Conic Assignment

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***Abstract*—This document contains the solution to Question 12 of Exercise 2 in Chapter 11 of the class 11 NCERT textbook.**

1) Find the equation of the parabola with vertex
$$\mathbf{P} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1}$$
and passing through the point
$$\mathbf{Q} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \tag{2}$$
and symmetric to the y-axis.
Solution: Let the equation of the conic with focus $\mathbf{F}$, directrix $\mathbf{n}^\top \mathbf{x} = c$ and eccentricity e be
$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{3}$$
where
$$\mathbf{V} \triangleq \|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n}\mathbf{n}^\top \tag{4}$$
$$\mathbf{u} \triangleq ce^2 \mathbf{n} - \|\mathbf{n}\|^2 \mathbf{F} \tag{5}$$
$$f \triangleq \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2 e^2 \tag{6}$$
Since the conic is a parabola symmetric to the y-axis, we have
$$\mathbf{n} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ e = 1 \tag{7}$$
and also that $\mathbf{F}$ lies on the y-axis. From (4),
$$\mathbf{V} = \mathbf{I} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{8}$$
$$\mathbf{u} = c\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \mathbf{F} \tag{9}$$
$$f = \|\mathbf{F}\|^2 - c^2 \tag{10}$$
Putting $\mathbf{x} = \mathbf{P}$ in (3) gives $f = 0$, thus
$$\|\mathbf{F}\|^2 = c^2 \tag{11}$$
Putting $\mathbf{x} = \mathbf{Q}$ in (3), using (8), (9) and noting that $f = 0$, we get
$$\begin{pmatrix} 5 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 5 \\ 2 \end{pmatrix} + 2\left(c\begin{pmatrix} 0 & 1 \end{pmatrix} - \mathbf{F}^\top\right)\begin{pmatrix} 5 \\ 2 \end{pmatrix} = 0 \tag{12}$$
$$\implies 25 + 4c - 2\mathbf{F}^\top \begin{pmatrix} 5 \\ 2 \end{pmatrix} = 0 \tag{13}$$
$$\implies \mathbf{F}^\top \begin{pmatrix} 5 \\ 2 \end{pmatrix} = \frac{25}{2} + 2c \tag{14}$$
$$\implies \mathbf{F} = \begin{pmatrix} 0 \\ \frac{25}{4} + c \end{pmatrix} \tag{15}$$
since $\mathbf{F}$ lies on the y-axis as remarked before. Using (11),
$$\frac{25}{4} + c \pm c = 0 \tag{16}$$
$$\implies c = -\frac{25}{8} \tag{17}$$
Thus,
$$\mathbf{F} = \begin{pmatrix} 0 \\ \frac{25}{8} \end{pmatrix} \tag{18}$$
Substituting into (5),
$$\mathbf{u} = -\frac{25}{8}\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ \frac{25}{8} \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{25}{4} \end{pmatrix} \tag{19}$$
And the equation of the conic is given by
$$\mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\mathbf{x} - \frac{25}{2}\begin{pmatrix} 0 & 1 \end{pmatrix}\mathbf{x} = 0 \tag{20}$$
The conic is plotted in Fig. 1 using the Python code `codes/conic.py`.

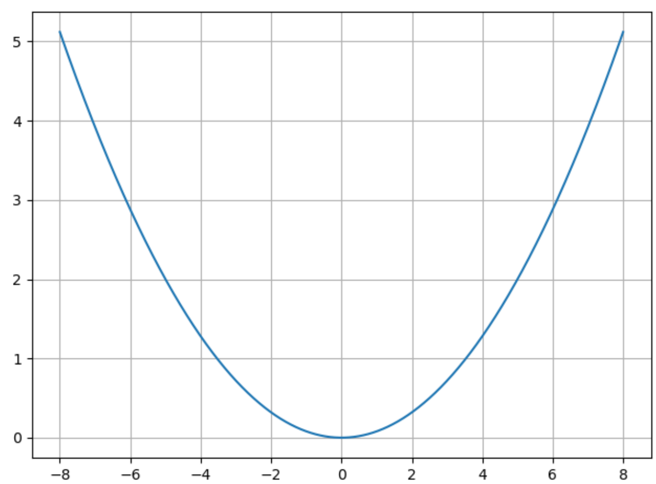

Fig. 1: Locus of the required parabola.