

# Line Assignment

Gautam Singh

**Abstract**—This document contains a general solution to Question 16 of Exercise 2 in Chapter 11 of the class 12 NCERT textbook.

- 1) Find the shortest distance between the lines whose vector equations are

$$L_1 : \mathbf{x} = \mathbf{x}_1 + \lambda_1 \mathbf{m}_1 \quad (1)$$

$$L_2 : \mathbf{x} = \mathbf{x}_2 + \lambda_2 \mathbf{m}_2 \quad (2)$$

**Solution:** Let  $\mathbf{A}$  and  $\mathbf{B}$  be points on lines  $L_1$  and  $L_2$  respectively such that  $AB$  is normal to both lines. Define

$$\mathbf{M} \triangleq (\mathbf{m}_1 \quad \mathbf{m}_2) \quad (3)$$

$$\lambda \triangleq \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} \quad (4)$$

$$\mathbf{x} \triangleq \mathbf{x}_2 - \mathbf{x}_1 \quad (5)$$

Then, we have the following equations:

$$\mathbf{A} = \mathbf{x}_1 + \lambda_1 \mathbf{m}_1 \quad (6)$$

$$\mathbf{B} = \mathbf{x}_2 + \lambda_2 \mathbf{m}_2 \quad (7)$$

From (6) and (7), define the real-valued function  $f$  as

$$f(\lambda) \triangleq \|\mathbf{A} - \mathbf{B}\| \quad (8)$$

$$= \|\mathbf{M}\lambda - \mathbf{x}\| \quad (9)$$

$$= \sqrt{(\mathbf{M}\lambda - \mathbf{x})^\top (\mathbf{M}\lambda - \mathbf{x})} \quad (10)$$

We show that  $f$  is convex. Indeed, consider  $\lambda_1$  and  $\lambda_2$  and let  $0 \leq \mu \leq 1$ . Then,

$$f(\mu\lambda_1 + (1 - \mu)\lambda_2) \quad (11)$$

$$= \|\mathbf{M}(\mu\lambda_1 + (1 - \mu)\lambda_2) - \mathbf{x}\| \quad (12)$$

$$= \|\mu(\mathbf{M}\lambda_1 - \mathbf{x}) + (1 - \mu)(\mathbf{M}\lambda_2 - \mathbf{x})\| \quad (13)$$

$$\leq \mu \|\mathbf{M}\lambda_1 - \mathbf{x}\| + (1 - \mu) \|\mathbf{M}\lambda_2 - \mathbf{x}\| \quad (14)$$

Where (14) follows from the triangle inequality. We need to minimize  $f$  as a function of  $\lambda$ . Thus, from (10),

$$\frac{df(\lambda)}{d\lambda} = \frac{\mathbf{M}^\top (\mathbf{M}\lambda - \mathbf{x})}{\|\mathbf{M}\lambda - \mathbf{x}\|} \quad (15)$$

using the chain rule. Setting (15) to zero gives

$$\mathbf{M}^\top \mathbf{M}\lambda = \mathbf{M}^\top \mathbf{x} \quad (16)$$

We have the following cases:

- a) There exists a  $\lambda$  satisfying

$$\mathbf{M}\lambda = \mathbf{x} \quad (17)$$

$$\implies \lambda_1 \mathbf{m}_1 - \lambda_2 \mathbf{m}_2 = \mathbf{x}_2 - \mathbf{x}_1 \quad (18)$$

$$\implies \mathbf{x}_1 + \lambda_1 \mathbf{m}_1 = \mathbf{x}_2 + \lambda_2 \mathbf{m}_2 \quad (19)$$

Thus, both lines intersect at a point and the shortest distance between them is 0. To check for the existence of such a  $\lambda$ , we can bring the augmented matrix  $(\mathbf{M} \quad \mathbf{x})$  into row-reduced echelon form and check whether there is a pivot in the last column.

- b)  $\mathbf{M}^\top \mathbf{M}$  is singular. Since  $\mathbf{M}^\top \mathbf{M}$  is a square matrix of order 2, its rank must be 1. Further,

$$\begin{aligned} \det(\mathbf{M}^\top \mathbf{M}) &= \begin{vmatrix} \mathbf{m}_1^\top \mathbf{m}_1 & \mathbf{m}_1^\top \mathbf{m}_2 \\ \mathbf{m}_1^\top \mathbf{m}_2 & \mathbf{m}_2^\top \mathbf{m}_2 \end{vmatrix} \quad (20) \\ &= (\|\mathbf{m}_1\| \cdot \|\mathbf{m}_2\|)^2 - (\mathbf{m}_1^\top \mathbf{m}_2)^2 \quad (21) \end{aligned}$$

Thus, equating the determinant to zero gives

$$\|\mathbf{m}_1\| \cdot \|\mathbf{m}_2\| = |\mathbf{m}_1^\top \mathbf{m}_2| \quad (22)$$

which implies that both lines are parallel to each other. Setting  $\mathbf{m}_2 = k\mathbf{m}_1, k \in \mathbb{R} \setminus \{0\}$ , we obtain one equation from (16).

$$\mathbf{m}_1^\top \mathbf{m}_1 (\lambda_1 - k\lambda_2) = \mathbf{m}_1^\top \mathbf{x} \quad (23)$$

$$\implies \lambda_1 - k\lambda_2 = \frac{\mathbf{m}_1^\top \mathbf{x}}{\|\mathbf{m}_1\|^2} \quad (24)$$

Therefore, the required shortest distance is

$$\|\mathbf{A} - \mathbf{B}\| = \left\| \frac{\mathbf{m}_1^\top \mathbf{x} \mathbf{m}_1}{\|\mathbf{m}_1\|^2} - \mathbf{x} \right\| \quad (25)$$

- c)  $\mathbf{M}^\top \mathbf{M}$  is nonsingular. This implies that the lines are skew. From (16),

$$\lambda = (\mathbf{M}^\top \mathbf{M})^{-1} \mathbf{M}^\top \mathbf{x} \quad (26)$$

and therefore, the shortest distance is

$$\|\mathbf{A} - \mathbf{B}\| = \left\| \left( \mathbf{M}(\mathbf{M}^\top \mathbf{M})^{-1} \mathbf{M}^\top - \mathbf{I}_n \right) \mathbf{x} \right\| \quad (27)$$

where  $\mathbf{I}_n$  is the identity matrix of order  $n$ .