

Random Numbers

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Abstract—This manual provides a simple introduction to the generation of random numbers. Commands that must be executed in a *NIX shell are preceded by a \$ symbol.

1 UNIFORM RANDOM NUMBERS

Let U be a uniform random variable between 0 and 1.

- 1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .

Solution: Download the following files

```
$ wget https://raw.githubusercontent.com/
goats-9/ai1110-probability/master/
manual/codes/1_1.c
$ wget https://raw.githubusercontent.com/
goats-9/ai1110-probability/master/
manual/codes/1_1.h
```

and compile and execute the C program using

```
$ gcc 1_1.c -lm -Wall -g
$ ./a.out
```

- 1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \quad (1.1)$$

Solution: The following code plots Fig. 1.1

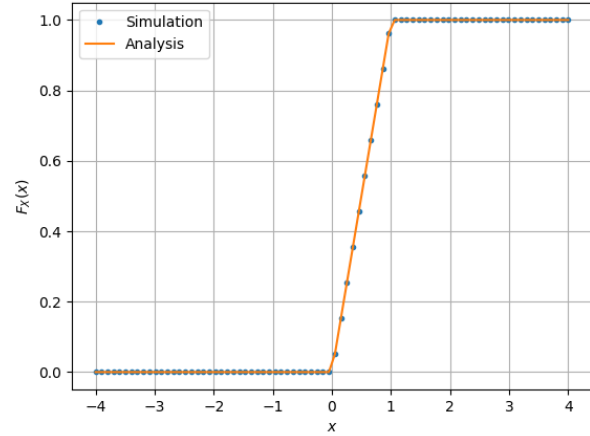


Fig. 1.1: The CDF of U

```
$ wget https://raw.githubusercontent.com/
goats-9/ai1110-probability/master/
manual/codes/1_2.py
```

It is executed with

```
$ python3 1_2.py
```

- 1.3 Find a theoretical expression for $F_U(x)$.

Solution: The CDF of U is given by

$$F_U(x) = \Pr(U \leq x) = \int_{-\infty}^x p_U(u) du \quad (1.2)$$

We now have three cases:

- a) $x < 0$: $p_U(x) = 0$, and hence $F_U(x) = 0$.
b) $0 \leq x < 1$: Here,

$$F_U(x) = \int_0^x du = x \quad (1.3)$$

- c) $x \geq 1$: Put $x = 1$ in (1.3) as U is uniform in $[0, 1]$ to get $F_U(x) = 1$.

Therefore,

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases} \quad (1.4)$$

This is verified in Figure (1.1)

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N} \sum_{i=1}^N U_i \quad (1.5)$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \quad (1.6)$$

Write a C program to find the mean and variance of U .

Solution: The C program can be downloaded using

```
$ wget https://raw.githubusercontent.com/
goats-9/ai1110-probability/master/
manual/codes/1_4.c
```

and compiled and executed with

```
$ gcc 1_4.c -lm -Wall -g
$ ./a.out
```

The calculated mean is 0.500007 and the calculated variance is 0.083301.

1.5 Verify your result theoretically given that

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \quad (1.7)$$

Solution: We write

$$E[U^2] = \int_{-\infty}^{\infty} x^2 dF_U(x) \quad (1.8)$$

$$= \int_{-\infty}^{\infty} x^2 p_U(x) dx \quad (1.9)$$

$$= \int_0^1 x^2 dx = \frac{1}{3} \quad (1.10)$$

and

$$E[U] = \int_{-\infty}^{\infty} x dF_U(x) \quad (1.11)$$

$$= \int_{-\infty}^{\infty} x p_U(x) dx \quad (1.12)$$

$$= \int_0^1 x dx = \frac{1}{2} \quad (1.13)$$

which checks out with the empirical mean on 0.500007. Now, using linearity of expectation,

$$\text{var}[U] = E[U - E[U]]^2 \quad (1.14)$$

$$= E[U^2 - 2UE[U] + (E[U])^2] \quad (1.15)$$

$$= E[U^2] - 2(E[U])^2 + (E[U])^2 \quad (1.16)$$

$$= E[U^2] - (E[U])^2 = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{12} \quad (1.17)$$

and this checks out with the empirical variance 0.083301 of the sample data.

2 CENTRAL LIMIT THEOREM

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \quad (2.1)$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

Solution: The sample data is generated by the C file in Question 1.1.

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?

Solution: The CDF of X is plotted in Fig. 2.1 Download the Python code using

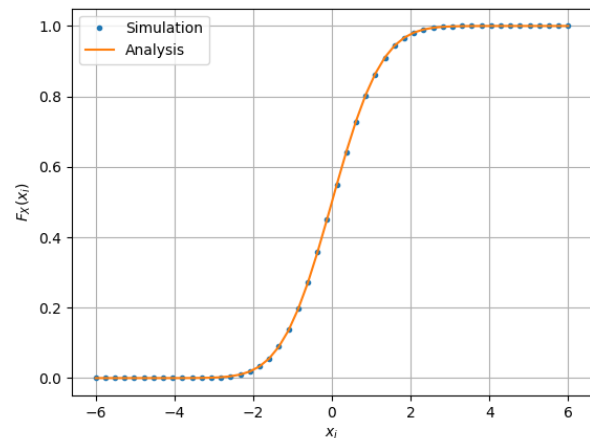


Fig. 2.1: The CDF of X

```
$ wget https://raw.githubusercontent.com/
goats-9/ai1110-probability/master/
manual/codes/2_2.py
```

and execute it with

```
$ python3 2_2.py
```

The CDF of a probability distribution has the following properties:

- a) It is non-decreasing
- b) It is right-continuous
- c) $\lim_{x \rightarrow -\infty} F_X(x) = 0$
- d) $\lim_{x \rightarrow \infty} F_X(x) = 1$

The CDF of the normal distribution is expressed in terms of the Q-function as $F_X(x) = 1 - Q(x)$.

- 2.3 Load `gau.dat` in python and plot the empirical PDF of X using the samples in `gau.dat`. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \quad (2.2)$$

What properties does the PDF have?

Solution: The PDF of X is plotted in Fig. 2.2 using the code below

```
$ wget https://raw.githubusercontent.com/
goats-9/ai1110-assignments/master/
manual/codes/2_3.py
```

The figure is generated using

```
$ python3 2_3.py
```

The properties of a PDF are as follows:

- a) $\forall x \in \mathbb{R}, p_X(x) \geq 0$
- b) $\int_{-\infty}^{\infty} p_X(x) dx = 1$
- c) For $a < b, a, b \in \mathbb{R}$

$$\Pr(a < X < b) = \Pr(a \leq X \leq b) \quad (2.3)$$

$$= \int_a^b p_X(x) dx \quad (2.4)$$

If we take $a = b$, then we get $\Pr(X = a) = 0$.

- 2.4 Find the mean and variance of X by writing a C program.

Solution: The mean and variance have been calculated using (1.5) and (1.6) respectively. The C program can be downloaded using

```
$ wget https://raw.githubusercontent.com/
goats-9/ai1110-assignments/master/
manual/codes/2_4.c
```

and compiled and executed with the following commands

```
$ gcc 2_4.c -lm -Wall -g
```

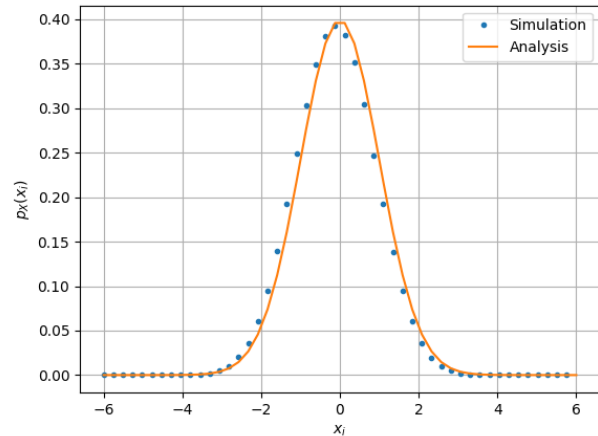


Fig. 2.2: The PDF of X

```
$ ./a.out
```

The calculated mean is 0.000326 and the calculated variance is 1.000906.

- 2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.5)$$

repeat the above exercise theoretically.

Solution: The mean is given by

$$E[X] = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx = 0 \quad (2.6)$$

as the integrand is odd. This checks out with the empirical mean of 0.000326. The variance is given by

$$\text{var}[X] = E[X^2] - (E[X])^2 \quad (2.7)$$

$$= \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (2.8)$$

$$= \int_0^{\infty} \frac{2}{\sqrt{2\pi}} \sqrt{2t} e^{-t} dt \quad (2.9)$$

$$= \frac{2}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}\right) \quad (2.10)$$

$$= \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}\right) = 1 \quad (2.11)$$

where we have used $t = \frac{x^2}{2}$ and so $dt = x dx$. We have also used the gamma function defined

as

$$\Gamma(n) = \int_{-\infty}^{\infty} x^{n-1} e^{-x} dx \quad (2.12)$$

$$\Gamma(n) = (n-1)\Gamma(n-1) \text{ for } n > 1 \quad (2.13)$$

and the fact that $\Gamma(1/2) = \sqrt{\pi}$. This agrees with the empirical variance of 1.000906.

3 FROM UNIFORM TO OTHER

3.1 Generate samples of

$$V = -2 \ln(1 - U) \quad (3.1)$$

and plot its CDF.

Solution: The relevant python code is at

```
$ wget https://raw.githubusercontent.com/
goats-9/ai1110-probability/master/
manual/codes/3_1.py
```

and can be executed with

```
$ python3 3_1.py
```

The CDF is plotted in Figure (3.1).

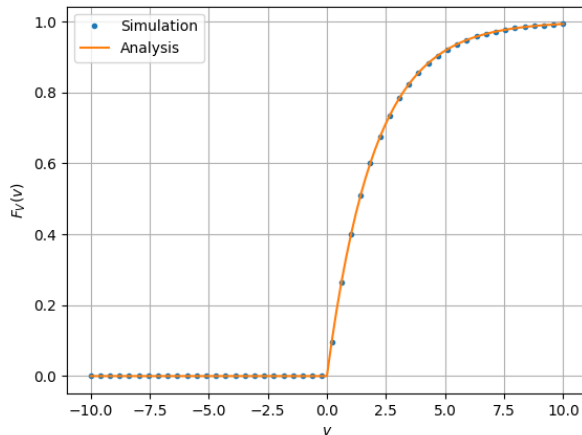


Fig. 3.1: The CDF of V

3.2 Find a theoretical expression for $F_V(x)$.

Solution: Note that the function

$$v = f(u) = -2 \ln(1 - u) \quad (3.2)$$

is monotonically increasing in $[0, 1]$ and $v \in \mathbb{R}^+$. Hence, it is invertible and the inverse function is given by

$$u = f^{-1}(v) = 1 - \exp\left(-\frac{v}{2}\right) \quad (3.3)$$

Therefore, from the monotonicity of v , and using (1.4),

$$F_V(v) = F_U\left(1 - \exp\left(-\frac{v}{2}\right)\right) \quad (3.4)$$

$$\Rightarrow F_V(v) = \begin{cases} 0 & v < 0 \\ 1 - \exp\left(-\frac{v}{2}\right) & v \geq 0 \end{cases} \quad (3.5)$$

4 TRIANGULAR DISTRIBUTION

4.1 Generate

$$T = U_1 + U_2 \quad (4.1)$$

Solution: The samples are generated in the C file exrand.c in 1.1 as the file tri.dat.

4.2 Find the CDF of T .

Solution: The Python code for the figure is at

```
$ wget https://raw.githubusercontent.com/
goats-9/ai1110-assignments/master/
manual/codes/4_2.py
```

and can be run using

```
$ python3 4_2.py
```

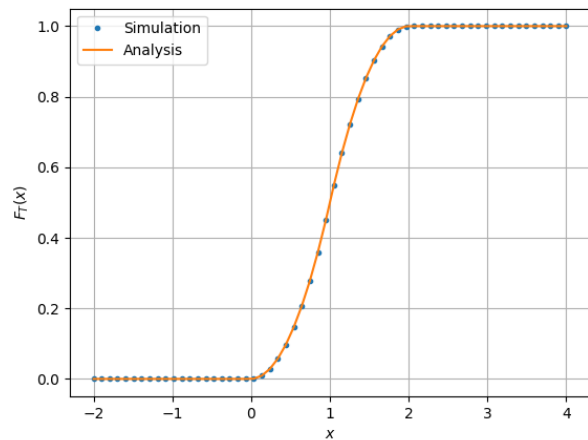


Fig. 4.1: The CDF of T

4.3 Find the PDF of T .

Solution: The Python code for the figure can be downloaded using

```
$ wget https://raw.githubusercontent.com/
goats-9/ai1110-assignments/master/
manual/codes/4_3.py
```

and run using

```
$ python3 4_3.py
```

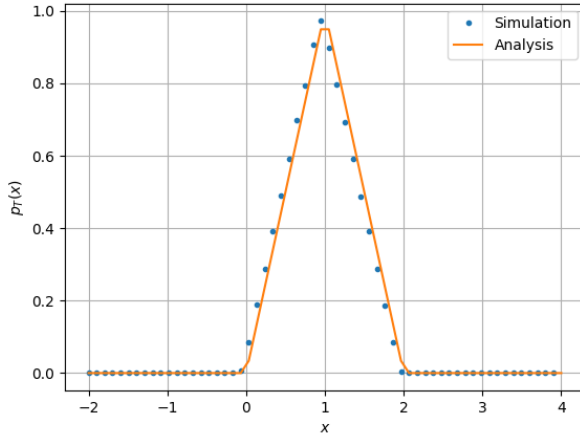


Fig. 4.2: The PDF of T

4.4 Find the theoretical PDF and CDF of T .

Solution: We write,

$$F_T(t) = \Pr(U_1 + U_2 \leq t) \quad (4.2)$$

$$= \Pr(U_1 \leq t - U_2) \quad (4.3)$$

$$= \int_0^1 F_{U_1}(t - x) p_{U_2}(x) dx \quad (4.4)$$

where U_1 and U_2 are uniform i.i.d. random variables in $[0, 1]$. Then, $0 \leq U_1 + U_2 \leq 2$. We have three cases:

- a) $t < 0$: Using Equation 1.4, $F_T(t) = 0$.
- b) $0 \leq t < 1$: We have,

$$F_T(t) = \int_0^t (t - x) dx = \frac{t^2}{2} \quad (4.5)$$

- c) $1 \leq t < 2$: Here, we get

$$F_T(t) = \int_0^{t-1} dx + \int_{t-1}^1 (t - x) dx \quad (4.6)$$

$$= t - 1 + t(2 - t) - \frac{1 - (t - 1)^2}{2} \quad (4.7)$$

$$= -\frac{t^2}{2} + 2t - 1 \quad (4.8)$$

- d) $t \geq 2$: Here, $F_T(t) = 1$.

Therefore,

$$F_T(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \leq t < 1 \\ -\frac{t^2}{2} + 2t - 1 & 1 \leq t < 2 \\ 1 & t \geq 2 \end{cases} \quad (4.9)$$

Using Equation 2.2,

$$p_T(t) = \begin{cases} t & 0 \leq t < 1 \\ 2 - t & 1 \leq t < 2 \\ 0 & \text{otherwise} \end{cases} \quad (4.10)$$

4.5 Verify your results through a plot.

Solution: This has been done in the plots (4.1) and (4.2).

5 MAXIMUM LIKELIHOOD

5.1 Generate

$$Y = AX + N \quad (5.1)$$

where $A = 5$ dB, $X \in \{1, -1\}$ is Bernoulli and $N \sim \mathcal{N}(0, 1)$.

Solution: The C file in Question 1.1 generates the numbers in the file data/gau_ber.dat

5.2 Plot Y.

Solution: The Python code can be downloaded using

```
$ wget https://raw.githubusercontent.com/goats-9/ai1110-probability/master/manual/codes/5_2.py
```

and run using

```
$ python3 5_2.py
```

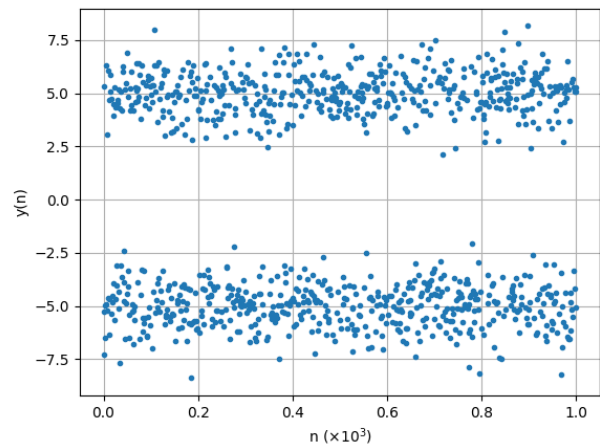


Fig. 5.1: Plot of Y

5.3 Guess how to estimate X from Y .

Solution: From the plot of Y , we see that $X = 1$ usually correlates to $Y > 0$ and $X = -1$ correlates to $Y < 0$.

5.4 Find

$$P_{e|0} = \Pr(\hat{X} = -1|X = 1) \quad (5.2)$$

and

$$P_{e|1} = \Pr(\hat{X} = 1|X = -1) \quad (5.3)$$

Solution: Letting $X = 1$ and $X = -1$ respectively, we see the number of mismatched data points to compute the error probabilities. The simulation is coded in

```
$ wget https://raw.githubusercontent.com/
goats-9/ai1110-assignments/master/
manual/codes/5_4.py
```

and can be run by typing

```
$ python3 5_4.py
```

The results are

$$P_{e|0} = 0 \quad (5.4)$$

$$P_{e|1} = 0 \quad (5.5)$$

5.5 Find P_e .

Solution: Here, we assume $\Pr(X = 1) = \Pr(X = -1) = 0.5$. Thus,

$$P_e = \frac{1}{2} (P_{e|0} + P_{e|1}) = 0 \quad (5.6)$$

5.6 Verify by plotting the theoretical P_e .

Solution: We note that

$$P_{e|0} = \Pr(\hat{X} = 1|X = -1) \quad (5.7)$$

$$= \Pr(Y > 0|X = -1) \quad (5.8)$$

$$= \Pr(AX + N > 0|X = -1) \quad (5.9)$$

$$= \Pr(N > A|X = -1) = Q(A) \quad (5.10)$$

since X and N are independent. Writing a similar expression for $P_{e|1}$ and noting that

$$\Pr(N < -A) = \Pr(N > A) = Q(A) \quad (5.11)$$

it follows that $P_e = Q(A)$. This is the idea used to plot the theoretical P_e . The plot is coded both in the rectangular axes and the semilog-y axes. Download the relevant codes using

```
$ wget https://raw.githubusercontent.com/
goats-9/ai1110-assignments/master/
manual/codes/5_6.py
$ wget https://raw.githubusercontent.com/
goats-9/ai1110-assignments/master/
manual/codes/5_6_semilog.py
```

and execute them using

```
$ python3 5_6.py
$ python3 5_6_semilog.py
```

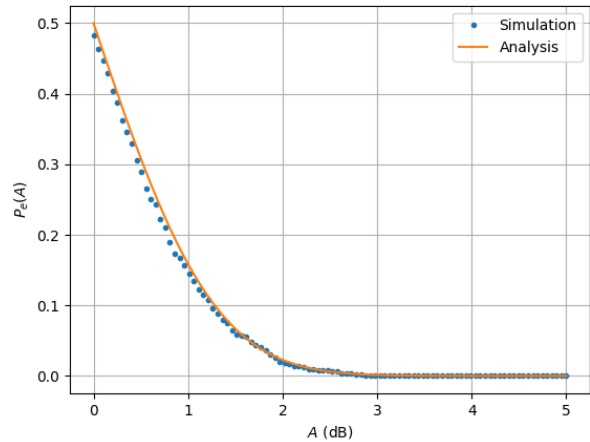


Fig. 5.2: $P_e(A)$ (rectangular axes)

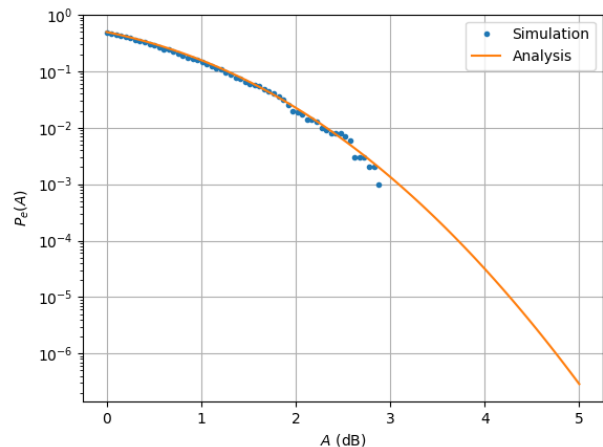


Fig. 5.3: $P_e(A)$ (semilog-y axes)

6 GAUSSIAN TO OTHER

6.1 Let $X_1 \sim \mathcal{N}(0, 1)$ and $X_2 \sim \mathcal{N}(0, 1)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \quad (6.1)$$

Solution: We transform the variables X_1 and X_2 as:

$$X_1 = R \cos \Theta \quad (6.2)$$

$$X_2 = R \sin \Theta \quad (6.3)$$

where $R \in [0, \infty)$, $\Theta \in [0, 2\pi)$. The Jacobian Matrix for this transformation is given by

$$\mathbf{J} = \begin{pmatrix} \frac{\partial X_1}{\partial R} & \frac{\partial X_2}{\partial R} \\ \frac{\partial X_1}{\partial \Theta} & \frac{\partial X_2}{\partial \Theta} \end{pmatrix} \quad (6.4)$$

$$= \begin{pmatrix} \cos \Theta & \sin \Theta \\ -R \sin \Theta & R \cos \Theta \end{pmatrix} \quad (6.5)$$

$$\Rightarrow |\mathbf{J}| = R \quad (6.6)$$

We also know that

$$|\mathbf{J}| p_{X_1, X_2}(x_1, x_2) = p_{R, \Theta}(r, \theta) \quad (6.7)$$

$$\Rightarrow p_{R, \Theta}(r, \theta) = R p_{X_1}(x_1) p_{X_2}(x_2) \quad (6.8)$$

$$= \frac{R}{2\pi} \exp\left(-\frac{X_1^2 + X_2^2}{2}\right) \quad (6.9)$$

$$= \frac{R}{2\pi} \exp\left(-\frac{R^2}{2}\right) \quad (6.10)$$

where (6.8) follows as X_1, X_2 are iid random variables. Thus,

$$p_R(r) = \int_0^{2\pi} p_{R, \Theta}(r, \theta) d\theta \quad (6.11)$$

$$= R \exp\left(-\frac{R^2}{2}\right) \quad (6.12)$$

However, $V = X_1^2 + X_2^2 = R^2 \geq 0$, thus $F_V(x) = 0$ for $x \geq 0$.

$$F_V(x) = F_R(\sqrt{x}) \quad (6.13)$$

$$= \int_0^{\sqrt{x}} r \exp\left(-\frac{r^2}{2}\right) dr \quad (6.14)$$

$$= \int_0^{\frac{x}{2}} e^{-t} dt = 1 - e^{-\frac{x}{2}} \quad (6.15)$$

where $t = \frac{r^2}{2}$ and so, for $x \geq 0$,

$$p_V(x) = \frac{1}{2} e^{-\frac{x}{2}} \quad (6.16)$$

Hence,

$$F_V(x) = \begin{cases} 1 - e^{-\frac{x}{2}} & x \geq 0 \\ 0 & x < 0 \end{cases} \quad (6.17)$$

$$p_V(x) = \begin{cases} \frac{1}{2} e^{-\frac{x}{2}} & x \geq 0 \\ 0 & x < 0 \end{cases} \quad (6.18)$$

The equations (6.17) and (6.18) have been used to generate the plots. The Python code can be downloaded from

`$ wget https://raw.githubusercontent.com/`

`goats-9/ai1110-assignments/master/
manual/codes/6_1_cdf.py
$ wget https://raw.githubusercontent.com/
goats-9/ai1110-assignments/master/
manual/codes/6_1_pdf.py`

Execute the codes by typing the commands

`$ python3 6_1_cdf.py
$ python3 6_1_pdf.py`

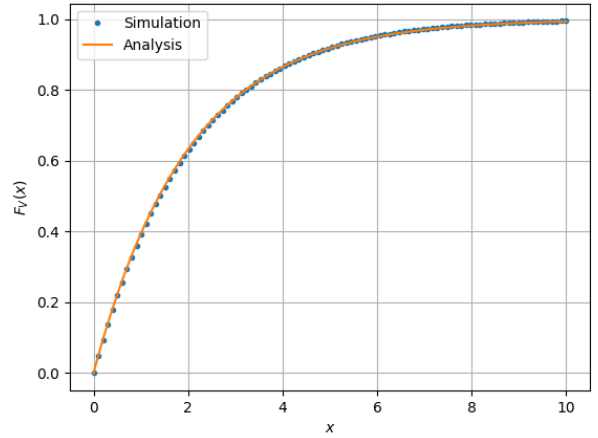


Fig. 6.1: CDF of V

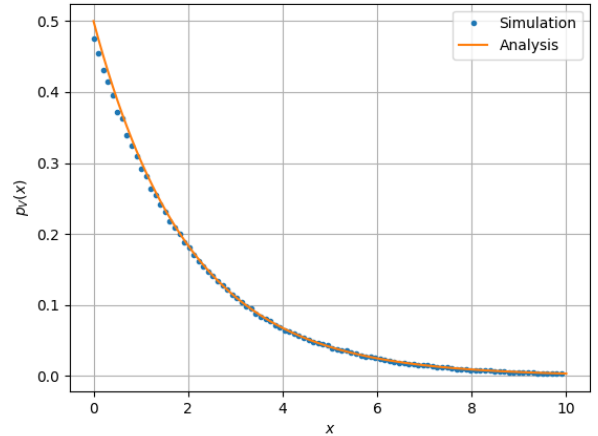


Fig. 6.2: PDF of V

6.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0 \end{cases} \quad (6.19)$$

find α .

Solution: From (6.17), it is clear that $\alpha = 0.5$.

6.3 Plot the CDF and PDF of

$$A = \sqrt{V} \quad (6.20)$$

Solution:

Note that for $x \geq 0$,

$$F_A(x) = \Pr(A \leq x) \quad (6.21)$$

$$= \Pr(\sqrt{V} \leq x) \quad (6.22)$$

$$= \Pr(V \leq x^2) \quad (6.23)$$

$$= F_V(x^2) = 1 - e^{-\frac{x^2}{2}} \quad (6.24)$$

and so,

$$p_A(x) = xe^{-\frac{x^2}{2}} \quad (6.25)$$

Thus, the CDF and PDF of A is given by

$$F_V(x) = \begin{cases} 1 - e^{-\frac{x^2}{2}} & x \geq 0 \\ 0 & x < 0 \end{cases} \quad (6.26)$$

$$p_V(x) = \begin{cases} xe^{-\frac{x^2}{2}} & x \geq 0 \\ 0 & x < 0 \end{cases} \quad (6.27)$$

The Python codes for the plots are at

```
$ wget https://raw.githubusercontent.com/
goats-9/ai1110-assignments/master/
manual/codes/6_3_cdf.py
$ wget https://raw.githubusercontent.com/
goats-9/ai1110-assignments/master/
manual/codes/6_3_pdf.py
```

and can be executed using

```
$ python3 6_3_cdf.py
$ python3 6_3_pdf.py
```

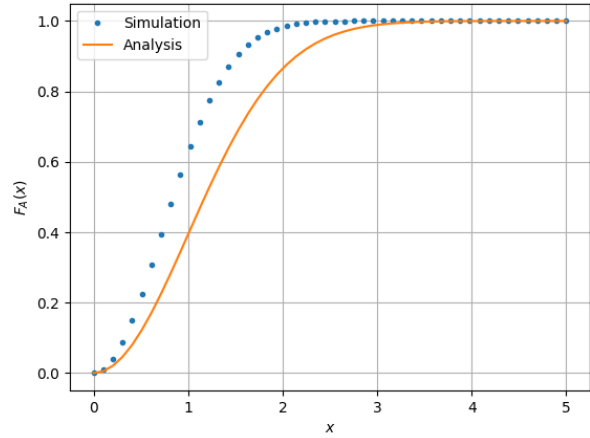


Fig. 6.3: CDF of A

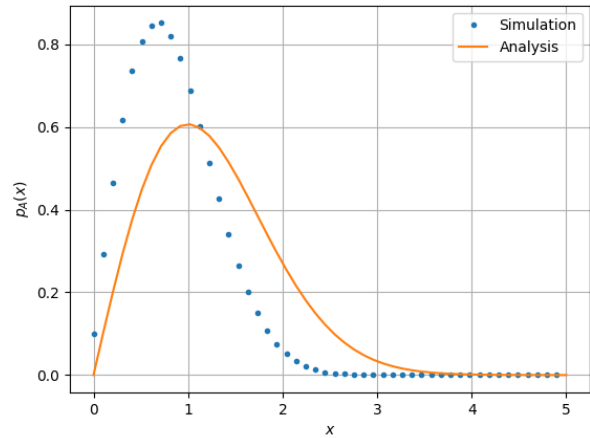


Fig. 6.4: PDF of A

and run it using

```
$ python3 7_1.py
```

7 CONDITIONAL PROBABILITY

7.1 Plot

$$P_e = \Pr(\hat{X} = -1 | X = 1) \quad (7.1)$$

for

$$Y = AX + N \quad (7.2)$$

where A is Rayleigh with $E[A^2] = \gamma$, $N \sim \mathcal{N}(0, 1)$, $X \in \{1, -1\}$ for $0 \leq \gamma \leq 10$ dB.

Solution: Download the relevant Python code

```
$ wget https://raw.githubusercontent.com/
goats-9/ai1110-assignments/master/
manual/codes/7_1.py
```

7.2 Assuming that N is a constant, find an expression for P_e . Call this $P_e(N)$.

Solution: We rewrite the previous expression for P_e as

$$P_e(N) = \Pr(A < -N) = F_A(-N) \quad (7.3)$$

$$= \begin{cases} 1 - e^{-\frac{N^2}{\gamma}} & N \leq 0 \\ 0 & N > 0 \end{cases} \quad (7.4)$$

7.3 For a function g ,

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)p_X(x)dx \quad (7.5)$$

Find $P_e = E[P_e(N)]$.

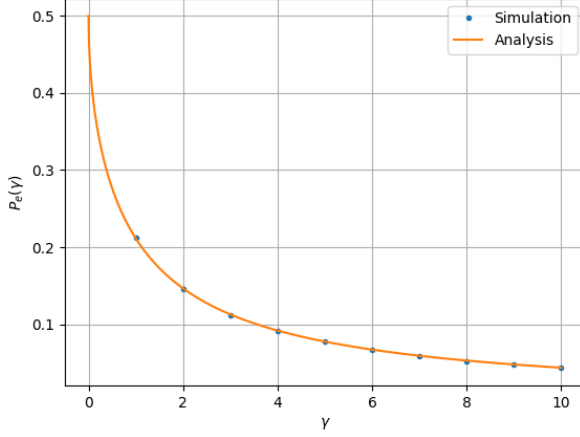


Fig. 7.1: P_e as a function of γ

Solution: We write,

$$P_e = \int_0^\infty F_A(x) f_N(x) dx \quad (7.6)$$

$$= \int_0^\infty (1 - e^{-\frac{x^2}{\gamma}}) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \quad (7.7)$$

$$= \frac{1}{2} - \frac{1}{\sqrt{2\pi}} \int_0^\infty \exp\left(-\frac{x^2}{\frac{2\gamma}{\gamma+2}}\right) dx \quad (7.8)$$

$$= \frac{1}{2} \left(1 - \sqrt{\frac{\gamma}{\gamma+2}}\right) \quad (7.9)$$

where we have integrated by parts and f_N denotes the standard normal distribution.

7.4 Plot P_e in problems 7.1 and 7.3 on the same graph wrt γ . Comment.

Solution: This has been done in Figure (7.1) using the result (7.9). We observe that $P_{e|0} = E[P_e(N)]$ i.e., the error rate is independent of the noise.