Conic Assignment

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Abstract—This document contains the solution to Question 12 of Exercise 2 in Chapter 11 of the class 11 NCERT textbook.

1) Find the equation of the parabola with vertex

$$\mathbf{P} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (1)$$

and passing through the point

$$\mathbf{Q} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \quad (2)$$

and symmetric to the *y*-axis.
Solution: Let the equation of the conic with focus $\mathbf{F}$, directrix $\mathbf{n}^\top\mathbf{x} = c$ and eccentricity e be

$$\mathbf{x}^\top\mathbf{V}\mathbf{x} + 2\mathbf{u}^\top\mathbf{x} + f = 0 \quad (3)$$

where

$$\mathbf{V} \triangleq \|\mathbf{n}\|^2\mathbf{I} - e^2\mathbf{n}\mathbf{n}^\top \quad (4)$$
$$\mathbf{u} \triangleq ce^2\mathbf{n} - \|\mathbf{n}\|^2\mathbf{F} \quad (5)$$
$$f \triangleq \|\mathbf{n}\|^2\|\mathbf{F}\|^2 - c^2e^2 \quad (6)$$

Since the conic is a parabola symmetric to the *y*-axis, we have

$$\mathbf{n} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ e = 1 \quad (7)$$

and also that $\mathbf{F}$ lies on the *y*-axis. From (4),

$$\mathbf{V} = \mathbf{I} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (8)$$

$$\mathbf{u} = c\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \mathbf{F} \quad (9)$$

$$f = \|\mathbf{F}\|^2 - c^2 \quad (10)$$

Putting $\mathbf{x} = \mathbf{P}$ in (3) gives $f = 0$, thus

$$\|\mathbf{F}\|^2 = c^2 \quad (11)$$

Putting $\mathbf{x} = \mathbf{Q}$ in (3), we get

$$\mathbf{F}^\top\begin{pmatrix} 5 \\ 2 \end{pmatrix} = \frac{25}{2} - 2c \quad (12)$$

$$\implies \mathbf{F} = \begin{pmatrix} 0 \\ \frac{25}{4} - c \end{pmatrix} \quad (13)$$

since $\mathbf{F}$ lies on the *y*-axis as remarked before. Using (11),

$$\frac{25}{4} - c \pm c = 0 \quad (14)$$

$$\implies c = \frac{25}{8} \quad (15)$$

Thus,

$$\mathbf{F} = \begin{pmatrix} 0 \\ \frac{25}{8} \end{pmatrix} \quad (16)$$

Substituting into (5) and (6),

$$\mathbf{u} = \frac{25}{8}\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ \frac{25}{8} \end{pmatrix} = \mathbf{O} \quad (17)$$

$$f = \|\mathbf{F}\|^2 - c^2 = 0 \quad (18)$$

And the equation of the conic is given by

$$\mathbf{x}^\top\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\mathbf{x} = 0 \quad (19)$$
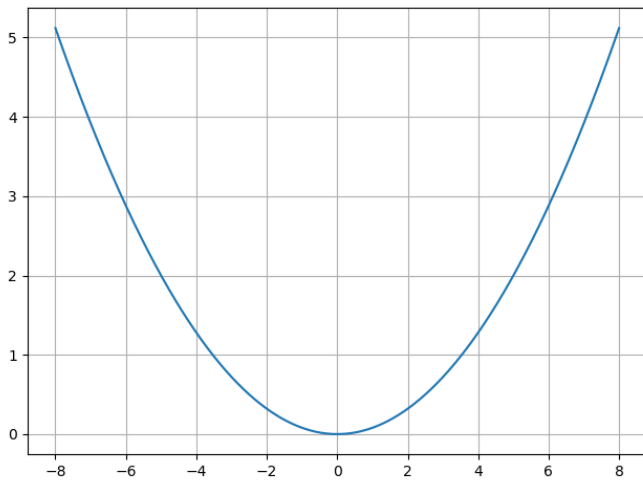
The conic is plotted in Fig. 1 using the Python code `codes/conic.py`.

Fig. 1: Locus of the required parabola.