

Circuits and Transforms

Gautam Singh

CONTENTS

1	Definitions	1
2	Laplace Transform	1
3	Initial Conditions	2

Abstract—This manual provides a simple introduction to Transforms

1 DEFINITIONS

1. The unit step function is

$$u(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases} \quad (1.1)$$

2. The Laplace transform of $g(t)$ is defined as

$$G(s) = \int_{-\infty}^{\infty} g(t)e^{-st} dt \quad (1.2)$$

2 LAPLACE TRANSFORM

1. In the circuit, the switch S is connected to position P for a long time so that the charge on the capacitor becomes $q_1 \mu C$. Then S is switched to position Q. After a long time, the charge on the capacitor is $q_2 \mu C$.

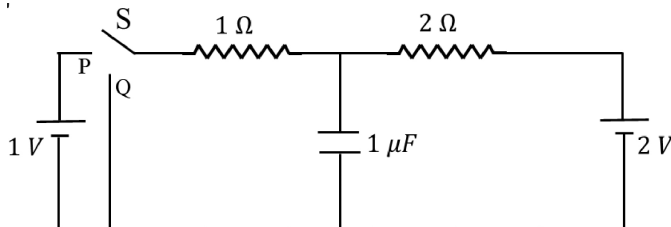


Fig. 2.1

2. Find q_1 .

Solution: The equivalent circuit at steady-state when the switch is at P is shown alongside. Assuming the circuit to be grounded at G and

the relative potential at point X to be V , we use KCL at X and get

$$\frac{V-1}{1} + \frac{V-2}{2} = 0 \quad (2.1)$$

$$\Rightarrow V = \frac{4}{3} \text{ V} \quad (2.2)$$

Hence,

$$q_1 = CV = \frac{4}{3} \mu C \quad (2.3)$$

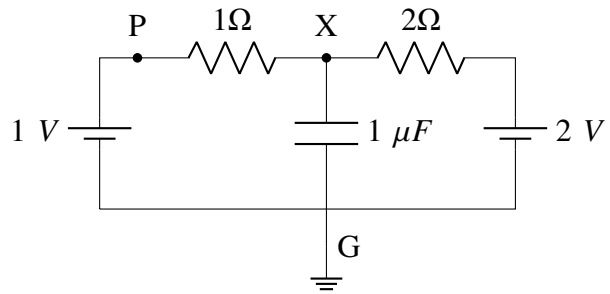


Fig. 2.2

3. Show that the Laplace transform of $u(t)$ is $\frac{1}{s}$ and find the ROC.

Solution: We have,

$$u(t) \xleftrightarrow{\mathcal{L}} \int_0^{\infty} u(t)e^{-st} dt \quad (2.4)$$

$$= \int_0^{\infty} \frac{1}{2} e^{-st} dt + \int_0^{\infty} e^{-st} dt \quad (2.5)$$

$$= \frac{1}{s}, \quad \Re(s) > 0 \quad (2.6)$$

4. Show that

$$e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a}, \quad a > 0 \quad (2.7)$$

and find the ROC.

Solution: Note that by substituting $s := s + a$ in (2.6), and considering $a \in \mathbb{R}$,

$$e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \int_0^{\infty} u(t)e^{-(s+a)t} dt \quad (2.8)$$

$$= \frac{1}{s+a}, \quad \Re(s) > -a \quad (2.9)$$

5. Now consider the following resistive circuit transformed from Fig. 2.1 where

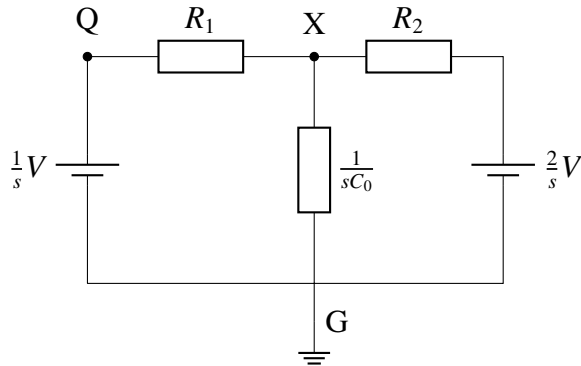


Fig. 2.3

$$u(t) \xleftrightarrow{\mathcal{L}} V_1(s) \quad (2.10)$$

$$2u(t) \xleftrightarrow{\mathcal{L}} V_2(s) \quad (2.11)$$

Find the voltage across the capacitor $V_{C_0}(s)$.

Solution: We see that

$$V_1(s) = \frac{1}{s} V_2(s) = \frac{2}{s} \quad (2.12)$$

Now, labelling points G and X as in Fig. 2.2, we use KCL at X.

$$\frac{V - \frac{1}{s}}{R_1} + \frac{V - \frac{2}{s}}{R_2} + sC_0 V = 0 \quad (2.13)$$

$$V \left(\frac{1}{R_1} + \frac{1}{R_2} + sC_0 \right) = \frac{1}{s} \left(\frac{1}{R_1} + \frac{2}{R_2} \right) \quad (2.14)$$

$$V(s) = \frac{\frac{1}{R_1} + \frac{2}{R_2}}{s \left(\frac{1}{R_1} + \frac{1}{R_2} + sC_0 \right)} \quad (2.15)$$

$$= \frac{\frac{1}{R_1} + \frac{2}{R_2}}{\frac{1}{R_1} + \frac{1}{R_2}} \left(\frac{1}{s} - \frac{1}{\frac{1}{C_0} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) + s} \right) \quad (2.16)$$

6. Find $v_{C_0}(t)$. Plot using python.

Solution: Taking the inverse Laplace transform

in (2.16),

$$V(s) \xleftrightarrow{\mathcal{L}} \frac{2R_1 + R_2}{R_1 + R_2} u(t) \left(1 - e^{-\left(\frac{1}{R_1} + \frac{1}{R_2} \right) \frac{t}{C_0}} \right) \quad (2.17)$$

$$= \frac{4}{3} \left(1 - e^{-(1.5 \times 10^6)t} \right) u(t) \quad (2.18)$$

The python code codes/2_6.py plots the graph below.

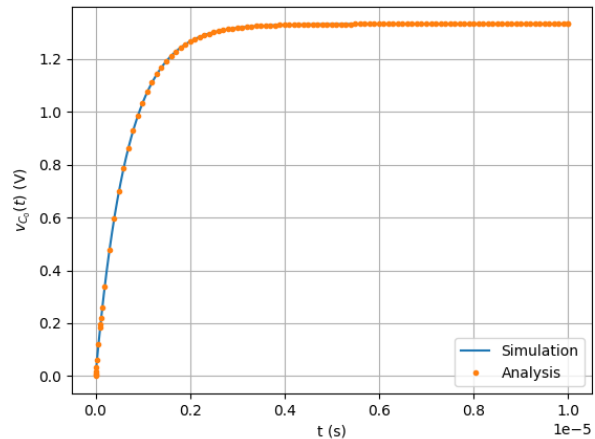


Fig. 2.4: $v_{C_0}(t)$ before the switch is flipped

7. Verify your result using ngspice.

Solution: The ngspice script codes/2_7.cir simulates the given circuit and the generated output is depicted in Fig. (2.4)

3 INITIAL CONDITIONS

1. Find q_2 in Fig. 2.1.

Solution: The equivalent circuit at steady state when the switch is at Q is shown below.

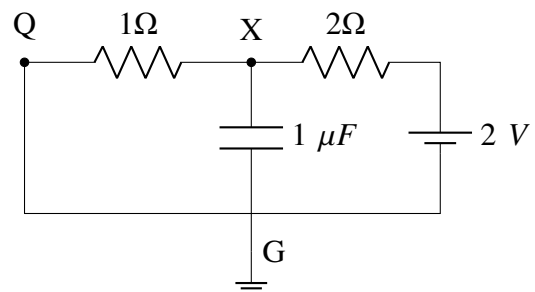


Fig. 3.1

Since capacitor behaves as an open circuit, we use KCL at X.

$$\frac{V-0}{1} + \frac{V-2}{2} = 0 \implies V = \frac{2}{3} \text{ V} \quad (3.1)$$

and hence, $q_2 = \frac{2}{3} \mu\text{C}$.

2. Draw the equivalent s -domain resistive circuit when S is switched to position Q. Use variables R_1, R_2, C_0 for the passive elements.

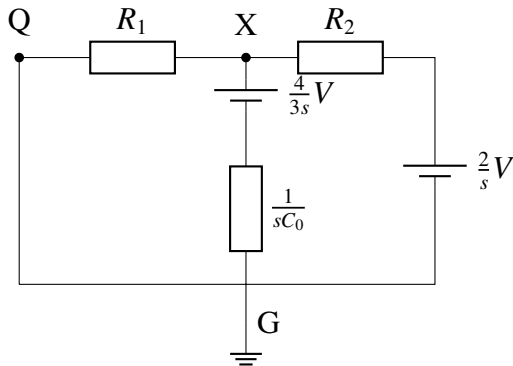


Fig. 3.2

3. $V_{C_0}(s) = ?$

Solution: Using KCL at node X in Fig. 3.2

$$\frac{V-0}{R_1} + \frac{V-\frac{2}{s}}{R_2} + sC_0 \left(V - \frac{4}{3s} \right) = 0 \quad (3.2)$$

$$\implies V_{C_0}(s) = \frac{\frac{2}{sR_2} + \frac{4C_0}{3}}{\frac{1}{R_1} + \frac{2}{R_2} + sC_0} \quad (3.3)$$

4. $v_{C_0}(t) = ?$ Plot using python.

Solution: From (3.3),

$$V_{C_0}(s) = \frac{4}{3} \left(\frac{1}{\frac{1}{C_0} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) + s} \right) + \frac{2}{R_2 \left(\frac{1}{R_1} + \frac{1}{R_2} \right)} \left(\frac{1}{s} - \frac{1}{\frac{1}{C_0} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) + s} \right) \quad (3.4)$$

Taking an inverse Laplace Transform,

$$v_{C_0}(t) = \frac{4}{3} e^{-\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{t}{C_0}} u(t) + \frac{2}{R_2 \left(\frac{1}{R_1} + \frac{1}{R_2} \right)} \left(1 - e^{-\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{t}{C_0}} \right) u(t) \quad (3.5)$$

Substituting values gives

$$v_{C_0}(t) = \frac{2}{3} \left(1 + e^{-(1.5 \times 10^6)t} \right) u(t) \quad (3.6)$$

The Python code `codes/3_4.py` plots the

graph below.

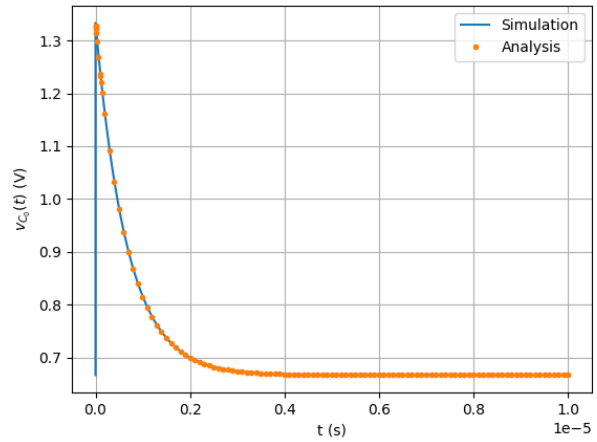


Fig. 3.3: $v_{C_0}(t)$ after the switch is flipped

5. Verify your result using ngspice.

Solution: The ngspice script `codes/3_5.cir` simulates the given circuit and the generated output is depicted in Fig. (3.3).

6. Find $v_{C_0}(0-)$, $v_{C_0}(0+)$ and $v_{C_0}(\infty)$.

Solution: From the initial conditions,

$$v_{C_0}(0-) = \frac{q_1}{C} = \frac{4}{3} \text{ V} \quad (3.7)$$

Using (3.6),

$$v_{C_0}(0+) = \lim_{t \rightarrow 0+} v_{C_0}(t) = \frac{4}{3} \text{ V} \quad (3.8)$$

$$v_{C_0}(\infty) = \lim_{t \rightarrow \infty} v_{C_0}(t) = \frac{2}{3} \text{ V} \quad (3.9)$$

7. Obtain the Fig. in problem 3.2 using the equivalent differential equations.

Solution: The equivalent circuit in the t -domain is shown below.

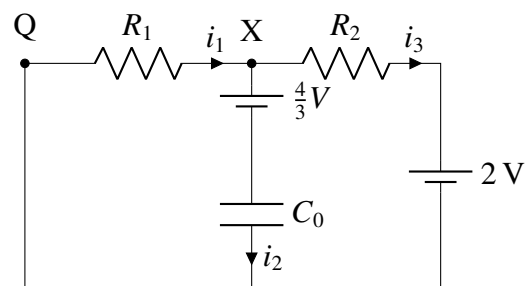


Fig. 3.4

From KCL and KVL,

$$i_1 = i_2 + i_3 \quad (3.10)$$

$$i_1 R_1 + \frac{4}{3} + \frac{1}{C_0} \int_0^t i_2 dt = 0 \quad (3.11)$$

$$\frac{4}{3} + \frac{1}{C_0} \int_0^t i_2 dt - i_3 R_2 - 2 = 0 \quad (3.12)$$

$$(3.13)$$

Taking Laplace Transforms on both sides and using the properties of Laplace Transforms,

$$I_1 = I_2 + I_3 \quad (3.14)$$

$$I_1 R_1 + \frac{4}{3} + \frac{1}{sC_0} I_2 = 0 \quad (3.15)$$

$$\frac{4}{3} + \frac{1}{sC_0} I_2 - I_3 R_2 - 2 = 0 \quad (3.16)$$

$$(3.17)$$

where $i(t) \xleftrightarrow{\mathcal{L}} I(s)$. Note that the capacitor is equivalent to a resistive element of resistance $R_C = \frac{1}{sC_0}$ in the s -domain. Equations (3.14) - (3.16) precisely describe Fig. 3.2.