

Lecture 10: 27 September 2023

Instructor: Shashank Vatedka

Scribe: Gautam Singh

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10.1 Unbiased Estimation in DRIVE

Claim 10.1. If \mathbf{R} is a uniform random rotation and $s = \frac{\|\mathbf{x}\|_2}{\|\mathbf{R}\mathbf{x}\|_1}$, then DRIVE is unbiased.

Proof. We want to show that $\mathbb{E}[\hat{\mathbf{X}}] = \mathbf{x} \forall \mathbf{x} \in \mathbb{R}^d$. Consider

$$\mathbf{x}' = (r \ 0 \ \dots \ 0)^\top \quad (10.1)$$

where $r = \|\mathbf{x}\|_2$. Note that any vector \mathbf{x} can be rotated to \mathbf{x}' , call the rotation matrix $\mathbf{R}_{\mathbf{x} \rightarrow \mathbf{x}'}$. We see immediately that

$$\mathbf{R}_{\mathbf{x} \rightarrow \mathbf{x}'} = \mathbf{R}_{\mathbf{x}' \rightarrow \mathbf{x}}^{-1}. \quad (10.2)$$

Now,

$$\hat{\mathbf{X}} = s \mathbf{R}^\top \text{sign}(\mathbf{R}\mathbf{x}) \quad (10.3)$$

$$= s \mathbf{R}_{\mathbf{x} \rightarrow \mathbf{x}'}^{-1} \mathbf{R}_{\mathbf{x} \rightarrow \mathbf{x}'} \mathbf{R}^\top \text{sign}(\mathbf{R} \mathbf{R}_{\mathbf{x} \rightarrow \mathbf{x}'}^{-1} \mathbf{R}_{\mathbf{x} \rightarrow \mathbf{x}'} \mathbf{x}) \quad (10.4)$$

$$= s \mathbf{R}_{\mathbf{x} \rightarrow \mathbf{x}'}^{-1} \mathbf{R}_{\mathbf{x}}^\top \text{sign}(\mathbf{R}_{\mathbf{x}} \mathbf{x}') \quad (10.5)$$

$$(10.6)$$

where we define

$$\mathbf{R}_{\mathbf{x}} \triangleq \mathbf{R} \mathbf{R}_{\mathbf{x} \rightarrow \mathbf{x}'}^{-1} = \begin{pmatrix} \mathbf{r}_{\mathbf{x}}^{(1)} & \dots & \mathbf{r}_{\mathbf{x}}^{(d)} \end{pmatrix}. \quad (10.7)$$

By (10.1), we have

$$\mathbf{R}_{\mathbf{x}} \mathbf{x}' = \|\mathbf{x}\|_2 \mathbf{r}_{\mathbf{x}}^{(1)}. \quad (10.8)$$

Define

$$\mathbf{y} \triangleq \text{sign}(\mathbf{R}_{\mathbf{x}} \mathbf{x}'). \quad (10.9)$$

Thus,

$$\mathbf{R}_{\mathbf{x}}^\top \mathbf{y} = \left(\langle \mathbf{r}_{\mathbf{x}}^{(1)} \rangle \ \dots \ \langle \mathbf{r}_{\mathbf{x}}^{(d)} \rangle \right)^\top. \quad (10.10)$$

But

$$s = \frac{\|\mathbf{x}\|_2^2}{\|\mathbf{R}\mathbf{x}\|_1} \quad (10.11)$$

$$= \frac{\|\mathbf{x}\|_2^2}{\|\mathbf{R}_{\mathbf{x}} \mathbf{x}'\|_1} \quad (10.12)$$

$$= \frac{\|\mathbf{x}\|_2^2}{\langle \mathbf{R}_{\mathbf{x}}, \text{sign}(\mathbf{R}_{\mathbf{x}} \mathbf{x}') \rangle} \quad (10.13)$$

$$= \frac{\|vecx\|_2^2}{\langle \mathbf{R}_x \mathbf{x}', \mathbf{y} \rangle} \quad (10.14)$$

$$= \frac{\|vecx\|_2^2}{\langle \mathbf{r}_x^{(1)}, \mathbf{y} \rangle} \quad (10.15)$$

Thus,

$$\hat{\mathbf{X}} = \mathbf{R}_{x \rightarrow x'}^{-1} \|\mathbf{x}\|_2 \left(\frac{\langle \mathbf{r}_x^{(1)} \rangle}{\langle \mathbf{r}_x^{(1)} \rangle} \quad \dots \quad \frac{\langle \mathbf{r}_x^{(d)} \rangle}{\langle \mathbf{r}_x^{(1)} \rangle} \right)^\top. \quad (10.16)$$

If $\mathbf{y} = \mathbf{r}_x^{(1)}$, then we would have

$$\hat{\mathbf{X}} = \mathbf{R}_{x \rightarrow x'}^{-1} \|\mathbf{x}\|_2 \mathbf{e}_1 = \mathbf{x}. \quad (10.17)$$

Define

$$\bar{\mathbf{R}}_x \triangleq \mathbf{R} \text{diag}(-1, 1, \dots, 1) \mathbf{R}_{x \rightarrow x'}^{-1} \quad (10.18)$$

Thus, if we use $\bar{\mathbf{R}}_x$ instead, we would get

$$\hat{\mathbf{X}}'' = \mathbf{R}_{x \rightarrow x'}^{-1} \|\mathbf{x}\|_2 \left(1 \quad -\frac{\langle \mathbf{r}_x^{(2)} \rangle}{\langle \mathbf{r}_x^{(1)} \rangle} \quad \dots \quad -\frac{\langle \mathbf{r}_x^{(d)} \rangle}{\langle \mathbf{r}_x^{(1)} \rangle} \right)^\top. \quad (10.19)$$

We see that

$$\mathbb{E} [\hat{\mathbf{X}}''] = \mathbb{E} [\hat{\mathbf{X}}] \quad (10.20)$$

$$\mathbb{E} [\hat{\mathbf{X}}'' + \hat{\mathbf{X}}] = 2\mathbf{x} \quad (10.21)$$

which implies that

$$\mathbb{E} [\hat{\mathbf{X}}] = \mathbf{x} \quad (10.22)$$

as required. \square

We generalize this scheme.

Claim 10.2. Suppose we replace $\text{sign}(\mathbf{R}\mathbf{x})$ with any scalar quantizer $Q(\mathbf{R}\mathbf{x})$, and take

$$s = \frac{\|\mathbf{x}\|_2^2}{\langle \mathbf{R}\mathbf{x}, Q(\mathbf{R}\mathbf{x}) \rangle}. \quad (10.23)$$

Then, DRIVE is unbiased with $MSE = \Theta(\|\mathbf{x}\|_2^2)$.

However, if *structured* random rotation is used, then we cannot make the scheme unbiased by choosing s .

Note that if \mathbf{R} is a uniform random rotation or a structured random rotation, then $\mathbf{y}_i = \mathbf{R}\mathbf{x}_i$ is approximately Gaussian distributed for all i identically. That is,

$$|F_{Y_i}(y) - F_G(y)| \rightarrow 0 \text{ as } d \rightarrow \infty \quad \forall y \in \mathbb{R}. \quad (10.24)$$