EE6367: Topics in Data Storage and Communications **2023**

Lecture 19: 08 November 2023

Instructor: Shashank Vatedka *Scribe: Gautam Singh*

Disclaimer: *These notes have not been subjected to the usual scrutiny reserved for formal publications. They may be distributed outside this class only with the permission of the Instructor.*

19.1 Notions of Convergence

Definition 19.1 (Almost Sure Convergence)**.** A sequence of random variables $X_1, X_2, \ldots, X_n$ converges **almost surely** to a random variable X if

$$\Pr\left(\lim_{n\to\infty} |X_n - X| > \epsilon\right) = 0,\ \forall\ \epsilon > 0. \tag{19.1}$$

and we denote this as $X_n \xrightarrow{\text{as}} X$.

Equivalently, we can write (19.1) as

$$\Pr\left(|X_n - X| > \epsilon\right) \to 0,\ n \geqslant N,\ N \to \infty. \tag{19.2}$$

Definition 19.2 (Convergence in Probability)**.** We say that X_n converges to X in probability and write $X_n \xrightarrow{\text{P}} X$ if

$$\lim_{n\to\infty} \Pr\left(|X_n - X| > \epsilon\right) = 0 \tag{19.3}$$

or equivalently

$$\Pr\left(|X_n - X| > \epsilon\right) \to 0,\ n \to 0. \tag{19.4}$$

Definition 19.3 (Convergence in Distribution)**.** We say that a sequence of random variables $\{X_n\}$ converges in distribution to another random variables X if for all x where F_X is continuous,

$$X_n \xrightarrow{\text{d}} X. \tag{19.5}$$

As an example, consider $X_1, X_2, \ldots, X_n$ which are iid samples drawn from an independent distribution f_X. Then, $F_{X_n}(x) = F_X(x)$ for all x and n, but X_n does not converge to X almost surely and in probability.

Theorem 19.4 (Strong Law of Large Numbers)**.** *If $X_1, X_2, \ldots, X_n$ are iid with $\mathbb{E}[X_1] = \mu < \infty$, then*

$$\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{\text{as}} \mu \tag{19.6}$$

We state two important results for these notions of convergence.

Theorem 19.5 (Weak Law of Large Numbers)**.** *If $X_1, X_2, \ldots, X_n$ are iid with $\mathbb{E}[X_1] = \mu < \infty$, then*

$$\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{\text{P}} \mu. \tag{19.7}$$

19.2 Mean Squared Error of DME With Location Families

In the DME problem,

$$\begin{aligned}\mathbb{E}\left[Y_i\right] &= \Pr\left(X_i \leqslant \theta\right) && (19.8)\\ &= F_X\left(\theta - \mu\right) && (19.9)\\ \implies \frac{1}{n}\sum_{i=1}^{n} Y_i &\xrightarrow{\text{as}} F_X\left(\theta-\mu\right). && (19.10)\end{aligned}$$

We quote another result for F_X^{-1}.

Theorem 19.6 (Continuous Mapping Theorem)**.** *If $g : \mathbb{R} \to \mathbb{R}$ is continuous and differentiable, then the following identities hold.*

1. $X_n \xrightarrow{\text{as}} X \implies g\left(X_n\right) \xrightarrow{\text{as}} g\left(X\right)$.
2. $X_n \xrightarrow{\text{P}} X \implies g\left(X_n\right) \xrightarrow{\text{P}} g\left(X\right)$.
3. $X_n \xrightarrow{\text{d}} X \implies g\left(X_n\right) \xrightarrow{\text{d}} g\left(X\right)$.

Using Theorem 19.6, we see that

$$\begin{aligned}F_X^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} Y_i\right) &\xrightarrow{\text{as}} F_X^{-1}\left(F_X\left(\theta-\mu\right)\right) && (19.11)\\ \implies \hat{\mu} &\xrightarrow{\text{as}} \mu. && (19.12)\end{aligned}$$

We also make use of the central limit theorem.

Theorem 19.7 (Central Limit Theorem)**.** *If $X_1, X_2, \ldots, X_n$ are iid samples with $\mathbb{E}\left[X_i\right] = \mu$ and $\mathrm{Var} X_i = \sigma^2$, then*

$$\frac{\sum_{i=1}^{n}\left(X_i - \mu\right)}{\sqrt{n\sigma^2}} \xrightarrow{\text{d}} \mathcal{N}\left(0,1\right). \quad (19.13)$$

From (19.13), we have

$$\begin{aligned}Z \triangleq \sqrt{n}\left(\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) - \mu\right) &\xrightarrow{\text{d}} \mathcal{N}\left(0,\sigma^2\right) && (19.14)\\ \implies \mathbb{E}\left[Z^2\right] = n\left(\text{MSE}\right) &\to \sigma^2 && (19.15)\\ \implies \text{MSE} &\to \frac{\sigma^2}{n}. && (19.16)\end{aligned}$$

However, we still cannot conclude anything about the MSE of $\hat{\mu}$. We require another tool.

19.3 Delta Method

Suppose $\phi : \mathbb{R} \to \mathbb{R}$ is differentiable $\{X_i\}_{i=1}^n$ is a sequence of random variables and $\{r_i\}_{i=1}^n$ is a sequence of real numbers such that $r_n \to \infty$ as $n \to \infty$. Further, suppose that

$$r_n \left(X_n - \theta\right) \xrightarrow{\mathrm{d}} X. \tag{19.17}$$

Then,

$$r_n \left(\phi\left(X_n\right) - \phi\left(\theta\right)\right) \xrightarrow{\mathrm{d}} \left(\frac{dt}{d\phi}\bigg|_{t=\theta}\right) X \tag{19.18}$$

Using the Delta Method for our problem, define

$$Y_n \triangleq \frac{1}{n} \sum_{i=1}^{n} X_i. \tag{19.19}$$

To estimate $\phi\left(\mu\right)$, we use the *plugin estimator*

$$\phi\left(\hat{\mu}\right) = \phi\left(\frac{1}{n}\sum_{i=1}^{n} Y_i\right). \tag{19.20}$$

Hence,

$$\begin{aligned}
\sqrt{n}\left(Y_n - \mu\right) &\xrightarrow{\mathrm{d}} Y \sim \mathcal{N}\left(0, \sigma^2\right) && (19.21)\\
\implies \sqrt{n}\left(\phi Y_n - \phi\left(\mu\right)\right) &\xrightarrow{\mathrm{d}} \left(\phi'\left(\mu\right)\right) Y && (19.22)\\
n\left(\mathrm{MSE}\right) &\xrightarrow{\mathrm{d}} 2\mu Y && (19.23)\\
n\mathrm{MSE}\left(\mu^2\right) &= 4\mu^2\sigma^2. && (19.24)
\end{aligned}$$