Circle Assignment

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***Abstract*—This document contains the solution to Question 12 of Exercise 5 in Chapter 10 of the class 9 NCERT textbook.**

1) Prove that a cyclic paralellogram is a rectangle.
Solution: Consider the points $\mathbf{P_i},\ 1 \le i \le 4$ on the unit circle. Thus, for $1 \le i \le 4$,

$$\|\mathbf{P_i}\| = 1 \tag{1}$$

Since $P_1P_2P_3P_4$ form a parallelogram,

$$\|\mathbf{P_1} - \mathbf{P_2}\|^2 = \|\mathbf{P_3} - \mathbf{P_4}\|^2 \tag{2}$$

$$\|\mathbf{P_1}\|^2 - 2\mathbf{P_1}^\top\mathbf{P_2} + \|\mathbf{P_2}\|^2 = \|\mathbf{P_3}\|^2 - 2\mathbf{P_3}^\top\mathbf{P_4} + \|\mathbf{P_4}\|^2 \tag{3}$$

$$\mathbf{P_1}^\top\mathbf{P_2} = \mathbf{P_3}^\top\mathbf{P_4} \tag{4}$$

Similarly,

$$\|\mathbf{P_1} - \mathbf{P_4}\|^2 = \|\mathbf{P_2} - \mathbf{P_3}\|^2 \tag{5}$$

$$\implies \mathbf{P_1}^\top\mathbf{P_4} = \mathbf{P_2}^\top\mathbf{P_3} \tag{6}$$

We know that

$$\cos\angle P_4P_1P_2 = \frac{(\mathbf{P_1} - \mathbf{P_2})^\top(\mathbf{P_1} - \mathbf{P_4})}{\|\mathbf{P_1} - \mathbf{P_2}\|\,\|\mathbf{P_1} - \mathbf{P_4}\|} \tag{7}$$

Using the fact that the opposite angles of a cyclic quadrilateral are supplementary, we see that the sum of the cosines of opposite angles is zero. Combining this with (2) and (5),

$$\frac{(\mathbf{P_1} - \mathbf{P_4})^\top(\mathbf{P_1} - \mathbf{P_2})}{\|\mathbf{P_1} - \mathbf{P_4}\|\,\|\mathbf{P_1} - \mathbf{P_2}\|} + \frac{(\mathbf{P_3} - \mathbf{P_4})^\top(\mathbf{P_3} - \mathbf{P_2})}{\|\mathbf{P_3} - \mathbf{P_4}\|\,\|\mathbf{P_3} - \mathbf{P_2}\|} = 0 \tag{8}$$

$$(\mathbf{P_1} - \mathbf{P_4})^\top(\mathbf{P_1} - \mathbf{P_2}) + (\mathbf{P_3} - \mathbf{P_4})^\top(\mathbf{P_3} - \mathbf{P_2}) = 0 \tag{9}$$

Using (4) and (6) in (9), and noting that $\mathbf{P_i}^\top\mathbf{P_i} = 1,\ 1 \le i \le 4$,

$$\mathbf{P_4}^\top\mathbf{P_1} - \mathbf{P_4}^\top\mathbf{P_2} + \mathbf{P_1}^\top\mathbf{P_2} - 1 = 0 \tag{10}$$

$$\mathbf{P_4}^\top\mathbf{P_1} - \mathbf{P_4}^\top\mathbf{P_2} + \mathbf{P_1}^\top\mathbf{P_2} - \mathbf{P_1}^\top\mathbf{P_1} = 0 \tag{11}$$

$$(\mathbf{P_4} - \mathbf{P_1})^\top(\mathbf{P_1} - \mathbf{P_2}) = 0 \tag{12}$$

Hence $P_1P_2 \perp P_1P_4$, and thus, the parallelogram is indeed a rectangle. The situation is demonstrated in Fig. 1, plotted by the Python code `codes/circle.py`.

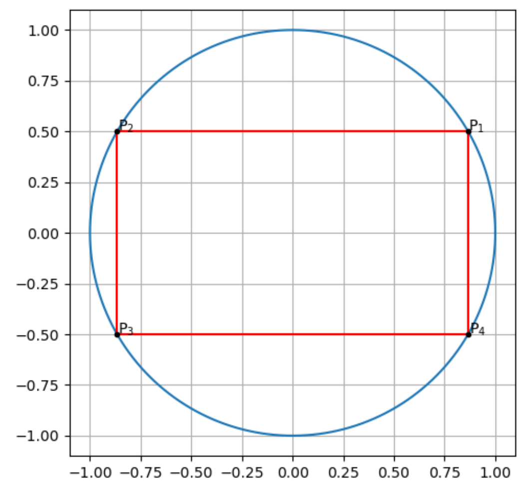

Fig. 1: $P_1P_2P_3P_4$ is a rectangle.