EE6367: Topics in Data Storage and Communications **2023**

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9.1 Performance of DRIVE

We have seen that

$$\mathbf{Rx} \sim \text{Unif}\left(\mathbb{S}\left(\mathbf{0}, \|\mathbf{x}\|_2^2\right)\right). \tag{9.1}$$

Consider

$$\mathbf{U} \triangleq \frac{\mathbf{Z}}{\|\mathbf{Z}\|_2} \tag{9.2}$$

where $\mathbf{Z} \sim \mathcal{N}(\mathbf{0}, 1)$. Then,

$$\mathbf{U} \sim \text{Unif}\left(\mathbb{S}\left(\mathbf{0}, 1\right)\right). \tag{9.3}$$

Notice that

$$f_{\mathbf{U}|\|\mathbf{Z}\|_2 = r}(\mathbf{u}) = f_{\mathbf{U}}(u) = \begin{cases} \frac{1}{\text{Ar}(\mathbb{S}(\mathbf{0},1))} & \|u\| = 1 \\ 0 & \text{else} \end{cases} \tag{9.4}$$

so that $\mathbf{U}$ and $\|\mathbf{Z}\|_2$ are statistically independent. Thus, from (9.2),

$$\begin{aligned} \mathbf{U}\|\mathbf{Z}\|_2 &= \mathbf{Z} & (9.5)\\ \|\mathbf{U}\|_1\|\mathbf{Z}\|_2 &= \|\mathbf{Z}\|_1 & (9.6)\\ \|\mathbf{U}\|_1 &= \frac{\|\mathbf{Z}\|_2}{\|\mathbf{Z}\|_1}. & (9.7) \end{aligned}$$

Clearly, the 2-norm of $\mathbf{Z}$ and 1-norm of $\mathbf{U}$ are statistically independent.

Note that

$$\mathbb{E}\left[\|\mathbf{Z}\|_1^2\right] = \mathbb{E}\left[\left(\sum_{i=1}^{d} |Z_i|\right)^2\right] \tag{9.8}$$

$$= \mathbb{E}\left[\sum_{i=1}^{d}\sum_{j=1}^{d}\left|Z_i\right|\left|Z_j\right|\right] \tag{9.9}$$

$$= \mathbb{E}\left[\sum_{i=1}^{d}\left|Z_i\right|^2 + \sum_{i=1}^{d}\sum_{j=1,\ j\neq i}^{d}\left|Z_i\right|\left|Z_j\right|\right] \tag{9.10}$$

$$= d + \sum_{i=1}^{d}\sum_{j=1,\ j\neq i}\mathbb{E}\left[\left|Z_i\right|\left|Z_j\right|\right] \tag{9.11}$$

$$= d + d\left(d-1\right)\frac{2}{\pi} \tag{9.12}$$

since

$$\mathbb{E}\left[\left|Z_i\right|\right] = \int_{-\infty}^{\infty}\left|z_i\right|\frac{e^{-\frac{z_i^2}{2}}}{\sqrt{2\pi}}dz_i \tag{9.13}$$

$$= 2\int_{0}^{\infty}z_i\frac{e^{-\frac{z_i^2}{2}}}{\sqrt{2\pi}}dz_i \tag{9.14}$$

$$= \sqrt{\frac{2}{\pi}}\int_{0}^{\infty}e^{-y}dy = \sqrt{\frac{2}{\pi}} \tag{9.15}$$

where we make the change of variables $y \triangleq \frac{z_i^2}{2}$. Using (9.12),

$$\mathbb{E}\left[\left\|\mathbf{U}\right\|_1^2\right] = \frac{\mathbb{E}\left[\left\|\mathbf{Z}\right\|_1^2\right]}{\left\|\mathbf{Z}\right\|_2^2} \tag{9.16}$$

$$= \frac{d + d\left(d-1\right)\frac{2}{pi}}{d} = 1 + \left(d-1\right)\frac{2}{\pi}. \tag{9.17}$$

Using (9.17), for any norm $\left\|\mathbf{x}\right\|_2^2$ and taking $d \to \infty$, we obtain

$$\text{MSE} = \left(1 - \frac{2}{\pi}\right)\left\|\mathbf{x}\right\|_2^2. \tag{9.18}$$

9.2 Generating a Uniform Rotation Matrix

We can generate

$$\mathbf{A}_{d\times d} \sim \mathcal{N}\left(0,1\right) \tag{9.19}$$

and perform Gram-Schmidt orthogonalization of take a QR-decomposition to obtain the orthonormal matrix Q.

Lemma 9.1. *If* $\mathbf{A}$ *is a randomly generated* $d \times d$ *matrix with all entries drawn independently from the standard normal distribution, then* $\mathbf{A} = \mathbf{QR}$ *where* $\mathbf{Q}$ *is a uniform rotation matrix.*

9.3 Structured Random Rotation Matrices

It is costly to share $\mathcal{O}\left(d^2\right)$ bits. We present an alternate choice of $\mathbf{R}$ that does incur a higher MSE but shares less randomness. We define

$$\mathbf{R} \triangleq \frac{1}{\sqrt{d}}\mathbf{H}_l\mathbf{D} \tag{9.20}$$

where we assume that $d = 2^l$ for some nonnegative integer l and $\mathbf{H}_l$ is the d-dimensional *Walsh-Hadamard* matrix, which is recursively defined as

$$\mathbf{H}_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{9.21}$$

$$\mathbf{H}_l = \begin{pmatrix} \mathbf{H}_{l-1} & \mathbf{H}_{l-1} \\ \mathbf{H}_{l-1} & -\mathbf{H}_{l-1} \end{pmatrix} \tag{9.22}$$

and $\mathbf{D}$ is a diagonal matrix with iid Rademacher $\{1, -1\}$ entries.

Using this choice of $\mathbf{R}$, the overall complexity reduces to $\mathcal{O}\left(d \log d\right)$ and the MSE is still $\Theta\left(\|\mathbf{x}\|_2^2\right)$.