Linear Programming

Engineering Mathematics 3

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7 October 2020

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Overview: Optimization Problems

- ***Optimization theory and methods*** *deal with selecting the* ***best solution (either maximize or minimize)*** *for a given function f called* ***objective function***.
- In most optimization problems, the **objective function** f depends on several variables, $\mathbf{x} = \{x_1, \ldots, x_n\}$ called **control variables** because we can choose their values.
- The choice of control variables usually restricted due to the nature of the problem, hence forming **constraints**.

$$\begin{aligned} &\textbf{Maximize} f(\mathbf{x}) \\ &\textbf{subject to}: \\ &\left.\begin{array}{l}\text{constraint 1,} \\ \text{constraint 2.}\end{array}\right\} \textbf{control variables} \text{ in the form of constraints} \end{aligned}$$

Various types of optimization classification

Control variables:

- Constrained
- Unconstrained

Convexity

- Convex
- Nonconvex

Approach:

- Deterministic:
  - Linear Prog. (LP)
  - NLP
  - Mixed-Integer (MILP)
- Heuristics Algo.
- Meta-Heuristic Algo.

- **Deterministic** approach employs analytic properties of the problem.
- **Heuristic algo** is trial-and-error approach mainly based on nature-inspired ideas.

Heuristic Classification [3]

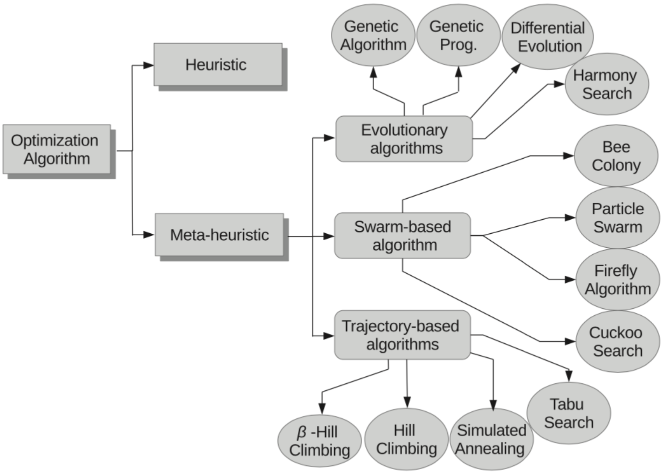


Figure: Heuristic Optimization Algorithm

Convexity Classification [2]

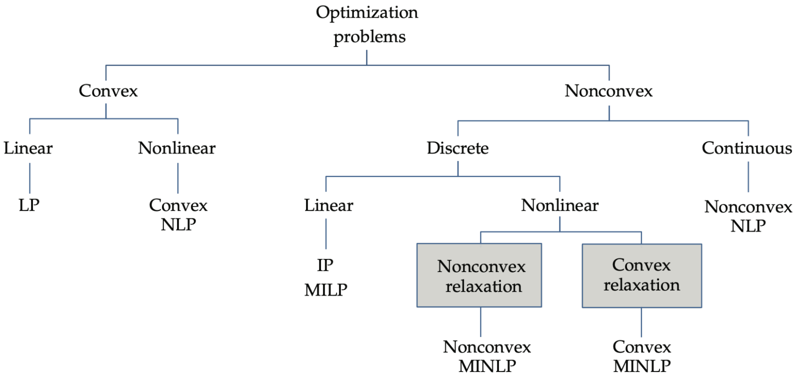


Figure: Convexity Based classification

Convexity Test

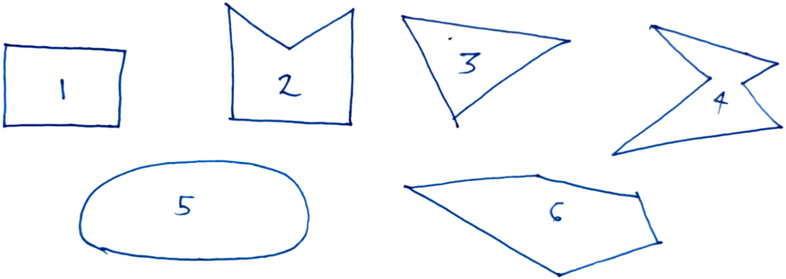


Figure: Which is a convex polygon?

Convexity Test: Drawing a line

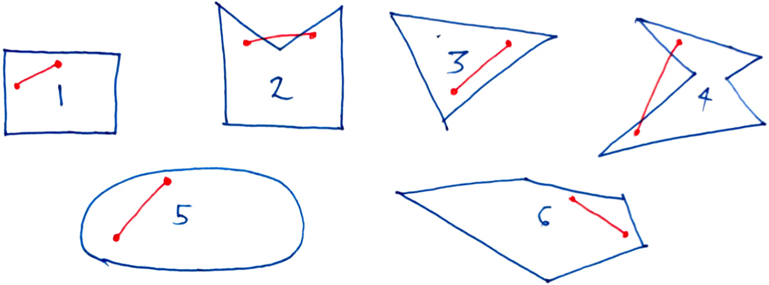


Figure: drawing a line for convexity identification

Convexity Test: Wrapping with a rubber-band via nails

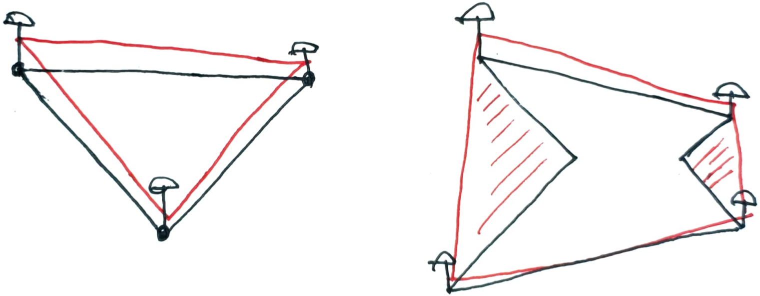


Figure: The area enclosed by the rubber-band is denoted as **convex hull**

Some constraint notations

Symbol	**Remark**
$<$	Less than, fewer
$>$	Greater than, over
$\leq$	Less than or equal to, at most
$\geq$	Greater than or equal to, at least
$s.t$	subject to

Linear Programming: two variables

Simple Linear Programming *or linear optimization consists of methods for solving optimization problems with constraints for finding maximum (or minimum)* $\mathbf{x} = [x_1, x_2]$ *of a* ***linear objective function****:*

maximize (or minimum) $\quad f(x_1, x_2) = x_1 + x_2 \;\}$ linear objective function,

s.t.

$$\left.\begin{aligned} &x_1 + 2x_2 \leq 10, \\ &3x_1 + 2x_2 \leq 18 \\ &x_1 \geq 0 \\ &x_2 \geq 0 \end{aligned}\right\} \text{linear constraints,} \tag{1}$$

Graphing the Constraints: Two constraints

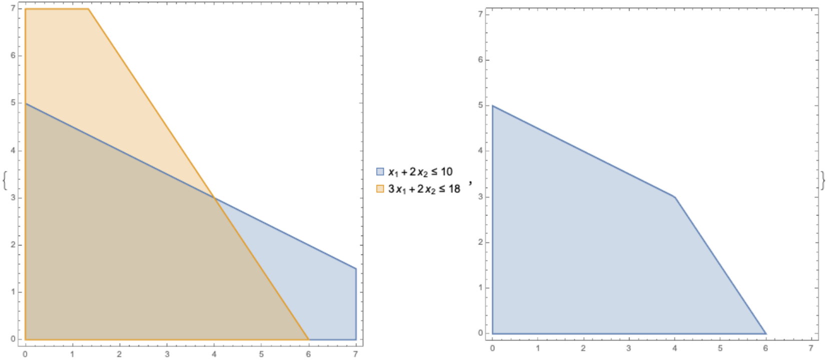


Figure: Right: Shaded area is where the solution lives. Each point in this region is called **feasibility region**, is the set of all feasible solution (maximum or minimum).

Graphing the Constraints:

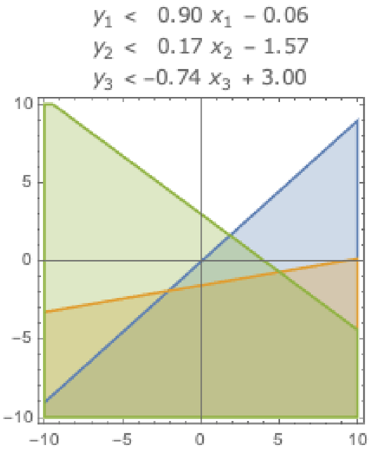


Figure: **feasibility region**

Think...

1. How many control variables?
2. How many constraints?

Graphing the Objective Function

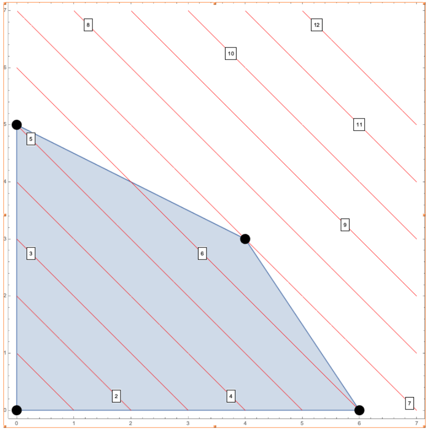


Figure: Feasible solution region.

Red line shows the corresponding value of the objective function.

Think...

What will be the maximum or minimum feasible solution?

Graphing the Objective Function

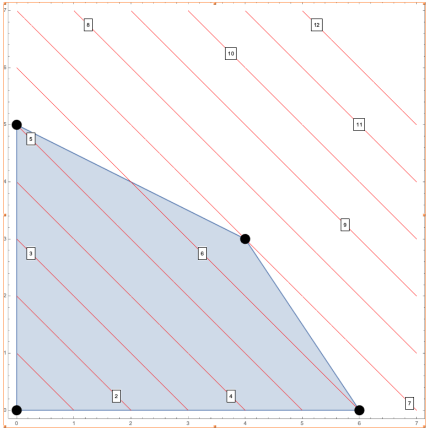


Figure: Feasible solution region.

Red line shows the corresponding value of the objective function.

Think...

What will be the maximum or minimum feasible solution?

Remarkable Theorem

The **maximum or minimum** of a linear program, if it exists, will necessarily occur at a vertex (corner point) of the constraint.

Why non-convex case cannot be the constraint set of a Linear Program?

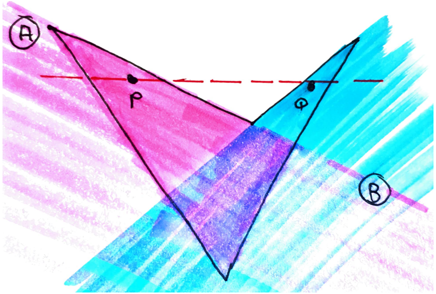


Figure: A set of constraints forming non-convex region

Remark

The line connecting P and Q shows this is a non-convex region.
ALL the region of constraint AB (pink) does not intersect with blue region. This is not a proper solution region!
For example Q does not satisfy constraint AB

Nonlinear Constraints

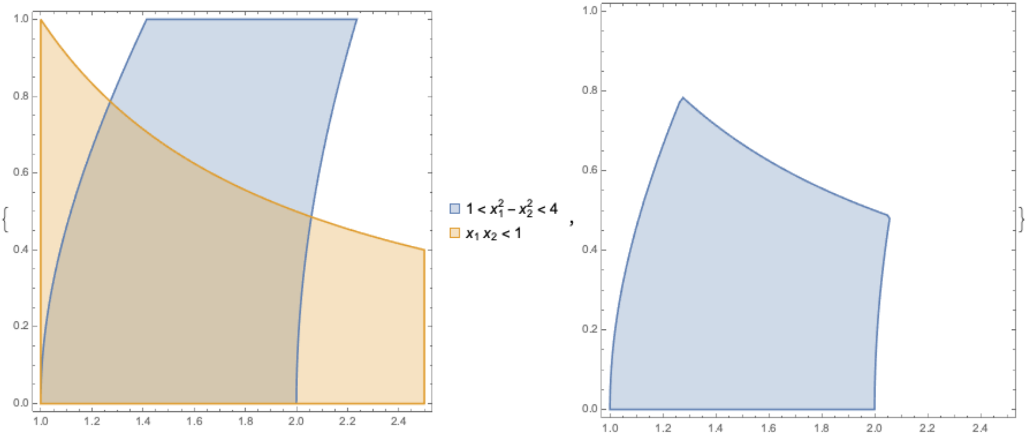


Figure: two variables

Feasible region vertices: Optimal solution

Maximum value: The point furthest from the origin containing at least one point on the line parallel to the isoline.

Minimum value: The point closest to the origin containing at least one point on the line parallel to the isoline.

Two methods:

- isoline,
- trial-and-error.
- **Simplex Method**.
- See **simulation** example for two variables.

Three control variables: Feasible solutions region

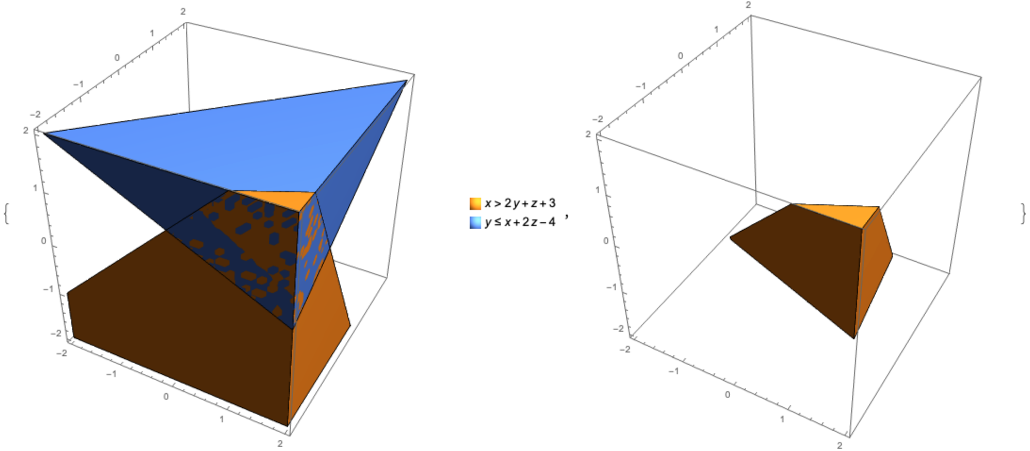


Figure: The solution region is a **polyhedron**

More than three constraints?

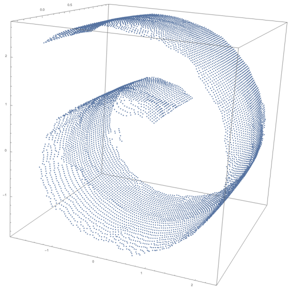


Figure: 3D cloud points

Play...

See simulation for better understanding.

Think...

What if i have more than three control variables?

General LP: More than three constraints

Linear Programming: *solve optimization problems with constraints for finding maximum (or minimum)* $\mathbf{x} = [x_1, \ldots, x_n]$ *of a* ***linear objective function***:

$$
\begin{array}{ll}
\textbf{Maximize:} & f(\mathbf{x}) = c_1 x_1 + \cdots + c_n x_n \\
\textbf{s.t.} & \\
& \left.\begin{array}{l} a_{11}x_1 + a_{12}x_1 + a_{1n}x_n \leq b_1, \\ \ldots \\ a_{m1}x_1 + a_{m2}x_1 + a_{mn}x_n \leq b_m, \\ x_i, \geq 0, \; (i = 1, ..., n) \end{array}\right\} \textbf{linear inequalities},
\end{array} \tag{2}
$$

Minimization: *simply means we maximize* $-f(\mathbf{x})$, *thus no need separate consideration.*

Conversion to normal form of LP

Simplex Method: *First, we need convert to* ***normal form***:

$$
\begin{array}{ll}
\textbf{Maximize} & f(\mathbf{x}) = c_1 x_1 + \cdots + c_n x_n \\
\textbf{s.t.} & \\
& \left.\begin{array}{l}
a_{11}x_1 + a_{12}x_1 + a_{1n}x_n = b_1, \\
\ldots \\
a_{m1}x_1 + a_{m2}x_1 + a_{mn}x_n = b_m, \\
x_i, \geq 0, \ (i = 1, ..., n)
\end{array}\right\} \textbf{linear eqns with extra variables,}
\end{array} \tag{3}
$$

Take note that $\mathbf{b_i} \geq \mathbf{0}$. If a component of b_i is negative, then we multiply the ith b_i by -1 to obtain a positive right-hand side.

General LP: keywords

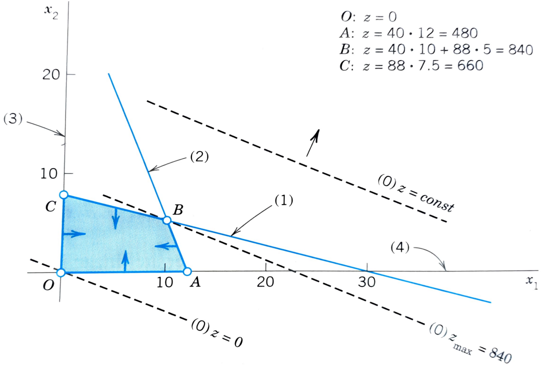
Feasible solution: *An n-tuple* $x_1, \ldots, x_n$ *that satisfies all the constraints in (3) .*

Optimal solution: *A feasible solution is called optiomal when the objective function becomes maximum among the rest of feasible solution.*

Basic Feasible Solution (BFS): *a feasible solution for which some of the control variables* $x_1, \ldots, x_n$ *are zero. e.g. LP with 2 control variables and 4 constraints: where BFS are the extreme points in the form of vertices.*

BFS: Example of LP [1]

Basic Feasible Solution (BFS): *a feasible solution for which some of the control variables* $x_1, \ldots, x_n$ *are zero where BFS are the extreme points . e.g. LP with 2 control variables and 4 constraints: the BFS are in the form of vertices* O, A, B, C.



Slack variable: constraints with: lesser than or equals to

Slack variable: *We introduce* ***slack variable*** *for the conversion to normal form.* For example:

$$
\begin{aligned}
&\textbf{Maximize} \qquad f(x_1, x_2) = x_1 + x_2 \\
&\textbf{s.t.} \\
&\qquad \left.\begin{aligned} x_1 + 2x_2 &\leq 10, \\ 3x_1 + 2x_2 &\leq 18 \\ x_1, x_2 &\geq 0 \end{aligned}\right\} \text{inequalities with } \leq, \text{except for } x_i
\end{aligned}
\tag{4}
$$

One Slack for each constraint (lesser than or equals to)

Constraint 1

$$x_1 + 2x_2 \leq 10,$$
$$\Rightarrow \underbrace{10 - x_1 - 2x_2}_{\text{always non-negative}} \geq 0$$

$$\Rightarrow x_3 = 10 - x_1 - 2x_2,$$
$$x_3 \geq 0$$

$$\therefore \; x_1 + 2x_2 + x_3 = 10.$$

Constraint 2

$$3x_1 + 2x_2 \leq 18,$$
$$\Rightarrow \underbrace{18 - 3x_1 - 2x_2}_{\text{always non-negative}} \geq 0$$

$$\Rightarrow x_4 = 18 - 3x_1 - 2x_2,$$
$$x_4 \geq 0$$

$$\therefore \; 3x_1 + 2x_2 + x_4 = 18.$$

Simple Simplex Table

Simplex Tableau: *We represent the equations to **simplex table**.*

Maximize $f(x_1, x_2) = x_1 + x_2$:

we re-write as $P - x_1 - x_2 = 0,$

where $P = f(x_1, x_2)$

s.t.

$$\left.\begin{aligned} x_1 + 2x_2 + s_1 &= 10, \\ 3x_1 + 2x_2 + s_2 &= 18 \\ x_1, x_2, s_1, s_2 &\geq 0 \end{aligned}\right\}$$

Simple Tableau:

	x_1	x_2	x_3	x_4	(b_i)
P	-1	-1	0	0	0
s_1	1	2	1	0	10
s_2	3	2	0	1	18

In Augmented Matrix:

$$\left(\begin{array}{cccc|c} -1 & -1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 10 \\ 3 & 2 & 0 & 1 & 18 \end{array}\right)$$

Augmented Matrix: stepwise pivot

In Augmented Matrix:

$$\left(\begin{array}{cccc|c} -1 & -1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 10 \\ 3 & 2 & 0 & 1 & 18 \end{array}\right)$$

Two types of variables :

1. **Basic variables** (columns with only one nonzero entry)
2. **nonbasic variables** (columns with many nonzero entries)

1. Every simplex table gives a **basic feasible solution**.
2. E.g., we let the **nonbasic variables** to zero; $x_1 = x_2 = 0$:

$$x_1 = 0, x_2 = 0,$$
$$s_1 = 10, s_2 = 18,$$
$$P = f(0,0) = 0$$

Optimal solution is obtained by **stepwise pivoting** until maximum of P is reached; slightly different than Gauss elimination.

A short review of Gauss elimination

$$
\begin{array}{rc}
 & \text{Augmented Matrix } \tilde{\mathbf{A}} \\
\text{Pivot 1} \longrightarrow & \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ \boxed{-1} & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{array}\right] \\
\text{Eliminate} \longrightarrow &
\end{array}
\qquad
\begin{array}{rl}
 & \text{Equations} \\
\text{Pivot 1} \longrightarrow & x_1 - x_2 + x_3 = 0 \\
 & -x_1 + x_2 - x_3 = 0 \\
\text{Eliminate} \longrightarrow & 10x_2 + 25x_3 = 90 \\
 & 20x_1 + 10x_2 = 80.
\end{array}
$$

Step 1. Elimination of x_1

Call the first row of **A** the **pivot row** and the first equation the **pivot equation.** Call the coefficient 1 of its x_1-term the **pivot** in this step. Use this equation to eliminate x_1 (get rid of x_1) in the other equations. For this, do:

Add 1 times the pivot equation to the second equation.

Add -20 times the pivot equation to the fourth equation.

This corresponds to **row operations** on the augmented matrix as indicated in BLUE behind the ***new matrix*** in (3). So the operations are performed on the ***preceding matrix.*** The result is

$$
(3)\qquad
\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 30 & -20 & 80 \end{array}\right]
\quad
\begin{array}{l} \\ \text{Row 2 + Row 1} \\ \\ \text{Row 4} - \text{20 Row 1} \end{array}
\quad
\begin{array}{r} x_1 - x_2 + x_3 = 0 \\ 0 = 0 \\ 10x_2 + 25x_3 = 90 \\ 30x_2 - 20x_3 = 80. \end{array}
$$

A short review of Gauss elimination

$$\begin{array}{r} \\ \text{Pivot 10} \longrightarrow \\ \text{Eliminate 30} \longrightarrow \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & \boxed{30} & -20 & 80 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{rrrrrrr} & & x_1 - & x_2 & + & x_3 & = 0 \\ \text{Pivot 10} \longrightarrow & & & 10x_2 & + & 25x_3 & = 90 \\ \text{Eliminate } 30x_2 \longrightarrow & & & \boxed{30x_2} & - & 20x_3 & = 80 \\ & & & & & 0 & = 0 \end{array}$$

To eliminate x_2, do:

Add -3 times the pivot equation to the third equation.

The result is

$$(4) \qquad \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & -95 & -190 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} \\ \\ \text{Row 3} - 3 \text{ Row 2} \\ \\ \end{array} \quad \begin{array}{rrrl} x_1 - & x_2 + & x_3 = & 0 \\ & 10x_2 + & 25x_3 = & 90 \\ & - & 95x_3 = & -190 \\ & & 0 = & 0 \end{array}$$

Stepwise pivot: three steps

The choice of **pivot equation** and **pivot** are different from the Gauss elimination because $x_1, x_2, s_1, s_2 \geq 0$.
The Augmented Matrix:

$$\left(\begin{array}{ccccc|c} 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 10 \\ 0 & 3 & 2 & 0 & 1 & 18 \end{array}\right)$$

The **three steps** moving to optimal solution:

1. Selection of the Column of the Pivot.
2. Selection of the Row of the Pivot
3. Elimination by Row Operation

Stepwise pivot

1. Selection of the Column to Pivot:
   - Select the first column with **negative** entry in Row 1.

$$\left(\begin{array}{ccccc|c} 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 10 \\ 0 & 3 & 2 & 0 & 1 & 18 \end{array}\right)$$

2. Selection of the Row of the Pivot:
   - Divide the right sides by the corresponding entries of the column which you selected earlier.
   - $(10/1 = 10, \mathbf{18/3 = 6})$
   - Take the pivot equation that gives the **smallest positive quotient**. Thus, the pivot is **3** because $18/3$ is the smallest.

$$\left(\begin{array}{ccccc|c} 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 10 \\ 0 & \boxed{3} & 2 & 0 & 1 & 18 \end{array}\right)$$

Stepwise pivot:

3. Elimination by Row Operations:

- This steps gives zeros above and below pivot.
- Below is the row operation with its details:

$$\left(\begin{array}{rrrrr|r} 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 10 \\ 0 & \boxed{3} & 1 & 0 & 1 & 18 \end{array}\right) \xrightarrow{r_1+\frac{1}{3}r_3} \left(\begin{array}{rrrrr|r} 1 & 0 & \frac{-1}{3} & 0 & \frac{1}{3} & 6 \\ 0 & 1 & 2 & 1 & 0 & 10 \\ 0 & \boxed{3} & 2 & 0 & 1 & 18 \end{array}\right)$$

$$\xrightarrow{r_2-\frac{1}{3}r_3} \left(\begin{array}{rrrrr|r} 1 & 0 & \frac{-1}{3} & 0 & \frac{1}{3} & 6 \\ 0 & 0 & \frac{4}{3} & 1 & \frac{-1}{3} & 4 \\ 0 & \boxed{3} & 1 & 0 & 1 & 18 \end{array}\right)$$

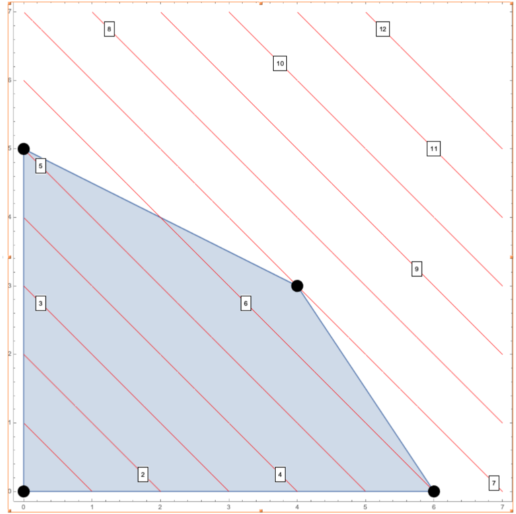
Optimal solution after first iteration

$$\left(\begin{array}{ccccc|c} 1 & 0 & \frac{-1}{3} & 0 & \frac{1}{3} & 6 \\ 0 & 0 & \frac{4}{3} & 1 & \frac{-1}{3} & 4 \\ 0 & 3 & 1 & 0 & 1 & 18 \end{array}\right)$$

The **basic variables** are x_1, x_3 and **nonbasic variables** are x_2, x_4. We set nonbasic variables as zero; $x_2 = 0, x_4 = 0$:

- row 3: $x_1 = 18/3 = 6$
- row 2: $x_3 = 4/1 = 4$
- row 1: $P = f(x_1, x_2) = 6 + 0 = 6$

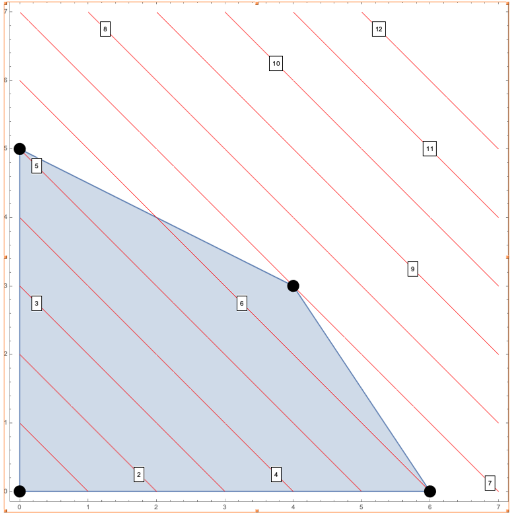
Check your optimal solution



Think...

Can you identify whether we have reached to optimal (maximum) solution?

Check your optimal solution



Think...

Can you identify whether we have reached to optimal (maximum) solution?

When $(x_1 = 6, x_2 = 0)$, $P = 6$. We are yet to reach optimal solution!

Stepwise pivot: second iteration

The basic feasible solution is not yet optimal because there is a negative entry: $\frac{-1}{3}$. We need to repeat again the three steps.

$$\left(\begin{array}{ccccc|c} 1 & 0 & -1/3 & 0 & \frac{1}{3} & 6 \\ 0 & 0 & \frac{4}{3} & 1 & \frac{-1}{3} & 4 \\ 0 & 3 & 1 & 0 & 1 & 18 \end{array}\right)$$

- Steps towards optimal solution:
  1. Column of the pivot: Select the **second column** with negative entry in Row 1.
  2. Row of the Pivot $(4 \div \frac{4}{3} = 3)$ or $(18 \div 1 = 18)$, choose smaller: $\frac{4}{3}$
  3. Next is row operation based on pivot at $\frac{4}{3}$.

$$\left(\begin{array}{ccccc|c} 1 & 0 & -1/3 & 0 & \frac{1}{3} & 6 \\ 0 & 0 & \boxed{4/3} & 1 & \frac{-1}{3} & 4 \\ 0 & 3 & 1 & 0 & 1 & 18 \end{array}\right)$$

Stepwise pivot: second iteration

3. Elimination by Row Operations:
   - This steps gives zeros above and below pivot.
   - Below is the row operation with its details:

$$\left(\begin{array}{ccccc|c} 1 & 0 & -1/3 & 0 & \frac{1}{3} & 6 \\ 0 & 0 & \boxed{4/3} & 1 & \frac{-1}{3} & 4 \\ 0 & 3 & 1 & 0 & 1 & 18 \end{array}\right) \xrightarrow{r_1+\frac{1}{4}r_2} \left(\begin{array}{ccccc|c} 1 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & 7 \\ 0 & 0 & \boxed{4/3} & 1 & \frac{-1}{3} & 4 \\ 0 & 3 & 1 & 0 & 1 & 18 \end{array}\right)$$

$$\xrightarrow{r_3-\frac{3}{2}r_2} \left(\begin{array}{ccccc|c} 1 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & 7 \\ 0 & 0 & \boxed{4/3} & 1 & -\frac{1}{3} & 4 \\ 0 & 3 & 0 & -\frac{3}{2} & -\frac{1}{2} & 12 \end{array}\right)$$

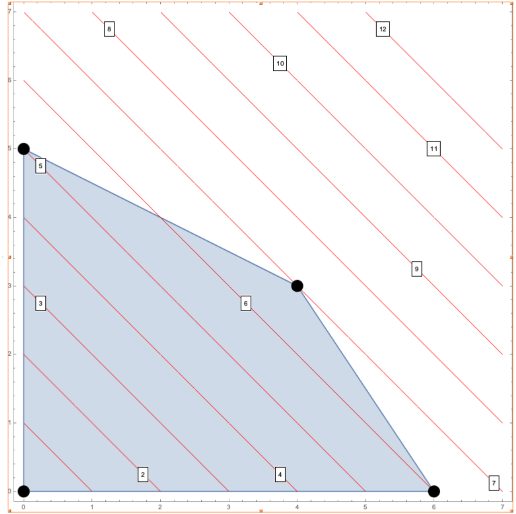
Optimal solution after second iteration

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & 7 \\ 0 & 0 & \frac{4}{3} & 1 & -\frac{1}{3} & 4 \\ 0 & 3 & 0 & -\frac{3}{2} & -\frac{1}{2} & 12 \end{array}\right)$$

The **basic variables** are x_1, x_2 and **nonbasic variables** are x_3, x_4. We set nonbasic variables as zero; $x_3 = 0, x_4 = 0$:

- row 3: $x_1 = 12 \div 3 = 4$
- row 2: $x_2 = 4 \div \frac{4}{3} = 3$
- row 1: $P = x_1 + x_2 = 7$

Check your optimal solution

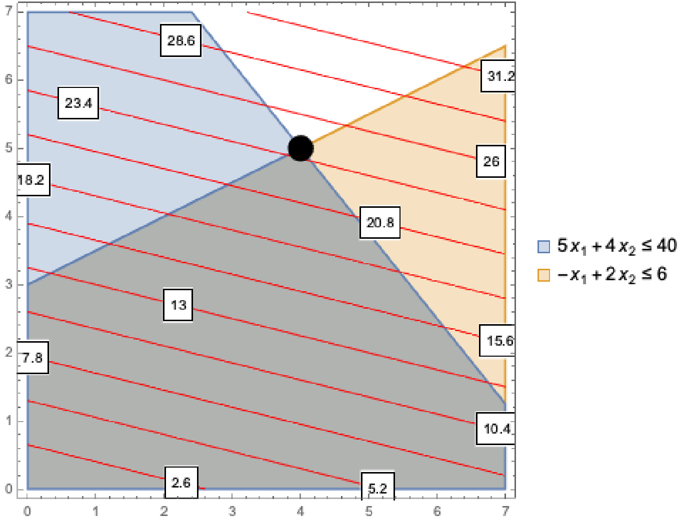


When $(x_1 = 4, x_2 = 3)$, $P = 7$. We reach to the vertex $(4, 3)$, which is indeed an optimal solution !

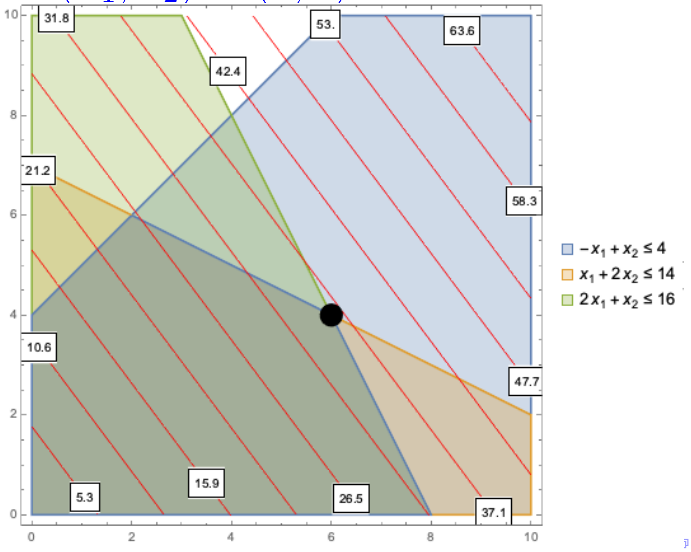
Simplex Method: Minimization

1. If we want to minimize $f(\mathbf{x})$ (instead of maximize), we take as columns of the pivot those entry in Row 1 **positive** (instead of negative).
2. In such a column k, we consider only **positive entries** t_{jk} and take the pivot a t_{jk} for which b_{jk}/t_{jk} is smallest (as before).

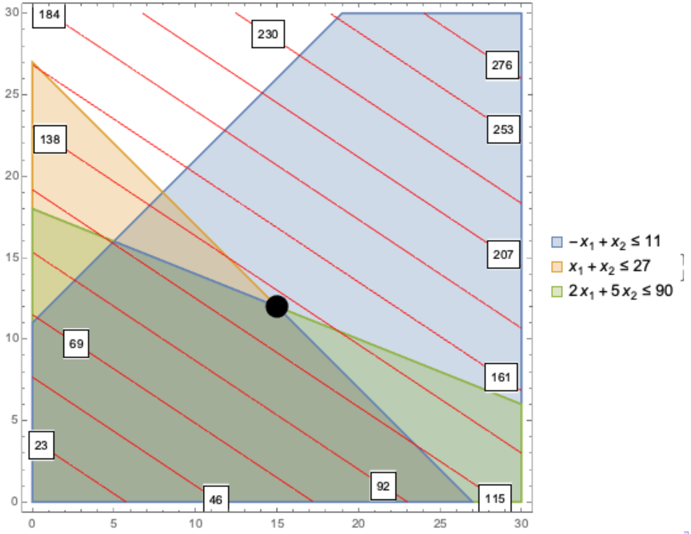
Examples 3.17: $(x_1, x_2) = (4, 5)$



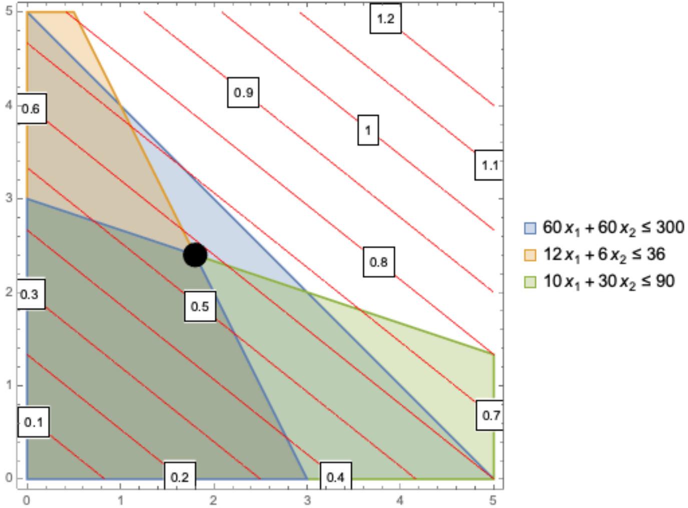
Examples 3.18: $(x_1, x_2) = (6, 4)$



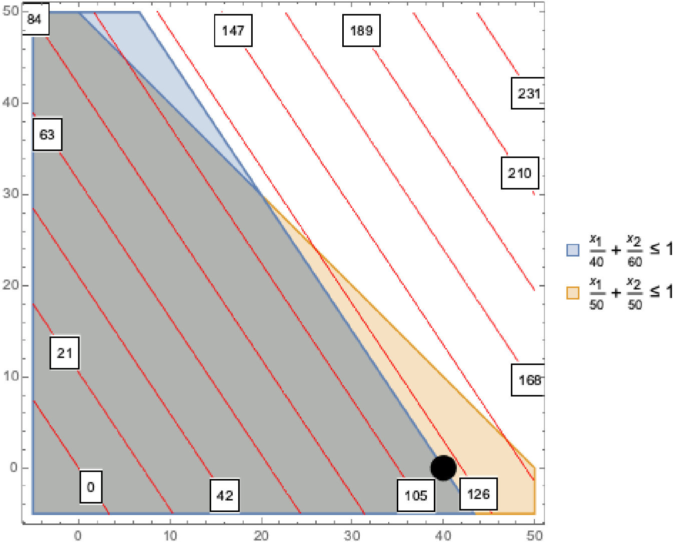
Examples 3.19: $(x_1, x_2) = (15, 12)$



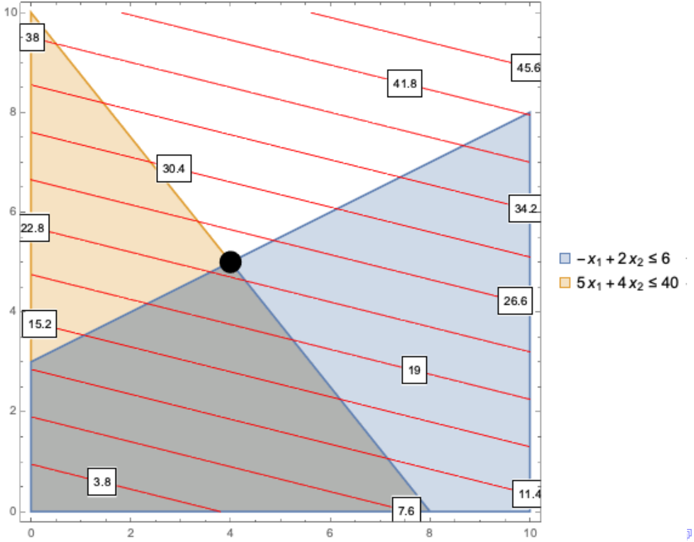
Examples 3.20: $(x_1, x_2) = (1.8, 2.4)$



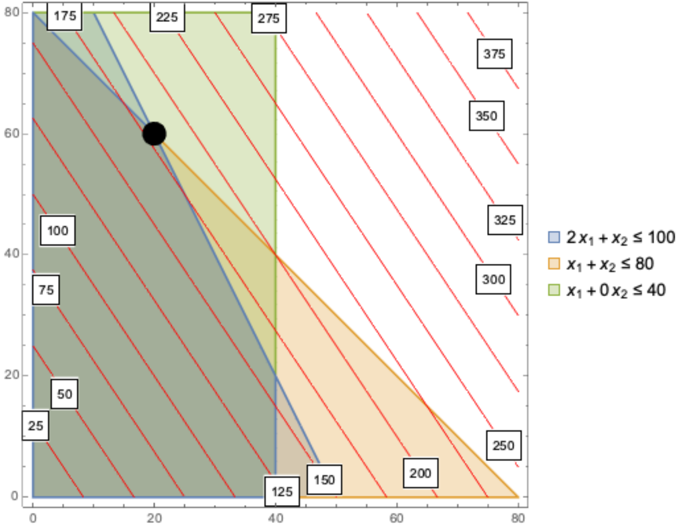
Examples 3.25: $(x_1, x_2) = (20, 30) = (40, 0)$



Examples 3.26: $(x_1, x_2) = (4, 5)$



Examples 3.27: $(x_1, x_2) = (20, 60)$



Summary Simplex method:

Three methods to solve:

- graphs
- trial-error of vertices
- Simplex method (after normal form conversion):
  1. Selection of the Column of the Pivot.
  2. Selection of the Row of the Pivot
  3. Elimination by Row Operation

  Make sure no more negative (for maximizing $f(\mathbf{x})$) entries in the first row to obtain maximum possible solution.

Thanks

Doubts and Suggestions?

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Appendices

Discussion on worked examples coming next!

Artificial variable: constraints with: greater than or equals to

Artificial variable: *We introduce **extra variables** for the conversion to normal form with inequality* $\geq$. For example:

$$
\begin{array}{ll}
\textbf{Maximize} & f(x_1, x_2) = 2x_1 + x_2 \\
\textbf{s.t.} & \\
& \left.\begin{array}{l}
\mathbf{x_1 - \frac{1}{2}x_2 \geq 1}, \\
x_1 - x_2 \leq 2 \\
x_1 + x_2 \leq 4 \\
x_1 \geq 0 \\
x_2 \geq 0
\end{array}\right\} \text{mixed type of inequalities,}
\end{array}
\tag{5}
$$

Artificial variable: constraints with: greater than or equals to

Constraint 1

$$\mathbf{x_1 - \frac{1}{2}x_2 \geq 1},$$

$$\Rightarrow \underbrace{x_1 - \frac{1}{2}x_2 - 1}_{\text{always non-negative}} \geq 0$$

$$\Rightarrow x_3 = x_1 - \frac{1}{2}x_2 - 1,$$

$$\therefore\ x_1 - \frac{1}{2}x_2 - x_3 = 1.$$

We cannot proceed solving the row operation as $x_3 < 0$ might not be in solution region. So we have to introduce extra variable:

$$x_3 = x_1 - \frac{1}{2}x_2 - 1 + \mathbf{x_6}$$

x_6 is called an artificial variable and is subject to the constraint $x_6 \geq 0$. Thus, need to modify $f(\mathbf{x})$!

Artificial variable: constraints with: greater than or equals to

with artificial variable:

$$x_3 = x_1 - \frac{1}{2}x_2 - 1 + \mathbf{x_6}$$
$$\Rightarrow x_6 = x_3 - x_1 + \frac{1}{2}x_2 + 1$$
$$\Rightarrow Mx_6 = x_3 - M(x_1 + \frac{1}{2}x_2 + 1)$$

where $M \in \Re$. Next, need to remove the artifical variable from objective function: $z = f(x_1, x_2) = 2x_1 + x_2$.

Modified objective function

$$\begin{aligned} \hat{z} &= 2x_1 + x_2 - Mx_6 \\ \therefore \hat{z} &= (2 + M)x_1 + (1 - \frac{1}{2}M)x_2 \\ & \quad -Mx_3 - M \end{aligned}$$

Next, we can represent simplex table and continue with three steps of Simplex method.