Exact Differential Equation and Their Solution

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2.1 Introduction

An exact differential equation, also known as a total differential equation, is a type of ordinary differential equation utilized in physics and engineering. In this chapter, we discuss the exact differential equation and its solution. This chapter also discusses how non-exact differential equations can be converted into exact differential equations to obtain the solution.

2.2 Exact Differential Equation

An exact differential equation is a 1st order differential equation that is obtained by differentiation of its general solution without any term elimination or reduction. It $g(x,y)=c$ is the general solution then $dg=0 \Rightarrow \frac{\partial g}{\partial x}dx+\frac{\partial g}{\partial y}dy=0$.

$M(x,y)dx+N(x,y)dy=0$ represents the exact differential equation where $\frac{\partial f}{\partial x}=M(x,y)$ and $\frac{\partial f}{\partial y}=N(x,y)$.

Theorem: To find the necessary and sufficient conditions for a first-order differential equation of first degree to be accurate.

Statement: The necessary and sufficient condition for the differential equation

$$Mdx+Ndy=0 \tag{2.1}$$

to be exact is

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}. \tag{2.2}$$

Proof: For necessary condition (2): Let (2.1) be exact. Hence by definition, there must exist a function $f(x, y)$ of x and y such that

$$d(f(x,y))=\left(\frac{\partial f}{\partial x}\right)dx+\left(\frac{\partial f}{\partial y}\right)dy=Mdx+Ndy \tag{2.3}$$

Equating coefficients of dx and dy in (1.187), we get

$$\frac{\partial f}{\partial x}=M \tag{2.4}$$

and

$$\frac{\partial f}{\partial y}=N \tag{2.5}$$

To remove the unknown function $f(x, y)$, we differentiate partially (2.4) and (2.5) w.r.t to y and x respectively, we get

$$\frac{\partial M}{\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial y \partial x}$$

$$\frac{\partial N}{\partial x}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial y \partial x}$$

Since, $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$, therefore $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Thus, if (2.1) is exact, M and N satisfy condition (2.2).

Sufficient condition: Assume that condition (2.2) i.e., $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ holds and prove that (2.1) $Mdx + Ndy = 0$ is an exact differential equation.

Let $\int M\,dx = u$ where y is supposed constant while performing integration

Then $\frac{\partial}{\partial x}\left(\int M\,dx\right) = \frac{\partial u}{\partial x} \Rightarrow M = \frac{\partial u}{\partial x}$

$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x}$ or $\frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$ $\left(Given\ \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}\ and\ \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}\right)$b

Integrating both sides w.r.t x (taking y as constant)

$N = \frac{\partial u}{\partial y} + f(y)$, where $f(y)$ is a function of y alone.

$$\therefore M\,dx + N\,dy = \frac{\partial u}{\partial x}dx + \left\{\frac{\partial u}{\partial y} + f(y)\right\}dy$$

$$= \left\{\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right\} + f(y)dy = du + f(y)dy = d\left[u + \int f(y)dy\right]$$

which shows that $Mdx + Ndy = 0$.

Working rule to find the solution of an exact differential equation

Step I: Compare the given differential equation with $Mdx + Ndy = 0$ and find the value of M and N.

Step II: Find $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ check $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then conclude that the given differential equation is exact.

Step III: Integrating M w.r.t x treating y as constant.

Step IV: Integrate w.r.t y only those terms of N which do not contain x.

Step V: Equate the sum of two integrals from step III and Step IV to an arbitrary constant and thus we obtain the required solution i.e., the solution of the exact differential equation $Mdx + Ndy = 0$ is

$$\int_{Treating\ y\ as\ constant} M\,dx + \int (terms\ in\ N\ not\ containing\ x)\,dy = c,$$

Where c is an arbitrary constant.

Example: Check whether given equation $xdy + 2y^2 dx = 0$ is exact or not.

Solution: Comparing the given equation $xdy + 2y^2 dx = 0$ with $Mdx + Ndy = 0$, we get $M = 2y^2$ and $N = x$

$\frac{\partial M}{\partial y} = 4y$ and $\frac{\partial N}{\partial x} = 1$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ therefore $xdy + 2y^2 dx = 0$ is not an exact differential equation.

Example: Solve: $e^y dx + (xe^y + 2y)dy = 0$.

Solution: Comparing the given equation $e^y\,dx + \left(xe^y + 2y\right)dy = 0$ with $Mdx + Ndy = 0$, we get

$M = e^y$ and $N = xe^y + 2y$ so that $\dfrac{\partial M}{\partial y} = e^y = \dfrac{\partial N}{\partial x}$

Hence the given equation is exact.
Thus, the solution of the exact differential equation $Mdx + Ndy = 0$ is $\int\limits_{Treating\ y\ as\ constant} M\,dx + \int(terms\ in\ N\ not\ containing\ x)\,dy = c$, where c is an arbitrary constant.

$$\Rightarrow \int\limits_{Treating\ y\ as\ constant} e^y\,dx + \int 2y\,dy = c$$

$$\Rightarrow xe^y + y^2 = c$$

Example: Solve: $(\cos x - x\cos y)dy - (\sin y + y\sin x)dx = 0$

Solution: Comparing the given equation with standard form (2.1), we get

$\boxed{M = 2y^2 \text{ and } N = x}$ so that

$$\frac{\partial M}{\partial y} = -\cos y - \sin x = \frac{\partial N}{\partial x}$$

Hence the given equation is exact.
Thus, the solution of the exact differential equation $Mdx + Ndy = 0$ is $\int\limits_{Treating\ y\ as\ constant} M\,dx + \int(terms\ in\ N\ not\ containing\ x)\,dy = c$, where c is an arbitrary constant.

$$\Rightarrow \int\limits_{Treating\ y\ as\ constant} -(\sin y + y\sin x)\,dx + \int 0\,dy = c$$

$\Rightarrow -x\sin y + y\cos x = c$ is the required solution.

Example: Solve: $\left(2x^3y - 3x^2y^2 - 5y^4\right)dy + \left(5x^4 + 3x^2y^2 - 2xy^3\right)dx = 0$.

Solution: Comparing the given equation with (2.1), we get

$M = 5x^4 + 3x^2y^2 - 2xy^3$ and $N = 2x^3y - 3x^2y^2 - 5y^4$ so that

$$\frac{\partial M}{\partial y} = 6x^2y - 6xy^2 = \frac{\partial N}{\partial x}$$

Hence the given equation is exact.
Thus, the solution of the exact differential equation $Mdx + Ndy = 0$ is $\int\limits_{Treating\ y\ as\ constant} M\,dx + \int(terms\ in\ N\ not\ containing\ x)\,dy = c$, where c is an arbitrary constant.

$$\Rightarrow \int\limits_{Treating\ y\ as\ constant} \left(5x^4 + 3x^2y^2 - 2xy^3\right)dx + \int -5y^4\,dy = c$$

$\Rightarrow x^5 + x^3y^2 - x^2y^3 - y^5 = c$ is the required solution.

Example : Solve: $\left(y^2e^{xy^2} + 4x^3\right)dx + \left(2x\,ye^{xy^2} - 3y^2\right)dy = 0$.

Solution: Comparing the given equation with $Mdx + Ndy = 0$, we get

$M = y^2e^{xy^2} + 4x^3$ and $N = 2x\,ye^{xy^2} - 3y^2$ so that

$$\frac{\partial M}{\partial y} = 2y\,e^{xy^2} + y^2e^{xy^2}.2xy = \frac{\partial N}{\partial x}$$

Hence the given equation is exact.
Thus, the solution of $Mdx + Ndy = 0$ is

$$\int_{\text{Treating } y \text{ as constant}} M\,dx + \int (\text{terms in } N \text{ not containing } x)\,dy = c,$$ where c is an arbitrary constant.

$$\Rightarrow \int_{\text{Treating } y \text{ as constant}} \left(y^2 e^{xy^2} + 4x^3\right)dx + \int \left(-3y^2\right)dy = c$$

$\Rightarrow e^{xy^2} + x^4 - y^3 = c$ is the required solution.

Example : Solve: $\left\{y\left(1+\frac{1}{x}\right)+\cos y\right\}dx + \left(x + \log x - x\sin y\right)dy = 0$.

Solution: Comparing the given equation with $Mdx + Ndy = 0$, we get

$M = y\left(1+\frac{1}{x}\right)+\cos y$ and $N = x + \log x - x\sin y$ so that

$$\frac{\partial M}{\partial y} = 1 + \frac{1}{x} - \sin y = \frac{\partial N}{\partial x}$$

Hence the given equation is exact.
Thus, the solution $Mdx + Ndy = 0$ is

$$\int_{\text{Treating } y \text{ as constant}} M\,dx + \int (\text{terms in } N \text{ not containing } x)\,dy = c,$$ where c is an arbitrary constant.

$$\Rightarrow \int_{\text{Treating } y \text{ as constant}} \left(y\left(1+\frac{1}{x}\right)+\cos y\right)dx + \int 0.dy = c$$

$\Rightarrow (x + \log x)y + x\cos y = c$ is the required solution.

Example: Show solution of $(4x+3y+1)dx + (3x+2y+1)dy = 0$ is a family of hyperbolas with a common axis and tangent at the vertex.

Solution: Comparing the given equation with $Mdx + Ndy = 0$, we get $M = 4x + 3y + 1$ and $N = 3x + 2y + 1$ so that $\frac{\partial M}{\partial y} = 3 = \frac{\partial N}{\partial x}$

. Hence the given equation is exact.
Thus, the solution of $Mdx + Ndy = 0$ is

$$\int_{\text{Treating } y \text{ as constant}} M\,dx + \int (\text{terms in } N \text{ not containing } x)\,dy = c,$$ where c is an arbitrary constant.

$$\Rightarrow \int_{\text{Treating } y \text{ as constant}} (4x + 3y + 1)dx + \int (2y+1).dy = c$$

$$\Rightarrow 2x^2 + 3xy + x + y^2 + y + c = 0 \qquad (2.6)$$

Comparing (2.6) with standard form of conic section

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ where $a=2,\ b=1,\ h=3/2,\ g=1/2,\ f=1/2,\ c=c$ (2.7)

Then $h^2 - ab = \left(\frac{9}{4}\right) - 2 = +ve$,

shows that (2.6) represents a family of hyperbolas, c being the parameter, with common axis and tangent at vertex.

Example: Solve: $(r+\sin\theta-\cos\theta)dr+r(\sin\theta+\cos\theta)d\theta=0$.

Solution: Here we have r and θ in place of usual variables x and y.

Comparing the given equation $(r+\sin\theta-\cos\theta)dr+r(\sin\theta+\cos\theta)d\theta=0$ with $Mdr+Nd\theta=0$, we get

$M=r+\sin\theta-\cos\theta$ and $N=r(\sin\theta+\cos\theta)$ so that

$$\frac{\partial M}{\partial \theta}=\cos\theta+\sin\theta=\frac{\partial N}{\partial r}$$

Hence the given equation is exact.

Thus, the solution of the exact differential equation $Mdx+Ndy=0$ is $\int\limits_{\text{Treating }\theta\text{ as constant}} M\,dr+\int(\text{terms in } N \text{ not containing } r)\,d\theta=c$, where c is an arbitrary constant.

$$\Rightarrow \int\limits_{\text{Treating }\theta\text{ as constant}} (r+\sin\theta-\cos\theta)\,dr+\int 0.d\theta=c$$

$\Rightarrow \frac{r^2}{2}+r(\sin\theta-\theta)=c$ is the required solution.

Example: Solve: $\left(1+e^{\frac{x}{y}}\right)+e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)\frac{dy}{dx}=0$

Solution: Here given equation can be expressed as

$$\left(1+e^{\frac{x}{y}}\right)dx+\left(e^{\frac{x}{y}}-e^{\frac{x}{y}}\frac{x}{y}\right)dy=0$$

Comparing the $\left(1+e^{\frac{x}{y}}\right)dx+\left(e^{\frac{x}{y}}-e^{\frac{x}{y}}\frac{x}{y}\right)dy=0$ with $Mdx+Ndy=0$, we get

$M=1+e^{\frac{x}{y}}$ and $N=e^{\frac{x}{y}}-e^{\frac{x}{y}}\frac{x}{y}$.

Now $\frac{\partial M}{\partial y}=-\frac{x}{y^2}e^{\frac{x}{y}}$

$\frac{\partial N}{\partial x}=\frac{1}{y}e^{\frac{x}{y}}-\frac{1}{y}e^{\frac{x}{y}}-\frac{x}{y^2}e^{\frac{x}{y}}=-\frac{x}{y^2}e^{\frac{x}{y}}$

$$\frac{\partial M}{\partial y}=-\frac{x}{y^2}e^{\frac{x}{y}}=\frac{\partial N}{\partial x}$$

Hence the given equation is exact.

Thus, the solution of the exact differential equation $Mdx+Ndy=0$ is $\int\limits_{\text{Treating } y \text{ as constant}} M\,dx+\int(\text{terms in } N \text{ not containing } x)\,dy=c$, where c is an arbitrary constant.

$$\Rightarrow \int\limits_{\text{Treating } y \text{ as constant}} \left(1+e^{\frac{x}{y}}\right)dx+\int 0.dy=c$$

$\Rightarrow x+y.e^{\frac{x}{y}}=c$ is the required solution.

Example: Solve: $x\,dx + y\,dy = \dfrac{a^2(xdy - ydx)}{x^2+y^2}$.

Solution: Here given equation can be expressed as

$$\left(x + \frac{a^2 y}{x^2+y^2}\right)dx + \left(y - \frac{a^2 x}{x^2+y^2}\right)dy = 0$$

Comparing the equation $\left(x + \frac{a^2 y}{x^2+y^2}\right)dx + \left(y - \frac{a^2 x}{x^2+y^2}\right)dy = 0$ with $Mdx + Ndy = 0$,

we get

$M = x + \dfrac{a^2 y}{x^2+y^2}$ and $N = y - \dfrac{a^2 x}{x^2+y^2}$.

Now $\dfrac{\partial M}{\partial y} = \dfrac{a^2(x^2-y^2)}{x^2+y^2}$

$\dfrac{\partial N}{\partial x} = \dfrac{a^2(x^2-y^2)}{x^2+y^2}$

$$\frac{\partial M}{\partial y} = \frac{a^2(x^2-y^2)}{x^2+y^2} = \frac{\partial N}{\partial x}$$

Hence the given equation is exact.

Thus, the solution of the exact differential equation $Mdx + Ndy = 0$ is

$$\int_{Treating\ y\ as\ constant} M\,dx + \int (terms\ in\ N\ not\ containing\ x)\,dy = c,$$ where c is an arbitrary constant.

$$\Rightarrow \int_{Treating\ y\ as\ constant} \left(x + \frac{a^2 y}{x^2+y^2}\right)dx + \int y.dy = c$$

$$\Rightarrow \frac{x^2}{2} + a^2 y.\frac{1}{y}\tan^{-1}\left(\frac{x}{y}\right) + \frac{y^2}{2} = c$$

$$\Rightarrow \left(\frac{x^2+y^2}{2}\right) + a^2\tan^{-1}\left(\frac{x}{y}\right) = c$$ is the required solution.

Example: For what value of λ, $(xy^2 + \lambda x^2 y)dx + (x+y)x^2 dy = 0$ is exact. Solve the equation for this value of λ.

Solution: Comparing the equation $(xy^2 + \lambda x^2 y)dx + (x+y)x^2 dy = 0$ with

$Mdx + Ndy = 0$, we get

$M = xy^2 + \lambda x^2 y$ and $N = (x+y)x^2$

Now $\dfrac{\partial M}{\partial y} = 2xy + \lambda x^2$ and $\dfrac{\partial N}{\partial x} = 3x^2 + 2xy$

Condition to be exact: $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

$$2xy + \lambda x^2 = 3x^2 + 2xy$$

$$\Rightarrow \lambda x^2 = 3x^2 \Rightarrow \lambda = 3$$

If $\lambda = 3$ then $(xy^2 + \lambda x^2 y)dx + (x+y)x^2 dy = 0$ is exact.

Thus, the solution of $Mdx+Ndy=0$ is

$$\int\limits_{\text{Treating } y \text{ as constant}} M\,dx + \int (\text{terms in } N \text{ not containing } x)\,dy = c,$$ where c is an arbitrary constant.

$$\Rightarrow \int\limits_{\text{Treating } y \text{ as constant}} \left(xy^2+3x^2y\right)dx + \int 0.dy = c$$

$$\Rightarrow \frac{x^2y^2}{2}+\frac{3x^3y}{3}=c \Rightarrow \frac{x^2y^2}{2}+x^3y=c$$

$x^2y^2+2x^3y=c$ is the required solution.

Example: Find the value of λ, for the differential equation, $\left(2xe^y+3y^2\right)\frac{dy}{dx}+\left(3x^2+\lambda e^y\right)=0$ is exact. Solve the equation for this value of λ.

Solution: Here $\left(2xe^y+3y^2\right)\frac{dy}{dx}+\left(3x^2+\lambda e^y\right)=0$ can be expressed as

$$\left(2xe^y+3y^2\right)dy+\left(3x^2+\lambda e^y\right)dx=0$$

Comparing the equation $\left(2xe^y+3y^2\right)dy+\left(3x^2+\lambda e^y\right)dx=0$ with $M\,dx+N\,dy=0$, we get

$M=\left(3x^2+\lambda e^y\right)$ and $N=2xe^y+3y^2$

Now $\frac{\partial M}{\partial y}=\lambda e^y$ and $\frac{\partial N}{\partial x}=2e^y$

Condition to be exact: $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$

$$\lambda e^y=2e^y$$

$$\Rightarrow \lambda=2$$

If $\lambda=2$ then $\left(2xe^y+3y^2\right)dy+\left(3x^2+\lambda e^y\right)dx=0$ is exact.

Thus, the solution of $M\,dx+N\,dy=0$ is

$$\int\limits_{\text{Treating } y \text{ as constant}} M\,dx + \int (\text{terms in } N \text{ not containing } x)\,dy = c,$$ where c is an arbitrary constant.

$$\Rightarrow \int\limits_{\text{Treating } y \text{ as constant}} \left(3x^2+2e^y\right)dx + \int 3y^2.dy = c$$

$\Rightarrow x^3+2e^x+y^3=c$ is the required solution.

Tutorial -3

Determine which of the following equations are exact and find the solution of following differential equation

1. $(x+2y-2)dx+(2x-y+3)dy=0$ (Ans $x^2+4xy-4x-y^2+6y=c$)
2. $\left(2xy+e^y\right)dx+\left(x^2+xe^y\right)dy=0$ (Ans $x^2y+xe^y=c$)
3. $\left(x^2+2ye^{2x}\right)dy+\left(2xy+2y^2e^{2x}\right)dx=0$ (Ans $x^2y+y^2e^{2x}=c$)

4. $(y^2 - x^2)dx + 2xy\,dy = 0$ (Ans $\frac{x^3}{3} = xy^2 + c$)

5. $(3x^2 + 4xy)dx + (2x^2 + 2y)dy = 0$ (Ans $x^3 + 2x^2y + y^2 = c$)

6. $(2ax + by)ydx + (ax + 2by)xdy = 0$ (Ans $ayx^2 + by^2x = c$)

7. $(2y - x)dy = (y - 2x)dx; y(1) = 2$ (Ans $ayx^2 + by^2x = c$)

8. $(\sin x.\tan y + 1)dx + (\cos.\sec^2 y)dy = 0$ (Ans Not Exact)

9. $(\sin x.\sin - xe^y)dy = (e^y + \cos x.\cos y)dy = 0$ (Ans $xe^y + \sin x.\cos y = c$)

10. $(\cos x \cos y - \cot x)dx - (\sin x.\sin y)dy = 0$ (Ans $\sin x \cos y = \ln(c \sin x)$)

2.3 Equation Reducible to Exact Differential Equation

Consider a differential equation of the form $Mdx + Ndy = 0$, which is not exact. Suppose there exists a function $F(x, y)$ such that $F(x, y)[M\,dx + N\,dy = 0]$ is exact, then F (x, y) is called an integrating factor (I.F.) of the differential equation (1). In this chapter, we use the abbreviation I. F for integrating factor. There is no general method of finding integrating factors for the differential equation $Mdx + Ndy = 0$. There exist an infinite number of integrating factors for an equation $Mdx + Ndy = 0$ as established in the following theorem.

The rules for finding integrating factors of the equation $Mdx + Ndy = 0$ are as follows:

Rule I: If $Mdx + Ndy = 0$ be a homogenous equation in x and y then $\frac{1}{Mx + Ny}$ is an integrating factor provided $Mdx + Ndy \neq 0$.

$$\therefore \frac{M\,dx+N\,dy}{Mx+Ny}=d\left[\frac{1}{2}\times\log xy+\frac{1}{2}\times\int g\left\{\log\left(\frac{x}{y}\right)\right\}d\left\{\log\left(\frac{x}{y}\right)\right\}\right]$$

which shows that $\frac{1}{Mx+Ny}$ is an integrating factor provided $Mdx+Ndy\neq 0$.

Note: If $Mx+Ny=0\Rightarrow\frac{M}{y}=-\frac{N}{x}$ then the equation $Mx+Ny=0$ reduced to

$x\,dy-y\,dy=0$ that is

$\frac{x\,dy-y\,dy}{x^2}=0\Rightarrow d\left(\frac{y}{x}\right)=0\Rightarrow y=cx$ is the solution.

Example: Solve: $\left(x^2y-2xy^2\right)dx-\left(x^3-3x^2y\right)dy=0$.

Solution: Comparing given differential equation with $Mdx+Ndy=0$, we get

$$M=x^2y-2xy^2 \text{ and } N=-\left(x^3-3x^2y\right).$$

Now $\frac{\partial M}{\partial y}=x^2-4xy$ and $\frac{\partial N}{\partial x}=-\left(3x^2-6xy\right)$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, therefore given differential equation is not exact. The given differential equation is a homogenous differential equation.

$Mx=x\left(x^2y-2xy^2\right)=x^3y-2x^2y^2$

$Ny=y\left(3x^2y-x^3\right)=3x^2y^2-x^3y$

$Mx+Ny=x^3y-2x^2y^2+3x^2y^2-x^3y=x^2y^2$

According to the theorem $\frac{1}{Mx+Ny}$ is an integrating factor provided $Mx+Ny\neq 0$.

$\therefore \frac{1}{Mx+Ny}=\frac{1}{x^2y^2}$ is an integrating factor for given differential equation.

Now $M^{*}=\frac{1}{Mx+Ny}\times M=\frac{1}{x^2y^2}\times\left(x^2y-2xy^2\right)=\frac{1}{y}-\frac{2}{x}$

and $N^{*}=\frac{1}{Mx+Ny}\times N=-\frac{1}{x^2y^2}\times\left(x^3-3x^2y\right)=-\left(\frac{x}{y^2}-\frac{3}{y}\right)$

$\frac{\partial M^{*}}{\partial y}=\frac{\partial}{\partial y}\left(\frac{1}{y}-\frac{2}{x}\right)=-\frac{1}{y^2}$; $\frac{\partial N}{\partial x}=-\frac{\partial}{\partial x}\left(\frac{x}{y^2}-\frac{3}{y}\right)=-\frac{1}{y^2}$

Since $\frac{\partial M^{*}}{\partial y}=\frac{\partial N^{*}}{\partial x}$, therefore $M^{*}dx+N^{*}dy=0=\left(\frac{1}{y}-\frac{2}{x}\right)dx+\left(\frac{x}{y^2}-\frac{3}{y}\right)dy=0$ is an exact differential equation. Thus, the solution is given by

$\int_{\text{Treating } y \text{ as constant}} M^{*}\,dx+\int\left(\text{terms in } N^{*} \text{ not containing } x\right)dy=c$, where c is an arbitrary constant.

$$\int_{\text{Treating } y \text{ as constant}}\left(\frac{1}{y}-\frac{2}{x}\right)dx+\int\left(\text{terms in}\left(\frac{x}{y^2}-\frac{3}{y}\right)\text{not containing } x\right)dy=c$$

$$\int_{\text{Treating } y \text{ as constant}}\left(\frac{1}{y}-\frac{2}{x}\right)dx+\int\left(-\frac{3}{y}\right)dy=c$$

$\frac{x}{y} - 2\log x + 3\log y = c$ is the required solution.

Example: Solve: $\frac{dy}{dx} = \frac{x^3+y^3}{xy^2}$.

Given differential equation $\frac{dy}{dx} = \frac{x^3+y^3}{xy^2}$ can be written as

$$\left(x^3+y^3\right)dx + xy^2 dy = 0$$

Comparing the given equation $\left(x^3+y^3\right)dx + xy^2 dy = 0$ with $Mdx+Ndy=0$, we get

$$M=x^3+y^3 \text{ and } N=-x^2y.$$

Now $\frac{\partial M}{\partial y} = 3y^2$ and $\frac{\partial N}{\partial x} = -2xy$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, therefore given differential equation id not exact. The given differential equation is a homogenous differential equation.

$Mx = x\left(x^3+y^3\right) = x^4 + xy^3$

$Ny = y\left(-xy^2\right) = -xy^3$

$Mx+Ny = x^4+xy^3+-xy^3 = x^4$

According to the theorem $\frac{1}{Mx+Ny}$ is an integrating factor provided $Mx+Ny \neq 0$.

$\therefore \frac{1}{Mx+Ny} = \frac{1}{x^4}$ is an integrating factor for given differential equation .

Now $M^* = \frac{1}{Mx+Ny} \times M = \frac{1}{x^4} \times \left(x^3+y^3\right) = \frac{1}{x} + \frac{y^3}{x^4}$

and $N^* = \frac{1}{Mx+Ny} \times N = \frac{1}{x^4} \times \left(-xy^2\right) = -\frac{y^2}{x^3}$

$\frac{\partial M^*}{\partial y} = \frac{\partial}{\partial y}\left(\frac{1}{x} + \frac{y^3}{x^4}\right) = \frac{3y^2}{x^4}$; $\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(-\frac{y^2}{x^3}\right) = \frac{3y^2}{x^4}$

Since $\frac{\partial M^*}{\partial y} = \frac{\partial N^*}{\partial x}$, therefore $M^* dx + N^* dy = 0 = \left(\frac{1}{x} + \frac{y^3}{x^4}\right)dx + \left(-\frac{y^2}{x^3}\right)dy = 0$ is an exact. Thus, the solution is given by

$\int\limits_{\text{Treating } y \text{ as constant}} M^* dx + \int \left(\text{terms in } N^* \text{ not containing } x\right) dy = c$, where c is an arbitrary constant.

$$\int\limits_{\text{Treating } y \text{ as constant}} \left(\frac{1}{x} + \frac{y^3}{x^4}\right)dx + \int \left(\text{terms in} \left(-\frac{y^2}{x^3}\right) \text{not containing } x\right)dy = c$$

$$\int\limits_{\text{Treating } y \text{ as constant}} \left(\frac{1}{x} + \frac{y^3}{x^4}\right)dx + \int 0.dy = c$$

$\log x - \frac{y^3}{3x^3} = c$ is the required solution.

Example : Solve: $\left(y^3-3xy^2\right)dx+\left(2x^2y-xy^2\right)dy=0$.

Solution: Comparing $\left(y^3-3xy^2\right)dx+\left(2x^2y-xy^2\right)dy=0$ with $Mdx+Ndy=0$, we get

$$M=y^3-3xy^2 \text{ and } N=2x^2y-xy^2 .$$

Now $\dfrac{\partial M}{\partial y}=3y^2-6xy$ and $\dfrac{\partial N}{\partial x}=4xy-y^2$

Since $\dfrac{\partial M}{\partial y}\neq\dfrac{\partial N}{\partial x}$, therefore given differential equation id not exact. The given differential equation is a homogenous differential equation.

$Mx=x\left(y^3-3xy^2\right)=xy^3-3x^2y^2$

$Ny=y\left(2x^2y-xy^2\right)=2x^2y^2-xy^3$

$Mx+Ny=xy^3-3x^2y^2+2x^2y^2-xy^3=-x^2y^2$

According to the theorem $\dfrac{1}{Mx+Ny}$ is an integrating factor provided $Mx+Ny\neq0$.

$\therefore \dfrac{1}{Mx+Ny}=-\dfrac{1}{x^2y^2}$ is an integrating factor for given differential equation.

Now $M^*=\dfrac{1}{Mx+Ny}\times M=-\dfrac{1}{x^2y^2}\times\left(y^3-3xy^2\right)=\dfrac{3}{x}-\dfrac{y}{x^2}$

and $N^*=\dfrac{1}{Mx+Ny}\times N=-\dfrac{1}{x^2y^2}\times\left(2x^2y-xy^2\right)=\dfrac{1}{x}-\dfrac{2}{y}$

$\dfrac{\partial M^*}{\partial y}=\dfrac{\partial}{\partial y}\left(\dfrac{3}{x}-\dfrac{y}{x^2}\right)=-\dfrac{1}{x^2}$; $\dfrac{\partial N}{\partial x}=\dfrac{\partial}{\partial x}\left(\dfrac{1}{x}-\dfrac{2}{y}\right)=-\dfrac{1}{x^2}$

Since $\dfrac{\partial M^*}{\partial y}=\dfrac{\partial N^*}{\partial x}$, therefore $M^*dx+N^*dy=0=\left(\dfrac{3}{x}-\dfrac{y}{x^2}\right)dx+\left(\dfrac{1}{x}-\dfrac{2}{y}\right)dy=0$ is an exact. Thus, the solution is given by

$\displaystyle\int_{\text{Treating } y \text{ as constant}} M^*dx+\int\left(\text{terms in } N^* \text{ not containing } x\right)dy=c$, where c is an arbitrary constant.

$$\int_{\text{Treating } y \text{ as constant}}\left(\frac{3}{x}-\frac{y}{x^2}\right)dx+\int\left(\text{terms in}\left(\frac{1}{x}-\frac{2}{y}\right)\text{not containing } x\right)dy=c$$

$$\int_{\text{Treating } y \text{ as constant}}\left(\frac{3}{x}-\frac{y}{x^2}\right)dx+\int\left(-\frac{2}{y}\right)dy=c$$

$3\log|x|+\dfrac{y}{x}-2\log|y|=c$ is the required solution.

Example : Solve: $y\left(y^2-2x^2\right)dx+x\left(2y^2-x^2\right)dy=0$.

Solution: Comparing $y\left(y^2-2x^2\right)dx+x\left(2y^2-x^2\right)dy=0$ with $Mdx+Ndy=0$, we get

$$M=y\left(y^2-2x^2\right)=y^3-2x^2y \text{ and } N=x\left(2y^2-x^2\right)=2y^2x-x^3 .$$

Now $\dfrac{\partial M}{\partial y}=3y^2-2x^2$ and $\dfrac{\partial N}{\partial x}=2y^2-3x^2$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, therefore given differential equation id not exact. The given differential equation is a homogenous differential equation.

$Mx = x\left(y^3 - 2x^2y\right) = xy^3 - 2x^3y$

$Ny = y\left(2y^2x - x^3\right) = 2y^3x - x^3y$

$Mx + Ny = xy^3 - 2x^3y + 2y^3x - x^3y = 3y^3x - 3x^3y = 3xy\left(y^2 - x^2\right)$

According to the theorem $\frac{1}{Mx + Ny}$ is an integrating factor provided $Mx + Ny \neq 0$.

$\therefore \frac{1}{Mx + Ny} = \frac{1}{3xy\left(y^2 - x^2\right)}$ is an integrating factor for given differential equation.

Now $M^{*} = \frac{1}{Mx + Ny} \times M = \frac{1}{3xy\left(y^2 - x^2\right)} \times \left(y^3 - 2x^2y\right) = \frac{\left(y^2 - 2x^2\right)}{3x\left(y^2 - x^2\right)}$

and $N^{*} = \frac{1}{Mx + Ny} \times N = \frac{1}{3xy\left(y^2 - x^2\right)} \times \left(2y^2x - x^3\right) = \frac{\left(2y^2 - x^2\right)}{3y\left(y^2 - x^2\right)}$

Since $\frac{\partial M^{*}}{\partial y} = \frac{\partial N^{*}}{\partial x}$, therefore

$$M^{*} dx + N^{*} dy = 0 = \left(\frac{\left(y^2 - 2x^2\right)}{3x\left(y^2 - x^2\right)}\right) dx + \left(\frac{\left(2y^2 - x^2\right)}{3y\left(y^2 - x^2\right)}\right) dy = 0 \; ..(1)$$

is an exact. Rewriting (1), we get

$$\left(\frac{\left(y^2 - x^2\right) - x^2}{3x\left(y^2 - x^2\right)}\right) dx + \left(\frac{y^2 + \left(y^2 - x^2\right)}{3y\left(y^2 - x^2\right)}\right) dy = 0$$

$$\Rightarrow \frac{\left(y^2 - x^2\right)}{3x\left(y^2 - x^2\right)} dx - \frac{x^2}{3x\left(y^2 - x^2\right)} dx + \frac{y^2}{3y\left(y^2 - x^2\right)} dy + \frac{\left(y^2 - x^2\right)}{3y\left(y^2 - x^2\right)} dy = 0$$

Regrouping above equation, we get

$$\left[\frac{1}{x} dx + \frac{1}{y} dy\right] - \left[\frac{x}{\left(y^2 - x^2\right)} dx - \frac{y}{\left(y^2 - x^2\right)} dy\right] = 0$$

$$d\left[\log xy\right] + \frac{1}{2} \frac{d\left(y^2 - x^2\right)}{\left(y^2 - x^2\right)} = 0$$

$$d\left(\log\left\{x^2y^2\left(y^2 - x^2\right)\right\}\right) = 0$$

Integrating above equation, we get

$$\log\left\{x^2y^2\left(y^2 - x^2\right)\right\} = c$$

$x^2y^2\left(y^2 - x^2\right) = e^c \Rightarrow x^2y^2\left(y^2 - x^2\right) = c_1$ where $e^c = c_1$ is the required solution.

Example : Solve: $x(x - y)\frac{dy}{dx} = y(x + y)$.

Solution: Given differential equation $x(x - y)\frac{dy}{dx} = y(x + y)$ can be written as

$$y(x+y)dx - x(x-y)dy = 0$$

Comparing $y(x+y)dx - x(x-y)dy = 0$ with $Mdx + Ndy = 0$, we get

$$M = y(x+y) \text{ and } N = -x(x-y).$$

Now $\frac{\partial M}{\partial y} = x + 2y$ and $\frac{\partial N}{\partial x} = -2x + y$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, therefore given differential equation id not exact. The given differential equation is a homogenous differential equation.

$Mx = x(y(x+y)) = x^2y + xy^2$

$Ny = y(-x(x-y)) = y^2x - x^2y$

$Mx + Ny = x^2y + xy^2 + y^2x - x^2y = 2xy^2$

According to the theorem $\frac{1}{Mx + Ny}$ is an integrating factor provided $Mx + Ny \neq 0$.

$\therefore \frac{1}{Mx + Ny} = \frac{1}{2xy^2}$ is an integrating factor for given differential equation.

Now $M^* = \frac{1}{Mx + Ny} \times M = \frac{1}{2xy^2} \times (y(x+y)) = \frac{1}{2y} + \frac{1}{2x}$

and $N^* = \frac{1}{Mx + Ny} \times N = \frac{1}{2xy^2} \times (-x(x-y)) = -\frac{x}{2y^2} + \frac{1}{2y}$

$\frac{\partial M^*}{\partial y} = \frac{\partial}{\partial y}\left(\frac{1}{2y} + \frac{1}{2x}\right) = -\frac{1}{2y^2}$; $\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(-\frac{x}{2y^2} + \frac{1}{2y}\right) = -\frac{1}{2y^2}$

Since $\frac{\partial M^*}{\partial y} = \frac{\partial N^*}{\partial x}$, therefore $M^* dx + N^* dy = 0 = \left(\frac{1}{2y} + \frac{1}{2x}\right)dx + \left(-\frac{x}{2y^2} + \frac{1}{2y}\right)dy = 0$ is an exact. Thus, the solution is given by

$\int_{\text{Treating } y \text{ as constant}} M^* dx + \int(\text{terms in } N^* \text{ not containing } x)\,dy = c$, where c is an arbitrary constant.

$$\int_{\text{Treating } y \text{ as constant}} \left(\frac{1}{2y} + \frac{1}{2x}\right)dx + \int\left(\text{terms in}\left(-\frac{x}{2y^2} + \frac{1}{2y}\right)\text{not containing } x\right)dy = c$$

$$\int_{\text{Treating } y \text{ as constant}} \left(\frac{1}{2y} + \frac{1}{2x}\right)dx + \int \frac{1}{2y}.dy = c$$

$$\frac{x}{2y} + \frac{1}{2}\log x + \frac{1}{2}\log y = c \Rightarrow \frac{x}{2y} + \frac{1}{2}\log|xy| = c \Rightarrow x + y\log|xy| = 2cy$$

EXERCISE

Solve the following:

1. $x^2y\,dx - (x^3 + y^3)dy = 0$ (Ans $y = ce^{\frac{x^3}{3y^3}}$)

2. $-(x^3 - 3x^2y)dy + (x^2y - 2xy^2)dx = 0$ (Ans $\frac{x}{y} - 2\log x + 3\log y = c$)

3. $(x^4 + y^4)dx - x^3dy = 0$ (Ans $y^4 = 4x^4 \ln x + cx^4$)

4. $y^2 dx + (x^2 - xy - y^2)\,dy = 0$ (Ans $(x-y)y^2 = c(x+y)$

5. $(y-x)\,dx + (y+x)\,dy$ (Ans $\ln(x^2+y^2)^{\frac{1}{2}} - \tan\left(\frac{x}{y}\right) = c$

Rule II: If $Mdx + Ndy = 0$ is of the form $f_1(xy)\,y\,dx + f_2(xy)\,x\,dx = 0$ then $\frac{1}{Mx - Ny}$ is an integrating factor provided $Mx - Ny \neq 0$.

Example: Solve: $y(xy + 2x^2y^2)\,dx + x(xy - x^2y^2)\,dy = 0$. (2.15)

Solution: Dividing (2.15) by xy, we get

$y(1+2xy)\,dx + x(1-xy)\,dy = 0$ (2.16)

Comparing the given equation $y(1+2xy)\,dx + x(1-xy)\,dy = 0$ with $Mdx + Ndy = 0$, we get

$$M = y(1+2xy) = y + 2xy^2 \text{ and } N = x(1-xy) = x - x^2y.$$

Now $\frac{\partial M}{\partial y} = 1 + 4xy$ and $\frac{\partial N}{\partial x} = 1 - 2xy$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ therefore, the given differential equation is not exact.t. Equation (2.16) is of the form $f_1(xy)\,y\,dx + f_2(xy)\,x\,dx = 0$

$Mx = x\left(y(1+2xy)\right) = xy + 2x^2y^2$

$Ny = y\left(x(1-xy)\right) = xy - x^2y^2$

$Mx - Ny = xy + 2x^2y^2 - xy + x^2y^2x^2y^2 = 3x^2y^2$

According to the theorem $\frac{1}{Mx - Ny}$ is an integrating factor provided $Mx - Ny \neq 0$.

$\therefore \frac{1}{Mx - Ny} = \frac{1}{3x^2y^2}$ is an integrating factor for given differential equation.

Now $M^* = \frac{1}{Mx + Ny} \times M = \frac{1}{3x^2y^2} \times (y + 2xy^2) = \frac{1}{3x^2y} + \frac{2}{3x}$

and $N^* = \frac{1}{Mx + Ny} \times N = \frac{1}{3x^2y^2} \times (x - x^2y) = \frac{1}{3xy^2} - \frac{1}{3y}$

$\frac{\partial M^*}{\partial y} = \frac{\partial}{\partial y}\left(\frac{1}{3x^2y} + \frac{2}{3x}\right) = -\frac{1}{3x^2y^2}$; $\frac{\partial N^*}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{3xy^2} - \frac{1}{3y}\right) = -\frac{1}{3x^2y^2}$

Since $\frac{\partial M^*}{\partial y} = \frac{\partial N^*}{\partial x}$, therefore $M^*\,dx + N^*\,dy = 0 = \left(\frac{1}{y} - \frac{2}{x}\right)dx + \left(\frac{x}{y^2} - \frac{3}{y}\right)dy = 0$ is an exact differential equation. Thus, the solution is given by

$$\int_{\text{Treating } y \text{ as constant}} M^*\,dx + \int\left(\text{terms in } N^* \text{ not containing } x\right)dy = c,$$ where c is an arbitrary constant.

$$\int_{\text{Treating } y \text{ as constant}} \left(\frac{1}{3x^2y} + \frac{2}{3x}\right)dx + \int\left(\text{terms in}\left(\frac{1}{3xy^2} - \frac{1}{3y}\right)\text{not containing } x\right)dy = c$$

$$\int_{\text{Treating } y \text{ as constant}} \left(\frac{1}{3x^2y}+\frac{2}{3x}\right)dx+\int\left(-\frac{1}{3y}\right)dy=c$$

$$-\frac{1}{3xy}+\frac{2}{3}\log x-\frac{1}{3}\log y=c \Rightarrow -\frac{1}{xy}+\log\left(\frac{x^2}{y}\right)=3c$$

$\Rightarrow -\frac{1}{xy}+\log\left(\frac{x^2}{y}\right)=c_1$ where $3c=c_1$ is the required solution.

Example : Solve: $(xy\sin xy+\cos xy)\,y\,dx+(xy\sin xy-\cos xy)\,x\,dy=0$.

Solution: Comparing $(xy\sin xy+\cos xy)\,y\,dx+(xy\sin xy-\cos xy)\,x\,dy=0$ (2.17)

with $Mdx+Ndy=0$, we get

$$M=(xy\sin xy+\cos xy)\,y=xy^2\sin xy+y\cos xy$$

and $N=(xy\sin xy-\cos xy)\,x=x^2y\sin xy-x\cos xy$.

Now $\frac{\partial M}{\partial y}=xy\sin xy+y^2x^2\cos xy+\cos xy$ and $\frac{\partial N}{\partial x}=3xy\sin xy+y^2x^2\cos xy-\cos xy$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, therefore given differential equation is not exact. Equation (2.18) is of the form $f_1(xy)\,y\,dx+f_2(xy)\,x\,dx=0$.

$$Mx=x(xy^2\sin xy+y\cos xy)=x^2y^2\sin xy+xy\cos xy$$

$$Ny=y(x^2y\sin xy-x\cos xy)=x^2y^2\sin xy-xy\cos xy$$

$$Mx-Ny=x^2y^2\sin xy+xy\cos xy-x^2y^2\sin xy+xy\cos xy=2xy\cos xy$$

According to the theorem $\frac{1}{Mx-Ny}$ is an integrating factor provided $Mx-Ny\neq 0$.

$\therefore \frac{1}{Mx-Ny}=\frac{1}{2xy\cos xy}$ is an integrating factor for given differential equation .

Now $M^*=\frac{1}{Mx+Ny}\times M=\frac{1}{2xy\cos xy}\times(xy^2\sin xy+y\cos xy)=\frac{y\tan xy}{2}+\frac{1}{2x}$

and $N^*=\frac{1}{Mx+Ny}\times N=\frac{1}{2xy\cos xy}\times(x^2y\sin xy-x\cos xy)=\frac{x\tan xy}{2}-\frac{1}{2y}$

$$\frac{\partial M^*}{\partial y}=\frac{\partial}{\partial y}\left(\frac{y\tan xy}{2}+\frac{1}{2x}\right)=\frac{1}{2}\left[yx\sec^2 xy+\tan xy\right];$$

$$\frac{\partial N}{\partial x}=\frac{\partial}{\partial x}\left(\frac{x\tan xy}{2}-\frac{1}{2y}\right)=\frac{1}{2}\left[yx\sec^2 xy+\tan xy\right]$$

Since $\frac{\partial M^*}{\partial y}=\frac{\partial N^*}{\partial x}$, therefore

$M^*dx+N^*dy=0=\left(\frac{y\tan xy}{2}+\frac{1}{2x}\right)dx+\left(\frac{x\tan xy}{2}-\frac{1}{2y}\right)dy=0$ is an exact differential equation. Thus, the solution is given by

$\int_{\text{Treating } y \text{ as constant}} M^*dx+\int(\text{terms in } N^* \text{ not containing } x)\,dy=c$, where c is an arbitrary constant.

$$\int_{\text{Treating } y \text{ as constant}} \left(\frac{y\tan xy}{2}+\frac{1}{2x}\right)dx+\int\left(\text{terms in}\left(\frac{x\tan xy}{2}-\frac{1}{2y}\right)\text{not containing } x\right)dy=c$$

$$\int_{\text{Treating } y \text{ as constant}} \left(\frac{y\tan xy}{2}+\frac{1}{2x}\right)dx+\int\left(-\frac{1}{2y}\right)dy=c$$

$$\frac{1}{2}(\log\sec xy+\log x)-\frac{1}{2}\log y=\frac{1}{2}\log c$$

$\Rightarrow\left(\frac{x}{y}\right)\sec xy=c$ is the required solution.

Example : Solve: $\left(x^3y^3+x^2y^2+xy+1\right)y\,dx+\left(x^3y^3-x^2y^2-xy+1\right)x\,dy=0$. (2.19)

Solution: Comparing $\left(x^3y^3+x^2y^2+xy+1\right)y\,dx+\left(x^3y^3-x^2y^2-xy+1\right)x\,dy=0$ with $Mdx+Ndy=0$, we get

$$M=\left(x^3y^3+x^2y^2+xy+1\right)y=x^3y^4+x^2y^3+xy^2+y$$

and $N=\left(x^3y^3-x^2y^2-xy+1\right)x=x^4y^3-x^3y^2-x^2y+x$.

Now $\frac{\partial M}{\partial y}=4x^3y^3+3x^2y^2+2xy+1$ and $\frac{\partial N}{\partial x}=4x^3y^3-3x^2y^2-2xy+1$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, therefore given differential equation id not exact. Equation (2.19) is of the form $f_1(xy)y\,dx+f_2(xy)x\,dx=0$.

$$Mx=x\left(x^3y^4+x^2y^3+xy^2+y\right)=x^4y^4+x^3y^3+x^2y^2+xy$$

$$Ny=y\left(x^4y^3-x^3y^2-x^2y+x\right)=x^4y^4-x^3y^3-x^2y^2+xy$$

$$Mx-Ny=x^4y^4+x^3y^3+x^2y^2+xy-x^4y^4+x^3y^3+x^2y^2-xy=2x^2y^2(xy+1)$$

According to the theorem $\frac{1}{Mx-Ny}$ is an integrating factor provided $Mx-Ny\neq 0$.

$\therefore\frac{1}{Mx-Ny}=\frac{1}{2x^2y^2(xy+1)}$ is an integrating factor for given differential equation .

$$\text{Now } M^*=\frac{1}{Mx+Ny}\times M=\frac{1}{2x^2y^2(xy+1)}\times\left(x^3y^4+x^2y^3+xy^2+y\right)$$

$$=\frac{y(xy+1)\left(x^2y^2+1\right)}{2x^2y^2(xy+1)}$$

$$=\frac{y\left(x^2y^2+1\right)}{2x^2y^2}$$

$$\text{and } N^*=\frac{1}{Mx+Ny}\times N=\frac{x^4y^3-x^3y^2-x^2y+x}{2x^2y^2(xy+1)}$$

$$=\frac{x\left(xy^3-x^2y^2-xy+1\right)}{2x^2y^2(xy+1)}$$

$$= \frac{x\left(\left(x^3y^3+1\right)-\left(x^2y^2+xy\right)\right)}{2x^2y^2\left(xy+1\right)}$$

$$= \frac{x\left(\left(xy+1\right)\left(x^2y^2-xy+1\right)-xy\left(xy+1\right)\right)}{2x^2y^2\left(xy+1\right)}$$

$$= \frac{x\left(\left(x^2y^2-xy+1\right)-xy\right)}{2x^2y^2}$$

Since $\frac{\partial M^*}{\partial y}=\frac{\partial N^*}{\partial x}$, therefore

$M^*dx+N^*dy=0=\left(\frac{y\tan xy}{2}+\frac{1}{2x}\right)dx+\left(\frac{x\tan xy}{2}-\frac{1}{2y}\right)dy=0$ is an exact

differential equation. Thus, the solution is given by

$\bullet \int\limits_{\text{Treating } y \text{ as constant}} M^*dx+\int\left(\text{terms in } N^* \text{ not containing } x\right)dy=c$, where c is an arbitrary constant.

$$\int\limits_{\text{Treating } y \text{ as constant}}\left(\frac{y\left(x^2y^2+1\right)}{2x^2y^2}\right)dx+\int\left(\text{terms in}\left(\frac{x\left(\left(x^2y^2-xy+1\right)-xy\right)}{2x^2y^2}\right)\text{not containing } x\right)dy=c$$

$$\int\limits_{\text{Treating } y \text{ as constant}}\left(\frac{y\left(x^2y^2+1\right)}{2x^2y^2}\right)dx+\int\left(\text{terms in}\left(\frac{x}{2}-\frac{1}{y}+\frac{1}{2xy^2}\right)\text{not containing } x\right)dy=c$$

$$\int\limits_{\text{Treating } y \text{ as constant}}\left(\frac{y}{2}+\frac{1}{2x^2y}\right)dx+\int\left(-\frac{1}{y}\right)dy=c$$

$\frac{xy}{2}-\frac{1}{2xy}-\log y-\log y=c\Rightarrow xy-\frac{1}{xy}-2\log y=c_1$ where $c_1=2c$ is the required solution.

Example: Solve: $\left(x^2y^2+2\right)y\,dx+\left(2-2x^2y^2\right)x\,dy=0$. (2.20)

Solution: Comparing the given equation with $Mdx+Ndy=0$, we get

$$M=\left(x^2y^2+2\right)y=x^2y^3+2y \text{ and } N=\left(2-2x^2y^2\right)x=2x-2x^3y^2 .$$

Now $\frac{\partial M}{\partial y}=3x^2y^2+2$ and $\frac{\partial N}{\partial x}=2-6x^2y^2$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, therefore given differential equation is not exact. Equation (2.20) is $f_1(xy)y\,dx+f_2(xy)x\,dx=0$.

$Mx=x\left(x^2y^3+2y\right)=x^3y^3+2xy$

$Ny=y\left(2x-2x^3y^2\right)=2xy-2x^3y^3$

$Mx-Ny=x^3y^3+2xy-2xy+2x^3y^3=3x^3y^3$

According to the theorem $\frac{1}{Mx-Ny}$ is an integrating factor provided $Mx-Ny\neq 0$.

$\therefore \dfrac{1}{Mx - Ny} = \dfrac{1}{3x^3y^3}$ is an integrating factor for given differential equation .

Now $M^* = \dfrac{1}{Mx + Ny} \times M = \dfrac{1}{3x^3y^3} \times \left(x^2y^3 + 2y\right) = \dfrac{1}{3x} + \dfrac{2}{3x^3y^2}$

and $N^* = \dfrac{1}{Mx + Ny} \times N = \dfrac{1}{3x^3y^3} \times \left(2x - 2x^3y^2\right) = \dfrac{2}{3x^2y^3} - \dfrac{2}{3y}$

$$\frac{\partial M^*}{\partial y} = \frac{\partial}{\partial y}\left(\frac{1}{3x} + \frac{2}{3x^3y^2}\right) = -\frac{4}{3x^3y^3}\ ; \ \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{2}{3x^2y^3} - \frac{2}{3y}\right) = -\frac{4}{3x^3y^3}$$

Since $\dfrac{\partial M^*}{\partial y} = \dfrac{\partial N^*}{\partial x}$, therefore

$M^*dx + N^*dy = 0 = \left(\dfrac{1}{3x} + \dfrac{2}{3x^3y^2}\right)dx + \left(\dfrac{2}{3x^2y^3} - \dfrac{2}{3y}\right)dy = 0$ is an exact differential equation. Thus, the solution is given by $\displaystyle\int_{\text{Treating } y \text{ as constant}} M^*dx + \int\left(\text{terms in } N^* \text{ not containing } x\right)dy = c$, where c is an arbitrary constant.

$$\int_{\text{Treating } y \text{ as constant}} \left(\frac{1}{3x} + \frac{2}{3x^3y^2}\right)dx + \int\left(\text{terms in}\left(\frac{2}{3x^2y^3} - \frac{2}{3y}\right)\text{not containing } x\right)dy = \log c$$

$$\int_{\text{Treating } y \text{ as constant}} \left(\frac{1}{3x} + \frac{2}{3x^3y^2}\right)dx + \int\left(-\frac{2}{3y}\right)dy = \log c$$

$\dfrac{1}{3}\log x + \dfrac{1}{3x^2y^3} - \dfrac{2}{3}\log y = \log c$ is the required solution.

Rule III: If $\dfrac{1}{N}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right)$ is function of x alone say f(x), then $e^{\int f(x)dx}$ is an integrating factor of $Mdx + Ndy = 0$.

Rule IV: If $\dfrac{1}{M}\left(\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}\right)$ is function of x alone say f(x), then $e^{\int f(y)dy}$ is an integrating factor of $Mdx + Ndy = 0$.

Example: Solve $(2x\log x - xy)dy + 2y\,dx = 0$. (2.27)

Solution: Comparing the given equation with $Mdx + Ndy = 0$, we get

$$M = 2y \text{ and } N = 2x\log x - xy .$$

Now $\dfrac{\partial M}{\partial y} = 2$ and $\dfrac{\partial N}{\partial x} = 2(1 + \log x) - y$

Since $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$, therefore given differential equation is not exact.

Now $\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x} = 2 - 2 - 2\log x + y = -(2\log x - y)$

$$\Rightarrow \frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\frac{-(2\log x-y)}{x(2\log x-y)}=-\frac{1}{x}=f(x)$$ is function of x alone therefore $e^{\int f(x)dx}$ is an integrating factor.

Integrating factor (I.F) $=e^{\int f(x)dx}=e^{\int -\frac{1}{x}dx}=e^{-\log x}=e^{\log x^{-1}}=x^{-1}=\frac{1}{x}$

Now $M^{*}=I.F\times M=\frac{1}{x}\times 2y=\frac{2y}{x}$

and $N^{*}=I.F\times N=\frac{1}{x}\times(2x\log x-xy)=2\log x-y$

$$\frac{\partial M^{*}}{\partial y}=\frac{\partial}{\partial y}\left(\frac{2y}{x}\right)=\frac{2}{x}\ ;\ \frac{\partial N^{*}}{\partial x}=\frac{\partial}{\partial x}(2\log x-y)=\frac{2}{x}$$

Since $\frac{\partial M^{*}}{\partial y}=\frac{\partial N^{*}}{\partial x}$, therefore $M^{*}dx+N^{*}dy=0=\left(\frac{2y}{x}\right)dx+(2\log x-y)dy=0$ is an exact differential equation. Thus, the solution is given by

$$\int_{\text{Treating } y \text{ as constant}} M^{*}dx+\int(\text{terms in } N^{*} \text{ not containing } x)\,dy=c,$$ where c is an arbitrary constant.

$$\int_{\text{Treating } y \text{ as constant}}\left(\frac{2y}{x}\right)dx+\int(\text{terms in}(2\log x-y)\text{ not containing } x)\,dy=\log c$$

$$\int_{\text{Treating } y \text{ as constant}}\left(\frac{2y}{x}\right)dx+\int -y\,dy=\log c$$

$2y\log x-\frac{1}{2}y^2=c$ is the required solution.

Example: Solve $\left(xy^2-e^{\frac{1}{x^3}}\right)dx-x^2y\,dy=0$.

Solution: Comparing the given equation with $Mdx+Ndy=0$, we get

$$M=xy^2-e^{\frac{1}{x^3}} \text{ and } N=-x^2y.$$

Now $\frac{\partial M}{\partial y}=2xy$ and $\frac{\partial N}{\partial x}=-2xy$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, therefore given differential equation is not exact.

Now $\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}=2xy-(-2xy)=4xy$

$$\Rightarrow \frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\frac{4xy}{-x^2y}=-\frac{4}{x}=f(x)$$ is function of x alone therefore $e^{\int f(x)dx}$ is an integrating factor.

Integrating factor (I.F) $=e^{\int f(x)dx}=e^{\int -\frac{4}{x}dx}=e^{-4\log x}=e^{\log x^{-4}}=x^{-4}=\frac{1}{x^4}$

Now $M^{*} = I.F \times M = \frac{1}{x^4} \times \left(xy^2 - e^{\frac{1}{x^3}} \right) = \frac{y^2}{x^3} - \frac{e^{\frac{1}{x^3}}}{x^4}$

and $N^{*} = I.F \times N = \frac{1}{x^4} \times (-x^2 y) = -\frac{y}{x^2}$

$$\frac{\partial M^{*}}{\partial y} = \frac{\partial}{\partial y}\left(\frac{y^2}{x^3} - \frac{e^{\frac{1}{x^3}}}{x^4} \right) = \frac{2y}{x^3} \,;\quad \frac{\partial N^{*}}{\partial x} = \frac{\partial}{\partial x}\left(-\frac{y}{x^2} \right) = \frac{2y}{x^3}$$

Since $\frac{\partial M^{*}}{\partial y} = \frac{\partial N^{*}}{\partial x}$, therefore $M^{*}\,dx + N^{*}\,dy = 0 = \left(\frac{y^2}{x^3} - \frac{e^{\frac{1}{x^3}}}{x^4} \right) dx + (2\log x - y)\,dy = 0$

is an exact differential equation. Thus, the solution is given by

$\int\limits_{\text{Treating } y \text{ as constant}} M^{*}\,dx + \int (\text{terms in } N^{*} \text{ not containing } x)\,dy = c$, where c is an arbitrary constant.

$$\int\limits_{\text{Treating } y \text{ as constant}} \left(\frac{y^2}{x^3} - \frac{e^{\frac{1}{x^3}}}{x^4} \right) dx + \int \left(\text{terms in} \left(-\frac{y}{x^2} \right) \text{not containing } x \right) dy = \log c$$

$$\int\limits_{\text{Treating } y \text{ as constant}} \left(\frac{y^2}{x^3} - \frac{e^{\frac{1}{x^3}}}{x^4} \right) dx + \int 0\,dy = \log c$$

$\int\limits_{\text{Treating } y \text{ as constant}} \frac{y^2}{x^3}\,dx - \int\limits_{\text{Treating } y \text{ as constan}} \frac{e^{\frac{1}{x^3}}}{x^4} = c$ (For second integral put $\frac{1}{x^3} = t \Rightarrow \frac{-3}{x^4}dx = dt$)

$-\frac{y^2}{2x^2} + \frac{1}{3}e^{\frac{1}{x^3}} = c$ is the required solution.

Example: Solve $y(2x^2 - xy + 1)\,dx + (x - y)\,dy = 0$. (2.29)

Solution: Comparing the given equation with $Mdx + Ndy = 0$, we get

$$M = y(2x^2 - xy + 1) = 2x^2 y - xy^2 + y \text{ and } N = x - y\,.$$

Now $\frac{\partial M}{\partial y} = 2x^2 - 2xy + 1$ and $\frac{\partial N}{\partial x} = 1$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, therefore given differential equation is not exact.

Now $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 2x^2 - 2xy + 1 - 1 = 2x^2 - 2xy = 2x(x - y)$

$\Rightarrow \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2x(x-y)}{x-y} = 2x = f(x)$ is function of x alone therefore $e^{\int f(x)dx}$ is an integrating factor.

Integrating factor (I.F) $= e^{\int f(x)dx} = e^{\int 2x\,dx} = e^{x^2}$

Now $M^{*} = I.F \times M = e^{x^2} \times (2x^2y - xy^2 + y)$

and $N^{*} = I.F \times N = e^{x^2} \times (x - y)$

$$\frac{\partial M^{*}}{\partial y} = \frac{\partial}{\partial y}\left(e^{x^2} \times (2x^2y - xy^2 + y)\right) = e^{x^2}\left(2x^2y - 2xy + 1\right);$$

$$\frac{\partial N^{*}}{\partial x} = \frac{\partial}{\partial x}\left(e^{x^2} \times (x - y)\right) = e^{x^2}\left(2x^2y - 2xy + 1\right)$$

Since $\dfrac{\partial M^{*}}{\partial y} = \dfrac{\partial N^{*}}{\partial x}$, therefore $M^{*}\,dx + N^{*}\,dy = 0 = \left(\dfrac{y^2}{x^3} - \dfrac{e^{\frac{1}{x^3}}}{x^4}\right)dx + (2\log x - y)\,dy = 0$

is an exact differential equation. Thus, the solution is given by

$$\int\limits_{\text{Treating } y \text{ as constant}} M^{*}\,dx + \int (\text{terms in } N^{*} \text{ not containing } x)\,dy = c,$$ where c is an arbitrary constant.

$$\int\limits_{\text{Treating } y \text{ as constant}} \left(e^{x^2} \times (2x^2y - xy^2 + y)\right)dx + \int \left(\text{terms in}\left(e^{x^2}\left(2x^2y - 2xy + 1\right)\right) \text{not containing } x\right)dy = \log c$$

$$\int\limits_{\text{Treating } y \text{ as constant}} \left(e^{x^2} \times (2x^2y - xy^2 + y)\right)dx + \int 0\,dy = \log c$$

$e^{x^2}\left(2xy - y^2\right) = c$ is the required solution.

Example: Solve $(x - y)\,dx - dy = 0,\ y(0) = 2.$ (2.30)

Solution: Comparing (2.30) with the standard form gives $M = x - y$ and $N = -1$.

Now $\dfrac{\partial M}{\partial y} = -1$ and $\dfrac{\partial N}{\partial x} = 0$

Since $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$, therefore given differential equation is not exact.

Now $\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x} = -1 - 0 = -1$

$\Rightarrow \dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \dfrac{-1}{-1} = 1 = f(x)$ is function of x alone therefore $e^{\int f(x)dx}$ is an integrating factor.

Integrating factor (I.F) $= e^{\int f(x)dx} = e^{\int dx} = e^{x}$

Now $M^{*} = I.F \times M = e^{x}(x - y)$

and $N^{*} = I.F \times N = e^{x} \times (-1) = -e^{x}$

$$\frac{\partial M^{*}}{\partial y} = \frac{\partial}{\partial y}\left(e^{x}(x - y)\right) = -e^{x};\quad \frac{\partial N^{*}}{\partial x} = \frac{\partial}{\partial x}\left(-e^{x}\right) = -e^{x}$$

Since $\frac{\partial M^*}{\partial y} = \frac{\partial N^*}{\partial x}$, therefore $M^* dx + N^* dy = 0 = \left(e^x\left(x-y\right)\right)dx - e^x dy = 0$ is an exact differential equation. Thus, the solution is given by

$\int_{\text{Treating } y \text{ as constant}} M^* dx + \int\left(\text{terms in } N^* \text{ not containing } x\right)dy = c,$ where c is an arbitrary constant.

$$\int_{\text{Treating } y \text{ as constant}} e^x\left(x-y\right)dx + \int\left(\text{terms in}\left(-e^x\right)\text{not containing } x\right)dy = c$$

$$\int_{\text{Treating } y \text{ as constant}} e^x\left(x-y\right)dx + \int 0.dy = c$$

$$\left(x-1\right)e^x - ye^x = c \qquad (2.31)$$

is general solution of given differential equation.
Now it is given that $y(0) = 2$ i.e. when $x=0$ then $y=2$.
Put the value $x=0$ and $y=2$ in (2.31), we get

$$\left(0-1\right)e^0 - 2e^0 = c \Rightarrow c = -3$$

$\therefore \left(x-1\right)e^x - ye^x = -3$ is a particular solution

Example: Solve $\left(x^2 + y^2 + x\right)dx + xy\,dy = 0$.

Solution: Comparing the given equation with $Mdx + Ndy = 0$, we get

$$M = x^2 + y^2 + x \text{ and } N = xy.$$

Now $\frac{\partial M}{\partial y} = 2y$ and $\frac{\partial N}{\partial x} = y$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, therefore given differential equation is not exact.

Now $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 2y - y = y$

$$\Rightarrow \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{y}{xy} = \frac{1}{x} = f(x)$$ is function of x alone therefore $e^{\int f(x)dx}$ is an integrating factor.

Integrating factor (I.F) $= e^{\int f(x)dx} = e^{\int \frac{1}{x}dx} = e^{\log x} = e^{\log x} = x$

Now $M^* = I.F \times M = x \times \left(x^2 + y^2 + x\right) = x^3 + y^2x + x^2$

and $N^* = I.F \times N = x \times (xy) = x^2y$

$$\frac{\partial M^*}{\partial y} = \frac{\partial}{\partial y}\left(x^3 + y^2x + x^2\right) = 2yx \;;\; \frac{\partial N^*}{\partial x} = \frac{\partial}{\partial x}\left(x^2y\right) = 2yx$$

Since $\frac{\partial M^*}{\partial y} = \frac{\partial N^*}{\partial x}$, therefore $M^* dx + N^* dy = 0 = \left(x^3 + y^2x + x^2\right)dx + x^2y\,dy = 0$ is an exact differential equation. Thus, the solution is given by

$\int_{\text{Treating } y \text{ as constant}} M^* dx + \int\left(\text{terms in } N^* \text{ not containing } x\right)dy = c,$ where c is an arbitrary constant.

$$\int_{\text{Treating } y \text{ as constant}} \left(x^3+y^2x+x^2\right)dx + \int\left(\text{terms in}\left(x^2y\right)\text{not containing } x\right)dy = c$$

$$\frac{x^4}{4}+\frac{x^2y^2}{2}+\frac{x^3}{3}=\frac{c}{6} \Rightarrow 3x^4+6x^2y^2+2x^3=c$$ is the required solution.

Example : Solve $\left(y+\frac{y^3}{3}+\frac{x^2}{2}\right)dx+\left(\frac{x+xy^2}{4}\right)dy=0$.

Solution: Comparing the given equation with $Mdx+Ndy=0$, we get

$$M = y+\frac{y^3}{3}+\frac{x^2}{2} \text{ and } N=\frac{x+xy^2}{4}.$$

Now $\frac{\partial M}{\partial y}=1+y^2$ and $\frac{\partial N}{\partial x}=\frac{1}{4}\left(1+y^2\right)$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, therefore given differential equation is not exact.

Now $\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}=\left(1+y^2\right)-\frac{1}{4}\left(1+y^2\right)=\frac{3}{4}\left(1+y^2\right)$

$$\Rightarrow \frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\frac{\frac{3}{4}\left(1+y^2\right)}{\frac{1}{4}x\left(1+y^2\right)}=\frac{3}{x}=f(x)$$ is function of x alone therefore $e^{\int f(x)dx}$ is an integrating factor.

Integrating factor (I.F) $=e^{\int f(x)dx}=e^{\int\frac{3}{x}dx}=e^{3\log x}=e^{\log x^3}=x^3$

Now $M^{*}=I.F\times M=x^3\times\left(y+\frac{y^3}{3}+\frac{x^2}{2}\right)=x^3y+\frac{x^3y^3}{3}+\frac{x^5}{2}$

and $N^{*}=I.F\times N=x^3\times\left(\frac{x+xy^2}{4}\right)=\frac{x^4+x^4y^2}{4}$

$$\frac{\partial M^{*}}{\partial y}=\frac{\partial}{\partial y}\left(x^3y+\frac{x^3y^3}{3}+\frac{x^5}{2}\right)=x^3+x^3y^2;\ \frac{\partial N^{*}}{\partial x}=\frac{\partial}{\partial x}\left(\frac{x^4+x^4y^2}{4}\right)=x^3+x^3y^2$$

Since $\frac{\partial M^{*}}{\partial y}=\frac{\partial N^{*}}{\partial x}$, therefore

$M^{*}dx+N^{*}dy=0=\left(x^3y+\frac{x^3y^3}{3}+\frac{x^5}{2}\right)dx+\left(\frac{x^4+x^4y^2}{4}\right)dy=0$ is an exact differential equation. Thus, the solution is given by

$\int_{\text{Treating } y \text{ as constant}} M^{*}dx+\int\left(\text{terms in } N^{*} \text{ not containing } x\right)dy=c$, where c is an arbitrary constant.

$$\int_{\text{Treating } y \text{ as constant}}\left(x^3y+\frac{x^3y^3}{3}+\frac{x^5}{2}\right)dx+\int\left(\text{terms in}\left(\frac{x^4+x^4y^2}{4}\right)\text{not containing } x\right)dy=c$$

$$\int_{\text{Treating } y \text{ as constant}}\left(x^3y+\frac{x^3y^3}{3}+\frac{x^5}{2}\right)dx+\int 0.dy=c$$

$$\frac{x^4y}{4}+\frac{x^4y^3}{12}+\frac{x^6}{12}=\frac{c}{12}\Rightarrow 3x^4y+x^4y^3+x^6=c$$ is the required solution.

Example: Solve $\left(2xy^4e^y+2xy^3+y\right)dx+\left(x^2y^4e^y-x^2y^2-3x\right)dy=0$. (2.33)

Solution: Comparing the given equation with $Mdx+Ndy=0$, we get

$M=2xy^4e^y+2xy^3+y$ and $N=x^2y^4e^y-x^2y^2-3x$.

Now $\frac{\partial M}{\partial y}=8xy^3e^y+2xy^4e^y+6xy^2+1$ and $\frac{\partial N}{\partial x}=2xy^4e^y-2xy^2-3$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, therefore given differential equation is not exact.

Now

$$\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}=2xy^4e^y-2xy^2-3-8xy^3e^y-2xy^4e^y-6xy^2-1=-4\left(2xy^3e^y+xy^2+1\right)=\frac{-4}{y}\left(2xy^4e^y+xy^3+y\right)$$

$$\Rightarrow\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M}=\frac{\frac{-4}{y}\left(2xy^4e^y+xy^3+y\right)}{2xy^4e^y+xy^3+y}=\frac{-4}{y}=f(y)$$ is function of y alone therefore

$e^{\int f(y)dy}$ is an integrating factor.

$$\text{Integrating factor (I.F)}=e^{\int f(y)dy}=e^{\int-\frac{4}{y}dy}=e^{-4\log y}=e^{\log y^{-4}}=\frac{1}{y^4}$$

Now $M^*=I.F\times M=\frac{1}{y^4}\times\left(2xy^4e^y+2xy^3+y\right)=2xe^y+\frac{2x}{y}+\frac{1}{y^3}$

and $N^*=I.F\times N=\frac{1}{y^4}\times\left(x^2y^4e^y-x^2y^2-3x\right)=x^2e^y-\frac{x^2}{y^2}-\frac{3x}{y^4}$

$$\frac{\partial M^*}{\partial y}=\frac{\partial}{\partial y}\left(2xe^y+\frac{2x}{y}+\frac{1}{y^3}\right)=2xe^y-\frac{2x}{y^2}-\frac{3}{y^4};$$

$$\frac{\partial N^*}{\partial x}=\frac{\partial}{\partial x}\left(x^2e^y-\frac{x^2}{y^2}-\frac{3x}{y^4}\right)=2xe^y-\frac{2x}{y^2}-\frac{3}{y^4}$$

Since $\frac{\partial M^*}{\partial y}=\frac{\partial N^*}{\partial x}$, therefore

$M^*dx+N^*dy=0=\left(2xe^y+\frac{2x}{y}+\frac{1}{y^3}\right)dx+\left(x^2e^y-\frac{x^2}{y^2}-\frac{3x}{y^4}\right)dy=0$ is an exact differential equation. Thus, the solution is given by

$\int\limits_{Treating\ y\ as\ constant} M^*dx+\int\left(terms\ in\ N^*\ not\ containing\ x\right)dy=c$, where c is an arbitrary constant.

$$\int\limits_{Treating\ y\ as\ constant}\left(2xe^y+\frac{2x}{y}+\frac{1}{y^3}\right)dx+\int\left(terms\ in\left(x^2e^y-\frac{x^2}{y^2}-\frac{3x}{y^4}\right)not\ containing\ x\right)dy=c$$

$$\int\limits_{Treating\ y\ as\ constant}\left(2xe^y+\frac{2x}{y}+\frac{1}{y^3}\right)dx+\int 0.dy=c$$

$x^2e^y+\frac{x^2}{y}+\frac{x}{y^3}=c$ is the required solution.

Example: Solve $\left(xy^2-x^2\right)dx+\left(3x^2y^2+x^2y-2x^3+y^2\right)dy=0$. (2.34)

Solution: Comparing the given equation with $Mdx+Ndy=0$, we get

$$M=xy^2-x^2 \text{ and } N=3x^2y^2+x^2y-2x^3+y^2.$$

Now $\frac{\partial M}{\partial y}=2xy$ and $\frac{\partial N}{\partial x}=6xy^2+2xy-6x^2$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, therefore given differential equation is not exact.

Now $\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}=6xy^2+2xy-6x^2-2xy=6xy^2-6x^2$

$\Rightarrow \frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M}=\frac{6xy^2-6x^2}{xy^2-x^2}=\frac{6x\left(y^2-x\right)}{x\left(y^2-x\right)}=6=f(y)$ is function of y alone therefore

$e^{\int f(y)dy}$ is an integrating factor.

Integrating factor (I.F) $=e^{\int f(y)dy}=e^{\int 6dy}=e^{6y}$

Now $M^*=I.F\times M=e^{6y}\left(xy^2-x^2\right)$

and $N^*=I.F\times N=e^{6y}\left(3x^2y^2+x^2y-2x^3+y^2\right)$

$$\frac{\partial M^*}{\partial y}=\frac{\partial}{\partial y}\left(e^{6y}\left(xy^2-x^2\right)\right)=\frac{1}{6}\left(y^2e^{6y}-\frac{ye^{6y}}{3}+\frac{e^{6y}}{18}-\frac{x^2}{6}\right)$$

$$\frac{\partial N^*}{\partial x}=\frac{\partial}{\partial x}\left(e^{6y}\left(3x^2y^2+x^2y-2x^3+y^2\right)\right)=\frac{1}{6}\left(y^2e^{6y}-\frac{ye^{6y}}{3}+\frac{e^{6y}}{18}-\frac{x^2}{6}\right)$$

Since $\frac{\partial M^*}{\partial y}=\frac{\partial N^*}{\partial x}$, therefore $M^*dx+N^*dy=0$ i.e.,

$e^{6y}\left(xy^2-x^2\right)dx+e^{6y}\left(3x^2y^2+x^2y-2x^3+y^2\right)dy=0$ is an exact differential equation.

Thus, the solution is given by

$\int_{\text{Treating } y \text{ as constant}} M^*\,dx+\int\left(\textit{terms in } N^* \textit{ not containing } x\right)dy=c$, where c is an arbitrary constant.

$$\int_{\text{Treating } y \text{ as constant}}\left(e^{6y}\left(xy^2-x^2\right)\right)dx+\int\left(\textit{terms in}\left(e^{6y}\left(3x^2y^2+x^2y-2x^3+y^2\right)\right)\textit{not containing } x\right)dy=c$$

$$\int_{\text{Treating } y \text{ as constant}}\left(e^{6y}\left(xy^2-x^2\right)\right)dx+\int e^{y}.y^2dy=c$$

$$e^{6y}\left[\frac{x^2y^2}{2}-\frac{x^3}{3}\right]+\frac{y^2e^{6y}}{6}-\frac{1}{6}\int(2y)e^{6y}dy=c$$

$$e^{6y}\left[\frac{x^2y^2}{2}-\frac{x^3}{3}\right]+\frac{y^2e^{6y}}{6}-\frac{1}{3}\left[\frac{ye^{6y}}{6}-\int\left(1\times\frac{e^{6y}}{6}\right)dy\right]=c$$

$$e^{6y}\left[\frac{x^2y^2}{2}-\frac{x^3}{3}\right]+\frac{y^2e^{6y}}{6}-\frac{1}{3}\left[\frac{ye^{6y}}{6}-\frac{e^{6y}}{36}\right]=c$$

$$e^{6y}\left(\frac{x^2y^2}{2}-\frac{x^3}{3}+\frac{y^2}{6}-\frac{y}{18}+\frac{1}{108}\right)=c$$ is the required solution.

Example: Solve $\left(3x^2y^4+2xy\right)dx+\left(2x^3y^3-x^2\right)dy=0$. (2.35)

Solution: Comparing the given equation with $Mdx+Ndy=0$, we get

$$M=3x^2y^4+2xy \text{ and } N=2x^3y^3-x^2.$$

Now $\frac{\partial M}{\partial y}=12x^2y^3+2x$ and $\frac{\partial N}{\partial x}=6x^2y^3-2x$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, therefore given differential equation is not exact.

Now $\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}=6x^2y^3-2x-12x^2y^3-2x=-2x\left(3xy^3+2\right)$

$$\Rightarrow \frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M}=\frac{6xy^2-6x^2}{xy^2-x^2}=\frac{-2x\left(3xy^3+2\right)}{yx\left(3xy^3+2\right)}=\frac{-2}{y}=f(y)$$ is function of y alone

therefore $e^{\int f(y)dy}$ is an integrating factor.

Integrating factor (I.F) $=e^{\int f(y)dy}=e^{\int\frac{-2}{y}dy}=e^{\log y^{-2}}=\frac{1}{y^2}$

Now $M^{*}=I.F\times M=\frac{1}{y^2}\left(3x^2y^4+2xy\right)=3x^2y^2+\frac{2x}{y}$

and $N^{*}=I.F\times N=\frac{1}{y^2}\left(2x^3y^3-x^2\right)=2x^3y-\frac{x^2}{y^2}$

$$\frac{\partial M^{*}}{\partial y}=\frac{\partial}{\partial y}\left(3x^2y^2+\frac{2x}{y}\right)=6x^2y-\frac{2x}{y^2}\quad \frac{\partial N^{*}}{\partial x}=\frac{\partial}{\partial x}\left(2x^3y-\frac{x^2}{y^2}\right)=6x^2y-\frac{2x}{y^2}$$

Since $\frac{\partial M^{*}}{\partial y}=\frac{\partial N^{*}}{\partial x}$, therefore $M^{*}dx+N^{*}dy=0$ i.e.,

$\left(3x^2y^2+\frac{2x}{y}\right)dx+\left(2x^3y-\frac{x^2}{y^2}\right)dy=0$ is an exact differential equation. Thus, the solution is given by

$$\int_{\text{Treating } y \text{ as constant}} M^{*}\,dx+\int\left(\text{terms in } N^{*} \text{ not containing } x\right)dy=c,$$ where c is an arbitrary constant.

$$\int_{\text{Treating } y \text{ as constant}}\left(3x^2y^2+\frac{2x}{y}\right)dx+\int\left(\text{terms in}\left(2x^3y-\frac{x^2}{y^2}\right)\text{not containing } x\right)dy=c$$

$$\int_{\text{Treating } y \text{ as constant}}\left(3x^2y^2+\frac{2x}{y}\right)dx+\int 0.dy=c$$

$x^3y^2 + \frac{x^2}{y} = c$ is the required solution.

Example : Solve $\left(xy^3 + y\right)dx + 2\left(x^2y^2 + x + y^4\right)dy = 0$. (2.36)

Solution: Comparing the given equation with $Mdx + Ndy = 0$, we get

$$M = xy^3 + y \text{ and } N = 2\left(x^2y^2 + x + y^4\right).$$

Now $\frac{\partial M}{\partial y} = 3xy^2 + 1$ and $\frac{\partial N}{\partial x} = 4xy^2 + 2$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, therefore given differential equation is not exact.

Now $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 4xy^2 + 2 - 3xy^2 - 1 = xy^2 + 1$

$$\Rightarrow \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{6xy^2 - 6x^2}{xy^2 - x^2} = \frac{xy^2 + 1}{y\left(xy^2 + 1\right)} = \frac{1}{y} = f(y)$$ is function of y alone therefore

$e^{\int f(y)dy}$ is an integrating factor.

Integrating factor (I.F) $= e^{\int f(y)dy} = e^{\int \frac{1}{y}dy} = e^{\log y} = y$

Now $M^* = I.F \times M = y\left(xy^3 + y\right) = xy^4 + y^2$

and $N^* = I.F \times N = y\left(2x^2y^2 + 2x + 2y^4\right) = 2x^2y^3 + 2xy + 2y^5$

$$\frac{\partial M^*}{\partial y} = \frac{\partial}{\partial y}\left(xy^4 + y^2\right) = 4xy^3 + 2y \quad \frac{\partial N^*}{\partial x} = \frac{\partial}{\partial x}\left(2x^2y^3 + 2xy + 2y^5\right) = 4xy^3 + 2y$$

Since $\frac{\partial M^*}{\partial y} = \frac{\partial N^*}{\partial x}$, therefore $M^*dx + N^*dy = 0$ i.e.,

$\left(xy^4 + y^2\right)dx + \left(2x^2y^3 + 2xy + 2y^5\right)dy = 0$ is an exact differential equation. Thus, the solution is given by

$$\int\limits_{\text{Treating } y \text{ as constant}} M^*\,dx + \int\left(\textit{terms in } N^* \textit{ not containing } x\right)dy = c,$$ where c is an arbitrary constant.

$$\int\limits_{\text{Treating } y \text{ as constant}} \left(xy^4 + y^2\right)dx + \int\left(\textit{terms in}\left(2x^2y^3 + 2xy + 2y^5\right)\textit{not containing } x\right)dy = c$$

$$\int\limits_{\text{Treating } y \text{ as constant}} \left(xy^4 + y^2\right)dx + \int 2y^5\,dy = c$$

$\frac{x^2y^4}{2} + xy^2 + \frac{y^6}{3} = c$ is the required solution.

Example : Solve $\left(y\log y\right)dx + \left(x - \log y\right)dy = 0$. (2.37)

Solution: Comparing the given equation with $Mdx + Ndy = 0$, we get

$$M = y\log y \text{ and } N = x - \log y.$$

Now $\frac{\partial M}{\partial y} = 1 + \log y$ and $\frac{\partial N}{\partial x} = 1$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, therefore given differential equation is not exact.

Now $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1 - 1 - \log y = -\log y$

$$\Rightarrow \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{-\log y}{y \log y} = -\frac{1}{y} = f(y)$$ is function of y alone therefore $e^{\int f(y)dy}$ is an integrating factor.

Integrating factor (I.F) $= e^{\int f(y)dy} = e^{\int -\frac{1}{y}dy} = e^{-\log y} = e^{\log y^{-1}} = \frac{1}{y}$

Now $M^* = I.F \times M = \frac{1}{y}(y \log y) = \log y$

and $N^* = I.F \times N = \frac{1}{y}(x - \log y) = \frac{x}{y} - \frac{\log y}{y}$

$\frac{\partial M^*}{\partial y} = \frac{\partial}{\partial y}(\log y) = \frac{1}{y}$; $\frac{\partial N^*}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{y} - \frac{\log y}{y}\right) = \frac{1}{y}$

Since $\frac{\partial M^*}{\partial y} = \frac{\partial N^*}{\partial x}$, therefore $M^* dx + N^* dy = 0$ i.e., $\log y\, dx + \left(\frac{x}{y} - \frac{\log y}{y}\right) dy = 0$ is an exact differential equation. Thus, the solution is given by

$$\int\limits_{\text{Treating } y \text{ as constant}} M^* dx + \int (\text{terms in } N^* \text{ not containing } x)\, dy = c,$$ where c is an arbitrary constant.

$$\int\limits_{\text{Treating } y \text{ as constant}} \log y\, dx + \int \left(\text{terms in} \left(\frac{x}{y} - \frac{\log y}{y}\right) \text{not containing } x \right) dy = c$$

$$\int\limits_{\text{Treating } y \text{ as constant}} \log y\, dx - \int \frac{\log y}{y} dy = c$$

$x \log y - \frac{1}{2}(\log y)^2 = c$ is the required solution.

EXERCISE

Solve the following differential equation:

1. $-(x + x^2 y)dy + (y - xy^2)dx = 0$ (Ans $\log\left(\frac{x}{y}\right) - xy = c$)

2. $x(1 - xy)dy + y(1 + xy)dx = 0$ (Ans $xy \log\left(\frac{y}{x}\right) = cxy - 1$)

3. $y(1 + xy)dx + x(1 + xy + x^2y^2)dy = 0$ (Ans $\frac{1}{2x^2y^2} + \frac{1}{xy} - \log y = c$)

4. $(x^2y^2 + xy + 1)y\,dx + (x^2y^2 - xy + 1)x\,dy = 0$ (Ans $\log\left(\frac{x}{y}\right) + xy - \frac{1}{xy} = c$)

5. $(x^4y^4 + x^2y^2 + xy)y\,dx + (x^4y^4 - x^2y^2 + xy)x\,dy = 0$ (Ans $\log\left(\frac{x}{y}\right) + \frac{1}{2x^2y^2} - \frac{1}{xy} = c$)

6. $(1-xy)y\,dx-(1+xy)x\,dy=0$ (Ans $\log\left(\frac{x}{y}\right)-xy=c$)

7. $\left(x^2y^3+2x^2y^3\right)dx+\left(x^2y-x^3y^2\right)x\,dy=0$ (Ans $\log\left(\frac{x^2}{y}\right)-\frac{1}{xy}=c$)

8. $\left(x^2+y^2+2x\right)dx+2y\,dy=0$ (Ans $e^x\left(x^2+y^2\right)=c$)

9. $\left(x^2+y^2+1\right)dx-2xy\,dy=0$ (Ans $x^2-1-y^2=c$)

10. $\left(5xy+4y^2+1\right)dx+\left(x^2+2xy\right)dy=0$ (Ans $x^5y+x^4y^2+\frac{x^4}{4}=c$)

11. $2\left(x^2y^2+x+y^4\right)dy+\left(xy^3+y\right)dx=0$ (Ans $3x^2y^4+6xy^2+2y=c$)

12. $\left(2xy^2-2y\right)dx+\left(3x^2y-4x\right)dy=0$ (Ans $x^2y^3+2xy^2=c$)

13. $\left(xy^3+2y^4-4x\right)dy+\left(y^4+2y\right)dx=0$ (Ans $x\left\{y+\frac{2}{y^2}\right\}+2xy^2=c$)

14. $\left(y+xy^2\right)dx-x\,dy=0$ (Ans $\frac{x}{y}+\frac{x^2}{2}=c$)

15. $2xy\,dx+\left(y^2-x^2\right)dy=0;\ y(2)=1$ (Ans $x^2+y^2=5y$)

Rule V: By Inspection: In some cases, we can find the integrating factor by just regrouping the terms of the given equation and /or by dividing by suitable function of x and y to convert the non-exact differential equation to an exact differential equation. The following differential which commonly occurs in selecting suitable integrating factors are as follows:

Table 1: Integrating Factor Table

Sr. No	Group of Terms	Integrating Factor	Exact Differential Equation
1	$x\,dy+y\,dx$	1	$d(xy)$
2	$x\,dy+y\,dx$	$\frac{1}{xy}$	$\frac{x\,dy+y\,dx}{xy}=d\{\ln(xy)\}$
3	$x\,dy+y\,dx$	$\frac{1}{(xy)^n}, n\neq 1$	$\frac{x\,dy+y\,dx}{(xy)^n}=d\left\{\frac{(xy)^{1-n}}{1-n}\right\}$
4	$x\,dy-y\,dx$	$\frac{1}{x^2}$	$\frac{x\,dy-y\,dx}{x^2}=d\left(\frac{y}{x}\right)$
5	$x\,dy-y\,dx$	$\frac{1}{y^2}$	$\frac{-(x\,dy-y\,dx)}{y^2}=-d\left(\frac{x}{y}\right)$
6	$x\,dy-y\,dx$	$\frac{1}{xy}$	$\frac{x\,dy-y\,dx}{xy}=d\left(\ln\frac{y}{x}\right)$
7	$y\,dx-x\,dy$	$\frac{1}{xy}$	$\frac{y\,dx-x\,dy}{xy}=d\left(\ln\frac{x}{y}\right)$
8	$x\,dy-y\,dx$	$\frac{1}{x^2+y^2}$	$\frac{x\,dy-y\,dx}{x^2+y^2}=d\left(\tan^{-1}\frac{y}{x}\right)$

9	$ydx - xdy$	$\frac{1}{x^2+y^2}$	$\frac{ydx - xdy}{x^2+y^2} = d\left(\tan^{-1}\frac{x}{y}\right)$
10	$2xydy - y^2dx$	$\frac{1}{x^2}$	$\frac{2xydy - y^2dx}{x^2} = d\left(\frac{y^2}{x}\right)$
11	$2xydy - y^2dx$	$\frac{1}{y^2}$	$\frac{2xydy - x^2dx}{y^2} = d\left(\frac{x^2}{y}\right)$
12	$2x^2ydy - 2xy^2dx$	$\frac{1}{x^4}$	$\frac{2x^2ydy - 2xy^2dx}{x^4} = d\left(\frac{x^2}{y}\right)$
13	$2xy^2dy - 2x^2ydx$	$\frac{1}{y^4}$	$\frac{2xy^2dy - 2x^2ydx}{y^4} = d\left(\frac{x^2}{y^2}\right)$
14	$xdx + ydy$	$\frac{1}{x^2+y^2}$	$\frac{xdx + ydy}{x^2+y^2} = d\left(\frac{1}{2}\log\left(x^2+y^2\right)\right)$
15	$ye^x dx - e^x dy$	$\frac{1}{y^2}$	$\frac{ye^x dx - e^x dy}{y^2} = d\left(\frac{e^x}{y}\right)$
16	$xdy + ydx$	$\frac{1}{\sqrt{1-x^2y^2}}$	$\frac{xdy + ydx}{\sqrt{1-x^2y^2}} = d\left(\sin^{-1}xy\right)$

Example: Solve $y\left(2xy + e^x\right)dx = e^x dy$.

Solution: In the given equation $y\left(2xy + e^x\right)dx = e^x dy$ we can pair the terms $e^x dy$ and $ye^x dx$. It is also observed that $2xy^2$ should not involve the term y^2. This suggests that $\frac{1}{y^2}$ is the integrating factor. So, rewriting the equation and multiplying by $\frac{1}{y^2}$, we get

$$\frac{ye^x dx - e^x dy}{y^2} + 2xdx = 0 \tag{2.38}$$

From the Integrating Table 1(from 15), equation (2.38) can be written as

$$d\left(\frac{e^x}{y^2}\right) + 2xdx = 0 \tag{2.39}$$

On integrating (2.29), we get

$\frac{e^x}{y^2} + x^2 = c$ which is the required solution.

Example: Solve $ydx - xdy + \left(1+x^2\right)dx + x^2 \sin y\, dy = 0$.

Solution: In given differential equation $ydx - xdy + \left(1+x^2\right)dx + x^2 \sin y\, dy = 0$

(2.40)

we can identify the integrating factor with the help of term $ydx - xdy$. From Table 1, $\frac{1}{x^2}$ is the integrating factor. So, multiplying (2.40) by $\frac{1}{x^2}$, we get

$$-\frac{(xdy-ydx)}{x^2}+\left(\frac{1+x^2}{x^2}\right)dx+\sin y\,dy=0 \tag{2.41}$$

From the Integrating Table 1(from 4), equation (2.41) can be written as

$$-d\left(\frac{y}{x}\right)+\left(\frac{1}{x^2}+1\right)dx+\sin y\,dy=0 \tag{2.42}$$

On integrating (2.42), we get

$-\frac{y}{x}+x-\frac{1}{x}-\cos y=c$,which is the required solution.