# Quadratic Programming Assignment

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*Abstract*—**This document contains the solution to a modification of Question 27 of Exercise 5 in Chapter 6 of the class 12 NCERT textbook.**

1) Show that the point on the curve

$$x^2 = 2y \quad (1)$$

which is nearest to the point $\mathbf{P} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is a convex optimization problem.
**Solution:** We need to find

$$\min_{\mathbf{x}} g(\mathbf{x}) = \|\mathbf{x} - \mathbf{P}\|^2 \quad (2)$$
$$\text{s.t. } h(\mathbf{x}) = \mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} = 0 \quad (3)$$

where

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ \mathbf{u} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \quad (4)$$

Since $\mathbf{P}$ lies *outside* the given curve, we can apply the following relaxation to make it a convex optimization problem.

$$\min_{\mathbf{x}} g(\mathbf{x}) = \|\mathbf{x} - \mathbf{P}\|^2 \quad (5)$$
$$\text{s.t. } h(\mathbf{x}) = \mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} \leq 0 \quad (6)$$

We now show that the optimization problem is indeed convex. Suppose $\mathbf{x_1}$ and $\mathbf{x_2}$ satisfy $h(\mathbf{x}) \leq 0$. Then,

$$\mathbf{x_1}^\top \mathbf{V} \mathbf{x_1} + 2\mathbf{u}^\top \mathbf{x_1} + f \leq 0 \quad (7)$$
$$\mathbf{x_2}^\top \mathbf{V} \mathbf{x_2} + 2\mathbf{u}^\top \mathbf{x_2} + f \leq 0 \quad (8)$$

Then, for any $0 \leq \lambda \leq 1$, substituting

$$\mathbf{x}_\lambda \leftarrow \lambda \mathbf{x_1} + (1 - \lambda) \mathbf{x_2} \quad (9)$$

into (6), and noting that $\mathbf{V}$ is positive semidefinite, we get

$$h(\mathbf{x}_\lambda) \leq \lambda h(\mathbf{x_1}) + (1 - \lambda) h(\mathbf{x_2})$$
$$+ 2\lambda(1 - \lambda)(\mathbf{x_1} - \mathbf{x_2})^\top \mathbf{V}(\mathbf{x_1} - \mathbf{x_2}) \quad (10)$$
$$\leq 2\lambda(1 - \lambda)(\mathbf{x_1} - \mathbf{x_2})^\top \mathbf{V}(\mathbf{x_1} - \mathbf{x_2}) \quad (11)$$
$$\leq 0 \quad (12)$$

Hence, the optimization problem is convex as the set of points on the parabola form a convex set. We can solve the above problem using quadratic programming (QP) or semidefinite programming (SDP). In the case of the latter, we apply the semidefinite relaxation to get the following problem.

$$\min_{\mathbf{X}} \text{tr}(\mathbf{CX}) \quad (13)$$
$$\text{s.t. } \text{tr}(\mathbf{AX}) \leq 0 \quad (14)$$
$$\mathbf{X} \geq 0 \quad (15)$$

where

$$\mathbf{C} = \begin{pmatrix} \mathbf{I} & -\mathbf{P} \\ -\mathbf{P}^\top & \|\mathbf{P}\|^2 \end{pmatrix} \quad (16)$$

$$\mathbf{A} = \begin{pmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^\top & f \end{pmatrix} \quad (17)$$

The problem is solved using *cvxpy* in the Python codes `codes/parab_qp.py` using QP and `codes/parab_sdp.py` using SDP.