Advanced DSP (EE5900) Homework Assignment 1

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The transfer function in the s-domain is

$$\frac{Y(s)}{X(s)} = H(s) = \frac{\frac{R}{L}s}{s^2 + \frac{R}{L}s + \frac{1}{LC}}. \tag{1}$$

1. To find the zeros, we set

$$Y(s) = \frac{R}{L}s = 0 \implies s = 0. \tag{2}$$

Thus, the only zero is

$$z_1 = 0. \tag{3}$$

To find the poles, we set

$$X(s) = s^2 + \frac{R}{L}s + \frac{1}{LC} = 0 \tag{4}$$

$$\left(s + \frac{R}{2L}\right)^2 - \left(\frac{R^2}{4L^2} - \frac{1}{LC}\right) = 0 \tag{5}$$

$$s = -\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}. \tag{6}$$

Thus, the poles of the system are

$$p_1, p_2 = -\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}. \tag{7}$$

For various cases, the pole-zero plots are shown in Figure 3, Figure 2, and Figure 1. Notice that $H(s)$ is rational and has conjugate poles, thus it will have a real-valued impulse response. The system is causal if we consider the region of convergence (ROC) which is to the right of all the poles, that is,

$$\operatorname{Re}(s) > \max\left(\operatorname{Re}(p_1),\ \operatorname{Re}(p_2)\right). \tag{8}$$

From (7), it is clear that

$$\operatorname{Re}(p_1) \geq \operatorname{Re}(p_2). \tag{9}$$

Now, for the system to be stable, we must consider the ROC which contains the imaginary axis in the s-domain. However,

$$\operatorname{Re}(p_1) \leq -\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} < 0. \tag{10}$$

with equality iff $\frac{R^2}{4L^2} \geq \frac{1}{LC}$. Therefore, the

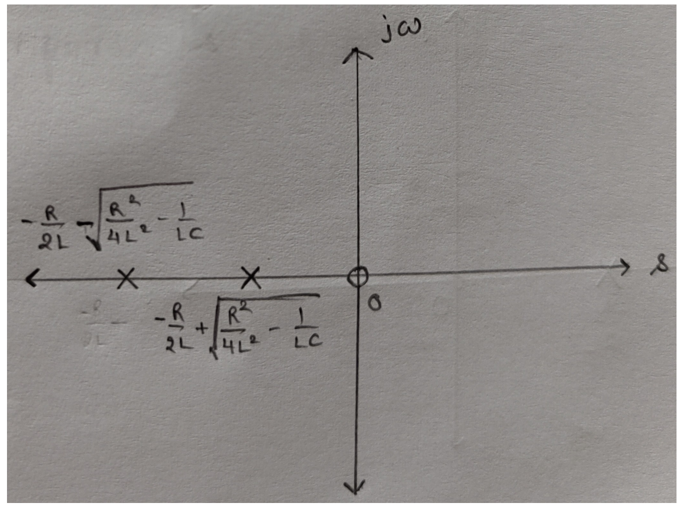

Fig. 1: Pole-zero plot when $\frac{R^2}{4L^2} > \frac{1}{LC}$.

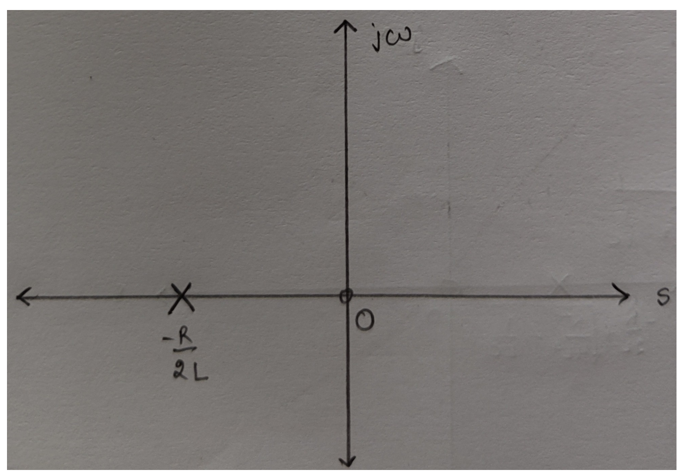

Fig. 2: Pole-zero plot when $\frac{R^2}{4L^2} = \frac{1}{LC}$. The two poles are repeated.

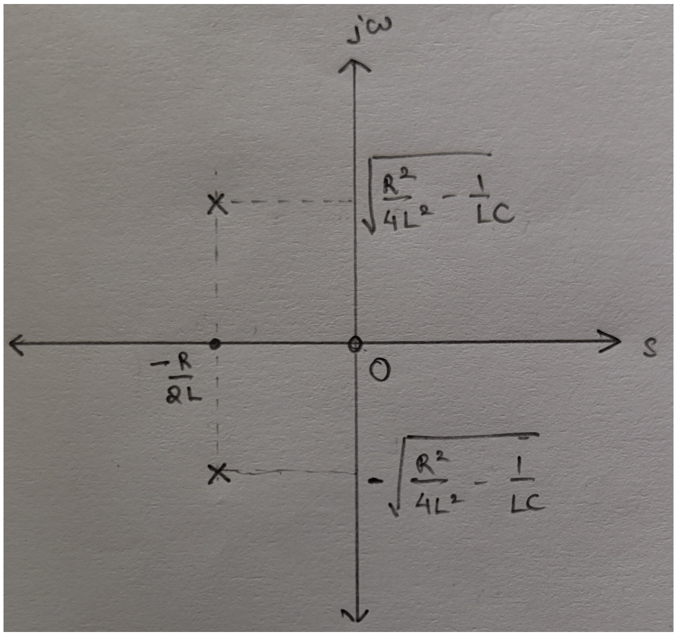

Fig. 3: Pole-zero plot when $\frac{R^2}{4L^2} < \frac{1}{LC}$.

required condition on s is

$$\operatorname{Re}(s) > \operatorname{Re}\left(-\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}\right). \tag{11}$$

2. Rewriting (1), we have

$$\left(sL + 1 + \frac{1}{sC}\right)\frac{Y(s)}{R} = X(s). \tag{12}$$

Taking the inverse Laplace transform on both sides of (12), and using the identities for the Laplace pair $f(t),\ F(s)$,

$$f'(t) \xleftrightarrow{\mathcal{L}} sF(s) - f(0), \tag{13}$$

$$\int_0^t f(t)\,dt \xleftrightarrow{\mathcal{L}} \frac{F(s)}{s}, \tag{14}$$

the required differential equation is

$$\frac{1}{R}\left(L\frac{dy}{dt} + y(t) + \frac{1}{C}\int_0^t y(t)\,dt\right) = x(t). \tag{15}$$

3. Defining

$$i(t) \triangleq \frac{y(t)}{R}, \tag{16}$$

$$v(t) \triangleq x(t), \tag{17}$$

(15) becomes

$$L\frac{di}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(t)\,dt = v(t). \tag{18}$$

This clearly represents a series RLC circuit with input voltage $v(t)$ and output voltage taken across R, as shown in Figure 4.

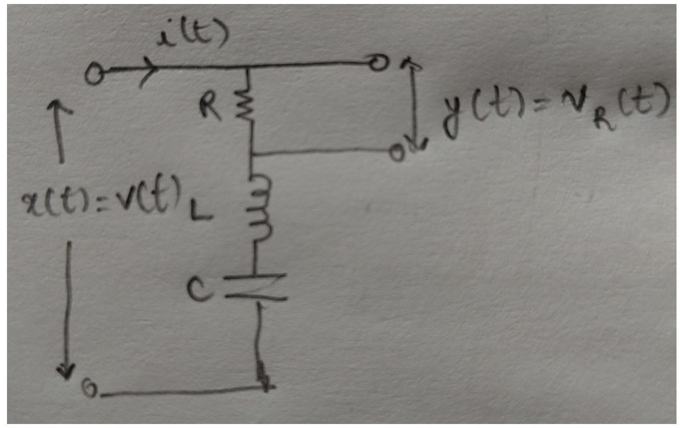

Fig. 4: Series RLC circuit depicting $x(t)$ and $y(t)$.

4. From (1), completing the square,

$$H(s) = \frac{\frac{R}{L}s}{\left(s + \frac{R}{2L}\right)^2 + \left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)}. \tag{19}$$

We define

$$\alpha \triangleq \frac{R}{2L}, \tag{20}$$

$$\omega_0 \triangleq \frac{1}{\sqrt{LC}}, \tag{21}$$

$$\tag{22}$$

where α and ω_0 are the damping coefficient and resonance frequency respectively. Thus, we rewrite (19) as

$$H(s) = \frac{2\alpha s}{(s+\alpha)^2 + \left(\omega_0^2 - \alpha^2\right)}. \tag{23}$$

We have three cases.

a) $\alpha < \omega_0$. This is called *underdamping*. Defining

$$\omega \triangleq \sqrt{\omega_0^2 - \alpha^2}, \tag{24}$$

from (23),

$$H(s) = \frac{2\alpha s}{(s+\alpha)^2 + \omega^2} \tag{25}$$

$$= \frac{2\alpha(s+\alpha)}{(s+\alpha)^2 + \omega^2} - \frac{2\alpha^2}{\omega^2}\frac{\omega^2}{(s+\alpha)^2 + \omega^2}. \tag{26}$$

Taking the inverse Laplace Transform on

both sides of (26),

$$h(t) = 2\alpha e^{-\alpha t}u(t)\left(\cos\omega t - \frac{\alpha}{\omega^2}\sin\omega t\right). \tag{27}$$

b) $\alpha = \omega_0$. This is called *critical damping*. Here, (23) becomes

$$H(s) = \frac{2\alpha s}{(s+\alpha)^2} \tag{28}$$

$$= \frac{2\alpha}{(s+\alpha)} - \frac{2\alpha^2}{(s+\alpha)^2}. \tag{29}$$

Taking the inverse Laplace transform on both sides of (29),

$$h(t) = 2\alpha e^{-\alpha t}u(t)(1-\alpha t). \tag{30}$$

c) $\alpha > \omega_0$. This is called *overdamping*. Defining

$$\beta \triangleq \sqrt{\alpha^2 - \omega_0^2}, \tag{31}$$

from (23),

$$H(s) = \frac{2\alpha s}{(s+\alpha)^2 - \beta^2} \tag{32}$$

$$= \frac{2\alpha(s+\alpha)}{(s+\alpha)^2-\beta^2} - \frac{2\alpha^2}{\beta^2}\frac{\beta^2}{(s+\alpha)^2-\beta^2}. \tag{33}$$

Taking the inverse Laplace transform on both sides of (33),

$$h(t) = 2\alpha e^{-\alpha t}u(t)\left(\cosh\beta t - \frac{\alpha}{\beta^2}\sinh\beta t\right). \tag{34}$$

5. Setting $s = j\omega$ in (1),

$$H(j\omega) = \frac{j\frac{R}{L}\omega}{\left(\frac{1}{LC}-\omega^2\right) + j\frac{R}{L}\omega} \tag{35}$$

$$\implies |H(j\omega)| = \frac{\frac{R}{L}\omega}{\sqrt{\left(\frac{1}{LC}-\omega^2\right)^2 + \left(\frac{R}{L}\omega\right)^2}}. \tag{36}$$

6. The magnitude response of the system is shown in Figure 5.
7. We rewrite (36) as

$$|H(j\omega)| = \frac{\frac{R}{L}}{\sqrt{\left(\frac{1}{LC\omega}-\omega\right)^2 + \left(\frac{R}{L}\right)^2}}. \tag{37}$$

Clearly, the maximum in (36) is maximized on minimizing the denominator, thus we must have

$$\frac{1}{LC\omega} - \omega = 0 \tag{38}$$

$$\implies \omega = \frac{1}{\sqrt{LC}} = \omega_0. \tag{39}$$

Thus, we obtain

$$H_{max} \triangleq |H(j\omega_0)| = 1. \tag{40}$$

This means that at resonance, the entire input voltage appears across the resistor.

8. We have to solve for ω_c, where

$$|H(j\omega_c)| = \frac{1}{\sqrt{2}}H_{max} = \frac{1}{\sqrt{2}}. \tag{41}$$

Using (36) and squaring on both sides,

$$\frac{\left(\frac{R}{L}\omega_c\right)^2}{\left(\frac{1}{LC}-\omega_c^2\right)^2 + \left(\frac{R}{L}\omega_c\right)^2} = \frac{1}{2} \tag{42}$$

$$\left(\frac{R}{L}\omega_c\right)^2 = \left(\frac{1}{LC}-\omega_c^2\right)^2 \tag{43}$$

$$\omega_c^2 \pm \frac{R}{L}\omega_c - \frac{1}{LC} = 0 \tag{44}$$

$$\omega_c = \pm\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} + \frac{1}{4LC}}. \tag{45}$$

Forcing ω_c to be positive, we get the 3dB-cutoff

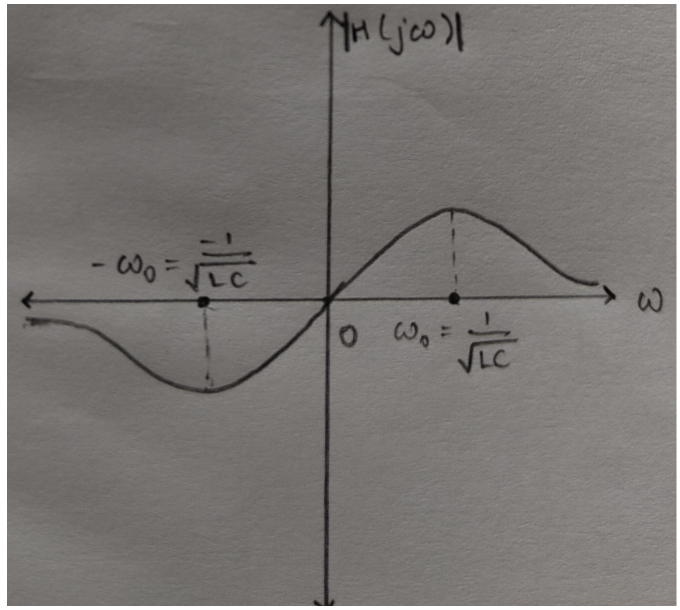

Fig. 5: Graph of magnitude response $|H(j\omega)|$.

frequencies to be

$$\omega_{c2}, \omega_{c1} = \sqrt{\frac{R^2}{4L^2} + \frac{1}{4LC}} \pm \frac{R}{2L}. \tag{46}$$

Therefore, the 3-dB bandwidth parameter,

$$\beta = \omega_{c2} - \omega_{c1} = \frac{R}{L}. \tag{47}$$

Hence, the Q-factor is

$$Q = \frac{\omega_0}{\beta} = \frac{1}{R}\sqrt{\frac{L}{C}}. \tag{48}$$

9. Define

$$\Omega_0 \triangleq 2\pi f_0. \tag{49}$$

The Laplace transform of the input waveform is

$$X(s) = \frac{s\cos\theta - \Omega_0 \sin\theta}{s^2 + \Omega_0^2}. \tag{50}$$

We have three cases.

a) $\alpha < \omega_0$. Using (25),

$$Y(s) = \frac{2\alpha s\,(s\cos\theta - \Omega_0\sin\theta)}{\left((s+\alpha)^2 + \omega^2\right)\left(s^2 + \Omega_0^2\right)} \tag{51}$$

$$= \frac{P_1}{s+\alpha-j\omega} + \frac{P_2}{s+\alpha+j\omega} + \frac{P_3}{s-j\Omega_0} + \frac{P_4}{s+j\Omega_0}, \tag{52}$$

where we have to solve for $P_i,\ i \in \{1,2,3,4\}$. From (52), we see that

$$\begin{aligned} &P_1(s+\alpha+j\omega)\left(s^2+\Omega_0^2\right) \\ &+ P_2(s+\alpha-j\omega)\left(s^2+\Omega_0^2\right) \\ &+ P_3(s+j\Omega_0)\left((s+\alpha)^2+\omega^2\right) \\ &+ P_4(s-j\Omega_0)\left((s+\alpha)^2+\omega^2\right) \\ &= 2\alpha s\,(s\cos\theta - \Omega_0\sin\theta) \end{aligned} \tag{53}$$

Setting $s = -\alpha + j\omega$ in (53),

$$P_1 = \frac{\alpha(\alpha-j\omega)\left((\alpha-j\omega)\cos\theta + \Omega_0\sin\theta\right)}{j\omega\left((\alpha-j\omega)^2 + \Omega_0^2\right)}. \tag{54}$$

Setting $s = -\alpha - j\omega$ in (53),

$$P_2 = \bar{P}_1 \tag{55}$$

$$= \frac{j\alpha(\alpha+j\omega)\left((\alpha+j\omega)\cos\theta + \Omega_0\sin\theta\right)}{\omega\left((\alpha+j\omega)^2 + \Omega_0^2\right)}. \tag{56}$$

Setting $s = j\Omega_0$ in (53),

$$P_3 = \frac{j\alpha\Omega_0 e^{j\theta}}{(\alpha+j\Omega_0)^2 + \omega^2} \tag{57}$$

Setting $s = -j\Omega_0$ in (53),

$$P_4 = \bar{P}_3 \tag{58}$$

$$= \frac{\alpha\Omega_0 e^{-j\theta}}{j\left((\alpha-j\Omega_0)^2 + \omega^2\right)} \tag{59}$$

Thus, taking the inverse Laplace transform of (52), and using (54), (56), (57), (59),

$$y(t) = 2\,\mathrm{Re}\left(P_1 e^{-(\alpha-j\omega)t} + P_3 e^{j\Omega_0 t}\right)u(t) \tag{60}$$

b) $\alpha = \omega_0$. Using (28),

$$Y(s) = \frac{2\alpha s\,(s\cos\theta - \Omega_0\sin\theta)}{(s+\alpha)^2\left(s^2+\Omega_0^2\right)} \tag{61}$$

$$= \frac{P_1}{s+\alpha} + \frac{P_2}{(s+\alpha)^2} + \frac{P_3}{s-j\Omega_0} + \frac{P_4}{s+j\Omega_0}, \tag{62}$$

where we have to solve for $P_i,\ i \in \{1,2,3,4\}$. From (62), we see that

$$\begin{aligned} &P_1(s+\alpha)\left(s^2+\Omega_0^2\right) + P_2\left(s^2+\Omega_0^2\right) \\ &+ P_3(s+\alpha)^2(s+j\Omega_0) \\ &+ P_4(s+\alpha)^2(s-j\Omega_0) \\ &= 2\alpha s\,(s\cos\theta - \Omega_0\sin\theta) \end{aligned} \tag{63}$$

Setting $s = -\alpha$ in (63),

$$P_2 = \frac{2\alpha^2(\alpha\cos\theta + \Omega_0\sin\theta)}{\alpha^2 + \Omega_0^2}. \tag{64}$$

Setting $s = j\Omega_0$ in (63),

$$P_3 = \frac{j\alpha\Omega_0 e^{j\theta}}{(\alpha+j\Omega_0)^2} \tag{65}$$

Setting $s = -j\Omega_0$ in (63),

$$P_4 = \bar{P}_3 \tag{66}$$

$$= \frac{\alpha\Omega_0 e^{-j\theta}}{j(\alpha-j\Omega_0)^2}. \tag{67}$$

Equating the coefficient of s^3 on both sides of (63),

$$P_1 = -(P_3 + P_4) \tag{68}$$

$$= \frac{2\alpha\Omega_0\left(2\alpha\Omega_0\cos\theta - \left(\alpha^2 - \Omega_0^2\right)\sin\theta\right)}{\left(\alpha^2 + \Omega_0^2\right)^2}. \tag{69}$$

Taking the inverse Laplace transform of (62), and using (69), (64), (65) and (67),

$$y(t) = \left(e^{-\alpha t}(P_1 + P_2 t) + 2\,\mathrm{Re}\left(P_3 e^{j\Omega_0 t}\right)\right)u(t). \tag{70}$$

c) $\alpha > \omega_0$. Using (32),

$$Y(s) = \frac{2\alpha s(s\cos\theta - \Omega_0\sin\theta)}{\left((s+\alpha)^2 - \beta^2\right)\left(s^2 + \Omega_0^2\right)} \tag{71}$$

$$= \frac{P_1}{s+\alpha-\beta} + \frac{P_2}{s+\alpha+\beta} + \frac{P_3}{s - j\Omega_0} + \frac{P_4}{s + j\Omega_0}. \tag{72}$$

Using (72),

$$\begin{aligned} &P_1(s+\alpha+\beta)\left(s^2+\Omega_0^2\right) \\ &+ P_2(s+\alpha-\beta)\left(s^2+\Omega_0^2\right) \\ &+ P_3(s+j\Omega_0)\left((s+\alpha)^2-\beta^2\right) \\ &+ P_4(s-j\Omega_0)\left((s+\alpha)^2-\beta^2\right) \\ &= 2\alpha s(s\cos\theta - \Omega_0\sin\theta). \end{aligned} \tag{73}$$

Setting $s = -\alpha + \beta$ in (73),

$$P_1 = \frac{\alpha(\alpha-\beta)((\alpha-\beta)\cos\theta + \Omega_0\sin\theta)}{\beta\left((\alpha-\beta)^2 + \Omega_0^2\right)}. \tag{74}$$

Setting $s = -\alpha - \beta$ in (73),

$$P_2 = \frac{\alpha(\alpha+\beta)((\alpha+\beta)\cos\theta + \Omega_0\sin\theta)}{\beta\left((\alpha+\beta)^2 + \Omega_0^2\right)}. \tag{75}$$

Setting $s = j\Omega_0$ in (73),

$$P_3 = \frac{j\alpha\Omega_0 e^{j\theta}}{(\alpha + j\Omega_0)^2 - \beta^2}. \tag{76}$$

Setting $s = -j\Omega_0$ in (73),

$$P_4 = \bar{P}_3 \tag{77}$$

$$= \frac{\alpha\Omega_0 e^{-j\theta}}{j\left((\alpha - j\Omega_0)^2 - \beta^2\right)}. \tag{78}$$

Taking the inverse Laplace transform of (72), and using (74), (75), (76) and (78),

$$y(t) = \left(e^{-\alpha t}\left(P_1 e^{\beta t} + P_2 e^{-\beta t}\right) + 2\,\mathrm{Re}\left(P_3 e^{j\Omega_0 t}\right)\right)u(t). \tag{79}$$

10. The Laplace transform of the given input is

$$X(s) = \frac{1}{s}. \tag{80}$$

Applying (80) in (1), and using (20), (21),

$$Y(s) = \frac{2\alpha}{(s+\alpha)^2 + \left(\omega_0^2 - \alpha^2\right)}. \tag{81}$$

Three cases arise.

a) $\alpha < \omega_0$. Using (24) and (25),

$$Y(s) = \frac{2\alpha}{\omega}\frac{\omega}{(s+\alpha)^2 + \omega^2} \tag{82}$$

$$\implies y(t) = \mathcal{L}^{-1}[Y(s)] = \frac{2\alpha}{\omega}e^{-\alpha t}u(t)\sin\omega t. \tag{83}$$

b) $\alpha = \omega_0$. Using (28),

$$Y(s) = \frac{2\alpha}{(s+\alpha)^2} \tag{84}$$

$$\implies y(t) = \mathcal{L}^{-1}[Y(s)] = 2\alpha e^{-\alpha t}u(t). \tag{85}$$

c) $\alpha > \omega_0$. Using (31) and (32),

$$Y(s) = \frac{2\alpha}{\beta}\frac{\beta}{(s+\alpha)^2 - \beta^2} \tag{86}$$

$$\implies y(t) = \mathcal{L}^{-1}[Y(s)] = \frac{2\alpha}{\beta}e^{-\alpha t}u(t)\sinh\beta t. \tag{87}$$