Probabilities

Engineering Mathematics 3

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Statistics & Probabilities

Statistics *is the scientific application of mathematical principles to the* ***collection****, analysis, and presentation of numerical data.*

Probabilities *is the branch of mathematics concerning numerical descriptions of how likely an* ***event*** *is to occur, or how likely it is that a proposition is true. The* ***probability of an event*** *is a number between 0 and 1, where, 0 indicates impossibility of the event and 1 indicates certainty.*

Introduction: Probability

Probability theory provides mathematical models of situations affected by ***chance effects***. For example:

- whether forecasting,
- life insurance,
- games of chance with cards or dice.

The accuracy of these models can be tested by suitable observations or experiments - this is the main purpose of **Statistics**.

Set Notation

Symbol	Remark
$\{\dots\}$	a collection of elements forms a set
$(\dots)$	n-tuples:order matters!
$a \in S$	a is an element of the set S
$A = \{x \in A \mid x \in \text{even numbers }\}$	$x \in A$ **such that** x is an even number
$\emptyset$	the empty set
$\cap$	intersection
$\cup$	union
$\overline{A}$	complement of A
$A^c = A'$	also the complement of A
$\lvert S \rvert = n(S)$	the size (number of elements) of S
$A \subseteq S$	A is a subset of (possibly equal to) S
$A \subsetneq S$	A is a strict subset (not equal to) S

Set Operations and Venn Diagrams

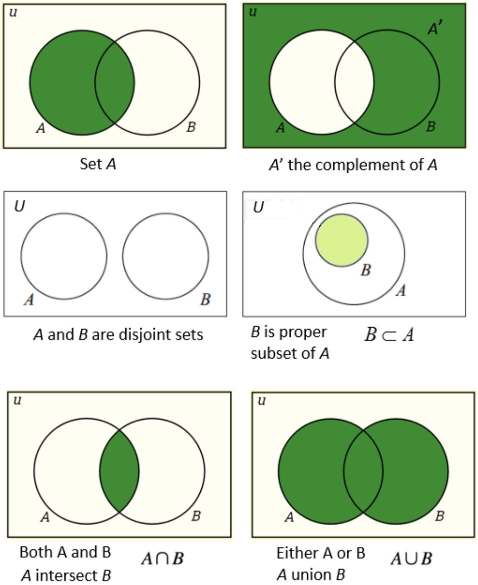


Figure: Experiment: Dice rolling

Remark 1:

$A \cap B = \emptyset$ means it is **disjoint**, also known as **mutually exclusive**

Preliminaries: Probability

$\boxed{\textit{Experiment}}$ is used to refer to any process whose outcome is not known in advance. Consider an experiment.

- Sample space $\mathcal{S}$: A collection of all possible outcomes.
- Sample point $x \in \mathcal{S}$: An element in $\mathcal{S}$.

$\boxed{\textit{Event } A}$: A subset of sample points, $A \subset \mathcal{S}$ for which a statement about an outcome is true.

Experiments & Events

Example (1)

A dice is rolled. Identify the following **events** :

1. Sample S and $|S| = n(S)$
2. $A = \{x \in A \mid x \text{ is even number } \}$
3. $B = \{x \in B \mid x \text{ is odd number}\}$
4. $C = \{x \in C \mid x \text{ can be divided by } 5\}$
5. $D = \{x \in D \mid x \text{ can be divided by } 7\}$

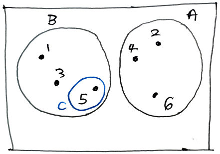


Figure: Venn Diagram: Dice rolling

Solution 1

1. $S = \{1, 2, 3, 4, 5, 6\}$, $n(S) = 6$.
2. $A = \{2, 4, 6\}$, $n(A) = 3$.
3. $B = \{1, 3, 5\}$, $n(B) = 3$.
4. $C = \{5\}$, $n(C) = 1$.
5. $D = \emptyset$, $n(D) = 0$.

Combination of events

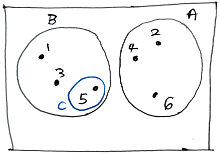


Figure: Venn Diagram: Dice rolling

Example (2)

1. A or $B = A \cup B =?$, $n(A \cup B) =?$
2. B and $C = B \cap C =?$, $n(B \cap C) =?$
3. A and $C = A \cap C =?$, $n(A \cap C) =?$
4. $C' =?$, $n(C') =?$

Solution 2:

$$\begin{aligned} A \text{ or } B &= A \cup B \\ &= \{2,4,6,1,3,5\} \\ &= S, \end{aligned}$$

$n(A \cup B) = 6.$

2. B and $C = B \cap C = \{5\}$,
   $n(B \cap C) = 1.$
3. A and $C = A \cap C = \{\} = \emptyset$,
   $n(A \cap C) = 0.$ **disjoint**
4. $C' = \{1,2,3,4,6\}$, $n(C') = 5.$

Introduction: Probability

One Event

If the sample space of S of an experiment consists of finitely many outcome that are equally likely, then the probability $P(A)$ of an event A is:

$$P(A) = \frac{\text{Nu. of elements in A}}{\text{Nu. of elements in S}} = \underbrace{\frac{n(A)}{n(S)}}_{\text{nu. of elements}} \tag{1}$$

Two Events

$$\text{Probability of } \mathbf{A \text{ or } B} \text{ happening: } P(A \cup B) = \frac{n(A \cup B)}{n(S)} \tag{2}$$

$$\text{Probability of } \mathbf{A \text{ and } B} \text{ happening: } P(A \cap B) = \frac{n(A \cap B)}{n(S)} \tag{3}$$

Dice Rolling : $S = \{1, 2, 3, 4, 5, 6\},\ n(S) = 6.$

Example (3: Even Numbers)

The probability of getting even number A:

$$\begin{aligned} A &= \{2, 4, 6\},\ n(A) = 3 \\ P(A) &= \frac{3}{6} \end{aligned}$$

Example (4 Two events $(B \cap C)$)

$B = \{1, 3, 5\}$, $C = \{5\}$. The event **B and C** $= \{5\}$, $n(B \cap C) = 1$. So The probability is

$$P(B \cap C) = \frac{1}{6}$$

Example (5: Two events ($A \cup B$))

The the probability of getting **A or B**:

$$\begin{aligned} A \cup B &= \{2, 4, 6, 1, 3, 5\} \\ &= S, \\ P(A \cup B) &= P(S) \\ &= \frac{6}{6} = 1. \end{aligned}$$

General Rules for Probability:

For any event A,

$$0 \le \mathsf{P}(A) \le 1.$$

Statements which are always false have **probability zero**, similarly, always-true statements have **probability one**. Some remarks:

1. In percentage: $\mathsf{P}(A) \times 100\%$
2. $P(\emptyset) = 0$: Always false(**empty set** has no element)
3. $P(S) = 1$: Always-true (**Any element of S** happening)
4. $P(A') = 1 - P(A)$: The probability of event A **not happening**; also known as **complementation rule.**

Introduction: Expectation

Expectation

The expectation E of an event A happening is defined as the product of the probability $p = P(A)$ of the event happening based on the number of attempts made n:

$$E = p \times n \tag{4}$$

Example (Expecting to get number 3)

Find the expectation of getting number 3 in five throws of a dice. Let the event of getting 3 is $A = \{3\}$.

$$\begin{aligned} P(A) &= \frac{1}{6},\ n = 3 \\ \therefore E &= \frac{1}{6} \times 5 = \frac{5}{6} \Rightarrow 83.3\% \text{ chance.} \end{aligned}$$

Introduction: Expectation

Example (Expecting to get number 3)

Find the expectation of getting number 3 in five throws of a dice. Let the event of getting 3 is $A = \{3\}$.

$$P(A) \ = \ \frac{1}{6}, \ n = 5, \quad \Rightarrow E = \frac{5}{6}$$

Think...

How many times do i need to roll (in average) the dice if i want to get 3 at least once?

Introduction: Expectation

Example (Expecting to get number 3)

Find the expectation of getting number 3 in five throws of a dice. Let the event of getting 3 is $A = \{3\}$.

$$P(A) = \frac{1}{6}, \; n = 5, \quad \Rightarrow E = \frac{5}{6}$$

Think...

How many times do i need to roll (in average) the dice if i want to get 3 at least once?

$$E = 1, \quad p = \frac{1}{6}, \qquad \therefore n = \frac{E}{p} = 6$$

Addition Rule: Either A or B

Addition Rule (arbitrary event)

$$\mathsf{P}(A \cup B) = \underbrace{\mathsf{P}(A) + \mathsf{P}(B) - \mathsf{P}(A \cap B).}_{\text{probability values , not nu. of elements}} \tag{5}$$

Addition Rule (mutually exclusive)

$$\begin{aligned} \text{mutually exclusive: } A \cup B &= \emptyset \\ \mathsf{P}(A \cup B) &= \mathsf{P}(A) + \mathsf{P}(B) - \cancel{\mathsf{P}(A \cap B)}. \\ \mathsf{P}(A \cup B) &= \mathsf{P}(A) + \mathsf{P}(B). \end{aligned} \tag{6}$$

Examples: Addition Rule

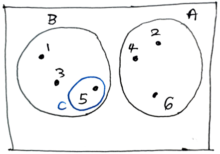


Figure: Dice rolling, A and B **mutually exclusive?**

Events and their elements

1. $S = \{1, 2, 3, 4, 5, 6\}$, $n(S) = 6$.
2. $A = \{2, 4, 6\}$, $n(A) = 3$.
3. $B = \{1, 3, 5\}$, $n(B) = 3$.
4. $C = \{5\}$, $n(C) = 1$.
5. $D = \emptyset$, $n(D) = 0$.

Example (6: mutually exclusive)

The the probability of getting **A or B** (mutually exclusive):

$$
\begin{aligned}
P(A) &= \frac{3}{6},\ P(B) = \frac{3}{6} \\
P(A \cap B) &= 0 \\
\therefore P(A \cup B) &= P(A) + P(B) \\
&= \frac{6}{6} = 1 = S.
\end{aligned}
$$

Example (7: arbitrary case)

The the probability of getting **B and C** (joint):

$$
\begin{aligned}
P(B) &= \frac{3}{6},\ P(C) = \frac{1}{6} \\
P(B \cap C) &= \frac{1}{6} \\
\therefore P(B \cup C) &= P(B) + \cancel{P(C) - P(B \cap C)} \\
&= \frac{3}{6}
\end{aligned}
$$

Joint set: Two events(See appendices for proof)

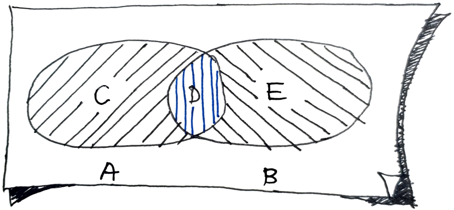


Figure: Joint set: $A \cap B = D$

$$\begin{aligned} \mathsf{P}(A \cup B) &= \mathsf{P}(A) + \mathsf{P}(B) - \mathsf{P}(A \cap B). \\ &= \mathsf{P}(C) + \mathsf{P}(E) + \mathsf{P}(D). \end{aligned}$$

Joint set: Three events (See appendices for proof)

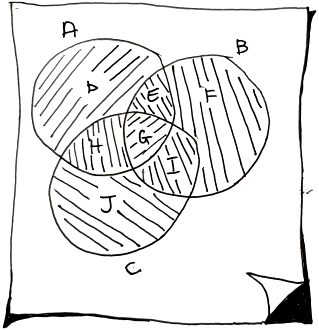


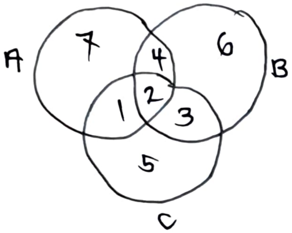
Figure: Joint set: $A \cap B \cap C = G$

$$
\begin{aligned}
\mathsf{P}(A \cup B \cup C) = \; & \mathsf{P}(A) + \mathsf{P}(B) + \mathsf{P}(C) \\
& + \mathsf{P}(A \cap B \cap C) \\
& - \mathsf{P}(A \cap B) \\
& - \mathsf{P}(A \cap C) \\
& - \mathsf{P}(B \cap C).
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{P}(A \cup B \cup C) = \; & \mathsf{P}(D) + \mathsf{P}(E) + \mathsf{P}(F) \\
& + \mathsf{P}(G) + \mathsf{P}(H) + \mathsf{P}(I) \\
& \mathsf{P}(J).
\end{aligned}
$$

3 Joint set: Example

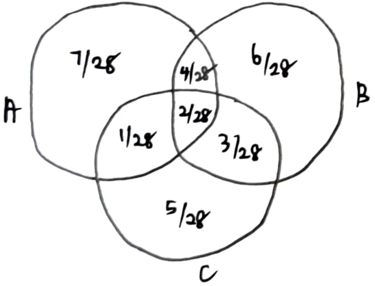
Numerical example



*Based on number of elements.

- $n(S) = 1+2+3+4+5+6+7 = 28$
  $= n(A \cup B \cup C)$ — (1)
- $n(A) = 7+4+2+1$
  $= 14$
- $n(B) = 6+4+3+2$
  $= 15$
- $n(C) = 5+3+2+1$
  $= 11.$
- $n(A \cap B) = 4+2 = 6$
- $n(A \cap C) = 1+2 = 3$
- $n(B \cap C) = 2+3 = 5$

3 Joint set: Example



* in probability.

$P(A) = 14/28$

$P(B) = 15/28$

$P(C) = 11/28$

$P(A \cap B) = 6/28$

$P(A \cap C) = 3/28$

$P(B \cap C) = 5/28$

$P(A \cap B \cap C) = 2/28$

From ①:

$$P(A \cup B \cup C) = \frac{n(A \cup B \cup C)}{n(S)}$$

$$= \frac{28}{28}$$

$$P(A \cup B \cup C) = 1 \quad \text{--- ②}$$

3 Joint set: Example

From derived equation of 3 events :

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) + P(A \cap B \cap C) - P(A \cap B) - P(A \cap C) - P(B \cap C)$$

$$= 14/28 + 15/28 + 11/28 + 2/28 - \frac{6}{28} - 3/28 - 5/28$$

$$= 42/28 - \frac{14}{28}$$

$$= \frac{28}{28}$$

$$= 1$$ # same as shown in 2

Product Rule: A and B

General Product Rule: independent events

If A and B in a sample space S and $P(A) \neq 0$, $P(B) \neq 0$, then

$$\mathsf{P}(A \cap B) = \underbrace{\mathsf{P}(A) \times \mathsf{P}(B).}_{\text{P reduces}} \tag{7}$$

Example (Independent Event)

Rolling two dices: The probability of getting odd numbers: $D_1 = \{1, 3, 5\}$ **and** number 5: $D_2 = \{5\}$. Note: $S = \{(\_, \_), \ldots, (\_, \_)\}, D_1 \cap D_2 = \{(1, 5), (3, 5), (5, 5)\}$.

$$\begin{aligned} P(D_1 \cap D_2) &= P(D_1) \times P(D_2) \\ &= \frac{3}{6} \times \frac{1}{6} = \frac{3}{36} \end{aligned}$$

Extension of Product Rule

Conditional Probability

Find the probability of an event happening under the **condition** another event occurs:

$$\text{Probability of } \mathbf{B} \textbf{ given } \mathbf{A}\text{: } P(B \mid A) = \frac{P(A \cap B)}{P(A)} \tag{8}$$

$$\text{Probability of } \mathbf{A} \textbf{ given } \mathbf{B}\text{: } P(A \mid B) = \frac{P(A \cap B)}{P(B)} \tag{9}$$

Remark

$P(A \cap B) = P(A)P(B \mid A) = P(B)P(A \mid B)$

Probability computation based on event types

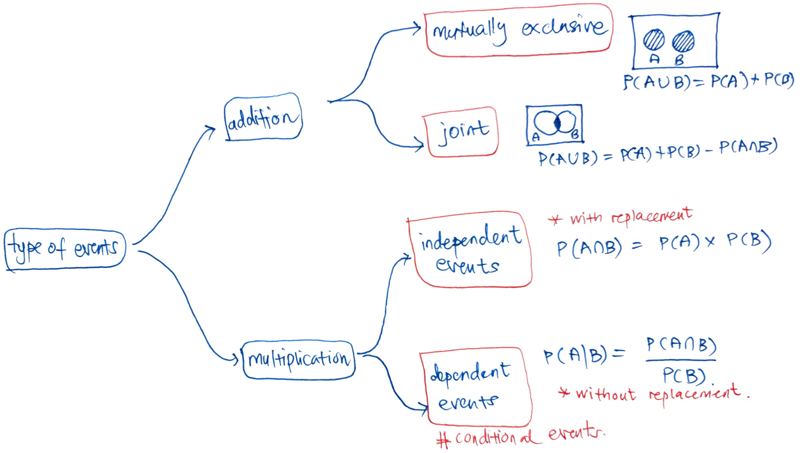


Figure: Probability of more than one event

Two types of sampling

Sampling from a population: We randomly draw objects, one at at time, from a given set of objects. There are two types of sampling:

1. **Sampling with replacement** : The object drawn is **replaced back** in the given set, Then, we draw the next object randomly again.
2. **Sampling without replacement**: the object that was drawn is **put aside**, not in the given set.

Independent events are in the category of "Sampling with replacement"; these events are not effected by other or previous events.

Examples: Sampling

Two types of sampling

A box contains 10 screws, 3 of which are defective. Two screws are drawn at random one by one. Let $P(A)$ denoted as the first draw and $P(B)$ be the second draw.

$\Rightarrow$ We sample at random, so each screw has the same probability, $\frac{1}{10}$.

Find the probability of getting nondefective screws are two draws with the following sampling:

1. with replacement.
2. without replacement.

Example (with replacement)

$\Rightarrow$ There are seven out of ten screws nondefective, so the probability of getting nondefective for the first draw is $P(A) = \frac{7}{10}$. The situation is similar for the second draw: $P(B) = \frac{7}{10}$. Hence the event is **independent:**

$$\begin{aligned} \therefore P(A \cap B) &= P(A) \times P(B) \\ &= 0.7 \times 0.7 = 0.49 = 49\% \end{aligned}$$

Example (without replacement)

After the first draw, we are left with 9 screws in the box, 3 are defective screws. Thus $P(B \mid A) = \frac{6}{9}$.

$$\begin{aligned} \therefore P(A \cap B) &= P(A)P(B \mid A) \\ &= \frac{7}{10} \times \frac{6}{9} = 46.7\% \end{aligned}$$

Tree Diagram Approach

Example (without replacement)

After the first draw, we are left with 9 screws in the box, 3 are defective screws. Thus $P(B \mid A) = \frac{6}{9}$.

$$\begin{aligned} \therefore P(A \cap B) &= P(A)P(B \mid A) \\ &= \frac{7}{10} \times \frac{6}{9} = 46.7\% \end{aligned}$$

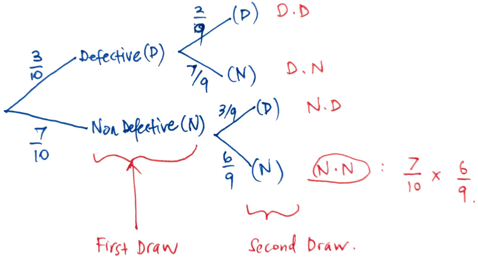


Figure: Tree Diagram: without replacement

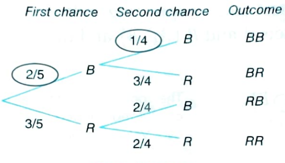
Think...

Can you identify the probability of identifying two defective screws?

There are two blue and three red stones in a box. Without replacement, what are the chances of getting two blue stones?

Solution

Using the following tree diagram, there is a $\frac{2}{5}$ chance followed by a $\frac{1}{4}$ cha



Tree diagram

So, the result for the chances of getting two blue stones:

$$\frac{2}{5}\times\frac{1}{4}=\frac{1}{10}$$

Figure: Example 1.25

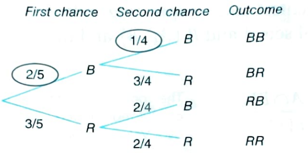
Think...

Can you identify the probability of getting only one blue stone?

There are two blue and three red stones in a box. Without replacement, what are the chances of getting two blue stones?

Solution

Using the following tree diagram, there is a $\frac{2}{5}$ chance followed by a $\frac{1}{4}$ cha



Tree diagram

So, the result for the chances of getting two blue stones:

$$\frac{2}{5} \times \frac{1}{4} = \frac{1}{10}$$

Figure: Example 1.25

Think...

Can you identify the probability of getting only one blue stone?

$$\begin{aligned} \therefore P(BR \cup RB) &= P(BR) + P(RB) \\ &= (\frac{2}{5} \times \frac{3}{4}) + (\frac{3}{5} \times \frac{2}{4}) \\ &= \frac{3}{5} = 60\% \end{aligned}$$

Probability computation based on event types

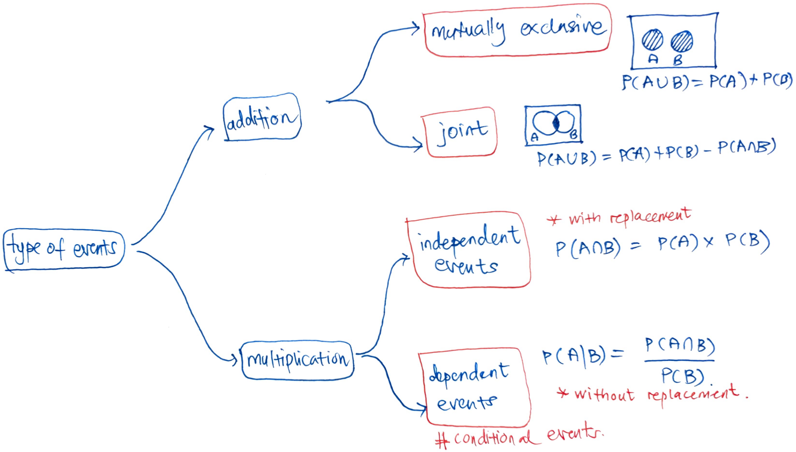


Figure: Probability of more than one event

Conclusion

1. Computation of probability of one event: $p = P(A)$.
2. Expectation: $E = p \times n$.
3. Computation of probability of more than one event: $\bigcap$ and $\bigcup$

Thanks

Doubts and Suggestions?

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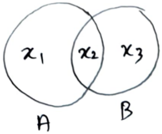
Proof 2 Joint set



joint :

$$\boxed{P(A\cup B) = P(A) + P(B) - P(A\cap B)}$$

$$P(A) = x_1 + x_2$$

$$P(A\cap B) = x_2 .$$

$$\Rightarrow \quad x_1+x_2+x_3 = (x_1+\cancel{x_2}) - (\cancel{x_2}) + P(B)$$

$$P(B) + x_1 = x_1 + x_2 + x_3$$

$$P(B) = x_2 + x_3 \quad \#.$$

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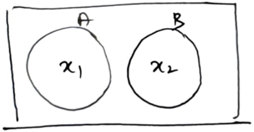
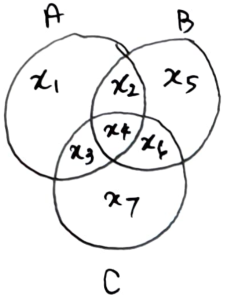
disjoint = mutual exclusive

$$P(A) + P(B) = P(A\cup B).$$



Proof 3 Joint set



$$P(A) = x_1 + x_2 + x_3 + x_4$$
$$P(B) = x_2 + x_4 + x_5 + x_6$$
$$P(C) = x_3 + x_4 + x_6 + x_7$$

$$\cancel{P(A \cup B) = x_2 + x_4}$$

$$\cancel{P(A \cap C) = x_3 + ?}$$

$$P(A \cap B) = x_2 + x_4$$
$$P(A \cap C) = x_3 + x_4$$
$$P(B \cap C) = x_6 + x_4$$

Figure: Joint set: $A \cap B = D$

Proof 3 Joint set

		x_1	x_2	x_3	x_4	x_5	x_6	x_7
$P(A)$	$+$	1	1	1	1			
$P(B)$	$+$		1		1	1	1	
$P(C)$	$+$			1	1		1	1
$P(A \cap B)$	$-$		-1		-1			
$P(A \cap C)$	$-$			-1	-1			
$P(B \cap C)$	$-$				-1		-1	
Total		1	1	1	0	1	1	1

$P(A \cap B \cap C)$

Proof 3 Joint set

$$\therefore P(A \cup B \cup C) = P(A) + P(B) + P(C) + P(A \cap B \cap C)$$
$$- [P(A \cap B) + P(A \cap C) + P(B \cap C)]$$