18.1

Chapter 18

Partial Differential Equations

INTRODUCTION

Real world problems in general involve functions of several (independent) variables giving rise to partial differential equations more often than ordinary differential equations. Thus most problems in engineering and science abound with first and second order linear non homogeneous partial differential equations. In this chapter, we consider methods of obtaining solutions by Lagrange's and Charpits method for first order. The general solution of non homogeneous second order linear P.D.E. with constant coefficients is obtained as the sum of complementary function and particular integral. Monge's method is also considered for solving nonlinear second order P.D.E.

18.1 PARTIAL DIFFERENTIAL EQUATIONS

A partial differential equation is an equation involving two (or more) independent variables x, y and a dependent variable z and its partial derivatives such as $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}$, etc.,

$$\text{i.e., } F\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}, \ldots\right) = 0$$

Standard notation

$$p = \frac{\partial z}{\partial x} = z_x, \quad q = \frac{\partial z}{\partial y} = z_y, \quad r = \frac{\partial^2 z}{\partial x^2} = z_{xx},$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = z_{xy}, \quad t = \frac{\partial^2 z}{\partial y^2} = z_{yy}$$

Order of a partial differential equation (P.D.E.) is the order of the highest ordered derivative appearing in the P.D.E.

Formation of Partial Differential Equation

By elimination of arbitrary constants

Let

$$f(x, y, z, a, b) = 0 \tag{1}$$

be an equation involving two arbitrary constants a and b. Differentiating this equation partially with respect to x and y, we get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = 0 \tag{2}$$

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y} = 0 \tag{3}$$

By eliminating a, b from (1), (2), (3), we get an equation of the form

$$F(x, y, z, p, q) = 0 \tag{4}$$

which is a partial differential equation of first order.

Note 1: If the number of arbitrary constants equals to the number of independent variables in (1), then the P.D.E. obtained by elimination is of first order.

Note 2: If the number of arbitrary constants is more than the number of independent variables then the P.D.E. obtained is of 2nd or higher orders.

By elimination of arbitrary functions of specific functions

a. *One arbitrary function* (resulting in first order P.D.E.):

Consider

$$z = f(u) \tag{5}$$

where $f(u)$ is an arbitrary function of u and u is a given (known) function of x, y, z i.e., $u = u(x, y, z)$.

Differentiating (5) partially w.r.t. x and y by chain rule

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial z}\frac{\partial z}{\partial x} \tag{6}$$

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial z}\frac{\partial z}{\partial y} \tag{7}$$

By eliminating the arbitrary function f from (5), (6), (7) we get a P.D.E. of first order.

b. Two arbitrary functions:

Differentiating twice or more, the elimination process results in a P.D.E. of 2nd or higher order.

Note: When n is the number of arbitrary functions, one may get several P.D. equations. But generally the one with the least order is chosen.

Example: For $z = x\,f(y) + y\,g(x)$ involving two arbitrary functions f and g, $\frac{\partial^4 z}{\partial x^2 \partial y^2} = 0$ is also a P.D.E. obtained by elimination. The other P.D.E. $xys = xp + yq - z$ of second order obtained by elimination may be chosen.

Elimination of Arbitrary Function F

from the equation

$$F(u, v) = 0 \tag{8}$$

where $u = u(x, y, z)$ and $v = v(x, y, z)$ are given functions of x, y, z.

Differentiating the Equation (8) partially w.r.t. x by chain rule, we get

$$\frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right) + \frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right) = 0 \tag{9}$$

Similarly, differentiating Equation (8) partially w.r.t. y, we get

$$\frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial y}\right) + \frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial y}\right) = 0 \tag{10}$$

Eliminating $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial v}$ from (9) and (10), we have

$$\begin{vmatrix} \frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x} & \frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x} \\ \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial y} & \frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial y} \end{vmatrix} = 0$$

Rewriting

$$\begin{bmatrix} u_x + pu_z & v_x + pv_z \\ u_y + qu_z & v_y + qv_z \end{bmatrix} = 0$$

Expansion of this determinant results in a P.D.E. which is free of the arbitrary function F as

$$(u_x + pu_z)(v_y + qv_z) - (u_y + qu_z)(v_x + pv_z) = 0$$

or

$$Pq + Qq = R$$

which is a first order linear P.D.E. Here

$$P = u_z v_y - u_y v_z = \frac{\partial(u, v)}{\partial(y, z)}$$

$$Q = u_x v_z - u_z v_x = \frac{\partial(u, v)}{\partial(z, x)}$$

$$R = u_y v_x - u_x v_y = \frac{\partial(u, v)}{\partial(x, y)}.$$

WORKED OUT EXAMPLES

Elimination of arbitrary constants

Form (obtain) partial differential equation by eliminating the arbitrary constants/functions:

Example 1: $z = ax^2 + by^2$

Solution: Differentiating partially w.r.t. x and y, we get

$$z_x = 2ax, z_y = 2by \quad \text{or} \quad a = \frac{z_x}{2x}, b = \frac{z_y}{2y}.$$

Eliminating the two arbitrary constants a and b

$$z = \frac{z_x}{2x}\cdot x^2 + \frac{z_y}{2y}\cdot y^2 \quad \text{or } 2z = xz_x + yz_y = xp + yq$$

Example 2: $(x-a)^2 + (y-b)^2 = z^2\cot^2\alpha$ where α is a parameter.

Solution: Differentiating partially w.r.t. x and y, we get

$$2(x-a) + 0 = 2z\cdot z_x\cdot\cot^2\alpha$$

$$2\cdot 0 + 2(y-b) = 2z\,z_y\cot^2\alpha$$

Substituting $(zz_x\cot^2\alpha)^2 + (zz_y\cot^2\alpha)^2 = z^2\cot^2\alpha$

$$p^2 + q^2 = \tan^2\alpha$$

Example 3: Find the differential equation of all spheres whose centres lie on the z-axis.

Solution: Equation $x^2 + y^2 + (z-a)^2 = b^2$ where a and b are arbitrary constants.

Differentiating

$$2x + 0 + 2(z-a)z_x = 0 \quad (1)$$

$$2y + 2(z-a)z_y = 0 \quad (2)$$

From (2),

$$(z-a) = -\frac{y}{z_y} \quad (3)$$

Substituting (3) in (1)

$$x + \left(-\frac{y}{z_y}\right)\cdot z_x = 0$$

$$xz_y - yz_x = 0$$

or

$$xq - yp = 0.$$

Example 4: $ax + by + cz = 1$

Solution: Differentiating w.r.t. x, $a + 0 + cz_x = 0$. Differentiating again w.r.t. x, $0 + cz_{xx} = 0$, since $c \neq 0$, $z_{xx} = 0$. Similarly by differentiating w.r.t. y and z twice, we get $z_{yy} = 0$, $z_{xy} = 0$ so $r = 0$ or $s = 0$ or $t = 0$. Thus we get 3 PDE.

Elimination of one arbitrary functions

Example 5:

$$z = (x+y)\,\phi\,(x^2 - y^2) \quad (1)$$

Solution: Differentiating

$$z_x = 1\cdot\phi + (x+y)2x\cdot\phi' \quad (2)$$

$$z_y = 1\cdot\phi + (x+y)(-2y)\phi' \quad (3)$$

From (3),

$$\phi' = \frac{\phi - z_y}{2y(x+y)} \quad (4)$$

Substituting (4) in (2)

$$z_x = \phi + 2x(x+y)\cdot\left[\frac{\phi - z_y}{2y(x+y)}\right]$$

$$p = \phi + \frac{x}{y}(\phi - q)$$

$$p = \left(\frac{x+y}{y}\right)\phi - \frac{x}{y}q$$

From the given Equation (1), $\phi = \frac{z}{(x+y)}$

Substituting ϕ,

$$p = \frac{(x+y)}{y}\cdot\frac{z}{x+y} - \frac{x}{y}q = \frac{z}{y} - \frac{x}{y}q$$

or $\quad yp + xq = z$

Example 6: $z = x^n f\left(\frac{y}{x}\right)$

Solution: By differentiation,

$$z_x = nx^{n-1}\cdot f + x^n\cdot\left(-\frac{y}{x^2}\right)\cdot f'$$

$$z_y = x^n\cdot\frac{1}{x}\cdot f' \quad \text{or } f' = \frac{z_y}{x^{n-1}}$$

Eliminating f',

$$z_x = nx^{n-1}f - x^{n-2}\cdot y\cdot\frac{z_y}{x^{n-1}}$$

$$xp = nx^n f - yq$$

or

$$xp = nz - yq$$

Example 7: $xyz = f(x+y+z)$

Solution: Differentiating w.r.t. x and y

$$yz + xy\,z_x = 1\cdot f' + f'\cdot z_x \quad (1)$$

$$xz + xy\,z_y = 1\cdot f' + f'\cdot z_y \quad (2)$$

From (2),

$$f' = \frac{xz + xy\,z_y}{1 + z_y} = \frac{xz + xyq}{1+q} \quad (3)$$

Put (3) in (1)

$$yz + xyp = (1+p)f' = (1+p)\left(\frac{xz+xyq}{1+q}\right)$$

$$(1+q)(yz+xyp) = (1+p)(xz+xyq)$$

or $$x(y-z)p + y(z-x)q = z(x-y)$$

Elimination of two arbitrary functions

Example 8: $z = f(x)\, g(y)$

Solution: Differentiating w.r.t. x and y, we get $z_x = f'g$, $z_y = fg'$

$$z_x \cdot z_y = f'g\, fg' = fg\, f'g' = z\, f'g'$$

But

$$z_{xy} = f'g' \quad \text{so}$$

$$z_x \cdot z_y = z \cdot z_{xy}$$

or $$pq = z \cdot s.$$

Example 9: $z = f(x+y) \cdot g(x-y)$

Solution: Differentiating partially w.r.t. x and y, we get

$$p = z_x = f' \cdot 1 \cdot g + f \cdot 1 \cdot g' \quad (1)$$

$$q = z_y = f' \cdot 1 \cdot g + f \cdot (-1)g' \quad (2)$$

$$r = z_{xx} = f''g + f'g' + f'g' + fg''$$

$$= f''g + 2f'g' + fg'' \quad (3)$$

$$t = z_{yy} = f''g + f'g'(-1)$$

$$-f'g' - fg''(-1)$$

$$= f''g - 2f'g' + fg'' \quad (4)$$

$$s = z_{xy} = f''g + f'g'(-1)$$

$$+f'g' + fg''(-1)$$

$$= f''g - fg'' \quad (5)$$

Adding (1) and (2),

$$f' = \frac{p+q}{2g} \quad (6)$$

Subtracting (2) from (1),

$$g' = \frac{p-q}{2f} \quad (7)$$

Adding (3) and (4)

$$r + t = 2(f''g + fg'')$$

From (5), $$2s = 2(f''g - fg'')$$

Adding $$r + t + 2s = 4f''g$$

Subtracting $$r + t - 2s = 4fg''$$

Substituting (6), (7), (8), (9) in (3)

$$r = \left(\frac{r+t+2s}{4}\right) + 2 \cdot \frac{(p+q)}{2g} \cdot \frac{(p-q)}{2f} + \left(\frac{r+t-2s}{4}\right)$$

$$(r-t)z = (p+q)(p-q)$$

Example 10: $z = xf(ax+by) + g(ax+by)$

Solution: Differentiating w.r.t. x and y, we get

$$z_x = f + xaf' + ag'$$

$$z_{xx} = af' + af' + ax\, f''a + a^2 g''$$

$$= a[2f' + axf'' + ag'']$$

$$z_y = bx\, f' + bg'$$

$$z_{yy} = b^2 xf'' + b^2 g'' = b^2[xf'' + g'']$$

$$z_{yx} = bf' + bx\, af'' + ba\, g''$$

$$= b[f' + a(xf'' + g'')]$$

Substituting (2) in (3)

$$\frac{z_{yx}}{b} = f' + a \cdot \frac{z_{yy}}{b^2}$$

Solving

$$f' = \frac{z_{yx}}{b} - \frac{a}{b^2} z_{yy} \quad (4)$$

Substituting (2) and (4) in (1)

$$z_{xx} = 2af' + a^2[xf'' + g'']$$

$$= 2a \cdot \left[\frac{z_{yx}}{b} - \frac{a}{b^2} z_{yy}\right] + a^2\left[\frac{z_{yy}}{b^2}\right]$$

$$b^2 z_{xx} + a^2 z_{yy} = 2ab\, z_{xy}$$

$$b^2 r + a^2 t = 2abs$$

Elimination of arbitrary function of specific functions

$F(u, v) = 0$ where u and v are given.

Example 11:

$$F(xy + z^2, x + y + z) = 0 \tag{1}$$

Solution:

$$\text{Let } u(x, y, z) = xy + z^2 \tag{2}$$

$$v(x, y, z) = x + y + z \tag{3}$$

Differentiating (1) partially w.r.t. x by chain rule

$$\frac{\partial F}{\partial u}\left[\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right] + \frac{\partial F}{\partial v}\left[\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right] = 0$$

i.e., $\quad F_u \cdot (y + 2z\, z_x) + F_v \cdot (1 + z_x) = 0$

Differentiating w.r.t. 'y', we get

$$F_u(x + 2z\, z_y) + F_v(1 + z_y) = 0$$

Eliminating F_u and F_v (i.e., the coefficient matrix should be singular)

$$\begin{vmatrix} y + 2z\, z_x & 1 + z_x \\ x + 2z\, z_y & 1 + z_y \end{vmatrix} = 0$$

or $\quad (1 + q)[y + 2zp] - (1 + p)[x + 2zq] = 0$

$$(2z - x)p + (y - 2z)q = x - y.$$

Example 12: $xyz = f(x + y + z)$

Solution: Put $u = x + y + z$, $v = xyz$ so that the given equation may be written as $F(u, v) = 0$

Differentiating w.r.t. x and y, we get

$$F_u \cdot (1 + z_x) + F_v(yz + xy\, z_x) = 0$$

$$F_u(1 + z_y) + F_v(xz + xy\, z_y) = 0$$

Eliminating F_u, F_v, we have

$$\begin{vmatrix} 1 + p & yz + xyp \\ 1 + q & xz + xyq \end{vmatrix} = 0$$

or $\quad (xz + xy\,q)(1 + p) - (1 + q)(yz + xyp) = 0$

$$x(z - y)p + (x - z)y\,q + z(x - y) = 0.$$

Example 13: $F(x^2 + y^2 + z^2, z^2 - 2xy) = 0$

Solution: Let $u = x^2 + y^2 + z^2$, $v = z^2 - 2xy$

Differentiating F partially w.r.t. x, we get

$$\frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial u}\frac{\partial u}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial F}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial z}\frac{\partial z}{\partial x} = 0$$

$$F_u \cdot 2x + F_u \cdot 2z \cdot p + F_v(-2y) + F_v\, 2z \cdot p = 0$$

Similarly w.r.t. y, we get

$$\frac{\partial F}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial F}{\partial u}\frac{\partial u}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial z}\frac{\partial z}{\partial y} = 0$$

$$F_u \cdot 2y + F_u\, 2z \cdot q + F_v(-2x) + F_v \cdot 2z \cdot q = 0$$

Solving

$$\begin{vmatrix} x + zp & -y + zp \\ y + zq & -x + zq \end{vmatrix} = 0$$

$$(x + zp)(zq - x) - (zp - y)(y + zq) = 0$$

$$xz(q - p) + yz(q - p) + (y^2 - x^2) = 0$$

$$(x + y)[z(q - p) + (y - x)] = 0.$$

Exercise

Form (obtain) partial differential equation by eliminating the arbitrary constants/functions:

Elimination of arbitrary constants

1. $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

Ans. $2z = xp + yq$

2. $z = ax + by + a^2 + b^2$

Ans. $z = px + qy + p^2 + q^2$

3. $z = (x^2 + a^2)(y^2 + b^2)$

Ans. $4xyz = pq$

4. $z = axy + b$

Ans. $px = qy$

5. $z = axe^y + \frac{1}{2}a^2e^{2y} + b$

Ans. $q = px + p^2$

6. $z = (x - a)^2 + (y - b)^2 + 1$

Ans. $4z = p^2 + q^2 + 4$

7. $z = a(x + y) + b(x - y) + abt + c$

Hint: Number of independent variables x, y, t are 3 = number of arbitrary constants a, b, c. So P.D.E. is of 1st order.

Ans. $z_x^2 - z_y^2 = 4z_t$

8. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Ans. $\quad pz = xp^2 + xz\,r$

or $\quad qz = yq^2 + zy\,t$

Note: Number of arbitrary constants a, b, c is three > number of independent variables x, y is two so P.D.E. is of 2nd order.

9. $z = ae^{-b^2 t}\cos bx$

Ans. $\frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2}$

10. $z = ae^{bx}\sin by$

Ans. $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$

11. $z = xy + y\sqrt{x^2 + a^2} + b$

Ans. $pq = py + qx$

Find the differential equation of:

12. All planes which are at a constant distance b from the origin.

Hint: Equation $lx + my + nz = b$ with $l^2 + m^2 + n^2 = 1$.

Ans. $z = px + qy + b\sqrt{1 + p^2 + q^2}$

13. All planes having equal x and y intercepts.

Hint: Equation $\frac{x}{a} + \frac{y}{a} + \frac{z}{b} = 1$.

Ans. $p = q$.

14. All spheres of given radius c having their centres in the xy-plane.

Hint: $(x-a)^2 + (y-b)^2 + z^2 = c^2$; a, b constants.

Ans. $z^2(p^2 + q^2 + 1) = c^2$

15. All cones with their vertices at the origin.

Ans. $px + qy = z$

Elimination of arbitrary functions

16. $z = f(x^2 - y^2)$

Ans. $yp + xq = 0$

17. $x + y + z = f(x^2 + y^2 + z^2)$

Ans. $(y - z)p + (z - x)q = x - y$

18. $z = y\,f\left(\frac{y}{x}\right)$

Ans. $z = px + qy$

19. $z = f(\sin x + \cos y)$

Ans. $p\sin y + q\cos x = 0$

20. $z = e^{ax+by}\cdot f(ax - by)$

Ans. $bp + aq = 2abz$

21. $z = y^2 + 2f\left(\frac{1}{x} + \ln y\right)$

Ans. $x^2 p + yq = 2y^2$

22. $z = x + y + f(xy)$

Ans. $xp - yq = x - y$

23. $z = f\left(\frac{xy}{z}\right)$

Ans. $xp = yq$

24. $z = f(x + at) + g(x - at)$

Ans. $z_{tt} = a^2 z_{xx}$

25. $z = f(x) + e^y g(x)$

Ans. $t = q$

26. $z = f(x + iy) + g(x - iy)$

Ans. $z_{xx} + z_{yy} = 0$

27. $z = yf(x) + xg(y)$

Ans. $xys = px + qy - z$

28. $z = xf\left(\frac{y}{x}\right) + yg(x)$

Ans. $x\frac{\partial^3 z}{\partial x \partial y^2} + \frac{\partial^2 z}{\partial y^2} + y\frac{\partial^3 z}{\partial y^3} = 0$

29. $z = f(xy) + g(x + y)$

Ans. $rx(y - x) - s(y^2 - x^2) + t\,y(y - x) + (p - q)(x + y) = 0$

30. $z = [f(r - at) + g(r + at)]/r$

Ans. $z_{tt} = \frac{a^2}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial z}{\partial r}\right)$

Elimination of arbitrary function F(u, v) = 0

31. $F(x^2 + y^2, z - xy) = 0$

Ans. $xq - yp = x^2 - y^2$

32. $F(x + y + z, x^2 + y^2 - z^2) = 0$

Ans. $(y + z)p - (z + x)q = x - y$

33. $F(x^2 + y^2, x^2 - z^2) = 0$

Ans. $yp - xq = \frac{xy}{z}$

34. $F(ax + by + cz, x^2 + y^2 + z^2) = 0$

Ans. $(bz - cy)p + (cx - az)q = ay - bx$

35. $z = x^2\phi(x - y)$

Hint: Rewrite the given equation in the form

$$F\left(\frac{z}{x^2}, x - y\right) = 0.$$

Ans. $2z = xp + xq$.

18.2 PARTIAL DIFFERENTIAL EQUATIONS OF FIRST ORDER

The general form of a first order partial differential equation is

$$F(x, y, z, p, q) = 0 \tag{1}$$

where x, y are the two independent variables, z is the dependent variable and $p = z_x, q = z_y$.

Complete Solution

Any function

$$f(x, y, z, a, b) = 0 \tag{2}$$

involving two arbitrary constants a, b and satisfying the P.D.E. (1) is known as complete solution or complete integral or primitive.

General Solution

of P.D.E. (1) is any arbitrary function F of specific (given) functions u, v

$$F(u, v) = 0 \tag{3}$$

satisfying P.D.E. (1).

Here $u = u(x, y, z)$ and $v = v(x, y, z)$ are known functions of x, y, z.

Linear

A partial differential equation is said to be linear (after rationalization and cleared of fractions) if the dependent variable z and its derivatives are of degree (power) one and products of z and its derivatives do not appear in the equation.

Quasi-linear

P.D.E. is said to be quasi-linear if degree of highest ordered derivative is one and no products of partial derivatives of the highest order are present.

Example: $x^2 p + y^2 q = z$
is linear in z and of first order.

Example: $z\, z_{xx} + (z_y)^2 = 0$ is a quasi-linear of second order.

Non-linear

A P.D.E. which is not linear is known as non-linear P.D.E.

Example: $\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + u^2\left(\frac{\partial u}{\partial y}\right) = f(x, y)$

is non-linear in u and of second order.

Homogeneous

if each term contains the dependent variable or its derivatives.

Otherwise non-homogeneous.

18.3 LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF FIRST ORDER

The general form of a quasi-linear partial differential equation of the first order is

$$P(x, y, z)z_x + Q(x, y, z)z_y = R(x, y, z) \tag{1}$$

This Equation (1) is known as **"Lagrange's linear equation"**.

If P and Q are independent of z and R is linear in z then (1) is a linear equation. The general solution of the Lagrange's linear P.D.E.

$$Pp + Qq = R \tag{1}$$

is given by the equation

$$F(u, v) = 0 \tag{2}$$

since the elimination of the arbitrary function F from (2) results in (1).

Here $u = u(x, y, z)$, $v = v(x, y, z)$ are specific (known) functions of x, y, z.

Method of Obtaining General Solution

1. Rewrite the equation in the standard form

$$Pp + Qq = R$$

2. Form the Lagrange's auxiliary equations (A.E.)

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \tag{3}$$

3. Nature of solution to the simultaneous equations of the form $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$: $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ are said to be the complete solution of the system of simultaneous equations (provided u_1 and u_2 are linearly independent i.e., $u_1/u_2 \neq$ constant).

Case 1: One of the variables is either absent or cancells out from the set of auxiliary equations.

Case 2: If $u = c_1$ is known but $v = c_2$ is not possible by case I, then use $u = c_1$ to get $v = c_2$.

Case 3: Introducing Lagrange's multipliers P_1, Q_1, R_1, which are functions of x, y, z or constants, each fraction in (3) is equal to

$$\frac{P_1 dx + Q_1 dy + R_1 dz}{P_1 P + Q_1 Q + R_1 R} \tag{4}$$

If P_1, Q_1, R_1 are so chosen that $P_1 P + Q_1 Q + R_1 R = 0$ then $P_1 dx + Q_1 dy + R_1 dz = 0$ which can be integrated.

Case 4: Multipliers may be chosen (more than once) such that the numerator $P_1 dx + Q_1 dy + R_1 dz$ is an exact differential of the denominator $P_1 P + Q_1 Q + R_1 R$. Now combine (4) with a fraction of (3) to get an integral.

4. General solution of (1) is

$$F(u, v) = 0 \quad \text{or} \quad v = \phi(u).$$

WORKED OUT EXAMPLES

Solve the following:

Example 1: $xp + yq = 3z$

Solution: This is a linear P.D.E. of first order $Pp + Qq = R$ with $P = x, Q = y$ and $R = 3z$. The Lagrange's auxiliary equations are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{3z}$$

Integrating the first two equations (or fractions) $\frac{dx}{x} = \frac{dy}{y}$; we get $\ln x = \ln y + c_1$ or $\frac{x}{y} = c$

Integrating first and the last equations $\frac{dx}{x} = \frac{dz}{3z}$, we have

$$3 \ln x = \ln z + c_2 \quad \therefore x^3 = c_1 z$$

Thus the required solution is $x^3 = zf\left(\frac{x}{y}\right)$. The general solution can also be written as $F(x^3/z, x/y) = 0$.

Note: By integrating 2nd and 3rd equations $\frac{dy}{y} = \frac{dz}{3z}$, we also get $y^3 = c_2 z$ so the general solution is also given by $y^3 = zf\left(\frac{x}{y}\right)$.

Example 2: $yzp - xzq = xy$

Solution: Auxiliary equations are

$$\frac{dx}{yz} = \frac{dy}{-xz} = \frac{dz}{xy}$$

From first and second fractions, we get

$$\frac{dx}{yz} = \frac{dy}{-xz}$$

or

$$\frac{dx}{y} = \frac{dy}{-x}$$

or

$$x dx + y dy = 0$$

Integrating $x^2 + y^2 = c_1$

From first and third fractions

$$\frac{dx}{yz} = \frac{dz}{xy}$$

or

$$\frac{dx}{z} = \frac{dz}{x}$$

Integrating $x^2 - z^2 = c_2$

Thus the general solution is

$$F(x^2 + y^2, x^2 - z^2) = 0.$$

Example 3: $p - q = \ln (x + y)$

Solution: Auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{\ln (x + y)}$$

Integrating the first two fractions $dx + dy = 0$ yields $x + y = c_1$

From first and last fractions $\ln (x + y) dx = dz$

Put $x + y = c_1$, then $\ln c_1 dx = dz$

Integrating $x \ln c_1 = z + c_2$

or $\quad x \cdot \ln(x+y) = z + c_2$

The general solution is

$$F(x+y, x\ln(x+y) - z) = 0.$$

Example 4: $z(z^2+xy)(px - qy) = x^4$

Solution: Auxiliary equations are

$$\frac{dx}{xz(z^2+xy)} = \frac{dy}{-yz(z^2+xy)} = \frac{dz}{x^4}$$

From first and second fractions, we get

$$\frac{dx}{x} = \frac{dy}{-y}$$

on integration $xy = c_1$

From first and third fractions

$$x^3 dx = (z^3 + xyz)dz$$

using $xy = c_1$, $\quad x^3 dx = (z^3 + c_1 z)dz$

Integrating $\frac{x^4}{4} = \frac{z^4}{4} + c_1\frac{z^2}{2} + c_2$

or $\quad x^4 - z^4 - 2c_1 z^2 = c_2$

Substituting for c_1, $x^4 - z^4 - 2(xy)z^2 = c_2$

The general solution is

$$F(xy, x^4 - z^4 - 2xyz^2) = 0$$

Example 5: $xzp + yzq = xy$

Solution: Auxiliary equations $\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{xy}$

From one and two $\frac{dx}{xz} = \frac{dy}{yz}$ or $\frac{dx}{x} = \frac{dy}{y}$

Integrating $x = c_1 y$

Choosing the multipliers as $y, x, 2z$

$$\frac{ydx + xdy}{yxz + xyz} = \frac{2zdz}{2zxy}$$

or $\quad ydx + xdy - 2z\,dz = 0$

$$d(xy) - d(z^2) = 0$$

Integrating $xy - z^2 = c_2$

The general solution is

$$F\left(\frac{x}{y}, xy - z^2\right) = 0.$$

Example 6: $(z-y)\,p + (x-z)\,q = y - x$

Solution: Auxiliary equations are

$$\frac{dx}{z-y} = \frac{dy}{x-z} = \frac{dz}{y-x}$$

Choosing multipliers as, 1, 1, 1

$$dx + dy + dz = (z-y) + (x-z) + (y-x) = 0$$

Integrating $x + y + z = c_1$

Choosing multipliers as x, y, z

$$xdx + ydy + zdz = x(z-y) + y(x-z) + z(y-x) = 0$$

Integrating $x^2 + y^2 + z^2 = c_2$

The general solution is

$$F(x+y+z, x^2+y^2+z^2) = 0$$

Example 7: $(y+zx)p - (x+yz)q = x^2 - y^2$

Solution: Auxiliary equations are

$$\frac{dx}{y+zx} = \frac{dy}{-(x+yz)} = \frac{dz}{x^2-y^2}$$

Choosing multipliers as $x, y, -z$

$$xdx + ydy - zdz$$
$$= x(y+zx) + y(-1)(x+yz) - z(x^2-y^2) = 0$$

Integrating $x^2 + y^2 - z^2 = c_1$

Choosing multipliers as $y, x, 1$, we get

$$ydx + xdy + dz$$
$$= y(y+zx) + x(-1)(x+yz) + (x^2-y^2) = 0$$

Integrating $xy + z = c_2$

The general solution is

$$F(x^2+y^2-z^2, xy+z) = 0.$$

Example 8: $(y^2+z^2)p - xyq + zx = 0$

Solution: Auxiliary equations are

$$\frac{dx}{y^2+z^2} = \frac{dy}{-xy} = \frac{dz}{-zx}$$

From the 2nd and 3rd fractions

$$\frac{dy}{y} = \frac{dz}{z} \quad \text{or} \quad \frac{y}{z} = c_1$$

Choosing multipliers as x, y, z

$$xdx + ydy + zdz = x(y^2+z^2) + y(-xy) + z(-zx) = 0$$

Integrating $x^2 + y^2 + z^2 = c_2$
The general solution is

$$F\left(\frac{y}{z}, x^2 + y^2 + z^2\right) = 0.$$

Example 9: $px(x + y) = qy(x + y)-$
$-(2x + 2y + z)(x - y)$

Solution: Auxiliary equations are

$$\frac{dx}{x(x + y)} = \frac{dy}{-y(x + y)} = \frac{dz}{-(x - y)(2x + 2y + z)}$$

From first two fractions, cancelling $(x + y)$, we get

$$\frac{dx}{x} = -\frac{dy}{y} \quad \text{or} \quad d(\ln x) + d(\ln y) = c$$

which on integration gives $xy = c_1$

$$\frac{dx + dy}{x(x + y) - y(x + y)} = \frac{dx + dy}{(x + y)(x - y)}$$

$$= \frac{dz}{-(x - y)(2x + 2y + z)}$$

Cancelling the $(x - y)$ term, we get

$$(2x + 2y + z)(dx + dy) + (x + y)dz = 0$$

or

$$(x+y+z)(dx+dy)+(x+y)(dx+dy)+(x+y)dz = 0$$

$$(x+y+z)\, d(x+y)+(x+y)\, d(x+y+z) = 0$$

i.e.,

$$d\,((x + y)(x + y + z)) = 0$$

Integrating $(x + y)(x + y + z) = c_2$
Thus the general solution is

$$F(xy, (x + y)(x + y + z)) = 0$$

Example 10: $(x^2 - y^2 - yz)p +$
$(x^2 - y^2 - zx)q = z(x - y)$

Solution: Auxiliary equations are

$$\frac{dx}{x^2 - y^2 - yz} = \frac{dy}{x^2 - y^2 - zx} = \frac{dz}{z(x - y)}$$

$$dx - dy - dz = x^2 - y^2 - yz - (x^2 - y^2 - zx)$$

$$-z(x - y) = 0$$

Integrating $x - y - z = c_1$

From first and second

$$\frac{xdx - ydy}{x^3 - xy^2 - x^2y - y^3} = \frac{dz}{z(x - y)}$$

or

$$\frac{xdx - ydy}{(x^2 - y^2)(x - y)} = \frac{dz}{z(x - y)}$$

i.e.,

$$\frac{1}{2}d(\ln(x^2 - y^2)) = d(\ln z)$$

$\therefore$

$$(x^2 - y^2)/z^2 = c_2.$$

$\therefore$ The general solution is

$$f(x - y - z; (x^2 - y^2)/z^2) = 0$$

EXERCISE

Solve the following:

1. $y^2zp + x^2zq = xy^2$

Ans. $F(x^3 - y^3, x^2 - z^2) = 0$

2. $p \tan x + q \tan y = \tan z$

Ans. $F\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right) = 0$

3. $xp + yq = z$

Ans. $F\left(\frac{x}{z}, \frac{y}{z}\right) = 0$

4. $2p + 3q = 1$

Ans. $F(3x - 2y, y - 3z) = 0$

5. $x^2p + y^2q = z^2$

Ans. $F\left(\frac{1}{x} - \frac{1}{y}, \frac{1}{y} - \frac{1}{z}\right) = 0$

6. $pyz + qzx = xy$

Ans. $F(x^2 - y^2, y^2 - z^2) = 0$

7. $x^2p + y^2q = (x + y)z$

Ans. $F\left(\frac{xy}{z}, \frac{x-y}{z}\right) = 0$

8. $p + 3q = 5z + \tan(y - 3x)$

Hint: Use the solution $y - 3x = c$ obtained from $\frac{dx}{1} = \frac{dy}{3}$.

Ans. $F(y - 3x, e^{-5x}\{5z + \tan(y - 3x)\}) = 0$

9. $z(p - q) = z^2 + (x + y)^2$

Hint: Use solution $x+y=c$ obtained from $\frac{dx}{1}=\frac{dy}{-1}$.

Ans. $F[x+y, e^{2y}\{z^2+(x+y)^2\}]=0$

10. $z(xp-yq)=y^2-x^2$

Hint: Use x, y, z as multipliers.

Ans. $F(xy, x^2+y^2+z^2)=0$

11. $(x^3+3xy^2)p+(y^3+3x^2y)q=2(x^2+y^2)z$

Hint: Choose multipliers $(1/x, 1/y, 0)$, $(1, 1, 0)$, $(1, -1, 0)$.

Ans. $F[(x-y)^{-2}-(x+y)^{-2}, xy/z^2]=0$

12. $x^2(y^3-z^3)p+y^2(z^3-x^3)q=z^2(x^3-y^3)$

Ans. $F(x^2+y^2+z^2, \frac{1}{x}+\frac{1}{y}+\frac{1}{z})=0$

13. $(2x^2+y^2+z^2-2yz-zx-xy)p+(x^2+2y^2+z-yz-2zx-xy)q=(x^2+y^2+2z^2-yz-2xy)$

Hint: $\frac{dx-dy}{x-y}=\frac{dy-dz}{y-z}=\frac{dz-dx}{z-x}$.

Ans. $F\left(\frac{x-y}{y-z}, \frac{y-z}{z-x}\right)=0$

14. $(mz-ny)p+(nx-lz)q=ly-mx$

Ans. $F(x^2+y^2+z^2, lx+my+nz)=0$

15. $(x^2-yz)p+(y^2-zx)q=z^2-xy$

Ans. $F\left(\frac{x-y}{y-z}, xy+yz+zx\right)=0$

16. $(x^2-y^2-z^2)p+2xyq=2xz$

Ans. $F(x^2+y^2+z^2, y/z)=0$

17. $(x+2z)p+(4zx-y)q=2x^2+y$

Hint: Multipliers $y, x, -2z$ and $2x, -1, -1$.

Ans. $F(xy-z^2, x^2-y-z)=0$

18. $x(y-z)p+y(z-x)q=z(x-y)$

Ans. $F(x+y+z, xyz)=0$

19. $(y-z)p+(x-y)q=(z-x)$

Ans. $F(x+y+z, \frac{x^2}{2}+yz)=0$

20. $(y+z)p+(z+x)q=x+y$

Ans. $F\left(\frac{x-y}{y-z}, \frac{y-z}{\sqrt{x+y+z}}\right)=0$

21. $x^2(y-z)p+y^2(z-x)q=z^2(x-y)$.

Hint: Multipliers: $1/x, 1/y, 1/z$ and $1/x^2, 1/y^2, 1/z^2$.

Ans. $F\left(xyz, \frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=0$

22. $(z^2-2yz-y^2)p+(xy+zx)q=xy-zx$

Ans. $F(x^2+y^2+z^2, y^2-2yz-z^2)=0$

23. $px(z-2y^2)=(z-qy)(z-y^2-2x^3)$

Hint: Multiply A.E. by $(z-y^2-2x^3)$, use multipliers $1, 2xy, -x$, divide by x^2 throughout.

Ans. $F\left(y/z, \frac{y^2}{x}-\frac{z}{x}-x^2\right)=0$

24. $x(y^2-z^2)p+y(z^2-x^2)q=z(x^2-y^2)$

Hint: Multipliers x, y, z and $1/x, 1/y, 1/z$.

Ans. $F(x^2+y^2+z^2, xyz)=0$.

18.4 NON-LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF FIRST ORDER

Non-linear P.D.E. of first order contains p and q of degree (power) other than one and/or product terms of p and q. Its complete solution is given by $f(x, y, z, a, b)=0$ where a and b are any two arbitrary constants. Some special types of non-linear first order P.D.E. are presented,

Form I: $f(p, q)=0$

i.e., equation contains only p and q (or x, y, z are absent)

Assume that $p=a$ then $f(a, q)=0$

Solving $q=\phi(a)$

Consider

$$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy=p\,dx+q\,dy$$

$$dz=a\,dx+\phi(a)dy$$

Integrating $z=ax+\phi(a)y+c$

where a and c are arbitrary constants.

Thus the complete solution is

$$z=ax+by+c$$

where a, b satisfy the equation $f(a, b)=0$

i.e.,
$$b=\phi(a)$$

Form II: $f(z, p, q) = 0$

i.e., equation does not involve the independent variables x and y.

Assume $q = ap$. Substituting q in the given equation $f(z, p, ap) = 0$ and solving for p, we get $p = \phi(z)$. Now

$$dz = p\,dx + q\,dy = p\,dx + ap\,dy$$

$$dz = p(dx + a\,dy) = \phi(z)(dx + a\,dy)$$

Integrating $x + ay = \int \frac{dz}{\phi(z)} + b$

where a and b are two arbitrary constants.

Form III: $f(x, p) = g(y, q)$

i.e., x, p and y, q are separable.

Assume $f(x, p) = g(y, q) = a =$ constant.

Solving each equation for p and q, we get

$$p = f_1(x, a) \quad \text{and} \quad q = g_1(y, a)$$

Now $dz = p\,dx + q\,dy = f_1(x, a)dx + g_1(y, a)dy$

Integrating $z = \int f_1(x, a)dx + \int g_1(y, a)dy + b.$

Form IV: Clairaut Equation:
$z = px + qy + f(p, q)$

The complete solution of this equation is

$$z = ax + by + f(a, b)$$

which is obtained by replacing p by a and q by b in the given clairaut equation.

Note: All these four forms can be solved by Charpit's method.

Worked Out Examples

Form I: $f(p, q) = 0$

Solve the following:

Example 1: $p^3 - q^3 = 0$

Solution: The complete solution is

$$z = ax + by + c$$

where a, b are connected by $a^3 - b^3 = 0$ or $a = b$

Thus $z = ax + ay + c$ is the complete integral.

Example 2: $p^2 + q^2 = npq$

Solution: The complete solution is

$$z = ax + by + c$$

where a, b satisfy the equation $a^2 + b^2 = nab$.

Solving for b

$$b = \frac{-na \pm \sqrt{n^2a^2 - 4a^2}}{2} = +\frac{a}{2}\left[+n \pm \sqrt{n^2 - 4}\right]$$

so complete integral:

$$z = ax + \tfrac{a}{2}\left(n \mp \sqrt{n^2 - 4}\right)y + c$$

Example 3: $(x + y)(p + q)^2 + (x - y)(p - q)^2 = 1$

Solution: To reduce this equation to $f(P, Q) = 0$ form, put $x + y = X^2$ and $x - y = Y^2$. Then

$$X = \sqrt{x + y}, \quad Y = \sqrt{x - y}$$

$$p = z_x = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\frac{\partial X}{\partial x} + \frac{\partial z}{\partial Y}\cdot\frac{\partial Y}{\partial x}$$

$$= \frac{1}{2X}\frac{\partial z}{\partial X} + \frac{1}{2Y}\frac{\partial z}{\partial Y}$$

since

$$\frac{\partial X}{\partial x} = \frac{1}{2}\frac{1}{\sqrt{x + y}} = \frac{1}{2X} \quad \text{and} \quad \frac{\partial Y}{\partial x} = \frac{1}{2}\frac{1}{\sqrt{x - y}} = \frac{1}{2Y}$$

Similarly,

$$q = z_y = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial X}\frac{\partial X}{\partial y} + \frac{\partial z}{\partial Y}\cdot\frac{\partial Y}{\partial y} = \frac{1}{2X}z_x - \frac{1}{2Y}z_y$$

Adding $p + q = z_x \cdot \frac{1}{X}$,

subtracting $p - q = z_y\frac{1}{Y}$.

Thus the given equation reduces to

$$X^2 \cdot \left(z_x\frac{1}{X}\right)^2 + Y^2\left(z_y\frac{1}{Y}\right)^2 = 1 \quad \text{or} \quad z_X^2 + z_Y^2 = 1$$

The complete solution is

$$z = aX + bY + c$$

where $a^2 + b^2 = 1$ or $b = \sqrt{1 - a^2}$

or $$z = a\sqrt{(x + y)} + \sqrt{1 - a^2}\sqrt{(x - y)} + c.$$

Example 4: $(x - y)(px - qy) = (p - q)^2$

Solution:

Put $$x + y = u \quad \text{and} \quad xy = v \tag{1}$$

Then $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = z_u + yz_v \tag{2}$$

Since $u_x = 1, u_y = 1, v_x = y, v_y = x$

Similarly, $q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = z_u + xz_v$ (3)

From (2) and (3)

$$p - q = (y - x)z_v \quad (4)$$

$$xp - qy = (x - y)z_u \quad (5)$$

Using (4), (5) the given equation transforms to

$$(x - y)(x - y)z_u = (y - x)^2 z_v^2$$

or

$$z_u = z_v^2$$

$$P = Q^2$$

Its complete solution is

$$z = au + bv + c$$

where $a = b^2$.

Using (1), replace u, v then

$$z = b^2(x + y) + bxy + c$$

where b, c are arbitrary constants.

Form II: $f(z, p, q) = 0$

Example 5: $zpq = p + q$

Solution: Assume $q = ap$.
Substituting in the given equation

$$zp \cdot ap = p + ap$$

Solving for p, we get

$$p = \frac{1 + a}{az}$$

We know that $dz = pdx + qdy = p(dx + ady)$

$dz = \frac{1 + a}{az}(dx + ady)$ or $az\,dz = (1 + a)(dx + ady)$

Integrating: $a\frac{z^2}{2} = (1 + a)[x + ay] + b$.

Example 6: $p^2z^2 + q^2 = p^2q$

Solution: Let $q = ap$ then the given equation reduces to

$$p^2z^2 + a^2p^2 = p^2 \cdot ap$$

Solving $p = (z^2 + a^2)/a$.

Then $$dz = pdx + qdy = p(dx + ady)$$

$$dz = \frac{(z^2 + a^2)}{a}(dx + ady)$$

or $$\frac{adz}{(z^2 + a^2)} = dx + ady$$

Integrating $\tan^{-1}\left(\frac{z}{a}\right) = x + ay + b$. Thus the complete solution is

$$z = a\tan(x + ay + b).$$

Example 7: $p^2x^2 = z(z - qy)$

Solution: To reduce this equation to $f(z, p, q) = 0$ form, put $X = \ln x$, $Y = \ln y$ so that

$$dx = \tfrac{dx}{x}, dy = \tfrac{dy}{y}$$

Rewriting

$$\left(x\frac{\partial z}{\partial x}\right)^2 = z\left(z - y\frac{\partial z}{\partial y}\right)$$

$$\left(\frac{\partial z}{\partial X}\right)^2 = z\left(z - \frac{\partial z}{\partial Y}\right)$$

Let $P = \frac{\partial z}{\partial X}$, $Q = \frac{\partial z}{\partial Y}$. Then the given equation reduces to

$$P^2 = z(z - Q)$$

which is of the form $f(z, P, Q) = 0$

Let $Q = aP$. Substituting this in the new equation

$$P^2 = z^2 - zQ = z^2 - zaP$$

or $$P^2 + azP - z^2 = 0$$

Solving $P = \frac{-az \pm \sqrt{a^2z^2 + 4z^2}}{2} = z \cdot k$

where $k = \left(-a \pm \sqrt{a^2 + 4}\right)/2$

now $dz = P\,dX + Q\,dY = P(dX + adY)$

or $dz = k \cdot z(dX + adY)$ since $P = kz$

Replacing X and Y by $\ln x$ and $\ln y$

$$\frac{1}{k}\frac{dz}{z} = d\ln x + a\,d\ln y$$

Integrating $z^{\frac{1}{k}} = xy^a b$
where a, b are arbitrary constants and

$$k = \left(-a \pm \sqrt{a^2 + 4}\right)/2$$

Form III: $f(x, p) = g(y, q)$

Example 8: $yp + xq + pq = 0$

Solution: Rewriting $(x + p)q = -yp$

or
$$\frac{x+p}{p} = -\frac{y}{q} = a, \quad \text{say}$$

Then solving for p and q

$$p = \frac{x}{a-1} \quad \text{and} \quad q = -\frac{y}{a}$$

Now $dz = pdx + qdy = \frac{x}{(a-1)}dx + \left(-\frac{y}{a}\right)dy$

Integrating $z = \frac{1}{(a-1)}\frac{x^2}{2} - \frac{y^2}{2a} + b.$

Example 9: $p^2q^2 + x^2y^2 = x^2q^2(x^2 + y^2)$

Solution: To transform this to the standard form, put $x^2 = X$ and $y^2 = Y$ so that

$$2x\,dx = dX, \quad 2y\,dy = dY$$

Then $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial x} = 2x\,z_x, q = 2xP$

Similarly, $q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y} \cdot \frac{\partial Y}{\partial y} = 2y\,z_y = 2yQ$

Substituting these values, the given equation reduces to

$$4x^2p^2 \cdot 4y^2Q^2 + x^2y^2 = x^24y^2Q^2(x^2 + y^2)$$

$$4XP^2 \cdot 4YQ^2 + XY = 4XYQ^2(X + Y)$$

or
$$16P^2Q^2 + 1 = 4Q^2(X + Y)$$

Now rewrite this as

$$16P^2Q^2 - 4XQ^2 = 4YQ^2 - 1$$

$$(4P^2 - X) = \frac{4YQ^2 - 1}{4Q^2} = a^2 \quad \text{say}$$

This is in the standard form

$$f(P, X) = g(Q, Y).$$

Solving for P and Q, we get

$$P = \frac{1}{2}(X + a^2)^{\frac{1}{2}}$$

$$Q = \frac{1}{2}\frac{1}{(Y - a^2)^{\frac{1}{2}}}$$

Now $dz = PdX + QdY$

$$dz = \frac{1}{2}(X + a^2)^{\frac{1}{2}}dX + \frac{1}{2}(Y - a^2)^{-\frac{1}{2}}dY$$

Integrating $z = \frac{1}{3}(X + a^2)^{\frac{3}{2}} + (Y - a^2)^{\frac{1}{2}} + b$

or
$$z = \frac{1}{3}(x^2 + a^2)^{\frac{3}{2}} + (y^2 - a^2)^{\frac{1}{2}} + b$$

where a, b are arbitrary constants.

Example 10: $zpy^2 = x(y^2 + z^2q^2).$

Solution: To get rid of z in the equation, put $Z = \frac{z^2}{2}$ or $dZ = zdz$. Then

$$zp = \frac{z\partial z}{\partial x} = \frac{\partial Z}{\partial x} = P, \text{ similarly, } zq = z\frac{\partial z}{\partial y} = \frac{\partial Z}{\partial y} = Q$$

Now the equation gets transformed to

$$Py^2 = x(y^2 + Q^2)$$

or
$$\frac{P}{x} = \frac{Q^2 + y^2}{y^2} = a \quad \text{say}$$

Solving $P = ax, \quad Q = \sqrt{a-1}\,y$

So
$$dz = Pdx + Qdy = ax\,dx + \sqrt{a-1}\,y\,dy$$

or
$$zdz = ax\,dx + \sqrt{a-1}\,y\,dy$$

Integrating $z^2 = ax^2 + \sqrt{a-1}\,y^2 + b$

where a, b are arbitrary constants.

Form IV: Clairaut's equation

Example 11: $z = px + qy + \ln pq$

Solution: The complete solution of this Clairaut's equation is

$$z = ax + by + \ln ab$$

Example 12: $(p - q)(z - px - qy) = 1$

Solution: Rewriting this in Clairaut's form

$$z = px + qy + \frac{1}{p - q}$$

The complete solution is

$$z = ax + by + \frac{1}{a - b}$$

EXERCISE

Find the complete solution of:

Form I: $f(p, q) = 0$

1. $pq = k$

Ans. $z = ax + k\frac{y}{a} + c$

2. $p^2 + q^2 = m^2$

Ans. $z = ax + \sqrt{m^2 - a^2}\, y + c$

3. $\sqrt{p} + \sqrt{q} = 1$

Ans. $z = ax + (1 - \sqrt{a})^2 y + c$

4. $p^2 - q^2 = 4$

Ans. $z = ax + \sqrt{a^2 - 4}\, y + c$

5. $p + q = pq$

Ans. $z = ax + ay/(a-1) + c$

6. $p = e^q$

Ans. $z = ax + y \ln a + c$

7. $2p^2 + 6p + 2q + 4 = 0$

Ans. $z = ax - (2 + 3a + a^2/2)y + c$

8. $x^2p^2 + y^2q^2 = z^2$

Hint: Put $Z = \ln z, X = \ln x,$ (where $a = \cos\alpha, b = \sqrt{1-a^2} = \sin\alpha,$) $Y = \ln y$, which transforms the given equation to $P^2 + Q^2 = 1$.

Ans. $z = c^* x^{\cos\alpha} \cdot y^{\sin\alpha}$

9. $(x^2 + y^2)(p^2 + q^2) = 1$

Hint: Put $x = r\cos\theta,\ y = r\sin\theta$, (where $r = \sqrt{x^2+y^2}$, $\tan\theta = y/x$, $R = \ln r$, then equation reduces to $\left(\frac{\partial z}{\partial R}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2 = 1$)

Ans. $z = a\ln r + \sqrt{1-a^2}\,\theta + c$

10. $pq = x^m y^n z^l$

Ans. $z^p = p\left(\frac{ax^q}{q} + \frac{1}{a}\frac{y^r}{r} + c\right)$

where $p = 1 - \frac{l}{2}, q = m + 1, r = n + 1$

Hint: Put $Z = \frac{z^p}{p}, X = \frac{x^q}{q}, Y = \frac{y^r}{r}$

Then equation reduces to

$$\frac{\partial Z}{\partial X} \cdot \frac{\partial Z}{\partial Y} = 1.$$

Form II: $f(z, p, q) = 0$

1. $p^2z^2 + q^2 = 1$

Ans. $az(1 + a^2z)^{\frac12} - \ln\left[az + (1 + a^2z^2)^{\frac12}\right] = 2a(ax + y + b)$

2. $p(1 + q) = qz$

Ans. $\ln(az - 1) = x + ay + b$

3. $q^2 = z^2p^2(1 - p^2)$

Ans. $a^2z^2 = (y + ax + c)^2 + 1$

4. $p^3 + q^3 = 27z$

Ans. $(1 + a^3)z^2 = 8(x + ay + b)^3$

5. $z^2(p^2 + q^2 + 1) = \alpha^2$

Ans. $(1 + b^2)(\alpha^2 - z^2) = (x + by + c)^2$

6. $z^2 = 1 + p^2 + q^2$

Ans. $z = \cosh(x + ay + c/\sqrt{1 + a^2})$

7. $p(1 + q^2) = q(z - \alpha)$

Ans. $4a(z - \alpha) = 4 + (x + ay + c)^2$

8. $9(p^2z + q^2) = 4$

Ans. $(z + a^2)^3 = (x + ay + b)^2$

9. $z^2(p^2x^2 + q^2) = 1$

Hint: Put $X = \ln x$, equation reduces to $z^2\left[\left(\frac{\partial z}{\partial X}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right] = 1.$

Ans. $z^2\sqrt{1 + a^2} = \pm 2(\ln x + ay) + b$

10. $q^2y^2 = z(z - px)$

Hint: Put $X = \ln x, Y = \ln y$ then equation reduces to $Q^2 = z^2 - zP$

Ans. $xy^a \cdot b = z^{\frac{1}{k}}$

where $k = \left(-1 \pm \sqrt{1 + 4a^2}\right)/(2a^2)$

Form III: $f(x, p) = g(y, q)$

1. $p^2 \pm q^2 = x \pm y$

Ans. $z = \frac{2}{3}(a + x)^{\frac32} + \frac{2}{3}(\mp a + y)^{\frac32} + b$

2. $\sqrt{p} + \sqrt{q} = x + y$

Ans. $z = \frac{(a+x)^3}{3} + \frac{(y-a)^3}{3} + b$

3. $p + q = \sin x + \sin y$

Ans. $z = ax - \cos x - \cos y - ay + b$

4. $p^2 y(1 + x^2) = qx^2$

Ans. $z = a\sqrt{1+x^2} + \frac{1}{2}a^2y^2 + b$

5. $pe^y = qe^x$

Ans. $z = a(e^x + e^y) + b$

6. $p - q = x^2 + y^2$

Ans. $z = \frac{1}{3}(x^3 - y^3) + a(x + y) + b$

7. $y^2q^2 - xp + 1 = 0$

Ans. $z = (a^2 + 1)\ln x + a\ln y + b$

8. $z^2(p^2 + q^2) = x^2 + y^2$

Hint: Put $Z = \frac{1}{2}z^2$ equation reduces to $P^2 + Q^2 = x^2 + y^2$ where $P = zp$, $Q = zq$.

Ans. $z^2 = x\sqrt{x^2+a} + y\sqrt{y^2-a} + a\ln\frac{x+\sqrt{x^2+a}}{y+\sqrt{y^2-a}} + 2b$

9. $z(xp - yq) = y^2 - x^2$

Hint: Put $Z = \frac{z^2}{2}$ equation reduces to $xP - yQ = y^2 - x^2$ where $P = \frac{\partial Z}{\partial x}$, $Q = \frac{\partial Z}{\partial y}$.

Ans. $z^2 = 2a\ln xy - (x^2 + y^2) + 2b$

10. $z(p^2 - q^2) = x - y$

Hint: Put $Z = \frac{2}{3}z^{\frac{3}{2}}$, equation reduces to $P^2 - Q^2 = x - y$ where $P = \frac{\partial Z}{\partial x}$, $Q = \frac{\partial Z}{\partial y}$

Ans. $z^{\frac{3}{2}} = (a + x)^{\frac{3}{2}} + (a + y)^{\frac{3}{2}} + c.$

Form IV: $z = px + qy + f(p, q)$: Clairaut's equation

1. $2q(z - px - qy) = 1 + q^2$

Ans. $z = ax + by + \frac{b^2+1}{2b}$

2. $pqz = p^2(xq + p^2) + q^2(yp + q^2)$

Ans. $z = ax + by + \left(\frac{a^3}{b} + \frac{b^3}{a}\right)$

3. $z = px + qy \pm pq$

Ans. $z = ax + by \pm ab$

4. $(px + qy - z)^2 = d\,(1 + p^2 + q^2)$

Ans. $z = ax + by \pm \left(\sqrt{1 + a^2 + b^2}\right)d$

5. $(p + q)(z - xp - yq) = 1$

Ans. $z = ax + by + \frac{1}{a+b}$

6. $4xyz = pq + 2px^2y + 2qxy^2$

Hint: Put $X = x^2$, $Y = y^2$, equation reduces to $z = PX + QY + PQ$.

Ans. $z = ax^2 + by^2 + ab.$

18.5 CHARPIT'S METHOD

Charpit's method is a general method to find the complete solution of the first order non-linear P.D.E. of the form

$$f(x, y, z, p, q) = 0 \tag{1}$$

We know that

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = p\,dx + q\,dy \tag{2}$$

Integrating (2), we get the complete solution of (1). In order to integrate (2), we must know p and q in terms of x, y, z. For this purpose, introduce another first order non-linear P.D.E. of the form

$$g(x, y, z, p, q, a) = 0 \tag{3}$$

involving an arbitrary constant "a" compatible with (1). Solving (1) and (3), we get

$$p = p(x, y, z, a), \quad q = q(x, y, z, a) \tag{4}$$

On substitution of (4) in (2), equation (2) becomes integrable, resulting in the complete solution of (1) in the form

$$F(x, y, z, a, b) = 0 \tag{5}$$

containing two arbitrary constants a and b.

Now differentiating (1) and (3) partially w.r.t. x and y and eliminating $\frac{\partial p}{\partial x}$ and $\frac{\partial q}{\partial x}$, we get after simplification, a Lagrange's linear equation of ϕ (as dependent variable) in terms of x, y, z, p, q (as independent variables) as

$$f_q\frac{\partial g}{\partial x} + f_q\frac{\partial g}{\partial y} + (pf_p + qf_q)\frac{\partial g}{\partial z} - (f_x + pf_z)\frac{\partial g}{\partial p} - (f_y + qf_z)\frac{\partial g}{\partial q} = 0 \tag{6}$$

47.4 SOLUTION OF EQUATION BY DIRECT INTEGRATION

Example 3. Solve $\dfrac{\partial^3 z}{\partial x^2 \partial y} = \cos(2x+3y)$

Solution. We have, $\dfrac{\partial^3 z}{\partial x^2 \partial y} = \cos(2x+3y)$

Integrating w.r.t. 'x', we get $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{1}{2}\sin(2x+3y) + f(y)$

Again, integrating w.r.t. x, we get $\dfrac{\partial z}{\partial y} = -\dfrac{1}{4}\cos(2x+3y) + x\int f(y)\,dx + g(y)$

$$= -\frac{1}{4}\cos(2x+3y) + x\phi(y) + g(y)$$

Integrating w.r.t. 'y', we get $z = -\dfrac{1}{12}\sin(2x+3y) + x\int \phi(y)\,dy + \int g(y)\,dy$

$$z = -\frac{1}{12}\sin(2x+3y) + x\phi_1(y) + \phi_2(y)$$ **Ans.**

Example 4. Solve $\dfrac{\partial^2 z}{\partial x \partial y} = x^2 y$

subject to the conditions $z(x, 0) = x^2$ *and* $z(1, y) = \cos y$.

Solution. We have, $\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right) = x^2 y$

On integrating w.r.t. x, we obtain

$$\frac{\partial z}{\partial y} = \frac{x^3}{3}y + f(y)$$

Integrating w.r.t. y, we obtain

$$z = \frac{x^3}{3}\cdot\frac{y^2}{2} + \int f(y)\,dy + g(x) \qquad \left[F(y) = \int f(y)\,dy\right]$$

$$\Rightarrow \qquad z = \frac{x^3 y^2}{6} + F(y) + g(x) \qquad \dots(1)$$

Condition 1 : Putting $z = x^2$ and $y = 0$ in (1), we get

$$x^2 = 0 + F(0) + g(x)$$

Putting the value of $g(x)$ in (1), we get

$$z = \frac{x^3y^2}{6} + F(y) + x^2 - F(0) \qquad \ldots (2)$$

Condition 2 : $z(1, y) = \cos y$

Putting $x = 1$ and $z = \cos y$ in (2), we get

$$\cos y = \frac{y^2}{6} + F(y) + 1 - F(0) \Rightarrow F(y) = \cos y - \frac{1}{6}y_2 - 1 + F(0)$$

Putting the value of $F(y)$ in (2), we get

$$z = \frac{1}{6}x^3y^2 + \cos y - \frac{1}{6}y^2 - 1 + F(0) + x^2 - F(0)$$

$$\Rightarrow \quad z = \frac{1}{6}x^3y^2 + \cos y - \frac{1}{6}y^2 - 1 + x^2 \qquad \textbf{Ans.}$$

Solve the following:

1. $\dfrac{\partial^2 z}{\partial x \partial y} = xy^2$ Ans. $z = \dfrac{x^2y^3}{6} + f(y) + \phi(x)$

2. $\dfrac{\partial^2 z}{\partial x \partial y} = e^y \cos x$ Ans. $z = e^y \sin x + f(y) + \phi(x)$

3. $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{y}{x} + 2$ Ans. $z = \dfrac{y^2}{2}\log x + 2xy + f(y) + \phi(x)$

47.5 LAGRANGE'S LINEAR EQUATION IS AN EQUATION OF THE TYPE

$$Pp + Qq = R$$

where P, Q, R are the functions of x, y, z and $p = \dfrac{\partial z}{\partial x}$, $q = \dfrac{\partial z}{\partial y}$

$$Pp + Qq = R \qquad \ldots (1)$$

47.6 WORKING RULE TO SOLVE Pp + Qq = R

Step 1. Write down the auxiliary equations $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$

Step 2. Solve the above auxiliary equations.
Let the two solutions be $u = c_1$ and $v = c_2$.

Step 3. Then $f(u, v) = 0$ or $u = \phi(v)$ is the required solution of

$$Pp + Qq = R.$$