Relational Incentive Contracts with Hidden Action and Unequal Discounting*

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September 3, 2024

Abstract

We analyze relational incentive contracts with hidden action when the principal and the agent have different discount factors, and identify a new agency cost arising from a trade-off between rent extraction and incentive provision, that exists only when they cannot commit themselves to long-term contracts and the principal is less patient than the agent. We characterize the condition under which the trade-off between rent extraction and incentive provision exists, as well as the condition for the optimality of non-stationary contracts.

1 Introduction

The purpose of this paper is to extend the theory of relational incentive contracts with hidden action to cases where the transacting parties have different discount factors. We consider an infinitely repeated principal-agent model in which the agent's binary effort level is unobservable to the principal, and the principal offers an incentive contract in a take-it-or-leave-it fashion to motivate the agent to exert effort. Both the principal and the agent are risk-neutral. The performance measure is non-verifiable, and hence the contract has to be dynamically enforceable, that is, it must satisfy self-enforcing constraints that require any bonus and

*We would like to thank the seminar participants at CTW, the 3rd Workshop on Relational Contracts (Northwestern University), Keio University, Japanese-German Workshop on Contracts and Incentives (Osaka), Queens University, and Waseda University for helpful comments. Itoh acknowledges the financial support from JSPS KAKENHI Grant Numbers 25245031 and 18H03640, as well as Waseda University.

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penalty specified to be bounded above by the principal and the agent's discounted continuation payoffs (net of their reservation utilities), respectively. We derive an optimal incentive contract in a way distinct from existing literature where the principal and the agent are equally patient.

The main contribution of the paper is to elucidate whether and how the standard results on relational contracts under the assumption of equal discounting, such as the optimality of stationary contracts and no rent left for the agent (Levin, 2003), change with the introduction of unequal discount factors. In particular, we identify a *new agency cost* that exists only under relational incentive contracting and unequal discount factors.

While the purpose of the paper is purely theoretical, considering unequal discount factors in the ongoing principal-agent relationship has its practical motivation. The parties usually have different time preferences such that one of them is more patient, expecting their relationship to be longer-lived, or facing a higher cost of capital (such as internal rate of return in capital budgeting). Our analysis, hence, possibly links characteristics of the contracting parties, such as their size, tenure, access to financial markets, and so on, to the optimal contracts between them and the resulting division of surplus.

We allow the principal's discount factor to be either higher or lower than that of the agent. Since patience is one source of bargaining power, one might think that the principal who makes a take-it-or-leave-it offer be more patient than the agent. We however think that situations where the impatient principal can make a take-it-or-leave-it offer are not unreasonable. As Lyon and Rasmusen (2004) argue, an "advantage of economic theory over looser thinking about bargaining is that this definition of bargaining power distinguishes strong bargaining power from a strong bargaining position. (p.151)" For example, the regulator and the CEO, who face elections and other turnover possibilities near future, may be more interested in short-term outcomes than, respectively, the regulated firms strongly interested in survival, and the employees who want to stay with the firm for a long period. And some features of the optimal relational contract we derive, such as "seniority-based," "promotion-like," and no termination, are consistent with the employment practices of the stylized, traditional Japanese firm where the typical CEO has short tenure and the long-term employment is prevailing (Itoh, 1994).

Furthermore, our analysis of the less patient principal applies with minor modifications to the case in which the less patient agent makes a take-it-or-leave-it offer to the principal. For example, the contracting relationship between an entrepreneur and investors is often modeled such that the entrepreneur has scarce and valuable business ideas and unique necessary skills, but can raise money only at a higher interest rate, while lenders behave competitively (see, for example, Tirole, 2006). Therefore in dynamic financial contracting literature, the entrepreneur is often assumed to be more impatient than the financiers but has all the bargaining power at the initial date (see, for example, Biais et al., 2013, for an overview).

We first confine our attention to stationary contracts, and derive the optimal stationary contract that maximizes the principal's expected payoff. When the principal's discount factor is at least as high as a threshold, the optimal contract simply consists of a bonus to provide effort incentives and a fixed salary (or, equivalently, an up-front payment) such that the agent's individual rationality constraint is binding and hence no rent is left to him every period. The principal will not renege on the bonus because she is so patient that the future payoff matters more than the benefit from reneging. While the optimal stationary contract is the same as the standard one under equal discount factors, the way we derive the optimal contract is distinct from that in existing literature because adding up the relevant constraints does *not* eliminate the monetary transfer parts when the discount factors are unequal.

When the principal's discount factor is lower than the threshold, bonuses alone are not enough to provide effort incentives because the principal's continuation payoff is too low for her not to renege on the payment of the bonus needed for effort provision. In this case, however, if instead the agent's discount factor is at least as high as the threshold (and hence the agent is more patient than the principal), it becomes worthwhile to use some penalty to supplement incentives, even though it requires leaving the agent some rent for him not to renege on the payment of the penalty. The trade-off between rent extraction and incentive provision arises even though the agent neither faces the limited-liability constraint nor has private information at the beginning of each period. Intuitively, when the principal is less patient than the agent, transferring payoff to the agent can *increase* the joint continuation payoff and making the implementation easier by relaxing the bound on penalty more than tightening the bound on bonus. Hence, when the principal is impatient enough (so that effort cannot be implemented without giving a rent to the agent) and the agent is patient enough, it becomes optimal for the principal to leave some rent to the agent by using penalty to make up the effort incentive that the bonus is not enough to provide by itself.

Finally, if both parties' discount factors are lower than the threshold, the agent reneges on the payment of the penalty even under the contract that consists of the penalty and a fixed salary only and hence leaves all the rent to the agent.[1]

Stationary contracts are in general suboptimal, however, when the principal

[1]Note that our analysis clearly distinguishes between bonuses and penalties as incentive devices, that are often pointed out as algebraically equivalent unless some psychological elements are taken into account (see, for example, Lazear, 1995). However, in this paper we do not intend to offer any explanation concerning under what conditions these contracts are used in practice.

and the agent have unequal discount factors. There are three reasons for this. The first, well-known reason is that while the principal has to leave a rent to the agent under the optimal stationary contract, she can extract the rent by selling the whole project via the first-period contract with a low enough payment.

Second, the principal and the agent benefit from engaging in intertemporal payoff trading via non-stationary contracts. Because of this second reason, the rent extraction by the initial contract mentioned above cannot be analyzed separately. The optimal intertemporal trading expands the Pareto frontier, so that it is no longer a straight line with slope minus one. Along the Pareto-optimal path, the stage payoff to the more patient party is increasing and that to the less patient party is decreasing over periods. This implies that while the stage payoff to the less patient player cannot drop below his/her reservation payoff, that to the more patient player *can* be lower than his/her reservation payoff for some periods.

If the principal is more patient than the agent, allowing non-stationary contracts does not alter the optimality of a stationary bonus contract because the principal must guarantee the less patient agent at least his reservation payoff every period, and she can attain the maximum stage payoff by a stationary bonus contract. If the principal is less patient, however, intertemporal payoff trading via a non-stationary contract benefits her. The optimal "semi-dynamic" contract, that explores intertemporal payoff trading but provides the agent with effort incentives only by current payments (bonus and penalty), entails a decreasing bonus plan, accompanied with an increasing penalty schedule, in order to ensure the agent's effort incentives. The fixed salary is an increasing, "seniority-based" plan in order to compensate the diminishing payment due to the decreasing bonuses and increasing penalties, as well as to back-load payoffs to the agent and guarantee him larger stage payoffs in later periods.

The third reason why stationary contracts are suboptimal is that the less patient principal may be able to extract more rent from the agent by making his continuation payoffs contingent on the current output, although they then have to deviate from the optimal intertemporal trading and hence their payoff vector is short of the Pareto frontier. We show that if the optimal semi-dynamic contract has to leave some rent to the agent, the optimal "dynamic" contract is a "promotion-like" one, under which his average payoff does not decrease with low output and one-time success (high output) moves his payoff to the one prescribed by the optimal semi-dynamic contract. And termination never occurs on the equilibrium path.

We also obtain several comparative statics results about the agent's rent. The rent is decreasing in the surplus of the relationship, and is increasing in the reservation utilities of both parties. An increase in the surplus of the relationship makes it easier to satisfy the self-enforcing constraints, and hence shifts the optimal contract toward more use of bonuses, thereby reducing the rent to the agent. An increase in the reservation utility of either party makes it harder to satisfy the self-enforcing constraints, and hence increases the rent to the agent via a change of the optimal contract toward more use of penalties. The agent weakly prefers less advanced contracting technology because his rent decreases as the optimal contracting form improves from stationary to semi-dynamic, and from semi-dynamic to dynamic contracts.

Related Literature

In the literature on repeated games, Lehrer and Pauzner (1999) study two-player repeated game with unequal discount factors to show how intertemporal payoff trading expands the Pareto frontier and prove a folk theorem result that outcomes on the frontier can be achieved as equilibria as discount factors converge to one with the relative patience between players fixed. Their folk theorem result is extended to n-player games (Chen and Takahashi, 2012) and to imperfect public monitoring (Sugaya, 2015). We also follow Lehrer and Pauzner (1999) to expand the Pareto frontier, but in our paper it is non-stationary contracts that are used for intertemporal payoff transfer. Fong and Surti (2009) confine their attention to repeated prisoners' dilemma with unequal discounting, but instead extend Lehrer and Pauzner (1999) by introducing the possibility of voluntary side payments at the beginning of each period. Side payments are used to provide incentives to cooperate as well as to trade payoffs across periods. They show that providing incentives for the impatient player and intertemporal payoff trading may conflict with each other, and full cooperation may be Pareto dominated by partial cooperation in which only the patient player chooses cooperation. In contrast to them, we study the principal-agent problem and solve the optimal relational contract to implement effort provision by the agent, where the impatient principal may face a conflict between providing incentives for the patient agent and transferring payoffs intertemporally.

Most literature on relational contracts assumes equal discounting: For example, a survey chapter Malcomson (2013) contains no discussion of unequal discounting. Some of the recent papers on dynamic principal-agent relationships assume that the principal and the agent have different discount factors (Biais et al., 2007, 2013; Opp and Zhu, 2015; Hoffmann et al., 2021; Krasikov et al., 2023). However, they exclusively focus on the case where the principal is more patient than the agent, and hence the principal benefits from *front-loading* rewards to the agent. In these papers, back-loading of the agent's payoffs arises from the features other than unequal discounting, such as limited liability (Biais et al., 2007, 2013), no hidden action (Opp and Zhu, 2015), persistent hidden action (Hoffmann et al., 2021), and persistent private information (Krasikov et al., 2023). And they do not consider the dynamic enforcement issue that the principal and the agent may renege on payments contingent on unverifiable performance measures. In contrast to them, in our model there is no reason to either front-load or back-load the payoffs to the agent if his discount factor is the same as that of the principal. We then highlight how the optimal incentive contract is affected when unequal discounting and the dynamic enforcement condition for incentive contracts interact.

In relational contract literature, the agent may enjoy a positive information rent when the agent's type is his private information (Yang, 2013; Ishihara, 2016; Malcomson, 2016) or a positive limited-liability rent when the agent is protected by the (ex post) limited liability (Fong and Li, 2017). The agent's rent in our model is clearly different from the information rent because there is no adverse selection in our model.

The agent's rent in our model is also different from the limited-liability rent. The latter exists because a bound is imposed on penalties exogenously, and, because there need to be enough difference between the bonus and the penalty to induce effort, the bonus is push up to the point where the agent obtains more than his reservation payoff. The agent enjoys the limited-liability rent under the optimal one-shot contract as well as long-term contract where the principal and the agent commit themselves to all payments (Fong and Li, 2017, Proposition 0). As a result, the rent continues to exist under relational incentive contracting when the principal and the agent are equally patient (their Proposition 1) or when the principal is more patient than the agent (their online appendix). In contrast, the agent receives no rent in all these cases in our paper (See Sections 3 and 6) because we do not assume an exogenous bound for penalties.[2] In our paper, the agent enjoys a positive rent only when the agent is more patient than the principal *and* they cannot commit themselves to long-term contracts, because if the surplus of the relationship is too low to satisfy the non-reneging constraint for rewards by letting the principal capture all the surplus, shifting some surplus to the agent relaxes the non-reneging constraint for penalties *more than* it tighten the non-reneging constraint for rewards, making a large enough gap between rewards and penalties possible.

Although our agency rent is distinct from theirs, there are still similarities between the optimal dynamic contract in our paper and that in Fong and Li (2017),

[2]We impose a lower bound on the *expected* total payment to the agent in order to make the optimization problem well-defined when we analyze non-stationary contracts. This assumption per se does not generate a positive rent.

because the dynamics helps the principal extract more rent in both papers. However, the different source of the rent entails some differences in the dynamics: The agency cost due to limited liability leads to inefficiency in the form of inefficient termination in Fong and Li (2017), whereas in our model, inefficient termination never happens and our new agency cost leads to inefficiency in the form of deviation from the efficient intertemporal payoff trading that expands the Pareto frontier.

The Structure of the Paper

The rest of the paper is organized as follows. Section 2 introduces the baseline model. Section 3 contains the analysis of the benchmark case when the discount factors are the same. Stationary contracts and non-stationary contracts under unequal discount factors are analyzed in Sections 4 and 5, respectively.

In Section 5, we first focus on the effect of intertemporal payoff transfer by restricting attention to "semi-dynamic contracts" in which continuation payoffs can vary only with time but not the revenues of the previous periods, and hence effort incentives are provided by current payments only (Subsection 5.1). We then allow fully dynamic contracts to be designed in Subsections 5.2 and 5.3. Most of the analysis assume that the principal is less patient than the agent. The other case where the principal is more patient is analyzed in Subsection 5.4.

Section 6 contains the analysis of another benchmark case where the parties can commit themselves to long-term contracts. Section 7 concludes. The appendix provides all the proofs that are not in the main text.

2 The Model

There are one principal and one agent, both are risk neutral. In each period $t \in \{0, 1, \ldots\}$, the agent chooses a binary effort $e_t \in \{0, 1\}$, not observable to the principal, by incurring cost ce_t where c is a positive constant. After the effort is chosen, the principal observes a non-verifiable output $y \in \{h, \ell\}$, where $h > \ell$.

The timing of each period t is as follows.

1. The principal offers a take-it-or-leave-it contract $\{w_t, b_{ht}, b_{\ell t}\}$, where w_t is a legally enforceable fixed salary (or equivalently, an up-front payment), b_{yt} is a discretionary payment from the principal to the agent, contingent on y. We say that the contract requires the principal to pay the agent a *bonus* if b_{yt} is positive and the agent to pay the principal a *penalty* if b_{yt} is negative. To simplify notation, we will use $\{W_t, b_{ht}, b_{\ell t}\}$ where W_t is the expected total

payment to the agent to represent the dynamic contract in the rest of the paper.

2. The agent accepts or rejects the contract. If he rejects the contract, the principal and the agent obtain, respectively, the reservation payoffs $\overline{\pi} \geq 0$ and $\overline{u} \geq 0$. Denote the total reservation payoff by $\overline{s} \equiv \overline{u} + \overline{\pi}$.

3. If the agent accepts the contract, then w_t is paid, and the agent chooses an effort e_t.

4. The output realizes. The principal and the agent decide whether to pay the bonus or penalty, respectively.

5. Both parties observe the realization of a random variable $x_t \in [0, 1]$.[3]

We denote δ_P and δ_A as the discount factors of the principal and the agent, respectively, and let $d_i \equiv \delta_i/(1-\delta_i)$ for $i = P, A$. To simplify notations, we will suppress the time subscript t when there is no risk of confusion. Denote $\Pr\{y = h \mid e = 1\} = P$, $\Pr\{y = h \mid e = 0\} = p$, and $q \equiv P - p > 0$. These are assumed to be time invariant. Let $R \equiv Ph + (1-P)\ell$ be the expected revenue and $s \equiv R - c$ be the total surplus, all conditional on $e = 1$. Following convention, we assume $s > \overline{s} > ph + (1-p)\ell$.

A *relational contract* is a complete plan for the relationship, that is, for each period and every public history, it describes the fixed salary the principal should offer the agent, whether or not the agent should accept the offer, the effort the agent should exert, the bonus the principal should pay, and the penalty the agent should pay. A relational contract is *self-enforcing* if it describes a perfect public equilibrium (PPE) of the repeated principal-agent game. We often refer to such a contract as the equilibrium contract.

We solve for the *optimal relational contract*, which is the perfect public equilibrium that maximizes the principal's payoff at the beginning of the initial period. For the agent to choose $e = 1$ in any given period, the equilibrium contract must satisfy the incentive compatibility constraint:

$$P\left(b_h + d_A u_h\right) + (1-P)\left(b_\ell + d_A u_\ell\right) - c \geq p\left(b_h + d_A u_h\right) + (1-p)\left(b_\ell + d_A u_\ell\right)$$

or

$$\frac{c}{q} \leq b_h - b_\ell + d_A\left(u_h - u_\ell\right), \tag{IC}$$

[3] In models of repeated games, it is common to assume the existence of a public randomization device to ensure the convexity of the set of equilibrium payoff. We also assume that the principal and the agent observe the realization of a public randomization device at the beginning of period 0.

where u_y is the average continuation payoff to the agent when the output is $y \in \{h, \ell\}$, respectively. For the payments not to be reneged, the contract needs to satisfy the dynamic enforcement constraints for the principal and the agent, respectively, i.e.

$$b_h \leq d_P\left(\pi_h - \overline{\pi}\right) \tag{DEP}$$

and

$$-b_\ell \leq d_A\left(u_\ell - \overline{u}\right), \tag{DEA}$$

where π_y is the average continuation payoff to the principal following output y. Adding the three constraints yields the following incentive-compatibility-dynamic-enforcement constraint:

$$\frac{c}{q} \leq d_P\left(\pi_h - \overline{\pi}\right) + d_A\left(u_h - \overline{u}\right), \tag{IC-DE}$$

which requires that the continuation payoff vector $(u, \pi) = (u_h, \pi_h)$ to lie on or above the line $(c/q) = d_P(\pi - \overline{\pi}) + d_A(u - \overline{u})$, which we refer to as the IC-DE line.

The optimal relational contract maximizes the principal's expected average payoff, that is,

$$\left(1 - \delta_P\right)\left(R - W\right) + \delta_P\left(P\pi_h + \left(1 - P\right)\pi_\ell\right), \tag{1}$$

To solve for the optimal relational contract, it would be helpful if the three constraints (IC), (DEP), and (DEA) can be replaced by one constraint (IC-DE). The following result shows that this is indeed the case.

Lemma 1 *Given that* (IC), (DEP), *and* (DEA) *are the only constraints for* b_ℓ *and* b_h*, it is without loss of generality to replace* (IC), (DEP), *and* (DEA) *with* (IC-DE) *in solving the optimal relational contract.*

Proof. Denote the equilibrium contract that maximizes the principal's payoff (1) subject to (IC), (DEP), and (DEA) and some other constraints (to be introduced later) that do not depend on b_ℓ or b_h, called Program I, as $C^I \equiv (W^I, b^I_\ell, b^I_h, u^I_h, u^I_\ell)$ and the contract that maximizes the principal's payoff (1) subject to (IC-DE) only and the same set of other constraints, called Program II, as $C^{II} \equiv (W^{II}, b^{II}_\ell, b^{II}_h, u^{II}_h, u^{II}_\ell)$. The lemma claims that the principal is indifferent between C^I and C^{II}. This can be proved by contradiction.

Suppose first that the principal is strictly better off with C^I. Because C^I satisfies all the constraints in Program I, it must also satisfy all the constraints in Program II. Given that the objective functions are the same in Programs I and II, the principal's strict preference for C^I contradicts the hypothesis that C^{II} is optimal for Program II.

Suppose next that the principal is strictly better off with C^{II}. Consider $C^{III} \equiv (W^{II}, b_\ell^{III}, b_h^{III}, u_h^{II}, u_\ell^{II})$ where b_h^{III} and b_ℓ^{III} satisfy (DEP) and (DEA) as equalities with u_h^{II} and u_ℓ^{II} plugged in, respectively. Since C^{II} satisfies (IC-DE), C^{III} satisfies (IC). Because C^{II} and C^{III} only differ in b_h and b_ℓ, and b_h and b_ℓ do not appear in the other constraints, C^{III} satisfies all the constraints in Program I. In addition, C^{II} and C^{III} gives the principal the same level of expected payoff because (1) does not depend on b_h and b_ℓ. It follows that C^{III} gives the principal a higher expected payoff than C^I, contradicting the hypothesis that C^I is optimal for Program I. ■

In each period, the contract also needs to satisfy the individual rationality constraints (IR) of the principal and the agent:[4]

$$\begin{aligned} \pi_t &= (1-\delta_P)\sum_{k=t}^{\infty}\delta_P^{(k-t)}\left(R - W_k\right) \geq \overline{\pi}\ \forall\ t \in \{0, 1, \ldots\}; \\ u_t &= (1-\delta_A)\sum_{k=t}^{\infty}\delta_A^{(k-t)}\left(W_k - c\right) \geq \overline{u}\ \forall\ t \in \{0, 1, \ldots\}, \end{aligned} \tag{IR}$$

where π_t and u_t are the average payoffs from period t on to the principal and the agent, respectively. We say a contract *implements* $e = 1$ if it satisfies (IC-DE) and (IR).

3 Benchmark: Equal Discounting

When the principal and the agent have the same discount factor, denoted as $\delta \in (0, 1)$, our model collapses to a special case of Levin (2003) with binary-effort moral hazard and no adverse selection. In this section, we review this benchmark case. Another important benchmark case, in which the principal and the agent may have different discount factors but can commit themselves to long-term contracts and hence (DEP) and (DEA) can be ignored, will be studied later in Section 6.

Under equal discounting, if the principal's objective is to implement $e = 1$ in every period, it is without loss of generality to restrict attention to *stationary contracts* in the sense that every period the principal offers an identical contract $\{W, b_h, b_\ell\}$ regardless of the output. This implies that $u_h = u_\ell$ and $u \equiv u_h = u_\ell = W - c$, and hence what are left to be decided are W, b_h, and b_ℓ. Given that $\delta = \delta_A = \delta_P$, the first-best payoff frontier is a straight line with a slope of -1.

[4] For simplicity, we present (IR) by not taking expectations with respect to public randomization.

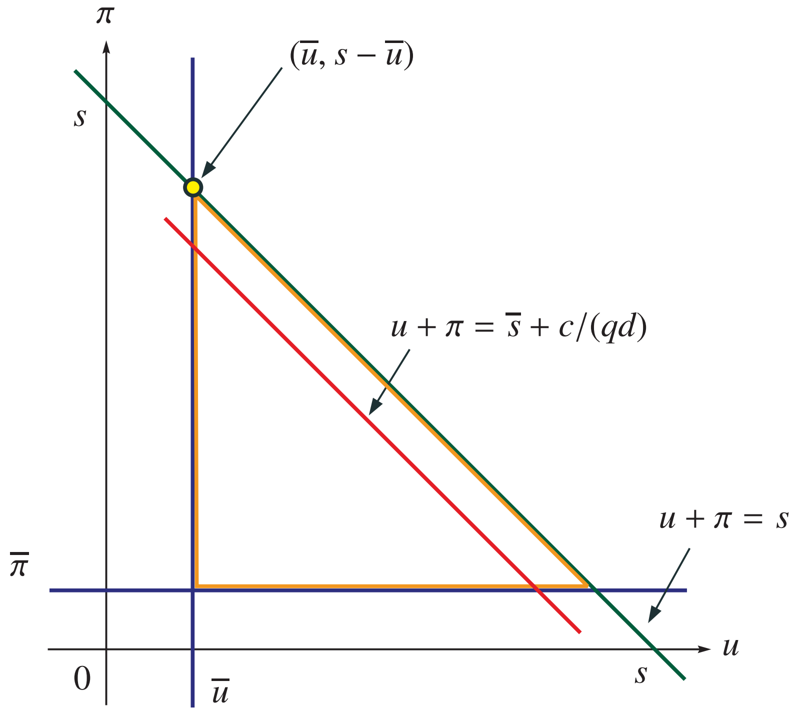

Figure 1: Benchmark: Equal Discounting

This means that (IC-DE) becomes

$$\frac{c}{q} \leq d(R-c-\overline{s}). \tag{IC-DE-B}$$

where $d = d_A = d_P$ and (IR) becomes

$$R - W \geq \overline{\pi} \quad \text{and} \quad W - c \geq \overline{u}. \tag{IR-B}$$

The intuition for this version of (IC-DE-B) is the same as that of Levin (2003)—the future gain from the relationship (the right-hand side) needs to be no less than the variation in the agent's payoff across different outcomes required to induce the effort (the left-hand side). In this case, the optimal contract will be stationary so the current payoff to the agent, u, will be the same as u_ℓ and u_h, and hence

$$u = (1-\delta)(W-c) + \delta u.$$

The principal's objective can be written as $\max(-W)$, which is independent of b_h and b_ℓ. It is hence without loss of generality to replace (IC), (DEP), and (DEA) with (IC-DE), by Lemma 1. The optimal contract thus solves

$$\max_{\{W, b_h, b_\ell\}} (-W) \quad \text{subject to (IC-DE-B) and (IR-B).}$$

We refer to the case when the effort is observable and enforceable as the first best, and the case when the effort is unobservable and contracts are relational as the second best.

In Figure 1, we give a geometric illustration of the optimal contract. The horizontal and vertical axes represent, respectively, the stage payoffs to the agent and the principal. The line $u + \pi = \overline{s} + c/(qd)$ is the IC-DE line. Implementing $e = 1$ results in a payoff vector (u, π) on the segment of the Pareto frontier $u+\pi = s$ that gives the principal at least $\overline{\pi}$ and the agent at least $\overline{u}$. Because all payoff vectors on this line segment lie above the IC-DE line, the principal can choose to implement any one of these payoffs by designing an appropriate contract. Clearly, the best for the principal is the one at the upper-left tip of this line segment. The payoff vector $(u, \pi) = (\overline{u}, s - \overline{u})$ characterizes the equilibrium payoffs to the principal and the agent. By setting the total expected payment to the agent as $W = \overline{u}+c$, the principal extracts all the rent from the agent. This implies, by (IC-DE-B), that $b_\ell = 0$, and hence there is no penalty and the agent is motivated by a bonus payment only. If (IC-DE-B) is slack, then there are multiple solutions for b_h, ranging from c/q to $d(s - \overline{s})$, combined with different levels of the fixed salary w. If (IC-DE-B) is binding, then the solution is unique and $b_h = c/q = d(s - \overline{s})$. Because the contract

$$\beta^* = \{\overline{u} + c, c/q, 0\}$$

is an equilibrium contract no matter (IC-DE-B) is slack or not, we will use it as a benchmark in the subsequent analysis. In particular, *the agent enjoys no rent when the discount factors of the principal and the agent are the same.*

One important feature of the analysis above depends crucially on the assumption of equal discount factors between the principal and the agent: When (DEP) and (DEA) are added together to obtain (IC-DE), the expected transfer W is canceled out, and hence disappears from (IC-DE) resulting in (IC-DE-B), i.e., *the payoffs to the principal and the agent are constant-sum.* Because the principal's objective function is to minimize W, which only appears as a positive term in the individual rationality constraint in (IR-B) for the agent, it is clear that the principal will not leave a rent to the agent. As we will show in the next section, this constant-sum feature disappears when discount factors are different.

4 Optimal Stationary Contracts

When the discount factors of the principal and the agent are different, it is conceivable that the optimal contract is non-stationary. However, the key feature that the agent earns rents does not depend on the non-stationarity and can be seen more clearly without it. We hence restrict attention to stationary contracts in this section.

Given that only stationary contracts are allowed, (IC) and (IR-B) of the benchmark remain the same. However, (DEP) and (DEA) become

$$b_h \leq d_P(R - W - \overline{\pi}) \tag{DEP-S}$$

$$-b_\ell \leq d_A(W - c - \overline{u}). \tag{DEA-S}$$

(IC-DE) hence becomes

$$\frac{c}{q} \leq d_P(R - W - \overline{\pi}) + d_A(W - c - \overline{u}). \tag{IC-DE-S}$$

The current payoff to the agent, u, is the same as u_ℓ and u_h, and hence

$$u = (1 - \delta_A)(W - c) + \delta_A u.$$

Since the principal's objective, $\max(-W)$, as well as (IC-DE-S), is independent of b_h and b_ℓ, Lemma 1 applies and we can replace (IC), (DEP-S), and (DEA-S) with (IC-DE-S).

Note that, in contrast to the case of equal discounting, the total expected payment W does not get canceled out when (DEP-S) and (DEA-S) are added together to obtain (IC-DE-S). If fact, if $\delta_P < \delta_A$, while increasing W reduces the continuation payoff to the principal, it raises that of the agent so much that it more than makes up for the former loss so the right-hand side of (IC-DE-S)—the join future gain of the relationship—is an *increasing* function of W. This means that giving more rents to the agent makes it *easier* to satisfy (IC-DE-S). Alternatively, if $\delta_P > \delta_A$, the right-hand side of (IC-DE-S) is an *decreasing* function of W, so giving more rents to the agent makes it harder to satisfy (IC-DE-S). This is a key feature of our model.

Define

$$d^* \equiv \frac{c}{q(s - \overline{s})}.$$

The following lemma provides the necessary and sufficient condition for $e = 1$ to be implemented by a stationary contract.

Lemma 2 *Effort $e = 1$ can be implemented by a stationary contract if and only if*

$$d^* \leq \max\{d_P, d_A\} \tag{2}$$

holds. If $d_P > d^*$, the implementation can be achieved by the bonus contract β^*. If $d_A > d^*$, the implementation can be achieved by a penalty contract $\rho^* = \{R - \overline{\pi}, 0, -c/q\}$.

Proof. *Necessity*: The result follows from (IC-DE-S) and the fact that

$$d_P(R - W - \overline{\pi}) + d_A(W - c - \overline{u}) \leq \max\{d_P, d_A\}(s - \overline{s}).$$

.

Sufficiency: If $d_P > d_A$, (2) implies that

$$\frac{c}{q} \leq d_P(s - \overline{s}).$$

Then the bonus contract β^* defined in the benchmark case can implement $e = 1$: (IC) and (DEA-S) bind, and then (DEP-S) becomes identical to (2).

Similarly, if $d_P < d_A$, (2) implies that and

$$\frac{c}{q} \leq d_A(s - \overline{s}) \tag{3}$$

holds. In this case, ρ^* can implement $e = 1$ because the payoffs to the principal and the agent are then $\pi = \overline{\pi}$ and $u = s - \overline{\pi}$, respectively, and hence individually rational. In addition, (IC) and (DEP-S) hold with equality, and (DEA-S) becomes identical to (3). ■

The optimal stationary contract is characterized by the solution to the following program:

$$\max_{\{W, b_h, b_\ell\}} -W \quad \text{subject to (IC-DE-S) and (IR-B)}$$

We can solve the program by analyzing three cases separately and illustrate them in Figure 2. The IC-DE line is obtained from (IC-DE-S) as

$$d_P\left(\pi - \overline{\pi}\right) + d_A\left(u - \overline{u}\right) = \frac{c}{q}. \tag{4}$$

First suppose that $d_P \geq d^*$. In this case, condition (IC-DE-S) can be satisfied without giving any rent to the agent. The IC-DE line (4) in this case can be illustrated by the line ICDEa in the figure which must cross the line $u = \overline{u}$ at a point *below* $(\overline{u}, s - \overline{u})$. According to Lemma 2, no matter $\delta_P < \delta_A$ or $\delta_P \geq \delta_A$, the pure-bonus contract β^* is optimal. This can be seen in the figure as follows: the point $(\overline{u}, s - \overline{u})$ that corresponds to β^* lies above line ICDEa and hence satisfies condition (IC-DE-S) and gives the principal the highest payoff. There can be other optimal contracts, but they are all pure-bonus contracts because all of them give the agent no rent and hence must entail $b_\ell = 0$.

Second, suppose that $d_P < d^*$ and $d_A \geq d^*$. Because $\delta_P < \delta_A$ must hold in this case, giving more rent makes it easier to satisfy (IC-DE-S) as we discussed

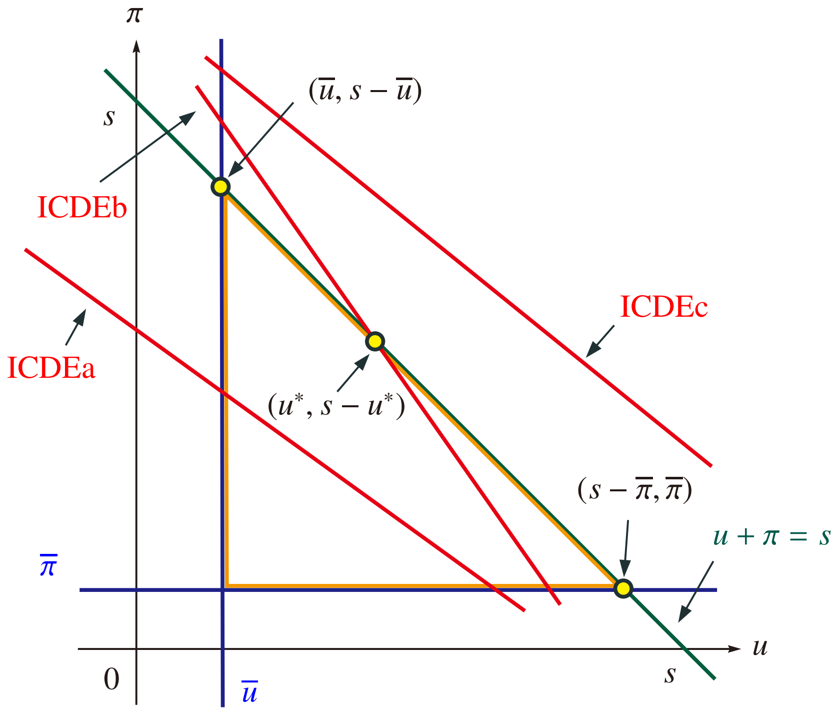

Figure 2: Optimal Stationary Contracts

above. Graphically, this means that the IC-DE line, as illustrated by line ICDEb in Figure 2, must be *steeper* than the Pareto frontier $u + \pi = s$. Hence, even though $(\overline{u}, s - \overline{u})$ lies below ICDEb and hence (IC-DE-S) cannot be satisfied without giving any rent to the agent, increasing W moves the point closer to ICDEb. The optimal contract, called γ^*, can be found by the intersection of ICDEb and the Pareto frontier because at this point the principal gives the agent just enough rent (as u^* indicated in the figure) to satisfy (IC-DE-S). The optimal contract generally involves a combination of bonus and penalty.

Finally, if $d_P < d^*$ and $d_A < d^*$, then by Lemma 2, (IC-DE-S) can never be satisfied so $e = 1$ can not be implemented. Graphically, this means that the whole Pareto frontier is below the IC-DE line, as illustrated by line ICDEc in Figure 2.

We summarize the discussion above in the following proposition.

Proposition 1 *The optimal stationary contract entails the following.*

(a) *If $d_P \geq d^*$, the agent receives no rent. The pure-bonus contract β^* is an optimal contract. While there may be multiple optimal contracts, they are all pure-bonus contracts.*

(b) *If $d_P < d^*$ and $d_A \geq d^*$, the agent enjoys a rent. The unique optimal contract*

$\gamma^* = \{W^*, b_h^*, b_\ell^*\}$ *entails that*

$$
\begin{aligned}
W^* &= \overline{u} + c + \frac{1}{d_A - d_P}\left[\frac{c}{q} - d_P(s - \overline{s})\right];\\
b_h^* &= d_P(R - W^* - \overline{\pi}) \geq 0;\\
b_\ell^* &= -d_A(W^* - c - \overline{u}) < 0.
\end{aligned}
$$

In particular, if $c/q = d_A(s - \overline{s})$, $\gamma^* = \rho^*$ *and hence only the penalty is used; if* $c/q < d_A(s - \overline{s})$, *the optimal contract involves both bonus and penalty.*

(c) *If* $d_P < d^*$ *and* $d_A < d^*$, $e = 1$ *cannot be implemented and the agent receives no rent.*

Note that if $d_P = d_A$, it must be in either (a) or (c) of Proposition 1 and hence the agent enjoys no rent. Since we do not impose the *ex post* limited liability constraint on the agent, the rent found in part (b) of Proposition 1 is different from the traditional limited-liability rent, and entirely due to differential discount factors (given the restriction to stationary contracts). From Proposition 1 (b), we can write down the formula for the agent's rent:

$$
u^* - \overline{u} = \frac{1}{d_A - d_P}\left[\frac{c}{q} - d_P(s - \overline{s})\right],
$$

which yields the following comparative statics results.

Corollary 1 *Suppose that* $d_P < d^*$ *and* $d_A \geq d^*$. *The agent's rent (agency cost) under the optimal stationary contract is increasing in* $(\overline{\pi}, \overline{u}, c)$, *and is decreasing in* (R, d_A, d_P), *and the relationships of the principal's payoff with these parameters are the opposite.*

The rent the agent enjoys when $d_P < d^*$ and $d_A \geq d^*$ ensures the agent will not renege the penalty needed to satisfy (IC-DE-S). Whether or not changing a parameter can increase or decrease the agent's rent discussed in Corollary 1 depends on whether it makes it harder or easier to satisfy (IC-DE-S). When $\overline{\pi}$ or $\overline{u}$ increases, it makes it harder to satisfy (IC-DE-S) so the agent's rent increases. An increase in the total surplus $s = R - c$ makes it easier to satisfy (IC-DE-S) so the agent's rent decreases. An increase in the discount factor of either the principal or the agent has the same effect.

In contrast to most existing agency models, Corollary 1 shows that principal's revenue and the reservation values of the principal and the agent can play important roles. Specifically, agency cost is decreasing in the revenue, exhibiting increasing return. In addition, Corollary 1 shows an increase in the agent's reservation value makes the agent *less* likely to quit, as opposed to the case in many existing models, and an increase in the principal's reservation value has the same effect. Since the limited-liability rent is independent of the principal's revenue and the reservation payoffs, the comparative statics result also shows that the rent we study here is very different from limited-liability rent.

In Proposition 1, both the cases with $d_P < d_A$ and $d_P \geq d_A$ are presented and the changing parameter is the discount factor. In the analysis of dynamic contracts, it is easier to characterize the different cases using the expected revenue R under $e = 1$, and the presentation is less messy by considering the cases with $d_P < d_A$ and $d_P \geq d_A$ separately (as in the main body of Section 5). For a better connection between the results here and that under dynamic contracts, we restate Proposition 1 in terms of R.

To this purpose define $\underline{R}_A$ as the solution to $d_A(R - c - \overline{s}) = c/q$, that is,

$$\underline{R}_A \equiv \overline{s} + c + \frac{c}{qd_A}. \tag{5}$$

$\underline{R}_A$ is the expected revenue under $e = 1$ such that $(s - \overline{\pi}, \overline{\pi})$ is on the IC-DE line. Similarly, define $\underline{R}_P$ as the expected revenue such that $(\overline{u}, s - \overline{u})$ is on the IC-DE line:

$$\underline{R}_P \equiv \overline{s} + c + \frac{c}{qd_P}. \tag{6}$$

Hence, $R < \underline{R}_A$ if and only if $d_A < d^*$ and $R < \underline{R}_P$ if and only if $d_P < d^*$. Given the assumption that $d_P < d_A$, the agent's rent exists under the optimal stationary contract if $\underline{R}_A \leq R < \underline{R}_P$ according to Proposition 1 (b). As $\underline{R}_P$ will be used as an important threshold that can be compared with the case of dynamic contract, denote $R^* = \underline{R}_P$.

We can now restate Proposition 1, focusing on the case with $d_P < d_A$, as the following corollary.[5]

Corollary 2 *Suppose the principal is less patient than the agent ($d_P < d_A$). The optimal stationary contract entails the following.*

(a) *If $R \geq R^*$, the agent receives no rent. The pure-bonus contract β^* is an optimal contract.*

[5]When the principal is more patient than the agent ($d_P > d_A$), cases (a) and (c) of Corollary 2 still hold with the definition of $R^* = \underline{R}_P$. However, case (b) disappears so that the more patient principal either extracts all the rent from the agent by the pure-bonus contract or finds no stationary contract implementing $e = 1$. In Subsection 5.4, we will show that this result continues to hold even if the more patient principal can design non-stationary contracts. In particular, we will show that threshold $R^* = \underline{R}_P$ does not change by allowing non-stationary contracts.

(b) *If* $\underline{R}_A \leq R < R^*$, *the agent enjoys a rent. The unique optimal contract is* γ^*.

(c) *If* $R < \underline{R}_A$, $e = 1$ *cannot be implemented and the agent receives no rent.*

5 Optimal Dynamic Contracts

As discussed in Section 1, dynamics brings three beneficial effects. The first, well-known effect is that the principal, if she is less patient and has to leave a rent to the agent under the optimal stationary contract, can be better off by modifying only the initial contract so as to extract the rent via a sufficiently low fixed payment. The second, more fundamental change is that, given the discount factors of the principal and the agent being different, they can increase the total surplus by engaging in intertemporal utility transfer (i.e., front-loading the payoff to the impatient one and back-loading that to the patient one), as discussed in Lehrer and Pauzner (1999). Importantly, the rent extraction via the initial contract and intertemporal payoff transfer cannot be separated and must be jointly analyzed.

In this section, we first focus on the case where the principal is less patient than the agent ($d_P < d_A$), and examine these two effects of intertemporal utility transfer by allowing continuation payoffs to vary with time but not history ("semi-dynamic contracts") in Subsection 5.1.

The third effect of dynamics is that allowing continuation payoffs to be contingent on the outputs of the previous periods helps providing incentive to the agent. This general case where the continuation payoff can vary with history ("dynamic contracts") is analyzed in Subsection 5.2 under the assumption that the principal induces the agent to work in every period. The possibility that the principal asks the agent to work in one period and shirk in another period, or shut down production in certain periods is considered in Subsection 5.3. Finally in Subsection 5.4, we consider the case in which the principal is more patient than the agent.

5.1 Semi-Dynamic Contracts

In this subsection, we study the effect of intertemporal utility transfer by restricting attention to "semi-dynamic contracts" in which continuation payoffs can vary only with time but not the revenues of the previous periods, and hence $u = u_\ell = u_h$ must hold for each period. The objective of the principal is

$$\max_{\{W, b_h, b_\ell, u\}} (1 - \delta_p)(R - W) + \delta_p f(u),$$

where $f(u)$ is the principal's highest average continuation payoff in PPE when the agent's payoff is u. Note that the objective function is independent of b_h and b_ℓ. Hence by Lemma 1, it is without loss of generality to replace (IC), (DEP), and (DEA) with (IC-DE). In the following, we first find the first-best Pareto frontier, denoted by $F(u)$, that gives the highest average payoff to the principal when the effort is observable and enforceable and the agent's average payoff is u, and then find its intersection with the IC-DE line. For u that satisfies (IC-DE), $f(u) = F(u)$ holds.

When the impatient principal and the patient agent engage in intertemporal payoff trading, the total expected payment W_t must be increasing in t, implying that along the Pareto-optimal path, the stage payoff $W_t - c$ to the agent is increasing and that to the principal $R - W_t$ is decreasing. Thus the principal's stage payoff cannot drop below $\overline{\pi}$ in any period because if it does in some period t', (IR) will be violated for all $t \geq t'$. In contrast, the agent's stage payoff *can* drop below $\overline{u}$ in some period because as long as (IR) is satisfied for $t = 0$, it holds for all t.

In order to make the Pareto frontier well-defined, we impose a lower bound on W_t, the *expected* total compensation to the agent, for all t.

Assumption 1 *$W_t \geq \underline{W}$ for all $t \in \{0, 1, \ldots\}$, where $\underline{W} \leq \overline{u} + c$.*

Assumption 1 ensures that the stage payoff to the agent is no lower than some finite level. While the standard (ex post) limited-liability constraint is a sufficient condition for Assumption 1 to hold, we emphasize that Assumption 1 does not directly give him a positive rent. In fact, since $\underline{W} \leq \overline{u} + c$, the agent receives no rent in the stage game and hence in the dynamic game as well if the discount factors are the same. Furthermore, we will show later in Section 6 that under the same condition, the agent earns no rent if the principal and the agent, whose discount factors may be different, can commit themselves to long-term contracts. In the following analysis, we will show that if they cannot make a commitment *and* the principal is less patient than the agent, he may enjoy rents even though $\underline{W} \leq \overline{u} + c$ holds.

The first step of deriving the Pareto frontier $F(u)$ is a direct application of Lehrer and Pauzner (1999). Given Assumption 1 and $d_P < d_A$, the set of stage payoffs that can be used along the Pareto-optimal path of the repeated game is the convex hull of $(\overline{u}, \overline{\pi})$, $(s - \overline{\pi}, \overline{\pi})$, and $(\underline{W} - c, R - \underline{W})$. The first-best Pareto frontier, which contains an infinite numbers of vertices, can be constructed as follows. The agent's most preferred vertex, apart from $(s - \overline{\pi}, \overline{\pi})$, is supported by playing $(\underline{W} - c, R - \underline{W})$ in the first period and playing $(s - \overline{\pi}, \overline{\pi})$ forever afterwards; the agent's second preferred vertex is supported by playing $(\underline{W} - c, R - \underline{W})$ in the first two periods and afterwards playing $(s - \overline{\pi}, \overline{\pi})$ forever, and so on. Thus the

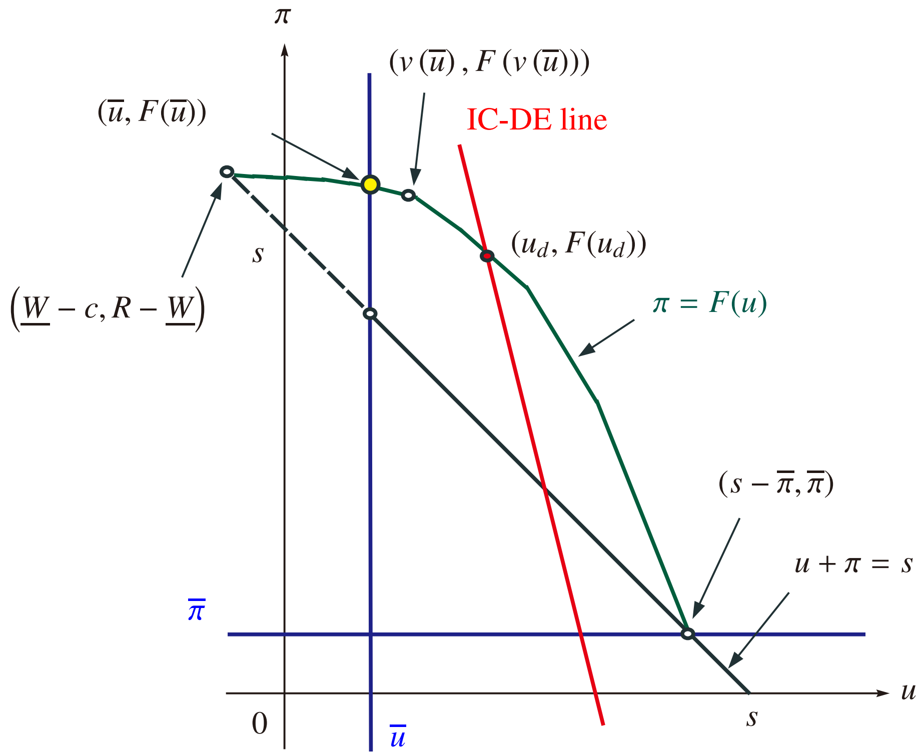

Figure 3: The Pareto Frontier

agent's n'th preferred vertex is given by

$$(\hat{u}_n, \hat{\pi}_n) = \left((1-\delta_A^n)\left(\underline{W} - c\right) + \delta_A^n\left(s - \overline{\pi}\right), (1-\delta_P^n)\left(R - \underline{W}\right) + \delta_P^n \overline{\pi}\right)$$

for $n \in \{0, 1, 2, \ldots\}$.

The access to public randomization device ensures that all the line segments connecting $(\hat{u}_n, \hat{\pi}_n)$ and $(\hat{u}_{n+1}, \hat{\pi}_{n+1})$ for $n \in \{0, 1, \ldots\}$ are also on the frontier. The slope of the line segment connecting $(\hat{u}_n, \hat{\pi}_n)$ and $(\hat{u}_{n+1}, \hat{\pi}_{n+1})$, denoted by λ_n, is

$$\lambda_n = -\left(\frac{\delta_P}{\delta_A}\right)^n \frac{1-\delta_P}{1-\delta_A}. \tag{7}$$

Note that λ_n is increasing in n, and $\lim_{n\to\infty} \lambda_n = 0$. The Pareto frontier $F(u)$ that satisfies (IR) is therefore characterized by the following functional form: For $\overline{u} \leq u \leq s - \overline{\pi}$,

$$\pi = F(u) = \hat{\pi}_{n+1} + \lambda_n \left(u - \hat{u}_{n+1}\right) \text{ if } \hat{u}_{n+1} < u \leq \hat{u}_n.$$

This frontier is depicted in Figure 3 as a green kinked curve $\pi = F(u)$. Starting from any initial average payoff pair $(u, F(u))$, where $\overline{u} \leq u < s - \overline{\pi}$, the average continuation payoff pairs gradually move towards the bottom-right until $(s - \overline{\pi}, \overline{\pi})$ is reached.

In the second step we go beyond Lehrer and Pauzner (1999) and analyze how the intertemporal payoff transfer affects the incentive and enforcement conditions for relational contracts, by imposing the IC-DE constraints on $F(u)$. Finding the optimal semi-dynamic contract requires finding a sequence of payoffs for the agent and a corresponding sequence of payoffs for the principal (on the Pareto frontier) for $t = 0, 1, ...$, that satisfy the IC-DE constraints for $t = 1, 2, ...$ (the IC-DE constraints are for continuation payoff, not initial payoff).

According to the way the first-best Pareto frontier is derived, the principal and the agent obtain $(\underline{W} - c, R - \underline{W})$ in every period before certain period $T \geq 1$ and then switch to $(s - \overline{\pi}, \overline{\pi})$ forever afterward. It then follows that, given a level of average payoff to the agent u in period $t = 0$, the subsequent average payoff, denoted as $v^t(u)$ for $t = 1, \ldots, T-1$ must satisfy the following equations:

$$\begin{aligned} u &= (1-\delta_A)(\underline{W} - c) + \delta_A v^1(u) \\ v^\tau(u) &= (1-\delta_A)(\underline{W} - c) + \delta_A v^{\tau+1}(u) \end{aligned}$$

for all $\tau = 1, \ldots, T-1$. Similarly,

$$\begin{aligned} F(u) &= (1-\delta_P)(R - \underline{W}) + \delta_P F\left(v^1(u)\right) \\ F(v^\tau(u)) &= (1-\delta_P)(R - \underline{W}) + \delta_P F\left(v^{\tau+1}(u)\right) \end{aligned}$$

for all $\tau = 1, \ldots, T-1$.

The sequence of the agent's payoff described above suggests that it can be characterized by a transition function

$$v : [\overline{u}, s - \overline{\pi}] \to [\overline{u}, s - \overline{\pi}] \tag{8}$$

that determines the transition from any current average payoff u to the agent to his average continuation payoff in the next period, such that $v(v(u)) = v^2(u)$, $v(v(v(u))) = v^3(u)$, and so on. It is clear that v is an increasing function so that v^τ is increasing in τ.

For a point $(u, F(u))$ in a given period to be attained with a semi-dynamic contract, Lemma 1 implies that the transition function associated with the contract must satisfy

$$\frac{c}{q} \leq d_P(F(v(u)) - \overline{\pi}) + d_A(v(u) - \overline{u}) \tag{IC-DED}$$

and

$$\frac{c}{q} \leq d_P\left(F\left(v^{\tau+1}(u)\right) - \overline{\pi}\right) + d_A\left(v^{\tau+1}(u) - \overline{u}\right) \quad \text{for } \tau = 1, \ldots, \infty. \tag{9}$$

Because this requires evaluating $d_P(F(u) - \overline{\pi}) + d_A(u - \overline{u})$ for any point $(u, F(u))$, the following result is useful.

Lemma 3 *Suppose that the principal is less patient than the agent and Assumption 1 holds. Then* (i) $d_P(F(u) - \overline{\pi}) + d_A(u - \overline{u})$ *is increasing in* u. (ii) *If constraint (IC-DED) is satisfied in a given period, then constraints (9) hold for all subsequent periods.*

Proof. (i) The conclusion follows from the fact that the slope of $\pi = F(\cdot)$ is larger than $(-d_A/d_P)$ by (7). (ii) The conclusion follows from (i) since v^τ is increasing in τ. ■

We now characterize the optimal semi-dynamic contract that implements $e = 1$ every period, for the different ranges of the expected revenue R. We first argue that if $R < \underline{R}_A$, no contract, whether stationary or non-stationary, can implement $e = 1$. Remember that if $R < \underline{R}_A$, $e = 1$ cannot be implemented by any stationary contract (Lemma 2 (c)), where $\underline{R}_A$ is defined by (5). This threshold level does *not* change even though we allow contracts to be non-stationary: $e = 1$ cannot be implemented by any contract if $R < \underline{R}_A$. To see this, note that in both Figures 2 and 3, point $(s - \overline{\pi}, \overline{\pi})$ is on the Pareto frontier. And it is on the IC-DE line (4) as well if $R = \underline{R}_A$. Furthermore, the slope of the IC-DE line, $-(d_A/d_P)$, is always steeper than that of the Pareto frontier, λ_n, defined by (7). Therefore, for $R < \underline{R}_A$, the whole first-best Pareto frontier is below the IC-DE line, and hence $e = 1$ is not implementable even by non-stationary contracts.

Second, note that the optimal stationary contract leaves no rent to the agent if $R \geq R^*$ (Lemma 2 (a)), where $R^* = \underline{R}_P$, defined by (6), is the expected revenue under which the IC-DE line intersects with $u + \pi = s$ at $(\overline{u}, s - \overline{u})$. Since $s - \overline{u} \leq F(\overline{u})$, the optimal semi-dynamic contract leaves no rent to the agent for $R \geq R^*$ as well.

From now on we focus on the intermediate case $R \in [\underline{R}_A, R^*)$. Referring to Figure 3, denote the intersection of the Pareto frontier with the IC-DE line as $(u_d, F(u_d))$, so that

$$d_P(F(u_d) - \overline{\pi}) + d_A(u_d - \overline{u}) = \frac{c}{q}. \tag{10}$$

While $(u_d, F(u_d))$ satisfies constraint (IC-DED), it is better for the principal to defer playing it to period-1 and play his favorite stage game payoff pair $(\underline{W} - c, R - \underline{W})$ in period 0, because constraint (IC-DED) only needs to be satisfied for continuation payoffs (in period 1). Doing so, the principal can lower the agent's payoff in period 0 further to u_e such that the agent's continuation payoff is u_d, i.e. $v(u_e) = u_d$ or

$$u_e = (1 - \delta_A)(\underline{W} - c) + \delta_A u_d, \tag{11}$$

and give herself

$$F(u_e) = (1-\delta_P)\left(R - \underline{W}\right) + \delta_P F\left(u_d\right)$$

provided that $u_e > \overline{u}$.

When $u_e > \overline{u}$ holds, the optimal semi-dynamic relational contract gives the principal $F(u_e)$ in the initial period because *(a)* $f(u_e) = F(u_e) > F(u)$ for all $u > u_e$ and *(b)* $f(u_e) = F(u_e) > F(u)$ for all $u < u_e$. To see (a), note that $F(u)$ is decreasing in $u \in (\overline{u}, s - \overline{\pi})$. To see (b), note that for $u < u_e$, constraint (IC-DED) is violated and $e = 1$ cannot be implemented. Furthermore, by $r < \overline{u} + \overline{\pi}$, f is a upward-sloping straight line connecting $(u_e, f(u_e))$ and $(\overline{u}, \overline{\pi})$.

On the other hand, if $u_e \leq \overline{u}$, the payoffs to the agent and the principal are $\overline{u}$ and $F(\overline{u})$, respectively, under the optimal semi-dynamic relational contract since $f(u) = F(u)$ is decreasing in $u \geq \overline{u}$. In this case, the agent receives no rent.

Whether $u_e > \overline{u}$ or $u_e \leq \overline{u}$ depends on R in a clear way because the relation between u_e and u_d in (11) and the relation between u_d and R imply that u_e is decreasing in R. And $u_e < u_d = \overline{u}$ for $R = R^*$ by (11). Hence if $u_e > \overline{u}$ is satisfied for $R = \underline{R}_A$, then there exists a threshold $R^{**} \in (\underline{R}_A, R^*)$ such that $u_e > \overline{u}$ if and only if $R < R^{**}$, so the agent can obtain a positive rent only when $R \in [\underline{R}_A, R^{**})$.

A condition for $u_e > \overline{u}$ to hold for $R = \underline{R}_A$ is derived as follows. When $R = \underline{R}_A$, $(s - \overline{\pi}, \overline{\pi})$ intersects with the IC-DE line, and hence by the definition of u_d given by (10), $u_d = \overline{u} + c/(qd_A)$. (11) then implies that $u_e > \overline{u}$ for $R = \underline{R}_A$ and thus R^{**} exists if and only if

$$\underline{W} > \overline{u} + c - \frac{c}{q}. \tag{12}$$

The dynamics of the optimal semi-dynamic contract goes as follows. First suppose $R \geq R^{**}$. Then in the initial period, $(\overline{u}, F(\overline{u}))$ can be attained by a semi-dynamic contract and the agent receives no rent. In this case, an optimal semi-dynamic contract entails

$$b^{**}_{ht} = d_P\left(F\left(v^{t+1}\left(\overline{u}\right)\right) - \overline{\pi}\right)$$

and

$$b^{**}_{\ell t} = -d_A\left(v^{t+1}\left(\overline{u}\right) - \overline{u}\right).$$

Both b^{**}_{ht} and $b^{**}_{\ell t}$ are decreasing in $t < T$, where T is the period when $(s - \overline{\pi}, \overline{\pi})$ is reached, since $F\left(v^t\left(\overline{u}\right)\right)$ is decreasing and $v^t\left(\overline{u}\right)$ is increasing in t. The fixed salary w^{**}_t is increasing in t for $t < T$ by

$$w^{**}_t + Pb^{**}_{ht} + (1-P)\,b^{**}_{\ell t} = \underline{W}.$$

Next suppose $\underline{R}_A \leq R < R^{**}$. This interval is non-degenerate if (12) holds, and the agent receives a rent $u_e - \overline{u} > 0$ in the initial period. The principal's maximum average payoff in period 0 is

$$F(u_e) = (1 - \delta_P)(R - \underline{W}) + \delta_P F(u_d) < F(\overline{u}).$$

An optimal semi-dynamic contract entails

$$b_{ht}^{**} = d_P \left(F\left(v^{t+1}(u_e)\right) - \overline{\pi} \right)$$

and

$$b_{\ell t}^{**} = -d_A \left(v^{t+1}(u_e) - \overline{u} \right).$$

Both b_{ht}^{**} and $b_{\ell t}^{**}$ are decreasing in t and the fixed salary w_t^{**} is "seniority-based," in the sense that it is increasing in t for $t < T$ to compensate for the decreasing bonuses and increasing penalties.

We summarize the preceding analysis as follows:

Proposition 2 *Suppose that the principal is less patient than the agent ($d_P < d_A$) and Assumption 1 holds.*

(a) *There is an optimal semi-dynamic contract $\{w_t^{**}, b_{ht}^{**}, b_{\ell t}^{**}\}$ implementing $e = 1$ every period such that b_{ht}^{**} and $b_{\ell t}^{**}$ are decreasing in t, and w_t^{**} is increasing in t.*

(b) *If* (12) *holds, then there exists $R^{**} \in (\underline{R}_A, R^*)$ such that under the optimal semi-dynamic contract the agent receives a rent in the initial period if and only if R, the expected revenue under $e = 1$, satisfies $R \in [\underline{R}_A, R^{**})$.*

Figure 4 illustrates Proposition 2 (b). The highest first-best Pareto frontier $\pi = F_a(u)$ corresponds to the case $R_a = R^{**}$ and hence the IC-DE line intersects with the Pareto frontier at $(v_a(\overline{u}), F_a(v_a(\overline{u})))$ where v_a is the transition function defined in (8) under $R = R_a$. In this case, $u_e = \overline{u}$ holds. The principal attains $F_a(\overline{u})$ and the agent enjoys no rent in the initial period for all $R \geq R^{**}$.

The middle first-best Pareto frontier $\pi = F_b(u)$ depicts the case $\underline{R} < R_b < R^{**}$, and hence $u_d > v_b(\overline{u})$, where v_b is the transition function defined in (8) under $R = R_b$. In this case, the agent enjoys a positive rent $u_e - \overline{u}$ in the initial period. Finally, the lowest first-best Pareto frontier in the figure depicts the case $R_c < \underline{R}_A$ so that the principal cannot implement $e = 1$.

Note that the rent the agent enjoy is not directly caused by the lower bound on the payments to the agent imposed by Assumption 1 because the rent only exist when the revenue is at an intermediate level and it disappears when the revenue is high enough.

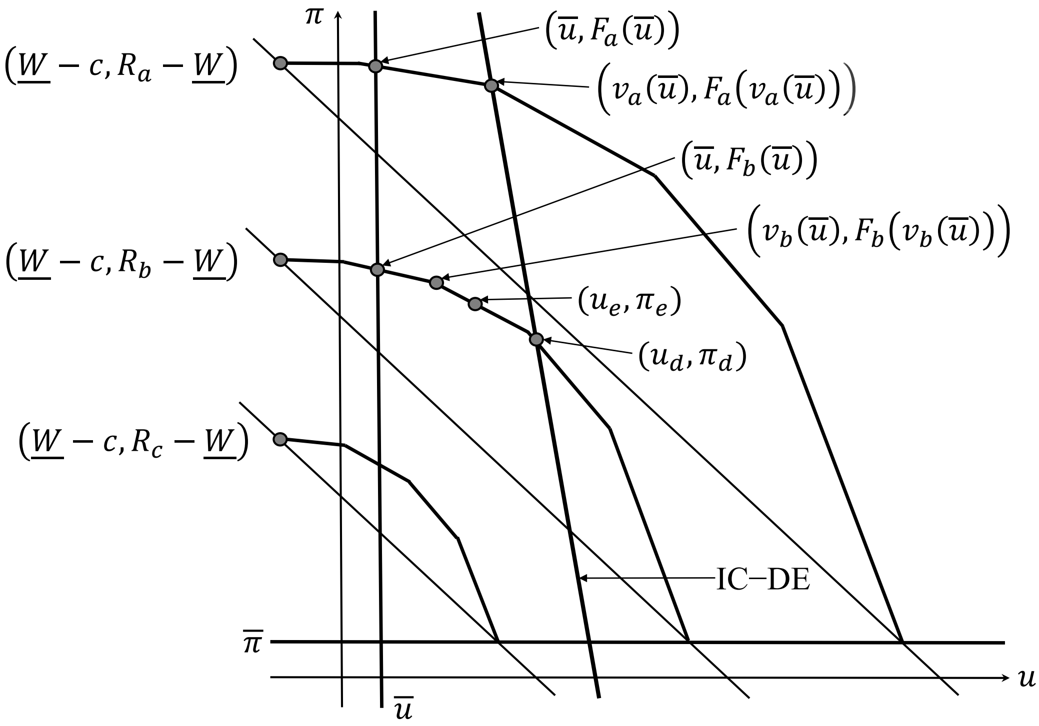

Figure 4: Optimal Semi-dynamic Contracts

The intuition behind Proposition 2 (a) is as follows. Because the principal is less patient than the agent, the join surplus can be increased if the principal "borrows" payoffs from the agent and then pays back in later periods. This kind of intertemporal payoff trade implies that the principal's continuation payoff will decrease and the agent's will increase over time. However, the principal's continuation payoff imposes an upper bound on the bonus he is willing to pay. Hence if the bonus keeps constant as in stationary contracts, the principal would renege on paying it in some point of time. To cope with this problem, the optimal semi-dynamic contract must entail a decreasing bonus plan, accompanied with an increasing penalty schedule, ensuring the agent's incentives to exert effort.

The optimal semi-dynamic contract clearly dominates the optimal stationary contract, as can be see by comparing (u_e, π_e) for the former and $(u^*, s - u^*)$ for the latter in Figure 4. The improvement of the principal's payoff comes from two sources. First, intertemporal payoff trading expands the first-best Pareto frontier. Second, the expansion of the first-best Pareto frontier makes it easier to satisfy (IC-DED), thereby increasing the principal's power to extract rents from the agent in the initial period. Note, however, that because of non-stationarity, the agent continues to earn rents beyond the initial period.

5.2 Dynamic Contracts

In the previous subsection 5.1, we show that if incentives are provided by current payments only, it is possible that the agent still enjoys a positive rent $u_e - \overline{u}$ in the initial period. This happens when $\underline{R}_A < R^{**}$ (equivalently $\underline{W} > \overline{u} + c - (c/q)$) and $\underline{R}_A \leq R < R^{**}$. In this subsection, we focus on this case, that is, $u_d \in (v(\overline{u}), s - \overline{\pi}]$ and $u_e \in (\overline{u}, u_d)$, and examine whether the principal can extract more rent if fully dynamic contracts are allowed.

Under either stationary or semi-dynamic contracts, the equilibrium effort is not allowed to vary across period. When fully dynamic contracts are allowed, however, it is in general possible that in equilibrium the principal asks the agent to work in one period and shirk in another period, or shut down production in certain periods. We say that the average payoff pair $(u, f(u))$ on the Pareto frontier is *supported with effort, shirking, or termination* if the agent puts in effort, shirks, or quit, respectively, in the initial period under the corresponding PPE.

For $u \geq \overline{u}$, the payoff pair $(u, f(u))$ is *supported with effort* if it satisfies the following Bellman equation:

$$f(u) = f_e(u) \equiv \max_{(W, b_\ell, b_h, u_h)} (1 - \delta_P)(R - W) + \delta_P \left[P f(u_h(u)) + (1 - P) f(u_\ell(u)) \right], \tag{PPe}$$

subject to (IC), (DEP), (DEA), $W \geq \underline{W}$, $u_h(u) \geq \overline{u}$, $u_\ell(u) \geq \overline{u}$, and

$$u = (1 - \delta_A)(W - c) + \delta_A (P u_h(u) + (1 - P) u_\ell(u)), \tag{UUe}$$

where $u_y(u)$ is the agent's average continuation payoff following the realization of $y \in \{h, \ell\}$. Because b_h and b_ℓ do not affect the objective function nor the constraints other than (IC), (DEP), and (DEA), Lemma 1 applies once again, and it is without loss of generality to replace (IC), (DEP), and (DEA) with (IC-DE), i.e.,

$$\frac{c}{q} \leq d_P (f(u_h(u)) - \overline{\pi}) + d_A (u_h(u) - \overline{u}), \tag{IC-DE}$$

Even though fully dynamic contracts are allowed, $f(u)$, the highest average payoff to the principal, coincides with the first-best Pareto frontier $F(u)$ for $u \geq u_e$, with $W = \underline{W}$, $u_h(u) = u_\ell(u)$, and $f_e(u) = f(u) = F(u)$. In other words, $(u, f(u))$ is supported with effort and $f(u) = f_e(u)$ for $u \geq u_e$. Furthermore, $u_h(u_e) = u_\ell(u_e) = u_d$ and $f(u_e) > \overline{\pi}$ hold by $u_e < u_d \leq s - \overline{\pi}$.

For $u < u_e$, however, $f(u)$ must be lower than the first-best Pareto frontier, and $(u, f(u))$ in this range may be supported not only with effort but also with shirking or termination. In this subsection, we focus on the analysis of problem (PPe), and relegate the analysis of payoff pairs to be supported with effort, shirking, or termination to Subsection 5.3.

The first lemma derives three properties that the solution to the maximization problem in (PPe) must satisfy.[6]

Lemma 4 *The solution to the maximization problem in* (PPe) *for* $u < u_e$ *satisfies the following three properties:* (a) $W = \underline{W}$*;* (b) $u_\ell(u) < u_h(u)$*; and* (c) *(IC-DE) binds.*

Lemma 4 (c) implies that (IC), (DEP), and (DEA) must bind at the solution. We can then use these binding constraints to determine b_ℓ, b_h, $u_\ell(u)$, $u_h(u)$, and $w = \underline{W} - Pb_h - (1-P)b_\ell$. The binding (DEP) and (DEA) yield

$$b_h - b_\ell = d_P \left(f\left(u_h(u)\right) - \overline{\pi}\right) + d_A \left(u_\ell(u) - \overline{u}\right).$$

By plugging this equation into the binding (IC) we obtain

$$d_P \left(f\left(u_h(u)\right) - \overline{\pi}\right) + d_A \left(u_h(u) - \overline{u}\right) = \frac{c}{q},$$

and hence $(u_h(u), f(u_h(u))) = (u_d, F(u_d))$ by the definition of u_d given by (10), Note that u_d is independent of u and hence so is $u_h = u_h(u)$. This implies that no matter how many times low output (failure) has happened, once the first high output realizes, the agent's average payoff increases to u_d, and then follows the path of u_d, $v^1(u_d)$, $v^2(u_d)$, and so on, entailed by the optimal semi-dynamic contract. In this sense, the contract only exhibit fully dynamic features until the first high output appears. After that, it is semi-dynamic. Since $R - \underline{W} > f(u_d) \geq \overline{\pi}$ and $f(u_\ell(u)) \geq \overline{\pi}$ by definition, $f_e(u) > \overline{\pi}$ holds for $u < u_e$ by (PPe).

On the other hand, $u_\ell(u)$ depends on u via (UUe), where the current payoff is replaced with $\underline{W} - c$. Hence, there exists a unique u_g such that that $u_\ell(u_g) = u_g$, that is,

$$u_g = \frac{(1-\delta_A)\left(\underline{W} - c\right) + \delta_A P u_d}{1 - \delta_A (1-P)}. \tag{13}$$

If $u_g > \overline{u}$, and $(u_g, f(u_g))$ is supported with effort, the principal's payoff corresponding to u_g is $f\left(u_g\right)$ given by

$$f\left(u_g\right) = (1-\delta_P)\left(R - \underline{W}\right) + \delta_P P f\left(u_d\right) + \delta_P (1-P) f\left(u_g\right),$$

or

$$f\left(u_g\right) = \frac{(1-\delta_P)\left(R - \underline{W}\right) + \delta_P P f\left(u_d\right)}{1 - \delta_P (1-P)}. \tag{14}$$

[6]Note that this lemma does *not* depend on whether or not the initial payoff pair $(u, f(u))$ is supported with effort. That is, whether or not $f(u) = f_e(u)$ holds.

Note that $f(u_g) > \overline{\pi}$ holds since $R - \underline{W} > \overline{\pi}$ and $f(u_d) \geq \overline{\pi}$. The solutions to w, b_ℓ, and b_h are obtained as follows:

$$\begin{aligned} b_\ell &= -d_A \left(u_\ell \left(u\right) - \overline{u}\right); \\ b_h &= d_P \left(f \left(u_d\right) - \overline{\pi}\right); \\ w &= \underline{W} - Pb_h - \left(1 - P\right) b_\ell. \end{aligned}$$

The next lemma determines the order of the magnitude between u_g and u_e, as well as that between $f\left(u_g\right)$ and $f\left(u_e\right)$.[7]

Lemma 5 *If $u_g > \overline{u}$ and $(u_g, f(u_g))$, defined by* (13) *and* (14)*, is supported with effort, then $u_g < u_e$ and $f\left(u_g\right) > f\left(u_e\right)$.*

Suppose $(u, f(u))$ is supported with effort. The dynamic of $u_\ell(u)$ can now be described as follows. At $u = u_g$, low output keeps the agent's average continuation payoff unchanged. If $u \in (u_g, u_e)$, the agent's payoff increases to $u_\ell(u) > u$ after low output, and if $u < u_g$, low output decreases his payoff to $u_\ell(u) < u$. These patterns arise because given that high output will raise the agent's average payoff from u to u_d, his payoff after low output must be adjusted to satisfy equation (UUe).

For $u \in (u_g, u_e)$, every time output is low, the agent's continuation payoff, as well as the penalty and base wage, increases. Although the resulting payoff pair is below the first-best Pareto frontier, the agent's average payoff is still increasing over time, which feature shares with that of the Pareto-optimal intertemporal payoff transfer.

However, for $u < u_g$, every time output is low, the agent's continuation payoff decreases and hence both the penalty and base wage do, too. The agent's average payoff is now too low to sustain with an increasing payoff stream. It now has to decrease in contrast to the Pareto-optimal intertemporal payoff transfer. Note that since $u_\ell(u) \geq \overline{u}$ must hold, $(u, f(u))$ cannot be supported with effort if u is such that (UUe) holds with $u_\ell(u) < \overline{u}$. This is the case for $u = \overline{u} < u_g$. Since $u_\ell(u)$ is decreasing, $(u, f(u))$ cannot be supported with effort for u sufficiently small if $\overline{u} < u_g$.

Define the payoff pair in the initial period under the optimal relational contract as $(\tilde{u}, f(\tilde{u}))$ such that $\tilde{u} \equiv \min\left\{u' | u' \in \arg\max_{u \geq \overline{u}} f(u)\right\}$. We call $(\tilde{u}, f(\tilde{u}))$ the optimal payoff pair. Because the optimal effort pair cannot be in the middle of a straight line but a vertex of the Pareto frontier, it must be supported by a pure-strategy equilibrium (effort, shirking, or termination). In this subsection, we analyze the optimal relational contract under the assumption that $(\tilde{u}, f(\tilde{u}))$

[7] Remember $u_e < u_d$ and $f(u_e) > f(u_d)$ as we have shown after the definition of u_e (11).

is supported with effort. Later in Subsection 5.3 we allow payoff pairs to be supported with shirking and termination as well, and obtain a sufficient condition for the payoff pair in the initial period under the optimal relational contract to be supported with effort only.

Given that the optimal payoff pair is supported with effort, the next lemma shows that the optimal relational contract gives the principal $f(u_g)$ if $u_g \geq \overline{u}$, or $f(\overline{u})$, otherwise.

Lemma 6 *Suppose $(\tilde{u}, f(\tilde{u}))$ is supported with effort. Then $\tilde{u} = \max\{\overline{u}, u_g\}$.*

Proof. We prove the lemma by ruling out the other three cases: $\tilde{u} = \overline{u} < u_g$, $\tilde{u} \in (\overline{u}, u_g)$, and $\tilde{u} > \max\{\overline{u}, u_g\}$.

First, $\tilde{u} = \overline{u} < u_g$ is not possible since, as we argue above, $(\tilde{u}, f(\tilde{u}))$ then cannot be supported with effort.

Second, if $\tilde{u} \in (\overline{u}, u_g)$, then $f'_+(u) > 0$ for u in the left neighborhood of $\tilde{u}$, and hence by the concavity of f, $f'_+(u) > 0$ for all $u < \tilde{u}$. In addition, $\tilde{u} < u_g$ implies that $u_\ell(\tilde{u}) < \tilde{u}$, and hence $f'_+(u_\ell(\tilde{u})) > 0$. Then by the definition of $\tilde{u}$, (PPe), and (UUe),

$$0 > f'_+(\tilde{u}) \geq f'_{e+}(\tilde{u}) = \frac{\delta_P}{\delta_A} f'_+(u_\ell(\tilde{u})) > 0.$$

A contradiction.

Finally, suppose $\tilde{u} > \max\{\overline{u}, u_g\}$. Then $f'_-(u) < 0$ for u in the right neighborhood of $\tilde{u}$, and hence by the concavity of f, $f'_-(u) < 0$ for all $u > \tilde{u}$. Since $u_\ell(\tilde{u}) > \tilde{u}$, $f'_-(u_\ell(\tilde{u})) < 0$ must hold. Then by the definition of $\tilde{u}$, (PPe), and (UUe),

$$0 < f'_-(\tilde{u}) \leq f'_{e-}(\tilde{u}) = \frac{\delta_P}{\delta_A} f'_-(u_\ell(\tilde{u})) < 0.$$

A contradiction. ■

When the optimal relational contract can be supported with effort, the question of our focus is whether the agent will receive a positive rent or no rent, which can be the case when $\tilde{u} = u_g > \overline{u}$ or $\tilde{u} = \overline{u} \geq u_g$, respectively. The relation between u_g and u_d in (13) and the relation between u_d and R imply that u_g is decreasing in R. And $u_g < u_e = \overline{u}$ for $R = R^{**}$ by (11) and (13). Hence if $u_g > \overline{u}$ holds for $R = \underline{R}_A$, there exists $R^{***} \in (\underline{R}_A, R^{**})$ such that $u_g > \overline{u}$ if and only if $R < R^{***}$.

To obtain a condition for $u_g > \overline{u}$ to be satisfied at $R = \underline{R}_A$, note that when $R = \underline{R}_A$, $u_d = \overline{u} + c/(qd_A)$, and hence (13) implies that $u_g > \overline{u}$ holds and R^{***} exists if and only if

$$\underline{W} > \overline{u} + c - \frac{P}{q} c. \tag{15}$$

Note that (15) implies (12) so that $R^{***} > \underline{R}_A$ implies $R^{**} > \underline{R}_A$.

If $R \geq R^{***}$, and hence $u_g \leq \overline{u}$, $\tilde{u} = \overline{u}$ holds, and the principal can extract all the rent from the agent in the initial period by designing a dynamic contract with the following, "promotion-like" feature: In period 0, the average payoff pair is $(\overline{u}, f(\overline{u}))$. The first time high output realizes, the average payoff pair becomes $(u_d, F(u_d))$, where u_d is defined by (10) as the intersection of the first-best Pareto frontier and the IC-DE line. After that, the pair follows the path of the optimal semi-dynamic contract in Proposition 2 and the agent continues to earn rents beyond the initial contract. When output is low, the average payoff to the agent either stays at u_g if $u_g = \overline{u}$, or increases with every failure if $u_g < \overline{u}$. In the latter case, the increment of the average payoff is increasing in the number of previous failures, until the average payoff goes beyond u_d. Once the average payoff goes beyond u_d, the payoff pair again follows the path of the optimal semi-dynamic contract in Proposition 2.

If $R < R^{***}$, and hence $u_g > \overline{u}$, the principal cannot extract all the rent in the initial period even though she designs the dynamic contract optimally ($\tilde{u} = u_g > \overline{u}$): In period 0, the average payoff pair is $(u_g, f(u_g))$. Until the first high output realizes, the average payoff pair remains at $(u_g, f(u_g))$. The average payoff pair becomes $(u_d, F(u_d))$ the first time high output appears. After that, the pair follows the path of the optimal semi-dynamic contract in Proposition 2. Note, however, that by exploiting the fully dynamic nature of the contract, the principal can reduce the rent in the initial period from $u_e - \overline{u}$, the rent under the optimal semi-dynamic contract, to $u_g - \overline{u}$.

We can summarize the preceding analysis as follows:

Proposition 3 *Suppose that the principal is less patient than the agent and the optimal payoff pair is supported with effort, and Assumption 1 holds.*

(a) *There is an optimal, "promotion-like" dynamic contract implementing* $e = 1$ *every period such that* (i) *the agent's continuation payoff will move to the path of the optimal semi-dynamic contract after high output realizes for the first time;* (ii) *his continuation payoff after every low output is constant or increasing; and* (iii) *his continuation payoff does not decrease as time goes on, regardless of the output, and there will never be termination.*

(b) *If* (15) *holds, then there exists* $R^{***} \in (\underline{R}_A, R^{**})$ *such that under the optimal dynamic contract the agent receives a rent in the initial period if and only if* R*, the expected revenue under* $e = 1$*, satisfies* $R \in [\underline{R}_A, R^{***})$*. This rent,* $u_g - \overline{u}$*, is smaller than the rent under the optimal semi-dynamic contract,* $u_e - \overline{u}$*.*

Our result in Proposition 3 (a) that there will never be termination is in sharp contrast with the result of Fong and Li (2017), where there is a positive probability

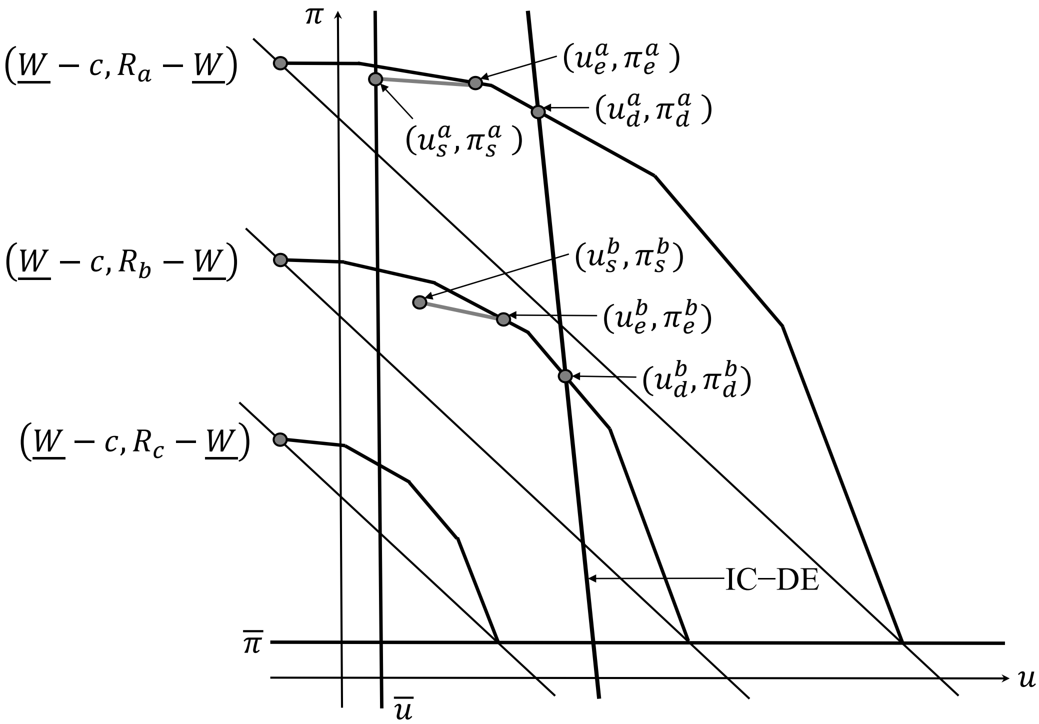

Figure 5: Optimal Dynamic Contracts

of inefficient termination in equilibrium. Under our framework, the inefficiency takes the form of deviation from the efficient intertemporal utility transfer. This is why the payoff pair $(u_g, f(u_g))$ does not lie on the first-best Pareto frontier, as illustrated in Figure 5.

Figure 5 illustrates Proposition 3 (b). The highest first-best Pareto frontier in the figure corresponds to the case $R_a = R^{***}$, and u_g is denoted by u_s^a, which is equal to $\overline{u}$. The optimal payoff for the principal in period 0 is $\pi_s^a = f(u_s^a) = f(\overline{u})$, and the agent obtains no rent. The expected revenue R_b satisfies $\underline{R}_A < R_b < R^{***}$ for the middle first-best Pareto frontier. In this case, u_g is denoted by u_s^b, which is greater than $\overline{u}$. Hence, the agent enjoy a positive rent $u_s^b - \overline{u}$, and the principal's payoff is $\pi_s^b = f(u_s^b)$.

Comparative Statics

We present how the agent's rent changes with some of the parameters in the model under the optimal dynamic contract.

Corollary 3 *Suppose that the principal is less patient than the agent and the optimal payoff pair is supported with effort, and Assumption 1 holds. Furthermore, suppose $\underline{R}_A < R < R^{**}$ is satisfied so that the optimal semi-dynamic contract leaves a rent $u_e - \overline{u}$ to the agent. The agent's rent under the optimal dynamic contract is increasing in $(\overline{\pi}, \overline{u}, \underline{W})$, and is decreasing in R. The opposite happens for the principal's payoff.*

The effects of the other parameters are difficult to pin down. For example, increasing the cost of effort c raises u_d by shifting F inward as well as by increasing the right-hand side of (10). However, u_g is decreasing in c by (13), and hence we cannot determine the net effect of c on the agent's rent. Similarly, the effects of the discount factors δ_A and δ_B are difficult to pin down.

Finally, we compare the agent's rent under optimal stationary, semi-dynamic, and dynamic contracts. The following result is immediate from the preceding analysis.

Corollary 4 *Suppose that the principal is less patient than the agent and the optimal payoff pair is supported with effort, and Assumption 1 holds. The agent's rent decreases as contracting technology advances from stationary contracts to semi-dynamic contracts if $R < R^*$ (so that he enjoys a rent under the former technology), and from semi-dynamic contracts to dynamic contracts if $R < R^{**}$.*

5.3 Possibility of Supporting with Termination and Shirking

In this subsection, we present the analysis on the possibility of supporting the optimal payoff pair with shirking or termination. If $(u, f(u))$ is *supported with shirking*, it must satisfy the following Bellman equation:

$$f(u) = f_s(u) \equiv \max_W (1 - \delta_P)(r - W) + \delta_P f(u_s(u)), \tag{PPs}$$

subject to $W \geq \underline{W}$, $u_s(u) \geq \overline{u}$, and

$$u = (1 - \delta_A) W + \delta_A u_s(u), \tag{UUs}$$

where $r = ph + (1 - p)\ell$ is the expected revenue under shirking, which we have assumed smaller than $\overline{s} = \overline{\pi} + \overline{u}$, and $u_s(u)$ is the average continuation payoff to the agent under shirking.

If $(u, f(u))$ is *supported with termination*, it must satisfy the following equations:

$$f(u) = f_x(u) \equiv (1 - \delta_P)\overline{\pi} + \delta_P f(u_x(u)), \tag{PPx}$$

subject to $u_x(u) \geq \overline{u}$ and

$$u = (1 - \delta_A)\overline{u} + \delta_A u_x(u), \tag{UUx}$$

where $u_x(u)$ is the expected continuation payoff to the agent under termination.

$(u, f(u))$ can also be supported with randomization, meaning that it is a convex combination of vertexes that are supported with either effort, shirking, or termination. The principal's highest payoff $f(u)$ is the concave upper envelope of $f_e(u)$, $f_s(u)$, and $f_x(u)$ for $u \geq \overline{u}$.

As in our analysis in Subsection 5.2, we focus on the optimal relational contract and the optimal payoff pair in the initial period, $(\tilde{u}, f(\tilde{u}))$. The first lemma shows that the optimal payoff pair cannot be supported with termination.

Lemma 7 *Given $R \geq \underline{R}_A$, $(\tilde{u}, f(\tilde{u}))$ cannot be supported with termination.*

The next lemma shows that if the optimal payoff pair is supported with shirking, the optimal payoff to the principal must be lower than $r - \underline{W}$.

Lemma 8 *If $(\tilde{u}, f(\tilde{u}))$ is supported with shirking, the agent must receive no rent ($\tilde{u} = \overline{u}$), and $f_s(\tilde{u}) < r - \underline{W}$ must hold.*

These characterizations has strong implications. In Fong and Li (2017), they implies that f must be linear between u and $u_s(u)$ if $(u, f(u))$ is supported with shirking, or between u and $u_x(u)$ if $(u, f(u))$ is supported with termination (see the proof of their Proposition 1). In the current paper, $\delta_P \neq \delta_A$, and in particular $\delta_P < \delta_A$ in this section, and they rule out the possibility that the equilibrium supported with termination, and the possibility that the equilibrium is supported with shirking under which the agent enjoys a positive rent.

We now obtain a sufficient condition for the optimal payoff pair $(\tilde{u}, f(\tilde{u}))$ to be supported with effort. By Lemma 7, it cannot be supported with termination. If it is supported with shirking, $f(\tilde{u}) = f_s(\overline{u})$ must be lower than $r - \underline{W}$. Hence the optimal payoff pair must be supported with effort if

$$r - \underline{W} \leq f_e(u_g)$$

holds. The left-hand side of the inequality is highest when $\underline{W} = \overline{u} + c - (Pc)/q$ (as required by Proposition 2 (b)). In that case,

$$r - \underline{W} = r - \overline{u} - c + \frac{P}{q}c = r - \overline{u} + \frac{p}{q}c. \tag{16}$$

The next lemma shows that $f_e(u_g)$, the right-hand side of the inequality, is increasing in R and decreasing in $\underline{W}$.

Lemma 9

$$\frac{d}{dR} f_e\left(u_g\right) > 0 \quad \textit{and} \quad \frac{d}{d\underline{W}} f_e\left(u_g\right) < 0 \quad \textit{hold.}$$

Hence the right-hand side of the inequality is lowest when $R = \underline{R}_A$ (in which case $u_g > \overline{u}$) and $\underline{W} = \overline{u}$,[8] which leads to

$$f_e(u_g) = \pi + \frac{1-\delta_P}{1-\delta_P(1-P)} \frac{1+d_A q}{d_A q} c. \tag{17}$$

Combining (17) with (16) yields a sufficient condition for $r - \underline{W} \leq f_e(u_g)$, which we present as the following assumption that the expected revenue under shirking r is sufficiently small.

Assumption 2

$$r \leq \overline{u} + \overline{\pi} + \left\{ \frac{1-\delta_P}{1-\delta_P(1-P)} \frac{1+d_A q}{d_A} - p \right\} \frac{c}{q}.$$

The result of this subsection is summarized as the following proposition.

Proposition 4 *If Assumption 2 is satisfied, the optimal payoff pair $(\tilde{u}, f(\tilde{u}))$ must be supported with effort.*

Equilibrium Supported with Shirking

We consider the parameter range in which equilibrium supported with shirking, i.e.

$$f_s(\overline{u}) = (1-\delta_P)(r - \underline{W}) + \delta_P f(u_s(\overline{u})) > f_e(u_g), \tag{18}$$

arises, by considering some extreme cases. First, we assume that both R and $\underline{W}$ are at their lowest possible value and r is at its highest possible value, i.e. $R = \underline{R}_A$, $\underline{W} = \overline{u} + c - (P/q)c$, and $r = \overline{u} + \overline{\pi}$. In this case, (14) yields

$$f\left(u_g\right) = \frac{\left(1-\delta_P\right)\left(\underline{R}_A - \underline{W}\right) + \delta_P P \overline{\pi}}{1-\delta_P\left(1-P\right)}.$$

In addition, $u_s(\overline{u}) = \overline{u} + ((1-\delta_A)/\delta_A)(p/q)c$. To find a lower bound for $f(u_s(\overline{u}))$, first note that $u_g = \overline{u} < u_s(\overline{u}) < u_d$. Second, the Pareto frontier is bounded below by a straight line between $(u_g, f(u_g)) = (\overline{u}, f(\overline{u}))$ and $(u_e, f(u_e))$, defined as $(u, L(u))$, and

$$\begin{aligned} L(u_s(\overline{u})) &= p\overline{\pi} + f(u_g(\overline{u})) \frac{(\overline{u} + c/(qd_A)) - u_s(\overline{u})}{c/(qd_A)} \\ &= p\overline{\pi} + f(u_g(\overline{u}))(1-p), \end{aligned}$$

[8]While $\underline{W} \leq \overline{u} + c$ in Assumption 1, we show in the proof of Lemma 8 in Appendix that if $(\overline{u}, f(\overline{u}))$ is supported with shirking, $\overline{W} \leq \overline{u}$ must hold.

Third,

$$(1 - \delta_P)(r - \underline{W}) + \delta_P L(\overline{u}) > f_e(u_g), \tag{19}$$

holds if and only if

$$p > \frac{1 - \delta_P(1 - p)}{1 - \delta_P(1 - P)} \frac{1 + Pd_A}{qd_A}, \tag{20}$$

It follows that (20) is a sufficient condition for (18). In (20), the first ratio on the right-hand side is smaller than 1 and the second ratio on the right-hand side is larger than 1. However, there can clearly be cases that (20) holds. The observation that (20) cannot hold if Assumption 2 holds and $r = \overline{u} + \overline{\pi}$ serves as a confirmation for (20) to be a sufficient condition for the equilibrium to be supported with shirking. Given this and the fact that $f_s(\overline{u})$ is continuous and increasing in r, and $f_e(u_g)$ is increasing in R the following result hold.

Lemma 10 *Suppose that* (20) *holds for* $R = \underline{R}_A$, $r = \overline{u} + \overline{\pi}$, *and* $\underline{W} = \overline{u} + c - (P/q)c$.

(i) *Given that* $R = \underline{R}_A$ *and* $\underline{W} = \overline{u} + c - (P/q)c$, *there is a cutoff* $\overline{r}_1$ *such that the equilibrium is supported with shirking if and only if* $r > \overline{r}_1$.

(ii) *Given that* $r > \overline{r}_1$ *and* $\underline{W} = \overline{u} + c - (P/q)c$, *there exists a cutoff* $\overline{R}_1$ *such that the equilibrium is supported with shirking if and only if* $R < \overline{R}_1$.

The second extreme point worth noting is when R and r remain the same as in the previous case and $\underline{W}$ is increased to its highest possible value, i.e. $\overline{u}$. In this case, it is clear that the equilibrium can not be supported with shirking because $u_s(\overline{u}) = \overline{u}$, so that supporting the equilibrium means the agent will shirk in every period. Note that in Fong and Li (2017), the possibility of shirking is ruled out under the assumption that the surplus from effort is high enough.

5.4 When The Principal Is More Patient

So far in this section we have focused on the case where the principal is less patient than the agent ($d_P < d_A$). In this subsection, we consider the other case in which the principal is more patient ($d_P > d_A$).

If $d_P > d_A$, the first-best Pareto frontier has to be modified from that under $d_P < d_A$. The procedure of finding it is similar to the preceding analysis except that the directions of movement for the payments are reversed: W_t must be decreasing in t. This means that the agent's stage payoff cannot drop below $\overline{u}$ in any period since otherwise (IR) will be violated for some t. Hence, the principal's maximum stage payoff is $s - \overline{u}$, which implies that $F(\overline{u}) = s - \overline{u}$. When (IC-DE) is taken into consideration, the result is the same whether contracts are restricted to

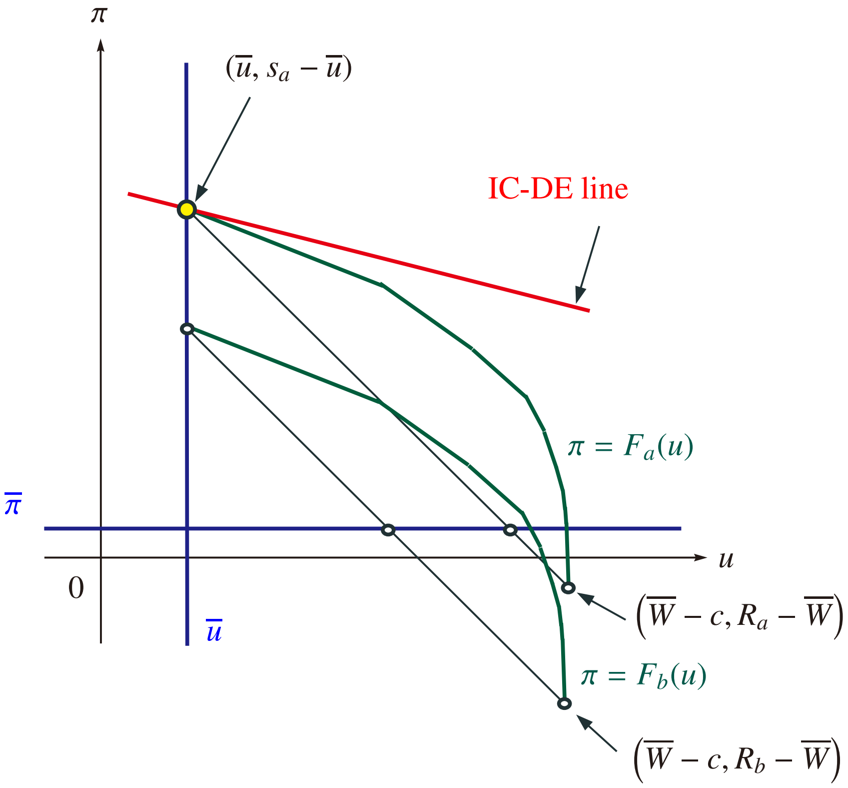

Figure 6: When The Principal Is More Patient

stationary ones, or semi-dynamic or dynamic contracts are feasible: It all depends on whether the payoff pair most preferred by the principal $(\overline{u}, s - \overline{u})$ satisfied the IC-DE constraint. The following result is immediate, where $\underline{R}_P$ is the expected revenue under $e = 1$, defined by (6), such that the IC-DE line intersects with the first-best Pareto frontier at $(\overline{u}, s - \overline{u})$.[9]

Proposition 5 *Suppose that the principal is more patient than the agent ($d_P > d_A$).*

(a) *If $R \geq \underline{R}_P$, then the stationary, pure-bonus contract β^* is optimal, and the agent receives no rent: The principal's stage payoff is $s - \overline{u}$ for all periods.*

(b) *If $R < \underline{R}_P$, $e = 1$ cannot be implemented and the agent receives no rent.*

Figure 6 illustrates cases (a) and (b) of Proposition 5, where $\overline{W}$ is an upper bound on the expected total payment to the agent. The first-best Pareto frontiers are represented by green kinked curves for two different levels of R. Case (a) corresponds to the higher frontier $\pi = F_a(u)$ under $R_a = \underline{R}_P$, and hence the stationary pure-bonus contract shown as $(\overline{u}, s_a - \overline{u})$ is optimal. If R increases from R_a, the Pareto frontier moves upward. Then the point corresponding to the pure-bonus contract remains above the IC-DE line, and hence optimal. It

[9]We denote $R^* = \underline{R}_P$ in the previous sections.

is important to observe that the slope of the Pareto frontier is anywhere steeper than that of the IC-DE line, and thus if there exists a point on the frontier above the IC-DE line, the one corresponding to the pure bonus contract is also above it.

Case (b) of the proposition is illustrated by the Pareto frontier $\pi = F_b(u)$ under $R_b < \underline{R}_P$. It is everywhere below the IC-DE line, and hence $e = 1$ cannot be implemented.

6 Commitment

In Section 3, we present a benchmark in which the principal and the agent are equally patient and engage in relational contracting. Another important benchmark is when they cam commit themselves to long-term contracts, that is, when payments specified by any contract signed between the principal and the agent cannot be reneged, so (DEP) and (DEA) can be ignored.

As long as W is allowed to be set below $\overline{u}+c$ in every period, the agent enjoys no rent in our model if the principal and the agent could make the long-term commitment. This result holds not only when the principal's discount factor is equal to that of the agent, based on the analysis of the benchmark in Section 3, but also whether her discount factor is higher or lower than that of the agent. Given that the set of stage payoffs that can be used along the Pareto-optimal path of the repeated game is a right triangle with a hypotenuse that has a negative one slope, the first-best Pareto frontier always has a negative slope (see, for example, (7) for the case of the impatient principal). Without the restriction of the enforcement constraints, the whole first-best Pareto frontier can be implemented with a current bonus and penalty satisfying (IC). The principal will clearly choose the tip $(\overline{u}, F(\overline{u}))$ and leave no rent to the agent. In sum, there will be no rent for the agent no matter how the the discount factors of the principal and the agent are ranked.

As we explain in Section 1, this benchmark distinguishes the current paper from the closely related paper on relational contracts by Fong and Li (2017). In their model, the agent enjoys the standard limited-liability rent under the optimal one-shot contract, and it remains so after the principal extracts some of the limited-liability rent back by using the optimal long-term contract or the optimal relational contract.

Based on Section 3 and the current section, we can argue that the rent we identify is not due to limited liability but due to *both* 1) limited revenue *and* 2) different discount factors between the principal and the agent. Our rent exists even though the expected total payment to the agent can be decreased below $\overline{u}+c$, unless it can be low enough to satisfy $\underline{W} \leq \overline{u}+c-(P/q)c$. Limited liability rent will not disappear when the revenue increases or when the discount factors are the same.

7 Conclusion

We solved for the optimal dynamic relational incentive contract when the principal and the agent have different discount factors, and identified a new agency cost arising from a trade-off between rent extraction and incentive provision that exists only when the parties cannot commit themselves to long-term contracts and the principal is less patient than the agent.

The analysis is restricted to the case the agent's effort choice is binary. The optimality of "one-step" payment schemes under the assumption of the validity of the first-order approach (Levin, 2003) suggests that we be able to extend our analysis to the continuous-effort case as long as the effort implemented is fixed. However, the optimal effort itself may change over time. Such an extended analysis is left for future research.

Another topic to be left for future research is an application to the relationship between an entrepreneur and investors, as we mentioned in Section 1. Most existing literature in financial contracting with different discount factors between them also assumes limited liability. Our model and analysis may be able to contribute to the literature by distinguishing effects of limited liability from those of different discount factors.

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Appendix

Proof of Lemma 4

(a) Suppose that $W > \underline{W}$ at the solution. Consider an alternative contract $\widehat{\mathbf{w}}$ with b_h and b_ℓ decreased by $\epsilon > 0$, that is small enough that $W \geq \underline{W}$ still holds, and with $u_h(u)$ and $u_\ell(u)$ increased by ϵ/d_A. This alternative contract satisfies (UUe) and, by Lemma 3, increases the right hand side of (IC-DE) and hence also satisfies it. The principal is better off with the alternative contract if the change in the principal's payoff from switching to $\widehat{\mathbf{w}}$, given by

$$(1-\delta_P)\,\epsilon + \delta_P \frac{\epsilon}{d_A}\left(P f'\left(u_h(u)\right) + (1-P)\, f'\left(u_\ell(u)\right)\right),$$

is positive. We show it is in fact positive, which is a contradiction.

To show this, note first that (7) implies $F'(z) > -d_A/d_P$ for all z. Because $f(z)$ is concave and coincides with $F(z)$ for the part of $F(z)$ that has the steepest slopes and $f(z) \leq F(z)$ for all z, we have

$$f'(z) \geq F'(z) > -\frac{d_A}{d_P}, \tag{A1}$$

which then implies

$$-\frac{d_A}{d_P} < P f'\left(u_h(u)\right) + (1-P)\, f'\left(u_\ell(u)\right),$$

and hence

$$\begin{aligned}&(1-\delta_P)\,\epsilon + \delta_P \frac{\epsilon}{d_A}\left(P f'\left(u_h(u)\right) + (1-P)\, f'\left(u_\ell(u)\right)\right),\\ &\quad > (1-\delta_P)\,\epsilon + \delta_P \frac{\epsilon}{d_A}\left(-\frac{d_A}{d_P}\right) = 0.\end{aligned}$$

(b) First note $u_h(u_e) = u_\ell(u_e) = u_d$. Furthermore, (IC-DE) and (A1) imply $u_h(u) \geq u_d$. Then (UUe), along with $W = \underline{W}$, implies that $u_\ell(u) < u_d$, that is, $u_\ell(u) < u_h(u)$ must hold for $u < u_e$.

(c) Note that after plugging $W = \underline{W}$ into the objective function, the decision variables W, b_ℓ, and b_h disappear from the program (PPe), and hence can be ignored. The program is then reduced to finding $u_h = u_h(u)$ and $u_\ell = u_\ell(u)$ so that $Pu_h+(1-P)u_\ell$ is equal to u (to satisfy (UUe)), u_h satisfies (IC-DE) (which no longer consists of u_ℓ), and $Pf(u_h)+(1-P)f(u_\ell)$ is maximized. Because $u_\ell < u_h$ as shown in part (b) and $f(z)$ is a concave function, reducing u_h to the point $u_h = u_d$, the lowest u_h satisfying (IC-DE), and then increasing u_ℓ to satisfy (UUe) increases the objective function. Hence, (IC-DE) must be binding.

Proof of Lemma 5

(13) implies that

$$u_e - u_g = \frac{\delta_A (1-\delta_A)(1-P)(u_d - (\underline{W}-c))}{1-\delta_A (1-P)} > 0.$$

and hence $u_g < u_e$.

Using $f(u_e) = (1-\delta_P)(R-\underline{W}) + \delta_P f(u_d)$ and (14) yields

$$f(u_e) - f(u_g) = \frac{\delta_P (1-\delta_P)(1-P)[f(u_d) - (R-\underline{W})]}{1-\delta_P (1-P)} < 0,$$

and hence $f(u_g) > f(u_e)$ holds.

Proof of Corollary 3

First, note that R and $\underline{W}$ affect the payoff frontier F: An increase in R shifts F outward, while $\underline{W}$ shifts F inward. Hence by (10), increasing R decreases u_d, which in turn increases $f(u_d)$. The conclusion then follows from (13) and (14). And by (10), increasing $\underline{W}$ raises u_d, which in turn reduces $f(u_d)$. Furthermore, u_g is increasing in $\underline{W}$ by (13) and $f(u_g)$ is decreasing in $\underline{W}$ by (14). The conclusion follows.

Next, increasing $\overline{u}$ does not affect F, but via (10) raises u_d, which in turn reduces $f(u_d)$. The conclusion then follows from (13) and (14).

Finally, an increase in $\overline{\pi}$ shifts F inward, and by (10), resulting in an increase in u_d and a decrease in $f(u_d)$. The conclusion then follows from (13) and (14).

Proof of Lemma 7

First suppose $\tilde{u} = \overline{u}$. Then if $(\overline{u}, f(\overline{u}))$ is supported by termination, $u_x(u) = \overline{u}$, and hence the agent's payoff is equal to $\overline{u}$ every period, implying that termination happens forever. Then $f_x(\tilde{u}) = \overline{\pi}$. This is impossible because the principal can obtain at least $f(u_e) > \overline{\pi}$ with effort. We then show below that $(\tilde{u}, f(\tilde{u}))$ cannot be supported with termination in the case of $\tilde{u} > \overline{u}$, either, and this completes the proof.

Suppose that $\tilde{u} > \overline{u}$ and $(\tilde{u}, f(\tilde{u}))$ can be supported with termination so $f(\tilde{u}) = f_x(\tilde{u})$. We prove the lemma by showing

$$f'_-(\tilde{u}) > f'_{x-}(\tilde{u}) > 0, \tag{A2}$$

which, along with $f_x(\tilde{u}) = f(\tilde{u})$ implies that f_x is higher than f in the left neighborhood of $\tilde{u}$, contradicting the definition of f as the Pareto frontier.

Equation (UUx) implies $u'_x(\tilde{u}) = 1/\delta_A$ and hence

$$f'_{x-}(\tilde{u}) = \delta_P f'_-(u_x(\tilde{u}))u'_x(\tilde{u}) = \frac{\delta_P}{\delta_A} f'_-(u_x(\tilde{u})) \tag{A3}$$

Because $f_x(\tilde{u}) = f(\tilde{u}) > \overline{\pi}$, (PPx) implies

$$f(\tilde{u}) = f_x(\tilde{u}) < f(u_x(\tilde{u})). \tag{A4}$$

In addition, (UUx) implies $\tilde{u} < u_x(\tilde{u})$, and hence

$$f'_-(\tilde{u}) \geq f'_-(u_x(\tilde{u})), \tag{A5}$$

by the concavity of f, and

$$f'_-(u_x(\tilde{u})) > 0, \tag{A6}$$

by (A4) and the concavity of f. (A3) and (A6) then lead to $f'_{x-}(\tilde{u}) > 0$, and hence, given $\delta_P < \delta_A$,

$$f'_-(u_x(\tilde{u})) > f'_{x-}(\tilde{u}). \tag{A7}$$

(A2) follows from (A5), (A6), and (A7).

Proof of Lemma 8

Before proving the lemma, we first prove the following two lemmas.

Lemma A1 *At the solution to the maximization problem* (PPs)*, $W = \underline{W}$ holds.*

Proof. Suppose that $W \geq \underline{W}$ is not binding at the optimum. Consider an alternative contract $\widehat{\mathbf{w}}$ with W decreased by $\epsilon > 0$, that is small enough that $W \geq \underline{W}$ still holds, and $u_s(u)$ increased by ϵ/d_A. This alternative contract satisfies (UUs). The principal is better off with the alternative contract if the change in her payoff from switching to $\widehat{\mathbf{w}}$,

$$(1 - \delta_P)\,\epsilon + \delta_P \frac{\epsilon}{d_A} f'\left(u_s(u)\right)$$

is positive. However, using (A1) in the proof of Lemma 4 yields

$$\begin{aligned} & (1-\delta_P)\,\epsilon + \delta_P \frac{\epsilon}{d_A} f'\left(u_s(u)\right) \\ > & (1-\delta_P)\,\epsilon + \delta_P \frac{\epsilon}{d_A}\left(-\frac{d_A}{d_P}\right) \\ = & 0. \end{aligned}$$

A contradiction. ■

Lemma A2 *If $(u, f(u))$ is supported with shirking, then* (a) $\underline{W} \leq u \leq u_s(u)$*;* (b) $f'_-(u) \leq 0$ *and* $f'_+(u) \leq 0$*; and* (c) $f(z) \leq r - \underline{W}$ *for all* $z \geq u$*.*

Proof. (a) Given (UUs), the definition of $u_s(u)$, the relationship between $\underline{W}$, u, and $u_s(u)$ can be either (i) $\underline{W} \leq u \leq u_s(u)$ or (ii) $u_s(u) < u < \underline{W}$. Suppose that (ii) is the case and hence $\underline{W} > \overline{u}$ and $r - \underline{W} < \overline{\pi}$, the latter of which is due to the assumption $r < \overline{s}$. Given that $f(u_s(u)) > \overline{\pi}$ by definition, this implies that

$$f(u) = f_s(u) = (1-\delta_P)(r - \underline{W}) + \delta_P f(u_s(u)) < f(u_s(u)).$$

Given $u_s(u) < u$, this implies $f'_+(u_s(u)) < 0$, which, combined with the fact that f is concave, implies that $f'_+(z) < 0$ holds for all $z > u_s(u)$. Hence using $u'_y(z) = 1/\delta_A$ yields

$$f'_{s+}(z) = \delta_P f'_+(u_s(z)) u'_y(z) = \frac{\delta_P}{\delta_A} f'_+(u_s(z)) > f'_+(u_s(z)) \geq f'_+(z)$$

for all $z \in [u, \underline{W}]$. This implies that

$$f(\underline{W}) \geq f_s(\underline{W}) = f_s(u) + \int_u^{\underline{W}} f'_{y+}(z)dz > f(u) + \int_u^{\underline{W}} f'_+(z)dz = f(\underline{W}),$$

which is a contradiction.

(b) It suffices to show that $f'_-(u) \leq 0$ because it implies $f'_+(u) \leq 0$ by the concavity of f. Suppose, on the contrary, that $f'_-(u) > 0$. Because $f(u) = f_s(u)$ and $f(z) \geq f_s(z)$ for all z, it follows that $f'_{y-}(u) > 0$. This implies $f'_-(u_s(u)) > 0$ since

$$f'_{y-}(z) = \frac{\delta_P}{\delta_A} f'_-(u_s(z)) \text{ for all } z. \tag{A8}$$

It then follows from the convexity of f that $f'_-(z) > 0$ for all $z \leq u_s(u)$ and hence, by (A8) again,

$$f'_{y-}(z) > 0 \text{ for all } z \leq u. \tag{A9}$$

This implies that

$$f'_{y-}(z) = \frac{\delta_P}{\delta_A} f'_-(u_s(z)) < f'_-(u_s(z)) \leq f'_-(z) \text{ for all } z \in [\underline{W}, u],$$

where the first equality is given by (A8), the second inequality is implied by (A9) and $\delta_P/\delta_A < 1$, and the third inequality is implied by the concavity of f. It follows that

$$f(\underline{W}) \geq f_s(\underline{W}) = f_s(u) - \int_{\underline{W}}^{u} f'_{y-}(z)dz > f(u) - \int_{\underline{W}}^{u} f'_-(z)dz = f(\underline{W}),$$

which is a contradiction.

(c) Because f is concave, (b) implies that $f'_+(z) \leq 0$ for all $z \geq u$. Hence, $f_s(u) = f(u)$ is no lower than any point on the second-best Pareto frontier to the right of u, and in particular, $f_s(u) \geq f(u_s(u))$. By definition (PPs), $f_s(u)$ is a weighted average between $r - \underline{W}$ and $f(u_s(u))$. This implies that $r - \underline{W} \geq f_s(u) = f(u) \geq f(u_s(u))$ and hence $f(z) \leq r - \underline{W}$ for all $z \geq u$. ■

We now prove Lemma 8. By Lemma 7, $(\tilde{u}, f(\tilde{u}))$ cannot be supported with termination. And if $\tilde{u} > \overline{u}$, $(\tilde{u}, f(\tilde{u}))$ cannot be supported with shirking because $f(\tilde{u}) = f_s(\tilde{u})$ implies that $f'_-(\tilde{u}) > 0$, violating Lemma A2 (b). Hence, the agent cannot enjoy a rent if it is supported with shirking.

Finally, if $\tilde{u} = \overline{u}$ and $(\overline{u}, f(\overline{u}))$ is supported with shirking,

$$f(\tilde{u}) = f_s(\tilde{u}) = (1 - \delta_P)(r - \underline{W}) + \delta_P f(u_s(\tilde{u}))$$

and

$$f(\tilde{u}) = f_s(\tilde{u}) > f(u_s(u)).$$

It follows that $f_s(\tilde{u}) < r - \underline{W}$.

Proof of Lemma 9

Proof.

$$\frac{d}{dR} f_e(u_g) = (1 - \delta_P) + \delta_P P \frac{d}{dR} f_e(u_d) + \delta_P (1 - P) \frac{d}{dR} f_e(u_g);$$
$$\frac{d}{d\underline{W}} f_e(u_g) = -(1 - \delta_P) + \delta_P P \frac{d}{d\underline{W}} f_e(u_d) + \delta_P (1 - P) \frac{d}{d\underline{W}} f_e(u_g).$$

Hence,

$$\begin{aligned}
\frac{d}{dR} f_e\left(u_g\right) &= \frac{1-\delta_P}{1-\delta_P(1-P)} + \frac{\delta_P P}{1-\delta_P(1-P)} \frac{d}{dR} f_e\left(u_d\right) > 0; \\
\frac{d}{d\underline{W}} f_e\left(u_g\right) &= -\frac{1-\delta_P}{1-\delta_P(1-P)} + \frac{\delta_P P}{1-\delta_P(1-P)} \frac{d}{d\underline{W}} f_e\left(u_d\right) < 0,
\end{aligned}$$

because $F(u)$ increases (decreases) as R ($\underline{W}$, respectively) increases. $\blacksquare$