Unit-V –Vector Calculus

Definition: Vectors and Scalars

Introduction: Physics is the study of natural phenomena. The study of any natural phenomenon involves measurements. For example, the distance between the planet earth and the Sun is finite. The study of speed of light involves the distance traveled by the ray of light and time consumed.

Any thing that is measurable is termed as 'quantity'. The quantities that come across in physics is referred to as a physical quantity.

Example: Mass, length, time, temperature, etc.,

Whenever we measure a physical quantity, the measured value is always a number. This number makes sense only when the relevant unit is specified.

Thus, the result of a measurement has a numerical value and a unit of measure. For example, the mass of a body is 3 kg. Here a quantity having numerical value 3 and the unit of measure kg are used. The numerical value together with the unit is called the 'magnitude'.

To describe certain physical quantities only magnitude is required. Apart from the mass of a body, distance to any place, time, temperature, height, the number of oscillations of a pendulum and the number of books in a bag are some examples of such numbers. They have no direction associated with them.

'Quantities which require only magnitude for their complete specifications and having no direction associated with them are called scalar quantities'.

To describe certain physical quantities like displacement along with the magnitude, the direction is essential. Consider a body moving from X to Y. XY is the displacement. On the contrary, if the body moves from Y to X, the displacement is YX.

'Quantities which require both magnitude and direction for their complete specification are called vectors'.

Example for vector quantities: momentum, force, torque, magnetic field etc., A vector is a quantity which has both a magnitude and a direction in space. Vectors are used to describe physical quantities such as velocity, momentum, acceleration and force, associated with an object. However, when we try to describe a system which consists of a large number of objects (e.g., moving water, snow, rain,...) we need to assign a vector to each individual object.

1.2 Scalar Fields

A *scalar field* is a function that gives us a single value of some variable for every point in space. As an example, the image in Figure 1.2.1 shows the nighttime temperatures measured by the Thermal Emission Spectrometer instrument on the Mars Global Surveyor (MGS). The data were acquired during the first 500 orbits of the MGS mapping mission. The coldest temperatures, shown in purple, are $-120°$ C while the warmest, shown in white, are $-65°$ C.

The view is centered on Isidis Planitia (15N, 270W), which is covered with warm material, indicating a sandy and rocky surface. The small, cold (blue) circular region to the right is the area of the Elysium volcanoes, which are covered in dust that cools off rapidly at night. At this season the north polar region is in full sunlight and is relatively warm at night. It is winter in the southern hemisphere and the temperatures are extremely low.

Figure 1.2.1 Nighttime temperature map for Mars

The various colors on the map represent the surface temperature. This map, however, is limited to representing only the temperature on a two-dimensional surface and thus, it does not show how temperature varies as a function of altitude. In principal, a scalar field provides values not only on a two-dimensional surface in space but for every point in space.

1.3 Vector Fields

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As an example, let's consider falling snowflakes, as shown in Figure 1.3.1. As snow falls, each snowflake moves in a specific direction. The motion of the snowflakes can be analyzed by taking a series of photographs. At any instant in time, we can assign, to each snowflake, a velocity vector which characterizes its movement.

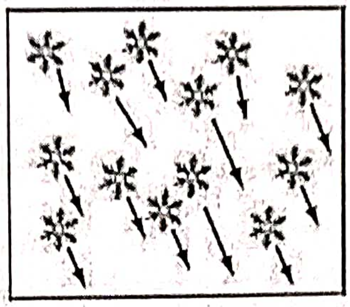


Figure 1.3.1 Falling snow.

The falling snow is an example of a collection of discrete bodies. On the other hand, if we try to analyze the motion of continuous bodies such as fluids, a velocity vector then needs to be assigned to every point in the fluid at any instant in time. Each vector describes the direction and magnitude of the velocity at a particular point and time. The collection of all the velocity vectors is called the velocity vector field. An important distinction between a vector field and a scalar field is that the former contains information about both the direction and the magnitude at every point in space, while only a single variable is specified for the latter. An example of a system of continuous bodies is air flow. Figure 1.3.2 depicts a scenario of the variation of the jet stream, which is the wind velocity as a function of position. Note that the value of the height is "fixed" at 34,000 ft.

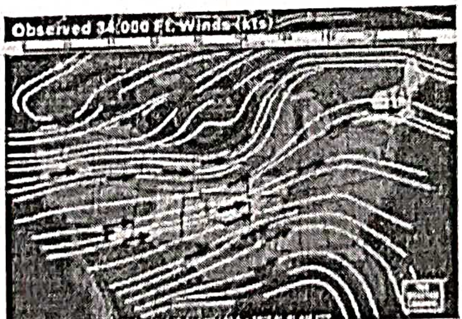
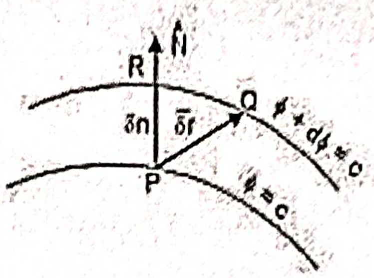


Figure 1.3.2 Jet stream with arrows indicating flow velocity. The "streamlines" are formed by joining arrows from head to tail.

Vector point Function and Scalar point function

Point function. A variable quantity whose value at any point in a region of space depends upon the position of the point, is called a *point function*. There are two types of point functions.



(*i*) **Scalar point function.** If to each point $P(x, y, z)$ of a region R in space there corresponds a unique scalar $f(P)$, then f is called a scalar point function. *For example*, the temperature distribution in a heated body, density of a body and potential due to gravity are the examples of a scalar point function.

(*ii*) **Vector point function.** If to each point $P(x, y, z)$ of a region R in space there corresponds a unique vector $f(P)$, then f is called a *vector point function*. The velocity of a moving fluid, gravitational force are the examples of vector point function.

(*U.P., I Semester, Winter 2000*)

Vector Differential Operator Del *i.e.* ∇

The vector differential operator Del is denoted by ∇. It is defined as

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

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GRADIENT OF A SCALAR FUNCTION

If $\phi(x, y, z)$ be a scalar function then $\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$ is called the gradient of the scalar function ϕ.

And is denoted by grad ϕ.

Thus, $$\text{grad } \phi = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$$

$$\text{gard } \phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi(x, y, z)$$

$$\text{gard } \phi = \nabla \phi \qquad (\nabla \text{ is read del or nebla})$$

Example *If $\phi = 3x^2y - y^3z^2$; find grad ϕ at the point $(1, -2, -1)$.*

Solution.

$$\text{grad } \phi = \nabla\phi$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(3x^2y - y^3z^2)$$

$$= \hat{i}\frac{\partial}{\partial x}(3x^2y - y^3z^2) + \hat{j}\frac{\partial}{\partial y}(3x^2y - y^3z^2) + \hat{k}\frac{\partial}{\partial z}(3x^2y - y^3z^2)$$

$$= \hat{i}(6xy) + \hat{j}(3x^2 - 3y^2z^2) + \hat{k}(-2y^3z)$$

$$\text{grad } \phi \text{ at } (1, -2, -1) = \hat{i}(6)(1)(-2) + \hat{j}[(3)(1) - 3(4)(1)] + \hat{k}(-2)(-8)(-1)$$

$$= -12\hat{i} - 9\hat{j} - 16\hat{k}$$

Ans.

Problem 13 : What is the greatest rate of increase of $\phi = xyz^2$ at the point $(1, 0, 3)$.

Solution : grad $\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$

$= \hat{i}yz^2 + \hat{j}xz^2 + \hat{k}\,2xyz$

$= \hat{i}0 + \hat{j}9 + \hat{k}0$ at $(1, 0, 3)$

$= 9\hat{j}$

Since we know the greatest rate of increase of $\phi = |\nabla\phi|$

$= \sqrt{(9)^2}$

$= 9$ Answer.

Example 3 :- Show that $\nabla(\vec{a}.\vec{r}) = \vec{a}$ where $\vec{a}$ is a constant vector.

Proof : Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

Then $\vec{a}.\vec{r} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}).(x\hat{i} + y\hat{j} + z\hat{k})$

$= a_1x + a_2y + a_3z$

Therefore $\nabla(\vec{a}.\vec{r}) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(a_1x + a_2y + a_3z)$

$= \hat{i}a_1 + \hat{j}a_2 + \hat{k}a_3$

$= \vec{a}$, hence proved.

Example 24. *If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, show that :*

(i) grad $r = \frac{\vec{r}}{r}$ (ii) grad $\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3}$. *(Nagpur University, Summer 2002)*

Solution. (i) $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \Rightarrow r = \sqrt{x^2 + y^2 + z^2} \Rightarrow r^2 = x^2 + y^2 + z^2$

$\therefore \quad 2r\frac{\partial r}{\partial x} = 2x \quad \Rightarrow \quad \frac{\partial r}{\partial x} = \frac{x}{r}$

Similarly, $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$

$$\text{grad } r = \nabla r = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)r = \hat{i}\frac{\partial r}{\partial x} + \hat{j}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r}{\partial z}$$

$$= \hat{i}\frac{x}{r} + \hat{j}\frac{y}{r} + \hat{k}\frac{z}{r} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{r} = \frac{\vec{r}}{r}$$ **Proved.**

(ii) grad $\left(\frac{1}{r}\right) = \nabla\left(\frac{1}{r}\right) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\frac{1}{r}\right) = \hat{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + \hat{j}\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + \hat{k}\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$

$$= \hat{i}\left(-\frac{1}{r^2}\frac{\partial r}{\partial x}\right) + \hat{j}\left(\frac{-1}{r^2}\frac{\partial r}{\partial y}\right) + \hat{k}\left(-\frac{1}{r^2}\frac{\partial r}{\partial z}\right)$$

$$= \hat{i}\left(-\frac{1}{r^2}\frac{x}{r}\right) + \hat{j}\left(-\frac{1}{r^2}\frac{y}{r}\right) + \hat{k}\left(-\frac{1}{r^2}\frac{z}{r}\right) = -\frac{x\hat{i} + y\hat{j} + z\hat{k}}{r^3} = -\frac{\vec{r}}{r^3}$$ **Proved.**

$= -12\hat{i} - 9\hat{j} - 18\hat{k}$ **Ans.**

Example 8. *If $u = x + y + z$, $v = x^2 + y^2 + z^2$, $w = yz + zx + xy$ prove that grad u, grad v and grad w are coplanar vectors.* [*U.P., I Semester, 2001*]

Solution. We have,

$$\text{grad } u = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x+y+z) = \hat{i} + \hat{j} + \hat{k}$$

$$\text{grad } v = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2) = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\text{grad } w = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(yz+zx+xy) = \hat{i}(z+y) + \hat{j}(z+x) + \hat{k}(y+x)$$

[For vectors to be coplanar, their scalar triple product is 0]

$$\text{Now, grad } u.(\text{grad } v \times \text{grad } w) = \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ z+y & z+x & y+x \end{vmatrix} = 2\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ z+y & z+x & y+x \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 1 & 1 \\ x+y+z & x+y+z & x+y+z \\ z+y & z+x & y+x \end{vmatrix} \qquad [\text{Applying } R_2 \to R_2 + R_3]$$

$$= 2(x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ y+z & z+x & x+y \end{vmatrix} = 0$$

Since the scalar product of grad u, grad v and grad w are zero, hence these vectors are coplanar vectors. **Proved.**

Example 25. *Prove that $\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$.* (*K. University, Dec. 2008*)

Solution.

$$\nabla f(r) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) f(r)$$

$$\left[r^2 = x^2 + y^2 + z^2 \Rightarrow 2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r},\ \frac{\partial r}{\partial y} = \frac{y}{r} \text{ and } \frac{\partial r}{\partial z} = \frac{z}{r}\right]$$

$$= i\, f'(r)\frac{\partial r}{\partial x} + j\, f'(r)\frac{\partial r}{\partial y} + k\, f'(r)\frac{\partial r}{\partial z} = f'(r)\left[i\frac{x}{r} + j\frac{y}{r} + k\frac{z}{r}\right]$$

$$= f'(r)\frac{xi + yj + zk}{r}$$

$$\nabla^2 f(r) = \nabla\,[\nabla f(r)] = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\left[f'(r)\frac{xi + yj + zk}{r}\right]$$

$$= \frac{\partial}{\partial x}\left[f'(r)\frac{x}{r}\right] + \frac{\partial}{\partial y}\left[f'(r)\frac{y}{r}\right] + \frac{\partial}{\partial z}\left[f'(r)\frac{z}{r}\right]$$

$$= \left(f''(r)\frac{\partial r}{\partial x}\right)\left(\frac{x}{r}\right) + f'(r)\frac{r1 - x\frac{\partial r}{\partial x}}{r^2} + \left(f''(r)\frac{\partial r}{\partial y}\right)\left(\frac{y}{r}\right) + f'(r)\frac{r.1 - y\frac{\partial r}{\partial y}}{r^2} +$$

$$\left(f''(r)\frac{\partial r}{\partial z}\right)\left(\frac{z}{r}\right) + f'(r)\frac{r.1 - z\frac{\partial z}{\partial r}}{r^2}$$

$$= \left(f''(r)\frac{x}{r}\right)\left(\frac{x}{r}\right) + f'(r)\frac{r - \frac{x^2}{r}}{r^2} + \left(f''(r)\frac{y}{r}\right)\left(\frac{y}{r}\right) + f'(r)\frac{r - \frac{y^2}{r}}{r^2} + \left(f''(r)\frac{z}{r}\right)\left(\frac{z}{r}\right) + f'(r)\frac{r - \frac{z^2}{r}}{r^3}$$

$$= \left(f''(r)\frac{x}{r}\right)\left(\frac{x}{r}\right) + f'(r)\frac{r^2 - x^2}{r^3} + \left(f''(r)\frac{y}{r}\right) + f'(r)\frac{r^2 - y^2}{r^3} + \left(f''(r)\frac{z}{r}\right)\left(\frac{z}{r}\right) + f'(r)\frac{r^2 - z^2}{r^3}$$

$$= f''(r)\frac{x^2}{r^2} + f'(r)\frac{y^2 + z^2}{r^3} + f''(r)\frac{y^2}{r^2} + f'(r)\frac{x^2 + z^2}{r^3} + f''(r)\frac{z^2}{r^2} + f'(r)\frac{x^2 + y^2}{r^3}$$

$$= f''(r)\left[\frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2}\right] + f'(r)\left[\frac{y^2 + z^2}{r^3} + \frac{z^2 + x^2}{r^3} + \frac{x^2 + y^2}{r^3}\right]$$

$$= f''(r)\frac{x^2 + y^2 + x^2}{r^2} + f'(r)\frac{2(x^2 + y^2 + z^2)}{r^3} = f''(r)\frac{r^2}{r^2} + f'(r)\frac{2r^2}{r^3}$$

$$= f''(r) + f'(r)\frac{2}{r}$$

Ans.

GEOMETRICAL MEANING OF GRADIENT, NORMAL

If a surface $\phi(x, y, z) = c$ passes through a point P. The value of the function at each point on the surface is the same as at P. Then such a surface is called a *level surface* through P. *For example*, If $\phi(x, y, z)$ represents potential at the point P, then *equipotential surface* $\phi(x, y, z) = c$ is a *level surface*.

Two level surfaces can not intersect.

Let the level surface pass through the point P at which the value of the function is ϕ. Consider another level surface passing through Q, where the value of the function is $\phi + d\phi$.

Let $\bar{r}$ and $\bar{r} + \delta\bar{r}$ be the position vector of P and Q then $\overrightarrow{PQ} = \delta\bar{r}$

$$\nabla\phi . d\bar{r} = \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right).(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$$

$$= \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = d\phi \qquad ...(1)$$

If Q lies on the level surface of P, then $d\phi = 0$

Equation (1) becomes $\nabla\phi . dr = 0$. Then $\bar{\nabla}\phi$ is $\perp$ to $d\bar{r}$ (tangent).

Hence, $\nabla\phi$ is normal to the surface $\phi(x, y, z) = c$

Let $\nabla\phi = |\nabla\phi|\hat{N}$, where $\hat{N}$ is a unit normal vector. ~~Let δn be the perpendicular distance between two level surfaces through P and R. Then the rate of change of ϕ in the direction of the~~