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F-24

TUTORIAL 8 : Beta & Gamma f^n

1. The beta f^n is defined as —

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \qquad m>0 \ \& \ n>0$$

The gamma f^n is defined as —

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\,dx \qquad n>0$$

Relation between Beta & Gamma f^n :

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Proof:

$$\Gamma(m) = \int_0^\infty e^{-t} t^{m-1}\,dt \qquad \text{Put } t = x^2, \quad dt = 2x\,dx$$

$$\Gamma(m) = 2\int_0^\infty e^{-x^2} x^{2m-1}\,dx \qquad -①$$

Similarly,

$$\Gamma(n) = 2\int_0^\infty e^{-y^2} y^{2n-1}\,dy \qquad -②$$

① × ② →

$$\Gamma(m)\Gamma(n) = 4\int_0^\infty e^{-x^2} x^{2m-1}\,dx \int_0^\infty e^{-y^2} y^{2n-1}\,dy$$

$$\Gamma(m)\Gamma(n) = 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)} x^{2m-1} y^{2n-1}\,dx\,dy \qquad -③$$

Put $x = r\cos\theta$, $y = r\sin\theta$ $\therefore dx\,dy = r\,d\theta\,dr$

LIMITS : $r - 0$ to ∞ $\quad \theta - 0$ to $\pi/2$

$$\therefore \Gamma(m)\Gamma(n) = 4\int_0^\infty\int_0^{\pi/2} e^{-r^2} r^{2(m+n)-1} \cos^{2m-1}\theta \sin^{2n-1}\theta\,d\theta\,dr$$

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$$\Gamma m\, \Gamma n = \left[2\int_0^{\pi/2} \cos^{2m-1}\theta \, \cancel{d\theta}\, \sin^{2n-1}\theta \, d\theta\right] \times \left[2\int_0^{\infty} e^{-r^2} r^{2(m+n)-1} dr\right]$$

Using ① gives

$$2\int_0^{\infty} e^{-r^2} r^{2(m+n)-1} = \Gamma(m+n)$$

$$\therefore \Gamma(m)\Gamma(n) = \underbrace{\left[2\int_0^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta \, d\theta\right]}_{I_1} \Gamma(m+n)$$

$$I_1 = 2\int_0^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta \, d\theta \qquad \text{put } \sin^2\theta = x$$

$$\therefore \sin\theta\, 2\cos\theta \, d\theta = dx$$

$$I_1 = \frac{2}{2}\int_0^1 \cancel{\cos\theta}\,(1-x)^{m-1} x^{n-1} dx = \beta(m,n)$$

$$\therefore \Gamma(m)\Gamma(n) = \beta(m,n)\,\Gamma(m+n)$$

$$\Rightarrow \boxed{\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}}$$

Proved

2. (i) To Prove : $\beta(m,n) = \beta(n,m)$

Proof : $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$

Put $x = 1-y$

$$\therefore \beta(m,n) = -\int_1^0 (1-y)^{m-1} (y)^{n-1} dy$$

$$\Rightarrow \beta(m,n) = \int_0^1 y^{n-1}(1-y)^{m-1} dy = \beta(n,m)$$

$$\boxed{\therefore \beta(m,n) = \beta(n,m)}$$

Proved

(ii) $B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$ → To Proved

→ Proof : $\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$

Put $x = \sin^2\theta$ $dx = 2\sin\theta\cos\theta \, d\theta$

$$\beta(m,n) = \int_0^{\pi/2} \sin^{2m-2}\theta \cos^{2n-2}\theta \, (2)\sin\theta\cos\theta \, d\theta$$

$$\Rightarrow \boxed{\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta}$$ → Proved

(iii) $n\beta(m+1,n) = m\beta(m,n+1)$ → To Prove

Proof :

We know, $\beta(m+1,n) = \dfrac{\Gamma(m+1)\Gamma(n)}{\Gamma(m+n+1)}$

Also, $\Gamma(m+1) = m\Gamma m$

$$\therefore \beta(m+1,n) = \frac{m\Gamma(m)\Gamma(n)}{\Gamma(m+n+1)} \quad -① $$

Similarly,

$$\beta(m,n+1) = \frac{n\Gamma(m)\Gamma(n)}{\Gamma(m+n+1)} \quad -②$$

From ① & ②,

$n \times ① = m \times ②$

$\therefore$ ~~AMT~~ $\boxed{n\beta(m+1,n) = m\beta(m,n+1)}$

Proved

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(iv) $\beta(m,n) = \beta(m,n+1) + \beta(m+1,n)$ → To Prove

Proof :

RHS => $\beta(m,n+1) + \beta(m+1,n)$

$\left[\text{Using } \beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\right]$

$\Rightarrow \frac{\Gamma(m)\Gamma(n+1)}{\Gamma(m+n+1)} + \frac{\Gamma(m+1)\Gamma(n)}{\Gamma(m+n+1)}$

$\Rightarrow \frac{n\Gamma m\Gamma n}{(m+n)\Gamma(m+n)} + \frac{m\Gamma(m)\Gamma(n)}{(m+n)\Gamma(m+n)}$ $\left[\text{Using } \Gamma(n+1) = n\Gamma(n)\right]$

$\Rightarrow \frac{(m+n)\Gamma(m)\Gamma(n)}{(m+n)\Gamma(m+n)}$

$\Rightarrow \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = \beta(m,n) = \text{LHS}$

Proved

3 (i) $\Gamma(n) = \int_0^1 \left(\log\frac{1}{y}\right)^{n-1} dy$ → To Prove

Proof :

RHS => $I = \int_0^1 \left(\log\frac{1}{y}\right)^{n-1} dy$ Put $-\log y = x$

$\Rightarrow I = -\int_\infty^0 x^{n-1} e^{-x} dx$ $y = e^{-x}$

$dy = -e^{-x} dx$

$I = \int_0^\infty e^{-x} x^{n-1} dx$ = Gamma f^n $\Gamma(n)$

$\therefore \boxed{\Gamma n = \int_0^1 \left(\log\frac{1}{y}\right)^{n-1} dy}$ PROVED

(iii) $\Gamma(n)\Gamma(1-n) = \dfrac{\pi}{\sin n\pi}$ → To Prove

Proof :

LHS → $\Gamma(n)\Gamma(1-n)$

$\Rightarrow \Gamma(n+1-n)\,\beta(n,1-n)$

$\Rightarrow \Gamma(1)\,\beta(n,1-n)$

$\Rightarrow \beta(n,1-n)$

$\Rightarrow \displaystyle\int_0^1 x^{n-1}(1-x)^{-n}\,dx$

$\Rightarrow \cancel{\displaystyle\int_0 a^{-n}(1-a)^{n-1}}$

$\Rightarrow \displaystyle\int_0^\infty \frac{y^{n-1}}{(y+1)^{n-1}(1+y)^{-n}(1+y)^2}\,dy$

$\Rightarrow \displaystyle\int_0^\infty \frac{y^{n-1}}{(1+y)}\,dy$

$\Rightarrow \dfrac{\pi}{\sin n\pi}$

Using $\beta(m,n) = \dfrac{\Gamma m\,\Gamma n}{\Gamma(m+n)}$

~~Put $1-x = a$~~

~~$-dx = da$~~

~~Put $\frac{1}{x} = a$~~

~~$-\frac{1}{x^2}dx = da$~~

Put $x = \dfrac{y}{1+y}$

$dx = \dfrac{1}{(1+y)^2}\,dy$

Using Euler's Reflection formula

$\displaystyle\int_0^\infty \frac{y^{n-1}}{(1+y)}\,dy = \frac{\pi}{\sin n\pi}$

$\therefore \boxed{\Gamma(n)\Gamma(1-n) = \dfrac{\pi}{\sin n\pi}}$ Proved

4. $\displaystyle\int_0^1 x^m(1-x^n)^p\,dx$

$\Rightarrow$ Put $x^n = a$ $\qquad nx^{n-1}dx = da$

$\Rightarrow \dfrac{1}{n}\displaystyle\int_0^1 n x^{n-1}\, x^{m-n+1}(1-x^n)^p\,dx$

$\Rightarrow \dfrac{1}{n}\displaystyle\int_0^1 a^{\frac{m+1}{n}-1}(1-a)^p\,da$

Using $\beta(m,n) = \displaystyle\int_0^1 x^{m-1}(1-x)^{n-1}d$

$\Rightarrow \dfrac{1}{n}\,\beta\left(\dfrac{m+1}{n}, p+1\right)$

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Evaluate : $\int_0^1 x^5(1-x^3)^{10} dx$

$\therefore m=5, \quad n=3, \quad p=10$

$\therefore \int_0^1 x^5(1-x^3)^{10} dx = \frac{1}{3} \beta\left(\frac{5+1}{3}, 10+1\right)$

$\Rightarrow \int_0^1 x^5(1-x^3)^{10} dx = \frac{1}{3} \beta(2, 11)$

$= \frac{1}{3} \frac{\Gamma 2 \, \Gamma 11}{\Gamma 13} = \frac{1}{3} \frac{1! \; 10!}{12!}$

$\Rightarrow \int_0^1 x^5(1-x^3)^{10} dx = \frac{1}{3(12)(11)} = \frac{1}{396}$

5. To Prove : $\int_0^\infty \frac{dx}{(e^x+e^{-x})^n} = \frac{1}{4} \beta\left(\frac{n}{2}, \frac{n}{2}\right)$

Proof : LHS $\Rightarrow \int_0^\infty \frac{(e^{-x})^n}{(e^{-2x}+1)^n} dx$

Put $e^{-2x} = a$

$-2e^{-2x} dx = da$

$\Rightarrow -\frac{1}{2} \int_1^0 \frac{1}{a} \frac{a^{n/2}}{(1+a)^n} da$

$\Rightarrow \frac{1}{2} \int_0^1 \frac{a^{n/2-1}}{(1+a)^n} da$

~~Put $a = \frac{y}{1-y}$~~

~~$da = \frac{1}{(1-y)^2} dy$~~

~~$\Rightarrow \frac{1}{2} \int_0^{1/2} \frac{y^{n/2-1}}{(1-y)^{n/2-1}} \frac{(1-y)^n}{(1-y)^2} dy$~~

~~$\Rightarrow \frac{1}{2} \int_0^{1/2} y^{n/2-1} (1-y)^{n/2-1} dy$~~

~~Put $2y = x$~~

~~$dy = \frac{1}{2} dx$~~

~~$\Rightarrow \frac{1}{4} \int_0^1 \frac{1}{2^{n/2-1}} x^{n/2-1}$~~

Using $B(m,n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx$

$\therefore \frac{1}{2}\int_0^1 \frac{a^{n/2-1}}{(1+a)^n} da = \frac{1}{4}\int_0^1 \frac{a^{n/2-1} + a^{n/2-1}}{(1+a)^n} da$

$\therefore LHS = \frac{1}{4}\int_0^1 \frac{a^{n/2-1} + a^{n/2-1}}{(1+a)^n} da = \frac{1}{4} B\left(\frac{n}{2}, \frac{n}{2}\right) = RHS$

Proved

~~6 $B(m,n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{}$~~

Evaluate: $\int_0^\infty \operatorname{sech}^8 x \, dx$

$\Rightarrow \int_0^\infty \frac{2^8}{(e^x + e^{-x})^8} dx$ here $n = 8$

$\therefore \int_0^\infty \frac{2^8}{(e^x+e^{-x})^8} dx = \frac{1}{4}(2^8) B\left(\frac{8}{2}, \frac{8}{2}\right)$

$= 2^6 B(4,4)$

$= 2^6 \frac{\Gamma 4 \, \Gamma 4}{\Gamma 8} = 2^6 \frac{3! \, 3!}{7!}$

$= \frac{2^6 \cdot 2^2 \cdot 3^2}{7 \times 6 \times 5 \times 24} = \frac{2^8 \cancel{3^2}}{35 \times 2^4 \times \cancel{3^2}}$

$\Rightarrow \int_0^\infty \operatorname{sech}^8 x \, dx = \frac{16}{35}$

6. To Prove: $B(m,n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx$

Proof:

Using $B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$

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$$\Rightarrow \beta(m,n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

$\hookrightarrow$ Put $x = 1/y$

$$\Rightarrow \beta(m,n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \left[-\int_1^0 \frac{1}{y^{m-1}}\left(\frac{1}{y^2}\right)\frac{y^{m+n}}{(y+1)^{m+n}}\,dy\right]$$

$$\beta(m,n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_0^1 \frac{y^{n-1}}{(y+1)^{m+n}}\,dy$$

$$\therefore \beta(m,n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} \quad \text{Proved}$$

7. To Prove : $\beta(m,m) \times \beta(m+\frac{1}{2}, m+\frac{1}{2}) = \frac{\pi}{m} \times 2^{1-4m}$

$\Rightarrow$ Proof : $\beta(m,m) = \frac{\Gamma(m)\,\Gamma(m)}{\Gamma(2m)}$ — ①

$\beta(m+\frac{1}{2}, m+\frac{1}{2}) = \frac{\Gamma(m+\frac{1}{2})\,\Gamma(m+\frac{1}{2})}{\Gamma(2m+1)}$ — ②

$$\therefore \beta(m,m) \times \beta(m+\tfrac{1}{2}, m+\tfrac{1}{2}) = \frac{\Gamma(m)\,\Gamma(m)\,\Gamma(m+\frac{1}{2})\,\Gamma(m+\frac{1}{2})}{\Gamma(2m)\quad\Gamma(2m+1)}$$

$$= \frac{1}{2m}\left[\frac{\Gamma(m)\,\Gamma(m+\frac{1}{2})}{\Gamma(2m)}\right]^2$$

Using Duplication Formula –

$$\Gamma(n)\,\Gamma(n+\tfrac{1}{2}) = \frac{\sqrt{\pi}\,\Gamma(2n)}{2^{2n-1}}$$

$$\therefore \beta(m,m)\,\beta(m+\tfrac{1}{2}, m+\tfrac{1}{2}) = \frac{1}{2m}\left[\frac{\sqrt{\pi}\,\Gamma(2m)}{2^{2m-1}\,\Gamma(2m)}\right]^2$$

$$= \frac{\pi}{m}\,2^{2-4m-1} = \frac{\pi}{m}\,2^{1-4m}$$

Proved

8. $\displaystyle\int_0^1 \frac{x}{\sqrt{1-x^5}}\,dx = \frac{1}{5}\beta\left(\frac{2}{5},\frac{1}{2}\right)$ → To Proved.

Put $x^5 = a$
$5x^4\,dx = da$
∴ Limits :
$X : 0$ to 1
so, $a : 0$ to 1

$\Rightarrow \displaystyle\frac{1}{5}\int_0^1 \frac{a^{1/5}}{a^{4/5}\sqrt{1-a}}\,da$

$\Rightarrow \displaystyle\frac{1}{5}\int_0^1 a^{2/5-1}\cdot(1-a)^{1/2-1}\,da = \frac{1}{5}\beta\left(\frac{2}{5},\frac{1}{2}\right)$

$\therefore \displaystyle\int_0^1 \frac{x}{\sqrt{1-x^5}}\,dx = \frac{1}{5}\beta\left(\frac{2}{5},\frac{1}{2}\right)$ Proved

9. $\displaystyle\int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}}\,dx = \frac{1}{a^n b^m}\beta(m,n)$ → To Prove

→ LHS :

Put $\dfrac{bx}{a} = \dfrac{y}{1-y}$ $\quad \therefore \dfrac{b\,dx}{a} = \dfrac{(1-y)+y}{(1-y)^2}\,dy$

Limits $x : 0 \to \infty$ ∴ $y : 0 \to 1$

$\Rightarrow \displaystyle\frac{a}{(a)^{m+n}b}\int_0^1 \frac{y^{m-1}}{(1-y)^{m-1}\left(1+\left(\frac{y}{1-y}\right)\right)^{m+n}}\frac{1}{(1-y)^2}\,dy$

$\Rightarrow \displaystyle\frac{1}{b^{m+n}}\int_0^1 \frac{y^{m-1}\,(1-y)^{m+n}}{(1-y)^{m-1}\left(\frac{a}{b}(1-y)+y\right)^{m+n}}\,dy$ X

$\Rightarrow \displaystyle\frac{1}{a^{m+n}}\int_0^1 \frac{a}{b}\left(\frac{a}{b}\right)^{m-1}\frac{y^{m-1}(1-y)^{m+n}}{(1-y)^{m-1}(1)}\frac{1}{(1-y)^2}\,dy$

$\Rightarrow \displaystyle\frac{1}{a^m b^n}\int_0^1 y^{m-1}(1-y)^{n-1}\,dy = \frac{1}{a^m b^n}\beta(m,n)$ Proved

10. (a) $\int_0^\infty \operatorname{sech}^6 x \, dx$

$\Rightarrow 2^6\int_0^\infty \frac{1}{(e^x+e^{-x})^6}\,dx = 2^6\int_0^\infty \frac{(e^{-x})^6}{(1+e^{-2x})^6}\,dx$

Put $e^{-2x} = y$

$-2e^{-2x}\,dx = dy$

Limits x: 0 to ∞ so, y: 1 to 0

$-\frac{2^6}{2}\int_1^0 \frac{y^{2}}{(1+y)^6}\,dy = \frac{2^6}{2}\int_0^1 \frac{y^{2}}{(1+y)^6}\,dy$

$\Rightarrow \frac{2^6}{4}\int_0^1 \frac{y^2+y^2}{(1+y)^6}\,dy$ $\left[\text{Using, } B(m,n) = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx\right.$

$\Rightarrow \frac{2^6}{4} B(3,3)$

$\Rightarrow \frac{2^6}{4}\,\frac{\Gamma 3\,\Gamma 3}{\Gamma 6} = \frac{(2^6)}{4}\,\frac{2!\,2!}{5!} = \frac{2^6}{120} = \frac{2^4}{30} = \frac{8}{15}$

(b) $\int_0^1 \frac{dx}{\sqrt{-\log x}} = \pi$

Put $-\log x = a$

$-\frac{1}{x}\,dx = da$

$\Rightarrow -\int_\infty^0 e^{-a}a^{-1/2}\,da = \int_0^\infty e^{-a}a^{1/2-1}\,da$

$\Rightarrow \left[\text{Using } \Gamma n = \int_0^\infty e^{-x}x^{n-1}\,dx\right]$

$\therefore \int_0^\infty e^{-a}a^{\frac{1}{2}-1}\,da = \Gamma\frac{1}{2} = \sqrt{\pi}$

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(c) $\int_0^\infty e^{-y^{+1/m}} dy = m\Gamma m$ $\quad$ [Put $y^{+1/m} = a$]

$\Rightarrow$ $\cancel{-\frac{1}{m} y^{\frac{-(1+m)}{m}} dy = da}$ $\quad$ $y = a^{+m}$

$dy = +m a^{+m-1} da$

Limits $y : 0$ to ∞ so $a : 0$ to ∞

$\Rightarrow +\int_0^\infty e^{-a} a^{+m-1} (m)\, da = m\int_0^\infty e^{-a} a^{m-1} da = m\Gamma m$

d) $\int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta = \frac{1}{2}\Gamma(1/4)\Gamma(3/4)$

RHS $\Rightarrow$ $\frac{1}{2}\Gamma(1/4)\Gamma(3/4) = \frac{1}{2}B(1/4, 3/4)\,\Gamma(1)$

$= \frac{1}{2}B(\frac{1}{4}, 3/4)$

LHS $\Rightarrow$ $\int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta = \int_0^{\pi/2} \frac{\sqrt{\cos\theta}}{\sqrt{\sin\theta}}\, d\theta$

Put $\sqrt{\sin\theta} = a$

$\frac{1}{2}\frac{\cos\theta\, d\theta}{\sqrt{\sin\theta}} = da$

Limits
$\theta : 0$ to $\pi/2$
$\therefore a : 0$ to 1

$\Rightarrow 2\int_0^1 \frac{da}{\sqrt{1-a^4}}$ $\quad$ Put $a = x^{1/4}$

$da = \frac{1}{4} x^{-3/4} dx$

$\Rightarrow \frac{1}{2}\int_0^1 \frac{x^{-3/4}}{\sqrt{1-x}}\, dx$

$\Rightarrow \frac{1}{2}\int_0^1 x^{1/4-1}(1-x)^{1/2-1} dx = \frac{1}{2}B(\frac{1}{4}, \frac{1}{2})$

$\Rightarrow \int_0^{\pi/2} \cos\theta^{1/2} \sin\theta^{-1/2} d\theta$ $\quad$ Using $B(m,n) = 2\int_0^{\pi/2} \sin\theta^{2m-1} \cos\theta^{2n-1} d\theta$

$\Rightarrow \frac{1}{2}B(\frac{1}{4}, 3/4)$

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11. $\int_{-\infty}^{\infty} e^{-k^2x^2}\,dx = \pi/2$ & $\int_0^{\infty} e^{-x^3}\,dx = \frac{1}{3}\Gamma(\frac{1}{3})$.

=> ~~Put~~

$$I_1 = \int_{-\infty}^{\infty} e^{-k^2x^2}\,dx = 2\int_0^{\infty} e^{-k^2x^2}\,dx$$

Put $k^2x^2 = a$ $\qquad 2k^2x\,dx = da$

$$I_1 = \frac{1}{2k^2}\int_0^{\infty} \frac{k}{\sqrt{a}} e^{-a}\,da$$

$$I_1 = \frac{1}{2k}\int_0^{\infty} e^{-a} a^{-1/2}\,da = \frac{1}{2k}\Gamma(1/2)$$

$$\boxed{I_1 = \frac{\sqrt{\pi}}{2k}}$$

=> $I_2 = \int_0^{\infty} e^{-x^3}\,dx$

Put $x^3 = a$
$3x^2\,dx = da$

=> $I_2 = \frac{1}{3}\int_0^{\infty} e^{-a} a^{-2/3}\,da$

$$I_2 = \frac{1}{3}\int_0^{\infty} e^{-a} a^{1/3-1}\,da = \frac{1}{3}\Gamma\tfrac{1}{3}$$

12. To Prove —

$$\int_a^b (x-a)^{l-1}(b-x)^{m-1}\,dx = (b-a)^{l+m-1}\,\beta(l,m)$$

Proof: Put $x-a = y$
$dx = dy$

=> $\int_0^{b-a} (y)^{l-1}(b-a-y)^{m-1}\,dy$

=> $(b-a)^{m-1}\int_0^{b-a} y^{(l-1)}\left(1-\frac{y}{b-a}\right)^{m-1}dy$

Put $\frac{y}{b-a} = x \mid \frac{dy}{b-a} = dx$,

$$\Rightarrow (b-a)(b-a)^{m-1}(b-a)^{l-1}\int_0^1 x^{l-1}(1-x)^{m-1}dx$$

$$\Rightarrow (b-a)^{l+m-1}\int_0^1 x^{l-1}(1-x)^{m-1}dx = (b-a)^{l+m-1}\beta(l,m)$$

Proved

13. To Prove: $\int_0^\infty x^{m-1}\cos ax\, dx = \frac{\Gamma(m)}{a^m}\cos\left(\frac{m\pi}{2}\right)$

Proof:

$$I = \int_0^\infty x^{m-1} e^{-iax} dx = \int_0^\infty x^{m-1}[\cos ax - i\sin ax]\,dx$$

We need real part here

Put $iax = y$

$ia\,dx = dy$

$$\Rightarrow \frac{1}{ia}\int_0^\infty e^{-y}\frac{y^{m-1}}{(ia)^{m-1}}dy = \frac{1}{(ia)^m}\int_0^\infty e^{-y}y^{m-1}dy$$

$$\Rightarrow \frac{1}{(ia)^m}\Gamma m \qquad \left[i = e^{i\pi/2} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right]$$

$$\Rightarrow \frac{\Gamma(m)}{a^m} i^{-m} = \frac{\Gamma(m)}{a^m}\left[\cos\frac{m\pi}{2} - i\sin\frac{m\pi}{2}\right]$$

$\therefore$ Real Part $= \frac{\Gamma(m)}{a^m}\cos m\pi/2$

Proved

14. Given: $\int_0^\infty \frac{x^{n-1}}{1+x}dx = \frac{\pi}{\sin n\pi}$ Show $\Gamma n\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}$

$$\Rightarrow \Gamma(n)\Gamma(1-n) = \beta(n,1-n)\,\Gamma(1) = \beta(n,1-n)$$

$$\beta(n,1-n) = \int_0^1 x^{n-1}(1-x)^{-n}dx$$

Put $x = \frac{y}{1+y}$ $dx = \frac{1}{(1+y)^2} dy$

Limits x : 0 to 1 $\therefore$ y : 0 to ∞

$$\Rightarrow \int_0^\infty \frac{1}{(1+y)^2} \frac{y^{n-1}}{(1+y)^{n-1}(1+y)^{-n}} dy$$

$$\Rightarrow \beta(n, 1-n) = \int_0^\infty \frac{y^{n-1}}{(1+y)} dy$$

$$\therefore \Gamma(n)\Gamma(1-n) = \int_0^\infty \frac{y^{n-1}}{1+y} dy$$

$$\because \int_0^\infty \frac{y^{n-1}}{1+y} dy = \frac{\pi}{\sin n\pi} \quad \text{given}$$

$$\therefore \Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}$$

Evaluate : $\int_0^\infty \frac{dy}{1+y^4}$

Put $y = x^{1/4}$ $dy = \frac{1}{4} x^{-3/4} dx$

$$\Rightarrow \frac{1}{4}\int_0^\infty \frac{x^{-3/4}}{1+x} dy = \frac{1}{4} B\left(\frac{1}{4}, \frac{3}{4}\right)$$

$$= \frac{1}{4}\Gamma(n)\Gamma(1-n)$$

$$= \frac{1}{4} \frac{\pi}{\sin\left(\frac{1}{4}\cdot\pi\right)} = \frac{\pi}{2\sqrt{2}}$$