Random Numbers

Gautam Singh

Contents

Abstract—**This manual provides a simple introduction to the generation of random numbers**

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following files and execute the C program.

curl https://github.com/gadepall/probability/
    raw/master/manual/codes/exrand.c
wget https://github.com/gadepall/probability/
    raw/master/manual/codes/coeffs.h

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \qquad (1.1)$$

Solution: The following code plots Fig. 1.2

wget https://github.com/gadepall/probability/
    raw/master/manual/codes/cdf_plot.py

1.3 Find a theoretical expression for $F_U(x)$.
Solution: The CDF of U is given by

$$F_U(x) = \Pr(U \leq x) = \int_{-\infty}^{x} p_U(u)du \qquad (1.2)$$

We now have three cases:
a) $x < 0$: $p_X(x) = 0$, and hence $F_U(x) = 0$.
b) $0 \leq x < 1$: Here,
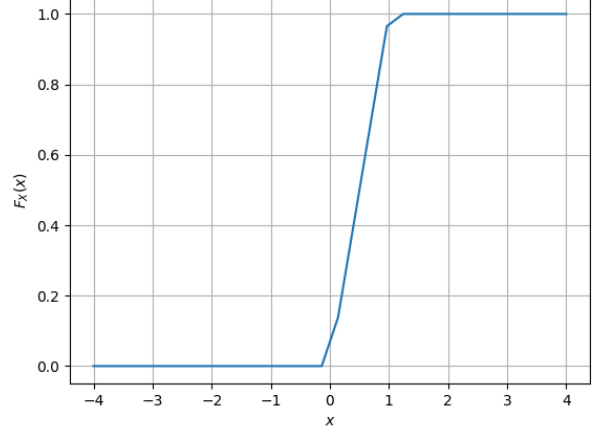
$$F_U(x) = \int_0^x du = x \qquad (1.3)$$

Fig. 1.2: The CDF of U

c) $x \geq 1$: Put $x = 1$ in (1.3) as U is uniform in [0, 1] to get $F_U(x) = 1$.
Therefore,

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases} \qquad (1.4)$$

This is verified in Figure (1.2)

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N} \sum_{i=1}^{N} U_i \qquad (1.5)$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \qquad (1.6)$$

Write a C program to find the mean and variance of U.

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x)dx \qquad (1.7)$$

Solution: We write

$$E\left[U^2\right] = \int_{-\infty}^{\infty} x^2 dF_U(x) \qquad (1.8)$$

$$= \int_{-\infty}^{\infty} x^2 p_U(x) dx \qquad (1.9)$$

$$= \int_0^1 x^2 dx = \frac{1}{3} \qquad (1.10)$$

and

$$E\left[U\right] = \int_{-\infty}^{\infty} x dF_U(x) \qquad (1.11)$$

$$= \int_{-\infty}^{\infty} x p_U(x) dx \qquad (1.12)$$

$$= \int_0^1 x dx = \frac{1}{2} \qquad (1.13)$$

Now, using linearity of expectation,

$$\text{var}\left[U\right] = E\left[U - E\left[U\right]\right]^2 \qquad (1.14)$$

$$= E\left[U^2 - 2UE\left[U\right] + (E\left[U\right])^2\right] \qquad (1.15)$$

$$= E\left[U^2\right] - 2(E\left[U\right])^2 + (E\left[U\right])^2 \qquad (1.16)$$

$$= E\left[U^2\right] - (E\left[U\right])^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \qquad (1.17)$$

$$(1.18)$$

and this answer checks out with the output of the C program.

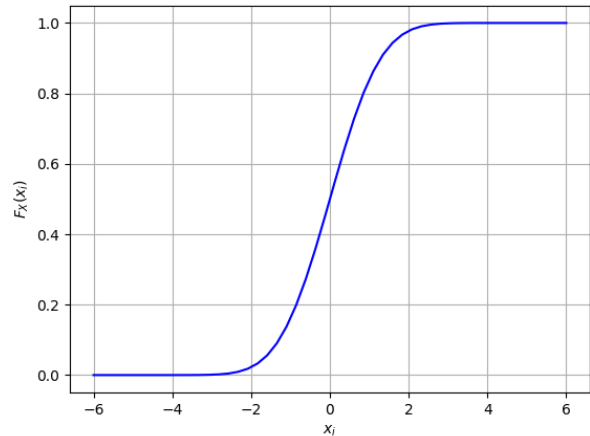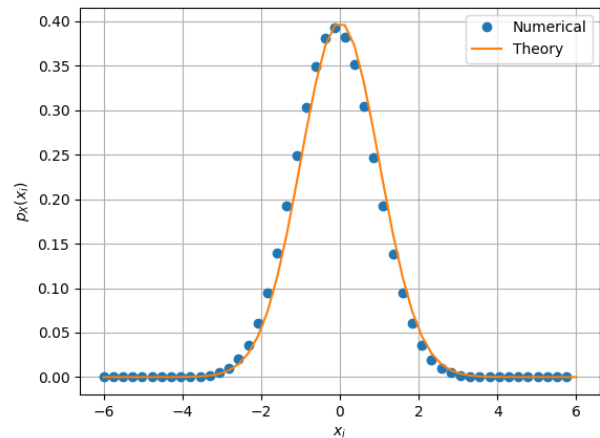## 2 CENTRAL LIMIT THEOREM

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \qquad (2.1)$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: The CDF of X is plotted in Fig. 2.2

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \qquad (2.2)$$

Fig. 2.2: The CDF of X

Fig. 2.3: The PDF of X

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. 2.3 using the code below

wget https://github.com/gadepall/probability/
raw/master/manual/codes/pdf_plot.py

2.4 Find the mean and variance of X by writing a C program.

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \qquad (2.3)$$

repeat the above exercise theoretically. **Solution:** The mean is given by

$$E\left[X\right] = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) = 0 \qquad (2.4)$$

as the integrand is odd. The variance is given by

$$var[X] = E[X^2] - (E[X])^2 \qquad (2.5)$$

$$= \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \qquad (2.6)$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp -\frac{x^2}{2} dx = 1 \qquad (2.7)$$

$$(2.8)$$

where we have integrated by parts.

3 FROM UNIFORM TO OTHER

3.1 Generate samples of

$$V = -2\ln(1 - U) \qquad (3.1)$$

and plot its CDF.

3.2 Find a theoretical expression for $F_V(x)$. **Solution:** Note that the function

$$v = f(u) = -2\ln 1 - u \qquad (3.2)$$

is monotonically increasing in [0, 1] and $v \in \mathbb{R}^+$. Hence, it is invertible and the inverse function is given by

$$u = f^{-1}(v) = 1 - \exp\left(-\frac{v}{2}\right) \qquad (3.3)$$

Therefore, from the monotonicity of v,

$$F_V(v) = F_U(1 - \exp(-fracv2)) = \begin{cases} 0 & v < 0 \\ 1 - \exp\left(-\frac{v}{2}\right) \end{cases}$$

$$(3.4)$$