Straight Lines Assignment

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Abstract—**This document contains the solution to Question 4 of Exercise 2 in Chapter 10 of the class 11 NCERT textbook.**

1) Find the coordinates of the foot of perpendicular from the point

$$\mathbf{P} = \begin{pmatrix} -1 \\ 3 \end{pmatrix} \tag{1}$$

to the line

$$\begin{pmatrix} 3 & -4 \end{pmatrix} \mathbf{x} = 16 \tag{2}$$

Solution: We present three methods to solve this problem.

a) *Using convex functions.* Any point on (2) is clearly of the form

$$\mathbf{Q} = \begin{pmatrix} 0 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 4 & 3 \end{pmatrix} \tag{3}$$

where $\lambda \in \mathbb{R}$. Thus, PQ becomes a function of λ, as shown in (5).

$$\|\mathbf{P} - \mathbf{Q}\|^2 = \left\| \begin{pmatrix} 4\lambda + 1 \\ 3\lambda - 7 \end{pmatrix} \right\|^2 \tag{4}$$

$$= 25\lambda^2 - 34\lambda + 50 = f(\lambda) \tag{5}$$

Since the coefficient of λ^2 in $f(\lambda)$ is positive, it follows that $f(\lambda)$ is convex. Hence, the minima is achieved at

$$f'(\lambda) = 50\lambda - 34 = 0 \tag{6}$$

$$\implies \lambda_m = \frac{17}{25} \tag{7}$$

Thus, substituting into (3), we get

$$\mathbf{Q_m} = \frac{1}{25} \begin{pmatrix} 68 \\ -49 \end{pmatrix} \tag{8}$$

b) *Using gradient descent.* Since (5) is convex, we use the gradient descent function on λ to converge at the minimum of $f(\lambda)$.

$$\lambda_{n+1} = \lambda_n - \alpha f'(\lambda_n) \tag{9}$$

$$= (1 - 50\alpha)\lambda_n + 34\alpha \tag{10}$$

Taking the one-sided Z-transform on both sides of (10),

$$z\Lambda(z) = (1 - 50\alpha)\Lambda(z) + \frac{34\alpha}{1 - z^{-1}} \tag{11}$$

$$\Lambda(z) = \frac{34\alpha z^{-1}}{(1 - z^{-1})(1 - (1 - 50\alpha) z^{-1})} \tag{12}$$

$$= \frac{17}{25} \left(\frac{1}{1 - (1 - 50\alpha) z^{-1}} - \frac{1}{1 - z^{-1}} \right) \tag{13}$$

$$= \frac{17}{25} \sum_{k=0}^{\infty} \left(1 - (1 - 50\alpha)^k \right) z^{-k} \tag{14}$$

From (12), the ROC is

$$|z| > \max\{1, 1 - 50\alpha\} \tag{15}$$

$$\implies 0 < 1 - 50\alpha < 1 \tag{16}$$

$$\implies \alpha \in (0, 0.02) \tag{17}$$

Thus, if $\alpha > 0$, then from (14) and (7)

$$\lim_{n\to\infty} \lambda_n = \lim_{n\to\infty} \frac{17}{25} \left(1 - (1 - 50\alpha)^n\right) \tag{18}$$

$$= \frac{17}{25} = \lambda_m \tag{19}$$

We select the following parameters to arrive at the optimal λ_m, where N is the number of iterations and ϵ is the convergence limit. The gradient descent is demonstrated in Fig. 1, plotted by the Python code `codes/grad_desc.py`. The relevant parameters are shown in Table 1.

Parameter	**Value**
λ_0	0
α	0.1
N	1000000
ϵ	10^{-6}

TABLE 1: Parameters for Gradient Descent

c) *Using Lagrange Multipliers.* We rewrite the

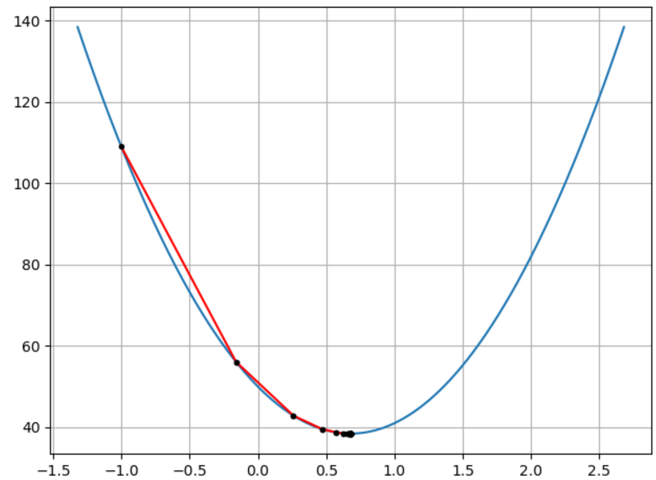

Fig. 1: Gradient descent to get the optimal λ.

problem as

$$\min_{\mathbf{x}} h(\mathbf{x}) \triangleq \|\mathbf{x} - \mathbf{P}\|^2 \tag{20}$$

$$\text{s.t. } g(\mathbf{x}) \triangleq \mathbf{n}^\top \mathbf{x} - c = 0 \tag{21}$$

Define

$$C(\mathbf{x}, \lambda) = h(\mathbf{x}) - \lambda g(\mathbf{x}) \tag{22}$$

and note that

$$\nabla h(\mathbf{x}) = 2(\mathbf{x} - \mathbf{P}) \tag{23}$$

$$\nabla g(\mathbf{x}) = \mathbf{n} \tag{24}$$

We are required to find $\lambda \in \mathbb{R}$ such that

$$\nabla C(\mathbf{x}, \lambda) = 0 \tag{25}$$

$$\implies 2(\mathbf{x} - \mathbf{P}) - \lambda \mathbf{n} = 0 \tag{26}$$

However, $\mathbf{x}$ lies on the line (2). Thus, from (26),

$$\mathbf{n}^\top \left(\frac{\lambda}{2}\mathbf{n} + \mathbf{P}\right) - c = 0 \tag{27}$$

$$\implies \frac{25\lambda}{2} - 15 - 16 = 0 \tag{28}$$

$$\implies \lambda = \frac{62}{25} \tag{29}$$

Substituting (29) in (26),

$$\mathbf{x_m} = \begin{pmatrix} -1 \\ 3 \end{pmatrix} + \frac{31}{25}\begin{pmatrix} 3 \\ -4 \end{pmatrix} \tag{30}$$

$$= \frac{1}{25}\begin{pmatrix} 68 \\ -49 \end{pmatrix} \tag{31}$$

which matches with the solutions from the above methods. To find $\mathbf{x_m}$ graphically from this method, we use constrained gradient descent, with $\alpha = 0.01$. The results are shown in Fig. 2, plotted using the Python code `codes/gd_lagrange.py`.

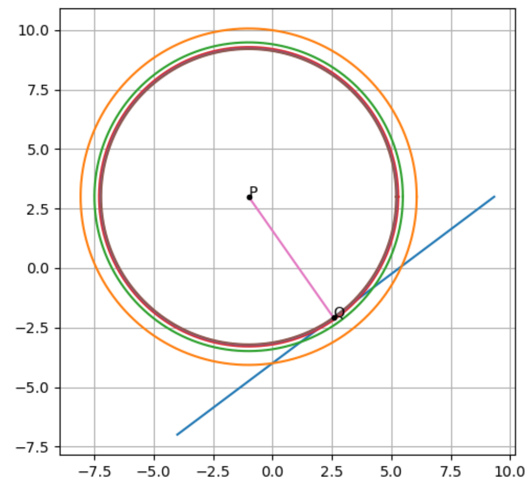

Fig. 2: Constrained gradient descent to find optimal $\mathbf{x_m}$.