Numerical Linear Algebra: Quiz 3

Due on Aug 20, 2014

Lecture time: 6:00 pm

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Problem 1

Since Q is an orthogonal matrix, then $Q^tQ = I$ or $Q^t = Q^{-1}$, therefore the characteristic polynomial of Q^tAQ is

$$\begin{aligned}
P_{Q^tAQ}(x) &= det(\lambda I - Q^tAQ) && (1)\\
&= det(Q^t\lambda IQ - Q^tAQ) && (2)\\
&= det(Q^t(\lambda I - A)Q) && (3)\\
&= det(Q^t)det(\lambda I - A)det(Q) && (4)\\
&= det(Q^{-1})det(Q)det(\lambda I - A) && (5)\\
&= det(\lambda I - A) = P_A(x) && (6)
\end{aligned}$$

which is the characteristic polynomial of A.

Problem 2

A is positive definite if and only if all of its n leading principal minors are strictly positive. There are four leading principal minors, one of order 1:

$$|a_{11}| = 1 > 0 \tag{7}$$

one of order 2:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 0.96 > 0 \tag{8}$$

one of order 3:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0.8775 > 0 \tag{9}$$

one of order 4:

$$|A| = 0.853275 > 0 \tag{10}$$

Therefore the matrix A is symmetric positive definite.

Problem 3

According to the description of C_N,

$$C_N(i,j) = \begin{cases} 1, & \text{if } i = j \\ -1, & \text{if } i = j - 1 \\ 0, & \text{otherwise} \end{cases} \tag{11}$$

for $1 \leq i \leq N, 1 \leq j \leq N+1$.
For its transpose, $C_N^t(i,j) = C_N(j,i)$ for any $1 \leq i \leq N+1, 1 \leq j \leq N$. Therefore,

$$C_N C_N^t(i,j) = \sum_k C_N(i,k)C_N^t(k,j) = \sum_k C_N(i,k)C_N(j,k) \tag{12}$$

In case $i = j$,

$$C_N C_N^t(i,j) = \sum_k C_N^2(i,k) = C_N^2(i,i) + C_N^2(i,i+1) = 2 \tag{13}$$

Problem 3 continued on next page...

In case $i = j + 1$,

$$C_N C_N^t(i,j) = \sum_k C_N(i,k)C_N(i-1,k) = C_N(i,i)C_N(i-1,i) = -1 \tag{14}$$

In case $i = j - 1$,

$$C_N C_N^t(i,j) = \sum_k C_N(i,k)C_N(i+1,k) = C_N(i,i+1)C_N(i+1,i+1) = -1 \tag{15}$$

Otherwise, $C_N C_N^t(i,j) = 0$ since the inner product (i.e., $\sum_k C_N(i,k)C_N(j,k)$) of any two rows in matrix C_N with index distance greater than 1 is 0. Given the entries above, it's exactly B_N given in the condition, so

$$B_N = C_N C_N^t \tag{16}$$

C_N is not the Choleskey decomposition of the matrix B_N because the Cholesky decomposition is a decomposition of a Hermitian, positive-definite matrix into the product of a lower triangular matrix and its conjugate transpose, where the lower triangular matrix must be a square matrix while C_N is not.

Problem 4

(i)

A symmetric matrix A is positive definite if and only if there exists a nonsingular matrix B such that $A = B^t B$, then for any column vector $v \neq 0$, we have

$$v^t M^t A M v = v^t M^t B^t B M v = v^t (BM)^t (BM) v = (BMv)^t \cdot (BMv) \geq 0 \tag{17}$$

therefore $M^t A M$ is semipositive definite. In addition, $M^t A M$ is symmetric because

$$(M^t A M)^t = M^t A^t M = M^t A M \tag{18}$$

where A is given symmetric, i.e. $A^t = A$. In conclude, $M^t A M$ is symmetric semipositive definite.

(ii)

If the columns of the matrix M is linearly independent, then M is nonsingular (i.e. $det(M) \neq 0$). BM is also nonsingular, where B is defined in part(i), because

$$det(BM) = det(B)det(M) \neq 0 \tag{19}$$

which implies that the columns of BM are linearly independent. We re-apply the formula in part(i) and find that, for any column vector $v \neq 0$,

$$v^t M^t A M v = (BMv)^t \cdot (BMv) > 0 \tag{20}$$

since any linear combination (i.e. v) of columns of BM, forming BMv cannot be a zero vector. In conclude, the matrix $M^t A M$ is symmetric positive definite from the formula above. (Symmetry proof is the same as part(i), $(M^t A M)^t = M^t A^t M = M^t A M$.)
On the other hand, if the matrix $M^t A M$ is symmetric positive definite, then $M^t A M$ is nonsingular. Assume by contradiction that the columns of the matrix M are not linearly independent (i.e.,$det(M) = det(M^t) = 0$), then

$$det(M^t A M) = det(M^t)det(A)det(M) = 0 \tag{21}$$

which contradicts to the fact that $M^t A M$ is nonsingular. So the columns of the matrix M are linearly independent.

Problem 5

A matrix can be a correlation matrix only if it is symmetric semidefinite with all the entries at the diagonal as 1. It is easy to verify the matrix given (denote as A) is symmetric with all diagonal entries being 1, for the semidefinite condition, we can check whether all its principle minors are 0:
3 of order 1:

$$|a_{11}| = |a_{22}| = |a_{33}| = 1 > 0 \tag{22}$$

3 of order 2:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 0.99 > 0, \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} = 0.96 > 0 \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = 0.91 > 0 \tag{23}$$

1 of order 3:

$$|A| = 0.848 > 0 \tag{24}$$

Therefore it is a correlation matrix.
The Cholesky factor of the matrix $A = LL^t$ is

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0.1 & 0.99498744 & 0 \\ 0.2 & -0.3216121 & 0.92550832 \end{pmatrix}, L^t = \begin{pmatrix} 1 & 0.1 & 0.2 \\ 0 & 0.99498744 & -0.3216121 \\ 0 & 0 & 0.92550832 \end{pmatrix} \tag{25}$$

using the formula

$$L_{kk} = \sqrt{a_{kk} - \sum_{j=1}^{k-1} L_{kj}^2}, L_{ik} = \frac{1}{L_{kk}} \left(a_{ik} - \sum_{j=1}^{k-1} L_{ij} L_{kj} \right) \tag{26}$$