GARVIT SHAH
F-24

PAGE NO.
DATE: / /

TUTORIAL 5: Taylor's Theorem & Maclaurin's Theorem for two variables, Tangent Plane & normal line.

1. Expand $e^x \log(1+y)$.

$\rightarrow f(x,y) = f(0,0) + [x f_x(0,0) + y f_y(0,0)] + \frac{1}{2!}[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)] + \ldots$

$\Rightarrow e^x \log(1+y) = 0 + [x[0] + y[\frac{1}{1}]] + \frac{1}{2}[x^2(0) + 2xy - y^2] + \frac{1}{3!}[x^3(0) + 3x^2y(1) + 3y^2x(-1) + y^3(2)]$

$\Rightarrow e^x \log(1+y) = y + xy - \frac{1}{2}y^2 + \frac{1}{2}(x^2y - y^2x) + \frac{1}{3}y^3 + \ldots$

2. $f(x,y) = \sin x \cos y$

$f_x(x,y) = \cos x \cos y \qquad f_y(x,y) = -\sin x \sin y$

$f_{xx}(x,y) = -\sin x \cos y$

$\therefore f(x,y) = f(0,0) + [x f_x(0,0) + y f_y(0,0)] + \frac{1}{2!}[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)] + \frac{1}{3!}[x^3 f_{xxx}(0,0) + 3x^2y f_{xxy}(0,0) + 3xy^2 f_{yxy} + y^3 f_{yyy}(0,0)] + \ldots$

$\Rightarrow \sin x \cos y = 0 + x + 0 + \frac{1}{2}[x^2(0) + 2xy(0) + y^2(0)] + \frac{1}{3!}[-x^3 - 3xy^2]$

$\sin x \cos y = x + \frac{1}{2}(0) + \frac{1}{3!}(-x^3 - 3y^2 x) + \dots$

$$\boxed{\sin x \cos y = x - \frac{1}{6}(x^3 + 3xy^2)}$$

3. If $f(x,y) = \tan^{-1} xy$ Find $f(0.9, -1.2)$

$\rightarrow$ ~~$f(x,y) = f(0,0) + x f_x(0,0) + y f_y(0,0)$~~

$\Rightarrow f(x+h, y+k) = f(x,y) + [h f_x(x,y) + k f_y(x,y)] +$

$\frac{1}{2!}[h^2 f_{xx}(x,y) + k^2 f_{yy}(x,y) + 2hk f_{xy}(x,y)] + \dots$

$\Rightarrow f(0.9, -1.2)$ $\therefore x = 1, y = -1, h = -0.1, k = -0.2$

$\therefore f(0.9, -1.2) = -\frac{\pi}{4} + \frac{1}{1}\left[\frac{(-0.1)(-1)}{1+1} + \frac{(-0.2)(1)}{1+1}\right] +$

$\frac{1}{2}\left[(0.01)\left[\frac{+2}{4}\right] + (0.04)\left[\frac{2}{4}\right] + 2(0.02)(0)\right]$

$= -0.78539 + \frac{1}{20} - \frac{2}{20} + \frac{0.005}{2} + \frac{0.02}{2}$

$= -0.8229$

4. $f(x,y) = f(a,b) + [(x-a) f_x(a,b) + (y-b) f_y(a,b)] +$

$\frac{1}{2!}[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)] \dots$

For $f(x,y) = \sin xy$ in terms of $(x-1)$ & $(y - \pi/2)$

$a = 1, \quad b = \pi/2$

$$\therefore \sin xy = \sin\left(\frac{\pi}{2}\right) + \left[(x-1)\,\cancel{y}\cos\left(\frac{\pi}{2}\right)\frac{\pi}{2} + \left(y-\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right)\right]$$

$$+ \frac{1}{2}\left[(x-1)^2\left(\frac{\pi}{2}\right)^2\left(-\sin\frac{\pi}{2}\right) + \left(y-\frac{\pi}{2}\right)^2\left(-\sin\frac{\pi}{2}\right) + 2(x-1)\left(y-\frac{\pi}{2}\right)\frac{\pi}{2}\left(-\sin\frac{\pi}{2}\right)\right]$$

$$\Rightarrow \sin xy = 1 - \frac{1}{2}\left[\left(\frac{\pi}{2}\right)^2(x-1)^2 + \left(y-\frac{\pi}{2}\right)^2\right] + \frac{2\pi}{2}(x-1)\left(y-\frac{\pi}{2}\right)$$

$$\sin xy = 1 - \frac{\pi^2}{8}(x-1)^2 - \frac{1}{2}\left(y-\frac{\pi}{2}\right)^2 + \frac{\pi}{2}(x-1)\left(y-\frac{\pi}{2}\right)$$

5. $f(x,y) = \tan^{-1}(y/x)$ in powers of $(x+1)$ & $(y-1)$

$$\Rightarrow f(x,y) = f(a,b) + \left[(x-a)f_x(a,b) + (y-b)f_y(a,b)\right] +$$

$$\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right]$$

$\Rightarrow f(x,y) = \tan^{-1}(y/x) \qquad a = -1, \; b = 1$

$$\therefore \tan^{-1}(y/x) = -\frac{\pi}{4} + \left[(x+1)\left(-\frac{1}{2}\right) + (y-1)\left(-\frac{1}{2}\right)\right] +$$

$$\frac{1}{2}\left[(x+1)^2\left(-\frac{1}{2}\right) + 2(x+1)(y-1)(0) + (y-1)^2\left(\frac{1}{2}\right)\right]$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) = \frac{3\pi}{4} - \frac{(x+1)}{2} - \frac{(y-1)}{2} - \frac{(x+1)^2}{4} + \frac{(y-1)^2}{4}$$

6. $f(x,y) = e^{x^2+y}$. Taylor polynomial about $(x,y) = 0$

$\rightarrow$ $f_x = 2x e^{x^2+y}$ $\quad f_y = e^{x^2+y}$

$f_{xx} = 4x^2 e^{x^2+y} + 2e^{x^2+y}$ | $f_{yy} = e^{x^2+y}$ | $f_{xy} = 2xe^{x^2+y}$

$$f(x,y) = f(0,0) + [x f_x(0,0) + y f_y(0,0)] + \frac{1}{2}[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)] + \dots$$

$$\Rightarrow e^{x^2+y} = 1 + x(0) + y(1) + \frac{1}{2}[2x^2 + y^2]$$

$$e^{x^2+y} = 1 + y + x^2 + \frac{y^2}{2} \dots$$

7. $f(x,y) = \sin x \cosh y$

- $f_x = \cos x \cosh y$, $f_x(0,0) = 1$
- $f_y = \sin x \sinh x$, $f_y(0,0) = 0$
- $f_{xx}(0,0) = 0$ $\quad f_{yy}(0,0) = 0$ $\quad f_{xy}(0,0) = 0$
- $f_{xxx}(0,0) = -1$ $\quad f_{yyy}(0,0) = 0$ $\quad f_{xxy}(0,0) = 0$ $\quad f_{yyx}(0,0) = 1$
- $f_{xxxx}(0,0) = 0$ $\quad f_{yyyy}(0,0) = 0$ $\quad f_{xxxy}(0,0) = 0$ $\quad f_{xxyy}(0,0) = 0$
  $f_{xyyy}(0,0) = 0$
- $f_{xxxxx}(0,0) = 1$, $f_{yyyyy}(0,0) = 0$, $f_{xxxxy}(0,0) = 0$, $f_{xxxyy}(0,0) = -1$
  $f_{xxyyy}(0,0) = 0$ $\quad f_{xyyyy}(0,0) = 1$ $\quad f_{yyyyy}(0,0) = 0$

[Using Maclaurin's Theorem -
$f(x,y) = f(0,0) + [x f_x(0,0) + y f_y(0,0)] + \dots$]

$$\rightarrow \sin x \cosh y = 0 + x(1) + y(0) + x^2(0) + 2xy(0) + y^2(0) + y^3(0) - \frac{x^3}{3!} + \frac{3y^2x}{3!} + \frac{x^5}{5!} + \frac{5xy^4}{5!} - \frac{10x^3y^2}{5!} + \dots$$

PAGE NO.

DATE: / /

$\Rightarrow \sin x \cosh y = x - \frac{1}{6}(x^3 - 3xy^2) + \frac{1}{120}(x^5 - 10x^3y^2 \; 5xy^4) + \ldots$

Proved

8. $f(x,y) = x^3 - 3xy^2$ show that $f(2+h, 1+k) = 2 + 9h - 12k + 6(h^2 - hk - k^2) + h^3 - 3hk^2$

$\rightarrow f(a+h, b+k) = f(a,b) + [h f_x(a,b) + k f_y(a,b)] +$

$\frac{1}{2!}[h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)] + \ldots$

$a = 2 \quad b = 1$

$f(2,1) = 8 - 3(2) = 2$

$\rightarrow f_x(2,1) = 3(4) - 3(1) = 9 \qquad f_y(2,1) = -6(2)(1) = -12$

$f_{xx}(x,y) = 6x \qquad f_{yy}(x,y) = -6x \qquad f_{xy}(x,y) = -6y$

$f_{xx}(2,1) = 12 \qquad f_{yy}(2,1) = -12 \qquad f_{xy}(2,1) = -6$

$\therefore f(2+h, 1+k) = 2 + [h(9) + k(-12)] + \frac{1}{2!}[h^2(12) + 2hk(-6) + k^2(-12)]$

$+ \frac{1}{3!}[h^3(6) + k^3(0) + 3h^2k(0) + 3hk^2(-6)]$

$\Rightarrow f(2+h, 1+k) = 2 + 9h - 12k + 6h^2 - 6hk - 6k^2 + h^3 - 3hk^2$

Proved

10. $F(x) = xyz - a^2$ at (x_1, y_1, z_1)

$\frac{dF}{dx} = y_1z_1 \;\Big|\; \frac{dF}{dy} = x_1z_1 \;\Big|\; \frac{dF}{dz} = x_1y_1$

Tangent Plane : $\frac{dF}{dx}(x - x_1) + \frac{dF}{dy}(y - y_1) + \frac{dF}{dz}(z - z_1) = 0$

$\Rightarrow 3x_1y_1z_1 - x y_1z_1 - y x_1z_1 - z x_1y_1 = 0$

$\Rightarrow \cancel{\frac{x_1}{x} + \frac{y_1}{y} + \frac{z_1}{z}}$. $\boxed{\frac{x}{x_1} + \frac{y}{y_1} + \frac{z}{z_1} = 3}$

Normal Line :

$$\frac{x-x_1}{1/x_1} = \frac{y-y_1}{1/y_1} = \frac{z-z_1}{1/z_1}$$

$\Rightarrow \boxed{x_1(x-x_1) = y_1(y-y_1) = z_1(z-z_1)}$

10. $F(x) = 2x^2 + y^2 - 3 + 2z$ at $(2, 1, -3)$

$\frac{dF}{dx} = 2(4) = 8$ $\quad \frac{dF}{dy} = 2$ $\quad \frac{dF}{dz} = 2$

- Tangent Plane :

$8(x-2) + (y-1)2 + 2(z+3) = 0$

$\cancel{8}\,4x - 8 + y - 1 + z + 3 = 0$

$\boxed{4x + y + z = 6}$

- Normal Line :

$$\frac{x-2}{4} = \frac{y-1}{1} = \frac{z+3}{1}$$

11 $F(x) = x^2 + y^2 + z^2 - a^2$

Normal Line $\rightarrow$ $\frac{X-x}{x} = \frac{Y-y}{y} = \frac{Z-z}{z}$

12. $F(x) = x^2 + y^2 + z^2 - 30$ at $(-1, 2, 5)$

$\frac{dF}{dx} = -2$ $\quad \frac{dF}{dy} = 4$ $\quad \frac{dF}{dz} = 10$

T : $-2(x+1) + 4(y-2) + 10(z-5) = 0$

N : $\frac{x+1}{-2} = \frac{y-2}{4} = \frac{z-5}{10}$

PAGE NO.

DATE: / /

13. $\cancel{E(x)}$ $F = x^2+y^2-z$ at $(1,-2,5)$

$\frac{dF}{dx} = 2$ | $\frac{dF}{dy} = -4$ | $\frac{dF}{dz} = -1$

T: $2(x-1) \cancel{+} -4(y+2) - 1(z-5) = 0$

N: $\frac{x-1}{2} = \frac{y+2}{-4} = \frac{z-5}{-1}$

14. (a) $F(x,y,z) = \ln(x+y+z) - 2$

at $P(-1, e^2, 1)$ $\frac{dF}{dx} = \frac{1}{e^2}$ | $\frac{dF}{dy} = \frac{1}{e^2}$ | $\frac{dF}{dz} = \frac{1}{e^2}$

∴ T: $\frac{1}{e^2}\left[(x+1) + (y-e^2) + (z-1)\right] = 0$

N: $\cancel{e}(x+1) = \cancel{e}(y-e^2) = \cancel{e}(z-1)$

(b) $F = x^2+y^2+z^2-1$

$\frac{dF}{dx} = \frac{2}{\sqrt{3}} = \frac{dF}{dy} = \frac{dF}{dz}$ at $P\left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$

T: $\frac{2}{\sqrt{3}}\left[\left(x-\frac{1}{\sqrt{3}}\right) + \left(y-\frac{1}{\sqrt{3}}\right) + \left(z-\frac{1}{\sqrt{3}}\right)\right] = 0$

$\boxed{x+y+z-\sqrt{3} = 0}$

N: $x - \frac{1}{\sqrt{3}} = y - \frac{1}{\sqrt{3}} = z - \frac{1}{\sqrt{3}}$