Advanced DSP (EE5900) Homework Assignment 2

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CS21BTECH11018

1. Given impulse response of digital filter is

$$h(n) = A\cos(\omega_0 n + \phi)\, u(n). \tag{1}$$

The difference equation of the digital filter is

$$y(n) = x(n) + a_1 y(n-1) + b_1 y(n-2). \tag{2}$$

Taking the Z-transform on both sides of (2),

$$Y(z) = X(z) + a_1 z^{-1} Y(z) + b_1 z^{-2} Y(z). \tag{3}$$

Therefore,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - \left(a_1 z^{-1} + b_1 z^{-2}\right)}. \tag{4}$$

a) Taking the Z-transform of (1),

$$H(z) = \sum_{k=0}^{\infty} A\cos(\omega_0 k + \phi)\, z^{-k}. \tag{5}$$

Expanding (4),

$$H(z) = \sum_{k=0}^{\infty} \left(a_1 z^{-1} + b_1 z^{-2}\right)^k \tag{6}$$

$$= 1 + a_1 z^{-1} + \left(a_1^2 + b_1\right) z^{-2} + \left(a_1^3 + 2a_1 b_1\right) z^{-3} + \dots. \tag{7}$$

Equating the coefficients of (5) and (7), we get

$$A\cos\phi = 1 \tag{8}$$

$$A\cos\left(\frac{\pi}{4} + \phi\right) = a_1 \tag{9}$$

$$A\cos\left(\frac{\pi}{2} + \phi\right) = a_1^2 + b_1 \tag{10}$$

$$A\cos\left(\frac{3\pi}{4} + \phi\right) = a_1^3 + 2a_1 b_1. \tag{11}$$

Using (8) in (9),

$$A(\cos\phi - \sin\phi) = \sqrt{2}a_1 \tag{12}$$

$$\implies A\sin\phi = 1 - \sqrt{2}a_1. \tag{13}$$

Using (13) in (10),

$$a_1^2 - \sqrt{2}a_1 + b_1 + 1 = 0 \tag{14}$$

$$\implies b_1 = -a_1^2 + \sqrt{2}a_1 - 1. \tag{15}$$

Using (8) and (13) in (11),

$$a_1^3 + 2a_1 b_1 + \frac{2 - \sqrt{2}a_1}{\sqrt{2}} = 0 \tag{16}$$

$$\implies a_1^3 + 2a_1 b_1 + \sqrt{2} - a_1 = 0. \tag{17}$$

Using (15) in (17) and simplifying,

$$a_1^3 - 2\sqrt{2}a_1^2 + 3a_1 - \sqrt{2} = 0 \tag{18}$$

$$\left(a_1 - \sqrt{2}\right)\left(a_1^2 - \sqrt{2}a_1 + 1\right) = 0 \tag{19}$$

$$\left(a_1 - \sqrt{2}\right)\left(\left(a_1 - \frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2}\right) = 0 \tag{20}$$

$$\implies a_1 = \sqrt{2}. \tag{21}$$

Substituting in (15), $b_1 = -1$.

b) We see that $h(1)$ in the unquantized digital filter is $a_1 = \sqrt{2}$. However, after quantization, using (8) to (10),

$$h(0) = A\cos\phi = 1 \tag{22}$$

$$h(1) = A\cos(\omega + \phi) = 1.375 \tag{23}$$

$$h(2) = A\cos(2\omega + \phi) = 1, \tag{24}$$

where the right hand side of (23) follows because $\sqrt{2} = (1.011\dots)_2$. Using (22) in (24),

$$\cos 2\omega - A\sin\phi\sin 2\omega = 1 \tag{25}$$

$$2\sin^2\omega + A\sin\phi\,(2\sin\omega\cos\omega) = 0 \tag{26}$$

$$\sin\omega\,(A\sin\phi\cos\omega + \sin\omega) = 0. \tag{27}$$

In (27), if $\sin\omega = 0$ or $\omega = n\pi$, (22) and (23) contradict each other. Thus, we must have

$$\tan\omega = -A\sin\phi. \tag{28}$$

Substituting (28) in (23),

$$\cos\omega + \tan\omega \sin\omega = 1.375 \tag{29}$$

$$\implies \frac{1}{\cos\omega} = \frac{11}{8} \tag{30}$$

$$\implies \cos\omega = \frac{8}{11}. \tag{31}$$

Thus, on quantization, $\omega = \cos^{-1}\left(\frac{8}{11}\right)$.

2. Given analog frequecy response,

$$H_a(j\Omega) = \begin{cases} j\Omega e^{-j\Omega\tau} & |\Omega| \le \Omega_c \\ 0 & \text{otherwise} \end{cases}. \tag{32}$$

a) Setting $\Omega = \frac{\omega}{T}$, the frequency response of the digital filter obtained is

$$H_d\left(e^{j\omega}\right) = \begin{cases} j\frac{\omega}{T} e^{-j\frac{\omega}{T}\tau} & |\omega| \le \Omega_c T \\ 0 & \text{otherwise} \end{cases}. \tag{33}$$

b) From (33), setting $\tau = 0$,

$$\hat{H}_d\left(e^{j\omega}\right) = j\frac{\omega}{T} \tag{34}$$

Note that

$$\hat{h}_d(n - n_\tau) \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega n_\tau}\hat{H}_d\left(e^{j\omega}\right) \tag{35}$$

$$= j\frac{\omega}{T}e^{-j\omega n_\tau} = j\frac{\omega}{T}e^{-j\omega\frac{\tau}{T}}. \tag{36}$$

From (36), since $n_\tau \in \mathbb{Z}$, we see that

$$\tau = kT,\ n_\tau = k\ \forall\ k \in \mathbb{Z}. \tag{37}$$

3. From Figure 1,

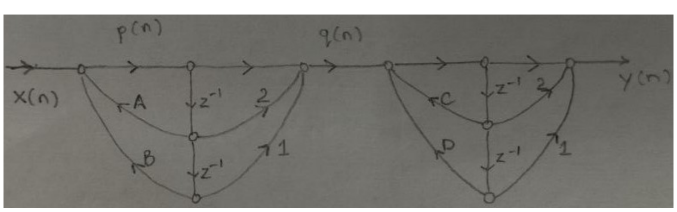

Fig. 1: Lowpass Digital Filter.

$$p(n) = x(n) + Ap(n-1) + Bp(n-2), \tag{38}$$

$$q(n) = p(n) + 2p(n-1) + p(n-2). \tag{39}$$

Taking the Z-transform on both sides of (38) and (39),

$$P(z)\left(1 - Az^{-1} - Bz^{-2}\right) = X(z), \tag{40}$$

$$Q(z) = P(z)\left(1 + 2z^{-1} + z^{-2}\right). \tag{41}$$

From (40) and (41),

$$\frac{Q(z)}{X(z)} = \frac{\left(1 + z^{-1}\right)^2}{1 - Az^{-1} - Bz^{-2}} \tag{42}$$

Since two similar blocks are cascaded, we see that the lowpass filter has transfer function

$$H_{LP}(z) = \frac{Y(z)}{X(z)} \tag{43}$$

$$= \frac{\left(1 + z^{-1}\right)^4}{\left(1 - Az^{-1} - Bz^{-2}\right)\left(1 - Cz^{-1} - Dz^{-2}\right)}. \tag{44}$$

We apply the lowpass to highpass transform to (44). The given cutoff frequency of the lowpass filter is $\omega_L = \frac{\pi}{2}$ and the required cutoff frequency of the highpass filter is $\omega_H = \frac{\pi}{2}$. Thus,

$$\alpha = -\frac{\cos\left(\frac{\omega_L + \omega_H}{2}\right)}{\cos\left(\frac{\omega_L - \omega_H}{2}\right)} = 0 \tag{45}$$

and we must use the transformation

$$z^{-1} \to -\frac{z^{-1} + \alpha}{1 + \alpha z^{-1}} = -z^{-1}. \tag{46}$$

Using (46) in (44), the required highpass transfer function is

$$H_{HP}(z) = \frac{\left(1 - z^{-1}\right)^4}{\left(1 + Az^{-1} - Bz^{-2}\right)\left(1 + Cz^{-1} - Dz^{-2}\right)} \tag{47}$$

$$= \frac{1 + (-2)z^{-1} + z^{-2}}{1 - (-A)z^{-1} - Bz^{-2}}\frac{1 + (-2)z^{-1} + z^{-2}}{1 - (-C)z^{-1} - Dz^{-2}}. \tag{48}$$

and the modified filter is shown in Figure 2.

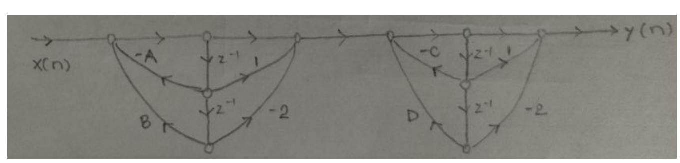

Fig. 2: Highpass Digital Filter.

4. We are given the following.

$$\omega_p = 0.2613\pi,\ \omega_s = 0.4018\pi \tag{49}$$

$$20\log_{10}\left|H_d\left(e^{j\omega_p}\right)\right| \ge -0.75\ \text{dB} \tag{50}$$

$$20\log_{10}\left|H_d\left(e^{j\omega_s}\right)\right| \le -20\ \text{dB} \tag{51}$$

We will use the bilinear transformation method to construct the required digital filter from the corresponding analog filter, for which we have

$$\Omega_p = \tan\left(\frac{\omega_p}{2}\right) = 0.435165 \tag{52}$$

$$\Omega_s = \tan\left(\frac{\omega_s}{2}\right) = 0.730871 \tag{53}$$

We also know that the minimum passband magnitude and maximum stopband ripple are given by

$$\left|H_a\left(j\Omega_p\right)\right|^2 = \frac{1}{1+\epsilon^2} \tag{54}$$

$$\left|H_a\left(j\Omega_s\right)\right|^2 = \frac{1}{A^2} \tag{55}$$

Since the bilinear transformation does not change magnitude characteristics, we substitute (54) in (50) and (55) in (51) to get

$$\epsilon^2 = 0.188502,\ A^2 = 100 \tag{56}$$

Hence, the order of the analog Butterworth filter that meets the specifications is

$$N = \frac{1}{2}\frac{\log_{10}\left(\frac{A^2-1}{\epsilon^2}\right)}{\log_{10}\left(\frac{\Omega_s}{\Omega_p}\right)} = 6.040140 \tag{57}$$

Taking the ceiling, the order of the analog filter is $N = 7$. From (55),

$$\left|H_a\left(j\Omega_s\right)\right|^2 = \frac{1}{1+\left(\frac{\Omega_s}{\Omega_c}\right)^{2N}} = \frac{1}{A^2} \tag{58}$$

$$\implies \Omega_c = \frac{\Omega_s}{\left(A^2-1\right)^{\frac{1}{2N}}} = 0.526375 \tag{59}$$

The transfer function of the normalized 7$^{\text{th}}$ order Butterworth analog filter is

$$H_{an}(s) = \prod_{i=1}^{7}\frac{1}{s-p_i} \tag{60}$$

where $p_i,\ 1 \le i \le 7$ are the poles of the normalized Butterworth analog filter. De-normalizing (60),

$$H_a(s) = H_{an}\left(\frac{s}{\Omega_c}\right) = \prod_{i=1}^{7}\frac{\Omega_c}{s-\Omega_c p_i} \tag{61}$$

Thus, the poles of the required Butterworth filter are tabulated in Table 1. The transfer function of the corresponding digital filter is obtained as

p_i	$p_i' = p_i\Omega_c$
$-0.9010 + j0.4339$	$-0.4743 + j0.2284$
$-0.9010 - j0.4339$	$-0.4743 - j0.2284$
$-0.6235 + j0.7818$	$-0.3283 + j0.4115$
$-0.6235 - j0.7818$	$-0.3283 - j0.4115$
$-0.2225 + j0.9749$	$-0.4743 + j0.2284$
$-0.2225 - j0.9749$	$-0.4743 - j0.2284$
-1.000	-0.5264

TABLE 1: Poles of the analog Butterworth filter.

$$H_d(z) = \left.H_a(s)\right|_{s=\frac{1-z^{-1}}{1+z^{-1}}} \tag{62}$$

$$= \prod_{i=1}^{7}\frac{\Omega_c\left(1+z^{-1}\right)}{\left(1-p_i'\right)-\left(1+p_i'\right)z^{-1}} \tag{63}$$

where p_i''s are as defined in Table 1.