Application of Integrals Assignment

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***Abstract*—This document contains the solution to Question 18 of Exercise 3 in Chapter 8 of the class 12 NCERT textbook.**

1) The area of the circle
$$x^2 + y^2 = 16 \tag{1}$$
exterior to the parabola
$$y^2 = 6x \tag{2}$$
is
a) $\frac{4}{3}\left(4\pi - \sqrt{3}\right)$
b) $\frac{4}{3}\left(4\pi + \sqrt{3}\right)$
c) $\frac{4}{3}\left(8\pi - \sqrt{3}\right)$
d) $\frac{4}{3}\left(8\pi + \sqrt{3}\right)$

Solution: We convert (1) and (2) into matrix form to find their points of intersection.
$$\mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - 16 = 0 \tag{3}$$
$$\mathbf{x}^\top \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + 2\begin{pmatrix} -3 & 0 \end{pmatrix} \mathbf{x} = 0 \tag{4}$$
Adding (3) to μ times (4), we get the locus of the intersection of the two conics.
$$\mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & \mu + 1 \end{pmatrix} \mathbf{x} + 2\begin{pmatrix} -3\mu & 0 \end{pmatrix} \mathbf{x} - 16 = 0 \tag{5}$$
For (5) to represent a pair of straight lines,
$$\begin{vmatrix} 1 & 0 & -3\mu \\ 0 & \mu + 1 & 0 \\ -3\mu & 0 & -16 \end{vmatrix} = 0 \tag{6}$$
$$\implies (\mu + 1)\left(9\mu^2 + 16\right) = 0 \tag{7}$$
$$\implies \mu = -1 \tag{8}$$
where (8) follows since $9\mu^2 + 16 \geq 0$. Hence, the straight lines are represented by
$$\mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + 2\begin{pmatrix} 3 & 0 \end{pmatrix} \mathbf{x} - 16 = 0 \tag{9}$$
$$\implies \mathbf{x}^\top \mathbf{e_1} \mathbf{e_1}^\top \mathbf{x} + 6\mathbf{e_1}^\top \mathbf{x} - 16 = 0 \tag{10}$$
$$\implies \left(\mathbf{e_1}^\top \mathbf{x}\right)^2 + 6\mathbf{e_1}^\top \mathbf{x} - 16 = 0 \tag{11}$$
$$\implies \left(\mathbf{e_1}^\top \mathbf{x} - 2\right)\left(\mathbf{e_1}^\top \mathbf{x} + 8\right) = 0 \tag{12}$$
Since $\mathbf{x}$ lies on (1), it follows that
$$-8 = \mathbf{e_1}^\top \mathbf{x} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 4\cos\theta \\ 4\sin\theta \end{pmatrix} = 4\cos\theta \geq -4 \tag{13}$$
$$\implies \mathbf{e_1}^\top \mathbf{x} = 2 \tag{14}$$
Substituting in (4),
$$\mathbf{x}^\top \mathbf{e_2} \mathbf{e_2}^\top \mathbf{x} = 6\mathbf{e_1}^\top \mathbf{x} = 12 \tag{15}$$
$$\implies \left(\mathbf{e_2}^\top \mathbf{x}\right)^2 = 12 \tag{16}$$
$$\implies \mathbf{e_2}^\top \mathbf{x} = \pm 2\sqrt{3} \tag{17}$$
Combining (14) and (17), the conics intersect at
$$\mathbf{x} = \begin{pmatrix} \mathbf{e_1}^\top \\ \mathbf{e_2}^\top \end{pmatrix} \mathbf{x} = \begin{pmatrix} 2 \\ \pm 2\sqrt{3} \end{pmatrix} \tag{18}$$
Hence, the area interior to the circle and parabola is
$$A = \int_{-2\sqrt{3}}^{2\sqrt{3}} \sqrt{16 - y^2} - \frac{y^2}{6}\, dy \tag{19}$$
$$= \frac{y\sqrt{16 - y^2}}{2} + \frac{16}{2}\sin^{-1}\left(\frac{y}{4}\right) - \frac{y^3}{18}\Bigg|_{-2\sqrt{3}}^{2\sqrt{3}} \tag{20}$$
$$= 4\sqrt{3} + \frac{16\pi}{3} - \frac{8\sqrt{3}}{3} \tag{21}$$
$$= \frac{4\sqrt{3} + 16\pi}{3} \tag{22}$$
Thus, the required exterior area is (where $r = 4$

is the radius of (1))

$$A' = \pi r^2 - A \tag{23}$$

$$= 16\pi - \frac{4\sqrt{3} + 16\pi}{3} \tag{24}$$

$$= \frac{4}{3}\left(8\pi + \sqrt{3}\right) \tag{25}$$

Hence, the correct answer is option **d)**.
The situation is demonstrated in Fig. 1, plotted by the Python code `codes/parab_circ.py`

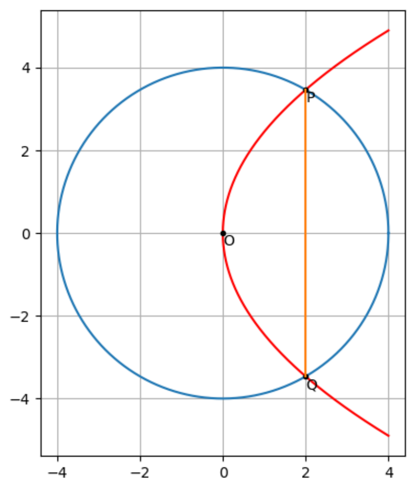

Fig. 1: The conics meet at points P and Q.