EE6367: Topics in Data Storage and Communications **2023**

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6.1 Unbiased Quantization of Real Numbers

The deterministic encoding scheme discussed before is *biased*. We consider a *randomized rounding* scheme where

$$\operatorname{Enc}(x) = \operatorname{Ber}(x) \tag{6.1}$$
$$\operatorname{Dec}(c) = c \tag{6.2}$$

Here, the cost is

$$\text{Cost} = \max_{x \in [0,1]} \mathbb{E}\left[(x-c)^2\right] \tag{6.3}$$
$$= \max_{x \in [0,1]} x(1-x) = \frac{1}{4}. \tag{6.4}$$

We claim that for any unbiased algorithm with no shared randomness, the cost is lower bounded by $\frac{1}{4}$.

6.2 Estimation Schemes With Shared Randomness

Consider an encoder and decoder which share a uniform random variable $U \in [0,1]$. The encoder and decoder are defined as follows.

$$\operatorname{Enc}(x) = c = \begin{cases} 1 & U \leqslant x \\ 0 & U > x \end{cases} \tag{6.5}$$
$$\operatorname{Dec}(c) = \hat{X} = c + U - \frac{1}{2}. \tag{6.6}$$

Clearly, $\mathbb{E}\left[\hat{X}\right] = x$, so the scheme is unbiased. The cost is

$$\mathbb{E}\left[\left(\hat{X} - x\right)^2\right] = \mathbb{E}\left[\left\{(c-x) + \left(U - \frac{1}{2}\right)\right\}^2\right] \tag{6.7}$$

$$= \mathrm{Var}\left(c\right) + \mathrm{Var}\left(U\right) + 2\mathbb{E}\left[\left(c - x\right)\left(U - \frac{1}{2}\right)\right] \tag{6.8}$$

$$= x\left(1 - x\right) + \frac{1}{12} + 2\left[\mathbb{E}\left[cU\right] - x\mathbb{E}\left[U - \frac{1}{2}\right] - \frac{1}{2}\mathbb{E}\left[c\right]\right] \tag{6.9}$$

$$= x\left(1 - x\right) + \frac{1}{12} - x + 2\int_0^1 cuf_U\left(u\right)du \tag{6.10}$$

$$= x\left(1 - x\right) + \frac{1}{12} - x + 2\int_0^x udu = \frac{1}{12} \tag{6.11}$$

and this scheme beats randomized rounding with a cost equal to the variance of $U \sim \mathrm{Unif}\left[0, 1\right]$.

In summary,

Shared Randomness	**Biased**	**Unbiased**
No	$\frac{1}{16}$	$\frac{1}{4}$
Yes	$0.0459\ldots$	$\frac{1}{12}$

Table 6.1: Cost of using various estimation schemes.

6.3 Generalization To More Than One Bit

If we have k bits for quantization, then

1. For *deterministic rounding*, split $[0, 1]$ into 2^k equal sized intervals. The cost is $\left(\frac{1}{2^{k+1}}\right)^2$.
2. For *randomized rounding*, split into $2^k - 1$ equal intervals, so that we have 2^k reconstruction points. If $x \in [l_i, r_i]$, then

$$\mathrm{Enc}\left(x\right) = c^k = \begin{cases} l_i & \mathrm{wp}\frac{x - l_i}{r_i - l_i} \\ r_i & \mathrm{else} \end{cases} \tag{6.12}$$

$$\mathrm{Dec}\left(c^k\right) = \mathrm{Real}\left(c^k\right). \tag{6.13}$$