EE6367: Topics in Data Storage and Communications **2023**

Lecture 14: 11 October 2023

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14.1 Correlated Quantization

We consider stochastic rounding in the case of n users. Let a random permutation π of $\{0, 1, \ldots, n-1\}$ be shared among the users. User i is given the number π_i (here the users are 0-indexed). Then, each user generates $\gamma_i \sim \text{Unif}\left[0, \frac{1}{m}\right]$. Note that γ_i is *private randomness*. Define

$$U_i \triangleq \frac{\pi_i}{m} + \gamma_i. \tag{14.1}$$

The CDF of U_i, for $u \in [0, 1]$, is

$$\begin{aligned}
\Pr\left(U_i \leqslant u\right) &= \Pr\left(\frac{\pi_i}{m} + \gamma_i \leqslant u\right) && (14.2)\\
&= \Pr\left(\pi_i + m\gamma_i \leqslant mu\right) && (14.3)\\
&= \Pr\left(\pi_i < \lfloor mu \rfloor\right) + \Pr\left(\pi_i = \lfloor mu \rfloor,\ m\gamma_i \leqslant \mu - \lfloor mu \rfloor\right) && (14.4)\\
&= \frac{\lfloor mu \rfloor}{m} + \frac{mu - \lfloor mu \rfloor}{m} = u && (14.5)
\end{aligned}$$

and hence $U_i \sim \text{Unif}\left[0, 1\right]$.

Now, each user i transmits

$$Y_i \triangleq \mathbb{1}_{\{U_i < x_i\}}. \tag{14.6}$$

Clearly, this scheme is unbiased because $\mathbb{E}\left[Y_i\right] = x_i$. We now present a claim for the MSE of this scheme.

Claim 14.1. *For the above scheme, the MSE is upper bounded by* $\frac{3}{m}\sigma_{md} + \frac{12}{m^2}$*, where*

$$\sigma_{md} \triangleq \frac{1}{m}\sum_{i=1}^{m} |x_i - \bar{x}| \leqslant \sqrt{\frac{1}{m}\sum_{i=1}^{m} \left(x_i - \bar{x}\right)^2}. \tag{14.7}$$

Proof. Define

$$z_i \triangleq \frac{\lfloor mx_i \rfloor}{m} \tag{14.8}$$

$$Y_i \triangleq Q_i\left(x_i\right) \tag{14.9}$$
$$Q_i\left(t\right) \triangleq \mathbb{1}_{\{U_i<t\}} \tag{14.10}$$

The MSE is given by

$$\begin{aligned}
\mathrm{MSE} &= \mathbb{E}\left[\left(\frac{1}{m}\sum_{i=1}^m Y_i - \frac{1}{m}\sum_{i=1}^m x_i\right)^2\right] &(14.11)\\
&= \frac{1}{m^2}\mathbb{E}\left[\left(\sum_{i=1}^m \left(Y_i - x_i\right)\right)^2\right] &(14.12)\\
&= \frac{1}{m^2}\mathbb{E}\left[\left(\sum_{i=1}^m \left(\left(x_i - z_i\right) + \left(z_i - Q_i\left(z_i\right)\right) + \left(Q_i\left(z_i\right) - Q_i\left(x_i\right)\right)\right)\right)^2\right] &(14.13)\\
&\leqslant \frac{3}{m^2}\left\{\mathbb{E}\left[\left(\sum_{i=1}^m \left(x_i - z_i\right)\right)^2\right] + \mathbb{E}\left[\left(\sum_{i=1}^m \left(z_i - Q_i\left(z_i\right)\right)\right)^2\right] + \mathbb{E}\left[\left(\sum_{i=1}^m \left(Q_i\left(z_i\right) - Q_i\left(x_i\right)\right)\right)^2\right]\right\} &(14.14)
\end{aligned}$$

where we use in (14.13) the inequality

$$\left(\alpha+\beta+\gamma\right)^2 \leqslant 3\left(\alpha^2+\beta^2+\gamma^2\right). \tag{14.15}$$

for reals α, β, γ.

Note that

$$x_i - z_i = x_i - \frac{\lfloor m x_i \rfloor}{m} = \frac{m x_i - \lfloor m x_i \rfloor}{m} \leqslant \frac{1}{m}, \tag{14.16}$$

hence

$$\mathbb{E}\left[\left(\sum_{i=1}^m \left(x_i - z_i\right)\right)^2\right] \leqslant 1. \tag{14.17}$$

Now,

$$\begin{aligned}
\mathbb{E}\left[\left(\sum_{i=1}^m z_i - Q_i\left(z_i\right)\right)^2\right] &= \sum_{i=1}^m \mathbb{E}\left[\left(z_i - Q_i\left(z_i\right)\right)^2\right] + \sum_{i=1}^m \sum_{\substack{j=1\\ j\neq i}}^m \mathbb{E}\left[\left(z_i - Q_i\left(z_i\right)\right)\left(z_j - Q_j\left(z_j\right)\right)\right] &(14.18)\\
&= \sum_{i=1}^m \mathbb{E}\left[\left(z_i - Q_i\left(z_i\right)\right)^2\right] + \sum_{i=1}^m \sum_{\substack{j=1\\ j\neq m}}^m \left(\mathbb{E}\left[Q_i\left(z_i\right) Q_j\left(z_j\right)\right] - z_i z_j\right) &(14.19)
\end{aligned}$$

Note that

$$\begin{aligned}
Q_i\left(z_i\right) &= \mathbb{1}_{\{\pi_i + m\gamma_i < m z_i\}} &(14.20)\\
&= \mathbb{1}_{\{\pi_i < m z_i\}} &(14.21)
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}\left[Q_i\left(z_i\right) Q_j\left(z_j\right)\right] &= \Pr\left(\pi_i < m z_i,\ \pi_j < m z_j\right) && (14.22)\\
&= \min\left\{z_i, z_j\right\} \frac{m \max\left\{z_i, z_j\right\} - 1}{m-1} && (14.23)\\
&= \frac{m z_i z_j - \min\left\{z_i, z_j\right\}}{m-1} && (14.24)\\
&= \frac{m z_i z_j}{m-1} - \frac{1}{m-1}\left(\frac{z_i + z_j}{2} - \frac{\left|z_i - z_j\right|}{2}\right). && (14.25)
\end{aligned}$$

Using (14.25) in (14.19),

$$\begin{aligned}
\mathbb{E}\left[\left(\sum_{i=1}^{m} z_i - Q_i\left(z_i\right)\right)^2\right] &= \sum_{i=1}^{m} z_i\left(1 - z_i\right) + \sum_{i=1}\sum_{\substack{j=1\\ j\neq i}}^{m}\left(\frac{z_i z_j}{m-1} - \frac{z_i + z_j}{2\left(m-1\right)} + \frac{\left|z_i - z_j\right|}{2\left(m-1\right)}\right) && (14.26)\\
&= \sum_{i=1}^{m} z_i\left(1 - z_i\right) + \frac{\left(\sum_i z_i\right)^2 - \sum_i z_i^2}{m-1} - \sum_i z_i + \sum_{i=1}^{m}\sum_{\substack{j=1\\ j\neq i}}^{m} \frac{\left|z_i - z_j\right|}{2\left(m-1\right)} && (14.27)\\
&\leqslant \frac{\left(\sum_i z_i\right)^2}{m} - \sum_i z_i^2 + \sum_{i=1}^{m}\sum_{\substack{j=1\\ j\neq i}}^{m} \frac{\left|z_i - z_j\right|}{2\left(m-1\right)} && (14.28)\\
&\leqslant \sum_{\substack{j=1\\ j\neq i}}^{m} \frac{\left|z_i - z_j\right|}{2\left(m-1\right)} && (14.29)\\
&\leqslant \sum_{\substack{j=1\\ j\neq i}}^{m} \frac{\left|x_i - x_j\right| + \frac{1}{m}}{2\left(m-1\right)} && (14.30)\\
&\leqslant \sum_{i=1}^{m}\sum_{\substack{j=1\\ j\neq i}}^{m} \frac{\left|x_i - x_j\right|}{2\left(m-1\right)} + \frac{1}{2}. && (14.31)
\end{aligned}$$

Finally, note that

$$Q_i\left(z_i\right) = 1 \implies U_i < z_i \leqslant x_i \implies Q_i\left(x_i\right) = 1 \qquad (14.32)$$

and thus

$$\mathbb{E}\left[\left(\sum_{i=1}^{m}\left(Q_i\left(z_i\right) - Q_i\left(x_i\right)\right)\right)^2\right] \leqslant 2. \qquad (14.33)$$

Putting (14.17), (14.31) and (14.33) together,

$$\text{MSE} \leqslant \frac{3}{m^2}\left(4 + \sum_{i=1}^{m}\sum_{\substack{j=1\\ j\neq i}}^{m}\frac{|x_i - x_j|}{2\,(m-1)}\right) \tag{14.34}$$

$$\leqslant \frac{12}{m^2} + \frac{3}{m}\sigma_{md} \tag{14.35}$$

as desired. It is left as an exercise to show that

$$\sum_{i=1}^{m}\sum_{\substack{j=1\\ j\neq i}}^{m}\frac{|x_i - x_j|}{2m\,(m-1)} \leqslant \sigma_{md} = \frac{1}{m}\sum_{i=1}^{m}|x_i - \bar{x}|\,. \tag{14.36}$$

$\square$