

## Lecture 5: 14 September 2023

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## 5.1 Deterministic Single-Bit Scalar Quantization

Suppose we have to encode  $x \in [0, 1]$  using a single bit  $c \in \{0, 1\}$ . Let the decoded output be  $\hat{x}$ . We require to minimize the *distortion* which is taken to be the maximum squared error

$$\text{Max} - \text{SE} = \max_{x \in [0, 1]} (x - \hat{x})^2. \quad (5.1)$$

One such set of encoding and decoding functions is

$$\text{Enc}(x) = \begin{cases} 0 & x \leq \frac{1}{2} \\ 1 & x > \frac{1}{2} \end{cases} \quad (5.2)$$

$$\text{Dec}(c) = \begin{cases} \frac{1}{4} & c = 0 \\ \frac{3}{4} & c = 1 \end{cases}. \quad (5.3)$$

This is called *deterministic rounding* and has a maximum squared error of  $\frac{1}{16}$  as the absolute maximum error is  $\frac{1}{4}$ .

## 5.2 Random Single-Bit Scalar Quantization

**Claim 5.1.** *The maximum MSE of a random quantizer under no shared randomness is at least  $\frac{1}{16}$ .*

*Proof.* In this case,  $c$  is a Bernoulli random variable and  $\hat{X} = \text{Dec}(c)$  is a random variable. Denote

$$\hat{X}_i \triangleq \text{Dec}(i), \quad i \in \{0, 1\}. \quad (5.4)$$

WLOG, let  $\mathbb{E}[\hat{X}_0] \leq \mathbb{E}[\hat{X}_1]$ . Then,

$$\mathbb{E}[\hat{X}_0] \leq p_C(0) \mathbb{E}[\hat{X}_0] + p_C(1) \mathbb{E}[\hat{X}_1] \quad (5.5)$$

$$= \mathbb{E}[\hat{X}] \leq \mathbb{E}[\hat{X}_1]. \quad (5.6)$$

Suppose that  $x = 0$  is encoded. Then,

$$\mathbb{E} \left[ \left( x - \hat{X} \right)^2 | x = 0 \right] = \mathbb{E} \left[ \hat{X}^2 | x = 0 \right] \quad (5.7)$$

$$\geq \left( \mathbb{E} \left[ \hat{X} | x = 0 \right] \right)^2 \quad (5.8)$$

$$= \left( \mathbb{E} \left[ \hat{X}_0 | x = 0 \right] p_{c|x=0}(0) + \mathbb{E} \left[ \hat{X}_1 | x = 0 \right] p_{c|x=0}(1) \right)^2 \quad (5.9)$$

$$\geq \left( \mathbb{E} \left[ \hat{X}_0 | x = 0 \right] \right)^2. \quad (5.10)$$

If the claim is not true, then (5.10) gives

$$\mathbb{E} \left[ \hat{X}_0 | x = 0 \right] \leq \frac{1}{4}. \quad (5.11)$$

Similarly, if  $x = 1$  is encoded,

$$\left( \mathbb{E} \left[ 1 - \hat{X} | x = 1 \right] \right)^2 = \left( 1 - \mathbb{E} \left[ \hat{X} | x = 1 \right] \right)^2 \quad (5.12)$$

$$\geq \left( 1 - \mathbb{E} \left[ \hat{X}_1 | x = 1 \right] \right)^2. \quad (5.13)$$

Again, if the claim does not hold,

$$\mathbb{E} \left[ \hat{X}_1 | x = 1 \right] > \frac{3}{4}. \quad (5.14)$$

Now, choosing  $x = \frac{1}{2}$ , and using (5.11) and (5.14),

$$\mathbb{E} \left[ \left( \hat{X} - \frac{1}{2} \right)^2 | x = \frac{1}{2} \right] = \mathbb{E} \left[ \left( \hat{X}_0 - \frac{1}{2} \right)^2 | x = \frac{1}{2}, c = 0 \right] p_c(0) + \mathbb{E} \left[ \left( \hat{X}_1 - \frac{1}{2} \right)^2 | x = \frac{1}{2}, c = 1 \right] p_c(1) \quad (5.15)$$

$$\geq \left( \mathbb{E} \left[ \hat{X}_0 - \frac{1}{2} | x = \frac{1}{2}, c = 0 \right] \right)^2 p_c(0) + \left( \mathbb{E} \left[ \hat{X}_1 - \frac{1}{2} | x = \frac{1}{2}, c = 1 \right] \right)^2 p_c(1) \quad (5.16)$$

$$\geq \left( \frac{1}{4} - \frac{1}{2} \right)^2 p_c(0) + \left( \frac{3}{4} - \frac{1}{2} \right)^2 p_c(1) \geq \frac{1}{16} \quad (5.17)$$

□