Fourier Series

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Contents

***Abstract*—This manual provides a simple introduction to Fourier Series.**

1 Periodic Function

Let

$$x(t) = A_0 \left|\sin\left(2\pi f_0 t\right)\right| \tag{1.1}$$

1.1 Plot $x(t)$.
Solution: The Python code `codes/1_1.py` plots $x(t)$ in Fig. 1.1.

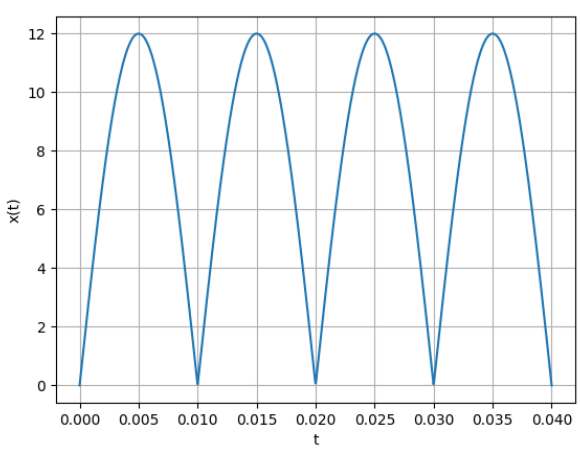

Fig. 1.1: $x(t)$

1.2 Show that $x(t)$ is periodic and find its period.

Solution: Note that,

$$x\left(t + \frac{1}{2f_0}\right) = A_0 \left|\sin\left(2\pi f_0 \left(t + \frac{1}{2f_0}\right)\right)\right| \tag{1.2}$$
$$= A_0 \left|\sin\left(2\pi f_0 t + \pi\right)\right| \tag{1.3}$$
$$= A_0 \left|\sin\left(2\pi f_0 t\right)\right| \tag{1.4}$$

Hence the period of $x(t)$ is $\frac{1}{2f_0}$.

2 Fourier Series

Consider $A_0 = 12$ and $f_0 = 50$ for all numerical calculations.

2.1 If

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{\jmath 2\pi k f_0 t} \tag{2.1}$$

show that

$$c_k = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-\jmath 2\pi k f_0 t}\, dt \tag{2.2}$$

Solution: For some $n \in \mathbb{Z}$,

$$x(t) e^{-\jmath 2\pi n f_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{\jmath 2\pi (k-n) f_0 t} \tag{2.3}$$

But

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{\jmath 2\pi k f_0 t}\, dt = \frac{1}{f_0}\delta_{0k} \tag{2.4}$$

where δ_{ij} denotes the Kronecker delta. Thus,

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-\jmath 2\pi n f_0 t}\, dt = \frac{c_n}{f_0} \tag{2.5}$$

$$\implies c_n = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-\jmath 2\pi n f_0 t}\, dt \tag{2.6}$$

2.2 Find c_k for (1.1)

Solution: Using (2.2),

$$c_n = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} A_0 \left|\sin\left(2\pi f_0 t\right)\right| e^{-\jmath 2\pi n f_0 t}\, dt \tag{2.7}$$

$$= f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} A_0 \left|\sin\left(2\pi f_0 t\right)\right| \cos\left(2\pi n f_0 t\right)\, dt$$

$$+ \jmath f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} A_0 \left|\sin\left(2\pi f_0 t\right)\right| \sin\left(2\pi n f_0 t\right)\, dt \tag{2.8}$$

$$= 2f_0 \int_0^{\frac{1}{2f_0}} A_0 \sin\left(2\pi f_0 t\right) \cos\left(2\pi n f_0 t\right)\, dt \tag{2.9}$$

$$= f_0 A_0 \int_0^{\frac{1}{2f_0}} \left(\sin\left(2\pi\left(n+1\right) f_0 t\right)\right)\, dt$$

$$- f_0 A_0 \int_0^{\frac{1}{2f_0}} \left(\sin\left(2\pi\left(n-1\right) f_0 t\right)\right)\, dt \tag{2.10}$$

$$= A_0 \frac{1+(-1)^n}{2\pi}\left(\frac{1}{n+1} - \frac{1}{n-1}\right) \tag{2.11}$$

$$= \begin{cases} \frac{2A_0}{\pi\left(1-n^2\right)} & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \tag{2.12}$$

2.3 Verify (2.1) using python.
Solution: The Python code `codes/2_3.py` verifies (2.13) by plotting Fig. 2.3.

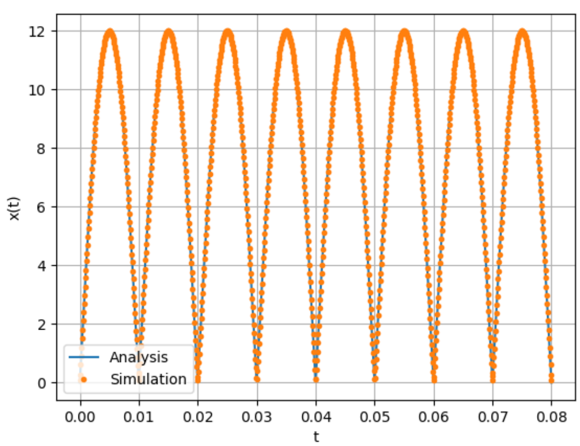

Fig. 2.3: Verification of (2.1).

2.4 Show that

$$x(t) = \sum_{k=0}^{\infty} \left(a_k \cos \jmath 2\pi k f_0 t + b_k \sin \jmath 2\pi k f_0 t\right) \tag{2.13}$$

and obtain the formulae for a_k and b_k.
Solution: From (2.1),

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{\jmath 2\pi k f_0 t} \tag{2.14}$$

$$= c_0 + \sum_{k=1}^{\infty} c_k e^{\jmath 2\pi k f_0 t} + c_{-k} e^{-\jmath 2\pi k f_0 t} \tag{2.15}$$

$$= c_0 + \sum_{k=1}^{\infty} \left(c_k + c_{-k}\right) \cos\left(2\pi k f_0 t\right)$$

$$+ \sum_{k=0}^{\infty} \jmath \left(c_k - c_{-k}\right) \sin\left(2\pi k f_0 t\right) \tag{2.16}$$

Hence, for $k \geq 0$,

$$a_k = \begin{cases} c_0 & k = 0 \\ c_k + c_{-k} & k > 0 \end{cases} \tag{2.17}$$

$$= \begin{cases} f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t)\, dt & k = 0 \\ 2f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) \cos\left(2\pi k f_0 t\right)\, dt & k > 0 \end{cases} \tag{2.18}$$

$$b_k = \frac{c_k - c_{-k}}{\jmath} = 2f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) \sin\left(2\pi k f_0 t\right)\, dt \tag{2.19}$$

2.5 Find a_k and b_k for (1.1)
Solution: From (2.1), we see that since $x(t)$ is even,

$$x(-t) = \sum_{k=-\infty}^{\infty} c_k e^{-\jmath 2\pi k f_0 t} \tag{2.20}$$

$$= \sum_{k=-\infty}^{\infty} c_{-k} e^{\jmath 2\pi k f_0 t} \tag{2.21}$$

$$= \sum_{k=-\infty}^{\infty} c_k e^{\jmath 2\pi k f_0 t} \tag{2.22}$$

where we substitute $k := -k$ in (2.21). Hence, we see that $c_k = c_{-k}$. So, from (2.17) and (2.19), for $k \geq 0$,

$$a_k = \begin{cases} \frac{2A_0}{\pi} & k = 0 \\ \frac{4A_0}{\pi\left(1-k^2\right)} & k > 0,\ k \text{ even} \\ 0 & \text{otherwise} \end{cases} \tag{2.23}$$

$$b_k = 0 \tag{2.24}$$

2.6 Verify (2.13) using python.
Solution: The Python code `codes/2_6.py`

verifies (2.13) by plotting Fig. 2.6.

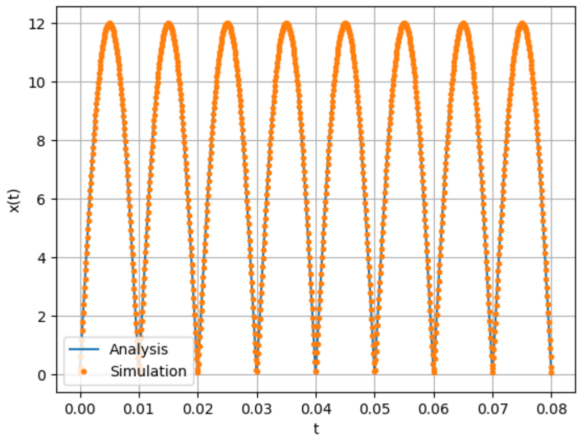

Fig. 2.6: Verification of (2.13).

3 Fourier Transform

3.1

$$\delta(t) = 0, \quad t \neq 0 \tag{3.1}$$
$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1 \tag{3.2}$$

3.2 The Fourier Transform of $g(t)$ is

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi f t}\, dt \tag{3.3}$$

3.3 Show that

$$g(t-t_0) \overset{\mathcal{F}}{\longleftrightarrow} G(f)e^{-j2\pi f t_0} \tag{3.4}$$

Solution: We write, substituting $u := t - t_0$,

$$g(t-t_0) \overset{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\infty} g(t-t_0) e^{-\jmath 2\pi f t}\, dt \tag{3.5}$$
$$= \int_{-\infty}^{\infty} g(u) e^{-\jmath 2\pi f (u+t_0)}\, du \tag{3.6}$$
$$= G(f)e^{-j2\pi f t_0} \tag{3.7}$$

where the last equality follows from (3.3).

3.4 Show that

$$G(t) \overset{\mathcal{F}}{\longleftrightarrow} g(-f) \tag{3.8}$$

Solution: Using the definition of the Inverse Fourier Transform,

$$g(t) = \int_{-\infty}^{\infty} G(f) e^{\jmath 2\pi f t}\, df \tag{3.9}$$

Hence, setting $t := -f$ and $f := t$, which implies $df = dt$,

$$g(-f) = \int_{-\infty}^{\infty} G(t) e^{-\jmath 2\pi f t}\, dt \tag{3.10}$$
$$\implies G(t) \overset{\mathcal{F}}{\longleftrightarrow} g(-f) \tag{3.11}$$

3.5 $\delta(t) \overset{\mathcal{F}}{\longleftrightarrow} ?$

Solution: We have, from the definition of $\delta(t)$,

$$\delta(t) \overset{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\infty} \delta(t) e^{-\jmath 2\pi f t}\, dt \tag{3.12}$$
$$= \int_{-\infty}^{\infty} \delta(0)\, dt \tag{3.13}$$
$$= \int_{-\infty}^{\infty} \delta(t)\, dt = 1 \tag{3.14}$$

3.6 $e^{-j2\pi f_0 t} \overset{\mathcal{F}}{\longleftrightarrow} ?$

Solution: Suppose $g(t) \overset{\mathcal{F}}{\longleftrightarrow} G(f)$. Then,

$$g(t)e^{\jmath 2\pi f_0 t} \overset{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\infty} g(t) e^{-\jmath 2\pi (f-f_0) t}\, dt \tag{3.15}$$
$$= F(f - f_0) \tag{3.16}$$

Using (3.9) in (3.14), $1 \overset{\mathcal{F}}{\longleftrightarrow} \delta(-f)$. Hence, applying (3.16),

$$e^{-\jmath 2\pi f_0 t} \overset{\mathcal{F}}{\longleftrightarrow} \delta(-(f+f_0)) = \delta(f+f_0) \tag{3.17}$$

3.7 $\cos(2\pi f_0 t) \overset{\mathcal{F}}{\longleftrightarrow} ?$

Solution: Using the linearity of the Fourier Transform and (3.17),

$$\cos(2\pi f_0 t) = \frac{1}{2}\left(e^{\jmath 2\pi f_0 t} + e^{-\jmath 2\pi f_0 t}\right) \tag{3.18}$$
$$\overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2}\left(\delta(f+f_0) + \delta(f-f_0)\right) \tag{3.19}$$

3.8 Find the Fourier Transform of $x(t)$ and plot it. Verify using python.

Solution: Substituting (2.12) in (2.1),

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} \sum_{k=-\infty}^{\infty} c_k \delta(f + k f_0) \tag{3.20}$$
$$= \frac{2A_0}{\pi} \sum_{k=-\infty}^{\infty} \frac{\delta(f + 2k f_0)}{1-4k^2} \tag{3.21}$$

The python code `codes/3_8.py` verifies (3.21) while plotting Fig. 3.8

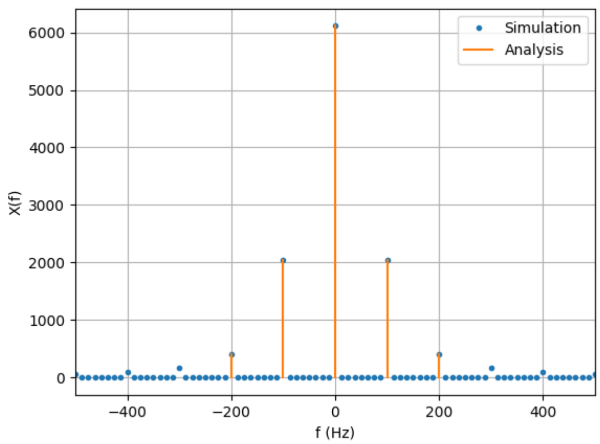

Fig. 3.8: Fourier Transform of $x(t)$.

3.9 Show that

$$\text{rect}\, t \overset{\mathcal{F}}{\longleftrightarrow} \text{sinc}\, f \tag{3.22}$$

Verify using python.
Solution: We write

$$\text{rect}\, t \overset{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\infty} \text{rect}\, t e^{-\jmath 2\pi f t}\, dt \tag{3.23}$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-\jmath 2\pi f t}\, dt \tag{3.24}$$

$$= \frac{e^{\jmath \pi f} - e^{-\jmath \pi f}}{\jmath 2\pi f} = \frac{\sin \pi f}{\pi f} = \text{sinc}\, f \tag{3.25}$$

The python code `codes/3_9.py` verifies (3.25) by plotting Fig. 3.9.

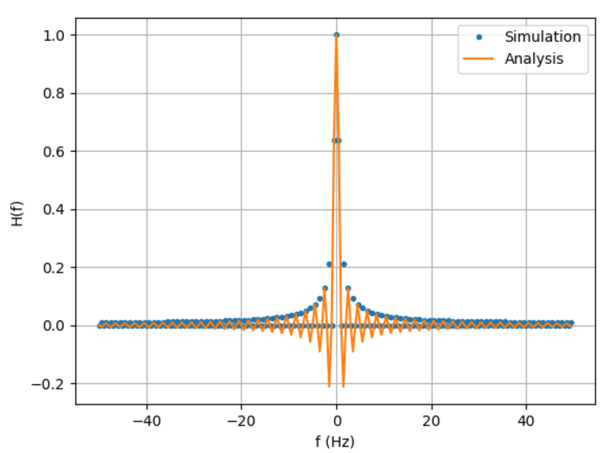

Fig. 3.9: Fourier Transform of rect(t).

3.10 $\text{sinc}\, t \overset{\mathcal{F}}{\longleftrightarrow}$? Verify using python.

Solution: From (3.9), we have

$$\text{sinc}\, t \overset{\mathcal{F}}{\longleftrightarrow} \text{rect}(-f) = \text{rect}\, f \tag{3.26}$$

Since $\text{rect}\, f$ is an even function. The python code `codes/3_10.py` verifies (3.26) by plotting Fig. 3.10.

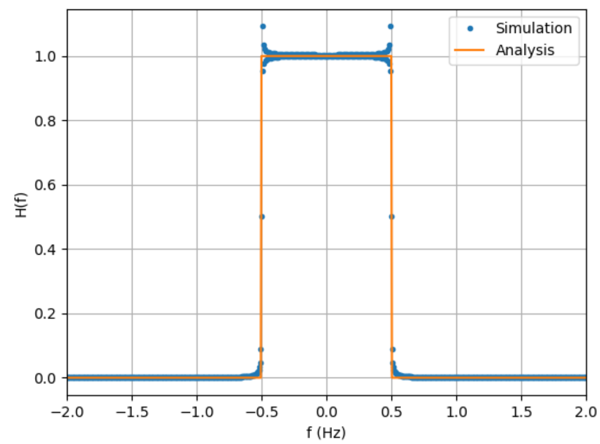

Fig. 3.10: Fourier Transform of sinc(t).

4 Filter

4.1 Find $H(f)$ which transforms $x(t)$ to DC 5V.
Solution: The function $H(f)$ is a low pass filter which filters out even harmonics and leaves the zero frequency component behind. The rectangular function represents an ideal low pass filter. Suppose the cutoff frequency is $f_c = 50$ Hz, then

$$H(f) = \text{rect}\, \frac{f}{2f_c} = \begin{cases} 1 & |f| < f_c \\ 0 & \text{otherwise} \end{cases} \tag{4.1}$$

Multiplying by a scaling factor to get DC 5V,

$$H(f) = \frac{\pi V_0}{2A_0} \text{rect}\left(\frac{f}{2f_c}\right) \tag{4.2}$$

where $V_0 = 5$ V.

4.2 Find $h(t)$.
Solution: Suppose $g(t) \overset{\mathcal{F}}{\longleftrightarrow} G(f)$. Then, for

some nonzero $a \in \mathbb{R}$

$$g(at) \xleftrightarrow{\mathcal{F}} \int_{-\infty}^{\infty} g(at) e^{-\jmath 2\pi f t}\, dt \tag{4.3}$$
$$= \frac{1}{a} \int_{-\infty}^{\infty} g(u) e^{\left(-\jmath 2\pi \frac{f}{a} t\right)}\, dt \tag{4.4}$$
$$= \frac{1}{a} G\left(\frac{f}{a}\right) \tag{4.5}$$

where we have substituted $u := at$. Using (4.5) of the Fourier Transform in (4.1),

$$h(t) = \frac{2\pi V_0}{A_0} f_c \operatorname{sinc}(2 f_c t) \tag{4.6}$$

4.3 Verify your result using convolution.
Solution: The Python code `codes/4_3.py` verifies the result by plotting the graph below.

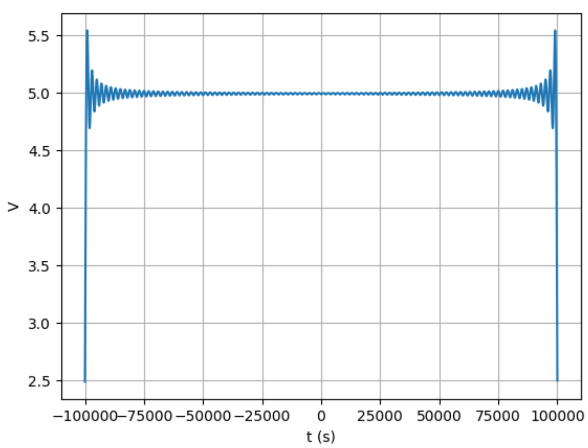

Fig. 4.3: Convolution of the two signals.

5 Filter Design

5.1 Design a Butterworth filter for $H(f)$.
Solution: The Butterworth filter has an amplitude response given by

$$|H(f)|^2 = \frac{1}{\left(1 + \left(\frac{f}{f_c}\right)^{2n}\right)} \tag{5.1}$$

where n is the order of the filter and f_c is the cutoff frequency. The attenuation at frequency f is given by

$$A = -10 \log_{10} |H(f)|^2 \tag{5.2}$$
$$= -20 \log_{10} |H(f)| \tag{5.3}$$

We consider the following design parameters for our lowpass analog Butterworth filter:

a) Passband edge, $f_p = 50$ Hz
b) Stopband edge, $f_s = 100$ Hz
c) Passband attenuation, $A_p = -1$ dB
d) Stopband attenuation, $A_s = -20$ dB

We are required to find a desriable order n and cutoff frequency f_c for the filter. From (5.3),

$$A_p = -10 \log_{10} \left[1 + \left(\frac{f_p}{f_c}\right)^{2n}\right] \tag{5.4}$$
$$A_s = -10 \log_{10} \left[1 + \left(\frac{f_s}{f_c}\right)^{2n}\right] \tag{5.5}$$

Thus,

$$\left(\frac{f_p}{f_c}\right)^{2n} = 10^{-\frac{A_p}{10}} - 1 \tag{5.6}$$
$$\left(\frac{f_s}{f_c}\right)^{2n} = 10^{-\frac{A_s}{10}} - 1 \tag{5.7}$$

Therefore, on dividing the above equations and solving for n,

$$n = \frac{\log\left(10^{-\frac{A_s}{10}} - 1\right) - \log\left(10^{-\frac{A_p}{10}} - 1\right)}{2\left(\log f_s - \log f_p\right)} \tag{5.8}$$

In this case, making appropriate susbstitutions gives $n = 4.29$. Hence, we take $n = 5$. Solving for f_c in (5.6) and (5.7),

$$f_{c1} = f_p \left[10^{-\frac{A_p}{10}} - 1\right]^{-\frac{1}{2n}} = 57.23\,\text{Hz} \tag{5.9}$$
$$f_{c2} = f_s \left[10^{-\frac{A_s}{10}} - 1\right]^{-\frac{1}{2n}} = 63.16\,\text{Hz} \tag{5.10}$$

Hence, we take $f_c = \sqrt{f_{c1} f_{c2}} = 60\,\text{Hz}$ approximately.

5.2 Design a Chebyshev filter for $H(f)$.
Solution: The Chebyshev filter has an amplitude response given by

$$|H(f)|^2 = \frac{1}{\left(1 + \epsilon^2 C_n^2 \left(\frac{f}{f_c}\right)^2\right)} \tag{5.11}$$

where

a) n is the order of the filter
b) ϵ is the ripple
c) f_c is the cutoff frequency
d) $C_n = \cosh^{-1}(n \cosh x)$ denotes the n^th order

Chebyshev polynomial, given by

$$c_n(x) = \begin{cases} \cos\left(n \cos^{-1} x\right) & |x| \le 1 \\ \cosh\left(n \cosh^{-1} x\right) & \text{otherwise} \end{cases} \tag{5.12}$$

We are given the following specifications:

a) Passband edge (which is equal to cutoff frequency), $f_p = f_c$
b) Stopband edge, f_s
c) Attenuation at stopband edge, A_s
d) Peak-to-peak ripple δ in the passband. It is given in dB and is related to ϵ as

$$\delta = 10 \log_{10}\left(1 + \epsilon^2\right) \tag{5.13}$$

and we must find a suitable n and ϵ. From (5.13),

$$\epsilon = \sqrt{10^{\frac{\delta}{10}} - 1} \tag{5.14}$$

At $f_s > f_p = f_c$, using (5.12), A_s is given by

$$A_s = -10 \log_{10}\left[1 + \epsilon^2 c_n^2\left(\frac{f_s}{f_p}\right)\right] \tag{5.15}$$

$$\implies c_n\left(\frac{f_s}{f_p}\right) = \frac{\sqrt{10^{-\frac{A_s}{10}} - 1}}{\epsilon} \tag{5.16}$$

$$\implies n = \frac{\cosh^{-1}\left(\frac{\sqrt{10^{-\frac{A_s}{10}} - 1}}{\epsilon}\right)}{\cosh^{-1}\left(\frac{f_s}{f_p}\right)} \tag{5.17}$$

We consider the following specifications:

a) Passband edge/cutoff frequency, $f_p = f_c = 60\,\text{Hz}$.
b) Stopband edge, $f_s = 100\,\text{Hz}$.
c) Passband ripple, $\delta = 0.5\,\text{dB}$
d) Stopband attenuation, $A_s = -20\,\text{dB}$

$\epsilon = 0.35$ and $n = 3.68$. Hence, we take $n = 4$ as the order of the Chebyshev filter.

5.3 Design a circuit for your Butterworth filter.
Solution: Looking at the table of normalized element values L_k, C_k, of the Butterworth filter for order 5, and noting that de-normalized values L'_k and C'_k are given by

$$C'_k = \frac{C_k}{\omega_c} \qquad L'_k = \frac{L_k}{\omega_c} \tag{5.18}$$

De-normalizing these values, taking $f_c = 60\,\text{Hz}$,

$$C'_1 = C'_5 = 1.64\,\text{mF} \tag{5.19}$$
$$L'_2 = L'_4 = 4.29\,\text{mH} \tag{5.20}$$
$$C'_3 = 5.31\,\text{mF} \tag{5.21}$$
$$\tag{5.22}$$

The L-C network is shown in Fig. 5.3.

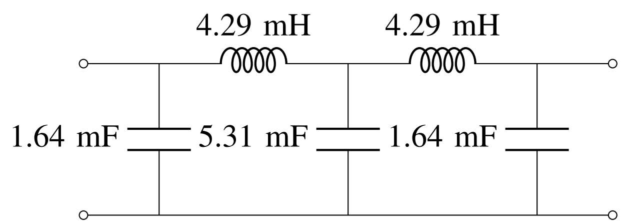

Fig. 5.3: L-C Butterworth Filter

This circuit is simulated in the ngspice code `codes/5_3.cir`. The Python code `codes/5_3.py` compares the amplitude response of the simulated circuit with the theoretical expression.

5.4 Design a circuit for your Chebyshev filter.
Solution: Looking at the table of normalized element values of the Chebyshev filter for order 3 and 0.5 dB ripple, and de-nomrmalizing those values, taking $f_c = 50\,\text{Hz}$,

$$C'_1 = 4.43\,\text{mF} \tag{5.23}$$
$$L'_2 = 3.16\,\text{mH} \tag{5.24}$$
$$C'_3 = 6.28\,\text{mF} \tag{5.25}$$
$$L'_4 = 2.23\,\text{mH} \tag{5.26}$$

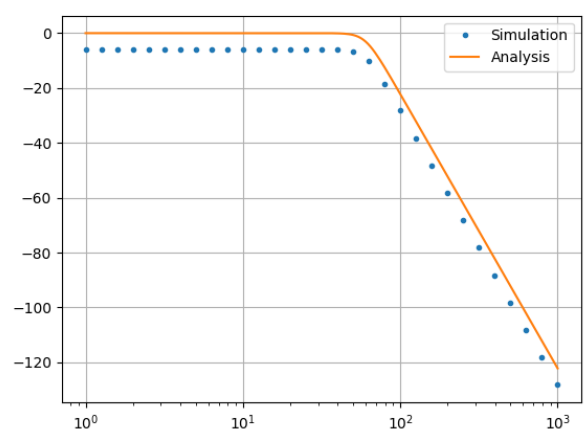

Fig. 5.4: Simulation of Butterworth filter.

The L-C network is shown in Fig. 5.4.

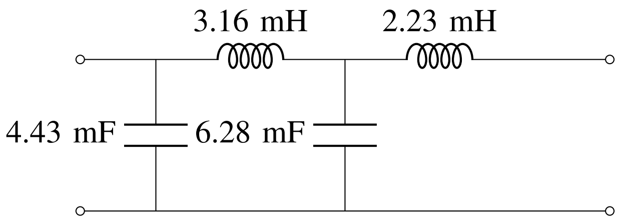

Fig. 5.4: L-C Chebyshev Filter

This circuit is simulated in the ngspice code `codes/5_4.cir`. The Python code `codes/5_4.py` compares the amplitude response of the simulated circuit with the theoretical expression.

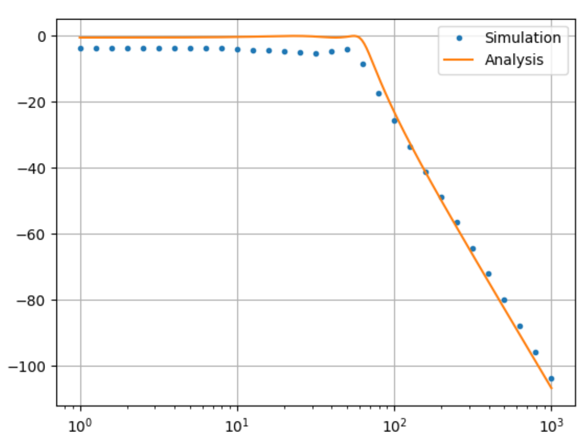

Fig. 5.4: Simulation of Chebyshev filter.