Circle Assignment

Gautam Singh

***Abstract*—This document contains the solution to Question 10 of Exercise 6 in Chapter 10 of the class 9 NCERT textbook.**

1) In any $\triangle ABC$, if the angle bisector of $\angle A$ and perpendicular bisector of BC intersect, prove that they intersect on the circumcircle of $\triangle ABC$.
Solution: Let the position vectors of the points be

$$\mathbf{A} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \mathbf{B} = \begin{pmatrix} \cos\beta \\ \sin\beta \end{pmatrix}, \ \mathbf{C} = \begin{pmatrix} \cos\gamma \\ \sin\gamma \end{pmatrix} \tag{1}$$

Then, the equation of the perpendicular bisector of BC is given by

$$\left\|\mathbf{x} - \mathbf{B}\right\|^2 = \left\|\mathbf{x} - \mathbf{C}\right\|^2 \tag{2}$$
$$\implies (\mathbf{B} - \mathbf{C})^\top \mathbf{x} = 0 \tag{3}$$

and the equation of the angle bisector of $\angle A$ is given by

$$\frac{(\mathbf{B} - \mathbf{A})^\top (\mathbf{x} - \mathbf{A})}{\left\|\mathbf{B} - \mathbf{A}\right\|} = \frac{(\mathbf{C} - \mathbf{A})^\top (\mathbf{x} - \mathbf{A})}{\left\|\mathbf{C} - \mathbf{A}\right\|} \tag{4}$$
$$\left(\frac{\mathbf{B} - \mathbf{A}}{\left\|\mathbf{B} - \mathbf{A}\right\|} - \frac{\mathbf{C} - \mathbf{A}}{\left\|\mathbf{C} - \mathbf{A}\right\|}\right)^\top (\mathbf{x} - \mathbf{A}) = 0 \tag{5}$$

Note that from (1),

$$\frac{\mathbf{B} - \mathbf{A}}{\left\|\mathbf{B} - \mathbf{A}\right\|} = \frac{1}{\sqrt{2(1 - \cos\beta)}} \begin{pmatrix} \cos\beta - 1 \\ \sin\beta \end{pmatrix} \tag{6}$$
$$= \begin{pmatrix} -\sin\frac{\beta}{2} \\ \cos\frac{\beta}{2} \end{pmatrix} \tag{7}$$

Therefore, using (1) in (3) and (7) in (5),

$$\begin{pmatrix} \cos\beta - \cos\gamma \\ \sin\beta - \sin\gamma \end{pmatrix}^\top \mathbf{x} = 0 \tag{8}$$
$$\begin{pmatrix} \sin\frac{\gamma}{2} - \sin\frac{\beta}{2} \\ \cos\frac{\beta}{2} - \cos\frac{\gamma}{2} \end{pmatrix}^\top \mathbf{x} = \sin\frac{\gamma}{2} - \sin\frac{\beta}{2} \tag{9}$$

Using (8) and (9), we form the matrix equation

$$\begin{pmatrix} \cos\beta - \cos\gamma & \sin\beta - \sin\gamma \\ \sin\frac{\gamma}{2} - \sin\frac{\beta}{2} & \cos\frac{\beta}{2} - \cos\frac{\gamma}{2} \end{pmatrix} \mathbf{x} = \begin{pmatrix} 0 \\ \sin\frac{\gamma}{2} - \sin\frac{\beta}{2} \end{pmatrix} \tag{10}$$

Solving (10) using the augmented matrix, and letting $\theta \triangleq \frac{\beta+\gamma}{2}$,

$$\begin{pmatrix} \cos\beta - \cos\gamma & \sin\beta - \sin\gamma & 0 \\ \sin\frac{\gamma}{2} - \sin\frac{\beta}{2} & \cos\frac{\beta}{2} - \cos\frac{\gamma}{2} & \sin\frac{\gamma}{2} - \sin\frac{\beta}{2} \end{pmatrix} \tag{11}$$

$$\xleftrightarrow[R_2 \leftarrow \frac{R_2}{\sin\frac{\gamma}{2} - \sin\frac{\beta}{2}}]{R_1 \leftarrow \frac{R_1}{\cos\beta - \cos\gamma}} \begin{pmatrix} 1 & -\cot\theta & 0 \\ 1 & \tan\frac{\theta}{2} & 1 \end{pmatrix} \tag{12}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & -\cot\theta & 0 \\ 0 & \tan\frac{\theta}{2} + \cot\theta & 1 \end{pmatrix} \tag{13}$$

$$= \begin{pmatrix} 1 & -\cot\theta & 0 \\ 0 & \csc\theta & 1 \end{pmatrix} \tag{14}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + R_2\cos\theta} \begin{pmatrix} 1 & 0 & \cos\theta \\ 0 & \csc\theta & 1 \end{pmatrix} \tag{15}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 \sin\theta} \begin{pmatrix} 1 & 0 & \cos\theta \\ 0 & 1 & \sin\theta \end{pmatrix} \tag{16}$$

Thus, the intersection of the lines in (8) and (9) is

$$\mathbf{D} \triangleq \begin{pmatrix} \cos\frac{\beta+\gamma}{2} \\ \sin\frac{\beta+\gamma}{2} \end{pmatrix} \tag{17}$$

Hence, it is clear from (17) that $\mathbf{D}$ lies on the circumcircle of $\triangle ABC$, as required.
The situation is illustrated in Fig. 1 plotted by the Python code `codes/angle_perp_bisector.py`.

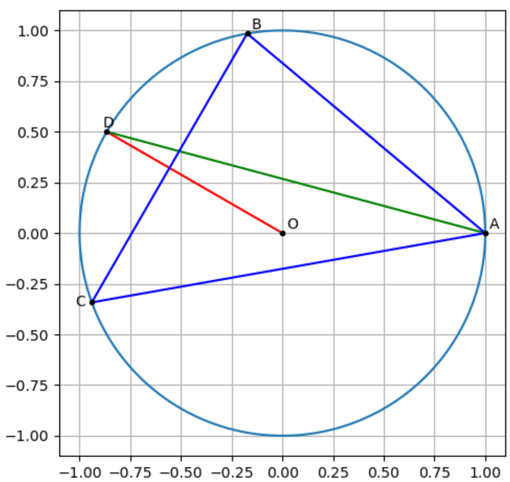

Fig. 1: The bisector of $\angle A$ and of BC meet on the circumcircle at D.