De Broglie Wave: Phase velocity (v_{Phase} or v_p)

A point marked on a wave can be regarded as representing a particular phase for the wave at that point. The velocity with which such a point would propagate is known as phase velocity (or) wave velocity.
It is represented by

$$\mathrm{v}_{phase} \quad \text{or} \quad \mathrm{v}_p = \frac{\omega}{k}$$

where, ω is angular frequency and k is the propagation constant or wave number.

Group Velocity (v_{group} or v_g)
The velocity with which the resultant envelops of the group of waves travels is called group velocity. It is denoted by v_{group} or v_g and is equal to the particle velocity v.

$$\mathrm{v}_{group} \quad or \quad \mathrm{v}_g = \frac{d\omega}{dk}$$

Relation between Group Velocity v_g and Phase Velocity v_p

$$v_g = v_p - \lambda \frac{dV_p}{d\lambda} \qquad or \qquad v_{group} = v_{phase} - \lambda\left(\frac{dv_{phase}}{d\lambda}\right)$$

Relation between Group Velocity v_g and Particle Velocity v

$$v_g = v \qquad or \qquad v_{group} = v_{particle}$$

Relation between group velocity v_g, phase velocity v_p & velocity of light c

$$v_p \text{ X } v_{group} = c^2 \qquad or \qquad v_{phase} \text{ X } v_{group} = c^2$$

Heisenberg's uncertainty principle

Statement: "it is impossible to determine simultaneously both position and momentum of a moving particle accurately at same time. The product of uncertainty in these quantities is always greater than or equal to h/4π". If Δx and ΔPx are the uncertainties in the measurement of position and momentum of a particle, then

$$\Delta x \,.\, \Delta p_x \geq \frac{h}{4\pi}$$

If Δx is small, ΔPx will be large and vice versa. That is if one quantity is measured accurately, the other quantity becomes less accurate. Similarly the other uncertainty relations for other physical variables pair are,

$$\Delta E . \Delta t \geq \frac{h}{4\pi}$$

$$\Delta L . \Delta \theta \geq \frac{h}{4\pi}$$

The group velocity of de-Broglie waves associated with a moving particle is

$$G = \frac{d\omega}{dk} = \frac{d\omega}{2\pi d\,(1/\lambda)} = -\frac{\lambda^2}{2\pi}\frac{d\omega}{d\lambda}. \quad ...(1)$$

As $$\lambda = \frac{h}{p}$$

$\therefore$ $$d\lambda = -h\frac{dp}{p^2}. \quad ..(2)$$

From (1) and (2), $$G = -\frac{h^2}{2\pi p^2}\frac{d\omega}{\left(-h\frac{dp}{p^2}\right)} = -\frac{h}{2\pi}\frac{d\omega}{d\rho} = -\hbar\frac{d\omega}{dp}$$

or Equivalently $$G = -\hbar\frac{\Delta\omega}{\Delta p} \Rightarrow G = \hbar\frac{\Delta\omega}{\Delta p} \text{ (numerically).} \quad ...(3)$$

As the group of waves represents a particle moving with velocity v (say along X-axis), then

$$v = \frac{\Delta x}{\Delta t} \quad ...(4)$$

As $$v = G \quad \therefore \quad \frac{\Delta x}{\Delta t} = \hbar\frac{\Delta\omega}{\Delta p}$$

This gives $$\Delta x\, \Delta p = \hbar\, \Delta\omega\, \Delta t.$$

If the angular frequency of the wave is to be measured, the least time of measurement will be time required for one complete wavelength to pass a reference point. If Δt is this time, then

$$\Delta t \geq \frac{1}{\Delta \omega}, \quad i.e., \quad \Delta t\, \Delta \omega \geq 1.$$

Using this, equation (5) gives $\Delta x\, \Delta p \geq \hbar$. ...(6)

This is *uncertainty principle.*

Applications of Uncertainty Principle:

Non-existence of electrons in the nucleus and its implications- non-relativistic approach

(i) The non-existence of the electrons in the nucleus : The radius of the nucleus of any atom is of the order of 10^{-14} m, so that if an electron is confined within nucleus, the uncertainty in its position must not be greater than 10^{-14} m.

According to uncertainty principle $\Delta q\, \Delta p \approx \hbar$. ...(1)

where Δq is the uncertainty in the position and Δp is the uncertainty in the momentum and

$$\hbar = \frac{h}{2\pi} = \frac{6{\cdot}625 \times 10^{-34}}{2 \times 3{\cdot}14} = 1{\cdot}055 \times 10^{-34} \text{ joule sec.}$$

Radius of nucleus $= 10^{-14}$ m.

$\therefore$

$$(\Delta q)_{\max} = 2 \times 10^{-14} \text{ m.}$$

Equation (1) gives $\Delta p = \dfrac{\hbar}{\Delta q}$

$$= \frac{1{\cdot}055 \times 10^{-34}}{2 \times 10^{-14}} = 5{\cdot}275 \times 10^{-21} \text{ kg-m/s.}$$

It this is the uncertainty in momentum of the electron, the momentum of the electron must be at least-comparable with its magnitude,

i.e., $$p \approx 5{\cdot}275 \times 10^{-21} \text{ kg-m/s}$$

This kinetic energy of the electron of mass m is given by

$$T = \frac{p^2}{2m}$$

(since $m = 9 \times 10^{-31}$ kg.)

$$\approx \frac{(5{\cdot}275 \times 10^{-21})^2}{2 \times 9 \times 10^{-31}} \text{ joule}$$

$$\approx \frac{(5{\cdot}275 \times 10^{-21})^2}{2 \times 9 \times 10^{-31} \times 1{\cdot}6 \times 10^{-19}} \text{ eV} = 9{\cdot}7 \times 10^7 \text{ eV} \approx 97 \text{ MeV.}$$

This means that if the electrons exist inside the nucleus, their kinetic energy must be of the order of 97 MeV. But experimental observations show that no electron in the atom posseses energy greater than 4 MeV. Clearly the inference is that the *electrons do not exist in the nucleus.*

If Δq and Δp are the uncertainties in determining *in the nucleus.*

(ii) Radius of Bohr's first orbit : If Δq and Δp are the uncertainties in determining the position and momentum of electron in first orbit, then $\Delta q\, \Delta p \approx \hbar$

or
$$\Delta p = \frac{\hbar}{\Delta p}.$$

The uncertainty in kinetic energy may calculated as follows :

kinetic energy,
$$T = \frac{p^2}{2m}.$$

$\therefore$ Uncertainty in kinetic energy of electron.

$\therefore$
$$\Delta T = \frac{(\Delta p)^2}{2m} = \frac{1}{2m}\left(\frac{\hbar}{\Delta q}\right)^2 = \frac{\hbar^2}{2m\,(\Delta q)^2}.$$

The potential energy of electron $V = \frac{1}{4\pi\varepsilon_0}\frac{(Ze)(-e)}{q}.$

The uncertainty in potential energy of same electron

$$\Delta V = +\frac{1}{4\pi\varepsilon_0}\frac{(Ze)(-e)}{\Delta q} = -\frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{\Delta q}.$$

$\therefore$ Uncertainty in total energy $\Delta E = \Delta T + \Delta V = \frac{\hbar^2}{2m\,(\Delta q)^2} - \frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{\Delta q}.$

Minimum energy
$$E_{\min} = \Delta E = \frac{\hbar^2}{2m\,(\Delta q)^2} - \frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{\Delta q}.$$

The uncertainty in energy will be minimum if $\frac{d\,(E_{\min})}{d\,(\Delta q)} = 0$

i.e.,

$$-\frac{\hbar^2}{m\,(\Delta q)^3} + \frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{(\Delta q)^2} = 0$$

This gives

$$\Delta q = 4\pi\varepsilon_0 \frac{\hbar^2}{mZe^2}.$$

Radius of first Bohr orbit is $r = \Delta q = 4\pi\varepsilon_0 \frac{\hbar^2}{mZe^2} = \frac{\varepsilon_0 h^2}{\pi\, mZe^2}.$

This is just the radius of Bohr's **I orbit.**

This is just the radius of Bohr's I orbit.

(iii) Minimum energy of a harmonic oscillator : Let a particle of mass m execute simple harmonic motion along x-axis. The uncertainty in position of particle = Δx.

According to uncertainty principle, $\Delta x\, \Delta p = \frac{\hbar}{2}$, therefore uncertainty in momentum

$$\Delta p = \frac{\hbar}{2\Delta x}.$$

Then the total energy of the system is of the order of

(E = Kinetic energy + Potential energy)

$$E = \frac{(\Delta p)^2}{2m} + \frac{1}{2}k\,(\Delta x)^2 = \left(\frac{\hbar}{2\Delta x}\right)^2 \cdot \frac{1}{2m} + \frac{1}{2}k\,(\Delta x)^2 = \frac{\hbar^2}{8m\,(\Delta x)^2} + \frac{1}{2}\,k\,(\Delta x)^2. \quad ...(1)$$

Minimising this energy with respect to Δx, *i.e.*, $\frac{\partial E}{\partial(\Delta x)} = 0$, we get

$$-\frac{\hbar^2}{4m\,(\Delta x)^3} + k\,\Delta x = 0 \quad \text{or} \quad \Delta x = \left(\frac{\hbar^2}{4mk}\right)^{1/4}. \quad ...(2)$$

Substituting this value in (1), we get

$$E_{\text{min}} = \frac{\hbar^2}{8m}\left(\frac{4mk}{\hbar^2}\right)^{1/2} + \frac{1}{2}k \cdot \left(\frac{\hbar^2}{4mk}\right)^{1/2} = \frac{\hbar^2}{2}\left(\frac{k}{m}\right)^{1/2}$$

But $\sqrt{\frac{k}{m}} = \omega$ = angular frequency.

$\therefore$ Minimum energy of harmonic oscillator $= \frac{1}{2}\hbar\,\omega$.

$$\ldots = \frac{1}{2}\hbar\,\omega.$$

(iv) **Energy of a particle in one-dimensional box :** Consider a particle of mass m in one dimensional box of length l. The maximum uncertainty in the position of the particle will be

$$(\Delta x)_{max} = l.$$

Therefore from uncertainty relation $\Delta x\, \Delta p \approx \hbar \Rightarrow \Delta p = \frac{\hbar}{\Delta x}$.

As uncertainty in momentum must be less than momentum itself, therefore minimum momentum of particle $= \frac{\hbar}{l}$.

Therefore kinetic energy of particle $T = \frac{p^2}{2m} = \frac{\hbar^2}{2ml^2}$.

This is minimum kinetic energy of a particle in a box.