Assignment 13

Gautam Singh (CS21BTECH11018)

May 30, 2022

Outline

Problem Statement

(Papoulis/Pillai, Exercise 5-48) The random variable X is $N(0;\sigma^2)$.

1. Using characteristic functions, show that if $g(x)$ is a function such that $g(x)\exp\left(-\frac{x^2}{2\sigma^2}\right) \to 0$ as $|x| \to \infty$, then (Price's Theorem)

$$\frac{dE\{g(X)\}}{dv} = \frac{1}{2}E\left\{\frac{d^2g(X)}{dX^2}\right\} \tag{1}$$

2. The moments μ_n of X are functions of v. Using (1), show that

$$\mu_n(v) = \frac{n(n-1)}{2}\int_0^v \mu_{n-2}(\beta)d\beta \tag{2}$$

Here, $v = \sigma^2$.

Solution

PMF and Characteristic Function of X

Since $X \sim N(\mu; \sigma^2)$, the PMF of X is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \tag{3}$$

The characteristic function of X is given by

$$\begin{aligned} \phi_X(\omega) = E[\exp(-j\omega X)] &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-j\omega x} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx && (4) \\ &= \exp\left(j\mu\omega + \frac{(\sigma j\omega)^2}{2}\right) \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-(\mu+\sigma^2 j\omega))^2}{2\sigma^2}}\, dx && (5) \\ &= \exp\left(j\mu\omega + \frac{(\sigma j\omega)^2}{2}\right) && (6) \end{aligned}$$

Inverse Transform of X

The inverse transform of $f(x)$ is given by

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_X(\omega) e^{-j\omega x} d\omega \tag{7}$$

Here, $f(x)$ and $\phi_X(\omega)$ form a Fourier transform pair.

To prove the first part, using (6) and (7), and noting that $\mu = 0$,

$$\frac{dE\{g(X)\}}{dv} = \frac{d}{dv} \int_{-\infty}^{\infty} g(x) \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_X(\omega) e^{-j\omega x} d\omega \right) dx \tag{8}$$

$$= \int_{-\infty}^{\infty} g(x) \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{\partial}{\partial v} \phi_X(\omega) \right) e^{-j\omega x} d\omega \right) dx \tag{9}$$

$$= \int_{-\infty}^{\infty} g(x) \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \left(-\frac{\omega^2}{2} \right) \phi_X(\omega) e^{-j\omega x} d\omega \right) dx \tag{10}$$

We can rewrite (10) using (7) as

$$\frac{dE\{g(X)\}}{dv} = \frac{1}{2}\int_{-\infty}^{\infty} g(x)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}(-j\omega)^2\phi_X(\omega)e^{-j\omega x}d\omega\right)dx \tag{11}$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} g(x)\frac{\partial^2 f(x)}{\partial x^2}dx \tag{12}$$

We assume that $g^{(k)}(x)\exp\left(-\frac{x^2}{2\sigma^2}\right) \to 0$ as $|x| \to \infty$ for $k = 0, 1, 2$. Repeatedly integrating (12) by parts gives

$$\frac{dE\{g(X)\}}{dv} = \frac{1}{2}\int_{-\infty}^{\infty} g\frac{\partial^2 f}{\partial x^2}dx \tag{13}$$

$$= \frac{1}{2}g\frac{\partial f}{\partial x}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{\partial g}{\partial x}\frac{\partial f}{\partial x}dx \tag{14}$$

$$= \frac{1}{2}\int_{-\infty}^{\infty}\frac{\partial^2 g}{\partial x^2}fdx - \frac{\partial g}{\partial x}f\Big|_{-\infty}^{\infty} = \frac{1}{2}E\left\{\frac{d^2 g(X)}{dX^2}\right\} \tag{15}$$

For the second part, observe that $\mu_n = E[X^n]$ and hence it is a function of v. Further, using the exponential power series, note that for any positive integer n,

$$\exp\left(\frac{x^2}{2\sigma^2}\right) > \frac{x^{2n}}{(2\sigma^2)^n n!} \tag{16}$$

$$\implies 0 < \frac{x^n}{\exp\left(\frac{x^2}{2\sigma^2}\right)} < \frac{(2\sigma^2)^n n!}{x^n} \tag{17}$$

and using the Sandwich Theorem, we can choose $g(x) = x^n$ to use in (1).

$$\mu_n'(v) = \frac{1}{2}E\left\{n(n-1)x^{n-2}\right\} = \frac{n(n-1)}{2}\mu_{n-2}(v) \tag{18}$$

However, note that if $v = 0$, then from (3), $x = 0$ and consequently $\mu_n(0) = 0$. Integrating (18) and changing variables, we get

$$\mu_n(v) = \frac{n(n-1)}{2}\int_0^v \mu_{n-2}(\beta)d\beta \tag{19}$$