# Handout: Causal Directed Acyclic Graphs

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### Elements

Table 1: Our variable naming conventions

**Variable naming conventions**

| **Symbol** | |
|---|---|
| $X$ | Any variable. |
| $A$ | The treatment or, equivalently, the exposure. |
| $Y$ | The outcome. |
| $L$ | Measured confounder(s): typically comprises a set of variables. |
| $U$ | Unmeasured confounder. |
| $Z$ | Effect-modifier (or 'moderator') of $A$ on $Y$. |
| $M$ | Mediator of $A$ on $Y$. |
| $\bar{X}$ | Sequential variables, e.g. $\bar{A} = \{A_1, A_2, A_3\}$; $\bar{L} = \{L_0, L_1, L_2\}$. |
| $\mathscr{R}$ | Denotes randomisation into treatment event. |

Table 2: Undirected Graphs

**Undirected Graphs Describe Conditional Dependence/Independence of Probability Distributions**

| Graph | Feature |
|---|---|
| X | Node. |
| —— | Edge/path. |
| $X \quad X'$ | Disconnected nodes are not associated. |
| $X — X'$ | Nodes connected by a path are associated. |
| $X — X' — X''$ | All nodes connected by a path are symmetrically associated. |
| $X — X' \quad X''$ | These nodes are symmetrically associated. |
| $X — X' \quad X'' \quad X'''$ | $X'''$ is not associated with other nodes. |

Table 3: Undirected Graphs

**Elements of a Causal Directed Acyclic Graph: D-separation Rules**

| Type | Graph | Explanation | d-Separation Rule |
|---|---|---|---|
| | | **Two variables** | |
| **Causality Absent** | $A \quad B$ | $A$ and $B$ have no causal effect on each other. | $A \coprod B$ (independent) |
| **Causality** | $A \rightarrow B$ | $A$ causally affects $B$, and they are associated. | $A \not\coprod B$ (dependent) |
| | | **Three variables** | |
| **Fork** | $A \rightarrow B \quad C$ | $A$ causally affects both $B$ and $C$; $B$ and $C$ are conditionally independent given $A$. | $B \coprod C \| A$ |
| **Chain** | $A \rightarrow B \rightarrow C$ | $C$ is affected by $B$ which is, in turn, affected by $A$; $A$ and $C$ are conditionally independent given $B$. | $A \coprod C \| B$ |
| **Collider** | $A \rightarrow C \leftarrow B$ | $C$ is affected by both $A$ and $B$, which are independent; conditioning on $C$ induces association between $A$ and $B$. | $A \not\coprod B \| C$. |

Nodes are indexed by temporal order, e.g., $X_t$ precedes $X_{t+1}$. A boxed node, $\boxed{X_t}$, indicates conditioning on that variable. $\longrightarrow$ denotes a directed causal path. The symbol $\coprod$ signifies conditional independence, assessed with the conditioning bar '|'. According to the rules of d-separation, in the fork graph, conditioning on $A$ blocks the association between the non-adjacent nodes $B$ and $C$; in a chain graph, conditioning on $B$ blocks the association between the non-adjacent nodes $A$ and $C$. In a collider graph, conditioning on ($\boxed{C}$) or its descendant ($\boxed{D}$) introduces an association between the non-adjacent nodes $A$ and $B$. These elementary graphs and rules help to clarify strategies for causal identification from the assumptions encoded in graphs.

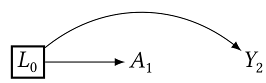
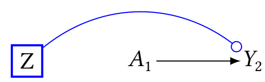
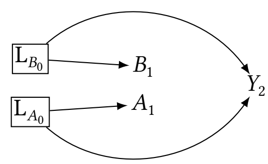
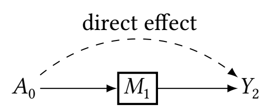
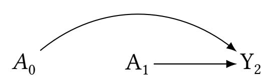
Table 4: Directed Graphs

**Causal Diagram Graphical Conventions**

| Symbol | Meaning | Example |
|---|---|---|
| | **Statistical Machinery** | |
| $A \coprod B$ | **Statistical independence (unconditional)** | $A \coprod Y(a)$ |
| $A \not\!\coprod B$ | **Statistical dependence (unconditional)** | $A \not\!\coprod Y(a)$ |
| $A \coprod B \mid C$ | **Conditional statistical independence** | $A \coprod Y(a) \mid L$ |
| $A \not\!\coprod B \mid C$ | **Conditional statistical dependence** | $A \not\!\coprod Y(a) \mid L$ |
| | **Graphical Machinery** | |
| $X$ | **Node:** variable, denoted by a letter | $A$ |
| $X_t$ | **Time-indexed node**: denotes relative chronology | $A_1 \quad Y_2$ |
| $X_{\phi_t}$ | **Time-indexed node assumed but not known**: relative chronology **asserted**. | $A_{\phi 1} \quad Y_{\phi 2}$ |
| $\rightarrow$ | **Path with an arrow** causal association | $A_1 \rightarrow Y_2$ |
| $\rightarrow$ | **Red arrow:** pathway of bias | $L_0 \rightarrow A_1 \quad Y_2$ |
| $\dashrightarrow$ | **Dashed arrow:** causal effect not through a mediator (direct effect). | direct effect<br>$A_0 \rightarrow \boxed{M_1} \rightarrow Y_2$ |
| $\dashrightarrow$ | **Dashed red arrow:** pathway of biased causal association. | attenuated total effect<br>$A_0 \rightarrow \boxed{L_1} \rightarrow Y_2$ |
| $-\!\circ$ | **Effect-modification path** We assume $A \rightarrow Y$ and focus on the modification within levels of another variable.**Blue path is not evaluated for causality and need have a causal interpretation**. | $Z \quad A_1 \rightarrow Y_2$ |
| $\boxed{X}$ | **Boxed variable:** conditioning/adjustment | $\boxed{L_0} \rightarrow A_1 \quad Y_2$ |
| $\boxed{X}$ | **Red boxed variable** variable that when conditioned upon induces bias. | $A_1 \rightarrow \boxed{L_3} \leftarrow Y_2$ |
| $X$ | **Dashed circle:** no adjustment for variable | unbiased total effect<br>$A_0 - L_1 \rightarrow Y_2$ |
| $\mathcal{R} \rightarrow A$ | **Randomisation into treatment:** such that $A \coprod Y(a) \mid \mathcal{R}$ | $\mathcal{R} \rightarrow A_1 \quad Y_2$ |

Table 5: Common causal interests

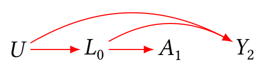
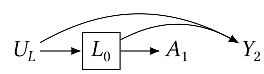
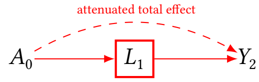
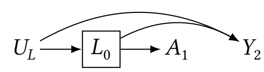
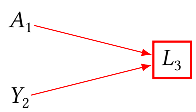
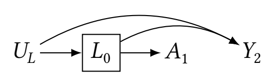
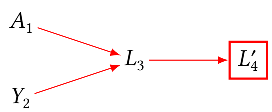
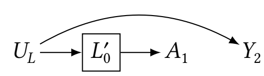
**Examples of Common Causal Questions Presented as Causal DAGs**

| **Interest** | **Causal Directed Acyclic Graph** |
|---|---|
| **Treatment effect**: after controlling for all common causes, what is the causal effect of $A \longrightarrow Y$? |   |
| **Effect Modification (Moderation)**: does the effect of $A \longrightarrow Y$ vary across levels of $Z$? |   |
| **Interaction (joint-intervention)**: after controlling for all common causes of $A \longrightarrow Y$ and $B \longrightarrow Y$, is the combined effect of $A$ and $B$ on $Y$ different from their individual effects? |   |
| **Mediation**: assuming unbiased confounding control, what are the separable effects of $A \longrightarrow Y$ (a): indirectly through $M$ and (b) directly not through $M$? |   |
| **Time-varying exposures**: What are the causal effects of clearly specified sequential treatments? $\bar{A} \longrightarrow Y$? |   |

**Key**:
$A$ denotes the treatment;
$B$ denotes a second treatment (as in interaction analysis).
$Y$ denotes the outcome;
$L$ denotes a confounder;
$Z$ denotes a modifier of for the effect of $A \longrightarrow Y$
$\boxed{X}$ indicates conditioning on variable $X$.
$\longrightarrow$ asserts causality
$-\!\!\circ$ indicates an interest in effect-modification (causality need not be asserted);
**Note 1**: when analysing effect-modification, our focus is on the $A \longrightarrow Y$ pathway, and whether this effect varies within levels of $Z$; we do not estimate $Z$'s causal effect (if any) on $Y$. We use the agnostic arrow $-\!\!\circ$ to denote this specific interest in $Z$ in relation to $A \longrightarrow Y$.
**Note 2**: interaction analysis requires identifying two causal effects ($A \longrightarrow Y$ and $B \longrightarrow Y$) and controlling for their common causes. Mediation analysis, not interaction analysis, is needed if $A$ and $B$ influence each other ($B \longrightarrow A$ or $A \longrightarrow B$).

Table 6: Elementary confounding scenarios

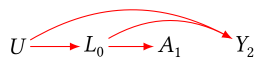
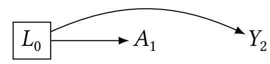
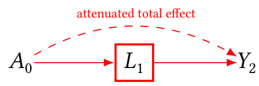
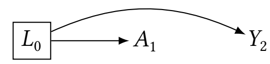
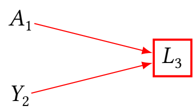
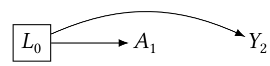
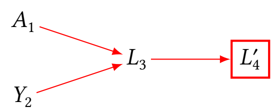
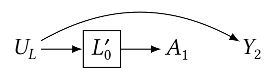
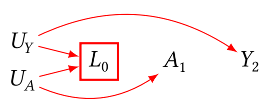
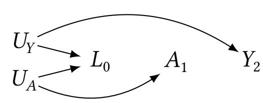
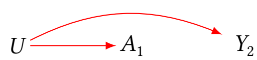
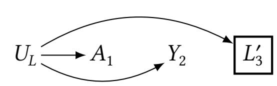
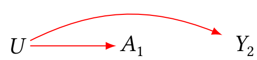
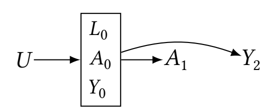
**Elementary Confounding**

| | Bias | Problem | Sequential data solution |
|---|---|---|---|
| 1 | **Paradigmatic confounding**: $A$ and $Y$ share both measured and unmeasured common causes; we condition to block the open backdoor path. |  |  |
| 2 | **Mediator bias**: inaccurate timing: $\boxed{L}$ blocks true causal association $A \to Y$. |   |  |
| 3 | **Collider**: inaccurate timing: $\boxed{L}$ creates path from $A \dashrightarrow Y$. |  |  |
| 4 | **Collider by descent**: inaccurate timing: confounder proxy $\boxed{L'}$ creates path from $A \dashrightarrow Y$. |  |  |

**Key**:
$A$ denotes the treatment;
$Y$ denotes the outcome;
$U$ denotes an unmeasured confounder;
$L$ denotes a confounder;
$\longrightarrow$ asserts causality
$\longrightarrow$ indicates a pathway for bias linking $A$ to $Y$ absent causation.
$\boxed{X}$ indicates that conditioning on $X$ introduces bias.
Where $\phi t$ denotes assumed confounding, examples 2-4 illustrate how errors in $L_{\phi_t} \neq L_t$ lead to confounding.

Table 7: Common confounding scenarios with timing

**Examples of Confounding Focussing on the Timing of Measured Confounders**

| | | | |
|---|---|---|---|
| | **Simple scenarios** | | |
| | **Bias** | **Confounding** | **Address with variable timing** |
| 1 | **Common cause**: common cause creates association absent causation. |  |  |
| 2 | **Mediator bias**: conditioning on a mediator biases total effect. |  |  |
| 3 | **Collider**: conditioning on a common effect creates association absent causation. |  |  |
| 4 | **Collider by descent**: conditioning on child of collider induces confounding-by-proxy. |  |  |
| | **More complex scenarios** | | |
| | **Bias** | **Confounding** | **Use DAG to find strategy** |
| 5 | **M-bias**: over-conditioning bias: do not condition on $L$. |  |  |
| 6 | **Unmeasured common cause**: conditioning on a measured proxy of unmeasured common cause enables creative confounding control. |  |  |
| 7 | **Unmeasured common cause**: condition on baseline treatment and outcome to reduce unmeasured confounding and assess incident effect. |  |  |

$A$ denotes the treatment;
$Y$ denotes the outcome;
$U$ denotes an unmeasured confounder;
$L$ denotes a confounder;
$L'$ denotes a proxy for an unmeasured confounder;
$\to$ indicates a pathway for bias linking $A$ to $Y$ absent causation. For example where $L_0 \to A_1 \to Y_2$, the biasing pathway flows from $A_1$——$L_0$——$Y_2$.
$\boxed{L}$ indicates conditioning on variable $L$ eliminates or reduces;
$\boxed{L}$ indicates that conditioning on $L$ introduces bias.
To denote relative timing we place subscripts on nodes (e.g., $L_0, A_1, Y_2$)
To denote assumed relative timing we add $\phi$ to these subscripts (e.g. $L_{\phi 0}, A_{\phi 1}, Y_{\phi 2}$)
**Examples 1-4** demonstrate how incorrect assumptions about variably timing $\phi t$ leads to bias.
**Examples 5-7** demonstrate possibilities for confounding that are not solved by longitudinal data collection.