Advanced DSP (EE5900) Homework Assignment 3

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1. a) Using the inverse Fourier transform,

$$x_a(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_a(j\Omega)\, e^{j\Omega t} d\Omega, \tag{1}$$

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\omega}\right) e^{j\omega n} d\omega. \tag{2}$$

However, we also know that

$$x(n) = x_a(nT) = x_a(t)|_{t=nT} \tag{3}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_a(j\Omega)\, e^{j\Omega nT} d\Omega \tag{4}$$

$$= \frac{1}{2\pi}\sum_{r=-\infty}^{\infty}\int_{(2r-1)\frac{\pi}{T}}^{(2r+1)\frac{\pi}{T}} X_a(j\Omega)\, e^{j\Omega nT} d\Omega. \tag{5}$$

Making a change of variables for each term in the summation, $\Omega \to \Omega + \frac{2\pi r}{T}$,

$$x(n) = \frac{1}{2\pi}\sum_{r=-\infty}^{\infty}\int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} X_a\left(j\Omega + j\frac{2\pi r}{T}\right) e^{j\Omega nT} e^{j2\pi rn} d\Omega. \tag{6}$$

Interchanging the order of summation and integration, and noting that $e^{j2\pi rn} = 1\ \forall\ n \in \mathbb{Z},\ r \in \mathbb{Z}$,

$$x(n) = \frac{1}{2\pi}\int_{-\frac{\pi}{T}}^{\frac{\pi}{T}}\left[\sum_{r=-\infty}^{\infty} X_a\left(j\Omega + j\frac{2\pi r}{T}\right)\right] e^{j\Omega nT} d\Omega. \tag{7}$$

Making the substitution $\Omega = \frac{\omega}{T}$,

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{1}{T}\sum_{r=-\infty}^{\infty} X_a\left(j\frac{\omega}{T} + j\frac{2\pi r}{T}\right)\right] e^{j\omega n} d\omega. \tag{8}$$

Comparing (2) with (8), we obtain

$$X\left(e^{j\omega}\right) = \frac{1}{T}\sum_{r=-\infty}^{\infty} X_a\left(\frac{j\omega}{T} + j\frac{2\pi r}{T}\right). \tag{9}$$

where $\Omega = \frac{\omega}{T}$.

b) Consider a signal $x_a(t) = A\cos(2\pi Ft + \phi)$ sampled every T seconds. The sampling frequency F_s is given by

$$F_s = \frac{1}{T} \tag{10}$$

and the sampled discrete time signal is

$$x(n) = x_a(nT) \tag{11}$$

$$= A\cos(2n\pi FT + \phi) \tag{12}$$

$$= A\cos\left(2n\pi\frac{F}{F_s} + \phi\right). \tag{13}$$

From (13), we see that the normalized digital frequency is given by

$$f \triangleq \frac{F}{F_s} = FT. \tag{14}$$

c) The ranges are given in Table 1.

Quantity	**Range**
ω	$(-\pi, \pi]$
Ω	$(-\infty, \infty)$
f	$\left(-\frac{1}{2}, \frac{1}{2}\right]$
F	$(-\infty, \infty)$
F_s	$(-\infty, \infty)$
T	$(-\infty, \infty)$

TABLE 1: Range of various quantities.

d) By sampling theorem, we know that all analog frequencies of the input signal are mapped to the range $\left(-\frac{F_s}{2}, \frac{F_s}{2}\right]$. Therefore, the range of digital frequency that can be obtained,

$$f = \frac{F}{F_s} \in \left(-\frac{1}{2}, \frac{1}{2} \right]. \tag{15}$$

Hence, the maximum obtainable digital frequency is $\frac{1}{2}$ at $F = (2n+1)\frac{F_s}{2},\ n \in \mathbb{Z}$.

e) We know that

$$f = \frac{F}{F_s} \leq \frac{1}{2} \tag{16}$$

$$\implies F \leq \frac{F_s}{2}. \tag{17}$$

Thus, for a given f and F_s, the maximum analog frequency is $\frac{F_s}{2}$.

f) The units are given in Table 2. Where no units are mentioned, it means that the quantity is dimensionless.

Quantity	Units
ω	
Ω	rad s^{-1}
f	
F	Hz
F_s	Hz
T	s

TABLE 2: Units of various quantities.

2. The given input signal is

$$x_a(t) = 3\cos(100\pi t) + 2\sin(250\pi t) \tag{18}$$

and the frequency response is

$$\begin{aligned} X_a(j\Omega) &= 3\pi\left(\delta(\Omega - 100\pi) + \delta(\Omega + 100\pi)\right) \\ &\quad - 2j\pi\left(\delta(\Omega - 250\pi) - \delta(\Omega + 250\pi)\right) \end{aligned} \tag{19}$$

as shown in Figure 1.
After the ADC stage, the output waveform is obtained by setting $t = nT_1$. Thus,

$$x(n) = x_a(nT_1) \tag{20}$$

$$= 3\cos\left(\frac{n\pi}{2}\right) + 2\sin\left(\frac{5n\pi}{4}\right) \tag{21}$$

$$= 3\cos\left(\frac{n\pi}{2}\right) + 2\sin\left(\frac{-3n\pi}{4}\right) \tag{22}$$

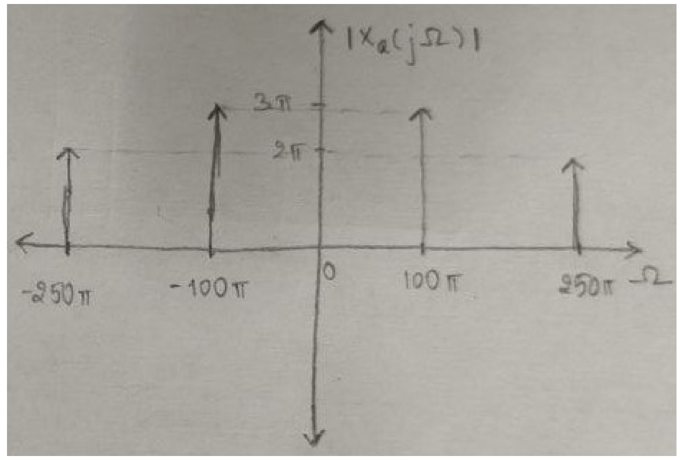

Fig. 1: Spectrum of input signal.

and the frequency response is (for $-\pi < \omega \leq \pi$)

$$\begin{aligned} X(\omega) &= 3\pi\left(\delta\left(\omega - \frac{\pi}{2}\right) + \delta\left(\omega + \frac{\pi}{2}\right)\right) \\ &\quad + \frac{2\pi}{j}\left(\delta\left(\omega + \frac{3\pi}{4}\right) - \delta\left(\omega - \frac{3\pi}{4}\right)\right) \end{aligned} \tag{23}$$

which is periodic with period 2π. The spectral content of the principal range is shown in Figure 2.

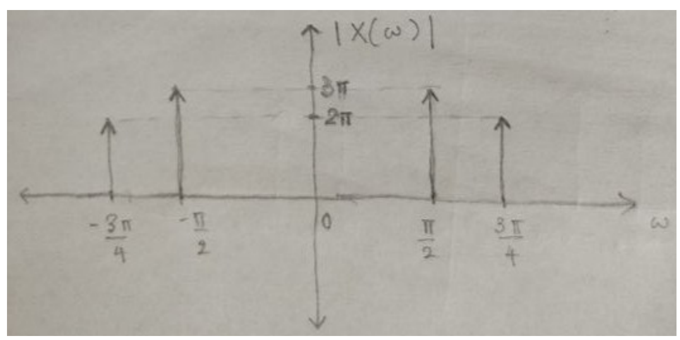

Fig. 2: Spectrum after ADC stage ($-\pi < \omega \leq \pi$).

At the DAC stage, we upsample the input digital signal by a factor of $\frac{T_1}{T_2} = 5$, by padding 4 zeros between each sample. Denote the zero-padded signal by $\tilde{x}(k)$. Then,

$$\tilde{x}(k) = \begin{cases} x(m) & k = 5m,\ m \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}. \tag{24}$$

The output waveform after the DAC stage will be

$$\tilde{x}(t) = \begin{cases} x(m) & 5mT_2 \leq t < (5m+1)T_2 \\ 0 & \text{otherwise,} \end{cases} \tag{25}$$

and the corresponding frequency response is, noting that $T_1 = 5T_2$,

$$\tilde{X}(j\Omega) = \sum_{m=-\infty}^{\infty} \int_{5mT_2}^{(5m+1)T_2} x(m)\, e^{-j\Omega t} dt \tag{26}$$

$$= \sum_{m=-\infty}^{\infty} \int_{mT_1}^{mT_1+T_2} x(m)\, e^{-j\Omega t} dt \tag{27}$$

$$= \frac{1-e^{-j\Omega T_2}}{j\Omega} \sum_{m=-\infty}^{\infty} x(m)\, e^{-j(\Omega T_1)m} \tag{28}$$

$$= \frac{1-e^{-j\Omega T_2}}{j\Omega} X(\Omega T_1) \tag{29}$$

as shown in Figure 3.

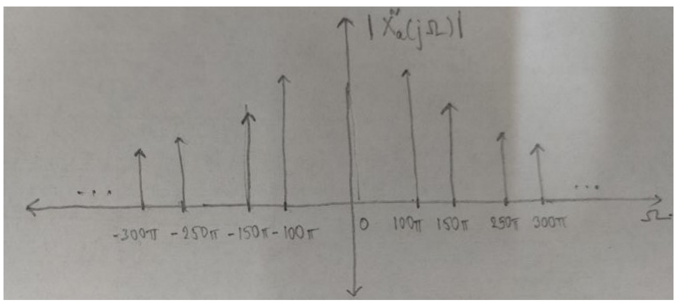

Fig. 3: Spectrum after upsampling and DAC stage.

After the filter stage, from (24) and (22),

$$Y_a(t) = \sum_{k=-\infty}^{\infty} \tilde{x}(k)\, h(t - kT_2) \tag{30}$$

$$= \sum_{r=0}^{4} \sum_{m=-\infty}^{\infty} \tilde{x}(5m+r)\, h(t-(5m+r)T_2) \tag{31}$$

$$= \sum_{m=-\infty}^{\infty} x(m)\, h(t - 5mT_2) \tag{32}$$

$$= \sum_{m=-\infty}^{\infty} x(m)\, h(t - mT_1) \tag{33}$$

$$= \sum_{m=-\infty}^{\infty} \left(3\cos\left(\frac{m\pi}{2}\right) + 2\sin\left(\frac{-3m\pi}{4}\right)\right) h(t - mT_1) \tag{34}$$

$$= 3\cos(100\pi t) + 2\sin(-150\pi t). \tag{35}$$

and the corresponding frequency response is

$$Y_a(j\Omega) = 3\pi(\delta(\Omega - 100\pi) + \delta(\Omega + 100\pi)) - 2j\pi(\delta(\Omega + 150\pi) - \delta(\Omega - 150\pi)) \tag{36}$$

as shown in Figure 4.

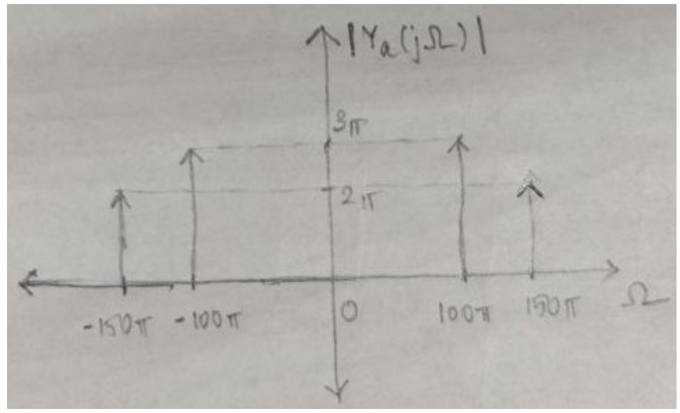

Fig. 4: Spectrum after the filter stage.

Notice that (35) and (18) are not equal. This is because the ADC undersampled the signal, leading to *aliasing* when upsampled by the DAC and filtered.

3. For a discrete time signal with input range A and resolution Δ, the number of bits required, N, is given by

$$2^N = \left\lceil \frac{A}{\Delta} \right\rceil + 1 \tag{37}$$

and so,

$$N = \left\lceil \log_2\left(\left\lceil \frac{A}{\Delta} \right\rceil + 1\right)\right\rceil. \tag{38}$$

For both cases, $A = 6.35 - (-6.35) = 13.7$.

a) $\Delta = 0.1$. Using (38), $N = 8$ bits.
b) $\Delta = 0.02$. Using (38), $N = 10$ bits.

4. To check the linearity of a recursive system, we consider two input signals $x_1(n)$, $x_2(n)$ and their respective outputs $y_1(n)$, $y_2(n)$. If for all reals $a_1,\ a_2$, we have

$$h(n) \circledast (a_1 x_1(n) + a_2 x_2(n)) = a_1 y(n) + a_2 y(n). \tag{39}$$

then the given system is linear. For the given case, we have

$$y_1(n) = a y_1(n-1) + x_1(n), \tag{40}$$

$$y_2(n) = a y_2(n-1) + x_2(n), \tag{41}$$

and so,

$$\begin{aligned}
&a_1 y_1(n) + a_2 y_2(n) \\
&= a_1 (a y_1(n-1) + x_1(n)) \\
&+ a_2 (a y_2(n-1) + x_2(n)) && (42) \\
&= a (a_1 y_1(n-1) + a_2 y_2(n-1)) \\
&+ (a_1 x_1(n) + a_2 x_2(n)). && (43)
\end{aligned}$$

This shows that the system is linear. To show that the system is stable, we must show that $|z| = 1$ lies in the region of convergence (ROC) of the system function $H(z)$. However, from the given recurrence,

$$Y(z) = a z^{-1} Y(z) + X(z), \tag{44}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - a z^{-1}}. \tag{45}$$

For (45) to converge, the required ROC is

$$\left| a z^{-1} \right| < 1 \implies |z| > |a|, \tag{46}$$

and since the ROC must contain the unit circle, we must have for stability

$$|a| < 1. \tag{47}$$

The homogeneous equation is

$$y(n) - a y(n-1) = 0. \tag{48}$$

Setting $y(n) = \lambda^n$, we get

$$\lambda^{n-1} (\lambda - a) = 0. \tag{49}$$

Therefore, $\lambda = a$. Hence, the homogeneous solution is

$$y(n) = a^n u(n). \tag{50}$$