EE6367: Topics in Data Storage and Communications	**2023**

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13.1 Stochastic Rounding

We consider stochastic rounding in the case of n users. Let a random permutation π of $\{0, 1, \ldots, n-1\}$ be shared among the users. User i is given the number π_i (here the users are 0-indexed). Then, each user generates $\gamma_i \sim \text{Unif}\left[0, \frac{1}{m}\right]$. Note that γ_i is *private randomness*. Define

$$U_i \triangleq \frac{\pi_i}{m} + \gamma_i. \tag{13.1}$$

The CDF of U_i, for $u \in [0,1]$, is

$$\begin{aligned}
\Pr\left(U_i \leqslant u\right) &= \Pr\left(\frac{\pi_i}{m} + \gamma_i \leqslant u\right) && (13.2)\\
&= \Pr\left(\pi_i + m\gamma_i \leqslant mu\right) && (13.3)\\
&= \Pr\left(\pi_i < \lfloor mu \rfloor\right) + \Pr\left(\pi_i = \lfloor mu \rfloor,\ m\gamma_i \leqslant \mu - \lfloor mu \rfloor\right) && (13.4)\\
&= \frac{\lfloor mu \rfloor}{m} + \frac{mu - \lfloor mu \rfloor}{m} = u && (13.5)
\end{aligned}$$

and hence $U_i \sim \text{Unif}\left[0,1\right]$.

Now, each user i transmits

$$Y_i \triangleq \mathbb{1}_{\{U_i \leqslant x_i\}}. \tag{13.6}$$

Clearly, this scheme is unbiased because $\mathbb{E}\left[Y_i\right] = x_i$. We now present a claim for the MSE of this scheme.

Claim 13.1. *For the above scheme, the MSE is upper bounded by $\frac{3}{m}\sigma_{md} + \frac{12}{m^2}$, where*

$$\sigma_{md} \triangleq \frac{1}{m}\sum_{i=1}^{m} \left|x_i - \bar{x}\right| \leqslant \sqrt{\frac{1}{m}\sum_{i=1}^{m} \left(x_i - \bar{x}\right)^2}. \tag{13.7}$$

Proof. Define

$$z_i \triangleq \frac{\lfloor mx_i \rfloor}{m} \tag{13.8}$$

$$Y_i \triangleq Q_i\left(x_i\right) \tag{13.9}$$
$$Q_i\left(t\right) \triangleq \mathbb{1}_{\{U_i<t\}} \tag{13.10}$$

The MSE is given by

$$\begin{aligned}
\mathrm{MSE} &= \mathbb{E}\left[\left(\frac{1}{m}\sum_{i=1}^{m} Y_i - \frac{1}{m}\sum_{i=1}^{m} x_i\right)^2\right] &(13.11)\\
&= \frac{1}{m^2}\mathbb{E}\left[\left(\sum_{i=1}^{m}\left(Y_i - x_i\right)\right)^2\right] &(13.12)\\
&= \frac{1}{m^2}\mathbb{E}\left[\left(\sum_{i=1}^{m}\left(\left(x_i - z_i\right) + \left(z_i - Q_i\left(z_i\right)\right) + \left(Q_i\left(z_i\right) - Q_i\left(x_i\right)\right)\right)\right)^2\right] &(13.13)\\
&\leqslant \frac{3}{m^2}\left\{\mathbb{E}\left[\left(\sum_{i=1}^{m}\left(x_i - z_i\right)\right)^2\right] + \mathbb{E}\left[\left(\sum_{i=1}^{m}\left(z_i - Q_i\left(z_i\right)\right)\right)^2\right] + \mathbb{E}\left[\left(\sum_{i=1}^{m}\left(Q_i\left(z_i\right) - Q_i\left(x_i\right)\right)\right)^2\right]\right\} &(13.14)\\
& &(13.15)
\end{aligned}$$

where we use in (13.13) the inequality

$$\left(\alpha + \beta + \gamma\right)^2 \leqslant 3\left(\alpha^2 + \beta^2 + \gamma^2\right). \tag{13.16}$$

for reals α, β, γ.

Note that

$$x_i - z_i = x_i - \frac{\lfloor m x_i \rfloor}{m} = \frac{m x_i - \lfloor m x_i \rfloor}{m} \leqslant \frac{1}{m}, \tag{13.17}$$

hence

$$\mathbb{E}\left[\left(\sum_{i=1}^{m}\left(x_i - z_i\right)\right)^2\right] \leqslant 1. \tag{13.18}$$

Now,

$$\mathbb{E}\left[\left(\sum_{i=1}^{m} z_i - Q_i\left(z_i\right)\right)^2\right] = \sum_{i=1}^{m}\mathbb{E}\left[\left(z_i - Q_i\left(z_i\right)\right)^2\right] \tag{13.19}$$

$\square$