Application of Integrals Assignment

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Abstract—This document contains the solution to Question 18 of Exercise 3 in Chapter 8 of the class 12 NCERT textbook.

1) The area of the circle

$$x^2 + y^2 = 16 \quad (1)$$

exterior to the parabola

$$y^2 = 6x \quad (2)$$

is
a) $\frac{4}{3}\left(4\pi - \sqrt{3}\right)$
b) $\frac{4}{3}\left(4\pi + \sqrt{3}\right)$
c) $\frac{4}{3}\left(8\pi - \sqrt{3}\right)$
d) $\frac{4}{3}\left(8\pi + \sqrt{3}\right)$

Solution: We convert (1) and (2) into matrix form to find their points of intersection.

$$\mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - 16 = 0 \quad (3)$$

$$\mathbf{x}^\top \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + 2\begin{pmatrix} -3 & 0 \end{pmatrix} \mathbf{x} = 0 \quad (4)$$

Adding (3) to μ times (4), we get the locus of the intersection of the two conics.

$$\mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & \mu + 1 \end{pmatrix} \mathbf{x} + 2\begin{pmatrix} -3\mu & 0 \end{pmatrix} \mathbf{x} - 16 = 0 \quad (5)$$

For (5) to represent a pair of straight lines,

$$\begin{vmatrix} 1 & 0 & -3\mu \\ 0 & \mu + 1 & 0 \\ -3\mu & 0 & -16 \end{vmatrix} = 0 \quad (6)$$

$$\implies (\mu + 1)\left(9\mu^2 + 16\right) = 0 \quad (7)$$

$$\implies \mu = -1 \quad (8)$$

where (8) follows since $9\mu^2 + 16 \geq 0$. Hence,

the straight lines are represented by

$$\mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + 2\begin{pmatrix} 3 & 0 \end{pmatrix} \mathbf{x} - 16 = 0 \quad (9)$$

$$\implies \mathbf{x}^\top \mathbf{e_1} \mathbf{e_1}^\top \mathbf{x} + 6\mathbf{e_1}^\top \mathbf{x} - 16 = 0 \quad (10)$$

$$\implies \left(\mathbf{e_1}^\top \mathbf{x}\right)^2 + 6\mathbf{e_1}^\top \mathbf{x} - 16 = 0 \quad (11)$$

$$\implies \left(\mathbf{e_1}^\top \mathbf{x} - 2\right)\left(\mathbf{e_1}^\top \mathbf{x} + 8\right) = 0 \quad (12)$$

Since $\mathbf{x}$ lies on (1), it follows that

$$-8 = \mathbf{e_1}^\top \mathbf{x} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 4\cos\theta \\ 4\sin\theta \end{pmatrix} = 4\cos\theta \geq -4 \quad (13)$$

$$\implies \mathbf{e_1}^\top \mathbf{x} = 2 \quad (14)$$

Substituting in (4),

$$\mathbf{x}^\top \mathbf{e_2} \mathbf{e_2}^\top \mathbf{x} = 6\mathbf{e_1}^\top \mathbf{x} = 12 \quad (15)$$

$$\implies \left(\mathbf{e_2}^\top \mathbf{x}\right)^2 = 12 \quad (16)$$

$$\implies \mathbf{e_2}^\top \mathbf{x} = \pm 2\sqrt{3} \quad (17)$$

Combining (14) and (17), the conics intersect at

$$\mathbf{x} = \begin{pmatrix} \mathbf{e_1}^\top \\ \mathbf{e_2}^\top \end{pmatrix} \mathbf{x} = \begin{pmatrix} 2 \\ \pm 2\sqrt{3} \end{pmatrix} \quad (18)$$

Hence, the area interior to the circle and parabola is

$$A = \int_{-2\sqrt{3}}^{2\sqrt{3}} \sqrt{16 - y^2} - \frac{y^2}{6} \, dy \quad (19)$$

$$= \frac{y\sqrt{16 - y^2}}{2} + \frac{16}{2}\sin^{-1}\left(\frac{y}{4}\right) - \frac{y^3}{18}\Bigg|_{-2\sqrt{3}}^{2\sqrt{3}} \quad (20)$$

$$= 4\sqrt{3} + \frac{16\pi}{3} - \frac{8\sqrt{3}}{3} \quad (21)$$

$$= \frac{4\sqrt{3} + 16\pi}{3} \quad (22)$$

Thus, the required exterior area is (where $r = 4$

is the radius of (1))

$$A' = \pi r^2 - A \tag{23}$$

$$= 16\pi - \frac{4\sqrt{3} + 16\pi}{3} \tag{24}$$

$$= \frac{4}{3}\left(8\pi + \sqrt{3}\right) \tag{25}$$

Hence, the correct answer is option **d)**.
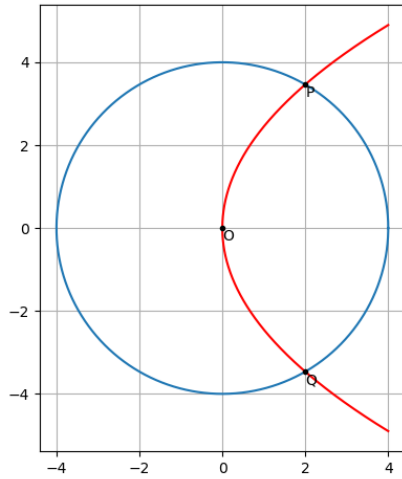The situation is demonstrated in Fig. 1, plotted by the Python code `codes/parab_circ.py`

Fig. 1: The conics meet at points P and Q.