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Random Numbers

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Abstract—This manual provides a simple introduction to the generation of random numbers. Commands that must be executed in a *NIX shell are preceded by a \$ symbol.

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .

Solution: Download the following files

- \$ wget https://raw.githubusercontent.com/ goats-9/ai1110-probability/master/ manual/codes/exrand.c
- \$ wget https://raw.githubusercontent.com/ goats-9/ai1110-probability/master/ manual/codes/coeffs.h

and compile and execute the C program using

- \$ gcc exrand.c -lm -Wall -g \$./a.out
- 1.2 Load the uni.dat file into python and plot the empirical CDF of *U* using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \le x) \tag{1.1}$$

Solution: The following code plots Fig. 1.2

\$ wget https://raw.githubusercontent.com/ goats-9/ai1110-probability/master/ manual/codes/cdf plot.py

It is executed with

\$ python3 cdf plot.py

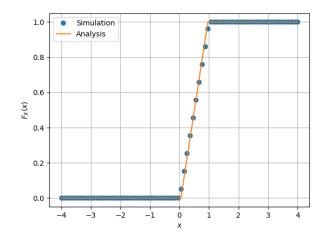


Fig. 1.2: The CDF of U

1.3 Find a theoretical expression for $F_U(x)$. **Solution:** The CDF of U is given by

$$F_U(x) = \Pr\left(U \le x\right) = \int_{-\infty}^x p_U(u) du \qquad (1.2)$$

We now have three cases:

- a) x < 0: $p_X(x) = 0$, and hence $F_U(x) = 0$.
- b) $0 \le x < 1$: Here,

$$F_U(x) = \int_0^x du = x$$
 (1.3)

c) $x \ge 1$: Put x = 1 in (1.3) as U is uniform in [0, 1] to get $F_U(x) = 1$.

Therefore,

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$
 (1.4)

This is verified in Figure (1.2)

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N} \sum_{i=1}^{N} U_i$$
 (1.5)

and its variance as

$$var[U] = E[U - E[U]]^2$$
 (1.6)

Write a C program to find the mean and variance of U.

Solution: The C program can be downloaded using

\$ wget https://raw.githubusercontent.com/ goats-9/ai1110-probability/master/ manual/codes/mean_var_uni.c

and compiled and executed with

The calculated mean is 0.500007 and the calculated variance is 0.083301.

1.5 Verify your result theoretically given that

$$E\left[U^{k}\right] = \int_{-\infty}^{\infty} x^{k} dF_{U}(x) dx \tag{1.7}$$

Solution: We write

$$E\left[U^{2}\right] = \int_{-\infty}^{\infty} x^{2} dF_{U}(x) \tag{1.8}$$

$$= \int_{-\infty}^{\infty} x^2 p_U(x) dx \tag{1.9}$$

$$= \int_0^1 x^2 dx = \frac{1}{3}$$
 (1.10)

and

$$E[U] = \int_{-\infty}^{\infty} x dF_U(x) \tag{1.11}$$

$$= \int_{-\infty}^{\infty} x p_U(x) dx \tag{1.12}$$

$$= \int_0^1 x dx = \frac{1}{2}$$
 (1.13)

which checks out with the empirical mean on 0.500007. Now, using linearity of expectation,

$$var[U] = E[U - E[U]]^2$$
 (1.14)

$$= E \left[U^2 - 2UE \left[U \right] + (E \left[U \right])^2 \right]$$
 (1.15)

$$= E[U^{2}] - 2(E[U])^{2} + (E[U])^{2}$$
 (1.16)

$$= E[U^2] - (E[U])^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \quad (1.17)$$

(1.18)

and this checks out with the empirical variance 0.083301 of the sample data.

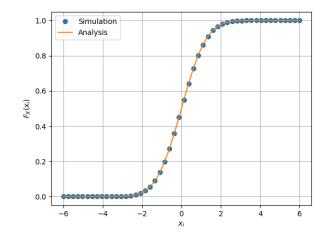


Fig. 2.2: The CDF of X

2 Central Limit Theorem

2.1 Generate 10⁶ samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where U_i , i = 1, 2, ..., 12 are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

Solution: The sample data is generated by the C file in Question 1.1

2.2 Load gau.dat in python and plot the empirical CDF of *X* using the samples in gau.dat. What properties does a CDF have?

Solution: The CDF of *X* is plotted in Fig. 2.2 The required python file can be downloaded using

\$ wget https://raw.githubusercontent.com/ goats-9/ai1110-probability/master/ manual/codes/cdf gauss plot.py

and executed using

\$ python3 cdf_gauss_plot.py

- a) The CDF is non-decreasing
- b) It is right-continuous.
- c) $\lim_{x\to -\infty} F_X(x) = 0$
- d) $\lim_{x\to\infty} F_X(x) = 1$
- 2.3 Load gau.dat in python and plot the empirical PDF of *X* using the samples in gau.dat. The

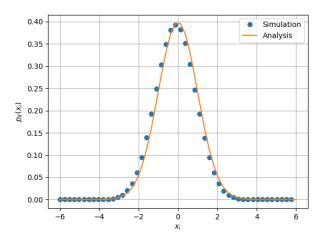


Fig. 2.3: The PDF of X

PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{2.2}$$

What properties does the PDF have?

Solution: The PDF of X is plotted in Fig. 2.3 using the code below

\$ wget https://raw.githubusercontent.com/ goats-9/ai1110-assignments/master/ manual/codes/pdf plot.py

The figure is generated using

\$ python pdf plot.py

The properties of a PDF $p_X(x)$ are as follows:

- a) $\forall x \in \mathbb{R}, \ p_X(x) \ge 0$ b) $\int_{-\infty}^{\infty} p_X(x) dx = 1$ c) For $a < b, \ a, b \in \mathbb{R}$

$$\Pr(a < X < b) = \Pr(a \le X \le b) \qquad (2.3)$$

$$= \int_{a}^{b} p_X(x)dx \tag{2.4}$$

If we take a = b, then we get Pr(X = a) = 0.

2.4 Find the mean and variance of X by writing a C program.

Solution: The mean and variance have been calculated using (1.5) and (1.6) respectively. The C program can be downloaded using

\$ wget https://raw.githubusercontent.com/ goats-9/ai1110-assignments/master/ manual/codes/mean var gau.c

and compiled and executed with the following commands

The calculated mean is 0.000326 and the calculated variance is 1.000906.

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.5)$$

repeat the above exercise theoretically.

Solution: The mean is given by

$$E[X] = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) = 0 \qquad (2.6)$$

as the integrand is odd. This checks out with the empirical mean of 0.000326. The variance is given by

$$\operatorname{var}[X] = E[X^{2}] - (E[X])^{2}$$
 (2.7)

$$= \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \qquad (2.8)$$

$$= \int_0^\infty \frac{2}{\sqrt{2\pi}} \sqrt{2t} e^{-t} dt \tag{2.9}$$

$$=\frac{2}{\sqrt{\pi}}\Gamma\left(\frac{3}{2}\right) \tag{2.10}$$

$$= \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}\right) = 1 \tag{2.11}$$

where we have used $t = \frac{x^2}{2}$ and so dt = xdx. We have also used the gamma function given as

$$\Gamma(n) = \int_{-\infty}^{\infty} x^{n-1} e^{-x} dx \qquad (2.12)$$

$$\Gamma(n) = (n-1)\Gamma(n-1) \text{ for } n > 1$$
 (2.13)

and the fact that $\Gamma(1/2) = \sqrt{\pi}$. This agrees with the empirical variance of 1.000906.

3 From Uniform to Other

3.1 Generate samples of

$$V = -2\ln(1 - U) \tag{3.1}$$

and plot its CDF.

Solution: The relevant python code is at

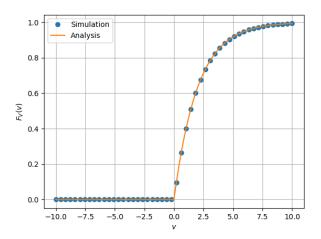


Fig. 3.1: The CDF of V

\$ wget https://raw.githubusercontent.com/ goats-9/ai1110-probability/master/ manual/codes/cdf exp_plot.py

and can be executed with

and the CDF is plotted in Figure (3.1).

3.2 Find a theoretical expression for $F_V(x)$.

Solution: Note that the function

$$v = f(u) = -2\ln(1 - u) \tag{3.2}$$

is monotonically increasing in [0, 1] and $v \in$ \mathbb{R}^+ . Hence, it is invertible and the inverse function is given by

$$u = f^{-1}(v) = 1 - \exp\left(-\frac{v}{2}\right)$$
 (3.3)

Therefore, from the monotonicity of v, and using (1.4),

$$F_V(v) = F_U \left(1 - \exp\left(-\frac{v}{2}\right) \right) \tag{3.4}$$

$$F_V(v) = F_U \left(1 - \exp\left(-\frac{v}{2}\right) \right)$$
 (3.4)

$$\implies F_V(v) = \begin{cases} 0 & v < 0 \\ 1 - \exp\left(-\frac{v}{2}\right) & v \ge 0 \end{cases}$$
 (3.5)