

Conic Assignment

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Abstract—This document contains the solution to Question 14 of Exercise 4 in Chapter 11 of the class 11 NCERT textbook.

- 1) Find the equation of the hyperbola eccentricity is $e = \frac{4}{3}$ and whose vertices are

$$\mathbf{P}_1 = \begin{pmatrix} 7 \\ 0 \end{pmatrix}, \mathbf{P}_2 = \begin{pmatrix} -7 \\ 0 \end{pmatrix} \quad (1)$$

Solution: Let the equation of the conic with focus \mathbf{F} , directrix $\mathbf{n}^\top \mathbf{x} = c$ and eccentricity e be

$$\mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \quad (2)$$

where

$$\mathbf{V} \triangleq \|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n} \mathbf{n}^\top \quad (3)$$

$$\mathbf{u} \triangleq ce^2 \mathbf{n} - \|\mathbf{n}\|^2 \mathbf{F} \quad (4)$$

$$f \triangleq \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2 e^2 \quad (5)$$

Since the conic is a hyperbola whose vertices are given by (1), the major axis is the x -axis and the directrix is parallel to the y -axis. Hence,

$$\mathbf{n} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (6)$$

Thus,

$$\mathbf{V} = \begin{pmatrix} 1 - e^2 & 0 \\ 0 & 1 \end{pmatrix} \quad (7)$$

$$\mathbf{u} = ce^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \mathbf{F} \quad (8)$$

$$f = \|\mathbf{F}\|^2 - c^2 e^2 \quad (9)$$

Substituting \mathbf{P}_1 and \mathbf{P}_2 in (2),

$$\mathbf{P}_1^\top \mathbf{V} \mathbf{P}_1 + 2\mathbf{u}^\top \mathbf{P}_1 + f = 0 \quad (10)$$

$$\mathbf{P}_2^\top \mathbf{V} \mathbf{P}_2 + 2\mathbf{u}^\top \mathbf{P}_2 + f = 0 \quad (11)$$

Subtracting (11) from (10), and noting that $\mathbf{P}_2 = -\mathbf{P}_1$,

$$\mathbf{u}^\top \mathbf{P}_1 = 0 \quad (12)$$

Hence, from (1), we see that \mathbf{u} lies on the y -axis. The general expression of the centre of the circle is given by

$$\mathbf{c} = -\mathbf{V}^{-1} \mathbf{u} \quad (13)$$

$$= \frac{1}{e^2 - 1} \begin{pmatrix} 1 & 0 \\ 0 & 1 - e^2 \end{pmatrix} \mathbf{u} \quad (14)$$

$$= -\mathbf{u} \quad (15)$$

and so, \mathbf{c} lies on the y -axis. However, \mathbf{c} also lies on the x -axis since it is the major axis. Therefore, $\mathbf{c} = -\mathbf{u} = \mathbf{0}$. Thus, from (8),

$$\mathbf{F} = \begin{pmatrix} ce^2 \\ 0 \end{pmatrix} \quad (16)$$

and so,

$$f = c^2 e^2 (e^2 - 1) \quad (17)$$

Putting $\mathbf{x} = \mathbf{P}_1$ or $\mathbf{x} = \mathbf{P}_2$ in (2) and using (16) and (17),

$$(\pm 7 \ 0) \begin{pmatrix} 1 - e^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \pm 7 \end{pmatrix} + f = 0 \quad (18)$$

$$\implies 49e^2 - f = 49 \quad (19)$$

Since $e = \frac{4}{3}$, (19) implies

$$f = 49(e^2 - 1) = \frac{343}{9} \quad (20)$$

Therefore, the equation of the conic is

$$\mathbf{x}^\top \begin{pmatrix} -\frac{7}{9} & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + \frac{343}{9} = 0 \quad (21)$$

The situation is illustrated in Fig. 1.

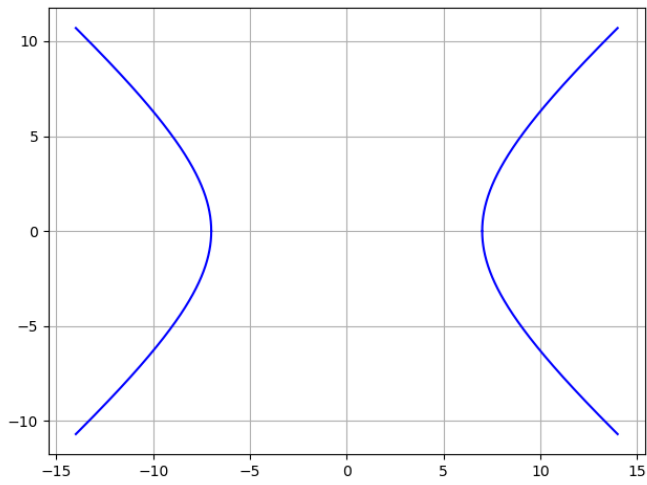


Fig. 1: Locus of the required hyperbola.