Geometric Group Theory

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1 Introduction

Geometric group theory is the study of how algebraic objects called groups can be represented geometrically, and what insights these representations might provide. We will approach this topic through the lens of how we can build a "Cayley graph" to represent a group given a set of generators. We will also discuss the word problem on groups, and about how thinking about a graph as a metric space can help us answer algorithmic questions.

"One of my personal beliefs is that fascination with symmetries and groups is one way of coping with frustrations of life's limitations: we like to recognize symmetries which allow us to recognize more than what we can see."

Pierre de la Harpe, *Topics in Geometric Group Theory*

2 Groups

The two mathematical objects that we are going to examine and relate are groups and graphs. Let's first familiarize ourselves with the notion of a group through examples.

A group is essentially a set in which the elements of the set have nice properties under some group "operation", that help to give the set more structure. For example, we've been thinking a lot about the integers, so take the set $\mathbb{Z}$ along with the operation "+", meaning integer addition.

One of the properties we can observe of this set in terms of our chosen operation is that of "closure". This means that given two elements of the set, under the operation "+", the result is still in the set.

For example, take the elements $3, -5 \in \mathbb{Z}$.

$$3 + (-5) = -2 \in \mathbb{Z}$$

For reference, below is a more complete list of conditions that a set and its corresponding operation need to satisfy in order to be considered a group. We won't go into these in detail.

Defn: A **group** $G = (S, \star)$ is a set S together with a binary operation $\star$ where the following properties hold:

1. Closure: Given any $a, b \in S$, we have $a \star b \in S$.
2. Associativity: Given any $a, b, c \in S$, we have $(a \star b) \star c = a \star (b \star c)$.
3. Identity: There exists some element $e \in S$ such that for any other element $a \in S$, we have $a \star e = e \star a = a$.
4. Inverses: For every $a \in S$, we have a unique $b \in S$ such that $a \star b = e$ where e is the identity element explained above.

Let's look at a more abstract example of a group.

Defn: The **free group** F_S over some generating set S is the set of all **words** from S, meaning concatenations of elements of S and/or their inverses.

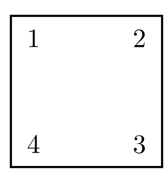
For example, take the free group over a generating set with two elements, $S = \{a, b\}$. Then some words in the free group over S might be:

$$ab,\ ababba,\ a^{-1}bba,\ b^{-1}a$$

This wording of a "generating" set is nice in the sense that by combining elements of the set, we can generate the full group (by definition). We can extend these notions of generators and words to groups in general.

Defn: A **generating set** $S \subset G$ for some group G is a subset of elements such that we can get any element of G using words from S.

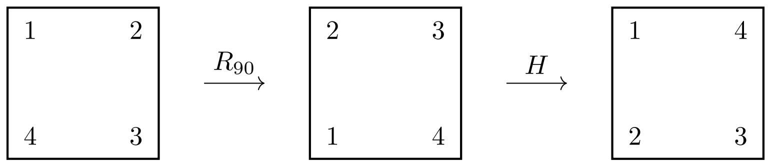
Now take for example, the finite group D_4, or the group of symmetries of the square. Consider a square with labels added to the corners for the purpose of keeping track of otherwise indistinguishable shifts in orientation:



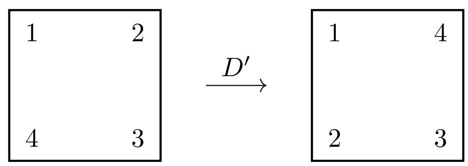
The elements of D_4 correspond to ways in which we can move the square by flipping or rotating it, so that we end up with a square of the same orientation. Using the numbers at the corners, we can construct a finite set of moves based on how the square has moved in relation to its starting position.

We define $D_4 = R_0, R_{90}, R_{180}, R_{270}, H, V, D, D'$, where R_i represents a counter-clockwise rotation of $i°$ and the other elements represent reflections across the horizontal, vertical, and diagonal axes.

Below is an example word, which we denote $R_{90}H$ (where the group operation is doing the moves corresponding to each element from left to right):



However, note that this result is the same as the element D', or a flip across the negative slope diagonal:



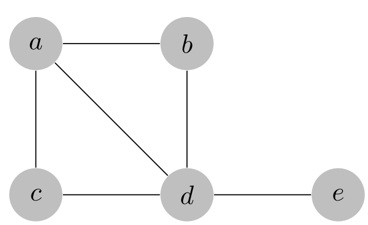
So, we can say that $R_{90}H = D'$. It turns out that we can also generate all of the other elements of D_4 in this way, using only the elements $\{R_{90}, H\}$, so this is a generating set.

<u>Note</u>: There are other generating sets for D_4, for example $\{R_{270}, H\}$. We will come back to this idea of having multiple generating sets for the same group later on...

3 Representing Groups as Graphs

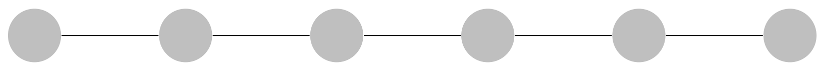
A graph is another useful mathematical object through which we can represent the relationships between entities in a network. The two components of a graph are a set of vertices (dots) and a set of edges (lines connecting pairs of dots).

<u>Defn</u>: A **graph** $G = (V, E)$ is a set of vertices V together with a set of edges E (unordered pairs of vertices from V).

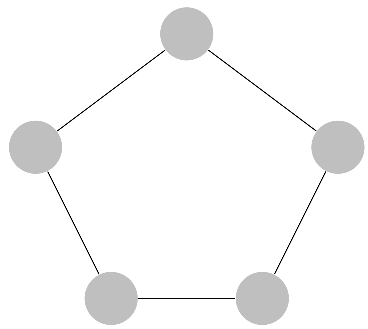


We say that two vertices are adjacent if they are adjoined by an edge. In the above example, a is adjacent to b, c, d. We can write the edge set of this graph as: $\{ab, ac, ad, bd, cd, de\}$.

We can also think about special families of graphs. The below graph is an example of a path, P_6:



... and the below graph is an example of a cycle, C_5:



Later, we'll think about these as subgraphs occurring in some larger graph.

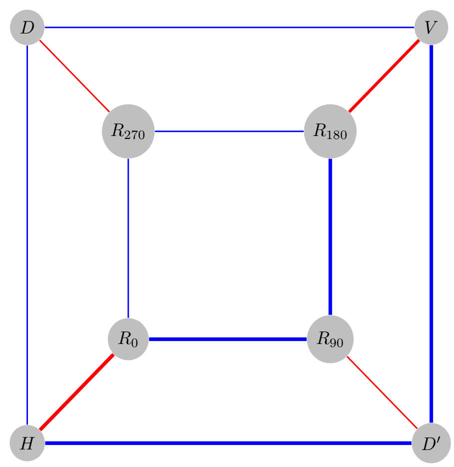
One of the ways that we can represent a group geometrically is through a Cayley graph. To generate such a graph, the group must be finitely presented, meaning it has some finite generating set as defined earlier.

<u>Defn</u>: Given some group G along with a generating set S, the **Cayley graph** $Cay(G, S)$ is defined as the graph where:

- The vertex set V is the set of elements in G.
- An edge $g_1 g_2$ is included in the edge set E iff $\exists\ s \in S$ such that $g_1 s = g_2$.
- Sometimes these edges are shown as directed, but we will ignore this. Thus, edges can be in either direction, so we can multiply any element by some s or s^{-1} to get a potentially new vertex.

In other words, we can construct the edges of the graph by starting with the identity of the group, and drawing out the edges of the graph by multiplying by every possible generating set element (or the inverse of such an element) at each new vertex.

Let's go back to our example of D_4 under the generating set $\{R_{90}, H\}$. Below is the associated Cayley graph, with the blue edges representing multiplication by R_{90} and the red edges representing multiplication by H. An example word, $R_{90}R_{90}HR_{90}R_{90}H$, is depicted through the bolded *path* (note that this is equivalent to the identity, as it returns to R_0).



4 The Word Problem

It turns out that these geometric representations can give us insight into certain algorithmic questions in groups.

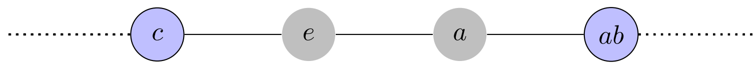
The **word problem** on groups is the question of whether, given some group G and some word w, we can find a finite-time algorithm to determine whether $w = e$ where e is the identity in G. This type of question is also known as *decidability*.

- Note that this is the same thing as finding an algorithm to tell us when two words are equivalent. For example, $g_1g_2g_3 = g_4g_5 \leftrightarrow g_1g_2g_3g_4^{-1}g_5^{-1} = e$.

For finite groups, the word problem is trivially solved by drawing out the Cayley graph for the group. This is because we can simply follow the path that the word forms in the graph and check whether we have arrives at the identity.

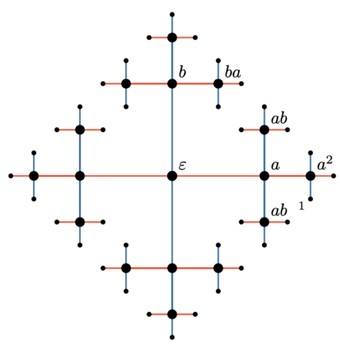
For infinite groups, do we always know what the Cayley graph will look like? In fact, all we need to do for some word of length w is to draw out the graph up to a radius of $\frac{w}{2}$ (meaning that we include all vertices which from the identity element have a path distance of at most $\frac{w}{2}$). This is essentially done using the Todd-Coxeter algorithm.

However, for infinite groups, there is not always a method through which we can determine the graph algorithmically. Let's look at a problem which can occur. Assume we have some generators a, b, c.



However, as we are doing so, suppose we have $c = ab$ in the graph above. If this was true, we would have *thought* we had succeeded in building the Cayley graph up to a radius of 2 as ab has length 2, but in reality, we might not have. The necessary step here while building the graph is repeatedly checking whether two such vertices are equivalent. However, since this step is algorithmically equivalent to solving the word problem itself for G, there is a possibility that this can occur repeatedly, causing the algorithm to run forever.

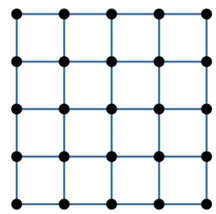
This algorithmic issue depends on the existence of a cycle as a subgraph. In our example, since c and ab are equivalent, we should have had the 3-cycle $(c = ab, e, a)$. Therefore, we have no problem when drawing out Cayley graphs up to a given radius for some group if the graph is a tree (no cycles). Take for example, the previously defined free group over two variables, $F_{\{a,b\}}$. This continues on infinitely in all directions:



This demonstrates a more general principle:

<u>Thm</u>: Given some group G with generating set S, the word problem is decidable in finite time for G, S if $Cay(G, S)$ is a tree.

However, what happens if we introduce a restriction? Consider the commutative group on two variables under the operation of concatenation, $F_{\{a,b\}} \mid ab = ba$. (This of course also continues on infinitely in all directions):



While this graph includes cycles, their structure is nice enough that we could imagine being able to draw out the graph up to a given radius without running into unforeseen repeated vertices. In fact, the word problem is solvable for this group as well. How extreme of a condition do we need to place on the group to ensure we don't get any unexpected loops?

Does this particular graph remind you of anything? Hold that thought...

5 Metric spaces

When thinking about traversing through a graph geometrically, it can be helpful to characterize the space we are moving through.

Perhaps a familiar geometric space one can think of is the plane, or $\mathbb{R}^2$. In $\mathbb{R}^2$, we conveniently associate two coordinates to each point so that we can find out where two points lie relative to one another. This existence of relative positions in space suggests the notion of distance as a means to quantify how far one point is from another.

In general, these kinds of spaces are called metric spaces, as they are equipped with a "distance" function (which can be defined however we want as long as it satisfies a few properties)...

Defn: A **metric space** is a set of points S together with a distance function $d : S \times S \to \mathbb{R}_{\geq 0}$, satisfying the following:

1. Distance from a point to itself is zero: $\forall\ s \in S$, $d(s, s) = 0$.
2. Distance between two distinct points is always positive: $\forall\ s_1, s_2 \in S$ where $s_1 \neq s_2$, $d(s_1, s_2) > 0$.
3. Symmetry: $\forall\ s_1, s_2 \in S$, $d(s_1, s_2) = d(s_2, s_1)$.
4. Triangle inequality: $\forall s_1, s_2, s_3 \in S$, $d(s_1, s_3) \leq d(s_1, s_2) + d(s_2, s_3)$

In a similar sense, we can consider the set of points in a graph to be a metric space, in which we define the distance between two vertices as the length of the shortest path of edges between them. For example, if some v_1 and v_2 are adjacent (connected by a single edge), we have $d(v_1, v_2) = 1$.

We call the shortest path which generates this distance the **geodesic** between the two points. Note that the nature of this path depends on what kind of space we are working in. For example, if we are in S^2, consider two non-antipodal points on the sphere. Then, we have two different shortest paths between the points. Similarly, in the metric space of a graph, we can have multiple distinct shortest paths between two points.

Defn: We call two metric spaces (M_1, d_1) and (M_2, d_2) **isometric** if there exists some *bijective* isometry (distance-preserving map) between them, meaning some $f : M_1 \to M_2$ such that:

$$d_1(a_1, a_2) = d_2(f(a_1), f(a_2)) \ \forall a_1, a_2 \in M_1$$

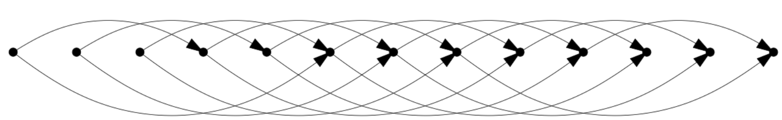
Ex: Think back to the Cayley graph we looked at for the commutative group of words on two variables. We can see that it is isometric to $\mathbb{Z}^2$.

Defn: We call two metric spaces (M_1, d_1) and (M_2, d_2) **quasi-isometric** if there exists some map $f : M_1 \to M_2$ such that the distance between two points is close to the distance between their images, up to some constant factors of shifting and scaling. In other words, there exist some constants $k_1, k_2 \in \mathbb{R}$ such that:

$$\frac{1}{k_1} \cdot d_1(a_1, a_2) - k_2 \leq d_2(f(a_1), f(a_2)) \leq k_1 \cdot d_1(a_1, a_2) + k_2 \ \forall a_1, a_2 \in M_1$$

Let's look more closely at the Cayley graphs of the infinite groups from Section 3. Ignoring the directions of the edges, take for example the Cayley graph of $\mathbb{Z}$ under the generating set $\{1\}$:

However, we could also use a different generating set for $\mathbb{Z}$, $\{3, 5\}$, which creates:



If we "zoom out" from these images infinitely, they seem to have the same structure (essentially a line). This means that these spaces are quasi-isometric, since distances between any two points are preserved up to some constant.

Ignoring optimality, let's claim that we can let the multiplicative constant be $k_1 = 5$ and let the additive constant be $k_2 = 0$. We can think of $(\mathbb{Z}, d_1)$ to be $\mathbb{Z}$ together with the distance metric of path length between two points in the first graph (generating set of $\{1\}$), and $\mathbb{Z}, d_2$ to be $\mathbb{Z}$ together with the distance metric of path length in the second (generating set of $\{3, 5\}$).

If $|a_1 - a_2| = n$, then $d_1(a_1, a_2) = n$ as this represents the number of steps of size 1 needed to get to move between the vertices.

For the second metric:

- In the best case, if $5|n$, we have a distance of $d_2(a_1, a_2) = \frac{n}{5}$. So in general, $d_2(a_1, a_2) \geq \frac{n}{5} = \frac{1}{5} \cdot d_1(a_1, a_2)$.
- Otherwise, we know that we can move forward one vertex by traveling along three edges : $3 + 3 - 5 = 1$, which takes a maximum of $3n$ moves. So, $d_2(a_1, a_2) \leq 3n$.

Therefore, since $3n \leq 5n$, we have:

$$\frac{1}{5} \cdot d_1(a_1, a_2) - 0 \leq d_2(f(a_1), f(a_2)) \leq 5 \cdot d_1(a_1, a_2) + 0$$

where $f(a_1) = a_1$ and $f(a_2) = a_2$ since $M_1 = M_2 = \mathbb{Z}$. Thus, these two spaces are quasi-isometric.

Thm: Given some group G with generating sets S_1 and S_2. Then, $Cay(G, S_1)$ and $Cay(G, S_2)$ are quasi-isometric under the metric of geodesic path length.

This is a powerful theorem because it means we can think of the metric space generated by some group as independent of some chosen generating set.

6 Gromov Hyperbolic Space

Thm: If $Cay(G, S)$ under some finite presentation is quasi-isometric to some hyperbolic space, the word problem is decidable in finite time for G.

Our motivation for discussing hyperbolic space is the previously described problem of not having a finite-time algorithm to check for cycles (or repeated vertices) when constructing the Cayley graph. The intuition here is that we need to be able to check every possible way that a given cycle might be reduced if some part of the word is equal to the identity due to the group relators. We can think of this as “filling” up the loop by breaking it up into triangles which represent potential paths back to the identity.

In hyperbolic space, there is a finite bound on how many of these combinations we need to check due to the “slim triangles” condition. If we guarantee that the triangles we use to break up a potential cycle back to the identity must be slim to some constant degree, any geodesics between two points must also stay close together up to a constant.

Hyperbolic space can be visualized as a space that curves inwards into itself (think the opposite of the outward-facing surface of an object like a sphere). This means that if we visualize a triangle, the sides curve inwards towards each other in terms of geodesic distance between points. We will formally define this idea using a more general distance metric.

<u>Defn</u>: Given some metric space (X, d), Gromov product of two points y, z in X with respect to some other point x is:

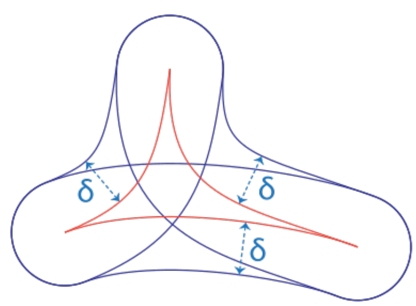
$$(y, z)_x = \frac{1}{2}(d(x, y) + d(x, z) - d(y, z))$$

Intuitively, this is a measure of how close together two sides of the triangle are in relation to the other side.

<u>Defn</u>: Some metric space X is a δ-hyperbolic space if for some constant δ, we have for all $w, x, y, z \in X$ that (with respect to some point w):

$$(x, z)_w \geq min((x, y)_w, (y, z)_w) - \delta$$

This condition is equivalent (up to a constant) to ensuring slimness of triangle in the space, which can be visualized as follows, in which every point on a given geodesic (side) is within δ of one of the other geodesics:



One observation we might make is that the graph of a tree under the metric of path distance is 0-hyperbolic. This makes sense, since we have previously reasoned that groups with Cayley graphs which are quasi-isometric to trees have a solvable word problem. Essentially, we can also think of the δ-hyperbolic condition as a measure of how "tree-like" a space is. If a space is "tree-like" up to some constant, we have a finite algorithm.

<u>Sidenote</u>: Hyperbolic groups are not the only groups with a solvable word problem. Other examples include (as previously mentioned) finite groups, free groups and free abelian groups, residually finite groups, and more. In fact, it is a bit difficult to actually construct a group which does not have a solvable word problem. Doing so often requires constructing some convoluted relations. For an explicit example, see [Collins 1986].

7 References

The following works are responsible for some paraphrased definitions, theorems, and ideas, as well as a few shamelessly stolen figures. In particular, Löh's *Geometric Group Theory* is an approachable read and gives a thorough deep dive into this topic.

1. *Dehn's Problems and Cayley Graphs* (pp. 157-162). In: The Geometry of the Word Problem for Finitely Generated Groups. Advanced Courses in Mathematics CRM Barcelona. Birkhäuser Basel, 2007.

2. *Geometric Group Theory: An Introduction.* Clara Löh, 2016.

3. *A Simple Presentation of a Group With Unsolvable Word Problem.* Donald Collins, 1986.