Poularikas A. D. "The Mellin Transform" The Handbook of Formulas and Tables for Signal Processing. Ed. Alexander D. Poularikas

Boca Raton: CRC Press LLC,1999

# The Mellin Transform

- 18.1 The Mellin Transform
- 18.2 Properties of Mellin Transform
- 18.3 Examples of Mellin Transform
- 18.4 Special Functions Frequency Occuring in Mellin Transforms
- 18.5 Tables of Mellin Transform

References

# 18.1 The Mellin Transform

#### 18.1.1 Definition

$$M\{f(t);s\} = F(s) = \int_0^\infty f(t)t^{s-1} dt$$

**Example** 

$$M\{e^{-at}u(t)\} = \int_0^\infty e^{-at} t^{s-1} dt = a^{-s}\Gamma(s)$$

### 18.1.2 Relation to Laplace Transform

By letting  $t = e^{-x}$ ,  $dt = -e^{-x} dx$ , the transform becomes

$$M\{f(t)\} = F(s) = \int_{-\infty}^{\infty} f(e^{-x})e^{-sx}dx = L\{f(e^{-x})\}\$$

#### 18.1.3 Relation to Fourier Transform

By setting  $s = \sigma + j2\pi\beta$  in 18.1.2 we obtain

$$F(s) = \int_{-\infty}^{\infty} f(e^{-x}) e^{-ax} e^{-j2\pi\beta x} dx$$

which implies that

$$M\{f(t); s = \sigma + j2\pi\beta\} = F\{f(e^{-x})e^{-ax}; \beta\}$$

where

$$F\{f(x);\beta\} = \int_{-\infty}^{\infty} f(x)e^{-j2\pi\beta x} dx = \text{ Fourier Transform}$$

#### 18.1.4 Inversion Formula

$$f(t) = \frac{1}{2\pi i} \int_{c-j\infty}^{c+j\infty} F(s)t^{-s} ds$$

where c is within the strip of analyticity a < Re s < b.

# 18.2 Properties of Mellin Transform

### 18.2.1 Scaling Property

$$\mathbb{M}\{f(at);s\} = \int_0^\infty f(at)t^{s-1} dt = a^{-s} \int_0^\infty f(x)x^{s-1} dx = a^{-s}F(s)$$

### 18.2.2 Multiplication by $t^a$

$$M\{t^{a}f(t);s\} = \int_{0}^{\infty} f(t)t^{(s+a)-1} dt = F(s+a)$$

# 18.2.3 Raising the Independent Variable to a Real Power

$$\mathbb{M}\{f(t^a);s\} = \int_0^\infty f(t^a)t^{s-1} dt = \int_0^\infty f(x)x^{\frac{s-1}{a-a}} \left(\frac{1}{a}x^{\frac{1}{a-1}} dx\right) = a^{-1}F\left(\frac{s}{a}\right), \ a > 0$$

### 18.2.4 Inverse of Independent Variable

$$M\{t^{-1} f(t^{-1}); s\} = F(1-s)$$

### 18.2.5 Multiplication by ln t

$$M\{\ln t \, f(t); s\} = \frac{d}{ds} F(s)$$

# 18.2.6 Multiplication by a Power of ln t

$$M\{(\ln t)^k f(t); s\} = \frac{d^k}{ds^k} F(s)$$

#### 18.2.7 Derivative

$$\mathbb{M}\left\{\frac{d^k}{ds^k}f(t);s\right\} = (-1)^k(s-k)_kF(s-k)$$

$$(s-k)_k \equiv (s-k)(s-k+1)\cdots(s-1) = \frac{(s-1)!}{(s-k-1)!} = \frac{\Gamma(s)}{\Gamma(s-k)}$$

### 18.2.8 Derivative Multiplied by Independent Variable

$$\mathbb{M}\left\{t^{k} \frac{d^{k}}{ds^{k}} f(t); s\right\} = (-1)^{k} (s)_{k} F(s) = (-1)^{k} \frac{\Gamma(s+k)}{\Gamma(s)} F(s), (s)_{k} = s(s+1) \cdots (s+k-1)$$

**Example** 

$$\mathbb{M}\left\{t^2 \frac{d^2 f(t)}{dt^2} + t \frac{df(t)}{dt}; s\right\} = s^2 F(s)$$

#### 18.2.9 Convolution

$$M\{f(t)g(t);s\} = \frac{1}{2\pi i} \int_{c-j\infty}^{c+j\infty} F(z)G(s-z)dz$$

# 18.2.10 Multiplicative Convolution

$$M\{f \lor g\} = M\left\{ \int_0^\infty f\left(\frac{t}{u}\right) g(u) \frac{du}{u}; s \right\} = F(s) G(s)$$

$$\mathsf{M}^{-1}\{F(s)\,G(s)\} = \int_0^\infty f\!\left(\frac{t}{u}\right)\!g(u)\frac{du}{u}$$

Properties of the Multiplicative Convolution

$$\int_0^\infty f\left(\frac{t}{u}\right)g(u)\frac{du}{u}$$

- 1.  $f \lor g = g \lor f$  commutative
- 2.  $(f \lor g) \lor h = f \lor (g \lor h)$  associative
- 3.  $f \lor \delta(t-1) = f$  unit element

4. 
$$\left(t\frac{d}{dt}\right)^k (f \vee g) = \left[\left(t\frac{d}{dt}\right)^k f\right] \vee g = f \vee \left[\left(t\frac{d}{dt}\right)^k g\right]$$

5. 
$$(\ln t)(f \vee g) = [(\ln t)f] \vee g + f \vee [(\ln t)g]$$

6. 
$$\delta(t-a) \vee f = a^{-1} f(a^{-1}t)$$

$$\delta(t-p) \lor \delta(t-p') = \delta(t-pp'), \quad p, p' > 0$$

$$\frac{d^k \delta(t-1)}{dt^k} \vee f = \left(\frac{d}{ds}\right)^k (t^k f)$$

### 18.2.11 Parseval's Formulas

$$\int_{0}^{\infty} f(t) g(t) = \frac{1}{2\pi i} \int_{c-j\infty}^{c+j\infty} M\{f; s\} M\{g; 1-s\} ds$$

$$\int_0^\infty f(t)g^*(t)t^{2r+1}dt = \int_{-\infty}^\infty M\{f\}(\beta)M^*\{g\}(\beta)d\beta$$

where

$$M\{f\}(\beta) = \int_0^\infty f(t) t^{2\pi i \beta + r} dt$$

# 18.3 Examples of Mellin Transform

### **18.3.1** Example

$$M\{t^{a}u(t-t_{0})\} = \int_{t_{0}}^{\infty} t^{a+s-1} dt = -\frac{t_{o}^{a+s}}{a+s}, \operatorname{Re}\{s\} < -a$$

# **18.3.2 Example**

$$M\left\{\frac{1}{1+t};s\right\} = \int_0^\infty \frac{1}{1+t} t^{s-1} dt.$$

Setting  $t+1=\frac{1}{1-x}$  we obtain:  $x=\frac{t}{t+1}$ ,  $dx=\frac{dt}{(1+t)^2}$ . Hence,

$$\mathbb{M}\left\{f;s\right\} = \int_{0}^{1} (1-x) \frac{x^{s-1}}{(1-x)^{s-1}} \frac{dx}{(1-x)^{2}} = \int_{0}^{1} x^{s-1} (1-x)^{-s} dx = B(s,1-s) = \Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}$$

$$0 < \operatorname{Re}\left\{s\right\} < 1.$$

# 18.3.3 Example

From  $\int_{0}^{1} (1-u)^{m-1} u^{s-1} du = \frac{\Gamma(m)\Gamma(s)}{\Gamma(m+s)}$ , Re $\{s\} > 0$ , Re $\{m\} > 0$ , with the setting u = x/(1+x), we obtain

$$\int_0^\infty \frac{x^{s-1}}{(1+x)^{m+s}} dx = \frac{\Gamma(m)\Gamma(s)}{\Gamma(m+s)}.$$

Hence,

$$M\{(1+t)^{-a}; s\} = \frac{\Gamma(s)\Gamma(a-s)}{\Gamma(a)}, \ 0 < \text{Re}\{s\} < \text{Re}\{a\}.$$

# **18.3.4** Example

Using 18.2.3 and 18.3.3 we obtain

$$\mathbb{M}\{(1+t^a)^{-b};s\} = \frac{\Gamma(s/a)\Gamma\left(b-\frac{s}{a}\right)}{a\Gamma(b)}, \ 0 < \operatorname{Re}\{s\} < \operatorname{Re}\{ab\}.$$

### **18.3.5** Example

$$M\{\delta(t-t_o);s\} = \int_0^\infty \delta(t-t_0)t^{s-1}dt = t_0^{s-1} \quad \text{(see 5.3.1)}$$

### **18.3.6** Example

From 18.3.1 and 18.3.5  $M\{t^a u(t-t_0)\} = -\frac{t_0^{a+s}}{a+s}$  and hence,

$$\begin{split} \mathbb{M}\bigg\{\frac{df}{dt};s\bigg\} &= -(s-1)F(s-1) = (s-1)\frac{t_0^{a+s-1}}{a+s-1} = -a\frac{t_0^{a+s-1}}{a+s-1} + t_0^{a+s-1} \\ &= \mathbb{M}\{au(t-t_0)t^{a+s-1};s\} + \mathbb{M}\{t_0^a \,\delta(t-t_0);s\} \end{split}$$

# 18.4 Special Functions Frequency Occurring in Mellin Transforms

#### 18.4.1 Definition

The gamma function  $\Gamma(s)$  is defined on the complex half-plane Re(s) > 0 by the integral

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt$$

# 18.4.2 Analytic Continuation

The analytical continued gamma function is holomorphic in the whole plane except at the points  $s = -n, n = 0, 1, 2, \dots$ , where it has a simple pole.

#### 18.4.3 Residues at the Poles

$$\operatorname{Re} s_{s=-n}(\Gamma(s)) = \frac{(-1)^n}{n!}$$

#### 18.4.4 Relation to the Factorial

$$\Gamma(n+1) = n!$$

### 18.4.5 Functional Relations

$$\Gamma(s+1) = s\Gamma(s)$$

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma(2s) = \pi^{-1/2} 2^{2s-1} \Gamma(s) \Gamma(s+1/2)$$

(Legendre's duplication formula)

$$\Gamma(ms) = m^{ms-1/2} (2\pi)^{(1-m)/2} \prod_{k=0}^{m-1} \Gamma(s+k/m), \quad m=2,3,\cdots,$$

(Gauss-Legendre's multiplication formula)

$$\Gamma(s) \sim \sqrt{2\pi} \ s^{s-1/2} \exp \left[ -s \left( 1 + \frac{1}{12 \, s} \right) + O(s^{-2}) \right], \ s \to \infty, \ \left| \arg(s) \right| < \pi$$

(Stirling asymptotic formula)

#### 18.4.6 The Beta Function

Definition:  $B(x,y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt$ 

Relation to the gamma function:  $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ 

# 18.4.7 The psi Function (logarithmic derivative of the gamma function)

Definition:  $\psi(s) \equiv \frac{d}{ds} \ln \Gamma(s)$ 

$$= -\gamma + \sum_{n=0}^{\infty} \left( \frac{1}{n+1} - \frac{1}{s+n} \right)$$

Euler's constant  $\gamma$ , also called C, is defined by

$$\gamma \equiv -\Gamma'(1)/\Gamma(1)$$

and has value  $\gamma \cong 0.577 \dots$ 

#### 18.4.8 Riemann's Zeta Function

$$\zeta(z,q) \equiv \sum_{n=0}^{\infty} \frac{1}{(q+n)^z}, \text{ Re}(z) > 1, q \neq 0, -1, -2, \dots$$

$$\zeta(z) \equiv \sum_{n=0}^{\infty} \frac{1}{n^z}, \operatorname{Re}(z) > 1$$

The function  $\zeta(z)$  is analytic in the whole complex z-plane except in z=1 where it has a simple pole with residue equal to +1.

# 18.5 Tables of Mellin Transform

# 18.5.1 Tables of Mellin Transform

TABLE 18.1 Some Standard Mellin Transform Pairs

Original Function	Mellin Transform		
f(t), t > 0	$M[f;s] \equiv \int_0^\infty f(t) t^{s-1} dt$	Strip of holomorphy	
Algebraic Functions			
$u(t-a)t^b, \ a>0$	$-\frac{a^{b+s}}{b+s}$	$\operatorname{Re}(s) < -\operatorname{Re}(b)$	
$(u(t-a)-u(t))t^b$	$-\frac{a^{b+s}}{b+s}$	$\operatorname{Re}(s) > -\operatorname{Re}(b)$	
$(1+t)^{-1}$	$\frac{\pi}{\sin(\pi s)}$	$0 < \operatorname{Re}(s) < 1$	
$(a+t)^{-1}, \ \left \arg a\right  < \pi$	$\pi a^{s-1} \csc(\pi s)$	$0 < \operatorname{Re}(s) < 1$	
$(1+t)^{-a}$	$\frac{\Gamma(s)\Gamma(a-s)}{\Gamma(a)}$	$0 < \operatorname{Re}(s) < \operatorname{Re}(a)$	
$(1-t)^{-1}$	$\pi \cot(\pi s)$	$0 < \operatorname{Re}(s) < 1$	
$(a-t)^{-1}, \ a>0$	$\pi a^{s-1} \cot(\pi s)$	$0 < \operatorname{Re}(s) < 1$	
$u(1-t)(1-t)^{b-1}$ , Re(b) > 0	$\frac{\Gamma(s)\Gamma(b)}{\Gamma(s+b)}$	Re(s) > 0	
$u(t-1)(t-1)^{-b}$	$\frac{\Gamma(b-s)\Gamma(1-b)}{\Gamma(1-s)}$	Re(s) < Re(b) < 1	
$(t^2 + a^2)^{-1}$ , Re(a) > 0	$\frac{1}{2}\pi a^{s-2}\csc\left(\frac{\pi s}{2}\right)$	$0 < \operatorname{Re}(s) < 2$	
$(t^n + a), \ \left  \arg a \right  < \pi$	$(\pi/n)\csc(\pi s/n)a^{s/n-1}$	$0 < \operatorname{Re}(s) < n$	
$\begin{cases} t^{v} & 0 < t < 1 \\ 0 & t > 1 \end{cases}$	$(s+v)^{-1}$	Re(s) > -Re(v)	
$\begin{cases} (1-t^{h})^{v-1} & 0 < t < 1 \\ 0 & t \ge 0 \\ h > 0 & \text{Re}(v) > 0 \end{cases}$	$h^{-1} \frac{\Gamma(v) \Gamma\left(\frac{s}{h}\right)}{\Gamma\left(v + \frac{s}{h}\right)}$	Re(s) > 0	
$(1-t^a)(1-t^{na})^{-1}$	$\frac{\pi}{na}\sin\left(\frac{\pi}{n}\right)\csc\left(\frac{\pi s}{na}\right)\csc\left(\frac{\pi s + \pi a}{na}\right)$	$0 < \operatorname{Re}(s) < (n-1)a$	
Exponential Functions			
$e^{-pt}, p>0$	$p^{-s}\Gamma(s)$	Re(s) > 0	
$(e^t-1)^{-1}$	$\Gamma(s)\zeta(s)$	Re(s) > 1	
	$(\zeta(s) = \text{zeta function})$		
$(e^{at} + 1)^{-1}$ , $Re(a) > 0$	$a^{-s} \Gamma(s)(1-2^{1-s}) \zeta(s)$	Re(s) > 0	
$(e^{at}-1)^{-1}, \operatorname{Re}(a) > 0$	$a^{-s}\Gamma(s)\zeta(s)$	Re(s) > 1	
$(e^{-at})(1-e^{-t})^{-1}$ , Re(a) > 0	$\Gamma(s)\zeta(s,a)$	Re(s) > 1	
$\left(e^{t}-1\right)^{-2}$	$\Gamma(s)[\zeta(s-1)-\zeta(s)]$	Re(s) > 2	

 TABLE 18.1 Some Standard Mellin Transform Pairs (continued)

Mellin Transform		
$M[f;s] \equiv \int_0^\infty f(t)t^{s-1} dt$	Strip of holomorphy	
$h^{-1}a^{-s/h}\Gamma(s/h)$	Re(s) > 0	
$\Gamma(1-s)$	$-\infty < \text{Re}(s) < 1$	
$\frac{1}{2}\Gamma(s/2)$	$0 < \text{Re}(s) < \infty$	
$-h^{-1}a^{-s/h}\Gamma(s/h)$	$-h < \operatorname{Re}(s) < 0$	
$a^{-s}\Gamma(s)e^{j\pi(s/2)}$	$0 < \operatorname{Re}(s) < 1$	
$\frac{\pi}{s\sin(\pi s)}$	$-1 < \operatorname{Re}(s) < 0$	
$\pi s^{-1}a^{-s}\csc(\pi s)$	$-1 < \operatorname{Re}(s) < 0$	
$-p^{s}s^{-1}[\psi(s+1)+p^{-1}\ln\gamma]$	Re(s) > 0	
$\frac{\pi}{(1-s)\sin(\pi s)}$	$0 < \operatorname{Re}(s) < 1$	
$(\pi/s)\tan(\pi s/2)$	$-1 < \operatorname{Re}(s) < 1$	
$s^{-1}a^{-s}(\ln a - s^{-1})$	Re(s) > 0	
$-(s+v)^{-2}$	Re(s) > -Re(v)	
$\frac{d^n\Gamma(s)}{ds^n}$	Re(s) > 0	
$\frac{a}{s^2 + a^2}$	$\operatorname{Re}(s) < - \big  \operatorname{Im}(a) \big $	
$\frac{a}{s^2 + a^2}$	$\operatorname{Re}(s) >  \operatorname{Im}(a) $	
$\frac{p^s}{s^2}$	Re(s) > 0	
<del></del>		
$a^{-s}\Gamma(s)\sin(\pi s/2)$	$-1 < \operatorname{Re}(s) < 1$	
$(a^2 + \beta^2)^{-s/2} \Gamma(s) \sin\left(s \tan^{-1} \frac{\beta}{a}\right)$	$\operatorname{Re}(s) > -1$	
$-2^{-s-1}a^{-s}\Gamma(s)\cos(\pi s/2)$	$-2 < \operatorname{Re}(s) < 0$	
$a^{-s}\Gamma(s)\cos(\pi s/2)$	$0 < \operatorname{Re}(s) < 1$	
$\frac{-\pi}{2s\cos(\pi s/2)}$	$-1 < \operatorname{Re}(s) < 0$	
$\frac{\pi}{2s\cos(\pi s/2)}$	$0 < \operatorname{Re}(s) < 1$	
	$M[f;s] = \int_{0}^{\infty} f(t)t^{s-1} dt$ $h^{-1}a^{-s/h}\Gamma(s/h)$ $\Gamma(1-s)$ $\frac{1}{2}\Gamma(s/2)$ $-h^{-1}a^{-s/h}\Gamma(s/h)$ $a^{-s}\Gamma(s)e^{j\pi(s/2)}$ $\frac{\pi}{s\sin(\pi s)}$ $\pi s^{-1}a^{-s}\csc(\pi s)$ $-p^{s}s^{-1}[\psi(s+1)+p^{-1}\ln\gamma]$ $\frac{\pi}{(1-s)\sin(\pi s)}$ $(\pi/s)\tan(\pi s/2)$ $s^{-1}a^{-s}(\ln a-s^{-1})$ $-(s+v)^{-2}$ $\frac{a^{n}\Gamma(s)}{ds^{n}}$ $\frac{a}{s^{2}+a^{2}}$ $\frac{a^{n}\Gamma(s)}{s^{2}}$ $a^{-s}\Gamma(s)\sin(\pi s/2)$ $(a^{2}+\beta^{2})^{-s/2}\Gamma(s)\sin\left(s\tan^{-1}\frac{\beta}{a}\right)$ $-2^{-s-1}a^{-s}\Gamma(s)\cos(\pi s/2)$ $a^{-s}\Gamma(s)\cos(\pi s/2)$ $\frac{-\pi}{2s\cos(\pi s/2)}$ $\frac{\pi}{s^{n}}$	

TABLE 18.1 Some Standard Mellin Transform Pairs (continued)

Original Function	Mellin Transform		
f(t), t > 0	$M[f;s] \equiv \int_0^\infty f(t)t^{s-1} dt$	Strip of holomorphy	
Other Functions			
$J_{v}(at), \ a > 0$	$\frac{2^{s-1}\Gamma\left(\frac{s}{2} + \frac{v}{2}\right)}{a^s\Gamma\left(\frac{v}{2} - \frac{s}{2} + 1\right)}$	$-\operatorname{Re}(v) < \operatorname{Re}(s) < 3/2$	
$\sin at J_{v}(at), \ a > 0$	$\frac{2^{\nu-1}\Gamma\left(\frac{1}{2}-s\right)\Gamma\left(\frac{1}{2}+\frac{\nu}{2}+\frac{s}{2}\right)}{a^{s}\Gamma(1+\nu-s)\Gamma\left(1-\frac{\nu}{2}-\frac{s}{2}\right)}$	-1 < Re(v) < Re(s) < 1/2	
$\delta(t-p), \ p > 0$	$p^{s-1}$	whole plane	
$\sum_{n=1}^{\infty} \delta(t - pn), \ p > 0$	$p^{s-1}\zeta(1-s)$	Re(s) < 0	
$J_{v}(t)$	$\frac{2^{s-1}\Gamma(s+v)/2}{\Gamma[(1/2)(v-s)+1]}$	$-v < \operatorname{Re}(s) < 3/2$	
$\sum_{n=-\infty}^{\infty} p^{-nr} \delta(t-p^n),$	$\frac{1}{\ln p} \sum_{n=-\infty}^{\infty} \delta \left( \beta - \frac{n}{\ln p} \right),$	$s = r + j\beta$	
p > 0, $r = real$	$\beta = \text{Im}(s)$		
$t^b$	$\delta(b+s)$	none (analytic functional)	

### References

Bertrand, Jacqueline, Pierre Bertrand, and Jean-Philippe Ovarlez, The Mellin Transform, in *Transforms and Applications Handbook*, ed. Alexander Poularikas, CRC Press, Boca Raton, Florida, 1996. Davies, G., *Integral Transforms and Their Applications*, 2nd ed., Springer-Verlag, New York, NY, 1984. Erdelyi, A., W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Tables of Integral Transfer*, McGraw-Hill Book Co., New York, NY, 1954.

Oberhettinger, F., *Tables of Mellin Transform*, 2nd ed., Springer-Verlag, New York, NY, 1974. Sneddon, Ian N., *The Use of Integral Transform*, McGraw-Hill Book Co., New York, NY, 1972.