

Assignment 11

Gautam Singh (CS21BTECH11018)

May 22, 2022

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Problem Statement

(Papoulis/Pillai Exercise 3-4) A coin with $\Pr(h) = p = 1 - q$ is tossed n times. Show that the probability that the number of heads is even equals $0.5[1 + (q - p)^n]$.

Solution

Let $Y \sim \text{Bin}(n, p)$ represent the binomial random variable representing the number of heads attained. We note the following binomial expansions:

$$1 = (q + p)^n = \sum_{k=0}^{k=n} \binom{n}{k} q^{(n-k)} p^k \quad (1)$$

$$= \binom{n}{0} q^n + \binom{n}{1} q^{(n-1)} p + \binom{n}{2} q^{(n-2)} p^2 + \dots \quad (2)$$

$$(q - p)^n = \sum_{k=0}^{k=n} \binom{n}{k} (-1)^k q^{(n-k)} p^k \quad (3)$$

$$= \binom{n}{0} q^n - \binom{n}{1} q^{(n-1)} p + \binom{n}{2} q^{(n-2)} p^2 + \dots \quad (4)$$

PMF of Y

$$\Pr(Y = k) = \begin{cases} \binom{n}{k} p^{(n-k)} q^k, & 0 \leq k \leq n \\ 0, & \text{otherwise} \end{cases} \quad (5)$$

$$\implies \Pr(Y \equiv 0 \pmod{2}) = \binom{n}{0} p^n + \binom{n}{2} p^{(n-2)} q^2 + \dots \quad (6)$$

Therefore, adding 2 and 4, we get,

$$1 + (q - p)^n = 2\left(\binom{n}{0} p^n + \binom{n}{2} p^{(n-2)} q^2 + \dots\right) \quad (7)$$

$$= 2 \Pr(Y \equiv 0 \pmod{2}) \quad (8)$$

$$\implies \Pr(Y \equiv 0 \pmod{2}) = 0.5[1 + (q - p)^n] \quad (9)$$

as desired. This is verified in codes/11_1.py.