

Lecture 13: 9 October 2023

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13.1 Stochastic Rounding

We consider stochastic rounding in the case of n users. Let a random permutation π of $\{0, 1, \dots, n-1\}$ be shared among the users. User i is given the number π_i (here the users are 0-indexed). Then, each user generates $\gamma_i \sim \text{Unif}[0, \frac{1}{m}]$. Note that γ_i is *private randomness*. Define

$$U_i \triangleq \frac{\pi_i}{m} + \gamma_i. \quad (13.1)$$

The CDF of U_i , for $u \in [0, 1]$, is

$$\Pr(U_i \leq u) = \Pr\left(\frac{\pi_i}{m} + \gamma_i \leq u\right) \quad (13.2)$$

$$= \Pr(\pi_i + m\gamma_i \leq mu) \quad (13.3)$$

$$= \Pr(\pi_i < \lfloor mu \rfloor) + \Pr(\pi_i = \lfloor mu \rfloor, m\gamma_i \leq \mu - \lfloor mu \rfloor) \quad (13.4)$$

$$= \frac{\lfloor mu \rfloor}{m} + \frac{mu - \lfloor mu \rfloor}{m} = u \quad (13.5)$$

and hence $U_i \sim \text{Unif}[0, 1]$.

Now, each user i transmits

$$Y_i \triangleq \mathbb{K}_{\{U_i \leq x_i\}}. \quad (13.6)$$

Clearly, this scheme is unbiased because $\mathbb{E}[Y_i] = x_i$. We now present a claim for the MSE of this scheme.

Claim 13.1. *For the above scheme, the MSE is upper bounded by $\frac{3}{m}\sigma_{md} + \frac{12}{m^2}$, where*

$$\sigma_{md} \triangleq \frac{1}{m} \sum_{i=1}^m |x_i - \bar{x}| \leq \sqrt{\frac{1}{m} \sum_{i=1}^m (x_i - \bar{x})^2}. \quad (13.7)$$

Proof. Define

$$z_i \triangleq \frac{\lfloor mx_i \rfloor}{m} \quad (13.8)$$

$$Y_i \triangleq Q_i(x_i) \quad (13.9)$$

$$Q_i(t) \triangleq \mathbb{1}_{\{U_i < t\}} \quad (13.10)$$

The MSE is given by

$$\text{MSE} = \mathbb{E} \left[\left(\frac{1}{m} \sum_{i=1}^m Y_i - \frac{1}{m} \sum_{i=1}^m x_i \right)^2 \right] \quad (13.11)$$

$$= \frac{1}{m^2} \mathbb{E} \left[\left(\sum_{i=1}^m (Y_i - x_i) \right)^2 \right] \quad (13.12)$$

$$= \frac{1}{m^2} \mathbb{E} \left[\left(\sum_{i=1}^m ((x_i - z_i) + (z_i - Q_i(z_i)) + (Q_i(z_i) - Q_i(x_i))) \right)^2 \right] \quad (13.13)$$

$$\leq \frac{3}{m^2} \left\{ \mathbb{E} \left[\sum_{i=1}^m (x_i - z_i) \right]^2 + \mathbb{E} \left[\sum_{i=1}^m (z_i - Q_i(z_i)) \right]^2 + \mathbb{E} \left[\sum_{i=1}^m (Q_i(z_i) - Q_i(x_i)) \right]^2 \right\} \quad (13.14)$$

$$(13.15)$$

where we use in (13.13) the inequality

$$(\alpha + \beta + \gamma)^2 \leq 3(\alpha^2 + \beta^2 + \gamma^2). \quad (13.16)$$

for reals α, β, γ . □