Numerical Linear Algebra: Homework 1

Due on Aug 18, 2014

Lecture time: 6:00 pm

Weiyi Chen

Problem 1

(i)

Suppose A is an invertible square matrix and u, v are vectors. Suppose furthermore that $1 + v^T A^{-1} u \neq 0$. Then the ShermanMorrison formula states that

$$(A + uv^T)^{-1} = A^{-1} - \frac{A^{-1}uv^T A^{-1}}{1 + v^T A^{-1} u}. \tag{1}$$

Here, uv^T is the outer product of two vectors u and v. **Proof.** We verify the properties of the inverse. A matrix Y (in this case the right-hand side of the ShermanMorrison formula) is the inverse of a matrix X (in this case $A + uv^T$) if and only if $XY = YX = I$.
We first verify that the right hand side (Y) satisfies XY = I.

$$\begin{aligned} XY &= (A + uv^T)\left(A^{-1} - \frac{A^{-1}uv^T A^{-1}}{1 + v^T A^{-1}u}\right) && (2)\\ &= AA^{-1} + uv^T A^{-1} - \frac{AA^{-1}uv^T A^{-1} + uv^T A^{-1}uv^T A^{-1}}{1 + v^T A^{-1}u} && (3)\\ &= I + uv^T A^{-1} - \frac{uv^T A^{-1} + uv^T A^{-1}uv^T A^{-1}}{1 + v^T A^{-1}u} && (4)\\ &= I + uv^T A^{-1} - \frac{u(1 + v^T A^{-1}u)v^T A^{-1}}{1 + v^T A^{-1}u} && (5) \end{aligned}$$

Note that $v^T A^{-1} u$ is a scalar, so $(1 + v^T A^{-1} u)$ can be factored out, leading to:

$$XY = I + uv^T A^{-1} - uv^T A^{-1} = I. \tag{6}$$

In the same way, it is verified that

$$YX = \left(A^{-1} - \frac{A^{-1}uv^T A^{-1}}{1 + v^T A^{-1}u}\right)(A + uv^T) = I. \tag{7}$$

In our problem, let $A = I, u = x, v = y$, we have

$$(I + xy^t)^{-1} = I - \frac{xy^t}{1 + y^t x} \tag{8}$$

(ii)

We have proved one direction, the other direction is to prove if $y^t x = -1$ then the matrix $I + xy^t$ is singular. Assume by contradiction that $I + xy^t$ is nonsingular, then

$$AA = (I + xy^t)(I + xy^t) = I + xy^t + xy^t + xy^t xy^t = I + xy^t = A \tag{9}$$

Since A is nonsingular, there exist A^-1, therefore

$$AA = A \Rightarrow AAA^{-1} = AA^{-1} \Rightarrow A = I \Rightarrow xy^t = 0 \tag{10}$$

which implies $\sum_i x_i y_i = 0$, however $y^t x = \sum_i x_i y_i = 0$ contradicts to the condition $y^t x = -1$. So the matrix $I + xy^t$ is singular.

Problem 2

(i)

When $n = 1$, obviously $A_1^t = A_1^t$.
Suppose when $n = k$ for $k \geq 1$, it satisfies the equation

$$(\prod_{i=1}^{k} A_i)^t = \prod_{i=1}^{k} A_{k+1-i}^t \tag{11}$$

When $n = k + 1$,

$$(\prod_{i=1}^{k+1} A_i)^t = ((\prod_{i=1}^{k} A_i)A_{k+1})^t = A_{k+1}^t(\prod_{i=1}^{k} A_i)^t = A_{k+1}^t \prod_{i=1}^{k} A_{k+1-i}^t = \prod_{i=1}^{k+1} A_{k+1-i}^t \tag{12}$$

satisfying the given formula.

(ii)

Similar to the proof above, the inverse of matrix also satisfies the equation

$$(AB)^{-1} = B^{-1}A^{-1} \tag{13}$$

similar to the transpose $(AB)^t = B^tA^t$, the reason is illustrated as follows, the inverse of a product AB of matrices A and B can be expressed in terms of A^{-1} and B^{-1}. Let $C = AB$. Then

$$B = A^{-1}AB = A^{-1}C \tag{14}$$

and

$$A = ABB^{-1} = CB^{-1} \tag{15}$$

Therefore,

$$C = AB = (CB^{-1})(A^{-1}C) = CB^{-1}A^{-1}C, \tag{16}$$

so

$$CB^{-1}A^{-1} = I, \tag{17}$$

where I is the identity matrix, and

$$B^{-1}A^{-1} = C^{-1} = (AB)^{-1} \tag{18}$$

Now we use induction to prove. When $n = 1$, obviously $A_1^{-1} = A_1^{-1}$.
Suppose when $n = k$ for $k \geq 1$, it satisfies the equation

$$(\prod_{i=1}^{k} A_i)^{-1} = \prod_{i=1}^{k} A_{k+1-i}^{-1} \tag{19}$$

When $n = k + 1$,

$$(\prod_{i=1}^{k+1} A_i)^{-1} = ((\prod_{i=1}^{k} A_i)A_{k+1})^{-1} = A_{k+1}^{-1}(\prod_{i=1}^{k} A_i)^{-1} = A_{k+1}^{-1} \prod_{i=1}^{k} A_{k+1-i}^{-1} = \prod_{i=1}^{k+1} A_{k+1-i}^{-1} \tag{20}$$

satisfying the given formula.

Problem 3

(i)

The result is

$$B^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 \\ 7 & -1 & 0 & 0 \end{pmatrix}, B^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 \end{pmatrix}, B^4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{21}$$

(ii)

According to the hint, the result is

$$C^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 6 & 1 & 0 & 0 \\ -1 & -2 & 1 & 0 \\ 5 & 3 & 2 & 1 \end{pmatrix}, C^3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 9 & 1 & 0 & 0 \\ -6 & -3 & 1 & 0 \\ 15 & 3 & 3 & 1 \end{pmatrix}, C^4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 12 & 1 & 0 & 0 \\ -14 & -4 & 1 & 0 \\ 26 & 2 & 4 & 1 \end{pmatrix} \tag{22}$$

In general (if $m \geq 3$),

$$C^m = \sum_{j=0}^{m} \binom{m}{j} B^{m-j} = I + mB + \frac{m(m-1)}{2}B^2 + \frac{m(m-1)(m-2)}{6}B^3 \tag{23}$$

where B, B^2, B^3 are given in part(i) and $B_i = 0$ for $i \geq 4$.

Problem 4

(i)

We will prove by induction that

$$L^m(i,j) = 0 \tag{24}$$

for any $1 \leq i \leq j + (m-1) \leq n$.
When $m = 1$, obviously it is true since we are given

$$L(i,j) = 0 \tag{25}$$

for any $1 \leq i \leq j \leq n$.
When $m = 2$,

$$L^2(i,j) = \sum_{k=1}^{n} L(i,k)L(k,j) = 0 \tag{26}$$

for any $1 \leq i \leq j+1 \leq n$, since $L(i,j) = 0$ when $k \geq i$ and $L(k,j) = 0$ when $k \leq j$.
Now suppose the formula holds when $m = K$, let's consider case $m = K+1$ when $i \leq j + K$,

$$L^{K+1}(i,j) = \sum_{k=1}^{n} L^K(i,k)L(k,j) = \sum_{k=1}^{j} L(k,j)L^K(i,k) + \sum_{k=j+1}^{n} L^K(i,k)L(k,j) \tag{27}$$

Note that $L(k,j) = 0$ when $k \leq j$ and

$$L^K(i,k) = 0 \tag{28}$$

when $i \leq k + (K-1)$. Note that $j+1 \geq (i-K)+1 = i-(K-1) \geq k$, so the equation above holds when k iterate from $j+1$ to n, that is

$$L^{K+1}(i,j) = \sum_{k=1}^{j} L(k,j)L^K(i,k) + \sum_{k=j+1}^{n} L^K(i,k)L(k,j) = \sum_{k=1}^{j} 0L^K(i,k) + \sum_{k=j+1}^{n} 0L(k,j) = 0 \tag{29}$$

We finish the proof. Now just let $m = n$, we find that

$$L^n(i,j) = 0 \tag{30}$$

for any $1 \leq i \leq j + (n-1) \leq n$ which implies all entries of L^n are 0, $L^n = 0$.

(ii)

Since $L^k = 0$ for $k \geq n$,

$$(1+L)^m = \sum_{j=0}^{m} \binom{m}{j} B^j = \sum_{j=0}^{n-1} \binom{m}{j} B^j \tag{31}$$

where $m \geq n$.

Problem 5

Let $A = L_1U_1 = L_2U_2$, decomposite L_1 as

$$L_1 = LD_1 \tag{32}$$

where L is a unit lower triangle matrix and D_1 is a diagonal matrix, with their entries as

$$L(i,j) = \frac{L_1(i,j)}{L_1(j,j)}, D_1(i,i) = L_1(i,i) \tag{33}$$

for $i = 1:n, j = 1:n$. Note that $L_1(j,j) \neq 0$ since L_1 is nonsingular, otherwise $det(L_1) = \prod_i L_1(i,i) = 0$. In the same way we have

$$L_2 = L'D_2 \tag{34}$$

where

$$L'(i,j) = \frac{L_2(i,j)}{L_2(j,j)}, D_2(i,i) = L_2(i,i) \tag{35}$$

According to the uniqueness property of LU-decomposition: if a non-singular matrix A has an LU-factorization in which L is a unit lower triangular matrix, then L and U are unique. In our case,

$$L = L', U = D_1U_1 = D_2U_2 \tag{36}$$

is the unique LU-decomposition of A. Therefore we obtain the D indicated in the problem,

$$L_2 = L'D_2 = LD_2 = L_1D_1^{-1}D_2 = L_1(D_2^{-1}D_1)^{-1}, U_2 = D_2^{-1}D_1U_1 \tag{37}$$

so $D = D_2^{-1}D_1$. It is easy to verify that D is nonsingular and diagonal because both D_1 and D_2 are nonsingular and diagonal.

Problem 6

Denote $C = AB$, according to the definition,

$$C(i,j) = \sum_k A(i,k)B(k,j) \tag{38}$$

Obviously all entries of C are nonnegative since all entries of A, B are nonnegative. Furthermore, the sum of the entries in any row i:

$$\sum_j C(i,j) = \sum_j \sum_k A(i,k)B(k,j) = \sum_k \left(A(i,k) \sum_j B(k,j) \right) = \sum_k A(i,k) = 1 \tag{39}$$

Therefore the matrix $C = AB$ has the same property.

Problem 7

$$det \begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix} = det \begin{pmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{pmatrix} \tag{40}$$

$$= (b-a)(c-a) det \begin{pmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{pmatrix} \tag{41}$$

$$= (b-a)(c-a) det \begin{pmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & c-b \end{pmatrix} \tag{42}$$

$$= (b-a)(c-a)(c-b) \tag{43}$$

Problem 8

(i)

The correlation matrix is

$$\Omega = D^{-1}\Sigma D^{-1} = \begin{pmatrix} 1. & -0.35 & 0.55 & -0.15 & -0.25 \\ -0.35 & 1. & 0.05 & 0.25 & -0.15 \\ 0.55 & 0.05 & 1. & 0.35 & -0.25 \\ -0.15 & 0.25 & 0.35 & 1. & 0.2 \\ -0.25 & -0.15 & -0.25 & 0.2 & 1. \end{pmatrix} \tag{44}$$

where

$$D = diag(\sqrt{\Sigma(i,i)})_{i=1:n} = \begin{pmatrix} 1. & 0. & 0. & 0. & 0. \\ 0. & 1.5 & 0. & 0. & 0. \\ 0. & 0. & 2.5 & 0. & 0. \\ 0. & 0. & 0. & 0.5 & 0. \\ 0. & 0. & 0. & 0. & 3. \end{pmatrix} \tag{45}$$

Problem 8 continued on next page...

(ii)-(i)

The covariance matrix is

$$\Sigma = D\Omega D \tag{46}$$

where $D = diag(\sqrt{\Sigma(i,i)})_{i=1:n}$. The result printed out from python is

```
[[ 0.0625  -0.03125  0.0375  -0.025   -0.3    ]
 [ -0.03125  0.25    -0.05    -0.25     0.2    ]
 [  0.0375  -0.05     1.       0.4      0.2    ]
 [ -0.025   -0.25     0.4      4.      -0.8    ]
 [ -0.3      0.2      0.2     -0.8     16.    ]]
```

(ii)-(ii)

Similarly, the printed result is

```
[[ 16.  -2.    0.6    -0.1     -0.3]
 [ -2.    4.   -0.2    -0.25     0.05]
 [  0.6  -0.2   1.      0.1      0.0125]
 [ -0.1  -0.25  0.1     0.25    -0.0125]
 [ -0.3   0.05  0.0125 -0.0125   0.0625]]
```

Problem 9

Following is my code to compute the covariance and correlation matrices in different cases.

```
def corr_given_cov(mat_cov):
    mat_D = np.diag(np.sqrt(np.diag(mat_cov)))
    mat_corr = inv(mat_D) * mat_cov * inv(mat_D)
    return mat_corr

def corr_and_cov_of_percent_ret_given_file(filename='indices-july2011.csv', delta_time=1, b_log =
    False):
    df_price = pd.read_csv(filename)
    del df_price['Date']
    if b_log == False:
        df_rets = df_price.shift(-delta_time) / df_price - 1
    else:
        df_rets = np.log(df_price.shift(-delta_time) / df_price)
    if delta_time > 1:
        df_rets = df_rets.drop(df_rets.index[-delta_time:-1])
    df_rets = df_rets.drop(df_rets.index[-1])
    df_norm = df_rets - df_rets.mean()
    N = df_norm.shape[0]
    mat_norm = np.matrix(df_norm)
    mat_cov = 1.0/(N-1) * mat_norm.transpose() * mat_norm
    mat_corr = corr_given_cov(mat_cov)
    return mat_cov, mat_corr
```

Problem 9 continued on next page...

(i)

The sample covariance matrix of the daily percentage returns of the indeces: (multiplied by 1 million then show 3 decimal places)

```
[[ 100.036  67.044  96.674  40.044  82.767  78.903  74.351  71.106  54.555]
 [  67.044  58.162  71.286  38.014  65.292  61.375  59.632  52.867  41.986]
 [  96.674  71.286 135.876  45.364  86.149  83.05   77.463  60.147  57.457]
 [  40.044  38.014  45.364  44.841  43.203  40.1    39.365  34.6    30.428]
 [  82.767  65.292  86.149  43.203  82.372  74.457  71.375  73.05   54.008]
 [  78.903  61.375  83.05   40.1    74.457  69.774  66.942  61.8    46.546]
 [  74.351  59.632  77.463  39.365  71.375  66.942  65.169  58.658  43.657]
 [  71.106  52.867  60.147  34.6    73.05   61.8    58.658 103.964  50.009]
 [  54.555  41.986  57.457  30.428  54.008  46.546  43.657  50.009 110.842]]
```

The corresponding sample corelation matrix:

```
[[ 1.    0.879 0.829 0.598 0.912 0.944 0.921 0.697 0.518]
 [ 0.879 1.    0.802 0.744 0.943 0.963 0.969 0.68  0.523]
 [ 0.829 0.802 1.    0.581 0.814 0.853 0.823 0.506 0.468]
 [ 0.598 0.744 0.581 1.    0.711 0.717 0.728 0.507 0.432]
 [ 0.912 0.943 0.814 0.711 1.    0.982 0.974 0.789 0.565]
 [ 0.944 0.963 0.853 0.717 0.982 1.    0.993 0.726 0.529]
 [ 0.921 0.969 0.823 0.728 0.974 0.993 1.    0.713 0.514]
 [ 0.697 0.68  0.506 0.507 0.789 0.726 0.713 1.    0.466]
 [ 0.518 0.523 0.468 0.432 0.565 0.529 0.514 0.466 1.   ]]
```

The sample covariance matrix for daily log returns:

```
[[ 100.4    67.27   97.249  40.152  83.128  79.212  74.627  71.384  54.597]
 [  67.27   58.323  71.573  38.091  65.533  61.576  59.82   53.059  41.992]
 [  97.249  71.573 136.667  45.459  86.589  83.46   77.815  60.39   57.572]
 [  40.152  38.091  45.459  44.9    43.337  40.201  39.465  34.758  30.555]
 [  83.128  65.533  86.589  43.337  82.724  74.761  71.652  73.328  54.041]
 [  79.212  61.576  83.46   40.201  74.761  70.033  67.178  62.037  46.575]
 [  74.627  59.82   77.815  39.465  71.652  67.178  65.387  58.877  43.665]
 [  71.384  53.059  60.39   34.758  73.328  62.037  58.877 104.159  50.058]
 [  54.597  41.992  57.572  30.555  54.041  46.575  43.665  50.058 111.05 ]]
```

The sample correlation matrix for daily log returns:

```
[[ 1.    0.879 0.83  0.598 0.912 0.945 0.921 0.698 0.517]
 [ 0.879 1.    0.802 0.744 0.943 0.963 0.969 0.681 0.522]
 [ 0.83  0.802 1.    0.58  0.814 0.853 0.823 0.506 0.467]
 [ 0.598 0.744 0.58  1.    0.711 0.717 0.728 0.508 0.433]
 [ 0.912 0.943 0.814 0.711 1.    0.982 0.974 0.79  0.564]
 [ 0.945 0.963 0.853 0.717 0.982 1.    0.993 0.726 0.528]
 [ 0.921 0.969 0.823 0.728 0.974 0.993 1.    0.713 0.512]
 [ 0.698 0.681 0.506 0.508 0.79  0.726 0.713 1.    0.465]
 [ 0.517 0.522 0.467 0.433 0.564 0.528 0.512 0.465 1.   ]]
```

Compare: slight difference between covariance matrices within 1, between correlation matrices within 10^{-3}.

(ii)

The sample covariance matrix of the weekly percentage returns of the indeces:

```
[[ 472.156 311.098 405.204 174.564 400.558 369.005 354.473 417.483 379.165]
 [ 311.098 271.755 298.043 155.098 307.855 282.528 277.279 288.047 272.618]
 [ 405.204 298.043 549.629 184.198 354.593 342.675 318.596 292.357 323.415]
 [ 174.564 155.098 184.198 184.696 179.838 169.515 162.684 157.188 164.829]
 [ 400.558 307.855 354.593 179.838 404.847 352.153 341.254 420.45  364.174]
 [ 369.005 282.528 342.675 169.515 352.153 320.149 310.003 345.018 312.412]
 [ 354.473 277.279 318.596 162.684 341.254 310.003 304.292 333.379 294.753]
 [ 417.483 288.047 292.357 157.188 420.45  345.018 333.379 645.487 357.972]
 [ 379.165 272.618 323.415 164.829 364.174 312.412 294.753 357.972 570.861]]
```

The corresponding sample corelation matrix:

```
[[ 1.    0.868 0.795 0.591 0.916 0.949 0.935 0.756 0.73 ]
 [ 0.868 1.    0.771 0.692 0.928 0.958 0.964 0.688 0.692]
 [ 0.795 0.771 1.    0.578 0.752 0.817 0.779 0.491 0.577]
 [ 0.591 0.692 0.578 1.    0.658 0.697 0.686 0.455 0.508]
 [ 0.916 0.928 0.752 0.658 1.    0.978 0.972 0.822 0.758]
 [ 0.949 0.958 0.817 0.697 0.978 1.    0.993 0.759 0.731]
 [ 0.935 0.964 0.779 0.686 0.972 0.993 1.    0.752 0.707]
 [ 0.756 0.688 0.491 0.455 0.822 0.759 0.752 1.    0.59 ]
 [ 0.73  0.692 0.577 0.508 0.758 0.731 0.707 0.59  1.   ]]
```

The sample covariance matrix for weekly log returns:

```
[[ 469.054 308.971 406.473 173.49  397.631 366.859 352.062 412.186 375.063]
 [ 308.971 269.846 297.68  154.142 305.898 280.934 275.516 284.464 269.219]
 [ 406.473 297.68  552.808 184.453 354.65  343.462 318.826 290.193 321.019]
 [ 173.49  154.142 184.453 183.75  178.502 168.686 161.666 154.738 162.31 ]
 [ 397.631 305.898 354.65  178.502 402.235 350.156 339.091 415.894 360.272]
 [ 366.859 280.934 343.462 168.686 350.156 318.703 308.319 340.815 308.794]
 [ 352.062 275.516 318.826 161.666 339.091 308.319 302.411 329.203 290.86 ]
 [ 412.186 284.464 290.193 154.738 415.894 340.815 329.203 639.595 353.58 ]
 [ 375.063 269.219 321.019 162.31  360.272 308.794 290.86  353.58  570.729]]
```

The sample correlation matrix for weekly log returns:

```
[[ 1.    0.868 0.798 0.591 0.915 0.949 0.935 0.753 0.725]
 [ 0.868 1.    0.771 0.692 0.928 0.958 0.964 0.685 0.686]
 [ 0.798 0.771 1.    0.579 0.752 0.818 0.78  0.488 0.572]
 [ 0.591 0.692 0.579 1.    0.657 0.697 0.686 0.451 0.501]
 [ 0.915 0.928 0.752 0.657 1.    0.978 0.972 0.82  0.752]
 [ 0.949 0.958 0.818 0.697 0.978 1.    0.993 0.755 0.724]
 [ 0.935 0.964 0.78  0.686 0.972 0.993 1.    0.749 0.7  ]
 [ 0.753 0.685 0.488 0.451 0.82  0.755 0.749 1.    0.585]
 [ 0.725 0.686 0.572 0.501 0.752 0.724 0.7   0.585 1.   ]]
```

Compare: slight difference between covariance matrices within 10^1, between correlation matrices within 10^{-2}.

Problem 9 continued on next page...

(iii)

The sample covariance matrix of the monthly percentage returns of the indeces

```
[[ 5.22613000e+02 6.52471000e+02 7.67191000e+02 1.76086000e+02
   5.82136000e+02 5.25557000e+02 5.45466000e+02 5.62990000e+01
   6.32730000e+01]
 [ 6.52471000e+02 8.20539000e+02 9.20218000e+02 2.00726000e+02
   7.41528000e+02 6.62624000e+02 6.87928000e+02 1.45713000e+02
   5.06490000e+01]
 [ 7.67191000e+02 9.20218000e+02 2.40318400e+03 2.49390000e+02
   7.37085000e+02 7.32447000e+02 6.85305000e+02 -3.92243000e+02
   4.77142000e+02]
 [ 1.76086000e+02 2.00726000e+02 2.49390000e+02 1.42265000e+02
   1.47259000e+02 1.51609000e+02 1.66419000e+02 -1.96005000e+02
   7.88350000e+01]
 [ 5.82136000e+02 7.41528000e+02 7.37085000e+02 1.47259000e+02
   6.92667000e+02 6.09023000e+02 6.36799000e+02 1.91163000e+02
  -3.28830000e+01]
 [ 5.25557000e+02 6.62624000e+02 7.32447000e+02 1.51609000e+02
   6.09023000e+02 5.43765000e+02 5.67473000e+02 7.44930000e+01
  -1.63200000e+00]
 [ 5.45466000e+02 6.87928000e+02 6.85305000e+02 1.66419000e+02
   6.36799000e+02 5.67473000e+02 6.00143000e+02 5.09470000e+01
  -5.01310000e+01]
 [ 5.62990000e+01 1.45713000e+02 -3.92243000e+02 -1.96005000e+02
   1.91163000e+02 7.44930000e+01 5.09470000e+01 1.48489800e+03
  -1.68390000e+01]
 [ 6.32730000e+01 5.06490000e+01 4.77142000e+02 7.88350000e+01
  -3.28830000e+01 -1.63200000e+00 -5.01310000e+01 -1.68390000e+01
   5.82484000e+02]]
```

The corresponding sample corelation matrix:

```
[[ 1.    0.996  0.685  0.646  0.968  0.986  0.974  0.064  0.115]
 [ 0.996 1.    0.655  0.587  0.984  0.992  0.98   0.132  0.073]
 [ 0.685 0.655 1.    0.427  0.571  0.641  0.571 -0.208  0.403]
 [ 0.646 0.587 0.427 1.    0.469  0.545  0.57  -0.426  0.274]
 [ 0.968 0.984 0.571 0.469 1.    0.992  0.988  0.188 -0.052]
 [ 0.986 0.992 0.641 0.545 0.992 1.    0.993  0.083 -0.003]
 [ 0.974 0.98  0.571 0.57  0.988 0.993 1.    0.054 -0.085]
 [ 0.064 0.132 -0.208 -0.426 0.188 0.083 0.054 1.   -0.018]
 [ 0.115 0.073 0.403 0.274 -0.052 -0.003 -0.085 -0.018 1. ]]
```

The sample covariance matrix for monthly log returns:

```
[[ 526.53   657.246  767.206  173.003  590.047  530.131  551.248   70.067   59.119]
 [ 657.246  826.617  920.074  196.427  751.563  668.259  695.011  162.352   44.837]
 [ 767.206  920.074 2319.89   249.781  743.912  733.515  692.423 -355.846  457.48 ]
 [ 173.003  196.427  249.781  138.707  145.146  149.207  163.998 -185.547   77.55 ]
 [ 590.047  751.563  743.912  145.146  705.064  617.134  646.112  204.788  -36.801]
 [ 530.131  668.259  733.515  149.207  617.134  548.464  573.24    88.39    -5.143]
 [ 551.248  695.011  692.423  163.998  646.112  573.24   606.755   66.388  -52.953]
 [  70.067  162.352 -355.846 -185.547  204.788   88.39    66.388 1402.348   -7.745]
```

```
 [  59.119   44.837  457.48    77.55   -36.801   -5.143  -52.953   -7.745  589.212]]
```

The sample correlation matrix for weekly log returns:

```
[[ 1.    0.996 0.694 0.64  0.968 0.986 0.975 0.082 0.106]
 [ 0.996 1.    0.664 0.58  0.984 0.992 0.981 0.151 0.064]
 [ 0.694 0.664 1.    0.44  0.582 0.65  0.584 -0.197 0.391]
 [ 0.64  0.58  0.44  1.    0.464 0.541 0.565 -0.421 0.271]
 [ 0.968 0.984 0.582 0.464 1.    0.992 0.988 0.206 -0.057]
 [ 0.986 0.992 0.65  0.541 0.992 1.    0.994 0.101 -0.009]
 [ 0.975 0.981 0.584 0.565 0.988 0.994 1.    0.072 -0.089]
 [ 0.082 0.151 -0.197 -0.421 0.206 0.101 0.072 1.   -0.009]
 [ 0.106 0.064 0.391 0.271 -0.057 -0.009 -0.089 -0.009 1. ]]
```

Compare: slight difference between covariance matrices within 10^2, between correlation matrices within 10^{-1}.

(iv)

The differences between the sample covariance and correlation matrices for daily, weekly, and monthly returns become larger and larger when observing the difference between covariance matrices changing from 10^0 to 10^2, and the difference between correlation matrices changing from 10^{-3} to 10^{-1}.

Problem 10

(i)

The eigenvalues:

$$\lambda_1 = -1, \lambda_2 = -3, \lambda_3 = 2 \tag{47}$$

The corresponding normalized eigenvectors (8 decimal places):

$$v_1 = (0, 0, 1)^T \tag{48}$$
$$v_2 = (0, 0.70710678, -0.70710678)^T \tag{49}$$
$$v_3 = (0.96225045, 0.19245009, -0.19245009)^T \tag{50}$$

(ii)

The eigenvalues:

$$\lambda_1 = -2, \lambda_2 = 1, \lambda_3 = 3 \tag{51}$$

The corresponding normalized eigenvectors (8 decimal places):

$$v_1 = (1, 0, 0)^T \tag{52}$$
$$v_2 = (-0.31622777, 0.9486833, 0)^T \tag{53}$$
$$v_3 = (0.27216553, 0.68041382, 0.68041382)^T \tag{54}$$

Problem 11

Suppose λ is the eigenvalue of A and v is the corresponded eigenvector, then $Av = \lambda v$, on the other hand,

$$Av = A^2 v = A(Av) = A(\lambda v) = \lambda A v = \lambda \lambda v = \lambda^2 v \tag{55}$$

Therefore we have

$$\lambda v = \lambda^2 v \tag{56}$$

since v is a non-zero vector, we can conclude that λ is either 0 or 1.

Problem 12

Suppose λ is the eigenvalue of A and v is the corresponded eigenvector, then $Av = \lambda v$, on the other hand,

$$A^n v = A^{n-1}(Av) = \lambda A^{n-1} v = \cdots = \lambda^n v \tag{57}$$

since $A^n = 0$, then with the condition that v is a non-zero vector,

$$\lambda^n = 0 \tag{58}$$

so we can conclude that λ must be equal to 0.

Problem 13

(i)

If $v \neq 0$, the number of non-zero eigenvalues is 1. Think about the rows of vv^t where $v^t = (v_1, \ldots, v_n)$. The first row is $v_1 v^t$, the second row is $v_2 v^t$, etc. so the rows are all multiples of v^t. Therefore the rank of vv^t is 1, the number of non-zero eigenvalues is 1.
If $v = 0$, then vv^t is a zero matrix, the number of non-zero eigenvalues is 0.

(ii)

Assume $v \neq 0$. Then v is an eigenvector with eigenvalue $|v|^2 > 0$, since

$$(vv^t)v = v(v^t v) = v|v|^2 = |v|^2 v \tag{59}$$

and any nonzero vector x in the orthogonal complement of v (which is of dimension $n-1$) is an eigenvector with eigenvalue zero, since

$$(vv^t)x = v(v^t x) = v(vx) = v0 = \mathbf{0} = 0x. \tag{60}$$

If $v = 0$, then vv^t is a zero matrix, all the eigenvalues are 0 and the eigenvectors can be any non-zero vectors.

Problem 14

Suppose the matrix given is A, to derive its eigenvalues is to solve

$$det(\lambda I - A) = det \begin{vmatrix} \lambda - d & -1 & \ldots & -1 \\ -1 & \lambda - d & \ldots & -1 \\ \ldots & \ldots & \ldots & \ldots \\ -1 & -1 & \ldots & \lambda - d \end{vmatrix} \tag{61}$$

If we add up all the rows and divide it by $\lambda - d - (n-1)$, we will generate a row with all values being 1. By add this row to each row in the matrix above, we derive a diagonal matrix

$$det(\lambda I - A) = (\lambda - d - n + 1) det \begin{vmatrix} \lambda - d + 1 & 0 & \dots & 0 \\ 0 & \lambda - d + 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{vmatrix} = (\lambda - d + 1)^{n-1} (\lambda - d - n + 1) \tag{62}$$

Therefore the eigenvalues are

$$\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = d - 1, \lambda_n = d + n - 1 \tag{63}$$

When the eigenvalue is $\lambda = d - 1$, the eigenvectors are

$$v_1 = (1, -1, 0, 0, \dots, 0) \tag{64}$$
$$v_2 = (1, 0, -1, 0, \dots, 0) \tag{65}$$
$$v_3 = (1, 0, 0, -1, \dots, 0) \tag{66}$$
$$\dots \tag{67}$$
$$v_{n-1} = (1, 0, 0, 0, \dots, -1) \tag{68}$$

When the eigenvalue is $\lambda = d + n - 1$, the eigenvectors are

$$v_n = (1, 1, 1, \dots, 1) \tag{69}$$

Problem 15

(i)

Let

$$M' = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{70}$$

where A, B, C, and D may be real or complex numbers. Furthermore, let $\tau = A + D$ be the trace of M, and $\delta = AD - BC$ be its determinant. Let s be such that $s^2 = \delta$, and t be such that $t^2 = \tau + 2s$. That is,

$$s = \pm\sqrt{\delta} \qquad t = \pm\sqrt{\tau + 2s} \tag{71}$$

Then, if $t \neq 0$, a square root of M is

$$R = \frac{1}{t} \begin{pmatrix} A + s & B \\ C & D + s \end{pmatrix} \tag{72}$$

Indeed, the square of R is

$$\begin{aligned} R^2 &= \frac{1}{t^2} \begin{pmatrix} (A+s)^2 + BC & (A+s)B + B(D+s) \\ C(A+s) + (D+s)C & (D+s)^2 + BC \end{pmatrix} \\ &= \frac{1}{A + D + 2s} \begin{pmatrix} A(A+D+2s) & (A+D+2s)B \\ C(A+D+2s) & D(A+D+2s) \end{pmatrix} = M \end{aligned} \tag{73}$$

The square root formula of matrix to our problem, the result is (8 decimal places):

$$M = \begin{pmatrix} 1.28989795 & -0.5797959 \\ -0.5797959 & 2.15959179 \end{pmatrix} \tag{74}$$

(ii)

One solution is just using Cholesky decomposition, which generate the result as

$$M = \begin{pmatrix} \sqrt{2} & -\sqrt{2} \\ 0 & \sqrt{3} \end{pmatrix} = \begin{pmatrix} 1.41421356 & -1.41421356 \\ 0 & 1.73205081 \end{pmatrix} \tag{75}$$

It's easy to verify that

$$M^t M = \begin{pmatrix} \sqrt{2} & 0 \\ -\sqrt{2} & \sqrt{3} \end{pmatrix} \begin{pmatrix} \sqrt{2} & -\sqrt{2} \\ 0 & \sqrt{3} \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix} \tag{76}$$

Problem 16

(i)

Define

$$C = \begin{pmatrix} xI_n & A \\ B & I_n \end{pmatrix}, D = \begin{pmatrix} I_n & 0 \\ -B & xI_n \end{pmatrix}, \tag{77}$$

we have

$$det(CD) = x^n|I_n - AB|, det(DC) = x^n|I_n - BA| \tag{78}$$

and we know $det(CD) = det(DC)$ then A,B have same characteristic polynomials.
Actually there is a much easier proof, since A is nonsingular then A is invertible, we have

$$A^{-1}(AB)A = BA \tag{79}$$

so AB and BA are similar, which implies (but is stronger than) AB and BA have the same characteristic polynomial.

(ii)

I didn't use the condition that A is nonsingular in my first method to solve part(i), therefore it can be applied to solve this part too.

(iii)

Denote $C = AB, D = BA$,

$$tr(AB) = \sum_k C(k,k) = \sum_k \sum_i A(k,i)B(i,k) = \sum_i \sum_k B(i,k)A(k,i) = \sum_i D(i,i) = tr(BA) \tag{80}$$

(iv)

According to the fact that the eigenvalues of a matrix are the roots of its characteristic polynomial. We've proved the matrices AB and BA have the same characteristic polynomial in part(ii), therefore their eigenvalues are the same.

Problem 17

As stated, the four vectors are

$$v_1(i) = \left[\sin(\frac{\pi}{5}), \sin(\frac{2\pi}{5}), \sin(\frac{3\pi}{5}), \sin(\frac{4\pi}{5})\right] \tag{81}$$

$$v_2(i) = \left[\sin(\frac{2\pi}{5}), \sin(\frac{4\pi}{5}), \sin(\frac{6\pi}{5}), \sin(\frac{8\pi}{5})\right] \tag{82}$$

$$v_3(i) = \left[\sin(\frac{3\pi}{5}), \sin(\frac{6\pi}{5}), \sin(\frac{9\pi}{5}), \sin(\frac{12\pi}{5})\right] \tag{83}$$

$$v_4(i) = \left[\sin(\frac{4\pi}{5}), \sin(\frac{8\pi}{5}), \sin(\frac{12\pi}{5}), \sin(\frac{16\pi}{5})\right] \tag{84}$$

respectively. Then we have

$$v^t v = \begin{pmatrix} 2.5 & 0 & 0 & 0 \\ 0 & 2.5 & 0 & 0 \\ 0 & 0 & 2.5 & 0 \\ 0 & 0 & 0 & 2.5 \end{pmatrix} \tag{85}$$

So

$$v_1^2 = v_2^2 = v_3^2 = v_4^2 = 2.5 \tag{86}$$

Therefore they are not of norm 1. Also we can continue to verify that

$$v_1 \cdot v_2 = v_1 \cdot v_3 = v_1 \cdot v_4 = v_2 \cdot v_3 = v_2 \cdot v_4 = v_3 \cdot v_4 = 0 \tag{87}$$

they are orthagonal.

Problem 18

Denote the matrix given as A. Then matrix A is weakly diagonally dominant if

$$|a_{ii}| \geq \sum_{j \neq i} |a_{ij}| \quad \text{for all } i, \tag{88}$$

where a_{ij} denotes the entry in the i-th row and j-th column.
Since

$$3 + 2 + 1 > |-1| + |-1| + 2 + 1 \tag{89}$$

it is weakly diagonally dominant.
In addition, it is easy to verify that

$$det(A) = 0 \tag{90}$$

so it is singular.

Problem 19

To derive the eigenvalues of A, we have

$$det(\lambda I - A) = \det \begin{vmatrix} \lambda - 2 & 0 & 0 & \dots & 0 \\ -1 & \lambda - 2 & 0 & \dots & 0 \\ \dots & & & & \\ 0 & 0 & \dots & -1 & \lambda - 2 \end{vmatrix} = (\lambda - 2)^n = 0 \tag{91}$$

Therefore,

$$\lambda_1 = \lambda_2 = \cdots = \lambda_n = 2 \tag{92}$$

The corresponded eigenvector is

$$v = (0, 0, \ldots, 0, 1)^T \tag{93}$$

Problem 20

This is similar to the previous problem,

$$det(\lambda I - A) = \det \begin{vmatrix} \lambda - a & -b & 0 & \ldots & 0 \\ 0 & \lambda - a & -b & \ldots & 0 \\ 0 & 0 & \lambda - b & \ldots & 0 \\ \ldots & & & & -b \\ 0 & 0 & \ldots & 0 & \lambda - a \end{vmatrix} = (\lambda - a)^n = 0 \tag{94}$$

Therefore,

$$\lambda_1 = \lambda_2 = \cdots = \lambda_n = a \tag{95}$$

If $b \neq 0$, the corresponded eigenvector is

$$(1, 0, \ldots, 0)^t \tag{96}$$

If $b = 0$, any nonzero vector is an eigenvector.