Semidefinite Programming Assignment

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Abstract—**This document contains the solution to Question 21 of Exercise 6 in Chapter 6 of the class 12 NCERT textbook using semidefinite programming.**

1) The line

$$y = mx + 1 \tag{1}$$

is a tangent to the curve

$$y^2 = 4x \tag{2}$$

if the value of m is

a) 1
b) 2
c) 3
d) $\frac{1}{2}$

Solution: Rewriting (1) and (2) in standard forms, we get,

$$\mathbf{n} = \begin{pmatrix} m \\ -1 \end{pmatrix}, \; c = -1 \tag{3}$$

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \; \mathbf{u} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}, \; f = 0 \tag{4}$$

Hence, the equivalent optimization problem is

$$\min_{\mathbf{x}} \frac{\left(\mathbf{n}^\top \mathbf{x} - c\right)^2}{\|\mathbf{n}\|^2} \tag{5}$$

$$\text{s.t. } \mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{6}$$

Using

$$\mathbf{y} \triangleq \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} \tag{7}$$

In quadratic form, (5) becomes

$$\min_{\mathbf{x}} \mathbf{x}^\top \left(\frac{\mathbf{n}\mathbf{n}^\top}{\|\mathbf{n}\|^2}\right) \mathbf{x} + 2\left(-\frac{c\mathbf{n}^\top}{\|\mathbf{n}\|^2}\right) \mathbf{x} + \frac{c^2}{\|\mathbf{n}\|^2} \tag{8}$$

We rewrite (5) and (6) in matrix form thus,

$$\min_{\mathbf{y}} \mathbf{y}^\top \mathbf{C} \mathbf{y} \tag{9}$$

$$\text{s.t. } \mathbf{y}^\top \mathbf{A} \mathbf{y} = 0 \tag{10}$$

where

$$\mathbf{A} \triangleq \begin{pmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^\top & f \end{pmatrix} \tag{11}$$

$$\mathbf{C} \triangleq \frac{1}{\|\mathbf{n}\|^2} \begin{pmatrix} \mathbf{n}\mathbf{n}^\top & -c\mathbf{n} \\ -c\mathbf{n}^\top & c^2 \end{pmatrix} \tag{12}$$

We solve this problem using semidefinite programming. Applying semidefinite relaxation (SDR), the optimization problem becomes

$$\min_{\mathbf{Y}} \operatorname{tr}\left(\mathbf{C}\mathbf{Y}\right) \tag{13}$$

$$\text{s.t. } \operatorname{tr}\left(\mathbf{A}\mathbf{Y}\right) = 0 \tag{14}$$

$$\mathbf{A} \succeq 0 \tag{15}$$

where

$$\mathbf{Y} \triangleq \left(yy\right)^\top \tag{16}$$

The Python code `codes/sdp.py` finds the value of m that has the least absolute cost by plotting Fig. 1. We see that the optimal slope

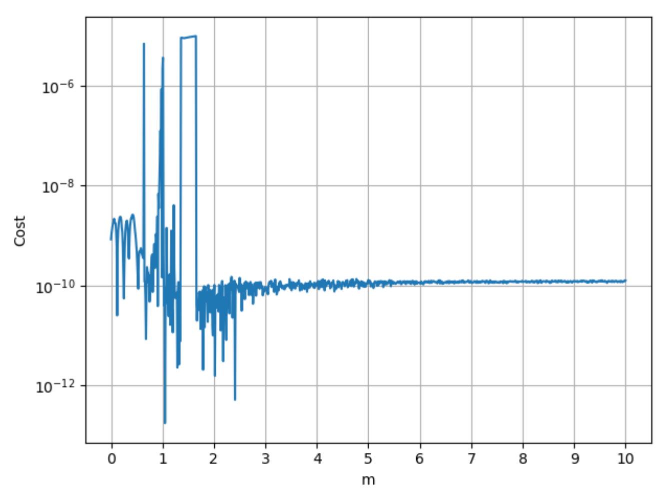

Fig. 1: Cost as a function of m.

is $m = 1$. Hence, **a)** is the correct answer.