Differential Equations Review Sheet 1, Fall 2018

I. First Order DEs

(a) Separable

Form: $\frac{dy}{dx} = f(x)g(y)$

To solve: arrange it like so: $\frac{dy}{g(y)} = f(x)\,dx$, integrate both sides!

Don't forget the lost solutions $y = c$, where $g(c) = 0$.

(b) Linear

General Form: $a(x)y' + b(x)y = c(x)$

Standard form: $y' + p(x)y = f(x)$

To solve:

- Put in standard form (by dividing by $a(x)$ if necessary).
- Compute the homogeneous solution: $y_h(x) = e^{-\int p(x)dx}$.
- Use the variation of parameters formula: $y(x) = y_h(x)(\int \frac{q(x)}{y_h(x)}\,dx + C)$.
- There is also the definite integral solution: $y(x) = y_h(x)(\int_a^x \frac{q(u)}{y_h(u)}\,du + C)$, where $C = y(a)/y_h(a)$.

II. Linear Constant Coefficient Homogeneous DEs

(a) Linear, Constant Coefficient

1. What does it look like? Well, it's linear, and has constant coefficients. :)
   We write it as $P(D)x = 0$, where $P(r)$ is a polynomial.
2. How do I solve it?
   Get the characteristic polynomial: replace y by 1, y' by r, y'' by r^2 etc. (This comes from guessing $y = e^{rt}$ as a trial solution.)
3. Solve for the roots of the equation containing r (= characteristic equation).
4. Take roots, r_1, r_2 etc. and arrange as: $y = c_1e^{r_1t} + c_2e^{r_2t} + \ldots$
5. If roots are complex in the form of $a \pm bi$, and you want a real valued solution, then make them: $y = c_1e^{at}\cos(bt) + c_2e^{at}\sin(bt) + \ldots$
6. If r is a double root, then e^{rt} and te^{rt} are both homogeneous solutions.

(b) Damping (for $my'' + by' + ky = 0$)

1. Underdamping when $b^2 - 4mk < 0$, so roots are complex, solutions oscillate.
2. Overdamping when $b^2 - 4mk > 0$, so roots are real, solutions are exponentials.
3. Critical damping when $b^2 - 4mk = 0$, so roots are repeated,
   solution is $y = c_1\mathrm{e}^{-bt/2m} + c_2t\,\mathrm{e}^{-bt/2m}$.

(c) Stability

1. $y(t) = 0$ is the equilibrium solution.
2. Physics: the system is stable (really asymptotically stable) if the output to the unforced system always goes to the equilibrium as $t \to \infty$.
3. Math: the system is stable if all charactericstic roots have negative real part.
4. Equivalently: the system is stable if all homogeneous solutions to the DE go to 0 as $t \to \infty$.
5. For $my'' + by' + ky = 0$ the system is stable exactly when m, b and k all have the same sign (usually positive).

III. Complex Numbers

(a) Euler formula: $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.

(b) Polar form: $a + ib = re^{i\theta},\ r = \sqrt{a^2 + b^2}, \tan\theta = b/a$. (Remember how to draw the polar triangle!)

(c) n^{th} roots of $re^{i\theta}$: $z = r^{\frac{1}{n}} e^{\frac{i\theta}{n} + \frac{i2\pi k}{n}}$, $k = 0, 1, 2, \ldots, n-1$.

(d) Complexification: e.g. to solve $P(D)x = F_0 cos(\omega t)$ solve $P(D)z = F_0 e^{i\omega t}$ and then decomplexify: $x = \text{Re}(z)$, or $\int e^{-x}\sin(\omega x)dx = \text{Im}(\int e^{(-1+i\omega)x}dx)$.

IV. Linear Constant Coefficient Inhomogenous DEs

(a) Preliminaries

1. I'm assuming you can solve the homogenous part, $P(D)y = 0$, already.
2. Inhomogenous CC linear DEs are of the form $P(D)y = f$, with a function $f(t) \neq 0$.
3. The general solution to $P(D)y = f$ is $y = y_p + y_h$.
   (y_p = particular solution, y_h = general homogeneous solution.)

(b) Exponential response formula (ERF), also called exponential input theorem

1. For solving $P(D)x = Be^{at}$.
2. Usual version: $x(t) = \dfrac{Be^{at}}{P(a)}$, if $P(a) \neq 0$.
   Note: a is allowed to be complex.
3. Extended version: if $P(a) = 0$ then the solution is $x(t) = \dfrac{Bte^{at}}{P'(a)}$, if $P'(a) \neq 0$.
4. How do you prove the ERF?
   • Try the solution ce^{at}. After substitution you find this works with $c = B/P(a)$.

(c) Sinusoidal response formula (SRF)

1. For solving $P(D)x = B\cos\omega t$.
2. Usual version: $x(t) = \dfrac{B\cos(\omega t - \phi)}{|p(i\omega)|}$, if $P(i\omega) \neq 0$.
   Here $\phi = \text{Arg}(P(i\omega))$. When writing ϕ using $\tan^{-1}$ don't forget to give the quadrants where $P(i\omega)$ might lie.
3. How do you prove the SRF?
   • Complexify $P(D)x = B\cos(\omega t)$ to $P(D)z = Be^{i\omega t}$. Then use the ERF.
4. Extended version: if $P(i\omega) = 0$ then you find the solution by complexifying $P(D)x = B\cos(\omega t)$ to $P(D)z = Be^{i\omega t}$. Then use the extended ERF.

(d) Undetermined coefficients

1. For solving $P(D)x$ = a polynomial.
2. Usual version: guess a solution $x(t)$ = a polynomial of the same degree. Then substitute and solve for the coefficients.
3. **Example.** Solve $x'' + 8x' + 7x = 2t$.
   answer: Try $x = At + B$.
   Substitution gives $7At + (8A + 7B) = 2t$
   Now equate coefficients: $7A = 2$, $8A + 7B = 0$. (So, $A = 2/7$, $B = -16/49$.)
4. Extended version: If the DE doesn't go all the way to x then multiply the guess by the right power of t
5. **Example.** Solve $x^{(4)} + 8x''' = 2t$.
   answer: This only goes to x''', so multiply the guess by t^3
   That is, guess $x = At^4 + Bt^3$.

V. Linear Operators in General

1. An operator T is linear if $T(c_1 f + c_2 g) = c_1 Tf + c_2 Tg$ for all functions f, g and constants c_1, c_2.

2. Our main examples of linear operators are D, $P(D)$.
3. Our main example of a non-linear operator is the squaring operator, $Tf = f^2$.
4. Linearity is almost always easy to check for.

VI. Physical Models

1. Exponential growth and decay: DE is $y' + ky = f(t)$.
2. Spring-mass-dashpot: DE is $my'' + by' + ky = F(t)$,
   where m = mass, b = damping, k = spring constant, F = external (driving) force.
3. LRC circuit: DE is $LI'' + RI' + \frac{1}{C}I = E'$,
   where L = inductance, R = resistance, C = capacitance, E = input voltage.
4. Mixing tanks
   Remember work with amounts not concentrations.
   Rate of change = rate in - rate out.

VII. Amplitude, Phase Lag, Resonance

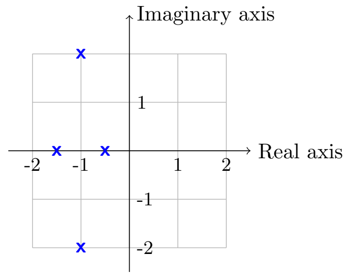
1. Consider the system
   $$my'' + by' + ky = kF_0 \cos(\omega t),$$
   where **we have declared** $F_0 \cos(\omega t)$ to be the input.
   - The characteristic polynomial is $P(r) = mr^2 + br + k$.
   - The input (angular) frequency is ω.
   - The periodic solution (response) is $y_p = g(\omega)F_0 \cos(\omega t - \phi)$.
   - The natural frequency of the system is $\omega_0 = \sqrt{k/m}$. This is the frequecy of oscillation of the the undamped unforced spring: $mx'' + kx = 0$.
   - $A = g(\omega)F_0$ is called the amplitude, where $g(\omega) = k/|P(i\omega)|$.
     The function $g(\omega)$ is called the gain or amplitude response of the system. It depends on ω (and m, b and k).
   - ϕ also depends on ω. The function $\phi(\omega)$ is called the phase lag or the phase response of the system.
   - Practical resonance occurs if $g(\omega)$ has a maximum value at ω_r (for $\omega_r > 0$).
     If there is no such maximum then the system does not have practical resonance.
   - Pure resonance can only happen if $b = 0$. In this case, at $\omega = \omega_0$ we say the gain $g(\omega_0)$ is infinite. Really, when $\omega = \omega_0$ the ERF gives $y_p = \dfrac{t \sin \omega_0 t}{2m\omega_0}$. This is not a sinusoid, rather it is a 'growing' oscillation.
2. Remember the gain depends on what we consider the input.
   For example, consider the DE: $my'' + by' + ky = bF_0 \cos(\omega t)'$,
   but still consider $F_0 \cos(\omega t)$ to be the input. Then the gain is $g(\omega) = \dfrac{b\omega}{|P(i\omega)|}$.
   There are many variations on this.

VIII. Pole diagrams

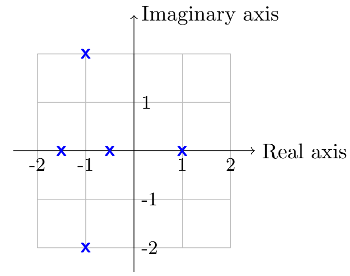
1. For our systems $P(D)x = f$, the pole diagram is drawn in the complex plane.
   The pole diagram tells us a lot about the homogeneous system.
   - We call the charactistic roots poles.
   - You put an $\times$ at each pole.
   - By counting the poles you can determine the order of the system.
   - If all the poles are in the left half-plane then the system is stable because all the exponents in the homogeneous solutions have negative real part.
   - If there are complex poles then the system is oscillatory.

- For a stable system the exponential rate that the unforced (homogeneous) system returns to equilibrium is determined by the real part of the right-most pole.

2. Examples



4 poles, stable, oscillatory



5 poles, unstable, oscillatory

End of review sheet 1

ODE Cheat Sheet

First Order Equations

Separable

$y'(x) = f(x)g(y)$
$\int \frac{dy}{g(y)} = \int f(x)\,dx + C$

Linear First Order

$y'(x) + p(x)y(x) = f(x)$
$\mu(x) = \exp \int^x p(\xi)\,d\xi$ Integrating factor.
$(\mu y)' = f\mu$ Exact Derivative.
Solution: $y(x) = \frac{1}{\mu(x)}\left(\int f(\xi)\mu(\xi)\,d\xi + C\right)$

Exact

$0 = M(x,y)\,dx + N(x,y)\,dy$
Solution: $u(x,y) = \text{const}$ where
$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$
$\frac{\partial u}{\partial x} = M(x,y), \quad \frac{\partial u}{\partial y} = N(x,y)$

Condition: $M_y = N_x$

Non-Exact Form

$\mu(x,y)\left(M(x,y)\,dx + N(x,y)\,dy\right) = du(x,y)$
$M_y = N_x$
$N\frac{\partial \mu}{\partial x} - M\frac{\partial \mu}{\partial y} = \mu\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right).$

Special cases

If $\frac{M_y - N_x}{M} = h(y)$, then $\mu(y) = \exp \int h(y)\,dy$
If $\frac{M_y - N_x}{N} = -h(x)$, then $\mu(y) = \exp \int h(x)\,dx$

Second Order Equations

Linear

$a(x)y''(x) + b(x)y'(x) + c(x)y(x) = f(x)$
$y(x) = y_h(x) + y_p(x)$
$y_h(x) = c_1y_1(x) + c_2y_2(x)$

Constant Coefficients

$ay''(x) + by'(x) + cy(x) = f(x)$
$y(x) = e^{rx} \Rightarrow ar^2 + br + c = 0$

Cases

Distinct, real roots: $r = r_{1,2}$, $y_h(x) = c_1e^{r_1x} + c_2e^{r_2x}$
One real root: $y_h(x) = (c_1 + c_2x)e^{rx}$
Complex roots: $r = \alpha \pm i\beta$, $y_h(x) = (c_1\cos\beta x + c_2\sin\beta x)e^{\alpha x}$

Cauchy-Euler Equations

$ax^2y''(x) + bxy'(x) + cy(x) = f(x)$
$y(x) = x^r \Rightarrow ar(r-1) + br + c = 0$

Cases

Distinct, real roots: $r = r_{1,2}$, $y_h(x) = c_1x^{r_1} + c_2x^{r_2}$
One real root: $y_h(x) = (c_1 + c_2\ln|x|)x^r$
Complex roots: $r = \alpha \pm i\beta$,
$y_h(x) = (c_1\cos(\beta\ln|x|) + c_2\sin(\beta\ln|x|))x^\alpha$

Nonhomogeneous Problems

Method of Undetermined Coefficients

$f(x)$	$y_p(x)$
$a_nx^n + \cdots + a_1x + a_0$	$A_nx^n + \cdots + A_1x + A_0$
ae^{bx}	Ae^{bx}
$a\cos\omega x + b\sin\omega x$	$A\cos\omega x + B\sin\omega x$

Modified Method of Undetermined Coefficients: if any term in the guess $y_p(x)$ is a solution of the homogeneous equation, then multiply the guess by x^k, where k is the smallest positive integer such that no term in $x^ky_p(x)$ is a solution of the homogeneous problem.

Reduction of Order

Homogeneous Case

Given $y_1(x)$ satisfies $L[y] = 0$, find second linearly independent solution as $v(x) = v(x)y_1(x)$. $z = v'$ satisfies a separable ODE.

Nonhomogeneous Case

Given $y_1(x)$ satisfies $L[y] = 0$, find solution of $L[y] = f$ as $v(x) = v(x)y_1(x)$. $z = v'$ satisfies a first order linear ODE.

Method of Variation of Parameters

$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x)$
$c_1'(x)y_1(x) + c_2'(x)y_2(x) = 0$
$c_1'(x)y_1'(x) + c_2'(x)y_2'(x) = \frac{f(x)}{a(x)}$

Applications

Free Fall

$x''(t) = -g$
$v'(t) = -g + f(v)$

Population Dynamics

$P'(t) = kP(t)$
$P'(t) = kP(t) - bP^2(t)$

Newton's Law of Cooling

$T'(t) = -k(T(t) - T_a)$

Oscillations

$mx''(t) + kx(t) = 0$
$mx''(t) + bx'(t) + kx(t) = 0$
$mx''(t) + bx'(t) + kx(t) = F(t)$

Types of Damped Oscillation

Overdamped, $b^2 > 4mk$
Critically Damped, $b^2 = 4mk$
Underdamped, $b^2 < 4mk$

Numerical Methods

Euler's Method

$y_0 = y(x_0),$
$y_n = y_{n-1} + \Delta x f(x_{n-1}, y_{n-1}), \quad n = 1, \ldots, N.$

Series Solutions

Taylor Method

$f(x) \sim \sum_{n=0}^\infty c_nx^n$, $c_n = \frac{f^{(n)}(0)}{n!}$

1. Differentiate DE repeatedly.
2. Apply initial conditions.
3. Find Taylor coefficients.
4. Insert coefficients into series form for $y(x)$.

Power Series Solution

1. Let $y(x) = \sum_{n=0}^\infty c_n(x-a)^n$.
2. Find $y'(x)$, $y''(x)$.
3. Insert expansions in DE.
4. Collect like terms using reindexing.
5. Find recurrence relation.
6. Solve for coefficients and insert in $y(x)$ series.

Ordinary and Singular Points

$y'' + a(x)y' + b(x)y = 0$. x_0 is a
Ordinary point: $a(x), b(x)$ real analytic in $|x - x_0| < R$
Regular singular point: $(x - x_0)a(x)$, $(x - x_0)^2b(x)$ have convergent Taylor series about $x = x_0$.
Irregular singular point: Not ordinary or regular singular point.

Frobenius Method

1. Let $y(x) = \sum_{n=0}^\infty c_n(x - x_0)^{n+r}$.
2. Obtain indicial equation $r(r-1) + a_0r + b_0$.
3. Find recurrence relation based on types of roots of indicial equation.
4. Solve for coefficients and insert in $y(x)$ series.

Laplace Transforms

Transform Pairs

c	$\frac{c}{s}$
e^{at}	$\frac{1}{s-a}$, $s > a$
t^n	$\frac{n!}{s^{n+1}}$, $s > 0$
$\sin\omega t$	$\frac{\omega}{s^2 + \omega^2}$
$\cos\omega t$	$\frac{s}{s^2 + \omega^2}$
$\sinh at$	$\frac{a}{s^2 - a^2}$
$\cosh at$	$\frac{s}{s^2 - a^2}$
$H(t-a)$	$\frac{e^{-as}}{s}$, $s > 0$
$\delta(t-a)$	e^{-as}, $a \ge 0, s > 0$

Laplace Transform Properties

$\mathcal{L}[af(t)+bg(t)] = aF(s)+bG(s)$
$\mathcal{L}[tf(t)] = -\dfrac{d}{ds}F(s)$
$\mathcal{L}\left[\dfrac{df}{dt}\right] = sF(s)-f(0)$
$\mathcal{L}\left[\dfrac{d^2f}{dt^2}\right] = s^2F(s)-sf(0)-f'(0)$
$\mathcal{L}[e^{at}f(t)] = F(s-a)$
$\mathcal{L}[H(t-a)f(t-a)] = e^{-as}F(s)$
$\mathcal{L}[(f*g)(t)] = \mathcal{L}[\displaystyle\int_0^t f(t-u)g(u)\,du] = F(s)G(s)$

Solve Initial Value Problem

1. Transform DE using initial conditions.
2. Solve for $Y(s)$.
3. Use transform pairs, partial fraction decomposition, to obtain $y(t)$.

Special Functions

Legendre Polynomials

$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$
$(1-x^2)y'' - 2xy' + n(n+1)y = 0.$
$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x), \quad n = 1, 2, \ldots.$
$g(x,t) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n, \quad |x| \le 1, |t| < 1.$

Bessel Functions, $J_p(x)$, $N_p(x)$

$x^2y'' + xy' + (x^2-p^2)y = 0.$

Gamma Functions

$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt, \quad x > 0.$
$\Gamma(x+1) = x\Gamma(x).$

Systems of Differential Equations

Planar Systems

$x' = ax + by$
$y' = cx + dy.$
$x'' - (a+d)x' + (ad-bc)x = 0.$

Matrix Form

$\mathbf{x}' = \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \equiv A\mathbf{x}.$
Guess $\mathbf{x} = \mathbf{v}e^{\lambda t} \Rightarrow A\mathbf{v} = \lambda\mathbf{v}.$

Eigenvalue Problem

$A\mathbf{v} = \lambda\mathbf{v}.$
Find Eigenvalues: $\det(A-\lambda I) = 0$
Find Eigenvectors $(A-\lambda I)\mathbf{v} = 0$ for each λ.

Cases

Real, Distinct Eigenvalues: $\mathbf{x}(t) = c_1e^{\lambda_1 t}\mathbf{v}_1 + c_2e^{\lambda_2 t}\mathbf{v}_2$
Repeated Eigenvalue: $\mathbf{x}(t) = c_1e^{\lambda t}\mathbf{v}_1 + c_2e^{\lambda t}(\mathbf{v}_2 + t\mathbf{v}_1)$, where $A\mathbf{v}_2 - \lambda\mathbf{v}_2 = \mathbf{v}_1$ for $\mathbf{v}_2$.
Complex Conjugate Eigenvalues: $\mathbf{x}(t) = c_1 Re(e^{\alpha t}(\cos\beta t + i\sin\beta t)\mathbf{v}) + c_2 Im(e^{\alpha t}(\cos\beta t + i\sin\beta t)\mathbf{v})$.

Solution Behavior

Stable Node: $\lambda_1, \lambda_2 < 0$.
Unstable Node: $\lambda_1, \lambda_2 > 0$.
Saddle: $\lambda_1\lambda_2 < 0$.
Center: $\lambda = i\beta$.
Stable Focus: $\lambda = \alpha + i\beta$, $\alpha < 0$.
Unstable Focus: $\lambda = \alpha + i\beta$, $\alpha > 0$.

Matrix Solutions

Let $\mathbf{x}' = A\mathbf{x}$.
Find eigenvalues λ_i
Find eigenvectors $\mathbf{v}_i = \begin{pmatrix} v_{i1} \\ v_{i2} \end{pmatrix}$
Form the Fundamental Matrix Solution:
$\Phi = \begin{pmatrix} v_{11}e^{\lambda_1 t} & v_{21}e^{\lambda_2 t} \\ v_{12}e^{\lambda_1 t} & v_{22}e^{\lambda_2 t} \end{pmatrix}$
General Solution: $\mathbf{x}(t) = \Phi(t)\mathbf{C}$ for $\mathbf{C}$
Find C: $\mathbf{x}_0 = \Phi(t_0)\mathbf{C} \Rightarrow \mathbf{C} = \Phi^{-1}(t_0)\mathbf{x}_0$
Particular Solution: $\mathbf{x}(t) = \Phi(t)\Phi^{-1}(t_0)\mathbf{x}_0$.
Principal Matrix solution: $\Psi(t) = \Phi(t)\Phi^{-1}(t_0)$.
Particular Solution: $\mathbf{x}(t) = \Psi(t)\mathbf{x}_0$.
Note: $\Psi' = A\Psi, \quad \Psi(t_0) = I$.

Nonhomogeneous Matrix Solutions

$\mathbf{x}(t) = \Phi(t)\mathbf{C} + \Phi(t)\int_{t_0}^t \Phi^{-1}(s)\mathbf{f}(s)\,ds$
$\mathbf{x}(t) = \Psi(t)\mathbf{x}_0 + \Psi(t)\int_{t_0}^t \Psi^{-1}(s)\mathbf{f}(s)\,ds$

2×2 Matrix Inverse

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Differential Equations Study Sheet

Matthew Chesnes
It's all about the Mathematics!
Kenyon College

Exam date: May 11, 2000
6:30 P.M.

1 First Order Differential Equations

- Differential equations can be used to explain and predict new facts for about everything that changes continuously.
- $\frac{d^2x}{dt^2} + a\frac{dx}{dt} + kx = 0$.
- t is the independent variable, x is the dependent variable, a and k are parameters.
- The order of a differential equation is the highest deriviative in the equation.
- A differential equation is linear if it is linear in parameters such that the coefficients on each deriviative of y term is a function of the independent variable (t).
- Solutions: Explicit $\rightarrow$ Written as a function of the independent variable. Implicit $\rightarrow$ Written as a function of both y and t. (defines one or more explicit solutions.

1.1 Population Model

- Model: $\frac{dP}{dt} = kP$.
- Equilibrium solution occurs when $\frac{dP}{dt} = 0$.
- Solution: $P(t) = Ae^{(kt)}$.
- If $k > 0$, then $\lim_{t\to\infty} P(t) = \infty$. If $k < 0$, then $\lim_{t\to\infty} P(t) = 0$.
- Redefine model so it doesn't blow up to infinity.
- $\frac{dP}{dt} = kP(1 - \frac{P}{N})$.
- N is the carrying capacity of the population.

1.2 Seperation of Variables Technique

- $\frac{dy}{dt} = g(t)h(y)$.
- $\frac{1}{h(y)}dy = g(t)dt$.
- Integrate both sides and solve for y.
- You might lose the solution $h(y) = 0$.

1.3 Mixing Problems

- $\frac{dQ}{dt}$ = Rate In - Rate Out.
- Consider a tank that initally contains 50 gallons of pure water. A salt solution containing 2 pounds of salt per gallon of water is poured into the tank at a rate of 3 gal/min. The solution leaves the tank also at 3 gal/min.
- Therefore Input = 2(lb/gal)*3(gal/min).
- Output = ?(lbs/gal)*3(gal/min).
- Salt in Tank = $\frac{Q(t)}{50}$.
- Therefore output of salt = $\frac{Q(t)}{50}$ (lbs/gal)*3(gal/min).
- $\frac{dQ}{dt}$ = Rate In - Rate Out = 2 lbs/gal*3gal/min - $\frac{Q(t)}{50}$ (lbs/gal)*3(gal/min).
- 6 lbs/min - $\frac{3Q}{50}$lbs/min.
- Solve via seperation of Variables.

1.4 Existance and Uniqueness

- Given $\frac{dy}{dt} = f(t, y)$. If f is continuous on some interval, then there exists at least one solution on that interval.
- If both $f(t, y)$ and $\frac{\partial}{\partial y} f(t, y)$ are continuous on some interval then an initial value problem on that interval is guaranteed to have exactly one Unique solution.

1.5 Phase Lines

- Takes all the information from a slope fields and captures it in a single vertical line.
- Draw a vertical line, label the equilibrium points, determine if the slope of y is positive or negative between each equilibrium and label up or down arrows.

1.6 Classifying Equilibria and the Linearization Theorem

- Source: solutions tend away from an equilibrium $\rightarrow f'(y_o) > 0$.
- Sink: solutions tend toward an equilibrium $\rightarrow f'(y_o) < 0$.
- Node: Nither a source or a sink $\rightarrow f'(y_o) = 0$ or DNE.

1.7 Bifurcations

- Bifurcations occur at parameters where the equilibrium profile changes.
- Draw phase lines (y) for several values of a.

1.8 Linear Differential Equations and Integrating Factors

- Properties of Linear DE: If y_p and y_h are both solutions to a differential equation, (particular and homogeneous), then $y_p + y_h$ is also a solution.
- Using the integrating factor to solve linear differential equations such that $\frac{dy}{dt} + P(t)y = f(t)$.
- The integrating factor is therefore $e^{(\int P(t)dt)}$.
- Multiply both sides by the integrating factor.
- $e^{(\int P(t)dt)}\frac{dy}{dt} + e^{(\int P(t)dt)}P(t)y = e^{(\int P(t)dt)}f(t)$.
- then via chain rule ...
- $\frac{d\{e^{(\int P(t)dt)}y\}}{dt}$ ((Integrating factor * y))= $e^{(\int P(t)dt)}f(t)$.
- Then integrate to find solution.

1.9 Integration by Parts

$\int udv = uv - \int vdu$.

2 Systems

- $\frac{dx}{dt} = ax - bxy, \frac{dy}{dt} = -cy + dxy$.
- Equilibrium occurs when both differential equations are equal to zero.
- a and c are growth effects and b and d are interaction effects.
- To verify that $x(t)$, $y(t)$ is a solution to a system, take the deriviative of each and compare them to the originial differerential equations with x and y plugged in.
- Converting a second order differential equation, $\frac{d^2y}{dt^2} = y$. Let $v = \frac{dy}{dt}$. Thus $dv = \frac{d^2y}{dt}$.

2.1 Vector Notation

- A system of the form $\frac{dx}{dt} = ax + bxy$ and $\frac{dy}{dt} = cy + exy$ can be written in vector notation.
- $$\frac{d}{dt}\mathbf{P}(t) = \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} ax + bxy \\ cy + exy \end{bmatrix}. \tag{1}$$

2.2 Decoupled System

- Completely decoupled: $\frac{dx}{dt} = f(x), \frac{dy}{dt} = g(y)$.
- Partially decoupled: $\frac{dx}{dt} = f(x), \frac{dy}{dt} = g(x, y)$.

3 Systems II

- Matrix form.
- Homogeneous = $\frac{d}{dt}\mathbf{X} = \mathbf{AX}$.
- Non-homogeneous = $\frac{d}{dt}\mathbf{X} = \mathbf{AX} + \mathbf{F}$.
- Linearity Principal
- Consider $\frac{d}{dt}\mathbf{X} = \mathbf{AX}$,where

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \tag{2}$$

- If $X_1(t)$ and $X_2(t)$ are solutions, then $k_1X_1(t) + k_2X_2(t)$ is also a solution provided $X_1(t)$ and $X_2(t)$ are linearly independent.
- Theorem: If $\mathbf{A}$ is a matrix wtih det $\mathbf{A}$ not equal to zero, then the only equilibrium piont for the system $\frac{d}{dt}\mathbf{X} = \mathbf{AX}$ is,

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{3}$$

3.1 Straightline Solutions, Eigencool Eigenvectors and Eigenvalues

- A straightline solution to the system $\frac{d}{dt}\mathbf{X} = \mathbf{AX}$ exists provided that,

$$\mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}. \tag{4}$$

- To determine λ, compute the det[($\mathbf{A}$ - λI)] =

$$det \begin{bmatrix} a-\lambda & b \\ c & e-\lambda \end{bmatrix} = (a-\lambda)(e-\lambda) - bc = 0. \tag{5}$$

.

- This expands to the characteristic polynomial =

$$\lambda^2 - (a-d)\lambda + ae - bc = 0.$$

- Solving the characteristic polynomial provides us with the eigenvalues of A.

3.2 Stability

Consider a linear 2 dimensional system with two nonzero, real, distinct eigenvalues, λ_1 and λ_2.

- If both eigenvalues are positive then the origin is a source (unstable).
- If both eigenvalues are negative then the origin is a sink (stable).
- If the eigenvalues have different signs, then the origin is a saddle (unstable).

3.3 Complex Eigenvalues

- Euler's Formula: $e^{a+ib} = e^a e^i b = e^a cos(b) + ie^a sin(b)$.
- Given real and complex parts of a solution, the two parts can be treated as seperate independent solutions and used in the linearization theorem to determine the general solution.
- Stability: consider a linear two dimensional system with complex eigenvalues $\lambda_1 = a+ib$ and $\lambda_2 = a - ib$.
  - If a is negative then solution spiral towards the origin (spiral sink).
  - If a is positive then the solutions spiral away from the origin (spiral source).
  - If $a = 0$ the solutions are periodic closed paths (neutral centers).

3.4 Repeated Eigenvalues

- Given the system, $\dfrac{d}{dt}\mathbf{X} = \mathbf{AX}$ with one repeated eigenvalue, λ_1.
- If **V1** is an eigenvector, then $X_1(t) = e^{\lambda t} V_1$ is a straight line solution.
- Another solution is of the form $X_2(t) = e^{\lambda t}(tV_1 + V_2)$.
- Where $V_1 = (A - \lambda I)V_2$.
- X_1 and X_2 will be independent and the general solution is formed in the usual manner.

3.5 Zero as an Eigenvalue

- If zero is an eigenvector, nothing changes but the form of the general solution is now

$$\mathbf{X}(t) = k_1 \mathrm{V}_1 + k_2 e^{\lambda_2 t} \mathbf{V}_2.$$

4 Second Order Differential Equations

- Form: $\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} = q(t)y = f(t)$.
- Homogeneous if $f(t) = 0$.
- given solutions y_1 and y_2 to the 2nd order differential equation, you must check the Wronskian if both solutions are from real roots of the characteristic.
- $$\mathbf{W} = det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}. \tag{6}$$
- If W is equal to 0 anywhere on the interval of consideration, then y_1 and y_2 are not linearly independent.
- General solution given y_1 and y_2 is found as usual by the linearization theorem.
- Characteristic polynomial of a 2nd order with constant coefficients: $as^2 + bs + c = 0$.
- Solutions of the form $y(t) = e^{st}$.
- $s = -\frac{b}{2a} + / - \frac{\sqrt{b^2 - 4ac}}{2a}$.
  - if $b^2 - 4ac > 0$, then two distinct real roots.
  - if $b^2 - 4ac < 0$, then complex roots.
  - $b^2 - 4ac = 0$, then repeated real roots.

4.1 Two real distinct Roots

- Two real roots, s_1 and s_2.
- General solution $= y(t) = k_1e^{s_1t} + k_2e^{s-2t}$.

4.2 Complex Roots

- Complex Roots, $s_1 = p + iq$ and $s_2 = p - iq$.
- General solution $= y(t) = k_1e^{pt}cos(qt) + k_2e^{pt}sin(qt)$.

4.3 Repeated Roots

- Repeated Root, s_1.
- General solution $= y(t) = k_1e^{-\frac{b}{2a}t} + k_2te^{-\frac{b}{a2}t}$.

4.4 Nonhomogeneous with constant coefficents

- General solution $= y(t) = y_h + y_p$.
- Polynomial $f(t)$.
  - Look for particular solution of the form $y_p = At^n + Bt^{n-1} + Ct^{n-2} + ... + Dt + E$.
- Exponential $f(t)$.
  - Look for particular solution of the form $y_p = Ae^{pt}$.
- Sine or Cosine $f(t)$.
  - Look for particular solution of the form $y_p = Asin(at) + Bcos(at)$.
- Combination $f(t)$.
  - $f(t) = P_n(t)e^{at}, \Rightarrow y_p = (At^n + Bt^{n-1} + Ct^{n-2} + ... + Dt + E)e^{at}$.
  - $f(t) = P_n(t)sin(at)$ or $P_n(t)cos(at), \Rightarrow y_p = (A1t^n + A2t^{n-1} + A3t^{n-2} + ... + A4t + A5)cos(at) + (B1t^n + B2t^{n-1} + B3t^{n-2} + ... + B4t + B5)sin(at)$.
  - $f(t) = e^{at}sin(bt)$ or $e^{at}cos(bt), \Rightarrow y_p = Ae^{at}cos(bt) + Be^{at}sin(bt)$.
  - $f(t) = P_n(t)e^{at}sin(bt)$ or $P_n(t)e^{at}cos(bt), \Rightarrow y_p = (A1t^n + A2t^{n-1} + A3t^{n-2} + ... + A4t + A5)e^{at}cos(bt) + (B1t^n + B2t^{n-1} + B3t^{n-2} + ... + B4t + B5)e^{at}sin(bt)$.
- Superposition $f(t)$.
  - If $f(t)$ is the sum of m terms of the forms previously described.
  - $y_p = y_{p1} + y_{p2} + y_{p3} + ... + y_{pm}$.

5 LaPlace Transformations

- Definition $L\{f(t)\} = \int_0^\infty e^{-st}f(t)dt = \lim_{T\to\infty}\int_0^T e^{-st}f(t)dt$.
- ONLY PROVIDED THAT THE INTEGRAL CONVERGES!!! MUST BE OF EXPONENTIAL ORDER!!!
- $L\{f(t)\} = F(s)$.
- $L\{1\} = \dfrac{1}{s}$.
- $L\{t\} = \dfrac{1}{s^2}$.
- $L\{e^{at}\} = \dfrac{1}{s-a}$.
- $L\{sin(\omega t)\} = \dfrac{\omega}{s^2+\omega^2}$.
- $L\{cos(\omega t)\} = \dfrac{s}{s^2+\omega^2}$.
- Linear: $L\{\alpha f(t) + \beta g(t)\} = \alpha F(s) + \beta G(s)$.

5.1 Inverse Laplace Transforms

- Linear: $L^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha f(t) + \beta g(t)$.

5.2 Transform of a derivative

- $L\{f'(t)\} = sL(f(t) - f(0)$.
- $L\{f''(t)\} = s^2L(f(t) - sf(0) - f'(0)$.

21-260: Differential Equations
Final Exam Formula Sheet Suggestions

1. For the Final Exam you are allowed to bring a 8.5×5.5 inches piece of paper with formulas written on both sides. This is half of a usual sheet of paper. You can write anything you want on this formula sheet. Here is what I would consider writing if I were a student in this course.

   By the way, when I omit information on this sheet, I am not implying that it will not be on the exam. I believe that the formulas below might be harder to memorize for some students.

2. What not to write: anything with the Laplace transform. The Laplace transform table will come with the exam.

3. The solution of $y' + p(t)y = g(t)$ is
$$y = \frac{1}{\mu(t)}\left(\int \mu(t)g(t)dt + C\right), \quad \text{where } \mu(t) = \exp\left(\int p(t)dt\right).$$

4. If $M(x,y)dx + N(x,y)dy = 0$ is exact (which happens when $M_y = N_x$), the solution is $\psi(x,y) = C$, where ψ is found by solving $\psi_x = M$ and $\psi_y = N$.

5. If $M(x,y)dx + N(x,y)dy = 0$ is not exact, you might be able to find and integrating factor that will make it exact. If $(M_y - N_x)/N$ is a function of x only, then there is an integrating factor $\mu(x)$ that is found by solving
$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu.$$
   If $(N_x - M_y)/M$ is a function of y only, then there is an integrating factor $\mu(y)$ that is found by solving
$$\frac{d\mu}{dy} = \frac{N_x - M_y}{M}\mu.$$

6. To solve $y' = f(t,y)$, where $y(0) = 0$, using the method of successive approximations, you go like this:
$$\phi_0(t) = 0,$$
$$\phi_1(t) = \int_0^t f(s, \phi_0(s))ds,$$
$$\vdots$$

$$\phi_{n+1}(t) = \int_0^t f(s, \phi_n(s))ds.$$

Then $y(t) = \lim_{n\to\infty} \phi_n(t)$.

7. When it comes to first order linear systems with constant coefficients, there are only three things I believe one can forget:

   (a) In the case of 2×2 matrices: the types of phase portraits and the stability of the origin. You might need four tiny pictures to remember how a saddle, a (proper) node, an improper node and a spiral look like.

   (b) In the cases of 2×2 and 3×3 real matrices, if there is an eigenvalue λ repeated twice, then λ has to be real, because complex eigenvalues of real matrices come in conjugate pairs. The two fundamental solutions that correspond to λ are

   $$\bar{v}e^{\lambda t} \text{ and } (t\bar{v} + \bar{u})e^{\lambda t},$$

   where $\bar{v}$ in the eigenvector and $\bar{u}$ is the generalized eigenvector corresponding to λ. The generalized eigenvector $\bar{u}$ is found from $(A - \lambda I)\bar{u} = \bar{v}$.

   (c) Non-homogeneous systems: let $\Psi(t)$ be the fundamental matrix of $\bar{x}' = A\bar{x} + \bar{g}(t)$. Then $\bar{u}'(t) = \Psi^{-1}(t)\bar{g}(t)$. Don't forget to integrate $\bar{u}'(t)$. Then $\bar{x}(t) = \Psi(t)\bar{u}(t)$.

8. Particular solutions in second order linear differential equations with constant coefficients: some might need the table from Section 3.5. I think that once you understand the method, you don't need the table. But anyway, it's up to you.

9. The Fourier series of $f(x)$ on $[-L, L]$ is

   $$\frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L})],$$

   where $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx$; $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{n\pi x}{L})dx$; $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{n\pi x}{L})dx$.

10. The sine Fourier series of $f(x)$ on $[0, L]$ is

    $$\sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{L}),$$

    where $b_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi x}{L})dx$.

11. The cosine Fourier series of $f(x)$ on $[0, L]$ is

    $$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{L}),$$

    where $a_0 = \frac{2}{L}\int_0^L f(x)dx$; $a_n = \frac{2}{L}\int_0^L f(x)\cos(\frac{n\pi x}{L})dx$.

12. Heat equation (bar ends kept at constant temperature):

$$\alpha^2 u_{xx} = u_t, \qquad 0 < x < L, \quad t > 0;$$
$$u(0,t) = T_1, \qquad u(L,t) = T_2, \quad t > 0;$$
$$u(x,0) = f(x), \qquad 0 < x < L.$$

Solution:

$$u(x,t) = (T_2 - T_1)\frac{x}{L} + T_1 + \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2\alpha^2 t/L^2} \sin(\frac{n\pi x}{L}),$$

where c_n is the coefficient of $\sin(\frac{n\pi x}{L})$ in the sine series of $f(x) - (T_2 - T_1)\frac{x}{L} - T_1$ over $[0, L]$. In other words $c_n = \frac{2}{L}\int_0^L (f(x) - (T_2 - T_1)\frac{x}{L} - T_1)\sin(\frac{n\pi x}{L})dx$.

Notice that if you take $T_1 = T_2 = 0$, you obtain the original formula for the case when both end are kept at temperature zero. So there is no reason to waste space on the case when both ends are kept at temperature zero.

13. Heat equation (bar ends insulated):

$$\alpha^2 u_{xx} = u_t, \qquad 0 < x < L, \quad t > 0;$$
$$u_x(0,t) = 0, \qquad u_x(L,t) = 0, \quad t > 0;$$
$$u(x,0) = f(x), \qquad 0 < x < L.$$

Solution:

$$u(x,t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2\alpha^2 t/L^2} \cos(\frac{n\pi x}{L}),$$

where c_n is the coefficient of $\cos(\frac{n\pi x}{L})$ in the cosine series of $f(x)$ over $[0, L]$. In other words $c_n = \frac{2}{L}\int_0^L f(x)\cos(\frac{n\pi x}{L})dx$ for $n \geq 0$.

14. Wave equation:

$$\alpha^2 u_{xx} = u_{tt}, \qquad 0 < x < L, \quad t > 0;$$
$$u(0,t) = 0, \qquad u(L,t) = 0, \quad t \geq 0;$$
$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 \leq x \leq L.$$

Solution:

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin(\frac{n\pi x}{L})\cos(\frac{n\pi\alpha t}{L}) + \sum_{n=1}^{\infty} k_n \sin(\frac{n\pi x}{L})\sin(\frac{n\pi\alpha t}{L})$$

Here c_n is the coefficient of $\sin(\frac{n\pi x}{L})$ in the sine series of $f(x)$ over $[0, L]$. In other words

$$c_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi x}{L})dx.$$

And $\frac{n\pi\alpha}{L}k_n$ is the coefficient of $\sin(\frac{n\pi x}{L})$ in the sine series of $g(x)$ over $[0, L]$. In other words

$$k_n = \frac{2}{n\pi\alpha}\int_0^L g(x)\sin(\frac{n\pi x}{L})dx.$$

15. Here are some trigonometric formulas you won't regret:

$$\sin(\alpha)\sin(\beta) = \tfrac{1}{2}(\cos(\alpha-\beta) - \cos(\alpha+\beta))$$
$$\cos(\alpha)\cos(\beta) = \tfrac{1}{2}(\cos(\alpha-\beta) + \cos(\alpha+\beta))$$
$$\sin(\alpha)\cos(\beta) = \tfrac{1}{2}(\sin(\alpha+\beta) + \sin(\alpha-\beta))$$

16. To be honest, I would write down only formulas 3 (?), 5 (?), 12, 13, 14, 15. Everything else most can memorize. You should not expect to be able to write down EVERYTHING.