Assignment 13

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Outline

Problem Statement

(Papoulis/Pillai, Exercise 5-48) The random variable X is $N(0;\sigma^2)$.

1. Using Laplace transforms, show that if $g(x)$ is a function such that $g(x)\exp\left(-\frac{x^2}{2\sigma^2}\right) \to 0$ as $|x| \to \infty$, then (Price's Theorem)

$$\frac{dE\{g(X)\}}{dv} = \frac{1}{2}E\left\{\frac{d^2g(X)}{dX^2}\right\} \tag{1}$$

2. The moments μ_n of X are functions of v. Using (1), show that

$$\mu_n(v) = \frac{n(n-1)}{2}\int_0^v \mu_{n-2}(\beta)d\beta \tag{2}$$

Here, $v = \sigma^2$.

PDF and Partial Derivatives of $N(\mu; \sigma^2)$

Since $X \sim N(\mu; \sigma^2)$, the PDF of X is given by

$$f(x) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \tag{3}$$

$$\implies f(x, v) = \frac{1}{\sqrt{2\pi v}} \exp\left(-\frac{(x-\mu)^2}{2v}\right) \tag{4}$$

Notice that

$$\frac{\partial f}{\partial v} = \frac{\exp\left(\frac{(x-\mu)^2}{2v}\right)\left((x-\mu)^2 - v\right)}{2\sqrt{2\pi v^5}} \tag{5}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\exp\left(\frac{(x-\mu)^2}{2v}\right)\left((x-\mu)^2 - v\right)}{\sqrt{2\pi v^5}} \tag{6}$$

$$\implies \frac{\partial f}{\partial v} = \frac{1}{2}\frac{\partial^2 f}{\partial x^2} \tag{7}$$

Definition and Properties of Laplace Transform

The Laplace Transform of a real function $f(t)$ is given by

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t) dt \tag{8}$$

where $s \in \mathbb{C}$. Here, $k(s,t) = e^{-st}$ is called the kernel function. A useful property of Laplace transforms is the following, where the n^{th} derivative of f is assumed to be of exponential type.

$$\mathcal{L}\{f^{(n)}\} = s^n \mathcal{L}\{f\} - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0^+) \tag{9}$$

For $n = 1$,

$$\mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0^+) \tag{10}$$

Solution

In this solution, we assume that $g^{(k)}(x)\exp\left(-\frac{x^2}{2\sigma^2}\right) \to 0$ as $|x| \to \infty$ for $k = 0, 1, 2$. Applying a Laplace transform, and noting that $f(x, 0^+) = 0$,

$$\frac{dE\{g(X)\}}{dv} \xleftrightarrow{\mathcal{L}} s\mathcal{L}\{E\{g(X)\}\} \tag{11}$$

$$= s\int_0^\infty e^{-sv}\left(\int_{-\infty}^{\infty} g(x)f(x,v)dx\right)dv \tag{12}$$

$$= \int_{-\infty}^{\infty} g(x)\left(\int_0^\infty se^{-sv}f(x,v)dv\right)dx \tag{13}$$

$$= -\int_{-\infty}^{\infty} g\left(\int_0^\infty \frac{\partial k}{\partial v} f dv\right)dx \tag{14}$$

$$= \int_{-\infty}^{\infty} g\left(\int_0^\infty e^{-sv}\frac{\partial f}{\partial v}dv\right)dx \tag{15}$$

Here, we integrated (14) by parts. Using (7) and our assumptions, we repeatedly integrate (16) by parts to get

$$\frac{dE\{g(X)\}}{dv} \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{2}\int_0^\infty e^{-sv}\left(\int_{-\infty}^{\infty} g\frac{\partial^2 f}{\partial x^2}dx\right)dv \tag{16}$$

$$= \frac{1}{2}\int_0^\infty e^{-sv}\left(\int_{-\infty}^{\infty} f\frac{\partial^2 g}{\partial x^2}dx\right)dv \tag{17}$$

$$= \mathcal{L}\{E\left\{\frac{d^2 g(X)}{dX^2}\right\}\}(s) \overset{\mathcal{L}^{-1}}{\longleftrightarrow} \frac{1}{2}E\left\{\frac{d^2 g(X)}{gX^2}\right\} \tag{18}$$

as required.

For the second part, observe that $\mu_n = E[X^n]$ and hence it is a function of v. Further, using the exponential power series, note that for any positive integer n,

$$\exp\left(\frac{x^2}{2\sigma^2}\right) > \frac{x^{2n}}{(2\sigma^2)^n n!} \tag{19}$$

$$\implies 0 < \frac{x^n}{\exp\left(\frac{x^2}{2\sigma^2}\right)} < \frac{(2\sigma^2)^n n!}{x^n} \tag{20}$$

and using the Sandwich Theorem, we can choose $g(x) = x^n$ to use in (1).

$$\mu_n'(v) = \frac{1}{2}E\left\{n(n-1)x^{n-2}\right\} = \frac{n(n-1)}{2}\mu_{n-2}(v) \tag{21}$$

However, note that if $v = 0$, then from (4), $f(x) = 0$ and consequently $\mu_n(0) = 0$. Integrating (21) and changing variables, we get

$$\mu_n(v) = \frac{n(n-1)}{2}\int_0^v \mu_{n-2}(\beta)d\beta \tag{22}$$