Sardar Vallabhbhai National Institute of Technology, Surat

Department of Applied Mathematics and Humanities

B.Tech - I   Sem - 1   Branch - All   Subject - Mathematics - I (MA 101 S1)

Tutorial - 8 : Beta and Gamma Function

1. Define Beta function and Gamma function. State and prove relation between Beta and Gamma function.

2. Prove the following properties of Beta function:

   i. $B(m,n) = B(n,m)$

   ii. $B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$

   iii. $nB(m+1,n) = mB(m,n+1)$

   iv. $B(m,n) = B(m,n+1) + B(m+1,n)$.

3. Prove the following:

   i. $\Gamma(n) = \int\limits_0^1 (\log \frac{1}{y})^{n-1} \, dy$.

   ii. $\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}$.

4. Express $\int_0^1 x^m(1-x^n)^p dx$ in terms of gamma function and hence evaluate the integral $\int_0^1 x^5(1-x^3)^{10}dx$.

5. Prove that $\int_0^\infty \frac{dx}{(e^x+e^{-x})^n} = \frac{1}{4}B\left(\frac{n}{2}, \frac{n}{2}\right)$. Hence evaluate $\int_0^\infty \sec h^8 x \, dx$. *Ans* : 16/35.

6. Prove that $B(m,n) = \int\limits_0^1 \frac{x^{m-1}+x^{n-1}}{\sqrt{1+x}^{m+n}} \, dx$.

7. Prove that $B(m,m)\cdot B(m+\frac{1}{2}, m+\frac{1}{2}) = \frac{\pi}{m}\cdot 2^{1-4m}$.

8. Prove that $\int\limits_0^1 \frac{x}{\sqrt{1-x^5}} \, dx = \frac{1}{5}\cdot B(\frac{2}{5}, \frac{1}{2})$.

9. Show that $\int\limits_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}} \, dx = \frac{1}{a^n b^m}\cdot B(m,n)$.

10. Prove that

    (a) $\int\limits_0^\infty \sec h^6 x \, dx = \frac{8}{15}$.

    (b) $\int\limits_0^1 \frac{dx}{\sqrt{-\log x}} = \sqrt{\pi}$

    (c) $\int\limits_0^\infty e^{y^{-\frac{1}{m}}} dy = m\cdot\Gamma(m)$

    (d) $\int\limits_0^{\frac{\pi}{2}} \sqrt{\cot\theta} d\theta = \frac{1}{2}\cdot\Gamma(\frac{1}{4})\cdot\Gamma(\frac{3}{4})$.

11. Show that $\int\limits_{-\infty}^{\infty} e^{-k^2x^2} \, dx = \frac{\pi}{2}$ and $\int\limits_0^\infty e^{-x^3} \, dx = \frac{1}{3}\cdot\Gamma(\frac{1}{3})$.

12. Prove that $\int_a^b (x-a)^{l-1}(b-x)^{m-1}dx = (b-a)^{l+m-1}B(l,m)$, where $l>0, m>0, b>a$.

13. Prove that $\int_0^\infty x^{m-1} \cos ax \; dx = \frac{\Gamma(m)}{a^m} \cos\left(\frac{m\pi}{2}\right).$

14. Given $\int_0^\infty \frac{x^{n-1}}{1+x} dx = \frac{\pi}{\sin n\pi}$, show that $\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}$. Hence evaluate $\int_0^\infty \frac{dy}{1+y^4}$.

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