GARVIT SHAH
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TUTORIAL 10 :

A.1 (a) $\int_0^a \int_0^{\sqrt{a^2-y^2}} (a^2-x^2-y^2)^{1/2}\, dx\, dy$

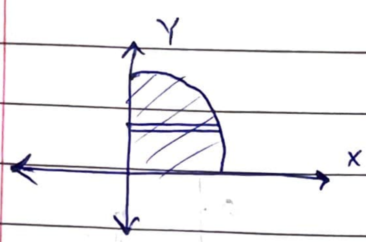
Put $x = r\cos\theta$ Limits :

$y = r\sin\theta$ $x : 0$ to $\sqrt{a^2-y^2}$

$y : 0$ to a

$\therefore r : 0$ to a

$\theta : 0$ to $\pi/2$



$|J| = r$ $\quad \therefore dx\,dy = r\,dr\,d\theta$

$$I = \int_0^{\pi/2}\int_0^a (a^2-r^2)^{1/2}\, r\, dr\, d\theta$$

Put $a^2 - r^2 = t$

$-2r\,dr = dt$

$$\therefore I = \frac{-1}{2}\int_0^{\pi/2}\int_{a^2}^0 t^{1/2}\, dt\, d\theta$$

$$I = -\frac{1}{2}\int_0^{\pi/2} \frac{2}{3}\left[t^{3/2}\right]_{a^2}^0 d\theta$$

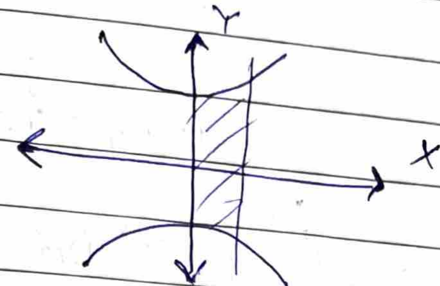
$$I = \frac{-1}{2}\left(\frac{2}{3}\right)\left[-a^3\right]\left(\frac{\pi}{2}\right) = \frac{a^3\pi}{6}$$

(c) $$\int_0^1\int_0^{\sqrt{1+x^2}} (1+x^2+y^2)^{-1}\,dx\,dy$$

Limits
$x : 0$ to 1
$y : 0$ to $\sqrt{1+x^2}$

Put $y = \sqrt{1+x^2}\,u$

$dy = \sqrt{1+x^2}\,du$



$$\Rightarrow I = \int_0^1\int_0^1 \frac{\sqrt{1+x^2}}{(1+x^2)(u^2+1)}\,du\,dx$$

$$\Rightarrow I = \int_0^1 \frac{1}{\sqrt{1+x^2}}\left[\tan^{-1}u\right]_0^1 dx$$

$$\Rightarrow I = \frac{\pi}{4}\int_0^1 \frac{1}{\sqrt{1+x^2}}\,dx = \frac{\pi}{4}\left[\log\left(x+\sqrt{1+x^2}\right)\right]_0^1$$

$$I = \frac{\pi}{4}\log(1+\sqrt{2})$$

Q.2. $$\int_0^a\int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} \frac{1}{\sqrt{a^2-(x^2+y^2)}}\,dy\,dx$$

Limits
$Y : \sqrt{ax-x^2}$ to $\sqrt{a^2-x^2}$
$X : 0$ to a

$x = r\cos\theta \quad y = r\sin\theta$

$|J| = r \quad \therefore dx\,dy = r\,dr\,d\theta$

$$\therefore I = \int_0^{\pi/2}\int_{a\cos\theta}^{a} \frac{r}{\sqrt{a^2-r^2}}\,dr\,d\theta$$

Put
$a^2 - r^2 = t$
$-2r\,dr = dt$

$$\Rightarrow I = \int_0^{\pi/2}\int_{a^2\sin^2\theta}^{0} \frac{-1}{2\sqrt{t}}\,dt\,d\theta$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{-1}{2}(2)\left[\sqrt{t}\right]_{a\sin\theta}^{0} d\theta = \int_0^{\pi/2} a\sin\theta\,d\theta$$

$$I = a$$

A-3. $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \, dx\,dy$

Limits
$X : 0$ to ∞
$Y : 0$ to ∞

Put $x = r\cos\theta$
$y = r\sin\theta$
$dx\,dy = r\,dr\,d\theta$

$$\Rightarrow I = \int_0^{\pi/2} \int_0^\infty r e^{-r^2} \, dr\,d\theta$$

Put $r^2 = t$
$2r\,dr = dt$

$$\Rightarrow I = \frac{1}{2} \int_0^{\pi/2} \int_0^\infty e^{-t} \, dt\,d\theta$$

$$\Rightarrow I = \frac{1}{2} \int_0^{\pi/2} -[0-1] \, d\theta = \frac{\pi}{4}$$

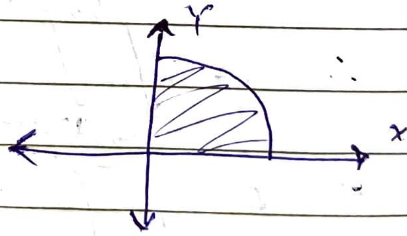
In $I = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\,dy$ x & y are independent of each other

$$\therefore I = \left[\int_0^\infty e^{-x^2} dx \right]^2$$

$$\therefore I_1 = \int_0^\infty e^{-x^2} dx = \sqrt{I} = \frac{\sqrt{\pi}}{2}$$

A.4 $\iint x^2 y^2 \, dx\,dy$ $x = 0$ $y = 0$ & $x^2+y^2 = 1$

$$\rightarrow I = \int_0^1 \int_0^{\sqrt{1-x^2}} x^2 y^2 \, dy\,dx$$



Limits
$Y : 0$ to $\sqrt{1-x^2}$
$X : 0$ to 1

$$\Rightarrow I = \int_0^1 \frac{x^2}{3} \left[(1-x^2)^{3/2} \right] dx$$

Put $x^2 = t$
$2x\,dx = dt$

$$\Rightarrow \quad I = \frac{1}{6}\int_0^1 \frac{\sqrt{t}\,(1-t)^{3/2}}{\sqrt{t}}\,dt$$

Using ~~Be~~ ~~Gamma~~ Beta f^n

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$\therefore \quad I = \frac{1}{6}\int_0^1 t^{\frac{3}{2}-1}(1-t)^{\frac{5}{2}-1}\,dt = \frac{1}{6}\beta\left(\frac{3}{2},\frac{5}{2}\right)$$

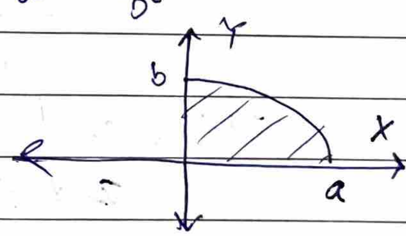
$$I = \frac{1}{6}\,\frac{\Gamma\frac{3}{2}\,\Gamma\frac{5}{2}}{\Gamma 4}$$

using $\beta(m,n) = \dfrac{\Gamma m\,\Gamma n}{\Gamma m+n}$

$$\therefore \quad I = \frac{\cancel{6}\;\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\,\frac{3}{2}\,\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)}{3!}$$

$$I = \frac{\pi}{8(6)(2)} = \frac{\pi}{96}$$

$\boxed{4.5}$ $$\iint \sqrt{\frac{a^2b^2 - b^2x^2 - a^2y^2}{a^2b^2 + b^2x^2 + a^2y^2}}\,dx\,dy \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



Put $\frac{x}{a} = X \qquad \frac{y}{b} = Y$

$$\Rightarrow I = \iint \sqrt{\frac{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}{1 + \frac{x^2}{a^2} + \frac{y^2}{b^2}}}\,dx\,dy$$

$$\Rightarrow \therefore I = \iint \sqrt{\frac{1 - X^2 - Y^2}{1 + X^2 + Y^2}}\;ab\,dY\,dX$$

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Limits

Put $x = r\cos\theta$ $y = r\sin\theta$ $\theta : 0$ to $\pi/2$

$r : 0$ to 1

$\therefore |J| = r$ $\therefore dx\,dy = r\,dr\,d\theta$

$$\Rightarrow I = ab \int_0^{\pi/2}\int_0^{1} \sqrt{\frac{1-r^2}{1+r^2}}\ \cancel{ab}\, r\,dr\,d\theta$$

$\cancel{\Rightarrow I = ab \iint}$ put $r^2 = t$

$2r\,dr = dt$

$$I = \frac{ab}{2}\int_0^{\pi/2}\int_0^{1}\sqrt{\frac{1-t}{1+t}}\,dt\,d\theta$$

Put $t = \cos\alpha$

$dt = -\sin\alpha\,d\alpha$

$$\Rightarrow I = -\frac{ab}{2}\int_0^{\pi/2}\int_{\pi/2}^{0}\sin\alpha\left(\frac{\cos\alpha/2}{\sin\alpha/2}\right)^{-1} d\alpha\,d\theta$$

$$\Rightarrow I = +\frac{ab}{2}\int_0^{\pi/2}\int_0^{\pi/2} \cancel{\frac{\sin^2\alpha}{\cos\alpha}\,d\alpha\,d\theta}\ 2\sin\alpha/2\,\cos\alpha/2\ \frac{\sin\alpha/2}{\cos\alpha/2}\,d\alpha\,d\theta$$

$$\Rightarrow I = \frac{ab}{\cancel{2}}\int_0^{\pi/2}\int_0^{\pi/2}\sin^2\alpha/2\,d\alpha\,d\theta$$

$$\Rightarrow I = \frac{ab}{2}\int_0^{\pi/2}\int_0^{\pi/2}(1-\cos\alpha)\,d\alpha\,d\theta$$

$$I = \frac{ab}{2}\int_0^{\pi/2}\left(\frac{\pi}{2}-1\right)d\theta = \frac{ab}{2}\left(\frac{\pi}{2}-1\right)\left(\frac{\pi}{2}\right)$$

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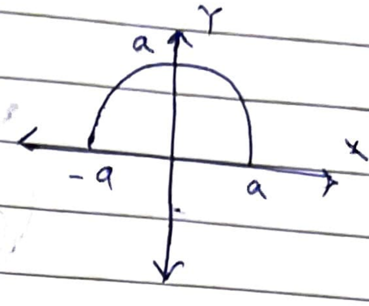
6. $\int_0^{\pi}\int_0^{a} r^3 \sin\theta \cos\theta \, dr\, d\theta$

$\Rightarrow$ Put $x = r\cos\theta$ $y = r\sin\theta$

$\therefore x^2+y^2 = a^2$ in

I^{st} & 2^{nd} quadrant



$\therefore I = \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} x^2 y \, dy\, dx$

$I = - \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} xy \, dy\, dx$

$\therefore 2I = 0$ $\therefore \underline{\underline{I = 0}}$

7. Area bounded by $y = x^2$ & $y = 2x+3$

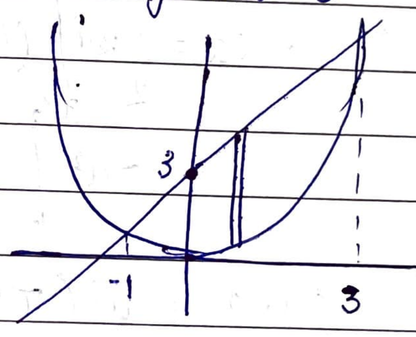
$\rightarrow$ $y = x^2$ & $y = 2x+3$

$\therefore$ For intersection pts.

$x^2 = 2x+3$

$x^2-2x-3 = 0$

$(x-3)(x+1) = 0$



$\therefore$ $\cancel{I}$ = Area $= \int_{-1}^{3}\int_{x^2}^{2x+3} dy\, dx$

Area $= \int_{-1}^{3} (2x+3-x^2)\, dx$

$= x^2 \Big|_{-1}^{3} + 3x \Big|_{-1}^{3} - \frac{1}{3} x^3 \Big|_{-1}^{3}$

$= (9-1) + 3(3+1) - \frac{1}{3}(28)$

$= 8 + 12 - \frac{28}{3}$

Area $= \underline{\underline{\frac{32}{2}}}$

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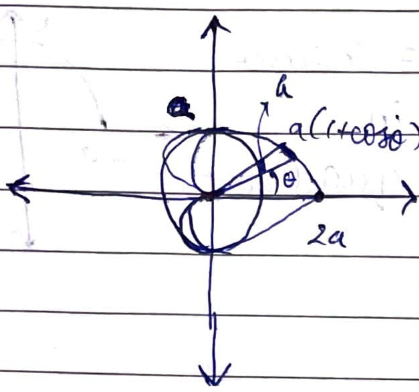
8. Area for -
Outside circle $r=a$ & inside $r=a(1+\cos\theta)$



$$\therefore I = 2\int_0^{\pi}\int_a^{a(1+\cos\theta)} r\,dr\,d\theta$$

$$\Rightarrow Area = \cancel{2}\int_{-\pi/2}^{\pi/2} \frac{1}{2}\left(a^2(1+\cos\theta)^2 - a^2\right)d\theta$$

$\Rightarrow$ ~~$Area = 2\int_0^{\pi/2} a^2(1+\cos^2\theta+2\cos\theta-1)\,d\theta$~~

~~$Area = 2a^2\left[\frac{1}{2}\cdot\frac{\pi}{2} + 2\right] = \frac{a^2(\pi+8)}{2}$~~

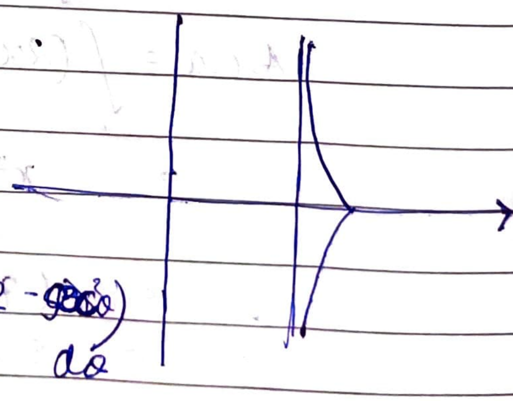
$$\Rightarrow Area = \frac{1}{2}\cancel{2}a^2\int_{-\pi/2}^{\pi/2}(\cos^2\theta + 2\cos\theta)\,d\theta$$

$=$ ~~$a^2\left[2(0) + 1+\cos 2\theta\right.$~~

$$Area = \frac{a^2(\pi+8)}{4}$$

9. $r = a(\sec\theta + \cos\theta)$ & its asymptote.

$$\rightarrow Area = \int_{-\pi/2}^{\pi/2}\int_{a\sec\theta}^{a(\sec\theta+\cos\theta)} r\,dr\,d\theta$$



$$\rightarrow Area = 2\int_0^{\pi/2} \frac{a^2}{2}\left(\cancel{\sec^2\theta}+\cos^2\theta+2-\cancel{\sec^2\theta}\right)d\theta$$

$$\text{Area} = a^2 \int_0^{\pi/2} (\cos^2\theta + 2)\, d\theta$$

$$a^2 \left(\frac{2\pi}{2} + \frac{\pi}{2}\left(\frac{1}{2}\right) \right)$$

$$\text{Area} = \frac{a^2 \pi (3)}{4}$$

10. Use transformation $x+y=u$ & $y=uv$ to show that $\int_0^1 \int_0^{1-x} e^{y/(x+y)}\, dy\, dx = \frac{e-1}{2}$.

→ $|J| = u$

$\therefore \; I = \iint e^v u \, du \, dv$

Initially limits —

X : 0 to 1

Y : 0 to $1-x$

New limits —

when $x = 0 \Rightarrow \boxed{u = y}$ $\therefore v = 1$

$x = 1 \Rightarrow 1 + y = u$ $\therefore u(1-v) = 1$

$y = 0 \;\therefore\; u = 0$ & $\underline{y = 1}$ $\underline{u = 1}$

$\therefore$ ~~at y = 0~~ $u = 0$ to 1

$v = 0$ to 1

$$\therefore I = \int_0^1 \int_0^1 e^v u \, du \, dv = \frac{1}{2} \int_0^1 \left[\frac{u^2}{2} \right]_0^1 e^v \, dv$$

$$I = \frac{1}{2}\left(\frac{1}{2}\right)(e-1) = \frac{e-1}{4}$$

11. Using $x+y=u$, $y=uv$

Show $\iint \sqrt{xy(1-x-y)}\;dx\,dy$

bounded by $x=0$, $y=0$, $x+y=1$

Limits :

X : 0 to 1

Y : 0 to $1-x$

$$\therefore I=\int_0^1\int_0^{1-x}\sqrt{xy(1-x-y)}\;dx\,dy$$

Using $x+y=u$ $\quad y=uv$

$|J|=u$ $\quad \therefore dx\,dy = u\,du\,dv$

for $x=0$ $\quad \therefore y=u$ $\quad \therefore v=1$

$y=0$ $\quad u=0$ $\quad \therefore v=0$

$y=1$ $\quad u=1$

for $x=1$ $\Rightarrow$ $y+1=u$ $\quad \therefore \cancel{d(u-1)=uv}$

$\therefore$ $u(v-1)=1$

$$\therefore I=\int_0^1\int_0^1\sqrt{u^2(1-v)v(1-u)}\;u\,du\,dv$$

$$\Rightarrow I=\int_0^1\int_0^1 u^2\sqrt{(1-v)(1-u)v}\;du\,dv$$

$$\Rightarrow I=\int_0^1 u^{3-1}(1-u)^{3/2-1}\,du\int_0^1 v^{3/2-1}(1-v)^{3/2-1}\,dv$$

Using Beta f^n $\quad B(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

& $\quad B(m,n)=\dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$

we get $I = \frac{\cancel{\Gamma 3}\,\Gamma 3/2}{\Gamma 9/2} \times \frac{\Gamma(3/2)\,\Gamma(3/2)}{\cancel{\Gamma(3)}}$

$I = \frac{1}{8}\pi^{3/2}\,\frac{1}{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}$

$I = \frac{2\pi}{105}$

12. To find the mass contained in a thin plate shape: $x = y-y^2$ & $x=-y$ & mass density $x+y$

→ $y^2 - y = -x$

$\Rightarrow (y-\tfrac12)^2 = -x+\frac{1}{4}$ $\Rightarrow (y-\tfrac12)^2 = -\frac{1}{4}\cdot 4\left(x-\frac{1}{4}\right)$

$\rho = x+y$ → mass density

$dm = \rho\, dx\, dy$

$\therefore m = \int_{0}^{2}\int_{-y}^{y-y^2} (x+y)\, dx\, dy$

$\Rightarrow m = \int_0^2 \left[\frac{1}{2}\left((y-y^2)^2 - y^2\right) + y\left[y-y^2+y\right]\right] dy$

$\Rightarrow m = \int_0^2 \frac{1}{2}\left[\cancel{y^2}+y^4-2y^3-\cancel{y^2}+(2y^2-y^3)2\right] dy$

$m = \frac{1}{2}\left[\frac{1}{5}(2^5) - \frac{4}{4}(2^4) + \frac{4}{3}(2^3)\right]$

$m = \left|\frac{2^4}{5} - 8 + \frac{2^4}{3}\right| = 8/15$

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13. Volume enclosed in $(x^2+y^2) = 2ax$ & $z^2 = 2ax$

$\rightarrow$ $x^2+y^2 = z^2$

& $(x-a)^2 + y^2 = a^2$

Put
$x = r\cos\theta$
$y = r\sin\theta$
$x^2+y^2 = 2ax$
$\therefore r^2 = 2ar\cos\theta$
$\therefore r = 2a\cos\theta$

~~Volume $= \int_{-\pi/2}^{\pi/2}\int_{0}^{2a\cos\theta}$~~

$\therefore z^2 = 2ar\cos\theta$

$\therefore z = \pm\sqrt{2ar\cos\theta}$

Symmetric about XY plane

Limits $r=0$ to $r = 2a\cos\theta$

$\theta = -\pi/2$ to $\pi/2$

$|J| = r$

$$\therefore V = 2\int_{-\pi/2}^{\pi/2}\int_{0}^{2a\cos\theta} \sqrt{2a\cos\theta}\ r\,dr\,d\theta$$

$$= 2\int_{-\pi/2}^{\pi/2}\int_{0}^{2a\cos\theta} r^{3/2}\sqrt{2a}\cos^{1/2}\theta\,dr\,d\theta$$

$$= 2\int_{-\pi/2}^{\pi/2} \frac{2}{5}(2a\cos\theta)^{5/2}\sqrt{2a}\cos^{1/2}\theta\,d\theta$$

$$= 2\cdot\frac{2}{5}\sqrt{2a}\int_{-\pi/2}^{\pi/2}(2a)^{5/2}\cos^{3}\theta\,d\theta$$

$$V = \frac{4}{5}(2a)^3(2)\int_{0}^{\pi/2}\cos^{3}\theta\,d\theta$$

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$$V = \frac{64a^3}{5}\left(\frac{\cancel{8}2\cdot 1}{3}\right) = \cancel{+8}\ \frac{128a^3}{15}$$

14. To find volume bounded by paraboloid $(x^2+y^2) = az$, the cylinder $(x^2+y^2) = 2ay$ & the plane $z=0$

→ The required volume is found by integrating $z = \frac{x^2+y^2}{a}$ over the circle $x^2+y^2 = 2ay$.

Changing to polar coordinates in the x-y plane we have $x = r\cos\theta$ & $y = r\sin\theta$, so that $z = \frac{r^2}{a}$ & the polar eq^n of the circle is $r = 2a\sin\theta$ to cover this circle, r varies from 0 to $2a\sin\theta$ & θ varies from 0 to π.

∴ Required vol. →

$$I = \int_0^{\pi}\int_0^{2a\sin\theta} z\cdot r\, d\theta\, dr$$

$$I = \frac{1}{a}\int_0^{\pi} d\theta \int_0^{2a\sin\theta} r^3\, dr$$

$$\cancel{I = \frac{1}{a}(\pi)\left(\frac{1}{4}\right)(2a\sin\theta)^4}$$

$$\cancel{I = 4a^3\pi \sin}$$

$$I = \frac{(2a)^4}{4a}\int_0^{\pi}\sin^4\theta\, d\theta = 4a^3(2)\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right)$$

$$I = \frac{3a^3\pi}{2}$$