Random Numbers

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Contents

***Abstract*—This manual provides a simple introduction to the generation of random numbers. Commands that must be executed in a *NIX shell are preceded by a $ symbol.**

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following files

```
$ wget https://raw.githubusercontent.com/
    goats-9/ai1110-probability/master/
    manual/codes/1_1.c
$ wget https://raw.githubusercontent.com/
    goats-9/ai1110-probability/master/
    manual/codes/1_1.h
```

and compile and execute the C program using

```
$ gcc 1_1.c -lm -Wall -g
$ ./a.out
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \tag{1.1}$$

Solution: The following code plots Fig. 1.2

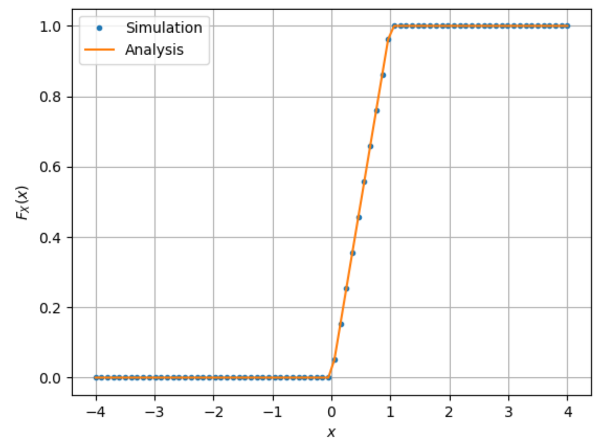

Fig. 1.2: The CDF of U

```
$ wget https://raw.githubusercontent.com/
    goats-9/ai1110-probability/master/
    manual/codes/1_2.py
```

It is executed with

```
$ python3 1_2.py
```

1.3 Find a theoretical expression for $F_U(x)$.
Solution: The CDF of U is given by

$$F_U(x) = \Pr(U \leq x) = \int_{-\infty}^{x} p_U(u)du \tag{1.2}$$

We now have three cases:

a) $x < 0$: $p_X(x) = 0$, and hence $F_U(x) = 0$.
b) $0 \leq x < 1$: Here,

$$F_U(x) = \int_0^x du = x \tag{1.3}$$

c) $x \geq 1$: Put $x = 1$ in (1.3) as U is uniform in $[0, 1]$ to get $F_U(x) = 1$.

Therefore,

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases} \tag{1.4}$$

This is verified in Figure (1.2)

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.5}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{1.6}$$

Write a C program to find the mean and variance of U.
Solution: The C program can be downloaded using

```
$ wget https://raw.githubusercontent.com/
    goats−9/ai1110−probability/master/
    manual/codes/1_4.c
```

and compiled and executed with

```
$ gcc 1_4.c −lm −Wall −g
$ ./a.out
```

The calculated mean is 0.500007 and the calculated variance is 0.083301.

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x)dx \tag{1.7}$$

Solution: We write

$$E\left[U^2\right] = \int_{-\infty}^{\infty} x^2 dF_U(x) \tag{1.8}$$
$$= \int_{-\infty}^{\infty} x^2 p_U(x)dx \tag{1.9}$$
$$= \int_{0}^{1} x^2 dx = \frac{1}{3} \tag{1.10}$$

and

$$E[U] = \int_{-\infty}^{\infty} x dF_U(x) \tag{1.11}$$
$$= \int_{-\infty}^{\infty} x p_U(x)dx \tag{1.12}$$
$$= \int_{0}^{1} x dx = \frac{1}{2} \tag{1.13}$$

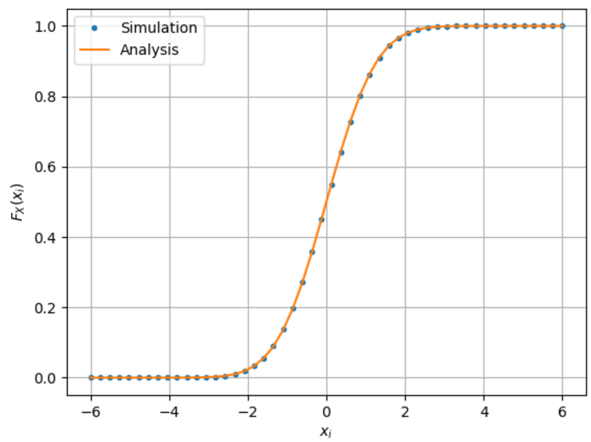

Fig. 2.2: The CDF of X

which checks out with the empirical mean on 0.500007. Now, using linearity of expectation,

$$\text{var}[U] = E[U - E[U]]^2 \tag{1.14}$$
$$= E\left[U^2 - 2UE[U] + (E[U])^2\right] \tag{1.15}$$
$$= E\left[U^2\right] - 2(E[U])^2 + (E[U])^2 \tag{1.16}$$
$$= E\left[U^2\right] - (E[U])^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \tag{1.17}$$
$$\tag{1.18}$$

and this checks out with the empirical variance 0.083301 of the sample data.

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat
Solution: The sample data is generated by the C file in Question 1.1.

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: The CDF of X is plotted in Fig. 2.2 The required python file can be downloaded using

```
$ wget https://raw.githubusercontent.com/
    goats−9/ai1110−probability/master/
    manual/codes/2_2.py
```

and executed using

```
$ python3 2_2.py
```

a) The CDF is non-decreasing
b) It is right-continuous.
c) $\lim_{x\to-\infty} F_X(x) = 0$
d) $\lim_{x\to\infty} F_X(x) = 1$

The CDF is expressed in terms of the Q-function as $F_X(x) = 1 - Q(x)$.

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx}F_X(x) \tag{2.2}$$

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. 2.3 using the code below

```
$ wget https://raw.githubusercontent.com/
    goats−9/ai1110−assignments/master/
    manual/codes/2_3.py
```

The figure is generated using

```
$ python3 2_3.py
```

The properties of a PDF $p_X(x)$ are as follows:
a) $\forall x \in \mathbb{R},\ p_X(x) \ge 0$
b) $\int_{-\infty}^{\infty} p_X(x)dx = 1$
c) For $a < b,\ a, b \in \mathbb{R}$

$$\Pr(a < X < b) = \Pr(a \le X \le b) \tag{2.3}$$
$$= \int_a^b p_X(x)dx \tag{2.4}$$

If we take $a = b$, then we get $\Pr(X = a) = 0$.

2.4 Find the mean and variance of X by writing a C program.
Solution: The mean and variance have been calculated using (1.5) and (1.6) respectively. The C program can be downloaded using

```
$ wget https://raw.githubusercontent.com/
    goats−9/ai1110−assignments/master/
    manual/codes/2_4.c
```

and compiled and executed with the following commands

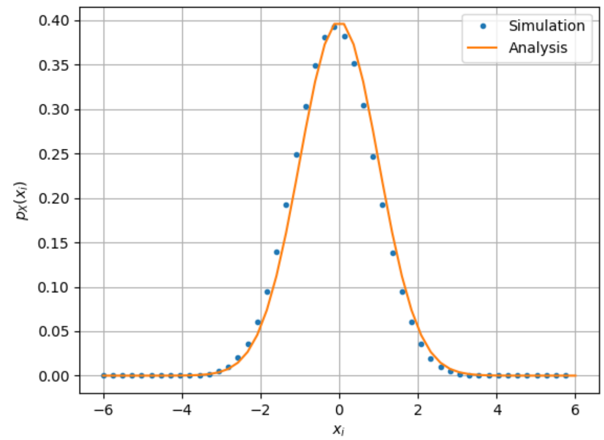

Fig. 2.3: The PDF of X

```
$ gcc 2_4.c −lm −Wall −g
$ ./a.out
```

The calculated mean is 0.000326 and the calculated variance is 1.000906.

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.5}$$

repeat the above exercise theoretically.
Solution: The mean is given by

$$E[X] = \int_{-\infty}^{\infty} x\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right) = 0 \tag{2.6}$$

as the integrand is odd. This checks out with the empirical mean of 0.000326. The variance is given by

$$\operatorname{var}[X] = E\left[X^2\right] - (E[X])^2 \tag{2.7}$$
$$= \int_{-\infty}^{\infty} x^2\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)dx \tag{2.8}$$
$$= \int_0^{\infty} \frac{2}{\sqrt{2\pi}}\sqrt{2t}e^{-t}dt \tag{2.9}$$
$$= \frac{2}{\sqrt{\pi}}\Gamma\left(\frac{3}{2}\right) \tag{2.10}$$
$$= \frac{1}{\sqrt{\pi}}\Gamma\left(\frac{1}{2}\right) = 1 \tag{2.11}$$

where we have used $t = \frac{x^2}{2}$ and so $dt = xdx$. We have also used the gamma function given

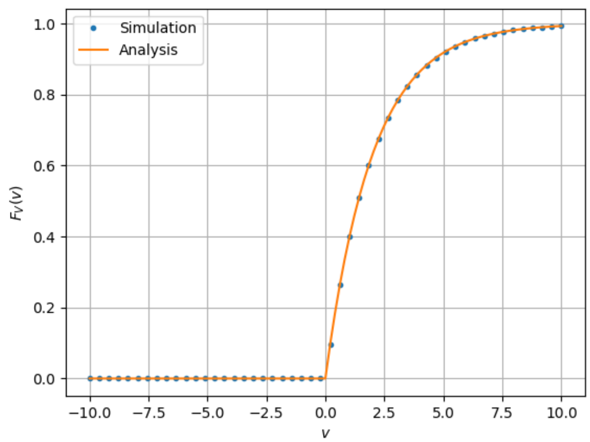

Fig. 3.1: The CDF of V

as

$$\Gamma(n) = \int_{-\infty}^{\infty} x^{n-1} e^{-x} dx \tag{2.12}$$

$$\Gamma(n) = (n-1)\Gamma(n-1) \text{ for } n > 1 \tag{2.13}$$

and the fact that $\Gamma(1/2) = \sqrt{\pi}$. This agrees with the empirical variance of 1.000906.

3 From Uniform to Other

3.1 Generate samples of

$$V = -2\ln(1-U) \tag{3.1}$$

and plot its CDF.
Solution: The relevant python code is at

```
$ wget https://raw.githubusercontent.com/
    goats−9/ai1110−probability/master/
    manual/codes/3_1.py
```

and can be executed with

```
$ python3 3_1.py
```

and the CDF is plotted in Figure (3.1).

3.2 Find a theoretical expression for $F_V(x)$.
Solution: Note that the function

$$v = f(u) = -2\ln(1-u) \tag{3.2}$$

is monotonically increasing in [0, 1] and $v \in \mathbb{R}^+$. Hence, it is invertible and the inverse function is given by

$$u = f^{-1}(v) = 1 - \exp\left(-\frac{v}{2}\right) \tag{3.3}$$

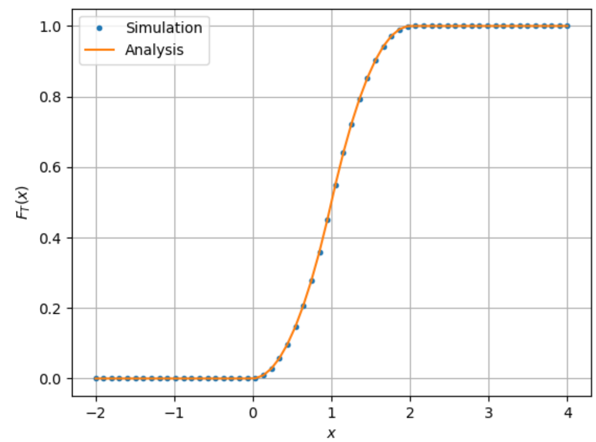

Fig. 4.2: The CDF of T

Therefore, from the monotonicity of v, and using (1.4),

$$F_V(v) = F_U\left(1 - \exp\left(-\frac{v}{2}\right)\right) \tag{3.4}$$

$$\implies F_V(v) = \begin{cases} 0 & v < 0 \\ 1 - \exp\left(-\frac{v}{2}\right) & v \geq 0 \end{cases} \tag{3.5}$$

4 Triangular Distribution

4.1 Generate

$$T = U_1 + U_2 \tag{4.1}$$

Solution: The samples are generated in the C file exrand.c in 1.1 as the file tri.dat.

4.2 Find the CDF of T.
Solution: The Python code for the figure is at

```
$ wget https://raw.githubusercontent.com/
    goats−9/ai1110−assignments/master/
    manual/codes/4_2.py
```

and can be run using

```
$ python3 4_2.py
```

4.3 Find the PDF of T.
Solution: The Python code for the figure can be downloaded using

```
$ wget https://raw.githubusercontent.com/
    goats−9/ai1110−assignments/master/
    manual/codes/4_3.py
```

and run using

```
$ python3 4_3.py
```

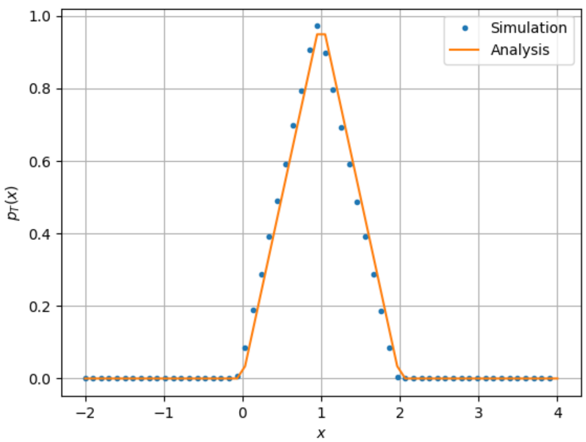

Fig. 4.3: The PDF of T

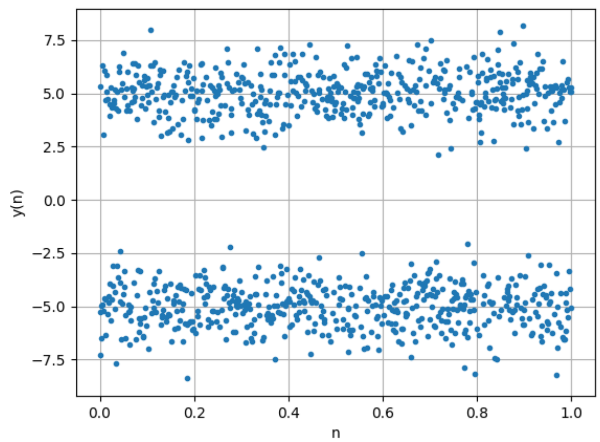

Fig. 5.2: Plot of Y

4.4 Find the theoretical PDF and CDF of T.
Solution: We write,

$$F_T(t) = \Pr(U_1 + U_2 \le t) \tag{4.2}$$
$$= \Pr(U_1 \le t - U_2) \tag{4.3}$$
$$= \int_0^1 F_{U_1}(t - x) p_{U_2}(x) dx \tag{4.4}$$

where U_1 and U_2 are uniform i.i.d. random variables in [0, 1]. Then, $0 \le U_1 + U_2 \le 2$. We have three cases:

a) $t < 0$: Using Equation 1.4, $F_T(t) = 0$.
b) $0 \le t < 1$: We have,

$$F_T(t) = \int_0^t (t - x) dx = \frac{t^2}{2} \tag{4.5}$$

c) $1 \le t < 2$: Here, we get

$$F_T(t) = \int_0^{t-1} dx + \int_{t-1}^1 (t - x) dx \tag{4.6}$$
$$= t - 1 + t(2 - t) - \frac{1 - (t-1)^2}{2} \tag{4.7}$$
$$= -\frac{t^2}{2} + 2t - 1 \tag{4.8}$$

d) $t \ge 2$: Here, $F_T(t) = 1$.
Therefore,

$$F_T(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \le t < 1 \\ -\frac{t^2}{2} + 2t - 1 & 1 \le t < 2 \\ 1 & t \ge 2 \end{cases} \tag{4.9}$$

Using Equation 2.2,

$$p_T(t) = \begin{cases} t & 0 \le t < 1 \\ 2 - t & 1 \le t < 2 \\ 0 & \text{otherwise} \end{cases} \tag{4.10}$$

4.5 Verify your results through a plot.
Solution: This has been done in the plots (4.3) and (4.2).

5 Maximum Likelihood

5.1 Generate

$$Y = AX + N \tag{5.1}$$

where $A = 5$ dB, $X \in \{1, -1\}$ is Bernoulli and $N \sim \mathcal{N}(0, 1)$.
Solution: The C file in Question 1.1 generates the numbers in the file gau_ber.dat

5.2 Plot Y.
Solution: The Python code can be downloaded using

```
$ wget https://raw.githubusercontent.com/
    goats-9/ai1110-probability/master/
    manual/codes/5_2.py
```

and run using

```
$ python3 5_2.py
```

5.3 Guess how to estimate X from Y.
Solution: From the plot of Y, we see that $X = 1$ usually correlates to $Y > 0$ and $X = -1$ correlates to $Y < 0$.

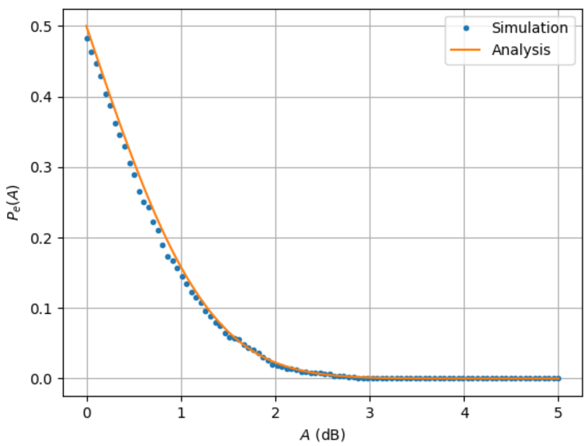

Fig. 5.6: P_e as a function of A

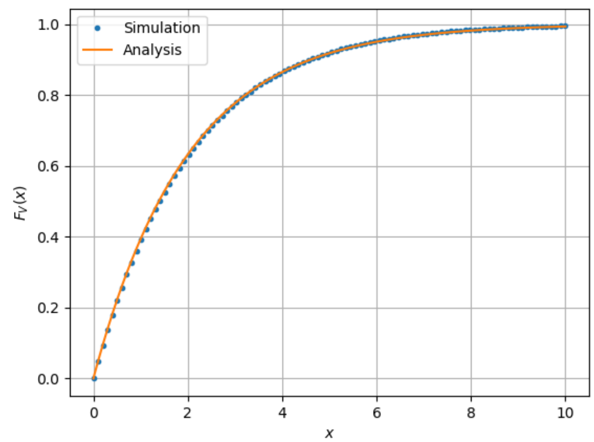

Fig. 6.1: CDF of V

5.4 Find

$$P_{e|0} = \Pr\left(\hat{X} = -1|X = 1\right) \tag{5.2}$$

and

$$P_{e|1} = \Pr\left(\hat{X} = 1|X = -1\right) \tag{5.3}$$

Solution: Letting $X = 1$ and $X = -1$ respectively, we see the number of mismatched data points to compute the error probabilities. The simulation is coded in

```
$ wget https://raw.githubusercontent.com/
    goats−9/ai1110−assignments/master/
    manual/codes/5_4.py
```

and can be run by typing

```
$ python3 5_4.py
```

The results are

$$P_{e|0} = 0 \tag{5.4}$$
$$P_{e|1} = 0 \tag{5.5}$$

5.5 Find P_e.
Solution: Here, we assume $\Pr(X = 1) = \Pr(X = -1) = 0.5$. Thus,

$$P_e = \frac{1}{2}\left(P_{e|0} + P_{e|1}\right) \tag{5.6}$$
$$= 0.502 \tag{5.7}$$

5.6 Verify by plotting the theoretical P_e.
Solution: The plot is coded in

```
$ wget https://raw.githubusercontent.com/
    goats−9/ai1110−assignments/master/
    manual/codes/5_6.py
```

and can be executed using

```
$ python3 5_6.py
```

We note that

$$P_{e|0} = \Pr\left(\hat{X} = 1|X = -1\right) = \Pr(Y > 0|X = -1) \tag{5.8}$$
$$= \Pr(AX + N > 0|X = -1) \tag{5.9}$$
$$= \Pr(N > A|X = -1) = Q(A) \tag{5.10}$$

since X and N are independent. Writing a similar expression for $P_{e|1}$ and noting that

$$\Pr(N < -A) = \Pr(N > A) = Q(A) \tag{5.11}$$

it follows that $P_e = Q(A)$. This is the idea used to plot the theoretical P_e.

6 Gaussian to Other

6.1 Let $X_1 \sim \mathcal{N}(0,1)$ and $X_2 \sim \mathcal{N}(0,1)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \tag{6.1}$$

Solution: We transform the variables X_1 and X_2 as:

$$X_1 = R\cos\Theta \tag{6.2}$$
$$X_2 = R\sin\Theta \tag{6.3}$$

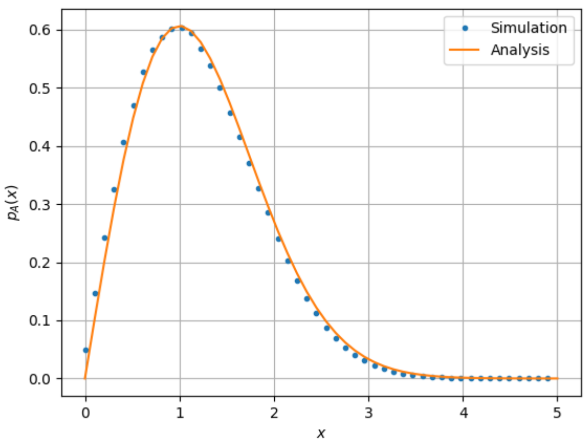

Fig. 6.1: PDF of V

where $R \in [0, \infty), \Theta \in [0, 2\pi)$. The Jacobian Matrix for this transformation is given by

$$\mathbf{J} = \begin{pmatrix} \frac{\partial X_1}{\partial R} & \frac{\partial X_2}{\partial R} \\ \frac{\partial X_1}{\partial \Theta} & \frac{\partial X_2}{\partial \Theta} \end{pmatrix} \tag{6.4}$$

$$= \begin{pmatrix} \cos\Theta & \sin\Theta \\ -R\sin\Theta & R\cos\Theta \end{pmatrix} \tag{6.5}$$

$$\implies |\mathbf{J}| = R \tag{6.6}$$

We also know that

$$|\mathbf{J}| p_{X_1,X_2}(x_1, x_2) = p_{R,\Theta}(r, \theta) \tag{6.7}$$

$$\implies p_{R,\Theta}(r, \theta) = R p_{X_1}(x_1) p_{X_2}(x_2) \tag{6.8}$$

$$= \frac{R}{2\pi} \exp\left(-\frac{X_1^2 + X_2^2}{2}\right) \tag{6.9}$$

$$= \frac{R}{2\pi} \exp\left(-\frac{R^2}{2}\right) \tag{6.10}$$

where (6.8) follows as X_1, X_2 are iid random variables. Thus,

$$p_R(r) = \int_0^{2\pi} p_{R,\Theta}(r, \theta) d\theta \tag{6.11}$$

$$= R \exp\left(-\frac{R^2}{2}\right) \tag{6.12}$$

However, $V = X_1^2 + X_2^2 = R^2 \geq 0$, thus $F_V(x) = 0$ for $x \geq 0$.

$$F_V(x) = F_R(\sqrt{x}) \tag{6.13}$$

$$= \int_0^{\sqrt{x}} r \exp\left(-\frac{r^2}{2}\right) dr \tag{6.14}$$

$$= \int_0^{\frac{x}{2}} e^{-t} dt = 1 - e^{-\frac{x}{2}} \tag{6.15}$$

and so, for $x \geq 0$,

$$p_V(x) = \frac{1}{2} e^{-\frac{x}{2}} \tag{6.16}$$

Hence,

$$F_V(x) = \begin{cases} 1 - e^{-\frac{x}{2}} & x \geq 0 \\ 0 & x < 0 \end{cases} \tag{6.17}$$

$$p_V(x) = \begin{cases} \frac{1}{2} e^{-\frac{x}{2}} & x \geq 0 \\ 0 & x < 0 \end{cases} \tag{6.18}$$

The equations (6.17) and (6.18) have been used to generate the plots. The Python code can be downloaded using

```
$ wget https://raw.githubusercontent.com/
    goats−9/ai1110−assignments/master/
    manual/codes/6_1_cdf.py
$ wget https://raw.githubusercontent.com/
    goats−9/ai1110−assignments/master/
    manual/codes/6_1_pdf.py
```

and run by typing the command

```
$ python3 6_1_cdf.py
$ python3 6_1_pdf.py
```

6.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0 \end{cases} \tag{6.19}$$

find α.

Solution: From (6.17), it is clear that $\alpha = 0.5$.

6.3 Plot the CDF and PDF of

$$A = \sqrt{V} \tag{6.20}$$

Solution: Note that for $x \geq 0$,

$$F_A(x) = \Pr(A \leq x) \tag{6.21}$$

$$= \Pr\left(\sqrt{V} \leq x\right) \tag{6.22}$$

$$= \Pr\left(V \leq x^2\right) \tag{6.23}$$

$$= F_V(x^2) = 1 - e^{-\frac{x^2}{2}} \tag{6.24}$$

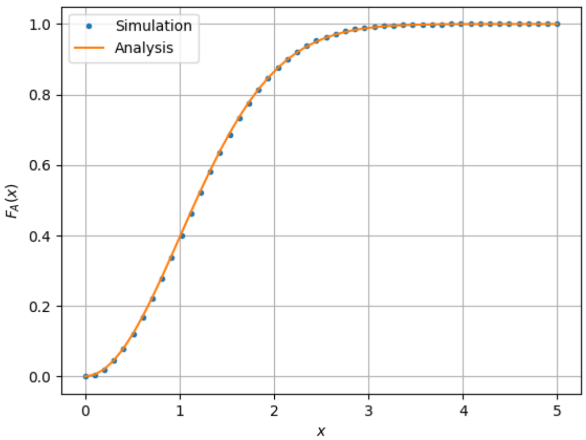

Fig. 6.3: CDF of A

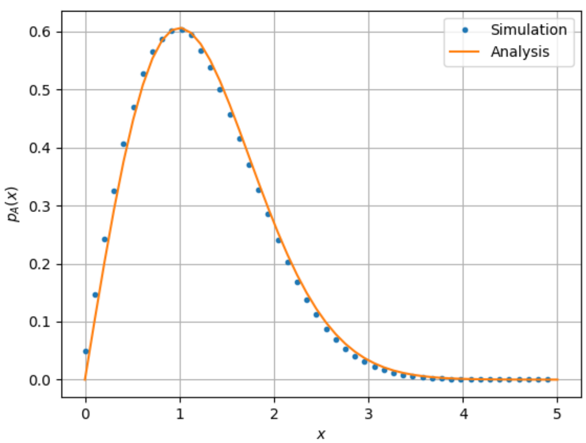

Fig. 6.3: PDF of A

and so,

$$p_A(x) = xe^{-\frac{x^2}{2}} \tag{6.25}$$

Thus, the CDF and PDF of A is given by

$$F_V(x) = \begin{cases} 1 - e^{-\frac{x^2}{2}} & x \geq 0 \\ 0 & x < 0 \end{cases} \tag{6.26}$$

$$p_V(x) = \begin{cases} xe^{-\frac{x}{2}} & x \geq 0 \\ 0 & x < 0 \end{cases} \tag{6.27}$$

The Python codes for the plots are at

```
$ wget https://raw.githubusercontent.com/
    goats-9/ai1110-assignments/master/
    manual/codes/6_3_cdf.py
$ wget https://raw.githubusercontent.com/
    goats-9/ai1110-assignments/master/
```

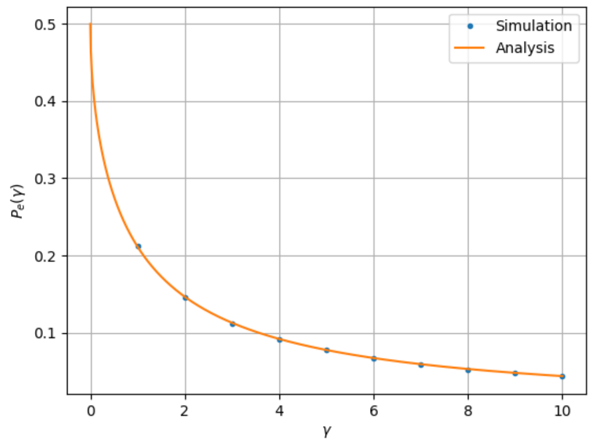

Fig. 7.1: P_e as a function of γ

```
    manual/codes/6_3_pdf.py
```

and can be executed using

```
$ python3 6_3_cdf.py
$ python3 6_3_pdf.py
```

7 Conditional Probability

7.1 Plot

$$P_e = \Pr\left(\hat{X} = -1|X = 1\right) \tag{7.1}$$

for

$$Y = AX + N \tag{7.2}$$

where A is Rayleigh with $E\left[A^2\right] = \gamma$, $N \sim \mathcal{N}(0, 1)$, $X \in \{1, -1\}$ for $0 \leq \gamma \leq 10$ dB.
Solution: Download the relevant Python code

```
$ wget https://raw.githubusercontent.com/
    goats-9/ai1110-assignments/master/
    manual/codes/7_1.py
```

and run it using

```
$ python3 7_1.py
```

7.2 Assuming that N is a constant, find an expression for P_e. Call this $P_e(N)$.
Solution: We rewrite the previous expression for P_e as

$$P_e(N) = \Pr(A < -N) = F_A(-N) \tag{7.3}$$

$$= \begin{cases} 1 - e^{-\frac{N^2}{\gamma}} & N \leq 0 \\ 0 & N > 0 \end{cases} \tag{7.4}$$

7.3 For a function g,

$$E\left[g(X)\right] = \int_{-\infty}^{\infty} g(x) p_X(x) dx \tag{7.5}$$

Find $P_e = E\left[P_e(N)\right]$.

Solution: We write,

$$P_e = \int_0^{\infty} F_A(x) f_N(x) dx \tag{7.6}$$

$$= \int_0^{\infty} (1 - e^{-\frac{x^2}{\gamma}}) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \tag{7.7}$$

$$= \frac{1}{2} - \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \exp\left(-\frac{x^2}{2\left(\gamma/\gamma + 2\right)}\right) dx \tag{7.8}$$

$$= \frac{1}{2}\left(1 - \sqrt{\frac{\gamma}{\gamma + 2}}\right) \tag{7.9}$$

where we have integrated by parts and f_N denotes the standard normal distribution.

7.4 Plot P_e in problems 7.1 and 7.3 on the same graph wrt γ. Comment.

Solution: This has been done in problem 7.1 using the result (7.9). We observe that $P_{e|0} = E\left[P_e(N)\right]$.