Line Assignment

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***Abstract*—This document contains a general solution to Question 16 of Exercise 2 in Chapter 11 of the class 12 NCERT textbook.**

1) Find the shortest distance between the lines whose vector equations are

$$L_1 : \mathbf{x} = \mathbf{x_1} + \lambda_1 \mathbf{m_1} \tag{1}$$
$$L_2 : \mathbf{x} = \mathbf{x_2} + \lambda_2 \mathbf{m_2} \tag{2}$$

Solution: Let $\mathbf{A}$ and $\mathbf{B}$ be points on lines L_1 and L_2 respectively such that AB is normal to both lines. Define

$$\mathbf{M} \triangleq \begin{pmatrix} \mathbf{m_1} & \mathbf{m_2} \end{pmatrix} \tag{3}$$
$$\boldsymbol{\lambda} \triangleq \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} \tag{4}$$
$$\mathbf{x} \triangleq \mathbf{x_2} - \mathbf{x_1} \tag{5}$$

Then, we have the following equations:

$$\mathbf{A} = \mathbf{x_1} + \lambda_1 \mathbf{m_1} \tag{6}$$
$$\mathbf{B} = \mathbf{x_2} + \lambda_2 \mathbf{m_2} \tag{7}$$

From (6) and (7), define the real-valued function f as

$$f(\boldsymbol{\lambda}) \triangleq \|\mathbf{A} - \mathbf{B}\|^2 \tag{8}$$
$$= \|\mathbf{M}\boldsymbol{\lambda} - \mathbf{x}\|^2 \tag{9}$$
$$= (\mathbf{M}\boldsymbol{\lambda} - \mathbf{x})^\top (\mathbf{M}\boldsymbol{\lambda} - \mathbf{x}) \tag{10}$$
$$= \boldsymbol{\lambda}^\top \left(\mathbf{M}^\top \mathbf{M}\right) \boldsymbol{\lambda} - 2\mathbf{x}^\top \mathbf{M}\boldsymbol{\lambda} + \|\mathbf{x}\|^2 \tag{11}$$

From (11), we see that f is quadratic in $\boldsymbol{\lambda}$. We now prove a useful lemma here.

Lemma 1. The quadratic form

$$q(\mathbf{x}) \triangleq \mathbf{x}^\top \mathbf{A}\mathbf{x} + \mathbf{b}^\top \mathbf{x} + c \tag{12}$$

is convex iff $\mathbf{A}$ is positive semi-definite.

Proof. Consider two points $\mathbf{x_1}$ and $\mathbf{x_2}$, and a real constant $0 \le \mu \le 1$. Then,

$$\mu f(\mathbf{x_1}) + (1-\mu) f(\mathbf{x_2}) - f(\mu\mathbf{x_1} + (1-\mu)\mathbf{x_2})$$
$$= \left(\mu - \mu^2\right) \mathbf{x_1}^\top \mathbf{A}\mathbf{x_1} + \left(1 - \mu - (1-\mu)^2\right) \mathbf{x_2}^\top \mathbf{A}\mathbf{x_2} - 2\mu(1-\mu)\mathbf{x_1}^\top \mathbf{A}\mathbf{x_2} \tag{13}$$
$$= \mu(1-\mu)\left(\mathbf{x_1}^\top \mathbf{A}\mathbf{x_1} - 2\mathbf{x_1}^\top \mathbf{A}\mathbf{x_2} + \mathbf{x_2}^\top \mathbf{A}\mathbf{x_2}\right) \tag{14}$$
$$= \mu(1-\mu)(\mathbf{x_1} - \mathbf{x_2})^\top \mathbf{A}(\mathbf{x_1} - \mathbf{x_2}) \tag{15}$$

Since $\mathbf{x_1}$ and $\mathbf{x_2}$ are arbitrary, it follows from (15) that

$$\mu f(\mathbf{x_1}) + (1-\mu) f(\mathbf{x_2}) \ge f(\mu\mathbf{x_1} + (1-\mu)\mathbf{x_2}) \tag{16}$$

iff $\mathbf{A}$ is positive semi-definite, as required. □

Using the above lemma, we show that f is convex by showing that $\mathbf{M}^\top\mathbf{M}$ is positive semi-definite. Indeed, for any $\mathbf{p} \triangleq \begin{pmatrix} x \\ y \end{pmatrix}$,

$$\mathbf{p}^\top \mathbf{M}^\top \mathbf{M}\mathbf{p} = \|\mathbf{M}\mathbf{p}\|^2 \ge 0 \tag{17}$$

and thus, f is convex.
We need to minimize f as a function of $\boldsymbol{\lambda}$. Differentiating (11) using the chain rule,

$$\frac{df(\boldsymbol{\lambda})}{d\boldsymbol{\lambda}} = \mathbf{M}^\top (\mathbf{M}\boldsymbol{\lambda} - \mathbf{x}) + \mathbf{M}(\mathbf{M}\boldsymbol{\lambda} - \mathbf{x})^\top \tag{18}$$
$$= 2\mathbf{M}^\top (\mathbf{M}\boldsymbol{\lambda} - \mathbf{x}) \tag{19}$$

Setting (19) to zero gives the equation

$$\mathbf{M}^\top \mathbf{M}\boldsymbol{\lambda} = \mathbf{M}^\top \mathbf{x} \tag{20}$$

We use singular value decomposition here. Let

$$\mathbf{M} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^\top \tag{21}$$

where $\mathbf{U}, \mathbf{V}$ are orthogonal and $\boldsymbol{\Sigma}$ is diagonal with nonnegative diagonal entries. Substituting in (20),

$$\mathbf{V}\boldsymbol{\Sigma}\mathbf{U}^\top\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^\top\boldsymbol{\lambda} = \mathbf{V}\boldsymbol{\Sigma}\mathbf{U}^\top\mathbf{x} \tag{22}$$

$$\implies \mathbf{V}\boldsymbol{\Sigma}^2\mathbf{V}^\top\boldsymbol{\lambda} = \mathbf{V}\boldsymbol{\Sigma}\mathbf{U}^\top\mathbf{x} \tag{23}$$

$$\implies \boldsymbol{\lambda} = \left(\mathbf{V}\boldsymbol{\Sigma}^2\mathbf{V}^\top\right)^{-1}\mathbf{V}\boldsymbol{\Sigma}\mathbf{U}^\top\mathbf{x} \tag{24}$$

$$\implies \boldsymbol{\lambda} = \mathbf{V}\boldsymbol{\Sigma}^{-2}\mathbf{V}^\top\mathbf{V}\boldsymbol{\Sigma}\mathbf{U}^\top\mathbf{x} \tag{25}$$

$$\implies \boldsymbol{\lambda} = \mathbf{V}\boldsymbol{\Sigma}^{-1}\mathbf{U}^\top\mathbf{x} \tag{26}$$

where $\boldsymbol{\Sigma}^{-1}$ is obtained by inverting the nonzero elements of $\boldsymbol{\Sigma}$. Thus, the shortest distance is given by using (21) and (26) in (11), and is given by

$$d = \left\|\left(\mathbf{U}\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{-1}\right)\mathbf{U}^\top - \mathbf{I}\right)\mathbf{x}\right\| \tag{27}$$