EE6367: Topics in Data Storage and Communications **2023**

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4.1 Lossy Compresssion, Quantization

We require to communicate $X^d \in \mathbb{R}^d$. To do so, it is encoded as $c^k \in \{0,1\}^k$. Suppose the decoded quantity is $\hat{X}^d \in \mathbb{R}^d$. Here, we have $k = dR$, that is, k is linearly dependent on d. The *average distortion* is given by

$$L = \sum_{i=1}^{k} \mathbb{E}\left[l\left(X_i, \hat{X}_i\right)\right] \tag{4.1}$$

and the *mean-squared-error* is given by

$$\text{MSE}\,(=) \sum_{i=1}^{n} \mathbb{E}\left[\left(X_i - \hat{X}_i\right)^2\right]. \tag{4.2}$$

4.1.1 Quantization of Real Numbers

Consider the case when $d = 1$. Let $X \sim f_X$. A possible encoding function could be

$$c = \text{Enc}\,(X) = \begin{cases} 1 & X \geqslant \mathbb{E}\,[X] \\ 0 & X < \mathbb{E}\,[X] \end{cases}. \tag{4.3}$$

The best decoding function is

$$\text{Dec}\,(X) = \begin{cases} \mathbb{E}\,[X | X \geqslant \mathbb{E}\,[X]] & c = 1 \\ \mathbb{E}\,[X | X < \mathbb{E}\,[X]] & c = 0 \end{cases}. \tag{4.4}$$

Theorem 4.1. *For any $\alpha \in \mathbb{R}$ and random variable X,*

$$\mathbb{E}\left[(X - \alpha)^2\right] \geqslant \text{Var}\,(X) \tag{4.5}$$

with equality iff $\alpha = \mathbb{E}\,[X]$.

Proof. We have,

$$\mathbb{E}\left[(X - \alpha)^2\right] = \mathbb{E}\left[\left[(X - \mathbb{E}\,[X]) + (\mathbb{E}\,[X] - \alpha)\right]^2\right] \tag{4.6}$$

$$= \mathrm{Var}(X) + (\mathbb{E}[X] - \alpha)^2 \geqslant \mathrm{Var}(X) \tag{4.7}$$

with equality iff $\alpha = \mathbb{E}[X]$. □

Using Theorem 4.1, we see that (4.4) is indeed the minimum mean-squared error (MMSE) estimator of X. This gives us the following.

Theorem 4.2. *Given intervals $I_i, 1 \leqslant i \leqslant k$, the optimal decoder is*

$$\hat{X} = \hat{x}_i = \mathbb{E}[X | X \in I_i]. \tag{4.8}$$

*The points $\hat{x}_i$ are called **reconstruction points**.*

4.1.2 Optimal Encoder Given Reconstruction Points

Suppose the reconstruction points $\{\hat{x}_i\}_{i=1}^{2^k}$ are given. We require to design an encoder that minimizes the MSE.

Theorem 4.3. *Given the reconstruction points $\{\hat{x}_i\}$, the encoder*

$$g^* \triangleq \arg\min_i (x - \hat{x}_i)^2 \tag{4.9}$$

is optimal in the MSE sense.

Proof. Consider any encoder g. Then, by the definition (4.9),

$$\mathrm{MSE}(g) - \mathrm{MSE}(g^*) = \int_{-\infty}^{\infty} f_X(x) \left[(x - g(x))^2 - (x - g^*(x))^2 \right] \geqslant 0 \tag{4.10}$$

which implies $\mathrm{MSE}(g) \geqslant \mathrm{MSE}(g^*)$. Additionally, the intervals are

$$I_j \triangleq \left(\frac{\hat{x}_{j-1} + \hat{x}_j}{2}, \frac{\hat{x}_j + \hat{x}_{j+1}}{2} \right) \tag{4.11}$$

where we define $\hat{x}_{-1} = -\infty$ and $\hat{x}_{2^k+1} = \infty$. □

4.2 Lloyd-Max Algorithm

Using Theorem 4.2 and Theorem 4.3, we obtain the Lloyd-Max Algorithm as follows.

1. Choose $I_i, 1 \leqslant i \leqslant 2^k$.
2. Choose the best $\hat{x}_i$ for the I_i.
3. Choose the best I_i for the $\hat{x}_i$.
4. Repeat until the decrease in MSE is small enough.