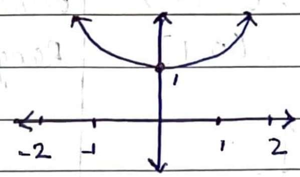
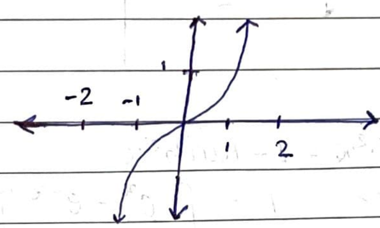
TUTORIAL 1 : Hyperbolic f^n, Successive differentiation & Leibnitz's Theorem

1. • $\cosh x = \dfrac{e^x + e^{-x}}{2}$



domain : $(-\infty, \infty)$
Range : $[1, \infty)$

• $\sinh x = \dfrac{e^x - e^{-x}}{2}$



domain : $(-\infty, \infty)$
Range : $(-\infty, \infty)$

(i) $\cosh^2 x - \sinh^2 x$

$\Rightarrow \left[\dfrac{e^x + e^{-x}}{2}\right]^2 - \left[\dfrac{e^x - e^{-x}}{2}\right]^2 \Rightarrow \dfrac{4\, e^x e^{-x}}{4} \Rightarrow \underline{\underline{1}}$

(ii) $\cosh 2x = 2\cosh^2 x - 1$

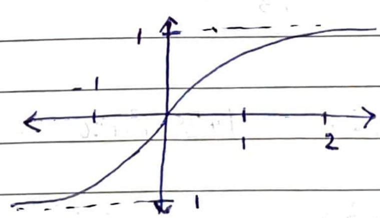
RHS $\Rightarrow 2\left[\dfrac{e^x + e^{-x}}{2}\right]^2 - 1 \Rightarrow \dfrac{e^{2x} + e^{-2x} + 2\cancel{x} - 2}{2}$

$\Rightarrow \underline{\underline{\cosh 2x}}$

(iii) $\cosh 2x = 1 + 2\sinh^2 x$

RHS $\Rightarrow 2\left[\dfrac{e^x - e^{-x}}{2}\right]^2 + 1 \Rightarrow \dfrac{e^{2x} + e^{-2x} - 2 + 2}{2} \Rightarrow \underline{\underline{\cosh 2x}}$

2. • $\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$



domain : $(-\infty, \infty)$
Range : $(-1, 1)$

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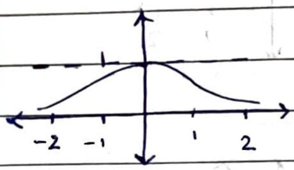
- $\operatorname{sech} x = \dfrac{2}{e^x + e^{-x}}$

domain : $(-\infty, \infty)$

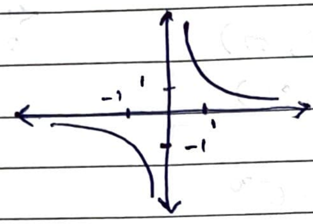
Range : $(0, 1]$



- $\operatorname{cosech} x = \dfrac{2}{e^x - e^{-x}}$

domain : $(-\infty, 0) \cup (0, \infty)$

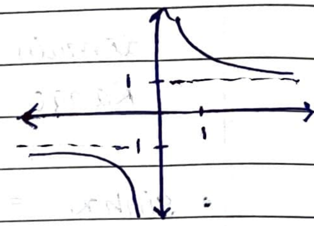
Range : $(-\infty, 0) \cup (0, \infty)$



- $\coth x = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

domain : $(-\infty, 0) \cup (0, \infty)$

range : $(-\infty, -1) \cup (1, \infty)$



(i) $\operatorname{sech}^2 x = 1 - \tanh^2 x$

RHS $\Rightarrow$ $1 - \left[\dfrac{e^x - e^{-x}}{e^x + e^{-x}}\right]^2 \Rightarrow \dfrac{4}{(e^x + e^{-x})^2} = \underline{\operatorname{sech}^2 x}$

(ii) $\coth^2 x = 1 + \operatorname{cosech}^2 x$

RHS $\Rightarrow$ $1 + \dfrac{2}{(e^x - e^{-x})^2} \Rightarrow \left(\dfrac{e^x + e^{-x}}{e^x - e^{-x}}\right)^2 = \underline{\coth^2 x}$

3. $\sinh x = \dfrac{5}{12}$

$\therefore$ (a) $\cosh^2 x = 1 + \sinh^2 x$

$= \dfrac{144 + 25}{144}$

$\boxed{\therefore \cosh x = \dfrac{13}{12}}$

(b) $\tanh^2 x = 1 - \operatorname{sech}^2 x$

$= 1 - \dfrac{144}{169} = \dfrac{25}{169}$ $\boxed{\therefore \tanh x = \dfrac{5}{13}}$

(c) $\boxed{\operatorname{sech} x = \dfrac{12}{13}}$ (d) $\boxed{\operatorname{cosech} x = \dfrac{12}{5}}$ (e) $\boxed{\coth x = \dfrac{13}{5}}$

(f) $\cosh 2x = 1 + 2\sinh^2 x = \cancel{\dfrac{169}{144}}\ 1 + \dfrac{50}{144} = \dfrac{194}{144}$

$\cosh^2 2x - 1 = \sinh^2 2x \Rightarrow \sinh^2 2x = \dfrac{(194)^2 - (144)^2}{(144)^2}$

$\rightarrow \sinh 2x = \dfrac{\sqrt{(50)(338)}}{144} = \dfrac{10\sqrt{169}}{144} = \dfrac{130}{144}$

$\Rightarrow \sinh x = \dfrac{e^x - e^{-x}}{\cancel{2}} = \dfrac{5}{\cancel{12}\, 6}$

$\Rightarrow 6(e^{2x} - 1) = 5e^x \quad \Rightarrow \quad 6e^{2x} - 5e^x - 6 = 0$

$\Rightarrow \quad e^x = \dfrac{5 \pm \sqrt{25 + 144}}{12} = \dfrac{5 \pm 13}{12}$

$= \dfrac{18}{12} = \dfrac{3}{2} \qquad \boxed{\therefore\ x = \log 3/2}$

4 • $\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right) \qquad [x \in \mathbb{R}]$

$\Rightarrow$ Let $y = \sinh x = \dfrac{e^x - e^{-x}}{2}$

$\Rightarrow \quad 2e^x \sinh x = e^{2x} - 1 \quad \Rightarrow \quad e^{2x} - 2e^x \sinh x - 1 = 0$

$\Rightarrow \quad e^x = \dfrac{\cancel{2}\sinh x \pm \sqrt{4\sinh^2 x + 4}}{\cancel{2}}$

$\ln(e^x) = \ln\left[\dfrac{\sinh x + \sqrt{\sinh^2 x + 1}}{1}\right]$

$x = \ln\left[\sinh x + \sqrt{\sinh^2 x + 1}\right]$

Put $x = \sinh^{-1} y$

$\boxed{\sinh^{-1} y = \ln\left[y + \sqrt{y^2+1}\right]} \qquad \boxed{y \in \mathbb{R}}$

• $\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right) \qquad [x \geq 1]$

$y = \cosh x = \dfrac{e^x + e^{-x}}{2}$

$\Rightarrow \quad e^{2x} - 2\cosh x \cdot e^x + 1 = 0 \qquad \therefore\ e^x = \cancel{2}\cosh x + \sqrt{\cosh^2 x - 1}$

$\Rightarrow \quad \boxed{y = \ln\left[y + \sqrt{y^2 - 1}\right]} \qquad \boxed{y \geq 1}$

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5 (i) $y = \sinh^{-1}(x^3)$ diff wrt x

$\rightarrow y_1 = \dfrac{3x^2}{\sqrt{x^6+1}}$

$\left[\dfrac{d(\sinh^{-1}x)}{dx} = \dfrac{1}{\sqrt{x^2+1}}\right.$

(ii) $y = \cosh^{-1}(2x+1)$ diff wrt x

$\rightarrow y_1 = \dfrac{2}{\sqrt{(2x+1)^2-1}} = \dfrac{1}{\sqrt{x^2+x}}$

$\left[\dfrac{d(\cosh^{-1}x)}{dx} = \dfrac{1}{\sqrt{x^2-1}}\right]$

6. $y = \dfrac{x^4}{(x-1)(x-2)}$, then find y_n

$y = x^4\left[\dfrac{1}{x-2} - \dfrac{1}{x-1}\right]$

$v = x^4$

$u = \left[\dfrac{1}{x-2} - \dfrac{1}{x-1}\right]$

~~$y_1 =$~~

$y_n = {}^nC_0\left[(-1)^n n!\left(\dfrac{1}{(x-2)^{n+1}} - \dfrac{1}{(x-1)^{n+1}}\right)\right]x^4 +$

${}^nC_1\left[(-1)^{n-1}(n-1)!\left(\dfrac{(x-2)}{(x-2)^{n+1}} - \dfrac{1}{(x-1)^n}\right)\right]4x^3 +$

${}^nC_2\left[(-1)^{n-2}(n-2)!\left[\dfrac{(x-2)^2}{(x-2)^{n+1}} - \dfrac{1}{(x-1)^{n-1}}\right]\right) 4\cdot3\cdot x^2 +$

${}^nC_3\left[(-1)^{n-3}(n-3)!\left[\dfrac{(x-2)^3}{(x-2)^{n+1}} - \dfrac{1}{(x-1)^{n-2}}\right]\right] 4\cdot3\cdot2\cdot x +$

${}^nC_3\left[(-1)^{n-4}(n-4)!\left[\dfrac{(x-2)^4}{(x-2)^{n+3}} - \dfrac{1}{(x-1)^{n-3}}\right]\right] 4!$

$y_n = (-1)^n(n!)\left[\dfrac{16}{(x-2)^{n+1}} - \dfrac{1}{(x-1)^{n+1}}\right]$

7. $y = e^{2x}\cos x \sin^2 2x = e^{2x}\cos x\left(\frac{1-\cos 4x}{2}\right)$

$$y = \frac{1}{2}\left[e^{2x}\cos x - e^{2x}\cos 4x\right]$$

diff wrt x

$$y_1 = \frac{1}{2}\left[e^{2x}(-\sin x + 4\sin 4x) + 2e^{2x}(\cos x - \cos 4x)\right]$$

$$y_1 = \frac{e^{2x}}{2}\left[4\sin 4x - 2\cos 4x + (-\sin x) + 2\cos x\right]$$

$$y_1 = \frac{e^{2x}}{2}\left[\frac{2}{\sqrt{5}}\sin(4x-\alpha) - \frac{1}{\sqrt{5}}\sin(x-\beta)\right]$$

$$\therefore y_n = \frac{e^{2x}}{2(5^{n/2})}\left[2\sin(4x-n\alpha) - \sin(x-n\beta)\right]$$

8. n^{th} derivative of $\frac{1}{x^2+a^2}$

$$y = \frac{1}{(x-ia)(x+ia)} = \left[\cancel{\frac{1}{x-ia}} \cancel{\pm} \cancel{\frac{1}{x+ia}}\right]\cancel{\frac{1}{2ia}}$$

$$= \left[\frac{1}{x-ia} - \frac{1}{x+ia}\right]\frac{1}{2ia}$$

$$= \frac{1}{2}\left[\frac{(-1)^n n!}{(x-ia)^{n+1}} - \frac{(-1)^n n!}{(x+ia)^{n+1}}\right]\frac{1}{ia}$$

$$\left[\begin{array}{l} x = r\cos\theta \\ a = r\sin\theta \\ \theta = \tan^{-1}(a/x) \end{array}\right]$$

$$= \frac{(-1)^n n!}{2ia}\left[r^{n+1}\left[\cos(n+1)\theta - i\sin(n+1)\theta\right] - r^{n+1}\left[\cos(n+1)\theta + i\sin(n+1)\theta\right]\right]$$

$$= \frac{(-1)^n n!\,(r^{n+1})^{-1}}{a}\left[\sin(n+1)\theta\right]$$

$$= \frac{(-1)^n n!}{a^{n+2}}\left[\sin(n+1)\theta\cdot(\sin\theta)^{n+1}\right]$$

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9. Show that $D^{2n}(x^2-1)^n = (2n)!$

$\rightarrow {}^nC_0 x^{2n} - {}^nC_1 (x^2)^{n-1} + {}^nC_2 (x^2)^{n-2} \dots$

diff wrt x

$\therefore y_{2n} = (2n)!$

10. Leibnitz Theorem - If u & v are f^n of x which posses derivative of nth order then -

$y_n = (uv)_n = {}^nC_0 u_n v + {}^nC_1 u_{n-1} v_1 \dots + {}^nC_n u v_n$

Proof : $y = uv$

$y_1 = u_1 v + u v_1 = {}^1C_0 u_1 v + {}^1C_1 u v_1$

$y_2 = (u_2 v + u_1 v_1) + (u_1 v_1 + u v_2)$

By using MI - $y_n = {}^nC_0 u_n v + {}^nC_1 u_{n-1} v_1 \dots {}^nC_n u v_n$

11. $y = x\log\left(\frac{x-1}{x+1}\right)$, show that $y_n = (-1)^{n-2}(n-2)!\left[\frac{x-n}{(x-1)^n} - \frac{x+n}{(x+1)^n}\right]$

~~$y_n = {}^nC_0$~~ $u = \log(x-1) - \log(x+1)$

$u_1 = \frac{1}{x-1} - \cancel{\log} \frac{1}{x+1}$

$u_{n+1} = \frac{(-1)^n (n!)}{(x-1)^{n+1}} - \frac{(-1)^n (n!)}{(x+1)^{n+1}}$

$u_n = \frac{(-1)^{n-1}(n-1)!}{(x-1)^n} - \frac{(-1)^{n-1}(n-1)!}{(x+1)^n}$

$v = x$

$\rightarrow y_n = {}^nC_0\left[\frac{(-1)^{n-1}(n-1)!}{(x-1)^n} - \frac{(-1)^{n-1}(n-1)!}{(x+1)^n}\right]x +$

${}^nC_1\left[\frac{(-1)^{n-2}(n-2)!}{(x-1)^{n-1}} - \frac{(-1)^{n-2}(n-2)!}{(x+1)^{n-1}}\right]$

$$\Rightarrow y_n = (-1)^{n-2}\left[\frac{-(n-1)x}{(x-1)^n} + \frac{(n+1)x}{(x+1)^n} + \frac{n(x-1)}{(x-1)^n} - \frac{n(x+1)}{(x+1)^n}\right](n-2)!$$

$$= (-1)^{n-2}(n-2)!\left[\frac{(x-n)}{(x-1)^n} - \frac{(x+n)}{(x+1)^n}\right]$$

$$\therefore y_n = (-1)^{n-2}(n-2)!\left[\frac{x-n}{(x-1)^n} - \frac{(x+n)}{(x+1)^n}\right]$$

12. $I_n = \frac{d^n}{dx^n}(x^n \log x)$, prove that $I_n = nI_{n-1} + (n-1)!$

$$\rightarrow I_n = {}^nC_0 (n!) \log x + {}^nC_1 \frac{(n!)}{\cancel{2}}\frac{x}{x} + {}^nC_2 \frac{(n!)}{2} x^2\left(\frac{-1}{x^2}\right)\cdots$$

$$I_{n-1} = {}^{n-1}C_0 (n-1)! \log x + {}^{n-1}C_1 (n-1)! \frac{x}{x} + {}^{n-1}C_2 \frac{(n-1)!}{2} x^2\left(\frac{-1}{x^2}\right)\cdots$$

$$n I_{n-1} = {}^{n-1}C_0 (n!) \log x + {}^{n-1}C_1 (n!) \frac{x}{x} + {}^{n-1}C_2 \frac{n!}{2} x^2\left(\frac{-1}{x^2}\right)\cdots$$

$$I_n - nI_{n-1} = n!\left[n-(n-1)\right] + \frac{n!}{2}\left[\frac{n(n-1)}{2} \mp \frac{(n-1)(n-2)}{2}\right]$$

$$I_n = \frac{d^n}{dx^n}\left[x^n \log x\right] \qquad \cancel{I_{n-1} = \frac{d^{n-1}}{dx}}$$

$$I_n = \frac{d^{n-1}}{dx^{n-1}}\left[d(x^n \log x)\right] = \frac{d^{n-1}}{dx^{n-1}}\left(nx^{n-1}\log x + x^{n-1}\right)$$

$$I_n = nI_{n-1} + (n-1)!$$

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13. $y = x^2 e^x$, prove that $y_n = \frac{1}{2} n(n-1) y_2 - n(n-2) y_1 + \frac{1}{2}(n-1)(n-2) y$

$y_n = {}^nC_0 e^x x^2 + {}^nC_1 e^x (2x) + {}^nC_2 e^x (2) + 0.$

$y_n = e^x x^2 + n e^x 2x + \frac{n(n-1)}{2} 2e^x$

$y_1 = e^x x^2 + 2xe^x = y + 2xe^x$

$\boxed{y_2 = y_1 + 2e^x + 2e^x x}$

$\Rightarrow y_n = y + n(y_1 - y) + \frac{n(n-1)}{2}[y_2 - y_1 - (y_1 - y)]$

$= y(n-1) + ny_1 + \frac{n(n-1)}{2}[y_2 - 2y_1 + y]$

$\Rightarrow y\left[\frac{(n-1)(n+2)}{2}\right] = y_1 (n)(n-2) + y_2 \frac{n(n-1)}{2}$

14. $y^{1/m} + y^{-1/m} = 2x$ then prove that $(x^2-1) y_{n+2} + (2n+1) x y_{n+1} + (n^2 - m^2) y_n = 0$

$\rightarrow \frac{y^{1/m} + y^{-1/m}}{2} = x$

Put $y^{1/m} = e^a$

$\frac{1}{m} y^{1/m} \cdot \frac{1}{y} \cdot \frac{dy}{da} = e^a$

$\therefore da = \frac{1}{my} \cdot dy$

$\Rightarrow \cosh a = x$

$a = \cosh^{-1} x$

$\frac{da}{dx} = \frac{1}{\sqrt{x^2-1}}$

$\frac{1}{my} \cdot y_1 = \frac{1}{\sqrt{x^2-1}}$

$\Rightarrow (x^2-1) y_1^2 = m^2 y^2$

diff wrt x $\rightarrow$ $2xy_1^2 + 2(x^2-1) y_1 y_2 = 2m^2 y \cdot y_1$

$\Rightarrow \boxed{xy_1 + (x^2-1) y_2 - m^2 y = 0}$

Using Leibnitz theorem, we get -

$\rightarrow$ $\left[y_{2+n}(x^2-1) + n\,y_{n+1}(2x) + \frac{n(n-1)}{2} y_n (2) \right] +$

$\left[y_{n+1}\, x + n y_n \right] + \left[-m^2 y_n \right] = 0$

$\Rightarrow y_{n+2}(x^2-1) + y_{n+1}\left[2xn + x \right] + y_n\left(n(n-1) + n - m^2 \right) = 0$

$\Rightarrow \boxed{y_{n+2}(x^2-1) + y_{n+1}\, x(2n+1) + y_n(n^2 - m^2) = 0}$

15. Determine $y_n(0)$ where $y = e^{m\cos^{-1}x}$

$\rightarrow$ $y = e^{m\cos^{-1}x}$ diff wrt x $\rightarrow$ $y_1 = \frac{-m\, e^{m\cos^{-1}x}}{\sqrt{1-x^2}}$

$\Rightarrow \left(-(1-x^2)^{1/2} y_1 \right) = my$ $\Rightarrow$ $(1-x^2) y_1^2 = m^2 y^2$

$\Rightarrow -2xy_1^2 + 2(1-x^2) y_1 \cdot y_2 = 2m^2 y \cdot y_1$

$\Rightarrow -xy_1 + (1-x^2) y_2 = m^2 y$

$\Rightarrow (x^2-1) y_2 + xy_1 + m^2 y = 0$

$\left[y_{n+2}(x^2-1) + n(2x) y_{n+1} + n(n-1) y_n \right] + \left[y_{n+1} x + n y_n \right]$

$+ m^2 y_n = 0$ $\left[\text{Put } x = 0 \right]$

$\Rightarrow -y_{n+2} + (n)(n-1) y_n + n y_n + m^2 y_n = 0$

$y_{n+2} = (n^2 + m^2) y_n$

$y_n = ((n-2)^2 + m^2)\, y_{n-2}$

if n is even, $y_n(0) = ((n-2)^2 + m^2)\ldots(4^2+m^2)(2^2+m^2)\, m^2$

if n is odd, $y_n(0) = ((n-2)^2 + m^2)\ldots(3^2+m^2)(1+m^2)$

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16 If $f(x) = \tan x$, prove that -

$$f^n(0) - {}^nC_2 f^{n-2}(0) + {}^nC_4 f^{n-4}(0) \dots = \sin(n\pi/2)$$

→ $f(x) = \tan x = \dfrac{\sin x}{\cos x}$ $\therefore f(x)\cos x = \sin x$

Use Leibnitz Theorem -

$$\therefore f^n(x)\cos x - {}^nC_1 f^{n-1}(x)\sin x - {}^nC_2 f^{n-2}(x)\cos x \dots = \sin\left(x + \frac{n\pi}{2}\right)$$

=> Put $x = 0$

$$\boxed{f^n(0) - {}^nC_2 f^{n-2}(0) + {}^nC_4 f^{n-4}(0) \dots = \sin(n\pi/2)}$$

Proved