Line Assignment

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***Abstract*—This document contains the solution to Question 16 of Exercise 2 in Chapter 11 of the class 12 NCERT textbook.**

1) Find the shortest distance between the lines whose vector equations are

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \tag{1}$$

and

$$\mathbf{x} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \tag{2}$$

Solution: Suppose there are two lines in general given by

$$\mathbf{x} = \mathbf{x_1} + \lambda_1 \mathbf{m_1} \tag{3}$$
$$\mathbf{x} = \mathbf{x_2} + \lambda_2 \mathbf{m_2} \tag{4}$$

If these lines intersect, then

$$\lambda_1 \mathbf{m_1} - \lambda_2 \mathbf{m_2} = \mathbf{x_2} - \mathbf{x_1} \tag{5}$$
$$\implies \mathbf{M}\boldsymbol{\lambda} = \mathbf{x_2} - \mathbf{x_1} \tag{6}$$

where

$$\mathbf{M} \triangleq \begin{pmatrix} \mathbf{m_1} & \mathbf{m_2} \end{pmatrix} \tag{7}$$
$$\boldsymbol{\lambda} \triangleq \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} \tag{8}$$

In this case,

$$\mathbf{x_1} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \mathbf{x_2} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \quad \mathbf{m_1} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \quad \mathbf{m_2} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \tag{9}$$

To check whether (6) has a solution in $\boldsymbol{\lambda}$, we use the augmented matrix.

$$\begin{pmatrix} 1 & 2 & 3 \\ -3 & 3 & 3 \\ 2 & 1 & 3 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 + 3R_1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 9 & 12 \\ 2 & 1 & 3 \end{pmatrix} \tag{10}$$
$$\xleftrightarrow{R_3 \leftarrow R_3 - 2R_1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 9 & 12 \\ 0 & -3 & -3 \end{pmatrix} \tag{11}$$
$$\xleftrightarrow{R_3 \leftarrow 3R_3 + R_2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 9 & 12 \\ 0 & 0 & 3 \end{pmatrix} \tag{12}$$

Clearly, the rank of this matrix is 3, and therefore, the lines are skew.
Now, suppose that the closest points on both lines are

$$\mathbf{A} = \mathbf{x_1} + \lambda_1 \mathbf{m_1} \tag{13}$$
$$\mathbf{B} = \mathbf{x_2} + \lambda_2 \mathbf{m_2} \tag{14}$$

Then, AB is perpendicular to both lines, hence

$$\mathbf{m_1}^\top (\mathbf{A} - \mathbf{B}) = 0 \tag{15}$$
$$\mathbf{m_2}^\top (\mathbf{A} - \mathbf{B}) = 0 \tag{16}$$
$$\implies \mathbf{M}^\top (\mathbf{A} - \mathbf{B}) = \mathbf{0} \tag{17}$$

Using (13) and (14) in (17),

$$\mathbf{M}^\top (\mathbf{x_1} - \mathbf{x_2} + \mathbf{M}\boldsymbol{\lambda}) = \mathbf{0} \tag{18}$$
$$\implies \mathbf{M}^\top \mathbf{M}\boldsymbol{\lambda} = \mathbf{M}^\top (\mathbf{x_2} - \mathbf{x_1}) \tag{19}$$

Substituting from (9) in (19) and forming the augmented matrix,

$$\begin{pmatrix} 14 & -5 & 0 \\ -5 & 14 & 18 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 + R_2} \begin{pmatrix} 9 & 9 & 18 \\ -5 & 14 & 18 \end{pmatrix} \tag{20}$$

$$\xleftrightarrow{R_1 \leftarrow \frac{R_1}{9}} \begin{pmatrix} 1 & 1 & 2 \\ -5 & 14 & 18 \end{pmatrix} \tag{21}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 + 5R_1} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 19 & 28 \end{pmatrix} \tag{22}$$

$$\xleftrightarrow{R_1 \leftarrow 19R_1 - R_2} \begin{pmatrix} 19 & 0 & 10 \\ 0 & 19 & 28 \end{pmatrix} \tag{23}$$

$$\xleftrightarrow[]{\substack{R_1 \leftarrow \frac{R_1}{19} \\ R_2 \leftarrow \frac{R_2}{9}}} \begin{pmatrix} 1 & 0 & \frac{10}{19} \\ 0 & 1 & \frac{28}{19} \end{pmatrix} \tag{24}$$

$$\implies \lambda = \frac{1}{19} \begin{pmatrix} 10 \\ 28 \end{pmatrix} \tag{25}$$

Hence, using (8) and substituing into (13) and (14),

$$\mathbf{A_m} = \frac{1}{19} \begin{pmatrix} 29 \\ 8 \\ 77 \end{pmatrix}, \ \mathbf{B_m} = \frac{1}{19} \begin{pmatrix} 20 \\ 11 \\ 86 \end{pmatrix} \tag{26}$$

Thus, the required distance is

$$\|\mathbf{B_m} - \mathbf{A_m}\| = \frac{\sqrt{9^2 + 3^2 + (-9)^2}}{19} = \frac{3}{\sqrt{19}} \tag{27}$$

The situation is depicted in Fig. 1.

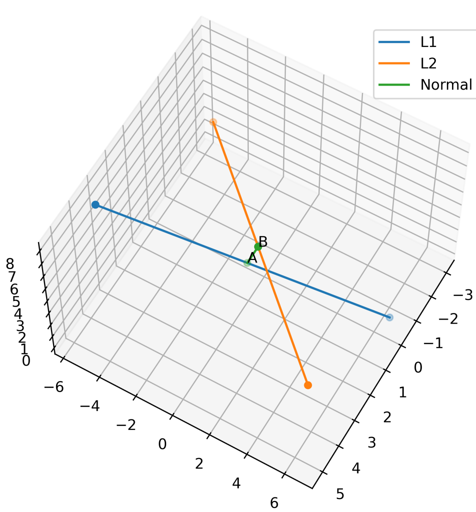

Fig. 1: AB is the required shortest distance.