

# Maxwells Electromagnetic equations:(diffrantial form)

$$\vec{\nabla} \cdot \vec{D} = \rho$$

or

$$\text{Div } \vec{D} = \rho$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

or

$$\text{Div } \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = - \frac{\partial \vec{B}}{\partial t}$$

or

$$\text{Curl } \vec{E} = - \frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

or

$$\text{Curl } \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \text{ where}$$

$\vec{D}$  = Electric displacement vector

$\vec{E}$  = Electric field intensity

$\vec{B}$  = Magnetic field intensity

$\vec{J}$  = Current density (conventional)

$\rho$  = Charge density

# Maxwell's Electromagnetic equations:(Integral form)

$$\oint_S \vec{D} \cdot d\vec{s} = \int_V \rho dv$$

$$\oint_S \vec{B} \cdot d\vec{s} = 0$$

$$\oint_S \vec{H} \cdot d\vec{l} = \int_S (\vec{J} + \frac{\partial \vec{D}}{\partial t}) \cdot d\vec{S}$$

$$\oint \vec{E} \cdot d\vec{l} = -\frac{\partial}{\partial t} \int \vec{B} \cdot d\vec{s} \quad \text{symbols have their usual meaning}$$

# Derivation of Maxwells Equation

- **Maxwells First Equation** (  $\text{div } \vec{D} = \rho$  or  $\vec{\nabla} \cdot \vec{D} = \rho$  ):

When a dielectric is placed in a uniform electric field , its molecule get polarized. Thus ,a dielectric in an electric field contains two type of charges- free charge and bound charge . if  $\rho$  and  $\rho_p$  be the free and bound charge densities respectively, at appoint in a small volume element  $dv$ , then for such a medium, Gausss law may be expressed as

$$\int_s \vec{E} \cdot d\vec{s} = \frac{1}{\epsilon_0} \int_v (\rho + \rho_p) dv \dots \dots \dots (1)$$

where  $\epsilon_0$  is the permittivity of free space.

now the bound charge density

$\rho_p = - \text{div } \vec{P}$ , where  $\vec{P}$  is electric polarization. therefore

$$\int_s \vec{E} \cdot d\vec{s} = \frac{1}{\epsilon_0} \int_v (\rho - \text{div } \vec{P}) dv$$

# Continued.....

- Using Gauss divergence theorem on left hand side of the above equation, we get

$$\int_s \vec{E} \cdot d\vec{s} = \int_v \text{div } \vec{E} \, dV = \frac{1}{\epsilon_0} \int_v \rho \, dV - \frac{1}{\epsilon_0} \int_v \text{div } \vec{P} \, dV$$

$$\text{or } \int_v \epsilon_0 \text{div } \vec{E} \, dv + \int_v \text{div } \vec{P} \, dv = \int_v \rho \, dv$$

$$\int_v \text{div } \epsilon_0 \vec{E} \, dv + \int_v \text{div } \vec{P} \, dv = \int_v \rho \, dv$$

$$\int_v \text{div} ( \epsilon_0 \vec{E} + \vec{P} ) \, dV = \int_v \rho \, dV$$

but  $(\epsilon_0 \vec{E} + \vec{P}) = \vec{D}$  is the electric displacement vector.

$$\int_v \text{div } \vec{D} \, dv = \int_v \rho \, dV \quad \text{or} \quad \int_v (\text{div } \vec{D} - \rho) \, dv = 0$$

therefore, for an arbitrary function, we have

$$\text{div } \vec{D} - \rho = 0 \quad \text{or} \quad \text{div } \vec{D} = \rho$$

$\vec{\nabla} \cdot \vec{D} = \rho$  this is the required equation.

# Maxwell's Second Equation

- The net magnetic flux through any closed surface is always zero.

$$\phi_B = \oint_S \vec{B} \cdot d\vec{s} = 0 \dots\dots\dots(1)$$

Using Gauss theorem

$$\oint_S \vec{B} \cdot d\vec{s} = \int_V \text{div } \vec{B} \, dV$$
$$\int_V \text{div } \vec{B} \, dV = 0$$

$\vec{\nabla} \cdot \vec{B} = 0$  The above expression shows that monopole or an isolated pole can not exist to serve as a source. this law is also known as Gauss law in magnetostatics. where V is the volume enclosed by surface S.

Hence ,for an arbitrary surface  $\text{div } B = 0$  or  $\vec{\nabla} \cdot \vec{B} = 0$

## Derivation of Maxwell's third Equation (faraday law of electromagnetic induction)

- According to faraday law of electromagnetic induction, induced emf around a closed circuit is equal to the negative time rate of change of magnetic flux i.e.

$$e = - \frac{\partial \phi_B}{\partial t} \dots\dots\dots(1)$$

if B is the magnetic field induction, then the magnetic flux linked with the area ds

$$\phi_B = \int_S \vec{B} \cdot d\vec{s} \dots\dots\dots(2)$$

On combining the equation (1) and (2) we get

$$e = - \int_S \vec{B} \cdot d\vec{s} \dots\dots\dots(3)$$

according to definition the induced emf is related to the corresponding field as

$$e = - \int_l \vec{E} \cdot d\vec{l} \dots\dots\dots(4)$$

## Continued.....

Therefore from (3) and (4) will give

$$\int_c \vec{E} \cdot d\vec{l} = - \int_s \frac{\partial}{\partial t} (\vec{B} \cdot d\vec{s})$$

$$\text{or } \int_c \vec{E} \cdot d\vec{l} = - \int_s \frac{\partial \vec{B}}{\partial t} d\vec{s}$$

now using the stoks theorem on the left hand side, we get

$$\int_c \vec{E} \cdot d\vec{l} = - \int_s \text{Curl } \vec{E} d\vec{s}$$

Thus , we have

$$\int_s \text{Curl } \vec{E} d\vec{s} = - \int_s \frac{\partial \vec{B}}{\partial t} d\vec{s}$$

$$\int_s \text{Curl } \vec{E} d\vec{s} + \frac{\partial \vec{B}}{\partial t} d\vec{s} = 0$$

For an arbitrary bfunction,  $\text{Curl } \vec{E} d\vec{s} + \frac{\partial \vec{B}}{\partial t} = 0$

$$\text{Curl } \vec{E} = - \frac{\partial \vec{B}}{\partial t} \quad \text{or} \quad \vec{\nabla} \times \vec{E} = - \frac{\partial \vec{B}}{\partial t}$$

# Maxwells fourth equation (modified amperes law)

- According to ampere law

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I$$

the formula  $I = \oint \vec{J} \cdot d\vec{s}$

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \int \vec{J} \cdot d\vec{s}$$

Using stokes theorem on left hand side of the above expression, we get

$$\oint_s \text{Curl } \vec{B} \cdot d\vec{s} = \mu_0 \int_s \vec{J} \cdot d\vec{s}$$

$$\frac{1}{\mu_0} \int_s \text{Curl } \vec{B} \cdot d\vec{s} = \int_s \vec{J} \cdot d\vec{s}$$

$$\int_s \text{Curl } \frac{\vec{B}}{\mu_0} \cdot d\vec{s} = \int_s \vec{J} \cdot d\vec{s} \quad \text{now from dielectric properties } \frac{\vec{B}}{\mu_0}$$

$$\int_s \text{Curl } \vec{H} \cdot d\vec{s} = \int_s \vec{J} \cdot d\vec{s}$$

or  $\int_s (\text{Curl } \vec{H} - \vec{J}) \cdot d\vec{s} = 0$  For an arbitrary surface,

$$\text{Curl } \vec{H} = \vec{J}$$



## Continued.....

- Taking div on both side we get ,

$$\text{div Curl } \mathbf{H} = \text{div } \mathbf{J}$$

but  $\text{div curl } \mathbf{H} = 0$  (from vector calculus )

Since  $\text{div } \mathbf{J} = 0$  from continuity equation, we have

$$\text{div } \mathbf{J} + \frac{\partial \rho}{\partial t} = 0$$

$$\text{Hence } \frac{\partial \rho}{\partial t} = 0$$

$\rho = \text{constant}$  (static)

This shows amperes law applicable for static charges, therefore Maxwell's suggested that ampere law must be modified by adding a quantity having dimension as that of current, produced due to polarization of charges. this physical quantity is called displacement current ( $J_d$ ).thus modified ampere law becomes

## Continued.....

 *Curl*  $\mathbf{H} = \mathbf{J} + \mathbf{J}_d$  taking divergence on both side, we get

$$\text{div curl } \mathbf{H} = \text{div} (\mathbf{J} + \mathbf{J}_d)$$

$$0 = \text{div } \mathbf{J} + \text{div } \mathbf{J}_d \quad (\text{div Curl } \mathbf{H} = 0)$$

$$0 = \text{div } \mathbf{J} + \text{div } \mathbf{J}_d$$

$$\text{div } \mathbf{J} = - \text{div } \mathbf{J}_d$$

$$\text{But } \text{div } \mathbf{J} = - \frac{\partial \rho}{\partial t}$$

$$\text{but } \rho = \text{div } \mathbf{D} \quad \text{since } \text{div } \dot{\mathbf{j}}_d = \frac{\partial}{\partial t} (\text{div } \mathbf{D})$$

$$\text{div } \mathbf{J}_d = \text{div} \left( \frac{\partial \mathbf{D}}{\partial t} \right) \quad \text{or} \quad \mathbf{J}_d = \frac{\partial \mathbf{D}}{\partial t}$$

*Therefore* modifie Ampere law becomes

$$\text{Curl } \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

Maxwell's equations have two major variants.

- The microscopic equations have universal applicability but are unwieldy for common calculations. They relate the electric and magnetic fields to total charge and total current, including the complicated charges and currents in materials at the atomic scale.
- The macroscopic equations describe the large-scale behavior of matter without having to consider atomic scale charges and quantum phenomena like spins. However, their use requires experimentally determined parameters for a phenomenological description of the electromagnetic response of materials.

In a region with no charges ( $\rho = 0$ ) and no currents ( $\mathbf{J} = 0$ ), such as in a vacuum, Maxwell's equations reduce to:

$$\begin{aligned}\nabla \cdot \mathbf{E} &= 0 & \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \\ \nabla \cdot \mathbf{B} &= 0 & \nabla \times \mathbf{B} &= \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}.\end{aligned}$$

- During Maxwell's lifetime, it was found that the known values for  $\epsilon_0$  and  $\mu_0$  give  $c \sim 2.998 \times 10^8 \text{ m/s}$  then already known to be the speed of light in free space. This led him to propose that light and radio waves were propagating electromagnetic waves.
- The above equations are the microscopic version of Maxwell's equations, expressing the electric and the magnetic fields in terms of the (possibly atomic-level) charges and currents present. The microscopic version is sometimes called "Maxwell's equations in a vacuum". The microscopic version was introduced by Lorentz, who tried to use it to derive the macroscopic properties of bulk matter from its microscopic constituents.
- "Maxwell's macroscopic equations", also known as **Maxwell's equations in matter**, are more similar to those that Maxwell introduced himself.

Name	Integral equations (SI convention)	Differential equations (SI convention)	Differential equations (Gaussian convention)
Gauss's law	$\oiint_{\partial\Omega} \mathbf{D} \cdot d\mathbf{S} = \iiint_{\Omega} \rho_f dV$	$\nabla \cdot \mathbf{D} = \rho_f$	$\nabla \cdot \mathbf{D} = 4\pi\rho_f$
Gauss's law for magnetism	$\oiint_{\partial\Omega} \mathbf{B} \cdot d\mathbf{S} = 0$	$\nabla \cdot \mathbf{B} = 0$	$\nabla \cdot \mathbf{B} = 0$
Maxwell–Faraday equation (Faraday's law of induction)	$\oint_{\partial\Sigma} \mathbf{E} \cdot d\boldsymbol{\ell} = -\frac{d}{dt} \iint_{\Sigma} \mathbf{B} \cdot d\mathbf{S}$	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}$
Ampère's circuital law (with Maxwell's addition)	$\oint_{\partial\Sigma} \mathbf{H} \cdot d\boldsymbol{\ell} = \iint_{\Sigma} \mathbf{J}_f \cdot d\mathbf{S} + \frac{d}{dt} \iint_{\Sigma} \mathbf{D} \cdot d\mathbf{S}$	$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$	$\nabla \times \mathbf{H} = \frac{1}{c} \left( 4\pi\mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \right)$

## Internal or local field

In dielectric solids, the atoms or molecules experience not only the external applied electric field but also the electric field produced by the dipoles. The resultant electric field acting on the atoms or molecules of dielectric substance is called the local field or an internal field. Consider a dielectric material and is subjected to external field of intensity  $E_1$ . The charges are induced on the dielectric plate and the induced electric field intensity is taken as  $E_2$ . Let  $E_3$  be the field at the center of the material.  $E_4$  be the induced field due to the charges on the spherical cavity. The total internal field of the material is

$$E_i = E + \frac{P}{3 \epsilon_0}$$

## **Types of Polarization**

There are three main polarisation mechanisms that can occur within a dielectric material:

- Electronic polarisation,
- Ionic polarisation (sometimes referred to as atomic polarisation) and
- Orientational polarisation.

- **Electronic polarization:**

It occurs in neutral atoms when an electric field displaces the nucleus with respect to the negative charge. Thus electronic polarization is an induced polarization effect.

- **Atomic/ionic polarization:**

It is observed when different atoms that comprise a molecule share their electrons asymmetrically, and cause the electron cloud to be shifted towards the stronger binding atom, the atoms acquire charges of opposite polarity and an external field acting on these net charges will tend to change the equilibrium positions of the atoms themselves, leading to the atomic polarization.



- **Dipolar/orientational polarization:**

When an ionic bond is formed between two molecules by the transfer of some valence electrons, a permanent dipole moment will originate in them. This permanent dipole moment is equal to the product of the charges of the transferred valence electrons and the inter-atomic distance between them. In the presence of an electric field  $E$ , the molecules carrying a permanent dipole moment will orient to align along the direction of the electric field  $E$ . This process is called the dipolar or orientational polarization. This occurs only in dipolar materials possessing permanent dipole moments.

## Polarization and dielectric constant

The ability of a dielectric material to store electric energy under the influence of an electric field, results from the field-induced separation and alignment of electric charges. Polarization occurs when the electric field causes a separation of the positive and negative charges in the material. The larger the dipole moment arms of this charge separation in the direction of a field and the larger the number of these dipoles, the higher the material's dielectric permittivity.

In the presence of electronic, ionic and dipolar polarization mechanisms, the average induced dipole moment per molecule  $P_{av}$  will be the sum of all the contributions in terms of the local field (effective field) acting on each individual molecule.

$$P_{av} = \alpha_e E_{loc} + \alpha_i E_{loc} + \alpha_d E_{loc}$$

Here,  $\alpha_e$ ,  $\alpha_i$ ,  $\alpha_d$  are the electronic, ionic and dipolar polarizabilities.  $E_{\text{loc}}$  is the local field or the effective field at the site of an individual molecule that causes the individual polarization. Each effect adds linearly to the net dipole moment of the molecule.

For simple dielectrics ( eg. gases) one can take the local field to be the same as the macroscopic field. This means that  $E_{\text{loc}} = E$  the applied field and therefore the polarization is,

$$P = \chi_e \epsilon_e E = (\epsilon_r - 1) \epsilon_0 E$$

$P = N \cdot P_{\text{av}}$  where  $N$  is the number of atoms or molecule per unit volume

$$\epsilon_r = 1 + N\alpha / \epsilon_0 \quad (\text{dielectric permittivity or dielectric constant})$$

$\alpha$  is the polarizability of the molecule.

\* the relationship between the electric susceptibility and the dielectric permittivity is given as:  $\chi' = \epsilon_r - 1$

# Clausius Mossotti's equation

It gives the relation between the dielectric constant and the ionic polarizability of atoms in dielectric material. If there are N number of atoms, the dipole moment per unit volume which is called Polarization is given by,

$$P = N \alpha_i E_i$$

we know internal field

$$E_i = \left( E + \frac{P}{3 \epsilon_0} \right)$$

From above equations

$$P = N \alpha_i \left( E + \frac{P}{3 \epsilon_0} \right)$$

$$P = \frac{N \alpha_i}{3 \epsilon_0} (3E \epsilon_0 + P)$$

$$\frac{N \alpha_i}{3 \epsilon_0} = \frac{P}{(3E \epsilon_0 + P)}$$

we know polarization from the relation between polarization and dielectric constant

$$P = E \epsilon_0 (\epsilon_r - 1)$$

from the above two equations we get,

$$\frac{N\alpha_e}{3 \epsilon_0} = \frac{E \epsilon_0 (\epsilon_r - 1)}{3E \epsilon_0 + E \epsilon_0 (\epsilon_r - 1)}$$

$$\frac{N\alpha}{3 \epsilon_0} = \frac{E \epsilon_0 (\epsilon_r - 1)}{E \epsilon_0 (\epsilon_r + 2)}$$

$$\frac{N\alpha}{3 \epsilon_0} = \frac{(\epsilon_r - 1)}{(\epsilon_r + 2)}$$

This is Clausius Mossotti's equation.