Line Assignment

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Abstract—**This document contains a general solution to Question 16 of Exercise 2 in Chapter 11 of the class 12 NCERT textbook.**

1) Find the shortest distance between the lines whose vector equations are

$$L_1 : \mathbf{x} = \mathbf{x_1} + \lambda_1 \mathbf{m_1} \tag{1}$$
$$L_2 : \mathbf{x} = \mathbf{x_2} + \lambda_2 \mathbf{m_2} \tag{2}$$

Solution: Let $\mathbf{A}$ and $\mathbf{B}$ be points on lines L_1 and L_2 respectively such that AB is normal to both lines. Define

$$\mathbf{M} \triangleq \begin{pmatrix} \mathbf{m_1} & \mathbf{m_2} \end{pmatrix} \tag{3}$$
$$\boldsymbol{\lambda} \triangleq \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} \tag{4}$$
$$\mathbf{x} \triangleq \mathbf{x_2} - \mathbf{x_1} \tag{5}$$

Then, we have the following equations:

$$\mathbf{A} = \mathbf{x_1} + \lambda_1 \mathbf{m_1} \tag{6}$$
$$\mathbf{B} = \mathbf{x_2} + \lambda_2 \mathbf{m_2} \tag{7}$$

From (6) and (7), define the real-valued function f as

$$f(\boldsymbol{\lambda}) \triangleq \left\|\mathbf{A} - \mathbf{B}\right\|^2 \tag{8}$$
$$= \left\|\mathbf{M}\boldsymbol{\lambda} - \mathbf{x}\right\|^2 \tag{9}$$
$$= \left(\mathbf{M}\boldsymbol{\lambda} - \mathbf{x}\right)^\top \left(\mathbf{M}\boldsymbol{\lambda} - \mathbf{x}\right) \tag{10}$$

We prove a useful lemma here.

Lemma 1. For any two vectors $\mathbf{a}$ and $\mathbf{b}$, we have

$$\mathbf{a}^\top \mathbf{b} \leq \left\|\mathbf{a}\right\| \left\|\mathbf{b}\right\| \tag{11}$$

Proof. Note that

$$\left\|\mathbf{a} - \frac{\left(\mathbf{a}^\top\mathbf{b}\right)^2}{\left\|\mathbf{b}\right\|^2}\mathbf{b}\right\| \geq 0 \tag{12}$$
$$\left\|\mathbf{a}\right\|^2 - 2\frac{\left(\mathbf{a}^\top\mathbf{b}\right)^2}{\left\|\mathbf{b}\right\|^2} + \frac{\left(\mathbf{a}^\top\mathbf{b}\right)^2}{\left\|\mathbf{b}\right\|^2} \geq 0 \left\|\mathbf{a}\right\|^2 \quad \geq \frac{\left(\mathbf{a}^\top\mathbf{b}\right)^2}{\left\|\mathbf{b}\right\|^2} \tag{13}$$
$$\left\|\mathbf{a}\mathbf{b}\right\| \geq \mathbf{a}^\top \mathbf{b} \tag{14}$$

as required. □

We show that f is convex by computing the Hessian matrix of f. For a real-valued function $f(\mathbf{x})$ where $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$,

$$\mathbf{H}_f \triangleq \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 x_2} \\ \frac{\partial^2 f}{\partial x_2 x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{pmatrix} \tag{15}$$

Here, taking $\mathbf{x} = \boldsymbol{\lambda} = \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix}$ and f as defined in (10), we compute the partial derivatives and double derivatives of f.

$$\frac{\partial f}{\partial \lambda_1} = 2\mathbf{m_1}^\top \left(\mathbf{M}\boldsymbol{\lambda} - \mathbf{x}\right) \tag{16}$$
$$\frac{\partial f}{\partial \left(-\lambda_2\right)} = 2\mathbf{m_2}^\top \left(\mathbf{M}\boldsymbol{\lambda} - \mathbf{x}\right) \tag{17}$$

Hence,

$$\frac{\partial^2 f}{\partial \lambda_1^2} = 2\left\|\mathbf{m_1}\right\|^2 \tag{18}$$
$$\frac{\partial^2 f}{\partial \lambda_1 \left(-\lambda_2\right)} = 2\mathbf{m_1}^\top \mathbf{m_2} \tag{19}$$
$$\frac{\partial^2 f}{\partial \left(-\lambda_2\right) \lambda_1} = 2\mathbf{m_2}^\top \mathbf{m_1} = 2\mathbf{m_1}^\top \mathbf{m_2} \tag{20}$$
$$\frac{\partial^2 f}{\partial \left(-\lambda_2\right)^2} = 2\left\|\mathbf{m_2}\right\|^2 \tag{21}$$

Therefore, the Hessian matrix of f is

$$\mathbf{H}_f = 2\begin{pmatrix} \left\|\mathbf{m_1}\right\|^2 & \mathbf{m_1}^\top \mathbf{m_2} \\ \mathbf{m_1}^\top \mathbf{m_2} & \left\|\mathbf{m_2}\right\|^2 \end{pmatrix} \tag{22}$$

The characteristic polynomial of $\mathbf{H}_f$ is

$$\mathrm{char}_x\left(\mathbf{H}_f\right) = x^2 - 2\left(\|\mathbf{m_1}\|^2 + \|\mathbf{m_2}\|^2\right)x + 4\left(\|\mathbf{m_1}\|^2\|\mathbf{m_2}\|^2 - (\mathbf{m_1}^\top\mathbf{m_2})\right) \quad (23)$$

Notice that the minima of the characteristic polynomial is at $x_m = \left(\|\mathbf{m_1}\|^2 + \|\mathbf{m_2}\|^2\right)$. Also, using Lemma 1,

$$\mathrm{char}_x\left(\mathbf{H}_f\right)(0) = 4\left(\|\mathbf{m_1}\|^2\|\mathbf{m_2}\|^2 - (\mathbf{m_1}^\top\mathbf{m_2})\right) \quad (24)$$

$$\geq 0 \quad (25)$$

Thus, the zeros of $\mathrm{char}_x\left(\mathbf{H}_f\right)$, or the eigenvalues of $\mathbf{H}_f$ are nonnegative. This implies that $\mathbf{H}_f$ is positive definite, and so f is convex.

We need to minimize f as a function of $\boldsymbol{\lambda}$. Thus, differentiating (10) using the chain rule,

$$\frac{df(\boldsymbol{\lambda})}{d\boldsymbol{\lambda}} = \frac{\mathbf{M}^\top(\mathbf{M}\boldsymbol{\lambda} - \mathbf{x}) + \mathbf{M}(\mathbf{M}\boldsymbol{\lambda} - \mathbf{x})^\top}{2\|\mathbf{M}\boldsymbol{\lambda} - \mathbf{x}\|} \quad (26)$$

$$= \frac{\mathbf{M}^\top(\mathbf{M}\boldsymbol{\lambda} - \mathbf{x})}{\|\mathbf{M}\boldsymbol{\lambda} - \mathbf{x}\|} \quad (27)$$

Setting (27) to zero gives

$$\mathbf{M}^\top\mathbf{M}\boldsymbol{\lambda} = \mathbf{M}^\top\mathbf{x} \quad (28)$$

We have the following cases:

a) There exists a $\boldsymbol{\lambda}$ satisfying

$$\mathbf{M}\boldsymbol{\lambda} = \mathbf{x} \quad (29)$$

$$\implies \lambda_1\mathbf{m_1} - \lambda_2\mathbf{m_2} = \mathbf{x_2} - \mathbf{x_1} \quad (30)$$

$$\implies \mathbf{x_1} + \lambda_1\mathbf{m_1} = \mathbf{x_2} + \lambda_2\mathbf{m_2} \quad (31)$$

Thus, both lines intersect at a point and the shortest distance between them is 0. To check for the existence of such a $\boldsymbol{\lambda}$, we can bring the augmented matrix $\begin{pmatrix}\mathbf{M} & \mathbf{x}\end{pmatrix}$ into row-reduced echelon form and check whether there is a pivot in the last column.

b) $\mathbf{M}^\top\mathbf{M}$ is singular. Since $\mathbf{M}^\top\mathbf{M}$ is a sqaure matrix of order 2, its rank must be 1. Further,

$$\det(\mathbf{M}^\top\mathbf{M}) = \begin{vmatrix}\mathbf{m_1}^\top\mathbf{m_1} & \mathbf{m_1}^\top\mathbf{m_2} \\ \mathbf{m_1}^\top\mathbf{m_2} & \mathbf{m_2}^\top\mathbf{m_2}\end{vmatrix} \quad (32)$$

$$= (\|\mathbf{m_1}\|\,.\,\|\mathbf{m_2}\|)^2 - (\mathbf{m_1}^\top\mathbf{m_2})^2 \quad (33)$$

Thus, equating the determinant to zero gives

$$\|\mathbf{m_1}\|\,.\,\|\mathbf{m_2}\| = \left|\mathbf{m_1}^\top\mathbf{m_2}\right| \quad (34)$$

which implies that both lines are parallel to each other. Setting $\mathbf{m_2} = k\mathbf{m_1}, k \in \mathbb{R}\setminus\{0\}$, we obtain one equation from (28).

$$\mathbf{m_1}^\top\mathbf{m_1}(\lambda_1 - k\lambda_2) = \mathbf{m_1}^\top\mathbf{x} \quad (35)$$

$$\implies \lambda_1 - k\lambda_2 = \frac{\mathbf{m_1}^\top\mathbf{x}}{\|\mathbf{m_1}\|^2} \quad (36)$$

Therefore, the required shortest distance is

$$\|\mathbf{A} - \mathbf{B}\| = \left\|\frac{\mathbf{m_1}^\top\mathbf{x}\mathbf{m_1}}{\|\mathbf{m_1}\|^2} - \mathbf{x}\right\| \quad (37)$$

c) $\mathbf{M}^\top\mathbf{M}$ is nonsinglar. This implies that the lines are skew. From (28),

$$\boldsymbol{\lambda} = (\mathbf{M}^\top\mathbf{M})^{-1}\mathbf{M}^\top\mathbf{x} \quad (38)$$

and therefore, the shortest distance is

$$\|\mathbf{A} - \mathbf{B}\| = \left\|\left(\mathbf{M}(\mathbf{M}^\top\mathbf{M})^{-1}\mathbf{M}^\top - \mathbf{I_n}\right)\mathbf{x}\right\| \quad (39)$$

where $\mathbf{I_n}$ is the identity matrix of order n.