

Line Assignment

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Abstract—This document contains a general solution to Question 16 of Exercise 2 in Chapter 11 of the class 12 NCERT textbook.

- 1) Find the shortest distance between the lines whose vector equations are

$$L_1 : \mathbf{x} = \mathbf{x}_1 + \lambda_1 \mathbf{m}_1 \quad (1)$$

$$L_2 : \mathbf{x} = \mathbf{x}_2 + \lambda_2 \mathbf{m}_2 \quad (2)$$

Solution: Let \mathbf{A} and \mathbf{B} be points on lines L_1 and L_2 respectively such that AB is normal to both lines. Define

$$\mathbf{M} \triangleq (\mathbf{m}_1 \quad \mathbf{m}_2) \quad (3)$$

$$\lambda \triangleq \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} \quad (4)$$

$$\mathbf{x} \triangleq \mathbf{x}_2 - \mathbf{x}_1 \quad (5)$$

Then, we have the following equations:

$$\mathbf{A} = \mathbf{x}_1 + \lambda_1 \mathbf{m}_1 \quad (6)$$

$$\mathbf{B} = \mathbf{x}_2 + \lambda_2 \mathbf{m}_2 \quad (7)$$

From (6) and (7), define the real-valued function f as

$$f(\lambda) \triangleq \|\mathbf{A} - \mathbf{B}\|^2 = \|\mathbf{M}\lambda - \mathbf{x}\|^2 \quad (8)$$

We need to minimize f as a function of λ . Thus,

$$\frac{df(\lambda)}{d\lambda} = \frac{d}{d\lambda} ((\mathbf{M}\lambda - \mathbf{x})^\top (\mathbf{M}\lambda - \mathbf{x})) \quad (9)$$

$$= \mathbf{M}^\top (\mathbf{M}\lambda - \mathbf{x}) + (\mathbf{M}\lambda - \mathbf{x})^\top \mathbf{M} \quad (10)$$

$$= 2\mathbf{M}^\top (\mathbf{M}\lambda - \mathbf{x}) \quad (11)$$

Setting (11) to zero gives

$$\mathbf{M}^\top \mathbf{M} \lambda = \mathbf{M}^\top \mathbf{x} \quad (12)$$

We have the following cases:

- a) There exists a λ satisfying

$$\mathbf{M}\lambda = \mathbf{x} \quad (13)$$

$$\implies \lambda_1 \mathbf{m}_1 - \lambda_2 \mathbf{m}_2 = \mathbf{x}_2 - \mathbf{x}_1 \quad (14)$$

$$\implies \mathbf{x}_1 + \lambda_1 \mathbf{m}_1 = \mathbf{x}_2 + \lambda_2 \mathbf{m}_2 \quad (15)$$

Thus, both lines intersect at a point and the shortest distance between them is 0. To check for the existence of such a λ , we can bring the augmented matrix $(\mathbf{M} \quad \mathbf{x})$ into row-reduced echelon form and check whether there is a pivot in the last column.

- b) $\mathbf{M}^\top \mathbf{M}$ is singular. Since $\mathbf{M}^\top \mathbf{M}$ is a square matrix of order 2, its rank must be 1. Further,

$$\begin{aligned} \det(\mathbf{M}^\top \mathbf{M}) &= \begin{vmatrix} \mathbf{m}_1^\top \mathbf{m}_1 & \mathbf{m}_1^\top \mathbf{m}_2 \\ \mathbf{m}_1^\top \mathbf{m}_2 & \mathbf{m}_2^\top \mathbf{m}_2 \end{vmatrix} \\ &= (\|\mathbf{m}_1\| \cdot \|\mathbf{m}_2\|)^2 - (\mathbf{m}_1^\top \mathbf{m}_2)^2 \end{aligned} \quad (16)$$

Thus, equating the determinant to zero gives

$$\|\mathbf{m}_1\| \cdot \|\mathbf{m}_2\| = |\mathbf{m}_1^\top \mathbf{m}_2| \quad (18)$$

which implies that both lines are parallel to each other. Setting $\mathbf{m}_2 = k\mathbf{m}_1, k \in \mathbb{R} \setminus \{0\}$, we obtain one equation from (12).

$$\mathbf{m}_1^\top \mathbf{m}_1 (\lambda_1 - k\lambda_2) = \mathbf{m}_1^\top \mathbf{x} \quad (19)$$

$$\implies \lambda_1 - k\lambda_2 = \frac{\mathbf{m}_1^\top \mathbf{x}}{\|\mathbf{m}_1\|^2} \quad (20)$$

Therefore, the required shortest distance is

$$\|\mathbf{A} - \mathbf{B}\| = \left\| \frac{\mathbf{m}_1^\top \mathbf{x} \mathbf{m}_1}{\|\mathbf{m}_1\|^2} - \mathbf{x} \right\| \quad (21)$$

- c) $\mathbf{M}^\top \mathbf{M}$ is nonsingular. This implies that the lines are skew. From (12),

$$\lambda = (\mathbf{M}^\top \mathbf{M})^{-1} \mathbf{M}^\top \mathbf{x} \quad (22)$$

and therefore, the shortest distance is

$$\|\mathbf{A} - \mathbf{B}\| = \left\| (\mathbf{M}(\mathbf{M}^\top \mathbf{M})^{-1} \mathbf{M}^\top - \mathbf{I}_n) \mathbf{x} \right\| \quad (23)$$

where \mathbf{I}_n is the identity matrix of order n .