

# QUIC-FL: Quick Unblased Compression for Federated Learning

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# The DME Problem

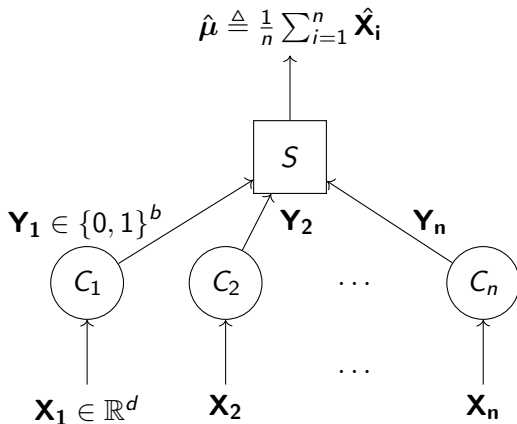


Figure 1: Illustration of the DME Problem. Here,  $\hat{\mathbf{x}}_i$  denotes the server estimate for  $\mathbf{x}_i$ .

# vNMSE and NMSE

## Definition 1 (vNMSE)

The *vector Normalized Mean Square Error* of  $\mathbf{x}$  is defined as

$$vNMSE \triangleq \frac{\mathbb{E} \left[ \|\hat{\mathbf{x}} - \mathbf{x}\|_2^2 \right]}{\|\mathbf{x}\|_2^2}. \quad (1)$$

## Definition 2 (NMSE)

The *Normalized Mean Square Error* in the case of the DME problem is defined as

$$NMSE \triangleq \frac{\mathbb{E} \left[ \|\hat{\boldsymbol{\mu}} - \boldsymbol{\mu}\|_2^2 \right]}{\frac{1}{n} \sum_{i=1}^n \|\mathbf{x}_i\|_2^2} = \frac{\mathbb{E} \left[ \left\| \hat{\boldsymbol{\mu}} - \frac{1}{n} \sum_{i=1}^n \mathbf{x}_i \right\|_2^2 \right]}{\frac{1}{n} \sum_{i=1}^n \|\mathbf{x}_i\|_2^2}. \quad (2)$$

# An Important Lemma

## Lemma 3

For  $n$  clients, we have [1]

$$NMSE = \frac{\sum_{i=1}^n vNMSE(i) \|\mathbf{x}_i\|_2^2}{n \sum_{i=1}^n \|\mathbf{x}_i\|_2^2} \quad (3)$$

## Corollary 4

If  $\|\mathbf{x}_i\| = r \ \forall \ 1 \leq i \leq n$ , then

$$NMSE = \frac{vNMSE}{n} \quad (4)$$

From the above lemma, we need the  $vNMSE$  to be a constant so that the  $NMSE$  is  $\mathcal{O}(1/n)$  for QUIC-FL.

# Randomized Hadamard Transform

## Definition 5 (Walsh-Hadamard Matrix)

The Walsh-Hadamard matrix  $\mathbf{H}_{2^k}$  with  $\mathbf{H}_1 = (1)$  is recursively defined as

$$\mathbf{H}_{2^k} = \begin{pmatrix} \mathbf{H}_{2^{k-1}} & \mathbf{H}_{2^{k-1}} \\ \mathbf{H}_{2^{k-1}} & -\mathbf{H}_{2^{k-1}} \end{pmatrix}. \quad (5)$$

## Definition 6 (Randomized Hadamard Transform)

Let  $\mathbf{H} \in \{-1, +1\}^{d \times d}$  be a Walsh-Hadamard Matrix and  $\mathbf{D}$  be a diagonal matrix with uniform iid Rademacher entries *i.e.*, entries that are  $\pm 1$  with equal probability. Then, the *randomized Hadamard transform* (RHT) of  $\mathbf{x} \in \mathbb{R}^d$  is given by

$$\mathcal{R}_{\mathbf{H}}(\mathbf{x}) \triangleq \left( \frac{1}{\sqrt{d}} \mathbf{H} \mathbf{D} \right) \mathbf{x}. \quad (6)$$

# Properties of RHT

## Definition 7 (Inverse RHT)

The *inverse randomized Hadamard transform* of  $\mathbf{x}$  is given by

$$\mathcal{R}_{\mathbf{H}}^{-1}(\mathbf{x}) \triangleq \left( \frac{1}{\sqrt{d}} \mathbf{D} \mathbf{H} \right) \mathbf{x}. \quad (7)$$

## Lemma 8

Define  $F_{i,d}(x)$  to be the CDF of  $\frac{1}{\sigma} \mathcal{R}_{\mathbf{H}}(\mathbf{x})_i$ , where  $\mathbb{E}[x_i^2] = \sigma^2$ , and  $\Phi(x)$  the CDF of the standard normal distribution. Then, as  $d \rightarrow \infty$ ,  $\forall 1 \leq j \leq d$ ,

$$F_{i,d}(x_j) \xrightarrow{d} \Phi(x_j). \quad (8)$$

# Design Goals of QUIC-FL

- 1 Less computational complexity compared to other methods, at *both* client and server.
- 2 Same (asymptotic) NMSE as other methods:  $\mathcal{O}\left(\frac{1}{n}\right)$ .
- 3 Better compression ratio.



# BSQ Algorithm

- ① Uses a parameter  $p \in (0, 1]$ , based on which a threshold  $[-t_p, t_p]$  is chosen such that at most  $dp$  values lie outside this interval.
- ②  $\mathbf{x}$  separated into two parts: *large* values that lie outside this interval and *small* values that lie inside this interval.
- ③ To encode  $\mathbf{x}$ ,
  - ① *Large* values are sent exactly, along with their indices. Costs  $\log \binom{d}{dp} \approx dp \log \left( \frac{1}{p} \right)$  bits. Can use delta encoding or no encoding at cost of  $dp \lceil \log d \rceil$  bits (Authors assume  $p \log d \ll 1$ ).
  - ② *Small* values are stochastically quantized and sent using  $b$  bits per coordinate. *This implies that BSQ is unbiased.*
- ④ To decode and compute  $\hat{\mathbf{x}}$ , decode the *large* values along with their indices. Then, estimate the quantized *small* values.

# Main Result of BSQ

## Lemma 9

*BSQ, without further assumptions, admits a worst-case NMSE of  $\frac{1}{np(2^b-1)^2}$  with  $\mathcal{O}(d)$  and  $\mathcal{O}(nd)$  complexity for encoding and decoding respectively.*

## Arbitrarily Large NMSE

Notice that the constant in the NMSE can be made arbitrarily large. For example,  $p = 2^{-10}$ ,  $b = 1$ .

## What QUIC-FL Does

The **random rotation preprocessing** applied by QUIC-FL guarantees smaller constant values for small  $p$ .

# Distribution-Aware Unbiased Quantization

## Key Idea

We have  $\forall 1 \leq i \leq d$ , where  $\mathbf{R}$  denotes a random rotation matrix [2],

$$\frac{\sqrt{d}}{\|\mathbf{x}\|_2} (\mathbf{R}\mathbf{x})_i \sim \mathcal{N}(0, 1). \quad (9)$$

Thus, we only need to optimize the reconstruction values for when the data is sampled from the *standard normal distribution*. This will yield a *near-optimal* quantization for the  $\mathbf{x}_i$ .

Notice that this is possible since

- 1 We know the distribution of the transformed vector by (9).
- 2 We know the range  $[-t_p, t_p]$  of  $\mathbf{x}_i$ .

# Distribution-Aware Unbiased Quantization

## Definitions

Notation	Definition
$p$	BSQ parameter
$t_p$	BSQ threshold
$b$	Number of bits per quantized value
$\mathcal{M}$	Message space $\{0, 1, \dots, 2^b - 1\}$
$R(x)$	Reconstruction point associated with $x$ at the receiver
$\mathcal{Q}_{b,p}$	Set of all reconstruction points $\{R(x) \mid x \in \mathcal{M}\}$
$\mathcal{I}_m$	$\{0, 1, \dots, m-1\}$
$\mathcal{A}_{p,m}(i)$	$\Pr(Z \leq \mathcal{A}_{p,m}(i) \mid Z \in [-t_p, t_p], Z \sim \mathcal{N}(0, 1)) = \frac{i}{m-1}$
$\mathcal{A}_{p,m}$	Right endpoints of quantiles $\{\mathcal{A}_{p,m}(i) \mid i \in \mathcal{I}_m\}$
$S(z, x)$	$\Pr(z \text{ is quantized to } R(x))$
$S'(i, x)$	$S(\mathcal{A}_{p,m}(i), x)$

Table 1: Notation for Distribution-Aware Unbiased Quantization.

# Distribution-Aware Unbiased Quantization

## Optimization Problem

We want to minimize the MSE of this scheme. The optimization problem becomes

$$\min_{S,R} \int_{-t_p}^{t_p} \sum_{x \in \mathcal{M}} S(z, x) (z - R(x))^2 e^{-\frac{z^2}{2}} dz \text{ s.t.} \quad (10)$$

$$\text{(Probability)} \quad \sum_{x \in \mathcal{M}} S(z, x) = 1 \quad (11)$$

$$\text{(Unbiasedness)} \quad \sum_{x \in \mathcal{M}} S(z, x) R(x) = z. \quad (12)$$

*This is not easy to solve!*

# Distribution-Aware Unbiased Quantization

## Discretization

Can now use solvers like APOPT [3] and IPOPT [4] to solve for  $\mathcal{Q}_{b,p}$ .

$$\min_{S', R} \sum_{\substack{i \in \mathcal{I}_m \\ x \in \mathcal{M}}} S'(i, x) (\mathcal{A}_{p,m}(i) - R(x))^2 \quad \text{s.t.} \quad (13)$$

$$(\text{Probability}) \sum_{x \in \mathcal{M}} S'(i, x) = 1 \quad (14)$$

$$(\text{Unbiasedness}) \sum_{x \in \mathcal{M}} S'(i, x) R(x) = \mathcal{A}_{p,m}(i). \quad (15)$$

The reconstruction points  $\mathcal{Q}_{b,p}$  obtained correspond to a *near-optimal* solution.

- ① Convergence of randomly rotated coordinates to normal distribution, from (9).
- ② As  $m \rightarrow \infty$ , the discrete problem converges to the continuous version.

# QUIC-FL Client

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## Algorithm 1 QUIC-FL Client

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**Require:** BSQ parameter  $p$  and threshold  $t_p$ , bits per symbol  $b$ , global randomness  $T$ , precomputed reconstruction values  $\mathcal{Q}_{b,p}$ , stochastic quantization algorithm  $Q$ .

- 1: **procedure** QUICFLCLIENT( $\mathbf{x}_c$ )
  - 2:    $\mathbf{Z}_c \leftarrow \frac{\sqrt{d}}{\|\mathbf{x}_c\|_2} T(\mathbf{x}_c)$
  - 3:    $\mathbf{U}_c \leftarrow \{Z_{ci} \mid |Z_{ci}| > t_p\}$
  - 4:    $\mathbf{I}_c \leftarrow \{i \mid |Z_{ci}| > t_p\}$
  - 5:    $\mathbf{V}_c \leftarrow \{Z_{ci} \mid |Z_{ci}| \leq t_p\}$
  - 6:    $\mathbf{X}_c \leftarrow Q(\mathbf{V}_c)$  using  $\mathcal{Q}_{b,p}$
  - 7:   Send  $(\|\mathbf{x}_c\|_2, \mathbf{U}_c, \mathbf{I}_c, \mathbf{X}_c)$  to server
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# QUIC-FL Server

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## Algorithm 2 QUIC-FL Server

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**Require:** BSQ parameter  $p$  and threshold  $t_p$ , bits per symbol  $b$ , global randomness  $T$ , precomputed reconstruction values  $\mathcal{Q}_{b,p}$ , stochastic quantization algorithm  $Q$ , subroutine MERGE to reconstruct quantized vector given its BSQ parts.

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1: procedure QUICFLSERVER
2:   for all  $1 \leq c \leq n$  do
3:      $\hat{\mathbf{V}}_c \leftarrow \{\mathcal{Q}_{b,p}(x) \text{ for } x \text{ in } \mathbf{X}_c\}$ 
4:      $\hat{\mathbf{Z}}_c \leftarrow \text{MERGE}(\hat{\mathbf{V}}_c, \mathbf{U}_c, \mathbf{I}_c)$ 
5:    $\hat{\boldsymbol{\mu}}_Z \leftarrow \frac{1}{n} \sum_{c=1}^n \frac{\|\mathbf{x}_c\|_2}{\sqrt{d}} \hat{\mathbf{Z}}_c$ 
6:    $\hat{\boldsymbol{\mu}} \leftarrow T^{-1}(\hat{\boldsymbol{\mu}}_Z)$ 

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# NMSE of QUIC-FL

## Theorem 10

Let  $Z \sim \mathcal{N}(0, 1)$  and  $\hat{Z}$  be its estimate using the distribution-aware unbiased quantization scheme. Then, for any number of clients  $n$  and any set of input vectors  $\{\mathbf{x}_i \in \mathbb{R}^d \mid i \in \{0, 1, \dots, n-1\}\}$ , the NMSE of QUIC-FL is given by

$$NMSE = \frac{1}{n} \mathbb{E} \left[ \left( Z - \hat{Z} \right)^2 \right] + \mathcal{O} \left( \frac{1}{n} \sqrt{\frac{\log d}{d}} \right) \quad (16)$$

When  $d \rightarrow \infty$ , the NMSE respects  $\mathcal{O} \left( \frac{1}{n} \right)$ .

# Optimizations

## Client-Specific Shared Randomness

Some amount of random bits are shared between each client and the server. This can reduce the MSE at the cost of generating more random bits.

Define the following quantities.

- ①  $l$ , the number of random bits used.
- ②  $\mathcal{H}_l \triangleq \{0, 1, \dots, 2^l - 1\}$ , the set of all possible random bitstrings of length  $l$ .
- ③  $H \sim \mathcal{U}[\mathcal{H}_l]$ , the random variable representing the shared randomness, selected uniformly from  $\mathcal{H}_l$ .
- ④  $S'(h, i, x) \triangleq \Pr(\mathcal{A}_{p,m}(i) \text{ is quantized to } R(H, x) \mid H = h)$ .
- ⑤  $R(h, x)$ , the reconstruction point for  $x \in \mathcal{M}$  given shared randomness  $h$ .

# Optimizations

## Client-Specific Shared Randomness

The new optimization problem becomes the following.

$$\min_{S', R} \sum_{\substack{h \in \mathcal{H}_I \\ i \in \mathcal{I}_m \\ x \in \mathcal{M}}} S'(\mathbf{h}, i, x) (\mathcal{A}_{p,m}(i) - R(\mathbf{h}, x))^2 \quad \text{s.t.} \quad (17)$$

$$\text{(Probability)} \quad \sum_{x \in \mathcal{M}} S'(\mathbf{h}, i, x) = 1 \quad (18)$$

$$\text{(Unbiasedness)} \quad \frac{1}{2^I} \sum_{\substack{h \in \mathcal{H}_I \\ x \in \mathcal{M}}} S'(\mathbf{h}, i, x) R(\mathbf{h}, x) = \mathcal{A}_{p,m}(i). \quad (19)$$

# Optimizations

## Using the RHT

- ① Accelerates both client and server algorithms.
- ② Same asymptotic guarantees but with larger constants.
  - ① For the RHT,  $\mathbb{E} \left[ \left( Z - \hat{Z} \right)^2 \right] = \mathcal{O}(p)$ .
  - ② Can compensate by slight increase in  $b$ .
  - ③ The guarantees on the constant are still better than DRIVE and EDEN.
- ③ Fastest complexity in comparison to other state-of-the-art methods, at both client and server.

# Complexity Analysis

Algorithm	Enc. Complexity	Dec. Complexity	NMSE
<b>QSGD</b>	$\mathcal{O}(d)$	$\mathcal{O}(nd)$	$\mathcal{O}(d/n)$
<b>Hadamard</b>	$\mathcal{O}(d \log d)$	$\mathcal{O}(nd + d \log d)$	$\mathcal{O}(\log d/n)$
<b>Kashin</b>	$\mathcal{O}(d \log d \log(nd))$	$\mathcal{O}(nd + d \log d)$	$\mathcal{O}(1/n)$
<b>EDEN</b>	$\mathcal{O}(d \log d)$	$\mathcal{O}(nd \log d)$	$\mathcal{O}(1/n)$
<b>QUIC-FL</b>	$\mathcal{O}(d \log d)$	$\mathcal{O}(nd + d \log d)$	$\mathcal{O}(1/n)$

Table 2: Complexities and NMSE of Unbiased DME Algorithms.

# Accuracy

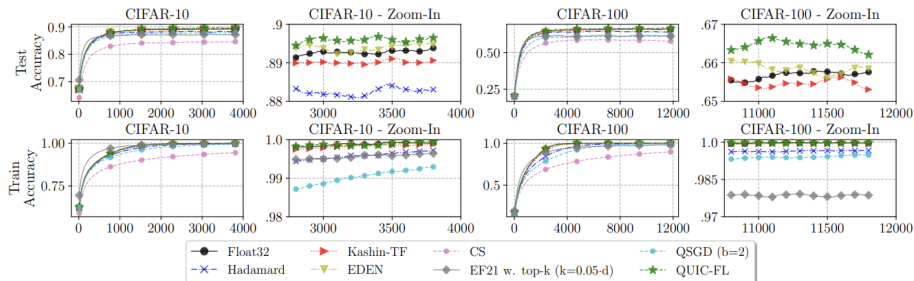







Figure 2: Cross-device Federated Learning with Various Methods of Unbiased Quantization. [5]

# Further Improvements in QUIC-FL

- ① (Open problem) Using more than one RHT can lead to better bounds?
- ② QUIC-FL for cases using limited bits  $b$  and dimensions  $d$ .
- ③ Space complexity per client? Better compression ratio?
  - ① Depends on  $d$ ,  $b$  and  $p$ .
- ④ Can the NMSE be lowered using a client-server feedback system?
- ⑤ Security concerns
  - ① Does the random transformation  $T$  act as a “secret key”?
  - ② What security guarantees does QUIC-FL have against various adversarial attacks?

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