Garvit ①

Q1

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(a) $2\cosh 2x + 10\sinh 2x = 5$

$\Rightarrow \frac{2}{2}\left(e^{2x}+e^{-2x}\right) + 5\left[e^{2x}-e^{-2x}\right] = 5$

$\Rightarrow 6e^{2x} - 4e^{-2x} = 5$

$\Rightarrow 6e^{4x} - 4 - 5e^{2x} = 0$

$\Rightarrow 6e^{4x} - 5e^{2x} - 4 = 0$

$\Rightarrow e^{2x} = \frac{5 \pm \sqrt{25+96}}{12} = \frac{5 \pm 11}{12}$

$\therefore e^{2x} = \frac{5+11}{12} = \frac{4}{3}$ $\quad \therefore 2x = \cancel{\log}\ \ln \frac{4}{3}$

$\boxed{\therefore x = \frac{1}{2}\ln 4/3}$

(b) $y = \frac{x}{(x+1)^4} = (x)\left(\frac{1}{(x+1)^4}\right)$

$y_n = (uv)_n = {}^nC_0 u_n v + {}^nC_1 u_{n-1} v_1 + \dots + {}^nC_n u v_n$

(using Leibnitz Theorem)

$\Rightarrow$ ~~y~~ $\therefore u = \frac{1}{(x+1)^4}$ $\quad v = x$

$\therefore y_n = (1)(x)\frac{n!\,(-1)^n}{3!\,(x+1)^{4+n}} + \frac{n(1)\,(n-1)!\,(-1)^{n-1}}{3!\,(x+1)^{3+n}}$

$y_n = \frac{n!\,(-1)^n}{3!\,(x+1)^{3+n}}\left[\frac{x}{x+1} - 1\right] = \frac{n!\,(-1)^{n+1}}{3!\,(x+1)^{4+n}}$

(c) $u = \tan^{-1}\left(\frac{x^3+y^3}{x+y}\right)$ $\quad$ To Prove : $x\frac{du}{dx} + y\frac{du}{dy} = \sin 2u$

$z = \tan u \rightarrow$ homogeneous of deg 2

$\therefore \; x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = n\frac{f(u)}{f'(u)}$ here $z = f(u)$, $n = 2$

$\therefore \; f(u) = \tan u \qquad f'(u) = \sec^2 u$

$\therefore \; x\frac{du}{dx} + y\frac{du}{dy} = \frac{2\tan u}{\sec^2 u} = \sin 2u$ $\therefore$ Proved

(d) $u(x+y) = x^2+y^2$

$\Rightarrow \cancel{ux + uy = x^2+y^2}$

To Prove: $\left(\frac{du}{dx} - \frac{du}{dy}\right)^2 = 4\left(1 - \frac{du}{dx} - \frac{du}{dy}\right)$

• diff wrt x

$\frac{du}{dx}(x+y) + u(1) = 2x \Rightarrow \frac{du}{dx} = \frac{2x - \left(\frac{x^2+y^2}{x+y}\right)}{x+y}$

• diff wrt y

$\frac{du}{dy}(x+y) + u(1) = 2y \Rightarrow \frac{du}{dy} = \frac{2y - \frac{(x^2+y^2)}{x+y}}{x+y}$

$\therefore$ RHS $\rightarrow 4\left[1 - \left(\frac{2x^2+2xy-x^2-y^2}{(x+y)^2}\right) - \left(\frac{2xy+2y^2-x^2-y^2}{(x+y)^2}\right)\right]$

$\Rightarrow 4\left[\frac{\cancel{x^2+y^2+2xy} - \cancel{2x^2} - \cancel{2xy} + x^2+y^2 - 2xy - \cancel{2y^2} + \cancel{x^2+y^2}}{(x+y)^2}\right]$

$\Rightarrow 4\left(\frac{x^2+y^2-2xy}{(x+y)^2}\right) = 4\frac{(x-y)^2}{(x+y)^2} = \left[\frac{2(x-y)}{(x+y)}\right]^2$

LHS $\rightarrow \left(\frac{du}{dx} - \frac{du}{dy}\right)^2 = \left[\frac{(2x^2 + \cancel{2xy} - \cancel{x^2} - \cancel{y^2} - \cancel{2xy} - 2y^2 + \cancel{x^2+y^2})}{(x+y)^2}\right]^2$

$= \left[\frac{2(x-y)}{(x+y)}\right]^2 \rightarrow$ RHS

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(c) $y^2(2a-x) = x^3$

SYMMETRY –

- Symmetric about x-axis $\because$ y as $-y$ doesn't change eq^n

ORIGIN –

- Passes through origin $\because$ (0,0) satisfies.

- $2ay^2 - xy^2 - x^3 = 0$ $\therefore$ Lowest degree term $2ay^2 = 0$ gives $y = 0$ $\therefore$ x-axis is a tangent at (0,0)

ASYMPTOTES –

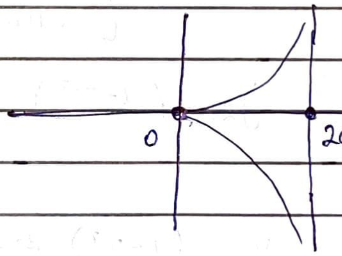
- $(2a - x) = 0$ $\rightarrow$ Highest degree of y coefficient

$\therefore \boxed{x = 2a}$

$y^2 = \dfrac{x^3}{2a-x}$

$\therefore y = \pm\sqrt{\dfrac{x^3}{2a-x}}$ $\therefore x > 0$ & $x < 2a$

$\therefore$ No curve in $x < 0$



CURVE

~~For x>0~~ $y \cancel{=} \cancel{\pm} + \cancel{\sqrt{\dfrac{x^3}{2a-x}}} = \cancel{\dfrac{x^{3/2}}{(2a-x)^{1/2}}}$

X

Q.2 (A) $y = e^{m\cos^{-1}x}$ diff wrt x

$y_1 = \dfrac{-me^{m\cos^{-1}x}}{\sqrt{1-x^2}}$ => $\sqrt{1-x^2}\, y_1 = -me^{m\cos^{-1}x}$

Squaring both sides

=> $(1-x^2)\, y_1^2 = m^2 e^{2m\cos^{-1}x}$

$\Rightarrow (1-x^2)y_1^2 = m^2 e^{2m\cos^{-1}x}$ diff w.r.t x

$\Rightarrow 2y_1y_2(1-x^2) + (-2x)y_1^2 = \frac{m^2(-2m)}{\sqrt{1-x^2}} e^{2m\cos^{-1}x}$

$\Rightarrow 2y_1y_2(1-x^2) - 2xy_1^2 = 2m^2 \frac{(-m) e^{m\cos^{-1}x}}{\sqrt{1-x^2}} \cdot e^{m\cos^{-1}x}$

$\Rightarrow \cancel{2}\cancel{y_1}y_2(1-x^2) - \cancel{2}xy_1^{\cancel{2}} = \cancel{2}m^2(\cancel{y_1})(y)$

$\Rightarrow \boxed{y_2(1-x^2) - xy_1 - m^2y = 0}$

Using Leibnitz theorem -

$\left[y_{2+n}(1-x^2) + ny_{1+n}(-2x) + y_n(-2)^nC_2\right] - \left[y_{1+n}x + ny_n\right] - m^2y_n = 0$

$\Rightarrow y_{2+n}(1-x^2) + y_{1+n}[-2xn - x] + y_n[-n(n-1) - n - m^2] = 0$

$\Rightarrow y_{2+n}(1-x^2) - y_{n+1}x(2n+1) - y_n[m^2 + \cancel{n} + n^2 - \cancel{n}] = 0$

$\Rightarrow \boxed{y_{2+n}(1-x^2) - y_{n+1}x(2n+1) - y_n(m^2+n^2) = 0}$ Proved

at $x=0$ $y = e^{m\pi/2}$ & $y_2 - m^2y = 0$

$y = \frac{1}{m^2}y_2$ $\therefore \cancel{y_n = \frac{1}{m^2}y_{n+2}}$

$y_{2+n}(1) - y_n(m^2+n^2) = 0$ $\boxed{\therefore y_n = \frac{1}{m^2+n^2}y_{2+n}}$

if n - even • $y_{n+2} = (m^2+n^2)^n y_2 = (m^2+n^2)^n m^2 e^{m\cos^{-1}x}$

n - odd • $y_{n+2} = (m^2+n^2)^n y_1 = (m^2+n^2)^n \frac{m e^{m\cos^{-1}x}}{\sqrt{1-x^2}}$

$\therefore y_n = (m^2+n^2)^{n-1} m^2 e^{m\cos^{-1}x}$ (n - even)

$= (m^2+n^2)^{n-1} m e^{m\cos^{-1}x}/\sqrt{1-x^2}$ (n - odd)

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(Q.2)

((B)) (i) ~~$y = \sin^2 x \cos^3 x \sin^3 x$~~

(ii) $x = a\left(t - \frac{t^3}{3}\right)$ $\quad y = at^2$

$\rho = \frac{(x_1^2 + y_1^2)^{3/2}}{x_1 y_2 - y_1 x_2}$ $\quad x_1 = a(1-t^2)$, $x_2 = a(-2t)$, $y_1 = 2at$, $y_2 = 2a$

$\therefore \rho = \frac{\left[a^2(1-t^2)^2 + (2at)^2\right]^{3/2}}{a(1-t^2)(2a) + (2at)(2at)}$

$\Rightarrow \rho = \frac{a^3\left[1 + t^4 - 2t^2 + 4t^2\right]^{3/2}}{a^2\left(2(1-t^2) + 4t^2\right)}$

$\Rightarrow \rho = \frac{a(1+t^2)^3}{2 - 2t^2 + 4t^2} = \frac{a(1+t^2)^3}{2(1+t^2)}$

$\Rightarrow \boxed{\rho = \frac{a}{2}(1+t^2)^2}$

(iii) $z = f(x,y)$ $\quad x = r\cos\theta$, $\quad y = r\sin\theta$

To show: $\left(\frac{df}{dx}\right)^2 + \left(\frac{df}{dy}\right)^2 = \left(\frac{df}{dr}\right)^2 + \frac{1}{r^2}\left(\frac{df}{d\theta}\right)^2$

$\therefore f(x,y) = x^2 + y^2 - r^2$

$\therefore \frac{df}{dx} = 2x \quad \Big| \quad \frac{df}{dy} = 2y \quad \Big| \quad \frac{df}{dr} = 2r \quad \Big| \quad \frac{dr}{d\theta} = x\sec\theta\tan\theta$

$\frac{df}{d\theta} = 2x\frac{dx}{d\theta} + 2y\frac{dy}{d\theta} \cancel{- 2r\frac{dr}{d\theta}} = 2r^2\left[-\cos\theta\sin\theta + \sin\theta\cos\theta\right] \cancel{- 2r x\sec\theta\tan\theta}$

~~$\frac{df}{d\theta} = \frac{-2r\, r\cos\theta \cdot \sin\theta}{\cos^2\theta} = -2r^2\tan\theta$~~

$\frac{df}{d\theta} = 0$ $\qquad \therefore (2x)^2 + (2y)^2 = (2r)^2$

$r^2\cos^2\theta + r^2\sin^2\theta = r^2$

Proved

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(iv) ~~$z = f(x,y)$~~ $x = \log u$ $y = \log v$
~~To show $\frac{d^2 z}{dx\,dy} = uv \frac{d^2 z}{du\,dv}$~~

~~$\rightarrow \frac{dz}{dx} = \frac{df}{dx}$~~

$\boxed{\cos 3\theta = 4\cos^3\theta - 3\cos\theta}$

~~$\theta = 4\left(\frac{3\sqrt{3}}{8} - \frac{3\sqrt{3}}{2}\right)$~~

$\boxed{\sin 3\theta = 3\sin\theta - 4\sin^3\theta}$

(i) $y = \sin^5 x \cos^3 x$

$y = \sin^2 x \sin^3 2x \left(\frac{1}{8}\right)$

$y = \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1}{8}\right)\left(\frac{3\sin 2x - \sin 6x}{4}\right]$

$y = \frac{1}{64}\left[3\sin 2x - \sin 6x - \frac{3}{2}\sin 4x + \frac{2}{2}\sin 6x \cos 2x\right]$

$y = \frac{1}{64}\left[3\sin 2x - \sin 6x - \frac{3}{2}\sin 4x + \frac{1}{2}(\sin 8x + \sin 4x)\right]$

$\therefore y_n = \frac{1}{64}\left[3(2^n)\sin(n\pi/2 + 2x) - 6^n \sin(n\pi/2 + 6x) - \frac{3}{2}(4)^n \sin(n\pi/2 + 4x) + \frac{1}{2}8^n \sin(n\pi/2 + 8x) + \frac{4^n}{2}\sin(n\pi/2 + 4x)\right]$

$\Rightarrow y_n = \frac{1}{64}\left[3(2^n)\sin(n\pi/2 + 2x) - \frac{4^n}{2}\sin(n\pi/2 + 4x) - 6^n \sin(n\pi/2 + 6x) + \frac{1}{2}8^n \sin(n\pi/2 + 8x)\right]$

Q.2 (c) To prove $\frac{d^{2n}}{dx^{2n}}(x^2-1)^n = (2n)!$

$\rightarrow (x^2-1)^n = x^{2n} - {}^nC_1 x^{2n-2} \ldots$ Using Binomial Theorem

$\therefore y_n = (2n)!$ Using Leibnitz Theorem

(Q.3) (A) Euler's Theorem states that for a homogeneous f^n u of degree n we can write

$$x\frac{du}{dx} + y\frac{du}{dy} = nu$$

$\rightarrow$ $u = \dfrac{x^2+y^2}{x+y}$ $\therefore n = 1$

Proof :

$$\frac{du}{dx} = \frac{2x(x+y) - (x^2+y^2)}{(x+y)^2} \quad -(1)$$

$$\frac{du}{dy} = \frac{2y(x+y) - (x^2+y^2)}{(x+y)^2} \quad -(2)$$

$\therefore$ (1) $\times x$ + (2) $\times y$

$$\Rightarrow \frac{1}{(x+y)^2}\left[2x^2(x+y) - x^3 - y^2x + 2y^2(x+y) - x^2y - y^3\right]$$

$$\Rightarrow \frac{1}{(x+y)^2}\left[x^3 + 2x^2y - y^2x + y^3 + 2y^2x - x^2y\right]$$

$$\Rightarrow \frac{x^3 + x^2y + y^2x + y^3}{(x+y)^2} = \frac{x^2(x+y) + y^2(x+y)}{(x+y)^2}$$

$$\Rightarrow \left(\frac{x^2+y^2}{x+y}\right)(1)$$

Proved

(B) (ii) $F(x) = x^2+y^2+z^2$ $\phi(x) = xyz - a^3$

$\therefore F(x) = x^2+y^2+z^2 + \lambda(xyz - a^3)$

$$\frac{dF}{dx} = 2x + \lambda(yz) = 0 \quad -(1)$$
$$\frac{dF}{dy} = 2y + \lambda(xz) = 0 \quad -(2)$$
$$\frac{dF}{dz} = 2z + \lambda(xy) = 0 \quad -(3)$$

$\left[\text{Put } \frac{dF}{dx} = 0,\ \frac{dF}{dy} = 0,\ \frac{dF}{dz} = 0\right]$

(1) $\times y$ − (2) $\times x$

$\Rightarrow \lambda y^2 z - \lambda x^2 z = 0$ $\qquad \boxed{\therefore (y-x)(y+x) = 0}$

(1) $\times z$ − (3) $\times x$

$\lambda y z^2 - \lambda x^2 y = 0$ $\qquad \boxed{\therefore (z-x)(z+x) = 0}$

$\therefore$ if $y = x$: $x = z$ or $x = -z$ $\qquad$ if $y = -x$: $x = -z$ or $x = z$

$\therefore$ For satisfying $xyz = a^3$ $\qquad x = y = z = a$

or $x = a,\ y = z = -a$

$\therefore$ so

$\therefore \min(x^2+y^2+z^2) = \underline{\underline{3a^2}}$

(i) $f(x,y) = 21 + x - 20y + 4x^2 + xy + 6y^2$

$a = -1 \quad b = 2 \quad f(a,b) = 21 - 1 - 40 + 4 - 2 + 24 - 20 + 26 = 6$

~~$f(x+h, y+k)$~~ $\boxed{f_x = 1 + 8x + y}$ $\boxed{f_y = -20 + x + 12y}$

$f(x,y) = f(a,b) + [x f_x(a,b) + y f_y(a,b)] +$

$\frac{1}{2!}\left(x^2 f_{xx}(a,y) + y^2 f_{yy}(a,b) + 2xy f_{xy} f(a,b)\right] \ldots$

$\Rightarrow f(x,y) = 6 + [x(1 + 8(-1) + (2)) + y(-20 - 1 + 24)]$

$+ \frac{1}{2}[x^2(8) + y^2(12) + 2xy(0)]$

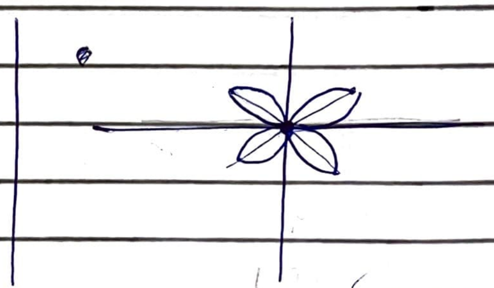
$\Rightarrow \boxed{f(x,y) = 6 - 5x + 3y + 4x^2 + 6y^2}$

(iv) ~~y~~ $r = a\sin 3\theta$

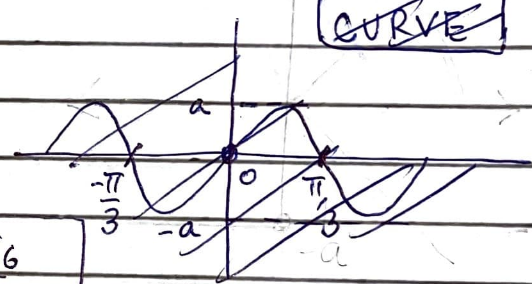
SYMMETRY : symmetric about OY
(on putting $(\pi-\theta)$ no change)

POLE : at $r=0$
$\theta = 0, \pi/3, 2\pi/3, \pi$

INTERSECTION with
OX → at $\theta = 0$, $r=0$
$\theta = \pi$, $r=0$
OY → at $\theta = \pi/2$ $r = -a$
$\theta = 3\pi/2$ $r = 0$



MAX & MIN value of r →
$r \in [-a, a]$



θ	0	$\pi/6$	$\pi/3$	~~$2\pi/3$~~ $5\pi/6$
r	0	a	0	a

(Q.3)
(C) Curve : $x^2y^2 = a^2(x^2+y^2)$

$F(x,y) = x^2y^2 - a^2x^2 - a^2y^2$

Highest deg terms in x : $x^2(y^2-a^2)$
Highest deg term in y : $y^2(x^2-a^2)$
∴ $\boxed{y = \pm a}$ → Parallel to x-axis
$\boxed{x = \pm a}$ → Parallel to Y axis