TUTORIAL 9 : Double Integration

1. $$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{(x^2+y^2)^3} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

To show – $$\int_0^2\int_0^1 f(x,y)\,dy\,dx \neq \int_0^1\int_0^2 f(x,y)\,dx\,dy$$

→ **Proof :**

LHS – $$\int_0^2\int_0^1 f(x,y)\,dy\,dx = \int_0^2\int_0^1 \frac{xy(x^2-y^2)}{(x^2+y^2)^3}\,dy\,dx$$

$$= \int_0^2 x\,dx \left\{ \int_0^1 \frac{yx^2\,dy}{(x^2+y^2)^3} - \int_0^1 \frac{y^3\,dy}{(x^2+y^2)^3} \right\}$$

Put $x^2+y^2 = a$

$dy = \frac{1}{2y}\,da$

$$= \int_0^2 x\,dx \left[\frac{x^2}{2}\int_{x^2}^{x^2+1} \frac{da}{a^3} - \int_{x^2}^{x^2+1} \frac{(a - x^2)\,da}{2a^3} \right]$$

$$= \int_0^2 x\,dx \left[\frac{x^2}{2}\left[\frac{-a^{-2}}{2}\right]_{x^2}^{x^2+1} - \frac{1}{2}\left[-\frac{1}{a}\right]_{x^2}^{x^2+1} + \frac{x^2}{2}\left[\frac{-a^{-2}}{2}\right]_{x^2}^{x^2+1} \right] dx$$

$$= \int_0^2 x\left[\frac{x^2}{2}\left(\frac{1}{2x^4} - \frac{1}{2(x^2+1)^2}\right) - \frac{1}{2}\left(\frac{1}{x^2} - \frac{1}{x^2+1}\right) + \frac{x^2}{4}\left(\frac{1}{x^4} - \frac{1}{(x^2+1)^2}\right) \right]$$

$$= \int_0^2 x\left[\frac{1}{4x^2} - \frac{x^2}{4(x^2+1)^2} - \frac{1}{2x^2} + \frac{1}{2(x^2+1)} + \frac{1}{4x^2} - \frac{x^2}{4(x^2+1)^2} \right] dx$$

$$\int_0^2 x\left[\frac{(x^2+1) - x^2}{2(x^2+1)^2} \right] dx = \frac{1}{2}\int_0^2 \frac{x}{(x^2+1)^2}\,dx$$

PAGE NO.

DATE: / /

$$RHS - \int_0^1\int_0^2 f(x,y)\, dx\, dy = \int_0^1\int_0^2 \frac{xy(x^2-y^2)}{(x^2+y^2)^3}\, dx\, dy$$

$$\Rightarrow \int_0^1 y\,dy\left[\int_0^2 \frac{x(x^2)}{(x^2+y^2)^3}\,dx - \int_0^2 \frac{xy^2\,dx}{(x^2+y^2)^3}\right]$$

Put $x^2+y^2 = t$

$dx = \frac{1}{2x} dt$

$$\Rightarrow \int_0^1 y\,dy\left[\int_0^{4+y^2} \frac{(t-y^2)\,dt}{2t^3} - \frac{y^2}{2}\int_0^{4+y^2} \frac{1}{t^3}\,dt\right]$$

$$= \int_0^1 y\,dy\left[\frac{1}{2}\left[\frac{-1}{t}\right]_{y^2}^{4+y^2} - \frac{y^2}{2}\left[\frac{-1}{2t^2}\right]_{y^2}^{4+y^2} - \frac{y^2}{2}\left[\frac{-1}{2t^2}\right]_{y^2}^{y^2+4}\right]$$

$$= \int_0^1 y\,dy\left[\frac{1}{2}\left[\frac{1}{y^2} - \frac{1}{4+y^2}\right] - \frac{y^2}{2}\left(\frac{1}{2y^4} - \frac{1}{2(4+y^2)^2}\right) - \frac{y^2}{2}\left[\frac{1}{2y^4} - \frac{1}{2(y^2+4)^2}\right]\right]$$

$$= \int_0^1 y\,dy\left[\frac{1}{2y^2} - \frac{1}{2(4+y^2)} - \frac{1}{4y^2} + \frac{y^2}{4(4+y^2)^2} - \frac{1}{4y^2} + \frac{y^2}{4(y^2+4)^2}\right]$$

$$= \int_0^1 y\,dy\left[\frac{y^2-(4+y^2)}{2(4+y^2)}\right] = \int_0^1 y\,dy\left(\frac{-2}{(4+y^2)^2}\right)$$

$$= -2\int_0^1 \frac{y}{(4+y^2)^2}\,dy$$

Put $y = 2x$

$dy = 2\,dx$

$$= -\frac{2^2}{4}\int_0^2 \frac{2x}{(1+x)^2}\,dx$$

$$= -2\int_0^2 \frac{x}{(1+x)^2}\,dx$$

$\therefore$ LHS $\neq$ RHS Proved

2\. (a) $\int_0^2 \int_0^{x^2} \exp\left(\frac{y}{x}\right) dy\, dx$

$\Rightarrow \int_0^2 \left[\cancel{\tfrac{1}{x}}\, x e^{y/x} \Big|_0^{x^2} \right] dx = \int_0^2 x \left[e^x - 1 \right] dx$

$\Rightarrow \int_0^2 x e^x dx - \frac{x^2}{2}\Big|_0^2 = \cancel{x e^x \Big|_0^2 - (e^2-1)x\Big|_0^2 - 2}$

$= \cancel{2e^2 - 2e^2 + 2} - 2$

$\Rightarrow x e^x \Big|_0^2 - e^x \Big|_0^2 - 2$

$\Rightarrow 2e^2 - e^2 + 1 - 2 = e^2 - 1$

(b) $\int_0^a \int_y^a \frac{x}{\sqrt{x^2+y^2}}\, dx\, dy$ Put $x^2+y^2 = p$, $2x\,dx = dp$

$\Rightarrow \int_0^a \int_{2y^2}^{a^2+y^2} \frac{1}{2\sqrt{p}}\, dp\, dy$

$= \int_0^a \frac{1}{2} \cdot 2 \left[\sqrt{a^2+y^2} - \sqrt{2y^2} \right] dy$

$= \int_0^a \sqrt{a^2+y^2} - \sqrt{2y^2}\, dy = \frac{y}{2}\sqrt{y^2+a^2}\Big|_0^a + \frac{a^2}{2}\ln\left(y + \sqrt{y^2+a^2}\right)\Big|_0^a$

$- \sqrt{2}\,\frac{y^2}{2}\Big|_0^a$

$\Rightarrow \frac{a}{2}\sqrt{2}a + \frac{a^2}{2} \ln \frac{(\sqrt{2}+1)a}{a} - \frac{\sqrt{2}}{2}a^2$

$\Rightarrow \frac{a^2}{2} \ln(\sqrt{2}+1)$

(c) $\int_0^{\pi}\int_0^{a(1-\cos\theta)} 2\pi r^2 \sin\theta \, d\theta \, dr$

$\Rightarrow \int_0^{\pi} \frac{2\pi \sin\theta}{3} \cdot r^3 \Big|_0^{a(1-\cos\theta)} d\theta$

$\Rightarrow \frac{2\pi}{3}\int_0^{\pi} \sin\theta \left[a^3 (1-\cos\theta)^3 \right] d\theta$

$\Rightarrow \frac{2\pi a^3}{3}\int_0^{\pi} \sin\theta \; 2^3 \sin^6 \frac{\theta}{2} \, d\theta$

$= \frac{2^4 \pi a^3}{3}\int_0^{\pi} 2 \sin^7 \frac{\theta}{2} \cos\frac{\theta}{2} \, d\theta$

Put $\frac{\theta}{2} = \alpha$

$d\theta = 2 d\alpha$

$= \frac{2^6 \pi a^3}{3}\int_0^{\pi/2} \sin^7\alpha \cos\alpha \, d\alpha$

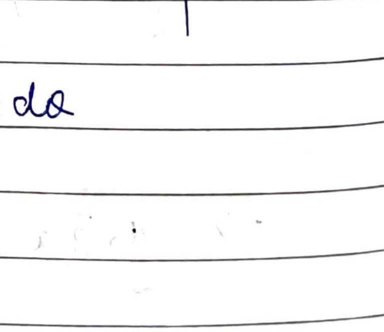
$\sin\alpha = a$

$\cos\alpha \, d\alpha = da$

$\frac{2^6 \pi a^3}{3}\int_0^{1} a^7 \, da = \frac{2^6 \pi a^3}{3}\left(\frac{1}{8}\right)$

$= \frac{2^3 \pi a^3}{3}$

3) $\iint r^2 \sin^2\theta \, d\theta \, dr$ $\qquad r = a\cos\theta$



r varies from 0 to a

θ varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$

$\Rightarrow$ Area $= 2\int_{-\pi/2}^{\pi/2}\int_0^{a\cos\theta} r^2 \sin^2\theta \, dr \, d\theta$

$\Rightarrow \int_{-\pi/2}^{\pi/2} \frac{1}{3}(a^3\cos^3\theta)\sin^2\theta \, d\theta$

$\Rightarrow \frac{2a^3}{3}\int_0^{\pi/2} \cos^3\theta \sin^2 d\theta = \frac{2a^3}{3}\left[-\frac{4}{5}\cdot\frac{2}{3}\cdot 1 + \frac{2}{3} \right]$

$\cos^5 - \cos^3$

$+\frac{2a^3}{3}\left(\frac{1}{5}\cdot\frac{2}{3}\right) = \frac{4a^3}{45}$

4. $$\int_1^{\log 8}\int_0^{\log y} e^x e^y \, dx\, dy$$

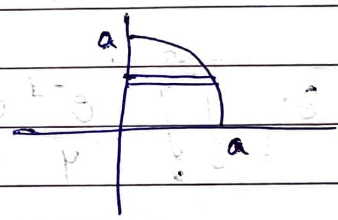
$$\Rightarrow \int_1^{\log 8} e^y (y-1)\, dy$$

$$\Rightarrow y e^y \Big|_1^{\log 8} - e^y \Big|_1^{\log 8} - e^y \Big|_1^{\log 8}$$

$$\Rightarrow 8\log 8 - e - \cancel{\log} 8 + e - 8 + e$$

$$= \underline{8\log 8 - 16 + e}$$

5. $\iint_A xy\, dx\, dy$ over +ve Quadrant of circle $x^2+y^2=a^2$



x varies 0 to a

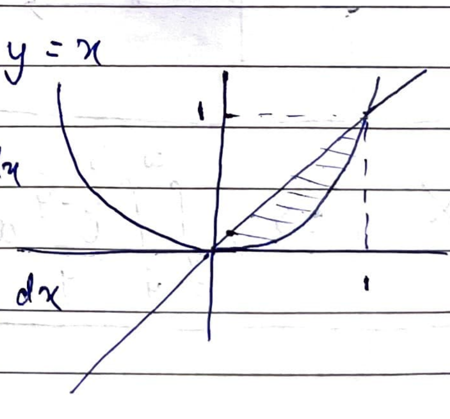
y varies 0 to a

$$\Rightarrow \int_0^a \int_0^{\sqrt{a^2-x^2}} xy\, dy\, dx$$

$$\Rightarrow \int_0^a \frac{x}{2}\left[a^2-x^2\right] dx = \int_0^a x\frac{a^2}{2} - \frac{x^3}{2}\, dx$$

$$\Rightarrow \frac{a^2}{4}a^2 - \frac{x^4}{8}\Big|_0^a = \frac{a^4}{8}$$

6. $\iint xy(x+y)\, dx\, dy$ $\quad y = x^2, \quad y = x$



$$\Rightarrow \int_0^1\int_0^x xy(x+y)\,dy\,dx - \int_0^1\int_0^{x^2} xy(x+y)\,dy\,dx$$

$$\Rightarrow \int_0^1 \left(\frac{x^2}{2}(x^2) + \frac{x}{3}(x^3) - \frac{x^2}{2}(x^4) - \frac{x}{3}x^6\right) dx$$

$$\Rightarrow \int_0^1 \left(\frac{x^4}{2} + \frac{x^4}{3} - \frac{x^6}{2} - \frac{x^7}{3}\right) dx = \frac{1}{5}\left(\frac{5}{6}\right) - \frac{1}{14} - \frac{1}{24}$$

$$= \frac{1}{6} - \frac{1}{14} - \frac{1}{24} = \frac{14-6}{6\cdot7\cdot2} - \frac{1}{6\cdot2\cdot2} = \frac{16-7}{6\cdot7\cdot2\cdot2}$$

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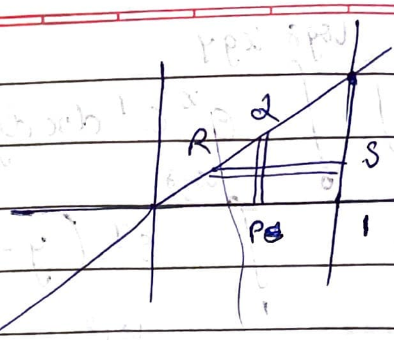
PAGE NO.
DATE: / /

7. $\int_0^1 \int_0^x f(x,y)\,dy\,dx$



$\Rightarrow$ X varies 0 to 1, Y varies 0 to x } in PQ

For RS, new limits

x varies y to 1

Y varies 0 to 1

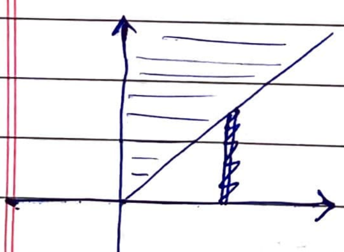
$\therefore \int_0^1 \int_0^x f(x,y)\,dy\,dx \longrightarrow \int_0^1 \int_1^y f(x,y)\,dx\,dy$

8. $\int_0^\infty \int_0^\infty \frac{e^{-y}}{y}\,dx\,dy$

Put ~~log y = p~~

~~$\frac{1}{y}\,dy = dp$~~

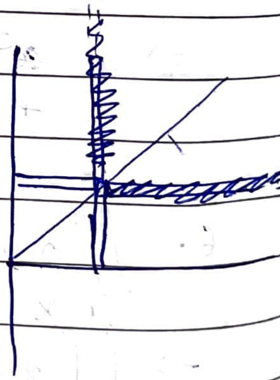
$\cancel{= \int_0^\infty \int_{-\infty}^{\infty} \frac{1}{p}\,dp\,dx}$



X varies 0 to ∞

Y varies ~~x to ∞~~ 0 to x

$I = \int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dy\,dx$



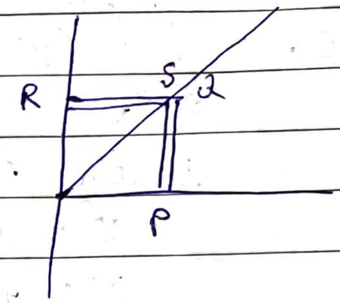
New Limits

X varies y to ∞

Y varies 0 to ∞

X $\Rightarrow \int_0^\infty \int_y^\infty \frac{e^{-y}}{y}\,dx\,dy = \int_0^\infty \frac{e^{-y}}{y}\, x\Big|_y$

8. $$\int_0^{\infty}\int_0^{x} \frac{e^{-y}}{y}\, dy\, dx$$



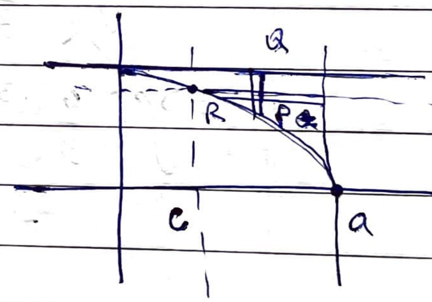
x varies 0 to ∞ } PQ
y varies 0 to x

New Limits : x varies 0 to y } RS
y varies 0 to ∞

$$\int_0^{\infty}\int_0^{y} \frac{e^{-y}}{y}\, dx\, dy = \int_0^{\infty} e^{-y}\, dy = \frac{-1}{-1} = 1$$

9. $$I = \int_c^{a}\int_{\frac{b}{a}\sqrt{a^2-x^2}}^{b} V\, dx\, dy \qquad c < a$$

PQ { y varies b till $\frac{b}{a}\sqrt{a^2-x^2}$
x varies c to a



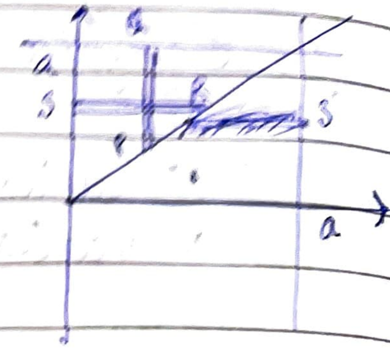
For new limits :

y varies $\frac{b}{a}\sqrt{a^2-c^2}$ to b & 0 to $\frac{b}{a}\sqrt{a^2-c^2}$

x varies c to a & $\frac{b}{a}\sqrt{a^2-y^2}$ to a

$$\Rightarrow I = \int_{\frac{b}{a}\sqrt{a^2-c^2}}^{b}\int_c^{a} V\, dx\, dy + \int_0^{\frac{b}{a}\sqrt{a^2-c^2}}\int_{\frac{b}{a}\sqrt{a^2-y^2}}^{a} V\, dx\, dy$$

10. $\int_0^a\int_x^a (x^2+y^2)\,dy\,dx$



Limits of PQ strip

X : 0 to a

Y : x to a

For new limits consider RS strip.

X : y to [illegible]

Y : 0 to a

$\therefore \int_0^a\int_0^y (x^2+y^2)\,dx\,dy = \int_0^a \left(\frac{x^3}{3}\Big|_0^y + y^2 x \Big|_0^y \right) dy$

$\Rightarrow \int_0^a \left(\frac{a^3}{3} - \frac{y^3}{3} + a^3 - y^3\right) dy$

$\Rightarrow \int_0^a \left(\frac{4a^3}{3} - \frac{4y^3}{3}\right) dy = \frac{4a^4}{3} - \frac{4a^4}{3(4)}$

$= \frac{4a^4}{3} - \frac{a^4}{3} = a^4$

$\Rightarrow \int_0^a \frac{y^3}{3} + y^3\,dy = \frac{a^4}{3(4)} + \frac{a^4}{4} = \frac{a^4}{3}$

11. $\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} (x^2+y^2)\,dx\,dy$ ~~Put~~

Y varies 0 to $\sqrt{2ax-x^2}$

$\therefore y^2 = -(x^2-2ax+a^2)+a^2$

$= y^2 + (x-a)^2 = a^2$

X varies 0 to 2a

Put $y = a\sin\theta$ $\therefore$ ~~$dy = a\,d\theta$~~

$x = \cancel{a}\,2a\cos\theta$

$dx\,dy = |J|\,dr\,d\theta$

$$J = \begin{vmatrix} 1+\sin\theta & -a\sin\theta \\ \sin\theta & a\cos\theta \end{vmatrix} = a\cos\theta + a\sin\theta\cos\theta + a\sin\theta\cos\theta$$

$$= a\cos\theta + a\sin 2\theta$$

$$\therefore \int_0^{\pi/2}\int_0^{a} \left(a^2(1+\cos\theta)^2 + a^2\sin^2\theta\right)(a)(\cos\theta)(1+\sin\theta)\,dr\,d\theta$$

$$\int_0^{\pi/2} a^3\left[1+1+2\cos\theta\right]\left[\cos\theta\right]\left[1+\sin\theta\right] r\Big|_0^a\,d\theta$$

$$\Rightarrow a^4\int_0^{\pi/2}$$

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = r \qquad \therefore dx\,dy = r\,dr\,d\theta$$

r varies 0 to $2a\cos\theta$

θ varies 0 to $\pi/2$

$$\therefore \int_0^{\pi/2}\int_0^{2a\cos\theta} \left(\cancel{a^2\sin^2\theta + 4a^2\cos^2\theta}\ r^2\right) r\,dr\,d\theta$$

$$\Rightarrow \int_0^{\pi/2} \frac{(2a\cos\theta)^2}{2}\left[a^2\right]\left(\cancel{\sin^2\theta}\ 1 + 3\cos^2\theta\right)d\theta$$

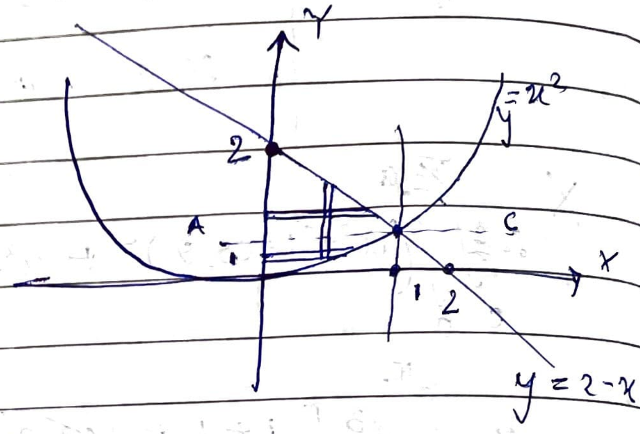
$$\Rightarrow 2a^4\int_0^{\pi/2} \cos^2\theta\,(1+3\cos^2\theta)\,d\theta = 2a^4\int_0^{\pi/2} \cos^2\theta + 3\cos^4\theta\,d\theta$$

$$\Rightarrow 2a^4\left[\frac{4}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} + 3\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right] = \frac{13\pi a^4}{8}$$

$$\int_0^{\pi/2}\int_0^{2a\cos\theta} r^3\,dr\,d\theta = \int_0^{\pi/2} \frac{1}{4}(2a)^4\cos^4\theta\,d\theta$$

$$= 2^2a^4\left(\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right) = \frac{3}{4}\pi a^4$$

12) (i) $\int_0^1\int_{x^2}^{2-x} xy\,dy\,dx$



$y = x^2$, $y = 2-x$

$\cancel{x^2 - x + 2 = 0}$

$\cancel{x = -2 \text{ or } 1}$

Y varies x^2 to $2-x$

X varies 0 to 1

New Limits

X varies 0 to $2-y$ & 0 to $\sqrt{y}$

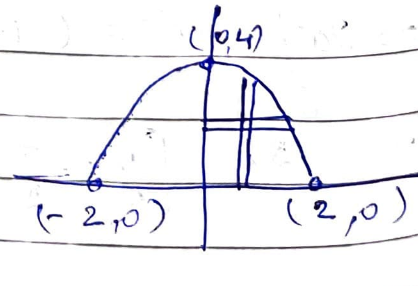
Y varies 1 to 2 & 0 to 1

$$I = \int_1^2\int_0^{2-y} xy\,dx\,dy + \int_0^1\int_0^{\sqrt{y}} xy\,dx\,dy$$

$$I = \int_1^2 \frac{y}{2}(2-y)^2\,dy + \int_0^1 \frac{y^2}{2}\,dy$$

$$I = \frac{3}{8}$$

(ii) $\int_0^2\int_0^{4-x^2} \frac{xe^{2y}}{4-y}\,dy\,dx$



$y = 4-x^2$

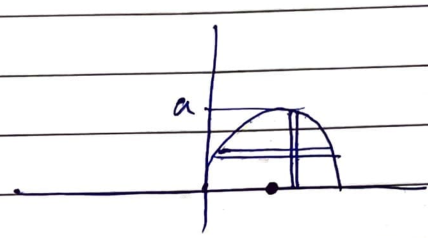
Change of order -

$$I = \int_0^4 \int_0^{\sqrt{4-y}} \frac{x e^{2y}}{4-y} \, dx \, dy$$

$$I = \int_0^4 \frac{e^{2y} (4-y)}{2(4-y)} dy = \frac{1}{2(2)} (e^8 - 1)$$

$$\boxed{I = \frac{e^8 - 1}{4}}$$

(iii) $$I = \int_0^a \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} dx\,dy$$



$x = a + \sqrt{a^2-y^2} \Rightarrow (x-a)^2 - (a^2-y^2) = 0$

$\Rightarrow (x-a)^2 + y^2 = a^2$

X varies $a - \sqrt{a^2-y^2}$ to $a + \sqrt{a^2-y^2}$

Y varies 0 to a

New limits,

X varies 0 to $2a$

Y varies 0 to $\sqrt{a^2 - (x-a)^2}$

$$I = \int_0^{2a} \int_0^{\sqrt{a^2-(x-a)^2}} dy\,dx = \int_0^{2a} \sqrt{a^2-(x-a)^2}\, dx$$

$$I = \frac{(x-a)\sqrt{a^2-(x-a)^2}}{2} \Big|_0^{2a} - \frac{a^2}{2} \sin^{-1}\left(\frac{x-a}{a}\right) \Big|_0^{2a}$$

$$I = \frac{a^2 \pi}{2}$$