EE6367: Topics in Data Storage and Communications **2023**

Lecture 13: 9 October 2023

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13.1 Encoding Sparse Vectors

Suppose that $\mathbf{x} \in \mathbb{R}^d$ is sparse and has s nonzero entries. To encode $\mathbf{x}$, we send the locations of the nonzero coordinates using $\log_2 \binom{d}{s}$ bits and separately encode the nonzero values.

The *Elias code* is used to encode a positive integer K. In this scheme of codes, we need $\lceil \log_2 K \rceil$ bits for the integer, and prepend it with $\lceil \log_2 \lceil \log_2 K \rceil \rceil$ bits for the length, and so on. The first set will be a *unary code*, where any positive integer L is encoded as $0^L 1$. The final 1 is to delimit the unary code.

The Elias code is faster than entropy-based codes such as Huffman or Arithmetic codes. Thus, it is preferred for cases like SGD where the time complexity is important.

13.2 Shared Randomness Among Users

Suppose that two users U_1 and U_2 use randomized rounding, where $U_2 = 1 - U_1$. Then, U_1 and U_2 are identically distributed.

Suppose that $x_1 = x_2 = x$. The MSE is then

$$\begin{aligned}
\mathbb{E}\left[\left(\frac{\hat{x_1} + \hat{x_2}}{2} - x\right)^2\right] &= \frac{1}{4}\mathbb{E}\left[\left((\hat{x_1} - x) + (\hat{x_2} - x)\right)^2\right] && (13.1) \\
&= \frac{1}{4}\left(2x\left(1 - x\right) + 2\mathbb{E}\left[\left(\hat{x_1} - x\right)\left(\hat{x_2} - x\right)\right]\right) && (13.2) \\
&= \frac{1}{2}\left(x\left(1 - x\right) + \left[2x - 1\right]_+ - x^2\right) && (13.3)
\end{aligned}$$

where we have

$$\begin{aligned}
\mathbb{E}\left[\hat{x_1}\hat{x_2}\right] &= \mathbb{E}\left[\mathbb{1}_{\{U_1 < x_1\}}\mathbb{1}_{\{U_2 < x_2\}}\right] && (13.4) \\
&= \Pr\left(U_1 < x_1,\ U_2 < x_2\right) && (13.5) \\
&= \Pr\left(1 - x < U_1 < x\right) = \left[2x - 1\right]_+ && (13.6)
\end{aligned}$$

Observe that this MSE is better than that produced by two iid uniform random variables.