Conic Assignment

Gautam Singh

***Abstract*—This document contains the solution to Question 12 of Exercise 2 in Chapter 11 of the class 11 NCERT textbook.**

1) Find the equation of the parabola with vertex

$$\mathbf{P} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1}$$

and passing through the point

$$\mathbf{Q} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \tag{2}$$

and symmetric to the y-axis.

Solution: Let the equation of the conic with focus $\mathbf{F}$, directrix $\mathbf{n}^\top \mathbf{x} = c$ and eccentricity e be

$$\mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{3}$$

where

$$\mathbf{V} \triangleq \|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n}\mathbf{n}^\top \tag{4}$$

$$\mathbf{u} \triangleq ce^2\mathbf{n} - \|\mathbf{n}\|^2 \mathbf{F} \tag{5}$$

$$f \triangleq \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2 e^2 \tag{6}$$

Since the conic is a parabola symmetric to the y-axis, we have

$$\mathbf{n} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \; e = 1 \tag{7}$$

and also that $\mathbf{F}$ lies on the y-axis. From (4),

$$\mathbf{V} = \mathbf{I} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{8}$$

$$\mathbf{u} = c\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \mathbf{F} \tag{9}$$

$$f = \|\mathbf{F}\|^2 - c^2 \tag{10}$$

Putting $\mathbf{x} = \mathbf{P}$ in (3) gives $f = 0$, thus

$$\|\mathbf{F}\|^2 = c^2 \tag{11}$$

Putting $\mathbf{x} = \mathbf{Q}$ in (3) and noting that $f = 0$, we get

$$\begin{pmatrix} 5 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 5 \\ 2 \end{pmatrix} + 2\left(c\begin{pmatrix} 0 & 1 \end{pmatrix} - \mathbf{F}^\top\right)\begin{pmatrix} 5 \\ 2 \end{pmatrix} = 0 \tag{12}$$

$$\implies 25 + 4c - 2\mathbf{F}^\top \begin{pmatrix} 5 \\ 2 \end{pmatrix} = 0 \tag{13}$$

$$\implies \mathbf{F}^\top \begin{pmatrix} 5 \\ 2 \end{pmatrix} = \frac{25}{2} + 2c \tag{14}$$

$$\implies \mathbf{F} = \begin{pmatrix} 0 \\ \frac{25}{4} + c \end{pmatrix} \tag{15}$$

since $\mathbf{F}$ lies on the y-axis as remarked before. Using (11),

$$\frac{25}{4} + c \pm c = 0 \tag{16}$$

$$\implies c = -\frac{25}{8} \tag{17}$$

Thus,

$$\mathbf{F} = \begin{pmatrix} 0 \\ \frac{25}{8} \end{pmatrix} \tag{18}$$

Substituting into (5),

$$\mathbf{u} = -\frac{25}{8}\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ \frac{25}{8} \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{25}{4} \end{pmatrix} \tag{19}$$

And the equation of the conic is given by

$$\mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} - \frac{25}{2}\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{20}$$

The conic is plotted in Fig. 1 using the Python code `codes/conic.py`.

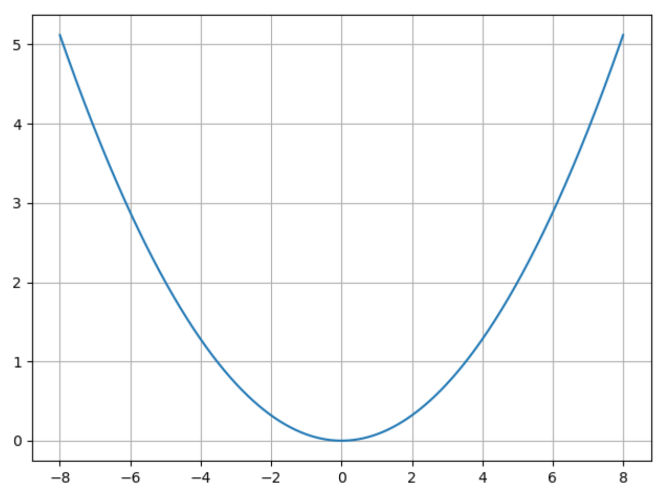

Fig. 1: Locus of the required parabola.