

Lecture 7: 20 September 2023

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7.1 Quantization of Vectors

In this situation, we must quantize high-dimensional vectors $\mathbf{x} \in \mathbb{R}^d$ using k bits, and recover a lossy version $\hat{\mathbf{X}}$. Further, we assume that $\|\mathbf{x}\|$ is upper bounded.

We use the algorithms for quantization of real numbers as follows. Define

$$x_{\min} \triangleq \min_{1 \leq i \leq d} \mathbf{x}^\top \mathbf{e}_i \quad (7.1)$$

$$x_{\max} \triangleq \max_{1 \leq i \leq d} \mathbf{x}^\top \mathbf{e}_i. \quad (7.2)$$

Now, since each coordinate lies between x_{\min} and x_{\max} , we can apply the previous algorithms. For example, using deterministic rounding,

$$\text{Cost} = \max_{\|\mathbf{x}\| \leq r} \mathbb{E} \left[\|\hat{\mathbf{X}} - \mathbf{x}\|^2 \right] \quad (7.3)$$

$$= \max_{\|\mathbf{x}\| \leq r} \sum_{i=1}^d \mathbb{E} \left[\left(\hat{X}_i - x_i \right)^2 \right] \quad (7.4)$$

$$= \frac{d(x_{\max} - x_{\min})^2}{2^{2k+2}} \leq \frac{d(2r^2)}{2^{2k+2}} \quad (7.5)$$

In general, for any scalar quantizer with MSE m , the cost is

$$\text{Cost} = 2r^2 dm \quad (7.6)$$

The *time complexity* of applying such algorithms is $\mathcal{O}(dT)$, where T is the time complexity for applying the scalar version of the algorithm, usually a constant-time algorithm. Thus, the time complexity usually grows linearly with d .

7.2 Lower Bound on Cost for Deterministic Algorithms

Assume in this situation that $k = \Theta(d) = d\alpha$. Since the algorithm is deterministic, each k -bit sequence represents one point in space. Denote the **codebook** of the chosen point vectors and their corresponding k -bit

sequence as \mathbb{C} . Clearly, given \mathbb{C} , the optimal encoding rule corresponds to searching for nearest neighbours, that is

$$\text{Enc}(\mathbf{x}) = \arg \min_{\mathbf{y} \in \mathbb{C}} \|\mathbf{x} - \mathbf{y}\|. \quad (7.7)$$

Denote the **cost** of a codebook \mathbb{C} as

$$\beta^2 \triangleq \text{Cost} = \max_{\mathbf{x} \in \mathcal{B}(r)} \min_{\mathbf{y} \in \mathbb{C}} \|\mathbf{x} - \mathbf{y}\|^2. \quad (7.8)$$

For every $\mathbf{x} \in \mathcal{B}(r)$, it follows that there exists at least one codeword at distance at most β . Hence, the union of the small balls of radius β covers the larger ball of radius r . Thus,

$$\bigcup_{\mathbf{y} \in \mathbb{C}} \mathcal{B}(\mathbf{y}, \beta) \supset \mathcal{B}(\mathbf{0}, r) \quad (7.9)$$

$$|\mathbb{C}| \geq \frac{\frac{\pi^{\frac{d}{2}} r^d}{\Gamma(\frac{d}{2}+1)}}{\frac{\pi^{\frac{d}{2}} \beta^d}{\Gamma(\frac{d}{2}+1)}} = \left(\frac{r}{\beta}\right)^d \quad (7.10)$$

$$2^{d\alpha} \geq \left(\frac{r}{\beta}\right)^d \quad (7.11)$$

$$\beta^2 \geq \frac{r^2}{2^{2\alpha}} \quad (7.12)$$

Comparing with (7.5), we see that (7.12) is better by a factor of d .