Conic Assignment

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***Abstract*—This document contains the solution to Question 27 of Exercise 5 in Chapter 6 of the class 12 NCERT textbook.**

1) The point on the curve
$$x^2 = 2y \tag{1}$$
which is nearest to the point $\mathbf{P} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$ is

a) $\begin{pmatrix} 2\sqrt{2} \\ 4 \end{pmatrix}$
b) $\begin{pmatrix} 2\sqrt{2} \\ 0 \end{pmatrix}$
c) $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
d) $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$

Solution: We rewrite the conic (1) in matrix form.
$$\mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + 2\begin{pmatrix} 0 & -1 \end{pmatrix} \mathbf{x} = 0 \tag{2}$$
Comparing with the general equation of the conic,
$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{3}$$
$$\mathbf{u} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \tag{4}$$
$$f = 0 \tag{5}$$
Therefore, the equation of the normal where $\mathbf{u}$ is the point of contact and $\mathbf{R} \triangleq \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is
$$(\mathbf{Vq} + \mathbf{u})^\top \mathbf{R} \left(\begin{pmatrix} 0 \\ 5 \end{pmatrix} - \mathbf{q} \right) = 0 \tag{6}$$
Substituting the appropriate values and simplifying, we get the equation
$$\mathbf{q}^\top \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mathbf{q} + \mathbf{q}^\top \begin{pmatrix} -4 \\ 0 \end{pmatrix} = 0 \tag{7}$$
We represent a quadratic equation by completing the squares. In the general case, where $\mathbf{A}$ is symmetric and invertible,
$$\mathbf{x}^\top \mathbf{A} \mathbf{x} + \mathbf{x}^\top \mathbf{b} + c = (\mathbf{x} - \mathbf{h})^\top \mathbf{A} (\mathbf{x} - \mathbf{h}) + k \tag{8}$$
Expanding (8) and comparing like terms,
$$-2\mathbf{Ah} = \mathbf{b} \tag{9}$$
$$k - \mathbf{h}^\top \mathbf{Ah} = c \tag{10}$$
From (9),
$$\mathbf{h} = -\frac{1}{2}\mathbf{A}^{-1}\mathbf{b} \tag{11}$$
Substituting (11) into (10),
$$k = c - \frac{1}{4}\mathbf{b}^\top \mathbf{A}^{-1} \mathbf{b} \tag{12}$$
In (7), $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is not symmetric. Replacing $\mathbf{A}$ with $\frac{\mathbf{A}^\top + \mathbf{A}}{2}$, we get
$$\mathbf{A} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \mathbf{b} = \begin{pmatrix} -4 \\ 0 \end{pmatrix}, \ c = 0 \tag{13}$$
Using (11) and (12),
$$\mathbf{A}^{-1} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \tag{14}$$
$$\implies \mathbf{h} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} \tag{15}$$
$$\implies k = 0 \tag{16}$$
and (7) becomes
$$\left(\mathbf{x} - \begin{pmatrix} 0 \\ 4 \end{pmatrix} \right)^\top \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} 0 \\ 4 \end{pmatrix} \right) = 0 \tag{17}$$
Let $\mathbf{S} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Note that
$$\mathbf{S}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{18}$$
$$\mathbf{S}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{19}$$
Since the eigenvectors of a real symmetric

matrix are orthogonal, we can decompose $\mathbf{S}$ as

$$\mathbf{S} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{20}$$

$$= \mathbf{P}\mathbf{\Lambda}\mathbf{P}^\top \tag{21}$$

where

$$\mathbf{P} := \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \mathbf{\Lambda} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{22}$$

Therefore, letting

$$\mathbf{y} \triangleq \mathbf{P}\left(\mathbf{x} - \mathbf{h}\right) \tag{23}$$

we rewrite (17) as

$$\mathbf{y}^\top \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{y} = 0 \tag{24}$$

The solutions to (24) are given by

$$\mathbf{y} = \begin{pmatrix} a \\ \pm a \end{pmatrix} \tag{25}$$

Using (23),

$$\mathbf{x} = \mathbf{h} + \mathbf{P}^{-1}\mathbf{y} \tag{26}$$

$$= \begin{pmatrix} 0 \\ 4 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} a \\ \pm a \end{pmatrix} \tag{27}$$

$$= \begin{pmatrix} \frac{a \pm a}{2} \\ \frac{a \mp a}{2} + 4 \end{pmatrix} \tag{28}$$

$$\implies \mathbf{x} \in \left\{ \begin{pmatrix} a \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ a + 4 \end{pmatrix} \right\} \tag{29}$$

In the first case, (1) implies $a^2 = 8$. In the second case, we have $a + 4 = 0$. Thus, the points of contact are

$$\mathbf{N} \in \left\{ \begin{pmatrix} \pm 2\sqrt{2} \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \tag{30}$$

The nearest point out of these three candidates for $\mathbf{N}$ is $\begin{pmatrix} \pm 2\sqrt{2} \\ 4 \end{pmatrix}$. Thus, the correct answer is **b)**.

The situation is depicted in Fig. 1 plotted by the Python code `codes/normal.py`.

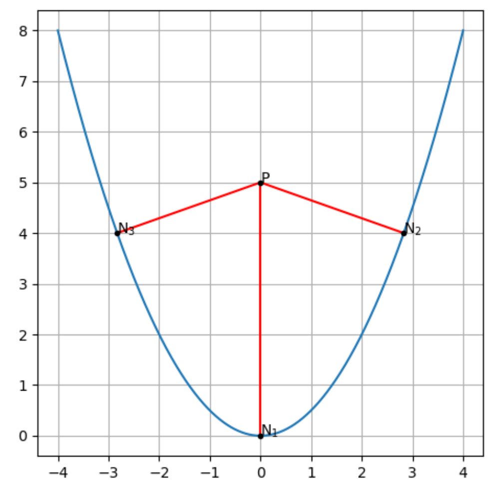

Fig. 1: N_1, N_2, N_3 are the points of contact of the normal from P to the parabola.