Lecture Note: Unit-II

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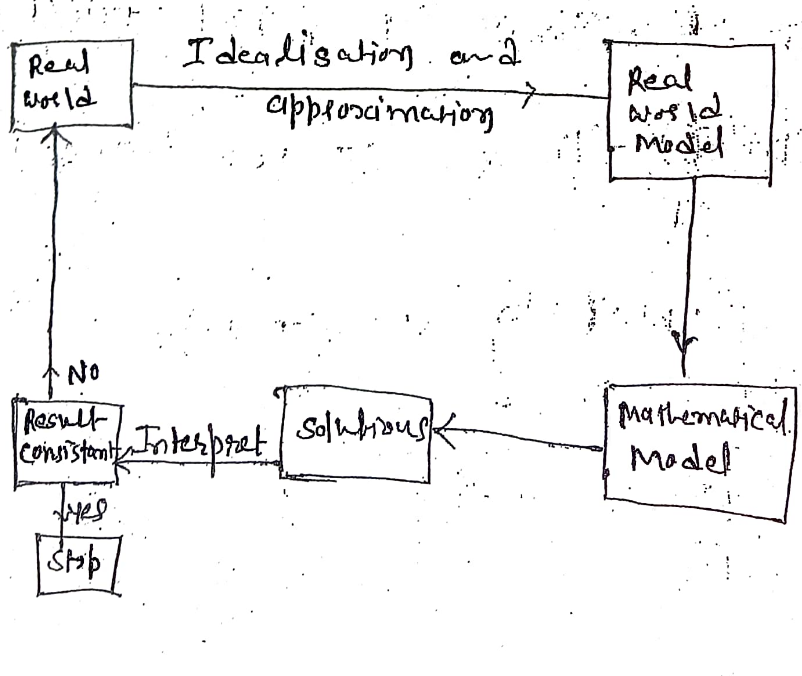
* Mathematical Model:

Mathematical modelling essentially consist of translating real world problems into mathematical problems, solving the mathematical problems and interpreting these solutions in the ~~large~~ language of real world



Model-I   Newton's Law of Cooling

According to Newton's law of cooling, the rate of change of temperature of a body ~~is~~ is proportional to the difference between the temperature of the body and temperature of the surrounding medium.

let T be the temperature of the body.

and T_s be the temperature of the surrounding medium

let $T(t)$ be the temperature at time t

and $T(t+\Delta t)$ be the temperature at time $t+\Delta t$

Hence,

$$(T(t+\Delta t)) - T(t) \propto (T-T_s).\Delta t + o(\Delta t)$$

$$\Rightarrow \frac{(T(t+\Delta t)) - T(t)}{\Delta t} \propto (T-T_s) + o(\Delta t)$$

$$\Rightarrow \lim_{\Delta t \to 0}\left[\frac{(T(t+\Delta t)) - T(t)}{\Delta t}\right] \propto (T-T_s)$$

$$\Rightarrow \frac{dT}{dt} \propto (T-T_s)$$

$$\Rightarrow \frac{dT}{dt} = K(T-T_s) \qquad K<0$$

Solutions

$$\frac{dT}{T-T_s} = k\,dt$$

$$\Rightarrow \int \frac{dT}{T-T_s} = \int k\,dt + c_1$$

$$\Rightarrow \log(T-T_s) = kt + c_1$$

$$\Rightarrow T-T_s = e^{kt+c_1}$$

$$\Rightarrow T-T_s = A e^{kt} \quad \text{--- (1)} \qquad [\because A = e^{c_1}]$$

Let, at $t=0$ $T(0)=T_0$ be the initial condition

$$\Rightarrow T_0 - T_s = A e^{0}$$

$$\Rightarrow T_0 - T_s = A \quad \text{--- (2)}$$

from (1) and (2)

$$T-T_s = (T_0 - T_s)\,e^{kt}$$

$$\Rightarrow \boxed{T = T_s + (T_0 - T_s)\,e^{kt}} \qquad k<0$$

This shows that the excess of the temperature of the body over that of the surrounding medium decays exponentially

Example-1 According to Newton's law of cooling, the rate at which a substance cools in moving air is proportional to the difference between the temperature of the substance and that of air. If the temperature of the air is 40°C and the substance cools from 80°C to 60°C in ...

Then find the temperature of substance at time 40 min...

Sol $T_s = 40\,°C$

Hence, $\frac{dT}{dt} = K(T-40)$

$T = T_s + (T_0 - T_s)e^{kt} \Rightarrow \cancel{T = T_s + (40 - T_s)e^{kt}}$

$T = 40 + (T_0 - 40)e^{kt}$

at $t=0$ $T = 80$

$\cancel{80 = T_s + (40 - T_s)}$

$\Rightarrow 80 = 40 + (T_0 - 40)e^{0}$

$\Rightarrow \boxed{80 = T_0}$

when $t = 20$ $T = 60$

$\Rightarrow 60 = 40 + (80 - 40)e^{20k}$

$\Rightarrow 20 = 40e^{20k}$

$\Rightarrow \frac{1}{2} = e^{20k}$

$\Rightarrow 20k = \log(1/2)$

$\Rightarrow k = \frac{1}{20}[\log 1 - \log 2]$

$\Rightarrow k = -\frac{1}{20}\log 2$

at $t = 40$

$T = 40 + (80-40)e^{(-\frac{1}{20}\log 2)\times 40}$

$= 40 + (40)\left[\cancel{e^{\log 2^{-1/20} \cdot 40}}\right] e^{-2\log 2}$

$= 40 + 40\left[\cancel{\quad}\right] e^{\log 2^{-2}}$

$T = 40 + 40 \times 2^{-2}$

$= 40 + \frac{40}{4}$

$= 40 + 10$

$\boxed{T = 50}$

Example: Water at temperature 100 °C cools in 10 (1

minutes to 88 °C in a room of temperature 25 °

Find the temperature of water after 20 minutes.

$T_0 = 100\ °C$

$T_s = 25\ °C$

$t = 10, \quad T = 88\ °C$

$\Rightarrow \quad T = T_s + (T_0 - T_s)\, e^{kt}$

$\Rightarrow \quad 88 = 25 + (100 - 25)\, e^{k10}$

$\Rightarrow \quad 88 - 25 = 75\, e^{k10}$

$\Rightarrow \quad \frac{63}{75} = e^{10k}$

$t = 20 \quad T = ?$

$\Rightarrow \quad T = T_s + (T_0 - T_s)\, e^{kt}$

$\Rightarrow \quad T = 25 + (100 - 25)\, e^{20k}$

$= 25 + 75\, (e^{10k})^2$

$T = 25 + 75 \left(\frac{63}{75}\right)^2$

$$\boxed{T = 77.92\ °C}$$

Example If the temperature of the air is 30 °C and the substance cools from 100 °C to 70 °C in 15 minute, find when the temperature will be 40 °C

[52.5 minute]

Example - A body where temperature T is initially 300 °C is placed in a large block of ice. Find its temperature at the end of 2 and 3 minutes ?

Model - II Simple Compartment Model

Let a vessel contain a volume V of a solution with concentration c(t) of a substance at time t. Let a solution with constant concentration c' in an overhead tank enter the vessel at a constant rate R and after mixing ~~thorou~~ thoroughly with the solution in the vessel, let the mixture with concentration c(t) leave the vessel at the same rate R, So that the volume of the solution in the vessel remains V.

Using the principle of Continuity

Let $c(t)$ be the concentration at t

and $c(t+\Delta t)$ be the concentration at $t+\Delta t$

we get,

$$V\,c(t+\Delta t) - V c(t) = R c_i \Delta t - R c(t)\Delta t + o(\Delta t)$$

$$\Rightarrow V\left[\frac{c(t+\Delta t) - c(t)}{\Delta t}\right] = R c_i - R c(t) + o(\Delta t)$$

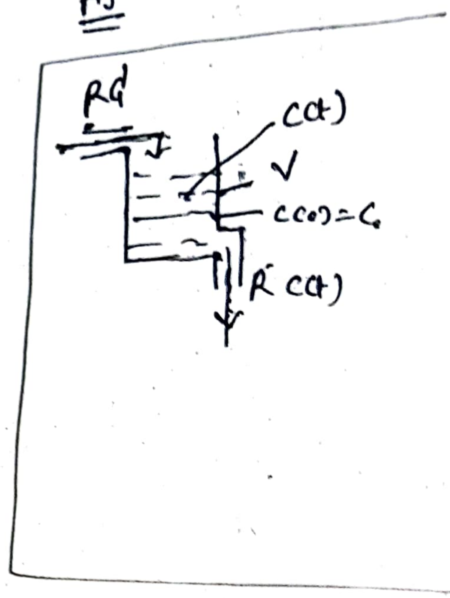
$$\Rightarrow V\left[\lim_{\Delta t \to 0} \frac{c(t+\Delta t) - c(t)}{\Delta t}\right] = R c_i - R c(t)$$

$$\Rightarrow V \frac{dc}{dt} = R c_i - R c(t)$$

$$\Rightarrow \frac{dc}{dt} = \frac{R}{V} c_i - \frac{R}{V} c(t)$$

$$\Rightarrow \boxed{\frac{dc}{dt} + \frac{R}{V} c(t) = \frac{R}{V} c_i}$$

Fig



~~at at t=0~~, ~~c(0) = c₀~~

Solution

$$I.F = e^{\int \frac{R}{V} dt} = e^{\frac{R}{V}t}$$

Hence,
$$c(t)\, e^{\frac{R}{V}t} = \int \frac{R}{V} c_i\, e^{\frac{R}{V}t} dt + A$$

$$\Rightarrow c(t)\, e^{\frac{R}{V}t} = \frac{R}{V} c_i \left[\frac{e^{\frac{R}{V}t}}{\frac{R}{V}}\right] + A$$

$$\Rightarrow c(t)\, e^{\frac{R}{V}t} = c_i\, e^{\frac{R}{V}t} + A$$

$$\Rightarrow c(t) = c_i + A e^{-\frac{R}{V}t} \quad \text{—— (1)}$$

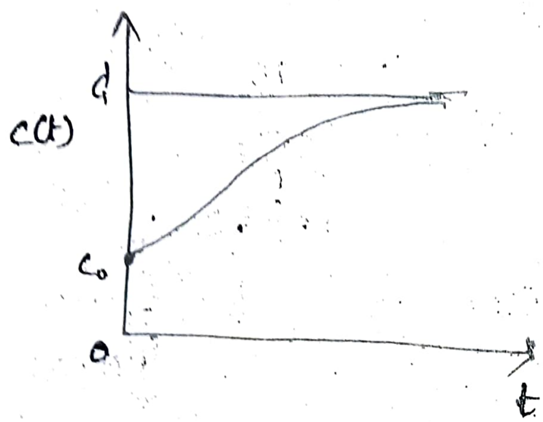
at t=0, $c(0) = c_0$

$$\Rightarrow \boxed{c_0 = c_i + A} \Rightarrow \boxed{A = c_0 - c_i} \quad \text{—— (2)}$$

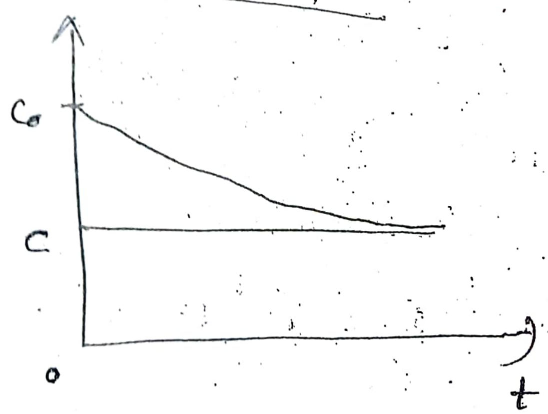
From ① and ② we get,

$$\boxed{C(t) = C_1 + [C_0 - C_1]\, e^{-\frac{R}{V}t}}$$

when ① $C_1 > C_0$ [Graphical Analysis]



② $C_1 < C_0$



③ $C = C_0$

example:-

A 2000 l

20 kg of d

solution with

Example:-1

A 2000 liters tank of water initially contains 20 kg of dissolved salt. A pipe brings salt-solution with concentration 0.04 kg/liters in to the tank at the rate 2 liters/second and a second pipe carries away the excess solution at the same rate. Calculate concentration of salt at any time with assumption that tank is well mixed.

Sol^n

$V = 2000$ liters

$C_0 = \frac{20}{2000} = 0.01$ kg/liters

$R = 2$, $C = 0.04$ kg/liters

$$C(t) = C + [C_0 - C]\, e^{-\frac{R}{V}t}$$

$$\Rightarrow C(t) = 0.04 + [0.01 - 0.04]\, e^{\frac{2}{2000}t}$$

$$\boxed{C(t) = 0.04 - 0.03\, e^{0.001t}}$$

Example:

A 8000 liter tank of water initially contains 5kg of dissolved potassium. A pipe brings a potassium solution with concentration 0.03 kg/liter in to the tank at the rate of 4 l/s and a second pipe carries away the excess solution with the same rate. Find the concentration of potassium at any time t in the tank.

$\Rightarrow$

$V = 8000$

$C_0 = 5/8000 = 0.000625$ kg/liter

$C' = 0.03$ kg/liter

$R = 4$

$$C(t) = C' + [C_0 - C']e^{-\frac{R}{V}t}$$

$$\Rightarrow \boxed{C(t) = 0.03 + [0.000625 - 0.03]e^{-\frac{4}{8000}t}}$$

~~Example~~:

Example: A 4000 liter tank of water ~~init~~ initially contains 4 kg of dissolved salt. A pipe brings a salt solution with concentration 0.02 kg/liter in to the tank at the rate of 2 l/s and a second pipe carries away the excess solution with the same rate. then Calculate concentration

Model: III

Differential Equation Model for Epidemi Sprade

Epidemic of infectious disease have always been a subject of great concern for the welfare of Community. Here, we introduce and discussed there simple differential equation model for the sprade of epidemic

(I) The SI Model (Susceptibles - Infective)

(II) The SIS Model (Susceptibles - Infective - Susceptibles)

(III) The SIR Model (Susceptibles - Infectives - Removed)

Problem Statement

" Consider a situation where a small group of people sufferening with infectious disease is inserted in to a large population which is Capable of catching diseases. Formulate a suitable differential Equation model and analyse it."

Step-I We partition the population N in to four mutually exclusive group.

(1) The susceptibles: (S): The person who are currently not infected but they are capable for

catching the disease

(2) The Latently Infected: (L):
Those person who are currently infected but not yet capable of transmitting the disease to other

(3) The Infectives: (I):
Those person who are currently afected and capable of transmitting it to others

(4) The Removeds (R):
Those person who have had the diseases and either dead or have acquired permanent immunity from the disease or isolated from the community.

Here we consider, S, L, I, R are function of time because the ~~num~~ number of person in this group changes during the period of epidemics.

Step-II We assume that—

(1) S, L, I, R are continious variable though they are in fact integer value

$$S(t) + I(t) + L(t) + R(t) = N \quad \text{—— (1)}$$

(2) The rate of change of susceptible population

$$\frac{ds}{dt} = -\beta\, I(t) \cdot S(t) \quad \forall\ t > 0 \qquad \text{(2)}$$

(14)

where β is positive constant called the infection rate

(3) The Latent period of disease is negligible that is

$$L(t) = 0 \quad \forall\ t > 0. \qquad \text{(3)}$$

(I) S-I Model

this is simplest model in which we make an additional assumption that no removal from the population is made during the epidemic

that is

$$R(t) = 0 \quad \forall\ t > 0 \qquad \text{(4)}$$

Hence, we get

$$S(t) + I(t) = N \qquad \text{(5)}$$

Assuming that the epidemic started with I_0 ie $t=0$, $I(0) = I_0$ infected person,

so we get, the following initial value problem

$$\frac{dI}{dt} = \beta I S \qquad I(0) = I_0$$

$$\Rightarrow \frac{dI}{dt} = \beta I (N - I), \quad I(0) = I_0 \qquad \text{(6)}$$

similarly If we are interested in number of suceptible the

$$\frac{ds}{dt} = -\beta I S \qquad S(0) = S_0$$

$$\Rightarrow \frac{ds}{dt} = -\beta S(N-S) \qquad S(0) = S_0 \qquad (7)$$

<u>solution of SI-Model</u>

From eq (1)

$$\frac{dI}{dt} = \beta I(N-I), \qquad I(0) = I_0$$

$$\Rightarrow \frac{dI}{I(N-I)} = \beta\, dt$$

$$\Rightarrow \frac{1}{N}\left[\frac{1}{N-I} + \frac{1}{I}\right] dI = \beta \cdot dt$$

$$\Rightarrow \left[\frac{1}{N-I} + \frac{1}{I}\right] dI = \beta N\, dt$$

$$\Rightarrow -\log(N-I) + \log I = \beta N t + A$$

$$\Rightarrow \log\left(\frac{I}{N-I}\right) = \beta N t + A$$

$$\Rightarrow \frac{I}{N-I} = e^{\beta N t + A}$$

$$\Rightarrow \frac{I}{N-I} = c\, e^{\beta N t} \qquad (8) \qquad [\because e^{A} = c]$$

similarly if we are interested in number of suceptible the

$$\frac{ds}{dt} = -\beta I S \qquad S(0) = S_0$$

$$\Rightarrow \frac{ds}{dt} = -\beta S (N-S) \qquad S(0) = S_0 \qquad (7)$$

Solution of SI-Model

From eq (1)

$$\frac{dI}{dt} = \beta I (N-I), \qquad I(0) = I_0$$

$$\Rightarrow \frac{dI}{I(N-I)} = \beta\, dt$$

$$\Rightarrow \frac{1}{N}\left[\frac{1}{N-I} + \frac{1}{I}\right] dI = \beta \cdot dt$$

$$\Rightarrow \left[\frac{1}{N-I} + \frac{1}{I}\right] dI = \beta N\, dt$$

$$\Rightarrow -\log(N-I) + \log I = \beta N t + A$$

$$\Rightarrow \log\left(\frac{I}{N-I}\right) = \beta N t + A$$

$$\Rightarrow \frac{I}{N-I} = e^{\beta N t + A}$$

$$\Rightarrow \frac{I}{N-I} = c\, e^{\beta N t} \qquad [\because e^{A} = c] \qquad (8)$$

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when $t=0$, $I(0)=I_0$

$$\Rightarrow \frac{I_0}{N-I_0} = c \quad \text{--- (9)}$$

from (9) and (8) we get,

$$\frac{I}{N-I} = \frac{I_0}{N-I_0} e^{BNt} \quad \text{--- (10)}$$

$$\Rightarrow I = (N-I) \frac{I_0}{N-I_0} e^{BNt}$$

$$\Rightarrow I + I\left(\frac{I_0}{N-I_0} e^{BNt}\right) = N\left(\frac{I_0}{N-I_0} e^{BNt}\right)$$

$$\Rightarrow \left[1 + \frac{I_0}{N-I_0} e^{BNt}\right] I = \frac{N \cdot I_0}{N-I_0} e^{BNt}$$

$$\Rightarrow \left[\frac{N-I_0 + I_0 e^{BNt}}{N-I_0}\right] I = N \cdot \frac{I_0}{N-I_0} e^{BNt}$$

$$\Rightarrow I(t) = \frac{N \cdot I_0 e^{BNt}}{[N - I_0 + I_0 e^{BNt}]}$$

$$I(t) = \frac{N}{\left[1 + \left(\frac{N}{I_0} - 1\right) e^{-BNt}\right]} \quad \text{--- (11)}$$

Again $S(t) = N - I(t)$

$$= N - \frac{N}{\left[1 + \left(\frac{N}{I_0} - 1\right) e^{-BNt}\right]} = \frac{N\left[1 + \left(\frac{N}{I_0} - 1\right) e^{-BNt}\right] - N}{\left[1 + \left(\frac{N}{I_0} - 1\right) e^{-BNt}\right]}$$

$$\therefore S(t) = \frac{N\left(\frac{N}{I_0}-1\right)e^{-BNt}}{\left[1+\left(\frac{N}{I_0}-1\right)e^{-BNt}\right]}$$

$$\Rightarrow S(t) = \frac{N\left(\frac{N}{I_0}-1\right)}{\left[e^{BNt}+\left(\frac{N}{I_0}-1\right)\right]} \quad \text{——— (12)}$$

<u>Analysis of SI model</u>

From eq. (11)

$$I(t) = \frac{N}{\left[1+\left(\frac{N}{I_0}-1\right)e^{-BNt}\right]}$$

$$\Rightarrow I'(t) = \frac{0 - N\left[0+\left(\frac{N}{I_0}-1\right)e^{-BNt}\times(-BN)\right]}{\left[1+\left(\frac{N}{I_0}-1\right)e^{-BNt}\right]^2}$$

$$= \frac{BN^2\left(\frac{N}{I_0}-1\right)e^{-BNt}}{\left[1+\left(\frac{N}{I_0}-1\right)e^{-BNt}\right]^2} \quad \text{——— (13)}$$

Again from eq (2)

$$I'(t) = BI(N-I) \quad \text{——— (14)}$$

$$\Rightarrow I''(t) = BI(0-I') + BI'(N-I)$$

$$= -BII' + BI'N - BII'$$

$$I''(t) = BI'N - 2BII'$$

$$I''(t) = BI'(N-2I) \quad \text{——— (15)}$$

* SIS Model

This is the slightly modified version of SI model. Here we consider the infected individual can recover and becomes susceptibles at a rate λ, where λ is positive constant.

Then model becomes

$$\frac{dI}{dt} = \beta I (N - I) - \lambda I \quad , \quad I(0) = I_0$$

$$\Rightarrow \frac{dI}{dt} = \beta I \left(N - I - \frac{\lambda}{\beta}\right)$$

$$\Rightarrow \frac{dI}{dt} = \beta I \left(\left(N - \frac{\lambda}{\beta}\right) - I\right)$$

$$\Rightarrow \frac{dI}{dt} = \beta I (a - I) \quad \text{where } a = N - \frac{\lambda}{\beta}$$

$$\Rightarrow \frac{dI}{dt} = \beta I (a - I), \quad I(0) = I_0 \qquad (1)$$

Solution of the model

$$\frac{dI}{I(a - I)} = \beta \cdot dt$$

$$\Rightarrow \frac{1}{a}\left[\frac{1}{a - I} + \frac{1}{I}\right] = \beta dt$$

essoy of
fected individual
δI

$$\Rightarrow \left[\frac{1}{a-I} + \frac{1}{I}\right] = a\beta . dt$$

$$\Rightarrow -\log(a-I) + \log I = a\beta t + A$$

$$\Rightarrow \log\left(\frac{I}{a-I}\right) = a\beta t + A$$

$$\Rightarrow \frac{I}{a-I} = e^{a\beta t + A} = e^{a\beta t} \cdot e^{A}$$

$$\Rightarrow \frac{I}{a-I} = c\, e^{a\beta t} \quad [\because e^{A} = c] \quad \text{--- (2)}$$

at $t = 0$, $I(0) = I_0$

$$\Rightarrow \frac{I_0}{a-I_0} = c \quad \text{--- (3)}$$

From (2) and (3) we get

$$\frac{I}{a-I} = \frac{I_0}{a-I_0} e^{a\beta t}$$

$$\Rightarrow I = (a-I)\frac{I_0}{a-I_0} e^{a\beta t}$$

$$\Rightarrow I + I\left(\frac{I_0}{a-I_0}\right)e^{a\beta t} = a \frac{I_0}{a-I_0} e^{a\beta t}$$

$$\Rightarrow \left[1 + \frac{I_0}{a-I_0} e^{a\beta t}\right] I = a \cdot \frac{I_0}{a-I_0} e^{a\beta t}$$

$$\Rightarrow \left[(a - I_0) + I_0 e^{a\beta t}\right] I = a I_0 e^{a\beta t}$$

$$\Rightarrow \left[(a-I_0) + I_0 e^{aBt} \right] I = a I_0 e^{aBt}$$

$$\Rightarrow I(t) = \frac{a I_0 e^{aBt}}{\left[(a-I_0) + I_0 e^{aBt} \right]}$$

$$I(t) = \frac{a}{\left[1 + \left(\frac{a}{I_0} - 1 \right) e^{-aBt} \right]} \quad \text{—— (4)}$$

similarly, we can find $S(t)$, and analyze the model as SI

SIR - model

This is the slightly modified ~~model~~ version of SI model. In this model we assume that the individuals are removed from the infective class at a rate γ which is proportional to the number of infectives.

Hence,

$$\frac{dR}{dt} = \gamma . I \quad \forall\ t > 0. \quad \text{—— (1)}$$

This removal may be on account of death or permanent immunity or isolation from the community

In this model we have

$$S(t) + I(t) + R(t) = N \quad \text{——— (2)}$$

and equations are

$$\frac{dI}{dt} = \beta I S - \gamma I \quad \text{——— (3)}$$

$$\frac{dS}{dt} = -\beta I S \quad \text{——— (4)}$$

$$\frac{dR}{dt} = \gamma I \quad \text{——— (5)}$$

& the model