GARVIT SHAH
F 24

TUTORIAL 4: Partial Differentiation, Euler's Theorem, & Modified Euler's Theorem

1. Euler's Theorem states that –

$$\boxed{x\frac{du}{dx} + y\frac{du}{dy} = nu} \quad -①$$

[u- homogeneous f^n in x & y of degree n]

Proof:

Let $u = f(x,y)$ · a homogeneous f^n $[u = x^n \phi(y/x)]$

– diff u wrt x

$$\frac{du}{dx} = n x^{n-1}\phi\left(\frac{y}{x}\right) + x^n \phi'\left(\frac{y}{x}\right)\left[\frac{-y}{x^2}\right] \quad -②$$

– diff u wrt y

$$\frac{du}{dy} = x^n \phi'\left(\frac{y}{x}\right)\left[\frac{1}{x}\right] \quad -③$$

using ② & ③

$\therefore$ LHS $\rightarrow$ $x\left[n x^{n-1}\phi\left(\frac{y}{x}\right) - x^{n-2}y\cdot\phi'\left(\frac{y}{x}\right)\right] + y\left[x^{n-1}\phi'\left(\frac{y}{x}\right)\right]$

$\Rightarrow n x^n \phi\left(\frac{y}{x}\right) \Rightarrow \boxed{nu}$ = RHS $\therefore$ PROVED

- To Prove: $\boxed{x^2\frac{d^2u}{dx^2} + 2xy\frac{d^2u}{dxdy} + y^2\frac{d^2u}{dy^2} = n(n-1)u}$

$\rightarrow$ diff ① wrt x

$$\frac{du}{dx} + x\frac{d^2u}{dx^2} + y\frac{d^2u}{dxdy} = n\frac{du}{dx} \quad -④$$

$\rightarrow$ diff ① wrt y

$$x\frac{d^2u}{dxdy} + \frac{du}{dy} + y\frac{d^2u}{dy^2} = n\frac{du}{dy} \quad -⑤$$

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$\therefore$ (4) $\times x$ + (5) $\times y$

$$\Rightarrow \boxed{x^2 \frac{d^2u}{dx^2} + 2xy \frac{d^2u}{dxdy} + y^2 \frac{d^2u}{dy^2} = n(n-1)\,u}$$

$\therefore$ Proved

2. Modified Euler's Theorem states :

$$\boxed{x\frac{du}{dx} + y\frac{du}{dy} = n\frac{f(u)}{f'(u)}} \quad - (1)$$

where z is a homogeneous f^n in x & y of deg n

$z = f(u)$

Proof : using Euler's theorem –

$$x\frac{dz}{dx} + y\frac{dz}{dy} = nz$$

$\frac{\delta z}{\delta x} = f'(u)\frac{\delta u}{\delta x}$ | $\frac{\delta z}{\delta y} = f'(u)\frac{du}{dy}$ $\therefore x\frac{\delta z}{\delta x} + y\frac{\delta z}{\delta y} = n\frac{f(u)}{f'(u)}$

To Prove : $\boxed{x^2\frac{d^2u}{\delta x^2} + 2xy\frac{\delta^2 u}{\delta x \delta y} + y^2\frac{\delta^2 u}{\delta y^2} = g(u)[g'(u)-1]}$

where $g(u) = n\frac{f(u)}{f'(u)}$

→ diff (1) wrt x

$$\frac{du}{\delta x} + x\frac{\delta^2 u}{\delta x^2} + y\frac{\delta^2 u}{\delta x \delta y} = g'(u)\frac{\delta u}{\delta x} \quad - (2)$$

→ diff (1) wrt y

$$x\frac{d^2u}{\delta y \delta x} + \frac{\delta u}{\delta y} + y\frac{\delta^2 u}{\delta y^2} = g'(u)\frac{\delta u}{\delta y} \quad - (3)$$

$\therefore$ (2) $\times x$ + (3) $\times y$

$$\Longrightarrow x^2\frac{\delta^2 u}{\delta x^2} + 2xy\frac{\delta^2 u}{\delta x \delta y} + y^2\frac{\delta^2 u}{\delta y^2} = g(u)[g'(u)-1]$$

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3. $u = f(r)$ $\quad r^2 = x^2 + y^2$ $\quad$ To show that : $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r)$

→ $r^2 = x^2 + y^2$ $\quad$ diff wrt x & y

$$\boxed{2r \frac{\partial r}{\partial x} = 2x} \qquad \boxed{2r \frac{\partial r}{\partial y} = 2y}$$

↓ diff again wrt x $\qquad$ ↓ diff again wrt y

$$\boxed{2\left(\frac{\partial r}{\partial x}\right)^2 + 2r \frac{\partial^2 r}{\partial x^2} = 2} \qquad \boxed{2\left(\frac{\partial r}{\partial y}\right)^2 + 2r \frac{\partial^2 r}{\partial y^2} = 2}$$

$$\boxed{\begin{aligned} \therefore \frac{\partial^2 r}{\partial x^2} &= \left[1 - \frac{x^2}{r^2}\right]\frac{1}{r} \\ \frac{\partial^2 r}{\partial y^2} &= \left[1 - \frac{y^2}{r^2}\right]\frac{1}{r} \end{aligned}}$$

$u = f(r)$

- diff wrt x $\qquad$ diff wrt y

$$\frac{\partial u}{\partial x} = f'(r) \frac{\partial r}{\partial x} \qquad \frac{\partial u}{\partial y} = f'(r) \frac{\partial r}{\partial y}$$

- diff wrt x $\qquad$ diff wrt y

$$\frac{\partial^2 u}{\partial x^2} = f''(r)\left[\frac{\partial r}{\partial x}\right]^2 + f'(r)\frac{\partial^2 r}{\partial x^2} \quad \text{①} \qquad \frac{\partial^2 u}{\partial y^2} = f''(r)\left(\frac{\partial r}{\partial y}\right)^2 + f'(r)\frac{\partial^2 r}{\partial y^2} \quad \text{②}$$

add ① & ②

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r)\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right] + f'(r)\left[\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2}\right]$$

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r)\left[\frac{1}{r^2}(x^2 + y^2)\right] + f'(r)\left[\frac{1}{r}\right]}$$

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4. $z(x+y) = x^2+y^2$ To show that :-

$$\left[\frac{dz}{dx} - \frac{dz}{dy}\right]^2 = 4\left[1 - \frac{dz}{dx} - \frac{dz}{dy}\right]$$

→ $xz + yz = x^2+y^2$

diff wrt x → $z + x\frac{dz}{dx} + y\frac{dz}{dx} = 2x$

$\therefore \frac{dz}{dx} = \frac{2x-z}{x+y}$

Similarly $\frac{dz}{dy} = \frac{2y-z}{x+y}$

$\therefore$ RHS → $4\left[1 - \left[\frac{2x+2y-2z}{x+y}\right]\right]$

$\Rightarrow 4\left[\frac{x+y-2x-2y+2\left(\frac{x^2+y^2}{x+y}\right)}{x+y}\right]$

$\Rightarrow 4\left[\frac{2(x^2+y^2) - (x+y)^2}{(x+y)^2}\right] = 4\left[\frac{x-y}{x+y}\right]^2$

LHS → $\left[\frac{2x-z}{x+y} - \frac{2y-z}{x+y}\right]^2 = 4\left(\frac{x-y}{x+y}\right)^2$ $\therefore$ LHS = RHS

PROVED

5. If $z = x f\left(\frac{y}{x}\right) + g\left(\frac{y}{x}\right)$, prove that -

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = 0$$

→ ~~From Euler's~~ diff z wrt x

$$\frac{dz}{dx} = f(y/x) + x f'(y/x)\left[\frac{-y}{x^2}\right] + g'\left(\frac{y}{x}\right)\left[\frac{-y}{x^2}\right] \quad -①$$

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diff z wrt y

$$\frac{dz}{dy} = xf'(y/x)\left[\frac{1}{x}\right] + g'(y/x)\left[\frac{1}{x}\right] \quad -(2)$$

(1) × x + (2) × y

$$\longrightarrow x\frac{dz}{dx} + y\frac{dz}{dy} = xf\left(\frac{y}{x}\right) + f'\left(\frac{y}{x}\right)\left[-y + y\right] + g'\left(\frac{y}{x}\right)\left[-\frac{y}{x} + \frac{y}{x}\right]$$

$$\Rightarrow \boxed{x\frac{dz}{dx} + y\frac{dz}{dy} = xf\left(\frac{y}{x}\right)} \quad -(3)$$

diff (3) wrt x

$$\Rightarrow \frac{dz}{dx} + x\frac{d^2z}{dx^2} + y\frac{d^2z}{dx\,dy} = f(y/x) + xf'\left(\frac{y}{x}\right)\left[-\frac{y}{x^2}\right] \quad -(4)$$

diff (3) wrt y

$$\Rightarrow \frac{dz}{dy} + y\frac{d^2z}{dy^2} + x\frac{d^2z}{dy\,dx} = xf'\left(\frac{y}{x}\right)\left[\frac{1}{x}\right] \quad -(5)$$

(4) × x + (5) × y

$$\Rightarrow x^2\frac{d^2z}{dx^2} + 2xy\frac{d^2z}{dy\,dx} + y^2\frac{d^2z}{dy^2} + \cancel{\left(x\frac{dz}{dx} + y\frac{dz}{dy}\right)} = \cancel{xf(y/x)}$$

$$\Rightarrow \boxed{x^2\frac{d^2z}{dx^2} + 2xy\frac{d^2z}{dy\,dy} + y^2\frac{d^2z}{dy^2} = 0}$$

∴ Proved

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6. $f(x,y) = \frac{1}{\sqrt{y}} e^{-\frac{(x-a)^2}{4y}}$, then Prove that $f_{xy} = f_{yx}$

$\rightarrow \frac{\delta f}{\delta x} = \frac{1}{\sqrt{y}} e^{-\frac{(x-a)^2}{4y}} \left[\frac{-2(x-a)}{4y}\right] = \frac{-(x-a)e^{-\frac{(x-a)^2}{4y}}}{2y^{3/2}}$

$f_{xy} = \frac{+(x-a)}{2}\left(\frac{+3}{2}\right)\frac{1}{y^{5/2}} e^{-\frac{(x-a)^2}{4y}} - \frac{(x-a)}{2y^{3/2}} e^{-\frac{(x-a)^2}{4y}} \left(\frac{(x-a)^2}{4y^2}\right)$

$\Rightarrow \boxed{f_{xy} = \frac{e^{-\frac{(x-a)^2}{4y}}(x-a)}{4y^{3/2}}\left[\frac{3}{y} - \frac{(x-a)^2}{2y^2}\right]}$

$\rightarrow \frac{\delta f}{\delta y} = \frac{-1}{2y^{3/2}} e^{-\frac{(x-a)^2}{4y}} + \frac{e^{-\frac{(x-a)^2}{4y}}}{\sqrt{y}}\left(\frac{(x-a)^2}{4y^2}\right)$

$f_{yx} = \frac{+1}{\cancel{2}y^{3/2}} e^{-\frac{(x-a)^2}{4y}}\left[\frac{+\cancel{2}(x-a)}{4y}\right] + \frac{2(x-a)\, e^{-\frac{(x-a)^2}{4y}}}{4y^2\sqrt{y}}$

$+ \frac{(x-a)^2 e^{-\frac{(x-a)^2}{4y}}}{4y^2\sqrt{y}}\left[\frac{-2(x-a)}{4y}\right]$

$\Rightarrow \boxed{f_{yx} = \frac{e^{-\frac{(x-a)^2}{4y}}(x-a)}{4y^{3/2}}\left[\frac{3}{y} - \frac{(x-a)^2}{2y^2}\right]}$ $\therefore f_{xy} = f_{yx}$

Proved

7. If $x = e^{r\cos\theta}\cos(r\sin\theta)$, $y = e^{r\cos\theta}\sin(r\sin\theta)$ then prove that - $\frac{\delta x}{\delta r} = \frac{1}{r}\frac{\delta y}{\delta \theta}$, $\frac{\delta y}{\delta r} = \frac{-1}{r}\frac{\delta x}{\delta \theta}$

$\frac{\delta x}{\delta r} = \cos\theta\, e^{r\cos\theta}\cos(r\sin\theta) + e^{r\cos\theta}(-\sin\theta\sin(r\sin\theta))$

$\boxed{\frac{\delta x}{\delta r} = e^{r\cos\theta}[\cos(\theta + r\sin\theta)]}$

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- $\frac{dy}{d\theta} = -r\sin\theta \, e^{r\cos\theta} \cdot \sin(r\sin\theta) + e^{r\cos\theta} r\cos\theta \cos(r\sin\theta)$

$$\boxed{\frac{dy}{d\theta} = r\left[e^{r\cos\theta}\right]\left[\cos(\theta + r\sin\theta)\right]}$$

$\therefore \frac{dx}{dr} = \frac{1}{r}\frac{dy}{d\theta}$ **Proved**

- $\frac{dy}{dr} = \cos\theta \, e^{r\cos\theta} \sin(r\sin\theta) + e^{r\cos\theta} \sin\theta \cos(r\sin\theta)$

$$\boxed{\frac{dy}{dr} = e^{r\cos\theta}\left[\sin(\theta + r\sin\theta)\right]}$$

- $\frac{dx}{d\theta} = -r\sin\theta \, e^{r\cos\theta} \cos(r\sin\theta) - r\cos\theta \, e^{r\cos\theta} \sin(r\sin\theta)$

$$\boxed{\frac{dx}{d\theta} = -r\left[e^{r\cos\theta}\right]\left[\sin(\theta + r\sin\theta)\right]}$$

$\therefore \frac{dy}{dr} = -\frac{1}{r}\frac{dx}{d\theta}$ **Proved**

8. $r^2 = x^2 + y^2 + z^2$ $V = r^m$ then Prove that
$V_{xx} + V_{yy} + V_{zz} = m(m+1) r^{m-2}$

$\rightarrow \frac{\partial V}{\partial x} = m r^{m-1} \frac{\partial r}{\partial x}$ $\left[2r\frac{\partial r}{\partial i} = 2i \quad \text{where } i = x, y, z\right]$

$\therefore V_{xx} = m(m-1) r^{m-2} \left(\frac{\partial r}{\partial x}\right)^2 + m r^{m-1} \frac{\partial^2 r}{\partial x^2}$

$\left[\left(\frac{\partial r}{\partial i}\right)^2 + 2r\frac{\partial^2 r}{\partial i^2} = 2 \quad \text{where } i = x, y, z\right]$

$\therefore V_{xx} = m(m-1) r^{m-2} \left(\frac{x^2}{r^2}\right) + m r^{m-1} \frac{1}{r}$

$\therefore V_{xx} + V_{yy} + V_{zz} = m(m-1) \frac{r^{m-2}}{r^2} (x^2 + y^2 + z^2) + m(r^{m-1}) \frac{1}{r}(2)$

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9. $\log u = \dfrac{x^2y^2}{x+y}$ then show that $xu_x + yu_y = 3u\log u$.

→ Let $z = \log u$ ∴ z is a homogeneous f^n in x, y of degree 3

∴ From modified Euler's Theorem

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz \qquad n\text{ - degree of } z$$

$$\therefore \; xz_x + yz_y = 3z$$

$$\Rightarrow \frac{x}{u}\frac{\partial u}{\partial x} + \frac{y}{u}\frac{\partial u}{\partial y} = 3\log u$$

$$\Rightarrow \boxed{xu_x + yu_y = 3u\log u}$$

10. If $u = F(x-y, y-z, z-x)$, Show that $u_x + u_y + u_z = 0$

→ $u_x + u_y + u_z$

$$\Rightarrow \frac{du}{dx} + \frac{dy}{\partial y} + \frac{\partial u}{\partial z}$$

$$\Rightarrow \frac{du}{\partial(x-y)}\left[\frac{\partial(x-y)}{\partial x} + \frac{\partial(x-y)}{\partial y} + \frac{\partial(x-y)}{\partial z}\right]$$

$$\Rightarrow 0$$

$$\therefore \; u_x + u_y + u_z = 0$$

11. $x^x y^y z^z = c$, then show that $\dfrac{\partial^2 z}{\partial x \partial y} = -(x\log ex)^{-1}$ at $x = y = z$

take log on both sides

$$x\log x + y\log y + z\log z = \log c$$

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$\therefore \quad z \log z = \log c - x \log x - y \log y$

diff wrt x -

$$\frac{dz}{dx} \log z + \frac{z}{z}\frac{dz}{dx} = -\log x - 1$$

$$\frac{dz}{dx}(\log z + 1) = -(\log x + 1) \qquad \boxed{\therefore \frac{dz}{dx} = -\frac{\log ex}{\log ez}}$$

diff wrt y

$$\frac{d^2 z}{dx\,dy}(\log z + 1) + \frac{dz}{dx}\left(\frac{1}{z}\frac{dz}{dy}\right) = 0$$

$$\Rightarrow \frac{d^2 z}{dx\,dy} = -\frac{1}{z}\left(\frac{dz}{dx}\right)\left(\frac{dz}{dy}\right)\frac{1}{(\log z + 1)}$$

if $x = y = z$

$$\therefore \frac{d^2 z}{dx\,dy} = -\frac{1}{x}\left(-\frac{\log ex}{\log ex}\right)\left(-\frac{\log ey}{\log ey}\right)\frac{1}{\log ex}$$

$$\Rightarrow \boxed{\frac{d^2 z}{dx\,dy} = -(x \log ex)^{-1}}$$

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12. $u = \log(x^3 + y^3 + z^3 - 3xyz)$ then prove that

$$\left[\frac{d}{dx} + \frac{d}{dy} + \frac{d}{dz}\right]^2 u = -\frac{9}{(x+y+z)^2}$$

$\rightarrow$ $\dfrac{du}{dx} = \dfrac{1}{R}(3x^2 - 3yz)$ $\qquad \left[R = x^3 + y^3 + z^3 - 3yxz\right]$

$R = (x+y+z)(x^2+y^2+z^2-xy-yz-zx)$

$$\therefore \left(\frac{d}{dx} + \frac{d}{dy} + \frac{d}{dz}\right)u = \frac{3(x^2+y^2+z^2-yz-xy-zx)}{R}$$

$$\left[\frac{d}{dx} + \frac{d}{dy} + \frac{d}{dz}\right]^2 u = \left(\frac{3}{(x+y+z)}\right)^2 = \frac{9}{(x+y+z)^2}$$

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13. $\frac{dy}{dx}$ when $y^{x^y} = \sin x$

(taking log)

$\rightarrow x^y \log y = \log\sin x$ (taking log again)

$\rightarrow y\log x + \log\log y = \log\log\sin x$

diff wrt x

$\Rightarrow \cancel{\frac{dy}{dx} = \frac{dy}{}}\ \frac{1}{x}\cdot y + \log x\cdot\frac{dy}{dx} + \frac{1}{y\log y}\cdot\frac{dy}{dx} = \frac{1}{\log\sin x}\cdot\frac{\cos x}{\sin x}$

$\Rightarrow \frac{dy}{dx}\left[\log x + \cancel{\log}\frac{1}{\cancel{y}\log y}\right] = \frac{\cot x}{\log\sin x} - \frac{y}{x}$

$\Rightarrow \frac{dy}{dx} = \cancel{\log y}\left[\frac{\cot x - y\cdot x^{-1}\cdot x^y\log y}{\log y\cdot x^y}\right]\frac{1}{(\log x\cdot\log y + \frac{1}{y})}$

$$\boxed{\frac{dy}{dx} = -\left[\frac{yx^{y-1}\log y - \cot x}{\log x\cdot\log y + \frac{1}{y}}\right]}$$

14. $z = f(x+ct) + g(x-ct)$

Prove that $\frac{\partial^2 z}{\partial t^2} = c^2\frac{\partial^2 z}{\partial x^2}$

$- \frac{dz}{dx} = \frac{df(x+ct)}{d(x+ct)}\cdot\frac{d(x+ct)}{dx} + \frac{dg(x-ct)}{d(x-ct)}\cdot\frac{d(x-ct)}{dx}$

$\frac{dz}{dx} = f'(x+ct) + g'(x-ct)$

$$\boxed{\therefore \frac{\partial^2 z}{\partial x^2} = f''(x+ct) + g''(x-ct)} \quad -①$$

$- \frac{\partial z}{\partial t} = c f'(x+ct) - c\, g'(x-ct)$

$$\boxed{\therefore \frac{\partial^2 z}{\partial t^2} = c^2\left[f''(x+ct) + g''(x-ct)\right]} \quad -②$$

from ① & ②

$\boxed{c^2\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial t^2}}$ Proved

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15. $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$; $x \neq y$ Prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \sin 4u - \sin 2u$$

$$= 2\sin u \cos 3u$$

→ Let $z = \tan u = \frac{x^3+y^3}{x-y}$ ∵ z is a homogeneous

$z = f(u)$ fn of deg 2 in x & y

∴ Using ~~Collo~~ corollary 3 –

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)\left[g'(u) - 1\right]$$

where $g(u) = 2\frac{f(u)}{f'(u)} = \frac{2\tan u}{\sec^2 u} = 2\sin u \cos u$

∴ $g(u) = \sin 2u$ & $g'(u) = \cancel{2\sin 2u}\ 2\cos 2u$

$$\therefore x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \sin 2u\left[2\cos 2u - 1\right]$$

$$= \cancel{\cos 4}\ \sin 4u - \sin 2u$$

PROVED.