Algebraic Combinatorics: Symmetric Functions, Young Tableaux, and Group Representations

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1 Introduction

Symmetric functions are combinatorial objects which are useful for encoding enumerative information in a way which can be manipulated algebraically. We will look at how to build symmetric functions, what it means to evaluate these polynomials over different bases, and will also work out an example application in graph theory. Then, we will look at symmetric polynomials generated from Young Tableaux, and will examine how these relate to classifying irreducible representations of the symmetric group S_n.

2 Symmetric Functions

To build up to polynomials, first we will define a smaller building block...

Defn: A **monomial** is a 1-term polynomial in n variables. If $\mathbf{x} = (x_1, x_2, ..., x_n)$ and $\mathbf{a} = (a_1, a_2, ..., a_n)$, we denote the monomial in vector notation as:

$$\mathbf{x}^{\mathbf{a}} = x_1^{a_1} x_2^{a_2} ... x_n^{a_n}$$

Why is this a combinatorial object? We can think of a monomial as a partition of the integer:

$$\sum_{i=1}^{n} a_n = a_1 + a_2 + ... + a_n$$

Defn: By a **partition** of some $n \in \mathbb{N}$, we mean a list $\lambda = (\lambda_1, \lambda_2, ..., \lambda_k)$ such that $\lambda \vdash n$, meaning:

$$\sum_{i=1}^{k} \lambda_i = n$$

Ex: Let's see how these ideas relate. For example, consider the monomial in 4 variables, $x_1^2 x_2^3 x_3 x_4^6$. This corresponds to a partition of 12, namely $\lambda = (2, 3, 1, 6) \vdash 12$. Questions such as enumerating all possible size k partitions of some number can be answered using combinatorial methods.

Another way we can view this object combinatorially is through the permutations of its variables. We can consider swapping the variables x_1 and x_2 in the previous example, resulting in the change:

$$x_1^2 x_2^3 x_3 x_4^6 \to x_2^2 x_1^3 x_3 x_4^6$$

This corresponds to a group action of S_n. Later, we will further explore the idea of how S_n can be more deeply represented by polynomials. For now, let's see how this can be defined as an action.

Defn: An **action** of a group G over some set X is a function $\sigma : G \times X \to X$ satisfying two properties:

- Identity: If e is the identity element of G, $\sigma(e, x) = x$.
- Compatibility: $\forall g_1, g_2 \in G,\ \sigma(g_1, \sigma(g_2, x)) = \sigma(g_1 g_2, x)$.

The convention is to write the action as a mapping to permutations of the exponents rather than the variables, but this becomes a bit confusing, so we will ignore convention. Given monomials $m = x_1^{a_1} x_2^{a_2} ... x_n^{a_n}$ and elements $\sigma \in S_n$, let's define the action of S_n on the set of n-variable monomials to be:

$$\sigma \cdot m = x_{\sigma(1)}^{a_1} x_{\sigma(2)}^{a_2} ... x_{\sigma(n)}^{a_n}$$

Now, define a **polynomial** to be a function of n variables, built from a sum of some number of monomials in n variables. Then, the action of S_n on the set of n-variable polynomials can be defined based on the action of S_n on each individual monomial terms in the sum. The set of all symmetric polynomials in n variables forms a ring for any n, which we will denote as Λ^n.

3 Bases for Λ^n

Given this ring of symmetric polynomials, a natural idea is to think about what a basis might look like. One possible basis can be constructed by considering all monomials in which the exponents of the variables correspond to a given partition of the total degree k. We call these objects **monomial symmetric polynomials**. Using vector notation for the set of variables and the set of exponents as before, we can denote:

$$m_\lambda(\mathbf{x}) = \sum_{(a_1 a_2 ... a_n) \vdash k} \mathbf{x}^{\mathbf{a}}$$

For example, if $\lambda = (2, 1, 1)$, we have:

$$m_{211} = x_1^2 x_2 x_3 + x_1 x_2^2 x_3 + x_1 x_2 x_3^2$$

The set of monomial symmetric polynomials in n variables forms a basis for Λ^n. This makes sense intuitively, because if we include some monomial term in a symmetric polynomial, we also need to include all other terms where the variables are permuted in order to preserve symmetry.

Defn: Using these objects, we can construct another basis. We define the **elementary symmetric polynomials** e_λ using the following conditions:

- If $\lambda = (\lambda_1, \lambda_2, ...\lambda_l)$, then $e_\lambda = e_{\lambda_1} e_{\lambda_2} ... e_{\lambda_l}$
- If $n \in \mathbb{N}$, $e_n = m_{11..1 \,\vdash \lambda}$

Ex: Note that we can easily write an expression of elementary symmetric polynomials in terms of monomial symmetric polynomials. Consider e_λ for $\lambda = (2, 1)$.

$$\begin{aligned} e_{21} = e_2 e_1 &= (x_1x_2 + x_1x_3 + x_2x_3 + \cdots)(x_1 + x_2 + x_3 + \cdots) \\ &= (x_1^2x_2 + x_1^2x_3 + \cdots) + (x_1x_2^2 + x_2^2x_3 + \cdots) + (x_1x_2x_3 + \cdots) \\ &= m_{21} + m_{111} \end{aligned}$$

The set of elementary symmetric polynomials in n variables also forms a basis for Λ^n. We can prove this by observing that the change of basis matrix from elementary symmetric polynomials to monomial symmetric polynomials is upper triangular, and is therefore invertible.

4 Application: Graph Theory

One motivation of looking at bases over these polynomials is determining information about their structure based on the coefficients we get. A way in which we can see how this works is through a method of extracting information from graphs using proper colorings.

Defn: A proper k-coloring of a graph $G = (V, E)$ is a function $\kappa : V \to S \subset \mathbb{N}$ such that if $uv \in E$, then $\kappa(u) \neq \kappa(v)$, and $|S| = k$.

- The subset S of $\mathbb{N}$ represents the possible set of distinct colors which we can assign to vertices.

The minimal number of colors in which we can color some graph, and the number of ways in which we can do so, gives insight to its structure. For example, as we add edges to a graph, it becomes more difficult to construct a proper coloring with a small set of colors. We can extend the idea of chromatic number by compiling the data from all possible k-colorings.

Defn: The Chromatic Symmetric Function (CSF) $X(G)$ of some graph G is the sum over all monomials generated by k-colorings of the graph, where the exponent of each x_i corresponds to the number of appearances of the color i.

Ex: $G = S_4$. Then, $X(G)$ can be separated into cases based on the number of possible colors used in the subset.

$$(x_1^3x_2 + x_1^3x_3 + x_1x_2^3 + x_1x_3^3)$$
$$+\binom{4}{2}(x_1^2x_2x_3 + x_1x_2^2x_3 + x_1x_2x_3^2 + x_1^2x_2x_4 + x_1x_2^2x_4 ...)$$
$$+4!(x_1x_2x_3x_4)$$

Note that for the case of 3 colors, we multiply each coloring monomial by $\binom{4}{2=6}$ since we need to choose which 2 of the 4 color options are the ones which are repeated in the outer vertices. For the case of 4 colors, we multiply the monomials by 4! since we can permute the vertex colors in any way. But this expression is equivalent to:

$$m_{31} + \binom{4}{2}m_{211} + (4!)m_{1111}$$

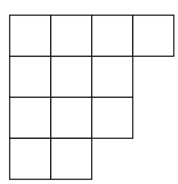
Since we can express this in terms of monomial symmetric functions, we can also express it in terms of the elementary basis. This can be useful in a few ways:

- Distinguishing graphs based on their corresponding CSFs (conjectured by Stanley to fully distinguish trees).
- [Stanley-Stembridge Thm] Characterizing which structural properties in a graph lead to fully positive coefficients over the elementary basis.

5 Young Tableaux

A tableau is a combinatorial object that we can use to add nuance to the idea of a partition. First, let's examine its simpler counterpart.

Defn: A **Young Diagram** (YD) is a collection of boxes which are left-justified and have the property that the number of boxes in each row is weakly decreasing. For example:



We can see that this diagram corresponds to the partition $\lambda = (4, 3, 3, 2) \vdash 12$. Such partitions are used to refer to the "shape" of a diagram, and we can define the "size" of a diagram to be the total number of boxes it contains.

Defn: Given some partition λ, a **Semi-Standard Young Tableau** (SSYT) T is a filling of a Young Diagram of shape λ using elements from $\mathbf{N}$, such that the following conditions hold (we will index elements similarly to elements in matrices, except starting from the upper left corner):

- Elements along rows weakly increase: $\forall i,\ j_1 < j_2 \implies T_{i,j_1} \leq T_{i,j_2}$
- Elements along columns strictly increase: $\forall j,\ i_1 < i_2 \implies T_{i_1,j} < T_{i_2,j}$
- We call some SSYT of size n a "Standard Young Tableau" (SYT) if all $\{1, 2, ..., n\}$ appear exactly once in the filling. Below are examples of an SSYT and a similar SYT, both of shape $\lambda = (4, 3, 2)$.

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 2 & 3 \\ \hline 2 & 3 & 5 & \multicolumn{1}{c}{} \\ \cline{1-3} 6 & 6 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array} \qquad \begin{array}{|c|c|c|c|}\hline 1 & 3 & 5 & 6 \\ \hline 2 & 7 & 8 & \multicolumn{1}{c}{} \\ \cline{1-3} 4 & 9 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array}$$

Given some tableau T, we can define a monomial x^T in m variables, where the variables represent the elements which appear in T. Let the number of times some number i appears in T be $\#(i)$, then define:

$$x^T = \prod_{i=1}^{m} x_i^{\#(i)}$$

<u>Defn</u>: Given some tableau shape λ, we can define its **Schur polynomial** s_λ in m variables as the sum over all such monomials over possible SSYT of shape λ, where the number of distinct elements ranges from 1 to m.

$$s_\lambda(x_1, x_2, ..., x_m) = \sum x^T$$

<u>Ex</u>: Consider $s_\lambda(x_1, x_2, x_3)$ for $\lambda = (3, 2)$. To compute this, we draw every possible SSYT of this shape, where the elements in the boxes are chosen from $\{1, 2, 3\}$.

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & 2 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 3 & 3 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & 3 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 2 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array} \qquad \cdots$$

Resulting in: $s_\lambda = x_1^3 + x_1^3x_3^2 + x_1^3x_2x_3 + x_1^2x_2^3 + x_1^2x_3^3 + 2x_1^2x_2x_3^2 + 2x_1^2x_2^2x_3 + x_1x_2x_3^3 + 2x_1x_2^2x_3^2 + x_1x_2^3x_3 + x_2^2x_3^3 + x_2^3x_3^2$.

It turns out that in general, Schur polynomials over some partition λ are both symmetric as we have previously defined, and also form a basis for the ring of symmetric polynomials Λ^n.

6 Representations of S_n

Let's pause and go over group representations.

<u>Defn</u>: Given a group G, a **representation** of G is a vector space V together with a homomorphism $\sigma : G \to GL(V)$. This condition can also be thought of as a linear action of G on V, defined as $g \cdot v = \sigma(g)(v)$.

Defn: A subspace W of V is a **subrepresentation** if W is invariant under the action of G. We call a subrepresentation **irreducible** if it has no nontrivial subrepresentations.

Ex: Consider a representation of S_3 over $\mathbb{C}$, call the homomorphism σ, where:

$$\sigma(\epsilon) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \sigma(12) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \sigma(13) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\sigma(23) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \sigma(123) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \sigma(132) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

A powerful fact from representation theory is that in any group, the number of conjugacy classes is equal to the number of irreducible representations of the group. In the case of S_n, this correlation can be explicitly shown through Young Tableau.

First, we claim that the number of conjugacy classes in S_n is the same as the number of cycle types, since two elements which are conjugates also have the same cycle type.

Pf: First consider some $\alpha \in S_n$ with cycle type $(k_1, k_2, ..., k_l)$, meaning $\alpha = \alpha_1\alpha_2...\alpha_l$ where each α_i is a cycle of length k_i.

Then consider any element in the same conjugacy class of α, which can be expressed as $\tau\alpha\tau^{-1}$ for some $\tau \in S_n$, we can write:

$$\tau(\alpha_1\alpha_2...\alpha_l)\tau^{-1} = (\tau\alpha_1\tau^{-1})(\tau\alpha_2\tau^{-1})...(\tau\alpha_l\tau^{-1})$$

We can easily show that each of these individual conjugates $\tau\alpha_i\tau^{-1}$ will have the same cycle length as α_i. Therefore, the original conjugate will have the same cycle type as α. We will leave the other direction of the proof as an exercise.

This is an exciting result because the possible cycle types of S_n correspond to ways in which we can partition n, which then correspond to Young Tableau with n boxes.

7 Tabloids and Modules

To relate representations of the symmetric group back to tableaux, we need to think about a slightly less restrictive object.

Defn: A **tabloid** $\{T\}$ is an equivalence class of fillings of a Young Diagram, in which equivalence is based on the rows of the diagram containing the same set of numbers. We will denote R_T and C_T as the row and column stabilizers, respectively. We will also define the vector space of all tabloids of shape λ to be M^λ.

Defn: We define a **polytabloid** as enforcing symmetry over colummns:

$$v_T = \sum_{q \in C_T} sgn(q)\{q \cdot T\}$$

As every polytabloid is a tabloid, we can define a module:

Defn: A Specht Module S^λ is a subspace of M^λ spanned by elements v_T:

$$S^\lambda = span_{\mathbb{C}}\{v_T | T \text{ is a Young Tableau of shape } \lambda\}$$

We can use these modules to construct the irreducible representations of S_n.

Ex: Consider S_3. Then the module $S^{(3)}$ corresponds to the tabloid:

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array}$$

Which corresponds to the trivial representation of S_3 (homomorphism is the identity function), as the actions of S_3 permute row entries, thus the polytabloid is fixed. The module $S^{(111)}$ corresponds to the tabloid:

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}$$

Which corresponds to the alternating representation of S_n (homomorphism is the sign function).

Exercise: Convince yourself that the findings of the above example holds in general for S_n.

Sidenote: One last interesting comment is that we can connect these different objects constructed from the idea of Young Tableaux. The fact that Specht modules correspond to irreducible representations is analogous to the fact that Schur polynomials form a basis over the ring of symmetric polynomials. This mapping is done through the Frobenius character map:

$$Ch : R^n \to \Lambda^n$$

where R^n is the representation ring of S^n and Λ^n is the ring of symmetric functions in n variables. This function maps irreducible representations S^λ of S_n to Schur polynomials s_λ for the same λ.

8 References

1. *Algebraic Combinatorics and Coinvariant Spaces* (chapters 1,2,4,5). Francois Bergeron, Canadian Mathematical Society, 2009.

2. *Young Tableaux* (chapters 6,7). William Fulton, *London Mathematical Society Student Texts: 35*, Cambridge University Press, 1997.