Mathematical Background for Cryptography

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January 19, 2024

Part II - AIgebraic Structures

ALGEBRAIC STRUCTURES

- Cryptography requires sets of integers and specific operations that are defined for those sets.
- The combination of the set and the operations that are applied to the elements of the set is called an algebraic structure.
- Three common algebraic structures: groups, rings, and fields.

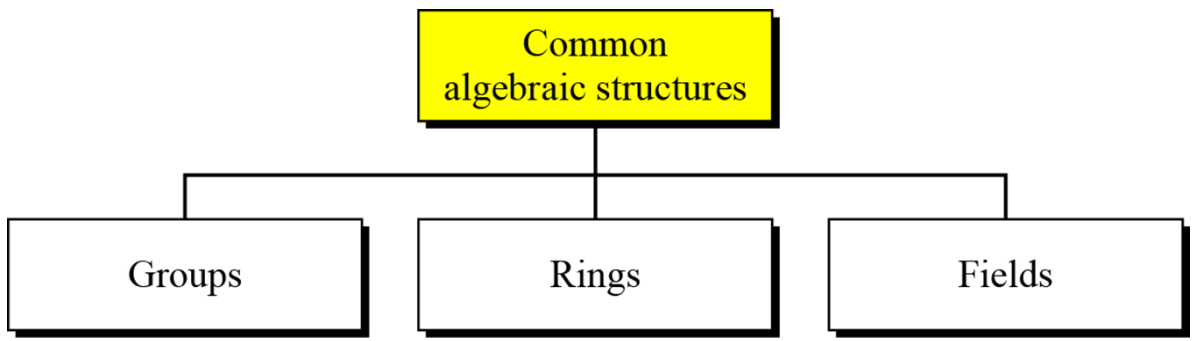


Figure: Common algebraic structure

Group

A group (G) is a set of elements with a binary operation ($\bullet$) that satisfies four properties (or axioms).

- Closure - If a and b are elements of G, then $c = a \bullet b$ is also an element of G.
- Associativity - If a, b and c are elements of G, then $(a \bullet b) \bullet c = a \bullet (b \bullet c)$
- Existence of identity - For all a in G, there exist an element e, called the identity element, such that $e \bullet a = a \bullet e = a$
- Existence of inverse - For each a in G, there exists an element a', called the inverse of a, such that $a \bullet a' = a' \bullet a = e$

A Commutative group (Abelian group) is group in which the operator satisfies four properties plus an extra property that is commutativity. For all a and b in G, we have $a \bullet b = b \bullet a$

Group(cont.)

Example 1 :
The set of residue integers with the addition operator,
$G =< Z_n, + >$, is a commutative group.
Check the properties. . . ..
Example 2 :
The set Z_{n^*} with the multiplication operator, G = Z_{n^*}, is also an abelian group.
Example 3 : Let us define a set $G =< a, b, c, d, \bullet >$ and the operation as shown in Table.

$\bullet$	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

Group(cont.)

Example 4 :

A very interesting group is the permutation group.
The set is the set of all permutations, and the operation is composition i.e. applying one permutation after another.
Check for properties. . . .
Is the group abelian????

$\circ$	[1 2 3]	[1 3 2]	[2 1 3]	[2 3 1]	[3 1 2]	[3 2 1]
[1 2 3]	[1 2 3]	[1 3 2]	[2 1 3]	[2 3 1]	[3 1 2]	[3 2 1]
[1 3 2]	[1 3 2]	[1 2 3]	[2 3 1]	[2 1 3]	[3 2 1]	[3 1 2]
[2 1 3]	[2 1 3]	[3 1 2]	[1 2 3]	[3 2 1]	[1 3 2]	[2 3 1]
[2 3 1]	[2 3 1]	[3 2 1]	[1 3 2]	[3 1 2]	[1 2 3]	[2 1 3]
[3 1 2]	[3 1 2]	[2 1 3]	[3 2 1]	[1 2 3]	[2 3 1]	[1 3 2]
[3 2 1]	[3 2 1]	[2 3 1]	[3 1 2]	[1 3 2]	[2 1 3]	[1 2 3]

Figure: Operation table for permutation group

Group(cont.)

- In the previous example, we showed that a set of permutations with the composition operation is a group.
- This implies that using two permutations one after another cannot strengthen the security of a cipher.
- Because we can always find a permutation that can do the same job because of the closure property.

Exercise :

Although a group involves a single operation, the properties imposed on the operation allow the use of a pair of operations!!!! How???

Group(cont.)

- Finite Group - If the set has a finite number of elements; otherwise, it is an infinite group.
- Order of a Group $|G|$ - The number of elements in the group. If the group is finite, its order is finite
- Subgroups - If $G =< S, \bullet >$ is a group, $H =< T, \bullet >$ is a group under the same operation, and T is a nonempty subset of S, then H is a subgroup of G.
  - If a and b are members of both groups, then $c = a \bullet b$ is also member of both groups
  - The group share the same identity element
  - If a is a member of both groups, the inverse of a is also a member of both groups
  - The group made of the identity element of G, $H =< e, \bullet >$, is a subgroup of G
  - Each group is a subgroup of itself

Group(cont.)

Exercise:
Is the group $H =< Z_{10}, + >$ a subgroup of the group $G =< Z_{12}, + >$?
Solution:
The answer is no. Although H is a subset of G, the operations defined for these two groups are different. The operation in H is addition modulo 10; the operation in G is addition modulo 12.
Exercise:
Is the group $H =< 0, 2, 4, 6, + >$ a subgroup of the group $G =< Z_8, + >$?

Cyclic Subgroups

If a subgroup of a group can be generated using the power of an element, the subgroup is called the **cyclic subgroup**.

$$a^n \rightarrow a \bullet a \bullet \ldots \bullet a \quad (n \text{ times})$$

Cyclic Subgroups (cont.)

Four cyclic subgroups can be made from the group $G =< Z_6, + >$.
They are $H_1 =< \{0\}, + >$, $H_2 =< \{0, 2, 4\}, + >$,
$H_3 =< \{0, 3\}, + >$, and $H_4 = G$.

$$0^0 \bmod 6 = 0$$

$$\begin{aligned}
1^0 \bmod 6 &= 0 \\
1^1 \bmod 6 &= 1 \\
1^2 \bmod 6 &= (1 + 1) \bmod 6 = 2 \\
1^3 \bmod 6 &= (1 + 1 + 1) \bmod 6 = 3 \\
1^4 \bmod 6 &= (1 + 1 + 1 + 1) \bmod 6 = 4 \\
1^5 \bmod 6 &= (1 + 1 + 1 + 1 + 1) \bmod 6 = 5
\end{aligned}$$

$$\begin{aligned}
2^0 \bmod 6 &= 0 \\
2^1 \bmod 6 &= 2 \\
2^2 \bmod 6 &= (2 + 2) \bmod 6 = 4
\end{aligned}$$

$$\begin{aligned}
3^0 \bmod 6 &= 0 \\
3^1 \bmod 6 &= 3
\end{aligned}$$

$$\begin{aligned}
4^0 \bmod 6 &= 0 \\
4^1 \bmod 6 &= 4 \\
4^2 \bmod 6 &= (4 + 4) \bmod 6 = 2
\end{aligned}$$

$$\begin{aligned}
5^0 \bmod 6 &= 0 \\
5^1 \bmod 6 &= 5 \\
5^2 \bmod 6 &= 4 \\
5^3 \bmod 6 &= 3 \\
5^4 \bmod 6 &= 2 \\
5^5 \bmod 6 &= 1
\end{aligned}$$

Cyclic Subgroups (cont.)

Three cyclic subgroups can be made from the group $G =< Z_{10}^*, x >$.
They are $H_1 =< \{1\}, x >$, $H_2 =< \{1, 9\}, x >$, $H_3 = G$.

$$1^0 \bmod 10 = 1$$

$$3^0 \bmod 10 = 1$$
$$3^1 \bmod 10 = 3$$
$$3^2 \bmod 10 = 9$$
$$3^3 \bmod 10 = 7$$

$$7^0 \bmod 10 = 1$$
$$7^1 \bmod 10 = 7$$
$$7^2 \bmod 10 = 9$$
$$7^3 \bmod 10 = 3$$

$$9^0 \bmod 10 = 1$$
$$9^1 \bmod 10 = 9$$

Cyclic Group

A cyclic group is a group that is its own cyclic subgroup.

$$\{e, g, g^2, \ldots, g^{n-1}\}, \text{where } g^n = e$$

Example:
Three cyclic subgroups can be made from the group $G =< Z_{10}^*, x >$.
They are $H_1 =< \{1\}, x >$, $H_2 =< \{1, 9\}, x >$, $H_3 = G$.
The group $G =< Z_{10}^*, x >$ is a cyclic group with two generators, g = 3 and g = 7.
The group $G =< Z_6, + >$ is a cyclic group with two generators, g = 1 and g = 5.

Group(cont.)

- **Lagrange's Theorem** - Assume that G is a group, and H is a subgroup of G. If the order of G and H are $|G|$ and $|H|$, respectively, then, based on this theorem, $|H|$ divides $|G|$.
- **Order of an Element** - The order of an element is the order of the cyclic group it generates.
- **Example:**
  In the group $G =< Z_6, + >$, the orders of the elements are: $\text{ord}(0) = 1$, $\text{ord}(1) = 6$, $\text{ord}(2) = 3$, $\text{ord}(3) = 2$, $\text{ord}(4) = 3$ and $\text{ord}(5) = 6$.
- In the group $G =< Z_{10}^*, x >$, the orders of the elements are : $\text{ord}(1) = 1$, $\text{ord}(3) = 4$, $\text{ord}(7) = 4$ and $\text{ord}(9) = 2$.

Ring

- A Ring, $R = < \{...\}, \bullet, \square >$, is an algebraic structure with two operations.
- First operation must satisfy all five properties
- operation must satisfy only the first two
- In addition, operation must be distributed over first i.e. for all a, b, and c elements of R, we have,
  $a\square(b \bullet c) = (a\square b) \bullet (a\square c)$ and $(a \bullet b)\square c = (a\square c) \bullet (a\square c)$

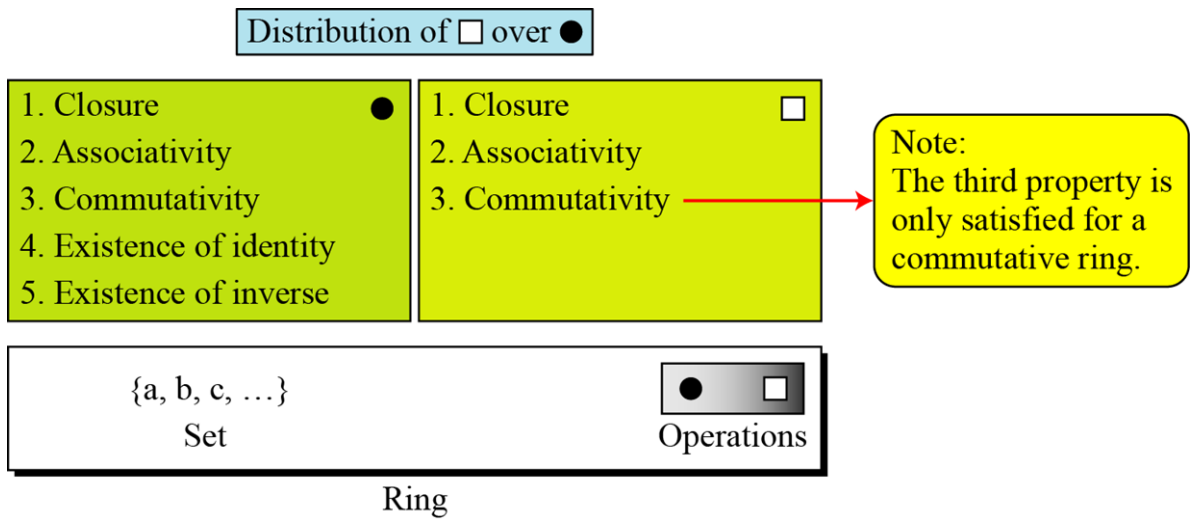


Figure: Commutative Ring

Ring(cont.)

- The set Z with two operations, addition and multiplication, is a commutative ring.
- We show it by R $= < Z, +, x >$.
- Addition satisfies all of the five properties; multiplication satisfies only three properties.

Field

A field, denoted by $F = < \{...\}, \bullet, \square >$ is a commutative ring in which the second operation satisfies all five properties defined for the first operation except that the identity of the first operation has no inverse.

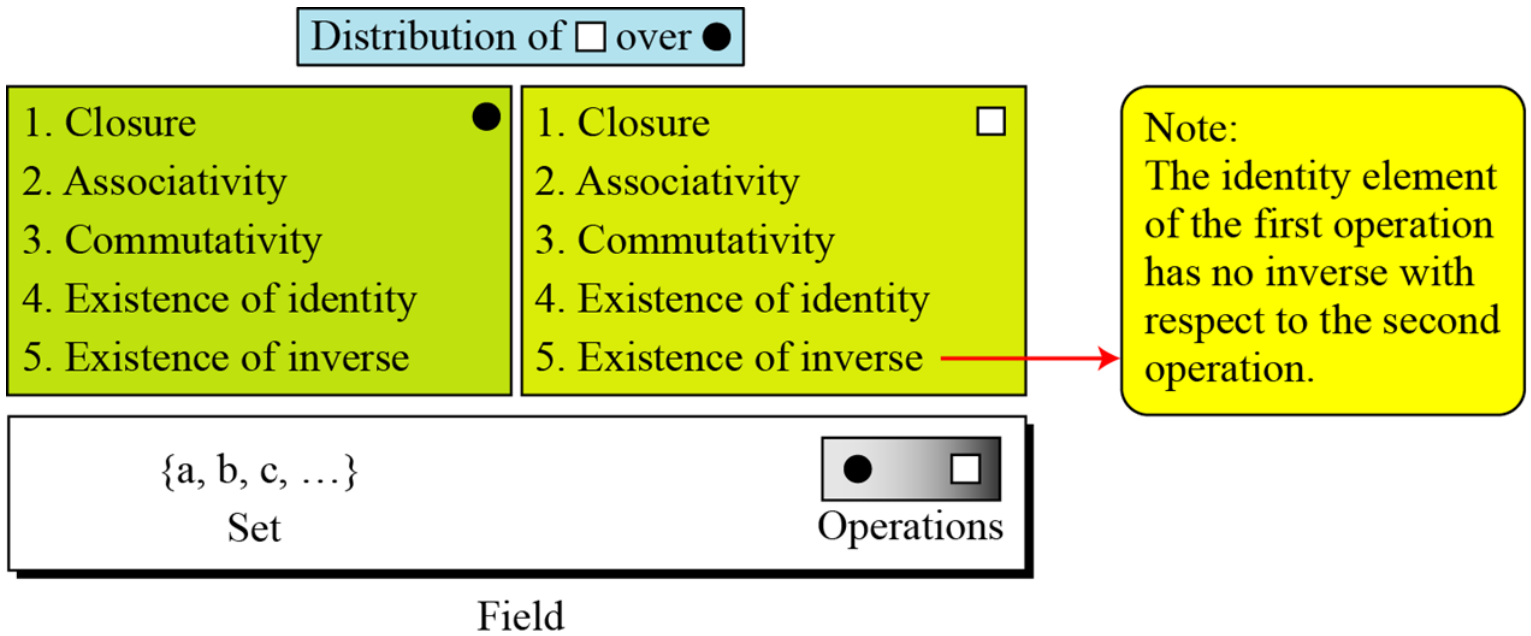


Figure: Field

Finite Fields

Galois showed that for a field to be finite, the number of elements should be p^n, where p is a prime and n is a positive integer.

A Galois field, GF(p^n), is a finite field with p^n elements.

- GF(p) Fields
- When n $= 1$, we have GF(p) field.
- This field can be the set Z_p, $\{0, 1, ..., p-1\}$, with two arithmetic operations.

Field(cont.)

A very common field in this category is GF(2) with the set $\{0, 1\}$ and two operations, addition and multiplication.

GF(2)

{0, 1} + ×

+	0	1
0	0	1
1	1	0

Addition

×	0	1
0	0	0
1	0	1

Multiplication

a	0	1
−a	1	0

a	0	1
a^{-1}	—	1

Inverses

Figure: GF(2) field

We can define GF(5) on the set Z_5 (5 is a prime) with addition and multiplication operators.

GF(5)

{0, 1, 2, 3, 4} + ×

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

Addition

×	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Multiplication

Additive inverse

a	0	1	2	3	4
−a	0	4	3	2	1

a	0	1	2	3	4
a^{-1}	—	1	3	2	4

Multiplicative inverse

Figure: GF(5) field

Algebraic Structures with typical oerations supported

Algebraic Structure	*Supported Typical Operations*	*Supported Typical Sets of Integers*
Group	$(+\ -)$ or $(\times\ \div)$	$\mathbf{Z}_n$ or $\mathbf{Z}_n{}^*$
Ring	$(+\ -)$ and $(\times)$	$\mathbf{Z}$
Field	$(+\ -)$ and $(\times\ \div)$	$\mathbf{Z}_p$

GF(2^n) FIELDS

- In cryptography, we often need to use four operations(addition, subtraction, multiplication and division).
- In other words, we need to use fields.
- However, when we work with computers, the positive integers are stored in the computers as n-bit words in which n is usually 8,16,32 and so on.
- Range of integers is 0 to 2^n–1
- Hence modulus is ?????
- What if we want to use field????

GF(2^n) FIELDS (cont.)

Solution 1

Use GF(p), with the set Z^p, where p is the largest prime number less than 2^n

But the problem ???

Solution 2

Use GF(2^n)

Use a set of 2^n words

The elements in this set are n-bit words

E.g. for n=3, the set is $\{000, 001, 010, 011, 100, 101, 110, 111\}$

What is the problem ?

2^n is not prime

Need to define operations on the set of elements in GF(2^n)

GF(2^n) FIELDS (cont.)

Let us define a GF(2^2) field in which the set has four 2-bit words: $\{00, 01, 10, 11\}$.

We can redefine addition and multiplication for this field in such a way that all properties of these operations are satisfied.

Addition

⊕	00	01	10	11
00	00	01	10	11
01	01	00	11	10
10	10	11	00	01
11	11	10	01	00

Identity: 00

Multiplication

⊗	00	01	10	11
00	00	00	00	00
01	00	01	10	11
10	00	10	11	01
11	00	11	01	10

Identity: 01

Figure: An example of GF(2^2) field

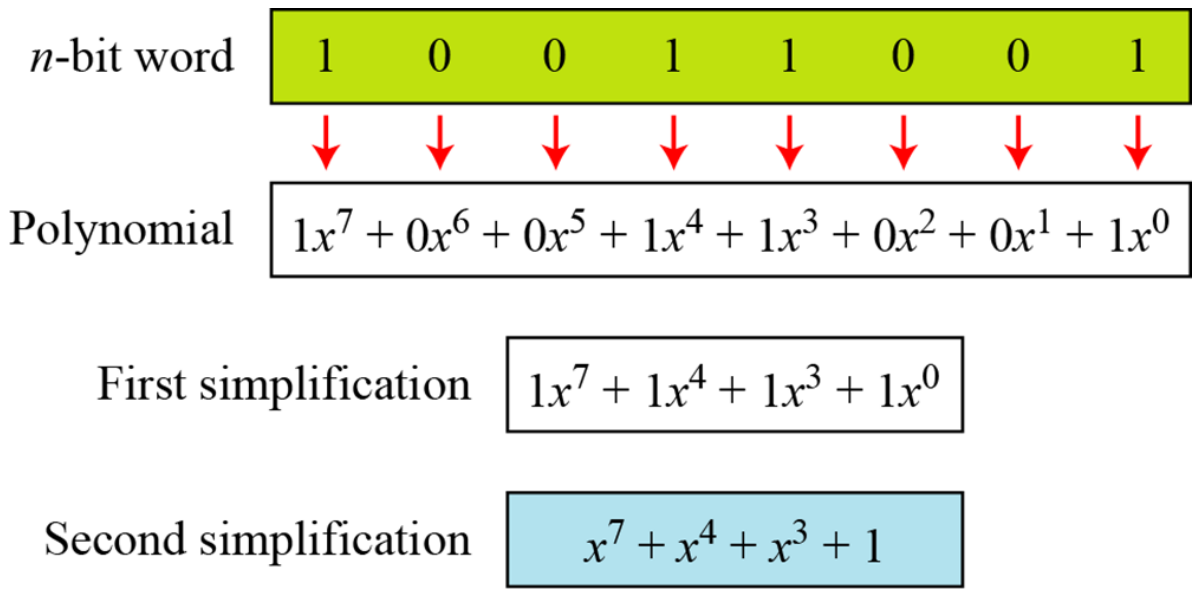
Polynomials

A polynomial of degree $n-1$ is an expression of the form,

$$f(x) = a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x^1 + a_0x^0$$

where x_i is called the i^{th} term and a^i is called coefficient of the i^{th} term.

We can represent the 8-bit word (10011001) using a polynomial.



Polynomials(cont.)

Find the 8-bit word related to the polynomial $x^5 + x^2 + x$.
We first supply the omitted terms.
Since n = 8, it means the polynomial is of degree 7.
The expanded polynomial is,

$$0x^7 + 0x^6 + 1x^5 + 0x^4 + 0x^3 + 1x^2 + 1x^1 + 0x^0$$

This is related to the 8-bit word 00100110.

- Operations on polynomials actually involves two operations
- Operation on coefficients and operation on polynomials
- Hence, need to define two fields
- GF(2) and GF(2^n)

Polynomials(cont.)

- **Modulus** - For the sets of polynomials in GF(2^n), a group of polynomials of degree n is defined as the modulus. Such polynomials are referred to as **irreducible polynomials**.
- **Irreducible polynomials** - No polynomial in the set can divide this polynomial, Can not be factored into a polynomial with degree of less than n

Degree	*Irreducible Polynomials*
1	$(x+1)$, (x)
2	(x^2+x+1)
3	(x^3+x^2+1), (x^3+x+1)
4	$(x^4+x^3+x^2+x+1)$, (x^4+x^3+1), (x^4+x+1)
5	(x^5+x^2+1), $(x^5+x^3+x^2+x+1)$, $(x^5+x^4+x^3+x+1)$, $(x^5+x^4+x^3+x^2+1)$, $(x^5+x^4+x^2+x+1)$

Polynomials(cont.)

Polynomial addition

Addition and subtraction operations on polynomials are the same operation.

Let us do $(x^5 + x^2 + x) \oplus (x^3 + x^2 + 1)$ in GF(2^8). We use the symbol $\oplus$ to show that we mean polynomial addition.
The following shows the procedure:

$$
\begin{array}{lll}
0x^7 + 0x^6 + 1x^5 + 0x^4 + 0x^3 + 1x^2 + 1x^1 + 0x^0 & \oplus & \\
0x^7 + 0x^6 + 0x^5 + 0x^4 + 1x^3 + 1x^2 + 0x^1 + 1x^0 & & \\
\text{---------- (50 dashes) ----------} & & \\
0x^7 + 0x^6 + 1x^5 + 0x^4 + 1x^3 + 0x^2 + 1x^1 + 1x^0 & \rightarrow & x^5 + x^3 + x + 1
\end{array}
$$

Polynomials(cont.)

- **Multiplication**
  The coefficient multiplication is done in GF(2).
  The multiplying x^i by x^j results in x^{i+j}.
  The multiplication may create terms with degree more than n - 1, which means the result needs to be reduced using a modulus polynomial.
- **Example**
  Find the result of $(x^5 + x^2 + x) \otimes (x^7 + x^4 + x^3 + x^2 + x)$ in GF(2^8) with irreducible polynomial $(x^8 + x^4 + x^3 + x + 1)$.
- To find the final result, divide the polynomial of degree 12 by the polynomial of degree 8 (the modulus) and keep only the remainder.

Polynomials(cont.)

Polynomial division with coefficients in GF(2)

$$
\begin{array}{r|l}
 & x^4 + 1 \\
\hline
x^8 + x^4 + x^3 + x + 1 & x^{12} + x^7 + x^2 \\
 & x^{12} + x^8 + x^7 + x^5 + x^4 \\
\hline
 & x^8 + x^5 + x^4 + x^2 \\
 & x^8 + x^4 + x^3 + x + 1 \\
\hline
\text{Remainder} & \boxed{x^5 + x^3 + x^2 + x + 1}
\end{array}
$$

Polynomials(cont.)

Example: In GF (2^4), find the inverse of $(x^2 + 1)$ modulo $(x^4 + x + 1)$.

Solution

The answer is $(x^3 + x + 1)$

q	r_1	r_2	r	t_1	t_2	t
$(x^2 + 1)$	$(x^4 + x + 1)$	$(x^2 + 1)$	(x)	(0)	(1)	$(x^2 + 1)$
(x)	$(x^2 + 1)$	(x)	(1)	(1)	$(x^2 + 1)$	$(x^3 + x + 1)$
(x)	(x)	(1)	(0)	$(x^2 + 1)$	$(x^3 + x + 1)$	(0)
	(1)	(0)		$(x^3 + x + 1)$	(0)	

Polynomials(cont.)

Example:
In GF(2^8), find the inverse of (x^5) modulo $(x^8 + x^4 + x^3 + x + 1)$.

q	r_1	r_2	r	t_1	t_2	t
(x^3)	$(x^8 + x^4 + x^3 + x + 1)$	(x^5)	$(x^4 + x^3 + x + 1)$	(0)	(1)	(x^3)
$(x + 1)$	(x^5)	$(x^4 + x^3 + x + 1)$	$(x^3 + x^2 + 1)$	(1)	(x^3)	$(x^4 + x^3 + 1)$
(x)	$(x^4 + x^3 + x + 1)$	$(x^3 + x^2 + 1)$	(1)	(x^3)	$(x^4 + x^3 + 1)$	$(x^5 + x^4 + x^3 + x)$
$(x^3 + x^2 + 1)$	$(x^3 + x^2 + 1)$	(1)	(0)	$(x^4 + x^3 + 1)$	$(x^5 + x^4 + x^3 + x)$	(0)
	(1)	(0)		$(x^5 + x^4 + x^3 + x)$	(0)	

References

1. Forouzan, Behrouz A. "Cryptography & Network Security. 2011."

Last updated by S J Patel on January 19, 2024