Chapter 1: Source Coding

Gautam Singh

1 Learning Objective 1

1.1 The source entropy is

$$H(X) = -\sum_{x \in X} p(x) \log p(x) \tag{1.1.1}$$
$$= 2.23 \text{ bits} \tag{1.1.2}$$

1.2 The source entropy is

$$H(X) = -\sum_{i=1}^{\infty} 2^{-i} (-i) \tag{1.2.1}$$
$$= \sum_{i=1}^{\infty} \frac{i}{2^i} = 2 \text{ bits} \tag{1.2.2}$$

1.3 The source entropy is

$$H(X) = -\sum_{i=1}^{\infty} p (1-p)^{i-1} \log\left(p (1-p)^{i-1}\right) \tag{1.3.1}$$
$$= -\left(\frac{p \log p}{1-(1-p)} + \frac{p \log (1-p) (1-p)}{(1-(1-p))^2}\right) \tag{1.3.2}$$
$$= -\left(\log p + \frac{1-p}{p} \log (1-p)\right) \text{ bits} \tag{1.3.3}$$

1.4 Since $p(H) = p$, we get $p(T) = 1 - p$. Let E denote the event of getting at least one head.

1.4.1 The probability of failure is clearly $p(E') = (1-p)^N$. Hence, the information conveyed by the failure is

$$I(E') = -N \log (1-p) \tag{1.4.1}$$

1.4.2 When $N \to \infty$ in (1.4.1), $I(E') \to \infty$ as $0 \leq 1 - p \leq 1$. This means that the event E conveys little information as N increases, which makes sense since the probability of E increases with N.

1.5 We compute the pmfs of X and Y first. Using Table 1.5.1 and Table 1.5.2, we can now compute the pmfs of $X|Y$ and $Y|X$.

x	x_1	x_2	x_3
$p(x)$	$\frac{5}{8}$	$\frac{1}{4}$	$\frac{1}{8}$

TABLE 1.5.1: PMF of X.

y	y_1	y_2
$p(y)$	$\frac{3}{4}$	$\frac{1}{4}$

TABLE 1.5.2: PMF of Y.

	x_1	x_2	x_3
y_1	$\frac{2}{3}$	$\frac{1}{6}$	$\frac{1}{6}$
y_2	$\frac{1}{2}$	$\frac{1}{2}$	0

TABLE 1.5.3: PMF of $X|Y$.

	x_1	x_2	x_3
y_1	$\frac{4}{5}$	$\frac{1}{2}$	1
y_2	$\frac{1}{5}$	$\frac{1}{2}$	0

TABLE 1.5.4: PMF of $Y|X$.

1.5.1 The entropy of X is

$$H(X) = -\left(\frac{5}{8} \log \frac{5}{8} + \frac{1}{4} \log \frac{1}{4} + \frac{1}{8} \log \frac{1}{8}\right) \tag{1.5.1}$$
$$= 1.30 \text{ bits} \tag{1.5.2}$$

1.5.2 The entropy of Y is

$$H(Y) = -\left(\frac{3}{4} \log \frac{3}{4} + \frac{1}{4} \log \frac{1}{4}\right) \tag{1.5.3}$$
$$= 0.81 \text{ bits} \tag{1.5.4}$$

1.5.3 The joint entropy of X and Y is

$$H(X, Y) = E\left[-\log p(X, Y)\right] \tag{1.5.5}$$
$$= 2 \text{ bits} \tag{1.5.6}$$

1.5.4 The conditional entropy of X given Y is

$$H(X|Y) = E\left[-\log p(X|Y)\right] \tag{1.5.7}$$
$$= 1.19 \text{ bits} \tag{1.5.8}$$

1.5.5 The conditional entropy of Y given X is

$$H(Y|X) = E\left[-\log(Y|X)\right] \tag{1.5.9}$$
$$= 0.70 \text{ bits} \tag{1.5.10}$$

1.6 The differential entropy of X is

$$h(X) = -\int_{-\infty}^{\infty} p(x)\log p(x)\ dx \tag{1.6.1}$$
$$= \int_{0}^{a} a^{-1}\log a\ dx = \log a \tag{1.6.2}$$

From Figure 1.6.1, the differential entropy monotonically increases with a. It is zero when $a = 1$, negative when $a < 1$ and positive when $a > 1$.

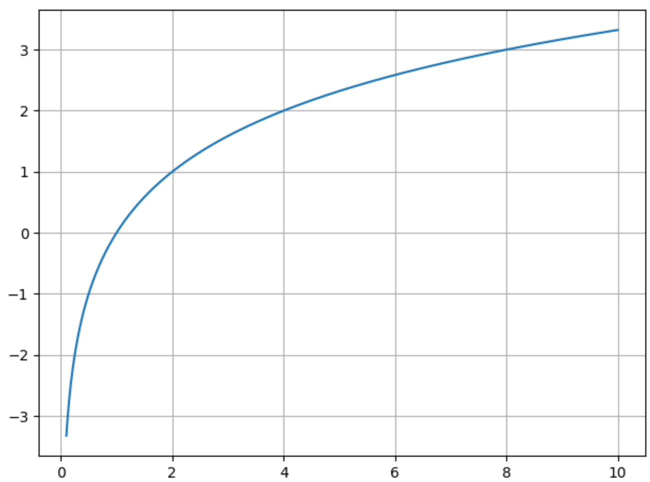

Fig. 1.6.1: $h(X)$ as a function of a.