Math 2ZZ3 Summary

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Review

Definition of Even/Odd

Even function: $f(x) = f(-x)$, $\int_{-p}^{p} \text{even} = 2\int_{0}^{p} \text{even}$

Odd function: $f(-x) = -f(x)$, $\int_{-p}^{p} \text{odd} = 0$

Even/Odd Operations

even · even = even
odd · odd = even
even · odd = odd
even ± even = even
odd ± odd = odd
even ± odd = neither (unless one of the functions is equal to zero over the given domain)

$f \text{ even} \Rightarrow \int_{-a}^{a} f(x)\,\mathrm{d}x = 2\int_{0}^{a} f(x)\,\mathrm{d}x$

$f \text{ odd} \Rightarrow \int_{-a}^{a} f(x)\,\mathrm{d}x = 0$

$$\cos(a)\cos(b) = \tfrac{1}{2}\left[\cos(a-b) + \cos(a+b)\right]$$

$$\cos(a)\sin(b) = \tfrac{1}{2}\left[\sin(a-b) + \sin(a+b)\right]$$

$$\sin(a)\sin(b) = \tfrac{1}{2}\left[\cos(a-b) - \cos(a+b)\right]$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\tan^{-1}(x) = \frac{1}{1+x^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\cos^{-1}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$$

$$\cosh(kt) = \frac{e^{kt} + e^{-kt}}{2}$$

$$\sinh(kt) = \frac{e^{kt} - e^{-kt}}{2}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\cosh(t)\right] = \sinh(t)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\sinh(t)\right] = \cosh(t)$$

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2} = \cosh(ix)$$
$$i\sin(x) = \frac{e^{ix} - e^{-ix}}{2} = \sinh(ix)$$

12 – Orthogonal Functions and Fourier Series

12.1 – Orthogonal Functions

Inner product of two functions is denoted by (f_1, f_2) and $(f_1, f_2) = \int_a^b f_1(x) f_2(x)\,\mathrm{d}x$.
If the inner product of two functions = 0, then they are said to be **orthogonal**.

An **orthogonal set** is one where, for each value, $m \neq n$, in set $[\phi_0, \phi_1, \ldots]$, $\int_a^b \phi_m(x)\phi_n(x)\,\mathrm{d}x = 0$, on the interval [$a$, b].
Norm of a set: $\|\phi(x)\| = \sqrt{\int_a^b \phi_n(x)\phi_m(x)\,\mathrm{d}x}$, where m can equal n.
The norm of any 2 elements in an orthogonal set is 1.
A function is said to be **periodic** with period T if $f(x) = f(x+T)$. The smallest T that is periodic is called the **fundamental period**.

12.2 – Fourier Series

The **Fourier Series** is an orthogonal representation of a function.
Fourier Equation: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{p}x\right) + b_n \sin\left(\frac{n\pi}{p}x\right)$
Your **interval** is defined from $(-p, p)$; n is your **harmonic**.
period: $T = \frac{2\pi}{p}$
For a given function, $f(x)$, $F(x)$ is the **Fourier approximation** of f. $F_1(x)$ is the Fourier approximation at the first harmonic of f. That means you plug in the harmonic into n. You'll notice that for odd harmonics, your $a_n = 0$ and for even harmonics, your $b_n = 0$.

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)\,\mathrm{d}x$$
$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\left(\frac{n\pi x}{p}\right)\mathrm{d}x$$
$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\left(\frac{n\pi x}{p}\right)\mathrm{d}x$$

If your function is piecewise, split up your integral into multiple pieces.

e.g. 1)

$$f(x)=\begin{cases}1, & -1<x<0\\ x, & 0\le x<1\end{cases}$$

$$a_0=\frac{1}{1}\left[\int_{-1}^{0}1\mathrm{d}x+\int_0^1 x\mathrm{d}x\right]$$

12.3 – Fourier cosine/sine

If a function is **odd**, expand using the **Fourier sine series**, since the cosines will be 0:

$$f(x)=\sum_{n=1}^{\infty}c_n\sin\left(\frac{n\pi}{p}x\right),\ c_n=\frac{2}{p}\int_0^p f(x)\sin\left(\frac{n\pi x}{p}\right)\mathrm{d}x$$

This includes the b_n's.

If a function is **even**, expand using the **Fourier cosine series**, since the sines will be 0:

$$f(x)=\sum_{n=0}^{\infty}c_n\cos\left(\frac{n\pi x}{p}\right),\ c_n=\frac{2}{p}\int_0^p f(x)\cos\left(\frac{n\pi x}{p}\right)\mathrm{d}x$$

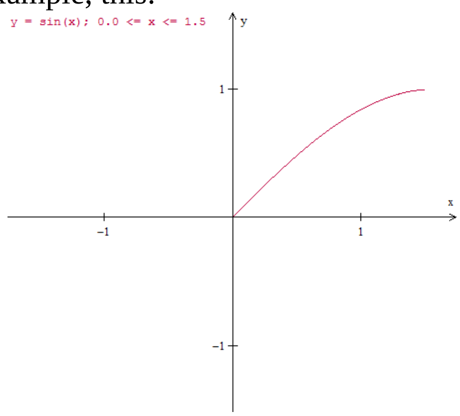
This includes a_0 and a_n.
How to remember it: "want even". Don't forget to memorize the products of odd/even functions.

$$f(x)\cdot\sin(x)=\text{even/odd?}$$

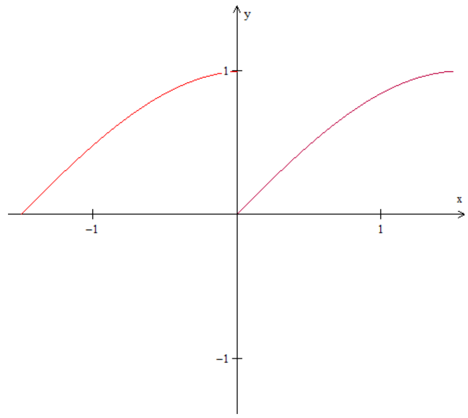
$$f(x)\cdot\cos(x)=\text{even/odd?}$$

Half-range Expansions

This is used when you have a function that has a range (0, p), instead of ($-p$, p). It is called "half-range" because you treat it as if it's only half of the full function, where the full function is the function repeated twice. For example, this:



Is treated like this:



Therefore, when you're taking the integrals to find a_0, a_n, b_n, multiply by 2.

Another way of thinking about it is that your period is:

$$\frac{p-0}{2} = \frac{p}{2}$$

$$= \frac{1}{\left(\frac{p}{2}\right)} \int_0^p$$

$$= \frac{2}{p} \int_0^p$$

The first way is probably easier to understand and the second is probably a more general way to find the expansion, given any range.

In summary, for half-range expansions, you use the regular Fourier Expansion, except you double the integrals:

$$a_0 = \frac{2}{p} \int_0^p f(x) \mathrm{d}x$$

$$a_n = \frac{2}{p} \int_0^p f(x) \cos\left(\frac{n\pi x}{p}\right) \mathrm{d}x$$

$$b_n = \frac{2}{p} \int_0^p f(x) \sin\left(\frac{n\pi x}{p}\right) \mathrm{d}x$$

The cosine and sine series are the same for half-range expansions.

Differential Equations

For differential equations look at my Math 2Z03 summary. Note that if you are doing undetermined coefficients, you are dealing with a sum of cosines and sines, so you guess

$$y = (A_0 + A_n) \cos\left(\frac{n\pi x}{p}\right) + B_n \sin\left(\frac{n\pi x}{p}\right)$$

12.4 – Complex Fourier

This is another representation of the Fourier Series that is useful when you are asked for it or if you are asked for frequency spectrum. It *can* be used in any situation that the regular Fourier Series can be used, but that is not recommended.

$$c_n = c_{-n} = \frac{1}{2p}\int_{-p}^{p} f(x)e^{-in\frac{\pi}{p}x}\mathrm{d}x$$

$$c_0 = \frac{1}{p}\int_{-p}^{p} f(x)\mathrm{d}x \leftarrow \text{same as } a_0$$

It is derived by plugging in Euler's formula into the sines and cosines. Since $c_n = c_{-n}$, you can summarize it as:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x\frac{1}{p}}$$

$$\left(1+n^2\right) = (1+in)(1-in)$$

Frequency Spectrum

Fundamental period: $T = 2p$

Frequency spectrum: $|c_n| = \|c_n\| = \sqrt{(\alpha)^2 + (\beta)^2} \leftarrow$ from $\alpha + \beta i$

Fundamental angular frequency: $\omega = \dfrac{2\pi}{T}$

Graphical representation

9 – Vector Calculus

9.1 – Vector Functions

Vectors have direction and are denoted by either an arrow over top ($\vec{u}$) or simply bold (**u**). We'll just use bold to make it easier to type.

The **norm** of a vector is notated: $\|\mathbf{r}(t)\| = \sqrt{[x(t)]^2 + [y(t)]^2 + [z(t)]^2}$

Unit vectors have a magnitude of 1 and are denoted with a hat over top of them ($\hat{\mathbf{u}}$). To make any vector a unit vector, divide it by its norm. $\hat{\mathbf{r}} = \dfrac{\mathbf{r}}{\|\mathbf{r}\|}$

Arc Length Formula: $L = \int_a^b \|\mathbf{r}'(t)\|\mathrm{d}t$

$$s(t) = \int_{t_0}^{t} \|\mathbf{r}'(x)\|\mathrm{d}x$$

Directional vector: Find the t where $\langle x_0, y_0, z_0 \rangle = \langle x(t), y(t), z(t) \rangle$

i.e. $x_0 = x(t), y_0 = y(t), z_0 = z(t)$. Call the *t*, t_0.

Your direction vector = $\langle x(t_0), y(t_0), z(t_0)\rangle$

Equation of a tangent line: $\langle x, y, z\rangle = \langle x_0, y_0, z_0\rangle + t\langle \text{direction vector}\rangle$

To find the curve of intersection of multiple surfaces, isolate *x*, *y*, and *z* for *t*, using each other.

Properties of Vector Functions

i) $\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathbf{r}_1(t)+\mathbf{r}_2(t)\right] = \mathbf{r}_1'(t)+\mathbf{r}_2'(t)$

ii) $\frac{\mathrm{d}}{\mathrm{d}t}\left[u(t)\mathbf{r}(t)\right] = u(t)\mathbf{r}'(t)+u'(t)\mathbf{r}(t)$

iii) $\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathbf{r}_1(t)\cdot\mathbf{r}_2(t)\right] = \mathbf{r}_1(t)\cdot\mathbf{r}_2'(t)+\mathbf{r}_1'(t)\cdot\mathbf{r}_2(t)$

iv) $\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathbf{r}_1(t)\times\mathbf{r}_2(t)\right] = \mathbf{r}_1(t)\times\mathbf{r}_2'(t)+\mathbf{r}_1'(t)+\mathbf{r}_2(t)$

9.2 – Motion on a Curve

Velocity: $\mathbf{v}(t) = \mathbf{r}'(t)$

In an ellipse, the cosines are the *x*'s and the sines are the *y*'s.

Speed of the particle is $\|\mathbf{v}(t)\| = \|\mathbf{r}'(t)\|$

Trajectory

If you are given an initial velocity at an angle, $\mathbf{v}_0 = v\cos(\theta)\hat{\mathbf{i}} + v\sin(\theta)\hat{\mathbf{j}}$

Stupid imperial fact to know: acceleration due to gravity is −32ft/s

Also, if you calculate velocity and they ask for speed, convert it: $v = \|\mathbf{v}\|$

9.3 – Curvature and Components of Acceleration

Unit tangent: $\hat{\mathbf{T}}(t) = \dfrac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$

Unit normal: $\hat{\mathbf{N}}(t) = \dfrac{\hat{\mathbf{T}}'(t)}{\|\hat{\mathbf{T}}'(t)\|}$

Unit Binormal: $\hat{\mathbf{B}}(t) = \hat{\mathbf{T}}(t)\times\hat{\mathbf{N}}(t)$

Call T, N, B the **Frenet frame**.

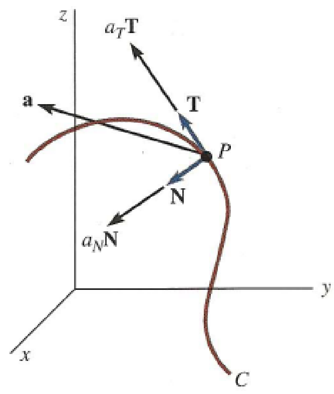


FIGURE 9.3.3 Components of acceleration

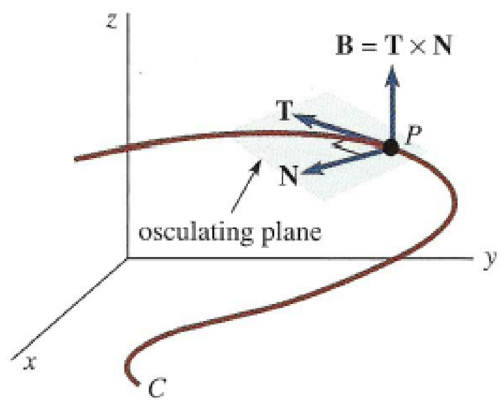


FIGURE 9.3.4 Osculating plane

Plane spanned by *T* and *N* is the **osculating plane** (important).
Plane spanned by *N* and *B* is the **normal plane**.
Plane spanned by *T* and *B* is the **rectifying plane**.

Identities:

$$B = T \times N$$
$$N = B \times T$$
$$T = N \times B$$

Curvature: magnitude of the acceleration vector; $\kappa(s) = \left\| \frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}s} \right\|$ ← not really useful, so we use the following formula:

$$\kappa(s) = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3} = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|^3} = \frac{\|\hat{\mathbf{T}}'(t)\|}{\|\mathbf{r}'(t)\|}$$

$$\begin{aligned} a(t) &= a_T \hat{\mathbf{T}} + a_N \hat{\mathbf{N}} \\ &= \mathbf{v}'\mathbf{T}(t) + \kappa \mathbf{v}^2 \mathbf{N}(t) \end{aligned}$$

$$a_T = \frac{\mathbf{r}' \cdot \mathbf{r}''}{\|\mathbf{v}\|} = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|} \leftarrow \text{Tangential component of acceleration}$$

$$a_N = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|} = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|} \leftarrow \text{Normal component of acceleration}$$

9.4 – Partial Derivatives

Partial derivatives use the operator, ∂. It is similar to the 'd'-operator when finding derivatives because we say $\frac{\partial y}{\partial x}$, like we would $\frac{\mathrm{d}y}{\mathrm{d}x}$. How you find a partial derivative is you treat all the variables as constants, except for the variable indicated in the denominator, e.g. $\frac{\partial}{\partial x} x^2 y^3 z = 2xy^3 z$, $\frac{\partial}{\partial x}\left(x^2 + y^2\right) = 2x$

Level curves are usually given in the following way: $f(x,y)=z=c$
These are useful for contour plots on elevation maps. They take a “level” 2D slice from a 3D object

Similarly, **level surfaces** are given in the following way: $f(x,y,z)=c$
They take a “level” 3D slice from a 4D object.

Concentric: shares the same centre
Hyperbola: $y^2-x^2=r^2$ or $x^2-y^2=r^2$
Quadric surfaces: highest power of any variable is 2
Cylinder if you have multiple variables for quadric (can be **spherical** or **elliptical**)

Chain Rule

If $z=f(x,y), u=g(x,y), v=h(x,y)$ and all have continuous 1st order partials, then

$$\frac{\partial z}{\partial x}:=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

Note: $x:=y$ means x is defined by y.

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$z=f(u_1,u_2,u_3,\ldots u), u_i$ are functions of $X_1, X_2, X_3, \ldots, X_k$

$$\frac{\partial z}{\partial x_j}=\frac{\partial z}{\partial u_1}\cdot\frac{\partial u_1}{\partial x_j}+\frac{\partial z}{\partial u_2}\cdot\frac{\partial u_2}{\partial x_j}+\frac{\partial z}{\partial u_3}\cdot\frac{\partial u_3}{\partial x_j}+\ldots+\frac{\partial z}{\partial u_n}\cdot\frac{\partial u_n}{\partial x_j}$$

GAH! Doesn’t this make regular derivatives confusing? Well here’s a new definition of derivative with your current understanding of partial derivatives:

$$\frac{\mathrm{d}f(x,y,z)}{\mathrm{d}x}=\frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}x}+\frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{\partial f}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}x}$$

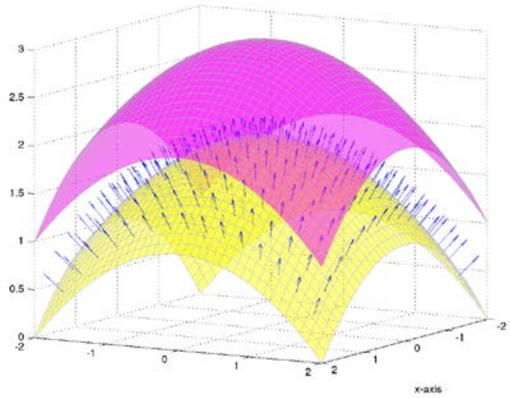
9.5 – Directional Derivatives

Gradient

The **nabla symbol**, ∇, denotes **gradient** of a scalar function, which is the normal vector to a surface.

$$\nabla=\frac{\partial}{\partial x}\hat{\mathbf{i}}+\frac{\partial}{\partial y}\hat{\mathbf{j}}+\frac{\partial}{\partial z}\hat{\mathbf{k}}$$

$$\nabla F=\frac{\partial F}{\partial x}\hat{\mathbf{i}}+\frac{\partial F}{\partial y}\hat{\mathbf{j}}+\frac{\partial F}{\partial z}\hat{\mathbf{k}}$$



The arrows in the above image represent the value of the gradient at each point.

Directional Derivatives

Directional derivatives are denoted by D_u

$D_{\hat{\mathbf{u}}} f = f_{\hat{\mathbf{u}}}$

2D: $D_{\hat{\mathbf{u}}} f(x, y) = f_{\hat{\mathbf{u}}} = \lim_{h \to 0} \dfrac{f(x + h\cos\theta, y + h\sin\theta) - f(x, y)}{h} = \nabla F(x, y) \cdot \hat{\mathbf{u}}$

$\hat{\mathbf{u}} = \cos(\theta)\hat{\mathbf{i}} + \sin(\theta)\hat{\mathbf{j}}$

3D: $D_{\hat{\mathbf{u}}} f(x, y, z) = \lim_{h \to 0} \dfrac{F(x + h\cos(\alpha), y + h\cos(\beta), z + h\cos(\gamma)) - F(x, y, z)}{h} = \nabla F(x, y, z) \cdot \hat{\mathbf{u}}$

$\hat{\mathbf{u}} = \cos(\alpha)\hat{\mathbf{i}} + \cos(\beta)\hat{\mathbf{j}} + \cos(\gamma)\hat{\mathbf{k}}$

When given a vector to find it in the direction of. Make it a unit vector. It is your **u**.

Maximum rate of change at a point, p: $\|\nabla F(p)\|$

Minimum rate of change at a point, p: $-\|\nabla F(p)\|$

Direction (unit vector): $\hat{\mathbf{u}} = \dfrac{\nabla F(p)}{\|\nabla F(p)\|}$

Note: if you have to find the directional derivative to a point (e.g. (a, b)) and you are given a point to find it in terms of (e.g. (c, d)), your vector that you need to make into **u** will be $\langle a - c, b - d \rangle$. Look out for the round brackets $(\ \ )$ amongst the square brackets $\langle\ \ \rangle$.

9.6 – Tangent Planes and Normal Lines

The **tangent plane** at P is the plane that is normal to ∇F evaluated at P.
Normal vectors are ∇f , where f equals your given equation, after you put all your variables to one side.
Implicit equation for tangent plane: $\nabla f(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0$

Given a point and a surface,
∇f

9.7 – Curl and Divergence

$\mathbf{F}(x,y,z)=P\mathbf{i}+Q\mathbf{j}+R\mathbf{k}$

Recall what the nabla symbol means from 9.5.

$$\operatorname{curl}\mathbf{F}=\nabla\times\mathbf{F}$$

$$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ P & Q & R\end{vmatrix}$$

$$=\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\mathbf{i}+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\mathbf{j}+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\mathbf{k}$$

$$\operatorname{div}\mathbf{F}=\nabla\cdot\mathbf{F}$$

$$=\left\langle\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right\rangle\cdot\langle P,Q,R\rangle$$

$$=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}$$

One practical application is fluid flow, modeled by **F**. Insert a paddle into the flow and measure tendency of paddle to be turned about vertical axis.

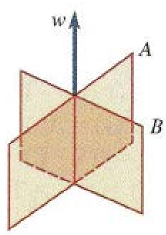


FIGURE 9.7.5 Paddle device

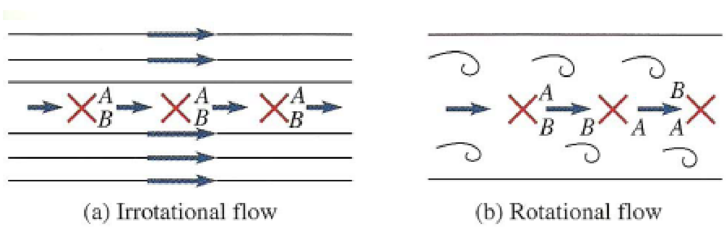


FIGURE 9.7.6 Irrotational flow in (a); rotational flow in (b)

$\operatorname{curl}(\nabla F)=0$

$\operatorname{div}(\operatorname{curl}\mathbf{F})=0$

Irrotational: curl **F** = 0

Incompressible: $\operatorname{div}\mathbf{F}=0$

Also, $\operatorname{div}\mathbf{F}=0$ if **F** is a solenoid

9.8 – Line Integrals

3 parts:

$$\oint_C G(x,y)\,\mathrm{d}x=\int_a^b G(f(t),g(t))\,f'(t)\,\mathrm{d}t$$

$$\oint_C G(x,y)\,\mathrm{d}y=\int_a^b G(f(t),g(t))\,g'(t)\,\mathrm{d}t$$

$$\oint_C G(x,y)\,\mathrm{d}s=\int_a^b G(f(t),g(t))\left\|G(x,y)\right\|\mathrm{d}t$$

Normal line is found by taking $\nabla f\left(p\right)\cdot\left\langle x-x_0, y-y_0, z-z_0\right\rangle=0$ and the coefficients in front of *x*, *y*, and *z* compose the direction of the vector and the point, *p*, composes the point.
Recall that the equation of a circle is $x=r\cos\left(t\right), y=r\sin\left(t\right)$

Note: $\int_C=$ lazy form of $\oint_C$

$-\int_C f\left(x,y\right)\mathrm{d}s=\int_{-C} f\left(x,y\right)\mathrm{d}s$

$W=\oint_C \mathbf{F}\cdot\mathrm{d}r$ ←hint: since d*r* is basically an incremental path, isn't it kinda like d*s*?

$W=\int_{t_1}^{t_2}\mathbf{F}\left(r\left(t\right)\right)\cdot r'\left(t\right)\mathrm{d}t$

9.9 – Independence of Path

The vector function, **F**, is **conservative** if there exists ϕ, such that $\nabla\phi=\mathbf{F}$ or $\operatorname{curl}\mathbf{F}=0$.
If ϕ exists, we say ϕ is the **potential function** of **F**.
The integral, $\oint_C\mathbf{F}\cdot\mathrm{d}\mathbf{r}$, is **independent of path** if $\oint_{C_1}\mathbf{F}\cdot\mathrm{d}\mathbf{r}=\oint_{C_2}\mathbf{F}\cdot\mathrm{d}\mathbf{r}$ for any two paths, C_1, C_2, that start / end at the same place.
If $\mathbf{F}=P\left(x,y\right)\hat{\mathbf{i}}+Q\left(x,y\right)\hat{\mathbf{j}}$ (2D) is conservative, with potential function, $\phi\left(x,y\right)$, then
When $\mathrm{d}\mathbf{r}=\mathrm{d}x\hat{\mathbf{i}}+\mathrm{d}y\hat{\mathbf{j}}$

$\oint_C\mathbf{F}\cdot\mathrm{d}\mathbf{r}=\oint_C P\mathrm{d}x+Q\mathrm{d}y$

Fundamental Theorem of Line Integrals:

$$\oint_C\mathbf{F}\cdot\mathrm{d}\mathbf{r}=\oint_C\nabla\phi\cdot\mathrm{d}\mathbf{r}$$
$$=\phi\left(B\right)-\phi\left(A\right)$$

$$\oint_C\mathbf{F}\cdot\mathrm{d}\mathbf{r}=\int_a^b\nabla f\cdot\mathbf{r}'\left(t\right)\mathrm{d}t$$

If you are given ∇f and 2 points, you can set: $\mathbf{r}\left(x,y,z\right)=\left\langle a\right\rangle+t\left\langle b-a\right\rangle$, as long as you specify the range is $0\le t\le 1$ (since at 1, you're at *b*). Now, expand **r**, so you know what its components are:

$$x=a_{\hat{\mathbf{i}}}+\left(b_{\hat{\mathbf{i}}}-a_{\hat{\mathbf{i}}}\right)t$$
$$y=a_{\hat{\mathbf{j}}}+\left(b_{\hat{\mathbf{j}}}-a_{\hat{\mathbf{j}}}\right)t$$
$$z=a_{\hat{\mathbf{k}}}+\left(b_{\hat{\mathbf{k}}}-a_{\hat{\mathbf{k}}}\right)t$$

Now plug-in *x*,*y*,*z* into ∇f and solve.

Conservative

How to tell when **F** is conservative?

If $\mathbf{F}(x,y)=P(x,y)\hat{\mathbf{i}}+Q(x,y)\hat{\mathbf{j}}$ is conservative on an open region, *D*, and *P*,*Q* are continuous, with continuous 1st order partials, then $\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$

Conversely, if *D* is a simply connected region and $\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$, **F** is conservative.

Finding ϕ

You can find the potential function of **F** in the following way (e.g. $\mathbf{F}=\left(y\cos(x)+y^2,\sin(x)+2xy-2y\right)$):

1) Partial integrate one of the components with respect to its place, like the **i** component by *x*, the **j** component by *y*, or the **k** component by *z*., $\left(\text{e.g. } y\cos(x)+y^2\right)$
$$\left(\text{e.g. } \frac{\mathrm{d}f}{\mathrm{d}x}=y\cos(x)+y^2\Rightarrow f=\int y\cos(x)+y^2\mathrm{d}x\Rightarrow f=y\sin(x)+y^2x+g(y)\right)$$
2) Now you need to get rid of your $g(y)$. Find the $\frac{\partial}{\partial y}$ of whatever you currently have. You should now be left with something similar to $\mathbf{F}_{\hat{\mathbf{j}}}$, except your last part.
e.g. $\begin{aligned}\frac{\partial f}{\partial y}&=\frac{\partial}{\partial y}\left(y\sin(x)+y^2x+g(y)\right)\\&=\sin(x)+2yx+\frac{\mathrm{d}g(y)}{\mathrm{d}y}\end{aligned}$, where $\sin(x)+2yx+\frac{\mathrm{d}g(y)}{\mathrm{d}y}\approx\underbrace{\sin(x)+2xy-2y}_{\text{2nd component of }\mathbf{F}}$
3) With that, equate the two equations and find out what $g(y)$ is by integrating.
$$\frac{\mathrm{d}g}{\mathrm{d}y}=-2y$$
$$g(y)=-y^2+k$$
$$\boxed{\therefore f=y\sin(x)+y^2x-y^2+k}$$
4) If you had a range given to you, you plug it in now, i.e. not before. $f(x_2,y_2)-f(x_1,y_1)$

Partial Integration

When you are doing partial integration, you apply regular integration rules, except you treat the other variables as constants.

Also, you don't add constants at the end of your result. Instead, you add $\begin{smallmatrix}+g(y) & \leftarrow\text{2D}\\ +g(y,z) & \leftarrow\text{3D}\end{smallmatrix}$ for all variables that you aren't partial integrating by because they could have been 'partial-differentiated-out' and you wouldn't even know it.

3D

In 3D space, everything is the same, except of the following:
F is conservative $\leftrightarrow$ $\operatorname{curl}\mathbf{F}=0$

Let's use this example: $\frac{\partial f}{\partial x} = -8x; \frac{\partial f}{\partial y} = -7y; \frac{\partial f}{\partial z} = 1$

1) Partial integrate one of the components with respect to its place, like the **i**-component by *x*, the **j**-component by *y*, or the **k**-component by *z*. Pick one and go. For the example, we're going to pick **i**.

$$\frac{\partial f}{\partial x} = -8x$$
$$\partial f = -8x\partial x$$
$$\int \partial f = \int -8x\partial x$$
$$f = -4x^2 + g(y,z)$$

2) Find *g* by partial integrating by the next variable and equating it to the next component to find d*g*/d*variable*. Let's pick *y*.

$$\int f\mathrm{d}y = \int -4x^2 + g(y,z)\mathrm{d}y$$
$$\frac{\partial f}{\partial y} = 0 + \frac{\mathrm{d}g}{\mathrm{d}y} = -7y$$
$$\frac{\mathrm{d}g}{\mathrm{d}y} = -7y$$

3) Now, find *g* by integrating.

$$\int \mathrm{d}g = \int -7y\mathrm{d}y$$
$$g = \tfrac{-7}{2}y^2 + h(z)$$

4) To find *h*, plug everything back into *f*, find the partial with respect to the last variable, and equate to your third component. Then, integrate.

$$f = -4x^2 - \frac{7}{2}y^2 + h(z)$$
$$\int f = \int -4x^2 - \frac{7}{2}y^2 + h(z)$$
$$\frac{\partial f}{\partial z} = 0 - 0 + \frac{\mathrm{d}h}{\mathrm{d}z} = 1$$
$$\frac{\mathrm{d}h}{\mathrm{d}z} = 1$$
$$\int \mathrm{d}h = \int \mathrm{d}z$$
$$h = z$$

Therefore, $\boxed{f = -4x^2 - \frac{7}{2}y^2 + z}$

5) If you had a range given to you, you plug it in <u>now</u>, $f(x_2, y_2, z_2) - f(x_1, y_1, z_1)$

9.10 – Double Integrals

When finding the volume between two plains subtract the top plain – the bottom plain.

Type I: dydx
Type II: dxdy
Area of a region, $D \subseteq \mathbb{R}^2$, is: $\iint_D (1)\,\mathrm{d}A$

Lamina: 2D region with a density function, $p\ (x, y)$
Mass of lamina: $m = \iint_R \rho(x, y)\,\mathrm{d}A$

$M_x = \iint_R y\rho\,\mathrm{d}A; M_y = \iint_R x\rho\,\mathrm{d}A \leftarrow 1^{\text{st}}$ **moments of mass** about x, y-axis

$I_x = \iint_R y^2\rho\,\mathrm{d}A; I_y = \iint_R x^2\rho\,\mathrm{d}A \leftarrow 2^{\text{nd}}$ moments of mass; 1^{st} **moment of inertia**

Centre of mass of lamina: $\overline{x} = \dfrac{M_x}{m}, \overline{y} = \dfrac{M_y}{m}$

9.11 – Double Integrals in Polar Coordinates

$\mathrm{d}x\mathrm{d}y \Rightarrow r\mathrm{d}r\mathrm{d}\theta$

Type I: $\iint_R f(r,\theta)\,\mathrm{d}A = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r,\theta)\, r\mathrm{d}r\mathrm{d}\theta$

Type II: $\iint_R f(r,\theta)\,\mathrm{d}A = \int_a^b \int_{f_1(r)}^{f_2(r)} f(r,\theta)\, r\mathrm{d}\theta\mathrm{d}r$

Area of a section of a circle: $A = \int_a^b \frac{1}{2} r^2 \mathrm{d}\theta$

9.12 – Green’s Theorem

Green’s Theorem is most useful for finding the area of regions (R) with holes.

$\oint_C = -\oint_{-C}$ ← reverse direction, reverse sign

$$\oint_C P\mathrm{d}x + Q\mathrm{d}y = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}A$$

Note, find the ranges of R and put it into the form

$$\int_c^d \int_a^b \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}y\mathrm{d}x$$

Ellipse: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ (i.e. if RS isn’t 1, divide both sides), $\begin{matrix} x = a\cos\theta \\ y = b\sin\theta \end{matrix}$; $a = x$ radius; $b = r$ radius

$A = \int y\mathrm{d}x$

Orientation: Positive orientation: counter-clockwise; negative orientation: clockwise

Green’s theorem can be used to find area by line integral:

$A = \iint_D \mathrm{d}A$, provided we choose L and M, such that $\dfrac{\partial M}{\partial x} - \dfrac{\partial L}{\partial y} = 1$

Holes

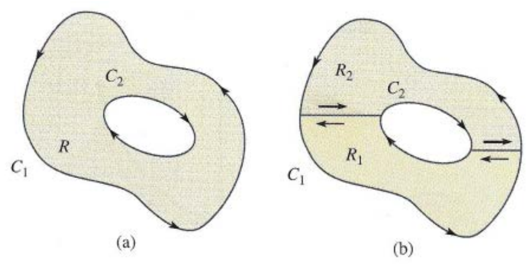


FIGURE 9.12.8 Boundary of R is $C = C_1 \cup C_2$

When you are dealing with Green's theorem with a hole, you need to take Green's theorem of the outside and subtract the inside because you can split up the region into R_1 and R_2, but the lines cancel each other out. Since the region you are dealing with should always be to your left, the hole is represented by a negatively-oriented line.

9.13 – Surface Integrals

Tangent plane equation: $\mathbf{r}_s \times \mathbf{r}_t$ or u, v.

Recall that vector equations are in terms of u and v or s and t:

$\mathbf{r}(u,v) = \langle \text{point} \rangle + u\langle \quad \rangle + v\langle \quad \rangle$

$\mathbf{r}(t)$ defines a curve

$\mathbf{r}(u,v)$ defines a surface

Surface Area: $SA = \iint\limits_R \|\mathbf{r}_s \times \mathbf{r}_t\| r\mathrm{d}r\mathrm{d}\theta$

If $z = f(x,y)$, surface area: $SA = \iint\limits_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,\mathrm{d}A$,

If $z = g(x,y)$, SA: $\iint\limits_S f(x,y,z)\,\mathrm{d}S = \iint\limits_D f(x,y,g(x,y))\sqrt{1+g_x^2+g_y^2}\,\mathrm{d}A$

If $z = g(x,y)$, $\mathbf{r}(x,y) = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + g(x,y)\hat{\mathbf{k}}$

ρ = radius of sphere

$x^2 + y^2 + z^2 = \rho^2$

$\iint\limits_S f(x,y,z)\,\mathrm{d}S = \iint\limits_D f(\mathbf{r}(u,v))\|\mathbf{r}_u \times \mathbf{r}_v\|\,\mathrm{d}A$

Orientable: no Möbius strips

Non-orientable: something containing a Möbius strip

Flux: $\iint\limits_S \mathbf{F}\cdot\mathrm{d}S = \iint\limits_S (\mathbf{F}\cdot\hat{\mathbf{n}})\,\mathrm{d}S = \iint\limits_D \mathbf{F}(\mathbf{r}(u,v))\cdot\|\mathbf{r}_u \times \mathbf{r}_v\|\,\mathrm{d}A$

When given points to find the flux of:

1. Find vectors to represent each of the lines.
2. Express each surface in terms of a point and two lines connected to that point
   $\mathbf{r}(u,v) = \langle A \rangle + u\langle AB \rangle + v\langle AC \rangle$
3. Find flux, regularly.

9.14 – Stokes' Theorem

Stokes's Theorem: the special case of Green's Theorem that involves vectors

$$\oint_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F}\cdot \hat{\mathbf{n}}\mathrm{d}S$$

Right Side

$\hat{\mathbf{n}}\mathrm{d}S = \mathrm{d}\mathbf{S} = \left(\mathbf{S}_u \times \mathbf{S}_v\right)\mathrm{d}A = \nabla f \mathrm{d}A$, where u and v are the other 2 variables

$$\iint_S \operatorname{curl}\mathbf{F}\cdot \hat{\mathbf{n}}\mathrm{d}S = \boxed{\iint_R \operatorname{curl}\mathbf{F}\cdot \nabla f \mathrm{d}A}$$

$$6 = 2x + y + 2z$$

$$f = 2x + y + 2z - 6$$

$$r(x,z) = \begin{cases} x = x \\ y = 6 - 2x - 2z \\ z = z \end{cases}$$

Now, to determine the bounds, set $y = 0$, so

$$0 = 6 - 2x - 2z$$

$$3 = x + z$$

Use this equation to define the boundary of the integral.

$$\boxed{\int_0^3 \int_0^{3-z} \operatorname{curl}\mathbf{F}\cdot \nabla f \mathrm{d}x\mathrm{d}z}$$

Left Side

Find your boundary curve, C, which should be derived by letting one variable be a function in terms of the other 2 variables.

$$\oint_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_a^b \mathbf{F}\left(\mathbf{r}(t)\right)\cdot \mathbf{r}'(t)\mathrm{d}t$$

9.15 – Triple Integrals

Cylindrical Coordinates:

$$(x,y,z) \to (r,\theta,z)$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$z = z$$

Jacobian: r

Spherical Coordinates:

$$(x,y,z) \to (\rho,\phi,\theta)$$

$$x = \rho\sin\phi\cos\theta$$

$$y = \rho\sin\phi\sin\theta$$

$$z = \rho\cos\phi$$

Jacobian: $\rho^2 \sin\phi$

9.16 – Divergence Theorem

Flux: $\iint_S \mathbf{F}\mathrm{d}S$

$$\iint_S \mathbf{F}\mathrm{d}S = \iiint_V \operatorname{div}\mathbf{F}\mathrm{d}V$$

9.17 – Change in Variables in Multiple Integrals

Jacobian $$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$
$$= \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$$

13 – Boundary Value Problems in Rectangular Coordinates

13.1 – Separable Partial Differential Equations

If non-homogeneous, go here. This chapter involves solving homogeneous, linear, second order partial differential equations.

Homogeneous 2nd Order PDE form

$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = 0$, where A to F are real constants

Separation Constant

Take this example: $\frac{X''}{FX} = \frac{Y'}{Y}$

Since the left side only depends on X and the right side only depends on Y, they must both be equal to the separation constant:

$$\frac{X''}{FX} = \frac{Y'}{Y} = -\lambda$$

Note: if you can't get the separation constant, you can't find the answer, such as if you have

$$\frac{\partial^2 u}{\partial x \partial y}$$

Also: if you notice that you have $\frac{\partial u}{\partial t}$'s instead of $\frac{\partial u}{\partial y}$'s, then use T in place of Y.

Separation of Variables

Goal: determine solutions in the form, $u(x,y) = X(x)Y(y)$. Thus, use this as your "guess", i.e. $u_x = X'(x)Y(y); u_y = X(x)Y'(y)$, etc.

Trivial solution: When your function equals 0, but it's WRONG because the purpose is to find a function that represents your thing

$X'' + \lambda F X = 0$ AND $Y' + \lambda Y = 0$

Once they are laid out, try each of the 3 cases because the λ could be in any of the three ranges. Also, you only have to do the cases for X or Y, since the λ would be the same for either one:

i) $\lambda = 0$

$X'' = 0, Y'' = 0$

Integrate 0 twice:

$X = c_1 x + c_2$

ii)

$\lambda < 0 , \lambda = -\alpha^2$

$X'' - k\lambda X = 0, Y'' - \lambda Y = 0$

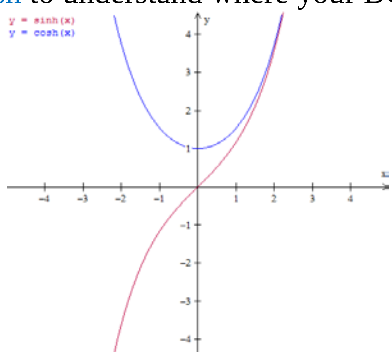
Auxiliary results in 2 real roots

$X = c_1 e^{-\alpha x} + c_2 e^{\alpha x}$

If you plug in the identities for hyperbolic cosine and sine, you'll get:

$X = c_1 \cosh(\alpha x) + c_2 \sinh(\alpha x)$

Picture the graphs of sinh and cosh to understand where your BC's fail to determine constants.



iii) Complex

$\lambda > 0 , \lambda = \alpha^2$

$X'' + 4\lambda X = 0, Y'' + \lambda Y = 0$

Auxiliary results in complex roots:

$X = c_1 \cos(\alpha x) + c_2 \sin(\alpha x)$

Superposition principle: If $u_1, u_2, ..., u_k$ are solutions to a homogeneous, linear PDE, then the linear combination, $u = c_1 u_1 + c_2 u_2 + ... + c_k u_k$, where c_i is a constant, is also a solution. Also works if solutions look like an infinite set $\{u_i\}_{i=1}^{\infty}$; then $u = \sum_{n=1}^{\infty} c_i u_i$ is also (formally) a solution.

Types of 2nd order PDE's

Hpe

>0<

Hyperbolic $B^2 - 4AC > 0$

Parabolic $B^2 - 4AC = 0$

Elliptic $B^2 - 4AC < 0$

13.2 – Classical PDE's and BVP's

Classifications

3 types of possible BC's $(\eta = x \text{ or } y)$

i) Dirachlet

Condition on u; hint: case 1 and case 2 are ALWAYS trivial.

ii) Neumann

Condition on $\frac{\partial u}{\partial \eta}$, normal derivative

iii) Robin

Condition on $\frac{\partial u}{\partial \eta} + hu$, h is a constant

a) If the heat is being lost to the right end: $\left.\frac{\partial u}{\partial x}\right|_{x=L} = -c\left(u(L,t) - u_{\text{medium}}\right)$

b) If heat is coming in on the left: $\left.\frac{\partial u}{\partial x}\right|_{x=0} = c\left(u(0,t) - u_{\text{medium}}\right)$

If you have heat transfer along lateral surface of rod into a medium at constant temperature, u_{medium}:

$$k\frac{\partial^2 u}{\partial x^2} - h\left(u - u_{\text{medium}}\right) = \frac{\partial u}{\partial t}$$

1D Heat equation

Parabolic function

Temperature function: $u(x,t)$

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

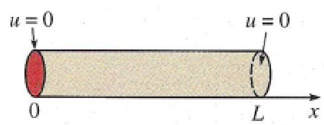


FIGURE 13.3.1 Find the temperature u in a finite rod

Constants:

- **Specific heat capacity** of the material of the rod: γ
- **Thermal conductivity** of the material of the rod: K
- Since it is assumed to have uniform density, **density**: ρ
- These are all summed up by **thermal diffusity**: $k = \dfrac{K}{\gamma\rho}$

Assumptions:

- Heat flows only in *x*-direction, i.e. $0 < x < L$
- Lateral surface is perfectly insulated, so heat can only leave through the ends.
- No heat generated within rod

Hint: when you put it in separation of variables format, group the *k* with the *T*'s.

1D Wave Equation

Hyperbolic function

Displacement function: $u(x,t)$

Wave equation: $a^2 \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}$

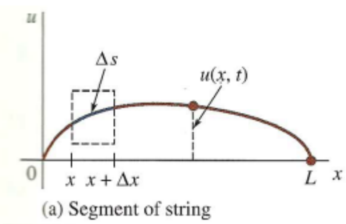


FIGURE 13.2.2 Taut string anchored at two points on the *x*-axis

Constants:

- Density: ρ
- Magnitude of tension: T

Assume:

- String is perfectly flexible, i.e. perfect elasticity
- u is small, compared to the length of the string
- Tension, **T**, is only tangent to string
- Ignore gravity and all other external forces

2D Laplace Equation

Elliptic function

The function, $u(x,y)$, describes a steady-state temperature distribution on a rectangular plate.

Steady-state means that the temperature at a given point doesn't change with time.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\Delta u = 0$$

$$\nabla^2 u = 0$$

Hint: when deciding whether to do *X* or *T*, choose the one with more boundary conditions.

Eigenvalues: $\lambda_n = \left(\frac{n\pi}{a}\right)^2, n = 1, 2, 3, \ldots$

Eigenfunctions: $\cos\left(\frac{n\pi x}{a}\right), n = 1, 2, 3, ..$

13.3 – Heat Equation

$$X = c_1 x + c_2$$

$$X = c_1 \cosh(\alpha x) + c_2 \sinh(\alpha x)$$

$$X = c_1 \cos(\alpha x) + c_2 \sin(\alpha x)$$

$$m^2 + 4n^2 = (m + 2ni)(m - 2ni)$$

13.4 – Wave Equation

Wave equation: $a^2 \frac{\partial^2 u}{\partial x^2} + \underbrace{f(x,t)}_{\text{external force}} - c \underbrace{\frac{\partial u}{\partial t}}_{\text{damping}} - \underbrace{ku}_{\text{restoring force}} = \frac{\partial^2 u}{\partial t^2}$

13.5 – Laplace

Be careful because there are 2 types of boundary conditions: one along the *x*-axis and the other along the *y*-axis.

13.6 – No-homo BVP's

A BVP is **non-homogeneous** when <u>either the PDE or the BCs</u> are non-homogeneous.
Warning: you cannot always perform separation of variables if the problem is inhomogeneous.

If you cannot perform separation of variables, you'll need to change the dependent variable. For example: $u(x,t) \Rightarrow v(x,t) + \psi(x)$

v: transient-state solution
ψ: steady-state solution

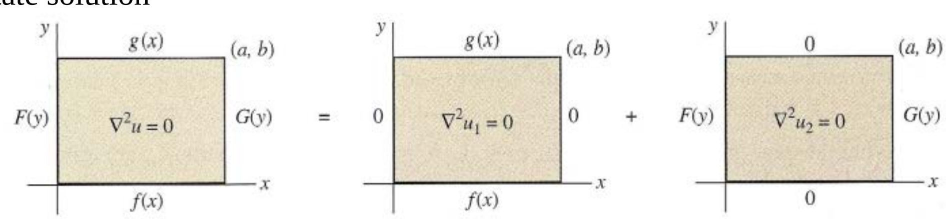


FIGURE 13.5.3 Solution u = Solution u_1 of Problem 1 + Solution u_2 of Problem 2

Gives 2 equations $\begin{cases} \frac{\partial^2 u}{\partial x^2} \Rightarrow \frac{\partial^2 v}{\partial x^2} + \psi''(x) \\ \frac{\partial u}{\partial t} \Rightarrow \frac{\partial v}{\partial t} + 0 \end{cases}$

Since ψ is not in term of t, it was treated as a constant and differentiated to 0.

$$k\frac{\partial^2 v}{\partial x^2}+\psi''(x)+F(x)=\frac{\partial v}{\partial t}$$

When heat is being produced inside a rod at a constant rate of, *r*, the heat equation becomes:

$$k\frac{\partial^2 u}{\partial x^2}+r=\frac{\partial u}{\partial t}\Rightarrow k\frac{\partial^2 v}{\partial x^2}+\psi''(x)+r=\frac{\partial v}{\partial t}$$

$k\psi''(x)+r=0$ Treat it like case 1. $k\psi''=-r$ $\psi''=-\frac{r}{k}$ $\psi'=-\frac{rx}{k}+c_1$ $\psi=-\frac{rx^2}{2k}+c_1x+c_2$	$k\frac{\partial^2 v}{\partial x^2}=\frac{\partial v}{\partial t}$ Now, evaluate it like this.

Once you've solved for both, using the BC's and IC's, add them together and you have your final answer.

Time Dependent

If your formula given looks like this: $k\frac{\partial^2 u}{\partial x^2}+\underbrace{F(x,t)}=\frac{\partial u}{\partial t}$, where the underlined portion is also in terms of *t*, you have been given a time-dependent question.

$u_0(t)=u(0,t)$

$u_1(t)=u(L,t)$

$\psi(x,t)=u_0(t)+\frac{x}{L}\left[u_1(t)-u_0(t)\right]$

$u(x,t)=v(x,t)+\psi(x,t)$

$G(x,t)=\psi''(t)-\frac{\partial\psi}{\partial t}$

$v(x,t)=\sum_{n=1}^{\infty}F_n(t)\sin\left(\frac{n\pi}{L}x\right)\Rightarrow\frac{\partial^2 v}{\partial^2 x}=\sum_{n=1}^{\infty}-\left(\frac{n\pi}{L}\right)^2F_n(t)\sin\left(\frac{n\pi}{L}x\right)$

$\frac{\partial v}{\partial t}=\sum_{n=1}^{\infty}F_n'(t)\sin\left(\frac{n\pi}{L}x\right)$

We want to express $G(x,t)$ as a Fourier summation, so you need to find G_n.

$G(x,t)=\sum_{n=1}^{\infty}G_n(t)\sin\left(\frac{n\pi}{L}x\right)\Rightarrow G_n(t)=\frac{2}{L}\int_0^L G(x,t)\sin\left(\frac{n\pi}{L}x\right)\mathrm{d}x$

Put your summation version of $G(x,t)$ into:

$$k\frac{\partial^2 v}{\partial x^2} + G\left(x,t\right) = \frac{\partial v}{\partial t}$$

Plug everything in:

$$\sum_{n=1}^{\infty} -\left(\frac{n\pi}{L}\right)^2 F_n\left(t\right)\sin\left(\frac{n\pi}{L}x\right) + \sum_{n=1}^{\infty} G_n\left(t\right)\sin\left(\frac{n\pi}{L}x\right) = \sum_{n=1}^{\infty} F_n'\left(t\right)\sin\left(\frac{n\pi}{L}x\right)$$

unfinished