

Conic Assignment

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Abstract—This document contains the solution to Question 20 of Exercise 3 in Chapter 11 of the class 11 NCERT textbook.

- 1) Find the equation of the ellipse whose major axis is the x -axis and passes through the points

$$\mathbf{P} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}, \quad \mathbf{Q} = \begin{pmatrix} 6 \\ 2 \end{pmatrix} \quad (1)$$

Solution: Let the equation of the conic with focus \mathbf{F} , directrix $\mathbf{n}^\top \mathbf{x} = c$ and eccentricity e be

$$\mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \quad (2)$$

where

$$\mathbf{V} \triangleq \|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n} \mathbf{n}^\top \quad (3)$$

$$\mathbf{u} \triangleq ce^2 \mathbf{n} - \|\mathbf{n}\|^2 \mathbf{F} \quad (4)$$

$$f \triangleq \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2 e^2 \quad (5)$$

Since the conic is an ellipse whose major axis is along the x -axis, we have

$$\mathbf{n} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (6)$$

Thus,

$$\mathbf{V} = \begin{pmatrix} 1 - e^2 & 0 \\ 0 & 1 \end{pmatrix} \quad (7)$$

$$\mathbf{u} = ce^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \mathbf{F} \quad (8)$$

$$f = \|\mathbf{F}\|^2 - c^2 e^2 \quad (9)$$

The centre of the conic is the point which bisects all chords passing through it. Suppose that the centre of the conic is \mathbf{c} . Then, if $\mathbf{c} + \mathbf{p}$ lies on the conic, so does $\mathbf{c} - \mathbf{p}$. Substituting these points in (2),

$$(\mathbf{c} + \mathbf{p})^\top \mathbf{V} (\mathbf{c} + \mathbf{p}) + 2\mathbf{u}^\top (\mathbf{c} + \mathbf{p}) + f = 0 \quad (10)$$

$$(\mathbf{c} - \mathbf{p})^\top \mathbf{V} (\mathbf{c} - \mathbf{p}) + 2\mathbf{u}^\top (\mathbf{c} - \mathbf{p}) + f = 0 \quad (11)$$

Subtracting (11) from (10), and noting from (3)

that \mathbf{V} is symmetric, we get

$$\mathbf{c}^\top \mathbf{V} \mathbf{p} + \mathbf{u}^\top \mathbf{p} = 0 \quad (12)$$

$$\implies (\mathbf{V} \mathbf{c} + \mathbf{u})^\top \mathbf{p} = 0 \quad (13)$$

$$\implies \mathbf{V} \mathbf{c} + \mathbf{u} = \mathbf{O} \quad (14)$$

$$\implies \mathbf{c} = -\mathbf{V}^{-1} \mathbf{u} \quad (15)$$

where (14) follows since \mathbf{p} can be arbitrary. Since $\mathbf{V} \neq \mathbf{O}$, it follows from (15) that $\mathbf{u} = \mathbf{O}$. Thus, from (8),

$$\mathbf{F} = \begin{pmatrix} ce^2 \\ 0 \end{pmatrix} \quad (16)$$

and so,

$$f = c^2 e^2 (e^2 - 1) \quad (17)$$

Putting $\mathbf{x} = \mathbf{P}$ in (2) and using (16) and (17),

$$\begin{pmatrix} 4 & 3 \end{pmatrix} \begin{pmatrix} 1 - e^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} + f = 0 \quad (18)$$

$$\implies 16e^2 - f = 25 \quad (19)$$

Putting $\mathbf{x} = \mathbf{Q}$ in (2), we get

$$\begin{pmatrix} 6 & 2 \end{pmatrix} \begin{pmatrix} 1 - e^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 6 \\ 2 \end{pmatrix} + f = 0 \quad (20)$$

$$\implies 36e^2 - f = 40 \quad (21)$$

The equations (19) and (21) can be solved using the augmented matrix as follows:

$$\begin{pmatrix} 16 & -1 & 25 \\ 36 & -1 & 40 \end{pmatrix} \xrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} -20 & 0 & -15 \\ 36 & -1 & 40 \end{pmatrix} \quad (22)$$

$$\xrightarrow{\begin{matrix} R_1 \leftarrow \frac{R_1}{-5} \\ R_2 \leftarrow -R_2 \end{matrix}} \begin{pmatrix} 4 & 0 & 3 \\ -36 & 1 & -40 \end{pmatrix} \quad (23)$$

$$\xrightarrow{R_2 \leftarrow R_2 + 9R_1} \begin{pmatrix} 4 & 0 & 3 \\ 0 & 1 & -13 \end{pmatrix} \quad (24)$$

$$\xrightarrow{R_1 \leftarrow \frac{R_1}{4}} \begin{pmatrix} 1 & 0 & \frac{3}{4} \\ 0 & 1 & -13 \end{pmatrix} \quad (25)$$

$$(26)$$

Thus,

$$e^2 = \frac{3}{4}, \quad f = -13 \quad (27)$$

And the equation of the conic is given by

$$\mathbf{x}^\top \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - 13 = 0 \quad (28)$$

The situation is illustrated in Fig. 1

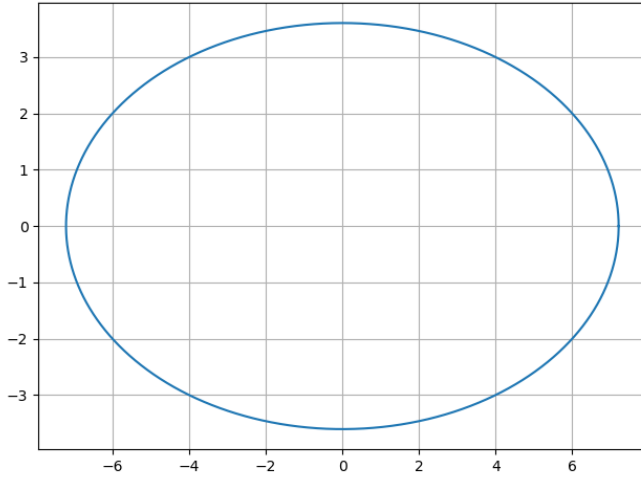


Fig. 1: Locus of the required ellipse.