Circle Assignment

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***Abstract*—This document contains the solution to Question 13 of Exercise 2 in Chapter 10 of the class 10 NCERT textbook.**

1) Prove that opposite sides of a quadrilateral circumscribing a circle subtend supplementary angles at the centre of the circle.
   Solution: We begin by proving a useful lemma.
   Lemma 0.1. The line joining the centre of the circle to an external point bisects the angle subtended by the tangent chord at the centre.
   Proof. Refer to Fig. 1, generated using the Python code `codes/tangent.py`. Set $\mathbf{O}$ to be the origin. Since $OA \perp AP$,
$$\mathbf{A}^\top(\mathbf{A}-\mathbf{P}) = 0 \tag{1}$$
$$\implies \mathbf{A}^\top\mathbf{P} = \|\mathbf{A}\|^2 \tag{2}$$
   Similarly,
$$\mathbf{B}^\top\mathbf{P} = \|\mathbf{B}\|^2 \tag{3}$$
   Since $\mathbf{A}$ and $\mathbf{B}$ lie on the circle, their norms are equal. Thus, from (2) and (3),
$$\mathbf{A}^\top\mathbf{P} = \mathbf{B}^\top\mathbf{P} \tag{4}$$
   and the lemma follows. □

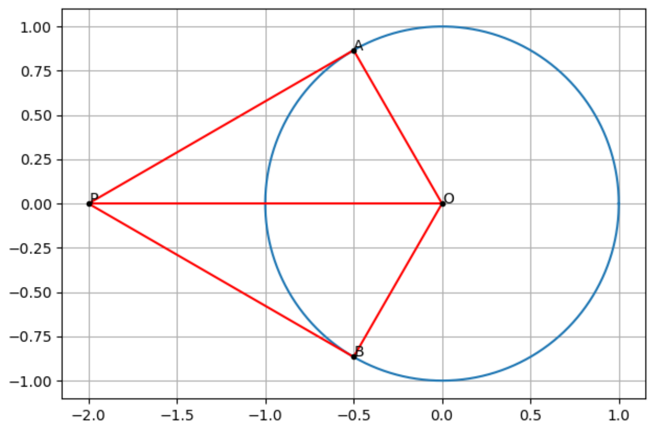

Fig. 1: OP bisects $\angle AOB$.

Call the quadrilateral $ABCD$, where
$$\mathbf{A} = \begin{pmatrix}-2\\0\end{pmatrix},\ \mathbf{C} = \begin{pmatrix}2\\0\end{pmatrix} \tag{5}$$
Suppose that $ABCD$ circumscribes the unit circle, given by
$$\mathbf{x}^\top\mathbf{x} - 1 = 0 \tag{6}$$
Comparing (6) with the general equation of the circle,
$$\mathbf{u} = \mathbf{0},\ f = -1 \tag{7}$$
To find the points of contact from $\mathbf{A}$, we have
$$\mathbf{\Sigma} = (\mathbf{A}+\mathbf{u})(\mathbf{A}+\mathbf{u})^\top - (\mathbf{A}^\top\mathbf{A} + 2\mathbf{u}^\top\mathbf{A} + f)\mathbf{I} \tag{8}$$
$$= \begin{pmatrix}1 & 0\\0 & -3\end{pmatrix} \tag{9}$$
The eigenvalues of $\mathbf{\Sigma}$ is
$$\lambda_1 = 1,\ \lambda_2 = -3 \tag{10}$$
and
$$\mathbf{n_1} = \begin{pmatrix}1\\\sqrt{3}\end{pmatrix},\ \mathbf{n_2} = \begin{pmatrix}1\\-\sqrt{3}\end{pmatrix} \tag{11}$$
Thus, the points of contact are given by
$$\mathbf{E} = \frac{1}{2}\begin{pmatrix}-1\\\sqrt{3}\end{pmatrix},\ \mathbf{H} = \frac{1}{2}\begin{pmatrix}-1\\-\sqrt{3}\end{pmatrix} \tag{12}$$
$$\mathbf{F} = \frac{1}{2}\begin{pmatrix}1\\\sqrt{3}\end{pmatrix},\ \mathbf{G} = \frac{1}{2}\begin{pmatrix}1\\-\sqrt{3}\end{pmatrix} \tag{13}$$
Using the lemma we proved above, the direction vectors of $\mathbf{B}$ and $\mathbf{D}$ are
$$\mathbf{d_B} = \mathbf{E}+\mathbf{F} = \begin{pmatrix}0\\\sqrt{3}\end{pmatrix} \tag{14}$$
$$\mathbf{d_D} = \mathbf{G}+\mathbf{H} = \begin{pmatrix}0\\-\sqrt{3}\end{pmatrix} \tag{15}$$

Clearly, from (5), (14) and (15),

$$\mathbf{A}^\top \mathbf{B} = 0 \implies \angle AOB = \frac{\pi}{2} \tag{16}$$

$$\mathbf{C}^\top \mathbf{D} = 0 \implies \angle COD = \frac{\pi}{2} \tag{17}$$

Hence, $\angle AOB + \angle COD = \pi$, as required.
The situation is illustrated in Fig. 2 plotted by the Python code `codes/quad_circ.py`.

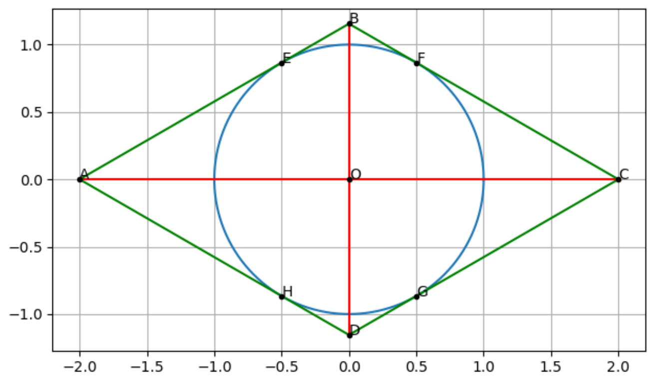

Fig. 2: Angles subtended by the opposite sides of a circumscribing quadrilateral at the center of its incircle are supplementary.