EE6367: Topics in Data Storage and Communications **2023**

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4.1 Lossy Compresssion, Quantization

We require to communicate $X^d \in \mathbb{R}^d$. To do so, it is encoded as $c^k \in \{0,1\}^k$. Suppose the decoded quantity is $\hat{X}^d \in \mathbb{R}^d$. Here, we have $k = dR$, that is, k is linearly dependent on d. The *average distortion* is given by

$$L = \sum_{i=1}^{k} \mathbb{E}\left[l\left(X_i, \hat{X}_i\right)\right] \tag{4.1}$$

and the *mean-squared-error* is given by

$$\text{MSE}\,(=) \sum_{i=1}^{n} \mathbb{E}\left[\left(X_i - \hat{X}_i\right)^2\right]. \tag{4.2}$$

4.1.1 Scalar Quantization

Consider the case when $d = 1$. Let $X \sim f_X$. A possible encoding function could be

$$c = \text{Enc}\,(X) = \begin{cases} 1 & X \geqslant \mathbb{E}\,[X] \\ 0 & X < \mathbb{E}\,[X] \end{cases}. \tag{4.3}$$

The best decoding function is

$$\text{Dec}\,(X) = \begin{cases} \mathbb{E}\,[X|X \geqslant \mathbb{E}\,[X]] & c = 1 \\ \mathbb{E}\,[X|X < \mathbb{E}\,[X]] & c = 0 \end{cases}. \tag{4.4}$$

Theorem 4.1. *For any $\alpha \in \mathbb{R}$ and random variable X,*

$$\mathbb{E}\left[(X-\alpha)^2\right] \geqslant \text{Var}\,(X) \tag{4.5}$$

with equality iff $\alpha = \mathbb{E}\,[X]$.

Proof. We have,

$$\mathbb{E}\left[(X-\alpha)^2\right] = \mathbb{E}\left[\left[(X - \mathbb{E}\,[X]) + (\mathbb{E}\,[X] - \alpha)\right]^2\right] \tag{4.6}$$

$$= \mathrm{Var}(X) + (\mathbb{E}[X] - \alpha)^2 \geqslant \mathrm{Var}(X) \tag{4.7}$$

with equality iff $\alpha = \mathbb{E}[X]$. $\square$

Using Theorem 4.1, we see that (4.4) is indeed the minimum mean-squared error (MMSE) estimator of X. This gives us the following.

Theorem 4.2. *Given intervals $I_i, 1 \leqslant i \leqslant k$, the optimal decoder is*

$$\hat{X} = \hat{x}_i = \mathbb{E}[X | X \in I_i]. \tag{4.8}$$

The points $\hat{x}_i$ are called ***reconstruction points****.*

4.1.2 Optimal Encoder Given Reconstruction Points

Suppose the reconstruction points $\{\hat{x}_i\}_{i=1}^{2^k}$ are given. We require to design an encoder that minimizes the MSE.

Theorem 4.3. *Given the reconstruction points $\{\hat{x}_i\}$, the encoder*

$$g^* \triangleq \arg\min_i (x - \hat{x}_i)^2 \tag{4.9}$$

is optimal in the MSE sense.

Proof. Consider any encoder g. Then, by the definition (4.9),

$$\mathrm{MSE}(g) - \mathrm{MSE}(g^*) = \int_{-\infty}^{\infty} f_X(x) \left[(x - g(x))^2 - (x - g^*(x))^2 \right] \geqslant 0 \tag{4.10}$$

which implies $\mathrm{MSE}(g) \geqslant \mathrm{MSE}(g^*)$. Additionally, the intervals are

$$I_j \triangleq \left(\frac{\hat{x}_{j-1} + \hat{x}_j}{2}, \ \frac{\hat{x}_j + \hat{x}_{j+1}}{2} \right) \tag{4.11}$$

where we define $\hat{x}_{-1} = -\infty$ and $\hat{x}_{2^k+1} = \infty$. $\square$

4.2 Lloyd-Max Algorithm

Using Theorem 4.2 and Theorem 4.3, we obtain the Lloyd-Max Algorithm as follows.

1. Choose $I_i, 1 \leqslant i \leqslant 2^k$.
2. Choose the best $\hat{x}_i$ for the I_i.
3. Choose the best I_i for the $\hat{x}_i$.
4. Repeat until the decrease in MSE is small enough.

Note that at every iteration, the mean squared error does decrease. However, it may not converge to the global MMSE and it may converge to a local minima.