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F-24

TUTORIAL 6 : Error & Approximations, Jacobians, Extreme values of f^n of two variables, Lagrange's Method of Undetermined multiplies

1. $D = \frac{a^2}{b} + \frac{c^2}{2}$ $\quad \frac{da}{a}\% = \frac{1}{2}\% \quad \frac{db}{b} = 1\% \quad \frac{dc}{c} = 1\%$

$$dD = \frac{\delta D}{\delta a} da + \frac{\delta D}{\delta b} db + \frac{\delta D}{\delta c} dc$$

$$dD = \left(\frac{2a}{b}\right) da + \left(\frac{-a^2}{b^2}\right) db + (c)\, dc$$

$$\frac{dD}{D} = \frac{2a^2 \cancel{2b}}{\cancel{b}(2a^2+c^2)} \frac{da}{a} - \frac{a^2\, 2\cancel{b}}{\cancel{b}(2a^2+c^2)} \frac{db}{b} + \frac{c^2(2b)}{2a^2+c^2} \frac{dc}{c}$$

$$\frac{dD}{D}\% = \frac{2}{(2a^2+c^2)} \left[\cancel{2}a^2 \frac{(1)}{\cancel{2}} - a^2(1) + bc^2(1) \right]$$

$$\frac{dD}{D}\% = \left(\frac{2}{2a^2+c^2}\right)\left[a^2 - c^2 + bc^2\right] = \frac{2bc^2}{2a^2+c^2}\%$$

2. $z = 2xy^2 - 3x^2y$ $\quad \frac{dx}{dt} = 2cm/s$ at $x = 3$ cm

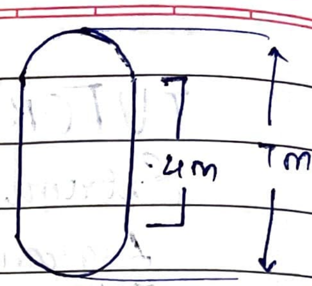
→ For z - constant

$$\frac{dz}{dt} = 0 = (2y^2 - 6xy)\frac{dx}{dt} + (4xy - 3x^2)\frac{dy}{dt}$$

at $(3, 1)$ → $\quad 0 = [2(1) - 6(3)](2) + (12 - 27)\frac{dy}{dt}$

$$\Rightarrow \boxed{-\left(+\frac{32}{15}\right) = \frac{dy}{dt}}$$

∴ y decreases at rate $-32/15$ cm/s at $y = 1$cm



3. $r = 1.5\,m \quad h = 4\,m$

$dr = 0.01\,m \quad dh = 0.05\,m$

$$V = \frac{4}{3}\pi r^3 + \pi r^2 h$$

$$dV = \frac{dV}{dr}dr + \frac{dV}{dh}\cdot dh = (4\pi r^2 + 2\pi r h)\,dr + (\pi r^2)\,dh$$

$$\cancel{(4\pi r^2)\,dr + (}$$

$$\cancel{dV = \pi\left[\left[4(2.25) + 2(1.5)(4)\right](0.01) + 2.25(0.05)\right]}$$

$$\therefore \frac{dV}{V}\% = \frac{\pi r}{\pi r^2\left(\frac{4r}{3} + h\right)}\left[(4r + 2h)\,dr + r\,dh\right]\%$$

$$\cancel{= \frac{3}{(4r^2 + 3h)}\left[(4r^2 + 2h)\,dr + hr\,\frac{dh}{h}\right]\%}$$

$$\cancel{= \frac{3}{[4(2.25) + 12]}\left[(4(2.25) + 8)\left(\frac{0.01}{1.5}\right) + \right.}$$

$$= \frac{3}{[4(2.25) + 12(1.5)]}\left[(4(1.5) + 8)(0.01) + 1.5(0.05)\right] \times 100$$

$$= \frac{3}{4(6.75)}(14 + 7.5) = \underline{2.389\,\%}$$

4. $(1.99)^2 (3.01)^3 (0.98)^{1/10}$

$f(x) = x^2 \qquad x = 2 \qquad \Delta x = -0.01$

$$f(x + \Delta x) = f(x) + f'(x)\cdot\Delta x$$
$$= 4 + 2(-0.01)(2)$$
$$= 4 - 0.04 = 3.96$$

$g(x) = x^3 \qquad x = 3 \qquad \Delta x = 0.01$

$$g(x + \Delta x) = g(x) + g'(x)\,\Delta x$$

$$(3.01)^3 = 3^3 + 3(3)^2(0.01)$$
$$= 27 + 0.27$$
$$= 27.27$$

$h(x) = x^{1/10}$ $\quad x = 1 \quad \Delta x = -0.02$

$$h(x+\Delta x) = h(x) + h'(x)\Delta x$$
$$= 1 + \frac{1}{10(1)^{9/10}}(-0.02)$$
$$= 1 - 0.002 = 0.998$$

$$\therefore (1.99)^2(3.01)^3(0.98)^{1/10} = (3.96)(27.27)(0.998)$$
$$= 107.77$$

5. (a) $x = a\cosh\theta\cos\phi \qquad y = a\sinh\theta\sin\phi$

$$J\left(\frac{x,\ y}{\theta,\ \phi}\right) = \begin{vmatrix} +a\sinh\theta\cos\phi & -a\cosh\theta\sin\phi \\ a\cosh\theta\sin\phi & a\sinh\theta\cos\phi \end{vmatrix}$$

$$= a^2\left[\sin^2h\theta\cos^2\phi + \cos^2h\theta\sin^2\phi\right]$$

$$= a^2(\cosh^2\theta - \cos^2\phi)$$

(b) $u = xyz \qquad v = x^2+y^2+z^2 \qquad w = x+y+z$

$$J\left(\frac{u\ v\ w}{x\ y\ z}\right) = \begin{vmatrix} yz & 2x & 1 \\ xz & 2y & 1 \\ xy & 2z & 1 \end{vmatrix} = 2\begin{vmatrix} y & -1 & 0 \\ -x & 1 & 0 \\ xy & z & 1 \end{vmatrix} \quad (z-x)(y-z)$$

$$= 2(z-x)(y-z)\left[y-x\right]$$

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6. $J \times J' = 1$ for f^n $x = u$, $y = u\tan v$, $z = w$

$\rightarrow J\left(\frac{x\ y\ z}{u\ v\ w}\right) = \begin{vmatrix} 1 & \tan v & 0 \\ 0 & u\sec^2 v & 0 \\ 0 & 0 & 1 \end{vmatrix} = u\sec^2 v$

$J' = \begin{vmatrix} \frac{du}{dx} & \frac{du}{dy} & \frac{du}{dz} \\ \frac{dv}{dx} & \frac{dv}{dy} & \frac{dv}{dz} \\ \frac{dw}{dx} & \frac{dw}{dy} & \frac{dw}{dz} \end{vmatrix} = \begin{vmatrix} 1 & \cot v & 0 \\ 0 & \frac{u^2}{u(u^2+y^2)} & 0 \\ 0 & 0 & 1 \end{vmatrix}$

$J' = \frac{1}{u\sec^2 v}$

$\boxed{\therefore J \times J' = 1}$ Proved

7. $u = e^x\cos y$ $v = e^x\sin y$ $x = lr + sm$ $y = mr - ls$ [l, m - constant

Chain Rule :

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \cdot \frac{\partial(r,s)}{\partial(x,y)}$$
$$J \quad = \quad J_1 \quad \quad J_2$$

LHS $\rightarrow J = \begin{vmatrix} e^x\cos y & -e^x\sin y \\ e^x\sin y & e^x\cos y \end{vmatrix} = e^{2x}[\cos^2 y + \sin^2 y] = e^{2x}$

$\frac{du}{dr} = \frac{du}{dx}\cdot\frac{dx}{dr} + \frac{du}{dy}\cdot\frac{dy}{dr}$

$r = \frac{x - sm}{l}$ or $\frac{y + ls}{m}$

$s = \frac{x - lr}{m}$ or $\frac{mr - y}{l}$

RHS → $J_1 = \begin{vmatrix} e^x\cos y(l) + m(-e^x\sin y) & e^x\cos y(m) + (+l)(e^x\sin y) \\ e^x\sin y(l) + m(e^x\cos y) & e^x\sin y(m) + (-l)(e^x\cos y) \end{vmatrix}$

$$J_1 = \frac{e^{2x}}{l^2+m^2}\begin{vmatrix} \sin(\theta-y) & \cos(\theta-y) \\ \cos(\theta-y) & -\sin(\theta-y) \end{vmatrix} = \frac{-e^{2x}}{(l^2+m^2)}$$

$$J_2 = \begin{vmatrix} 1/l & 1/m \\ 1/m & -1/l \end{vmatrix} = -\left(\frac{1}{l^2}+\frac{1}{m^2}\right)$$

7. $u = e^x\cos y \qquad v = e^x\sin y \qquad x = lr + sm$

$y = mr - ls$

Chain rule :

$$\underset{(J)}{\frac{\partial(u,v)}{\partial(r,s)}} = \underset{(J_1)}{\frac{\partial(u,v)}{\partial(x,y)}} \cdot \underset{(J_2)}{\frac{\partial(x,y)}{\partial(r,s)}}$$

$$J = \frac{e^{2x}}{l^2+m^2}\begin{vmatrix} \sin(\theta-y) & \cos(\theta-y) \\ \cos(\theta-y) & -\sin(\theta-y) \end{vmatrix} = \frac{-e^{2x}}{l^2+m^2} \qquad \left[\tan\theta = l/m\right.$$

$$J_1 = \begin{vmatrix} e^x\cos y & -e^x\sin y \\ e^x\sin y & e^x\cos y \end{vmatrix} = e^{2x}\left[\cos^2 y + \sin^2 y\right] = e^{2x}$$

$$J_2 = \begin{vmatrix} l & m \\ m & -l \end{vmatrix} = -(l^2+m^2)$$

$\therefore\ J = J_1 \times J_2$ Proved

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8. a) $u = x^3 + y^3 - 63(x+y) + 12xy$

$\frac{du}{dx} = 3x^2 - 63 + 12y \qquad \frac{du}{dy} = 3y^2 - (63) + 12x$

Put $\frac{du}{dx} = 0$ & $\frac{du}{dy} = 0$

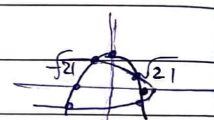
$\therefore \cancel{3x^2 + 12y = 3y^2 + 12x}$

$\Rightarrow \cancel{3x(x-4) - 3y(y-4)} = 0$

$\Rightarrow$

$x^2 = -\frac{1}{3}(12y - 63) = -4\left(y - \frac{21}{4}\right)$ — (1)

$y^2 = -4\left(x - \frac{21}{4}\right)$ — (2)



From $y = \frac{63 - 3x^2}{12}$ put in (2)

$\frac{3^2}{12^2}(21 - x^2)^2$

$\frac{du}{dx} = 3x^2 - 63 + 12y = 0$ & $\frac{du}{dy} = 3y^2 - 63 + 12x = 0$

$y = \frac{(63 - 3x^2)}{12} \quad \therefore \quad \frac{3(63 - 3x^2)^2}{12^2} + 12x - 63 = 0$

$\Rightarrow \quad \frac{3^2}{12^2}(21 - x^2)^2 + 4x - 21 = 0$

$\Rightarrow \quad 441 + x^4 - 42x^2 + 64x - 336 = 0$

$\Rightarrow \quad x^4 - 42x^2 + 64x + 105 = 0$

at $\boxed{\begin{array}{l} x = -1 \\ y = 5 \end{array}}$ $\quad \therefore (x+1)(x^3 - x^2 - 41x + 105) = 0$

$(x+1)(x-3)(x^2 + 2x - 35) = 0$

$(x+1)(x-3)(x+7)(x-5) = 0$

$\therefore \boxed{\begin{array}{l} x = -1 \\ y = 5 \end{array}} \quad \boxed{\begin{array}{l} x = 3 \\ y = 3 \end{array}} \quad \boxed{\begin{array}{l} x = -7 \\ y = -7 \end{array}} \quad \boxed{\begin{array}{l} x = 5 \\ y = -1 \end{array}}$

$\frac{d^2u}{dx^2} = r = 6x$ | $\frac{d^2u}{dy^2} = 6y = t$ | $\frac{d^2u}{dxdy} = 12 = s$

x	y	$rt - s^2$	r	Nature of pt.
-1	5	< 0	< 0	Not extreme
3	3	> 0	> 0	Minima
-7	-7	> 0	< 0	Maxima
5	-1	< 0	> 0	Not extreme

$\therefore f_{max}(-7,-7) = 2(7)^3 + 63(2)(7) + 12(7)^2$
$= 2(7)^3 + 18(7)^2 + 12(7)^2$
$= (14+30)7^2 = 44 \times 49 = \underline{2156}$

$\therefore f_{min}(3,3) = 2(3)^3 - 63(6) + 12(3)^2$
$= 8(3)^3 - 14(3)^3$
$= -8(3)^3 = \cancel{218}\ \underline{-216}$

9. Find the stationary value of $xy(a-x-y)$.

→ Put $F = axy - x^2y - xy^2$

$\frac{dF}{dx} = ay - 2xy - y^2$ $\qquad$ $\frac{dF}{dy} = ax - x^2 - 2xy$

Put $\frac{dF}{dx} = 0$ & $\frac{dF}{dy} = 0$

$\therefore$ ~~$ay - y^2 = ax - x^2$~~
~~$a(y-x) = (y-x)(y+x)$~~
~~$\therefore (y-x)(x+y-a) = 0$~~

$a - 2x - y = 0$ & $a - x - 2y = 0$

$y + 2x - a = 0$ $\qquad$ $x + 2y - a = 0$

$\therefore$ ~~$y = a$~~

$2y + 4x - 2a = 0$
$-2y + x - a = 0$

$3x - a = 0$

$\boxed{x = \frac{a}{3}}$

$\therefore y = a - \frac{2a}{3} = \frac{a}{3}$

$\therefore F_{extreme} = a^3/27$

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10. $z^2 = xy + 1$ to minimise
$f = [z^2 + x^2 + y^2]^{1/2}$

$\therefore$ $F = (z^2+x^2+y^2)^{1/2} + \lambda(z^2 - xy + 1)$

$\frac{dF}{dx} = \frac{1}{\cancel{2}}(z^2+x^2+y^2)(\cancel{2}x) + (-\lambda y) = 0$ — ①

$\frac{dF}{dy} = (z^2+y^2+x^2)y - \lambda x = 0$ — ②

$\frac{dF}{dz} = (z^2+y^2+x^2)z + 2\lambda z = 0$ — ③

① × x − ② y

$\Rightarrow x^2(z^2+x^2+y^2) - y^2(z^2+x^2+y^2) = 0$

$\Rightarrow z^2(x^2-y^2) + x^4 - y^4 = 0$

$\Rightarrow z^2(x^2-y^2) + (x^2-y^2)(x^2+y^2) = 0$

$\therefore \boxed{(z^2+x^2+y^2)(x-y)(x+y) = 0}$

$\therefore x = y$ or $x = \pm y$

① × 2z + ③ y

$\Rightarrow 2xz(z^2+x^2+y^2) + zy(z^2+y^2+x^2) = 0$

Put $x = y$

$2(z^3x + x^3z + x^3z) + z^3x + zx^3 + zx^3 = 0$

$\Rightarrow 3z^3x + 6zx^3 = 0$

$\Rightarrow zx(z^2 + 2x^2) = 0$

$\therefore$ $x = 0$, $y = 0$, $z = \pm 1$

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11. $ax + by + cz = p$ $\quad f = x^2 + y^2 + z^2$

$F = x^2 + y^2 + z^2 + \lambda(ax + by + cz - p)$

$\frac{dF}{dx} = 2x + \lambda a = 0 \quad \therefore x = \frac{-\lambda a}{2}$

$\frac{dF}{dy} = 2y + \lambda b = 0 \quad y = \frac{-\lambda b}{2}$

$\frac{dF}{dz} = 2z + \lambda c = 0 \quad z = \frac{-\lambda c}{2}$

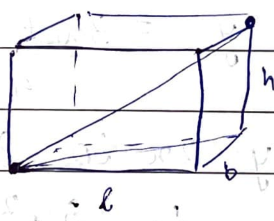
$\therefore \frac{-\lambda}{2}[a^2 + b^2 + c^2] = p \quad \therefore \lambda = \frac{-2p}{a^2 + b^2 + c^2}$

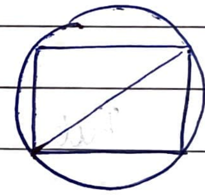
$\therefore f_{min} = \frac{4p^2}{(a^2+b^2+c^2)^2} \frac{1}{4}(a^2 + b^2 + c^2) = \frac{p^2}{a^2 + b^2 + c^2}$

12. Volume $= lbh$

$2R = \sqrt{h^2 + l^2 + b^2}$

$\Rightarrow 4R^2 = h^2 + l^2 + b^2$





$\therefore F = lbh + \lambda(l^2 + b^2 + h^2 - 4R^2)$

$\frac{dF}{dl} = bh + 2\lambda l = 0 \quad - ①$

$\frac{dF}{db} = lh + 2\lambda b = 0 \quad - ②$

$\frac{dF}{dh} = bl + 2\lambda h = 0 \quad - ③$

$\therefore ① \times b - ② \times l$

$\Rightarrow \cancel{2\lambda l^2}\ b^2 h - l^2 h = 0$

$\therefore h(l - b)(l + b) = 0$

$\therefore \boxed{l = b}$ & $b = h$

$\therefore$ Rectangular solid is cube

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13. $3x^2 + 4xy + 6y^2 = 140$

$d = (x^2+y^2)^{1/2}$

$F = -(x^2+y^2)^{1/2} + \lambda(3x^2+4xy+6y^2-140)$

$\frac{dF}{dx} = \frac{1}{\cancel{2}}(x^2+y^2)(\cancel{2}x) + 6\lambda x + 4y\lambda = 0$ —(1)

$\frac{dF}{dy} = \frac{1}{\cancel{2}}(x^2+y^2)(\cancel{2}y) + 4x\lambda + 12y\lambda = 0$ —(2)

(1) × y − (2) x

$\Rightarrow 6\cancel{\lambda}yx + 4y^2\cancel{\lambda} - 4x^2\cancel{\lambda} - 12yx\cancel{\lambda} = 0$

$\Rightarrow 4y^2 - 6yx - 4x^2 = 0$

$2y^2 - 3yx - 2x^2 = 0$

$y = \frac{3x \pm \sqrt{9x^2 + 4(4x^2)}}{4}$

$y = \frac{3x \pm 5x}{4} = 2x$ or $-\frac{x}{2}$

Put $y = 2x$ in $3x^2+4xy+6y^2 = 140$

$\therefore 3x^2 + 8x^2 + 24x^2 = 140$

$35x^2 = 140$

$x^2 = \frac{20}{5}$

$\boxed{\therefore x = \pm 2 \quad \& \quad y = \pm 4}$

Put $y = -x/2$ in $3x^2+4xy+6y^2 = 140$

$\therefore 3x^2 - 2x^2 + \frac{3}{2}x^2 = 140$

$5x^2 = 280$

$x^2 = 56$

$x = \pm 2\sqrt{14}$

$y = \pm\sqrt{14}$

$\therefore d_1 = (4+16)^{1/2} = \sqrt{20}$

$d_2 = 4(5\times14)^{1/2} = \sqrt{70}$

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14. $V = 8xyz$ $\quad E: \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$\therefore F = 8xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$

$\frac{dF}{dx} = 8yz + \lambda\left(\frac{2x}{a^2}\right) = 0$ ①

$\frac{dF}{dy} = 8xz + \lambda\left(\frac{2y}{b^2}\right) = 0$ ②

$\frac{dF}{dz} = 8xy + \lambda\left(\frac{2z}{c^2}\right) = 0$ ③

① × x − ② × y

$\therefore \frac{\cancel{2}\cancel{\lambda}x^2}{a^2} - \frac{\cancel{2}\cancel{\lambda}y^2}{b^2} = 0$

$\therefore \boxed{\frac{x}{a} = \pm\frac{y}{b}}$ Similarly, $\frac{x}{a} = \pm\frac{z}{c}$

$\therefore$ Let $x = \pm ka$, $y = \pm kb$, $z = \pm kc$

$\therefore 3k^2 = 1$ $\quad k = \pm\frac{1}{\sqrt{3}}$

$\therefore$ Volume $= 8\left(\frac{1}{3\sqrt{3}}\right)abc$