An Application of Machine Learning to Grading

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Outline

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3. Which method reflects student efforts better? What about failing students?
4. Which method can be extended to assess based on other factors?

Resources

Marks datasheet and relevant Python codes can be found here.

1. Marks of students: `marks.xlsx`
2. Python code using Gaussian method: `grades_norm.py`
3. Python code using K-means method: `grades.py`

Data Visualization

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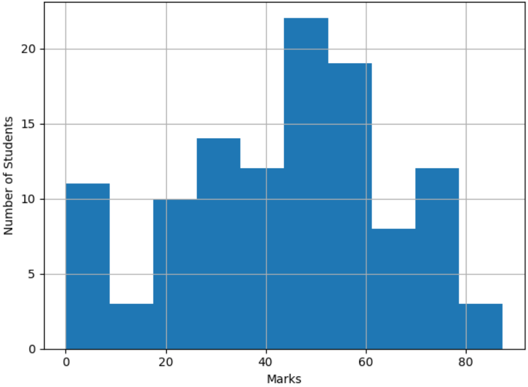

Figure: Histogram showing distribution of marks of the students.

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Note that the sample measures are **unbiased estimators** of their corresponding population measures.

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Statistical Note

If the population is large, computing population parameters directly is cumbersome. In this case, use the sample parameters to calculate the Z-score.

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3. The marks were scaled relative to the highest scoring student.

Grading Scheme

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Interval	**Grade**
($-\infty$, -3]	F
(-3, -2]	D
(-2, 1]	C
(-1, 0]	B-
(0, 1]	B
(1, 2]	A-
(2, 3]	A
(3,∞)	A+

Table: Grading scheme used for calculation of Z-scores

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3. It is an **EM** algorithm (explained ahead).

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2. The **cost function** is given by

$$J \triangleq \sum_{n=1}^{N}\sum_{k=1}^{K} r_{nk}\left\|\mathbf{x}_n - \boldsymbol{\mu}_k\right\|^2 \tag{8}$$

$$= \operatorname{tr}\left(\mathbf{X}\mathbf{R}^\top\right) \tag{9}$$

Definitions (Contd́...)

1. Here $\mathbf{X}$ and $\mathbf{R}$ are matrices such that

$$[\mathbf{X}]_{nk} \triangleq \left\|\mathbf{x}_n - \boldsymbol{\mu}_k\right\|^2 \tag{10}$$

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2. We are required to find $\{\boldsymbol{\mu}_k\}_{k=1}^{K}$ such that (9) is *minimized*.

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2. Selecting a point “far away” from all clusters.

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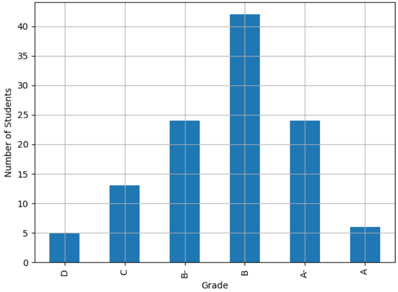
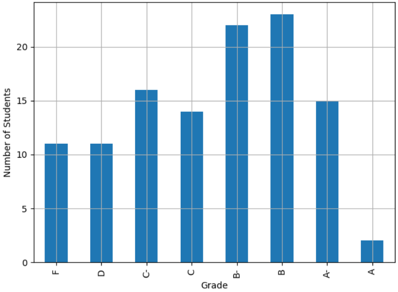


(a) Standard Normal Distribution



(b) K-Means Algorithm

Figure: Comparison of grading distributions using both algorithms.

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3. The K-Means algorithm gives a better idea of the performance of the class, especially when it is skew.
4. The K-Means algorithm can be extended to involve other factors such as attendance, prerequisites completed, and so on.