Chapter 2

lattices, complemented & Bounded Lattice, Complete lattice, Boolean Algebra.

Set Theory

INTRODUCTION

Most of mathematics is based upon the theory of sets that was originated in 1895 by the German mathematician G. Cantor who defined a set as a collection or aggregate of definite and distinguishable objects selected by means of some rules or description. The language of sets is a means to study such collections in an organised manner. We now provide a formal definition of a set.

BASIC CONCEPTS AND NOTATIONS

Definition

A *set* is a well-defined collection of objects, called the *elements* or *members* of the set.

The adjective 'well-defined' means that we should be able to determine if a given element is contained in the set under scrutiny. For example, the states in India, the self-financing engineering colleges in a state, the students who have joined the computer science branch in a college are sets.

Capital letters $A, B, C, \ldots$ are generally used to denote sets and lower case letters $a, b, c, \ldots$ to denote elements. If x is an element of the set A or x belongs to A, it is represented as $x \in A$. Similarly $y \notin A$ means that y is not an element of A.

Notations

Usually a set is represented in two ways, namely, (1) roster notation and (2) set builder notation.

In roster notation, all the elements of the set are listed, if possible, separated by commas and enclosed within braces. A few examples of sets in roster notation are given as follows:

1. The set V of all vowels in the English alphabet: $V = \{a, e, i, o, u\}$
2. The set E of even positive integers less than or equal to 10: $E = \{2, 4, 6, 8, 10\}$
3. The set P of positive integers less than 100: $P = \{1, 2, 3, \ldots, 99\}$

Note The order in which the elements of a set are listed is not important. Thus $\{1, 2, 3\}$, $\{2, 1, 3\}$ and $\{3, 2, 1\}$ represent the same set.

In set builder notation, we define the elements of the set by specifying a property that they have in common.

A few examples of sets in set builder notation are given as follows:

1. The set $V = \{x | x$ is a vowel in the English alphabet$\}$ is the same as $V = \{a, e, i, o, u\}$
2. The set $A = \{x | x = n^2$ where n is a positive integer less than 6$\}$ is the same as $A = \{1, 4, 9, 16, 25\}$
3. The set $B = \{x | x$ is an even positive integer not exceeding 10$\}$ is the same as $B = \{2, 4, 6, 8, 10\}$

Note The set V in example (1) is read as "The set of all x such that …"

The following sets play an important role in discrete mathematics:

$N = \{0, 1, 2, 3, \ldots]$, the set of *natural numbers*

$Z = \{\ldots, -2, -1, 0, 1, 2, \ldots]$, the set of *integers*

$Z^+ = \{1, 2, 3, \ldots\}$, the set of *positive integers*

$Q = \left\{\frac{p}{q} \mid p \in z, q \in z, q \neq 0\right\}$, the set of *rational numbers*

R = the set of *real numbers*.

Some More Definitions

The set which contains all the objects under consideration is called the *Universal set* and denoted as U.

A set which contains no elements at all is called the *Null set* or *Empty set* and is denoted by the symbol ϕ or $\{\ \}$.

For example, the set $A = \{x | x^2 + 1 = 0,\ x$ real$\}$ and the set $B = \{x | x > x^2, x \in z^+\}$ are null sets.

A set which contains only one element is called a *Singleton set*. For example, $A = \{0\}$ and $B = \{n\}$ are singleton sets.

A set which contains a finite number of elements is called a *finite set* and a set with infinite number of elements is called an *infinite set*.

For example, the set $A = \{x^2 | x \in z^+, x^2 < 100\}$ is a finite set as $A = \{1, 4, 9, 16, 25, 36, 49, 64, 81\}$. The set $B = \{x | x$ is an even positive integer$\}$ is an infinite set as $B = \{2, 4, 6, 8, \ldots\}$

If a set A is a finite set, then the number of elements in A is called the *cardinality* or *size* of A and is denoted by $|A|$. In the example given above, $|A| = 9$. Clearly $|\phi| = 0$.

The set A is said to be a *subset* of B, if and only if every element of A is also an element of B and it is denoted as $A \subseteq B$. For example, the set of all even positive integers between 1 and 100 is a subset of all positive integers between 1 and 100.

If A is not a subset of B, i.e., if $A \not\subseteq B$, at least one element of A does not belong to B.

Notes

1. The null set ϕ is considered as a subset of any set A. i.e., $\phi \subseteq A$.
2. Every set A is a subset of itself, i.e., $A \subseteq A$.
3. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.
4. If A is a subset of B, then B is called the *superset* of A and is written as $B \supseteq A$.

Any subset A of the set B is called the *proper subset* of B, if there is at least one element of B which does not belong to A, i.e., if $A \subseteq B$, but $A \neq B$. It is denoted as $A \subset B$.

For example, if $A = \{a, b\}$, $B = \{a, b, c\}$ and $C = \{b, c, a\}$, then A and B are subsets of C, but A is a proper subset of C, while B is not, since $B = C$.

Two sets A and B are said to be *equal*, i.e., $A = B$, if $A \subseteq B$ and $B \subseteq A$.

Given a set S, the set of all subsets of the set S is called the *power set* of S and is denoted by $P(S)$.

For example, if $S = \{a, b, c\}$, $P(S)$ is the set of all subsets of $\{a, b, c\}$. i.e., $P(S) = [\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}, \{a, b, c\}.]$

In this example, we note that $|P(S)| = 8 = 2^3$. This result is only a particular case of a more general property, given as follows:

Property

If a set S has n elements, then its power set has 2^n elements, viz., if $|S| = n$, then $|P(S)| = 2^n$.

Proof

Number of subsets of S having no element, i.e., the null sets $= 1$ or $C(n, 0)$

Number of subsets of S having 1 element $= C(n, 1)$

In general, the number of subsets of S having k elements = the number of ways of choosing k elements from n elements $= C(n, k)$; $0 \leq k \leq n$.

$$\therefore \quad |P(S)| = \text{total number of subsets of } S$$

$$= C(n, 0) + C(n, 1) + C(n, 2) + \ldots + C(n, n) \qquad (1)$$

Now $$(a + b)^n = C(n, 0)a^n + C(n, 1)\, a^{n-1}b + C(n, 2)a^{n-2}b^2 + \ldots + C(n, n)b^n \qquad (2)$$

Putting $a = b = 1$ in (2), we get

$$C(n, 0) + C(n, 1) + C(n, 2) + \ldots + C(n, n) = (1 + 1)^n = 2^n \qquad (3)$$

Using (3) in (1), we get $|P(S)| = 2^n$.

ORDERED PAIRS AND CARTESIAN PRODUCT

A pair of objects whose components occur in a specific order is called an order, separating by a comma and enclosing them in parentheses. For example, (a, b), $(1, 2)$ are ordered pairs.

The ordered pairs (a, b) and (c, d) are equal, if and only if $a = c$ and $b = d$. It is to be noted that (a, b) and (b, a) are not equal unless $a = b$.

If A and B are sets, the set of all ordered pairs whose first component belongs to A and second component belongs to B is called the *cartesian product* of A and B and is denoted by $A \times B$. In other words,

$$A \times B = \{(a, b) | a \in A \text{ and } b \in B\}$$

For example, if $A = \{a, b, c\}$ and $B = \{1, 2\}$,

then $A \times B = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}$

and $B \times A = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$

Note $A \times B$ and $B \times A$ are not equal, unless $A = \phi$ or $B = \phi$ (so that $A \times B = \phi$) or unless $A = B$.

The cartesian product of more than two sets can also be defined as follows:

The cartesian product of the sets $A_1, A_2, \ldots, A_n$, denoted by $A_1 \times A_2 \times \ldots \times A_n$ is the set of ordered n-tuples $(a_1, a_2, \ldots, a_n)$ where a_i belongs to A_i for $i = 1, 2, 3, \ldots n$. In other words,

$$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) | a_i \in A_i \text{ for } i = 1, 2, \ldots n\}$$

For example, if $A = \{a, b\}$, $B = \{1, 2\}$, $C = (\alpha, \beta, \gamma)$, then $A \times B \times C = \{(a, 1, \alpha), (a, 1, \beta), (a, 1, \gamma), (a, 2, \alpha), (a, 2, \beta), (a, 2, \gamma), (b, 1, \alpha), (b, 1, \beta), (b, 1, \gamma), (b, 2, \alpha), (b, 2, \beta), (b, 2, \gamma)\}$.

SET OPERATIONS

Two or more sets can be combined using set operations to give rise to new sets. These operations that play an important role in many applications are discussed as follows:

Definition

The *union* of two sets A and B, denoted by $A \cup B$, is the set of elements that belong to A or to B or to both, viz., $A \cup B = \{x | x \in A \text{ or } x \in B\}$.

Venn Diagram

Sets can also be represented graphically by means of Venn diagrams in which the universal set is represented by the interior of a rectangle and other sets are represented by the interiors of circles that lie inside the rectangle. If a set A is a subset of another set B, the circle representing A is drawn inside the circle representing B.

The union of two sets A and B is represented by the hatched area within the circle representing A or the circle representing B, as shown in the Fig. 2.1.

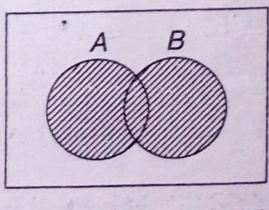


Fig. 2.1 $[A \cup B]$

For example, if $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$ and $C = \{3, 4, 5\}$, then $A \cup B = \{1, 2, 3, 4\}$,

$B \cup C = \{2, 3, 4, 5\}$ and $A \cup C = \{1, 2, 3, 4, 5\}$

Definition

The *intersection* of two sets A and B, denoted by $A \cap B$, is the set of elements that belong to both A and B.
viz., $A \cap B = \{x | x \in A \text{ and } x \in B\}$.
In the Venn diagram, the intersection of two sets A and B is represented by the hatched area that is within both the circles representing the sets A and B (Refer to Fig. 2.2).

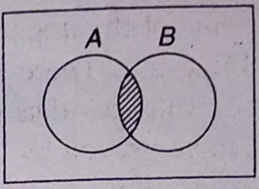


Fig. 2.2 $[A \cap B]$

In the example given earlier,

$$A \cap B = \{2, 3\},\ B \cap C = \{3, 4\} \text{ and } A \cap C = \{3\}.$$

Definition

If $A \cap B$ is the empty set, viz., if A and B do not have any element in common, then the sets A and B are said to be *disjoint*. For example, if $A = \{1, 3, 5\}$ and $B = \{2, 4, 6, 8\}$, then $A \cap B = \phi$ and hence A and B are disjoint.

Definition

If U is the universal set and A is any set, then the set of elements which belong to U but which do not belong to A is called the *complement of A* and is denoted by A' or A^c or $\bar{A}$
viz., $A' = \{x | x \in U \text{ and } x \notin A\}$
For example, if $U = \{1, 2, 3, 4, 5\}$ and $A = \{1, 3, 5\}$, then $\bar{A} = \{2, 4\}$.

Definition

If A and B are any two sets, then the set of elements that belong to A but do not belong to B is called the *difference* of A and B or *relative complement* of B with respect to A and is denoted by $A - B$ or $A \backslash B$.
viz., $A - B = \{x | x \in A \text{ and } x \notin B\}$
For example, if $A = \{1, 2, 3\}$ and $B = \{1, 3, 5, 7\}$, then $A - B = \{2\}$ and $B - A = \{5, 7\}$.

Definition

If A and B are any two sets, the set of elements that belong to A or B, but not to both is called the *symmetric difference* of A and B and is denoted by $A \oplus B$ or $A \Delta B$ or $A + B$. It is obvious that $A \oplus B = (A - B) \cup (B - A)$.
For example, if $A = \{a, b, c, d\}$ and $B = \{c, d, e, f\}$ then $A \oplus B = \{a, b, e, f\}$

The sets $\bar{A}$, $A - B$ and $A \oplus B$ are represented by the hatched areas shown in Figs. (2.3), (2.4) and (2.5) respectively.

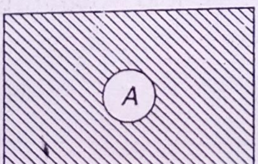


Fig. 2.3 $\bar{A}$

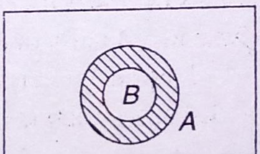


Fig. 2.4 $A - B$

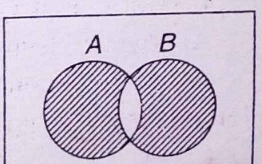


Fig. 2.5 $A \oplus B$

The Algebraic Laws of Set Theory

Some of the important set identities or algebraic laws of set theory are listed in Table 2.1. There is a marked similarity between these identities and logical equivalences discussed in the chapter on Mathematical Logic. All these laws can be proved by basic arguments or by using Venn diagrams and truth tables. We shall prove some of these laws and leave the proofs of the remaining laws as exercise to the reader.

Table 2.1 Set Identities

Identity	*Name of the law*
1. (a) $A \cup \phi = A$	Identity laws
1. (b) $A \cap U = A$	
2. (a) $A \cup U = U$	Domination laws
2. (b) $A \cap \phi = \phi$	
3. (a) $A \cup A = A$	Idempotent laws
3. (b) $A \cap A = A$	
4. (a) $A \cup \bar{A} = U$	Inverse laws or Complement laws
4. (b) $A \cap \bar{A} = \phi$	
5. $\bar{\bar{A}} = A$	Double Complement law or Involution law
6. (a) $A \cup B = B \cup A$	Commutative laws
6. (b) $A \cap B = B \cap A$	
7. (a) $A \cup (B \cup C) = (A \cup B) \cup C$	Associative laws
7. (b) $A \cap (B \cap C) = (A \cap B) \cap C$	
8. (a) $A \cap (B \cup C) = (A \cap B) \cup (A \cup C)$	Distributive laws
8. (b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	
9. (a) $A \cup (A \cap B) = A$	Absorption laws
9. (b) $A \cap (A \cup B) = A$	
10. (a) $\overline{A \cup B} = \bar{A} \cap \bar{B}$	De Morgan's laws
10. (b) $\overline{A \cap B} = \bar{A} \cup \bar{B}$	

Dual Statement and Principle of Duality

If s is a statement of equality of two set expressions each of which may contain the sets A, B, $\bar{A}$, $\bar{B}$ etc., ϕ and U and the only set operation symbols $\cup$ and $\cap$, then the *dual* of s, denoted by s^d, is obtained from s by replacing (1) each occurrence of ϕ and U (in s) by U and ϕ respectively and (2) each occurrence of $\cup$ and $\cap$ (in s) by $\cap$ and $\cup$ respectively.

The *principle of duality* states that whenever s, a statement of equality of two set expressions, is a valid theorem, then its dual s^d is also a valid theorem.

Note All the set identities given in (b) parts of various laws are simply the duals of the corresponding set identities in (a) parts.

Now let us establish some of the

Now $\quad x \in A \cup A \Rightarrow x \in A$ or $x \in A$
$\Rightarrow x \in A$

$\therefore \quad A \cup A \subseteq A$ (1)

$x \in A \Rightarrow x \in A$ or $x \in A$
$\Rightarrow x \in A \cup A$

$\therefore \quad A \subseteq A \cup A$ (2)

From (1) and (2), we get $A \cup A = A$

(ii) $A \cap B = B \cap A$

Let us use the set builder notation to establish this identity.

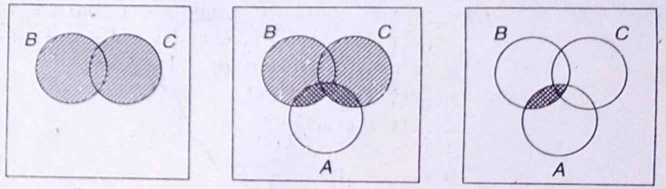
$$\begin{aligned} A \cap B &= \{x|x \in A \cap B\} \\ &= \{x|x \in A \text{ and } x \in B\} \\ &= \{x|x \in B \text{ and } x \in A\} \\ &= \{x|x \in B \cap A\} \\ &= B \cap A \end{aligned}$$

(iii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$$\begin{aligned} A \cup (B \cap C) &= \{x|x \in A \text{ or } x \in (B \cap C)\} \\ &= \{x|x \in A \text{ or } (x \in B \text{ and } x \in C)\} \\ &= \{x|(x \in A \text{ or } x \in B) \text{ and } (x \in A \text{ or } x \in C)\} \\ &= \{x|x \in A \cup B \text{ and } x \in A \cup C\} \\ &= \{x|x \in (A \cup B) \cap (A \cup C)\} \\ &= (A \cup B) \cap (A \cup C) \end{aligned}$$

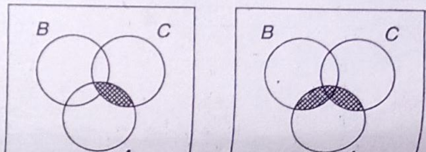
(iv) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Let us use Venn diagram to establish this identity.



$B \cup C$ (a) $\quad A \cap (B \cup C)$ (b) $\quad A \cap B$ (c)

Fig: 2.6



(v) $\overline{A \cap B} = \bar{A} \cup \bar{B}$

$$
\begin{aligned}
\overline{A \cap B} &= \{x | x \notin A \cap B\} \\
&= \{x | x \notin A \text{ or } x \notin B\} \\
&= \{x | x \in \bar{A} \text{ or } x \in \bar{B}\} \\
&= \{x | x \in \bar{A} \cup \bar{B}\} \\
&= \bar{A} \cup \bar{B}
\end{aligned}
$$

(vi) $\overline{A \cup B} = \bar{A} \cap \bar{B}$

$\therefore \quad \overline{A \cup B} = \bar{A} \cap \bar{B}$

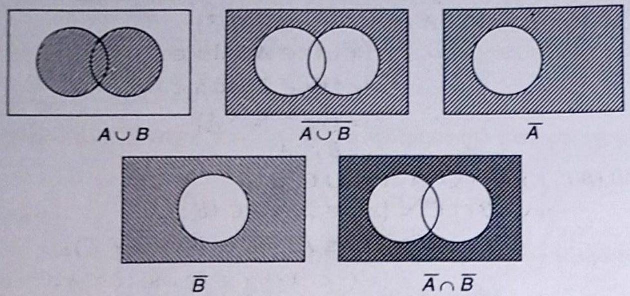


Fig. 2.7

WORKED EXAMPLES 2(A)

Example 2.1 Prove that $(A - C) \cap (C - B) = \phi$ analytically, where A, B, C are sets. Verify graphically

$$
\begin{aligned}
(A - C) \cap (C - B) &= \{x | x \in A \text{ and } x \notin C \text{ and } x \in C \text{ and } x \notin B\} \\
&= \{x | x \in A \text{ and } (x \in C \text{ and } x \in \bar{C}) \text{ and } x \in \bar{B}\} \\
&= \{x | (x \in A \text{ and } x \in \phi) \text{ and } x \in \bar{B}\} \\
&= \{x | x \in A \cap \phi \text{ and } x \in \bar{B}\} \\
&= \{x | x \in \phi \text{ and } x \in \bar{B}\} \\
&= \{x | x \in \phi \cap \bar{B}\} \\
&= \{x | x \in \phi\} \\
&= \phi
\end{aligned}
$$

Let us now use Venn diagrams to verify the result.

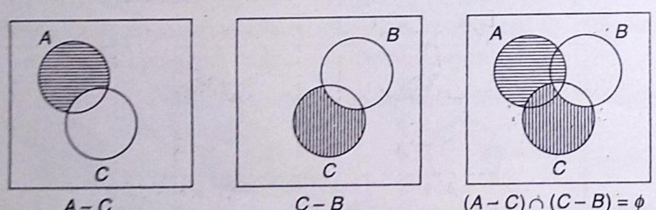


Fig. 2.8

Example 2.2 If A, B and C are sets, prove, both analytically and graphically, that $A - (B \cap C) = (A - B) \cup (A - C)$.

$$
\begin{aligned}
A - (B \cap C) &= \{x | x \in A \text{ and } x \notin (B \cap C)\} \\
&= \{x | x \in A \text{ and } (x \notin B \text{ or } x \notin C)\} \\
&= \{x | (x \in A \text{ and } x \notin B) \text{ or } (x \in A \text{ and } x \notin C)\} \\
&= \{x | x \in (A - B) \text{ or } x \in (A - C)\} \\
&= \{x | x \in (A - B) \cup (A - C)\} \\
&= (A - B) \cup (A - C)
\end{aligned}
$$

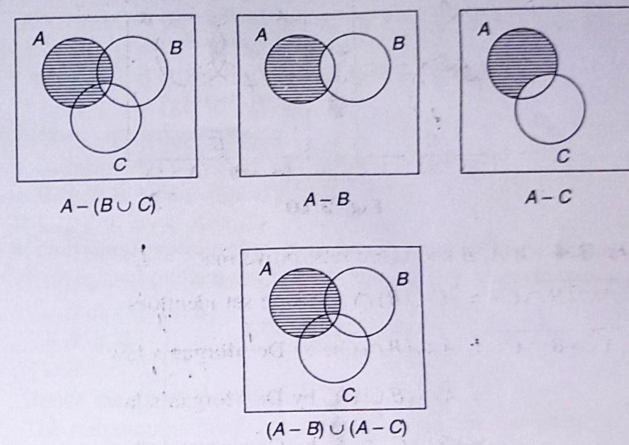


Fig. 2.9

Example 2.3 If A, B and C are sets, prove, both analytically and graphically, that $A \cap (B - C) = (A \cap B) - (A \cap C)$.

$$
\begin{aligned}
A \cap (B - C) &= \{x | x \in A \text{ and } x \in (B - C)\} \\
&= \{x | x \in A \text{ and } (x \in B \text{ and } x \notin C)\} \\
&= \{x | x \in A \text{ and } (x \in B \text{ and } x \in \bar{C})\} \\
&= \{x | x \in (A \cap B \cap \bar{C})\} \\
&= A \cap B \cap \bar{C}
\end{aligned}
$$

Now

$$
\begin{aligned}
(A \cap B) - (A \cap C) &= \{x | x \in (A \cap B) \text{ and } x \in \overline{A \cap C}\} \\
&= \{x | x \in (A \cap B) \text{ and } x \in (\bar{A} \cup \bar{C}\}, \text{ by De Morgan's law} \\
&= \{x | x \in (A \cap B) \text{ and } (x \in \bar{A} \text{ or } x \in \bar{C})\} \\
&= \{x | [x \in (A \cap B) \text{ and } x \in \bar{A}] \text{ or } [x \in (A \cap B) \text{ and } x \in \bar{C}]\} \\
&= \{x | x \in (A \cap \bar{A} \cap B) \text{ or } x \in (A \cap B \cap \bar{C})\} \\
&= \{x | x \in \phi \text{ or } x(A \cap B \cap \bar{C})\} \\
&= \{x | x \in A \cap B \cap \bar{C}\} \\
&= A \cap B \cap \bar{C}
\end{aligned}
$$

Hence the result.

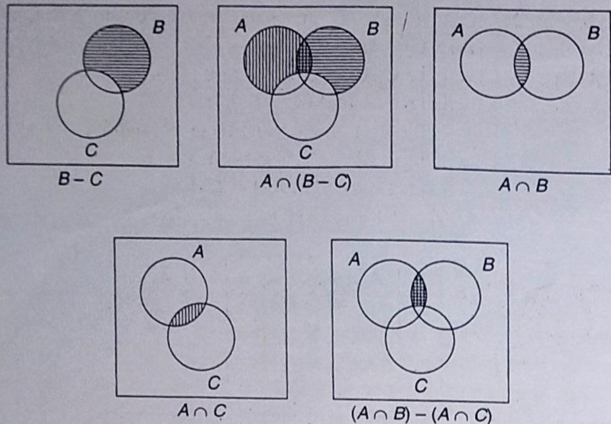


Fig. 2.10

Example 2.4 If A, B and C are sets, prove that

$$\overline{A \cup (B \cap C)} = (\bar{C} \cup \bar{B}) \cap \bar{A}, \text{ using set identities}$$

$$\begin{aligned} \text{L.S.} = \overline{A \cup (B \cap C)} &= \bar{A} \cap (\overline{B \cap C}), \text{ by De Morgan's law} \\ &= \bar{A} \cap (\bar{B} \cup \bar{C}), \text{ by De Morgan's law} \\ &= (\bar{B} \cup \bar{C}) \cap \bar{A}, \text{ by Commutative law} \\ &= (\bar{C} \cup \bar{B}) \cap \bar{A}, \text{ by Commutative law} \\ &= \text{R.S.} \end{aligned}$$

Example 2.5 If A, B and C are sets, prove algebraically that $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

$$\begin{aligned} A \times (B \cap C) &= \{(x, y) | x \in A \text{ and } y \in (B \cap C)\} \\ &= \{(x, y) | x \in A \text{ and } (y \in B \text{ and } y \in C)\} \\ &= \{(x, y) | (x \in A \text{ and } y \in B) \text{ and } (x \in A \text{ and } y \in C)\} \\ &= \{(x, y) | (x, y) \in A \times B \text{ and } (x, y) \in A \times C\} \\ &= \{(x, y) | (x, y) \in (A \times B) \cap (A \times C)\} \\ &= (A \times B) \cap (A \times C) \end{aligned}$$

Example 2.6 If A, B, C and D are sets, prove algebraically that $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$. Give an example to support this result.

$$\begin{aligned} (A \cap B) \times (C \cap D) &= \{(x, y) | x \in (A \cap B) \text{ and } y \in (C \cap D)\} \\ &= \{(x, y) | (x \in A \text{ and } x \in B) \text{ and } (y \in C \text{ and } y \in D)\} \\ &= \{(x, y) | (x \in A \text{ and } y \in C) \text{ and } (x \in B \text{ and } y \in D)\} \\ &= \{(x, y) | (x, y) \in (A \times C) \text{ and } (x, y) \in (B \times D)\} \\ &= \{(x, y) | (x, y) \in (A \times C) \cap (B \times D)\} \end{aligned}$$

Example Let $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$, $C = \{5, 6, 7\}$ and $D = \{6, 7, 8\}$.

Then $A \cap B = \{2, 3\}$ and $C \cap D = \{6, 7\}$

$\therefore \quad (A \cap B) \times (C \cap D) = \{(2, 6), (2, 7), (3, 6), (3, 7)\}$

Now $A \times C = \{(1, 5), (1, 6), (1, 7), (2, 5), (2, 6), (2, 7), (3, 5), (3, 6), (3, 7)\}$

$B \times D = \{(2, 6), (2, 7), (2, 8), (3, 6), (3, 7), (3, 8), (4, 6), (4, 7), (4, 8)\}$

$\therefore \quad (A \times C) \cap (B \times D) = \{(2, 6), (2, 7), (3, 6), (3, 7)\}$

Hence $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$

Example 2.7 Use Venn diagram to prove that $\oplus$ is an associative operation, viz., $(A \oplus B) \oplus C = A \oplus (B \oplus C)$.

Instead of shading or hatching the regions in the Venn diagram, let us label the various regions as follows:

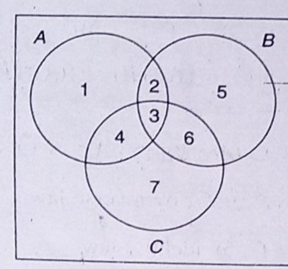


Fig. 2.11

Set A consists of the points in the regions labeled 1, 2, 3, 4; set B consists of the points in the region labeled 2, 3, 5, 6; set C consists of the points in the region labeled 3, 4, 6, 7.

$$\begin{aligned} \text{Now} \quad A \oplus B &= (A \cup B) - (A \cap B) \\ &= \{R_1, R_2, R_3, R_4, R_5, R_6\} - \{R_2, R_3\}, \end{aligned}$$

where R_i represents the region labeled i

$$\begin{aligned} &= \{R_1, R_4, R_5, R_6\} \\ (A \oplus B) \oplus C &= \{R_1, R_3, R_4, R_5, R_6, R_7\} - \{R_4, R_6\} \\ &= \{R_1, R_3, R_5, R_7\} \end{aligned}$$

$$\begin{aligned} \text{Now} \quad B \oplus C &= \{R_2, R_3, R_4, R_5, R_6, R_7\} - \{R_3, R_6\} \\ &= \{R_2, R_4, R_5, R_7\} \\ A \oplus (B \oplus C) &= \{R_1, R_2, R_3, R_4, R_5, R_7\} - \{R_2, R_4\} \\ &= \{R_1, R_3, R_5, R_7\} \end{aligned}$$

Hence $(A \oplus B) \oplus C = A \oplus (B \oplus C)$

Example 2.8 Use Venn diagram to prove that $(A \oplus B) \times C = (A \times C) \oplus (B \times C)$, where A, B, C are sets.

Using the same assumptions about A, B, and C and the Fig. 2.11 in the Example (8), we have $A \oplus B = \{R_1, R_4, R_5, R_6\}$.

$$
\begin{aligned}
(A \oplus B) \times C &= \{R_1, R_4, R_5, R_6\} \times \{R_3, R_4, R_6, R_7\} \\
&= \{R_1 \times R_3, R_1 \times R_4, \ldots, R_6 \times R_7\} \\
A \times C &= \{R_1, R_2, R_3, R_4\} \times \{R_3, R_4, R_6, R_7] \\
&= \{R \times R_3, R_1 \times R_4, \ldots, R_4 \times R_7\} \\
B \times C &= \{R_2, R_3, R_5, R_6\} \times \{R_3, R_4, R_6, R_7\}
\end{aligned}
$$

It is easily verified that

$$
\begin{aligned}
(A \oplus B) \times C &= (A \times C) \oplus (B \times C) \\
&= \{(R_1 \times R_i), (R_4 \times R_i), (R_5 \times R_i), (R_6 \times R_i)\}
\end{aligned}
$$

where $\quad i = 3, 4, 6, 7$

Example 2.9 Simplify the following sets, using set identities:

(a) $\bar{A} \cup \bar{B} \cup (A \cap B \cap \bar{C})$

(b) $(A \cap B) \cup [B \cap ((C \cap D) \cup (C \cap \bar{D}))]$

(a) $\bar{A} \cup \bar{B} \cup (A \cap B \cap \bar{C}) = \overline{(A \cap B)} \cup [(A \cap B) \cap \bar{C}]$, by De Morgan's law

$= [\overline{(A \cap B)} \cup (A \cap B)] \cap [\overline{A \cap B} \cup \bar{C}]$, by distributive law

$= U \cap \overline{A \cap B} \cup \bar{C}$, by inverse law

$= \overline{A \cap B} \cup \bar{C}$, by identity law

$= \bar{A} \cup \bar{B} \cup \bar{C}$, by De Morgan's law

(b) $(A \cap B) \cup [B \cap ((C \cap D) \cup (C \cap \bar{D}))]$

$= (A \cap B) \cup [B \cap \{C \cap (D \cup \bar{D})\}]$, by distributive law
$= (A \cap B) \cup [B \cap (C \cap U)]$, by inverse law
$= (A \cap B) \cup [B \cap C]$, by identity law
$= (B \cap A) \cup (B \cap C)$, by commutative law
$= B \cap (A \cup C)$, by distributive law

Example 2.10 Write the dual of each of the following statements:

(a) $A = (\bar{B} \cap A) \cup (A \cap B)$

(b) $(A \cap B) \cup (\bar{A} \cap B) \cup (A \cap \bar{B}) \cup (\bar{A} \cap \bar{B}) = U$

(a) Recalling that the dual of any statement is obtained by replacing $\cup$ by $\cap$ and $\cap$ by $\cup$, the dual of the statement in (a) is got as

$A = (\bar{B} \cup A) \cap (A \cup B)$, which can be easily seen to be a valid statement.

(b) The dual of the statement in (b) is

$$(A \cup B) \cap (\bar{A} \cup B) \cap (A \cup \bar{B}) \cap (\bar{A} \cup \bar{B}) = \phi$$

Example 2.11 For each of the following statements in which A, B, C and D are arbitrary sets, either prove that it is true or give a counter example to

(c) $A - (B \times C) = (A - B) \times (A - C)$
(d) $(A \cup B) \times (C \cup D) = (A \times C) \cup (B \times D)$
(e) $A \oplus C = B \oplus C \to A = B$
(f) $A \oplus (B \cap C) = (A \oplus B) \cap (A \oplus C)$

(a) The statement is false, as in the following counter example:

Let $A = \{1\}$, $B = \{2\}$ and $C = \{1, 2\}$

Now $A \cup C = B \cup C = \{1, 2\}$

But $A \neq B$

(b)
$$
\begin{aligned}
A &= \{x | x \in A\} \\
&= \{x | x \in A \cup C\} \\
&= \{x | x \in B \cup C\} \quad \text{(given)} \\
&= \{x | x \in B \text{ or } x \in C\} \\
&= \{x | x \in B\} \text{ or } \{x | x \in C\} \\
&= \{x | x \in B\} \text{ or } \{x | x \in A \text{ and } x \in C\} \\
&= B \text{ or } \{x | x \in A \cap C\} \\
&= B \text{ or} \{x | x \in B \cap C\} \quad \text{(given)} \\
&= B \text{ or } \{x | x \in B \text{ and } x \in C\} \\
&= B \text{ or } \{x | x \in B\} \\
&= B \text{ or } B \\
&= B
\end{aligned}
$$

Hence the given statement is true.

(c) The statement is false, as in the following counter example:

Let $A = \{1, 2, 3, 4, 5\}$, $B = \{1, 2\}$, $C = \{3, 4\}$

Then $B \times C = \{(1, 3), (1, 4), (2, 3), (2, 4)$

$\therefore \quad A - (B \times C) = \{1, 2, 3, 4, 5\}$

Now $A - B = \{3, 4, 5\}$ and $A - C = \{1, 2, 5\}$

$\therefore \quad (A - B) \times (A - C) = \{(3, 1), (3, 2), (3, 5), (4, 1), (4, 2), (4, 5), (5, 1), (5, 2), (5, 5)\}$

Hence $\quad A - (B \times C) \neq (A - B) \times (A - C)$

(d) The statement is false, as in the following counter example:

Let $A = \{1, 2\}$, $B = \{2, 3\}$, $C = \{4, 5)$, $D = \{5, 6\}$

Then $A \cup B = \{1, 2, 3\}$, $C \cup D = \{4, 5, 6\}$

$\therefore \quad (A \cup B) \times (C \cup D) = \{(1, 4), (1, 5), (1, 6), (2, 4), (2, 5), (2, 6), (3, 4)$ $(3, 5), (3, 6)\}$

Now $\quad A \times C = \{(1, 4), (1, 5), (2, 4), (2, 5)\}$

and $\quad B \times D = \{(2, 5), (2, 6), (3, 5), (3, 6)\}$

$\therefore \quad (A \times C) \cup (B \times D) = \{(1, 4), (1, 5), (2, 4), (2, 5), (2, 6), (3, 5), (3, 6)\}$

Thus $\quad (A \cup B) \times (C \cup D) \neq (A \times C) \cup (B \times D)$

(e) $x \in A$ and $x \in C \Rightarrow x \notin A \oplus C$, by definition of $A \oplus C$

$\Rightarrow x \notin B \oplus C$ (given)

$\Rightarrow x \in B$ and $x \in C$

$\Rightarrow x \in B$ $\quad$ (1)

Also $x \in A$ and $x \notin C \Rightarrow x \in A \oplus C$

$\Rightarrow x \in B \oplus C$ (given)

$\Rightarrow x \in B$ (2)

From (1) and (2), it follows that $A \subseteq B$

Similarly we can prove that $B \subseteq A$

Hence $A = B$

i.e., the given statement is true.

(f) The statement is false, as in the following counter example:

Let $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$ and $C = \{2, 3, 5, 7\}$

$B \cap C = \{3, 5\}$ and $A \oplus (B \cap C) = \{1, 2, 5\}$

$A \oplus B = \{1, 2, 5, 6\}$ and $A \oplus C = \{1, 4, 5, 7\}$

$\therefore (A \oplus B) \cap (A \oplus C) = \{1, 5\}$

Hence $A \oplus (B \cap C) \neq (A \oplus B) \cap (A \oplus C)$

Example 2.12 Find the sets A and B, if

(a) $A - B = \{1, 3, 7, 11\}$, $B - A = \{2, 6, 8\}$ and $A \cap B = \{4, 9\}$

(b) $A - B = \{1, 2, 4\}$, $B - A = \{7, 8\}$ and $A \cup B = \{1, 2, 4, 5, 7, 8, 9\}$

(a) From the Venn diagram, it is clear that

$A = \{1, 3, 4, 7, 9, 11\}$

and $B = \{2, 4, 6, 8, 9\}$

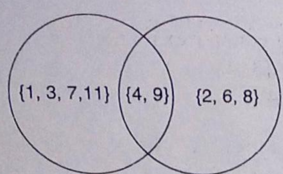


Fig. 2.12

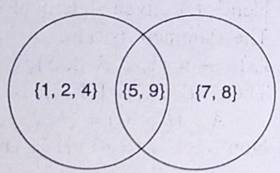


Fig. 2.13

(b) From the Venn diagram, it is clear that

$A = \{1, 2, 4, 5, 9\}$s

and $B = \{5, 7, 8, 9\}$

EXERCISE 2(A)

Part (A): (Short answer questions)

1. Explain the roster notation and set builder notation of sets with examples.
2. Define null set and singleton set.
3. Define finite and infinite sets. What is cardinality of a set?
4. Define subset and proper subset. When are two sets said to be equal?
5. What is a power set? State the relation between the cardinalities of a

18. For each of the following statements in which A, B, ...

6. Define the cartesian product of two sets and give an example.
7. Define union and intersection of two sets. Give their Venn diagram representation.
8. When are two sets said to be disjoint?
9. Define complement and relative complement of a set. Give examples.
10. Define the symmetric difference of two sets.
11. State the identity, domination, idempotent and inverse laws of set theory.
12. State the commutative, associative and distributive laws of set theory.
13. State De Morgan's laws of set theory.
14. State the principle of duality in set theory. Give an example.
15. Given that $U = \{1, 2, 3, \ldots, 9, 10\}$, $A = \{1, 2, 3, 4, 5\}$, $B = \{1, 2, 4, 8\}$, $C = \{1, 2, 3, 5, 7\}$ and $D = \{2, 4, 6, 8\}$, find each of the following ((a)–(l)):

(a) $(A \cup B) \cap C$ (b) $A \cup (B \cap C)$

(c) $\bar{C} \cup \bar{D}$ (d) $\overline{C \cup D}$

(e) $(A \cup B) - C$ (f) $A \cup (B - C)$

(g) $(B - C) - D$ (h) $B - (C - D)$

(i) $(A \cup B) - (C \cap D)$ (j) $(A - B) \cup (C - D)$

(k) $A \oplus (B \cap C)$ (l) $A \cup (B \oplus C)$

16. Prove the following analytically or graphically:

(a) $A - B = A \cap \bar{B}$ (b) $A - (A \cap B) = A - B$

(c) $(A \cap B) \cup (A \cap \bar{B}) = A$ (d) $A \cup (A \cap B) = A$

(e) $(A \cup B) \cap (A \cup \phi) = A$ (f) $(A \cap B) \cup (B - A) = B$

(g) $\overline{(A - B)} = \bar{A} \cup B$ (h) $A \cap (B - A) = \phi$

(i) $A - B = \bar{B} - \bar{A}$ (j) $(A - B) \cup (A \cap B) = A$

(k) $(A - B) \cap (B - A) = \phi$

(l) $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$

(m) $A \oplus B = (A \cup B) - (A \cap B)$ (n) $A \oplus B = (A - B) \cup (B - A)$

(o) $(A \cap B) \subset A \subset (A \cup B)$ (p) $(A \cap B) \subset B \subset (A \cup B)$

Part B

17. Prove the following statements analytically, where A, B and C are sets. Verify them graphically also.

(a) $A \cup B = (A \cap B) \cup (A \cap \bar{B}) \cup (\bar{A} \cap B)$

(b) $(A \cap B) - C = (A - C) \cap (B - C)$

(c) $A - (B \cup C) = (A - B) \cap (A - C)$

(d) $(B \cup C) - A = (B - A) \cup (C - A)$

(e) $(A - B) - C = A - (B \cup C)$

(f) $(A - B) - C = (A - C) - (B - C)$

(g) $A \cap (B - C) = (A \cap B) - (A \cap C)$

(h) $\overline{A \oplus B} = \bar{A} \oplus B = A \oplus \bar{B}$

notation $a R b$ and read it as "a is related to b by R". If $(a, b) \notin R$, it is denoted sets, either prove that it is true or give a counter example to show that it is false.

(a) $A \cap C = B \cap C \rightarrow A = B$
(b) $A \cap B = A \cap C$ and $\bar{A} \cap B = \bar{A} \cap C \rightarrow B = C$
(c) $(A - C) = (B - C) \rightarrow A = B$
(d) $A \cap C = B \cap C$ and $A - C = B - C \rightarrow A = B$
(e) $A \cup C = B \cup C$ and $A - C = B - C \rightarrow A = B$
(f) $A \times (B \cup C) = (A \times B) \cup (A \times C)$
(g) $A \cap (B \times C) = (A \cap B) \times (A \cap C)$
(h) $(A \cap B) \times C = (A \times C) \cap (B \times C)$
(i) $(A - B) \times C = (A \times C) - (B \times C)$
(j) $(A - B) \times (C - D) = (A \times C) - (B \times D)$
(k) $A \oplus (B \oplus C) = (A \oplus B) \oplus C$
(l) $(A \oplus B) \times C = (A \times C) \oplus (B \times C)$

19. Simplify the following set expressions, using set identities:
(a) $\overline{(A \cup B)} \cap \overline{(\bar{A} \cup \bar{C})} \cap \overline{(\bar{B} \cup C)}$
(b) $(A \cap B) \cup (A \cap B \cap \bar{C} \cap D) \cup (\bar{A} \cap B)$
(c) $(A - B) \cup (A \cap B)$

20. Write the dual of each of the following statements:
(a) $(A \cup B) \cap (A \cup \phi) = A$
(b) $A \cup B = (A \cap B) \cup (A \cap \bar{B}) \cup (\bar{A} \cap B)$
(c) $\overline{(A \cap B \cap C)} = \overline{(A \cap C)} \cup \overline{(A \cap B)}$.

RELATIONS

Introduction

A *relation* can be thought of as a structure (for example, a table) that represents the relationship of elements of a set to the elements of another set. We come across many situations where relationships between elements of sets, such as those between roll numbers of students in a class and their names, industries and their telephone numbers, employees in an organization and their salaries occur. Relations can be used to solve problems such as producing a useful way to store information in computer databases.

The simplest way to express a relationship between elements of two sets is to use ordered pairs consisting of two related elements. Due to this reason, sets of ordered pairs are called *binary relations*. In this section, we introduce the basic terminology used to describe binary relations, discuss the mathematics of relations defined on sets and explore the various properties of relations.

Definition

When A and B are sets, a subset R of the Cartesian product $A \times B$ is called a *binary relation* from A to B. viz., If R is a binary relation from A to B, R is a set of ordered pairs (a, b), where $a \in A$ and $b \in B$. When $(a, b) \in R$, we use the notation as $a \mathrel{R} b$.

Note Mostly we will deal with relationships between the elements of two sets. Hence the word 'binary' will be omitted hereafter.

Relation

If R is a relation from a set A to itself, viz., if R is a subset of $A \times A$, then R is called *a relation on the set A*.

domain

The set $\{a \in A | a \mathrel{R} b$, for some $b \in B\}$ is called *the domain of R* and denoted by $D(R)$.

Range

The set $\{b \in B | a \mathrel{R} b$, for some $a \in A\}$ is called *the range of R* and denoted by $R(R)$.

Examples

1. Let $A = \{0, 1, 2, 3, 4\}$, $B = \{0, 1, 2, 3\}$ and $a \mathrel{R} b$ if and only if $a + b = 4$.
Then $R = \{(1, 3), (2, 2), (3, 1), (4, 0)\}$
The domain of $R = \{1, 2, 3, 4\}$ and the image of $R = \{0, 1, 2, 3\}$
2. Let R be the relation on $A = \{1, 2, 3, 4\}$, defined by $a \mathrel{R} b$ if $a \leq b$; $a, b \in A$.
Then $R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$
The domain and range of R are both equal to A.

TYPES OF RELATIONS

A relation R on a set A is called a *universal* relation, if $R = A \times A$. For example if $A = \{1, 2, 3\}$, then $R = A \times A = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$ is the universal relation on A.

A relation R on a set A is called a *void relation*, if R is the null set ϕ. For example if $A = \{3, 4, 5\}$ and R is defined as $a \mathrel{R} b$ if and only if $a + b > 10$, then R is a null set, since no element in $A \times A$ satisfies the given condition.

Note The entire Cartesian product $A \times A$ and the empty set are subsets of $A \times A$.

A relation R on a set A is called an *identity relation*, if $R = \{(a, a) | a \in A\}$ and is denoted by I_A.

For example, if $A = \{1, 2, 3\}$, then $R = \{(1, 1), (2, 2), (3, 3)\}$ is the identity relation on A.

When R is any relation from a set A to a set B, *the inverse* of R, denoted by R^{-1}, is the relation from B to A which consists of those ordered pairs got by interchanging the elements of the ordered pairs in R.

viz., $R^{-1} = \{(b, a) | (a, b) \in R\}$
viz., if $a \mathrel{R} b$, then $b \mathrel{R^{-1}} a$.

For example, if $A = \{2, 3, 5\}$, $B = \{6, 8, 10\}$ and $a \mathrel{R} b$ if and only if $a \in A$ divides $b \in B$, then $R = \{(2, 6), (2, 8), (2, 10), (3, 6), (5, 10)\}$

Now $R^{-1} = \{(6, 2), (8, 2), (10, 2), (6, 3), (10, 5)\}$

We note that $b \mathrel{R^{-1}} a$, if and only if $b \in B$ is a multiple of $a \in A$. Also we note that

$$D(R) = R(R^{-1}) = \{2, 3, 5\} \text{ and}$$

$R(R) = D(R^{-1}) = \{6, 8, 10\}$

$\vee \rightarrow$ disjunction

 $\wedge \rightarrow$ conjunction

SOME OPERATIONS ON RELATIONS

As binary relations are sets of ordered pairs, all set operations can be done on relations. The resulting sets are ordered pairs and hence are relations.

If R and S denote two relations, *the intersection of R and S* denoted by $R \cap S$, is defined by

conjunction

$$\boxed{a\,(R \cap S)b = a \text{ R } b \wedge a \text{ S } b}$$

and *the union of R and S*, denoted by $R \cup S$, is defined by $a\,(R \cup S)\,b = a \text{ R } b \vee a \text{ S } b$.

disjunction *The difference of R and S*, denoted by $R - S$, is defined by $a\,(R - S)\,b = a \text{ R } b \wedge a\ \cancel{S}\ b$.

The *complement of R*, denoted by R' or $\sim R$ is defined by $a(R')b = a\ \cancel{R}\ b$.

For example, let $A = \{x, y, z\}$, $B = \{1, 2, 3\}$, $C = \{x, y\}$ and $D = \{2, 3\}$. Let R be a relation from A to B defined by $R = \{(x, 1), (x, 2), (y, 3)\}$ and let S be a relation from C to D defined by $S = \{(x, 2), (y, 3)\}$.

Then $\quad R \cap S = \{(x, 2), (y, 3)\}$ and $R \cup S = R$.

$$R - S = \{(x, 1)\}$$

$$R' = \{(x, 3), (y, 1), (y, 2), (z, 1), (z, 2), (z, 3)\}$$

COMPOSITION OF RELATIONS

If R is a relation from set A to set B and S is a relation from set B to set C, viz., R is a subset of $A \times B$ and S is a subset of $B \times C$, then *the composition of R and S*, denoted by $R \bullet S$, [some authors use the notation $S \bullet R$ instead of $R \bullet S$] is defined by

$a(R \bullet S)\, c$, if for some $b \in B$, we have $a \text{ R } b$ and $b \text{ R } c$.

viz., $R \bullet S = \{(a, c) |$ there exists some $b \in B$ for which $(a, b) \in R$ and $(b, c) \in S\}$

Note 1. For the relation $R \bullet S$, the domain is a subset of A and the range is a subset of C.

2. $R \bullet S$ is empty, if the intersection of the range of R and the domain of S is empty.
3. If R is a relation on a set A, then $R \bullet R$, the composition of R with itself is always defined and sometimes denoted as R^2.

For example, let $R = \{(1, 1), (1, 3), (3, 2), (3, 4), (4, 2)\}$ and $S = \{(2, 1), (3, 3), (3, 4), (4, 1)\}$.

Any member (ordered pair) of $R \bullet S$ can be obtained only if the second element in the ordered pair of R agrees with the first element in the ordered pair of S.

Thus $(1, 1)$ cannot combine with any member of S.

$(1, 3)$ of R can combine with $(3, 3)$ and $(3, 4)$ of S producing the members $(1, 3)$ and $(1, 4)$ respectively of $R \bullet S$. Similarly the other members of $R \bullet S$ are obtained.

Thus $R \bullet S = \{(1, 3), (1, 4), (3, 1), (4, 1)\}$

$$(R \bullet S) \bullet R = \{(1, 2), (1, 4), (3, 1), (3, 3), (4, 1), (4, 3)\}$$
$$R \bullet (S \bullet R) = \{(1, 2), (1, 4), (3, 1), (3, 3), (4, 1), (4, 3)\}$$
$$R^3 = R \bullet R \bullet R = (R \bullet R) \bullet R = R \bullet (R \bullet R)$$
$$= \{(1, 1), (1, 3), (1, 2), (1, 4)\}$$

PROPERTIES OF RELATIONS

(i) A relation R on a set A is said to be *reflexive*, if $a \text{ R } a$ for every $a \in A$, viz., if $(a, a) \in R$ for every $a \in A$.
For example, if R is the relation on $A = \{1, 2, 3\}$ defined by $(a, b) \in R$ if $a \le b$, where $a, b \in A$, then $R = \{(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)\}$.
Now R is reflexive, since each of the elements of A is related to itself, as $(1, 1)$, $(2, 2)$ and $(3, 3)$ are members in R.

Note A relation R on a set A is irreflexive, if, for every $a \in A$, $(a, a) \notin R$. viz., if there is no $a \in A$ such that $a \text{ R } a$.
For example, R, $\{(1, 2), (2, 3), (1, 3)\}$ in the above example is irreflexive.

(ii) A relation R on a set A is said to be *symmetric*, if whenever $a \text{ R } b$ then $b \text{ R } a$, viz., if whenever $(a, b) \in R$ then (b, a) also $\in R$.
Thus a relation R on A is not symmetric if there exist $a, b \in A$ such that $(a, b) \in R$, but $(b, a) \notin R$.

(iii) A relation R on a set A is said to be *antisymmetric*, whenever (a, b) and $(b, a) \in R$ then $a = b$. If there exist $a, b \in A$ such that (a, b) and $(b, a) \in R$, but $a \ne b$, then R is not antisymmetric.
For example, the relation of perpendicularity on a set of lines in the plane is symmetric, since if a line a is perpendicular to the line b, then b is perpendicular to a.
The relation $\le$ on the set Z of integers is not symmetric, since, for example, $4 \le 5$, but $5 \not\le 4$.
The relation of divisibility on N is antisymmetric, since whenever m is divisible by n and n is divisible by m then $m = n$.

Note Symmetry and antisymmetry are not negative of each other. For example, the relation $R = \{(1, 3), (3, 1), (2, 3)\}$ is neither symmetric nor antisymmetric, whereas the relation $S = \{(1, 1), (2, 2)\}$ is both symmetric and antisymmetric.

(iv) A relation R on a set A is said to be *transitive*, if whenever $a \text{ R } b$ and $b \text{ R } c$ then $a \text{ R } c$. viz., if whenever (a, b) and $(b, c) \in R$ then $(a, c) \in R$.
Thus if there exist $a, b, c \in A$ such that (a, b) and $(b, c) \in R$ but $(a, c) \notin R$, then R is not transitive.
For example, the relation of set inclusion on a collection of sets is transitive, since if $A \subseteq B$ and $B \subseteq C$, $A \subseteq C$.

(v) A relation R on a set A is called an *equivalence relation, if R is reflexive,*

Theorem

1. a R a, for every a ∈ A.
2. If a R b, then b R a
3. If a R b and b R c, then a R c

For example, the relation of similarity with respect to a set of triangles T is an equivalence relation, since if T_1, T_2, T_3 are elements of the set T, then

$T_1 \parallel T_1$, i.e., T_1 R T_1 for every $T_1 \in T$,
$T_1 \parallel T_2$ implies $T_2 \parallel T_1$ and
$T_1 \parallel T_2$ and $T_2 \parallel T_3$ simplify $T_1 \parallel T_3$

viz., the relation of similarity of triangles is reflexive, symmetric and transitive.

(vi) A relation R on a set A is called a *partial ordering* or *partial order relation*, if R is reflexive, antisymmetric and transitive.

viz., R is a partial order relation on A if it has the following three properties:
(a) a R a, for every $a \in A$
(b) a R b and b R $a \Rightarrow a = b$
(c) a R b and b R $c \Rightarrow a$ R c

A set A together with a partial order relation R is called *a partially ordered set* or *poset*. For example, the greater than or equal to (≥) relation is a partial ordering on the set of integers Z, since
(a) $a \geq a$ for every integer a, i.e. ≥ is reflexive
(b) $a \geq b$ and $b \geq a \Rightarrow a = b$, i.e. ≥ is antisymmetric
(c) $a \geq b$ and $b \geq c \Rightarrow a \geq c$, i.e. ≥ is transitive

Thus $(Z, \geq)$ is a poset.

EQUIVALENCE CLASSES

Definition

If R is an equivalence relation on a set A, the set of all elements of A that are related to an element a of A is called the *equivalence class of a* and denoted by $[a]_R$.

When there is no ambiguity regarding the relation, viz., when we deal with only one relation, the equivalence class of a is denoted by just $[a]$.

In other words, the equivalence class of a under the relation R is defined as

$$[a] = \{x|(a, x) \in R\}$$

Any element $b \in [a]$ is called *a representative* of the equivalence class $[a]$.

The collection of all equivalence classes of elements of A under an equivalence relation R is denoted by A/R and is called the *quotient set* of A by R.

viz. $$A/R = \{[a]|a \in A\}$$

For example, the relation R on the set $A = \{1, 2, 3\}$ defined by $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}$ is an equivalence relation, since R is reflexive symmetric and transitive.

Now $[1] = \{1, 2\}$, $[2] = \{1, 2\}$ and $[3] = \{3\}$

Thus [1], [2] and [3] are the equivalence classes of A under R and hence form A/R.

Theorem

If R is an equivalence relation on non-empty set A and if a and $b \in A$ are arbitrary, then
(i) $a \in [a]$, for every $a \in A$
(ii) $[a] = [b]$, if and only if $(a, b) \in R$
(iii) If $[a] \cap [b] \neq \phi$, then $[a] = [b]$

Proof:
(i) Since R is reflexive, $(a, a) \in R$ for every $a \in A$.
Hence $a \in [a]$. (1)
(ii) Let us assume that $(a, b) \in R$ or a R b (2)
Let $x \in [b]$. Then $(b, x) \in R$ or b R x
From (1) and (2), it follows that a R x or $(a, x) \in R$ (∵ R is transitive)
∴ $x \in [a]$ (3)
Thus $x \in [b] \Rightarrow x \in [a]$ ∴ $[b] \subseteq [a]$ (4)
Let $y \in [a]$. Then a R y (5)
From (1), we have b R a, since R is symmetric.
From (5) and (4), we get b R y, since R is transitive.
∴ $y \in [b]$ (6)
Thus $y \in [a] \Rightarrow y \in [b]$ ∴ $[a] \subseteq [b]$
From (3) and (6), we get $[a] = [b]$
Conversely, let $[a] = [b]$
Now $b \in [b]$ by (i)
i.e., $b \in [a]$ ∴ $(a, b) \in R$
(iii) Since $[a] \cap [b] \neq \phi$, there exists an element $x \in A$ such that $x \in [a] \cap [b]$
Hence $x \in [a]$ and $x \in [b]$
i.e., x R a and x R b
or a R x and x R b
∴ a R b, since R is transitive
Hence, by (ii), $[a] = [b]$
Equivalently, if $[a] \neq [b]$, then $[a] \cap [b] = \phi$.

Note From (ii) and (iii) of the above theorem, it follows that the equivalence classes of two arbitrary elements under R are identical or disjoint.)

PARTITION OF A SET

Definition

If S is a non empty set, a collection of disjoint non empty subsets of S whose union is S is called *a partition* of S. In other words, the collection of subsets A_i is a partition of S if and only if
(i) $A_i \neq \phi$, for each i
(ii) $A_i \cap A_j = \phi$, for $i \neq j$ and
(iii) $\bigcup_i A_i = S$, where $\bigcup_i A_i$ represents the union of the subsets A_i for all i.

Note The subsets in a partition are also called *blocks* of the partition.
For example, if $S = \{1, 2, 3, 4, 5, 6\}$
(i) $\{\{1, 3, 5\}, \{2, 4\}\}$ is not a partition since the union of the subsets is not S, as the element 6 is missing.

(ii) $[\{1, 3\}, \{3, 5\}, \{2, 4, 6\}]$ is not a partition, since $\{1, 3\}$ and $\{3, 5\}$ are not disjoint.

(iii) $[\{1, 2, 3\}, \{4, 5\}, \{6\}]$ is a partition.

PARTITIONING OF A SET INDUCED BY AN EQUIVALENCE RELATION

Let R be an equivalence relation of a non-empty set A.

Let $A_1, A_2, \ldots, A_k$ be the distinct equivalence classes of A under R. For every $a \in A_i$, $a \in [a]_R$, by the above theorem.

$$\therefore \quad A_i = [a]_R$$

$$\therefore \quad \bigcup_{a \in A_i} [a]_R = \bigcup_i A_i = A$$

Also by the above theorem, when $[a]_R \neq [b]_R$, then

$$[a]_R \cap [b]_R = \phi. \text{ viz., } A_i \cap A_j = \phi, \text{ if } [a]_R = A_i \text{ and } [b]_R = A_j$$

$\therefore$ The equivalence classes of A form a partition of A.

In other words, the quotient set A/R is a partition of A.

For example, let $A \equiv$ {blue, brown, green, orange, pink, red, white, yellow} and R be the equivalence relation of A defined by "has the same number of letters", then

$$A/R = [\{\text{red}\}, \{\text{blue, pink}\}, \{\text{brown, green, white}\}, \{\text{orange, yellow}\}]$$

The equivalence classes contained in A/R form a partition of A.

MATRIX REPRESENTATION OF A RELATION

If R is a relation from the set $A = \{a_1, a_2, \ldots, a_m\}$ to the set $B = \{b_1, b_2, \ldots, b_n\}$, where the elements of A and B are assumed to be in a specific order, the relation R can be represented by the matrix

$$M_R = [m_{ij}], \text{ where}$$

$$m_{ij} = \begin{cases} 1, & \text{if } (a_i, b_j) \in R \\ 0, & \text{if } (a_i, b_j) \notin R. \end{cases}$$

In other words, the zero-one matrix M_R has a 1 in its $(i-j)$th position when a_i is related to b_j and a 0 in this position when a_i is not related by b_j.

For example, if $A = [a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4\}$ and $R = \{(a_1, b_2), (a_2, b_1), (a_2, b_3), (a_2, b_4), (a_3, b_2), (a_3, b_4)\}$, then the matrix of R is given by

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Conversely, if R is the relation on $A = \{1, 3, 4\}$ represented by

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

then $R = \{(1, 1), (1, 3), (3, 3), (4, 4)\}$, ... of A is related to the jth element of A.

1. If R and S are relations on a set A, represented by M_R and M_S respectively, then the matrix representing $R \cup S$ is the *join* of M_R and M_S obtained by putting 1 in the positions where either M_R or M_S has a 1 and denoted by $M_R \vee M_S$ i.e., $M_{R \cup S} = M_R \vee M_S$.
2. The matrix representing $R \cap S$ is the *meet* of M_R and M_S obtained by putting 1 in the positions where both M_R and M_S have a 1 and denoted by $M_R \wedge M_S$ i.e., $M_{R \cap S} = M_R \wedge M_S$.

Note The operations 'join' and 'meet', denoted by $\vee$ and $\wedge$ respectively are Boolean operations, which will be discussed later in the topic on Boolean Algebra.

For example, if R and S are relations on a set A represented by the matrices

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } M_S = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \text{ respectively,}$$

then $M_{R \cup S} = M_R \vee M_S$

$$= \begin{bmatrix} 1 \vee 1 & 0 \vee 0 & 1 \vee 1 \\ 0 \vee 1 & 1 \vee 0 & 1 \vee 0 \\ 1 \vee 0 & 0 \vee 1 & 0 \vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

and $M_{R \cap S} = M_R \wedge M_S$

$$= \begin{bmatrix} 1 \wedge 1 & 0 \wedge 0 & 1 \wedge 1 \\ 0 \wedge 1 & 1 \wedge 0 & 1 \wedge 0 \\ 1 \wedge 0 & 0 \wedge 1 & 0 \wedge 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

3. If R is a relation from a set A to a set B represented by M_R, then the matrix representing R^{-1} (the inverse of R) is M_R^T, the transpose of M_R.

For example, if $A = \{2, 4, 6, 8\}$ and $B = \{3, 5, 7\}$ and if, R is defined by $\{(2, 3), (2, 5), (4, 5), (4, 7), (6, 3), (6, 7), (8, 7)\}$, then

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

R^{-1} is defined by $\{(3, 2), (5, 2), (5, 4), (7, 4), (3, 6), (7, 6), (7, 8)\}$

Now $$M_{R^{-1}} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} = M_R^T.$$

4. If R is a relation from A to B and S is a relation from B to C, then the composition of the relations R and S (if defined), viz., $R \bullet S$ is represented by the Boolean product of the matrices M_R and M_S, denoted by $M_R \bullet M_S$.

Note The Boolean product of two matrices is obtained in a way similar to the ordinary product, but with multiplication replaced by the Boolean operation $\wedge$ and with addition replaced by the Boolean operation $\vee$.

For example, the matrix representing $R \bullet S$

where $M_R = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ and $M_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$$M_{R \bullet S} = M_R \odot M_S = \begin{bmatrix} 0 \vee 0 \vee 0 & 0 \vee 1 \vee 0 & 0 \vee 1 \vee 0 \\ 0 \vee 0 \vee 1 & 1 \vee 1 \vee 1 & 0 \vee 1 \vee 1 \\ 0 \vee 0 \vee 0 & 1 \vee 0 \vee 0 & 0 \vee 0 \vee 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

5. Since the relation R on the set $A = \{a_1, a_2, \ldots, a_n\}$ is reflexive if and only if $(a_i, a_i) \in R$ for $i = 1, 2, \ldots, n$, $m_{ii} = 1$ for $i = 1, 2, \ldots, n$. In other words, R is reflexive if all the elements in the principal diagonal of M_R are equal to 1.
6. Since the relation R on the set $A = \{a_2, a_2, \ldots, a_n\}$ is symmetric if and only if $(a_j, a_i) \in R$ whenever $(a_i, a_j) \in R$, we will have $m_{ji} = 1$ whenever $m_{ij} = 1$ (or equivalently $m_{ji} = 0$ whenever $m_{ij} = 0$). In other words, R is symmetric if and only if $m_{ij} = m_{ji}$, for all pairs of integers i and j $(i, j = 1, 2, \ldots, n)$. This means that R is symmetric, if $M_R = (M_R)^T$, viz., M_R is a symmetric matrix.

Note The matrix of an antisymmetric relation has the property that if $m_{ij} = 1$ $(i \neq j)$, then $m_{ji} = 0$.

7. There is no simple way to test whether a relation R on a set A is transitive by examining the matrix M_R. However, we can easily verify that a relation R is transitive if and only if $R^n \subseteq R$ for $n \geq 1$.

REPRESENTATION OF RELATIONS BY GRAPHS

Let R be a relation on a set A. To represent R graphically, each element of A is represented by a point. These points are called *nodes* or *vertices*. Whenever the element a is related to the element b, an arc is drawn from the point 'a' to the point 'b'. These arcs are called *arcs* or *edges*. The arcs start from the first element of the related pair and go to the second element. The direction is indicated by an arrow. The resulting diagram is called the *directed graph* or

The edge of the form (a, a), represented by using an arc from the vertex a back to itself, is called a *loop*.

For example, if $A = \{2, 3, 4, 6\}$ and R is defined by $a\,R\,b$ if a divides b, then

$R = (2, 2), (2, 4), (2, 6), (3, 3), (3, 6), (4, 4), (6, 6)\}$

The digraph representing the relation R is given in Fig. 2.14.

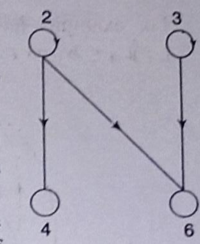


Fig. 2.14

Note The digraph of R^{-1}, the inverse of R, has exactly the same edges of the digraph of R, but the directions of the edges are reversed.

The digraph representing a relation can be used to determine whether the relation has the standard properties explained as follows:

(i) A relation R is reflexive if and only if there is a loop at every vertex of the digraph of the relation R, so that every ordered pair of the form (a, a) occurs in R. If no vertex has a loop, then R is irreflexive.

(ii) A relation R is symmetric if and only if for every edge between distinct vertices in its digraph there is an edge in the opposite direction, so that (b, a) is in R whenever (a, b) is in R.

(iii) A relation R is antisymmetric if and only if there are never two edges in opposite directions between distinct vertices.

(iv) A relation R is transitive if and only if whenever there is an edge from a vertex a to a vertex b and from the vertex b to a vertex c, there is an edge from a to c.

HASSE DIAGRAMS FOR PARTIAL ORDERINGS

The simplified form of the digraph of a partial ordering on a finite set that contains sufficient information about the partial ordering is called a *Hasse diagram*, named after the twentieth-century mathematician Helmut Haasse.

The simplification of the digraph as a Hasse diagram is achieved in three ways:

(i) Since the partial ordering is a reflexive relation, its digraph has loops at all vertices. We need not show these loops since they must be present.

(ii) Since the partial ordering is transitive, we need not show those edges that must be present due to transitivity. For example, if (1, 2) and (2, 3) are edges in the digraph of a partial ordering, (1, 3) will also be an edge due to transitivity. This edge (1, 3) need not be shown in the corresponding Hasse diagram.

(iii) If we assume that all edges are directed upward, we need not show the directions of the edges.

Thus the Hasse diagram representing a partial ordering can be obtained from its digraph, by removing all the loops, by removing all edges that are

For example, let us construct the Hasse diagram for the partial ordering $\{(a, b) | a \leq b\}$ on the set $\{1, 2, 3, 4\}$ starting from its digraph. (Fig. 2.15)

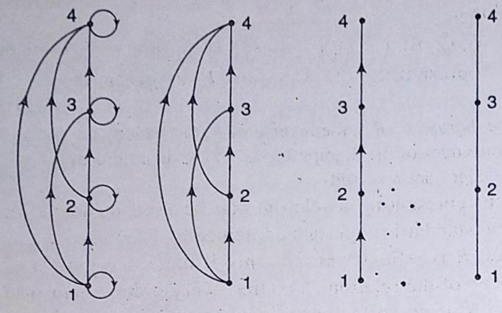


Fig. 2.15

TERMINOLOGY RELATED TO POSETS

We have already defined *poset* as a set S together with a partial order relation R. In a poset the notation $a \leq b$ (or equivalently $a \preccurlyeq b$) denotes that $(a, b) \in R$. $a \leq b$ is read as "a precedes b" or "b succeeds a".

ULb

Definitions

When $\{P, \leq\}$ is a poset, an element $a \in P$ is called a *maximal member* of P, if there is no element $b \in P$ such that $a < b$ (viz., a strictly precedes b).

b<a

Similarly, an element $a \in P$ is called a *minimal member* of P, if there is no element $b \in P$ such that $b < a$.

If there exists an element $a \in P$ such that $b \leq a$ for all $b \in P$, then a is called the *greatest member* of the poset $\{P, \leq\}$.

Similarly if there exists an element $a \in P$ such that $a \leq b$ for all $b \in P$, then a is called the *least member* of the poset $\{P, \leq\}$.

Note 1. The maximal, minimal, the greatest and least members of a poset can be easily identified using the Hasse diagram of the poset. They are the top and bottom elements in the diagram.

2. A poset can have more than one maximal member and more than one minimal member, whereas the greatest and least members, when they exist, are unique.

For example, let us consider the Hasse diagrams of four posets given in Fig. 2.16.

For the poset with Hasse diagram 2.16(a), a and b are minimal elements and d and e are maximal elements, but the poset has neither the greatest nor the least element.

For the poset with Hasse diagram (b), a and b are minimal elements and d is the greatest element (also the only maximal element). There is no least element.

For the poset with Hasse diagram (c), a is the least element (also the only minimal element) and c & d are maximal element. There is no greatest element

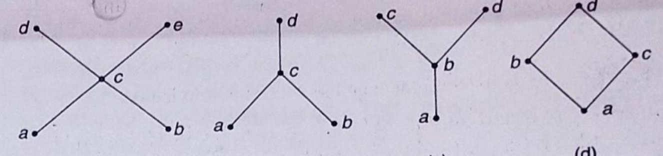


Fig. 2.16

For the poset with Hasse diagram (d), a is the least element and d is the greatest element.

Definitions

When A is a subset of a poset $\{P, \leq\}$ and if u is an element of P such that $a \leq u$ for all elements $a \in A$, then u is called an *upper bound* of A. Similarly if l is an element of P such that $l \leq a$ for all elements $a \in A$, then l is called a *lower bound* of A.

Note The upper and lower bounds of a subset of a poset are not necessarily unique.

The element x is called the *least upper bound* (LUB) or *supremum* of the subset A of a poset $\{P, \leq\}$, if x is an upper bound that is less than every other upper bound of A.

Similarly the element y is called the *greatest lower bound* (GLB) or *infimum* of the subset A of a poset $\{P, \leq\}$, if y is a lower bound that is greater than every other lower bound of A.

Note The LUB and GLB of a subset of a poset, if they exist, are unique.

For example, let us consider the poset with the Hasse diagram given in Fig. 2.17.

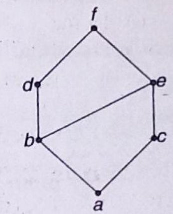


Fig. 2.17

The upper bounds of the subset $\{a, b, c\}$ are e and f. [*Note:* d is not an upper bound, since c is not related to d] and LUB of $\{a, b, c\}$ is e.

The lower bounds of the subset $\{d, e\}$ are a and b and GLB of $\{d, e\}$ is b.

Note c is not a lower bound, since c is not related to d.

WORKED EXAMPLES 2(B)

Example 2.1 List the ordered pairs in the relation R from $A = \{0, 1, 2, 3, 4\}$ to $B = \{0, 1, 2, 3\}$ where $(a, b) \in R$ if and only if (i) $a = b$, (ii) $a + b = 4$, (iii) $a > b$, (iv) $a | b$ (viz., a divides b), (v) $gcd(a, b) = 1$ and (vi) $lcm\ (a, b) = 2$.

(i) Since $a \in A$ and $b \in B$ and $a\ R\ b$ when $a = b$, $R = \{(0, 0), (1, 1), (2, 2), (3, 3)\}$.

(ii) Since $a\ R\ b$ if and only if $a + b = 4$, $R = \{(1, 3), (2, 2), (3, 1), (4, 0)\}$.

(iii) Since $a\ R\ b$, if and only if $a > b$, $R = \{(1, 0), (2, 0), (2, 1), (3, 0), (3, 1),$ (3, 2), (4, 0), (4, 1), (4, 2), (4, 3) }.

(iv) Since a R b, if and only if $a|b$, $R = \{(1, 0), (1, 1), (1, 2), (1, 3), (2, 0), (2, 2), (3, 0), (3, 3), (4, 0)\}$.

Note $\frac{0}{0}$ is indeterminate and so 0 does not divide 0.

(v) Since a R b, if and only if $gcd\ (a, b) = 1$, $R = \{(0, 1), (1, 0), (1, 1), (1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2), (4, 1), (4, 3)\}$.

(vi) Since a R b, if and only if $lcm\ (a, b) = 2$, $R = \{(1, 2), (2, 1), (2, 2)\}$.

Example 2.2 The relation R on the set $A = \{1, 2, 3, 4, 5\}$ is defined by the rule $(a, b) \in R$, if 3 divides $a - b$.

(i) List the elements of R and R^{-1},
(ii) Find the domain and range of R.
(iii) Find the domain and range of R^{-1}.
(iv) List the elements of the complement of R.

The Cartesian product $A \times A$ consists of $\{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 1), (2, 2), \ldots, (2, 5), (3, 1), (3, 2) \ldots, (3, 5), (4, 1), (4, 2), \ldots, (4, 5), (5, 1), (5, 2), \ldots, (5, 5)\}$

(i) Since $(a, b) \in R$, if 3 divides $(a - b)$, $R = \{(1, 1), (1, 4), (2, 2), (2, 5), (3, 3), (4, 1), (4, 4), (5, 2), (5, 5)\}$

R^{-1} (the inverse of R) $= \{(1, 1), (4, 1), (2, 2), (5, 2), (3, 3), (1, 4), (4, 4), (2, 5), (5, 5)\}$

We note that $R^{-1} = R$

(ii) Domain of R = Range of $R = \{1, 2, 3, 4, 5\}$
(iii) Domain of R^{-1} = Range of $R^{-1} = \{1, 2, 3, 4, 5\}$
(iv) R' (the complement of R) = the elements of $A \times A$, that are not in $R = \{(1, 2), (1, 3), (1, 5), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (3, 4), (3, 5), (4, 2), (4, 3), (4, 5), (5, 1), (5, 3), (5, 4)\}$

Example 2.3 If $R = \{(1, 2), (2, 4), (3, 3)\}$ and $S = \{(1, 3), (2, 4), (4, 2)\}$, find (i) $R \cup S$, (ii) $R \cap S$, (iii) $R - S$, (iv) $S - R$, (v) $R \oplus S$. Also verify that dom $(R \cup S)$ = dom$(R) \cup$ dom (S) and range $(R \cap S) \subseteq$ range $(R) \cap$ range (S).

(i) $R \cup S = \{(1, 2), (1, 3), (2, 4), (3, 3), (4, 2)\}$
(ii) $R \cap S = \{(2, 4)\}$
(iii) $R - S = \{(1, 2), (3, 3)\}$
(iv) $S - R = \{(1, 3), (4, 2)\}$
(v) $R \oplus S = (R \cup S) - (R \cap S)$

$= \{(1, 2), (1, 3), (3, 3), (4, 2)\}$

dom $(R) = \{1, 2, 3\}$; dom $(S) = \{1, 2, 4\}$

Now dom $(R) \cup$ dom $(S) = \{1, 2, 3, 4\}$

= domain $(R \cup S)$

Range (R) $= \{2, 3, 4\}$; Range $(S) = \{2, 3, 4\}$

Range $(R \cap S) = \{4\}$

Clearly $\{4\} \subseteq \{2, 3, 4\} \cap \{2, 3, 4\}$

relations respectively on the set of integers. That is
$R = \{(a, b) \mid a \equiv b \pmod 3\}$ and $S = \{(a, b) \mid a \equiv b \pmod 4\}$

Find (i) $R \cup S$, (ii) $R \cap S$, (iii) $R - S$, (iv) $S - R$, (v) $R \oplus S$.

$R = \{(a, b)$, where $(a - b)$ is a multiple of 3

i.e. $a - b = \ldots, -9, -6, -3, 0, 3, 6, 9, \ldots$

i.e. $a - b = \{\ldots, -9, 3, 15, 27, 39, \ldots\}, \{\ldots, -6, 6, 18, 30, \ldots\}, \{\ldots, -3, 9, 21, 33, \ldots\}, \{\ldots, 0, 12, 24, 36, \ldots\}$

i.e. $a - b = 3 \pmod{12}$ or $6 \pmod{12}$ or $9 \pmod{12}$ or $0 \pmod{12}$ (1)

$S = \{(a, b)\}$, where $(a - b)$ is a multiple of 4

i.e. $a - b = \ldots, -12, -8, -4, 0, 4, 8, 12, \ldots$

i.e. $a - b = \{\ldots, -8, 4, 16, 28, \ldots\}, \{\ldots, -16, -4, 8, 20, \ldots\}, \{\ldots, -24, -12, 0, 12, 24, \ldots\}$

i.e. $a - b = 4 \pmod{12}$ or $8 \pmod{12}$ or $0 \pmod{12}$ (2)

$\therefore$ $R \cup S = \{(a, b) | a - b = 0 \pmod{12}, 3 \pmod{12}, 4 \pmod{12}, 6 \pmod{12}, 8 \pmod{12}$ or $9 \pmod{12}\}$

$R \cap S = \{(a, b) | a - b = 0 \pmod{12}$, from (1) and (2)

$R - S = \{(a, b) | a - b = 3 \pmod{12}, 6 \pmod{12}$ or $9 \pmod{12}\}$

$S - R = \{(a, b) | a - b = 4 \pmod{12}$ or $8 \pmod{12}\}$

$R \oplus S = \{(a, b) | a - b = 3 \pmod{12}, 4 \pmod{12}, 6 \pmod{12}, 8 \pmod{12}$ or $9 \pmod{12}\}$.

Example 2.5 If the relations $R_1, R_2, \ldots, R_6$ are defined on the set of real numbers as given below,

$R_1 = \{(a, b) | a > b\}$, $R_2 = \{(a, b) | a \geq b\}$,
$R_3 = \{(a, b) | a < b\}$, $R_4 = \{(a, b) | a \leq b\}$,
$R_5 = \{(a, b) | a = b\}$, $R_6 = \{(a, b) | a \neq b\}$,

find the following composite relations:

$R_1 \bullet R_2$, $R_2 \bullet R_2$, $R_1 \bullet R_4$, $R_3 \bullet R_5$, $R_5 \bullet R_3$, $R_6 \bullet R_3$, $R_6 \bullet R_4$ and $R_6 \bullet R_6$

(i) $R_1 \bullet R_2 = R_1$. For example, let $(5, 3) \in R_1$ and let $(3, 1), (3, 2), (3, 3) \in R_2$

Then $R_1 \bullet R_2$ consists of $(5, 1), (5, 2), (5, 3)$ which belong to R_1

(ii) $R_2 \bullet R_2 = R_2$. For example, let $(5, 5), (5, 3), (5, 2) \in R_2$

Then $R_2 \bullet R_2 = \{(5, 5), (5, 3), (5, 2)\} = R_2$

(iii) $R_1 \bullet R_4 = R^2$ (the entire 2 dimensional vector space). For example, let $R_1 = \{(5, 4), (5, 3)\}$ and $R_4 = \{(4, 4), (4, 6), (3,3), (3, 5)\}$

Then $R_1 \bullet R_4 = \{(5, 4), (5, 6), (5, 3), (5, 5)\}$

Thus $R_1 \bullet R_4 = \{(a, b) | a > b, a = b$ and $a < b\}$

(iv) $R_3 \bullet R_5 = R_3$. For example, let $R_3 = \{(3, 4), (2, 4), (2, 5)\}$ and $R_5 = \{(3, 3), (4, 4), (5, 5)\}$

Then $R_3 \bullet R_5 = \{(3, 4), (2, 4), (2, 5)\} = R_3$

(v) $R_5 \bullet R_3 = R_3$. For example, let $R_5 = \{(3, 3), (4, 4), (5, 5)\}$ and $R_3 = \{(3, 4), (4, 6), (5, 7)\}$

Then $R_5 \bullet R_3 = \{(3, 4), (4, 6), (5, 7)\} = R_3$

(vii) $R_6 \bullet R_4 = R^2$. For example, let $R_6 = \{(1, 2), (4, 3), (5, 2)\}$ and $R_4 = \{(2, 3), (2, 5), (3, 3)\}$

Then $R_6 \bullet R_4 = \{(1, 3), (1, 5), (4, 3), (5, 3), (5, 5)\} \rightarrow R^2$

(viii) $R_6 \bullet R_6 = R^2$. For example, let $R_6 = \{(1, 2), (2, 1), (2, 3), (3, 2), (3, 4)\}$

Then $R_6 \bullet R_6 = \{(1, 1), (1, 3), (2, 2), (2, 4), (3, 1), (3, 3)\} \rightarrow R^2$

Example 2.6 Determine whether the relation R on the set of all integers is reflexive, symmetric, antisymmetric and/or transitive, where a R b if and only if (i) $a \neq b$, (ii) $ab \geq 0$, (iii) $ab \geq 1$, (iv) a is a multiple of b, (v) $a \equiv b$ (mod 7), (vi) $|a - b| = 1$, (vii) $a = b^2$, (viii) $a \geq b^2$.

(i) '$a \neq a$' is not true. Hence R is not reflexive

$a \neq b \Rightarrow b \neq a$. $\therefore$ R is symmetric

$a \neq b$ and $b \neq c$ does not necessarily imply that $a \neq c$. $\therefore$ R is not transitive

Hence R is symmetric only.

(ii) $a^2 \geq 0$. $\therefore$ R is reflexive.

$ab \geq 0 \Rightarrow ba \geq 0$. $\therefore$ R is symmetric.

Consider (2, 0) and (0, –3), that belong to R. But $(2, -3) \notin R$, as $2(-3) < 0$. $\therefore$ R is not transitive.

$\therefore$ R is reflexive, symmetric and not transitive.

(iii) '$a^2 \geq 1$' need not be true, since a may be zero. $\therefore$ R is not reflexive.

$ab \geq 1 \Rightarrow ba \geq 1$ $\therefore$ R is symmetric.

$ab \geq 1$ and $bc \geq 1 \Rightarrow$ all of $a, b, c > 0$ or < 0

If all of $a, b, c > 0$, least a = least b = least c = 1

$\therefore$ $ac \geq 1$

If all of $a, b, c < 0$, greatest a = greatest b = greatest $c = -1$

$\therefore$ $ac \geq 1$. Hence R is transitive.

$\therefore$ R is symmetric and transitive.

(iv) a is a multiple of a. $\therefore$ R is reflexive. If a is a multiple of b, b is not a multiple of a in general. But if a is a multiple of b and b is a multiple of a, then $a = b$.

$\therefore$ R is antisymmetric.

When a is a multiple of b and b is a multiple of c, then a is a multiple of c.

$\therefore$ R is transitive.

Thus R is reflexive, antisymmetric and transitive.

(v) $(a - a)$ is a multiple of 7 $\therefore$ R is reflexive. When $(a - b)$ is a multiple of 7, $(b - a)$ is also a multiple of 7. $\therefore$ R is symmetric.

When $(a - b)$ and $(b - c)$ are multiples of 7, $(a - b) + (b - c) = (a - c)$ is also a multiple of 7.

$\therefore$ R is transitive.

Hence R is reflexive, symmetric and transitive.

(vi) $|a - a| \neq 1$. $\therefore$ R is not reflexive

$|a - b| = 1 \Rightarrow |b - a| = 1$. $\therefore$ R is symmetric.

$|a - b| = 1 \Rightarrow a - b = 1$ or -1 (1)

$|b - c| = 1 \Rightarrow b - c = 1$ or -1 (2)

(1) + (2) gives $a - c = \pm 2$ or 0

i.e. $|a - c| = 2$ or 0

i.e. $|a - c| \neq 1$

Hence R is symmetric only.

(vii) '$a = a^2$' is not true for all integers.

$\therefore$ R is not reflexive.

$a = b^2$ and $b = a^2$, for $a = b = 0$ or 1

$\therefore$ R is antisymmetric.

$a = b^2$ and $b = c^2$ does not simply $a = c^2$

$\therefore$ R is not transitive

Hence R is antisymmetric only.

(viii) '$a \geq a^2$' is not true for all integers.

$\therefore$ R is not reflexive.

$a \geq b^2$ and $b \geq a^2$ imply that $a = b$

$\therefore$ R is antisymmetric

When $a \geq b^2$ and $b \geq c^2$, $a \geq c^2$

$\therefore$ R is transitive

Hence R is antisymmetric and transitive.

Example 2.7 Which of the following relations on {0, 1, 2, 3} are equivalence relations? Find the properties of an equivalence relation that the others lack.

(a) $R_1 = \{(0, 0), (1, 1), (2, 2), (3, 3)\}$

(b) $R_2 = \{(0, 0), (0, 2), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3)$

(c) $R_3 = \{(0, 0), (1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}$

(d) $R_4 = \{(0, 0), (1, 1), (1, 3), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$

(e) $R_5 = \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (3, 3)\}$

(a) R_1 is reflexive, symmetric and transitive.

$\therefore$ R_1 is an equivalence relation.

(b) R_2 is reflexive

R_2 is symmetric, but not transitive, since (3, 2) and $(2, 0) \in R_2$, but $(3, 0) \notin R_2$

$\therefore$ R_2 is not an equivalence relation.

(c) R_3 is reflexive, symmetric and transitive. $\therefore$ R_3 is an equivalence relation.

(d) R_4 is reflexive and symmetric, but not transitive, since (1, 3) and $(3, 2) \in R_4$, but $(1, 2) \notin R_4$. $\therefore$ R_4 is not an equivalence relation.

(e) R_5 is reflexive, but not symmetric since $(1, 2) \in R$, but $(2, 1) \notin R$. Also R_5 is not transitive, since (2, 0) and $(0, 1) \in R$, but $(2, 1) \notin R$.

$\therefore$ R_5 is not an equivalence relation.

Example 2.8 Show that the following relations are equivalence relations:

(i) R_1 is the relation on the set of integers such that aR_1b if and only if $a = b$ or $a = -b$.

(ii) R_2 is the relation on the set of integers such that aR_2b if and only if $a \equiv b$ (mod m), where m is a positive integer > 1.

(iii) R_3 is the relation on the set of real numbers such that aR_3b if and only if $(a-b)$ is an integer

(i) $a = a$ or $a = -a$, which is true for all integers.
$\therefore$ R_1 is reflexive.
When $a = b$ or $a = -b$, $b = a$ or $b = -a$.
$\therefore$ R_1 is symmetric
When $a, b, c \geq 0$, $a = b$ and $b = c$, if aR_1b and bR_1c
$\therefore$ $a = c$, i.e., aRc
Similarly when $a \geq 0, b \leq 0, c \leq 0$, we have $a = -b$ and $b = c$, if aR_1b and bR_1c.
$\therefore$ $a = -c$. i.e., aR_1c.
The result is true for all positive and negative value combinations of a, b, c.
$\therefore$ R_1 is transitive.
Hence R_1 is an equivalence relation.

(ii) $(a - a)$ is multiple of m
$\therefore$ $a \equiv a \pmod m$ i.e., R_2 is reflexive.
When $a - b$ is multiple of m, $b - a$ is also a multiple of m.
i.e. $a \equiv b \pmod m \Rightarrow b \equiv a \pmod m$
$\therefore$ R_2 is symmetric.
When $(a - b) = k_1m$ and $b - c = k_2m$, we get $a - c = (k_1 + k_2)m$ (by addition)
$\therefore$ When $a \equiv b \pmod m$ and $b \equiv c \pmod m$, $a \equiv c \pmod m$
$\therefore$ R_2 is transitive.
Hence R_2 is an equivalence relation.

(iii) $(a - a)$ is an integer. $\therefore$ R_3 is reflexive.
When $(a - b)$ is an integer, $(b - a)$ is an integer.
$\therefore$ R_3 is symmetric.
When $(a - b)$ and $(b - c)$ are integers, clearly $(a - c)$ is also an integer (by addition)
$\therefore$ R_3 is transitive.
Hence R_3 is an equivalence relation.

Example 2.9

(i) If R is the relation on the set of ordered pairs of positive integers such that $(a, b), (c, d) \in R$ whenever $ad = bc$, show that R is an equivalence relation.

(ii) if R is the relation on the set of positive integers such that $(a, b) \in R$ if and only if ab is a perfect square, show that R is an equivalence relation.

(i) $(a, b)\ R\ (a, b)$, since $ab = ba$
$\therefore$ R is reflexive.
When $(a, b)\ R\ (c, d)$, $ad = bc$ i.e., $cb = da$
This means that $(c, d)\ R\ (a, b)$
$\therefore$ R is symmetric.
When $(a, b)\ R\ (c, d)$, $ad = bc$ (1)

This means that $(a, b)\ R\ (e, f)$
~~$\therefore R$ is transitive.~~
Hence, R is an equivalence relation.

(ii) $(a, a) \in R_1$, since a^2 is a perfect square
$\therefore$ R is reflexive.
When ab is a perfect square, ba is also a perfect square.
i.e. $aRb \Rightarrow bRa$
$\therefore$ R is symmetric.
If, $a\ R\ b$, let $ab = x^2$ (1)
If $b\ R\ c$, let $bc = y^2$ (2)
(1) $\times$ (2) gives $ab^2c = x^2y^2$

$$\therefore \quad ac = \left(\frac{xy}{b}\right)^2 = \text{a present square.}$$

$\therefore$ aRc. i.e. R is transitive.
Hence R is an equivalence relation.

Example 2.10

(i) If R is the relation on the set of positive integers such that $(a, b) \in R$ if and only if $a^2 + b$ is even, prove that R is an equivalence relation.

(ii) If R is the relation on the set of integers such that $(a, b) \in R$, if and only if $3a + 4b = 7n$ for some integer n, prove that R is an equivalence relation.

(i) $a^2 + a = a(a + 1) =$ even, since a and $(a + 1)$ are consecutive positive integers.
$\therefore$ $(a, a) \in R$
Hence R is reflexive.
When $a^2 + b$ is even, a and b must be both even or both odd.
In either case, $b^2 + a$ is even
$\therefore$ $(a, b) \in R$ implies $(b, a) \in R$
Hence R is symmetric.
When a, b, c are even, $a^2 + b$ and $b^2 + c$ are even. Also $a^2 + c$ is even.
When a, b, c are odd, $a^2 + b$ and $b^2 + c$ are even. Also $a^2 + c$ is even.
Then $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ i.e., R is transitive.
$\therefore$ R is an equivalence relation.

(ii) $3a + 4a = 7a$, when a is an integer.
$\therefore$ $(a, a) \in R$. i.e., R is reflexive.

$$\begin{aligned} 3b + 4a &= 7a + 7b - (3a + 4b) \\ &= 7(a + b) - 7n \\ &= 7(a + b - n), \text{ where } a + b - n \text{ is an integer} \end{aligned}$$

$\therefore$ $(b, a) \in R$ when $(a, b) \in R$.
i.e. R is symmetric.
Let (a, b) and $(b, c) \in R$.
i.e. let $3a + 4b = 7m$ (1)
and $3b + 4c = 7n$ (2)

Example 2.11

(i) Prove that the relation $\subseteq$ of set inclusion is a partial ordering on any collection of sets.

(ii) If R is the relation on the set of integers such that $(a, b) \in R$ if and only if $b = a^m$ for some positive integer m, show that R is a partial ordering.

(i) $(A, B) \in R$, if and only if $A \subseteq B$, where A and B are any two sets.
Now $A \subseteq A$ $\therefore$ $(A, A) \in R$. i.e. R is reflexive.
If $A \subseteq B$ and $B \subseteq A$, then $A = B$.
i.e. R is antisymmetric.
If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$
i.e. $(A, B) \in R$ and $(B, C) \in R \Rightarrow (A, C) \in R$
$\therefore$ R is transitive
Hence R is a partial ordering.

(ii) $a = a^1$ $\therefore$ $(a, a) \in R$.
Let $(a, b) \in R$ and $(b, a) \in R$
i.e. $b = a^m$ and $a = b^n$ (1)
where m and n are positive integers.
$\therefore$ $a = (a^m)^n = a^{mn}$.
This means that $mn = 1$ or $a = 1$ or $a = -1$
Case (1): If $mn = 1$, then $m = 1$ and $n = 1$
$\therefore a = b$ [from (1)]
Case (2): If $a = 1$, then, from (1), $b = 1^m = 1 = a$
If $b = 1$, then, from (1), $a = 1^n = 1 = b$
Either way, $a = b$.
Case (3): If $a = -1$, then $b = -1$
Thus in all the three cases, $a = b$.
$\therefore$ R is antisymmetric.
Let $(a, b) \in R$ and $(b, c) \in R$
i.e. $b = a^m$ and $c = b^n$
$\therefore$ $c = (a^m)^n = a^{mn}$
$\therefore$ $(a, c) \in R$. i.e. R is transitive.
$\therefore$ R is a partial ordering.

Example 2.12

(i) If R is the equivalence relation on the set $A = \{1, 2, 3, 4, 5, 6\}$ given below, find the partition of A induced by R:
$R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4), (4, 5), (5, 4), (5, 5), (6, 6)\}$

(ii) If R is the equivalence relation on the set $A = \{(-4, -20), (-3, -9), (-2, -4), (-1, -11), (-1, -3), (1, 2), (1, 5), (2, 10), (2, 14), (3, 6), (4, 8), (4, 12)\}$, where $(a, b)\ R\ (c, d)$ if $ad = bc$, find the equivalent classes of R.

(i) The elements related to 1 are 1 and 2.
$\therefore$ $[1]_R = \{1, 2\}$
Also $[2]_R = \{1, 2\}$
The element related to 3 is 3 only
i.e. $[3]_R = \{3\}$ The elements related to 4 are $\{4, 5\}$
i.e. $[4]_R = \{4, 5\} = [5]_R$
The element related to 6 is 6 only
i.e. $[6]_R = \{6\}$
$\therefore$ $\{1, 2\}, \{3\}, \{4, 5\}, \{6\}$ is the partition induced by R.

(ii) The elements related to $(-4, -20)$ are $(1, 5)$ and $(2, 10)$
i.e. $[(-4, -20)] = \{(-4, -20), (1, 5), (2, 10)\}$
The elements related to $(-3, -9)$ are $(-1, -3)$ and $(4, 12)$
i.e. $[(-3, -9)] = \{(-3, -9), (-1, -3), (4, 12)\}$
The elements related to $(-2, -4)$ are $(-2, -4), (1, 2), (3, 6)$ and $(4, 8)$
i.e. $[(-2, -4)] = \{(-2, -4), (1, 2), (3, 6), (4, 8)\}$.
The element related to $(-1, -11)$ is itself only.
The element related to $(2, 14)$ is itself only.
$\therefore$ The partition induced by R consists of the cells
$[(-4, -20)], [(-3, -9)], [(-2, -4)], [(-1, -11]$ and $[(2, 14)]$.

Example 2.13

(i) If $A = \{1, 2, 3, 4\} \times \{1, 2, 3, 4\}$ and the relation R is defined on A by $(a, b)\ R\ (c, d)$ if $a + b = c + d$, verify that A is an equivalence relation on A and also find the quotient set of A by R.

(ii) If the relation R on the set of integers Z is defined by $a\ R\ b$ if $a \equiv b$ (mod 4), find the partition induced by R.

(i) $A = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3\ 2), (3, 3), (3, 4), (4, 1), (4, 2), (4, 3), (4, 4)\}$
If we take $R \equiv A$, it can be verified that R is an equivalence relation.
The quotient set A/R is the collection of equivalence classes of R.
It is easily seen that

$$[(1, 1)] = \{(1, 1)\}$$
$$[(1, 2)] = \{(1, 2), (2, 1)\}$$
$$[(1, 3)] = \{(1, 3), (2, 2), (3, 1)\}$$
$$[(1, 4)] = \{(1, 4), (2, 3), (3, 2), (4, 1)\}$$
$$[(2, 4)] = \{(2, 4), (3, 3), (4, 2)\}$$
$$[(3, 4)] = \{(3, 4), (4, 3)\}$$
$$[(4, 4)] = \{(4, 4)\}$$

Thus $[(1, 1)], [(1, 2)], [(1, 3)], [(1, 4)], [(2, 4)], [(3, 4)], [(4, 4)]$ form the quotient set A/R.

(ii) The equivalence classes of R are the following:

$$[0]_R = \{\cdots, -8, -4, 0, 4, 8, 12, \cdots\}$$
$$[1]_R = \{\cdots, -7, -3, 1, 5, 9, 13, \cdots\}$$
$$[2]_R = \{\cdots, -6, -2, 2, 6, 10, 14, \cdots\}$$
$$[3]_R = \{\cdots, -5, -1, 3, 7, 11, 15, \cdots\}$$

Thus $[0]_R$, $[1]_R$, $[2]_R$ and $[3]_R$ form the partition of R.

Note These equivalence classes are also called *the congruence classes* modulo 4 and also denoted $[0]_4$, $[1]_4$, $[2]_4$ and $[3]_4$.

Example 2.14 If R is the relation on $A = \{1, 2, 3\}$ such that $(a, b) \in R$, if and only if $a + b$ = even, find the relational matrix M_R. Find also the relational matrices R^{-1}, $\bar{R}$ and R^2.

$$R = \{(1, 1), (1, 3), (2, 2), (3, 1), (3, 3)\}$$

$$\therefore \quad M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Now $\quad M_{R^{-1}} = (M_R)^T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

$\bar{R}$ is the complement R that consists of elements of $A \times A$ that are not in R.

Thus $\quad \bar{R} = \{(1, 2), (2, 1), (2, 3), (3, 2)\}$

$\therefore \quad M_{\bar{R}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, which is the same as the matrix obtained from M_R by changing 0's to 1's and 1's to 0's.

$$M_{R^2} = M_R \bullet M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \vee 0 \vee 1 & 0 \vee 0 \vee 0 & 1 \vee 0 \vee 1 \\ 0 \vee 0 \vee 0 & 0 \vee 1 \vee 0 & 0 \vee 0 \vee 0 \\ 1 \vee 0 \vee 1 & 0 \vee 0 \vee 0 & 1 \vee 0 \vee 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

It can be found that $R^2 = R \bullet R = R$. Hence $M_{R^2} = M_R$

Example 2.15 If R and S be relations on a set A represented by the matrices

$\begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$

find the matrices that represent

(a) $R \cup S$ (b) $R \cap S$ (c) $R \bullet S$ (d) $S \bullet R$ (e) $R \oplus S$

(a) $M_{R \cup S} = M_R \vee M_S$

$$= \begin{bmatrix} 0 \vee 0 & 1 \vee 1 & 0 \vee 0 \\ 1 \vee 0 & 1 \vee 1 & 1 \vee 1 \\ 1 \vee 1 & 0 \vee 1 & 0 \vee 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

(b) $M_{R \cap S} = M_R \wedge M_S$

$$= \begin{bmatrix} 0 \wedge 0 & 1 \wedge 1 & 0 \wedge 0 \\ 1 \wedge 0 & 1 \wedge 1 & 1 \wedge 1 \\ 1 \wedge 1 & 0 \wedge 1 & 0 \wedge 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

(c) $M_{R \bullet S} = M_R \bullet M_S$

$$= \begin{bmatrix} 0 \vee 0 \vee 0 & 0 \vee 1 \vee 0 & 0 \vee 1 \vee 0 \\ 0 \vee 0 \vee 1 & 1 \vee 1 \vee 1 & 0 \vee 1 \vee 1 \\ 0 \vee 0 \vee 0 & 1 \vee 0 \vee 0 & 0 \vee 0 \vee 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

(d) $M_{S \bullet R} = M_S \bullet M_R$

$$= \begin{bmatrix} 0 \vee 1 \vee 0 & 0 \vee 1 \vee 0 & 0 \vee 1 \vee 0 \\ 0 \vee 1 \vee 1 & 0 \vee 1 \vee 0 & 0 \vee 1 \vee 0 \\ 0 \vee 1 \vee 1 & 1 \vee 1 \vee 0 & 0 \vee 1 \vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

(e) $M_{R \oplus S} = M_{R \cup S} - M_{R \cap S}$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Example 2.16 Examine if the relation R represented by $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ is an equivalence relation, using the properties of M_R.

Since all the elements in the main diagonal of M_R and equals to 1 each, R is a reflexive relation.

Since M_R is a symmetric matrix, R is a symmetric relation.

$\begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$

viz. $R^2 \subseteq R$

$\therefore$ R is a transitive relation.

Hence R is an equivalence relation.

Example 2.17 List the ordered pairs in the relation on $\{1, 2, 3, 4\}$ corresponding to the following matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Also draw the directed graph representing this relation. Use the graph to find if the relation is reflexive, symmetric and/or transitive.

The ordered pairs in the given relation are $\{(1, 1), (1, 2), (1, 3), (2, 2), (3, 3), (3, 4), (4, 1), (4, 4)\}$. The directed graph representing the relation is given in Fig. 2.18.

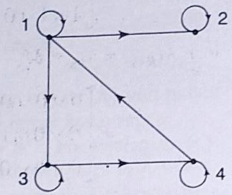


Fig. 2.18

Since there is a loop at every vertex of the digraph, the relation is reflexive. The relation is not symmetric. For example, there is an edge from 1 to 2, but there is no edge in the opposite direction, i.e. from 2 to 1. The relation is not transitive. For example, though there are edges from 1 to 3 and 3 to 4, there is no edge from 1 to 4.

Example 2.18 List the ordered pairs in the relation represented by the digraph given in Fig. 2.19. Also use the graph to prove that the relation is a partial ordering. Also draw the directed graphs representing R^{-1} and $\bar{R}$.

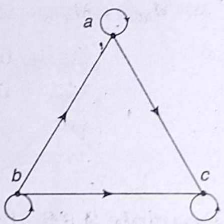


Fig. 2.19

The ordered pairs in the relation are $\{(a, a), (a, c), (b, a), (b, b), (b, c), (c, c)\}$.

Since there is a loop at every vertex, the relation is reflexive.

Though there are edges $b - a$, $a - c$ and $b - c$, the edges $a - b$, $c - a$ and $c - b$ are not present in the digraph. Hence the relation is antisymmetric.

When edges $b - a$ and $a - c$ are present in the digraph, the edge $b - c$ is also present (for example). Hence the relation is transitive.

Hence the relation is a partially ordering. The digraph of R^{-1} is got by reversing the directions of the edges (Fig. 2.20). The digraph of $\bar{R}$ contains the edges (a, b), (c, a), and (c, b) as shown in Fig. 2.21.

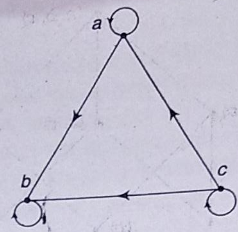


Fig. 2.20

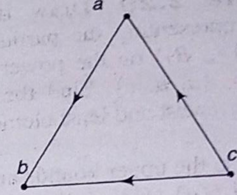


Fig. 2.21

Example 2.19 Draw the digraph representing the partial ordering $\{(a, b)\} \mid a$ divides $b\}$ on the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$. Reduce it to the Hasse diagram representing the given partial ordering.

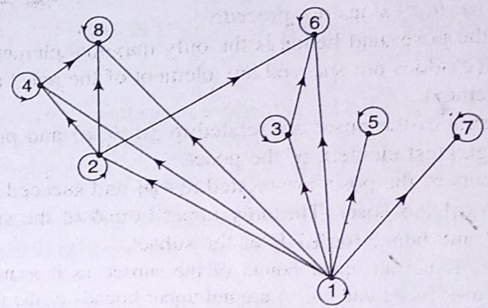


Fig. 2.22

Deleting all the loops at the vertices, deleting all the edges occurring due to transitivity, arranging all the edges to point upward and deleting all arrows, we get the corresponding Hasse diagram as given in Fig. 2.23.

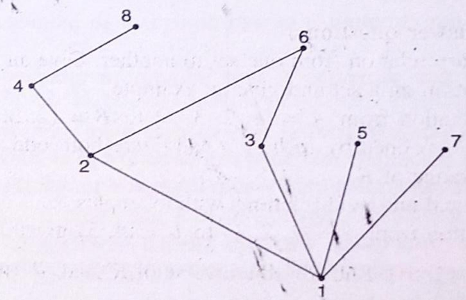


Fig. 2.23

Example 2.20 Draw the Hasse diagram representing the partial ordering $\{(A, B) | A \subseteq B\}$ on the power set $P(S)$, where $S = \{a, b, c\}$. Find the maximal, minimal, greatest and least elements of the poset.

Find also the upper bounds and LUB of the subset $(\{a\}, \{b\}, \{c\})$ and the lower bounds and GLB of the subset $(\{a, b\}, \{a, c\}, \{b, c\})$.

Here $P(S) = (\{\phi\}, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}$.

By using the usual procedure (as in the previous example), the Hasse diagram is shown, as shown in Fig. 2.24.

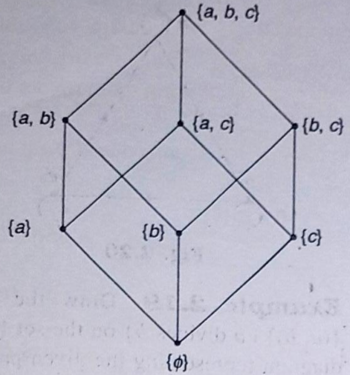


Fig. 2.24

The element $\{a, b, c\}$ does not precede any element of the poset and hence is the only maximal element of the poset.

The element $\{\phi\}$ does not succeed any element of the poset and hence is the only minimal element.

All the elements of the poset are related to $\{a, b, c\}$ and precede it. Hence $\{a, b, c\}$ is the greatest element of the poset.

All the elements of the poset are related to $\{\phi\}$ and succeed it. Hence $\{\phi\}$ is the least element of the poset. The only upper bound of the subset $(\{a\}, \{b\}, \{c\})$ is $\{a, b, c\}$ and hence the LUB of the subset.

Note $\{a, b\}$ is not an upper bound of the subset, as it is not related to $\{c\}$. Similarly $\{a, c\}$ and $\{b, c\}$ are not upper bounds of the given subset.

The only lower bound of the subset $(\{a, b\}, \{a, c\}, \{b, c\})$ is $\{\phi\}$ and hence GLB of the given subset.

Note $\{a\}, \{b\}, \{c\}$ are not lower bounds of the given subset.

EXERCISE 2(B)

Part A: (Short answer questions)

1. Define a binary relation from one set to another. Give an example.
2. Define a relation on a set and give an example.
3. If R is the relation from $A = \{1, 2, 3, 4\}$ to $B = \{2, 3, 4, 5\}$, list the elements in R, defined by aRb, if a and b are both odd. Write also the domain and range of R.
4. Define universal and void relations with examples.
5. If R is a relation from $A = \{1, 2, 3\}$ to $B = \{4, 5\}$ given by $R = \{(1, 4), (2, 4), (1, 5), (3, 5)\}$, find R^{-1} (the inverse of R) and $\bar{R}$ (the complement

7. Define composition of relations with an example.
8. When is a relation said to be reflexive, symmetric, antisymmetric and transitive?
9. Give an example of a relation that is both symmetric and antisymmetric.
10. Give an example of a relation that is neither symmetric nor antisymmetric.
11. Give an example of a relation that is reflexive and symmetric but not transitive.
12. Give an example of relation that is reflexive and transitive but not symmetric.
13. Give an example of a relation that is symmetric and transitive but not reflexive.
14. Define an equivalence relation with an example.
15. Define a partial ordering with an example.
16. Define a poset and give an example.
17. Define equivalence class.
18. Define quotient set of a set under an equivalence relation.
19. Find the quotient set of $\{1, 2, 3\}$ under the relation $\{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}$.
20. Define partition of a set and give an example.
21. What do you mean by partitioning of a set induced by an equivalence relation?
22. If R is a relation from $A = \{1, 2, 3\}$ to $B = \{1, 2\}$ such that aRb if $a > b$, write down the matrix representation of R.
23. If the matrix representation of a relation R on $\{1, 2, 3, 4\}$ is given by
$$M_R = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$
list the ordered pairs in the relation.
24. If the relations R and S on a set A are represented by the matrices
$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } M_S = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$
What are the matrices representing $R \cup S$ and $R \cap S$?
25. Draw the directed graph representing the relation on $\{1, 2, 3, 4\}$ given by the ordered pairs $\{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$.
26. Draw the directed graph representing the relation on $\{1, 2, 3, 4\}$ whose matrix representation is

27. What is Hasse diagram? Draw the Hasse diagram for $\leq$ relation on $\{0, 2, 5, 10, 11, 15\}$.

28. Define maximal and minimal members of a poset. Are they the same as the greatest and least members of the poset?

29. Define the greatest and least members of a poset. Are they different from the maximal and minimal members of the poset?

30. Define supremum and infimum of a subset of a poset.

Part B

31. Show that there are 2^{n^2} relations on a set with n elements. List all possible relations on the set $\{1, 2\}$.

Hint: When a set A has n elements, $A \times A$ has n^2 elements and hence the number of subsets of $A \times A = 2^{n^2}$.

32. Which of the ordered pairs given by $\{1, 2, 3\} \times \{1, 2, 3\}$ belong to the following relations?

(a) $a\,R\,b$ iff $a \leq b$, (b) $a\,R\,b$ iff $a > b$,
(c) $a\,R\,b$ iff $a = b$, (d) $a\,R\,b$ iff $a = b + 1$ and
(e) $a\,R\,b$ iff $a + b \leq 4$.

33. If R is a relation on the set $\{1, 2, 3, 4, 5\}$, list the ordered pairs in R when (a) aRb if 3 divides $a - b$, (b) aRb if $a + b = 6$, (c) aRb if $a - b$ is even, (d) aRb if *lcm* (a, b) is odd, (e) aRb if $a^2 = b$.

34. If R is the relation on the set $\{1, 2, 3, 4, 5\}$ defined by $(a, b) \in R$ if $a + b \leq 6$,

(a) list the elements of R, R^{-1} and $\bar{R}$.
(b) the domain and range of R and R^{-1}.
(c) the domain and range of $\bar{R}$.

35. If $R_1 = \{(1, 2), (2, 3), (3, 4)\}$ and $R_2 = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3), (3, 4)\}$ be the relations from $\{1, 2, 3\}$ to $\{1, 2, 3, 4\}$. Find

(a) $R_1 \cup R_2$, (b) $R_1 \cap R_2$, (c) $R_1 - R_2$,
(d) $R_2 - R_1$, (e) $R_1 \oplus R_2$.

36. If $R = \{(x, x^2)\}$ and $S = \{(x, 2x)\}$, where x is a non-negative integer, find

(a) $R \cup S$, (b) $R \cap S$, (c) $R - S$,
(d) $S - R$, (e) $R \oplus S$.

37. If R_1 and R_2 are relations on the set of all positive integers defined by $R_1 = \{(a, b) | a$ divides $b\}$ and $R_2 = \{(a, b) | a$ is a multiple of $b\}$, find

(a) $R_1 \cup R_2$, (b) $R_1 \cap R_2$, (c) $R_1 - R_2$,
(d) $R_2 - R_1$ (e) $R_1 \oplus R_2$.

38. If the relations R_1, R_2, R_3, R_4, R_5 are defined on the set of real numbers as given below,
$R_1 = \{(a, b) | a \geq b\}$, $R_2 = \{(a, b) | a < b\}$,
$R_3 = \{(a, b) | a \leq b\}$, $R_4 = \{(a, b) | a = b\}$, $R_5 = \{(a, b) | a \neq b\}$, find (a) $R_2 \cup R_5$, (b) $R_3 \cap R_5$, (c) $R_2 - R_5$, (d) $R_1 \oplus R_5$, (e) $R_2 \oplus R_4$.

39. If the relations R and S are given by
$R = \{(1, 2), (2, 2), (3, 4)\}$, $S = \{(1, 3), (2, 5), (3, 1), (4, 2)\}$, find $R \bullet S$.
S•R, R•R, S•S, R•S•R) (R•S)•R and R•R•R.

40. If R, S, T are relations on the set $A = \{0, 1, 2, 3\}$ defined by $R = \{(a, b) | a + b = 3\}$, $S = \{(a, b) |$ 3 is a divisor of $(a + b)\}$ and $T = \{(a, b) | \max(a, b) = 3\}$, find (a) $R \bullet T$, (b) $T \bullet R$ and (c) $S \bullet S$.

41. If the relations $R_1, R_2, R_3, R_4, R_5, R_6$ are defined on the set of real numbers as given below,
$R_1 = \{(a, b) | a > b\}$, $R_2 = \{(a, b) | a \geq b\}$, $R_3 = \{(a, b) | a < b\}$,
$R_4 = \{(a, b) | a \leq b\}$, $R_5 = \{(a, b) | a = b\}$, $R_6 = \{(a, b) | a \neq b\}$,
find $R_1 \bullet R_1, R_2 \bullet R_1, R_3 \bullet R_1, R_4 \bullet R_1, R_5 \bullet R_1, R_6 \bullet R_1, R_3 \bullet R_2$ and $R_3 \bullet R_3$.

42. Determine whether the relation R on the set of all real numbers is reflexive, symmetric, antisymmetric and/or transitive, where $(a, b) \in R$ if and only if

(a) $a + b = 0$ (b) $a = \pm b$
(c) $a - b$ is a rational number (d) $a = 2b$
(e) $ab \geq 0$ (f) $ab = 0$
(g) $a = 1$ (h) $a = 1$ or $b = 1$

43. For each of the following relations, determine whether the relation is reflexive, symmetric, antisymmetric and/or transitive:

(a) $R \subseteq Z^+ \times Z^+$, where aRb if a divides b.
(b) $R \subseteq Z \times Z$, where aRb if a divides b.
(c) R is the relation on Z, where aRb if $a + b$ is odd.
(d) R is the relation on Z, where aRb if $a - b$ is even.
(e) R is the relation on the set of lines in a plane such that aRb if a perpendicular to b.

44. Determine whether the relation R on the set of people is reflexive, symmetric, antisymmetric and/or transitive, where aRb if
(a) a is taller than b, (b) a and b were born on the same day, (c) a has the same first name as b, (d) a is a spouse of b, (e) a and b have a common grand parent.

45. Which of the following relations on the set $\{1, 2, 3, 4\}$ is/are equivalent relations? Find the properties of an equivalent relation that the others lack.

(a) $\{(2, 4), (4, 2)\}$
(b) $\{(1, 1), (2, 2), (3, 3), (4, 4)\}$
(c) $\{(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)$
(d) $\{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$
(e) $\{(1, 3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 4)\}$

46. If $A = \{1, 2, 3, \ldots, 9\}$ and R be the relation defined by $(a, b), (c, d) \in R$ if $a + d = b + c$, prove that R is an equivalence relation.

47. If R is a relation on Z defined by
(a) aRb, if and only if $2a + 3b = 5n$ for some integer n.
(b) aRb if and only if $3a + b$ is a multiple of 4, prove that R is an equivalence relation.

48. If R is a relation defined by
  (a) $(a, b)\ R\ (c, d)$ if and only if $a^2 + b^2 = c^2 + d^2$, where a, b, c and d are real.
  (b) $(a, b)\ R\ (c, d)$ if and only if $a + 2b = c + 2d$, where a, b, c and d are real, prove that R is an equivalence relation.
49. (a) If R is the relation defined on Z such that aRb if and only if $a^2 - b^2$ is divisible by 3, show that R is an equivalence relation.
  (b) If R is the relation on N defined by aRb if and only if $\frac{a}{b}$ is a power 2, show that R is an equivalence relation.
50. If R is the relation the set $A = \{1, 2, 4, 6, 8\}$ defined by aRb if and only if $\frac{b}{a}$ is an integer, show that R is a partial ordering on A.
51. (a) If R is the equivalence relation on $A = \{0, 1, 2, 3, 4\}$ given by $\{(0, 0), (0, 4), (1, 1), (1, 3), (2, 2), (3, 1), (3, 3), (4, 0), (4, 4)\}$, find the distinct equivalence classes of R.
  (b) If R is the equivalence relation on $A = \{1, 2, 3, 4, 5, 6\}$ given by $\{(1, 1), (1, 5), (2, 2), (2, 3), (2, 6), (3, 2), (3, 3), (3, 6), (4, 4), (5, 1), (5, 5), (6, 2), (6, 3), (6, 6)\}$, find the partition of A induced by R.
52. If R is the equivalence relation on the set $A = \{1, 2, 3, 4, 5, 6, 7\}$ defined by aRb if $a - b$ is a multiple of 3, find the partition of A induced by R.
53. If R is the equivalence relation on Z defined by aRb if $a^2 = b^2$ (or, $a = \pm b$), find the partition of Z.
54. If R and S are equivalence relations on $A = \{a, b, c, d, e\}$ given by $R = \{(a, a), (a, b), (b, a), (b, b), (c, c), (d, d), (d, e), (e, d), (e, e)\}$ and $S = \{(a, a), (b, b), (c, c), (d, d), (e, e), (a, c), (c, a), (d, e), (e, d)\}$, determine the partitions of A induced by (a) R^{-1}, (b) $R \cap S$.
55. List the ordered pairs in the equivalence relations R and S produced by the partitions of $\{0, 1, 2, 3, 4, 5\}$ and $\{1, 2, 3, 4, 5, 6, 7\}$ respectively that are given as follows:
  (a) $[\{0\}, \{1, 2\}, \{3, 4, 5\}]$ (b) $[\{1, 2\}, \{3\}, \{4, 5, 7\}, \{6\}]$
  Hint: $R = \{0\} \times \{0\} \cup \{1, 2\} \times \{1, 2\} \cup \{3, 4, 5\} \times \{3, 4, 5\}$
56. If R is the relation on $A = \{1, 2, 3\}$ represented by the matrix

$$M_R = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix},$$

find the matrix representing (a) R^{-1}, (b) $\bar{R}$ and R^2 and also express them as ordered pairs.
57. If R and S are relations on $A = \{1, 2, 3\}$ represented by the matrices

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ \end{bmatrix} \text{ and } M_S = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ \end{bmatrix}$$

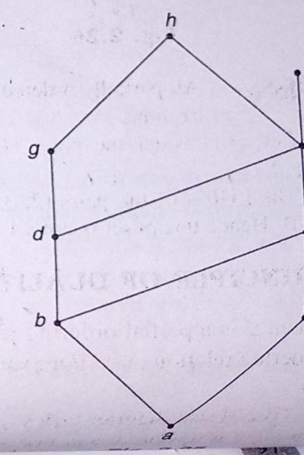
58. Examine if the relations R and S represented by M_R and M_S given below are equivalent relations:

(a) $M_R = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$ (b) $M_S = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

59. List the ordered pairs in the relations R and S whose matrix representations are given as follows:

(a) $M_R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$; (b) $M_S = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$

Also draw the directed graphs representing R and S. Use the graphs to find if R and S are equivalence relations.
60. Draw the directed graphs of the relations $R = \{(1, 1), (1, 3), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (4, 1)\}$ and $S = \{(1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (3, 1), (3, 3), (4, 1), (4, 3)\}$. Use these graphs to draw the graphs of (a) R^{-1}, S^{-1} and (b) $\bar{R}$ and $\bar{S}$.
61. Draw the Hasse diagram representing the partial ordering $P = \{(a, b) | a$ divides $b\}$ on $\{1, 2, 3, 4, 6, 8, 12\}$, starting from the digraph of P.
62. Draw the Hasse diagram for the divisibility relation on $\{2, 4, 5, 10, 12, 20, 25\}$ starting from the digraph.
63. Draw the Hasse diagram for the "less than or equal to" relation on $\{0, 2, 5, 10, 11, 15\}$ starting from the digraph.
64. Find the lower and upper bounds of the subsets $\{a, b, c\}$, $(j, h\}$ and $\{a, c, d, f\}$ in the poset with the Hasse diagram in Fig. 2.25. Find also the LUB and GLB of the subset $\{b, d, g\}$, if they exist.



65. For the poset $[\{(3, 5, 9, 15, 24, 45\}$; divisor of], find
  (a) the maximal and minimal elements
  (b) the greatest and the least elements
  (c) the upper bounds and LUB of $\{3, 5\}$
  (d) the lower bounds and GLB of $\{15, 45\}$

LATTICES

Definitions

A partially ordered set $\{L, \leq\}$ in which every pair of elements has a least upper bound and a greatest lower bound is called a *lattice*.

The LUB (supremum) of a subset $\{a, b\} \subseteq L$ is denoted by $a \vee b$ [or $a \oplus b$ or $a + b$ or $a \cup b$] and is called the *join* or *sum* of a and b.

The GLB (infemum) of a subset $\{a, b\} \subseteq L$ is denoted by $a \wedge b$ [or $a * b$ or $a \bullet b$ or $a \cap b$] is called the *meet* or *product* of a and b.

Note Since the LUB and GLB of any subset of a poset are unique, both $\wedge$ and $\vee$ are binary operations on a lattice.

For example, let us consider the poset $(\{1, 2, 4, 8, 16\}|)$, where | means 'divisor of'. The Hasse diagram of this poset is given in Fig. 2.26.

The LUB of any two elements of this poset is obviously the larger of them and the GLB of any two elements is the smaller of them. Hence this poset is a lattice.

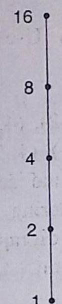


Fig. 2.26

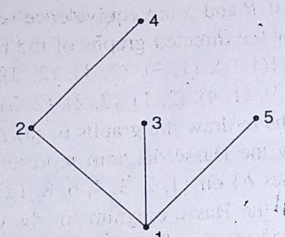


Fig. 2.27

Note All partially ordered sets are not lattices, as can be seen from the following example.

Let us consider the poset $(\{1, 2, 3, 4, 5\}, |)$ whose Hasse diagram is given in Fig. 2.27.

The LUB's of the pairs (2, 3) and (3, 5) do not exist and hence they do not have LUB. Hence this poset is not a Lattice.

PRINCIPLE OF DUALITY

When $\leq$ is a partial ordering relation on a set S, the converse $\geq$ is also a partial ordering relation on S. For example if $\leq$ denotes 'divisor of', $\geq$ denotes 'multiple of'.

The Hasse diagram of $(S, \geq)$ can be obtained from that of $(S, \leq)$ by simply turning it upside down. For example the Hasse diagram of the poset $(\{1, 2, 4,$ 8, 16}

multiple of | obtained from 2.26 will be easy Fig 2.28. in Fig 2.28.

From this example, it is obvious that LUB(A) with respect to $\leq$ is the same as GLB(A) with respect to $\geq$ and vice versa, where $A \subseteq S$. viz. LUB and GLB are interchanged, when $\leq$ and $\geq$ are interchanged.

In the case of lattices, if $\{L, \leq\}$ is a lattice, so also is $\{L, \geq\}$. Also the operations of join and meet on $\{L, \leq\}$ become the operations of meet and join respectively on $\{L, \geq\}$.

From the above observations, the following statement, known as *the principle of duality* follows:

Any statement in respect of lattices involving the operations $\vee$ and $\wedge$ and the relations $\leq$ and $\geq$ remains true, if $\vee$ is replaced by $\wedge$ and $\wedge$ is replaced by $\vee$, $\leq$ by $\geq$ and $\geq$ by $\leq$.

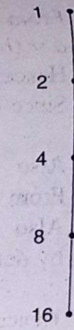


Fig. 2.28

The lattices $\{L, \leq\}$ and $\{L, \geq\}$ are called the *duals* of each other. Similarly the operations $\vee$ and $\wedge$ are duals of each other and the relations $\leq$ and $\geq$ are duals of each other.

PROPERTIES OF LATTICES

Property 1

If $\{L, \leq\}$ is a lattice, then for any $a, b, c \in L$,

L_1: $a \vee a = a$	$(L_1)'$: $a \wedge a = a$	(Idempotency)
L_2: $a \vee b = b \vee a$	$(L_2)'$: $a \wedge b = b \wedge a$	(Commutativity)
L_3: $a \vee (b \vee c) = (a \vee b) \vee c$	$(L_3)'$: $a \wedge (b \wedge c) = (a \wedge b) \wedge c$	(Associativity)
L_4: $a \vee (a \wedge b) = a$	$(L_4)'$: $a \wedge (a \vee b) = a$	(Absorption)

Proof

(i) $a \vee a = \text{LUB}(a, a) = \text{LUB}(a) = a$. Hence L_1 follows.

(ii) $a \vee b = \text{LUB}(a, b) = \text{LUB}(b, a) = b \vee a$ {$\because$ LUB (a, b) is unique.} Hence L_2 follows.

(iii) Since $(a \vee b) \vee c$ is the LUB $\{(a \vee b), c\}$, we have

$a \vee b \leq (a \vee b) \vee c$ (1)

and $c \leq (a \vee b) \vee c$ (2)

Since $a \vee b$ is the LUB $\{a, b\}$, we have

$a \leq a \vee b$ (3)

and $b \leq a \vee b$ (4)

From (1) and (3), $a \leq (a \vee b) \vee c$ by transitivity (5)

From (1) and (4), $b \leq (a \vee b) \vee c$ by transitivity (6)

From (2) and (6), $b \vee c \leq (a \vee b) \vee c$ by definition of join (7)

From (5) and (7), $a \vee (b \vee c) \leq (a \vee b) \vee c$ by definition of join (8)

Similarly, $a \leq a \vee (b \vee c)$ (9)

$b \leq b \vee c \leq a \vee (b \vee c)$ (10)

and $c \leq b \vee c \leq a \vee (b \vee c)$ (11)

From (9) & (10) $a \vee b \leq a \vee (b \vee c)$ — (12)

From (11) & (12) $(a \vee b) \vee c \leq a \vee (b \vee c)$ (13)

From (8) and (13), by antisymmetry of $\leq$, we get
$a \vee (b \vee c) = (a \vee b) \vee c$.
Hence L_3 follows.

(iv) Since $a \wedge b$ is the GLB $\{a, b\}$, we have

$a \wedge b \leq a$ (1)

Also $a \leq a$ (2)

From (1) and (2), $a \vee (a \wedge b) \leq a$ (3)

Also $a \leq a \vee (a \wedge b)$ (4)

by definition of LUB

$\therefore$ From (3) and (4), by antisymmetry, we get $a \vee (a \wedge b) = a$.
Hence L_4 follows.
Now the identities $(L_1)'$ to $(L_4)'$ follow from the principle of duality.

Property 2

If $\{L, \leq\}$ is a lattice in which $\vee$ and $\wedge$ denote the operations of join and meet respectively, then for $a, b \in L$,

$$a \leq b \Leftrightarrow a \vee b = b \Leftrightarrow a \wedge b = a.$$

In other words,
(i) $a \vee b = b$, if and only if $a \leq b$.
(ii) $a \wedge b = a$, if and only if $a \leq b$.
(iii) $a \wedge b = a$, if and only if $a \vee b = b$.

Proof

(i) Let $a \leq b$.
Now $b \leq b$ (by reflexivity).

$\therefore$ $a \vee b \leq b$ (1)

Since $a \vee b$ is the LUB (a, b),

$b \leq a \vee b$ (2)

From (1) and (2), we get $a \vee b = b$ (3)

Let $a \vee b = b$.

(1≤

Since $a \vee b$ is the LUB (a, b),

$a \leq a \vee b$

i.e., $a \leq b$, by the data (4)

From (3) and (4), result (i) follows. Result (ii) can be probed in a way similar to the proof (i).
From (i) and (ii), result (iii) follows.

Note Property (2) gives a connection between the partial ordering relation $\leq$ and the two binary operations $\vee$ and $\wedge$ in a lattice $\{L, \leq\}$.

Property 3 (Isotonic Property)

If $\{L, \leq\}$ is a lattice, then for any $a, b, c, \in L$, the following properties hold good:

If $b \leq c$, then (i) $a \vee b \leq a \vee c$ and (ii) $a \wedge b \leq a \wedge c$.

Proof

Since $b \leq c$, $b \vee c = c$, by property 2(i) Also $a \wedge a = a$ by idempotent

Now $a \vee c = (a \vee a) \vee (b \vee c)$, by the above steps
$= a \vee (a \vee b) \vee c$, by associativity
$= a \vee (b \vee a) \vee c$, by commutativity
$= (a \vee b) \vee (a \vee c)$, by associativity
This is of the form $x \vee y = y$. $\therefore$ $x \leq y$, by property 2(i).
i.e. $a \vee b \leq a \vee c$, which is the required result (i).
Similarly, result (ii) can be proved.

Property 4 (Distributive Inequalities)

If $\{L, \leq\}$ is a lattice, then for any $a, b, c, \in L$,
(i) $a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$
(ii) $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$.

Proof

Since $a \wedge b$ is the GLB(a, b), $a \wedge b \leq a$ (1)

Also $a \wedge b \leq b \leq b \vee c$ (2)

since $b \vee c$ is the LUB of b and c.
From (1) and (2), we have $a \wedge b$ is a lower bound of $\{a, b \vee c\}$

$\therefore$ $a \wedge b \leq a \wedge (b \vee c)$ (3)

Similarly $a \wedge c \leq a$

and $a \wedge c \leq c \leq b \vee c$

$\therefore$ $a \wedge c \leq a \wedge (b \vee c)$ (4)

From (3) and (4), we get

$$(a \wedge b) \vee (a \wedge c) \leq a \wedge (b \vee c)$$

i.e. $a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$, which is result (i).
Result (ii) follows by the principle of duality.

Property 5 (Modular Inequality)

If $\{L, \leq\}$ is a lattice, then for any $a, b, c, \in L$, $a \leq c \Leftrightarrow a \vee (b \wedge c) \leq (a \vee b) \wedge c$. (1)

Proof

Since $a \leq c$, $a \vee c = c$ (1), by property 2(i)

$a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$ (2), by property 4(ii)

i.e. $a \vee (b \wedge c) \leq (a \vee b) \wedge c$ (3), by (1)

Now $a \vee (b \wedge c) \leq (a \vee b) \wedge c$

$\therefore$ $a \leq a \vee (b \wedge c) \leq (a \vee b) \wedge c \leq c$, by the definitions of LUB and GLB

i.e. $a \leq c$ (4)

From (3) and (4), we get

$$a \leq c \Leftrightarrow a \vee (b \wedge c) \leq (a \vee b) \wedge c.$$

LATTICE AS ALGEBRAIC SYSTEM

A set t[illegible]ther with certain operations (rules) for combining the elements of set to form other elements ...

ordered set in which for every

pair of elements $a, b \in L$, $LUB(a, b) = a \vee b$ and $GLB(a, b) = a \wedge b$ exist in the set. That is, in a Lattice $\{L, \leq\}$, for every pair of elements a, b of L, the two elements $a \vee b$ and $a \wedge b$ of L are obtained by means of the operations $\vee$ and $\wedge$. Due to this, the operations $\vee$ and $\wedge$ are considered as binary operations on L. Moreover we have seen that $\vee$ and $\wedge$ satisfy certain properties such as commutativity, associativity and absorption. The formal definition of a lattice as an algebraic system is given as follows:

Definition

A lattice is an algebraic system $(L, \vee, \wedge)$ with two binary operations $\vee$ and $\wedge$ on L which satisfy the commutative, associative and absorption laws.

Note We have not explicity included the idempotent law in the definition, since the absorption law implies the idempotent law as follows:

$a \vee a = a \vee [a \wedge (a \vee a)]$, by using $a \vee a$ for $a \vee b$ in $(L_4)'$ of property 1

$= a$, by using $a \vee a$ for b in L_4 of property 1.

$a \wedge a = a$ follows by duality.

Though the above definition does not assume the existence of any partial ordering on L, it is implied by the properties of the operations $\vee$ and $\wedge$ as explained below:

Let us assume that there exists a relation R on L such that for $a, b \in L$,

$$aRb \text{ if and only if } a \vee b = b$$

For any $a \in L$, $a \vee a = a$, by idempotency

$\therefore$ aRa or R is reflexive.

Now for any $a, b, \in L$, let us assume that aRb and bRa.

$\therefore$ $a \vee b = b$ and $b \vee a = a$

Since $a \vee b = b \vee a$ by commutativity, we have $a = b$ and so R is antisymmetric.

Finally let us assume that aRb and bRc

$\therefore$ $a \vee b = b$ and $b \vee c = c$.

Now $a \vee c = a \vee (b \vee c) = (a \vee b) \vee c = b \vee c = c$

viz. aRc and so R is transitive.

Hence R is a partial ordering.

Thus the two definitions given for a lattice are equivalent.

SUBLATTICES

Definition

A non-empty subset M of a lattice $\{L, \vee, \wedge\}$ is called a *sublattice* of L, iff M is closed under both the operations $\vee$ and $\wedge$, viz. if $a, b, \in M$, then $a \vee b$ and $a \wedge b$ also $\in M$.

From the definition, it is obvious that the sublattice itself is a lattice with respect to $\vee$ and $\wedge$.

For example if aRb whenever a divides b, where $a, b \in Z^+$ (the set of all positive integers) then $\{Z^+, R\}$ is a lattice in which $a \vee b = \text{LCM}(a, b)$ and $a \wedge b = \text{GCD}(a, b)$.

If $\{S_n, R\}$ is the lattice of divisors of any positive integer n, then $\{S_m, R\}$ is a sublattice of $\{Z^+, R\}$.

Definition

If $\{L_1, \vee, \wedge\}$ and $\{L_2, \oplus, *\}$ are two lattices, a mapping $f: L_1 \to L_2$ is called a *lattice homomorphism* from L_1 to L_2, if for any $a, b \in L_1$,

$$f(a \vee b) = f(a) \oplus f(b) \text{ and } f(a \wedge b) = f(a) * f(b).$$

If a homomorphism $f: L_1 \to L_2$ of two lattices $\{L_1, \vee, \wedge\}$ and $\{L_2, \oplus *\}$ is objective, i.e. one-to-one onto, then f is called an *isomorphism*. If there exists an isomorphism between two lattices, then the lattices are said to be *isomorphic*.

SOME SPECIAL LATTICES

(a) A lattice L is said to have a *lower bound* denoted by 0, if $0 \leq a$ for all $a \in L$. Similarly L is said to have an *upper bound* denoted by 1, if $a \leq 1$ for all $a \in L$. The lattice L is said to be *bounded*, if it has both a lower bound 0 and an upper bound 1.

The bounds 0 and 1 of a lattice $\{L, \vee, \wedge, 0, 1\}$ satisfy the following identities, which are seen to be true by the meanings of $\vee$ and $\wedge$.

For any $a \in L$, $a \vee 1 = 1$; $a \wedge 1 = a$ and $a \vee 0 = a$; $a \wedge 0 = 0$.

Since $a \vee 0 = a$ and $a \wedge 1 = a$, 0 is the identity of the operation $\vee$ and 1 is the identity of the operation $\wedge$.

Since $a \vee 1 = 1$ and $a \wedge 0 = 0$, 1 and 0 are the zeros of the operations $\vee$ and $\wedge$ respectively.

Note 1 If we treat 1 and 0 as duals of each other in a bounded lattice, the principle of duality can be extended to include the interchange of 0 and 1. Thus the identities $a \vee 1 = 1$ and $a \wedge 0 = 0$ are duals of each other; so also are $a \vee 0 = a$ and $a \wedge 1 = a$.

Note 2 If $L = \{a_1, a_2, \ldots, a_n\}$ is a finite lattice, then $a_1 \vee a_2 \vee a_3 \ldots \vee a_n$ and $a_1 \wedge a_2 \wedge a_3 \wedge \ldots \wedge a_n$ are upper and lower bounds of L respectively and hence we conclude that every finite lattice is bounded.

(ii) A lattice $\{L, \vee, \wedge\}$ is called a *distributive lattice*, if for any elements $a, b, c \in L$,

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \text{ and}$$
$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

In other words if the operations $\vee$ and $\wedge$ distribute over each other in a lattice, it is said to be distributive. Otherwise it is said to be *non distributive*.

(iii) If $\{L, \vee, \wedge, 0, 1\}$ is a bounded lattice and $a \in L$, then an element $b \in L$ is called a *complement* of a, if

$$a \vee b = 1 \text{ and } a \wedge b = 0$$

Since $0 \vee 1 = 1$ and $0 \wedge 1 = 0$, 0 and 1 are complements of each other.

When $a \vee b = 1$, we know that $b \vee a = 1$ and when $a \wedge b = 0$, $b \wedge a = 0$. Hence when b is the complement of a, a is the complement of b.

An element $a \in L$ may have no complement. Similarly an element, other than 0 and 1, may have more than one complement in L as seen from Fig. 2.28.

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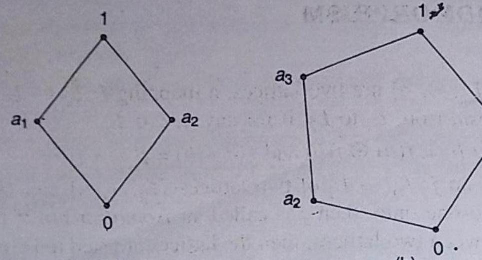

(a) (b)

Fig. 2.28

In Fig. 2.28(a), complement of a_1 is a_2, whereas in (b), complement of a_1 is a_2 and a_3. It is to be noted that 1 is the only complement of 0. If possible, let $x \neq 1$ be another complement of 0, where $x \in L$.

Then $\quad 0 \vee x = 1 \quad$ and $\quad 0 \wedge x = 0$

But $\quad 0 \vee x = x \quad \therefore \quad x = 1$, which contradicts the assumption $x \neq 1$.

Similarly we can prove that 0 is the only complement of 1.

Now a lattice $\{L, \vee, \wedge, 0, 1\}$ is called a *complemented lattice* if every element of L has at least one complement.

The following property holds good for a distributive lattice.

Property

In a distributive lattice $\{L, \vee, \wedge\}$ if an element $a \in L$ has a complement, then it is unique.

Proof

If possible, let b and c be the complements of $a \in L$.

Then $\quad a \vee b = a \vee c = 1 \qquad (1)$

and $\quad a \wedge b = a \wedge c = 0 \qquad (2)$

Now
$$\begin{aligned} b &= b \vee 0 = b \vee (a \wedge c), \text{ by } (2) \\ &= (b \vee a) \wedge (b \vee c), \text{ since } L \text{ is distributive} \\ &= 1 \wedge (b \vee c), \text{ by } (1) \\ &= b \vee c \qquad (3) \end{aligned}$$

Similarly,
$$\begin{aligned} c &= c \vee 0 = c \vee (a \wedge b), \text{ by } (2) \\ &= (c \vee a) \wedge (c \vee b), \text{ since } L \text{ is distributive} \\ &= 1 \wedge (c \vee b), \text{ by } (1) \\ &= c \vee b \qquad (4) \end{aligned}$$

From (3) and (4), since $b \vee c = c \vee b$, we get $b = c$.

Note From the definition of complemented lattice and the previous property, it follows that every element a of a complemented and distributive lattice has a unique complement denoted by a'.

BOOLEAN ALGEBRA

Definition

A lattice which is complemented and distributive is called a Boolean Algebra, (which is named after the mathematician George Boole). Alternatively, Boolean Algebra can be defined as follows:

Definition

If B is a nonempty set with two binary operations + and •, two distinct elements 0 and 1 and a unary operation ′, then B is called a *Boolean Algebra* if the following basic properties hold for all a, b, c in B:

B1: $\left.\begin{aligned} a + 0 &= a \\ a \cdot 1 &= a \end{aligned}\right\}$ Identity laws

B2: $\left.\begin{aligned} a + b &= b + a \\ a \cdot b &= b \cdot a \end{aligned}\right\}$ Commutative laws

B3: $\left.\begin{aligned} (a + b) + c &= a + (b + c) \\ (a \cdot b) \cdot c &= a \cdot (b \cdot c) \end{aligned}\right\}$ Associative laws

B4 $\left.\begin{aligned} a + (b \cdot c) &= (a + b) \cdot (a + c) \\ a \cdot (b + c) &= (a \cdot b) + (a \cdot c) \end{aligned}\right\}$ Distributive laws

B5: $\left.\begin{aligned} a + a' &= 1 \\ a \cdot a' &= 0 \end{aligned}\right\}$ Complement laws.

Note

1. We have switched over to the symbols + and • instead of ∨ (join) and ∧ (meet) used in the study of lattices. The operations + and •, that will be used hereafter in Boolean algebra, are called *Boolean sum* and *Boolean product* respectively. We may even drop the symbol • and instead use juxtaposition. That is $a \bullet b$ may be written as ab.
2. If B is the set $\{0, 1\}$ and the operations +, •, ′ are defined for the elements of B as follows:
$$0 + 0 = 0;\ 0 + 1 = 1 + 0 = 1 + 1 = 1$$
$$0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0;\ 1 \cdot 1 = 1$$
$$0' = 1 \text{ and } 1' = 0,$$
then the algebra $\{B, +, \bullet, ', 0, 1\}$ satisfies all the 5 properties given above and is the simplest Boolean algebra called a two-element Boolean algebra. It can be proved that two element Boolean algebra is the only Boolean algebra.
If a variable x takes on only the values 0 and 1, it is called a *Boolean variable*.
3. 0 and 1 are merely symbolic names and, in general, have nothing to do with the numbers 0 and 1. Similarly + and • are merely binary operators and, in general, have nothing to do with ordinary addition and multiplication.

ADDITIONAL PROPERTIES OF BOOLEAN ALGEBRA

If $\{B, +, \bullet, ', 0, 1\}$ is a Boolean algebra, the following properties hold good. They can be pro[illegible]d by using the basic properties of Boolean algebra liste[illegible] the definition.

(i) Idempotent Laws

$$a + a = a \quad \text{and} \quad a \cdot a = a, \quad \text{for all } a \in B$$

Proof

$$\begin{aligned} a &= a + 0, \text{ by } B1 \\ &= a + a \cdot a', \text{ by } B5 \\ &= (a + a) \cdot (a + a'), \text{ by } B4 \\ &= (a + a) \cdot 1, \text{ by } B5 \\ &= a + a, \text{ by } B1 \end{aligned}$$

Now,

$$\begin{aligned} a &= a \cdot 1, \text{ by } B1 \\ &= a \cdot (a + a'), \text{ by } B5 \\ &= a \cdot a + a \cdot a', \text{ by } B4 \\ &= a \cdot a + 0, \text{ by } B5 \\ &= a \cdot a, \text{ by } B1. \end{aligned}$$

(ii) Dominance Laws

$$a + 1 = 1 \text{ and } a \cdot 0 = 0, \text{ for all } a \in B.$$

Proof

$$\begin{aligned} a + 1 &= (a + 1) \cdot 1, \text{ by } B1 \\ &= (a + 1) \cdot (a + a'), \text{ by } B5 \\ &= a + 1 \cdot a', \text{ by } B4 \\ &= a + a' \cdot 1, \text{ by } B2 \\ &= a + a', \text{ by } B1 \\ &= 1, \text{ by } B5. \end{aligned}$$

Now

$$\begin{aligned} a \cdot 0 &= a \cdot 0 + 0, \text{ by } B1 \\ &= a \cdot 0 + a \cdot a', \text{ by } B5 \\ &= a \cdot (0 + a'), \text{ by } B4 \\ &= a \cdot (a' + 0), \text{ by } B2 \\ &= a \cdot a', \text{ by } B1 \\ &= 0, \text{ by } B5 \end{aligned}$$

(iii) Absorption Laws

$$a \cdot (a + b) = a \text{ and } a + a \cdot b = a, \text{ for all } a, b \in B.$$

Proof

$$\begin{aligned} a \cdot (a + b) &= (a + 0) \cdot (a + b), \text{ by } B1 \\ &= a + 0 \cdot b, \text{ by } B4 \\ &= a + b \cdot 0, \text{ by } B2 \\ &= a + 0, \text{ by dominance law} \\ &= a, \text{ by } B1. \end{aligned}$$

Now

$$\begin{aligned} a + a \cdot b &= a \cdot 1 + a \cdot b, \text{ by } B1 \\ &= a \cdot (1 + b), \text{ by } B4 \\ &= a \cdot (b + 1), \text{ by } B2 \\ &= a \cdot 1 \text{ by dominance law} \\ &= a \text{ by } B1 \end{aligned}$$

...e following properties hold good.
...asic properties of Boolean algeb...

(iv) De Morgan's Laws

$$(a + b)' = a' \cdot b' \text{ and } (a \cdot b)' = a' + b', \text{ for all } a, b \in B.$$

Proof

Note If y is to be the complement of x, by definition, we must show that $x + y = 1$ and $x \cdot y = 0$.

$$\begin{aligned} (a + b) + a'b' &= \{(a + b) + a'\} \cdot \{(a + b) + b'\}, \text{ by } B4 \\ &= \{(b + a) + a'\} \cdot \{(a + b) + b'\}, \text{ by } B2 \\ &= \{b + (a + a')\} \cdot \{a + (b + b')\}, \text{ by } B3 \\ &= (b + 1) \cdot (a + 1), \text{ by } B5 \\ &= 1 \cdot 1, \text{ by dominance law} \\ &= 1, \text{ by } B1. \qquad (1) \end{aligned}$$

Now

$$\begin{aligned} (a + b) \cdot a'b' &= a'b' \cdot (a + b), \text{ by } B2 \\ &= a'b' \cdot a + a'b' \cdot b, \text{ by } B4 \\ &= a \cdot (a'b') + a' \cdot b'b, \text{ by } B3 \\ &= (a \cdot a') \cdot b' + a' \cdot (bb'), \text{ by } B_3 \text{ and } B_2 \\ &= 0 \cdot b' + a' \cdot 0, \text{ by } B5 \\ &= b' \cdot 0 + a' \cdot 0, \text{ by } B2 \\ &= 0 + 0, \text{ by dominance law} \\ &= 0, \text{ by } B1. \qquad (2) \end{aligned}$$

From (1) and (2), we get $a'b'$ is the complement of $(a + b)$. i.e. $(a + b)' = a'b'$.
[$\because$ the complement is unique]

Note The students are advised to give the proof for the other part in a similar manner.

(v) Double Complement or Involution Law

$$(a')' = a, \text{ for all } a \in B.$$

Proof

$$a + a' = 1 \text{ and } a \cdot a' = 0, \text{ by } B5$$

i.e. $a' + a = 1$ and $a' \cdot a = 0$, by $B2$

$\therefore$ a is the complement of a'

i.e. $(a')' = a$, by the uniqueness of the complement of a'. [See example (14)]

(vi) Zero and One Law

$$0' = 1 \text{ and } 1' = 0$$

Proof

$$\begin{aligned} 0' &= (aa')', \text{ by } B5 \\ &= a' + (a')', \text{ by De Morgan's law} \\ &= a' + a, \text{ by involution law} \\ &= a + a', \text{ by } B2 \\ &= 1, \text{ by } B5 \end{aligned}$$

Now $(0')' = 1'$

i.e. $0 = 1'$ or $1' = 0$.

DUAL AND PRINCIPLE OF DUALITY

Definition

The *dual* of any statement in a Boolean algebra B is the statement obtained by interchanging the operations + and • and interchanging the elements 0 and 1 in the original statement.

For example, the dual of $a + a(b + 1) = a$ is $a \bullet (a + b \bullet 0) = a$.

PRINCIPLE OF DUALITY

The dual of a theorem in a Boolean algebra is also theorem.

For example, $(a \cdot b)' = a' + b'$ is a valid result, since it is the dual of the valid statement $(a + b)' = a' \cdot b'$ [De Morgan's laws]. If a theorem in Boolean algebra is proved by using the axioms of Boolean algebra, the dual theorem can be proved by using the dual of each step of the proof of the original theorem. This is obvious from the proofs of additional properties of Boolean algebra.

SUBALGEBRA

If C is a nonempty subset of a Boolean algebra such that C itself is a Boolean algebra with respect to the operations of B, then C is called a *subalgebra* of B.

It is obvious that C is a subalgebra of B if and only if C is closed under the three operations of B, namely, +, • and ′ and contains the element 0 and 1.

BOOLEAN HOMOMORPHISM

If $\{B +, \bullet, ', 0, 1\}$ and $\{C, \cup, \cap, -, \alpha, \beta\}$ are two Boolean algebras, then a mapping $f : B \to C$ is called a *Boolean homomorphism*, if all the operations of Boolean algebra are preserved. viz., for any $a, b \in B$,

$$f(a + b) = f(a) \cup f(b),\ f(a \cdot b) = f(a) \cap f(b),$$

$$f(a') = \overline{f(a)},\ f(0) = \alpha \text{ and } f(1) = \beta,$$

where α and β are the zero and unit elements of C.

ISOMORPHIC BOOLEAN ALGEBRAS

Two Boolean algebras B and B' are said to be *isomorphic* if there is one-to-one correspondence between B and B' with respect to the three operations, viz. there exists a mapping $f: B \to B'$ such that $f(a + b) = f(a) + f(b)$, $f(a \cdot b) = f(a) \cdot f(b)$ and $f(a') = \{f(a)\}'$.

BOOLEAN EXPRESSIONS AND BOOLEAN FUNCTIONS

Definitions

A *Boolean expression* in n Boolean variables $x_1, x_2, \dots x_n$ is a finite string of symbols formed recursively as follows:

1. 0, 1, $x_1, x_2, \dots x_n$ are Boolean expressions.

2. If E_1 and E_2 are Boolean expressions, then $E_1 \cdot E_2$ and $E_1 + E_2$ are also Boolean expressions.
3. If E is a Boolean expression, E' is also a Boolean expression.

Note A Boolean exprssion in n variables may or may not contain all the n *literals*, viz., variables or their complements.

If $x_1, x_2, \dots, x_n$ are Boolean variables, a function from $B^n = \{(x_1, x_2, \dots, x_n\}$ to $B = \{0, 1\}$ is called a *Boolean function of degree n*. Each Boolean expression represents a Boolean function, which is evaluated by substituting the value 0 or 1 for each variable. The values of a Boolean function for all possible combinations of values of the variables in the function are often displayed in truth tables.

For example, the values of the Boolean function $f(a, b, c) = ab + c'$ are displayed in the following truth table:

$B^n \to \{0,1\}$ $(x_1, x_2, \dots x_n)$

a	b	c	ab	c'	$ab + c'$
1	1	1	1	0	1
1	1	0	1	1	1
1	0	1	0	0	0
1	0	0	0	1	1
0	1	1	0	0	0
0	1	0	0	1	1
0	0	1	0	0	0
0	0	0	0	1	1

Note Although the order of the variable values may be random, a symmetric way of writing them in a cyclic manner which will be advantageous is as follows:

If there be n variables in the Boolean function, there will obviously be 2^n rows in the truth table corresponding to all possible combinations of the values 0 and 1 of the variables.

We write $\frac{1}{2} \times 2^n$ ones followed by $\frac{1}{2} \times 2^n$ zeros in the first column representing the values of the first variable.

Then in the second column, we write $\frac{1}{4} \times 2^n$ ones and $\frac{1}{4} \times 2^n$ zeros alternately, representing the values of the second variable. Next in the third column, we write $\frac{1}{8} \times 2^n$ ones and $\frac{1}{8} \times 2^n$ zeros alternately, representing the values of the third variable. We continue this procedure and in the final column, we write $\frac{1}{2^n} \times 2^n (=1)$ one and 1 zero alternately, representing the values of the n^{th} variable.]

Definitions of

1. A *minterm* of n Boolean variables is a Boolean product of the n literals (variables of complements) in which each literal appears exactly once.