EE6367: Topics in Data Storage and Communications **2023**

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10.1 Unbiased Estimation in DRIVE

Claim 10.1. *If $\mathbf{R}$ is a uniform random rotation and $s = \frac{\|\mathbf{x}\|_2}{\|\mathbf{R}\mathbf{x}\|_1}$, then DRIVE is unbiased.*

Proof. We want to show that $\mathbb{E}\left[\hat{\mathbf{X}}\right] = \mathbf{x}\ \forall \mathbf{x} \in \mathbb{R}^d$. Consider

$$\mathbf{x}' = \begin{pmatrix} r & 0 & \dots & 0 \end{pmatrix}^\top \tag{10.1}$$

where $r = \|\mathbf{x}\|_2$. Note that any vector $\mathbf{x}$ can be rotated to $\mathbf{x}'$, call the rotation matrix $\mathbf{R}_{\mathbf{x}\to\mathbf{x}'}$. We see immediately that

$$\mathbf{R}_{\mathbf{x}\to\mathbf{x}'} = \mathbf{R}^{-1}_{\mathbf{x}'\to\mathbf{x}}. \tag{10.2}$$

Now,

$$\begin{aligned}
\hat{\mathbf{X}} &= s\mathbf{R}^\top \operatorname{sign}\left(\mathbf{R}\mathbf{x}\right) && (10.3)\\
&= s\mathbf{R}^{-1}_{\mathbf{x}\to\mathbf{x}'}\mathbf{R}_{\mathbf{x}\to\mathbf{x}'}\mathbf{R}^\top \operatorname{sign}\left(\mathbf{R}\mathbf{R}^{-1}_{\mathbf{x}\to\mathbf{x}'}\mathbf{R}_{\mathbf{x}\to\mathbf{x}'}\mathbf{x}\right) && (10.4)\\
&= s\mathbf{R}^{-1}_{\mathbf{x}\to\mathbf{x}'}\mathbf{R}_{\mathbf{x}}^\top \operatorname{sign}\left(\mathbf{R}_{\mathbf{x}}\mathbf{x}'\right) && (10.5)\\
& && (10.6)
\end{aligned}$$

where we define

$$\mathbf{R}_{\mathbf{x}} \triangleq \mathbf{R}\mathbf{R}^{-1}_{\mathbf{x}\to\mathbf{x}'} = \begin{pmatrix} \mathbf{r}_{\mathbf{x}}^{(1)} & \dots & \mathbf{r}_{\mathbf{x}}^{(d)} \end{pmatrix}. \tag{10.7}$$

By (10.1), we have

$$\mathbf{R}_{\mathbf{x}}\mathbf{x}' = \|\mathbf{x}\|_2\, \mathbf{r}_{\mathbf{x}}^{(1)}. \tag{10.8}$$

Define

$$\mathbf{y} \triangleq \operatorname{sign}\left(\mathbf{R}_{\mathbf{x}}\mathbf{x}'\right). \tag{10.9}$$

Thus,

$$\mathbf{R}_{\mathbf{x}}^\top \mathbf{y} = \begin{pmatrix} \left\langle \mathbf{r}_{\mathbf{x}}^{(1)} \right\rangle & \dots & \left\langle \mathbf{r}_{\mathbf{x}}^{(d)} \right\rangle \end{pmatrix}^\top. \tag{10.10}$$

But

$$\begin{aligned}
s &= \frac{\|\mathbf{x}\|_2^2}{\|\mathbf{R}\mathbf{x}\|_1} && (10.11)\\
&= \frac{\|\mathbf{x}\|_2^2}{\|\mathbf{R}_{\mathbf{x}}\mathbf{x}'\|_1} && (10.12)\\
&= \frac{\|\mathbf{x}\|_2^2}{\left\langle \mathbf{R}_{\mathbf{x}}, \operatorname{sign}\left(\mathbf{R}_{\mathbf{x}}\mathbf{x}'\right)\right\rangle} && (10.13)
\end{aligned}$$

$$= \frac{\|vecx\|_2^2}{\langle \mathbf{R}_{\mathbf{x}}\mathbf{x}', \mathbf{y} \rangle} \tag{10.14}$$

$$= \frac{\|vecx\|_2}{\langle \mathbf{r}_{\mathbf{x}}^{(1)}, \mathbf{y} \rangle} \tag{10.15}$$

Thus,

$$\hat{\mathbf{X}} = \mathbf{R}_{\mathbf{x}\to\mathbf{x}'}^{-1} \|\mathbf{x}\|_2 \begin{pmatrix} \frac{\langle \mathbf{r}_{\mathbf{x}}^{(1)} \rangle}{\langle \mathbf{r}_{\mathbf{x}}^{(1)} \rangle} & \dots & \frac{\langle \mathbf{r}_{\mathbf{x}}^{(d)} \rangle}{\langle \mathbf{r}_{\mathbf{x}}^{(1)} \rangle} \end{pmatrix}^\top . \tag{10.16}$$

If $\mathbf{y} = \mathbf{r}_{\mathbf{x}}^{(1)}$, then we would have

$$\hat{\mathbf{X}} = \mathbf{R}_{\mathbf{x}\to\mathbf{x}'}^{-1} \|\mathbf{x}\|_2 \, \mathbf{e_1} = \mathbf{x}. \tag{10.17}$$

Define

$$\bar{\mathbf{R}_{\mathbf{x}}} \triangleq \mathbf{R}\text{diag}\left(-1, 1, \dots, 1\right) \mathbf{R}_{\mathbf{x}\to\mathbf{x}'}^{-1} \tag{10.18}$$

Thus, if we use $\bar{\mathbf{R}_{\mathbf{x}}}$ instead, we would get

$$\hat{\mathbf{X}}'' = \mathbf{R}_{\mathbf{x}\to\mathbf{x}'}^{-1} \|\mathbf{x}\|_2 \begin{pmatrix} 1 & -\frac{\langle \mathbf{r}_{\mathbf{x}}^{(2)} \rangle}{\langle \mathbf{r}_{\mathbf{x}}^{(1)} \rangle} & \dots - \frac{\langle \mathbf{r}_{\mathbf{x}}^{(d)} \rangle}{\langle \mathbf{r}_{\mathbf{x}}^{(1)} \rangle} \end{pmatrix}^\top . \tag{10.19}$$

We see that

$$\mathbb{E}\left[\hat{\mathbf{X}}''\right] = \mathbb{E}\left[\hat{\mathbf{X}}\right] \tag{10.20}$$

$$\mathbb{E}\left[\hat{\mathbf{X}}'' + \hat{\mathbf{X}}\right] = 2\mathbf{x} \tag{10.21}$$

which implies that

$$\mathbb{E}\left[\hat{\mathbf{X}}\right] = \mathbf{x} \tag{10.22}$$

□

We generalize this scheme.

Claim 10.2. *Suppose we replace* sign$(\mathbf{Rx})$ *with any scalar quantizer* $Q(\mathbf{Rx})$*, and take*

$$s = \frac{\|\mathbf{x}\|_2^2}{\langle \mathbf{Rx}, Q(\mathbf{Rx}) \rangle}. \tag{10.23}$$

Then, DRIVE is unbiased with MSE $= \Theta\left(\|\mathbf{x}\|_2^2\right)$.

However, if structured random rotation is used, then we cannot make the scheme unbiased by choosing s.

Note that if $\mathbf{R}$ is a uniform random rotation or a structured random rotation, then $\mathbf{y}_i = \mathbf{R}\mathbf{x}_i$ is approximately Gaussian distributed for all i identically. That is,

$$\left|F_{Y_i}(y) - F_G(y)\right| \to 0 \ d \to \infty \ \forall y \in \mathbb{R} \tag{10.24}$$