EE6367: Topics in Data Storage and Communications **2023**

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5.1 Deterministic Single-Bit Scalar Quantization

Suppose we have to encode $x \in [0,1]$ using a single bit $c \in \{0,1\}$. Let the decoded output be $\hat{x}$. We require to minimize the *distortion* which is taken to be the maximum squared error

$$\mathrm{Max} - \mathrm{SE} = \max_{x \in [0,1]} (x - \hat{x})^2 . \tag{5.1}$$

One such set of encoding and decoding functions is

$$\mathrm{Enc}(x) = \begin{cases} 0 & x \leqslant \frac{1}{2} \\ 1 & x > \frac{1}{2} \end{cases} \tag{5.2}$$

$$\mathrm{Dec}(c) = \begin{cases} \frac{1}{4} & c = 0 \\ \frac{3}{4} & c = 1 \end{cases} . \tag{5.3}$$

This is called *deterministic rounding* and has a maximum squared error of $\frac{1}{16}$ as the absolute maximum error is $\frac{1}{4}$.

5.2 Random Single-Bit Scalar Quantization

Claim 5.1. *The maximum MSE of a random quantizer under no shared randomness is at least* $\frac{1}{16}$.

Proof. In this case, c is a Bernoulli random variable and $\hat{X} = \mathrm{Dec}(c)$ is a random variable. Denote

$$\hat{X}_i \triangleq \mathrm{Dec}(i), \; i \in \{0,1\} . \tag{5.4}$$

WLOG, let $\mathbb{E}\left[\hat{X}_0\right] \leqslant \mathbb{E}\left[\hat{X}_1\right]$. Then,

$$\mathbb{E}\left[\hat{X}_0\right] \leqslant p_C(0)\,\mathbb{E}\left[\hat{X}_0\right] + p_C(1)\,\mathbb{E}\left[\hat{X}_1\right] \tag{5.5}$$

$$= \mathbb{E}\left[\hat{X}\right] \leqslant \mathbb{E}\left[\hat{X}_1\right] . \tag{5.6}$$

Suppose that $x = 0$ is encoded. Then,

$$\begin{aligned}
\mathbb{E}\left[\left(x-\hat{X}\right)^2|x=0\right] &= \mathbb{E}\left[\hat{X}^2|x=0\right] && (5.7)\\
&\geqslant \left(\mathbb{E}\left[\hat{X}|x=0\right]\right)^2 && (5.8)\\
&= \left(\mathbb{E}\left[\hat{X}_0|x=0\right]p_{c|x=0}\left(0\right)+\mathbb{E}\left[\hat{X}_1|x=0\right]p_{c|x=0}\left(1\right)\right)^2 && (5.9)\\
&\geqslant \left(\mathbb{E}\left[\hat{X}_0|x=0\right]\right)^2. && (5.10)
\end{aligned}$$

If the claim is not true, then (5.10) gives

$$\mathbb{E}\left[\hat{X}_0|x=0\right] \leqslant \frac{1}{4}. \tag{5.11}$$

Similarly, if $x = 1$ is encoded,

$$\begin{aligned}
\left(\mathbb{E}\left[1-\hat{X}|x=1\right]\right)^2 &= \left(1-\mathbb{E}\left[\hat{X}|x=1\right]\right)^2 && (5.12)\\
&\geqslant \left(1-\mathbb{E}\left[\hat{X}_1|x=1\right]\right)^2. && (5.13)
\end{aligned}$$

Again, if the claim does not hold,

$$\mathbb{E}\left[\hat{X}_1|x=1\right] > \frac{3}{4}. \tag{5.14}$$

Now, choosing $x = \frac{1}{2}$, and using (5.11) and (5.14),

$$\begin{aligned}
\mathbb{E}\left[\left(\hat{X}-\frac{1}{2}\right)|x=\frac{1}{2}\right] \geqslant \left(\mathbb{E}\left[\hat{X}-\frac{1}{2}|x=\frac{1}{2}\right]\right)^2 & && (5.15)\\
= \left(\mathbb{E}\left[\hat{X}_0\right]p_{c|x=\frac{1}{2}}\left(0\right)+\mathbb{E}\left[\hat{X}_1\right]p_{c|x=\frac{1}{2}}\left(1\right)-\frac{1}{2}\right)^2 & && (5.16)
\end{aligned}$$

$\square$