The relativistic Force law and the Dynamics of a single Particle.

Now, the generalised form of Newton's second law is,

$$\vec{F} = \frac{d}{dt}(\vec{p})$$

$$\vec{F} = \frac{d}{dt}\left(\frac{m_0\vec{u}}{\sqrt{1-u^2/c^2}}\right) \quad \text{——— (1)}$$

Kinetic energy in classical mechanics and relativistic mechanics.

in classical,

$$K = \int_{u=0}^{u=u} \vec{F}\cdot d\vec{l}$$

$$= \int_0^u F\,dx \qquad \text{(one dimension)}$$

$$= \int_0^u m_0 \frac{du}{dt}\,dx = \int_0^u m_0\,du\left(\frac{dx}{dt}\right) = \int_0^u m_0 u\,du$$

$$\boxed{K = m_0 \frac{u^2}{2}\Big|_0^u = \frac{1}{2}m_0u^2} \quad \text{——— (2)}$$

Now relativistic :

$$K = \int_0^u \frac{d}{dt}(mu)\,dx = \int_0^u d(mu)\,\frac{dx}{dt}$$

$$K = \int_0^u (m\,du + u\,dm)\,u$$

$$K = \int_0^u (mu\,du + u^2 dm) \quad \text{——— (3)}$$

as $m = \dfrac{m_0}{\sqrt{1-u^2/c^2}} \Rightarrow c^2m^2 - m^2u^2 = m_0^2c^2$

$$c^2m^2 - m^2u^2 = m_0^2c^2$$

on differentiation,

$$2c^2m\,dm - 2m^2u\,du - 2u^2m\,dm = 0$$

or $$c^2dm - mu\,du - u^2dm = 0$$

or $$u^2dm + mu\,du = c^2dm.$$

$$K = \int_0^u (mu\,du + u^2dm)$$

$$= \int_{m_0}^{m} c^2dm = c^2m\Big|_{m_0}^{m} = c^2m - m_0c^2.$$

$$K = mc^2 - m_0c^2 \qquad \text{——— (4)}$$

$$K = \frac{m_0c^2}{\sqrt{1-u^2/c^2}} - m_0c^2$$

$$\boxed{K = m_0c^2\left[\frac{1}{\sqrt{1-u^2/c^2}} - 1\right]} \qquad \text{— (5)}$$

or $$K = \underbrace{mc^2}_{E} - m_0c^2$$

E = total energy.

$$K = E - m_0c^2$$

$$\boxed{\underbrace{E}_{\text{total Energy}} = \underbrace{m_0c^2}_{\text{rest energy}} + \underbrace{K}_{\text{K.E}}} \qquad \text{——— (6)}$$

As $K = m_0 c^2 \left[(1 - u^2/c^2)^{-1/2} - 1 \right]$

as $\frac{u}{c} << 1$

$$K = m_0 c^2 \left[\cancel{1} + \frac{1}{2} \frac{u^2}{c^2} + \cdots - \cancel{1} \right]$$

$$K = m_0 \cancel{c^2} \cdot \frac{1}{2} \frac{u^2}{\cancel{c^2}}$$

$$\boxed{K = \frac{1}{2} m_0 u^2} \quad - (7)$$

(Classical definition of. K.E is retained under approximation $u/c << 1$)

*

$$K = \frac{m_0 c^2}{\sqrt{1 - u^2/c^2}} - m_0 c^2$$

On squaring,

$$(K + m_0 c^2)^2 = \frac{m_0^2 c^4}{1 - u^2/c^2}$$

as $\vec{p} = \frac{m_0 \vec{u}}{\sqrt{1 - u^2/c^2}}$

$p^2 c^2 = \frac{m_0^2 u^2 c^2}{1 - u^2/c^2}$

$$= \frac{m_0^2 c^4}{1 - u^2/c^2} + p^2 c^2 - p^2 c^2$$

$$= p^2 c^2 + \frac{m_0^2 c^4}{1 - u^2/c^2} - \frac{m_0^2 u^2 c^2}{1 - u^2/c^2}$$

$$= p^2 c^2 + \frac{m_0^2 c^4 - m_0^2 u^2 c^2}{1 - u^2/c^2}$$

$$= p^2 c^2 + \frac{m_0^2 c^4 \cancel{(1 - u^2/c^2)}}{\cancel{(1 - u^2/c^2)}}$$

$$\boxed{(K + m_0 c^2)^2 = p^2 c^2 + m_0^2 c^4} \quad - (8)$$

or

$$\boxed{E^2 = p^2c^2 + m_0^2c^4} \quad \text{———} \quad (9)$$

or

$$\boxed{E = c\sqrt{p^2 + m_0^2c^2}} \quad \text{———} \quad (10)$$

on differentiation,

$$\frac{dE}{dp} = c\,\frac{1}{\cancel{2}}\left(p^2 + m_0^2c^2\right)^{\frac{1}{2}-1}(\cancel{2}p + 0)$$

$$= \frac{pc}{\sqrt{p^2 + m_0^2c^2}} = \frac{pc^2}{c\sqrt{p^2+m_0^2c^2}} = \frac{pc^2}{E}$$

$$\frac{dE}{dp} = \frac{pc^2}{E}$$

As $\vec{p} = m\vec{u}$

$\vec{E} = mc^2$

$$\frac{dE}{dp} = \frac{\cancel{m}u\cancel{c^2}}{\cancel{mc^2}} = u$$

$$\boxed{\frac{dE}{dp} = u} \quad \text{———} \quad (11)$$

$$\vec F = \frac{d\vec p}{dt} = \frac{d}{dt}(m\vec u)$$

$$\vec F = m\frac{d\vec u}{dt} + \vec u\frac{dm}{dt} \quad \text{———} \quad (12)$$

as $E = mc^2 \Rightarrow m = E/c^2$

$$\frac{dm}{dt} = \frac{1}{c^2}\frac{dE}{dt} = \frac{1}{c^2}\frac{d}{dt}\left(K + m_0c^2\right) = \frac{1}{c^2}\frac{dK}{dt}$$

$$\frac{dm}{dt} = \frac{1}{c^2}\,\frac{\vec F\cdot d\vec l}{dt} = \frac{1}{c^2}\,\vec F\cdot\frac{d\vec l}{dt}$$

$$\frac{dm}{dt} = \frac{\vec F\cdot\vec u}{c^2} \quad \text{———} \quad (13)$$

So

$$\boxed{\vec F = m\frac{d\vec u}{dt} + \vec u\left(\frac{\vec F\cdot\vec u}{c^2}\right)} \quad \text{———} \quad (14)$$

$\vec a$ (acceleration)

$$\boxed{\vec a = \frac{d\vec u}{dt} = \frac{\vec F}{m} - \frac{\vec u}{mc^2}(\vec F\cdot\vec u)} \quad \text{———} \quad (15)$$

if $\vec F$ & $\vec a$ are parallel to $\vec u$ then

$$\vec a\cdot\vec u = \frac{\vec F\cdot\vec u}{m} - \frac{\vec u\cdot\vec u}{mc^2}(\vec F\cdot\vec u) = \frac{\vec F\cdot\vec u}{m}\left(1 - \frac{u^2}{c^2}\right)$$

$$a_{\parallel} = \frac{F_{\parallel}}{m}\left(1 - \frac{u^2}{c^2}\right)$$

$$a_{\parallel} = \frac{F_{\parallel}}{m_0}\left(1-u^2/c^2\right)^{1/2}\left(1-u^2/c^2\right)$$

$$a_{\parallel} = \frac{F_{\parallel}}{m_0}\left(1-u^2/c^2\right)^{3/2}$$

$$F_{\parallel} = \frac{a_{\parallel}\, m_0}{\left(1-u^2/c^2\right)^{3/2}}$$

$$\boxed{F_{\parallel} = \frac{m_0}{\left(1-u^2/c^2\right)^{3/2}}\, a_{\parallel}} \quad \text{--- (16)}$$

"longitudinal mass"

Another case in which $\vec{a}$ is parallel to $\vec{F}$ is that the $\vec{F}$ is $\perp \vec{u}$ means $\vec{F}\cdot\vec{u} = 0$

$$\vec{a} = \frac{\vec{F}}{m}$$

$$a_{\perp} = \frac{F_{\perp}}{m_0}\sqrt{1-u^2/c^2}$$

$$F_{\perp} = \frac{m_0\, a_{\perp}}{\sqrt{1-u^2/c^2}}$$

$$\boxed{F_{\perp} = \frac{m_0}{\sqrt{1-u^2/c^2}}\, a_{\perp}} \quad \text{--- (17)}$$

"transverse mass"