Assignment 14

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Outline

Problem Statement

(Papoulis/Pillai, Exercise 8-25) We are given a random variable X with mean η and standard deviation $\sigma = 2$, and we wish to test the hypothesis $\eta_0 = 8$ against $\eta = 8.7$ with $\alpha = 0.01$ using as the test statistic the sample mean $\bar{x}$ of n samples.

1. Find the critical region R_c of the test and the resulting β if $n = 64$.
2. Find n and R_c if $\beta = 0.05$.

Definitions

Random Variable Used

When using the test statistic as the mean $(\bar{X})$, we consider the random variable

$$Y = \frac{\bar{X} - \eta_0}{\frac{\sigma}{\sqrt{n}}} \tag{1}$$

We assume $\bar{X} \sim N(\eta_0; \frac{\sigma}{\sqrt{n}})$. Hence, $Y \sim N(\eta_y; 1)$, where η is the new parameter of $\bar{X}$

$$\eta_y = \frac{\eta - \eta_0}{\frac{\sigma}{\sqrt{n}}} \tag{2}$$

Observe that for the null hypothesis $H_0 : \eta = \eta_0$, we have $Y \sim N(0; 1)$ so we can use the standard normal percentiles.

Solution

1 For $H_1 : \eta > \eta_0$, the critical region is the half-line $y > c$, where $c = Q^{-1}(\alpha) = 2.326$. In terms of $\bar{X}$, the critical region is the half-line $\bar{x} > C$, where from (2),

$$C = \eta_0 + Q^{-1}(\alpha)\frac{\sigma}{\sqrt{n}} = 8 + 2.326 \times \frac{2}{8} = 8.58 \tag{3}$$

Hence, R_c is given by $\bar{x} > 8.58$ (refer Figure (1)). Now, if $H_1 : \eta > \eta_0$, then

$$\eta_q = \frac{8.7 - 8}{\frac{2}{8}} = 2.8 \tag{4}$$

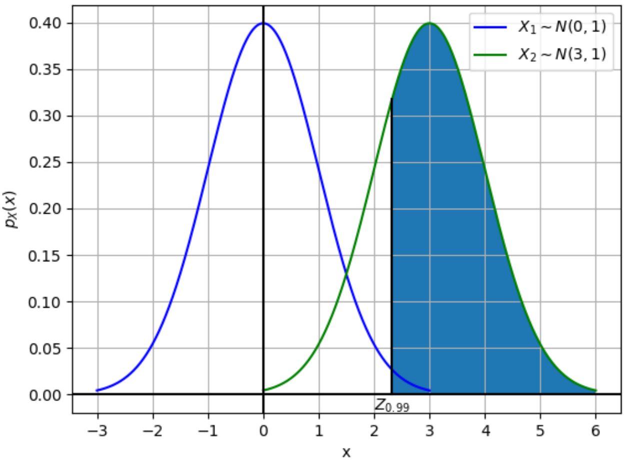

Figure 1: Critical region where null hypothesis is not accepted. Here, $\eta_0 = 0$ and $\eta = 3$.

1 (Cont'd...) β is a function of η given by

$$\beta(\eta) = H_1 : \Pr\left(Y \notin R_c\right) = \Pr\left(Y < c\right) \tag{5}$$
$$= Q(Q^{-1}(\alpha) - \eta_y) = 0.32 \tag{6}$$

2 Proceeding in reverse from (6), we have from the definition of β,

$$Q^{-1}(\alpha) - \eta_y = Q^{-1}(1 - \beta) \tag{7}$$
$$\implies \eta_y = Q^{-1}(\alpha) - Q^{-1}(1 - \beta) \tag{8}$$
$$= Q^{-1}(0.01) - Q^{-1}(0.95) = 4.97 \tag{9}$$

Given that $\beta(\eta = 8.7) = 0.05$, we can also use (2) to get

$$n = \left(\frac{\sigma\eta_q}{\eta - \eta_0}\right)^2 = 201 \tag{10}$$

Thus, using (3), we get

$$C = 8 + Q^{-1}(0.99)\frac{2}{\sqrt{201}} = 8.33 \tag{11}$$