# Predicate Logic

**Example:**
All men are mortal.
Socrates is a man.
$\therefore$ Socrates is mortal.

**Note:** We need logic laws that work for statements involving quantities like "some" and "all".

In English, the **predicate** is the part of the sentence that tells you something about the subject.

## More on predicates

**Example:** Nate is a student at UT.
What is the subject? What is the predicate?

**Example:** We can form two different predicates.
Let P(x) be "x is a student at UT".
Let Q(x, y) be "x is a student at y".

**Definition:** A **predicate** is a property that a variable or a finite collection of variables can have. A predicate becomes a proposition when specific values are assigned to the variables. $P(x_1, x_2, ..., x_n)$ is called a predicate of n variables or n arguments.

**Example:** She lives in the city.
P(x,y): x lives in y.
P(Mary, Austin) is a proposition: Mary lives in Austin.

**Example:** Predicates are often used in if statements and loop conditions.
if($x > 100$)
then y:= $x * x$

predicate T(x): $x > 100$

## Domains and Truth Sets

**Definition:** The **domain** or **universe** or **universe of discourse** for a predicate variable is the set of values that may be assigned to the variable.

**Definition:** If P(x) is a predicate and x has domain U, the **truth set** of P(x) is the set of all elements t of U such that P(t) is true, ie $\{t \in U | P(t) \text{ is true}\}$

**Example:** $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
P(x): "x is even".
The truth set is: $\{2, 4, 6, 8, 10\}$

## The Universal Quantifier: ∀

Turn predicates into propositions by assigning values to all variables:
Predicate P(x): "x is even"
Proposition P(6): "6 is even"

The other way to turn a predicate into a proposition: add a *quantifier* like "all" or "some" that indicates the number of values for which the predicate is true.

**Definition:** The symbol ∀ is called the **universal quantifier**. The **universal quantification** of P(x) is the statement "P(x) for all values x in the universe", which is written in logical notation as: $\forall x P(x)$ or sometimes $\forall x \in D, P(x)$.

**Ways to read $\forall x P(x)$:**
For every x, P(x)
For every x, P(x) is true
For all x, P(x)

## More on the universal quantifier

**Definition:** A **counterexample** for $\forall x P(x)$ is any $t \in U$, where U is the universe, such that P(t) is false.

**Some Examples**
**Example:** P(x, y): x + y = 8
Assign x to be 1, and y to be 7. We get proposition P(1, 7) which is true.
Proposition P(2, 5) is false since $2 + 5 \neq 8$.

**Example:** $\forall x[x \geq 0]$
$U = \mathbb{N}$ (non-negative integers)
We could re-write this proposition as: $\forall x \in \mathbb{N}, x \geq 0$
Is the proposition true?
What if the universe is $\mathbb{R}$?

**Example:** $\forall x \forall y[x + y > x]$
Is this proposition true if:

1. If $U = \mathbb{N}$?
2. If $U = \mathbb{R}$?

**Example:** $\forall x \forall y[x > y]$
True if:
universe for x = the non-negative integers
universe for y = the non-positive integers

## The Existential Quantifier: $\exists$

**Definition:** The symbol $\exists$ is call the **existential quantifier** and represents the phrase "there exists" or "for some". The **existential quantification** of P(x) is the statement "P(x) for some values x in the universe", or equivalently, "There exists a value for x such that P(x) is true", which is written $\exists x P(x)$.

**Note:** If P(x) is true for at least one element in the domain, then $\exists x P(x)$ is true. Otherwise it is false.

**Note:** Let P(x) be a predicate and $c \in U$ (U = domain).
The following implications are true:
$\forall x P(x) \rightarrow P(c)$
$P(c) \rightarrow \exists x P(x)$

**Example:** $\exists x$ [x is prime] where $U = \mathbb{Z}$
Is this proposition true or false?

**Example:** $\exists x[x^2 < 0]$ where $U = \mathbb{R}$
True or false?

**Exercises:** True or false? Prove your answer.

1. $\exists n[n^2 = n]$ where $U = \mathbb{Z}$.
2. $\exists n[n^2 = n]$ where $U = \{4, 5, 6, 7\}$.

## Translating Quantified Statements

Translate the following into English.

1. $\forall x[x^2 \geq 0]$ where $U = \mathbb{R}$.
2. $\exists t[(t > 3) \wedge (t^3 > 27)]$ where $U = \mathbb{R}$.
3. $\forall x[(2|x) \vee (2 \not| x)]$ where $U = \mathbb{N}$

Translate the following into logic statements.

1. There is an integer whose square is twice itself.
2. No school buses are purple.
3. If a real number is even, then its square is even.

**Note:** Let $U = \{1, 2, 3\}$.
Proposition $\forall x P(x)$ is equivalent to $P(1) \wedge P(2) \wedge P(3)$.
Proposition $\exists x P(x)$ is equivalent to $P(1) \vee P(2) \vee P(3)$.

## Bound and Free Variables

**Definition:** All variables in a predicate must be **bound** to turn a predicate into a proposition. We **bind** a variable by assigning it a value or quantifying it. Variables which are not bound are **free**.

**Note:** If we bind one variable in a predicate $P(x, y, z)$ with 3 variables, say by setting $z = 4$, we get a predicate with 2 variables: $P(x, y, 4)$.

**Example:** Let $U = \mathbb{N}$.
$P(x, y, z) : x + y = z \leftarrow 3$ free variables
Let $Q(y, z) = P(2, y, z) : 2 + y = z \leftarrow 2$ free variables

### Examples with Quantifiers

**Example:** $U = \mathbb{Z}$
N(x): x is a non-negative integer
E(x): x is even
O(x): x is odd
P(x): x is prime
Translate into logical notation.

1. There exists an even integer.
2. Every integer is even or odd.
3. All prime integers are non-negative.
4. The only even prime is 2.
5. Not all integers are odd.
6. Not all primes are odd.
7. If an integer is not odd, then it is even.

## Examples with Nested Quantifiers

**Note about nested quantifiers:** For predicate $P(x, y)$:
$\forall x \forall y P(x, y)$ has the same meaning as $\forall y \forall x P(x, y)$.
$\exists x \exists y P(x, y)$ has the same meaning as $\exists y \exists x P(x, y)$.
We can **not** interchange the position of $\forall$ and $\exists$ like this!

**Example:** $U$ = set of married people. True or false?

1. $\forall x \exists y$[x is married to y]
2. $\exists y \forall x$[x is married to y]

**Example:** $U = \mathbb{Z}$. True or false?

1. $\forall x \exists y[x + y = 0]$
2. $\exists y \forall x[x + y = 0]$

**Exercise:** $U = \mathbb{N}$.
$L(x, y) : x < y$
S(x, y, z): x + y = z
$P(x, y, z) : xy = z$
Rewrite the following in logic notation.

1. For every x and y, there is a z such that $x + y = z$.
2. No x is less than 0.
3. For all x, $x + 0 = x$.
4. There is some x such that $xy = y$ for all y.

### Negating Quantified Statements

**Precedence of logical operators**
1. $\forall, \exists$
2. $\neg$
3. $\wedge, \vee$
4. $\rightarrow, \leftrightarrow$

**Example:** Statement: "All dogs bark."
Negation: "One or more dogs do not bark" or "some dogs do not bark".
NOT "No dogs bark".
If at least one dog does not bark, then the original statement is false.

**One example of DeMorgan's laws for quantifiers:**
$\neg \forall x P(x) \equiv \exists x \neg P(x)$

**Example:** Some cats purr.
Negation: No cats purr.
I.e., if it is false that some cats purr, then no cat purrs.

**DeMorgan's laws for quantifiers:**
$\neg \forall x P(x) \equiv \exists x \neg P(x)$
$\neg \exists x P(x) \equiv \forall x \neg P(x)$

## More Examples - Negating Statements with Quantifiers

**Example:** Write the statements in logical notation. Then negate the statements.

1. Some drivers do not obey the speed limit.
2. All dogs have fleas.

**Example:** Using DeMorgan's laws to push negation through multiple quantifiers:

$$\begin{aligned}\neg\exists x\forall y\forall z P(x,y,z) &\equiv \forall x\neg\forall y\forall z P(x,y,z)\\ &\equiv \forall x\exists y\neg\forall z P(x,y,z)\\ &\equiv \forall x\exists y\exists z\neg P(x,y,z).\end{aligned}$$

**Example:** Write the following statement in logical notation and then negate it.
For every integer x and every integer y, there exists an integer z such that $y - z = x$.
Logical notation:
Negation (apply DeMorgan's laws):
Let $U = \mathbb{N}$. Show the original statement is false by showing the negation is true.

## Some Definitions

**Definition:** Let $U$ be the universe of discourse and $P(x_1, ..., x_n)$ be a predicate. If $P(x_1, ..., x_n)$ is true for every choice of $x_1, ..., x_n \in U$, then we say P is **valid** in universe $U$. If $P(x_1, ..., x_n)$ is true for some (not necessarily all) choices of arguments from $U$, then we say that P is **satisfiable** in $U$. If P is not satisfiable in $U$, we say P is **unsatisfiable** in $U$.

**Definition:** The **scope** if a quantifier is the part of a statement in which variables are bound by the quantifier.

**Example:** $R \vee \exists(P(x) \vee Q(x))$
Scope of $\exists$: $P(x) \vee Q(x)$.

**Note:** We can use parentheses to change the scope, but otherwise the scope is the smallest expression possible.

**Example:** $\forall x P(x) \wedge Q(x)$
Scope of $\forall$: $P(x)$.
Note that this is a predicate, not a proposition, since the variable in $Q(x)$ is not bound. It is confusing to have 2 variables which are both denoted $x$. Rewrite as: $\forall x P(x) \wedge Q(z)$.

## Quantifiers plus $\land$ and $\lor$

**Example:** $\forall x(P(x) \land Q(x)) \equiv \forall x P(x) \land \forall x Q(x)$ (That is, no matter what the domain is, these two propositions always have the same truth value)

**Proof:** at end of notes, after some required techniques are discussed.

**Terminology:** We say that $\forall$ distributes over $\land$

This shouldn't be surprising, since we know that for a finite domain, say {1,2,3}, $\forall x P(x) \equiv P(1) \land P(2) \land P(3)$. We also know that $\land$ is commutative and associative, so:

$\forall x \in \{1,2,3\}(P(x) \land Q(x))$
$\equiv (P(1) \land Q(1)) \land (P(2) \land Q(2)) \land (P(3) \land Q(3))$ {for this example domain}
$\equiv (P(1) \land P(2) \land P(3)) \land (Q(1) \land Q(2) \land Q(3))$ {$\land$ commutativity/associativity}
$\equiv \forall x \in \{1,2,3\} P(x) \land \forall x \in \{1,2,3\} Q(x)$ {for this specific example}

Though this is only an example domain, the intuition extends to other domains as well, including infinite domains.

## Distributing $\exists$ over $\land$

**Note:** The existential quantifier $\exists$ does **not** distribute over $\land$. That is, $\exists x(P(x) \land Q(x)) \not\equiv \exists x P(x) \land \exists x Q(x)$.

**Proof:** We must find a counterexample - a universe and predicates P and Q such that one of the propositions is true and the other is false.

Let $U = \mathbb{N}$. Set P(x): "x is prime" and Q(x): "x is composite" (ie not prime). Then $\exists x(P(x) \land Q(x))$ is false, but $\exists x P(x) \land \exists x Q(x)$ is true. $\square$

**Note:** The following is true though:
$\exists x(P(x) \land Q(x)) \rightarrow \exists x P(x) \land \exists x Q(x)$.

**Proof:** exercise

**Note:** With $\lor$, the situation is reversed. $\exists$ distributes over $\lor$, but $\forall$ does not.

## Distributing the Existential Quantifier

**Recall:** $\forall x(P(x) \land Q(x)) \equiv \forall xP(x) \land \forall xQ(x)$

This rule holds for arbitrary $P$ and $Q$. Replace $P$ by $\neg S$ and $Q$ by $\neg R$ and negate both sides to see that:

$\exists x(S(x) \lor R(x)) \equiv \exists xS(x) \lor \exists xR(x)$.

**Exercise:** Show that
1. $(\forall xP(x) \lor \forall xQ(x)) \to \forall x(P(x) \lor Q(x))$ is true.
2. $\forall xP(x) \lor \forall xQ(x) \not\equiv \forall x(P(x) \lor Q(x))$

## $\exists$ does not distribute over $\to$

**Note:** $\exists$ does not distribute over $\to$. I.e.,
$\exists x(P(x) \to Q(x)) \not\equiv \exists xP(x) \to \exists xQ(x)$.

**Proof:**
$\exists x(P(x) \to Q(x)) \equiv \exists x(Q(x) \lor \neg P(x))$ by implication
$\equiv \exists xQ(x) \lor \exists x\neg P(x)$ by distributivity of $\exists$ over $\lor$
$\equiv \exists xQ(x) \lor \neg\forall xP(x)$ by DeMorgan's law
$\equiv \forall xP(x) \to \exists xQ(x)$ by implication law

So we need to show that $\forall xP(x) \to \exists xQ(x)$ is not logically equivalent to $\exists xP(x) \to \exists xQ(x)$. Note that if $\exists xQ(x)$ is false, $\forall xP(x)$ is false, and $\exists xP(x)$ is true, then we would have a counterexample, since one of the implications is true and the other is false. So let $U = \mathbb{N}$, and set P(x) to be "x is even" and Q(x) to be "x is negative". In this case $\forall xP(x) \to \exists xQ(x)$ and $\exists xP(x) \to \exists xQ(x)$ have different truth values.

## Logical Relationships with Quantifiers

| Law | Name |
|---|---|
| $\neg\forall xP(x) \equiv \exists x\neg P(x)$<br>$\neg\exists xP(x) \equiv \forall x\neg P(x)$ | DeMorgan's laws for quantifiers |
| $\forall xP(x) \land \forall xQ(x) \equiv \forall x(P(x) \land Q(x))$ | distributivity of $\forall$ over $\land$ |
| $\exists x(P(x) \lor Q(x)) \equiv \exists xP(x) \lor \exists xQ(x)$ | distributivity of $\exists$ over $\lor$ |

## Compact Notation

**Example:** For every $x > 0$, P(x) is true.
Current notation: $\forall x[(x > 0) \rightarrow P(x)]$.
More compact notation: $\forall x_{x>0} P(x)$ (or $\forall x > 0, P(x)$).

**Example:** There exists an x such that $x \neq 0$ and P(x) is true.
Compact notation: $\exists x_{x \neq 0} P(x)$, instead of $\exists x[(x \neq 0) \wedge P(x)]$. The compact notation is more readable.

**Example:**
**Definition:** The **limit** of f(x) as x approaches c is k (denoted $lim_{x\rightarrow c} f(x) = k$) if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all x, if $|x - c| < \delta$, then $|f(x) - k| < \varepsilon$.

Notation: $lim_{x\rightarrow c} f(x) = k$ if $\forall \varepsilon_{\varepsilon>0} \exists \delta_{\delta>0} \forall x[|x - c| < \delta \rightarrow |f(x) - k| < \varepsilon]$.

## Arguments with Quantified Statements

### Rules of Inference with Quantifiers

### Rule of Universal Instantiation

$\forall x P(x)$

$\therefore P(c)$ (where c is some element of P's domain)

**Example:** $U =$ all men
All men are mortal.
Dijkstra is a man.
$\therefore$ Dijkstra is mortal.

P(x): x is mortal.
Argument:
$\forall x P(x)$
$\therefore$ P(Dijkstra)

### Universal Modus Ponens

$\forall x(P(x) \rightarrow Q(x))$
$P(c)$
$\therefore$ Q(c)

**Example:**
All politicians are crooks.
Joe Lieberman is a politician.
$\therefore$ Joe Lieberman is a crook.

P(x): x is a politician, Q(x): x is a crook, $U$ =all people.

**Example:** If x is an even number, then $x^2$ is an even number.
206 is an even number.
$\therefore 206^2$ is an even number.

**Universal Modus Tollens**

$\forall x(P(x) \to Q(x))$
$\neg Q(c)$
$\therefore \neg P(c)$

**Example:**
All dogs bark.
Otis does not bark.
$\therefore$ Otis is not a dog.

$U$ = all living creatures, P(x): x is a dog, Q(x): x barks.

**Universal Hypothetical Syllogism**

$\forall x(P(x) \to Q(x))$
$\forall x(Q(x) \to R(x))$
$\therefore \forall x(P(x) \to R(x))$

**Example:**
If integer x is even, then 2x is even.
If 2x is even, then $4x^2$ is even.
$\therefore$ If x is even, then $4x^2$ is even.

## Universal Generalization

**Universal Generalization:**
$P(c)$ for arbitrary $c \in U$

$\therefore \forall x P(x)$

**Example:** For an arbitrary real number $x$, $x^2$ is non-negative. Therefore, the square of any real number is non-negative.

**Note:** We will be using this rule a lot. We use it to prove statements of the form $\forall x P(x)$. We assume that some $c$ is an arbitrary element of the domain, and prove that $P(c)$ is true. Then we use the rule of Universal Generalization to conclude that $\forall x P(x)$.

**Prove:** The square of every even integer $n$ is even.
**Formally:** $\forall n \in \mathbb{Z}[(n \text{ is even}) \to (n^2 \text{ is even})]$
**Proof:**
1. Let $n \in \mathbb{Z}$ {$n$ is an arbitrary integer}
2. Assume $n$ is even {Premise}
3. $\exists k \in \mathbb{Z}(n = 2k)$ {Definition of even}
4. $\exists k \in \mathbb{Z}(n^2 = (2k)^2)$ {Square both sides of equation}
5. $\exists k \in \mathbb{Z}(n^2 = 2(2k^2))$ {Simplify, Factor out 2}
6. $\exists q \in \mathbb{Z}(n^2 = 2q)$ {$q = 2k^2$, $q \in \mathbb{Z}$ because $\mathbb{Z}$ is closed under multiplication}
7. $n^2$ is even {Definition of even}
8. We have proven $(n \text{ is even}) \to (n^2 \text{ is even})$
9. $\therefore \forall n \in \mathbb{Z}[(n \text{ is even}) \to (n^2 \text{ is even})]$ {Universal Generalization: 1,8}

**Existential Instantiation and Existential Generalization**

**Existential Instantiation**

$\exists x P(x)$

$\therefore$ P(c) for some c

**Existential Generalization**

P(c) for some element c

$\therefore \exists x P(x)$

## Arguments with Quantifiers

**Def:** An argument with quantifiers is **valid** if the conclusion is true whenever the premises are all true.

**Example:** A horse that is registered for today's race is not a throughbred. Every horse registered for today's race has won a race this year. Therefore a horse that has won a race this year is not a thoroughbred.

P(x): x is registered for today's race.
Q(x): x is a thoroughbred.
R(x): x has won a race this year.
U = all horses

$\exists x(P(x) \wedge \neg Q(x))$

$\forall x(P(x) \rightarrow R(x))$

$\therefore \exists x(R(x) \wedge \neg Q(x))$

**Proof:**

| Step | Reason |
|---|---|
| 1. $\exists x(P(x) \wedge \neg Q(x))$ | premise |
| 2. $P(a) \wedge \neg Q(a)$ for some a | step 1, existential instantiation |
| 3. $P(a)$ | simplification, step 2 |
| 4. $\forall x(P(x) \rightarrow R(x))$ | premise |
| 5. $P(a) \rightarrow R(a)$ | universal instantiation, step 4 |
| 6. R(a) | modus ponens, steps 3 and 5 |
| 7. $\neg Q(a)$ | step 2, simplification |
| 8. $R(a) \wedge \neg Q(a)$ | conjunction, steps 6 and 7 |
| 9. $\exists x(R(x) \wedge \neg Q(x))$ | existential generalization, step 8 |

# More Proofs

**Prove:** $\forall x(P(x) \wedge Q(x)) \equiv \forall x P(x) \wedge \forall x Q(x)$ (from earlier in notes)
**Recall:** Two formulas are "equivalent" if replacing the $\equiv$ symbol with a $\leftrightarrow$ (bi-conditional) results in a tautology, i.e. a formula that is always true. So, we want to show that $\forall x(P(x) \wedge Q(x)) \leftrightarrow \forall x P(x) \wedge \forall x Q(x)$ is true. This can be accomplished with two inference proofs.

($\rightarrow$) Prove $\forall x(P(x) \wedge Q(x)) \rightarrow \forall x P(x) \wedge \forall x Q(x)$
1. $\forall x(P(x) \wedge Q(x))$ {Premise}
2. Let $c \in U$ {$c$ is an arbitrary member of the universe}
3. $P(c) \wedge Q(c)$ {Universal instantiation: 1,2}
4. $P(c)$ {Simplification: 3}
5. $\forall x P(x)$ {Universal Generalization: 2,4 (because $c$ was arbitrary)}
6. $Q(c)$ {Simplification: 3}
7. $\forall x Q(x)$ {Universal Generalization: 2,6 (because $c$ was arbitrary)}
8. $\forall x P(x) \wedge \forall x Q(x)$ {Conjunction: 5,7}

Therefore $\forall x(P(x) \wedge Q(x)) \rightarrow \forall x P(x) \wedge \forall x Q(x)$ ■

($\leftarrow$) Prove $\forall x P(x) \wedge \forall x Q(x) \rightarrow \forall x(P(x) \wedge Q(x))$
1. $\forall x P(x) \wedge \forall x Q(x)$ {Premise}
2. Let $c \in U$ {$c$ is an arbitrary member of the universe}
3. $\forall x P(x)$ {Simplification: 1}
4. $P(c)$ {Universal instantiation: 2,3}
5. $\forall x Q(x)$ {Simplification: 1}
6. $Q(c)$ {Universal instantiation: 2,5}
7. $P(c) \wedge Q(c)$ {Conjunction: 4,6}
8. $\forall x(P(x) \wedge Q(x))$ {Universal Generalization: 2,7 (because $c$ was arbitrary)}

Therefore $\forall x P(x) \wedge \forall x Q(x) \rightarrow \forall x(P(x) \wedge Q(x))$ ■

Proving $\forall x(P(x) \wedge Q(x)) \equiv \forall x P(x) \wedge \forall x Q(x)$, continued

Now we have two separate proofs, which can be used to prove the original equivalence:
1. $\forall x(P(x) \wedge Q(x)) \rightarrow \forall x P(x) \wedge \forall x Q(x)$ {Proven}
2. $\forall x P(x) \wedge \forall x Q(x) \rightarrow \forall x(P(x) \wedge Q(x))$ {Proven}
3. $(\forall x(P(x) \wedge Q(x)) \rightarrow \forall x P(x) \wedge \forall x Q(x)) \wedge$
   $(\forall x P(x) \wedge \forall x Q(x) \rightarrow \forall x(P(x) \wedge Q(x)))$ {Conjunction: 1,2}
4. $\forall x(P(x) \wedge Q(x)) \leftrightarrow \forall x P(x) \wedge \forall x Q(x)$ {Equivalence/Biconditional: 3}

Since we have proven that $\forall x(P(x) \wedge Q(x)) \leftrightarrow \forall x P(x) \wedge \forall x Q(x)$ is true, can also say that $\forall x(P(x) \wedge Q(x)) \equiv \forall x P(x) \wedge \forall x Q(x)$, thus completing our proof.

In fact, these last few steps are so common that they are generally left out of any proof of the form $A \leftrightarrow B$ (more on this later).

| | |
|---|---|
| $P \equiv P \wedge P$<br>$P \equiv P \vee P$ | Idempotence of $\wedge$<br>Idempotence of $\vee$ |
| $P \wedge Q \equiv Q \wedge P$<br>$P \vee Q \equiv Q \vee P$ | Commutativity of $\wedge$<br>Commutativity of $\vee$ |
| $(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$<br>$(P \vee Q) \vee R \equiv P \vee (Q \vee R)$ | Associativity of $\wedge$<br>Associativity of $\vee$ |
| $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$<br>$\neg(P \vee Q) \equiv \neg P \wedge \neg Q$ | De Morgan's Laws |
| $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$<br>$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$ | Distributivity of $\wedge$ over $\vee$<br>Distributivity of $\vee$ over $\wedge$ |
| $P \wedge \mathbf{F} \equiv \mathbf{F}$<br>$P \vee \mathbf{T} \equiv \mathbf{T}$ | $\wedge$ Domination<br>$\vee$ Domination |
| $P \wedge \mathbf{T} \equiv P$<br>$P \vee \mathbf{F} \equiv P$ | $\wedge$ Identity<br>$\vee$ Identity |
| $P \wedge \neg P \equiv \mathbf{F}$<br>$P \vee \neg P \equiv \mathbf{T}$ | $\wedge$ Negation<br>$\vee$ Negation |
| $\neg\neg P \equiv P$ | Double Negation |
| $P \wedge (P \vee Q) \equiv P$<br>$P \vee (P \wedge Q) \equiv P$ | Absorbtion Laws |
| $P \to Q \equiv \neg P \vee Q$ | Implication |
| $P \to Q \equiv \neg Q \to \neg P$ | Contrapositive |
| $P \leftrightarrow Q \equiv (P \to Q) \wedge (Q \to P)$ | Equivalence/Biconditional |
| $(P \wedge Q) \to R \equiv P \to (Q \to R)$ | Exportation |
| $\neg \forall x P(x) \equiv \exists x \neg P(x)$<br>$\neg \exists x P(x) \equiv \forall x \neg P(x)$ | De Morgan's Laws for Quantifiers |
| $\forall x P(x) \wedge \forall x Q(x) \equiv \forall x (P(x) \wedge Q(x))$<br>$\exists x P(x) \vee \exists x Q(x) \equiv \exists x (P(x) \vee Q(x))$ | Distributivity of $\forall$ over $\wedge$<br>Distributivity of $\exists$ over $\vee$ |
| $[P \to Q] \wedge [P] \Rightarrow Q$ | Modus ponens |
| $[P \to Q] \wedge [\neg Q] \Rightarrow \neg P$ | Modus tollens |
| $[P \to Q] \wedge [Q \to R] \Rightarrow (P \to R)$ | Hypothetical syllogism |
| $[P \vee Q] \wedge [\neg P] \Rightarrow Q$ | Disjunctive syllogism |
| $[P] \Rightarrow P \vee Q$ | Addition |
| $[P \wedge Q] \Rightarrow P$ | Simplification |
| $[P] \wedge [Q] \Rightarrow P \wedge Q$ | Conjunction (Adding a premise) |
| $[P \vee Q] \wedge [\neg P \vee R] \Rightarrow Q \vee R$ | Resolution |
| $[P \to Q] \wedge [R \to S] \wedge [P \vee R] \Rightarrow Q \vee S$ | Constructive dilemma |
| $[P \to Q] \wedge [R \to S] \wedge [\neg Q \vee \neg S] \Rightarrow \neg P \vee \neg R$ | Destructive dilemma |
| $[\forall x \in U P(x)] \wedge [c \in U] \Rightarrow P(c)$ | Universal instantiation |
| $[\forall x (P(x) \to Q(x))] \wedge [P(c)] \Rightarrow Q(c)$ | Universal modus ponens |
| $[\forall x (P(x) \to Q(x))] \wedge [\neg Q(c)] \Rightarrow \neg P(c)$ | Universal modus tollens |
| $[\forall x (P(x) \to Q(x))] \wedge [\forall x (Q(x) \to R(x))] \Rightarrow \forall x (P(x) \to R(x))$ | Universal hypothetical syllogism |
| $[P(c)$ for arbitrary $c \in U] \Rightarrow [\forall x \in U P(x)]$ | Universal generalization |
| $[\exists x \in U P(x)] \Rightarrow P(c)$ for some $c \in U$ | Existential instantiation |
| $[P(c)] \wedge [c \in U] \Rightarrow \exists x \in U P(x)$ | Existential generalization |