Line Assignment

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***Abstract*—This document contains a general solution to Question 16 of Exercise 2 in Chapter 11 of the class 12 NCERT textbook.**

1) Find the shortest distance between the lines whose vector equations are

$$L_1 : \mathbf{x} = \mathbf{x_1} + \lambda_1 \mathbf{m_1} \tag{1}$$
$$L_2 : \mathbf{x} = \mathbf{x_2} + \lambda_2 \mathbf{m_2} \tag{2}$$

Solution: Let $\mathbf{A}$ and $\mathbf{B}$ be points on lines L_1 and L_2 respectively such that AB is normal to both lines. Define

$$\mathbf{M} \triangleq \begin{pmatrix} \mathbf{m_1} & \mathbf{m_2} \end{pmatrix} \tag{3}$$
$$\boldsymbol{\lambda} \triangleq \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} \tag{4}$$
$$\mathbf{x} \triangleq vecx_2 - \mathbf{x_1} \tag{5}$$

Then, we have the following equations:

$$\mathbf{A} = \mathbf{x_1} + \lambda_1 \mathbf{m_1} \tag{6}$$
$$\mathbf{B} = \mathbf{x_2} + \lambda_2 \mathbf{m_2} \tag{7}$$
$$\mathbf{m_1}^\top (\mathbf{A} - \mathbf{B}) = 0 \tag{8}$$
$$\mathbf{m_2}^\top (\mathbf{A} - \mathbf{B}) = 0 \tag{9}$$

(8) and (9) give

$$\mathbf{M}^\top (\mathbf{A} - \mathbf{B}) = \mathbf{O} \tag{10}$$

where $\mathbf{O} \triangleq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Substituting (6) and (7) into (10), and using (3) and (4),

$$\mathbf{M}^\top \mathbf{M} \boldsymbol{\lambda} = \mathbf{M}^\top \mathbf{x} \tag{11}$$

We have the following cases:

a) There exists a $\boldsymbol{\lambda}$ satisfying

$$\mathbf{M} \boldsymbol{\lambda} = \mathbf{x} \tag{12}$$
$$\implies \lambda_1 \mathbf{m_1} - \lambda_2 \mathbf{m_2} = \mathbf{x_2} - \mathbf{x_1} \tag{13}$$
$$\implies \mathbf{x_1} + \lambda_1 \mathbf{m_1} = \mathbf{x_2} + \lambda_2 \mathbf{m_2} \tag{14}$$

Thus, both lines intersect at a point in this case, and the shortest distance between them is trivially 0. To check whether such a $\boldsymbol{\lambda}$ exists, we can bring the augmented matrix $\begin{pmatrix} \mathbf{M} & \mathbf{x} \end{pmatrix}$ into row-reduced echelon form and check whether there is a pivot in the last column.

b) $\mathbf{M}^\top \mathbf{M}$ is singular. Since $\mathbf{M}^\top \mathbf{M}$ is a sqaure matrix of order 2, its rank must be 1. Further,

$$\det(\mathbf{M}^\top \mathbf{M}) = \begin{vmatrix} \mathbf{m_1}^\top \mathbf{m_1} & \mathbf{m_1}^\top \mathbf{m_2} \\ \mathbf{m_1}^\top \mathbf{m_2} & \mathbf{m_2}^\top \mathbf{m_2} \end{vmatrix} \tag{15}$$
$$= (\|\mathbf{m_1}\| . \|\mathbf{m_2}\|)^2 - (\mathbf{m_1}^\top \mathbf{m_2})^2 \tag{16}$$

Thus, equating the determinant to zero gives

$$\|\mathbf{m_1}\| . \|\mathbf{m_2}\| = \left| \mathbf{m_1}^\top \mathbf{m_2} \right| \tag{17}$$

which implies that both lines are parallel to each other. Setting $\mathbf{m_2} = k\mathbf{m_1}, k \in \mathbb{R} \setminus \{0\}$, we obtain one equation from (11).

$$\mathbf{m_1}^\top \mathbf{m_1} (\lambda_1 - k\lambda_2) = \mathbf{m_1}^\top \mathbf{x} \tag{18}$$
$$\implies \lambda_1 - k\lambda_2 = \frac{\mathbf{m_1}^\top \mathbf{x}}{\|\mathbf{m_1}\|^2} \tag{19}$$

Therefore, the required shortest distance is

$$\|\mathbf{A} - \mathbf{B}\| = \left\| \frac{\mathbf{m_1}^\top \mathbf{x} \mathbf{m_1}}{\|\mathbf{m_1}\|^2} - \mathbf{x} \right\| \tag{20}$$

c) $\mathbf{M}^\top \mathbf{M}$ is nonsinglar. This implies that the lines are skew. From (11),

$$\boldsymbol{\lambda} = (\mathbf{M}^\top \mathbf{M})^{-1} \mathbf{M}^\top \mathbf{x} \tag{21}$$

and therefore, the shortest distance is

$$\|\mathbf{A} - \mathbf{B}\| = \left\| \left(\mathbf{M} (\mathbf{M}^\top \mathbf{M})^{-1} \mathbf{M}^\top - \mathbf{I_n} \right) \mathbf{x} \right\| \tag{22}$$

where $\mathbf{I_n}$ denotes the identity matrix of order n.