General Relativity

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Chapter 1

Introduction

1.1 The Idea behind General Relativity

There was no need for general relativity when Einstein started working on it. There was no experimental data signalling any failure of the Newtonian theory of gravity, except perhaps for the minute advance of the perihelion of Mercury's orbit by $43''$ per *century*, which researchers at the time tried to explain by perturbations not included yet into the calculations of celestial mechanics in the Solar System.

Essentially, Einstein found general relativity because he was deeply dissatisfied with some of the concepts of the Newtonian theory, in particular the concept of an inertial system, for which no experimental demonstration could be given.

After special relativity, he was convinced quite quickly that trying to build a relativistic theory of gravitation led to conclusions which were in conflict with experiments. Action at a distance is impossible in special relativity because the absolute meaning of space and time had to be given up. The most straightforward way to combine special relativity with Newtonian gravity seemed to start from Poisson's equation for the gravitational potential and to add time derivatives to it so as to make it relativistically invariant.

However, it was then unclear how the law of motion should be modified because, according to special relativity, energy and mass are equivalent and thus the mass of a body should depend on its position in a gravitational field.

This led Einstein to a result which raised his suspicion. In Newtonian theory, the vertical acceleration of a body in a vertical gravitational field is independent of its horizontal motion. In a special-relativistic extension of Newton's theory, this would no longer be the case: the vertical gravitational acceleration would depend on the kinetic energy of a body, and thus not be independent of its horizontal motion.

This was in striking conflict with experiment, which says that all bodies experience the same gravitational acceleration. At this point, the equivalence of inertial and gravitational mass struck Einstein as a law of deep significance. It became the heuristic guiding principle in the construction of general relativity.

This line of thought leads to the fundamental concept of general relativity. It says that it must be possible to introduce local, freely-falling frames of reference in which gravity is locally "transformed away". The directions of motion of different freely-falling reference frames will generally not be parallel: Einstein elevators released at the same height above the Earth's surface but over different locations will fall towards the Earth's centre and thus approach each other.

This leads to the idea that space-time is a four-dimensional manifold instead of the "rigid", four-dimensional Euclidean space. As will be explained in the following two chapters, manifolds can locally be mapped onto Euclidean space. In a freely-falling reference frame, special relativity must hold, which implies that the Minkowskian metric of special relativity must locally be valid. The same operation must be possible in all freely-falling reference frames individually, but not globally, as is illustrated by the example of the Einstein elevators falling towards the Earth.

Thus, general relativity considers the metric of the space-time manifold as a dynamical field. The necessity to match it with the Minkowski metric in freely-falling reference frames means that the signature of the metric must be $(-,+,+,+)$ or $(+,-,-,-)$. A manifold with a metric which is not positive definite is called *pseudo-Riemannian*, or *Lorentzian* if the metric has the signature of the Minkowski metric.

The lecture starts with an introductory chapter describing the fundamental characteristics of gravity, their immediate consequences and the failure of a specially-relativistic theory of gravity. It then introduces in two chapters the mathematical apparatus necessary for general relativity, which are the basics of differential geometry, i.e. the geometry on curved manifolds. After this necessary mathematical digression, we shall return to physics when we introduce Einstein's field equations in chapter 4.

1.2 Fundamental Properties of Gravity

1.2.1 Scales

The first remarkable property of gravity is its weakness. It is by far the weakest of the four known fundamental interactions. To see this, compare the gravitational and electrostatic forces acting between two protons at a distance r. We have

$$\frac{\text{gravity}}{\text{electrostatic force}} = \left(\frac{Gm_{\rm p}^2}{r^2}\right)\left(\frac{e^2}{r^2}\right)^{-1} = \frac{Gm_{\rm p}^2}{e^2} = 8.1 \times 10^{-37}\ ! \tag{1.1}$$

This leads to an interesting comparison of scales. In quantum physics, a particle of mass m can be assigned the Compton wavelength

$$\lambda = \frac{\hbar}{mc}\ , \tag{1.2}$$

where Planck's constant h is replaced by $\hbar$ merely for conventional reasons. We ask what the mass of the particle must be such that its gravitational potential energy equals its rest mass mc^2, and set

$$\frac{Gm^2}{\lambda} \stackrel{!}{=} mc^2\ . \tag{1.3}$$

The result is the *Planck mass*,

$$m = M_{\rm Pl} = \sqrt{\frac{\hbar c}{G}} = 2.2 \times 10^{-5}\,\text{g} = 1.2 \times 10^{19}\,\frac{\text{GeV}}{c^2}\ , \tag{1.4}$$

which, inserted into (1.2), yields the *Planck length*

$$\lambda_{\rm Pl} = \sqrt{\frac{\hbar G}{c^3}} = 1.6 \times 10^{-33}\,\text{cm} \tag{1.5}$$

and the *Planck time*

$$t_{\rm Pl} = \frac{\lambda_{\rm Pl}}{c} = \sqrt{\frac{\hbar G}{c^5}} = 5.3 \times 10^{-44}\,\text{s}\ . \tag{1.6}$$

As Max Planck noted already in 1900[1], these are the only scales for mass, length and time that can be assigned an objective meaning.

The Planck mass is huge in comparison to the mass scales of elementary particle physics. The Planck length and time are commonly interpreted as the scales where our "classical" description of space-time is expected to break down and must be replaced by an unknown theory combining relativity and quantum physics.

Using the Planck mass, the ratio from (1.1) can be written as

$$\frac{Gm_{\rm p}^2}{e^2} = \frac{1}{\alpha}\frac{m_{\rm p}^2}{M_{\rm Pl}^2}\ , \tag{1.7}$$

where $\alpha = e^2/\hbar c \approx 1/137$ is the fine-structure constant.

[1] Über irreversible Strahlungsvorgänge, Annalen der Physik 306 (1900) 69

This suggests that gravity will dominate all other interactions once the mass of an object is sufficiently large. A mass scale important for the astrophysics of stars is set by the ratio

$$M_{\rm Pl}\,\frac{M_{\rm Pl}^2}{m_{\rm p}^2} = 1.7\times 10^{38}\,M_{\rm Pl} = 3.7\times 10^{33}\,{\rm g}\;, \tag{1.8}$$

which is almost two solar masses. We shall see at the end of this lecture that stellar cores of this mass cannot be stabilised against gravitational collapse.

1.2.2 The Equivalence Principle

The observation that inertial and gravitational mass cannot be experimentally distinguished is a highly remarkable finding. It is by no means obvious that the ratio between any force exerted on a body and its consequential acceleration should have anything to do with the ratio between the gravitational force and the body's acceleration in a gravitational field.

The experimentally well-established fact that inertial and gravitational mass are the same at least within our measurement accuracy was raised to a guiding principle by Einstein, the *principle of equivalence*, which can be formulated in several different ways.

The weaker and less precise statement is that *the motion of a test body in a gravitational field is independent of its mass and composition*,

which can be cast into the more precise form that *in an arbitrary gravitational field, no local non-gravitational experiment can distinguish a freely falling, non-rotating system from a uniformly moving system in absence of the gravitational field.*

The latter is *Einstein's Equivalence Principle*, which is the heuristic guiding principle for the construction of general relativity.

It is important to note the following remarkable conceptual advance: Newtonian mechanics starts from Newton's axioms, which introduce the concept of an inertial reference frame, saying that force-free bodies in inertial systems remain at rest or move at constant velocity, and that bodies in inertial systems experience an acceleration which is given by the force acting on them, divided by their mass.

Firstly, inertial systems are a deeply unsatisfactory concept because they cannot be realised in any strict sense. Approximations to inertial systems are possible, but the degree to which a reference frame will approximate an inertial system will depend on the precise circumstances of the experiment or the observation made.

Secondly, Newton's second axiom is, strictly speaking, circular in the sense that it defines forces if one is willing to accept inertial systems, while it defines inertial systems if one is willing to accept the relation between force and acceleration. A satisfactory, non-circular definition of force is not given in Newton's theory. The existence of inertial frames is *postulated*.

Special relativity replaces the rigid Newtonian concept of absolute space and time by a space-time which carries the peculiar light-cone structure demanded by the universal value of the light speed. Newtonian space-time can be considered as the Cartesian product $\mathbb{R} \times \mathbb{R}^3$. An instant $t \in \mathbb{R}$ in time uniquely identifies the three-dimensional Euclidean space of all simultaneous events.

Of course, it remains possible in special relativity to define simultaneous events, but the three-dimensional hypersurface from four-dimensional Euclidean space $\mathbb{R}^4$ identified in this way depends on the motion of the observer relative to another observer. Independent of their relative motion, however, is the light-cone structure of Minkowskian space-time. The future light cone encloses events in the future of a point p in space-time which can be reached by material particles, and its boundary is defined by events which can be reached from p by light signals. The past light cone encloses events in the past of p from which material particles can reach p, and its boundary is defined by events from which light signals can reach p.

Yet, special relativity makes use of the concept of inertial reference frames. Physical laws are required to be invariant under transformations from the Poincaré group, which translate from one inertial system to another.

General relativity keeps the light-cone structure of special relativity, even though its rigidity is given up: the orientation of the light cones can vary across space-time. Thus, the relativity of distances in space and time remains within the theory. However, it is one of the great achievements of general relativity that it finally replaces the concept of inertial systems by something else which can be experimentally demonstrated: the principle of equivalence replaces inertial systems by non-rotating, freely-falling frames of reference.

1.3 Immediate Consequences of the Equivalence Principle

Without any specific form of the theory, the equivalence principle immediately allows us to draw conclusions on some of the consequences any theory must have which is built upon it. We discuss two here to illustrate its general power, namely the gravitational redshift and gravitational light deflection.

1.3.1 Gravitational Redshift

We enter an Einstein elevator which is at rest in a gravitational field at $t = 0$. The elevator is assumed to be small enough for the gravitational field to be considered as homogeneous within it, and the (local) gravitational acceleration be g.

According to the equivalence principle, the *downward* gravitational acceleration felt in the elevator cannot locally be distinguished from a constant *upward* acceleration of the elevator with the same acceleration g. Adopting the equivalence principle, we thus assume that the gravitational field is absent and that the elevator is constantly accelerated upward instead.

At $t = 0$, a photon is emitted by a light source at the bottom of the elevator, and received some time Δt later by a detector at the ceiling. The time interval Δt is determined by

$$h + \frac{g}{2}\Delta t^2 = c\Delta t \,, \tag{1.9}$$

where h is the height of elevator. This equation has the solution

$$\Delta t_\pm = \frac{1}{g}\left[c \pm \sqrt{c^2 - 2gh}\right] = \frac{c}{g}\left[1 - \sqrt{1 - \frac{2gh}{c^2}}\right] \approx \frac{h}{c} \,; \tag{1.10}$$

the other branch makes no physical sense.

When the photon is received at the ceiling, the ceiling moves with the velocity

$$\Delta v = g\Delta t \approx \frac{gh}{c} \tag{1.11}$$

compared to the floor when the photon was emitted. The photon is thus Doppler shifted with respect to its emission, and is received with the longer wavelength

$$\lambda' \approx \left(1 + \frac{\Delta v}{c}\right)\lambda \approx \left(1 + \frac{gh}{c^2}\right)\lambda \,. \tag{1.12}$$

The gravitational acceleration is given by the gravitational potential Φ through

$$g = |\vec{\nabla}\Phi| \quad \Rightarrow \quad gh \approx \Delta\Phi \,, \tag{1.13}$$

where $\Delta\Phi \approx |\vec{\nabla}\Phi|\, h$ is the change in Φ from the floor to the ceiling of the elevator. Thus, the equivalence principle demands a gravitational redshift of

$$z \equiv \frac{\lambda' - \lambda}{\lambda} \approx \frac{\Delta\Phi}{c^2} \,. \tag{1.14}$$

1.3.2 Gravitational Light Deflection

Similarly, it can be concluded from the equivalence principle that light rays should be curved in gravitational fields. To see this, consider again the Einstein elevator from above which is at rest in a gravitational field $g = |\vec{\nabla}\Phi|$ at $t = 0$.

As before, the equivalence principle asserts that we can consider the elevator as being accelerated upwards with the acceleration g.

Suppose now that a horizontal light ray enters the elevator at $t = 0$ from the left and leaves it at a time $\Delta t = w/c$ to the right, if w is the horizontal width of the elevator.

As the light ray leaves the elevator, the elevator's velocity has increased to

$$\Delta v = g\Delta t = \frac{|\vec{\nabla}\Phi|w}{c} \tag{1.15}$$

such that, in the rest frame of the elevator, it leaves at an angle

$$\Delta\alpha = \frac{\Delta v}{c} = \frac{|\vec{\nabla}\Phi|w}{c^2} \tag{1.16}$$

downward from the horizontal because of the aberration due to the finite light speed. Since the *upward* accelerated elevator corresponds to an elevator at rest in a *downward* gravitational field, this leads to the expectation that light will be deflected towards gravitational fields.

Although it is possible to construct theories of gravity which obey the equivalence principle and do not lead to gravitational light deflection, the bending of light in gravitational fields is by now a well-established experimental fact.

1.4 Impossibility of a Theory of Gravity with Flat Spacetime

1.4.1 Gravitational Redshift

We have seen before that the equivalence principle implies a gravitational redshift, which has been demonstrated experimentally. We must thus require from a theory of gravity that it does lead to gravitational redshift.

Suppose we wish to construct a theory of gravity which retains the Minkowski metric $\eta_{\mu\nu}$. In such a theory, how ever it may look in detail, the proper time measured by observers moving along a world line $x^\mu(\lambda)$ from λ_1 to λ_2 is

$$\Delta\tau = \int_{\lambda_1}^{\lambda_2} \mathrm{d}\lambda \,\sqrt{-\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} \;, \tag{1.17}$$

where the minus sign under the square root appears because we choose the signature of $\eta_{\mu\nu}$ to be $(-1, 1, 1, 1)$.

Now, let a light ray propagate from the floor to the ceiling of the elevator in which we have measured gravitational redshift before. Specifically, let the light source shine between coordinates times t_1 and t_2. The emitted photons will propagate to the receiver at the ceiling along world lines which may be curved, but must be *parallel* because the metric is constant. The time interval within which the photons arrive at the receiver must thus equal the time interval $t_2 - t_1$ within which they left the emitter. Thus there cannot be gravitational redshift in a theory of gravity in flat spacetime.

1.4.2 A Scalar Theory of Gravity and the Perihelion Shift

Let us now try and construct a scalar theory of gravity starting from the field equation

$$\Box\Phi = -4\pi G T \ , \tag{1.18}$$

where Φ is the gravitational potential and $T = T^\mu_\mu$ is the trace of the energy-momentum tensor. Note that Φ is made dimensionless here by dividing it by c^2.

In the limit of weak fields and non-relativistic matter, this reduces to Poisson's equation

$$\vec{\nabla}^2\Phi = 4\pi G\rho \ , \tag{1.19}$$

since then the time derivatives in d'Alembert's operator and the pressure contributions to T can be neglected.

Let us further adopt the Lagrangian

$$\mathcal{L}(x^\mu, \dot{x}^\mu) = -mc\sqrt{-\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}(1+\Phi) \ , \tag{1.20}$$

which is the ordinary Lagrangian of a free particle in special relativity, multiplied by the factor $(1 + \Phi)$. This is the *only* possible Lagrangian that yields the right weak-field (Newtonian) limit.

We can write the square root in (1.20) as

$$\sqrt{-\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} = \sqrt{c^2 - \vec{v}^2} = c\sqrt{1-\vec{\beta}^2} \ , \tag{1.21}$$

where $\vec{\beta} = \vec{v}/c$ is the velocity in units of c. The weak-field limit of (1.20) for non-relativistic particles is thus

$$\mathcal{L}(x^\mu, \dot{x}^\mu) \approx -mc^2\left(1 - \frac{\vec{v}^2}{2c^2}\right)(1+\Phi) \approx -mc^2 + \frac{m}{2}\vec{v}^2 - mc^2\Phi \ , \tag{1.22}$$

which is the right expression in Newtonian gravity.

The equations of motion can now be calculated inserting (1.20) into Euler's equations,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{x}^\alpha} = \frac{\partial\mathcal{L}}{\partial x^\alpha} \ . \tag{1.23}$$

On the right-hand side, we find

$$\frac{\partial\mathcal{L}}{\partial x^\alpha} = -mc^2\sqrt{1-\beta^2}\frac{\partial\Phi}{\partial x^\alpha} \ . \tag{1.24}$$

On the left-hand side, we first have

$$\frac{\partial\mathcal{L}}{\partial\dot{x}^\alpha} = \frac{1}{c}\frac{\partial\mathcal{L}}{\partial\beta^\alpha} = \frac{mc\beta^\alpha}{\sqrt{1-\beta^2}}(1+\Phi) \ , \tag{1.25}$$

and thus

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{x}^\alpha} &= mc(1+\Phi)\left(\frac{\dot{\beta}^\alpha}{\sqrt{1-\beta^2}} + \frac{\beta^\alpha\vec{\beta}\dot{\vec{\beta}}}{(1-\beta^2)^{3/2}}\right) \\ &+ \frac{mc\beta^\alpha}{\sqrt{1-\beta^2}}\dot{\Phi} \ . \end{aligned}\tag{1.26}$$

We shall now simplify these equations assuming that the potential is static, $\dot{\Phi} = 0$, and that the motion is non-relativistic, $\beta \ll 1$. Then, (1.26) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\vec{v}} \approx mc(1+\Phi)\dot{\vec{\beta}}\left(1+\frac{\beta^2}{2}\right) \approx m(1+\Phi)\ddot{\vec{x}} \ , \tag{1.27}$$

and (1.24) turns into

$$\frac{\partial\mathcal{L}}{\partial\vec{x}} \approx -mc^2\left(1-\frac{\beta^2}{2}\right)\vec{\nabla}\Phi \approx -mc^2\vec{\nabla}\Phi \ . \tag{1.28}$$

The equation of motion thus reads, in this approximation,

$$(1+\Phi)\ddot{\vec{x}} = -c^2\vec{\nabla}\Phi \tag{1.29}$$

or

$$\ddot{\vec{x}} = -c^2\vec{\nabla}\Phi(1-\Phi) = -c^2\vec{\nabla}\left(\Phi - \frac{\Phi^2}{2}\right) \ . \tag{1.30}$$

Compared to the equation of motion in Newtonian gravity, therefore, the potential is augmented by a quadratic perturbation.

For a static potential and non-relativistic matter, the potential is given by Poisson's equation.

We now proceed to work out the perihelion shift expected for planetary orbits around the Sun in such a theory of gravity. As we know from the discussion of Kepler's problem in classical mechanics, the radius r and the polar angle ϕ of such orbits are characterised by

$$\frac{\mathrm{d}r}{\mathrm{d}\phi} = \frac{mr^2}{L}\sqrt{\frac{2}{m}(E - V_\mathrm{L})}\;, \tag{1.31}$$

where V_L is the effective potential energy

$$V_\mathrm{L} = V + \frac{L^2}{2mr^2}\;, \tag{1.32}$$

and L is the (orbital) angular momentum. Thus,

$$\frac{\mathrm{d}r}{\mathrm{d}\phi} = \frac{r^2}{L}\sqrt{2m(E - V) - \frac{L^2}{r^2}}\;. \tag{1.33}$$

The perihelion shift is the change in ϕ upon integrating once around the orbit, or integrating twice from the perihelion radius r_0 to the aphelion radius r_1,

$$\Delta\phi = 2\int_{r_0}^{r_1}\mathrm{d}r\,\frac{\mathrm{d}\phi}{\mathrm{d}r}\;. \tag{1.34}$$

Inverting (1.33), it is easily seen that (1.34) can be written as

$$\Delta\phi = -2\frac{\partial}{\partial L}\int_{r_0}^{r_1}\mathrm{d}r\,\sqrt{2m(E - V) - \frac{L^2}{r^2}}\;. \tag{1.35}$$

Now, we split the potential energy V into the Newtonian contribution V_0 and a perturbation $\delta V \ll V$ and expand the integrand to lowest order in δV, which yields

$$\Delta\phi \approx -2\frac{\partial}{\partial L}\int_{r_0}^{r_1}\mathrm{d}r\,\sqrt{A_0}\left(1 - \frac{m\delta V}{A_0}\right) \tag{1.36}$$

where the abbreviation

$$A_0 \equiv 2m(E - V_0) - \frac{L^2}{r^2} \tag{1.37}$$

was inserted for convenience.

We know that orbits in Newtonian gravity are closed, so that the first term in the integrand of (1.36) must vanish. Thus, we can write

$$\Delta\phi \approx 2\frac{\partial}{\partial L}\int_{r_0}^{r_1}\mathrm{d}r\,\frac{m\delta V}{\sqrt{A_0}}\;. \tag{1.38}$$

Next, we transform the integration variable from r to ϕ, using that

$$\frac{\mathrm{d}r}{\mathrm{d}\phi} \approx \frac{r^2}{L}\sqrt{A_0} \tag{1.39}$$

to leading order in δV, according to (1.31). Thus, (1.38) can be written as

$$\Delta\phi \approx \frac{\partial}{\partial L}\frac{2m}{L}\int_0^\pi \mathrm{d}\phi\, r^2 \delta V \; . \tag{1.40}$$

Finally, we specialise the potential energy. Since Poisson's equation for the gravitational potential remains valid, we have

$$V_0 = mc^2\Phi = -\frac{GM_\odot m}{r} \; , \tag{1.41}$$

where $M_\odot$ is the Sun's mass, and following (1.30), the potential perturbation is

$$\delta V = mc^2\frac{\Phi^2}{2} = \frac{mc^2}{2}\frac{V_0^2}{m^2c^4} = \frac{G^2M_\odot^2 m}{2c^2r^2} \; . \tag{1.42}$$

Inserting this into (1.40) yields the perihelion shift

$$\Delta\phi = \frac{\partial}{\partial L}\frac{m}{L}\frac{\pi G^2 M_\odot^2 m}{c^2} = -\frac{\pi G^2 M_\odot^2 m^2}{c^2L^2} \; . \tag{1.43}$$

The angular momentum L can be expressed by the semi-major axis a and the eccentricity e of the orbit,

$$L^2 = GM_\odot m^2 a(1-e^2) \; , \tag{1.44}$$

which allows us to write (1.43) in the form

$$\Delta\phi = -\frac{\pi GM_\odot}{ac^2(1-e^2)} \; . \tag{1.45}$$

For the Sun, $M_\odot = 2\times 10^{33}$ g, thus

$$\frac{GM_\odot}{c^2} = 1.5\times 10^5\,\mathrm{cm} \; . \tag{1.46}$$

For Mercury, $a = 5.8\times 10^{12}$ cm and the eccentricity $e = 0.2$ can be neglected because it appears quadratic in (1.45). Thus, we find

$$\Delta\phi = -8.1\times 10^{-8}\,\text{radian} = -0.017'' \tag{1.47}$$

per orbit. Mercury's orbital time is 88 d, i.e. it completes about 415 orbits per century, so that the perihelion shift predicted by the scalar theory of gravity is

$$\Delta\phi_{100} = -7'' \tag{1.48}$$

per century.

This turns out to be wrong: Mercury's perihelion shift is six times as large, and not even the sign is right. Therefore, our scalar theory of gravity fails in its first comparison with observations.

Chapter 2

Differential Geometry I

2.1 Differentiable Manifolds

By the preceding discussion of how a theory of gravity may be constructed which is compatible with special relativity, we are led to the concept of a space-time which "looks like" Minkowskian space-time locally, but may globally be curved. This concept is cast into a mathematically precise form by the introduction of a *manifold*.

An n-dimensional *manifold* M is a topological Hausdorff space with a countable base, which is locally homeomorphic to $\mathbb{R}^n$. This means that for every point $p \in M$, an open neighbourhood U of p exists together with a homeomorphism h which maps U onto an open subset U' of $\mathbb{R}^n$,

$$h : U \to U' \ . \tag{2.1}$$

A topological space is a set M together with a collection T of open subsets $T_i \subset M$ with the properties (i) $\emptyset \subset T$ and $M \subset T$; (ii) $\cap_{i=1}^n T_i \subset T$ for any finite n; (iii) $\cup_{i=1}^n T_i \subset T$ for any n. In a Hausdorff space, any two points $x, y \in M$ with $x \neq y$ can be surrounded by disjoint neighbourhoods.

A trivial example for an n-dimensional manifold is the $\mathbb{R}^n$ itself, on which h may be the identity map id. Thus, h is a specialisation of a map ϕ from one manifold M to another manifold N, $\phi : M \to N$.

A homeomorphism is a bijective, continuous map whose inverse is also continuous.

The homeomorphism h is called a *chart* or a *coordinate system* in the language of physics. U is the *domain* or the *coordinate neighbourhood* of the chart. The image $h(p)$ of a point $p \in M$ under the chart h is expressed by the n real numbers $(x^1, \ldots x^n)$, the coordinates of p in the chart h.

A set of charts h_α is called an *atlas* of M if the domains of the charts cover M completely.

An example for a manifold is the n-sphere S^n, for which the two-sphere S^2 is a particular specialisation. It cannot be continuously mapped to $\mathbb{R}^2$, but pieces of it can.

We can embed the two-sphere into $\mathbb{R}^3$ and describe it as the point set

$$S^2 = \left\{(x^1, x^2, x^3) \in \mathbb{R}^3 \,\middle|\, (x^1)^2 + (x^2)^2 + (x^3)^2 = 1\right\} ; \qquad (2.2)$$

then, the six half-spheres $U_i^\pm$ defined by

$$U_i^\pm = \left\{(x^1, x^2, x^3) \in S^2 \,\middle|\, \pm x^i > 0\right\} \qquad (2.3)$$

can be considered as domains of maps whose union covers S^2 completely, and the charts can be the projections of the half-spheres onto open disks

$$D_{ij} = \left\{(x^i, x^j) \in \mathbb{R}^2 \,\middle|\, (x^i)^2 + (x^j)^2 < 1\right\} , \qquad (2.4)$$

such as

$$f_1^+ : U_1^+ \to D_{23} , \quad f_1^+(x^1, x^2, x^3) = (x^2, x^3) . \qquad (2.5)$$

Thus, the six charts $f_i^\pm$, $i \in \{1, 2, 3\}$, together form an atlas of the two-sphere.

Let now h_α and h_β be two charts, and $U_{\alpha\beta} \equiv U_\alpha \cap U_\beta \neq \emptyset$ be the intersection of their domains. Then, the composition of charts $h_\beta \circ h_\alpha^{-1}$ exists and defines a map between two open sets in $\mathbb{R}^n$ which describes the *change of coordinates* or a *coordinate transform* on the intersection of domains U_α and U_β. An atlas of a manifold is called *differentiable* if the coordinate changes between all its charts are differentiable. A manifold, combined with a differentiable atlas, is called a *differentiable manifold*.

Using charts, it is possible to define *differentiable maps between manifolds*. Let M and N be differentiable manifolds of dimension m and n, respectively, and $\phi : M \to N$ be a map from one manifold to the other. Introduce further two charts $h : M \to M' \subset \mathbb{R}^m$ and $k : N \to N' \subset \mathbb{R}^n$ whose domains cover a point $p \in M$ and its image $\phi(p) \in N$. Then, the combination $k \circ \phi \circ h^{-1}$ is a map from the domain M' to the domain N', for which it is clear from advanced calculus what differentiability means. Unless stated otherwise, we shall generally assume that coordinate changes and maps between manifolds are C^∞, i.e. their derivatives of all orders exist and are continuous.

We return to the two-sphere S^2 and the atlas of the six projection charts $\mathcal{A} = \{f_1^\pm, f_2^\pm, f_3^\pm\}$ described above and investigate whether it is differentiable. For doing so, we arbitrarily pick the charts f_3^+ and f_1^+, whose domains are the "northern" and "eastern" half-spheres, respectively, which overlap on the "north-eastern" quarter-sphere. Let therefore $p = (p^1, p^2, p^3)$ be a point in the domain overlap, then

$$\begin{aligned} f_3^+(p) &= (p^1, p^2) , \quad f_1^+(p) = (p^2, p^3) , \\ (f_3^+)^{-1}(p^1, p^2) &= \left(p^1, p^2, \sqrt{1 - (p^1)^2 - (p^2)^2}\right) , \\ f_1^+ \circ (f_3^+)^{-1}(p^1, p^2) &= \left(p^2, \sqrt{1 - (p^1)^2 - (p^2)^2}\right) , \end{aligned} \qquad (2.6)$$

which is obviously differentiable. The same applies to all other coordinate changes between charts of $\mathcal{A}$, and thus S^2 is a differentiable manifold.

As an example for a differentiable map, let $\phi : S^2 \to S^2$ be a map which rotates the sphere by 45° around its z axis. Let us further choose a point p on the positive quadrant of S^2 in which all coordinates are positive. We can combine ϕ with the charts f_3^+ and f_1^+ to define the map

$$\left(f_1^+ \circ \phi \circ (f_3^+)^{-1}\right)(p^1, p^2) = \left(\frac{p^1 + p^2}{\sqrt{2}}, \sqrt{1 - (p^1)^2 - (p^2)^2}\right) , \tag{2.7}$$

which is also evidently differentiable.

Finally, we introduce *product manifolds* in a straightforward way. Given two differentiable manifolds M and N of dimension m and n, respectively, we can turn the product space $M \times N$ consisting of all pairs (p, q) with $p \in M$ and $q \in N$ into an $(m + n)$-dimensional manifold as follows: if $h : M \to M'$ and $k : N \to N'$ are charts of M and N, a chart $h \times k$ can be defined on $M \times N$ such that

$$h \times k : M \times N \to M' \times N' , \quad (h \times k)(p, q) = [h(p), k(q)] . \tag{2.8}$$

In other words, pairs of points from the product manifold are mapped to pairs of points from the two open subsets $M' \subset \mathbb{R}^m$ and $N' \subset \mathbb{R}^n$.

Many manifolds which are relevant in General Relativity can be expressed as product manifolds of the Euclidean space $\mathbb{R}^m$ with spheres S^n. For example, we can construct the product manifold $\mathbb{R} \times S^2$ composed of the real line and the two-sphere. Points on this product manifold can be mapped onto $\mathbb{R} \times \mathbb{R}^2$ for instance using the identical chart id on $\mathbb{R}$ and the chart f_3^+ on the "northern" half-sphere of S^2,

$$(\mathrm{id} \times f_3^+) : \mathbb{R} \times S^2 \to \mathbb{R} \times D_{12} , \quad (p, q) \to (p, q^2, q^3) . \tag{2.9}$$

2.2 Vectors and Tensors

2.2.1 The tangent space

Now we have essentially introduced ways how to construct local coordinate systems, or charts, on a manifold, how to change between them, and how to use charts to define what differentiable functions on the manifold are. We now proceed to see how vectors can be introduced on a manifold.

Recall the definition of a vector space: a set V, combined with a field (*Körper* in German) F, an addition,

$$+ : V \times V \to V , \quad (v, w) \mapsto v + w , \tag{2.10}$$

and a multiplication,

$$\cdot : F \times V \to V \ , \quad (\lambda, v) \mapsto \lambda v \ , \tag{2.11}$$

is an F-vector space if V is an Abelian group under the addition + and the multiplication is distributive and associative. In other words, a vector space is a set of elements which can be added and multiplied with scalars (i.e. numbers from the field F).

On a curved manifold, this vector space structure is lost because it is not clear how vectors at different points on the manifold should be added. However, it still makes sense to define vectors locally in terms of infinitesimal displacements within a sufficiently small neighbourhood of a point p, which are "tangential" to the manifold at p.

This leads to the concept of the tangential space of a manifold, whose elements are tangential vectors, or directional derivatives of functions. We denote by $\mathcal{F}$ the set of C^∞ functions f from the manifold into $\mathbb{R}$.

An example for a function from the manifold $S^2 \to \mathbb{R}$ could be the average temperature on Earth.

Generally, the tangent space V_p of a differentiable manifold M at a point p is the set of *derivations* of $\mathcal{F}(p)$. A derivation v is a map from $\mathcal{F}(p)$ into $\mathbb{R}$,

$$v : \mathcal{F}(p) \to \mathbb{R} \ , \tag{2.12}$$

which is linear,

$$v(\lambda f + \mu g) = \lambda v(f) + \mu v(g) \tag{2.13}$$

for $f, g \in \mathcal{F}(p)$ and $\lambda, \mu \in \mathbb{R}$, and satisfies the product rule (or Leibniz rule)

$$v(fg) = v(f)g + fv(g) \ . \tag{2.14}$$

Note that this definition immediately implies that the derivation of a constant vanishes: let $h \in \mathcal{F}$ be a constant function, $h(p) = c$ for all $p \in M$, then $v(h^2) = 2cv(h)$ from (2.14) and $v(h^2) = v(ch) = cv(h)$ from (2.13), which is possible only if $v(h) = 0$.

Together with the real numbers $\mathbb{R}$ and their addition and multiplication laws, V_p does indeed have the structure of a vector space, with

$$(v + w)(f) = v(f) + w(f) \quad \text{and} \quad (\lambda v)(f) = \lambda v(f) \tag{2.15}$$

for $v, w \in V_p$, $f \in \mathcal{F}$ and $\lambda \in \mathbb{R}$.

2.2.2 Coordinate basis and transformation under coordinate changes

We now construct a basis for the vector space V_p, i.e. we provide a complete set $\{e_i\}$ of linearly independent basis vectors. For doing so, let

$h : U \to U' \subset \mathbb{R}^n$ be a chart with $p \in U$ and $f \in \mathcal{F}(p)$ a function. Then, $f \circ h^{-1} : U' \to \mathbb{R}$ is C^∞ by definition, and we introduce n vectors $e_i \in V_p$, $1 \leq i \leq n$, by

$$e_i(f) = \frac{\partial}{\partial x^i}(f \circ h^{-1})\Big|_{h(p)} , \tag{2.16}$$

where x^i are the usual cartesian coordinates of $\mathbb{R}^n$.

The function $(f \circ h^{-1})$ is applied to the image $h(p) \in \mathbb{R}^n$ of p under the chart h, i.e. $(f \circ h^{-1})$ "carries" the function f from the manifold M to the locally isomorphic manifold $\mathbb{R}^n$.

To show that these vectors span V_p, we first state that for any C^∞ function $F : U' \to \mathbb{R}$ defined on an open neighbourhood U' of the origin of $\mathbb{R}^n$, there exist n C^∞ functions $H_i : U' \to \mathbb{R}$ such that

$$F(x) = F(0) + \sum_{i=1}^{n} x^i H_i(x) . \tag{2.17}$$

Note the equality! This is *not* a Taylor expansion. This is easily seen using the identity

$$\begin{aligned} F(x) - F(0) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} F(tx^1, \ldots, tx^n)\mathrm{d}t \\ &= \sum_{i=1}^{n} x^i \int_0^1 D_i F(tx^1, \ldots, tx^n)\mathrm{d}t , \end{aligned} \tag{2.18}$$

where D_i is the partial derivative with respect to the i-th argument of F. Thus, it suffices to set

$$H_i(x) = \int_0^1 D_i F(tx^1, \ldots, tx^n)\mathrm{d}t \tag{2.19}$$

to prove (2.17). For $x = 0$ in particular, we find

$$H_i(0) = \int_0^1 \frac{\partial F}{\partial x^i}\Big|_0 \mathrm{d}t = \frac{\partial F}{\partial x^i}\Big|_0 . \tag{2.20}$$

Now we substitute $F = f \circ h^{-1}$ and choose a chart $h : U \to U'$ such that $h(q) = x$ and $h(p) = 0$, i.e. $q = h^{-1}(x)$. Then, we first obtain from (2.17)

$$f(q) = f(p) + \sum_{i=1}^{n} (x^i \circ h)(q)\,(H_i \circ h)(q) , \tag{2.21}$$

and from (2.20)

$$H_i(0) = (H_i \circ h)(p) = \frac{\partial}{\partial x^i}(f \circ h^{-1})\Big|_{h(p)} = e_i(f) . \tag{2.22}$$

Next, we apply a tangent vector $v \in V_p$ to (2.21),

$$\begin{aligned} v(f) &= v[f(p)] + \sum_{i=1}^{n} \left[v(x^i \circ h)\,(H_i \circ h)|_p + (x^i \circ h)|_p\, v(H_i \circ h) \right] \\ &= \sum_{i=1}^{n} v(x^i \circ h)\, e_i(f) \ , \end{aligned} \tag{2.23}$$

where we have used that v applied to the constant $f(p)$ vanishes, that $(x^i \circ h)(p) = 0$ and that $H_i(0) = e_i(f)$ according to (2.22). Thus, setting $v^i = v(x^i \circ h)$, we find that any $v \in V_p$ can be written as a linear combination of the basis vectors e_i. This also demonstrates that the dimension of the tangent space V_p equals that of the manifold itself.

The basis $\{e_i\}$, which is often simply denoted as $\{\partial/\partial x^i\}$ or $\{\partial_i\}$, is called a coordinate basis of V_p. If we choose a different chart h' instead of h, we obtain of course a different coordinate basis $\{e'_i\}$. Denoting the i-th coordinate of the map $h' \circ h^{-1}$ with x', the chain rule applied to $f \circ h^{-1} = (f \circ h'^{-1}) \circ (h' \circ h^{-1})$ yields

$$e_i = \sum_{j=1}^{n} \frac{\partial x'^j}{\partial x^i}\, e'_j =: J'^j_i\, e'_j \ . \tag{2.24}$$

which shows that the two different coordinate bases are related by the *Jacobian matrix* of the coordinate change, which has the elements $J'^j_i = \partial x'^j / \partial x^i$. Its inverse has the elements $J^i_j = \partial x^i / \partial x'^j$. This relates the present definition of a tangent vector to the traditional definition of a vector as a quantity whose components transform as

$$v'^i = v(x^i \circ h') = \sum_{j=1}^{n} v^j e_j(x^i \circ h') = \sum_{j=1}^{n} \frac{\partial x'^i}{\partial x^j}\, v^j = J'^i_j\, v^j \ . \tag{2.25}$$

Repeating the construction of a tangent space at another point $q \in M$, we obtain a tangent space V_q which cannot be identified in any way with the tangent space V_p given only the structure of a differentiable manifold that we have so far.

Consequently, a *vector field* is defined as a map $v : p \mapsto v_p$ which assigns a tangent vector $v_p \in V_p$ to every point $p \in M$. If we apply a vector field v to a C^∞ function f, its result $(v(f))(p)$ is a number for each point p. The vector field is called smooth if the function $(v(f))(p)$ is also smooth.

Since we can write $v = v^i \partial_i$ with components v^i in a local coordinate neighbourhood, the function $v(f)$ is

$$(v(f))(p) = v^i(p) \partial_i f(p) \ , \tag{2.26}$$

and thus it is called the *derivative of f with respect to the vector field v*.

2.2.3 Tangent vectors, curves, and infinitesimal transformations

We can give a geometrical meaning to tangent vectors as "infinitesimal displacements" on the manifold. First, we define a *curve* on M through $p \in M$ as a map from an open interval $I \subset \mathbb{R}$ with $0 \in I$ into M,

$$\gamma : I \to M \ , \tag{2.27}$$

such that $\gamma(0) = p$.

Next, we introduce a *one-parameter group of diffeomorphisms* γ_t as a C^∞ map,

$$\gamma_t : \mathbb{R} \times M \to M \ , \tag{2.28}$$

such that for a fixed $t \in \mathbb{R}$, $\gamma_t : M \to M$ is a diffeomorphism and, for all $t, s \in \mathbb{R}$, $\gamma_t \circ \gamma_s = \gamma_{t+s}$. Note the latter requirement implies that γ_0 is the identity map.

A diffeomorphism is a continuously differentiable, bijective map with a continously differentiable inverse.

For a fixed t, γ_t maps points $p \in M$ to other points $q \in M$ in a differentiable way. As an example on the two-sphere S^2, γ_t could be the map which rotates the sphere about an (arbitrary) z axis by an angle parameterised by t, such that γ_0 is the rotation by zero degrees.

We can now associate a vector field v to γ_t as follows: For a fixed point $p \in M$, the map $\gamma_t : \mathbb{R} \to M$ is a curve as defined above which passes through p at $t = 0$. This curve is called an *orbit of* γ_t. Then, we assign to p the tangent vector v_p to this curve at $t = 0$. Repeating this operation for all points $p \in M$ defines a vector field v on M which is associated with γ_t and can be considered as the *infinitesimal generator* of the transformations γ_t.

In our example on S^2, we fix a point p on the sphere whose orbit under the map γ_t is a part of the "latitude circle" through p. The tangent vector to this curve in p defines the local "direction of motion" under the rotation expressed by γ_t. Applying this to all points $p \in S^2$ defines a vector field v on S^2.

Conversely, given a vector field v on M, we can construct curves through all points $p \in M$ whose tangent vectors are v_p. This is most easily seen in a local coordinate neighbourhood, $h(p) = (x^1, \dots x^n)$, in which the curves are the unique solutions of the system

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = v^i(x^1, \dots x^n) \tag{2.29}$$

of ordinary, first-order differential equations. Thus, tangent vectors can be identified with infinitesimal transformations of the manifold.

Given two vector fields v, w and a function f on M, we can define the *commutator* of the two fields as

$$[v, w](f) = vw(f) - wv(f) \ . \tag{2.30}$$

In coordinates, we can write $v = v^i \partial_i$ and $w = w^j \partial_j$, and the commutator can be written as

$$[v, w] = \left(v^i \partial_i w^j - w^i \partial_i v^j\right) \partial_j \ . \tag{2.31}$$

It can easily be shown to have the following properties (where v, w, x are vector fields and f, g are functions on M):

$$\begin{aligned} [v + w, x] &= [v, x] + [w, x] \\ [v, w] &= -[w, v] \\ [fv, gw] &= fg[v, w] + fv(g)w - gw(f)v \\ [v, [w, x]] &+ [x, [v, w]] + [w, [x, v]] = 0 \ , \end{aligned} \tag{2.32}$$

where the latter equation is called the *Jacobi identity*.

2.2.4 Dual vectors and tensors

We had introduced the tangent space V as the set of derivations of functions $\mathcal{F}$ on M, which were certain linear maps from $\mathcal{F}$ into $\mathbb{R}$. We now introduce the *dual vector space* V^* to V as the set of linear maps

$$f : V \to \mathbb{R} \tag{2.33}$$

from V into $\mathbb{R}$. Defining addition of elements of V^* and their multiplication with scalars in the obvious way, V^* obtains the structure of a vector space; the elements of V^* are called *dual vectors*.

Let now f be a C^∞ function on M and $v \in V$ an arbitrary tangent vector. Then, we define the *differential* of f by

$$\mathrm{d}f : V \to \mathbb{R} \ , \quad \mathrm{d}f(v) = v(f) \ . \tag{2.34}$$

It is obvious that, by definition of the dual space V^*, $\mathrm{d}f$ is an element of V^* and thus a dual vector. Choosing a coordinate representation, we see that

$$\mathrm{d}f(v) = v^i \partial_i f \ . \tag{2.35}$$

Specifically letting $f = x^i$ be the ith coordinate function, we see that

$$\mathrm{d}x^i(\partial_j) = \partial_j x^i = \delta^i_j \ , \tag{2.36}$$

which shows that the n-tuple $\{e^{*i}\} = \{\mathrm{d}x^i\}$ forms a basis of V^*, which is called the *dual basis* to the basis $\{e_i\} = \{\partial_i\}$ of the tangent space V.

Starting the same operation leading from V to the dual space V^* with V^* instead, we arrive at the double-dual vector space V^{**} as the vector space of all linear maps from $V^* \to \mathbb{R}$. It can be shown that V^{**} is isomorphic to V and can thus be identified with V.

Tensors T of rank (r, s) can now be defined as multilinear maps

$$T : \underbrace{V^* \times \ldots \times V^*}_{r} \times \underbrace{V \times \ldots \times V}_{s} \to \mathbb{R} \,, \tag{2.37}$$

in other words, given r dual vectors and s tangent vectors, T returns a real number, and if all but one vector or dual vector are fixed, the map is linear in the remaining argument. If a tensor of rank (r, s) is assigned to every point $p \in M$, we have a *tensor field* of rank (r, s) on M.

According to this definition, tensors of rank $(0, 1)$ are simply dual vectors, and tensors of rank $(1, 0)$ are elements of V^{**} and can thus be identified with tangent vectors.

For one specific example, a tensor of rank $(1, 1)$ is a bilinear map from $V^* \times V \to \mathbb{R}$. If we fix a vector $v \in V$, $T(\cdot, v)$ is a linear map $V^* \to \mathbb{R}$ and thus an element of V^{**}, which can be identified with a vector. In this way, given a vector $v \in V$, a tensor of rank $(1, 1)$ produces another vector $\in V$, and vice versa for dual vectors. Thus, tensors of rank $(1, 1)$ can be seen as linear maps from $V \to V$, or from $V^* \to V^*$.

With the obvious rules for adding linear maps and multiplying them with scalars, the set of tensors $\mathcal{T}^r_s$ of rank (r, s) attains the structure of a vector space of dimension n^{r+s}.

Given a tensor t of rank (r, s) and another tensor t' of rank (r', s'), we can construct a tensor of rank $(r + r', s + s')$ called the *outer product* $t \otimes t'$ of t and t' by simply multiplying their results on the $r + r'$ dual vectors v^{*i} and the $s + s'$ vectors v_j, thus

$$\begin{aligned} &(t \otimes t')(v^{*1}, \ldots, v^{*r+r'}, v_1, \ldots, v_{s+s'}) = \\ &t(v^{*1}, \ldots, v^{*r}, v_1, \ldots, v_s)\, t'(v^{*r+1}, \ldots, v^{*r+r'}, v_{s+1}, \ldots, v_{s+s'}) \,. \end{aligned} \tag{2.38}$$

In particular, it is thus possible to construct a basis for tensors of type (r, s) out of the bases $\{e_i\}$ of the tangent space and $\{e^{*j}\}$ of the dual space by taking the tensor products. Thus, a tensor of rank (r, s) can be written in the form

$$t = t^{i_1 \ldots i_r}_{j_1 \ldots j_s} \left(\partial_{i_1} \otimes \ldots \otimes \partial_{i_r}\right) \otimes \left(\mathrm{d}x^{j_1} \otimes \ldots \otimes \mathrm{d}x^{j_s}\right) \,, \tag{2.39}$$

where the numbers $t^{i_1 \ldots i_r}_{j_1 \ldots j_s}$ are its components with respect to the coordinate system h.

The transformation law (2.24) for the basis vectors under coordinate changes implies that the tensor components transform as

$$t'^{i_1 \ldots i_r}_{j_1 \ldots j_s} = J'^{i_1}_{k_1} \ldots J'^{i_r}_{k_r} J^{l_1}_{j_1} \ldots J^{l_s}_{j_s} \, t^{k_1 \ldots k_r}_{l_1 \ldots l_s} \,, \tag{2.40}$$

a property which is often used to define tensors in the first place.

The *contraction* Ct of a tensor of rank (r, s) is a map which reduces both r and s by unity,

$$Ct : \mathcal{T}^r_s \to \mathcal{T}^{r-1}_{s-1} \,, \quad Ct = t(\ldots, e^{*k}, \ldots, e_k, \ldots) \,, \tag{2.41}$$

where $\{e_k\}$ and $\{e^{*k}\}$ are bases of the tangent and dual spaces, as before, and the summation over all $1 \leq k \leq n$ is implied. The basis vectors e_k and e^{*k} are inserted as the ith and jth arguments of the tensor t.

Expressing the tensor in a coordinate basis, we can write the tensor in the form (2.39), and thus its contraction with respect to the i_ath and j_bth arguments reads

$$\begin{aligned} Ct &= t^{i_1 \ldots i_r}_{j_1 \ldots j_s} \mathrm{d}x^{i_k}(\partial_{j_k}) \\ &\quad (\partial_{i_1} \otimes \ldots \otimes \partial_{i_a-1} \otimes \partial_{i_a+1} \otimes \ldots \otimes \partial_{i_r}) \\ &\quad \left(\mathrm{d}x^{j_1} \otimes \ldots \otimes \mathrm{d}x^{j_b-1} \otimes \mathrm{d}x^{j_b+1} \otimes \ldots \otimes \mathrm{d}x^{j_s}\right) \\ &= t^{i_1 \ldots i_{a-1} i_k i_{a+1} \ldots i_r}_{j_1 \ldots j_{b-1} i_k j_{b+1} \ldots j_s} \\ &\quad (\partial_{i_1} \otimes \ldots \otimes \partial_{i_a-1} \otimes \partial_{i_a+1} \otimes \ldots \otimes \partial_{i_r}) \\ &\quad \left(\mathrm{d}x^{j_1} \otimes \ldots \otimes \mathrm{d}x^{j_b-1} \otimes \mathrm{d}x^{j_b+1} \otimes \ldots \otimes \mathrm{d}x^{j_s}\right) \; . \end{aligned} \tag{2.42}$$

For a simple example, let $v \in V$ be a tangent vector and $w \in V^*$ a dual vector, and $t = v \otimes w$ a tensor of rank $(1, 1)$. Its contraction results in a tensor of rank $(0, 0)$, i.e. a real number, which is

$$Ct = C(v \otimes w) = \mathrm{d}x^k(v)\, w(\partial_k) = v^k w_k \; . \tag{2.43}$$

At the same time, this can be written as

$$\begin{aligned} Ct &= C(v \otimes w) = w(v) & (2.44) \\ &= (w_j \mathrm{d}x^j)(v^i \partial_i) = w_j v^i \mathrm{d}x^j(\partial_i) = w_j v^i \partial_i x^j = w_j v^i \delta^j_i = w_i v^i \; . \end{aligned}$$

In that sense, the contraction amounts to applying the tensor (partially) "on itself".

2.2.5 The metric

We need some way to define and measure the "distance" between two points on a manifold. A metric is introduced as the infinitesimal squared distance between two neighbouring points on the manifold.

We have seen above that tangent vectors $v \in V_p$ are closely related to infinitesimal displacements around a point p on the manifold. Moreover, the infinitesimal squared distance between two neighbouring points p and q should be quadratic in the displacement leading from one point to the other. Thus, we construct the metric g as a bi-linear map

$$g : V \times V \to \mathbb{R} \; , \tag{2.45}$$

which means that the g is a tensor of rank $(0, 2)$. The metric thus assigns a number to two elements of a vector field V on M. The metric g thus assigns to two vectors their scalar product, which is *not necessarily positive*. We abbreviate the scalar product of two vectors $v, w \in V$ by

$$g(v, w) \equiv \langle v, w \rangle \; . \tag{2.46}$$

In addition, we require that the metric be symmetric and non-degenerate, which means

$$\begin{aligned} g(v,w) &= g(w,v) \quad \forall \quad v,w \in V_p \ , \\ g(v,w) &= 0 \quad \forall \quad v \in V_p \quad \Leftrightarrow \quad w = 0 \ . \end{aligned} \tag{2.47}$$

In a coordinate basis, the metric can be written in components as

$$g = g_{ij}\,\mathrm{d}x^i \otimes \mathrm{d}x^j \tag{2.48}$$

or, omitting the tensor-product sign and introducing the *line element* $\mathrm{d}s^2$,

$$\mathrm{d}s^2 = g_{ij}\mathrm{d}x^i\mathrm{d}x^j \ . \tag{2.49}$$

Given a coordinate basis $\{e_i\}$, the metric g can always be chosen such that

$$g(e_i, e_j) = \langle e_i, e_j \rangle = \pm\delta_{ij} \ , \tag{2.50}$$

where the number of positive and negative signs is independent of the coordinate choice and is called the *signature* of the metric. Positive-(semi)definite metrics, which have only positive signs, are called *Riemannian*, and *pseudo-Riemannian* metrics have positive and negative signs.

Perhaps the most common pseudo-Riemannian metric is the Minkowski metric known from special relativity, which can be chosen to have the signature $(-,+,+,+)$ and has the line element

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2 \ . \tag{2.51}$$

A metric with the same signature as for the space-time is called *Lorentzian*.

Given a tangent vector v, the metric can also be seen as a linear map from V into V^*,

$$g : V \to V^* \ , \quad v \mapsto g(\cdot, v) \ . \tag{2.52}$$

This is an element of V^* because it linearly maps vectors into $\mathbb{R}$. Since the metric is non-degenerate, the inverse map g^{-1} also exists, and the metric can be used to establish a one-to-one correspondence between vectors and dual vectors, and thus between the tangent space V and its dual space V^*.

Chapter 3

Differential Geometry II

3.1 Connections, Covariant Derivatives and Geodesics

3.1.1 Linear Connections

The curvature of the two-dimensional sphere S^2 can be described by embedding the sphere into a Euclidean space of the next-higher dimension, $\mathbb{R}^3$. However, (as far as we know) there is no natural embedding of our four-dimensional curved space-time into $\mathbb{R}^5$, and thus we need a description of curvature which is intrinsic to the manifold.

There is a close correspondence between the curvature of a manifold and the transport of vectors along curves.

As we have seen before, the structure of a manifold does not trivially allow to compare vectors which are elements of tangent spaces at two different points. We will thus have to introduce additional structure which allows us to meaningfully shift vectors from one point to another on the manifold.

Even before we do so, it is intuitively clear how vectors can be transported along closed paths in flat Euclidean space, say $\mathbb{R}^3$. There, the vector arriving at the starting point after the transport will be identical to the vector before the transport.

However, this will no longer be so on the two-sphere: starting on the equator with a vector pointing north, we can shift it along a meridian to the north pole, then back to the equator along a different meridian, and finally back to its starting point on the equator. There, it will point into a different direction than the original vector.

Curvature can thus be defined from this misalignment of vectors after transport along closed curves. In order to work this out, we thus first need some way for transporting vectors along curves.

We start by generalising the concept of a directional derivative from $\mathbb{R}^n$ by defining a *linear* or *affine connection* or *covariant differentiation* on a manifold as a mapping ∇ which assigns to every pair v, y of C^∞ vector fields another vector field $\nabla_v y$ which is bilinear in v and y and satisfies

Elwin Christoffel (1829–1900)

$$\begin{aligned} \nabla_{fv} y &= f \nabla_v y \\ \nabla_v (fy) &= f \nabla_v y + v(f) y \, , \end{aligned} \tag{3.1}$$

where $f \in \mathcal{F}$ is a C^∞ function on M.

In a local coordinate basis $\{e_i\}$, we can describe the linear connection by its action on the basis vectors,

$$\nabla_{\partial_i}(\partial_j) \equiv \Gamma^k{}_{ij} \, \partial_k \, , \tag{3.2}$$

where the n^3 numbers $\Gamma^k{}_{ij}$ are called the *Christoffel symbols* or *connection coefficients* of the connection ∇ in the given chart.

The Christoffel symbols are *not* the components of a tensor, which is seen from their transformation under coordinate changes. Let x^i and x'^i be two different coordinate systems, then we have on the one hand, by definition,

$$\nabla_{\partial'_a}(\partial'_b) = \Gamma'^c{}_{ab} \, \partial'_c = \Gamma'^c{}_{ab} \, \frac{\partial x^k}{\partial x'^c} \, \partial_k = \Gamma'^c{}_{ab} \, J^k_c \, \partial_k \, , \tag{3.3}$$

where J^k_c is the Jacobian matrix of the coordinate transform as defined in (2.24). On the other hand, the axioms (3.1) imply, with f represented by the elements J^k_i of the Jacobian matrix,

$$\begin{aligned} \nabla_{\partial'_a}(\partial'_b) &= \nabla_{J^i_a \partial_i}(J^j_b \partial_j) = J^i_a \nabla_{\partial_i}(J^j_b \partial_j) \\ &= J^i_a \left[J^j_b \nabla_{\partial_i} \partial_j + \partial_i J^j_b \partial_j \right] \\ &= J^i_a J^j_b \Gamma^k{}_{ij} \, \partial_k + J^i_a \partial_i J^k_b \partial_k \, . \end{aligned} \tag{3.4}$$

Comparison of the two results (3.3) and (3.4) shows that

$$\Gamma'^c{}_{ab} \, J^k_c = J^i_a J^j_b \Gamma^k{}_{ij} + J^i_a \partial_i J^k_b \, , \tag{3.5}$$

or, after multiplying with the inverse Jacobian matrix J'^c_k,

$$\Gamma'^c{}_{ab} = J^i_a J^j_b J'^c_k \Gamma^k{}_{ij} + J'^c_k J^i_a \partial_i J^k_b \, . \tag{3.6}$$

While the first term on the right-hand side reflects the tensor transformation law (2.40), the second term differs from it.

Let now y and v be vector fields on M and w a dual vector field, then the *covariant derivative* ∇y is a tensor field of rank $(1, 1)$ which is defined by

$$\nabla y(v, w) \equiv w[\nabla_v(y)] \, . \tag{3.7}$$

In a coordinate basis $\{\partial_i\}$, we write

$$y = y^i \partial_i \quad \text{and} \quad \nabla y \equiv y^i_{\,;j}\, \mathrm{d}x^j \otimes \partial_i \;, \tag{3.8}$$

and have

$$\begin{aligned} y^i_{\,;j} &= \nabla y(\partial_j, \mathrm{d}x^i) = \mathrm{d}x^i \left(\nabla_{\partial_j}(y^k \partial_k)\right) \\ &= \mathrm{d}x^i \left(y^k_{\,,j}\partial_k + y^k \Gamma^l_{\ jk}\, \partial_l\right) \\ &= y^k_{\,,j}\delta^i_k + y^k \Gamma^l_{\ jk}\, \delta^i_l \\ &= y^i_{\,,j} + y^k \Gamma^i_{\ jk} \;. \end{aligned} \tag{3.9}$$

Indices separated by a comma denote ordinary partial differentiations with respect to coordinates, $y_{,i} \equiv \partial_i y$.

An affine connection is symmetric if

$$\nabla_v w - \nabla_w v = [v, w] \;, \tag{3.10}$$

which a short calculation shows to be equivalent to the symmetry property

$$\Gamma^k_{\ ij} = \Gamma^k_{\ ji} \tag{3.11}$$

of the Christoffel symbols in a coordinate basis.

3.1.2 Parallel Transport and Geodesics

Given a linear connection, it is now straightforward to introduce parallel transport . To begin, let $\gamma : I \to M$ with $I \subset \mathbb{R}$ a curve in M with tangent vector $\dot{\gamma}(t)$. A vector field v is called *parallel along* γ if

$$\nabla_{\dot{\gamma}} v = 0 \;. \tag{3.12}$$

The vector $\nabla_{\dot{\gamma}} v$ is the covariant derivative of v along γ, and it is often denoted by

$$\nabla_{\dot{\gamma}} v = \frac{\mathrm{D}v}{\mathrm{d}t} = \frac{\nabla v}{\mathrm{d}t} \;. \tag{3.13}$$

In the coordinate basis $\{\partial_i\}$, the covariant derivative along γ reads

$$\begin{aligned} \nabla_{\dot{\gamma}} v &= \nabla_{\dot{x}^i \partial_i}(v^j \partial_j) = \dot{x}^i \nabla_{\partial_i}(v^j \partial_j) \\ &= \dot{x}^i \left[v^j \nabla_{\partial_i}(\partial_j) + \partial_i v^j \partial_j\right] = \left(\dot{v}^k + \Gamma^k_{\ ij}\, \dot{x}^i v^j\right) \partial_k \;, \end{aligned} \tag{3.14}$$

and if this is to vanish identically, (3.12) implies

$$\dot{v}^k + \Gamma^k_{\ ij}\, \dot{x}^i v^j = 0 \;. \tag{3.15}$$

The existence and uniqueness theorems for ordinary differential equations imply that (3.15) has a unique solution once v is given at one point along the curve $\gamma(t)$. The *parallel transport* of a vector along a curve is then uniquely defined.

If the tangent vector $\dot{\gamma}$ of a curve γ is *autoparallel* along γ,

$$\nabla_{\dot{\gamma}}\dot{\gamma} = 0 \,, \tag{3.16}$$

the curve is called a *geodesic*. In a local coordinate system, this condition reads

$$\ddot{x}^k + \Gamma^k{}_{ij}\,\dot{x}^i\dot{x}^j = 0 \,. \tag{3.17}$$

In flat Euclidean space, geodesics are straight lines. Quite intuitively, the condition (3.16) generalises the concept of straight lines to manifolds.

3.1.3 Normal Coordinates

Geodesics allow the introduction of a special coordinate system in the neighbourhood of a point $p \in M$. First, given a point $p = \gamma(0)$ and a vector $\dot{\gamma}(0) \in V_p$ from the tangent space in p, the existence and uniqueness theorems for ordinary differential equations ensure that (3.17) has a unique solution, which implies that a unique geodesic exists through p into the direction $\dot{\gamma}(0)$.

Obviously, if $\gamma_v(t)$ is a geodesic with "initial velocity" $v = \dot{\gamma}(0)$, then $\gamma_v(at)$ is also a geodesic with initial velocity $av = a\dot{\gamma}(0)$, or

$$\gamma_{av}(t) = \gamma_v(at) \,. \tag{3.18}$$

Thus, given some neighbourhood $U \subset V_p$ of $p = \gamma(0)$, unique geodesics $\gamma(t)$ with $t \in [0, 1]$ can be constructed through p into any direction $v \in U$, i.e. such that $\gamma(0) = p$ and $\dot{\gamma}(0) = v \in U$.

Using this, we define the *exponential map* at p,

$$\exp_p : V_p \supset U \to M \,, \quad v \mapsto \exp_p(v) = \gamma_v(1) \,, \tag{3.19}$$

which maps any vector v from $U \subset V_p$ into a point along the geodesic through p into direction v at distance $t = 1$.

Now, we choose a coordinate basis $\{e_i\}$ of V_p and use the n basis vectors in the exponential mapping (3.19). Then, the neighbourhood of p can uniquely be represented by the exponential mapping along the basis vectors, $\exp_p(x^i e_i)$, and the x^i are called *normal coordinates*.

Since $\exp_p(tv) = \gamma_{tv}(1) = \gamma_v(t)$, the curve $\gamma_v(t)$ has the normal coordinates $x^i = tv^i$, with $v = v^i e_i$. In these coordinates, x^i is linear in t, thus $\ddot{x}^i = 0$, and (3.17) implies

$$\Gamma^k{}_{ij}\,v^i v^j = 0 \,, \tag{3.20}$$

and thus

$$\Gamma^k{}_{ij} + \Gamma^k{}_{ji} = 0 \,. \tag{3.21}$$

If the connection is symmetric as defined in (3.11), the connection coefficients must vanish,

$$\Gamma^k{}_{ij} = 0 \,. \tag{3.22}$$

Thus, at every point $p \in M$, local coordinates can uniquely be introduced by means of the exponential map, the normal coordinates, in which the coefficients of a symmetric connection vanish. This will turn out to be important.

3.1.4 Covariant derivative of tensor fields

Extending the concept of the covariant derivative to tensor fields, we start with a simple tensor of rank $(1, 1)$ which is the tensor product of a vector field v and a dual vector field w,

$$t = v \otimes w \,, \tag{3.23}$$

and we require that ∇_x satisfy the Leibniz rule,

$$\nabla_x(v \otimes w) = \nabla_x v \otimes w + v \otimes \nabla_x w \,, \tag{3.24}$$

and commute with the contraction,

$$C\,[\nabla_x(v \otimes w)] = \nabla_x[w(v)] \,. \tag{3.25}$$

We now contract (3.24) and use (3.25) to find

$$\begin{aligned} C\,[\nabla_x(v \otimes w)] &= C\,(\nabla_x v \otimes w) + C\,(v \otimes \nabla_x w) \\ &= w(\nabla_x v) + (\nabla_x w)(v) \\ &= \nabla_x[w(v)] = xw(v) \,, \end{aligned} \tag{3.26}$$

where (3.1) was used in the final step (note that $w(v)$ is a real-valued function). Thus, we find an expression for the covariant derivative of a dual vector,

$$(\nabla_x w)(v) = xw(v) - w(\nabla_x v) \,. \tag{3.27}$$

Introducing the coordinate basis $\{\partial_i\}$, it is straightforward to show (and a useful exercise!) that this result can be expressed as

$$(\nabla_x w)(v) = \left(w_{j,i} - \Gamma^k_{\;ij}\, w_k\right) x^i v^j \,. \tag{3.28}$$

Specialising $x = \partial_i$, $w = \mathrm{d}x^j$ and $v = \partial_k$, hence $x^a = \delta^a_i$, $w_b = \delta^j_b$ and $v^c = \delta^c_k$, we see that this implies for the covariant derivatives of the dual basis vectors $\mathrm{d}x^j$

$$(\nabla_{\partial_i}\mathrm{d}x^j)(\partial_k) = -\Gamma^j_{\;ik} \quad \text{or} \quad \nabla_{\partial_i}\mathrm{d}x^j = -\Gamma^j_{\;ik}\,\mathrm{d}x^k \,. \tag{3.29}$$

As before, we now define the covariant derivative ∇t of a tensor field as a map from the tensor fields of rank (r, s) to the tensor fields of rank $(r, s+1)$,

$$\nabla : \mathcal{T}^r_s \to \mathcal{T}^r_{s+1} \tag{3.30}$$

by setting

$$\begin{aligned}(\nabla t)(w_1, \ldots, w_r, v_1, \ldots, v_s, v_{s+1}) \equiv \\ (\nabla_{v_{s+1}} t)(w_1, \ldots, w_r, v_1, \ldots, v_s) \ ,\end{aligned} \tag{3.31}$$

where the v_i are vector fields and the w_j dual vector fields.

We find a general expression for ∇t, with $t \in \mathcal{T}^r_s$, by taking the tensor product of t with s vector fields v_i and r dual vector fields w_j and applying ∇_x to the result, using the Leibniz rule,

$$\begin{aligned} & \nabla_x \left(w_1 \otimes \ldots \otimes w_r \otimes v_1 \otimes \ldots \otimes v_s \otimes t\right) \\ = \ & (\nabla_x w_1) \otimes \ldots \otimes t + \ldots w_1 \otimes \ldots \otimes (\nabla_x v_1) \otimes \ldots \otimes t \\ + \ & w_1 \otimes \ldots \otimes (\nabla_x t) \ , \end{aligned} \tag{3.32}$$

and then taking the total contraction, using that it commutes with the covariant derivative, which yields

$$\begin{aligned} & \nabla_x \left[t(w_1, \ldots, w_r, v_1, \ldots, v_s)\right] \\ = \ & t(\nabla_x w_1, \ldots, w_r, v_1, \ldots, v_s) + \ldots + t(w_1, \ldots, w_r, v_1, \ldots, \nabla_x v_s) \\ + \ & (\nabla_x t)(w_1, \ldots, w_r, v_1, \ldots, v_s) \ . \end{aligned} \tag{3.33}$$

Therefore, the covariant derivative $\nabla_x t$ of t is

$$\begin{aligned} & (\nabla_x t)(w_1, \ldots, w_r, v_1, \ldots, v_s) \\ = \ & x t(w_1, \ldots, w_r, v_1, \ldots, v_s) \\ - \ & t(\nabla_x w_1, \ldots, v_s) - \ldots - t(w_1, \ldots, \nabla_x v_s) \ . \end{aligned} \tag{3.34}$$

We now work out the last expression for the covariant derivative of a tensor field in a local coordinate basis $\{\partial_i\}$ and its dual basis $\{\mathrm{d}x^j\}$ for the special case of a tensor field t of rank $(1, 1)$. The result for tensor fields of higher rank are then easily found by induction.

We can write the tensor field t as

$$t = t^i{}_j \left(\partial_i \otimes \mathrm{d}x^j\right) \ , \tag{3.35}$$

and the result of its application to $w_1 = \mathrm{d}x^a$ and $v_1 = \partial_b$ is

$$t(\mathrm{d}x^a, \partial_b) = t^i{}_j \, \mathrm{d}x^a(\partial_i)\mathrm{d}x^j(\partial_b) = t^a{}_b \ . \tag{3.36}$$

Therefore, we can write (3.34) as

$$\begin{aligned} & (\nabla_x t)(\mathrm{d}x^a, \partial_b) \\ = \ & x^c \partial_c t^a{}_b - t^i{}_j \, (\nabla_x \mathrm{d}x^a)(\partial_i)\mathrm{d}x^j(\partial_b) - t^i{}_j \, \mathrm{d}x^a(\partial_i)\mathrm{d}x^j(\nabla_x \partial_b) \ . \end{aligned} \tag{3.37}$$

According to (3.29), the second term on the right-hand side is

$$t^i{}_j \, \delta^j_b x^c (\nabla_{\partial_c} \mathrm{d}x^a)(\partial_i) = -x^c t^i{}_b \, \Gamma^a{}_{ci} \ , \tag{3.38}$$

while the third term is

$$t^i{}_j\,\delta^a_i\,x^c(\nabla_{\partial_c}\partial_b)(x^j) = x^c t^a{}_j\,\Gamma^k{}_{cb}\,\partial_k x^j = x^c t^a{}_j\,\Gamma^j{}_{cb} \ . \tag{3.39}$$

Summarising, the components of $\nabla_x t$ are

$$t^a{}_{b;c} = t^a{}_{b,c} + \Gamma^a{}_{ci}\,t^i{}_b - \Gamma^j{}_{cb}\,t^a{}_j \ , \tag{3.40}$$

showing that the covariant indices are transformed with the negative, the contravariant indices with the positive Christoffel symbols.

In particular, the covariant derivatives of tensors of rank $(0,1)$ (dual vectors w) and of tensors of rank $(1,0)$ (vectors v) have components

$$\begin{aligned} w_{i;k} &= w_{i,k} - \Gamma^j{}_{ki}\,w_j \ , \\ v^i_{;k} &= v^i_{,k} + \Gamma^i{}_{kj}\,v^j \ . \end{aligned} \tag{3.41}$$

3.2 Curvature

3.2.1 The Torsion and Curvature Tensors

The *torsion* T maps two vector fields x and y into another vector field,

$$T : V \times V \to V \ , \tag{3.42}$$

such that

$$T(x,y) = \nabla_x y - \nabla_y x - [x,y] \ ; \tag{3.43}$$

obviously, the torsion vanishes if and only if the connection is symmetric, cf. (3.10).

The torsion is antisymmetric,

$$T(x,y) = -T(y,x) \ , \tag{3.44}$$

and satisfies

$$T(fx, gy) = fg\,T(x,y) \tag{3.45}$$

with arbitrary C^∞ functions f and g.

The map

$$V^* \times V \times V \to \mathbb{R} \ , \quad (w,x,y) \to w[T(x,y)] \tag{3.46}$$

with $w \in V^*$ and $x, y \in V$ is a tensor of rank $(1,2)$ called the *torsion tensor* .

According to (3.46), the components of the torsion tensor in the coordinate basis $\{\partial_i\}$ and its dual basis $\{\mathrm{d}x^i\}$ are

$$T^k{}_{ij} = \mathrm{d}x^k\left[T(\partial_i, \partial_j)\right] = \Gamma^k{}_{ij} - \Gamma^k{}_{ji} \ . \tag{3.47}$$

The *curvature* R maps three vector fields x, y and v into a vector field,

$$R : V \times V \times V \to V \;, \tag{3.48}$$

such that

$$R(x, y)v = \nabla_x(\nabla_y v) - \nabla_y(\nabla_x v) - \nabla_{[x,y]} v \;. \tag{3.49}$$

Since the covariant derivatives ∇_x and ∇_y represent the infinitesimal parallel transports along the integral curves of the vector fields x and y, the curvature R directly quantifies the change of the vector v when it is parallel-transported around an infinitesimal, closed loop.

Exchanging x and y and using the antisymmetry of the commutator $[x, y]$, we see that R is antisymmetric in x and y,

$$R(x, y) = -R(y, x) \;. \tag{3.50}$$

Also, if f, g and h are C^∞ functions on M,

$$R(f x, gy)hv = fgh\, R(x, y)v \;, \tag{3.51}$$

which follows immediately from the defining properties (3.1) of the connection.

Obviously, the map

$$V^* \times\; V \times V \times V \to \mathbb{R} \;, \quad (w, x, y, v) = w[R(x, y)v] \tag{3.52}$$

with $w \in V^*$ and $x, y, v \in V$ defines a tensor of rank $(1, 3)$ called the *curvature tensor*.

To work out the components of R in a local coordinate basis $\{\partial_i\}$, we first note that

$$\begin{aligned} \nabla_{\partial_i}(\nabla_{\partial_j}\partial_k) &= \nabla_{\partial_i}\left[\nabla_{\partial_j}(\partial_k)\right] = \nabla_{\partial_i}\left(\Gamma^l{}_{jk}\,\partial_l\right) \\ &= \Gamma^l{}_{jk,i}\,\partial_l + \Gamma^l{}_{jk}\,\Gamma^m{}_{il}\,\partial_m \;. \end{aligned} \tag{3.53}$$

Interchanging i and j yields the coordinate expression for $\nabla_y(\nabla_x v)$. Since the commutator of the basis vectors vanishes, $[\partial_i, \partial_j] = 0$, the components of the curvature tensor are

$$\begin{aligned} R^i{}_{jkl} &= \mathrm{d}x^i[R(\partial_k, \partial_l)\partial_j] \\ &= \Gamma^i{}_{lj,k} - \Gamma^i{}_{kj,l} + \Gamma^m{}_{lj}\,\Gamma^i{}_{km} - \Gamma^m{}_{kj}\,\Gamma^i{}_{lm} \;. \end{aligned} \tag{3.54}$$

The *Ricci tensor* is a contraction of the curvature tensor whose components are

$$R_{jl} = R^i{}_{jil} = \Gamma^i{}_{lj,i} - \Gamma^i{}_{ij,l} + \Gamma^m{}_{lj}\,\Gamma^i{}_{im} - \Gamma^m{}_{ij}\,\Gamma^i{}_{lm} \;. \tag{3.55}$$

3.2.2 The Bianchi Identities

The curvature and the torsion together satisfy the two Bianchi identities. The *first Bianchi identity* is

$$\sum_{\text{cyclic}} [R(x, y)z] = \sum_{\text{cyclic}} \{T[T(x, y), z] + (\nabla_x T)(y, z)\} \;, \tag{3.56}$$

where the sums extend over all cyclic permutations of the vectors x, y and z. The *second Bianchi identity* is

$$\sum_{\text{cyclic}} \{(\nabla_x R)(y, z) + R[T(x, y), z]\} = 0 \;. \tag{3.57}$$

They are important because they define symmetry relations of the curvature and the curvature tensor. In particular, for a symmetric connection, $T = 0$ and the Bianchi identities reduce to

$$\sum_{\text{cyclic}} [R(x, y)z] = 0 \;, \quad \sum_{\text{cyclic}} (\nabla_x R)(y, z) = 0 \;. \tag{3.58}$$

Before we go on, we have to clarify the meaning of the covariant derivatives of the torsion and the curvature. We have seen that T defines a tensor field $\tilde{T}$ of rank $(1, 2)$. Given a dual vector field $w \in V^*$, we define the covariant derivative of the torsion T such that

$$w\,[\nabla_v T(x, y)] = (\nabla_v \tilde{T})(w, x, y) \;. \tag{3.59}$$

Using (3.34), we can write the right-hand side as

$$\begin{aligned} (\nabla_v \tilde{T})(w, x, y) &= v\tilde{T}(w, x, y) - \tilde{T}(\nabla_v w, x, y) \\ &- \tilde{T}(w, \nabla_v x, y) - \tilde{T}(w, x, \nabla_v y) \;. \end{aligned} \tag{3.60}$$

The first two terms on the right-hand side can be combined using (3.27),

$$\begin{aligned} \tilde{T}(\nabla_v w, x, y) &= \nabla_v w[T(x, y)] = vw[T(x, y)] - w[\nabla_v T(x, y)] \\ &= v\tilde{T}(w, x, y) - w[\nabla_v T(x, y)] \;, \end{aligned} \tag{3.61}$$

which yields

$$(\nabla_v \tilde{T})(w, x, y) = w[\nabla_v T(x, y)] - \tilde{T}(w, \nabla_v x, y) - \tilde{T}(w, x, \nabla_v y) \tag{3.62}$$

or, dropping the common argument w from all terms,

$$(\nabla_v T)(x, y) = \nabla_v[T(x, y)] - T(\nabla_v x, y) - T(x, \nabla_v y) \;. \tag{3.63}$$

Similarly, we find that

$$(\nabla_v R)(x, y) = \nabla_v[R(x, y)] - R(\nabla_v x, y) - R(x, \nabla_v y) - R(x, y)\nabla_v \;. \tag{3.64}$$

For symmetric connections, $T = 0$, the first Bianchi identity is easily proven. Its left-hand side reads

$$\begin{aligned}
& \nabla_x\nabla_y z - \nabla_y\nabla_x z + \nabla_y\nabla_z x - \nabla_z\nabla_y x + \nabla_z\nabla_x y - \nabla_x\nabla_z y \\
- \; & \nabla_{[x,y]}z - \nabla_{[y,z]}x - \nabla_{[z,x]}y \\
= \; & \nabla_x(\nabla_y z - \nabla_z y) + \nabla_y(\nabla_z x - \nabla_x z) + \nabla_z(\nabla_x y - \nabla_y x) \\
- \; & \nabla_{[x,y]}z - \nabla_{[y,z]}x - \nabla_{[z,x]}y \\
= \; & \nabla_x[y,z] - \nabla_{[y,z]}x + \nabla_y[z,x] - \nabla_{[z,x]}y + \nabla_z[x,y] - \nabla_{[x,y]}z \\
= \; & [x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0 \ , \qquad (3.65)
\end{aligned}$$

where we have used the relation (3.10) and the Jacobi identity (2.32).

3.3 Riemannian Connections

3.3.1 Definition and Uniqueness

Up to now, the affine connection ∇ is not uniquely defined. We shall now see that a unique connection can be introduced on each pseudo-Riemannian manifold (M, g).

A connection is called *metric* if the parallel transport along any smooth curve γ in M leaves the inner product of two autoparallel vector fields x and y unchanged. This is the case if and only if the covariant derivative ∇ of g vanishes,

$$\nabla g = 0 \ . \qquad (3.66)$$

Because of (3.34), this condition is equivalent to the *Ricci identity*

$$xg(y,z) = g(\nabla_x y, z) + g(y, \nabla_x z) \ , \qquad (3.67)$$

where x, y, z are vector fields.

It can now be shown that a unique connection ∇ can be introduced on each pseudo-Riemannian manifold such that ∇ is symmetric or torsion-free, and metric, i.e. $\nabla g = 0$. Such a connection is called the *Riemannian* or *Levi-Civita connection*.

Suppose first that such a connection exists, then (3.67) and the symmetry of ∇ allow us to write

$$xg(y,z) = g(\nabla_y x, z) + g([x,y], z) + g(y, \nabla_x z) \ . \qquad (3.68)$$

Taking the cyclic permutations of this equation, summing the second and the third and subtracting the first (3.68), we obtain the *Koszul formula*

$$\begin{aligned}
2g(\nabla_z y, x) &= -xg(y,z) + yg(z,x) + zg(x,y) \qquad (3.69) \\
&+ g([x,y],z) - g([y,z],x) - g([z,x],y) \ .
\end{aligned}$$

Since the right-hand side is independent of ∇, and g is non-degenerate, this result implies the *uniqueness* of ∇. The *existence* of an affine, symmetric and metric connection can be proven by explicit construction.

The Christoffel symbols for a Riemannian connection can now be determined specialising the Koszul formula (3.69) to the basis vectors $\{\partial_i\}$ of a local coordinate system. We choose $x = \partial_k$, $y = \partial_j$ and $z = \partial_i$ and use that their commutator vanishes, $[\partial_i, \partial_j] = 0$, and that $g(\partial_i, \partial_j) = g_{ij}$.

Then, (3.69) implies

$$2g(\nabla_{\partial_i}\partial_j, \partial_k) = -\partial_k g_{ij} + \partial_j g_{ik} + \partial_i g_{jk} \ , \tag{3.70}$$

thus

$$g_{mk}\Gamma^m{}_{ij} = \frac{1}{2}\left(g_{ik,j} + g_{jk,i} - g_{ij,k}\right) \ . \tag{3.71}$$

If (g^{ij}) denotes the matrix inverse to (g_{ij}), we can write

$$\Gamma^l{}_{ij} = \frac{1}{2}g^{lk}\left(g_{ik,j} + g_{jk,i} - g_{ij,k}\right) \ . \tag{3.72}$$

3.3.2 Symmetries. The Einstein Tensor

In addition to (3.50), the curvature tensor of a Riemannian connection has the following symmetry properties:

$$\langle R(x,y)v, w\rangle = -\langle R(x,y)w, v\rangle \ , \quad \langle R(x,y)v, w\rangle = \langle R(v,w)x, y\rangle \ . \tag{3.73}$$

The first of these relations is easily seen noting that the antisymmetry is equivalent to

$$\langle R(x,y)v, v\rangle = 0 \ . \tag{3.74}$$

From the definition of R and the antisymmetry (3.50), we first have

$$\langle v, R(x,y)v\rangle = \langle v, \nabla_x\nabla_y v - \nabla_y\nabla_x v - \nabla_{[x,y]}v\rangle \ . \tag{3.75}$$

Replacing y by $\nabla_y v$ and z by v, the Ricci identity (3.67) allows us to write

$$\langle v, \nabla_x\nabla_y v\rangle = x\langle\nabla_y v, v\rangle - \langle\nabla_y v, \nabla_x v\rangle \tag{3.76}$$

and, replacing x by y and both y and z by v,

$$\langle\nabla_y v, v\rangle = \frac{1}{2}\,y\langle v, v\rangle \ . \tag{3.77}$$

Hence, the first two terms on the right-hand side of (3.75) yield

$$\begin{aligned}\langle v, \nabla_x\nabla_y v - \nabla_y\nabla_x v\rangle &= \langle v, x\langle\nabla_y v, v\rangle - y\langle\nabla_x v, v\rangle\rangle \\ &= \frac{1}{2}\langle v, xy\langle v, v\rangle - yx\langle v, v\rangle\rangle \\ &= \frac{1}{2}\langle v, [x,y]\langle v, v\rangle\rangle \ . \end{aligned} \tag{3.78}$$

Bernhard Riemann (1826–1866)

By (3.77), this is the negative of the third term on the right-hand side of (3.75), which proves (3.74).

The symmetries (3.50) and (3.73) imply

$$R_{ijkl} = -R_{jikl} = -R_{ijlk}\ , \quad R_{ijkl} = R_{klij}\ , \tag{3.79}$$

where $R_{ijkl} \equiv g_{im}R^m{}_{jkl}$. Of the $4^4 = 256$ components of the Riemann tensor in four dimensions, the first symmetry relation (3.79) leaves $6 \times 6 = 36$ independent components, while the second symmetry relation (3.79) reduces their number to $6+5+4+3+2+1 = 21$.

In a coordinate basis, the Bianchi identities (3.58) for the curvature tensor of a Riemannian connection read

$$\sum_{(jkl)} R^i{}_{jkl} = 0\ , \quad \sum_{(klm)} R^i{}_{jkl;m} = 0\ , \tag{3.80}$$

where (jkl) denotes the cyclic permutations of the indices enclosed in parentheses. In four dimensions, the first Bianchi identity establishes one further relation between the components of the Riemann tensor which is not covered yet by the symmetry relations (3.79), namely

$$R_{0123} + R_{0231} + R_{0312} = 0\ , \tag{3.81}$$

and thus leaves 20 independent components of the Riemann tensor. These are

$$\begin{pmatrix} R_{0101} & R_{0102} & R_{0103} & R_{0112} & R_{0113} & \\ & R_{0202} & R_{0203} & R_{0212} & R_{0213} & R_{0223} \\ & & R_{0303} & R_{0312} & R_{0313} & R_{0323} \\ & & & R_{1212} & R_{1213} & R_{1223} \\ & & & & R_{1313} & R_{1323} \\ & & & & & R_{2323} \end{pmatrix}\ , \tag{3.82}$$

where R_{0123} is determined by (3.81).

Using the symmetries (3.79) and the second Bianchi identity from (3.80), we can obtain an important result. We first contract

$$R^i{}_{jkl;m} + R^i{}_{jlm;k} + R^i{}_{jmk;l} = 0 \tag{3.83}$$

by multiplying with δ_i^k and use the symmetry relations (3.79) to find

$$R_{jl;m} + R_j{}^i{}_{ml;i} - R_{jm;l} = 0\ . \tag{3.84}$$

Next, we contract again by multiplying with g^{jm}, which yields

$$R^m{}_{l;m} + R^i{}_{l;i} - R_{;l} = 0\ , \tag{3.85}$$

where R_{ij} are the components of the Ricci tensor and $R = R^i{}_i$ is the *Ricci scalar* or the *scalar curvature*. Renaming dummy indices, the last equation can be brought into the form

$$\left(R^i{}_j - \frac{R}{2}\delta^i_j\right)_{;i} = 0\ , \tag{3.86}$$

which is the *contracted Bianchi identity*. Moreover, the Ricci tensor can easily be shown to be symmetric,

$$R_{ij} = R_{ji} \; . \tag{3.87}$$

We finally introduce the symmetric *Einstein tensor* by

$$G_{ij} \equiv R_{ij} - \frac{R}{2} g_{ij} \; , \tag{3.88}$$

which has vanishing divergence because of the contracted Bianchi identity,

$$G^{i}{}_{j;i} = 0 \; . \tag{3.89}$$

Chapter 4

Physical Laws in External Gravitational Fields

4.1 Motion of Particles and Light

4.1.1 Freely Falling Particles

In special relativity, the action of a free particle was

$$S = -mc^2 \int_a^b \mathrm{d}\tau = -mc \int_a^b \mathrm{d}s = -mc \int_a^b \sqrt{-\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu}\ , \tag{4.1}$$

where we have introduced the Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. This can be rewritten as follows: first, we parameterise the trajectory of the particle as a curve $\gamma(\tau)$ and write the four-vector $\mathrm{d}x = \dot{\gamma}\mathrm{d}\tau$. Second, we use the notation (2.46)

$$\eta(\dot{\gamma}, \dot{\gamma}) = \langle \dot{\gamma}, \dot{\gamma} \rangle \tag{4.2}$$

to cast the action into the form

$$S = -mc \int_a^b \sqrt{-\langle \dot{\gamma}, \dot{\gamma} \rangle}\, \mathrm{d}\tau\ . \tag{4.3}$$

In general relativity, the metric η is replaced by the dynamic metric g. We thus expect that the motion of a free particle will be described by the action

$$S = -mc \int_a^b \sqrt{-\langle \dot{\gamma}, \dot{\gamma} \rangle}\, \mathrm{d}\tau = -mc \int_a^b \sqrt{-g(\dot{\gamma}, \dot{\gamma})}\, \mathrm{d}\tau\ . \tag{4.4}$$

To see what this equation implies, we now carry out the variation of S and set it to zero,

$$\delta S = -mc\, \delta \int_a^b \sqrt{-g(\dot{\gamma}, \dot{\gamma})}\, \mathrm{d}\tau = 0\ . \tag{4.5}$$

Since the curve is assumed to be parameterised by the proper time τ, we must have

$$c\mathrm{d}\tau = \mathrm{d}s = \sqrt{-\langle\dot{\gamma},\dot{\gamma}\rangle}\mathrm{d}\tau\,, \tag{4.6}$$

which implies that the four-velocity $\dot{\gamma}$ must satisfy

$$\langle\dot{\gamma},\dot{\gamma}\rangle = -c^2\,. \tag{4.7}$$

This allows us to write the variation (4.5) as

$$\delta S = \frac{mc}{2}\int_a^b \mathrm{d}\tau\left[\partial_\lambda g_{\mu\nu}\delta x^\lambda \dot{x}^\mu \dot{x}^\nu + 2g_{\mu\nu}\delta\dot{x}^\mu \dot{x}^\nu\right] = 0\,. \tag{4.8}$$

We can integrate the second term by parts to find

$$\begin{aligned}2\int_a^b \mathrm{d}\tau\, g_{\mu\nu}\delta\dot{x}^\mu\dot{x}^\nu &= -2\int_a^b \mathrm{d}\tau\,\frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{\mu\nu}\dot{x}^\nu\right)\delta x^\mu \\ &= -2\int_a^b \mathrm{d}\tau\left(\partial_\lambda g_{\mu\nu}\dot{x}^\lambda\dot{x}^\nu + g_{\mu\nu}\ddot{x}^\nu\right)\delta x^\mu\,.\end{aligned} \tag{4.9}$$

Interchanging the summation indices λ and μ and inserting the result into (4.8) yields

$$\left(\partial_\lambda g_{\mu\nu} - 2\partial_\mu g_{\lambda\nu}\right)\dot{x}^\mu\dot{x}^\nu - 2g_{\lambda\nu}\ddot{x}^\nu = 0 \tag{4.10}$$

or, after multiplication with $g^{\alpha\lambda}$,

$$\ddot{x}^\alpha + \frac{1}{2}g^{\alpha\lambda}\left(2\partial_\mu g_{\lambda\nu} - \partial_\lambda g_{\mu\nu}\right)\dot{x}^\mu\dot{x}^\nu = 0\,. \tag{4.11}$$

Comparing the result (4.11) to (3.17) and using the symmetry of the Christoffel symbols (3.70), we see that trajectories extremising the action (4.4) are geodesic curves. We thus arrive at the conclusion that *freely falling particles follow geodesic curves.*

4.1.2 Maxwell's Equations

As an example for how physical laws can be carried from special to general relativity, we now formulate the equations of classical electrodynamics in a gravitational field.

In terms of the field tensor F, Maxwell's equations read

$$\begin{aligned}\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} &= 0\,, \\ \partial_\nu F^{\mu\nu} &= \frac{4\pi}{c}j^\mu\,,\end{aligned} \tag{4.12}$$

where j^μ is the current four-vector. The homogeneous equations are identically satisfied introducing the potentials A^μ, in terms of which the field tensor is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,. \tag{4.13}$$

We can impose a gauge condition, such as the Lorenz gauge

$$\partial_\mu A^\mu = 0 \,, \tag{4.14}$$

which allows to write the inhomogeneous Maxwell equation in the form

$$\partial_\nu \partial^\nu A^\mu = -\frac{4\pi}{c} j^\mu \,. \tag{4.15}$$

Indices are raised with the (inverse) Minkowski metric,

$$F^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}F_{\alpha\beta} \,. \tag{4.16}$$

Finally, the equation for the Lorentz force can be written as

$$m\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} = \frac{q}{c}F^\mu{}_\nu u^\nu \,, \tag{4.17}$$

where $u^\mu = \mathrm{d}x^\mu/\mathrm{d}\tau$ is the four-velocity.

Moving to general relativity, we first replace the partial by covariant derivatives in Maxwell's equations and find

$$\begin{aligned} \nabla_\lambda F_{\mu\nu} + \nabla_\mu F_{\nu\lambda} + \nabla_\nu F_{\lambda\mu} &= 0 \,, \\ \nabla_\nu F^{\mu\nu} &= \frac{4\pi}{c} j^\mu \,. \end{aligned} \tag{4.18}$$

However, it is easy to see that the identity

$$\nabla_\lambda F_{\mu\nu} + \text{cyclic} \equiv \partial_\lambda F_{\mu\nu} + \text{cyclic} \tag{4.19}$$

holds because of the antisymmetry of the field tensor F and the symmetry of the connection ∇.

Indices have to be raised with the inverse metric g^{-1} now,

$$F^{\mu\nu} = g^{\mu\alpha}g^{\nu\beta}F_{\alpha\beta} \,. \tag{4.20}$$

Equation (4.17) for the Lorentz force has to be replaced by

$$m\left(\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} + \Gamma^\mu{}_{\alpha\beta}u^\alpha u^\beta\right) = \frac{q}{c}F^\mu{}_\nu u^\nu \,. \tag{4.21}$$

We thus arrive at the following *general rule*: In the presence of a gravitational field, the physical laws of special relativity are changed simply by substituting the covariant derivative for the partial derivative, $\partial \to \nabla$, by raising indices with $g^{\mu\nu}$ instead of $\eta^{\mu\nu}$ and by lowering them with $g_{\mu\nu}$ instead of $\eta_{\mu\nu}$, and by replacing the motion of free particles along straight lines by the motion along geodesics. Note that this is a rule, not a law, because ambiguities may occur in presence of second derivatives, as we shall see shortly.

We can impose a gauge condition such as the generalised Lorenz gauge

$$\nabla_\mu A^\mu = 0 \;, \tag{4.22}$$

but now the inhomogeneous wave equation (4.15) becomes more complicated. We first note that

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu \equiv \partial_\mu A_\nu - \partial_\nu A_\mu \tag{4.23}$$

identically. Inserting (4.23) into the inhomogeneous Maxwell equation first yields

$$\nabla_\nu \left(\nabla^\mu A^\nu - \nabla^\nu A^\mu\right) = \frac{4\pi}{c} j^\mu \;, \tag{4.24}$$

but now the term $\nabla_\nu \nabla^\mu A^\nu$ does not vanish despite the Lorenz gauge condition because the covariant derivatives do not commute. Instead, we have to use

$$\left(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu\right) A^\alpha = R^\alpha{}_{\beta\mu\nu} A^\beta \tag{4.25}$$

by definition of the curvature tensor, and thus

$$\nabla_\nu \nabla^\mu A^\nu = \nabla^\mu \nabla_\nu A^\nu + R^\mu{}_\beta A^\beta = R^\mu{}_\beta A^\beta \tag{4.26}$$

inserting the Lorenz gauge condition. Thus, the inhomogeneous wave equation in general relativity reads

$$\nabla_\nu \nabla^\nu A^\mu - R^\mu{}_\nu A^\nu = -\frac{4\pi}{c} j^\mu \;. \tag{4.27}$$

Had we started directly from the wave equation (4.15) from special relativity, we would have missed the curvature term! This illustrates the ambiguities that may occur applying the rule $\partial \to \nabla$ when second derivatives are involved.

4.1.3 Geometrical Optics

We now study how light rays propagate in a gravitational field. As in geometrical optics, we assume that the wavelength λ of the electromagnetic field is very much smaller compared to the scale L of the space within which we study light propagation. In a gravitational field, which causes space-time to curve on another scale R, we have to further assume that λ is also very small compared to R, thus

$$\lambda \ll L \quad \text{and} \quad \lambda \ll R \;. \tag{4.28}$$

An example could be an astronomical source at a distance of several million light-years from which light with optical wavelengths travels to the observer. The scale L would then be of order 10^{24} cm or larger, the scale R would be the curvature radius of the Universe, of order 10^{28} cm, while the light would have wavelengths of order 10^{-6} cm.

Consequently, we introduce an expansion of the four-potential in terms of a small parameter $\epsilon \equiv \lambda/\min(L, R) \ll 1$ and write the four-potential as a product of a slowly varying amplitude and a quickly varying phase,

$$A^\mu = \mathrm{Re}\left\{(a^\mu + \epsilon b^\mu)\mathrm{e}^{\mathrm{i}\psi/\epsilon}\right\} , \tag{4.29}$$

where the amplitude is understood as the two leading-order terms in the expansion, and the phase ψ carries the factor ϵ^{-1} because it is inversely proportional to the wave length. The real part is introduced because the amplitude is complex.

As in ordinary geometrical optics, the wave vector is the gradient of the phase, thus $k_\mu = \partial_\mu\psi$. We further introduce the scalar amplitude $a \equiv (a_\mu a^{*\mu})^{1/2}$, where the asterisk denotes complex conjugation, and the polarisation vector $e_\mu \equiv a_\mu/a$.

We first impose the Lorenz gauge and find the condition

$$\mathrm{Re}\left\{\left[\nabla_\mu(a^\mu + \epsilon b^\mu) + (a^\mu + \epsilon b^\mu)\frac{\mathrm{i}}{\epsilon}k_\mu\right]\mathrm{e}^{\mathrm{i}\psi/\epsilon}\right\} = 0 . \tag{4.30}$$

To leading order (ϵ^{-1}), this implies

$$k_\mu a^\mu = 0 , \tag{4.31}$$

which shows that the wave vector is perpendicular to the polarisation vector. The next-higher order yields

$$\nabla_\mu a^\mu + \mathrm{i}k_\mu b^\mu = 0 . \tag{4.32}$$

Next, we insert the *ansatz* (4.29) into Maxwell's equation (4.27) in vacuum, i.e. setting the right-hand side to zero. This yields

$$\begin{aligned} &\mathrm{Re}\left\{\left[\nabla_\nu\nabla^\nu(a^\mu + \epsilon b^\mu) + \frac{2\mathrm{i}}{\epsilon}k^\nu\nabla_\nu(a^\mu + \epsilon b^\mu)\right.\right. \\ +\; &\frac{\mathrm{i}}{\epsilon}(a^\mu + \epsilon b^\mu)\nabla_\nu k^\nu - \frac{1}{\epsilon^2}k_\nu k^\nu(a^\mu + \epsilon b^\mu) \\ -\; &\left.\left. R^\mu{}_\nu\,(a^\nu + \epsilon b^\nu)\right]\mathrm{e}^{\mathrm{i}\psi/\epsilon}\right\} = 0 . \end{aligned} \tag{4.33}$$

To leading order (ϵ^{-2}), this implies

$$k_\nu k^\nu = 0 , \tag{4.34}$$

which yields the general-relativistic *eikonal equation*

$$g^{\mu\nu}\partial_\mu\psi\partial_\nu\psi = 0 . \tag{4.35}$$

Trivially, (4.34) implies

$$0 = \nabla_\mu(k_\nu k^\nu) = 2k^\nu\nabla_\mu k_\nu . \tag{4.36}$$

Recall that the wave vector is the gradient of the scalar phase ψ. The second covariant derivatives of ψ commute,

$$\nabla_\mu \nabla_\nu \psi = \nabla_\nu \nabla_\mu \psi \tag{4.37}$$

as is easily seen by direct calculation, using the symmetry of the connection. Thus,

$$\nabla_\mu k_\nu = \nabla_\nu k_\mu \ , \tag{4.38}$$

which, inserted into (4.36), leads to

$$k^\nu \nabla_\nu k^\mu = 0 \quad \text{or} \quad \nabla_k k = 0 \ . \tag{4.39}$$

In other words, in the limit of geometrical optics, Maxwell's equations imply that *light rays follow null geodesics*.

The next-higher order (ϵ^{-1}) gives

$$2\mathrm{i}\left(k^\nu \nabla_\nu a^\mu + \frac{1}{2} a^\mu \nabla_\nu k^\nu\right) - k_\nu k^\nu b^\mu = 0 \tag{4.40}$$

and, with (4.34), this becomes

$$k^\nu \nabla_\nu a^\mu + \frac{1}{2} a^\mu \nabla_\nu k^\nu = 0 \ . \tag{4.41}$$

We use this to derive a propagation law for the amplitude a. Obviously, we can write

$$2ak^\nu \partial_\nu a = 2ak^\nu \nabla_\nu a = k^\nu \nabla_\nu (a^2) = k^\nu \left(a^{*\mu} \nabla_\nu a_\mu + a_\mu \nabla_\nu a^{*\mu}\right) \ . \tag{4.42}$$

By (4.41), this can be transformed to

$$k^\nu \left(a^{*\mu} \nabla_\nu a^\mu + a^\mu \nabla_\nu a^{*\mu}\right) = -\frac{1}{2} \nabla_\nu k^\nu (a^{*\mu} a_\mu + a_\mu a^{*\mu}) = -a^2 \nabla_\nu k^\nu \ . \tag{4.43}$$

Combining (4.43) with (4.42) yields

$$k^\nu \partial_\nu a = -\frac{a}{2} \nabla_\nu k^\nu \ , \tag{4.44}$$

which shows how the amplitude is transported along light rays: the change of the amplitude in the direction of the wave vector is proportional to the negative divergence of the wave vector, which is a very intuitive result.

Finally, we obtain a law for the propagation of the polarisation. Using $a^\mu = ae^\mu$ in (4.41) gives

$$\begin{aligned} 0 &= k^\nu \nabla_\nu (ae^\mu) + \frac{1}{2} ae^\mu \nabla_\nu k^\nu \\ &= ak^\nu \nabla_\nu e^\mu + e^\mu \left(k^\nu \partial_\nu a + \frac{a}{2} \nabla_\nu k^\nu\right) = ak^\nu \nabla_\nu e^\mu \ , \end{aligned} \tag{4.45}$$

where (4.44) was used in the last step. This shows that

$$k^\nu \nabla_\nu e^\mu = 0 \quad \text{or} \quad \nabla_k e = 0 \ , \tag{4.46}$$

in other words, the polarisation is parallel-transported along light rays.

4.1.4 Redshift

Suppose now that a light source moving with four-velocity u_s is sending a light ray to an observer moving with four-velocity u_o, and another light ray after a proper-time interval $\delta\tau_\mathrm{s}$. The phases of the first and second light rays be ψ_1 and $\psi_2 = \psi_1 + \delta\psi$, respectively.

Clearly, the phase difference measured at the source and at the observer must equal, thus

$$u_\mathrm{s}^\mu(\partial_\mu\psi)_\mathrm{s}\delta\tau_\mathrm{s} = \delta\psi = u_\mathrm{o}^\mu(\partial_\mu\psi)_\mathrm{o}\delta\tau_\mathrm{o} \ . \tag{4.47}$$

Using $k_\mu = \partial_\mu\psi$, and assigning frequencies ν_s and ν_o to the light rays which are indirectly proportional to the time intervals $\delta\tau_\mathrm{s}$ and $\delta\tau_\mathrm{o}$, we find

$$\frac{\nu_\mathrm{o}}{\nu_\mathrm{s}} = \frac{\delta\tau_\mathrm{s}}{\delta\tau_\mathrm{o}} = \frac{\langle k, u\rangle_\mathrm{o}}{\langle k, u\rangle_\mathrm{s}} \ , \tag{4.48}$$

which gives the combined gravitational redshift and the Doppler shift of the light rays. Any distinction between Doppler shift and gravitational redshift has no invariant meaning in general relativity.

4.2 Energy-Momentum “Conservation”

4.2.1 Contracted Christoffel Symbols

From (3.72), we see that the contracted Christoffel symbol can be written as

$$\Gamma^\mu{}_{\mu\nu} = \frac{1}{2}g^{\mu\alpha}\left(g_{\alpha\nu,\mu} + g_{\mu\alpha,\nu} - g_{\mu\nu,\alpha}\right) \ . \tag{4.49}$$

Exchanging the arbitrary dummy indices α and μ and using the symmetry of the metric, we can simplify this to

$$\Gamma^\mu{}_{\mu\nu} = \frac{1}{2}g^{\mu\alpha}g_{\mu\alpha,\nu} \ . \tag{4.50}$$

We continue by using Cramer’s rule from linear algebra, which states that the inverse of a matrix A has the components

$$(A^{-1})_{ij} = \frac{C_{ji}}{\det A} \ , \tag{4.51}$$

where the C_{ji} are the cofactors (signed minors) of the matrix A. Thus, the cofactors are

$$C_{ji} = \det A(A^{-1})_{ij} \ . \tag{4.52}$$

The determinant of A can be expressed using the cofactors as

$$\det A = \sum_{j=1}^{n} C_{ji}A_{ji} \tag{4.53}$$

for any fixed i, where n is the dimension of the (square) matrix. This so-called *Laplace expansion* of the determinant follows after multiplying (4.52) with the matrix A_{jk}.

By definition of the cofactors, any cofactor C_{ji} does not contain the element A_{ji} of the matrix A. Therefore, we can use (4.52) and the Laplace expansion (4.53) to conclude

$$\frac{\partial \det A}{\partial A_{ji}} = C_{ji} = \det A (A^{-1})_{ij} \; . \tag{4.54}$$

The metric is represented by the matrix $g_{\mu\nu}$, its inverse by $g^{\mu\nu}$. We abbreviate its determinant by g here. Cramer's rule (4.52) then implies that the cofactors of $g_{\mu\nu}$ are $C^{\mu\nu} = g\, g^{\mu\nu}$, and we can immediately conclude from (4.54) that

$$\frac{\partial g}{\partial g_{\mu\nu}} = g g^{\mu\nu} \tag{4.55}$$

and thus

$$\partial_\lambda g = \frac{\partial g}{\partial g_{\mu\nu}} \partial_\lambda g_{\mu\nu} = g g^{\mu\nu} \partial_\lambda g_{\mu\nu} \; . \tag{4.56}$$

Comparing this with the expression (4.50) for the contracted Christoffel symbol, we see that

$$\begin{aligned} g g^{\mu\nu} g_{\mu\nu,\lambda} &= 2g\Gamma^\mu{}_{\mu\lambda} \; , \\ \Gamma^\mu{}_{\mu\lambda} &= \frac{1}{2} g^{\mu\nu} g_{\mu\nu,\lambda} = \frac{1}{2g} g_{,\lambda} = \frac{1}{\sqrt{-g}} \partial_\lambda \sqrt{-g} \; , \end{aligned} \tag{4.57}$$

which is a very convenient expression for the contracted Christoffel symbol, as we shall see.

4.2.2 Covariant Divergences

The covariant derivative of a vector with components v^μ has the components

$$\nabla_\nu v^\mu = \partial_\nu v^\mu + \Gamma^\mu{}_{\nu\alpha} v^\alpha \; . \tag{4.58}$$

Using (4.57), the covariant divergence of this vector can thus be written

$$\nabla_\mu v^\mu = \partial_\mu v^\mu + \frac{1}{\sqrt{-g}} v^\mu \partial_\mu \sqrt{-g} = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g}\, v^\mu) \; . \tag{4.59}$$

Similarly, for a tensor A of rank $(2, 0)$ with components $A^{\mu\nu}$, we have

$$\nabla_\nu A^{\mu\nu} = \partial_\nu A^{\mu\nu} + \Gamma^\mu{}_{\alpha\nu} A^{\alpha\nu} + \Gamma^\nu{}_{\nu\alpha} A^{\mu\alpha} \; . \tag{4.60}$$

Again, by means of (4.57), we can combine the first and third terms on the right-hand side to write

$$\nabla_\nu A^{\mu\nu} = \frac{1}{\sqrt{-g}} \partial_\nu (\sqrt{-g} A^{\mu\nu}) + \Gamma^\mu{}_{\alpha\nu} A^{\alpha\nu} \; . \tag{4.61}$$

If the tensor $A^{\mu\nu}$ is antisymmetric, the second term on the right-hand side vanishes because then the symmetric Christoffel symbol $\Gamma^{\mu}{}_{\alpha\nu}$ is contracted with the antisymmetric tensor $A^{\alpha\nu}$. If $A^{\mu\nu}$ is symmetric, however, this final term remains, with important consequences.

4.2.3 Charge Conservation

Since the electromagnetic field tensor $F^{\mu\nu}$ is antisymmetric, (4.61) implies

$$\nabla_\nu F^{\mu\nu} = \frac{1}{\sqrt{-g}}\partial_\nu(\sqrt{-g}F^{\mu\nu}) . \tag{4.62}$$

On the other hand, replacing the vector v^μ by $\nabla_\nu F^{\mu\nu}$ in (4.59), we see that

$$\nabla_\mu\nabla_\nu F^{\mu\nu} = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\nabla_\nu F^{\mu\nu}) = \frac{1}{\sqrt{-g}}\partial_\mu\partial_\nu(\sqrt{-g}F^{\mu\nu}) , \tag{4.63}$$

where we have used (4.62) in the final step. But the partial derivatives commute, so that once more the antisymmetric tensor $F^{\mu\nu}$ is contracted with the symmetric symbol $\partial_\mu\partial_\nu$. Thus, the result must vanish, allowing us to conclude

$$\nabla_\mu\nabla_\nu F^{\mu\nu} = 0 . \tag{4.64}$$

However, by Maxwell's equation (4.18),

$$\nabla_\mu\nabla_\nu F^{\mu\nu} = \frac{4\pi}{c}\nabla_\mu j^\mu , \tag{4.65}$$

which implies, by (4.59)

$$\partial_\mu(\sqrt{-g}j^\mu) = 0 . \tag{4.66}$$

This is the continuity equation of the electric four-current, implying charge conservation. We thus see that the antisymmetry of the electromagnetic field tensor is necessary for charge conservation.

4.2.4 Energy-Momentum "Conservation"

In special relativity, energy-momentum conservation can be expressed by the vanishing four-divergence of the energy-momentum tensor T,

$$\partial_\nu T^{\mu\nu} = 0 . \tag{4.67}$$

For example, the energy-momentum tensor of the electromagnetic field is, in special relativity

$$T^{\mu\nu} = \frac{1}{4\pi}\left[-F^{\mu\lambda}F^{\nu}{}_{\lambda} + \frac{1}{4}\eta^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right] , \tag{4.68}$$

and for $\mu = 0$, the vanishing divergence (4.67) yields the energy conservation equation

$$\frac{\partial}{\partial t}\left(\frac{\vec{E}^2 + \vec{B}^2}{8\pi}\right) + \vec{\nabla} \cdot \left[\frac{c}{4\pi}(\vec{E} \times \vec{B})\right] = 0 \,, \tag{4.69}$$

in which the Poynting vector

$$\vec{S} = \frac{c}{4\pi}\left(\vec{E} \times \vec{B}\right) \tag{4.70}$$

represents the energy current density.

According to our general rule for moving results from special relativity to general relativity, we can replace the partial derivative in (4.67) by the covariant derivative,

$$\nabla_\nu T^{\mu\nu} = 0 \,, \tag{4.71}$$

and obtain an equation which is covariant and thus valid in all reference frames. Moreover, we would have to replace the Minkowski metric η in (4.68) by the metric g if we wanted to consider the energy-momentum tensor of the electromagnetic field.

From our general result (4.61), we know that we can rephrase (4.71) as

$$\frac{1}{\sqrt{-g}}\partial_\nu(\sqrt{-g}T^{\mu\nu}) + \Gamma^\mu_{\ \lambda\nu} T^{\lambda\nu} = 0 \,. \tag{4.72}$$

If the second term on the left-hand side was absent, this equation would imply a conservation law. It remains there, however, because the energy-momentum tensor is symmetric. In presence of this term, we cannot convert (4.72) to a conservation law any more. This result expresses the fact that *energy is not conserved in general relativity*. This is not surprising because energy can now be exchanged with the gravitational field.

4.3 The Newtonian Limit

4.3.1 Metric and Gravitational Potential

Finally, we want to see under which conditions for the metric the Newtonian limit for the equation of motion in a gravitational field is reproduced, which is

$$\ddot{\vec{x}} = -\vec{\nabla}\Phi \tag{4.73}$$

to very high precision in the Solar System.

We first restrict the gravitational field to be weak and to vary slowly with time. This implies that the Minkowski metric of flat space is perturbed by a small amount,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \,, \tag{4.74}$$

with $|h_{\mu\nu}| \ll 1$.

Moreover, we restrict the consideration to bodies moving much slower than the speed of light, such that

$$\frac{dx^i}{d\tau} \ll \frac{dx^0}{d\tau} \approx 1 \ . \tag{4.75}$$

Under these conditions, the geodesic equation for the i-th spatial component reduces to

$$\frac{d^2x^i}{c^2dt^2} \approx \frac{d^2x^i}{d\tau^2} = -\Gamma^i_{\ \alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} \approx -\Gamma^i_{\ 00} \ . \tag{4.76}$$

By definition (3.72), the remaining Christoffel symbols read

$$\Gamma^i_{\ 00} = h_0{}^i{}_{,0} - \frac{1}{2}h_{00}{}^{,i} \approx -\frac{1}{2}h_{00}{}^{,i} \tag{4.77}$$

due to the assumption that the metric changes slowly in time so that its time derivative can be ignored compared to its spatial derivatives. Equation (4.76) can thus be reduced to

$$\frac{d^2\vec{x}}{dt^2} \approx \frac{c^2}{2}\vec{\nabla}h_{00} \ , \tag{4.78}$$

which agrees with the Newtonian equation of motion (4.73) if we identify

$$h_{00} \approx -\frac{2\Phi}{c^2} + \text{const.} \ . \tag{4.79}$$

The constant can be set to zero because both the deviation from the Minkowski metric and the gravitational potential vanish at large distance from the source of gravity. Therefore, the metric in the Newtonian limit has the 00 element

$$g_{00} \approx -1 - \frac{2\Phi}{c^2} \ . \tag{4.80}$$

4.3.2 Gravitational Light Deflection

Based on this result, we might speculate that the metric in Newtonian approximation could be written as

$$g = \mathrm{diag}\left[-\left(1 + \frac{2\Phi}{c^2}\right), 1, 1, 1\right] \ . \tag{4.81}$$

We shall now work out the gravitational light deflection by the Sun in this metric, which was one of the first observational tests of general relativity.

Since light rays propagate along null geodesics, we have

$$\nabla_k k = 0 \quad \text{or} \quad k^\nu \partial_\nu k^\mu + \Gamma^\mu_{\ \nu\lambda} k^\nu k^\lambda = 0 \;, \tag{4.82}$$

where $k = (\omega/c, \vec{k})$ is the wave four-vector which satisfies

$$\langle k, k \rangle = 0 \quad \text{thus} \quad \omega = c|\vec{k}| \;, \tag{4.83}$$

which is the ordinary dispersion relation for electromagnetic waves in vacuum. We introduce the unit vector $\vec{e}$ in the direction of $\vec{k}$ by $\vec{k} = |\vec{k}|\,\vec{e} = \omega\,\vec{e}/c$.

Assuming that the gravitational potential Φ does not vary with time, $\partial_0\Phi = 0$, the only non-vanishing Christoffel symbols of the metric (4.81) are

$$\Gamma^0_{\ 0i} \approx \frac{1}{c^2}\partial_i\Phi \approx \Gamma^i_{\ 00} \;. \tag{4.84}$$

For $\mu = 0$, (4.82) yields

$$\left(\frac{1}{c}\partial_t + \vec{e}\cdot\vec{\nabla}\right)\omega + \omega\frac{\vec{e}\cdot\vec{\nabla}\Phi}{c^2} = 0 \;, \tag{4.85}$$

which shows that the frequency changes with time only because the light path can run through a spatially varying gravitational potential. Thus, if the potential is constant in time, the frequencies of the incoming and the outgoing light must equal.

Using this result, the spatial components of (4.82) read

$$\left(\frac{1}{c}\partial_t + \vec{e}\cdot\vec{\nabla}\right)\vec{e} = \frac{\mathrm{d}\vec{e}}{c\mathrm{d}t} = -\frac{1}{c^2}\left[\vec{\nabla} - \vec{e}(\vec{e}\cdot\vec{\nabla})\right]\Phi = -\frac{1}{c^2}\vec{\nabla}_\perp\Phi \;; \tag{4.86}$$

in other words, the total time derivative of the unit vector in the direction of the light ray equals the negative perpendicular gradient of the gravitational potential.

For calculating the light deflection, we need to know the total change in $\vec{e}$ as the light ray passes the Sun. This is obtained by integrating (4.86) along the *actual (curved)* light path, which is quite complicated. However, due to the weakness of the gravitational field, the deflection will be very small, and we can evaluate the integral along the *unperturbed (straight)* light path.

We choose a coordinate system centred on the Sun and rotated such that the light ray propagates parallel to the z axis from $-\infty$ to ∞ at an impact parameter b. Outside the Sun, its gravitational potential is

$$\frac{\Phi}{c^2} = -\frac{GM_\odot}{c^2 r} = -\frac{GM_\odot}{c^2\sqrt{b^2 + z^2}} \;. \tag{4.87}$$

The perpendicular gradient of Φ is

$$\vec{\nabla}_\perp \Phi = \frac{\partial \Phi}{\partial b} \vec{e}_b = \frac{GM\, b}{c^2 (b^2 + z^2)^{3/2}} \vec{e}_b \;, \tag{4.88}$$

where $\vec{e}_b$ is the radial unit vector in the x-y plane from the Sun to the light ray.

Thus, the deflection angle is

$$\delta \vec{e} = -\vec{e}_b \int_{-\infty}^{\infty} \mathrm{d}z \, \frac{GMb}{c^2 (b^2 + z^2)^{3/2}} = -\frac{2GM}{c^2 b} \, \vec{e}_b \;. \tag{4.89}$$

Evaluating (4.89) at the rim of the Sun, we insert $M_\odot = 2 \times 10^{33}$ g and $R_\odot = 7 \times 10^{10}$ cm to find

$$|\delta \vec{e}| = 0.87'' \;. \tag{4.90}$$

For several reasons, this is a remarkable result. First, it had already been derived by the German astronomer Soldner in the 19th century who had assumed that light was a stream of material particles to which celestial mechanics could be applied just as well as to planets. Before general relativity, a strict physical meaning could not be given to the trajectory of light in the presence of a gravitational field because the interaction between electromagnetic fields and gravity was entirely unclear. The statement of general relativity that light propagates along null geodesics for the first time provided a physical law for the propagation of light rays in gravitational fields.

Second, the result (4.90) is experimentally found to be wrong. In fact, the measured value is twice as large. This is a consequence of our assumption that the metric in the Newtonian limit is given by (4.81), while the true Newtonian limit is

$$g = \mathrm{diag}[-(1 + 2\Phi/c^2), 1 - 2\Phi/c^2, 1 - 2\Phi/c^2, 1 - 2\Phi/c^2] \;. \tag{4.91}$$

Chapter 5

Differential Geometry III

5.1 The Lie Derivative

5.1.1 The Pull-Back

Following (2.27), we considered one-parameter groups of diffeomorphisms

$$\gamma_t : \mathbb{R} \times M \to M \tag{5.1}$$

such that points $p \in M$ can be considered as transported along curves

$$\gamma : \mathbb{R} \to M \tag{5.2}$$

with $\gamma(0) = p$. Similarly, the diffeomorphism γ_t can be taken at fixed $t \in \mathbb{R}$, defining a diffeomorphism

$$\gamma_t : M \to M \tag{5.3}$$

which maps the manifold onto itself and satisfies $\gamma_t \circ \gamma_s = \gamma_{s+t}$.

We have seen the relationship between vector fields and one-parameter groups of diffeomorphisms before. Let now v be a vector field on M and γ from (5.2) be chosen such that the tangent vector $\dot{\gamma}(t)$ defined by

$$(\dot{\gamma}(t))(f) = \frac{\mathrm{d}}{\mathrm{d}t}(f \circ \gamma)(t) \tag{5.4}$$

is identical with v, $\dot{\gamma} = v$. Then γ is called an *integral curve* of v.

If this is true for *all* curves γ obtained from γ_t by specifying initial points $\gamma(0)$, the result is called the *flow* of v.

The domain of definition $\mathcal{D}$ of γ_t can be a subset of $\mathbb{R} \times M$. If $\mathcal{D} = \mathbb{R} \times M$, the vector field is said to be *complete* and γ_t is called the *global flow* of v.

If $\mathcal{D}$ is restricted to open intervals $I \subset \mathbb{R}$ and open neighbourhoods $U \subset M$, thus $\mathcal{D} = I \times U \subset \mathbb{R} \times M$, the flow is called *local*.

Let now M and N be two manifolds and $\phi : M \to N$ a map from M onto N. A function f defined at a point $q \in N$ can be defined at a point $p \in M$ with $q = \phi(p)$ by

$$\phi^* f : M \to \mathbb{R} \;, \quad (\phi^* f)(p) := (f \circ \phi)(p) = f[\phi(p)] \;. \tag{5.5}$$

The map ϕ^* "pulls" functions f on N "back" to M and is thus called the *pull-back*.

Similarly, the pull-back allows to map vectors v from the tangent space V_p of M in p onto vectors from the tangent space V_q of N in q. We can first pull-back the function f defined in $q \in N$ to $p \in M$ and then apply v on it, and identify the result as a vector $\phi_* v$ applied to f,

$$\phi_* : V_p \to V_q \;, \quad v \mapsto \phi_* v = v \circ \phi^* \;, \tag{5.6}$$

such that $(\phi_* v)(f) = v(\phi^* f) = v(f \circ \phi)$. This defines a vector from the tangent space of N in $q = \phi(p)$. The map ϕ_* "pushes" vectors from the tangent space of M in p to the tangent space of N in q and is thus called the *push-forward*.

In a natural generalisation to dual vectors, we define their pull-back ϕ^* by

$$\phi^* : V_q^* \to V_p^* \;, \quad w \mapsto \phi^* w = w \circ \phi_* \;, \tag{5.7}$$

such that $(\phi^* w)(v) = w(\phi_* v) = w(v \circ \phi^*)$, where $w \in V_q^*$ is an element of the dual space of N in q. This operation "pulls back" the dual vector w from the dual space in $q = \phi(p) \in N$ to $p \in M$.

The pull-back ϕ^* and the push-forward ϕ_* can now be extended to tensors. Let T be a tensor field of rank $(0, r)$ on N, then its pull-back is defined by

$$\phi^* : \mathcal{T}_r^0(N) \to \mathcal{T}_r^0(M) \;, \quad T \mapsto \phi^* T = T \circ \phi_* \;, \tag{5.8}$$

such that $(\phi^* T)(v_1 \; \ldots, v_r) = T(\phi_* v_1, \ldots, \phi_* v_r)$. Similarly, we can define the pull-back of a tensor field of rank $(r, 0)$ on N by

$$\phi^* : \mathcal{T}_0^r(N) \to \mathcal{T}_0^r(M) \;, \quad T \mapsto \phi^* T \tag{5.9}$$

such that $(\phi^* T)(\phi^* w_1, \ldots, \phi^* w_r) = T(w_1, \ldots, w_r)$.

If the pull-back ϕ^* is a diffeomorphism, which implies in particular that the dimensions of M and N are equal, the pull-back and the push-forward are each other's inverses,

$$\phi_* = (\phi^*)^{-1} \;. \tag{5.10}$$

Irrespective of the rank of a tensor, we now denote by ϕ^* the pull-back of the tensor and by ϕ_* its inverse, i.e.

$$\begin{aligned} \phi^* : \mathcal{T}_s^r(N) &\to \mathcal{T}_s^r(M) \;, \\ \phi_* : \mathcal{T}_s^r(M) &\to \mathcal{T}_s^r(N) \;. \end{aligned} \tag{5.11}$$

The important point is that if $\phi^* : M \to M$ is a diffeomorphism and T is a tensor field on M, then $\phi^* T$ can be compared to T. If $\phi^* T = T$, ϕ^* is a *symmetry transformation* of T because T stays the same even though it was "moved" by ϕ^*. If the tensor field is the metric g, such a symmetry transformation of g is called an *isometry*.

5.1.2 The Lie Derivative

Let now v be a vector field on M and γ_t be the flow of v. Then, for an arbitrary tensor $T \in \mathcal{T}^r_s$, the expression

$$\mathcal{L}_v T := \lim_{t\to 0} \frac{\gamma_t^* T - T}{t} \tag{5.12}$$

is called the *Lie derivative* of the tensor T with respect to v.

Note that this definition naturally generalises the ordinary derivative with respect to "time" t. The manifold M is infinitesimally transformed by one element γ_t of a one-parameter group of diffeomorphisms. This could, for instance, represent an infinitesimal rotation of the two-sphere S^2. The tensor T on the manifold *after* the transformation is pulled back to the manifold before the transformation, where it can be compared to the original tensor T before the transformation.

Obviously, the Lie derivative of a rank-(r, s) tensor is itself a rank-(r, s) tensor. It is linear,

$$\mathcal{L}_v(t_1 + t_2) = \mathcal{L}_v(t_1) + \mathcal{L}_v(t_2) \;, \tag{5.13}$$

satisfies the Leibniz rule

$$\mathcal{L}_v(t_1 \otimes t_2) = \mathcal{L}_v(t_1) \otimes t_2 + t_1 \otimes \mathcal{L}_v(t_2) \;, \tag{5.14}$$

and it commutes with contractions. So far, these properties are easy to verify in particular after choosing local coordinates.

The application of the Lie derivative to a function f follows directly from the definition (5.4) of the tangent vector $\dot\gamma$,

$$\begin{aligned}\mathcal{L}_v f &= \lim_{t\to 0} \frac{\gamma_t^* f - f}{t} = \lim_{t\to 0} \frac{(f\circ\gamma_t) - (f\circ\gamma_0)}{t} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}(f\circ\gamma) = \dot\gamma f = vf = \mathrm{d}f(v) \;.\end{aligned} \tag{5.15}$$

The additional convenient property

$$\mathcal{L}_x y = [x, y] \tag{5.16}$$

for vector fields y is non-trivial to prove.

Given two vector fields x and y, the Lie derivative further satisfies the linearity relations

$$\mathcal{L}_{x+y} = \mathcal{L}_x + \mathcal{L}_y \ , \quad \mathcal{L}_{\lambda x} = \lambda \mathcal{L}_x \ , \tag{5.17}$$

with $\lambda \in \mathbb{R}$, and the commutation relation

$$\mathcal{L}_{[x,y]} = [\mathcal{L}_x, \mathcal{L}_y] = \mathcal{L}_x \circ \mathcal{L}_y - \mathcal{L}_y \circ \mathcal{L}_x \ . \tag{5.18}$$

If and only if two vector fields x and y commute, so do the respective Lie derivatives,

$$[x, y] = 0 \quad \Leftrightarrow \quad \mathcal{L}_x \circ \mathcal{L}_y = \mathcal{L}_y \circ \mathcal{L}_x \ . \tag{5.19}$$

If ϕ and ψ are the flows of x and y, the following commutation relation is equivalent to (5.19),

$$\phi_s \circ \psi_t = \psi_t \circ \phi_s \ . \tag{5.20}$$

Let $t \in \mathcal{T}^0_r$ be a rank-$(0, r)$ tensor field and $v_1, \ldots, v_r$ be vector fields, then

$$\begin{aligned} (\mathcal{L}_x t)(v_1, \ldots, v_r) &= x(t(v_1, \ldots, v_r)) \\ &- \sum_{i=1}^{r} t(v_1, \ldots, [x, v_i], \ldots, v_r) \ . \end{aligned} \tag{5.21}$$

To demonstrate this, we apply the Lie derivative to the tensor product of t and all v_i and use the Leibniz rule (5.14),

$$\begin{aligned} \mathcal{L}_x(t \otimes v_1 \otimes \ldots \otimes v_r) &= \mathcal{L}_x t \otimes v_1 \otimes \ldots \otimes v_r \\ &+ t \otimes \mathcal{L}_x v_1 \otimes \ldots \otimes v_r + \ldots \\ &+ t \otimes v_1 \otimes \ldots \otimes \mathcal{L}_x v_r \ . \end{aligned} \tag{5.22}$$

Then, we take the complete contraction and use the fact that the Lie derivative commutes with contractions, which yields

$$\begin{aligned} \mathcal{L}_x(t(v_1, \ldots, v_r)) &= (\mathcal{L}_x t)(v_1, \ldots, v_r) \\ &+ t(\mathcal{L}_x v_1, \ldots, v_r) + \ldots + t(v_1, \ldots, \mathcal{L}_x v_r) \ . \end{aligned} \tag{5.23}$$

Inserting (5.16), we now obtain (5.21).

As an example, we apply (5.21) to a tensor of rank $(0, 1)$, i.e. a dual vector w:

$$(\mathcal{L}_x w)(y) = x w(y) - w([x, y]) \ . \tag{5.24}$$

One particular dual vector is the differential of a function f, defined in (2.34). Inserting $\mathrm{d}f$ for w in (5.24) yields the useful relation

$$\begin{aligned} (\mathcal{L}_x \mathrm{d}f)(y) &= x\mathrm{d}f(y) - \mathrm{d}f([x, y]) \\ &= xy(f) - [x, y](f) = yx(f) \\ &= y\mathcal{L}_x f = \mathrm{d}\mathcal{L}_x f(y) \ , \end{aligned} \tag{5.25}$$

and since this holds for any vector field y, we find

$$\mathcal{L}_x \mathrm{d}f = \mathrm{d}\mathcal{L}_x f \ . \tag{5.26}$$

Using the latter expression, we can derive coordinate expressions for the Lie derivative. We introduce the coordinate basis $\{\partial_i\}$ and its dual basis $\{\mathrm{d}x^i\}$ and apply (5.26) to $\mathrm{d}x^i$,

$$\mathcal{L}_v \mathrm{d}x^i = \mathrm{d}\mathcal{L}_v x^i = \mathrm{d}v(x^i) = \mathrm{d}v^j \partial_j x^i = \mathrm{d}v^i = \partial_j v^i \, \mathrm{d}x^j \ . \tag{5.27}$$

The Lie derivative of the basis vectors ∂_i is

$$\mathcal{L}_v \partial_i = [v, \partial_i] = -(\partial_i v^j)\partial_j \ , \tag{5.28}$$

where (2.31) was used in the second step.

To illustrate the components of the Lie derivative of a tensor, we take a tensor t of rank $(1,1)$ and apply the Lie derivative to the tensor product $t \otimes \mathrm{d}x^i \otimes \partial x_j$,

$$\begin{aligned} \mathcal{L}_v(t \otimes \mathrm{d}x^i \otimes \partial_j) &= (\mathcal{L}_v t) \otimes \mathrm{d}x^i \otimes \partial_j \\ &+ t \otimes \mathcal{L}_v \mathrm{d}x^i \otimes \partial_j + t \otimes \mathrm{d}x^i \otimes \mathcal{L}_v \partial_j \ , \end{aligned} \tag{5.29}$$

and now contract completely. This yields

$$\begin{aligned} \mathcal{L}_v t^i_j &= (\mathcal{L}_v t)^i_j + t(\partial_k v^i \mathrm{d}x^k, \partial_j) - t(\mathrm{d}x^i, \partial_j v^k \partial_k) \\ &= (\mathcal{L}_v t)^i_j + t^k_j \partial_k v^i - t^i_k \partial_j v^k \ . \end{aligned} \tag{5.30}$$

Solving for the components of the Lie derivative of t, we thus obtain

$$(\mathcal{L}_v t)^i_j = v^k \partial_k t^i_j - t^k_j \partial_k v^i + t^i_k \partial_j v^k \ , \tag{5.31}$$

and similarly for tensors of higher ranks.

In particular, for a tensor of rank $(0,1)$, i.e. a dual vector w,

$$(\mathcal{L}_v w)_i = v^k \partial_k w_i + w_k \partial_i v^k \ . \tag{5.32}$$

5.1.3 Killing Vector Fields

A Killing vector field K is a vector field along which the Lie derivative of the metric vanishes,

$$\mathcal{L}_K g = 0 \ . \tag{5.33}$$

This implies that the flow of a Killing vector field defines a symmetry transformation of the metric, i.e. an *isometry*.

To find a coordinate expression, we use (5.31) to write

$$\begin{aligned} (\mathcal{L}_K g)_{ij} &= K^k \partial_k g_{ij} + g_{kj} \partial_i K^k + g_{ik} \partial_j K^k \\ &= K^k (\partial_k g_{ij} - \partial_i g_{kj} - \partial_j g_{ik}) + \partial_i (g_{kj} K^k) + \partial_j (g_{ik} K^k) \\ &= \nabla_i K_j + \nabla_j K_i = 0 \ , \end{aligned} \tag{5.34}$$

where we have identified the Christoffel symbols (3.72) in the last step. This is the *Killing equation*.

Let γ be a geodesic, i.e. a curve satisfying

$$\nabla_{\dot\gamma}\dot\gamma = 0 \,, \tag{5.35}$$

then the projection of a Killing vector K on the tangent to the geodesic $\dot\gamma$ is constant along the geodesic,

$$\nabla_{\dot\gamma}\langle\dot\gamma, K\rangle = 0 \,. \tag{5.36}$$

This is easily seen as follows. First,

$$\nabla_{\dot\gamma}\langle\dot\gamma, K\rangle = \langle\nabla_{\dot\gamma}\dot\gamma, K\rangle + \langle\dot\gamma, \nabla_{\dot\gamma}K\rangle = \langle\dot\gamma, \nabla_{\dot\gamma}K\rangle \tag{5.37}$$

because of the geodesic equation (5.35).

Writing the last expression explicitly in components yields

$$\langle\dot\gamma, \nabla_{\dot\gamma}K\rangle = g_{ik}\dot\gamma^i\dot\gamma^j\nabla_j K^k = \dot\gamma^i\dot\gamma^j\nabla_j K_i \,, \tag{5.38}$$

changing indices and using the symmetry of the metric, we can also write it as

$$\langle\dot\gamma, \nabla_{\dot\gamma}K\rangle = g_{jk}\dot\gamma^j\dot\gamma^i\nabla_i K^k = \dot\gamma^j\dot\gamma^i\nabla_i K_j \,. \tag{5.39}$$

Adding the latter two equations and using the Killing equation (5.34) shows

$$2\langle\dot\gamma, \nabla_{\dot\gamma}K\rangle = \dot\gamma^i\dot\gamma^j\left(\nabla_i K_j + \nabla_j K_i\right) = 0 \,, \tag{5.40}$$

which proves (5.36). More elegantly, we have contracted the symmetric tensor $\dot\gamma^i\dot\gamma^j$ with the tensor $\nabla_i K_j$ which is antisymmetric because of the Killing equation, thus the result must vanish.

Equation (5.36) has a profound meaning: freely-falling particles and light rays both follow geodesics. The constancy of $\langle\dot\gamma, K\rangle$ along geodesics means that each Killing vector field gives rise to a conserved quantity for freely-falling particles and light rays. Since a Killing vector field generates an isometry, this shows that symmetry transformations of the metric give rise to conservation laws.

5.2 Differential Forms

5.2.1 Definition

Differential p-forms are totally antisymmetric tensors of rank $(0, p)$. The most simple example are dual vectors $w \in V^*$ since they are tensors of rank $(0, 1)$. A general tensor t of rank $(0, 2)$ is not antisymmetric, but can be antisymmetrised defining the two-form

$$\tau(v_1, v_2) \equiv \frac{1}{2}\left[t(v_1, v_2) - t(v_2, v_1)\right] \,, \tag{5.41}$$

with two vectors $v_1, v_2 \in V$.

To generalise this for tensors of arbitrary ranks $(0, r)$, we first define the *alternation operator* by

$$(\mathcal{A}t)(v_1, \dots, v_r) := \frac{1}{r!} \sum_\pi \mathrm{sgn}(\pi) t(v_{\pi(1)}, \dots, v_{\pi(r)}) \;, \tag{5.42}$$

where the sum extends over all permutations π of the integer numbers from 1 to r. The sign of a permutation, $\mathrm{sgn}(\pi)$, is negative if the permutation is odd and positive otherwise.

In components, we briefly write

$$(\mathcal{A}t)_{i_1 \dots i_r} = t_{[i_1 \dots i_r]} \tag{5.43}$$

so that p-forms ω are defined by the relation

$$\omega_{i_1 \dots i_p} = \omega_{[i_1 \dots i_p]} \tag{5.44}$$

between their components. For example, for a 2-form ω we have

$$\omega_{ij} = \omega_{[ij]} = \frac{1}{2} \left(\omega_{ij} - \omega_{ji} \right) \;. \tag{5.45}$$

The vector space of p-forms is denoted by $\bigwedge^p$. Taking the product of two differential forms $\omega \in \bigwedge^p$ and $\eta \in \bigwedge^q$ yields a tensor of rank $(0, p + q)$ which is not antisymmetric, but can be antisymmetrised by means of the alternation operator. The result

$$\omega \wedge \eta \equiv \frac{(p+q)!}{p!q!} \mathcal{A}(\omega \otimes \eta) \tag{5.46}$$

is called the *exterior product* . Evidently, it turns the tensor $\omega \otimes \eta \in \mathcal{T}^0_{p+q}$ into a $(p + q)$-form.

The definition of the exterior product implies that it is bilinear, associative, and satisfies

$$\omega \wedge \eta = (-1)^{pq} \eta \wedge \omega \;. \tag{5.47}$$

A basis for the vector space $\bigwedge^p$ can be constructed from the basis $\{\mathrm{d}x^i\}$, $1 \le i \le n$, of the dual space V^* by taking

$$\mathrm{d}x^{i_1} \wedge \dots \wedge \mathrm{d}x^{i_p} \quad \text{with} \quad 1 \le i_1 < \dots < i_p \le n \;, \tag{5.48}$$

which shows that the dimension of $\bigwedge^p$ is

$$\begin{pmatrix} n \\ p \end{pmatrix} \equiv \frac{n!}{p!(n-p)!} \tag{5.49}$$

for $p \le n$ and zero otherwise. The skewed commutation relation (5.47) implies

$$\mathrm{d}x^i \wedge \mathrm{d}x^j = -\mathrm{d}x^j \wedge \mathrm{d}x^i \;. \tag{5.50}$$

Given two vector spaces V and W above the same field F, the Cartesian product $V \times W$ of the two spaces can be turned into a vector space by defining the vector-space operations component-wise. Let $v, v_1, v_2 \in V$ and $w, w_1, w_2 \in W$, then the operations

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2)\ , \quad \lambda(v, w) = (\lambda v, \lambda w) \tag{5.51}$$

with $\lambda \in F$ give $V \times W$ the structure of a vector space $V \oplus W$ which is called the *direct sum* of V and W. Similarly, we define the vector space of differential forms

$$\bigwedge \equiv \bigoplus_{p=0}^{n} \bigwedge^{p} \tag{5.52}$$

as the direct sum of the vector spaces of p-forms with arbitrary $p \le n$.

A Graßmann algebra (Hermann Graßmann, 1809-1877) is an associative, skew-symmetric, graduated algebra with an identity element.

Recalling that a vector space V attains the structure of an *algebra* by defining a vector-valued product between two vectors,

$$\times : V \times V \to V\ , \quad (v, w) \mapsto v \times w\ , \tag{5.53}$$

we see that the exterior product $\wedge$ gives the vector space $\bigwedge$ of differential forms the structure of a *Graßmann algebra*,

$$\wedge : \bigwedge \times \bigwedge \to \bigwedge\ , \quad (\omega, \eta) \mapsto \omega \wedge \eta\ . \tag{5.54}$$

The *interior product* of a p-form ω with a vector $v \in V$ is a mapping

$$V \times \bigwedge^{p} \to \bigwedge^{p-1}\ , \quad (v, \omega) \mapsto i_v\omega \tag{5.55}$$

defined by

$$(i_v\omega)(v_1, \ldots, v_{p-1}) \equiv \omega(v, v_1, \ldots, v_{p-1}) \tag{5.56}$$

and $i_v\omega = 0$ if ω is 0-form (a number or a function).

5.2.2 The Exterior Derivative

If a connection ∇ is defined on the manifold, we can use it to define a map

$$\bigwedge^{p} \to \bigwedge^{p+1}\ , \quad \omega \mapsto \mathrm{d}\omega \tag{5.57}$$

with

$$\mathrm{d}\omega \equiv (p+1)\mathcal{A}(\nabla\omega) \tag{5.58}$$

such that

$$\mathrm{d}\omega(v_1, \ldots, v_{p+1}) = (p+1)\mathcal{A}\left(\nabla_{v_1}\omega(v_2, \ldots, v_{p+1})\right)\ . \tag{5.59}$$

This is called the *exterior derivative* of ω.

The result (3.34) for the covariant derivative of a tensor field and the antisymmetrisation imply the convenient formula for the exterior derivative of a p-form ω

$$\begin{aligned} \mathrm{d}\omega(v_1, \ldots, v_{p+1}) &= \sum_{i=1}^{p+1} (-1)^{i+1} v_i \omega(v_1, \ldots, \hat{v}_i, \ldots, v_{p+1}) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([v_i, v_j], v_1, \ldots, \hat{v}_i, \ldots, \hat{v}_j, \ldots, v_{p+1}) , \end{aligned} \tag{5.60}$$

where the hat over a symbol means that this object is to be left out.

If the connection ∇ is symmetric, in particular if we use the Riemannian connection, it is straightforward to conclude that all terms in (5.59) containing Christoffel symbols cancel because of the antisymmetrisation and the symmetry of the connection. Thus, we can replace the covariant by ordinary partial differentiation, $\nabla \to \partial$. The components of the exterior derivative of a p-form ω can thus be written

$$(\mathrm{d}\omega)_{i_1 \ldots i_{p+1}} = (p+1) \partial_{[i_1} \omega_{i_2 \ldots i_{p+1}]} . \tag{5.61}$$

Since $\omega_{i_2 \ldots i_{p+1}}$ is itself antisymmetric, this last expression can be brought into the simpler form

$$(\mathrm{d}\omega)_{i_1 \ldots i_{p+1}} = \sum_{k=1}^{p+1} (-1)^{k+1} \partial_{i_k} \omega_{i_1, \ldots, \hat{i}_k, \ldots i_{p+1}} , \tag{5.62}$$

with $1 \le i_1 < \ldots < i_p < i_{p+1} \le n$.

For an example, let us apply these relations to a 1-form ω. For it, the definition (5.58) implies

$$(\mathrm{d}\omega)(v_1, v_2) = (\nabla_{v_1} \omega)(v_2) - (\nabla_{v_2} \omega)(v_1) , \tag{5.63}$$

while (5.60) specialises to

$$\mathrm{d}\omega(v_1, v_2) = v_1 \omega(v_2) - v_2 \omega(v_1) - \omega([v_1, v_2]) . \tag{5.64}$$

With (5.61) or (5.62), we find the components

$$\mathrm{d}\omega_{ij} = \partial_i \omega_j - \partial_j \omega_i . \tag{5.65}$$

An alternative way of calculating the exterior derivative proceeds as follows. A p-form ω can be expanded in the basis (5.48) as

$$\omega = \omega_{i_1 \ldots i_p} \, \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_p} . \tag{5.66}$$

Its exterior derivative is thus

$$\mathrm{d}\omega = \mathrm{d}\omega_{i_1 \ldots i_p} \wedge \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_p} . \tag{5.67}$$

For example, for the 1-form $\omega = \omega_i\,\mathrm{d}x^i$, we find

$$\mathrm{d}\omega = \mathrm{d}\omega_i \wedge \mathrm{d}x^i = \partial_j\omega_i\,\mathrm{d}x^j \wedge \mathrm{d}x^i\ . \tag{5.68}$$

In $\mathbb{R}^3$, this turns into

$$\begin{aligned}\mathrm{d}\omega &= (\partial_1\omega_2 - \partial_2\omega_1)\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 + (\partial_1\omega_3 - \partial_3\omega_1)\,\mathrm{d}x^1 \wedge \mathrm{d}x^3 \\ &+ (\partial_2\omega_3 - \partial_3\omega_2)\,\mathrm{d}x^2 \wedge \mathrm{d}x^3\ .\end{aligned} \tag{5.69}$$

As (5.61) shows, we can calculate the exterior derivative using the ordinary partial derivative instead of the covariant derivative. This implies immediately that $\mathrm{d} \circ \mathrm{d}$ must vanish identically because the partial derivatives commute and thus cancel by the antisymmetrisation.

A differential p-form α is called *exact* if a $p-1$-form β exists such that $\alpha = \mathrm{d}\beta$. If $\mathrm{d}\alpha = 0$, the p-form α is called *closed*. Obviously, an exact form is closed because of $\mathrm{d} \circ \mathrm{d} = 0$.

5.3 Integration

5.3.1 The Volume Form and the Codifferential

An atlas of a differentiable manifold is called *oriented* if for every pair of charts h_1 on $U_1 \subset M$ and h_2 on $U_2 \subset M$ with $U_1 \cap U_2 \neq 0$, the Jacobi determinant of the coordinate change $h_2 \circ h_1^{-1}$ is positive.

An n-dimensional, paracompact manifold M is orientable if and only if a C^∞, n-form exists on M which vanishes nowhere. This is called a *volume form*.

The *canonical volume form* on a pseudo-Riemannian manifold (M, g) is defined by

$$\eta \equiv \sqrt{|g|}\,\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n\ . \tag{5.70}$$

This definition is independent of the coordinate system because it transforms proportional to the Jacobian determinant upon coordinate changes.

Equation (5.70) implies that the components of the canonical volume form in n dimensions are proportional to the n-dimensional Levi-Civita symbol,

$$\eta_{i_1\ldots i_n} = \sqrt{|g|}\,\epsilon_{i_1\ldots i_n}\ , \tag{5.71}$$

which is defined such that it is $+1$ for even permutations of the $i_1, \ldots, i_n$, -1 for odd permutations, and vanishes if any two of its indices are equal. A very useful relation is

$$\epsilon^{j_1\ldots j_q k_1\ldots k_p}\epsilon_{j_1\ldots j_q i_1\ldots i_p} = p!q!\,\delta^{k_1}_{[i_1}\delta^{k_2}_{i_2}\ldots\delta^{k_p}_{i_p]}\ , \tag{5.72}$$

where the square brackets again denote the complete antisymmetrisation. In three dimensions, one specific example for (5.72) is the familiar formula

$$\epsilon^{ijk}\epsilon_{klm} = \epsilon^{kij}\epsilon_{klm} = \delta^i_l\delta^j_m - \delta^i_m\delta^j_l \ . \tag{5.73}$$

Note that $p = 1$ and $q = 2$ here, but the factor $2! = 2$ is cancelled by the antisymmetrisation.

The *Hodge star operator* ($*$-operation) turns a p form ω into an $(n-p)$-form $(*\omega)$,

$$* : \bigwedge^p \to \bigwedge^{n-p} \ , \quad \omega \mapsto *\omega \ , \tag{5.74}$$

which is uniquely defined by its application to the dual basis. For the basis $\{\mathrm{d}x^i\}$ of the dual space V^*,

$$*(\mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_p}) := \frac{\sqrt{|g|}}{(n-p)!}\epsilon^{i_1\ldots i_p}{}_{i_{p+1}\ldots i_n} \ \mathrm{d}x^{i_{p+1}} \wedge \ldots \wedge \mathrm{d}x^{i_n} \ . \tag{5.75}$$

If the dual basis $\{e^i\}$ is orthonormal, this simplifies to

$$*(e^{i_1} \wedge \ldots \wedge e^{i_p}) = e^{i_{p+1}} \wedge \ldots \wedge e^{i_n} \ . \tag{5.76}$$

In components, we can write

$$(*\omega)_{i_{p+1}\ldots i_n} = \frac{1}{p!}\,\eta_{i_1\ldots i_n}\omega^{i_1\ldots i_p} \ , \tag{5.77}$$

i.e. $(*\omega)$ is the volume form η contracted with the p-form ω. A straightforward calculation shows that

$$*(*\omega) = \mathrm{sgn}(g)(-1)^{p(n-p)}\omega \ . \tag{5.78}$$

For a 1-form $\omega = \omega_i \mathrm{d}x^i$ in $\mathbb{R}^3$, we can use

$$*\mathrm{d}x^1 = \mathrm{d}x^2 \wedge \mathrm{d}x^3 \ , \quad *\mathrm{d}x^2 = \mathrm{d}x^3 \wedge \mathrm{d}x^1 \ , \quad *\mathrm{d}x^3 = \mathrm{d}x^1 \wedge \mathrm{d}x^2 \tag{5.79}$$

to find the Hodge-dual 2-form

$$*\omega = \omega_1\mathrm{d}x^2 \wedge \mathrm{d}x^3 - \omega_2\mathrm{d}x^1 \wedge \mathrm{d}x^3 + \omega_3\mathrm{d}x^1 \wedge \mathrm{d}x^2 \ , \tag{5.80}$$

while the 2-form $\mathrm{d}\omega$ (5.69) has the Hodge dual 1-form

$$\begin{aligned} *\mathrm{d}\omega &= (\partial_2\omega_3 - \partial_3\omega_2)\mathrm{d}x^1 - (\partial_1\omega_3 - \partial_3\omega_1)\mathrm{d}x^2 \\ &+ (\partial_1\omega_2 - \partial_2\omega_1)\mathrm{d}x^3 = \epsilon_i{}^{jk}\partial_j\omega_k\mathrm{d}x^i \ . \end{aligned} \tag{5.81}$$

The *codifferential* is a map

$$\delta : \bigwedge^p \to \bigwedge^{p-1} \ , \quad \omega \mapsto \delta\omega \tag{5.82}$$

defined by

$$\delta\omega \equiv \mathrm{sgn}(g)(-1)^{n(p+1)}(*\mathrm{d}*)\omega \ . \tag{5.83}$$

$\mathrm{d} \circ \mathrm{d} = 0$ immediately implies $\delta \circ \delta = 0$.

By successive application of (5.72) and (5.62), it can be shown that the coordinate expression for the codifferential is

$$(\delta\omega)^{i_1 \ldots i_{p-1}} = \frac{1}{\sqrt{|g|}} \partial_k \left(\sqrt{|g|} \omega^{k i_1 \ldots i_{p-1}} \right) . \tag{5.84}$$

Comparing this with (4.59), we see that this is the divergence of ω. To see this more explicitly, let us work out the codifferential of a 1-form in $\mathbb{R}^3$ by first taking the exterior derivative of $*\omega$ from (5.80),

$$\mathrm{d}{*\omega} = (\partial_1 \omega_1 + \partial_2 \omega_2 + \partial_3 \omega_3)\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 , \tag{5.85}$$

whose Hodge dual is

$$\delta\omega = \partial_1 \omega_1 + \partial_2 \omega_2 + \partial_3 \omega_3 . \tag{5.86}$$

5.3.2 Example: Maxwell's Equations

The Faraday 2-form is defined by

$$F \equiv \frac{1}{2} F_{\mu\nu}\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu . \tag{5.87}$$

Application of (5.62) shows that

$$\begin{aligned} (\mathrm{d}F)_{\lambda\mu\nu} &= \partial_\lambda F_{\mu\nu} - \partial_\mu F_{\lambda\nu} + \partial_\nu F_{\lambda\mu} \\ &= \partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0 , \end{aligned} \tag{5.88}$$

i.e. the homogeneous Maxwell equations can simply be expressed by

$$\mathrm{d}F = 0 . \tag{5.89}$$

Similarly, the components of the codifferential of the Faraday form are, according to (5.84) and (4.62)

$$(\delta F)^\mu = \frac{1}{\sqrt{-g}} \partial_\nu \left(\sqrt{-g} F^{\nu\mu} \right) = \nabla_\nu F^{\nu\mu} = -\frac{4\pi}{c}\, j^\mu . \tag{5.90}$$

Introducing further the current 1-form by $j = j_\mu \mathrm{d}x^\mu$, we can thus write the inhomogeneous Maxwell equations as

$$\delta F = -\frac{4\pi}{c}\, j . \tag{5.91}$$

5.3.3 Integrals and Integral Theorems

The integral over an n-form ω,

$$\int_M \omega \,, \tag{5.92}$$

is defined in the following way: Suppose first that the support $U \subset M$ of ω is contained in a single chart which defines positive coordinates $(x^1, \ldots, x^n)$ on U. Then, if $\omega = f\,\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n$ with a function $f \in \mathcal{F}(U)$,

$$\int_M \omega = \int_U f(x^1, \ldots, x^n)\mathrm{d}x^1 \ldots \mathrm{d}x^n \,. \tag{5.93}$$

Note that this definition is *independent* of the coordinate system because upon changes of the coordinate system, both f and the volume element $\mathrm{d}x^1 \ldots \mathrm{d}x^n$ change in proportion to the Jacobian determinant of the coordinate change.

If the domain of the n-form ω is contained in multiple maps, the integral (5.93) needs to be defined piece-wise, but the principle remains the same.

The integration of functions $f \in \mathcal{F}(M)$ is achieved using the canonical volume form η,

$$\int_M f \equiv \int_M f\eta \,. \tag{5.94}$$

Stokes' theorem can now be formulated as follows: let M be an n-dimensional manifold and the region $D \subset M$ have a smooth boundary ∂D such that $\bar{D} \equiv D \cup \partial D$ is compact. Then, for every $n-1$-form ω, we have

$$\int_D \mathrm{d}\omega = \int_{\partial D} \omega \,. \tag{5.95}$$

Likewise, *Gauss' theorem* can be brought into the form

$$\int_D \delta x^\flat \eta = \int_{\partial D} *x^\flat \,, \tag{5.96}$$

where $x \in V$ is a vector field on M and $x^\flat$ is the 1-form belonging to this vector field.

Generally, the *musical operators* $\flat$ and $\sharp$ are isomorphisms between the tangent spaces of a manifold and their dual spaces given by the metric,

$$\flat : TM \to T^*M \,, \quad v \mapsto v^\flat \,, \quad v^\flat_i = g_{ij}v^j \tag{5.97}$$

and similarly

$$\sharp : T^*M \to TM \,, \quad w \mapsto w^\sharp \,, \quad (w^\sharp)^i = g^{ij}w_j \,. \tag{5.98}$$

Chapter 6

Einstein's Field Equations

6.1 The physical meaning of the curvature tensor

6.1.1 Congruences of time-like geodesics

Having walked through the introductory chapters, we are now ready to introduce Einstein's field equations, i.e. the equations describing the dynamics of the gravitational field; Einstein searched for these equations essentially between early August 1912, when he moved back from Prague to Zurich, and November 25, 1915, when he published them in their final form. We shall give a heuristic argument for the form of the field equations, which should not be mistaken for a derivation, and later show that these equations follow from a suitable Lagrangian.

First, however, we shall investigate into the physical role of the curvature tensor. As we have seen, gravitational fields can locally be transformed away by choosing normal coordinates, in which the Christoffel symbols (the connection coefficients) all vanish. by its nature, this does not hold for the curvature tensor which, as we shall see, is related to the gravitational *tidal field*. thus, in this sense, the gravitational tidal field has a more profound physical significance as the gravitational field itself.

Let us begin with a *congruence of geodesics*. This is a bundle of time-like geodesics imagined to run through every point of a small environment $U \subset M$ of a point $p \in M$.

Let the geodesics be parameterised by the proper time τ along them, and introduce a curve γ transversal to the congruence, parameterised by a curve parameter λ. *Transversal* means that the curve γ is nowhere parallel to the congruence.

When normalised, the tangent vector to one of the time-like geodesics can be written as

$$u = \partial_\tau \quad \text{with} \quad \langle u, u \rangle = -1 \; . \tag{6.1}$$

Since it is tangent to a geodesic, it is parallel-transported along the geodesic,

$$\nabla_u u = 0 \; . \tag{6.2}$$

Similarly, we introduce a unit tangent vector v along the curve γ,

$$v = \dot{\gamma} = \partial_\lambda \; . \tag{6.3}$$

Since the partial derivatives with respect to the curve parameters τ and λ commute, so do the vectors u and v, and thus v is Lie-transported (or Lie-invariant) along u,

$$0 = [u, v] = \mathcal{L}_u v \; . \tag{6.4}$$

Now, we project v on u and define a vector n which is perpendicular to u,

$$n = v + \langle v, u \rangle u \; , \tag{6.5}$$

which does indeed satisfy $\langle n, u \rangle = 0$ because of $\langle u, u \rangle = -1$. This vector is also Lie-transported along u, as we shall verify now.

First, we have

$$\begin{aligned} \mathcal{L}_u n &= [u, n] = [u, v] + [u, \langle v, u \rangle u] \\ &= u(\langle v, u \rangle)u = (\partial_\tau \langle v, u \rangle)u \; , \end{aligned} \tag{6.6}$$

where (6.4) was used in the first step. Since $\langle u, u \rangle = -1$, we have

$$0 = \partial_\lambda \langle u, u \rangle = v\langle u, u \rangle = 2\langle \nabla_v u, u \rangle \; , \tag{6.7}$$

if we use the Ricci identity (3.67).

But the vanishing commutator between u and v and the symmetry of the connection imply $\nabla_v u = \nabla_u v$, and thus

$$\partial_\tau \langle u, v \rangle = u\langle u, v \rangle = \langle \nabla_u u, v \rangle + \langle u, \nabla_u v \rangle = \langle u, \nabla_v u \rangle = 0 \; , \tag{6.8}$$

where the Ricci identity was used in the second step, the geodesic property (6.2) in the third, and (6.7) in the last. Returning to (6.6), this proves that n is Lie-transported,

$$\mathcal{L}_u n = 0 \; . \tag{6.9}$$

The perpendicular separation vector between neighbouring geodesics of the congruence is thus Lie-invariant along the congruence.

6.1.2 The curvature tensor and the tidal field

Now, we take the second derivative of v along u,

$$\nabla_u^2 v = \nabla_u \nabla_u v = \nabla_u \nabla_v u = (\nabla_u \nabla_v - \nabla_v \nabla_u) u \ , \tag{6.10}$$

where we have used again that u and v commute and that u is a geodesic. With $[u, v] = 0$, the curvature (3.49) applied to u and v reads

$$R(u, v)u = (\nabla_u \nabla_v - \nabla_v \nabla_u) u \ , \tag{6.11}$$

and thus we see that the second derivative of v along u is determined by the curvature tensor through the *Jacobi equation*

$$\nabla_u^2 v = R(u, v)u \ . \tag{6.12}$$

Let us now use this result to find a similar equation for n. First, we observe that

$$\nabla_u n = \nabla_u v + \nabla_u (\langle v, u \rangle u) = \nabla_u v + (\partial_\tau \langle v, u \rangle) u = \nabla_u v \tag{6.13}$$

because of (6.8). Thus $\nabla_u^2 n = \nabla_u^2 v$ and

$$\nabla_u^2 n = R(u, v)u \ . \tag{6.14}$$

Next, we use

$$R(u, n) = R(u, v + \langle u, v \rangle u) = R(u, v) + \langle u, v \rangle R(u, u) = R(u, v) \tag{6.15}$$

to find

$$\nabla_u^2 n = R(u, n)u \ . \tag{6.16}$$

This is called the *equation of geodesic deviation* because it describes directly how the separation between neighbouring geodesics evolves along the geodesics according to the curvature.

Finally, let us introduce a coordinate basis $\{e_i\}$ in the subspace perpendicular to u which is parallel-transported along u. Since n is confined to that subspace, we can write

$$n = n^i e_i \tag{6.17}$$

and thus

$$\nabla_u n = (u n^i) e_i + (n^i \nabla_u) e_i = \frac{\mathrm{d}n^i}{\mathrm{d}\tau} e_i \ . \tag{6.18}$$

Since u is normalised and perpendicular to the space spanned by the triad $\{e_i\}$, we can form a tetrad from the e_i and $e_0 = u$. The equation of geodesic deviation (6.16) then implies

$$\frac{\mathrm{d}^2 n^i}{\mathrm{d}\tau^2} e_i = R(e_0, n^j e_j) e_0 = n^j R(e_0, e_j) e_0 = n^j R^i{}_{00j} \, e_i \ . \tag{6.19}$$

Thus, defining a matrix K by

$$\frac{\mathrm{d}^2 n^i}{\mathrm{d}\tau^2} = R^i{}_{00j}\, n^j \equiv K^i_j n^j \,, \tag{6.20}$$

we can write (6.19) in matrix form

$$\frac{\mathrm{d}^2 \vec{n}}{\mathrm{d}\tau^2} = K\vec{n} \,. \tag{6.21}$$

Note that K is symmetric because of the symmetries (3.79) of the curvature tensor.

Moreover, the trace of K is

$$\mathrm{tr}K = R^i{}_{00i} = R^\mu{}_{00\mu} = -R_{00} = -R_{\mu\nu}u^\mu u^\nu \,, \tag{6.22}$$

where we have inserted $R^0{}_{000} = 0$ and the definition of the Ricci tensor (3.55).

Let us now compare this result to the motion of test bodies in Newtonian theory. At two neighbouring points $\vec{x}$ and $\vec{x} + \vec{n}$, we have the equations of motion

$$\ddot{x}^i = -\left.(\partial_i \Phi)\right|_{\vec{x}} \tag{6.23}$$

and, to first order in a Taylor expansion,

$$\ddot{x}^i + \ddot{n}^i = -\left.(\partial_i \Phi)\right|_{\vec{x}+\vec{n}} \approx -\left.(\partial_i \Phi)\right|_{\vec{x}} - \left.(\partial_i \partial_j \Phi)\right|_{\vec{x}} n^j \,. \tag{6.24}$$

Subtracting (6.23) from (6.24) yields the evolution equation for the separation vector

$$\ddot{n}^i = -(\partial_i \partial_j \Phi) n^j \,, \tag{6.25}$$

which we can compare to the result (6.21).

Taking into account that

$$\frac{\mathrm{d}^2 n^i}{\mathrm{d}\tau^2} = \frac{\ddot{n}^i}{c^2} = -\left(\frac{\partial_i \partial_j \Phi}{c^2}\right) n^j \,, \tag{6.26}$$

we see that the matrix K in Newton's theory is

$$K^{(\mathrm{N})}_{ij} = -\frac{\partial_i \partial_j \Phi}{c^2} \,, \tag{6.27}$$

and its trace is

$$\mathrm{tr}K^{(\mathrm{N})} = -\frac{\vec{\nabla}^2 \Phi}{c^2} = -\frac{\Delta \Phi}{c^2} \,, \tag{6.28}$$

i.e. the negative Laplacian of the Newtonian potential, scaled by the squared light speed.

Thus, we observe the correspondences

$$R^i{}_{0j0} \leftrightarrow \frac{\partial_i \partial_j \Phi}{c^2} \tag{6.29}$$

and

$$R_{\mu\nu}u^\mu u^\nu \leftrightarrow \frac{\vec{\nabla}^2\Phi}{c^2}\ . \tag{6.30}$$

this confirms the assertion that the curvature represents the gravitational *tidal* field, describing the *relative* accelerations of freely-falling test bodies; (6.29) and (6.30) will provide useful guidance in guessing the field equations.

6.2 Einstein's field equations

6.2.1 Heuristic "derivation"

We start from the field equation from Newtonian gravity, i.e. the Poisson equation

$$4\pi G\rho = \vec{\nabla}^2\Phi = -c^2 \mathrm{tr}K^{(\mathrm{N})}\ . \tag{6.31}$$

The density ρ can be expressed by the energy-momentum tensor. For an ideal fluid, we have

$$T^{\mu\nu} = (\rho c^2 + p)u^\mu u^\nu + pg^{\mu\nu}\ , \tag{6.32}$$

from which we find because of $\langle u, u\rangle = -1$

$$T_{\mu\nu}u^\mu u^\nu = \rho c^2\ . \tag{6.33}$$

Moreover, its trace is

$$\mathrm{tr}T \equiv T = T^\mu_\mu = -\rho c^2 + 3p \approx -\rho c^2 \tag{6.34}$$

because $p \ll \rho c^2$ under Newtonian conditions (the pressure is much less than the *energy* density).

Hence, let us take a constant λ, put

$$\rho c^2 = \lambda T_{\mu\nu}u^\mu u^\nu + (1-\lambda)g_{\mu\nu}u^\mu u^\nu T \tag{6.35}$$

and insert this into the field equation (6.31), using (6.22) for the trace of K. We thus obtain

$$R_{\mu\nu}u^\mu u^\nu = \frac{4\pi G}{c^4}\left(\lambda T_{\mu\nu} + (1-\lambda)g_{\mu\nu}T\right)u^\mu u^\nu\ . \tag{6.36}$$

Since this equation should hold for any observer and thus for arbitrary four-velocities u, we find

$$R_{\mu\nu} = \frac{4\pi G}{c^4}\left(\lambda T_{\mu\nu} + (1-\lambda)g_{\mu\nu}T\right)\ , \tag{6.37}$$

where $\lambda \in \mathbb{R}$ remains to be determined.

We take the trace of (6.37), obtain

$$R = \frac{4\pi G}{c^4}\left(\lambda T + 4(1-\lambda)T\right) \tag{6.38}$$

and combine this with (6.37) to assemble the Einstein tensor (3.88),

$$\begin{aligned} G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \\ &= \frac{4\pi G}{c^4}\left(\lambda T_{\mu\nu} - \frac{2-\lambda}{2}g_{\mu\nu}T\right) . \end{aligned} \tag{6.39}$$

We have seen in (3.89) that the Einstein tensor G satisfies the contracted Bianchi identity

$$\nabla_\nu G^{\mu\nu} = 0 . \tag{6.40}$$

Likewise, the divergence of the energy-momentum tensor must vanish in order to guarantee local energy-momentum conservation,

$$\nabla_\nu T^{\mu\nu} = 0 . \tag{6.41}$$

These two conditions are generally compatible with (6.39) only if we choose $\lambda = 2$, from which we find the field equations either in the form

$$G_{\mu\nu} = \frac{8\pi G}{c^4}\,T_{\mu\nu} \tag{6.42}$$

or, from (6.37),

$$R_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) . \tag{6.43}$$

These are Einstein's field equations as he published them on November 25, 1915.

6.2.2 Uniqueness

In the appropriate limit, Einstein's equations satisfy Newton's theory by construction and are thus *one possible* set of gravitational field equations. A remarkable theorem due to Lovelock states that they are the *only possible* field equations under certain very general conditions.

It is reasonable to assume that the gravitational field equations can be written in the form

$$\mathcal{D}[g] = T \ , \tag{6.44}$$

where the tensor $\mathcal{D}[g]$ is a functional of the metric tensor g and T is the energy-momentum tensor. This equation says that the source of the gravitational field is assumed to be expressed by the energy-momentum tensor of all matter and energy contained in space-time. Now, Lovelock's theorem says, if $\mathcal{D}[g]$ depends on g and its derivatives only up to second order, then it must be a linear combination of the Einstein and metric tensors,

$$\mathcal{D}[g] = \alpha G + \beta g \; , \tag{6.45}$$

with $\alpha, \beta \in \mathbb{R}$. This absolutely remarkable theorem says that G must be of the form

$$G = \kappa T + \Lambda g \; , \tag{6.46}$$

with κ and Λ are constants. The correct Newtonian limit then requires that $\kappa = 8\pi G c^{-4}$, and Λ is the "cosmological constant" introduced by Einstein for reasons which will become clear later.

6.2.3 Lagrangian formulation

The remarkable uniqueness of the tensor $\mathcal{D}$ shown by Lovelock's theorem lets us suspect that a Lagrangian formulation of general relativity should be possible starting from a scalar constructed from $\mathcal{D}$, most naturally its contraction $\mathcal{D}^{\mu}_{\mu}$, which is simply proportional to the Ricci scalar $R = R^{\mu}_{\mu}$ if we ignore the cosmological term proportional to Λ for now.

Writing down the action, we have to take into account that we require an invariant volume element, which we obtain from the canonical volume form η introduced in (5.70). Then, according to (5.93) and (5.94), we can represent volume integrals as

$$\int_M \eta = \int_U \sqrt{-g}\,\mathrm{d}^4 x \; , \tag{6.47}$$

where $\sqrt{-g}$ is the square root of the determinant of g, and $U \subset M$ admits a single chart. Recall that, if we need integrate over a domain covered by multiple charts, a sum over the domains of the individual charts is understood.

Thus, we suppose that the action of general relativity in a compact region $D \subset M$ with smooth boundary ∂D is

$$S_{\mathrm{GR}}[g] = \int_D R[g]\eta = \int_D R[g]\sqrt{-g}\,\mathrm{d}^4 x \; . \tag{6.48}$$

Working out the variation of this action with respect to the metric components $g_{\mu\nu}$, we write explicitly

$$R = g^{\mu\nu} R_{\mu\nu} \tag{6.49}$$

and thus

$$\begin{aligned} \delta S_{\mathrm{GR}} &= \int_D \delta\left(g^{\mu\nu} R_{\mu\nu}\sqrt{-g}\right)\mathrm{d}^4 x \qquad (6.50)\\ &= \int_D \delta R_{\mu\nu}\, g^{\mu\nu}\sqrt{-g}\,\mathrm{d}^4 x + \int_D R_{\mu\nu}\delta\left(g^{\mu\nu}\sqrt{-g}\right)\mathrm{d}^4 x \; . \end{aligned}$$

We evaluate the variation of the Ricci tensor first, using its expression (3.55) in terms of the Christoffel symbols. Matters simplify considerably if we introduce normal coordinates, which allow us to ignore the terms in (3.55) which are quadratic in the Christoffel symbols. Then, the Ricci tensor specialises to

$$R_{\mu\nu} = \partial_\alpha \Gamma^\alpha{}_{\mu\nu} - \partial_\nu \Gamma^\alpha{}_{\mu\alpha} \ , \tag{6.51}$$

and its variation is

$$\delta R_{\mu\nu} = \partial_\alpha (\delta\Gamma^\alpha{}_{\mu\nu}) - \partial_\nu (\delta\Gamma^\alpha{}_{\mu\alpha}) \ . \tag{6.52}$$

Although the Christoffel symbols do not transform as tensors, their variation does, as the transformation law (3.6) shows. Thus, we can locally replace the partial by the covariant derivatives and write

$$\delta R_{\mu\nu} = \nabla_\alpha (\delta\Gamma^\alpha{}_{\mu\nu}) - \nabla_\nu (\delta\Gamma^\alpha{}_{\mu\alpha}) \ , \tag{6.53}$$

which is a tensor identity, called the *Palatini identity*, and thus holds in all coordinate systems everywhere. It implies

$$g^{\mu\nu}\delta R_{\mu\nu} = \nabla_\alpha \left(g^{\mu\nu}\delta\Gamma^\alpha{}_{\mu\nu} - g^{\mu\alpha}\delta\Gamma^\nu{}_{\mu\nu} \right) \ , \tag{6.54}$$

where the indices α and ν were swapped in the last term. Thus, the variation of the Ricci tensor, contracted with the metric, can be expressed by the divergence of a vector W,

$$g^{\mu\nu}\delta R_{\mu\nu} = \nabla_\alpha W^\alpha \ , \tag{6.55}$$

whose components W^α are defined by the term in parentheses on the right-hand side of (6.54).

From Cramer's rule in the form (4.55), we see that

$$\delta g = \frac{\partial g}{\partial g_{\mu\nu}} \delta g_{\mu\nu} = g g^{\mu\nu} \delta g_{\mu\nu} \ . \tag{6.56}$$

Moreover, since $g^{\mu\nu} g_{\mu\nu} = \text{const.} = 4$, we conclude

$$g^{\mu\nu}\delta g_{\mu\nu} = -g_{\mu\nu}\delta g^{\mu\nu} \ . \tag{6.57}$$

Using these expressions, we obtain for the variation of $\sqrt{-g}$

$$\delta\sqrt{-g} = -\frac{\delta g}{2\sqrt{-g}} = \frac{g g_{\mu\nu}\delta g^{\mu\nu}}{2\sqrt{-g}} = -\frac{\sqrt{-g}}{2} g_{\mu\nu}\delta g^{\mu\nu} \ , \tag{6.58}$$

or, in terms of the canonical volume form η,

$$\delta\eta = -\frac{1}{2} g_{\mu\nu}\delta g^{\mu\nu}\,\eta = \frac{1}{2} g^{\mu\nu}\delta g_{\mu\nu}\,\eta \ . \tag{6.59}$$

Now, we put (6.58) and (6.55) back into (6.50) and obtain

$$\begin{aligned}\delta S_{\mathrm{GR}} &= \int_D \nabla_\alpha W^\alpha \eta + \int_D \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}\right) \delta g^{\mu\nu} \eta \\ &= \int_D G_{\mu\nu} \delta g^{\mu\nu} \eta + \int_D \nabla_\alpha W^\alpha \eta \stackrel{!}{=} 0 \; . \end{aligned} \tag{6.60}$$

Varying $g^{\mu\nu}$ only in the interior of D, the divergence term vanishes by Gauß' theorem, and admitting arbitrary variations $\delta g^{\mu\nu}$ implies

$$G_{\mu\nu} = 0 \; . \tag{6.61}$$

Including the cosmological constant and using (6.58) once more, we see that Einstein's vacuum equations, $G - \Lambda g = 0$, follow from the variational principle

$$\delta \int_D (R + 2\Lambda) \eta = 0 \; . \tag{6.62}$$

The complete Einstein equations including the energy momentum tensor cannot yet be obtained here because no matter or energy contribution to the Lagrange density has been included yet into the action.

6.2.4 The energy-momentum tensor

In order to include matter (where "matter" summarises all kinds of matter and non-gravitational energy) into the field equations, we assume that the matter fields ψ are described by a Lagrangian $\mathcal{L}$ depending on ψ, its gradient $\nabla\psi$ and the metric g,

$$\mathcal{L}(\psi, \nabla\psi, g) \; , \tag{6.63}$$

where ψ may be a scalar or tensor field.

The field equations are determined by the variational principle

$$\delta \int_D \mathcal{L} \eta = 0 \; , \tag{6.64}$$

where the Lagrangian is varied with respect to the fields ψ and their derivatives $\nabla\psi$. Thus,

$$\delta \int_D \mathcal{L} \eta = \int_D \left(\frac{\partial \mathcal{L}}{\partial \psi} \delta\psi + \frac{\partial \mathcal{L}}{\partial \nabla\psi} \delta\nabla\psi \right) \eta = 0 \; . \tag{6.65}$$

As usual, we can express the second term by the difference

$$\frac{\partial \mathcal{L}}{\partial \nabla\psi} \delta\nabla\psi = \nabla \cdot \left(\frac{\partial \mathcal{L}}{\partial \nabla\psi} \delta\psi \right) - \left(\nabla \cdot \frac{\partial \mathcal{L}}{\partial \nabla\psi} \right) \delta\psi \; , \tag{6.66}$$

of which the first term is a divergence which vanishes according to Gauß' theorem upon volume integration. Combining (6.66) with (6.65), and allowing arbitrary variations $\delta\psi$ of the matter fields, then yields the Euler-Lagrange equations for the matter fields,

$$\frac{\partial\mathcal{L}}{\partial\psi} - \nabla\cdot\frac{\partial\mathcal{L}}{\partial\nabla\psi} = 0\ . \tag{6.67}$$

To give an example, suppose we describe a neutral scalar field ψ with the Lagrangian

$$\mathcal{L} = -\frac{1}{2}\langle\nabla\psi,\nabla\psi\rangle - \frac{m^2}{2}\psi^2 = -\frac{1}{2}\nabla_\mu\psi\nabla^\mu\psi - \frac{m^2}{2}\psi^2\ , \tag{6.68}$$

where $m\psi^2/2$ is a self-interaction potential with a constant parameter m. The Euler-Lagrange equations then imply the field equations

$$\left(-\Box + m^2\right)\psi = \left(-\nabla_\mu\nabla^\mu + m^2\right)\psi = 0\ , \tag{6.69}$$

which can be interpreted as the Klein-Gordon equation for a particle with mass m.

Similarly, we can vary the action with respect to the *metric*, which requires care because the Lagrangian may depend on the metric explicitly and implicitly through the covariant derivatives $\nabla\psi$ of the fields, and the canonical volume form η depends on the metric as well because of (6.59). Thus,

$$\delta\int_D\mathcal{L}\eta = \int_D\left[(\delta\mathcal{L})\eta + \mathcal{L}\delta\eta\right] = \int_D\left(\delta\mathcal{L} - \frac{1}{2}g_{\mu\nu}\mathcal{L}\delta g^{\mu\nu}\right)\eta\ . \tag{6.70}$$

If the Lagrangian does not implicitly depend on the metric, we can write

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial g^{\mu\nu}}\delta g^{\mu\nu}\ . \tag{6.71}$$

If there is an implicit dependence on the metric, we can introduce normal coordinates to evaluate the variation of the Christoffel symbols,

$$\delta\Gamma^\alpha{}_{\mu\nu} = \frac{1}{2}g^{\alpha\sigma}\left(\nabla_\nu\delta g_{\sigma\mu} + \nabla_\mu\delta g_{\sigma\nu} - \nabla_\sigma\delta g_{\mu\nu}\right)\ , \tag{6.72}$$

which is a tensor, as remarked above, whence (6.72) holds in all coordinate frames everywhere. The derivatives can then be moved away from the variations of the metric by partial integration, and expressions proportional to $\delta g^{\mu\nu}$ remain.

Thus, it is possible to write the variation of the action with respect to the *metric* in the form

$$\delta\int_D\mathcal{L}\eta = -\frac{1}{2}\int_D T_{\mu\nu}\delta g^{\mu\nu}\eta\ , \tag{6.73}$$

in which the tensor T is the energy-momentum tensor. If there are no implicit dependences on the metric, its components are

$$T_{\mu\nu} = -2\frac{\partial\mathcal{L}}{\partial g^{\mu\nu}} + \mathcal{L}g_{\mu\nu} \ . \tag{6.74}$$

Let us show by an example that this identification does indeed make sense. We start from the Lagrangian of the free electromagnetic field,

$$\mathcal{L} = -\frac{1}{16\pi}F^{\alpha\beta}F_{\alpha\beta} = -\frac{1}{16\pi}F_{\alpha\beta}F_{\gamma\delta}g^{\alpha\gamma}g^{\beta\delta} \ . \tag{6.75}$$

We know from (4.23) that the covariant derivatives in the field tensor F can be replaced by partial derivatives, thus there is no implicit dependence on the metric. Then, the variation $\delta\mathcal{L}$ is

$$\delta\mathcal{L} = -\frac{1}{8\pi}F_{\mu\alpha}F_{\nu\beta}g^{\alpha\beta}\delta g^{\mu\nu} \tag{6.76}$$

With (6.70), this implies

$$\delta\int_D \mathcal{L}\eta = \frac{1}{8\pi}\int_D\left(F_{\mu\alpha}F^{\alpha}{}_{\nu} + \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}g_{\mu\nu}\right)\delta g^{\mu\nu}\eta \tag{6.77}$$

and, from (6.73), the familiar energy-momentum tensor

$$T^{\mu\nu} = \frac{1}{4\pi}\left(F^{\mu\lambda}F^{\nu}{}_{\lambda} - \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right) \tag{6.78}$$

of the electromagnetic field.

Therefore, Einstein's field equations *and* the matter equations follow from the variational principle

$$\delta\int_D\left(\frac{Rc^4}{16\pi G} + \frac{\Lambda c^4}{8\pi G} + \mathcal{L}\right)\eta = 0 \tag{6.79}$$

Since, as we have seen before, the variation of the first two terms yields $G - \Lambda g$, and the variation of the third term yields minus one-half of the energy-momentum tensor, $-T/2$. In components, the variation yields

$$G_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} + \Lambda g_{\mu\nu} \ . \tag{6.80}$$

This shows that the cosmological constant can be considered as part of the energy-momentum tensor,

$$T_{\mu\nu} \to T_{\mu\nu} + T^{\Lambda}_{\mu\nu} \ , \quad T^{\Lambda}_{\mu\nu} \equiv \frac{\Lambda c^4}{8\pi G}g_{\mu\nu} \ . \tag{6.81}$$

6.2.5 Equations of motion

Suppose space-time is filled with an ideal fluid whose pressure p can be neglected compared to the energy density ρc^2. Then, the energy-momentum tensor (6.32) can be written as

$$T^{\mu\nu} = \rho c^2 \, u^\mu u^\nu \; . \tag{6.82}$$

Conservation of the fluid can be expressed in the following way: the amount of matter contained in a domain D of space-time must remain the same, even if the domain is mapped into another domain $\phi_t(D)$ by the flow ϕ_t of the vector field u with the time t. Thus

$$\int_D \rho\eta = \int_{\phi_t(D)} \rho\eta \; . \tag{6.83}$$

This expression just says that, if the domain D is mapped along the flow lines of the fluid flow, it will encompass a constant amount of material independent of time t.

Now, we can use the pull-back to write

$$\int_{\phi_t(D)} \rho\eta = \int_D \phi_t^*(\rho\eta) \; , \tag{6.84}$$

and take the limit $t \to 0$ to see the equivalence of (6.83) and (6.84) with the vanishing Lie derivative of $\rho\eta$ along u,

$$\mathcal{L}_u(\rho\eta) = 0 \; . \tag{6.85}$$

The Leibniz rule (5.14) yields

$$(\mathcal{L}_u\rho)\eta + \rho\mathcal{L}_u\eta = 0 \; . \tag{6.86}$$

Due to (5.15), the first term yields

$$(\mathcal{L}_u\rho)\eta = (u\rho)\eta = (u^i\partial_i\rho)\eta = (u^i\nabla_i\rho)\eta = (\nabla_u\rho)\eta \; . \tag{6.87}$$

For the second term, we can apply equation (5.30) for the components of the Lie derivative of a rank-$(0,4)$ tensor, and use the antisymmetry of η to see that

$$\mathcal{L}_u\eta = (\nabla_\mu u^\mu)\eta = (\nabla\cdot u)\eta \; . \tag{6.88}$$

Accordingly, (6.86) can be written as

$$0 = (\nabla_u\rho + \rho\nabla\cdot u)\,\eta = \nabla\cdot(\rho u)\eta \; , \tag{6.89}$$

or

$$\nabla_\mu(\rho u^\mu) = 0 \; . \tag{6.90}$$

At the same time, the divergence of T must vanish, hence

$$0 = \nabla_\nu T^{\mu\nu} = \nabla_\nu \left(\rho u^\mu u^\nu\right) = \nabla_\nu \left(\rho u^\nu\right) u^\mu + \rho u^\nu \nabla_\nu u^\mu \ . \tag{6.91}$$

The first term vanishes because of (6.90), and the second implies

$$u^\nu \nabla_\nu u^\mu = 0 \quad \Leftrightarrow \quad \nabla_u u = 0 \ . \tag{6.92}$$

In other words, the flow lines have to be geodesics. For an ideal fluid, the equation of motion thus follows directly from the vanishing divergence of the energy-momentum tensor, which is required in general relativity by the contracted Bianchi identity (3.89).

Chapter 7

Weak Gravitational Fields

7.1 Linearised Theory of Gravity

7.1.1 The linearised field equations

We begin our study of solutions for the field equations with situations in which the metric is almost Minkowskian, writing

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \ , \tag{7.1}$$

where $h_{\mu\nu}$ is considered as a perturbation of the Minkowski metric $\eta_{\mu\nu}$ such that

$$|h_{\mu\nu}| \ll 1 \ . \tag{7.2}$$

This condition is excellently satisfied e.g. in the Solar System, where

$$|h_{\mu\nu}| \approx \frac{\Phi}{c^2} \approx 10^{-6} \ . \tag{7.3}$$

Note that small perturbations of the *metric* do not necessarily imply small perturbations of the *matter density*, as the Solar System illustrates. Also, the metric perturbations may change rapidly in time.

First, we write down the Christoffel symbols for this kind of metric. Starting from (3.72) and ignoring quadratic terms in $h_{\mu\nu}$, we can write

$$\begin{aligned} \Gamma^{\alpha}{}_{\mu\nu} &= \frac{1}{2}\eta^{\alpha\beta}\left(\partial_\nu h_{\mu\beta} + \partial_\mu h_{\beta\nu} - \partial_\beta h_{\mu\nu}\right) \\ &= \frac{1}{2}\left(\partial_\nu h^{\alpha}_{\mu} + \partial_\mu h^{\alpha}_{\nu} - \partial^{\alpha} h_{\mu\nu}\right) \ . \end{aligned} \tag{7.4}$$

Next, we can ignore the terms quadratic in the Christoffel symbols in the components of the Ricci tensor (3.54) and find

$$R_{\mu\nu} = \partial_\lambda \Gamma^{\lambda}{}_{\mu\nu} - \partial_\nu \Gamma^{\lambda}{}_{\lambda\mu} \ . \tag{7.5}$$

Inserting (7.4) yields

$$\begin{aligned} R_{\mu\nu} &= \frac{1}{2}\left(\partial_\lambda\partial_\nu h^\lambda_\mu + \partial_\lambda\partial_\mu h^\lambda_\nu - \partial_\lambda\partial^\lambda h_{\mu\nu} - \partial_\mu\partial_\nu h^\lambda_\lambda\right) \\ &= \frac{1}{2}\left(\partial_\lambda\partial_\nu h^\lambda_\mu + \partial_\lambda\partial_\mu h^\lambda_\nu - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h\right) , \end{aligned} \tag{7.6}$$

where we have introduced the d'Alembert operator and abbreviated the trace of the metric perturbation,

$$\Box = \partial_\lambda\partial^\lambda \ , \quad h \equiv h^\lambda_\lambda \ . \tag{7.7}$$

The Ricci scalar is the contraction of $R_{\mu\nu}$,

$$R = \partial_\lambda\partial_\mu h^{\lambda\mu} - \Box h \ , \tag{7.8}$$

and the Einstein tensor is

$$\begin{aligned} G_{\mu\nu} &= \frac{1}{2}\Big(\partial_\lambda\partial_\nu h^\lambda_\mu + \partial_\lambda\partial_\mu h^\lambda_\nu - \eta_{\mu\nu}\partial_\lambda\partial_\sigma h^{\lambda\sigma} \\ &- \partial_\mu\partial_\nu h - \Box h_{\mu\nu} + \eta_{\mu\nu}\Box h\Big) \ . \end{aligned} \tag{7.9}$$

Neglecting terms of order $|h_{\mu\nu}|^2$, the contracted Bianchi identity reduces to

$$\partial_\nu G^{\mu\nu} = 0 \ , \tag{7.10}$$

which, together with the field equations, implies

$$\partial_\nu T^{\mu\nu} = 0 \ . \tag{7.11}$$

One could now insert the Minkowski metric in $T^{\mu\nu}$, search for a first solution $h^{(0)}_{\mu\nu}$ of the linearised field equations and iterate replacing $\eta_{\mu\nu}$ by $\eta_{\mu\nu} + h^{(0)}_{\mu\nu}$ in $T^{\mu\nu}$ to find a corrected solution $h^{(1)}_{\mu\nu}$, and so forth. This procedure is useful as long as the back-reaction of the metric on the energy-momentum tensor is small.

If we specialise (7.11) for pressure-less dust and insert (6.82), we find the equation of motion

$$u^\nu\partial_\nu u^\mu = 0 \ , \tag{7.12}$$

which means that the fluid elements follow straight lines.

The field equations simplify considerably when we substitute

$$\gamma_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\, h \tag{7.13}$$

for $h_{\mu\nu}$. Since $\gamma \equiv \gamma^\mu_\mu = -h$, we can solve (7.13) for $h_{\mu\nu}$ and insert

$$h_{\mu\nu} = \gamma_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\,\gamma \tag{7.14}$$

into (7.9) to obtain the linearised field equations

$$\partial^\lambda\partial_\nu\gamma_{\lambda\mu} + \partial^\lambda\partial_\mu\gamma_{\lambda\nu} - \eta_{\mu\nu}\partial^\lambda\partial^\sigma\gamma_{\lambda\sigma} - \Box\gamma_{\mu\nu} = \frac{16\pi G}{c^4}\,T_{\mu\nu} \ . \tag{7.15}$$

7.1.2 Gauge transformations

Let ϕ be a diffeomorphism of M, then the metric is physically equivalent to the pulled-back metric $\phi^* g$. In particular, this holds for a one-parameter group ϕ_t of diffeomorphisms which is the (local) flow of some vector field v. By the definition of the Lie derivative, we have, to first order in t,

$$\phi^* g = g + t\mathcal{L}_v g \ . \tag{7.16}$$

Now, set $g = \eta + h$ and define the infinitesimal vector $\xi \equiv tv$. Then, the transformation (7.16) implies

$$h \to \phi^* h = h + t\mathcal{L}_v \eta + t\mathcal{L}_v h = h + \mathcal{L}_\xi \eta + \mathcal{L}_\xi h \ . \tag{7.17}$$

For weak fields, the third term on the right-hand side can be neglected. Using (5.31), we see that

$$(\mathcal{L}_\xi \eta)_{\mu\nu} = \eta_{\lambda\nu}\partial_\mu \xi^\lambda + \eta_{\mu\lambda}\partial_\nu \xi^\lambda = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \ . \tag{7.18}$$

Thus, the weak metric perturbation $h_{\mu\nu}$ admits the *gauge transformation*

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \ . \tag{7.19}$$

which transforms the tensor $\gamma_{\mu\nu}$ as

$$\gamma_{\mu\nu} \to \gamma_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu}\partial_\lambda \xi^\lambda \ . \tag{7.20}$$

We can now arrange matters to enforce the *Hilbert gauge*

$$\partial_\nu \gamma^{\mu\nu} = 0 \ . \tag{7.21}$$

The gauge transformation (7.20) shows that the divergence of $\gamma^{\mu\nu}$ is transformed as

$$\partial_\nu \gamma^{\mu\nu} \to \partial_\nu \gamma^{\mu\nu} + \partial_\nu \partial^\mu \xi^\nu + \Box \xi^\mu - \partial^\mu \partial_\lambda \xi^\lambda = \partial_\nu \gamma^{\mu\nu} + \Box \xi^\mu \ , \tag{7.22}$$

such that, if (7.21) is not satisfied yet, it can be achieved by choosing for ξ^μ a solution of the inhomogeneous wave equation

$$\Box \xi^\mu = -\partial_\nu \gamma^{\mu\nu} \ , \tag{7.23}$$

which, as we know from electrodynamics, can be obtained by means of the retarded Greens function of the d'Alembert operator.

Enforcing the Hilbert gauge in this way simplifies the linearised field equation (7.15) dramatically,

$$\Box \gamma^{\mu\nu} = -\frac{16\pi G}{c^4} T^{\mu\nu} \ . \tag{7.24}$$

These equations are formally identical to Maxwell's equations in Lorentz gauge, and therefore admit the same solutions. Defining the Greens function of the d'Alembert operator $\Box$ by

$$\Box G(x, x') = \Box G(t, t', \vec{x}, \vec{x}') = 4\pi\delta_{\mathrm{D}}(t - t', \vec{x} - \vec{x}') \tag{7.25}$$

and using $x^0 = ct$ instead of t, we find the retarded Greens function

$$G(x, x') = \frac{1}{|\vec{x} - \vec{x}'|}\delta_{\mathrm{D}}\left(x^0 - x'^0 - |\vec{x} - \vec{x}'|\right) . \tag{7.26}$$

Using it, we arrive at the particular solution

$$\gamma_{\mu\nu}(x) = -\frac{4G}{c^4}\int\frac{T_{\mu\nu}\left(x^0 - |\vec{x} - \vec{x}'|, \vec{x}'\right)}{|\vec{x} - \vec{x}'|}\mathrm{d}^3x' \tag{7.27}$$

for the linearised field equation. Of course, arbitrary solutions of the homogeneous (vacuum) wave equation can be added.

Thus, similar to electrodynamics, the metric perturbation consists of the field generated by the source plus wave-like vacuum solutions propagating at the speed of light.

7.2 Nearly Newtonian gravity

7.2.1 Newtonian approximation of the metric

A nearly Newtonian source of gravity can be described by the approximations $T_{00} \gg |T_{0j}|$ and $T_{00} \gg |T_{ij}|$, which express that mean velocities are small, and the rest-mass energy dominates the kinetic energy. Then, we can also neglect retardation effects and write

$$\gamma_{00}(\vec{x}) = -\frac{4G}{c^2}\int\frac{\rho(\vec{x}')\,\mathrm{d}^3x'}{|\vec{x} - \vec{x}'|} = -4\frac{\Phi(\vec{x})}{c^2} , \tag{7.28}$$

where $\Phi(\vec{x})$ is the ordinary Newtonian gravitational potential. All other components of the metric perturbation $\gamma_{\mu\nu}$ vanish,

$$\gamma_{0j} = 0 = \gamma_{ij} . \tag{7.29}$$

Then, the full metric

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} = \eta_{\mu\nu} + \left(\gamma_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\gamma\right) \tag{7.30}$$

has the components

$$g_{00} = -\left(1 + \frac{2\Phi}{c^2}\right) , \quad g_{0j} = 0 , \quad g_{ij} = \left(1 - \frac{2\Phi}{c^2}\right)\delta_{ij} , \tag{7.31}$$

creating the line element

$$\mathrm{d}s^2 = -\left(1 + \frac{2\Phi}{c^2}\right)c^2\mathrm{d}t^2 + \left(1 - \frac{2\Phi}{c^2}\right)(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)\ . \tag{7.32}$$

Far away from the source, the monopole term in (7.28) dominates, which yields

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{rc^2}\right)c^2\mathrm{d}t^2 + \left(1 + \frac{2GM}{rc^2}\right)(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)\ . \tag{7.33}$$

7.2.2 Gravitational lensing and the Shapiro delay

Two interesting conclusions can be drawn directly from (7.32). Since light follows null geodesics, light propagation is characterised by $\mathrm{d}s^2 = 0$ or

$$\left(1 + \frac{2\Phi}{c^2}\right)c^2\mathrm{d}t^2 = \left(1 - \frac{2\Phi}{c^2}\right)\mathrm{d}\vec{x}^2\ , \tag{7.34}$$

which implies that the light speed in a (weak) gravitational field is

$$c' = \frac{|\mathrm{d}\vec{x}|}{\mathrm{d}t} = \left(1 + \frac{2\Phi}{c^2}\right)c \tag{7.35}$$

to first order in Φ.

Since $\Phi \le 0$ if normalised such that $\Phi \to 0$ at infinity, $c' \le c$, which we can express by the *index of refraction for a weak gravitational field*,

$$n = \frac{c}{c'} = 1 - \frac{2\Phi}{c^2}\ . \tag{7.36}$$

This can be used to calculate light deflection using Fermat's principle, which asserts that light follows a path along which the light-travel time between a fixed source and a fixed observer is extremal, thus

$$\delta\int \mathrm{d}t = \delta\int\frac{\mathrm{d}x}{c'} \quad \Rightarrow \quad \delta\int n(\vec{x})|\mathrm{d}\vec{x}| = 0\ . \tag{7.37}$$

Introducing a curve parameter λ, we can write $\vec{x} = \vec{x}(\lambda)$, thus $|\mathrm{d}\vec{x}| = (\dot{\vec{x}}^2)^{1/2}\mathrm{d}\lambda$ and

$$\delta\int n(\vec{x})(\dot{\vec{x}}^2)^{1/2}\,\mathrm{d}\lambda = 0\ , \tag{7.38}$$

where the overdot denotes derivation with respect to λ.

The variation leads to the Euler-Lagrange equation

$$\frac{\partial L}{\partial\vec{x}} - \frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial L}{\partial\dot{\vec{x}}} = 0 \quad \text{with} \quad L \equiv n(\vec{x})(\dot{\vec{x}}^2)^{1/2}\ . \tag{7.39}$$

Thus, we find

$$(\dot{\vec{x}}^2)^{1/2}\,\vec{\nabla}n - \frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{n\dot{\vec{x}}}{\sqrt{\dot{\vec{x}}}} = 0\,. \tag{7.40}$$

We can simplify this expression by choosing the curve parameter such that $\dot{\vec{x}}$ is a unit vector $\vec{e}$, hence

$$\vec{\nabla}n - \vec{e}\cdot\vec{\nabla}n - n\dot{\vec{e}} = 0\,. \tag{7.41}$$

The first two terms are the component of $\vec{\nabla}n$ perpendicular to $\vec{e}$, and $\dot{\vec{e}}$ is the change of direction of the tangent vector along the light ray. Thus,

$$\dot{\vec{e}} = \vec{\nabla}_\perp \ln n = -\frac{2}{c^2}\vec{\nabla}_\perp\Phi \tag{7.42}$$

to first order in Φ. The total deflection angle is obtained by integrating $\dot{\vec{e}}$ along the light path.

As a second consequence, we see that the light travel time along an infinitesimal path length $\mathrm{d}l$ is

$$\mathrm{d}t = \frac{\mathrm{d}l}{c'} = n\,\frac{\mathrm{d}l}{c} = \left(1 - \frac{2\Phi}{c^2}\right)\frac{\mathrm{d}l}{c}\,. \tag{7.43}$$

Compared to light propagation in vacuum, there is thus a time delay

$$\Delta(\mathrm{d}t) = \mathrm{d}t - \frac{\mathrm{d}l}{c} = -\frac{2\Phi}{c^3}\,\mathrm{d}l\,, \tag{7.44}$$

which is called the *Shapiro delay*.

7.2.3 The gravitomagnetic field

At next order in powers of c^{-1}, the current terms in the energy-momentum tensor appear, but no stresses yet. That is, we now approximate $T_{ij} = 0$ and use the field equations

$$\Box\gamma_{ij} = 0\,, \quad \Box\gamma_{0\mu} = -\frac{16\pi G}{c^4}T_{0\mu}\,. \tag{7.45}$$

Now, we set $A_\mu \equiv \gamma_{0\mu}/4$ and obtain the Maxwell-type equations

$$\Box A_\mu = -\frac{4\pi}{c^2}\,j_\mu\,, \tag{7.46}$$

where the current density $j_\mu \equiv GT_{0\mu}/c^2$ was introduced. According to our earlier result (7.28), $A_0 = -\Phi/c^2$. This similarity to electromagnetic theory naturally leads to the introduction of "electric" and "magnetic" components of the gravitational field.

Suppose now that the field is quasi-stationary, so that time derivatives of the metric $\gamma_{\mu\nu}$ can be neglected. Then, $\vec{\nabla}^2\gamma_{ij} = 0$ everywhere because $T_{ij} = 0$ was assumed, thus $\gamma_{ij} = 0$, and the potentials A_μ determine the field completely. They are

$$A_0 = -\frac{\Phi}{c^2} , \quad A_i = \frac{G}{c^4}\int \frac{T_{0i}(\vec{x}')\,\mathrm{d}^3x'}{|\vec{x}-\vec{x}'|} , \tag{7.47}$$

and the components of the metric g are, according to (7.30),

$$g_{00} = -1 + 2A_0 , \quad g_{0i} = \gamma_{0i} = 4A_i , \quad g_{ij} = (1+2A_0)\delta_{ij} . \tag{7.48}$$

The most direct approach to the equations of motion starts from the variational principle (4.5), or

$$\delta \int \left(-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu\right)^{1/2} \mathrm{d}t = 0 , \tag{7.49}$$

where the dot now denotes the derivative with respect to the coordinate time t. The radicand is

$$c^2 - 2c^2A_0 - 8c\vec{A}\cdot\vec{v} - \vec{v}^2 , \tag{7.50}$$

where we have neglected terms of order $\Phi\vec{v}^2$ since the velocities are assumed small compared to the speed of light.

Using (7.50), we can reduce the least-action principle (7.49) to the Euler-Lagrange equations with the effective Lagrangian

$$\mathcal{L} = \frac{\vec{v}^2}{2} + A_0c^2 + 4c\vec{A}\cdot\vec{v} . \tag{7.51}$$

We first find

$$\frac{\partial\mathcal{L}}{\partial\vec{v}} = \vec{v} + 4c\vec{A} \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\vec{v}} = \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} + 4c(\vec{v}\cdot\vec{\nabla})\vec{A} . \tag{7.52}$$

Using the vector identity

$$\vec{\nabla}(\vec{a}\cdot\vec{b}) = (\vec{a}\cdot\vec{\nabla})\vec{b} + (\vec{b}\cdot\vec{\nabla})\vec{a} + \vec{a}\times\vec{\nabla}\times\vec{b} + \vec{b}\times\vec{\nabla}\times\vec{a} , \tag{7.53}$$

we further obtain

$$\frac{\partial\mathcal{L}}{\partial\vec{x}} = c^2\vec{\nabla}A_0 + 4c\left[(\vec{v}\cdot\vec{\nabla})\vec{A} + \vec{v}\times(\vec{\nabla}\times\vec{A})\right] , \tag{7.54}$$

from which we obtain the equations of motion

$$\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} \equiv \vec{f} = c^2\vec{\nabla}A_0 + 4c\vec{v}\times(\vec{\nabla}\times\vec{A}) , \tag{7.55}$$

in which the (specific) force term on the right-hand side corresponds to the Lorentz force in electrodynamics.

Let us consider now a small body characterised by its density suspended in a gravitomagnetic field; "small" means that the field can be considered constant across it. It experiences the torque about its centre-of-mass

$$\begin{aligned}\vec{M} &= \int \mathrm{d}^3x\, \vec{x} \times \rho \vec{f} \\ &= -c^2 \vec{\nabla} A_0 \times \int \mathrm{d}^3x\, \vec{x}\rho + 4c \int \mathrm{d}^3x\, \vec{x} \times (\vec{j} \times \vec{B})\ ,\end{aligned} \tag{7.56}$$

where $\vec{j} = \rho\vec{v}$ is the matter current density and $\vec{B} = \vec{\nabla} \times \vec{A}$ is the gravitomagnetic field. Since the coordinates are centred on the centre-of-mass, the first term vanishes, and the second gives

$$\vec{M} = 4c \left(\int \mathrm{d}^3x\, \vec{x} \times \vec{j} \right) \times \vec{B} = 4c\vec{s} \times \vec{B}\ , \tag{7.57}$$

where $\vec{s}$ is the intrinsic angular momentum of the body, i.e. its spin.

Thus, the body's spin changes according to

$$\dot{\vec{s}} = \vec{M} = 4c\vec{s} \times \vec{B}\ . \tag{7.58}$$

Let us now orient the coordinate frame such that $\vec{B} = B\vec{e}_3$, i.e. $B_1 = 0 = B_2$. Then,

$$\dot{s}_1 = 4cBs_2\ , \quad \dot{s}_2 = -4cs_1B\ . \tag{7.59}$$

Introducing $\sigma = s_1 + \mathrm{i}s_2$ turns this into the single equation

$$\dot{\sigma} = -4cB\mathrm{i}\sigma\ , \tag{7.60}$$

which is solved by the ansatz $\sigma = \sigma_0 \exp(\mathrm{i}\omega t)$ if $\omega = -4cB$. This shows that a spinning body in a gravitomagnetic field will experience *spin precession* with the angular frequency

$$\vec{\omega} = -4c\vec{B} = -4c\vec{\nabla} \times \vec{A}\ , \tag{7.61}$$

which is called the *Lense-Thirring* effect.

7.3 Gravitational Waves

7.3.1 Polarisation states

As shown in (7.24), the linearised field equations in vacuum are

$$\Box\gamma^{\mu\nu} = 0\ , \tag{7.62}$$

if the Hilbert gauge condition (7.21) is enforced,

$$\partial_\nu\gamma^{\mu\nu} = 0\ . \tag{7.63}$$

Within the Hilbert gauge class, we can further require that the trace of $\gamma^{\mu\nu}$ vanish,

$$\gamma = \gamma^{\mu}_{\mu} = 0 \ . \tag{7.64}$$

To see this, we return to the gauge transformation (7.20), which implies

$$\gamma \to \gamma + 2\partial_\mu \xi^\mu - 4\partial_\mu \xi^\mu = \gamma - 2\partial_\mu \xi^\mu \ , \tag{7.65}$$

i.e. if $\gamma \neq 0$, we can choose the vector ξ^μ such that

$$2\partial_\mu \xi^\mu = \gamma \ . \tag{7.66}$$

Moreover, (7.22) shows that the Hilbert gauge condition remains preserved if ξ^μ satisfies the d'Alembert equation

$$\Box \xi^\mu = 0 \tag{7.67}$$

at the same time. It can be generally shown that vector fields ξ^μ can be constructed which indeed satisfy (7.66) and (7.67) at the same time. If we arrange things in this way, (7.14) shows that then $h_{\mu\nu} = \gamma_{\mu\nu}$.

All functions propagating with the light speed satisfy the d'Alembert equation (7.62). In particular, we can describe them as superpositions of plane waves

$$\gamma_{\mu\nu} = h_{\mu\nu} = \mathrm{Re}\left(\epsilon_{\mu\nu} \mathrm{e}^{\mathrm{i}\langle k,x\rangle}\right) \tag{7.68}$$

with amplitudes given by the so-called *polarisation tensor* $\epsilon_{\mu\nu}$. They satisfy the d'Alembert equation if

$$k^2 = \langle k, k\rangle = k_\mu k^\mu = 0 \ . \tag{7.69}$$

The Hilbert gauge condition then requires

$$0 = \partial_\nu h^{\mu\nu} \quad \Rightarrow \quad k_\nu \epsilon^{\mu\nu} = 0 \ , \tag{7.70}$$

and (7.64) is satisfied if the trace of $\epsilon_{\mu\nu}$ vanishes,

$$\epsilon^{\mu}_{\mu} = 0 \ . \tag{7.71}$$

The five conditions (7.70) and (7.71) imposed on the originally ten independent components of $\epsilon_{\mu\nu}$ leave five independent components. Without loss of generality, suppose the wave propagates into the positive z direction, then

$$k^\mu = (k, 0, 0, k) \ , \tag{7.72}$$

and (7.70) implies

$$\epsilon^{0\mu} = \epsilon^{3\mu} \ ; \tag{7.73}$$

specifically,

$$\epsilon^{00} = \epsilon^{30} = \epsilon^{03} = \epsilon^{33} \quad \text{and} \quad \epsilon^{01} = \epsilon^{31} \ , \quad \epsilon^{02} = \epsilon^{32} \ , \tag{7.74}$$

while (7.71) means

$$-\epsilon^{00} + \epsilon^{11} + \epsilon^{22} + \epsilon^{33} = 0 \ . \tag{7.75}$$

Since $\epsilon^{33} = \epsilon^{00}$, this last equation means

$$\epsilon^{11} + \epsilon^{22} = 0 \ . \tag{7.76}$$

Therefore, all components of $\epsilon^{\mu\nu}$ can be expressed by five of them, as follows:

$$\epsilon^{\mu\nu} = \begin{pmatrix} \epsilon^{00} & \epsilon^{01} & \epsilon^{02} & \epsilon^{00} \\ \epsilon^{01} & \epsilon^{11} & \epsilon^{12} & \epsilon^{01} \\ \epsilon^{02} & \epsilon^{12} & -\epsilon^{11} & \epsilon^{02} \\ \epsilon^{00} & \epsilon^{01} & \epsilon^{02} & \epsilon^{00} \end{pmatrix} . \tag{7.77}$$

Now, a gauge transformation belonging to a vector field

$$\xi^\mu = \mathrm{Re}\left(\mathrm{i}\epsilon^\mu \mathrm{e}^{\mathrm{i}\langle k,x\rangle}\right) \tag{7.78}$$

which keeps the metric perturbation $h_{\mu\nu}$ trace-less,

$$\partial_\mu \xi^\mu = 0 \ , \tag{7.79}$$

changes the polarisation tensor according to

$$\epsilon_{\mu\nu} \to \epsilon_{\mu\nu} + k_\mu \epsilon_\nu + k_\nu \epsilon_\mu \tag{7.80}$$

for the k vector specified in (7.72), we thus have

$$\begin{aligned} &\epsilon^{00} \to \epsilon^{00} + 2k\epsilon^0 \ , \quad \epsilon^{01} \to \epsilon^{01} + k\epsilon^1 \ , \quad \epsilon^{02} \to \epsilon^{02} + k\epsilon^2 \ , \\ &\epsilon^{11} \to \epsilon^{11} \ , \quad \epsilon^{12} \to \epsilon^{12} \ . \end{aligned} \tag{7.81}$$

The condition (7.79) implies that $k_\mu \epsilon^\mu = 0$, hence $\epsilon^0 = \epsilon^3$. We can then use (7.81) to make ϵ^{00}, ϵ^{01} and ϵ^{02} vanish, and only the gauge-invariant components ϵ^{11} and ϵ^{12} are left. Then

$$\epsilon^\mu = \frac{1}{2k}\left(-\epsilon^{00}, -2\epsilon^{01} - 2\epsilon^{02}, -\epsilon^{00}\right) \tag{7.82}$$

fixes the gauge transformation, and the polarisation tensor is reduced to

$$\epsilon^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \epsilon^{11} & \epsilon^{12} & 0 \\ 0 & \epsilon^{12} & -\epsilon^{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{7.83}$$

Thus, there are only two linearly independent polarisation states, as for light.

Under rotations about the z axis by an arbitrary angle ϕ, the polarisation tensor changes according to

$$\epsilon'^{\mu\nu} = R^\mu_\alpha R^\nu_\beta \epsilon^{\alpha\beta} \ , \tag{7.84}$$

where R is the rotation matrix with the components

$$R(\phi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & \sin\phi & 0 \\ 0 & -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{7.85}$$

Carrying out the matrix multiplication yields

$$\begin{aligned} \epsilon'^{11} &= \epsilon^{11}\cos 2\phi + \epsilon^{12}\sin 2\phi \\ \epsilon'^{12} &= -\epsilon^{11}\sin 2\phi + \epsilon^{12}\cos 2\phi \ . \end{aligned} \tag{7.86}$$

Defining $\epsilon_\pm \equiv \epsilon^{11} \mp \mathrm{i}\epsilon^{12}$, this can be written as

$$\epsilon'_\pm = \mathrm{e}^{\pm 2\mathrm{i}\phi}\epsilon_\pm \ , \tag{7.87}$$

which shows that the two polarisation states $\epsilon_\pm$ have helicity ± 2, and thus that they correspond to left and right-handed circular polarisation.

7.3.2 Generation of gravitational waves

We return to (7.27) to see how gravitational waves can be emitted. From the start, we introduce the two simplifications that the source is far away and changing with a velocity small compared to the speed-of-light. Then, we can replace the distance $|\vec{x} - \vec{x}'|$ by

$$|\vec{x} - \vec{x}'| \approx |\vec{x}| = r \tag{7.88}$$

because "far away" means that the source is small compared to its distance. Moreover, we can approximate the retarded time coordinate x^0 as follows:

$$\begin{aligned} x^0 - |\vec{x} - \vec{x}'| &= x^0 - \sqrt{(\vec{x} - \vec{x}')^2} = x^0 - \sqrt{\vec{x}^2 + \vec{x}'^2 - 2\vec{x}\cdot\vec{x}'} \\ &\approx x^0 - r + \vec{x}'\cdot\vec{e}_r \ , \end{aligned} \tag{7.89}$$

where $\vec{e}_r$ is the unit vector in radial direction. Then, we obtain

$$\gamma_{\mu\nu}(t, \vec{x}) = -\frac{4G}{rc^4}\int T_{\mu\nu}\left(t - \frac{r - \vec{x}'\cdot\vec{e}_r}{c}, \vec{x}'\right)\mathrm{d}^3x' \ . \tag{7.90}$$

Under the assumption of slow motion, we can further ignore the directional dependence of the retarted time, thus approximate $\vec{x}'\cdot\vec{e}_r = 0$, and write

$$\gamma_{\mu\nu}(t, \vec{x}) = -\frac{4G}{rc^4}\int T_{\mu\nu}\left(t - \frac{r}{c}, \vec{x}'\right)\mathrm{d}^3x' \ . \tag{7.91}$$

By means of the local conservation law $\partial_\nu T^{\mu\nu} = 0$, we can now simplify the integral on the right-hand side of (7.91):

$$\begin{aligned} 0 &= \int x^k \partial_\nu T^{\mu\nu}\mathrm{d}^3x = \frac{1}{c}\partial_t \int x^k T^{0\mu}\mathrm{d}^3x + \int x^k \partial_l T^{l\mu}\mathrm{d}^3x \\ &= \frac{1}{c}\partial_t \int x^k T^{0\mu}\mathrm{d}^3x - \int T^{l\mu}\delta^k_l \mathrm{d}^3x \ , \end{aligned} \tag{7.92}$$

where the second term on the right-hand side was partially integrated, assuming that boundary terms vanish (i.e. enclosing the source completely in the integration boundary). Thus, we see that

$$\int T^{k\mu} \mathrm{d}^3 x = \frac{1}{c} \partial_t \int x^k T^{0\mu} \mathrm{d}^3 x \,. \tag{7.93}$$

According to Gauß' theorem, we know that the volume integral over a divergence within the same surface must vanish,

$$\int \partial_j (T^{j0} x^l x^k) \mathrm{d}^3 x = 0 \,. \tag{7.94}$$

This enables us to write

$$\begin{aligned} \frac{1}{c} \partial_t \int T^{00} x^l x^k \mathrm{d}^3 x &= \int \partial_\nu (T^{\nu 0} x^l x^k) \mathrm{d}^3 x = \int T^{\nu 0} \partial_\nu (x^l x^k) \mathrm{d}^3 x \\ &= \int T^{\nu 0} (\delta^k_\nu x^l + x^k \delta^l_\nu) \mathrm{d}^3 x \\ &= \int (T^{k0} x^l + T^{l0} x^k) \mathrm{d}^3 x \,. \end{aligned} \tag{7.95}$$

Taking the partial time derivative of (7.95) yields

$$\begin{aligned} \frac{1}{2c^2} \partial_t^2 \int (T^{00} x^k x^l) \mathrm{d}^3 x &= \frac{1}{2c} \partial_t \int (T^{k0} x^l + T^{l0} x^k) \mathrm{d}^3 x \\ &= \frac{1}{2} \int (T^{kl} + T^{lk}) \mathrm{d}^3 x = \int T^{kl} \mathrm{d}^3 x \,, \end{aligned} \tag{7.96}$$

where (7.93) was used. In other words, the spatial components of the metric perturbation $\gamma_{\mu\nu}$ are

$$\begin{aligned} \gamma^{jk}(t, \vec{x}) &= -\frac{2G}{rc^6} \partial_t^2 \int T^{00} \left(t - \frac{r}{c}, \vec{x}'\right) x'^j x'^k \mathrm{d}^3 x' \\ &= -\frac{2G}{rc^4} \partial_t^2 \int \rho \left(t - \frac{r}{c}, \vec{x}'\right) x'^j x'^k \mathrm{d}^3 x' \,, \end{aligned} \tag{7.97}$$

where we have used that the T^{00} component of the energy-momentum tensor is well approximated by the matter density if the source's material is moving slowly.

The source's quadrupole tensor is

$$Q^{jk} = \int \left(3x^j x^k - r^2 \delta^{jk}\right) \rho(\vec{x}) \mathrm{d}^3 x \,, \tag{7.98}$$

which allows us to write the metric perturbation as

$$\gamma_{jk} = -\frac{2G}{3rc^4} \left[\partial_t^2 Q_{jk} \left(t - \frac{r}{c}, \vec{x}\right) + \delta_{jk} \partial_t^2 \int r'^2 \rho \left(t - \frac{r}{c}, \vec{x}'\right) \mathrm{d}^3 x' \right] \,. \tag{7.99}$$

7.3.3 Energy transport by gravitational waves

It can be shown that the energy-momentum tensor in gravitational waves propagating into the x^1-direction is

$$T^{01}_{\mathrm{GW}} = \frac{1}{32\pi G}\left\langle 2\dot{\gamma}^2_{23} + \frac{1}{2}\left(\dot{\gamma}_{22} - \dot{\gamma}_{33}\right)^2\right\rangle , \tag{7.100}$$

which can be written with the help of (7.99) as

$$T^{01}_{\mathrm{GW}} = \frac{G}{72\pi r^2}\left\langle 2\dddot{Q}^2_{23} + \frac{1}{2}\left(\dddot{Q}_{22} - \dddot{Q}_{33}\right)^2\right\rangle , \tag{7.101}$$

showing one of the rare cases of a third time derivative in physics.

The transversal quadrupole tensor is

$$Q^{\mathrm{T}} = \begin{pmatrix} Q_{22} & Q_{23} \\ Q_{32} & Q_{33} \end{pmatrix} \tag{7.102}$$

because the direction of propagation was chosen as the x^1 axis. Defining the transversal trace-free quadrupole tensor by

$$Q^{\mathrm{TT}} := Q^{\mathrm{T}} - \frac{\mathcal{I}}{2}\mathrm{tr}Q^{\mathrm{T}} = \frac{1}{2}\begin{pmatrix} Q_{22} - Q_{33} & 2Q_{23} \\ 2Q_{23} & -(Q_{22} - Q_{33}) \end{pmatrix} , \tag{7.103}$$

we see that an invariant expression for the right-hand side of (7.101) is given by

$$\mathrm{tr}(Q^{\mathrm{TT}}Q^{\mathrm{TT}}) = \frac{1}{2}\left(Q_{22} - Q_{33}\right)^2 + 2Q^2_{23} , \tag{7.104}$$

and thus the energy current in gravitational waves is

$$T^{0i}_{\mathrm{GW}} = \frac{G}{72\pi r^2}\left\langle \mathrm{tr}(Q^{\mathrm{TT}}Q^{\mathrm{TT}})\right\rangle . \tag{7.105}$$

A final integration over all directions yields Einstein's famous quadrupole formula for the gravitational-wave "luminosity",

$$L_{\mathrm{GW}} = \frac{G}{45}\left\langle \dddot{Q}_{kl}\dddot{Q}_{kl}\right\rangle . \tag{7.106}$$

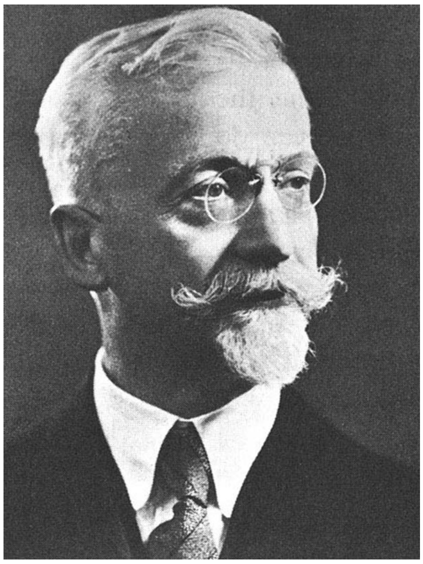
Chapter 8

The Schwarzschild Solution

8.1 Cartan's structure equations

8.1.1 The curvature and torsion forms

Karl Schwarzschild (1873–1916)



Élie Cartan (1869–1951)

This section deals with a generalisation of the connection coefficients, and the torsion and curvature tensor components, to arbitrary bases. This will prove enormously efficient in our further discussion of the Schwarzschild solution.

Let M be a differentiable manifold, $\{e_i\}$ an arbitrary basis for vector fields and $\{\theta^i\}$ an arbitrary basis for dual vector fields, or 1-forms. In analogy to the Christoffel symbols, we introduce the *connection forms* by

$$\nabla_v e_i = \omega_i^j(v) e_j \ . \tag{8.1}$$

Since $\nabla_v e_i$ is a vector, $\omega_i^j(v) \in \mathbb{R}$ is a real number, and thus $\omega_i^j \in \bigwedge^1$ is a dual vector, or a one-form.

Since, by definition (3.2) of the Christoffel symbols

$$\nabla_{\partial_k}\partial_j = \Gamma^i{}_{kj}\,\partial_i = \omega^i_j(\partial_k)\partial_i \tag{8.2}$$

in the coordinate basis $\{\partial_i\}$, we have in that particular basis,

$$\omega^i_j = \Gamma^i{}_{kj}\,\mathrm{d}x^k \ . \tag{8.3}$$

Since $\langle\theta^i, e_j\rangle$ is a constant (which is either zero or unity), we must have

$$\begin{aligned} 0 &= \nabla_v\langle\theta^i, e_j\rangle = \langle\nabla_v\theta^i, e_j\rangle + \langle\theta^i, \nabla_v e_j\rangle \\ &= \langle\nabla_v\theta^i, e_j\rangle + \langle\theta^i, \omega^k_j(v)e_k\rangle \\ &= \langle\nabla_v\theta^i, e_j\rangle + \omega^i_j(v) \ . \end{aligned} \tag{8.4}$$

Thus, we conclude

$$\nabla_v\theta^i = -\omega^i_j(v)\theta^j \tag{8.5}$$

or, without specifying the vector v,

$$\nabla\theta^i = -\theta^j \otimes \omega^i_j \ . \tag{8.6}$$

Let now $\alpha \in \bigwedge^1$ be a one-form such that $\alpha = \alpha_i\theta^i$ with arbitrary functions α_i. Then, the equations we have derived so far imply

$$\nabla_v\alpha = v(\alpha_i)\theta^i + \alpha_i\nabla_v\theta^i = \langle \mathrm{d}\alpha_i - \alpha_k\omega^k_i, v\rangle\theta^i \ , \tag{8.7}$$

where we have used the differential of the function α_i, defined in (2.34) by $\mathrm{d}\alpha_i(v) = v(\alpha_i)$, together with the notation $\langle w, v\rangle = w(v)$ for a vector v and a dual vector w. This expression can be generally written as

$$\nabla\alpha = \theta^i \otimes (\mathrm{d}\alpha_i - \alpha_k\omega^k_i) \ . \tag{8.8}$$

Similarly, for a vector field $x = x^i e_i$, we find

$$\nabla_v(x) = \langle \mathrm{d}x^i + x^k\omega^i_k, v\rangle e_i \tag{8.9}$$

or

$$\nabla x = e_i \otimes (\mathrm{d}x^i + \omega^i_k x^k) \ . \tag{8.10}$$

We are now in a position to introduce the torsion and curvature forms. By definition, the torsion $T(x, y)$ is a vector, which can be written in terms of the *torsion forms* Θ^i as

$$T(x, y) = \Theta^i(x, y)e_i \ . \tag{8.11}$$

Obviously, $\Theta^i \in \bigwedge^2$ is a two-form, such that $\Theta^i(x, y) \in \mathbb{R}$ is a real number.

In the same manner, we express the curvature by the *curvature forms* $\Omega^i_j \in \bigwedge^2$,

$$R(x, y)e_j = \Omega^i_j(x, y)e_i \ . \tag{8.12}$$

The torsion and curvature 2-forms satisfy *Cartan's structure equations*,

$$\begin{aligned} \Theta^i &= \mathrm{d}\theta^i + \omega^i_j \wedge \theta^j \\ \Omega^i_j &= \mathrm{d}\omega^i_j + \omega^i_k \wedge \omega^k_j \ . \end{aligned} \tag{8.13}$$

Their proof is straightforward.

To prove the first structure equation, we insert the definition (3.43) of the torsion to obtain as a first step

$$\begin{aligned} \Theta^i(x, y) &= \nabla_x y - \nabla_y x - [x, y] \\ &= \nabla_x(\theta^i(y)e_i) - \nabla_y(\theta^i(x)e_i) - \theta^i([x, y])e_i \ , \end{aligned} \tag{8.14}$$

where we have expanded the vectors x, y and $[x, y]$ in the basis $\{e_i\}$ according to $x = \langle x, \theta^i\rangle e_i = \theta^i(x)e_i$. Then, we continue by using the connection forms,

$$\begin{aligned}
\Theta^i(x, y) &= \nabla_x(\theta^i(y)e_i) - \nabla_y(\theta^i(x)e_i) - \theta^i([x, y])e_i \\
&= x\theta^i(y)e_i + \theta^i(y)\omega_i^j(x)e_j \\
&- y\theta^i(x)e_i - \theta^i(x)\omega_i^j(y)e_j - \theta^i([x, y])e_i \\
&= \left[x\theta^i(y) - y\theta^i(x) - \theta^i([x, y])\right] e_i \\
&+ \left[\theta^i(y)\omega_i^j(x) - \theta^i(x)\omega_i^j(y)\right] e_j \ .
\end{aligned} \tag{8.15}$$

According to (5.64), the first term can be expressed by the exterior derivative of the θ^i, and since the second term is antisymmetric in x and y, we can write this as

$$\Theta^i(x, y) = \mathrm{d}\theta^i(x, y)e_i + (\omega^i_j \wedge \theta^j)(x, y)e_i \ , \tag{8.16}$$

from which the first structure equation follows immediately.

The proof of the second structure equation proceeds similarly, using the definition (3.49) of the curvature. Thus,

$$\begin{aligned}
\Omega^i_j(x, y)e_i &= \nabla_x\nabla_y e_j - \nabla_y\nabla_x e_j - \nabla_{[x,y]}e_j \\
&= \nabla_x(\omega^i_j(y)e_i) - \nabla_y(\omega^i_j(x)e_i) - \omega^i_j([x, y])e_i \\
&= x\omega^i_j(y)e_i + \omega^i_j(y)\nabla_x e_i \\
&- y\omega^i_j(x)e_i - \omega^i_j(x)\nabla_y e_i - \omega^i_j([x, y])e_i \\
&= \left[x\omega^i_j(y) - y\omega^i_j(x) - \omega^i_j([x, y])\right] e_i \\
&+ \left[\omega^i_j(y)\omega^k_i(x) - \omega^i_j(x)\omega^k_i(y)\right] e_k \\
&= \mathrm{d}\omega^i_j(x, y)e_i + (\omega^k_i \wedge \omega^i_j)(x, y)e_k \ ,
\end{aligned} \tag{8.17}$$

which proves the second structure equation.

Now, let us use the curvature forms Ω^i_j to define tensor components $R^i{}_{jkl}$ by

$$\Omega^i_j \equiv \frac{1}{2}R^i{}_{jkl}\,\theta^k \wedge \theta^l \ , \tag{8.18}$$

whose antisymmetry in the last two indices is obvious by definition,

$$R^i{}_{jkl} = -R^i{}_{jlk} \ . \tag{8.19}$$

In an arbitrary basis $\{e_i\}$, we then have

$$\langle\theta^i, R(e_k, e_l)e_j\rangle = \langle\theta^i, \Omega^s_j(e_k, e_l)e_s\rangle = \Omega^i_j(e_k, e_l) = R^i{}_{jkl} \ . \tag{8.20}$$

Comparing this to the components of the curvature tensor in the coordinate basis $\{\partial_i\}$ given by (3.54) shows that the functions $R^i{}_{jkl}$ are indeed the components of the curvature tensor in the arbitrary basis $\{e_i\}$.

A similar operation shows that the functions $T^i{}_{jk}$ defined by

$$\Theta^i \equiv \frac{1}{2} T^i{}_{jk}\, \theta^j \wedge \theta^k \tag{8.21}$$

are the elements of the torsion tensor in the basis $\{e_i\}$, since

$$\langle \theta^i, T(e_j, e_k) \rangle = \langle \theta^i, \Theta^s(e_j, e_k) e_s \rangle = \Theta^i(e_j, e_k) = T^i{}_{jk} \ . \tag{8.22}$$

Thus, Cartan's structure equations allow us to considerably simplify the computation of curvature and torsion for an arbitrary metric, provided we find a base in which the metric appears simple (e.g. diagonal and constant).

We mention without proof that the connection ∇ is metric if and only if

$$\omega_{ij} + \omega_{ji} = \mathrm{d}g_{ij} \ , \tag{8.23}$$

where the definitions

$$\omega_{ij} \equiv g_{ik}\omega^k_j \quad \text{and} \quad g_{ij} \equiv g(e_i, e_j) \tag{8.24}$$

were used, i.e. the g_{ij} are the components of the metric in the arbitrary basis $\{e_i\}$.

8.2 Stationary and static space-times

Stationary space-times (M, g) are defined to be space-times which have a time-like Killing vector field K. This means that observers moving along the integral curves of K do not notice any change.

This definition implies that we can introduce coordinates in which the components $g_{\mu\nu}$ of the metric do not depend on time. To see that, suppose we choose a space-like hypersurface $\Sigma \subset M$ and construct the integral curves of K through Σ.

We further introduce arbitrary coordinates on Σ and carry them into M as follows: let ϕ_t be the flow of K, $p_0 \in \Sigma$ and $p = \phi_t(p_0)$, then the coordinates of p are chosen as $(t, x^1(p_0), x^2(p_0), x^3(p_0))$. These are the so-called *Lagrange coordinates* of p.

In these coordinates, $K = (\partial_0, 0, 0, 0)$. From (5.34), we further have that the components of the Lie derivative of the metric are

$$\begin{aligned} (\mathcal{L}_K g)_{\mu\nu} &= K^\lambda \partial_\lambda g_{\mu\nu} + g_{\lambda\nu}\partial_\mu K^\lambda + g_{\mu\lambda}\partial_\nu K^\lambda \\ &= \partial_0 g_{\mu\nu} = 0 \ , \end{aligned} \tag{8.25}$$

which proves that the $g_{\mu\nu}$ do not depend on time in these so-called *adapted* coordinates.

We can straightforwardly assign a one-form ω to the vector K by defining $\omega_\mu = g_{\mu\nu}K^\nu$. This one-form obviously satisfies

$$\omega(K) = \langle K, K \rangle \neq 0 \; . \tag{8.26}$$

Suppose that we have a *stationary* space-time in which we have introduced adapted coordinates and in which also $g_{0i} = 0$. Then, the Killing vector field is orthogonal to the spatial sections, for which $t = \text{const.}$ Then, the one-form ω is obviously

$$\omega = g_{00}\, c\mathrm{d}t = \langle K, K \rangle\, c\mathrm{d}t \; , \tag{8.27}$$

because $K = \partial_0$. This then trivially implies the *Frobenius condition*

$$\omega \wedge \mathrm{d}\omega = 0 \tag{8.28}$$

because the exterior derivative d satisfies $\mathrm{d} \circ \mathrm{d} \equiv 0$.

Conversely, it can be shown that if the Frobenius condition holds, the one-form ω can be written in the form (8.27). For a vector field v tangential to a spacelike section defined by $t = \text{const.}$, we have

$$\langle K, v \rangle = \omega(v) = \langle K, K \rangle\, c\mathrm{d}t(v) = \langle K, K \rangle v(t) = 0 \tag{8.29}$$

because $t = \text{const.}$, and thus K is then perpendicular to the spatial section. Thus, $K = \partial_0$ and

$$g_{0i} = \langle \partial_0, \partial_i \rangle = \langle K, \partial_i \rangle = 0 \; . \tag{8.30}$$

Thus, in a stationary space-time with time-like Killing vector field K, the Frobenius condition (8.28) for the one-form ω corresponding to K is equivalent to the condition $g_{0i} = 0$ in adapted coordinates. Such space-times are called *static*. In other words, stationary space-times are static if and only if the Frobenius condition holds.

In *static* space-times, the metric can thus be written in the form

$$g = g_{00}(\vec{x})c^2\mathrm{d}t^2 + g_{ij}(\vec{x})\mathrm{d}x^i\mathrm{d}x^j \; . \tag{8.31}$$

8.3 The Schwarzschild solution

8.3.1 Form of the metric

Technically speaking, the Schwarzschild solution is a static, spherically symmetric solution of Einstein's field equations for vacuum space-time.

From our earlier considerations, we know that a *static* space-time is a *stationary* space-time whose (time-like) Killing vector field satisfies the Frobenius condition (8.28).

As the space-time is (globally) stationary, we know that we can introduce spatial hypersurfaces Σ perpendicular to the Killing vector field which, in adapted coordinates, is $K = \partial_0$. The manifold (M, g) can thus be *foliated* as $M = \mathbb{R} \times \Sigma$.

From (8.31), we then know that, also in adapted coordinates, the metric assumes the form

$$g = -\phi^2\, c^2 \mathrm{d}t^2 + h \;, \tag{8.32}$$

where ϕ is a smoothly varying function on Σ and h is the metric of the spatial sections Σ. Under the assumption that K is the only time-like Killing vector field which the space-time admits, t is a uniquely distinguished time coordinate, and we can write

$$-\,\phi^2 = \langle K, K\rangle \;. \tag{8.33}$$

The stationarity of the space-time, expressed by the existence of the single Killing vector field K, thus allows a convenient foliation of the space-time.

Furthermore, the spatial hypersurfaces Σ are expected to be spherically symmetric. This means that the group $SO(3)$ (i.e. the group of rotations in three dimensions) must be an isometry group of the metric h. The orbits of $SO(3)$ are two-dimensional, space-like surfaces on Σ. Thus, $SO(3)$ foliates the space-time (Σ, h) into invariant two-spheres.

Let the surface of these two-spheres be A, then we *define* a radial coordinate for the Schwarzschild metric requiring

$$4\pi r^2 = A \tag{8.34}$$

as in Euclidean geometry. Moreover, the spherical symmetry implies that we can introduce spherical polar coordinates (ϑ, ϕ) on one particular orbit of $SO(3)$ which can then be transported along geodesic lines perpendicular to the orbits. Then, the spatial metric h can be written in the form

$$h = \mathrm{e}^{2b(r)}\mathrm{d}r^2 + r^2\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\phi^2\right) \;, \tag{8.35}$$

where the exponential factor was introduced to allow a scaling of the radial coordinate.

Due to the stationarity of the metric and the spherical symmetry of the spatial sections, $\langle K, K\rangle$ can only depend on r. We set

$$\phi^2 = -\langle K, K\rangle = \mathrm{e}^{2a(r)} \;. \tag{8.36}$$

The full metric g is thus characterised by two radial functions $a(r)$ and $b(r)$ which we need to determine. The exponential functions in (8.35) and (8.36) are chosen to ensure that the prefactors e^a and e^b are always positive.

The spatial sections Σ are now foliated according to

$$\Sigma = I \times S^2 , \quad I \subset \mathbb{R}^+ , \tag{8.37}$$

the metric is

$$g = -e^{2a(r)} c^2 dt^2 + e^{2b(r)} dr^2 + r^2 (d\vartheta^2 + \sin^2 \vartheta d\phi^2) \tag{8.38}$$

in the Schwarzschild coordinates (t, r, θ, ϕ). The functions $a(r)$ and $b(r)$ are constrained by the requirement that the metric should asymptotically turn flat, which means

$$a(r) \to 0 , \quad b(r) \to 0 \quad \text{for} \quad r \to \infty . \tag{8.39}$$

They must be determined by inserting the metric (8.38) into the vacuum field equations, $G = 0$.

8.3.2 Connection and curvature forms

In order to evaluate Einstein's field equations for the Schwarzschild metric, we need to compute the Riemann, Ricci, and Einstein tensors. Traditionally, one would begin this step with computing all Christoffel symbols of the metric (8.38). This very lengthy and error-prone procedure can be considerably shortened using Cartan's structure equations (8.13) for the torsion and curvature forms Θ^i and $\Omega^i_{\ j}$.

To do so, we need to introduce a suitable basis, or *tetrad* $\{e_i\}$ or dual basis $\{\theta^i\}$. Guided by the form of the metric (8.38), we choose

$$\theta^0 = e^a c dt , \quad \theta^1 = e^b dr , \quad \theta^2 = r d\vartheta , \quad \theta^3 = r \sin\vartheta d\phi . \tag{8.40}$$

In terms of these, the metric attains the simple diagonal, Minkowskian form

$$g = g_{\mu\nu} \theta^\mu \otimes \theta^\nu , \quad g_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1) . \tag{8.41}$$

Obviously, $dg = 0$, and thus (8.23) implies that the connection forms $\omega_{\mu\nu}$ are antisymmetric,

$$\omega_{\mu\nu} = -\omega_{\nu\mu} . \tag{8.42}$$

Given the dual tetrad $\{\theta^\mu\}$, we must take their exterior derivatives. For this purpose, we apply the expression (5.67) with $p = 1$ and find, for $d\theta^0$,

$$d\theta^0 = -\partial_r \theta^0_0 dt \wedge dr = -a' e^a dt \wedge dr \tag{8.43}$$

because $\theta^0 = (e^a, 0, 0, 0)$ according to (8.40), so that $i_0 = 0$ and $i_1 = 1$ are the only possible indices to be summed over.

Similarly, we find

$$\mathrm{d}\theta^1 = 0 , \tag{8.44}$$

further

$$\mathrm{d}\theta^2 = \partial_r\theta^2_2\,\mathrm{d}r \wedge \mathrm{d}\vartheta = \mathrm{d}r \wedge \mathrm{d}\vartheta \tag{8.45}$$

and

$$\begin{aligned} \mathrm{d}\theta^3 &= \partial_r\theta^3_3\,\mathrm{d}r \wedge \mathrm{d}\phi + \partial_\vartheta\theta^3_3\,\mathrm{d}\vartheta \wedge \mathrm{d}\phi \\ &= \sin\vartheta\,\mathrm{d}r \wedge \mathrm{d}\phi + r\cos\vartheta\,\mathrm{d}\vartheta \wedge \mathrm{d}\phi . \end{aligned} \tag{8.46}$$

Using (8.40), we can also express the coordinate differentials by the dual tetrad,

$$\mathrm{d}t = \mathrm{e}^{-a}\theta^0 , \quad \mathrm{d}r = \mathrm{e}^{-b}\theta^1 , \quad \mathrm{d}\vartheta = \frac{\theta^2}{r} , \quad \mathrm{d}\phi = \frac{\theta^3}{r\sin\vartheta} , \tag{8.47}$$

so that we can write the exterior derivatives of the dual tetrad as

$$\begin{aligned} &\mathrm{d}\theta^0 = a'\mathrm{e}^{-b}\,\theta^1 \wedge \theta^0 , \quad \mathrm{d}\theta^1 = 0 , \quad \mathrm{d}\theta^2 = \frac{1}{r\mathrm{e}^b}\,\theta^1 \wedge \theta^2 , \\ &\mathrm{d}\theta^3 = \frac{1}{r\mathrm{e}^b}\,\theta^1 \wedge \theta^3 + \frac{\cot\vartheta}{r}\,\theta^2 \wedge \theta^3 . \end{aligned} \tag{8.48}$$

Since the torsion must vanish, $\Theta^i = 0$, Cartan's first structure equation from (8.13) implies

$$\mathrm{d}\theta^\mu = -\omega^\mu_\nu \wedge \theta^\nu . \tag{8.49}$$

With (8.48), this suggests that the connection forms of the Schwarzschild metric are

$$\begin{aligned} &\omega^0_1 = \omega^1_0 = \frac{a'\theta^0}{\mathrm{e}^b} , \quad \omega^0_2 = \omega^2_0 = 0 , \quad \omega^0_3 = \omega^3_0 = 0 , \\ &\omega^2_1 = -\omega^1_2 = \frac{\theta^2}{r\mathrm{e}^b} , \quad \omega^3_1 = -\omega^1_3 = \frac{\theta^3}{r\mathrm{e}^b} , \\ &\omega^3_2 = -\omega^2_3 = \frac{\cot\vartheta\,\theta^3}{r} . \end{aligned} \tag{8.50}$$

They satisfy the antisymmetry condition (8.42) and Cartan's first structure equation (8.49) for a torsion-free connection.

For evaluating the curvature forms Ω^μ_ν, we first recall that the exterior derivative of a one-form ω multiplied by a function f is

$$\begin{aligned} \mathrm{d}(f\omega) &= \mathrm{d}f \wedge \omega + f\mathrm{d}\omega \\ &= (\partial_i f)\mathrm{d}x^i \wedge \omega + f\mathrm{d}\omega \end{aligned} \tag{8.51}$$

according to the Leibniz rule.

Thus, we have for $\mathrm{d}\omega^0_1$

$$\begin{aligned} \mathrm{d}\omega^0_1 &= (a'\mathrm{e}^{-b})'\,\mathrm{d}r \wedge \theta^0 + a'\mathrm{e}^{-b}\,\mathrm{d}\theta^0 \\ &= (a''\mathrm{e}^{-b} - a'b'\mathrm{e}^{-b})\mathrm{e}^{-b}\,\theta^1 \wedge \theta^0 + (a'\mathrm{e}^{-b})^2\,\theta^1 \wedge \theta^0 \\ &= -\mathrm{e}^{-2b}(a'' - a'b' + a'^2)\,\theta^0 \wedge \theta^1 , \end{aligned} \tag{8.52}$$

where we have used (8.47) and (8.48).

In much the same way, we find

$$\begin{aligned}
\mathrm{d}\omega^2_1 &= -\frac{b'}{r\mathrm{e}^{2b}}\,\theta^1\wedge\theta^2\ ,\\
\mathrm{d}\omega^3_1 &= -\frac{b'}{r\mathrm{e}^{2b}}\,\theta^1\wedge\theta^3+\frac{\cot\vartheta}{r^2\mathrm{e}^b}\,\theta^2\wedge\theta^3\ ,\\
\mathrm{d}\omega^3_2 &= -\frac{1}{r^2}\,\theta^2\wedge\theta^3\ .
\end{aligned}\tag{8.53}$$

This yields the curvature two-forms according to (8.13),

$$\begin{aligned}
\Omega^0_1 &= \mathrm{d}\omega^0_1=-\frac{a''-a'b'+a'^2}{e^{2b}}\,\theta^0\wedge\theta^1=\Omega^1_0\\
\Omega^0_2 &= \omega^0_1\wedge\omega^1_2=-\frac{a'}{r\mathrm{e}^{2b}}\,\theta^0\wedge\theta^2=\Omega^2_0\\
\Omega^0_3 &= \omega^0_1\wedge\omega^1_3=-\frac{a'}{r\mathrm{e}^{2b}}\,\theta^0\wedge\theta^3=\Omega^3_0\\
\Omega^1_2 &= \mathrm{d}\omega^1_2+\omega^1_3\wedge\omega^3_2=\frac{b'}{r\mathrm{e}^{2b}}\,\theta^1\wedge\theta^2=-\Omega^2_1\\
\Omega^1_3 &= \mathrm{d}\omega^1_3+\omega^1_2\wedge\omega^2_3=\frac{b'}{r\mathrm{e}^{2b}}\,\theta^1\wedge\theta^3=-\Omega^3_1\\
\Omega^2_3 &= \mathrm{d}\omega^2_3+\omega^2_1\wedge\omega^1_3=\frac{1-\mathrm{e}^{-2b}}{r^2}\,\theta^2\wedge\theta^3=-\Omega^3_2\ .
\end{aligned}\tag{8.54}$$

The other curvature two-forms follow from antisymmetry since

$$\Omega_{\mu\nu}=g_{\mu\lambda}\Omega^\lambda_\nu=-\Omega_{\nu\mu}\ ,\tag{8.55}$$

because of the antisymmetry properties of the curvature.

8.3.3 Components of the Ricci and Einstein tensors

The components of the curvature tensor are given by (8.20), and thus the components of the Ricci tensor in the tetrad $\{e_\alpha\}$ are

$$R_{\mu\nu}=R^\lambda{}_{\mu\lambda\nu}=\Omega^\lambda_\mu(e_\lambda,e_\nu)\ .\tag{8.56}$$

Thus, the components of the Ricci tensor in the Schwarzschild tetrad are

$$\begin{aligned}
R_{00} &= \Omega^1_0(e_1,e_0)+\Omega^2_0(e_2,e_0)+\Omega^3_0(e_3,e_0)\\
&= \frac{a''-a'b'+a'^2+2a'/r}{\mathrm{e}^{2b}}\ ,\\
R_{11} &= \Omega^0_1(e_0,e_1)+\Omega^2_1(e_2,e_1)+\Omega^3_1(e_3,e_1)\\
&= -\frac{a''-a'b'+a'^2-2b'/r}{\mathrm{e}^{2b}}
\end{aligned}\tag{8.57}$$

and

$$\begin{aligned}
R_{22} &= \Omega^0_2(e_0, e_2) + \Omega^1_2(e_1, e_2) + \Omega^3_2(e_3, e_2) \\
&= -\frac{a' - b'}{r\mathrm{e}^{2b}} + \frac{\mathrm{e}^{2b} - 1}{r^2\mathrm{e}^{2b}} \\
R_{33} &= \Omega^0_3(e_0, e_3) + \Omega^1_3(e_1, e_3) + \Omega^2_3(e_2, e_3) = R_{22} \qquad (8.58)
\end{aligned}$$

The Ricci scalar becomes

$$R = -2\frac{a'' - a'b' + a'^2}{\mathrm{e}^{2b}} - 4\frac{a' - b'}{r\mathrm{e}^{2b}} + 2\frac{\mathrm{e}^{2b} - 1}{r^2\mathrm{e}^{2b}} \;, \qquad (8.59)$$

such that the components of the Einstein tensor in the tetrad $\{e_\alpha\}$ read

$$\begin{aligned}
G_{00} &= R_{00} - \frac{R}{2}g_{00} = \frac{1}{r^2} - \mathrm{e}^{-2b}\left(\frac{1}{r^2} - \frac{2b'}{r}\right) \\
G_{11} &= -\frac{1}{r^2} + \mathrm{e}^{-2b}\left(\frac{1}{r^2} + \frac{2a'}{r}\right) \\
G_{22} &= \mathrm{e}^{-2b}\left(a'' - a'b' + a'^2 + \frac{a' - b'}{r}\right) = G_{33} \;, \qquad (8.60)
\end{aligned}$$

and all other components of $G_{\mu\nu}$ vanish identically.

8.3.4 Solution of the vacuum field equations

The vacuum field equations now require that all components of the Einstein tensor vanish. In particular, then,

$$0 = G_{00} + G_{11} = \frac{2\mathrm{e}^{2b}}{r}(a' + b') \qquad (8.61)$$

shows that $a' + b' = 0$. Since $a + b \to 0$ asymptotically for $r \to \infty$, integrating $a + b$ from $r \to \infty$ indicates that $a + b = 0$ everywhere, or $b = -a$.

The equation $G_{00} = 0$ itself implies that

$$\mathrm{e}^{-2b}\left(\frac{2b'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} = 0 \;. \qquad (8.62)$$

Multiplication by $-r^2$ gives

$$\mathrm{e}^{-2b}(1 - 2rb') = 1 \quad \Leftrightarrow \quad \left(r\mathrm{e}^{-2b}\right)' = 1 \;. \qquad (8.63)$$

Therefore, (8.62) is equivalent to

$$r\mathrm{e}^{-2b} = r - 2m \quad \Leftrightarrow \quad \mathrm{e}^{-2b} = 1 - \frac{2m}{r} \;, \qquad (8.64)$$

where $-2m$ has been chosen as an integration constant.

Since $a = -b$, this also means that

$$e^{2a} = e^{-2b} = 1 - \frac{2m}{r} \ . \tag{8.65}$$

We thus obtain the *Schwarzschild solution* for the metric,

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \frac{dr^2}{1 - \frac{2m}{r}} + r^2\left(d\vartheta^2 + \sin^2\vartheta d\phi^2\right) \ . \tag{8.66}$$

We have seen before in (4.80) that the 00 element of the metric must be related to the Newtonian gravitational potential as $g_{00} = -(1 + 2\Phi/c^2)$ in order to meet the Newtonian limit. The Newtonian potential of a point mass M at a distance r is

$$\Phi = -\frac{GM}{r} \ . \tag{8.67}$$

Together with (8.66), this shows that the Newtonian limit is reached by the Schwarzschild solution if the integration constant m has the value

$$\frac{2m}{r} = \frac{2GM}{rc^2} \quad \Rightarrow \quad m = \frac{GM}{c^2} \approx 1.5\,\mathrm{km}\left(\frac{M}{M_\odot}\right) \ . \tag{8.68}$$

The Schwarzschild metric (8.66) has an (apparent) singularity at $r = 2m$ or

$$r = R_s \equiv \frac{2GM}{c^2} \ , \tag{8.69}$$

the so-called *Schwarzschild radius*. We shall clarify later the meaning of this singularity.

In order to illustrate the geometrical meaning of the spatial part of the Schwarzschild metric, we need to find a geometrical interpretation for its radial dependence. Specialising to the equatorial plane of the Schwarzschild solution, $\vartheta = \pi/2$ and $t = 0$, we find the induced spatial line element

$$dl^2 = \frac{dr^2}{1 - 2m/r} + r^2 d\phi^2 \tag{8.70}$$

on that plane.

On the other hand, consider a surface of rotation in the three-dimensional Euclidean space E^3. If we introduce the adequate cylindrical coordinates (r, ϕ, z) on E^3 and rotate a curve $z(r)$ about the z axis, we find the induced line element

$$\begin{aligned} dl^2 &= dz^2 + dr^2 + r^2 d\phi^2 = \left(\frac{dz}{dr}\right)^2 dr^2 + dr^2 + r^2 d\phi^2 \\ &= (1 + z'^2)dr^2 + r^2 d\phi^2 \ . \end{aligned} \tag{8.71}$$

We can now try and identify the two induced line elements from (8.70) and (8.71) and find that this is possible if

$$z' = \left(\frac{1}{1-2m/r} - 1\right)^{1/2} = \sqrt{\frac{2m}{r-2m}}\;, \tag{8.72}$$

which is readily integrated to yield

$$z = \sqrt{8m(r-2m)} + \text{const.} \quad \text{or} \quad z^2 = 8m(r-2m)\;, \tag{8.73}$$

if we set the integration constant to zero.

This shows that the geometry on the equatorial plane of the spatial section of the Schwarzschild solution can be identified with a rotational paraboloid in E^3. In other words, the dependence of radial distances on the radius r is equivalent to that on a rotational paraboloid.

8.3.5 Birkhoff's theorem

Suppose now we had started from a spherically symmetric vacuum space-time, but with explicit time dependence of the functions a and b, such that the space-time could either expand or contract. Then, a repetition of the derivation of the connection and curvature forms, and the components of the Einstein tensor following from them, had resulted in the new components $\bar{G}_{\mu\nu}$

$$\begin{aligned}
\bar{G}_{00} &= G_{00}\;, \quad \bar{G}_{11} = G_{11} \\
\bar{G}_{22} &= G_{22} - \mathrm{e}^{-2a}\left(\dot{b}^2 - \dot{a}\dot{b} - \ddot{b}\right) = \bar{G}_{33} \\
\bar{G}_{10} &= \frac{2\dot{b}}{r}\mathrm{e}^{-a-b}
\end{aligned} \tag{8.74}$$

and $\bar{G}_{\mu\nu} = 0$ for all other components.

The vacuum field equations imply $\bar{G}_{10} = 0$ and thus $\dot{b} = 0$, hence b must be independent of time. From $\bar{G}_{00} = 0$, we can again conclude (8.64), i.e. b retains the same form as before. Similarly, since $\bar{G}_{00} + \bar{G}_{11} = G_{00} + G_{11}$, the requirement $a' + b' = 0$ must continue to hold, but now the time dependence of a allows us to conclude only that

$$a = -b + f(t)\;, \tag{8.75}$$

where $f(t)$ is an otherwise unconstrained function of time only. Thus, the line element then reads

$$\mathrm{d}s^2 = -\mathrm{e}^{2f}\left(1 - \frac{2m}{r}\right)\mathrm{d}t^2 + \mathrm{d}l^2\;, \tag{8.76}$$

where $\mathrm{d}l^2$ is the unchanged line element of the spatial sections.

Introducing the new time coordinate t' by

$$t' = \int e^{f} dt \tag{8.77}$$

converts (8.76) back to the original form (8.66) of the Schwarzschild metric.

This is *Birkhoff's theorem*, which says that a spherically symmetric solution of Einstein's vacuum equations is *necessarily static* for $r > 2m$.

Chapter 9

Physics in the Schwarzschild Space-Time

9.1 Motion in the Schwarzschild space-time

9.1.1 Equation of motion

According to (4.3), the motion of a particle in the Schwarzschild space-time is determined by the Lagrangian

$$\mathcal{L} = \sqrt{-\langle\dot\gamma,\dot\gamma\rangle}\,\mathrm{d}\tau\ , \tag{9.1}$$

where τ is the proper time defined by (4.6), satisfying

$$\mathrm{d}\tau = \sqrt{-\langle\dot\gamma,\dot\gamma\rangle}\,\mathrm{d}\tau \tag{9.2}$$

and thus requiring that the four-velocity $\dot\gamma$ be normalised,

$$\langle\dot\gamma,\dot\gamma\rangle = -1\ . \tag{9.3}$$

Note that we have to differentiate and integrate with respect to the proper time τ rather than the coordinate time t because the latter has no invariant physical meaning. In the Newtonian limit, $\tau = ct$.

This implies that, instead of varying the action

$$S = -mc\int_a^b \sqrt{-\langle\dot\gamma,\dot\gamma\rangle}\,\mathrm{d}\tau\ , \tag{9.4}$$

we can just as well require that the variation of

$$\bar S = \frac{1}{2}\int_a^b \langle\dot\gamma,\dot\gamma\rangle\,\mathrm{d}\tau \tag{9.5}$$

vanish. In fact, from $\delta S = 0$, we have

$$\begin{aligned} 0 &= -\delta \int_a^b \sqrt{-\langle\dot\gamma,\dot\gamma\rangle}\,\mathrm{d}\tau = \frac{1}{2}\int_a^b \frac{\delta\langle\dot\gamma,\dot\gamma\rangle}{\sqrt{-\langle\dot\gamma,\dot\gamma\rangle}}\,\mathrm{d}\tau \\ &= \delta\left[\frac{1}{2}\int_a^b \langle\dot\gamma,\dot\gamma\rangle\,\mathrm{d}\tau\right] . \end{aligned} \tag{9.6}$$

because of the normalisation condition (9.3).

Thus, we can obtain the equation of motion just as well from the Lagrangian

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\langle\dot\gamma,\dot\gamma\rangle = \frac{1}{2}g_{\mu\nu}\dot x^\mu \dot x^\nu \\ &= \frac{1}{2}\left[-(1-2m/r)\dot t^2 + \frac{\dot r^2}{1-2m/r} + r^2\left(\dot\vartheta^2 + \sin^2\vartheta\,\dot\phi^2\right)\right] , \end{aligned} \tag{9.7}$$

where it is important to recall that the overdot denotes differentiation with respect to proper time τ. In addition, (9.3) immediately implies that $2\mathcal{L} = -1$ for material particles, but $2\mathcal{L} = 0$ for light, which will be discussed later.

The Euler-Lagrange equation for ϑ is

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial\mathcal{L}}{\partial\dot\vartheta} - \frac{\partial\mathcal{L}}{\partial\vartheta} = 0 = \frac{\mathrm{d}}{\mathrm{d}\tau}(r^2\dot\vartheta) - r^2\dot\phi^2\sin\vartheta\cos\vartheta . \tag{9.8}$$

Suppose the motion starts in the equatorial plane, $\vartheta = \pi/2$ and $\dot\vartheta = 0$. Should this not be the case, we can always rotate the coordinate frame so that this is satisfied. Then, (9.8) shows that

$$r^2\dot\vartheta = \text{const.} = 0 . \tag{9.9}$$

Without loss of generality, we can thus restrict the discussion to motion in the equatorial plane, which simplifies the Lagrangian to

$$\mathcal{L} = \frac{1}{2}\left[-(1-2m/r)\dot t^2 + \frac{\dot r^2}{1-2m/r} + r^2\dot\phi^2\right] . \tag{9.10}$$

Obviously, t and ϕ are cyclic, thus

$$\frac{\partial\mathcal{L}}{\partial\dot\phi} = r^2\dot\phi \equiv L = \text{const.} \tag{9.11}$$

and

$$\frac{\partial\mathcal{L}}{\partial\dot t} = -(1-2m/r)\dot t \equiv E = \text{const.} . \tag{9.12}$$

Now, we insert

$$\dot\phi = \frac{L}{r^2} , \quad \dot t = -\frac{E}{1-2m/r} \tag{9.13}$$

into the Lagrangian (9.10), use $2\mathcal{L} = -1$ and find

$$-1 = -(1-2m/r)\dot{t}^2 + \frac{\dot{r}^2}{1-2m/r} + r^2\dot{\phi}^2 = \frac{\dot{r}^2 - E^2}{1-2m/r} + \frac{L^2}{r^2}\,, \tag{9.14}$$

which can be cast into the form

$$\dot{r}^2 + V(r) = E^2\,, \tag{9.15}$$

where $V(r)$ is the effective potential

$$V(r) \equiv \left(1 - \frac{2m}{r}\right)\left(1 + \frac{L^2}{r^2}\right)\,. \tag{9.16}$$

Since it is our primary goal to find the orbit $r(\phi)$, we use $r' = \mathrm{d}r/\mathrm{d}\phi = \dot{r}/\dot{\phi}$ to transform (9.15) to

$$\dot{r}^2 + V(r) = \dot{\phi}^2 r'^2 + V(r) = \frac{L^2}{r^4} r'^2 + V(r) = E^2\,. \tag{9.17}$$

Now, we substitute $u \equiv 1/r$ and $u' = -r'/r^2 = -u^2 r'$ and find

$$L^2 u^4 \frac{u'^2}{u^4} + V(1/u) = L^2 u'^2 + (1-2mu)(1+L^2u^2) = E^2 \tag{9.18}$$

or, after dividing by L^2 and rearranging terms,

$$u'^2 + u^2 = \frac{E^2 - 1}{L^2} + \frac{2m}{L^2}u + 2mu^3\,. \tag{9.19}$$

Differentiation with respect to ϕ cancels the constant first term on the right-hand side and yields

$$2u'u'' + 2uu' = \frac{2m}{L^2}u' + 6mu^2u'\,. \tag{9.20}$$

the trivial solution is $u' = 0$, which implies a circular orbit. If $u' \neq 0$, this equation can be simplified to read

$$u'' + u = \frac{m}{L^2} + 3mu^2\,. \tag{9.21}$$

The fact that t and ϕ are cyclic coordinates in the Schwarzschild space-time can be studied from a more general point of view. Let $\gamma(\tau)$ be a geodesic curve with tangent vector $\dot{\gamma}(\tau)$, and let further ξ be a Killing vector field of the metric. Then, we know from (5.36) that the projection of the Killing vector field on the geodesic is constant along the geodesic,

$$\nabla_{\dot{\gamma}}\langle\dot{\gamma},\xi\rangle = 0 \quad \Rightarrow \quad \langle\dot{\gamma},\xi\rangle = \text{constant along}\,\gamma \tag{9.22}$$

Due to its stationarity and the spherical symmetry, the Schwarzschild space-time has the Killing vector fields ∂_t and ∂_ϕ. Thus,

$$\langle\dot\gamma, \partial_t\rangle = \langle\dot\gamma^t\partial_t, \partial_t\rangle = \dot\gamma^t\langle\partial_t, \partial_t\rangle = g_{00}\dot\gamma^t = -\left(1 - \frac{2m}{r}\right)\dot t = \text{const.} \tag{9.23}$$

and

$$\langle\dot\gamma, \partial_\phi\rangle = \dot\gamma^\phi\langle\partial_\phi, \partial_\phi\rangle = g_{\phi\phi}\dot\gamma^\phi = r^2\sin^2\vartheta\,\dot\phi = r^2\dot\phi = \text{const.}\ , \tag{9.24}$$

where we have used $\vartheta = \pi/2$ without loss of generality. This reproduces (9.11) and (9.12).

9.1.2 Comparison to the Kepler problem

It is instructive to compare this to the Newtonian case. There, the Lagrangian is

$$\mathcal{L} = \frac{1}{2}\left(\dot r^2 + r^2\dot\phi^2\right) - \Phi(r)\ , \tag{9.25}$$

where $\Phi(r)$ is some centrally-symmetric potential and the dots denote the derivative with respect to the *coordinate time t* now instead of the proper time τ. In the Newtonian limit, $\tau = ct$. For later comparison of results obtained in this and the previous sections, we thus replace coordinate-time derivatives by derivatives with respect to proper time and introduce the appropriate factors of c^2, thus

$$\mathcal{L} \to \frac{\mathcal{L}}{c^2} = \frac{1}{2}\left(\dot r^2 + r^2\dot\phi^2\right) - \frac{\Phi(r)}{c^2}\ , \tag{9.26}$$

where the dots now mean derivatives with respect to τ, as in the previous section.

Since ϕ is cyclic,

$$\frac{\partial\mathcal{L}}{\partial\dot\phi} = r^2\dot\phi \equiv L = \text{const}\ . \tag{9.27}$$

The Euler-Lagrange equation for r is

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot r} - \frac{\partial\mathcal{L}}{\partial r} = 0 = \ddot r - r\dot\phi^2 + \frac{1}{c^2}\frac{\mathrm{d}\Phi}{\mathrm{d}r}\ . \tag{9.28}$$

Since

$$\dot r = \frac{\mathrm{d}r}{\mathrm{d}t} = r'\dot\phi = r'\frac{L}{r^2} = -Lu'\ , \tag{9.29}$$

we can write the second time derivative of r as

$$\ddot r = -L\frac{\mathrm{d}u'}{\mathrm{d}t} = -L\frac{\mathrm{d}u'}{\mathrm{d}\phi}\dot\phi = -Lu''\frac{L}{r^2} = -L^2u^2u''\ . \tag{9.30}$$

Thus, the equation of motion (9.28) can be written as

$$-L^2u^2u'' - r\frac{L^2}{r^4} + \frac{1}{c^2}\frac{\mathrm{d}\Phi}{\mathrm{d}r} = 0 \tag{9.31}$$

or, after dividing by $-u^2L^2$,

$$u'' + u = \frac{1}{L^2u^2c^2}\frac{\mathrm{d}\Phi}{\mathrm{d}r} \; . \tag{9.32}$$

In the Newtonian limit of the Schwarzschild solution,

$$\Phi = -\frac{GM}{r} \; , \quad \frac{1}{c^2}\frac{\mathrm{d}\Phi}{\mathrm{d}r} = \frac{GM}{c^2r^2} = \frac{GM}{c^2}u^2 = mu^2 \; , \tag{9.33}$$

so that the equation of motion becomes

$$u'' + u = \frac{m}{L^2} \; . \tag{9.34}$$

Compared to this, the equation of motion in the Schwarzschild case (9.21) has the additional term $3mu^2$. We have seen in (8.68) that $m \approx 1.5\,\mathrm{km}$ in the Solar System. There, the ratio of the two terms on the right-hand side of (9.21) is

$$\frac{3mu^2}{m/L^2} = 3u^2L^2 = \frac{3}{r^2}r^4\dot{\phi}^2 = 3(r\dot{\phi})^2 = \frac{3v_\perp^2}{c^2} \approx 7.7 \times 10^{-8} \tag{9.35}$$

for the innermost planet Mercury. Here, $v_\perp$ is the tangential velocity along the orbit, $v_\perp = r\dot{\phi}$.

The equation of motion (9.21) in the Schwarzschild space-time can thus be reduced to a Kepler problem with a potential which, according to (9.32), is given by

$$\frac{1}{L^2u^2c^2}\frac{\mathrm{d}\Phi(r)}{\mathrm{d}r} = \frac{m}{L^2} + 3mu^2 \tag{9.36}$$

or

$$\frac{1}{c^2}\frac{\mathrm{d}\Phi(r)}{\mathrm{d}r} = mu^2 + 3mL^2u^4 = \frac{m}{r^2} + \frac{3mL^2}{r^4} \; , \tag{9.37}$$

which leads to

$$\frac{\Phi(r)}{c^2} = -\frac{m}{r} - \frac{mL^2}{r^3} \tag{9.38}$$

if we set the integration constant such that $\Phi(r) \to 0$ for $r \to \infty$.

As a function of $x \equiv r/R_\mathrm{s} = r/2m$, the effective potential $V(r)$ from (9.16) depends in an interesting way on $L/R_\mathrm{s} = L/2m \equiv \lambda$. The function

$$V(x) = \left(1 - \frac{1}{x}\right)\left(1 + \frac{\lambda^2}{x^2}\right) \tag{9.39}$$

corresponding to the effective potential (9.16) asymptotically behaves as $V(x) \to 1$ for $x \to \infty$ and $V(x) \to -\infty$ for $x \to 0$.

For the potential to have a minimum, $V(x)$ must have a vanishing derivative, $V'(x) = 0$. This is the case where

$$0 = V'(x) = \frac{1}{x^2}\left(1 + \frac{\lambda^2}{x^2}\right) - \left(1 - \frac{1}{x}\right)\frac{2\lambda^2}{x^3} \tag{9.40}$$

or, after multiplication with x^4,

$$x^2 - 2\lambda^2 x + 3\lambda^2 = 0 \quad \Rightarrow \quad x_\pm = \lambda^2 \pm \lambda\sqrt{\lambda^2 - 3}\ . \tag{9.41}$$

Real solutions require $\lambda \geq \sqrt{3}$. If $\lambda < \sqrt{3}$, particles with $E^2 < 1$ will crash directly towards $r = R_\mathrm{s}$.

The last stable orbit must thus have $\lambda = \sqrt{3}$ and thus $x_\pm = 3$, i.e. at $r = 3R_\mathrm{s}$, or three Schwarzschild radii. There, the effective potential is

$$V(x = 3) = \frac{2}{3}\left(1 + \frac{3}{9}\right) = \frac{8}{9}\ . \tag{9.42}$$

For $\lambda > \sqrt{3}$, the effective potential has a minimum at x_+ and a maximum at x_- which reaches the height $V = 1$ for $\lambda = 2$ at $x_- = 2$ and is higher for larger λ. This means that particles with $E \geq 1$ and $L < 2R_\mathrm{s}$ will fall unimpededly towards $r = R_\mathrm{s}$.

9.1.3 The perihelion shift

The treatment of the Kepler problem in classical mechanics shows that closed orbits in the Newtonian limit are described by

$$u_0(\phi) = \frac{1}{p}(1 + e\cos\phi)\ , \tag{9.43}$$

where the parameter p is

$$p = a(1 - e^2) = \frac{L^2}{m} \tag{9.44}$$

in terms of the semi-major axis a and the eccentricity e of the orbit.

Assuming that the perturbation $3mu^2$ in the equation of motion (9.21) is small, we can approximate it by $3mu_0^2$, thus

$$u'' + u = \frac{m}{L^2} + \frac{3m}{p^2}(1 + e\cos\phi)^2\ . \tag{9.45}$$

The solution of this equation turns out to be simple because differential equations of the sort

$$u'' + u = \begin{cases} A \\ B\cos\phi \\ C\cos^2\phi \end{cases}\ , \tag{9.46}$$

which are driven harmonic-oscillator equations, have the particular solutions

$$u_1 = A\ , \quad u_2 = \frac{B}{2}\phi\sin\phi\ , \quad u_3 = \frac{C}{2}\left(1 - \frac{1}{3}\cos 2\phi\right)\ . \tag{9.47}$$

Since the unperturbed equation $u'' + u = m/L^2$ has the Keplerian solution $u = u_0$, the complete solution is thus the sum

$$\begin{aligned} u &= u_0 + u_1 + u_2 + u_3 && (9.48) \\ &= \frac{1}{p}(1 + e\cos\phi) + \frac{3m}{p^2}\left[1 + e\phi\sin\phi + \frac{e^2}{2}\left(1 - \frac{1}{3}\cos 2\phi\right)\right] . \end{aligned}$$

This solution of (9.45) has its perihelion at $\phi = 0$ because the unperturbed solution u_0 was chosen to have it there. This can be seen by taking the derivative with respect to ϕ,

$$u' = -\frac{e}{p}\sin\phi + \frac{3me}{p^2}\left[\sin\phi + \phi\cos\phi + \frac{e}{3}\sin 2\phi\right] \tag{9.49}$$

and verifying that $u' = 0$ at $\phi = 0$, i.e. the orbital radius $r = 1/u$ still has an extremum at $\phi = 0$.

We now use equation (9.49) in the following way. Starting at the perihelion at $\phi = 0$, we wait for approximately one revolution at $\phi = 2\pi + \delta\phi$ and see what $\delta\phi$ has to be for u' to vanish again. Thus, the condition for the next perihelion is

$$0 = -\sin\delta\phi + \frac{3m}{p}\left[\sin\delta\phi + (2\pi + \delta\phi)\cos\delta\phi + \frac{e}{3}\sin 2\delta\phi\right] \tag{9.50}$$

or, to first order in the small angle $\delta\phi$,

$$\delta\phi \approx \frac{3m}{p}\left[2\delta\phi + 2\pi + \frac{2e}{3}\delta\phi\right] . \tag{9.51}$$

Sorting terms, we find

$$\delta\phi\left[1 - \frac{6m}{p}\left(1 + \frac{e}{3}\right)\right] \approx \frac{6\pi m}{p} = \frac{6\pi m}{a(1 - e^2)} . \tag{9.52}$$

Substituting the Schwarzschild radius from (8.69), we can write this result as

$$\delta\phi \approx \frac{3\pi R_\mathrm{s}}{a(1 - e^2)} . \tag{9.53}$$

This turns out to be −6 times the result (1.45) from the scalar theory of gravity discussed in § 1.4.2, or

$$\delta\phi \approx 43'' \tag{9.54}$$

per century for Mercury's orbit, which reproduces the measurement extremely well.

9.1.4 Light deflection

For light rays, the condition $2\mathcal{L} = -1$ that we had before for material particles is replaced by $2\mathcal{L} = 0$. Then, (9.14) changes to

$$\frac{\dot{r}^2 - E^2}{1 - 2m/r} + \frac{L^2}{r^2} = 0 \tag{9.55}$$

or

$$\dot{r}^2 + \frac{L^2}{r^2}\left(1 - \frac{2m}{r}\right) = E^2 \ . \tag{9.56}$$

Changing again the independent variable to ϕ and substituting $u = 1/r$ leads to the equation of motion for light rays in the Schwarzschild space-time

$$u'^2 + u^2 = \frac{E^2}{L^2} + 2mu^3 \ , \tag{9.57}$$

which should be compared to the equation of motion for material particles, (9.19). Differentiation finally yields

$$u'' + u = 3mu^2 \ . \tag{9.58}$$

Compared to u on the left-hand side, the term $3mu^2$ is very small. In the Solar System,

$$\frac{3mu^2}{u} = 3mu = \frac{3R_\mathrm{s}}{2r} \le \frac{R_\mathrm{s}}{R_\odot} \approx 10^{-6} \ . \tag{9.59}$$

Thus, the light ray is almost given by the homogeneous solution of the harmonic-oscillator equation $u'' + u = 0$, which is $u_0 = A\sin\phi + B\cos\phi$. We require that the closest impact at $u_0 = 1/b$ be reached when $\phi = \pi/2$, which implies $B = 0$ and $A = 1/b$, or

$$u_0 = \frac{\sin\phi}{b} \quad \Rightarrow \quad r_0 = \frac{b}{\sin\phi} \ . \tag{9.60}$$

Note that this is a straight line in plane polar coordinates, as it should be!

Inserting this lowest-order solution as a perturbation into the right-hand side of (9.58) gives

$$u'' + u = \frac{3m}{b^2}\sin^2\phi = \frac{3m}{b^2}(1 - \cos^2\phi) \ , \tag{9.61}$$

for which particular solutions can be found using (9.46) and (9.47). Combining this with the unperturbed solution (9.60) gives

$$u = \frac{\sin\phi}{b} + \frac{3m}{b^2} - \frac{3m}{2b^2}\left(1 - \frac{1}{3}\cos 2\phi\right) \ . \tag{9.62}$$

Given the orientation of our coordinate system, i.e. with the closest approach at $\phi = \pi/2$, we have $\phi \approx 0$ for a ray incoming from the left at large distances. Then, $\sin\phi \approx \phi$ and $\cos 2\phi \approx 1$, and (9.62) yields

$$u \approx \frac{\phi}{b} + \frac{2m}{b^2} \ . \tag{9.63}$$

In the asymptotic limit $r \to \infty$, or $u \to 0$, this gives the angle

$$|\phi| \approx \frac{2m}{b} \ . \tag{9.64}$$

The total deflection angle is twice this result,

$$\alpha = 2|\phi| \approx \frac{4m}{b} = 2\frac{R_\mathrm{s}}{b} \approx 1.74'' \ . \tag{9.65}$$

This is *twice* the result from our simple consideration leading to (4.90) which did not take the field equations into account yet.

9.1.5 Spin precession

Let us now study how a gyroscope with spin s is moving along a geodesic γ in the Schwarzschild space-time. Without loss of generality, we assume that the orbit falls into the equatorial plane $\vartheta = \pi/2$, and we restrict the motion to circular orbits.

Then, the four-velocity of the gyroscope is characterised by $u^1 = 0 = u^2$ because both $r = x^1$ and $\vartheta = x^2$ are constant.

The equations that the spin s and the tangent vector $u = \dot\gamma$ of the orbit have to satisfy are

$$\langle s, u\rangle = 0 \ , \quad \nabla_u s = 0 \ , \quad \nabla_u u = 0 \ . \tag{9.66}$$

The first is because s falls into a spatial hypersurface perpendicular to the time-like four-velocity u, the second because the spin is parallel transported, and the third because the gyroscope is moving along a geodesic curve.

We work in the same tetrad $\{\theta^\mu\}$ (8.40) that we used to derive the Schwarzschild solution. From (8.9), we know that

$$\begin{aligned} (\nabla_u s)^\mu &= \langle \mathrm{d}s^\mu + s^\nu\omega^\mu_\nu, u\rangle = u(s^\mu) + \omega^\mu_\nu(u)s^\nu \\ &= \dot s^\mu + \omega^\mu_\nu(u)s^\nu = 0 \ , \end{aligned} \tag{9.67}$$

where the overdot marks the derivative with respect to the proper time τ.

With the connection forms in the Schwarzschild tetrad given in (8.50), and taking into account that $a = -b$ and $\cot\vartheta = 0$, we find for the components of $\dot{s}$

$$\begin{aligned}
\dot{s}^0 &= -\omega^0_1(u)s^1 = b'\mathrm{e}^{-b}u^0s^1 \;, \\
\dot{s}^1 &= -\omega^1_0(u)s^0 - \omega^1_2(u)s^2 - \omega^1_3(u)s^3 = b'\mathrm{e}^{-b}u^0s^0 + \frac{\mathrm{e}^{-b}}{r}u^3s^3 \;, \\
\dot{s}^2 &= -\omega^2_1(u)s^1 - \omega^2_3(u)s^3 = 0 \;, \\
\dot{s}^3 &= -\omega^3_1(u)s^1 - \omega^3_2(u)s^2 = -\frac{\mathrm{e}^{-b}}{r}u^3s^1 \;, \qquad (9.68)
\end{aligned}$$

where we have repeatedly used that

$$\theta^1(u) = u^1 = 0 = u^2 = \theta^2(u) \qquad (9.69)$$

and $\omega^2_3 = 0$ because $\cot\vartheta = 0$.

Similarly, the geodesic equation $\nabla_u u = 0$, specialised to $u^1 = 0 = u^2$, leads to

$$\begin{aligned}
\dot{u}^0 &= b'\mathrm{e}^{-b}u^0u^1 = 0 \;, \\
\dot{u}^1 &= -b'\mathrm{e}^{-b}(u^0)^2 - \frac{\mathrm{e}^{-b}}{r}(u^3)^2 = 0 \;, \\
\dot{u}^2 &= 0 \;, \\
\dot{u}^3 &= -\frac{\mathrm{e}^{-b}}{r}u^1u^3 = 0 \;. \qquad (9.70)
\end{aligned}$$

The second of these equations implies

$$\left(\frac{u^0}{u^3}\right)^2 = -\frac{1}{b'r} \;. \qquad (9.71)$$

We now introduce a set of basis vectors orthogonal to u, namely

$$\bar{e}_1 = e_1 \;, \quad \bar{e}_2 = e_2 \;, \quad \bar{e}_3 = u^3e_0 + u^0e_3 \;. \qquad (9.72)$$

The orthogonality of $\bar{e}_1$ and $\bar{e}_2$ to u is obvious because of $u^1 = 0 = u^2$, and

$$\langle u, \bar{e}_3\rangle = u^3u_0 + u^0u_3 = 0 \qquad (9.73)$$

shows the orthogonality of u and $\bar{e}_3$. Recall that $u_0 = -u^0$, but $u_3 = u^3$ because the metric is $g = \mathrm{diag}(-1, 1, 1, 1)$ in this basis.

Since the basis $\{\bar{e}_i\}$ spans the three-space orthogonal to u, the spin s of the gyroscope can be expanded into this basis as $s = \bar{s}^i\bar{e}_i$. We find

$$s^0 = \langle\bar{s}^i\bar{e}_i, e_0\rangle = u^3\bar{s}^3 \;, \quad s^1 = \bar{s}^1 \;, \quad s^2 = \bar{s}^2 \;, \quad s^3 = u^0\bar{s}^3 \;, \qquad (9.74)$$

which we can insert into (9.68) to find

$$\begin{aligned}
\dot{u}^3\bar{s}^3 + u^3\dot{\bar{s}}^3 &= u^3\dot{\bar{s}}^3 = b'\mathrm{e}^{-b}u^0\bar{s}^1 \;, \\
\dot{\bar{s}}^1 &= \left(b'\mathrm{e}^{-b} + \frac{\mathrm{e}^{-b}}{r}\right)u^0u^3\bar{s}^3 \;, \\
\dot{\bar{s}}^2 &= 0 \;, \\
\dot{u}^0\bar{s}^3 + u^0\dot{\bar{s}}^3 &= u^0\dot{\bar{s}}^3 = -\frac{\mathrm{e}^{-b}}{r}u^3\bar{s}^1 \;. \qquad (9.75)
\end{aligned}$$

Note that $\dot{u}^\mu = 0$ for all μ according to (9.70). Using (9.71) and the normalisation relation $(u^0)^2 - (u^3)^2 = 1$, we obtain

$$\begin{aligned}
\dot{\bar{s}}^1 &= b' \mathrm{e}^{-b} \left[1 - \frac{(u^0)^2}{(u_3)^2}\right] u^0 u^3 \bar{s}^3 = -b' \mathrm{e}^{-b} \frac{u^0}{u^3} \bar{s}^3 \;, \\
\dot{\bar{s}}^2 &= 0 \;, \\
\dot{\bar{s}}^3 &= b' \mathrm{e}^{-b} \frac{u^0}{u^3} \bar{s}^1 \;.
\end{aligned} \tag{9.76}$$

From now on, we shall drop the overbar, understanding that the s^i denote the components of the spin with respect to the basis $\bar{e}_i$.

Next, we transform the time derivative from the proper time τ to the coordinate time t. Since

$$u^0 = \langle \theta^0, u \rangle = \langle \mathrm{e}^a \mathrm{d}t, u \rangle = \frac{\mathrm{d}t}{\mathrm{d}\tau} \mathrm{e}^a \;, \tag{9.77}$$

we have

$$\dot{t} = \frac{\mathrm{d}t}{\mathrm{d}\tau} = u^0 \mathrm{e}^{-a} = u^0 \mathrm{e}^b \;, \tag{9.78}$$

or

$$\frac{\mathrm{d}s^i}{\mathrm{d}t} = \frac{\dot{s}^i}{\dot{t}} = \frac{\dot{s}^i}{u^0} \mathrm{e}^{-b} \;. \tag{9.79}$$

Inserting this into (9.76) yields

$$\frac{\mathrm{d}s^1}{\mathrm{d}t} = -\frac{b'}{u^3} \mathrm{e}^{-2b} s^3 \;, \quad \frac{\mathrm{d}s^2}{\mathrm{d}t} = 0 \;, \quad \frac{\mathrm{d}s^3}{\mathrm{d}t} = \frac{b'}{u^3} \mathrm{e}^{-2b} s^1 \;. \tag{9.80}$$

Finally, using (8.40), we have

$$u^3 = \langle \theta^3, u \rangle = \langle r \sin\vartheta \mathrm{d}\phi, u \rangle = r \langle \mathrm{d}\phi, u^\phi \partial_\phi \rangle = r\dot{\phi} \;, \tag{9.81}$$

which yields the angular frequency

$$\omega \equiv \frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{\dot{\phi}}{\dot{t}} = \frac{\mathrm{e}^{-b}}{r} \frac{u^3}{u^0} \;, \tag{9.82}$$

which can be rewritten by means of (9.71),

$$\omega^2 = \left(\frac{u^3}{u^0}\right)^2 \frac{1}{r^2} \mathrm{e}^{-2b} = -\frac{b'}{r} \mathrm{e}^{-2b} = \frac{1}{2r} \left(\mathrm{e}^{-2b}\right)' \;. \tag{9.83}$$

Now, the exponential factor was

$$\mathrm{e}^{-2b} = \left(1 - \frac{2m}{r}\right) \quad \Rightarrow \quad (\mathrm{e}^{-2b})' = \frac{2m}{r^2} \;, \tag{9.84}$$

whence

$$\omega^2 = \frac{m}{r^3} \;, \tag{9.85}$$

which is Kepler's third law.

Taking another time derivative of (9.80), we can use $\dot r = 0$ for circular orbits and $\dot u^3 = 0$ because of (9.70). Thus,

$$\frac{\mathrm{d}^2 s^1}{\mathrm{d}t^2} = -\frac{b'}{u^3}\mathrm{e}^{-2b}\,\frac{\mathrm{d}s^3}{\mathrm{d}t} = -\frac{b'^2\mathrm{e}^{-4b}}{(u^3)^2}\,s^1 \tag{9.86}$$

and likewise for s^3. This is an oscillator equation for s^1 with the squared angular frequency

$$\begin{aligned}\Omega^2 &= \frac{b'^2\mathrm{e}^{-4b}}{(u^3)^2} = b'^2\mathrm{e}^{-4b}\frac{(u^0)^2-(u^3)^2}{(u^3)^2}\\ &= b'^2\mathrm{e}^{-4b}\left(-1-\frac{1}{b'r}\right) = -\frac{b'\mathrm{e}^{-4b}}{r}(1+b'r)\ .\end{aligned} \tag{9.87}$$

Now, we use (9.82) to substitute the factor out front and find

$$\Omega^2 = \omega^2\mathrm{e}^{-2b}(b'r+1)\ . \tag{9.88}$$

From (8.63), we further know that

$$rb' = \frac{1}{2}\left(1-\mathrm{e}^{2b}\right) = \frac{1}{2}\left(1-\frac{1}{1-2m/r}\right) = -\frac{m}{r}\frac{1}{1-2m/r}\ , \tag{9.89}$$

thus

$$rb'+1 = \frac{r-3m}{r-2m} = \frac{1-3m/r}{1-2m/r} \tag{9.90}$$

and

$$\Omega^2 = \omega^2\mathrm{e}^{-2b}\frac{1-3m/r}{1-2m/r} = \omega^2\left(1-\frac{3m}{r}\right)\ . \tag{9.91}$$

In vector notation, we can write (9.80) as

$$\frac{\mathrm{d}\vec s}{\mathrm{d}t} = \vec\Omega\times\vec s\ ,\quad \vec\Omega = \begin{pmatrix}0\\ \Omega\\ 0\end{pmatrix}\ . \tag{9.92}$$

Recall that we have projected the spin into the three-dimensional space perpendicular to the direction of motion. Thus, the result (9.92) shows that $\vec s$ precesses retrograde in that space about an axis perpendicular to the plane of the orbit, since $u^2 = 0$.

After a complete orbit, the projection of $\vec s$ onto the plane of the orbit has advanced by an angle

$$2\pi\left(1-\frac{3m}{r}\right) \equiv \frac{2\pi}{\omega}\omega_\mathrm{s}\ , \tag{9.93}$$

and thus the *geodetic precession frequency* is

$$\begin{aligned}\omega_\mathrm{s} &= \sqrt{\frac{m}{r^3}}\left[1-\left(1-\frac{3m}{r}\right)^{1/2}\right]\\ &\approx \left(\frac{GM}{r^3}\right)^{1/2}\frac{3GM}{2r} = \frac{3}{2}\frac{(GM)^{3/2}}{r^{5/2}}\ .\end{aligned} \tag{9.94}$$

If we insert the Earth's mass and radius here, we find

$$\omega_{\rm s} \approx 8.4'' \left(\frac{R_{\rm earth}}{r}\right)^{5/2} . \tag{9.95}$$

Chapter 10

Schwarzschild Black Holes

10.1 The singularity at $r = 2m$

10.1.1 Free fall towards the centre

Before we can continue discussing the physical meaning of the Schwarzschild metric, we need to clarify the nature of the singularity at the Schwarzschild radius, $r = 2m$. Upon closer inspection, it seems to lead to contradictory conclusions.

Let us begin with an observer falling freely towards the centre of the Schwarzschild space-time along a radial orbit. Since $\dot{\varphi} = 0$, the angular momentum vanishes, $L = 0$, and the equation of motion (9.15) reads

$$\dot{r}^2 + \left(1 - \frac{2m}{r}\right) = E^2 \; . \tag{10.1}$$

Suppose the observer was at rest at $r = R$, then $E^2 = 1 - 2m/R$ and $E^2 < 1$, and we have

$$\dot{r}^2 = \left(1 - \frac{2m}{R}\right) - \left(1 - \frac{2m}{r}\right) = 2m\left(\frac{1}{r} - \frac{1}{R}\right) \; , \tag{10.2}$$

which yields

$$\frac{\mathrm{d}r}{\sqrt{2m\left(\frac{1}{r} - \frac{1}{R}\right)}} = \mathrm{d}\tau \; , \tag{10.3}$$

where τ is the proper time.

This equation admits a parametric solution. Starting from

$$r = \frac{R}{2}(1 + \cos\eta) \; , \quad \mathrm{d}r = -\frac{R}{2}\sin\eta\,\mathrm{d}\eta \; , \tag{10.4}$$

we first see that

$$\begin{aligned}\frac{1}{r}-\frac{1}{R} &= \frac{2}{R(1+\cos\eta)}-\frac{1}{R}=\frac{1}{R}\left(\frac{1-\cos\eta}{1+\cos\eta}\right)\\ &= \frac{1}{R}\frac{(1-\cos\eta)^2}{\sin^2\eta}\,,\end{aligned} \tag{10.5}$$

where we have used in the last step that $1-\cos\eta^2=\sin^2\eta$. This result allows us to translate (10.3) into

$$\begin{aligned}\frac{\sqrt{R}\sin\eta\mathrm{d}r}{\sqrt{2m}(1-\cos\eta)} &= -\frac{R\sqrt{R}}{2\sqrt{2m}}\frac{\sin^2\eta\mathrm{d}\eta}{1-\cos\eta}\\ &= -\sqrt{\frac{R^3}{8m}}(1+\cos\eta)\mathrm{d}\eta\,.\end{aligned} \tag{10.6}$$

Integrating, we find that this solves (10.3) if

$$\tau=\sqrt{\frac{R^3}{8m}}(\eta+\sin\eta)\,,\quad \mathrm{d}\tau=\sqrt{\frac{R^3}{8m}}(1+\cos\eta)\mathrm{d}\eta\,. \tag{10.7}$$

At $\eta=0$, the proper time is $\tau=0$ and $r=R$, i.e. the proper time starts running when the free fall begins. The centre $r=0$ is reached when $\eta=\pi$, i.e. after the proper time

$$\tau_0=\pi\sqrt{\frac{R^3}{8m}}\,. \tag{10.8}$$

This indicates that the observer falls freely within finite time "through" the singularity at $r=2m$ without encountering any (kinematic) problem.

10.1.2 Problems with the Schwarzschild coordinates

However, let us now describe the radial coordinate r as a function of the coordinate time t. Using (9.13), we first find

$$\dot{r}=\frac{\mathrm{d}r}{\mathrm{d}t}\dot{t}=-\frac{\mathrm{d}r}{\mathrm{d}t}\frac{E}{1-2m/r}\,. \tag{10.9}$$

Next, we introduce a new, convenient radial coordinate $\bar{r}$ such that

$$\mathrm{d}\bar{r}=\frac{\mathrm{d}r}{1-2m/r}\,, \tag{10.10}$$

which can be integrated as follows,

$$\begin{aligned}\frac{\mathrm{d}r}{1-2m/r} &= \frac{r/2m-1+1}{r/2m-1}\mathrm{d}r=\mathrm{d}r+\frac{\mathrm{d}r}{r/2m-1}\\ &= \mathrm{d}r+2m\,\mathrm{d}\ln\left(\frac{r}{2m}-1\right)\,,\end{aligned} \tag{10.11}$$

giving

$$\bar{r} = r + 2m \ln\left(\frac{r}{2m} - 1\right) . \tag{10.12}$$

With this, we find

$$\dot{r} = -\frac{E}{1 - 2m/r}\frac{\mathrm{d}r}{\mathrm{d}t} = -E\frac{\mathrm{d}\bar{r}}{\mathrm{d}t} , \tag{10.13}$$

and thus, from the equation of motion (10.1),

$$E^2\left(\frac{\mathrm{d}\bar{r}}{\mathrm{d}t}\right)^2 = E^2 + \frac{2m}{r} - 1 . \tag{10.14}$$

Approaching the Schwarzschild radius from outside, i.e. in the limit $r \to 2m+$, we have

$$\lim_{r\to 2m+} \bar{r} = \lim_{r\to 2m+} 2m\left[1 + \ln\left(\frac{r}{2m} - 1\right)\right] = -\infty , \tag{10.15}$$

However, in the same limit, the equation of motion says

$$E^2\left(\frac{\mathrm{d}\bar{r}}{\mathrm{d}t}\right)^2 \to E^2 , \tag{10.16}$$

and thus

$$\frac{\mathrm{d}\bar{r}}{\mathrm{d}t} \to \pm 1 . \tag{10.17}$$

Of the two signs, we have to select the negative because of $\bar{r} \to -\infty$, as (10.15) shows. Therefore, an approximate solution of the equation of motion near the singularity is

$$\bar{r} \approx -t + \text{const.} \tag{10.18}$$

or, substituting r for $\bar{r}$,

$$\begin{aligned} -t + \text{const.} &= r + 2m \ln\left(\frac{r}{2m} - 1\right) \\ &\approx 2m\left[1 + \ln\left(\frac{r}{2m} - 1\right)\right] . \end{aligned} \tag{10.19}$$

Solving for r, we find

$$\ln\left(\frac{r}{2m} - 1\right) \approx \frac{-t + \text{const.}}{2m} - 1 \tag{10.20}$$

or

$$r \approx 2m + \text{const.}\, \mathrm{e}^{-t/2m} , \tag{10.21}$$

showing that the Schwarzschild radius is only reached after infinitely long coordinate time!

Finally, radial light rays are described by radial null geodesics, thus satisfying

$$0 = \mathrm{d}s^2 = -\left(1 - \frac{2m}{r}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - \frac{2m}{r}} \tag{10.22}$$

or

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \pm\left(1 - \frac{2m}{r}\right) , \tag{10.23}$$

suggesting that the light cones become infinitely narrow as $r \to 2m+$.

These results are obviously dissatisfactory: while a freely falling observer reaches the Schwarzschild radius and even the centre of the Schwarzschild space-time after finite proper time, the coordinate time becomes infinite even for reaching the Schwarzschild radius, and the flattening of the light cones as one approaches the Schwarzschild radius is totally unwanted because causality cannot be assessed when the light cone degenerates to a line.

10.1.3 Curvature at $r = 2m$

Moreover, consider the components of the Ricci tensor given in (8.57) and (8.58) near the Schwarzschild radius. Since $a = -b$ and

$$b = -\frac{1}{2}\ln\left(1 - \frac{2m}{r}\right) = -a \tag{10.24}$$

from (8.65), the needed derivatives are

$$a' = \frac{m}{r(r-2m)} = -b' , \quad a'' = -\frac{2m(r-m)}{r^2(r-2m)^2} = -b'' . \tag{10.25}$$

Thus,

$$R_{00} = -\left(a'' + 2a'^2 + \frac{2a'}{r}\right)\left(1 - \frac{2m}{r}\right) = 0 = -R_{11} \tag{10.26}$$

and

$$\begin{aligned} R_{22} &= -\frac{2a'}{r}\left(1 - \frac{2m}{r}\right) + \frac{1}{r^2}\left(1 - \mathrm{e}^{-2b}\right) \\ &= -\frac{2m(r-2m)}{r^3(r-2m)} + \frac{2m}{r^3} = 0 = R_{33} , \end{aligned} \tag{10.27}$$

i.e. the components of the Ricci tensor in the Schwarzschild tetrad remain perfectly regular at the Schwarzschild radius!

10.2 The Kruskal Continuation of the Schwarzschild Space-Time

10.2.1 Transformation to Kruskal coordinates

We shall now try to remove the obvious problems with the Schwarzschild coordinates by transforming (t, r) to new coordinates (u, v), leaving ϑ and φ, requiring that the metric can be written as

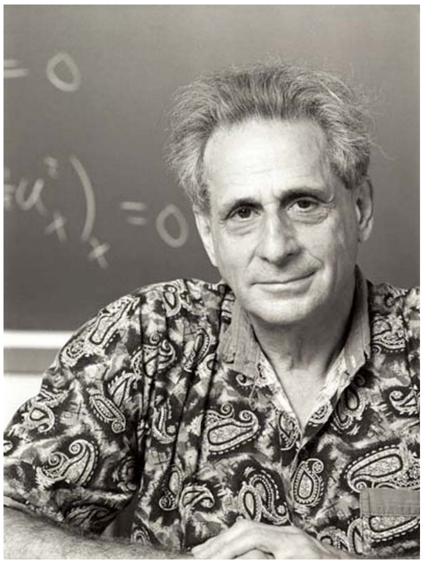
$$g = -f^2(u, v)(\mathrm{d}v^2 - \mathrm{d}u^2) + r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}\varphi^2) \tag{10.28}$$

with a function $f(u, v)$ to be determined.

Provided $f(u, v) \neq 0$, radial light rays propagate as in a two-dimensional Minkowski metric according to

$$\mathrm{d}v^2 = \mathrm{d}u^2 \ , \quad \left(\frac{\mathrm{d}u}{\mathrm{d}v}\right)^2 = 1 \ , \tag{10.29}$$

which shows that the light cones remain undeformed in the new coordinates.



Martin Kruskal (1925–2006)

The Jacobian matrix of the transformation from the Schwarzschild coordinates $(t, r, \vartheta, \varphi)$ to the new coordinates $(v, u, \vartheta, \varphi)$ is

$$J^\alpha_\beta = \begin{pmatrix} v_t & u_t & 0 & 0 \\ v_r & u_r & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \tag{10.30}$$

where subscripts denote derivatives,

$$v_t = \partial_t v \ , \quad v_r = \partial_r v \tag{10.31}$$

and likewise for u. The metric $\bar{g}$ in the new coordinates,

$$\bar{g} = \mathrm{diag}(-f^2, f^2, r^2, r^2 \sin^2\vartheta) \ , \tag{10.32}$$

is transformed into the original Schwarzschild coordinates by

$$\begin{aligned} g &= J\bar{g}J^T \qquad (10.33) \\ &= \begin{pmatrix} -f^2\left(v_t^2 - u_t^2\right) & -f^2\left(v_t v_r - u_t u_r\right) & 0 & 0 \\ -f^2\left(v_t v_r - u_t u_r\right) & -f^2\left(v_r^2 - u_r^2\right) & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\vartheta \end{pmatrix} , \end{aligned}$$

which, by comparison with our requirement (10.28), yields the three equations

$$\begin{aligned} -\left(1 - \frac{2m}{r}\right) &= -f^2\left(v_t^2 - u_t^2\right) \ , \\ \frac{1}{1 - 2m/r} &= -f^2\left(v_r^2 - u_r^2\right) \ , \\ 0 &= v_t v_r - u_t u_r \ . \end{aligned} \tag{10.34}$$

For convenience, we now fall back to the radial coordinates $\bar{r}$ and introduce the function

$$F(\bar{r}) \equiv \frac{1 - 2m/r}{f^2(r)} \ , \tag{10.35}$$

assuming that f will turn out to depend on r only since any dependence on time and on the angles ϑ and φ is forbidden in a static, spherically-symmetric space-time. Then,

$$v_{\bar{r}} = \frac{\mathrm{d}r}{\mathrm{d}\bar{r}} v_r = \left(1 - \frac{2m}{r}\right) v_r \tag{10.36}$$

and the same for u.

The equations (10.34) then transform to

$$F(\bar{r}) = v_t^2 - u_t^2 \ , \quad -F(\bar{r}) = v_{\bar{r}}^2 - u_{\bar{r}}^2 \ , \quad v_t v_{\bar{r}} - u_t u_{\bar{r}} = 0 \ . \tag{10.37}$$

Now, we add the two equations containing $F(\bar{r})$ and then add and subtract from the result twice the third equation from (10.37). This yields

$$(v_t \pm v_{\bar{r}})^2 = (u_t \pm u_{\bar{r}})^2 \ . \tag{10.38}$$

Taking the square root of this equation, we can choose the signs. The choice

$$v_t + v_{\bar{r}} = u_t + u_{\bar{r}} \ , \quad v_t - v_{\bar{r}} = -(u_t - u_{\bar{r}}) \tag{10.39}$$

avoids that the Jacobian matrix could become singular, $\det J = 0$.

Adding and subtracting the equations (10.39), we find

$$v_t = u_{\bar{r}} \ , \quad u_t = v_{\bar{r}} \ . \tag{10.40}$$

Taking partial derivatives once with respect to t and once with respect to $\bar{r}$ allows us to combine these equations to find the wave equations

$$v_{tt} - v_{\bar{r}\bar{r}} = 0 \ , \quad u_{tt} - u_{\bar{r}\bar{r}} = 0 \ , \tag{10.41}$$

which are solved by any two functions $h_\pm$ propagating with unit velocity,

$$v = h_+(\bar{r} + t) + h_-(\bar{r} - t) \ , \quad u = h_+(\bar{r} + t) - h_-(\bar{r} - t) \ , \tag{10.42}$$

where the signs were chosen such as to satisfy the sign choice in (10.39).

Now, since

$$\begin{aligned} v_t &= h'_+ - h'_- \ , \quad u_t = h'_+ + h'_- \ , \\ v_{\bar{r}} &= h'_+ + h'_- \ , \quad u_{\bar{r}} = h'_+ - h'_- \ , \end{aligned} \tag{10.43}$$

where the primes denote derivatives with respect to the functions' arguments, we find from (10.37)

$$F(\bar{r}) = (h'_+ - h'_-)^2 - (h'_+ + h'_-)^2 = -4h'_+ h'_- \ . \tag{10.44}$$

We start from outside the Schwarzschild radius, assuming $r > 2m$, where also $F(\bar{r}) > 0$ according to (10.35). The derivative of (10.44) with respect to $\bar{r}$ yields

$$F'(\bar{r}) = -4\left(h''_+ h'_- + h'_+ h''_-\right) \tag{10.45}$$

or, with (10.44),

$$\frac{F'}{F} = \frac{h''_+}{h'_+} + \frac{h''_-}{h'_-} \ . \tag{10.46}$$

the derivative of (10.44) with respect to time yields

$$0 = -4\left(h''_+ h'_- - h'_+ h''_-\right) \quad \Rightarrow \quad \frac{h''_+}{h'_+} - \frac{h''_-}{h'_-} = 0 \ . \tag{10.47}$$

The sum of these two equations gives

$$(\ln F)' = 2(\ln h'_+)' \ . \tag{10.48}$$

Now, the left-hand side depends on $\bar{r}$, the right-hand side on the independent variable $\bar{r} + t$. Thus, the two sides of this equation must equal the same constant, which we call $2C$:

$$(\ln F)' = 2C = 2(\ln h'_+)' \ . \tag{10.49}$$

The left of these equations yields

$$\ln F = 2C\bar{r} + \text{const.} \quad \Rightarrow \quad F = \text{const.}\mathrm{e}^{2C\bar{r}} \ , \tag{10.50}$$

while the right equation gives

$$\ln h'_+ = C(\bar{r} + t) + \text{const.} \tag{10.51}$$

or

$$h_+ = \text{const.}\mathrm{e}^{C(\bar{r}+t)} \ . \tag{10.52}$$

For later convenience, we choose the remaining constants in (10.50) and (10.52) such that

$$F(\bar{r}) = C^2\mathrm{e}^{2C\bar{r}} \ , \quad h_+(\bar{r} + t) = \frac{1}{2}\mathrm{e}^{C(\bar{r}+t)} \ , \tag{10.53}$$

and (10.47) gives

$$h_-(\bar{r} - t) = -\frac{1}{2}\mathrm{e}^{C(\bar{r}-t)} \ , \tag{10.54}$$

where the negative sign must be chosen to satisfy both (10.44) and $F > 0$.

Working our way back, we find

$$\begin{aligned} u &= h_+(\bar{r} + t) - h_-(\bar{r} - t) = \frac{1}{2}\left[\mathrm{e}^{C(\bar{r}+t)} + \mathrm{e}^{C(\bar{r}-t)}\right] \\ &= \mathrm{e}^{C\bar{r}}\cosh(Ct) = \left(\frac{r}{2m} - 1\right)^{2mC} \mathrm{e}^{Cr}\cosh(Ct) \ , \end{aligned} \tag{10.55}$$

using (10.12) for $\bar{r}$. Similarly, we find

$$v = \left(\frac{r}{2m} - 1\right)^{2mC} \mathrm{e}^{Cr} \sinh(Ct) , \tag{10.56}$$

and the function f follows from (10.35),

$$\begin{aligned} f^2 &= \frac{1 - 2m/r}{F} = \frac{1 - 2m/r}{C^2} \mathrm{e}^{-2C\bar{r}} \\ &= \frac{1 - 2m/r}{C^2} \mathrm{e}^{-2Cr} \exp\left[-4mC \ln\left(\frac{r}{2m} - 1\right)\right] \\ &= \frac{2m}{rC^2} \left(\frac{r}{2m} - 1\right)^{1-4mC} \mathrm{e}^{-2Cr} . \end{aligned} \tag{10.57}$$

Now, since we want f to be non-zero and regular at $r = 2m$, we must require $4mC = 1$, which finally fixes the *Kruskal transformation* of the Schwarzschild metric,

$$\begin{aligned} u &= \sqrt{\frac{r}{2m} - 1}\, \mathrm{e}^{r/4m} \cosh\left(\frac{t}{4m}\right) , \\ v &= \sqrt{\frac{r}{2m} - 1}\, \mathrm{e}^{r/4m} \sinh\left(\frac{t}{4m}\right) , \\ f^2 &= \frac{32m^3}{r} \mathrm{e}^{-r/2m} . \end{aligned} \tag{10.58}$$

We have thus achieved our goal to replace the Schwarzschild coordinates by others in which the Schwarzschild metric remains prefectly regular at $r = 2m$.

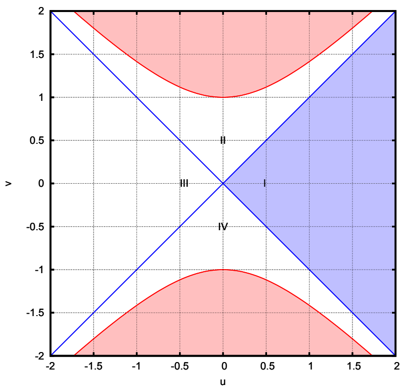


Illustration of the Kruskal continuation in the u-v plane. The Schwarzschild domain $r > 2m$ is shaded in blue, the forbidden region $r < 0$ is shaded in red.

10.2.2 Physical meaning of the Kruskal continuation

Since $\cosh^2(x) - \sinh^2(x) = 1$, eqs. (10.58) imply

$$u^2 - v^2 = \left(\frac{r}{2m} - 1\right) \mathrm{e}^{r/2m} , \quad \frac{v}{u} = \tanh\left(\frac{t}{4m}\right) . \tag{10.59}$$

This means $u = |v|$ for $r = 2m$, which is reached for $t \to \pm\infty$. Lines of constant coordinate time t are straight lines through the origin in the (u, v) plane with slope $\tanh(t/4m)$, and lines of constant radial coordinate r are hyperbolae.

The metric in Kruskal coordinates (10.28) is regular as long as $r(u, v) > 0$, which is the case for

$$u^2 - v^2 > -1 , \tag{10.60}$$

as (10.59) shows. The hyperbola limiting the regular domain in the Kruskal manifold is thus given by $v^2 - u^2 = 1$. If (10.60) is satisfied, r is uniquely defined, because the equation

$$\rho(x) \equiv (x - 1)\mathrm{e}^x = u^2 - v^2 > -1 \tag{10.61}$$

is monotonic for $x > 0$:

$$\rho'(x) = e^x(x-1) + e^x = xe^x > 0 \quad (x > 0)\ . \tag{10.62}$$

The domain of the original Schwarzschild solution is restricted to $u > 0$ and $|v| < u$, but this is a consequence of our choice for the relative signs of $h_\pm$ in (10.54). We could as well have chosen $h_+ < 0$ and $h_- > 0$, which would correspond to the replacement $(u, v) \to (-u, -v)$.

The original Schwarzschild solution for $r < 2m$ also satisfies Einstein's vacuum field equations. There, the Schwarzschild metric shows that r then behaves like a time coordinate because $g_{rr} < 0$, and t behaves like a spatial coordinate.

Looking at the definition of $F(\bar{r})$ in (10.35), we see that $r < 2m$ corresponds to $F < 0$, which implies that h_+ and h_- must have the same (rather than opposite) signs because of (10.44). This interchanges the functions u and v from (10.58), i.e. $u \to v$ and $v \to u$. Then, the condition $|v| < u$ derived for $r > 2m$ changes to $|v| > u$.

In summary, the exterior of the Schwarzschild radius corresponds to the domain $u > 0$, $|v| < u$, and its interior is bounded in the (u, v) plane by the lines $u > 0$, $|v| = u$ and $v^2 - u^2 = 1$.

Radial light rays propagate according to $ds^2 = 0$ or $dv = du$, i.e. they are straight diagonal lines in the (u, v) plane. This shows that light rays can propagate freely into the region $r < 2m$, but there is no causal connection from within $r < 2m$ to the outside.

The Killing vector field $K = \partial_t$ for the Schwarzschild space-time outside $r = 2m$ becomes space-like for $r < 2m$, which means that the space-time cannot be static any more inside the Schwarzschild radius.

10.2.3 Eddington-Finkelstein coordinates

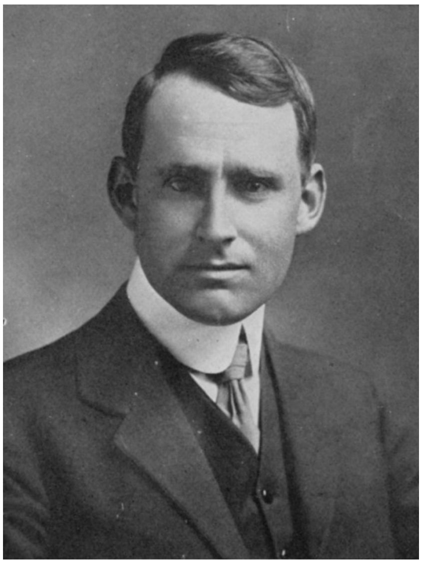
We now want to study the collapse of an object, e.g. a star. For this purpose, coordinates originally introduced by Eddington and re-discovered by Finkelstein are convenient, which are defined by

$$\begin{aligned} r = r'\ , \quad \vartheta = \vartheta'\ , \quad \varphi = \varphi' \\ t = t' - 2m \ln\left(\pm\frac{r}{2m} \mp 1\right) \end{aligned} \tag{10.63}$$

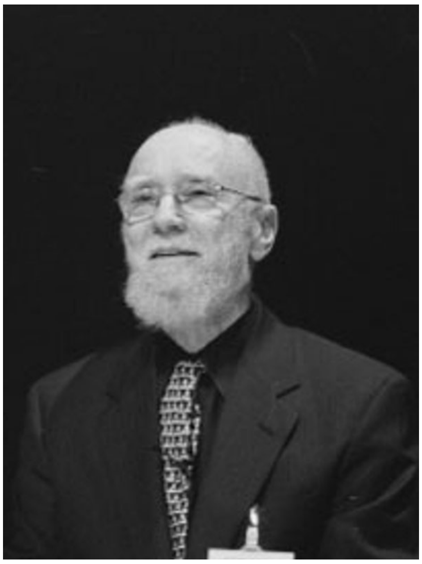
in analogy to the radial coordinate $\bar{r}$ from (10.12), where the upper and lower signs in the second line are valid for $r > 2m$ and $r < 2m$, respectively.

Since

$$e^{\pm t/4m} = e^{\pm t'/4m} \begin{cases} \left(\frac{r}{2m} - 1\right)^{\mp 1/2} & (r > 2m) \\ \left(-\frac{r}{2m} + 1\right)^{\mp 1/2} & (r < 2m) \end{cases}\ , \tag{10.64}$$



Arthur S. Eddington (1882–1944)



David R. Finkelstein (1929–)

inserting these expressions into the Kruskal coordinates (10.58) shows that they are related by

$$\begin{aligned} u &= \frac{e^{r/4m}}{2}\left(\mathrm{e}^{t'/4m} + \frac{r-2m}{2m}\mathrm{e}^{-t'/4m}\right) \\ v &= \frac{e^{r/4m}}{2}\left(\mathrm{e}^{t'/4m} - \frac{r-2m}{2m}\mathrm{e}^{-t'/4m}\right) , \end{aligned} \tag{10.65}$$

such that

$$\frac{r-2m}{2m}\mathrm{e}^{r/2m} = u^2 - v^2 , \quad \mathrm{e}^{t'/2m} = \frac{r-2m}{2m}\frac{u+v}{u-v} . \tag{10.66}$$

The first of these equations shows again that r can be uniquely determined from u and v if $u^2 - v^2 > -1$. The second equation determines t' uniquely provided $r > 2m$ and $(u+v)/(u-v) > 0$, or $r < 2m$ and $(u+v)/(u-v) < 0$. This is possible if $v > -u$.

Using

$$\mathrm{d}t = \mathrm{d}t' - 2m\frac{1}{\pm r/2m \mp 1}\frac{\pm\mathrm{d}r}{2m} = \mathrm{d}t' - \frac{2m}{r}\frac{\mathrm{d}r}{1-2m/r} , \tag{10.67}$$

we find

$$-\left(1-\frac{2m}{r}\right)\mathrm{d}t^2 = -\left(1-\frac{2m}{r}\right)\mathrm{d}t'^2 + \frac{4m}{r}\mathrm{d}t'\mathrm{d}r - \frac{4m^2}{r^2}\frac{\mathrm{d}r^2}{1-2m/r} , \tag{10.68}$$

and thus the metric in Eddington-Finkelstein coordinates reads

$$\begin{aligned} g &= -\left(1-\frac{2m}{r}\right)\mathrm{d}t'^2 + \left(1-\frac{4m^2}{r^2}\right)\frac{\mathrm{d}r^2}{1-2m/r} \\ &+ \frac{4m}{r}\mathrm{d}t'\mathrm{d}r + r^2\mathrm{d}\Omega^2 \\ &= -\left(1-\frac{2m}{r}\right)\mathrm{d}t'^2 + \left(1+\frac{2m}{r}\right)\mathrm{d}r^2 + \frac{4m}{r}\mathrm{d}t'\mathrm{d}r + r^2\mathrm{d}\Omega^2 . \end{aligned} \tag{10.69}$$

Thus, the metric acquires off-diagonal elements such that it no longer depends on t' and r separately.

For radial light rays, $\mathrm{d}\Omega = 0$ and $\mathrm{d}s^2 = 0$, which implies from (10.69)

$$\left(1-\frac{2m}{r}\right)\mathrm{d}t'^2 - \left(1+\frac{2m}{r}\right)\mathrm{d}r^2 - \frac{4m}{r}\mathrm{d}t'\mathrm{d}r = 0 , \tag{10.70}$$

which can be factorised as

$$\left[\left(1-\frac{2m}{r}\right)\mathrm{d}t' - \left(1+\frac{2m}{r}\right)\mathrm{d}r\right](\mathrm{d}t' + \mathrm{d}r) = 0 . \tag{10.71}$$

Thus, the light cones are defined either by

$$\frac{\mathrm{d}r}{\mathrm{d}t'} = -1 \quad \Rightarrow \quad r = -t' + \text{const.} \tag{10.72}$$

or by

$$\frac{\mathrm{d}r}{\mathrm{d}t'} = \frac{r-2m}{r+2m} \;, \tag{10.73}$$

which shows that $\mathrm{d}r/\mathrm{d}t' \to -1$ for $r \to 0$, $\mathrm{d}r/\mathrm{d}t' = 0$ for $r = 2m$, and $\mathrm{d}r/\mathrm{d}t' = 1$ for $r \to \infty$. Due to the vanishing derivative of r with respect to t' at $r = 2m$, geodesics cannot cross the Schwarzschild radius from inside, but they can from outside because of (10.72).

10.2.4 Redshift approaching the Schwarzschild radius

Suppose a light-emitting source (e.g. an astronaut with a torch) is falling towards a (Schwarzschild) black hole, what does a distant observer see? Let v and u be the four-velocities of the astronaut and the observer, respectively. Then according to (4.48) the redshift of the light from the torch as seen by the observer is

$$1+z = \frac{\nu_{\mathrm{em}}}{\nu_{\mathrm{obs}}} = \frac{\langle k, v\rangle}{\langle k, u\rangle} \;, \tag{10.74}$$

where k is the wave vector of the light.

We transform to the retarded time $t_{\mathrm{ret}} \equiv t - \bar{r}$, with $\bar{r}$ given by (10.12). Then, (10.10) implies that

$$\mathrm{d}t_{\mathrm{ret}} = \mathrm{d}t - \frac{\mathrm{d}r}{1-2m/r} \;, \tag{10.75}$$

thus

$$\mathrm{d}t^2 = \mathrm{d}t_{\mathrm{ret}}^2 + \frac{\mathrm{d}r^2}{(1-2m/r)^2} + \frac{2\mathrm{d}t_{\mathrm{ret}}\mathrm{d}r}{1-2m/r} \tag{10.76}$$

and the Schwarzschild metric transforms to

$$g = -\left(1-\frac{2m}{r}\right)\mathrm{d}t_{\mathrm{ret}}^2 - 2\mathrm{d}t_{\mathrm{ret}}\mathrm{d}r + r^2\mathrm{d}\Omega^2 \;. \tag{10.77}$$

For radial light rays, $\mathrm{d}\Omega = 0$, this means

$$0 = -\left(1-\frac{2m}{r}\right)\mathrm{d}t_{\mathrm{ret}}^2 - 2\mathrm{d}t_{\mathrm{ret}}\mathrm{d}r \;, \tag{10.78}$$

which is possible only if $\mathrm{d}t_{\mathrm{ret}} = 0$. This shows that light rays must propagate along r, or $k \propto \partial_r$, which is of course a consequence of our using the retarded time t_{ret}.

For a distant observer at a fixed distance $r \gg 2m$, the metric (10.77) simplifies to

$$g \approx -\mathrm{d}t_{\mathrm{ret}}^2 \;, \tag{10.79}$$

which shows that t_{ret} is also that distant observer's proper time.

Expanding the astronaut's velocity as

$$v = \dot{t}_{\rm ret}\partial_{t_{\rm ret}} + \dot{r}\partial_r \;, \tag{10.80}$$

we find

$$\langle k, v\rangle = \langle \partial_r, v\rangle = g_{t_{\rm ret}r}\dot{t}_{\rm ret} = -\dot{t}_{\rm ret} \tag{10.81}$$

because according to the metric (10.77) $\langle \partial_r, \partial_r\rangle = 0$. The dots here indicate derivatives with respect to the astronaut's proper time.

Similarly, we have for a fixed distant observer $u = \dot{t}\partial_t$. Expanding ∂_t into $\partial_{t_{\rm ret}}$ and ∂_r, we find

$$\partial_{t_{\rm ret}} = a_t\partial_t + a_r\partial_r \;, \qquad \begin{aligned} a_t &= {\rm d}t_{\rm ret}\partial_t = \left({\rm d}t - \frac{{\rm d}r}{1-2m/r}\right)\partial_t = 1 \\ a_r &= {\rm d}t_{\rm ret}\partial_r = -\frac{1}{1-2m/r} \end{aligned} \tag{10.82}$$

using (10.75), and thus

$$\langle k, u\rangle = \dot{t}\left\langle \partial_r, \partial_{t_{\rm ret}} + \frac{\partial_r}{1-2m/r}\right\rangle = -\dot{t} \approx -1 \tag{10.83}$$

because of (10.79). Here, the dots indicate derivatives with respect to the astronaut's proper time. This gives the redshift

$$1 + z \approx \dot{t}_{\rm ret} = \dot{t} - \frac{\dot{r}}{1-2m/r} \;. \tag{10.84}$$

When restricted to radial motion, $\dot{\varphi} = 0 = L$, the equation of motion (9.15) is

$$\dot{r}^2 + 1 - \frac{2m}{r} = E^2 \;, \tag{10.85}$$

where E was defined as

$$E = -\dot{t}\left(1 - \frac{2m}{r}\right) \;, \tag{10.86}$$

see (9.12). Requiring that the astronaut's proper time increases with the coordinate time, $\dot{t} > 0$ and $E < 0$. Since $\dot{r} < 0$ for the infalling astronaut,

$$\dot{r} = E\sqrt{1 - \frac{\delta}{E^2}} \;, \tag{10.87}$$

with $\delta \equiv 1 - 2m/r$.

The redshift (10.84) can now be written

$$1 + z = -\frac{E}{\delta}\left(1 + \sqrt{1 - \frac{\delta}{E^2}}\right) \approx \frac{-2E}{\delta} \tag{10.88}$$

near the Schwarzschild radius, where $\delta \to 0+$.

The change in the astronauts radius with the observer's proper time is

$$\frac{\mathrm{d}r}{\mathrm{d}t_{\mathrm{ret}}} = \frac{\dot{r}}{\dot{t}_{\mathrm{ret}}} = \frac{\dot{r}}{1+z} \approx -\frac{E\delta}{2E} = -\frac{\delta}{2} \,. \tag{10.89}$$

Since, by definition,

$$-\frac{\delta}{2} = -\frac{1}{2}\left(1 - \frac{2m}{r}\right) \approx -\frac{r-2m}{4m} \,, \tag{10.90}$$

we have

$$\mathrm{d}(r-2m) = \mathrm{d}r = -\frac{\delta}{2}\mathrm{d}t_{\mathrm{ret}} \approx -\frac{r-2m}{4m}\,\mathrm{d}t_{\mathrm{ret}} \,, \tag{10.91}$$

thus

$$\frac{\mathrm{d}(r-2m)}{r-2m} \approx -\frac{\mathrm{d}t_{\mathrm{ret}}}{4m} \quad \Rightarrow \quad r - 2m \propto \mathrm{e}^{-t_{\mathrm{ret}}/4m} \tag{10.92}$$

and

$$\delta \propto \frac{\mathrm{e}^{-t_{\mathrm{ret}}/4m}}{r} \approx \frac{\mathrm{e}^{-t_{\mathrm{ret}}/4m}}{4m} \,, \tag{10.93}$$

showing that the astronaut's redshift

$$1 + z \approx -\frac{2E}{\delta} \propto -8mE\mathrm{e}^{t_{\mathrm{ret}}/4m} \tag{10.94}$$

grows exponentially as he approaches the Schwarzschild radius.

Chapter 11

Charged, Rotating Black Holes

11.1 Metric and electromagnetic field

11.1.1 The Reissner-Nordström solution

Hans J. Reissner (1874–1967)

Gunnar Nordström (1881–1923)

The Schwarzschild solution is a very important exact solution of Einstein's vacuum equations, but we expect that real objects collapsing to become black holes may be charged and rotating. We shall now generalise the Schwarzschild solution in these two directions.

First, we consider a static, axially-symmetric solution of Einstein's equations in the presence of an electromagnetic charge q at the origin of the Schwarzschild coordinates, i.e. at $r = 0$. The electromagnetic field will then also be static and axially symmetric.

Expressing the field tensor in the Schwarzschild tetrad, we thus expect the Faraday 2-form (5.87) to be

$$F = -\frac{q}{r^2}\,\mathrm{d}t \wedge \mathrm{d}r = -\frac{q}{r^2}\,\mathrm{e}^{-a-b}\,\theta^0 \wedge \theta^1 \ . \tag{11.1}$$

We shall verify below that $a = -b$ also for a Schwarzschild solution with charge, so that the exponential factor will become unity later.

The electromagnetic energy-momentum tensor

$$T^{\mu\nu} = \frac{1}{4\pi}\left[F^{\mu\lambda}F^{\nu}{}_{\lambda} - \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right] \tag{11.2}$$

is now easily evaluated. Since the only non-vanishing component of $F_{\mu\nu}$ is F_{01} and the metric is diagonal in the Schwarzschild tetrad, $g = \mathrm{diag}(-1, 1, 1, 1)$, we have

$$F_{\alpha\beta}F^{\alpha\beta} = F_{01}F^{01} + F_{10}F^{10} = -2F_{01}^2 \ . \tag{11.3}$$

Using this, we find the components of the energy-momentum tensor

$$
\begin{aligned}
T^{00} &= \frac{1}{4\pi}\left[F^{01}F^{0}{}_{1} - \frac{1}{2}F_{01}^{2}\right] = \frac{1}{8\pi}F_{01}^{2} = \frac{q^2}{8\pi r^4}\mathrm{e}^{-2(a+b)} \;, \\
T^{11} &= \frac{1}{4\pi}\left[F^{10}F^{1}{}_{0} + \frac{1}{2}F_{01}^{2}\right] = -\frac{q^2}{8\pi r^4}\mathrm{e}^{-2(a+b)} = -T^{00} \;, \\
T^{22} &= \frac{1}{8\pi}F_{01}^{2} = \frac{q^2}{8\pi r^4}\mathrm{e}^{-2(a+b)} = T^{33} \;.
\end{aligned}
\tag{11.4}
$$

Inserting these expressions instead of zero into the right-hand side of Einstein's field equations yields, with (8.60),

$$
\begin{aligned}
G_{00} &= \frac{1}{r^2} - \mathrm{e}^{-2b}\left(\frac{1}{r^2} - \frac{2b'}{r}\right) = \frac{8\pi G}{c^4}T_{00} = \frac{Gq^2}{c^4 r^4}\mathrm{e}^{-2(a+b)} \;, \\
G_{11} &= -\frac{1}{r^2} + \mathrm{e}^{-2b}\left(\frac{1}{r^2} + \frac{2a'}{r}\right) = -\frac{8\pi G}{c^4}T_{00} = -G_{00} \;.
\end{aligned}
\tag{11.5}
$$

Adding these two equations, we find $a' + b' = 0$, which implies $a + b = 0$ because the functions have to tend to zero at infinity. This confirms that we can identify $\mathrm{d}t \wedge \mathrm{d}r = \theta^0 \wedge \theta^1$ and write $F_{01} = q/r^2$.

Analogous to (8.63), we note that the first of equations (11.5) with $a = -b$ is equivalent to

$$
\left(r\mathrm{e}^{-2b}\right)' = 1 - \frac{Gq^2}{c^4 r^2} \;, \tag{11.6}
$$

which gives

$$
\mathrm{e}^{-2b} = \mathrm{e}^{2a} = 1 - \frac{2m}{r} + \frac{Gq^2}{c^4 r^2} \;, \tag{11.7}
$$

if we use $-2m$ as the integration constant as for the neutral Schwarzschild solution.

Defining

$$
\Delta \equiv r^2 - 2mr + \frac{Gq^2}{c^4} \;, \tag{11.8}
$$

we thus obtain the metric of a charged Schwarzschild black hole,

$$
g = -\frac{\Delta}{r^2}\mathrm{d}t^2 + \frac{r^2\mathrm{d}r^2}{\Delta} + r^2\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2\right) \;. \tag{11.9}
$$

This is the *Reissner-Nordström solution*. Of course, for $q = 0$ it returns to the Schwarzschild solution. Recall that the unit of the electric charge in the cgs system is $\mathrm{g}^{1/2}\mathrm{cm}^{3/2}\mathrm{s}^{-1}$, so that the quantity Gq^2/c^4 has the dimension of a squared length.

Before we proceed, we should verify that Maxwell's equations are satisfied. First, we note that the Faraday 2-form (11.1) is exact because it is the exterior derivative of the 1-form

$$
A = -\frac{q}{r}\mathrm{d}t \;, \quad \mathrm{d}A = \frac{q}{r^2}\,\mathrm{d}r \wedge \mathrm{d}t = -\frac{q}{r^2}\,\mathrm{d}t \wedge \mathrm{d}r = -\frac{q}{r^2}\,\theta^0 \wedge \theta^1 \;. \tag{11.10}
$$

Thus, since $\mathrm{d} \circ \mathrm{d} = 0$, $\mathrm{d}F = \mathrm{d}^2 A = 0$, so that the homogeneous Maxwell equations are satisfied.

Moreover, we notice that

$$*F = \frac{q}{r^2}\,\theta^2 \wedge \theta^3 \,, \tag{11.11}$$

which is easily verified using (5.76),

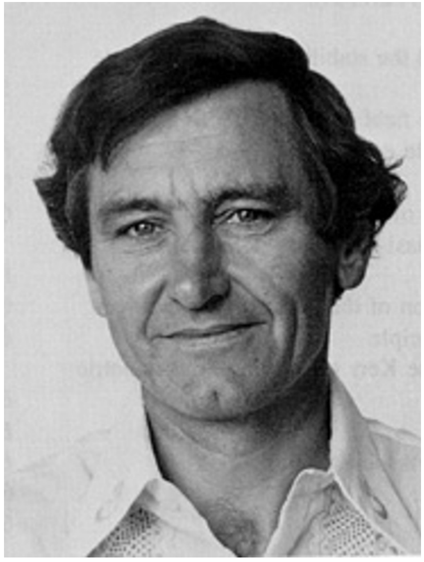
$$\begin{aligned} *(\theta^0 \wedge \theta^1) &= \frac{1}{2} g^{00} g^{11} \epsilon_{01\alpha\beta}\, \theta^\alpha \wedge \theta^\beta \\ &= -\frac{1}{2}\left(\theta^2 \wedge \theta^3 - \theta^3 \wedge \theta^2\right) = -\theta^2 \wedge \theta^3 \,. \end{aligned} \tag{11.12}$$

Inserting the Schwarzschild tetrad from (8.40) yields

$$*F = q \sin\vartheta\, \mathrm{d}\vartheta \wedge \mathrm{d}\varphi = -\mathrm{d}\left(q \cos\vartheta \mathrm{d}\varphi\right) \,, \tag{11.13}$$

which shows, again by $\mathrm{d} \circ \mathrm{d} = 0$, that $\mathrm{d}(*F) = 0$, hence also $(*\mathrm{d}*)F = 0$ and $\delta F = 0$, so that also the inhomogeneous Maxwell equations (in vacuum!) are satisfied.

11.1.2 The Kerr-Newman solution



Roy Kerr (* 1934)

The formal derivation of the metric of a rotating black hole is a formidable task which we cannot possibly demonstrate during this lecture. We thus start with general remarks on the expected form of the metric and then immediately quote the metric coefficients without deriving them.

In presence of angular momentum, we expect the spherical symmetry of the Schwarzschild solution to be broken. Instead, we expect that the solution must be *axisymmetric*, with the axis fixed by the angular momentum. Moreover, we seek to find a stationary solution.

Then, the group $\mathbb{R} \times SO(2)$ must be an isometry of the metric, where $\mathbb{R}$ represents the stationarity and $SO(2)$ the (two-dimensional) rotations about the symmetry axis. Expressing these symmetries, there must be a time-like Killing vector field k and another Killing vector field m which is tangential to the orbits of $SO(2)$.

These two Killing vector fields span the tangent spaces of the two-dimensional submanifolds which are the orbits of $\mathbb{R} \times SO(2)$, i.e. cylinders.

We can choose adapted coordinates t and φ such that $k = \partial_t$ and $m = \partial_\varphi$. Then, the metric can be decomposed as

$$g = g_{ab}(x^i)\mathrm{d}x^a \mathrm{d}x^b + g_{ij}(x^k)\mathrm{d}x^i \mathrm{d}x^j \,, \tag{11.14}$$

where indices $a, b = 0, 1$ indicate the coordinates on the orbits of $\mathbb{R} \times SO(2)$, and indices $i, j, k = 2, 3$ the others. Note that, due to the symmetry imposed, the remaining metric coefficients can only depend on the coordinates x^i.

A stationary, axi-symmetric space-time (M, g) can thus be foliated into $M = \Sigma \times \Gamma$, where Σ is diffeomorphic to the orbits of $\mathbb{R} \times SO(2)$, and the metric coefficients in adapted coordinates can only depend on the coordinates of Γ. We write

$$ {}^{(4)}g = \sigma + g \tag{11.15} $$

and have

$$ \sigma = \sigma_{ab}(x^i)\mathrm{d}x^a \mathrm{d}x^b \ . \tag{11.16} $$

The coefficients σ_{ab} are scalar products of the two Killing vector fields k and m,

$$ (\sigma_{ab}) = \begin{pmatrix} -\langle k, k\rangle & \langle k, m\rangle \\ \langle k, m\rangle & \langle m, m\rangle \end{pmatrix} , \tag{11.17} $$

and we abbreviate the determinant of σ by

$$ \rho \equiv \sqrt{-\det \sigma} = \sqrt{\langle k, k\rangle\langle m, m\rangle + \langle k, m\rangle^2} \ . \tag{11.18} $$

Without proof, we now quote the metric of a stationary, axially-symmetric solution of Einstein's field equations for either vacuum or an electromagnetic field. We first define the auxiliary quantities

$$ \begin{aligned} \Delta &:= r^2 - 2mr + Q^2 + a^2 \ , \quad \rho^2 := r^2 + a^2\cos^2\vartheta \ , \\ \Sigma^2 &:= (r^2 + a^2)^2 - a^2\Delta\sin^2\vartheta \ . \end{aligned} \tag{11.19} $$

Moreover, we need appropriately scaled expressions Q and a for the charge q and the angular momentum L of the black hole, which are given by

$$ Q^2 := \frac{Gq^2}{c^4} \ , \quad a := \frac{L}{Mc} = \frac{GL}{mc^3} \tag{11.20} $$

and both have the dimension of a length.

With these definitions, we can write the coefficients of the metric as

$$ \begin{aligned} g_{tt} &= -1 + \frac{2mr - Q^2}{\rho^2} = \frac{a^2\sin^2\vartheta - \Delta}{\rho^2} \ , \\ g_{t\varphi} &= -\frac{2mr - Q^2}{\rho^2} a\sin^2\vartheta = -\frac{r^2 + a^2 - \Delta}{\rho^2} a\sin^2\vartheta \ , \\ g_{rr} &= \frac{\rho^2}{\Delta} \ , \quad g_{\vartheta\vartheta} = \rho^2 \ , \quad g_{\varphi\varphi} = \frac{\Sigma^2}{\rho^2}\sin^2\vartheta \ . \end{aligned} \tag{11.21} $$

Evidently, for $a = 0 = Q$, $\rho = r$, $\Delta = r^2 - 2mr$ and $\Sigma = r^2$ and we return to the Schwarzschild solution (8.66). For $a = 0$, we still have $\rho = r$ and $\Sigma = r^2$, but $\Delta = r^2 - 2mr + Q^2$ as in (11.8), and we return to the Reissner-Nordström solution (11.9). For $Q = 0$, we obtain the *Kerr solution* for a rotating, uncharged black hole, and for $a \neq 0$ and $Q \neq 0$, the solution is called *Kerr-Newman solution*.

Also without derivation, we quote that the vector potential of the rotating, charged black hole is given by the 1-form

$$A = -\frac{qr}{\rho^2}\left(\mathrm{d}t - a\sin^2\vartheta\mathrm{d}\varphi\right) , \tag{11.22}$$

from which we obtain the Faraday 2-form

$$\begin{aligned} F &= \mathrm{d}A = \frac{q}{\rho^4}\left(r^2 - a^2\cos^2\vartheta\right)\mathrm{d}r \wedge \left(\mathrm{d}t - a\sin^2\vartheta\mathrm{d}\varphi\right) \\ &+ \frac{2qra}{\rho^4}\sin\vartheta\cos\vartheta\mathrm{d}\vartheta \wedge \left[\left(r^2 + a^2\right)\mathrm{d}\varphi - a\mathrm{d}t\right] . \end{aligned} \tag{11.23}$$

For $a = 0$, this trivially returns to the field (11.1) for the Reissner-Nordström solution. Sufficiently far away from the black hole, such that $a \ll r$, we can approximate to first order in a/r and write

$$F = \frac{q}{r^2}\mathrm{d}r \wedge \left(\mathrm{d}t - a\sin^2\vartheta\mathrm{d}\varphi\right) + \frac{2qa}{r}\sin\vartheta\cos\vartheta\mathrm{d}\vartheta \wedge \mathrm{d}\varphi . \tag{11.24}$$

The field components far away from the black hole can now be read off the result (11.24). Using the orthonormal basis

$$e_t = \partial_t , \quad e_r = \partial_r , \quad e_\vartheta = \frac{1}{r}\partial_\vartheta , \quad e_\varphi = \frac{1}{r\sin\vartheta}\partial_\varphi , \tag{11.25}$$

we find

$$\begin{aligned} E_r &= F(e_r, e_t) = \frac{q}{r^2} , \\ B_r &= F(e_\vartheta, e_\varphi) = \frac{2qa}{r^3}\cos\vartheta , \\ B_\vartheta &= F(e_\varphi, e_r) = \frac{qa}{r^3}\sin\vartheta , \end{aligned} \tag{11.26}$$

while all other field components vanish. In the limit of large r, the electric field thus becomes that of a point charge q at the origin, and the magnetic field has a characteristic dipolar structure.

It is known from electrodynamics that the magnetic dipole moment of a sphere with charge q, mass M and angular momentum $\vec{L}$ is

$$\vec{\mu} = g\,\frac{q\vec{L}}{2Mc} , \tag{11.27}$$

where g is the *gyromagnetic moment*. A comparison of the magnetic field in (11.26) with the magnetic dipole field from electrodynamics,

$$\vec{B} = \frac{3(\vec{\mu}\cdot\vec{e}_r)\vec{e}_r - \vec{\mu}}{r^3} , \tag{11.28}$$

shows that the magnetic dipole moment of a charged, rotating black holes has the modulus

$$\mu = qa = \frac{qL}{Mc} = 2\,\frac{qL}{2Mc}\;, \tag{11.29}$$

showing that its gyromagnetic moment is $g = 2$.

11.1.3 Schwarzschild horizon, ergosphere and Killing horizon

By construction, the Kerr-Newman metric (11.21) has the two Killing vector fields $k = \partial_t$, expressing the stationarity of the solution, and $m = \partial_\varphi$, which expresses its axial symmetry.

Since the metric coefficients in adapted coordinates satisfy

$$g_{tt} = \langle k, k\rangle\;,\quad g_{\varphi\varphi} = \langle m, m\rangle\;,\quad g_{t\varphi} = \langle k, m\rangle\;, \tag{11.30}$$

they have an invariant meaning which will now be clarified.

Let us consider an observer moving with $r =$ const. and $\vartheta =$ const. with uniform angular velocity ω. If her four-velocity is u, then

$$\omega = \frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{\dot\varphi}{\dot t} = \frac{u^\varphi}{u^t} \tag{11.31}$$

for a static observer at infinity, whose proper time can be identified with the coordinate time t. Correspondingly, we can expand the four-velocity as

$$u = u^t\partial_t + u^\varphi\partial_\varphi = u^t\left(\partial_t + \omega\partial_\varphi\right) = u^t(k + \omega m)\;, \tag{11.32}$$

inserting the Killing vector fields. Let

$$|k + \omega m| \equiv \left(-\langle k + \omega m, k + \omega m\rangle\right)^{1/2} \tag{11.33}$$

define the norm of $k + \omega m$, then the four-velocity is

$$u = \frac{k + \omega m}{|k + \omega m|}\;. \tag{11.34}$$

Obviously, $k+\omega m$ is a time-like Killing vector field, at least at sufficiently large distances from the black hole. Since then

$$\begin{aligned}\langle k + \omega m, k + \omega m\rangle &= \langle k, k\rangle + \omega^2\langle m, m\rangle + 2\omega\langle k, m\rangle \\ &= g_{tt} + \omega^2 g_{\varphi\varphi} + 2\omega g_{t\varphi} < 0\;,\end{aligned} \tag{11.35}$$

$k + \omega m$ becomes light-like for angular velocities

$$\omega_\pm = \frac{-g_{t\varphi} \pm \sqrt{g_{t\varphi}^2 - g_{tt}g_{\varphi\varphi}}}{g_{\varphi\varphi}}\;. \tag{11.36}$$

If we define

$$\Omega \equiv -\frac{g_{t\varphi}}{g_{\varphi\varphi}} = -\frac{\langle k, m\rangle}{\langle m, m\rangle}\ , \tag{11.37}$$

we can write (11.36) as

$$\omega_\pm = \Omega \pm \sqrt{\Omega^2 - \frac{g_{tt}}{g_{\varphi\varphi}}}\ . \tag{11.38}$$

For an interpretation of Ω, we note that freely-falling test particles on radial orbits have zero angular momentum and thus $\langle u, m\rangle = 0$. By (11.34), this implies

$$0 = \langle k + \omega m, m\rangle = g_{t\varphi} + \omega g_{\varphi\varphi} \tag{11.39}$$

and thus

$$\omega = -\frac{g_{t\varphi}}{g_{\varphi\varphi}} = \Omega \tag{11.40}$$

according to the definition (11.37). This shows that Ω is the angular velocity of a test particle falling freely towards the black hole on a radial orbit.

The minimum angular velocity ω_- from (11.38) vanishes if and only if $g_{tt} = \langle k, k\rangle = 0$, i.e. if the Killing vector field k turns light-like. With (11.21), this is so where

$$0 = a^2 \sin^2\vartheta - \Delta = 2mr - r^2 - Q^2 - a^2\cos^2\vartheta\ , \tag{11.41}$$

i.e. at the radius

$$r_0 = m + \sqrt{m^2 - Q^2 - a^2\cos^2\vartheta}\ . \tag{11.42}$$

This is the *static limit*: for an observer at this radius to remain static with respect to observers at infinity (i.e. with respect to the “fixed stars”), she would have to move with the speed of light. At smaller radii, observers cannot remain static against the drag of the rotating black hole.

We have seen in (4.48) that the light emitted by a source with four-velocity u_s is seen by an observer with four-velocity u_o with a redshift

$$\frac{\nu_\text{o}}{\nu_\text{s}} = \frac{\langle \tilde{k}, u_\text{o}\rangle}{\langle \tilde{k}, u_\text{s}\rangle}\ , \tag{11.43}$$

where $\tilde{k}$ is the wave vector of the light.

Observers at rest in a stationary space-time have four-velocities proportional to the Killing vector field k,

$$u = \frac{k}{\sqrt{-\langle k, k\rangle}}\ , \quad \text{hence} \quad k = \sqrt{-\langle k, k\rangle}\, u\ . \tag{11.44}$$

Moreover, the projection of the wave vector $\tilde{k}$ on the Killing vector field k does not change along the light ray, since

$$\begin{aligned}\nabla_{\tilde{k}}\langle\tilde{k},k\rangle &= \langle\nabla_{\tilde{k}}\tilde{k},k\rangle+\langle\tilde{k},\nabla_{\tilde{k}}k\rangle \\ &= \tilde{k}^\alpha\tilde{k}^\beta\nabla_\beta k_\alpha = 0\;,\end{aligned}\tag{11.45}$$

because light rays are null geodesics, $\nabla_{\tilde{k}}\tilde{k}=0$, and because the Killing equation (5.34) implies that the last expression is the contraction of the symmetric tensor field $\tilde{k}^\alpha\tilde{k}^\beta$ with the antisymmetric tensor field $\nabla_\beta k_\alpha$.

Using this in a combination of (11.43) and (11.44), we obtain

$$\frac{\nu_\mathrm{o}}{\nu_\mathrm{s}} = \frac{\langle\tilde{k},k\rangle_\mathrm{o}}{\langle\tilde{k},k\rangle_\mathrm{s}}\frac{\sqrt{-\langle k,k\rangle_\mathrm{s}}}{\sqrt{-\langle k,k\rangle_\mathrm{o}}} = \frac{\sqrt{-\langle k,k\rangle_\mathrm{s}}}{\sqrt{-\langle k,k\rangle_\mathrm{o}}}\;.\tag{11.46}$$

For an observer at rest far away from the black hole, $\langle k,k\rangle_\mathrm{o}\approx-1$, and the redshift becomes

$$1+z=\frac{\nu_\mathrm{s}}{\nu_\mathrm{o}}\approx\frac{1}{\sqrt{-\langle k,k\rangle_\mathrm{s}}}\;,\tag{11.47}$$

which tends to infinity as the source approaches the static limit.

The minimum and maximum angular velocities $\omega_\pm$ from (11.38) both become equal to Ω for

$$\Omega^2=\left(\frac{g_{t\varphi}}{g_{\varphi\varphi}}\right)^2=\frac{g_{tt}}{g_{\varphi\varphi}}\quad\Rightarrow\quad g_{t\varphi}^2-g_{tt}g_{\varphi\varphi}=0\;.\tag{11.48}$$

This equation means that the Killing field $\xi\equiv k+\Omega m$ turns light-like,

$$\begin{aligned}\langle\xi,\xi\rangle &= \langle k,k\rangle+2\Omega\langle k,m\rangle+\Omega^2\langle m,m\rangle \\ &= g_{tt}+2\Omega g_{t\varphi}+\Omega^2 g_{\varphi\varphi}=g_{tt}-2\frac{g_{t\varphi}^2}{g_{\varphi\varphi}}+\frac{g_{t\varphi}^2}{g_{\varphi\varphi}} \\ &= \frac{g_{tt}g_{\varphi\varphi}-g_{t\varphi}^2}{g_{\varphi\varphi}}=0\;.\end{aligned}\tag{11.49}$$

Interestingly, writing the expression from (11.48) with the metric coefficients (11.21) leads to the simple result

$$g_{t\varphi}^2-g_{tt}g_{\varphi\varphi}=\Delta\sin^2\vartheta\;,\tag{11.50}$$

so that the condition (11.48) is equivalent to

$$0=\Delta=r^2-2mr+Q^2+a^2\;,\tag{11.51}$$

which describes a spherical hypersurface with radius

$$r_H=m+\sqrt{m^2-Q^2-a^2}\;,\tag{11.52}$$

for which we choose the larger of the two solutions of (11.51).

By its definition (11.37), the angular velocity Ω on this hypersurface H can be written as

$$\Omega_H = -\left.\frac{g_{t\varphi}}{g_{\varphi\varphi}}\right|_H = \left.\frac{a(2mr - Q^2)}{\Sigma^2}\right|_H = \frac{a(2mr_H - Q^2)}{(r_H^2 + a^2)^2} , \tag{11.53}$$

since $\Sigma^2 = (r^2 + a^2)^2$ because of $\Delta = 0$ at r_H. Because of (11.51), the numerator is $a(r_H^2 + a^2)$, and we find

$$\Omega_H = \frac{a}{r_H^2 + a^2} . \tag{11.54}$$

This means remarkably that the hypersurface H is rotating at constant angular velocity like a solid body.

Since the hypersurface H is defined by the condition $\Delta = 0$, its normal vectors are given by

$$\mathrm{grad}\Delta = \mathrm{d}\Delta^\sharp , \quad \mathrm{d}\Delta = 2(r - m)\mathrm{d}r . \tag{11.55}$$

Thus, the norm of the normal vectors is

$$\langle \mathrm{grad}\Delta, \mathrm{grad}\Delta \rangle = 4g^{rr}(r - m)^2 , \tag{11.56}$$

now, according to (11.21), $g^{rr} \propto \Delta = 0$ on the hypersurface, showing that H is a *null hypersurface*.

Because of this fact, the tangent space to the null hypersurface H at any of its points is orthogonal to a null vector, and hence it does not contain time-like vectors. H is called a *Killing horizon*.

The hypersurface defined by the static limit is time-like, which means that it can be crossed in both directions, in contrast to the horizon H. The region in between the static limit and the Killing horizon is the *ergosphere*, in which k is space-like and no observer can be prevented from following the rotation of the black hole.

Formally, the Kerr solution is singular where $\Delta = 0$, but this singularity can be lifted by a transformation to coordinates similar to the Eddington-Finkelstein coordinates for a Schwarzschild black hole.

11.2 Motion near a Kerr black hole

11.2.1 Kepler's third law

We shall now assume $q = 0$ and consider motion on a circular orbit in the equatorial plane. Thus $\dot{r} = 0$ and $\vartheta = \pi/2$, and

$$\Delta = r^2 - 2mr + a^2 \quad \text{and} \quad \rho = r , \tag{11.57}$$

further

$$\Sigma^2 = (r^2 + a^2)^2 - a^2\Delta = r^4 + a^2r^2 + 2ma^2r \;, \tag{11.58}$$

and the coefficients of the metric (11.21) become

$$\begin{aligned} g_{tt} &= -1 + \frac{2m}{r} \;, \quad g_{t\varphi} = -\frac{2ma}{r} \;, \\ g_{rr} &= \frac{r^2}{\Delta} \;, \quad g_{\vartheta\vartheta} = r^2 \;, \\ g_{\varphi\varphi} &= \frac{\Sigma^2}{r^2} = r^2 + a^2 + \frac{2ma^2}{r} \;. \end{aligned} \tag{11.59}$$

Since $\dot{\vartheta} = 0$ and $\dot{r} = 0$, the Lagrangian reduces to

$$2\mathcal{L} = -\left(1 - \frac{2m}{r}\right)\dot{t}^2 - \frac{4ma}{r}\dot{t}\dot{\varphi} + \left(r^2 + a^2 + \frac{2ma^2}{r}\right)\dot{\varphi}^2 \;. \tag{11.60}$$

By the Euler-Lagrange equation for r and due to $\dot{r} = 0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{r}} = 0 = \frac{\partial\mathcal{L}}{\partial r} \;, \tag{11.61}$$

which yields, after multiplying with $r^2/\dot{t}^2$,

$$-m + 2ma\omega + (r^3 - ma^2)\omega^2 = 0 \tag{11.62}$$

where we have introduced the angular frequency ω according to (11.31). Noticing that

$$r^3 - ma^2 = (r^{3/2} - m^{1/2}a)(r^{3/2} + m^{1/2}a) \;, \tag{11.63}$$

we can write the solutions as

$$\omega_\pm = \pm\frac{m^{1/2}}{r^{3/2} \pm m^{1/2}a} \;. \tag{11.64}$$

This is Kepler's third law for a Kerr black hole: The angular velocity of a test particle depends on whether it is co-rotating with or counter-rotating against the black hole.

11.2.2 Accretion flow onto a Kerr black hole

We now consider a stationary, axially-symmetric flow of a perfect fluid onto a Kerr black hole. Because of the symmetry constraints, the Lie derivatives of all physical quantities in the direction of the Killing vector fields $k = \partial_t$ and $m = \partial_\varphi$ need to vanish.

As in (11.32), the four-velocity of the flow is

$$u = u^t(k + \omega m) \;. \tag{11.65}$$

We introduce

$$e \equiv -\langle u, k\rangle = -u_t\ , \quad j \equiv \langle u, m\rangle = u_\varphi\ , \quad l \equiv \frac{j}{e} = -\frac{u_\varphi}{u_t} \tag{11.66}$$

and use

$$\begin{aligned} u_\varphi &= g_{t\varphi}u^t + g_{\varphi\varphi}u^\varphi = u^t\left(g_{t\varphi} + g_{\varphi\varphi}\omega\right) \\ u_t &= g_{tt}u^t + g_{t\varphi}u^\varphi = u^t\left(g_{tt} + g_{t\varphi}\omega\right) \end{aligned} \tag{11.67}$$

to see that

$$l = -\frac{g_{t\varphi} + \omega g_{\varphi\varphi}}{g_{tt} + \omega g_{t\varphi}} \quad \Leftrightarrow \quad \omega = -\frac{g_{t\varphi} + l g_{tt}}{g_{\varphi\varphi} + l g_{t\varphi}}\ . \tag{11.68}$$

Moreover, by the definition of l and ω, and using $-1 = \langle u, u\rangle = u^t u_t + u^\varphi u_\varphi$, we see that

$$\omega l = -\frac{u^\varphi u_\varphi}{u^t u_t} \quad \Rightarrow \quad u_t u^t = -\frac{1}{1-\omega l}\ . \tag{11.69}$$

Finally, using (11.67) and (11.69), we have

$$-u_t^2 = -u_t u^t(g_{tt} + g_{t\varphi}\omega) = \frac{g_{tt} + g_{t\varphi}\omega}{1-\omega l}\ . \tag{11.70}$$

If we substitute ω from (11.68) here, we obtain after a short calculation

$$u_t^2 = e^2 = \frac{g_{t\varphi}^2 - g_{tt}g_{\varphi\varphi}}{g_{\varphi\varphi} + 2l g_{t\varphi} + l^2 g_{tt}}\ . \tag{11.71}$$

It was shown on problem sheet 5 that the relativistic Euler equation reads

$$(\rho + p)\nabla_u u = -\mathrm{grad}p - u\nabla_u p\ , \tag{11.72}$$

where ρ and p are the density and the pressure of the ideal fluid, further

$$\mathrm{grad}p = \mathrm{d}p^\sharp\ , \tag{11.73}$$

where $\mathrm{d}p^\sharp$ is the vector belonging to the dual vector $\mathrm{d}p$.

We first observe that, since the flow is stationary,

$$0 = \mathcal{L}_u p = up = u^\mu\partial_\mu p = u^\mu\nabla_\mu p \quad \Rightarrow \quad \nabla_u p = 0\ , \tag{11.74}$$

thus the second term on the right-hand side of (11.72) vanishes.

Next, we introduce the dual vector $u^\flat$ belonging to the four-velocity u. In components, $u^\flat_\mu = g_{\mu\nu}u^\nu = u_\mu$. Then, from (5.32),

$$\left(\mathcal{L}_u u^\flat\right)_\mu = u^\nu\partial_\nu u_\mu + u_\nu\partial_\mu u^\nu = u^\nu\nabla_\nu u_\mu + u_\nu\nabla_\mu u^\nu\ , \tag{11.75}$$

where we have employed the symmetry of the connection ∇. This shows that

$$\mathcal{L}_u u^\flat = \nabla_u u^\flat \ . \tag{11.76}$$

Then, we introduce $f \equiv 1/u^t$ and compute $\mathcal{L}_{fu} u^\flat$ in two different ways. First, according to (5.32), we can write

$$\left(\mathcal{L}_{fu} u^\flat\right)_\mu = f u^\nu \partial_\nu u_\mu + u_\nu \partial_\mu (f u^\nu) = f\left(u^\nu \partial_\nu u_\mu + u_\nu \partial_\mu u^\nu\right) - \partial_\mu f \tag{11.77}$$

hence

$$\mathcal{L}_{fu} u^\flat = f \mathcal{L}_u u^\flat - \mathrm{d}f = f \nabla_u u^\flat - \mathrm{d}f \ , \tag{11.78}$$

where (11.76) was inserted.

On the other hand, $fu = u/u^t = k + \omega m$ because of (11.65), which allows us to write

$$\mathcal{L}_{fu} u^\flat = \underbrace{\mathcal{L}_k u^\flat}_{=0} + \mathcal{L}_{\omega m} u^\flat \ . \tag{11.79}$$

Now, in components,

$$\begin{aligned}\left(\mathcal{L}_{\omega m} u^\flat\right)_\mu &= \omega m^\nu \partial_\nu u_\mu + u_\nu \partial_\mu (\omega m^\nu) \\ &= \omega\left(m^\nu \partial_\nu u_\mu + u_\nu \partial_\mu m^\nu\right) + u_\nu m^\nu \partial_\mu \omega \ . \end{aligned} \tag{11.80}$$

But the Lie derivative of $u^\flat$ in the direction m must vanish because of the axisymmetry, thus

$$0 = \left(\mathcal{L}_m u^\flat\right)_\mu = m^\nu \partial_\nu u_\mu + u_\nu \partial_\mu m^\nu \ , \tag{11.81}$$

and the first term on the right-hand side of (11.80) must vanish. Therefore,

$$\mathcal{L}_{fu} u^\flat = \mathcal{L}_{\omega m} u^\flat = \langle u, m\rangle \mathrm{d}\omega = j \mathrm{d}\omega \ . \tag{11.82}$$

Equating this to (11.78) gives

$$f \nabla_u u^\flat = \mathrm{d}f + j \mathrm{d}\omega \ . \tag{11.83}$$

However, from (11.69),

$$f = \frac{1}{u^t} = -u_t(1 - \omega l) = e(1 - \omega l) \ . \tag{11.84}$$

When inserted into (11.83), this yields

$$\begin{aligned} e(1 - \omega l)\nabla_u u^\flat &= (1 - \omega l)\mathrm{d}e - el\mathrm{d}\omega - e\omega \mathrm{d}l + j\mathrm{d}\omega \\ &= (1 - \omega l)\mathrm{d}e - e\omega \mathrm{d}l \ , \end{aligned} \tag{11.85}$$

where $el = j$ was used. Thus,

$$\nabla_u u^\flat = \mathrm{d}\ln e - \frac{\omega \mathrm{d}l}{1 - \omega l} \ . \tag{11.86}$$

Inserting this into Euler's equation (11.72), we obtain

$$\frac{\mathrm{d}p}{\rho + p} = -\mathrm{d}\ln e + \frac{\omega \mathrm{d}l}{1 - \omega l} , \qquad (11.87)$$

which shows that surfaces of constant pressure are given by

$$\ln e - \int \frac{\omega \mathrm{d}l}{1 - \omega l} = \text{const.} . \qquad (11.88)$$

Setting $l = 0$ makes the second term on the left-hand side vanish, and we find from (11.71)

$$\frac{g_{\varphi\varphi} + 2lg_{t\varphi} + l^2 g_{tt}}{g_{t\varphi}^2 - g_{tt}g_{\varphi\varphi}} = \frac{1}{e^2} = \text{const.} . \qquad (11.89)$$

If $a = 0$, we obtain the isobaric surfaces of the accretion flow onto a Schwarzschild black hole. With

$$g_{tt} = -1 + \frac{2m}{r} , \quad g_{\varphi\varphi} = r^2 \sin^2\vartheta , \quad g_{t\varphi} = 0 , \qquad (11.90)$$

we find

$$\frac{r}{r - 2m} - \frac{l^2}{r^2 \sin^2\vartheta} = \text{const.} \qquad (11.91)$$

This describes toroidal surfaces around the black hole, the so-called accretion torus.

11.3 Entropy and temperature of a black hole

It was realised by Hawking, Penrose and Christodoulou that the area of a possibly charged and rotating black hole, defined by

$$A := 4\pi\alpha := 4\pi\left(r_+^2 + a^2\right) \qquad (11.92)$$

cannot shrink. Here, r_+ is the positive branch of the two solutions of (11.51),

$$r_\pm = m \pm \sqrt{m^2 - Q^2 - a^2} . \qquad (11.93)$$

This led Bekenstein (1973) to the following consideration. If A cannot shrink, it reminds of the entropy as the only other quantity known in physics that cannot shrink. Could the area A have anything to do with an entropy that could be assigned to a black hole? In fact, this is much more plausible than it may appear at first sight. Suppose radiation disappears in a black hole. Without accounting for a possible entropy of the black hole, its entropy would be gone, violating the second law of thermodynamics.

The same holds for gas accreted by the black hole: Its entropy would be removed from the outside world, leaving the entropy there lower than before.

If, however, the increased mass of the black hole led to a suitably increased entropy of the black hole itself, this violation of the second law could be remedied. Any mass and angular momentum swallowed by a black hole leads to an increase of the area (11.92), which makes it appear plausible that the area of a black hole might be related to its entropy. Following Bekenstein (1973), we shall now work out this relation.

First, (11.93) implies

$$r_+ + r_- = 2m \;, \quad r_+ r_- = Q^2 + a^2 \;, \quad r_+ - r_- = 2\sqrt{m^2 - Q^2 - a^2} \;, \tag{11.94}$$

thus

$$2mr_+ = (r_+ + r_-)r_+ = r_+^2 + Q^2 + a^2 \tag{11.95}$$

and

$$\alpha = \frac{A}{4\pi} = 2mr_+ - Q^2 \;. \tag{11.96}$$

The differential of α is

$$\mathrm{d}\alpha = 2r_+\mathrm{d}r_+ + 2a\mathrm{d}a \;, \tag{11.97}$$

which leads to

$$\mathrm{d}\alpha = \frac{4\alpha}{\delta r}\mathrm{d}m - \frac{4r_+ Q}{\delta r}\mathrm{d}Q - \frac{4\vec{a}\cdot\mathrm{d}\vec{L}}{\delta r} \;, \tag{11.98}$$

where $\delta r := r_+ - r_-$ and the relation (11.20) between $\vec{a}$ and the angular momentum $\vec{L}$ was used. Solving this equation for dm yields

$$\mathrm{d}m = \Theta\mathrm{d}\alpha + \Phi\mathrm{d}Q + \vec{\Omega}\cdot\mathrm{d}\vec{L} \tag{11.99}$$

with the definitions

$$\Theta := \frac{\delta r}{4\alpha} \;, \quad \Phi := \frac{r_+ Q}{\alpha} \;, \quad \vec{\Omega} := \frac{\vec{a}}{\alpha} \;. \tag{11.100}$$

This reminds of the first law of thermodynamics if we tentatively identify m with the internal energy, α with the entropy and the remaining terms with external work.

Let us now see whether a linear relation between the entropy S and the area α will lead to consistent results. Thus, assume $S = \gamma\alpha$ with some constant γ. Then, a change $\delta\alpha$ will lead to a change $\delta S = \gamma\delta\alpha$ in the entropy.

Bekenstein showed that the minimal change of the effective area is twice the squared Planck length (1.5), thus

$$\delta\alpha = \frac{2\hbar G}{c^3} \;. \tag{11.101}$$

On the other hand, he identified the minimal entropy change of the black hole with the minimal change of the Shannon entropy, which is derived from information theory and is

$$\delta S = \ln 2 \ . \tag{11.102}$$

This could e.g. correspond to the minimal information loss when a single particle disappears in a black hole. Requiring

$$\ln 2 = \delta S = \gamma \delta\alpha = \gamma \frac{2\hbar G}{c^3} \tag{11.103}$$

fixes γ,

$$\gamma = \frac{\ln 2}{2} \frac{c^3}{\hbar G} \ , \tag{11.104}$$

and thus the Bekenstein entropy becomes

$$S = \frac{\ln 2}{8\pi} \frac{c^3 k_\mathrm{B}}{\hbar G} A \ , \tag{11.105}$$

where A is the area of the black hole and the Boltzmann constant k_B was inserted to arrive at conventional units for the entropy.

The quantity Θ from (11.100) must then correspond to the temperature of the black hole. If this should be consistent, it must agree with the thermodynamic definition of temperature,

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_V \ . \tag{11.106}$$

For E, we can use the mass or rather

$$E = Mc^2 = \frac{mc^4}{G} \ . \tag{11.107}$$

Then,

$$\frac{\partial S}{\partial E} = \frac{G}{c^4} \frac{\partial S}{\partial m} = \frac{\ln 2}{8\pi} \frac{k_\mathrm{B}}{\hbar c} \frac{\partial A}{\partial m} \ . \tag{11.108}$$

Taking A from (11.92) yields

$$\frac{\partial A}{\partial m} = \frac{4\pi}{\Theta} \tag{11.109}$$

and therefore

$$T = \frac{2}{\ln 2} \frac{\hbar c}{k_\mathrm{B}} \Theta = \frac{2\pi}{\ln 2} \frac{\hbar c}{k_\mathrm{B}} \frac{\delta r}{A} \ . \tag{11.110}$$

This result leads to a remarkable conclusion. If black holes have a temperature, they will radiate and thus lose energy or its mass equivalent. They can therefore evaporate. By the Stefan-Boltzmann law, the luminosity radiated by a black body of area A and temperature T is

$$L = \sigma A T^4 \ , \quad \sigma = \frac{\pi^2 k_\mathrm{B}^4}{60\hbar^3 c^2} \ . \tag{11.111}$$

For an uncharged and non-rotating black hole, $\delta r = 2m$ and $A = 16\pi m^2$, thus its temperature is

$$T = \frac{1}{4\ln 2}\frac{\hbar c}{k_B m} = \frac{1}{4\ln 2}\frac{\hbar c^3}{k_B G M} = \frac{1}{4\ln 2}\frac{m_{\rm Pl}^2 c^2}{k_B M} \tag{11.112}$$

with the Planck mass $m_{\rm Pl} = 2.2 \times 10^{-5}$ g from (1.4). For a black hole of solar mass,

$$T = 5.7 \times 10^{-7}\,\mathrm{K}\,. \tag{11.113}$$

Chapter 12

Homogeneous, Isotropic Cosmology

12.1 The generalised Birkhoff theorem

12.1.1 Spherically-symmetric space-times

Generally, a space-time (M, g) is called *spherically symmetric* if it admits the group $SO(3)$ as an isometry such that the group's orbits are two-dimensional, space-like surfaces.

For any point $p \in M$, we can then select the orbit $\Omega(p)$ of $SO(3)$ through p. In other words, we construct the spatial two-sphere containing p which is compatible with the spherical symmetry.

Next, we construct the set of all geodesics $N(p)$ through p which are orthogonal to $\Omega(p)$. Locally, $N(p)$ forms a two-dimensional surface which we also call $N(p)$. Repeating this construction for all $p \in M$ yields the surfaces N.

We can now introduce coordinates (r, t) on N and (ϑ, ϕ) on Ω, i.e. such that the group orbits Ω are given by $(r, t) = \text{const.}$ and the surfaces N by $(\vartheta, \phi) = \text{const.}$

Then, the metric of the space-time M can be written in the form

$$g = \tilde{g} + R^2(t, r)\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\phi^2\right) \ , \tag{12.1}$$

where $\tilde{g}$ is an indefinite metric in the coordinates (t, r) on N.

Without loss of generality, we can now choose t and r such that $\tilde{g}$ is diagonal, which we denote by

$$\tilde{g} = -\mathrm{e}^{2a(t,r)}\mathrm{d}t^2 + \mathrm{e}^{2b(t,r)}\mathrm{d}r^2 \ , \tag{12.2}$$

where $a(t, r)$ and $b(t, r)$ are functions to be determined.

We introduce the dual basis

$$\theta^0 = \mathrm{e}^a \mathrm{d}t \;, \quad \theta^1 = \mathrm{e}^b \mathrm{d}r \;, \quad \theta^2 = R\mathrm{d}\vartheta \;, \quad \theta^3 = R\sin\vartheta\mathrm{d}\phi \tag{12.3}$$

and find its exterior derivatives

$$\begin{aligned}
\mathrm{d}\theta^0 &= a'\mathrm{e}^{-b}\,\theta^1\wedge\theta^0 \;, \quad \mathrm{d}\theta^1 = \dot b\mathrm{e}^{-a}\,\theta^0\wedge\theta^1 \;, \\
\mathrm{d}\theta^2 &= \frac{\dot R}{R}\mathrm{e}^{-a}\,\theta^0\wedge\theta^2 + \frac{R'}{R}\mathrm{e}^{-b}\,\theta^1\wedge\theta^2 \;, \\
\mathrm{d}\theta^3 &= \frac{\dot R}{R}\mathrm{e}^{-a}\,\theta^0\wedge\theta^3 + \frac{R'}{R}\mathrm{e}^{-b}\,\theta^1\wedge\theta^3 + \frac{\cot\vartheta}{R}\,\theta^2\wedge\theta^3 \;,
\end{aligned} \tag{12.4}$$

where the overdots and primes denote derivatives with respect to t and r, respectively.

In the dual basis (12.3), the metric is $g = \mathrm{diag}(-1,1,1,1)$, thus $\mathrm{d}g = 0$, and Cartan's first structure equation (8.13) implies

$$\omega^i_{\ j}\wedge\theta^j = -\mathrm{d}\theta^i \tag{12.5}$$

for the connection 1-forms $\omega^i_{\ j}$. This yields

$$\begin{aligned}
\omega^0_{\ 1} &= \omega^1_{\ 0} = a'\mathrm{e}^{-b}\,\theta^0 + \dot b\mathrm{e}^{-a}\,\theta^1 \;, \\
\omega^0_{\ 2} &= \omega^2_{\ 0} = \frac{\dot R}{R}\mathrm{e}^{-a}\,\theta^2 \;, \quad \omega^0_{\ 3} = \omega^3_{\ 0} = \frac{\dot R}{R}\mathrm{e}^{-a}\,\theta^3 \;, \\
\omega^1_{\ 2} &= -\omega^2_{\ 1} = -\frac{R'}{R}\mathrm{e}^{-b}\,\theta^2 \;, \\
\omega^1_{\ 3} &= -\omega^3_{\ 1} = -\frac{R'}{R}\mathrm{e}^{-b}\,\theta^3 \;, \quad \omega^2_{\ 3} = -\omega^3_{\ 2} = -\frac{\cot\vartheta}{R}\,\theta^3 \;.
\end{aligned} \tag{12.6}$$

Cartan's second structure equation (8.13) then yields the curvature 2-forms $\Omega^i_{\ j}$,

$$\begin{aligned}
\Omega^0_{\ 1} &= \mathrm{d}\omega^0_{\ 1} \equiv E\,\theta^0\wedge\theta^1 \;, \\
\Omega^0_{\ 2} &= \mathrm{d}\omega^0_{\ 2} + \omega^0_{\ 1}\wedge\omega^1_{\ 2} \equiv \tilde E\,\theta^0\wedge\theta^2 + H\,\theta^1\wedge\theta^2 \;, \\
\Omega^0_{\ 3} &= \mathrm{d}\omega^0_{\ 3} + \omega^0_{\ 1}\wedge\omega^1_{\ 3} + \omega^0_{\ 2}\wedge\omega^2_{\ 3} = \tilde E\,\theta^0\wedge\theta^3 + H\,\theta^1\wedge\theta^3 \;, \\
\Omega^1_{\ 2} &= \mathrm{d}\omega^1_{\ 2} + \omega^1_{\ 0}\wedge\omega^0_{\ 2} \equiv -H\,\theta^0\wedge\theta^2 + \tilde F\,\theta^1\wedge\theta^2 \;, \\
\Omega^1_{\ 3} &= \mathrm{d}\omega^1_{\ 3} + \omega^1_{\ 0}\wedge\omega^0_{\ 3} + \omega^1_{\ 2}\wedge\omega^2_{\ 3} = \tilde F\,\theta^1\wedge\theta^3 - H\,\theta^0\wedge\theta^3 \;, \\
\Omega^2_{\ 3} &= \mathrm{d}\omega^2_{\ 3} + \omega^2_{\ 0}\wedge\omega^0_{\ 3} + \omega^2_{\ 1}\wedge\omega^1_{\ 3} \equiv F\,\theta^2\wedge\theta^3 \;,
\end{aligned} \tag{12.7}$$

where the functions

$$\begin{aligned}
E &= \mathrm{e}^{-2a}\left(\ddot b - \dot a\dot b + \dot b^2\right) - \mathrm{e}^{-2b}\left(a'' - a'b' + a'^2\right) \;, \\
\tilde E &= \frac{\mathrm{e}^{-2a}}{R}\left(\ddot R - \dot a\dot R\right) - \frac{\mathrm{e}^{-2b}}{R}a'R' \;, \\
H &= \frac{\mathrm{e}^{-a-b}}{R}\left(\dot R' - a'\dot R - \dot b R'\right) \;, \\
F &= \frac{1}{R^2}\left(1 - R'^2\mathrm{e}^{-2b} + \dot R^2\mathrm{e}^{-2a}\right) \;, \\
\tilde F &= \frac{\mathrm{e}^{-2a}}{R}\dot b\dot R + \frac{\mathrm{e}^{-2b}}{R}\left(b'R' - R''\right)
\end{aligned} \tag{12.8}$$

were defined.

According to (8.20), the curvature forms imply the components

$$R_{\alpha\beta} = \Omega^{\lambda}_{\alpha}(e_\lambda, e_\beta) \tag{12.9}$$

of the Ricci tensor, for which we obtain

$$\begin{aligned} R_{00} &= -E - 2\tilde{E}\ , \quad R_{01} = -2H\ , \quad R_{02} = 0 = R_{03}\ , \\ R_{11} &= E + 2\tilde{F}\ , \quad R_{12} = 0 = R_{13} \\ R_{22} &= \tilde{E} + \tilde{F} + F = R_{33}\ , \quad R_{23} = 0 \end{aligned} \tag{12.10}$$

and the Ricci scalar

$$\begin{aligned} R &= (E + 2\tilde{E}) + (E + 2\tilde{F}) + 2(\tilde{E} + \tilde{F} + F) \\ &= 2(E + F) + 4(\tilde{E} + \tilde{F})\ , \end{aligned} \tag{12.11}$$

and finally for the Einstein tensor

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{R}{2} g_{\alpha\beta} \tag{12.12}$$

the components

$$\begin{aligned} G_{00} &= F + 2\tilde{F}\ , \quad G_{01} = -2H\ , \quad G_{02} = 0 = G_{03}\ , \\ G_{11} &= -2\tilde{E} - F\ , \quad G_{12} = 0 = G_{13}\ , \\ G_{22} &= -E - \tilde{E} - \tilde{F} = G_{33}\ , \quad G_{23} = 0\ . \end{aligned} \tag{12.13}$$

12.1.2 Birkhoff's theorem

We can now state and prove Birkhoff's theorem in its general form: *Every C^2 solution of Einstein's vacuum equations which is spherically symmetric in an open subset U is locally isometric to a domain of the Schwarzschild-Kruskal solution.*

The proof proceeds in four steps:

1. If the surfaces $\{R(t, r) = \text{const.}\}$ are time-like in U and $\mathrm{d}R \neq 0$, we can choose $R(t, r) = r$, thus $\dot{R} = 0$ and $R' = 1$. Since $H = -\dot{b}\mathrm{e}^{-a-b}/R$ then, the requirement $G_{01} = 0$ implies $\dot{b} = 0$. The sum $G_{00} + G_{11} = 2(\tilde{F} - \tilde{E})$ must also vanish, thus

   $$\frac{\mathrm{e}^{-2b}}{R}(b' + a') = 0\ , \tag{12.14}$$

   which means $a(t, r) = -b(r) + f(t)$. By a suitable choice of a new time coordinate, a can therefore be made time-independent as well. Moreover, we see that

   $$0 = G_{00} = F + 2\tilde{F} = \frac{1 - \mathrm{e}^{-2b}}{R^2} + \frac{2b'\mathrm{e}^{-2b}}{R} \tag{12.15}$$

is identical to the condition (8.62) for the function b in the Schwarzschild space-time. Thus, we have $\mathrm{e}^{-2b} = 1 - 2m/r$ as there, further $a(r) = -b(r)$, and the metric turns into the Schwarzschild metric.

2. If the surfaces $\{R(t,r) = \text{const.}\}$ are space-like in U and $\mathrm{d}R \neq 0$, we can choose $R(t,r) = t$ and proceed in an analogous way. Then, $\dot{R} = 1$ and $R' = 0$, thus $H = -a'\mathrm{e}^{-a-b}/R$, hence $G_{01} = 0$ implies $a' = 0$ and, again through $G_{00} + G_{11} = 0$, the condition $\dot{a} + \dot{b} = 0$ or $b(t,r) = -a(t) + f(r)$. This allows us to change the radial coordinate appropriately so that $b(t,r)$ also becomes independent of r. Then, $G_{00} = 0$ implies

$$0 = G_{00} = \frac{1}{R^2}\left(1 + \mathrm{e}^{-2a}\right) - 2\dot{a}\frac{\mathrm{e}^{-2a}}{R} \;, \tag{12.16}$$

where $\dot{b} = -\dot{a}$ was used. Since $R = t$, this is equivalent to

$$\partial_t(t\mathrm{e}^{-2a}) = -1 \quad \Rightarrow \quad \mathrm{e}^{-2a} = \mathrm{e}^{2b} = \frac{2m}{t} - 1 \;, \tag{12.17}$$

with $t < 2m$. This is the Schwarzschild solution for $r < 2m$ because r and t change roles inside the Schwarzschild horizon.

3. If $\{R(t,r) = \text{const.}\}$ are space-like in some part of U and time-like in another, we obtain the respective different domains of the Schwarzschild space-time.

4. Assume finally $\langle \mathrm{d}R, \mathrm{d}R\rangle = 0$ on U. If R is constant in U, $G_{00} = R^{-2} = 0$ implies $R = \infty$. Therefore, suppose $\mathrm{d}R$ is not zero, but light-like. Then, r and t can be chosen such that $R = t - r$ and $\mathrm{d}R = \mathrm{d}t - \mathrm{d}r$. For $\mathrm{d}R$ to be light-like,

$$\langle \mathrm{d}R, \mathrm{d}R\rangle = \tilde{g}(\mathrm{d}R, \mathrm{d}R) = -\mathrm{e}^{2a} + \mathrm{e}^{2b} = 0 \;, \tag{12.18}$$

we require $a = b$. Then, $G_{00} + G_{11} = 0$ or

$$-\frac{e^{-2a}}{R}\left(\dot{a} + \dot{b} - a' - b'\right) = 0 \;, \tag{12.19}$$

implies $\dot{a} = a'$, which again leads to $R = \infty$ through $G_{00} = 0$.

This shows that the metric reduces to the Schwarzschild metric in all relevant cases.

It is a corollary to Birkhoff's theorem that a spherical cavity in a spherically-symmetric spacetime has the Minkowski metric. Indeed, Birkhoff's theorem says that the cavity must have a Schwarzschild metric with mass zero, which is the Minkowski metric.

12.2 Cosmological solutions

12.2.1 Homogeneity and isotropy

There are good reasons to believe that the Universe at large is *isotropic* around our position. The most convincing observational data are provided by the cosmic microwave background, which is a sea of blackbody radiation at a temperature of 2.726 K whose intensity is almost exactly independent of the direction into which it is observed.

There is furthermore no good reason to believe that our position in the Universe is in any sense prefered compared to others. We must therefore conclude that any observer sees the cosmic microwave background as an isotropic source such as we do. Then, the Universe must also be *homogeneous*.

We are thus led to the expectation that our Universe at large may be described by a homogeneous and isotropic space-time. Let us now give these terms a precise mathematical meaning.

A space-time (M, g) is called *spatially homogeneous* if there exists a one-parameter family of space-like hypersurfaces Σ_t that foliate the space-time such that for each t and any two points $p, q \in \Sigma_t$, there exists an isometry ϕ of g which takes p into q.

Before we can define isotropy, we have to note that isotropy requires that the state of motion of the observer needs to be specified first because two observers moving with different velocities through a given point in space-time will generally observe different redshifts in different directions.

Therefore, we define a space-time (M, g) as *spatially isotropic* about a point p if there exists a congruence of time-like geodesics through p with tangents u such that for any two vectors $v_1, v_2 \in V_p$ orthogonal to u, there exists an isometry of g taking v_1 into v_2 but leaving u and p invariant. In other words, if the space-time is spatially isotropic, no prefered spatial direction orthogonal to u can be identified.

Isotropy thus identifies a special class of observers, with four-velocities u, who cannot identify a prefered spatial direction. The spatial hypersurfaces Σ_t must then be orthogonal to u because otherwise a prefered direction could be identified through the misalignment of the normal direction to Σ_t and u, breaking isotropy.

We thus arrive at the following conclusions: a homogeneous and isotropic space-time (M, g) is foliated into space-like hypersurfaces Σ_t on which g induces a metric h. There must be isometries of h carrying any point $p \in \Sigma_t$ into any other point $q \in \Sigma_t$. Because of isotropy, it must furthermore be impossible to identify prefered spatial directions on Σ_t. These are very restrictive requirements which we shall now exploit.

12.2.2 Spaces of constant curvature

Consider now the curvature tensor ${}^{(3)}R$ induced on Σ_t (i.e. the curvature tensor belonging to the induced metric h). We shall write it in components with its first two indices lowered and the following two indices raised,

$$ {}^{(3)}R = {}^{(3)}R_{ij}{}^{kl} \; . \tag{12.20} $$

In this way, ${}^{(3)}R$ represents a linear map from the vector space of 2-forms $\bigwedge^2$ into $\bigwedge^2$, because of the antisymmetry of ${}^{(3)}R$ with respect to permutations of the first and the second pairs of indices. Thus, it defines an endomorphism

$$ L : \bigwedge^2 \to \bigwedge^2 \; , \quad (L\omega)_{ij} = {}^{(3)}R_{ij}{}^{kl}\omega_{kl} \; . \tag{12.21} $$

Due to the symmetry (3.79) of ${}^{(3)}R$ upon swapping the first with the second pair of indices the endomorphism L is self-adjoint. In fact, for any pair of 2-forms $\alpha, \beta \in \bigwedge^2$,

$$ \begin{aligned} \langle \alpha, L\beta \rangle &= {}^{(3)}R_{ij}{}^{kl}\alpha^{ij}\beta_{kl} = {}^{(3)}R_{ijkl}\alpha^{ij}\beta^{kl} = {}^{(3)}R_{klij}\alpha^{ij}\beta^{kl} \\ &= {}^{(3)}R_{kl}{}^{ij}\alpha_{ij}\beta^{kl} = \langle \beta, L\alpha \rangle \; , \end{aligned} \tag{12.22} $$

which defines a self-adjoint endomorphism.

We can now use the theorem stating that the eigenvectors of a self-adjoint endomorphism provide an orthonormal basis for the vector space it is operating on. *Isotropy* now requires us to conclude that the eigenvalues of these eigenvectors need to be equal because we could otherwise define a prefered direction (e.g. by the eigenvector belonging to the largest eigenvalue). Then, however, the endomorphism L must be proportional to the identical map

$$ L = 2kI \; , \tag{12.23} $$

with some $k \in \mathbb{R}$.

By the definition (12.21) of L, this implies for the coefficients of the curvature tensor

$$ {}^{(3)}R_{ij}{}^{kl} = k\left(\delta^k_i\delta^l_j - \delta^k_j\delta^l_i\right) \tag{12.24} $$

because ${}^{(3)}R$ must be antisymmetrised. Lowering the indices by means of the induced metric h yields

$$ {}^{(3)}R_{ijkl} = k\left(h_{ik}h_{jl} - h_{jk}h_{il}\right) \; . \tag{12.25} $$

The Ricci tensor is

$$ \begin{aligned} {}^{(3)}R_{jl} &= {}^{(3)}R^i{}_{jil} = kh^{is}\left(h_{si}h_{jl} - h_{ji}h_{sl}\right) = k\left(3h_{jl} - h_{jl}\right) \\ &= 2kh_{jl} \; , \end{aligned} \tag{12.26} $$

and the Ricci scalar becomes

$$ {}^{(3)}R = {}^{(3)}R^j_j = 6k \ . \tag{12.27} $$

In the coordinate-free representation, the curvature is

$$ R(x,y)v = k\left(\langle x,v\rangle y - \langle y,v\rangle x\right) \ . \tag{12.28} $$

from (8.18) and (12.25), we find the curvature forms

$$ \begin{aligned} \Omega^i_j &= \frac{1}{2}{}^{(3)}R^i{}_{jkl}\,\theta^k \wedge \theta^l = \frac{k}{2}h^{is}\left(h_{sk}h_{jl} - h_{jk}h_{sl}\right)\theta^k \wedge \theta^l \\ &= k\,\theta^i \wedge \theta_j \end{aligned} \tag{12.29} $$

in a so far arbitrary dual basis θ^i.

The *curvature parameter k* must be (spatially) constant because of homogeneity. Space-times with constant curvature can be shown to be *conformally flat*, which means that coordinates can be introduced in which the metric h reads

$$ h = \frac{1}{\psi^2}\sum_{i=1}^{3}(\mathrm{d}x^i)^2 \ , \tag{12.30} $$

with a yet unknown arbitrary function $\psi = \psi(x^j)$. This leads us to introduce the dual basis

$$ \theta^i \equiv \frac{1}{\psi}\,\mathrm{d}x^i \ , \tag{12.31} $$

from which we find

$$ \mathrm{d}\theta^i = -\frac{\partial_j\psi}{\psi^2}\,\mathrm{d}x^j \wedge \mathrm{d}x^i = (\partial_j\psi)\,\theta^i \wedge \theta^j \ . \tag{12.32} $$

In this basis, the metric h can be represented by $h = \mathrm{diag}(1,1,1)$. Therefore, we do not need to distinguish between raised and lowered indices, and $\mathrm{d}h = 0$. Hence Cartan's first structure equation (8.13) implies the connection forms

$$ \omega_{ij} = (\partial_i\psi)\,\theta_j - (\partial_j\psi)\,\theta_i \ . \tag{12.33} $$

According to Cartan's second structure equation, the curvature forms are

$$ \begin{aligned} \Omega_{ij} &= \mathrm{d}\omega_{ij} + \omega_{ik} \wedge \omega^k_j \qquad (12.34) \\ &= \psi\left(\partial_i\partial_k\psi\,\theta^k \wedge \theta_j - \partial_j\partial_k\psi\,\theta^k \wedge \theta_i\right) - (\partial_k\psi\partial^k\psi)\,\theta_i \wedge \theta_j \ , \end{aligned} $$

but at the same time we must satisfy (12.29). This immediately implies

$$ \partial_i\partial_k\psi = 0 \quad (i \neq k) \ , \tag{12.35} $$

thus ψ has to be of the form

$$\psi = \sum_{k=1}^{3} f_k(x^k) \tag{12.36}$$

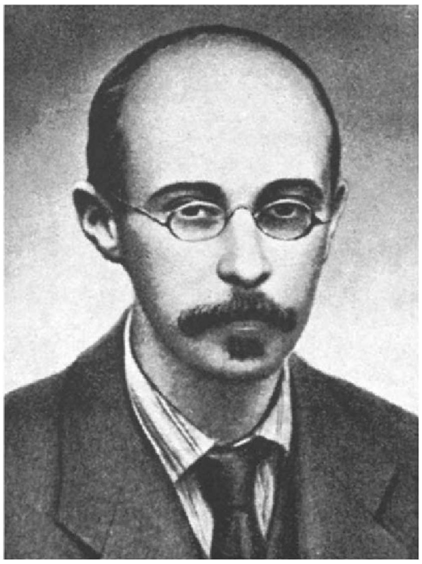
because otherwise the mixed derivatives could not vanish.

Inserting this result into (12.34) shows

$$\Omega_{ij} = \psi\left(f_i'' + f_j'' - \frac{f_k' f'^k}{\psi}\right)\theta_i \wedge \theta_j \ . \tag{12.37}$$

In order to satisfy (12.29), we must have

$$f_i'' + f_j'' = \frac{k + f_k' f'^k}{\psi} \ . \tag{12.38}$$



Alexander Friemann (1888–1925)

Since the right-hand side does not depend on x^i or x^j, the second derivatives f_i'' and f_j'' must all be equal and constant, and thus the f_i must be quadratic in x^i with a coefficient of $(x^i)^2$ which is independent of x^i. Therefore, we can write

$$\psi = 1 + \frac{k}{4}\sum_{i=1}^{3}(x^i)^2 \tag{12.39}$$

because, if the linear term is non-zero, it can be made zero by translating the coordinate origin, and a constant factor on ψ is irrelevant because it simply scales the coordinates.

12.2.3 Form of the metric and Friedmann's equations

According to the preceding discussion, the homogeneous and isotropic spatial hypersurfaces Σ_t must have a metric of the form

$$h = \frac{\sum_{i=1}^{3}(\mathrm{d}x^i)^2}{(1 + kr^2/4)^2} \ , \quad r^2 \equiv \sum_{i=1}^{3}(x^i)^2 \ . \tag{12.40}$$

By a suitable choice of the time coordinate t, the metric of a spatially homogeneous and isotropic space-time can then be written as

$$g = -c^2\mathrm{d}t^2 + a^2(t)h \ , \tag{12.41}$$

because the scaling function $a(t)$ must not depend on the x^i in order to preserve isotropy and homogeneity.

We choose the appropriate dual basis

$$\theta^0 = c\mathrm{d}t \ , \quad \theta^i = \frac{a(t)\,\mathrm{d}x^i}{1 + kr^2/4} \ , \tag{12.42}$$

in terms of which the metric coefficients are $g = \mathrm{diag}(-1, 1, 1, 1)$.

The exterior derivatives of the dual basis are

$$\begin{aligned}
\mathrm{d}\theta^0 &= 0\,, \\
\mathrm{d}\theta^i &= \frac{\dot{a}\,\mathrm{d}t \wedge \mathrm{d}x^i}{1 + kr^2/4} - \frac{a}{(1 + kr^2/4)^2}\,\frac{k}{2} x_j\,\mathrm{d}x^j \wedge \mathrm{d}x^i \\
&= \frac{\dot{a}}{ca}\,\theta^0 \wedge \theta^i + \frac{kx_j}{2a}\,\theta^i \wedge \theta^j\,. \qquad (12.43)
\end{aligned}$$

Since the exterior derivative of the metric is $\mathrm{d}g = 0$, Cartan's first structure equation (8.13) implies

$$\omega^i_{\ j} \wedge \theta^j = -\mathrm{d}\theta^i\,, \qquad (12.44)$$

suggesting the curvature forms

$$\begin{aligned}
\omega^0_{\ i} &= \omega^i_{\ 0} = \frac{\dot{a}}{ca}\,\theta^i\,, \\
\omega^i_{\ j} &= -\omega^j_{\ i} = \frac{k}{2a}\left(x_i\,\theta^j - x_j\,\theta^i\right)\,, \qquad (12.45)
\end{aligned}$$

which evidently satisfy (12.44).

Their exterior derivatives are

$$\mathrm{d}\omega^0_{\ i} = \frac{\ddot{a}a - \dot{a}^2}{c^2a^2}\,\theta^0 \wedge \theta^i + \frac{\dot{a}}{ca}\,\mathrm{d}\theta^i = \frac{\ddot{a}}{c^2a}\,\theta^0 \wedge \theta^i + \frac{k\dot{a}x_j}{2ca^2}\,\theta^i \wedge \theta^j \qquad (12.46)$$

and

$$\begin{aligned}
\mathrm{d}\omega^i_{\ j} &= -\frac{k\dot{a}}{2a^2}\,\theta^0 \wedge \left(x_i\theta^j - x_j\theta^i\right) \qquad (12.47) \\
&+ \frac{k}{2a}\left(\mathrm{d}x_i \wedge \theta^j - \mathrm{d}x_j \wedge \theta^i + x_i\mathrm{d}\theta^j - x_j\mathrm{d}\theta^i\right) \\
&= \frac{k}{a^2}\left(1 + \frac{k}{4}r^2\right)\theta^i \wedge \theta^j + \frac{k^2}{4a^2}\left(x_ix_k\,\theta^j \wedge \theta^k - x_jx_k\,\theta^i \wedge \theta^k\right)
\end{aligned}$$

Cartan's second structure equation (8.13) then gives the curvature forms

$$\begin{aligned}
\Omega^0_{\ i} &= \mathrm{d}\omega^0_{\ i} + \omega^0_{\ k} \wedge \omega^k_{\ i} = \frac{\ddot{a}}{c^2a}\,\theta^0 \wedge \theta^i\,, \qquad (12.48) \\
\Omega^i_{\ j} &= \mathrm{d}\omega^i_{\ j} + \omega^i_{\ 0} \wedge \omega^0_{\ j} + \omega^i_{\ k} \wedge \omega^k_{\ j} = \frac{k + \dot{a}^2/c^2}{a^2}\,\theta^i \wedge \theta^j\,,
\end{aligned}$$

from which we obtain the components of the Ricci tensor

$$R_{jk} = R^i_{\ jik} = \Omega^i_{\ j}(e_i, e_k) \qquad (12.49)$$

as

$$R_{00} = -3\frac{\ddot{a}}{c^2a}\,, \quad R_{11} = R_{22} = R_{33} = \frac{\ddot{a}}{c^2a} + 2\frac{k + \dot{a}^2/c^2}{a^2}\,. \qquad (12.50)$$

The Ricci scalar is then

$$R = R^i_i = 6\left(\frac{\ddot{a}}{c^2 a} + \frac{k + \dot{a}^2/c^2}{a^2}\right) , \tag{12.51}$$

and the Einstein tensor gets the components

$$G_{00} = 3\frac{k + \dot{a}^2/c^2}{a^2} , \quad G_{11} = G_{22} = G_{33} = -\frac{2\ddot{a}}{c^2 a} - \frac{k + \dot{a}^2/c^2}{a^2} . \tag{12.52}$$

For Einstein's field equations to be satisfied, the energy-momentum tensor must be diagonal, and its components must not depend on the spatial coordinates in order to preserve isotropy and homogeneity. We set $T_{00} = \rho c^2$, which is the total energy density, and $T_{ij} = p\delta_{ij}$, where p is the pressure.

This corresponds to the energy-momentum tensor of an ideal fluid,

$$T^{\mu\nu} = (\rho c^2 + p)u^\mu u^\nu + pg^{\mu\nu} \tag{12.53}$$

as seen by a fundamental observer (i.e. an observer for whom the spatial hypersurfaces are isotropic). For such an observer, $u^\mu = \delta^\mu_0$, and the components of $T^{\mu\nu}$ are as given.

Then, Einstein's field equations in the form (6.80) *with* the cosmological constant Λ reduce to

$$\begin{aligned} 3\frac{k + \dot{a}^2/c^2}{a^2} &= \frac{8\pi G}{c^2}\rho + \Lambda , \\ -\frac{2\ddot{a}}{c^2 a} - \frac{k + \dot{a}^2/c^2}{a^2} &= \frac{8\pi G}{c^4}p - \Lambda . \end{aligned} \tag{12.54}$$

Adding a third of the first equation to the second, and re-writing the first equation, we find *Friedmann's equations*

$$\begin{aligned} \frac{\dot{a}^2}{a^2} &= \frac{8\pi G}{3}\rho + \frac{\Lambda c^2}{3} - \frac{kc^2}{a^2} \\ \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda c^2}{3} . \end{aligned} \tag{12.55}$$

12.2.4 Density evolution

After multiplication with $3a^2$ and differentiation with respect to t, Friedmann's first equation gives

$$6\dot{a}\ddot{a} = 8\pi G(\dot{\rho}a^2 + 2\rho a\dot{a}) + 2\Lambda c^2 a\dot{a} . \tag{12.56}$$

If we eliminate

$$6\ddot{a} = -8\pi G a\left(\rho + \frac{3p}{c^2}\right) + 2\Lambda c^2 a \tag{12.57}$$

by means of Friedmann's second equation, we find

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0 \tag{12.58}$$

for the evolution of the density ρ with time.

This equation has a very intuitive meaning. To see it, let us consider the energy contained in a volume V_0, which changes over time in proportion to $V_0 a^3$, and employ the first law of thermodynamics,

$$\mathrm{d}(\rho c^2 V_0 a^3) + p\mathrm{d}(V_0 a^3) = 0 \quad \Rightarrow \quad \mathrm{d}(\rho c^2 a^3) + p\mathrm{d}(a^3) = 0 \; . \tag{12.59}$$

We can use the first law of thermodynamics here because isotropy forbids any energy currents, thus no energy can flow into or out of the volume a^3.

Equation (12.59) yields

$$a^3\dot{\rho} + 3\rho a^2\dot{a} + \frac{3p}{c^2}a^2\dot{a} = 0 \; , \tag{12.60}$$

which is identical to (12.58). This demonstrates that (12.58) simply expresses energy-momentum conservation. Consequently, one can show that it also follows from the contracted Bianchi identity, $\nabla \cdot T = 0$.

Two limits are typically considered for (12.58). First, if matter moves non-relativistically, $p \ll \rho c^2$, and we can assume $p \approx 0$. Then,

$$\frac{\dot{\rho}}{\rho} = -3\frac{\dot{a}}{a} \; , \tag{12.61}$$

which implies

$$\rho = \rho_0 a^{-3} \tag{12.62}$$

if ρ_0 is the density when $a = 1$.

Second, relativistic matter has $p = \rho c^2/3$, with which we obtain

$$\frac{\dot{\rho}}{\rho} = -4\frac{\dot{a}}{a} \tag{12.63}$$

and thus

$$\rho = \rho_0 a^{-4} \; . \tag{12.64}$$

This shows that the density of non-relativistic matter drops as expected in proportion to the inverse volume, but the density of relativistic matter drops faster by one order of the scale factor. An explanation will be given below.

12.2.5 Cosmological redshift

We can write the metric (12.41) in the form

$$g = -c^2\mathrm{d}t^2 + a^2(t)\mathrm{d}\sigma^2 \; , \tag{12.65}$$

where $\mathrm{d}\sigma^2$ is the metric of a three-space with constant curvature k. Since light propagates on null geodesics, (12.65) implies

$$c\mathrm{d}t = \pm a(t)\mathrm{d}\sigma \ . \tag{12.66}$$

Suppose a light signal leaves the source at the coordinate time t_0 and reaches the observer at t_1, then (12.66) shows that the coordinate time satisfies the equation

$$\int_{t_0}^{t_1} \frac{c\mathrm{d}t}{a(t)} = \int_{\text{source}}^{\text{observer}} \mathrm{d}\sigma \ , \tag{12.67}$$

whose right-hand side is time-independent. Thus, for another light signal leaving the source at $t_0 + \mathrm{d}t_0$ and reaching the observer at $t_1 + \mathrm{d}t_1$, we have

$$\int_{t_0}^{t_1} \frac{c\mathrm{d}t}{a(t)} = \int_{t_0+\mathrm{d}t_0}^{t_1+\mathrm{d}t_1} \frac{c\mathrm{d}t}{a(t)} \ . \tag{12.68}$$

Since this implies

$$\int_{t_0}^{t_0+\mathrm{d}t_0} \frac{\mathrm{d}t}{a(t)} = \int_{t_1}^{t_1+\mathrm{d}t_1} \frac{\mathrm{d}t}{a(t)} \ , \tag{12.69}$$

we find for sufficiently small $\mathrm{d}t_{0,1}$ that

$$\frac{\mathrm{d}t_0}{a(t_0)} = \frac{\mathrm{d}t_1}{a(t_1)} \ . \tag{12.70}$$

We can now identify the time intervals $\mathrm{d}t_{0,1}$ with the inverse frequencies of the emitted and observed light, $\mathrm{d}t_i = \nu_i^{-1}$ for $i = 0, 1$. This shows that the emitted and observed frequencies are related by

$$\frac{\nu_0}{\nu_1} = \frac{a(t_1)}{a(t_0)} \ . \tag{12.71}$$

Since the redshift z is defined in terms of the wavelengths as

$$z = \frac{\lambda_1 - \lambda_0}{\lambda_0} = \frac{\nu_0 - \nu_1}{\nu_1} \ , \tag{12.72}$$

we find that light emitted at t_0 and observed at t_1 is redshifted by

$$1 + z = \frac{\lambda_1}{\lambda_0} = \frac{a(t_1)}{a(t_0)} \ . \tag{12.73}$$

Thus, the wavelength of light is increased or decreased in the same proportion as the universe itself expands or contracts.

We can now interpret the result (12.64) that the density of relativistic matter drops by one power of a more than expected by mere dilution: as the universe expands, relativistic particles are redshifted by another factor a and thus loose energy in addition to their dilution.

12.2.6 Alternative forms of the metric

Before we proceed, we bring the spatial metric (12.40) into a different form. We first write it in terms of spherical polar coordinates as

$$h = \frac{\mathrm{d}r^2 + r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\phi^2)}{(1 + kr^2/4)^2} \tag{12.74}$$

and introduce a new radial coordinate u defined by

$$u = \frac{r}{1 + kr^2/4} \ . \tag{12.75}$$

Requiring that $r \approx u$ for small r and u, we can uniquely solve (12.75) to find

$$r = \frac{2}{ku}\left(1 - \sqrt{1 - ku^2}\right) \ , \tag{12.76}$$

which gives the differential

$$\mathrm{d}r = \frac{r}{u\sqrt{1 - ku^2}}\mathrm{d}u \ . \tag{12.77}$$

by (12.75), we have

$$\frac{1}{1 + kr^2/4} = \frac{u}{r} \ , \tag{12.78}$$

and thus

$$\frac{\mathrm{d}r}{1 + kr^2/4} = \frac{\mathrm{d}u}{\sqrt{1 - ku^2}} \ . \tag{12.79}$$

In terms of the new radial coordinate u, we can thus write the spatial part of the metric in the frequently used form

$$h = \frac{\mathrm{d}u^2}{1 - ku^2} + u^2\mathrm{d}\Omega^2 \ , \tag{12.80}$$

where $\mathrm{d}\Omega$ abbreviates the solid-angle element. The constant k can be positive, negative or zero, but its absolute value does not matter since it merely scales the coordinates. Therefore, we can normalise the coordinates such that $k = 0, \pm 1$.

Yet another form of the metric is found by introducing a radial coordinate w such that

$$\mathrm{d}w = \frac{\mathrm{d}u}{\sqrt{1 - ku^2}} \ . \tag{12.81}$$

Integrating both sides, we find that this is satisfied if

$$u = f_k(w) \equiv \begin{cases} \sin w & (k = 1) \\ w & (k = 0) \\ \sinh w & (k = -1) \end{cases} \ . \tag{12.82}$$

We thus find that the homogeneous and isotropic class of cosmological models based on Einstein's field equations are characterised by the metric

$$g = -c^2 \mathrm{d}t^2 + a^2(t)\left[\mathrm{d}w^2 + f_k^2(w)\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}\phi^2\right)\right] \tag{12.83}$$

which is equivalent to

$$g = -c^2 \mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}u^2}{1-ku^2} + u^2\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}\phi^2\right)\right] \tag{12.84}$$

with u related to w by (12.82), and the scale factor $a(t)$ satisfies the Friedmann equations (12.55).

Metrics of the form (12.83) or (12.84) are called *Robertson-Walker metrics*, and *Friedmann-Lemaître-Robertson-Walker metrics* if their scale factor satisfies Friedmann's equations.

Chapter 13

Two Examples of Relativistic Astrophysics

13.1 Light Propagation

13.1.1 Arbitrary spacetimes

We had seen in (6.16) that the separation vector n between two geodesics out of a congruence evolves in a way determined by the equation of geodesic deviation or *Jacobi equation*

$$\nabla_u^2 n = R(u,n)u \;, \tag{13.1}$$

where u is the tangent vector to the geodesics and $R(u,n)u$ is the curvature as defined in (3.49).

We apply this now to a light bundle, i.e. a congruence of light rays or null geodesics propagating from a source moving with a four-velocity u_s. Let now k be the wave vector of the light rays, then the frequency of the light at the source is

$$\omega_\mathrm{s} = \langle k, u_\mathrm{s}\rangle \;, \tag{13.2}$$

and we introduce the normalised wave vector $\tilde{k} = k/\omega_\mathrm{s}$ which satisfies $\langle \tilde{k}, u_\mathrm{s}\rangle = 1$. Since k is a null vector, so is $\tilde{k}$.

Next, we introduce a *screen* perpendicular to k and to u. It thus falls into the local three-space of the source, where it is perpendicular to the light rays. Since it is two-dimensional, it can be spanned by two orthonormal vectors $E_{1,2}$, which are parallel-transported along the light bundle such that

$$\nabla_k E_i = 0 = \nabla_{\tilde{k}} E_i \;. \tag{13.3}$$

Notice that the parallel transport along a null geodesic implies that the E_i remain perpendicular to $\tilde{k}$,

$$\nabla_{\tilde{k}}\langle \tilde{k}, E_i\rangle = \langle \nabla_{\tilde{k}}\tilde{k}, E_i\rangle + \langle \tilde{k}, \nabla_{\tilde{k}} E_i\rangle = 0 \;. \tag{13.4}$$

In a coordinate basis $\{e_\alpha\}$ and its conjugate dual basis $\{\theta^i\}$, they can be written as

$$E_i = E_i^\alpha e_\alpha \;, \quad E_i^\alpha = \theta^\alpha(E_i) \;. \tag{13.5}$$

The separation vector n between rays of the bundle can now be expanded into the $E_{1,2}$,

$$n = n^\alpha e_\alpha = N^i E_i \;, \tag{13.6}$$

showing that its components n^α in the basis $\{e_\alpha\}$ are

$$n^\alpha = \theta^\alpha(n) = \theta^\alpha(N^i E_i) = N^i E_i^\alpha \;. \tag{13.7}$$

Substituting the normalised wave vector $\tilde{k}$ for the four-velocity u in the Jacobi equation (**??**), we first have

$$\nabla_{\tilde{k}}^2 n = R(\tilde{k}, n)\tilde{k} \;. \tag{13.8}$$

Writing $n = N^i E_i$ and using (**??**), we find

$$\nabla_{\tilde{k}} n = E_i \nabla_{\tilde{k}} N^i \;, \quad \nabla_{\tilde{k}}^2 n = E_i \nabla_{\tilde{k}}^2 N^i \;, \tag{13.9}$$

and thus

$$E_i \nabla_{\tilde{k}}^2 N^i = R(\tilde{k}^\gamma e_\gamma, n^\delta e_\delta)\tilde{k}^\beta e_\beta \;. \tag{13.10}$$

Next, we apply θ^α to the vectors on either side of this equation and find

$$E_i^\alpha \,\nabla_{\tilde{k}}^2 N^i = R^\alpha{}_{\beta\gamma\delta}\; \tilde{k}^\beta \tilde{k}^\gamma n^\delta = R^\alpha{}_{\beta\gamma\delta}\; \tilde{k}^\beta \tilde{k}^\gamma E_j^\delta\, N^j \;, \tag{13.11}$$

where we have used the components of the curvature tensor for the arbitrary basis $\{e_\alpha\}$ as given in (8.20). Finally, we multiply this equation with $E_{i\alpha}$ from the left and use the orthonormality of the vectors E_i,

$$E_i^\alpha E_{i\alpha} = 1 \;, \tag{13.12}$$

to find the equation

$$\nabla_{\tilde{k}}^2 N_i = R^\alpha{}_{\beta\gamma\delta}\; E_{i\alpha} \tilde{k}^\beta \tilde{k}^\gamma E_j^\delta\, N^j = R_{\alpha\beta\gamma\delta}\, E_i^\alpha \tilde{k}^\beta \tilde{k}^\gamma E_j^\delta\, N^j \tag{13.13}$$

describing how the perpendicular cross section of a light bundle changes along the bundle.

It will turn out convenient to introduce the *Weyl tensor* C, whose components are determined by

$$\begin{aligned} R_{\alpha\beta\gamma\delta} &= C_{\alpha\beta\gamma\delta} + g_{\alpha[\gamma}R_{\delta]\beta} - g_{\beta[\gamma}R_{\delta]\alpha} - \frac{R}{3} g_{\alpha[\gamma}g_{\delta]\beta} \\ &= C_{\alpha\beta\gamma\delta} + \frac{1}{2}\left[g_{\alpha\gamma}R_{\delta\beta} - g_{\alpha\delta}R_{\gamma\beta} - g_{\beta\gamma}R_{\delta\alpha} + g_{\beta\delta}R_{\gamma\alpha}\right] \\ &- \frac{R}{6}\left[g_{\alpha\gamma}g_{\delta\beta} - g_{\alpha\delta}g_{\gamma\beta}\right] \;. \end{aligned} \tag{13.14}$$

Inserting this into (**??**) gives

$$\begin{aligned}
\nabla_{\tilde{k}}^2 N^i &= C_{\alpha\beta\gamma\delta}\, E_i^\alpha \tilde{k}^\beta \tilde{k}^\gamma E_j^\delta\, N^j \\
&+ \frac{1}{2}\Big[\langle\tilde{k}, E_i\rangle\, R_{\beta\delta}\, \tilde{k}^\beta E_j^\delta\, N^j - \langle E_i, E_j\rangle\, R_{\beta\gamma}\, \tilde{k}^\beta \tilde{k}^\gamma\, N^j \\
&- \langle\tilde{k}, \tilde{k}\rangle\, R_{\alpha\delta}\, E_i^\alpha E_j^\delta\, N^j + \langle\tilde{k}, E_j\rangle\, R_{\alpha\gamma}\, E_i^\alpha \tilde{k}^\gamma\, N^j\Big] \\
&- \frac{R}{6}\Big[\langle\tilde{k}, E_i\rangle\langle\tilde{k}, E_j\rangle - \langle\tilde{k}, \tilde{k}\rangle\langle E_i, E_j\rangle\Big] \ .
\end{aligned}$$

The second, the fifth and the sixth terms on the right-hand side vanish because the E_i are perpendicular to $\tilde{k}$, and the fourth and the last terms vanishes because $\tilde{k}$ is a null vector. Using the orthonormality of the E_i, we thus obtain

$$\begin{aligned}
\nabla_{\tilde{k}}^2 N^i &= -\frac{1}{2}\,\delta_j^i\, R_{\beta\gamma}\, \tilde{k}^\beta \tilde{k}^\gamma\, N^j + C_{\alpha\beta\gamma\delta}\, E_i^\alpha \tilde{k}^\beta \tilde{k}^\gamma E_j^\delta\, N^j \\
&= -\frac{1}{2} R_{\beta\gamma}\, \tilde{k}^\beta \tilde{k}^\gamma\, N^i + C_{\alpha\beta\gamma\delta}\, E_i^\alpha \tilde{k}^\beta \tilde{k}^\gamma E_j^\delta\, N^j \ . \qquad (13.15)
\end{aligned}$$

The evolution of the bundle's perpendicular cross section can thus be described by a matrix $\mathcal{T}$,

$$\nabla_{\tilde{k}}^2 \begin{pmatrix} N^1 \\ N^2 \end{pmatrix} = \mathcal{T} \begin{pmatrix} N^1 \\ N^2 \end{pmatrix} \ , \qquad (13.16)$$

which can be written in the form

$$\mathcal{T} = -\frac{1}{2} R_{\alpha\beta} \tilde{k}^\alpha \tilde{k}^\beta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + C_{\alpha\beta\gamma\delta}\, \tilde{k}^\beta \tilde{k}^\gamma \begin{pmatrix} E_1^\alpha E_1^\delta & E_1^\alpha E_2^\delta \\ E_2^\alpha E_1^\delta & E_2^\alpha E_2^\delta \end{pmatrix} \ . \qquad (13.17)$$

Further insight can be gained by extracting the trace-free part from $\mathcal{T}$. Since the trace is

$$\mathrm{tr}\,\mathcal{T} = -R_{\alpha\beta} \tilde{k}^\alpha \tilde{k}^\beta + C_{\alpha\beta\gamma\delta}\, \tilde{k}^\beta \tilde{k}^\gamma \left(E_1^\alpha E_1^\delta + E_2^\alpha E_2^\delta\right) \ , \qquad (13.18)$$

the trace-free part of $\mathcal{T}$ is

$$\begin{aligned}
&\mathcal{T} - \frac{1}{2}\mathrm{tr}\,\mathcal{T}\,\mathcal{I} = \\
&\frac{1}{2}\, C_{\alpha\beta\gamma\delta}\, \tilde{k}^\beta \tilde{k}^\gamma \begin{pmatrix} E_1^\alpha E_1^\delta - E_2^\alpha E_2^\delta & 2E_1^\alpha E_2^\delta \\ 2E_2^\alpha E_1^\delta & E_2^\alpha E_2^\delta - E_1^\alpha E_1^\delta \end{pmatrix} \ . \qquad (13.19)
\end{aligned}$$

The Weyl tensor has the same symmetries as the curvature tensor, thus we can transform

$$\begin{aligned}
C_{\alpha\beta\gamma\delta}\, E_i^\alpha \tilde{k}^\beta \tilde{k}^\gamma E_j^\delta &= C_{\beta\alpha\delta\gamma}\, E_i^\alpha \tilde{k}^\beta \tilde{k}^\gamma E_j^\delta = C_{\delta\gamma\beta\alpha}\, E_i^\alpha \tilde{k}^\beta \tilde{k}^\gamma E_j^\delta \\
&= C_{\alpha\beta\gamma\delta}\, E_j^\alpha \tilde{k}^\beta \tilde{k}^\gamma E_i^\delta \ , \qquad (13.20)
\end{aligned}$$

where we have renamed the summation indices in the last step. Thus, the trace-free part of $\mathcal{T}$ is symmetric.

In a last step, we introduce the complex vector

$$\epsilon \equiv E_1 + \mathrm{i}E_2 \,, \tag{13.21}$$

in terms of which we can write

$$\begin{aligned} E_1^\alpha E_1^\delta + E_2^\alpha E_2^\delta &= \epsilon^\alpha \epsilon^{*\delta} \\ E_1^\alpha E_1^\delta - E_2^\alpha E_2^\delta &= \mathrm{Re}\left(\epsilon^\alpha \epsilon^\delta\right) \,, \\ 2E_1^\alpha E_2^\delta &= \mathrm{Im}\left(\epsilon^\alpha \epsilon^\delta\right) \,. \end{aligned} \tag{13.22}$$

Summarising, we define the two scalars

$$\begin{aligned} \mathcal{R} &\equiv -\frac{1}{2} R_{\alpha\beta}\tilde{k}^\alpha \tilde{k}^\beta + \frac{1}{2} C_{\alpha\beta\gamma\delta}\epsilon^\alpha \tilde{k}^\beta \tilde{k}^\gamma \epsilon^{*\delta} \,, \\ \mathcal{F} &\equiv \frac{1}{2} C_{\alpha\beta\gamma\delta}\, \epsilon^\alpha \tilde{k}^\beta \tilde{k}^\gamma \epsilon^\delta \,, \end{aligned} \tag{13.23}$$

in terms of which the matrix $\mathcal{T}$ can be brought into the form

$$\mathcal{T} = \begin{pmatrix} \mathcal{R} + \mathrm{Re}(\mathcal{F}) & \mathrm{Im}(\mathcal{F}) \\ \mathrm{Im}(\mathcal{F}) & \mathcal{R} - \mathrm{Re}(\mathcal{F}) \end{pmatrix} \,. \tag{13.24}$$

this is called the *optical tidal matrix*.

The effect of the tidal matrix becomes obvious if we start with a light bundle with circular cross section, for which the components N^i of the distance vector can be written as

$$\begin{pmatrix} N^1 \\ N^2 \end{pmatrix} = \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} \,, \tag{13.25}$$

where ϕ is the polar angle on the screen spanned by the vectors $E_{1,2}$. Before we apply the optical tidal matrix, we rotate it into its principal-axis frame,

$$\begin{pmatrix} \mathcal{R} + \mathrm{Re}(\mathcal{F}) & \mathrm{Im}(\mathcal{F}) \\ \mathrm{Im}(\mathcal{F}) & \mathcal{R} - \mathrm{Re}(\mathcal{F}) \end{pmatrix} \rightarrow \begin{pmatrix} \mathcal{R} + |\mathcal{F}| & 0 \\ 0 & \mathcal{R} - |\mathcal{F}| \end{pmatrix} \,, \tag{13.26}$$

which shows that it maps the circle onto a curve outlined by the vector

$$\begin{pmatrix} x \\ y \end{pmatrix} \equiv \begin{pmatrix} (\mathcal{R} + |\mathcal{F}|)\cos\phi \\ (\mathcal{R} - |\mathcal{F}|)\sin\phi \end{pmatrix} \,. \tag{13.27}$$

This is an ellipse with semi-major axis $\mathcal{R} + |\mathcal{F}|$ and semi-minor axis $\mathcal{R} - |\mathcal{F}|$, because obviously

$$\frac{x^2}{(\mathcal{R} + |\mathcal{F}|)^2} + \frac{y^2}{(\mathcal{R} - |\mathcal{F}|)^2} = \cos^2\phi + \sin^2\phi = 1 \,. \tag{13.28}$$

Thus, for $\mathcal{F} = 0$, the originally circular cross section remains circular, with $\mathcal{R}$ being responsible for isotropically expanding or shrinking it, while the light bundle is elliptically deformed if $\mathcal{F} \neq 0$.

13.1.2 Specialisation to homogeneous and isotropic spacetimes

In an isotropic spacetime, it must be impossible to single out preferred directions. This implies that $\mathcal{F} = 0$ then, because otherwise the principal-axis frame of the optical tidal matrix would break isotropy. If the spacetime is homogeneous, this must hold everywhere, so that we can specialise

$$\mathcal{T} = \begin{pmatrix} \mathcal{R} & 0 \\ 0 & \mathcal{R} \end{pmatrix} , \tag{13.29}$$

with $\mathcal{R}$ defined in (**??**).

Moreover, we see that

$$G_{\alpha\beta}\,\tilde{k}^\alpha\tilde{k}^\beta = \left(R_{\alpha\beta} - \frac{R}{2}g_{\alpha\beta}\right)\tilde{k}^\alpha\tilde{k}^\beta = R_{\alpha\beta}\,\tilde{k}^\alpha\tilde{k}^\beta \tag{13.30}$$

because $\tilde{k}$ is a null vector. Thus, we can put

$$\mathcal{R} = -\frac{1}{2}\,G_{\alpha\beta}\,\tilde{k}^\alpha\tilde{k}^\beta = -\frac{4\pi G}{c^4}T_{\alpha\beta}\,\tilde{k}^\alpha\tilde{k}^\beta\ , \tag{13.31}$$

using Einstein's field equations in the second step.

Next, we can insert the energy-momentum tensor (12.53) for a perfect fluid,

$$T_{\alpha\beta} = (\rho c^2 + p)u_\alpha u_\beta + p g_{\alpha\beta}\ , \tag{13.32}$$

and use the fact that fundamental observers (i.e. observers for whom the universe appears isotropic) have $u^\mu = -u_\mu = \partial_t$.

Since the frequency measured by an observer moving with four-velocity u is $\langle k, u\rangle$, due to our definition of $\tilde{k} = k/\omega_\mathrm{s}$, and because of the cosmological redshift (12.73), we can write

$$u_\alpha\tilde{k}^\alpha = \frac{1}{\omega_\mathrm{s}}\langle k, u\rangle = \frac{\omega_\mathrm{o}}{\omega_\mathrm{s}} = 1 + z\ , \tag{13.33}$$

where ω_o is the frequency measured by the observer, and z is the redshift of the source relative to the observer.

Thus, we find

$$\mathcal{R} = -\frac{4\pi G}{c^2}\left(\rho + \frac{p}{c^2}\right)(1+z)^2 = -\frac{4\pi G}{c^2}\,\frac{\rho + p/c^2}{a^2}\ , \tag{13.34}$$

where a is the scale factor of the metric inserted according to (12.73), setting $a = 1$ at the time of observation.

If $p \ll \rho c^2$, the density scales like a^{-3} as shown in (12.62), and then

$$\mathcal{R} = -\frac{4\pi G}{c^2}\rho_0 a^{-5}\ . \tag{13.35}$$

We are still free to choose a suitable parameter τ along the fiducial light ray. Since the tangent vector $\tilde{k}$ is given by

$$\tilde{k} = \frac{\mathrm{d}x}{\mathrm{d}\tau} \tag{13.36}$$

and $\langle\tilde{k}, u\rangle = 1 + z$ as shown above, we must have

$$\left\langle \frac{\mathrm{d}x}{\mathrm{d}\tau}, u \right\rangle = \frac{\mathrm{d}x^0}{\mathrm{d}\tau} = \frac{c\mathrm{d}t}{\mathrm{d}\tau} = 1 + z = \frac{1}{a} , \tag{13.37}$$

where $u = \partial_t$ was used in the second step. Thus, we must have $\mathrm{d}\tau = ca\mathrm{d}t$. Then, observing that $\mathrm{d}a = \dot{a}\mathrm{d}t$, we find

$$\mathrm{d}\tau = ca\mathrm{d}t = \frac{ca\mathrm{d}a}{\dot{a}} = \frac{c\mathrm{d}a}{\dot{a}/a} . \tag{13.38}$$

With this result, we can rewrite

$$\nabla_{\tilde{k}} N^i = \tilde{k}^\alpha \nabla_\alpha N^i = \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\partial_\alpha N^i = \frac{\mathrm{d}N^i}{\mathrm{d}\tau} , \tag{13.39}$$

and the optical tidal matrix becomes

$$\frac{\mathrm{d}^2 N^i}{\mathrm{d}\tau^2} = -\frac{4\pi G}{c^2}\rho_0 a^{-5}\, N^i . \tag{13.40}$$

This equation can be substantially simplified to reveal its very intuitive meaning. From the metric in the form (12.83), we see that radially propagating light rays must satisfy

$$c\mathrm{d}t = \pm a\mathrm{d}w , \tag{13.41}$$

where w is the radial distance coordinate defined in (12.81). The sign can be chosen depending on whether the distance should grow with increasing time (i.e. into the future) or with decreasing time (i.e. into the past), but it is irrelevant for our consideration. We choose $c\mathrm{d}t = a\mathrm{d}w$ and therefore, with (**??**),

$$\mathrm{d}\tau = ca\mathrm{d}t = a^2\mathrm{d}w . \tag{13.42}$$

Due to the isotropy of the light bundle's deformation, we can replace its two diameters $N^{1,2}$ by a single diameter D. Moreover, we consider the *comoving* diameter D/a, i.e. the diameter with the expansion of the universe divided out. Substituting $\mathrm{d}w$ for $\mathrm{d}\tau$ as a measure of length, we first see that

$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}w^2}\left(\frac{D}{a}\right) &= a^2\frac{\mathrm{d}}{\mathrm{d}\tau}\left[a^2\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{D}{a}\right)\right] = a^2\frac{\mathrm{d}}{\mathrm{d}\tau}(aD' - a'D) \\ &= a^2(aD'' - a''D) , \end{aligned} \tag{13.43}$$

where the prime denotes the derivative with respect to the parameter τ.

Next, we use (**??**) to write

$$\frac{\mathrm{d}a'}{\mathrm{d}\tau} = \frac{1}{c}\left(\frac{\dot{a}}{a}\right)\frac{\mathrm{d}a'}{\mathrm{d}a} = \frac{1}{c^2}\left(\frac{\dot{a}}{a}\right)\frac{\mathrm{d}(\dot{a}/a)}{\mathrm{d}a} = \frac{1}{2c^2}\frac{\mathrm{d}(\dot{a}/a)^2}{\mathrm{d}a}\,, \tag{13.44}$$

which enables us to insert Friedmann's equation (12.55) in the form

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\frac{\rho_0}{a^3} + \frac{\Lambda c^2}{3} - \frac{kc^2}{a^2} \tag{13.45}$$

to find

$$a'' = \frac{\mathrm{d}a'}{\mathrm{d}\tau} = -\frac{4\pi G}{c^2}\rho_0 a^{-4} + ka^{-3}\,. \tag{13.46}$$

Now, we substitute $D'' = (N^i)''$ from (**??**) and a'' from (**??**) into (**??**) and obtain

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}w^2}\left(\frac{D}{a}\right) &= -\frac{4\pi G}{c^2}\rho_0 a^{-2}D + \frac{4\pi G}{c^2}\rho_0 a^{-2}D - kDa^{-1}\\ &= -k\left(\frac{D}{a}\right)\,,\end{aligned} \tag{13.47}$$

which is a simple oscillator equation for the *comoving* bundle diameter D.

Equation (**??**) is now easily solved. We set the boundary conditions that the bundle emerges from a source point, hence $D = 0$ at the source, and that it initially expands linearly with the radial distance w, hence $\mathrm{d}(D/a)/\mathrm{d}w = 1$ there. Then, the solution of (**??**) is

$$D = af_k(w) = a\begin{cases}\frac{1}{\sqrt{k}}\sin(\sqrt{k}w) & (k > 0)\\ w & (k = 0)\\ \frac{1}{\sqrt{-k}}\sinh(\sqrt{-k}w) & (k < 0)\end{cases}\,, \tag{13.48}$$

with $f_k(w)$ defined in (12.82).

This shows that the diameter of the bundle increases linearly if space is flat, diverges hyperbolically if space is negatively curved, and expands and shrinks as a sine if space is positively curved.

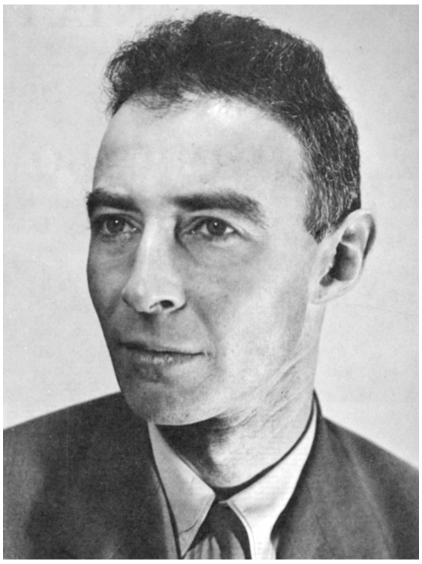
13.2 Relativistic Stellar Structure Equations

13.2.1 The Tolman-Oppenheimer-Volkoff equation

We now consider an axially symmetric, static solution of Einstein's field equations in presence of matter. As usual for an axisymmetric solution, we can work in the Schwarzschild tetrad (8.40), in which the energy-momentum tensor of a perfect fluid,

$$T = T_{\mu\nu}\,\theta^\mu\otimes\theta^\nu \quad \text{with} \quad T_{\mu\nu} = (\rho c^2 + p)u_\mu u_\nu + pg_{\mu\nu}\,, \tag{13.49}$$

Richard C. Tolman (1881–1948)



J. Robert Oppenheimer (1904–1967)

simplifies to

$$T_{\mu\nu} = \mathrm{diag}(\rho c^2, p, p, p) \tag{13.50}$$

because $u = u^0 e_0 = e_0$ in the static situation we are considering.

It had been shown as part of the problems that the relativistic Euler equation is

$$(\rho c^2 + p)\nabla_u u = -\mathrm{grad} p - u \nabla_u p \ , \tag{13.51}$$

which had been derived by contracting the local conservation equation

$$\nabla_\nu T^{\mu\nu} = 0 \tag{13.52}$$

with the projection tensor $h = g + u \otimes u$.

Specialising (**??**) to our situation, we use (8.9) to see that

$$\nabla_u u = \langle \mathrm{d}u^\alpha + u^\beta \omega^\alpha_\beta, u \rangle e_\alpha = \langle u^0 \omega^1_0, u \rangle e_1 = a' \mathrm{e}^{-b} e_1 \ , \tag{13.53}$$

because the only non-vanishing of the connection forms ω^α_0 for a static, axially symmetric spacetime is

$$\omega^1_0 = a' \mathrm{e}^{-b} \, \theta^0 \ , \tag{13.54}$$

as shown in (8.50). Moreover, in the static situation, $\nabla_u p = 0$.

The gradient grad$p$ was defined as

$$\mathrm{grad} p = \mathrm{d}p^\flat = p' \mathrm{d}r^\flat = p' \mathrm{e}^{-b} \, (\theta^1)^\flat = p' \mathrm{e}^{-b} \, e_1 \ . \tag{13.55}$$

Substituting (**??**) and (**??**) into (**??**) yields

$$(\rho c^2 + p) a' = -p' \quad \Rightarrow \quad a' = -\frac{p'}{\rho c^2 + p} \ , \tag{13.56}$$

which is the relativistic hydrostatic equation.

With the components of the Einstein tensor given in (8.60) and the energy-momentum tensor (**??**), the two independent field equations read

$$\begin{aligned} -\frac{1}{r^2} + \mathrm{e}^{-2b}\left(\frac{1}{r^2} - \frac{2b'}{r}\right) &= -\frac{8\pi G}{c^2}\rho \\ -\frac{1}{r^2} + \mathrm{e}^{-2b}\left(\frac{1}{r^2} + \frac{2a'}{r}\right) &= \frac{8\pi G}{c^4} p \ . \end{aligned} \tag{13.57}$$

The first of these equations is equivalent to

$$\left(r\mathrm{e}^{-2b}\right)' = 1 - \frac{8\pi G}{c^2}\rho r^2 \ . \tag{13.58}$$

Integrating, and using the mass

$$M(r) = 4\pi \int_0^r \rho(r') r'^2 \mathrm{d}r' \ , \tag{13.59}$$

000)

shows that the function b is determined by

$$e^{-2b} = 1 - \frac{2GM}{rc^2} . \tag{13.60}$$

If we subtract the first from the second field equation (**??**), we find

$$\frac{2e^{-2b}}{r}(a' + b') = \frac{8\pi G}{c^4}(\rho c^2 + p) \tag{13.61}$$

or

$$a' = -b' + \frac{4\pi G}{c^4} e^{2b}(\rho c^2 + p) r . \tag{13.62}$$

On the other hand, (**??**) gives

$$-2b' e^{-2b} = \frac{2GM}{r^2c^2} - \frac{2GM'}{rc^2} = \frac{2GM}{r^2c^2} - \frac{8\pi G}{c^2}\rho r , \tag{13.63}$$

or

$$b' = \left(\frac{4\pi G}{c^2}\rho r - \frac{GM}{r^2c^2}\right) e^{2b} , \tag{13.64}$$

which allows us to write (**??**) as

$$a' = \left(\frac{GM}{r^2c^2} + \frac{4\pi G}{c^4} pr\right) e^{2b} = \frac{G(M + 4\pi p r^3/c^2)}{r(rc^2 - 2GM)} . \tag{13.65}$$

But the hydrostatic equation demands (**??**), which we combine with (**??**) to find

$$-p' = \frac{G(\rho c^2 + p)(M + 4\pi p r^3/c^2)}{r(rc^2 - 2GM)} . \tag{13.66}$$

This is the Tolman-Oppenheimer-Volkoff equation for the pressure gradient in a relativistic star.

This equation generalises the hydrostatic Euler equation in Newtonian physics, which reads for a spherically-symmetric configuration

$$-p' = \frac{GM\rho}{r^2} . \tag{13.67}$$

This shows that gravity acts on $\rho c^2 + p$ instead of ρ alone, the pressure itself adds to the source of gravity, and gravity increases more strongly than $\propto r^{-2}$ towards the centre of the star.

13.2.2 The mass of non-rotating neutron stars

Neutron stars are a possible end product of the evolution of massive stars. When such stars explode as supernovae, they may leave behind an object with a density so high that protons and electrons combine to neutrons in the process of inverse β decay. Objects thus form which consist of matter with nuclear density.

Let us assume that we can describe such objects with an energy-momentum tensor of the form (**??**) in which anisotropic stresses are unimportant. The pressure p is related to the density ρ by an equation-of-state $p(\rho)$.

Furthermore, let the density be positive, $\rho > 0$, the pressure be a monotonically increasing function of ρ,

$$\frac{\mathrm{d}p}{\mathrm{d}\rho} \geq 0 \,, \tag{13.68}$$

and the equation-of-state be known for densities $\rho \leq \rho_0$, where ρ_0 reaches nuclear densities.

We must satisfy the Tolman-Oppenheimer-Volkoff equation (**??**), equation (**??**) for the function $a(r)$ in the metric, and the equation

$$M' = 4\pi\rho r^2 \tag{13.69}$$

determining the mass.

The boundary conditions are

$$\begin{aligned} p(r=0) &= p(\rho_\mathrm{c}) \equiv p_\mathrm{c} \,, \quad M(0) = 0 \\ \mathrm{e}^{a(R)} &= 1 - \frac{2GM}{Rc^2} \,, \end{aligned} \tag{13.70}$$

where ρ_c is the central density. The latter equation is valid at the stellar radius R and follows from Birkhoff's theorem.

In order to prevent infinite pressure gradient forces, we require

$$\frac{2GM}{c^2} = 2m < r \tag{13.71}$$

due to the Tolman-Oppenheimer-Volkoff equation. Assuming that the core of a neutron star reaches nuclear densities,

$$\rho_0 \approx 5 \times 10^{14}\,\mathrm{g\,cm^{-3}} \,, \tag{13.72}$$

and that the core has radius r_0, the core mass is limited by

$$m_0 < \frac{r_0}{2} \,. \tag{13.73}$$

By the assumption that the density does not increase outward,

$$\rho \geq \rho_0 \quad \Rightarrow \quad m_0 \geq \frac{4\pi G}{3c^2} r_0^3 \rho_0 \,, \tag{13.74}$$

which yields the bound

$$r_0 \leq \left(\frac{3m_0c^2}{4\pi G\rho_0}\right)^{1/3} \,. \tag{13.75}$$

Combining this with (**??**), we find

$$m_0 < \frac{1}{2}\left(\frac{3m_0c^2}{4\pi G\rho_0}\right)^{1/3} , \tag{13.76}$$

which gives

$$m_0 < \frac{1}{2}\left(\frac{3c^2}{8\pi G\rho_0}\right)^{1/2} , \quad r_0 < \left(\frac{3c^2}{8\pi G\rho_0}\right)^{1/2} . \tag{13.77}$$

Inserting the density (**??**) yields

$$r_0 < 18\,\mathrm{km} , \quad M_0 = \frac{m_0c^2}{G} < 6M_\odot . \tag{13.78}$$

Masses exceeding this limit will collapse into black holes.

Appendix A

Electrodynamics

A.1 Electromagnetic field tensor

Electric and magnetic fields are components of the antisymmetric field tensor

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{A.1}$$

formed from the four-potential

$$A^\mu = \begin{pmatrix} \Phi \\ \vec{A} \end{pmatrix} . \tag{A.2}$$

The field tensor can be conveniently summarised as

$$(F^{\mu\nu}) = \begin{pmatrix} 0 & \vec{E} \\ -\vec{E} & \mathcal{B} \end{pmatrix} \tag{A.3}$$

with

$$\mathcal{B}_{ij} = \epsilon_{ija} B^a . \tag{A.4}$$

Given the signature $(-, +, +, +)$ of the Minkowski metric, its associated rank-$(0, 2)$ tensor has the components

$$(F_{\mu\nu}) = \begin{pmatrix} 0 & -\vec{E} \\ \vec{E} & \mathcal{B} \end{pmatrix} . \tag{A.5}$$

A.2 Maxwell's equations

The homogeneous Maxwell equations read

$$\partial_{[\alpha} F_{\beta\gamma]} = 0 . \tag{A.6}$$

For $\alpha = 0$, $(\beta, \gamma) = (1, 2)$, $(1, 3)$ and $(2, 3)$, this gives

$$\dot{\vec{B}} + c\vec{\nabla} \times \vec{E} = 0 , \tag{A.7}$$

and for $\alpha = 1$, $(\beta, \gamma) = (2, 3)$, we find

$$\vec{\nabla} \cdot \vec{B} = 0 \,. \tag{A.8}$$

The inhomogeneous Maxwell equations are

$$\partial_\nu F^{\mu\nu} = \frac{4\pi}{c} j^\mu \,, \tag{A.9}$$

where

$$j^\mu = \begin{pmatrix} \rho c \\ \vec{j} \end{pmatrix} \tag{A.10}$$

is the four-current density. For $\mu = 0$ and $\mu = i$, (A.9) gives

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho \,, \quad c\vec{\nabla} \times \vec{B} - \dot{\vec{E}} = 4\pi\vec{j} \,, \tag{A.11}$$

respectively.

With the definition (A.1) and the Lorenz gauge condition $\partial_\mu A^\mu = 0$, the inhomogeneous equations (A.9) can be written as

$$\Box A^\mu = -\frac{4\pi}{c} j^\mu \,, \tag{A.12}$$

where $\Box = -\partial_0^2 + \vec{\nabla}^2$ is the d'Alembert operator. The particular solution of the homogeneous equation is given by the convolution of the source with the retarded Greens function

$$G(t, t', \vec{x}, \vec{x}') = \frac{1}{\left|\vec{x} - \vec{x}'\right|} \delta\left(t - t' - \frac{\left|\vec{x} - \vec{x}'\right|}{c}\right) \,, \tag{A.13}$$

i.e. by

$$A^\mu(t, \vec{x}) = \frac{1}{c} \int \mathrm{d}^3x' \int \mathrm{d}t' G(t, t', \vec{x}, \vec{x}') \, j^\mu(t, \vec{x}') \,. \tag{A.14}$$

A.3 Lagrange density and energy-momentum tensor

The Lagrange density of the electromagnetic field coupled to matter is

$$\mathcal{L} = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} - \frac{1}{c} A_\mu j^\mu \,, \tag{A.15}$$

from which Maxwell's equations follow by the Euler-Lagrange equations,

$$\partial^\nu \frac{\partial \mathcal{L}}{\partial(\partial^\nu A^\mu)} - \frac{\partial \mathcal{L}}{\partial A^\mu} = 0 \,. \tag{A.16}$$

From the Lagrange density of the free electromagnetic field,

$$\mathcal{L} = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} \ , \tag{A.17}$$

we find the energy-momentum tensor

$$T^{\mu\nu} = -2\frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} + g^{\mu\nu}\mathcal{L} = \frac{1}{4\pi}\left(F^{\mu\lambda}F^{\nu}{}_{\lambda} - \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right) \ . \tag{A.18}$$

From (A.3), we find first

$$F^{\alpha\beta}F_{\alpha\beta} = -2(\vec{E}^2 - \vec{B}^2) \ , \tag{A.19}$$

and the energy-momentum tensor can be written as

$$T^{\mu\nu} = \frac{1}{8\pi}\begin{pmatrix} \vec{E}^2 + \vec{B}^2 & 2\vec{E}\mathcal{B}^T \\ 2\mathcal{B}\vec{E} & -\vec{E}^2 - \vec{B}^2 + \mathcal{B}\mathcal{B}^T \end{pmatrix} \ . \tag{A.20}$$

This yields the energy density

$$T^{00} = \frac{\vec{E}^2 + \vec{B}^2}{8\pi} \tag{A.21}$$

of the electromagnetic field and the Poynting vector

$$cT^{0i} = \frac{c}{4\pi}\vec{E} \times \vec{B} \ . \tag{A.22}$$

Appendix B

Summary of Differential Geometry

B.1 Manifold

An n-dimensional manifold M is a suitably well-behaved space that is locally homeomorphic to $\mathbb{R}^n$, i.e. that locally "looks like" $\mathbb{R}^n$.

A chart h, or a coordinate system, is a homeomorphism from $D \subset M$ to $U \subset \mathbb{R}^n$,

$$h : D \to U \ , \quad p \mapsto h(p) = (x^1, \ldots, x^n) \ , \tag{B.1}$$

i.e. it assigns an n-tupel of coordinates $\{x^i\}$ to a point $p \in D$.

An atlas is a collection of charts whose domains cover the entire manifold. If all coordinate changes between charts of the atlas with overlapping domains are differentiable, the manifold and the atlas themselves are called differentiable.

B.2 Tangent and Dual Spaces

The tangent space V_p at a point $p \in M$ is the vector space of all derivations. A derivation v is a map from the space $\mathcal{F}_p$ of C^∞ functions in p into the real numbers,

$$v : \mathcal{F}_p \to \mathbb{R} \ , \quad f \mapsto v(f) \ . \tag{B.2}$$

A derivation is a linear map which satisfies the Leibniz rule,

$$v(\lambda f + \mu g) = \lambda v(f) + \mu v(g) \ , \quad v(fg) = v(f)g + fv(g) \ . \tag{B.3}$$

Tangent vectors generalise directional derivatives of functions.

A coordinate basis of the tangent vector space is given by the partial derivatives $\{\partial_i\}$. Tangent vectors can then be expanded in this basis,

$$v = v^i \partial_i \;, \quad v(f) = v^i \partial_i f \;. \tag{B.4}$$

A dual vector w is a linear map assigning a real number to a vector,

$$w : V \to \mathbb{R} \;, \quad v \mapsto w(v) \;. \tag{B.5}$$

The space of dual vectors to a tangent vector space V is the dual space V^*.

Specifically, the differential of a function $f \in \mathcal{F}$ is a dual vector defined by

$$\mathrm{d}f : V \to \mathbb{R} \;, \quad v \mapsto \mathrm{d}f(v) = v(f) \;. \tag{B.6}$$

Accordingly, the differentials of the coordinate functions x^i form a basis $\{\mathrm{d}x^i\}$ of the dual space which is orthonormal to the coordinate basis $\{\partial_i\}$ of the tangent space,

$$\mathrm{d}x^i(\partial_j) = \partial_j(x^i) = \delta^i_j \;. \tag{B.7}$$

B.3 Tensors

A tensor $t \in \mathcal{T}^r_s$ of rank (r, s) is a multilinear mapping of r dual vectors and s vectors into the real numbers. For example, a tensor of rank $(0, 2)$ is a bilinear mapping of 2 vectors into the real numbers,

$$t : V \times V \to \mathbb{R} \;, \quad (x, y) \mapsto t(x, y) \;. \tag{B.8}$$

The tensor product is defined component-wise. For example, two dual vectors $w_{1,2}$ can be multiplied to form a rank-$(0, 2)$ tensor $w_1 \otimes w_2$

$$(v_1, v_2) \mapsto (w_1 \otimes w_2)(v_1, v_2) = w_1(v_1)\, w_2(v_2) \;. \tag{B.9}$$

A basis for tensors of arbitrary rank is obtained by the tensor product of suitably many elements of the bases $\{\partial_i\}$ of the tangent space and $\{\mathrm{d}x^j\}$ of the dual space. For example, a tensor $t \in \mathcal{T}^0_2$ can be expanded as

$$t = t_{ij}\, \mathrm{d}x^i \otimes \mathrm{d}x^j \;. \tag{B.10}$$

If applied to two vectors $x = x^k \partial_k$ and $y = y^l \partial_l$, the result is

$$t(x, y) = t_{ij}\, \mathrm{d}x^i(x^k \partial_k)\, \mathrm{d}x^j(y^l \partial_l) = t_{ij} x^k y^l \delta^i_k \delta^j_l = t_{ij} x^i y^j \;. \tag{B.11}$$

The contraction of a tensor $t \in \mathcal{T}^r_s$ is defined by

$$C : \mathcal{T}^r_s \to \mathcal{T}^{r-1}_{s-1} \;, \quad t \mapsto Ct \tag{B.12}$$

such that one of the dual vector arguments and one of the vector arguments are filled with pairs of basis elements and summed over all pairs. For example, the contraction of a tensor $t \in \mathcal{T}_1^1$ is

$$Ct = t(\mathrm{d}x^k, \partial_k) = (t^i_j\, \partial_i \otimes \mathrm{d}x^j)(\mathrm{d}x^k, \partial_k) = t^i_j \delta^k_i \delta^j_k = t^k_k \; . \tag{B.13}$$

The metric $g \in \mathcal{T}_2^0$ is a symmetric, non-degenerate tensor field of rank $(0, 2)$, i.e. it satisfies

$$g(x, y) = g(y, x) \; , \quad g(x, y) = 0 \; \forall \; y \quad \Rightarrow \quad x = 0 \; . \tag{B.14}$$

The metric defines the scalar product between two vectors,

$$\langle x, y \rangle = g(x, y) \; . \tag{B.15}$$

B.4 Covariant Derivative

The covariant derivative or a connection linearly maps a pair of vectors to a vector,

$$\nabla : V \times V \to V \; , \quad (x, y) \mapsto \nabla_x y \tag{B.16}$$

such that for a function $f \in \mathcal{F}$

$$\nabla_{fx} y = f \nabla_x y \; , \quad \nabla_x (f y) = f \nabla_x y + x(f) y \; . \tag{B.17}$$

The covariant derivative of a function f is its differential,

$$\nabla_v f = v f = \mathrm{d}f(v) \; . \tag{B.18}$$

Due to the linearity, it is completely specified by the covariant derivatives of the basis vectors,

$$\nabla_{\partial_i} \partial_j = \Gamma^k_{ij} \partial_k \; . \tag{B.19}$$

The functions Γ^k_{ij} are called connection coefficients or Christoffel symbols. They are not tensors.

By means of the exponential map, so-called normal coordinates can always be introduced locally in which the Christoffel symbols all vanish.

The covariant derivative ∇y of a vector y is a rank-$(1, 1)$ tensor field defined by

$$\nabla y : V^* \times V \to \mathbb{R} \; , \quad \nabla y(w, v) = w(\nabla_v y) \; . \tag{B.20}$$

In components,

$$(\nabla y)^i_j = \nabla y(\mathrm{d}x^i, \partial_j) = \mathrm{d}x^i(\nabla_{\partial_j} y^k \partial_k) = \partial_j y^i + \Gamma^i_{jk} y^k \; . \tag{B.21}$$

The covariant derivative of a tensor field is defined to obey the Leibniz rule and to commute with contractions. Specifically, the covariant derivative of a dual vector field $w \in V^*$ is a tensor of rank $(0, 2)$ with components

$$(\nabla w)_{ij} = \partial_j w_i - \Gamma^k_{ij} w_k \; . \tag{B.22}$$

B.5 Parallel Transport and Geodesics

A curve γ is defined as a map from some interval $I \subset \mathbb{R}$ to the manifold,

$$\gamma : I \to M \;, \quad t \mapsto \gamma(t) \;. \tag{B.23}$$

Its tangent vector is $\dot\gamma(t)$.

A vector v is said to be parallel transported along γ if

$$\nabla_{\dot\gamma} v = 0 \;. \tag{B.24}$$

A geodesic curve is defined as a curve whose tangent vector is parallel transported along γ,

$$\nabla_{\dot\gamma} \dot\gamma = 0 \;. \tag{B.25}$$

In coordinates, let $u = \dot\gamma$ be the tangent vector to γ and with components $\dot x^i = u^i$, then

$$\ddot x^k + \Gamma^k_{ij} \dot x^i \dot x^j = 0 \;. \tag{B.26}$$

B.6 Torsion and Curvature

The torsion of a connection is defined by

$$T : V \times V \to V \;, \quad (x, y) \mapsto T(x, y) = \nabla_x y - \nabla_y x - [x, y] \;. \tag{B.27}$$

It vanishes if and only if the connection is symmetric.

On a manifold M with a metric g, a symmetric connection can always be uniquely definedby requiring that $\nabla g = 0$. This is the Riemannian connection, whose Christoffel symbols are

$$\Gamma^i_{jk} = \frac{1}{2} g^{ia} \left(\partial_j g_{ak} + \partial_k g_{ja} - \partial_a g_{jk} \right) \;. \tag{B.28}$$

From now on, we shall assume that we are working with the Riemannian connection whose torsion vanishes.

The curvature is defined by

$$\begin{aligned} &R : V \times V \times V \to V \;, \\ &(x, y, z) \mapsto R(x, y)z = \left(\nabla_x \nabla_y - \nabla_y \nabla_x - \nabla_{[x,y]} \right) z \;. \end{aligned} \tag{B.29}$$

The curvature or Riemann tensor $R \in \mathcal{T}^1_3$ is given by

$$R : V^* \times V \times V \times V \to \mathbb{R} \;, \quad (w, x, y, z) \mapsto w[R(x, y)z] \;. \tag{B.30}$$

Its components are

$$R^i_{jkl} = \mathrm{d}x^i[R(\partial_k, \partial_l)\partial_j] = \partial_k \Gamma^i_{jl} - \partial_l \Gamma^i_{jk} + \Gamma^a_{jl} \Gamma^i_{ak} - \Gamma^a_{jk} \Gamma^i_{al} \;. \tag{B.31}$$

The Riemann tensor obeys three important symmetries,

$$R_{ijkl} = -R_{jikl} = R_{jilk} \ , \quad R_{ijkl} = R_{klij} \ , \tag{B.32}$$

which reduce its $4^4 = 256$ components in four dimensions to 21.

In addition, the Bianchi identities hold,

$$\sum_{(x,y,z)} R(x,y)z = 0 \ , \quad \sum_{(x,y,z)} \nabla_x R(y,z) = 0 \ , \tag{B.33}$$

where the sums extend over all cyclic permutations of x, y, z. The first Bianchi identity reduces the number of independent components of the Riemann tensor to 20. In components, the second Bianchi identity can be written

$$R^i_{\ j[kl;m]} = 0 \ , \tag{B.34}$$

where the indices in brackets need to be antisymmetrised.

The Ricci tensor is the contraction of the Riemann tensor over its first and third indices, thus its components are

$$R_{jl} = R^i_{\ jil} = R_{lj} \ . \tag{B.35}$$

A further contraction yields the Ricci scalar,

$$R = R^i_i \ . \tag{B.36}$$

The Einstein tensor is the combination

$$G_{ij} = R_{ij} - \frac{R}{2} g_{ij} \ . \tag{B.37}$$

Contracting the second Bianchi identity, we find the contracted Bianchi identity,

$$\nabla_j G^{ij} = 0 \ . \tag{B.38}$$

B.7 Pull-Back, Lie Derivative and Killing Vector Fields

A differentiable curve $\gamma_t(p)$ defined at every point $p \in M$ defines a diffeomorphic map $\phi_t : M \to M$. If $\dot{\gamma}_t = v$ for a vector field $v \in V$, ϕ_t is called the flow of v.

The pull-back of a function f defined on the target manifold at $\phi_t(p)$ is given by

$$(\phi_t^* f)(p) = (f \circ \phi_t)(p) \ . \tag{B.39}$$

This allows vectors defined at p to be pushed forward to $\phi_t(p)$ by

$$(\phi_{t*} v)(f) = v(\phi_t^* f) = v(f \circ \phi_t) \ . \tag{B.40}$$

Dual vectors w can then be pulled back by

$$(\phi_t^* w)(v) = w(\phi_{t*} v) \ . \tag{B.41}$$

For diffeomorphisms ϕ_t, the pull-back and the push-forward are inverse, $\phi_t^* = \phi_{t*}^{-1}$.

As for vectors and dual vectors, the pull-back and the push-forward can also be defined for tensors of arbitrary rank.

The Lie derivative of a tensor field T into direction v is given by the limit

$$\mathcal{L}_v T = \lim_{t\to 0} \frac{\phi_t^* T - T}{t} \ , \tag{B.42}$$

where ϕ_t is the flow of v. The Lie derivative quantifies how a tensor changes as the manifold is transformed by the flow of a vector field.

The Lie derivative is linear and obeys the Leibniz rule,

$$\mathcal{L}_x(y+z) = \mathcal{L}_x y + \mathcal{L}_x z \ , \quad \mathcal{L}_x(y \otimes z) = \mathcal{L}_x y \otimes z + y \otimes \mathcal{L}_x z \ . \tag{B.43}$$

It commutes with the contraction. Further important properties are

$$\mathcal{L}_{x+y} = \mathcal{L}_x + \mathcal{L}_y \ , \quad \mathcal{L}_{\lambda x} = \lambda \mathcal{L}_x \ , \quad \mathcal{L}_{[x,y]} = [\mathcal{L}_x, \mathcal{L}_y] \ . \tag{B.44}$$

The Lie derivative of a function f is the ordinary differential

$$\mathcal{L}_v f = v(f) = \mathrm{d}f(v) \ . \tag{B.45}$$

The Lie derivative and the differential commute,

$$\mathcal{L}_v \mathrm{d}f = \mathrm{d}\mathcal{L}_v f \ . \tag{B.46}$$

The Lie derivative of a vector x is the commutator

$$\mathcal{L}_v x = [v, x] \ . \tag{B.47}$$

By its commutation with contractions and the Leibniz rule, the Lie derivative of a dual vector w turns out to be

$$(\mathcal{L}_x w)(v) = x[w(v)] - w([x, v]) \ . \tag{B.48}$$

Lie derivatives of arbitrary tensors can be similarly derived. For example, if $g \in \mathcal{T}_2^0$, we find

$$(\mathcal{L}_x g)(v_1, v_2) = x[g(v_1, v_2)] - g([x, v_1], v_2) - g(v_1, [x, v_2]) \tag{B.49}$$

with $v_{1,2} \in V$.

Killing vector fields K define isometries of the metric, i.e. the metric does not change under the flow of K. This implies the Killing equation

$$\mathcal{L}_K g = 0 \quad \Rightarrow \quad \nabla_i K_j + \nabla_j K_i = 0 \ . \tag{B.50}$$

B.8 Differential Forms

Differential p-forms $\omega \in \bigwedge^p$ are totally antisymmetric tensor fields of rank $(0, p)$. Their components satisfy

$$\omega_{i_1 \dots i_p} = \omega_{[i_1 \dots i_p]} \ . \tag{B.51}$$

The exterior product $\wedge$ is defined by

$$\wedge : \bigwedge^p \times \bigwedge^q \to \bigwedge^{p+q} \ , \quad (\omega, \eta) \mapsto \omega \wedge \eta = \frac{(p+q)!}{p!q!} \mathcal{A}(\omega \otimes \eta) \ , \tag{B.52}$$

where $\mathcal{A}$ is the alternation operator

$$(\mathcal{A}t)(v_1, \dots, v_p) = \frac{1}{p!} \sum_\pi \operatorname{sgn}(\pi) t(v_{\pi(1)}, \dots, v_{\pi(p)}) \ . \tag{B.53}$$

On the vector space $\bigwedge$ of differential forms, the wedge product defines an associative, skew-commutative Grassmann algebra,

$$\omega \wedge \eta = (-1)^{pq} \, \eta \wedge \omega \ , \tag{B.54}$$

with $\omega \in \bigwedge^p$ and $\eta \in \bigwedge^q$.

A basis for the p-forms is

$$\mathrm{d}x^{i_1} \wedge \dots \wedge \mathrm{d}x^{i_p} \ , \tag{B.55}$$

which shows that the dimension of $\bigwedge^p$ is

$$\dim \bigwedge^p = \begin{pmatrix} n \\ p \end{pmatrix} \ . \tag{B.56}$$

The interior product i_v is defined by

$$i : V \times \bigwedge^p \to \bigwedge^{p-1} \ , \quad (v, \omega) \mapsto i_v(\omega) = \omega(v, \dots) \ . \tag{B.57}$$

In components, the interior product is given by

$$(i_v \omega)_{i_2 \dots i_p} = v^j \omega_{j i_2 \dots i_p} \ . \tag{B.58}$$

The exterior derivative turns p-forms ω into $(p+1)$-forms $\mathrm{d}\omega$,

$$\mathrm{d} : \bigwedge^p \to \bigwedge^{p+1} \ , \quad \omega \mapsto \mathrm{d}\omega = (p+1)\mathcal{A}(\nabla \omega) \ . \tag{B.59}$$

Due to the symmetry of the Riemannian connection, its components are given by partial derivatives,

$$(\mathrm{d}\omega)_{i_1, \dots, i_{p+1}} = (p+1) \partial_{[i_1} \omega_{i_2 \dots i_{p+1}]} \tag{B.60}$$

A differential form α is called exact if a differential form β exists such that $\alpha = \mathrm{d}\beta$. It is called closed if $\mathrm{d}\alpha = 0$.

B.9 Cartan's Structure Equations

Let $\{e_i\}$ be an arbitrary basis and $\{\theta^i\}$ its dual basis such that

$$\langle \theta^i, \mathrm{e}_j \rangle = \delta^i_j \ . \tag{B.61}$$

The connection forms $\omega^i_j \in \bigwedge^1$ are defined by

$$\nabla_v e_i = \omega^j_i(v)\, e_j \ . \tag{B.62}$$

In terms of Christoffel symbols, they can be expressed as

$$\omega^i_j = \Gamma^i{}_{kj}\ \theta^k \ . \tag{B.63}$$

They satisfy the antisymmetry relation

$$\mathrm{d}g_{ij} = \omega_{ij} + \omega_{ji} \ . \tag{B.64}$$

The covariant derivative of a dual basis vector is

$$\nabla_v \theta^i = -\omega^i_j(v)\, \theta^j \ . \tag{B.65}$$

Covariant derivatives of arbitrary vectors x and dual vectors α are then given by

$$\nabla_v x = \langle \mathrm{d}x^i + x^j \omega^i_j, v \rangle\, e_i \ , \quad \nabla_v \alpha = \langle \mathrm{d}\alpha_i - \alpha_j \omega^j_i, v \rangle\, \theta^i \tag{B.66}$$

or

$$\nabla x = e_i \otimes (\mathrm{d}x^i + x^j \omega^j_i) \ , \quad \nabla \alpha = \theta^i \otimes (\mathrm{d}\alpha_i - \alpha_j \omega^j_i) \ . \tag{B.67}$$

Torsion and curvature are expressed by the torsion 2-form $\Theta^i \in \bigwedge^2$ and the curvature 2-form $\Omega^i_j \in \bigwedge^2$ as

$$T(x, y) = \Theta^i(x, y)\, e_i \ , \quad R(x, y)e_i = \Omega^j_i(x, y)\, e_j \ . \tag{B.68}$$

The torsion and curvature forms are related to the connection forms and the dual basis vectors by Cartan's structure equations

$$\Theta^i = \mathrm{d}\theta^i + \omega^i_k \wedge \theta^k \ , \quad \Omega^i_j = \mathrm{d}\omega^i_j + \omega^i_k \wedge \omega^k_j \ . \tag{B.69}$$

The components of the torsion and curvature tensors are determined by

$$\Theta^i = T^i_{jk}\, \theta^j \wedge \theta^k \ , \quad \Omega^i_j = R^i{}_{jkl}\ \theta^k \wedge \theta^l \ . \tag{B.70}$$

B.10 Differential Operators and Integration

The Hodge star operator turns a p-form into an $(n-p)$-form,

$$* : \bigwedge^p \to \bigwedge^{n-p} , \quad \omega \mapsto *\omega . \tag{B.71}$$

If $\{e^i\}$ is an orthonormal basis of the dual space, the Hodge star operator is uniquely defined by

$$* \left(e^1 \wedge \ldots \wedge e^{i_p}\right) = e^{i_{p+1}} \wedge \ldots \wedge e^{i_n} , \tag{B.72}$$

where the indices $i_1 \ldots i_n$ appear in their natural order or a cyclic permutation thereof. For example, the coordinate differentials $\{\mathrm{d}x^i\}$ are an orthonormal dual basis in $\mathbb{R}^3$, and

$$*\mathrm{d}x^1 = \mathrm{d}x^2 \wedge \mathrm{d}x^3 , \quad *\mathrm{d}x^2 = \mathrm{d}x^3 \wedge \mathrm{d}x^1 , \quad *\mathrm{d}x^3 = \mathrm{d}x^1 \wedge \mathrm{d}x^2 . \tag{B.73}$$

The codifferential is a differentiation lowering the order of a p-form by one,

$$\delta : \bigwedge^p \to \bigwedge^{p-1} , \quad \omega \mapsto \delta\omega , \tag{B.74}$$

which is defined by

$$\delta\omega = \mathrm{sgn}(g) * (-1)^{n(p+1)} *\mathrm{d}*\omega . \tag{B.75}$$

It generalises the divergence of a vector field and thus has the components

$$(\delta\omega)_{i_2 \ldots i_p} = \frac{1}{\sqrt{|g|}} \partial_{i_1} \left(\sqrt{|g|} \omega^{i_1 i_2 \ldots i_p} \right) . \tag{B.76}$$

The Laplace-de Rham operator

$$\mathrm{d} \circ \delta + \delta \circ \mathrm{d} \tag{B.77}$$

generalises the Laplace operator.

The canonical volume form is an n-form given by

$$\eta = \sqrt{|g|}\, \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n . \tag{B.78}$$

The integration of n-forms $\omega = f(x_1, \ldots, x_n)\, \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n$ over domains $D \subset M$ is defined by

$$\int_D \omega = \int_D f(x_1, \ldots, x_n)\, \mathrm{d}x^1 \ldots \mathrm{d}x^n . \tag{B.79}$$

Functions f are integrated by means of the canonical volume form,

$$\int_D f\eta = \int_D f \sqrt{|g|}\, \mathrm{d}x^1 \ldots \mathrm{d}x^n . \tag{B.80}$$

The theorems of Stokes and Gauss can be expressed as

$$\int_D \mathrm{d}\alpha = \int_{\partial D} \alpha , \quad \int_D \delta\beta\, \eta = \int_{\partial D} *\beta , \tag{B.81}$$

where α is an $(n-1)$-form, β is a 1-form.