Circle Assignment

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***Abstract*—This document contains the solution to Question 13 of Exercise 2 in Chapter 10 of the class 10 NCERT textbook.**

1) Prove that opposite sides of a quadrilateral circumscribing a circle subtend supplementary angles at the centre of the circle.
Solution: We begin by proving a useful lemma.
Lemma 0.1. *The line joining the centre of the circle to an external point bisects the angle subtended by the tangent chord at the centre.*
Proof. Refer to Fig. 1, generated using the Python code `codes/tangent.py`. Set $\mathbf{O}$ to be the origin. Since $OA \perp AP$,

$$\mathbf{A}^\top (\mathbf{A} - \mathbf{P}) = 0 \tag{1}$$
$$\implies \mathbf{A}^\top \mathbf{P} = \|\mathbf{A}\|^2 \tag{2}$$

Similarly,

$$\mathbf{B}^\top \mathbf{P} = \|\mathbf{B}\|^2 \tag{3}$$

Since $\mathbf{A}$ and $\mathbf{B}$ lie on the circle, their norms are equal. Thus, from (2) and (3),

$$\mathbf{A}^\top \mathbf{P} = \mathbf{B}^\top \mathbf{P} \tag{4}$$

and the lemma follows. □

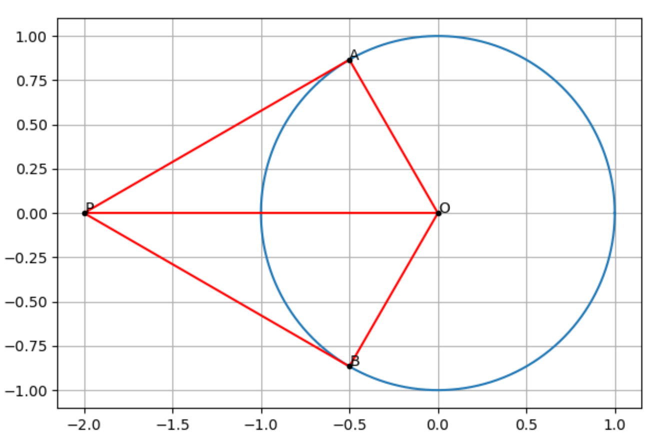

Fig. 1: OP bisects $\angle AOB$.

Call the quadrilateral $ABCD$, where

$$\mathbf{A} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}, \ \mathbf{C} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{5}$$

Suppose that $ABCD$ circumscribes the unit circle, given by

$$\mathbf{x}^\top \mathbf{x} - 1 = 0 \tag{6}$$

Comparing (6) with the general equation of the circle,

$$\mathbf{u} = \mathbf{0}, \ f = -1 \tag{7}$$

To find the points of contact from $\mathbf{A}$, we have

$$\mathbf{\Sigma_A} = (\mathbf{A} + \mathbf{u})(\mathbf{A} + \mathbf{u})^\top - (\mathbf{A}^\top \mathbf{A} + 2\mathbf{u}^\top \mathbf{A} + f)\mathbf{I} \tag{8}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix} \tag{9}$$

The eigenparameters of $\mathbf{\Sigma_A}$ are

$$\lambda_1 = 1, \ \lambda_2 = -3, \ \mathbf{P_A} = \mathbf{I} \tag{10}$$

Thus, the normals to the tangents are

$$\mathbf{n_1} = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}, \ \mathbf{n_2} = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} \tag{11}$$

The points of contact are given by

$$\mathbf{E} = -\frac{r\mathbf{n_2}}{\|\mathbf{n_2}\|} - \mathbf{u} \tag{12}$$
$$= \frac{1}{2}\begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix} \tag{13}$$
$$\mathbf{H} = -\frac{r\mathbf{n_1}}{\|\mathbf{n_1}\|} - \mathbf{u} \tag{14}$$
$$= \frac{1}{2}\begin{pmatrix} -1 \\ -\sqrt{3} \end{pmatrix} \tag{15}$$

Similarly for $\mathbf{C}$,

$$\mathbf{\Sigma_C} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - \mathbf{I} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{16}$$

Notice that

$$\mathbf{\Sigma_C}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} \tag{17}$$

$$\mathbf{\Sigma_C}\begin{pmatrix}1\\-1\end{pmatrix} = -\begin{pmatrix}1\\-1\end{pmatrix} \tag{18}$$

$$\tag{19}$$

Thus, the eigenparameters of $\mathbf{C}$ are

$$\mu_1 = 1,\ \mu_2 = -1,\ \mathbf{P_C} = \begin{pmatrix}1 & 1\\1 & -1\end{pmatrix} \tag{20}$$

The normals to the tangents are given by

$$\mathbf{m_1} = \mathbf{P_C}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}2\\0\end{pmatrix} \tag{21}$$

$$\mathbf{m_2} = \mathbf{P_C}\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}0\\2\end{pmatrix} \tag{22}$$

Therefore, the points of contact of $\mathbf{C}$ are

$$\mathbf{F} = \frac{r\mathbf{m_1}}{\|\mathbf{m_1}\|} - \mathbf{u} \tag{23}$$

$$= \begin{pmatrix}1\\0\end{pmatrix} \tag{24}$$

$$\mathbf{G} = \frac{r\mathbf{m_2}}{\|\mathbf{m_2}\|} - \mathbf{u} \tag{25}$$

$$= \begin{pmatrix}0\\1\end{pmatrix} \tag{26}$$

Using the lemma we proved above, the direction vectors of $\mathbf{B}$ and $\mathbf{D}$ are

$$\mathbf{d_B} = \mathbf{E} + \mathbf{F} = \frac{\sqrt{3}}{2}\begin{pmatrix}\sqrt{3}\\1\end{pmatrix} \tag{27}$$

$$\mathbf{d_D} = \mathbf{G} + \mathbf{H} = \frac{1}{2}\begin{pmatrix}-1\\2-\sqrt{3}\end{pmatrix} \tag{28}$$

Clearly,

$$\|\mathbf{d_B}\| = \sqrt{3} \tag{29}$$

$$\|\mathbf{d_D}\| = \sqrt{2-\sqrt{3}} \tag{30}$$

and from (5), (27) and (28),

$$\cos\angle AOD = \frac{\mathbf{A}^\top\mathbf{d_D}}{\|A\|\,\|\mathbf{d_D}\|} \tag{31}$$

$$= \frac{-1}{2\sqrt{2\sqrt{3}}} \tag{32}$$

$$= -\frac{\sqrt{2+\sqrt{3}}}{2} \tag{33}$$

$$= -\frac{\sqrt{3}+1}{2\sqrt{2}} \tag{34}$$

$$\cos\angle BOC = \frac{\mathbf{C}^\top\mathbf{d_B}}{\|C\|\,\|\mathbf{d_B}\|} \tag{35}$$

$$= \frac{\sqrt{3}+1}{2\sqrt{2}} \tag{36}$$

Hence,

$$\cos\angle AOD + \cos\angle BOC = 0 \tag{37}$$

which implies $\angle AOD + \angle BOC = \pi$, as required. The situation is illustrated in Fig. 2 plotted by the Python code `codes/quad_circ.py`. The numerical parameters used in the construction are shown in Table 1.

Parameter	Value
r	1
$\mathbf{A}$	$\begin{pmatrix}-2\\0\end{pmatrix}$
$\mathbf{C}$	$\begin{pmatrix}1\\1\end{pmatrix}$

TABLE 1: Parameters used in the construction of Fig. 2.

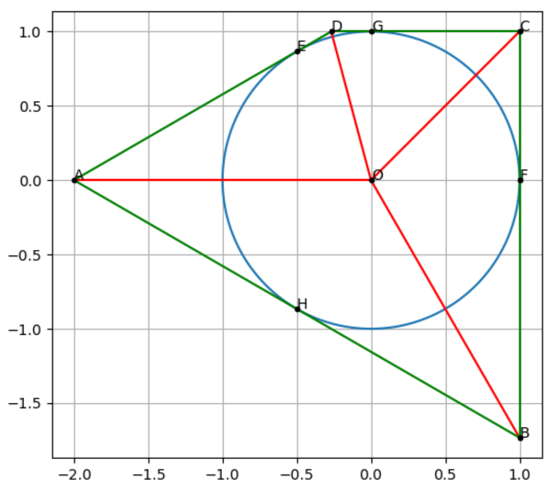

Fig. 2: Angles subtended by the opposite sides of a circumscribing quadrilateral at the center of its incircle are supplementary.