18. For each of the following statements in which A, B, C and D are arbitrary sets, either prove that it is true or give a counter example to show that it is false.
    (a) $A \cap C = B \cap C \rightarrow A = B$
    (b) $A \cap B = A \cap C$ and $\bar{A} \cap B = \bar{A} \cap C \rightarrow B = C$
    (c) $(A - C) = (B - C) \rightarrow A = B$
    (d) $A \cap C = B \cap C$ and $A - C = B - C \rightarrow A = B$
    (e) $A \cup C = B \cup C$ and $A - C = B - C \rightarrow A = B$
    (f) $A \times (B \cup C) = (A \times B) \cup (A \times C)$
    (g) $A \cap (B \times C) = (A \cap B) \times (A \cap C)$
    (h) $(A \cap B) \times C = (A \times C) \cap (B \times C)$
    (i) $(A - B) \times C = (A \times C) - (B \times C)$
    (j) $(A - B) \times (C - D) = (A \times C) - (B \times D)$
    (k) $A \oplus (B \oplus C) = (A \oplus B) \oplus C$
    (l) $(A \oplus B) \times C = (A \times C) \oplus (B \times C)$
19. Simplify the following set expressions, using set identities:
    (a) $\overline{(A \cup B)} \cap \overline{(\bar{A} \cup \bar{C})} \cap \overline{(\bar{B} \cup C)}$
    (b) $(A \cap B) \cup (A \cap B \cap \bar{C} \cap D) \cup (\bar{A} \cap B)$
    (c) $(A - B) \cup (A \cap B)$
20. Write the dual of each of the following statements:
    (a) $(A \cup B) \cap (A \cup \phi) = A$
    (b) $A \cup B = (A \cap B) \cup (A \cap \bar{B}) \cup (\bar{A} \cap B)$
    (c) $\overline{(A \cap B \cap C)} = \overline{(A \cap C)} \cup \overline{(A \cap B)}$.

RELATIONS

Introduction

A *relation* can be thought of as a structure (for example, a table) that represents the relationship of elements of a set to the elements of another set. We come across many situations where relationships between elements of sets, such as those between roll numbers of students in a class and their names, industries and their telephone numbers, employees in an organization and their salaries occur. Relations can be used to solve problems such as producing a useful way to store information in computer databases.

The simplest way to express a relationship between elements of two sets is to use ordered pairs consisting of two related elements. Due to this reason, sets of ordered pairs are called *binary relations*. In this section, we introduce the basic terminology used to describe binary relations, discuss the mathematics of relations defined on sets and explore the various properties of relations.

Definition

When A and B are sets, a subset R of the Cartesian product $A \times B$ is called a *binary relation* from A to B. viz., If R is a binary relation from A to B, R is a set of ordered pairs (a, b), where $a \in A$ and $b \in B$. When $(a, b) \in R$, we use the notation a R b and read it as "a is related to b by R". If $(a, b) \notin R$, it is denoted as $a \not{R}\ b$.

Note Mostly we will deal with relationships between the elements of two sets. Hence the word 'binary' will be omitted hereafter.

If R is a relation from a set A to itself, viz., if R is a subset of $A \times A$, then R is called *a relation on the set A*.

The set $\{a \in A | a \text{ R } b, \text{ for some } b \in B\}$ is called *the domain* of R and denoted by $D(R)$.

The set $\{b \in B | a \text{ R } b, \text{ for some } a \in A\}$ is called *the range* of R and denoted by $R(R)$.

Examples

1. Let $A = \{0, 1, 2, 3, 4\}$, $B = \{0, 1, 2, 3\}$ and a R b if and only if $a + b = 4$.
   Then $R = \{(1, 3), (2, 2), (3, 1), (4, 0)\}$
   The domain of $R = \{1, 2, 3, 4\}$ and the image of $R = \{0, 1, 2, 3\}$
2. Let R be the relation on $A = \{1, 2, 3, 4\}$, defined by a R b if $a \le b$; $a, b \in A$.
   Then $R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$
   The domain and range of R are both equal to A.

TYPES OF RELATIONS

A relation R on a set A is called a *universal* relation, if $R = A \times A$.
For example if $A = \{1, 2, 3\}$, then $R = A \times A = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$ is the universal relation on A.

A relation R on a set A is called a *void relation*, if R is the null set ϕ. For example if $A = \{3, 4, 5\}$ and R is defined as a R b if and only if $a + b > 10$, then R is a null set, since no element in $A \times A$ satisfies the given condition.

Note The entire Cartesian product $A \times A$ and the empty set are subsets of $A \times A$.

A relation R on a set A is called an *identity relation*, if $R = \{(a, a) | a \in A\}$and is denoted by I_A.

For example, if $A = \{1, 2, 3\}$, then $R = \{(1, 1), (2, 2), (3, 3)\}$ is the identity relation on A.

When R is any relation from a set A to a set B, *the inverse* of R, denoted by R^{-1}, is the relation from B to A which consists of those ordered pairs got by interchanging the elements of the ordered pairs in R.

viz., $R^{-1} = \{(b, a) | (a, b) \in R\}$

viz., if a R b, then $b\ R^{-1}\ a$.

For example, if $A = \{2, 3, 5\}$, $B = \{6, 8, 10\}$ and a R b if and only if $a \in A$ divides $b \in B$, then $R = \{(2, 6), (2, 8), (2, 10), (3, 6), (5, 10)\}$

Now $R^{-1} = \{(6, 2), (8, 2), (10, 2), (6, 3), (10, 5)\}$

We note that $b\ R^{-1}\ a$, if and only if $b \in B$ is a multiple of $a \in A$. Also we note that

$$D(R) = R(R^{-1}) = \{2, 3, 5\} \text{ and}$$
$$R(R) = D(R^{-1}) = \{6, 8, 10\}$$

SOME OPERATIONS ON RELATIONS

As binary relations are sets of ordered pairs, all set operations can be done on relations. The resulting sets are ordered pairs and hence are relations.

If R and S denote two relations, *the intersection of R and S* denoted by $R \cap S$, is defined by

$$a\ (R \cap S)b = a \text{ R } b \wedge a \text{ S } b$$

and *the union of R and S*, denoted by $R \cup S$, is defined by $a\ (R \cup S)\ b = a \text{ R } b \vee a \text{ S } b$.

The difference of R and S, denoted by $R - S$, is defined by $a\ (R - S)\ b = a \text{ R } b \wedge a \not{S}\ b$.

The *complement of R*, denoted by R' or $\sim R$ is defined by $a(R')b = a \not{R}\ b$. For example, let $A = \{x, y, z\}$, $B = \{1, 2, 3\}$, $C = \{x, y\}$ and $D = \{2, 3\}$. Let R be a relation from A to B defined by $R = \{(x, 1), (x, 2), (y, 3)\}$ and let S be a relation from C to D defined by $S = \{(x, 2), (y, 3)\}$.

Then $R \cap S = \{(x, 2), (y, 3)\}$ and $R \cup S = R$.

$R - S = \{(x, 1)\}$

$R' = \{(x, 3), (y, 1), (y, 2), (z, 1), (z, 2), (z, 3)\}$

COMPOSITION OF RELATIONS

If R is a relation from set A to set B and S is a relation from set B to set C, viz., R is a subset of $A \times B$ and S is a subset of $B \times C$, then *the composition of R and S*, denoted by $R \bullet S$, [some authors use the notation $S \bullet R$ instead of $R \bullet S$] is defined by

$a(R \bullet S)\ c$, if for some $b \in B$, we have $a \text{ R } b$ and $b \text{ R } c$.

viz., $R \bullet S = \{(a, c) |$ there exists some $b \in B$ for which $(a, b) \in R$ and $(b, c) \in S\}$

Note

1. For the relation $R \bullet S$, the domain is a subset of A and the range is a subset of C.
2. $R \bullet S$ is empty, if the intersection of the range of R and the domain of S is empty.
3. If R is a relation on a set A, then $R \bullet R$, the composition of R with itself is always defined and sometimes denoted as R^2.

For example, let $R = \{(1, 1), (1, 3), (3, 2), (3, 4), (4, 2)\}$ and $S = \{(2, 1), (3, 3), (3, 4), (4, 1)\}$.

Any member (ordered pair) of $R \bullet S$ can be obtained only if the second element in the ordered pair of R agrees with the first element in the ordered pair of S.

Thus (1, 1) cannot combine with any member of S.

(1, 3) of R can combine with (3, 3) and (3, 4) of S producing the members (1, 3) and (1, 4) respectively of $R \bullet S$. Similarly the other members of $R \bullet S$ are obtained.

Thus $R \bullet S = \{(1, 3), (1, 4), (3, 1), (4, 1)\}$

Similarly, $S \bullet R = \{(2, 1), (2, 3), (3, 2), (3, 4), (4, 1), (4, 3)\}$

$R \bullet R = \{(1, 1), (1, 3), (1, 2), (1, 4), (3, 2)\}$

$S \bullet S = \{(3, 3), (3, 4), (3, 1)\}$

$$(R \bullet S) \bullet R = \{(1, 2), (1, 4), (3, 1), (3, 3), (4, 1), (4, 3)\}$$
$$R \bullet (S \bullet R) = \{(1, 2), (1, 4), (3, 1), (3, 3), (4, 1), (4, 3)\}$$
$$R^3 = R \bullet R \bullet R = (R \bullet R) \bullet R = R \bullet (R \bullet R)$$
$$= \{(1, 1), (1, 3), (1, 2), (1, 4)\}$$

PROPERTIES OF RELATIONS

(i) A relation R on a set A is said to be *reflexive*, if a R a for every $a \in A$, viz., if $(a, a) \in R$ for every $a \in A$.
For example, if R is the relation on $A = \{1, 2, 3\}$ defined by $(a, b) \in R$ if $a \leq b$, where $a, b \in A$, then $R = \{(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)\}$. Now R is reflexive, since each of the elements of A is related to itself, as $(1, 1)$, $(2, 2)$ and $(3, 3)$ are members in R.

Note A relation R on a set A is irreflexive, if, for every $a \in A$, $(a, a) \notin R$. viz., if there is no $a \in A$ such that a R a.
For example, R, $\{(1, 2), (2, 3), (1, 3)\}$ in the above example is irreflexive.

(ii) A relation R on a set A is said to be *symmetric*, if whenever a R b then b R a, viz., if whenever $(a, b) \in R$ then (b, a) also $\in R$.
Thus a relation R on A is not symmetric if there exist $a, b \in A$ such that $(a, b) \in R$, but $(b, a) \notin R$.

(iii) A relation R on a set A is said to be *antisymmetric*, whenever (a, b) and $(b, a) \in R$ then $a = b$. If there exist $a, b \in A$ such that (a, b) and $(b, a) \in R$, but $a \neq b$, then R is not antisymmetric.
For example, the relation of perpendicularity on a set of lines in the plane is symmetric, since if a line a is perpendicular to the line b, then b is perpendicular to a.
The relation $\leq$ on the set Z of integers is not symmetric, since, for example, $4 \leq 5$, but $5 \not\leq 4$.
The relation of divisibility on N is antisymmetric, since whenever m is divisible by n and n is divisible by m then $m = n$.

Note Symmetry and antisymmetry are not negative of each other. For example, the relation $R = \{(1, 3), (3, 1), (2, 3)\}$ is neither symmetric nor antisymmetric, whereas the relation $S = \{(1, 1), (2, 2)\}$ is both symmetric and antisymmetric.

(iv) A relation R on a set A is said to be *transitive*, if whenever a R b and b R c then a R c. viz., if whenever (a, b) and $(b, c) \in R$ then $(a, c) \in R$.
Thus if there exist $a, b, c \in A$ such that (a, b) and $(b, c) \in R$ but $(a, c) \notin R$, then R is not transitive.
For example, the relation of set inclusion on a collection of sets is transitive, since if $A \subseteq B$ and $B \subseteq C$, $A \subseteq C$.

(v) A relation R on a set A is called an *equivalence relation*, if R is reflexive, symmetric and transitive.
viz., R is an equivalence relation on a set A, if it has the following three properties:

1. a R a, for every $a \in A$

2. If a R b, then b R a
3. If a R b and b R c, then a R c

For example, the relation of similarity with respect to a set of triangles T is an equivalence relation, since if T_1, T_2, T_3 are elements of the set T, then

$T_1 \,|||\, T_1$, i.e., T_1 R T_1 for every $T_1 \in T$,
$T_1 \,|||\, T_2$ implies $T_2 \,|||\, T_1$ and
$T_1 \,|||\, T_2$ and $T_2 \,|||\, T_3$ simplify $T_1 \,|||\, T_3$

viz., the relation of similarity of triangles is reflexive, symmetric and transitive.

(vi) A relation R on a set A is called a *partial ordering* or *partial order relation*, if R is reflexive, antisymmetric and transitive.

viz., R is a partial order relation on A if it has the following three properties:

(a) a R a, for every $a \in A$
(b) a R b and b R $a \Rightarrow a = b$
(c) a R b and b R $c \Rightarrow a$ R c

A set A together with a partial order relation R is called *a partially ordered set* or *poset*. For example, the greater than or equal to ($\geq$) relation is a partial ordering on the set of integers Z, since

(a) $a \geq a$ for every integer a, i.e. $\geq$ is reflexive
(b) $a \geq b$ and $b \geq a \Rightarrow a = b$, i.e. $\geq$ is antisymmetric
(c) $a \geq b$ and $b \geq c \Rightarrow a \geq c$, i.e. $\geq$ is transitive

Thus $(Z, \geq)$ is a poset.

EQUIVALENCE CLASSES

Definition

If R is an equivalence relation on a set A, the set of all elements of A that are related to an element a of A is called the *equivalence class of a* and denoted by $[a]_R$.

When there is no ambiguity regarding the relation, viz., when we deal with only one relation, the equivalence class of a is denoted by just $[a]$.

In other words, the equivalence class of a under the relation R is defined as

$$[a] = \{x | (a, x) \in R\}$$

Any element $b \in [a]$ is called *a representative* of the equivalence class $[a]$.

The collection of all equivalence classes of elements of A under an equivalence relation R is denoted by A/R and is called the *quotient set* of A by R.

viz. $$A/R = \{[a] | a \in A\}$$

For example, the relation R on the set $A = \{1, 2, 3\}$ defined by $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}$ is an equivalence relation, since R is reflexive symmetric and transitive.

Now $[1] = \{1, 2\}$, $[2] = \{1, 2\}$ and $[3] = \{3\}$

Thus [1], [2] and [3] are the equivalence classes of A under R and hence form A/R.

Theorem

If R is an equivalence relation on non-empty set A and if a and $b \in A$ are arbitrary, then

(i) $a \in [a]$, for every $a \in A$

(ii) $[a] = [b]$, if and only if $(a, b) \in R$

(iii) If $[a] \cap [b] \neq \phi$, then $[a] = [b]$

Proof:

(i) Since R is reflexive, $(a, a) \in R$ for every $a \in A$.
Hence $a \in [a]$

(ii) Let us assume that $(a, b) \in R$ or a R b (1)
Let $x \in [b]$. Then $(b, x) \in R$ or b R x (2)
From (1) and (2), it follows that a R x or $(a, x) \in R$ ($\because$ R is transitive)
$\therefore$ $x \in [a]$
Thus $x \in [b] \Rightarrow x \in [a]$ $\therefore$ $[b] \subseteq [a]$ (3)
Let $y \in [a]$. Then a R y (4)
From (1), we have b R a, since R is symmetric. (5)
From (5) and (4), we get b R y, since R is transitive.
$\therefore$ $y \in [b]$
Thus $y \in [a] \Rightarrow y \in [b]$ $\therefore$ $[a] \subseteq [b]$ (6)
From (3) and (6), we get $[a] = [b]$
Conversely, let $[a] = [b]$
Now $b \in [b]$ by (i)
i.e., $b \in [a]$ $\therefore$ $(a, b) \in R$

(iii) Since $[a] \cap [b] \neq \phi$, there exists an element $x \in A$ such that $x \in [a] \cap [b]$
Hence $x \in [a]$ and $x \in [b]$
i.e., x R a and x R b
or a R x and x R b
$\therefore$ a R b, since R is transitive
Hence, by (ii), $[a] = [b]$
Equivalently, if $[a] \neq [b]$, then $[a] \cap [b] = \phi$.

Note From (ii) and (iii) of the above theorem, it follows that the equivalence classes of two arbitrary elements under R are identical or disjoint.)

PARTITION OF A SET

Definition

If S is a non empty set, a collection of disjoint non empty subsets of S whose union is S is called *a partition* of S. In other words, the collection of subsets A_i is a partition of S if and only if

(i) $A_i \neq \phi$, for each i

(ii) $A_i \cap A_j = \phi$, for $i \neq j$ and

(iii) $\bigcup_i A_i = S$, where $\bigcup_i A_i$ represents the union of the subsets A_i for all i.

Note The subsets in a partition are also called *blocks* of the partition. For example, if $S = \{1, 2, 3, 4, 5, 6\}$

(i) $[\{1, 3, 5\}, \{2, 4\}]$ is not a partition, since the union of the subsets is not S, as the element 6 is missing.

(ii) [{1, 3}, {3, 5}, {2, 4, 6}] is not a partition, since {1, 3} and {3, 5} are not disjoint.

(iii) [{1, 2, 3}, {4, 5}, {6}] is a partition.

PARTITIONING OF A SET INDUCED BY AN EQUIVALENCE RELATION

Let R be an equivalence relation of a non-empty set A.

Let $A_1, A_2, \ldots, A_k$ be the distinct equivalence classes of A under R. For every $a \in A_i$, $a \in [a]_R$, by the above theorem.

$$\therefore \qquad A_i = [a]_R$$

$$\therefore \qquad \bigcup_{a \in A_i} [a]_R = \bigcup_i A_i = A$$

Also by the above theorem, when $[a]_R \neq [b]_R$, then

$$[a]_R \cap [b]_R = \phi. \text{ viz., } A_i \cap A_j = \phi, \text{ if } [a]_R = A_i \text{ and } [b]_R = A_j$$

$\therefore$ The equivalence classes of A form a partition of A.
In other words, the quotient set A/R is a partition of A.
For example, let $A \equiv$ {blue, brown, green, orange, pink, red, white, yellow} and R be the equivalence relation of A defined by "has the same number of letters", then

A/R = [{red}, {blue, pink}, {brown, green, white}, {orange, yellow}]

The equivalence classes contained in A/R form a partition of A.

MATRIX REPRESENTATION OF A RELATION

If R is a relation from the set $A = \{a_1, a_2, \ldots, a_m\}$ to the set $B = \{b_1, b_2, \ldots, b_n\}$, where the elements of A and B are assumed to be in a specific order, the relation R can be represented by the matrix

$$M_R = [m_{ij}], \text{ where}$$

$$m_{ij} = \begin{cases} 1, & \text{if } (a_i, b_j) \in R \\ 0, & \text{if } (a_i, b_j) \notin R. \end{cases}$$

In other words, the zero-one matrix M_R has a 1 in its $(i - j)$th position when a_i is related to b_j and a 0 in this position when a_i is not related by b_j.

For example, if $A = [a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4\}$ and $R = \{(a_1, b_2), (a_2, b_1), (a_2, b_3), (a_2, b_4), (a_3, b_2), (a_3, b_4)\}$, then the matrix of R is given by

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Conversely, if R is the relation on $A = \{1, 3, 4\}$ represented by

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

then $R = \{(1, 1), (1, 3), (3, 3), (4, 4)\}$, since $m_{ij} = 1$ means that the ith element of A is related to the jth element of A.

1. If R and S are relations on a set A, represented by M_R and M_S respectively, then the matrix representing $R \cup S$ is the *join* of M_R and M_S obtained by putting 1 in the positions where either M_R or M_S has a 1 and denoted by $M_R \vee M_S$ i.e., $M_{R \cup S} = M_R \vee M_S$.
2. The matrix representing $R \cap S$ is the *meet* of M_R and M_S obtained by putting 1 in the positions where both M_R and M_S have a 1 and denoted by $M_R \wedge M_S$ i.e., $M_{R \cap S} = M_R \wedge M_S$.

Note The operations 'join' and 'meet', denoted by $\vee$ and $\wedge$ respectively are Boolean operations which will be discussed later in the topic on Boolean Algebra.

For example, if R and S are relations on a set A represented by the matrices

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad M_S = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \text{ respectively,}$$

then $M_{R \cup S} = M_R \vee M_S$

$$= \begin{bmatrix} 1 \vee 1 & 0 \vee 0 & 1 \vee 1 \\ 0 \vee 1 & 1 \vee 0 & 1 \vee 0 \\ 1 \vee 0 & 0 \vee 1 & 0 \vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

and $M_{R \cap S} = M_R \wedge M_S$

$$= \begin{bmatrix} 1 \wedge 1 & 0 \wedge 0 & 1 \wedge 1 \\ 0 \wedge 1 & 1 \wedge 0 & 1 \wedge 0 \\ 1 \wedge 0 & 0 \wedge 1 & 0 \wedge 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

3. If R is a relation from a set A to a set B represented by M_R, then the matrix representing R^{-1} (the inverse of R) is M_R^T, the transpose of M_R. For example, if $A = \{2, 4, 6, 8\}$ and $B = \{3, 5, 7\}$ and if R is defined by $\{(2, 3), (2, 5), (4, 5), (4, 7), (6, 3), (6, 7), (8, 7)\}$, then

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

R^{-1} is defined by $\{(3, 2), (5, 2), (5, 4), (7, 4), (3, 6), (7, 6), (7, 8)$

Now $$M_{R^{-1}} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} = M_R^T.$$

4. If R is a relation from A to B and S is a relation from B to C, then the composition of the relations R and S (if defined), viz., $R \bullet S$ is represented by the Boolean product of the matrices M_R and M_S, denoted by $M_R \bullet M_S$.

   Note The Boolean product of two matrices is obtained in a way similar to the ordinary product, but with multiplication replaced by the Boolean operation $\wedge$ and with addition replaced by the Boolean operation $\vee$.

   For example, the matrix representing $R \bullet S$

   $$\text{where} \quad M_R = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad M_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

   $$M_{R \bullet S} = M_R \odot M_S = \begin{bmatrix} 0 \vee 0 \vee 0 & 0 \vee 1 \vee 0 & 0 \vee 1 \vee 0 \\ 0 \vee 0 \vee 1 & 1 \vee 1 \vee 1 & 0 \vee 1 \vee 1 \\ 0 \vee 0 \vee 0 & 1 \vee 0 \vee 0 & 0 \vee 0 \vee 0 \end{bmatrix}$$

   $$= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

5. Since the relation R on the set $A = \{a_1, a_2, \ldots, a_n\}$ is reflexive if and only if $(a_i, a_i) \in R$ for $i = 1, 2, \ldots, n$, $m_{ii} = 1$ for $i = 1, 2, \ldots, n$. In other words, R is reflexive if all the elements in the principal diagonal of M_R are equal to 1.
6. Since the relation R on the set $A = \{a_2, a_2, \ldots, a_n\}$ is symmetric if and only if $(a_j, a_i) \in R$ whenever $(a_i, a_j) \in R$, we will have $m_{ji} = 1$ whenever $m_{ij} = 1$ (or equivalently $m_{ji} = 0$ whenever $m_{ij} = 0$). In other words, R is symmetric if and only if $m_{ij} = m_{ji}$, for all pairs of integers i and j ($i, j = 1, 2, \ldots, n$). This means that R is symmetric, if $M_R = (M_R)^T$, viz., M_R is a symmetric matrix.

   Note The matrix of an antisymmetric relation has the property that if $m_{ij} = 1$ $(i \neq j)$, then $m_{ji} = 0$.
7. There is no simple way to test whether a relation R on a set A is transitive by examining the matrix M_R. However, we can easily verify that a relation R is transitive if and only if $R^n \subseteq R$ for $n \geq 1$.

REPRESENTATION OF RELATIONS BY GRAPHS

Let R be a relation on a set A. To represent R graphically, each element of A is represented by a point. These points are called *nodes* or *vertices*. Whenever the element a is related to the element b, an arc is drawn from the point 'a' to the point 'b'. These arcs are called *arcs* or *edges*. The arcs start from the first element of the related pair and go to the second element. The direction is indicated by an arrow. The resulting diagram is called the *directed graph* or *digraph* of R.

The edge of the form (a, a), represented by using an arc from the vertex a back to itself, is called a *loop*.

For example, if $A = \{2, 3, 4, 6\}$ and R is defined by a R b if a divides b, then

$R = (2, 2), (2, 4), (2, 6), (3, 3), (3, 6), (4, 4), (6, 6)\}$

The digraph representing the relation R is given in Fig. 5.14.

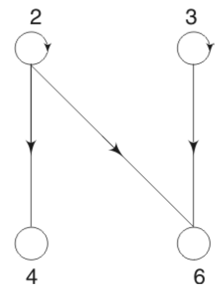

Fig. 5.14

Note The digraph of R^{-1}, the inverse of R, has exactly the same edges of the digraph of R, but the directions of the edges are reversed.

The digraph representing a relation can be used to determine whether the relation has the standard properties explained as follows:

(i) A relation R is reflexive if and only if there is a loop at every vertex of the digraph of the relation R, so that every ordered pair of the form (a, a) occurs in R. If no vertex has a loop, then R is irreflexive.

(ii) A relation R is symmetric if and only if for every edge between distinct vertices in its digraph there is an edge in the opposite direction, so that (b, a) is in R whenever (a, b) is in R.

(iii) A relation R is antisymmetric if and only if there are never two edges in opposite directions between distinct vertices.

(iv) A relation R is transitive if and only if whenever there is an edge from a vertex a to a vertex b and from the vertex b to a vertex c, there is an edge from a to c.

HASSE DIAGRAMS FOR PARTIAL ORDERINGS

The simplified form of the digraph of a partial ordering on a finite set that contains sufficient information about the partial ordering is called a *Hasse diagram*, named after the twentieth-century mathematician Helmut Haasse.

The simplification of the digraph as a Hasse diagram is achieved in three ways:

(i) Since the partial ordering is a reflexive relation, its digraph has loops at all vertices. We need not show these loops since they must be present.

(ii) Since the partial ordering is transitive, we need not show those edges that must be present due to transitivity. For example, if (1, 2) and (2, 3) are edges in the digraph of a partial ordering, (1, 3) will also be an edge due to transitivity. This edge (1, 3) need not be shown in the corresponding Hasse diagram.

(iii) If we assume that all edges are directed upward, we need not show the directions of the edges.

Thus the Hasse diagram representing a partial ordering can be obtained from its digraph, by removing all the loops, by removing all edges that are present due to transitivity and by drawing each edge without arrow so that its initial vertex is below its terminal vertex.

For example, let us construct the Hasse diagram for the partial ordering $\{(a, b)|\ a \le b\}$ on the set $\{1, 2, 3, 4\}$ starting from its digraph. (Fig. 5.15)

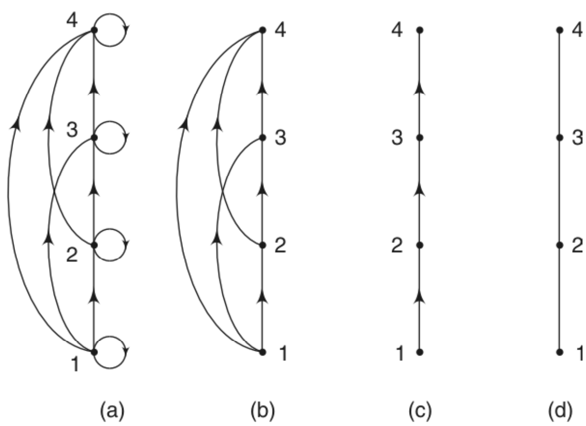

Fig. 5.15

TERMINOLOGY RELATED TO POSETS

We have already defined *poset* as a set S together with a partial order relation R. In a poset the notation $a \le b$ (or equivalently $a \preccurlyeq b$) denotes that $(a, b) \in R$. $a \le b$ is read as "a precedes b" or "b succeeds a".

Definitions

When $\{P, \le\}$ is a poset, an element $a \in P$ is called a *maximal member* of P, if there is no element $b \in P$ such that $a < b$ (viz., a strictly precedes b).

Similarly, an element $a \in P$ is called a *minimal member* of P, if there is no element $b \in P$ such that $b < a$.

If there exists an element $a \in P$ such that $b \le a$ for all $b \in P$, then a is called the *greatest member* of the poset $\{P, \le\}$.

Similarly if there exists an element $a \in P$ such that $a \le b$ for all $b \in P$, then a is called the *least member* of the poset $\{P, \le\}$.

Note

1. The maximal, minimal, the greatest and least members of a poset can be easily identified using the Hasse diagram of the poset. They are the top and bottom elements in the diagram.
2. A poset can have more than one maximal member and more than one minimal member, whereas the greatest and least members, when they exist, are unique.

For example, let us consider the Hasse diagrams of four posets given in Fig. 5.16.

For the poset with Hasse diagram 5.16(a), a and b are minimal elements and d and e are maximal elements, but the poset has neither the greatest nor the least element.

For the poset with Hasse diagram (b), a and b are minimal elements and d is the greatest element (also the only maximal element). There is no least element.

For the poset with Hasse diagram (c), a is the least element (also the only minimal element) and c and d are maximal elements. There is no greatest element.

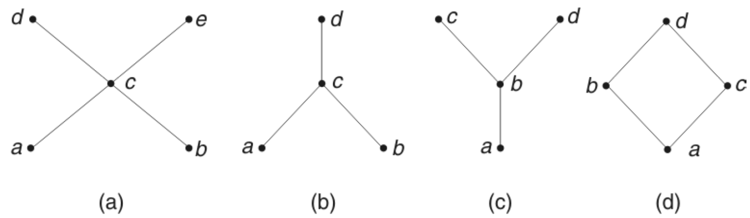

Fig. 5.16

For the poset with Hasse diagram (d), a is the least element and d is the greatest element.

Definitions

When A is a subset of a poset $\{P, \leq\}$ and if u is an element of P such that $a \leq u$ for all elements $a \in A$, then u is called an *upper bound* of A. Similarly if l is an element of P such that $l \leq a$ for all elements $a \in A$, then l is called a *lower bound* of A.

Note The upper and lower bounds of a subset of a poset are not necessarily unique.

The element x is called the *least upper bound* (LUB) or *supremum* of the subset A of a poset $\{P, \leq\}$, if x is an upper bound that is less than every other upper bound of A.

Similarly the element y is called the *greatest lower bound* (GLB) or *infimum* of the subset A of a poset $\{P, \leq\}$, if y is a lower bound that is greater than every other lower bound of A.

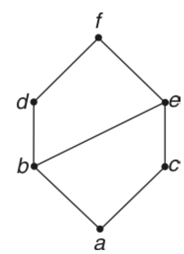

Fig. 5.17

Note The LUB and GLB of a subset of a poset, if they exist, are unique.

For example, let us consider the poset with the Hasse diagram given in Fig. 5.17.

The upper bounds of the subset $\{a, b, c\}$ are e and f. [*Note:* d is not an upper bound, since c is not related to d] and LUB of $\{a, b, c\}$ is e.

The lower bounds of the subset $\{d, e\}$ are a and b and GLB of $\{d, e\}$ is b.

Note c is not a lower bound, since c is not related to d.

WORKED EXAMPLES 5(B)

Example 5.1 List the ordered pairs in the relation R from $A = \{0, 1, 2, 3, 4\}$ to $B = \{0, 1, 2, 3\}$ where $(a, b) \in R$ if and only if (i) $a = b$, (ii) $a + b = 4$, (iii) $a > b$, (iv) $a|b$ (viz., a divides b), (v) $gcd(a, b) = 1$ and (vi) $lcm\,(a, b) = 2$.

(i) Since $a \in A$ and $b \in B$ and a R b when $a = b$, $R = \{(0, 0), (1, 1), (2, 2), (3, 3)\}$.

(ii) Since a R b if and only if $a + b = 4$, $R = \{(1, 3), (2, 2), (3, 1), (4, 0)\}$.

(iii) Since a R b, if and only if $a > b$, $R = \{(1, 0), (2, 0), (2, 1), (3, 0), (3, 1), (3, 2), (4, 0), (4, 1), (4, 2), (4, 3)\}$.

(iv) Since a R b, if and only if $a|b$, $R = \{(1, 0), (1, 1), (1, 2), (1, 3), (2, 0), (2, 2), (3, 0), (3, 3), (4, 0)\}$.

Note $\frac{0}{0}$ is indeterminate and so 0 does not divide 0.

(v) Since a R b, if and only if $gcd\ (a, b) = 1$, $R = \{(0, 1), (1, 0), (1, 1), (1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2), (4, 1), (4, 3)\}$.

(vi) Since a R b, if and only if $lcm\ (a, b) = 2$, $R = \{(1, 2), (2, 1), (2, 2)\}$.

Example 5.2 The relation R on the set $A = \{1, 2, 3, 4, 5\}$ is defined by the rule $(a, b) \in R$, if 3 divides $a - b$.

(i) List the elements of R and R^{-1},
(ii) Find the domain and range of R.
(iii) Find the domain and range of R^{-1}.
(iv) List the elements of the complement of R.

The Cartesian product $A \times A$ consists of $\{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 1), (2, 2), \ldots, (2, 5), (3, 1), (3, 2) \ldots, (3, 5), (4, 1), (4, 2), \ldots, (4, 5), (5, 1), (5, 2), \ldots, (5, 5)\}$

(i) Since $(a, b) \in R$, if 3 divides $(a - b)$, $R = \{(1, 1), (1, 4), (2, 2), (2, 5), (3, 3), (4, 1), (4, 4), (5, 2), (5, 5)\}$

R^{-1} (the inverse of R) $= \{(1, 1), (4, 1), (2, 2), (5, 2), (3, 3), (1, 4), (4, 4), (2, 5), (5, 5)\}$

We note that $R^{-1} = R$

(ii) Domain of R = Range of $R = \{1, 2, 3, 4, 5\}$
(iii) Domain of R^{-1} = Range of $R^{-1} = \{1, 2, 3, 4, 5\}$
(iv) R' (the complement of R) = the elements of $A \times A$, that are not in $R = \{(1, 2), (1, 3), (1, 5), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (3, 4), (3, 5), (4, 2), (4, 3), (4, 5), (5, 1), (5, 3), (5, 4)\}$

Example 5.3 If $R = \{(1, 2), (2, 4), (3, 3)\}$ and $S = \{(1, 3), (2, 4), (4, 2)\}$, find (i) $R \cup S$, (ii) $R \cap S$, (iii) $R - S$, (iv) $S - R$, (v) $R \oplus S$. Also verify that dom $(R \cup S) = \text{dom}(R) \cup \text{dom}\ (S)$ and range $(R \cap S) \subseteq$ range $(R) \cap$ range (S).

(i) $R \cup S = \{(1, 2), (1, 3), (2, 4), (3, 3), (4, 2)\}$
(ii) $R \cap S = \{(2, 4)\}$
(iii) $R - S = \{(1, 2), (3, 3)\}$
(iv) $S - R = \{(1, 3), (4, 2)\}$
(v) $R \oplus S = (R \cup S) - (R \cap S)$

$$= \{(1, 2), (1, 3), (3, 3), (4, 2)\}$$

$$\text{dom}\ (R) = \{1, 2, 3\};\ \text{dom}\ (S) = \{1, 2, 4\}$$

Now dom $(R) \cup$ dom $(S) = \{1, 2, 3, 4\}$
= domain $(R \cup S)$

Range $(R) = \{2, 3, 4\}$; Range $(S) = \{2, 3, 4\}$

Range $(R \cap S) = \{4\}$

Clearly $\{4\} \subseteq \{2, 3, 4\} \cap \{2, 3, 4\}$

i.e., Range $(R \cap S) \subseteq$ Range $(R) \cap$ Range (S).

Example 5.4 R and S are "Congruent modulo 3" and "Congruent modulo 4" relations respectively on the set of integers. That is $R = \{(a, b)|a \equiv b \pmod 3\}$ and $S = \{a, b)|a \equiv b \pmod 4\}$.

Find (i) $R \cup S$, (ii) $R \cap S$, (iii) $R - S$, (iv) $S - R$, (v) $R \oplus S$.

$R = \{(a, b)$, where $(a - b)$ is a multiple of 3

i.e. $a - b = \ldots, -9, -6, -3, 0, 3, 6, 9, \ldots$

i.e. $a - b = \{\ldots, -9, 3, 15, 27, 39, \ldots\}, \{\ldots, -6, 6, 18, 30, \ldots\}, \{\ldots, -3, 9, 21, 33, \ldots\}, \{\ldots, 0, 12, 24, 36, \ldots\}$

i.e. $a - b = 3 \pmod{12}$ or $6 \pmod{12}$ or $9 \pmod{12}$ or $0 \pmod{12}$ (1)

$S = \{(a, b)\}$, where $(a - b)$ is a multiple of 4

i.e. $a - b = \ldots, -12, -8, -4, 0, 4, 8, 12, \ldots$

i.e. $a - b = \{\ldots, -8, 4, 16, 28, \ldots\}, \{\ldots, -16, -4, 8, 20, \ldots\}, \{\ldots, -24, -12, 0, 12, 24, \ldots\}$

i.e. $a - b = 4 \pmod{12}$ or $8 \pmod{12}$ or $0 \pmod{12}$ (2)

$\therefore$ $R \cup S = \{(a, b) | a - b = 0 \pmod{12}, 3 \pmod{12}, 4 \pmod{12}, 6 \pmod{12}, 8 \pmod{12}$ or $9 \pmod{12}\}$

$R \cap S = \{(a, b) | a - b = 0 \pmod{12}$, from (1) and (2)

$R - S = \{(a, b) | a - b = 3 \pmod{12}, 6 \pmod{12}$ or $9 \pmod{12}\}$

$S - R = \{(a, b) | \; a - b = 4 \pmod{12}$ or $8 \pmod{12}\}$

$R \oplus S = \{(a, b) | a - b = 3 \pmod{12}, 4 \pmod{12}, 6 \pmod{12}, 8 \pmod{12}$ or $9 \pmod{12}\}$.

Example 5.5 If the relations $R_1, R_2, \ldots, R_6$ are defined on the set of real numbers as given below,

$$R_1 = \{(a, b) | a > b\}, \quad R_2 = \{(a, b) | a \geq b\},$$
$$R_3 = \{(a, b) | a < b\}, \quad R_4 = \{(a, b) | a \leq b\},$$
$$R_5 = \{(a, b) | a = b\}, \quad R_6 = \{(a, b) | a \neq b\},$$

find the following composite relations:

$R_1 \bullet R_2$, $R_2 \bullet R_2$, $R_1 \bullet R_4$, $R_3 \bullet R_5$, $R_5 \bullet R_3$, $R_6 \bullet R_3$, $R_6 \bullet R_4$ and $R_6 \bullet R_6$

(i) $R_1 \bullet R_2 = R_1$. For example, let $(5, 3) \in R_1$ and let $(3, 1), (3, 2), (3, 3) \in R_2$
Then $R_1 \bullet R_2$ consists of $(5, 1), (5, 2), (5, 3)$ which belong to R_1

(ii) $R_2 \bullet R_2 = R_2$. For example, let $(5, 5), (5, 3), (5, 2) \in R_2$
Then $R_2 \bullet R_2 = \{(5, 5), (5, 3), (5, 2)\} = R_2$

(iii) $R_1 \bullet R_4 = R^2$ (the entire 2 dimensional vector space). For example, let $R_1 = \{(5, 4), (5, 3)\}$ and $R_4 = \{(4, 4), (4, 6), (3,3), (3, 5)\}$
Then $R_1 \bullet R_4 = \{(5, 4), (5, 6), (5, 3), (5, 5)\}$
Thus $R_1 \bullet R_4 = \{(a, b) | a > b, a = b$ and $a < b\}$

(iv) $R_3 \bullet R_5 = R_3$. For example, let $R_3 = \{(3, 4), (2, 4), (2, 5)\}$ and $R_5 = \{(3, 3), (4, 4), (5, 5)\}$
Then $R_3 \bullet R_5 = \{(3, 4), (2, 4), (2, 5)\} = R_3$

(v) $R_5 \bullet R_3 = R_3$. For example, let $R_5 = \{(3, 3), (4, 4), (5, 5)\}$ and $R_3 = \{(3, 4), (4, 6), (5, 7)\}$
Then $R_5 \bullet R_3 = \{(3, 4), (4, 6), (5, 7)\} = R_3$

(vi) $R_6 \bullet R_3 = R^2$. For exasmple, let $R_6 = \{(1, 2), (4, 3), (5, 2)\}$ and $R_3 = \{(2, 5), (3, 4), (2, 3)\}$
Then $R_6 \bullet R_3 = \{(1, 5), (1, 3), (4, 4), (5, 5), (5, 3)\}$
Thus $R_6 \bullet R_3 = \{(a, b) | a > b, a = b$ and $a < b\}$

(vii) $R_6 \bullet R_4 = R^2$. For example, let $R_6 = \{(1, 2), (4, 3), (5, 2)\}$ and $R_4 = \{(2, 3), (2, 5), (3, 3)\}$

Then $R_6 \bullet R_4 = \{(1, 3), (1, 5), (4, 3), (5, 3), (5, 5)\} \to R^2$

(viii) $R_6 \bullet R_6 = R^2$. For example, let $R_6 = \{(1, 2), (2, 1), (2, 3), (3, 2), (3, 4)\}$

Then $R_6 \bullet R_6 = \{(1, 1), (1, 3), (2, 2), (2, 4), (3, 1), (3, 3)\} \to R^2$

Example 5.6 Determine whether the relation R on the set of all integers is reflexive, symmetric, antisymmetric and/or transitive, where a R b if and only if (i) $a \neq b$, (ii) $ab \geq 0$, (iii) $ab \geq 1$, (iv) a is a multiple of b, (v) $a \equiv b$ (mod 7), (vi) $|a - b| = 1$, (vii) $a = b^2$, (viii) $a \geq b^2$.

(i) '$a \neq a$' is not true. Hence R is not reflexive

$a \neq b \Rightarrow b \neq a$. $\therefore$ R is symmetric

$a \neq b$ and $b \neq c$ does not necessarily imply that $a \neq c$. $\therefore$ R is not transitive

Hence R is symmetric only.

(ii) $a^2 \geq 0$. $\therefore$ R is reflexive.

$ab \geq 0 \Rightarrow ba \geq 0$. $\therefore$ R is symmetric.

Consider (2, 0) and (0, –3), that belong to R. But $(2, -3) \notin R$, as $2(-3) < 0$. $\therefore$ R is not transitive.

$\therefore$ R is reflexive, symmetric and not transitive.

(iii) '$a^2 \geq 1$' need not be true, since a may be zero. $\therefore$ R is not reflexive.

$ab \geq 1 \Rightarrow ba \geq 1$ $\therefore$ R is symmetric.

$ab \geq 1$ and $bc \geq 1 \Rightarrow$ all of $a, b, c > 0$ or < 0

If all of $a, b, c > 0$, least a = least b = least $c = 1$

$\therefore$ $ac \geq 1$

If all of $a, b, c < 0$, greatest a = greatest b = greatest $c = -1$

$\therefore$ $ac \geq 1$. Hence R is transitive.

$\therefore$ R is symmetric and transitive.

(iv) a is a multiple of a. $\therefore$ R is reflexive. If a is a multiple of b, b is not a multiple of a in general. But if a is a multiple of b and b is a multiple of a, then $a = b$.

$\therefore$ R is antisymmetric.

When a is a multiple of b and b is a multiple of c, then a is a multiple of c.

$\therefore$ R is transitive.

Thus R is reflexive, antisymmetric and transitive.

(v) $(a - a)$ is a multiple of 7 $\therefore$ R is reflexive. When $(a - b)$ is a multiple of 7, $(b - a)$ is also a multiple of 7. $\therefore$ R is symmetric.

When $(a - b)$ and $(b - c)$ are multiples of 7, $(a - b) + (b - c) = (a - c)$ is also a multiple of 7.

$\therefore$ R is transitive.

Hence R is reflexive, symmetric and transitive.

(vi) $|a - a| \neq 1$. $\therefore$ R is not reflexive

$|a - b| = 1 \Rightarrow |b - a| = 1$. $\therefore$ R is symmetric.

$|a - b| = 1 \Rightarrow a - b = 1$ or -1 (1)

$|b - c| = 1 \Rightarrow b - c = 1$ or -1 (2)

(1) + (2) gives $a - c = \pm 2$ or 0
i.e. $|a - c| = 2$ or 0
i.e. $|a - c| \neq 1$
Hence R is symmetric only.

(vii) '$a = a^2$' is not true for all integers.
$\therefore$ R is not reflexive.
$a = b^2$ and $b = a^2$, for $a = b = 0$ or 1
$\therefore$ R is antisymmetric.
$a = b^2$ and $b = c^2$ does not simply $a = c^2$
$\therefore$ R is not transitive
Hence R is antisymmetric only.

(viii) '$a \geq a^2$' is not true for all integers.
$\therefore$ R is not reflexive.
$a \geq b^2$ and $b \geq a^2$ imply that $a = b$
$\therefore$ R is antisymmetric
When $a \geq b^2$ and $b \geq c^2$, $a \geq c^2$
$\therefore$ R is transitive
Hence R is antisymmetric and transitive.

Example 5.7 Which of the following relations on {0, 1, 2, 3} are equivalence relations? Find the properties of an equivalence relation that the others lack.

(a) $R_1 = \{(0, 0), (1, 1), (2, 2), (3, 3)\}$
(b) $R_2 = \{(0, 0), (0, 2), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3)$
(c) $R_3 = \{(0, 0), (1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}$
(d) $R_4 = \{(0, 0), (1, 1), (1, 3), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$
(e) $R_5 = \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (3, 3)\}$

(a) R_1 is reflexive, symmetric and transitive.
$\therefore$ R_1 is an equivalence relation.
(b) R_2 is reflexive
R_2 is symmetric, but not transitive, since (3, 2) and (2, 0) $\in R_2$, but (3, 0) $\notin R_2$
$\therefore$ R_2 is not an equivalence relation.
(c) R_3 is reflexive, symmetric and transitive. $\therefore$ R_3 is an equivalence relation.
(d) R_4 is reflexive and symmetric, but not transitive, since (1, 3) and (3, 2) $\in R_4$, but (1, 2) $\notin R_4$. $\therefore$ R_4 is not an equivalence relation.
(e) R_5 is reflexive, but not symmetric since (1, 2) $\in R$, but (2, 1) $\notin R$.
Also R_5 is not transitive, since (2, 0) and (0, 1) $\in R$, but (2, 1) $\notin R$.
$\therefore$ R_5 is not an equivalence relation.

Example 5.8 Show that the following relations are equivalence relations:

(i) R_1 is the relation on the set of integers such that aR_1b if and only if $a = b$ or $a = -b$.
(ii) R_2 is the relation on the set of integers such that aR_2b if and only if $a \equiv b$ (mod m), where m is a positive integer > 1.
(iii) R_3 is the relation on the set of real numbers such that aR_3b if and only if $(a - b)$ is an integer.

(i) $a = a$ or $a = -a$, which is true for all integers.

$\therefore$ R_1 is reflexive.

When $a = b$ or $a = -b$, $b = a$ or $b = -a$.

$\therefore$ R_1 is symmetric

When $a, b, c \geq 0$, $a = b$ and $b = c$, if aR_1b and bR_1c

$\therefore$ $a = c$, i.e., aRc

Similarly when $a \geq 0$, $b \leq 0$, $c \leq 0$, we have $a = -b$ and $b = c$, if aR_1b and bR_1c.

$\therefore$ $a = -c$. i.e., aR_1c.

The result is true for all positive and negative value combinations of a, b, c.

$\therefore$ R_1 is transitive.

Hence R_1 is an equivalence relation.

(ii) $(a - a)$ is multiple of m

$\therefore$ $a \equiv a \pmod{m}$ i.e., R_2 is reflexive.

When $a - b$ is multiple of m, $b - a$ is also a multiple of m.

i.e. $a \equiv b \pmod{m} \Rightarrow b \equiv a \pmod{m}$

$\therefore$ R_2 is symmetric.

When $(a - b) = k_1m$ and $b - c = k_2m$, we get $a - c = (k_1 + k_2)m$ (by addition)

$\therefore$ When $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$, $a \equiv c \pmod{m}$

$\therefore$ R_2 is transitive.

Hence R_2 is an equivalence relation.

(iii) $(a - a)$ is an integer. $\therefore$ R_3 is reflexive.

When $(a - b)$ is an integer, $(b - a)$ is an integer.

$\therefore$ R_3 is symmetric.

When $(a - b)$ and $(b - c)$ are integers, clearly $(a - c)$ is also an integer (by addition)

$\therefore$ R_3 is transitive.

Hence R_3 is an equivalence relation.

Example 5.9

(i) If R is the relation on the set of ordered pairs of positive integers such that $(a, b), (c, d) \in R$ whenever $ad = bc$, show that R is an equivalence relation.

(ii) if R is the relation on the set of positive integers such that $(a, b) \in R$ if and only if ab is a perfect square, show that R is an equivalence relation.

(i) $(a, b)\ R\ (a, b)$, since $ab = ba$

$\therefore$ R is reflexive.

When $(a, b)\ R\ (c, d)$, $ad = bc$ i.e., $cb = da$

This means that $(c, d)\ R\ (a, b)$

$\therefore$ R is symmetric.

When $(a, b)\ R\ (c, d)$, $ad = bc$ (1)

When $(c, d)\ R\ (e, f)$, $cf = de$ (2)

(1) and (2) gives $af = be$ ($\because$ c and d are > 0)

This means that $(a, b)\ R\ (e, f)$

$\therefore$ R is transitive

Hence R is an equivalence relation.

(ii) $(a, a) \in R_1$, since a^2 is a perfect square

$\therefore$ R is reflexive.

When ab is a perfect square, ba is also a perfect square.

i.e. $aRb \Rightarrow bRa$

$\therefore$ R is symmetric.

If, $a\ R\ b$, let $ab = x^2$ (1)

If $b\ R\ c$, let $bc = y^2$ (2)

(1) × (2) gives $ab^2c = x^2y^2$

$\therefore$ $ac = \left(\frac{xy}{b}\right)^2$ = a present square.

$\therefore$ aRc. i.e. R is transitive.

Hence R is an equivalence relation.

Example 5.10

(i) If R is the relation on the set of positive integers such that $(a, b) \in R$ if and only if $a^2 + b$ is even, prove that R is an equivalence relation.

(ii) If R is the relation on the set of integers such that $(a, b) \in R$, if and only if $3a + 4b = 7n$ for some integer n, prove that R is an equivalence relation.

(i) $a^2 + a = a(a + 1)$ = even, since a and $(a + 1)$ are consecutive positive integers.

$\therefore$ $(a, a) \in R$

Hence R is reflexive.

When $a^2 + b$ is even, a and b must be both even or both odd.

In either case, $b^2 + a$ is even

$\therefore$ $(a, b) \in R$ implies $(b, a) \in R$

Hence R is symmetric.

When a, b, c are even, $a^2 + b$ and $b^2 + c$ are even. Also $a^2 + c$ is even.

When a, b, c are odd, $a^2 + b$ and $b^2 + c$ are even. Also $a^2 + c$ is even.

Then $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ i.e., R is transitive.

$\therefore$ R is an equivalence relation.

(ii) $3a + 4a = 7a$, when a is an integer.

$\therefore$ $(a, a) \in R$. i.e., R is reflexive.

$$\begin{aligned} 3b + 4a &= 7a + 7b - (3a + 4b) \\ &= 7(a + b) - 7n \\ &= 7(a + b - n), \text{ where } a + b - n \text{ is an integer} \end{aligned}$$

$\therefore$ $(b, a) \in R$ when $(a, b) \in R$.

i.e. R is symmetric.

Let (a, b) and $(b, c) \in R$.

i.e. let $3a + 4b = 7m$ (1)

and $3b + 4c = 7n$ (2)

(1) and (2) gives, $3a + 4c = 7(m + n - b)$, where $m + n - b$ is an integer.

$\therefore$ $(a, c) \in R$

i.e. R is transitive

$\therefore$ R is an equivalence relation.

Example 5.11

(i) Prove that the relation $\subseteq$ of set inclusion is a partial ordering on any collection of sets.

(ii) If R is the relation on the set of integers such that $(a, b) \in R$ if and only if $b = a^m$ for some positive integer m, show that R is a partial ordering.

(i) $(A, B) \in R$, if and only if $A \subseteq B$, where A and B are any two sets.
Now $A \subseteq A$ $\therefore$ $(A, A) \in R$. i.e. R is reflexive.
If $A \subseteq B$ and $B \subseteq A$, then $A = B$.
i.e. R is antisymmetric.
If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$
i.e. $(A, B) \in R$ and $(B, C) \in R \Rightarrow (A, C) \in R$
$\therefore$ R is transitive
Hence R is a partial ordering.

(ii) $a = a^1$ $\therefore$ $(a, a) \in R$.
Let $(a, b) \in R$ and $(b, a) \in R$
i.e. $b = a^m$ and $a = b^n$
where m and n are positive integers. (1)
$\therefore$ $a = (a^m)^n = a^{mn}$.
This means that $mn = 1$ or $a = 1$ or $a = -1$
Case (1): If $mn = 1$, then $m = 1$ and $n = 1$
$\therefore$ $a = b$ [from (1)]
Case (2): If $a = 1$, then, from (1), $b = 1^m = 1 = a$
If $b = 1$, then, from (1), $a = 1^n = 1 = b$
Either way, $a = b$.
Case (3): If $a = -1$, then $b = -1$
Thus in all the three cases, $a = b$.
$\therefore$ R is antisymmetric.
Let $(a, b) \in R$ and $(b, c) \in R$
i.e. $b = a^m$ and $c = b^n$
$\therefore$ $c = (a^m)^n = a^{mn}$
$\therefore$ $(a, c) \in R$. i.e. R is transitive.
$\therefore$ R is a partial ordering.

Example 5.12

(i) If R is the equivalence relation on the set $A = \{1, 2, 3, 4, 5, 6\}$ given below, find the partition of A induced by R:
$R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4), (4, 5), (5, 4), (5, 5), (6, 6)\}$

(ii) If R is the equivalence relation on the set $A = \{(-4, -20), (-3, -9), (-2, -4), (-1, -11), (-1, -3), (1, 2), (1, 5), (2, 10), (2, 14), (3, 6), (4, 8), (4, 12)\}$, where $(a, b)\ R\ (c, d)$ if $ad = bc$, find the equivalent classes of R.

(i) The elements related to 1 are 1 and 2.
$\therefore$ $[1]_R = \{1, 2\}$
Also $[2]_R = \{1, 2\}$
The element related to 3 is 3 only
i.e. $[3]_R = \{3\}$
The elements related to 4 are $\{4, 5\}$

i.e. $[4]_R = \{4, 5\} = [5]_R$

The element related to 6 is 6 only

i.e. $[6]_R = \{6\}$

$\therefore$ $\{1, 2\}, \{3\}, \{4, 5\}, \{6\}$ is the partition induced by R.

(ii) The elements related to $(-4, -20)$ are $(1, 5)$ and $(2, 10)$

i.e. $$[(-4, -20)] = \{(-4, -20), (1, 5), (2, 10)\}$$

The elements related to $(-3, -9)$ are $(-1, -3)$ and $(4, 12)$

i.e. $$[(-3, -9)] = \{(-3, -9), (-1, -3), (4, 12)\}$$

The elements related to $(-2, -4)$ are $(-2, -4), (1, 2), (3, 6)$ and $(4, 8)$

i.e. $$[(-2, -4)] = \{(-2, -4), (1, 2), (3, 6), (4, 8)\}.$$

The element related to $(-1, -11)$ is itself only.

The element related to $(2, 14)$ is itself only.

$\therefore$ The partition induced by R consists of the cells

$[(-4, -20)], [(-3, -9)], [(-2, -4)], [(-1, -11]$ and $[(2, 14)]$.

Example 5.13

(i) If $A = \{1, 2, 3, 4\} \times \{1, 2, 3, 4\}$ and the relation R is defined on A by $(a, b)\ R\ (c, d)$ if $a + b = c + d$, verify that A is an equivalence relation on A and also find the quotient set of A by R.

(ii) If the relation R on the set of integers Z is defined by $a\ R\ b$ if $a \equiv b$ (mod 4), find the partition induced by R.

(i) $A = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (4, 1), (4, 2), (4, 3), (4, 4)\}$

If we take $R \equiv A$, it can be verified that R is an equivalence relation.

The quotient set A/R is the collection of equivalence classes of R.

It is easily seen that

$$[(1, 1)] = \{(1, 1)\}$$
$$[(1, 2)] = \{(1, 2), (2, 1)\}$$
$$[(1, 3)] = \{(1, 3), (2, 2), (3, 1)\}$$
$$[(1, 4)] = \{(1, 4), (2, 3), (3, 2), (4, 1)\}$$
$$[(2, 4)] = \{(2, 4), (3, 3), (4, 2)\}$$
$$[(3, 4)] = \{(3, 4), (4, 3)\}$$
$$[(4, 4)] = \{(4, 4)\}$$

Thus $[(1, 1)], [(1, 2)], [(1, 3)], [(1, 4)], [(2, 4)], [(3, 4)], [(4, 4)]$ form the quotient set A/R.

(ii) The equivalence classes of R are the following:

$$[0]_R = \{\cdots, -8, -4, 0, 4, 8, 12, \cdots\}$$
$$[1]_R = \{\cdots, -7, -3, 1, 5, 9, 13, \cdots\}$$
$$[2]_R = \{\cdots, -6, -2, 2, 6, 10, 14, \cdots\}$$
$$[3]_R = \{\cdots, -5, -1, 3, 7, 11, 15, \cdots\}$$

Thus $[0]_R$, $[1]_R$, $[2]_R$ and $[3]_R$ form the partition of R.

Note These equivalence classes are also called *the congruence classes modulo 4* and also denoted $[0]_4$, $[1]_4$, $[2]_4$ and $[3]_4$.

Example 5.14 If R is the relation on $A = \{1, 2, 3\}$ such that $(a, b) \in R$, if and only if $a + b$ = even, find the relational matrix M_R. Find also the relational matrices R^{-1}, $\bar{R}$ and R^2.

$$R = \{(1, 1), (1, 3), (2, 2), (3, 1), (3, 3)\}$$

$$\therefore \qquad M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Now
$$M_{R^{-1}} = (M_R)^T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$\bar{R}$ is the complement R that consists of elements of $A \times A$ that are not in R.

Thus $\bar{R} = \{(1, 2), (2, 1), (2, 3), (3, 2)\}$

$\therefore \quad M_{\bar{R}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, which is the same as the matrix obtained from M_R by changing 0's to 1's and 1's to 0's.

$$M_{R^2} = M_R \bullet M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \vee 0 \vee 1 & 0 \vee 0 \vee 0 & 1 \vee 0 \vee 1 \\ 0 \vee 0 \vee 0 & 0 \vee 1 \vee 0 & 0 \vee 0 \vee 0 \\ 1 \vee 0 \vee 1 & 0 \vee 0 \vee 0 & 1 \vee 0 \vee 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

It can be found that $R^2 = R \bullet R = R$. Hence $M_{R^2} = M_R$

Example 5.15 If R and S be relations on a set A represented by the matrices

$$M_R = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad M_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

find the matrices that represent

(a) $R \cup S$ (b) $R \cap S$ (c) $R \bullet S$ (d) $S \bullet R$ (e) $R \oplus S$

(a) $M_{R \cup S} = M_R \vee M_S$

$$= \begin{bmatrix} 0 \vee 0 & 1 \vee 1 & 0 \vee 0 \\ 1 \vee 0 & 1 \vee 1 & 1 \vee 1 \\ 1 \vee 1 & 0 \vee 1 & 0 \vee 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

(b) $M_{R \cap S} = M_R \wedge M_S$

$$= \begin{bmatrix} 0 \wedge 0 & 1 \wedge 1 & 0 \wedge 0 \\ 1 \wedge 0 & 1 \wedge 1 & 1 \wedge 1 \\ 1 \wedge 1 & 0 \wedge 1 & 0 \wedge 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

(c) $M_{R \bullet S} = M_R \bullet M_S$

$$= \begin{bmatrix} 0 \vee 0 \vee 0 & 0 \vee 1 \vee 0 & 0 \vee 1 \vee 0 \\ 0 \vee 0 \vee 1 & 1 \vee 1 \vee 1 & 0 \vee 1 \vee 1 \\ 0 \vee 0 \vee 0 & 1 \vee 0 \vee 0 & 0 \vee 0 \vee 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

(d) $M_{S \bullet R} = M_S \bullet M_R$

$$= \begin{bmatrix} 0 \vee 1 \vee 0 & 0 \vee 1 \vee 0 & 0 \vee 1 \vee 0 \\ 0 \vee 1 \vee 1 & 0 \vee 1 \vee 0 & 0 \vee 1 \vee 0 \\ 0 \vee 1 \vee 1 & 1 \vee 1 \vee 0 & 0 \vee 1 \vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

(e) $M_{R \oplus S} = M_{R \cup S} - M_{R \cap S}$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Example 5.16 Examine if the relation R represented by $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ is an equivalence relation, using the properties of M_R.

Since all the elements in the main diagonal of M_R and equals to 1 each, R is a reflexive relation.

Since M_R is a symmetric matrix, R is a symmetric relation.

$$M_{R^2} = M_R \bullet M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = M_R$$

viz. $R^2 \subseteq R$

$\therefore$ R is a transitive relation.

Hence R is an equivalence relation.

Example 5.17 List the ordered pairs in the relation on {1, 2, 3, 4} corresponding to the following matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Also draw the directed graph representing this relation. Use the graph to find if the relation is reflexive, symmetric and/or transitive.

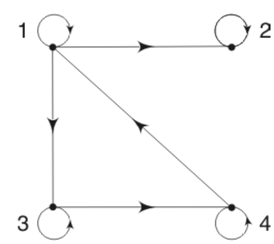

Fig. 5.18

The ordered pairs in the given relation are {(1, 1), (1, 2), (1, 3), (2, 2), (3, 3), (3, 4), (4, 1), (4, 4)}. The directed graph representing the relation is given in Fig. 5.18.

Since there is a loop at every vertex of the digraph, the relation is reflexive. The relation is not symmetric. For example, there is an edge from 1 to 2, but there is no edge in the opposite direction, i.e. from 2 to 1. The relation is not transitive. For example, though there are edges from 1 to 3 and 3 to 4, there is no edge from 1 to 4.

Example 5.18 List the ordered pairs in the relation represented by the digraph given in Fig. 5.19. Also use the graph to prove that the relation is a partial ordering. Also draw the directed graphs representing R^{-1} and $\bar{R}$.

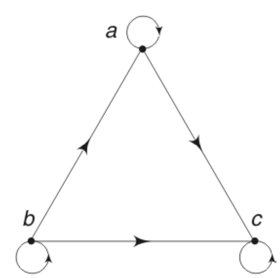

Fig. 5.19

The ordered pairs in the relation are {(a, a), (a, c), (b, a), (b, b), (b, c), (c, c)}.

Since there is a loop at every vertex, the relation is reflexive.

Though there are edges $b - a$, $a - c$ and $b - c$, the edges $a - b$, $c - a$ and $c - b$ are not present in the digraph. Hence the relation is antisymmetric.

When edges $b - a$ and $a - c$ are present in the digraph, the edge $b - c$ is also present (for example). Hence the relation is transitive.

Hence the relation is a partially ordering. The digraph of R^{-1} is got by reversing the directions of the edges (Fig. 5.20). The digraph of $\bar{R}$ contains the edges (a, b), (c, a), and (c, b) as shown in Fig. 5.21.

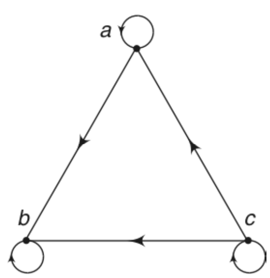

Fig. 5.20

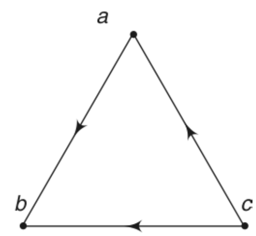

Fig. 5.21

Example 5.19 Draw the digraph representing the partial ordering $\{(a, b)\} \mid a$ divides $b\}$ on the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$. Reduce it to the Hasse diagram representing the given partial ordering.

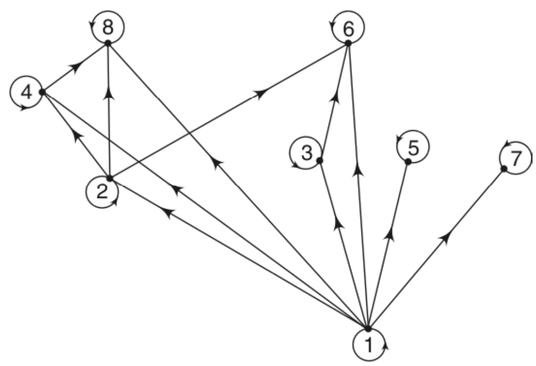

Fig. 5.22

Deleting all the loops at the vertices, deleting all the edges occurring due to transitivity, arranging all the edges to point upward and deleting all arrows, we get the corresponding Hasse diagram as given in Fig. 5.23.

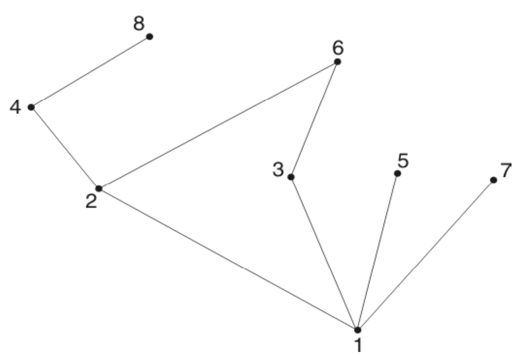

Fig. 5.23

Example 5.20 Draw the Hasse diagram representing the partial ordering $\{(A, B) | (A \subseteq B)\}$ on the power set $P(S)$, where $S = \{a, b, c\}$. Find the maximal, minimal, greatest and least elements of the poset.

Find also the upper bounds and LUB of the subset $(\{a\}, \{b\}, \{c\})$ and the lower bounds and GLB of the subset $(\{a, b\}, \{a, c\}, \{b, c\})$.

Here $P(S) = (\{\phi\}, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}$.

By using the usual procedure (as in the previous example), the Hasse diagram is shown, as shown in Fig. 5.24.

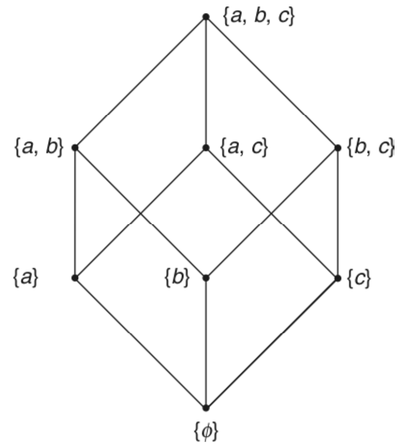

Fig. 5.24

The element $\{a, b, c\}$ does not precede any element of the poset and hence is the only maximal element of the poset.

The element $\{\phi\}$ does not succeed any element of the poset and hence is the only minimal element.

All the elements of the poset are related to $\{a, b, c\}$ and precede it. Hence $\{a, b, c\}$ is the greatest element of the poset.

All the elements of the poset are related to $\{\phi\}$ and succeed it. Hence $\{\phi\}$ is the least element of the poset. The only upper bound of the subset $(\{a\}, \{b\}, \{c\})$ is $\{a, b, c\}$ and hence the LUB of the subset.

Note $\{a, b\}$ is not an upper bound of the subset, as it is not related to $\{c\}$. Similarly $\{a, c\}$ and $\{b, c\}$ are not upper bounds of the given subset.

The only lower bound of the subset $(\{a, b\}, \{a, c\}, \{b, c\})$ is $\{\phi\}$ and hence GLB of the given subset.

Note $\{a\}, \{b\}, \{c\}$ are not lower bounds of the given subset.

EXERCISE 5(B)

Part A: (Short answer questions)

1. Define a binary relation from one set to another. Give an example.
2. Define a relation on a set and give an example.
3. If R is the relation from $A = \{1, 2, 3, 4\}$ to $B = \{2, 3, 4, 5\}$, list the elements in R, defined by aRb, if a and b are both odd. Write also the domain and range of R.
4. Define universal and void relations with examples.
5. If R is a relation from $A = \{1, 2, 3\}$ to $B = \{4, 5\}$ given by $R = \{(1, 4), (2, 4), (1, 5), (3, 5)\}$, find R^{-1} (the inverse of R) and $\bar{R}$ (the complement of R).
6. If $R = \{(1, 1), (2, 2), (3, 3)\}$ and $S = \{(1, 1), (1, 2), (1, 3), (1, 4)\}$ find $R \oplus S$.

7. Define composition of relations with an example.
8. When is a relation said to be reflexive, symmetric, antisymmetric and transitive?
9. Give an example of a relation that is both symmetric and antisymmetric.
10. Give an example of a relation that is neither symmetric nor antisymmetric.
11. Give an example of a relation that is reflexive and symmetric but not transitive.
12. Give an example of relation that is reflexive and transitive but not symmetric.
13. Give an example of a relation that is symmetric and transitive but not reflexive.
14. Define an equivalence relation with an example.
15. Define a partial ordering with an example.
16. Define a poset and give an example.
17. Define equivalence class.
18. Define quotient set of a set under an equivalence relation.
19. Find the quotient set of $\{1, 2, 3\}$ under the relation $\{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}$.
20. Define partition of a set and give an example.
21. What do you mean by partitioning of a set induced by an equivalence relation?
22. If R is a relation from $A = \{1, 2, 3\}$ to $B = \{1, 2\}$ such that aRb if $a > b$, write down the matrix representation of R.
23. If the matrix representation of a relation R on $\{1, 2, 3, 4\}$ is given by

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

list the ordered pairs in the relation.

24. If the relations R and S on a set A are represented by the matrices

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } M_S = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

What are the matrices representing $R \cup S$ and $R \cap S$?

25. Draw the directed graph representing the relation on $\{1, 2, 3, 4\}$ given by the ordered pairs $\{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$.
26. Draw the directed graph representing the relation on $\{1, 2, 3, 4\}$ whose matrix representation is

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

27. What is Hasse diagram? Draw the Hasse diagram for ≤ relation on {0, 2, 5, 10, 11, 15}.
28. Define maximal and minimal members of a poset. Are they the same as the greatest and least members of the poset?
29. Define the greatest and least members of a poset. Are they different from the maximal and minimal members of the poset?
30. Define supremum and infimum of a subset of a poset.

Part B

31. Show that there are 2^{n^2} relations on a set with n elements. List all possible relations on the set {1, 2}.

    Hint: When a set A has n elements, $A \times A$ has n^2 elements and hence the number of subsets of $A \times A = 2^{n^2}$.
32. Which of the ordered pairs given by {1, 2, 3} × {1, 2, 3} belong to the following relations?
    (a) $a\ R\ b$ iff $a \le b$, (b) $a\ R\ b$ iff $a > b$,
    (c) $a\ R\ b$ iff $a = b$, (d) $a\ R\ b$ iff $a = b + 1$ and
    (e) $a\ R\ b$ iff $a + b \le 4$.
33. If R is a relation on the set {1, 2, 3, 4, 5}, list the ordered pairs in R when (a) aRb if 3 divides $a - b$, (b) aRb if $a + b = 6$, (c) aRb if $a - b$ is even, (d) aRb if $lcm\ (a, b)$ is odd, (e) aRb if $a^2 = b$.
34. If R is the relation on the set {1, 2, 3, 4, 5} defined by $(a, b) \in R$ if $a + b \le 6$,
    (a) list the elements of R, R^{-1} and $\overline{R}$.
    (b) the domain and range of R and R^{-1}.
    (c) the domain and range of $\overline{R}$.
35. If R_1 = {(1, 2), (2, 3), (3, 4)} and R_2 = {(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3), (3, 4)} be the relations from {1, 2, 3} to {1, 2, 3, 4}. Find
    (a) $R_1 \cup R_2$, (b) $R_1 \cap R_2$, (c) $R_1 - R_2$,
    (d) $R_2 - R_1$, (e) $R_1 \oplus R_2$.
36. If $R = \{(x, x^2)\}$ and $S = \{(x, 2x)\}$, where x is a non-negative integer, find
    (a) $R \cup S$, (b) $R \cap S$, (c) $R - S$,
    (d) $S - R$, (e) $R \oplus S$.
37. If R_1 and R_2 are relations on the set of all positive integers defined by $R_1 = \{(a, b) | a$ divides $b\}$ and $R_2 = \{(a, b) | a$ is a multiple of $b\}$, find
    (a) $R_1 \cup R_2$, (b) $R_1 \cap R_2$, (c) $R_1 - R_2$,
    (d) $R_2 - R_1$ (e) $R_1 \oplus R_2$.
38. If the relations R_1, R_2, R_3, R_4, R_5 are defined on the set of real numbers as given below,
    $R_1 = \{(a, b) | a \ge b\}$, $R_2 = \{(a, b) | a < b)\}$,
    $R_3 = \{(a, b) | a \le b\}$, $R_4 = \{(a, b) | a = b\}$, $R_5 = \{(a, b) | a \ne b\}$, find (a) $R_2, \cup R_5$, (b) $R_3 \cap R_5$, (c) $R_2 - R_5$, (d) $R_1 \oplus R_5$, (e) $R_2 \oplus R_4$.
39. If the relations R and S are given by
    R = {(1, 2), (2, 2), (3, 4)}, S = {(1, 3), (2, 5), (3, 1), (4, 2)}, find $R \bullet S$, $S \bullet R$, $R \bullet R$, $S \bullet S$, $R \bullet (S \bullet R)$, $(R \bullet S) \bullet R$ and $R \bullet R \bullet R$.

40. If R, S, T are relations on the set $A = \{0, 1, 2, 3\}$ defined by $R = \{(a, b)| a + b = 3\}$, $S = \{(a, b)|$ 3 is a divisor of $(a + b)$ and $T = \{(a, b)|\max(a, b) = 3$, find (a) $R \bullet T$, (b) $T \bullet R$ and (c) $S \bullet S$.
41. If the relations R_1, R_2, R_3, R_4, R_5, R_6 are defined on the set of real numbers as given below,

    $R_1 = \{(a, b)|a > b\}$, $R_2 = \{(a, b)|a \geq b\}$, $R_3 = \{(a, b)|a < b\}$,

    $R_4 = \{(a, b)|a \leq b\}$, $R_5 = \{(a, b)|a = b\}$, $R_6 = \{(a, b)|a \neq b\}$,

    find $R_1 \bullet R_1$, $R_2 \bullet R_1$, $R_3 \bullet R_1$, $R_4 \bullet R_1$, $R_5 \bullet R_1$, $R_6 \bullet R_1$, $R_3 \bullet R_2$ and $R_3 \bullet R_3$.
42. Determine whether the relation R on the set of all real numbers is reflexive, symmetric, antisymmetric and/or transitive, where $(a, b) \in R$ if and only if
    (a) $a + b = 0$ (b) $a = \pm b$
    (c) $a - b$ is a rational number (d) $a = 2b$
    (e) $ab \geq 0$ (f) $ab = 0$
    (g) $a = 1$ (h) $a = 1$ or $b = 1$
43. For each of the following relations, determine whether the relation is reflexive, symmetric, antisymmetric and/or transitive:
    (a) $R \subseteq Z^+ \times Z^+$, where aRb if a divides b.
    (b) $R \subseteq Z \times Z$, where aRb if a divides b.
    (c) R is the relation on Z, where aRb if $a + b$ is odd.
    (d) R is the relation on Z, where aRb if $a - b$ is even.
    (e) R is the relation on the set of lines in a plane such that aRb if a perpendicular to b.
44. Determine whether the relation R on the set of people is reflexive, symmetric, antisymmetric and/or transitive, where aRb if
    (a) a is taller than b, (b) a and b were born on the same day, (c) a has the same first name as b, (d) a is a spouse of b, (e) a and b have a common grand parent.
45. Which of the following relations on the set $\{1, 2, 3, 4\}$ is/are equivalent relations? Find the properties of an equivalent relation that the others lack.
    (a) $\{(2, 4), (4, 2)$
    (b) $\{(1, 1), (2, 2), (3, 3), (4, 4)\}$
    (c) $\{(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)$
    (d) $\{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$
    (e) $\{(1, 3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 4)\}$
46. If $A = \{1, 2, 3, \ldots, 9\}$ and R be the relation defined by $(a, b), (c, d) \in R$ if $a + d = b + c$, prove that R is an equivalence relation.
47. If R is a relation on Z defined by
    (a) aRb, if and only if $2a + 3b = 5n$ for some integer n.
    (b) aRb if and only if $3a + b$ is a multiple of 4, prove that R is an equivalence relation.

48. If R is a relation defined by
    (a) $(a, b)\ R\ (c, d)$ if and only if $a^2 + b^2 = c^2 + d^2$, where a, b, c and d are real.
    (b) $(a, b)\ R\ (c, d)$ if and only if $a + 2b = c + 2d$, where a, b, c and d are real, prove that R is an equivalence relation.
49. (a) If R is the relation defined on Z such that aRb if and only if $a^2 - b^2$ is divisible by 3, show that R is an equivalence relation.
    (b) If R is the relation on N defined by aRb if and only if $\frac{a}{b}$ is a power 2, show that R is an equivalence relation.
50. If R is the relation the set $A = \{1, 2, 4, 6, 8\}$ defined by aRb if and only if $\frac{b}{a}$ is an integer, show that R is a partial ordering on A.
51. (a) If R is the equivalence relation on $A = \{0, 1, 2, 3, 4\}$ given by $\{(0, 0), (0, 4), (1, 1), (1, 3), (2, 2), (3, 1), (3, 3), (4, 0), (4, 4)\}$, find the distinct equivalence classes of R.
    (b) If R is the equivalence relation on $A = \{1, 2, 3, 4, 5, 6\}$ given by $\{(1, 1), (1, 5), (2, 2), (2, 3), (2, 6), (3, 2), (3, 3), (3, 6), (4, 4), (5, 1), (5, 5), (6, 2), (6, 3), (6, 6)\}$, find the partition of A induced by R.
52. If R is the equivalence relation on the set $A = \{1, 2, 3, 4, 5, 6, 7\}$ defined by aRb if $a - b$ is a multiple of 3, find the partition of A induced by R.
53. If R is the equivalence relation on Z defined by aRb if $a^2 = b^2$ (or, $a = \pm b$), find the partition of Z.
54. If R and S are equivalence relations on $A = \{a, b, c, d, e\}$ given by $R = \{a, a), (a, b), (b, a), (b, b), (c, c), (d, d), (d, e), (e, d), (e, e)\}$ and $S = \{(a, a), (b, b), (c, c), (d, d), (e, e), (a, c), (c, a), (d, e), (e, d)\}$, determine the partitions of A induced by (a) R^{-1}, (b) $R \cap S$.
55. List the ordered pairs in the equivalence relations R and S produced by the partitions of $\{0, 1, 2, 3, 4, 5\}$ and $\{1, 2, 3, 4, 5, 6, 7\}$ respectively that are given as follows:
    (a) $[\{0\}, \{1, 2\}, \{3, 4, 5\}]$ (b) $[\{1, 2\}, \{3\}, \{4, 5, 7\}, \{6\}]$
    Hint: $R = \{0\} \times \{0\} \cup \{1, 2\} \times \{1, 2\} \cup \{3, 4, 5\} \times \{3, 4, 5\}$
56. If R is the relation on $A = \{1, 2, 3\}$ represented by the matrix

$$M_R = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix},$$

    find the matrix representing (a) R^{-1}, (b) $\bar{R}$ and R^2 and also express them as ordered pairs.
57. If R and S are relations on $A = \{1, 2, 3\}$ represented by the matrices

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } M_S = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

    find the matrices that represent (a) $R \cup S$, (ii) $R \cap S$, (c) $R \bullet S$, (d) $S \bullet R$, (e) $R \oplus S$.

58. Examine if the relations R and S represented by M_R and M_S given below are equivalent relations:

(a) $M_R = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$ (b) $M_S = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

59. List the ordered pairs in the relations R and S whose matrix representations are given as follows:

(a) $M_R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$; (b) $M_S = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$

Also draw the directed graphs representing R and S. Use the graphs to find if R and S are equivalence relations.

60. Draw the directed graphs of the relations $R = \{(1, 1), (1, 3), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (4, 1)\}$ and $S = \{(1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (3, 1), (3, 3), (4, 1), (4, 3)\}$. Use these graphs to draw the graphs of (a) R^{-1}, S^{-1} and (b) $\bar{R}$ and $\bar{S}$.
61. Draw the Hasse diagram representing the partial ordering $P = \{(a, b) | a$ divides $b\}$ on $\{1, 2, 3, 4, 6, 8, 12\}$, starting from the digraph of P.
62. Draw the Hasse diagram for the divisibility relation on $\{2, 4, 5, 10, 12, 20, 25\}$ starting from the digraph.
63. Draw the Hasse diagram for the "less than or equal to" relation on $\{0, 2, 5, 10, 11, 15\}$ starting from the digraph.
64. Find the lower and upper bounds of the subsets $\{a, b, c\}$, $(j, h\}$ and $\{a, c, d, f\}$ in the poset with the Hasse diagram in Fig. 5.25. Find also the LUB and GLB of the subset $\{b, d, g\}$, if they exist.
65. For the poset $[\{(3, 5, 9, 15, 24, 45\}$; divisor of], find

(a) the maximal and minimal elements

(b) the greatest and the least elements

(c) the upper bounds and LUB of $\{3, 5\}$

(d) the lower bounds and GLB of $(15, 45\}$

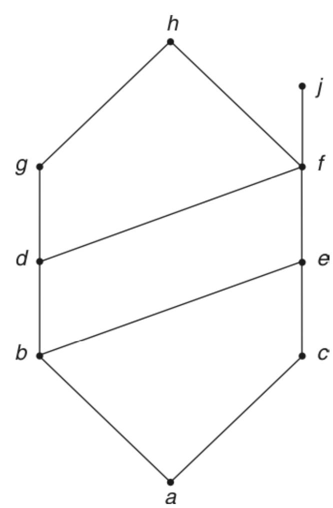

Fig. 5.25

LATTICES

Definitions

A partially ordered set $\{L, \leq\}$ in which every pair of elements has a least upper bound and a greatest lower bound is called a *lattice*.

The LUB (supremum) of a subset $\{a, b\} \subseteq L$ is denoted by $a \vee b$ [or $a \oplus b$ or $a + b$ or $a \cup b$] and is called the *join* or *sum* of a and b.

The GLB (infemum) of a subset $\{a, b\} \subseteq L$ is denoted by $a \wedge b$ [or $a * b$ or $a \bullet b$ or $a \cap b$] is called the *meet* or *product* of a and b.

Note Since the LUB and GLB of any subset of a poset are unique, both $\wedge$ and $\vee$ are binary operations on a lattice.

For example, let us consider the poset ($\{1, 2, 4, 8, 16\}|$), where | means 'divisor of'. The Hasse diagram of this poset is given in Fig. 5.26.

The LUB of any two elements of this poset is obviously the larger of them and the GLB of any two elements is the smaller of them. Hence this poset is a lattice.

Fig. 5.26

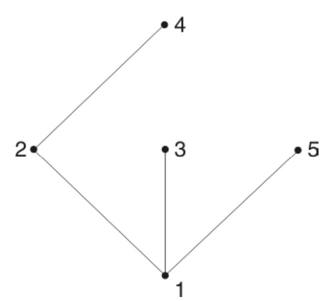

Fig. 5.27

Note All partially ordered sets are not lattices, as can be seen from the following example.

Let us consider the poset ($\{1, 2, 3, 4, 5\}$, |) whose Hasse diagram is given in Fig. 5.27.

The LUB's of the pairs (2, 3) and (3, 5) do not exist and hence they do not have LUB. Hence this poset is not a Lattice.

PRINCIPLE OF DUALITY

When $\leq$ is a partial ordering relation on a set S, the converse $\geq$ is also a partial ordering relation on S. For example if $\leq$ denotes 'divisor of', $\geq$ denotes 'multiple of'.

The Hasse diagram of $(S, \geq)$ can be obtained from that of $(S, \leq)$ by simply turning it upside down. For example the Hasse diagram of the poset ($\{1, 2, 4, 8, 16\}$, multiple of), obtained from Fig. 5.26 will be as given in Fig. 5.28.

From this example, it is obvious that LUB(A) with respect to $\le$ is the same as GLB(A) with respect to $\ge$ and vice versa, where $A \subseteq S$. viz. LUB and GLB are interchanged, when $\le$ and $\ge$ are interchanged.

In the case of lattices, if $\{L, \le\}$ is a lattice, so also is $\{L, \ge\}$. Also the operations of join and meet on $\{L, \le\}$ become the operations of meet and join respectively on $\{L, \ge\}$.

From the above observations, the following statement, known as *the principle of duality* follows:

Any statement in respect of lattices involving the operations $\vee$ and $\wedge$ and the relations $\le$ and $\ge$ remains true, if $\vee$ is replaced by $\wedge$ and $\wedge$ is replaced by $\vee$, $\le$ by $\ge$ and $\ge$ by $\le$.

Fig. 5.28

The lattices $\{L, \le\}$ and $\{L, \ge\}$ are called the *duals* of each other. Similarly the operations $\vee$ and $\wedge$ are duals of each other and the relations $\le$ and $\ge$ are duals of each other.

PROPERTIES OF LATTICES

Property 1

If $\{L, \le\}$ is a lattice, then for any $a, b, c \in L$,

L_1: $a \vee a = a$	$(L_1)'$: $a \wedge a = a$	(Idempotency)
L_2: $a \vee b = b \vee a$	$(L_2)'$: $a \wedge b = b \wedge a$	(Commutativity)
L_3: $a \vee (b \vee c) = (a \vee b) \vee c$	$(L_3)'$: $a \wedge (b \wedge c) = (a \wedge b) \wedge c$	(Associativity)
L_4: $a \vee (a \wedge b) = a$	$(L_4)'$: $a \wedge (a \vee b) = a$	(Absorption)

Proof

(i) $a \vee a$ = LUB (a, a) = LUB $(a) = a$. Hence L_1 follows.

(ii) $a \vee b$ = LUB (a, b) = LUB $(b, a) = b \vee a$ $\{\because$ LUB (a, b) is unique.$\}$
Hence L_2 follows.

(iii) Since $(a \vee b) \vee c$ is the LUB $\{(a \vee b), c\}$, we have

$$a \vee b \le (a \vee b) \vee c \tag{1}$$

and
$$c \le (a \vee b) \vee c \tag{2}$$

Since $a \vee b$ is the LUB $\{a, b\}$, we have

$a \le a \vee b$ (3)

and $b \le a \vee b$ (4)

From (1) and (3), $a \le (a \vee b) \vee c$ by transitivity (5)

From (1) and (4), $b \le (a \vee b) \vee c$ by transitivity (6)

From (2) and (6), $b \vee c \le (a \vee b) \vee c$ by definition of join (7)

From (5) and (7), $a \vee (b \vee c) \le (a \vee b) \vee c$ by definition of join (8)

Similarly, $a \le a \vee (b \vee c)$ (9)

$b \le b \vee c \le a \vee (b \vee c)$ (10)

and $c \le b \vee c \le a \vee (b \vee c)$ (11)

From (9) and (10), $a \vee b \le a \vee (b \vee c)$ (12)

From (11) and (12), $(a \vee b) \vee c \le a \vee (b \vee c)$ (13)

From (8) and (13), by antisymmetry of $\leq$, we get
$a \vee (b \vee c) = (a \vee b) \vee c$.
Hence L_3 follows.

(iv) Since $a \wedge b$ is the GLB $\{a, b\}$, we have

$$a \wedge b \leq a \tag{1}$$

Also $$a \leq a \tag{2}$$

From (1) and (2), $a \vee (a \wedge b) \leq a$ (3)

Also $$a \leq a \vee (a \wedge b) \tag{4}$$

by definition of LUB

$\therefore$ From (3) and (4), by antisymmetry, we get $a \vee (a \wedge b) = a$.
Hence L_4 follows.
Now the identities $(L_1)'$ to $(L_4)'$ follow from the principle of duality.

Property 2

If $\{L, \leq\}$ is a lattice in which $\vee$ and $\wedge$ denote the operations of join and meet respectively, then for $a, b \in L$,

$$a \leq b \Leftrightarrow a \vee b = b \Leftrightarrow a \wedge b = a.$$

In other words,

(i) $a \vee b = b$, if and only if $a \leq b$.
(ii) $a \wedge b = a$, if and only if $a \leq b$.
(iii) $a \wedge b = a$, if and only if $a \vee b = b$.

Proof

(i) Let $a \leq b$.
Now $b \leq b$ (by reflexivity).

$\therefore$ $$a \vee b \leq b \tag{1}$$

Since $a \vee b$ is the LUB (a, b),

$$b \leq a \vee b \tag{2}$$

From (1) and (2), we get $a \vee b = b$ (3)

Let $a \vee b = b$.
Since $a \vee b$ is the LUB (a, b),

$$a \leq a \vee b$$

i.e., $a \leq b$, by the data (4)

From (3) and (4), result (i) follows. Result (ii) can be probed in a way similar to the proof (i).
From (i) and (ii), result (iii) follows.

Note Property (2) gives a connection between the partial ordering relation $\leq$ and the two binary operations $\vee$ and $\wedge$ in a lattice $\{L, \leq\}$.

Property 3 (Isotonic Property)

If $\{L, \leq\}$ is a lattice, then for any $a, b, c, \in L$, the following properties hold good:

If $b \leq c$, then (i) $a \vee b \leq a \vee c$ and (ii) $a \wedge b \leq a \wedge c$.

Proof

Since $b \leq c$, $b \vee c = c$, by property 2(i).
Also $a \vee a = a$, by idempotent property

Now $a \vee c \quad = (a \vee a) \vee (b \vee c)$, by the above steps

$= a \vee (a \vee b) \vee c$, by associativity

$= a \vee (b \vee a) \vee c$, by commutativity

$= (a \vee b) \vee (a \vee c)$, by associativity

This is of the form $x \vee y = y$. $\therefore$ $x \leq y$, by property 2(i).

i.e. $a \vee b \leq a \vee c$, which is the required result (i).

Similarly, result (ii) can be proved.

Property 4 (Distributive Inequalities)

If $\{L, \leq\}$ is a lattice, then for any $a, b, c, \in L$,

(i) $a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$

(ii) $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$.

Proof

Since $a \wedge b$ is the GLB(a, b), $a \wedge b \leq a$ (1)

Also $a \wedge b \leq b \leq b \vee c$ (2)

since $b \vee c$ is the LUB of b and c.

From (1) and (2), we have $a \wedge b$ is a lower bound of $\{a, b \vee c\}$

$\therefore$ $a \wedge b \leq a \wedge (b \vee c)$ (3)

Similarly $a \wedge c \leq a$

and $a \wedge c \leq c \leq b \vee c$

$\therefore$ $a \wedge c \leq a \wedge (b \vee c)$ (4)

From (3) and (4), we get

$$(a \wedge b) \vee (a \wedge c) \leq a \wedge (b \vee c)$$

i.e. $a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$, which is result (i).

Result (ii) follows by the principle of duality.

Property 5 (Modular Inequality)

If $\{L, \leq\}$ is a lattice, then for any $a, b, c, \in L$, $a \leq c \Leftrightarrow a \vee (b \wedge c) \leq (a \vee b) \wedge c$. (1)

Proof

Since $a \leq c$, $a \vee c = c$ (1), by property 2(i)

$a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$ (2), by property 4(ii)

i.e. $a \vee (b \wedge c) \leq (a \vee b) \wedge c$ (3), by (1)

Now $a \vee (b \wedge c) \leq (a \vee b) \wedge c$

$\therefore$ $a \leq a \vee (b \wedge c) \leq (a \vee b) \wedge c \leq c$, by the definitions of LUB and GLB

i.e. $a \leq c$ (4)

From (3) and (4), we get

$$a \leq c \Leftrightarrow a \vee (b \wedge c) \leq (a \vee b) \wedge c.$$

LATTICE AS ALGEBRAIC SYSTEM

A set together with certain operations (rules) for combining the elements of the set to form other elements of the set is usually referred to as an *algebraic system*. Lattice L was introduced as a partially ordered set in which for every pair of elements $a, b \in L$, LUB$(a, b) = a \vee b$ and GLB$(a, b) = a \wedge b$ exist in the set. That is, in a Lattice $\{L, \leq\}$, for every pair of elements a, b of L, the two elements $a \vee b$ and $a \wedge b$ of L are obtained by means of the operations $\vee$ and $\wedge$. Due to this, the operations $\vee$ and $\wedge$ are considered as binary operations on L. Moreover we have seen that $\vee$ and $\wedge$ satisfy certain properties such as commutativity, associativity and absorption. The formal definition of a lattice as an algebraic system is given as follows:

Definition

A lattice is an algebraic system $(L, \vee, \wedge)$ with two binary operations $\vee$ and $\wedge$ on L which satisfy the commutative, associative and absorption laws.

Note We have not explicity included the idempotent law in the definition, since the absorption law implies the idempotent law as follows:

$a \vee a = a \vee [a \wedge (a \vee a)]$, by using $a \vee a$ for $a \vee b$ in $(L_4)'$ of property 1

$= a$, by using $a \vee a$ for b in L_4 of property 1.

$a \wedge a = a$ follows by duality.

Though the above definition does not assume the existence of any partial ordering on L, it is implied by the properties of the operations $\vee$ and $\wedge$ as explained below:

Let us assume that there exists a relation R on L such that for $a, b \in L$,

$$aRb \text{ if and only if } a \vee b = b$$

For any $a \in L$, $a \vee a = a$, by idempotency

$\therefore$ aRa or R is reflexive.

Now for any $a, b, \in L$, let us assume that aRb and bRa.

$\therefore$ $a \vee b = b$ and $b \vee a = a$

Since $a \vee b = b \vee a$ by commutativity, we have $a = b$ and so R is antisymmetric.

Finally let us assume that aRb and bRc

$\therefore$ $a \vee b = b$ and $b \vee c = c$.

Now $a \vee c = a \vee (b \vee c) = (a \vee b) \vee c = b \vee c = c$

viz. aRc and so R is transitive.

Hence R is a partial ordering.

Thus the two definitions given for a lattice are equivalent.

SUBLATTICES

Definition

A non-empty subset M of a lattice $\{L, \vee, \wedge\}$ is called a *sublattice* of L, iff M is closed under both the operations $\vee$ and $\wedge$. viz. if $a, b, \in M$, then $a \vee b$ and $a \wedge b$ also $\in M$.

From the definition, it is obvious that the sublattice itself is a lattice with respect to $\vee$ and $\wedge$.

For example if aRb whenever a divides b, where $a, b \in Z^+$ (the set of all positive integers) then $\{Z^+, R\}$ is a lattice in which $a \vee b$ = LCM (a, b) and $a \wedge b$ = GCD(a, b).

If $\{S_n, R\}$ is the lattice of divisors of any positive integer n, then $\{S_n, R\}$ is a sublattice of $\{Z^+, R\}$.

LATTICE HOMOMORPHISM

Definition

If $\{L_1, \vee, \wedge\}$ and $\{L_2, \oplus, *\}$ are two lattices, a mapping $f: L_1 \to L_2$ is called *a lattice homomorphism* from L_1 to L_2, if for any $a, b \in L_1$,

$$f(a \vee b) = f(a) \oplus f(b) \text{ and } f(a \wedge b) = f(a) * f(b).$$

If a homomorphism $f: L_1 \to L_2$ of two lattices $\{L_1, \vee, \wedge\}$ and $\{L_2, \oplus *\}$ is objective, i.e. one-to-one onto, then f is called an *isomorphism*. If there exists an isomorphism between two lattices, then the lattices are said to be *isomorphic*.

SOME SPECIAL LATTICES

(a) A lattice L is said to have a *lower bound* denoted by 0, if $0 \le a$ for all $a \in L$. Similarly L is said to have an *upper bound* denoted by 1, if $a \le 1$ for all $a \in L$. The lattice L is said to be *bounded*, if it has both a lower bound 0 and an upper bound 1.

The bounds 0 and 1 of a lattice $\{L, \vee, \wedge, 0, 1\}$ satisfy the following identities, which are seen to be true by the meanings of $\vee$ and $\wedge$.

For any $a \in L$, $a \vee 1 = 1$; $a \wedge 1 = a$ and $a \vee 0 = a$; $a \wedge 0 = 0$.

Since $a \vee 0 = a$ and $a \wedge 1 = a$, 0 is the identity of the operation $\vee$ and 1 is the identity of the operation $\wedge$.

Since $a \vee 1 = 1$ and $a \wedge 0 = 0$, 1 and 0 are the zeros of the operations $\vee$ and $\wedge$ respectively.

Note 1 If we treat 1 and 0 as duals of each other in a bounded lattice, the principle of duality can be extended to include the interchange of 0 and 1. Thus the identities $a \vee 1 = 1$ and $a \wedge 0 = 0$ are duals of each other; so also are $a \vee 0 = a$ and $a \wedge 1 = a$.

Note 2 If $L = \{a_1, a_2, \ldots, a_n\}$ is a finite lattice, then $a_1 \vee a_2 \vee a_3 \ldots \vee a_n$ and $a_1 \wedge a_2 \wedge a_3 \wedge \ldots \wedge a_n$ are upper and lower bounds of L respectively and hence we conclude that every finite lattice is bounded.

(ii) A lattice $\{L, \vee, \wedge\}$ is called a *distributive lattice*, if for any elements a, b, $c \in L$,

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \text{ and}$$
$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

In other words if the operations $\vee$ and $\wedge$ distribute over each other in a lattice, it is said to be distributive. Otherwise it is said to be *non distributive*.

(iii) If $\{L, \vee, \wedge, 0, 1\}$ is a bounded lattice and $a \in L$, then an element $b \in L$ is called a *complement* of a, if

$$a \vee b = 1 \text{ and } a \wedge b = 0$$

Since $0 \vee 1 = 1$ and $0 \wedge 1 = 0$, 0 and 1 are complements of each other. When $a \vee b = 1$, we know that $b \vee a = 1$ and when $a \wedge b = 0$, $b \wedge a = 0$. Hence when b is the complement of a, a is the complement of b.

An element $a \in L$ may have no complement. Similarly an element, other than 0 and 1, may have more than one complement in L as seen from Fig. 5.28.

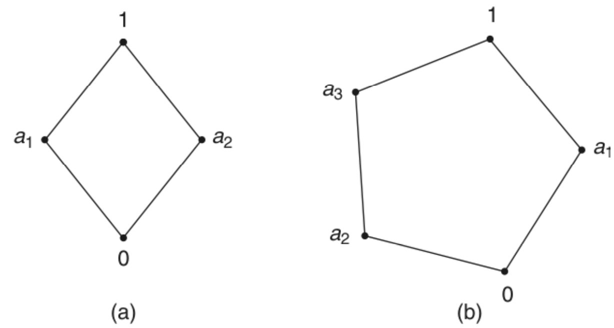

Fig. 5.28

In Fig. 5.28(a), complement of a_1 is a_2, whereas in (b), complement of a_1 is a_2 and a_3. It is to be noted that 1 is the only complement of 0. If possible, let $x \neq 1$ be another complement of 0, where $x \in L$.

Then $\quad 0 \vee x = 1 \quad$ and $\quad 0 \wedge x = 0$

But $\quad 0 \vee x = x \quad \therefore \quad x = 1$, which contradicts the assumption $x \neq 1$.

Similarly we can prove that 0 is the only complement of 1.

Now a lattice $\{L, \vee, \wedge, 0, 1\}$ is called a *complemented lattice* if every element of L has at least one complement.

The following property holds good for a distributive lattice.

Property

In a distributive lattice $\{L, \vee, \wedge\}$ if an element $a \in L$ has a complement, then it is unique.

Proof

If possible, let b and c be the complements of $a \in L$.

Then $\quad a \vee b = a \vee c = 1 \quad (1)$

and $\quad a \wedge b = a \wedge c = 0 \quad (2)$

Now
$$\begin{aligned} b &= b \vee 0 = b \vee (a \wedge c), \text{ by (2)} \\ &= (b \vee a) \wedge (b \vee c), \text{ since } L \text{ is distributive} \\ &= 1 \wedge (b \vee c), \text{ by (1)} \\ &= b \vee c \qquad (3) \end{aligned}$$

Similarly,
$$\begin{aligned} c &= c \vee 0 = c \vee (a \wedge b), \text{ by (2)} \\ &= (c \vee a) \wedge (c \vee b), \text{ since } L \text{ is distributive} \\ &= 1 \wedge (c \vee b), \text{ by (1)} \\ &= c \vee b \qquad (4) \end{aligned}$$

From (3) and (4), since $b \vee c = c \vee b$, we get $b = c$.

Note From the definition of complemented lattice and the previous property, it follows that every element a of a complemented and distributive lattice has a unique complement denoted by a'.

BOOLEAN ALGEBRA

Definition

A lattice which is complemented and distributive is called a Boolean Algebra, (which is named after the mathematician George Boole). Alternatively, Boolean Algebra can be defined as follows:

Definition

If B is a nonempty set with two binary operations + and •, two distinct elements 0 and 1 and a unary operation ′, then B is called a *Boolean Algebra* if the following basic properties hold for all a, b, c in B:

$B1$: $\left.\begin{aligned} a + 0 &= a \\ a \cdot 1 &= a \end{aligned}\right\}$ Identity laws

$B2$: $\left.\begin{aligned} a + b &= b + a \\ a \cdot b &= b \cdot a \end{aligned}\right\}$ Commutative laws

B3: $\left.\begin{aligned} (a + b) + c &= a + (b + c) \\ (a \cdot b) \cdot c &= a \cdot (b \cdot c) \end{aligned}\right\}$ Associative laws

B4 $\left.\begin{aligned} a + (b \cdot c) &= (a + b) \cdot (a + c) \\ a \cdot (b + c) &= (a \cdot b) + (a \cdot c) \end{aligned}\right\}$ Distributive laws

B5: $\left.\begin{aligned} a + a' &= 1 \\ a \cdot a' &= 0 \end{aligned}\right\}$ Complement laws.

Note

1. We have switched over to the symbols + and • instead of ∨ (join) and ∧ (meet) used in the study of lattices. The operations + and •, that will be used hereafter in Boolean algebra, are called *Boolean sum* and *Boolean product* respectively. We may even drop the symbol • and instead use juxtaposition. That is $a \bullet b$ may be written as ab.
2. If B is the set {0, 1} and the operations +, •, ′ are defined for the elements of B as follows:

$$0 + 0 = 0;\ 0 + 1 = 1 + 0 = 1 + 1 = 1$$
$$0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0;\ 1 \cdot 1 = 1$$
$$0' = 1 \text{ and } 1' = 0,$$

   then the algebra $\{B, +, \bullet, ', 0, 1\}$ satisfies all the 5 properties given above and is the simplest Boolean algebra called a two-element Boolean algebra. It can be proved that two element Boolean algebra is the only Boolean algebra.

   If a variable x takes on only the values 0 and 1, it is called a *Boolean variable*.
3. 0 and 1 are merely symbolic names and, in general, have nothing to do with the numbers 0 and 1. Similarly + and • are merely binary operators and, in general, have nothing to do with ordinary addition and multiplication.

ADDITIONAL PROPERTIES OF BOOLEAN ALGEBRA

If $\{B, +, \bullet, ', 0, 1\}$ is a Boolean algebra, the following properties hold good. They can be proved by using the basic properties of Boolean algebra listed in the definition.

(i) Idempotent Laws

$$a + a = a \quad \text{and} \quad a \cdot a = a, \quad \text{for all } a \in B$$

Proof

$$\begin{aligned} a &= a + 0, \text{ by } B1 \\ &= a + a \cdot a', \text{ by } B5 \\ &= (a + a) \cdot (a + a'), \text{ by } B4 \\ &= (a + a) \cdot 1, \text{ by } B5 \\ &= a + a, \text{ by } B1 \end{aligned}$$

Now,

$$\begin{aligned} a &= a \cdot 1, \text{ by } B1 \\ &= a \cdot (a + a'), \text{ by } B5 \\ &= a \cdot a + a \cdot a', \text{ by } B4 \\ &= a \cdot a + 0, \text{ by } B5 \\ &= a \cdot a, \text{ by } B1. \end{aligned}$$

(ii) Dominance Laws

$$a + 1 = 1 \text{ and } a \cdot 0 = 0, \text{ for all } a \in B.$$

Proof

$$\begin{aligned} a + 1 &= (a + 1) \cdot 1, \text{ by } B1 \\ &= (a + 1) \cdot (a + a'), \text{ by } B5 \\ &= a + 1 \cdot a', \text{ by } B4 \\ &= a + a' \cdot 1, \text{ by } B2 \\ &= a + a', \text{ by } B1 \\ &= 1, \text{ by } B5. \end{aligned}$$

Now

$$\begin{aligned} a \cdot 0 &= a \cdot 0 + 0, \text{ by } B1 \\ &= a \cdot 0 + a \cdot a', \text{ by } B5 \\ &= a \cdot (0 + a'), \text{ by } B4 \\ &= a \cdot (a' + 0), \text{ by } B2 \\ &= a \cdot a', \text{ by } B1 \\ &= 0, \text{ by } B5 \end{aligned}$$

(iii) Absorption Laws

$$a \cdot (a + b) = a \text{ and } a + a \cdot b = a, \text{ for all } a, b \in B.$$

Proof

$$\begin{aligned} a \cdot (a + b) &= (a + 0) \cdot (a + b), \text{ by } B1 \\ &= a + 0 \cdot b, \text{ by } B4 \\ &= a + b \cdot 0, \text{ by } B2 \\ &= a + 0, \text{ by dominance law} \\ &= a, \text{ by } B1. \end{aligned}$$

Now

$$\begin{aligned} a + a \cdot b &= a \cdot 1 + a \cdot b, \text{ by } B1 \\ &= a \cdot (1 + b), \text{ by } B4 \\ &= a \cdot (b + 1), \text{ by } B2 \\ &= a \cdot 1, \text{ by dominance law} \\ &= a, \text{ by } B1 \end{aligned}$$

(iv) De Morgan's Laws

$$(a + b)' = a' \cdot b' \text{ and } (a \cdot b)' = a' + b', \text{ for all } a, b \in B.$$

Proof

Note If y is to be the complement of x, by definition, we must show that $x + y = 1$ and $x \cdot y = 0$.

$(a + b) + a'b' = \{(a + b) + a'\} \cdot \{(a + b) + b'\}$, by $B4$
$= \{(b + a) + a'\} \cdot \{(a + b) + b'\}$, by $B2$
$= \{b + (a + a')\} \cdot \{a + (b + b')\}$, by $B3$
$= (b + 1) \cdot (a + 1)$, by $B5$
$= 1 \cdot 1$, by dominance law
$= 1$, by $B1$. (1)

Now $(a + b) \cdot a'b' = a'b' \cdot (a + b)$, by $B2$
$= a'b' \cdot a + a'b' \cdot b$, by $B4$
$= a \cdot (a'b') + a' \cdot b'b$, by $B3$
$= (a \cdot a') \cdot b' + a' \cdot (bb')$, by B_3 and B_2
$= 0 \cdot b' + a' \cdot 0$, by $B5$
$= b' \cdot 0 + a' \cdot 0$, by $B2$
$= 0 + 0$, by dominance law
$= 0$, by $B1$. (2)

From (1) and (2), we get $a'b'$ is the complement of $(a + b)$. i.e. $(a + b)' = a'b'$.
[$\because$ the complement is unique]

Note The students are advised to give the proof for the other part in a similar manner.

(v) Double Complement or Involution Law

$$(a')' = a, \text{ for all } a \in B.$$

Proof

$a + a' = 1$ and $a \cdot a' = 0$, by $B5$

i.e. $a' + a = 1$ and $a' \cdot a = 0$, by $B2$

$\therefore$ a is the complement of a'

i.e. $(a')' = a$, by the uniqueness of the complement of a'. [See example (14)]

(vi) Zero and One Law

$$0' = 1 \text{ and } 1' = 0$$

Proof

$0' = (aa')'$, by $B5$
$= a' + (a')'$, by De Morgan's law
$= a' + a$, by involution law
$= a + a'$, by $B2$
$= 1$, by $B5$

Now $(0')' = 1'$

i.e. $0 = 1'$ or $1' = 0$.

DUAL AND PRINCIPLE OF DUALITY

Definition

The *dual* of any statement in a Boolean algebra B is the statement obtained by interchanging the operations + and • and interchanging the elements 0 and 1 in the original statement.

For example, the dual of $a + a(b + 1) = a$ is $a \bullet (a + b \bullet 0) = a$.

PRINCIPLE OF DUALITY

The dual of a theorem in a Boolean algebra is also theorem.

For example, $(a \cdot b)' = a' + b'$ is a valid result, since it is the dual of the valid statement $(a + b)' = a' \cdot b'$ [De Morgan's laws]. If a theorem in Boolean algebra is proved by using the axioms of Boolean algebra, the dual theorem can be proved by using the dual of each step of the proof of the original theorem. This is obvious from the proofs of additional properties of Boolean algebra.

SUBALGEBRA

If C is a nonempty subset of a Boolean algebra such that C itself is a Boolean algebra with respect to the operations of B, then C is called a *subalgebra* of B.

It is obvious that C is a subalgebra of B if and only if C is closed under the three operations of B, namely, +, • and ′ and contains the element 0 and 1.

BOOLEAN HOMOMORPHISM

If $\{B +, \bullet, ', 0, 1\}$ and $\{C, \cup, \cap, -, \alpha, \beta\}$ are two Boolean algebras, then a mapping $f : B \to C$ is called a *Boolean homomorphism*, if all the operations of Boolean algebra are preserved. viz., for any $a, b \in B$,

$$f(a + b) = f(a) \cup f(b),\ f(a \cdot b) = f(a) \cap f(b),$$

$$f(a') = \overline{f(a)},\ f(0) = \alpha \text{ and } f(1) = \beta,$$

where α and β are the zero and unit elements of C.

ISOMORPHIC BOOLEAN ALGEBRAS

Two Boolean algebras B and B' are said to be *isomorphic* if there is one-to-one correspondence between B and B' with respect to the three operations, viz. there exists a mapping f: $B \to B'$ such that $f(a + b) = f(a) + f(b)$, $f(a \cdot b) = f(a) \cdot f(b)$ and $f(a') = \{f(a)\}'$.

BOOLEAN EXPRESSIONS AND BOOLEAN FUNCTIONS

Definitions

A *Boolean expression* in n Boolean variables $x_1, x_2, \ldots x_n$ is a finite string of symbols formed recursively as follows:

1. $0, 1, x_1, x_2, \ldots x_n$ are Boolean expressions.

2. If E_1 and E_2 are Boolean expressions, then $E_1 \cdot E_2$ and $E_1 + E_2$ are also Boolean expressions.
3. If E is a Boolean expression, E' is also a Boolean expression.

Note A Boolean exprssion in n variables may or may not contain all the n *literals*, viz., variables or their complements.

If $x_1, x_2, \ldots, x_n$ are Boolean variables, a function from $B^n = \{(x_1, x_2, \ldots, x_n\}$ to $B = \{0, 1\}$ is called a *Boolean function of degree n*. Each Boolean expression represents a Boolean function, which is evaluated by substituting the value 0 or 1 for each variable. The values of a Boolean function for all possible combinations of values of the variables in the function are often displayed in truth tables.

For example, the values of the Boolean function $f(a, b, c) = ab + c'$ are displayed in the following truth table:

a	b	c	ab	c'	$ab + c'$
1	1	1	1	0	1
1	1	0	1	1	1
1	0	1	0	0	0
1	0	0	0	1	1
0	1	1	0	0	0
0	1	0	0	1	1
0	0	1	0	0	0
0	0	0	0	1	1

Note Although the order of the variable values may be random, a symmetric way of writing them in a cyclic manner which will be advantageous is as follows:

If there be n variables in the Boolean function, there will obviously be 2^n rows in the truth table corresponding to all possible combinations of the values 0 and 1 of the variables.

We write $\frac{1}{2} \times 2^n$ ones followed by $\frac{1}{2} \times 2^n$ zeros in the first column representing the values of the first variable.

Then in the second column, we write $\frac{1}{4} \times 2^n$ ones and $\frac{1}{4} \times 2^n$ zeros alternately, representing the values of the second variable. Next in the third column, we write $\frac{1}{8} \times 2^n$ ones and $\frac{1}{8} \times 2^n$ zeros alternately, representing the values of the third variable. We continue this procedure and in the final column, we write $\frac{1}{2^n} \times 2^n$ (=1) one and 1 zero alternately, representing the values of the n^{th} variable.]

Definitions

1. A *minterm* if n Boolean variables is a Boolean product of the n literals (variables of complements) in which each literal appears exactly once.

For example, ab, $a'b$, ab' and $a'b'$ form the complete set of minterms of two variables a and b, abc, abc', $ab'c$, $a'bc$, $ab'c'$, $a'bc'$, $a'b'c$ and $a'b'c'$ form the complete set of minterms of three variables a, b, c.

2. A *maxterm* of n Boolean variables is a Boolean sum of the n literals in which each literal appears exactlty once.
   For example, $a + b$, $a' + b$, $a + b'$ and $a' + b'$ form the complete set of maxterms in two variables a and b.
3. When a Boolean function is expressed as a sum of minterms, it is called its *sum of products expansion* or it is said to be in the *disjunctive normal form* (DNF).
4. When a Boolean function is expressed as a product of maxterms, it is called its *product of sums expansion* or it is said to be in the *conjunctive normal form* (CNF).
5. Boolean function expressed in the DNF or CNF are said to be in *canonical form*.
6. If a Boolean function in n variables is expressed as the sum (product) of all the 2^n minterms (maxterms), it is said to be in *complete DNF* (*complete CNF*).
7. Boolean functions expressed in complete DNF or complete CNF are said to be *complete canonical form*.

EXPRESSION OF A BOOLEAN FUNCTION IN CANONICAL FORM

1. Truth Table Method

If the Boolean function $f(x, y, z)$ is represented by a truth table, we express $f(x, y, z)$ in DNF as follows:

We note down the rows in which 'f' column entry is 1. The DNF of f is the Boolean sum of the minterms corresponding to the literals in those rows. While forming the minterm corresponding to a row, 1 entry is replaced by the corresponding variable and 0 entry is replaced by the complement of the variable concerned.

For example, let us consider the function $f(x, y, z)$ whose truth table representation is given as follows:

x	y	z	f
1	1	1	0
1	1	0	1
1	0	1	1
1	0	0	1
0	1	1	0
0	1	0	0
0	0	1	0
0	0	0	0

1's occur in the 'f' column against the second, third and fourth rows. The minterm corrsponding to the second row is xyz', since 1 occurs in each of the x column and y column and 0 occurs in the z column. Similarly the minterms corresponding to the third and fourth rows are $xy'z$ and $xy'z'$ respectively.

Since f is the Boolean sum of these three minterms, the required DNF of f is

$$xyz' + xy'z + xy'z'$$

The CNF of $f(x, y, z)$ represented by a truth table is obtained as follows:

We note down the rows in which the 'f' column entry is 0. The CNF of f is the Boolean product of the maxterms corresponding to the literals in those rows. While forming the maxterm corresponding to a row, 0 entry is replaced by the corresponding variable and 1 entry is replaced by the complement of the variable concerned.

In the above example, 0's occur in the 'f' column against the 1st row and the fifth to the eighth rows. The maxterm corresponding to the first row is $(x' + y' + z')$, since 1 occurs under each of x, y, z.

Similarly the maxterms corrsponding to the other rows are written.

The CNF of f is the Boolean product of these maxterms.

$\therefore \quad f = (x' + y' + z')\,(x + y' + z')\,(x + y' + z)\,(x + y + z')\,(x + y + z).$

2. Algebraic Method

To get the DNF of a given Boolean function, we express it as a sum of products. Then each product is multiplied in Boolean sense by $a + a'$, which is equal to 1, if a is the missing literal and simplified. In the end if a product term is repeated, the repetition is avoided since $a + a = a$.

To get the CNF of a given Boolean function, we express it as a product of sums. Then to each sum is added in Boolean sense the term aa', which is equal to 0, if a is the missing literal and simplified. In the end if a sum factor is repeated, the repetition is avoided since $a \cdot a = a$. For example, let us consider the Boolean function $f(x, y, z) = x(y' + z')$ and express it in the sum of products canonical form:

$$\begin{aligned} f &= xy' + xz' \\ &= xy' \cdot (z + z') + xz'(y + y') \ \{\because\ z \text{ is the missing literal in the first} \\ &\qquad \text{product and } y \text{ is the missing literal in the second product.}\} \\ &= xy'z + xy'z' + xyz' + xy'z' \\ &= xy'z + xy'z' + xyz' \ (\because\ xy'z' \text{ is repeated}) \end{aligned}$$

Now let us express the same function in the product of sums canonical form.

$$\begin{aligned} f &= x \cdot (y' + z') \\ &= (x + yy') \cdot (y' + z' + xx') \\ &= (x + y) \cdot (x + y') \cdot (y' + z' + x)\,(y' + z' + x') \\ &= (x + y + zz')\,(x + y' + zz')\,(x + y' + z')\,(x' + y' + z') \\ &= (x + y + z) \cdot (x + y + z') \cdot (x + y' + z)\,(x + y' + z') \\ &\quad\ (x + y' + z')\,(x' + y' + z') \\ &= (x + y + z) \cdot (x + y + z') \cdot (x + y' + z) \cdot (x + y' + z')\,(x' + y' + z') \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{(repetition avoided)} \end{aligned}$$

LOGIC GATES

A computer or any other electronic device is made up of a number of circuits. Boolean algebra can be used to design the circuits of electronic devices. The basic elements of circuits are solid state devices called *gates*, that implement Boolean operations. The circuits that we consider in this section give the output that depends only on the input and not on the current state of the circuit. In other words these circuits have no memory capabilities. Such circuits are called *combinational circuits gating networks*.

We shall now consider three basic types of gates that are used to construct combinational circuits:

1. *OR gate:* This gate receives two or more inputs (Boolean variables) and produces an output equal to the Boolean sum of the values of the input variables. The symbol used for an OR gate is shown in Fig. 5.29(a). The inputs are shown on the left side entering the symbol and the output on the right side leaving the symbol.

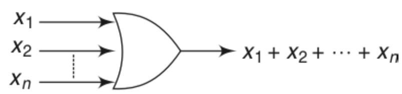

Fig. 5.29(a)

2. *AND gate:* This gate receives two or more inputs (Boolean variables) and produces an outut equal to their Boolean product. The symbol used for an AND gate is shown in Fig. 5.29(b).

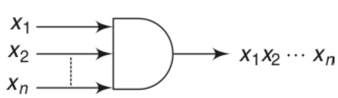

Fig. 5.29(b)

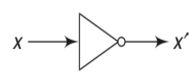

Fig. 5.29(c)

3. *NOT gate or Invertor:* This gate accepts only one input (value of one Boolean variable) and produces the complement of this value as the output. The symbol for this NOT gate is shown in Fig. 5.29(c).

COMBINATION OF GATES

Combinational circuits are formed by interconnecting the basic gates. When such circuits are formed, some gates may share inputs. One method is to indicate the inputs separately for each gate [Fig. 5.30(a)]. The other method is to use branchings that indicate all the gates that use a given input [Fig. 5.30(b)].

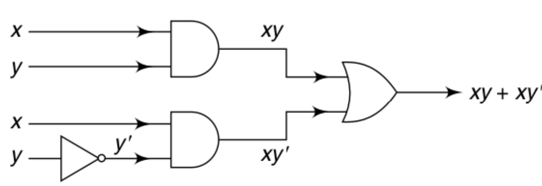

Fig. 5.30(a)

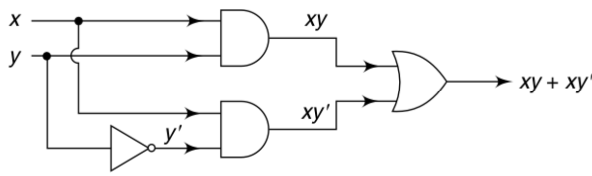

Fig. 5.30(b)

Thus we can compute the value of the output y by tracing the flow through the circuit symbolically from left to right as in the following example. [See Fig. 5.30(c)].

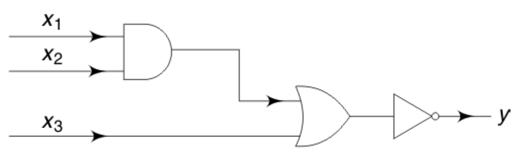

Fig. 5.30(c)

First the Boolean product of x_1 and x_2 is obatined as $x_1 \cdot x_2$. This output is Boolean added with x_3 to produce $x_1x_2 + x_3$. This output is conmplemented to produce the final output $y = (x_1x_2 + x_3)'$.

ADDERS

We shall consider two examples of circuits that perform some useful functions. First we consider a *half adder* that is a logic circuit used to find $x + y$, where x and y are two bits each of which has the value 0 or 1. The output will consist of two bits, namely the sum bits and carry bit c. Circuits of this type having more than one output are called *multiple output circuits*. The truth table for the half adder is given as follows:

Inputs		*Outputs*	
x	y	s	c
0	0	0	0
0	1	1	0
1	0	1	0
1	1	0	1

From the truth table, we get $s = xy' + x'y$ and $c = xy$. The half adder circuit is given in Fig. 5.31(a).

If we observe that

$$\begin{aligned}(x + y)\ (xy)' &= (x + y)\ (x' + y')\\ &= xx' + xy' + x'y + yy'\\ &= xy' + x'y\end{aligned}$$

the half adder circuit can be simplified with only four gates as shown in Fig. 5.31(b).

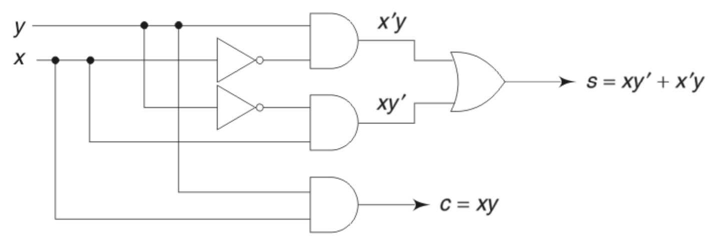

Fig. 5.31(a)

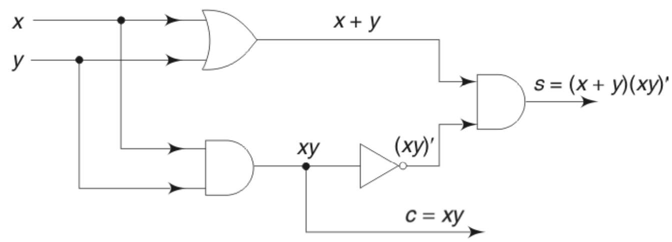

Fig. 5.31(b)

A full adder accepts three bits x, y, z as input and produces two output bits s (sum bit) and c (carry bit). The truth table for the full order is given as follows:

Inputs			*Outputs*	
x	y	z	c	s
1	1	1	1	1
1	1	0	1	0
1	0	1	1	0
1	0	0	0	1
0	1	1	1	0
0	1	0	0	1
0	0	1	0	1
0	0	0	0	0

From the truth table, we get

$$s = xyz + xy'z' + x'yz' + x'y'z$$

and $$c = xyz + xyz' + xy'z + x'yz$$

If we observe that

$$\begin{aligned} c &= xyz + xyz' + xy'z + x'yz \\ &= (xyz + xyz') + (xyz + xy'z) + (xyz + x'yz) \\ &= xy\,(z + z') + zx(y + y') + yz(x + x') \\ &= xy + yz + zx, \end{aligned}$$

the circuit for the full adder can be drawn as given in Fig. 5.32.

Note If we simplify s, we can draw the circuit for full adder in a simpler way using lesser number of gates.

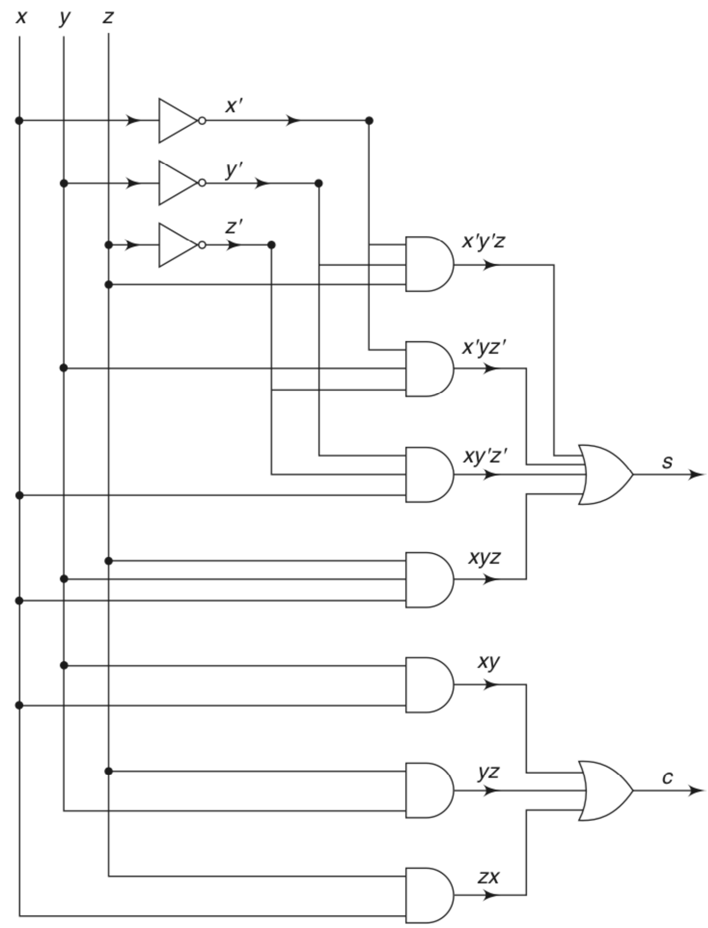

Fig. 5.32

Minimisation of Circuits/Boolean Functions

The important use of Boolean algebra is to express circuit design problems in a simplified form that is more readily understood. The efficiency of a combinatorial circuit depends on the number of gates used and on the manner of arranging them, because the cost of a circuit depends on the number of gates in the circuit to a certain extent.

For example, let us consider the following circuit, the output of which is $xyz + xyz'$, that is in the sum of products form.

Since the two products in this example differ in only one variable, namely z, they can be combined as follows:

$$\begin{aligned} xyz + xyz' &= xy(z + z') \\ &= xy \cdot 1 \\ &= xy \end{aligned}$$

The circuit for the simplified function xy is shown in Fig. 5.33(b).

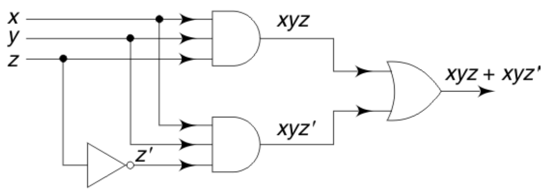

Fig. 5.33(a)

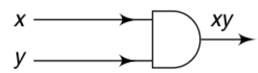

Fig. 5.33(b)

The second circuit uses only one gate, whereas the first circuit used three gates and an inverter. Thus the second circuit is a simplified or minimised version of the first circuit.

From this example, we see that combining terms in the *S* of *P* expansion of a circuit leads to a simpler expression for the circuit. Though simplification of *S* of *P* expansions can be done algebraically using laws of Boolean algebra, there are two other procedures which are more elegant and which will be described as follows. The goal of all these procedures is to obtain Boolean sums of Boolean products that contain the least number of products of least number of literals.

KARNAUGH MAP METHOD

Karnaugh Map method is a graphical method for simplifying Boolean expressions involving six or fewer variables that are expressed in the sum of products form and that represent combinational circuits. Simplification requires identification of terms in the Boolean expression which can be combined (as in the previous example). The terms which can be combined can be easily found out from Karnaugh maps.

A Karnaugh map (K-map) is a diagram consisting of squares. If the Boolean expression contains n variables, the corresponding K-map will have 2^n squares, each of which represents a minterm. A '1' is placed in the square representing a minterm if it is present in the given expression. A '0' is placed in the square that corresponds to the minterm not present in the expression. The simplified Boolean expression that represents the output is then obtained by combining or grouping adjacent squares that contain 1. Adjacent squares are those that represent minterms differing by only one litreral.

To identify adjacent cells (squares) in the K-map for grouping, the following points may be borne in mind:

1. The number of cells in a group must be a power of 2, i.e., 2, 4, 8, 16, etc.
2. A cell containing 1 may be included in any number of groups.
3. To minimise the expression to the maximum possible extent, largest possible groups must be preferred. viz., a group of two cells should not be considered, if these cells can be included in a group of four cells and so on.
4. Adjacent cells exist not only within the interior of the K-map, but also at the extremes of each column and each row viz. the top cell in any column is adjacent to the bottom cell in the same column. The left most cell in

any row is adjacent to the rightmost cell in that row. [see Fig. 5.37 and 5.38]

Karnaugh maps for 2, 3 and 4 variables in two forms for each are given in Figs. 5.34, 5.35 and 5.36. The minterms which the cells represent are written within the cells.

	y'	y
x'	$x'y'$	$x'y$
x	xy'	xy

(a)

$x \backslash y$	0	1
0	$x'y'$	$x'y$
1	xy'	xy

(b)

Fig. 5.34 K-map for 2 variables

	$x'y'$	$x'y$	xy	xy'
z'	$x'y'z'$	$x'yz'$	xyz'	$xy'z'$
z	$x'y'z$	$x'yz$	xyz	$xy'z$

(a)

$z \backslash xy$	00	01	11	10
0	$x'y'z'$	$x'yz'$	xyz'	$xy'z'$
1	$x'y'z$	$x'yz$	xyz	$xy'z$

(b)

Fig. 5.35 K-map for 3 variables

	$y'z'$	$y'z$	yz	yz'
$w'x'$	$w'x'y'z'$	$w'x'y'z$	$w'x'yz$	$w'x'yz'$
$w'x$	$w'xy'z'$	$w'xy'z$	$w'xyz$	$w'xyz'$
wx	$wxy'z'$	$wxy'z$	$wxyz$	$wxyz'$
wx'	$wx'y'z'$	$wx'y'z$	$wx'yz$	$wx'yz'$

(a)

$wx \backslash yz$	00	01	11	10
00	$w'x'y'z'$	$w'x'y'z$	$w'x'yz$	$w'x'yz'$
01	$w'xy'z'$	$w'xy'z$	$w'xyz$	$w'xyz'$
11	$wxy'z'$	$wxy'z$	$wxyz$	$wxyz'$
10	$wx'y'z'$	$wx'y'z$	$wx'yz$	$wx'yz'$

(b)

Fig. 5.36 K-map for 4 Variables

While minimising Boolean expressions by K-map method, it will be advantageous if we are familiar with patterns of adjacent cells and groups of 1's, that will be enclosed by loops. All the basic patterns are given as follows for 3 and 4 variable K-maps:

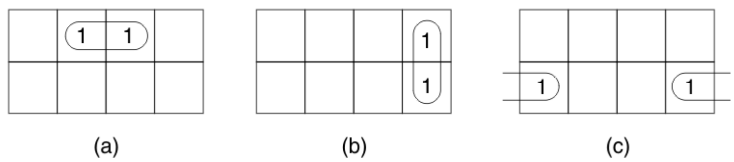

Fig. 5.37(a) All possible forms of basic loops of 2 cells or 3 variables

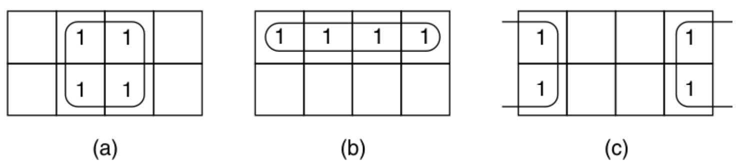

Fig. 5.37(b) All possible forms of 4 cell basic loops for 3 variables

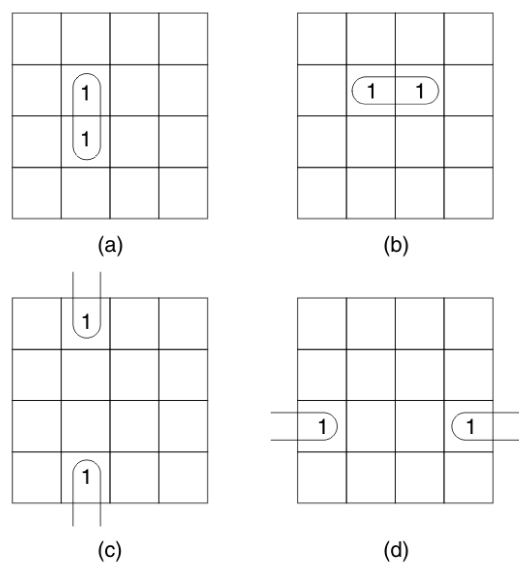

Fig. 5.38(a) All possible forms of 2 cell basic loops for 4 variables

Note A loop of 2, 4 and 8 cells will eliminate from the simplified Boolean expression 1, 2 and 3 variables.

Procedure for minimisation of Boolean expressions using K-maps

1. K-map is first constructed by placing 1's in those squares corresponding to the minterms present in the expression and 0's in other squares.
2. All those 1's that cannot be combined with any other 1's are identified and looped.
3. All those 1's that combine in a loop of two but do not make a loop of four are looped.
4. All those 1's that combine in a loop of four but do not make a loop of eight are looped.
5. The process is stopped when all the 1's have been covered.

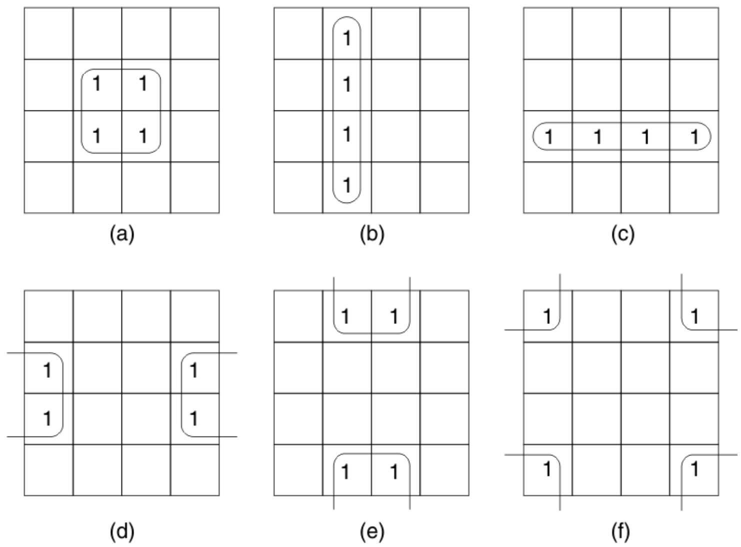

Fig. 5.38(b) All possible forms of 4 cell basic loops for 4 variables

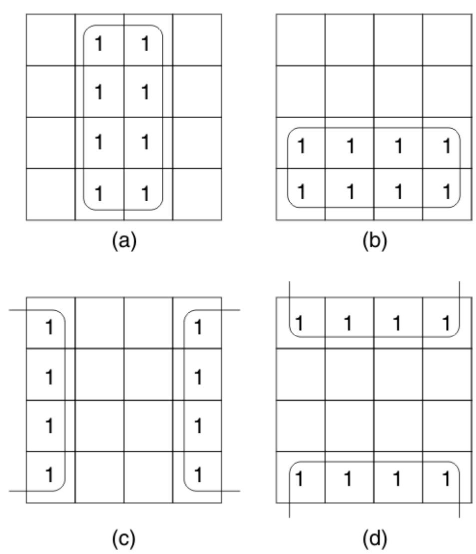

Fig. 5.38(c) All possible forms of 8 cell basic loops for 4 variables

6. The simplified expression is the sum of all the terms corresponding to various loops.

Note Minimisation of Boolean expressions with 5 or 6 variables by the K-map method is beyond the scope of this book.)

Alternative notation for S of P form of Boolean expressions

In each minterm of the sum of products form of a Boolean expression, a variable is replaced by 1 and a complemented variable is replaced by 0. Thus we get the binary equivalent of the minterm. Then the decimal equivalent of the binary number is found out. All the decimal numbers corresponding to the minterms are written after *a* Σ separated by commas.

For example let us consider the Boolean expression $(xy'z + xy'z' + x'yz + x'y'z + x'y'z')$ in 3 variables.

The binary equivalents of the minterms in the given order are 101, 100, 011, 001, 000.

The decimal equivalents of the binary numbers in the given order are 5, 4, 3, 1, 0. The alternative notation for the given Boolean expression is $f(x, y, z) = \Sigma(0, 1, 3, 4, 5)$.

On the other hand, if the Boolean expression is given as $f(a, b, c, d) = \Sigma(0, 2, 6, 7, 8, 9, 13, 15)$, the binary equivalents to the given decimal numbers are written as 4 digit numbers, as f is a 4 variable expression.

They are 0000, 0010, 0110, 0111, 1000, 1001, 1101 and 1111.

The minterms corresponding to these binary numbers are $a'b'c'd'$, $a'b'cd'$, $a'bcd'$, $a'bcd$, $ab'c'd'$, $ab'c'd$, $abc'd$.

Thus the given Boolean expression is $f(a, b, c, d) = a'b'c'd' + a'b'cd' + a'bcd' + a'bcd + ab'c'd' + ab'c'd + abc'd$.

DON'T CARE TERMS

The Boolean function to be simplified will contain two groups of minterms— one group of minterms which are to be necessarily included in the function and the other group of minterms which may or may not be included in the function. The second group of minterms are called *Don't care terms*.

In the K-map representation, the cells corresponding to 'don't care minterms' will be filled up with ϕ – *a* '0' and *a* '1' superimposed or with the letter *d*. Those minterms in the don't care group which when included with the regular input terms will simplify the output to the maximum, viz. will yield the most economical circuit are assigned the value 1 and others are assigned the value 0.

Thus a don't care term of a Boolean function is a minterm whose value is not of any consequence and as such its value can be chosen either as a 0 or as a 1 at our convenience.

For example, let the Boolean function to be simplified be $f(a, b, c) = \Sigma(3, 5) + \sum_{\phi}(0, 7)$. The regular terms in $f(a, b, c)$ are $a'bc$ and $ab'c$ and the don't care terms are $a'b'c'$ and abc.

The K-map representation of $f(a, b, c)$ is given in Fig. 5.39.

The most simplified output will be obtained if we include '*abc*' as a regular input minterm. The ouptput function in this case is $(ac + bc)$.

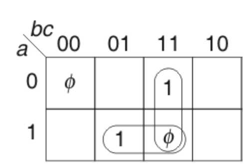

Fig. 5.39

QUINE-McCLUSKEY'S TABULATION METHOD

This method provides a mechanical procedure for simplifying Boolean expressions in the sum of products form. K-map method is cumbersome when there are five or six variables in the expression, whereas the Quine-McCluskey's method can be used to simplify Boolean functions in any number of variables. When the K-map method is used, one has to depend on visual inspection to identify adjacent cells that are to be looped, whereas the tabulation method uses a step-by-step procedure, which is described as follows:

Step-by-step procedure of Quine-McCluskey's method

1. The given Boolean function is first expressed in its canonical sum form.
2. Then each term in the function is converted to a binary form by replacing x_i in it by 1 and x_i' by 0.
3. Then the terms are separated into groups, according to the number 1's in each. (column 1)
4. The binary numbers are then converred to the decimal form and the decimal numbers are arranged in ascending order of their values within the groups. (column 2)
5. The smallest decimal number in the uppermost group in column 2 is compared successively with all numerically greater numbers that appear in the next group in that column. When the two numbers under comparison differ by a power of 2 [viz., 2^0, 2^1, 2^2, etc.] the pair is placed is a new 3rd column along with the value by which they differ in brackets. The second number (next smaller number) in the first group is then compared with all numerically greater numbers in the second group. The process is continued until the first group is exhausted. A line is then placed under the last entry in the 3rd column.

   Now the first number in the second group is compared with all numerically greater numbers in the third group. This procedure is continued until the entire list in column 2 is exhausted.

   Any decimal number that fails to combine with any other number is noted for later reference. The Boolean term that corresponds to such a number is called a *prime implicant*.
6. The second comparison is performed on column 3. This comparison is almost identical with the procedure used on column 2, except that both the decimal numbers in the brackets must be the same before checking the difference of the leading number in each row.

   For example, let us consider the following:

Column 3
0, 2 (2)
0, 8 (8)
2, 10 (8)
8, 10 (2)

   The first row numbers in the first group are compared with the second row numbers in the second group, since the difference numbers in the brackets are the same, namely 2. Similarly the second row numbers in the first group are compared with the first row numbers in the second group, since the bracketed difference numbers are equal, namely 8. The third column entries will then be

   $$0,\ 2,\ 8,\ 10\ (2,\ 8)$$

   and

   $$0,\ 8,\ 2,\ 10\ (8,\ 2)$$

The first entry in the brackets is the difference in the previous column just carried over and the second entry is the new difference between the leading terms in the rows compared. As the order of the digits has no significance, the two rows in column 4 are listed only once in the column 4 as 0, 2, 8, 10 (2, 8). Again the terms that fail to compare are recorded.

7. A new comparison is now performed on column 4. Again all the terms in the brackets must be identical before a comparison is made. Only the leading decimal numbers in the rows are actually checked to determine if the compared numbers differ by a power of 2. A new comparison is performed on each new column generated until further comparisons are not possible.
8. A graphical method (Prime implicant chart method) is now used to eliminate unnecessary prime implicants and to show all possible answers. All the decimal numbers corresponding to the terms in the given Boolean function are entered in the first row of the chart. All the prime implicants chosen are entered in the first column of the chart. Check marks (×) are now placed in the body of the chart below those decimal numbers in the first row which also occur in the first column. Numbers in the brackets are not considered for this purpose.
9. Columns that contain only one check mark are noted. The term in the first column that produces that ckeck mark is required in the answer and is called irredundant prime implicant. The check mark is now encircled.
10. The first decimal number in each irredundant prime implicant is converted to its binary form. The bit positions in the binary number corresponding to the decimal numbers in the brackets are crossed out. The remaining bits are then converted to their Boolean (alphabetic) variables.

WORKED EXAMPLES 5(C)

Example 5.1 Determine whether the posets represented by the Hasse diagrams given in Fig. 5.40 are lattices.

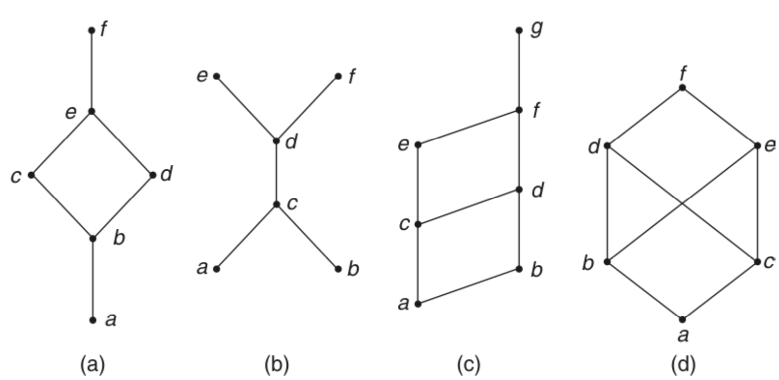

Fig. 5.40

(a) The poset represented by the Hasse diagram in Fig. 5.40(a) is a lattice, since every pair of elements of this poset has both an LUB and a GLB.
(b) The pair of elements a, b does not have a GLB and the pair e, f does not have an LUB. Hence the poset in Fig. 5.40(b) is not a lattice.
(c) Since every pair of elements of the poset in Fig. 5.40(c) has both an LUB and a GLB, it is a lattice.
(d) Though the pair of elements $\{b, c\}$ has 3 upper bounds d, e, f, none of these precedes the other two i.e. $\{b, c\}$ does not have an LUB. Hence the poset in Fig. 5.40(d) is not a lattice.

Example 5.2 If $P(S)$ is the power set of a set S and $\cup$ and $\cap$ are taken as the join and meet, prove that $\{P(S), \subseteq\}$ is a lattice.

Let A and B be any two elements of $P(S)$, i.e. any two subsets of S.

Then an upper bound of $\{A, B\}$ is a subset of S that contains both A and B and the least among them is $A \cup B \in P(S)$, as can be seen from the following:

We know $A \subseteq A \cup B$ and $B \subseteq A \cup B$. i.e. $A \cup B$ is an upper bound of $\{A, B\}$. If we assume that $A \subseteq C$ and $B \subseteq C$, then $A \cup B \subseteq C$.

Thus the LUB $\{A, B\} = A \cup B$.

Similarly $A \cap B \subseteq A$ and $A \cap B \subseteq B$

i.e. $A \cap B$ is a lower bound of $\{A, B\}$.

If we assume that $C \subseteq A$ and $C \subseteq B$, then $C \subseteq A \cap B$.

Thus the GLB $\{A, B\} = A \cap B$.

i.e. every pair of elements of $P(S)$ has both an LUB and a GLB under set inclusion relation.

Hence $\{P(S), \subseteq\}$ is a lattice.

Note Refer to the example 20 of the previous section in which the Hasse diagram of $\{P(S), \subseteq\}$, where $S \equiv \{a, b, c\}$ is given.

Example 5.3 If L is the collection of 12 partitions of $S = \{1, 2, 3, 4\}$ ordered such that $P_i \le P_j$ if each block of P_i is a subset of a block P_j, show that L is a bounded lattice and draw its Hasse diagram.

The 12 partitions of $S = \{1, 2, 3, 4\}$ are

$P_1 = \{(1), (2), (3), (4)\}$ i.e. $[1, 2, 3, 4]$, $P_2 = \{(1, 2), (3), (4)\}$ i.e. $[12, 3, 4]$

$P_3 = [13, 2, 4]$, $P_4 = [14, 2, 3]$, $P_5 = [23, 1, 4]$, $P_6 = [24, 1, 3]$, $P_7 = [34, 1, 2]$, $P_8 = [123, 4]$, $P_9 = [124, 3]$, $P_{10} = [134, 2]$, $P_{11} = [234, 1]$ and $P_{12} = [1234]$.

Using the ordering relation, the Hasse diagram of L has been drawn as in Fig. 5.41.

Since $P_1 \le P_j$, for $j = 2, 3, \ldots, 12$, P_1 is a lower bound of the lattice.

Similarly since $P_j \le P_{12}$ for $j = 1, 2, \ldots, 11$, P_{12} is an upper bound of the lattice.

Since L has both a lower bound and an upper bound, it is a bounded lattice.

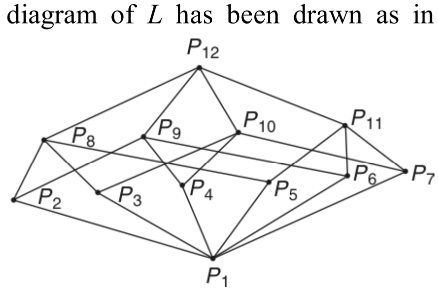

Fig. 5.41

Example 5.4 Draw the Hasse diagram of the lattice $\{P(S), \subseteq\}$ in which the join and meet are the operations $\cup$ and $\cap$ respectively, where $S = \{a, b, c\}$.

Identify *a* sublattice of this lattice with 4 elements and a subset of this lattice with 4 elements which is not a sublattice.

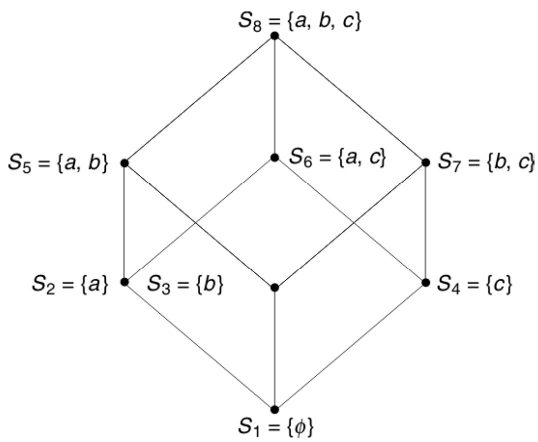

Fig. 5.42

$L_1 = \{S_1, S_2, S_4, S_6\}$ is a sublattice of L, by the argument given below:

$$S_1 \cup S_2 = S_2 \in L_1,\ S_1 \cup S_4 = S_4 \in L_1,\ S_1 \cup S_6 = S_6 \in L_1,$$
$$S_2 \cup S_4 = S_6 \in L_1,\ S_2 \cup S_6 = S_6 \in L_1 \text{ and } S_4 \cup S_6 = S_6 \in L_1$$

Thus L_1 is closed under the operation $\cup$.

Now $S_1 \cap S_2 = S_1 \in L_1,\ S_1 \cap S_4 = S_1 \in L_1,\ S_1 \cap S_6 = S_1 \in L_1,$
$$S_2 \cap S_4 = S_1 \in L_1,\ S_2 \cap S_6 = S_2 \in L_1,\ S_4 \cap S_6 = S_4 \in L_1.$$

Thus L_1 is closed under the operation $\cap$.

Let us now consider $L_2 = \{S_1, S_5, S_7, S_8\}$.

$S_5 \cap S_7 = b = S_3 \notin L_2$. Hence L_2 is not a sublattice of L.

Example 5.5 If S_n is the set of all divisors of the positive integer n and D is the relation of 'division', viz., aDb if and only if a divides b, prove that $\{S_{24}, D\}$ is a lattice. Find also all the sublattices of D_{24} [= $\{S_{24}, D\}$] that contain 5 or more elements.

Clearly $\{S_{24}, D\} = \{(1, 2, 3, 4, 6, 8, 12, 24), D\}$ is a lattice whose Hasse diagram is given in Fig. (5.43).

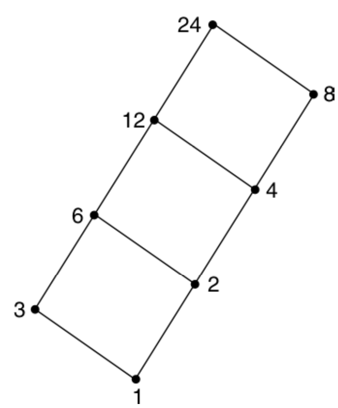

Fig. 5.43

The sublattices containing 5 elements are {1, 2, 3, 6, 12}, {1, 2, 3, 12, 24}, {1, 2, 6, 12, 24}, {1, 3, 6, 12, 24} and {1, 2, 4, 8, 24}

The sublattice containing 6 elements is {1, 2, 3, 6, 12, 24}.

Example 5.6 If a and b are elements of a lattice L such that $a \le b$ and if the interval $[a, b]$ is defined as the set of all $x \in L$ such that $a \le x \le b$, show that $[a, b]$ is a sublattice of L.

Let x, y be in $[a, b]$. Then $x, y \in L$.

$\therefore$ $x \vee y$ and $x \wedge y \in L$, since L is a lattice.

Now $a \le x \le x \vee y \le b$ $\quad \therefore x \vee y \in [a, b]$

Also $a \le x \wedge y \le x \le b$ $\quad \therefore x \wedge y \in [a, b]$

Hence $[a, b]$ is a sublattice.

Example 5.7 Verify whether the lattice given by the Hasse diagram in Fig. 5.44 is distributive.

$a \wedge (b \vee c) = a \wedge b = 0$

Also $(a \wedge b) \vee (a \wedge c) = 0 \vee 0 = 0$

$\therefore$ $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ (1)

Now $c \wedge (a \vee b) = c \wedge 1 = c$

Also $(c \wedge a) \vee (c \wedge b) = 0 \vee c = c$

$\therefore$ $c \wedge (a \vee b) = (c \wedge a) \vee (c \wedge b)$ (2)

Steps (1) and (2) do not mean that the lattice is distributive.

Now let us consider

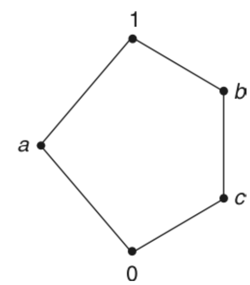

Fig. 5.44

$b \wedge (c \vee a) = b \wedge 1 = b$

But $(b \wedge c) \vee (b \wedge a) = c \vee 0 = c$

This means that $b \wedge (c \vee A) \neq (b \wedge c) \vee (b \wedge a)$

Hence the given lattice is not distributive.

Example 5.8 Prove that $D_{42} \equiv \{S_{42}, D\}$ is a complemented lattice by finding the complements of all the elements.

$$D_{42} = \{1, 2, 3, 6, 7, 14, 21, 42\}$$

The Hasse diagram of D_{42} is given in Fig. 5.45.

The zero element of the lattice is 1 and the unit element of the lattice is 42.

$1 \vee 42 = \text{LCM } \{1, 42\} = 42 \equiv \text{'1'}$

and $1 \wedge 42 = \text{GCD } \{1, 42\} = 1 \equiv \text{'0'}$

$\therefore$ $1' = 42$

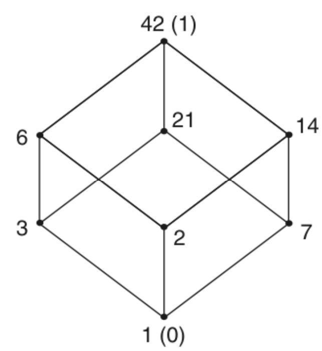

Fig. 5.45

Similarly we can find that $2' = 21$, $3' = 14$, $6' = 7$, $7' = 6$, $14' = 3$, $21' = 2$ and $42' = 1$.

Since every element of D_{42} has a complements, it is a complemented lattice.

Example 5.9 Find the complements, if they exist, of the elements a, b, c of the lattice, whose Hasse diagram is given in Fig. 5.46. Can the lattice be complemented?

From the Hasse diagram, it is seen that $a \vee e = 1$ and $a \wedge e = 0$.

$\therefore$ The complement of a is e.

Similarly $b \vee d = 1$ and $b \wedge d = 0$

$\therefore$ The complement of b is d.

But $\quad c \vee a = c$ and $c \wedge a = a$

$\quad c \vee b = c$ and $c \wedge b = b$

$\quad c \vee d = 1$ and $c \wedge d = a$

$\quad c \vee e = 1$ and $c \wedge e = b$

$\therefore$ c has no complement.

Since one of the elements of the lattice, namely c has no complement, the lattice is not complemented.

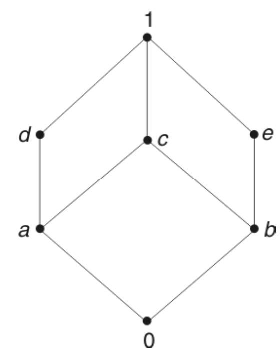

Fig. 5.46

Example 5.10 Prove that cancellation law holds good in a distributive lattice, viz. if $\{L, \vee, \wedge\}$ is a distributive lattice such that $a \vee b = a \vee c$ and $a \wedge b = a \wedge c$, where $a, b, c \in L$, then $b = c$.

$$\begin{aligned}
c \wedge (a \vee b) &= (c \wedge a) \vee (c \wedge b), \text{ since } L \text{ is distributive}\\
&= (a \wedge c) \vee (c \wedge b), \text{ by commutativity}\\
&= (a \wedge b) \vee (c \wedge b), \text{ given}\\
&= (b \wedge a) \vee (b \wedge c), \text{ by commutativity}\\
&= b \wedge (a \vee b), \text{ by distributivity}\\
&= b \wedge (b \vee a), \text{ by commutativity}\\
&= b, \text{ by absorption law} \qquad (1)
\end{aligned}$$

Also
$$\begin{aligned}
c \wedge (a \vee b) &= c \wedge (a \vee c), \text{ given}\\
&= c \wedge (c \vee a), \text{ by commutativity}\\
&= c, \text{ by absorption law} \qquad (2)
\end{aligned}$$

From (1) and (2), it follows that $b = c$.

Example 5.11 Prove that De Morgan's laws hold good for a complemented distributive lattice $\{L, \vee, \wedge\}$, viz. $(a \vee b)' = a' \wedge b'$ and $(a \wedge b)' = a' \vee b'$, where $a, b \in L$.

Since the lattice is complemented, the complements of a and b exist. Let them be a' and b'.

Now
$$\begin{aligned}
(a \vee b) \vee (a' \wedge b') &= \{(a \vee b) \vee a'\} \wedge \{(a \vee b) \vee b'\}, \text{ by distributivity}\\
&= \{a \vee (b \vee a')\} \wedge \{a \vee (b \vee b')\}, \text{ by associativity}\\
&= \{a \vee (a' \vee b)\} \wedge \{a \vee 1\}, \text{ by commutativity}\\
&= \{(a \vee a') \vee b\} \wedge \{a \vee 1\}, \text{ by associativity}\\
&= (1 \vee b) \wedge (a \vee 1)\\
&= 1 \wedge 1\\
&= 1 \qquad (1)
\end{aligned}$$

$$(a \vee b) \wedge (a' \wedge b') = \{a \wedge (a' \wedge b')\} \vee \{b \wedge (a' \wedge b')\}, \text{ by distributivity}$$
$$= \{(a \wedge a') \wedge b'\} \vee \{b \wedge (b' \wedge a')\},$$
by associativity and commutativity
$$= \{(a \wedge a') \wedge b'\} \vee \{(b \wedge b') \wedge a'\}, \text{ by associativity}$$
$$= (0 \wedge b') \vee (0 \wedge a')$$
$$= 0 \vee 0$$
$$= 0 \quad (2)$$

From (1) and (2), we get
$a' \wedge b'$ is the complement of $a \vee b$
or $$(a \vee b)' = a' \wedge b' \quad (3)$$
By principle of duality, it follows from (3) that
$$(a \wedge b)' = a' \vee b'.$$

Example 5.12 If $P(S)$ is the power set of a non-empty set S, prove that $\{P(S), \cup, \cap, \backslash, \phi, S\}$ is a Boolean algebra, where the complement of any set $A \subseteq S$ is taken as $S \backslash A$ or $S - A$ that is the relative complement of A with respect to S.

Let X, Y and Z be any three elements of $P(S)$.

Now $X \cup \phi = X$ and $X \cap S = X$

Thus ϕ and S play the roles of 0 and 1 and the identity laws are satisfied (1)
$$X \cup Y = Y \cup X \text{ and } X \cap Y = Y \cap X$$
i.e. the commutative laws are satisfied (2)
$$(X \cup Y) \cup Z = X \cup (Y \cup Z) \text{ and } (X \cap Y) \cap Z = X \cap (Y \cap Z)$$
i.e. the associative laws hold good (3)
$$X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z) \text{ and}$$
$$X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$$
i.e. the distributive laws hold good (4)
$$X \cup (S - X) = S \text{ and } X \cap (S - X) = \phi$$
i.e. the complement laws hold good (5)

Thus all the 5 axioms of Boolean algebra hold good.

Hence $\{P(S), \cup, \cap, \backslash, \phi, S\}$ is a Boolean algebra.

Example 5.13

(i) If $a, b \in S = \{1, 2, 3, 6\}$ and $a + b = \text{LCM}(a, b)$, $a \cdot b = \text{GCD}(a, b)$ and $a' = \dfrac{6}{a}$, show that $\{S, +, \cdot, ', 1, 6\}$ is a Boolean algebra.

(ii) If $a, b \in S = \{1, 2, 4, 8\}$ and $a + b = \text{LCM}(a, b)$, $a, b = \text{GCD}(a, b)$ and $a' = \dfrac{8}{a}$, show that $\{S, +, \cdot, ', 1, 8\}$ is not a Boolean algebra.

(i) 1 and 6 are the zero element and unit element of $\{S, +, \cdot, ', 1, 6\}$
If a represents any of the elements 1, 2, 3, 6 of S, clearly a + '0' = LCM $(a, 1) = a$ and a · '1' = GCD$(a, 6) = a$
i.e. identity laws hold good.

Similarly commutative, associative and distributive laws can be verified.

$$a + a' = \text{LCM}\left(a, \frac{6}{a}\right)$$
$$= 6 = \text{‘1’}$$

and
$$a \cdot a' = \text{GCD}\left(a, \frac{6}{a}\right)$$
$$= 1 = \text{‘0’}$$

i.e. the complement laws hold good.

Hence $\{S, +, \cdot, ', 1, 6\}$ is a Boolean algebra.

(ii) 1 and 8 are the zero element and unit element of $\{S, +, \cdot, ', 1, 8\}$

The first 4 axioms can be verified to be true.

When $a = 2$,
$$a + a' = \text{LCM}\left(2, \frac{8}{2}\right)$$
$$= 4 \neq 8$$

Similarly
$$a \cdot a' = \text{GCD}\left(2, \frac{8}{2}\right)$$
$$= 2 \neq 1$$

Hence the complement laws do not hold good.

Hence $\{S, +, \cdot, ', 1, 8\}$ is not a Boolean algebra.

Example 5.14 In Boolean algebra, if $a + b = 1$ and $a \cdot b = 0$, show that $b = a'$, viz., the complement of every element a is unique.

$$\begin{aligned}
b &= b \cdot 1 \\
&= b \cdot (a + a'), \text{ by } B5 \\
&= b \cdot a + b \cdot a', \text{ by } B4 \\
&= a \cdot b + b \cdot a', \text{ by } B2 \\
&= 0 + b \cdot a', \text{ given} \\
&= a \cdot a' + b \cdot a', \text{ by } B5 \\
&= a' \cdot a + a' \cdot b, \text{ by } B2 \\
&= a' \cdot (a + b), \text{ by } B4 \\
&= a' \cdot 1, \text{ given} \\
&= a', \text{ by } B1
\end{aligned}$$

Example 5.15 In a Boolean algebra, prove that the following statements are equivalent:

(1) $a + b = b$, (2) $a \cdot b = a$, (3) $a' + b = 1$, (4) $a \cdot b' = 0$.

Let (1) be true.

Then
$$\begin{aligned}
a \cdot b &= a \cdot (a + b), \text{ by } (1) \\
&= a, \text{ by absorption law.}
\end{aligned}$$

i.e. (1) $\Rightarrow$ (2)

Now
$$\begin{aligned}
a + b &= a \cdot b + b, \text{ by } (2) \\
&= b + b \cdot a \\
&= b
\end{aligned}$$

i.e. (2) $\Rightarrow$ (1)

$\therefore$ (1) and (2) are equivalent.

$$\begin{aligned} a' + b &= a' + (a + b), \text{ by (1)} \\ &= (a + a') + b \\ &= 1 + b \\ &= 1, \text{ by dominance law.} \end{aligned}$$

i.e. (1) $\Rightarrow$ (3)

Also
$$\begin{aligned} a + b &= (a + b) \cdot 1 \\ &= (a + b) \cdot (a' + b), \text{ by (3)} \\ &= a \cdot a' + b \\ &= 0 + b \\ &= b \end{aligned}$$

i.e. (3) $\Rightarrow$ (1)

$\therefore$ (1) and (3) are equivalent.

Given: $a' + b = 1$ (3)

$\therefore$ $(a' + b)' = 1'$

i.e. $(a')' \cdot b' = 0$, by De Morgan's law

i.e. $a \cdot b' = 0$

i.e. (3) $\Rightarrow$ (4)

Given: $a \cdot b' = 0$ (4)

$\therefore$ $a' + (b')' = 0'$, by De Morgan's law

i.e. $a' + b = 1$

i.e. (4) $\Rightarrow$ (3)

$\therefore$ (3) and (4) are equivalent.

Hence all the 4 statements are equivalent.

Example 5.16 The Hasse diagram of a Boolean algebra B is given in Fig. 5.47. Which of the following subsets are subalgebras of B, just Boolean algebras and neither?

$S_1 = \{0, a, a', 1\}$; $S_2 = \{0, a' + b, a \cdot b', 1\}$;

$S_3 = \{a, a \cdot b', b, 1\}$;

$S_4 = \{0, b', a \cdot b', a'\}$; $S_5 = \{0, a, b', 1\}$

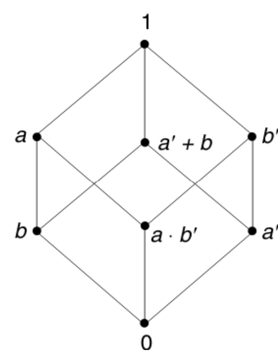

Fig. 5.47

Note To test whether S is a subalgebra of B, it is not necessary to check for closure with respect to all the three operations $+$, $\cdot$ and $'$, nor is it necessary to check whether 0 and 1 are in S_1. Equivalently it is enough to test the closure with respect to $\{+, '\}$ or $\{\cdot, '\}$]

$0 + a = a$, $0 + a' = a'$, $0 + 1 = 1$, $a + a' = 1$, $a + 1 = 1$ and $a' + 1 = 1$ are in S_1.

$$0' = 1,\ a',\ (a')' = a,\ 1' = 0 \text{ are in } S_1$$

$\therefore$ S_1 is a subalgebra of B.

In fact, the general form of a 4-element subalgebra is $(0, a, a', 1)$.

Accordingly, $(a' + b)' = a \cdot b'$. Hence $S_2 = \{0, a' + b, a \cdot b', 1\}$ is also a subalgebra of B.

Though S_3 and S_4 satisfy the axioms of Boolean algebra, they are not closed with respect $(+, ')$.

Hence S_3 and S_4 are Boolean algebras, but not subalgebras of B.

In S_5, a' and $(b')'$ are not present. Hence S_5 is not even a Boolean algebra, but it is only a subset of B.

Example 5.17 Simplify the Boolean expression

$a' \cdot b' \cdot c + a \cdot b' \cdot c + a' \cdot b' \cdot c'$, using Boolean algebra identities.

$$\begin{aligned} & a' \cdot b' \cdot c + a \cdot b' \cdot c + a \cdot b' \cdot c' \\ &= a' \cdot b' \cdot c + a \cdot b' \cdot (c + c') \\ &= a' \cdot b' \cdot c + a \cdot b' \cdot 1 \\ &= b' \cdot (a + a' \cdot c) \\ &= b' \cdot (a + a') \cdot (a + c) \\ &= b' \cdot 1 \cdot (a + c) \\ &= a \cdot b' + b' \cdot c \end{aligned}$$

Example 5.18 In any Boolean algebra, show that

$ab' + a'b = 0$ if and only if $a = b$.

(i) Let $a = b$.

Then $ab' + a'b = aa' + a'a$

$= 0 + 0$

$= 0.$

(ii) Let $ab' + a'b = 0$ (1)

Then $a + ab' + a'b = a$

i.e. $a + a'b = a$, by absorption law

i.e. $(a + a') \cdot (a + b) = a$

i.e. $1 \cdot (a + b) = a$

i.e. $a + b = a$ (2)

Similarly, from (1), $ab' + a'b + b = b$

i.e. $ab' + b = b$, by absorption law.

i.e. $(a + b) \cdot (b + b') = b$

i.e. $(a + b) \cdot 1 = b$

i.e. $a + b = b$ (3)

From (2) and (3), it follows that $a = b$.

Example 5.19 In any Boolean algebra, show that

$(a + b')(b + c')(c + a') = (a' + b)(b' + c)(c' + a)$

$$\begin{aligned} \text{L.S.} &= (a + b' + 0)(b + c' + 0)(c + a' + 0) \\ &= (a + b' + c \cdot c') \cdot (b + c' + aa') \cdot (c + a' + bb') \\ &= (a + b' + c) \cdot (a + b' + c') \cdot (b + c' + a) \cdot (b + c' + a') \\ &\quad \cdot (c + a' + b) \cdot (c + a' + b') \\ &= \{(a' + b + c)(a' + b + c')\} \cdot \{(b' + c + a)(b' + c + a')\} \\ &\quad \cdot \{(c' + a + b)(c' + a + b')\} \\ &= (a' + b + cc') \cdot (b' + c + aa') \cdot (c' + a + bb') \end{aligned}$$

$= (a' + b + 0) \cdot (b' + c + 0) \cdot (c' + a + 0)$
$= (a' + b) \cdot (b' + c) \cdot (c' + a)$
$=$ R.S.

Example 5.20 In any Boolean algebra, prove that

(i) $x + wy + uvz = (x + u + w)(x + u + y)(x + v + w)(x + v + y)(x + w + z)(x + y + z)$

(ii) $ab + abc + a'b + ab'c = b + ac$.

(i) R.S. $= (x + u + wy) \cdot (x + v + wy)(x + z + wy)$
$= \{(x + wy) + uv\} \cdot (x + wy + z)$
$= x + wy + uvz$
$=$ L.S.

(ii) L.S. $= (ab + a'b) + (abc + ab'c)$
$= (a + a') \cdot b + (b + b') \cdot ac$
$= 1 \cdot b + 1 \cdot ac$
$= b + ac$
$=$ R.S.

Example 5.21 Find the output of the network given in Fig. 5.48(a) and design a simpler network having the same output.

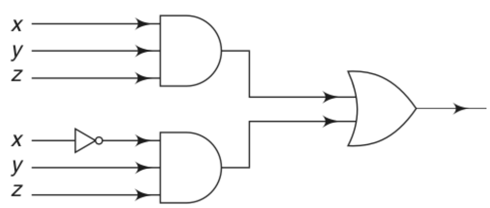

Fig. 5.48(a)

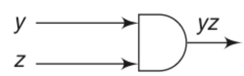

Fig. 5.48(b)

The output of the upper AND gate is xyz. The output of the inverter before the lower AND gate is x' and so the output of the lower AND gate is $x'yz$.

Consequently, the output of the OR gate is $xyz + x'yz$.

Now $xyz + x'yz = (x + x') \cdot yz$
$= 1 \cdot yz$
$= yz$

Thus the simplified Boolean expression is yz which is reresented by the simplified circuit diagram given in Fig. 5.48(b).

Example 5.22 Find the output of the network given in Fig. 5.49(a) and design a simpler network having the same output.

The outputs of the AND gates from top to bottom are xyz', $xy'z'$, $x'yz'$ and $x'y'z'$.

Hence the output of the OR gate is

$$xyz' + xy'z' + x'yz' + x'y'z'.$$

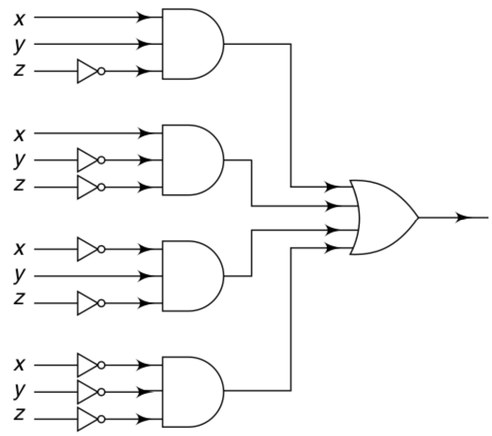

Fig. 5.49(a)

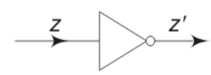

Fig. 5.49(b)

Simplfying algebraically, the output

$$= xz'(y + y') + x'z'(y + y')$$
$$= xz' \cdot 1 + x'z' \cdot 1$$
$$= (x + x')z' = 1 \cdot z' = z'$$

The simplified output is represented by the network [Fig. (5.49(b)].

Example 5.23 Find the output of the combinational circuit given in Fig. 5.50(a) and design a simpler circuit having the same output.

Proceeding backwards from the output f, we have

$$f = f_1 + f_2 + f_3$$
$$= (f_4 \cdot f_5) + f_2 + (f_6 \cdot y)$$
$$= (yz)'\,(wx)' + w + x + y + (f_7 + w)y$$
$$= (yz)' \cdot (wx)' + w + x + y + (x + z + w)y$$

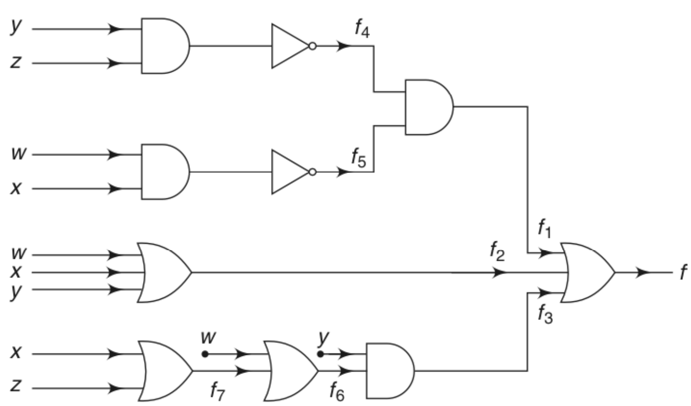

Fig. 5.50(a)

Rewriting f using Boolean algebra rules, we have

$$f = (yz)' \cdot (wx)' + w + x + y + xy + yz + yw$$
$$= (yz)' \cdot (wx)' + (w + yw) + (x + xy) + (y + yz)$$
$$= (yz)' \cdot (wx)' + w + x + y$$

Note $(yz)' \cdot (wx)'$ is not rewritten as $(y' + z') \cdot (w' + x') = y'w' + x'y' + z'w' + z'x'$, as the modified form requires more gates and more inverters than the original form.

The simpler circuit corresponding to the modified f is given in Fig. 5.50(b).

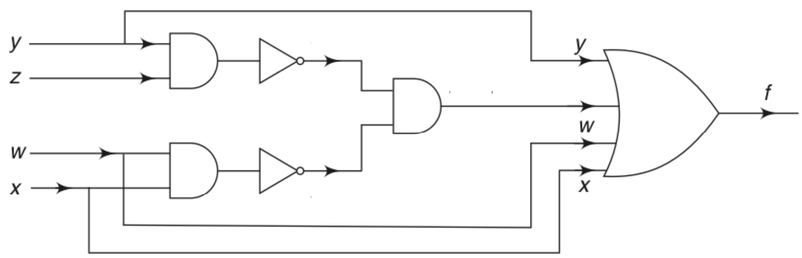

Fig. 5.50(b)

Example 5.24 Simplify the following Boolean expressions using Boolean algebra:

(i) $(x + y + xy)(x + z)$
(ii) $x[y + z(xy + xz)']$
(iii) $xy' + z + (x' + y)z'$.

(i) $(x + y + xy)(x + z) = (x + y)(x + z)$ $[\because y + xy = y]$
$= x \cdot x + xz + xy + yz$
$= x + xz + xy + yz$ $[\because x \cdot x = x]$
$= x + xy + yz$ $[\because x + xz = x]$
$= x + yz$ $[\because x + xy = x]$

(ii) $x[y + z(xy + xz)']$
$= x[y + z(xy)' \cdot (xz)']$ [by De Morgan's law]
$= x[y + z(x' + y')(x' + z')]$ [by De Morgan's law]
$= x[y + z(x' + x'z' + x'y' + y'z')]$ $[\because x' \cdot x' = x']$
$= x[y + z(x' + x'y' + y'z')]$ $[\because x' + x'z' = x']$
$= x[y + z(x' + y'z')]$ $[\because x' + x'y' = x']$
$= x[y + zx' + y'zz']$
$= x(y + zx')$ $[\because zz' = 0]$
$= xy + zxx'$
$= xy$ $[\because xx' = 0]$

(iii) $xy' + z + (x' + y)z' = (xy' + z) + (xy' + z)'$, by De Morgan's law
$= 1$ $[\because a + a' = 1]$

Example 5.25 Simplify the following expressions using Boolean algebra:

(i) $a'b(a' + c) + ab'(b' + c)$
(ii) $a + a'bc' + (b + c)'$

(i) $a'b(a' + c) + ab'(b' + c) \quad = a'b + a'bc + ab' + ab'c$

$(\because\ a' \cdot a' = a' \text{ and } b' \cdot b' = b')$

$= (a'b + ab') + (a'b + ab')c$

$= a'b + ab' \quad [\because\ x + xy = x]$

(ii) $a + a'bc' + (b + c)'$

$= a + a'bc' + b'c'$, by De Morgan's law

$= a + (a'b + b') \cdot c'$

$= a + [(a' + b') \cdot (b + b')c']$

$= a + (a' + b')c' \quad (\because\ b + b' = 1)$

$= (a + a'c') + b'c'$

$= (a + a') (a + c') + b'c'$

$= a + (c' + b'c') \quad [\because\ a + a' = 1]$

$= a + c' \quad [\because\ x + xy = x]$

Example 5.26 In any Boolean algebra, show that

(i) $(x + y) (x' + z) = xz + x'y + yz = xz + x'y$

(ii) $(xy'z' + xy'z + xyz + xyz') (x + y) = x$

(i) $(x + y) (x' + z)$

$= xx' + xz + x'y + yz$

$= xz + x'y + yz \quad (\because\ xx' = 0)$

Now $xz + x'y + yz$

$= xz + x'y + yz(x + x')$

$= xz + x'y + xyz + x'yz$

$= (xz + xzy) + (x'y + x'yz)$

$= xz + x'y$

(ii) L.S. $= [xy'(z + z') + xy(z + z')] \cdot (x + y)$

$= (xy' + xy) (x + y) \quad [\because\ a + a' = 1]$

$= x(y + y') (x + y)$

$= x(x + y) \quad [\because\ y + y' = 1]$

$= x + xy$

$= x$

$=$ R.S.

Example 5.27 Find the disjunctive normal forms of the following Boolean expressions by (i) truth table method and (ii) algebraic method:

(a) $f(x, y, z) = xy + yz'$

(b) $f(x, y, z) = y' + [z' + x + (yz)'] \quad (z + x'y)$

(c) $f(x, y, z, w) = xy + yzw'$

(a) (i) Truth Table Method

x	y	z	xy	yz'	f
0	0	0	0	0	0
0	0	1	0	0	0
0	1	0	0	1	1
0	1	1	0	0	0
1	0	0	0	0	0
1	0	1	0	0	0
1	1	0	1	1	1
1	1	1	1	0	1

The minterms corresponding to the 3 rows for which 1 occurs in the f column are $x'yz'$, xyz' and xyz.

$\therefore$ DNF of $f(x, y, z) = x'yz' + xyz' + xyz$.

(ii) *Algebraic method*

$$\begin{aligned} f &= xy + yz' = xy(z + z') + (x + x')yz' \\ &= xyz + xyz' + xyz' + x'yz' \\ &= xyz + xyz' + x'yz' \quad (\because\ a + a = a) \end{aligned}$$

(b) (i) Truth Table Method

x	y	z	yz	$(yz)'$	$g = z' + x + (yz)'$	$x'y$	$h = z + x'y$	gh	$f = y' + gh$
0	0	0	0	1	1	0	0	0	1
0	0	1	0	1	1	0	1	1	1
0	1	0	0	1	1	1	1	1	1
0	1	1	1	0	0	1	1	0	0
1	0	0	0	1	1	0	0	0	1
1	0	1	0	1	1	0	1	1	1
1	1	0	0	1	1	0	0	0	0
1	1	1	1	0	1	0	1	1	1

The minterms correspond to all the rows except the 4th and 7th rows.

$\therefore$ DNF of $f(x, y, z) = x'y'z' + x'y'z + x'yz' + xy'z' + xy'z + xyz$.

(ii) *Algebraic method*

$$\begin{aligned} f(x, y, z) &= y' + [z' + x + y' + z'](z + x'y), \text{ by De Morgan's law} \\ &= y' + (x + y' + z')(z + x'y) \quad (\because\ z' + z' = z') \\ &= y' + xz + y'z + x'y'z \quad (\because\ xx' = yy' = zz' = 0) \\ &= y'(x + x') + xz(y + y') + y'z(x + x') + x'yz' \\ &= xy'(z + z') + x'y'(z + z') + xyz + xy'z + x'y'z + x'yz' \\ &= xy'z + xy'z' + x'y'z + x'y'z' + xyz + x'yz' \end{aligned}$$

(avoiding repetition of terms).

(c) (i) Truth Table Method

x	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
y	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
z	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
w	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
xy	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
yzw'	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0
f	0	0	0	0	0	0	1	0	0	0	0	0	1	1	1	1

DNF of $f = x'yzw' + xyz'w' + xyz'w + xyzw' + xyzw$

(ii) *Algebraic method*

$$\begin{aligned} f(x, y, z) &= xy(z + z') + (x + x')\, yzw' \\ &= xyz(w + w') + xyz'\,(w + w') + xyzw' + x'yzw' \\ &= xyzw + xyzw' + xyz'w + xyz'w' + xyzw' + x'yzw' \\ &= xyzw + xyzw' + xyz'w + xyz'w + x'yzw' \end{aligned}$$

(repetition of $xyzw'$ avoided).

Example 5.28 Find the conjuctive normal forms of the following Boolean expressions using (i) truth table method and (ii) algebraic method:

(a) $f(x, y, z) = (x + z)y$;

(b) $f(x, y, z) = x$;

(c) $f(x, y, z) = (yz + xz')\,(xy' + z)'$.

(a) (i) Truth Table Method

x	y	z	$x + z$	$f = (x + z)y$
0	0	0	0	0
0	0	1	1	0
0	1	0	0	0
0	1	1	1	1
1	0	0	1	0
1	0	1	1	0
1	1	0	1	1
1	1	1	1	1

The maxterms corresponding to the rows for which 0 occurs in the f column are

$(x + y + z)$, $(x + y + z')$, $(x + y' + z)$, $(x' + y + z)$ and $(x' + y + z')$

$\therefore$ The required CNF of $f(x, y, z)$ is

$$(x + y + z)\,(x + y + z')\,(x + y' + z)\,(x' + y + z)\,(x' + y + z').$$

(ii) *Algebraic method*

$$\begin{aligned} f &= (x + z)y = (x + z + yy')y \\ &= (x + y + z)\,(x + y' + z)\,(y + xx') \\ &= (x + y + z)\,(x + y' + z)\,(x + y)\,(x' + y) \\ &= (x + y + z)\,(x + y' + z)\,(x + y + zz')\,(x' + y + zz') \end{aligned}$$

$= (x + y + z)\ (x + y' + z)\ (x + y + z)\ (x + y + z')$
$(x' + y + z)\ (x' + y + z')$
$= (x + y + z)\ (x + y' + z)\ (x + y + z')\ (x' + y + z)$
$(x' + y + z')\quad (\because\ aa = a).$

(b) (i) *Truth Table Method*

Since $f(x, y, z) = x$, 0's occur in the first rows of the f column. The maxterms corresponding to three rows are

$(x + y + z)$, $(x + y + z')$, $(x + y' + z)$ and $(x + y' + z')$

$\therefore$ DNF of $f = (x + y + z)\ (x + y' + z)\ (x + y + z')\ (x + y' + z')$

(ii) *Algebraic method*

$f(x, y, z) = x = x + yy' = (x + y)\ (x + y')$
$= (x + y + zz')\ (x + y' + zz')$
$= (x + y + z)\ (x + y + z')\ (x + y' + z)\ (x + y' + z')$

(c) (i) Truth Table Method

x	y	z	yz	xz'	$g = yz + xz'$	xy'	$h = xy' + z$	h'	$f = gh'$
0	0	0	0	0	0	0	0	1	0
0	0	1	0	0	0	0	1	0	0
0	1	0	0	0	0	0	0	1	0
0	1	1	1	0	1	0	1	0	0
1	0	0	0	1	1	1	1	0	0
1	0	1	0	0	0	1	1	0	0
1	1	0	0	1	1	0	0	1	1
1	1	1	1	0	1	0	1	0	0

By Boolean multiplication of the maxterms corresponding to the 0's in f column, we get

DNF of $f = (x + y + z)\ (x + y + z')\ (x + y' + z)\ (x + y' + z')$
$(x' + y + z)\ (x' + y + z')\ (x' + y' + z')$

(ii) *Algebraic method*

$f(x, y, z) = (yz + xz')\ (xy' + z)'$
$= (yz + xz')\ (x' + y)z'$, by De Morgan's laws.
$= (yz + xz')\ (x'z' + yz')$
$= (yz + x)\ (yz + z')\ (x'z' + y)\ (x'z' + z')$
$= (x + y)\ (x + z)\ (y + z')\ (x' + y)\ (y + z')\ (x' + z')z'$
$[\because\ z + z' = 1 \text{ and } z' + z' = z']$
$= (x + y + zz')\ (x + z + yy')\ (y + z' + xx')\ (x' + y + zz')$
$(y + z' + xx')\ (x' + z' + yy')\ (z' + xx')$
$= (x + y + z)\ (x + y + z')\ (x + y + z)\ (x + y' + z)\ (x + \text{y} + z')$
$(x' + y + z')\ (x' + y + z)\ (x' + y + z')\ (x + y + z')\ (x' + y + z')$
$(x' + y + z')\ (x' + y' + z')\ (z' + x + yy')\ (z' + x' + yy')$
$= (x + y + z)\ (x + y + z')\ (x + y' + z)\ (x' + y + z)\ (x' + y + z')$
$(x' + y' + z')\ (x + y' + z')\quad$ [avoiding repetition of factors]