ELLIPTIC CURVE CRYPTOGRAPHY

Elliptic Curve Cryptography: Motivation

- Public key cryptographic algorithms (asymmetric key algorithms) play an important role in providing security services:
  - Key management
  - Confidentiality
  - User authentication
  - Signature
- Public key cryptography systems are constructed by relying on the hardness of mathematical problems
  - RSA: based on the integer factorization problem
  - DH: based on the discrete logarithm problem
- The main problem of conventional public key cryptography systems
  - key size has to be sufficient large in order to meet the high-level security requirement.
- This results in lower speed and consumption of more bandwidth
  - Solution: Elliptic Curve Cryptography system

Introduction to Elliptic Curves

- Lets start with a puzzle…

What is the number of balls that may be piled as a square pyramid and also rearranged into a square array?

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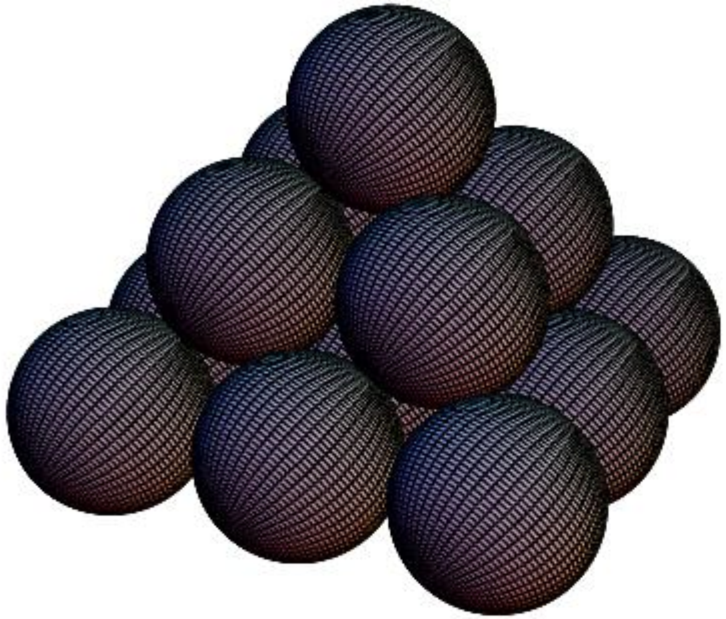
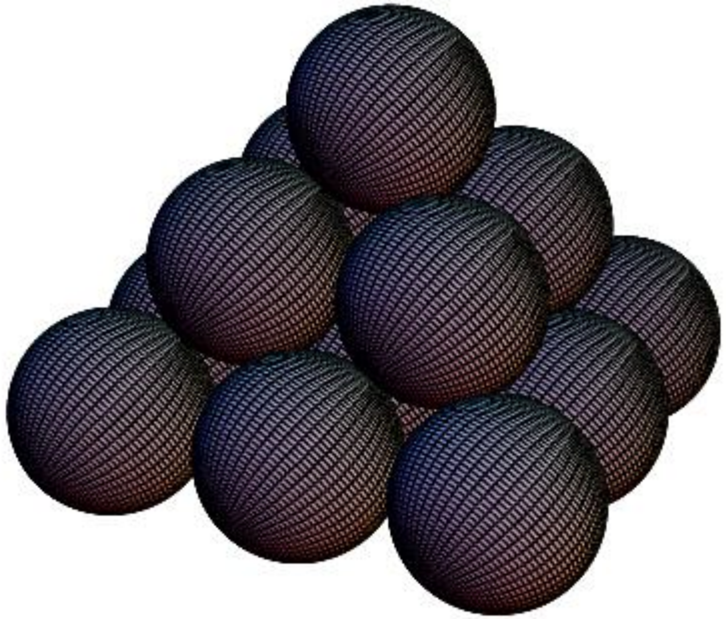


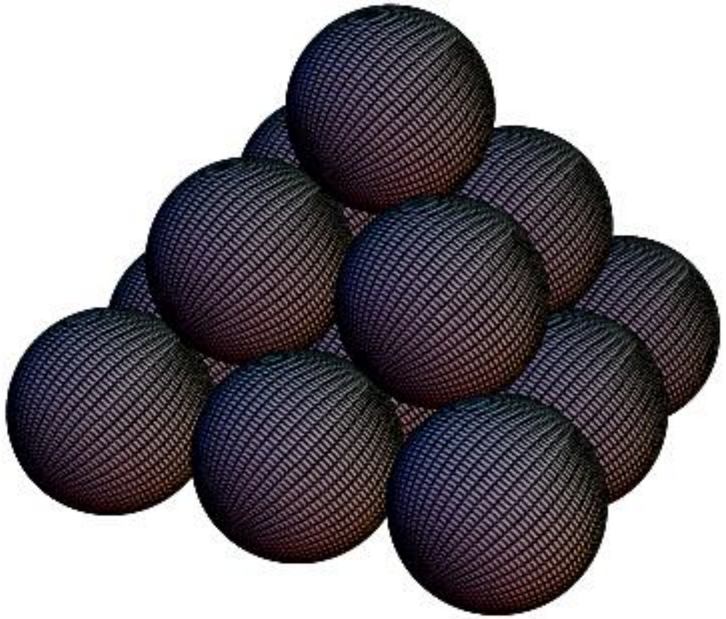
Introduction to Elliptic Curves…

- What about the figure shown?
- *Does it fulfil our requirements?*



Introduction to Elliptic Curves…

- What about the figure shown?
- *Does it fulfil our requirements???*
- *Can you find solutions to this problem???*



Introduction to Elliptic Curves…

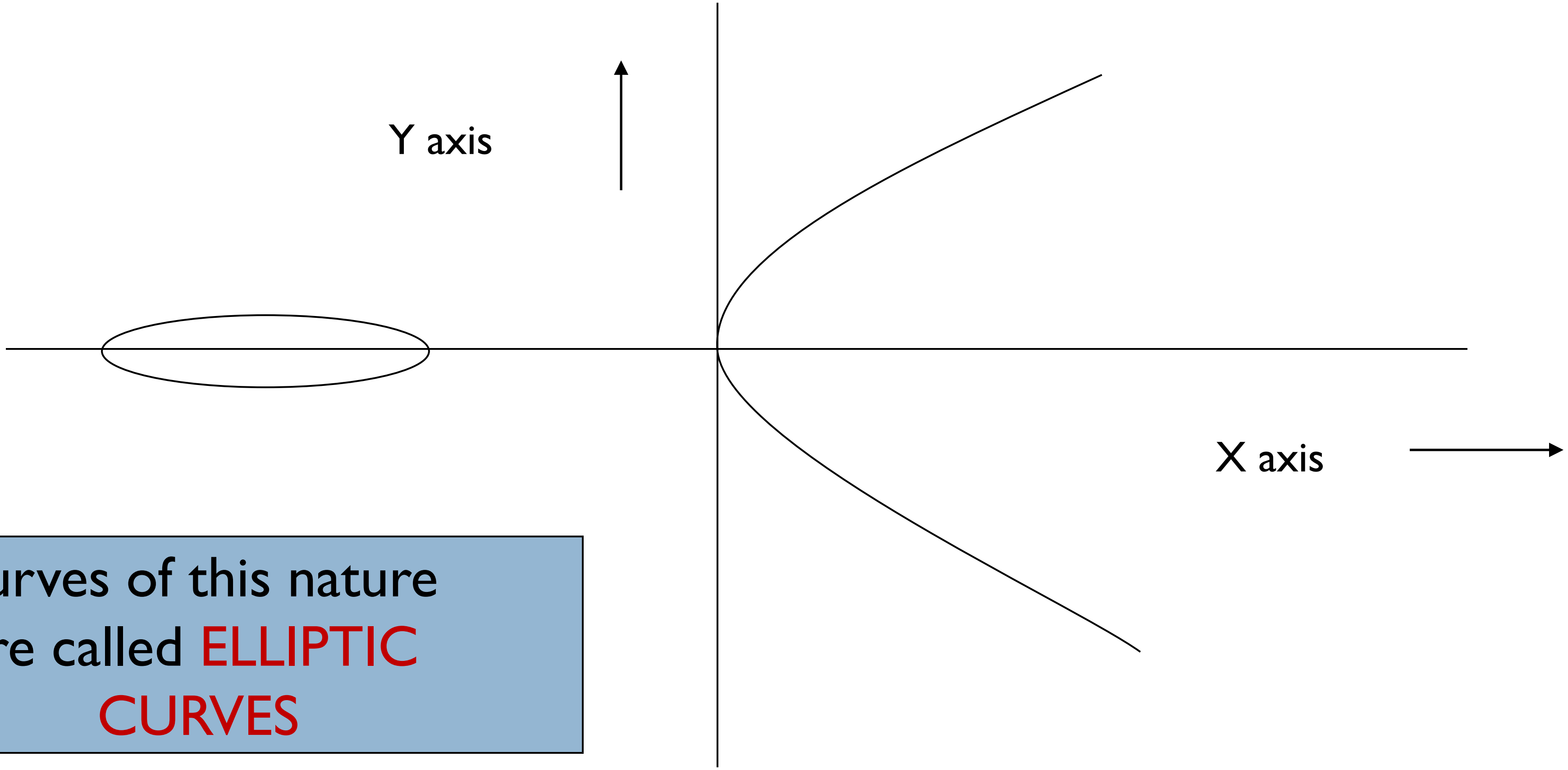
- Let x be the height of the pyramid, then the number of balls in pyramid is,

$$1^2+2^2+3^2+\ldots+x^2=\frac{x(x+1)(2x+1)}{6}$$

- We also want this to be a square. Hence,

$$y^2=\frac{x(x+1)(2x+1)}{6}$$

Graphical Representation



Curves of this nature are called ELLIPTIC CURVES

Method of Diophantus

- Uses a set of known points to produce new points
- (0,0) and (1,1) are two trivial solutions
- Equation of line through these points is y=x.
- Intersecting with the curve and rearranging terms:

$$x^3 - \frac{3}{2}x^2 + \frac{1}{2}x = 0$$

- What are the roots of this equation???

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- What are the roots of this equation???
  - Two trivial roots x=0 and x=1…… But what about third one????

Method of Diophantus…

- We know that, for any numbers a,b,c, we have,

$$(x-a)(x-b)(x-c) = x^3 - (a+b+c)x^2 + (ab+bc+ac)x - abc$$

- Hence, for the equation

$$x^3 - \frac{3}{2}x^2 + \frac{1}{2}x = 0$$

- We have,

$$a+b+x = \frac{3}{2} \rightarrow 0+1+x = \frac{3}{2} \rightarrow x = \frac{1}{2}$$

- Hence, one more point (½ , ½) and because of the symmetry , another (½ , -½)

Method of Diophantus… : Exercise

- Can you find out another point on curve using Diophantus's method ???

Consider two points (½ , -½) and (1,1) and find out another point on the curve …..

Method of Diophantus… : Exercise solution

- Consider the line through (1/2,-1/2) and (1,1) => y=3x-2
- Intersecting with the curve we have:

$$x^3 - \frac{51}{2}x^2 + \ldots = 0$$

- Thus ½ + 1 + x = 51/2 or x = 24 and y=70
- Thus if we have 4900 balls we may arrange them in either way

Weierstrass Equation

- For most situations, an elliptic curve E is the graph of an equation of the form:

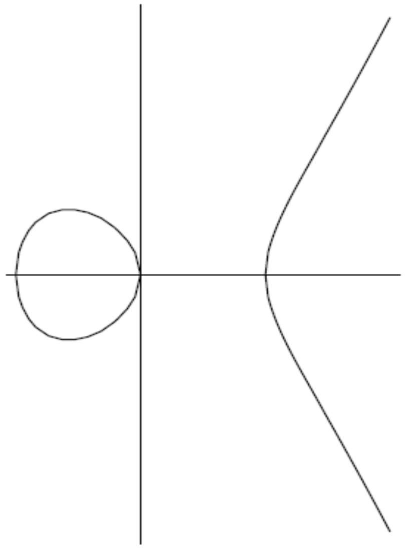
$$y^2 = x^3 + Ax + B$$

  where A and B are constants. This refers to the Weierstrass Equation of Elliptic Curve.
- Here, A, B, x and y all belong to a field of say rational numbers, complex numbers, finite fields (F_p) or Galois Fields ($GF(2^n)$).
- If K is the field where $A,B \in K$, then we say that the Elliptic Curve E is defined over K

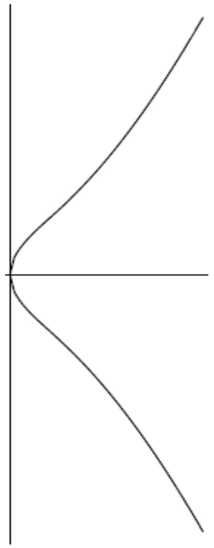
Points on Elliptic Curve

- If we want to consider points with coordinates in some field *L*, *we* write *E(L).* By definition, this set always contains the point at infinity O

$$E(L) = \{O\} \cup \{(x, y) \in L \times L \mid y^2 = x^3 + Ax + B\}$$



$(a) \quad y^2 = x^3 - x$



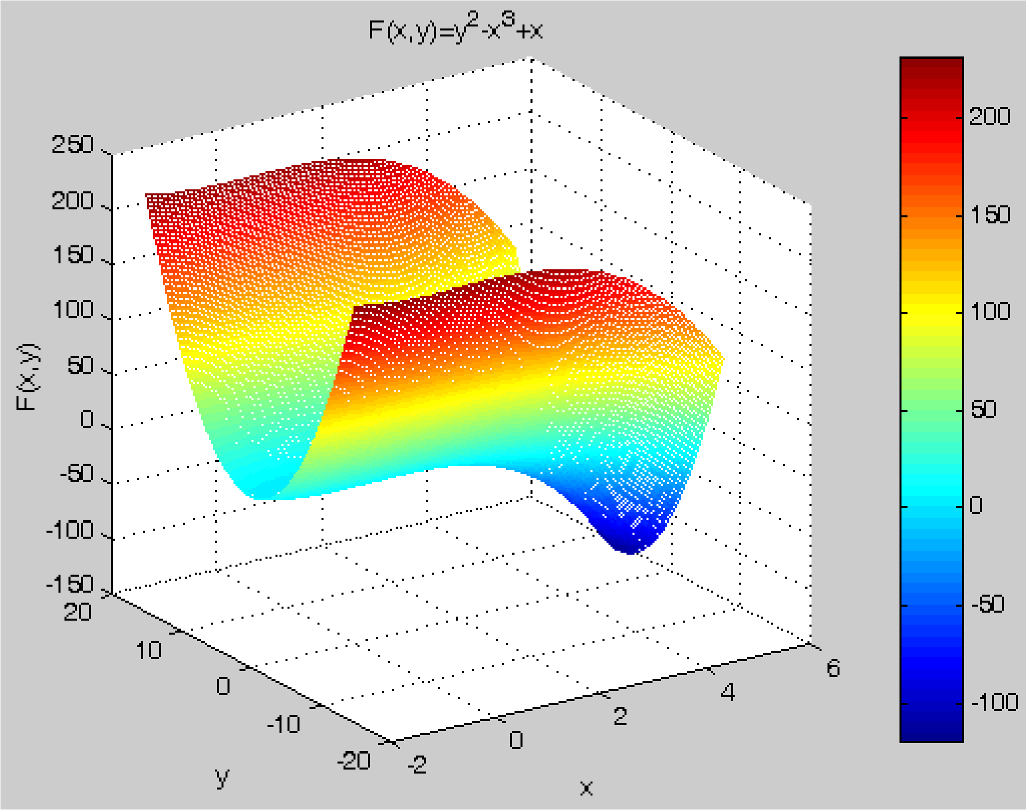
$(b) \quad y^2 = x^3 + x$

What about the roots of these curves ????

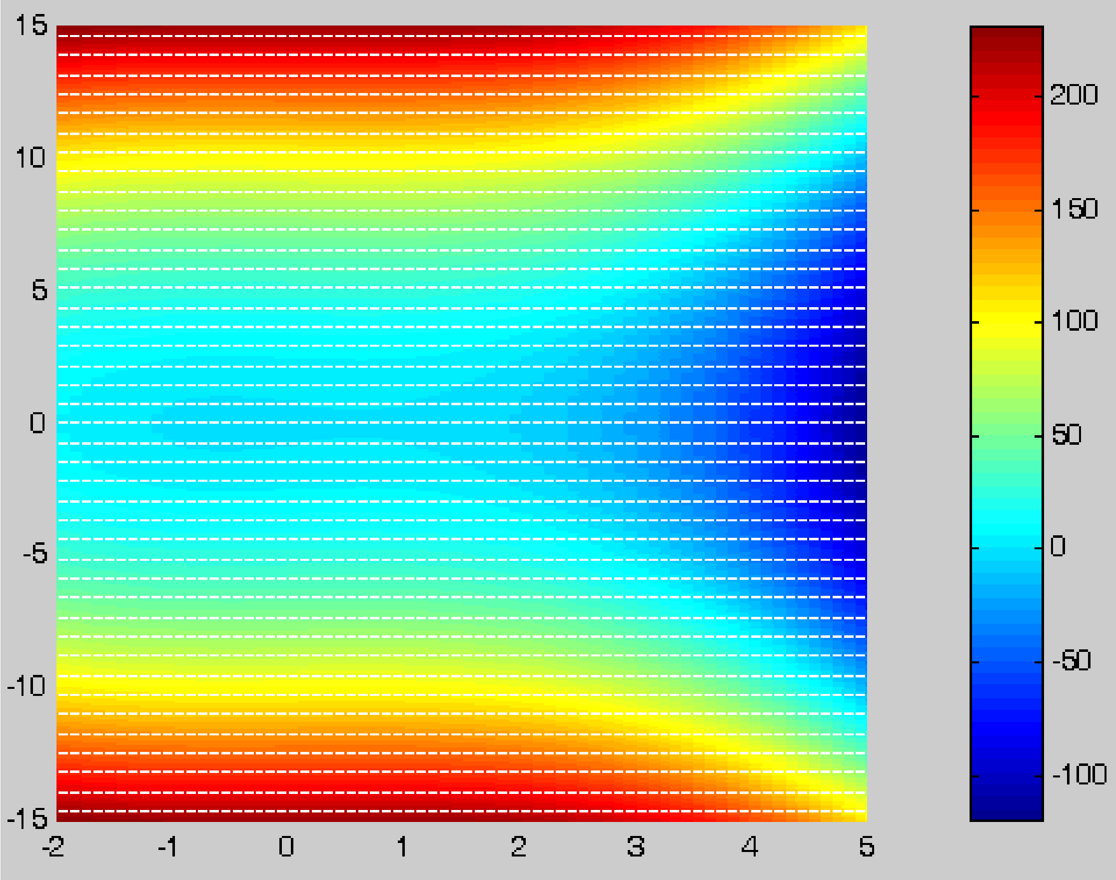
We must have the equation $4A^3 + 27B^2 \neq 0$ satisfied

A condition for an Elliptic curve to be a group !!!!!

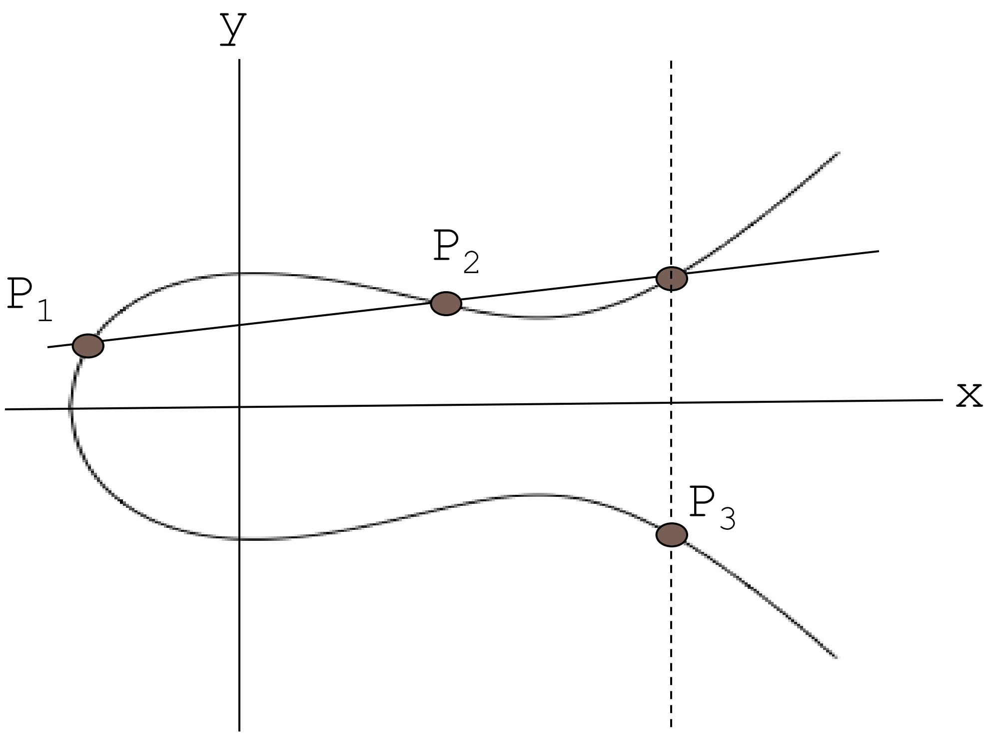
Points on Elliptic Curve…



Points on Elliptic Curve…



Adding points on Elliptic Curve…



- Consider elliptic curve

  $E:\ y^2 = x^3 - x + 1$

- Start with two points : $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ on elliptic curve
- To get a new point P_3 ,
  - Draw a line L through P_1 and P_2
  - Get the intersection P_3'
  - Reflect across x-axis to get P_3
- We define $P_1 + P_2 = P_3$

Adding points on Elliptic Curve…

- Case 1: $P_1 \neq P_2$ and neither point is O
  - For $x_1 \neq x_2$
  - For $x_1 = x_2$????
    - We get $P_1 + P_2 = O$

Slope of the line L passing through P_1 and P_2 is,

$$m = \frac{(y_2 - y_1)}{(x_2 - x_1)}$$

For $x_1 \neq x_2$, equation of line L is,

$$y = m(x - x_1) + y_1$$

To find intersection with E, substitute to get,

$$(m(x - x_1) + y_1)^2 = x^3 + Ax + B$$

Rearrange to form,

$$0 = x^3 - m^2 x^2 + \ldots$$

Given two roots x_1 and x_2, third root can be calculated,

$$(a + b + c) = m^2 \Rightarrow (x_1 + x_2 + x) = m^2$$

$$\Rightarrow x = m^2 - x_1 - x_2$$

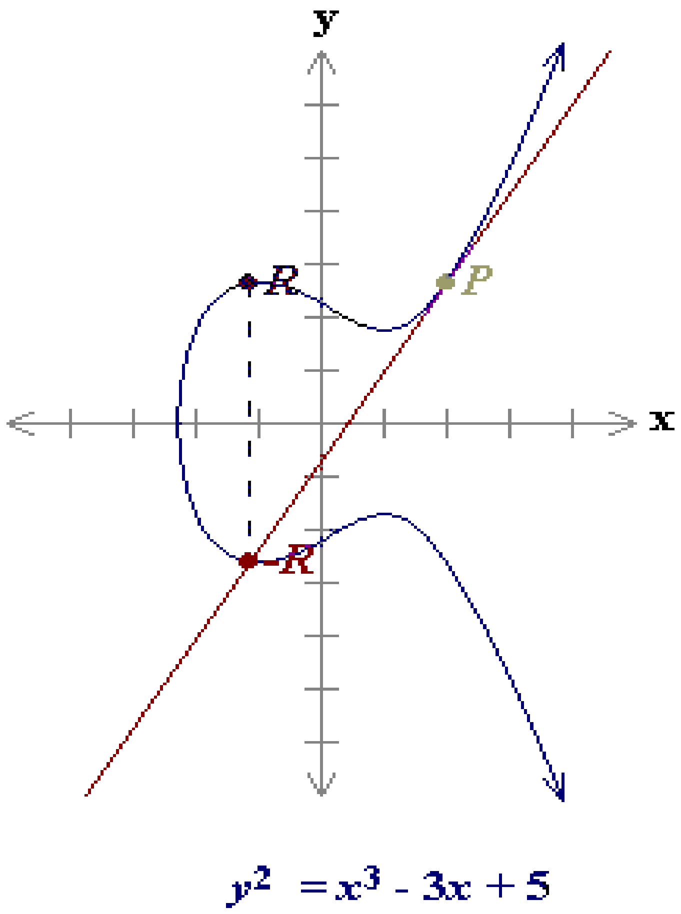
and $y = m(x - x_1) + y_1$

refecting across the x-axis to obtain the point $P_3 = (x_3, y_3)$:

$$x_3 = m^2 - x_1 - x_2 \text{ and } y_3 = m(x_1 - x_3) - y_1$$

Adding points on Elliptic Curve…

- Case II : $P_1 = P_2 = (x_1, y_1)$
  - When two points on a curve are very close to each other, the line through them approximates a tangent line. Therefore, when the two points coincide, we take the line *L* through them to be the tangent line.
  - Implicit differentiation allows us to find the slope *m of L*



P (2, 2.65)

$-R$ (-1.11, -2.64)

R (-1.11, 2.64)

$2P = R =$ (-1.11, 2.64).

Adding points on Elliptic Curve…

- Case II : $P_1 = P_2 = (x_1, y_1)$
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$$2y\frac{dy}{dx} = 3x^2 + A, \text{so } m = \frac{dy}{dx} = \frac{3x_1^2 + A}{2y_1}$$

If $y_1 \neq 0$, the equation of L is,

$$y = m(x - x_1) + y_1$$

We find the cubic equation,

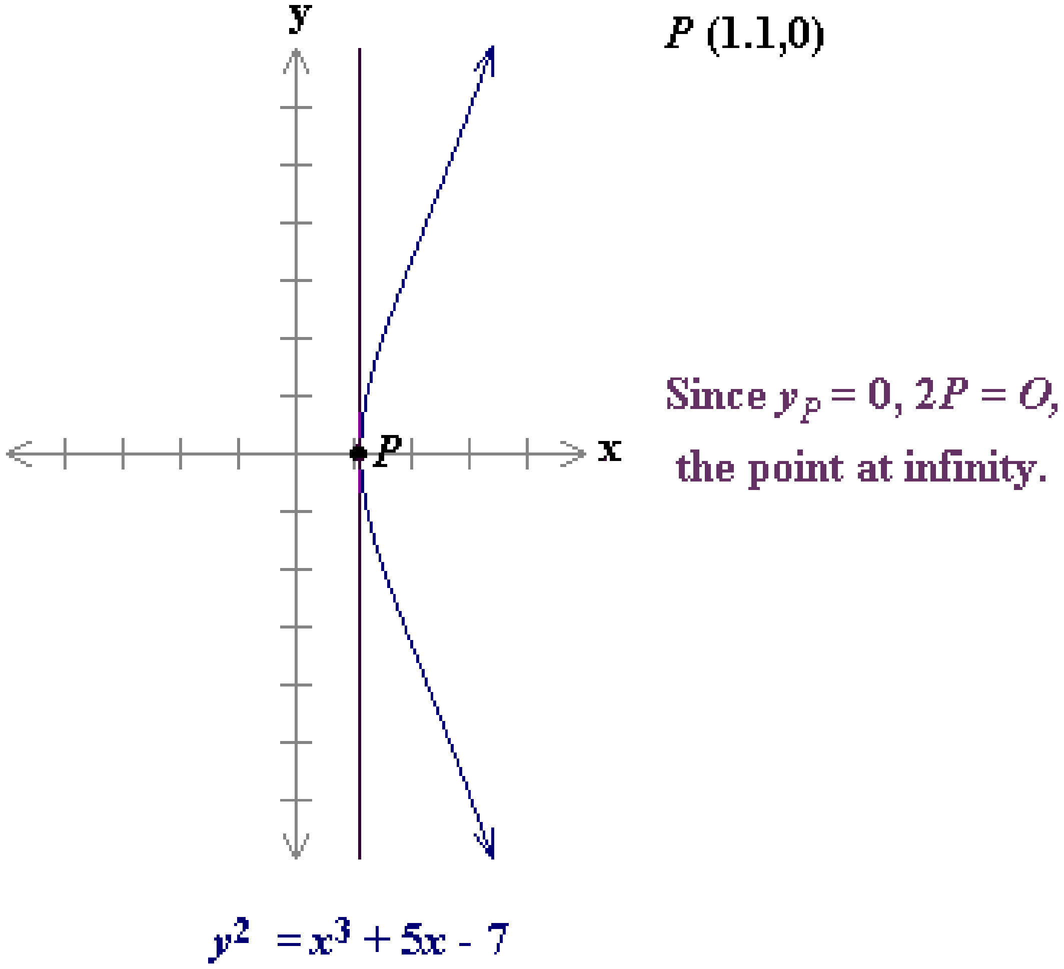
$$0 = x^3 - m^2x^2 + \dots$$

This time we know only one root x_1, we obtain :

$$x_3 = m^2 - 2x_1, \quad y_3 = \mathrm{m}(x_1 - x_3) - y_1$$

Adding points on Elliptic Curve…

- Case II : $P_1 = P_2 = (x_1, y_1)$
  - If $y_1 = 0$
    - We get $P_1 + P_2 = O$
- Case III: $P_2 = O$
  - What about $P_1 + P_2$????
  - Do we get $P_1 + P_2 = P_1$??
  - In other words, $P_1 + O = P_1$



Group Law

- The addition of points on an elliptic curve E satisfies the following properties:
  - (Commutativity) : $P_1 + P_2 = P_2 + P_1$ for all P_1, P_2 on E
  - (Existence of identity) : P + O = P for all P on E
  - (Existence of inverses) : Given P on E, there exists P' on E with P + P' = O. This point P' will usually be denoted as –P
  - (Associatively) : $(P_1 + P_2)+ P_3 = P_1 + (P_2+ P_3)$ for all P_1, P_2, P_3 on E

\`

The points on E form an additive abelian group with O as the identity element.

Integer times a point

- Let k be a positive integer and let P be a point on an elliptic curve, then
  - kP denotes P + P + · · · + P (with k summands)
- Efficient computation for large k
  - Successive doubling method
    - For example, to compute 19*P, we compute*
      - *2P, 4P = 2P+2P, 8P = 4P+4P, 16P = 8P+8P, 19P = 16P+2P+P.*
- But, the only difficulty is....
  - The size of the coordinates of the points increases very rapidly if we are working over the rational numbers
  - What about finite fields ????

ELLIPTIC CURVES IN CRYPTOGRAPHY

Elliptic curves in Cryptography

- Elliptic Curve (EC) systems as applied to cryptography were first proposed in 1985 independently by Neal Koblitz and Victor Miller.
- The discrete logarithm problem on elliptic curve groups
  - More difficult than the corresponding problem in (the multiplicative group of nonzero elements of) the underlying finite field.

Why finite field?

- Elliptic curves over real numbers
  - Calculations prove to be slow
  - Inaccurate due to rounding error
  - Infinite field
- Cryptographic schemes need fast and accurate arithmetic
- In the cryptographic schemes, elliptic curves over two finite fields are mostly used.
  - Prime field F_p , where p is a prime.
  - Binary field F_{2^m}, where m is a positive integer

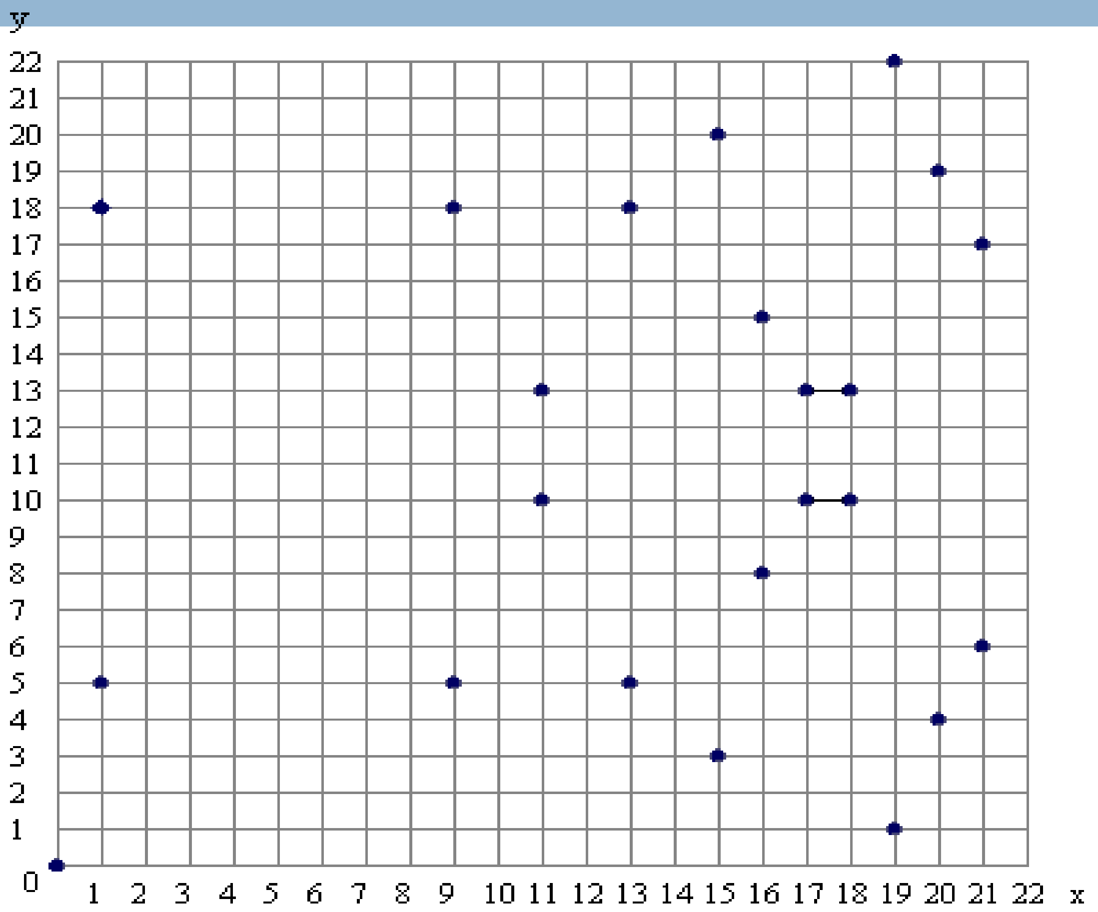
Elliptic Curve over finite field F_{23}

- As a very small example, consider an elliptic curve over the field F_{23}. With A = 1 and B = 0, the elliptic curve equation is $y^2 = x^3 + x$.
- The point (9,5) satisfies this equation since:
  $y^2 \bmod p = x^3 + x \bmod p$

  25 mod 23 = 729 + 9 mod 23
  25 mod 23 = 738 mod 23
  2 = 2
- The 23 points which satisfy this equation are:

  (0,0) (1,5) (1,18) (9,5) (9,18) (11,10) (11,13) (13,5) (13,18) (15,3) (15,20) (16,8) (16,15) (17,10) (17,13) (18,10) (18,13) (19,1) (19,22) (20,4) (20,19) (21,6) (21,17)

Elliptic Curve over finite field F_{23}…



Elliptic curve equation: $y^2 = x^3 + x$ **over** F_{23}

Elliptic curves over finite fields

- Let us do an exercise....
- Let E be the curve $y^2 = x^3+x+1$ over F_5, find all the points on E

Therefore, $E(F_5)$ has order 9.

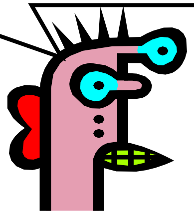
Can you show that $E(F_5)$ is cyclic??? What is the generator??

x	x^3+x+1	y	Points
0	1	± 1	(0,1),(0,4)
1	3	-	-
2	1	± 1	(2,1),(2,4)
3	1	± 1	(3,1),(3,4)
4	4	± 2	(4,2),(4,3)
O		O	O

Discrete logarithms in Finite Fields

F={1,2,3,…,p-1}

Pick secret, random X from F

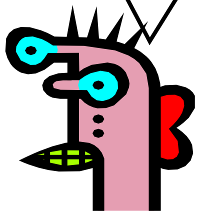


Alice

g^x mod p →

← g^y mod p

Pick secret, random Y from F



Bob

Compute $k=(g^y)^x=g^{xy}$ mod p

Compute $k=(g^x)^y=g^{xy}$ mod p

Eve has to compute g^{xy} from g^x and g^y without knowing x and y…
She faces the Discrete Logarithm Problem in finite fields

Elliptic Curve Discrete Logarithm Problem (ECDLP)

If we are working over a large finite field and are given points P and kP, it is computationally hard to determine the value of k. This is called the ***discrete logarithm problem for elliptic curves (ECDLP)*** *and is the basis for the cryptographic applications.*

What Is Elliptic Curve Cryptography (ECC)?

- Elliptic curve cryptography [ECC] is a **public-key** cryptosystem just like RSA, El Gamal.
- Every user has a **public** and a **private** key.
  - Public key is used for encryption/signature verification.
  - Private key is used for decryption/signature generation.
- Elliptic curves are used as an extension to other current cryptosystems.
  - Elliptic Curve El-Gamal Encryption
  - Elliptic Curve Diffie-Hellman Key Exchange
  - Elliptic Curve Digital Signature Algorithm

Using Elliptic Curves In Cryptography

- The central part of any cryptosystem involving elliptic curves is the **elliptic group**.
- All public-key cryptosystems have some underlying mathematical operation.
  - RSA has exponentiation (raising the message or ciphertext to the public or private values)
  - ECC has point multiplication (repeated addition of two points).

Discrete Logarithm Key pair generation

- A key pair is associated with a set of public domain parameters *(p, q, g). Here, p is a prime, and g ∈ [1, p−1] has order q*

INPUT: D L ⇄ domain parameters (p,q,g).
OUTPUT: Public key y and private key x.
1. Select x $\in_R [1, q-1]$.
2. Compute y $= g^x \bmod p$
3. Return (y,x).

ECC Key pair generation

- Let E be an elliptic curve defined over a finite field F_p.
- Let P be a point in $E(F_p)$, and suppose that P has prime order n. Then the cyclic subgroup of $E(F_p)$ generated by P is,

$$P = \{O, P, 2P, 3P, \ldots, (n-1)P\}.$$

The public domain parameters are : The prime *p, the equation of the elliptic curve E, and the point P and its order n* : *(p,E,P,n)*

A private key is an integer d that is selected uniformly at random from the interval $[1, n-1]$, *and the corresponding public key is* $Q = dP$.

Basic Elgamal encryption scheme

Basic ElGamal Encryption

INPUT : DLdomain parameters (p, q, g), public key y, plaintext $m \in [0, p-1]$.

OUTPUT : Ciphertext (c_1, c_2).

1. Select $k \in_R [1, q-1]$.

2. Compute $c_1 = g^k \bmod p$

3. Compute $c_2 = m \cdot y^k \bmod p$

2. Return (c_1, c_2).

Basic ElGamal Decryption

INPUT : DLdomain parameters (p, q, g), private key x, ciphertext (c_1, c_2).

OUTPUT : Plaintext m.

1. Compute $m = c_2 \bullet c_1^{-x} \bmod p$.

2. Return (m).

ECC Analog to El Gamal : ECEG

EC-ElGamal Encryption

INPUT : Elliptic curve domain parameters (p, E, P, n), public key Q, plaintext m.

OUTPUT : Ciphertext (C_1, C_2)

1. Represent the message m as a point M in $E(F_p)$
2. Select $k \in_R [1, n-1]$.
3. Compute $C_1 = kP$.
4. Compute $C_2 = M + kQ$.
5. Return (C_1, C_2).

EC-ElGamal Decryption

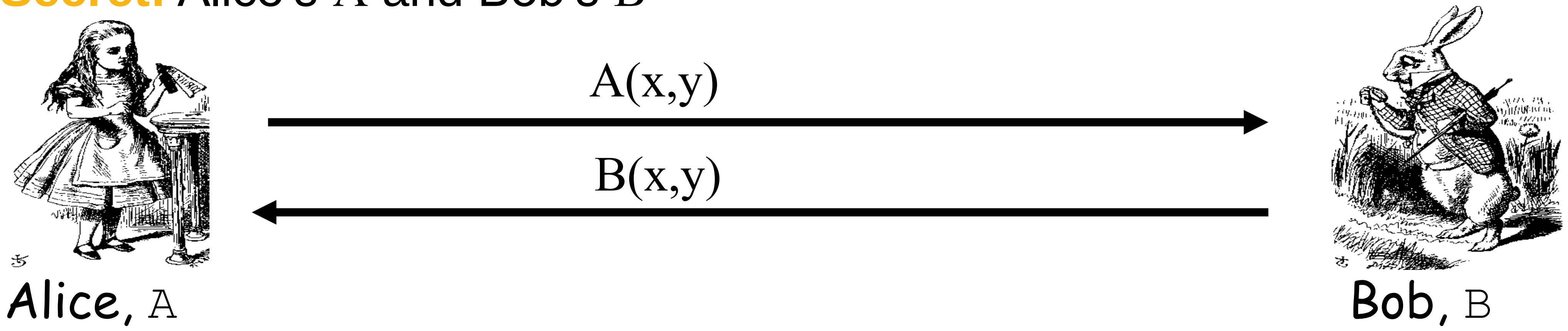
INPUT : Elliptic curve domain parameters (p, E, P, n), private key d, ciphertext (C_1, C_2)

OUTPUT : Plaintext m.

1. Compute $M = C_2 - dC_1$, and extract m from M
2. Return M.

ECC Diffie-Hellman: ECDH

- **Public:** Elliptic curve and point G=(x,y) on curve
- **Secret:** Alice's A and Bob's B



- Alice computes A(B(x,y))
- Bob computes B(A(x,y))
- These are the same since AB = BA

ECC Diffie-Hellman: ECDH…

- **Public:** Curve $y^2 = x^3 + 7x + b \pmod{37}$ and point G=$(2,5)$
- **Alice's secret:** $A = 4$
- **Bob's secret:** $B = 7$
- Alice sends Bob: $4(2,5) = (7,32)$
- Bob sends Alice: $7(2,5) = (18,35)$
- Alice computes: $4(18,35) = (22,1)$
- Bob computes: $7(7,32) = (22,1)$

Why use ECC?

- Criteria to be considered while selecting PKC for application
  - Functionality: Does the public-key family provide the desired capabilities?
  - Security: What assurances are available that the protocols are secure?
  - Performance: For the desired level of security, do the protocols meet performance objectives?
  - Also some misc. factors such as existence of best-practice standards developed by accredited standards organizations, the availability of commercial cryptographic products, and patent coverage.

Why use ECC?...

- The RSA, DL and EC families all provide the basic functionality expected of public-key cryptography
- But…… How do we analyze these Cryptosystems?
  - How difficult is the underlying problem that it is based upon
    - RSA – Integer Factorization
    - DH – Discrete Logarithms
    - ECC - Elliptic Curve Discrete Logarithm problem

Why use ECC?…

- How do we measure difficulty?
  - We examine the algorithms used to solve these problems
  - Integer factorization
    - Number Field Sieve (NFS) : Sub exponential running time
  - Discrete Logarithm
    - Number Field Sieve (NFS) : Sub exponential running time
    - Pollard’s rho algorithm
  - Elliptic Curve Discrete Logarithm Problem(ECDLP)
    - Pollard’s rho algorithm : Fully exponential running time

Why use ECC?...

- To **protect** a 128 bit AES key it would take a:
  - RSA Key Size: 3072 bits
  - ECC Key Size: 256 bits
- How do we strengthen RSA?
  - Increase the key length
- **Impractical?**

NIST guidelines for public key sizes for AES

ECC KEY SIZE (Bits)	RSA KEY SIZE (Bits)	KEY SIZE RATIO	AES KEY SIZE (Bits)
163	1024	1 : 6	
256	3072	1 : 12	128
384	7680	1 : 20	192
512	15 360	1 : 30	256

Supplied by NIST to ANSI X9F1

Applications of ECC

- Many devices are small and have limited storage and computational power
- Where can we apply ECC?
  - **Wireless communication devices**
  - Smart cards
  - Web servers that need to handle many encryption sessions
  - **Any application where security is needed but lacks the power, storage and computational power that is necessary for our current cryptosystems**

key references

- Elliptic Curves: Number Theory and Cryptography, by Lawrence C. Washington
- Guide to Elliptic Curve Cryptography, Alfred J. Menezes
- Guide to Elliptic Curve Cryptography, Darrel R. Hankerson, A. Menezes and A. Vanstone
- For Tutorials: www.certicom.com