# Assignment 11

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# Outline

# Problem Statement

**(Papoulis/Pillai Exercise 3-4)** A coin with $\Pr(h) = p = 1 - q$ is tossed n times. Show that the probability that the number of heads is even equals $0.5[1 + (q - p)^n]$.

# Solution

Let $Y \sim \text{Bin}(n, p)$ represent the binomial random variable representing the number of heads attained. We note the following binomial expansions:

$$
\begin{aligned}
1 = (q+p)^n &= \sum_{k=0}^{k=n} \binom{n}{k} q^{(n-k)} p^k && (1) \\
&= \binom{n}{0} q^n + \binom{n}{1} q^{(n-1)} p + \binom{n}{2} q^{(n-2)} p^2 + \dots && (2) \\
(q-p)^n &= \sum_{k=0}^{k=n} \binom{n}{k} (-1)^k q^{(n-k)} p^k && (3) \\
&= \binom{n}{0} q^n - \binom{n}{1} q^{(n-1)} p + \binom{n}{2} q^{(n-2)} p^2 + \dots && (4)
\end{aligned}
$$

## PMF of Y

$$\Pr(Y = k) = \begin{cases} \binom{n}{k} p^{(n-k)} q^k, & 0 \le k \le n \\ 0, & \text{otherwise} \end{cases} \tag{5}$$

$$\implies \Pr(Y \equiv 0 \pmod 2) = \binom{n}{0} p^n + \binom{n}{2} p^{(n-2)} q^2 + \dots \tag{6}$$

Therefore, adding 2 and 4, we get,

$$1 + (q - p)^n = 2(\binom{n}{0} p^n + \binom{n}{2} p^{(n-2)} q^2 + \dots) \tag{7}$$

$$= 2 \Pr(Y \equiv 0 \pmod 2) \tag{8}$$

$$\implies \Pr(Y \equiv 0 \pmod 2) = 0.5[1 + (q - p)^n] \tag{9}$$

as deisred. This is verified in `codes/11_1.py`.