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F-24

TUTORIAL 11: Triple Integration & its application

A.1 (a) $I = \int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)\,dx\,dy\,dz$

$\Rightarrow \quad I = \int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)\,dy\,dx\,dz$

$\Rightarrow \quad I = \int_{-1}^{1}\int_{0}^{z}\left[2x+z^2+\frac{1}{2}\left((x+z)^2-(x-z)^2\right)\right]dx\,dz$

$\Rightarrow \quad I = \int_{-1}^{1}\int_{0}^{z}(x+z)+\frac{1}{2}\left[(2x)(2z)\right]dx\,dz$

$\Rightarrow \quad I = \int_{-1}^{1}\int_{0}^{z}(x+z+2xz)\,dx\,dz$

$\Rightarrow \quad I = \int_{-1}^{1}\left(\frac{1}{2}(z^2)+z^2+z^3\right)dz$ ×

$\Rightarrow \quad I = \frac{1}{2}\cdot\frac{1}{3}[1+1] + \frac{2}{2}\cdot\frac{1}{3}(1+1) + 0$

$I = \frac{6}{6} = 1$

$I = \int_{-1}^{1}\int_{0}^{z}\left[(x+z)(2z)+\frac{1}{2}\left[(x+z)^2-(x-z)^2\right]\right]dx\,dz$

$I = \int_{-1}^{1}\int_{0}^{z}\left[(x+z)(2z)+2xz\right]dx\,dz$

$I = \int_{-1}^{1}\int_{0}^{z}(2z^2+4xz)\,dx\,dz$

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$$I = \int_{-1}^{1} (2z^3 + 2z^3)\, dz = 0$$

(b) $$I = \int_0^{\pi/2}\int_{x}^{\pi/2}\int_0^{xy} \cos\left(\frac{z}{x}\right) dz\, dy\, dx$$

$$\Rightarrow I = \int_0^{\pi/2}\int_{x}^{\pi/2}\int_0^{xy} \cos(z/x)\, dz\, dy\, dx$$

$$\Rightarrow \cancel{I = \int_0^{\pi/2}\int_x^{\pi/2}\int_0^{xy}} \qquad I = \int_0^{\pi/2}\int_x^{\pi/2} x\left[\sin y - \sin 0\right] dy\, dx$$

$$\Rightarrow \cancel{I = \int_0^{\pi/2} \sin \frac{\pi y}{2x}\, dx} \qquad I = \int_0^{\pi/2}\int_x^{\pi/2} x \sin y\, dy\, dx$$

$$\Rightarrow I = -\int_0^{\pi/2} x\left[\cos y\right]_x^{\pi/2} dx$$

$$\Rightarrow I = +\int_0^{\pi/2} x\cos x\, dx \qquad \Rightarrow I = x\sin x\Big|_0^{\pi/2} + \cos x\Big|_0^{\pi/2}$$

$$\Rightarrow I = \frac{\pi}{2} - 1$$

(c) $$I = \int_0^{\log 2}\int_0^{x}\int_0^{x+\log y} e^{x+y+z}\, dz\, dy\, dx$$

$$\Rightarrow \cancel{I = \int_0^{\log 2}\int_0^{x} \left(e^{x+y+x+\log y} - 1\right) dy\, dx} \qquad I = \int_0^{\log 2}\int_0^{x} e^{x+y}\left[y e^{x} - 1\right] dy\, dx$$

$$\cancel{I = \int_0^{\log 2}\int_0^{x} \left(e^{2x+y}\cdot y - 1\right) dy\, dx} \qquad I = \int_0^{\log 2} e^{x}\left[e^{x}\left(x e^{x} - e^{x} + 1\right) - (e^{x} + 1)\right] dx$$

$$\cancel{I = \int_0^{\log 2}\left[(-x) + x e^{3x} - e^{3x} + 1\right] dx}$$

$$I = \int_0^{\log 2} e^{x}\left(e^{2x}(x-1) + 1\right) dx$$

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$$I = \int_0^{\log 2} e^{3x}(x-1) + e^x \, dx$$

$$I = (2-1) + \frac{8(\log 2 - 1)}{3} - \frac{1}{9}[2^3 - 1] + 1$$

$$I = \frac{8\log 2}{3} - \frac{8\cdot 3+9}{3\cdot 3 \;\; 9} - \frac{8}{9} + \frac{1}{9} + \frac{1}{3}$$

$$I = \frac{8\log 2}{3} - \frac{32+10+3}{9}$$

$$I = \frac{8\log 2}{3} - \frac{19}{9}$$

2. Using $u = x+y+z$, $uv = y+z$, $uvw = z$ & enclosed by ~~x=y=~~ $x=0, y=0, z=0, x+y+z=1$

(a) $\iiint \sqrt{xyz(1-x-y-z)}\, dx\,dy\,dz$

$z = uvw$

$x = u - (y+z) = u - uv = u(1-v)$

$y = uv - uvw = uv(1-w)$

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 1-v & -u & 0 \\ v-vw & u-uw & -uv \\ vw & uw & uv \end{vmatrix}$$

$J = (1-v)[uv(u-uw) + u^2vw] + u[uv^2(1-w)]$

$J = u^2v$

$\rightarrow I = \int_0^1\int_0^1\int_0^1 \sqrt{uvw\, u(1-v)\, uv(1-w)\, (1-u)}\; u^2v\, du\,dv\,dw$

$\Rightarrow I = \int_0^1\int_0^1\int_0^1 \sqrt{u^3v^2w(1-u)(1-v)(1-w)}\; u^2v\, du\,dv\,dw$

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$$I = \int_0^1\int_0^1\int_0^1 u^{\frac{9}{2}-1}(1-u)^{\frac{3}{2}-1} v^{3-1}(1-v)^{\frac{3}{2}-1} w^{\frac{3}{2}-1}(1-w)^{\frac{3}{2}-1}\,du\,dv\,dw$$

$$I = \beta\left(\frac{9}{2},\frac{3}{2}\right) \times \beta\left(3,\frac{3}{2}\right) \times \beta\left(\frac{3}{2},\frac{3}{2}\right)$$

$$I = \frac{\cancel{\Gamma(9/2)}\,\Gamma(3/2)}{\Gamma(6)} \times \frac{\cancel{\Gamma(3)}\,\Gamma(3/2)}{\cancel{\Gamma(9/2)}} \times \frac{\Gamma(3/2)\,\Gamma(3/2)}{\cancel{\Gamma(3)}}$$

$$I = \frac{[\Gamma(3/2)]^4}{\Gamma(6)} = \left(\frac{1}{2}\sqrt{\pi}\right)^4 \frac{1}{120}$$

$$I = \frac{\pi^2}{16(120)}$$

(b) $I = \iiint xyz(x+y+z)^2\,dx\,dy\,dz$

$\Rightarrow$ $z = uvw$

$x = u-(y+z) = u-uv = u(1-v)$

$y = uv(1-w)$

$\therefore J = u^2 v$

$$\therefore I = \int_0^1\int_0^1\int_0^1 uvw\,u(1-v)\,uv(1-w)\,u^2\,u^2v\,du\,dv\,dw$$

$$I = \int_0^1\int_0^1\int_0^1 u^7v^3w(1-v)(1-w)\,du\,dv\,dw$$

$$I = \frac{1}{8}u^8\Big|_0^1 \int_0^1 v^{4-1}(1-v)^{2-1}\,dv \int_0^1 w^{2-1}(1-w)^{2-1}\,dw$$

$$I = \left[\frac{1}{8}\right]\beta(4,2)\,\beta(2,2) = \frac{1}{8}\,\frac{\cancel{\Gamma(4)}\,\Gamma(2)}{\Gamma(6)}\,\frac{\Gamma(2)\,\Gamma(2)}{\cancel{\Gamma(4)}}$$

$$I = \frac{1}{8(120)}$$

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(c) $I = \int_0^1\int_0^1\int_0^1 e^{(x+y+z)^3}\, dx\, dy\, dz$

$\Rightarrow I = \int_0^1\int_0^1\int_0^1 e^{u^3} u^2 v\, du\, dv\, dw$ Put $u^3 = t$, $3u^2\, du = dt$

$\Rightarrow I = \int_0^1\int_0^1\int_0^1 \frac{1}{3} e^t v\, dt\, dv\, dw$

$\Rightarrow I = \frac{1}{3}(e-1) \times \frac{1}{2}(1) + 1$

$I = \frac{e-1}{6}$

3. $I = \int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} \frac{dz\, dy\, dx}{\sqrt{1-z^2-y^2-x^2}}$

$|J| = r^2 \sin\theta$

$I = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \frac{r^2 \sin\theta\, d\theta\, d\phi\, dr}{\sqrt{1-r^2}}$

$I = \frac{\pi}{2} \times (1) \int_0^1 \frac{r^2}{\sqrt{1-r^2}}\, dr$

$I = \frac{\pi}{2}\left(-\frac{1}{2}\right) \int_1^0 \frac{(1-t)\, t^{-1/2}}{\sqrt{1-t}}\, dt$ Put $1-r^2 = t$, $-2r\, dr = dt$

$I = \frac{\pi}{4} \int_0^1 t^{1/2 - 1} (1-t)^{3/2 - 1}\, dt$

$I = \frac{\pi}{4} \beta\left(\frac{1}{2}, \frac{3}{2}\right) = \frac{\pi}{4} \cdot \frac{1}{2} \frac{\sqrt{\pi}\sqrt{\pi}}{1} = \frac{\pi^2}{8}$

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4. Volume surrounded by $\left(\frac{x}{a}\right)^{2/3}+\left(\frac{y}{b}\right)^{2/3}+\left(\frac{z}{c}\right)^{2/3}=1$

Let $\frac{x}{a}=X^3 \quad \frac{y}{b}=Y^3 \quad \frac{z}{c}=Z^3$

$\therefore X^2+Y^2+Z^2=1 \quad \rightarrow$ Sphere

$$|J| = \begin{vmatrix} 3X^2a & 0 & 0 \\ 0 & 3Y^2b & 0 \\ 0 & 0 & 3Z^2c \end{vmatrix} = 27\,abc\,X^2Y^2Z^2$$

$$V = \int_0^1\int_0^{\sqrt{1-X^2}}\int_0^{\sqrt{1-X^2-Y^2}} dZ\,dY\,dX\,(27abc\,X^2Y^2Z^2)$$

$$V = \int_0^1\int_0^{\sqrt{1-X^2}} \frac{1}{3}(1-X^2-Y^2)^{3/2}\,dY\,dX\,(27abc\,X^2Y^2)$$

$$V = \int_0^1\int_0^{\sqrt{1-X^2}} 9abc\,(1-X^2-Y^2)^{3/2}\,X^2Y^2\,dY\,dX$$

$$V = \int_0^1\int_0^1 \frac{9abc\left[(1-X^2)(1-t)\right]^{3/2} X^2(1-X^2)t}{2\sqrt{1-X^2}\sqrt{t}}\,dt\,dX$$

Put $Y^2=(1-X^2)t$

$2Y\,dY=(1-X^2)\,dt$

$$V = \int_0^1\int_0^1 \frac{9abc}{2}(1-X^2)^3(1-t)^{3/2}X^2t^{1/2}\,dt\,dX$$

$$V = \frac{9abc}{2}\,\beta\left(\tfrac{3}{2},\tfrac{5}{2}\right)\int_0^1 (1-X^2)^3 X^2\,dX$$

Put $X^2=p$

$2X\,dX=dp$

$$V = \frac{9abc}{2}\,\frac{\frac{1}{2}\sqrt{\pi}\,\frac{3}{2}\,\frac{1}{2}\sqrt{\pi}}{3!}\int_0^1 \frac{(1-p)^3\,p}{2\sqrt{p}}\,dp$$

$$V = \frac{9abc\,\pi}{16\,(2)}\,\frac{1}{2}\,B\left(\frac{3}{2},4\right) = \frac{9\,abc\,\pi}{64\cdot 2}\,\frac{1}{2}\,\frac{\frac{1}{2}\sqrt{\pi}\cdot 3!}{\frac{9}{2}\times\frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\sqrt{\pi}}$$

$$V = \frac{abc\,\pi}{2(35)} \rightarrow \text{for I octant}$$

$\therefore$ Total Vol. $= \frac{4\pi abc}{35}$

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5. Vol. of tetrahedron bounded by $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ & coordinate planes.

$\rightarrow$

$$V = \int_0^a \int_0^{b(1-\frac{x}{a})} \int_0^{c(1-\frac{x}{a}-\frac{y}{b})} dz\, dy\, dx$$

$$V = \int_0^a \int_0^{b(1-\frac{x}{a})} c\left(1-\frac{x}{a}-\frac{y}{b}\right) dy\, dx$$

$$V = \int_0^a \left[c - \frac{xc}{a}\, b\left(1-\frac{x}{a}\right) - \frac{c\, b^2\left(1-\frac{x}{a}\right)^2}{2b} \right] dx$$

$$V = \int_0^a \left(c - x\frac{bc}{a^2}(1-x) - \frac{b}{2a^2}(a-x)^2 \right) dx$$

$$V = ac - \frac{bc}{a^2}\left[\frac{x^2}{2}\Big|_0^a - \frac{x^3}{3}\Big|_0^a \right] + \frac{b}{2a^2}\frac{(a-x)^3}{3}\Big|_0^a$$

$$V = ac - \frac{bc}{a^2}\left[\frac{a^2}{2}-\frac{a^3}{3}\right] - \frac{b}{2a^2}\frac{(a^3)}{3}$$

$$V = ac - \frac{bc}{6a^3}(a^3) -$$

$\times$

$$V = \int_0^a c\left[b\left(1-\frac{x}{a}\right) - \frac{b}{a}\left(1-\frac{x}{a}\right)x - \frac{1}{2}\frac{b^2\left(1-\frac{x}{a}\right)^2}{b} \right] dx$$

$$V = \int_0^a bc\left[1-\frac{x}{a} - \cancel{\frac{x}{a}} + \frac{x^2}{a^2} - \frac{1}{2}\left(1+\frac{x^2}{a^2} - 2\frac{x}{a}\right) \right] dx$$

$$V = \int_0^a bc\left[\frac{1}{2} - \frac{x}{a} + \frac{x^2}{2a^2} \right] dx$$

$$V = bc\left[\frac{a}{2} - \frac{a}{2} + \frac{a}{6}\right] = \frac{abc}{6}$$

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6. Volume of solid bounded by plane $2x+3y+4z=12$, XY plane & cylinder $x^2+y^2=1$

$$V = 4\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\frac{12-3y-2x}{4}} dx\,dy\,dz$$

$$V = 4\int_0^1\int_0^{\sqrt{1-x^2}}\left(\frac{12-3y-2x}{4}\right) dy\,dx$$

$$V = 4\int_0^1\left(3\sqrt{1-x^2} - \frac{3}{8}(1-x^2) - \frac{1}{2}x\sqrt{1-x^2}\right) dx$$

$$V = (I_1 - I_2 - I_3)$$

$$I_1 = 4\int_0^1 3\sqrt{1-x^2}\,dx$$

$$= 12\left[\frac{x}{2}\sqrt{1-x^2}\Big|_0^1 + \frac{1}{2}\sin^{-1}x\Big|_0^1\right] = 12\left[0+\frac{\pi}{4}\right]$$

$$\boxed{I_1 = 3\pi}$$

$$I_2 = 4\left(\frac{3}{8}\right)\int_0^1 1-x^2\,dx$$

$$I_2 = \frac{3}{2}\left[1-\frac{1}{3}\right] = \frac{3}{2}\left(\frac{2}{3}\right) = \boxed{1}$$

$$I_3 = 4\left(\frac{1}{2}\right)\int_0^1 x\sqrt{1-x^2}\,dx$$

Put $1-x^2 = t$

$-2x\,dx = dt$

$$I_3 = -\frac{2}{2}\int \sqrt{t}\,dt = \int_0^1 \sqrt{t}\,dt$$

$$\boxed{I_3 = \frac{2}{3}(1)}$$

$$\therefore I = 3\pi - 1 - \frac{2}{3} = 3\pi - \frac{5}{3}$$

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7. Vol. of solid bounded by ~~pl~~ spherical surface $x^2+y^2+z^2=4a^2$ & the cylinder $x^2+y^2-2ay=0$

$$\rightarrow \quad V = 4\int_0^{2a}\int_0^{\sqrt{2ay-y^2}}\int_0^{\sqrt{4a^2-y^2-x^2}} dz\, dx\, dy$$

$$V = 4\int_0^{2a}\int_0^{\sqrt{2ay-y^2}} \sqrt{4a^2-y^2-x^2}\; dx\, dy$$

Put $x = r\cos\theta$

$y = r\sin\theta$

$$\therefore V = 4\int_0^{\pi/2}\int_0^{2a\sin\theta} \sqrt{4a^2-r^2}\; r\,dr\,d\theta$$

Put $4a^2-r^2=t$

$-2r\,dr = dt$

$$V = -4\int_0^{\pi/2}\int_{4a^2}^{4a^2\cos^2\theta} \sqrt{t}\; dt\, d\theta$$

$$V = +2\int_0^{\pi/2} \frac{2}{3}\left[(4a^2)^{3/2} - (4a^2\cos^2\theta)^{3/2}\right] d\theta$$

$$V = \frac{4}{3}\int_0^{\pi/2} \left[(8a^3)(1-\cos^3\theta)\right] d\theta$$

$$V = \frac{4}{3}(8a^3)\left[\frac{\pi}{2}-\frac{2}{3}\right]$$

$$V = \frac{32a^3}{3}\left[\frac{\pi}{2}-\frac{2}{3}\right]$$

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8. Vol. bounded by paraboloid $x^2+y^2=az$ cylinder $x^2+y^2=2ay$ & plane $z=0$

→ $$V = \int_0^{\pi}\int_0^{2a\sin\theta}\int_0^{r^2/a} r\,dz\,dr\,d\theta$$ using cylindrical form

$$V = \int_0^{\pi}\int_0^{2a\sin\theta} r\left[\frac{r^2}{a}\right] dr\,d\theta$$

$$V = \int_0^{\pi} \frac{1}{4a}\left(2a\sin\theta\right)^4 d\theta$$

$$V = 4a^3(2)\int_0^{\pi/2} \sin^4\theta\,d\theta = 8a^3\left(\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2}\right)$$

$$V = \frac{3a^3\pi}{2}$$

9. Vol. cut from sphere $x^2+y^2+z^2=a^2$ by cone $z^2=x^2+y^2$

→ Intersection of both curves - $z^2 = \frac{a^2}{2}$

$\therefore z = \pm \frac{a}{\sqrt{2}}$

$\therefore$ circle eqn is -

$$x^2+y^2 = a^2/2$$

$$V = \int_{-a/\sqrt{2}}^{a/\sqrt{2}}\int_{-\sqrt{\frac{a^2}{2}-y^2}}^{\sqrt{\frac{a^2}{2}-y^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{a^2-x^2-y^2}} dz\,dy\,dx$$

Convert into cylindrical polar form.

→ $$V = \int_0^{2\pi}\int_0^{a/\sqrt{2}}\int_r^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta$$

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$$\Rightarrow V = \int_0^{2\pi}\int_0^{a/\sqrt{2}} r\left(\sqrt{a^2-r^2} - r\right) dr\, d\theta$$

$$V = \int_0^{2\pi}\left[\int_0^{a/\sqrt{2}} r\sqrt{a^2-r^2} - r^2 \, dr\right] d\theta$$

$$V = \int_0^{2\pi}\left[\left.\frac{-r^3}{3}\right|_0^{a/\sqrt{2}} + \int_{a^2}^{a^2/2} -\frac{\sqrt{t}\, dt}{2}\right] d\theta$$

$$V = \int_0^{2\pi} -\frac{1}{3}\left(\frac{a^3}{2\sqrt{2}}\right) + \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{3}\left(\frac{-a^3}{2\sqrt{2}} + a^3\right) d\theta$$

$$\cancel{V = \frac{a^3}{3}\int_0^{2\pi}\left(\frac{7}{8} - \frac{1}{2\sqrt{2}}\right) d\theta}$$

$$V = \frac{a^3}{3}\int_0^{2\pi}\left(\frac{-1}{2\sqrt{2}} + 1 - \frac{1}{2\sqrt{2}}\right) d\theta$$

$$V = \frac{a^3}{3}\left(1 - \frac{1}{\sqrt{2}}\right) 2\pi = \frac{2\pi a^3}{3}\left(1 - \frac{1}{\sqrt{2}}\right)$$

$$\boxed{V = \frac{\pi a^3}{3}(2-\sqrt{2})}$$

10. Volume enclosed by cylinders $x^2+y^2 = 2ax$ & $z^2 = 2ax$

$\rightarrow$

$$\therefore z = \pm\sqrt{2ax}$$

$$V = \int_0^{2a}\int_{-\sqrt{2ax-x^2}}^{\sqrt{2ax-x^2}}\int_{-\sqrt{2ax}}^{\sqrt{2ax}} dx\, dy\, dz$$

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$$V = 4\int_0^{2a}\int_0^{\sqrt{2ax-x^2}}\int_0^{\sqrt{2ax}} dz\,dy\,dx$$

$$V = 4\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} \sqrt{2ax}\,dy\,dx$$

$$V = 4\int_0^{2a} \sqrt{2ax}\,\sqrt{2ax-x^2}\,dx$$

$$V = 4\int_0^{2a} 2ax\sqrt{1-\frac{x}{2a}}\,dx$$

Put $\frac{x}{2a} = t$

$\frac{1}{2a}dx = dt$

$$V = 4\int_0^{\cancel{2a}\,1} 2a(2at)(2a)\sqrt{1-t}\,dt$$

$$V = 4(2a)^3\int_0^{\cancel{2a}\,1} t\sqrt{1-t}\,dt$$

$$V = 4a^3(8)\ \beta(2, 3/2)$$

$$V = 32a^3\ \frac{(1)\,\frac{1}{\cancel{2}}\sqrt{\pi}}{\frac{5}{2}\times\frac{3}{2}\times\frac{1}{\cancel{2}}\sqrt{\pi}} = \frac{32a^3(4)}{15}$$

$$V = \frac{128a^3}{15}$$