Circuits and Transforms

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***Abstract*—This manual provides a simple introduction to Transforms**

1 Definitions

1. The unit step function is

$$u(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases} \tag{1.1}$$

2. The Laplace transform of $g(t)$ is defined as

$$G(s) = \int_{-\infty}^{\infty} g(t)e^{-st}\,dt \tag{1.2}$$

2 Laplace Transform

1. In the circuit, the switch S is connected to position P for a long time so that the charge on the capacitor becomes $q_1\,\mu C$. Then S is switched to position Q. After a long time, the charge on the capacitor is $q_2\,\mu C$.

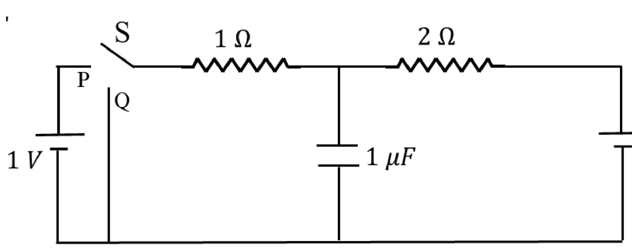

Fig. 2.1

2. Find q_1.
Solution: The equivalent circuit at steady-state when the switch is at P is shown alongside. Assuming the circuit to be grounded at G and the relative potential at point X to be V, we use KCL at X and get

$$\frac{V-1}{1} + \frac{V-2}{2} = 0 \tag{2.1}$$

$$\implies V = \frac{4}{3}\,\mathrm{V} \tag{2.2}$$

Hence,

$$q_1 = CV = \frac{4}{3}\,\mu\mathrm{C} \tag{2.3}$$

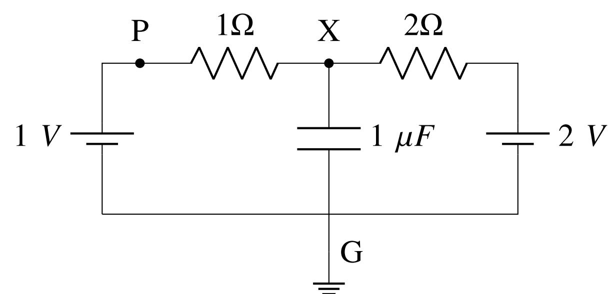

Fig. 2.2

3. Show that the Laplace transform of $u(t)$ is $\frac{1}{s}$ and find the ROC.
Solution: We have,

$$u(t) \xleftrightarrow{\mathcal{L}} \int_0^{\infty} u(t)e^{-st}dt \tag{2.4}$$

$$= \int_0^0 \frac{1}{2}e^{-st}dt + \int_0^{\infty} e^{-st}dt \tag{2.5}$$

$$= \frac{1}{s}, \quad \Re(s) > 0 \tag{2.6}$$

4. Show that

$$e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a}, \quad a > 0 \tag{2.7}$$

and find the ROC.
Solution: Note that by substituting $s := s + a$ in (2.6), and considering $a \in \mathbb{R}$,

$$e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \int_0^\infty u(t)e^{-(s+a)t}dt \tag{2.8}$$

$$= \frac{1}{s+a}, \quad \Re(s) > -a \tag{2.9}$$

5. Now consider the following resistive circuit transformed from Fig. 2.1 where

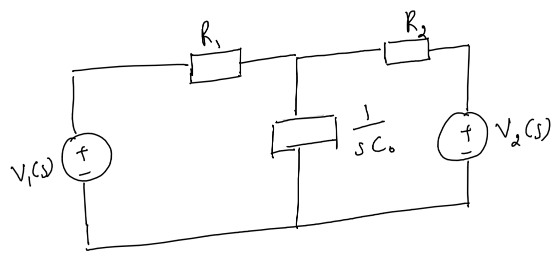

Fig. 2.3

$$u(t) \xleftrightarrow{\mathcal{L}} V_1(s) \tag{2.10}$$

$$2u(t) \xleftrightarrow{\mathcal{L}} V_2(s) \tag{2.11}$$

Find the voltage across the capacitor $V_{C_0}(s)$.
Solution: We see that

$$V_1(s) = \frac{1}{s}V_2(s) = \frac{2}{s} \tag{2.12}$$

Now, labelling points G and X as in Fig. 2.2, we use KCL at X.

$$\frac{V - \frac{1}{s}}{R_1} + \frac{V - \frac{2}{s}}{R_2} + sC_0V = 0 \tag{2.13}$$

$$V\left(\frac{1}{R_1} + \frac{1}{R_2} + sC_0\right) = \frac{1}{s}\left(\frac{1}{R_1} + \frac{2}{R_2}\right) \tag{2.14}$$

$$V(s) = \frac{\frac{1}{R_1} + \frac{2}{R_2}}{s\left(\frac{1}{R_1} + \frac{1}{R_2} + sC_0\right)} \tag{2.15}$$

$$= \frac{\frac{1}{R_1} + \frac{2}{R_2}}{\frac{1}{R_1} + \frac{1}{R_2}}\left(\frac{1}{s} - \frac{1}{\frac{1}{C_0}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + s}\right) \tag{2.16}$$

6. Find $v_{C_0}(t)$. Plot using python.
Solution: Taking the inverse Laplace transform in (2.16),

$$V(s) \xleftrightarrow{\mathcal{L}} \frac{2R_1 + R_2}{R_1 + R_2}u(t)\left(1 - e^{-\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{t}{C_0}}\right) \tag{2.17}$$

$$= \frac{4}{3}\left(1 - e^{-\left(1.5\times10^6\right)t}\right)u(t) \tag{2.18}$$

7. Verify your result using ngspice.

3 Initial Conditions

1. Find q_2 in Fig. 2.1.
Solution: The equivalent circuit at steady state when the switch is at Q is shown below.

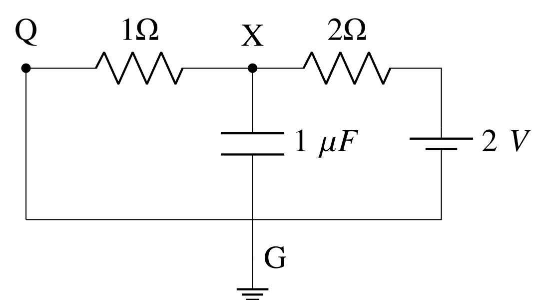

Fig. 3.1

2. Draw the equivalent s-domain resistive circuit when S is switched to position Q. Use variables R_1, R_2, C_0 for the passive elements.

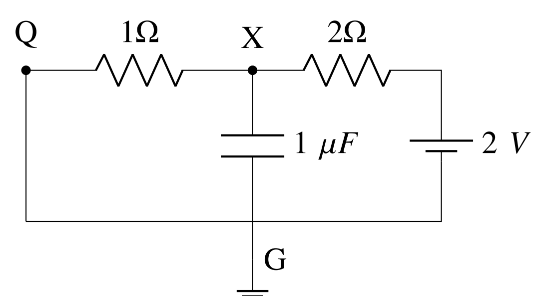

Fig. 3.2

3. $V_{C_0}(s) = ?$
4. $v_{C_0}(t) = ?$ Plot using python.
5. Verify your result using ngspice.
6. Find $v_{C_0}(0-), v_{C_0}(0+)$ and $v_{C_0}(\infty)$.