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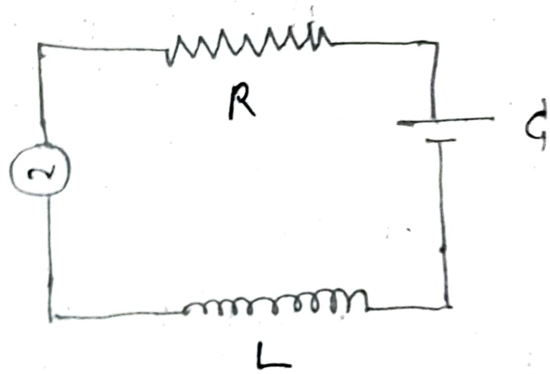
Model: IV - LCR Model -

* **Problem statement**

"Consider a series circuit consisting of an EMF source E, a resistor R, a capacitor C and a inductor L, Formulate a suitable differential Equation model and analyze it."



* **Formulation of model:**

Step-I (Identification of Variable)

Here the dependent variable is either the current i or the charge q and the independent variable is the time t.

Step-II (Assumption)

We assume that

(1) The characteristic parameters of the resistor, capacitor and inductor are constant.

(iii) The flow of current in the closed circuit is given by the Kirchoff's voltage law

$$E_R + E_C + E_L = E \quad \text{——— (1)}$$

Step-III As we know that $E_R = R \cdot i$, $E_L = L \cdot \frac{di}{dt}$, $E_C = q/C$ and $i = \frac{dq}{dt}$

we get;

$$E_R + E_C + E_L = E$$

$$\Rightarrow R \cdot i + q/C + L \cdot \frac{di}{dt} = E \quad \text{——— (2)}$$

$$\Rightarrow L \frac{d^2q}{dt^2} + R \cdot \frac{dq}{dt} + q/C = E \quad \text{——— (3)}$$

$$\left[\because i = \frac{dq}{dt} \right]$$

Again from eq. (2) we get,

$$R \cdot \frac{di}{dt} + \frac{1}{C} \frac{dq}{dt} + L \frac{d^2i}{dt^2} = E' \quad \text{——— (4)}$$

$$\Rightarrow L \frac{d^2i}{dt^2} + R \frac{di}{dt} + \frac{1}{C} i = E' \quad \text{——— (5)}$$

$$\left[i = \frac{dq}{dt} \right]$$

Eq. (3) and (6) are second order differential equation with single dependent variable represent differential equation model for LCR network

* **Analysis of Mathematical model:**

We analyze the LCR model in two steps. In Step-I we obtain the mathematical solution of the

Model and step-II the interpretation of the result obtained. Due to practical importance we consider following four cases. (2)

<u>Case-I</u> LC circuit without voltage source

[R = 0, E(t) = 0]

<u>Part (I)</u> From eq (1) we get,

$$L \frac{d^2 i}{dt^2} + \frac{1}{C} i = 0$$

$$\Rightarrow \frac{d^2 i}{dt^2} + \frac{1}{LC} i = 0$$

$$\Rightarrow \frac{d^2 i}{dt^2} + \omega_0^2 i = 0 \qquad \text{—— (6)}$$

where $\omega_0^2 = \frac{1}{LC}$.

<u>Solution of eq (10)</u>

A.E is

$$m^2 + \omega_0^2 = 0$$

$$\Rightarrow m = \pm i \omega_0$$

Hence,

$$i(t) = (c_1 \cos\omega_0 t + c_2 \sin\omega_0 t) \qquad \text{—— (7)}$$

Suppose $c_1 = c \cos\alpha$, $c_2 = c \sin\alpha$

$$\Rightarrow c = \sqrt{c_1^2 + c_2^2}, \quad \alpha = \tan^{-1}\left(\frac{c_2}{c_1}\right)$$

$$\Rightarrow i(t) = c \cos\alpha \cos\omega_0 t + c \sin\alpha \sin\omega_0 t$$

$$= c\,[\cos\alpha \cos\omega_0 t + \sin\alpha \sin\omega_0 t)$$

$$i(t) = c \cos(\cancel{\omega_0 t} \ \alpha + \omega_0 t)$$

Part-III

Eq (8) described a simple harmonic motion of period T is given by

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{LC}$$

It's frequency is $\frac{1}{T} = \frac{1}{2\pi\sqrt{LC}}$.

amplitude c and

phase angle α

Case-II LC - circuit with voltage source $(R=0,\ E(t) \neq 0)$

From eq (5) we get,

$$L\frac{d^2 i}{dt^2} + \frac{1}{C} i = E'(t)$$

$$\Rightarrow \frac{d^2 i}{dt^2} + \frac{1}{LC} i = \frac{E'(t)}{L}$$

$$\Rightarrow \frac{d^2 i}{dt^2} + \omega_0^2 i = \frac{E'(t)}{L} \quad \text{—— (9)}$$

where $\omega_0^2 = \frac{1}{LC}$

Suppose $E(t) = E_0 \sin\omega t$, then eq (9) becomes

$$\Rightarrow \frac{d^2 i}{dt^2} + \omega_0^2 i = \frac{E_0 \omega \cos\omega t}{L}$$

$$\Rightarrow \frac{d^2 i}{dt^2} + \omega_0^2 i = F_0 \cos\omega t \quad \text{where } F_0 = \frac{E_0 \omega}{L}$$

Solution of eq (iii)

The solution of eq (iii) is given by (3)

$i(t) = i_c(t) + i_p(t)$

For $i_c(t)$

A.E $\quad \dfrac{d^2 i}{dt^2} + \omega_0^2 i = 0$

A.E $\quad m^2 + \omega_0^2 = 0$

$\Rightarrow i_c(t) = (c_1 \cos \omega_0 t + c_2 \sin \omega_0 t)$

Suppose $c_1 = c\cos\alpha$, $c_2 = c\sin\alpha$ then

$i_c(t) = c\cos(\omega_0 t - \alpha)$ where $c = \sqrt{c_1^2 + c_2^2}$

$\alpha = \tan^{-1}\left(\frac{c_2}{c_1}\right)$

For $i_p(t)$

$$i_p(t) = \frac{1}{(D^2 + \omega_0^2)} F_0 \cos\omega t$$

$$= F_0 \frac{1}{D^2 + \omega_0^2} \cos\omega t$$

When $\omega \neq \omega_0$ then

$$i_p(t) = F_0 \frac{1}{-\omega^2 + \omega_0^2} \cos\omega t$$

$$\Rightarrow i_p(t) = \frac{F_0}{\omega_0^2 - \omega^2} \cos\omega t \quad \text{——— (I)}$$

When $\omega = \omega_0$ then

$$i_p(t) = F_0 \cdot t \frac{1}{2D} \cos\omega t \qquad \left[\frac{1}{D^2 + a^2}\cos ax = x \cdot \frac{1}{2D}\cos ax\right.$$

$$= \frac{F_0 \cdot t}{2} \frac{\sin\omega t}{\omega}$$

$$i_p(t) = F_0 \left(\frac{t}{2\omega}\right) \sin\omega t \quad \text{———}$$

Thus, the general solution is given by

$$i_g(t) = i_c(t) + i_p(t)$$

$$\Rightarrow \quad i(t) = C \cdot \cos(\omega_0 t - \alpha) + \frac{F_0 \cos \omega t}{\omega_0^2 - \omega^2} \quad [\omega \neq \omega_0] \quad (13)$$

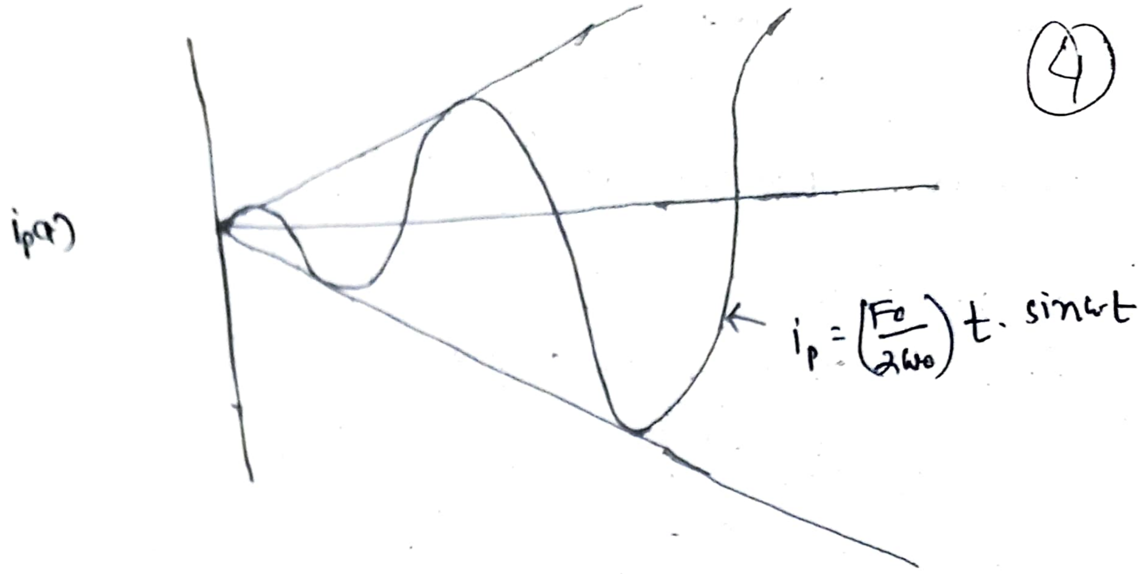
or

$$i(t) = C \cdot \cos(\omega_0 t - \alpha) + F_0 \cdot \left(\frac{t}{2\omega_0}\right) \sin \omega_0 t \quad (14)$$

Part - II Interpretation

From eq (13) it is clear that the resulting motion is the superposition of two oscillation one with natural circular frequency ω_0 and the other with external circular frequency ω.

From eq (14) it is clear that the resulting motion is the reinforcement of the natural vibrations of the system given by $\cos(\omega_0 t - \alpha)$ by externally impressed vibrations at the same frequency ω_0 but by every increasing amplitude given by $\left(\frac{F_0}{2\omega_0}\right) t \sin \omega_0 t$. The graph of $i_p(t)$ given below shows clearly how the amplitude of the oscillations theoretically ~~incr~~ increase without limit in the case $\omega = \omega_0$.



Case - III LCR - Network without voltage source $(E(t) = 0)$

From eq (5) we get.

$$L\frac{d^2 i}{dt^2} + R \cdot \frac{di}{dt} + \frac{1}{C} i = 0$$

$$\Rightarrow \frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC} i = 0$$

$$\Rightarrow \frac{d^2 i}{dt^2} + 2p\frac{di}{dt} + w_0^2 i = 0 \qquad (15)$$

where $2p = \frac{R}{L}$, $w_0^2 = \frac{1}{LC}$

solution of eq (15)

The auxiliary equation of $\frac{di^2}{dt^2} + 2p\frac{di}{dt} + w_0^2 i = 0$ is

$$m^2 + 2pm + w_0^2 = 0,$$

whose roots ~~are~~ m_1 and m_2 are given by

$m_1 = -p + \sqrt{p^2 - w_0^2}$ ~~Cther~~ and

$m_2 = -p - \sqrt{p^2 - w_0^2}$

We note that s_1, s_2 are real and distinct, repeted or complex conjugate according as

$$p^2 \gtreqless w_0^2$$

$$\Rightarrow \left(\frac{R}{2L}\right)^2 \gtreqless \frac{1}{LC} \qquad \left[\because p = \frac{R}{2L}, \; w_0^2 = \frac{1}{LC}\right]$$

$$\Rightarrow \frac{R}{2L} \gtreqless \frac{1}{\sqrt{LC}}$$

$$\Rightarrow R \gtreqless 2\sqrt{L/C} \qquad (16)$$

Let us set $2\sqrt{L/C} = R_c$ and designate it as a critical path resistance.

If $\underline{R > R_c}$ [overdamped Case]

Then m_1 and m_2 are real and distinct root. Hence

$$\boxed{i(t) = Ae^{m_1 t} + Be^{m_2 t}} \qquad (17)$$

where A and B are arbitrary Constant.

If $\underline{R = R_c}$ [Critically damped case]

They $m_1 = m_2 = -p$, Hence

$$\boxed{i(t) = (A + Bt)\, e^{-pt}} \qquad (18)$$

If $R < R_c$ (Underdamped case) ⑤

Then the roots are complex conjugate given by

$$m_1 = -p + i\sqrt{\omega_0^2 - p^2},$$
$$m_2 = -p - i\sqrt{\omega_0^2 - p^2}$$

Hence,

$$\cancel{i(t) = e^{-pt}(c_1 \cos\omega_1 t + B_2 \sin\omega_1 t}$$
$$i(t) = e^{-pt}\left(A\cos\left(\sqrt{\omega_0^2 - p^2}\right)t + B\sin\left(\sqrt{\omega_0^2 - p^2}\right)t\right)$$
$$i(t) = e^{-pt}(A\cos\omega_1 t + B\sin\omega_1 t) \quad \text{——— (19)}$$

where $\omega_1 = \sqrt{\omega_0^2 - p^2}$

$$= \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

if we take

$$A = c\cos\alpha, \quad B = c\sin\alpha$$

$$\Rightarrow \quad i(t) = ce^{-pt}(\cos\alpha\cos\omega_1 t + \sin\alpha\sin\omega_1 t)$$
$$i(t) = ce^{-pt}\cos(\omega_1 t - \alpha) \quad \text{——— (20)}$$

where $c = \sqrt{A^2 + B^2}$

$\tan\alpha = (B/A) \Rightarrow \alpha = \tan^{-1}(B/A)$

(II) Interpretation

① $R > R_c$ (Overdamped case)

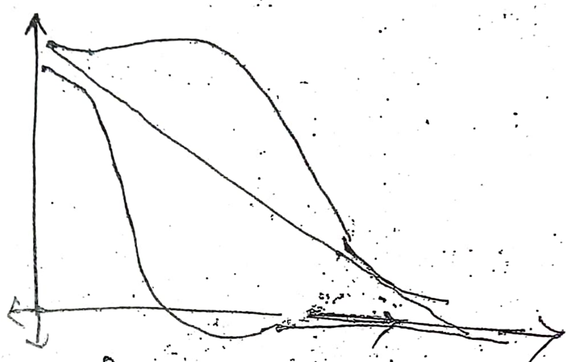
From (17) it is clear that as $t \to \infty$

$i(t) \to 0$

Thus, the system settles to its equilibrium position without any oscillations.

We choose $i(0) = I_0$ a fixed positive number as the initial point and illustrate the effects of changing the initial slope $i'(0)$ in figure. We notice that in every case there would be oscillations which are damped out

$i(t)$



fig

(2) $R = R_c$ [Critically damped case]

From eq(18) we note the following facts.

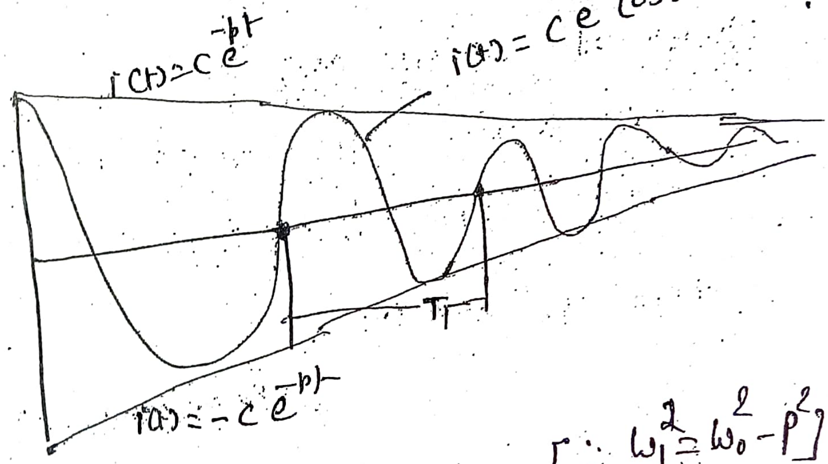
(a) As $t \to \infty$, $i(t) \to 0$

(b) since $e^{-pt} > 0$ and $(A+Bt)$ has almost the positive zero therefore the system passes through its equilibrium position almost once. The graph in this case is similar to that as in (1).

③ $R < R_c$

eq.(23) represents exponentially damped oscillation of the system about its equilibrium position. The graph of $i(t)$ lies between the curve $i(t) = c\,e^{-pt}$ and $i(t) = -c\,e^{-pt}$ (∵ Range of cos is $[-1, 1]$.)

In this case the motion is said to be pseudoperiodic with $c\,e^{-pt}$ as it's time varying amplitude, ω_1 as its circular frequency and $T_1 = \frac{2\pi}{\omega_1}$ as its pseudo-period.



Further we note that $\omega_1 < \omega_0$ [∵ $\omega_1^2 = \omega_0^2 - p^2$]

therefore $T_1 > T$

Thus, the damp exhibits the following three effects.

(a) It exponentially damps the oscillation according to the time varying amplitude.

(b) It slows the motion (since $\omega_1 < \omega_0$)

(c) it delays the motion.

Case-IV LCR Model with Voltage Source.

(I) From equation (5) we get,

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = E' \quad \text{--- (21)}$$

Suppose $E(t) = E_0 \sin\omega t$, then (21) becomes

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}i = \frac{E_0\omega \cdot \cos\omega t}{L}$$

$$\Rightarrow \frac{d^2i}{dt^2} + 2p\frac{di}{dt} + \omega_0^2 i = F_0 \cos\omega t \quad \text{--- (22)}$$

where $2p = R/L$, $\omega_0^2 = \frac{1}{LC}$, $F_0 = \frac{E_0\omega}{L}$

Solution of equation (22)

The solution of eq(22) is obtained as

$$i(t) = i_c(t) + i_p(t)$$

From the case (3)

$$i_c(t) = c\,e^{-pt}\cos(\omega_1 t - \alpha) \quad \text{--- (23)}$$

For particular integral

$$i_p(t) = \frac{1}{(D^2 + 2pD + \omega_0^2)} F_0 \cos\omega t$$

$$= F_0 \frac{1}{[2pD + \omega_0^2 - \omega^2]} \cos\omega t$$

$$= F_0 \frac{1}{(2pD - (\omega^2 - \omega_0^2))} \cos\omega t$$

$$= F_0 \frac{1}{(2PD - (\omega^2-\omega_0^2)) \times (2PD + (\omega^2-\omega_0^2))} (2PD + (\omega - \omega_0)) \cos\omega t$$ (7)

$$= F_0 \frac{(2PD + (\omega^2-\omega_0^2))}{[4P^2D^2 - (\omega^2-\omega_0^2)^2]} \cos\omega t.$$

$$= \frac{F_0}{[4p^2(-\omega^2) - (\omega^2-\omega_0^2)^2]} [-2p\omega\sin\omega t - (\omega_0^2-\omega^2)\cos\omega t]$$

$$= \frac{F_0}{4p^2\omega^2 + (\omega^2-\omega_0^2)} [2p\omega\sin\omega t + (\omega_0^2-\omega^2)\cos\omega t] \quad (24)$$

Now, putting $2p\omega = \rho\cos\phi$ and

$\omega_0^2 - \omega^2 = -\rho\sin\phi$

in eq (24)

we get

$$i_p(t) = \frac{F_0}{4p^2\omega^2 + (\omega_0^2-\omega^2)^2} \times [\rho\cos\phi\sin\omega t - \rho\sin\phi\cos\omega t]$$

$$i_p(t) = \frac{F_0}{4p^2\omega^2 + (\omega_0^2-\omega^2)^2} \rho\sin(\omega t - \phi) \quad (25)$$

$$\Rightarrow i_p(t) = I_0 \sin(\omega t - \phi) \quad (26.)$$

where $I_0 = \frac{F_0}{4p^2\omega^2 + (\omega_0^2-\omega^2)^2}$

Hence, solution is

$$\boxed{i(t) = i_c(t) + i_p(t)}$$

II Interpretation

Notwithstanding the specific form of $i_c(t)$ that is, whether it is given by eq. (17), (18) or (21) we note that $i_c(t) \to 0$ at $t \to \infty$.

Thus $I_c(t)$ as a transient solution and gives the transient current which dies out with the passage of time.

From eq (21), it is clear that $i_p(t)$ represent the simple Harmonic motion of period $2/\omega$ and amplitude $E_0/2$.

The Expression for $i_p(t)$, being a sine term of constant amplitude, continue to contribute to the motion in periodic, oscillatory manner. Thus $i_p(t)$ gives the steady periodic current.

— x — x —