EE6367: Topics in Data Storage and Communications **2023**

Homework 1: September 2023

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Disclaimer: *Any results proved in class may be used without proof. All other arguments must be justified. Provide clear details for all your steps.*

For the programming questions, please submit py or ipynb files. Your code must be commented properly.

You are allowed to collaborate with your friends, but you must declare who you collaborated with. Solutions must be written independently though. Copying is not allowed.

Suppose that X^n is drawn according to $f_{X^n|\theta}$ where $\theta \in \Theta$ is the parameter of interest. Then, the maximum likelihood (ML) estimator for θ is defined

$$\hat{\theta}_{ML}(x^n) = \arg\max_{\theta\in\Theta} f_{X^n|\theta}(x^n) = \arg\max_{\theta\in\Theta} \log f_{X^n|\theta}(x^n)$$

Here, $\log f_{X^n|\theta}(x^n)$ is called the log-likelihood function and $f_{X^n|\theta}(x^n)$ is called the likelihood function.

Exercise 1.1. Suppose that X is a single sample drawn from the Gaussian parametric family $\mathcal{N}(0, \sigma^2)$, with unknown variance $\sigma^2 \in \mathbb{R}$.

1. Find an unbiased estimator for σ^2 (2 points)
2. Find the maximum likelihood estimator for σ (2 points)
3. Is the log-likelihood function convex? (2 points)

Exercise 1.2. Suppose that $x_1, \ldots, x_n$ are known real numbers, $Y_1, Y_2, \ldots, Y_n$ are random variables satisfying

$$Y_i = \beta x_i + Z_i$$

where $Z_1, \ldots, Z_n$ are iid random variables that are Gaussian with mean zero and variance 1. The parameter β is unknown and needs to be estimated given Y^n and x^n.

1. Find the maximum likelihood estimator of β from Y^n. Is this unbiased? (4 points)
2. Note that the ML estimator of β is actually a random variable. Find the distribution of the ML estimator. Also find the MSE of the estimator. (4 points)
3. Find an unbiased estimator for β (different from the ML estimator, if it is unbiased). Also find the MSE for your estimator. (4 points)

Exercise 1.3. Let $X_1, \ldots, X_n$ be iid Bernoulli random variables with $p_{X_i}(1) = p$, where $p \in [0, 1]$ is the unknown parameter to be estimated. Let $\hat{X}$ denote the fraction of 1's in X^n.

1. Is this an unbiased estimator? (1 point)
2. Prove that this is the best (minimum MSE) unbiased estimator for p. (4 points)

Exercise 1.4 (10 points)**.** Implement general k-bit deterministic rounding and randomized rounding in python. The template files for evaluating your code is provided (see `hw1_dr_rr_test.py`). You are only allowed to modify the functions in the files `custom_quantizers.py`.

Exercise 1.5 (10 points)**.** Numerically design a 4-bit quantizer for Gaussian $\mathcal{N}(0, 1)$ random variables using the Lloyd-Max algorithm. Compute the empirical MSE.