Numerical Linear Algebra: Quiz 4

Due on Aug 21, 2014

Lecture time: 6:00 pm

Weiyi Chen

Problem 1

Since the linear combination of normal distribution is still normal distribute, we let

$$X_1 = w_1^t Z = (w_{11}, w_{12})(Z_1, Z_2)^t, X_2 = w_2^t Z = (w_{21}, w_{22})(Z_1, Z_2)^t \tag{1}$$

Then according to the covariance matrix, we have

$$var(X_1) = w_{11}^2 + w_{12}^2 = w_1^2 = 4 \tag{2}$$
$$var(X_2) = w_{21}^2 + w_{22}^2 = w_2^2 = 9 \tag{3}$$
$$cov(X_1, X_2) = w_{11}w_{21} + w_{21}w_{22} = w_1^t w_2 = -1 \tag{4}$$

where $w_1 = (w_{11}, w_{12})^t$, $w_2 = (w_{21}, w_{22})^t$. Let $W = col(w_i)$ for $i = 1, 2$, then

$$W^t W = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix} \begin{pmatrix} w_{11} & w_{21} \\ w_{12} & w_{22} \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ -1 & 9 \end{pmatrix} \tag{5}$$

One solution is just apply Cholesky decomposition (which implies $w_{12} = 0$), we get

$$W = \begin{pmatrix} 2 & -0.5 \\ 0 & 2.95803989 \end{pmatrix} \tag{6}$$

Therefore $X_1 = 2Z_1$ and $X_2 = -0.5Z_1 + 2.95803989Z_2$ is a possible pair of normal variables satisfying the covariance matrix.

Problem 2

Similarly to last problem, let

$$W = \begin{pmatrix} w_{11} & w_{12} & w_{13} \\ w_{21} & w_{22} & w_{23} \\ w_{31} & w_{32} & w_{33} \end{pmatrix} \tag{7}$$

satisfying $WZ = W(Z_1, Z_2, Z_3)^t = (X_1, X_2, X_3)^t$. Then the covariance matrix and the correlation matrix are

$$\Sigma_X = WW^t, \Omega_X = D_{\sigma_X}^{-1} \Sigma_X D_{\sigma_X}^{-1} \tag{8}$$

where

$$D_{\sigma_X}^{-1} = diag(\Sigma_X)^{-1} = diag(|w_1|, |w_2|, |w_3|)^{-1} \tag{9}$$

with $w_i = (w_{i1}, w_{i2}, w_{i3})^t$ and $|w_i| = (w_{i1}^2 + w_{i2}^2 + w_{i3}^2)^{1/2}$.
Since $(D_{\sigma_X}^{-1})^t = D_{\sigma_X}^{-1}$, then

$$\Omega_X = D_{\sigma_X}^{-1} \Sigma_X D_{\sigma_X}^{-1} = (W^t D_{\sigma_X}^{-1})^t (W^t D_{\sigma_X}^{-1}) \tag{10}$$

where

$$W^t D_{\sigma_X}^{-1} = col(w_i)_{i=1:3} diag(|w_i|^{-1})_{i=1:3} = col(\frac{w_i}{|w_i|})_{i=1:3} \tag{11}$$

Again we apply Cholesky decomposition to derive $W^t D_{\sigma_X}^{-1}$ since we are given Ω_X, the result is

$$W^t D_{\sigma_X}^{-1} = \begin{pmatrix} 1 & 0.3 & 0.4 \\ 0 & 0.9539392 & 0.39834824 \\ 0 & 0 & 0.82542031 \end{pmatrix} \tag{12}$$

One solution of W is just take the values above since $D_{\sigma_X}^{-1}$ is just to normalize each columns of W^t. In other words,

$$W^t = \begin{pmatrix} w_{11} & w_{21} & w_{31} \\ w_{12} & w_{22} & w_{32} \\ w_{13} & w_{23} & w_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0.3 & 0.4 \\ 0 & 0.9539392 & 0.39834824 \\ 0 & 0 & 0.82542031 \end{pmatrix} \tag{13}$$

Therefore

$$X_1 = Z_1 \tag{14}$$
$$X_2 = 0.3Z_1 + 0.9539392Z_2 \tag{15}$$
$$X_3 = 0.4Z_1 + 0.39834824Z_2 + 0.82542031Z_3 \tag{16}$$

is a possible combination of normal variables satisfying the given correlation matrix.

Problem 3

According to the distribution of the 3-dimensional multivariate normal random variable,

$$X_1 + 2X_2 + 3X_3 \sim N\left(1 - 4 + 3, (1, 2, 3)\begin{pmatrix} 1 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\right) \tag{17}$$

or

$$X_1 + 2X_2 + 3X_3 \sim N(0, 12) \tag{18}$$

Therefore the probability that $X_1 + 2X_2 + 3X_3$ is positive is $p = 0.5$.