Gaussian process regression methods and extensions for stock market prediction

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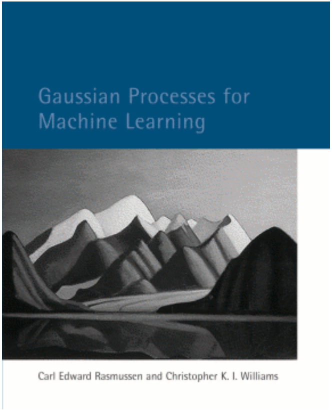
Overview

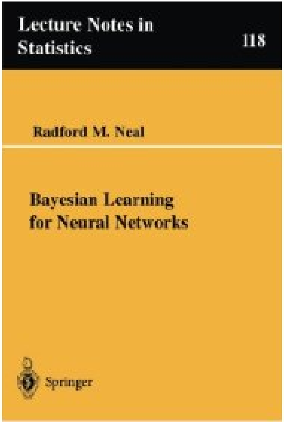
1. Thesis structure
2. Gaussian process regression (GPR)
   - Kernel and mean function
   - Parameter estimation
3. Gaussian process regression extensions
   - GPR extensions
   - Applications of GPR and its extensions
4. Further extensions for multi-output prediction
   - Multivariate Gaussian process regression (MV-GPR)
   - Multivariate Student$-t$ process regression (MV-TPR)
   - Applications of MV-GPR and MV-TPR
5. Summary

Thesis structure

- Gaussian process regression (GPR)
    - Model derivations, covariance function (kernel) and mean function
    - **Parameter estimation**
        - **Initial hyper-parameter selection (my work)**
- Gaussian process regression extensions and applications
    - Model extensions, including Gaussian process regression with Student$-t$ likelihood (GPRT), Student$-t$ process regression (TPR), state-space GPR (SSGPR), state-space TPR (SSTPR)
    - **Equity index prediction (my work)**
- **GPR extensions for multi-output prediction (my work)**
    - Multivariate Gaussian process regression (MV-GPR)
    - Multivariate Student$-t$ process regression (MV-TPR)
    - Simulated examples and stock market investment

Gaussian process for machine learning





Gaussian process

A Gaussian process is a collection of random variables, any finite number of which have (consistent) Gaussian distribution.

Theorem (Gaussian process)

For any set S, any mean function $\mu : S \mapsto \mathbb{R}$ and any covariance function $k : S \times S \mapsto \mathbb{R}$, there exists a GP $f(t)$ on S, s.t., $\mathbb{E}[f(t)] = \mu(t)$, $\mathrm{cov}(f(s), f(t)) = k(s, t), \forall s, t \in S$. It denotes $f(t) \sim \mathcal{GP}(\mu, k)$.

A collection of functions $[f(t_1), f(t_2), \ldots, f(t_n)]$ are realised through a multivariate Gaussian distribution

$$[f(t_1), f(t_2), \ldots, f(t_n)]^{\mathrm{T}} \sim \mathcal{N}(\boldsymbol{\mu}, K)$$

where $\mu_i = \mu(t_i)$ and $K_{ij} = k(t_i, t_j)$.

Gaussian process regression

- Consider a noisy model $y = f(x) + \varepsilon$, where f is drawn from a GP and ε is Gaussian noise. Then, we wish to predict new observations $\boldsymbol{y}_*$ for new test points Z.
- The joint distribution of training observations $\boldsymbol{y}$ and test data $\boldsymbol{y}_*$ is

$$\begin{bmatrix} \boldsymbol{y} \\ \boldsymbol{y}_* \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \mu(X) \\ \mu(Z) \end{bmatrix}, \begin{bmatrix} K(X,X) + \sigma_n^2 I & K(Z,X)^{\mathrm{T}} \\ K(Z,X) & K(Z,Z) \end{bmatrix} \right),$$

- The predictive distribution is $p(\boldsymbol{y}_* | X, \boldsymbol{y}, Z) \sim \mathcal{N}(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\Sigma}})$, where

$$\hat{\boldsymbol{\mu}} = K(Z,X)^{\mathrm{T}}(K(X,X) + \sigma_n^2 I)^{-1}(\boldsymbol{y} - \mu(X)) + \mu(Z), \tag{1}$$

$$\hat{\boldsymbol{\Sigma}} = K(Z,Z) - K(Z,X)^{\mathrm{T}}(K(X,X) + \sigma_n^2 I)^{-1} K(Z,X). \tag{2}$$

Kernel and mean function

Kernels

- Basic kernels, such as, linear (LIN), squared exponential (SE), periodic (PER), etc.
- Composite kernels, such as local periodic (LP), spectral mixture (SM), etc. [Wilson, 2014]

Mean functions

- Zero-offset (mainstream) [Roberts et al., 2013]
- Linear
- Others

Maximum marginal likelihood

Sensitivity of initial hyper-parameters

- Negative log marginal likelihood (nlml) is not always convex, it may suffer a local optima
- Initial hyper-parameters selection may affect the final result

A common approach to overcome the sensitivity

- Randomly select several initial hyper-parameters in a reasonable range
- Compute nlmls for these initial hyper-parameters
- Select the best initial hyper-parameters with the smallest nlml

Problem: how to randomly select ?

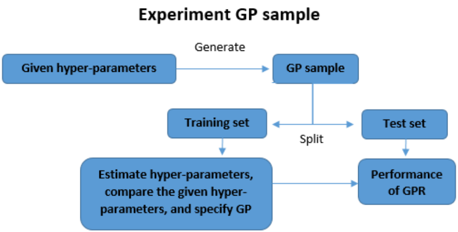
Problem: how to randomly select ?

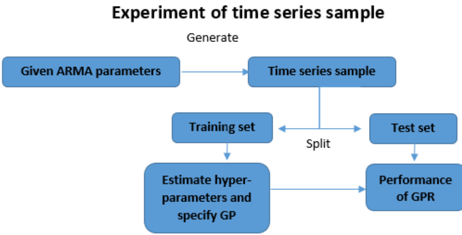
- Researchers' expert opinions and experiences
- Uniform (0,1) (mainstream)
- $\mathcal{N}(0,1)$
- Other simple distributions

My work

- **Put different priors on initial hyper-parameters**
- **Explore how priors of initial parameters affect the estimation**
- **Study how priors of initial parameters affect the performance of GPR model**

How priors of initial hyper-parameters affect GPR models





Experimental results [Chen and Wang, 2016]

- The hyper-parameter estimation depends on the choice of kernels
    - SE: robust
    - PER: very sensitive
- The performance of GPR is considerable regardless of the prior choice

GPR extensions

Fat-tailed distributions

- Student$-t$ distribution
- Pareto distribution
- Lévy distribution (Lévy walk [Fedotov and Korabel, 2017])
- Other stable distributions

Gaussian process regression with Student$-t$ likelihood (GPRT)

$y = f + \varepsilon$, where f follows a GP and ε follows Student$-t$ distribution [Vanhatalo et al., 2009].

Student$-t$ process regression (TPR)

$y = f$, where f follows a Student$-t$ process (TP) [Shah et al., 2014].

State-space GPR (SSGPR) and state-space TPR (SSTPR)

Consider the historical prices/indices as state in GPR and TPR model.

Some applications (my work)

Equity index series predictions with fixed forecasting horizon

- INDU, NDX, SPX, and UKX predictions using GPR, GPRT, and TPR
- DAX, HSI, INDU, NDX, NKY, SENSEX, SPX, UKX predictions using GPR, GPRT, TPR, and ARMA model

Result: GPR and TPR perform relatively better overall

Model validation based on equity index predictions

- Objective: ten equity indices, including DAX, HSI, INDU, NDX, NKY, SENSEX, SHSZ300, SPX, UKX, XU100.
- Methods: leave-one-out cross-validation, k-fold validation ($k = 10$) and sliding windows (using GPR and TPR).
- **Result: Overall both GPR and TPR perform well and the superiority of them depends on the data set.**

Some applications (my work)

State-space models for equity index predictions

- Objective: ten equity indices, including DAX, HSI, INDU, NDX, NKY, SENSEX, SHSZ300, SPX, UKX, XU100.
- Methods: sliding windows using State-space Gaussian process regression (SSGPR) and State-space Student$-t$ process regression (SSTPR) [Brahim-Belhouari and Bermak, 2004, Girard et al., 2003]
- **Result: SSTPR performs better than SSGPR overall**

Stock market efficiency analysis

- Data: sliding windows predictions using SSGPR and SSTPR
- **Result: the developed markets are efficient on the whole while sometimes some emerging markets are also unpredictable, for example, Chinese market**

Multi-output prediction

- Chinese company in NASDAQ
- Internet sector

How can we make multi-output predictions by extending GPR?

Gaussian process for vector-valued function

Main steps in the existing methods [Boyle and Frean, 2005, Alvarez et al., 2011, Wang and Chen, 2015]

- Vectorise the multi-output variables(convert a matrix to a vector) $\mathrm{vec}[(\boldsymbol{f}^{\mathrm{T}}(\boldsymbol{x}_1), \cdots, \boldsymbol{f}^{\mathrm{T}}(\boldsymbol{x}_n))^{\mathrm{T}}]$, where $\boldsymbol{f} : \mathcal{X} \mapsto \mathbb{R}^d$ is a vector-valued function and $\mathrm{vec}[\cdot]$ is vector operator
- Construct a new "big" covariance matrix, usually Kronecker product, $\Omega \otimes K$), where $K \in \mathbb{R}^{n \times n}$ is still a covariance matrix and $\Omega \in \mathbb{R}^{d \times d}$ captures the covariance of outputs
- Assume $\mathrm{vec}[(\boldsymbol{f}^{\mathrm{T}}(\boldsymbol{x}_1), \cdots, \boldsymbol{f}^{\mathrm{T}}(\boldsymbol{x}_n))^{\mathrm{T}}] \sim \mathcal{N}(\mathbf{0}, \Omega \otimes K)$
- Operate the remaining steps similar to the classical GPR

Essential assumption

$$(\boldsymbol{f}^{\mathrm{T}}(\boldsymbol{x}_1), \cdots, \boldsymbol{f}^{\mathrm{T}}(\boldsymbol{x}_n))^{\mathrm{T}} \sim \mathcal{MN}(\mathbf{0}, K, \Omega).$$

Problem and solution

Essential property

Matrix-variate Gaussian distribution can be reformulated as multivariate Gaussian distribution [Gupta and Nagar, 1999].

Can we use this method to extend multi-output TPR? **NO !**
Can we have another way to derive multi-output GPR, which can be easily extended to multi-output TPR? **YES !**

Standard solution of GPR and TPR for multi-output prediction (my work)

- Define a multivariate Gaussian (Student$-t$) process
- Carry out regression model (matrix form) with the assumption of the multivariate Gaussian (Student$-t$) process directly
- Estimate the parameters directly (matrix computation)
- Make predictions directly (matrix computation)

Multivariate Gaussian process regression (MV-GPR)

Given n pairs of observations $\{(\boldsymbol{x}_i, \boldsymbol{y}_i)\}_{i=1}^n, \boldsymbol{x}_i \in \mathbb{R}^p, \boldsymbol{y}_i \in \mathbb{R}^d$, assuming

$$\begin{aligned} \boldsymbol{f} &\sim \mathcal{MGP}(\mathbf{0}, k', \Omega), \\ \boldsymbol{y}_i &= \boldsymbol{f}(\boldsymbol{x}_i), \text{for } i = 1, \cdots, n, \end{aligned}$$

where $k' = k(\boldsymbol{x}_i, \boldsymbol{x}_j) + \delta_{ij}\sigma_n^2$, $\delta_{ij} = 1$ if $i = j$, otherwise $\delta_{ij} = 0$. The joint distribution of the training observations Y and the predictive targets $\boldsymbol{y}_*$

$$\begin{bmatrix} Y \\ \boldsymbol{y}_* \end{bmatrix} \sim \mathcal{MN}\left(\mathbf{0}, \begin{bmatrix} K'(X,X) & K'(Z,X)^{\mathrm{T}} \\ K'(Z,X) & K'(Z,Z) \end{bmatrix}, \Omega\right), \tag{3}$$

Therefore, the predictive distribution is

$$p(\boldsymbol{y}_*|X, Y, Z) = \mathcal{MN}(\hat{M}, \hat{\Sigma}, \hat{\Omega}), \tag{4}$$

where

$$\begin{aligned} \hat{M} &= K'(Z,X)^{\mathrm{T}} K'(X,X)^{-1} Y, & (5) \\ \hat{\Sigma} &= K'(Z,Z) - K'(Z,X)^{\mathrm{T}} K'(X,X)^{-1} K'(Z,X), & (6) \\ \hat{\Omega} &= \Omega. & (7) \end{aligned}$$

Comparison

Table 1: The comparison of existing multi-output GPR and MV-GPR

	Existing multi-output GPR	MV-GPR
Predictive distribution	Same	
Formulation	Complicated	Concise
Extension	Non-exist	MV-TPR

Multivariate Student−t process regression (MV-TPR)

Given n pairs of observations $\{(\boldsymbol{x}_i, \boldsymbol{y}_i)\}_{i=1}^{n}, \boldsymbol{x}_i \in \mathbb{R}^p, \boldsymbol{y}_i \in \mathbb{R}^d$, we assume

$$\begin{aligned} \boldsymbol{f} &\sim \mathcal{MTP}(\nu, \boldsymbol{u}, k', \Omega), \nu > 2, \\ \boldsymbol{y}_i &= \boldsymbol{f}(\boldsymbol{x}_i), \text{for } i = 1, \cdots, n, \end{aligned}$$

Therefore the predictive distribution is

$$p(\boldsymbol{f}_*|X, Y, Z) = \mathcal{MT}(\hat{\nu}, \hat{M}, \hat{\Sigma}, \hat{\Omega}), \tag{8}$$

where

$$\begin{aligned} \hat{\nu} &= \nu + n, && (9) \\ \hat{M} &= K'(Z, X)^{\mathrm{T}} K'(X, X)^{-1} \boldsymbol{y}, && (10) \\ \hat{\Sigma} &= K'(Z, Z) - K'(Z, X)^{\mathrm{T}} K'(X, X)^{-1} K'(Z, X), && (11) \\ \hat{\Omega} &= \Omega + Y^{\mathrm{T}} K'(X, X)^{-1} Y. && (12) \end{aligned}$$

Experiments and results

- Simulated data prediction

	Simulated data with correlated	
	Gaussian noise	Student$-t$ noise
Perform	MV-TPR>MV-GPR>TPR>GPR	MV-TPR>MV-GPR>TPR>GPR

- Stock market investments

(1)	Three Chinese stocks investment in NASDAQ		
	BIDU	NTES	CTRP
Result	Insignificant difference	Insignificant difference	MV-TPR best

(2)	Stocks investment from diverse sectors in Dow 30						
	Oil&Gas	Industrial	Goods	Health Care	Services	Financials	Technology
Best model	TPR	TPR	MV-GPR	MV-TPR	MV-GPR	MV-GPR	GPR

What have I done in this thesis?

- Reviews
  - GPR (model derivation, kernel, mean function, parameter estimation)
  - GPR extensions, including TPR, GPRT, SSGPR and SSTPR
- Further studies and developments
  - Initial hyper-parameters selection (put different priors on the initials)
  - Another derivation of the existing dependent Gaussian process (MV-GPR)
  - New considerable method for multi-output prediction (MV-TPR)
- Applications
  - Predict many equity indices (GPR and its extensions)
  - Make stock market investment based on Buy&Sell strategy (MV-GPR and MV-TPR)

What have I achieved in this thesis?

- Study the sensitivity of initial hyper-parameters and find that
  - the initial hyper-parameters' estimates depend on the choice of kernel
  - the performance of GPR is satisfactory in terms of predictability regardless of the prior selection
- Apply GPR and its extension to equity index prediction and find that
  - GPR and TPR have relatively considerable capability of predicting performances
  - the developed markets are efficient overall
- Propose MV-GPR and MV-TPR and find that both MV-GPR and MV-TPR
  - have considerable predictions on the simulated data, particularly MV-TPR performs better
  - are shown profitable in stock market investment

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