

Circle Assignment

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Abstract—This document contains the solution to Question 10 of Exercise 6 in Chapter 10 of the class 9 NCERT textbook.

- 1) In any $\triangle ABC$, if the angle bisector of $\angle A$ and perpendicular bisector of BC intersect, prove that they intersect on the circumcircle of $\triangle ABC$.

Solution: Let the position vectors of the points be

$$\mathbf{A} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} \cos \beta \\ \sin \beta \end{pmatrix}, \mathbf{C} = \begin{pmatrix} \cos \gamma \\ \sin \gamma \end{pmatrix} \quad (1)$$

Then, the equation of the perpendicular bisector of BC is given by

$$\|\mathbf{x} - \mathbf{B}\|^2 = \|\mathbf{x} - \mathbf{C}\|^2 \quad (2)$$

$$\Rightarrow (\mathbf{B} - \mathbf{C})^\top \mathbf{x} = 0 \quad (3)$$

and the equation of the angle bisector of $\angle A$ is given by

$$\frac{(\mathbf{B} - \mathbf{A})^\top (\mathbf{x} - \mathbf{A})}{\|\mathbf{B} - \mathbf{A}\|} = \frac{(\mathbf{C} - \mathbf{A})^\top (\mathbf{x} - \mathbf{A})}{\|\mathbf{C} - \mathbf{A}\|} \quad (4)$$

$$\left(\frac{\mathbf{B} - \mathbf{A}}{\|\mathbf{B} - \mathbf{A}\|} - \frac{\mathbf{C} - \mathbf{A}}{\|\mathbf{C} - \mathbf{A}\|} \right)^\top (\mathbf{x} - \mathbf{A}) = 0 \quad (5)$$

Note that from (1),

$$\frac{\mathbf{B} - \mathbf{A}}{\|\mathbf{B} - \mathbf{A}\|} = \frac{1}{\sqrt{2(1 - \cos \beta)}} \begin{pmatrix} \cos \beta - 1 \\ \sin \beta \end{pmatrix} \quad (6)$$

$$= \begin{pmatrix} -\sin \frac{\beta}{2} \\ \cos \frac{\beta}{2} \end{pmatrix} \quad (7)$$

Therefore, using (1) in (3) and (7) in (5),

$$\begin{pmatrix} \cos \beta - \cos \gamma \\ \sin \beta - \sin \gamma \end{pmatrix}^\top \mathbf{x} = 0 \quad (8)$$

$$\begin{pmatrix} \sin \frac{\gamma}{2} - \sin \frac{\beta}{2} \\ \cos \frac{\beta}{2} - \cos \frac{\gamma}{2} \end{pmatrix}^\top \mathbf{x} = \sin \frac{\gamma}{2} - \sin \frac{\beta}{2} \quad (9)$$

Using (8) and (9), we form the matrix equation

$$\begin{pmatrix} \cos \beta - \cos \gamma & \sin \beta - \sin \gamma \\ \sin \frac{\gamma}{2} - \sin \frac{\beta}{2} & \cos \frac{\beta}{2} - \cos \frac{\gamma}{2} \end{pmatrix} \mathbf{x} = \begin{pmatrix} 0 \\ \sin \frac{\gamma}{2} - \sin \frac{\beta}{2} \end{pmatrix} \quad (10)$$

Solving (10) using the augmented matrix, and letting $\theta \triangleq \frac{\beta + \gamma}{2}$,

$$\begin{pmatrix} \cos \beta - \cos \gamma & \sin \beta - \sin \gamma & 0 \\ \sin \frac{\gamma}{2} - \sin \frac{\beta}{2} & \cos \frac{\beta}{2} - \cos \frac{\gamma}{2} & \sin \frac{\gamma}{2} - \sin \frac{\beta}{2} \end{pmatrix} \quad (11)$$

$$\begin{array}{l} R_1 \leftarrow \frac{R_1}{\cos \beta - \cos \gamma} \\ R_2 \leftarrow \frac{R_2}{\sin \frac{\gamma}{2} - \sin \frac{\beta}{2}} \end{array} \begin{pmatrix} 1 & -\cot \theta & 0 \\ 1 & \tan \frac{\theta}{2} & 1 \end{pmatrix} \quad (12)$$

$$R_2 \leftarrow R_2 - R_1 \begin{pmatrix} 1 & -\cot \theta & 0 \\ 0 & \tan \frac{\theta}{2} + \cot \theta & 1 \end{pmatrix} \quad (13)$$

$$= \begin{pmatrix} 1 & -\cot \theta & 0 \\ 0 & \csc \theta & 1 \end{pmatrix} \quad (14)$$

$$R_1 \leftarrow R_1 + R_2 \cos \theta \begin{pmatrix} 1 & 0 & \cos \theta \\ 0 & \csc \theta & 1 \end{pmatrix} \quad (15)$$

$$R_2 \leftarrow R_2 \sin \theta \begin{pmatrix} 1 & 0 & \cos \theta \\ 0 & 1 & \sin \theta \end{pmatrix} \quad (16)$$

Thus, the intersection of the lines in (8) and (9) is

$$\mathbf{D} \triangleq \begin{pmatrix} \cos \frac{\beta + \gamma}{2} \\ \sin \frac{\beta + \gamma}{2} \end{pmatrix} \quad (17)$$

Hence, it is clear from (17) that \mathbf{D} lies on the circumcircle of $\triangle ABC$, as required.

The situation is illustrated in Fig. 1 plotted by the Python code `codes/bisector.py`. The parameters used in the construction are shown in Table I.

Parameter	Value
r	1
β	100°
γ	200°

TABLE I: Parameters used in the construction of Fig. 1.

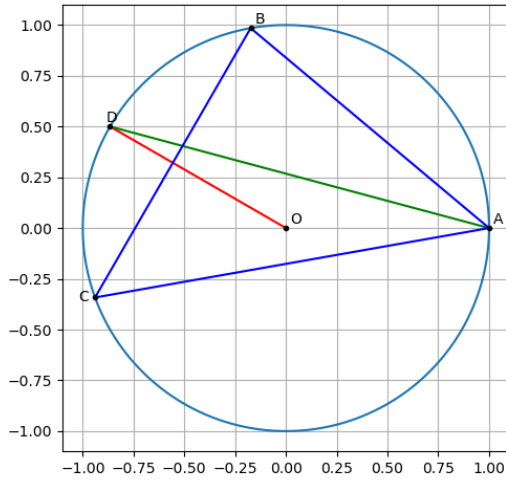


Fig. 1: The bisector of $\angle A$ and of BC meet on the circumcircle at D .