Conic Assignment

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***Abstract*—This document contains the solution to Question 20 of Exercise 3 in Chapter 11 of the class 11 NCERT textbook.**

1) Find the equation of the ellipse whose major axis is the x-axis and center is the origin and passes through the points

$$\mathbf{P} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}, \ \mathbf{Q} = \begin{pmatrix} 6 \\ 2 \end{pmatrix} \tag{1}$$

Solution: Let the equation of the conic with focus $\mathbf{F}$, directrix $\mathbf{n}^\top\mathbf{x} = c$ and eccentricity e be

$$\mathbf{x}^\top\mathbf{V}\mathbf{x} + 2\mathbf{u}^\top\mathbf{x} + f = 0 \tag{2}$$

where

$$\mathbf{V} \triangleq \|\mathbf{n}\|^2\mathbf{I} - e^2\mathbf{n}\mathbf{n}^\top \tag{3}$$

$$\mathbf{u} \triangleq ce^2\mathbf{n} - \|\mathbf{n}\|^2\mathbf{F} \tag{4}$$

$$f \triangleq \|\mathbf{n}\|^2\|\mathbf{F}\|^2 - c^2e^2 \tag{5}$$

Since the conic is an ellipse whose major axis is along the x-axis, we have

$$\mathbf{n} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{6}$$

Thus,

$$\mathbf{V} = \begin{pmatrix} 1-e^2 & 0 \\ 0 & 1 \end{pmatrix} \tag{7}$$

$$\mathbf{u} = ce^2\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \mathbf{F} \tag{8}$$

$$f = \|\mathbf{F}\|^2 - c^2e^2 \tag{9}$$

The centre of the conic is given by

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \tag{10}$$

Since $\mathbf{c} = \mathbf{0}$ and $\mathbf{V}^{-1} \neq \mathbf{0}$, it follows from (10) that $\mathbf{u} = \mathbf{0}$. Thus, from (8),

$$\mathbf{F} = \begin{pmatrix} ce^2 \\ 0 \end{pmatrix} \tag{11}$$

and so,

$$f = c^2e^2\left(e^2 - 1\right) \tag{12}$$

Putting $\mathbf{x} = \mathbf{P}$ in (2) and using (11) and (12),

$$\begin{pmatrix} 4 & 3 \end{pmatrix}\begin{pmatrix} 1-e^2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 3 \end{pmatrix} + f = 0 \tag{13}$$

$$\implies 16e^2 - f = 25 \tag{14}$$

Putting $\mathbf{x} = \mathbf{Q}$ in (2), we get

$$\begin{pmatrix} 6 & 2 \end{pmatrix}\begin{pmatrix} 1-e^2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 6 \\ 2 \end{pmatrix} + f = 0 \tag{15}$$

$$\implies 36e^2 - f = 40 \tag{16}$$

The equations (14) and (16) can be formulated as a matrix equation

$$\begin{pmatrix} 16 & -1 \\ 36 & -1 \end{pmatrix}\begin{pmatrix} e^2 \\ f \end{pmatrix} = \begin{pmatrix} 25 \\ 40 \end{pmatrix} \tag{17}$$

and can be solved using the augmented matrix.

$$\begin{pmatrix} 16 & -1 & 25 \\ 36 & -1 & 40 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} -20 & 0 & -15 \\ 36 & -1 & 40 \end{pmatrix} \tag{18}$$

$$\xleftrightarrow[R_2 \leftarrow -R_2]{R_1 \leftarrow \frac{R_1}{-5}} \begin{pmatrix} 4 & 0 & 3 \\ -36 & 1 & -40 \end{pmatrix} \tag{19}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 + 9R_1} \begin{pmatrix} 4 & 0 & 3 \\ 0 & 1 & -13 \end{pmatrix} \tag{20}$$

$$\xleftrightarrow{R_1 \leftarrow \frac{R_1}{4}} \begin{pmatrix} 1 & 0 & \frac{3}{4} \\ 0 & 1 & -13 \end{pmatrix} \tag{21}$$

Thus,

$$e^2 = \frac{3}{4}, \ f = -13 \tag{22}$$

And the equation of the conic is given by

$$\mathbf{x}^\top\begin{pmatrix} \frac{1}{4} & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} - 13 = 0 \tag{23}$$

The situation is illustrated in Fig. 1

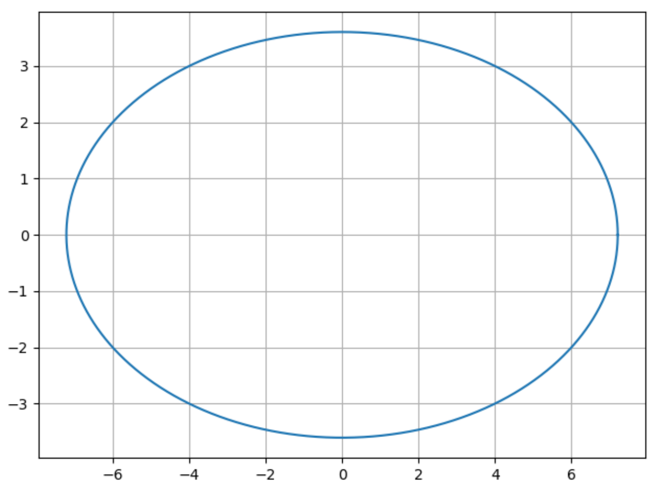

Fig. 1: Locus of the required ellipse.