

Assignment 11

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Outline

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Problem Statement

(Papoulis/Pillai Exercise 5-52) A box contains n white and m black marbles. Let X represent the number of draws needed for the r^{th} white marble.

- ① If sampling is done with replacement, show that X has a negative binomial distribution with parameters r and $p = \frac{n}{m+n}$.
- ② If sampling is done without replacement, then show that for $r \leq k \leq m+n$,

$$\Pr(X = k) = \binom{k-1}{r-1} \frac{\binom{m+n-k}{n-r}}{\binom{m+n}{n}} \quad (1)$$

- ③ For a given k and r , show that the probability distribution in (1) tends to a negative binomial distribution as $n+m \rightarrow \infty$. Thus, for large population size, sampling with or without replacement is the same.

Solution

- 1 Suppose $X = k$, $r \leq k \leq m + n$. Then, the k^{th} marble drawn must be white. The other $r - 1$ white marbles can be drawn in any of the previous draws, and we must also have $k - r$ black marbles drawn. Since marbles are drawn with replacement, the probability of drawing a white marble at any instant is given by

$$p = \frac{n}{m + n} \quad (2)$$

We can now get the PMF of X (here, $q = 1 - p$)

PMF of X , With Replacement

$$\Pr(X = k) = \begin{cases} \binom{k-1}{r-1} p^r q^{k-r}, & r \leq k \leq m + n \\ 0, & \text{otherwise} \end{cases} \quad (3)$$

- 2 If sampling is done without replacement, the value of p in (2) keeps changing with each draw. We then write out the PMF as follows

PMF of X , Without Replacement

For $r \leq k \leq m + n$,

$$\Pr(X = k) = \binom{k-1}{r-1} \frac{[n \dots (n-r+1)][m \dots (m-k+r-1)]}{[(m+n) \dots (m+n-k+1)]} \quad (4)$$

$$= \binom{k-1}{r-1} \frac{n!}{(n-r)!} \frac{m!}{(m+r-k)!} \frac{(m+n-k)!}{(m+n)!} \quad (5)$$

$$= \binom{k-1}{r-1} \frac{\binom{m+n-k}{n-r}}{\binom{m+n}{n}} \quad (6)$$

$$(7)$$

Therefore, we can write the PMF of X completely as follows

$$\Pr(X = k) = \begin{cases} \binom{k-1}{r-1} \frac{\binom{m+n-k}{n-r}}{\binom{m+n}{n}}, & r \leq k \leq m+n \\ 0, & \text{otherwise} \end{cases} \quad (8)$$

3 Letting $m+n \rightarrow \infty$, and using (2) and (4),

$$\Pr(X = k) \approx \binom{k-1}{r-1} \frac{n^r m^{k-r}}{(m+n)^k} \quad (9)$$

$$= \binom{k-1}{r-1} p^r q^{k-r} \sim NB(r, p) \quad (10)$$

as desired.