Quadratic Programming Assignment

Gautam Singh

***Abstract*—This document contains the solution to a modification of Question 27 of Exercise 5 in Chapter 6 of the class 12 NCERT textbook.**

1) Show that the point on the curve
$$x^2 = 2y \tag{1}$$
which is nearest to the point $\mathbf{P} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is a convex optimization problem.
Solution: We need to find
$$\min_{\mathbf{x}} g(\mathbf{x}) = \|\mathbf{x} - \mathbf{P}\|^2 \tag{2}$$
$$\text{s.t. } h(\mathbf{x}) = \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} = 0 \tag{3}$$
where
$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ \mathbf{u} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \tag{4}$$
Since $\mathbf{P}$ lies *outside* the given curve, we can apply the following relaxation to make it a convex optimization problem.
$$\min_{\mathbf{x}} g(\mathbf{x}) = \|\mathbf{x} - \mathbf{P}\|^2 \tag{5}$$
$$\text{s.t. } h(\mathbf{x}) = \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} \le 0 \tag{6}$$
We now show that the optimization problem is indeed convex. Suppose $\mathbf{x_1}$ and $\mathbf{x_2}$ satisfy $h(\mathbf{x}) \le 0$. Then,
$$\mathbf{x_1}^\top \mathbf{V}\mathbf{x_1} + 2\mathbf{u}^\top \mathbf{x_1} + f \le 0 \tag{7}$$
$$\mathbf{x_2}^\top \mathbf{V}\mathbf{x_2} + 2\mathbf{u}^\top \mathbf{x_2} + f \le 0 \tag{8}$$
Then, for any $0 \le \lambda \le 1$, substituting
$$\mathbf{x}_\lambda \leftarrow \lambda\mathbf{x_1} + (1-\lambda)\mathbf{x_2} \tag{9}$$
into (6), and noting that $\mathbf{V}$ is positive semi-definite, we get
$$\begin{aligned} h(\mathbf{x}_\lambda) &\le \lambda h(\mathbf{x_1}) + (1-\lambda) h(\mathbf{x_2}) \\ &\quad + 2\lambda(1-\lambda)(\mathbf{x_1} - \mathbf{x_2})^\top \mathbf{V}(\mathbf{x_1} - \mathbf{x_2}) && (10) \\ &\le 2\lambda(1-\lambda)(\mathbf{x_1} - \mathbf{x_2})^\top \mathbf{V}(\mathbf{x_1} - \mathbf{x_2}) && (11) \\ &\le 0 && (12) \end{aligned}$$
Hence, the optimization problem is convex as the set of points on the parabola form a convex set. The problem is solved using *cvxpy* in the Python code `codes/parab_cvx.py`.