GARVIT SHAH
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TUTORIAL 7: Curve Tracing - V
Cartesian, Polar & Parametric

1. Draw the circle : $x^2+y^2-4x+4y+4=0$

→ $F(x,y) = x^2+y^2-4x+4y+4=0$

$= (x-2)^2+(y+2)^2-4-4+4=0$

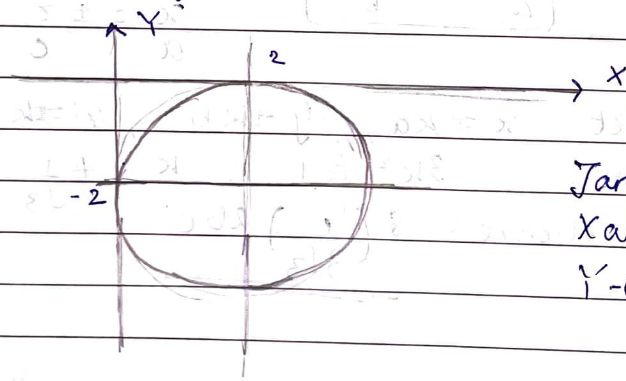
$= (x-2)^2+(y+2)^2=4$

∴ Eq^n of circle : $(x-2)^2+(y+2)^2=4$

Gen form of circle having center (a,b) & radius r is $(x-a)^2+(y-b)^2=r^2$

∴ For given eq^n $a=2$, $b=-2$, $r=2$

∴



Tangents :
X axis at (2,0)
Y-axis at (0,-2)

2. (i) $y^2+4x-4y+8=0$

→ $y^2-4y+4=-4x-4$

⇒ $(y-2)^2=-4(x+1)$

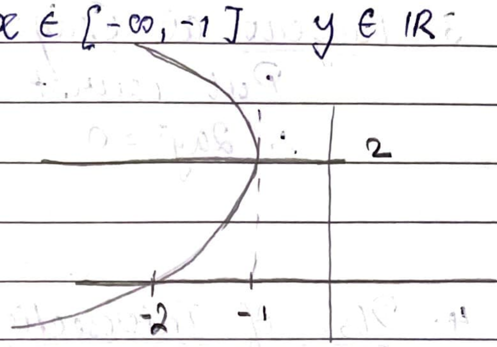
Gen form of Parabola : $(y-a)^2=4p(x-b)^2$

∴ $a=2$, $p=-1$, & $b=-1$

which means it is symmetric about $y=2$

Region of curve is $x \in [-\infty, -1]$ $y \in \mathbb{R}$

Tangent at $(-1,0)$ is
$x = -1$



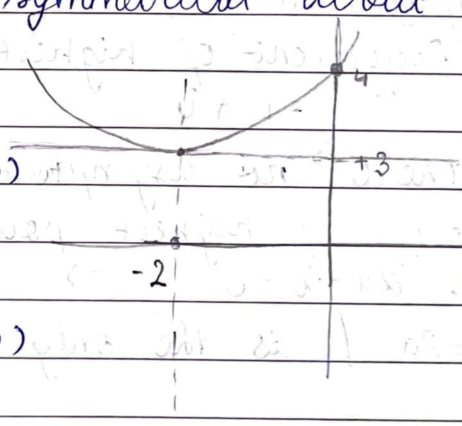
(ii) $x^2+4x-4y+16=0$

$\rightarrow$ $x^2+4x+4 = 4y-8-4$
$\Rightarrow$ $(x+2)^2 = 4(y-3)$

Gen form of Parabola : $(x-a)^2 = 4p(y-b)^2$

- $\therefore$ Curve is symmetrical about $x=a$

$a=-2$, $b=3$, $p=1$



- Tangent at vertex $(-2,3)$ is ~~x=~~ $y=-3$

- Region is $x \in (-\infty, \infty)$
  $y \in [+3, \infty)$

3. Trace : $y^2(2a-x) = x^3$

1. SYMMETRY –
   The eq^n is symmetric about X-axis

2. ORIGIN –
   Eq^n passes through $(0,0)$ $\because$ it passes through $x=0, y=0$

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3. Tangent at Origin -
Put lowest degree terms = 0
$\therefore 2ay^2 = 0$ $\therefore y = 0$ $\therefore$ X-axis is tangent at origin

4. Pts. of Intersection -
For pts. of X-axis : put $y = 0$
$\therefore x = 0$
For pts. of Y-axis : put $x = 0$
$y = 0$ $\therefore (0,0)$

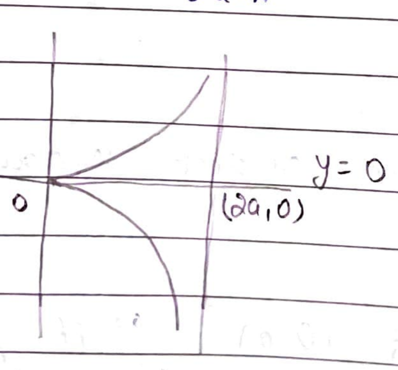
$\therefore$ Curve meets x-axis & y-axis at pt. $(0,0)$.

5. Assymptotes -
- Coefficient of highest power terms of $x = 0$
$-1 \neq 0$

$\therefore$ There is no asymptote $\parallel$ to X-axis
- Coeff. of highest power terms of $y = 0$
$\therefore 2a - x = 0 \Rightarrow x = 2a$
$[x = 2a]$ is the only asymptote $\parallel$ to Y-axis

6. Region -
$$y^2 = \frac{x^3}{2a-x}$$
Curve doesn't exist for $x > 2a$ & $x < 0$

(Cissiod)



A4 Trace the curve : $y^2x = a^2(a-x)$

$y^2x - a^3 + a^2x$

1. Symmetry -

Curve is symmetric about x-axis

2. Origin -

Putting (0,0) in Eq^n satisfies it ∴ curve passes through origin.

3. Tangent at origin -

Put lowest degree term = 0

$a^2x^0 \neq 0$ ∴ No tangent at origin

4. Tangent at (a,0)

taking $x-a=p$

$\therefore y^2(a+p) - a^3 + a^2(a+p) = 0$

$\Rightarrow y^2a + y^2p - a^3 + a^3 + a^2p = 0$

Put lowest degree ~~pe~~ term = 0 ∴ $a^2p = 0$

$\therefore p = 0$

∴ $x = a$ is a tangent at (a,0)

5. ~~Region~~ Asymptote -

Put coeff $y^2x = 0$ ∴ $x = 0$ ∴ $\boxed{\text{Y-axis an ~~tangent~~ assymptote}}$

coeff$(x(y^2+a^2)) = 0$ ∴ $y^2+a^2 \neq 0$

6. Region -

$y^2 = \frac{a^2(a-x)}{x}$

For $x > a$, $y^2 < 0$ → False

∴ No curve in $x > a$

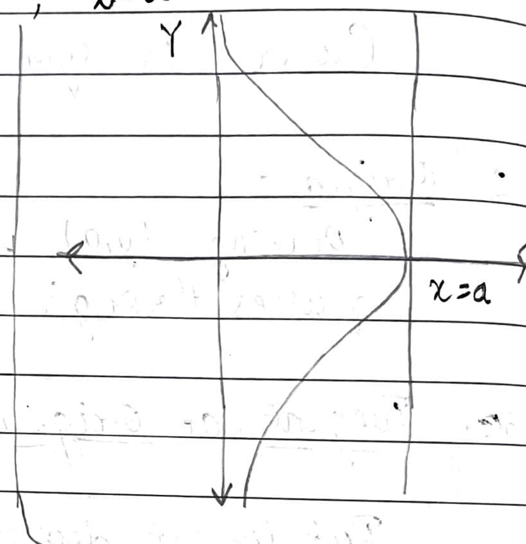
For $x < 0$, $y^2 < 0$ → False

∴ No curve in $x < 0$

7 Pt. of Intersection -

On Y-axis : Put $x = 0$, $y \to \infty$ or $y \to -\infty$

On x-axis : Put $y = 0$, $x = a$



5. $\sqrt{x} + \sqrt{y} = \sqrt{a}$

Put $a\sin^4 t = \cancel{x}\ y$

$a\cos^4 t = \cancel{y}\ x$

for $t = -t$ $\quad x = x$

$y = +y$

1. $\therefore$ Curve is symmetric about both axes.

2. ORIGIN :

Do not passe through (0,0)

3. Pts of Intersection :

X axis : at $y = 0$ $\quad x = a$ $\quad \therefore t = 0, \pi, 2\pi$

Y-axis : at $x = 0$ $\quad y = a$ $\quad \therefore t = \pi/2, 3\pi/2$

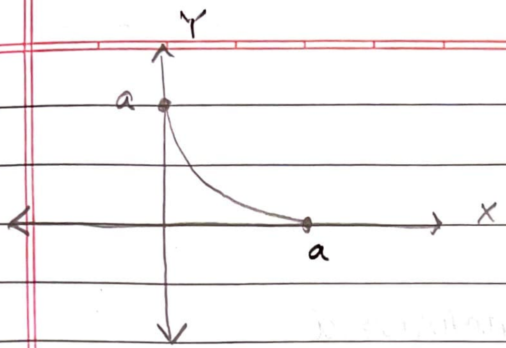
4. Region :

$x = a\cos^4 t$ $\quad \therefore x \in [0, a]$

$y = a\sin^4 t$ $\quad \therefore y \in [0, a]$

5. $$\frac{dy}{dx} = \frac{-4\sin^3 t \cos t}{4\cos^4 t \sin t} = \frac{-\sin^2 t}{\cos^2 t} = -\tan^2 t < 0$$

$$\frac{d^2y}{dx^2} = \frac{-2\tan t \sec^2 t}{4a\cos^3 t \sin t} = \frac{-\sec^6 t}{2a} < 0$$



6) $y = 2x + \frac{x^2}{2} - \frac{x^3}{3}$

① Symmetry: Non-symmetric

② Origin: Passes through origin (0,0)

③ Tangents at origin: $2x - y = 0$ $\boxed{\therefore y = 2x}$

④ Pts. of Intersection:

X axis: at $y = 0$ $\quad x\left(2 + \frac{x}{2} - \frac{x^2}{3}\right) = 0$

$-x(2x^2 - 3x - 12) = 0$

$\therefore x = 0$ or $x = \frac{3 \pm \sqrt{105}}{4}$ $\qquad 9 + 4(24)$, $\sqrt{105}$

Y axis: at $x = 0$, $y = 0$

⑤ Asymptotes:

Highest deg. of $x \to -\frac{x^3}{3}$ & $y \to -y$

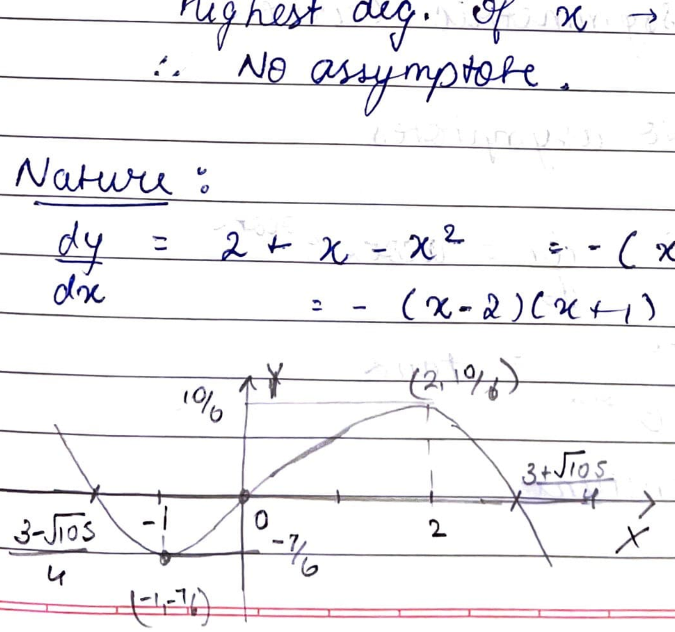
$\therefore$ No assymptote.

⑥ Nature:

$\frac{dy}{dx} = 2 + x - x^2 = -(x^2 - x - 2)$ $\qquad 1+8$

$= -(x-2)(x+1)$

− + −, −1, 2



at $x = -1$

$y = -2 + \frac{1}{2} + \frac{1}{3}$

$= -\frac{7}{3}$

at $x = 2$

$y = \frac{10}{6}$

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7. $r = a(1+\cos\theta)$

- ~~Symmetry~~ Symmetry :

For, θ & $-\theta$, r unchanged

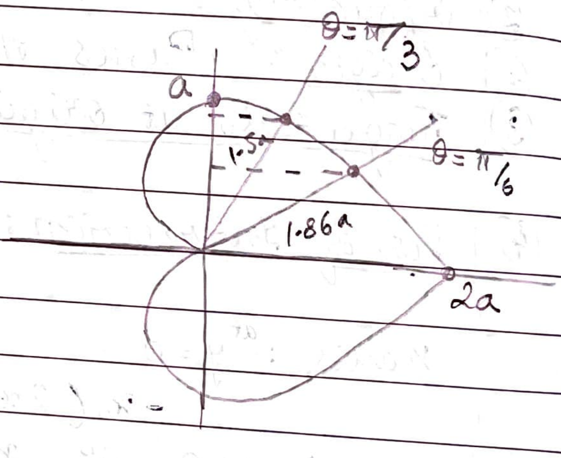
$\therefore$ Symmetric about $\theta = 0$

- ~~Origin~~ Pole :

For $r = 0$ $\theta = \pi$,

-

θ	r
0	$2a$
$\pi/6$	$a\left(1+\frac{\sqrt{3}}{2}\right) = 1.86a$
$\pi/3$	$a(3/2) = 1.5a$
$\pi/4$	$a(1+1/\sqrt{2}) = 1.71a$
$\pi/2$	a
$2\pi/3$	$a(1-1/2) = a/2$
$5\pi/6$	$a\left(1-\frac{\sqrt{3}}{2}\right) = 0.13a$
π	0



8.) $r = ae^{\theta\cot\alpha}$; $\alpha > 0$ Equiangular Spiral

① Symmetry : Assymmetric Curve

② Assymptotes : No asymptotes

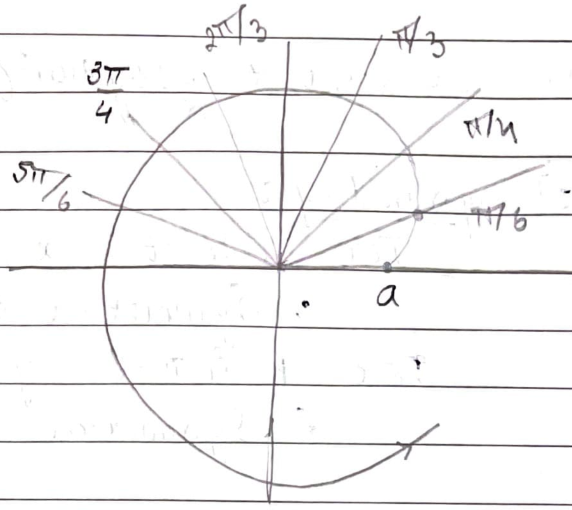
③ $\tan\phi = \frac{r}{dr/d\theta}$ $\frac{dr}{d\theta} = a\cot\alpha\, e^{\theta\cot\alpha}$

$\tan\phi = \frac{ae^{\theta\cot\alpha}}{a\cot\alpha\, e^{\theta\cot\alpha}} = \tan\alpha$

$\therefore \phi = \alpha \rightarrow$ constant

(4)

θ	r
0	a
$\pi/6$	$ae^{\pi/6 \cot\alpha}$
$\pi/4$	$ae^{\pi/4 \cot\alpha}$
$\pi/3$	$ae^{\pi/3 \cot\alpha}$
$\pi/2$	$ae^{\pi/2 \cot\alpha}$
π	$ae^{\pi \cot\alpha}$



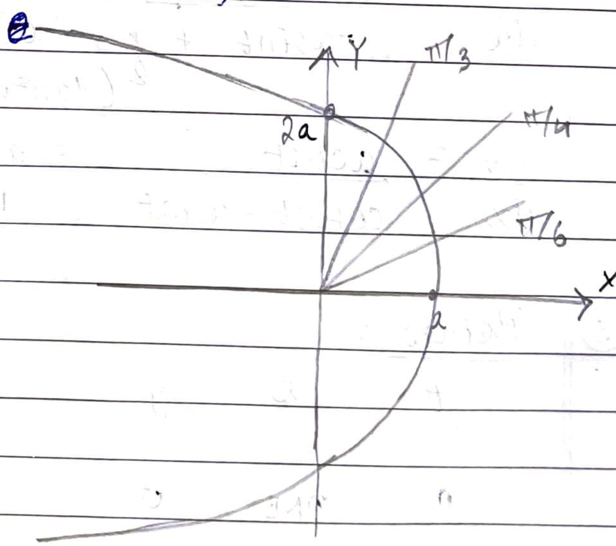
9. $r(1+\cos\theta) = 2a$ (Parabola opening on left)

(1) Symmetry : Symmetric about $\theta = 0$

(2) Pole : at $r=0$, ~~$\theta = \pi$~~ θ doesn't exist
∴ doesn't pass through pole.

(3) $r = \dfrac{2a}{1+\cos\theta}$ at $\theta = \pi$ $r \to \infty$

θ	r
0	a
$\pi/6$	$\dfrac{2a}{1+\sqrt{3}/2}$
$\pi/4$	$1.17a$
$\pi/3$	$1.33a$
$\pi/2$	$2a$
$2\pi/3$	$4a$



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10. $x = a\cos t + \frac{1}{2}\log\tan^2\left(\frac{t}{2}\right)$ $y = a\sin t$

① Symmetry :

For t & $-t$ x is unchanged

∴ Symmetric about X-axis

For t & $\pi - t$ y is unchanged

∴ Symmetric about Y-axis

∴ Curve is symmetric about both axes.

② ~~Pole~~ Origin : doesn't passe through pole

③ Pts. of Intersection :

For $x = 0$ ∴ $t = \pi/2, 3\pi/2$

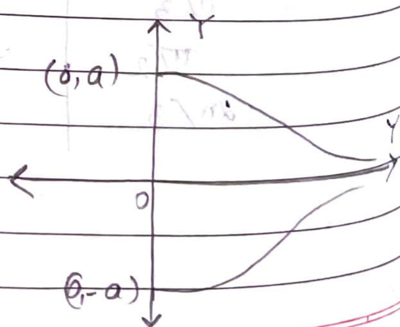
So, $y = a, -a$

④ $\frac{dy}{dx} = \frac{a\cos t}{-a\sin t + \frac{1}{2}\frac{\sec^2 t/2}{(\tan t/2)}\left(\frac{1}{2}\right)} = \frac{a\cos t}{\frac{1}{2\sin t/2 \cos t/2} - a\sin t}$

$\frac{dy}{dx} = \frac{a\cos t}{\operatorname{cosec} t - a\sin t} = \frac{a\cos t \sin t}{1 - a\sin^2 t}$

⑤ Points :

t	x	y	dy/dx
0	DNE	0	0
$\pi/2$	0	a	0
π	DNE	0	0



11. (i) $x = a\sin 2\theta\,(1+\cos 2\theta)$ $\quad y = a\cos 2\theta\,(1-\cos 2\theta)$

$\Rightarrow x = \cancel{a\sin 2\theta\,(2\cos^2\theta)}$ $\quad y = a\cos 2\theta\,(2\sin^2\theta)$

$x = a\sin 2\theta + \frac{a}{2}\sin 4\theta$ $\quad y = a\cos 2\theta - a\cos^2 2\theta$

① Symmetries :

For θ & $-\theta$ y is unchanged & $x \to -x$

$\therefore$ Symmetric about Y-axis

② Pts. of Intersection :

X axis : $y=0$ at $\theta = 0, \pi, \pi/4, 3\pi/4$

$\therefore x = 0, a, -a$

$\therefore (0,0), (a,0), (-a,0)$

Y axis : $x=0$ at $\theta = 0, \pi, \frac{\pi}{2}, \frac{3\pi}{2}$,

$\therefore y = 0, -2a,$

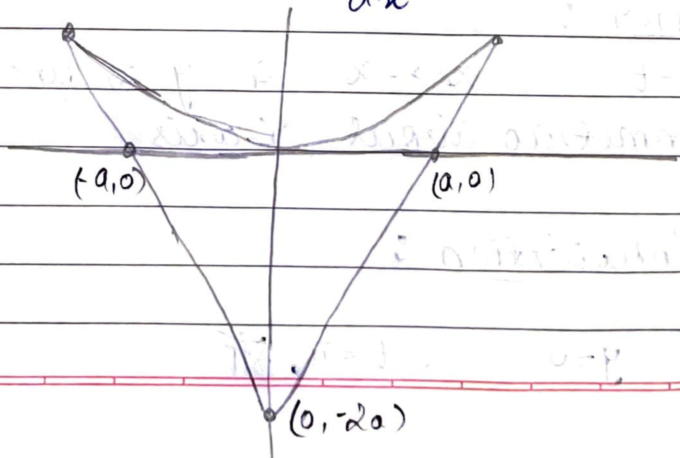
$\therefore (0,0), (0,-2a)$

③ $\frac{dy}{dx} = \frac{-2a\sin 2\theta + 2a\cos 2\theta\sin 2\theta(2)}{2a\cos 2\theta + 2a\cos 4\theta} = \frac{\sin 4\theta - \sin 2\theta}{\cos 4\theta + \cos 2\theta}$

$= \frac{2\cos 3\theta\sin\theta}{2\cos 3\theta\cos\theta} = \underline{\tan\theta}$

For $\frac{dy}{dx} = 0$ $\quad \tan\theta = 0$ $\quad \therefore \theta = 0, \pi$

$\therefore$ at $(0,0)$, $\frac{dy}{dx} = 0$



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(ii) $r = \dfrac{\sin^2\theta}{\cos\theta}$

① Symmetries :

For $\theta \to -\theta$, r is unchanged

$\therefore$ Symmetric about $\theta = 0$

② Asymptote :

For $\theta = \pi/2$, $r \to \infty$

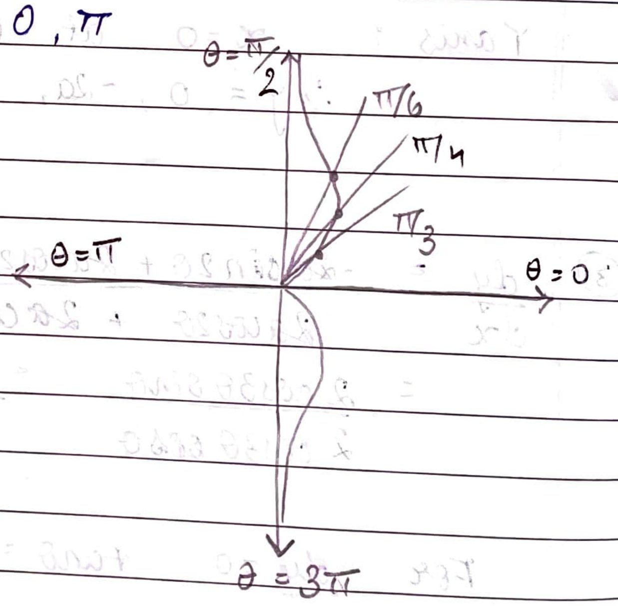
$\therefore$ $\theta = \pi/2$ is asymptote

③ Pole :

at $r = 0$, $\theta = 0, \pi$

④

θ	r
0	0
$\pi/6$	0.29
$\pi/4$	0.7
$\pi/3$	1.5



12) $x = a(t + \sin t)$

$y = a(1 + \cos t)$

SYMMETRY :

① for $t \to -t$, $x \to -x$ & y is unchanged

$\therefore$ Symmetric about Y-axis

② Pts. of Intersection :

X-axis : $y = 0$ $\therefore t = \pi, 3\pi$

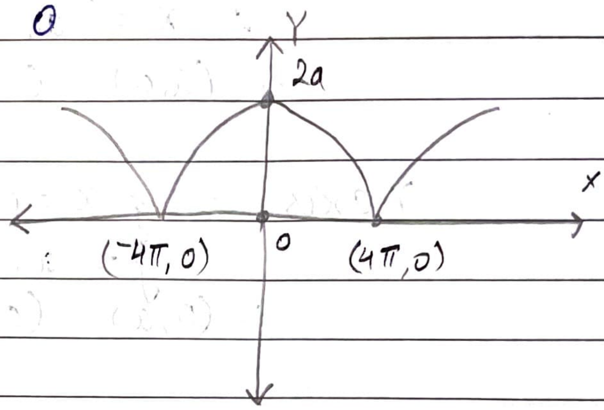
$\therefore x = a\pi,\ 3\pi a$

$\therefore$ ~~θ_z~~ $(a\pi, 0)$ $\quad (3\pi a, 0)$ $\rightarrow$ X-axis

Y axis : $x = 0$ $\quad \therefore t = 0$

$\therefore y = 2a$

$\therefore (0, 2a)$



(3) $y_{max} = 2a$

$y_{min} = 0$

(4) $\dfrac{dy}{dx} = \dfrac{a(-\sin t)}{a(1+\cos t)} = -\tan t/_2$

$\therefore$ for $t = 0$ $\quad \dfrac{dy}{dx} = 0$ $\quad \therefore x = 0,\ y = 2a$

$t = 2\pi$ $\quad \dfrac{dy}{dx} = 0$ $\quad \therefore x = 2a\pi,\ y = 2a$

$t = 4\pi$ $\quad \dfrac{dy}{dx} = 0$ $\quad \therefore x = 4a\pi,\ y = 2a$

13) $x = a\cos^3 t,\ y = a\sin^3 t$ $\quad$ [Cartesian form : $x^{2/3} + y^{2/3} = a^{2/3}$]

(2) Symmetry :

for $t \rightarrow -t$ $\quad y \rightarrow -y$ but x is unchanged

$\therefore$ Symmetric about X-axis

for $t \rightarrow \pi - t$ $\quad y \rightarrow y$ but $x \rightarrow -x$

$\therefore$ Symmetric about Y-axis

$\therefore$ Symmetric ~~of~~ about X & Y axes

② Pts. of Intersection :

X-axis : $y = 0$ $\therefore t = 0, \pi, \cancel{3\pi}\ -\pi$

$\therefore x = a, -a$

$\therefore (a, 0) \quad (-a, 0)$

Y axis : $x = 0$ $\therefore t = \pi/2, -\pi/2$

$y = a, -a$

$\therefore (0, a) \quad (0, -a)$

③ Region -

$x \in [-a, a]$

$y \in [-a, a]$

④ $\dfrac{dy}{dx} = \dfrac{3a\sin^2 t \cos t}{-3a\cos^2 t \sin t} = -\tan t$

$\therefore$ Curve is decreasing in $(0, \pi/2)$ & increasing in $(\pi/2, \pi)$

$t = 0, \pi, -\pi$ $\therefore$ Pts. $(a, 0) \quad (-a, 0)$

$\dfrac{d^2y}{dx^2} = \dfrac{-\sec^2 t}{-3a\cos^2 t \sin t} = \dfrac{1}{3a\sin t\cos^4 t}$