EE6367: Topics in Data Storage and Communications **2023**

Lecture 14: 11 October 2023

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Note: *The discussion started in Lecture 14 continues on in Lecture 15. This document contains the content of both lectures clubbed together for the sake of continuity.*

14.1 Correlated Quantization

We consider stochastic rounding in the case of n users. Let a random permutation π of $\{0, 1, \ldots, n-1\}$ be shared among the users. User i is given the number π_i (here the users are 0-indexed). Then, each user generates $\gamma_i \sim \text{Unif}\left[0, \frac{1}{m}\right]$. Note that γ_i is *private randomness.* Define

$$U_i \triangleq \frac{\pi_i}{m} + \gamma_i. \tag{14.1}$$

The CDF of U_i, for $u \in [0, 1]$, is

$$\begin{aligned}
\Pr\left(U_i \leqslant u\right) &= \Pr\left(\frac{\pi_i}{m} + \gamma_i \leqslant u\right) && (14.2)\\
&= \Pr\left(\pi_i + m\gamma_i \leqslant mu\right) && (14.3)\\
&= \Pr\left(\pi_i < \lfloor mu \rfloor\right) + \Pr\left(\pi_i = \lfloor mu \rfloor,\ m\gamma_i \leqslant \mu - \lfloor mu \rfloor\right) && (14.4)\\
&= \frac{\lfloor mu \rfloor}{m} + \frac{mu - \lfloor mu \rfloor}{m} = u && (14.5)
\end{aligned}$$

and hence $U_i \sim \text{Unif}\left[0, 1\right]$.

Now, each user i transmits

$$Y_i \triangleq \mathbb{1}_{\{U_i < x_i\}}. \tag{14.6}$$

Clearly, this scheme is unbiased because $\mathbb{E}\left[Y_i\right] = x_i$. We now present a claim for the MSE of this scheme.

Claim 14.1. *For the above scheme, the MSE is upper bounded by $\frac{3}{m}\sigma_{md} + \frac{12}{m^2}$, where*

$$\sigma_{md} \triangleq \frac{1}{m}\sum_{i=1}^{m}\left|x_i - \bar{x}\right| \leqslant \sqrt{\frac{1}{m}\sum_{i=1}^{m}\left(x_i - \bar{x}\right)^2}. \tag{14.7}$$

Proof. Define

$$z_i \triangleq \frac{\lfloor m x_i \rfloor}{m} \tag{14.8}$$

$$Y_i \triangleq Q_i(x_i) \tag{14.9}$$

$$Q_i(t) \triangleq \mathbb{1}_{\{U_i < t\}} \tag{14.10}$$

The MSE is given by

$$\begin{aligned}
\mathrm{MSE} &= \mathbb{E}\left[\left(\frac{1}{m}\sum_{i=1}^{m} Y_i - \frac{1}{m}\sum_{i=1}^{m} x_i\right)^2\right] && (14.11)\\
&= \frac{1}{m^2}\mathbb{E}\left[\left(\sum_{i=1}^{m} (Y_i - x_i)\right)^2\right] && (14.12)\\
&= \frac{1}{m^2}\mathbb{E}\left[\left(\sum_{i=1}^{m} ((x_i - z_i) + (z_i - Q_i(z_i)) + (Q_i(z_i) - Q_i(x_i)))\right)^2\right] && (14.13)\\
&\leqslant \frac{3}{m^2}\left\{\mathbb{E}\left[\left(\sum_{i=1}^{m}(x_i - z_i)\right)^2\right] + \mathbb{E}\left[\left(\sum_{i=1}^{m}(z_i - Q_i(z_i))\right)^2\right] + \mathbb{E}\left[\left(\sum_{i=1}^{m}(Q_i(z_i) - Q_i(x_i))\right)^2\right]\right\} && (14.14)
\end{aligned}$$

where we use in (14.13) the inequality

$$(\alpha + \beta + \gamma)^2 \leqslant 3\left(\alpha^2 + \beta^2 + \gamma^2\right). \tag{14.15}$$

for reals α, β, γ.

Note that

$$x_i - z_i = x_i - \frac{\lfloor m x_i \rfloor}{m} = \frac{m x_i - \lfloor m x_i \rfloor}{m} \leqslant \frac{1}{m}, \tag{14.16}$$

hence

$$\mathbb{E}\left[\left(\sum_{i=1}^{m}(x_i - z_i)\right)^2\right] \leqslant 1. \tag{14.17}$$

Now,

$$\begin{aligned}
\mathbb{E}\left[\left(\sum_{i=1}^{m} z_i - Q_i(z_i)\right)^2\right] &= \sum_{i=1}^{m}\mathbb{E}\left[(z_i - Q_i(z_i))^2\right] + \sum_{i=1}^{m}\sum_{\substack{j=1\\ j\neq i}}^{m}\mathbb{E}\left[(z_i - Q_i(z_i))(z_j - Q_j(z_j))\right] && (14.18)\\
&= \sum_{i=1}^{m}\mathbb{E}\left[(z_i - Q_i(z_i))^2\right] + \sum_{i=1}^{m}\sum_{\substack{j=1\\ j\neq m}}^{m}\left(\mathbb{E}\left[Q_i(z_i) Q_j(z_j)\right] - z_i z_j\right) && (14.19)
\end{aligned}$$

Note that

$$Q_i(z_i) = \mathbb{1}_{\{\pi_i + m\gamma_i < mz_i\}} \tag{14.20}$$
$$= \mathbb{1}_{\{\pi_i < mz_i\}} \tag{14.21}$$

and

$$\mathbb{E}[Q_i(z_i) Q_j(z_j)] = \Pr(\pi_i < mz_i,\ \pi_j < mz_j) \tag{14.22}$$
$$= \min\{z_i, z_j\} \frac{m \max\{z_i, z_j\} - 1}{m-1} \tag{14.23}$$
$$= \frac{mz_i z_j - \min\{z_i, z_j\}}{m-1} \tag{14.24}$$
$$= \frac{mz_i z_j}{m-1} - \frac{1}{m-1}\left(\frac{z_i + z_j}{2} - \frac{|z_i - z_j|}{2}\right). \tag{14.25}$$

Using (14.25) in (14.19),

$$\mathbb{E}\left[\left(\sum_{i=1}^{m} z_i - Q_i(z_i)\right)^2\right] = \sum_{i=1}^{m} z_i(1-z_i) + \sum_{i=1}\sum_{\substack{j=1\\ j\neq i}}^{m}\left(\frac{z_i z_j}{m-1} - \frac{z_i + z_j}{2(m-1)} + \frac{|z_i - z_j|}{2(m-1)}\right) \tag{14.26}$$
$$= \sum_{i=1}^{m} z_i(1-z_i) + \frac{\left(\sum_i z_i\right)^2 - \sum_i z_i^2}{m-1} - \sum_i z_i + \sum_{i=1}^{m}\sum_{\substack{j=1\\ j\neq i}}^{m} \frac{|z_i - z_j|}{2(m-1)} \tag{14.27}$$
$$\leqslant \frac{\left(\sum_i z_i\right)^2}{m} - \sum_i z_i^2 + \sum_{i=1}^{m}\sum_{\substack{j=1\\ j\neq i}}^{m} \frac{|z_i - z_j|}{2(m-1)} \tag{14.28}$$
$$\leqslant \sum_{\substack{j=1\\ j\neq i}}^{m} \frac{|z_i - z_j|}{2(m-1)} \tag{14.29}$$
$$\leqslant \sum_{\substack{j=1\\ j\neq i}}^{m} \frac{|x_i - x_j| + \frac{1}{m}}{2(m-1)} \tag{14.30}$$
$$\leqslant \sum_{i=1}^{m}\sum_{\substack{j=1\\ j\neq i}}^{m} \frac{|x_i - x_j|}{2(m-1)} + \frac{1}{2}. \tag{14.31}$$

Finally, note that

$$Q_i(z_i) = 1 \implies U_i < z_i \leqslant x_i \implies Q_i(x_i) = 1 \tag{14.32}$$

and thus

$$\mathbb{E}\left[\left(\sum_{i=1}^{m} (Q_i(z_i) - Q_i(x_i))\right)^2\right] \leqslant 2. \tag{14.33}$$

Putting (14.17), (14.31) and (14.33) together,

$$\mathrm{MSE} \leqslant \frac{3}{m^2}\left(4 + \sum_{i=1}^{m}\sum_{\substack{j=1\\ j\neq i}}^{m} \frac{|x_i - x_j|}{2\,(m-1)}\right) \tag{14.34}$$

$$\leqslant \frac{12}{m^2} + \frac{3}{m}\sigma_{md} \tag{14.35}$$

as desired. It is left as an exercise to show that

$$\sum_{i=1}^{m}\sum_{\substack{j=1\\ j\neq i}}^{m} \frac{|x_i - x_j|}{2m\,(m-1)} \leqslant \sigma_{md} = \frac{1}{m}\sum_{i=1}^{m} |x_i - \bar{x}|\,. \tag{14.36}$$

□