Circle Assignment

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***Abstract*—This document contains the solution to Question 12 of Exercise 5 in Chapter 10 of the class 9 NCERT textbook.**

1) Prove that a cyclic paralellogram is a rectangle.
Solution: Consider the points $\mathbf{P_i}$, $1 \le i \le 4$ on the unit circle, where

$$\mathbf{P_i} \triangleq \begin{pmatrix} \cos\theta_i \\ \sin\theta_i \end{pmatrix} \tag{1}$$

Now, since $P_1P_2P_3P_4$ form a parallelogram,

$$\|\mathbf{P_1} - \mathbf{P_2}\| = \|\mathbf{P_3} - \mathbf{P_4}\| \tag{2}$$
$$\implies \mathbf{P_1}^\top\mathbf{P_2} = \mathbf{P_3}^\top\mathbf{P_4} \tag{3}$$

and

$$\|\mathbf{P_1} - \mathbf{P_4}\| = \|\mathbf{P_2} - \mathbf{P_3}\| \tag{4}$$
$$\implies \mathbf{P_1}^\top\mathbf{P_4} = \mathbf{P_2}^\top\mathbf{P_3} \tag{5}$$

Also, it is given that the parallelogram is cyclic. So,

$$\angle P_2P_1P_4 + \angle P_2P_3P_4 = \pi \tag{6}$$
$$\implies \angle P_2P_1P_4 = \pi - \angle P_2P_3P_4 \tag{7}$$
$$\implies \cos\angle P_2P_1P_4 = -\cos\angle P_2P_3P_4 \tag{8}$$
$$\implies \cos\angle P_2P_1P_4 + \cos\angle P_2P_3P_4 = 0 \tag{9}$$

Using (2) and (4) and also the fact that

$$\cos\theta = \frac{\mathbf{a}^\top\mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|} \tag{10}$$

where θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$, (9) implies

$$(\mathbf{P_4} - \mathbf{P_1})^\top(\mathbf{P_1} - \mathbf{P_2}) + (\mathbf{P_4} - \mathbf{P_3})^\top(\mathbf{P_3} - \mathbf{P_2}) = 0 \tag{11}$$

Using (3) and (5) in (11), and noting that $\mathbf{P_i}^\top\mathbf{P_i} = 1$, $1 \le i \le 4$,

$$\mathbf{P_4}^\top\mathbf{P_1} - \mathbf{P_4}^\top\mathbf{P_2} + \mathbf{P_1}^\top\mathbf{P_2} - 1 = 0 \tag{12}$$
$$\mathbf{P_4}^\top\mathbf{P_1} - \mathbf{P_4}^\top\mathbf{P_2} + \mathbf{P_1}^\top\mathbf{P_2} - \mathbf{P_1}^\top\mathbf{P_1} = 0 \tag{13}$$
$$(\mathbf{P_4} - \mathbf{P_1})^\top(\mathbf{P_1} - \mathbf{P_2}) = 0 \tag{14}$$

Hence $P_1P_2 \perp P_1P_4$, and thus, the parallelogram is indeed a rectangle. The situation is demonstrated in Fig. 1, plotted by the Python code `codes/circle.py`.

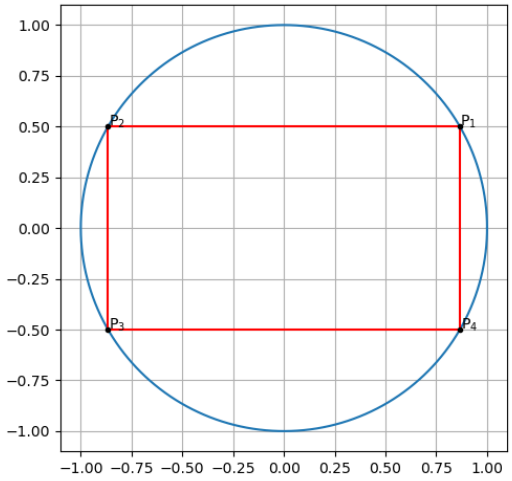

Fig. 1: $P_1P_2P_3P_4$ is a rectangle.