

# Optimization Assignment

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**Abstract**—This document contains the solution to Question 4 of Exercise 2 in Chapter 10 of the class 11 NCERT textbook.

- 1) Find the coordinates of the foot of perpendicular from the point

$$\mathbf{P} = \begin{pmatrix} -1 \\ 3 \end{pmatrix} \quad (1)$$

to the line

$$(3 \quad -4)\mathbf{x} = 16 \quad (2)$$

**Solution:** Any point on (2) is clearly of the form

$$\mathbf{Q} = \mathbf{A} + \lambda \mathbf{m} \quad (3)$$

where  $\lambda \in \mathbb{R}$  and

$$\mathbf{A} = \begin{pmatrix} 0 \\ -4 \end{pmatrix}, \quad \mathbf{m} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \quad (4)$$

Thus,

$$f(\lambda) = \|\mathbf{Q} - \mathbf{P}\|^2 \quad (5)$$

$$= \|\mathbf{A} - \mathbf{P} + \lambda \mathbf{m}\|^2 \quad (6)$$

$$= \|\mathbf{m}\|^2 \lambda^2 + 2\mathbf{m}^\top (\mathbf{A} - \mathbf{P}) \lambda + \|\mathbf{A} - \mathbf{P}\|^2 \quad (7)$$

Since (7) is convex, we use the gradient descent function on  $\lambda$  to converge at the minimum of  $f(\lambda)$ .

$$\lambda_{n+1} = \lambda_n - \alpha f'(\lambda_n) \quad (8)$$

$$= (1 - 2\alpha \|\mathbf{m}\|^2) \lambda_n + 2\alpha \mathbf{m}^\top (\mathbf{A} - \mathbf{P}) \quad (9)$$

Taking the one-sided Z-transform on both sides of (9),

$$z\Lambda(z) = (1 - 2\alpha \|\mathbf{m}\|^2) \Lambda(z) - \frac{2\alpha \mathbf{m}^\top (\mathbf{A} - \mathbf{P})}{1 - z^{-1}} \quad (10)$$

Solving (10)

$$\Lambda(z) = -\frac{2\alpha \mathbf{m}^\top (\mathbf{A} - \mathbf{P}) z^{-1}}{(1 - z^{-1})(1 - (1 - 2\alpha \|\mathbf{m}\|^2) z^{-1})} \quad (11)$$

$$= -\frac{\mathbf{m}^\top (\mathbf{A} - \mathbf{P})}{\|\mathbf{m}\|^2} \left( \frac{1}{1 - z^{-1}} \right) \quad (12)$$

$$- \frac{1}{1 - (1 - 2\alpha \|\mathbf{m}\|^2) z^{-1}} \quad (13)$$

$$= -\frac{\mathbf{m}^\top (\mathbf{A} - \mathbf{P})}{\|\mathbf{m}\|^2} \sum_{k=0}^{\infty} (1 - (1 - 2\alpha \|\mathbf{m}\|^2)^k) z^{-k} \quad (14)$$

From (11), the ROC is

$$|z| > \max \{1, |1 - 2\alpha \|\mathbf{m}\|^2|\} \quad (15)$$

$$\Rightarrow -1 < 1 - 2\alpha \|\mathbf{m}\|^2 < 1 \quad (16)$$

$$\Rightarrow 0 < \alpha < \frac{1}{\|\mathbf{m}\|^2} \quad (17)$$

Thus, if  $\alpha$  satisfies (17), then from (14), substituting from (4),

$$\lim_{n \rightarrow \infty} \lambda_n = -\frac{\mathbf{m}^\top (\mathbf{A} - \mathbf{P})}{\|\mathbf{m}\|^2} = \frac{17}{25} \quad (18)$$

We select the following parameters to arrive at the optimal  $\lambda$ , where  $N$  is the number of iterations and  $\epsilon$  is the convergence limit. The gradient descent is demonstrated in Fig. 1, plotted by the Python code `codes/grad_desc.py`. The relevant parameters are shown in Table 1.

| Parameter   | Value     |
|-------------|-----------|
| $\lambda_0$ | 0         |
| $\alpha$    | 0.1       |
| $N$         | 1000000   |
| $\epsilon$  | $10^{-6}$ |

TABLE 1: Parameters for Gradient Descent

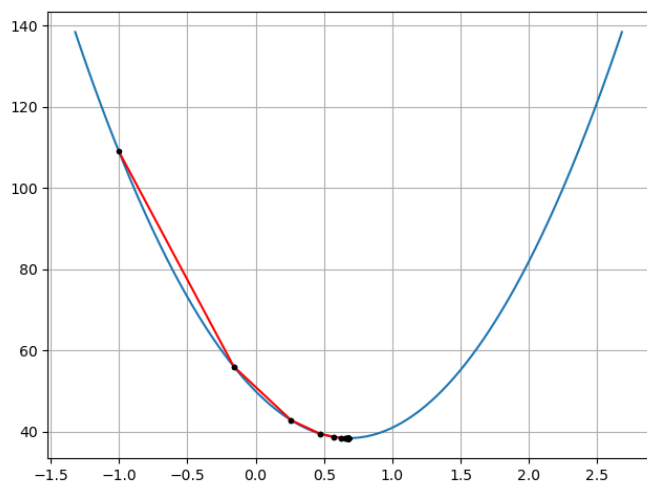


Fig. 1: Gradient descent to get the optimal  $\lambda$ .