# Assignment 12

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# Outline

## Problem Statement

**(Papoulis/Pillai, Exercise 5-48)** The random variable $X$ is $N(0; \sigma^2)$.

1. Using characteristic functions, show that if $g(x)$ is a function such that $g(x)\exp\left(-\frac{x^2}{2\sigma^2}\right) \to 0$ as $|x| \to \infty$, then (Price's Theorem)

$$\frac{dE\{g(X)\}}{dv} = \frac{1}{2}E\left\{\frac{d^2g(X)}{dX^2}\right\} \tag{1}$$

2. The moments $\mu_n$ of $X$ are functions of $v$. Using (1), show that

$$\mu_n(v) = \frac{n(n-1)}{2}\int_0^v \mu_{n-2}(\beta)d\beta \tag{2}$$

Here, $v = \sigma^2$.

# Solution

## PMF and Characteristic Function of $X$

Since $X \sim N(\mu; \sigma^2)$, the PMF of $X$ is given by

$$f(x) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \tag{3}$$

The characteristic function of $X$ is given by

$$\phi_X(\omega) = E[\exp(-j\omega X)] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma}} e^{-j\omega x} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \tag{4}$$

$$= \exp\left(j\mu\omega + \frac{(\sigma j\omega)^2}{2}\right) \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-(\mu+\sigma^2 j\omega))^2}{2\sigma^2}} dx \tag{5}$$

$$= \exp\left(j\mu\omega + \frac{(\sigma j\omega)^2}{2}\right) \tag{6}$$

## Inverse Transform of $X$

The inverse transform of $f(x)$ is given by

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi_X(\omega)e^{-j\omega x}\,d\omega \tag{7}$$

To prove the first part, using (6) and (7), and noting that $\mu = 0$,

$$\begin{aligned}
\frac{dE\{g(X)\}}{dv} &= \frac{d}{dv}\int_{-\infty}^{\infty} g(x)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \phi_X(\omega)e^{-j\omega x}\,d\omega\right)dx && (8)\\
&= \int_{-\infty}^{\infty} g(x)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\frac{\partial}{\partial v}\phi_X(\omega)\right)e^{-j\omega x}\,d\omega\right)dx && (9)\\
&= \int_{-\infty}^{\infty} g(x)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \left(-\frac{\omega^2}{2}\right)\phi_X(\omega)e^{-j\omega x}\,d\omega\right)dx && (10)\\
& && (11)
\end{aligned}$$

We can rewrite (11) using (7) as

$$\frac{dE\{g(X)\}}{dv} = \frac{1}{2}\int_{-\infty}^{\infty} g(x)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}(-j\omega)^2\phi_X(\omega)e^{-j\omega x}d\omega\right)dx \tag{12}$$

$$= \int_{-\infty}^{\infty} g(x)\frac{\partial^2 f(x)}{\partial x^2}dx \tag{13}$$

We assume that $g^{(k)}(x)\exp\left(-\frac{x^2}{2\sigma^2}\right) \to 0$ as $|x| \to \infty$ for $k = 0, 1, 2$. Repeatedly integrating (13) by parts gives

$$\frac{dE\{g(X)\}}{dv} = \frac{1}{2}\int_{-\infty}^{\infty} g\frac{\partial^2 f}{\partial x^2}dx \tag{14}$$

$$= \frac{1}{2}g\frac{\partial f}{\partial x}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{\partial g}{\partial x}\frac{\partial f}{\partial x}dx \tag{15}$$

$$= \frac{1}{2}\int_{-\infty}^{\infty}\frac{\partial^2 g}{\partial x^2}fdx - \frac{\partial g}{\partial x}f\Big|_{-\infty}^{\infty} = \frac{1}{2}E\left\{\frac{d^2 g(X)}{dX^2}\right\} \tag{16}$$

To prove the second part, we begin by observing that $\mu_n = E[X^n]$ and hence it must be a function of $v$. Using (1),

$$\mu_n'(v) = \frac{1}{2}E\left\{n(n-1)x^{n-2}\right\} = \frac{n(n-1)}{2}\mu_{n-2}v \tag{17}$$

However, note that if $v = 0$, then from (3), $x = 0$ and consequently $\mu_n(0) = 0$. Integrating (17), we get

$$\mu_n(v) = \frac{n(n-1)}{2}\int_0^v \mu_{n-2}(\beta)d\beta \tag{18}$$