Optimization Assignment

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***Abstract*—This document contains the solution to Question 4 of Exercise 2 in Chapter 10 of the class 11 NCERT textbook.**

1) Find the coordinates of the foot of perpendicular from the point

$$\mathbf{P} = \begin{pmatrix} -1 \\ 3 \end{pmatrix} \tag{1}$$

to the line

$$\begin{pmatrix} 3 & -4 \end{pmatrix} \mathbf{x} = 16 \tag{2}$$

Solution: Any point on (2) is clearly of the form

$$\mathbf{Q} = \mathbf{A} + \lambda \mathbf{m} \tag{3}$$

where $\lambda \in \mathbb{R}$ and

$$\mathbf{A} = \begin{pmatrix} 0 \\ -4 \end{pmatrix}, \ \mathbf{m} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \tag{4}$$

Thus,

$$f(\lambda) = \left\|\mathbf{Q} - \mathbf{P}\right\|^2 \tag{5}$$

$$= \left\|\mathbf{A} - \mathbf{P} + \lambda \mathbf{m}\right\|^2 \tag{6}$$

$$= \left\|\mathbf{m}\right\|^2 \lambda^2 + 2\mathbf{m}^\top (\mathbf{A} - \mathbf{P}) \lambda + \left\|\mathbf{A} - \mathbf{P}\right\|^2 \tag{7}$$

Since (7) is convex, we use the gradient descent function on λ to converge at the minimum of $f(\lambda)$.

$$\lambda_{n+1} = \lambda_n - \alpha f'(\lambda_n) \tag{8}$$

$$= \left(1 - 2\alpha \left\|\mathbf{m}\right\|^2\right) \lambda_n + 2\alpha \mathbf{m}^\top (\mathbf{A} - \mathbf{P}) \tag{9}$$

Taking the one-sided Z-transform on both sides of (9),

$$z\Lambda(z) = \left(1 - 2\alpha \left\|\mathbf{m}\right\|^2\right) \Lambda(z) - \frac{2\alpha \mathbf{m}^\top (\mathbf{A} - \mathbf{P})}{1 - z^{-1}} \tag{10}$$

Solving (10)

$$\Lambda(z) = -\frac{2\alpha \mathbf{m}^\top (\mathbf{A} - \mathbf{P}) z^{-1}}{\left(1 - z^{-1}\right)\left(1 - \left(1 - 2\alpha \left\|\mathbf{m}\right\|^2\right) z^{-1}\right)} \tag{11}$$

$$= -\frac{\mathbf{m}^\top (\mathbf{A} - \mathbf{P})}{\left\|\mathbf{m}\right\|^2} \left(\frac{1}{1 - z^{-1}}\right. \tag{12}$$

$$\left. - \frac{1}{1 - \left(1 - 2\alpha \left\|\mathbf{m}\right\|^2\right) z^{-1}}\right) \tag{13}$$

$$= -\frac{\mathbf{m}^\top (\mathbf{A} - \mathbf{P})}{\left\|\mathbf{m}\right\|^2} \sum_{k=0}^{\infty} \left(1 - \left(1 - 2\alpha \left\|\mathbf{m}\right\|^2\right)^k\right) z^{-k} \tag{14}$$

From (11), the ROC is

$$|z| > \max\left\{1, 1 - 2\alpha \left\|\mathbf{m}\right\|^2\right\} \tag{15}$$

$$\implies 0 < 1 - 2\alpha \left\|\mathbf{m}\right\|^2 < 1 \tag{16}$$

$$\implies 0 < \alpha < \frac{1}{2\left\|\mathbf{m}\right\|^2} \tag{17}$$

Thus, if α satisfies (17), then from (14), substituting from (4),

$$\lim_{n \to \infty} \lambda_n = -\frac{\mathbf{m}^\top (\mathbf{A} - \mathbf{P})}{\left\|\mathbf{m}^2\right\|} = \frac{17}{25} \tag{18}$$

We select the following parameters to arrive at the optimal λ, where N is the number of iterations and ϵ is the convergence limit. The gradient descent is demonstrated in Fig. 1, plotted by the Python code `codes/grad_desc.py`. The relevant parameters are shown in Table 1.

Parameter	Value
λ_0	0
α	0.1
N	1000000
ϵ	10^{-6}

TABLE 1: Parameters for Gradient Descent

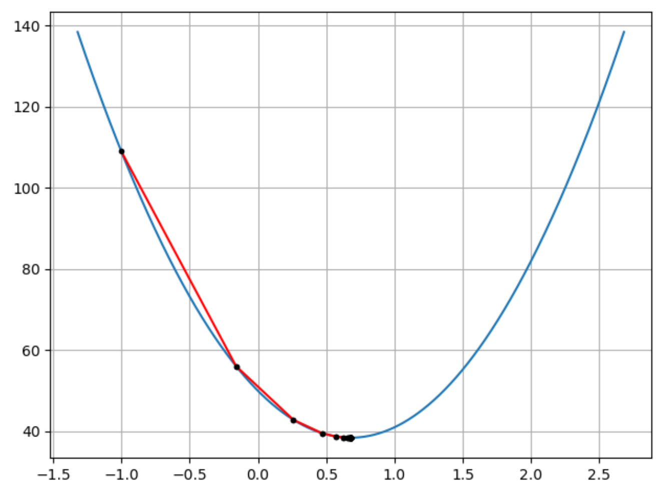

Fig. 1: Gradient descent to get the optimal λ.