# Line Assignment

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***Abstract*—This document contains a general solution to Question 16 of Exercise 2 in Chapter 11 of the class 12 NCERT textbook.**

1) Find the shortest distance between the lines whose vector equations are

$$L_1 : \mathbf{x} = \mathbf{x_1} + \lambda_1 \mathbf{m_1} \tag{1}$$
$$L_2 : \mathbf{x} = \mathbf{x_2} + \lambda_2 \mathbf{m_2} \tag{2}$$

**Solution:** Let $\mathbf{A}$ and $\mathbf{B}$ be points on lines $L_1$ and $L_2$ respectively such that $AB$ is normal to both lines. Define

$$\mathbf{M} \triangleq \begin{pmatrix} \mathbf{m_1} & \mathbf{m_2} \end{pmatrix} \tag{3}$$
$$\boldsymbol{\lambda} \triangleq \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} \tag{4}$$
$$\mathbf{x} \triangleq \mathbf{x_2} - \mathbf{x_1} \tag{5}$$

Then, we have the following equations:

$$\mathbf{A} = \mathbf{x_1} + \lambda_1 \mathbf{m_1} \tag{6}$$
$$\mathbf{B} = \mathbf{x_2} + \lambda_2 \mathbf{m_2} \tag{7}$$

From (6) and (7), define the real-valued function $f$ as

$$f(\boldsymbol{\lambda}) \triangleq \|\mathbf{A} - \mathbf{B}\| \tag{8}$$
$$= \|\mathbf{M}\boldsymbol{\lambda} - \mathbf{x}\| \tag{9}$$
$$= \sqrt{(\mathbf{M}\boldsymbol{\lambda} - \mathbf{x})^\top (\mathbf{M}\boldsymbol{\lambda} - \mathbf{x})} \tag{10}$$

Note that the norm function obeys the triangle inequality, which will be used later. To prove this, note that for vectors $\mathbf{a}$ and $\mathbf{b}$,

$$\mathbf{a}^\top \mathbf{b} \le \|\mathbf{a}\| \|\mathbf{b}\| \tag{11}$$
$$\|\mathbf{a}\|^2 + 2\mathbf{a}^\top \mathbf{b} + \|\mathbf{b}\|^2 \le \|\mathbf{a}\|^2 + 2\|\mathbf{a}\|\|\mathbf{b}\| + \|\mathbf{b}\|^2 \tag{12}$$
$$\|\mathbf{a} + \mathbf{b}\|^2 \le (\|\mathbf{a}\| + \|\mathbf{b}\|)^2 \tag{13}$$
$$\|\mathbf{a} + \mathbf{b}\| \le \|\mathbf{a}\| + \|\mathbf{b}\| \tag{14}$$

We now show that $f$ is convex. Indeed, consider $\boldsymbol{\lambda_1}$ and $\boldsymbol{\lambda_2}$ and let $0 \le \mu \le 1$. Then,

$$f(\mu\boldsymbol{\lambda_1} + (1-\mu)\boldsymbol{\lambda_2}) \tag{15}$$
$$= \|\mathbf{M}(\mu\boldsymbol{\lambda_1} + (1-\mu)\boldsymbol{\lambda_2}) - \mathbf{x}\| \tag{16}$$
$$= \|\mu(\mathbf{M}\boldsymbol{\lambda_1} - \mathbf{x}) + (1-\mu)(\mathbf{M}\boldsymbol{\lambda_2} - \mathbf{x})\| \tag{17}$$
$$\le \mu\|\mathbf{M}\boldsymbol{\lambda_1} - \mathbf{x}\| + (1-\mu)\|\mathbf{M}\boldsymbol{\lambda_2} - \mathbf{x}\| \tag{18}$$

Where (18) follows from (14). We need to minimize $f$ as a function of $\boldsymbol{\lambda}$. Thus, differentiating (10),

$$\frac{df(\boldsymbol{\lambda})}{d\boldsymbol{\lambda}} = \frac{\mathbf{M}^\top(\mathbf{M}\boldsymbol{\lambda} - \mathbf{x})}{\|\mathbf{M}\boldsymbol{\lambda} - \mathbf{x}\|} \tag{19}$$

using the chain rule. Setting (19) to zero gives

$$\mathbf{M}^\top\mathbf{M}\boldsymbol{\lambda} = \mathbf{M}^\top\mathbf{x} \tag{20}$$

We have the following cases:

a) There exists a $\boldsymbol{\lambda}$ satisfying

$$\mathbf{M}\boldsymbol{\lambda} = \mathbf{x} \tag{21}$$
$$\implies \lambda_1\mathbf{m_1} - \lambda_2\mathbf{m_2} = \mathbf{x_2} - \mathbf{x_1} \tag{22}$$
$$\implies \mathbf{x_1} + \lambda_1\mathbf{m_1} = \mathbf{x_2} + \lambda_2\mathbf{m_2} \tag{23}$$

Thus, both lines intersect at a point and the shortest distance between them is 0. To check for the existence of such a $\boldsymbol{\lambda}$, we can bring the augmented matrix $\begin{pmatrix} \mathbf{M} & \mathbf{x} \end{pmatrix}$ into row-reduced echelon form and check whether there is a pivot in the last column.

b) $\mathbf{M}^\top\mathbf{M}$ is singular. Since $\mathbf{M}^\top\mathbf{M}$ is a sqaure matrix of order 2, its rank must be 1. Further,

$$\det(\mathbf{M}^\top\mathbf{M}) = \begin{vmatrix} \mathbf{m_1}^\top\mathbf{m_1} & \mathbf{m_1}^\top\mathbf{m_2} \\ \mathbf{m_1}^\top\mathbf{m_2} & \mathbf{m_2}^\top\mathbf{m_2} \end{vmatrix} \tag{24}$$
$$= (\|\mathbf{m_1}\| . \|\mathbf{m_2}\|)^2 - (\mathbf{m_1}^\top\mathbf{m_2})^2 \tag{25}$$

Thus, equating the determinant to zero gives

$$\|\mathbf{m_1}\| . \|\mathbf{m_2}\| = \left|\mathbf{m_1}^\top\mathbf{m_2}\right| \tag{26}$$

which implies that both lines are parallel to each other. Setting $\mathbf{m_2} = k\mathbf{m_1}, k \in \mathbb{R}\setminus\{0\}$, we obtain one equation from (20).

$$\mathbf{m_1}^\top \mathbf{m_1} \left(\lambda_1 - k\lambda_2\right) = \mathbf{m_1}^\top \mathbf{x} \tag{27}$$

$$\implies \lambda_1 - k\lambda_2 = \frac{\mathbf{m_1}^\top \mathbf{x}}{\|\mathbf{m_1}\|^2} \tag{28}$$

Therefore, the required shortest distance is

$$\|\mathbf{A} - \mathbf{B}\| = \left\| \frac{\mathbf{m_1}^\top \mathbf{x} \mathbf{m_1}}{\|\mathbf{m_1}\|^2} - \mathbf{x} \right\| \tag{29}$$

c) $\mathbf{M}^\top \mathbf{M}$ is nonsinglar. This implies that the lines are skew. From (20),

$$\boldsymbol{\lambda} = \left(\mathbf{M}^\top \mathbf{M}\right)^{-1} \mathbf{M}^\top \mathbf{x} \tag{30}$$

and therefore, the shortest distance is

$$\|\mathbf{A} - \mathbf{B}\| = \left\| \left( \mathbf{M} \left(\mathbf{M}^\top \mathbf{M}\right)^{-1} \mathbf{M}^\top - \mathbf{I_n} \right) \mathbf{x} \right\| \tag{31}$$

where $\mathbf{I_n}$ is the identity matrix of order $n$.