Line Assignment

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Abstract—**This document contains a general solution to Question 16 of Exercise 2 in Chapter 11 of the class 12 NCERT textbook.**

1) Find the shortest distance between the lines whose vector equations are

$$L_1 : \mathbf{x} = \mathbf{x_1} + \lambda_1 \mathbf{m_1} \tag{1}$$
$$L_2 : \mathbf{x} = \mathbf{x_2} + \lambda_2 \mathbf{m_2} \tag{2}$$

Solution: Let $\mathbf{A}$ and $\mathbf{B}$ be points on lines L_1 and L_2 respectively such that AB is normal to both lines. Define

$$\mathbf{M} \triangleq \begin{pmatrix} \mathbf{m_1} & \mathbf{m_2} \end{pmatrix} \tag{3}$$
$$\boldsymbol{\lambda} \triangleq \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} \tag{4}$$
$$\mathbf{x} \triangleq \mathbf{x_2} - \mathbf{x_1} \tag{5}$$

Then, we have the following equations:

$$\mathbf{A} = \mathbf{x_1} + \lambda_1 \mathbf{m_1} \tag{6}$$
$$\mathbf{B} = \mathbf{x_2} + \lambda_2 \mathbf{m_2} \tag{7}$$

From (6) and (7), define the real-valued function f as

$$f(\boldsymbol{\lambda}) \triangleq \|\mathbf{A} - \mathbf{B}\|^2 \tag{8}$$
$$= \|\mathbf{M}\boldsymbol{\lambda} - \mathbf{x}\|^2 \tag{9}$$
$$= (\mathbf{M}\boldsymbol{\lambda} - \mathbf{x})^\top (\mathbf{M}\boldsymbol{\lambda} - \mathbf{x}) \tag{10}$$
$$= \boldsymbol{\lambda}^\top (\mathbf{M}^\top \mathbf{M}) \boldsymbol{\lambda} - 2\mathbf{x}^\top \mathbf{M}\boldsymbol{\lambda} + \|\mathbf{x}\|^2 \tag{11}$$

From (11), we see that f is quadratic in $\boldsymbol{\lambda}$. Thus, we show that f is convex by showing that $\mathbf{M}^\top\mathbf{M}$ is positive semi-definite. Indeed, for any $\mathbf{p} \triangleq \begin{pmatrix} x \\ y \end{pmatrix}$,

$$\mathbf{p}^\top \mathbf{M}^\top \mathbf{M}\mathbf{p} = \|\mathbf{M}\mathbf{p}\|^2 \geq 0 \tag{12}$$

and thus, f is convex.
We need to minimize f as a function of $\boldsymbol{\lambda}$. Thus, differentiating (11) using the chain rule,

$$\frac{df(\boldsymbol{\lambda})}{d\boldsymbol{\lambda}} = \mathbf{M}^\top (\mathbf{M}\boldsymbol{\lambda} - \mathbf{x}) + \mathbf{M}(\mathbf{M}\boldsymbol{\lambda} - \mathbf{x})^\top \tag{13}$$
$$= 2\mathbf{M}^\top (\mathbf{M}\boldsymbol{\lambda} - \mathbf{x}) \tag{14}$$

Setting (14) to zero gives

$$\mathbf{M}^\top \mathbf{M}\boldsymbol{\lambda} = \mathbf{M}^\top \mathbf{x} \tag{15}$$

We have the following cases:

a) There exists a $\boldsymbol{\lambda}$ satisfying

$$\mathbf{M}\boldsymbol{\lambda} = \mathbf{x} \tag{16}$$
$$\implies \lambda_1 \mathbf{m_1} - \lambda_2 \mathbf{m_2} = \mathbf{x_2} - \mathbf{x_1} \tag{17}$$
$$\implies \mathbf{x_1} + \lambda_1 \mathbf{m_1} = \mathbf{x_2} + \lambda_2 \mathbf{m_2} \tag{18}$$

Thus, both lines intersect at a point and the shortest distance between them is 0. To check for the existence of such a $\boldsymbol{\lambda}$, we can bring the augmented matrix $\begin{pmatrix} \mathbf{M} & \mathbf{x} \end{pmatrix}$ into row-reduced echelon form and check whether there is a pivot in the last column.

b) $\mathbf{M}^\top\mathbf{M}$ is singular. Since $\mathbf{M}^\top\mathbf{M}$ is a sqaure matrix of order 2, its rank must be 1. Further,

$$\det(\mathbf{M}^\top \mathbf{M}) = \begin{vmatrix} \mathbf{m_1}^\top \mathbf{m_1} & \mathbf{m_1}^\top \mathbf{m_2} \\ \mathbf{m_1}^\top \mathbf{m_2} & \mathbf{m_2}^\top \mathbf{m_2} \end{vmatrix} \tag{19}$$
$$= (\|\mathbf{m_1}\| \,.\, \|\mathbf{m_2}\|)^2 - (\mathbf{m_1}^\top \mathbf{m_2})^2 \tag{20}$$

Thus, equating the determinant to zero gives

$$\|\mathbf{m_1}\| \,.\, \|\mathbf{m_2}\| = \left|\mathbf{m_1}^\top \mathbf{m_2}\right| \tag{21}$$

which implies that both lines are parallel to each other. Setting $\mathbf{m_2} = k\mathbf{m_1}, k \in \mathbb{R} \setminus \{0\}$, we obtain one equation from (15).

$$\mathbf{m_1}^\top \mathbf{m_1} (\lambda_1 - k\lambda_2) = \mathbf{m_1}^\top \mathbf{x} \tag{22}$$
$$\implies \lambda_1 - k\lambda_2 = \frac{\mathbf{m_1}^\top \mathbf{x}}{\|\mathbf{m_1}\|^2} \tag{23}$$

Therefore, the required shortest distance is

$$\|\mathbf{A} - \mathbf{B}\| = \left\| \frac{\mathbf{m_1}^\top \mathbf{x} \mathbf{m_1}}{\|\mathbf{m_1}\|^2} - \mathbf{x} \right\| \tag{24}$$

c) $\mathbf{M}^{\top}\mathbf{M}$ is nonsinglar. This implies that the lines are skew. From (15),

$$\boldsymbol{\lambda} = \left(\mathbf{M}^{\top}\mathbf{M}\right)^{-1}\mathbf{M}^{\top}\mathbf{x} \tag{25}$$

and therefore, the shortest distance is

$$\|\mathbf{A}-\mathbf{B}\| = \left\|\left(\mathbf{M}\left(\mathbf{M}^{\top}\mathbf{M}\right)^{-1}\mathbf{M}^{\top}-\mathbf{I_n}\right)\mathbf{x}\right\| \tag{26}$$

where $\mathbf{I_n}$ is the identity matrix of order n.