An Application of Machine Learning to Grading

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Outline

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4. Which method can be extended to assess based on other factors?

Resources

Marks datasheet and relevant Python codes can be found here.

1. Marks of students: `marks.xlsx`
2. Python code using Gaussian method: `grades_norm.py`
3. Python code using K-means method: `grades.py`

Data Visualization

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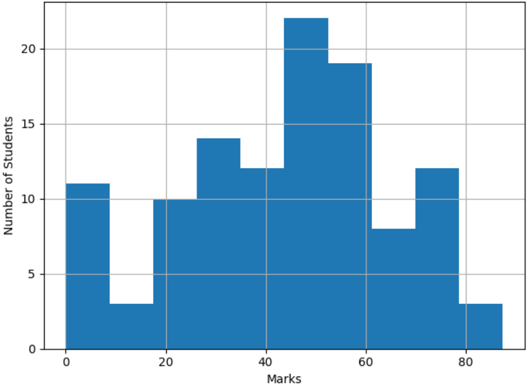

Figure: Histogram showing distribution of marks of the students.

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Note that the sample measures are **unbiased estimators** of their corresponding population measures.

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Statistical Note

If the population is large, computing population parameters directly is cumbersome. In this case, use the sample parameters to calculate the Z-score.

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3. The marks were scaled relative to the highest scoring student.

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Interval	Grade
$(-\infty$, -3]	F
(-3, -2]	D
(-2, 1]	C
(-1, 0]	B-
(0, 1]	B
(1, 2]	A-
(2, 3]	A
$(3,\infty)$	A+

Table: Grading scheme used for calculation of Z-scores

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3. It is an **EM** algorithm (explained ahead).

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3. We are required to find $\{\boldsymbol{\mu}_k\}_{k=1}^K$ such that (8) is *minimized*.

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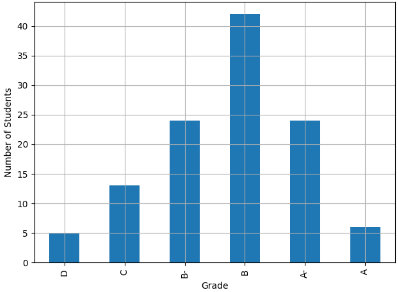
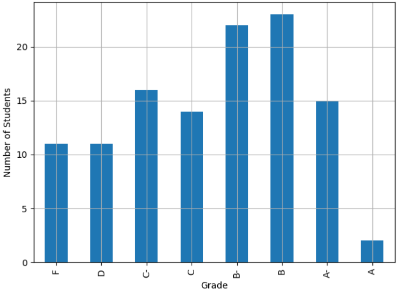


(a) Standard Normal Distribution



(b) K-Means Algorithm

Figure: Comparison of grading distributions using both algorithms.

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2. Grading on a Gaussian curve is faster for larger datasets, and both algorithms would have very little difference.
3. The K-Means algorithm gives a better idea of the performance of the class, especially when it is skew.
4. The K-Means algorithm can be extended to involve other factors such as attendance, prerequisites completed, and so on.