Random Numbers

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CONTENTS

***Abstract*—This manual provides a simple introduction to the generation of random numbers. Commands that must be executed in a *NIX shell are preceded by a $ symbol.**

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following files

```
$ wget https://raw.githubusercontent.com/
    goats-9/ai1110-assignments/master/
    manual/codes/1_1.c
$ wget https://raw.githubusercontent.com/
    goats-9/ai1110-assignments/master/
    manual/codes/1_1.h
```

and compile and execute the C program using

```
$ gcc 1_1.c -lm -Wall -g
$ ./a.out
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \le x) \tag{1.1}$$

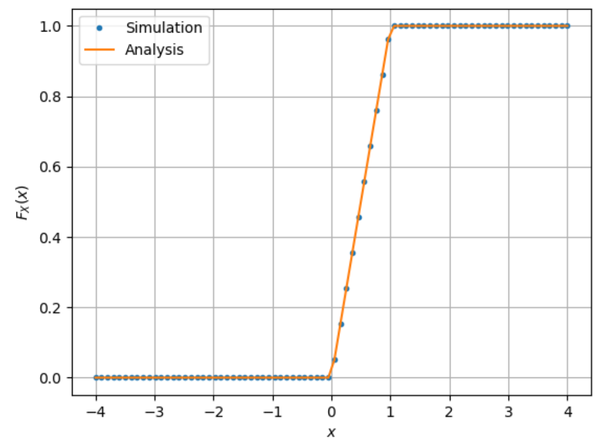

Fig. 1.1: The CDF of U

Solution: The following code plots Fig. 1.1

```
$ wget https://raw.githubusercontent.com/
    goats-9/ai1110-assignments/master/
    manual/codes/1_2.py
```

It is executed with

```
$ python3 1_2.py
```

1.3 Find a theoretical expression for $F_U(x)$.
Solution: The CDF of U is given by

$$F_U(x) = \Pr(U \le x) = \int_{-\infty}^{x} p_U(u)du \tag{1.2}$$

We now have three cases:

a) $x < 0$: $p_X(x) = 0$, and hence $F_U(x) = 0$.
b) $0 \le x < 1$: Here,

$$F_U(x) = \int_0^x du = x \tag{1.3}$$

c) $x \ge 1$: Put $x = 1$ in (1.3) as U is uniform in $[0, 1]$ to get $F_U(x) = 1$.

Therefore,

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases} \tag{1.4}$$

This is verified in Figure (1.1)

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.5}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{1.6}$$

Write a C program to find the mean and variance of U.

Solution: The C program can be downloaded using

```
$ wget https://raw.githubusercontent.com/
    goats-9/ai1110-assignments/master/
    manual/codes/1_4.c
```

and compiled and executed with

```
$ gcc 1_4.c -lm -Wall -g
$ ./a.out
```

The calculated mean is 0.500007 and the calculated variance is 0.083301.

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) dx \tag{1.7}$$

Solution: We write

$$E\left[U^2\right] = \int_{-\infty}^{\infty} x^2 dF_U(x) \tag{1.8}$$

$$= \int_{-\infty}^{\infty} x^2 p_U(x) dx \tag{1.9}$$

$$= \int_{0}^{1} x^2 dx = \frac{1}{3} \tag{1.10}$$

and

$$E[U] = \int_{-\infty}^{\infty} x dF_U(x) \tag{1.11}$$

$$= \int_{-\infty}^{\infty} x p_U(x) dx \tag{1.12}$$

$$= \int_{0}^{1} x dx = \frac{1}{2} \tag{1.13}$$

which checks out with the empirical mean on 0.500007. Now, using linearity of expectation,

$$\text{var}[U] = E[U - E[U]]^2 \tag{1.14}$$

$$= E\left[U^2 - 2UE[U] + (E[U])^2\right] \tag{1.15}$$

$$= E\left[U^2\right] - 2(E[U])^2 + (E[U])^2 \tag{1.16}$$

$$= E\left[U^2\right] - (E[U])^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \tag{1.17}$$

and this checks out with the empirical variance 0.083301 of the sample data.

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

Solution: The sample data is generated by the C file in Question 1.1.

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?

Solution: The CDF of X is plotted in Fig. 2.1 Download the Python code using

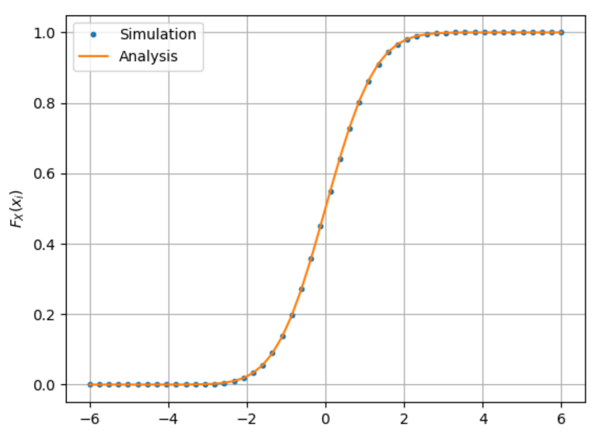

Fig. 2.1: The CDF of X

```
$ wget https://raw.githubusercontent.com/
    goats-9/ai1110-assignments/master/
    manual/codes/2_2.py
```

and execute it with

```
$ python3 2_2.py
```

The CDF of a probability distribution has the following properties:

a) It is non-decreasing
b) It is right-continuous
c) $\lim_{x\to-\infty} F_X(x) = 0$
d) $\lim_{x\to\infty} F_X(x) = 1$

The CDF of the normal distribution is expressed in terms of the Q-function as $F_X(x) = 1 - Q(x)$.

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{2.2}$$

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. 2.2 using the code below

```
$ wget https://raw.githubusercontent.com/
    goats-9/ai1110-assignments/master/
    manual/codes/2_3.py
```

The figure is generated using

```
$ python3 2_3.py
```

The properties of a PDF are as follows:

a) $\forall x \in \mathbb{R},\ p_X(x) \geq 0$
b) $\int_{-\infty}^{\infty} p_X(x)dx = 1$
c) For $a < b,\ a, b \in \mathbb{R}$

$$\Pr(a < X < b) = \Pr(a \leq X \leq b) \tag{2.3}$$

$$= \int_a^b p_X(x)dx \tag{2.4}$$

If we take $a = b$, then we get $\Pr(X = a) = 0$.

2.4 Find the mean and variance of X by writing a C program.
Solution: The mean and variance have been calculated using (1.5) and (1.6) respectively. The C program can be downloaded using

```
$ wget https://raw.githubusercontent.com/
    goats-9/ai1110-assignments/master/
    manual/codes/2_4.c
```

and compiled and executed with the following commands

```
$ gcc 2_4.c -lm -Wall -g
$ ./a.out
```

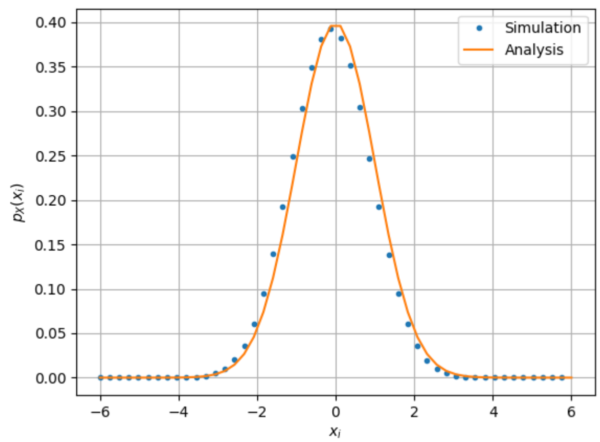

Fig. 2.2: The PDF of X

The calculated mean is 0.000326 and the calculated variance is 1.000906.

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.5}$$

repeat the above exercise theoretically.
Solution: The mean is given by

$$E[X] = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) = 0 \tag{2.6}$$

as the integrand is odd. This checks out with the empirical mean of 0.000326. The variance is given by

$$\text{var}[X] = E\left[X^2\right] - (E[X])^2 \tag{2.7}$$

$$= \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \tag{2.8}$$

$$= \int_0^{\infty} \frac{2}{\sqrt{2\pi}} \sqrt{2t} e^{-t} dt \tag{2.9}$$

$$= \frac{2}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}\right) \tag{2.10}$$

$$= \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}\right) = 1 \tag{2.11}$$

where we have used $t = \frac{x^2}{2}$ and so $dt = xdx$. We have also used the gamma function defined as

$$\Gamma(n) = \int_{-\infty}^{\infty} x^{n-1} e^{-x} dx \quad (2.12)$$

$$\Gamma(n) = (n-1)\Gamma(n-1) \text{ for } n > 1 \quad (2.13)$$

and the fact that $\Gamma(1/2) = \sqrt{\pi}$. This agrees with the empirical variance of 1.000906.

3 From Uniform to Other

3.1 Generate samples of

$$V = -2\ln(1-U) \quad (3.1)$$

and plot its CDF.
Solution: The relevant python code is at

```
$ wget https://raw.githubusercontent.com/
    goats-9/ai1110-assignments/master/
    manual/codes/3_1.py
```

and can be executed with

```
$ python3 3_1.py
```

The CDF is plotted in Figure (3.1).

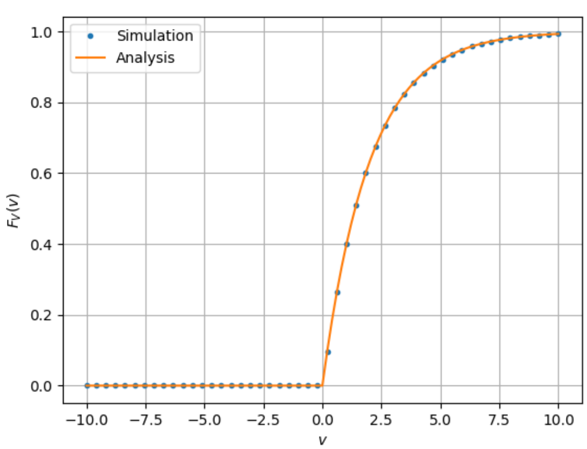

Fig. 3.1: The CDF of V

3.2 Find a theoretical expression for $F_V(x)$.
Solution: Note that the function

$$v = f(u) = -2\ln(1-u) \quad (3.2)$$

is monotonically increasing in [0, 1] and $v \in \mathbb{R}^+$. Hence, it is invertible and the inverse function is given by

$$u = f^{-1}(v) = 1 - \exp\left(-\frac{v}{2}\right) \quad (3.3)$$

Therefore, from the monotonicity of v, and using (1.4),

$$F_V(v) = F_U\left(1 - \exp\left(-\frac{v}{2}\right)\right) \quad (3.4)$$

$$\implies F_V(v) = \begin{cases} 0 & v < 0 \\ 1 - \exp\left(-\frac{v}{2}\right) & v \geq 0 \end{cases} \quad (3.5)$$

4 Triangular Distribution

4.1 Generate

$$T = U_1 + U_2 \quad (4.1)$$

Solution: The samples are generated in the C file exrand.c in 1.1 as the file tri.dat.

4.2 Find the CDF of T.
Solution: The Python code for the figure is at

```
$ wget https://raw.githubusercontent.com/
    goats-9/ai1110-assignments/master/
    manual/codes/4_2.py
```

and can be run using

```
$ python3 4_2.py
```

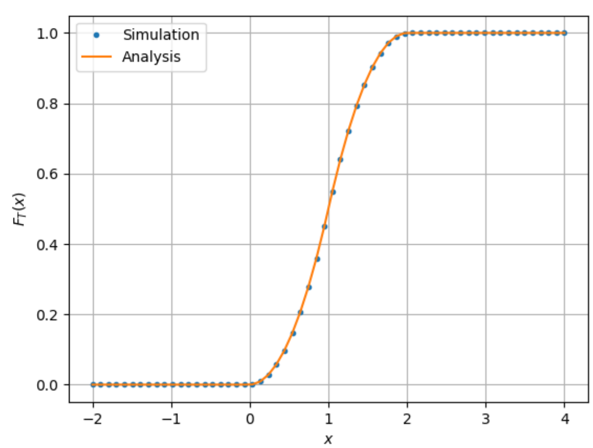

Fig. 4.1: The CDF of T

4.3 Find the PDF of T.
Solution: The Python code for the figure can be downloaded using

```
$ wget https://raw.githubusercontent.com/
    goats-9/ai1110-assignments/master/
    manual/codes/4_3.py
```

and run using

```
$ python3 4_3.py
```

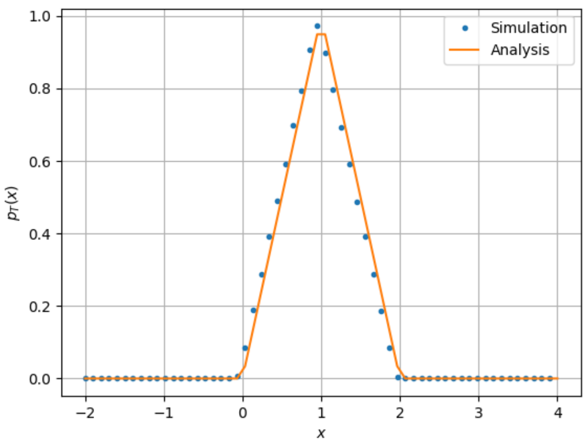

Fig. 4.2: The PDF of T

4.4 Find the theoretical PDF and CDF of T.
Solution: We write,

$$F_T(t) = \Pr(U_1 + U_2 \le t) \tag{4.2}$$
$$= \Pr(U_1 \le t - U_2) \tag{4.3}$$
$$= \int_0^1 F_{U_1}(t-x) p_{U_2}(x) dx \tag{4.4}$$

where U_1 and U_2 are uniform i.i.d. random variables in [0, 1]. Then, $0 \le U_1 + U_2 \le 2$. We have three cases:

a) $t < 0$: Using Equation 1.4, $F_T(t) = 0$.
b) $0 \le t < 1$: We have,

$$F_T(t) = \int_0^t (t-x) dx = \frac{t^2}{2} \tag{4.5}$$

c) $1 \le t < 2$: Here, we get

$$F_T(t) = \int_0^{t-1} dx + \int_{t-1}^1 (t-x) dx \tag{4.6}$$
$$= t - 1 + t(2-t) - \frac{1-(t-1)^2}{2} \tag{4.7}$$
$$= -\frac{t^2}{2} + 2t - 1 \tag{4.8}$$

d) $t \ge 2$: Here, $F_T(t) = 1$.

Therefore,

$$F_T(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \le t < 1 \\ -\frac{t^2}{2} + 2t - 1 & 1 \le t < 2 \\ 1 & t \ge 2 \end{cases} \tag{4.9}$$

Using Equation 2.2,

$$p_T(t) = \begin{cases} t & 0 \le t < 1 \\ 2 - t & 1 \le t < 2 \\ 0 & \text{otherwise} \end{cases} \tag{4.10}$$

4.5 Verify your results through a plot.
Solution: This has been done in the plots (4.1) and (4.2).

5 Maximum Likelihood

5.1 Generate equiprobable $X \in \{1, -1\}$.
Solution: The C file in Question 1.1 generates samples of X in the file data/ber.dat.

5.2 Generate

$$Y = AX + N \tag{5.1}$$

where $A = 5$ dB, $X \in \{1, -1\}$ is Bernoulli and $N \sim \mathcal{N}(0, 1)$.
Solution: The C file in Question 1.1 generates the numbers in the file data/gau_ber.dat

5.3 Plot Y using a scatter plot.
Solution: The Python code can be downloaded using

```
$ wget https://raw.githubusercontent.com/
    goats−9/ai1110−assignments/master/
    manual/codes/5_2.py
```

and run using

```
$ python3 5_2.py
```

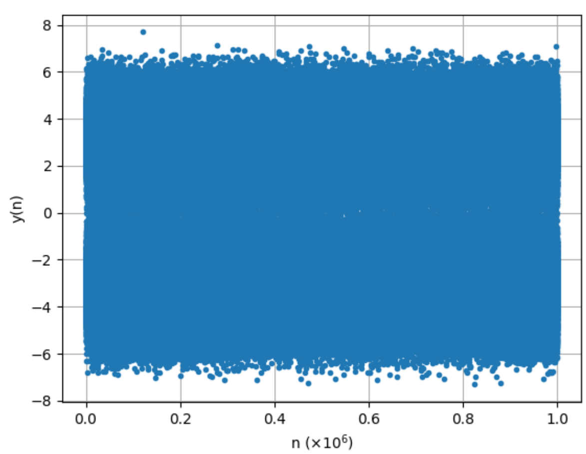

Fig. 5.1: Plot of Y

5.4 Guess how to estimate X from Y.

Solution: From the plot of Y, we see that the estimate model can be written as

$$\hat{X} = \begin{cases} 1 & Y > 0 \\ 0 & Y < 0 \end{cases} \tag{5.2}$$

5.5 Find

$$P_{e|0} = \Pr\left(\hat{X} = -1|X = 1\right) \tag{5.3}$$

and

$$P_{e|1} = \Pr\left(\hat{X} = 1|X = -1\right) \tag{5.4}$$

Solution: Letting $X = 1$ and $X = -1$ respectively, we see the number of mismatched data points to compute the error probabilities. The simulation is coded in

```
$ wget https://raw.githubusercontent.com/
    goats−9/ai1110−assignments/master/
    manual/codes/5_4.py
```

and can be run by typing

```
$ python3 5_4.py
```

The results are

$$P_{e|0} = 5.33 \times 10^{-4} \tag{5.5}$$
$$P_{e|1} = 5.57 \times 10^{-4} \tag{5.6}$$

5.6 Find P_e assuming that X has equiprobable symbols.
Solution: Here, $\Pr(X = 1) = \Pr(X = -1) = 0.5$. Thus,

$$P_e = \Pr(X = 1) P_{e|1} + \Pr(X = -1) P_{e|0} \tag{5.7}$$
$$= \frac{1}{2}\left(P_{e|0} + P_{e|1}\right) = 5.45 \times 10^{-4} \tag{5.8}$$

5.7 Verify by plotting the theoretical P_e wrt A from 0 dB to 10 dB.
Solution: We note that

$$P_{e|0} = \Pr\left(\hat{X} = 1|X = -1\right) \tag{5.9}$$
$$= \Pr(Y > 0|X = -1) \tag{5.10}$$
$$= \Pr(AX + N > 0|X = -1) \tag{5.11}$$
$$= \Pr(N > A|X = -1) = Q(A) \tag{5.12}$$

since X and N are independent. Writing a similar expression for $P_{e|1}$ and noting that

$$\Pr(N < -A) = \Pr(N > A) = Q(A) \tag{5.13}$$

it follows that $P_e = Q(A)$. This is the idea used to plot the theoretical P_e. The plot is coded both in the rectangular axes and the semilog-y axes. Download the relevant codes using

```
$ wget https://raw.githubusercontent.com/
    goats−9/ai1110−assignments/master/
    manual/codes/5_6.py
$ wget https://raw.githubusercontent.com/
    goats−9/ai1110−assignments/master/
    manual/codes/5_6_semilog.py
```

and execute them using

```
$ python3 5_6.py
$ python3 5_6_semilogy.py
```

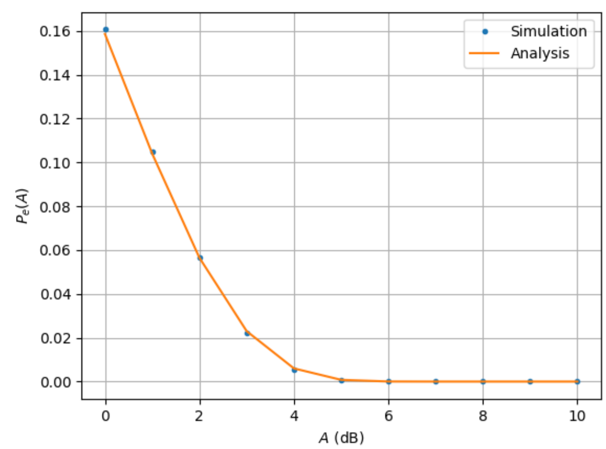

Fig. 5.2: $P_e(A)$ (rectangular axes)

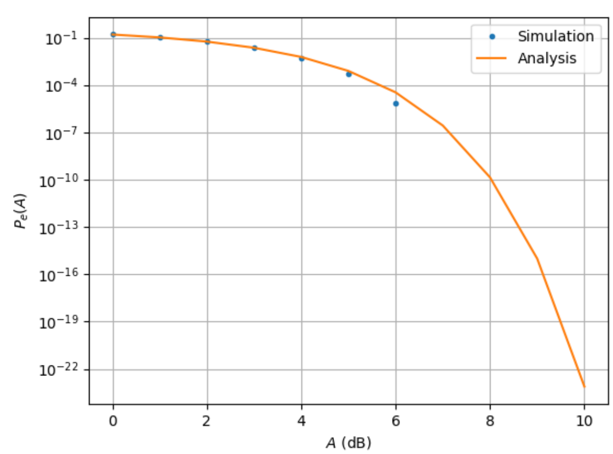

Fig. 5.3: $P_e(A)$ (semilog-y axes)

5.8 Now, consider a threshold δ while estimating X from Y. Find the value of δ that maximizes the theoretical P_e.

Solution: Replacing the 0 in (5.11) with δ and performing a similar operation for $P_{e|1}$, we get

$$P_e = \Pr(X = -1)\, Q(A+\delta) + \Pr(X = 1)\, Q(A-\delta) \tag{5.14}$$
$$= \frac{1}{2}\left(Q(A+\delta) + Q(A-\delta)\right) \tag{5.15}$$

Differentiating with respect to δ leads to the equation (here f_N denotes standard normal distibution)

$$f_N(A+\delta) = f_N(A-\delta) \tag{5.16}$$

which implies that for $A \neq 0,\ \delta = 0$ and for $A = 0,\ \delta \in \mathbb{R}$.

5.9 Repeat the above exercise when

$$p_X(1) = p \tag{5.17}$$

Solution: Using (5.14) and following a similar procedure as in the previous question, we see that

$$p f_N(A-\delta) = (1-p)\, f_N(A+\delta) \tag{5.18}$$
$$pe^{-\frac{(A-\delta)^2}{2}} = (1-p)\, e^{-\frac{(A+\delta)^2}{2}} \tag{5.19}$$
$$\implies \delta = \frac{1}{2A}\ln\left(\frac{1-p}{p}\right) \tag{5.20}$$

5.10 Repeat the above exercise using MAP criterion.
Solution: Using Bayes' Theorem, we get

$$\Pr(X = 1|Y = y)$$
$$= \frac{\Pr(N = y - A|X = 1)\Pr(X = 1)}{p_Y(y)} \tag{5.21}$$
$$= \frac{p f_N(y-A)}{p f_N(y-A) + (1-p)\, f_N(y+A)} \tag{5.22}$$
$$= \frac{p}{p + (1-p)\, e^{-2yA}} \tag{5.23}$$

and

$$\Pr(X = -1|Y = y)$$
$$= \frac{\Pr(N = y + A|X = -1)\Pr(X = -1)}{p_Y(y)} \tag{5.24}$$
$$= \frac{(1-p)\, f_N(y+A)}{p f_N(y-A) + (1-p)\, f_N(y+A)} \tag{5.25}$$
$$= \frac{1-p}{(1-p) + pe^{2yA}} \tag{5.26}$$

Hence,

$$\frac{p}{p + (1-p)\, e^{-2yA}} \gtrless \frac{1-p}{(1-p) + pe^{2yA}} \tag{5.27}$$
$$\implies p^2 e^{2yA} \gtrless (1-p)^2\, e^{-2yA} \tag{5.28}$$
$$\implies y \gtrless \frac{1}{2A}\ln\left(\frac{1-p}{p}\right) \tag{5.29}$$

6 Gaussian to Other

6.1 Let $X_1 \sim \mathcal{N}(0,1)$ and $X_2 \sim \mathcal{N}(0,1)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \tag{6.1}$$

Solution: We transform the variables X_1 and X_2 as:

$$X_1 = R\cos\Theta \tag{6.2}$$
$$X_2 = R\sin\Theta \tag{6.3}$$

where $R \in [0,\infty), \Theta \in [0, 2\pi)$. The Jacobian Matrix for this transformation is given by

$$\mathbf{J} = \begin{pmatrix} \frac{\partial X_1}{\partial R} & \frac{\partial X_2}{\partial R} \\ \frac{\partial X_1}{\partial \Theta} & \frac{\partial X_2}{\partial \Theta} \end{pmatrix} \tag{6.4}$$
$$= \begin{pmatrix} \cos\Theta & \sin\Theta \\ -R\sin\Theta & R\cos\Theta \end{pmatrix} \tag{6.5}$$
$$\implies |\mathbf{J}| = R \tag{6.6}$$

We also know that

$$|\mathbf{J}| p_{X_1,X_2}(x_1,x_2) = p_{R,\Theta}(r,\theta) \tag{6.7}$$
$$\implies p_{R,\Theta}(r,\theta) = R p_{X_1}(x_1) p_{X_2}(x_2) \tag{6.8}$$
$$= \frac{R}{2\pi}\exp\left(-\frac{X_1^2 + X_2^2}{2}\right) \tag{6.9}$$
$$= \frac{R}{2\pi}\exp\left(-\frac{R^2}{2}\right) \tag{6.10}$$

where (6.8) follows as X_1, X_2 are iid random variables. Thus,

$$p_R(r) = \int_0^{2\pi} p_{R,\Theta}(r,\theta) d\theta \tag{6.11}$$
$$= R\exp\left(-\frac{R^2}{2}\right) \tag{6.12}$$

However, $V = X_1^2 + X_2^2 = R^2 \geq 0$, thus $F_V(x) = 0$

for $x < 0$ and

$$F_V(x) = F_R(\sqrt{x}) \tag{6.13}$$

$$= \int_0^{\sqrt{x}} r \exp\left(-\frac{r^2}{2}\right) dr \tag{6.14}$$

$$= \int_0^{\frac{x}{2}} e^{-t} dt = 1 - e^{-\frac{x}{2}} \tag{6.15}$$

where $t = \frac{r^2}{2}$ for $x \geq 0$, thus

$$p_V(x) = \frac{1}{2} e^{-\frac{x}{2}} \tag{6.16}$$

Hence,

$$F_V(x) = \begin{cases} 1 - e^{-\frac{x}{2}} & x \geq 0 \\ 0 & x < 0 \end{cases} \tag{6.17}$$

$$p_V(x) = \begin{cases} \frac{1}{2} e^{-\frac{x}{2}} & x \geq 0 \\ 0 & x < 0 \end{cases} \tag{6.18}$$

The equations (6.17) and (6.18) have been used to generate the plots. The Python code can be downloaded from

```
$ wget https://raw.githubusercontent.com/
    goats−9/ai1110−assignments/master/
    manual/codes/6_1_cdf.py
$ wget https://raw.githubusercontent.com/
    goats−9/ai1110−assignments/master/
    manual/codes/6_1_pdf.py
```

Execute the codes by typing the commands

```
$ python3 6_1_cdf.py
$ python3 6_1_pdf.py
```

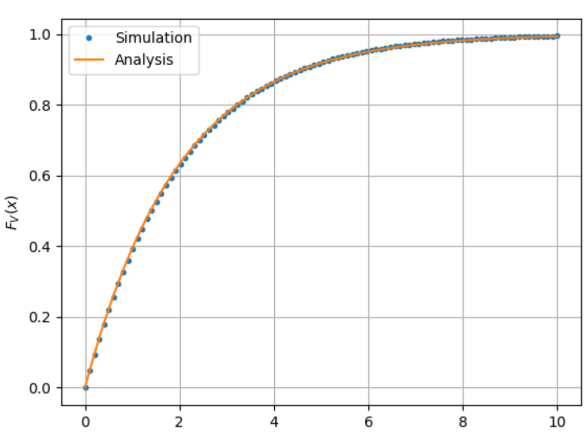

Fig. 6.1: CDF of V

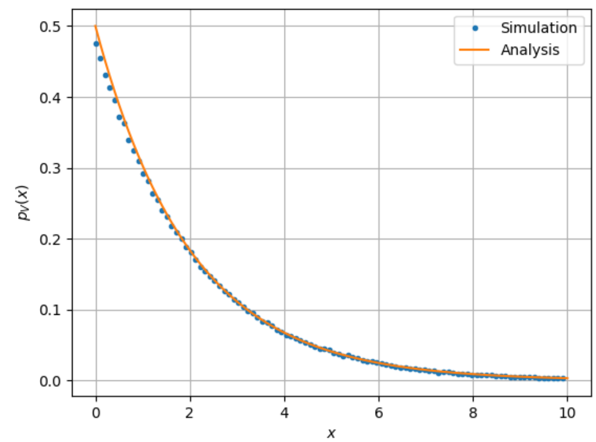

Fig. 6.2: PDF of V

6.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0 \end{cases} \tag{6.19}$$

find α.

Solution: From (6.17), it is clear that $\alpha = 0.5$.

6.3 Plot the CDF and PDF of

$$A = \sqrt{V} \tag{6.20}$$

Solution:

Note that for $x \geq 0$,

$$F_A(x) = \Pr(A \leq x) \tag{6.21}$$

$$= \Pr\left(\sqrt{V} \leq x\right) \tag{6.22}$$

$$= \Pr\left(V \leq x^2\right) \tag{6.23}$$

$$= F_V(x^2) = 1 - e^{-\frac{x^2}{2}} \tag{6.24}$$

and so,

$$p_A(x) = x e^{-\frac{x^2}{2}} \tag{6.25}$$

Thus, the CDF and PDF of A is given by

$$F_V(x) = \begin{cases} 1 - e^{-\frac{x^2}{2}} & x \geq 0 \\ 0 & x < 0 \end{cases} \tag{6.26}$$

$$p_V(x) = \begin{cases} x e^{-\frac{x}{2}} & x \geq 0 \\ 0 & x < 0 \end{cases} \tag{6.27}$$

The Python codes for the plots are at

```
$ wget https://raw.githubusercontent.com/
    goats−9/ai1110−assignments/master/
    manual/codes/6_3_cdf.py
```

```
$ wget https://raw.githubusercontent.com/
    goats-9/ai1110-assignments/master/
    manual/codes/6_3_pdf.py
```

and can be executed using

```
$ python3 6_3_cdf.py
$ python3 6_3_pdf.py
```

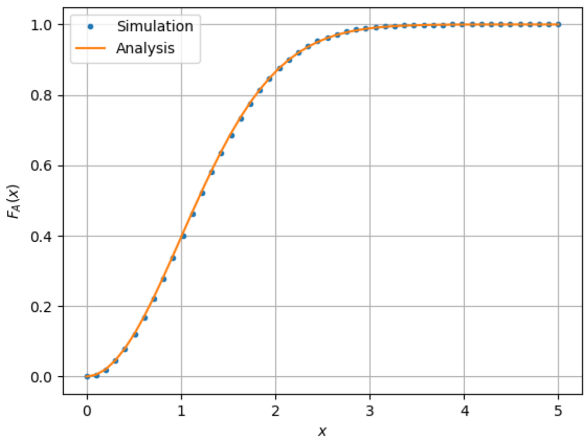

Fig. 6.3: CDF of A

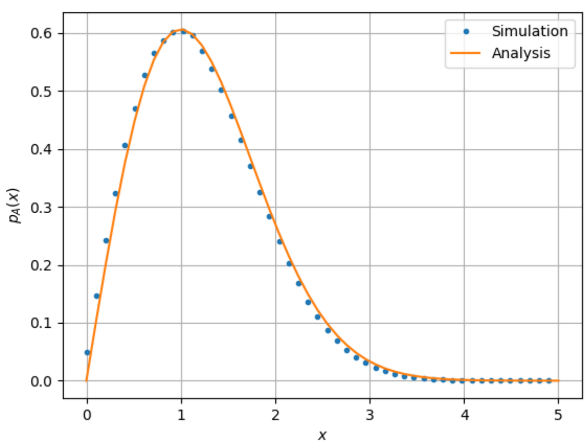

Fig. 6.4: PDF of A

7 Conditional Probability

7.1 Plot

$$P_e = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{7.1}$$

for

$$Y = AX + N \tag{7.2}$$

where A is Rayleigh with $E\left[A^2\right] = \gamma$, $N \sim \mathcal{N}(0, 1)$, $X \in \{1, -1\}$ for $0 \le \gamma \le 10$ dB.

Solution: Download the relevant Python code

```
$ wget https://raw.githubusercontent.com/
    goats-9/ai1110-assignments/master/
    manual/codes/7_1.py
```

and run it using

```
$ python3 7_1.py
```

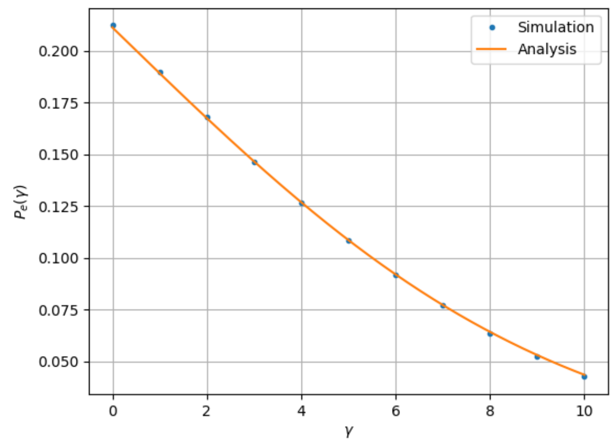

Fig. 7.1: P_e as a function of γ (rectangular axes)

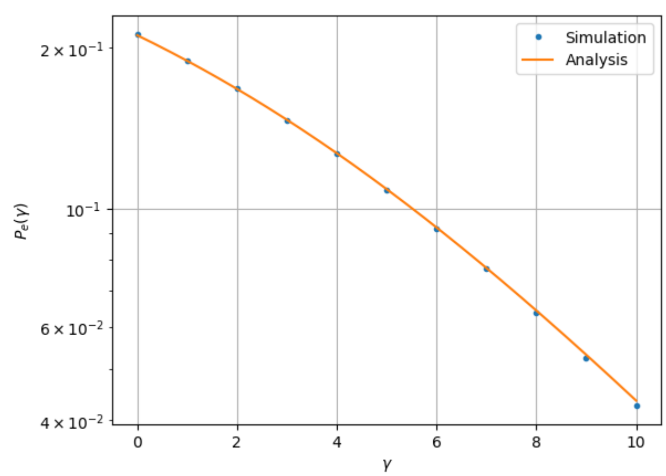

Fig. 7.2: P_e as a function of γ (semilog-y axes)

7.2 Assuming that N is a constant, find an expression for P_e. Call this $P_e(N)$.

Solution: We rewrite the previous expression for P_e as

$$P_e(N) = \Pr(A < -N) = F_A(-N) \tag{7.3}$$

$$= \begin{cases} 1 - e^{-\frac{N^2}{\gamma}} & N \le 0 \\ 0 & N > 0 \end{cases} \tag{7.4}$$

7.3 For a function g,

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) p_X(x) dx \tag{7.5}$$

Find $P_e = E[P_e(N)]$.
Solution: We write,

$$P_e = \int_0^{\infty} F_A(x) f_N(x) dx \tag{7.6}$$

$$= \int_0^{\infty} (1 - e^{-\frac{x^2}{\gamma}}) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \tag{7.7}$$

$$= \frac{1}{2} - \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \exp\left(-\frac{x^2}{\frac{2\gamma}{\gamma+2}}\right) dx \tag{7.8}$$

$$= \frac{1}{2}\left(1 - \sqrt{\frac{\gamma}{\gamma+2}}\right) \tag{7.9}$$

where f_N denotes the standard normal distribution.

7.4 Plot P_e in problems 7.1 and 7.3 on the same graph wrt γ. Comment.
Solution: This has been done in Figure (7.1) using the result (7.9). We observe that $P_{e|0} = E[P_e(N)]$ i.e., the error rate is independent of the noise.

8 Two Dimensions

Let

$$\mathbf{y} = A\mathbf{x} + \mathbf{n} \tag{8.1}$$

where

$$\mathbf{x} \in (\mathbf{s}_0, \mathbf{s}_1), \mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{8.2}$$

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, n_1, n_2 \sim \mathcal{N}(0, 1) \tag{8.3}$$

8.1 Plot $\mathbf{y}|\mathbf{s_0}$ and $\mathbf{y}|\mathbf{s_1}$ on the same graph using a scatter plot.
Solution: The samples are generated in the C file in Question 1.1 as the files data/ber_2D.dat and data/gau_2D.dat. The generated random samples are then plotted taking $A = 5$dB in the Python code

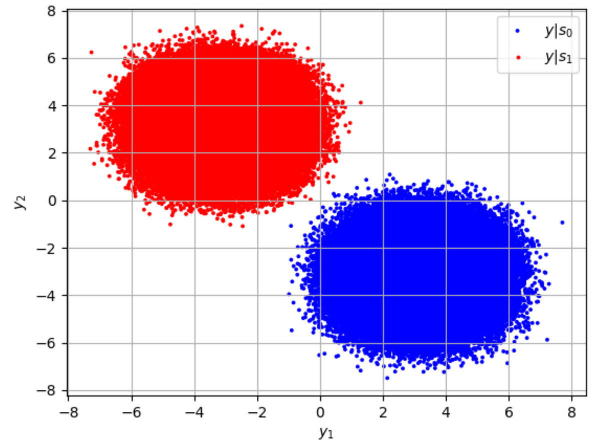

Fig. 8.1: Scatterplot of $\mathbf{y}$, $(A = 5\text{dB})$

```
$ wget https://raw.githubusercontent.com/
    goats−9/ai1110−assignments/master/
    manual/codes/8_1.py
```

and the plot can be generated by running the code using

```
$ python3 8_1.py
```

8.2 For the above problem, find a decision rule for detecting the symbols $\mathbf{s}_0$ and $\mathbf{s}_1$.
Solution: The decision rule is

$$\hat{\mathbf{x}} = \begin{cases} \mathbf{s}_0 & y_1 > y_2 \\ \mathbf{s}_1 & y_1 < y_2 \end{cases} \tag{8.4}$$

where $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

8.3 Plot

$$P_e = \Pr(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0) \tag{8.5}$$

with respect to the SNR from 0 to 10 dB.
Solution: Download the Python codes

```
$ wget https://raw.githubusercontent.com/
    goats−9/ai1110−assignments/master/
    manual/codes/8_3.py
$ wget https://raw.githubusercontent.com/
    goats−9/ai1110−assignments/master/
    manual/codes/8_3_semilogy.py
```

and run them with

```
$ python3 8_3.py
$ python3 8_3_semilogy.py
```

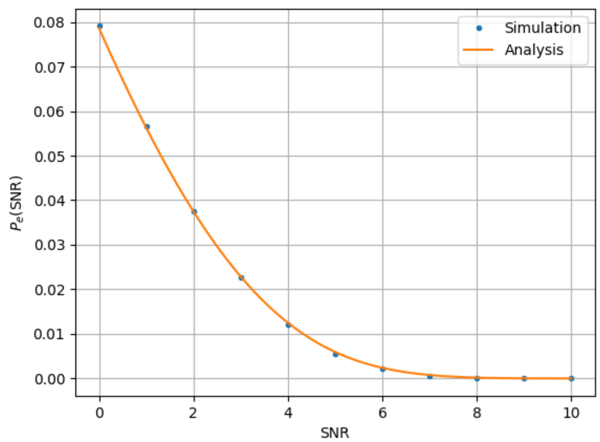

Fig. 8.2: P_e as a function of SNR (rectangular axes)

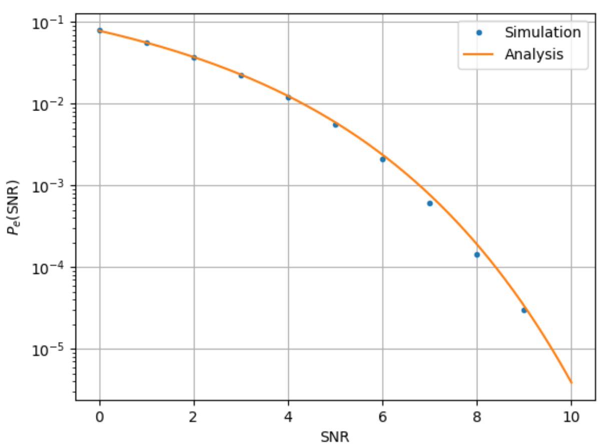

Fig. 8.3: P_e as a function of SNR (semilog-y axes)

8.4 Obtain an expression for P_e. Verify this by comparing the theory and simulation plots on the same graph.
Solution: We have,

$$P_e = \Pr\left(\hat{\mathbf{x}} = \mathbf{s}_1|\mathbf{x} = \mathbf{s}_0\right) \tag{8.6}$$
$$= \Pr\left(y_1 < y_2|\mathbf{x} = \mathbf{s}_0\right) \tag{8.7}$$
$$= \Pr\left(A + n_1 < -A + n_2\right) \tag{8.8}$$
$$= \Pr\left(n_2 - n_1 > 2A\right) \tag{8.9}$$
$$= \Pr\left(N > 2A\right) = Q\left(\sqrt{2}A\right) \tag{8.10}$$

where $N \triangleq n_2 - n_1 \sim \mathcal{N}(0, 2)$ and SNR $= \frac{E[A^2]}{\sigma_N^2}$.

8.5 Find the optimal decision boundary using MAP criterion.
Solution: Note that in this solution, we write the bivariate gaussian PDF as

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{\exp\left(-\frac{1}{2}\left((\mathbf{x}-\boldsymbol{\mu})^\top \boldsymbol{\Sigma}_{\mathbf{x}}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right)\right)}{2\pi\sqrt{|\boldsymbol{\Sigma}_{\mathbf{x}}|}} \tag{8.11}$$

where

$$\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{8.12}$$
$$\boldsymbol{\mu} = E\left(\mathbf{X}\right) \tag{8.13}$$
$$\boldsymbol{\Sigma}_{\mathbf{x}} = E\left[(\mathbf{x}-\boldsymbol{\mu})(\mathbf{x}-\boldsymbol{\mu})^\top\right] \tag{8.14}$$

Here, since the variables are normalized and i.i.d., $\boldsymbol{\Sigma}_{\mathbf{x}} = \mathbf{I}$ and $\boldsymbol{\Sigma}$ We write using Bayes' theorem,

$$\Pr\left(\mathbf{x} = \mathbf{s}_0|\mathbf{Y} = \mathbf{y}\right)$$
$$= \frac{\Pr\left(\mathbf{Y} = \mathbf{y}|\mathbf{x} = \mathbf{s}_0\right)\Pr\left(\mathbf{x} = \mathbf{s}_0\right)}{p_{\mathbf{Y}}(\mathbf{y})} \tag{8.15}$$
$$= \frac{f_{\mathbf{Y}}\left(\mathbf{y} - A\mathbf{s}_0\right)\Pr\left(\mathbf{x} = \mathbf{s}_0\right)}{p_{\mathbf{Y}}(\mathbf{y})} \tag{8.16}$$

and

$$\Pr\left(\mathbf{x} = \mathbf{s}_1|\mathbf{Y} = \mathbf{y}\right)$$
$$= \frac{\Pr\left(\mathbf{y} = y|\mathbf{x} = \mathbf{s}_1\right)\Pr\left(\mathbf{x} = \mathbf{s}_1\right)}{p_{\mathbf{Y}}(\mathbf{y})} \tag{8.17}$$
$$= \frac{f_{\mathbf{Y}}\left(\mathbf{y} - A\mathbf{s}_1\right)\Pr\left(\mathbf{x} = \mathbf{s}_1\right)}{p_{\mathbf{Y}}(\mathbf{y})} \tag{8.18}$$

Therefore, the decision boundary is given by, assuming $\Pr\left(\mathbf{x} = \mathbf{s}_0\right) = p$,

$$\Pr\left(\mathbf{x} = \mathbf{s}_0|\mathbf{Y} = \mathbf{y}\right) \gtrless \Pr\left(\mathbf{x} = \mathbf{s}_1|\mathbf{Y} = \mathbf{y}\right) \tag{8.19}$$
$$p f_{\mathbf{X}}\left(\mathbf{y} - A\mathbf{s}_0\right) \gtrless (1-p) f_{\mathbf{X}}\left(\mathbf{y} - A\mathbf{s}_1\right) \tag{8.20}$$
$$\exp\left(-\frac{1}{2}4A\left(y_1 - y_2\right)\right) \gtrless \frac{1-p}{p} \tag{8.21}$$
$$2\left(y_2 - y_1\right) \gtrless \ln\left(\frac{1-p}{p}\right) \tag{8.22}$$
$$y_2 - y_1 \gtrless \frac{1}{2}\ln\left(\frac{1-p}{p}\right) \tag{8.23}$$

or $y - x = \frac{1}{2}\ln\left(\frac{1-p}{p}\right)$.