Assignment 14

Gautam Singh (CS21BTECH11018)

June 13, 2022

Outline

Problem Statement

(Papoulis/Pillai, Exercise 8-25) We are given a random variable X with mean η and standard deviation $\sigma = 2$, and we wish to test the hypothesis $\eta_0 = 8$ against $\eta = 8.7$ with $\alpha = 0.01$ using as the test statistic the sample mean $\bar{x}$ of n samples.

1. Find the critical region R_c of the test and the resulting β if $n = 64$.
2. Find n and R_c if $\beta = 0.05$.

Definitions

Random Variable Used

When using the test statistic as the mean $(\bar{X})$, we consider the random variable

$$Q = \frac{\bar{X} - \eta}{\frac{\sigma}{\sqrt{n}}} \tag{1}$$

We assume $\bar{X} \sim N(\eta_0; \frac{\sigma}{\sqrt{n}})$. Hence, $Q \sim N(\eta_q; 0)$, where

$$\eta_q = \frac{\eta - \eta_0}{\frac{\sigma}{\sqrt{n}}} \tag{2}$$

Observe that for the null hypothesis $H_0 : \eta = \eta_0$, we have $Q \sim N(0; 1)$ so we can use the standard normal percentiles.

Solution

Note that in this solution, writing $\Pr(\ldots|H_0)$ denotes that the hypothesis H_0 is true. Also note that percentile notation is used. In particular, z_p denotes the point of the p^{th} percentile of the standard normal disribution.

1 For $H_1 : \eta > \eta_0$, the critical region is the half-line $q > c$, where

$$\Pr(Q > c|H_0) = \alpha \tag{3}$$

$$\implies c = q_{1-\alpha} \tag{4}$$

$$\implies c = \eta + z_{1-\alpha}\frac{\sigma}{\sqrt{n}} = 8 + 2.326 \times \frac{2}{8} = 8.58 \tag{5}$$

Hence, R_c is given by $\bar{x} > 8.58$ (refer Figure (1)). Now, if $H_1 : \eta > \eta_0$, then

$$\eta_q = \frac{8.7 - 8}{\frac{2}{8}} = 2.8 \tag{6}$$

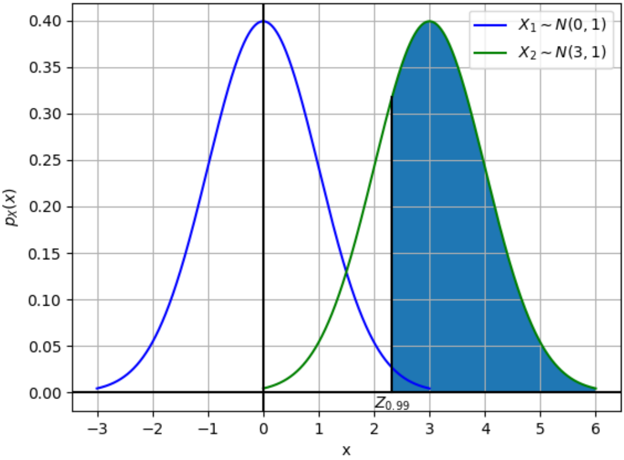

Figure 1: Critical region where null hypothesis is not accepted. Here, $\eta_0 = 0$ and $\eta = 3$.

1 (Cont'd...) β is a function of η given by

$$\beta(\eta) = \Pr\left(Q \notin R_c | H_1\right) = \Pr\left(Q < c\right) \tag{7}$$

$$= \text{erf}(z_{1-\alpha} - \eta_q) = 0.32 \tag{8}$$

2 Proceeding in reverse from (8), we have from the definition of β,

$$z_{1-\alpha} - \eta_q = z_\beta \tag{9}$$

$$\implies \eta_q = z_{1-\alpha} - z_\beta \tag{10}$$

$$= z_{0.99} - z_{0.05} = 4.97 \tag{11}$$

Given that $\beta(\eta = 8.7) = 0.05$, we can also use (2) to get

$$n = \left(\frac{\sigma\eta_q}{\eta - \eta_0}\right)^2 = 201 \tag{12}$$

Thus, using (5), we get

$$c = 8 + z_{0.99}\frac{2}{\sqrt{201}} = 8.33 \tag{13}$$