for 4 variables, we do not expect to necessarily find a solution, still we can proceed solving the system of the first four equations and checking whether the solution is valid for the last equation.

$$\left[\begin{array}{cccc|c} 2 & 3 & 1 & 2 & 4 \\ 4 & 3 & 1 & 1 & 5 \\ 5 & 11 & 3 & 2 & 2 \\ 2 & 5 & 1 & 1 & 1 \end{array}\right] \quad \begin{array}{c} \mathrm{I} \\ \mathrm{II} \\ \mathrm{III} \\ \mathrm{IV} \end{array}$$

we can reduce the problem to solving for $x_2, x_3, and x_4$

$$\left[\begin{array}{ccc|c} 3 & 1 & 3 & 3 \\ 7 & 1 & -6 & -16 \\ 2 & 0 & -1 & -3 \end{array}\right] \quad \begin{array}{c} \mathrm{V} = 2 \times \mathrm{I} - \mathrm{II} \\ \mathrm{VI} = 2 \times \mathrm{III} - 5 \times \mathrm{I} \\ \mathrm{VII} = \mathrm{IV} - \mathrm{I} \end{array}$$

which is the same as solving:

$$\left[\begin{array}{cc|c} 4 & 39 & 69 \\ 2 & 9 & 15 \end{array}\right] \quad \begin{array}{c} \mathrm{VIII} = 7 \times \mathrm{V} - 3 \times \mathrm{VI} \\ \mathrm{IX} = 2 \times \mathrm{V} - 3 \times \mathrm{VII} \end{array}$$

from which it is simple to find

$$x_4 = \frac{39}{21} = \frac{13}{7}$$

therefore using IX

$$x_3 = \frac{105 - 117}{14} = -\frac{13}{14}$$

using VII we find:

$$x_2 = \frac{13 - 21}{14} = -\frac{4}{7}$$

and inserting in I we find:

$$x_1 = \frac{12}{14} + \frac{13}{28} - \frac{26}{14} + 2 = \frac{24+13-52+54}{28} = \frac{39}{28}$$

now testing this in the last equation of the exercise, we get:

$$\frac{39}{28} + 4 + \frac{13}{14} + \frac{26}{7} = \frac{39+102+26+104}{28} = \frac{271}{28} \neq 7$$

Hence the solution of the reduced system of equations does not solve the entire system and the solution set is

$$S = \emptyset$$

a2

$$\begin{bmatrix} 9 & -3 & 5 & 6 \\ 6 & -2 & 3 & 1 \\ 3 & -1 & 3 & 14 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ -8 \end{bmatrix}$$

since there are 3 constrains in four dimensions, we expect the solution set to be at least a hyperplane of dimension 1 lets set $x_4 = a \in \mathbb{R}$

$$\left[\begin{array}{cccc|c} 9 & -3 & 5 & 6 & 4 \\ 6 & -2 & 3 & 1 & 5 \\ 3 & -1 & 3 & 14 & -8 \\ 0 & 0 & 0 & 1 & a \end{array}\right] \begin{array}{c} \mathrm{I} \\ \mathrm{II} \\ \mathrm{III} \\ \mathrm{IV} \end{array}$$

$$\left[\begin{array}{cccc|c} 9 & -3 & 5 & 6 & 4 \\ 0 & 0 & -4 & -36 & 28 \\ 0 & 0 & -3 & -27 & 21 \\ 0 & 0 & 0 & 1 & a \end{array}\right] \begin{array}{c} \mathrm{I} \\ \mathrm{V} = \mathrm{I} - 3 \times \mathrm{III} \\ \mathrm{VI} = \mathrm{II} - 2 \times \mathrm{III} \\ \mathrm{IV} \end{array}$$

$$\left[\begin{array}{cccc|c} 9 & -3 & 5 & 6 & 4 \\ 0 & 0 & -4 & -36 & 28 \\ 0 & 0 & 1 & 0 & -7-9a \\ 0 & 0 & 0 & 1 & a \end{array}\right] \begin{array}{c} \\ \\ \mathrm{VII} = -\frac{1}{3}\mathrm{VI} - 7 \times \mathrm{IV} \\ \\ \end{array}$$

since the second row does not depend depend on x_2 we can choose $x_2 = b \in \mathbb{R}$

$$\left[\begin{array}{cccc|c} 9 & -3 & 5 & 6 & 4 \\ 0 & 1 & 0 & 0 & b \\ 0 & 0 & 1 & 0 & -7-9a \\ 0 & 0 & 0 & 1 & a \end{array}\right] \begin{array}{c} \\ \mathrm{VIII} = -\frac{1}{4}\mathrm{V} - 9 \times \mathrm{IV} \\ \mathrm{VI} = \mathrm{II} - 2 \times \mathrm{III} \\ \mathrm{IV} \end{array}$$

which leaves us with

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{13+13a+b}{3} \\ 0 & 1 & 0 & 0 & b \\ 0 & 0 & 1 & 0 & -7-9a \\ 0 & 0 & 0 & 1 & a \end{array}\right]$$

so the solution set is of the form

$$S = \begin{bmatrix} 13/3 \\ 0 \\ -7 \\ 0 \end{bmatrix} + a \begin{bmatrix} 13/3 \\ 0 \\ -9 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1/3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \; a, b \in \mathbb{R}$$

b

solve for rational numbers:

$$\left[\begin{array}{ccccc|c} 24 & 14 & 30 & 40 & 41 & 28 \\ 36 & 21 & 45 & 61 & 62 & 43 \\ 48 & 28 & 60 & 82 & 83 & 58 \\ 60 & 35 & 75 & 99 & 102 & 69 \end{array}\right]$$

choose $x_5 = a \in \mathbb{Q}$

$$\left[\begin{array}{ccccc|c} 24 & 14 & 30 & 40 & 41 & 28 \\ 36 & 21 & 45 & 61 & 62 & 43 \\ 48 & 28 & 60 & 82 & 83 & 58 \\ 60 & 35 & 75 & 99 & 102 & 69 \\ 0 & 0 & 0 & 0 & 1 & a \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} 24 & 14 & 30 & 40 & 0 & 28-41a \\ 36 & 21 & 45 & 61 & 0 & 43-62a \\ 48 & 28 & 60 & 82 & 0 & 58-83a \\ 60 & 35 & 75 & 99 & 0 & 69-102a \\ 0 & 0 & 0 & 0 & 1 & a \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} 24 & 14 & 30 & 40 & 0 & 28-41a \\ 36 & 21 & 45 & 61 & 0 & 43-62a \\ 48 & 28 & 60 & 82 & 0 & 58-83a \\ 0 & 0 & 0 & -2 & 0 & 2(69-102a)-5(28-41a) \\ 0 & 0 & 0 & 0 & 1 & a \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} 24 & 14 & 30 & 40 & 0 & 28-41a \\ 36 & 21 & 45 & 61 & 0 & 43-62a \\ 48 & 28 & 60 & 82 & 0 & 58-83a \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & a \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} 24 & 14 & 30 & 0 & 0 & 28-40-41a \\ 36 & 21 & 45 & 0 & 0 & 43-61-62a \\ 48 & 28 & 60 & 0 & 0 & 58-82-83a \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & a \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} 24 & 14 & 30 & 0 & 0 & 12-41a \\ 36 & 21 & 45 & 0 & 0 & 18-62a \\ 48 & 28 & 60 & 0 & 0 & 24-83a \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & a \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} 24 & 14 & 30 & 0 & 0 & 12-41a \\ 36 & 21 & 45 & 0 & 0 & 18-62a \\ 0 & 0 & 0 & 0 & 0 & 24-24+82a-83a \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & a \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} 24 & 14 & 30 & 0 & 0 & 12-41a \\ 36 & 21 & 45 & 0 & 0 & 18-62a \\ 0 & 0 & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & a \end{array}\right]$$

we follow that $a = 0$ and see that the second row is a multiple of the first, hence choose $x_3 = b \in \mathbb{Q} and x_2 = c \in \mathbb{Q}$

$$\left[\begin{array}{ccccc|c} 12 & 0 & & 0 & 0 & 34-15b-7c \\ 0 & 1 & 0 & 0 & 0 & c \\ 0 & 0 & 1 & 0 & 0 & b \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

the solution set is of the form:

$$S = \begin{bmatrix} \frac{17}{6} \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} -15 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} -7 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad b, c \in \mathbb{Q}$$

c

solve for complex numbers

$$\begin{array}{cc} \left[\begin{array}{ccc|c} -1 & i & 1+i & i \\ 3+i & 0 & -4+2i & -1-i \\ 1-i & 2 & -1+2i & 1+i \end{array}\right] & \begin{array}{c} \mathrm{I} \\ \mathrm{II} \\ \mathrm{III} \end{array} \end{array}$$

$$\begin{array}{cc} \left[\begin{array}{ccc|c} 3+i & 0 & -4+2i & -1-i \\ -1 & i & 1+i & i \\ (1-i+3+i-4) & 2+4i & (-1+2i-4+2i+4+4i) & 1+i-1-i+4i \end{array}\right] & \begin{array}{c} \mathrm{II} \\ \mathrm{I} \\ \mathrm{IV} = \mathrm{III} + \mathrm{II} + 4 \times \mathrm{I} \end{array} \end{array}$$

$$\begin{array}{cc} \left[\begin{array}{ccc|c} 3+i & 0 & -4+2i & -1-i \\ -1 & i & 1+i & i \\ 0 & 2+4i & -1+8i & 4i \end{array}\right] & \begin{array}{c} \mathrm{II} \\ \mathrm{I} \\ \mathrm{IV} \end{array} \end{array}$$

$$\begin{array}{cc} \left[\begin{array}{ccc|c} 3+i & 0 & -4+2i & -1-i \\ 0 & i & (3+i)(1+i)-4+2i & (3+i)i-1-i \\ 0 & 2+4i & -1+8i & 4i \end{array}\right] & \begin{array}{c} \mathrm{II} \\ \mathrm{V} = \mathrm{II} + (3+i) \times \mathrm{I} \\ \mathrm{IV} \end{array} \end{array}$$

$$\left[\begin{array}{ccc|c} 3+i & 0 & -4+2i & -1-i \\ 0 & i & -2+6i & 2i-2 \\ 0 & 2+4i & -1+8i & 4i \end{array}\right]$$

$$\begin{array}{cc} \left[\begin{array}{ccc|c} 3+i & 0 & -4+2i & -1-i \\ 0 & i & -2+6i & 2i-2 \\ 0 & 0 & (-1+8i+(4-2i)(-2+6i)) & 4i+(4-2i)(2i-2) \end{array}\right] & \begin{array}{c} \mathrm{II} \\ \mathrm{V} \\ \mathrm{VI} = \mathrm{IV} + (4-2i) \times \mathrm{V} \end{array} \end{array}$$

$$\left[\begin{array}{ccc|c} 3+i & 0 & -4+2i & -1-i \\ 0 & i & -2+6i & 2i-2 \\ 0 & 0 & 3+36i & -4+16i \end{array}\right]$$

therefore we find x_3 as

$$x_3 = \frac{-4+16i}{3+36i} = \frac{(-4+16i)(3-36i)}{(3+36i)(3-36i)} = \frac{+192i+564}{3^2+6^4}$$

$$\left[\begin{array}{ccc|c} 3+i & 0 & & -1-i-\frac{(+192i+564)(-4+2i)}{3^2+6^4} \\ 0 & i & 0 & 2i-2-\frac{(+192i+564)(-2+6i)}{3^2+6^4} \\ 0 & 0 & 1 & \frac{+192i+564}{3^2+6^4} \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 3+i & 0 & 0 & -1-i-\frac{(+192i+564)(-4+2i)}{1305} \\ 0 & i & 0 & 2i-2-\frac{(+192i+564)(-2+6i)}{1305} \\ 0 & 0 & 1 & \frac{+192i+564}{1305} \end{array}\right]$$

we calculate x_2 as

$$x_2 = -i\left(\frac{(2i-2)(3^2+6^4)-(+192i+564)(-2+6i)}{3^2+6^4}\right) = \frac{-330-390i}{1305}$$

and x_1 as:

$$x_1 = \frac{\left[(-1-i)(1305)-(+192i+564)(-4+2i)\right](3-i)}{1305(3+i)(3-i)}$$

$$= \frac{\left[(-1-i)(1305)-(+192i+564)(-4+2i)\right](3-i)}{13050}$$

$$= \frac{\left[-1305-1305i+768i+2256+384-1128i\right](3-i)}{13050}$$

$$= \frac{\left[-1305-1305i-360i+2640\right](3-i)}{13050}$$

$$= \frac{(1335-1665i)(3-i)}{13050}$$

$$= \frac{2340-6330i}{13050}$$

whith these amazing numbers we can write the solution set in the form:

$$S = \frac{1}{1305}\begin{bmatrix} \frac{2340-6330i}{10} \\ -330-390i \\ 192i+564 \end{bmatrix}$$

2 solve in $\mathbb{R}$ in dependence on $\lambda \in \mathbb{R}$

2

$$\left[\begin{array}{cccc|c} \lambda & 1 & 1 & 1 & 1 \\ 1 & \lambda & 1 & 1 & 1 \\ 1 & 1 & \lambda & 1 & 1 \\ 1 & 1 & 1 & \lambda & 1 \end{array}\right]$$

we notice that the problem is symmetric, subtracting any two rows yields:

$$(1-\lambda)x_i = (1-\lambda)x_j$$

so in the case $\lambda = 1$ we have three free variables and the solutionset is:

$$S = a \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \quad a, b, c \in \mathbb{R}.$$

in the case $\lambda \neq 1$ we get $x_1 = x_2 = x_3 = x_4$ which in turn constrains λ to:

$$x = \frac{1}{\lambda + 3}$$

which gives the solutionset

$$S = \frac{1}{\lambda + 3} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

3 check if the following matrices in $\mathbb{R}$ are invertible and compute their inverse if posible

3.1 a

$$\begin{bmatrix} 23 & 21 \\ 435 & 45 \end{bmatrix}$$

$$det \begin{bmatrix} 23 & 21 \\ 435 & 45 \end{bmatrix} = 23 \times 45 - 435 \times 21 = 8100$$

by staring at the 21 it is obvious that the determinant is nonzero, hence we start to compute eigenvalues: set

$$\det \mathbf{A} - \lambda \mathbf{I}$$

$$(23 - \lambda)(45 - \lambda) = 21 \times 435$$

$$\lambda^2 - 69\lambda + 1035 - 9135 = 0$$

$$\lambda^2 - 69\lambda - 90^2$$

$$\lambda_{1,2} = \frac{69}{2} \pm \sqrt{(\frac{4761 + 32400}{4})} = \frac{69}{2} \pm \sqrt{(\frac{4761 + 32400}{4})}$$

$$\mathbf{A}^{-1} = \frac{1}{det\mathbf{A}} \begin{bmatrix} 45 & -435 \\ -21 & 23 \end{bmatrix} = \begin{bmatrix} \frac{45}{8100} & -\frac{435}{8100} \\ -\frac{21}{8100} & \frac{23}{8100} \end{bmatrix}$$

b

the inverse matrix of $\begin{bmatrix} 37 \end{bmatrix}$ exists and is $\begin{bmatrix} \frac{1}{37} \end{bmatrix}$

c

$$det \begin{bmatrix} 17 & 24 & 215 \\ 15 & -10 & 119 \\ 1 & 17 & 48 \end{bmatrix} = (17 \times (-10) \times 119) + (24 \times 119 \times 1) + (215 \times 15 \times 17) - (17 \times 119 \times 17) - (24 \times 15 \times 48) - (215 \times ($$

$$= -20230 + 2856 + 54825$$