Posets and Lattices

CHAPTER 9

9.1. Introduction

In the chapter 7 we discussed various types of relations that can be defined on a set. Now, we narrow down our interest to partial order relation which is defined on a set called a partially ordered set. This would finally lead the concepts of lattices. We shall show they could equivalently be introduced as algebraic systems possesing some specific properties.

9.2. Partially Ordered Sets

A relation R on a set S is called a partial ordering if it is reflexive, antisymmetric and transitive. That is

1. $a\text{R}a$ for all $a \in$ S (reflexivity)
2. $a\text{R}b$ and $b\text{R}a \Rightarrow a = b$ (antisymmetry)
3. $a\text{R}b$ and $b\text{R}c \Rightarrow a\text{R}c$ (transitivity)

A set S together with a partial order R is called a **partially ordered set** or a **poset**. It is denoted by (S, R).

The most familiar order relation, called the usual order is the relation ≤ on the positive integer Z^+ or, more generally, on any subset of the real numbers R. For this reason, the symbol ≤ is often used to refer to a general partial order relation. If there is already a notation, such ≤ or ⊆, for a partial order, then we will generally use it in preference to ≤.

Example 1. Show that the relation ≥ is a partial ordering on the set of integers.

Solution. Since (1) $a \geq a$ for every a, ≥ is reflexive.

(2) $a \geq b$ and $b \geq a$ imply $a = b$, ≥ is antisymmetric

(3) $a \geq b$ and $b \geq c$ imply $a \geq c$, ≥ is transitive

it follows that ≥ is a partial ordering on the set of integers and $(\mathbb{Z}, \geq)$ is a poset.

Example 2. Consider P (S) as the power set *i.e.* the set of all subsets of a given set S. Show that the inclusion relation ⊆ is a partial ordering on the power set P (S).

Solution. Since (1) $A \subseteq A$ for all $A \subseteq S$, ⊆ is reflexive.

(2) $A \subseteq B$ and $B \subseteq A$ imply $A = B$, ⊆ is antisymmetric.

(3) $A \subseteq B$ and $B \subseteq C$ imply $A \subseteq C$, ⊆ is transitive.

it follows that ⊆ is a partial ordering on P (S) and (P (S), ⊆) is a poset.

Example 3. Show that the set Z^+ of all positive integers under divisibility forms a poset.

Solution. Since (1) $n \mid n$ for all $n \in Z^+$, 1 is reflexive.

(2) $n \mid m$ and $m \mid n$ imply $n = m$, 1 is antisymmetric.

(3) $n \mid m$ and $m \mid p$ imply $n \mid p$, 1 is transitive.

it follows that 1 is a partial ordering on Z^+ and (Z^+, 1) is a poset.

Note. On the set of all integers, the above relation is not a partial order as a and $(-a)$ both divide each other without being equal.

Comparability

The elements a and b of a poset (S, ≤) are called comparable if either $a \leq b$ or $b \leq a$. When a and b are elements of S such that neither $a \leq b$ nor $b \leq a$, a and b are called incomparable. For example, in the poset (Z^+,/) the integers 2 and 4 are comparable, since 2/4 but 3 and 5 are incomparable, because neither 3 × 5 nor 5 × 3.

If (S, ≤) is a poset and every two elements of S are comparable, S is called a **totally ordered** or **linearly ordered set**, and ≤ is called a **total order** or a **linear order**. A totally ordered set is also called a **chain**. For example, in the poset (z, ≤). $a \leq b$ or $b \geq a$ for all integers a and b, hence ($\mathbb{Z}$, ≤) is totally ordered but the poset ($\mathbb{Z}^+$, /) is not totally ordered since it contains elements that are incomparable such as 3 and 5.

Although an ordered set S may not be totally ordered, it is still possible for a subset A of S to be totally ordered. Clearly, every subset of a totally ordered set S must also be totally ordered. Consider the poset (Z^+, /) which is not totally ordered but A = {2, 6, 12, 36} is a totally ordered subset of Z^+ since 2/6, 6/12 and 12/36.

Product and Lexicographic Order

One way to combine two sets into a new one is to form their product. Suppose that (S, $\preceq$) and (T, $\preceq$) are posets, where we use the subscripts to keep track of which partial order is which. There is more than one natural way to make S × T into a poset. Two of these ways follow.

(*a*) **Product order.** For $s, s' \in$ S and $t, t' \in$ T, the product set S × T defined by

$(s, t) \preceq (s', t')$ if $s \preceq s'$ in s and $t \preceq_2 t'$ in T, where (s, t) and $(s', t') \in$ S × T

is called the product order.

Theorem 9.1. If $(A, \preceq)$ and $(B, \preceq)$ are posets, then $(A \times B, \preceq)$ is a poset, with partial order $\preceq$ defined by

$(a, b)' \preceq (a', b')$ if $a \preceq a'$ in A and $b \preceq b'$ in B.

Proof. If $(a, b) \in A \times B$, then (1) $(a, b) \preceq (a, b)$ since $a \preceq a$ in A and $b \preceq b$ in B. Hence, $\preceq$ is reflexive in $A \times B$.

(2) For $(a, b) \preceq (a', b')$ and $(a', b') \preceq (a, b)$ where a and $a' \in A$ and b and $b' \in B$, we have

$a \preceq a'$ and $a' \preceq a$ in A imply $a = a'$

and $b \preceq b'$ and $b' \preceq b$ in B imply $b = b'$

since A and B are posets.

Hence, $\preceq$ is antisymmetric in $A \times B$.

(3) for $(a, b) \preceq (a', b')$ and $(a', b') \preceq (a'', b'')$ where $a, a', a'' \in A$ and $b, b', b'' \in B$, we have

$a \preceq a'$ and $a' \preceq a''$ in A imply $a \preceq a''$

and $b \preceq b'$ and $b' \preceq b''$ in B imply $b \preceq b''$

since A and B are posets.

Hence, $(a, b) \preceq (a'', b'')$ which shows that transitive property holds in $A \times B$.

This proves that $(A \times B, \preceq)$ is a poset.

The same concept can be extended for any finite of posets. If $(A_1, \preceq_1)$, $(A_2, \preceq_2)$, ..., $(A_n, \preceq_n)$ are n posets, then $(A_1 \times A_2 \times \cdots\cdots A_n, \preceq)$ is a poset, where $\preceq$ is defined on $A_1 \times A_2 \times \cdots\cdots \times A_n$ as

$(a_1, a_2, \ldots\ldots, a_n) \preceq (b_1, b_2, \ldots\ldots, b_n)$ if and only if $a_1 \preceq b_1$ in A_1; $a_2 \preceq b_2$ in A_2, ..., as $\preceq b_n$ in An.

Lexicographic Order : Suppose $(S, \preceq_1)$ and $(T, \preceq_2)$ are are posets. The cartesian product $S \times T$ defined by $(s, t) < (s', t')$ either if $s < s'$ or if both $s = s'$ and $t <_2 t'$, i.e. one pair is less than a second pair if the first entry of the first pair is less than (in s) the first entry of the second pair or if the first entries are equal, but the second entry of this pair is less than (in T) the second entry of the second pair.

This order can be extended to $S_1 \times S_2 \times \ldots\ldots \times S_n$ as follows :

$(s_1, s_2, \ldots\ldots, s_n) < (s_1', s_2', \ldots\ldots, s_n')$ if $s_i = s_i'$ for $i = 1, 2, \ldots\ldots, k-1$ and $s_k < s_k'$.

Thus, the first coordinate dominates except for equality, in which case we consider the second coordinate. If equality holds again, we pass to be next coordinate, and so on.

Note that the lexicographic order is linear.

Example 4. *Determine whether (a) (3, 7) < (4, 8) and (b) (3, 9) < (3, 11) in the poset* ($\mathbb{Z} \times \mathbb{Z}$, $\leq$) where $\leq$ is the lexicographic ordering constructed from the relation $\leq$ on $\mathbb{Z}$.

Solution. (*a*) Since 3 < 4, it follows that (3, 7) < (4, 8)

(*b*) Since 3 = 3 and 9 < 11, it follows that (3, 9) < (3, 11)

We now define lexicographic ordering of strings. Consider the strings $w = a_1, a_2, \ldots\ldots a_m$ and $w' = b_1, b_2, \ldots\ldots b_n$ on a partially ordered set S. Let $K = \min(m, n)$. From the definition of lexicographic ordering the string w is less than w' if and only if

$(a_1, a_2, \ldots\ldots a_k) < (b_1, b_2, \ldots\ldots, b_k)$ or

$(a_1, a_2, \ldots\ldots a_k) = (b_1, b_2, \ldots\ldots, b_k)$ and $m < n$.

The idea is to compute two words $A = \{a_1, a_2, \ldots\ldots, a_m\}$ and $B = \{b_1, b_2, \ldots\ldots, b_n\}$ character by character, passing over equal characters. If at any point the passing over equal characters. If at any point the A - cahracter alphabetically preceds the corresponding B- character, then, precedes B; if all characters in A are equal to the corresponding B characters but we run out of characters in A before characters in B, then A precedes B. Otherwise B precedes A.

Example. *Suppose the lowercase English alphabet*

A = {a, b, c, ..., x, y, z} is ordered with usual order and A denotes the set of all strings from the elements of A. Sort the following elements :*

(i) Lexicorgraphically order (ii) usual alphabetical order from, were, forget, so, toast, she, discrete, for solution (i) First order the elements by length and then order them lexicographically.

so, for, she, from, were, taost, forget, discrete

(ii) discrete, for, forget, from, she, so, toast, were.

Theorem 9.2. Let $(S_1, \preceq_1), (S_2, \preceq_2), \ldots\ldots\ldots\ldots (S_n, \preceq_n)$ be chains, then the lexicographic order $<$ on $S_1 \times S_2 \times \ldots\ldots\ldots\ldots\ldots \times S_n$ is also a chain.

Proof. We know that $<$ is a partial order on $S_1 \times S_2 \times \ldots\ldots\ldots\ldots\ldots \times S_n$.

Now let $(s_1, \ldots\ldots\ldots, s_n)$ and $(t_1, \ldots\ldots\ldots, t_n)$ be distinct elements in $S_1 \times \ldots\ldots\ldots \times S_n$. Since $s_r \neq t_r$ for some r, there is a first r for which $s_r \neq 1$. Since $(S_r, \preceq_r)$ is a chain, either $s_r \prec_r t_r$ or $t_r \prec_r s_r$. In the first case $(s_1, \ldots\ldots\ldots, s_n) \prec (t_1, \ldots\ldots\ldots\ldots, t_n)$; in the second $(t_1, \ldots\ldots\ldots, t_n) \prec (s_1, \ldots\ldots\ldots, s_n)$. In either case, the two elements of $S_1 \times \ldots \times S_n$ are comparable.

Definition. Let $(X, \preceq)$ be a poset and suppose $x, y \in X$, then y is said to be **immediate successor** of x if $x \prec y$ and $x \preceq z \preceq y \Rightarrow x = z$ or $z = y$.

We also say y covers x or x is an immediate predecessor of y. What it says that there is no intermediate element between x and y which is distinct from both. Now immediate successor or predecessor may or may not exist for any given elements.

For example if X = Q or R ur der the natural order.

We know that no element has an immediate predecssor or successor. But if X = Z the every element has such successor or predecessor. It is clear that if $(X, \preceq)$ is a finite set, then an element x either does not have a successor *i.e.* there is no y such that $x < y$ (in this case x is said to be a maximal element), or else x must have immediate successor, for if not, then we would get an infinite chain of successor of x contradicting the finiteness of X. Similarly for predecessors.

Representation and Hasse diagrams

A partial order $\preceq$ on a set X can be represented by means of a diagram known as Hasse diagram of (X, $\preceq$).

This gives a method of representing finite posets which works well for posets with relatively few elements. We represent the elements of X by points and if y is an immediate successor of x, we take y at a higher level than x and join x and y by a straight line. A diagram formed as above is known as a Hasse diagram. Thus there will not be any horizontal lines in the diagram of a poset.

Note that the Hasse diagram for a total order relation can be drawn as a single vertical chain.

Example 5. Let X = {1, 2, 3, 4, 5, 6}, then / is a partial order relation on X. Draw the, Hasse diagram of (X, /).

Solution. The diagram in Fig 9.2 is a Hasse diagram of the poset (X, /). There is no segment between 1 and 6 because 6 does not cover 1. One can see 1/6 through chain of segments corresponding to 1/2 and 2/6 because the relation is transitive.

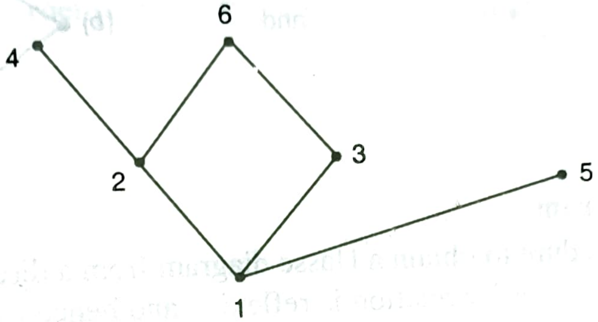
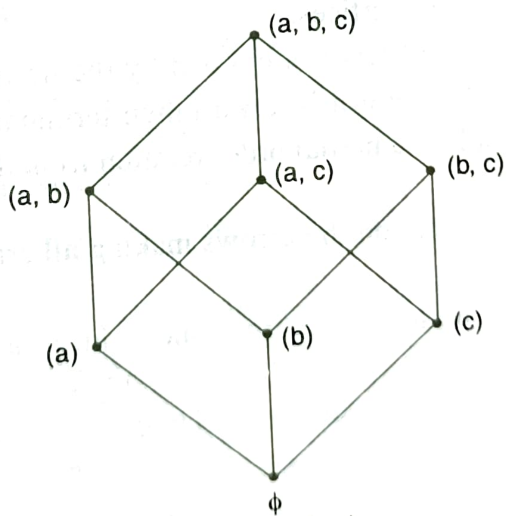


Fig. 9.2

Example 6. Draw the Hasse diagram for the partial ordering {(A, B) | A $\subseteq$ B} on the power set P (S) where S = $\{a, b, c\}$.

Solution. Here P (S) = $\{\varphi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$. The Hasse diagram of the poset (P (S), $\subseteq$) is shown in Fig. 9.3.



(P {a, b, c}, ⊂)

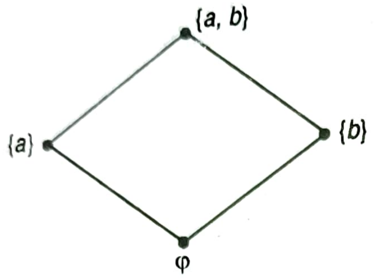
Fig. 9.3.

Example 7. Let A = {1, 3, 9, 27, 81}, draw the Hasse diagram of the poset (A, /).

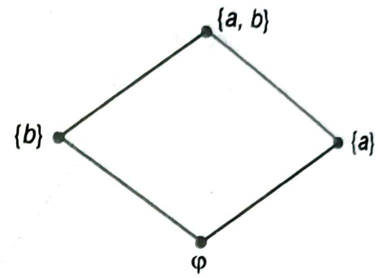
Solution. The Hasse diagram of the poset is shown in Fig. 9.4.

Fig. 9.4

Note that Hasse diagram for a given set is not unique. For example, Hasse diagram of (P(A), $\subseteq$) for A = $\{a, b\}$ are



and



Constructing a Hasse Diagram

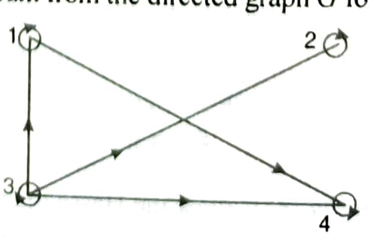
Here we discuss a procedure to obtain a Hasse diagram from a directed graph of a partial order relation. We know that a partial order relation is reflexive and hence its directed graph has loop on each vertex in the diagram. Consequently these loops can be frcm the directed graph of a partial order relation since they must be present. Because a partial ordering is transitive, the edges showing the implied transitivity can be. If we assume that all the edges are pointed upwards then we do not have to show the direction of the edges. In general we can represent a partial ordering on a finite set using the following procedure.

1. Start with a directed graph of the relation.
2. Remove the loops at all the vertices.
3. Remove all edges whose existence is implied by the transitive property.
4. Arrange all arrows pointing upwards toward their terminal vertex. Remove all the arrows.

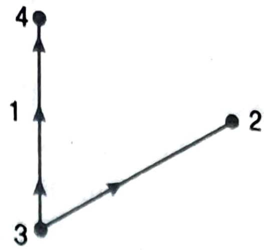
To recover the directed graph of a partial order relation from the Hasse diagram, the following procedure can be used.

1. Reinsert the direction makers on the arrows making all arrows point upward.
2. Add loop at each vertex.
3. For each sequence of pointing edges from one point to a second point and from second point to third point, add an edge from first to the third pointing toward third point.

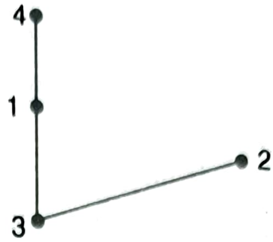
Example. Draw a Hasse diagram from the directed graph G for a relation on a set $A = \{1, 2, 3, 4\}$.



Solution. First we remove all the loops at the vertices 1, 2, 3 and 4 and the edge 3 and 4 which is transitively implied. Arranging all the arrows pointing upward we get the diagram as

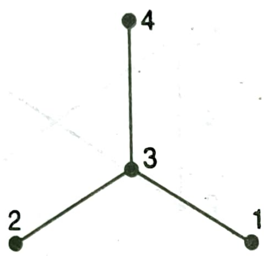


Now, removing all the arrows we get the required Hasse diagram as shown below.



Example. Describe the order pairs in the relation determined by the Hasse diagram on the set $A = \{1, 2, 3, 4\}$.

(a)



(b)

Solution. The ordered pairs in the relation represented by Hasse diagram (*a*) is

$$\{(11), (2, 2), (3, 3), (4, 4), (1, 3), (2, 3), (1, 4), (2, 4), (3, 4)\}.$$

The ordered pairs in the relation represented by Hasse diagram (*b*) is

$$\{(11), (2, 2), (3, 3), (4, 4), (1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}.$$

Special Elements in Posets

Let $(P, \preceq)$ be a poset. An element a is the **greatest element** of P if $x \preceq a$ for all $x \in P$. The greatest element, if it exists, is unique. For if a and a' are two greatest elements of P, then we should have $a' \preceq a$ and $a \preceq a'$. Hence by antisymmetry $a = a'$. Note that the greatest element, if it exists, will be comparable with all elements of the poset.

Similarly, an element $b \in P$ is called the **least element** if $b \preceq x$ for all $x \in P$. The least element is unique if it exists.

An element a in the poset is called a **maximal element** of P if $a < x$ for no x in P, that is, if no element of P strictly suceeds a. Similarly an element b in P is called a minimal element of P if $x < b$ for no x in P. Maximal and minimal elements are easy to spot using a Hasse diagram. They are the top and bottom elements in the diagram. That is, a maximal element has no connections leading up and a minimal element has no conncections leading down. A greatest element is connected to every other element by a path leading down and a least element is connected to every other element by a path leading up. The following points are to be noted :

(*a*) A poset may not have a maximal element. For instance, the natural numbers under usual $\leq$ have no maximal element.

(*b*) A poset may have more than one maximal element. In the poset $\{2, 3, 4, 6\}$ under divisibility, 4 and 6 are both maximal elements.

(*c*) Maximal element may not be the greatest element. In above 4 and 6 are maximal but neither 4 nor 6 is the greatest element.

(*d*) In a totally ordered set the concepts of minimal and least coincide, as do those of maximal and greatest.

Example 8. Let X = $\{a, b, c\}$. Then (P (S), $\subseteq$) is a poset.

(*a*) Let
$$A = \{\varphi, \{b\}, \{c\}, \{a, c\}\}$$
Then (A, $\subseteq$) is a poset with φ as least element and A has no greatest element.

(*b*) Let
$$B = \{\{b\}, \{c\}, \{a, b\}, \{a, b, c\}\}.$$
Then (B, $\subseteq$) is a poset with $\{a, b, c\}$ as greatest element and B has no least element.

(*c*) Let
$$C = \{\varphi, \{a\}, \{b\}, \{a, b\}\}$$
Then (C, $\subseteq$) is a poset with φ as least and $\{a, b\}$ as greatest element.

Example 9. Find the least and greatest element in the poset (Z^+, /), if they exist.

Solution. The least element of the poset (Z^+, /) is 1 since $1/n$ whenever n is a positive integer. There is no greatest element since there is no integer which is divisible by all positive integers.

Example 10. Determine whether the posets represented by Hasse diagrams in Fig. 9.5 have a greatest element, least element, minimal element and maximal elements.

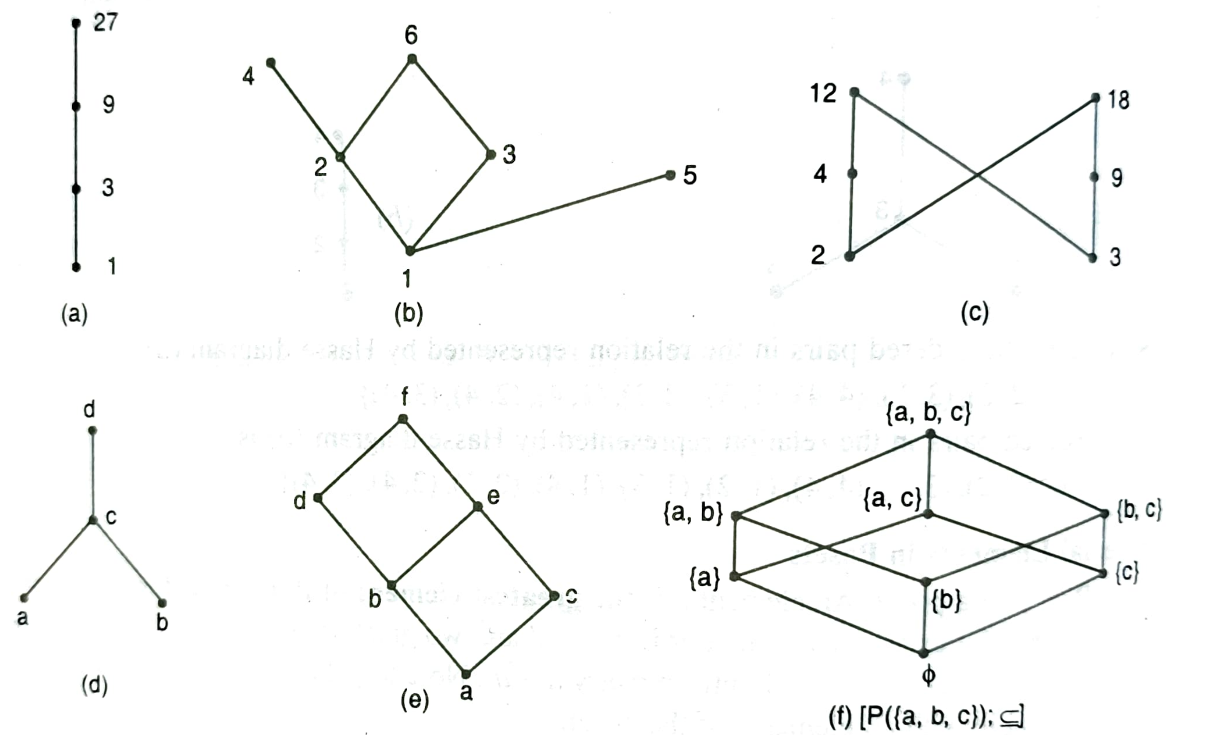


Fig. 9.5

Solution. (*a*) The least element of the poset with Hasse diagram (*a*) is 1 and is the only minimal element. The greatest element is 27 and is the only maximal element.

(*b*) The least element of the poset with Hasse diagram (*b*) is 1 and is the only minimal element. There is no greatest element and 4, 6 and 5 are maximal elements.

(*c*) The poset with Hasse diagram (*c*) has neither greatest element nor least element. Two minimal elements are 2 and 3 and two maximals are 12 and 18.

(*d*) The greatest element of the poset with Hasse diagram (*d*) is *d* and is the only maximal element. There is no least element and *a* and *b* are minimal elements.

(*e*) The greatest element of the poset with Hasse diagram (*e*) is *f* and is the only maximal element. The least element is a and is the only minimal element.

(*f*) The greatest element of the poset with Hasse diagram (*f*) is $\{a, b, c\}$ and is the only maximal element. The Least element is φ and is the only minimal element.

Well-Ordered Set

A set with an ordering relation is well-order if every non-empty subset of the set has a least element. For example, the set of natural numbers is well-ordered. The set of integers $\mathbb{Z}$ is not well-ordered since the set of negative integers, which is a subset of $\mathbb{Z}$, has no least element.

Upper and lower bound

Let B be a subset of a poset (A, $\preceq$). An element $u \in$ A is called an upper bound of B if u succeeds every element of B *i.e.* $x \preceq u$ for all $x \in$ B. An element $l \in$ A is called a lower bound of B if l precedes every element of B *i.e.* $l \preceq x$ for all $x \in$ B.

An element $a \in$ A is called a **least upper bound** (lub) of B if a is an upper bound of B and $a \preceq a'$, when ever a' is the upper bound of B. Thus a = lub (B) if $b \preceq a$ for all $b \in$ B, and if whenever $a' \in$ A is also an upper bound of B, then $a \preceq a'$.

Similary, an element $a \in$ A is called the **greatest lower bound** (glb) of B if a is a lower bound of B and $a' \preceq a$, whenever a' is a lower bound of B. Thus a = glb (B) if $a \preceq b$ for all $b \in$ B, and if whenever $a' \in$ A is also a lower bound of B, then $a' \preceq a$.

Some texts use the term **supremum** of A instead of the term least upper bound and write sup(A) instead of lub(A) and use the term **infimum** of A instead of the term greatest lower bound and write inf(A) instead of glb(A).

The following points are to be noted

1. A subset B of a poset may or may not have upper or lower bounds.
2. An upper or lower bound may or may not belong to B itself.
3. There can be more than one upper bound and lower bound of a set.
4. The greatest element is always the supremum but the converse is not true. In fact a = sup (B) is the greatest element *iff* $a \in$ B.
5. The least element is always the infimum but the converse is not true. In fact b = inf(B) is the least element iff $b \in$ B.

Example 11. In the poset A = ({1, 2, 3, 4, ..., 10}, 1) the subset {2, 7} has no upper bound since there is no integer in A which is divisible by both 2 and 7. The lower bound of the subset is 1 since 2 and 7 are divisible by only 1. Hence, 1 is the greatest lower bound for {2, 7} *i.e.* glb {2, 7} = 1.

The subset {1, 2, 3} has 6 and 1 as unique upper and lower bounds. Hence lub {1, 2, 3} = 6 and glb {1, 2, 3} = 1.

The subset {1, 2, 4} has 4 and 8 as upper bounds. Hence lub {1, 2, 4} = 4.

Definition. Let *A* be a set that is partially ordered with respect to a relation. A subset *B* of *A* is called a chain if, and only if, each pair of elements in *B* is comparable. Thus, if *A* is a chain and $x, y \in A$ then either $x \le y$ or $y \le x$. The **length of a chain** is one less than the number of elements in the chain (*i.e.*, length is the number of links that the chain has.) A **maximal chain** is a chain that is not a subset of a larger chain. A subset of a poset is called an **antichain** if every two elements of this subset are incomparable.

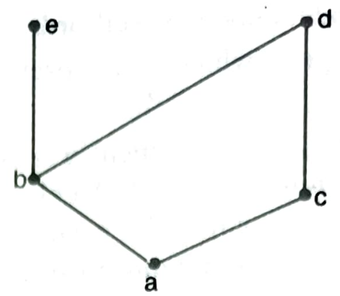
Example 12. *Consider the poset [P ({a, b, c}), ≤] with the Hasse diagram shown in Fig. 9.3, find a chain of length 3 in the poset.*

Solution. Since $\varphi \le \{a\} \le \{a, b\} \le \{a, b, c\}$, the set $S = \{\varphi\ \{a\}\ \{a, b\}\ \{a, b, c\}\}$ is a chain of length 3 in [P ({a, b, c}), ≤]. There are total six chains of length 3.

Example 13. *Consider the pentagon lattice with the Hasse diagram shown in Fig. 9.12 (b), find all the chains. Mention also the maximal chains.*

Solution. The lattice has five chains since $0 < 1$, $0 < a_1 < 1$, $0 < a_2 < 1$, $0 < a_3 < 1$, $0 < a_2 < a_1 < 1$ from 0 to 1. Hence the five chains are $\{0, 1\}$, $\{0, a_1, 1\}$, $\{0, a_2, 1\}$, $\{0, a_3, 1\}$, $\{0, a_2, a_1, 1\}$. The chains have length 1, 2, 2, 2, 3. The last two chains are maximal chains.

Example 14. *Find the antichain with the greatest number of elements in the poset with the following Hasse diagram.*



Solution. The only antichains with more than one element are $\{b, c\}$, $\{c, e\}$, and $\{d, e\}$.

Isomorphism

Let (A, $\preceq$) and (B, $\preceq$) be posets. A one-one onto map f: A $\to$ B is called an iosmorphism from (A, $\preceq$) to (B, $\preceq$) if $x \preceq y \Leftrightarrow f(x) \preceq f(y)$ for all $x, y \in$ A. The posets (A, $\preceq$) and (B, $\preceq$) are isomorphic posets. The mapping is also called a similarity mapping. Hasse diagram of both the posets are identical except for the labelling of vertices.

Example 15. Let A = $\{1, 2, 4, 8\}$ with divisibility and B = $\{0, 1, 2, 3\}$ with usual $\preceq$ are partial order. Consider the function f: A $\to$ B defined by $f(1) = 0, f(2) = 1, f(4) = 2, f(8) = 3$. Show, that f is an isomorphism from A onto B. Also verify that A and B have identical Hasse diagrams.

Solution. Here | A | = | B | = 4. It is evident the given function f is one-to-one and onto. Again for all $x, y \in$ A.,

$$x/y \Leftrightarrow f(x) \preceq f(y)$$

Hence the given function is an isomorphism from (A, 1) onto (B, $\leq$) Hasse diagrams of A and B are shown below:

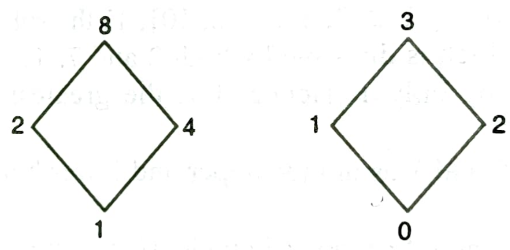


Fig. 9.6

The function f preserves the order structure and hence may be viewed simply as a relabeling of the vertices in the diagram. Two diagrams are identical.

Example 16. Prove that a finite partial ordered set has

(a) at most one greatest element.

(b) at most one least element.

Solution. (a) Assume a and b are greatest of (A, $\preceq$). Since a is the greatest element, we have $b \preceq a$. Also, since b is the greatest element, we have $a \preceq b$. Thus, $b \preceq a$ and $a \preceq b$. Since $\preceq$ is an antisymmetric for partial ordered set, it follows that $a = b$. Thus there can not be two different greatest elements of (A, $\preceq$), if it exists. Hence a finite partial ordered set has at most one greatest element.

Example 17. Consider the poset A = ($\{1, 2, 3, 4, 6, 9, 12, 18, 36\}$, 1) find the greatest lower bound and the least upper bound of the sets $\{6, 18\}$ and $\{4, 6, 9\}$.

Solution. An integer is a lower bound of $\{6, 18\}$ if 6 and 18 are divisible by this integer. Only such integers are 1 and 6. Since 1/6, 6 is the greatest lower bound of $\{6, 18\}$ is glb $\{6, 18\} = 6$.

An integer is an upper bound of {6, 18} if and only if it is divisible by 6 and 18 which is 18. Hence lub {6, 18} = 18.

The only lower bound of {4, 6, 9} is 1. Hence glb {4, 6, 9} = 1. The only upper bound of {4, 6, 9} is 36. Hence lub {4, 6, 9} = 36.

Note that gcd (a, b) = glb $\{a, b\}$ and lcm (a, b) = lub $\{a, b\}$.

Example 18. In the poset P shown in Fig. 9.7, find upper bound and least upper bound for A = {2, 3} and B = {4, 6}.

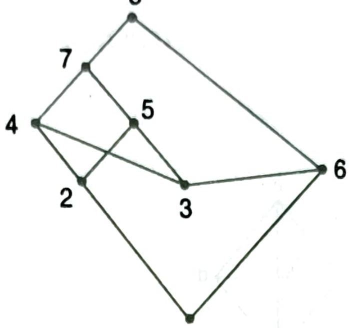


Fig. 9.7

Solution. From the given Fig. it is clear that upper bounds of {2, 3} are 4, 5, 7 and 8 since they are successors of 2 and 3. The least upper bound does not exist since 4 is not related to upper bound 5. The upper bounds of {4, 6} is 8 and lub {4, 6} = 8.

Example 19. Consider $A = \{x \in R : 1 < x < 2\}$ with $\leq$ as the partial order find

(*i*) All the upper and lower bounds of A.

(*ii*) Greatest lower bound and least upper bound of A.

Solution. (*i*) Every real number ≥ 2 is an upper bound of A and every real number ≤ 1 is a lower bound of A.

(*ii*) 1 is a greatest lower bound and 2 is the least upper bound of A.

9.3. Lattice

A poset $(P, \preceq)$ is called a lattice if every 2-element subset of P has both a least upper bound a greatest lower bound *i.e.* if lub (x, y) and glb (x, y) exist for every x and y in P. In this case, we denote

$$x \vee y = \text{lub } \{x, y\} \quad \text{(read as } x \text{ join } y)$$
$$x \wedge y = \text{glb } \{x, y\} \quad \text{(read as } x \text{ meet } y)$$

Note that every chain is a lattice. Since any two elements a, b of a chain are comparable, we find

$$x \vee y = \text{lub } (x, y) = y$$
$$x \wedge y = \text{glb } (x, y) = x$$

Example 20. The power set $(P(S), \subseteq)$ is a lattice because if $A, B \subseteq S$ then an upper bound of {A, B} is a subset of S which contains both A and B and the least among them is $A \cup B, \in P(S)$. To see this, note that $A \subseteq A \cup B$, $B \subseteq A \cup B$ and if $A \subseteq C$ and $B \subseteq C$, then it follows that $A \cup B \subseteq C$.

Similarly, the greatest lower bound of A and B is $A \cap B, \in P(S)$. To see this, note that $A \cap B \subseteq B$, so that $A \cap B$ is a lower bound of {A, B}. On the other hand, if $C \subseteq A$ and $C \subseteq B$, then $C \subseteq A \cap B$. Then $A \cap B$ is the greatest lower bound of {A, B}.

$\therefore$
$$A \vee B = A \cup B$$
$$A \wedge B = A \cap B.$$

For instance in the poset $(P\{a, b, c\}, \subseteq)$

$$\{a\} \vee \{b\} = \{a, b\}$$
$$\{a, b\} \vee \{b, c\} = \{a, b, c\}$$
$$\{a, b\} \wedge \{b, c\} = \{b\}$$
$$\{a\} \wedge \{b\} = \phi$$

Example 21. The poset $(Z^+, /)$ is a lattice because any upper bound of $\{a, b\}$ where $a, b \in Z^+$ is nothing but an element which is divisible by both a and b *i.e.* a common multiple of a and b which is an upper bound of a and b. The least upper bound is thus the least common multiple (lcm). Similarly, the greatest lower bound is the greatest common divisor (gcd) of a and b.

$$a \vee b = l\text{cm}(a, b)$$
$$a \wedge b = \text{gcd}(a, b)$$

For instance
$$6 \vee 4 = \text{lcm}(6, 4) = 12$$
$$6 \wedge 4 = \text{gcd}(6, 4) = 2$$

Example 22. Determine whether the posets represented by each of the Hasse-diagram in Fig. 9.8. are lattices.

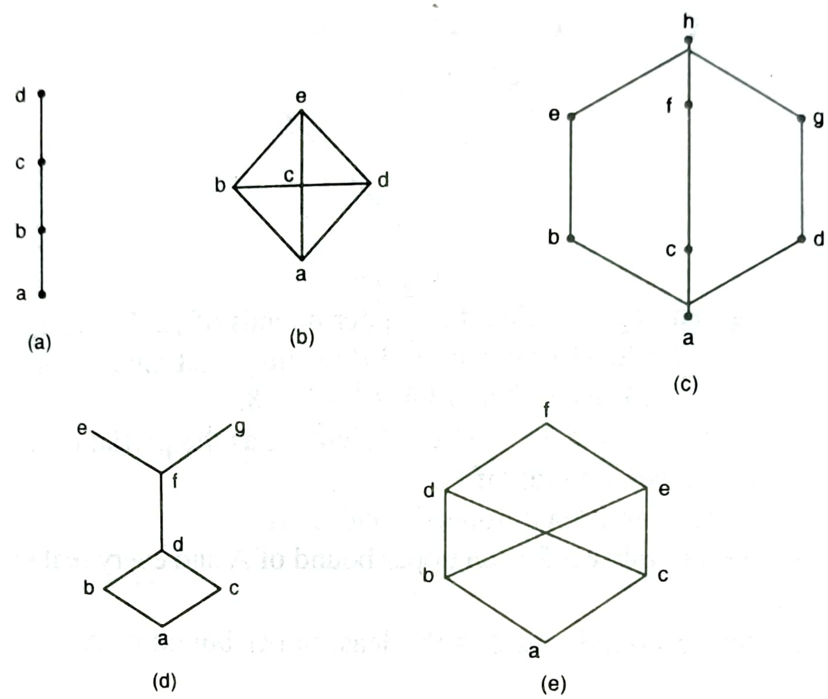


Fig. 9.8

Solution. The posets represented by Hasse diagrams in (a), (b) and (c) are lattices because in each poset there exists both lub and glb for each pair of elements.

The poset with Hasse diagram shown in (d) is not a lattice since $e \vee g$ does not exist. The poset with Hasse diagram shown in (e) is not lattice since $\{b, c\}$ has three upper bounds d, e and f but no one of them precedes the other two *i.e.* lub $\{b, c\}$ does not exist.

Properties of Lattices

A number of properties of Lattices is given below.

Theorem 9.3. If L be a lattice, then for every a and b in L.

(a) $a \vee b = b$ if and only if $a \preceq b$

(b) $a \wedge b = a$ if and only if $a \preceq b$

(c) $a \wedge b = a$ if and only if $a \vee b = b$

Proof. (a) Let $a \vee b = b$. Since $a \preceq a \vee b = b$, we get $a \preceq b$. Conversely, if $a \preceq b$, then, since $b \preceq b$, b is an upper bound of a and b. So, by definition of lub, we have $a \vee b \preceq b$. Again, since $a \vee b$ is an upper bound, $b \preceq a \vee b$, so $a \vee b = b$. Hence proved.

(b) Similar to (a)

(c) Combining (a) and (b), we get (c).

Theorem 9.4. If L be any lattice, then for any $a, b, c \in$ L,

1.	(a) $a \vee a = a$	(b) $a \wedge a = a$	Idempotency
2.	(a) $a \vee b = b \vee a$	(b) $a \wedge b = b \wedge a$	Commutativity

3. (*a*) $a \vee (b \vee c) = (a \vee b) \vee c$ (*b*) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ Associativity

4. (*a*) $a \vee (a \wedge b) = a$ (*b*) $a \wedge (a \vee b) = a$ Absorption

Proof. 1. (*a*) $a \vee a = \text{lub}\ \{a, a\} = \text{lub}\ \{a\} = a$

(*b*) $a \wedge a = \text{glb}\ \{a, a\} = \text{glb}\ \{a\} = a$

2. (*a*) $a \vee b = \text{lub}\ \{a, b\} = \text{lub}\ \{b, a\} = b \vee a$

(*b*) $a \wedge b = \text{glb}\ \{a, b\} = \text{glb}\ \{b, a\} = b \wedge a$

3. (*a*) We have $(a \vee b) \vee c = \text{lub}\ \{a \vee b, c\}$. So $a \vee b \preceq (a \vee b) \vee c$ and $c \preceq (a \vee b) \vee c$. Also $a \preceq a \vee b$ and $b \preceq a \vee b$

Hence, $a \preceq (a \vee b) \vee c$ and $b \preceq (a \vee b) \vee c$ by transitivity.

Thus $(a \vee b) \vee c$ is an upper bound of a and b. By definition of lub of b and c *viz* $b \vee c$.

$$a,\ b \vee c \preceq (a \vee b) \vee c$$

So it is a upper bound of a and $b \vee c$. Thus by definition of $\vee$ we have

$$a \vee (b \vee c) \preceq (a \vee b) \vee c$$

Similarly, $$(a \vee b) \vee c \preceq a \vee (b \vee c)$$

By antisymmetry of $\preceq$ we see that

$$(a \vee b) \vee c = a \vee (b \vee c).$$

(*b*) The proof is similar to the proof of part (*a*).

4(*a*) Since $a \wedge b \preceq a$ and $a \preceq a$, a is an upper bound of $a \wedge b$ and a, so $a \vee (a \wedge b) \preceq a$. By definition of lub, we have $a \preceq a \vee (a \wedge b)$. Hence by antisymmetry of $\preceq$, we see that

$$a \vee (a \wedge b) = a$$

(*b*) The proof is similar to the proof of (*a*).

Theorem 9.5. In any lattice the distributive inequalities

(*i*) $a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$

(*ii*) $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$

hold for any a, b, c.

Proof. (*i*) $$a \wedge b \leq a$$

$$a \wedge b \leq b \leq b \vee c$$

$\Rightarrow a \wedge b$ is lower bound of $\{a, b \vee c\}$

$\Rightarrow a \wedge b \leq a\,(b \vee c)$ ...(1)

Again $$a \wedge c \leq a$$

$$a \wedge c \leq c \leq b \vee c$$

$\Rightarrow$ $$a \wedge c \leq a \wedge (b \vee c) \quad \text{...(2)}$$

(1) and (2) show that $a \wedge (b \vee c)$ is an upper bound of $\{a \wedge b, a \wedge c\}$

$\Rightarrow$ $$(a \wedge b) \vee (a \wedge c) \leq a \wedge (b \vee c)$$

Hence $$a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$$

(*ii*) The proof is similar to (*i*)

Note : The above inequalities become equalities in many lattices (including the power set lattice) where the operation of union and inter-section are distributive over each other. But they do not hold in general.

Principle of Duality

We observe that if $\leq$ is a partial order on any set, then its inverse relation $\geq$ is also a partial order. Also it follows from the definitions, that the lub of a and b with respect to $\leq$ is the same as the glb with respect to the relation and vice versa. This observations can be developed into a formal principle of duality for lattices, which says that if we interchange $\vee$ with $\wedge$ and $\leq$ with $\geq$ in a true statement about lattices, we get another true statement. And the corresponding statements are called dual of each other. In the theorem 9.4, the statements and the dual statements holding in a lattice is grouped together.

Definition. Let L and M be lattices. The set of ordered pairs

$$\{(x, y) : x \in L, y \in M\}$$

with operations and defined by

$$(x_1, y_1) \vee (x_2, y_2) = (x_1 \vee x_2, y_1 \vee y_2)$$

$$(x_1, y_1) \wedge (x_2, y_2) = (x_1 \wedge x_2, y_1 \wedge y_2)$$

is the direct product of L and M are denoted by $L \times M$. It is also called the product partial order.

Theorem. Prove that product of two lattices is a lattice.

Proof. Let L and M be two lattices. We already know from theorem is a poset with partial order defined $(x_1, y_1) \leq (x_2, y_2)$ if $x_1 \leq x_2$ in L and $y_1 \leq y_2$ in M.

Now if we prove that $(x_1, y_1) \vee (x_2, y_2) = (x_1 \vee x_2, y_1 \vee y_2)$ and

$$(x_1, y_1) \wedge (x_2, y_2) = (x_1 \wedge x_2, y_1 \wedge y_2)$$

then we can conclude that for any two ordered pairs (x_1, y_1) and $(x_2, y_2) \in L$ its join and meet exist and hence $L \times M$ is a lattice.

Let $(x_1, y_1), (x_2, y_2) \varepsilon\ L \times M$ be any two elements. Then $x_1, x_2 \varepsilon\ L$ and $y_1, y_2 \varepsilon\ M$. Since L and M are lattices $\{x_1, x_2\}$ and $\{y_1, y_2\}$ have Sup and Inf in L and M respectively.

Let $x_1 \vee x_2 = \sup \{x_1, x_2\}$ $y_1 \vee y_2 = \sup \{y_1, y_2\}$ then $x_1 \vee x_2 \geq x_1, x_1 \vee x_2 \geq x_2$ and $y_1 \vee y_2 \geq y_1, y_1 \vee y_2 \geq y_2$

This implies $(x_1 \vee x_2, y_1 \vee y_2) \geq (x_1, y_1)$ and

$$(x_1 \vee x_2, y_1 \vee y_2) \geq (x_2, y_2)$$

Hence $(x_1 \vee x_2, y_1 \vee y_2)$ is an upper bound of $\{(x_1, y_1), (x_2, y_2)\}$. Suppose (z_1, z_2) is any upper bound of $\{(x_1, y_1), (x_2, y_2)\}$, then

$$(z_1, z_2) \geq (x_1, y_1), (z_1, z_2) \geq (x_2, y_2)$$

$$\Rightarrow \quad z_1 \geq x_1, z_1 \geq x_2, z_2 \geq y_1, z_2 \geq y_2$$

This shows that z_1 is an upper bound of $\{x_1, x_2\}$ in L and z_2 is an upper bound of $\{y_1, y_2\}$ in M. Hence $z_1 \geq x_1 \vee x_2 = \sup \{x_1, x_2\}$ and $z_2 \geq y_1 \vee y_2 = \sup \{y_1, y_2\}$.

Thus, $\quad (z_1, z_2) \geq (x_1 \vee x_2, y_1 \vee y_2)$

Therefore, $\quad (x_1 \vee x_2, y_1 \vee y_2)$ is lub $\{(x_1, y_1), (x_2, y_2)\}$ i.e.,

$$(x_1, y_1) \vee (x_2, y_2) = (x_1 \vee x_2, y_1 \vee y_2)$$

By duality we can write

$$(x_1, y_1) \wedge (x_2, y_2) = (x_1 \wedge x_2, y_1 \wedge y_2)$$

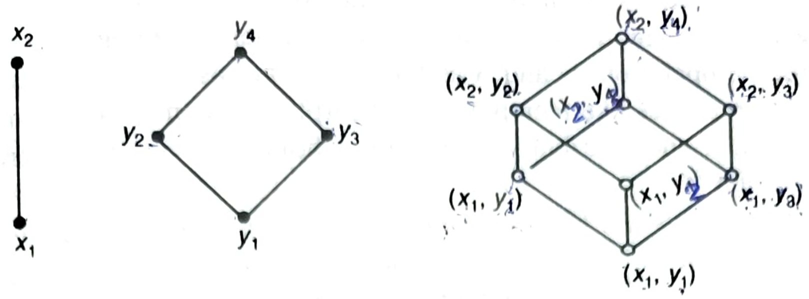
Hence $L \times M$ is a lattice.

One can use the direct product of lattices to construct larger lattices from smaller ones. If L is a lattice we can form lattices.

$L \times L$, $L \times L \times L$, $L \times L \times L \times L$, ... which are denoted by L^2, L^3, L^4, ... respectively.

The direct product of the lattives L and M can graphically be described in terms of the Hasse diagram.

Example 23. *Let L and M be two lattices shown Fig. (a) and (b) respectively. Then, L × M is the lattice shown Fig. (c).*



L (a) M (b) L × M (c)

Definitions. Let L and M be lattices. A mapping $f : L \to M$ is called a :

(*i*) **Join - homomorphism** if $f(x \vee y) = f(x) \vee f(y)$;

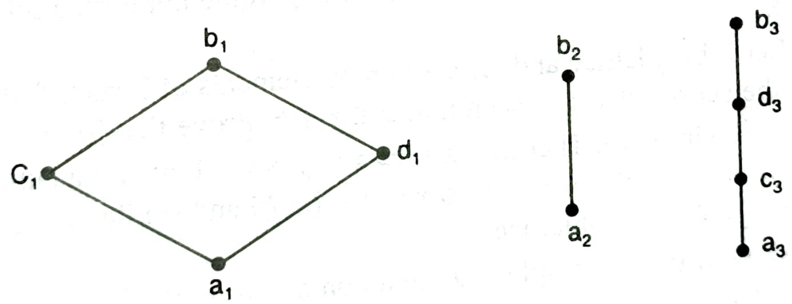
(*ii*) **meet - homomorphism** if $f(x \wedge y) = f(x) \wedge f(y)$;

(*iii*) **order - homomorphism** if $x \leq y \Rightarrow f(x) \leq f(y)$ *i.e.* it preserves the partial order; hold for all $x, y \in L$.

The mapping f is called a **homomorphism** (or **lattice homomorphism**) if it is both a join and meet homomorphism. If f is homomorphism from L and M then $f(L)$ is called a **homomorphic image** of L. It can be easily shown that every join (or meet) homomorphism is an order – homomorphism. But, the converse is not true.

If a homomorphism f is bijective, that, is one-to-one and onto, then f is called an **isomorphism**. If there is an isomorphism from L to M, then we say that L and M are **isomorphic.** If f is an isomorphism from L to L, we call it an **automorphism.**

Example 24. Let L, M and N be the lattices with Hasse diagrams of Fig. respectively



If we define $f : L \to M; f(a_1) = f(c_1) = f(d_1) = a_2, f(b_1) = b_2$, then one can easily verify that f is both order-homomorphism and meet homomorphism but f is not homomorphism.

If we define $g : L \to M; g(a_1) = a_2, g(b_1) = g(c_1) = g(d_1) = b_2$, then g is a join homomorphism but not a homomorphism.

Again if we define $h : L \to N; h(a_1) = a_3, h(c_1) = c_3, h(d_1) = d_3, h(b_1) = b_3$, then h is neither a meet nor a join homomorphism as

$$h(c_1 \wedge d_1) = h(a_1) = a_3 \text{ and } h(c_1) \wedge h(d_1) = c_3 \wedge d_3 = c_3$$

$$h(c_1 \vee d_1) = h(b_1) = a_3 \text{ and } h(c_1) \vee h(d_1) = c_3 \vee d_3 = d_3$$

9.4. Lattice as Algebraic System

By an algebraic system, we mean a set together with a few rules for combining elements of the set to form other elements of the set.

We had introduced lattices as partially ordered sets in which for each $a, b \in A$,

$$\text{lub}(a, b) = a \vee b; \quad \text{glb}(a, b) = a \wedge b$$

exists in the set.

By virtue of the definitions, we note that, in a lattice (A, $\preceq$), with every two elements a, b of A there is associated an element $a \vee b$ of A and there is associated an element $a \wedge b$ of A. In view of this, we interpret the operations $\vee$ and $\wedge$ as binary operations on A. Also we have seen that $\wedge$ and $\vee$ satisfy certain properties like idempotence, absorption, commutativity and associativity and other results like modular and distributive inequalities hold involving the operations $\vee$ and $\wedge$, as well as the order relation $\preceq$.

Any set L together with two binary operations satisfying the commutative, associative and absorption laws is actually a lattice in the sense that a partial order relation can be defined in terms of either of these operations with respect to which it becomes a lattice with sup and inf with respect to this order coinciding with the given binary operations.

Thus any lattice can be viewed as an a algebraic system and hence one can apply many concepts associated with algebraic system to lattices.

9.5. Sublattices

Just an in other algebraic structures like groups or vector spaces. We have the concepts of substructures like subgroups and subspaces of groups and vector spaces respectively, we have also the concept of sublattices.

Definition. A non empty subset L′ of a lattice L is called a sublattice of L if $a, b \in L' \Rightarrow a \vee b, a \wedge b \in L'$ *i.e.* the algebra (L′, $\vee$, $\wedge$) is a sublattice of (L, $\wedge$, $\vee$) iff L′ is closed under both operations $\wedge$ and $\vee$.

From the definition it follows that a sub lattice itself is a lattice and every singleton of a lattice L is a sublattice of L. However, any subset of L, which is a lattice need not be a sublattice.

Example 25. If S be a set and L = (P (S), $\cap$, $\cup$). Then only those sub collections of P (S) are sublattices of L which are closed under $\cap$ and $\cup$. If S = $\{a, b, c, d\}$, then $\{\phi\}$, $\{a\}$, $\{a, c\}$, $\{c\}$, $\{a, b, c\}$ would be sublattice but $\{\varphi, \{a\}, \{b\}, \{c\}, \{b, c\}\}$ would not form a sublattice as $\{a\} \cup \{c\} = \{a, c\}$ is not there.

Example 26. Let L be a lattice and let a and b be elements of L such that $a \le b$. The interval $[a, b]$ is defined as the set of all $x \in$ L such that $a \le x \le b$. Prove that $[a, b]$ is a sublattice of L.

Solution. Let x, y be in $[a, b]$. Then $x, y \in L \Rightarrow x \vee, y \wedge x \in L$ as L is a lattice. Now a is a lower bound of x and y. Then $a \preceq x \preceq x \vee y \le b$, so $x \vee y \in [a, b]$ and similarly $a \preceq x \wedge y \preceq x \preceq b$ so $x \wedge y \in [a, b]$. Hence $[a, b]$ is a sublattice.

Example 27. Show with an example that the union of two sublattices may not be a sub lattice.

Solution. Consider the lattice L = $\{1, 2, 3, 4, 6, 12\}$ of factors of 12 under divisibility. Then S = $\{1, 2\}$ and T = $\{1, 3\}$ are sublattices of L. But the union of S and T i.e. S $\cup$ T = $\{1, 2, 3\}$ is not a sublattice since 2, 3 $\in$ S $\cup$ T but $2 \vee 3 = 6 \notin$ S $\cup$ T.

Example 28. Let D(n) denote the set of all positive divisors of n. If L = D (30) = $\{1, 2, 3, 5, 6, 10, 15, 30\}$ having the Hasse diagram shown in Fig. 9.9

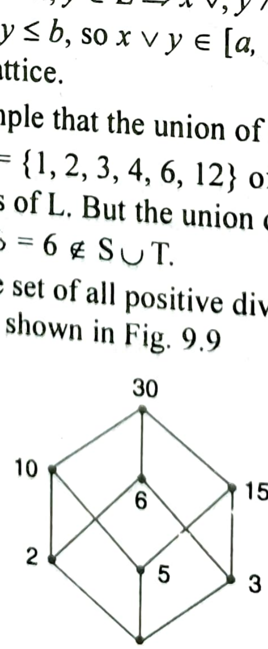


Fig. 9.9

then $D(6) = \{1, 2, 3, 6\}$, $D(10) = \{1, 2, 5, 10\}$, $D(15) = \{1, 3, 5, 15\}$ are sublattices. Another sublattice is $\{5, 10, 15, 30\}$. In general if $m \mid n$ then $D(m)$ is a sublattice of $D(n)$. Also all the multiples of m in $D(n)$ form a sublattice.

9.6. Some Special Lattices

In a lattice every pair of elements has a least upper bound (lub) and a greatest lower bound (glb). As a consequence of this fact, one can show by using the principle of mathematical induction that every finite subset of a lattice has a lub and glb. For example, $n = 3$, given a_1, a_2, a_3, in lattice L, $(a_1 \vee a_2) \in L$ and then $(a_1 \vee a_2) \vee a_3 \in L$ by definition of a lattice.

Now, $a_1, a_2 \preceq (a_1 \vee a_2) \preceq (a_1 \vee a_2) \vee a_3$ and if $a_1, a_2, a_3 \preceq c$ then $a_1 \vee a_2, a_3 \preceq c$ is an upper bound of $\{a_1, a_2\}$, $\therefore (a_1 \vee a_2) \vee a_3 \preceq c$ as c is an upper bound of $\{a_1 \vee a_2, a_3)$ too.

Thus $(a_1 \vee a_2) \vee a_3 =$ l.u.b. $\{a_1, a_2, a_3\}$. By associativity and commutativity of the operation $\vee$, the above may be written in any order. Thus l.u.b. $\{a_1, a_2, a_3\} = \{a_1 \vee a_2\} \vee a_3 = a_1 \vee (a_2 \vee a_3)$

$= a_2 \vee \{a_1 \vee a_3\} = \dots\dots\dots = \bigvee_{i=1}^{3} a_i$.

Similarly $\bigwedge_{i=1}^{3} a_i = a_1 \wedge (a_2 \wedge a_3) = (a_1 \wedge a_2) \wedge a_3$ etc. In general. Let $(L, \vee, \wedge)$ be a lattice and $S \subseteq L$ be a finite subset of L where $S = \{a_1, a_2, \dots\dots\dots, a_n\}$. The lub and glb of S can be expressed as

lub S $= \bigvee_{i=1}^{n} a_i$ and glb S $= \bigwedge_{i=1}^{n} a_i$

where $\bigvee_{i=1}^{n} a_i = a_1 \vee a_2 \vee \dots\dots\dots \vee a_n$ and $\bigwedge_{i=1}^{n} a_i = a_1 \wedge a_2 \wedge \dots\dots\dots \wedge a_n$.

Note: A lattice may fail to possess a glb or lub for arbitrary subsets.

There may not be lub and glb for an infinite subsets of a lattice. For example, the lattice $(Z^+, \leq)$ where Z^+ is the set of positive integers, the subset consisting of even positive integers has no least upper bound.

Complete Lattice. A lattice is called complete if each of its non empty subsets has a least upper bound and a greatest lower bound.

Clearly, every finite lattice is complete because every subset here is finite. Also every complete lattice must have a least element and a greatest element. The least and the greatest elements of a lattice are called **bounds (units, universal bounds)** of the lattice and are denoted by 0 and 1 respectively.

A lattice which has both elements 0 and 1 is called a **bounded lattice.**

Therefore (1) every finite lattice $(L, \vee, \wedge)$ with $L_n = \{a_1, a_2, \dots\dots\dots, a_n\}$ is bounded. Here $\bigwedge_{i=1}^{n} a_i$ $= 0$ and $\bigvee_{i=1}^{n} a_i = 1$.

(2) $(P(S), \cup, \cap)$ is bounded with $\phi = 0$ and $S = 1$.

In a bounded lattice, the bounds 0 and 1 clearly satisfy :

$$0 \wedge a = 0 = a \wedge 0 \text{ for all } a \in L$$
$$0 \vee a = a = a \vee 0 \text{ for all } a \in L$$
$$1 \wedge a = a = a \wedge 1 \text{ for all } a \in L$$
$$1 \vee a = 1 = a \vee 1 \text{ for all } a \in L$$

In a bounded lattice, a complement of an element can be defined in the following manner.

Defintion. In a bounded lattice $(L, \vee, \wedge, 0, 1)$, an element $b \in L$, is called a complement of an element $a \in L$ if

$$a \wedge b = 0 \quad \text{and} \quad a \vee b = 1$$

where 0 and 1 are lower and upper bound of L.

Note: (1) The defintion of a complement is symmetric in a and b, so that b is complement of a if a is a complement in b.

(2) Any element $a \in L$ may or may not have a complement.

(3) An element $a \in L$ may have more than one complement in L, so complements are not unique.

(4) In any bounded lattice, the bound 0 and 1 are unique complements of each other because

$$0 \vee 1 = 1 \quad \text{and} \quad 0 \wedge 1 = 0$$

A bounded lattice is said to be complemented if every element has at least one complement in the lattice.

Example 29. In Fig. 9.10. (*a*), complement of a_1 is a_2. In Fig. 9.10. (*b*) both a_1 and a_2 are complements of a_3 and so here complements are not unique.

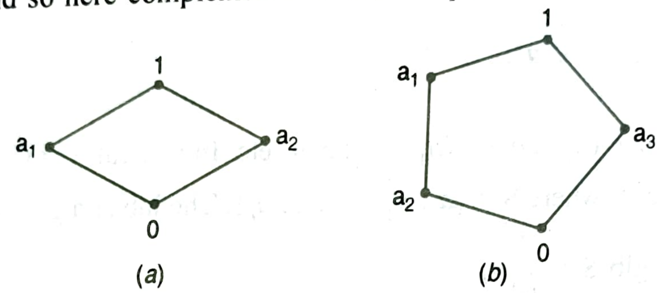


Fig. 9.10

Example 30. Show that the lattice $(L^3, \preceq_3)$ of 3 tuples of 0 and 1 is complemented.

Solution. The Hasse-diagram of the lattice $(L^3, \preceq_3)$ is shown in Fig. 9.11.

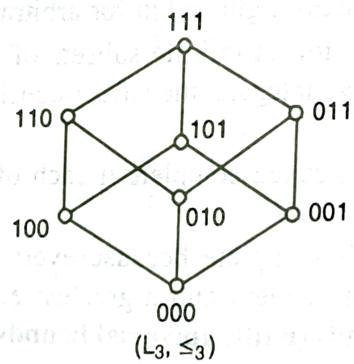


Fig. 9.11

The bound of the lattice are (1, 0, 1) and (1, 1, 1). This is a complemnted lattice in which every element has a unique complement. The complement of (1, 0, 1) is (0, 1, 0).

In general, the lattice $(L^n, \preceq_n)$ is a complemented lattice. The complement of an element of L^n can be obtained by interchanging 1 by 0 and 0 by 1.

Distributive lattice

We have seen that in any lattice the distributive inequalities

$a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$

$a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$

hold for all a, b and c. In many important lattices, including Boolean Algebras, the above inequalities are actually equalities. Wherever this holds, the lattices are called distributive. We first observe that equality of the two sides for all a, b, c for either of the above distributive inequalities imply, that of the other.

Theorem 9.6. In any lattice (L, ∧, ∨) the following statements are equivalent :

(*i*) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)\ \forall\ a, b, c \in L$

(*ii*) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)\ \forall\ a, b, c \in L$

Proof : (*i*) ⇒ (*ii*)

$$\begin{aligned}
\text{R.H.S. of } (ii) &= [(a \vee b) \wedge a] \vee [(a \vee b) \wedge c] && \text{by } (i) \\
&= a \vee [c \wedge (a \vee b)] && \text{by commutativity and absorption} \\
&= [a \vee (c \wedge a) \vee (c \wedge b)] && \text{by } (i) \\
&= [a \vee (c \wedge a)] \vee (c \wedge b) && \text{by associativity} \\
&= a \vee (b \wedge c) && \text{by absorption and commutativity} \\
&= \text{L.H.S. of } (ii)
\end{aligned}$$

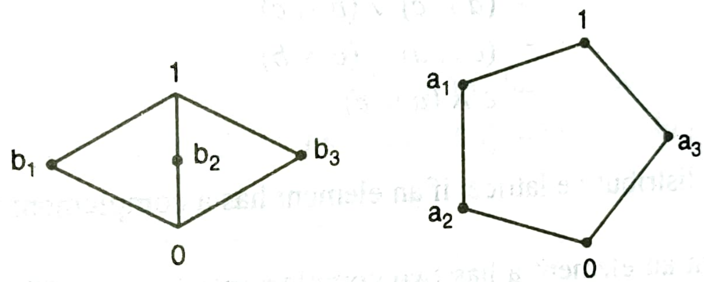
(*ii*) ⇒ (*i*) may be proved in a dual manner.

Definition. A lattice (L, ∨, ∧) is called a distributive lattice if for any $a, b, c \in L$

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$
$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

Example 31. L = (P (S), ∪, ∩) is distributive lattice, since the distributive laws for ∪ over ∩ and ∩ over ∪ are well known facts of set theory.

Example 32. Show that the lattices given in the following diagrams are not distributive.



(a) Diamond

(b) Pentagon

Fig. 9.12

Solution. In the lattice shown in (*a*)

$$b_1 \wedge (b_2 \vee b_3) = b_1 \wedge 1 = b_1$$

but $(b_1 \wedge b_2) \vee (b_1 \wedge b_3) = 0 \vee 0 = 0$

Hence the lattice shown in (*a*) is not distributive.

In the lattice shown in (*b*)

$$a_1 \wedge (a_2 \vee a_3) = a_1 \wedge 1 = a_1$$

but $(a_1 \wedge a_2) \vee (a_1 \wedge a_3) = a_2 \vee 0 = a_2 \neq a_1$

Hence the lattice shown in (*b*) is not distributive.

Observe that the distributive equalties may be satisfied for a particular combination in a particular order but this does not guarantee that the lattice is distributive. For example in the diagram (*b*)

$$a_3 \wedge (a_1 \vee a_2) = 0 = (a_3 \wedge a_1) \vee (a_1 \wedge a_2).$$

In particular if any lattice contains at least one of these five elements lattices as a sublattice, then it cannot be distributive. These are the smallest non distributive lattice the next theorem which we state without proof, states, that any non-distributive lattice must contain one of these as sublatices.

Theorem : A lattice (L, ∧, ∨) is distributive if and only if it does not contain the five element pentagonal or, the diamond lattice given above as one of its sublattices (or an isomorphic copy thereof).

Theorem 9.7. Show that every chain is a distributive lattice.

Proof. Let $(L, \preceq)$ be a chain and $a, b, c \in L$. Since L is a chain, either $a \preceq b$ or $b \preceq a$. If $a \preceq b$, then $a \vee b = b$ and $a \wedge b = a$. Hence, for any two elements $a, b \in L$, $a \wedge b$ and $a \vee b$ exist in L. Suppose $a \preceq b$.

Case 1: $b \preceq c$.

Now $a \wedge (b \vee c) = a \wedge c = a$ and $(a \wedge b) \vee (a \wedge c) = a \vee a = a$. Hence, we have $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

Case 2: $c \preceq b$

Subcase 2a: $a \preceq c$.

In this case, we have $a \preceq c \preceq b$. Now $a \wedge (b \vee c) = a \wedge b = a$ and $(a \wedge b) \vee (a \wedge c) = a \vee a = a$. Hence, $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

Similarly, if $b \leq a$, then $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

Theorem 9.8. In a distributive lattice $(L, \preceq)$,

$$a \wedge b = a \wedge c \text{ and } a \vee b = a \vee c \text{ imply that } b = c.$$

Proof. Here

$$\begin{aligned}
b &= b \wedge (a \vee b) && \text{by absorption} \\
&= b \wedge (a \vee c) && (\because a \vee b = a \vee c) \\
&= (b \wedge a) \vee (b \wedge c) && \text{by distributivity} \\
&= (a \wedge b) \vee (b \wedge c) && \text{by commutativity} \\
&= (a \wedge c) \vee (b \wedge c) && (\because a \wedge b = a \wedge c) \\
&= (c \wedge a) \vee (c \wedge b) && \text{by commutativity} \\
&= c \wedge (a \vee c) && \text{by distributivity} \\
&= c && \text{by absorption}
\end{aligned}$$

Theorem 9.9. In a distributive lattice, if an element has a complement then this complement is unique.

Proof. Suppose that an element a has two complements b and c. Then

$$a \vee b = 1 \qquad a \wedge b = 0$$
$$a \vee c = 1 \qquad a \wedge c = 0$$

We have

$$\begin{aligned}
b &= b \wedge 1 \\
&= b \wedge (a \vee c) && [\because a \vee c = 1] \\
&= (b \wedge a) \vee (b \wedge c) && \text{by distributivity} \\
&= 0 \vee (b \wedge c) \\
&= (a \wedge c) \vee (b \wedge c) \\
&= (a \vee b) \wedge c \\
&= 1 \wedge c \\
&= c
\end{aligned}$$

Modular Lattice

A lattice $(L, \preceq)$ is said to be modular if $a \vee (b \wedge c) = (a \vee b) \wedge c$ when ever $a \preceq c$ for all $a, b, c, \in L$.

Theorem 9.10. Every distributive lattice is modular.

Proof. Let $(L, \preceq)$ be a distributive lattice and $a, b, c \in L$ be such that $a \preceq c$. Thus if $a \preceq c$ then $a \vee c = c$. Now

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) = (a \vee b) \wedge c$$

Hence, every distributive lattice is modular.

However, the converse is not true, as shown by the following example.

(Note : Any chain is modular)

Example 33. Prove that the lattice given by the following diagram is modular.

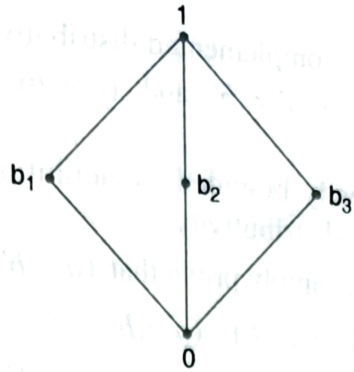


Fig. 9.13

Solution. We have already shown in example 27 that the diamond lattice is nondistributive. Since the lattice is symmetric with respect to b_1, b_2, b_3, the only situations are $b_1 < 1$ and $0 < b_1$ when $a < c$.

Thus taking $a = b_1$, and $c = 1$.

$a \vee (b \wedge c) = b_1 \vee (b \wedge 1) = b_1 \vee b$

and $(a \vee b) \wedge c = (b_1 \vee b) \wedge 1 = b_1 \vee b$

whatever be b.

Similarly for $a = 0$, $c = b_1$,

$a \vee (b \wedge c) = 0 \vee (b \wedge b_1) = b \wedge b_1$

while $(a \vee b) \wedge c = (0 \vee b) \wedge b_1 = b \wedge b_1$ and so modularity holds.

Example 34. Prove that a lattice (L, $\preceq$) is modular if and only if $(a \wedge b) \vee (a \wedge c) = a \wedge (b \vee (a \wedge c))$ for all $a, b, c \in$ L.

Solution. Suppose (L, $\preceq$) is modular. Then

$$\begin{aligned} (a \wedge b) \vee (a \wedge c) &= (a \wedge c) \vee (a \wedge b) \\ &= (a \wedge c) \vee (b \wedge a) \\ &= ((a \wedge c) \vee b) \wedge a \qquad \text{by modularity since } a \wedge c \preceq a) \\ &= a \wedge (b \vee (a \wedge c)). \end{aligned}$$

Conversely, suppose that $(a \wedge b) \vee (a \wedge c) = a \wedge (b \vee (a \wedge c))$, for all $a, b, c \in$ L. Let a, b, c $\in$ L be such that $a \leq c$. Then $a \wedge c = a$. Now $(c \wedge b) \vee (c \wedge a) = c \wedge (b \vee (a \wedge c))$.

Hence, $(c \wedge b) \vee a = c \wedge (b \vee a)$, *i.e.*, $a \vee (b \wedge c) = (a \vee b) \wedge c$.

Example 35. The pentagonal lattice is not modular.

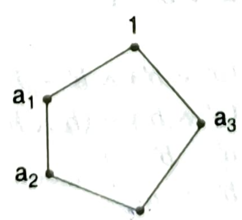


Fig. 9.14

Solution. We take $a = a_2$, $b = a_3$, $c = a_1$ then $a < c$. But

$a \vee (b \wedge c) = a_2 \vee (a_3 \wedge a_1) = a_2 \vee 0 = a_2$

whereas $(a \vee b) \wedge c = (a_2 \vee a_3) \wedge a_1 = 1 \wedge a_1 = a_1$ and these are unequal.

The following result stated without proof says that the root cause behind non-modularity is the above pentagonal lattice.

Theorem: A lattice (A, $\wedge$, $\vee$) is modular if and only if it does not have a sublattice isomorphic with the pentagonal lattice.

Theorem 9.11. If (L, $\vee$, $\wedge$) is a complemented distributive lattice, then De Morgam laws

$$(a \vee b)' = a' \wedge b' \quad \text{and} \quad (a \wedge b)' = a' \vee b'$$

hold for all $a, b \in$ L.

Proof: Observe that L is already bounded by definition of a complemented lattice. Also complements are unique because of distributivity.

To prove $(a \vee b)' = a' \wedge b'$, we simply prove that $(a' \wedge b')$ is complement of $(a \vee b)$ *i.e.*,

$$(a \vee b) \vee (a' \wedge b') = 1 \quad \text{and} \quad (a \vee b) \wedge (a' \wedge b') = 0$$

We have
$$\begin{aligned}(a \vee b) \vee (a' \wedge b') &= [(a \vee b) \vee a'] \wedge [(a \vee b) \vee b'] \text{ by distributivity}\\ &= [(a \vee a') \vee b] \wedge [a \vee (b \vee b')]\\ &= [I \vee b] \wedge [a \vee I]\\ &= I \vee 1 = 1\end{aligned}$$

and
$$\begin{aligned}(a \vee b) \wedge (a' \wedge b') &= [(a \wedge (a' \wedge b')] \vee [b \wedge (a' \wedge b')]\\ &= [(a \wedge a') \wedge b'] \vee [a' \wedge (b \wedge b')]\\ &= [0 \wedge b'] \vee [a' \vee 0]\\ &= 0 \vee 0 = 0\end{aligned}$$

That $(a' \wedge b')$ behaves as the complement of $(a \vee b)$. By uniqueness of complements, it is the only complement of $a \vee b$.

$$(a \vee b)' = a' \wedge b'.$$

The other follows dually.

Example 36. Show that in a complemented, distributive lattice, the followings are equivalent.

(*i*) $a \preceq b$ (*ii*) $a \wedge b' = 0$ (*iii*) $a' \vee b = 1$ (*iv*) $b' \preceq a'$.

Solution. (*i*)
$$\begin{aligned} a \preceq b &\Rightarrow a \vee b = b\\ &\Rightarrow (a \vee b) \wedge b' = 0 && [\because b \wedge b' = 0]\\ &\Rightarrow (a \wedge b') \vee (b \wedge b') = 0 && \text{by distributivity}\\ &\Rightarrow a \wedge b' = 0\end{aligned}$$

Hence (*i*) $\Rightarrow$ (*ii*)

Again
$$\begin{aligned} a \wedge b' = 0 &\Rightarrow (a \wedge b')' = 1\\ &\Rightarrow a' \vee (b')' = 1\\ &\Rightarrow a' \vee b = 1\end{aligned}$$

Hence (*ii*) $\Rightarrow$ (*iii*)

Now
$$\begin{aligned} a' \vee b = 1 &\Rightarrow (a' \vee b') \wedge b' = b' && [\because 1 \wedge b' = b']\\ &\Rightarrow (a' \wedge b') \vee (b \wedge b') = b' && \text{by distributivity}\\ &\Rightarrow a' \wedge b' = b' && [\because b \wedge b' = 0]\\ &\Rightarrow b' \preceq a'\end{aligned}$$

Hence (*iii*) $\Rightarrow$ (*iv*)

Now
$$\begin{aligned} b' \preceq a' &\Rightarrow a' \wedge b' = b'\\ &\Rightarrow (a' \wedge b')' = b && \text{by Demorgan's law}\\ &\Rightarrow a \vee b = b\\ &\Rightarrow a \preceq b\end{aligned}$$

Hence (*iv*) $\Rightarrow$ (*i*)

Thus (*i*), (*ii*), (*iii*) and (*iv*) are equivalent.

Example 37. Consider the lattice L in Fig. 9.15.

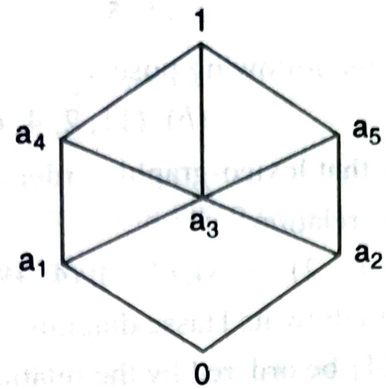


Fig. 9.15

(*a*) Which of the followings are sublattices of L ?

$$L_1 = \{0, a_1, a_2, I\}, \quad L_2 = \{0, a_1, a_5, I\}$$
$$L_3 = \{a_1, a_3, a_4, I\}$$

(*b*) Find complements, if exist, for elements, a_1, a_2 and a_3.

(*c*) Is L distributive ?

(*d*) Is L a complemented Lattice ?

Solution. (*a*) A subset L′ is a sublattice if it is closed under $\wedge$ and $\vee$. L_1 is not a sublattice since $a_1 \vee a_2 = a_3$ which does not belong to L_1. L_2 is a sublattice since $a_1 \vee a_5 = I$ and $a_1 \wedge a_5 = 0$ belong to L_2. L_3 is a sublattice since $a_3 \vee a_4 = I$ and $a_3 \wedge a_4 = a_1$, $a_1 \vee a_3 = I$ and $a_1 \wedge a_3 = a_4$, $a_1 \vee a_4 = I$, $a_1 \wedge a_4 = a_3$.

(*b*) We have $a_1 \wedge a_5 = 0$ and $a_1 \vee a_5 = I$, so a_1 and a_5 are complements. Also, $a_2 \wedge a_4 = 0$ and $a_2 \vee a_4 = I$, so a_2 and a_4 are complements. But a_3 has no complement.

(*c*) L is not distributive since K = $\{0, a_1, a_4, a_5, 1\}$ is a sublattice which forms pentagon.

(*d*) L is not complemented since a_3 has no complement.

A complemented distributive lattice is called a Boolean algebra (or a Boolean lattice). Distributivity in a Boolean algebra guarantee the uniqueness of compliments. In Boolean algebra $(B, +, \cdot, ', 0, 1)$, $(B, +, \cdot)$ is a lattice with two binary operatons + and $\cdot$ called the join the meet respectively.

POSETS AND LATICE PROBLEM SET