Circle Assignment

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***Abstract*—This document contains the solution to Question 12 of Exercise 5 in Chapter 10 of the class 9 NCERT textbook.**

1) Prove that a cyclic paralellogram is a rectangle.
Solution: Consider the points $\mathbf{P_i}$, $1 \le i \le 4$ in anticlockwise order on the unit circle. Thus, for $1 \le i \le 4$,
$$\mathbf{P_i} = \begin{pmatrix} \cos\theta_i \\ \sin\theta_i \end{pmatrix} \tag{1}$$
where
$$\theta_i \in [0, 2\pi),\ i \neq j \iff \theta_i \neq \theta_j \tag{2}$$
Without loss of generality, suppose that P_1P_2 and P_3P_4 are parallel to the x-axis. Since
$$\mathbf{P_1} - \mathbf{P_2} = \begin{pmatrix} \cos\theta_1 - \cos\theta_2 \\ \sin\theta_1 - \sin\theta_2 \end{pmatrix} \tag{3}$$
$$\mathbf{P_3} - \mathbf{P_4} = \begin{pmatrix} \cos\theta_4 - \cos\theta_4 \\ \sin\theta_3 - \sin\theta_4 \end{pmatrix} \tag{4}$$
we have
$$\sin\theta_1 - \sin\theta_2 = 0 \tag{5}$$
$$\implies \sin\frac{\theta_1 - \theta_2}{2}\cos\frac{\theta_1 + \theta_2}{2} = 0 \tag{6}$$
However, from (2), we see that
$$\theta_1 - \theta_2 \in (-2\pi, 2\pi) \tag{7}$$
$$\implies \sin\frac{\theta_1 - \theta_2}{2} \neq 0 \tag{8}$$
$$\implies \cos\frac{\theta_1 + \theta_2}{2} = 0 \tag{9}$$
$$\implies \theta_1 + \theta_2 \in (\pi, 3\pi) \tag{10}$$
Similarly,
$$\theta_3 + \theta_4 \in (\pi, 3\pi) \tag{11}$$
Since $P_1P_2P_3P_4$ is a parallelogram, its diagonals bisect each other. Thus, using (10) and (11),
$$\frac{\mathbf{P_1} + \mathbf{P_3}}{2} = \frac{\mathbf{P_2} + \mathbf{P_4}}{2} \tag{12}$$
$$\implies \mathbf{P_1} + \mathbf{P_3} = \mathbf{P_2} + \mathbf{P_4} \tag{13}$$
$$\implies \begin{pmatrix} \cos\theta_1 + \cos\theta_3 \\ \sin\theta_1 + \sin\theta_3 \end{pmatrix} = \begin{pmatrix} \cos\theta_2 + \cos\theta_4 \\ \sin\theta_2 + \sin\theta_4 \end{pmatrix} \tag{14}$$
$$\implies \cos\theta_1 + \cos\theta_3 = \cos\theta_2 + \cos\theta_4 \tag{15}$$
$$= -(\cos\theta_1 + \cos\theta_3) \tag{16}$$
$$\implies \cos\theta_1 + \cos\theta_3 = \cos\theta_2 + \cos\theta_4 = 0 \tag{17}$$
Using (17), (10) and (11), we have
$$\cos\theta_1 = -\cos\theta_3 = \cos\theta_4 \tag{18}$$
$$\cos\theta_2 = -\cos\theta_4 = \cos\theta_3 \tag{19}$$
Thus,
$$\mathbf{P_1} - \mathbf{P_4} = \begin{pmatrix} \cos\theta_1 - \cos\theta_4 \\ \sin\theta_1 - \sin\theta_4 \end{pmatrix} \tag{20}$$
$$= \begin{pmatrix} 0 \\ \sin\theta_1 - \sin\theta_4 \end{pmatrix} \tag{21}$$
Thus, from (21),
$$(\mathbf{P_1} - \mathbf{P_2})^\top (\mathbf{P_1} - \mathbf{P_4}) = \begin{pmatrix} \cos\theta_1 - \cos\theta_2 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \sin\theta_1 - \sin\theta_4 \end{pmatrix} = 0 \tag{22}$$
From (22), we see that $P_1P_2 \perp P_1P_4$. Hence, $P_1P_2P_3P_4$ is a rectangle.
The situation is demonstrated in Fig. 1, plotted by the Python code `codes/circle.py`. The various input parameters are shown in Table I.

Parameter	Value
r	1
θ_1	$\frac{\pi}{6}$
θ_2	$\frac{5\pi}{6}$
θ_3	$\frac{7\pi}{6}$
θ_4	$\frac{11\pi}{6}$

TABLE I: Parameters used in the construction of Fig. 1.

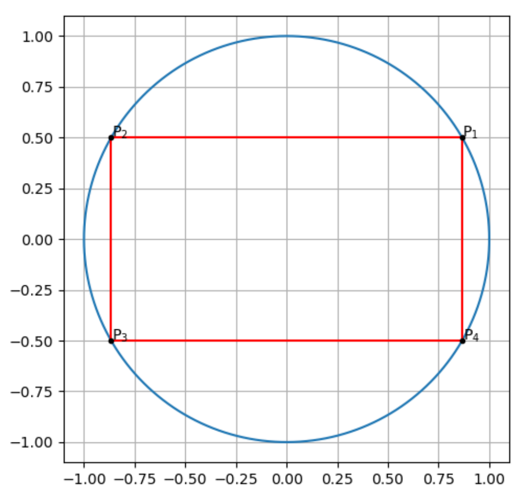

Fig. 1: $P_1P_2P_3P_4$ is a rectangle.