Semidefinite Programming Assignment

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***Abstract*—This document contains the solution to Question 21 of Exercise 6 in Chapter 6 of the class 12 NCERT textbook.**

1) The line
$$y = mx + 1 \tag{1}$$
is a tangent to the curve
$$y^2 = 4x \tag{2}$$
if the value of m is
a) 1
b) 2
c) 3
d) $\frac{1}{2}$

Solution: Rewriting (1) and (2) in standard forms, we get,
$$\mathbf{n} = \begin{pmatrix} m \\ -1 \end{pmatrix}, c = -1 \tag{3}$$
$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}, f = 0 \tag{4}$$
Now, we rotate the axes by applying the transformation
$$\mathbf{x} \leftarrow \mathbf{P}\mathbf{x} \tag{5}$$
such that
$$\mathbf{P}\mathbf{n} = \mathbf{e_1} \tag{6}$$
We find $\mathbf{P}$ by forming the augmented matrix
$$\begin{pmatrix} 1 & 0 & m \\ 0 & 1 & -1 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R_1}{m}} \begin{pmatrix} \frac{1}{m} & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} \tag{7}$$
$$\xleftrightarrow{R_2 \leftarrow R_1 + R_2} \begin{pmatrix} \frac{1}{m} & 0 & 1 \\ \frac{1}{m} & 1 & 0 \end{pmatrix} \tag{8}$$
giving
$$\mathbf{P} = \begin{pmatrix} \frac{1}{m} & 0 \\ \frac{1}{m} & 1 \end{pmatrix} \tag{9}$$
Thus, using (5), the optimization problem becomes
$$\min_{\mathbf{x}} \|\mathbf{x}\|^2 \tag{10}$$
$$\text{s.t. } \mathbf{x}^\top \mathbf{P}^\top \mathbf{V} \mathbf{P} \mathbf{x} + 2\mathbf{u}^\top \mathbf{P} \mathbf{x} + f = 0 \tag{11}$$
We use semidefinite programming to solve this problem. Setting
$$\mathbf{y} = \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} \tag{12}$$
we rewrite the optimization problem as
$$\min_{\mathbf{y}} \mathbf{y}^\top \mathbf{C} \mathbf{y} \tag{13}$$
$$\text{s.t. } \mathbf{y}^\top \mathbf{A} \mathbf{y} = 0 \tag{14}$$
Using semidefinite relaxation, the problem becomes
$$\min_{\mathbf{Y}} \operatorname{tr}(\mathbf{C}\mathbf{Y}) \tag{15}$$
$$\text{s.t. } \operatorname{tr}(\mathbf{A}\mathbf{Y}) = 0 \tag{16}$$
$$\mathbf{A} \succeq \mathbf{0} \tag{17}$$
The Python code `codes/sdp.py` solves this problem by plotting the values of (15) in Fig. 1 and choosing m appropriately using *cvxpy*. The cost (15) is minimized when $m = \pm 1$. Hence, option **a)** is correct.

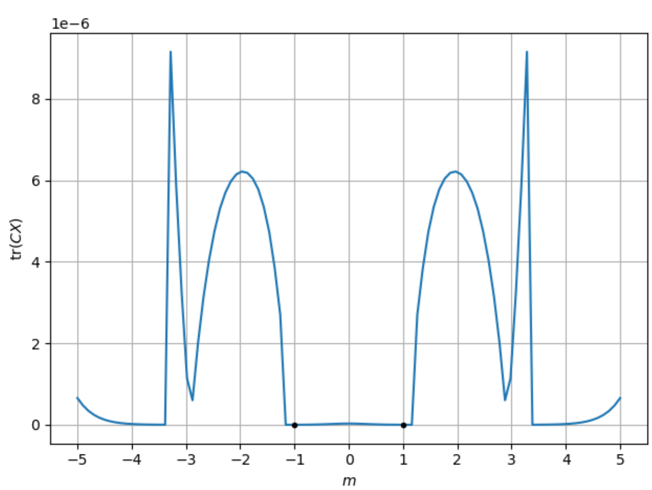

Fig. 1: Cost for various values of m.