EE6367: Topics in Data Storage and Communications **2023**

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Instructor: Shashank Vatedka *Scribe: Gautam Singh*

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7.1 Quantization of Vectors

In this situation, we must quantize high-dimensional vectors $\mathbf{x} \in \mathbb{R}^d$ using k bits, and recover a lossy version $\hat{\mathbf{X}}$. Further, we assume that $\|\mathbf{x}\|$ is upper bounded.

We use the algorithms for quantization of real numbers as follows. Define

$$x_{\min} \triangleq \min_{1 \leqslant i \leqslant d} \mathbf{x}^\top \mathbf{e_i} \tag{7.1}$$

$$x_{\max} \triangleq \max_{1 \leqslant i \leqslant d} \mathbf{x}^\top \mathbf{e_i}. \tag{7.2}$$

Now, since each coordinate lies between $x_{\min}$ and $x_{\max}$, we can apply the previous algorithms. For example, using deterministic rounding,

$$\text{Cost} = \max_{\|\mathbf{x}\| \leqslant r} \mathbb{E}\left[\left\|\hat{\mathbf{X}} - \mathbf{x}\right\|^2\right] \tag{7.3}$$

$$= \max_{\|\mathbf{x}\| \leqslant r} \sum_{i=1}^{d} \mathbb{E}\left[\left(\hat{X}_i - x_i\right)^2\right] \tag{7.4}$$

$$= \frac{d\left(x_{\max} - x_{\min}\right)^2}{2^{2k+2}} \leqslant \frac{d\left(2r^2\right)}{2^{2k+2}} \tag{7.5}$$

In general, for any scalar quantizer with MSE m, the cost is

$$\text{Cost} = 2r^2 dm \tag{7.6}$$

The *time complexity* of applying such algorithms is $\mathcal{O}\left(dT\right)$, where T is the time complexity for applying the scalar version of the algorithm, usually a constant-time algorithm. Thus, the time complexity usually grows linearly with d.

7.2 Lower Bound on Cost for Deterministic Algorithms

Assume in this situation that $k = \Theta\left(d\right) = d\alpha$. Since the algorithm is deterministic, each k-bit sequence represents one point in space. Denote the **codebook** of the chosen point vectors and their corresponding k-bit sequence as $\mathbb{C}$. Clearly, given $\mathbb{C}$, the optimal encoding rule corresponds to searching for nearest neighbours, that is

$$\mathrm{Enc}\left(\mathbf{x}\right) = \arg\min_{\mathbf{y}\in\mathbb{C}} \left\|\mathbf{x}-\mathbf{y}\right\|. \tag{7.7}$$

Denote the cost of a codebook $\mathbb{C}$ as

$$\beta^2 \triangleq \mathrm{Cost} = \max_{\mathbf{x}\in\mathcal{B}(r)} \min_{\mathbf{y}\in\mathbb{C}} \left\|\mathbf{x}-\mathbf{y}\right\|^2. \tag{7.8}$$

For every $\mathbf{x} \in \mathcal{B}\left(r\right)$, it follows that there exists at least one codeword at distance at most β. Hence, the union of the small balls of radius β covers the larger ball of radius r. Thus,

$$\bigcup_{\mathbf{y}\in\mathbb{C}} \mathcal{B}\left(\mathbf{y},\beta\right) \supset \mathcal{B}\left(\mathbf{0},r\right) \tag{7.9}$$

$$|\mathbb{C}| \geqslant \frac{\frac{\pi^{\frac{d}{2}} r^d}{\Gamma\left(\frac{d}{2}+1\right)}}{\frac{\pi^{\frac{d}{2}} \beta^d}{\Gamma\left(\frac{d}{2}+1\right)}} = \left(\frac{r}{\beta}\right)^d \tag{7.10}$$

$$2^{d\alpha} \geqslant \left(\frac{r}{\beta}\right)^d \tag{7.11}$$

$$\beta^2 \geqslant \frac{r^2}{2^{2\alpha}} \tag{7.12}$$

Comparing with (7.5), we see that (7.12) is better by a factor of d.