PROJECTILE MOTION WITH AIR RESISTANCE, WIND AND CORIOLIS FORCE

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Contents

1 Introduction **3**

2 Air Resistance **4**
2.1 Findings 7

3 Wind **9**
3.1 Findings 12

4 Coriolis Force **13**
4.1 Findings 15

5 Conclusion **18**

6 Appendix **19**
6.1 Range at launch angle $\phi = 60°$ 19
6.2 Maximum range launch angle 21
6.3 Size of the surface area of hits at launch angle $\phi = 60°$ 23
6.4 Displacement because of the Coriolis effect at launch angle $\phi = 60°$ 25

7 References **27**

1 Introduction

The aim of this project was to analyse projectile motion under different conditions. First, I analysed projectile motion with quadratic (Newton) air resistance, then I examined how the projectile behaves under windy conditions and lastly, I analysed the effect of the Coriolis force on the motion. I used Python to numerically solve the defined problems as well as to visualize the obtained results.

Parameters used in the analysis are: initial velocity v_0 (in the xz plane), launch angle to the horizontal ϕ (i.e. perpendicular to the xy plane), effective surface area of the projectile S, mass of the projectile m, maximal wind speed c_{max}, air density ρ and gravitational acceleration g. Their values are $v_0 = 500\,\frac{m}{s}$, $S = 45\,cm^2$, $m = 40\,kg$, $c_{max} = 8\,\frac{m}{s}$, $\rho = 1.2\,\frac{kg}{m^3}$ and $g = 10\,\frac{m}{s^2}$.

An example of the motion trajectory in a constant horizontal wind (in the xy plane) is shown for three different winds in Figure 1.

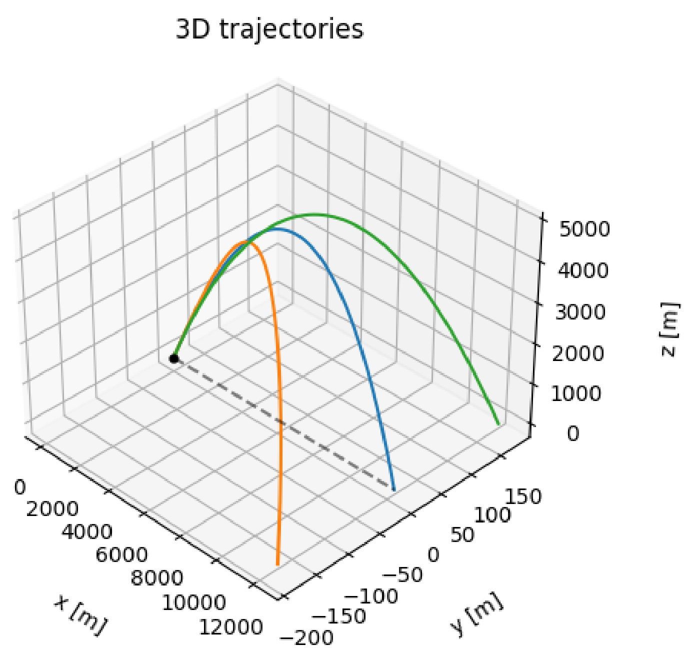

Figure 1: Projectile motion example

2 Air Resistance

I analysed the projectile motion with incrementation in very small time steps Δt. The force that acts on the body in a short enough time interval is constant. Thus it is possible to calculate the acting force in every time interval and from that the change in the body's velocity and displacement. In order to obtain the desired accuracy, it is necessary to appropriately choose the value of Δt. When using a value too large, the results obtained can be inaccurate and of little value. On the other hand, using smaller values of Δt means more incrementations in the calculation which results in longer computation time. It is therefore necessary to choose values small enough so as not to compromise the quality of the results and at the same time large enough to ensure an adequate computational time. In my case, setting the time interval to $0.001s$ brought satisfying results.

The Newton drag force that acts on the projectile is written as:

$$\vec{F} = -\frac{1}{2}\rho S v \vec{v},$$

where $\vec{v}$ is the current velocity of the body relative to the air. As the air is not moving in this case, $\vec{v}$ equals the projectile's velocity. Notice that the Newton drag force always acts in the opposite direction to the relative velocity of the projectile to the air.

If n is the number of passed time intervals Δt, the projectile's velocity v_n in time interval t_n is given by

$$v_n = \sqrt{(v_{x_{n-1}})^2 + (v_{y_{n-1}}))^2 + (v_{z_{n-1}}))^2}. \tag{1}$$

The components of the acting force in each time interval Δt are obtained as follows:

$$F_{x_n} = -\frac{1}{2}\rho S v_{n-1} v_{x_{n-1}},$$

$$F_{y_n} = 0,$$

$$F_{z_n} = -mg - \frac{1}{2}\rho S v_{n-1} v_{z_{n-1}}.$$

Then, it is possible to calculate the new velocity and the new position on the x axis:

$$v_{x_n} = v_{x_{n-1}} + \frac{F_{x_n}}{m}\Delta t, \tag{2}$$

$$x_n = x_{n-1} + v_{x_{n-1}}\Delta t. \tag{3}$$

The calculation of velocities and positions along y and z axis is similar. The process is repeated as long as $z > 0$, i.e. until the projectile falls to the ground (I assumed the same launch and landing height).

The comparison of projectile motion trajectories in a vacuum and with air resistance is shown in Figure 2. In the case of air resistance, one can notice the lower maximum height as well as the shorter range. Besides, the shape of the trajectory is not parabolic which is better visualized in Figure 3.

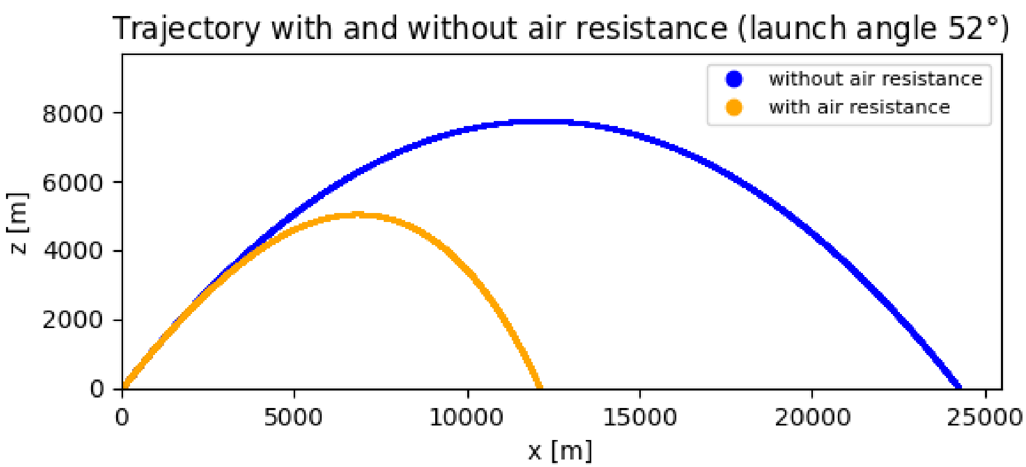

Figure 2: Comparison of trajectories with and without air resistance

Trajectories of the projectile for different launch angles in the presence of air resistance are shown in Figure 3.

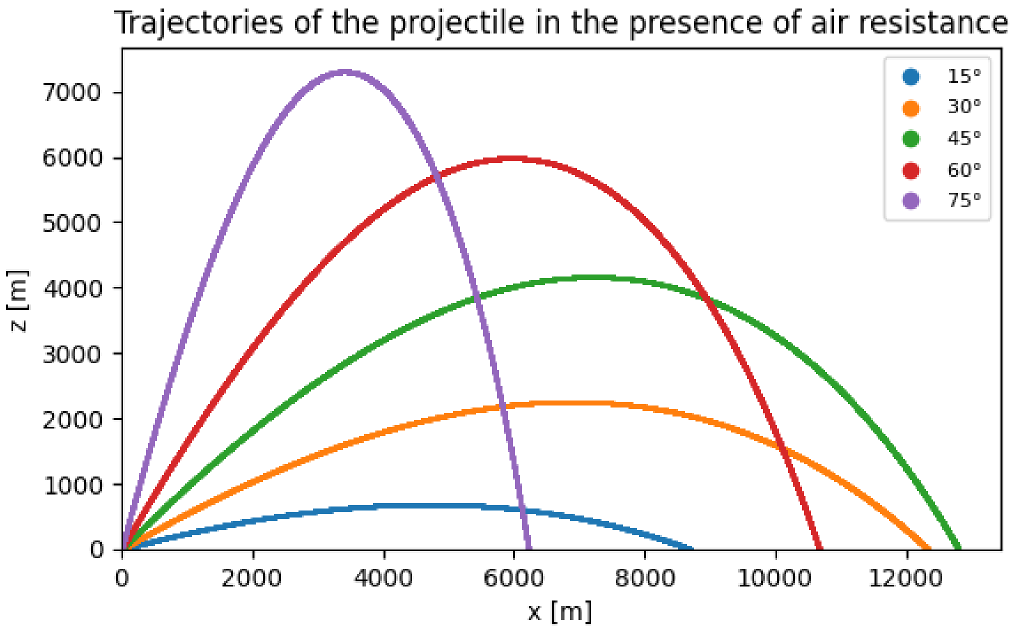

Figure 3: Trajectories of the projectile for different launch angles

I calculated the range of the projectile using different launch angles ϕ and obtained a graph Range vs. launch angle which is shown in Figure 4.

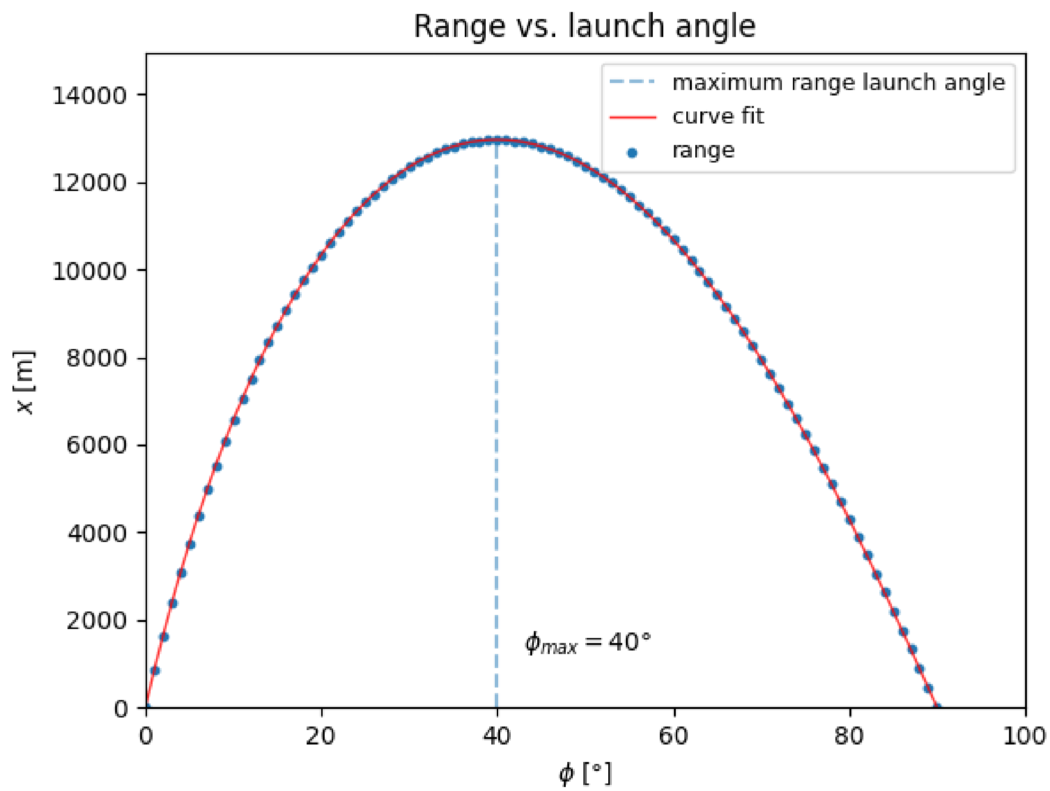

Figure 4: Range x vs. launch angle ϕ

2.1 Findings

The range of the projectile along the x axis in the case of projectile motion without air resistance is calculated as

$$D = \frac{v_0^2 sin(2\phi)}{g}. \tag{4}$$

The range is therefore maximal, when $sin(2\phi) = 1$ or $\phi = 45°$.

When taking into account air resistance, the optimal angle of launch for achieving maximum range depends on the initial velocity v_0, as well as on the properties of the projectile, i.e. the effective surface area S and mass m. Increasing m and decreasing S and v_0 makes the optimal angle converge to

$\phi_{max} = 45°$, which is reasonable as the conditions are then more similar to the projectile motion without air resistance. Larger mass gives the projectile more inertia and the motion is therefore less affected by external forces. Smaller surface area undermines the effect of air resistance. Higher speed means a much stronger air resistance since the drag force depends on the square of the projectile's speed. For the curious reader, the relationship between the optimal angle and the parameters v_0, S and m is presented with graphs in section Appendix.

In the example shown in Figure 4, the optimal angle of launch for maximum range is $\phi_{max} = 40°$. It is interesting to note that the optimal angle never exceeds 45°. In general, the larger the deviation from ideal, non-drag conditions, the smaller the optimal launch angle. From the graph it is also clear that the same range can be achieved with two different launch angles one of which is bigger and the other that is smaller.

3 Wind

Adding a constant horizontal wind in the xy plane makes the projectile motion especially amusing. I chose two angles of launch which under non-windy conditions give approximately the same range, using parameters defined in Air Resistance, and analysed the dispersion of landings in windy conditions. The selected angles are $\alpha = 23°$ in $\beta = 58°$ and the corresponding range achieved in non-windy conditions is $x = 11\,100\,m$.

An example of the motion trajectory in a constant horizontal wind is shown for three different winds in Figure 5 (each trajectory corresponds to a different windy situation).

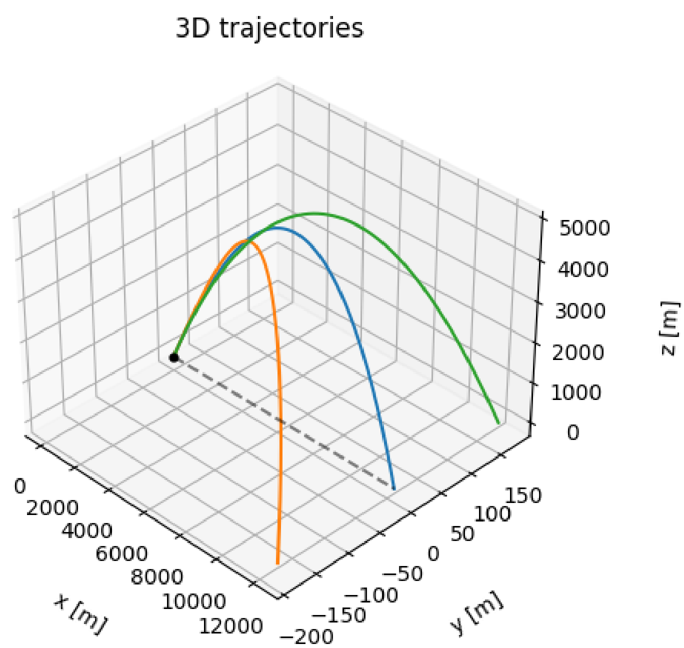

Figure 5: Projectile motion in horizontal wind for three different winds

To get enough different windy situations, I randomly generated 10 000 wind vectors in the xy plane which are uniformly distributed across the surface area of a circle with a radius of c_{max}. First, I generated 10 000 random angles between 0 and 2π and 10 000 random numbers between 0 and 1. Then I applyed square root function on the latter set of numbers, multiplied the results by c_{max} and randomly assigned the generated angles to these numbers. In this way, I created vectors in polar coordinates, given by their length and angle. In other words, I generated two-dimensional vectors of wind $\vec{c} = (c_x, c_y)$, where $c_x, c_y \in [0, c_{max}]$ and at the same time $\sqrt{c_x^2 + c_y^2} \in [0, c_{max}]$. Given this data, it is possible to calculate coordinates of these vectors in the cartesian coordinate system which is more useful in this analysis. Figure 6 shows the uniform distribution of the generated wind vectors across the surface area of a circle with radius c_{max}.

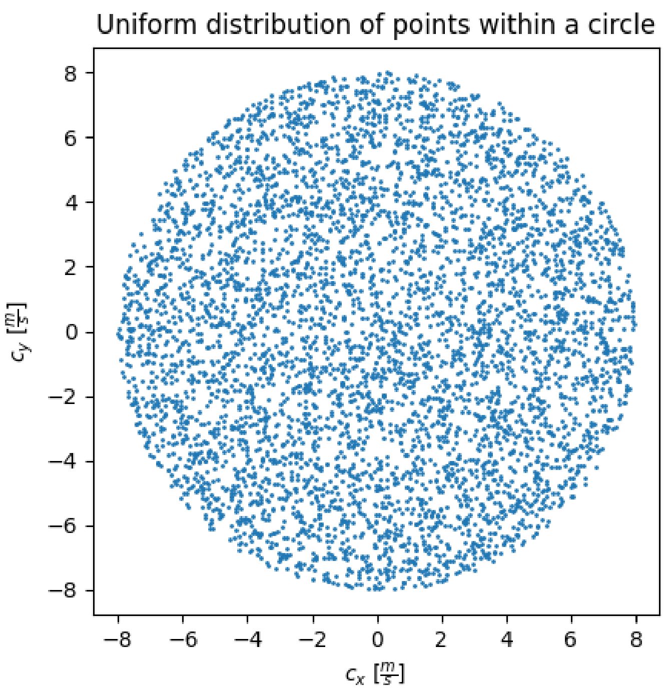

Figure 6: Uniform distribution of wind vectors across the surface area of a circle with radius c_{max}

I used the generated vectors to simulate 10 000 different flight situations. The process of solving the problem is very similar to that in section Air Resistance.
Velocity in every time interval t_n is defined as in Equation 1. The components of the force acting on the projectile in time interval t_n are

$$F_{x_n} = -\frac{1}{2}\rho S v_{n-1}(v_{x_{n-1}} - c_x),$$

$$F_{y_n} = -\frac{1}{2}\rho S v_{n-1}(v_{y_{n-1}} - c_y)_{n-1},$$

$$F_{z_n} = -mg - \frac{1}{2}\rho S v_{n-1} v_{z_{n-1}}.$$

Components of the new velocity and position are then calculated using Equations 2 and 3.

The dispersion of hits in the xy plane is shown in Figure 7. The area of hits has an elliptic shape, which has a greater surface area if the angle of launch ϕ is greater. The centre of the area corresponds to the point of landing for launch in non-windy conditions. The two angles used achieve the same range in non-windy conditions.

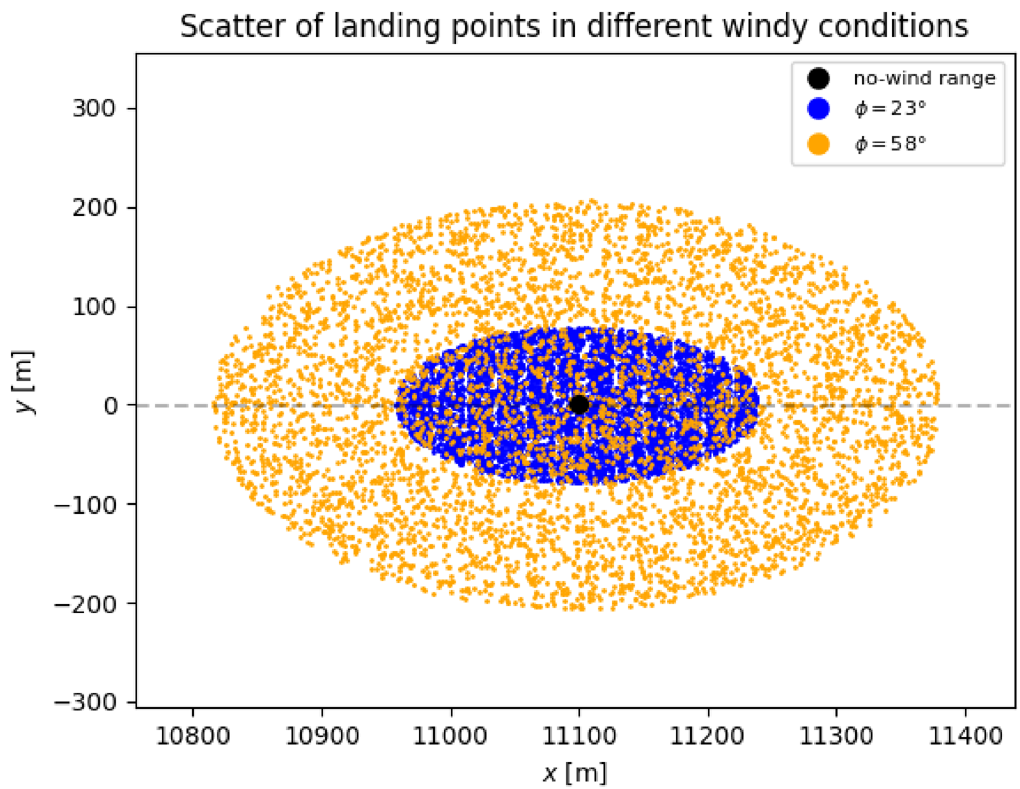

Figure 7: Scatter of landing points in windy conditions

3.1 Findings

Using more wind vectors yields a more uniform distribution of landing points. To show the effect of wind on the projectile, however, 10 000 points are enough. Using too many points also does not add much value to the graph as it is then harder to distinguish between different hits as individual points.

The dispersion of landing points is dependent on the parameters v_0, m in S. These dependences are presented with graphs in section Appendix.

4 Coriolis Force

When considering the Coriolis effect, the calculation is similar as in section Wind. Here, it is necessary to take into account the Coriolis force which acts on a body that has a non-zero velocity relative to the axis of Earth's rotation (moving to or away from the axis). I analysed a launch to the South, with initial launch latitude of $\gamma = 46°$ North. Coriolis force is defined as

$$\vec{F_{co}} = 2m\vec{v} \times \vec{\omega},$$

where m is the body's mass, $\vec{v}$ its velocity and $\vec{\omega}$ angular velocity of Earth. First, I transformed the angular velocity of the Earth $\vec{\omega}$ into the projectile's coordinate system (reference frame) which I am using implicitly throughout the analysis and is defined as follows: the x axis is parallel to the ground and intersects the axis of rotation of the Earth, the z axis is perpendicular to the ground and the y axis is perpendicular to both x and z and is pointing towards East. Figure 8 shows the projectile's coordinate system as viewed from Earth's point of view.

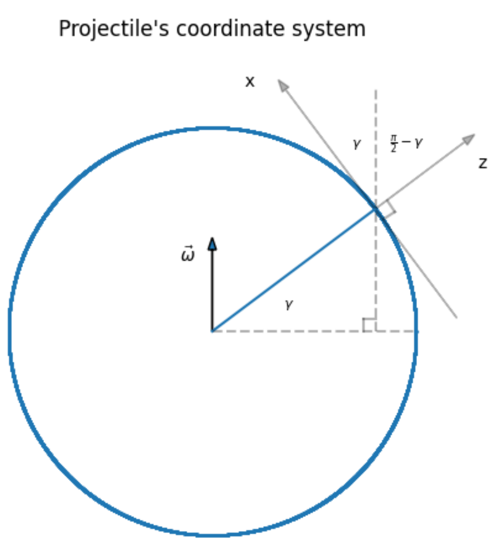

Figure 8: Projectile's coordinate system as seen from space

The angular velocity of the Earth's rotation is expressed in the projectile's coordinate system as

$$\vec{\omega} = \omega(\cos\gamma, 0, \sin\gamma),$$

where γ is the latitude of the launch location.

The velocity of the projectile in every time interval is calculated using Equation 1. The Coriolis force is then defined as follows:

$$\begin{aligned}\vec{F}_{co} &= 2m\vec{v} \times \vec{\omega} = 2m(v_x, v_y, v_z) \times (v_x\omega\cos\gamma, 0, \omega\sin\gamma) \\ &= (v_y\omega sin\gamma,\ v_z\omega\cos\gamma - v_x\omega\sin\gamma,\ -v_y\omega\cos\gamma).\end{aligned} \tag{5}$$

The components of the total force on the projectile are:

$$\begin{aligned}F_{x_n} &= -\frac{1}{2}\rho S v_{n-1}(v_{x_{n-1}} - c_x) + F_{co_x}, \\ F_{y_n} &= -\frac{1}{2}\rho S v_{n-1}(v_{y_{n-1}} - c_y) + F_{co_y}, \\ F_{z_n} &= -mg - \frac{1}{2}\rho S v_{n-1} v_{z_{n-1}} + F_{co_z}.\end{aligned}$$

Again, the new velocity and displacement along all three axes are calculated using Equations 2 in 3.
Figure 9 shows the dispersion of hits in the xy plane, in the same manner as in section Wind.

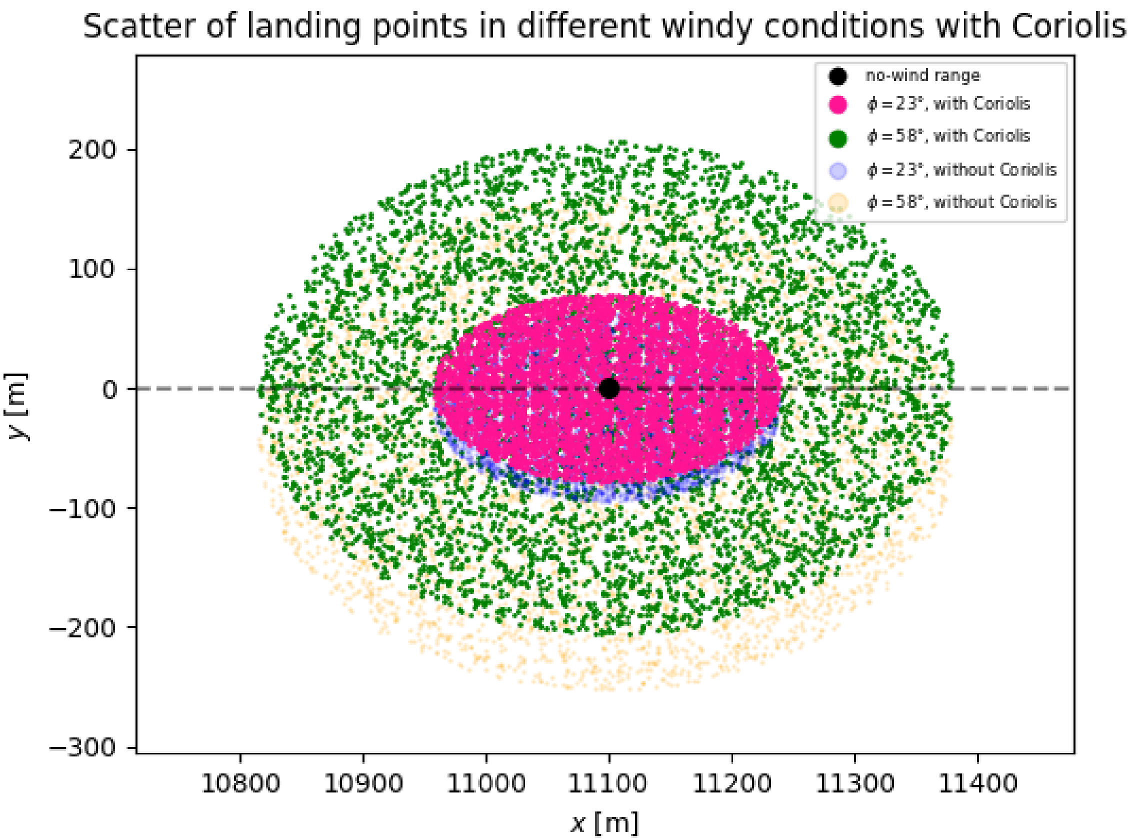

Figure 9: Scatter of projectile's landing points in wind when adding the Coriolis effect

4.1 Findings

The Coriolis effect causes a displacement of the area of hits to the East or West, depending on the hemisphere the projectile is initially located in (Northern or Southern) and the direction of the motion (towards North or South). In my case, launching the projectile in the Northern hemisphere towards South causes a displacement of the area of hits to the West. It is important to note that the area of hits would also be displaced significantly along the x axis, i.e. to the North or South, if the projectile's speed along the

y axis were comparable to the speed along the x axis, since the x component of the Coriolis force depends on the projectile's speed along the y axis.

Longitude of the launch location does not play a role in the motion because the axial symmetry (the axis being the axis of Earth's rotation) makes it irrelevant. For the Coriolis force, the thing that matters is the size of the velocity relative to the axis; it is not important from which side of the axis the projectile comes. Therefore, only latitude is important. The crucial assumption here is that the Earth is an ideal sphere. If it were not, changing the latitude may change the distance to the axis of Earth's rotation which would have effect on the Coriolis force.

Throughout the analysis I made some implicit, though important assumptions that simplify the calculation. Let's justify them.

1. First of all, I assumed that the Earth is a perfect sphere. This assumption simplifies the calculation of the latitude of the launch location. The assumption is legitimate since the deviation from the perfect sphere for Earth is about 0.3 %[1] and therefore lies within the desired accuracy.

2. Because the range achieved by the projectile is much smaller than the Earth's circumference, namely only about 0.025 % of Earth's circumference (the range being about 10 km and the circumference 40 000 km), it is acceptable to neglect the Earth's curvature. This allows one to assume a flat ground as well as to use a constant direction of the gravitational acceleration, which simplifies the calculation.

3. For the same reason as in point 2, it is acceptable to take latitude as constant during the flight. The change in latitude during the flight is approximately 1° per 111 km displacement to the North or South. In my case, the maximum range is approximately 10 km which causes a change in latitude of approximately 0.1°.

4. I used constant gravitational acceleration at all altitudes which is acceptable since the projectile reaches heights on the order of 10 km, where the gravitational acceleration is 0.3 % smaller than at sea level.

5. I assumed that the air density is constant at all altitudes. This is not true, as the air density falls from 1.2 $\frac{kg}{m^3}$ at sea level to only 0.6 $\frac{kg}{m^3}$ at

7000 m above sea level[9]. This causes a decrease in the magnitude of the acting drag force at these altitudes by about a factor of two which brings significantly different results. Therefore, taking into account this effect of decreasing air density with height would yield more accurate results.

It is interesting to observe how Coriolis displacement depends on parameters v_0, S, m. These dependences are presented with graphs in section Appendix.

There are some useful simulations of the projectile motion on the internet. One of them is PhET's simulation, created by University of Colorado Boulder, and is accessible at https://phet.colorado.edu/en/simulations/projectile-motion. One can play with it, change the parameters and see how the trajectory of the motion itself changes.

5 Conclusion

Analysis of projectile motion under windy conditions and under the effect of the Coriolis force offers plenty of space to explore. The most straightforward way of dealing with the problem is by solving it numerically with a computer. I used an incremental approach, dividing the time of flight into many short time intervals and updating the quantities in every interval. This approach allowed me to obtain results in a relatively simple and timely matter. There are also other, more advanced numerical approaches to this problem. For example, it is possible to obtain the same results with the use of the Python library SciPy, which offers the possibility to numerically integrate ordinary differential equations. Projectile motion with quadratic air resistance in general has no analytical solution[5].

There is a lot of room for further research on the topic of projectile motion. It would definitely be appropriate to take into account the effect of decreasing air density with altitude. Additionally, it would be interesting to analyse higher-speed flights, where it is necessary to take into account the Earth's curvature and change of latitude. One could also use variable wind, whirlwinds, and tornados. Another potential topic would be to add thrust to the projectile. Additionally, one could involve the Magnus effect[3], where the path of a spinning object deflects when moving through a fluid.

It would be interesting to present dependences of different quantities such as range, Coriolis displacement etc., on the parameters of the projectile.

6 Appendix

This section presents the reader with different graphs which give a deeper insight into the nature of projectile motion. Graphs present dependences of different quantities on v_0, S and m. In every graph, the variable quantities are those on the graph, all other quantities are fixed and their values are equal to the ones defined in Introduction. Note: some graphs have logaritmic scales.

6.1 Range at launch angle $\phi = 60°$

(referring to section Air Resistance)

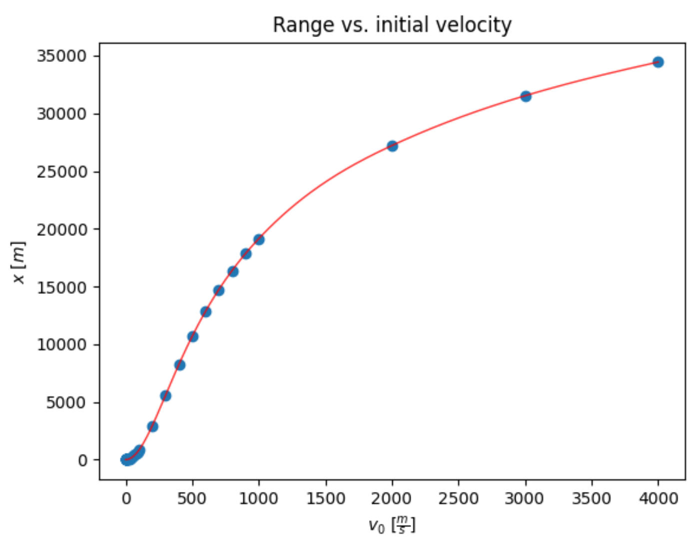

Figure 10: Range x vs. initial velocity v_0

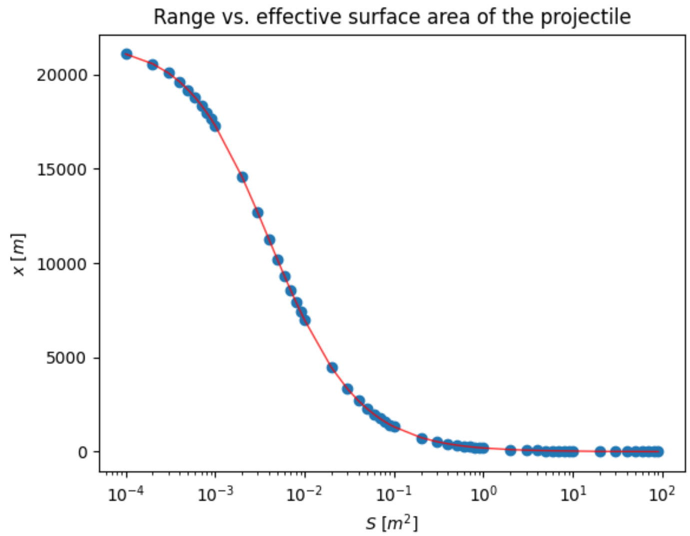

Figure 11: Range x vs. effective surface area S

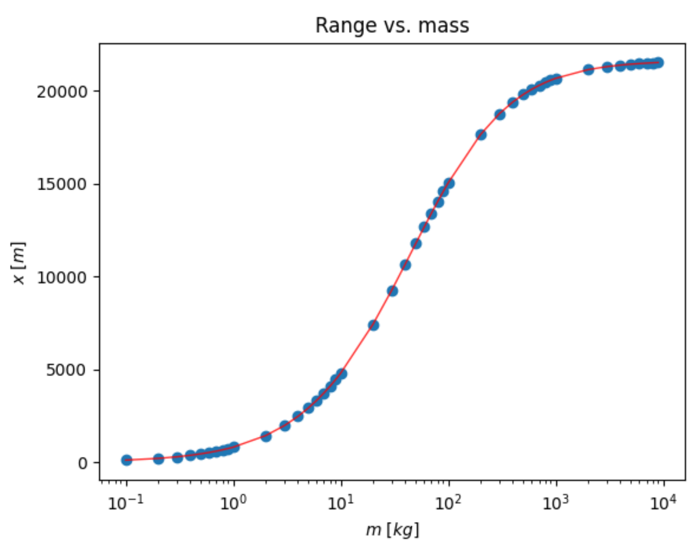

Figure 12: Range x vs. mass m

6.2 Maximum range launch angle

(referring to section Air Resistance)

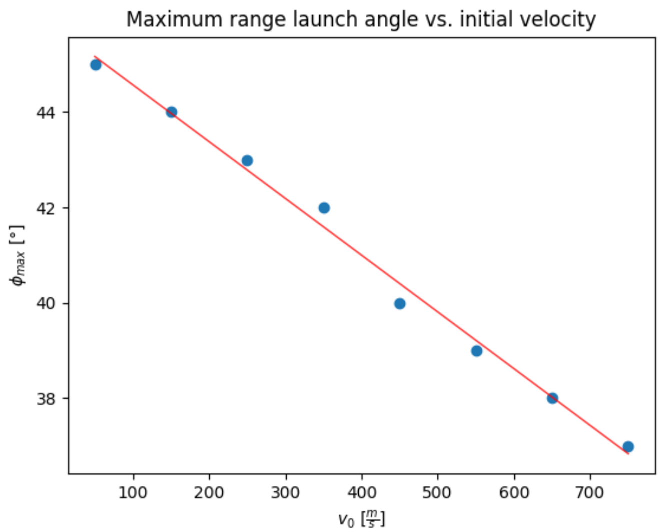

Figure 13: Maximum range launch angle ϕ_{max} vs. initial velocity v_0

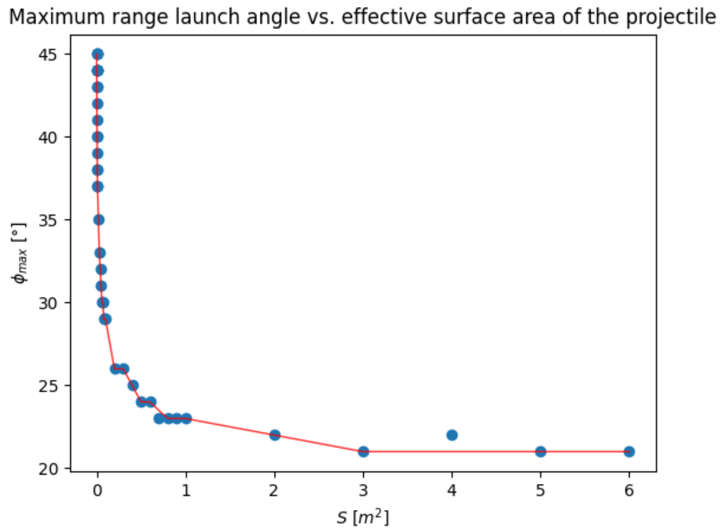

Figure 14: Maximum range launch angle ϕ_{max} vs. effective surface area of the projectile S

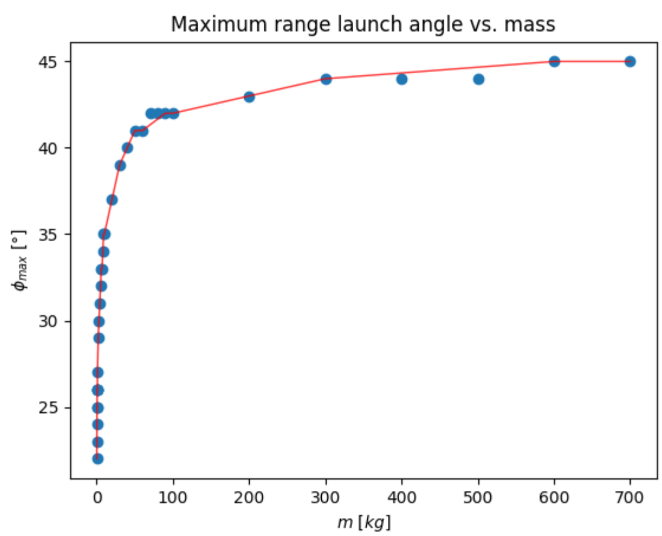

Figure 15: Maximum range launch angle ϕ_{max} vs. mass m

6.3 Size of the surface area of hits at launch angle $\phi = 60°$

(referring to section Wind)

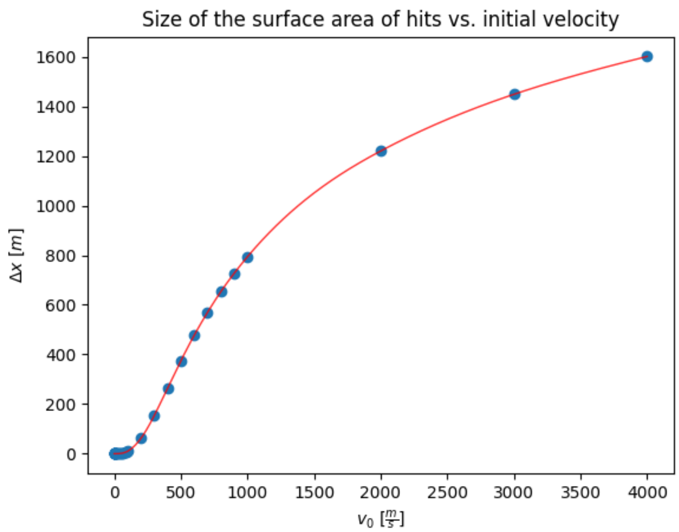

Figure 16: Size of the surface area of hits Δx along x axis vs. initial velocity v_0

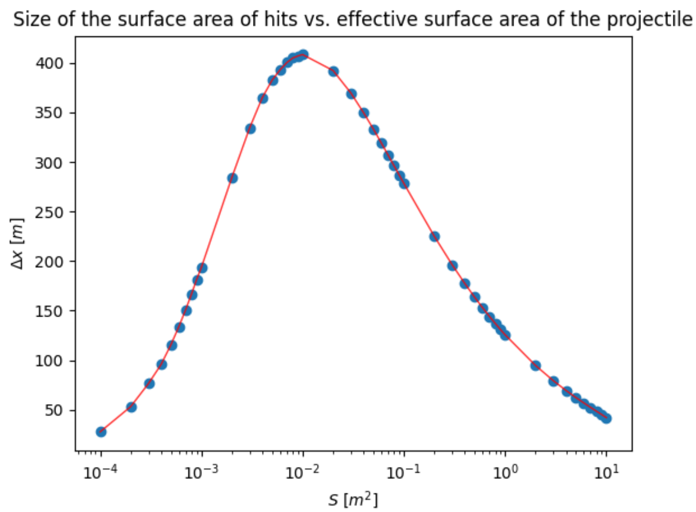

Figure 17: Size of the surface area of hits Δx along x axis vs. effective surface area of the projectile S

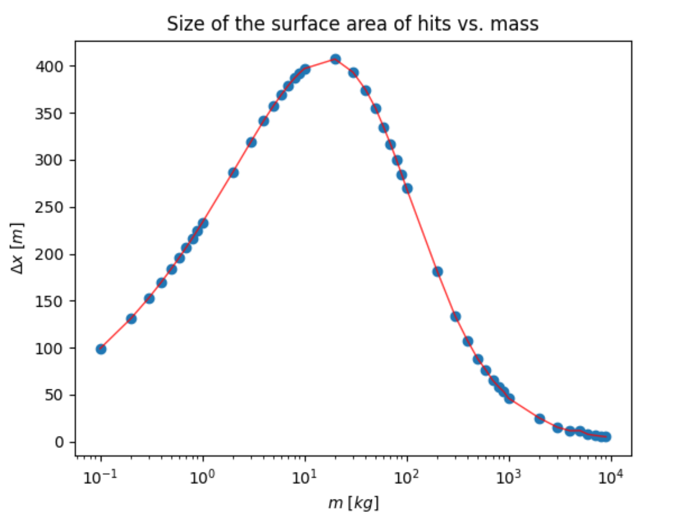

Figure 18: Size of the surface area of hits Δx along x axis vs. mass m

6.4 Displacement because of the Coriolis effect at launch angle $\phi = 60°$

(referring to section Coriolis Force)

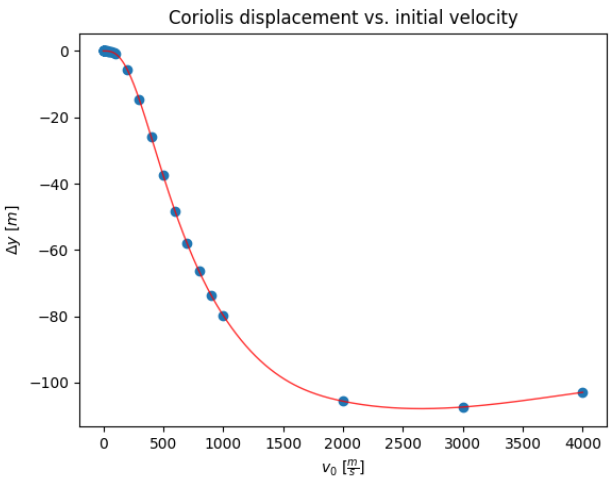

Figure 19: Displacement because of the Coriolis effect Δy along y axis vs. initial velocity v_0

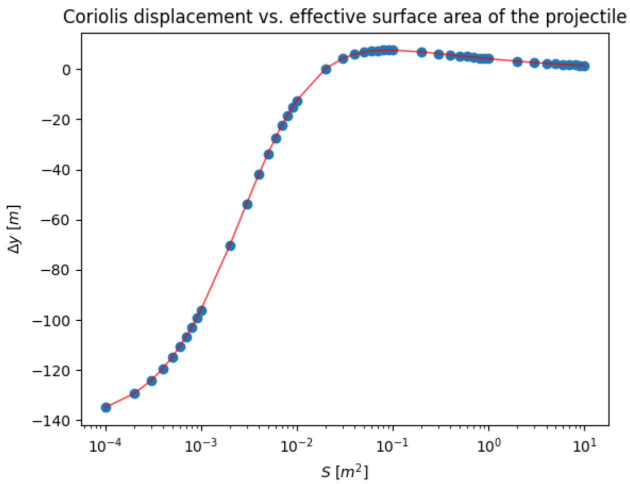

Figure 20: Displacement because of the Coriolis effect Δy along y axis vs. effective surface area of the projectile S

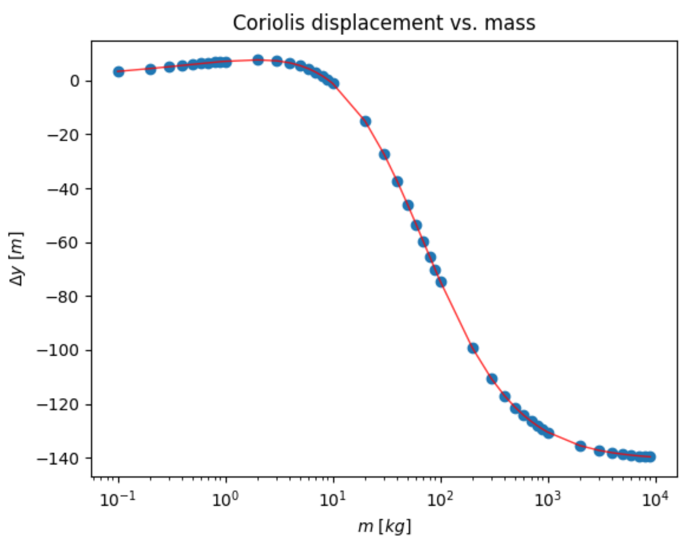

Figure 21: Displacement because of the Coriolis effect Δy along y axis vs. mass m

7 References

[1] *'Figure of the Earth'*. Wikipedia. Available at https://en.wikipedia.org/wiki/Figure_of_the_Earth (Accessed: 27 August 2022).

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[9] *'U.S. Standard Atmosphere vs. Altitude'*. Available at https://www.engineeringtoolbox.com/standard-atmosphere-d_604.html (Accessed: 28 August 2022).