Partial Differential Equations

Lecture Notes

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Version October, 2012

Contents

Preface

These lecture notes are intented as a straightforward introduction to partial differential equations which can serve as a textbook for undergraduate and beginning graduate students.

For additional reading we recommend following books: W. I. Smirnov [21], I. G. Petrowski [17], P. R. Garabedian [8], W. A. Strauss [23], F. John [10], L. C. Evans [5] and R. Courant and D. Hilbert[4] and D. Gilbarg and N. S. Trudinger [9]. Some material of these lecture notes was taken from some of these books.

Chapter 1

Introduction

Ordinary and partial differential equations occur in many applications. An ordinary differential equation is a special case of a partial differential equation but the behaviour of solutions is quite different in general. It is much more complicated in the case of partial differential equations caused by the fact that the functions for which we are looking at are functions of more than one independent variable.

Equation

$$F(x, y(x), y'(x), \dots, y^{(n)}) = 0$$

is an *ordinary differential equation* of n-th order for the unknown function $y(x)$, where F is given.

An important problem for ordinary differential equations is the *initial value problem*

$$\begin{aligned} y'(x) &= f(x, y(x)) \\ y(x_0) &= y_0 \ , \end{aligned}$$

where f is a given real function of two variables $x,\ y$ and $x_0,\ y_0$ are given real numbers.

Picard-Lindelöf Theorem. *Suppose*
(i) $f(x, y)$ *is continuous in a rectangle*

$$Q = \{(x, y) \in \mathbb{R}^2 : \ |x - x_0| < a,\ |y - y_0| < b\}.$$

(ii) There is a constant K *such that* $|f(x, y)| \leq K$ *for all* $(x, y) \in Q$.
(ii) Lipschitz condition: There is a constant L *such that*

$$|f(x, y_2) - f(x, y_1)| \leq L|y_2 - y_1|$$

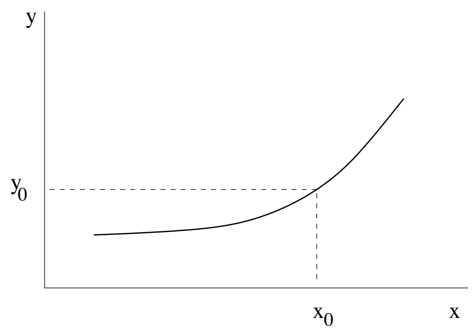

Figure 1.1: Initial value problem

for all $(x,y_1),(x,y_2)$.

Then there exists a unique solution $y\in C^1(x_0-\alpha,x_0+\alpha)$ *of the above initial value problem, where* $\alpha=\min(b/K,a)$.

The linear ordinary differential equation

$$y^{(n)}+a_{n-1}(x)y^{(n-1)}+\ldots a_1(x)y'+a_0(x)y=0,$$

where a_j are continuous functions, has exactly n linearly independent solutions. In contrast to this property the partial differential $u_{xx}+u_{yy}=0$ in $\mathbb{R}^2$ has infinitely many linearly independent solutions in the linear space $C^2(\mathbb{R}^2)$.

The ordinary differential equation of second order

$$y''(x)=f(x,y(x),y'(x))$$

has in general a family of solutions with two free parameters. Thus, it is naturally to consider the associated *initial value problem*

$$\begin{aligned} y''(x) &= f(x,y(x),y'(x)) \\ y(x_0) &= y_0,\ y'(x_0)=y_1, \end{aligned}$$

where y_0 and y_1 are given, or to consider the *boundary value problem*

$$\begin{aligned} y''(x) &= f(x,y(x),y'(x)) \\ y(x_0) &= y_0,\ y(x_1)=y_1. \end{aligned}$$

Initial and boundary value problems play an important role also in the theory of partial differential equations. A *partial differential equation* for

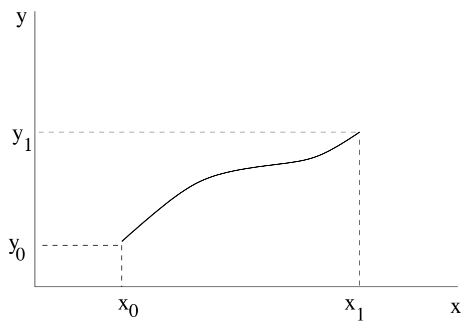

Figure 1.2: Boundary value problem

the unknown function $u(x,y)$ is for example

$$F(x,y,u,u_x,u_y,u_{xx},u_{xy},u_{yy})=0,$$

where the function F is given. This equation is of second order.

An equation is said to be of *n-th order* if the highest derivative which occurs is of order n.

An equation is said to be *linear* if the unknown function and its derivatives are linear in F. For example,

$$a(x,y)u_x+b(x,y)u_y+c(x,y)u=f(x,y),$$

where the functions a, b, c and f are given, is a linear equation of first order.

An equation is said to be *quasilinear* if it is linear in the highest derivatives. For example,

$$a(x,y,u,u_x,u_y)u_{xx}+b(x,y,u,u_x,u_y)u_{xy}+c(x,y,u,u_x,u_y)u_{yy}=0$$

is a quasilinear equation of second order.

1.1 Examples

1. $u_y=0$, where $u=u(x,y)$. All functions $u=w(x)$ are solutions.

2. $u_x=u_y$, where $u=u(x,y)$. A change of coordinates transforms this equation into an equation of the first example. Set $\xi=x+y,\ \eta=x-y$, then

$$u(x,y)=u\left(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2}\right)=:v(\xi,\eta).$$

Assume $u \in C^1$, then
$$v_\eta = \frac{1}{2}(u_x - u_y).$$
If $u_x = u_y$, then $v_\eta = 0$ and vice versa, thus $v = w(\xi)$ are solutions for arbitrary C^1-functions $w(\xi)$. Consequently, we have a large class of solutions of the original partial differential equation: $u = w(x + y)$ with an *arbitrary* C^1-function w.

3. A necessary and sufficient condition such that for given C^1-functions $M,\ N$ the integral
$$\int_{P_0}^{P_1} M(x,y)dx + N(x,y)dy$$
is independent of the curve which connects the points P_0 with P_1 in a simply connected domain $\Omega \subset \mathbb{R}^2$ is the partial differential equation (condition of integrability)
$$M_y = N_x$$
in Ω.

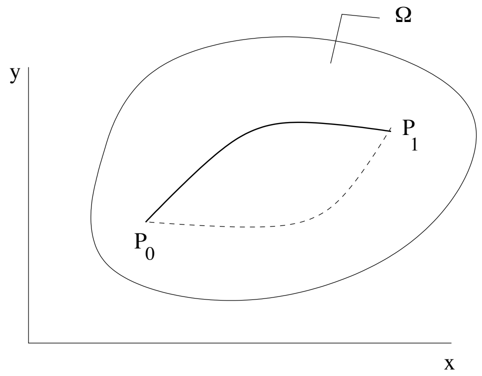

Figure 1.3: Independence of the path

This is one equation for two functions. A large class of solutions is given by $M = \Phi_x,\ N = \Phi_y$, where $\Phi(x,y)$ is an arbitrary C^2-function. It follows from Gauss theorem that these are all C^1-solutions of the above differential equation.

4. *Method of an integrating multiplier for an ordinary differential equation.* Consider the ordinary differential equation
$$M(x,y)dx + N(x,y)dy = 0$$

for given C^1-functions $M,\ N$. Then we seek a C^1-function $\mu(x,y)$ such that $\mu M dx + \mu N dy$ is a total differential, i. e., that $(\mu M)_y = (\mu N)_x$ is satisfied. This is a linear partial differential equation of first order for μ:

$$M\mu_y - N\mu_x = \mu(N_x - M_y).$$

5. Two C^1-functions $u(x,y)$ and $v(x,y)$ are said to be *functionally dependent* if

$$\det\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = 0,$$

which is a linear partial differential equation of first order for u if v is a given C^1-function. A large class of solutions is given by

$$u = H(v(x,y)),$$

where H is an *arbitrary* C^1-function.

6. *Cauchy-Riemann equations.* Set $f(z) = u(x,y)+iv(x,y)$, where $z = x+iy$ and $u,\ v$ are given $C^1(\Omega)$-functions. Here is Ω a domain in $\mathbb{R}^2$. If the function $f(z)$ is differentiable with respect to the complex variable z then $u,\ v$ satisfy the Cauchy-Riemann equations

$$u_x = v_y,\quad u_y = -v_x.$$

It is known from the theory of functions of one complex variable that the real part u and the imaginary part v of a differentiable function $f(z)$ are solutions of the *Laplace equation*

$$\triangle u = 0,\quad \triangle v = 0,$$

where $\triangle u = u_{xx} + u_{yy}$.

7. The *Newton potential*

$$u = \frac{1}{\sqrt{x^2+y^2+z^2}}$$

is a solution of the Laplace equation in $\mathbb{R}^3 \setminus (0,0,0)$, i. e., of

$$u_{xx} + u_{yy} + u_{zz} = 0.$$

8. *Heat equation.* Let $u(x,t)$ be the temperature of a point $x \in \Omega$ at time t, where $\Omega \subset \mathbb{R}^3$ is a domain. Then $u(x,t)$ satisfies in $\Omega \times [0,\infty)$ the *heat equation*

$$u_t = k\triangle u,$$

where $\triangle u = u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3}$ and k is a positive constant. The condition

$$u(x,0) = u_0(x), \quad x \in \Omega,$$

where $u_0(x)$ is given, is an *initial condition* associated to the above heat equation. The condition

$$u(x,t) = h(x,t), \quad x \in \partial\Omega,\ t \geq 0,$$

where $h(x,t)$ is given is a *boundary condition* for the heat equation.

If $h(x,t) = g(x)$, that is, h is independent of t, then one expects that the solution $u(x,t)$ tends to a function $v(x)$ if $t \to \infty$. Moreover, it turns out that v is the solution of the *boundary value problem* for the Laplace equation

$$\begin{aligned} \triangle v &= 0 \text{ in } \Omega \\ v &= g(x) \text{ on } \partial\Omega. \end{aligned}$$

9. *Wave equation.* The wave equation

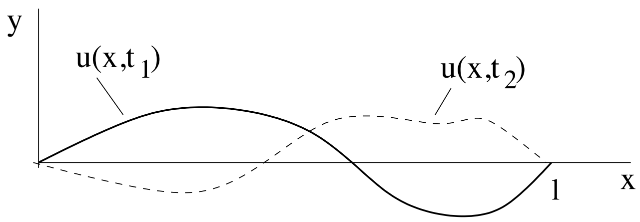

Figure 1.4: Oscillating string

$$u_{tt} = c^2\triangle u,$$

where $u = u(x,t)$, c is a positive constant, describes oscillations of membranes or of three dimensional domains, for example. In the one-dimensional case

$$u_{tt} = c^2 u_{xx}$$

describes oscillations of a string.

Associated *initial conditions* are

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x),$$

where $u_0,\ u_1$ are given functions. Thus the initial position and the initial velocity are prescribed.

If the string is finite one describes additionally *boundary conditions*, for example

$$u(0,t) = 0, \quad u(l,t) = 0 \ \text{ for all } t \geq 0.$$

1.2 Equations from variational problems

A large class of ordinary and partial differential equations arise from variational problems.

1.2.1 Ordinary differential equations

Set

$$E(v) = \int_a^b f(x, v(x), v'(x))\ dx$$

and for given $u_a,\ u_b \in \mathbb{R}$

$$V = \{v \in C^2[a,b] :\ v(a) = u_a,\ v(b) = u_b\},$$

where $-\infty < a < b < \infty$ and f is sufficiently regular. One of the basic problems in the calculus of variation is

$$(P) \qquad\qquad \min_{v\in V} E(v).$$

Euler equation. *Let $u \in V$ be a solution of (P), then*

$$\frac{d}{dx} f_{u'}(x, u(x), u'(x)) = f_u(x, u(x), u'(x))$$

in (a,b).

Proof. Exercise. Hints: For fixed $\phi \in C^2[a,b]$ with $\phi(a) = \phi(b) = 0$ and real ϵ, $|\epsilon| < \epsilon_0$, set $g(\epsilon) = E(u + \epsilon\phi)$. Since $g(0) \leq g(\epsilon)$ it follows $g'(0) = 0$. Integration by parts in the formula for $g'(0)$ and the following basic lemma in the calculus of variations imply Euler's equation.

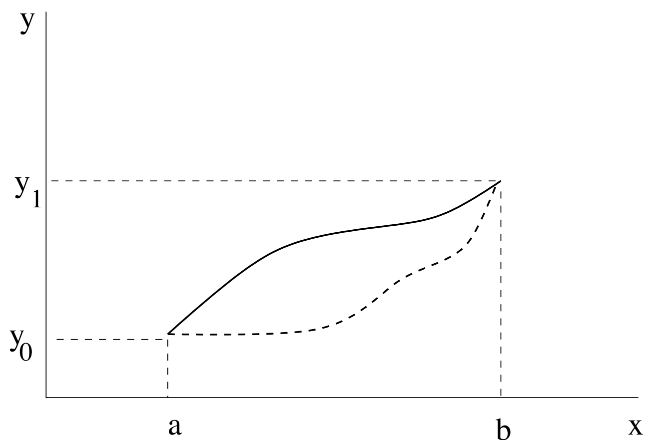

Figure 1.5: Admissible variations

Basic lemma in the calculus of variations. *Let* $h \in C(a,b)$ *and*

$$\int_a^b h(x)\phi(x)\ dx = 0$$

for all $\phi \in C_0^1(a,b)$. *Then* $h(x) \equiv 0$ *on* (a,b).

Proof. Assume $h(x_0) > 0$ for an $x_0 \in (a,b)$, then there is a $\delta > 0$ such that $(x_0-\delta, x_0+\delta) \subset (a,b)$ and $h(x) \geq h(x_0)/2$ on $(x_0-\delta, x_0+\delta)$. Set

$$\phi(x) = \begin{cases} \left(\delta^2 - |x-x_0|^2\right)^2 & \text{if} \quad x \in (x_0-\delta, x_0+\delta) \\ 0 & \text{if} \quad x \in (a,b) \setminus [x_0-\delta, x_0+\delta] \end{cases} .$$

Thus $\phi \in C_0^1(a,b)$ and

$$\int_a^b h(x)\phi(x)\ dx \geq \frac{h(x_0)}{2} \int_{x_0-\delta}^{x_0+\delta} \phi(x)\ dx > 0,$$

which is a contradiction to the assumption of the lemma. □

1.2.2 Partial differential equations

The same procedure as above applied to the following multiple integral leads to a second-order quasilinear partial differential equation. Set

$$E(v) = \int_\Omega F(x, v, \nabla v)\ dx,$$

where $\Omega \subset \mathbb{R}^n$ is a domain, $x = (x_1, \ldots, x_n)$, $v = v(x) : \Omega \mapsto \mathbb{R}$, and $\nabla v = (v_{x_1}, \ldots, v_{x_n})$. Assume that the function F is sufficiently regular in its arguments. For a given function h, defined on $\partial\Omega$, set

$$V = \{v \in C^2(\overline{\Omega}) : \ v = h \text{ on } \partial\Omega\}.$$

Euler equation. *Let $u \in V$ be a solution of (P), then*

$$\sum_{i=1}^n \frac{\partial}{\partial x_i} F_{u_{x_i}} - F_u = 0$$

in Ω.

Proof. Exercise. Hint: Extend the above fundamental lemma of the calculus of variations to the case of multiple integrals. The interval $(x_0 - \delta, x_0 + \delta)$ in the definition of ϕ must be replaced by a ball with center at x_0 and radius δ.

Example: **Dirichlet integral**

In two dimensions the Dirichlet integral is given by

$$D(v) = \int_\Omega \left(v_x^2 + v_y^2\right)\ dxdy$$

and the associated Euler equation is the Laplace equation $\triangle u = 0$ in Ω.

Thus, there is natural relationship between the boundary value problem

$$\triangle u = 0 \ \text{ in } \Omega, \ u = h \ \text{ on } \ \partial\Omega$$

and the variational problem

$$\min_{v \in V} \ D(v).$$

But these problems are not equivalent in general. It can happen that the boundary value problem has a solution but the variational problem has no solution, see for an example Courant and Hilbert [4], Vol. 1, p. 155, where h is a continuous function and the associated solution u of the boundary value problem has no finite Dirichlet integral.

The problems are equivalent, provided the given boundary value function h is in the class $H^{1/2}(\partial\Omega)$, see Lions and Magenes [14].

Example: **Minimal surface equation**

The non-parametric minimal surface problem in two dimensions is to find a minimizer $u = u(x_1, x_2)$ of the problem

$$\min_{v\in V} \int_\Omega \sqrt{1 + v_{x_1}^2 + v_{x_2}^2}\; dx,$$

where for a given function h defined on the boundary of the domain Ω

$$V = \{v \in C^1(\overline{\Omega}) :\ v = h \text{ on } \partial\Omega\}.$$

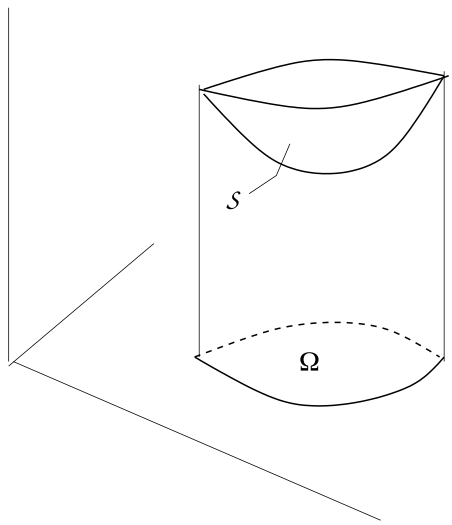

Figure 1.6: Comparison surface

Suppose that the minimizer satisfies the regularity assumption $u \in C^2(\Omega)$, then u is a solution of the *minimal surface equation* (Euler equation) in Ω

$$\frac{\partial}{\partial x_1}\left(\frac{u_{x_1}}{\sqrt{1 + |\nabla u|^2}}\right) + \frac{\partial}{\partial x_2}\left(\frac{u_{x_2}}{\sqrt{1 + |\nabla u|^2}}\right) = 0. \tag{1.1}$$

In fact, the additional assumption $u \in C^2(\Omega)$ is superfluous since it follows from regularity considerations for quasilinear elliptic equations of second order, see for example Gilbarg and Trudinger [9].

Let $\Omega = \mathbb{R}^2$. Each linear function is a solution of the minimal surface equation (1.1). It was shown by Bernstein [2] that there are no other solutions of the minimal surface quation. This is true also for higher dimensions

$n \le 7$, see Simons [19]. If $n \ge 8$, then there exists also other solutions which define cones, see Bombieri, De Giorgi and Giusti [3].

The linearized minimal surface equation over $u \equiv 0$ is the Laplace equation $\triangle u = 0$. In $\mathbb{R}^2$ linear functions are solutions but also many other functions in contrast to the minimal surface equation. This striking difference is caused by the strong nonlinearity of the minimal surface equation.

More general minimal surfaces are described by using parametric representations. An example is shown in Figure 1.7[1]. See [18], pp. 62, for example, for rotationally symmetric minimal surfaces.

Figure 1.7: Rotationally symmetric minimal surface

Neumann type boundary value problems

Set $V = C^1(\overline{\Omega})$ and

$$E(v) = \int_\Omega F(x, v, \nabla v)\ dx - \int_{\partial\Omega} g(x, v)\ ds,$$

where F and g are given sufficiently regular functions and $\Omega \subset \mathbb{R}^n$ is a bounded and sufficiently regular domain. Assume u is a minimizer of $E(v)$ in V, that is

$$u \in V : \quad E(u) \le E(v) \ \text{ for all } v \in V,$$

[1] An experiment from Beutelspacher's Mathematikum, Wissenschaftsjahr 2008, Leipzig

then

$$\int_\Omega \Big(\sum_{i=1}^n F_{u_{x_i}}(x,u,\nabla u)\phi_{x_i} \quad + \quad F_u(x,u,\nabla u)\phi\Big)\ dx$$
$$- \quad \int_{\partial\Omega} g_u(x,u)\phi\ ds = 0$$

for all $\phi \in C^1(\overline{\Omega})$. Assume additionally $u \in C^2(\Omega)$, then u is a solution of the Neumann type boundary value problem

$$\begin{aligned} \sum_{i=1}^n \frac{\partial}{\partial x_i} F_{u_{x_i}} - F_u &= 0 \ \text{ in } \Omega \\ \sum_{i=1}^n F_{u_{x_i}} \nu_i - g_u &= 0 \ \text{ on } \partial\Omega, \end{aligned}$$

where $\nu = (\nu_1, \ldots, \nu_n)$ is the exterior unit normal at the boundary $\partial\Omega$. This follows after integration by parts from the basic lemma of the calculus of variations.

Example: **Laplace equation**

Set
$$E(v) = \frac{1}{2}\int_\Omega |\nabla v|^2\ dx - \int_{\partial\Omega} h(x)v\ ds,$$
then the associated boundary value problem is

$$\begin{aligned} \triangle u &= 0 \ \text{ in } \Omega \\ \frac{\partial u}{\partial \nu} &= h \ \text{ on } \partial\Omega. \end{aligned}$$

Example: **Capillary equation**

Let $\Omega \subset \mathbb{R}^2$ and set
$$E(v) = \int_\Omega \sqrt{1 + |\nabla v|^2}\ dx + \frac{\kappa}{2}\int_\Omega v^2\ dx - \cos\gamma \int_{\partial\Omega} v\ ds.$$

Here κ is a positive constant (capillarity constant) and γ is the (constant) boundary contact angle, i. e., the angle between the container wall and the capillary surface, defined by $v = v(x_1, x_2)$, at the boundary. Then the related boundary value problem is

$$\begin{aligned} \mathrm{d}iv\ (Tu) &= \kappa u \ \ \text{in } \Omega \\ \nu \cdot Tu &= \cos\gamma \text{ on } \partial\Omega, \end{aligned}$$

where we use the abbreviation

$$Tu = \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}},$$

div (Tu) is the left hand side of the minimal surface equation (1.1) and it is twice the mean curvature of the surface defined by $z = u(x_1, x_2)$, see an exercise.

The above problem describes the ascent of a liquid, water for example, in a vertical cylinder with cross section Ω. Assume the gravity is directed downwards in the direction of the negative x_3-axis. Figure 1.8 shows that liquid can rise along a vertical wedge which is a consequence of the strong nonlinearity of the underlying equations, see Finn [7]. This photo was taken

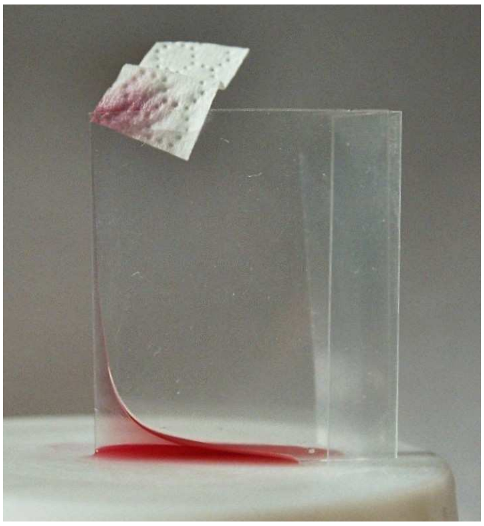

Figure 1.8: Ascent of liquid in a wedge

from [15].

1.3 Exercises

1. Find nontrivial solutions u of
$$u_x y - u_y x = 0 \ .$$

2. Prove: In the linear space $C^2(\mathbb{R}^2)$ there are infinitely many linearly independent solutions of $\triangle u = 0$ in $\mathbb{R}^2$.

   Hint: Real and imaginary part of holomorphic functions are solutions of the Laplace equation.

3. Find all radially symmetric functions which satisfy the Laplace equation in $\mathbb{R}^n \setminus \{0\}$ for $n \geq 2$. A function u is said to be radially symmetric if $u(x) = f(r)$, where $r = (\sum_i^n x_i^2)^{1/2}$.

   Hint: Show that a radially symmetric u satisfies $\triangle u = r^{1-n}\left(r^{n-1} f'\right)'$ by using $\nabla u(x) = f'(r)\frac{x}{r}$.

4. Prove the basic lemma in the calculus of variations: Let $\Omega \subset \mathbb{R}^n$ be a domain and $f \in C(\Omega)$ such that
$$\int_\Omega f(x)h(x) \ dx = 0$$
for all $h \in C_0^2(\Omega)$. Then $f \equiv 0$ in Ω.

5. Write the minimal surface equation (1.1) as a quasilinear equation of second order.

6. Prove that a sufficiently regular minimizer in $C^1(\overline{\Omega})$ of
$$E(v) = \int_\Omega F(x, v, \nabla v) \ dx - \int_{\partial\Omega} g(v, v) \ ds,$$
is a solution of the boundary value problem
$$\begin{aligned} \sum_{i=1}^n \frac{\partial}{\partial x_i} F_{u_{x_i}} - F_u &= 0 \ \text{ in } \Omega \\ \sum_{i=1}^n F_{u_{x_i}} \nu_i - g_u &= 0 \ \text{ on } \partial\Omega, \end{aligned}$$
where $\nu = (\nu_1, \ldots, \nu_n)$ is the exterior unit normal at the boundary $\partial\Omega$.

7. Prove that $\nu \cdot Tu = \cos\gamma$ on $\partial\Omega$, where γ is the angle between the container wall, which is here a cylinder, and the surface S, defined by $z = u(x_1, x_2)$, at the boundary of S, ν is the exterior normal at $\partial\Omega$.

   Hint: The angle between two surfaces is by definition the angle between the two associated normals at the intersection of the surfaces.

8. Let Ω be bounded and assume $u \in C^2(\overline{\Omega})$ is a solution of
$$\begin{aligned} \mathrm{d}iv\ Tu &= C \text{ in } \Omega \\ \nu \cdot \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} &= \cos\gamma \text{ on } \partial\Omega, \end{aligned}$$
   where C is a constant.

   Prove that
$$C = \frac{|\partial\Omega|}{|\Omega|}\cos\gamma\ .$$
   Hint: Integrate the differential equation over Ω.

9. Assume $\Omega = B_R(0)$ is a disc with radius R and the center at the origin. Show that radially symmetric solutions $u(x) = w(r)$, $r = \sqrt{x_1^2 + x_2^2}$, of the capillary boundary value problem are solutions of
$$\begin{aligned} \left(\frac{rw'}{\sqrt{1+w'^2}}\right)' &= \kappa r w \quad \text{in } 0 < r < R \\ \frac{w'}{\sqrt{1+w'^2}} &= \cos\gamma \quad \text{if } r = R. \end{aligned}$$
   Remark. It follows from a maximum principle of Concus and Finn [7] that a solution of the capillary equation over a disc must be radially symmetric.

10. Find all radially symmetric solutions of
$$\begin{aligned} \left(\frac{rw'}{\sqrt{1+w'^2}}\right)' &= Cr \quad \text{in } 0 < r < R \\ \frac{w'}{\sqrt{1+w'^2}} &= \cos\gamma \quad \text{if } r = R. \end{aligned}$$
   Hint: From an exercise above it follows that
$$C = \frac{2}{R}\cos\gamma.$$

11. Show that $\mathrm{d}iv\ Tu$ is twice the mean curvature of the surface defined by $z = u(x_1, x_2)$.

Chapter 2

Equations of first order

For a given sufficiently regular function F the general equation of first order for the unknown function $u(x)$ is

$$F(x, u, \nabla u) = 0$$

in $\Omega \in \mathbb{R}^n$. The main tool for studying related problems is the theory of ordinary differential equations. This is quite different for systems of partial differential of first order.

The general linear partial differential equation of first order can be written as

$$\sum_{i=1}^{n} a_i(x)u_{x_i} + c(x)u = f(x)$$

for given functions a_i, c and f. The general quasilinear partial differential equation of first order is

$$\sum_{i=1}^{n} a_i(x, u)u_{x_i} + c(x, u) = 0.$$

2.1 Linear equations

Let us begin with the linear homogeneous equation

$$a_1(x, y)u_x + a_2(x, y)u_y = 0. \tag{2.1}$$

Assume there is a C^1-solution $z = u(x, y)$. This function defines a surface S which has at $P = (x, y, u(x, y))$ the normal

$$\mathbf{N} = \frac{1}{\sqrt{1 + |\nabla u|^2}}(-u_x, -u_y, 1)$$

and the tangential plane defined by

$$\zeta - z = u_x(x,y)(\xi - x) + u_y(x,y)(\eta - y).$$

Set $p = u_x(x,y)$, $q = u_y(x,y)$ and $z = u(x,y)$. The tuple (x,y,z,p,q) is called *surface element* and the tuple (x,y,z) *support* of the surface element. The tangential plane is defined by the surface element. On the other hand, differential equation (2.1)

$$a_1(x,y)p + a_2(x,y)q = 0$$

defines at each support (x,y,z) a bundle of planes if we consider all (p,q) satisfying this equation. For fixed (x,y), this family of planes $\Pi(\lambda) = \Pi(\lambda; x, y)$ is defined by a one parameter family of ascents $p(\lambda) = p(\lambda; x, y)$, $q(\lambda) = q(\lambda; x, y)$. The envelope of these planes is a line since

$$a_1(x,y)p(\lambda) + a_2(x,y)q(\lambda) = 0,$$

which implies that the normal $\mathbf{N}(\lambda)$ on $\Pi(\lambda)$ is perpendicular on $(a_1, a_2, 0)$.

Consider a curve $\mathbf{x}(\tau) = (x(\tau), y(\tau), z(\tau))$ on $\mathcal{S}$, let $T_{\mathbf{x}_0}$ be the tangential plane at $\mathbf{x}_0 = (x(\tau_0), y(\tau_0), z(\tau_0))$ of $\mathcal{S}$ and consider on $T_{\mathbf{x}_0}$ the line

$$L: \quad l(\sigma) = \mathbf{x}_0 + \sigma \mathbf{x}'(\tau_0), \quad \sigma \in \mathbb{R},$$

see Figure 2.1.

We assume L coincides with the envelope, which is a line here, of the family of planes $\Pi(\lambda)$ at (x,y,z). Assume that $T_{\mathbf{x}_0} = \Pi(\lambda_0)$ and consider two planes

$$\begin{aligned}
\Pi(\lambda_0): \quad z - z_0 &= (x - x_0)p(\lambda_0) + (y - y_0)q(\lambda_0) \\
\Pi(\lambda_0 + h): \quad z - z_0 &= (x - x_0)p(\lambda_0 + h) + (y - y_0)q(\lambda_0 + h).
\end{aligned}$$

At the intersection $l(\sigma)$ we have

$$(x - x_0)p(\lambda_0) + (y - y_0)q(\lambda_0) = (x - x_0)p(\lambda_0 + h) + (y - y_0)q(\lambda_0 + h).$$

Thus,

$$x'(\tau_0)p'(\lambda_0) + y'(\tau_0)q'(\lambda_0) = 0.$$

From the differential equation

$$a_1(x(\tau_0), y(\tau_0))p(\lambda) + a_2(x(\tau_0), y(\tau_0))q(\lambda) = 0$$

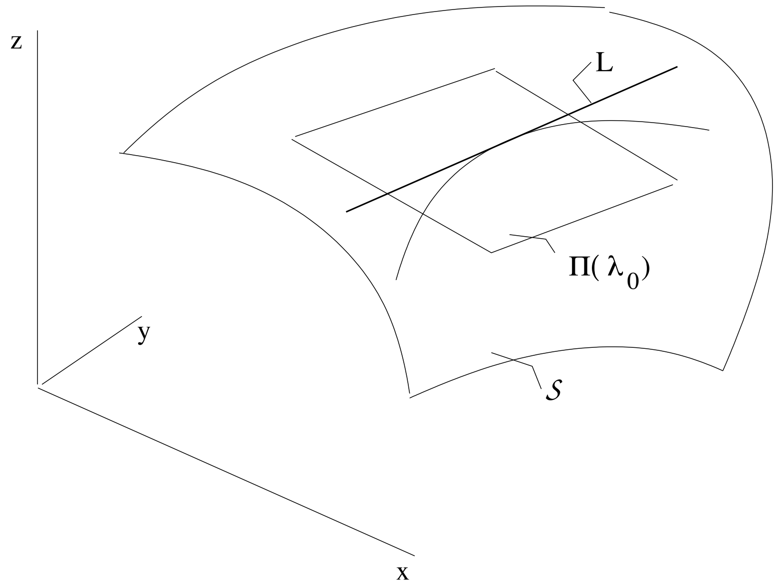

Figure 2.1: Curve on a surface

it follows

$$a_1 p'(\lambda_0) + a_2 q'(\lambda_0) = 0.$$

Consequently

$$(x'(\tau), y'(\tau)) = \frac{x'(\tau)}{a_1(x(\tau, y(\tau))}(a_1(x(\tau), y(\tau)), a_2(x(\tau), y(\tau)),$$

since τ_0 was an arbitrary parameter. Here we assume that $x'(\tau) \neq 0$ and $a_1(x(\tau), y(\tau)) \neq 0$.

Then we introduce a new parameter t by the inverse of $\tau = \tau(t)$, where

$$t(\tau) = \int_{\tau_0}^{\tau} \frac{x'(s)}{a_1(x(s), y(s))}\, ds.$$

It follows $x'(t) = a_1(x, y)$, $y'(t) = a_2(x, y)$. We denote $\mathbf{x}(\tau(t))$ by $\mathbf{x}(t)$ again.

Now we consider the initial value problem

$$x'(t) = a_1(x, y), \quad y'(t) = a_2(x, y), \quad x(0) = x_0, \; y(0) = y_0. \qquad (2.2)$$

From the theory of ordinary differential equations it follows (Theorem of Picard-Lindelöf) that there is a unique solution in a neighbouhood of $t = 0$ provided the functions a_1, a_2 are in C^1. From this definition of the curves

$(x(t), y(t))$ is follows that the field of directions $(a_1(x_0, y_0), a_2(x_0, y_0))$ defines the slope of these curves at $(x(0), y(0))$.

Definition. The differential equations in (2.2) are called *characteristic equations* or characteristic system and solutions of the associated initial value problem are called *characteristic curves.*

Definition. A function $\phi(x, y)$ is said to be an *integral* of the characteristic system if $\phi(x(t), y(t)) = const.$ for each characteristic curve. The constant depends on the characteristic curve considered.

Proposition 2.1. *Assume $\phi \in C^1$ is an integral, then $u = \phi(x, y)$ is a solution of (2.1).*

Proof. Consider for given (x_0, y_0) the above initial value problem (2.2). Since $\phi(x(t), y(t)) = const.$ it follows

$$\phi_x x' + \phi_y y' = 0$$

for $|t| < t_0$, $t_0 > 0$ and sufficiently small. Thus

$$\phi_x(x_0, y_0)a_1(x_0, y_0) + \phi_y(x_0, y_0)a_2(x_0, y_0) = 0.$$

□

Remark. If $\phi(x, y)$ is a solution of equation (2.1) then also $H(\phi(x, y))$, where $H(s)$ is a given C^1-function.

Examples

1. Consider

$$a_1 u_x + a_2 u_y = 0,$$

where $a_1,\ a_2$ are constants. The system of characteristic equations is

$$x' = a_1,\ y' = a_2.$$

Thus the characteristic curves are parallel straight lines defined by

$$x = a_1 t + A,\ y = a_2 t + B,$$

where $A,\ B$ are arbitrary constants. From these equations it follows that

$$\phi(x,y) := a_2x - a_1y$$

is constant along each characteristic curve. Consequently, see Proposition 2.1, $u = a_2x - a_1y$ is a solution of the differential equation. From an exercise it follows that

$$u = H(a_2x - a_1y), \tag{2.3}$$

where $H(s)$ is an arbitrary C^1-function, is also a solution. Since u is constant when $a_2x - a_1y$ is constant, equation (2.3) defines cylinder surfaces which are generated by parallel straight lines which are parallel to the (x,y)-plane, see Figure 2.2.

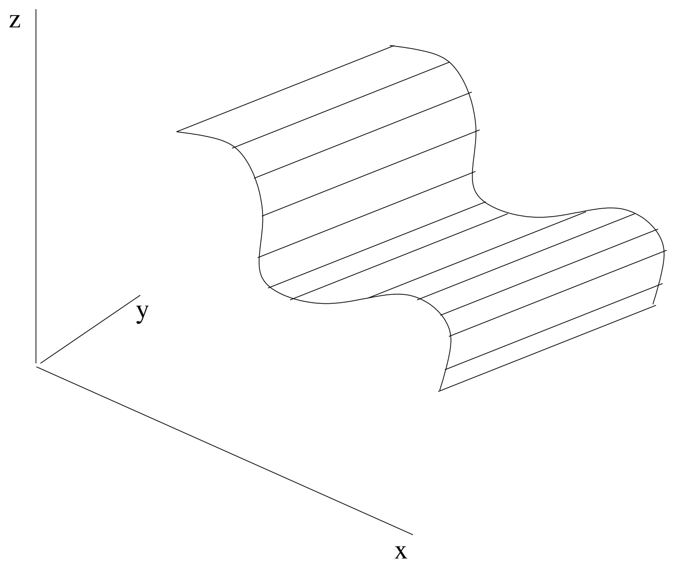

Figure 2.2: Cylinder surfaces

2. Consider the differential equation

$$xu_x + yu_y = 0.$$

The characteristic equations are

$$x' = x,\ y' = y,$$

and the characteristic curves are given by

$$x = A\mathrm{e}^t,\ y = B\mathrm{e}^t,$$

where $A,\ B$ are arbitrary constants. Thus, an integral is y/x, $x \neq 0$, and for a given C^1-function the function $u = H(x/y)$ is a solution of the differential equation. If $y/x = const.$, then u is constant. Suppose that $H'(s) > 0$, for example, then u defines right helicoids (in German: Wendelflächen), see Figure 2.3

Figure 2.3: Right helicoid, $a^2 < x^2 + y^2 < R^2$ (Museo Ideale Leonardo da Vinci, Italy)

3. Consider the differential equation

$$yu_x - xu_y = 0.$$

The associated characteristic system is

$$x' = y,\ y' = -x.$$

If follows

$$x'x + yy' = 0,$$

or, equivalently,

$$\frac{d}{dt}(x^2 + y^2) = 0,$$

which implies that $x^2 + y^2 = const.$ along each characteristic. Thus, rotationally symmetric surfaces defined by $u = H(x^2 + y^2)$, where $H' \neq 0$, are solutions of the differential equation.

4. The associated characteristic equations to

$$ayu_x + bxu_y = 0,$$

where $a,\ b$ are positive constants, are given by

$$x' = ay,\ y' = bx.$$

It follows $bxx' - ayy' = 0$, or equivalently,

$$\frac{d}{dt}(bx^2 - ay^2) = 0.$$

Solutions of the differential equation are $u = H(bx^2 - ay^2)$, which define surfaces which have a hyperbola as the intersection with planes parallel to the (x, y)-plane. Here $H(s)$ is an arbitrary C^1-function, $H'(s) \neq 0$.

2.2 Quasilinear equations

Here we consider the equation

$$a_1(x, y, u)u_x + a_2(x, y, u)u_y = a_3(x, y, u). \tag{2.4}$$

The inhomogeneous linear equation

$$a_1(x, y)u_x + a_2(x, y)u_y = a_3(x, y)$$

is a special case of (2.4).

One arrives at characteristic equations $x' = a_1,\ y' = a_2,\ z' = a_3$ from (2.4) by the same arguments as in the case of homogeneous linear equations in two variables. The additional equation $z' = a_3$ follows from

$$\begin{aligned} z'(\tau) &= p(\lambda)x'(\tau) + q(\lambda)y'(\tau) \\ &= pa_1 + qa_2 \\ &= a_3, \end{aligned}$$

see also Section 2.3, where the general case of nonlinear equations in two variables is considered.

2.2.1 A linearization method

We can transform the inhomogeneous equation (2.4) into a homogeneous linear equation for an unknown function of three variables by the following trick.

We are looking for a function $\psi(x, y, u)$ such that the solution $u = u(x, y)$ of (2.4) is defined implicitly by $\psi(x, y, u) = const$. Assume there is such a function ψ and let u be a solution of (2.4), then

$$\psi_x + \psi_u u_x = 0, \quad \psi_y + \psi_u u_y = 0.$$

Assume $\psi_u \neq 0$, then

$$u_x = -\frac{\psi_x}{\psi_u}, \quad u_y = -\frac{\psi_y}{\psi_u}.$$

From (2.4) we obtain

$$a_1(x, y, z)\psi_x + a_2(x, y, z)\psi_y + a_3(x, y, z)\psi_z = 0, \tag{2.5}$$

where $z := u$.

We consider the associated system of characteristic equations

$$\begin{aligned} x'(t) &= a_1(x, y, z) \\ y'(t) &= a_2(x, y, z) \\ z'(t) &= a_3(x, y, z). \end{aligned}$$

One arrives at this system by the same arguments as in the two-dimensional case above.

Proposition 2.2. (i) *Assume $w \in C^1$, $w = w(x, y, z)$, is an integral, i. e., it is constant along each fixed solution of (2.5), then $\psi = w(x, y, z)$ is a solution of (2.5).*
(ii) *The function $z = u(x, y)$, implicitly defined through $\psi(x, u, z) = const.$, is a solution of (2.4), provided that $\psi_z \neq 0$.*
(iii) *Let $z = u(x, y)$ be a solution of (2.4) and let $(x(t), y(t))$ be a solution of*

$$x'(t) = a_1(x, y, u(x, y)), \quad y'(t) = a_2(x, y, u(x, y)),$$

then $z(t) := u(x(t), y(t))$ satisfies the third of the above characteristic equations.

Proof. Exercise.

2.2.2 Initial value problem of Cauchy

Consider again the quasilinear equation

$$(\star) \qquad a_1(x,y,u)u_x + a_2(x,y,u)u_y = a_3(x,y,u).$$

Let

$$\Gamma: \quad x = x_0(s),\ y = y_0(s),\ z = z_0(s),\ s_1 \leq s \leq s_2,\ -\infty < s_1 < s_2 < +\infty$$

be a regular curve in $\mathbb{R}^3$ and denote by $\mathcal{C}$ the orthogonal projection of Γ onto the (x,y)-plane, i. e.,

$$\mathcal{C}: \quad x = x_0(s), \quad y = y_0(s).$$

Initial value problem of Cauchy: *Find a C^1-solution $u = u(x,y)$ of $(\star)$ such that $u(x_0(s), y_0(s)) = z_0(s)$, i. e., we seek a surface $\mathcal{S}$ defined by $z = u(x,y)$ which contains the curve Γ.*

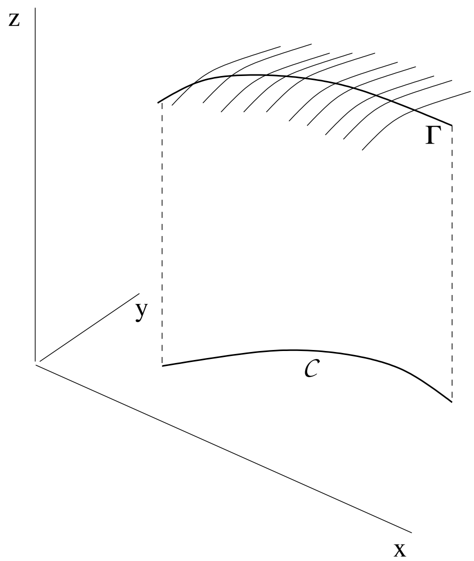

Figure 2.4: Cauchy initial value problem

Definition. The curve Γ is said to be *noncharacteristic* if

$$x_0'(s)a_2(x_0(s), y_0(s)) - y_0'(s)a_1(x_0(s), y_0(s)) \neq 0.$$

Theorem 2.1. *Assume $a_1,\ a_2,\ a_2 \in C^1$ in their arguments, the initial data $x_0,\ y_0,\ z_0 \in C^1[s_1, s_2]$ and Γ is noncharacteristic.*

Then there is a neighbourhood of $\mathcal{C}$ such that there exists exactly one solution u of the Cauchy initial value problem.

Proof. (i) Existence. Consider the following initial value problem for the system of characteristic equations to $(\star)$:

$$\begin{aligned} x'(t) &= a_1(x,y,z) \\ y'(t) &= a_2(x,y,z) \\ z'(t) &= a_3(x,y,z) \end{aligned}$$

with the initial conditions

$$\begin{aligned} x(s,0) &= x_0(s) \\ y(s,0) &= y_0(s) \\ z(s,0) &= z_0(s). \end{aligned}$$

Let $x = x(s,t)$, $y = y(s,t)$, $z = z(s,t)$ be the solution, $s_1 \le s \le s_2$, $|t| < \eta$ for an $\eta > 0$. We will show that this set of strings sticked onto the curve Γ, see Figure 2.4, defines a surface. To show this, we consider the inverse functions $s = s(x,y)$, $t = t(x,y)$ of $x = x(s,t)$, $y = y(s,t)$ and show that $z(s(x,y),t(x,y))$ is a solution of the initial problem of Cauchy. The inverse functions s and t exist in a neighbourhood of $t = 0$ since

$$\det \frac{\partial(x,y)}{\partial(s,t)}\Big|_{t=0} = \begin{vmatrix} x_s & x_t \\ y_s & y_t \end{vmatrix}_{t=0} = x_0'(s)a_2 - y_0'(s)a_1 \neq 0,$$

and the initial curve Γ is noncharacteristic by assumption.

Set

$$u(x,y) := z(s(x,y),t(x,y)),$$

then u satisfies the initial condition since

$$u(x,y)|_{t=0} = z(s,0) = z_0(s).$$

The following calculation shows that u is also a solution of the differential equation $(\star)$.

$$\begin{aligned} a_1u_x + a_2u_y &= a_1(z_ss_x + z_tt_x) + a_2(z_ss_y + z_tt_y) \\ &= z_s(a_1s_x + a_2s_y) + z_t(a_1t_x + a_2t_y) \\ &= z_s(s_xx_t + s_yy_t) + z_t(t_xx_t + t_yy_t) \\ &= a_3 \end{aligned}$$

since $0 = s_t = s_x x_t + s_y y_t$ and $1 = t_t = t_x x_t + t_y y_t$.

(ii) Uniqueness. Suppose that $v(x, y)$ is a second solution. Consider a point (x', y') in a neighbourhood of the curve $(x_0(s), y(s))$, $s_1 - \epsilon \le s \le s_2 + \epsilon$, $\epsilon > 0$ small. The inverse parameters are $s' = s(x', y')$, $t' = t(x', y')$, see Figure 2.5.

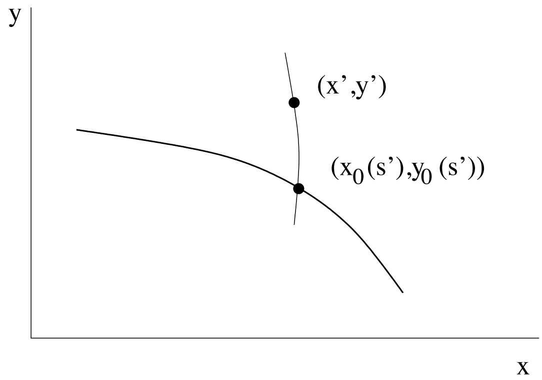

Figure 2.5: Uniqueness proof

Let
$$\mathcal{A}: \quad x(t) := x(s', t),\ y(t) := y(s', t),\ z(t) := z(s', t)$$
be the solution of the above initial value problem for the characteristic differential equations with the initial data
$$x(s', 0) = x_0(s'),\ y(s', 0) = y_0(s'),\ z(s', 0) = z_0(s').$$
According to its construction this curve is on the surface $\mathcal{S}$ defined by $u = u(x, y)$ and $u(x', y') = z(s', t')$. Set
$$\psi(t) := v(x(t), y(t)) - z(t),$$
then
$$\begin{aligned} \psi'(t) &= v_x x' + v_y y' - z' \\ &= x_x a_1 + v_y a_2 - a_3 = 0 \end{aligned}$$
and
$$\psi(0) = v(x(s', 0), y(s', 0)) - z(s', 0) = 0$$
since v is a solution of the differential equation and satisfies the initial condition by assumption. Thus, $\psi(t) \equiv 0$, i. e.,
$$v(x(s', t), y(s', t)) - z(s', t) = 0.$$

Set $t = t'$, then

$$v(x', y') - z(s', t') = 0,$$

which shows that $v(x', y') = u(x', y')$ because of $z(s', t') = u(x', y')$. $\square$

Remark. In general, there is no uniqueness if the initial curve Γ is a characteristic curve, see an exercise and Figure 2.6 which illustrates this case.

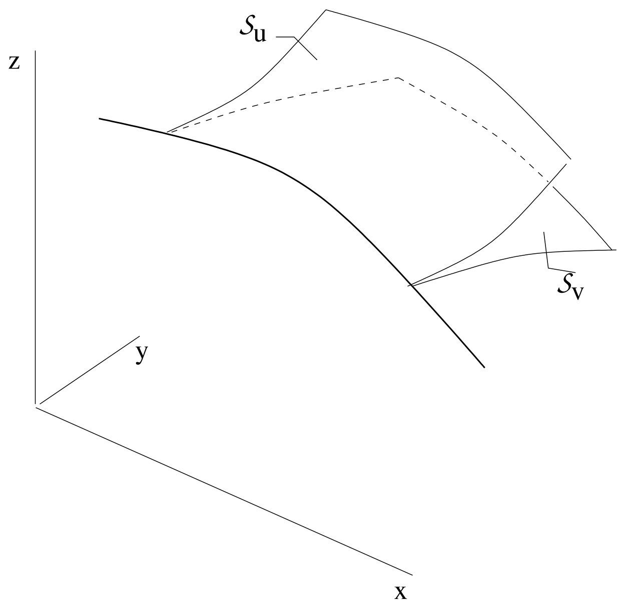

Figure 2.6: Multiple solutions

Examples

1. Consider the Cauchy initial value problem

$$u_x + u_y = 0$$

with the initial data

$$x_0(s) = s,\ y_0(s) = 1,\ z_0(s) \text{ is a given } C^1\text{-function}.$$

These initial data are noncharacteristic since $y_0' a_1 - x_0' a_2 = -1$. The solution of the associated system of characteristic equations

$$x'(t) = 1,\ y'(t) = 1,\ u'(t) = 0$$

with the initial conditions

$$x(s,0)=x_0(s),\ y(s,0)=y_0(s),\ z(s,0)=z_0(s)$$

is given by

$$x=t+x_0(s),\ y=t+y_0(s),\ z=z_0(s),$$

i. e.,

$$x=t+s,\ y=t+1,\ z=z_0(s).$$

It follows $s=x-y+1,\ t=y-1$ and that $u=z_0(x-y+1)$ is the solution of the Cauchy initial value problem.

2. A problem from kinetics in chemistry. Consider for $x\ge 0,\ y\ge 0$ the problem

$$u_x+u_y=\left(k_0\mathrm{e}^{-k_1x}+k_2\right)(1-u)$$

with initial data

$$u(x,0)=0,\ x>0,\ \text{and } u(0,y)=u_0(y),\ y>0.$$

Here the constants k_j are positive, these constants define the velocity of the reactions in consideration, and the function $u_0(y)$ is given. The variable x is the time and y is the hight of a tube, for example, in which the chemical reaction takes place, and u is the concentration of the chemical substance.

In contrast to our previous assumptions, the initial data are not in C^1. The projection $\mathcal{C}_1\cup\mathcal{C}_2$ of the initial curve onto the (x,y)-plane has a corner at the origin, see Figure 2.7.

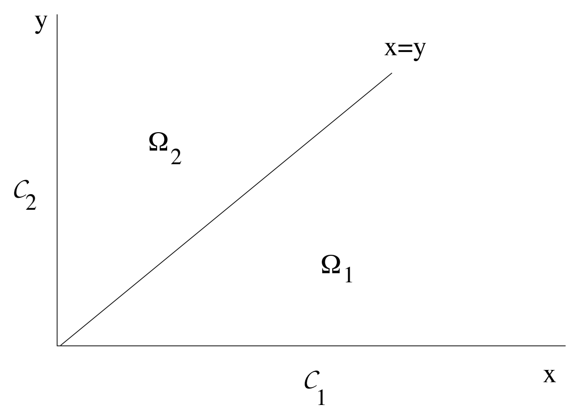

Figure 2.7: Domains to the chemical kinetics example

The associated system of characteristic equations is

$$x'(t) = 1,\ y'(t) = 1,\ z'(t) = \left(k_0 \mathrm{e}^{-k_1 x} + k_2\right)(1 - z).$$

It follows $x = t + c_1$, $y = t + c_2$ with constants c_j. Thus the projection of the characteristic curves on the (x, y)-plane are straight lines parallel to $y = x$. We will solve the initial value problems in the domains Ω_1 and Ω_2, see Figure 2.7, separately.

(i) *The initial value problem in* Ω_1. The initial data are

$$x_0(s) = s,\ y_0(s) = 0,\ z_0(0) = 0,\ s \geq 0.$$

It follows

$$x = x(s, t) = t + s,\ y = y(s, t) = t.$$

Thus

$$z'(t) = (k_0 \mathrm{e}^{-k_1(t+s)} + k_2)(1 - z),\ z(0) = 0.$$

The solution of this initial value problem is given by

$$z(s, t) = 1 - \exp\left(\frac{k_0}{k_1}\mathrm{e}^{-k_1(s+t)} - k_2 t - \frac{k_0}{k_1}\mathrm{e}^{-k_1 s}\right).$$

Consequently

$$u_1(x, y) = 1 - \exp\left(\frac{k_0}{k_1}\mathrm{e}^{-k_1 x} - k_2 y - k_0 k_1 \mathrm{e}^{-k_1(x-y)}\right)$$

is the solution of the Cauchy initial value problem in Ω_1. If time x tends to ∞, we get the limit

$$\lim_{x \to \infty} u_1(x, y) = 1 - \mathrm{e}^{-k_2 y}.$$

(ii) *The initial value problem in* Ω_2. The initial data are here

$$x_0(s) = 0,\ y_0(s) = s,\ z_0(0) = u_0(s),\ s \geq 0.$$

It follows

$$x = x(s, t) = t,\ y = y(s, t) = t + s.$$

Thus

$$z'(t) = (k_0 \mathrm{e}^{-k_1 t} + k_2)(1 - z),\ z(0) = 0.$$

The solution of this initial value problem is given by

$$z(s,t) = 1 - (1 - u_0(s)) \exp\left(\frac{k_0}{k_1}e^{-k_1 t} - k_2 t - \frac{k_0}{k_1}\right).$$

Consequently

$$u_2(x,y) = 1 - (1 - u_0(y-x)) \exp\left(\frac{k_0}{k_1}e^{-k_1 x} - k_2 x - \frac{k_0}{k_1}\right)$$

is the solution in Ω_2.

If $x = y$, then

$$\begin{aligned} u_1(x,y) &= 1 - \exp\left(\frac{k_0}{k_1}e^{-k_1 x} - k_2 x - \frac{k_0}{k_1}\right) \\ u_2(x,y) &= 1 - (1 - u_0(0)) \exp\left(\frac{k_0}{k_1}e^{-k_1 x} - k_2 x - \frac{k_0}{k_1}\right). \end{aligned}$$

If $u_0(0) > 0$, then $u_1 < u_2$ if $x = y$, i. e., there is a jump of the concentration of the substrate along its burning front defined by $x = y$.

Remark. Such a problem with discontinuous initial data is called *Riemann problem.* See an exercise for another Riemann problem.

The case that a solution of the equation is known

Here we will see that we get immediately a solution of the Cauchy initial value problem if a solution of the *homogeneous linear equation*

$$a_1(x,y)u_x + a_2(x,y)u_y = 0$$

is known.

Let

$$x_0(s),\ y_0(s),\ z_0(s),\ s_1 < s < s_2$$

be the initial data and let $u = \phi(x,y)$ be a solution of the differential equation. We assume that

$$\phi_x(x_0(s), y_0(s))x_0'(s) + \phi_y(x_0(s), y_0(s))y_0'(s) \neq 0$$

is satisfied. Set $g(s) = \phi(x_0(s), y_0(s))$ and let $s = h(g)$ be the inverse function.

The solution of the Cauchy initial problem is given by $u_0\left(h(\phi(x,y))\right)$.

This follows since in the problem considered a composition of a solution is a solution again, see an exercise, and since

$$u_0\left(h(\phi(x_0(s), y_0(s))\right) = u_0(h(g)) = u_0(s).$$

Example: Consider equation

$$u_x + u_y = 0$$

with initial data

$$x_0(s) = s,\ y_0(s) = 1,\ u_0(s) \text{ is a given function.}$$

A solution of the differential equation is $\phi(x, y) = x - y$. Thus

$$\phi((x_0(s), y_0(s)) = s - 1$$

and

$$u_0(\phi + 1) = u_0(x - y + 1)$$

is the solution of the problem.

2.3 Nonlinear equations in two variables

Here we consider equation

$$F(x, y, z, p, q) = 0, \tag{2.6}$$

where $z = u(x, y)$, $p = u_x(x, y)$, $q = u_y(x, y)$ and $F \in C^2$ is given such that $F_p^2 + F_q^2 \neq 0$.

In contrast to the quasilinear case, this general nonlinear equation is more complicated. Together with (2.6) we will consider the following system of ordinary equations which follow from considerations below as necessary conditions, in particular from the assumption that there is a solution of (2.6).

$$\begin{aligned}
x'(t) &= F_p && (2.7)\\
y'(t) &= F_q && (2.8)\\
z'(t) &= pF_p + qF_q && (2.9)\\
p'(t) &= -F_x - F_u p && (2.10)\\
q'(t) &= -F_y - F_u q. && (2.11)
\end{aligned}$$

Definition. Equations (2.7)–(2.11) are said to be *characteristic equations* of equation (2.6) and a solution

$$(x(t), y(t), z(t), p(t), q(t))$$

of the characteristic equations is called *characteristic strip* or *Monge curve.*

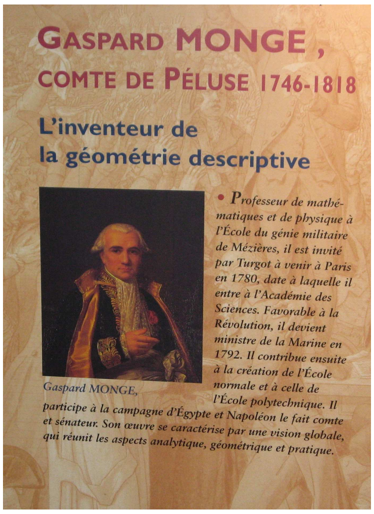

Figure 2.8: Gaspard Monge (Panthéon, Paris)

We will see, as in the quasilinear case, that the strips defined by the characteristic equations build the solution surface of the Cauchy initial value problem.

Let $z = u(x, y)$ be a solution of the general nonlinear differential equation (2.6).

Let (x_0, y_0, z_0) be fixed, then equation (2.6) defines a set of planes given by (x_0, y_0, z_0, p, q), i. e., planes given by $z = v(x, y)$ which contain the point (x_0, y_0, z_0) and for which $v_x = p$, $v_y = q$ at (x_0, y_0). In the case of quasilinear equations these set of planes is a bundle of planes which all contain a fixed straight line, see Section 2.1. In the general case of this section the situation is more complicated.

Consider the example

$$p^2 + q^2 = f(x, y, z), \tag{2.12}$$

where f is a given positive function. Let E be a plane defined by $z = v(x, y)$ and which contains (x_0, y_0, z_0). Then the normal on the plane E directed downward is

$$\mathbf{N} = \frac{1}{\sqrt{1 + |\nabla v|^2}}(p, q, -1),$$

where $p = v_x(x_0, y_0)$, $q = v_y(x_0, y_0)$. It follows from (2.12) that the normal $\mathbf{N}$ makes a constant angle with the z-axis, and the z-coordinate of $\mathbf{N}$ is constant, see Figure 2.9.

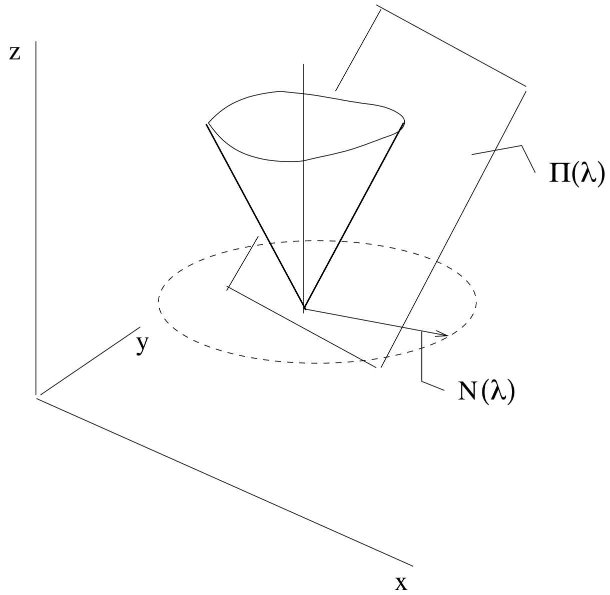

Figure 2.9: Monge cone in an example

Thus the endpoints of the normals fixed at (x_0, y_0, z_0) define a circle parallel to the (x, y)-plane, i. e., there is a cone which is the envelope of all these planes.

We assume that the general equation (2.6) defines such a Monge cone at each point in $\mathbb{R}^3$. Then we seek a surface S which touches at each point its Monge cone, see Figure 2.10.

More precisely, we assume there exists, as in the above example, a one parameter C^1-family

$$p(\lambda) = p(\lambda; x, y, z),\ \ q(\lambda) = q(\lambda; x, y, z)$$

of solutions of (2.6). These $(p(\lambda), q(\lambda))$ define a family $\Pi(\lambda)$ of planes.

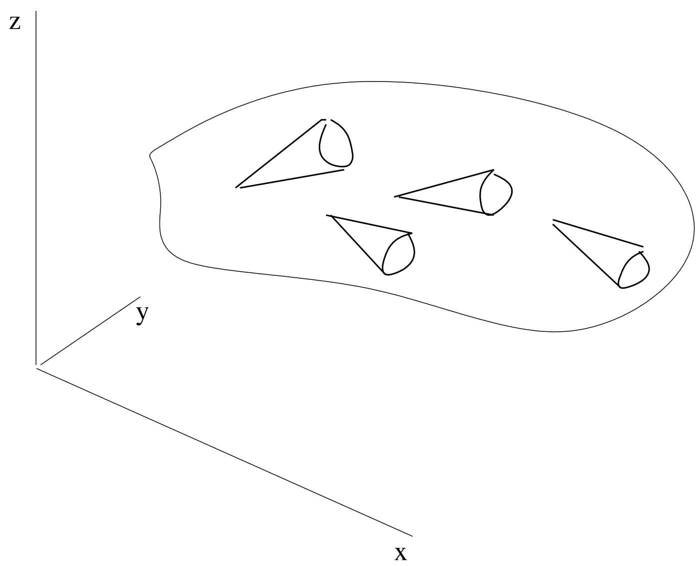

Figure 2.10: Monge cones

Let

$$\mathbf{x}(\tau) = (x(\tau), y(\tau), z(\tau))$$

be a curve on the surface S which touches at each point its Monge cone, see Figure 2.11. Thus we assume that at each point of the surface $\mathcal{S}$ the associated tangent plane coincides with a plane from the family $\Pi(\lambda)$ at this point. Consider the tangential plane $T_{\mathbf{x}_0}$ of the surface S at $\mathbf{x}_0 = (x(\tau_0), y(\tau_0), z(\tau_0))$. The straight line

$$\mathbf{l}(\sigma) = \mathbf{x}_0 + \sigma \mathbf{x}'(\tau_0), \qquad -\infty < \sigma < \infty,$$

is an apothem (in German: Mantellinie) of the cone by assumption and is contained in the tangential plane $T_{\mathbf{x}_0}$ as the tangent of a curve on the surface S. It is defined through

$$\mathbf{x}'(\tau_0) = \mathbf{l}'(\sigma). \tag{2.13}$$

The straight line $\mathbf{l}(\sigma)$ satisfies

$$l_3(\sigma) - z_0 = (l_1(\sigma) - x_0)p(\lambda_0) + (l_2(\sigma) - y_0)q(\lambda_0),$$

since it is contained in the tangential plane $T_{\mathbf{x}_0}$ defined by the slope (p, q). It follows

$$l_3'(\sigma) = p(\lambda_0)l_1'(\sigma) + q(\lambda_0)l_2'(\sigma).$$

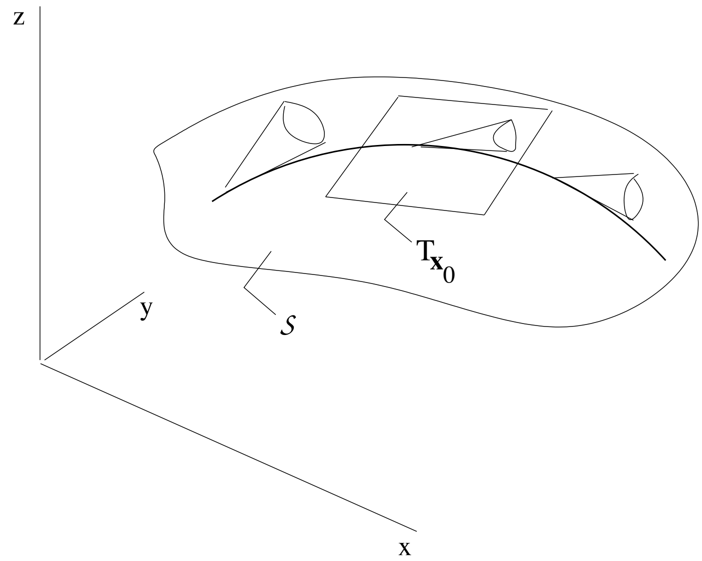

Figure 2.11: Monge cones along a curve on the surface

Together with (2.13) we obtain

$$z'(\tau) = p(\lambda_0)x'(\tau) + q(\lambda_0)y'(\tau). \tag{2.14}$$

The above straight line $\mathbf{l}$ is the limit of the intersection line of two neighbouring planes which envelopes the Monge cone:

$$\begin{aligned} z - z_0 &= (x - x_0)p(\lambda_0) + (y - y_0)q(\lambda_0) \\ z - z_0 &= (x - x_0)p(\lambda_0 + h) + (y - y_0)q(\lambda_0 + h). \end{aligned}$$

On the intersection one has

$$(x - x_0)p(\lambda) + (y - y_0)q(\lambda_0) = (x - x_0)p(\lambda_0 + h) + (y - y_0)q(\lambda_0 + h).$$

Let $h \to 0$, it follows

$$(x - x_0)p'(\lambda_0) + (y - y_0)q'(\lambda_0) = 0.$$

Since $x = l_1(\sigma)$, $y = l_2(\sigma)$ in this limit position, we have

$$p'(\lambda_0)l_1'(\sigma) + q'(\lambda_0)l_2'(\sigma) = 0,$$

and it follows from (2.13) that

$$p'(\lambda_0)x'(\tau) + q'(\lambda_0)y'(\tau) = 0. \tag{2.15}$$

From the differential equation $F(x_0, y_0, z_0, p(\lambda), q(\lambda)) = 0$ we see that

$$F_p p'(\lambda) + F_q q'(\lambda) = 0. \tag{2.16}$$

Assume $x'(\tau_0) \neq 0$ and $F_p \neq 0$, then we obtain from (2.15), (2.16)

$$\frac{y'(\tau_0)}{x'(\tau_0)} = \frac{F_q}{F_p},$$

and from (2.14) (2.16) that

$$\frac{z'(\tau_0)}{x'(\tau_0)} = p + q\frac{F_q}{F_p}.$$

It follows, since τ_0 was an arbitrary fixed parameter,

$$\begin{aligned}
\mathbf{x}'(\tau) &= (x'(\tau), y'(\tau), z'(\tau)) \\
&= \left(x'(\tau), x'(\tau)\frac{F_q}{F_p}, x'(\tau)\left(p + q\frac{F_q}{F_p}\right)\right) \\
&= \frac{x'(\tau)}{F_p}(F_p, F_q, pF_p + qF_q),
\end{aligned}$$

i. e., the tangential vector $\mathbf{x}'(\tau)$ is proportional to $(F_p, F_q, pF_p + qF_q)$. Set

$$a(\tau) = \frac{x'(\tau)}{F_p},$$

where $F = F(x(\tau), y(\tau), z(\tau), p(\lambda(\tau)), q(\lambda(\tau)))$. Introducing the new parameter t by the inverse of $\tau = \tau(t)$, where

$$t(\tau) = \int_{\tau_0}^{\tau} a(s)\ ds,$$

we obtain the characteristic equations (2.7)–(2.9). Here we denote $\mathbf{x}(\tau(t))$ by $\mathbf{x}(t)$ again. From the differential equation (2.6) and from (2.7)–(2.9) we get equations (2.10) and (2.11). Assume the surface $z = u(x, y)$ under consideration is in C^2, then

$$\begin{aligned}
F_x + F_z p + F_p p_x + F_q p_y &= 0, \quad (q_x = p_y) \\
F_x + F_z p + x'(t)p_x + y'(t)p_y &= 0 \\
F_x + F_z p + p'(t) &= 0
\end{aligned}$$

since $p = p(x, y) = p(x(t), y(t))$ on the curve $\mathbf{x}(t)$. Thus equation (2.10) of the characteristic system is shown. Differentiating the differential equation (2.6) with respect to y, we get finally equation (2.11).

Remark. In the previous quasilinear case

$$F(x, y, z, p, q) = a_1(x, y, z)p + a_2(x, y, z)q - a_3(x, y, z)$$

the first three characteristic equations are the same:

$$x'(t) = a_1(x, y, z),\ y'(t) = a_2(x, y, z),\ z'(t) = a_3(x, y, z).$$

The point is that the right hand sides are independent on p or q. It follows from Theorem 2.1 that there exists a solution of the Cauchy initial value problem provided the initial data are noncharacteristic. That is, we do not need the other remaining two characteristic equations.

The other two equations (2.10) and (2.11) are satisfied in this quasilinear case automatically if there is a solution of the equation, see the above derivation of these equations.

The geometric meaning of the first three characteristic differential equations (2.7)–(2.11) is the following one. Each point of the curve $\mathcal{A}:\quad (x(t), y(t), z(t))$ corresponds a tangential plane with the normal direction $(-p, -q, 1)$ such that

$$z'(t) = p(t)x'(t) + q(t)y'(t).$$

This equation is called *strip condition*. On the other hand, let $z = u(x, y)$ defines a surface, then $z(t) := u(x(t), y(t))$ satisfies the strip condition, where $p = u_x$ and $q = u_y$, that is, the "scales" defined by the normals fit together.

Proposition 2.3. *$F(x, y, z, p, q)$ is an integral, i. e., it is constant along each characteristic curve.*

Proof.

$$\begin{aligned}
\frac{d}{dt}F(x(t), y(t), z(t), p(t), q(t)) &= F_x x' + F_y y' + F_z z' + F_p p' + F_q q' \\
&= F_x F_p + F_y F_q + pF_z F_p + qF_z F_q \\
&\quad -F_p f_x - F_p F_z p - F_q F_y - F_q F_z q \\
&= 0.
\end{aligned}$$

□

Corollary. Assume $F(x_0, y_0, z_0, p_0, q_0) = 0$, then $F = 0$ along characteristic curves with the initial data $(x_0, y_0, z_0, p_0, q_0)$.

Proposition 2.4. *Let $z = u(x, y)$, $u \in C^2$, be a solution of the nonlinear equation (2.6). Set*

$$z_0 = u(x_0, y_0,)\ p_0 = u_x(x_0, y_0),\ \ q_0 = u_y(x_0, y_0).$$

Then the associated characteristic strip is in the surface $\mathcal{S}$, defined by $z = u(x, y)$. Thus

$$\begin{aligned} z(t) &= u(x(t), y(t)) \\ p(t) &= u_x(x(t), y(t)) \\ q(t) &= u_y(x(t), y(t)), \end{aligned}$$

where $(x(t), y(t), z(t), p(t), q(t))$ is the solution of the characteristic system (2.7)–(2.11) with initial data $(x_0, y_0, z_0, p_0, q_0)$

Proof. Consider the initial value problem

$$\begin{aligned} x'(t) &= F_p(x, y, u(x, y), u_x(x, y), u_y(x, y)) \\ y'(t) &= F_q(x, y, u(x, y), u_x(x, y), u_y(x, y)) \end{aligned}$$

with the initial data $x(0) = x_0$, $y(0) = y_0$. We will show that

$$(x(t), y(t), u(x(t), y(t)), u_x(x(t), y(t)), u_y(x(t), y(t)))$$

is a solution of the characteristic system. We recall that the solution exists and is uniquely determined.

Set $z(t) = u(x(t), y(t))$, then $(x(t), y(t), z(t)) \subset \mathcal{S}$, and

$$z'(t) = u_x x'(t) + u_y y'(t) = u_x F_p + u_y F_q.$$

Set $p(t) = u_x(x(t), y(t))$, $q(t) = u_y(x(t), y(t))$, then

$$\begin{aligned} p'(t) &= u_{xx} F_p + u_{xy} F_q \\ q'(t) &= u_{yx} F_p + u_{yy} F_q. \end{aligned}$$

Finally, from the differential equation $F(x, y, u(x, y), u_x(x, y), u_y(x, y)) = 0$ it follows

$$\begin{aligned} p'(t) &= -F_x - F_u p \\ q'(t) &= -F_y - F_u q. \end{aligned}$$

□

2.3.1 Initial value problem of Cauchy

Let

$$x = x_0(s),\ y = y_0(s),\ z = z_0(s),\ p = p_0(s),\ q = q_0(s),\ s_1 < s < s_2, \tag{2.17}$$

be a given *initial strip* such that the *strip condition*

$$z_0'(s) = p_0(s)x_0'(s) + q_0(s)y_0'(s) \tag{2.18}$$

is satisfied. Moreover, we assume that the initial strip satisfies the nonlinear equation, that is,

$$F(x_0(s), y_0(s), z_0(s), p_0(s), q_0(s)) = 0. \tag{2.19}$$

Initial value problem of Cauchy: *Find a C^2-solution $z = u(x, y)$ of $F(x, y, z, p, q) = 0$ such that the surface $\mathcal{S}$ defined by $z = u(x, y)$ contains the above initial strip.*

Similar to the quasilinear case we will show that the set of strips defined by the characteristic system which are sticked at the initial strip, see Figure 2.12, fit together and define the surface for which we are looking at.

Definition. A strip $(x(\tau), y(\tau), z(\tau), p(\tau), q(\tau))$, $\tau_1 < \tau < \tau_2$, is said to be *noncharacteristic* if

$$x'(\tau)F_q(x(\tau), y(\tau), z(\tau), p(\tau), q(\tau)) - y'(\tau)F_p(x(\tau), y(\tau), z(\tau), p(\tau), q(\tau)) \neq 0.$$

Theorem 2.2. *For a given noncharacteristic initial strip (2.17), $x_0,\ y_0,\ z_0 \in C^2$ and $p_0,\ q_0 \in C^1$ which satisfies the strip condition (2.18) and the differential equation (2.19) there exists exactly one solution $z = u(x, y)$ of the Cauchy initial value problem in a neighbourhood of the initial curve $(x_0(s), y_0(s), z_0(s))$, i. e., $z = u(x, y)$ is the solution of the differential equation (2.6) and $u(x_0(s), y_0(s)) = z_0(s)$, $u_x(x_0(s), y_0(s)) = p_0(s)$, $u_y(x_0(s), y_0(s)) = q_0(s)$.*

Proof. Consider the system (2.7)–(2.11) with initial data

$$x(s, 0) = x_0(s),\ y(s, 0) = y_0(s),\ z(s, 0) = z_0(s),\ p(s, 0) = p_0(s),\ q(s, 0) = q_0(s).$$

We will show that the surface defined by $x = x(s, t),\ y(s, t)$ is the surface defined by $z = u(x, y)$, where u is the solution of the Cauchy initial value

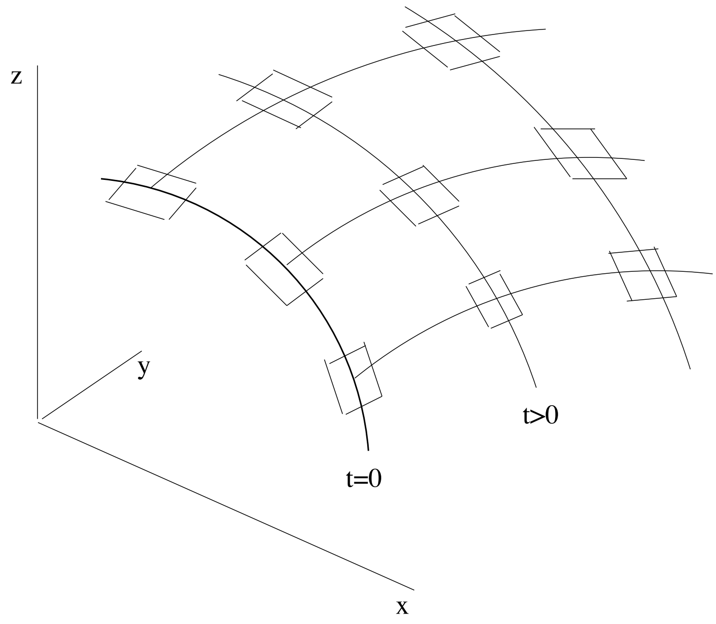

Figure 2.12: Construction of the solution

problem. It turns out that $u(x,y) = z(s(x,y),t(x,y))$, where $s = s(x,y)$, $t = t(x,y)$ is the inverse of $x = x(s,t)$, $y = y(s,t)$ in a neigbourhood of $t = 0$. This inverse exists since the initial strip is noncharacteristic by assumption:

$$\det \frac{\partial(x,y)}{\partial(s,t)}\Big|_{t=0} = x_0 F_q - y_0 F_q \neq 0.$$

Set

$$P(x,y) = p(s(x,y),t(x,y)), \quad Q(x,y) = q(s(x,y),t(x,y)).$$

From Proposition 2.3 and Proposition 2.4 it follows $F(x,y,u,P,Q) = 0$. We will show that $P(x,y) = u_x(x,y)$ and $Q(x,y) = u_y(x,y)$. To see this, we consider the function

$$h(s,t) = z_s - px_s - qy_s.$$

One has

$$h(s,0) = z_0'(s) - p_0(s)x_0'(s) - q_0(s)y_0'(s) = 0$$

since the initial strip satisfies the strip condition by assumption. In the following we will find that for fixed s the function h satisfies a linear homogeneous ordininary differential equation of first order. Consequently,

$h(s,t)=0$ in a neighbourhood of $t=0$. Thus the strip condition is also satisfied along strips transversally to the characteristic strips, see Figure 2.18. Thaen the set of "scales" fit together and define a surface like the scales of a fish.

From the definition of $h(s,t)$ and the characteristic equations we get

$$\begin{aligned}
h_t(s,t) &= z_{st}-p_tx_s-q_ty_s-px_{st}-qy_{st}\\
&= \frac{\partial}{\partial s}(z_t-px_t-qy_t)+p_sx_t+q_sy_t-q_ty_s-p_tx_s\\
&= (px_s+qy_s)F_z+F_xx_s+F_yz_s+F_pp_s+F_qq_s.
\end{aligned}$$

Since $F(x(s,t),y(s,t),z(s,t),p(s,t),q(s,t))=0$, it follows after differentiation of this equation with respect to s the differential equation

$$h_t=-F_zh.$$

Hence $h(s,t)\equiv 0$, since $h(s,0)=0$.

Thus we have

$$\begin{aligned}
z_s &= px_s+qy_s\\
z_t &= px_t+qy_t\\
z_s &= u_xx_s+u_yy_s\\
z_t &= u_xy_t+u_yy_t.
\end{aligned}$$

The first equation was shown above, the second is a characteristic equation and the last two follow from $z(s,t)=u(x(s,t),y(s,t))$. This system implies

$$\begin{aligned}
(P-u_x)x_s+(Q-u_y)y_s &= 0\\
(P-u_x)x_t+(Q-u_y)y_t &= 0.
\end{aligned}$$

It follows $P=u_x$ and $Q=u_y$.

The initial conditions

$$\begin{aligned}
u(x(s,0),y(s,0)) &= z_0(s)\\
u_x(x(s,0),y(s,0)) &= p_0(s)\\
u_y(x(s,0),y(s,0)) &= q_0(s)
\end{aligned}$$

are satisfied since

$$\begin{aligned}
u(x(s,t),y(s,t)) &= z(s(x,y),t(x,y))=z(s,t)\\
u_x(x(s,t),y(s,t)) &= p(s(x,y),t(x,y))=p(s,t)\\
u_y(x(s,t),y(s,t)) &= q(s(x,y),t(x,y))=q(s,t).
\end{aligned}$$

The uniqueness follows as in the proof of Theorem 2.1. □

Example. A differential equation which occurs in the geometrical optic is

$$u_x^2 + u_y^2 = f(x,y),$$

where the positive function $f(x,y)$ is the index of refraction. The level sets defined by $u(x,y) = const.$ are called *wave fronts*. The characteristic curves $(x(t), y(t))$ are the rays of light. If n is a constant, then the rays of light are straight lines. In $\mathbb{R}^3$ the equation is

$$u_x^2 + u_y^2 + u_z^2 = f(x,y,z).$$

Thus we have to extend the previous theory from $\mathbb{R}^2$ to $\mathbb{R}^n$, $n \geq 3$.

2.4 Nonlinear equations in $\mathbb{R}^n$

Here we consider the nonlinear differential equation

$$F(x,z,p) = 0, \tag{2.20}$$

where

$$x = (x_1, \ldots, x_n),\ z = u(x):\ \Omega \subset \mathbb{R}^n \mapsto \mathbb{R},\ p = \nabla u.$$

The following system of $2n+1$ ordinary differential equations is called *characteristic system*.

$$\begin{aligned} x'(t) &= \nabla_p F \\ z'(t) &= p \cdot \nabla_p F \\ p'(t) &= -\nabla_x F - F_z p. \end{aligned}$$

Let

$$x_0(s) = (x_{01}(s), \ldots, x_{0n}(s)),\ s = (s_1, \ldots, s_{n-1}),$$

be a given regular (n-1)-dimensional C^2-hypersurface in $\mathbb{R}^n$, i. e., we assume

$$\text{rank}\frac{\partial x_0(s)}{\partial s} = n - 1.$$

Here $s \in D$ is a parameter from an $(n-1)$-dimensional parameter domain D.

For example, $x = x_0(s)$ defines in the three dimensional case a regular surface in $\mathbb{R}^3$.

Assume

$$z_0(s): \ D \mapsto \mathbb{R}, \ p_0(s) = (p_{01}(s), \ldots, p_{0n}(s))$$

are given sufficiently regular functions.

The $(2n+1)$-vector

$$(x_0(s), z_0(s), p_0(s))$$

is called *initial strip manifold* and the condition

$$\frac{\partial z_0}{\partial s_l} = \sum_{i=1}^{n-1} p_{0i}(s) \frac{\partial x_{0i}}{\partial s_l},$$

$l = 1, \ldots, n-1$, *strip condition.*

The initial strip manifold is said to be *noncharacteristic* if

$$\det \begin{pmatrix} F_{p_1} & F_{p_2} & \cdots & F_{p_n} \\ \frac{\partial x_{01}}{\partial s_1} & \frac{\partial x_{02}}{\partial s_1} & \cdots & \frac{\partial x_{0n}}{\partial s_1} \\ \multicolumn{4}{c}{\dotfill} \\ \frac{\partial x_{01}}{\partial s_{n-1}} & \frac{\partial x_{02}}{\partial s_{n-1}} & \cdots & \frac{\partial x_{0n}}{\partial s_{n-1}} \end{pmatrix} \neq 0,$$

where the argument of F_{p_j} is the initial strip manifold.

Initial value problem of Cauchy. *Seek a solution $z = u(x)$ of the differential equation (2.20) such that the initial manifold is a subset of $\{(x, u(x), \nabla u(x)) : \ x \in \Omega\}$.*

As in the two dimensional case we have under additional regularity assumptions

Theorem 2.3. *Suppose the initial strip manifold is not characteristic and satisfies differential equation (2.20), that is, $F(x_0(s), z_0(s), p_0(s)) = 0$. Then there is a neighbourhood of the initial manifold $(x_0(s), z_0(s))$ such that there exists a unique solution of the Cauchy initial value problem.*

Sketch of proof. Let

$$x = x(s,t), \ z = z(s,t), \ p = p(s,t)$$

be the solution of the characteristic system and let

$$s = s(x), \ t = t(x)$$

be the inverse of $x = x(s,t)$ which exists in a neighbourhood of $t = 0$. Then, it turns out that

$$z = u(x) := z(s_1(x_1, \ldots, x_n), \ldots, s_{n-1}(x_1, \ldots, x_n), t(x_1, \ldots, x_n))$$

is the solution of the problem.

2.5 Hamilton-Jacobi theory

The nonlinear equation (2.20) of previous section in one more dimension is

$$F(x_1, \ldots, x_n, x_{n+1}, z, p_1, \ldots, p_n, p_{n+1}) = 0.$$

The content of the Hamilton[1]-Jacobi[2] theory is the theory of the special case

$$F \equiv p_{n+1} + H(x_1, \ldots, x_n, x_{n+1}, p_1, \ldots, p_n) = 0, \tag{2.21}$$

i. e., the equation is linear in p_{n+1} and does not depend on z explicitly.

Remark. Formally, one can write equation (2.20)

$$F(x_1, \ldots, x_n, u, u_{x_1}, \ldots, u_{x_n}) = 0$$

as an equation of type (2.21). Set $x_{n+1} = u$ and seek u implicitely from

$$\phi(x_1, \ldots, x_n, x_{n+1}) = const.,$$

where ϕ is a function which is defined by a differential equation.
Assume $\phi_{x_{n+1}} \neq 0$, then

$$\begin{aligned} 0 &= F(x_1, \ldots, x_n, u, u_{x_1}, \ldots, u_{x_n}) \\ &= F(x_1, \ldots, x_n, x_{n+1}, -\frac{\phi_{x_1}}{\phi_{x_{n+1}}}, \ldots, -\frac{\phi_{x_n}}{\phi_{x_{n+1}}}) \\ &=: G(x_1, \ldots, x_{n+1}, \phi_1, \ldots, \phi_{x_{n+1}}). \end{aligned}$$

Suppose that $G_{\phi_{x_{n+1}}} \neq 0$, then

$$\phi_{x_{n+1}} = H(x_1, \ldots, x_n, x_{n+1}, \phi_{x_1}, \ldots, \phi_{x_{n+1}}).$$

[1]Hamilton, William Rowan, 1805–1865
[2]Jacobi, Carl Gustav, 1805–1851

The associated characteristic equations to (2.21) are

$$\begin{aligned}
x'_{n+1}(\tau) &= F_{p_{n+1}} = 1 \\
x'_k(\tau) &= F_{p_k} = H_{p_k}, \qquad k = 1, \ldots, n \\
z'(\tau) &= \sum_{l=1}^{n+1} p_l F_{p_l} = \sum_{l=1}^{n} p_l H_{p_l} + p_{n+1} \\
&= \sum_{l=1}^{n} p_l H_{p_l} - H \\
p'_{n+1}(\tau) &= -F_{x_{n+1}} - F_z p_{n+1} \\
&= -F_{x_{n+1}} \\
p'_k(\tau) &= -F_{x_k} - F_z p_k \\
&= -F_{x_k}, \qquad k = 1, \ldots, n.
\end{aligned}$$

Set $t := x_{n+1}$, then we can write partial differential equation (2.21) as

$$u_t + H(x, t, \nabla_x u) = 0 \tag{2.22}$$

and $2n$ of the characteristic equations are

$$\begin{aligned}
x'(t) &= \nabla_p H(x, t, p) &(2.23)\\
p'(t) &= -\nabla_x H(x, t, p). &(2.24)
\end{aligned}$$

Here is

$$x = (x_1, \ldots, x_n),\ \ p = (p_1, \ldots, p_n).$$

Let $x(t),\ p(t)$ be a solution of (2.23) and (2.24), then it follows $p'_{n+1}(t)$ and $z'(t)$ from the characteristic equations

$$\begin{aligned}
p'_{n+1}(t) &= -H_t \\
z'(t) &= p \cdot \nabla_p H - H.
\end{aligned}$$

Definition. The function $H(x, t, p)$ is called *Hamilton function*, equation (2.21) *Hamilton-Jacobi equation* and the system (2.23), (2.24) *canonical system to H.*

There is an interesting interplay between the Hamilton-Jacobi equation and the canonical system. According to the previous theory we can construct a solution of the Hamilton-Jacobi equation by using solutions of the canonical system. On the other hand, one obtains from solutions of the Hamilton-Jacobi equation also solutions of the canonical system of ordinary differential equations.

Definition. A solution $\phi(a;x,t)$ of the Hamilton-Jacobi equation, where $a=(a_1,\ldots,a_n)$ is an n-tuple of real parameters, is called a *complete integral* of the Hamilton-Jacobi equation if

$$\det(\phi_{x_i a_l})_{i,l=1}^n \neq 0.$$

Remark. If u is a solution of the Hamilton-Jacobi equation, then also $u + const.$

Theorem 2.4 (Jacobi). *Assume*

$$u = \phi(a;x,t) + c,\ \ c = const.,\ \ \phi \in C^2 \ \textit{in its arguments},$$

is a complete integral. Then one obtains by solving of

$$b_i = \phi_{a_i}(a;x,t)$$

with respect to $x_l = x_l(a,b,t)$*, where* b_i $i=1,\ldots,n$ *are given real constants, and then by setting*

$$p_k = \phi_{x_k}(a;x(a,b;t),t)$$

a 2n-parameter family of solutions of the canonical system.

Proof. Let

$$x_l(a,b;t),\ \ l=1,\ldots,n,$$

be the solution of the above system. The solution exists since ϕ is a complete integral by assumption. Set

$$p_k(a,b;t) = \phi_{x_k}(a;x(a,b;t),t),\ \ k=1,\ldots,n.$$

We will show that x and p solves the canonical system. Differentiating $\phi_{a_i} = b_i$ with respect to t and the Hamilton-Jacobi equation $\phi_t + H(x,t,\nabla_x\phi) = 0$ with respect to a_i, we obtain for $i=1,\ldots,n$

$$\begin{aligned}
\phi_{ta_i} + \sum_{k=1}^n \phi_{x_k a_i}\frac{\partial x_k}{\partial t} &= 0\\
\phi_{ta_i} + \sum_{k=1}^n \phi_{x_k a_i} H p_k &= 0.
\end{aligned}$$

Since ϕ is a complete integral it follows for $k=1,\ldots,n$

$$\frac{\partial x_k}{\partial t}=H_{p_k}.$$

Along a trajectory, i. e., where $a,\ b$ are fixed, it is $\frac{\partial x_k}{\partial t}=x_k'(t)$. Thus

$$x_k'(t)=H_{p_k}.$$

Now we differentiate $p_i(a,b;t)$ with respect to t and $\phi_t+H(x,t,\nabla_x\phi)=0$ with respect to x_i, and obtain

$$\begin{aligned}
p_i'(t) &= \phi_{x_it}+\sum_{k=1}^n\phi_{x_ix_k}x_k'(t)\\
0 &= \phi_{x_it}+\sum_{k=1}^n\phi_{x_ix_k}H_{p_k}+H_{x_i}\\
0 &= \phi_{x_it}+\sum_{k=1}^n\phi_{x_ix_k}x_k'(t)+H_{x_i}
\end{aligned}$$

It follows finally that $p_i'(t)=-H_{x_i}$. □

Example: **Kepler problem**

The motion of a mass point in a central field takes place in a plane, say the (x,y)-plane, see Figure 2.13, and satisfies the system of ordinary differential equations of second order

$$x''(t)=U_x,\ \ y''(t)=U_y,$$

where

$$U(x,y)=\frac{k^2}{\sqrt{x^2+y^2}}.$$

Here we assume that k^2 is a positive constant and that the mass point is attracted of the origin. In the case that it is pushed one has to replace U by $-U$. See Landau and Lifschitz [12], Vol 1, for example, for the related physics.

Set

$$p=x',\ \ q=y'$$

and

$$H=\frac{1}{2}(p^2+q^2)-U(x,y),$$

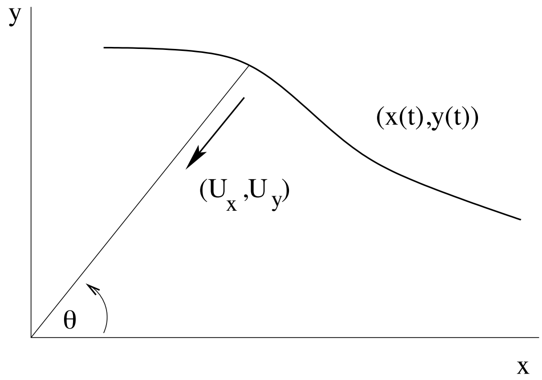

Figure 2.13: Motion in a central field

then

$$\begin{aligned} x'(t) &= H_p,\ y'(t) = H_q \\ p'(t) &= -H_x,\ q'(t) = -H_y. \end{aligned}$$

The associated Hamilton-Jacobi equation is

$$\phi_t + \frac{1}{2}(\phi_x^2 + \phi_y^2) = \frac{k^2}{\sqrt{x^2+y^2}}.$$

which is in polar coordinates (r, θ)

$$\phi_t + \frac{1}{2}(\phi_r^2 + \frac{1}{r^2}\phi_\theta^2) = \frac{k^2}{r}. \tag{2.25}$$

Now we will seek a complete integral of (2.25) by making the ansatz

$$\phi_t = -\alpha = const. \ \ \phi_\theta = -\beta = const. \tag{2.26}$$

and obtain from (2.25) that

$$\phi = \pm \int_{r_0}^{r} \sqrt{2\alpha + \frac{2k^2}{\rho} - \frac{\beta^2}{\rho^2}}\, d\rho + c(t, \theta).$$

From ansatz (2.26) it follows

$$c(t, \theta) = -\alpha t - \beta\theta.$$

Therefore we have a two parameter family of solutions

$$\phi = \phi(\alpha, \beta; \theta, r, t)$$

of the Hamilton-Jacobi equation. This solution is a complete integral, see an exercise. According to the theorem of Jacobi set

$$\phi_\alpha = -t_0, \quad \phi_\beta = -\theta_0.$$

Then

$$t - t_0 = -\int_{r_0}^{r} \frac{d\rho}{\sqrt{2\alpha + \frac{2k^2}{\rho} - \frac{\beta^2}{\rho^2}}}.$$

The inverse function $r = r(t)$, $r(0) = r_0$, is the r-coordinate depending on time t, and

$$\theta - \theta_0 = \beta \int_{r_0}^{r} \frac{d\rho}{\rho^2\sqrt{2\alpha + \frac{2k^2}{\rho} - \frac{\beta^2}{\rho^2}}}.$$

Substitution $\tau = \rho^{-1}$ yields

$$\begin{aligned}
\theta - \theta_0 &= -\beta \int_{1/r_0}^{1/r} \frac{d\tau}{\sqrt{2\alpha + 2k^2\tau - \beta^2\tau^2}} \\
&= -\arcsin\left(\frac{\frac{\beta^2}{k^2}\frac{1}{r} - 1}{\sqrt{1 + \frac{2\alpha\beta^2}{k^4}}}\right) + \arcsin\left(\frac{\frac{\beta^2}{k^2}\frac{1}{r_0} - 1}{\sqrt{1 + \frac{2\alpha\beta^2}{k^4}}}\right).
\end{aligned}$$

Set

$$\theta_1 = \theta_0 + \arcsin\left(\frac{\frac{\beta^2}{k^2}\frac{1}{r_0} - 1}{\sqrt{1 + \frac{2\alpha\beta^2}{k^4}}}\right)$$

and

$$p = \frac{\beta^2}{k^2}, \quad \epsilon^2 = \sqrt{1 + \frac{2\alpha\beta^2}{k^4}},$$

then

$$\theta - \theta_1 = -\arcsin\left(\frac{\frac{p}{r} - 1}{\epsilon^2}\right).$$

It follows

$$r = r(\theta) = \frac{p}{1 - \epsilon^2\sin(\theta - \theta_1)},$$

which is the polar equation of conic sections. It defines an ellipse if $0 \le \epsilon < 1$, a parabola if $\epsilon = 1$ and a hyperbola if $\epsilon > 1$, see Figure 2.14 for the case of an ellipse, where the origin of the coordinate system is one of the focal points of the ellipse.

For another application of the Jacobi theorem see Courant and Hilbert [4], Vol. 2, pp. 94, where geodedics on an ellipsoid are studied.

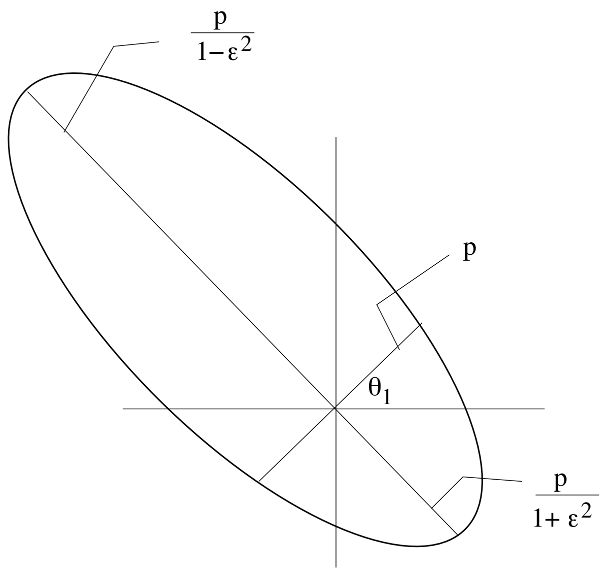

Figure 2.14: The case of an ellipse

2.6 Exercises

1. Suppose $u: \mathbb{R}^2 \mapsto \mathbb{R}$ is a solution of

$$a(x,y)u_x + b(x,y)u_y = 0.$$

   Show that for arbitrary $H \in C^1$ also $H(u)$ is a solution.

2. Find a solution $u \not\equiv const.$ of

$$u_x + u_y = 0$$

   such that

$$\text{graph}(u) := \{(x,y,z) \in \mathbb{R}^3 : \ z = u(x,y), \ (x,y) \in \mathbb{R}^2\}$$

   contains the straight line $(0,0,1) + s(1,1,0), \ s \in \mathbb{R}$.

3. Let $\phi(x,y)$ be a solution of

$$a_1(x,y)u_x + a_2(x,y)u_y = 0 \ .$$

   Prove that level curves $S_C := \{(x,y) : \ \phi(x,y) = C = const.\}$ are characteristic curves, provided that $\nabla\phi \neq 0$ and $(a_1,a_2) \neq (0,0)$.

4. Prove Proposition 2.2.

5. Find two different solutions of the initial value problem

$$u_x + u_y = 1,$$

where the initial data are $x_0(s) = s,\ y_0(s) = s,\ z_0(s) = s$.

Hint: (x_0, y_0) is a characteristic curve.

6. Solve the initial value problem

$$xu_x + yu_y = u$$

with initial data $x_0(s) = s,\ y_0(s) = 1,\ z_0(s)$, where z_0 is given.

7. Solve the initial value problem

$$-xu_x + yu_y = xu^2,$$

$x_0(s) = s,\ y_0(s) = 1,\ z_0(s) = \mathrm{e}^{-s}$.

8. Solve the initial value problem

$$uu_x + u_y = 1,$$

$x_0(s) = s,\ y_0(s) = s,\ z_0(s) = s/2$ if $0 < s < 1$.

9. Solve the initial value problem

$$uu_x + uu_y = 2,$$

$x_0(s) = s,\ y_0(s) = 1,\ z_0(s) = 1 + s$ if $0 < s < 1$.

10. Solve the initial value problem $u_x^2 + u_y^2 = 1 + x$ with given initial data $x_0(s) = 0,\ y_0(s) = s,\ u_0(s) = 1,\ p_0(s) = 1,\ q_0(s) = 0,\ -\infty < s < \infty$.

11. Find the solution $\Phi(x, y)$ of

$$(x - y)u_x + 2yu_y = 3x$$

such that the surface defined by $z = \Phi(x, y)$ contains the curve

$$C: \quad x_0(s) = s,\ y_0(s) = 1,\ z_0(s) = 0,\ s \in \mathbb{R}.$$

12. Solve the following initial problem of chemical kinetics.

$$u_x + u_y = \left(k_0 \mathrm{e}^{-k_1 x} + k_2\right)(1-u)^2,\ x > 0,\ y > 0$$

with the initial data $u(x,0) = 0,\ u(0,y) = u_0(y)$, where u_0, $0 < u_0 < 1$, is given.

13. Solve the Riemann problem

$$\begin{aligned} u_{x_1} + u_{x_2} &= 0 \\ u(x_1, 0) &= g(x_1) \end{aligned}$$

in $\Omega_1 = \{(x_1, x_2) \in \mathbb{R}^2 :\ x_1 > x_2\}$ and in $\Omega_2 = \{(x_1, x_2) \in \mathbb{R}^2 :\ x_1 < x_2\}$, where

$$g(x_1) = \left\{ \begin{array}{lcl} u_l & : & x_1 < 0 \\ u_r & : & x_1 > 0 \end{array} \right.$$

with constants $u_l \neq u_r$.

14. Determine the opening angle of the Monge cone, i. e., the angle between the axis and the apothem (in German: Mantellinie) of the cone, for equation

$$u_x^2 + u_y^2 = f(x, y, u),$$

where $f > 0$.

15. Solve the initial value problem

$$u_x^2 + u_y^2 = 1,$$

where $x_0(\theta) = a\cos\theta,\ y_0(\theta) = a\sin\theta,\ z_0(\theta) = 1,\ p_0(\theta) = \cos\theta,$ $q_0(\theta) = \sin\theta$ if $0 \le \theta < 2\pi$, $a = const. > 0$.

16. Show that the integral $\phi(\alpha, \beta; \theta, r, t)$, see the Kepler problem, is a complete integral.

17. a) Show that $S = \sqrt{\alpha}\ x + \sqrt{1-\alpha}\ y + \beta$, $\alpha,\ \beta \in \mathbb{R},\ 0 < \alpha < 1$, is a complete integral of $S_x - \sqrt{1 - S_y^2} = 0$.
b) Find the envelope of this family of solutions.

18. Determine the length of the half axis of the ellipse

$$r = \frac{p}{1 - \varepsilon^2 \sin(\theta - \theta_0)},\ \ 0 \le \varepsilon < 1.$$

19. Find the Hamilton function $H(x,p)$ of the Hamilton-Jacobi-Bellman differential equation if $h = 0$ and $f = Ax + B\alpha$, where A, B are constant and real matrices, $A:\ \mathbb{R}^m \mapsto \mathbb{R}^n$, B is an orthogonal real $n \times n$-Matrix and $p \in \mathbb{R}^n$ is given. The set of admissible controls is given by

$$U = \{\alpha \in \mathbb{R}^n :\ \sum_{i=1}^{n} \alpha_i^2 \leq 1\}\ .$$

Remark. The Hamilton-Jacobi-Bellman equation is formally the Hamilton-Jacobi equation $u_t + H(x, \nabla u) = 0$, where the Hamilton function is defined by

$$H(x,p) := \min_{\alpha \in U} \left(f(x,\alpha) \cdot p + h(x,\alpha) \right),$$

$f(x,\alpha)$ and $h(x,\alpha)$ are given. See for example, Evans [5], Chapter 10.

Chapter 3

Classification

Different types of problems in physics, for example, correspond different types of partial differential equations. The methods how to solve these equations differ from type to type.

The classification of differential equations follows from one single question: Can we calculate formally the solution if sufficiently many initial data are given? Consider the initial problem for an ordinary differential equation $y'(x) = f(x, y(x))$, $y(x_0) = y_0$. Then one can determine formally the solution, provided the function $f(x, y)$ is sufficiently regular. The solution of the initial value problem is formally given by a power series. This formal solution is a solution of the problem if $f(x, y)$ is real analytic according to a theorem of Cauchy. In the case of partial differential equations the related theorem is the Theorem of Cauchy-Kowalevskaya. Even in the case of ordinary differential equations the situation is more complicated if y' is implicitly defined, i. e., the differential equation is $F(x, y(x), y'(x)) = 0$ for a given function F.

3.1 Linear equations of second order

The general nonlinear partial differential equation of second order is

$$F(x, u, Du, D^2u) = 0,$$

where $x \in \mathbb{R}^n$, $u : \ \Omega \subset \mathbb{R}^n \mapsto \mathbb{R}$, $Du \equiv \nabla u$ and D^2u stands for all second derivatives. The function F is given and sufficiently regular with respect to its $2n + 1 + n^2$ arguments.

In this section we consider the case

$$\sum_{i,k=1}^{n} a^{ik}(x)u_{x_i x_k} + f(x,u,\nabla u) = 0. \tag{3.1}$$

The equation is *linear* if

$$f = \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u + d(x).$$

Concerning the classification the *main part*

$$\sum_{i,k=1}^{n} a^{ik}(x)u_{x_i x_k}$$

plays the essential role. Suppose $u \in C^2$, then we can assume, without restriction of generality, that $a^{ik} = a^{ki}$, since

$$\sum_{i,k=1}^{n} a^{ik}u_{x_i x_k} = \sum_{i,k=1}^{n} (a^{ik})^\star u_{x_i x_k},$$

where

$$(a^{ik})^\star = \frac{1}{2}(a^{ik} + a^{ki}).$$

Consider a hypersurface $\mathcal{S}$ in $\mathbb{R}^n$ defined implicitly by $\chi(x) = 0$, $\nabla\chi \neq 0$, see Figure 3.1

Assume u and ∇u are given on $\mathcal{S}$.

Problem: *Can we calculate all other derivatives of u on $\mathcal{S}$ by using differential equation (3.1) and the given data?*

We will find an answer if we map $\mathcal{S}$ onto a hyperplane $\mathcal{S}_0$ by a mapping

$$\begin{aligned} \lambda_n &= \chi(x_1,\ldots,x_n) \\ \lambda_i &= \lambda_i(x_1,\ldots,x_n),\ i = 1,\ldots,n-1, \end{aligned}$$

for functions λ_i such that

$$\det \frac{\partial(\lambda_1,\ldots,\lambda_n)}{\partial(x_1,\ldots,x_n)} \neq 0$$

in $\Omega \subset \mathbb{R}^n$. It is assumed that χ and λ_i are sufficiently regular. Such a mapping $\lambda = \lambda(x)$ exists, see an exercise.

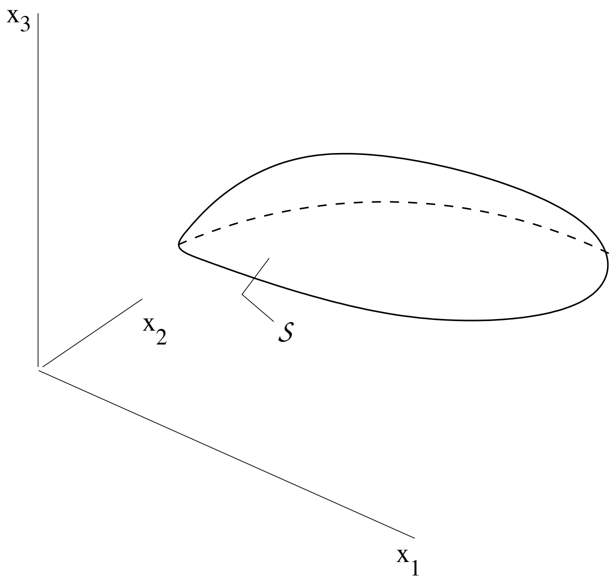

Figure 3.1: Initial manifold $\mathcal{S}$

The above transform maps $\mathcal{S}$ onto a subset of the hyperplane defined by $\lambda_n = 0$, see Figure 3.2.

We will write the differential equation in these new coordinates. Here we use Einstein's convention, i. e., we add terms with repeating indices. Since

$$u(x) = u(x(\lambda)) =: v(\lambda) = v(\lambda(x)),$$

where $x = (x_1, \ldots, x_n)$ and $\lambda = (\lambda_1, \ldots, \lambda_n)$, we get

$$\begin{aligned} u_{x_j} &= v_{\lambda_i} \frac{\partial \lambda_i}{\partial x_j}, \\ u_{x_j x_k} &= v_{\lambda_i \lambda_l} \frac{\partial \lambda_i}{\partial x_j} \frac{\partial \lambda_l}{\partial x_k} + v_{\lambda_i} \frac{\partial^2 \lambda_i}{\partial x_j \partial x_k}. \end{aligned} \tag{3.2}$$

Thus, differential equation (3.1) in the new coordinates is given by

$$a^{jk}(x) \frac{\partial \lambda_i}{\partial x_j} \frac{\partial \lambda_l}{\partial x_k} v_{\lambda_i \lambda_l} + \text{terms known on } \mathcal{S}_0 = 0.$$

Since $v_{\lambda_k}(\lambda_1, \ldots, \lambda_{n-1}, 0)$, $k = 1, \ldots, n$, are known, see (3.2), it follows that $v_{\lambda_k \lambda_l}$, $l = 1, \ldots, n-1$, are known on $\mathcal{S}_0$. Thus we know all second derivatives $v_{\lambda_i \lambda_j}$ on $\mathcal{S}_0$ with the only exception of $v_{\lambda_n \lambda_n}$.

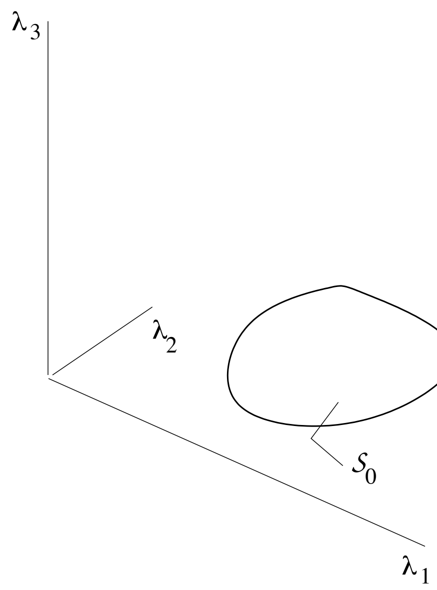

Figure 3.2: Transformed flat manifold $\mathcal{S}_0$

We recall that, provided v is sufficiently regular,

$$v_{\lambda_k\lambda_l}(\lambda_1,\ldots,\lambda_{n-1},0)$$

is the limit of

$$\frac{v_{\lambda_k}(\lambda_1,\ldots,\lambda_l+h,\lambda_{l+1},\ldots,\lambda_{n-1},0)-v_{\lambda_k}(\lambda_1,\ldots,\lambda_l,\lambda_{l+1},\ldots,\lambda_{n-1},0)}{h}$$

as $h\to 0$.

Thus the differential equation can be written as

$$\sum_{j,k=1}^{n} a^{jk}(x)\frac{\partial\lambda_n}{\partial x_j}\frac{\partial\lambda_n}{\partial x_k}v_{\lambda_n\lambda_n} = \text{terms known on } \mathcal{S}_0.$$

It follows that we can calculate $v_{\lambda_n\lambda_n}$ if

$$\sum_{i,j=1}^{n} a^{ij}(x)\chi_{x_i}\chi_{x_j}\neq 0 \tag{3.3}$$

on $\mathcal{S}$. This is a condition for the given equation and for the given surface $\mathcal{S}$.

Definition. The differential equation

$$\sum_{i,j=1}^{n} a^{ij}(x)\chi_{x_i}\chi_{x_j} = 0$$

is called *characteristic differential equation* associated to the given differential equation (3.1).

If χ, $\nabla\chi \neq 0$, is a solution of the characteristic differential equation, then the surface defined by $\chi = 0$ is called *characteristic surface*.

Remark. The condition (3.3) is satisfied for each χ with $\nabla\chi \neq 0$ if the quadratic matrix $(a^{ij}(x))$ is positive or negative definite for each $x \in \Omega$, which is equivalent to the property that all eigenvalues are different from zero and have the same sign. This follows since there is a $\lambda(x) > 0$ such that, in the case that the matrix (a^{ij}) is poitive definite,

$$\sum_{i,j=1}^{n} a^{ij}(x)\zeta_i\zeta_j \geq \lambda(x)|\zeta|^2$$

for all $\zeta \in \mathbb{R}^n$. Here and in the following we assume that the matrix (a^{ij}) is real and symmetric.

The characterization of differential equation (3.1) follows from the signs of the eigenvalues of $(a^{ij}(x))$.

Definition. Differential equation (3.1) is said to be of *type* (α, β, γ) *at* $x \in \Omega$ if α eigenvalues of $(a^{ij})(x)$ are positive, β eigenvalues are negative and γ eigenvalues are zero ($\alpha + \beta + \gamma = n$).
In particular, equation is called
elliptic if it is of type $(n, 0, 0)$ or of type $(0, n, 0)$, i. e., all eigenvalues are different from zero and have the same sign,
parabolic if it is of type $(n-1, 0, 1)$ or of type $(0, n-1, 1)$, i. e., one eigenvalue is zero and all the others are different from zero and have the same sign,
hyperbolic if it is of type $(n-1, 1, 0)$ or of type $(1, n-1, 0)$, i. e., all eigenvalues are different from zero and one eigenvalue has another sign than all the others.

Remarks:

1. According to this definition there are other types aside from elliptic, parabolic or hyperbolic equations.

2. The classification depends in general on $x \in \Omega$. An example is the Tricomi equation, which appears in the theory of transsonic flows,

$$yu_{xx} + u_{yy} = 0.$$

This equation is elliptic if $y > 0$, parabolic if $y = 0$ and hyperbolic for $y < 0$.

Examples:

1. The *Laplace equation* in $\mathbb{R}^3$ is $\triangle u = 0$, where

$$\triangle u := u_{xx} + u_{yy} + u_{zz}.$$

This equation is elliptic. Thus for each manifold $\mathcal{S}$ given by $\{(x, y, z) : \chi(x, y, z) = 0\}$, where χ is an arbitrary sufficiently regular function such that $\nabla\chi \neq 0$, all derivatives of u are known on $\mathcal{S}$, provided u and ∇u are known on $\mathcal{S}$.

2. The *wave equation* $u_{tt} = u_{xx} + u_{yy} + u_{zz}$, where $u = u(x, y, z, t)$, is hyperbolic. Such a type describes oscillations of mechanical structures, for example.

3. The *heat equation* $u_t = u_{xx}+u_{yy}+u_{zz}$, where $u = u(x, y, z, t)$, is parabolic. It describes, for example, the propagation of heat in a domain.

4. Consider the case that the (real) coefficients a^{ij} in equation (3.1) are *constant*. We recall that the matrix $A = (a^{ij})$ is symmetric, i. e., $A^T = A$. In this case, the transform to principle axis leads to a normal form from which the classification of the equation is obviously. Let U be the associated orthogonal matrix, then

$$U^T AU = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}.$$

Here is $U = (z_1, \ldots, z_n)$, where z_l, $l = 1, \ldots, n$, is an orthonormal system of eigenvectors to the eigenvalues λ_l.

Set $y = U^T x$ and $v(y) = u(Uy)$, then

$$\sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} = \sum_{i=1}^{n} \lambda_i v_{y_i y_j}. \tag{3.4}$$

3.1.1 Normal form in two variables

Consider the differential equation

$$a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} + \text{terms of lower order} = 0 \tag{3.5}$$

in $\Omega \subset \mathbb{R}^2$. The associated characteristic differential equation is

$$a\chi_x^2 + 2b\chi_x\chi_y + c\chi_y^2 = 0. \tag{3.6}$$

We show that an appropriate coordinate transform will simplify equation (3.5) sometimes in such a way that we can solve the transformed equation explicitly.

Let $z = \phi(x,y)$ be a solution of (3.6). Consider the level sets $\{(x,y) : \phi(x,y) = const.\}$ and assume $\phi_y \neq 0$ at a point (x_0, y_0) of the level set. Then there is a function $y(x)$ defined in a neighbourhood of x_0 such that $\phi(x, y(x)) = const.$ It follows

$$y'(x) = -\frac{\phi_x}{\phi_y},$$

which implies, see the characteristic equation (3.6),

$$ay'^2 - 2by' + c = 0. \tag{3.7}$$

Then, provided $a \neq 0$, we can calculate $\mu := y'$ from the (known) coefficients a, b and c:

$$\mu_{1,2} = \frac{1}{a}\left(b \pm \sqrt{b^2 - ac}\right). \tag{3.8}$$

These solutions are real if and only of $ac - b^2 \leq 0$.

Equation (3.5) is hyperbolic if $ac - b^2 < 0$, parabolic if $ac - b^2 = 0$ and elliptic if $ac - b^2 > 0$. This follows from an easy discussion of the eigenvalues of the matrix

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix},$$

see an exercise.

Normal form of a hyperbolic equation

Let ϕ and ψ are solutions of the characteristic equation (3.6) such that

$$\begin{aligned} y_1' \equiv \mu_1 &= -\frac{\phi_x}{\phi_y} \\ y_2' \equiv \mu_2 &= -\frac{\psi_x}{\psi_y}, \end{aligned}$$

where μ_1 and μ_2 are given by (3.8). Thus ϕ and ψ are solutions of the linear homogeneous equations of first order

$$\phi_x + \mu_1(x,y)\phi_y = 0 \tag{3.9}$$
$$\psi_x + \mu_2(x,y)\psi_y = 0. \tag{3.10}$$

Assume $\phi(x,y)$, $\psi(x,y)$ are solutions such that $\nabla\phi \neq 0$ and $\nabla\psi \neq 0$, see an exercise for the existence of such solutions.

Consider two families of level sets defined by $\phi(x,y) = \alpha$ and $\psi(x,y) = \beta$, see Figure 3.3.

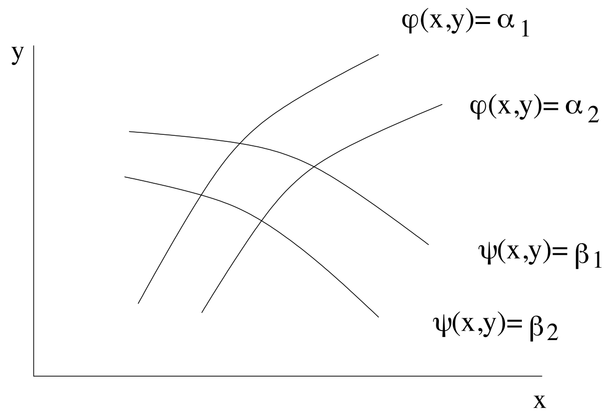

Figure 3.3: Level sets

These level sets are characteristic curves of the partial differential equations (3.9) and (3.10), respectively, see an exercise of the previous chapter.

Lemma. (i) *Curves from different families can not touch each other.*

(ii) $\phi_x\psi_y - \phi_y\psi_x \neq 0$.

Proof. (i):

$$y_2' - y_1' \equiv \mu_2 - \mu_1 = -\frac{2}{a}\sqrt{b^2 - ac} \neq 0.$$

(ii):

$$\mu_2 - \mu_1 = \frac{\phi_x}{\phi_y} - \frac{\psi_x}{\psi_y}.$$

□

Proposition 3.1. *The mapping $\xi = \phi(x,y)$, $\eta = \psi(x,y)$ transforms equation (3.5) into*

$$v_{\xi\eta} = \textit{lower order terms}, \tag{3.11}$$

where $v(\xi,\eta) = u(x(\xi,\eta), y(\xi,\eta))$.

Proof. The proof follows from a straightforward calculation.

$$\begin{aligned}
u_x &= v_\xi\phi_x + v_\eta\psi_x\\
u_y &= v_\xi\phi_y + v_\eta\psi_y\\
u_{xx} &= v_{\xi\xi}\phi_x^2 + 2v_{\xi\eta}\phi_x\psi_x + v_{\eta\eta}\psi_x^2 + \text{lower order terms}\\
u_{xy} &= v_{\xi\xi}\phi_x\phi_y + v_{\xi\eta}(\phi_x\psi_y + \phi_y\psi_x) + v_{\eta\eta}\psi_x\psi_y + \text{lower order terms}\\
u_{yy} &= v_{\xi\xi}\phi_y^2 + 2v_{\xi\eta}\phi_y\psi_y + v_{\eta\eta}\psi_y^2 + \text{lower order terms.}
\end{aligned}$$

Thus

$$au_{xx} + 2bu_{xy} + cu_{yy} = \alpha v_{\xi\xi} + 2\beta v_{\xi\eta} + \gamma v_{\eta\eta} + l.o.t.,$$

where

$$\begin{aligned}
\alpha : &= a\phi_x^2 + 2b\phi_x\phi_y + c\phi_y^2\\
\beta : &= a\phi_x\psi_x + b(\phi_x\psi_y + \phi_y\psi_x) + c\phi_y\psi_y\\
\gamma : &= a\psi_x^2 + 2b\psi_x\psi_y + c\psi_y^2.
\end{aligned}$$

The coefficients α and γ are zero since ϕ and ψ are solutions of the characteristic equation. Since

$$\alpha\gamma - \beta^2 = (ac - b^2)(\phi_x\psi_y - \phi_y\psi_x)^2,$$

it follows from the above lemma that the coefficient β is different from zero.
□

Example: Consider the differential equation

$$u_{xx} - u_{yy} = 0.$$

The associated characteristic differential equation is

$$\chi_x^2 - \chi_y^2 = 0.$$

Since $\mu_1 = -1$ and $\mu_2 = 1$, the functions ϕ and ψ satisfy differential equations

$$\begin{aligned} \phi_x + \phi_y &= 0 \\ \psi_x - \psi_y &= 0. \end{aligned}$$

Solutions with $\nabla\phi \neq 0$ and $\nabla\psi \neq 0$ are

$$\phi = x - y, \quad \psi = x + y.$$

Thus the mapping

$$\xi = x - y, \quad \eta = x + y$$

leads to the simple equation

$$v_{\xi\eta}(\xi, \eta) = 0.$$

Assume $v \in C^2$ is a solution, then $v_\xi = f_1(\xi)$ for an arbitrary C^1 function $f_1(\xi)$. It follows

$$v(\xi, \eta) = \int_0^\xi f_1(\alpha)\ d\alpha + g(\eta),$$

where g is an arbitrary C^2 function. Thus each C^2-solution of the differential equation can be written as

$$(\star) \qquad v(\xi, \eta) = f(\xi) + g(\eta),$$

where $f,\ g \in C^2$. On the other hand, for arbitrary C^2-functions f, g the function $(\star)$ is a solution of the differential equation $v_{\xi\eta} = 0$. Consequently each C^2-solution of the original equation $u_{xx} - u_{yy} = 0$ is given by

$$u(x, y) = f(x - y) + g(x + y),$$

where $f,\ g \in C^2$.

3.2 Quasilinear equations of second order

Here we consider the equation

$$\sum_{i,j=1}^{n} a^{ij}(x,u,\nabla u)u_{x_ix_j} + b(x,u,\nabla u) = 0 \tag{3.12}$$

in a domain $\Omega \subset \mathbb{R}^n$, where $u:\ \Omega \mapsto \mathbb{R}$. We assume that $a^{ij} = a^{ji}$.

As in the previous section we can derive the characteristic equation

$$\sum_{i,j=1}^{n} a^{ij}(x,u,\nabla u)\chi_{x_i}\chi_{x_j} = 0.$$

In contrast to linear equations, solutions of the characteristic equation depend on the solution considered.

3.2.1 Quasilinear elliptic equations

There is a large class of quasilinear equations such that the associated characteristic equation has no solution χ, $\nabla\chi \neq 0$.

Set

$$U = \{(x,z,p):\ x \in \Omega,\ z \in \mathbb{R},\ p \in \mathbb{R}^n\}.$$

Definition. The quasilinear equation (3.12) is called *elliptic* if the matrix $(a^{ij}(x,z,p))$ is positive definite for each $(x,z,p) \in U$.

Assume equation (3.12) is elliptic and let $\lambda(x,z,p)$ be the minimum and $\Lambda(x,z,p)$ the maximum of the eigenvalues of (a^{ij}), then

$$0 < \lambda(x,z,p)|\zeta|^2 \le \sum_{i,j=1}^{n} a^{ij}(x,z,p)\zeta_i\zeta_j \le \Lambda(x,z,p)|\zeta|^2$$

for all $\zeta \in \mathbb{R}^n$.

Definition. Equation (3.12) is called *uniformly elliptic* if Λ/λ is uniformly bounded in U.

An important class of elliptic equations which are not uniformly elliptic (nonuniformly elliptic) is

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{u_{x_i}}{\sqrt{1+|\nabla u|^2}}\right) + \text{lower order terms} = 0. \tag{3.13}$$

The main part is the minimal surface operator (left hand side of the minimal surface equation). The coefficients a^{ij} are

$$a^{ij}(x,z,p) = \left(1+|p|^2\right)^{-1/2}\left(\delta_{ij} - \frac{p_i p_j}{1+|p|^2}\right),$$

δ_{ij} denotes the Kronecker delta symbol. It follows that

$$\lambda = \frac{1}{(1+|p|^2)^{3/2}}, \quad \Lambda = \frac{1}{(1+|p|^2)^{1/2}}.$$

Thus equation (3.13) is not uniformly elliptic.

The behaviour of solutions of uniformly elliptic equations is similar to linear elliptic equations in contrast to the behaviour of solutions of nonuniformly elliptic equations. Typical examples for nonuniformly elliptic equations are the minimal surface equation and the capillary equation.

3.3 Systems of first order

Consider the quasilinear system

$$\sum_{k=1}^{n} A^k(x,u)u_{u_k} + b(x,u) = 0, \tag{3.14}$$

where A^k are $m \times m$-matrices, sufficiently regular with respect to their arguments, and

$$u = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix}, \quad u_{x_k} = \begin{pmatrix} u_{1,x_k} \\ \vdots \\ u_{m,x_k} \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}.$$

We ask the same question as above: can we calculate all derivatives of u in a neighbourhood of a given hypersurface $\mathcal{S}$ in $\mathbb{R}^n$ defined by $\chi(x) = 0$, $\nabla\chi \neq 0$, provided $u(x)$ is given on $\mathcal{S}$?

For an answer we map $\mathcal{S}$ onto a flat surface $\mathcal{S}_0$ by using the mapping $\lambda = \lambda(x)$ of Section 3.1 and write equation (3.14) in new coordinates. Set $v(\lambda) = u(x(\lambda))$, then

$$\sum_{k=1}^{n} A^k(x,u)\chi_{x_k} v_{\lambda_n} = \text{terms known on } \mathcal{S}_0.$$

We can solve this system with respect to v_{λ_n}, provided that

$$\det\left(\sum_{k=1}^{n} A^k(x,u)\chi_{x_k}\right) \neq 0$$

on $\mathcal{S}$.

Definition. Equation

$$\det\left(\sum_{k=1}^{n} A^k(x,u)\chi_{x_k}\right) = 0$$

is called *characteristic equation* associated to equation (3.14) and a surface $\mathcal{S}$: $\chi(x) = 0$, defined by a solution χ, $\nabla\chi \neq 0$, of this characteristic equation is said to be *characteristic surface.*

Set

$$C(x,u,\zeta) = \det\left(\sum_{k=1}^{n} A^k(x,u)\zeta_k\right)$$

for $\zeta \in \mathbb{R}^n$.

Definition. (i) The system (3.14) is *hyperbolic* at $(x, u(x))$ if there is a regular linear mapping $\zeta = Q\eta$, where $\eta = (\eta_1, \ldots, \eta_{n-1}, \kappa)$, such that there exists m *real* roots $\kappa_k = \kappa_k(x, u(x), \eta_1, \ldots, \eta_{n-1})$, $k = 1, \ldots, m$, of

$$D(x, u(x), \eta_1, \ldots, \eta_{n-1}, \kappa) = 0$$

for all $(\eta_1, \ldots, \eta_{n-1})$, where

$$D(x, u(x), \eta_1, \ldots, \eta_{n-1}, \kappa) = C(x, u(x), x, Q\eta).$$

(ii) System (3.14) is *parabolic* if there exists a regular linear mapping $\zeta = Q\eta$ such that D is independent of κ, i. e., D depends on less than n parameters.

(iii) System (3.14) is *elliptic* if $C(x, u, \zeta) = 0$ only if $\zeta = 0$.

Remark. In the elliptic case all derivatives of the solution can be calculated from the given data and the given equation.

3.3.1 Examples

1. Beltrami equations

$$\begin{aligned} Wu_x - bv_x - cv_y &= 0 \qquad (3.15)\\ Wu_y + av_x + bv_y &= 0, \qquad (3.16) \end{aligned}$$

where $W,\ a,\ b,\ c$ are given functions depending of (x, y), $W \neq 0$ and the matrix

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

is positive definite.

The Beltrami system is a generalization of Cauchy-Riemann equations. The function $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$, is called a *quasiconform mapping*, see for example [9], Chapter 12, for an application to partial differential equations.

Set

$$A^1 = \begin{pmatrix} W & -b \\ 0 & a \end{pmatrix}, \quad A^2 = \begin{pmatrix} 0 & -c \\ W & b \end{pmatrix}.$$

Then the system (3.15), (3.16) can be written as

$$A^1 \begin{pmatrix} u_x \\ v_x \end{pmatrix} + A^2 \begin{pmatrix} u_y \\ v_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Thus,

$$C(x, y, \zeta) = \begin{vmatrix} W\zeta_1 & -b\zeta_1 - c\zeta_2 \\ W\zeta_2 & a\zeta_1 + b\zeta_2 \end{vmatrix} = W(a\zeta_1^2 + 2b\zeta_1\zeta_2 + c\zeta_2^2),$$

which is different from zero if $\zeta \neq 0$ according to the above assumptions. Thus the *Beltrami system is elliptic.*

2. Maxwell equations

The Maxwell equations in the isotropic case are

$$\begin{aligned} c \text{ rot}_x\ H &= \lambda E + \epsilon E_t \qquad (3.17)\\ c \text{ rot}_x\ E &= -\mu H_t, \qquad (3.18) \end{aligned}$$

where
$E=(e_1,e_2,e_3)^T$ electric field strength, $e_i=e_i(x,t)$, $x=(x_1,x_2,x_3)$,
$H=(h_1,h_2,h_3)^T$ magnetic field strength, $h_i=h_i(x,t)$,
c speed of light,
λ specific conductivity,
ϵ dielectricity constant,
μ magnetic permeability.
Here $c,\ \lambda,\ \epsilon$ and μ are positive constants.

Set $p_0=\chi_t,\ p_i=\chi_{x_i},\ i=1,\ldots 3$, then the characteristic differential equation is

$$\begin{vmatrix} \epsilon p_0/c & 0 & 0 & 0 & p_3 & -p_2 \\ 0 & \epsilon p_0/c & 0 & -p_3 & 0 & p_1 \\ 0 & 0 & \epsilon p_0/c & p_2 & -p_1 & 0 \\ 0 & -p_3 & p_2 & \mu p_0/c & 0 & 0 \\ p_3 & 0 & -p_1 & 0 & \mu p_0/c & 0 \\ -p_2 & p_1 & 0 & 0 & 0 & \mu p_0/c \end{vmatrix} = 0.$$

The following manipulations simplifies this equation:
(i) multiply the first three columns with $\mu p_0/c$,
(ii) multiply the 5th column with $-p_3$ and the the 6th column with p_2 and add the sum to the 1st column,
(iii) multiply the 4th column with p_3 and the 6th column with $-p_1$ and add the sum to the 2th column,
(iv) multiply the 4th column with $-p_2$ and the 5th column with p_1 and add the sum to the 3th column,
(v) expand the resulting determinant with respect to the elements of the 6th, 5th and 4th row.

We obtain

$$\begin{vmatrix} q+p_1^2 & p_1p_2 & p_1p_3 \\ p_1p_2 & q+p_2^2 & p_2p_3 \\ p_1p_3 & p_2p_3 & q+p_3^2 \end{vmatrix} = 0,$$

where

$$q := \frac{\epsilon\mu}{c^2}p_0^2 - g^2$$

with $g^2 := p_1^2+p_2^2+p_3^2$. The evaluation of the above equation leads to $q^2(q+g^2)=0$, i. e.,

$$\chi_t^2\left(\frac{\epsilon\mu}{c^2}\chi_t^2 - |\nabla_x\chi|^2\right) = 0.$$

It follows immediately that *Maxwell equations* are a *hyperbolic system*, see an exercise. There are two solutions of this characteristic equation. The first one are characteristic surfaces $\mathcal{S}(t)$, defined by $\chi(x,t) = 0$, which satisfy $\chi_t = 0$. These surfaces are called *stationary waves*. The second type of characteristic surfaces are defined by solutions of

$$\frac{\epsilon\mu}{c^2}\chi_t^2 = |\nabla_x \chi|^2.$$

Functions defined by $\chi = f(n\cdot x - Vt)$ are solutions of this equation. Here is $f(s)$ an arbitrary function with $f'(s) \neq 0$, n is a unit vector and $V = c/\sqrt{\epsilon\mu}$. The associated characteristic surfaces $\mathcal{S}(t)$ are defined by

$$\chi(x,t) \equiv f(n\cdot x - Vt) = 0,$$

here we assume that 0 is in he range of $f:\ \mathbb{R} \mapsto \mathbb{R}$. Thus, $\mathcal{S}(t)$ is defined by $n\cdot x - Vt = c$, where c is a fixed constant. It follows that the planes $\mathcal{S}(t)$ with normal n move with speed V in direction of n, see Figure 3.4

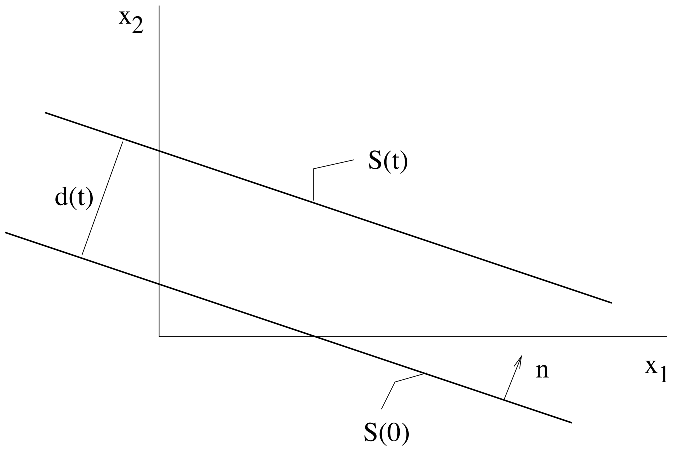

Figure 3.4: $d'(t)$ is the speed of plane waves

V is called *speed* of the plane wave $\mathcal{S}(t)$.

Remark. According to the previous discussions, singularities of a solution of Maxwell equations are located at most on characteristic surfaces.

A special case of Maxwell equations are the **telegraph equations**, which follow from Maxwell equations if div $E = 0$ and div $H = 0$, i. e., E and

H are fields free of sources. In fact, it is sufficient to assume that this assumption is satisfied at a fixed time t_0 only, see an exercise.
Since

$$\text{rot}_x\ \text{rot}_x\ A = \text{grad}_x\ \text{div}_x\ A - \triangle_x A$$

for each C^2-vector field A, it follows from Maxwell equations the uncoupled system

$$\begin{aligned}\triangle_x E &= \frac{\epsilon\mu}{c^2}E_{tt} + \frac{\lambda\mu}{c^2}E_t\\ \triangle_x H &= \frac{\epsilon\mu}{c^2}H_{tt} + \frac{\lambda\mu}{c^2}H_t.\end{aligned}$$

3. Equations of gas dynamics

Consider the following quasilinear equations of first order.

$$v_t + (v \cdot \nabla_x)\ v + \frac{1}{\rho}\nabla_x p = f \quad \text{(Euler equations)}.$$

Here is
$v = (v_1, v_2, v_3)$ the vector of speed, $v_i = v_i(x,t)$, $x = (x_1, x_2, x_3)$,
p pressure, $p = (x,t)$,
ρ density, $\rho = \rho(x,t)$,
$f = (f_1, f_2, f_3)$ density of the external force, $f_i = f_i(x,t)$,
$(v \cdot \nabla_x)v \equiv (v \cdot \nabla_x v_1, v \cdot \nabla_x v_2, v \cdot \nabla_x v_3))^T$.

The second equation is

$$\rho_t + v \cdot \nabla_x \rho + \rho\ \text{div}_x\ v = 0 \quad \text{(conservation of mass)}.$$

Assume the gas is compressible and that there is a function (state equation)

$$p = p(\rho),$$

where $p'(\rho) > 0$ if $\rho > 0$. Then the above system of four equations is

$$v_t + (v \cdot \nabla)v + \frac{1}{\rho}p'(\rho)\nabla\rho = f \tag{3.19}$$

$$\rho_t + \rho\ \text{div}\ v + v \cdot \nabla\rho = 0, \tag{3.20}$$

where $\nabla \equiv \nabla_x$ and $\text{div} \equiv \text{div}_x$, i. e., these operators apply on the spatial variables only.

The characteristic differential equation is here

$$\begin{vmatrix} \frac{d\chi}{dt} & 0 & 0 & \frac{1}{\rho}p'\chi_{x_1} \\ 0 & \frac{d\chi}{dt} & 0 & \frac{1}{\rho}p'\chi_{x_2} \\ 0 & 0 & \frac{d\chi}{dt} & \frac{1}{\rho}p'\chi_{x_3} \\ \rho\chi_{x_1} & \rho\chi_{x_2} & \rho\chi_{x_3} & \frac{d\chi}{dt} \end{vmatrix} = 0,$$

where

$$\frac{d\chi}{dt} := \chi_t + (\nabla_x\chi)\cdot v.$$

Evaluating the determinant, we get the characteristic differential equation

$$\left(\frac{d\chi}{dt}\right)^2 \left(\left(\frac{d\chi}{dt}\right)^2 - p'(\rho)|\nabla_x\chi|^2\right) = 0. \tag{3.21}$$

This equation implies consequences for the speed of the characteristic surfaces as the following consideration shows.

Consider a family $\mathcal{S}(t)$ of surfaces in $\mathbb{R}^3$ defined by $\chi(x,t) = c$, where $x \in \mathbb{R}^3$ and c is a fixed constant. As usually, we assume that $\nabla_x\chi \neq 0$. One of the two normals on $\mathcal{S}(t)$ at a point of the surface $\mathcal{S}(t)$ is given by, see an exercise,

$$\mathbf{n} = \frac{\nabla_x\chi}{|\nabla_x\chi|}. \tag{3.22}$$

Let $Q_0 \in \mathcal{S}(t_0)$ and let $Q_1 \in \mathcal{S}(t_1)$ be a point on the line defined by $Q_0 + s\mathbf{n}$, where $\mathbf{n}$ is the normal (3.22 on $\mathcal{S}(t_0)$ at Q_0 and $t_0 < t_1$, $t_1 - t_0$ small, see Figure 3.5.

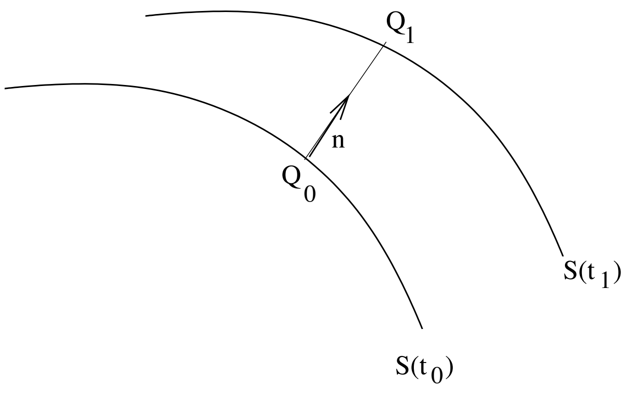

Figure 3.5: Definition of the speed of a surface

Definition. The limit
$$P = \lim_{t_1 \to t_0} \frac{|Q_1 - Q_0|}{t_1 - t_0}$$
is called *speed* of the surface $\mathcal{S}(t)$.

Proposition 3.2. *The speed of the surface $\mathcal{S}(t)$ is*
$$P = -\frac{\chi_t}{|\nabla_x \chi|}. \tag{3.23}$$

Proof. The proof follows from $\chi(Q_0, t_0) = 0$ and $\chi(Q_0 + d\mathbf{n}, t_0 + \triangle t) = 0$, where $d = |Q_1 - Q_0|$ and $\triangle t = t_1 - t_0$.

□

Set $v_n := v \cdot \mathbf{n}$ which is the component of the velocity vector in direction $\mathbf{n}$. From (3.22) we get
$$v_n = \frac{1}{|\nabla_x \chi|} v \cdot \nabla_x \chi.$$

Definition. $V := P - v_n$, the difference of the speed of the surface and the speed of liquid particles, is called *relative speed.*

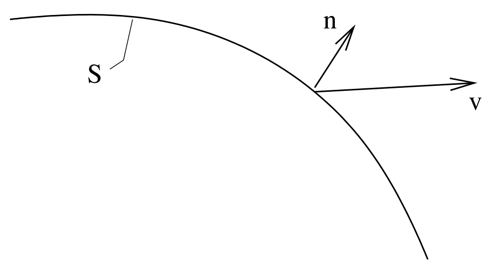

Figure 3.6: Definition of relative speed

Using the above formulas for P and v_n it follows
$$V = P - v_n = -\frac{\chi_t}{|\nabla_x \chi|} - \frac{v \cdot \nabla_x \chi}{|\nabla_x \chi|} = -\frac{1}{|\nabla_x \chi|} \frac{d\chi}{dt}.$$
Then, we obtain from the characteristic equation (3.21) that
$$V^2 |\nabla_x \chi|^2 \left(V^2 |\nabla_x \chi|^2 - p'(\rho) |\nabla_x \chi|^2 \right) = 0.$$

An interesting conclusion is that there are two relative speeds: $V = 0$ or $V^2 = p'(\rho)$.

Definition. $\sqrt{p'(\rho)}$ is called *speed of sound* .

3.4 Systems of second order

Here we consider the system

$$\sum_{k,l=1}^{n} A^{kl}(x, u, \nabla u)u_{x_k x_l} + \text{lower order terms} = 0, \tag{3.24}$$

where A^{kl} are $(m \times m)$ matrices and $u = (u_1, \ldots, u_m)^T$. We assume $A^{kl} = A^{lk}$, which is no restriction of generality provided $u \in C^2$ is satisfied. As in the previous sections, the classification follows from the question whether or not we can calculate formally the solution from the differential equations, if sufficiently many data are given on an initial manifold. Let the initial manifold $\mathcal{S}$ be given by $\chi(x) = 0$ and assume that $\nabla\chi \neq 0$. The mapping $x = x(\lambda)$, see previous sections, leads to

$$\sum_{k,l=1}^{n} A^{kl}\chi_{x_k}\chi_{x_l}v_{\lambda_n\lambda_n} = \text{terms known on } \mathcal{S},$$

where $v(\lambda) = u(x(\lambda))$.

The characteristic equation is here

$$\det\left(\sum_{k,l=1}^{n} A^{kl}\chi_{x_k}\chi_{x_l}\right) = 0.$$

If there is a solution χ with $\nabla\chi \neq 0$, then it is possible that second derivatives are not continuous in a neighbourhood of $\mathcal{S}$.

Definition. The system is called *elliptic* if

$$\det\left(\sum_{k,l=1}^{n} A^{kl}\zeta_k\zeta_l\right) \neq 0$$

for all $\zeta \in \mathbb{R}^n$, $\zeta \neq 0$.

3.4.1 Examples

1. Navier-Stokes equations

The Navier-Stokes system for a viscous incompressible liquid is

$$\begin{aligned} v_t + (v \cdot \nabla_x)v &= -\frac{1}{\rho}\nabla_x p + \gamma \triangle_x v \\ \mathrm{d}iv_x\ v &= 0, \end{aligned}$$

where ρ is the (constant and positive) density of liquid,
γ is the (constant and positive) viscosity of liquid,
$v = v(x,t)$ velocity vector of liquid particles, $x \in \mathbb{R}^3$ or in $\mathbb{R}^2$,
$p = p(x,t)$ pressure.

The problem is to find solutions $v,\ p$ of the above system.

2. Linear elasticity

Consider the system

$$\rho \frac{\partial^2 u}{\partial t^2} = \mu \triangle_x u + (\lambda + \mu)\nabla_x(\mathrm{d}iv_x\ u) + f. \tag{3.25}$$

Here is, in the case of an elastic body in $\mathbb{R}^3$,
$u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ displacement vector,
$f(x,t)$ density of external force,
ρ (constant) density,
$\lambda,\ \mu$ (positive) Lamé constants.

The characteristic equation is $\det C = 0$, where the entries of the matrix C are given by

$$c_{ij} = (\lambda + \mu)\chi_{x_i}\chi_{x_j} + \delta_{ij}\left(\mu|\nabla_x \chi|^2 - \rho\chi_t^2\right).$$

The characteristic equation is

$$\left((\lambda + 2\mu)|\nabla_x \chi|^2 - \rho\chi_t^2\right)\left(\mu|\nabla_x \chi|^2 - \rho\chi_t^2\right)^2 = 0.$$

It follows that two different speeds P of characteristic surfaces $\mathcal{S}(t)$, defined by $\chi(x,t) = const.$, are possible, namely

$$P_1 = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \quad \text{and} \quad P_2 = \sqrt{\frac{\mu}{\rho}}.$$

We recall that $P = -\chi_t/|\nabla_x \chi|$.

3.5 Theorem of Cauchy-Kovalevskaya

Consider the quasilinear system of first order (3.14) of Section 3.3. Assume an initial manifolds $\mathcal{S}$ is given by $\chi(x) = 0$, $\nabla\chi \neq 0$, and suppose that χ is not characteristic. Then, see Section 3.3, the system (3.14) can be written as

$$\begin{aligned} u_{x_n} &= \sum_{i=1}^{n-1} a^i(x,u)u_{x_i} + b(x,u) && (3.26) \\ u(x_1,\ldots,x_{n-1},0) &= f(x_1,\ldots,x_{n-1}) && (3.27) \end{aligned}$$

Here is $u = (u_1,\ldots,u_m)^T$, $b = (b_1,\ldots,b_n)^T$ and a^i are $(m\times m)$-matrices. We assume a^i, b and f are in C^∞ with respect to their arguments. From (3.26) and (3.27) it follows that we can calculate formally all derivatives $D^\alpha u$ in a neigbourhood of the plane $\{x:\ x_n = 0\}$, in particular in a neighbourhood of $0 \in \mathbb{R}^n$. Thus we have a formal power series of $u(x)$ at $x = 0$:

$$u(x) \sim \sum \frac{1}{\alpha!} D^\alpha u(0) x^\alpha.$$

For notations and definitions used here and in the following see the appendix to this section.

Then, as usually, two questions arise:

(i) Does the power series converge in a neighbourhood of $0 \in \mathbb{R}^n$?

(ii) Is a convergent power series a solution of the initial value problem (3.26), (3.27)?

Remark. Quite different to this power series method is the method of *asymptotic expansions*. Here one is interested in a good approximation of an unknown solution of an equation by a finite sum $\sum_{i=0}^N \phi_i(x)$ of functions ϕ_i. In general, the infinite sum $\sum_{i=0}^\infty \phi_i(x)$ does not converge, in contrast to the power series method of this section. See [15] for some asymptotic formulas in capillarity.

Theorem 3.1 (Cauchy-Kovalevskaya). *There is a neighbourhood of* $0 \in \mathbb{R}^n$ *such there is a real analytic solution of the initial value problem (3.26), (3.27). This solution is unique in the class of real analytic functions.*

Proof. The proof is taken from F. John [10]. We introduce $u-f$ as the new solution for which we are looking at and we add a new coordinate $u^\star$ to the solution vector by setting $u^\star(x)=x_n$. Then

$$u^\star_{x_n}=1,\ u^\star_{x_k}=0,\ k=1,\ldots,n-1,\ u^\star(x_1,\ldots,x_{n-1},0)=0$$

and the extended system (3.26), (3.27) is

$$\begin{pmatrix} u_{1,x_n} \\ \vdots \\ u_{m,x_n} \\ u^\star_{x_n} \end{pmatrix} = \sum_{i=1}^{n-1} \begin{pmatrix} a^i & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_{1,x_i} \\ \vdots \\ u_{m,x_i} \\ u^\star_{x_i} \end{pmatrix} + \begin{pmatrix} b_1 \\ \vdots \\ b_m \\ 1 \end{pmatrix},$$

where the associated initial condition is $u(x_1,\ldots,x_{n-1},0)=0$. The new u is $u=(u_1,\ldots,u_m)^T$, the new a^i are $a^i(x_1,\ldots,x_{n-1},u_1,\ldots,u_m,u^\star)$ and the new b is $b=(x_1,\ldots,x_{n-1},u_1,\ldots,u_m,u^\star)^T$.

Thus we are led to an initial value problem of the type

$$\begin{aligned} u_{j,x_n} &= \sum_{i=1}^{n-1}\sum_{k=1}^{N} a^i_{jk}(z)u_{k,x_i}+b_j(z),\ j=1,\ldots,N && (3.28)\\ u_j(x) &= 0 \ \text{ if } x_n=0, && (3.29)\end{aligned}$$

where $j=1,\ldots,N$ and $z=(x_1,\ldots,x_{n-1},u_1,\ldots,u_N)$.

The point here is that a^i_{jk} and b_j are independent of x_n. This fact simplifies the proof of the theorem.

From (3.28) and (3.29) we can calculate formally all $D^\beta u_j$. Then we have formal power series for u_j:

$$u_j(x)\sim\sum_\alpha c^{(j)}_\alpha x^\alpha,$$

where

$$c^{(j)}_\alpha=\frac{1}{\alpha!}D^\alpha u_j(0).$$

We will show that these power series are (absolutely) convergent in a neighbourhood of $0\in\mathbb{R}^n$, i. e., they are real analytic functions, see the appendix for the definition of real analytic functions. Inserting these functions into the left and into the right hand side of (3.28) we obtain on the right and on the left hand side real analytic functions. This follows since compositions of real analytic functions are real analytic again, see Proposition A7 of the appendix to this section. The resulting power series on the left and on the right have the same coefficients caused by the calculation of the derivatives $D^\alpha u_j(0)$ from (3.28). It follows that $u_j(x)$, $j = 1, \ldots, n$, defined by its formal power series are solutions of the initial value problem (3.28), (3.29).

Set
$$d = \left(\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_{N+n-1}}\right)$$

Lemma A. *Assume* $u \in C^\infty$ *in a neighbourhood of* $0 \in \mathbb{R}^n$. *Then*
$$D^\alpha u_j(0) = P_\alpha\left(d^\beta a^i_{jk}(0), d^\gamma b_j(0)\right),$$
where $|\beta|,\ |\gamma| \le |\alpha|$ *and* P_α *are polynomials in the indicated arguments with* **nonnegative** *integers as coefficients which are* **independent** *of* a^i *and of* b.

Proof. It follows from equation (3.28) that
$$D_n D^\alpha u_j(0) = P_\alpha(d^\beta a^i_{jk}(0), d^\gamma b_j(0), D^\delta u_k(0)). \tag{3.30}$$
Here is $D_n = \partial/\partial x_n$ and $\alpha,\ \beta,\ \gamma,\ \delta$ satisfy the inequalities
$$|\beta|,\ |\gamma| \le |\alpha|, \quad |\delta| \le |\alpha| + 1,$$
and, which is essential in the proof, the last coordinates in the multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\delta = (\delta_1, \ldots, \delta_n)$ satisfy $\delta_n \le \alpha_n$ since the right hand side of (3.28) is independent of x_n. Moreover, it follows from (3.28) that the polynomials P_α have integers as coefficients. The initial condition (3.29) implies
$$D^\alpha u_j(0) = 0, \tag{3.31}$$
where $\alpha = (\alpha_1, \ldots, \alpha_{n-1}, 0)$, that is, $\alpha_n = 0$. Then, the proof is by induction with respect to α_n. The induction starts with $\alpha_n = 0$, then we replace $D^\delta u_k(0)$ in the right hand side of (3.30) by (3.31), that is by zero. Then it follows from (3.30) that
$$D^\alpha u_j(0) = P_\alpha(d^\beta a^i_{jk}(0), d^\gamma b_j(0), D^\delta u_k(0)),$$
where $\alpha = (\alpha_1, \ldots, \alpha_{n-1}, 1)$. □

Definition. Let $f = (f_1, \ldots, f_m)$, $F = (F_1, \ldots, F_m)$, $f_i = f_i(x)$, $F_i = F_i(x)$, and $f,\ F \in C^\infty$. We say f is *majorized* by F if
$$|D^\alpha f_k(0)| \le D^\alpha F_k(0), \quad k = 1, \ldots, m$$

for all α. We write $f << F$, if f is majorized by F.

Definition. The initial value problem

$$
\begin{aligned}
U_{j,x_n} &= \sum_{i=1}^{n-1}\sum_{k=1}^{N} A^i_{jk}(z)U_{k,x_i} + B_j(z) && (3.32)\\
U_j(x) &= 0 \qquad \text{if } x_n = 0, && (3.33)
\end{aligned}
$$

$j = 1, \ldots, N$, A^i_{jk}, B_j real analytic, ist called *majorizing problem* to (3.28), (3.29) if

$$a^i_{jk} << A^i_{jk} \ \text{ and } b_j << B_j.$$

Lemma B. *The formal power series*

$$\sum_\alpha \frac{1}{\alpha!} D^\alpha u_j(0)x^\alpha,$$

where $D^\alpha u_j(0)$ are defined in Lemma A, is convergent in a neighbourhood of $0 \in \mathbb{R}^n$ if there exists a majorizing problem which has a real analytic solution U in $x = 0$, and

$$|D^\alpha u_j(0)| \le D^\alpha U_j(0).$$

Proof. It follows from Lemma A and from the assumption of Lemma B that

$$
\begin{aligned}
|D^\alpha u_j(0)| &\le P_\alpha\left(|d^\beta a^i_{jk}(0)|, |d^\gamma b_j(0)|\right)\\
&\le P_\alpha\left(|d^\beta A^i_{jk}(0)|, |d^\gamma B_j(0)|\right) \equiv D^\alpha U_j(0).
\end{aligned}
$$

The formal power series

$$\sum_\alpha \frac{1}{\alpha!} D^\alpha u_j(0)x^\alpha,$$

is convergent since

$$\sum_\alpha \frac{1}{\alpha!}|D^\alpha u_j(0)x^\alpha| \le \sum_\alpha \frac{1}{\alpha!} D^\alpha U_j(0)|x^\alpha|.$$

The right hand side is convergent in a neighbourhood of $x \in \mathbb{R}^n$ by assumption. □

Lemma C. *There is a majorising problem which has a real analytic solution.*

Proof. Since $a^i_{ij}(z)$, $b_j(z)$ are real analytic in a neighbourhood of $z = 0$ it follows from Proposition A5 of the appendix to this section that there are positive constants M and r such that all these functions are majorized by

$$\frac{Mr}{r - z_1 - \ldots - z_{N+n-1}}.$$

Thus a majorizing problem is

$$\begin{aligned} U_{j,x_n} &= \frac{Mr}{r - x_1 - \ldots - x_{n-1} - U_1 - \ldots - U_N} \left(1 + \sum_{i=1}^{n-1} \sum_{k=1}^{N} U_{k,x_i} \right) \\ U_j(x) &= 0 \ \text{ if } x_n = 0, \end{aligned}$$

$j = 1, \ldots, N$.

The solution of this problem is

$$U_j(x_1, \ldots, x_{n-1}, x_n) = V(x_1 + \ldots + x_{n-1}, x_n), \ j = 1, \ldots, N,$$

where $V(s,t)$, $s = x_1 + \ldots + x_{n-1}$, $t = x_n$, is the solution of the Cauchy initial value problem

$$\begin{aligned} V_t &= \frac{Mr}{r - s - NV} \left(1 + N(n-1)V_s\right), \\ V(s,0) &= 0. \end{aligned}$$

which has the solution, see an exercise,

$$V(s,t) = \frac{1}{Nn} \left(r - s - \sqrt{(r-s)^2 - 2nMNrt} \right).$$

This function is real analytic in (s,t) at $(0,0)$. It follows that $U_j(x)$ are also real analytic functions. Thus the Cauchy-Kovalevskaya theorem is shown. □

Examples:

1. Ordinary differential equations

Consider the initial value problem

$$\begin{aligned} y'(x) &= f(x, y(x)) \\ y(x_0) &= y_0, \end{aligned}$$

where $x_0 \in \mathbb{R}$ and $y_0 \in \mathbb{R}^n$ are given. Assume $f(x, y)$ is real analytic in a neighbourhood of $(x_0, y_0) \in \mathbb{R} \times \mathbb{R}^n$. Then it follows from the above theorem that there exists an analytic solution $y(x)$ of the initial value problem in a neighbourhood of x_0. This solution is unique in the class of analytic functions according to the theorem of Cauchy-Kovalevskaya. From the Picard-Lindelöf theorem it follows that this analytic solution is unique even in the class of C^1-functions.

2. Partial differential equations of second order

Consider the boundary value problem for two variables

$$\begin{aligned} u_{yy} &= f(x, y, u, u_x, u_y, u_{xx}, u_{xy}) \\ u(x, 0) &= \phi(x) \\ u_y(x, 0) &= \psi(x). \end{aligned}$$

We assume that ϕ, ψ are analytic in a neighbourhood of $x = 0$ and that f is real analytic in a neighbourhood of

$$(0, 0, \phi(0), \phi'(0), \psi(0), \psi'(0)).$$

There exists a real analytic solution in a neigbourhood of $0 \in \mathbb{R}^2$ *of the above initial value problem.*

In particular, there is a real analytic solution in a neigbourhood of $0 \in \mathbb{R}^2$ of the initial value problem

$$\begin{aligned} \triangle u &= 1 \\ u(x, 0) &= 0 \\ u_y(x, 0) &= 0. \end{aligned}$$

The proof follows by writing the above problem as a system. Set $p = u_x$, $q = u_y$, $r = u_{xx}$, $s = u_{xy}$, $t = u_{yy}$, then

$$t = f(x, y, u, p, q, r, s).$$

Set $U = (u, p, q, r, s, t)^T$, $b = (q, 0, t, 0, 0, f_y + f_u q + f_q t)^T$ and

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & f_p & 0 & f_r & f_s \end{pmatrix}.$$

Then the rewritten differential equation is the system $U_y = AU_x + b$ with the initial condition

$$U(x, 0) = \left(\phi(x), \phi'(x), \psi(x), \phi''(x), \psi'(x), f_0(x)\right),$$

where $f_0(x) = f(x, 0, \phi(x), \phi'(x), \psi(x), \phi''(x), \psi'(x))$.

3.5.1 Appendix: Real analytic functions

Multi-index notation

The following multi-index notation simplifies many presentations of formulas. Let $x = (x_1, \ldots, x_n)$ and

$$u: \ \Omega \subset \mathbb{R}^n \mapsto \mathbb{R} \ \ (\text{or } \mathbb{R}^m \text{ for systems}).$$

The n-tuple of nonnegative integers (including zero)

$$\alpha = (\alpha_1, \ldots, \alpha_n)$$

is called *multi-index*. Set

$$\begin{aligned} |\alpha| &= \alpha_1 + \ldots + \alpha_n \\ \alpha! &= \alpha_1!\alpha_2! \cdot \ldots \cdot \alpha_n! \\ x^\alpha &= x_1^{\alpha_1} x_2^{\alpha_2} \cdot \ldots \cdot x_n^{\alpha_n} \quad (\text{for a monom}) \\ D_k &= \frac{\partial}{\partial x_k} \\ D &= (D_1, \ldots, D_n) \\ Du &= (D_1 u, \ldots, D_n u) \equiv \nabla u \equiv grad\ u \\ D^\alpha &= D_1^{\alpha_1} D_2^{\alpha_2} \cdot \ldots \cdot D_n^{\alpha_n} \equiv \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}}. \end{aligned}$$

Define a partial order by

$$\alpha \geq \beta \ \text{ if and only if } \ \alpha_i \geq \beta_i \ \text{ for all } i.$$

Sometimes we use the notations

$$\mathbf{0} = (0, 0 \ldots, 0), \quad \mathbf{1} = (1, 1 \ldots, 1),$$

where $\mathbf{0},\ \mathbf{1} \in \mathbb{R}^n$.

Using this multi-index notion, we have

1.

$$(x+y)^\alpha = \sum_{\substack{\beta, \gamma \\ \beta + \gamma = \alpha}} \frac{\alpha!}{\beta!\gamma!} x^\beta y^\gamma,$$

where $x,\ y \in \mathbb{R}^n$ and $\alpha,\ \beta,\ \gamma$ are multi-indices.

2. Taylor expansion for a *polynomial* $f(x)$ of degree m:

$$f(x) = \sum_{|\alpha| \leq m} \frac{1}{\alpha!} \left(D^\alpha f(0) \right) x^\alpha,$$

here is $D^\alpha f(0) := (D^\alpha f(x))\,|_{x=0}$.

3. Let $x = (x_1, \ldots, x_n)$ and $m \geq 0$ an integer, then

$$(x_1 + \ldots + x_n)^m = \sum_{|\alpha| = m} \frac{m!}{\alpha!} x^\alpha.$$

4.

$$\alpha! \leq |\alpha|! \leq n^{|\alpha|} \alpha!.$$

5. Leibniz's rule:

$$D^\alpha (fg) = \sum_{\substack{\beta, \gamma \\ \beta + \gamma = \alpha}} \frac{\alpha!}{\beta!\gamma!} (D^\beta f)(D^\gamma g).$$

6.

$$\begin{aligned} D^\beta x^\alpha &= \frac{\alpha!}{(\alpha-\beta)!} x^{\alpha-\beta} \ \text{ if } \alpha \geq \beta, \\ D^\beta x^\alpha &= 0 \ \text{ otherwise.} \end{aligned}$$

7. Directional derivative:

$$\frac{d^m}{dt^m} f(x+ty) = \sum_{|\alpha|=m} \frac{|\alpha|!}{\alpha!} \left(D^\alpha f(x+ty) \right) y^\alpha,$$

where $x,\ y \in \mathbb{R}^n$ and $t \in \mathbb{R}$.

8. Taylor's theorem: Let $u \in C^{m+1}$ in a neighbourhood $N(y)$ of y, then, if $x \in N(y)$,

$$u(x) = \sum_{|\alpha| \leq m} \frac{1}{\alpha!} \left(D^\alpha u(y) \right) (x-y)^\alpha + R_m,$$

where

$$R_m = \sum_{|\alpha|=m+1} \frac{1}{\alpha!} \left(D^\alpha u(y + \delta(x-y)) \right) x^\alpha,\ 0 < \delta < 1,$$

$\delta = \delta(u, m, x, y)$, or

$$R_m = \frac{1}{m!} \int_0^1 (1-t)^m \Phi^{(m+1)}(t)\ dt,$$

where $\Phi(t) = u(y + t(x-y))$. It follows from **7.** that

$$R_m = (m+1) \sum_{|\alpha|=m+1} \frac{1}{\alpha!} \left(\int_0^1 (1-t) D^\alpha u(y + t(x-y))\ dt \right) (x-y)^\alpha.$$

9. Using multi-index notation, the general linear partial differential equation of order m can be written as

$$\sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha u = f(x) \ \text{ in } \Omega \subset \mathbb{R}^n.$$

Power series

Here we collect some definitions and results for power series in $\mathbb{R}^n$.

Definition. Let $c_\alpha \in \mathbb{R}$ (or $\in \mathbb{R}^m$). The series

$$\sum_\alpha c_\alpha \equiv \sum_{m=0}^{\infty}\left(\sum_{|\alpha|=m} c_\alpha\right)$$

is said to be convergent if

$$\sum_\alpha |c_\alpha| \equiv \sum_{m=0}^{\infty}\left(\sum_{|\alpha|=m} |c_\alpha|\right)$$

is convergent.

Remark. According to the above definition, a convergent series is absolutely convergent. It follows that we can rearrange the order of summation.

Using the above multi-index notation and keeping in mind that we can rearrange convergent series, we have

10. Let $x \in \mathbb{R}^n$, then

$$\begin{aligned}\sum_\alpha x^\alpha &= \prod_{i=1}^{n}\left(\sum_{\alpha_i=0}^{\infty} x_i^{\alpha_i}\right)\\ &= \frac{1}{(1-x_1)(1-x_2)\cdot\ldots\cdot(1-x_n)}\\ &= \frac{1}{(\mathbf{1}-x)^{\mathbf{1}}},\end{aligned}$$

provided $|x_i| < 1$ is satisfied for each i.

11. Assume $x \in \mathbb{R}^n$ and $|x_1| + |x_2| + \ldots + |x_n| < 1$, then

$$\begin{aligned}\sum_\alpha \frac{|\alpha|!}{\alpha!}x^\alpha &= \sum_{j=0}^{\infty}\sum_{|\alpha|=j}\frac{|\alpha|!}{\alpha!}x^\alpha\\ &= \sum_{j=0}^{\infty}(x_1+\ldots+x_n)^j\\ &= \frac{1}{1-(x_1+\ldots+x_n)}.\end{aligned}$$

12. Let $x \in \mathbb{R}^n$, $|x_i| < 1$ for all i, and β is a given multi-index. Then

$$\begin{aligned}\sum_{\alpha \geq \beta} \frac{\alpha!}{(\alpha-\beta)!} x^{\alpha-\beta} &= D^\beta \frac{1}{(\mathbf{1}-x)^{\mathbf{1}}} \\ &= \frac{\beta!}{(\mathbf{1}-x)^{\mathbf{1}+\beta}} .\end{aligned}$$

13. Let $x \in \mathbb{R}^n$ and $|x_1| + \ldots + |x_n| < 1$. Then

$$\begin{aligned}\sum_{\alpha \geq \beta} \frac{|\alpha|!}{(\alpha-\beta)!} x^{\alpha-\beta} &= D^\beta \frac{1}{1 - x_1 - \ldots - x_n} \\ &= \frac{|\beta|!}{(1 - x_1 - \ldots - x_n)^{1+|\beta|}} .\end{aligned}$$

Consider the power series

$$\sum_\alpha c_\alpha x^\alpha \tag{3.34}$$

and assume this series is convergent for a $z \in \mathbb{R}^n$. Then, by definition,

$$\mu := \sum_\alpha |c_\alpha||z^\alpha| < \infty$$

and the series (3.34) is uniformly convergent for all $x \in Q(z)$, where

$$Q(z): \quad |x_i| \leq |z_i| \ \text{ for all } \ i.$$

Thus the power series (3.34) defines a continuous function defined on $Q(z)$, according to a theorem of Weierstrass.

The interior of $Q(z)$ is not empty if and only if $z_i \neq 0$ for all i, see Figure 3.7. For given x in a fixed compact subset D of $Q(z)$ there is a $q,\ 0 < q < 1$, such that

$$|x_i| \leq q|z_i| \ \text{ for all } i.$$

Set

$$f(x) = \sum_\alpha c_\alpha x^\alpha.$$

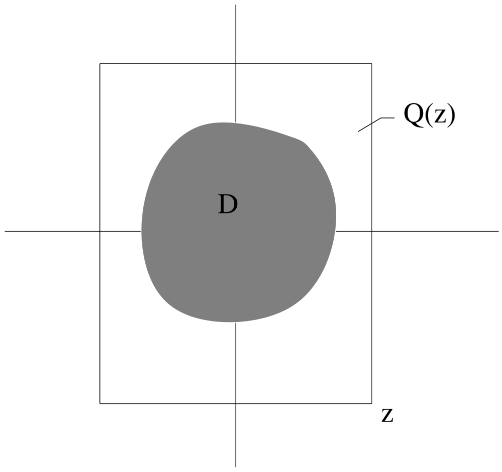

Figure 3.7: Definition of $D \in Q(z)$

Proposition A1. (i) *In every compact subset D of $Q(z)$ one has $f \in C^\infty(D)$ and the formal differentiate series, that is $\sum_\alpha D^\beta c_\alpha x^\alpha$, is uniformly convergent on the closure of D and is equal to $D^\beta f$.*

(ii)

$$|D^\beta f(x)| \le M|\beta|!r^{-|\beta|} \quad \textit{in } D,$$

where

$$M = \frac{\mu}{(1-q)^n}, \qquad\qquad r = (1-q)\min_i |z_i|.$$

Proof. See F. John [10], p. 64. Or an exercise. Hint: Use formula **12.** where x is replaced by $(q, \ldots, q)$.

Remark. From the proposition above it follows

$$c_\alpha = \frac{1}{\alpha!} D^\alpha f(0).$$

Definition. Assume f is defined on a domain $\Omega \subset \mathbb{R}^n$, then f is said to be *real analytic in* $y \in \Omega$ if there are $c_\alpha \in \mathbb{R}$ and if there is a neighbourhood $N(y)$ of y such that

$$f(x) = \sum_\alpha c_\alpha (x - y)^\alpha$$

for all $x \in N(y)$, and the series converges (absolutely) for each $x \in N(y)$. A function f is called *real analytic in* Ω if it is real analytic for each $y \in \Omega$. We will write $f \in C^\omega(\Omega)$ in the case that f is real analytic in the domain Ω. A vector valued function $f(x) = (f_1(x), \ldots, f_m)$ is called real analytic if each coordinate is real analytic.

Proposition A2. (i) *Let $f \in C^\omega(\Omega)$. Then $f \in C^\infty(\Omega)$.*

(ii) *Assume $f \in C^\omega(\Omega)$. Then for each $y \in \Omega$ there exists a neighbourhood $N(y)$ and positive constants M, r such that*

$$f(x) = \sum_\alpha \frac{1}{\alpha!}(D^\alpha f(y))(x-y)^\alpha$$

for all $x \in N(y)$, and the series converges (absolutely) for each $x \in N(y)$, and

$$|D^\beta f(x)| \leq M|\beta|!r^{-|\beta|}.$$

The proof follows from Proposition A1.

An open set $\Omega \in \mathbb{R}^n$ is called *connected* if Ω is not a union of two nonempty open sets with empty intersection. An open set $\Omega \in \mathbb{R}^n$ is connected if and only if its path connected, see [11], pp. 38, for example. We say that Ω is *path connected* if for any $x,\ y \in \Omega$ there is a continuous curve $\gamma(t) \in \Omega$, $0 \leq t \leq 1$, with $\gamma(0) = x$ and $\gamma(1) = y$. From the theory of one complex variable we know that a continuation of an analytic function is uniquely determined. The same is true for real analytic functions.

Proposition A3. Assume $f \in C^\omega(\Omega)$ and Ω is connected. Then f is uniquely determined if for one $z \in \Omega$ all $D^\alpha f(z)$ are known.

Proof. See F. John [10], p. 65. Suppose $g,\ h \in C^\omega(\Omega)$ and $D^\alpha g(z) = D^\alpha h(z)$ for every α. Set $f = g - h$ and

$$\begin{aligned}\Omega_1 &= \{x \in \Omega :\ D^\alpha f(x) = 0 \ \text{ for all } \alpha\},\\ \Omega_2 &= \{x \in \Omega :\ D^\alpha f(x) \neq 0 \ \text{ for at least one } \alpha\}.\end{aligned}$$

The set Ω_2 is open since $D^\alpha f$ are continuous in Ω. The set Ω_1 is also open since $f(x) = 0$ in a neighbourhood of $y \in \Omega_1$. This follows from

$$f(x) = \sum_\alpha \frac{1}{\alpha!}(D^\alpha f(y))(x-y)^\alpha.$$

Since $z \in \Omega_1$, i. e., $\Omega_1 \neq \emptyset$, it follows $\Omega_2 = \emptyset$. □

It was shown in Proposition A2 that derivatives of a real analytic function satisfy estimates. On the other hand it follows, see the next proposition, that a function $f \in C^\infty$ is real analytic if these estimates are satisfied.

Definition. Let $y \in \Omega$ and M, r positive constants. Then f is said to be in the class $C_{M,r}(y)$ if $f \in C^\infty$ in a neighbourhood of y and if

$$|D^\beta f(y)| \leq M|\beta|!r^{-|\beta|}$$

for all β.

Proposition A4. *$f \in C^\omega(\Omega)$ if and only if $f \in C^\infty(\Omega)$ and for every compact subset $S \subset \Omega$ there are positive constants M, r such that*

$$f \in C_{M,r}(y) \quad \text{for all } y \in S.$$

Proof. See F. John [10], pp. 65-66. We will prove the local version of the proposition, that is, we show it for each fixed $y \in \Omega$. The general version follows from Heine-Borel theorem. Because of Proposition A3 it remains to show that the Taylor series

$$\sum_\alpha \frac{1}{\alpha!} D^\alpha f(y)(x-y)^\alpha$$

converges (absolutely) in a neighbourhood of y and that this series is equal to $f(x)$.

Define a neighbourhood of y by

$$N_d(y) = \{x \in \Omega : \quad |x_1 - y_1| + \ldots + |x_n - y_n| < d\},$$

where d is a sufficiently small positive constant. Set $\Phi(t) = f(y + t(x-y))$. The one-dimensional Taylor theorem says

$$f(x) = \Phi(1) = \sum_{k=0}^{j-1} \frac{1}{k!}\Phi^{(k)}(0) + r_j,$$

where

$$r_j = \frac{1}{(j-1)!}\int_0^1 (1-t)^{j-1}\Phi^{(j)}(t)\ dt.$$

From formula **7.** for directional derivatives it follows for $x \in N_d(y)$ that

$$\frac{1}{j!}\frac{d^j}{dt^j}\Phi(t) = \sum_{|\alpha|=j} \frac{1}{\alpha!} D^\alpha f(y + t(x-y))(x-y)^\alpha .$$

From the assumption and the multinomial formula **3.** we get for $0 \le t \le 1$

$$\begin{aligned} \left| \frac{1}{j!}\frac{d^j}{dt^j}\Phi(t) \right| &\le M \sum_{|\alpha|=j} \frac{|\alpha|!}{\alpha!} r^{-|\alpha|} \left| (x-y)^\alpha \right| \\ &= M r^{-j} \left(|x_1 - y_1| + \ldots + |x_n - y_n| \right)^j \\ &\le M \left(\frac{d}{r} \right)^j . \end{aligned}$$

Choose $d > 0$ such that $d < r$, then the Taylor series converges (absolutely) in $N_d(y)$ and it is equal to $f(x)$ since the remainder satisfies, see the above estimate,

$$|r_j| = \left| \frac{1}{(j-1)!} \int_0^1 (1-t)^{j-1} \Phi^j(t)\ dt \right| \le M \left(\frac{d}{r} \right)^j .$$

$\square$

We recall that the notation $f << F$ (f is majorized by F) was defined in the previous section.

Proposition A5. (i) $f = (f_1, \ldots, f_m) \in C_{M,r}(0)$ *if and only if* $f << (\Phi, \ldots, \Phi)$, *where*

$$\Phi(x) = \frac{Mr}{r - x_1 - \ldots - x_n} .$$

(ii) $f \in C_{M,r}(0)$ *and* $f(0) = 0$ *if and only if*

$$f << (\Phi - M, \ldots, \Phi - M),$$

where

$$\Phi(x) = \frac{M(x_1 + \ldots + x_n)}{r - x_1 - \ldots - x_n} .$$

Proof.

$$D^\alpha \Phi(0) = M|\alpha|! r^{-|\alpha|} .$$

□

Remark. The definition of $f << F$ implies, trivially, that $D^\alpha f << D^\alpha F$.

The next proposition shows that compositions majorize if the involved functions majorize. More precisely, we have

Proposition A6. *Let $f,\ F:\ \mathbb{R}^n \mapsto \mathbb{R}^m$ and $g,\ G$ maps a neighbourhood of $0 \in \mathbb{R}^m$ into $\mathbb{R}^p$. Assume all functions $f(x),\ F(x),\ g(u),\ G(u)$ are in C^∞, $f(0) = F(0) = 0$, $f << F$ and $g << G$. Then $g(f(x)) << G(F(x))$.*

Proof. See F. John [10], p. 68. Set

$$h(x) = g(f(x)), \quad H(x) = G(F(x)).$$

For each coordinate h_k of h we have, according to the chain rule,

$$D^\alpha h_k(0) = P_\alpha(\delta^\beta g_l(0), D^\gamma f_j(0)),$$

where P_α are polynomials with *nonnegative* integers as coefficients, P_α are independent on g or f and $\delta := (\partial/\partial u_1, \ldots, \partial/\partial u_m)$. Thus,

$$\begin{aligned} |D^\alpha h_k(0)| &\le P_\alpha(|\delta^\beta g_l(0)|, |D^\gamma f_j(0)|) \\ &\le P_\alpha(\delta^\beta G_l(0), D^\gamma F_j(0)) \\ &= D^\alpha H_k(0). \end{aligned}$$

□

Using this result and Proposition A4, which characterizes real analytic functions, it follows that compositions of real analytic functions are real analytic functions again.

Proposition A7. *Assume $f(x)$ and $g(u)$ are real analytic, then $g(f(x))$ is real analytic if $f(x)$ is in the domain of definition of g.*

Proof. See F. John [10], p. 68. Assume that f maps a neighbourhood of $y \in \mathbb{R}^n$ in $\mathbb{R}^m$ and g maps a neighbourhood of $v = f(y)$ in $\mathbb{R}^m$. Then $f \in C_{M,r}(y)$ and $g \in C_{\mu,\rho}(v)$ implies

$$h(x) := g(f(x)) \in C_{\mu,\rho r/(mM+\rho)}(y).$$

Once one has shown this inclusion, the proposition follows from Proposition A4. To show the inclusion, we set

$$h(y + x) := g(f(y + x)) \equiv g(v + f(y + x) - f(x)) =: g^*(f^*(x)),$$

where $v = f(y)$ and

$$\begin{aligned} g^*(u): &= g(v+u) \in C_{\mu,\rho}(0) \\ f^*(x): &= f(y+x) - f(y) \in C_{M,r}(0). \end{aligned}$$

In the above formulas $v,\ y$ are considered as fixed parameters. From Proposition A5 it follows

$$\begin{aligned} f^*(x) &<< (\Phi - M, \ldots, \Phi - M) =: F \\ g^*(u) &<< (\Psi, \ldots, \Psi) =: G, \end{aligned}$$

where

$$\begin{aligned} \Phi(x) &= \frac{Mr}{r - x_1 - x_2 - \ldots - x_n} \\ \Psi(u) &= \frac{\mu\rho}{\rho - x_1 - x_2 - \ldots - x_n}. \end{aligned}$$

From Proposition A6 we get

$$h(y+x) << (\chi(x), \ldots, \chi(x)) \equiv G(F),$$

where

$$\begin{aligned} \chi(x) &= \frac{\mu\rho}{\rho - m(\Phi(x) - M)} \\ &= \frac{\mu\rho(r - x_1 - \ldots - x_n)}{\rho r - (\rho + mM)(x_1 + \ldots + x_n)} \\ &<< \frac{\mu\rho r}{\rho r - (\rho + mM)(x_1 + \ldots + x_n)} \\ &= \frac{\mu\rho r/(\rho + mM)}{\rho r/(\rho + mM) - (x_1 + \ldots x_n)}. \end{aligned}$$

See an exercise for the "<<"-inequality. □

3.6 Exercises

1. Let χ: $\mathbb{R}^n \to \mathbb{R}$ in C^1, $\nabla\chi \neq 0$. Show that for given $x_0 \in \mathbb{R}^n$ there is in a neighbourhood of x_0 a local diffeomorphism $\lambda = \Phi(x)$, Φ : $(x_1, \ldots, x_n) \mapsto (\lambda_1, \ldots, \lambda_n)$, such that $\lambda_n = \chi(x)$.

2. Show that the differential equation

$$a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} + \text{lower order terms} = 0$$

   is elliptic if $ac - b^2 > 0$, parabolic if $ac - b^2 = 0$ and hyperbolic if $ac - b^2 < 0$.

3. Show that in the hyperbolic case there exists a solution of $\phi_x + \mu_1\phi_y = 0$, see equation (3.9), such that $\nabla\phi \neq 0$.

   Hint: Consider an appropriate Cauchy initial value problem.

4. Show equation (3.4).

5. Find the type of

$$Lu := 2u_{xx} + 2u_{xy} + 2u_{yy} = 0$$

   and transform this equation into an equation with vanishing mixed derivatives by using the orthogonal mapping (transform to principal axis) $x = Uy$, U orthogonal.

6. Determine the type of the following equation at $(x,y) = (1, 1/2)$.

$$Lu := xu_{xx} + 2yu_{xy} + 2xyu_{yy} = 0.$$

7. Find all C^2-solutions of

$$u_{xx} - 4u_{xy} + u_{yy} = 0.$$

   Hint: Transform to principal axis and stretching of axis lead to the wave equation.

8. Oscillations of a beam are described by

$$\begin{aligned} w_x - \frac{1}{E}\sigma_t &= 0 \\ \sigma_x - \rho w_t &= 0, \end{aligned}$$

where σ stresses, w deflection of the beam and $E,\ \rho$ are positive constants.
a) Determine the type of the system.
b) Transform the system into two uncoupled equations, that is, $w,\ \sigma$ occur only in one equation, respectively.
c) Find non-zero solutions.

9. Find nontrivial solutions ($\nabla\chi \neq 0$) of the characteristic equation to
$$x^2 u_{xx} - u_{yy} = f(x, y, u, \nabla u),$$
where f is given.

10. Determine the type of
$$u_{xx} - x u_{yx} + u_{yy} + 3u_x = 2x,$$
where $u = u(x, y)$.

11. Transform equation
$$u_{xx} + (1 - y^2)u_{xy} = 0,$$
$u = u(x, y)$, into its normal form.

12. Transform the Tricomi-equation
$$y u_{xx} + u_{yy} = 0,$$
$u = u(x, y)$, where $y < 0$, into its normal form.

13. Transform equation
$$x^2 u_{xx} - y^2 u_{yy} = 0,$$
$u = u(x, y)$, into its normal form.

14. Show that
$$\lambda = \frac{1}{(1 + |p|^2)^{3/2}}, \quad \Lambda = \frac{1}{(1 + |p|^2)^{1/2}}.$$
are the minimum and maximum of eigenvalues of the matrix (a^{ij}), where
$$a^{ij} = \left(1 + |p|^2\right)^{-1/2} \left(\delta_{ij} - \frac{p_i p_j}{1 + |p|^2}\right).$$

15. Show that Maxwell equations are a hyperbolic system.

16. Consider Maxwell equations and prove that div $E = 0$ and div $H = 0$ for all t if these equations are satisfied for a fixed time t_0.

    Hint. div rot $A = 0$ for each C^2-vector field $A = (A_1, A_2, A_3)$.

17. Assume a characteristic surface $\mathcal{S}(t)$ in $\mathbb{R}^3$ is defined by $\chi(x, y, z, t) =$ $const.$ such that $\chi_t = 0$ and $\chi_z \neq 0$. Show that $\mathcal{S}(t)$ has a nonparametric representation $z = u(x, y, t)$ with $u_t = 0$, that is $\mathcal{S}(t)$ is independent of t.

18. Prove formula (3.22) for the normal on a surface.

19. Prove formula (3.23) for the speed of the surface $\mathcal{S}(t)$.

20. Write the Navier-Stokes system as a system of type (3.24).

21. Show that the following system (linear elasticity, stationary case of (3.25) in the two dimensional case) is elliptic

    $$\mu\triangle u + (\lambda + \mu)\ \text{grad}(\text{div}\ u) + f = 0,$$

    where $u = (u_1, u_2)$. The vector $f = (f_1, f_2)$ is given and $\lambda,\ \mu$ are positive constants.

22. Discuss the type of the following system in stationary gas dynamics (isentrop flow) in $\mathbb{R}^2$.

    $$\begin{aligned} \rho u u_x + \rho v u_y + a^2 \rho_x &= 0 \\ \rho u v_x + \rho v v_y + a^2 \rho_y &= 0 \\ \rho(u_x + v_y) + u\rho_x + v\rho_y &= 0. \end{aligned}$$

    Here are (u, v) velocity vector, ρ density and $a = \sqrt{p'(\rho)}$ the sound velocity.

23. Show formula **7.** (directional derivative).

    Hint: Induction with respect to m.

24. Let $y = y(x)$ be the solution of:

    $$\begin{aligned} y'(x) &= f(x, y(x)) \\ y(x_0) &= y_0, \end{aligned}$$

where f is real analytic in a neighbourhood of $(x_0, y_0) \in \mathbb{R}^2$. Find the polynomial P of degree 2 such that

$$y(x) = P(x - x_0) + O(|x - x_0|^3)$$

as $x \to x_0$.

25. Let u be the solution of

$$\begin{aligned} \triangle u &= 1 \\ u(x,0) &= u_y(x,0) = 0. \end{aligned}$$

Find the polynomial P of degree 2 such that

$$u(x,y) = P(x,y) + O((x^2 + y^2)^{3/2})$$

as $(x,y) \to (0,0)$.

26. Solve the Cauchy initial value problem

$$\begin{aligned} V_t &= \frac{Mr}{r - s - NV}(1 + N(n-1)V_s) \\ V(s,0) &= 0. \end{aligned}$$

Hint: Multiply the differential equation with $(r - s - NV)$.

27. Write $\triangle^2 u = -u$ as a system of first order.
Hint: $\triangle^2 u \equiv \triangle(\triangle u)$.

28. Write the minimal surface equation

$$\frac{\partial}{\partial x}\left(\frac{u_x}{\sqrt{1 + u_x^2 + u_y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{u_y}{\sqrt{1 + u_x^2 + u_y^2}}\right) = 0$$

as a system of first order.
Hint: $v_1 := u_x/\sqrt{1 + u_x^2 + u_y^2}$, $v_2 := u_y/\sqrt{1 + u_x^2 + u_y^2}$.

29. Let $f:\ \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ be real analytic in (x_0, y_0). Show that a real analytic solution in a neighbourhood of x_0 of the problem

$$\begin{aligned} y'(x) &= f(x,y) \\ y(x_0) &= y_0 \end{aligned}$$

exists and is equal to the unique $C^1[x_0 - \epsilon, x_0 + \epsilon]$-solution from the Picard-Lindelöf theorem, $\epsilon > 0$ sufficiently small.

30. Show (see the proof of Proposition A7)

$$\frac{\mu\rho(r - x_1 - \ldots - x_n)}{\rho r - (\rho + mM)(x_1 + \ldots + x_n)} << \frac{\mu\rho r}{\rho r - (\rho + mM)(x_1 + \ldots + x_n)}.$$

Hint: Leibniz's rule.

Chapter 4

Hyperbolic equations

Here we consider hyperbolic equations of second order, mainly wave equations.

4.1 One-dimensional wave equation

The one-dimensional wave equation is given by

$$\frac{1}{c^2}u_{tt} - u_{xx} = 0, \tag{4.1}$$

where $u = u(x,t)$ is a scalar function of two variables and c is a positive constant. According to previous considerations, all C^2-solutions of the wave equation are

$$u(x,t) = f(x+ct) + g(x-ct), \tag{4.2}$$

with arbitrary C^2-functions f and g

The *Cauchy initial value problem* for the wave equation is to find a C^2-solution of

$$\begin{aligned}\frac{1}{c^2}u_{tt} - u_{xx} &= 0\\ u(x,0) &= \alpha(x)\\ u_t(x,0) &= \beta(x),\end{aligned}$$

where $\alpha,\ \beta \in C^2(-\infty,\infty)$ are given.

Theorem 4.1. *There exists a unique $C^2(\mathbb{R}\times\mathbb{R})$-solution of the Cauchy initial value problem, and this solution is given by d'Alembert's[1] formula*

$$u(x,t)=\frac{\alpha(x+ct)+\alpha(x-ct)}{2}+\frac{1}{2c}\int_{x-ct}^{x+ct}\beta(s)\ ds. \tag{4.3}$$

Proof. Assume there is a solution $u(x,t)$ of the Cauchy initial value problem, then it follows from (4.2) that

$$\begin{aligned} u(x,0) &= f(x)+g(x)=\alpha(x) &(4.4)\\ u_t(x,0) &= cf'(x)-cg'(x)=\beta(x). &(4.5)\end{aligned}$$

From (4.4) we obtain

$$f'(x)+g'(x)=\alpha'(x),$$

which implies, together with (4.5), that

$$\begin{aligned} f'(x) &= \frac{\alpha'(x)+\beta(x)/c}{2}\\ g'(x) &= \frac{\alpha'(x)-\beta(x)/c}{2}.\end{aligned}$$

Then

$$\begin{aligned} f(x) &= \frac{\alpha(x)}{2}+\frac{1}{2c}\int_0^x\beta(s)\ ds+C_1\\ g(x) &= \frac{\alpha(x)}{2}-\frac{1}{2c}\int_0^x\beta(s)\ ds+C_2.\end{aligned}$$

The constants C_1, C_2 satisfy

$$C_1+C_2=f(x)+g(x)-\alpha(x)=0,$$

see (4.4). Thus each C^2-solution of the Cauchy initial value problem is given by d'Alembert's formula. On the other hand, the function $u(x,t)$ defined by the right hand side of (4.3) is a solution of the initial value problem. □

Corollaries. 1. The solution $u(x,t)$ of the initial value problem depends on the values of α at the endpoints of the interval $[x-ct,x+ct]$ and on the values of β on this interval only, see Figure 4.1. The interval $[x-ct,x+ct]$ is called *domain of dependence*.

[1] d'Alembert, Jean Babtiste le Rond, 1717-1783

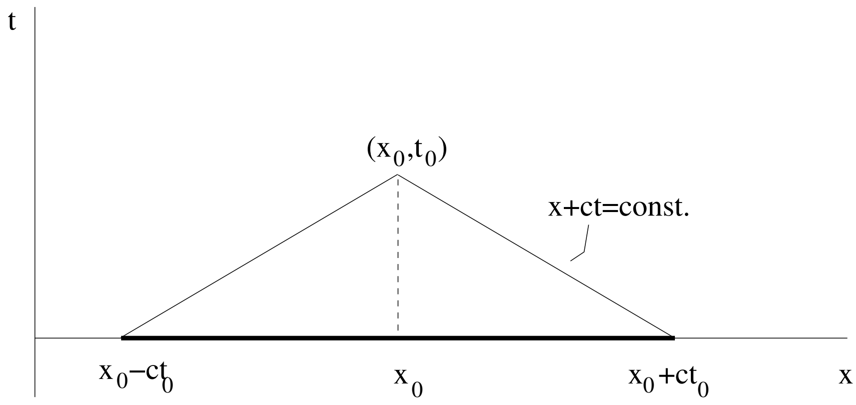

Figure 4.1: Interval of dependence

2. Let P be a point on the x-axis. Then we ask which points (x,t) need values of α or β at P in order to calculate $u(x,t)$? From the d'Alembert formula it follows that this domain is a cone, see Figure 4.2. This set is called *domain of influence*.

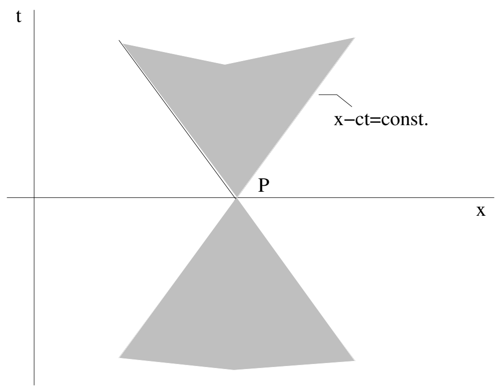

Figure 4.2: Domain of influence

4.2 Higher dimensions

Set

$$\Box u = u_{tt} - c^2 \triangle u, \quad \triangle \equiv \triangle_x = \partial^2/\partial x_1^2 + \ldots + \partial^2/\partial x_n^2,$$

and consider the initial value problem

$$\begin{aligned}\Box u &= 0 \quad \text{in } \mathbb{R}^n \times \mathbb{R} && (4.6)\\ u(x,0) &= f(x) && (4.7)\\ u_t(x,0) &= g(x), && (4.8)\end{aligned}$$

where f and g are given $C^2(\mathbb{R}^2)$-functions.

By using spherical means and the above d'Alembert formula we will derive a formula for the solution of this initial value problem.

Method of spherical means

Define the spherical mean for a C^2-solution $u(x,t)$ of the initial value problem by

$$M(r,t) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x)} u(y,t)\ dS_y, \tag{4.9}$$

where

$$\omega_n = (2\pi)^{n/2}/\Gamma(n/2)$$

is the area of the n-dimensional sphere, $\omega_n r^{n-1}$ is the area of a sphere with radius r.

From the mean value theorem of the integral calculus we obtain the function $u(x,t)$ for which we are looking at by

$$u(x,t) = \lim_{r\to 0} M(r,t). \tag{4.10}$$

Using the initial data, we have

$$\begin{aligned}M(r,0) &= \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x)} f(y)\ dS_y =: F(r) && (4.11)\\ M_t(r,0) &= \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x)} g(y)\ dS_y =: G(r), && (4.12)\end{aligned}$$

which are the spherical means of f and g.

The next step is to derive a partial differential equation for the spherical mean. From definition (4.9) of the spherical mean we obtain, after the mapping $\xi = (y-x)/r$, where x and r are fixed,

$$M(r,t) = \frac{1}{\omega_n} \int_{\partial B_1(0)} u(x + r\xi, t)\ dS_\xi.$$

It follows

$$\begin{aligned} M_r(r,t) &= \frac{1}{\omega_n}\int_{\partial B_1(0)}\sum_{i=1}^n u_{y_i}(x+r\xi,t)\xi_i\ dS_\xi \\ &= \frac{1}{\omega_n r^{n-1}}\int_{\partial B_r(x)}\sum_{i=1}^n u_{y_i}(y,t)\xi_i\ dS_y. \end{aligned}$$

Integration by parts yields

$$\frac{1}{\omega_n r^{n-1}}\int_{B_r(x)}\sum_{i=1}^n u_{y_iy_i}(y,t)\ dy$$

since $\xi \equiv (y-x)/r$ is the exterior normal at $\partial B_r(x)$. Assume u is a solution of the wave equation, then

$$\begin{aligned} r^{n-1}M_r &= \frac{1}{c^2\omega_n}\int_{B_r(x)} u_{tt}(y,t)\ dy \\ &= \frac{1}{c^2\omega_n}\int_0^r\int_{\partial B_c(x)} u_{tt}(y,t)\ dS_y dc. \end{aligned}$$

The previous equation follows by using spherical coordinates. Consequently

$$\begin{aligned} (r^{n-1}M_r)_r &= \frac{1}{c^2\omega_n}\int_{\partial B_r(x)} u_{tt}(y,t)\ dS_y \\ &= \frac{r^{n-1}}{c^2}\frac{\partial^2}{\partial t^2}\left(\frac{1}{\omega_n r^{n-1}}\int_{\partial B_r(x)} u(y,t)\ dS_y\right) \\ &= \frac{r^{n-1}}{c^2}M_{tt}. \end{aligned}$$

Thus we arrive at the differential equation

$$(r^{n-1}M_r)_r = c^{-2}r^{n-1}M_{tt},$$

which can be written as

$$M_{rr}+\frac{n-1}{r}M_r = c^{-2}M_{tt}. \tag{4.13}$$

This equation (4.13) is called *Euler-Poisson-Darboux equation.*

4.2.1 Case n=3

The Euler-Poisson-Darboux equation in this case is

$$(rM)_{rr} = c^{-2}(rM)_{tt}.$$

Thus rM is the solution of the one-dimensional wave equation with initial data

$$(rM)(r,0) = rF(r) \quad (rM)_t(r,0) = rG(r). \tag{4.14}$$

From the d'Alembert formula we get formally

$$\begin{aligned} M(r,t) &= \frac{(r+ct)F(r+ct)+(r-ct)F(r-ct)}{2r} \\ &\quad +\frac{1}{2cr}\int_{r-ct}^{r+ct} \xi G(\xi)\ d\xi. \end{aligned} \tag{4.15}$$

The right hand side of the previous formula is well defined if the domain of dependence $[x-ct, x+ct]$ is a subset of $(0,\infty)$. We can extend F and G to F_0 and G_0 which are defined on $(-\infty,\infty)$ such that rF_0 and rG_0 are $C^2(\mathbb{R})$-functions as follows. Set

$$F_0(r) = \begin{cases} F(r) & : & r > 0 \\ f(x) & : & r = 0 \\ F(-r) & : & r < 0 \end{cases}$$

The function $G_0(r)$ is given by the same definition where F and f are replaced by G and g, respectively.

Lemma. $rF_0(r),\ rG_0(r) \in C^2(\mathbb{R}^2)$.

Proof. From definition of $F(r)$ and $G(r)$, $r > 0$, it follows from the mean value theorem

$$\lim_{r\to+0} F(r) = f(x), \quad \lim_{r\to+0} G(r) = g(x).$$

Thus $rF_0(r)$ and $rG_0(r)$ are $C(\mathbb{R})$-functions. These functions are also in

$C^1(\mathbb{R})$. This follows since F_0 and G_0 are in $C^1(\mathbb{R})$. We have, for example,

$$\begin{aligned} F'(r) &= \frac{1}{\omega_n}\int_{\partial B_1(0)} \sum_{j=1}^n f_{y_j}(x+r\xi)\xi_j\ dS_\xi \\ F'(+0) &= \frac{1}{\omega_n}\int_{\partial B_1(0)} \sum_{j=1}^n f_{y_j}(x)\xi_j\ dS_\xi \\ &= \frac{1}{\omega_n}\sum_{j=1}^n f_{y_j}(x)\int_{\partial B_1(0)} n_j\ dS_\xi \\ &= 0. \end{aligned}$$

Then, $rF_0(r)$ and $rG_0(r)$ are in $C^2(\mathbb{R})$, provided F'' and G'' are bounded as $r \to +0$. This property follows from

$$F''(r) = \frac{1}{\omega_n}\int_{\partial B_1(0)} \sum_{i,j=1}^n f_{y_iy_j}(x+r\xi)\xi_i\xi_j\ dS_\xi.$$

Thus

$$F''(+0) = \frac{1}{\omega_n}\sum_{i,j=1}^n f_{y_iy_j}(x)\int_{\partial B_1(0)} n_in_j\ dS_\xi.$$

We recall that $f, g \in C^2(\mathbb{R}^2)$ by assumption. □

The solution of the above initial value problem, where F and G are replaced by F_0 and G_0, respectively, is

$$\begin{aligned} M_0(r,t) &= \frac{(r+ct)F_0(r+ct)+(r-ct)F_0(r-ct)}{2r} \\ &\quad +\frac{1}{2cr}\int_{r-ct}^{r+ct} \xi G_0(\xi)\ d\xi. \end{aligned}$$

Since F_0 and G_0 are even functions, we have

$$\int_{r-ct}^{ct-r} \xi G_0(\xi)\ d\xi = 0.$$

Thus

$$\begin{aligned} M_0(r,t) &= \frac{(r+ct)F_0(r+ct)-(ct-r)F_0(ct-r)}{2r} \\ &\quad +\frac{1}{2cr}\int_{ct-r}^{ct+r} \xi G_0(\xi)\ d\xi, \end{aligned} \tag{4.16}$$

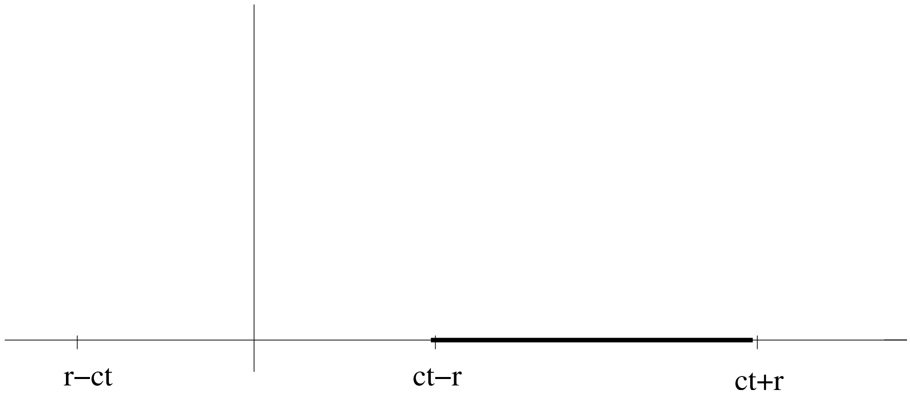

Figure 4.3: Changed domain of integration

see Figure 4.3. For fixed $t > 0$ and $0 < r < ct$ it follows that $M_0(r,t)$ is the solution of the initial value problem with given initially data (4.14) since $F_0(s) = F(s)$, $G_0(s) = G(s)$ if $s > 0$. Since for fixed $t > 0$

$$u(x,t) = \lim_{r\to 0} M_0(r,t),$$

it follows from d'Hospital's rule that

$$\begin{aligned} u(x,t) &= ctF'(ct) + F(ct) + tG(ct) \\ &= \frac{d}{dt}\left(tF(ct)\right) + tG(ct). \end{aligned}$$

Theorem 4.2. *Assume $f \in C^3(\mathbb{R}^3)$ and $g \in C^2(\mathbb{R}^3)$ are given. Then there exists a unique solution $u \in C^2(\mathbb{R}^3 \times [0,\infty))$ of the initial value problem (4.6)-(4.7), where $n = 3$, and the solution is given by the Poisson's formula*

$$\begin{aligned} u(x,t) &= \frac{1}{4\pi c^2}\frac{\partial}{\partial t}\left(\frac{1}{t}\int_{\partial B_{ct}(x)} f(y)\ dS_y\right) \\ &\quad + \frac{1}{4\pi c^2 t}\int_{\partial B_{ct}(x)} g(y)\ dS_y. \end{aligned} \tag{4.17}$$

Proof. Above we have shown that a C^2-solution is given by Poisson's formula. Under the additional assumption $f \in C^3$ it follows from Poisson's formula that this formula defines a solution which is in C^2, see F. John [10], p. 129. □

Corollary. From Poisson's formula we see that the domain of dependence for $u(x,t_0)$ is the intersection of the cone defined by $|y-x|=c|t-t_0|$ with the hyperplane defined by $t=0$, see Figure 4.4

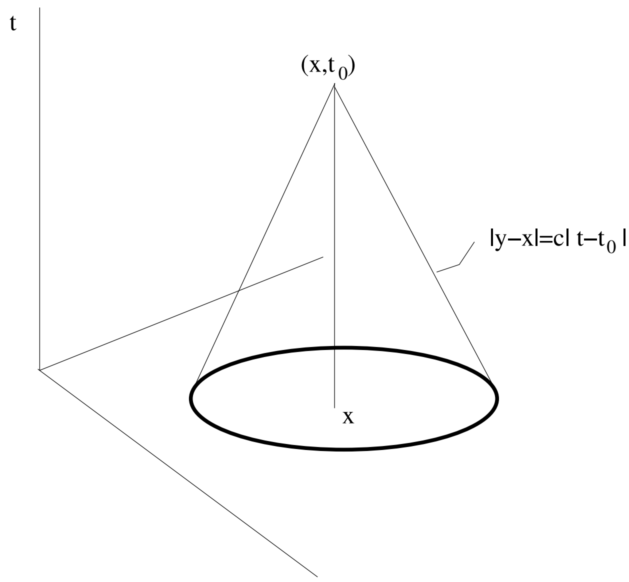

Figure 4.4: Domain of dependence, case $n=3$

4.2.2 Case $n=2$

Consider the initial value problem

$$
\begin{aligned}
v_{xx}+v_{yy} &= c^{-2}v_{tt} &(4.18)\\
v(x,y,0) &= f(x,y) &(4.19)\\
v_t(x,y,0) &= g(x,y), &(4.20)
\end{aligned}
$$

where $f\in C^3,\ g\in C^2$.

Using the formula for the solution of the three-dimensional initial value problem we will derive a formula for the two-dimensional case. The following consideration is called *Hadamard's method of decent.*

Let $v(x,y,t)$ be a solution of (4.18)-(4.20), then

$$u(x,y,z,t):=v(x,y,t)$$

is a solution of the three-dimensional initial value problem with initial data $f(x,y)$, $g(x,y)$, independent of z, since u satisfies (4.18)-(4.20). Hence, since $u(x,y,z,t) = u(x,y,0,t) + u_z(x,y,\delta z,t)z$, $0<\delta<1$, and $u_z = 0$, we have

$$v(x,y,t) = u(x,y,0,t).$$

Poisson's formula in the three-dimensional case implies

$$\begin{aligned} v(x,y,t) &= \frac{1}{4\pi c^2}\frac{\partial}{\partial t}\left(\frac{1}{t}\int_{\partial B_{ct}(x,y,0)} f(\xi,\eta)\ dS\right) \\ &\quad +\frac{1}{4\pi c^2 t}\int_{\partial B_{ct}(x,y,0)} g(\xi,\eta)\ dS. \qquad (4.21) \end{aligned}$$

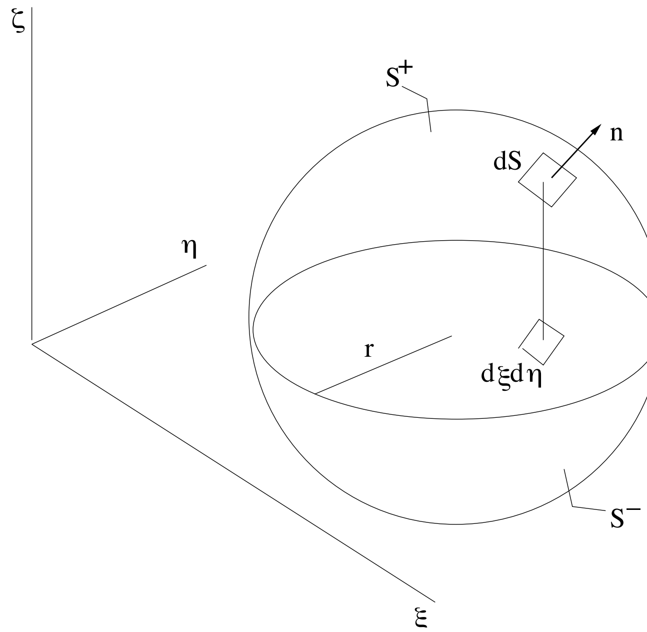

Figure 4.5: Domains of integration

The integrands are independent on ζ. The surface S is defined by $\chi(\xi,\eta,\zeta) := (\xi-x)^2+(\eta-y)^2+\zeta^2-c^2t^2 = 0$. Then the exterior normal n at S is $n = \nabla\chi/|\nabla\chi|$ and the surface element is given by $dS = (1/|n_3|)d\xi d\eta$, where the third coordinate of n is

$$n_3 = \pm\frac{\sqrt{c^2t^2-(\xi-x)^2-(\eta-y)^2}}{ct}.$$

The positive sign applies on S^+, where $\zeta > 0$ and the sign is negative on S^- where $\zeta < 0$, see Figure 4.5. We have $S = S^+ \cup \overline{S^-}$.

Set $\rho = \sqrt{(\xi - x)^2 + (\eta - y)^2}$. Then it follows from (4.21)

Theorem 4.3. *The solution of the Cauchy initial value problem (4.18)-(4.20) is given by*

$$\begin{aligned} v(x, y, t) &= \frac{1}{2\pi c}\frac{\partial}{\partial t}\int_{B_{ct}(x,y)} \frac{f(\xi, \eta)}{\sqrt{c^2t^2 - \rho^2}}\, d\xi d\eta \\ &\quad + \frac{1}{2\pi c}\int_{B_{ct}(x,y)} \frac{g(\xi, \eta)}{\sqrt{c^2t^2 - \rho^2}}\, d\xi d\eta. \end{aligned}$$

Corollary. In contrast to the three dimensional case, the domain of dependence is here the disk $B_{ct_o}(x_0, y_0)$ and not the boundary only. Therefore, see formula of Theorem 4.3, if $f,\ g$ have supports in a compact domain $D \subset \mathbb{R}^2$, then these functions have influence on the value $v(x, y, t)$ for *all* time $t > T$, T sufficiently large.

4.3 Inhomogeneous equation

Here we consider the initial value problem

$$\begin{aligned} \Box u &= w(x, t) \ \text{ on } x \in \mathbb{R}^n,\ t \in \mathbb{R} && (4.22) \\ u(x, 0) &= f(x) && (4.23) \\ u_t(x, 0) &= g(x), && (4.24) \end{aligned}$$

where $\Box u := u_{tt} - c^2\triangle u$. We assume $f \in C^3$, $g \in C^2$ and $w \in C^1$, which are given.

Set $u = u_1 + u_2$, where u_1 is a solution of problem (4.22)-(4.24) with $w := 0$ and u_2 is the solution where $f = 0$ and $g = 0$ in (4.22)-(4.24). Since we have explicit solutions u_1 in the cases $n = 1$, $n = 2$ and $n = 3$, it remains to solve

$$\begin{aligned} \Box u &= w(x, t) \ \text{ on } x \in \mathbb{R}^n,\ t \in \mathbb{R} && (4.25) \\ u(x, 0) &= 0 && (4.26) \\ u_t(x, 0) &= 0. && (4.27) \end{aligned}$$

The following method is called *Duhamel's principle* which can be considered as a generalization of the method of variations of constants in the theory of ordinary differential equations.

To solve this problem, we make the ansatz

$$u(x,t)=\int_0^t v(x,t,s)\ ds, \tag{4.28}$$

where v is a function satisfying

$$\Box v=0 \ \text{ for all } s \tag{4.29}$$

and

$$v(x,s,s)=0. \tag{4.30}$$

From ansatz (4.28) and assumption (4.30) we get

$$\begin{aligned} u_t &= v(x,t,t)+\int_0^t v_t(x,t,s)\ ds, \\ &= \int_0^t v_t(x,t,s). \end{aligned} \tag{4.31}$$

It follows $u_t(x,0)=0$. The initial condition $u(x,t)=0$ is satisfied because of the ansatz (4.28). From (4.31) and ansatz (4.28) we see that

$$\begin{aligned} u_{tt} &= v_t(x,t,t)+\int_0^t v_{tt}(x,t,s)\ ds, \\ \triangle_x u &= \int_0^t \triangle_x v(x,t,s)\ ds. \end{aligned}$$

Therefore, since u is an ansatz for (4.25)-(4.27),

$$\begin{aligned} u_{tt}-c^2\triangle_x u &= v_t(x,t,t)+\int_0^t (\Box v)(x,t,s)\ ds \\ &= w(x,t). \end{aligned}$$

Thus necessarily $v_t(x,t,t)=w(x,t)$, see (4.29). We have seen that the ansatz provides a solution of (4.25)-(4.27) if for all s

$$\Box v=0,\ \ v(x,s,s)=0,\ \ v_t(x,s,s)=w(x,s). \tag{4.32}$$

Let $v^*(x,t,s)$ be a solution of

$$\Box v=0,\ \ v(x,0,s)=0,\ \ v_t(x,0,s)=w(x,s), \tag{4.33}$$

then

$$v(x,t,s):=v^*(x,t-s,s)$$

is a solution of (4.32). In the case $n = 3$, where v^* is given by, see Theorem 4.2,

$$v^*(x,t,s) = \frac{1}{4\pi c^2 t} \int_{\partial B_{ct}(x)} w(\xi,s)\ dS_\xi .$$

Then

$$\begin{aligned} v(x,t,s) &= v^*(x,t-s,s) \\ &= \frac{1}{4\pi c^2 (t-s)} \int_{\partial B_{c(t-s)}(x)} w(\xi,s)\ dS_\xi . \end{aligned}$$

from ansatz (4.28) it follows

$$\begin{aligned} u(x,t) &= \int_0^t v(x,t,s)\ ds \\ &= \frac{1}{4\pi c^2} \int_0^t \int_{\partial B_{c(t-s)}(x)} \frac{w(\xi,s)}{t-s}\ dS_\xi ds . \end{aligned}$$

Changing variables by $\tau = c(t-s)$ yields

$$\begin{aligned} u(x,t) &= \frac{1}{4\pi c^2} \int_0^{ct} \int_{\partial B_\tau(x)} \frac{w(\xi, t-\tau/c)}{\tau}\ dS_\xi d\tau \\ &= \frac{1}{4\pi c^2} \int_{B_{ct}(x)} \frac{w(\xi, t-r/c)}{r}\ d\xi , \end{aligned}$$

where $r = |x - \xi|$.

Formulas for the cases $n = 1$ and $n = 2$ follow from formulas for the associated homogeneous equation with inhomogeneous initial values for these cases.

Theorem 4.4. *The solution of*

$$\Box u = w(x,t), \quad u(x,0) = 0, \quad u_t(x,0) = 0,$$

where $w \in C^1$, is given by:

Case $n = 3$:

$$u(x,t) = \frac{1}{4\pi c^2} \int_{B_{ct}(x)} \frac{w(\xi, t-r/c)}{r}\ d\xi ,$$

where $r = |x - \xi|$, $x = (x_1, x_2, x_3)$, $\xi = (\xi_1, \xi_2, \xi_3)$.

Case $n = 2$:

$$u(x,t) = \frac{1}{4\pi c}\int_0^t \left(\int_{B_{c(t-\tau)}(x)} \frac{w(\xi,\tau)}{\sqrt{c^2(t-\tau)^2 - r^2}}\ d\xi \right)\ d\tau,$$

$x = (x_1, x_2)$, $\xi = (\xi_1, \xi_2)$.

Case $n = 1$:

$$u(x,t) = \frac{1}{2c}\int_0^t \left(\int_{x-c(t-\tau)}^{x+c(t-\tau)} w(\xi,\tau)\ d\xi \right)\ d\tau.$$

Remark. The integrand on the right in formula for $n = 3$ is called *retarded potential*. The integrand is taken not at t, it is taken at an *earlier* time $t - r/c$.

4.4 A method of Riemann

Riemann's method provides a formula for the solution of the following Cauchy initial value problem for a hyperbolic equation of second order in two variables. Let

$$\mathcal{S}: \quad x = x(t), y = y(t), \quad t_1 \le t \le t_2,$$

be a regular curve in $\mathbb{R}^2$, that is, we assume $x,\ y \in C^1[t_1, t_2]$ and $x'^2 + y'^2 \neq 0$. Set

$$Lu := u_{xy} + a(x,y)u_x + b(x,y)u_y + c(x,y)u,$$

where $a,\ b \in C^1$ and $c,\ f \in C$ in a neighbourhood of $\mathcal{S}$. Consider the initial value problem

$$\begin{aligned} Lu &= f(x,y) && (4.34)\\ u_0(t) &= u(x(t),y(t)) && (4.35)\\ p_0(t) &= u_x(x(t),y(t)) && (4.36)\\ q_0(t) &= u_y(x(t),y(t)), && (4.37) \end{aligned}$$

where $f \in C$ in a neighbourhood of $\mathcal{S}$ and $u_0,\ p_0,\ q_0 \in C^1$ are given.

We assume:

(i) $u_0'(t) = p_0(t)x'(t) + q_0(t)y'(t)$ (strip condition),

(ii) $\mathcal{S}$ is not a characteristic curve. Moreover assume that the characteristic curves, which are lines here and are defined by $x = const.$ and $y = const.$, have at most one point of intersection with $\mathcal{S}$, and such a point is not a touching point, i. e., tangents of the characteristic and $\mathcal{S}$ are different at this point.

We recall that the characteristic equation to (4.34) is $\chi_x\chi_y = 0$ which is satisfied if $\chi_x(x, y) = 0$ or $\chi_y(x, y) = 0$. One family of characteristics associated to these first partial differential of first order is defined by $x'(t) = 1,\ y'(t) = 0$, see Chapter 2.

Assume $u,\ v \in C^1$ and that $u_{xy},\ v_{xy}$ exist and are continuous. Define the adjoint differential expression by

$$Mv = v_{xy} - (av)_x - (bv)_y + cv.$$

We have

$$2(vLu - uMv) = (u_xv - v_xu + 2buv)_y + (u_yv - v_yu + 2auv)_x. \tag{4.38}$$

Set

$$\begin{aligned} P &= -(u_xv - x_xu + 2buv) \\ Q &= u_yv - v_yu + 2auv. \end{aligned}$$

From (4.38) it follows for a domain $\Omega \in \mathbb{R}^2$

$$\begin{aligned} 2\int_\Omega (vLu - uMv)\ dxdy &= \int_\Omega (-P_y + Q_x)\ dxdy \\ &= \oint Pdx + Qdy, \end{aligned} \tag{4.39}$$

where integration in the line integral is anticlockwise. The previous equation follows from Gauss theorem or after integration by parts:

$$\int_\Omega (-P_y + Q_x)\ dxdy = \int_{\partial\Omega} (-Pn_2 + Qn_1)\ ds,$$

where $n = (dy/ds, -dx/ds)$, s arc length, $(x(s), y(s))$ represents $\partial\Omega$.

Assume u is a solution of the initial value problem (4.34)-(4.37) and suppose that v satisfies

$$Mv = 0 \ \text{ in } \Omega.$$

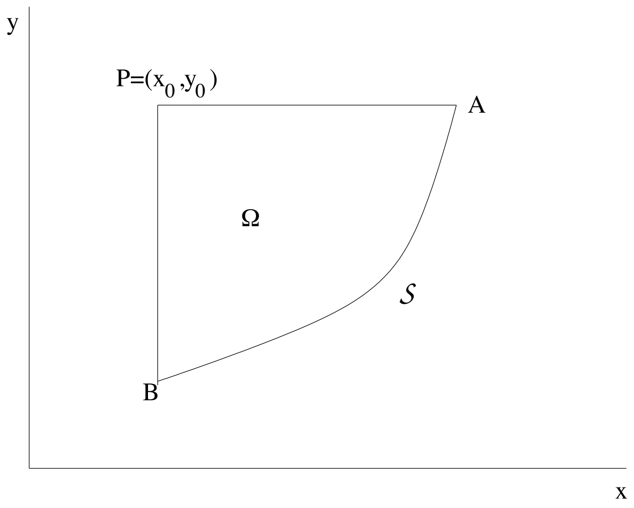

Figure 4.6: Riemann's method, domain of integration

Then, if we integrate over a domain Ω as shown in Figure 4.6, it follows from (4.39) that

$$2\int_\Omega vf\ dxdy = \int_{BA} Pdx+Qdy+\int_{AP} Pdx+Qdy+\int_{PB} Pdx+Qdy. \quad (4.40)$$

The line integral from B to A is known from initial data, see the definition of P and Q.

Since

$$u_xv - v_xu + 2buv = (uv)_x + 2u(bv - v_x),$$

it follows

$$\begin{aligned}\int_{AP} Pdx + Qdy &= -\int_{AP} ((uv)_x + 2u(bv - v_x))\ dx \\ &= -(uv)(P) + (uv)(A) - \int_{AP} 2u(bv - v_x)\ dx.\end{aligned}$$

By the same reasoning we obtain for the third line integral

$$\begin{aligned}\int_{PB} Pdx + Qdy &= \int_{PB} ((uv)_y + 2u(av - v_y))\ dy \\ &= (uv)(B) - (uv)(P) + \int_{PB} 2u(av - v_y)\ dy.\end{aligned}$$

Combining these equations with (4.39), we get

$$\begin{aligned} 2v(P)u(P) &= \int_{BA} (u_x v - v_x + 2buv)\ dx - (u_y v - v_y u + 2auv)\ dy \\ &\quad + u(A)v(A) + u(B)v(B) + 2\int_{AP} u(bv - v_x)\ dx \\ &\quad + 2\int_{PB} u(av - v_y)\ dy - 2\int_\Omega fv\ dxdy. \end{aligned} \tag{4.41}$$

Let v be a solution of the initial value problem, see Figure 4.7 for the definition of domain $D(P)$,

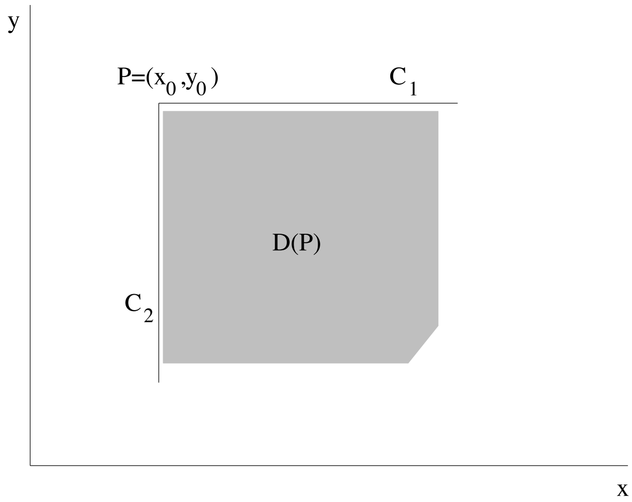

Figure 4.7: Definition of Riemann's function

$$Mv = 0 \text{ in } D(P) \tag{4.42}$$
$$bv - v_x = 0 \text{ on } C_1 \tag{4.43}$$
$$av - v_y = 0 \text{ on } C_2 \tag{4.44}$$
$$v(P) = 1. \tag{4.45}$$

Assume v satisfies (4.42)-(4.45), then

$$\begin{aligned} 2u(P) &= u(A)v(A) + u(B)v(B) - 2\int_\Omega fv\ dxdy \\ &= \int_{BA} (u_x v - v_x + 2buv)\ dx - (u_y v - v_y u + 2auv)\ dy, \end{aligned}$$

where the right hand side is known from given data.

A function $v = v(x, y; x_0, y_0)$ satisfying (4.42)-(4.45) is called *Riemann's function*.

Remark. Set $w(x, y) = v(x, y; x_0, y_0)$ for fixed $x_0,\ y_0$. Then (4.42)-(4.45) imply

$$\begin{aligned} w(x, y_0) &= \exp\left(\int_{x_0}^{x} b(\tau, y_0)\ d\tau\right) \quad \text{on } C_1, \\ w(x_0, y) &= \exp\left(\int_{y_0}^{y} a(x_0, \tau)\ d\tau\right) \quad \text{on } C_2. \end{aligned}$$

Examples

1. $u_{xy} = f(x, y)$, then a Riemann function is $v(x, y) \equiv 1$.

2. Consider the telegraph equation of Chapter 3

$$\varepsilon\mu u_{tt} = c^2 \triangle_x u - \lambda\mu u_t,$$

where u stands for one coordinate of electric or magnetic field. Introducing

$$u = w(x, t)\mathrm{e}^{\kappa t},$$

where $\kappa = -\lambda/(2\varepsilon)$, we arrive at

$$w_{tt} = \frac{c^2}{\varepsilon\mu}\triangle_x w - \frac{\lambda^2}{4\epsilon^2}.$$

Stretching the axis and transform the equation to the normal form we get finally the following equation, the new function is denoted by u and the new variables are denoted by $x,\ y$ again,

$$u_{xy} + cu = 0,$$

with a positive constant c. We make the ansatz for a Riemann function

$$v(x, y; x_0, y_0) = w(s), \quad s = (x - x_0)(y - y_0)$$

and obtain

$$sw'' + w' + cw = 0.$$

Substitution $\sigma = \sqrt{4cs}$ leads to Bessel's differential equation

$$\sigma^2 z''(\sigma) + \sigma z'(\sigma) + \sigma^2 z(\sigma) = 0,$$

where $z(\sigma) = w(\sigma^2/(4c))$. A solution is

$$J_0(\sigma) = J_0\left(\sqrt{4c(x-x_0)(y-y_0)}\right)$$

which defines a Riemann function since $J_0(0) = 1$.

Remark. Bessel's differential equation is

$$x^2 y''(x) + xy'(x) + (x^2 - n^2)y(x) = 0,$$

where $n \in \mathbb{R}$. If $n \in \mathbb{N} \cup \{0\}$, then solutions are given by Bessel functions. One of the two linearly independent solutions is bounded at 0. This bounded solution is the Bessel function $J_n(x)$ of first kind and of order n, see [1], for example.

4.5 Initial-boundary value problems

In previous sections we looked at solutions defined for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. In this and in the following section we seek solutions $u(x,t)$ defined in a bounded domain $\Omega \subset \mathbb{R}^n$ and for all $t \in \mathbb{R}$ and which satisfy additional boundary conditions on $\partial\Omega$.

4.5.1 Oscillation of a string

Let $u(x,t)$, $x \in [a,b]$, $t \in \mathbb{R}$, be the deflection of a string, see Figure 1.4 from Chapter 1. Assume the deflection occurs in the (x,u)-plane. This problem is governed by the initial-boundary value problem

$$\begin{aligned} u_{tt}(x,t) &= u_{xx}(x,t) \ \text{ on } (0,l) && (4.46)\\ u(x,0) &= f(x) && (4.47)\\ u_t(x,0) &= g(x) && (4.48)\\ u(0,t) &= u(l,t) = 0. && (4.49) \end{aligned}$$

Assume the initial data f, g are sufficiently regular. This implies compatibility conditions $f(0) = f(l) = 0$ and $g(0) = g(l)$.

Fourier's method

To find solutions of differential equation (4.46) we make the *separation of variables* ansatz

$$u(x,t) = v(x)w(t).$$

Inserting the ansatz into (4.46) we obtain

$$v(x)w''(t) = v''(x)w(t),$$

or, if $v(x)w(t) \neq 0$,

$$\frac{w''(t)}{w(t)} = \frac{v''(x)}{v(x)}.$$

It follows, provided $v(x)w(t)$ is a solution of differential equation (4.46) and $v(x)w(t) \neq 0$,

$$\frac{w''(t)}{w(t)} = const. =: -\lambda$$

and

$$\frac{v''(x)}{v(x)} = -\lambda$$

since $x,\ t$ are independent variables.

Assume $v(0) = v(l) = 0$, then $v(x)w(t)$ satisfies the boundary condition (4.49). Thus we look for solutions of the eigenvalue problem

$$-v''(x) = \lambda v(x) \ \text{ in } (0,l) \tag{4.50}$$
$$v(0) = v(l) = 0, \tag{4.51}$$

which has the eigenvalues

$$\lambda_n = \left(\frac{\pi}{l}n\right)^2, \quad n = 1, 2, \ldots,$$

and associated eigenfunctions are

$$v_n = \sin\left(\frac{\pi}{l}nx\right).$$

Solutions of

$$-w''(t) = \lambda_n w(t)$$

are

$$\sin(\sqrt{\lambda_n}t), \quad \cos(\sqrt{\lambda_n}t).$$

Set

$$w_n(t) = \alpha_n \cos(\sqrt{\lambda_n}t) + \beta_n \sin(\sqrt{\lambda_n}t),$$

where α_n, $\beta_n \in \mathbb{R}$. It is easily seen that $w_n(t)v_n(x)$ is a solution of differential equation (4.46), and, since (4.46) is linear and homogeneous, also (principle of superposition)

$$u_N = \sum_{n=1}^{N} w_n(t)v_n(x)$$

which satisfies the differential equation (4.46) and the boundary conditions (4.49). Consider the formal solution of (4.46), (4.49)

$$u(x,t) = \sum_{n=1}^{\infty} \left(\alpha_n \cos(\sqrt{\lambda_n}t) + \beta_n \sin(\sqrt{\lambda_n}t)\right) \sin\left(\sqrt{\lambda_n}x\right). \tag{4.52}$$

"Formal" means that we know here neither that the right hand side converges nor that it is a solution of the initial-boundary value problem. Formally, the unknown coefficients can be calculated from initial conditions (4.47), (4.48) as follows. We have

$$u(x,0) = \sum_{n=1}^{\infty} \alpha_n \sin(\sqrt{\lambda_n}x) = f(x).$$

Multiplying this equation by $\sin(\sqrt{\lambda_k}x)$ and integrate over $(0,l)$, we get

$$\alpha_n \int_0^l \sin^2(\sqrt{\lambda_k}x)\ dx = \int_0^l f(x)\sin(\sqrt{\lambda_k}x)\ dx.$$

We recall that

$$\int_0^l \sin(\sqrt{\lambda_n}x)\sin(\sqrt{\lambda_k}x)\ dx = \frac{l}{2}\delta_{nk}.$$

Then

$$\alpha_k = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{\pi k}{l}x\right)\ dx. \tag{4.53}$$

By the same argument it follows from

$$u_t(x,0) = \sum_{n=1}^{\infty} \beta_n\sqrt{\lambda_n}\sin(\sqrt{\lambda_n}x) = g(x)$$

that

$$\beta_k = \frac{2}{k\pi}\int_0^l g(x)\sin\left(\frac{\pi k}{l}x\right)\ dx. \tag{4.54}$$

Under additional assumptions $f \in C_0^4(0,l)$, $g \in C_0^3(0,l)$ it follows that the right hand side of (4.52), where α_n, β_n are given by (4.53) and (4.54), respectively, defines a classical solution of (4.46)-(4.49) since under these assumptions the series for u and the formal differentiate series for u_t, u_{tt}, u_x, u_{xx} converges uniformly on $0 \le x \le l$, $0 \le t \le T$, $0 < T < \infty$ fixed, see an exercise.

4.5.2 Oscillation of a membrane

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. We consider the initial-boundary value problem

$$\begin{aligned} u_{tt}(x,t) &= \triangle_x u \quad \text{in } \Omega \times \mathbb{R}, & (4.55)\\ u(x,0) &= f(x), \quad x \in \overline{\Omega}, & (4.56)\\ u_t(x,0) &= g(x), \quad x \in \overline{\Omega}, & (4.57)\\ u(x,t) &= 0 \quad \text{on } \partial\Omega \times \mathbb{R}. & (4.58) \end{aligned}$$

As in the previous subsection for the string, we make the ansatz (separation of variables)

$$u(x,t) = w(t)v(x)$$

which leads to the eigenvalue problem

$$\begin{aligned} -\triangle v &= \lambda v \quad \text{in } \Omega, & (4.59)\\ v &= 0 \quad \text{on } \partial\Omega. & (4.60) \end{aligned}$$

Let λ_n are the eigenvalues of (4.59), (4.60) and v_n a complete associated orthonormal system of eigenfunctions. We assume Ω is sufficiently regular such that the eigenvalues are countable, which is satisfied in the following examples. Then the formal solution of the above initial-boundary value problem is

$$u(x,t) = \sum_{n=1}^{\infty} \left(\alpha_n \cos(\sqrt{\lambda_n} t) + \beta_n \sin(\sqrt{\lambda_n} t) \right) v_n(x),$$

where

$$\begin{aligned} \alpha_n &= \int_\Omega f(x) v_n(x)\ dx \\ \beta_n &= \frac{1}{\sqrt{\lambda_n}} \int_\Omega g(x) v_n(x)\ dx. \end{aligned}$$

Remark. In general, eigenvalues of (4.59), (4.59) are not known explicitly. There are numerical methods to calculate these values. In some special cases, see next examples, these values are known.

Examples

1. Rectangle membrane. Let

$$\Omega = (0, a) \times (0, b).$$

Using the method of separation of variables, we find all eigenvalues of (4.59), (4.60) which are given by

$$\lambda_{kl} = \sqrt{\frac{k^2}{a^2} + \frac{l^2}{b^2}}, \quad k, l = 1, 2, \ldots$$

and associated eigenfunctions, not normalized, are

$$u_{kl}(x) = \sin\left(\frac{\pi k}{a} x_1\right) \sin\left(\frac{\pi l}{b} x_2\right).$$

2. Disk membrane. Set

$$\Omega = \{x \in \mathbb{R}^2 : \ x_1^2 + x_2^2 < R^2\}.$$

In polar coordinates, the eigenvalue problem (4.59), (4.60) is given by

$$-\frac{1}{r}\left((r u_r)_r + \frac{1}{r} u_{\theta\theta}\right) = \lambda u \tag{4.61}$$

$$u(R, \theta) = 0, \tag{4.62}$$

here is $u = u(r, \theta) := v(r \cos\theta, r \sin\theta)$. We will find eigenvalues and eigenfunctions by separation of variables

$$u(r, \theta) = v(r) q(\theta),$$

where $v(R) = 0$ and $q(\theta)$ is periodic with period 2π since $u(r, \theta)$ is single valued. This leads to

$$-\frac{1}{r}\left((r v')' q + \frac{1}{r} v q''\right) = \lambda v q.$$

Dividing by vq, provided $vq \neq 0$, we obtain

$$-\frac{1}{r}\left(\frac{(r v'(r))'}{v(r)} + \frac{1}{r}\frac{q''(\theta)}{q(\theta)}\right) = \lambda, \tag{4.63}$$

which implies

$$\frac{q''(\theta)}{q(\theta)} = const. =: -\mu.$$

Thus, we arrive at the eigenvalue problem

$$\begin{aligned} -q''(\theta) &= \mu q(\theta) \\ q(\theta) &= q(\theta + 2\pi). \end{aligned}$$

It follows that eigenvalues μ are real and nonnegative. All solutions of the differential equation are given by

$$q(\theta) = A\sin(\sqrt{\mu}\theta) + B\cos(\sqrt{\mu}\theta),$$

where A, B are arbitrary real constants. From the periodicity requirement

$$A\sin(\sqrt{\mu}\theta) + B\cos(\sqrt{\mu}\theta) = A\sin(\sqrt{\mu}(\theta + 2\pi)) + B\cos(\sqrt{\mu}(\theta + 2\pi))$$

it follows[2]

$$\sin(\sqrt{\mu}\pi)\,(A\cos(\sqrt{\mu}\theta + \sqrt{\mu}\pi) - B\sin(\sqrt{\mu}\theta + \sqrt{\mu}\pi)) = 0,$$

which implies, since A, B are not zero simultaneously, because we are looking for q not identically zero,

$$\sin(\sqrt{\mu}\pi)\sin(\sqrt{\mu}\theta + \delta) = 0$$

for all θ and a $\delta = \delta(A, B, \mu)$. Consequently the eigenvalues are

$$\mu_n = n^2, \quad n = 0, 1, \ldots \ .$$

Inserting $q''(\theta)/q(\theta) = -n^2$ into (4.63), we obtain the boundary value problem

$$\begin{aligned} r^2v''(r) + rv'(r) + (\lambda r^2 - n^2)v &= 0 \ \text{ on } (0, R) && (4.64) \\ v(R) &= 0 && (4.65) \\ \sup_{r\in(0,R)} |v(r)| &< \infty. && (4.66) \end{aligned}$$

Set $z = \sqrt{\lambda}r$ and $v(r) = v(z/\sqrt{\lambda}) =: y(z)$, then, see (4.64),

$$z^2y''(z) + zy'(z) + (z^2 - n^2)y(z) = 0,$$

[2]

$$\begin{aligned} \sin x - \sin y &= 2\cos\frac{x+y}{2}\sin\frac{x-y}{2} \\ \cos x - \cos y &= -2\sin\frac{x+y}{2}\sin\frac{x-y}{2} \end{aligned}$$

where $z > 0$. Solutions of this differential equations which are bounded at zero are Bessel functions of first kind and n-th order $J_n(z)$. The eigenvalues follows from boundary condition (4.65), i. e., from $J_n(\sqrt{\lambda}R) = 0$. Denote by τ_{nk} the zeros of $J_n(z)$, then the eigenvalues of (4.61)-(4.61) are

$$\lambda_{nk} = \left(\frac{\tau_{nk}}{R}\right)^2$$

and the associated eigenfunctions are

$$\begin{array}{ll} J_n(\sqrt{\lambda_{nk}}r)\sin(n\theta), & n = 1, 2, \ldots \\ J_n(\sqrt{\lambda_{nk}}r)\cos(n\theta), & n = 0, 1, 2, \ldots. \end{array}$$

Thus the eigenvalues λ_{0k} are simple and $\lambda_{nk},\ n \geq 1$, are double eigenvalues.

Remark. For tables with zeros of $J_n(x)$ and for much more properties of Bessel functions see [25]. One has, in particular, the asymptotic formula

$$J_n(x) = \left(\frac{2}{\pi x}\right)^{1/2} \left(\cos(x - n\pi/2 - \pi/5) + O\left(\frac{1}{x}\right)\right)$$

as $x \to \infty$. It follows from this formula that there are infinitely many zeros of $J_n(x)$.

4.5.3 Inhomogeneous wave equations

Let $\Omega \subset \mathbb{R}^n$ be a bounded and sufficiently regular domain. In this section we consider the initial-boundary value problem

$$\begin{aligned} u_{tt} &= Lu + f(x,t) \ \text{ in } \Omega \times \mathbb{R} && (4.67) \\ u(x,0) &= \phi(x) \ \ x \in \overline{\Omega} && (4.68) \\ u_t(x,0) &= \psi(x) \ \ x \in \overline{\Omega} && (4.69) \\ u(x,t) &= 0 \ \text{ for } x \in \partial\Omega \text{ and } t \in \mathbb{R}^n, && (4.70) \end{aligned}$$

where $u = u(x,t)$, $x = (x_1, \ldots, x_n)$, $f,\ \phi,\ \psi$ are given and L is an elliptic differential operator. Examples for L are:

1. $L = \partial^2/\partial x^2$, oscillating string.

2. $L = \triangle_x$, oscillating membrane.

3.

$$Lu = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a^{ij}(x)u_{x_i}\right),$$

where $a^{ij} = a^{ji}$ are given sufficiently regular functions defined on $\overline{\Omega}$. We assume L is uniformly elliptic, that is, there is a constant $\nu > 0$ such that

$$\sum_{i,j=1}^{n} a^{ij}\zeta_i\zeta_j \geq \nu|\zeta|^2$$

for all $x \in \Omega$ and $\zeta \in \mathbb{R}^n$.

4. Let $u = (u_1, \ldots, u_m)$ and

$$Lu = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(A^{ij}(x)u_{x_i}\right),$$

where $A^{ij} = A^{ji}$ are given sufficiently regular $(m \times m)$-matrices on $\overline{\Omega}$. We assume that L defines an elliptic system. An example for this case is the linear elasticity.

Consider the eigenvalue problem

$$\begin{aligned} -Lv &= \lambda v \ \text{ in } \Omega && (4.71)\\ v &= 0 \ \text{ on } \partial\Omega. && (4.72) \end{aligned}$$

Assume there are infinitely many eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq \ldots \ \to \infty$$

and a system of associated eigenfunctions $v_1,\ v_2, \ldots$ which is complete and orthonormal in $L^2(\Omega)$. This assumption is satisfied if Ω is bounded and if $\partial\Omega$ is sufficiently regular.

For the solution of (4.67)-(4.70) we make the ansatz

$$u(x,t) = \sum_{k=1}^{\infty} v_k(x)w_k(t), \tag{4.73}$$

with functions $w_k(t)$ which will be determined later. It is assumed that all series are convergent and that following calculations make sense. Let

$$f(x,t) = \sum_{k=1}^{\infty} c_k(t)v_k(x) \tag{4.74}$$

be Fourier's decomposition of f with respect to the eigenfunctions v_k. We have

$$c_k(t) = \int_\Omega f(x,t)v_k(x)\ dx, \tag{4.75}$$

which follows from (4.74) after multiplying with $v_l(x)$ and integrating over Ω.

Set

$$\langle \phi, v_k\rangle = \int_\Omega \phi(x)v_k(x)\ dx,$$

then

$$\begin{aligned} \phi(x) &= \sum_{k=1}^\infty \langle \phi, v_k\rangle v_k(x) \\ \psi(x) &= \sum_{k=1}^\infty \langle \psi, v_k\rangle v_k(x) \end{aligned}$$

are Fourier's decomposition of ϕ and ψ, respectively.

In the following we will determine $w_k(t)$, which occurs in ansatz (4.73), from the requirement that $u = v_k(x)w_k(t)$ is a solution of

$$u_{tt} = Lu + c_k(t)v_k(x)$$

and that the initial conditions

$$w_k(0) = \langle \phi, v_k\rangle, \quad w_k'(0) = \langle \psi, v_k\rangle$$

are satisfied. From the above differential equation it follows

$$w_k''(t) = -\lambda_k w_k(t) + c_k(t).$$

Thus

$$\begin{aligned} w_k(t) &= a_k\cos(\sqrt{\lambda_k}t) + b_k\sin(\sqrt{\lambda_k}t) \\ &\quad + \frac{1}{\sqrt{\lambda_k}}\int_0^t c_k(\tau)\sin(\sqrt{\lambda_k}(t-\tau))\ d\tau, \end{aligned} \tag{4.76}$$

where

$$a_k = \langle \phi, v_k\rangle, \quad b_k = \frac{1}{\sqrt{\lambda_k}}\langle \psi, v_k\rangle.$$

Summarizing, we have

Proposition 4.2. *The (formal) solution of the initial-boundary value problem (4.67)-(4.70) is given by*

$$u(x,t)=\sum_{k=1}^{\infty} v_k(x)w_k(t),$$

where v_k is a complete orthonormal system of eigenfunctions of (4.71), (4.72) and the functions w_k are defined by (4.76).

The resonance phenomenon

Set in (4.67)-(4.70) $\phi=0$, $\psi=0$ and assume that the external force f is periodic and is given by

$$f(x,t)=A\sin(\omega t)v_n(x),$$

where A, ω are real constants and v_n is one of the eigenfunctions of (4.71), (4.72). It follows

$$c_k(t)=\int_\Omega f(x,t)v_k(x)\ dx=A\delta_{nk}\sin(\omega t).$$

Then the solution of the initial value problem (4.67)-(4.70) is

$$\begin{aligned} u(x,t) &= \frac{Av_n(x)}{\sqrt{\lambda_n}}\int_0^t \sin(\omega\tau)\sin(\sqrt{\lambda_n}(t-\tau))\ d\tau \\ &= Av_n(x)\frac{1}{\omega^2-\lambda_n}\left(\frac{\omega}{\sqrt{\lambda_n}}\sin(\sqrt{\lambda_k}t)-\sin(\omega t)\right), \end{aligned}$$

provided $\omega\neq\sqrt{\lambda_n}$. It follows

$$u(x,t)\to\frac{A}{2\sqrt{\lambda_n}}v_n(x)\left(\frac{\sin(\sqrt{\lambda_n}t)}{\sqrt{\lambda_n}}-t\cos(\sqrt{\lambda_n}t)\right)$$

if $\omega\to\sqrt{\lambda_n}$. The right hand side is also the solution of the initial-boundary value problem if $\omega=\sqrt{\lambda_n}$.

Consequently $|u|$ can be arbitrarily large at some points x and at some times t if $\omega=\sqrt{\lambda_n}$. The frequencies $\sqrt{\lambda_n}$ are called *critical frequencies* at which resonance occurs.

A uniqueness result

The solution of of the initial-boundary value problem (4.67)-(4.70) is unique in the class $C^2(\overline{\Omega}\times\mathbb{R})$.

Proof. Let u_1, u_2 are two solutions, then $u = u_2 - u_1$ satisfies

$$\begin{aligned} u_{tt} &= Lu \ \text{ in } \Omega \times \mathbb{R} \\ u(x,0) &= 0 \ \ x \in \overline{\Omega} \\ u_t(x,0) &= 0 \ \ x \in \overline{\Omega} \\ u(x,t) &= 0 \ \text{ for } x \in \partial\Omega \text{ and } t \in \mathbb{R}^n. \end{aligned}$$

As an example we consider Example 3 from above and set

$$E(t) = \int_\Omega (\sum_{i,j=1}^n a^{ij}(x) u_{x_i} u_{x_j} + u_t u_t) \ dx.$$

Then

$$\begin{aligned} E'(t) &= 2 \int_\Omega (\sum_{i,j=1}^n a^{ij}(x) u_{x_i} u_{x_j t} + u_t u_{tt}) \ dx \\ &= 2 \int_{\partial\Omega} (\sum_{i,j=1}^n a^{ij}(x) u_{x_i} u_t n_j) \ dS \\ &\quad +2 \int_\Omega u_t(-Lu + u_t t) \ dx \\ &= 0. \end{aligned}$$

It follows $E(t) = const.$ From $u_t(x,0) = 0$ and $u(x,0) = 0$ we get $E(0) = 0$. Consequently $E(t) = 0$ for all t, which implies, since L is elliptic, that $u(x,t) = const.$ on $\overline{\Omega} \times \mathbb{R}$. Finally, the homogeneous initial and boundary value conditions lead to $u(x,t) = 0$ on $\overline{\Omega} \times \mathbb{R}$. □

4.6 Exercises

1. Show that $u(x,t) \in C^2(\mathbb{R}^2)$ is a solution of the one-dimensional wave equation
$$u_{tt} = c^2 u_{xx}$$
if and only if
$$u(A) + u(C) = u(B) + u(D)$$
holds for all parallelograms $ABCD$ in the (x,t)-plane, which are bounded by characteristic lines, see Figure 4.8.

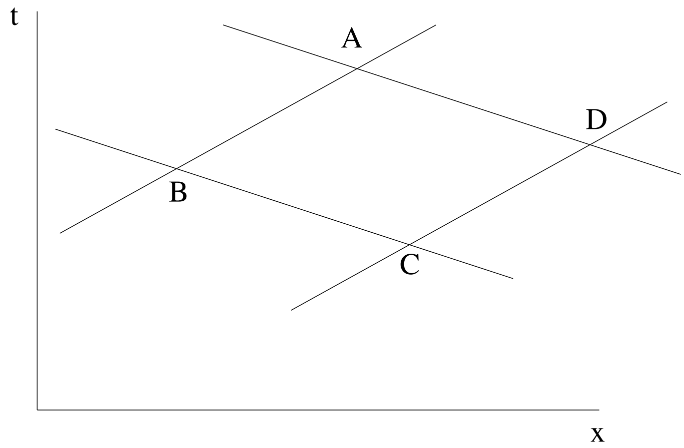

Figure 4.8: Figure to the exercise

2. Method of separation of variables: Let $v_k(x)$ be an eigenfunction to the eigenvalue of the eigenvalue problem $-v''(x) = \lambda v(x)$ in $(0,l)$, $v(0) = v(l) = 0$ and let $w_k(t)$ be a solution of differential equation $-w''(t) = \lambda_k w(t)$. Prove that $v_k(x)w_k(t)$ is a solution of the partial differential equation (wave equation) $u_{tt} = u_{xx}$.

3. Solve for given $f(x)$ and $\mu \in \mathbb{R}$ the initial value problem
$$\begin{aligned} u_t + u_x + \mu u_{xxx} &= 0 \quad \text{in} \quad \mathbb{R} \times \mathbb{R}_+ \\ u(x,0) &= f(x)\ . \end{aligned}$$

4. Let $S := \{(x,t);\ t = \gamma x\}$ be spacelike, i. e., $|\gamma| < 1/c^2$) in (x,t)-space, $x = (x_1, x_2, x_3)$. Show that the Cauchy initial value problem $\Box u = 0$

with data for u on S can be transformed using the Lorentz-transform

$$x_1 = \frac{x_1 - \gamma c^2 t}{\sqrt{1-\gamma^2 c^2}},\ x_2' = x_2,\ x_3' = x_3,\ t' = \frac{t - \gamma x_1}{\sqrt{1-\gamma^2 c^2}}$$

into the initial value problem, in new coordinates,

$$\begin{aligned} \Box u &= 0 \\ u(x',0) &= f(x') \\ u_{t'}(x',0) &= g(x')\ . \end{aligned}$$

Here we denote the transformed function by u again.

5. (i) Show that

$$u(x,t) := \sum_{n=1}^{\infty} \alpha_n \cos\left(\frac{\pi n}{l} t\right) \sin\left(\frac{\pi n}{l} x\right)$$

is a C^2-solution of the wave equation $u_{tt} = u_{xx}$ if $|\alpha_n| \le c/n^4$, where the constant c is independent of n.

(ii) Set

$$\alpha_n := \int_0^l f(x) \sin\left(\frac{\pi n}{l} x\right)\, dx.$$

Prove $|\alpha_n| \le c/n^4$, provided $f \in C_0^4(0,l)$.

6. Let Ω be the rectangle $(0,a) \times (0,b)$. Find all eigenvalues and associated eigenfunctions of $-\triangle u = \lambda u$ in Ω, $u = 0$ on $\partial\Omega$.

   Hint: Separation of variables.

7. Find a solution of Schrödinger's equation

$$i\hbar\psi_t = -\frac{\hbar^2}{2m}\triangle_x \psi + V(x)\psi \quad \text{in } \mathbb{R}^n \times \mathbb{R},$$

which satisfies the side condition

$$\int_{\mathbb{R}}^n |\psi(x,t)|^2 dx = 1\ ,$$

provided $E \in \mathbb{R}$ is an (eigenvalue) of the elliptic equation

$$\triangle u + \frac{2m}{\hbar^2}(E - V(x))u = 0 \quad \text{in } \ \mathbb{R}^n$$

under the side condition $\int_{\mathbb{R}}^n |u|^2 dx = 1$, $u:\ \mathbb{R}^n \mapsto \mathbb{C}$.
Here is $\psi:\ \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{C}$, $\hbar$ Planck's constant (a small positive constant), $V(x)$ a given potential.

Remark. In the case of a hydrogen atom the potential is $V(x) = -e/|x|$, e is here a positive constant. Then eigenvalues are given by $E_n = -me^4/(2\hbar^2 n^2)$, $n \in \mathbb{N}$, see [22], pp. 202.

8. Find nonzero solutions by using separation of variables of $u_{tt} = \triangle_x u$ in $\Omega \times (0,\infty)$, $u(x,t) = 0$ on $\partial\Omega$, where Ω is the circular cylinder $\Omega = \{(x_1,x_2,x_3) \in \mathbb{R}^n:\ x_1^2 + x_2^2 < R^2,\ 0 < x_3 < h\}$.

9. Solve the initial value problem
$$\begin{aligned} 3u_{tt} - 4u_{xx} &= 0 \\ u(x,0) &= \sin x \\ u_t(x,0) &= 1\ . \end{aligned}$$

10. Solve the initial value problem
$$\begin{aligned} u_{tt} - c^2 u_{xx} &= x^2,\ t > 0,\ x \in \mathbb{R} \\ u(x,0) &= x \\ u_t(x,0) &= 0\ . \end{aligned}$$

    Hint: Find a solution of the differential equation independent on t, and transform the above problem into an initial value problem with homogeneous differential equation by using this solution.

11. Find with the method of separation of variables nonzero solutions $u(x,t)$, $0 \le x \le 1,\ 0 \le t < \infty$, of
$$u_{tt} - u_{xx} + u = 0\ ,$$
    such that $u(0,t) = 0$, and $u(1,t) = 0$ for all $t \in [0,\infty)$.

12. Find solutions of the equation
$$u_{tt} - c^2 u_{xx} = \lambda^2 u,\ \lambda = const.$$
    which can be written as
$$u(x,t) = f(x^2 - c^2t^2) = f(s),\ s := x^2 - c^2t^2$$

with $f(0) = K$, K a constant.

Hint: Transform equation for $f(s)$ by using the substitution $s := z^2/A$ with an appropriate constant A into Bessel's differential equation

$$z^2 f''(z) + zf'(z) + (z^2 - n^2)f = 0,\ z > 0$$

with $n = 0$.

Remark. The above differential equation for u is the transformed telegraph equation (see Section 4.4).

13. Find the formula for the solution of the following Cauchy initial value problem $u_{xy} = f(x, y)$, where S: $y = ax + b$, $a > 0$, and the initial conditions on S are given by

$$\begin{aligned} u &= \alpha x + \beta y + \gamma, \\ u_x &= \alpha, \\ u_y &= \beta, \end{aligned}$$

$a,\ b,\ \alpha,\ \beta,\ \gamma$ constants.

14. Find all eigenvalues μ of

$$\begin{aligned} -q''(\theta) &= \mu q(\theta) \\ q(\theta) &= q(\theta + 2\pi)\ . \end{aligned}$$

Chapter 5

Fourier transform

Fourier's transform is an integral transform which can simplify investigations for linear differential or integral equations since it transforms a differential operator into an algebraic equation.

5.1 Definition, properties

Definition. Let $f \in C_0^s(\mathbb{R}^n)$, $s = 0, 1, \ldots$. The function $\hat{f}$ defined by

$$\widehat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \mathrm{e}^{-i\xi\cdot x} f(x)\ dx, \tag{5.1}$$

where $\xi \in \mathbb{R}^n$, is called *Fourier transform* of f, and the function $\widetilde{g}$ given by

$$\widetilde{g}(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \mathrm{e}^{i\xi\cdot x} g(\xi)\ d\xi \tag{5.2}$$

is called *inverse Fourier transform*, provided the integrals on the right hand side exist.

From (5.1) it follows by integration by parts that differentiation of a function is changed to multiplication of its Fourier transforms, or an analytical operation is converted into an algebraic operation. More precisely, we have

Proposition 5.1.

$$\widehat{D^\alpha f}(\xi) = i^{|\alpha|} \xi^\alpha \widehat{f}(\xi),$$

where $|\alpha| \leq s$.

The following proposition shows that the Fourier transform of f decreases rapidly for $|\xi| \to \infty$, provided $f \in C_0^s(\mathbb{R}^n)$. In particular, the right hand side of (5.2) exists for $g := \hat{f}$ if $f \in C_0^{n+1}(\mathbb{R}^n)$.

Proposition 5.2. Assume $g \in C_0^s(\mathbb{R}^n)$, then there is a constant $M = M(n, s, g)$ such that

$$|\widehat{g}(\xi)| \leq \frac{M}{(1+|\xi|)^s}.$$

Proof. Let $\xi = (\xi_1, \ldots, \xi_n)$ be fixed and let j be an index such that $|\xi_j| = \max_k |\xi_k|$. Then

$$|\xi| = \left(\sum_{k=1}^n \xi_k^2\right)^{1/2} \leq \sqrt{n}|\xi_j|$$

which implies

$$\begin{aligned}
(1+|\xi|)^s &= \sum_{k=0}^s \binom{s}{k} |\xi|^k \\
&\leq 2^s \sum_{k=0}^s n^{k/2} |\xi_j|^k \\
&\leq 2^s n^{s/2} \sum_{|\alpha| \leq s} |\xi^\alpha|.
\end{aligned}$$

This inequality and Proposition 5.1 imply

$$\begin{aligned}
(1+|\xi|)^s |\widehat{g}(\xi)| &\leq 2^s n^{s/2} \sum_{|\alpha| \leq s} |(i\xi)^\alpha \widehat{g}(\xi)| \\
&\leq 2^s n^{s/2} \sum_{|\alpha| \leq s} \int_{\mathbb{R}^n} |D^\alpha g(x)|\ dx =: M.
\end{aligned}$$

□

The notation *inverse* Fourier transform for (5.2) is justified by

Theorem 5.1. $\widetilde{\widehat{f}} = f$ *and* $\widehat{\widetilde{f}} = f$.

Proof. See [27], for example. We will prove the first assertion

$$(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\xi \cdot x} \widehat{f}(\xi)\ d\xi = f(x) \tag{5.3}$$

here. The proof of the other relation is left as an exercise. All integrals appearing in the following exist, see Proposition 5.2 and the special choice of g.

(i) Formula

$$\int_{\mathbb{R}^n} g(\xi)\widehat{f}(\xi)e^{ix\cdot\xi}\, d\xi = \int_{\mathbb{R}^n} \widehat{g}(y)f(x+y)\, dy \tag{5.4}$$

follows by direct calculation:

$$\begin{aligned}
&\int_{\mathbb{R}^n} g(\xi)\left((2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-ix\cdot y}f(y)\, dy\right)e^{ix\cdot\xi}\, d\xi \\
&\quad = (2\pi)^{-n/2}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} g(\xi)e^{-i\xi\cdot(y-x)}\, d\xi\right)f(y)\, dy \\
&\quad = \int_{\mathbb{R}^n} \widehat{g}(y-x)f(y)\, dy \\
&\quad = \int_{\mathbb{R}^n} \widehat{g}(y)f(x+y)\, dy.
\end{aligned}$$

(ii) Formula

$$(2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-iy\cdot\xi}g(\varepsilon\xi)\, d\xi = \varepsilon^{-n}\widehat{g}(y/\varepsilon) \tag{5.5}$$

for each $\varepsilon > 0$ follows after substitution $z = \varepsilon\xi$ in the left hand side of (5.1).

(iii) Equation

$$\int_{\mathbb{R}^n} g(\varepsilon\xi)\widehat{f}(\xi)e^{ix\cdot\xi}\, d\xi = \int_{\mathbb{R}^n} \widehat{g}(y)f(x+\varepsilon y)\, dy \tag{5.6}$$

follows from (5.4) and (5.5). Set $G(\xi) := g(\varepsilon\xi)$, then (5.4) implies

$$\int_{\mathbb{R}^n} G(\xi)\widehat{f}(\xi)e^{ix\cdot\xi}\, d\xi = \int_{\mathbb{R}^n} \widehat{G}(y)f(x+y)\, dy.$$

Since, see (5.5),

$$\begin{aligned}
\widehat{G}(y) &= (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-iy\cdot\xi}g(\varepsilon\xi)\, d\xi \\
&= \varepsilon^{-n}\widehat{g}(y/\varepsilon),
\end{aligned}$$

we arrive at

$$\begin{aligned}\int_{\mathbb{R}^n} g(\varepsilon\xi)\widehat{f}(\xi) &= \int_{\mathbb{R}^n} \varepsilon^{-n}\widehat{g}(y/\varepsilon)f(x+y)\ dy \\ &= \int_{\mathbb{R}^n} \widehat{g}(z)f(x+\varepsilon z)\ dz.\end{aligned}$$

Letting $\varepsilon \to 0$, we get

$$g(0)\int_{\mathbb{R}^n} \widehat{f}(\xi)e^{ix\cdot\xi}\ d\xi = f(x)\int_{\mathbb{R}^n} \widehat{g}(y)\ dy. \tag{5.7}$$

Set

$$g(x) := \mathrm{e}^{-|x|^2/2},$$

then

$$\int_{\mathbb{R}^n} \widehat{g}(y)\ dy = (2\pi)^{n/2}. \tag{5.8}$$

Since $g(0) = 1$, the first assertion of Theorem 5.1 follows from (5.7) and (5.8). It remains to show (5.8).

(iv) *Proof of (5.8).* We will show

$$\begin{aligned}\widehat{g}(y) : &= (2\pi)^{-n/2}\int_{\mathbb{R}^n} \mathrm{e}^{-|x|^2/2}\mathrm{e}^{-ix\cdot x}\ dx \\ &= \mathrm{e}^{-|y|^2/2}.\end{aligned}$$

The proof of

$$\int_{\mathbb{R}^n} \mathrm{e}^{-|y|^2/2}\ dy = (2\pi)^{n/2}$$

is left as an exercise. Since

$$-\left(\frac{x}{\sqrt{2}} + i\frac{y}{\sqrt{2}}\right)\cdot\left(\frac{x}{\sqrt{2}} + i\frac{y}{\sqrt{2}}\right) = -\left(\frac{|x|^2}{2} + ix\cdot y - \frac{|y|^2}{2}\right)$$

it follows

$$\begin{aligned}\int_{\mathbb{R}^n} \mathrm{e}^{-|x|^2/2}\mathrm{e}^{-ix\cdot y}\ dx &= \int_{\mathbb{R}^n} \mathrm{e}^{-\eta^2}\mathrm{e}^{-|y|^2/2}\ dx \\ &= \mathrm{e}^{-|y|^2/2}\int_{\mathbb{R}^n} \mathrm{e}^{-\eta^2}\ dx \\ &= 2^{n/2}\mathrm{e}^{-|y|^2/2}\int_{\mathbb{R}^n} \mathrm{e}^{-\eta^2}\ d\eta\end{aligned}$$

where
$$\eta := \frac{x}{\sqrt{2}} + i\frac{y}{\sqrt{2}}.$$
Consider first the one-dimensional case. According to Cauchy's theorem we have
$$\oint_C e^{-\eta^2}\ d\eta = 0,$$
where the integration is along the curve C which is the union of four curves as indicated in Figure 5.1.

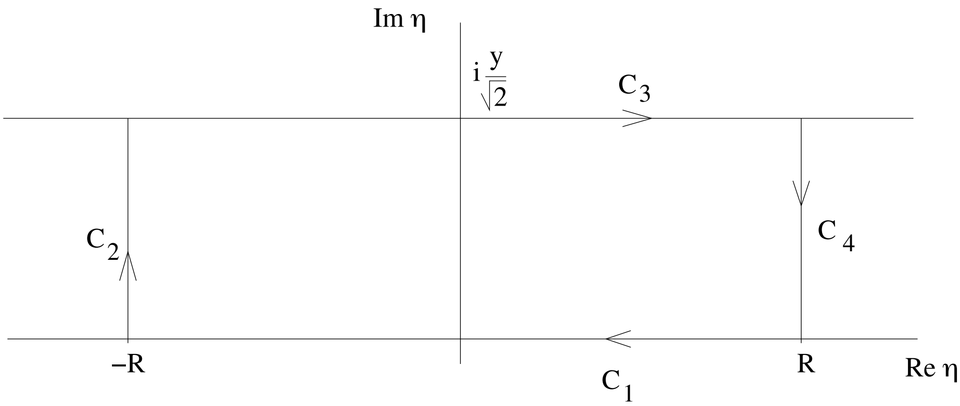

Figure 5.1: Proof of (5.8)

Consequently
$$\int_{C_3} e^{-\eta^2}\ d\eta = \frac{1}{\sqrt{2}}\int_{-R}^{R} e^{-x^2/2}\ dx - \int_{C_2} e^{-\eta^2}\ d\eta - \int_{C_4} e^{-\eta^2}\ d\eta.$$
It follows
$$\lim_{R\to\infty}\int_{C_3} e^{-\eta^2}\ d\eta = \sqrt{\pi}$$
since
$$\lim_{R\to\infty}\int_{C_k} e^{-\eta^2}\ d\eta = 0, \quad k = 2,\ 4.$$
The case $n > 1$ can be reduced to the one-dimensional case as follows. Set
$$\eta = \frac{x}{\sqrt{2}} + i\frac{y}{\sqrt{2}} = (\eta_1, \ldots, \eta_n),$$
where
$$\eta_l = \frac{x_l}{\sqrt{2}} + i\frac{y_l}{\sqrt{2}}.$$

From $d\eta = d\eta_1 \dots d\eta_l$ and

$$\mathrm{e}^{-\eta^2} = \mathrm{e}^{-\sum_{l=1}^n \eta_l^2} = \prod_{l=1}^n \mathrm{e}^{-\eta_l^2}$$

it follows

$$\int_{\mathbb{R}^n} e^{-\eta^2}\ d\eta = \prod_{l=1}^n \int_{\Gamma_l} \mathrm{e}^{-\eta_l^2}\ d\eta_l,$$

where for fixed y

$$\Gamma_l = \{z \in \mathbb{C} :\ z = \frac{x_l}{\sqrt{2}} + i\frac{y_l}{\sqrt{2}}, -\infty < x_l < +\infty\}.$$

□

There is a useful class of functions for which the integrals in the definition of $\widehat{f}$ and $\widetilde{f}$ exist.

For $u \in C^\infty(\mathbb{R}^n)$ we set

$$q_{j,k}(u) := \max_{\alpha:\ |\alpha|\le k} \left(\sup_{\mathbb{R}^n} \left((1+|x|^2)^{j/2} |D^\alpha u(x)| \right) \right).$$

Definition. The *Schwartz class* of rapidly degreasing functions is

$$\mathcal{S}(\mathbb{R}^n) = \{u \in C^\infty(\mathbb{R}^n) :\ q_{j,k}(u) < \infty \text{ for any } j,k \in \mathbb{N}\cup\{0\}\}.$$

This space is a Frechét space.

Proposition 5.3. *Assume $u \in \mathcal{S}(\mathbb{R}^n)$, then $\widehat{u}$ and $\widetilde{u} \in \mathcal{S}(\mathbb{R}^n)$.*

Proof. See [24], Chapter 1.2, for example, or an exercise.

5.1.1 Pseudodifferential operators

The properties of Fourier transform lead to a general theory for linear partial differential or integral equations. In this subsection we define

$$D_k = \frac{1}{i}\frac{\partial}{\partial x_k}, \quad k = 1, \dots, n,$$

and for each multi-index α as in Subsection 3.5.1

$$D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}.$$

Thus

$$D^\alpha = \frac{1}{i^{|\alpha|}} \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}.$$

Let

$$p(x, D) := \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha,$$

be a linear partial differential of order m, where a_α are given sufficiently regular functions.

According to Theorem 5.1 and Proposition 5.3, we have, at least for $u \in \mathcal{S}(\mathbb{R}^n)$,

$$u(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \mathrm{e}^{ix\cdot\xi} \widehat{u}(\xi)\ d\xi,$$

which implies

$$D^\alpha u(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \mathrm{e}^{ix\cdot\xi} \xi^\alpha \widehat{u}(\xi)\ d\xi.$$

Consequently

$$p(x, D)u(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \mathrm{e}^{ix\cdot\xi} p(x, \xi) \widehat{u}(\xi)\ d\xi, \tag{5.9}$$

where

$$p(x, \xi) = \sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha.$$

The right hand side of (5.9) makes sense also for more general functions $p(x, \xi)$, not only for polynomials.

Definition. The function $p(x, \xi)$ is called *symbol* and

$$(Pu)(x) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} \mathrm{e}^{ix\cdot\xi} p(x, \xi) \widehat{u}(\xi)\ d\xi$$

is said to be *pseudodifferential operator*.

An important class of symbols for which the right hand side in this definition of a pseudodifferential operator is defined is S^m which is the subset of $p(x, \xi) \in C^\infty(\Omega \times \mathbb{R}^n)$ such that

$$|D_x^\beta D_\xi^\alpha p(x, \xi)| \leq C_{K,\alpha,\beta}(p)\, (1 + |\xi|)^{m-|\alpha|}$$

for each compact $K \subset \Omega$.

Above we have seen that linear differential operators define a class of pseudodifferential operators. Even integral operators can be written (formally) as pseudodifferential operators. Let

$$(Pu)(x) = \int_{\mathbb{R}^n} K(x,y)u(y)\ dy$$

be an integral operator. Then

$$\begin{aligned}(Pu)(x) &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} K(x,y) \int_{\mathbb{R}^n} \mathrm{e}^{ix\cdot\xi} \xi^\alpha \widehat{u}(\xi)\ d\xi \\ &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} \mathrm{e}^{ix\cdot\xi} \left(\int_{\mathbb{R}^n} \mathrm{e}^{i(y-x)\cdot\xi} K(x,y)\ dy \right) \widehat{u}(\xi).\end{aligned}$$

Then the symbol associated to the above integral operator is

$$p(x,\xi) = \int_{\mathbb{R}^n} \mathrm{e}^{i(y-x)\cdot\xi} K(x,y)\ dy.$$

5.2 Exercises

1. Show
$$\int_{\mathbb{R}^n} \mathrm{e}^{-|y|^2/2}\, dy = (2\pi)^{n/2}.$$

2. Show that $u \in \mathcal{S}(\mathbb{R}^n)$ implies $\hat{u},\ \widetilde{u} \in \mathcal{S}(\mathbb{R}^n)$.

3. Give examples for functions $p(x,\xi)$ which satisfy $p(x,\xi) \in S^m$.

4. Find a formal solution of Cauchy's initial value problem for the wave equation by using Fourier's transform.

Chapter 6

Parabolic equations

Here we consider linear parabolic equations of second order. An example is the heat equation

$$u_t = a^2 \triangle u,$$

where $u = u(x,t)$, $x \in \mathbb{R}^3$, $t \geq 0$, and a^2 is a positive constant called conductivity coefficient. The heat equation has its origin in physics where $u(x,t)$ is the temperature at x at time t, see [20], p. 394, for example.

Remark 1. After scaling of axis we can assume $a = 1$.

Remark 2. By setting $t := -t$, the heat equation changes to an equation which is called backward equation. This is the reason for the fact that the heat equation describes irreversible processes in contrast to the wave equation $\Box u = 0$ which is invariant with respect the mapping $t \mapsto -t$. Mathematically, it means that it is not possible, in general, to find the distribution of temperature at an earlier time $t < t_0$ if the distribution is given at t_0.

Consider the initial value problem for $u = u(x,t)$, $u \in C^\infty(\mathbb{R}^n \times R_+)$,

$$\begin{aligned} u_t &= \triangle u \ \text{ in } x \in \mathbb{R}^n,\ t \geq 0, && (6.1)\\ u(x,0) &= \phi(x), && (6.2) \end{aligned}$$

where $\phi \in C(\mathbb{R}^n)$ is given and $\triangle \equiv \triangle_x$.

6.1 Poisson's formula

Assume u is a solution of (6.1), then, since Fourier transform is a linear mapping,

$$\widehat{u_t - \triangle u} = \hat{0}.$$

From properties of the Fourier transform, see Proposition 5.1, we have

$$\widehat{\triangle u} = \sum_{k=1}^{n} \widehat{\frac{\partial^2 u}{\partial x_k^2}} = \sum_{k=1}^{n} i^2 \xi_k^2 \widehat{u}(\xi),$$

provided the transforms exist. Thus we arrive at the ordinary differential equation for the Fourier transform of u

$$\frac{d\widehat{u}}{dt} + |\xi|^2 \widehat{u} = 0,$$

where ξ is considered as a parameter. The solution is

$$\widehat{u}(\xi, t) = \widehat{\phi}(\xi) \mathrm{e}^{-|\xi|^2 t}$$

since $\widehat{u}(\xi, 0) = \widehat{\phi}(\xi)$. From Theorem 5.1 it follows

$$\begin{aligned} u(x,t) &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} \widehat{\phi}(\xi) \mathrm{e}^{-|\xi|^2 t} \mathrm{e}^{i\xi\cdot x} \, d\xi \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} \phi(y) \left(\int_{\mathbb{R}^n} \mathrm{e}^{i\xi\cdot(x-y) - |\xi|^2 t} \, d\xi \right) dy. \end{aligned}$$

Set

$$K(x,y,t) = (2\pi)^{-n} \int_{\mathbb{R}^n} \mathrm{e}^{i\xi\cdot(x-y) - |\xi|^2 t} \, d\xi.$$

By the same calculations as in the proof of Theorem 5.1, step (vi), we find

$$K(x,y,t) = (4\pi t)^{-n/2} \mathrm{e}^{-|x-y|^2/4t}. \tag{6.3}$$

Thus we have

$$u(x,t) = \frac{1}{\left(2\sqrt{\pi t}\right)^n} \int_{\mathbb{R}^n} \phi(z) \mathrm{e}^{-|x-z|^2/4t} \, dz. \tag{6.4}$$

Definition. Formula (6.4) is called *Poisson's formula* and the function K defined by (6.3) *heat kernel* or *fundamental solution* of the heat equation.

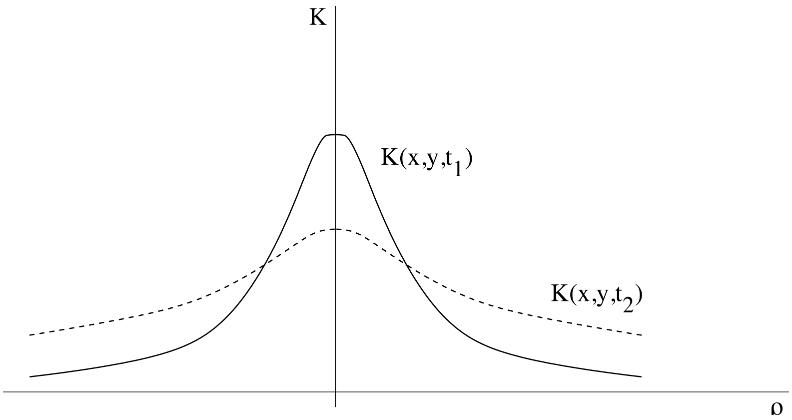

Figure 6.1: Kernel $K(x, y, t)$, $\rho = |x - y|$, $t_1 < t_2$

Proposition 6.1 *The kernel K has following properties:*

(i) $K(x, y, t) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+)$,

(ii) $(\partial/\partial t \ - \triangle)K(x, y, t) = 0, \ t > 0$,

(iii) $K(x, y, t) > 0, \ t > 0$,

(iv) $\int_{\mathbb{R}^n} \ K(x, y, t) \ dy = 1, \ x \in \mathbb{R}^n, \ t > 0$,

(v) For each fixed $\delta > 0$:

$$\lim_{\substack{t \to 0 \\ t > 0}} \int_{\mathbb{R}^n \setminus B_\delta(x)} \ K(x, y, t) \ dy = 0$$

uniformly for $x \in \mathbb{R}^n$.

Proof. (i) and (iii) are obviously, and (ii) follows from the definition of K. Equations (iv) and (v) hold since

$$\begin{aligned}
\int_{\mathbb{R}^n \setminus B_\delta(x)} \ K(x, y, t) \ dy &= \int_{\mathbb{R}^n \setminus B_\delta(x)} \ (4\pi t)^{-n/2} \mathrm{e}^{-|x-y|^2/4t} \ dy \\
&= \pi^{-n/2} \int_{\mathbb{R}^n \setminus B_{\delta/\sqrt{4t}}(0)} \ e^{-|\eta|^2} \ d\eta
\end{aligned}$$

by using the substitution $y = x + (4t)^{1/2}\eta$. For fixed $\delta > 0$ it follows (v) and for $\delta := 0$ we obtain (iv). $\Box$

Theorem 6.1. *Assume $\phi \in C(\mathbb{R}^n)$ and $\sup_{\mathbb{R}^n} |\phi(x)| < \infty$. Then $u(x,t)$ given by Poisson's formula (6.4) is in $C^\infty(\mathbb{R}^n \times \mathbb{R}_+)$, continuous on $\mathbb{R}^n \times [0,\infty)$ and a solution of the initial value problem (6.1), (6.2).*

Proof. It remains to show

$$\lim_{\substack{x \to \xi \\ t \to 0}} u(x,t) = \phi(\xi).$$

Since ϕ is continuous there exists for given $\varepsilon > 0$ a $\delta = \delta(\varepsilon)$ such that $|\phi(y) -$

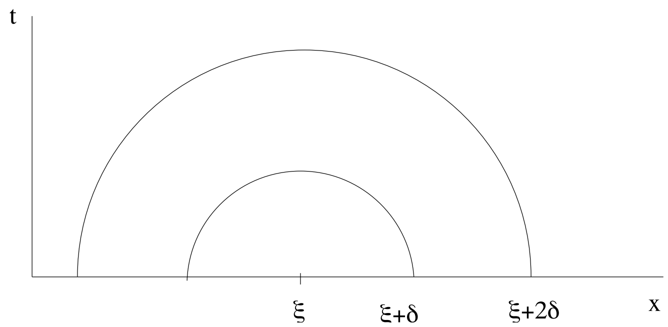

Figure 6.2: Figure to the proof of Theorem 6.1

$\phi(\xi)| < \varepsilon$ if $|y - \xi| < 2\delta$. Set $M := \sup_{\mathbb{R}^n} |\phi(y)|$. Then, see Proposition 6.1,

$$u(x,t) - \phi(\xi) = \int_{\mathbb{R}^n} K(x,y,t)\,(\phi(y) - \phi(\xi))\ dy.$$

It follows, if $|x-\xi|<\delta$ and $t>0$, that

$$\begin{aligned}
|u(x,t)-\phi(\xi)| &\le \int_{B_\delta(x)} K(x,y,t)\,|\phi(y)-\phi(\xi)|\ dy \\
&\quad +\int_{\mathbb{R}^n\setminus B_\delta(x)} K(x,y,t)\,|\phi(y)-\phi(\xi)|\ dy \\
&\le \int_{B_{2\delta}(x)} K(x,y,t)\,|\phi(y)-\phi(\xi)|\ dy \\
&\quad +2M\int_{\mathbb{R}^n\setminus B_\delta(x)} K(x,y,t)\ dy \\
&\le \varepsilon\int_{\mathbb{R}^n} K(x,y,t)\ dy+2M\int_{\mathbb{R}^n\setminus B_\delta(x)} K(x,y,t)\ dy \\
&< 2\varepsilon
\end{aligned}$$

if $0<t\le t_0$, t_0 sufficiently small. □

Remarks. 1. Uniqueness follows under the additional growth assumption

$$|u(x,t)|\le M\mathrm{e}^{a|x|^2}\ \text{ in } D_T,$$

where M and a are positive constants, see Proposition 6.2 below.
In the one-dimensional case, one has uniqueness in the class $u(x,t)\ge 0$ in D_T, see [10], pp. 222.

2. $u(x,t)$ defined by Poisson's formula depends on all values $\phi(y)$, $y\in\mathbb{R}^n$. That means, a perturbation of ϕ, even far from a fixed x, has influence to the value $u(x,t)$. In physical terms, this means that heat travels with infinite speed, in contrast to the experience.

6.2 Inhomogeneous heat equation

Here we consider the initial value problem for $u=u(x,t)$, $u\in C^\infty(\mathbb{R}^n\times R_+)$,

$$\begin{aligned}
u_t-\triangle u &= f(x,t)\ \text{ in } x\in\mathbb{R}^n,\ t\ge 0,\\
u(x,0) &= \phi(x),
\end{aligned}$$

where ϕ and f are given. From

$$\widehat{u_t-\triangle u}=\widehat{f(x,t)}$$

we obtain an initial value problem for an ordinary differential equation:

$$\begin{aligned}\frac{d\widehat{u}}{dt}+|\xi|^2\widehat{u} &= \widehat{f}(\xi,t)\\ \widehat{u}(\xi,0) &= \widehat{\phi}(\xi).\end{aligned}$$

The solution is given by

$$\widehat{u}(\xi,t)=\mathrm{e}^{-|\xi|^2t}\widehat{\phi}(\xi)+\int_0^t \mathrm{e}^{-|\xi|^2(t-\tau)}\widehat{f}(\xi,\tau)\ d\tau.$$

Applying the inverse Fourier transform and a calculation as in the proof of Theorem 5.1, step (vi), we get

$$\begin{aligned}u(x,t) &= (2\pi)^{-n/2}\int_{\mathbb{R}^n} \mathrm{e}^{ix\cdot\xi}\Big(\mathrm{e}^{-|\xi|^2t}\widehat{\phi}(\xi)\\ &\qquad +\int_0^t \mathrm{e}^{-|\xi|^2(t-\tau)}\widehat{f}(\xi,\tau)\ d\tau\Big)\ d\xi.\end{aligned}$$

From the above calculation for the homogeneous problem and calculation as in the proof of Theorem 5.1, step (vi), we obtain the formula

$$\begin{aligned}u(x,t) &= \frac{1}{(2\sqrt{\pi t})^n}\int_{\mathbb{R}^n} \phi(y)\mathrm{e}^{-|y-x|^2/(4t)}\ dy\\ &\quad +\int_0^t\int_{\mathbb{R}^n} f(y,\tau)\frac{1}{\Big(2\sqrt{\pi(t-\tau)}\Big)^n}\ \mathrm{e}^{-|y-x|^2/(4(t-\tau))}\ dy\ d\tau.\end{aligned}$$

This function $u(x,t)$ is a solution of the above inhomogeneous initial value problem provided

$$\phi\in C(\mathbb{R}^n),\quad \sup_{\mathbb{R}^n}|\phi(x)|<\infty$$

and if

$$f\in C(\mathbb{R}^n\times[0,\infty)),\quad M(\tau):=\sup_{\mathbb{R}^n}|f(y,\tau)|<\infty,\ 0\le\tau<\infty.$$

6.3 Maximum principle

Let $\Omega\subset\mathbb{R}^n$ be a *bounded* domain. Set

$$\begin{aligned}D_T &= \Omega\times(0,T),\quad T>0,\\ S_T &= \{(x,t):\ (x,t)\in\Omega\times\{0\}\ \text{or}\ (x,t)\in\partial\Omega\times[0,T]\},\end{aligned}$$

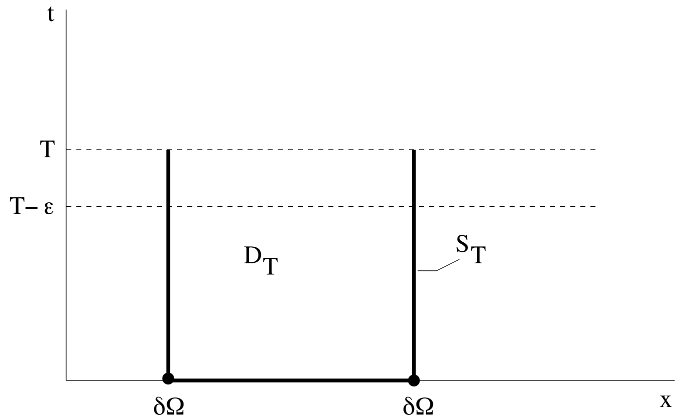

Figure 6.3: Notations to the maximum principle

see Figure 6.3

Theorem 6.2. *Assume* $u \in C(\overline{D_T})$, *that* u_t, $u_{x_i x_k}$ *exist and are continuous in* D_T, *and*

$$u_t - \triangle u \le 0 \quad \textit{in } D_T.$$

Then

$$\max_{\overline{D_T}} u(x,t) = \max_{S_T} u.$$

Proof. Assume initially $u_t - \triangle u < 0$ in D_T. Let $\varepsilon > 0$ be small and $0 < \varepsilon < T$. Since $u \in C(\overline{D_{T-\varepsilon}})$, there is an $(x_0, t_0) \in \overline{D_{T-\varepsilon}}$ such that

$$u(x_0, t_0) = \max_{\overline{D_{T-\varepsilon}}} u(x,t).$$

Case (i). Let $(x_0, t_0) \in D_{T-\varepsilon}$. Hence, since $D_{T-\varepsilon}$ is open, $u_t(x_0, t_0) = 0$, $u_{x_l}(x_0, t_0) = 0$, $l = 1, \ldots, n$ and

$$\sum_{l,k=1}^{n} u_{x_l x_k}(x_0, t_0)\zeta_l \zeta_k \le 0 \quad \text{for all } \zeta \in \mathbb{R}^n.$$

The previous inequality implies that $u_{x_k x_k}(x_0, t_0) \le 0$ for each k. Thus we arrived at a contradiction to $u_t - \triangle u < 0$ in D_T.

Case (ii). Assume $(x_0,t_0)\in\Omega\times\{T-\varepsilon\}$. Then it follows as above $\triangle u\le 0$ in (x_0,t_0), and from $u(x_0,t_0)\ge u(x_0,t)$, $t\le t_0$, one concludes that $u_t(x_0,t_0)\ge 0$. We arrived at a contradiction to $u_t-\triangle u<0$ in D_T again.

Summarizing, we have shown that

$$\max_{\overline{D_{T-\varepsilon}}} u(x,t)=\max_{T-\varepsilon} u(x,t).$$

Thus there is an $(x_\varepsilon,t_\varepsilon)\in S_{T-\varepsilon}$ such that

$$u(x_\varepsilon,t_\varepsilon)=\max_{\overline{D_{T-\varepsilon}}} u(x,t).$$

Since u is continuous on $\overline{D}_T$, we have

$$\lim_{\varepsilon\to 0}\max_{\overline{D_{T-\varepsilon}}} u(x,t)=\max_{\overline{D_T}} u(x,t).$$

It follows that there is $(\overline{x},\overline{t})\in S_T$ such that

$$u(\overline{x},\overline{t})=\max_{\overline{D_T}} u(x,t)$$

since $S_{T-\varepsilon}\subset S_T$ and S_T is compact. Thus, theorem is shown under the assumption $u_t-\triangle u<0$ in D_T. Now assume $u_t-\triangle u\le 0$ in D_T. Set

$$v(x,t):=u(x,t)-kt,$$

where k is a positive constant. Then

$$v_t-\triangle v=u_t-\triangle u-k<0.$$

From above we have

$$\begin{aligned}\max_{\overline{D_T}} u(x,t)&=\max_{\overline{D_T}}(v(x,t)+kt)\\ &\le \max_{\overline{D_T}} v(x,t)+kT\\ &=\max_{S_T} v(x,t)+kT\\ &\le \max_{S_T} u(x,t)+kT,\end{aligned}$$

Letting $k\to 0$, we obtain

$$\max_{\overline{D_T}} u(x,t)\le\max_{S_T} u(x,t).$$

Since $S_T \subset \overline{D_T}$, the theorem is shown. $\Box$

If we replace in the above theorem the bounded domain Ω by $\mathbb{R}^n$, then the result remains true provided we assume an *additional* growth assumption for u. More precisely, we have the following result which is a corollary of the previous theorem. Set for a fixed T, $0 < T < \infty$,

$$D_T = \{(x,t) : \ x \in \mathbb{R}^n, \ 0 < t < T\}.$$

Proposition 6.2. *Assume $u \in C(\overline{D_T})$, that u_t, $u_{x_i x_k}$ exist and are continuous in D_T,*

$$u_t - \triangle u \le 0 \quad in \ D_T,$$

and additionally that u satisfies the growth condition

$$u(x,t) \le M e^{a|x|^2},$$

where M and a are positive constants. Then

$$\max_{\overline{D_T}} u(x,t) = \max_{S_T} u.$$

It follows immediately the

Corollary. *The initial value problem $u_t - \triangle u = 0$ in D_T, $u(x,0) = f(x)$, $x \in \mathbb{R}^n$, has a unique solution in the class defined by $u \in C(\overline{D_T})$, u_t, $u_{x_i x_k}$ exist and are continuous in D_T and $|u(x,t)| \le M e^{a|x|^2}$.*

Proof of Proposition 6.2. See [10], pp. 217. We can assume that $4aT < 1$, since the *finite* interval can be divided into finite many intervals of equal length τ with $4a\tau < 1$. Then we conclude successively for k that

$$u(x,t) \le \sup_{y \in \mathbb{R}^n} u(y, k\tau) \le \sup_{y \in \mathbb{R}^n} u(y,0)$$

for $k\tau \le t \le (k+1)\tau$, $k = 0, \ldots, N-1$, where $N = T/\tau$.

There is an $\epsilon > 0$ such that $4a(T+\epsilon) < 1$. Consider the comparison function

$$\begin{aligned} v_\mu(x,t) : &= u(x,t) - \mu\,(4\pi(T+\epsilon-t))^{-n/2}\, \mathrm{e}^{|x-y|^2/(4(T+\epsilon-t))} \\ &= u(x,t) - \mu K(ix, iy, T+\epsilon-t) \end{aligned}$$

for fixed $y \in \mathbb{R}^n$ and for a constant $\mu > 0$. Since the heat kernel $K(ix, iy, t)$ satisfies $K_t = \triangle K_x$, we obtain

$$\frac{\partial}{\partial t} v_\mu - \triangle v_\mu = u_t - \triangle u \le 0.$$

Set for a constant $\rho > 0$

$$D_{T,\rho} = \{(x,t) :\ |x-y| < \rho,\ 0 < t < T\}.$$

Then we obtain from Theorem 6.2 that

$$v_\mu(y,t) \le \max_{S_{T,\rho}} v_\mu,$$

where $S_{T,\rho} \equiv S_T$ of Theorem 6.2 with $\Omega = B_\rho(y)$, see Figure 6.3. On the bottom of $S_{T,\rho}$ we have, since $\mu K > 0$,

$$v_\mu(x,0) \le u(x,0) \le \sup_{z \in \mathbb{R}^n} f(z).$$

On the cylinder part $|x-y| = \rho$, $0 \le t \le T$, of $S_{T,\rho}$ it is

$$\begin{array}{rcl} v_\mu(x,t) & \le & M \mathrm{e}^{a|x|^2} - \mu \left(4\pi(T+\epsilon-t)\right)^{-n/2} \mathrm{e}^{\rho^2/(4(T+\epsilon-t))} \\ & \le & M \mathrm{e}^{a(|y|+\rho)^2} - \mu \left(4\pi(T+\epsilon)\right)^{-n/2} \mathrm{e}^{\rho^2/(4(T+\epsilon))} \\ & \le & \sup\limits_{z \in \mathbb{R}^n} f(z) \end{array}$$

for all $\rho > \rho_0(\mu)$, ρ_0 sufficiently large. We recall that $4a(T+\epsilon) < 1$. Summarizing, we have

$$\max_{S_{T,\rho}} v_\mu(x,t) \le \sup_{z \in \mathbb{R}^n} f(z)$$

if $\rho > \rho_0(\mu)$. Thus

$$v_\mu(y,t) \le \max_{S_{T,\rho}} v_\mu(x,t) \le \sup_{z \in \mathbb{R}^n} f(z)$$

if $\rho > \rho_0(\mu)$. Since

$$v_\mu(y,t) = u(y,t) - \mu \left(4\pi(T+\epsilon-t)\right)^{-n/2}$$

it follows

$$u(y,t) - \mu \left(4\pi(T+\epsilon-t)\right)^{-n/2} \le \sup_{z \in \mathbb{R}^n} f(z).$$

Letting $\mu \to 0$, we obtain the assertion of the proposition. $\square$

The above maximum principle of Theorem 6.2 holds for a large class of parabolic differential operators, even for degenerate equations. Set

$$Lu = \sum_{i,j=1}^{n} a^{ij}(x,t)u_{x_ix_j},$$

where $a^{ij} \in C(D_T)$ are real, $a^{ij} = a^{ji}$, and the matrix (a^{ij}) is nonnegative, that is,

$$\sum_{i,j=1}^{n} a^{ij}(x,t)\zeta_i\zeta_j \geq 0 \;\; \text{for all } \zeta \in \mathbb{R}^n,$$

and $(x,t) \in D_T$.

Theorem 6.3. *Assume $u \in C(\overline{D_T})$, that u_t, $u_{x_ix_k}$ exist and are continuous in D_T, and*

$$u_t - Lu \leq 0 \;\; \text{in } D_T.$$

Then

$$\max_{\overline{D_T}} u(x,t) = \max_{S_T} u.$$

Proof. (i) One proof is a consequence of the following lemma: Let A, B real, symmetric and nonnegative matrices. Nonnegative means that all eigenvalues are nonnegative. Then trace $(AB) \equiv \sum_{i,j=1}^{n} a^{ij}b_{ij} \geq 0$, see an exercise.

(ii) Another proof exploits transform to principle axis directly: Let $U = (z_1, \ldots, z_n)$, where z_l is an orthonormal system of eigenvectors to the eigenvalues λ_l of the matrix $A = (a^{i,j}(x_0,t_0))$. Set $\zeta = U\eta$, $x = U^T(x - x_0)y$ and $v(y) = u(x_0 + Uy, t_0)$, then

$$\begin{aligned} 0 \;&\leq\; \sum_{i,j=1}^{n} a^{ij}(x_0,t_0)\zeta_i\zeta_j = \sum_{i=1}^{n} \lambda_i\eta_i^2 \\ 0 \;&\geq\; \sum_{i,j=1}^{n} u_{x_ix_j}\zeta_i\zeta_j = \sum_{i=1}^{n} v_{y_iy_i}\eta_i^2. \end{aligned}$$

It follows $\lambda_i \geq 0$ and $v_{y_iy_i} \leq 0$ for all i. Consequently

$$\sum_{i,j=1}^{n} a^{ij}(x_0,t_0)u_{x_ix_j}(x_0,t_0) = \sum_{i=1}^{n} \lambda_i v_{y_iy_i} \leq 0.$$

$\square$

6.4 Initial-boundary value problem

Consider the initial-boundary value problem for $c = c(x,t)$

$$\begin{aligned} c_t &= D\triangle c \ \text{ in } \Omega\times(0,\infty) && (6.5)\\ c(x,0) &= c_0(x) \ \ x\in\overline{\Omega} && (6.6)\\ \frac{\partial c}{\partial n} &= 0 \ \text{ on } \partial\Omega\times(0,\infty). && (6.7) \end{aligned}$$

Here is $\Omega\subset\mathbb{R}^n$, n the exterior unit normal at the smooth parts of $\partial\Omega$, D a positive constant and $c_0(x)$ a given function.

Remark. In application to diffusion problems, $c(x,t)$ is the concentration of a substance in a solution, $c_0(x)$ its initial concentration and D the coefficient of diffusion.
First Fick's rule says that $w = D\partial c/\partial n$, where w is the flow of the substance through the boundary $\partial\Omega$. Thus according to the Neumann boundary condition (6.7), we assume that there is no flow through the boundary.

6.4.1 Fourier's method

Separation of variables ansatz $c(x,t) = v(x)w(t)$ leads to the eigenvalue problem, see the arguments of Section 4.5,

$$\begin{aligned} -\triangle v &= \lambda v \ \text{ in } \Omega && (6.8)\\ \frac{\partial v}{\partial n} &= 0 \ \text{ on } \partial\Omega, && (6.9) \end{aligned}$$

and to the ordinary differential equation

$$w'(t) + \lambda D w(t) = 0. \qquad (6.10)$$

Assume Ω is bounded and $\partial\Omega$ sufficiently regular, then the eigenvalues of (6.8), (6.9) are countable and

$$0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \ldots \to \infty.$$

Let $v_j(x)$ be a complete system of orthonormal (in $L^2(\Omega)$) eigenfunctions. Solutions of (6.10) are

$$w_j(t) = C_j \mathrm{e}^{-D\lambda_j t},$$

where C_j are arbitrary constants.

According to the superposition principle,

$$c_N(x,t) := \sum_{j=0}^{N} C_j \mathrm{e}^{-D\lambda_j t} v_j(x)$$

is a solution of the differential equation (6.8) and

$$c(x,t) := \sum_{j=0}^{\infty} C_j \mathrm{e}^{-D\lambda_j t} v_j(x),$$

with

$$C_j = \int_\Omega c_0(x) v_j(x)\ dx,$$

is a formal solution of the initial-boundary value problem (6.5)-(6.7).

Diffusion in a tube

Consider a solution in a tube, see Figure 6.4. Assume the initial concentra-

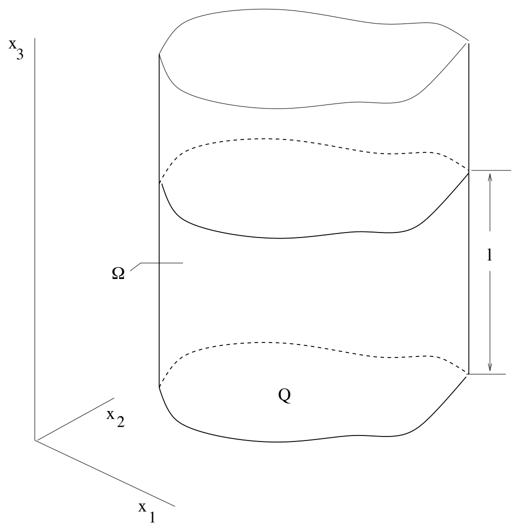

Figure 6.4: Diffusion in a tube

tion $c_0(x_1, x_2, x_3)$ of the substrate in a solution is constant if $x_3 = const.$ It follows from a uniqueness result below that the solution of the initial-boundary value problem $c(x_1, x_2, x_3, t)$ is independent of x_1 and x_2.

Set $z = x_3$, then the above initial-boundary value problem reduces to

$$\begin{aligned} c_t &= Dc_{zz} \\ c(z,0) &= c_0(z) \\ c_z &= 0, \quad z = 0,\ z = l. \end{aligned}$$

The (formal) solution is

$$c(z,t) = \sum_{n=0}^{\infty} C_n \mathrm{e}^{-D\left(\frac{\pi}{l}n\right)^2 t} \cos\left(\frac{\pi}{l}nz\right),$$

where

$$\begin{aligned} C_0 &= \frac{1}{l}\int_0^l c_0(z)\ dz \\ C_n &= \frac{2}{l}\int_0^l c_0(z)\cos\left(\frac{\pi}{l}nz\right)\ dz, \quad n \geq 1. \end{aligned}$$

6.4.2 Uniqueness

Sufficiently regular solutions of the initial-boundary value problem (6.5)-(6.7) are uniquely determined since from

$$\begin{aligned} c_t &= D\triangle c \ \text{ in } \Omega \times (0,\infty) \\ c(x,0) &= 0 \\ \frac{\partial c}{\partial n} &= 0 \ \text{ on } \partial\Omega \times (0,\infty). \end{aligned}$$

it follows that for each $\tau > 0$

$$\begin{aligned} 0 &= \int_0^\tau \int_\Omega (c_t c - D(\triangle c)c)\ dxdt \\ &= \int_\Omega \int_0^\tau \frac{1}{2}\frac{\partial}{\partial t}(c^2)\ dtdx + D\int_\Omega \int_0^\tau |\nabla_x c|^2\ dxdt \\ &= \frac{1}{2}\int_\Omega c^2(x,\tau)\ dx + D\int_\Omega \int_0^\tau |\nabla_x c|^2\ dxdt. \end{aligned}$$

6.5 Black-Scholes equation

Solutions of the Black-Scholes equation define the value of a derivative, for example of a call or put option, which is based on an asset. An asset can be a stock or a derivative again, for example. In principle, there are infinitely many such products, for example n-th derivatives. The Black-Scholes equation for the value $V(S,t)$ of a derivative is

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV = 0 \ \text{ in } \Omega, \tag{6.11}$$

where for a fixed T, $0 < T < \infty$,

$$\Omega = \{(S,t) \in \mathbb{R}^2 : \ 0 < S < \infty, \ 0 < t < T\},$$

and σ, r are positive constants. More precisely,
σ is the volatility of the underlying asset S,
r is the guaranteed interest rate of a risk-free investment.

If $S(t)$ is the value of an asset at time t, then $V(S(t),t)$ is the value of the derivative at time t, where $V(S,t)$ is the solution of an appropriate initial-boundary value problem for the Black-Scholes equation, see below.

The Black-Scholes equation follows from Ito's Lemma under some assumptions on the random function associated to $S(t)$, see [26], for example.

Call option

Here is $V(S,t) := C(S,t)$, where $C(S,t)$ is the value of the (European) call option. In this case we have following side conditions to (6.11):

$$\begin{aligned} C(S,T) &= \max\{S-E,0\} && (6.12) \\ C(0,t) &= 0 && (6.13) \\ C(S,t) &= S + o(S) \text{ as } S \to \infty, \text{ uniformly in } t, && (6.14) \end{aligned}$$

where E and T are positive constants, E is the exercise price and T the expiry.

Side condition (6.12) means that the value of the option has no value at time T if $S(T) \leq E$,
condition (6.13) says that it makes no sense to buy assets if the value of the asset is zero,
condition (6.14) means that we buy assets if its value becomes large, see Figure 6.5, where the side conditions are indicated.

Theorem 6.4 (Black-Scholes formula for European call options). *The solution $C(S,t)$, $0 \leq S < \infty$, $0 \leq t \leq T$, of the initial-boundary value problem (6.11)-(6.14) is explicitly known and is given by*

$$C(S,t) = SN(d_1) - Ee^{-r(T-t)}N(d_2),$$

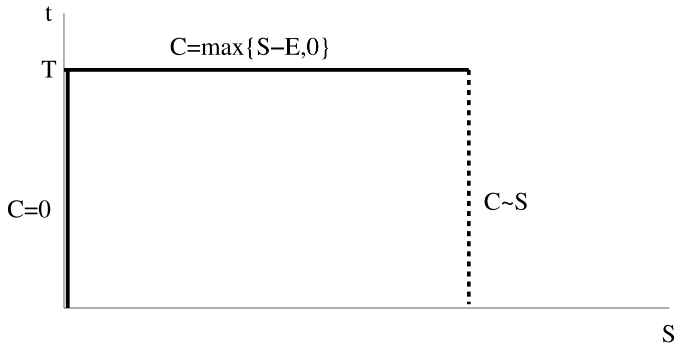

Figure 6.5: Side conditions for a call option

where

$$
\begin{aligned}
N(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\,dy, \\
d_1 &= \frac{\ln(S/E) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \\
d_2 &= \frac{\ln(S/E) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}.
\end{aligned}
$$

Proof. Substitutions

$$S = E\mathrm{e}^x, \quad t = T - \frac{\tau}{\sigma^2/2}, \quad C = Ev(x,\tau)$$

change equation (6.11) to

$$v_\tau = v_{xx} + (k-1)v_x - kv, \tag{6.15}$$

where

$$k = \frac{r}{\sigma^2/2}.$$

Initial condition (6.15) implies

$$v(x,0) = \max\{\mathrm{e}^x - 1, 0\}. \tag{6.16}$$

For a solution of (6.15) we make the ansatz

$$v = \mathrm{e}^{\alpha x + \beta\tau} u(x,\tau),$$

where α and β are constants which we will determine as follows. Inserting the ansatz into differential equation (6.15), we get

$$\beta u + u_\tau = \alpha^2 u + 2\alpha u_x + u_{xx} + (k-1)(\alpha u + u_x) - ku.$$

Set $\beta = \alpha^2 + (k-1)\alpha - k$ and choose α such that $0 = 2\alpha + (k-1)$, then $u_\tau = u_{xx}$. Thus

$$v(x,\tau) = \mathrm{e}^{-(k-1)x/2-(k+1)^2\tau/4}u(x,\tau), \tag{6.17}$$

where $u(x,\tau)$ is a solution of the initial value problem

$$\begin{aligned} u_\tau &= u_{xx}, \quad -\infty < x < \infty,\ \tau > 0 \\ u(x,0) &= u_0(x), \end{aligned}$$

with

$$u_0(x) = \max\left\{\mathrm{e}^{(k+1)x/2} - \mathrm{e}^{(k-1)x/2}, 0\right\}.$$

A solution of this initial value problem is given by Poisson's formula

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{+\infty} u_0(s)\mathrm{e}^{-(x-s)^2/(4\tau)}\ ds.$$

Changing variable by $q = (s-x)/(\sqrt{2\tau})$, we get

$$\begin{aligned} u(x,\tau) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} u_0(q\sqrt{2\tau} + x)\mathrm{e}^{-q^2/2}\ dq \\ &= I_1 - I_2, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \frac{1}{\sqrt{2\pi}} \int_{-x/(\sqrt{2\tau})}^{\infty} \mathrm{e}^{(k+1)(x+q\sqrt{2\tau})}\mathrm{e}^{-q^2/2}\ dq \\ I_2 &= \frac{1}{\sqrt{2\pi}} \int_{-x/(\sqrt{2\tau})}^{\infty} \mathrm{e}^{(k-1)(x+q\sqrt{2\tau})}\mathrm{e}^{-q^2/2}\ dq. \end{aligned}$$

An elementary calculation shows that

$$\begin{aligned} I_1 &= \mathrm{e}^{(k+1)x/2+(k+1)^2\tau/4}N(d_1) \\ I_2 &= \mathrm{e}^{(k-1)x/2+(k-1)^2\tau/4}N(d_2), \end{aligned}$$

where

$$\begin{aligned} d_1 &= \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k+1)\sqrt{2\tau} \\ d_2 &= \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k-1)\sqrt{2\tau} \\ N(d_i) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_i} \mathrm{e}^{-s^2/2}\, ds, \quad i = 1,\ 2. \end{aligned}$$

Combining the formula for $u(x,\tau)$, definition (6.17) of $v(x,\tau)$ and the previous settings $x = \ln(S/E)$, $\tau = \sigma^2(T-t)/2$ and $C = Ev(x,\tau)$, we get finally the formula of of Theorem 6.4.

In general, the solution u of the initial value problem for the heat equation is not uniquely defined, see for example [10], pp. 206.

Uniqueness. The uniqueness follows from the growth assumption (6.14). Assume there are two solutions of (6.11), (6.12)-(6.14), then the difference $W(S,t)$ satisfies the differential equation (6.11) and the side conditions

$$W(S,T) = 0,\ W(0,t) = 0,\ W(S,t) = O(S) \text{ as } S \to \infty$$

uniformly in $0 \le t \le T$.

From a maximum principle consideration, see an exercise, it follows that $|W(S,t)| \le cS$ on $S \ge 0$, $0 \le t \le T$. The constant c is independent on S and t. From the definition of u we see that

$$u(x,\tau) = \frac{1}{E}\mathrm{e}^{-\alpha x - \beta\tau} W(S,t),$$

where $S = E\mathrm{e}^x$, $t = T - 2\tau/(\sigma^2)$. Thus we have the growth property

$$|u(x,\tau)| \le M\mathrm{e}^{a|x|}, \quad x \in \mathbb{R}, \tag{6.18}$$

with positive constants M and a. Then the solution of $u_\tau = u_{xx}$, in $-\infty < x < \infty$, $0 \le \tau \le \sigma^2 T/2$, with the initial condition $u(x,0) = 0$ is uniquely defined in the class of functions satisfying the growth condition (6.18), see Proposition 6.2 of this chapter. That is, $u(x,\tau) \equiv 0$. $\square$

Put option

Here is $V(S,t) := P(S,t)$, where $P(S,t)$ is the value of the (European) put option. In this case we have following side conditions to (6.11):

$$\begin{aligned} P(S,T) &= \max\{E - S, 0\} && (6.19) \\ P(0,t) &= E\mathrm{e}^{-r(T-t)} && (6.20) \\ P(S,t) &= o(S) \text{ as } S \to \infty, \text{ uniformly in } 0 \le t \le T. && (6.21) \end{aligned}$$

Here E is the exercise price and T the expiry.

Side condition (6.19) means that the value of the option has no value at time T if $S(T) \ge E$,
condition (6.20) says that it makes no sense to sell assets if the value of the asset is zero,
condition (6.21) means that it makes no sense to sell assets if its value becomes large.

Theorem 6.5 (Black-Scholes formula for European put options). *The solution $P(S,t)$, $0 < S < \infty$, $t < T$ of the initial-boundary value problem (6.11), (6.19)-(6.21) is explicitly known and is given by*

$$P(S,t) = Ee^{-r(T-t)}N(-d_2) - SN(-d_1)$$

where $N(x)$, d_1, d_2 are the same as in Theorem 6.4.

Proof. The formula for the put option follows by the same calculations as in the case of a call option or from the put-call parity

$$C(S,t) - P(S,t) = S - E\mathrm{e}^{-r(T-t)}$$

and from

$$N(x) + N(-x) = 1.$$

Concerning the put-call parity see an exercise. See also [26], pp. 40, for a heuristic argument which leads to the formula for the put-call parity. □

6.6 Exercises

1. Show that the solution $u(x,t)$ given by Poisson's formula satisfies
$$\inf_{z\in\mathbb{R}^n}\varphi(z)\le u(x,t)\le\sup_{z\in\mathbb{R}^n}\varphi(z)\ ,$$
provided $\varphi(x)$ is continuous and bounded on $\mathbb{R}^n$.

2. Solve for given $f(x)$ and $\mu\in\mathbb{R}$ the initial value problem
$$\begin{aligned} u_t+u_x+\mu u_{xxx} &= 0 \quad \text{in} \quad \mathbb{R}\times\mathbb{R}_+\\ u(x,0) &= f(x)\ . \end{aligned}$$

3. Show by using Poisson's formula:
(i) Each function $f\in C([a,b])$ can be approximated uniformly by a sequence $f_n\in C^\infty[a,b]$.
(ii) In (i) we can choose polynomials f_n (Weierstrass's approximation theorem).

   Hint: Concerning (ii), replace the kernel $K=exp(-\frac{|y-x|^2}{4t})$ by a sequence of Taylor polynomials in the variable $z=-\frac{|y-x|^2}{4t}$.

4. Let $u(x,t)$ be a positive solution of
$$u_t=\mu u_{xx},\ t>0,$$
where μ is a constant. Show that $\theta:=-2\mu u_x/u$ is a solution of Burger's equation
$$\theta_t+\theta\theta_x=\mu\theta_{xx},\ t>0.$$

5. Assume $u_1(s,t),...,u_n(s,t)$ are solutions of $u_t=u_{ss}$. Show that $\prod_{k=1}^n u_k(x_k,t)$ is a solution of the heat equation $u_t-\triangle u=0$ in $\mathbb{R}^n\times(0,\infty)$.

6. Let A, B are real, symmetric and nonnegative matrices. Nonnegative means that all eigenvalues are nonnegative. Prove that trace $(AB)\equiv\sum_{i,j=1}^n a^{ij}b_{ij}\ge 0$.

   Hint: (i) Let $U=(z_1,\ldots,z_n)$, where z_l is an orthonormal system of eigenvectors to the eigenvalues λ_l of the matrix B. Then
$$X=U\begin{pmatrix}\sqrt{\lambda_1} & 0 & \cdots & 0\\ 0 & \sqrt{\lambda_2} & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & \cdots & \sqrt{\lambda_n}\end{pmatrix}U^T$$

is a square root of B. We recall that

$$U^TBU=\begin{pmatrix} \lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & \cdots & 0\\ \multicolumn{4}{c}{\dotfill}\\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}.$$

(ii) trace (QR) =trace (RQ).
(iii) Let $\mu_1(C),\ldots\mu_n(C)$ are the eigenvalues of a real symmetric $n\times n$-matrix. Then trace $C=\sum_{l=1}^n\mu_l(C)$, which follows from the fundamental lemma of algebra:

$$\begin{aligned}\det\,(\lambda I-C) &= \lambda^n-(c_{11}+\ldots+c_{nn})\lambda^{n-1}+\ldots\\ &\equiv (\lambda-\mu_1)\cdot\ldots\cdot(\lambda-\mu_n)\\ &= \lambda^n-(\mu_1+\ldots+\mu_n)\lambda^{n+1}+\ldots\end{aligned}$$

7. Assume Ω is bounded, u is a solution of the heat equation and u satisfies the regularity assumptions of the maximum principle (Theorem 6.2). Show that u achieves its maximum and its minimum on S_T.

8. Prove the following comparison principle: Assume Ω is bounded and u,v satisfy the regularity assumptions of the maximum principle. Then

$$\begin{aligned}u_t-\triangle u &\le v_t-\triangle v \ \text{ in } D_T\\ u &\le v \ \text{ on } S_T\end{aligned}$$

imply that $u\le v$ in D_T.

9. Show that the comparison principle implies the maximum principle.

10. Consider the boundary-initial value problem

$$\begin{aligned}u_t-\triangle u &= f(x,t) \ \text{ in } D_T\\ u(x,t) &= \phi(x,t) \ \text{ on } S_T,\end{aligned}$$

where f, ϕ are given.
Prove uniqueness in the class $u,\ u_t,\ u_{x_ix_k}\in C(\overline{D_T})$.

11. Assume $u,\ v_1,\ v_2\in C^2(D_T)\cap C(\overline{D_T})$, and u is a solution of the previous boundary-initial value problem and v_1, v_2 satisfy

$$\begin{aligned}(v_1)_t-\triangle v_1 &\le f(x,t)\le (v_2)_t-\triangle v_2 \ \text{ in } D_T\\ v_1 &\le \phi\le v_2 \ \text{ on } S_T.\end{aligned}$$

Show that (inclusion theorem)

$$v_1(x,t) \le u(x,t) \le v_2(x,t) \quad \text{on } \overline{D_T}.$$

12. Show by using the comparison principle: let u be a sufficiently regular solution of

$$\begin{aligned} u_t - \triangle u &= 1 \quad \text{in} \quad D_T \\ u &= 0 \quad \text{on} \quad S_T, \end{aligned}$$

then $0 \le u(x,t) \le t \quad \text{in} \quad D_T$.

13. Discuss the result of Theorem 6.3 for the case

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x,t)u_{x_i x_j} + \sum_{i}^{n} b_i(x,t)u_{x_i} + c(x,t)u(x,t).$$

14. Show that

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2 t} \sin(nx),$$

where

$$c_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\ dx,$$

is a solution of the initial-boundary value problem

$$\begin{aligned} u_t &= u_{xx},\ x \in (0,\pi),\ t > 0, \\ u(x,0) &= f(x), \\ u(0,t) &= 0, \\ u(\pi,t) &= 0, \end{aligned}$$

if $f \in C^4(\mathbb{R})$ is odd with respect to 0 and 2π-periodic.

15. (i) Find the solution of the diffusion problem $c_t = Dc_{zz}$ in $0 \le z \le l$, $0 \le t < \infty$, $D = const. > 0$, under the boundary conditions $c_z(z,t) = 0$ if $z = 0$ and $z = l$ and with the given initial concentration

$$c(z,0) = c_0(z) := \begin{cases} c_0 = const. & \text{if} \quad 0 \le z \le h \\ 0 & \text{if} \quad h < z \le l. \end{cases}$$

(ii) Calculate $\lim_{t\to\infty} c(z,t)$.

16. Prove the Black-Scholes Formel for an European put option.

    Hint: Put-call parity.

17. Prove the put-call parity for European options

$$C(S,t) - P(S,t) = S - E\mathrm{e}^{-r(T-t)}$$

    by using the following uniqueness result: Assume W is a solution of (6.11) under the side conditions $W(S,T) = 0$, $W(0,t) = 0$ and $W(S,t) = O(S)$ as $S \to \infty$, uniformly on $0 \le t \le T$. Then $W(S,t) \equiv 0$.

18. Prove that a solution $V(S,t)$ of the initial-boundary value problem (6.11) in Ω under the side conditions (i) $V(S,T) = 0$, $S \ge 0$, (ii) $V(0,t) = 0$, $0 \le t \le T$, (iii) $\lim_{S\to\infty} V(S,t) = 0$ uniformly in $0 \le t \le T$, is uniquely determined in the class $C^2(\Omega) \cap C(\overline{\Omega})$.

19. Prove that a solution $V(S,t)$ of the initial-boundary value problem (6.11) in Ω, under the side conditions (i) $V(S,T) = 0$, $S \ge 0$, (ii) $V(0,t) = 0$, $0 \le t \le T$, (iii) $V(S,t) = S + o(S)$ as $S \to \infty$, uniformly on $0 \le t \le T$, satisfies $|V(S,t)| \le cS$ for all $S \ge 0$ and $0 \le t \le T$.

Chapter 7

Elliptic equations of second order

Here we consider linear elliptic equations of second order, mainly the Laplace equation

$$\triangle u = 0.$$

Solutions of the Laplace equation are called *potential functions* or *harmonic functions*. The Laplace equation is called also potential equation.

The general elliptic equation for a scalar function $u(x)$, $x \in \Omega \subset \mathbb{R}^n$, is

$$Lu := \sum_{i,j=1}^{n} a^{ij}(x) u_{x_i x_j} + \sum_{j=1}^{n} b^j(x) u_{x_j} + c(x)u = f(x),$$

where the matrix $A = (a^{ij})$ is real, symmetric and positive definite. If A is a constant matrix, then a transform to principal axis and stretching of axis leads to

$$\sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} = \triangle v,$$

where $v(y) := u(Ty)$, T stands for the above composition of mappings.

7.1 Fundamental solution

Here we consider particular solutions of the Laplace equation in $\mathbb{R}^n$ of the type

$$u(x) = f(|x - y|),$$

where $y \in \mathbb{R}^n$ is fixed and f is a function which we will determine such that u defines a solution if the Laplace equation.

Set $r = |x - y|$, then

$$\begin{aligned} u_{x_i} &= f'(r)\frac{x_i - y_i}{r} \\ u_{x_i x_i} &= f''(r)\frac{(x_i - y_i)^2}{r^2} + f'(r)\left(\frac{1}{r} - \frac{(x_i - y_i)^2}{r^3}\right) \\ \triangle u &= f''(r) + \frac{n-1}{r} f'(r). \end{aligned}$$

Thus a solution of $\triangle u = 0$ is given by

$$f(r) = \begin{cases} c_1 \ln r + c_2 & : \quad n = 2 \\ c_1 r^{2-n} + c_2 & : \quad n \geq 3 \end{cases}$$

with constants c_1, c_2.

Definition. Set $r = |x - y|$. The function

$$s(r) := \begin{cases} -\frac{1}{2\pi} \ln r & : \quad n = 2 \\ \frac{r^{2-n}}{(n-2)\omega_n} & : \quad n \geq 3 \end{cases}$$

is called *singularity function* associated to the Laplace equation. Here is ω_n the area of the n-dimensional unit sphere which is given by $\omega_n = 2\pi^{n/2}/\Gamma(n/2)$, where

$$\Gamma(t) := \int_0^\infty \mathrm{e}^{-\rho} \rho^{t-1}\, d\rho, \quad t > 0,$$

is the Gamma function.

Definition. A function

$$\gamma(x, y) = s(r) + \phi(x, y)$$

is called *fundamental solution* associated to the Laplace equation if $\phi \in C^2(\Omega)$ and $\triangle_x \phi = 0$ for each fixed $y \in \Omega$.

Remark. The fundamental solution γ satisfies for each fixed $y \in \Omega$ the relation

$$-\int_\Omega \gamma(x, y) \triangle_x \Phi(x)\, dx = \Phi(y) \quad \text{for all } \Phi \in C_0^2(\Omega),$$

see an exercise. This formula follows from considerations similar to the next section.

In the language of distribution, this relation can be written by definition as

$$-\triangle_x \gamma(x,y) = \delta(x-y),$$

where δ is the Dirac distribution, which is called δ-function.

7.2 Representation formula

In the following we assume that Ω, the function ϕ which appears in the definition of the fundamental solution and the potential function u considered are sufficiently regular such that the following calculations make sense, see [6] for generalizations. This is the case if Ω is bounded, $\partial\Omega$ is in C^1, $\phi \in C^2(\overline{\Omega})$ for each fixed $y \in \Omega$ and $u \in C^2(\overline{\Omega})$.

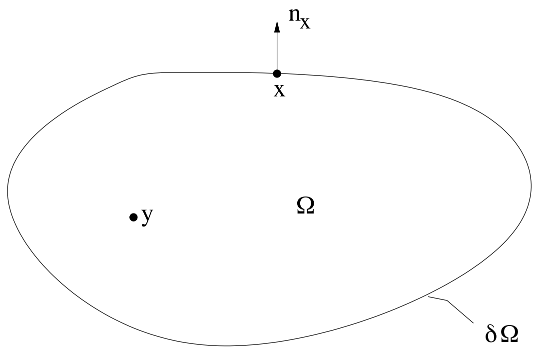

Figure 7.1: Notations to Green's identity

Theorem 7.1. *Let u be a potential function and γ a fundamental solution, then for each fixed $y \in \Omega$*

$$u(y) = \int_{\partial\Omega} \left(\gamma(x,y) \frac{\partial u(x)}{\partial n_x} - u(x) \frac{\partial \gamma(x,y)}{\partial n_x} \right) \, dS_x.$$

Proof. Let $B_\rho(y) \subset \Omega$ be a ball. Set $\Omega_\rho(y) = \Omega \setminus B_\rho(y)$. See Figure 7.2 for notations. From Green's formula, for $u,\ v \in C^2(\overline{\Omega})$,

$$\int_{\Omega_\rho(y)} (v \triangle u - u \triangle v) \, dx = \int_{\partial\Omega_\rho(y)} \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) \, dS$$

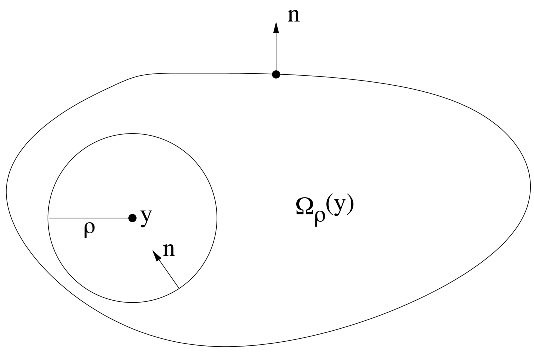

Figure 7.2: Notations to Theorem 7.1

we obtain, if v is a fundamental solution and u a potential function,

$$\int_{\partial\Omega_\rho(y)} \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right)\ dS = 0.$$

Thus we have to consider

$$\begin{aligned}\int_{\partial\Omega_\rho(y)} v\frac{\partial u}{\partial n}\ dS &= \int_{\partial\Omega} v\frac{\partial u}{\partial n}\ dS + \int_{\partial B_\rho(y)} v\frac{\partial u}{\partial n}\ dS\\ \int_{\partial\Omega_\rho(y)} u\frac{\partial v}{\partial n}\ dS &= \int_{\partial\Omega} u\frac{\partial v}{\partial n}\ dS + \int_{\partial B_\rho(y)} u\frac{\partial v}{\partial n}\ dS.\end{aligned}$$

We estimate the integrals over $\partial B_\rho(y)$:

(i)

$$\begin{aligned}\left|\int_{\partial B_\rho(y)} v\frac{\partial u}{\partial n}\ dS\right| &\le M\int_{\partial B_\rho(y)} |v|\ dS\\ &\le M\left(\int_{\partial B_\rho(y)} s(\rho)\ dS + C\omega_n\rho^{n-1}\right),\end{aligned}$$

where

$$\begin{aligned}M &= M(y) = \sup_{B_{\rho_0}(y)} |\partial u/\partial n|,\quad \rho\le\rho_0,\\ C &= C(y) = \sup_{x\in B_{\rho_0}(y)} |\phi(x,y)|.\end{aligned}$$

From the definition of $s(\rho)$ we get the estimate as $\rho \to 0$

$$\int_{\partial B_\rho(y)} v\frac{\partial u}{\partial n}\, dS = \begin{cases} O(\rho|\ln\rho|) & : \quad n=2 \\ O(\rho) & : \quad n\geq 3. \end{cases} \tag{7.1}$$

(ii) Consider the case $n \geq 3$, then

$$\begin{aligned} \int_{\partial B_\rho(y)} u\frac{\partial v}{\partial n}\, dS &= \frac{1}{\omega_n}\int_{\partial B_\rho(y)} u\frac{1}{\rho^{n-1}}\, dS + \int_{\partial B_\rho(y)} u\frac{\partial\phi}{\partial n}\, dS \\ &= \frac{1}{\omega_n\rho^{n-1}}\int_{\partial B_\rho(y)} u\, dS + O(\rho^{n-1}) \\ &= \frac{1}{\omega_n\rho^{n-1}}u(x_0)\int_{\partial B_\rho(y)} dS + O(\rho^{n-1}), \\ &= u(x_0) + O(\rho^{n-1}). \end{aligned}$$

for an $x_0 \in \partial B_\rho(y)$.

Combining this estimate and (7.1), we obtain the representation formula of the theorem. □

Corollary. Set $\phi \equiv 0$ and $r = |x-y|$ in the representation formula of Theorem 7.1, then

$$u(y) = \frac{1}{2\pi}\int_{\partial\Omega}\left(\ln r\frac{\partial u}{\partial n_x} - u\frac{\partial(\ln r)}{\partial n_x}\right) dS_x, \quad n=2, \tag{7.2}$$

$$u(y) = \frac{1}{(n-2)\omega_n}\int_{\partial\Omega}\left(\frac{1}{r^{n-2}}\frac{\partial u}{\partial n_x} - u\frac{\partial(r^{2-n})}{\partial n_x}\right) dS_x, \quad n\geq 3. \tag{7.3}$$

7.2.1 Conclusions from the representation formula

Similar to the theory of functions of one complex variable, we obtain here results for harmonic functions from the representation formula, in particular from (7.2), (7.3). We recall that a function u is called *harmonic* if $u \in C^2(\Omega)$ and $\triangle u = 0$ in Ω.

Proposition 7.1. *Assume u is harmonic in Ω. Then $u \in C^\infty(\Omega)$.*

Proof. Let $\Omega_0 \subset\subset \Omega$ be a domain such that $y \in \Omega_0$. It follows from representation formulas (7.2), (7.3), where $\Omega := \Omega_0$, that $D^l u(y)$ exist and are continuous for all l since one can change differentiation with integration in right hand sides of the representation formulae. $\Box$

Remark. In fact, a function which is harmonic in Ω is even real analytic in Ω, see an exercise.

Proposition 7.2 (Mean value formula for harmonic functions). *Assume u is harmonic in Ω. Then for each $B_\rho(x) \subset\subset \Omega$*

$$u(x) = \frac{1}{\omega_n \rho^{n-1}} \int_{\partial B_\rho(x)} u(y)\ dS_y.$$

Proof. Consider the case $n \geq 3$. The assertion follows from (7.3) where $\Omega := B_\rho(x)$ since $r = \rho$ and

$$\begin{aligned}
\int_{\partial B_\rho(x)} \frac{1}{r^{n-2}} \frac{\partial u}{\partial n_y}\ dS_y &= \frac{1}{\rho^{n-2}} \int_{\partial B_\rho(x)} \frac{\partial u}{\partial n_y}\ dS_y \\
&= \frac{1}{\rho^{n-2}} \int_{B_\rho(x)} \triangle u\ dy \\
&= 0.
\end{aligned}$$

$\Box$

We recall that a domain $\Omega \in \mathbb{R}^n$ is called connected if Ω is not the union of two nonempty open subsets Ω_1, Ω_2 such that $\Omega_1 \cap \Omega_2 = \emptyset$. A domain in $\mathbb{R}^n$ is connected if and only if its path connected.

Proposition 7.3 (Maximum principle). *Assume u is harmonic in a connected domain and achieves its supremum or infimum in Ω. Then $u \equiv const.$ in Ω.*

Proof. Consider the case of the supremum. Let $x_0 \in \Omega$ such that

$$u(x_0) = \sup_\Omega u(x) =: M.$$

Set $\Omega_1 := \{x \in \Omega :\ u(x) = M\}$ and $\Omega_2 := \{x \in \Omega :\ u(x) < M\}$. The set Ω_1 is not empty since $x_0 \in \Omega_1$. The set Ω_2 is open since $u \in C^2(\Omega)$. Consequently, Ω_2 is empty if we can show that Ω_1 is open. Let $\overline{x} \in \Omega_1$, then there is a $\rho_0 > 0$ such that $\overline{B_{\rho_0}(\overline{x})} \subset \Omega$ and $u(x) = M$ for all $x \in B_{\rho_0}(\overline{x})$.

If not, then there exists $\rho > 0$ and $\widehat{x}$ such that $|\widehat{x} - \overline{x}| = \rho$, $0 < \rho < \rho_0$ and $u(\widehat{x}) < M$. From the mean value formula, see Proposition 7.2, it follows

$$M = \frac{1}{\omega_n \rho^{n-1}} \int_{\partial B_\rho(\overline{x})} u(x)\ dS < \frac{M}{\omega_n \rho^{n-1}} \int_{\partial B_\rho(\overline{x})} dS = M,$$

which is a contradiction. Thus, the set Ω_2 is empty since Ω_1 is open. $\Box$

Corollary. Assume Ω is connected and bounded, and $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is harmonic in Ω. Then u achieves its minimum and its maximum on the boundary $\partial\Omega$.

Remark. The previous corollary fails if Ω is not bounded as simple counterexamples show.

7.3 Boundary value problems

Assume $\Omega \subset \mathbb{R}^n$ is a connected domain.

7.3.1 Dirichlet problem

The *Dirichlet problem* (first boundary value problem) is to find a solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ of

$$\begin{aligned} \triangle u &= 0 \ \text{ in } \Omega && (7.4) \\ u &= \Phi \ \text{ on } \partial\Omega, && (7.5) \end{aligned}$$

where Φ is given and continuous on $\partial\Omega$.

Proposition 7.4. *Assume Ω is bounded, then a solution to the Dirichlet problem is uniquely determined.*

Proof. Maximum principle.

Remark. The previous result fails if we take away in the boundary condition (7.5) one point from the the boundary as the following example shows. Let $\Omega \subset \mathbb{R}^2$ be the domain

$$\Omega = \{x \in B_1(0) :\ x_2 > 0\},$$

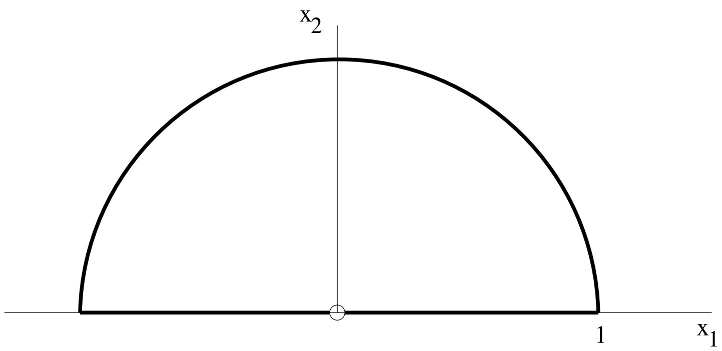

Figure 7.3: Counterexample

Assume $u \in C^2(\Omega) \cap C(\overline{\Omega} \setminus \{0\})$ is a solution of

$$\begin{aligned} \triangle u &= 0 \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \setminus \{0\}. \end{aligned}$$

This problem has solutions $u \equiv 0$ *and* $u = \text{Im}(z + z^{-1})$, where $z = x_1 + ix_2$. Another example see an exercise.

In contrast to this behaviour of the Laplace equation, one has uniqueness if $\triangle u = 0$ is replaced by the minimal surface equation

$$\frac{\partial}{\partial x_1}\left(\frac{u_{x_1}}{\sqrt{1+|\nabla u|^2}}\right) + \frac{\partial}{\partial x_2}\left(\frac{u_{x_2}}{\sqrt{1+|\nabla u|^2}}\right) = 0.$$

7.3.2 Neumann problem

The *Neumann problem* (second boundary value problem) is to find a solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ of

$$\triangle u = 0 \text{ in } \Omega \tag{7.6}$$

$$\frac{\partial u}{\partial n} = \Phi \text{ on } \partial\Omega, \tag{7.7}$$

where Φ is given and continuous on $\partial\Omega$.

Proposition 7.5. *Assume Ω is bounded, then a solution to the Dirichlet problem is in the class $u \in C^2(\overline{\Omega})$ uniquely determined up to a constant.*

Proof. Exercise. Hint: Multiply the differential equation $\triangle w = 0$ by w and integrate the result over Ω.

Another proof under the weaker assumption $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ follows from the Hopf boundary point lemma, see Lecture Notes: Linear Elliptic Equations of Second Order, for example.

7.3.3 Mixed boundary value problem

The *Mixed boundary value problem* (third boundary value problem) is to find a solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ of

$$\begin{aligned} \triangle u &= 0 \ \text{ in } \Omega \qquad (7.8)\\ \frac{\partial u}{\partial n} + hu &= \Phi \ \text{ on } \partial\Omega, \qquad (7.9) \end{aligned}$$

where Φ and h are given and continuous on $\partial\Omega$.e Φ and h are given and continuous on $\partial\Omega$.

Proposition 7.6. *Assume Ω is bounded and sufficiently regular, then a solution to the mixed problem is uniquely determined in the class $u \in C^2(\overline{\Omega})$ provided $h(x) \geq 0$ on $\partial\Omega$ and $h(x) > 0$ for at least one point $x \in \partial\Omega$.*

Proof. Exercise. Hint: Multiply the differential equation $\triangle w = 0$ by w and integrate the result over Ω.

7.4 Green's function for $\triangle$

Theorem 7.1 says that each harmonic function satisfies

$$u(x) = \int_{\partial\Omega} \left(\gamma(y,x) \frac{\partial u(y)}{\partial n_y} - u(y) \frac{\partial \gamma(y,x)}{\partial n_y} \right) \, dS_y, \qquad (7.10)$$

where $\gamma(y,x)$ is a fundamental solution. In general, u does not satisfies the boundary condition in the above boundary value problems. Since $\gamma = s + \phi$, see Section 7.2, where ϕ is an *arbitrary* harmonic function for each fixed x, we try to find a ϕ such that u satisfies also the boundary condition.

Consider the Dirichlet problem, then we look for a ϕ such that

$$\gamma(y,x) = 0, \quad y \in \partial\Omega, \ x \in \Omega. \qquad (7.11)$$

Then

$$u(x) = -\int_{\partial\Omega} \frac{\partial \gamma(y,x)}{\partial n_y} u(y) \, dS_y, \quad x \in \Omega.$$

Suppose that u achieves its boundary values Φ of the Dirichlet problem, then

$$u(x) = -\int_{\partial\Omega} \frac{\partial\gamma(y,x)}{\partial n_y}\Phi(y)\ dS_y, \tag{7.12}$$

We claim that this function solves the Dirichlet problem (7.4), (7.5).

A function $\gamma(y,x)$ which satisfies (7.11), and some additional assumptions, is called *Green's* function. More precisely, we define a Green function as follows.

Definition. A function $G(y,x)$, $y,\ x \in \overline{\Omega}$, $x \neq y$, is called *Green function* associated to Ω and to the Dirichlet problem (7.4), (7.5) if for fixed $x \in \Omega$, that is we consider $G(y,x)$ as a function of y, the following properties hold:

(i) $G(y,x) \in C^2(\Omega \setminus \{x\}) \cap C(\overline{\Omega} \setminus \{x\})$, $\qquad \triangle_y G(y,x) = 0,\ \ x \neq y$.

(ii) $G(y,x) - s(|x-y|) \in C^2(\Omega) \cap C(\overline{\Omega})$.

(iii) $G(y,x) = 0$ if $y \in \partial\Omega$, $x \neq y$.

Remark. We will see in the next section that a Green function exists at least for some domains of simple geometry. Concerning the existence of a Green function for more general domains see [13].

It is an interesting fact that we get from (i)-(iii) of the above definition two further important properties. We assume that Ω is bounded, sufficiently regular and connected.

Proposition 7.7. *A Green function has the following properties. In the case* $n = 2$ *we assume* $\operatorname{diam}\Omega < 1$.

(A) $\quad G(x,y) = G(y,x)$ *(symmetry).*

(B) $\quad 0 < G(x,y) < s(|x-y|),\ \ x,\ y \in \Omega,\ x \neq y$.

Proof. (A) Let $x^{(1)},\ x^{(2)} \in \Omega$. Set $B_i = B_\rho(x^{(i)})$, $i = 1,\ 2$. We assume $\overline{B_i} \subset \Omega$ and $B_1 \cap B_2 = \emptyset$. Since $G(y,x^{(1)})$ and $G(y,x^{(2)})$ are harmonic in $\Omega \setminus \left(\overline{B_1} \cup \overline{B_2}\right)$ we obtain from Green's identity, see Figure 7.4 for notations,

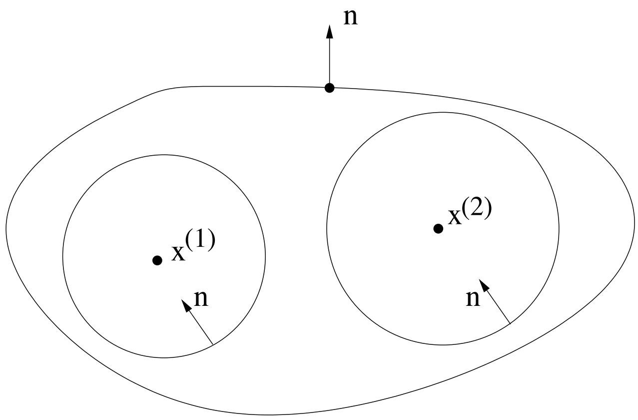

Figure 7.4: Proof of Proposition 7.7

$$\begin{aligned}
0 &= \int_{\partial\left(\Omega\setminus(\overline{B_1}\cup\overline{B_2})\right)} \Bigg(G(y,x^{(1)})\frac{\partial}{\partial n_y}G(y,x^{(2)}) \\
&\qquad\qquad -G(y,x^{(2)})\frac{\partial}{\partial n_y}G(y,x^{(1)})\Bigg)dS_y \\
&= \int_{\partial\Omega}\left(G(y,x^{(1)})\frac{\partial}{\partial n_y}G(y,x^{(2)}) - G(y,x^{(2)})\frac{\partial}{\partial n_y}G(y,x^{(1)})\right)dS_y \\
&+ \int_{\partial B_1}\left(G(y,x^{(1)})\frac{\partial}{\partial n_y}G(y,x^{(2)}) - G(y,x^{(2)})\frac{\partial}{\partial n_y}G(y,x^{(1)})\right)dS_y \\
&+ \int_{\partial B_2}\left(G(y,x^{(1)})\frac{\partial}{\partial n_y}G(y,x^{(2)}) - G(y,x^{(2)})\frac{\partial}{\partial n_y}G(y,x^{(1)})\right)dS_y.
\end{aligned}$$

The integral over $\partial\Omega$ is zero because of property (iii) of a Green function, and

$$\begin{aligned}
\int_{\partial B_1}\Bigg(G(y,x^{(1)})\frac{\partial}{\partial n_y}G(y,x^{(2)}) &- G(y,x^{(2)})\frac{\partial}{\partial n_y}G(y,x^{(1)})\Bigg)dS_y \\
&\to G(x^{(1)},x^{(2)}), \\
\int_{\partial B_2}\Bigg(G(y,x^{(1)})\frac{\partial}{\partial n_y}G(y,x^{(2)}) &- G(y,x^{(2)})\frac{\partial}{\partial n_y}G(y,x^{(1)})\Bigg)\,dS_y \\
&\to -G(x^{(2)},x^{(1)})
\end{aligned}$$

as $\rho\to 0$. This follows by considerations as in the proof of Theorem 7.1.

(B) Since

$$G(y,x) = s(|x-y|) + \phi(y,x)$$

and $G(y,x)=0$ if $y\in\partial\Omega$ and $x\in\Omega$ we have for $y\in\partial\Omega$

$$\phi(y,x)=-s(|x-y|).$$

From the definition of $s(|x-y|)$ it follows that $\phi(y,x)<0$ if $y\in\partial\Omega$. Thus, since $\triangle_y\phi=0$ in Ω, the maximum-minimum principle implies that $\phi(y,x)<0$ for all $y,\ x\in\Omega$. Consequently

$$G(y,x)<s(|x-y|),\quad x,\ y\in\Omega,\ x\neq y.$$

It remains to show that

$$G(y,x)>0,\quad x,\ y\in\Omega,\ x\neq y.$$

Fix $x\in\Omega$ and let $B_\rho(x)$ be a ball such that $B_\rho(x)\subset\Omega$ for all $0<\rho<\rho_0$. There is a sufficiently small $\rho_0>0$ such that for each ρ, $0<\rho<\rho_0$,

$$G(y,x)>0\ \text{ for all } y\in\overline{B_\rho(x)},\ x\neq y,$$

see property (iii) of a Green function. Since

$$\begin{aligned}\triangle_y G(y,x)&=0\ \text{ in } \Omega\setminus\overline{B_\rho(x)}\\ G(y,x)&>0\ \text{ if } y\in\partial B_\rho(x)\\ G(y,x)&=0\ \text{ if } y\in\partial\Omega\end{aligned}$$

it follows from the maximum-minimum principle that

$$G(y,x)>0\ \text{ on } \Omega\setminus\overline{B_\rho(x)}.$$

□

7.4.1 Green's function for a ball

If $\Omega=B_R(0)$ is a ball, then Green's function is explicitly known.

Let $\Omega=B_R(0)$ be a ball in $\mathbb{R}^n$ with radius R and the center at the origin. Let $x,\ y\in B_R(0)$ and let y' the reflected point of y on the sphere $\partial B_R(0)$, that is, in particular $|y||y'|=R^2$, see Figure 7.5 for notations. Set

$$G(x,y)=s(r)-s\left(\frac{\rho}{R}r_1\right),$$

where s is the singularity function of Section 7.1, $r=|x-y|$ and

$$\rho^2=\sum_{i=1}^n y_i^2,\quad r_1=\sum_{i=1}^n\left(x_i-\frac{R^2}{\rho^2}y_i\right)^2.$$

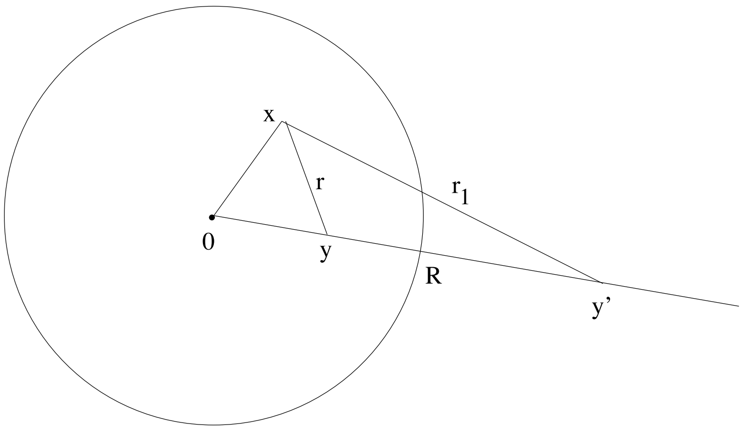

Figure 7.5: Reflection on $\partial B_R(0)$

This function $G(x,y)$ satisfies (i)-(iii) of the definition of a Green function. We claim that

$$u(x) = -\int_{\partial B_R(0)} \frac{\partial}{\partial n_y} G(x,y)\Phi \ dS_y$$

is a solution of the Dirichlet problem (7.4), (7.5). This formula is also true for a large class of domains $\Omega \subset \mathbb{R}^n$, see [13].

Lemma.

$$-\frac{\partial}{\partial n_y} G(x,y)\bigg|_{|y|=R} = \frac{1}{R\omega_n} \frac{R^2 - |x|^2}{|y-x|^n}.$$

Proof. Exercise.

Set

$$H(x,y) = \frac{1}{R\omega_n} \frac{R^2 - |x|^2}{|y-x|^n}, \tag{7.13}$$

which is called *Poisson's kernel*.

Theorem 7.2. *Assume $\Phi \in C(\partial\Omega)$. Then*

$$u(x) = \int_{\partial B_R(0)} H(x,y)\Phi(y) \ dS_y$$

is the solution of the first boundary value problem (7.4), (7.5) in the class $C^2(\Omega) \cap C(\overline{\Omega})$.

Proof. The proof follows from following properties of $H(x,y)$:

(i) $H(x,y) \in C^\infty, \ \ |y| = R, \ |x| < R, \ x \neq y,$

(ii) $\triangle_x H(x,y) = 0, \ \ |x| < R, \ |y| = R,$

(iii) $\int_{\partial B_R(0)} H(x,y) \, dS_y = 1, \ \ |x| < R,$

(iv) $H(x,y) > 0, \ \ |y| = R, \ |x| < R,$

(v) Fix $\zeta \in \partial B_R(0)$ and $\delta > 0$, then $\lim_{x \to \zeta, |x|<R} H(x,y) = 0$ uniformly in $y \in \partial B_R(0), \ |y - \zeta| > \delta.$

(i), (iv) and (v) follow from the definition (7.13) of H and (ii) from (7.13) or from

$$H = -\left.\frac{\partial G(x,y)}{\partial n_y}\right|_{y \in \partial B_R(0)},$$

G harmonic and $G(x,y) = G(y,x)$.
Property (iii) is a consequence of formula

$$u(x) = \int_{\partial B_R(0)} H(x,y) u(y) \, dS_y,$$

for each harmonic function u, see calculations to the representation formula above. We obtain (ii) if we set $u \equiv 1$.

It remains to show that u, given by Poisson's formula, is in $C(\overline{B_R(0)})$ and that u achieves the prescribed boundary values. Fix $\zeta \in \partial B_R(0)$ and let $x \in B_R(0)$. Then

$$\begin{aligned} u(x) - \Phi(\zeta) &= \int_{\partial B_R(0)} H(x,y) \left(\Phi(y) - \Phi(\zeta)\right) \, dS_y \\ &= I_1 + I_2, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \int_{\partial B_R(0), \ |y-\zeta|<\delta} H(x,y) \left(\Phi(y) - \Phi(\zeta)\right) \, dS_y \\ I_2 &= \int_{\partial B_R(0), \ |y-\zeta|\geq\delta} H(x,y) \left(\Phi(y) - \Phi(\zeta)\right) \, dS_y. \end{aligned}$$

For given (small) $\epsilon > 0$ there is a $\delta = \delta(\epsilon) > 0$ such that

$$|\Phi(y) - \Phi(\zeta)| < \epsilon$$

for all $y \in \partial B_R(0)$ with $|y - \zeta| < \delta$. It follows $|I_1| \le \epsilon$ because of (iii) and (iv).
Set $M = \max_{\partial B_R(0)} |\phi|$. From (v) we conclude that there is a $\delta' > 0$ such that

$$H(x,y) < \frac{\epsilon}{2M\omega_n R^{n-1}}$$

if x and y satisfy $|x-\zeta| < \delta'$, $|y-\zeta| > \delta$, see Figure 7.6 for notations. Thus

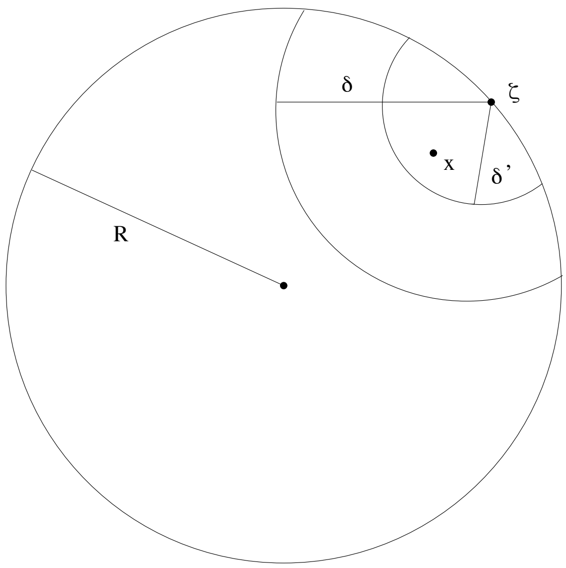

Figure 7.6: Proof of Theorem 7.2

$|I_2| < \epsilon$ and the inequality

$$|u(x) - \Phi(\zeta)| < 2\epsilon$$

for $x \in B_R(0)$ such that $|x - \zeta| < \delta'$ is shown. $\square$

Remark. Define $\delta \in [0, \pi]$ through $\cos\delta = x \cdot y/(|x||y|)$, then we write Poisson's formula of Theorem 7.2 as

$$u(x) = \frac{R^2 - |x|^2}{\omega_n R} \int_{\partial B_R(0)} \Phi(y) \frac{1}{(|x|^2 + R^2 - 2|x|R\cos\delta)^{n/2}}\, dS_y.$$

In the case $n = 2$ we can expand this integral in a power series with respect to $\rho := |x|/R$ if $|x| < R$, since

$$\begin{aligned} \frac{R^2 - |x|^2}{|x| + R^2 - 2|x|R\cos\delta} &= \frac{1-\rho^2}{\rho^2 - 2\rho\cos\delta + 1} \\ &= 1 + 2\sum_{n=1}^{\infty} \rho^n \cos(n\delta), \end{aligned}$$

see [16], pp. 18 for an easy proof of this formula, or [4], Vol. II, p. 246.

7.4.2 Green's function and conformal mapping

For two-dimensional domains there is a beautiful connection between conformal mapping and Green's function. Let $w = f(z)$ be a conformal mapping from a sufficiently regular connected domain in $\mathbb{R}^2$ onto the interior of the unit circle, see Figure 7.7. Then the Green function of Ω is, see for exam-

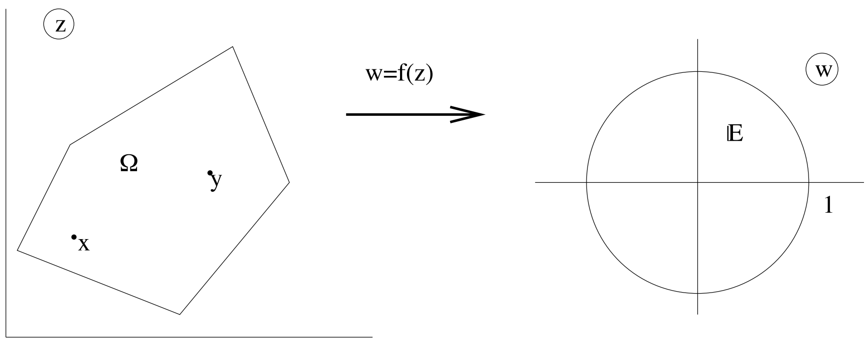

Figure 7.7: Conformal mapping

ple [16] or other text books about the theory of functions of one complex variable,

$$G(z, z_0) = \frac{1}{2\pi} \ln \left| \frac{1 - f(z)\overline{f(z_0)}}{f(z) - f(z_0)} \right|,$$

where $z = x_1 + ix_2$, $z_0 = y_1 + iy_2$.

7.5 Inhomogeneous equation

Here we consider solutions $u \in C^2(\Omega) \cap C(\overline{\Omega})$ of

$$\begin{aligned} -\triangle u &= f(x) \ \text{ in } \Omega && (7.14) \\ u &= 0 \quad \text{on } \partial\Omega, && (7.15) \end{aligned}$$

where f is given.

We need the following lemma concerning volume potentials. We assume that Ω is bounded and sufficiently regular such that all the following integrals exist. See [6] for generalizations concerning these assumptions.

Let for $x \in \mathbb{R}^n$, $n \geq 3$,

$$V(x) = \int_\Omega f(y) \frac{1}{|x-y|^{n-2}}\ dy$$

and set in the two-dimensional case

$$V(x) = \int_\Omega f(y) \ln\left(\frac{1}{|x-y|}\right)\ dy.$$

We recall that $\omega_n = |\partial B_1(0)|$.

Lemma.
(i) *Assume $f \in C(\Omega)$. Then $V \in C^1(\mathbb{R}^n)$ and*

$$\begin{aligned} V_{x_i}(x) &= \int_\Omega f(y) \frac{\partial}{\partial x_i}\left(\frac{1}{|x-y|^{n-2}}\right)\ dy, \quad \textit{if } n \geq 3, \\ V_{x_i}(x) &= \int_\Omega f(y) \frac{\partial}{\partial x_i}\left(\ln\left(\frac{1}{|x-y|}\right)\right)\ dy \quad \textit{if } n = 2. \end{aligned}$$

(ii) *If $f \in C^1(\Omega)$, then $V \in C^2(\Omega)$ and*

$$\begin{aligned} \triangle V &= -(n-2)\omega_n f(x),\ x \in \Omega,\ n \geq 3 \\ \triangle V &= -2\pi f(x),\ x \in \Omega,\ n = 2. \end{aligned}$$

Proof. To simplify the presentation, we consider the case $n = 3$.
(i) The first assertion follows since we can change differentiation with integration since the differentiate integrand is weakly singular, see an exercise.
(ii) We will differentiate at $x \in \Omega$. Let B_ρ be a fixed ball such that $x \in B_\rho$, ρ sufficiently small such that $B_\rho \subset \Omega$. Then, according to (i) and since we have the identity

$$\frac{\partial}{\partial x_i}\left(\frac{1}{|x-y|}\right) = -\frac{\partial}{\partial y_i}\left(\frac{1}{|x-y|}\right)$$

which implies that

$$f(y)\frac{\partial}{\partial x_i}\left(\frac{1}{|x-y|}\right) = -\frac{\partial}{\partial y_i}\left(f(y)\frac{1}{|x-y|}\right) + f_{y_i}(y)\frac{1}{|x-y|},$$

we obtain

$$
\begin{aligned}
V_{x_i}(x) &= \int_\Omega f(y)\frac{\partial}{\partial x_i}\left(\frac{1}{|x-y|}\right)\ dy \\
&= \int_{\Omega\setminus B_\rho} f(y)\frac{\partial}{\partial x_i}\left(\frac{1}{|x-y|}\right)\ dy + \int_{B_\rho} f(y)\frac{\partial}{\partial x_i}\left(\frac{1}{|x-y|}\right)\ dy \\
&= \int_{\Omega\setminus B_\rho} f(y)\frac{\partial}{\partial x_i}\left(\frac{1}{|x-y|}\right)\ dy \\
&\quad + \int_{B_\rho} \left(-\frac{\partial}{\partial y_i}\left(f(y)\frac{1}{|x-y|}\right) + f_{y_i}(y)\frac{1}{|x-y|}\right)\ dy \\
&= \int_{\Omega\setminus B_\rho} f(y)\frac{\partial}{\partial x_i}\left(\frac{1}{|x-y|}\right)\ dy \\
&\quad + \int_{B_\rho} f_{y_i}(y)\frac{1}{|x-y|}\ dy - \int_{\partial B_\rho} f(y)\frac{1}{|x-y|}n_i\ dS_y,
\end{aligned}
$$

where n is the exterior unit normal at ∂B_ρ. It follows that the first and second integral is in $C^1(\Omega)$. The second integral is also in $C^1(\Omega)$ according to (i) and since $f \in C^1(\Omega)$ by assumption.

Because of $\triangle_x(|x-y|^{-1}) = 0,\ x \neq y$, it follows

$$
\begin{aligned}
\triangle V &= \int_{B_\rho} \sum_{i=1}^n f_{y_i}(y)\frac{\partial}{\partial x_i}\left(\frac{1}{|x-y|}\right)\ dy \\
&\quad - \int_{\partial B_\rho} f(y)\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\frac{1}{|x-y|}\right)n_i\ dS_y.
\end{aligned}
$$

Now we choose for B_ρ a ball with the center at x, then

$$
\triangle V = I_1 + I_2,
$$

where

$$
\begin{aligned}
I_1 &= \int_{B_\rho(x)} \sum_{i=1}^n f_{y_i}(y)\frac{y_i - x_i}{|x-y|^3}\ dy \\
I_2 &= -\int_{\partial B_\rho(x)} f(y)\frac{1}{\rho^2}\ dS_y.
\end{aligned}
$$

We recall that $n \cdot (y-x) = \rho$ if $y \in \partial B_\rho(x)$. It is $I_1 = O(\rho)$ as $\rho \to 0$ and for I_2 we obtain from the mean value theorem of the integral calculus that for a $\overline{y} \in \partial B_\rho(x)$

$$\begin{aligned} I_2 &= -\frac{1}{\rho^2} f(\overline{y}) \int_{\partial B_\rho(x)} dS_y \\ &= -\omega_n f(\overline{y}), \end{aligned}$$

which implies that $\lim_{\rho \to 0} I_2 = -\omega_n f(x)$. □

In the following we assume that Green's function exists for the domain Ω, which is the case if Ω is a ball.

Theorem 7.3. *Assume $f \in C^1(\Omega) \cap C(\overline{\Omega})$. Then*

$$u(x) = \int_\Omega G(x, y) f(y)\ dy$$

is the solution of the inhomogeneous problem (7.14), (7.15).

Proof. For simplicity of the presentation let $n = 3$. We will show that

$$u(x) := \int_\Omega G(x, y) f(y)\ dy$$

is a solution of (7.4), (7.5). Since

$$G(x, y) = \frac{1}{4\pi |x - y|} + \phi(x, y),$$

where ϕ is a potential function with respect to x or y, we obtain from the above lemma that

$$\begin{aligned} \triangle u &= \frac{1}{4\pi} \triangle \int_\Omega f(y) \frac{1}{|x - y|}\ dy + \int_\Omega \triangle_x \phi(x, y) f(y)\ dy \\ &= -f(x), \end{aligned}$$

where $x \in \Omega$. It remains to show that u achieves its boundary values. That is, for fixed $x_0 \in \partial\Omega$ we will prove that

$$\lim_{x \to x_0,\ x \in \Omega} u(x) = 0.$$

Set

$$u(x) = I_1 + I_2,$$

where

$$\begin{aligned} I_1(x) &= \int_{\Omega\setminus B_\rho(x_0)} G(x,y)f(y)\ dy, \\ I_2(x) &= \int_{\Omega\cap B_\rho(x_0)} G(x,y)f(y)\ dy. \end{aligned}$$

Let $M=\max_{\overline{\Omega}}|f(x)|$. Since

$$G(x,y)=\frac{1}{4\pi}\frac{1}{|x-y|}+\phi(x,y),$$

we obtain, if $x\in B_\rho(x_0)\cap\Omega$,

$$\begin{aligned} |I_2| &\le \frac{M}{4\pi}\int_{\Omega\cap B_\rho(x_0)}\frac{dy}{|x-y|}+O(\rho^2) \\ &\le \frac{M}{4\pi}\int_{B_{2\rho(x)}}\frac{dy}{|x-y|}+O(\rho^2) \\ &= O(\rho^2) \end{aligned}$$

as $\rho\to 0$. Consequently for given ϵ there is a $\rho_0=\rho_0(\epsilon)>0$ such that

$$|I_2|<\frac{\epsilon}{2}\ \text{ for all }\ 0<\rho\le\rho_0.$$

For each fixed ρ, $0<\rho\le\rho_0$, we have

$$\lim_{x\to x_0,\ x\in\Omega} I_1(x)=0$$

since $G(x_0,y)=0$ if $y\in\Omega\setminus B_\rho(x_0)$ and $G(x,y)$ is uniformly continuous in $x\in B_{\rho/2}(x_0)\cap\Omega$ and $y\in\Omega\setminus B_\rho(x_0)$, see Figure 7.8. $\Box$

Remark. For the proof of (ii) in the above lemma it is sufficient to assume that f is Hölder continuous. More precisely, let $f\in C^\lambda(\Omega)$, $0<\lambda<1$, then $V\in C^{2,\lambda}(\Omega)$, see for example [9].

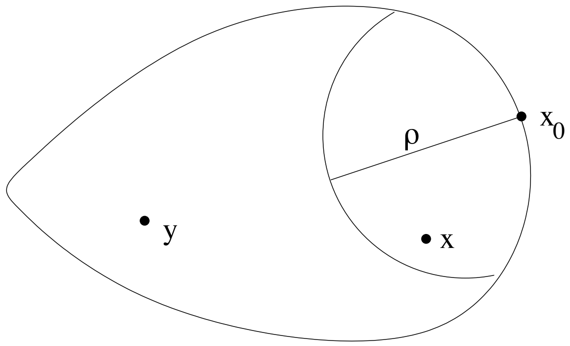

Figure 7.8: Proof of Theorem 7.3

7.6 Exercises

1. Let $\gamma(x,y)$ be a fundamental solution to $\triangle$, $y \in \Omega$. Show that

$$-\int_\Omega \gamma(x,y)\ \triangle\Phi(x)\ dx = \Phi(y) \quad \text{for all} \ \ \Phi \in C_0^2(\Omega)\ .$$

   Hint: See the proof of the representation formula.

2. Show that $|x|^{-1}\sin(k|x|)$ is a solution of the Helmholtz equation

$$\triangle u + k^2 u = 0 \text{ in } \mathbb{R}^n \setminus \{0\}.$$

3. Assume $u \in C^2(\overline{\Omega})$, Ω bounded and sufficiently regular, is a solution of

$$\begin{aligned} \triangle u &= u^3 \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega. \end{aligned}$$

   Show that $u = 0$ in Ω.

4. Let $\Omega_\alpha = \{x \in \mathbb{R}^2 :\ x_1 > 0,\ 0 < x_2 < x_1 \tan\alpha\},\ 0 < \alpha \le \pi$. Show that

$$u(x) = r^{\frac{\pi}{\alpha}k} \sin\left(\frac{\pi}{\alpha}k\theta\right)$$

   is a harmonic function in Ω_α satisfying $u = 0$ on $\partial\Omega_\alpha$, provided k is an integer. Here (r,θ) are polar coordinates with the center at $(0,0)$.

5. Let $u \in C^2(\overline{\Omega})$ be a solution of $\triangle u = 0$ on the quadrangle $\Omega = (0,1) \times (0,1)$ satisfying the boundary conditions $u(0,y) = u(1,y) = 0$ for all $y \in [0,1]$ and $u_y(x,0) = u_y(x,1) = 0$ for all $x \in [0,1]$. Prove that $u \equiv 0$ in $\overline{\Omega}$.

6. Let $u \in C^2(\mathbb{R}^n)$ be a solution of $\triangle u = 0$ in $\mathbb{R}^n$ satisfying $u \in L^2(\mathbb{R}^n)$, i. e., $\int_{\mathbb{R}^n} u^2(x)\ dx < \infty$.
Show that $u \equiv 0$ in $\mathbb{R}^n$.

   Hint: Prove
$$\int_{B_R(0)} |\nabla u|^2\ dx \le \frac{const.}{R^2} \int_{B_{2R}(0)} |u|^2\ dx,$$
   where c is a constant independent of R.
To show this inequality, multiply the differential equation by $\zeta := \eta^2 u$, where $\eta \in C^1$ is a cut-off function with properties: $\eta \equiv 1$ in $B_R(0)$, $\eta \equiv 0$ in the exterior of $B_{2R}(0)$, $0 \le \eta \le 1$, $|\nabla \eta| \le C/R$. Integrate the product, apply integration by parts and use the formula $2ab \le \epsilon a^2 + \frac{1}{\epsilon} b^2$, $\epsilon > 0$.

7. Show that a bounded harmonic function defined on $\mathbb{R}^n$ must be a constant (a theorem of Liouville).

8. Assume $u \in C^2(B_1(0)) \cap C(\overline{B_1(0)} \setminus \{(1,0)\})$ is a solution of
$$\begin{aligned} \triangle u &= 0 \ \text{ in } B_1(0) \\ u &= 0 \ \text{ on } \partial B_1(0) \setminus \{(1,0)\}. \end{aligned}$$
   Show that there are at least two solutions.

   Hint: Consider
$$u(x,y) = \frac{1-(x^2+y^2)}{(1-x)^2+y^2}.$$

9. Assume $\Omega \subset \mathbb{R}^n$ is bounded and $u,\ v \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfy $\triangle u = \triangle v$ and $\max_{\partial\Omega} |u - v| \le \epsilon$ for given $\epsilon > 0$. Show that $\max_{\overline{\Omega}} |u - v| \le \epsilon$.

10. Set $\Omega = \mathbb{R}^n \setminus \overline{B_1(0)}$ and let $u \in C^2(\overline{\Omega})$ be a harmonic function in Ω satisfying $\lim_{|x| \to \infty} u(x) = 0$. Prove that
$$\max_{\overline{\Omega}} |u| = \max_{\partial\Omega} |u| \ .$$
    Hint: Apply the maximum principle to $\Omega \cap B_R(0)$, R large.

11. Let $\Omega_\alpha = \{x \in \mathbb{R}^2 : \ x_1 > 0,\ 0 < x_2 < x_1 \tan\alpha\}$, $0 < \alpha \leq \pi$, $\Omega_{\alpha,R} = \Omega_\alpha \cap B_R(0)$, and assume f is given and bounded on $\overline{\Omega_{\alpha,R}}$.

    Show that for each solution $u \in C^1(\overline{\Omega_{\alpha,R}}) \cap C^2(\Omega_{\alpha,R})$ of $\triangle u = f$ in $\Omega_{\alpha,R}$ satisfying $u = 0$ on $\partial\Omega_{\alpha,R} \cap B_R(0)$, holds:

    For given $\epsilon > 0$ there is a constant $C(\epsilon)$ such that

    $$|u(x)| \leq C(\epsilon)\ |x|^{\frac{\pi}{\alpha}-\epsilon} \qquad \text{in } \Omega_{\alpha,R}.$$

    Hint: (a) Comparison principle (a consequence from the maximum principle): Assume Ω is bounded, $u, v \in C^2(\overline{\Omega}) \cap C(\overline{\Omega})$ satisfying $-\triangle u \leq -\triangle v$ in Ω and $u \leq v$ on $\partial\Omega$. Then $u \leq v$ in Ω.

    (b) An appropriate comparison function is

    $$v = Ar^{\frac{\pi}{\alpha}-\epsilon}\sin(B(\theta + \eta))\ ,$$

    $A,\ B,\ \eta$ appropriate constants, $B,\ \eta$ positive.

12. Let Ω be the quadrangle $(-1, 1) \times (-1, 1)$ and $u \in C^2(\Omega) \cap C(\overline{\Omega})$ a solution of the boundary value problem $-\triangle u = 1$ in Ω, $u = 0$ on $\partial\Omega$. Find a lower and an upper bound for $u(0, 0)$.

    Hint: Consider the comparison function $v = A(x^2 + y^2)$, $A = const.$

13. Let $u \in C^2(B_a(0)) \cap C(\overline{B_a(0)})$ satisfying $u \geq 0,\ \triangle u = 0$ in $B_a(0)$. Prove (Harnack's inequality):

    $$\frac{a^{n-2}(a - |\zeta|)}{(a + |\zeta|)^{n-1}}u(0) \leq u(\zeta) \leq \frac{a^{n-2}(a + |\zeta|)}{(a - |\zeta|)^{n-1}}u(0)\ .$$

    Hint: Use the formula (see Theorem 7.2)

    $$u(y) = \frac{a^2 - |y|^2}{a\omega_n}\int_{|x|=a}\frac{u(x)}{|x - y|^n}\ dS_x$$

    for $y = \zeta$ and $y = 0$.

14. Let $\phi(\theta)$ be a 2π-periodic C^4-function with the Fourier series

    $$\phi(\theta) = \sum_{n=0}^{\infty}(a_n\ cos(n\theta) + b_n \sin(n\theta))\ .$$

    Show that

    $$u = \sum_{n=0}^{\infty}(a_n\ cos(n\theta) + b_n \sin(n\theta))\, r^n$$

    solves the Dirichlet problem in $B_1(0)$.

15. Assume $u \in C^2(\Omega)$ satisfies $\triangle u = 0$ in Ω. Let $B_a(\zeta)$ be a ball such that its closure is in Ω. Show that
$$|D^\alpha u(\zeta)| \le M \left(\frac{|\alpha|\gamma_n}{a}\right)^{|\alpha|},$$
where $M = \sup_{x\in B_a(\zeta)} |u(x)|$ and $\gamma_n = 2n\omega_{n-1}/((n-1)\omega_n)$.

Hint: Use the formula of Theorem 7.2, successively to the k th derivatives in balls with radius $a(|\alpha| - k)/m$, $k = o, 1, \ldots, m-1$.

16. Use the result of the previous exercise to show that $u \in C^2(\Omega)$ satisfying $\triangle u = 0$ in Ω is real analytic in Ω.

Hint: Use Stirling's formula
$$n! = n^n \mathrm{e}^{-n} \left(\sqrt{2\pi n} + O\left(\frac{1}{\sqrt{n}}\right)\right)$$
as $n \to \infty$, to show that u is in the class $C_{K,r}(\zeta)$, where $K = cM$ and $r = a/(\mathrm{e}\gamma_n)$. The constant c is the constant in the estimate $n^n \le c\mathrm{e}^n n!$ which follows from Stirling's formula. See Section 3.5 for the definition of a real analytic function.

17. Assume Ω is connected and $u \in C^2(\Omega)$ is a solution of $\triangle u = 0$ in Ω. Prove that $u \equiv 0$ in Ω if $D^\alpha u(\zeta) = 0$ for all α, for a point $\zeta \in \Omega$. In particular, $u \equiv 0$ in Ω if $u \equiv 0$ in an open subset of Ω.

18. Let $\Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3 :\ x_3 > 0\}$, which is a half-space of $\mathbb{R}^3$. Show that
$$G(x,y) = \frac{1}{4\pi|x-y|} - \frac{1}{4\pi|x-\overline{y}|},$$
where $\overline{y} = (y_1, y_2, -y_3)$, is the Green function to Ω.

19. Let $\Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3 :\ x_1^2 + x_2^2 + x_3^2 < R^2,\ x_3 > 0\}$, which is half of a ball in $\mathbb{R}^3$. Show that
$$\begin{aligned} G(x,y) &= \frac{1}{4\pi|x-y|} - \frac{R}{4\pi|y||x-y^\star|} \\ &\quad - \frac{1}{4\pi|x-\overline{y}|} + \frac{R}{4\pi|y||x-\overline{y}^\star|}, \end{aligned}$$
where $\overline{y} = (y_1, y_2, -y_3)$, $y^\star = R^2 y/(|y|^2)$ and $\overline{y}^\star = R^2\overline{y}/(|y|^2)$, is the Green function to Ω.

20. Let $\Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \ x_2 > 0, \ x_3 > 0\}$, which is a wedge in $\mathbb{R}^3$. Show that

$$\begin{aligned} G(x,y) &= \frac{1}{4\pi|x-y|} - \frac{1}{4\pi|x-\overline{y}|} \\ &\quad - \frac{1}{4\pi|x-y'|} + \frac{1}{4\pi|x-\overline{y}'|}, \end{aligned}$$

where $\overline{y} = (y_1, y_2, -y_3)$, $y' = (y_1, -y_2, y_3)$ and $\overline{y}' = (y_1, -y_2, -y_3)$, is the Green function to Ω.

21. Find Green's function for the exterior of a disk, i. e., of the domain $\Omega = \{x \in \mathbb{R}^2 : \ |x| > R\}$.

22. Find Green's function for the angle domain $\Omega = \{z \in \mathbb{C} : \ 0 < \arg z < \alpha\pi\}$, $0 < \alpha < \pi$.

23. Find Green's function for the slit domain $\Omega = \{z \in \mathbb{C} : \ 0 < \arg z < 2\pi\}$.

24. Let for a sufficiently regular domain $\Omega \in \mathbb{R}^n$, a ball or a quadrangle for example,

$$F(x) = \int_\Omega K(x,y)\ dy,$$

where $K(x,y)$ is continuous in $\overline{\Omega} \times \overline{\Omega}$ where $x \neq y$, and which satisfies

$$|K(x,y)| \leq \frac{c}{|x-y|^\alpha}$$

with a constants c and α, $\alpha < n$.

Show that $F(x)$ is continuous on $\overline{\Omega}$.

25. Prove (i) of the lemma of Section 7.5.

Hint: Consider the case $n \geq 3$. Fix a function $\eta \in C^1(\mathbb{R})$ satisfying $0 \leq \eta \leq 1$, $0 \leq \eta' \leq 2$, $\eta(t) = 0$ for $t \leq 1$, $\eta(t) = 1$ for $t \geq 2$ and consider for $\epsilon > 0$ the regularized integral

$$V_\epsilon(x) := \int_\Omega f(y)\eta_\epsilon \frac{dy}{|x-y|^{n-2}},$$

where $\eta_\epsilon = \eta(|x-y|/\epsilon)$. Show that V_ϵ converges uniformly to V on compact subsets of $\mathbb{R}^n$ as $\epsilon \to 0$, and that $\partial V_\epsilon(x)/\partial x_i$ converges uniformly on compact subsets of $\mathbb{R}^n$ to

$$\int_\Omega f(y) \frac{\partial}{\partial x_i}\left(\frac{1}{|x-y|^{n-2}}\right)\ dy$$

as $\epsilon \to 0$.

26. Consider the inhomogeneous Dirichlet problem $-\triangle u = f$ in Ω, $u = \phi$ on $\partial\Omega$. Transform this problem into a Dirichlet problem for the Laplace equation.

    Hint: Set $u = w + v$, where $w(x) := \int_\Omega \, s(|x-y|)f(y)\, dy$.

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Index

Partial Differential Equations: Graduate Level Problems and Solutions

Igor Yanovsky

Disclaimer: This handbook is intended to assist graduate students with qualifying examination preparation. Please be aware, however, that the handbook might contain, and almost certainly contains, typos as well as incorrect or inaccurate solutions. I can not be made responsible for any inaccuracies contained in this handbook.

Contents

1 Trigonometric Identities

$$\begin{aligned}
\cos(a+b) &= \cos a \cos b - \sin a \sin b \\
\cos(a-b) &= \cos a \cos b + \sin a \sin b \\
\sin(a+b) &= \sin a \cos b + \cos a \sin b \\
\sin(a-b) &= \sin a \cos b - \cos a \sin b
\end{aligned}$$

$$\begin{aligned}
\cos a \cos b &= \frac{\cos(a+b)+\cos(a-b)}{2} \\
\sin a \cos b &= \frac{\sin(a+b)+\sin(a-b)}{2} \\
\sin a \sin b &= \frac{\cos(a-b)-\cos(a+b)}{2}
\end{aligned}$$

$$\begin{aligned}
\cos 2t &= \cos^2 t - \sin^2 t \\
\sin 2t &= 2\sin t \cos t
\end{aligned}$$

$$\begin{aligned}
\cos^2 \frac{1}{2}t &= \frac{1+\cos t}{2} \\
\sin^2 \frac{1}{2}t &= \frac{1-\cos t}{2}
\end{aligned}$$

$$\begin{aligned}
1+\tan^2 t &= \sec^2 t \\
\cot^2 t + 1 &= \csc^2 t
\end{aligned}$$

$$\begin{aligned}
\cos x &= \frac{e^{ix}+e^{-ix}}{2} \\
\sin x &= \frac{e^{ix}-e^{-ix}}{2i}
\end{aligned}$$

$$\begin{aligned}
\cosh x &= \frac{e^{x}+e^{-x}}{2} \\
\sinh x &= \frac{e^{x}-e^{-x}}{2}
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dx}\cosh x &= \sinh(x) \\
\frac{d}{dx}\sinh x &= \cosh(x)
\end{aligned}$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$\begin{aligned}
\int \frac{du}{a^2+u^2} &= \frac{1}{a}\tan^{-1}\frac{u}{a} + C \\
\int \frac{du}{\sqrt{a^2-u^2}} &= \sin^{-1}\frac{u}{a} + C
\end{aligned}$$

$$\begin{aligned}
\int_{-L}^{L} \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx &= \begin{cases} 0 & n \neq m \\ L & n = m \end{cases} \\
\int_{-L}^{L} \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx &= \begin{cases} 0 & n \neq m \\ L & n = m \end{cases} \\
\int_{-L}^{L} \sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx &= 0
\end{aligned}$$

$$\begin{aligned}
\int_{0}^{L} \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx &= \begin{cases} 0 & n \neq m \\ \frac{L}{2} & n = m \end{cases} \\
\int_{0}^{L} \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx &= \begin{cases} 0 & n \neq m \\ \frac{L}{2} & n = m \end{cases}
\end{aligned}$$

$$\begin{aligned}
\int_{0}^{L} e^{inx}\overline{e^{imx}}\,dx &= \begin{cases} 0 & n \neq m \\ L & n = m \end{cases} \\
\int_{0}^{L} e^{inx}\,dx &= \begin{cases} 0 & n \neq 0 \\ L & n = 0 \end{cases}
\end{aligned}$$

$$\begin{aligned}
&\int \sin^2 x\,dx = \frac{x}{2} - \frac{\sin x \cos x}{2} \\
&\int \cos^2 x\,dx = \frac{x}{2} + \frac{\sin x \cos x}{2} \\
&\int \tan^2 x\,dx = \tan x - x \\
&\int \sin x \cos x\,dx = -\frac{\cos^2 x}{2}
\end{aligned}$$

$$\begin{aligned}
\ln(xy) &= \ln(x) + \ln(y) \\
\ln\frac{x}{y} &= \ln(x) - \ln(y) \\
\ln x^r &= r \ln x
\end{aligned}$$

$$\begin{aligned}
\int \ln x\,dx &= x\ln x - x \\
\int x\ln x\,dx &= \frac{x^2}{2}\ln x - \frac{x^2}{4}
\end{aligned}$$

$$\begin{aligned}
\int_{\mathbb{R}} e^{-z^2}\,dz &= \sqrt{\pi} \\
\int_{\mathbb{R}} e^{-\frac{z^2}{2}}\,dz &= \sqrt{2\pi}
\end{aligned}$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

2 Simple Eigenvalue Problem

$$X'' + \lambda X = 0$$

Boundary conditions	Eigenvalues λ_n	Eigenfunctions X_n	
$X(0) = X(L) = 0$	$\left(\frac{n\pi}{L}\right)^2$	$\sin \frac{n\pi}{L}x$	$n = 1, 2, \ldots$
$X(0) = X'(L) = 0$	$\left[\frac{(n-\frac{1}{2})\pi}{L}\right]^2$	$\sin \frac{(n-\frac{1}{2})\pi}{L}x$	$n = 1, 2, \ldots$
$X'(0) = X(L) = 0$	$\left[\frac{(n-\frac{1}{2})\pi}{L}\right]^2$	$\cos \frac{(n-\frac{1}{2})\pi}{L}x$	$n = 1, 2, \ldots$
$X'(0) = X'(L) = 0$	$\left(\frac{n\pi}{L}\right)^2$	$\cos \frac{n\pi}{L}x$	$n = 0, 1, 2, \ldots$
$X(0) = X(L),\ X'(0) = X'(L)$	$\left(\frac{2n\pi}{L}\right)^2$	$\sin \frac{2n\pi}{L}x$	$n = 1, 2, \ldots$
		$\cos \frac{2n\pi}{L}x$	$n = 0, 1, 2, \ldots$
$X(-L) = X(L),\ X'(-L) = X'(L)$	$\left(\frac{n\pi}{L}\right)^2$	$\sin \frac{n\pi}{L}x$	$n = 1, 2, \ldots$
		$\cos \frac{n\pi}{L}x$	$n = 0, 1, 2, \ldots$

$$X'''' - \lambda X = 0$$

Boundary conditions	Eigenvalues λ_n	Eigenfunctions X_n	
$X(0) = X(L) = 0,\ X''(0) = X''(L) = 0$	$\left(\frac{n\pi}{L}\right)^4$	$\sin \frac{n\pi}{L}x$	$n = 1, 2, \ldots$
$X'(0) = X'(L) = 0,\ X'''(0) = X'''(L) = 0$	$\left(\frac{n\pi}{L}\right)^4$	$\cos \frac{n\pi}{L}x$	$n = 0, 1, 2, \ldots$

3 Separation of Variables: Quick Guide

Laplace Equation: $\triangle u = 0$.

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda.$$
$$X'' + \lambda X = 0.$$

$$\frac{X''(t)}{X(t)} = -\frac{Y''(\theta)}{Y(\theta)} = \lambda.$$
$$Y''(\theta) + \lambda Y(\theta) = 0.$$

Wave Equation: $u_{tt} - u_{xx} = 0$.

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)} = -\lambda.$$
$$X'' + \lambda X = 0.$$

$u_{tt} + 3u_t + u = u_{xx}$.

$$\frac{T''}{T} + 3\frac{T'}{T} + 1 = \frac{X''}{X} = -\lambda.$$
$$X'' + \lambda X = 0.$$

$u_{tt} - u_{xx} + u = 0$.

$$\frac{T''}{T} + 1 = \frac{X''}{X} = -\lambda.$$
$$X'' + \lambda X = 0.$$

$u_{tt} + \mu u_t = c^2 u_{xx} + \beta u_{xxt}, \qquad (\beta > 0)$

$$\frac{X''}{X} = -\lambda,$$
$$\frac{1}{c^2}\frac{T''}{T} + \frac{\mu}{c^2}\frac{T'}{T} = \left(1 + \frac{\beta}{c^2}\frac{T'}{T}\right)\frac{X''}{X}.$$

4th Order: $u_{tt} = -k\, u_{xxxx}$.

$$-\frac{X''''}{X} = \frac{1}{k}\frac{T''}{T} = -\lambda.$$
$$X'''' - \lambda X = 0.$$

Heat Equation: $u_t = k u_{xx}$.

$$\frac{T'}{T} = k\frac{X''}{X} = -\lambda.$$
$$X'' + \frac{\lambda}{k} X = 0.$$

4th Order: $u_t = -u_{xxxx}$.

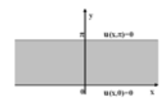
$$\frac{T'}{T} = -\frac{X''''}{X} = -\lambda.$$
$$X'''' - \lambda X = 0.$$

4 Eigenvalues of the Laplacian: Quick Guide

Laplace Equation: $u_{xx} + u_{yy} + \lambda u = 0$.

$$\frac{X''}{X} + \frac{Y''}{Y} + \lambda = 0. \quad (\lambda = \mu^2 + \nu^2)$$
$$X'' + \mu^2 X = 0, \qquad Y'' + \nu^2 Y = 0.$$

$u_{xx} + u_{yy} + k^2 u = 0.$



$$-\frac{X''}{X} = \frac{Y''}{Y} + k^2 = c^2.$$
$$X'' + c^2 X = 0,$$
$$Y'' + (k^2 - c^2)Y = 0.$$

$u_{xx} + u_{yy} + k^2 u = 0.$

$$-\frac{Y''}{Y} = \frac{X''}{X} + k^2 = c^2.$$
$$Y'' + c^2 Y = 0,$$
$$X'' + (k^2 - c^2)X = 0.$$

5 First-Order Equations

5.1 Quasilinear Equations

Consider the Cauchy problem for the quasilinear equation in two variables

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u),$$

with Γ parameterized by $(f(s), g(s), h(s))$. The characteristic equations are

$$\frac{dx}{dt} = a(x, y, z), \qquad \frac{dy}{dt} = b(x, y, z), \qquad \frac{dz}{dt} = c(x, y, z),$$

with initial conditions

$$x(s, 0) = f(s), \qquad y(s, 0) = g(s), \qquad z(s, 0) = h(s).$$

In a quasilinear case, the characteristic equations for $\frac{dx}{dt}$ and $\frac{dy}{dt}$ need not decouple from the $\frac{dz}{dt}$ equation; this means that we must take the z values into account even to find the projected characteristic curves in the xy-plane. In particular, this allows for the possibility that the projected characteristics may cross each other.

The condition for solving for s and t in terms of x and y requires that the Jacobian matrix be nonsingular:

$$J \equiv \begin{pmatrix} x_s & y_s \\ x_t & y_t \end{pmatrix} = x_s y_t - y_s x_t \neq 0.$$

In particular, at $t = 0$ we obtain the condition

$$f'(s) \cdot b(f(s), g(s), h(s)) - g'(s) \cdot a(f(s), g(s), h(s)) \neq 0.$$

Burger's Equation. *Solve the Cauchy problem*

$$\begin{cases} u_t + uu_x = 0, \\ u(x, 0) = h(x). \end{cases} \tag{5.1}$$

The characteristic equations are

$$\frac{dx}{dt} = z, \quad \frac{dy}{dt} = 1, \quad \frac{dz}{dt} = 0,$$

and Γ may be parametrized by $(s, 0, h(s))$.

$$x = h(s)t + s, \quad y = t, \quad z = h(s).$$

$$u(x, y) = h(x - uy) \tag{5.2}$$

The characteristic projection in the xt-plane[1] passing through the point $(s, 0)$ is the line

$$x = h(s)t + s$$

along which u has the constant value $u = h(s)$. Two characteristics $x = h(s_1)t + s_1$ and $x = h(s_2)t + s_2$ intersect at a point (x, t) with

$$t = -\frac{s_2 - s_1}{h(s_2) - h(s_1)}.$$

[1] y and t are interchanged here

From (5.2), we have

$$u_x = h'(s)(1 - u_x t) \quad \Rightarrow \quad u_x = \frac{h'(s)}{1 + h'(s)t}$$

Hence for $h'(s) < 0$, u_x becomes infinite at the positive time

$$t = \frac{-1}{h'(s)}.$$

The smallest t for which this happens corresponds to the value $s = s_0$ at which $h'(s)$ has a minimum (i.e.$-h'(s)$ has a maximum). At time $T = -1/h'(s_0)$ the solution u experiences a "gradient catastrophe".

5.2 Weak Solutions for Quasilinear Equations

5.2.1 Conservation Laws and Jump Conditions

Consider shocks for an equation

$$u_t + f(u)_x = 0, \tag{5.3}$$

where f is a smooth function of u. If we integrate (5.3) with respect to x for $a \le x \le b$, we obtain

$$\frac{d}{dt}\int_a^b u(x,t)\,dx \;+\; f(u(b,t)) - f(u(a,t)) = 0. \tag{5.4}$$

This is an example of a *conservation law.* Notice that (5.4) implies (5.3) if u is C^1, but (5.4) makes sense for more general u.

Consider a solution of (5.4) that, for fixed t, has a jump discontinuity at $x = \xi(t)$. We assume that u, u_x, and u_t are continuous up to ξ. Also, we assume that $\xi(t)$ is C^1 in t.

Taking $a < \xi(t) < b$ in (5.4), we obtain

$$\begin{aligned}
&\frac{d}{dt}\Big(\int_a^\xi u\,dx + \int_\xi^b u\,dx\Big) \;+\; f(u(b,t)) - f(u(a,t)) \\
&\quad= \xi'(t)u_l(\xi(t),t) - \xi'(t)u_r(\xi(t),t) + \int_a^\xi u_t(x,t)\,dx + \int_\xi^b u_t(x,t)\,dx \\
&\qquad\quad + f(u(b,t)) - f(u(a,t)) = 0,
\end{aligned}$$

where u_l and u_r denote the limiting values of u from the left and right sides of the shock. Letting $a \uparrow \xi(t)$ and $b \downarrow \xi(t)$, we get the **Rankine-Hugoniot jump condition**:

$$\xi'(t)(u_l - u_r) + f(u_r) - f(u_l) = 0,$$

$$\boxed{\xi'(t) = \frac{f(u_r) - f(u_l)}{u_r - u_l}.}$$

5.2.2 Fans and Rarefaction Waves

For Burgers' equation

$$u_t + \Big(\frac{1}{2}u^2\Big)_x = 0,$$

$$\text{we have } f'(u) = u, \quad f'\Big(\tilde{u}\Big(\frac{x}{t}\Big)\Big) = \frac{x}{t} \quad\Rightarrow\quad \tilde{u}\Big(\frac{x}{t}\Big) = \frac{x}{t}.$$

For a rarefaction fan emanating from $(s,0)$ on xt-plane, we have:

$$u(x,t) = \begin{cases} u_l, & \frac{x-s}{t} \le f'(u_l) = u_l, \\ \frac{x-s}{t}, & u_l \le \frac{x-s}{t} \le u_r, \\ u_r, & \frac{x-s}{t} \ge f'(u_r) = u_r. \end{cases}$$

5.3 General Nonlinear Equations

5.3.1 Two Spatial Dimensions

Write a general nonlinear equation $F(x, y, u, u_x, u_y) = 0$ as

$$F(x, y, z, p, q) = 0.$$

Γ is parameterized by

$$\Gamma : \big(\underbrace{f(s)}_{x(s,0)}, \underbrace{g(s)}_{y(s,0)}, \underbrace{h(s)}_{z(s,0)}, \underbrace{\phi(s)}_{p(s,0)}, \underbrace{\psi(s)}_{q(s,0)}\big)$$

We need to complete Γ to a strip. Find $\phi(s)$ and $\psi(s)$, the initial conditions for $p(s,t)$ and $q(s,t)$, respectively:

- $F(f(s), g(s), h(s), \phi(s), \psi(s)) \; = \; 0$
- $h'(s) \; = \; \phi(s)f'(s) + \psi(s)g'(s)$

The characteristic equations are

$$\frac{dx}{dt} = F_p \qquad \frac{dy}{dt} = F_q$$
$$\frac{dz}{dt} = pF_p + qF_q$$
$$\frac{dp}{dt} = -F_x - F_z p \qquad \frac{dq}{dt} = -F_y - F_z q$$

We need to have the Jacobian condition. That is, in order to solve the Cauchy problem in a neighborhood of Γ, the following condition must be satisfied:

$$f'(s) \cdot F_q[f, g, h, \phi, \psi](s) - g'(s) \cdot F_p[f, g, h, \phi, \psi](s) \neq 0.$$

5.3.2 Three Spatial Dimensions

Write a general nonlinear equation $F(x_1, x_2, x_3, u, u_{x_1}, u_{x_2}, u_{x_3}) = 0$ as

$$F(x_1, x_2, x_3, z, p_1, p_2, p_3) = 0.$$

Γ is parameterized by

$$\Gamma : \big(\underbrace{f_1(s_1, s_2)}_{x_1(s_1,s_2,0)}, \underbrace{f_2(s_1, s_2)}_{x_2(s_1,s_2,0)}, \underbrace{f_3(s_1, s_2)}_{x_3(s_1,s_2,0)}, \underbrace{h(s_1, s_2)}_{z(s_1,s_2,0)}, \underbrace{\phi_1(s_1, s_2)}_{p_1(s_1,s_2,0)}, \underbrace{\phi_2(s_1, s_2)}_{p_2(s_1,s_2,0)}, \underbrace{\phi_3(s_1, s_2)}_{p_3(s_1,s_2,0)}\big)$$

We need to complete Γ to a strip. Find $\phi_1(s_1, s_2)$, $\phi_2(s_1, s_2)$, and $\phi_3(s_1, s_2)$, the initial conditions for $p_1(s_1, s_2, t)$, $p_2(s_1, s_2, t)$, and $p_3(s_1, s_2, t)$, respectively:

- $F\big(f_1(s_1, s_2), f_2(s_1, s_2), f_3(s_1, s_2), h(s_1, s_2), \phi_1, \phi_2, \phi_3\big) \; = \; 0$
- $\dfrac{\partial h}{\partial s_1} \; = \; \phi_1 \dfrac{\partial f_1}{\partial s_1} + \phi_2 \dfrac{\partial f_2}{\partial s_1} + \phi_3 \dfrac{\partial f_3}{\partial s_1}$
- $\dfrac{\partial h}{\partial s_2} \; = \; \phi_1 \dfrac{\partial f_1}{\partial s_2} + \phi_2 \dfrac{\partial f_2}{\partial s_2} + \phi_3 \dfrac{\partial f_3}{\partial s_2}$

The characteristic equations are

$$\frac{dx_1}{dt} = F_{p_1} \qquad \frac{dx_2}{dt} = F_{p_2} \qquad \frac{dx_3}{dt} = F_{p_3}$$
$$\frac{dz}{dt} = p_1 F_{p_1} + p_2 F_{p_2} + p_3 F_{p_3}$$
$$\frac{dp_1}{dt} = -F_{x_1} - p_1 F_z \qquad \frac{dp_2}{dt} = -F_{x_2} - p_2 F_z \qquad \frac{dp_3}{dt} = -F_{x_3} - p_3 F_z$$

6 Second-Order Equations

6.1 Classification by Characteristics

Consider the second-order equation in which the derivatives of second-order all occur linearly, with coefficients only depending on the independent variables:

$$a(x,y)u_{xx} + b(x,y)u_{xy} + c(x,y)u_{yy} = d(x,y,u,u_x,u_y). \tag{6.1}$$

The *characteristic* equation is

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}.$$

- $b^2 - 4ac > 0 \;\Rightarrow\;$ *two* characteristics, and (6.1) is called *hyperbolic*;
- $b^2 - 4ac = 0 \;\Rightarrow\;$ *one* characteristic, and (6.1) is called *parabolic*;
- $b^2 - 4ac < 0 \;\Rightarrow\;$ *no* characteristics, and (6.1) is called *elliptic*.

These definitions are all taken at a point $x_0 \in \mathbb{R}^2$; unless a, b, and c are all constant, the *type* may change with the point x_0.

6.2 Canonical Forms and General Solutions

① $u_{xx} - u_{yy} = 0$ is hyperbolic (one-dimensional wave equation).
② $u_{xx} - u_y = 0$ is parabolic (one-dimensional heat equation).
③ $u_{xx} + u_{yy} = 0$ is elliptic (two-dimensional Laplace equation).

By the introduction of new coordinates μ and η in place of x and y, the equation (6.1) may be transformed so that its principal part takes the form ①, ②, or ③.
If (6.1) is *hyperbolic*, *parabolic*, or *elliptic*, there exists a change of variables $\mu(x,y)$ and $\eta(x,y)$ under which (6.1) becomes, respectively,

$$\begin{aligned}
&u_{\mu\eta} = \tilde{d}(\mu,\eta,u,u_\mu,u_\eta) \qquad \Leftrightarrow \qquad u_{\bar{x}\bar{x}} - u_{\bar{y}\bar{y}} = \bar{d}(\bar{x},\bar{y},u,u_{\bar{x}},u_{\bar{y}}),\\
&u_{\mu\mu} = \tilde{d}(\mu,\eta,u,u_\mu,u_\eta),\\
&u_{\mu\mu} + u_{\eta\eta} = \tilde{d}(\mu,\eta,u,u_\mu,u_\eta).
\end{aligned}$$

Example 1. *Reduce to canonical form and find the general solution:*

$$u_{xx} + 5u_{xy} + 6u_{yy} = 0. \tag{6.2}$$

Proof. $a = 1,\ b = 5,\ c = 6 \quad \Rightarrow \quad b^2 - 4ac = 1 > 0 \quad \Rightarrow$ **hyperbolic** $\Rightarrow$ two characteristics.
The characteristics are found by solving

$$\frac{dy}{dx} = \frac{5 \pm 1}{2} = \begin{cases} 3 \\ 2 \end{cases}$$

to find $y = 3x + c_1$ and $y = 2x + c_2$.

$$\begin{aligned}
\text{Let} \quad & \mu(x,y) = 3x - y, \quad \eta(x,y) = 2x - y. \\
& \mu_x = 3, \qquad \eta_x = 2, \\
& \mu_y = -1, \qquad \eta_y = -1.
\end{aligned}$$

$$\begin{aligned}
u &= u(\mu(x,y), \eta(x,y)); \\
u_x &= u_\mu \mu_x + u_\eta \eta_x = 3u_\mu + 2u_\eta, \\
u_y &= u_\mu \mu_y + u_\eta \eta_y = -u_\mu - u_\eta, \\
u_{xx} &= (3u_\mu + 2u_\eta)_x = 3(u_{\mu\mu}\mu_x + u_{\mu\eta}\eta_x) + 2(u_{\eta\mu}\mu_x + u_{\eta\eta}\eta_x) = 9u_{\mu\mu} + 12u_{\mu\eta} + 4u_{\eta\eta}, \\
u_{xy} &= (3u_\mu + 2u_\eta)_y = 3(u_{\mu\mu}\mu_y + u_{\mu\eta}\eta_y) + 2(u_{\eta\mu}\mu_y + u_{\eta\eta}\eta_y) = -3u_{\mu\mu} - 5u_{\mu\eta} - 2u_{\eta\eta}, \\
u_{yy} &= -(u_\mu + u_\eta)_y = -(u_{\mu\mu}\mu_y + u_{\mu\eta}\eta_y + u_{\eta\mu}\mu_y + u_{\eta\eta}\eta_y) = u_{\mu\mu} + 2u_{\mu\eta} + u_{\eta\eta}.
\end{aligned}$$

Inserting these expressions into (6.2) and simplifying, we obtain

$$\begin{aligned}
u_{\mu\eta} &= 0, \qquad \text{which is the } \textbf{Canonical form}, \\
u_\mu &= f(\mu), \\
u &= F(\mu) + G(\eta), \\
u(x,y) &= F(3x - y) + G(2x - y), \qquad \textbf{General solution}.
\end{aligned}$$

□

Example 2. *Reduce to canonical form and find the general solution:*

$$y^2 u_{xx} - 2y u_{xy} + u_{yy} = u_x + 6y. \tag{6.3}$$

Proof. $a = y^2$, $b = -2y$, $c = 1$ $\Rightarrow$ $b^2 - 4ac = 0$ $\Rightarrow$ **parabolic** $\Rightarrow$ one characteristic. The characteristics are found by solving

$$\frac{dy}{dx} = \frac{-2y}{2y^2} = -\frac{1}{y}$$

$$\text{to find } -\frac{y^2}{2} + c = x.$$

Let $\mu = \frac{y^2}{2} + x$. We must choose a second constant function $\eta(x,y)$ so that η is not parallel to μ. Choose $\eta(x,y) = y$.

$$\begin{aligned}
& \mu_x = 1, \qquad \eta_x = 0, \\
& \mu_y = y, \qquad \eta_y = 1.
\end{aligned}$$

$$\begin{aligned}
u &= u(\mu(x,y), \eta(x,y)); \\
u_x &= u_\mu \mu_x + u_\eta \eta_x = u_\mu, \\
u_y &= u_\mu \mu_y + u_\eta \eta_y = y u_\mu + u_\eta, \\
u_{xx} &= (u_\mu)_x = u_{\mu\mu}\mu_x + u_{\mu\eta}\eta_x = u_{\mu\mu}, \\
u_{xy} &= (u_\mu)_y = u_{\mu\mu}\mu_y + u_{\mu\eta}\eta_y = y u_{\mu\mu} + u_{\mu\eta}, \\
u_{yy} &= (y u_\mu + u_\eta)_y = u_\mu + y(u_{\mu\mu}\mu_y + u_{\mu\eta}\eta_y) + (u_{\eta\mu}\mu_y + u_{\eta\eta}\eta_y) \\
&= u_\mu + y^2 u_{\mu\mu} + 2y u_{\mu\eta} + u_{\eta\eta}.
\end{aligned}$$

Inserting these expressions into (6.3) and simplifying, we obtain

$$\begin{aligned}
u_{\eta\eta} &= 6y, \\
u_{\eta\eta} &= 6\eta, \qquad \text{which is the } \textbf{Canonical form}, \\
u_{\eta} &= 3\eta^2 + f(\mu), \\
u &= \eta^3 + \eta f(\mu) + g(\mu), \\
u(x,y) &= y^3 + y \cdot f\Big(\frac{y^2}{2} + x\Big) + g\Big(\frac{y^2}{2} + x\Big), \qquad \textbf{General solution}.
\end{aligned}$$

$\square$

Problem (F'03, #4). *Find the characteristics of the partial differential equation*

$$xu_{xx} + (x-y)u_{xy} - yu_{yy} = 0, \qquad x>0,\ y>0, \tag{6.4}$$

and then show that it can be transformed into the canonical form

$$(\xi^2 + 4\eta)u_{\xi\eta} + \xi u_\eta = 0$$

whence ξ and η are suitably chosen canonical coordinates. Use this to obtain the general solution in the form

$$u(\xi,\eta) = f(\xi) + \int^{\eta} \frac{g(\eta')\,d\eta'}{(\xi^2+4\eta')^{\frac{1}{2}}}$$

where f and g are arbitrary functions of ξ and η.

Proof. $a = x,\quad b = x-y,\quad c = -y \quad\Rightarrow\quad b^2-4ac = (x-y)^2+4xy > 0$ for $x>0$, $y>0 \quad\Rightarrow$ **hyperbolic** $\Rightarrow$ two characteristics.

① The **characteristics** are found by solving

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2-4ac}}{2a} = \frac{x-y\pm\sqrt{(x-y)^2+4xy}}{2x} = \frac{x-y\pm(x+y)}{2x} = \begin{cases} \frac{2x}{2x} = 1 \\ -\frac{2y}{2x} = -\frac{y}{x} \end{cases}$$

$$\Rightarrow \qquad y = x + c_1, \qquad\qquad \frac{dy}{y} = -\frac{dx}{x},$$

$$ln\ y = ln\ x^{-1} + \tilde{c_2},$$

$$y = \frac{c_2}{x}.$$

② Let $\mu = x - y$ and $\eta = xy$

$$\mu_x = 1, \qquad \eta_x = y,$$
$$\mu_y = -1, \qquad \eta_y = x.$$

$$\begin{aligned}
u &= u(\mu(x,y),\eta(x,y));\\
u_x &= u_\mu\mu_x + u_\eta\eta_x = u_\mu + yu_\eta,\\
u_y &= u_\mu\mu_y + u_\eta\eta_y = -u_\mu + xu_\eta,\\
u_{xx} &= (u_\mu + yu_\eta)_x = u_{\mu\mu}\mu_x + u_{\mu\eta}\eta_x + y(u_{\eta\mu}\mu_x + u_{\eta\eta}\eta_x) = u_{\mu\mu} + 2yu_{\mu\eta} + y^2u_{\eta\eta},\\
u_{xy} &= (u_\mu + yu_\eta)_y = u_{\mu\mu}\mu_y + u_{\mu\eta}\eta_y + u_\eta + y(u_{\eta\mu}\mu_y + u_{\eta\eta}\eta_y) = -u_{\mu\mu} + xu_{\mu\eta} + u_\eta - yu_{\eta\mu} + xyu_{\eta\eta},\\
u_{yy} &= (-u_\mu + xu_\eta)_y = -u_{\mu\mu}\mu_y - u_{\mu\eta}\eta_y + x(u_{\eta\mu}\mu_y + u_{\eta\eta}\eta_y) = u_{\mu\mu} - 2xu_{\mu\eta} + x^2u_{\eta\eta},
\end{aligned}$$

Inserting these expressions into (6.4), we obtain

$$x(u_{\mu\mu} + 2yu_{\mu\eta} + y^2u_{\eta\eta}) + (x-y)(-u_{\mu\mu} + xu_{\mu\eta} + u_\eta - yu_{\eta\mu} + xyu_{\eta\eta}) - y(u_{\mu\mu} - 2xu_{\mu\eta} + x^2u_{\eta\eta}) = 0,$$
$$(x^2 + 2xy + y^2)u_{\mu\eta} + (x-y)u_\eta = 0,$$
$$\big((x-y)^2 + 4xy\big)u_{\mu\eta} + (x-y)u_\eta = 0,$$
$$(\mu^2 + 4\eta)u_{\mu\eta} + \mu u_\eta = 0, \qquad \text{which is the \textbf{Canonical form}}.$$

③ We need to integrate twice to get the general solution:

$$
\begin{aligned}
&(\mu^2+4\eta)(u_\eta)_\mu + \mu u_\eta = 0, \\
&\int \frac{(u_\eta)_\mu}{u_\eta}\, d\mu = -\int \frac{\mu}{\mu^2+4\eta}\, d\mu, \\
&ln\ u_\eta = -\frac{1}{2} ln\ (\mu^2+4\eta) + \tilde{g}(\eta), \\
&ln\ u_\eta = ln\ (\mu^2+4\eta)^{-\frac{1}{2}} + \tilde{g}(\eta), \\
&u_\eta = \frac{g(\eta)}{(\mu^2+4\eta)^{\frac{1}{2}}}, \\
&u(\mu,\eta) = f(\mu) + \int \frac{g(\eta)\, d\eta}{(\mu^2+4\eta)^{\frac{1}{2}}}, \qquad \textbf{General solution}.
\end{aligned}
$$

□

6.3 Well-Posedness

Problem (S'99, #2). *In $\mathbb{R}^2$ consider the unit square Ω defined by $0 \le x, y \le 1$. Consider*

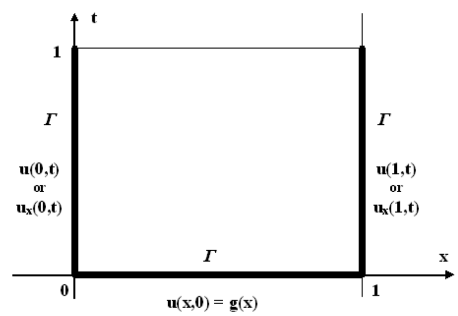
$$\begin{aligned} a) &\quad u_x + u_{yy} = 0; \\ b) &\quad u_{xx} + u_{yy} = 0; \\ c) &\quad u_{xx} - u_{yy} = 0. \end{aligned}$$

*Prescribe data for each problem separately on the boundary of Ω so that each of these problems is **well-posed**. Justify your answers.*

Proof. • The initial / boundary value problem for the **HEAT EQUATION** is *well-posed*:

$$\begin{cases} u_t = \triangle u & x \in \Omega,\ t > 0, \\ u(x,0) = g(x) & x \in \overline{\Omega}, \\ u(x,t) = 0 & x \in \partial\Omega,\ t > 0. \end{cases}$$

Existence - by eigenfunction expansion.
Uniqueness and *continuous dependence on the data* - by maximum principle.
The method of eigenfunction expansion and maximum principle give well-posedness for more general problems:



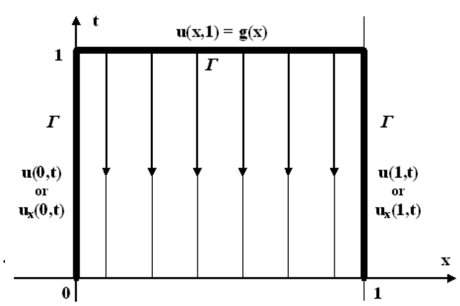
$$\begin{cases} u_t = \triangle u + f(x,t) & x \in \Omega,\ t > 0, \\ u(x,0) = g(x) & x \in \overline{\Omega}, \\ u(x,t) = h(x,t) & x \in \partial\Omega,\ t > 0. \end{cases}$$

It is also possible to replace the Dirichlet boundary condition $u(x,t) = h(x,t)$ by a Neumann or Robin condition, provided we replace λ_n, ϕ_n by the eigenvalues and eigenfunctions for the appropriate boundary value problem.

a) • Relabel the variables $(x \to t,\ y \to x)$.
We have the **BACKWARDS HEAT EQUATION**:

$$u_t + u_{xx} = 0.$$

Need to define initial conditions $u(x,1) = g(x)$, and either Dirichlet, Neumann, or Robin boundary conditions.



b) • The solution to the **LAPLACE EQUATION**

$$\begin{cases} \triangle u = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega \end{cases}$$

exists if g is continuous on $\partial\Omega$, by Perron's method. Maximum principle gives *uniqueness*.
To show the *continuous dependence on the data*, assume

$$\begin{cases} \triangle u_1 = 0 & \text{in } \Omega, \\ u_1 = g_1 & \text{on } \partial\Omega; \end{cases} \qquad \begin{cases} \triangle u_2 = 0 & \text{in } \Omega, \\ u_2 = g_2 & \text{on } \partial\Omega. \end{cases}$$

Then $\triangle(u_1 - u_2) = 0$ in Ω. Maximum principle gives

$$\begin{aligned} \max_{\overline{\Omega}}(u_1 - u_2) &= \max_{\partial\Omega}(g_1 - g_2). \qquad \text{Thus,} \\ \max_{\overline{\Omega}}|u_1 - u_2| &= \max_{\partial\Omega}|g_1 - g_2|. \end{aligned}$$

Thus, $|u_1 - u_2|$ is bounded by $|g_1 - g_2|$, i.e. continuous dependence on data.

- Perron's method gives *existence* of the solution to the **POISSON EQUATION**

$$\begin{cases} \triangle u = f & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = h & \text{on } \partial\Omega \end{cases}$$

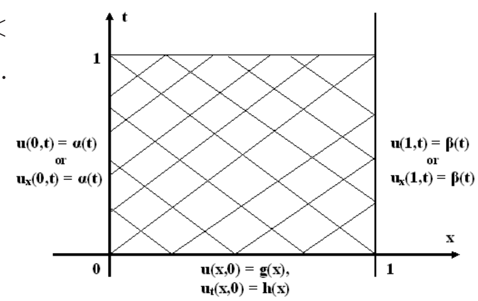
for $f \in C^\infty(\overline{\Omega})$ and $h \in C^\infty(\partial\Omega)$, satisfying the compatibility condition $\int_{\partial\Omega} h\, dS = \int_\Omega f\, dx$. It is *unique* up to an additive constant.

c) • Relabel the variables $(y \to t)$.
The solution to the **WAVE EQUATION**

$$u_{tt} - u_{xx} = 0,$$

is of the form $u(x, y) = F(x + t) + G(x - t)$.
The *existence* of the solution to the initial/boundary value problem

$$\begin{cases} u_{tt} - u_{xx} = 0 & 0 < x < 1, \ t > 0 \\ u(x,0) = g(x), \quad u_t(x,0) = h(x) & 0 < x < \\ u(0,t) = \alpha(t), \quad u(1,t) = \beta(t) & t \geq 0. \end{cases}$$



is given by the method of separation of variables (expansion in eigenfunctions)
and by the parallelogram rule.
Uniqueness is given by the energy method.
Need initial conditions $u(x, 0)$, $u_t(x, 0)$.
Prescribe u or u_x for each of the two boundaries.

$\square$

Problem (F'95, #7). *Let a, b be real numbers. The PDE*

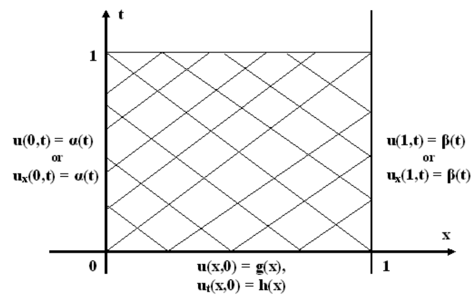
$$u_y + au_{xx} + bu_{yy} = 0$$

is to be solved in the box $\Omega = [0,1]^2$.
Find data, given on an appropriate part of $\partial\Omega$, that will make this a ***well-posed*** *problem.*
Cover all cases according to the possible values of a and b. Justify your statements.

Proof.
① $ab < 0 \;\Rightarrow\;$ two sets of characteristics $\;\Rightarrow\;$ **hyperbolic**.
Relabeling the variables $(y \to t)$, we have

$$u_{tt} + \frac{a}{b}u_{xx} = -\frac{1}{b}u_t.$$

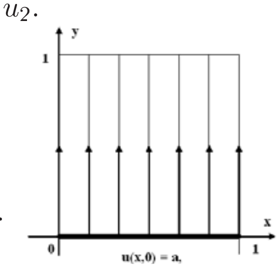
The solution of the equation is of the form
$u(x,t) = F(x + \sqrt{-\frac{a}{b}}t) + G(x - \sqrt{-\frac{a}{b}}t)$.
Existence of the solution to the initial/boundary value problem is given by the method of separation of variables (expansion in eigenfunctions) and by the parallelogram rule.
Uniqueness is given by the energy method.
Need initial conditions $u(x,0)$, $u_t(x,0)$.
Prescribe u or u_x for each of the two boundaries.



② $ab > 0 \;\Rightarrow\;$ no characteristics $\;\Rightarrow\;$ **elliptic**.
The solution to the Laplace equation with boundary conditions $u = g$ on $\partial\Omega$ *exists* if g is continuous on $\partial\Omega$, by Perron's method.
To show uniqueness, we use maximum principle. Assume there are two solutions u_1 and u_2 with with $u_1 = g(x)$, $u_2 = g(x)$ on $\partial\Omega$. By maximum principle

$$\max_{\overline{\Omega}}(u_1 - u_2) \;=\; \max_{\partial\Omega}(g(x) - g(x)) = 0. \qquad \text{Thus, } u_1 = u_2.$$

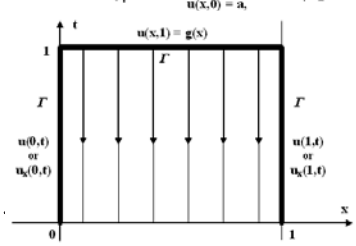
③ $ab = 0 \;\Rightarrow\;$ one set of characteristics $\;\Rightarrow\;$ **parabolic**.
- $a = b = 0$. We have $u_y = 0$, a first-order ODE.
u must be specified on $y = 0$, i.e. x -axis.
- $a = 0$, $b \neq 0$. We have $u_y + bu_{yy} = 0$, a second-order ODE.
u and u_y must be specified on $y = 0$, i.e. x -axis.



- $a > 0$, $b = 0$. We have a Backwards Heat Equation.

$$u_t = -au_{xx}.$$

Need to define initial conditions $u(x,1) = g(x)$, and either Dirichlet, Neumann, or Robin boundary conditions.

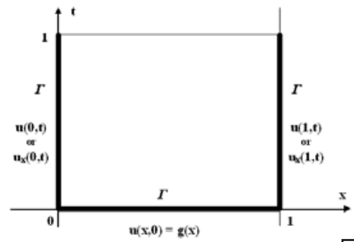


- $a < 0$, $b = 0$. We have a Heat Equation.

$$u_t = -a u_{xx}.$$

The initial / boundary value problem for the heat equation is *well-posed*:

$$\begin{cases} u_t = \triangle u & x \in \Omega,\ t > 0, \\ u(x,0) = g(x) & x \in \overline{\Omega}, \\ u(x,t) = 0 & x \in \partial\Omega,\ t > 0. \end{cases}$$



Existence - by eigenfunction expansion.
Uniqueness and *continuous dependence on the data* - by maximum principle. $\square$

7 Wave Equation

The *one-dimensional wave equation* is

$$u_{tt} - c^2 u_{xx} = 0. \tag{7.1}$$

The characteristic equation with $a = -c^2$, $b = 0$, $c = 1$ would be

$$\frac{dt}{dx} = \frac{b \pm \sqrt{b^2 - 4ac}}{2a} = \pm\frac{\sqrt{4c^2}}{-2c^2} = \pm\frac{1}{c},$$

and thus

$$\begin{array}{ccc} t = -\frac{1}{c}x + c_1 & \text{and} & t = \frac{1}{c}x + c_2, \\ \mu = x + ct & & \eta = x - ct, \end{array}$$

which transforms (7.1) to

$$u_{\mu\eta} = 0. \tag{7.2}$$

The general solution of (7.2) is $u(\mu, \eta) = F(\mu) + G(\eta)$, where F and G are C^1 functions. Returning to the variables x, t we find that

$$u(x,t) = F(x+ct) + G(x-ct) \tag{7.3}$$

solves (7.1). Moreover, u is C^2 provided that F and G are C^2.
If $F \equiv 0$, then u has constant values along the lines $x - ct = \text{const}$, so may be described as a wave moving in the positive x-direction with speed $dx/dt = c$; if $G \equiv 0$, then u is a wave moving in the negative x-direction with speed c.

7.1 The Initial Value Problem

For an initial value problem, consider the *Cauchy problem*

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, \\ u(x,0) = g(x), \quad u_t(x,0) = h(x). \end{cases} \tag{7.4}$$

Using (7.3) and (7.4), we find that F and G satisfy

$$F(x) + G(x) = g(x), \qquad cF'(x) - cG'(x) = h(x). \tag{7.5}$$

If we integrate the second equation in (7.5), we get $cF(x) - cG(x) = \int_0^x h(\xi)\,d\xi + C$. Combining this with the first equation in (7.5), we can solve for F and G to find

$$\begin{cases} F(x) = \frac{1}{2}g(x) + \frac{1}{2c}\int_0^x h(\xi)\,d\xi + C_1 \\ G(x) = \frac{1}{2}g(x) - \frac{1}{2c}\int_0^x h(\xi)\,d\xi - C_1, \end{cases}$$

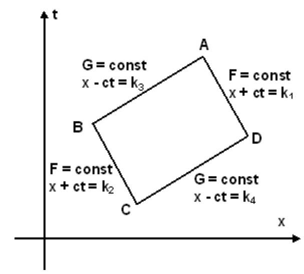
Using these expressions in (7.3), we obtain **d'Alembert's Formula** for the solution of the initial value problem (7.4):

$$\boxed{u(x,t) = \frac{1}{2}(g(x+ct) + g(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} h(\xi)\,d\xi.}$$

If $g \in C^2$ and $h \in C^1$, then d'Alembert's Formula defines a C^2 solution of (7.4).

7.2 Weak Solutions

Equation (7.3) defines a *weak* solution of (7.1) when F and G are not C^2 functions. Consider the parallelogram with sides that are segments of characteristics. Since $u(x,t) = F(x+ct) + G(x-ct)$, we have



$$\begin{aligned}
&u(A) + u(C) = \\
&= F(k_1) + G(k_3) + F(k_2) + G(k_4) \\
&= u(B) + u(D),
\end{aligned}$$

which is the *parallelogram rule.*

7.3 Initial/Boundary Value Problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & 0 < x < L, \ t > 0 \\ u(x,0) = g(x), \quad u_t(x,0) = h(x) & 0 < x < L \\ u(0,t) = \alpha(t), \quad u(L,t) = \beta(t) & t \geq 0. \end{cases} \tag{7.6}$$

Use separation of variables to obtain an expansion in eigenfunctions. Find $u(x,t)$ in the form

$$u(x,t) = \frac{a_0(t)}{2} + \sum_{n=1}^{\infty} a_n(t) \cos\frac{n\pi x}{L} + b_n(t) \sin\frac{n\pi x}{L}.$$

7.4 Duhamel's Principle

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) \\ u(x,0) = 0 \\ u_t(x,0) = 0. \end{cases} \quad \Rightarrow \quad \begin{cases} U_{tt} - c^2 U_{xx} = 0 \\ U(x,0,s) = 0 \\ U_t(x,0,s) = f(x,s) \end{cases} \qquad u(x,t) = \int_0^t U(x, t-s, s)\, ds.$$

$$\begin{cases} a_n'' + \lambda_n a_n = f_n(t) \\ a_n(0) = 0 \\ a_n'(0) = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} \tilde{a}_n'' + \lambda_n \tilde{a}_n = 0 \\ \tilde{a}_n(0,s) = 0 \\ \tilde{a}_n'(0,s) = f_n(s) \end{cases} \qquad a_n(t) = \int_0^t \tilde{a}_n(t-s, s)\, ds.$$

7.5 The Nonhomogeneous Equation

Consider the *nonhomogeneous wave equation* with *homogeneous initial conditions*:

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t), \\ u(x,0) = 0, \quad u_t(x,0) = 0. \end{cases} \tag{7.7}$$

Duhamel's Principle provides the solution of (7.7):

$$\boxed{u(x,t) = \frac{1}{2c}\int_0^t \left(\int_{x-c(t-s)}^{x+c(t-s)} f(\xi, s)\, d\xi \right) ds.}$$

If $f(x,t)$ is C^1 in x and C^0 in t, then Duhamel's Principle provides a C^2 solution of (7.7).

We can solve (7.7) with *nonhomogeneous initial conditions*,

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t), \\ u(x,0) = g(x), \quad u_t(x,0) = h(x), \end{cases} \tag{7.8}$$

by adding together d'Alembert's formula and Duhamel's principle gives the solution:

$$\boxed{u(x,t) = \frac{1}{2}(g(x+ct) + g(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} h(\xi)\,d\xi + \frac{1}{2c}\int_0^t \left(\int_{x-c(t-s)}^{x+c(t-s)} f(\xi,s)\,d\xi\right) ds.}$$

7.6 Higher Dimensions

7.6.1 Spherical Means

For a continuous function $u(x)$ on $\mathbb{R}^n$, its *spherical mean* or *average on a sphere of radius* r *and center* x is

$$M_u(x,r) = \frac{1}{\omega_n}\int_{|\xi|=1} u(x+r\xi)dS_\xi,$$

where ω_n is the area of the unit sphere $S^{n-1} = \{\xi \in \mathbb{R}^n : |\xi| = 1\}$ and dS_ξ is surface measure. Since u is continuous in x, $M_u(x,r)$ is continuous in x and r, so

$$M_u(x,0) = u(x).$$

Using the chain rule, we find

$$\frac{\partial}{\partial r}M_u(x,r) = \frac{1}{\omega_n}\int_{|\xi|=1}\sum_{i=1}^{n} u_{x_i}(x+r\xi)\,\xi_i\,dS_\xi \;=\; \circledast$$

To compute the RHS, we apply the divergence theorem in $\Omega = \{\xi \in \mathbb{R}^n : |\xi| < 1\}$, which has boundary $\partial\Omega = S^{n-1}$ and exterior unit normal $n(\xi) = \xi$. The integrand is $V \cdot n$ where $V(\xi) = r^{-1}\nabla_\xi u(x+r\xi) = \nabla_x u(x+r\xi)$. Computing the divergence of V, we obtain

$$\begin{aligned}
\operatorname{div} V(\xi) &= r\sum_{i=1}^{n} u_{x_i x_i}(x+r\xi) \;=\; r\triangle_x u(x+r\xi), \qquad \text{so,} \\
\circledast &= \frac{1}{\omega_n}\int_{|\xi|<1} r\triangle_x u(x+r\xi)\,d\xi \;=\; \frac{r}{\omega_n}\triangle_x\int_{|\xi|<1} u(x+r\xi)\,d\xi \qquad (\xi' = r\xi) \\
&= \frac{r}{\omega_n}\frac{1}{r^n}\triangle_x\int_{|\xi'|<r} u(x+\xi')\,d\xi' \qquad \text{(spherical coordinates)} \\
&= \frac{1}{\omega_n r^{n-1}}\triangle_x\int_0^r \rho^{n-1}\int_{|\xi|=1} u(x+\rho\xi)\,dS_\xi\,d\rho \\
&= \frac{1}{\omega_n r^{n-1}}\,\omega_n\triangle_x\int_0^r \rho^{n-1}M_u(x,\rho)\,d\rho \;=\; \frac{1}{r^{n-1}}\triangle_x\int_0^r \rho^{n-1}M_u(x,\rho)\,d\rho.
\end{aligned}$$

If we multiply by r^{n-1}, differentiate with respect to r, and then divide by r^{n-1}, we obtain the **Darboux equation:**

$$\boxed{\left(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}\right)M_u(x,r) \;=\; \triangle_x M_u(x,r).}$$

Note that for a *radial* function $u = u(r)$, we have $M_u = u$, so the equation provides the Laplacian of u in spherical coordinates.

7.6.2 Application to the Cauchy Problem

We want to solve the equation

$$u_{tt} = c^2\triangle u \qquad x \in \mathbb{R}^n,\ t > 0, \tag{7.9}$$

$$u(x,0) = g(x), \quad u_t(x,0) = h(x) \qquad x \in \mathbb{R}^n.$$

We use *Poisson's method of spherical means* to reduce this problem to a partial differential equation in the two variables r and t.

Suppose that $u(x,t)$ solves (7.9). We can view t as a parameter and take the spherical mean to obtain $M_u(x,r,t)$, which satisfies

$$\frac{\partial^2}{\partial t^2} M_u(x,r,t) \;=\; \frac{1}{\omega_n}\int_{|\xi|=1} u_{tt}(x+r\xi,t)\,dS_\xi \;=\; \frac{1}{\omega_n}\int_{|\xi|=1} c^2 \triangle u(x+r\xi,t)\,dS_\xi \;=\; c^2 \triangle M_u(x,r,t).$$

Invoking the Darboux equation, we obtain the **Euler-Poisson-Darboux equation:**

$$\boxed{\frac{\partial^2}{\partial t^2} M_u(x,r,t) \;=\; c^2\left(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}\right) M_u(x,r,t).}$$

The initial conditions are obtained by taking the spherical means:

$$M_u(x,r,0) \;=\; M_g(x,r), \qquad\qquad \frac{\partial M_u}{\partial t}(x,r,0) = M_h(x,r).$$

If we find $M_u(x,r,t)$, we can then recover $u(x,t)$ by:

$$u(x,t) = \lim_{r\to 0} M_u(x,r,t).$$

7.6.3 Three-Dimensional Wave Equation

When $n=3$, we can write the Euler-Poisson-Darboux equation as [2]

$$\frac{\partial^2}{\partial t^2}\Big(rM_u(x,r,t)\Big) \;=\; c^2\,\frac{\partial^2}{\partial r^2}\Big(rM_u(x,r,t)\Big).$$

For each fixed x, consider $V^x(r,t) = rM_u(x,r,t)$ as a solution of the one-dimensional wave equation in r, $t>0$:

$$\begin{aligned}
\frac{\partial^2}{\partial t^2} V^x(r,t) &= c^2\,\frac{\partial^2}{\partial r^2} V^x(r,t), \\
V^x(r,0) &= rM_g(x,r) \;\equiv\; G^x(r), && \text{(IC)} \\
V_t^x(r,0) &= rM_h(x,r) \;\equiv\; H^x(r), && \text{(IC)} \\
V^x(0,t) &= \lim_{r\to 0} rM_u(x,r,t) = 0\cdot u(x,t) = 0. && \text{(BC)} \\
G^x(0) &= H^x(0) \;=\; 0.
\end{aligned}$$

We may extend G^x and H^x as odd functions of r and use d'Alembert's formula for $V^x(r,t)$:

$$V^x(r,t) \;=\; \frac{1}{2}\Big(G^x(r+ct) + G^x(r-ct)\Big) \;+\; \frac{1}{2c}\int_{r-ct}^{r+ct} H^x(\rho)\,d\rho.$$

Since G^x and H^x are odd functions, we have for $r<ct$:

$$G^x(r-ct) \;=\; -G^x(ct-r) \qquad \text{and} \qquad \int_{r-ct}^{r+ct} H^x(\rho)\,d\rho = \int_{ct-r}^{ct+r} H^x(\rho)\,d\rho.$$

After some more manipulations, we find that the solution of (7.9) is given by the **Kirchhoff's formula:**

$$\boxed{u(x,t) \;=\; \frac{1}{4\pi}\frac{\partial}{\partial t}\left(t\int_{|\xi|=1} g(x+ct\xi)\,dS_\xi\right) + \frac{t}{4\pi}\int_{|\xi|=1} h(x+ct\xi)\,dS_\xi.}$$

If $g\in C^3(\mathbb{R}^3)$ and $h\in C^2(\mathbb{R}^3)$, then Kirchhoff's formula defines a C^2-solution of (7.9).

[2] It is seen by expanding the equation below.

7.6.4 Two-Dimensional Wave Equation

This problem is solved by Hadamard's *method of descent*, namely, view (7.9) as a special case of a three-dimensional problem with initial conditions independent of x_3.
We need to convert surface integrals in $\mathbb{R}^3$ to domain integrals in $\mathbb{R}^2$.

$$\boxed{u(x_1,x_2,t) = \frac{1}{4\pi}\frac{\partial}{\partial t}\left(2t\int_{\xi_1^2+\xi_2^2<1}\frac{g(x_1+ct\xi_1,x_2+ct\xi_2)d\xi_1 d\xi_2}{\sqrt{1-\xi_1^2-\xi_2^2}}\right)+\frac{t}{4\pi}\left(2\int_{\xi_1^2+\xi_2^2<1}\frac{h(x_1+ct\xi_1,x_2+ct\xi_2)d\xi_1 d\xi_2}{\sqrt{1-\xi_1^2-\xi_2^2}}\right.}$$

If $g \in C^3(\mathbb{R}^2)$ and $h \in C^2(\mathbb{R}^2)$, then this equation defines a C^2-solution of (7.9).

7.6.5 Huygen's Principle

Notice that $u(x,t)$ depends only on the Cauchy data g, h on the surface of the hypersphere $\{x+ct\xi : |\xi|=1\}$ in $\mathbb{R}^n$, $n=2k+1$; in other words we have *sharp signals.*
If we use the method of descent to obtain the solution for $n=2k$, the hypersurface integrals become domain integrals. This means that there are *no sharp signals.*
The fact that sharp signals exist only for *odd dimensions* $n \geq 3$ is known as *Huygen's principle.*
[3]

[3]For $x \in \mathbb{R}^n$:

$$\frac{\partial}{\partial t}\left(\int_{|\xi|=1} f(x+t\xi)dS_\xi\right) = \frac{1}{t^{n-1}}\int_{|y|\leq t}\triangle f(x+y)dy$$

$$\frac{\partial}{\partial t}\left(\int_{|y|\leq t} f(x+y)dy\right) = t^{n-1}\left(\int_{|\xi|=1} f(x+t\xi)dS_\xi\right)$$

7.7 Energy Methods

Suppose $u \in C^2(\mathbb{R}^n \times (0,\infty))$ solves

$$\begin{cases} u_{tt} = c^2 \triangle u & x \in \mathbb{R}^n,\ t > 0, \\ u(x,0) = g(x), \quad u_t(x,0) = h(x) & x \in \mathbb{R}^n, \end{cases} \tag{7.10}$$

where g and h have compact support.
Define **energy** for a function $u(x,t)$ at time t by

$$E(t) = \frac{1}{2}\int_{\mathbb{R}^n} (u_t^2 + c^2|\nabla u|^2)\,dx.$$

If we differentiate this energy function, we obtain

$$\begin{aligned} \frac{dE}{dt} &= \frac{d}{dt}\bigg[\frac{1}{2}\int_{\mathbb{R}^n} \big(u_t^2 + c^2\sum_{i=1}^n u_{x_i}^2\big)\,dx\bigg] = \int_{\mathbb{R}^n} \big(u_t u_{tt} + c^2 \sum_{i=1}^n u_{x_i} u_{x_i t}\big)\,dx \\ &= \int_{\mathbb{R}^n} u_t u_{tt}\,dx + c^2\Big[\sum_{i=1}^n u_{x_i} u_t\Big]_{\partial\mathbb{R}^n} - \int_{\mathbb{R}^n} c^2 \sum_{i=1}^n u_{x_i x_i} u_t\,dx \\ &= \int_{\mathbb{R}^n} u_t(u_{tt} - c^2\triangle u)\,dx = 0, \end{aligned}$$

or

$$\begin{aligned} \frac{dE}{dt} &= \frac{d}{dt}\bigg[\frac{1}{2}\int_{\mathbb{R}^n} \big(u_t^2 + c^2\sum_{i=1}^n u_{x_i}^2\big)\,dx\bigg] = \int_{\mathbb{R}^n} \big(u_t u_{tt} + c^2 \sum_{i=1}^n u_{x_i} u_{x_i t}\big)\,dx \\ &= \int_{\mathbb{R}^n} \big(u_t u_{tt} + c^2 \nabla u \cdot \nabla u_t\big)\,dx \\ &= \int_{\mathbb{R}^n} u_t u_{tt}\,dx + c^2\bigg[\int_{\partial\mathbb{R}^n} u_t \frac{\partial u}{\partial n}\,ds - \int_{\mathbb{R}^n} u_t \triangle u\,dx\bigg] \\ &= \int_{\mathbb{R}^n} u_t(u_{tt} - c^2\triangle u)\,dx = 0. \end{aligned}$$

Hence, $E(t)$ is constant, or $E(t) \equiv E(0)$.
In particular, if u_1 and u_2 are two solutions of (7.10), then $w = u_1 - u_2$ has zero Cauchy data and hence $E_w(0) = 0$. By discussion above, $E_w(t) \equiv 0$, which implies $w(x,t) \equiv$ const. But $w(x,0) = 0$ then implies $w(x,t) \equiv 0$, so the solution is **unique**.

7.8 Contraction Mapping Principle

Suppose X is a complete metric space with distance function represented by $d(\cdot,\cdot)$.
A mapping $T : X \to X$ is a *strict contraction* if there exists $0 < \alpha < 1$ such that

$$d(Tx, Ty) \leq \alpha\, d(x, y) \qquad \forall\, x, y \in X.$$

An obvious example on $X = \mathbb{R}^n$ is $Tx = \alpha x$, which shrinks all of $\mathbb{R}^n$, leaving 0 fixed.

The Contraction Mapping Principle. *If X is a complete metric space and $T : X \to X$ is a strict contraction, then T has a **unique** fixed point.*

The process of replacing a differential equation by an integral equation occurs in time-evolution partial differential equations.
The Contraction Mapping Principle is used to establish the local existence and **uniqueness** of solutions to various nonlinear equations.

8 Laplace Equation

Consider the *Laplace equation*

$$\triangle u = 0 \qquad \text{in} \;\; \Omega \subset \mathbb{R}^n \tag{8.1}$$

and the *Poisson equation*

$$\triangle u = f \qquad \text{in} \;\; \Omega \subset \mathbb{R}^n. \tag{8.2}$$

Solutions of (8.1) are called *harmonic functions* in Ω.
Cauchy problems for (8.1) and (8.2) are not well posed. We use separation of variables for some special domains Ω to find boundary conditions that are appropriate for (8.1), (8.2).

$$\begin{aligned}
&\textit{Dirichlet problem:} \qquad && u(x) = g(x), && x \in \partial\Omega \\
&\textit{Neumann problem:} && \frac{\partial u(x)}{\partial n} = h(x), && x \in \partial\Omega \\
&\textit{Robin problem:} && \frac{\partial u}{\partial n} + \alpha u = \beta, && x \in \partial\Omega
\end{aligned}$$

8.1 Green's Formulas

$$\int_\Omega \nabla u \cdot \nabla v \, dx \;=\; \int_{\partial\Omega} v \, \frac{\partial u}{\partial n} \, ds - \int_\Omega v \, \triangle u \, dx \tag{8.3}$$

$$\int_{\partial\Omega} (v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}) ds = \int_\Omega (v\triangle u - u\triangle v) \, dx$$

$$\int_{\partial\Omega} \frac{\partial u}{\partial n} \, ds \;=\; \int_\Omega \triangle u \, dx \qquad\qquad (v = 1 \text{ in } (8.3))$$

$$\int_\Omega |\nabla u|^2 \, dx \;=\; \int_{\partial\Omega} u \, \frac{\partial u}{\partial n} \, ds - \int_\Omega u \, \triangle u \, dx \qquad\qquad (u = v \text{ in } (8.3))$$

$$\int_\Omega u_x v_x \, dxdy \;=\; \int_{\partial\Omega} v u_x n_1 \, ds \;-\; \int_\Omega v u_{xx} \, dxdy \qquad\qquad \vec{n} = (n_1, n_2) \in \mathbb{R}^2$$

$$\int_\Omega u_{x_k} v \, dx \;=\; \int_{\partial\Omega} u v n_k \, ds \;-\; \int_\Omega u v_{x_k} \, dx \qquad\qquad \vec{n} = (n_1, \ldots, n_n) \in \mathbb{R}^n.$$

$$\int_\Omega u \, \triangle^2 v \, dx \;=\; \int_{\partial\Omega} u\frac{\partial \triangle v}{\partial n} \, ds \;-\; \int_{\partial\Omega} \triangle v \frac{\partial u}{\partial n} \, ds \;+\; \int_\Omega \triangle u \triangle v \, dx.$$

$$\int_\Omega \big(u \, \triangle^2 v - v \, \triangle^2 u\big) \, dx \;=\; \int_{\partial\Omega} \big(u\frac{\partial \triangle v}{\partial n} - v\frac{\partial \triangle u}{\partial n}\big) \, ds \;+\; \int_{\partial\Omega} \big(\triangle u\frac{\partial v}{\partial n} \;-\; \triangle v\frac{\partial u}{\partial n}\big) \, ds.$$

8.2 Polar Coordinates

Polar Coordinates. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous. Then*

$$\int_{\mathbb{R}^n} f\,dx \;=\; \int_0^\infty \Big(\int_{\partial B_r(x_0)} f\,dS\Big)\,dr$$

for each $x_0 \in \mathbb{R}^n$. In particular

$$\frac{d}{dr}\Big(\int_{B_r(x_0)} f\,dx\Big) \;=\; \int_{\partial B_r(x_0)} f\,dS$$

for each $r > 0$.

$$\begin{aligned} u &= u(x(r,\theta), y(r,\theta)) \\ x(r,\theta) &= r\cos\theta \\ y(r,\theta) &= r\sin\theta \end{aligned}$$

$$\begin{aligned}
u_r &= u_x x_r + u_y y_r = u_x\cos\theta + u_y\sin\theta, \\
u_\theta &= u_x x_\theta + u_y y_\theta = -u_x r\sin\theta + u_y r\cos\theta, \\
u_{rr} &= (u_x\cos\theta + u_y\sin\theta)_r = (u_{xx}x_r + u_{xy}y_r)\cos\theta + (u_{yx}x_r + u_{yy}y_r)\sin\theta \\
&= u_{xx}\cos^2\theta + 2u_{xy}\cos\theta\sin\theta + u_{yy}\sin^2\theta, \\
u_{\theta\theta} &= (-u_x r\sin\theta + u_y r\cos\theta)_\theta \\
&= (-u_{xx}x_\theta - u_{xy}y_\theta)r\sin\theta - u_x r\cos\theta + (u_{yx}x_\theta + u_{yy}y_\theta)r\cos\theta - u_y r\sin\theta \\
&= (u_{xx}r\sin\theta - u_{xy}r\cos\theta)r\sin\theta - u_x r\cos\theta + (-u_{yx}r\sin\theta + u_{yy}r\cos\theta)r\cos\theta - u_y r\sin\theta \\
&= r^2(u_{xx}\sin^2\theta - 2u_{xy}\cos\theta\sin\theta + u_{yy}\cos^2\theta) - r(u_x\cos\theta + u_y\sin\theta).
\end{aligned}$$

$$\begin{gathered}
u_{rr} + \tfrac{1}{r^2}u_{\theta\theta} \\
= u_{xx}\cos^2\theta + 2u_{xy}\cos\theta\sin\theta + u_{yy}\sin^2\theta + u_{xx}\sin^2\theta - 2u_{xy}\cos\theta\sin\theta + u_{yy}\cos^2\theta - \tfrac{1}{r}(u_x\cos\theta + u_y\sin\theta) \\
= u_{xx} + u_{yy} - \tfrac{1}{r}u_r.
\end{gathered}$$

$$\boxed{u_{xx} + u_{yy} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}.}$$

$$\boxed{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}.}$$

8.3 Polar Laplacian in $\mathbb{R}^2$ for Radial Functions

$$\boxed{\triangle u \;=\; \frac{1}{r}\big(ru_r\big)_r \;=\; \Big(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\Big)u.}$$

8.4 Spherical Laplacian in $\mathbb{R}^3$ and $\mathbb{R}^n$ for Radial Functions

$$\boxed{\triangle u \;=\; \Big(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}\Big)u.}$$

In $\mathbb{R}^3$: [4]

$$\boxed{\triangle u \;=\; \frac{1}{r^2}\big(r^2u_r\big)_r \;=\; \frac{1}{r}\big(ru\big)_{rr} \;=\; \Big(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\Big)u.}$$

[4] These formulas are taken from S. Farlow, p. 411.

8.5 Cylindrical Laplacian in $\mathbb{R}^3$ for Radial Functions

$$\boxed{\triangle u \;=\; \frac{1}{r}\big(ru_r\big)_r \;=\; \Big(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\Big)u.}$$

8.6 Mean Value Theorem

Gauss Mean Value Theorem. *If $u \in C^2(\Omega)$ is harmonic in Ω, let $\xi \in \Omega$ and pick $r > 0$ so that $\overline{B_r(\xi)} = \{x : |x-\xi| \le r\} \subset \Omega$. Then*

$$u(\xi) = M_u(\xi, r) \equiv \frac{1}{\omega_n}\int_{|x|=1} u(\xi + rx)\, dS_x,$$

where ω_n is the measure of the $(n-1)$-dimensional sphere in $\mathbb{R}^n$.

8.7 Maximum Principle

Maximum Principle. *If $u \in C^2(\Omega)$ satisfies $\triangle u \ge 0$ in Ω, then either u is a constant, or*

$$u(\xi) < \sup_{x\in\Omega} u(x)$$

for all $\xi \in \Omega$.

Proof. We may assume $A = \sup_{x\in\Omega} u(x) \le \infty$, so by continuity of u we know that $\{x \in \Omega : u(x) = A\}$ is relatively closed in Ω. But since

$$u(\xi) \le \frac{n}{\omega_n}\int_{|x|\le 1} u(\xi + rx)\, dx,$$

if $u(\xi) = A$ at an interior point ξ, then $u(x) = A$ for all x in a ball about ξ, so $\{x \in \Omega : u(x) = A\}$ is open. The connectedness of Ω implies $u(\xi) < A$ or $u(\xi) \equiv A$ for all $\xi \in \Omega$. $\square$

The maximum principle shows that $u \in C^2(\Omega)$ with $\triangle u \ge 0$ can attain an interior maximum only if u is constant. In particular, if $\overline{\Omega}$ is compact, and $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies $\triangle u \ge 0$ in Ω, we have the **weak maximum principle**:

$$\max_{x\in\overline{\Omega}} u(x) = \max_{x\in\partial\Omega} u(x).$$

8.8 The Fundamental Solution

A fundamental solution $K(x)$ for the Laplace operator is a distribution satisfying

$$\triangle K(x) = \delta(x) \tag{8.4}$$

where δ is the delta distribution supported at $x = 0$. In order to solve (8.4), we should first observe that $\triangle$ is symmetric in the variables $x_1, \ldots, x_n$, and $\delta(x)$ is also **radially symmetric** (i.e., its value only depends on $r = |x|$). Thus, we try to solve (8.4) with a radially symmetric function $K(x)$. Since $\delta(x) = 0$ for $x \neq 0$, we see that (8.4) requires K to be harmonic for $r > 0$. For the radially symmetric function K, Laplace equation becomes ($K = K(r)$):

$$\frac{\partial^2 K}{\partial r^2} + \frac{n-1}{r}\frac{\partial K}{\partial r} \;=\; 0. \tag{8.5}$$

The general solution to (8.5) is

$$K(r) = \begin{cases} c_1 + c_2 \log r & \text{if } n = 2 \\ c_1 + c_2 r^{2-n} & \text{if } n \geq 3. \end{cases} \tag{8.6}$$

After we determine c_2, we find the **fundamental solution for the Laplace operator**:

$$K(x) = \begin{cases} \frac{1}{2\pi} \log r & \text{if } n = 2 \\ \frac{1}{(2-n)\omega_n} r^{2-n} & \text{if } n \geq 3. \end{cases}$$

- We can derive, (8.6) for any given n. For intance, when $n = 3$, we have:

$$K'' + \frac{2}{r}K' \;=\; 0. \qquad \circledast$$

Let

$$\begin{aligned} K &= \frac{1}{r}w(r), \\ K' &= \frac{1}{r}w' - \frac{1}{r^2}w, \\ K'' &= \frac{1}{r}w'' - \frac{2}{r^2}w' + \frac{2}{r^3}w. \end{aligned}$$

Plugging these into $\circledast$, we obtain:

$$\begin{aligned} \frac{1}{r}w'' &= 0, \qquad \text{or} \\ w'' &= 0. \end{aligned}$$

Thus,

$$\begin{aligned} w &= c_1 r + c_2, \\ K &= \frac{1}{r}w(r) \;=\; c_1 + \frac{c_2}{r}. \qquad \checkmark \end{aligned}$$

See the similar problem, F'99, #2, where the fundamental solution for $(\triangle - I)$ is found in the process.

Find the Fundamental Solution of the Laplace Operator for $n = 3$
We found that starting with the Laplacian in $\mathbb{R}^3$ for a radially symmetric function K,

$$K'' + \frac{2}{r}K' = 0,$$

and letting $K = \frac{1}{r}w(r)$, we obtained the equation: $w = c_1 r + c_2$, which implied:

$$K = c_1 + \frac{c_2}{r}.$$

We now find the constant c_2 that ensures that for $v \in C_0^\infty(\mathbb{R}^3)$, we have

$$\int_{\mathbb{R}^3} K(|x|)\,\triangle v(x)\,dx = v(0).$$

Suppose $v(x) \equiv 0$ for $|x| \geq R$ and let $\Omega = B_R(0)$; for small $\epsilon > 0$ let

$$\Omega_\epsilon = \Omega - B_\epsilon(0).$$

$K(|x|)$ is harmonic ($\triangle K(|x|) = 0$) in Ω_ϵ. Consider Green's identity ($\partial\Omega_\epsilon = \partial\Omega \cup \partial B_\epsilon(0)$):

$$\int_{\Omega_\epsilon} K(|x|)\triangle v\,dx = \underbrace{\int_{\partial\Omega}\Big(K(|x|)\,\frac{\partial v}{\partial n} - v\,\frac{\partial K(|x|)}{\partial n}\Big)\,dS}_{=0,\ since\ v\equiv 0\ for\ x\geq R} + \int_{\partial B_\epsilon(0)}\Big(K(|x|)\,\frac{\partial v}{\partial n} - v\,\frac{\partial K(|x|)}{\partial n}\Big)\,dS.$$

$$\lim_{\epsilon\to 0}\Big[\int_{\Omega_\epsilon} K(|x|)\triangle v\,dx\Big] = \int_\Omega K(|x|)\triangle v\,dx. \qquad \Big(\text{Since } K(r) = c_1 + \frac{c_2}{r} \text{ is integrable at } x = 0.\Big)$$

On $\partial B_\epsilon(0)$, $K(|x|) = K(\epsilon)$. Thus, [5]

$$\Big|\int_{\partial B_\epsilon(0)} K(|x|)\,\frac{\partial v}{\partial n}\,dS\Big| = \big|K(\epsilon)\big|\int_{\partial B_\epsilon(0)}\Big|\frac{\partial v}{\partial n}\Big|\,dS \leq \Big|c_1 + \frac{c_2}{\epsilon}\Big|4\pi\epsilon^2 \max\big|\nabla v\big| \to 0, \quad \text{as } \epsilon \to 0.$$

$$\begin{aligned}
\int_{\partial B_\epsilon(0)} v(x)\,\frac{\partial K(|x|)}{\partial n}\,dS &= \int_{\partial B_\epsilon(0)} \frac{c_2}{\epsilon^2}v(x)\,dS \\
&= \int_{\partial B_\epsilon(0)} \frac{c_2}{\epsilon^2}v(0)\,dS + \int_{\partial B_\epsilon(0)} \frac{c_2}{\epsilon^2}[v(x) - v(0)]\,dS \\
&= \frac{c_2}{\epsilon^2}\,v(0)\,4\pi\epsilon^2 + \underbrace{4\pi c_2 \max_{x\in\partial B_\epsilon(0)}\big|v(x) - v(0)\big|}_{\to 0,\ (v\ is\ continuous)} \\
&= 4\pi c_2\,v(0) \to -v(0).
\end{aligned}$$

Thus, taking $4\pi c_2 = -1$, i.e. $c_2 = -\frac{1}{4\pi}$, we obtain

$$\int_\Omega K(|x|)\triangle v\,dx = \lim_{\epsilon\to 0}\int_{\Omega_\epsilon} K(|x|)\triangle v\,dx = v(0),$$

that is $K(r) = -\frac{1}{4\pi r}$ is the fundamental solution of $\triangle$.

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[5] In $\mathbb{R}^3$, for $|x| = \epsilon$,

$$K(|x|) = K(\epsilon) = c_1 + \frac{c_2}{\epsilon}.$$

$$\frac{\partial K(|x|)}{\partial n} = -\frac{\partial K(\epsilon)}{\partial r} = \frac{c_2}{\epsilon^2}, \qquad \text{(since } n \text{ points inwards.)}$$

n points *toward* 0 on the sphere $|x| = \epsilon$ (i.e., $n = -x/|x|$).

Show that the Fundamental Solution of the Laplace Operator is given by.

$$K(x) = \begin{cases} \frac{1}{2\pi}\log r & \textit{if } n = 2 \\ \frac{1}{(2-n)\omega_n} r^{2-n} & \textit{if } n \geq 3. \end{cases} \tag{8.7}$$

Proof. For $v \in C_0^\infty(\mathbb{R}^n)$, we want to show

$$\int_{\mathbb{R}^n} K(|x|)\,\triangle v(x)\,dx \;=\; v(0).$$

Suppose $v(x) \equiv 0$ for $|x| \geq R$ and let $\Omega = B_R(0)$; for small $\epsilon > 0$ let

$$\Omega_\epsilon = \Omega - B_\epsilon(0).$$

$K(|x|)$ is harmonic ($\triangle K(|x|) = 0$) in Ω_ϵ. Consider Green's identity ($\partial\Omega_\epsilon = \partial\Omega \cup \partial B_\epsilon(0)$):

$$\int_{\Omega_\epsilon} K(|x|)\triangle v\,dx \;=\; \underbrace{\int_{\partial\Omega}\Big(K(|x|)\,\frac{\partial v}{\partial n} \;-\; v\,\frac{\partial K(|x|)}{\partial n}\Big)\,dS}_{=0,\ since\ v\equiv 0\ for\ x\geq R} \;+\; \int_{\partial B_\epsilon(0)}\Big(K(|x|)\,\frac{\partial v}{\partial n} \;-\; v\,\frac{\partial K(|x|)}{\partial n}\Big)\,dS.$$

$$\lim_{\epsilon\to 0}\Big[\int_{\Omega_\epsilon} K(|x|)\triangle v\,dx\Big] \;=\; \int_\Omega K(|x|)\triangle v\,dx. \qquad \Big(\text{Since } K(r) \text{ is integrable at } x = 0.\Big)$$

On $\partial B_\epsilon(0)$, $K(|x|) = K(\epsilon)$. Thus, [6]

$$\Big|\int_{\partial B_\epsilon(0)} K(|x|)\,\frac{\partial v}{\partial n}\,dS\Big| \;=\; \big|K(\epsilon)\big|\int_{\partial B_\epsilon(0)}\Big|\frac{\partial v}{\partial n}\Big|\,dS \;\leq\; \big|K(\epsilon)\big|\omega_n\epsilon^{n-1}\max\big|\nabla v\big| \;\to\; 0, \quad \text{as} \quad \epsilon \to 0.$$

$$\begin{aligned}
\int_{\partial B_\epsilon(0)} v(x)\,\frac{\partial K(|x|)}{\partial n}\,dS \;&=\; \int_{\partial B_\epsilon(0)} -\frac{1}{\omega_n\epsilon^{n-1}}v(x)\,dS \\
&=\; \int_{\partial B_\epsilon(0)} -\frac{1}{\omega_n\epsilon^{n-1}}v(0)\,dS + \int_{\partial B_\epsilon(0)} -\frac{1}{\omega_n\epsilon^{n-1}}[v(x)-v(0)]\,dS \\
&=\; -\frac{1}{\omega_n\epsilon^{n-1}}\,v(0)\,\omega_n\epsilon^{n-1} - \underbrace{\max_{x\in\partial B_\epsilon(0)}\big|v(x)-v(0)\big|}_{\to 0,\ (v\ is\ continuous)} \\
&=\; -v(0).
\end{aligned}$$

Thus,

$$\int_\Omega K(|x|)\triangle v\,dx \;=\; \lim_{\epsilon\to 0}\int_{\Omega_\epsilon} K(|x|)\triangle v\,dx \;=\; v(0).$$

□

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[6] Note that for $|x| = \epsilon$,

$$K(|x|) \;=\; K(\epsilon) \;=\; \begin{cases} \frac{1}{2\pi}\log\epsilon & \text{if } n = 2 \\ \frac{1}{(2-n)\omega_n}\epsilon^{2-n} & \text{if } n \geq 3. \end{cases}$$

$$\frac{\partial K(|x|)}{\partial n} \;=\; -\frac{\partial K(\epsilon)}{\partial r} \;=\; -\begin{cases} \frac{1}{2\pi\epsilon} & \text{if } n = 2 \\ \frac{1}{\omega_n\epsilon^{n-1}} & \text{if } n \geq 3, \end{cases} \;=\; -\frac{1}{\omega_n\epsilon^{n-1}}, \quad \text{(since } n \text{ points inwards.)}$$

n points *toward* 0 on the sphere $|x| = \epsilon$ (i.e., $n = -x/|x|$).

8.9 Representation Theorem

Representation Theorem, n = 3.
Let Ω be bounded domain in $\mathbb{R}^3$ and let n be the unit exterior normal to $\partial\Omega$. Let $u \in C^2(\overline{\Omega})$. Then the value of u at any point $x \in \Omega$ is given by the formula

$$u(x) \;=\; \frac{1}{4\pi}\int_{\partial\Omega}\left[\frac{1}{|x-y|}\frac{\partial u(y)}{\partial n} - u(y)\frac{\partial}{\partial n}\frac{1}{|x-y|}\right]dS \;-\; \frac{1}{4\pi}\int_{\Omega}\frac{\triangle u(y)}{|x-y|}\,dy. \quad (8.8)$$

Proof. Consider the Green's identity:

$$\int_{\Omega}(u\triangle w - w\triangle u)\,dy \;=\; \int_{\partial\Omega}\Big(u\frac{\partial w}{\partial n} - w\frac{\partial u}{\partial n}\Big)\,dS,$$

where w is the harmonic function

$$w(y) = \frac{1}{|x-y|},$$

which is singular at $x \in \Omega$. In order to be able to apply Green's identity, we consider a new domain Ω_ϵ:

$$\Omega_\epsilon = \Omega - B_\epsilon(x).$$

Since $u,\, w \in C_2(\overline{\Omega}_\epsilon)$, Green's identity can be applied. Since w is harmonic ($\triangle w = 0$) in Ω_ϵ and since $\partial\Omega_\epsilon = \partial\Omega \cup \partial B_\epsilon(x)$, we have

$$-\int_{\Omega_\epsilon}\frac{\triangle u(y)}{|x-y|}dy \;=\; \int_{\partial\Omega}\left[u(y)\frac{\partial}{\partial n}\frac{1}{|x-y|} - \frac{1}{|x-y|}\frac{\partial u(y)}{\partial n}\right]dS \quad (8.9)$$

$$+ \int_{\partial B_\epsilon(x)}\left[u(y)\frac{\partial}{\partial n}\frac{1}{|x-y|} - \frac{1}{|x-y|}\frac{\partial u(y)}{\partial n}\right]dS. \quad (8.10)$$

We will show that formula (8.8) is obtained by letting $\epsilon \to 0$.

$$\lim_{\epsilon\to 0}\left[-\int_{\Omega_\epsilon}\frac{\triangle u(y)}{|x-y|}\,dy\right] \;=\; -\int_{\Omega}\frac{\triangle u(y)}{|x-y|}\,dy. \qquad \Big(\text{Since } \frac{1}{|x-y|} \text{ is integrable at } x = y.\Big)$$

The first integral on the right of (8.10) does not depend on ϵ. Hence, the limit as $\epsilon \to 0$ of the second integral on the right of (8.10) exists, and in order to obtain (8.8), need

$$\lim_{\epsilon\to 0}\int_{\partial B_\epsilon(x)}\left[u(y)\frac{\partial}{\partial n}\frac{1}{|x-y|} - \frac{1}{|x-y|}\frac{\partial u(y)}{\partial n}\right]dS \;=\; 4\pi u(x).$$

$$\begin{aligned}
\int_{\partial B_\epsilon(x)}\left[u(y)\frac{\partial}{\partial n}\frac{1}{|x-y|} - \frac{1}{|x-y|}\frac{\partial u(y)}{\partial n}\right]dS \;&=\; \int_{\partial B_\epsilon(x)}\left[\frac{1}{\epsilon^2}u(y) - \frac{1}{\epsilon}\frac{\partial u(y)}{\partial n}\right]dS\\
&=\; \int_{\partial B_\epsilon(x)}\frac{1}{\epsilon^2}u(x)\,dS + \int_{\partial B_\epsilon(x)}\left[\frac{1}{\epsilon^2}[u(y)-u(x)] - \frac{1}{\epsilon}\frac{\partial u(y)}{\partial n}\right]dS\\
&=\; 4\pi u(x) + \int_{\partial B_\epsilon(x)}\left[\frac{1}{\epsilon^2}[u(y)-u(x)] - \frac{1}{\epsilon}\frac{\partial u(y)}{\partial n}\right]dS.
\end{aligned}$$

[7] The last integral tends to 0 as $\epsilon \to 0$:

$$\begin{aligned}
\bigg|\int_{\partial B_\epsilon(x)}\bigg[\frac{1}{\epsilon^2}[u(y)-u(x)]-\frac{1}{\epsilon}\frac{\partial u(y)}{\partial n}\bigg]dS\bigg| &\le \frac{1}{\epsilon^2}\int_{\partial B_\epsilon(x)}\Big|u(y)-u(x)\Big|+\frac{1}{\epsilon}\int_{\partial B_\epsilon(x)}\Big|\frac{\partial u(y)}{\partial n}\Big|dS \\
&\le \underbrace{4\pi\max_{y\in\partial B_\epsilon(x)}\Big|u(y)-u(x)\Big|}_{\to 0,\ (u\ continuous\ in\ \overline{\Omega})}+\underbrace{4\pi\epsilon\max_{y\in\overline{\Omega}}\Big|\nabla u(y)\Big|}_{\to 0,\ (|\nabla u|\ is\ finite)}.
\end{aligned}$$

$\square$

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[7]Note that for points y on $\partial B_\epsilon(x)$,

$$\frac{1}{|x-y|}=\frac{1}{\epsilon} \qquad \text{and} \qquad \frac{\partial}{\partial n}\frac{1}{|x-y|}=\frac{1}{\epsilon^2}.$$

Representation Theorem, n = 2.
Let Ω be bounded domain in $\mathbb{R}^2$ and let n be the unit exterior normal to $\partial\Omega$. Let $u \in C^2(\overline{\Omega})$. Then the value of u at any point $x \in \Omega$ is given by the formula

$$u(x) = \frac{1}{2\pi}\int_\Omega \triangle u(y)\log|x-y|\,dy + \frac{1}{2\pi}\int_{\partial\Omega}\Big[u(y)\frac{\partial}{\partial n}\log|x-y| - \log|x-y|\frac{\partial u(y)}{\partial n}\Big]\,dS \quad (8.11)$$

Proof. Consider the Green's identity:

$$\int_\Omega (u\triangle w - w\triangle u)\,dy = \int_{\partial\Omega}\Big(u\frac{\partial w}{\partial n} - w\frac{\partial u}{\partial n}\Big)\,dS,$$

where w is the harmonic function

$$w(y) = \log|x-y|,$$

which is singular at $x \in \Omega$. In order to be able to apply Green's identity, we consider a new domain Ω_ϵ:

$$\Omega_\epsilon = \Omega - B_\epsilon(x).$$

Since u, $w \in C_2(\overline{\Omega}_\epsilon)$, Green's identity can be applied. Since w is harmonic ($\triangle w = 0$) in Ω_ϵ and since $\partial\Omega_\epsilon = \partial\Omega \cup \partial B_\epsilon(x)$, we have

$$\begin{aligned} &-\int_{\Omega_\epsilon}\triangle u(y)\log|x-y|\,dy \qquad (8.12)\\ &= \int_{\partial\Omega}\Big[u(y)\frac{\partial}{\partial n}\log|x-y| - \log|x-y|\frac{\partial u(y)}{\partial n}\Big]dS \\ &\quad + \int_{\partial B_\epsilon(x)}\Big[u(y)\frac{\partial}{\partial n}\log|x-y| - \log|x-y|\frac{\partial u(y)}{\partial n}\Big]dS. \end{aligned}$$

We will show that formula (8.11) is obtained by letting $\epsilon \to 0$.

$$\lim_{\epsilon\to 0}\Big[-\int_{\Omega_\epsilon}\triangle u(y)\log|x-y|\,dy\Big] = -\int_\Omega \triangle u(y)\log|x-y|\,dy. \qquad \Big(\text{since } \log|x-y| \text{ is integrable at } x=y.\Big)$$

The first integral on the right of (8.12) does not depend on ϵ. Hence, the limit as $\epsilon \to 0$ of the second integral on the right of (8.12) exists, and in order to obtain (8.11), need

$$\lim_{\epsilon\to 0}\int_{\partial B_\epsilon(x)}\Big[u(y)\frac{\partial}{\partial n}\log|x-y| - \log|x-y|\frac{\partial u(y)}{\partial n}\Big]dS = 2\pi u(x).$$

$$\begin{aligned} \int_{\partial B_\epsilon(x)}\Big[u(y)\frac{\partial}{\partial n}\log|x-y| - \log|x-y|\frac{\partial u(y)}{\partial n}\Big]dS &= \int_{\partial B_\epsilon(x)}\Big[\frac{1}{\epsilon}u(y) - \log\epsilon\frac{\partial u(y)}{\partial n}\Big]dS \\ &= \int_{\partial B_\epsilon(x)}\frac{1}{\epsilon}u(x)\,dS + \int_{\partial B_\epsilon(x)}\Big[\frac{1}{\epsilon}[u(y)-u(x)] - \log\epsilon\frac{\partial u(y)}{\partial n}\Big]dS \\ &= 2\pi u(x) + \int_{\partial B_\epsilon(x)}\Big[\frac{1}{\epsilon}[u(y)-u(x)] - \log\epsilon\frac{\partial u(y)}{\partial n}\Big]dS. \end{aligned}$$

[8] The last integral tends to 0 as $\epsilon \to 0$:

$$\begin{aligned}
\Big| \int_{\partial B_\epsilon(x)} \Big[\frac{1}{\epsilon}[u(y) - u(x)] - \log\epsilon \frac{\partial u(y)}{\partial n} \Big] dS \Big| &\le \frac{1}{\epsilon} \int_{\partial B_\epsilon(x)} \Big| u(y) - u(x) \Big| + \log\epsilon \int_{\partial B_\epsilon(x)} \Big| \frac{\partial u(y)}{\partial n} \Big| dS \\
&\le \underbrace{2\pi \max_{y \in \partial B_\epsilon(x)} \Big| u(y) - u(x) \Big|}_{\to 0,\ (u\ continuous\ in\ \overline{\Omega})} + \underbrace{2\pi\epsilon \log\epsilon \max_{y \in \overline{\Omega}} \Big| \nabla u(y) \Big|}_{\to 0,\ (|\nabla u|\ is\ finite)}.
\end{aligned}$$

□

[8]Note that for points y on $\partial B_\epsilon(x)$,

$$\log|x-y| = \log\epsilon \qquad \text{and} \qquad \frac{\partial}{\partial n}\log|x-y| = \frac{1}{\epsilon}.$$

Representation Theorems, n > 3 can be obtained in the same way. We use the Green's identity with

$$w(y) = \frac{1}{|x-y|^{n-2}},$$

which is a harmonic function in $\mathbb{R}^n$ with a singularity at x.

The fundamental solution for the Laplace operator is ($r = |x|$):

$$K(x) = \begin{cases} \frac{1}{2\pi}\log r & \text{if } n = 2 \\ \frac{1}{(2-n)\omega_n} r^{2-n} & \text{if } n \geq 3. \end{cases}$$

Representation Theorem. *If $\Omega \in \mathbb{R}^n$ is bounded, $u \in C^2(\overline{\Omega})$, and $x \in \Omega$, then*

$$u(x) = \int_\Omega K(x-y)\triangle u(y)\,dy + \int_{\partial\Omega}\left[u(y)\frac{\partial K(x-y)}{\partial n} - K(x-y)\frac{\partial u(y)}{\partial n}\right]dS \quad (8.13)$$

Proof. Consider the Green's identity:

$$\int_\Omega (u\triangle w - w\triangle u)\,dy = \int_{\partial\Omega}\Big(u\frac{\partial w}{\partial n} - w\frac{\partial u}{\partial n}\Big)\,dS,$$

where w is the harmonic function

$$w(y) = K(x-y),$$

which is singular at $y = x$. In order to be able to apply Green's identity, we consider a new domain Ω_ϵ:

$$\Omega_\epsilon = \Omega - B_\epsilon(x).$$

Since u, $K(x-y) \in C_2(\overline{\Omega}_\epsilon)$, Green's identity can be applied. Since $K(x-y)$ is harmonic ($\triangle K(x-y) = 0$) in Ω_ϵ and since $\partial\Omega_\epsilon = \partial\Omega \cup \partial B_\epsilon(x)$, we have

$$-\int_{\Omega_\epsilon} K(x-y)\triangle u(y)\,dy = \int_{\partial\Omega}\left[u(y)\frac{\partial K(x-y)}{\partial n} - K(x-y)\frac{\partial u(y)}{\partial n}\right]dS \quad (8.14)$$

$$+ \int_{\partial B_\epsilon(x)}\left[u(y)\frac{\partial K(x-y)}{\partial n} - K(x-y)\frac{\partial u(y)}{\partial n}\right]dS \quad (8.15)$$

We will show that formula (8.13) is obtained by letting $\epsilon \to 0$.

$$\lim_{\epsilon\to 0}\left[-\int_{\Omega_\epsilon} K(x-y)\triangle u(y)\,dy\right] = -\int_\Omega K(x-y)\triangle u(y)\,dy. \qquad \Big(\text{since } K(x-y) \text{ is integrable at } x = y.\Big)$$

The first integral on the right of (8.15) does not depend on ϵ. Hence, the limit as $\epsilon \to 0$ of the second integral on the right of (8.15) exists, and in order to obtain (8.13), need

$$\lim_{\epsilon\to 0}\int_{\partial B_\epsilon(x)}\left[u(y)\frac{\partial K(x-y)}{\partial n} - K(x-y)\frac{\partial u(y)}{\partial n}\right]dS = -u(x).$$

$$\begin{aligned}
\int_{\partial B_\epsilon(x)} \Big[u(y)\frac{\partial K(x-y)}{\partial n} - K(x-y)\frac{\partial u(y)}{\partial n}\Big] dS \;&=\; \int_{\partial B_\epsilon(x)} \Big[u(y)\frac{\partial K(\epsilon)}{\partial n} - K(\epsilon)\frac{\partial u(y)}{\partial n}\Big] dS \\
&= \int_{\partial B_\epsilon(x)} u(x)\frac{\partial K(\epsilon)}{\partial n}\, dS + \int_{\partial B_\epsilon(x)} \Big[\frac{\partial K(\epsilon)}{\partial n}[u(y)-u(x)] - K(\epsilon)\frac{\partial u(y)}{\partial n}\Big] dS \\
= -\frac{1}{\omega_n\epsilon^{n-1}} \int_{\partial B_\epsilon(x)} u(x)\, dS - \frac{1}{\omega_n\epsilon^{n-1}} &\int_{\partial B_\epsilon(x)} [u(y)-u(x)]\, dS - \int_{\partial B_\epsilon(x)} K(\epsilon)\frac{\partial u(y)}{\partial n}\, dS \\
= \underbrace{-\frac{1}{\omega_n\epsilon^{n-1}} u(x)\omega_n\epsilon^{n-1}}_{-u(x)} - \frac{1}{\omega_n\epsilon^{n-1}} &\int_{\partial B_\epsilon(x)} [u(y)-u(x)]\, dS - \int_{\partial B_\epsilon(x)} K(\epsilon)\frac{\partial u(y)}{\partial n}\, dS.
\end{aligned}$$

[9] The last two integrals tend to 0 as $\epsilon \to 0$:

$$\begin{aligned}
&\Big| -\frac{1}{\omega_n\epsilon^{n-1}} \int_{\partial B_\epsilon(x)} [u(y)-u(x)]\, dS - \int_{\partial B_\epsilon(x)} K(\epsilon)\frac{\partial u(y)}{\partial n}\, dS \Big| \\
&\le \underbrace{\frac{1}{\omega_n\epsilon^{n-1}} \max_{y\in\partial B_\epsilon(x)} \big|u(y)-u(x)\big|\omega_n\epsilon^{n-1}}_{\to 0,\ (u\ continuous\ in\ \overline{\Omega})} + \underbrace{\big|K(\epsilon)\big| \max_{y\in\overline{\Omega}} \big|\nabla u(y)\big|\omega_n\epsilon^{n-1}}_{\to 0,\ (|\nabla u|\ is\ finite)}.
\end{aligned}$$

□

8.10 Green's Function and the Poisson Kernel

With a slight change in notation, the Representation Theorem has the following special case.

Theorem. *If $\Omega \in \mathbb{R}^n$ is bounded, $u \in C^2(\Omega)\bigcap C^1(\overline{\Omega})$ is **harmonic**, and $\xi \in \Omega$, then*

$$u(\xi) \;=\; \int_{\partial\Omega} \Big[u(x)\frac{\partial K(x-\xi)}{\partial n} - K(x-\xi)\frac{\partial u(x)}{\partial n}\Big]\, dS. \tag{8.16}$$

Let $\omega(x)$ be any **harmonic** function in Ω, and for x, $\xi \in \Omega$ consider

$$G(x,\xi) = K(x-\xi) + \omega(x).$$

If we use the Green's identity (with $\triangle u = 0$ and $\triangle\omega = 0$), we get:

$$0 \;=\; \int_{\partial\Omega} (u\frac{\partial\omega}{\partial n} - \omega\frac{\partial u}{\partial n}) ds. \tag{8.17}$$

Adding (8.16) and (8.17), we obtain:

$$u(\xi) \;=\; \int_{\partial\Omega} \Big[u(x)\frac{\partial G(x,\xi)}{\partial n} - G(x,\xi)\frac{\partial u(x)}{\partial n}\Big]\, dS. \tag{8.18}$$

Suppose that for each $\xi \in \Omega$ we can find a function $\omega_\xi(x)$ that is harmonic in Ω and satisfies $\omega_\xi(x) = -K(x-\xi)$ for all $x \in \partial\Omega$. Then $G(x,\xi) = K(x-\xi) + \omega_\xi(x)$ is a fundamental solution such that

$$G(x,\xi) = 0 \qquad x \in \partial\Omega.$$

[9]Note that for points y on $\partial B_\epsilon(x)$,

$$K(x-y) \;=\; K(\epsilon) \;=\; \begin{cases} \frac{1}{2\pi}\log\epsilon & \text{if } n=2 \\ \frac{1}{(2-n)\omega_n}\epsilon^{2-n} & \text{if } n\ge 3. \end{cases}$$

$$\frac{\partial K(x-y)}{\partial n} \;=\; -\frac{\partial K(\epsilon)}{\partial r} \;=\; -\begin{cases} \frac{1}{2\pi\epsilon} & \text{if } n=2 \\ \frac{1}{\omega_n\epsilon^{n-1}} & \text{if } n\ge 3, \end{cases} \;=\; -\frac{1}{\omega_n\epsilon^{n-1}}, \quad \text{(since } n \text{ points inwards.)}$$

G is called the Green's function and is useful in satisfying Dirichlet boundary conditions. The Green's function is difficult to construct for a general domain Ω since it requires solving the Dirichlet problem $\triangle\omega_\xi = 0$ in Ω, $\omega_\xi(x) = -K(x-\xi)$ for $x \in \partial\Omega$, for each $\xi \in \Omega$.
From (8.18) we find [10]

$$\boxed{u(\xi) \ = \ \int_{\partial\Omega} u(x)\frac{\partial G(x,\xi)}{\partial n}\,dS.}$$

Thus *if* we know that the Dirichlet problem has a solution $u \in C^2(\overline{\Omega})$, then we can calculate u from the Poisson integral formula (provided of course that we can compute $G(x,\xi)$).

[10] If we did not assume $\triangle u = 0$ in our derivation, we would have (8.13) instead of (8.16), and an extra term in (8.17), which would give us a more general expression:

$$\boxed{u(\xi) \ = \ \int_{\Omega} G(x,\xi)\,\triangle u\,dx \ + \ \int_{\partial\Omega} u(x)\frac{\partial G(x,\xi)}{\partial n}\,dS.}$$

8.11 Properties of Harmonic Functions

Liouville's Theorem. *A bounded harmonic function defined on all of $\mathbb{R}^n$ must be a constant.*

8.12 Eigenvalues of the Laplacian

Consider the equation

$$\begin{cases} \triangle u + \lambda u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{8.19}$$

where Ω is a bounded domain and λ is a (complex) number. The values of λ for which (8.19) admits a nontrivial solution u are called the **eigenvalues** of $\triangle$ in Ω, and the solution u is an **eigenfunction associated to the eigenvalue** λ. (The convention $\triangle u + \lambda u = 0$ is chosen so that all eigenvalues λ will be positive.)

Properties of the Eigenvalues and Eigenfunctions for (8.19):
1. The eigenvalues of (8.19) form a countable set $\{\lambda_n\}_{n=1}^{\infty}$ of positive numbers with $\lambda_n \to \infty$ as $n \to \infty$.
2. For each eigenvalue λ_n there is a finite number (called the *multiplicity* of λ_n) of linearly independent eigenfunctions u_n.
3. The first (or *principal*) eigenvalue, λ_1, is simple and u_1 does not change sign in Ω.
4. Eigenfunctions corresponding to distinct eigenvalues are orthogonal.
5. The eigenfunctions may be used to expand certain functions on Ω in an infinite series.

9 Heat Equation

The *heat equation* is

$$u_t = k\triangle u \qquad \text{for } x \in \Omega,\ t > 0, \tag{9.1}$$

with initial and boundary conditions.

9.1 The Pure Initial Value Problem

9.1.1 Fourier Transform

If $u \in C_0^\infty(\mathbb{R}^n)$, define its **Fourier transform** $\widehat{u}$ by

$$\widehat{u}(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-ix\cdot\xi}u(x)\,dx \qquad \text{for } \ \xi \in \mathbb{R}^n.$$

We can differentiate $\hat{u}$:

$$\frac{\partial}{\partial\xi_j}\widehat{u}(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-ix\cdot\xi}(-ix_j)u(x)\,dx = \big[\widehat{(-ix_j)\,u}\big](\xi).$$

Iterating this computation, we obtain

$$\left(\frac{\partial}{\partial\xi_j}\right)^k\widehat{u}(\xi) = \big[\widehat{(-ix_j)^k\,u}\big](\xi). \tag{9.2}$$

Similarly, integrating by parts shows

$$\begin{aligned}
\left(\widehat{\frac{\partial u}{\partial x_j}}\right)(\xi) &= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-ix\cdot\xi}\frac{\partial u}{\partial x_j}(x)\,dx \ = \ -\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\frac{\partial}{\partial x_j}(e^{-ix\cdot\xi})u(x)\,dx \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}(i\xi_j)e^{-ix\cdot\xi}u(x)\,dx \\
&= (i\xi_j)\widehat{u}(\xi).
\end{aligned}$$

Iterating this computation, we obtain

$$\left(\widehat{\frac{\partial^k u}{\partial x_j^k}}\right)(\xi) = (i\xi_j)^k\widehat{u}(\xi). \tag{9.3}$$

Formulas (9.2) and (9.3) express the fact that *Fourier transform interchanges differentiation and multiplication by the coordinate function.*

9.1.2 Multi-Index Notation

A multi-index is a vector $\alpha = (\alpha_1, \ldots, \alpha_n)$ where each α_i is a nonnegative integer. The order of the multi-index is $|\alpha| = \alpha_1 + \ldots + \alpha_n$. Given a multi-index α, define

$$D^\alpha u = \frac{\partial^{|\alpha|}u}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}} = \partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n}u.$$

We can generalize (9.3) in multi-index notation:

$$\begin{aligned}
\widehat{D^\alpha u}(\xi) &= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-ix\cdot\xi}D^\alpha u(x)\,dx \ = \ \frac{(-1)^{|\alpha|}}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} D_x^\alpha(e^{-ix\cdot\xi})u(x)\,dx \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}(i\xi)^\alpha e^{-ix\cdot\xi}u(x)\,dx \\
&= (i\xi)^\alpha\widehat{u}(\xi). \\
(i\xi)^\alpha &= (i\xi_1)^{\alpha_1}\cdots(i\xi_n)^{\alpha_n}.
\end{aligned}$$

Parseval's theorem (Plancherel's theorem).
Assume $u \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. *Then* $\widehat{u}, u^\vee \in L^2(\mathbb{R}^n)$ *and*

$$||\widehat{u}||_{L^2(\mathbb{R}^n)} = ||u^\vee||_{L^2(\mathbb{R}^n)} = ||u||_{L^2(\mathbb{R}^n)}, \qquad or$$

$$\int_{-\infty}^{\infty} |u(x)|^2\,dx = \int_{-\infty}^{\infty} |\widehat{u}(\xi)|^2\,d\xi.$$

Also,

$$\int_{-\infty}^{\infty} u(x)\,\overline{v(x)}\,dx = \int_{-\infty}^{\infty} \widehat{u}(\xi)\,\overline{\widehat{v}(\xi)}\,d\xi.$$

The properties (9.2) and (9.3) make it very natural to consider the fourier transform on a subspace of $L^1(\mathbb{R}^n)$ called the *Schwartz class of functions*, S, which consists of the smooth functions whose derivatives of all orders decay faster than any polynomial, i.e.

$$S = \{u \in C^\infty(\mathbb{R}^n) : \text{ for every } k \in \mathbb{N} \text{ and } \alpha \in \mathbb{N}^n,\ |x|^k |D^\alpha u(x)| \text{ is bounded on } \mathbb{R}^n\}.$$

For $u \in S$, the Fourier transform $\widehat{u}$ exists since u decays rapidly at ∞.

Lemma. *(i) If* $u \in L^1(\mathbb{R}^n)$, *then* $\widehat{u}$ *is bounded. (ii) If* $u \in S$, *then* $\widehat{u} \in S$.

Define the **inverse Fourier transform** for $u \in L^1(\mathbb{R}^n)$:

$$\begin{aligned} u^\vee(\xi) &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\,u(x)\,dx \qquad \text{for } \xi \in \mathbb{R}^n, \qquad \text{or} \\ u(x) &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\,\widehat{u}(\xi)\,d\xi \qquad \text{for } x \in \mathbb{R}^n. \end{aligned}$$

Fourier Inversion Theorem (McOwen). *If* $u \in S$, *then* $(\widehat{u})^\vee = u$; *that is,*

$$\boxed{u(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\,\widehat{u}(\xi)\,d\xi = \frac{1}{(2\pi)^n} \int\int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi}\,u(y)\,dy\,d\xi = (\widehat{u})^\vee(x).}$$

Fourier Inversion Theorem (Evans). *Assume* $u \in L^2(\mathbb{R}^n)$. *Then,* $u = (\widehat{u})^\vee$.

Shift: Let $u(\underbrace{x-a}_{y}) = v(x)$, and determinte $\widehat{v}(\xi)$:

$$\begin{aligned}
\widehat{u(x-a)}(\xi) = \widehat{v}(\xi) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ix\xi}v(x)\,dx \;=\; \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-i(y+a)\xi}u(y)\,dy \\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-iy\xi}e^{-ia\xi}u(y)\,dy \;=\; e^{-ia\xi}\widehat{u}(\xi).
\end{aligned}$$

$$\boxed{\widehat{u(x-a)}(\xi) \;=\; e^{-ia\xi}\widehat{u}(\xi).}$$

Delta function:

$$\begin{aligned}
\widehat{\delta(x)}(\xi) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ix\xi}\,\delta(x)\,dx \;=\; \frac{1}{\sqrt{2\pi}}, && \Big(\text{since } u(x) = \int_{\mathbb{R}}\delta(x-y)\,u(y)\,dy\Big). \\
\widehat{\delta(x-a)}(\xi) &= e^{-ia\xi}\,\widehat{\delta}(\xi) \;=\; \frac{1}{\sqrt{2\pi}}\,e^{-ia\xi}. && \text{(using result from `Shift')}
\end{aligned}$$

Convolution:

$$\begin{aligned}
(f*g)(x) &= \int_{\mathbb{R}^n} f(x-y)g(y)\,dy, \\
\widehat{(f*g)}(\xi) &= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-ix\cdot\xi}\int_{\mathbb{R}^n} f(x-y)\,g(y)\,dy\,dx \;=\; \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ix\cdot\xi}f(x-y)\,g(y)\,dy\,dx \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\Big[e^{-i(x-y)\cdot\xi}f(x-y)\,dx\Big]\Big[e^{-iy\cdot\xi}g(y)\,dy\Big] \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-iz\cdot\xi}f(z)\,dz\cdot\int_{\mathbb{R}^n} e^{-iy\cdot\xi}g(y)\,dy \;=\; (2\pi)^{\frac{n}{2}}\widehat{f}(\xi)\widehat{g}(\xi).
\end{aligned}$$

$$\boxed{\widehat{(f*g)}(\xi) \;=\; (2\pi)^{\frac{n}{2}}\,\widehat{f}(\xi)\,\widehat{g}(\xi).}$$

Gaussian: (completing the square)

$$\begin{aligned}
\widehat{\big(e^{-\frac{x^2}{2}}\big)}(\xi) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ix\xi}\,e^{-\frac{x^2}{2}}\,dx \;=\; \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-\frac{x^2+2ix\xi}{2}}\,dx \;=\; \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-\frac{x^2+2ix\xi-\xi^2}{2}}\,dx\;e^{-\frac{\xi^2}{2}} \\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-\frac{(x+i\xi)^2}{2}}\,dx\;e^{-\frac{\xi^2}{2}} \;=\; \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{\frac{-y^2}{2}}\,dy\;e^{-\frac{\xi^2}{2}} = \frac{1}{\sqrt{2\pi}}\sqrt{2\pi}e^{-\frac{\xi^2}{2}} \;=\; e^{-\frac{\xi^2}{2}}.
\end{aligned}$$

$$\boxed{\widehat{\big(e^{-\frac{x^2}{2}}\big)}(\xi) \;=\; e^{-\frac{\xi^2}{2}}.}$$

Multiplication by x:

$$\widehat{-ixu}(\xi) \;=\; \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ix\xi}\big(-ixu(x)\big)\,dx \;=\; \frac{d}{d\xi}\widehat{u}(\xi).$$

$$\boxed{\widehat{xu(x)}(\xi) \;=\; i\frac{d}{d\xi}\widehat{u}(\xi).}$$

Multiplication of $\mathbf{u_x}$ by x: (using the above result)

$$\begin{aligned}
\widehat{xu_x(x)}(\xi) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ix\xi}\big(xu_x(x)\big)\,dx \;=\; \underbrace{\frac{1}{\sqrt{2\pi}}\Big[e^{-ix\xi}xu\Big]_{-\infty}^{\infty}}_{=0} - \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\Big((-i\xi)e^{-ix\xi}x + e^{-ix\xi}\Big)\,u\,dx \\
&= \frac{1}{\sqrt{2\pi}}\,i\xi\int_{\mathbb{R}} e^{-ix\xi}\,x\,u\,dx - \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ix\xi}\,u\,dx \\
&= i\xi\,\widehat{xu(x)}(\xi) - \widehat{u}(\xi) \;=\; i\xi\Big[i\frac{d}{d\xi}\widehat{u}(\xi)\Big] - \widehat{u}(\xi) \;=\; -\xi\frac{d}{d\xi}\widehat{u}(\xi) - \widehat{u}(\xi).
\end{aligned}$$

$$\boxed{\widehat{xu_x(x)}(\xi) \;=\; -\xi\frac{d}{d\xi}\widehat{u}(\xi) - \widehat{u}(\xi).}$$

Table of Fourier Transforms: [11]

$$\begin{aligned}
\widehat{\big(e^{-\frac{ax^2}{2}}\big)}(\xi) &= \frac{1}{\sqrt{a}}\,e^{-\frac{\xi^2}{2a}}, && \text{(Gaussian)} \\
\widehat{e^{ibx}f(ax)}(\xi) &= \frac{1}{a}\widehat{f}\Big(\frac{\xi-b}{a}\Big), \\
f(x) &= \begin{cases} 1, & |x|\le L \\ 0, & |x|>L, \end{cases} && \widehat{f(x)}(\xi) \;=\; \frac{1}{\sqrt{2\pi}}\frac{2\sin(\xi L)}{\xi}, \\
\widehat{e^{-a|x|}}(\xi) &= \frac{1}{\sqrt{2\pi}}\,\frac{2a}{a^2+\xi^2}, && (a>0) \\
\widehat{\frac{1}{a^2+x^2}}(\xi) &= \frac{\sqrt{2\pi}}{2a}\,e^{-a|\xi|}, && (a>0) \\
\widehat{H(a-|x|)}(\xi) &= \sqrt{\frac{2}{\pi}}\,\frac{1}{\xi}\,\sin a\xi, && \circledast \\
\widehat{H(x)}(\xi) &= \frac{1}{\sqrt{2\pi}}\Big(\pi\delta(\xi) + \frac{1}{i\xi}\Big), && \circledast \\
\widehat{\big(H(x)-H(-x)\big)}(\xi) &= \sqrt{\frac{2}{\pi}}\,\frac{1}{i\xi}, \quad \text{(sign)} && \circledast \\
\widehat{1}(\xi) &= \sqrt{2\pi}\delta(\xi). && \circledast
\end{aligned}$$

[11] Results with marked with $\circledast$ were taken from W. Strauss, where the definition of Fourier Transform is different. An extra multiple of $\frac{1}{\sqrt{2\pi}}$ was added to each of these results.

9.1.3 Solution of the Pure Initial Value Problem

Consider the pure initial value problem

$$\begin{cases} u_t = \triangle u & \text{for } t > 0,\ x \in \mathbb{R}^n \\ u(x,0) = g(x) & \text{for } x \in \mathbb{R}^n. \end{cases} \tag{9.4}$$

We take the Fourier transform of the heat equation in the x-variables.

$$\begin{aligned} \widehat{(u_t)}(\xi,t) &= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-ix\cdot\xi}u_t(x,t)\,dx = \frac{\partial}{\partial t}\widehat{u}(\xi,t) \\ \widehat{\triangle u}(\xi,t) &= \sum_{j=1}^{n}(i\xi_j)^2\widehat{u}(\xi,t) = -|\xi|^2\widehat{u}(\xi,t). \end{aligned}$$

The heat equation therefore becomes

$$\frac{\partial}{\partial t}\widehat{u}(\xi,t) = -|\xi|^2\widehat{u}(\xi,t),$$

which is an ordinary differential equation in t, with the solution $\widehat{u}(\xi,t) = Ce^{-|\xi|^2 t}$. The initial condition $\widehat{u}(\xi,0) = \widehat{g}(\xi)$ gives

$$\begin{aligned} \widehat{u}(\xi,t) &= \widehat{g}(\xi)\,e^{-|\xi|^2 t}, \\ u(x,t) &= \left(\widehat{g}(\xi)\,e^{-|\xi|^2 t}\right)^{\vee} = \frac{1}{(2\pi)^{\frac{n}{2}}}\left[g * \left(e^{-|\xi|^2 t}\right)^{\vee}\right] \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}}\,g * \left[\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-|\xi|^2 t}e^{ix\cdot\xi}\,d\xi\right] \\ &= \frac{1}{(4\pi^2)^{\frac{n}{2}}}\,g * \left[\int_{\mathbb{R}^n} e^{ix\cdot\xi-|\xi|^2 t}\,d\xi\right] = \frac{1}{(4\pi^2)^{\frac{n}{2}}}\,g * \left[e^{-\frac{|x|^2}{4t}}\left(\frac{\pi}{t}\right)^{\frac{n}{2}}\right] \\ &= \frac{1}{(4\pi t)^{\frac{n}{2}}}\,g * \left[e^{-\frac{|x|^2}{4t}}\right] = \frac{1}{(4\pi t)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}}g(y)\,dy. \end{aligned}$$

Thus, [12] solution of the initial value problem (9.4) is

$$\boxed{u(x,t) = \int_{\mathbb{R}^n} K(x,y,t)\,g(y)\,dy = \frac{1}{(4\pi t)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}}g(y)\,dy.}$$

Uniqueness of solutions for the pure initial value problem *fails*: there are nontrivial solutions of (9.4) with $g = 0$. [13] Thus, the pure initial value problem for the heat equation is *not* well-posed, as it was for the wave equation. However, the nontrivial solutions are unbounded as functions of x when $t > 0$ is fixed; uniqueness can be regained by adding a boundedness condition on the solution.

---

[12] Identity (Evans, p. 187.) :

$$\int_{\mathbb{R}^n} e^{ix\cdot\xi-|\xi|^2 t}\,d\xi = e^{-\frac{|x|^2}{4t}}\left(\frac{\pi}{t}\right)^{\frac{n}{2}}.$$

[13] The following function u satisfies $u_t = u_{xx}$ for $t > 0$ with $u(x,0) = 0$:

$$u(x,t) = \sum_{k=0}^{\infty}\frac{1}{(2k)!}x^{2k}\frac{d^k}{dt^k}e^{-1/t^2}.$$

9.1.4 Nonhomogeneous Equation

Consider the pure initial value problem with homogeneous initial condition:

$$\begin{cases} u_t = \triangle u + f(x,t) & \text{for } \ t > 0, \ x \in \mathbb{R}^n \\ u(x,0) = 0 & \text{for } \ x \in \mathbb{R}^n. \end{cases} \tag{9.5}$$

Duhamel's principle gives the solution:

$$\boxed{u(x,t) = \int_0^t \int_{\mathbb{R}^n} \tilde{K}(x-y, t-s)\, f(y,s)\, dy\, ds.}$$

9.1.5 Nonhomogeneous Equation with Nonhomogeneous Initial Conditions

Combining two solutions above, we find that the solution of the initial value problem

$$\begin{cases} u_t = \triangle u + f(x,t) & \text{for } \ t > 0, \ x \in \mathbb{R}^n \\ u(x,0) = g(x) & \text{for } \ x \in \mathbb{R}^n. \end{cases} \tag{9.6}$$

is given by

$$\boxed{u(x,t) = \int_{\mathbb{R}^n} \tilde{K}(x-y,t)\, g(y)\, dy + \int_0^t \int_{\mathbb{R}^n} \tilde{K}(x-y, t-s)\, f(y,s)\, dy\, ds.}$$

9.1.6 The Fundamental Solution

Suppose we want to solve the Cauchy problem

$$\begin{cases} u_t = Lu & x \in \mathbb{R}^n, \ t > 0 \\ u(x,0) = g(x) & x \in \mathbb{R}^n. \end{cases} \tag{9.7}$$

where L is a differential operator in $\mathbb{R}^n$ with constant coefficients. Suppose $K(x,t)$ is a distribution in $\mathbb{R}^n$ for each value of $t \geq 0$, K is C^1 in t and satisfies

$$\begin{cases} K_t - LK = 0, \\ K(x,0) = \delta(x). \end{cases} \tag{9.8}$$

We call K a **fundamental solution** for the initial value problem. The solution of (9.7) is then given by convolution in the space variables:

$$u(x,t) \ = \ \int_{\mathbb{R}^n} K(x-y,t)\, g(y)\, dy.$$

For operators of the form $\partial_t - L$, the fundamental solution of the initial value problem, $K(x,t)$ as defined in (9.8), coincides with the "free space" fundamental solution, which satisfies

$$\Big(\partial_t - L\Big)K(x,t) \;=\; \delta(x,t),$$

provided we extend $K(x,t)$ by zero to $t<0$. For the heat equation, consider

$$\tilde{K}(x,t) \;=\; \begin{cases} \frac{1}{(4\pi t)^{n/2}}e^{-\frac{|x|^2}{4t}} & t>0 \\ 0 & t\le 0. \end{cases} \tag{9.9}$$

Notice that $\tilde{K}$ is smooth for $(x,t)\neq(0,0)$.

$\tilde{K}$ defined as in (9.9), is the fundamental solution of the "free space" heat equation.

Proof. We need to show:

$$\Big(\partial_t - \triangle\Big)K(x,t) \;=\; \delta(x,t). \tag{9.10}$$

To verify (9.10) as distributions, we must show that for any $v\in C_0^\infty(\mathbb{R}^{n+1})$: [14]

$$\int_{\mathbb{R}^{n+1}} \tilde{K}(x,t)\Big(-\partial_t - \triangle\Big)\, v\, dx\, dt \;=\; \int_{\mathbb{R}^{n+1}} \delta(x,t)\, v(x,t)\, dx\, dt \;\equiv\; v(0,0).$$

To do this, let us take $\epsilon>0$ and define

$$\tilde{K}_\epsilon(x,t) \;=\; \begin{cases} \frac{1}{(4\pi t)^{n/2}}e^{-\frac{|x|^2}{4t}} & t>\epsilon \\ 0 & t\le \epsilon. \end{cases}$$

Then $\tilde{K}_\epsilon \to \tilde{K}$ as distributions, so it suffices to show that $(\partial_t - \triangle)\tilde{K}_\epsilon \to \delta$ as distributions. Now

$$\begin{aligned}
\int \tilde{K}_\epsilon\Big(-\partial_t-\triangle\Big)\, v\, dx\, dt \;&=\; \int_\epsilon^\infty\bigg(\int_{\mathbb{R}^n}\tilde{K}(x,t)\Big(-\partial_t-\triangle\Big)\, v(x,t)\, dx\bigg)dt \\
=\; -\int_\epsilon^\infty\bigg(\int_{\mathbb{R}^n}\tilde{K}(x,t)\;\partial_t v(x,t)\, dx\bigg)dt &- \int_\epsilon^\infty\bigg(\int_{\mathbb{R}^n}\tilde{K}(x,t)\;\triangle v(x,t)\, dx\bigg)dt \\
=\; -\bigg[\int_{\mathbb{R}^n}\tilde{K}(x,t)\, v(x,t)\, dx\bigg]_{t=\epsilon}^{t=\infty} &+ \int_\epsilon^\infty\bigg(\int_{\mathbb{R}^n}\partial_t\tilde{K}(x,t)\; v(x,t)\, dx\bigg)dt - \int_\epsilon^\infty\bigg(\int_{\mathbb{R}^n}\triangle\tilde{K}(x,t)\; v(x,t)\, dx\bigg)dt \\
=\; \int_\epsilon^\infty\bigg(\int_{\mathbb{R}^n}\Big(\partial_t-\triangle\Big)\tilde{K}(x,t)\, v(x,t)\, dx\bigg)dt &+ \int_{\mathbb{R}^n}\tilde{K}(x,\epsilon)\, v(x,\epsilon)\, dx.
\end{aligned}$$

But for $t>\epsilon$, $(\partial_t-\triangle)\tilde{K}(x,t)=0$; moreover, since $\lim_{t\to 0^+}\tilde{K}(x,t)=\delta_0(x)=\delta(x)$, we have $\tilde{K}(x,\epsilon)\to\delta_0(x)$ as $\epsilon\to 0$, so the last integral tends to $v(0,0)$. ∎

[14] Note, for the operator $L=\partial/\partial t$, the **adjoint operator** is $L^*=-\partial/\partial t$.

10 Schrödinger Equation

Problem (F'96, #5). *The Gauss kernel*

$$G(t,x,y) = \frac{1}{(4\pi t)^{\frac{1}{2}}} e^{-\frac{(x-y)^2}{4t}}$$

*is the **fundamental solution** of the **heat equation**, solving*

$$G_t = G_{xx}, \qquad G(0,x,y) = \delta(x-y).$$

*By analogy with the heat equation, find the fundamental solution $H(t,x,y)$ of the **Schrödinger equation***

$$H_t = iH_{xx}, \qquad H(0,x,y) = \delta(x-y).$$

Show that your expression $H(x)$ is indeed the fundamental solution for the Schrödinger equation. You may use the following special integral

$$\int_{-\infty}^{\infty} e^{\frac{-ix^2}{4}}\, dx = \sqrt{-i4\pi}.$$

Proof. • **Remark:** Consider the initial value problem for the Schrödinger equation

$$\begin{cases} u_t = i\triangle u & x \in \mathbb{R}^n,\ t > 0, \\ u(x,0) = g(x) & x \in \mathbb{R}^n. \end{cases}$$

If we formally replace t by it in the heat kernel, we obtain the **Fundamental Solution of the Schrödinger Equation**: [15]

$$H(x,t) = \frac{1}{(4\pi it)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4it}} \qquad (x \in \mathbb{R}^n,\ t \neq 0)$$

$$u(x,t) = \frac{1}{(4\pi it)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4it}} g(y)\, dy.$$

In particular, the Schrödinger equation is *reversible in time*, whereas the heat equation is not.

• **Solution:** We have already found the fundamental solution for the heat equation using the Fourier transform. For the Schrödinger equation is one dimension, we have

$$\frac{\partial}{\partial t}\widehat{u}(\xi,t) = -i\xi^2\widehat{u}(\xi,t),$$

which is an ordinary differential equation in t, with the solution $\widehat{u}(\xi,t) = Ce^{-i\xi^2 t}$. The initial condition $\widehat{u}(\xi,0) = \widehat{g}(\xi)$ gives

$$\begin{aligned}
\widehat{u}(\xi,t) &= \widehat{g}(\xi)\, e^{-i\xi^2 t}, \\
u(x,t) &= \left(\widehat{g}(\xi)\, e^{-i\xi^2 t}\right)^{\vee} = \frac{1}{\sqrt{2\pi}}\left[g * \left(e^{-i\xi^2 t}\right)^{\vee}\right] \\
&= \frac{1}{\sqrt{2\pi}}\, g * \left[\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi^2 t} e^{ix\cdot\xi}\, d\xi\right] \\
&= \frac{1}{2\pi}\, g * \left[\int_{\mathbb{R}} e^{ix\cdot\xi - i\xi^2 t}\, d\xi\right] \qquad = \text{(need some work)} = \\
&= \frac{1}{\sqrt{4\pi it}}\, g * \left[e^{-\frac{|x|^2}{4it}}\right] = \frac{1}{\sqrt{4\pi it}} \int_{\mathbb{R}} e^{-\frac{|x-y|^2}{4it}} g(y)\, dy.
\end{aligned}$$

[15] Evans, p. 188, Example 3.

- For the Schrödinger equation, consider

$$\tilde{\Psi}(x,t) \;=\; \begin{cases} \frac{1}{(4\pi i t)^{n/2}} e^{-\frac{|x|^2}{4it}} & t>0 \\ 0 & t \leq 0. \end{cases} \tag{10.1}$$

Notice that $\tilde{\Psi}$ is smooth for $(x,t) \neq (0,0)$.

$\tilde{\Psi}$ defined as in (10.1), is the fundamental solution of the Schrödinger equation. We need to show:

$$\Big(\partial_t - i\triangle\Big)\Psi(x,t) \;=\; \delta(x,t). \tag{10.2}$$

To verify (10.2) as distributions, we must show that for any $v \in C_0^\infty(\mathbb{R}^{n+1})$: [16]

$$\int_{\mathbb{R}^{n+1}} \tilde{\Psi}(x,t)\Big(-\partial_t - i\triangle\Big)\, v\, dx\, dt \;=\; \int_{\mathbb{R}^{n+1}} \delta(x,t)\, v(x,t)\, dx\, dt \;\equiv\; v(0,0).$$

To do this, let us take $\epsilon > 0$ and define

$$\tilde{\Psi}_\epsilon(x,t) \;=\; \begin{cases} \frac{1}{(4\pi i t)^{n/2}} e^{-\frac{|x|^2}{4it}} & t>\epsilon \\ 0 & t \leq \epsilon. \end{cases}$$

Then $\tilde{\Psi}_\epsilon \to \tilde{\Psi}$ as distributions, so it suffices to show that $(\partial_t - i\triangle)\tilde{\Psi}_\epsilon \to \delta$ as distributions. Now

$$\begin{aligned} \int \tilde{\Psi}_\epsilon\Big(-\partial_t - i\triangle\Big)\, v\, dx\, dt \;&=\; \int_\epsilon^\infty \bigg(\int_{\mathbb{R}^n} \tilde{\Psi}(x,t)\Big(-\partial_t - i\triangle\Big)\, v(x,t)\, dx\bigg) dt \\ &=\; \int_\epsilon^\infty \bigg(\int_{\mathbb{R}^n} \Big(\partial_t - i\triangle\Big)\tilde{\Psi}(x,t)\, v(x,t)\, dx\bigg) dt \;+\; \int_{\mathbb{R}^n} \tilde{\Psi}(x,\epsilon)\, v(x,\epsilon)\, dx. \end{aligned}$$

But for $t > \epsilon$, $(\partial_t - i\triangle)\tilde{\Psi}(x,t) = 0$; moreover, since $\lim_{t\to 0^+} \tilde{\Psi}(x,t) = \delta_0(x) = \delta(x)$, we have $\tilde{\Psi}(x,\epsilon) \to \delta_0(x)$ as $\epsilon \to 0$, so the last integral tends to $v(0,0)$. $\square$

[16] Note, for the operator $L = \partial/\partial t$, the **adjoint operator** is $L^* = -\partial/\partial t$.

11 Problems: Quasilinear Equations

Problem (F'90, #7). *Use the method of characteristics to find the solution of the first order partial differential equation*

$$x^2u_x + xyu_y = u^2$$

which passes through the curve $u = 1$, $x = y^2$. Determine where this solution becomes singular.

Proof. We have a condition $u(x = y^2) = 1$. Γ is parametrized by $\Gamma:\ (s^2, s, 1)$.

$$\frac{dx}{dt} = x^2 \Rightarrow x = \frac{1}{-t - c_1(s)} \Rightarrow x(0, s) = \frac{1}{-c_1(s)} = s^2 \Rightarrow x = \frac{1}{-t + \frac{1}{s^2}} = \frac{s^2}{1 - ts^2},$$

$$\frac{dy}{dt} = xy \Rightarrow \frac{dy}{dt} = \frac{s^2y}{1 - ts^2} \Rightarrow y = \frac{c_2(s)}{1 - ts^2} \Rightarrow y(s, 0) = c_2(s) = s \Rightarrow y = \frac{s}{1 - ts^2},$$

$$\frac{dz}{dt} = z^2 \Rightarrow z = \frac{1}{-t - c_3(s)} \Rightarrow z(0, s) = \frac{1}{-c_3(s)} = 1 \Rightarrow z = \frac{1}{1 - t}.$$

Thus,

$$\frac{x}{y} = s \Rightarrow y = \frac{\frac{x}{y}}{1 - t\frac{x^2}{y^2}} \Rightarrow t = \frac{y^2}{x^2} - \frac{1}{x}.$$

$$\Rightarrow u(x, y) = \frac{1}{1 - \frac{y^2}{x^2} + \frac{1}{x}} = \frac{x^2}{x^2 + x - y^2}.$$

The solution becomes singular when $y^2 = x^2 + x$.
It can be checked that the solution satisfies the PDE and $u(x = y^2) = \frac{y^4}{y^4 + y^2 - y^2} = 1$. □

Problem (S'91, #7). *Solve the first order PDE*

$$f_x + x^2yf_y + f = 0$$
$$f(x = 0, y) = y^2$$

using the method of characteristics.

Proof. Rewrite the equation

$$u_x + x^2yu_y = -u,$$
$$u(0, y) = y^2.$$

Γ is parameterized by $\Gamma:\ (0, s, s^2)$.

$$\frac{dx}{dt} = 1 \Rightarrow x = t,$$

$$\frac{dy}{dt} = x^2y \Rightarrow \frac{dy}{dt} = t^2y \Rightarrow y = se^{\frac{t^3}{3}},$$

$$\frac{dz}{dt} = -z \Rightarrow z = s^2e^{-t}.$$

Thus, $x = t$ and $s = ye^{-\frac{t^3}{3}} = ye^{-\frac{x^3}{3}}$, and

$$u(x, y) = (ye^{-\frac{x^3}{3}})^2e^{-x} = y^2e^{-\frac{2}{3}x^3 - x}.$$

The solution satisfies both the PDE and initial conditions. □

Problem (S'92, #1). *Consider the Cauchy problem*

$$\begin{aligned} &u_t = xu_x - u + 1 && -\infty < x < \infty,\ t \geq 0 \\ &u(x,0) = \sin x && -\infty < x < \infty \end{aligned}$$

and solve it by the method of characteristics. Discuss the properties of the solution; in particular investigate the behavior of $|u_x(\cdot,t)|_\infty$ *for* $t \to \infty$.

Proof. Γ is parametrized by $\Gamma:\ (s, 0, \sin s)$. We have

$$\begin{aligned} \frac{dx}{dt} &= -x \quad \Rightarrow \quad x = se^{-t}, \\ \frac{dy}{dt} &= 1 \quad \Rightarrow \quad y = t, \\ \frac{dz}{dt} &= 1 - z \quad \Rightarrow \quad z = 1 - \frac{1 - \sin s}{e^t}. \end{aligned}$$

Thus, $t = y$, $s = xe^y$, and

$$u(x,y) = 1 - \frac{1}{e^y} + \frac{\sin(xe^y)}{e^y}.$$

It can be checked that the solution satisfies the PDE and the initial condition.
As $t \to \infty$, $u(x,t) \to 1$. Also,

$$|u_x(x,y)|_\infty = |\cos(xe^y)|_\infty = 1.$$

$u_x(x,y)$ oscillate between -1 and 1. If $x = 0$, $u_x = 1$. $\square$

Problem (W'02, #6). *Solve the Cauchy problem*

$$\begin{aligned} &u_t + u^2 u_x = 0, \qquad t > 0, \\ &u(0,x) = 2 + x. \end{aligned}$$

Proof. **Solved** $\square$

Problem (S'97, #1). *Find the solution of the Burgers' equation*

$$u_t + uu_x = -x, \qquad t \geq 0$$
$$u(x,0) = f(x), \qquad -\infty < x < \infty.$$

Proof. Γ is parameterized by $\Gamma: \ (s, 0, f(s))$.

$$\frac{dx}{dt} = z,$$
$$\frac{dy}{dt} = 1 \quad \Rightarrow \quad y = t,$$
$$\frac{dz}{dt} = -x.$$

Note that we have a coupled system:

$$\begin{cases} \dot{x} = z, \\ \dot{z} = -x, \end{cases}$$

which can be written as a second order ODE:

$$\ddot{x} + x = 0, \qquad x(s,0) = s, \ \ \dot{x}(s,0) = z(0) = f(s).$$

Solving the equation, we get

$$\begin{aligned} x(s,t) &= s\cos t + f(s)\sin t, \qquad \text{and thus,} \\ z(s,t) &= \dot{x}(t) = -s\sin t + f(s)\cos t. \end{aligned}$$

$$\begin{cases} x = s\cos y + f(s)\sin y, \\ u = -s\sin y + f(s)\cos y. \end{cases} \quad \Rightarrow \quad \begin{cases} x\cos y = s\cos^2 y + f(s)\sin y\cos y, \\ u\sin y = -s\sin^2 y + f(s)\cos y\sin y. \end{cases}$$
$$\Rightarrow \quad x\cos y - u\sin y = s(\cos^2 y + \sin^2 y) = s.$$

$$\Rightarrow \quad u(x,y) = f(x\cos y - u\sin y)\cos y - (x\cos y - u\sin y)\sin y. \qquad \square$$

Problem (F'98, #2). *Solve the partial differential equation*

$$u_y - u^2 u_x = 3u, \qquad u(x,0) = f(x)$$

using method of characteristics. (Hint: find a parametric representation of the solution.)

Proof. Γ is parameterized by $\Gamma: \ (s, 0, f(s))$.

$$\frac{dx}{dt} = -z^2 \quad \Rightarrow \quad \frac{dx}{dt} = -f^2(s)e^{6t} \quad \Rightarrow \quad x = -\frac{1}{6}f^2(s)e^{6t} + \frac{1}{6}f^2(s) + s,$$
$$\frac{dy}{dt} = 1 \quad \Rightarrow \quad y = t,$$
$$\frac{dz}{dt} = 3z \quad \Rightarrow \quad z = f(s)e^{3t}.$$

Thus,

$$\begin{cases} x = -\frac{1}{6} f^2(s) e^{6y} + \frac{1}{6} f^2(s) + s, \\ f(s) = \frac{z}{e^{3y}} \end{cases} \Rightarrow \quad x = -\frac{1}{6}\frac{z^2}{e^{6y}} e^{6y} + \frac{1}{6}\frac{z^2}{e^{6y}} + s = \frac{z^2}{6e^{6y}} - \frac{z^2}{6} + s,$$

$$\Rightarrow \quad s = x - \frac{z^2}{6e^{6y}} + \frac{z^2}{6}.$$

$$\Rightarrow \quad z = f\Big(x - \frac{z^2}{6e^{6y}} + \frac{z^2}{6}\Big) e^{3y}.$$

$$\Rightarrow \quad u(x,y) = f\Big(x - \frac{u^2}{6e^{6y}} + \frac{u^2}{6}\Big) e^{3y}.$$

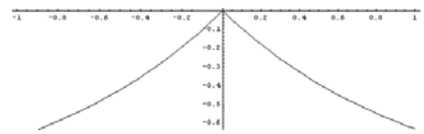
$\square$

Problem (S'99, #1) Modified Problem. ***a)*** *Solve*

$$u_t + \left(\frac{u^3}{3}\right)_x = 0 \tag{11.1}$$

for $t > 0$, $-\infty < x < \infty$ *with initial data*

$$u(x,0) = h(x) = \begin{cases} -a(1-e^x), & x < 0 \\ -a(1-e^{-x}), & x > 0 \end{cases}$$



where $a > 0$ *is constant. Solve until the first appearance of discontinuous derivative and determine that critical time.*
b) *Consider the equation*

$$u_t + \left(\frac{u^3}{3}\right)_x = -cu. \tag{11.2}$$

How large does the constant $c > 0$ *has to be, so that a smooth solution (with no discontinuities) exists for all* $t > 0$*? Explain.*

Proof. **a)** Characteristic form: $u_t + u^2 u_x = 0$. $\Gamma:\ (s, 0, h(s))$.

$$\frac{dx}{dt} = z^2, \quad \frac{dy}{dt} = 1, \quad \frac{dz}{dt} = 0.$$

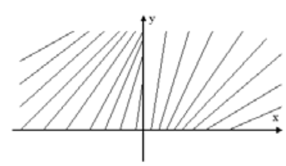
$$x = h(s)^2 t + s, \quad y = t, \quad z = h(s).$$

$$u(x,y) = h(x - u^2 y) \tag{11.3}$$

The characteristic projection in the xt-plane[17] passing through the point $(s,0)$ is the line

$$x = h(s)^2 t + s$$

along which u has the constant value $u = h(s)$.



The derivative of the initial data is discontinuous, and that leads to a rarefaction-like behavior at $t = 0$. However, if the question meant to ask to determine the first time when a shock forms, we proceed as follows.
Two characteristics $x = h(s_1)^2 t + s_1$ and $x = h(s_2)^2 t + s_2$ intersect at a point (x,t) with

$$t = -\frac{s_2 - s_1}{h(s_2)^2 - h(s_1)^2}.$$

From (11.3), we have

$$u_x = h'(s)(1 - 2uu_x t) \quad \Rightarrow \quad u_x = \frac{h'(s)}{1 + 2h(s)h'(s)t}$$

Hence for $2h(s)h'(s) < 0$, u_x becomes infinite at the positive time

$$t = \frac{-1}{2h(s)h'(s)}.$$

The smallest t for which this happens corresponds to the value $s = s_0$ at which $h(s)h'(s)$ has a minimum (i.e.$-h(s)h'(s)$ has a maximum). At time $T = -1/(2h(s_0)h'(s_0))$ the

[17] y and t are interchanged here

solution u experiences a "gradient catastrophe".
Therefore, need to find a minimum of

$$f(x) = 2h(x)h'(x) = \begin{cases} -2a(1-e^x)\cdot ae^x \\ -2a(1-e^{-x})\cdot(-ae^{-x}) \end{cases} = \begin{cases} -2a^2e^x(1-e^x), & x<0 \\ 2a^2e^{-x}(1-e^{-x}), & x>0 \end{cases}$$

$$f'(x) = \begin{cases} -2a^2e^x(1-2e^x), & x<0 \\ -2a^2e^{-x}(1-2e^{-x}), & x>0 \end{cases} = 0 \quad \Rightarrow \quad \begin{cases} x = \ln(\frac{1}{2}) = -\ln(2), & x<0 \\ x = \ln(2), & x>0 \end{cases}$$

$$\Rightarrow \quad \begin{cases} f(\ln(\frac{1}{2})) = -2a^2e^{\ln(\frac{1}{2})}(1-e^{\ln(\frac{1}{2})}) = -2a^2(\frac{1}{2})(\frac{1}{2}) = \frac{-a^2}{2}, & x<0 \\ f(\ln(2)) = 2a^2(\frac{1}{2})(1-\frac{1}{2}) = \frac{a^2}{2}, & x>0 \end{cases}$$

$$\Rightarrow \quad t = -\frac{1}{\min\{2h(s)h'(s)\}} = \frac{2}{a^2}$$

$\square$

Proof. **b)** Characteristic form: $u_t + u^2u_x = -cu$. $\quad \Gamma: \ (s, 0, h(s))$.

$$\begin{aligned} \frac{dx}{dt} &= z^2 = h(s)^2e^{-2ct} \quad \Rightarrow \quad x = s + \frac{1}{2c}h(s)^2(1-e^{-2ct}), \\ \frac{dy}{dt} &= 1 \qquad \Rightarrow \quad y = t, \\ \frac{dz}{dt} &= -cz \quad \Rightarrow \quad z = h(s)e^{-ct} \qquad\qquad (\Rightarrow\ h(s) = ue^{cy}). \end{aligned}$$

Solving for s and t in terms of x and y, we get:

$$t = y, \qquad s = x - \frac{1}{2c}h(s)^2(1-e^{-2cy}).$$

Thus,

$$\begin{aligned} u(x,y) &= h\big(x - \frac{1}{2c}u^2e^{2cy}(1-e^{-2cy})\big)\cdot e^{-cy}. \\ u_x &= h'(s)e^{-cy}\cdot(1 - \frac{1}{c}uu_xe^{2cy}(1-e^{-2cy})), \\ u_x &= \frac{h'(s)e^{-cy}}{1+\frac{1}{c}h'(s)e^{cy}u\cdot(1-e^{-2cy})} = \frac{h'(s)e^{-cy}}{1+\frac{1}{c}h'(s)h(s)(1-e^{-2cy})}. \end{aligned}$$

Thus, $c > 0$ that would allow a smooth solution to exist for all $t > 0$ should satisfy

$$1 + \frac{1}{c}h'(s)h(s)(1-e^{-2cy}) \neq 0.$$

We can perform further calculations taking into account the result from part (a):

$$\min\{2h(s)h'(s)\} = -\frac{a^2}{2}.$$

$\square$

Problem (S'99, #1). Original Problem. a). *Solve*

$$u_t + \frac{u_x^3}{3} = 0 \tag{11.4}$$

for $t > 0$, $-\infty < x < \infty$ *with initial data*

$$u(x,0) = h(x) = \begin{cases} -a(1-e^x), & x < 0 \\ -a(1-e^{-x}), & x > 0 \end{cases}$$

where $a > 0$ *is constant.*

Proof. Rewrite the equation as

$$\begin{aligned} F(x,y,u,u_x,u_y) &= \frac{u_x^3}{3} + u_y = 0, \\ F(x,y,z,p,q) &= \frac{p^3}{3} + q = 0. \end{aligned}$$

Γ is parameterized by $\Gamma: \ (s, 0, h(s), \phi(s), \psi(s))$.
We need to complete Γ to a strip. Find $\phi(s)$ and $\psi(s)$, the initial conditions for $p(s,t)$ and $q(s,t)$, respectively:

- $$\begin{aligned} &F(f(s), g(s), h(s), \phi(s), \psi(s)) = 0, \\ &\frac{\phi(s)^3}{3} + \psi(s) = 0, \\ &\psi(s) = -\frac{\phi(s)^3}{3}. \end{aligned}$$

- $$h'(s) = \phi(s)f'(s) + \psi(s)g'(s)$$
$$\begin{cases} ae^s = \phi(s), & x < 0 \\ -ae^{-s} = \phi(s), & x > 0 \end{cases} \quad \Rightarrow \quad \begin{cases} \psi(s) = -\frac{a^3e^{3s}}{3}, & x < 0 \\ \psi(s) = \frac{a^3e^{-3s}}{3}, & x > 0 \end{cases}$$

Therefore, now Γ is parametrized by

$$\begin{cases} \Gamma: \ (s, 0, \ -a(1-e^s), \quad ae^s, \ -\frac{a^3e^{3s}}{3}), & x < 0 \\ \Gamma: \ (s, 0, \ -a(1-e^{-s}), -ae^{-s}, \ \frac{a^3e^{-3s}}{3}), & x > 0 \end{cases}$$

$$\frac{dx}{dt} = F_p = p^2 = \begin{cases} a^2e^{2s} \\ a^2e^{-2s} \end{cases} \Rightarrow x(s,t) = \begin{cases} a^2e^{2s}t + c_4(s) \\ a^2e^{-2s}t + c_5(s) \end{cases} \Rightarrow x = \begin{cases} a^2e^{2s}t + s \\ a^2e^{-2s}t + s \end{cases}$$

$$\frac{dy}{dt} = F_q = 1 \quad \Rightarrow \quad y(s,t) = t + c_1(s) \quad \Rightarrow \quad y = t$$

$$\frac{dz}{dt} = pF_p + qF_q = p^3 + q = \begin{cases} a^3e^{3s} - \frac{a^3e^{3s}}{3} = \frac{2}{3}a^3e^{3s}, & x < 0 \\ -a^3e^{-3s} + \frac{a^3e^{-3s}}{3} = -\frac{2}{3}a^3e^{-3s}, & x > 0 \end{cases}$$

$$\Rightarrow z(s,t) = \begin{cases} \frac{2}{3}a^3e^{3s}t + c_6(s), & x < 0 \\ -\frac{2}{3}a^3e^{-3s}t + c_7(s), & x > 0 \end{cases} \Rightarrow z = \begin{cases} \frac{2}{3}a^3e^{3s}t - a(1-e^s), & x < 0 \\ -\frac{2}{3}a^3e^{-3s}t - a(1-e^{-s}), & x > 0 \end{cases}$$

$$\frac{dp}{dt} = -F_x - F_z p = 0 \quad \Rightarrow \quad p(s,t) = c_2(s) \quad \Rightarrow \quad p = \begin{cases} ae^s, & x < 0 \\ -ae^{-s}, & x > 0 \end{cases}$$

$$\frac{dq}{dt} = -F_y - F_z q = 0 \quad\Rightarrow\quad q(s,t) = c_3(s) \quad\Rightarrow\quad q = \begin{cases} -\frac{a^3e^{3s}}{3}, & x<0 \\ \frac{a^3e^{-3s}}{3}, & x>0 \end{cases}$$

Thus,

$$\boxed{u(x,y) = \begin{cases} \frac{2}{3}a^3e^{3s}y \;-\; a(1-e^s), & x<0 \\ -\frac{2}{3}a^3e^{-3s}y - a(1-e^{-s}), & x>0 \end{cases}}$$

where s is defined as

$$x = \begin{cases} a^2e^{2s}y + s, & x<0 \\ a^2e^{-2s}y + s, & x>0. \end{cases}$$

□

b). *Solve the equation*

$$u_t + \frac{u_x^3}{3} = -cu. \tag{11.5}$$

Proof. Rewrite the equation as

$$F(x,y,u,u_x,u_y) = \frac{u_x^3}{3} + u_y + cu = 0,$$
$$F(x,y,z,p,q) = \frac{p^3}{3} + q + cz = 0.$$

Γ is parameterized by $\Gamma:\ (s,0,h(s),\phi(s),\psi(s))$.
We need to complete Γ to a strip. Find $\phi(s)$ and $\psi(s)$, the initial conditions for $p(s,t)$ and $q(s,t)$, respectively:

- $$F(f(s),g(s),h(s),\phi(s),\psi(s)) = 0,$$
  $$\frac{\phi(s)^3}{3} + \psi(s) + ch(s) = 0,$$
  $$\psi(s) = -\frac{\phi(s)^3}{3} - ch(s) = \begin{cases} -\frac{\phi(s)^3}{3} + ca(1-e^x), & x<0 \\ -\frac{\phi(s)^3}{3} + ca(1-e^{-x}), & x>0 \end{cases}$$

- $$h'(s) = \phi(s)f'(s) + \psi(s)g'(s)$$
  $$\begin{cases} ae^s = \phi(s), & x<0 \\ -ae^{-s} = \phi(s), & x>0 \end{cases} \quad\Rightarrow\quad \begin{cases} \psi(s) = -\frac{a^3e^{3s}}{3} + ca(1-e^x), & x<0 \\ \psi(s) = \frac{a^3e^{-3s}}{3} + ca(1-e^{-x}), & x>0 \end{cases}$$

Therefore, now Γ is parametrized by

$$\begin{cases} \Gamma:\ (s,0,\ -a(1-e^s), & ae^s, & -\frac{a^3e^{3s}}{3} + ca(1-e^x), & x<0 \\ \Gamma:\ (s,0,\ -a(1-e^{-s}), & -ae^{-s}, & \frac{a^3e^{-3s}}{3} + ca(1-e^{-x}), & x>0 \end{cases}$$

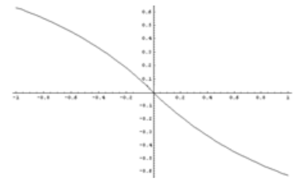
$$\begin{aligned}
\frac{dx}{dt} &= F_p = p^2 \\
\frac{dy}{dt} &= F_q = 1 \\
\frac{dz}{dt} &= pF_p + qF_q = p^3 + q \\
\frac{dp}{dt} &= -F_x - F_z p = -cp \\
\frac{dq}{dt} &= -F_y - F_z q = -cq
\end{aligned}$$

We can proceed solving the characteristic equations with initial conditions above. $\square$

Problem (S'95, #7). ***a)*** *Solve the following equation, using characteristics,*

$$u_t + u^3 u_x = 0,$$

$$u(x,0) = \begin{cases} a(1-e^x), & \text{for } x < 0 \\ -a(1-e^{-x}), & \text{for } x > 0 \end{cases}$$



where $a > 0$ is a constant. Determine the first time when a shock forms.

Proof. **a)** Γ is parameterized by $\Gamma: \ (s, 0, h(s))$.

$$\frac{dx}{dt} = z^3, \quad \frac{dy}{dt} = 1, \quad \frac{dz}{dt} = 0.$$

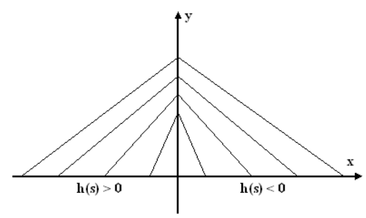
$$x = h(s)^3 t + s, \quad y = t, \quad z = h(s).$$

$$u(x,y) = h(x - u^3 y) \tag{11.6}$$

The characteristic projection in the xt-plane[18] passing through the point $(s, 0)$ is the line

$$x = h(s)^3 t + s$$

along which u has a constant value $u = h(s)$.



Characteristics $x = h(s_1)^3 t + s_1$ and $x = h(s_2)^3 t + s_2$ intersect at a point (x, t) with

$$t = -\frac{s_2 - s_1}{h(s_2)^3 - h(s_1)^3}.$$

From (11.6), we have

$$u_x = h'(s)(1 - 3u^2 u_x t) \quad \Rightarrow \quad u_x = \frac{h'(s)}{1 + 3h(s)^2 h'(s) t}$$

Hence for $3h(s)^2 h'(s) < 0$, u_x becomes infinite at the positive time

$$t = \frac{-1}{3h(s)^2 h'(s)}.$$

The smallest t for which this happens corresponds to the value $s = s_0$ at which $h(s)^2 h'(s)$ has a minimum (i.e.$-h(s)^2 h'(s)$ has a maximum). At time $T = -1/(3h(s_0)^2 h'(s_0))$ the solution u experiences a "gradient catastrophe".
Therefore, need to find a **minimum** of

$$f(x) = 3h(x)^2 h'(x) = \begin{cases} -3a^2(1-e^x)^2 a e^x & = -3a^3 e^x (1-e^x)^2, & x < 0 \\ -3a^2(1-e^{-x})^2 a e^{-x} & = -3a^3 e^{-x}(1-e^{-x})^2, & x > 0 \end{cases}$$

$$f'(x) = \begin{cases} -3a^3\big[e^x(1-e^x)^2 - e^x 2(1-e^x)e^x\big] & = -3a^3 e^x (1-e^x)(1-3e^x), & x < 0 \\ -3a^3\big[-e^{-x}(1-e^{-x})^2 + e^{-x}2(1-e^{-x})e^{-x}\big] & = -3a^3 e^{-x}(1-e^{-x})(-1+3e^{-x}), & x > 0 \end{cases} = 0$$

The zeros of $f'(x)$ are $\begin{cases} x = 0, \ \ x = -\ln 3, & x < 0, \\ x = 0, \ \ x = \ \ \ln 3, & x > 0. \end{cases}$ We check which ones give the minimum of $f(x)$:

$$\Rightarrow \quad \begin{cases} f(0) = -3a^3, \ \ f(-\ln 3) = -3a^3 \frac{1}{3}(1-\frac{1}{3})^2 = -\frac{4a^3}{9}, & x < 0 \\ f(0) = -3a^3, \ \ \ \ f(\ln 3) \ \ = -3a^3 \frac{1}{3}(1-\frac{1}{3})^2 = -\frac{4a^3}{9}, & x > 0 \end{cases}$$

[18] y and t are interchanged here

$$\Rightarrow \quad t = -\frac{1}{\min\{3h(s)^2 h'(s)\}} = -\frac{1}{\min f(s)} = \frac{1}{3a^3}.$$

$\square$

b) *Now consider*

$$u_t + u^3 u_x + cu = 0$$

with the same initial data and a positive constant c. How large does c need to be in order to prevent shock formation?

b) Characteristic form: $u_t + u^3 u_x = -cu$. $\Gamma: \ (s, 0, h(s))$.

$$\begin{aligned}
\frac{dx}{dt} &= z^3 = h(s)^3 e^{-3ct} \quad \Rightarrow \quad x = s + \frac{1}{3c} h(s)^3 (1 - e^{-3ct}), \\
\frac{dy}{dt} &= 1 \quad \Rightarrow \quad y = t, \\
\frac{dz}{dt} &= -cz \quad \Rightarrow \quad z = h(s) e^{-ct} \qquad (\Rightarrow \ h(s) = u e^{cy}). \\
&\Rightarrow \ z(s,t) = h\Big(x - \frac{1}{3c} h(s)^3 (1 - e^{-3ct})\Big) e^{-ct}, \\
&\Rightarrow \ u(x,y) = h\Big(x - \frac{1}{3c} u^3 e^{3cy} (1 - e^{-3cy})\Big) e^{-cy}.
\end{aligned}$$

$$\begin{aligned}
u_x &= h'(s) \cdot e^{-cy} \cdot \Big(1 - \frac{1}{c} u^2 u_x e^{3cy} (1 - e^{-3cy})\Big), \\
u_x &= \frac{h'(s) e^{-cy}}{1 + \frac{1}{c} h'(s) u^2 e^{2cy} (1 - e^{-3cy})} = \frac{h'(s) e^{-cy}}{1 + \frac{1}{c} h'(s) h(s)^2 (1 - e^{-3cy})}.
\end{aligned}$$

Thus, we need

$$1 + \frac{1}{c} h'(s) h(s)^2 (1 - e^{-3cy}) \neq 0.$$

We can perform further calculations taking into account the result from part (a):

$$\min\{3h(s)^2 h'(s)\} = -3a^3.$$

Problem (F'99, #4). *Consider the Cauchy problem*

$$u_y + a(x)u_x = 0,$$
$$u(x,0) = h(x).$$

*Give an example of an (unbounded) smooth $a(x)$ for which the solution of the Cauchy problem is **not** unique.*

Proof. Γ is parameterized by $\Gamma:\ (s, 0, h(s))$.

$$\begin{aligned}
\frac{dx}{dt} &= a(x) \quad \Rightarrow \quad x(t) - x(0) = \int_0^t a(x)dt \quad \Rightarrow \quad x = \int_0^t a(x)dt + s,\\
\frac{dy}{dt} &= 1 \quad \Rightarrow \quad y(s,t) = t + c_1(s) \quad \Rightarrow \quad y = t,\\
\frac{dz}{dt} &= 0 \quad \Rightarrow \quad z(s,t) = c_2(s) \quad \Rightarrow \quad z = h(s).
\end{aligned}$$

Thus,

$$u(x,t) = h\Big(x - \int_0^y a(x)dy\Big)$$

$\square$

Problem (F'97, #7). ***a)*** *Solve the Cauchy problem*

$$\begin{aligned}
u_t - xuu_x &= 0 \qquad -\infty < x < \infty,\ t \geq 0,\\
u(x,0) &= f(x) \qquad -\infty < x < \infty.
\end{aligned}$$

b) *Find a class of initial data such that this problem has a global solution for all t. Compute the critical time for the existence of a smooth solution for initial data, f, which is not in the above class.*

Proof. **a)** Γ is parameterized by $\Gamma:\ (s, 0, f(s))$.

$$\begin{aligned}
\frac{dx}{dt} &= -xz \quad \Rightarrow \quad \frac{dx}{dt} = -xf(s) \quad \Rightarrow \quad x = se^{-f(s)t},\\
\frac{dy}{dt} &= 1 \quad \Rightarrow \quad y = t,\\
\frac{dz}{dt} &= 0 \quad \Rightarrow \quad z = f(s).\\
&\Rightarrow \quad z = f\big(xe^{f(s)t}\big),\\
&\Rightarrow \quad u(x,y) = f\big(xe^{uy}\big).
\end{aligned}$$

Check:

$$\begin{cases} u_x = f'(s)\cdot(e^{uy} + xe^{uy}u_x y) \\ u_y = f'(s)\cdot xe^{uy}(u_y y + u) \end{cases} \Rightarrow \begin{cases} u_x - f'(s)xe^{uy}u_x y = f'(s)e^{uy} \\ u_y - f'(s)xe^{uy}u_y y = f'(s)xe^{uy}u \end{cases}$$

$$\Rightarrow \begin{cases} u_x = \frac{f'(s)e^{uy}}{1-f'(s)xye^{uy}} \\ u_y = \frac{f'(s)e^{uy}xu}{1-f'(s)xye^{uy}} \end{cases} \Rightarrow u_y - xuu_x = \frac{f'(s)e^{uy}xu}{1-f'(s)xye^{uy}} - xu\frac{f'(s)e^{uy}}{1-f'(s)xye^{uy}} = 0. \ \checkmark$$

$$u(x,0) = f(x). \ \checkmark$$

b) The characteristics would intersect when $1 - f'(s)xye^{uy} = 0$. Thus,

$$t_c = \frac{1}{f'(s)xe^{ut_c}}.$$

$\square$

Problem (F'96, #6). *Find an implicit formula for the solution u of the initial-value problem*

$$u_t = (2x-1)tu_x + \sin(\pi x) - t,$$
$$u(x, t=0) = 0.$$

Evaluate u explicitly at the point $(x = 0.5,\ t = 2)$.

Proof. Rewrite the equation as

$$u_y + (1-2x)yu_x = \sin(\pi x) - y.$$

Γ is parameterized by $\Gamma:\ (s, 0, 0)$.

$$\begin{aligned}
\frac{dx}{dt} &= (1-2x)y = (1-2x)t \quad\Rightarrow\quad x = \frac{1}{2}(2s-1)e^{-t^2} + \frac{1}{2}, \quad \Big(\Rightarrow\ s = (x-\frac{1}{2})e^{t^2} + \frac{1}{2}\Big),\\
\frac{dy}{dt} &= 1 \quad\Rightarrow\quad y = t,\\
\frac{dz}{dt} &= \sin(\pi x) - y \ = \ \sin\Big(\frac{\pi}{2}(2s-1)e^{-t^2} + \frac{\pi}{2}\Big) - t.
\end{aligned}$$

$$\begin{aligned}
\Rightarrow\quad z(s,t) &= \int_0^t \Big[\sin\Big(\frac{\pi}{2}(2s-1)e^{-t^2} + \frac{\pi}{2}\Big) - t\Big]\,dt + z(s,0),\\
z(s,t) &= \int_0^t \Big[\sin\Big(\frac{\pi}{2}(2s-1)e^{-t^2} + \frac{\pi}{2}\Big) - t\Big]\,dt.\\
\Rightarrow\quad u(x,y) &= \int_0^y \Big[\sin\Big(\frac{\pi}{2}(2s-1)e^{-y^2} + \frac{\pi}{2}\Big) - y\Big]\,dy\\
&= \int_0^y \Big[\sin\Big(\frac{\pi}{2}(2x-1)e^{y^2}e^{-y^2} + \frac{\pi}{2}\Big) - y\Big]\,dy\\
&= \int_0^y \Big[\sin\Big(\frac{\pi}{2}(2x-1) + \frac{\pi}{2}\Big) - y\Big]\,dy = \int_0^y \Big[\sin(\pi x) - y\Big]\,dy,\\
\Rightarrow\quad u(x,y) &= y\sin(\pi x) - \frac{y^2}{2}.
\end{aligned}$$

Note: This solution does **not** satisfy the PDE. ☐

Problem (S'90, #8). *Consider the Cauchy problem*

$$\begin{aligned}
u_t &= xu_x - u, && -\infty < x < \infty,\ t \ge 0,\\
u(x,0) &= f(x), && f(x) \in C^\infty.
\end{aligned}$$

Assume that $f \equiv 0$ for $|x| \ge 1$.
Solve the equation by the method of characteristics and discuss the behavior of the solution.

Proof. Rewrite the equation as

$$u_y - xu_x = -u,$$

Γ is parameterized by $\Gamma:\ (s, 0, f(s))$.

$$\begin{aligned}
\frac{dx}{dt} &= -x \quad\Rightarrow\quad x = se^{-t}, \qquad \frac{dy}{dt} = 1 \quad\Rightarrow\quad y = t,\\
\frac{dz}{dt} &= -z \quad\Rightarrow\quad z = f(s)e^{-t}.
\end{aligned}$$

$$\Rightarrow\ u(x,y) = f(xe^y)e^{-y}.$$

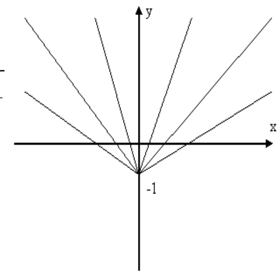
The solution satisfies the PDE and initial conditions.
As $y \to +\infty$, $u \to 0$. $\quad u = 0$ for $|xe^y| \geq 1 \quad \Rightarrow \quad u = 0$ for $|x| \geq \frac{1}{e^y}$. $\square$

Problem (F'02, #4). *Consider the nonlinear hyperbolic equation*

$$u_y + uu_x = 0 \qquad -\infty < x < \infty.$$

a) *Find a smooth solution to this equation for initial condition* $u(x,0) = x$.
b) *Describe the breakdown of smoothness for the solution if* $u(x,0) = -x$.

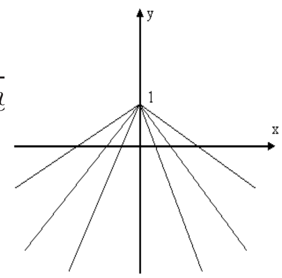
Proof. **a)** Γ is parameterized by $\Gamma: \ (s,0,s)$.



$$\frac{dx}{dt} = z = s \quad \Rightarrow \quad x = st + s \quad \Rightarrow \quad s = \frac{x}{t+1} = \frac{x}{y+1}$$
$$\frac{dy}{dt} = 1 \quad \Rightarrow \quad y = t,$$
$$\frac{dz}{dt} = 0 \quad \Rightarrow \quad z = s.$$

$$\Rightarrow \quad u(x,y) = \frac{x}{y+1}; \text{ solution is smooth for all positive time } y.$$

b) Γ is parameterized by $\Gamma: \ (s,0,-s)$.



$$\frac{dx}{dt} = z = -s \quad \Rightarrow \quad x = -st + s \ \Rightarrow \ s = \frac{x}{1-t} = \frac{x}{1-y}$$
$$\frac{dy}{dt} = 1 \quad \Rightarrow \quad y = t,$$
$$\frac{dz}{dt} = 0 \quad \Rightarrow \quad z = -s.$$

$$\Rightarrow \quad u(x,y) = \frac{x}{y-1}; \text{ solution blows up at time } y = 1.$$

□

Problem (F'97, #4). *Solve the initial-boundary value problem*

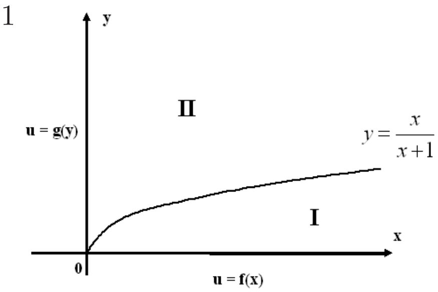
$$\begin{aligned} u_t + (x+1)^2 u_x &= x \quad \textit{for } \; x>0,\, t>0 \\ u(x,0) &= f(x) \quad\quad 0<x<+\infty \\ u(0,t) &= g(t) \quad\quad 0<t<+\infty. \end{aligned}$$

Proof. Rewrite the equation as

$$\begin{aligned} u_y + (x+1)^2 u_x &= x \quad \text{for } \; x>0,\, y>0 \\ u(x,0) &= f(x) \quad\quad 0<x<+\infty \\ u(0,y) &= g(y) \quad\quad 0<y<+\infty. \end{aligned}$$

• For region I, we solve the following characteristic equations with Γ is parameterized [19] by $\Gamma: \; (s, 0, f(s))$.

$$\begin{aligned} \frac{dx}{dt} &= (x+1)^2 \quad \Rightarrow \quad x = -\frac{s+1}{(s+1)t-1} - 1 \\ \frac{dy}{dt} &= 1 \quad \Rightarrow \quad y = t, \\ \frac{dz}{dt} &= x = -\frac{s+1}{(s+1)t-1} - 1, \\ &\Rightarrow \; z = -\ln|(s+1)t-1| - t + c_1(s), \\ &\Rightarrow \; z = -\ln|(s+1)t-1| - t + f(s). \end{aligned}$$



In region I, characteristics are of the form

$$x = -\frac{s+1}{(s+1)y-1} - 1.$$

Thus, region I is bounded above by the line

$$x = -\frac{1}{y-1} - 1, \qquad \text{or} \qquad y = \frac{x}{x+1}.$$

Since $t = y$, $s = \frac{x-xy-y}{xy+y+1}$, we have

$$\begin{aligned} u(x,y) &= -\ln\Big|\Big(\frac{x-xy-y}{xy+y+1}+1\Big)y - 1\Big| - y + f\Big(\frac{x-xy-y}{xy+y+1}\Big), \\ \Rightarrow u(x,y) &= -\ln\Big|\frac{-1}{xy+y+1}\Big| - y + f\Big(\frac{x-xy-y}{xy+y+1}\Big). \end{aligned}$$

• For region II, Γ is parameterized by $\Gamma: \; (0, s, g(s))$.

$$\begin{aligned} \frac{dx}{dt} &= (x+1)^2 \quad \Rightarrow \quad x = -\frac{1}{t-1} - 1, \\ \frac{dy}{dt} &= 1 \quad \Rightarrow \quad y = t+s, \\ \frac{dz}{dt} &= x = -\frac{1}{t-1} - 1, \\ &\Rightarrow \; z = -\ln|t-1| - t + c_2(s), \\ &\Rightarrow \; z = -\ln|t-1| - t + g(s). \end{aligned}$$

[19] Variable t as a third coordinate of u and variable t used to parametrize characteristic equations are two different entities.

Since $t = \frac{x}{x+1}$, $s = y - \frac{x}{x+1}$, we have

$$u(x,y) = -\ln\Big|\frac{x}{x+1} - 1\Big| - \frac{x}{x+1} + g\Big(y - \frac{x}{x+1}\Big).$$

Note that on $y = \frac{x}{x+1}$, both solutions are equal if $f(0) = g(0)$. $\square$

Problem (S'93, #3). *Solve the following equation*

$$u_t + u_x + yu_y = \sin t$$

for $0 \le t, \ \ 0 \le x, \ \ -\infty < y < \infty$ *and with*

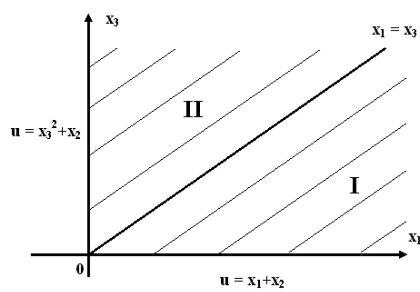
$$\begin{aligned} u &= x + y \qquad &&\text{for } t = 0,\ x \ge 0 \qquad \text{and} \\ u &= t^2 + y \qquad &&\text{for } x = 0,\ t \ge 0. \end{aligned}$$

Proof. Rewrite the equation as $(x \leftrightarrow x_1,\ y \leftrightarrow x_2,\ t \leftrightarrow x_3)$:

$$\begin{aligned} & u_{x_3} + u_{x_1} + x_2 u_{x_2} = \sin x_3 \qquad \text{for } \ 0 \le x_3,\ 0 \le x_1,\ -\infty < x_2 < \infty, \\ & u(x_1, x_2, 0) = x_1 + x_2, \\ & u(0, x_2, x_3) = x_3^2 + x_2. \end{aligned}$$

• For region I, we solve the following characteristic equations with Γ is parameterized[20] by $\Gamma:\ (s_1, s_2, 0, s_1 + s_2)$.

$$\begin{aligned} \frac{dx_1}{dt} &= 1 \quad \Rightarrow \quad x_1 = t + s_1, \\ \frac{dx_2}{dt} &= x_2 \quad \Rightarrow \quad x_2 = s_2 e^t, \\ \frac{dx_3}{dt} &= 1 \quad \Rightarrow \quad x_3 = t, \\ \frac{dz}{dt} &= \sin x_3 = \sin t \\ &\Rightarrow \ z = -\cos t + s_1 + s_2 + 1. \end{aligned}$$



Since in region I, in x_1x_3-plane, characteristics are of the form $x_1 = x_3 + s_1$, region I is bounded above by the line $x_1 = x_3$. Since $t = x_3,\ s_1 = x_1 - x_3,\ s_2 = x_2 e^{-x_3}$, we have

$$\begin{aligned} u(x_1, x_2, x_3) &= -\cos x_3 + x_1 - x_3 + x_2 e^{-x_3} + 1, \qquad \text{or} \\ u(x, y, t) &= -\cos t + x - t + ye^{-t} + 1, \qquad x \ge t. \end{aligned}$$

• For region II, we solve the following characteristic equations with Γ is parameterized by $\Gamma:\ (0, s_2, s_3, s_2 + s_3^2)$.

$$\begin{aligned} \frac{dx_1}{dt} &= 1 \quad \Rightarrow \quad x_1 = t, \\ \frac{dx_2}{dt} &= x_2 \quad \Rightarrow \quad x_2 = s_2 e^t, \\ \frac{dx_3}{dt} &= 1 \quad \Rightarrow \quad x_3 = t + s_3, \\ \frac{dz}{dt} &= \sin x_3 = \sin(t + s_3) \quad \Rightarrow \quad z = -\cos(t + s_3) + \cos s_3 + s_2 + s_3^2. \end{aligned}$$

Since $t = x_1,\ s_3 = x_3 - x_1,\ s_2 = x_2 e^{-x_3}$, we have

$$\begin{aligned} u(x_1, x_2, x_3) &= -\cos x_3 + \cos(x_3 - x_1) + x_2 e^{-x_3} + (x_3 - x_1)^2, \qquad \text{or} \\ u(x, y, t) &= -\cos t + \cos(t - x) + ye^{-t} + (t - x)^2, \qquad x \le t. \end{aligned}$$

Note that on $x = t$, both solutions are $u(x = t, y) = -\cos x + ye^{-x} + 1$. $\square$

[20] Variable t as a third coordinate of u and variable t used to parametrize characteristic equations are two different entities.

Problem (W'03, #5). *Find a solution to*

$$xu_x + (x+y)u_y = 1$$

which satisfies $u(1,y) = y$ *for* $0 \le y \le 1$. *Find a region in* $\{x \ge 0,\ y \ge 0\}$ *where* u *is uniquely determined by these conditions.*

Proof. Γ is parameterized by $\Gamma: (1, s, s)$.

$$\begin{aligned}
\frac{dx}{dt} &= x \quad \Rightarrow \quad x = e^t. \qquad \circledast \\
\frac{dy}{dt} &= x + y \quad \Rightarrow \quad y' - y = e^t. \\
\frac{dz}{dt} &= 1 \quad \Rightarrow \quad z = t + s.
\end{aligned}$$

The homogeneous solution for the second equation is $y_h(s,t) = c_1(s)e^t$. Since the right hand side and y_h are linearly dependent, our guess for the particular solution is $y_p(s,t) = c_2(s)te^t$. Plugging in y_p into the differential equation, we get

$$c_2(s)te^t + c_2(s)e^t - c_2(s)te^t = e^t \quad \Rightarrow \quad c_2(s) = 1.$$

Thus, $y_p(s,t) = te^t$ and

$$y(s,t) = y_h + y_p = c_1(s)e^t + te^t.$$

Since $y(s,0) = s = c_1(s)$, we get

$$y = se^t + te^t. \qquad \circledcirc$$

With $\circledast$ and $\circledcirc$, we can solve for s and t in terms of x and y to get

$$\begin{aligned}
&& t &= ln\, x, \\
y = sx + x\, ln\, x \quad &\Rightarrow & s &= \frac{y - x\, ln\, x}{x}.
\end{aligned}$$

$$u(x,y) = t + s = ln\, x + \frac{y - x\, ln\, x}{x}.$$

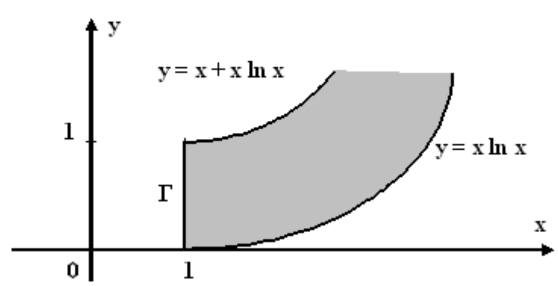
$$\boxed{u(x,y) = \frac{y}{x}.}$$

We have found that the characteristics in the xy-plane are of the form

$$y = sx + x\, ln\, x,$$

where s is such that $0 \le s \le 1$. Also, the characteristics originate from Γ.
Thus, u is **uniquely determined** in the region between the graphs:

$$\begin{aligned}
y &= x\, ln\, x, \\
y &= x + x\, ln\, x.
\end{aligned}$$



□

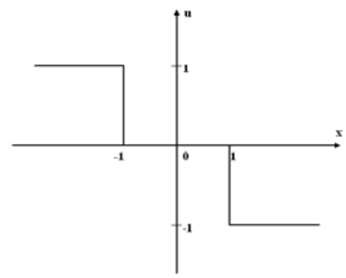
12 Problems: Shocks

Example 1. *Determine the exact solution to Burgers' equation*

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0, \qquad t > 0$$

with initial data

$$u(x,0) = h(x) = \begin{cases} 1 & \text{if } x < -1, \\ 0 & \text{if } -1 < x < 1, \\ -1 & \text{if } x > 1. \end{cases}$$



Proof. Characteristic form: $u_t + uu_x = 0$.
The characteristic projection in xt-plane passing through the point $(s,0)$ is the line

$$x = h(s)t + s.$$

- Rankine-Hugoniot shock condition at $\mathbf{s = -1}$:

$$\text{shock speed:} \qquad \xi'(t) = \frac{F(u_r) - F(u_l)}{u_r - u_l} = \frac{\frac{1}{2}u_r^2 - \frac{1}{2}u_l^2}{u_r - u_l} = \frac{0 - \frac{1}{2}}{0 - 1} = \frac{1}{2}.$$

The "1/slope" of the shock curve = 1/2. Thus,

$$x = \xi(t) = \frac{1}{2}t + s,$$

and since the jump occurs at $(-1,0)$, $\xi(0) = -1 = s$. Therefore,

$$x = \frac{1}{2}t - 1.$$

- Rankine-Hugoniot shock condition at $\mathbf{s = 1}$:

$$\text{shock speed:} \qquad \xi'(t) = \frac{F(u_r) - F(u_l)}{u_r - u_l} = \frac{\frac{1}{2}u_r^2 - \frac{1}{2}u_l^2}{u_r - u_l} = \frac{\frac{1}{2} - 0}{-1 - 0} = -\frac{1}{2}.$$

The "1/slope" of the shock curve = $-1/2$. Thus,

$$x = \xi(t) = -\frac{1}{2}t + s,$$

and since the jump occurs at $(1,0)$, $\xi(0) = 1 = s$. Therefore,

$$x = -\frac{1}{2}t + 1.$$

- At $\mathbf{t = 2}$, Rankine-Hugoniot shock condition at $\mathbf{s = 0}$:

$$\text{shock speed:} \qquad \xi'(t) = \frac{F(u_r) - F(u_l)}{u_r - u_l} = \frac{\frac{1}{2}u_r^2 - \frac{1}{2}u_l^2}{u_r - u_l} = \frac{\frac{1}{2} - \frac{1}{2}}{-1 - 1} = 0.$$

The "1/slope" of the shock curve = 0. Thus,

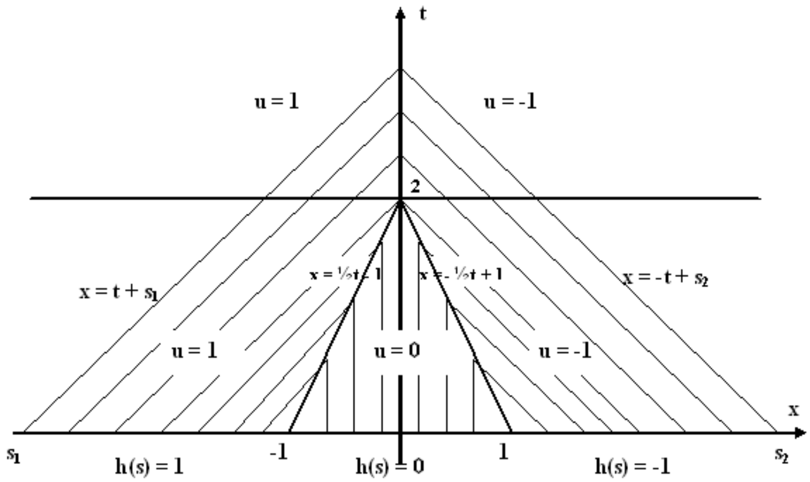
$$x = \xi(t) = s,$$

and since the jump occurs at $(x,t) = (0,2)$, $\xi(2) = 0 = s$. Therefore,

$$x = 0.$$

➡ For $t < 2$, $$u(x,t) = \begin{cases} 1 & \text{if } \ x < \frac{1}{2}t - 1, \\ 0 & \text{if } \ \frac{1}{2}t - 1 < x < -\frac{1}{2}t + 1, \\ -1 & \text{if } \ x > -\frac{1}{2}t + 1. \end{cases}$$

➡ and for $t > 2$, $$u(x,t) = \begin{cases} 1 & \text{if } \ x < 0, \\ -1 & \text{if } \ x > 0. \end{cases}$$



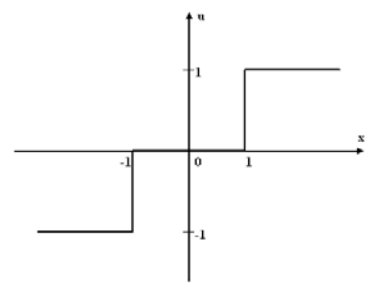
□

Example 2. *Determine the exact solution to Burgers' equation*

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0, \qquad t > 0$$

with initial data

$$u(x,0) = h(x) = \begin{cases} -1 & \text{if } \ x < -1, \\ 0 & \text{if } \ -1 < x < 1, \\ 1 & \text{if } \ x > 1. \end{cases}$$



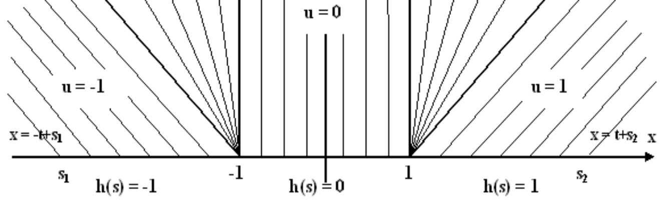
Proof. Characteristic form: $u_t + uu_x = 0$.
The characteristic projection in xt-plane passing through the point $(s,0)$ is the line

$$x = h(s)t + s.$$

For Burgers' equation, for a rarefaction fan emanating from $(s,0)$ on xt-plane, we have:

$$u(x,t) = \begin{cases} u_l, & \frac{x-s}{t} \le u_l, \\ \frac{x-s}{t}, & u_l \le \frac{x-s}{t} \le u_r, \\ u_r, & \frac{x-s}{t} \ge u_r. \end{cases}$$

$$\Rightarrow \ u(x,t) = \begin{cases} -1, & x < -t-1, \\ \frac{x+1}{t}, & -t-1 < x < -1, \qquad \text{i.e.} \quad -1 < \frac{x+1}{t} < 0 \\ 0, & -1 < x < 1, \\ \frac{x-1}{t}, & 1 < x < t+1, \qquad \text{i.e.} \quad 0 < \frac{x-1}{t} < 1 \\ 1, & x > t+1. \end{cases}$$



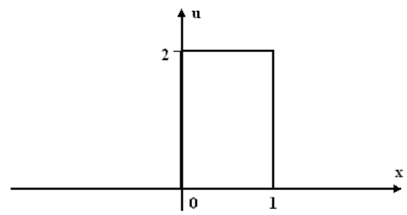
$\square$

Example 3. *Determine the exact solution to Burgers' equation*

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0, \qquad t > 0$$

with initial data

$$u(x,0) = h(x) = \begin{cases} 2 & \text{if } 0 < x < 1, \\ 0 & \text{if otherwise.} \end{cases}$$



Proof. Characteristic form: $u_t + uu_x = 0$.
The characteristic projection in xt-plane passing through the point $(s,0)$ is the line

$$x = h(s)t + s.$$

- **Shock:** Rankine-Hugoniot shock condition at $\mathbf{s = 1}$:

$$\text{shock speed:} \qquad \xi'(t) = \frac{F(u_r) - F(u_l)}{u_r - u_l} = \frac{\frac{1}{2}u_r^2 - \frac{1}{2}u_l^2}{u_r - u_l} = \frac{0-2}{0-2} = 1.$$

The "1/slope" of the shock curve = 1. Thus,

$$x = \xi(t) = t + s,$$

and since the jump occurs at $(1,0)$, $\xi(0) = 1 = s$. Therefore,

$$x = t + 1.$$

- **Rarefaction:** A rarefaction emanates from $(0,0)$ on xt-plane.

$$\text{➡ For } 0 < t < 1, \qquad u(x,t) = \begin{cases} 0 & \text{if } x < 0, \\ \frac{x}{t} & \text{if } 0 < x < 2t, \\ 2 & \text{if } 2t < x < t+1. \\ 0 & \text{if } x > t+1. \end{cases}$$

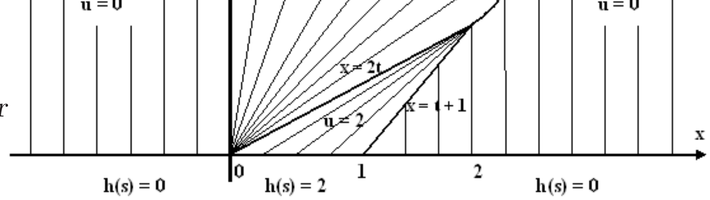
Rarefaction catches up to shock at $t = 1$.

- **Shock:** At $(\mathbf{x}, \mathbf{t}) = (\mathbf{2}, \mathbf{1})$, $u_l = x/t$, $u_r = 0$. Rankine-Hugoniot shock condition:

$$\begin{aligned} \xi'(t) &= \frac{F(u_r) - F(u_l)}{u_r - u_l} = \frac{\frac{1}{2}u_r^2 - \frac{1}{2}u_l^2}{u_r - u_l} = \frac{0 - \frac{1}{2}(\frac{x}{t})^2}{0 - \frac{x}{t}} = \frac{1}{2}\frac{x}{t}, \\ \frac{dx_s}{dt} &= \frac{x}{2t}, \\ x &= c\sqrt{t}, \end{aligned}$$

and since the jump occurs at $(x,t) = (2,1)$, $x(1) = 2 = c$. Therefore, $x = 2\sqrt{t}$.

$$\text{➡ For } t > 1, \qquad u(x,t) = \begin{cases} 0 & \text{if } x < 0, \\ \frac{x}{t} & \text{if } 0 < x < 2\sqrt{t}, \\ 0 & \text{if } x > 2\sqrt{t}. \end{cases}$$


u = 0
u = 0
x = 2t
u = 2
x = t + 1
x
0
1
2
h(s) = 0
h(s) = 2
h(s) = 0

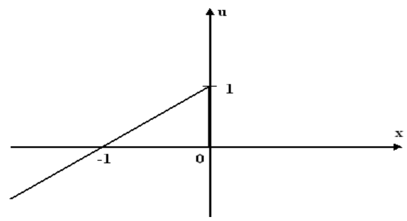
□

Example 4. *Determine the exact solution to Burgers' equation*

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0, \qquad t > 0$$

with initial data

$$u(x,0) = h(x) = \begin{cases} 1+x & \text{if } x < 0, \\ 0 & \text{if } x > 0. \end{cases}$$



Proof. Characteristic form: $u_t + uu_x = 0$.
The characteristic projection in xt-plane passing through the point $(s,0)$ is the line

$$x = h(s)t + s.$$

① For $s > 0$, the characteristics are $x = s$.
② For $s < 0$, the characteristics are $x = (1+s)t + s$.
• There are two ways to look for the solution on the left half-plane. One is to notice that the characteristic at $s = 0^-$ is $x = t$ and characteristic at $s = -1$ is $x = -1$ and that characteristics between $s = -\infty$ and $s = 0^-$ are intersecting at $(x,t) = (-1,-1)$. Also, for a fixed t, u is a linear function of x, i.e. for $t = 0$, $u = 1 + x$, allowing a continuous change of u with x. Thus, the solution may be viewed as an 'implicit' rarefaction, originating at $(-1,-1)$, thus giving rise to the solution

$$u(x,t) = \frac{x+1}{t+1}.$$

Another way to find a solution on the left half-plane is to solve ② for s to find

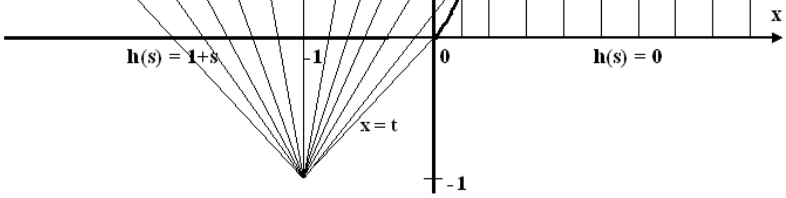
$$s = \frac{x-t}{1+t}. \qquad \text{Thus,} \quad u(x,t) = h(s) = 1 + s = 1 + \frac{x-t}{1+t} = \frac{x+1}{t+1}.$$

• **Shock:** At $(\mathbf{x},\mathbf{t}) = (\mathbf{0},\mathbf{0})$, $u_l = \frac{x+1}{t+1}$, $u_r = 0$. Rankine-Hugoniot shock condition:

$$\begin{aligned}
\xi'(t) &= \frac{F(u_r) - F(u_l)}{u_r - u_l} = \frac{\frac{1}{2}u_r^2 - \frac{1}{2}u_l^2}{u_r - u_l} = \frac{0 - \frac{1}{2}(\frac{x+1}{t+1})^2}{0 - \frac{x+1}{t+1}} = \frac{1}{2}\frac{x+1}{t+1}, \\
\frac{dx_s}{dt} &= \frac{1}{2}\frac{x+1}{t+1}, \\
x &= c\sqrt{t+1} - 1,
\end{aligned}$$

and since the jump occurs at $(x,t) = (0,0)$, $x(0) = 0 = c - 1$, or $c = 1$. Therefore, the shock curve is $x = \sqrt{t+1} - 1$.

$$\Rightarrow \quad u(x,t) = \begin{cases} \frac{x+1}{t+1} & \text{if } x < \sqrt{t+1} - 1, \\ 0 & \text{if } x > \sqrt{t+1} - 1. \end{cases}$$


x
h(s) = 1+s
-1
0
h(s) = 0
x = t
-1

∎

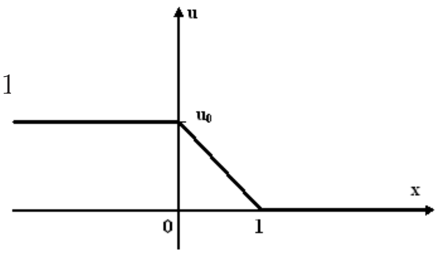
Example 5. *Determine the exact solution to Burgers' equation*

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0, \qquad t > 0$$

with initial data

$$u(x,0) = h(x) = \begin{cases} u_0 & \text{if } x < 0, \\ u_0 \cdot (1-x) & \text{if } 0 < x < 1 \\ 0 & \text{if } x \geq 1, \end{cases}$$

where $u_0 > 0$.



Proof. Characteristic form: $u_t + uu_x = 0$.
The characteristic projection in xt-plane passing through the point $(s,0)$ is the line

$$x = h(s)t + s.$$

① For $s > 1$, the characteristics are $x = s$.
② For $0 < s < 1$, the characteristics are $x = u_0(1-s)t + s$.
③ For $s < 0$, the characteristics are $x = u_0 t + s$.
The characteristics emanating from $(s,0)$, $0 < s < 1$ on xt-plane intersect at $(1, \frac{1}{u_0})$.
Also, we can check that the characteristics do not intersect before $t = \frac{1}{u_0}$ for this problem:

$$t_c = \min\left(\frac{-1}{h'(s)}\right) = \frac{1}{u_0}.$$

- To find solution in a triangular domain between $x = u_0 t$ and $x = 1$, we note that characteristics there are $x = u_0 \cdot (1-s)t + s$. Solving for s we get

$$s = \frac{x - u_0 t}{1 - u_0 t}. \qquad \text{Thus, } u(x,t) = h(s) = u_0 \cdot (1-s) = u_0 \cdot \left(1 - \frac{x - u_0 t}{1 - u_0 t}\right) = \frac{u_0 \cdot (1-x)}{1 - u_0 t}.$$

We can also find a solution in the triangular domain as follows. Note, that the characteristics are the straight lines

$$\frac{dx}{dt} = u = \text{const}.$$

Integrating the equation above, we obtain

$$x = ut + c$$

Since all characteristics in the triangular domain meet at $(1, \frac{1}{u_0})$, we have $c = 1 - \frac{u}{u_0}$, and

$$x = ut + \left(1 - \frac{u}{u_0}\right) \qquad \text{or} \qquad u = \frac{u_0 \cdot (1-x)}{1 - u_0 t}.$$

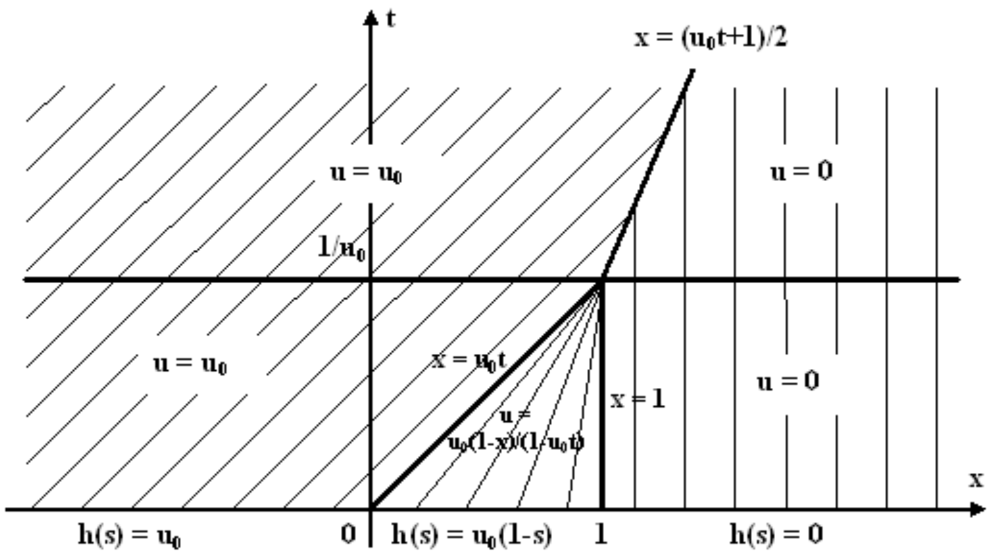
$$\Rightarrow \text{ For } 0 < t < \frac{1}{u_0}, \qquad u(x,t) = \begin{cases} u_0 & \text{if } \quad x < u_0 t, \\ \frac{u_0 \cdot (1-x)}{1-u_0 t} & \text{if } \quad u_0 t < x < 1, \\ 0 & \text{if } \quad x > 1. \end{cases}$$

- **Shock:** At $(\mathbf{x}, \mathbf{t}) = (\mathbf{1}, \frac{\mathbf{1}}{\mathbf{u_0}})$, Rankine-Hugoniot shock condition:

$$\begin{aligned}
\xi'(t) &= \frac{F(u_r) - F(u_l)}{u_r - u_l} = \frac{\frac{1}{2}u_r^2 - \frac{1}{2}u_l^2}{u_r - u_l} = \frac{0 - \frac{1}{2}u_0^2}{0 - u_0} = \frac{1}{2}u_0, \\
\xi(t) &= \frac{1}{2}u_0 t + c,
\end{aligned}$$

and since the jump occurs at $(x,t) = (1, \frac{1}{u_0})$, $x\big(\frac{1}{u_0}\big) = 1 = \frac{1}{2} + c$, or $c = \frac{1}{2}$. Therefore, the shock curve is $x = \frac{u_0 t + 1}{2}$.

➡ For $t > \dfrac{1}{u_0}$, $\quad u(x,t) = \begin{cases} u_0 & \text{if} \quad x < \frac{u_0 t+1}{2}, \\ 0 & \text{if} \quad x > \frac{u_0 t+1}{2}. \end{cases}$



□

Problem. *Show that for $u = f(x/t)$ to be a nonconstant solution of $u_t + a(u)u_x = 0$, f must be the inverse of the function a.*

Proof. If $u = f(x/t)$,

$$u_t = -f'\Big(\frac{x}{t}\Big) \cdot \frac{x}{t^2} \qquad \text{and} \qquad u_x = f'\Big(\frac{x}{t}\Big) \cdot \frac{1}{t}.$$

Hence, $u_t + a(u)u_x = 0$ implies that

$$-f'\Big(\frac{x}{t}\Big) \cdot \frac{x}{t^2} + a\Big(f\Big(\frac{x}{t}\Big)\Big)f'\Big(\frac{x}{t}\Big) \cdot \frac{1}{t} = 0$$

or, assuming f' is not identically 0 to rule out the constant solution, that

$$a\Big(f\Big(\frac{x}{t}\Big)\Big) = \frac{x}{t}.$$

This shows the functions a and f to be inverses of each other. □

13 Problems: General Nonlinear Equations

13.1 Two Spatial Dimensions

Problem (S'01, #3). *Solve the initial value problem*

$$\frac{1}{2}u_x^2 - u_y = -\frac{x^2}{2},$$
$$u(x,0) = x.$$

You will find that the solution blows up in finite time. Explain this in terms of the characteristics for this equation.

Proof. Rewrite the equation as

$$F(x,y,z,p,q) = \frac{p^2}{2} - q + \frac{x^2}{2} = 0.$$

Γ is parameterized by $\Gamma: \ (s, 0, s, \phi(s), \psi(s))$.
We need to complete Γ to a strip. Find $\phi(s)$ and $\psi(s)$, the initial conditions for $p(s,t)$ and $q(s,t)$, respectively:

- $$F(f(s), g(s), h(s), \phi(s), \psi(s)) = 0,$$
  $$F(s, 0, s, \phi(s), \psi(s)) = 0,$$
  $$\frac{\phi(s)^2}{2} - \psi(s) + \frac{s^2}{2} = 0,$$
  $$\psi(s) = \frac{\phi(s)^2 + s^2}{2}.$$

- $$h'(s) = \phi(s) f'(s) + \psi(s) g'(s),$$
  $$1 = \phi(s).$$
  $$\Rightarrow \quad \psi(s) = \frac{s^2+1}{2}.$$

Therefore, now Γ is parametrized by $\Gamma: \ (s, 0, s, 1, \frac{s^2+1}{2})$.

$$\begin{aligned}
\frac{dx}{dt} &= F_p = p, \\
\frac{dy}{dt} &= F_q = -1 \quad \Rightarrow \quad y(s,t) = -t + c_1(s) \quad \Rightarrow \quad y = -t, \\
\frac{dz}{dt} &= pF_p + qF_q = p^2 - q, \\
\frac{dp}{dt} &= -F_x - F_z p = -x, \\
\frac{dq}{dt} &= -F_y - F_z q = 0 \quad \Rightarrow \quad q(s,t) = c_2(s) \quad \Rightarrow \quad q = \frac{s^2+1}{2}.
\end{aligned}$$

Thus, we found y and q in terms of s and t. Note that we have a coupled system:

$$\begin{cases} x' = p, \\ p' = -x, \end{cases}$$

which can be written as two second order ODEs:

$$\begin{aligned}
x'' + x &= 0, \qquad x(s,0) = s, \qquad x'(s,0) = p(s,0) = 1, \\
p'' + p &= 0, \qquad p(s,0) = 1, \qquad p'(s,0) = -x(s,0) = -s.
\end{aligned}$$

Solving the two equations separately, we get

$$\begin{aligned}
x(s,t) &= s\cdot\cos t+\sin t,\\
p(s,t) &= \cos t - s\cdot \sin t.
\end{aligned}$$

From this, we get

$$\begin{aligned}
\frac{dz}{dt}=p^2-q &= \big(\cos t - s\cdot\sin t\big)^2-\frac{s^2+1}{2} \;=\; \cos^2 t-2s\cos t\sin t+s^2\sin^2 t-\frac{s^2+1}{2}.\\
z(s,t) &= \int_0^t \Big[\cos^2 t-2s\cos t\sin t+s^2\sin^2 t-\frac{s^2+1}{2}\Big]\,dt \;+\; z(s,0),\\
z(s,t) &= \Big[\frac{t}{2}+\frac{\sin t\cos t}{2}+s\cos^2 t+\frac{s^2 t}{2}-\frac{s^2\sin t\cos t}{2}-\frac{t(s^2+1)}{2}\Big]_0^t+s,\\
&= \Big[\frac{\sin t\cos t}{2}+s\cos^2 t-\frac{s^2\sin t\cos t}{2}\Big]_0^t+s,\\
&= \frac{\sin t\cos t}{2}+s\cos^2 t-\frac{s^2\sin t\cos t}{2}-s+s=\\
&= \frac{\sin t\cos t}{2}+s\cos^2 t-\frac{s^2\sin t\cos t}{2}.
\end{aligned}$$

Plugging in x and y found earlier for s and t, we get

$$\begin{aligned}
u(x,y) &= \frac{\sin(-y)\cos(-y)}{2}+\frac{x-\sin(-y)}{\cos(-y)}\cos^2(-y)-\frac{(x-\sin(-y))^2}{\cos^2(-y)}\cdot\frac{\sin(-y)\cos(-y)}{2}\\
&= -\frac{\sin y\cos y}{2}+\frac{x+\sin y}{\cos y}\cos^2 y+\frac{(x+\sin y)^2}{\cos^2 y}\cdot\frac{\sin y\cos y}{2}\\
&= -\frac{\sin y\cos y}{2}+(x+\sin y)\cos y+\frac{(x+\sin y)^2\sin y}{2\cos y}\\
&= x\cos y+\frac{\sin y\cos y}{2}+\frac{(x+\sin y)^2\sin y}{2\cos y}.
\end{aligned}$$

$\square$

Problem (S'98, #3). *Find the solution of*

$$u_t + \frac{u_x^2}{2} = \frac{-x^2}{2}, \qquad t \geq 0, \ -\infty < x < \infty$$
$$u(x,0) = h(x), \qquad -\infty < x < \infty,$$

where $h(x)$ is smooth function which vanishes for $|x|$ large enough.

Proof. Rewrite the equation as

$$F(x,y,z,p,q) = \frac{p^2}{2} + q + \frac{x^2}{2} = 0.$$

Γ is parameterized by $\Gamma: \ (s, 0, h(s), \phi(s), \psi(s))$.
We need to complete Γ to a strip. Find $\phi(s)$ and $\psi(s)$, the initial conditions for $p(s,t)$ and $q(s,t)$, respectively:

- $$F(f(s), g(s), h(s), \phi(s), \psi(s)) \ = \ 0,$$
  $$F(s, 0, h(s), \phi(s), \psi(s)) = 0,$$
  $$\frac{\phi(s)^2}{2} + \psi(s) + \frac{s^2}{2} = 0,$$
  $$\psi(s) = -\frac{\phi(s)^2 + s^2}{2}.$$

- $$h'(s) \ = \ \phi(s)f'(s) + \psi(s)g'(s),$$
  $$h'(s) = \phi(s).$$
  $$\Rightarrow \quad \psi(s) = -\frac{h'(s)^2 + s^2}{2}.$$

Therefore, now Γ is parametrized by $\Gamma: \ (s, 0, s, h'(s), -\frac{h'(s)^2+s^2}{2})$.

$$\frac{dx}{dt} = F_p = p,$$
$$\frac{dy}{dt} = F_q = 1 \quad \Rightarrow \quad y(s,t) = t + c_1(s) \quad \Rightarrow \quad y = t,$$
$$\frac{dz}{dt} = pF_p + qF_q = p^2 + q,$$
$$\frac{dp}{dt} = -F_x - F_z p = -x,$$
$$\frac{dq}{dt} = -F_y - F_z q = 0 \quad \Rightarrow \quad q(s,t) = c_2(s) \quad \Rightarrow \quad q = -\frac{h'(s)^2 + s^2}{2}.$$

Thus, we found y and q in terms of s and t. Note that we have a coupled system:

$$\begin{cases} x' = p, \\ p' = -x, \end{cases}$$

which can be written as a second order ODE:

$$x'' + x = 0, \qquad x(s,0) = s, \qquad x'(s,0) = \ p(s,0) \ = \ h'(s).$$

Solving the equation, we get

$$x(s,t) \ = \ s \cos t + h'(s) \sin t,$$
$$p(s,t) \ = \ x'(s,t) = h'(s) \cos t - s \sin t.$$

From this, we get

$$
\begin{aligned}
\frac{dz}{dt} = p^2 + q &= \big(h'(s)\cos t - s\sin t\big)^2 - \frac{h'(s)^2 + s^2}{2} \\
&= h'(s)^2\cos^2 t - 2sh'(s)\cos t\sin t + s^2\sin^2 t - \frac{h'(s)^2 + s^2}{2}. \\
z(s,t) &= \int_0^t \Big[h'(s)^2\cos^2 t - 2sh'(s)\cos t\sin t + s^2\sin^2 t - \frac{h'(s)^2 + s^2}{2}\Big]\,dt \;+\; z(s,0) \\
&= \int_0^t \Big[h'(s)^2\cos^2 t - 2sh'(s)\cos t\sin t + s^2\sin^2 t - \frac{h'(s)^2 + s^2}{2}\Big]\,dt \;+\; h(s).
\end{aligned}
$$

We integrate the above expression similar to $S'01\#3$ to get an expression for $z(s,t)$. Plugging in x and y found earlier for s and t, we get $u(x,y)$. $\square$

Problem (S'97, #4).
Describe the ***method of the bicharacteristics*** *for solving the initial value problem*

$$\begin{aligned}\Big(\frac{\partial}{\partial x}u(x,y)\Big)^2+\Big(\frac{\partial}{\partial y}u(x,y)\Big)^2 &= 2+y,\\ u(x,0)=u_0(x) &= x.\end{aligned}$$

Assume that $|\frac{\partial}{\partial x}u_0(x)|<2$ *and consider the solution such that* $\frac{\partial u}{\partial y}>0$.
Apply all general computations for the particular case $u_0(x)=x$.

Proof. We have

$$\begin{aligned}u_x^2+u_y^2 &= 2+y\\ u(x,0) &= u_0(x)=x.\end{aligned}$$

Rewrite the equation as

$$F(x,y,z,p,q)=p^2+q^2-y-2=0.$$

Γ is parameterized by $\Gamma:\ (s,0,s,\phi(s),\psi(s))$.
We need to complete Γ to a strip. Find $\phi(s)$ and $\psi(s)$, the initial conditions for $p(s,t)$ and $q(s,t)$, respectively:

- $$\begin{aligned}&F(f(s),g(s),h(s),\phi(s),\psi(s)) \ = \ 0,\\ &F(s,0,s,\phi(s),\psi(s))=0,\\ &\phi(s)^2+\psi(s)^2-2=0,\\ &\phi(s)^2+\psi(s)^2=2.\end{aligned}$$

- $$\begin{aligned}&h'(s) \ = \ \phi(s)f'(s)+\psi(s)g'(s),\\ &1=\phi(s).\\ &\Rightarrow \quad \psi(s)=\pm 1.\end{aligned}$$

Since we have a condition that $q(s,t)>0$, we choose $q(s,0)=\psi(s)=1$.
Therefore, now Γ is parametrized by $\Gamma:\ (s,0,s,1,1)$.

$$\begin{aligned}\frac{dx}{dt} &= F_p=2p \quad\Rightarrow\quad \frac{dx}{dt}=2 \quad\Rightarrow\quad x=2t+s,\\ \frac{dy}{dt} &= F_q=2q \quad\Rightarrow\quad \frac{dy}{dt}=2t+2 \quad\Rightarrow\quad y=t^2+2t,\\ \frac{dz}{dt} &= pF_p+qF_q=2p^2+2q^2=2y+4 \quad\Rightarrow\quad \frac{dz}{dt}=2t^2+4t+4,\\ &\Rightarrow\ z=\frac{2}{3}t^3+2t^2+4t+s=\frac{2}{3}t^3+2t^2+4t+x-2t=\frac{2}{3}t^3+2t^2+2t+x,\\ \frac{dp}{dt} &= -F_x-F_zp=0 \quad\Rightarrow\quad p=1,\\ \frac{dq}{dt} &= -F_y-F_zq=1 \quad\Rightarrow\quad q=t+1.\end{aligned}$$

We solve $y=t^2+2t$, a quadratic equation in t, $t^2+2t-y=0$, for t in terms of y to get:

$$\begin{aligned}&t=-1\pm\sqrt{1+y}.\\ \Rightarrow\quad &u(x,y)=\frac{2}{3}(-1\pm\sqrt{1+y})^3+2(-1\pm\sqrt{1+y})^2+2(-1\pm\sqrt{1+y})+x.\end{aligned}$$

Both $u_\pm$ satisfy the PDE. $u_x=1$, $u_y=\pm\sqrt{y+1} \quad\Rightarrow\quad u_x^2+u_y^2=y+2$ ✓
u_+ satisfies $u_+(x,0)=x$ ✓. However, u_- does not satisfy IC, i.e. $u_-(x,0)=x-\frac{4}{3}$. □

Problem (S'02, #6). *Consider the equation*

$$u_x + u_x u_y = 1,$$
$$u(x,0) = f(x).$$

Assuming that f is differentiable, what conditions on f insure that the problem is noncharacteristic? If f satisfies those conditions, show that the solution is

$$u(x,y) = f(r) - y + \frac{2y}{f'(r)},$$

where r must satisfy $y = (f'(r))^2(x-r)$.
Finally, show that one can solve the equation for (x,y) in a sufficiently small neighborhood of $(x_0,0)$ with $r(x_0,0) = x_0$.

Proof. **Solved**.
In order to solve the Cauchy problem in a neighborhood of Γ, need:

$$\begin{aligned} f'(s)\cdot F_q[f,g,h,\phi,\psi](s) - g'(s)\cdot F_p[f,g,h,\phi,\psi](s) &\neq 0, \\ 1\cdot h'(s) - 0\cdot\Big(1+\frac{1-h'(s)}{h'(s)}\Big) &\neq 0, \\ h'(s) &\neq 0. \end{aligned}$$

Thus, $h'(s)\neq 0$ ensures that the problem is noncharacteristic.
To show that one can solve $y = (f'(s))^2(x-s)$ for (x,y) in a sufficiently small neighborhood of $(x_0,0)$ with $s(x_0,0)=x_0$, let

$$\begin{aligned} G(x,y,s) &= (f'(s))^2(x-s) - y = 0, \\ G(x_0,0,x_0) &= 0, \\ G_r(x_0,0,x_0) &= -(f'(s))^2. \end{aligned}$$

Hence, if $f'(s)\neq 0$, $\forall s$, then $G_s(x_0,0,x_0)\neq 0$ and we can use the implicit function theorem in a neighborhood of $(x_0,0,x_0)$ to get

$$G(x,y,h(x,y)) = 0$$

and solve the equation in terms of x and y. $\square$

Problem (S'00, #1). *Find the solutions of*

$$(u_x)^2 + (u_y)^2 = 1$$

in a neighborhood of the curve $y = \frac{x^2}{2}$ *satisfying the conditions*

$$u\Big(x, \frac{x^2}{2}\Big) = 0 \qquad \textit{and} \qquad u_y\Big(x, \frac{x^2}{2}\Big) > 0.$$

Leave your answer in parametric form.

Proof. Rewrite the equation as

$$F(x, y, z, p, q) = p^2 + q^2 - 1 = 0.$$

Γ is parameterized by $\Gamma: \ (s, \frac{s^2}{2}, 0, \phi(s), \psi(s))$.
We need to complete Γ to a strip. Find $\phi(s)$ and $\psi(s)$, the initial conditions for $p(s,t)$ and $q(s,t)$, respectively:

- $$\begin{aligned} F(f(s), g(s), h(s), \phi(s), \psi(s)) &= 0, \\ F\big(s, \frac{s^2}{2}, 0, \phi(s), \psi(s)\big) &= 0, \\ \phi(s)^2 + \psi(s)^2 &= 1. \end{aligned}$$

- $$\begin{aligned} h'(s) &= \phi(s) f'(s) + \psi(s) g'(s), \\ 0 &= \phi(s) + s\psi(s), \\ \phi(s) &= -s\psi(s). \end{aligned}$$

$$\text{Thus,} \quad s^2\psi(s)^2 + \psi(s)^2 = 1 \quad \Rightarrow \quad \psi(s)^2 = \frac{1}{s^2+1}.$$

Since, by assumption, $\psi(s) > 0$, we have $\psi(s) = \frac{1}{\sqrt{s^2+1}}$.
Therefore, now Γ is parametrized by $\Gamma: \ \big(s, \frac{s^2}{2}, 0, \frac{-s}{\sqrt{s^2+1}}, \frac{1}{\sqrt{s^2+1}}\big)$.

$$\begin{aligned}
\frac{dx}{dt} &= F_p = 2p = \frac{-2s}{\sqrt{s^2+1}} \quad \Rightarrow \quad x = \frac{-2st}{\sqrt{s^2+1}} + s, \\
\frac{dy}{dt} &= F_q = 2q = \frac{2}{\sqrt{s^2+1}} \quad \Rightarrow \quad y = \frac{2t}{\sqrt{s^2+1}} + \frac{s^2}{2}, \\
\frac{dz}{dt} &= pF_p + qF_q = 2p^2 + 2q^2 = 2 \quad \Rightarrow \quad z = 2t, \\
\frac{dp}{dt} &= -F_x - F_z p = 0 \quad \Rightarrow \quad p = \frac{-s}{\sqrt{s^2+1}}, \\
\frac{dq}{dt} &= -F_y - F_z q = 0 \quad \Rightarrow \quad q = \frac{1}{\sqrt{s^2+1}}.
\end{aligned}$$

Thus, in parametric form,

$$\begin{aligned}
z(s,t) &= 2t, \\
x(s,t) &= \frac{-2st}{\sqrt{s^2+1}} + s, \\
y(s,t) &= \frac{2t}{\sqrt{s^2+1}} + \frac{s^2}{2}.
\end{aligned}$$

$\square$

13.2 Three Spatial Dimensions

Problem (S'96, #2). *Solve the following Cauchy problem*[21]*:*

$$\begin{aligned} u_x + u_y^2 + u_z^2 &= 1, \\ u(0,y,z) &= y \cdot z. \end{aligned}$$

Proof. Rewrite the equation as

$$\begin{aligned} &u_{x_1} + u_{x_2}^2 + u_{x_3}^2 = 1, \\ &u(0,x_2,x_3) = x_2 \cdot x_3. \end{aligned}$$

Write a general nonlinear equation

$$F(x_1,x_2,x_3,z,p_1,p_2,p_3) = p_1 + p_2^2 + p_3^2 - 1 = 0.$$

Γ is parameterized by

$$\Gamma : \Big(\underbrace{0}_{x_1(s_1,s_2,0)}, \underbrace{s_1}_{x_2(s_1,s_2,0)}, \underbrace{s_2}_{x_3(s_1,s_2,0)}, \underbrace{s_1 s_2}_{z(s_1,s_2,0)}, \underbrace{\phi_1(s_1,s_2)}_{p_1(s_1,s_2,0)}, \underbrace{\phi_2(s_1,s_2)}_{p_2(s_1,s_2,0)}, \underbrace{\phi_3(s_1,s_2)}_{p_3(s_1,s_2,0)} \Big)$$

We need to complete Γ to a strip. Find $\phi_1(s_1,s_2)$, $\phi_2(s_1,s_2)$, and $\phi_3(s_1,s_2)$, the initial conditions for $p_1(s_1,s_2,t)$, $p_2(s_1,s_2,t)$, and $p_3(s_1,s_2,t)$, respectively:

- $$\begin{aligned} &F\big(f_1(s_1,s_2), f_2(s_1,s_2), f_3(s_1,s_2), h(s_1,s_2), \phi_1, \phi_2, \phi_3\big) \;=\; 0, \\ &F\big(0, s_1, s_2, s_1 s_2, \phi_1, \phi_2, \phi_3\big) \;=\; \phi_1 + \phi_2^2 + \phi_3^2 - 1 \;=\; 0, \\ &\Rightarrow \quad \phi_1 + \phi_2^2 + \phi_3^2 = 1. \end{aligned}$$

- $$\begin{aligned} \frac{\partial h}{\partial s_1} &= \phi_1 \frac{\partial f_1}{\partial s_1} + \phi_2 \frac{\partial f_2}{\partial s_1} + \phi_3 \frac{\partial f_3}{\partial s_1}, \\ \Rightarrow \quad s_2 &= \phi_2. \end{aligned}$$

- $$\begin{aligned} \frac{\partial h}{\partial s_2} &= \phi_1 \frac{\partial f_1}{\partial s_2} + \phi_2 \frac{\partial f_2}{\partial s_2} + \phi_3 \frac{\partial f_3}{\partial s_2}, \\ \Rightarrow \quad s_1 &= \phi_3. \end{aligned}$$

Thus, we have: $\phi_2 = s_2$, $\phi_3 = s_1$, $\phi_1 = -s_1^2 - s_2^2 + 1$.

$$\Gamma : \Big(\underbrace{0}_{x_1(s_1,s_2,0)}, \underbrace{s_1}_{x_2(s_1,s_2,0)}, \underbrace{s_2}_{x_3(s_1,s_2,0)}, \underbrace{s_1 s_2}_{z(s_1,s_2,0)}, \underbrace{-s_1^2 - s_2^2 + 1}_{p_1(s_1,s_2,0)}, \underbrace{s_2}_{p_2(s_1,s_2,0)}, \underbrace{s_1}_{p_3(s_1,s_2,0)} \Big)$$

---

[21] This problem is very similar to an already hand-written solved problem F'95 #2.

The characteristic equations are

$$\begin{aligned}
\frac{dx_1}{dt} &= F_{p_1} = 1 \quad \Rightarrow \quad x_1 = t, \\
\frac{dx_2}{dt} &= F_{p_2} = 2p_2 \quad \Rightarrow \quad \frac{dx_2}{dt} = 2s_2 \quad \Rightarrow \quad x_2 = 2s_2 t + s_1, \\
\frac{dx_3}{dt} &= F_{p_3} = 2p_3 \quad \Rightarrow \quad \frac{dx_3}{dt} = 2s_1 \quad \Rightarrow \quad x_3 = 2s_1 t + s_2, \\
\frac{dz}{dt} &= p_1 F_{p_1} + p_2 F_{p_2} + p_3 F_{p_3} = p_1 + 2p_2^2 + 2p_3^2 = -s_1^2 - s_2^2 + 1 + 2s_2^2 + 2s_1^2 \\
&= s_1^2 + s_2^2 + 1 \quad \Rightarrow \quad z = (s_1^2 + s_2^2 + 1)t + s_1 s_2, \\
\frac{dp_1}{dt} &= -F_{x_1} - p_1 F_z = 0 \quad \Rightarrow \quad p_1 = -s_1^2 - s_2^2 + 1, \\
\frac{dp_2}{dt} &= -F_{x_2} - p_2 F_z = 0 \quad \Rightarrow \quad p_2 = s_2, \\
\frac{dp_3}{dt} &= -F_{x_3} - p_3 F_z = 0 \quad \Rightarrow \quad p_3 = s_1.
\end{aligned}$$

Thus, we have

$$\begin{cases} x_1 = t \\ x_2 = 2s_2 t + s_1 \\ x_3 = 2s_1 t + s_2 \\ z = (s_1^2 + s_2^2 + 1)t + s_1 s_2 \end{cases} \Rightarrow \begin{cases} t = x_1 \\ s_1 = x_2 - 2s_2 t \\ s_2 = x_3 - 2s_1 t \\ z = (s_1^2 + s_2^2 + 1)t + s_1 s_2 \end{cases} \Rightarrow \begin{cases} t = x_1 \\ s_1 = \frac{x_2 - 2x_1 x_3}{1 - 4x_1^2} \\ s_2 = \frac{x_3 - 2x_1 x_2}{1 - 4x_1^2} \\ z = (s_1^2 + s_2^2 + 1)t + s_1 s_2 \end{cases}$$

$$\Rightarrow \quad u(x_1, x_2, x_3) = \left[\left(\frac{x_2 - 2x_1 x_3}{1 - 4x_1^2}\right)^2 + \left(\frac{x_3 - 2x_1 x_2}{1 - 4x_1^2}\right)^2 + 1\right] x_1 + \left(\frac{x_2 - 2x_1 x_3}{1 - 4x_1^2}\right)\left(\frac{x_3 - 2x_1 x_2}{1 - 4x_1^2}\right).$$

□

Problem (F'95, #2). *Solve the following Cauchy problem*

$$\begin{aligned}
u_x + u_y + u_z^3 &= x + y + z, \\
u(x, y, 0) &= xy.
\end{aligned}$$

Proof. **Solved** □

Problem (S'94, #1). *Solve the following PDE for* $f(x, y, t)$*:*

$$f_t + xf_x + 3t^2 f_y = 0$$
$$f(x, y, 0) = x^2 + y^2.$$

Proof. Rewrite the equation as $(x \to x_1,\ y \to x_2,\ t \to x_3,\ f \to u)$:

$$x_1 u_{x_1} + 3x_3^2 u_{x_2} + u_{x_3} = 0,$$
$$u(x_1, x_2, 0) = x_1^2 + x_2^2.$$

$$F(x_1, x_2, x_3, z, p_1, p_2, p_3) = x_1 p_1 + 3x_3^2 p_2 + p_3 = 0.$$

Γ is parameterized by

$$\Gamma : \Big(\underbrace{s_1}_{x_1(s_1,s_2,0)}, \underbrace{s_2}_{x_2(s_1,s_2,0)}, \underbrace{0}_{x_3(s_1,s_2,0)}, \underbrace{s_1^2 + s_2^2}_{z(s_1,s_2,0)}, \underbrace{\phi_1(s_1, s_2)}_{p_1(s_1,s_2,0)}, \underbrace{\phi_2(s_1, s_2)}_{p_2(s_1,s_2,0)}, \underbrace{\phi_3(s_1, s_2)}_{p_3(s_1,s_2,0)} \Big)$$

We need to complete Γ to a strip. Find $\phi_1(s_1, s_2)$, $\phi_2(s_1, s_2)$, and $\phi_3(s_1, s_2)$, the initial conditions for $p_1(s_1, s_2, t)$, $p_2(s_1, s_2, t)$, and $p_3(s_1, s_2, t)$, respectively:

- $$\begin{aligned} F\big(f_1(s_1, s_2), f_2(s_1, s_2), f_3(s_1, s_2), h(s_1, s_2), \phi_1, \phi_2, \phi_3\big) &= 0, \\ F\big(s_1, s_2, 0, s_1^2 + s_2^2, \phi_1, \phi_2, \phi_3\big) &= s_1\phi_1 + \phi_3 = 0, \\ \Rightarrow \quad \phi_3 = s_1\phi_1. \end{aligned}$$
- $$\begin{aligned} \frac{\partial h}{\partial s_1} &= \phi_1 \frac{\partial f_1}{\partial s_1} + \phi_2 \frac{\partial f_2}{\partial s_1} + \phi_3 \frac{\partial f_3}{\partial s_1}, \\ \Rightarrow \quad 2s_1 &= \phi_1. \end{aligned}$$
- $$\begin{aligned} \frac{\partial h}{\partial s_2} &= \phi_1 \frac{\partial f_1}{\partial s_2} + \phi_2 \frac{\partial f_2}{\partial s_2} + \phi_3 \frac{\partial f_3}{\partial s_2}, \\ \Rightarrow \quad 2s_2 &= \phi_2. \end{aligned}$$

Thus, we have: $\phi_1 = 2s_1, \quad \phi_2 = 2s_2, \quad \phi_3 = 2s_1^2.$

$$\Gamma : \Big(\underbrace{s_1}_{x_1(s_1,s_2,0)}, \underbrace{s_2}_{x_2(s_1,s_2,0)}, \underbrace{0}_{x_3(s_1,s_2,0)}, \underbrace{s_1^2 + s_2^2}_{z(s_1,s_2,0)}, \underbrace{2s_1}_{p_1(s_1,s_2,0)}, \underbrace{2s_2}_{p_2(s_1,s_2,0)}, \underbrace{2s_1^2}_{p_3(s_1,s_2,0)} \Big)$$

The characteristic equations are

$$\begin{aligned}
\frac{dx_1}{dt} &= F_{p_1} = x_1 \quad \Rightarrow \quad x_1 = s_1 e^t, \\
\frac{dx_2}{dt} &= F_{p_2} = 3x_3^2 \quad \Rightarrow \quad \frac{dx_2}{dt} = 3t^2 \quad \Rightarrow \quad x_2 = t^3 + s_2, \\
\frac{dx_3}{dt} &= F_{p_3} = 1 \quad \Rightarrow \quad x_3 = t, \\
\frac{dz}{dt} &= p_1 F_{p_1} + p_2 F_{p_2} + p_3 F_{p_3} = p_1 x_1 + p_2 3x_3^2 + p_3 = 0 \quad \Rightarrow \quad z = s_1^2 + s_2^2, \\
\frac{dp_1}{dt} &= -F_{x_1} - p_1 F_z = -p_1 \quad \Rightarrow \quad p_1 = 2s_1 e^{-t}, \\
\frac{dp_2}{dt} &= -F_{x_2} - p_2 F_z = 0 \quad \Rightarrow \quad p_2 = 2s_2, \\
\frac{dp_3}{dt} &= -F_{x_3} - p_3 F_z = -6x_3 p_2 \quad \Rightarrow \quad \frac{dp_3}{dt} = -12ts_2 \quad \Rightarrow \quad p_3 = -6t^2 s_2 + 2s_1^2.
\end{aligned}$$

With $t = x_3,\ s_1 = x_1 e^{-x^3},\ s_2 = x_2 - x_3^3,$ we have

$$u(x_1, x_2, x_3) = x_1^2 e^{-2x^3} + (x_2 - x_3^3)^2. \qquad \Big(f(x, y, t) = x^2 e^{-2t} + (y - t^3)^2.\Big)$$

The solution satisfies the PDE and initial condition. $\square$

Problem (F'93, #3). *Find the solution of the following equation*

$$f_t + xf_x + (x+t)f_y = t^3$$
$$f(x,y,0) = xy.$$

Proof. Rewrite the equation as ($x \to x_1,\ y \to x_2,\ t \to x_3,\ f \to u$):

$$x_1 u_{x_1} + (x_1+x_3)u_{x_2} + u_{x_3} = x^3,$$
$$u(x_1,x_2,0) = x_1 x_2.$$

Method I: Treat the equation as a QUASILINEAR equation.
Γ is parameterized by $\Gamma:\ (s_1, s_2, 0, s_1 s_2)$.

$$\begin{aligned}
\frac{dx_1}{dt} &= x_1 \quad\Rightarrow\quad x_1 = s_1 e^t,\\
\frac{dx_2}{dt} &= x_1 + x_3 \quad\Rightarrow\quad \frac{dx_2}{dt} = s_1 e^t + t \quad\Rightarrow\quad x_2 = s_1 e^t + \frac{t^2}{2} + s_2 - s_1,\\
\frac{dx_3}{dt} &= 1 \quad\Rightarrow\quad x_3 = t,\\
\frac{dz}{dt} &= x_3^3 \quad\Rightarrow\quad \frac{dz}{dt} = t^3 \quad\Rightarrow\quad z = \frac{t^4}{4} + s_1 s_2.
\end{aligned}$$

Since $t = x_3,\ s_1 = x_1 e^{-x_3},\ s_2 = x_2 - s_1 e^t - \frac{t^2}{2} + s_1 = x_2 - x_1 - \frac{x_3^2}{2} + x_1 e^{-x_3}$, we have

$$\begin{aligned}
u(x_1,x_2,x_3) &= \frac{x_3^4}{4} + x_1 e^{-x_3}\Big(x_2 - x_1 - \frac{x_3^2}{2} + x_1 e^{-x_3}\Big), \qquad \text{or}\\
f(x,y,t) &= \frac{t^4}{4} + xe^{-t}\Big(y - x - \frac{t^2}{2} + xe^{-t}\Big).
\end{aligned}$$

The solution satisfies the PDE and initial condition.

Method II: Treat the equation as a fully NONLINEAR equation.

$$F(x_1,x_2,x_3,z,p_1,p_2,p_3) = x_1 p_1 + (x_1+x_3)p_2 + p_3 - x_3^3 = 0.$$

Γ is parameterized by

$$\Gamma:\ \Big(\underbrace{s_1}_{x_1(s_1,s_2,0)}, \underbrace{s_2}_{x_2(s_1,s_2,0)}, \underbrace{0}_{x_3(s_1,s_2,0)}, \underbrace{s_1 s_2}_{z(s_1,s_2,0)}, \underbrace{\phi_1(s_1,s_2)}_{p_1(s_1,s_2,0)}, \underbrace{\phi_2(s_1,s_2)}_{p_2(s_1,s_2,0)}, \underbrace{\phi_3(s_1,s_2)}_{p_3(s_1,s_2,0)}\Big)$$

We need to complete Γ to a strip. Find $\phi_1(s_1,s_2)$, $\phi_2(s_1,s_2)$, and $\phi_3(s_1,s_2)$, the initial conditions for $p_1(s_1,s_2,t)$, $p_2(s_1,s_2,t)$, and $p_3(s_1,s_2,t)$, respectively:

- $$\begin{aligned}
&F\big(f_1(s_1,s_2), f_2(s_1,s_2), f_3(s_1,s_2), h(s_1,s_2), \phi_1,\phi_2,\phi_3\big) = 0,\\
&F\big(s_1, s_2, 0, s_1 s_2, \phi_1,\phi_2,\phi_3\big) = s_1\phi_1 + s_1\phi_2 + \phi_3 = 0,\\
&\Rightarrow \quad \phi_3 = -s_1(\phi_1+\phi_2).
\end{aligned}$$
- $$\begin{aligned}
&\frac{\partial h}{\partial s_1} = \phi_1\frac{\partial f_1}{\partial s_1} + \phi_2\frac{\partial f_2}{\partial s_1} + \phi_3\frac{\partial f_3}{\partial s_1},\\
&\Rightarrow \quad s_2 = \phi_1.
\end{aligned}$$
- $$\begin{aligned}
&\frac{\partial h}{\partial s_2} = \phi_1\frac{\partial f_1}{\partial s_2} + \phi_2\frac{\partial f_2}{\partial s_2} + \phi_3\frac{\partial f_3}{\partial s_2},\\
&\Rightarrow \quad s_1 = \phi_2.
\end{aligned}$$

Thus, we have: $\phi_1 = s_2,\quad \phi_2 = s_1,\quad \phi_3 = -s_1^2 - s_1 s_2.$

$$\Gamma:\ \Big(\underbrace{s_1}_{x_1(s_1,s_2,0)}, \underbrace{s_2}_{x_2(s_1,s_2,0)}, \underbrace{0}_{x_3(s_1,s_2,0)}, \underbrace{s_1 s_2}_{z(s_1,s_2,0)}, \underbrace{s_2}_{p_1(s_1,s_2,0)}, \underbrace{s_1}_{p_2(s_1,s_2,0)}, \underbrace{-s_1^2 - s_1 s_2}_{p_3(s_1,s_2,0)}\Big)$$

The characteristic equations are

$$
\begin{aligned}
\frac{dx_1}{dt} &= F_{p_1} = x_1 \quad \Rightarrow \quad x_1 = s_1 e^t, \\
\frac{dx_2}{dt} &= F_{p_2} = x_1 + x_3 \quad \Rightarrow \quad \frac{dx_2}{dt} = s_1 e^t + t \quad \Rightarrow \quad x_2 = s_1 e^t + \frac{t^2}{2} + s_2 - s_1, \\
\frac{dx_3}{dt} &= F_{p_3} = 1 \quad \Rightarrow \quad x_3 = t, \\
\frac{dz}{dt} &= p_1 F_{p_1} + p_2 F_{p_2} + p_3 F_{p_3} = p_1 x_1 + p_2(x_1 + x_3) + p_3 = x_3^3 = t^3 \quad \Rightarrow \quad z = \frac{t^4}{4} + s_1 s_2, \\
\frac{dp_1}{dt} &= -F_{x_1} - p_1 F_z = -p_1 - p_2 = -p_1 - s_1 \quad \Rightarrow \quad p_1 = 2s_1 e^{-t} - s_1, \\
\frac{dp_2}{dt} &= -F_{x_2} - p_2 F_z = 0 \quad \Rightarrow \quad p_2 = s_1, \\
\frac{dp_3}{dt} &= -F_{x_3} - p_3 F_z = 3x_3^2 - p_2 = 3t^2 - s_1 \quad \Rightarrow \quad p_3 = t^3 - s_1 t - s_1^2 - s_1 s_2.
\end{aligned}
$$

With $t = x_3,\ s_1 = x_1 e^{-x^3},\ s_2 = x_2 - s_1 e^t - \frac{t^2}{2} + s_1 = x_2 - x_1 - \frac{x_3^2}{2} + x_1 e^{-x^3},$ we have

$$
\begin{aligned}
u(x_1, x_2, x_3) &= \frac{x_3^4}{4} + x_1 e^{-x^3}\Big(x_2 - x_1 - \frac{x_3^2}{2} + x_1 e^{-x^3}\Big), \qquad \text{or} \\
f(x, y, t) &= \frac{t^4}{4} + x e^{-t}\Big(y - x - \frac{t^2}{2} + x e^{-t}\Big).
\end{aligned}
$$

[22] The solution satisfies the PDE and initial condition. $\square$

[22] Variable t in the derivatives of characteristics equations and t in the solution $f(x, y, t)$ are different entities.

Problem (F'92, #1). *Solve the initial value problem*

$$\begin{aligned} u_t + \alpha u_x + \beta u_y + \gamma u = 0 \qquad & \textit{for} \;\; t > 0 \\ u(x,y,0) = \varphi(x,y), & \end{aligned}$$

in which α, β and γ are real constants and φ is a smooth function.

Proof. Rewrite the equation as $(x \to x_1,\; y \to x_2,\; t \to x_3)$[23]:

$$\begin{aligned} \alpha u_{x_1} + \beta u_{x_2} + u_{x_3} &= -\gamma u, \\ u(x_1, x_2, 0) &= \varphi(x_1, x_2). \end{aligned}$$

Γ is parameterized by $\Gamma: \; (s_1, s_2, 0, \varphi(s_1, s_2))$.

$$\begin{aligned} \frac{dx_1}{dt} &= \alpha \quad \Rightarrow \quad x_1 = \alpha t + s_1, \\ \frac{dx_2}{dt} &= \beta \quad \Rightarrow \quad x_2 = \beta t + s_2, \\ \frac{dx_3}{dt} &= 1 \quad \Rightarrow \quad x_3 = t, \\ \frac{dz}{dt} &= -\gamma z \quad \Rightarrow \quad \frac{dz}{z} = -\gamma dt \quad \Rightarrow \quad z = \varphi(s_1, s_2) e^{-\gamma t}. \end{aligned}$$

$$J \equiv \det\left(\frac{\partial(x_1, x_2, x_3)}{\partial(s_1, s_2, t)}\right) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \alpha & \beta & 1 \end{vmatrix} = 1 \neq 0 \qquad \Rightarrow \qquad J \text{ is invertible.}$$

Since $t = x_3,\; s_1 = x_1 - \alpha x_3,\; s_2 = x_2 - \beta x_3$, we have

$$\begin{aligned} u(x_1, x_2, x_3) &= \varphi(x_1 - \alpha x_3, x_2 - \beta x_3) e^{-\gamma x_3}, \qquad \text{or} \\ u(x, y, t) &= \varphi(x - \alpha t, y - \beta t) e^{-\gamma t}. \end{aligned}$$

The solution satisfies the PDE and initial condition.[24] $\square$

[23] Variable t as a third coordinate of u and variable t used to parametrize characteristic equations are two different entities.

[24] Chain Rule: $u(x_1, x_2, x_3) = \varphi(f(x_1, x_2, x_3), g(x_1, x_2, x_3))$, then $u_{x_1} = \frac{\partial \varphi}{\partial f}\frac{\partial f}{\partial x_1} + \frac{\partial \varphi}{\partial g}\frac{\partial g}{\partial x_1}$.

Problem (F'94, #2). *Find the solution of the Cauchy problem*

$$
\begin{aligned}
u_t(x,y,t) + au_x(x,y,t) + bu_y(x,y,t) + c(x,y,t)u(x,y,t) = 0 \\
u(x,y,0) = u_0(x,y),
\end{aligned}
$$

where $0 < t < +\infty, \quad -\infty < x < +\infty, \quad -\infty < y < +\infty,$
a, b *are constants,* $c(x,y,t)$ *is a continuous function of* (x,y,t)*, and* $u_0(x,y)$ *is a continuous function of* (x,y).

Proof. Rewrite the equation as $(x \to x_1,\ y \to x_2,\ t \to x_3)$:

$$
\begin{aligned}
au_{x_1} + bu_{x_2} + u_{x_3} &= -c(x_1,x_2,x_3)u, \\
u(x_1,x_2,0) &= u_0(x_1,x_2).
\end{aligned}
$$

Γ is parameterized by $\Gamma:\ (s_1, s_2, 0, u_0(s_1,s_2))$.

$$
\begin{aligned}
\frac{dx_1}{dt} &= a \quad \Rightarrow \quad x_1 = at + s_1, \\
\frac{dx_2}{dt} &= b \quad \Rightarrow \quad x_2 = bt + s_2, \\
\frac{dx_3}{dt} &= 1 \quad \Rightarrow \quad x_3 = t, \\
\frac{dz}{dt} &= -c(x_1,x_2,x_3)z \quad \Rightarrow \quad \frac{dz}{dt} = -c(at+s_1, bt+s_2, t)z \quad \Rightarrow \quad \frac{dz}{z} = -c(at+s_1, bt+s_2, t)dt \\
&\Rightarrow \ \ln z = -\int_0^t c(a\xi + s_1, b\xi + s_2, \xi)d\xi + c_1(s_1,s_2), \\
&\Rightarrow \ z(s_1,s_2,t) = c_2(s_1,s_2)e^{-\int_0^t c(a\xi+s_1,b\xi+s_2,\xi)d\xi} \quad \Rightarrow \quad z(s_1,s_2,0) = c_2(s_1,s_2) = u_0(s_2,s_2), \\
&\Rightarrow \ z(s_1,s_2,t) = u_0(s_1,s_2)e^{-\int_0^t c(a\xi+s_1,b\xi+s_2,\xi)d\xi}.
\end{aligned}
$$

$$
J \equiv \det\left(\frac{\partial(x_1,x_2,x_3)}{\partial(s_1,s_2,t)}\right) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & 1 \end{vmatrix} = 1 \neq 0 \quad \Rightarrow \quad J \text{ is invertible.}
$$

Since $t = x_3,\ s_1 = x_1 - ax_3,\ s_2 = x_2 - bx_3$, we have

$$
\begin{aligned}
u(x_1,x_2,x_3) &= u_0(x_1 - ax_3, x_2 - bx_3)e^{-\int_0^{x_3} c(a\xi + x_1 - ax_3, b\xi + x_2 - bx_3, \xi)d\xi} \\
&= u_0(x_1 - ax_3, x_2 - bx_3)e^{-\int_0^{x_3} c(x_1 + a(\xi - x_3), x_2 + b(\xi - x_3), \xi)d\xi}, \qquad \text{or} \\
u(x,y,t) &= u_0(x - at, y - bt)e^{-\int_0^t c(x + a(\xi - t), y + b(\xi - t), \xi)d\xi}.
\end{aligned}
$$

$\square$

Problem (F'89, #4). *Consider the first order partial differential equation*

$$u_t + (\alpha + \beta t)u_x + \gamma e^t u_y = 0 \tag{13.1}$$

in which α, β and γ are constants.
a) *For this equation, solve the initial value problem with initial data*

$$u(x, y, t = 0) = \sin(xy) \tag{13.2}$$

for all x and y and for $t \geq 0$.
b) *Suppose that this initial data is prescribed only for $x \geq 0$ (and all y) and consider (13.1) in the region $x \geq 0$, $t \geq 0$ and all y. For which values of α, β and γ is it possible to solve the initial-boundary value problem (13.1), (13.2) with $u(x = 0, y, t)$ given for $t \geq 0$?*
For non-permissible values of α, β and γ, where can boundary values be prescribed in order to determine a solution of (13.1) in the region ($x \geq 0$, $t \geq 0$, all y).

Proof. **a)** Rewrite the equation as ($x \to x_1$, $y \to x_2$, $t \to x_3$):

$$\begin{aligned}(\alpha + \beta x_3)u_{x_1} + \gamma e^{x_3} u_{x_2} + u_{x_3} = 0,\\ u(x_1, x_2, 0) = \sin(x_1 x_2).\end{aligned}$$

Γ is parameterized by $\Gamma: \ (s_1, s_2, 0, \sin(s_1 s_2))$.

$$\begin{aligned}
\frac{dx_1}{dt} &= \alpha + \beta x_3 \quad \Rightarrow \quad \frac{dx_1}{dt} = \alpha + \beta t \quad \Rightarrow \quad x_1 = \frac{\beta t^2}{2} + \alpha t + s_1,\\
\frac{dx_2}{dt} &= \gamma e^{x_3} \quad \Rightarrow \quad \frac{dx_2}{dt} = \gamma e^t \quad \Rightarrow \quad x_2 = \gamma e^t - \gamma + s_2,\\
\frac{dx_3}{dt} &= 1 \quad \Rightarrow \quad x_3 = t,\\
\frac{dz}{dt} &= 0 \quad \Rightarrow \quad z = \sin(s_1 s_2).
\end{aligned}$$

$$J \equiv \det\left(\frac{\partial(x_1, x_2, x_3)}{\partial(s_1, s_2, t)}\right) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \beta t + \alpha & \gamma e^t & 1 \end{vmatrix} = 1 \neq 0 \qquad \Rightarrow \qquad J \text{ is invertible.}$$

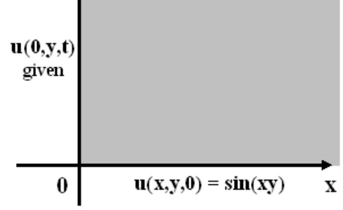
Since $t = x_3$, $s_1 = x_1 - \frac{\beta x_3^2}{2} - \alpha x_3$, $s_2 = x_2 - \gamma e^{x_3} + \gamma$, we have

$$\begin{aligned}
u(x_1, x_2, x_3) &= \sin((x_1 - \frac{\beta x_3^2}{2} - \alpha x_3)(x_2 - \gamma e^{x_3} + \gamma)), \qquad \text{or}\\
u(x, y, t) &= \sin((x - \frac{\beta t^2}{2} - \alpha t)(y - \gamma e^t + \gamma)).
\end{aligned}$$

The solution satisfies the PDE and initial condition.

b) We need a *compatibility condition* between the initial and boundary values to hold on y-axis ($x = 0, t = 0$):

$$\begin{aligned}
u(x = 0, y, 0) &= u(0, y, t = 0),\\
0 &= 0.
\end{aligned}$$



$\square$

14 Problems: First-Order Systems

Problem (S'01, #2a). *Find the solution* $u = \begin{pmatrix} u_1(x,t) \\ u_2(x,t) \end{pmatrix}$, $(x,t) \in \mathbb{R} \times \mathbb{R}$, *to the (strictly) hyperbolic equation*

$$u_t - \begin{pmatrix} 1 & 0 \\ 5 & 3 \end{pmatrix} u_x = 0,$$

satisfying $\begin{pmatrix} u_1(x,0) \\ u_2(x,0) \end{pmatrix} = \begin{pmatrix} e^{ixa} \\ 0 \end{pmatrix}$, $a \in \mathbb{R}$.

Proof. Rewrite the equation as

$$U_t + \begin{pmatrix} -1 & 0 \\ -5 & -3 \end{pmatrix} U_x = 0,$$
$$U(x,0) = \begin{pmatrix} u^{(1)}(x,0) \\ u^{(2)}(x,0) \end{pmatrix} = \begin{pmatrix} e^{ixa} \\ 0 \end{pmatrix}.$$

The eigenvalues of the matrix A are $\lambda_1 = -1$, $\lambda_2 = -3$ and the corresponding eigenvectors are $e_1 = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Thus,

$$\Lambda = \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix}, \qquad \Gamma = \begin{pmatrix} 2 & 0 \\ -5 & 1 \end{pmatrix}, \qquad \Gamma^{-1} = \frac{1}{\det \Gamma} \cdot \Gamma = \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{5}{2} & 1 \end{pmatrix}.$$

Let $U = \Gamma V$. Then,

$$U_t + AU_x = 0,$$
$$\Gamma V_t + A\Gamma V_x = 0,$$
$$V_t + \Gamma^{-1} A \Gamma V_x = 0,$$
$$V_t + \Lambda V_x = 0.$$

Thus, the transformed problem is

$$V_t + \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix} V_x = 0,$$
$$V(x,0) = \Gamma^{-1} U(x,0) = \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{5}{2} & 1 \end{pmatrix} \begin{pmatrix} e^{ixa} \\ 0 \end{pmatrix} = \frac{1}{2} e^{ixa} \begin{pmatrix} 1 \\ 5 \end{pmatrix}.$$

We have two initial value problems

$$\begin{cases} v_t^{(1)} - v_x^{(1)} = 0, \\ v^{(1)}(x,0) = \frac{1}{2} e^{ixa}; \end{cases} \qquad \begin{cases} v_t^{(2)} - 3v_x^{(2)} = 0, \\ v^{(2)}(x,0) = \frac{5}{2} e^{ixa}, \end{cases}$$

which we solve by characteristics to get

$$v^{(1)}(x,t) = \frac{1}{2} e^{ia(x+t)}, \qquad v^{(2)}(x,t) = \frac{5}{2} e^{ia(x+3t)}.$$

We solve for U: $\quad U = \Gamma V = \Gamma \begin{pmatrix} v^{(1)} \\ v^{(2)} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -5 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} e^{ia(x+t)} \\ \frac{5}{2} e^{ia(x+3t)} \end{pmatrix}.$

Thus, $\quad U = \begin{pmatrix} u^{(1)}(x,t) \\ u^{(2)}(x,t) \end{pmatrix} = \begin{pmatrix} e^{ia(x+t)} \\ -\frac{5}{2} e^{ia(x+t)} + \frac{5}{2} e^{ia(x+3t)} \end{pmatrix}.$

Can check that this is the correct solution by plugging it into the original equation. $\square$

Part (b) of the problem is solved in the Fourier Transform section.

Problem (S'96, #7). *Solve the following initial-boundary value problem in the domain $x > 0$, $t > 0$, for the unknown vector $U = \begin{pmatrix} u^{(1)} \\ u^{(2)} \end{pmatrix}$:*

$$U_t + \begin{pmatrix} -2 & 3 \\ 0 & 1 \end{pmatrix} U_x = 0. \tag{14.1}$$

$$U(x,0) = \begin{pmatrix} \sin x \\ 0 \end{pmatrix} \quad \textit{and} \quad u^{(2)}(0,t) = t.$$

Proof. The eigenvalues of the matrix A are $\lambda_1 = -2$, $\lambda_2 = 1$ and the corresponding eigenvectors are $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Thus,

$$\Lambda = \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \Gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \Gamma^{-1} = \frac{1}{\det \Gamma} \cdot \Gamma = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Let $U = \Gamma V$. Then,

$$\begin{aligned} &U_t + AU_x = 0, \\ &\Gamma V_t + A\Gamma V_x = 0, \\ &V_t + \Gamma^{-1} A \Gamma V_x = 0, \\ &V_t + \Lambda V_x = 0. \end{aligned}$$

Thus, the transformed problem is

$$V_t + \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix} V_x = 0, \tag{14.2}$$

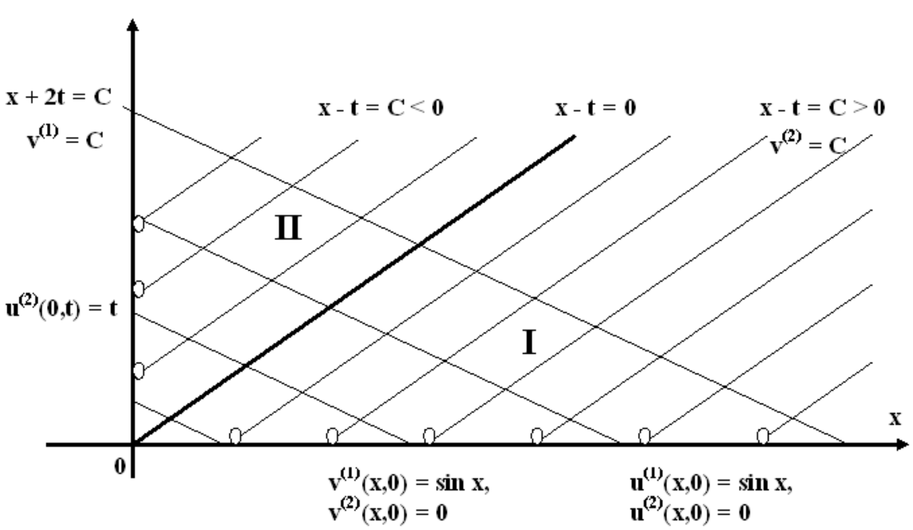
$$V(x,0) = \Gamma^{-1} U(x,0) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sin x \\ 0 \end{pmatrix} = \begin{pmatrix} \sin x \\ 0 \end{pmatrix}. \tag{14.3}$$

Equation (14.2) gives **traveling wave solutions** of the form

$$v^{(1)}(x,t) = F(x+2t), \qquad v^{(2)}(x,t) = G(x-t).$$

We can write U in terms of V:

$$U = \Gamma V = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v^{(1)} \\ v^{(2)} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} F(x+2t) \\ G(x-t) \end{pmatrix} = \begin{pmatrix} F(x+2t) + G(x-t) \\ G(x-t) \end{pmatrix}. \tag{14.4}$$



• For region I, (14.2) and (14.3) give two initial value problems (since any point in region I can be traced back along both characteristics to initial conditions):

$$\begin{cases} v_t^{(1)} - 2v_x^{(1)} = 0, \\ v^{(1)}(x,0) = \sin x; \end{cases} \qquad \begin{cases} v_t^{(2)} + v_x^{(2)} = 0, \\ v^{(2)}(x,0) = 0. \end{cases}$$

which we solve by characteristics to get *traveling wave solutions*:

$$v^{(1)}(x,t) = \sin(x+2t), \qquad v^{(2)}(x,t) = 0.$$

➡ Thus, for region I,

$$\boxed{U = \Gamma V = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sin(x+2t) \\ 0 \end{pmatrix} = \begin{pmatrix} \sin(x+2t) \\ 0 \end{pmatrix}.}$$

• For region II, solutions of the form $F(x+2t)$ can be traced back to initial conditions. Thus, $v^{(1)}$ is the same as in region I. Solutions of the form $G(x-t)$ can be traced back to the boundary. Since from (14.4), $u^{(2)} = v^{(2)}$, we use boundary conditions to get

$$u^{(2)}(0,t) = t = G(-t).$$

Hence, $G(x-t) = -(x-t)$.
➡ Thus, for region II,

$$\boxed{U = \Gamma V = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sin(x+2t) \\ -(x-t) \end{pmatrix} = \begin{pmatrix} \sin(x+2t) - (x-t) \\ -(x-t) \end{pmatrix}.}$$

Solutions for regions I and II satisfy (14.1).
Solution for region I satisfies both initial conditions.
Solution for region II satisfies given boundary condition. □

Problem (S'02, #7). *Consider the system*

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 2 & 2 \end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix} u \\ v \end{pmatrix}. \tag{14.5}$$

Find an explicit solution for the following mixed problem for the system (14.5):

$$\begin{aligned}\begin{pmatrix} u(x,0) \\ v(x,0) \end{pmatrix} &= \begin{pmatrix} f(x) \\ 0 \end{pmatrix} \quad \text{for} \ \ x>0, \\ u(0,t) &= 0 \qquad \text{for} \ \ t>0.\end{aligned}$$

You may assume that the function f is smooth and vanishes on a neighborhood of $x=0$.

Proof. Rewrite the equation as

$$\begin{aligned}&U_t + \begin{pmatrix} 1 & -2 \\ -2 & -2 \end{pmatrix} U_x = 0, \\ &U(x,0) = \begin{pmatrix} u^{(1)}(x,0) \\ u^{(2)}(x,0) \end{pmatrix} = \begin{pmatrix} f(x) \\ 0 \end{pmatrix}.\end{aligned}$$

The eigenvalues of the matrix A are $\lambda_1 = -3$, $\lambda_2 = 2$ and the corresponding eigenvectors are $e_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $e_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$. Thus,

$$\Lambda = \begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix}, \qquad \Gamma = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}, \qquad \Gamma^{-1} = \frac{1}{\det\Gamma}\cdot\Gamma = \frac{1}{5}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}.$$

Let $U = \Gamma V$. Then,

$$\begin{aligned}&U_t + AU_x = 0, \\ &\Gamma V_t + A\Gamma V_x = 0, \\ &V_t + \Gamma^{-1}A\Gamma V_x = 0, \\ &V_t + \Lambda V_x = 0.\end{aligned}$$

Thus, the transformed problem is

$$V_t + \begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix} V_x = 0, \tag{14.6}$$

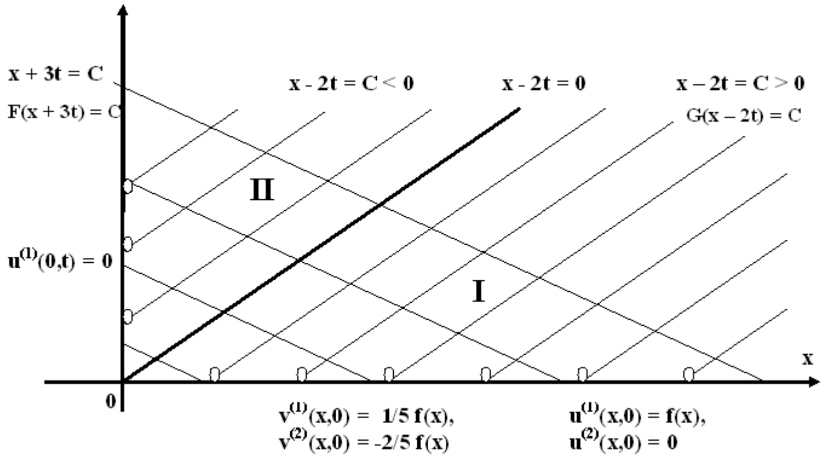
$$V(x,0) = \Gamma^{-1}U(x,0) = \frac{1}{5}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} f(x) \\ 0 \end{pmatrix} = \frac{f(x)}{5}\begin{pmatrix} 1 \\ -2 \end{pmatrix}. \tag{14.7}$$

Equation (14.6) gives **traveling wave solutions** of the form:

$$v^{(1)}(x,t) = F(x+3t), \qquad v^{(2)}(x,t) = G(x-2t). \tag{14.8}$$

We can write U in terms of V:

$$U = \Gamma V = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} v^{(1)} \\ v^{(2)} \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} F(x+3t) \\ G(x-2t) \end{pmatrix} = \begin{pmatrix} F(x+3t) - 2G(x-2t) \\ 2F(x+3t) + G(x-2t) \end{pmatrix}. \tag{14.9}$$



• For region I, (14.6) and (14.7) give two initial value problems (since value at any point in region I can be traced back along both characteristics to initial conditions):

$$\begin{cases} v_t^{(1)} - 3v_x^{(1)} = 0, \\ v^{(1)}(x,0) = \frac{1}{5}f(x); \end{cases} \qquad \begin{cases} v_t^{(2)} + 2v_x^{(2)} = 0, \\ v^{(2)}(x,0) = -\frac{2}{5}f(x). \end{cases}$$

which we solve by characteristics to get *traveling wave solutions*:

$$v^{(1)}(x,t) = \frac{1}{5}f(x+3t), \qquad v^{(2)}(x,t) = -\frac{2}{5}f(x-2t).$$

➡ Thus, for region I, $U = \Gamma V = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{5}f(x+3t) \\ -\frac{2}{5}f(x-2t) \end{pmatrix} = \begin{pmatrix} \frac{1}{5}f(x+3t) + \frac{4}{5}f(x-2t) \\ \frac{2}{5}f(x+3t) - \frac{2}{5}f(x-2t) \end{pmatrix}$.

• For region II, solutions of the form $F(x+3t)$ can be traced back to initial conditions. Thus, $v^{(1)}$ is the same as in region I. Solutions of the form $G(x-2t)$ can be traced back to the boundary. Since from (14.9),

$$u^{(1)} = v^{(1)} - 2v^{(2)}, \qquad \text{we have}$$

$$u^{(1)}(x,t) = F(x+3t) - 2G(x-2t) = \frac{1}{5}f(x+3t) - 2G(x-2t).$$

The boundary condition gives

$$u^{(1)}(0,t) = 0 = \frac{1}{5}f(3t) - 2G(-2t),$$

$$2G(-2t) = \frac{1}{5}f(3t),$$

$$G(t) = \frac{1}{10}f\Big(-\frac{3}{2}t\Big),$$

$$G(x-2t) = \frac{1}{10}f\Big(-\frac{3}{2}(x-2t)\Big).$$

➡ Thus, for region II, $U = \Gamma V = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{5}f(x+3t) \\ \frac{1}{10}f(-\frac{3}{2}(x-2t)) \end{pmatrix} = \begin{pmatrix} \frac{1}{5}f(x+3t) - \frac{1}{5}f(-\frac{3}{2}(x-2t)) \\ \frac{2}{5}f(x+3t) + \frac{1}{10}f(-\frac{3}{2}(x-2t)) \end{pmatrix}$.

Solutions for regions I and II satisfy (14.5).
Solution for region I satisfies both initial conditions.
Solution for region II satisfies given boundary condition. $\square$

Problem (F'94, #1; S'97, #7). *Solve the initial-boundary value problem*

$$u_t + 3v_x = 0,$$
$$v_t + u_x + 2v_x = 0$$

in the quarter plane $0 \le x, t < \infty$, *with initial conditions* [25]

$$u(x,0) = \varphi_1(x), \quad v(x,0) = \varphi_2(x), \quad 0 < x < +\infty$$

and boundary condition

$$u(0,t) = \psi(t), \quad t > 0.$$

Proof. Rewrite the equation as $U_t + AU_x = 0$:

$$U_t + \begin{pmatrix} 0 & 3 \\ 1 & 2 \end{pmatrix} U_x = 0, \tag{14.10}$$

$$U(x,0) = \begin{pmatrix} u^{(1)}(x,0) \\ u^{(2)}(x,0) \end{pmatrix} = \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix}.$$

The eigenvalues of the matrix A are $\lambda_1 = -1$, $\lambda_2 = 3$ and the corresponding eigenvectors are $e_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$, $e_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Thus,

$$\Lambda = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}, \qquad \Gamma = \begin{pmatrix} -3 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \Gamma^{-1} = \frac{1}{\det \Gamma} \cdot \Gamma = \frac{1}{4} \begin{pmatrix} -1 & 1 \\ 1 & 3 \end{pmatrix}.$$

Let $U = \Gamma V$. Then,

$$U_t + AU_x = 0,$$
$$\Gamma V_t + A\Gamma V_x = 0,$$
$$V_t + \Gamma^{-1} A \Gamma V_x = 0,$$
$$V_t + \Lambda V_x = 0.$$

Thus, the transformed problem is

$$V_t + \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix} V_x = 0, \tag{14.11}$$

$$V(x,0) = \Gamma^{-1} U(x,0) = \frac{1}{4} \begin{pmatrix} -1 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix} = \frac{1}{4} \begin{pmatrix} -\varphi_1(x) + \varphi_2(x) \\ \varphi_1(x) + 3\varphi_2(x) \end{pmatrix}. \tag{14.12}$$

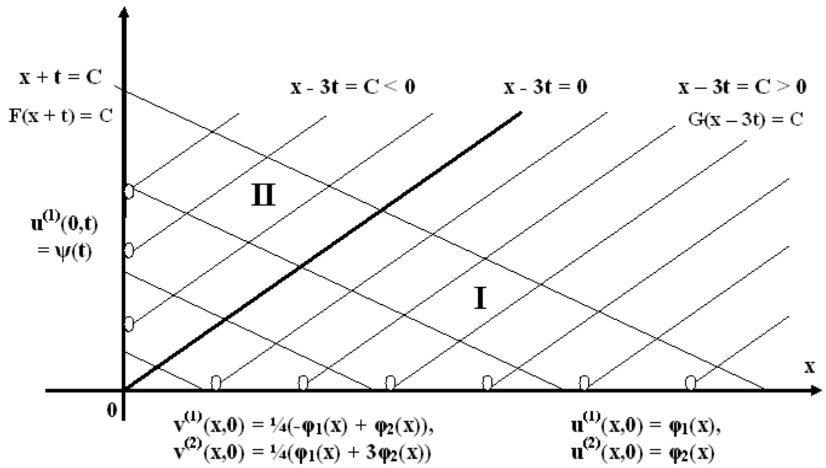
Equation (14.11) gives **traveling wave solutions** of the form:

$$v^{(1)}(x,t) = F(x+t), \qquad v^{(2)}(x,t) = G(x-3t). \tag{14.13}$$

We can write U in terms of V:

$$U = \Gamma V = \begin{pmatrix} -3 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} v^{(1)} \\ v^{(2)} \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} F(x+t) \\ G(x-3t) \end{pmatrix} = \begin{pmatrix} -3F(x+t) + G(x-3t) \\ F(x+t) + G(x-3t) \end{pmatrix}. \tag{14.14}$$

[25] In S'97, #7, the zero initial conditions are considered.



• For region I, (14.11) and (14.12) give two initial value problems (since value at any point in region I can be traced back along characteristics to initial conditions):

$$\begin{cases} v_t^{(1)} - v_x^{(1)} = 0, \\ v^{(1)}(x,0) = -\frac{1}{4}\varphi_1(x) + \frac{1}{4}\varphi_2(x); \end{cases} \qquad \begin{cases} v_t^{(2)} + 3v_x^{(2)} = 0, \\ v^{(2)}(x,0) = \frac{1}{4}\varphi_1(x) + \frac{3}{4}\varphi_2(x), \end{cases}$$

which we solve by characteristics to get *traveling wave solutions*:

$$v^{(1)}(x,t) = -\frac{1}{4}\varphi_1(x+t) + \frac{1}{4}\varphi_2(x+t), \qquad v^{(2)}(x,t) = \frac{1}{4}\varphi_1(x-3t) + \frac{3}{4}\varphi_2(x-3t).$$

➡ Thus, for region I,

$$\begin{aligned} U &= \Gamma V = \begin{pmatrix} -3 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{4}\varphi_1(x+t) + \frac{1}{4}\varphi_2(x+t) \\ \frac{1}{4}\varphi_1(x-3t) + \frac{3}{4}\varphi_2(x-3t) \end{pmatrix} \\ &= \frac{1}{4}\begin{pmatrix} 3\varphi_1(x+t) - 3\varphi_2(x+t) + \varphi_1(x-3t) + 3\varphi_2(x-3t) \\ -\varphi_1(x+t) + \varphi_2(x+t) + \varphi_1(x-3t) + 3\varphi_2(x-3t) \end{pmatrix}. \end{aligned}$$

• For region II, solutions of the form $F(x+t)$ can be traced back to initial conditions. Thus, $v^{(1)}$ is the same as in region I. Solutions of the form $G(x-3t)$ can be traced back to the boundary. Since from (14.14),

$$u^{(1)} = -3v^{(1)} + v^{(2)}, \qquad \text{we have}$$

$$u^{(1)}(x,t) = \frac{3}{4}\varphi_1(x+t) - \frac{3}{4}\varphi_2(x+t) + G(x-3t).$$

The boundary condition gives

$$\begin{aligned} &u^{(1)}(0,t) = \psi(t) = \frac{3}{4}\varphi_1(t) - \frac{3}{4}\varphi_2(t) + G(-3t), \\ &G(-3t) = \psi(t) - \frac{3}{4}\varphi_1(t) + \frac{3}{4}\varphi_2(t), \\ &G(t) = \psi\Big(-\frac{t}{3}\Big) - \frac{3}{4}\varphi_1\Big(-\frac{t}{3}\Big) + \frac{3}{4}\varphi_2\Big(-\frac{t}{3}\Big), \\ &G(x-3t) = \psi\Big(-\frac{x-3t}{3}\Big) - \frac{3}{4}\varphi_1\Big(-\frac{x-3t}{3}\Big) + \frac{3}{4}\varphi_2\Big(-\frac{x-3t}{3}\Big). \end{aligned}$$

➡ Thus, for region II,

$$\begin{aligned} U &= \Gamma V = \begin{pmatrix} -3 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{4}\varphi_1(x+t) + \frac{1}{4}\varphi_2(x+t) \\ \psi(-\frac{x-3t}{3}) - \frac{3}{4}\varphi_1(-\frac{x-3t}{3}) + \frac{3}{4}\varphi_2(-\frac{x-3t}{3}) \end{pmatrix} \\ &= \begin{pmatrix} \frac{3}{4}\varphi_1(x+t) - \frac{3}{4}\varphi_2(x+t) + \psi(-\frac{x-3t}{3}) - \frac{3}{4}\varphi_1(-\frac{x-3t}{3}) + \frac{3}{4}\varphi_2(-\frac{x-3t}{3}) \\ -\frac{1}{4}\varphi_1(x+t) + \frac{1}{4}\varphi_2(x+t) + \psi(-\frac{x-3t}{3}) - \frac{3}{4}\varphi_1(-\frac{x-3t}{3}) + \frac{3}{4}\varphi_2(-\frac{x-3t}{3}) \end{pmatrix}. \end{aligned}$$

Solutions for regions I and II satisfy (14.10).
Solution for region I satisfies both initial conditions.
Solution for region II satisfies given boundary condition. $\square$

Problem (F'91, #1). *Solve explicitly the following initial-boundary value problem for linear 2×2 hyperbolic system*

$$\begin{aligned} u_t &= u_x + v_x \\ v_t &= 3u_x - v_x, \end{aligned}$$

where $0 < t < +\infty$, $0 < x < +\infty$ *with initial conditions*

$$u(x,0) = u_0(x), \qquad v(x,0) = v_0(x), \qquad 0 < x < +\infty,$$

and the boundary condition

$$u(0,t) + bv(0,t) = \varphi(t), \qquad 0 < t < +\infty,$$

where $b \neq \frac{1}{3}$ *is a constant.*
What happens when $b = \frac{1}{3}$*?*

Proof. Let us change the notation ($u \leftrightarrow u^{(1)}$, $v \leftrightarrow u^{(2)}$). Rewrite the equation as

$$U_t + \begin{pmatrix} -1 & -1 \\ -3 & 1 \end{pmatrix} U_x = 0, \tag{14.15}$$

$$U(x,0) = \begin{pmatrix} u^{(1)}(x,0) \\ u^{(2)}(x,0) \end{pmatrix} = \begin{pmatrix} u_0^{(1)}(x) \\ u_0^{(2)}(x) \end{pmatrix}.$$

The eigenvalues of the matrix A are $\lambda_1 = -2$, $\lambda_2 = 2$ and the corresponding eigenvectors are $e_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $e_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$. Thus,

$$\Lambda = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}, \qquad \Gamma = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}, \qquad \Gamma^{-1} = \frac{1}{4}\begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}.$$

Let $U = \Gamma V$. Then,

$$\begin{aligned} &U_t + AU_x = 0, \\ &\Gamma V_t + A\Gamma V_x = 0, \\ &V_t + \Gamma^{-1}A\Gamma V_x = 0, \\ &V_t + \Lambda V_x = 0. \end{aligned}$$

Thus, the transformed problem is

$$V_t + \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix} V_x = 0, \tag{14.16}$$

$$V(x,0) = \Gamma^{-1}U(x,0) = \frac{1}{4}\begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} u^{(1)}(x,0) \\ u^{(2)}(x,0) \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 3u_0^{(1)}(x) + u_0^{(2)}(x) \\ u_0^{(1)}(x) - u_0^{(2)}(x) \end{pmatrix}. \tag{14.17}$$

Equation (14.16) gives **traveling wave solutions** of the form:

$$v^{(1)}(x,t) = F(x+2t), \qquad v^{(2)}(x,t) = G(x-2t). \tag{14.18}$$

We can write U in terms of V:

$$U = \Gamma V = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} v^{(1)} \\ v^{(2)} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} F(x+2t) \\ G(x-2t) \end{pmatrix} = \begin{pmatrix} F(x+2t) + G(x-2t) \\ F(x+2t) - 3G(x-2t) \end{pmatrix}. \tag{14.19}$$

• For region I, (14.16) and (14.17) give two initial value problems (since value at any point in region I can be traced back along characteristics to initial conditions):

$$\begin{cases} v_t^{(1)} - 2v_x^{(1)} = 0, \\ v^{(1)}(x,0) = \frac{3}{4}u_0^{(1)}(x) + \frac{1}{4}u_0^{(2)}(x); \end{cases} \qquad \begin{cases} v_t^{(2)} + 2v_x^{(2)} = 0, \\ v^{(2)}(x,0) = \frac{1}{4}u_0^{(1)}(x) - \frac{1}{4}u_0^{(2)}(x), \end{cases}$$

which we solve by characteristics to get *traveling wave solutions*:

$$v^{(1)}(x,t) = \frac{3}{4}u_0^{(1)}(x+2t) + \frac{1}{4}u_0^{(2)}(x+2t); \qquad v^{(2)}(x,t) = \frac{1}{4}u_0^{(1)}(x-2t) - \frac{1}{4}u_0^{(2)}(x-2t).$$

➡ Thus, for region I,

$$\begin{aligned} U &= \Gamma V = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} \frac{3}{4}u_0^{(1)}(x+2t) + \frac{1}{4}u_0^{(2)}(x+2t) \\ \frac{1}{4}u_0^{(1)}(x-2t) - \frac{1}{4}u_0^{(2)}(x-2t) \end{pmatrix} \\ &= \begin{pmatrix} \frac{3}{4}u_0^{(1)}(x+2t) + \frac{1}{4}u_0^{(2)}(x+2t) + \frac{1}{4}u_0^{(1)}(x-2t) - \frac{1}{4}u_0^{(2)}(x-2t) \\ \frac{3}{4}u_0^{(1)}(x+2t) + \frac{1}{4}u_0^{(2)}(x+2t) - \frac{3}{4}u_0^{(1)}(x-2t) + \frac{3}{4}u_0^{(2)}(x-2t) \end{pmatrix}. \end{aligned}$$

• For region II, solutions of the form $F(x+2t)$ can be traced back to initial conditions. Thus, $v^{(1)}$ is the same as in region I. Solutions of the form $G(x-2t)$ can be traced back to the boundary. The boundary condition gives

$$u^{(1)}(0,t) + bu^{(2)}(0,t) = \varphi(t).$$

Using (14.19),

$$\begin{aligned} & v^{(1)}(0,t) + G(-2t) + bv^{(1)}(0,t) - 3bG(-2t) = \varphi(t), \\ & (1+b)v^{(1)}(0,t) + (1-3b)G(-2t) = \varphi(t), \\ & (1+b)\Big(\frac{3}{4}u_0^{(1)}(2t) + \frac{1}{4}u_0^{(2)}(2t)\Big) + (1-3b)G(-2t) = \varphi(t), \\ & G(-2t) = \frac{\varphi(t) - (1+b)\Big(\frac{3}{4}u_0^{(1)}(2t) + \frac{1}{4}u_0^{(2)}(2t)\Big)}{1-3b}, \\ & G(t) = \frac{\varphi(-\frac{t}{2}) - (1+b)\Big(\frac{3}{4}u_0^{(1)}(-t) + \frac{1}{4}u_0^{(2)}(-t)\Big)}{1-3b}, \\ & G(x-2t) = \frac{\varphi(-\frac{x-2t}{2}) - (1+b)\Big(\frac{3}{4}u_0^{(1)}(-(x-2t)) + \frac{1}{4}u_0^{(2)}(-(x-2t))\Big)}{1-3b}. \end{aligned}$$

➡ Thus, for region II,

$$\begin{aligned} U &= \Gamma V = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} \frac{3}{4}u_0^{(1)}(x+2t) + \frac{1}{4}u_0^{(2)}(x+2t) \\ \frac{\varphi(-\frac{x-2t}{2}) - (1+b)\left(\frac{3}{4}u_0^{(1)}(-(x-2t)) + \frac{1}{4}u_0^{(2)}(-(x-2t))\right)}{1-3b} \end{pmatrix} \\ &= \begin{pmatrix} \frac{3}{4}u_0^{(1)}(x+2t) + \frac{1}{4}u_0^{(2)}(x+2t) + \frac{\varphi(-\frac{x-2t}{2}) - (1+b)\left(\frac{3}{4}u_0^{(1)}(-(x-2t)) + \frac{1}{4}u_0^{(2)}(-(x-2t))\right)}{1-3b} \\ \frac{3}{4}u_0^{(1)}(x+2t) + \frac{1}{4}u_0^{(2)}(x+2t) - \frac{3\varphi(-\frac{x-2t}{2}) - 3(1+b)\left(\frac{3}{4}u_0^{(1)}(-(x-2t)) + \frac{1}{4}u_0^{(2)}(-(x-2t))\right)}{1-3b} \end{pmatrix}. \end{aligned}$$

The following were performed, but are arithmetically complicated:
Solutions for regions I and II satisfy (14.15).

Solution for region I satisfies both initial conditions.
Solution for region II satisfies given boundary condition.

If $b = \frac{1}{3}$, $u^{(1)}(0,t) + \frac{1}{3}u^{(2)}(0,t) = F(2t) + G(-2t) + \frac{1}{3}F(2t) - G(-2t) = \frac{4}{3}F(2t) = \varphi(t)$. Thus, the solutions of the form $v^{(2)} = G(x - 2t)$ are not defined at $x = 0$, which leads to ill-posedness. $\square$

Problem (F'96, #8). *Consider the system*

$$\begin{aligned} u_t &= 3u_x + 2v_x \\ v_t &= -v_x - v \end{aligned}$$

in the region $x \geq 0$, $t \geq 0$. Which of the following sets of initial and boundary data make this a well-posed problem?

$$\begin{aligned}
&\textbf{a)} \quad u(x,0) = 0, \quad x \geq 0 \\
&\qquad\;\; v(x,0) = x^2, \quad x \geq 0 \\
&\qquad\;\; v(0,t) = t^2, \quad t \geq 0. \\
&\textbf{b)} \quad u(x,0) = 0, \quad x \geq 0 \\
&\qquad\;\; v(x,0) = x^2, \quad x \geq 0 \\
&\qquad\;\; u(0,t) = t, \quad t \geq 0. \\
&\textbf{c)} \quad u(x,0) = 0, \quad x \geq 0 \\
&\qquad\;\; v(x,0) = x^2, \quad x \geq 0 \\
&\qquad\;\; u(0,t) = t, \quad t \geq 0 \\
&\qquad\;\; v(0,t) = t^2, \quad t \geq 0.
\end{aligned}$$

Proof. Rewrite the equation as $U_t + AU_x = BU$. Initial conditions are same for (a),(b),(c):

$$\begin{aligned}
&U_t + \begin{pmatrix} -3 & -2 \\ 0 & 1 \end{pmatrix} U_x = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} U, \\
&U(x,0) = \begin{pmatrix} u^{(1)}(x,0) \\ u^{(2)}(x,0) \end{pmatrix} = \begin{pmatrix} 0 \\ x^2 \end{pmatrix}.
\end{aligned}$$

The eigenvalues of the matrix A are $\lambda_1 = -3$, $\lambda_2 = 1$, and the corresponding eigenvectors are $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$. Thus,

$$\Lambda = \begin{pmatrix} -3 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \Gamma = \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix}, \qquad \Gamma^{-1} = \frac{1}{2}\begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}.$$

Let $U = \Gamma V$. Then,

$$\begin{aligned}
&U_t + AU_x = BU, \\
&\Gamma V_t + A\Gamma V_x = B\Gamma V, \\
&V_t + \Gamma^{-1}A\Gamma V_x = \Gamma^{-1}B\Gamma V, \\
&V_t + \Lambda V_x = \Gamma^{-1}B\Gamma V.
\end{aligned}$$

Thus, the transformed problem is

$$V_t + \begin{pmatrix} -3 & 0 \\ 0 & 1 \end{pmatrix} V_x = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} V, \tag{14.20}$$

$$V(x,0) = \Gamma^{-1}U(x,0) = \frac{1}{2}\begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ x^2 \end{pmatrix} = \frac{x^2}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{14.21}$$

Equation (14.20) gives **traveling wave solutions** of the form

$$v^{(1)}(x,t) = F(x+3t), \qquad v^{(2)}(x,t) = G(x-t). \tag{14.22}$$

We can write U in terms of V:

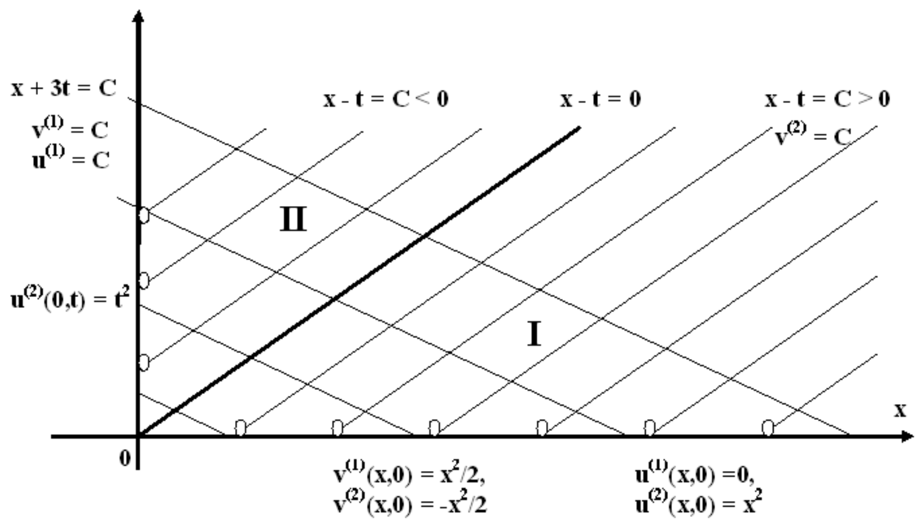
$$U = \Gamma V = \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} v^{(1)} \\ v^{(2)} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} F(x+3t) \\ G(x-t) \end{pmatrix} = \begin{pmatrix} F(x+3t) + G(x-t) \\ -2G(x-t) \end{pmatrix}. \tag{14.23}$$

• For region I, (14.20) and (14.21) give two initial value problems (since a value at any point in region I can be traced back along both characteristics to initial conditions):

$$\begin{cases} v_t^{(1)} - 3v_x^{(1)} = v^{(2)}, \\ v^{(1)}(x,0) = \frac{x^2}{2}; \end{cases} \qquad \begin{cases} v_t^{(2)} + v_x^{(2)} = -v^{(2)}, \\ v^{(2)}(x,0) = -\frac{x^2}{2}, \end{cases}$$

which we do not solve here. Thus, initial conditions for $v^{(1)}$ and $v^{(2)}$ have to be defined. Since (14.23) defines $u^{(1)}$ and $u^{(2)}$ in terms of $v^{(1)}$ and $v^{(2)}$, we need to define two initial conditions for U.

• For region II, solutions of the form $F(x+3t)$ can be traced back to initial conditions. Thus, $v^{(1)}$ is the same as in region I. Solutions of the form $G(x-t)$ are traced back to the boundary at $x = 0$. Since from (14.23), $u^{(2)}(x,t) = -2v^{(2)}(x,t) = -2G(x-t)$, i.e. $u^{(2)}$ is written in term of $v^{(2)}$ only, $u^{(2)}$ requires a boundary condition to be defined on $x = 0$.



Thus,
a) $u^{(2)}(0,t) = t^2, \quad t \geq 0.$ **Well-posed.**
b) $u^{(1)}(0,t) = t, \quad t \geq 0.$ **Not well-posed.**
c) $u^{(1)}(0,t) = t$, $u^{(2)}(0,t) = t^2, \quad t \geq 0.$ **Not well-posed.** ☐

Problem (F'02, #3). *Consider the first order system*

$$\begin{aligned} u_t + u_x + v_x &= 0 \\ v_t + u_x - v_x &= 0 \end{aligned}$$

on the domain $0 < t < \infty$ *and* $0 < x < 1$. *Which of the following sets of initial-boundary data are well posed for this system? Explain your answers.*
a) *u(x,0) = f(x), v(x,0) = g(x);*
b) *u(x,0) = f(x), v(x,0) = g(x), u(0,t) = h(x), v(0,t) = k(x);*
c) *u(x,0) = f(x), v(x,0) = g(x), u(0,t) = h(x), v(1,t) = k(x).*

Proof. Rewrite the equation as $U_t + AU_x = 0$. Initial conditions are same for (a),(b),(c):

$$\begin{aligned} &U_t + \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} U_x = 0, \\ &U(x,0) = \begin{pmatrix} u^{(1)}(x,0) \\ u^{(2)}(x,0) \end{pmatrix} = \begin{pmatrix} f(x) \\ g(x) \end{pmatrix}. \end{aligned}$$

The eigenvalues of the matrix A are $\lambda_1 = \sqrt{2}$, $\lambda_2 = -\sqrt{2}$ and the corresponding eigenvectors are $e_1 = \begin{pmatrix} 1 \\ -1+\sqrt{2} \end{pmatrix}$, $e_2 = \begin{pmatrix} 1 \\ -1-\sqrt{2} \end{pmatrix}$. Thus,

$$\Lambda = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & -\sqrt{2} \end{pmatrix}, \qquad \Gamma = \begin{pmatrix} 1 & 1 \\ -1+\sqrt{2} & -1-\sqrt{2} \end{pmatrix}, \qquad \Gamma^{-1} = \frac{1}{2\sqrt{2}} \begin{pmatrix} 1+\sqrt{2} & 1 \\ -1+\sqrt{2} & -1 \end{pmatrix}.$$

Let $U = \Gamma V$. Then,

$$\begin{aligned} &U_t + AU_x = 0, \\ &\Gamma V_t + A\Gamma V_x = 0, \\ &V_t + \Gamma^{-1}A\Gamma V_x = 0, \\ &V_t + \Lambda V_x = 0. \end{aligned}$$

Thus, the transformed problem is

$$V_t + \begin{pmatrix} \sqrt{2} & 0 \\ 0 & -\sqrt{2} \end{pmatrix} V_x = 0, \tag{14.24}$$

$$V(x,0) = \Gamma^{-1}U(x,0) = \frac{1}{2\sqrt{2}} \begin{pmatrix} 1+\sqrt{2} & 1 \\ -1+\sqrt{2} & -1 \end{pmatrix} \begin{pmatrix} f(x) \\ g(x) \end{pmatrix} = \frac{1}{2\sqrt{2}} \begin{pmatrix} (1+\sqrt{2})f(x) + g(x) \\ (-1+\sqrt{2})f(x) - g(x) \end{pmatrix}. \tag{14.25}$$

Equation (14.24) gives **traveling wave solutions** of the form:

$$v^{(1)}(x,t) = F(x - \sqrt{2}t), \qquad v^{(2)}(x,t) = G(x + \sqrt{2}t). \tag{14.26}$$

However, we can continue and obtain the solutions. We have two initial value problems

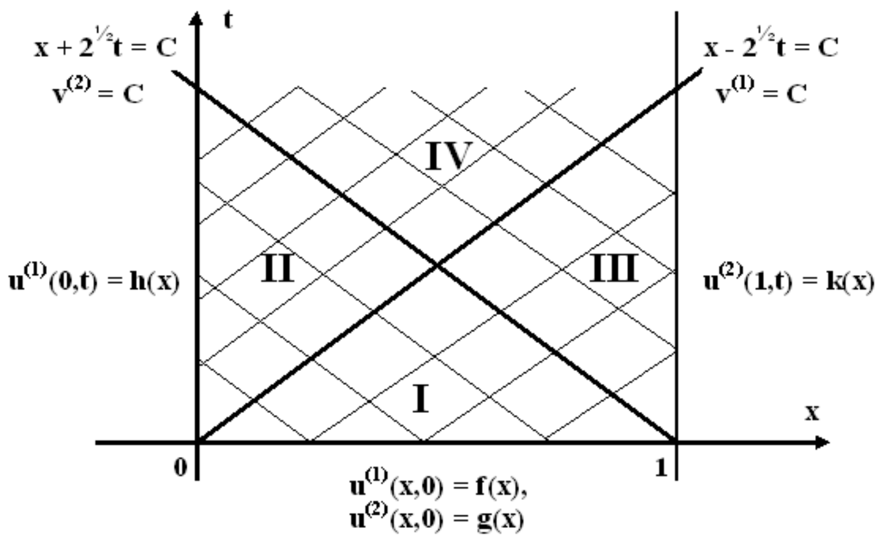
$$\begin{cases} v_t^{(1)} + \sqrt{2}v_x^{(1)} = 0, \\ v^{(1)}(x,0) = \frac{(1+\sqrt{2})}{2\sqrt{2}} f(x) + \frac{1}{2\sqrt{2}} g(x); \end{cases} \qquad \begin{cases} v_t^{(2)} - \sqrt{2}v_x^{(2)} = 0, \\ v^{(2)}(x,0) = \frac{(-1+\sqrt{2})}{2\sqrt{2}} f(x) - \frac{1}{2\sqrt{2}} g(x), \end{cases}$$

which we solve by characteristics to get *traveling wave solutions*:

$$\begin{aligned} v^{(1)}(x,t) &= \frac{(1+\sqrt{2})}{2\sqrt{2}} f(x - \sqrt{2}t) + \frac{1}{2\sqrt{2}} g(x - \sqrt{2}t), \\ v^{(2)}(x,t) &= \frac{(-1+\sqrt{2})}{2\sqrt{2}} f(x + \sqrt{2}t) - \frac{1}{2\sqrt{2}} g(x + \sqrt{2}t). \end{aligned}$$

We can obtain general solution U by writing U in terms of V:

$$U = \Gamma V = \Gamma \begin{pmatrix} v^{(1)} \\ v^{(2)} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1+\sqrt{2} & -1-\sqrt{2} \end{pmatrix} \frac{1}{2\sqrt{2}} \begin{pmatrix} (1+\sqrt{2})f(x-\sqrt{2}t) + g(x-\sqrt{2}t) \\ (-1+\sqrt{2})f(x+\sqrt{2}t) - g(x+\sqrt{2}t) \end{pmatrix}. \tag{14.27}$$



• In region I, the solution is obtained by solving two initial value problems(since a value at any point in region I can be traced back along both characteristics to initial conditions).
• In region II, the solutions of the form $v^{(2)} = G(x+\sqrt{2}t)$ can be traced back to initial conditions and those of the form $v^{(1)} = F(x-\sqrt{2}t)$, to left boundary. Since by (14.27), $u^{(1)}$ and $u^{(2)}$ are written in terms of both $v^{(1)}$ and $v^{(2)}$, one initial condition and one boundary condition at $x = 0$ need to be prescribed.
• In region III, the solutions of the form $v^{(2)} = G(x+\sqrt{2}t)$ can be traced back to right boundary and those of the form $v^{(1)} = F(x-\sqrt{2}t)$, to initial condition. Since by (14.27), $u^{(1)}$ and $u^{(2)}$ are written in terms of both $v^{(1)}$ and $v^{(2)}$, one initial condition and one boundary condition at $x = 1$ need to be prescribed.
• To obtain the solution for region IV, two boundary conditions, one for each boundary, should be given.

Thus,
a) No boundary conditions. **Not well-posed.**
b) $u^{(1)}(0,t) = h(x),\ u^{(2)}(0,t) = k(x)$. **Not well-posed.**
c) $u^{(1)}(0,t) = h(x),\ u^{(2)}(1,t) = k(x)$. **Well-posed.** ☐

Problem (S'94, #3). *Consider the system of equations*

$$f_t + g_x = 0$$
$$g_t + f_x = 0$$
$$h_t + 2h_x = 0$$

on the set $x \geq 0$, $t \geq 0$, *with the following initial-boundary values:*

a) f, g, h *prescribed on* $t = 0$, $x \geq 0$; f, h *prescribed on* $x = 0$, $t \geq 0$.
b) f, g, h *prescribed on* $t = 0$, $x \geq 0$; $f - g$, h *prescribed on* $x = 0$, $t \geq 0$.
c) $f + g$, h *prescribed on* $t = 0$, $x \geq 0$; f, g, h *prescribed on* $x = 0$, $t \geq 0$.

For each of these 3 sets of data, determine whether or not the system is ***well-posed****. Justify your conclusions.*

Proof. The third equation is decoupled from the first two and can be considered separately. Its solution can be written in the form

$$h(x,t) = H(x - 2t),$$

and therefore, h must be prescribed on $t = 0$ and on $x = 0$, since the characteristics propagate from both the x and t axis.
We rewrite the first two equations as ($f \leftrightarrow u_1$, $g \leftrightarrow u_2$):

$$U_t + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} U_x = 0,$$
$$U(x,0) = \begin{pmatrix} u^{(1)}(x,0) \\ u^{(2)}(x,0) \end{pmatrix}.$$

The eigenvalues of the matrix A are $\lambda_1 = -1, \lambda_2 = 1$ and the corresponding eigenvectors are $e_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, $e_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Thus,

$$\Lambda = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \Gamma = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \Gamma^{-1} = \frac{1}{2}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Let $U = \Gamma V$. Then,

$$U_t + AU_x = 0,$$
$$\Gamma V_t + A\Gamma V_x = 0,$$
$$V_t + \Gamma^{-1}A\Gamma V_x = 0,$$
$$V_t + \Lambda V_x = 0.$$

Thus, the transformed problem is

$$V_t + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} V_x = 0, \tag{14.28}$$

$$V(x,0) = \Gamma^{-1}U(x,0) = \frac{1}{2}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} u^{(1)}(x,0) \\ u^{(1)}(x,0) \end{pmatrix}. \tag{14.29}$$

Equation (14.28) gives **traveling wave solutions** of the form:

$$v^{(1)}(x,t) = F(x+t), \qquad v^{(2)}(x,t) = G(x-t). \tag{14.30}$$

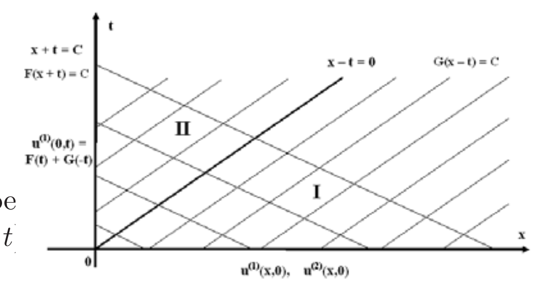
We can write U in terms of V:

$$U = \Gamma V = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} v^{(1)} \\ v^{(2)} \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} F(x+t) \\ G(x-t) \end{pmatrix} = \begin{pmatrix} -F(x+t)+G(x-t) \\ F(x+t)+G(x-t) \end{pmatrix}. \tag{14.31}$$

- For region I, (14.28) and (14.29) give two initial value problems (since a value at any point in region I can be traced back along both characteristics to initial conditions). Thus, initial conditions for $v^{(1)}$ and $v^{(2)}$ have to be defined. Since (14.31) defines $u^{(1)}$ and $u^{(2)}$ in terms of $v^{(1)}$ and $v^{(2)}$, we need to define two initial conditions for U.
- For region II, solutions of the form $F(x+t)$ can be traced back to initial conditions. Thus, $v^{(1)}$ is the same as in region I. Solutions of the form $G(x-t)$ are traced back to the boundary at $x=0$. Since from (14.31), $u^{(2)}(x,t) = v^{(1)}(x,t) + v^{(2)}(x,t) = F(x+t) + G(x-t)$, i.e. $u^{(2)}$ is written in terms of $v^{(2)} = G(x-t)$, $u^{(2)}$ requires a boundary condition to be defined on $x=0$.

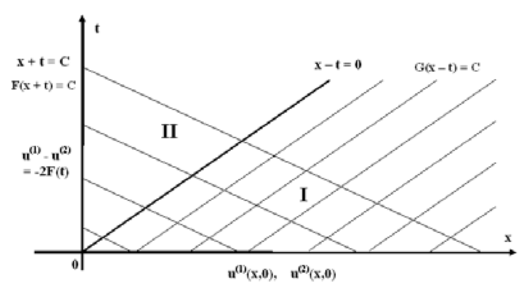
a) $u^{(1)}$, $u^{(2)}$ *prescribed on* $t=0$; $u^{(1)}$ *prescribed on* $x=0$.

Since $u^{(1)}(x,t) = -F(x+t) + G(x-t)$, $u^{(2)}(x,t) = F(x+t) + G(x-t)$, i.e. both $u^{(1)}$ and $u^{(2)}$ are written in terms of $F(x+t)$ and $G(x-t)$, we need to define two initial conditions for U (on $t=0$).
A boundary condition also needs to be prescribe on $x=0$ to be able to trace back $v^{(2)} = G(x-t$
Well-posed.



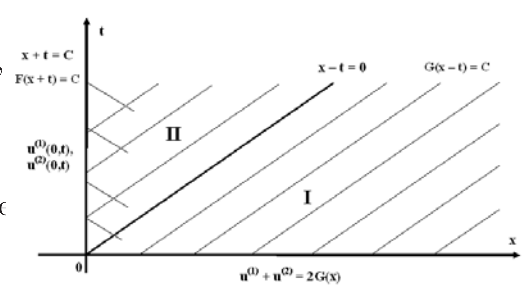
b) $u^{(1)}$, $u^{(2)}$ *prescribed on* $t=0$; $u^{(1)} - u^{(2)}$ *prescribed on* $x=0$.

As in part (a), we need to define two initial conditions for U.
Since $u^{(1)} - u^{(2)} = -2F(x+t)$, its definition on $x=0$ leads to ill-posedness. On the contrary, $u^{(1)} + u^{(2)} = 2G(x-t)$ should be defined on $x=0$ in order to be able to trace back the values through characteristics.
Ill-posed.



c) $u^{(1)} + u^{(2)}$ *prescribed on* $t=0$; $u^{(1)}$, $u^{(2)}$ *prescribed on* $x=0$.

Since $u^{(1)} + u^{(2)} = 2G(x-t)$, another initial condition should be prescribed to be able to trace back solutions of the form $v^{(2)} = F(x+t)$, without which the problem is ill-posed.
Also, two boundary conditions for both $u^{(1)}$ and $u^{(2)}$ define solutions of both $v^{(1)} = G(x-t)$ and $v^{(2)} = F(x+t)$ on the boundary. The forme boundary condition leads to ill-posedness.
Ill-posed.



$\square$

Problem (F'92, #8). *Consider the system*

$$u_t + u_x + av_x = 0$$
$$v_t + bu_x + v_x = 0$$

for $0 < x < 1$ *with boundary and initial conditions*

$$u = v = 0 \qquad \textit{for} \;\; x = 0$$
$$u = u_0, \quad v = v_0 \qquad \textit{for} \;\; t = 0.$$

a) *For which values of a and b is this a* ***well-posed*** *problem?*
b) *For this class of a, b, state conditions on u_0 and v_0 so that the solution u, v will be continuous and continuously differentiable.*

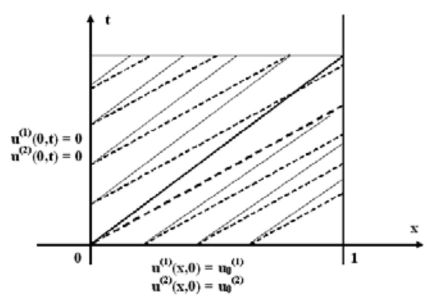
Proof. **a)** Let us change the notation ($u \leftrightarrow u^{(1)},\ v \leftrightarrow u^{(2)}$). Rewrite the equation as

$$U_t + \begin{pmatrix} 1 & a \\ b & 1 \end{pmatrix} U_x = 0, \tag{14.32}$$

$$U(x,0) = \begin{pmatrix} u^{(1)}(x,0) \\ u^{(2)}(x,0) \end{pmatrix} = \begin{pmatrix} u_0^{(1)}(x) \\ u_0^{(2)}(x) \end{pmatrix},$$
$$U(0,t) = \begin{pmatrix} u^{(1)}(0,t) \\ u^{(2)}(0,t) \end{pmatrix} = 0.$$

The eigenvalues of the matrix A are $\lambda_1 = 1 - \sqrt{ab}, \quad \lambda_2 = 1 + \sqrt{ab}.$

$$\Lambda = \begin{pmatrix} 1 - \sqrt{ab} & 0 \\ 0 & 1 + \sqrt{ab} \end{pmatrix}.$$



Let $U = \Gamma V$, where Γ is a matrix of eigenvectors. Ther

$$U_t + AU_x = 0,$$
$$\Gamma V_t + A\Gamma V_x = 0,$$
$$V_t + \Gamma^{-1} A \Gamma V_x = 0,$$
$$V_t + \Lambda V_x = 0.$$

Thus, the transformed problem is

$$V_t + \begin{pmatrix} 1 - \sqrt{ab} & 0 \\ 0 & 1 + \sqrt{ab} \end{pmatrix} V_x = 0, \tag{14.33}$$

$$V(x,0) = \Gamma^{-1} U(x,0).$$

The equation (14.33) gives traveling wave solutions of the form:

$$v^{(1)}(x,t) = F(x - (1 - \sqrt{ab})t), \qquad v^{(2)}(x,t) = G(x - (1 + \sqrt{ab})t). \tag{14.34}$$

We also have $U = \Gamma V$, i.e. both $u^{(1)}$ and $u^{(2)}$ (and their initial and boundary conditions) are combinations of $v^{(1)}$ and $v^{(2)}$.
In order for this problem to be well-posed, both sets of characteristics should emanate from the boundary at $x = 0$. Thus, the eigenvalues of the system are real ($ab > 0$) and $\lambda_{1,2} > 0 \;\; (ab < 1)$. Thus,

$$0 < ab < 1.$$

b) For U to be C^1, we require the compatibility condition, $u_0^{(1)}(0) = 0, \;\; u_0^{(2)}(0) = 0.$ □

Problem (F'93, #2). *Consider the initial-boundary value problem*

$$u_t + u_x = 0$$
$$v_t - (1 - cx^2)v_x + u_x = 0$$

on $-1 \le x \le 1$ *and* $0 \le t$, *with the following prescribed data:*

$$u(x,0), \qquad v(x,0),$$
$$u(-1,t), \qquad v(1,t).$$

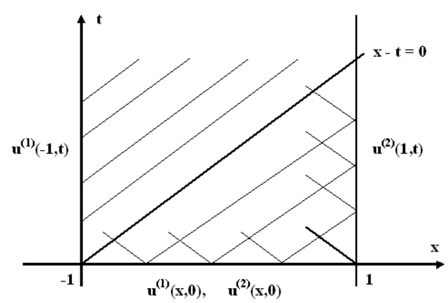
For which values of c is this a ***well-posed*** *problem?*

Proof. Let us change the notation ($u \leftrightarrow u^{(1)}$, $v \leftrightarrow u^{(2)}$).
The first equation can be solved with $u^{(1)}(x,0) = F(x)$ to get a solution in the form $u^{(1)}(x,t) = F(x-t)$, which requires $u^{(1)}(x,0)$ and $u^{(1)}(-1,t)$ to be defined.
With $u^{(1)}$ known, we can solve the second equation

$$u_t^{(2)} - (1 - cx^2)u_x^{(2)} + F(x-t) = 0.$$

Solving the equation by characteristics, we obtain the characteristics in the xt-plane are of the form

$$\frac{dx}{dt} = cx^2 - 1.$$



We need to determine c such that the prescribed data $u^{(2)}(x,0)$ and $u^{(2)}(1,t)$ makes the problem to be well-posed. The boundary condition for $u^{(2)}(1,t)$ requires the characteristics to propagate to the left with t increasing. Thus, $x(t)$ is a decreasing function, i.e.

$$\frac{dx}{dt} < 0 \qquad \Rightarrow \qquad cx^2 - 1 < 0 \quad \text{for} \quad -1 < x < 1 \qquad \Rightarrow \qquad c < 1.$$

We could also do similar analysis we have done in other problems on first order systems involving finding eigenvalues/eigenvectors of the system and using the fact that $u^{(1)}(x,t)$ is known at both boundaries (i.e. values of $u^{(1)}(1,t)$ can be traced back either to initial conditions or to boundary conditions on $x = -1$). $\square$

Problem (S'91, #4). *Consider the first order system*

$$u_t + au_x + bv_x = 0$$
$$v_t + cu_x + dv_x = 0$$

for $0 < x < 1$, *with prescribed initial data:*

$$u(x,0) = u_0(x)$$
$$v(x,0) = v_0(x).$$

a) *Find conditions on* a, b, c, d *such that there is a full set of characteristics and, in this case, find the characteristic speeds.*
b) *For which values of* a, b, c, d *can boundary data be prescribed on* $x = 0$ *and for which values can it be prescribed on* $x = 1$*? How many pieces of data can be prescribed on each boundary?*

Proof. **a)** Let us change the notation ($u \leftrightarrow u^{(1)},\ v \leftrightarrow u^{(2)}$). Rewrite the equation as

$$U_t + \begin{pmatrix} a & b \\ c & d \end{pmatrix} U_x = 0, \tag{14.35}$$

$$U(x,0) = \begin{pmatrix} u^{(1)}(x,0) \\ u^{(2)}(x,0) \end{pmatrix} = \begin{pmatrix} u_0^{(1)}(x) \\ u_0^{(2)}(x) \end{pmatrix}.$$

The system is **hyperbolic** if for each value of $u^{(1)}$ and $u^{(2)}$ the eigenvalues are real and the matrix is diagonalizable, i.e. there is a complete set of linearly independent eigenvectors. The eigenvalues of the matrix A are

$$\lambda_{1,2} = \frac{a + d \pm \sqrt{(a+d)^2 - 4(ad - bc)}}{2} = \frac{a + d \pm \sqrt{(a-d)^2 + 4bc}}{2}.$$

We **need** $(a-d)^2 + 4bc > 0$. This also makes the problem to be diagonalizable.
Let $U = \Gamma V$, where Γ is a matrix of eigenvectors. Then,

$$U_t + AU_x = 0,$$
$$\Gamma V_t + A\Gamma V_x = 0,$$
$$V_t + \Gamma^{-1} A \Gamma V_x = 0,$$
$$V_t + \Lambda V_x = 0.$$

Thus, the transformed problem is

$$V_t + \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} V_x = 0, \tag{14.36}$$

Equation (14.36) gives **traveling wave solutions** of the form:

$$v^{(1)}(x,t) = F(x - \lambda_1 t), \qquad v^{(2)}(x,t) = G(x - \lambda_2 t). \tag{14.37}$$

The characteristic speeds are $\frac{dx}{dt} = \lambda_1,\ \ \frac{dx}{dt} = \lambda_2$.

b) We assume $(a+d)^2 - 4(ad - bc) > 0$.
$a + d > 0,\ \ ad - bc > 0 \quad \Rightarrow \quad \lambda_1, \lambda_2 > 0 \quad \Rightarrow \quad$ 2 B.C. on $x = 0$.
$a + d > 0,\ \ ad - bc < 0 \quad \Rightarrow \quad \lambda_1 < 0, \lambda_2 > 0 \quad \Rightarrow \quad$ 1 B.C. on $x = 0$, 1 B.C. on $x = 1$.
$a + d < 0,\ \ ad - bc > 0 \quad \Rightarrow \quad \lambda_1, \lambda_2 < 0 \quad \Rightarrow \quad$ 2 B.C. on $x = 1$.

$a+d<0, \quad ad-bc<0 \quad \Rightarrow \quad \lambda_1<0, \lambda_2>0 \quad \Rightarrow \quad$ 1 B.C. on $x=0$, 1 B.C. on $x=1$.
$a+d>0, \quad ad-bc=0 \quad \Rightarrow \quad \lambda_1=0, \lambda_2>0 \quad \Rightarrow \quad$ 1 B.C. on $x=0$.
$a+d<0, \quad ad-bc=0 \quad \Rightarrow \quad \lambda_1=0, \lambda_2<0 \quad \Rightarrow \quad$ 1 B.C. on $x=1$.
$a+d=0, \quad ad-bc<0 \quad \Rightarrow \quad \lambda_1<0, \lambda_2>0 \quad \Rightarrow \quad$ 1 B.C. on $x=0$, 1 B.C. on $x=1$. $\square$

Problem (S'94, #2). *Consider the differential operator*

$$L\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_t + 9v_x - u_{xx} \\ v_t - u_x - v_{xx} \end{pmatrix}$$

on $0 \le x \le 2\pi$, $t \ge 0$, *in which the vector* $\begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix}$ *consists of two functions that are periodic in* x.
a) *Find the* ***eigenfunctions and eigenvalues*** *of the operator* L.
b) *Use the results of (a) to solve the initial value problem*

$$L\begin{pmatrix} u \\ v \end{pmatrix} = 0 \qquad \text{for } t \ge 0,$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} e^{ix} \\ 0 \end{pmatrix} \qquad \text{for } t = 0.$$

Proof. **a)** We find the "space" eigenvalues and eigenfunctions. We rewrite the system as

$$U_t + \begin{pmatrix} 0 & 9 \\ -1 & 0 \end{pmatrix} U_x + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} U_{xx} = 0,$$

and find eigenvalues

$$\begin{pmatrix} 0 & 9 \\ -1 & 0 \end{pmatrix} U_x + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} U_{xx} = \lambda U. \tag{14.38}$$

Set $U = \begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix} = \begin{pmatrix} \sum_{n=-\infty}^{n=\infty} u_n(t)e^{inx} \\ \sum_{n=-\infty}^{n=\infty} v_n(t)e^{inx} \end{pmatrix}$. Plugging this into (14.38), we get

$$\begin{pmatrix} 0 & 9 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \sum inu_n(t)e^{inx} \\ \sum inv_n(t)e^{inx} \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \sum -n^2u_n(t)e^{inx} \\ \sum -n^2v_n(t)e^{inx} \end{pmatrix} = \lambda\begin{pmatrix} \sum u_n(t)e^{inx} \\ \sum v_n(t)e^{inx} \end{pmatrix},$$

$$\begin{aligned}
\begin{pmatrix} 0 & 9 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} inu_n(t) \\ inv_n(t) \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -n^2u_n(t) \\ -n^2v_n(t) \end{pmatrix} &= \lambda\begin{pmatrix} u_n(t) \\ v_n(t) \end{pmatrix}, \\
\begin{pmatrix} 0 & 9in \\ -in & 0 \end{pmatrix}\begin{pmatrix} u_n(t) \\ v_n(t) \end{pmatrix} + \begin{pmatrix} n^2 & 0 \\ 0 & n^2 \end{pmatrix}\begin{pmatrix} u_n(t) \\ v_n(t) \end{pmatrix} &= \lambda\begin{pmatrix} u_n(t) \\ v_n(t) \end{pmatrix}, \\
\begin{pmatrix} n^2-\lambda & 9in \\ -in & n^2-\lambda \end{pmatrix}\begin{pmatrix} u_n(t) \\ v_n(t) \end{pmatrix} &= 0, \\
(n^2-\lambda)^2 - 9n^2 = 0,
\end{aligned}$$

which gives $\lambda_1 = n^2+3n$, $\lambda_2 = n^2-3n$, are eigenvalues, and $v_1 = \begin{pmatrix} 3i \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 3i \\ -1 \end{pmatrix}$, are corresponding eigenvectors.

b) We want to solve $\begin{pmatrix} u \\ v \end{pmatrix}_t + L\begin{pmatrix} u \\ v \end{pmatrix} = 0, \quad L\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 9v_x - u_{xx} \\ -u_x - v_{xx} \end{pmatrix}$. We have $\begin{pmatrix} u \\ v \end{pmatrix}_t = -L\begin{pmatrix} u \\ v \end{pmatrix} = -\lambda\begin{pmatrix} u \\ v \end{pmatrix}$, i.e. $\begin{pmatrix} u \\ v \end{pmatrix} = e^{-\lambda t}$. We can write the solution as

$$\begin{aligned}
U(x,t) &= \begin{pmatrix} \sum u_n(t)e^{inx} \\ \sum v_n(t)e^{inx} \end{pmatrix} = \sum_{n=-\infty}^{\infty} a_n e^{-\lambda_1 t} v_1 e^{inx} + b_n e^{-\lambda_2 t} v_2 e^{inx} \\
&= \sum_{n=-\infty}^{\infty} a_n e^{-(n^2+3n)t}\begin{pmatrix} 3i \\ 1 \end{pmatrix} e^{inx} + b_n e^{-(n^2-3n)t}\begin{pmatrix} 3i \\ -1 \end{pmatrix} e^{inx}. \\
U(x,0) &= \sum_{n=-\infty}^{\infty} a_n \begin{pmatrix} 3i \\ 1 \end{pmatrix} e^{inx} + b_n \begin{pmatrix} 3i \\ -1 \end{pmatrix} e^{inx} = \begin{pmatrix} e^{ix} \\ 0 \end{pmatrix}, \\
&\Rightarrow\ a_n = b_n = 0,\ n \neq 1; \\
&\quad\ a_1 + b_1 = \frac{1}{3i} \ \text{ and } \ a_1 = b_1 \qquad \Rightarrow \quad a_1 = b_1 = \frac{1}{6i}. \\
\Rightarrow \quad U(x,t) &= \frac{1}{6i}e^{-4t}\begin{pmatrix} 3i \\ 1 \end{pmatrix} e^{ix} + \frac{1}{6i}e^{2t}\begin{pmatrix} 3i \\ -1 \end{pmatrix} e^{ix} \\
&= \begin{pmatrix} \frac{1}{2}(e^{-4t} + e^{2t}) \\ \frac{1}{6i}(e^{-4t} - e^{2t}) \end{pmatrix} e^{ix}.
\end{aligned}$$

[26] [27] $\square$

[26] ChiuYen's and Sung-Ha's solutions give similar answers.

[27] Questions about this problem:

1. Needed to find eigenfunctions, not eigenvectors.
2. The notation of L was changed. The problem statement incorporates the derivatives wrt. t into L.
3. Why can we write the solution in this form above?

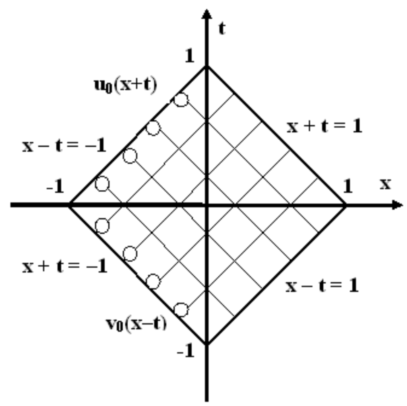
Problem (W'04, #6). *Consider the first order system*

$$u_t - u_x = v_t + v_x = 0$$

in the diamond shaped region $-1 < x + t < 1, \quad -1 < x - t < 1.$ *For each of the following boundary value problems state whether this problem is well-posed. If it is* ***well-posed****, find the solution.*
a) $u(x+t) = u_0(x+t)$ *on* $x - t = -1, \quad v(x-t) = v_0(x-t)$ *on* $x + t = -1.$
b) $v(x+t) = v_0(x+t)$ *on* $x - t = -1, \quad u(x-t) = u_0(x-t)$ *on* $x + t = -1.$

Proof. We have

$$u_t - u_x = 0,$$
$$v_t + v_x = 0.$$



- u is constant along the characteristics: $x + t = c_1(s)$.

Thus, its solution is $u(x,t) = u_0(x+t)$.
It the initial condition is prescribed at $x - t = -1$, the solution can be determined in the entire region by tracing back through the characteristics.

- v is constant along the characteristics: $x - t = c_2(s)$.

Thus, its solution is $v(x,t) = v_0(x-t)$.
It the initial condition is prescribed at $x + t = -1$, the solution can be determined in the entire region by tracing forward through the characteristics. $\square$

15 Problems: Gas Dynamics Systems

15.1 Perturbation

Problem (S'92, #3). [28] [29] *Consider the gas dynamic equations*

$$u_t + uu_x + (F(\rho))_x = 0,$$
$$\rho_t + (u\rho)_x = 0.$$

Here $F(\rho)$ is a given C^∞-smooth function of ρ. At $t = 0$, 2π-periodic initial data

$$u(x,0) = f(x), \qquad \rho(x,0) = g(x).$$

a) *Assume that*

$$f(x) = U_0 + \varepsilon f_1(x), \qquad g(x) = R_0 + \varepsilon g_1(x)$$

where $U_0, R_0 > 0$ are constants and $\varepsilon f_1(x)$, $\varepsilon g_1(x)$ are "small" perturbations. ***Linearize*** *the equations and given conditions for F such that the linearized problem is* ***well-posed****.*
b) *Assume that $U_0 > 0$ and consider the above linearized equations for $0 \le x \le 1$, $t \ge 0$. Construct boundary conditions such that the initial-boundary value problem is* ***well-posed****.*

Proof. **a)** We write the equations in characteristic form:

$$u_t + uu_x + F'(\rho)\rho_x = 0, \qquad \circledast$$
$$\rho_t + u_x\rho + u\rho_x = 0.$$

Consider the special case of nearly constant initial data

$$u(x,0) = u_0 + \varepsilon u_1(x,0),$$
$$\rho(x,0) = \rho_0 + \varepsilon \rho_1(x,0).$$

Then we can approximate nonlinear equations by linear equations. Assuming

$$u(x,t) = u_0 + \varepsilon u_1(x,t),$$
$$\rho(x,t) = \rho_0 + \varepsilon \rho_1(x,t)$$

remain valid with $u_1 = O(1)$, $\rho_1 = O(1)$, we find that

$$u_t = \varepsilon u_{1t}, \qquad \rho_t = \varepsilon \rho_{1t},$$
$$u_x = \varepsilon u_{1x}, \qquad \rho_x = \varepsilon \rho_{1x},$$
$$F'(\rho) = F'(\rho_0 + \varepsilon\rho_1(x,t)) = F'(\rho_0) + \varepsilon\rho_1 F''(\rho_0) + O(\varepsilon^2).$$

Plugging these into $\circledast$, gives

$$\varepsilon u_{1t} + (u_0 + \varepsilon u_1)\varepsilon u_{1x} + \big(F'(\rho_0) + \varepsilon\rho_1 F''(\rho_0) + O(\varepsilon^2)\big)\varepsilon\rho_{1x} = 0,$$
$$\varepsilon\rho_{1t} + \varepsilon u_{1x}(\rho_0 + \varepsilon\rho_1) + (u_0 + \varepsilon u_1)\varepsilon\rho_{1x} = 0.$$

Dividing by ε gives

$$u_{1t} + u_0 u_{1x} + F'(\rho_0)\rho_{1x} = -\varepsilon u_1 u_{1x} - \varepsilon\rho_1\rho_{1x} F''(\rho_0) + O(\varepsilon^2),$$
$$\rho_{1t} + u_{1x}\rho_0 + u_0\rho_{1x} = -\varepsilon u_{1x}\rho_1 - \varepsilon u_1\rho_{1x}.$$

[28] See LeVeque, Second Edition, Birkhäuser Verlag, 1992, p. 44.
[29] This problem has similar notation with S'92, #4.

For small ε, we have

$$\boxed{\begin{cases} u_{1t} + u_0 u_{1x} + F'(\rho_0)\rho_{1x} = 0, \\ \rho_{1t} + u_{1x}\rho_0 + u_0\rho_{1x} = 0. \end{cases}}$$

This can be written as

$$\begin{pmatrix} u_1 \\ \rho_1 \end{pmatrix}_t + \begin{pmatrix} u_0 & F'(\rho_0) \\ \rho_0 & u_0 \end{pmatrix} \begin{pmatrix} u_1 \\ \rho_1 \end{pmatrix}_x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$$\begin{vmatrix} u_0 - \lambda & F'(\rho_0) \\ \rho_0 & u_0 - \lambda \end{vmatrix} = (u_0 - \lambda)(u_0 - \lambda) - \rho_0 F'(\rho_0) = 0,$$

$$\lambda^2 - 2u_0\lambda + u_0^2 - \rho_0 F'(\rho_0) = 0,$$

$$\lambda_{1,2} = u_0 \pm \sqrt{\rho_0 F'(\rho_0)}, \quad u_0 > 0,\ \rho_0 > 0.$$

For well-posedness, need $\lambda_{1,2} \in \mathbb{R}$ or $F'(\rho_0) \geq 0$.

b) We have $u_0 > 0$, and $\lambda_1 = u_0 + \sqrt{\rho_0 F'(\rho_0)}$, $\lambda_2 = u_0 - \sqrt{\rho_0 F'(\rho_0)}$.
- If $u_0 > \sqrt{\rho_0 F'(\rho_0)} \Rightarrow \lambda_1 > 0, \lambda_2 > 0 \Rightarrow$ 2 BC at $x = 0$.
- If $u_0 = \sqrt{\rho_0 F'(\rho_0)} \Rightarrow \lambda_1 > 0, \lambda_2 = 0 \Rightarrow$ 1 BC at $x = 0$.
- If $0 < u_0 < \sqrt{\rho_0 F'(\rho_0)} \Rightarrow \lambda_1 > 0, \lambda_2 < 0 \Rightarrow$ 1 BC at $x = 0$, 1 BC at $x = 1$. □

15.2 Stationary Solutions

Problem (S'92, #4). [30] *Consider*

$$u_t + uu_x + \rho_x = \nu u_{xx},$$
$$\rho_t + (u\rho)_x = 0$$

for $t \geq 0$, $-\infty < x < \infty$.
Give conditions for the states U_+, U_-, R_+, R_-, such that the system has ***stationary solutions*** *(i.e. $u_t = \rho_t = 0$) satisfying*

$$\lim_{x\to+\infty} \begin{pmatrix} u \\ \rho \end{pmatrix} = \begin{pmatrix} U_+ \\ R_+ \end{pmatrix}, \qquad \lim_{x\to-\infty} \begin{pmatrix} u \\ \rho \end{pmatrix} = \begin{pmatrix} U_- \\ R_- \end{pmatrix}. \qquad \circledast$$

Proof. For stationary solutions, we need

$$u_t = -\Big(\frac{u^2}{2}\Big)_x - \rho_x + \nu u_{xx} = 0,$$
$$\rho_t = -(u\rho)_x = 0.$$

Integrating the above equations, we obtain

$$-\frac{u^2}{2} - \rho + \nu u_x = C_1,$$
$$-u\rho = C_2.$$

[30] This problem has similar notation with S'92, #3.

Conditions ⊛ give $u_x = 0$ at $x = \pm\infty$. Thus

$$\begin{aligned} \frac{U_+^2}{2} + R_+ &= \frac{U_-^2}{2} + R_-, \\ U_+ R_+ &= U_- R_-. \end{aligned}$$

$\square$

15.3 Periodic Solutions

Problem (F'94, #4). *Let $u(x,t)$ be a solution of the Cauchy problem*

$$u_t = -u_{xxxx} - 2u_{xx}, \qquad -\infty < x < +\infty, \ 0 < t < +\infty,$$
$$u(x,0) = \varphi(x),$$

where $u(x,t)$ and $\varphi(x)$ are C^∞ functions periodic in x with period 2π;
i.e. $u(x+2\pi,t) = u(x,t)$, $\forall x, \forall t$.
Prove that

$$||u(\cdot,t)|| \le Ce^{at}||\varphi||$$

where $||u(\cdot,t)|| = \sqrt{\int_0^{2\pi} |u(x,t)|^2\,dx}$, $||\varphi|| = \sqrt{\int_0^{2\pi} |\varphi(x)|^2\,dx}$, C, a are some constants.

Proof. **METHOD I:** Since u is 2π-periodic, let

$$u(x,t) = \sum_{n=-\infty}^{\infty} a_n(t)e^{inx}.$$

Plugging this into the equation, we get

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} a_n'(t)e^{inx} &= -\sum_{n=-\infty}^{\infty} n^4 a_n(t)e^{inx} + 2\sum_{n=-\infty}^{\infty} n^2 a_n(t)e^{inx}, \\
a_n'(t) &= (-n^4+2n^2)a_n(t), \\
a_n(t) &= a_n(0)e^{(-n^4+2n^2)t}.
\end{aligned}$$

Also, initial condition gives

$$\begin{aligned}
u(x,0) &= \sum_{n=-\infty}^{\infty} a_n(0)e^{inx} = \varphi(x), \\
&\left|\sum_{n=-\infty}^{\infty} a_n(0)e^{inx}\right| = |\varphi(x)|.
\end{aligned}$$

$$\begin{aligned}
||u(x,t)||_2^2 &= \int_0^{2\pi} u^2(x,t)\,dx = \int_0^{2\pi} \Big(\sum_{n=-\infty}^{\infty} a_n(t)e^{inx}\Big)\Big(\sum_{m=-\infty}^{\infty} a_n(t)\overline{e^{imx}}\Big)\,dx \\
&= \sum_{n=-\infty}^{\infty} a_n^2(t)\int_0^{2\pi} e^{inx}e^{-inx}\,dx = 2\pi\sum_{n=-\infty}^{\infty} a_n^2(t) = 2\pi\sum_{n=-\infty}^{\infty} a_n^2(0)e^{2(-n^4+2n^2)t} \\
&\le \left|2\pi\sum_{n=-\infty}^{\infty} a_n^2(0)\right|\left|\sum_{n=-\infty}^{\infty} e^{2(-n^4+2n^2)t}\right| = 2\pi\underbrace{\left|\sum_{n=-\infty}^{\infty} a_n^2(0)\right|}_{||\varphi||^2} e^{2t}\underbrace{\sum_{n=-\infty}^{\infty} e^{-2(n^2-1)^2t}}_{=C_1,\text{(convergent)}} \\
&= C_2e^{2t}||\varphi||^2.
\end{aligned}$$

$$\Rightarrow \quad ||u(x,t)|| \le Ce^t||\varphi||.$$

METHOD II: Multiply this equation by u and integrate

$$
\begin{aligned}
uu_t &= -uu_{xxxx} - 2uu_{xx}, \\
\frac{1}{2}\frac{d}{dt}(u^2) &= -uu_{xxxx} - 2uu_{xx}, \\
\frac{1}{2}\frac{d}{dt}\int_0^{2\pi} u^2\,dx &= -\int_0^{2\pi} uu_{xxxx}\,dx - \int_0^{2\pi} 2uu_{xx}\,dx, \\
\frac{1}{2}\frac{d}{dt}||u||_2^2 &= \underbrace{-uu_{xxx}\Big|_0^{2\pi}}_{=0} + \underbrace{u_x u_{xx}\Big|_0^{2\pi}}_{=0} - \int_0^{2\pi} u_{xx}^2\,dx - \int_0^{2\pi} 2uu_{xx}\,dx, \\
\frac{1}{2}\frac{d}{dt}||u||_2^2 &= -\int_0^{2\pi} u_{xx}^2\,dx - \int_0^{2\pi} 2uu_{xx}\,dx \qquad (-2ab \le a^2 + b^2) \\
&\le -\int_0^{2\pi} u_{xx}^2\,dx + \int_0^{2\pi} (u^2 + u_{xx}^2)\,dx = \int_0^{2\pi} u^2\,dx = ||u||^2, \\
\Rightarrow \quad & \frac{d}{dt}||u||^2 \le 2||u||^2, \\
& ||u||^2 \le ||u(0)||^2 e^{2t}, \\
& ||u|| \le ||u(0)||e^t. \ \checkmark
\end{aligned}
$$

METHOD III: Can use Fourier transform. See ChiuYen's solutions, that have both Method II and III. $\square$

Problem (S'90, #4).
Let $f(x) \in C^\infty$ be a 2π-periodic function, i.e., $f(x) = f(x+2\pi)$ and denote by

$$||f||^2 = \int_0^{2\pi} |f(x)|^2\,dx$$

the L_2-norm of f.
a) *Express $||d^p f/dx^p||^2$ in terms of the* ***Fourier coefficients*** *of f.*
b) *Let $q > p > 0$ be integers. Prove that $\forall\, \epsilon > 0$, $\exists K = N(\varepsilon, p, q)$, constant, such that*

$$\Big|\Big|\frac{d^p f}{dx^p}\Big|\Big|^2 \le \varepsilon \Big|\Big|\frac{d^q f}{dx^q}\Big|\Big|^2 + K||f||^2.$$

c) *Discuss how K depends on ε.*

Proof. **a)** Let [31]

$$\begin{aligned}
f(x) &= \sum_{-\infty}^{\infty} f_n e^{inx}, \\
\frac{d^p f}{dx^p} &= \sum_{-\infty}^{\infty} f_n (in)^p e^{inx}, \\
\Big|\Big|\frac{d^p f}{dx^p}\Big|\Big|^2 &= \int_0^{2\pi} \Big|\sum_{-\infty}^{\infty} f_n (in)^p e^{inx}\Big|^2\,dx \;=\; \int_0^{2\pi} |i^2|^p \,\Big|\sum_{-\infty}^{\infty} f_n n^p e^{inx}\Big|^2\,dx \\
&= \int_0^{2\pi} \Big|\sum_{-\infty}^{\infty} f_n n^p e^{inx}\Big|^2\,dx \;=\; 2\pi \sum_{n=0}^{\infty} f_n^2 n^{2p}.
\end{aligned}$$

b) We have

$$\begin{aligned}
\Big|\Big|\frac{d^p f}{dx^p}\Big|\Big|^2 &\le \varepsilon \Big|\Big|\frac{d^q f}{dx^q}\Big|\Big|^2 + K||f||^2, \\
2\pi \sum_{n=0}^{\infty} f_n^2 n^{2p} &\le \varepsilon\, 2\pi \sum_{n=0}^{\infty} f_n^2 n^{2q} + K\, 2\pi \sum_{n=0}^{\infty} f_n^2, \\
n^{2p} - \varepsilon n^{2q} &\le K, \\
n^{2p} \underbrace{(1 - \varepsilon n^{q'})}_{<0,\ for\ n\ large} &\le K, \qquad \text{some } q' > 0.
\end{aligned}$$

Thus, the above inequality is true for n large enough. The statement follows. ☐

[31] Note:

$$\int_0^L e^{inx}\overline{e^{imx}}\,dx = \begin{cases} 0 & n \ne m \\ L & n = m \end{cases}$$

Problem (S'90, #5). [32] *Consider the flame front equation*

$$u_t + uu_x + u_{xx} + u_{xxxx} = 0 \qquad \circledast$$

with 2π*-periodic initial data*

$$u(x,0) = f(x), \qquad f(x) = f(x+2\pi) \in C^\infty.$$

a) *Determine the solution, if* $f(x) \equiv f_0 =$ *const.*
b) *Assume that*

$$f(x) = 1 + \varepsilon g(x), \quad 0 < \varepsilon \ll 1, \qquad |g|_\infty = 1, \qquad g(x) = g(x+2\pi).$$

Linearize *the equation. Is the Cauchy problem well-posed for the linearized equation, i.e., do its solutions* v *satisfy an estimate*

$$||v(\cdot,t)|| \le Ke^{\alpha(t-t_0)}||v(\cdot,t_0)||?$$

c) *Determine the best possible constants* K, α.

Proof. **a)** The solution to

$$u_t + uu_x + u_{xx} + u_{xxxx} = 0,$$
$$u(x,0) = f_0 = \text{const},$$

is $u(x,t) = f_0 = \text{const}$.

b) We consider the special case of nearly constant initial data

$$u(x,0) = 1 + \varepsilon u_1(x,0).$$

Then we can approximate the nonlinear equation by a linear equation. Assuming

$$u(x,t) = 1 + \varepsilon u_1(x,t),$$

remain valid with $u_1 = O(1)$, from $\circledast$, we find that

$$\varepsilon u_{1t} + (1+\varepsilon u_1)\varepsilon u_{1x} + \varepsilon u_{1xx} + \varepsilon u_{1xxxx} = 0.$$

Dividing by ε gives

$$u_{1t} + u_{1x} + \varepsilon u_1 u_{1x} + u_{1xx} + u_{1xxxx} = 0.$$

For small ε, we have

$$\boxed{u_{1t} + u_{1x} + u_{1xx} + u_{1xxxx} = 0.}$$

Multiply this equation by u_1 and integrate

$$
\begin{aligned}
u_1u_{1t} + u_1u_{1x} + u_1u_{1xx} + u_1u_{1xxxx} &= 0,\\
\frac{d}{dt}\Big(\frac{u_1^2}{2}\Big) + \Big(\frac{u_1^2}{2}\Big)_x + u_1u_{1xx} + u_1u_{1xxxx} &= 0,\\
\frac{1}{2}\frac{d}{dt}\int_0^{2\pi} u_1^2\,dx + \underbrace{\frac{u_1^2}{2}\Big|_0^{2\pi}}_{=0} + \int_0^{2\pi} u_1u_{1xx}\,dx + \int_0^{2\pi} u_1u_{1xxxx}\,dx &= 0,\\
\frac{1}{2}\frac{d}{dt}||u_1||_2^2 + \underbrace{u_1u_{1x}\Big|_0^{2\pi}}_{=0} - \int_0^{2\pi} u_{1x}^2\,dx + \underbrace{u_1u_{1xxx}\Big|_0^{2\pi}}_{=0} - \underbrace{u_{1x}u_{1xx}\Big|_0^{2\pi}}_{=0} + \int_0^{2\pi} u_{1xx}^2\,dx &= 0,\\
\frac{1}{2}\frac{d}{dt}||u_1||_2^2 &= \int_0^{2\pi} u_{1x}^2\,dx - \int_0^{2\pi} u_{1xx}^2\,dx.
\end{aligned}
$$

[32] S'90 #5, #6, #7 all have similar formulations.

Since u_1 is 2π-periodic, let

$$u_1 = \sum_{n=-\infty}^{\infty} a_n(t) e^{inx}. \qquad \text{Then,}$$

$$u_{1x} = i \sum_{n=-\infty}^{\infty} n a_n(t) e^{inx} \quad \Rightarrow \quad u_{1x}^2 = -\Big(\sum_{n=-\infty}^{\infty} n a_n(t) e^{inx} \Big)^2,$$

$$u_{1xx} = - \sum_{n=-\infty}^{\infty} n^2 a_n(t) e^{inx} \quad \Rightarrow \quad u_{1xx}^2 = \Big(\sum_{n=-\infty}^{\infty} n^2 a_n(t) e^{inx} \Big)^2.$$

Thus,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}||u_1||_2^2 &= \int_0^{2\pi} u_{1x}^2\, dx - \int_0^{2\pi} u_{1xx}^2\, dx \\
&= -\int_0^{2\pi} \Big(\sum n a_n(t) e^{inx}\Big)^2 dx - \int_0^{2\pi} \Big(\sum n^2 a_n(t) e^{inx}\Big)^2 dx \\
&= -2\pi \sum n^2 a_n(t)^2 - 2\pi \sum n^4 a_n(t)^2 = -2\pi \sum a_n(t)^2 (n^2 + n^4) \le 0.
\end{aligned}$$

$$\boxed{\Rightarrow \quad ||u_1(\cdot, t)||_2 \le ||u_1(\cdot, 0)||_2,}$$

where $K = 1$, $\alpha = 0$. $\square$

Problem (W'03, #4). *Consider the PDE*

$$\begin{aligned}
u_t = u_x + u^4 \quad &\textit{for} \quad t > 0 \\
u = u_0 \quad &\textit{for} \quad t = 0
\end{aligned}$$

for $0 < x < 2\pi$. *Define the set* $A = \{u = u(x) : \hat{u}(k) = 0 \quad if \quad k < 0\}$, *in which* $\{\hat{u}(k, t)\}_{-\infty}^{\infty}$ *is the* ***Fourier series*** *of* u *in* x *on* $[0, 2\pi]$.
a) *If* $u_0 \in A$, *show that* $u(t) \in A$.
b) *Find differential equations for* $\hat{u}(0, t)$, $\hat{u}(1, t)$, *and* $\hat{u}(2, t)$.

Proof. **a)** Solving

$$\begin{aligned}
u_t &= u_x + u^4 \\
u(x, 0) &= u_0(x)
\end{aligned}$$

by the method of characteristics, we get

$$u(x, t) = \frac{u_0(x + t)}{(1 - 3t(u_0(x + t))^3)^{\frac{1}{3}}}.$$

Since $u_0 \in A$, $\widehat{u}_{0k} = 0$ if $k < 0$. Thus,

$$u_0(x) = \sum_{k=0}^{\infty} \widehat{u}_{0k}\, e^{i\frac{kx}{2}}.$$

Since

$$\widehat{u}_k = \frac{1}{2\pi} \int_0^{2\pi} u(x, t)\, e^{-i\frac{kx}{2}}\, dx,$$

we have

$$u(x,t) = \sum_{k=0}^{\infty} \widehat{u}_k \, e^{i\frac{kx}{2}},$$

that is, $u(t) \in A$. $\square$

15.4 Energy Estimates

Problem (S'90, #6). *Let $U(x,t) \in C^\infty$ be 2π-periodic in x. Consider the **linear** equation*

$$\begin{aligned}
&u_t + Uu_x + u_{xx} + u_{xxxx} = 0, \\
&u(x,0) = f(x), \qquad f(x) = f(x+2\pi) \in C^\infty.
\end{aligned}$$

a) *Derive an **energy estimate** for u.*
b) *Prove that one can estimate all derivatives $||\partial^p u/\partial x^p||$.*
c) *Indicate how to prove existence of solutions.* [33]

Proof. **a)** Multiply the equation by u and integrate

$$\begin{aligned}
&uu_t + Uuu_x + uu_{xx} + uu_{xxxx} = 0, \\
&\frac{1}{2}\frac{d}{dt}(u^2) + \frac{1}{2}U(u^2)_x + uu_{xx} + uu_{xxxx} = 0, \\
&\frac{1}{2}\frac{d}{dt}\int_0^{2\pi} u^2\,dx + \frac{1}{2}\int_0^{2\pi} U(u^2)_x\,dx + \int_0^{2\pi} uu_{xx}\,dx + \int_0^{2\pi} uu_{xxxx}\,dx = 0, \\
&\frac{1}{2}\frac{d}{dt}||u||^2 + \underbrace{\frac{1}{2}Uu^2\Big|_0^{2\pi}}_{=0} - \frac{1}{2}\int_0^{2\pi} U_x u^2\,dx + uu_x\Big|_0^{2\pi} - \int_0^{2\pi} u_x^2\,dx \\
&\qquad + uu_{xxx}\Big|_0^{2\pi} - u_x u_{xx}\Big|_0^{2\pi} + \int_0^{2\pi} u_{xx}^2\,dx = 0, \\
&\frac{1}{2}\frac{d}{dt}||u||^2 - \frac{1}{2}\int_0^{2\pi} U_x u^2\,dx - \int_0^{2\pi} u_x^2\,dx + \int_0^{2\pi} u_{xx}^2\,dx = 0, \\
&\frac{1}{2}\frac{d}{dt}||u||^2 = \frac{1}{2}\int_0^{2\pi} U_x u^2\,dx + \int_0^{2\pi} u_x^2\,dx - \int_0^{2\pi} u_{xx}^2\,dx \ \le \ \text{(from S'90, \#5)} \ \le \\
&\quad \le \ \frac{1}{2}\int_0^{2\pi} U_x u^2\,dx \ \le \ \frac{1}{2}\max_x U_x \int_0^{2\pi} u^2\,dx. \\
&\Rightarrow \quad \frac{d}{dt}||u||^2 \le \max_x U_x ||u||^2, \\
&\qquad\quad ||u(x,t)||^2 \le ||u(x,0)||^2 e^{(\max_x U_x)t}.
\end{aligned}$$

This can also been done using Fourier Transform. See ChiuYen's solutions where the above method and the Fourier Transform methods are used. ☐

[33] S'90 #5, #6, #7 all have similar formulations.

Problem (S'90, #7). [34] *Consider the* ***nonlinear*** *equation*

$$u_t + uu_x + u_{xx} + u_{xxxx} = 0, \qquad \circledast$$
$$u(x,0) = f(x), \qquad f(x) = f(x+2\pi) \in C^\infty.$$

a) *Derive an* ***energy estimate*** *for u.*
b) *Show that there is an interval $0 \le t \le T$, T depending on f, such that also $||\partial u(\cdot,t)/\partial x||$ can be bounded.*

Proof. **a)** Multiply the above equation by u and integrate

$$\begin{aligned}
&uu_t + u^2u_x + uu_{xx} + uu_{xxxx} = 0,\\
&\frac{1}{2}\frac{d}{dt}(u^2) + \frac{1}{3}(u^3)_x + uu_{xx} + uu_{xxxx} = 0,\\
&\frac{1}{2}\frac{d}{dt}\int_0^{2\pi} u^2\,dx + \frac{1}{3}\int_0^{2\pi}(u^3)_x\,dx + \int_0^{2\pi} uu_{xx}\,dx + \int_0^{2\pi} uu_{xxxx}\,dx = 0,\\
&\frac{1}{2}\frac{d}{dt}||u||^2 + \underbrace{\frac{1}{3}u^3\Big|_0^{2\pi}}_{=0} - \int_0^{2\pi} u_x^2\,dx + \int_0^{2\pi} u_{xx}^2\,dx = 0,\\
&\frac{1}{2}\frac{d}{dt}||u||^2 = \int_0^{2\pi} u_x^2\,dx - \int_0^{2\pi} u_{xx}^2\,dx \le 0, \qquad \text{(from S'90, \#5)}\\
&\Rightarrow \quad ||u(\cdot,t)|| \le ||u(\cdot,0)||.
\end{aligned}$$

b) In order to find a bound for $||u_x(\cdot,t)||$, differentiate $\circledast$ with respect to x:

$$u_{tx} + (uu_x)_x + u_{xxx} + u_{xxxxx} = 0,$$

Multiply the above equation by u_x and integrate:

$$\begin{aligned}
&u_xu_{tx} + u_x(uu_x)_x + u_xu_{xxx} + u_xu_{xxxxx} = 0,\\
&\frac{1}{2}\frac{d}{dt}\int_0^{2\pi}(u_x)^2\,dx + \int_0^{2\pi} u_x(uu_x)_x\,dx + \int_0^{2\pi} u_xu_{xxx}\,dx + \int_0^{2\pi} u_xu_{xxxxx}\,dx = 0.
\end{aligned}$$

We evaluate one of the integrals in the above expression using the periodicity:

$$\begin{aligned}
\int_0^{2\pi} u_x(uu_x)_x\,dx &= -\int_0^{2\pi} u_{xx}uu_x = \int_0^{2\pi} u_x(u_x^2 + uu_{xx}) = \int_0^{2\pi} u_x^3 + \int_0^{2\pi} uu_xu_{xx},\\
&\Rightarrow \quad \int_0^{2\pi} u_{xx}uu_x = -\frac{1}{2}\int_0^{2\pi} u_x^3,\\
&\Rightarrow \quad \int_0^{2\pi} u_x(uu_x)_x = \frac{1}{2}\int_0^{2\pi} u_x^3.
\end{aligned}$$

We have

$$\frac{1}{2}\frac{d}{dt}||u_x||^2 + \int_0^{2\pi} u_x^3\,dx + \int_0^{2\pi} u_xu_{xxx}\,dx + \int_0^{2\pi} u_xu_{xxxxx}\,dx = 0.$$

---

[34] S'90 #5, #6, #7 all have similar formulations.

Let $w = u_x$, then

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}||w||^2 &= -\int_0^{2\pi} w^3\,dx - \int_0^{2\pi} ww_{xx}\,dx - \int_0^{2\pi} ww_{xxxx}\,dx \\
&= -\int_0^{2\pi} w^3\,dx + \int_0^{2\pi} w_x^2\,dx - \int_0^{2\pi} w_{xx}^2\,dx \leq -\int_0^{2\pi} w^3\,dx, \\
\Rightarrow \quad \frac{d}{dt}||u_x||^2 &= -\int_0^{2\pi} u_x^3\,dx.
\end{aligned}$$

$\square$

16 Problems: Wave Equation

16.1 The Initial Value Problem

Example (McOwen 3.1 #1). *Solve the initial value problem:*

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, \\ u(x,0) = \underbrace{x^3}_{g(x)}, \quad u_t(x,0) = \underbrace{\sin x}_{h(x)}. \end{cases}$$

Proof. D'Alembert's formula gives the solution:

$$\begin{aligned} u(x,t) &= \frac{1}{2}(g(x+ct)+g(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} h(\xi)\,d\xi \\ &= \frac{1}{2}(x+ct)^3 + \frac{1}{2}(x-ct)^3 + \frac{1}{2c}\int_{x-ct}^{x+ct} \sin\xi\,d\xi \\ &= x^3 + 2xc^2t^2 - \frac{1}{2c}\cos(x+ct) + \frac{1}{2c}\cos(x-ct) = \\ &= x^3 + 2xc^2t^2 + \frac{1}{c}\sin x \sin ct. \end{aligned}$$

□

Problem (S'99, #6). *Solve the Cauchy problem*

$$\begin{cases} u_{tt} = a^2 u_{xx} + \cos x, \\ u(x,0) = \sin x, \quad u_t(x,0) = 1+x. \end{cases} \tag{16.1}$$

Proof. We have a nonhomogeneous PDE with nonhomogeneous initial conditions:

$$\begin{cases} u_{tt} - c^2 u_{xx} = \underbrace{\cos x}_{f(x,t)}, \\ u(x,0) = \underbrace{\sin x}_{g(x)}, \quad u_t(x,0) = \underbrace{1+x}_{h(x)}. \end{cases}$$

The solution is given by d'Alembert's formula and Duhamel's principle.[35]

$$\begin{aligned} u^A(x,t) &= \frac{1}{2}(g(x+ct)+g(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} h(\xi)\,d\xi \\ &= \frac{1}{2}(\sin(x+ct)+\sin(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} (1+\xi)\,d\xi \\ &= \sin x\,\cos ct + \frac{1}{2c}\Big[\xi + \frac{\xi^2}{2}\Big]_{\xi=x-ct}^{\xi=x+ct} = \sin x\,\cos ct + xt + t. \end{aligned}$$

$$\begin{aligned} u^D(x,t) &= \frac{1}{2c}\int_0^t \bigg(\int_{x-c(t-s)}^{x+c(t-s)} f(\xi,s)\,d\xi\bigg)\,ds = \frac{1}{2c}\int_0^t \bigg(\int_{x-c(t-s)}^{x+c(t-s)} \cos\xi\,d\xi\bigg)\,ds \\ &= \frac{1}{2c}\int_0^t \Big(\sin[x+c(t-s)] - \sin[x-c(t-s)]\Big)\,ds = \frac{1}{c^2}(\cos x - \cos x\cos ct). \end{aligned}$$

$$u(x,t) = u^A(x,t) + u^D(x,t) = \sin x\,\cos ct + xt + t + \frac{1}{c^2}(\cos x - \cos x \cos ct).$$

[35]Note the relationship: $x \leftrightarrow \xi$, $t \leftrightarrow s$.

We can check that the solution satisfies equation (16.1). Can also check that u^A, u^D satisfy

$$\begin{cases} u^A_{tt} - c^2 u^A_{xx} = 0, \\ u^A(x,0) = \sin x, \quad u^A_t(x,0) = 1 + x; \end{cases} \qquad \begin{cases} u^D_{tt} - c^2 u^D_{xx} = \cos x, \\ u^D(x,0) = 0, \quad u^D_t(x,0) = 0. \end{cases}$$

$\square$

16.2 Initial/Boundary Value Problem

Problem 1. *Consider the initial/boundary value problem*

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & 0 < x < L,\ t > 0 \\ u(x,0) = g(x), \quad u_t(x,0) = h(x) & 0 < x < L \\ u(0,t) = 0, \quad u(L,t) = 0 & t \geq 0. \end{cases} \tag{16.2}$$

Proof. Find $u(x,t)$ in the form

$$u(x,t) = \frac{a_0(t)}{2} + \sum_{n=1}^{\infty} a_n(t) \cos\frac{n\pi x}{L} + b_n(t) \sin\frac{n\pi x}{L}.$$

- Functions $a_n(t)$ and $b_n(t)$ are determined by the boundary conditions:

$$0 = u(0,t) = \frac{a_0(t)}{2} + \sum_{n=1}^{\infty} a_n(t) \quad \Rightarrow \quad a_n(t) = 0. \qquad \text{Thus,}$$

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin\frac{n\pi x}{L}. \tag{16.3}$$

- If we substitute (16.3) into the equation $u_{tt} - c^2 u_{xx} = 0$, we get

$$\sum_{n=1}^{\infty} b_n''(t) \sin\frac{n\pi x}{L} + c^2 \sum_{n=1}^{\infty} \Big(\frac{n\pi}{L}\Big)^2 b_n(t) \sin\frac{n\pi x}{L} = 0, \qquad \text{or}$$

$$b_n''(t) + \Big(\frac{n\pi c}{L}\Big)^2 b_n(t) = 0,$$

whose general solution is

$$b_n(t) = c_n \sin\frac{n\pi c t}{L} + d_n \cos\frac{n\pi c t}{L}. \tag{16.4}$$

Also, $b_n'(t) = c_n(\frac{n\pi c}{L}) \cos\frac{n\pi c t}{L} - d_n(\frac{n\pi c}{L}) \sin\frac{n\pi c t}{L}$.

- The constants c_n and d_n are determined by the initial conditions:

$$\begin{aligned} g(x) = u(x,0) &= \sum_{n=1}^{\infty} b_n(0) \sin\frac{n\pi x}{L} = \sum_{n=1}^{\infty} d_n \sin\frac{n\pi x}{L}, \\ h(x) = u_t(x,0) &= \sum_{n=1}^{\infty} b_n'(0) \sin\frac{n\pi x}{L} = \sum_{n=1}^{\infty} c_n \frac{n\pi c}{L} \sin\frac{n\pi x}{L}. \end{aligned}$$

By orthogonality, we may multiply by $\sin(m\pi x/L)$ and integrate:

$$\begin{aligned} \int_0^L g(x) \sin\frac{m\pi x}{L}\,dx &= \int_0^L \sum_{n=1}^{\infty} d_n \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L}\,dx = d_m \frac{L}{2}, \\ \int_0^L h(x) \sin\frac{m\pi x}{L}\,dx &= \int_0^L \sum_{n=1}^{\infty} c_n \frac{n\pi c}{L} \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L}\,dx = c_m \frac{m\pi c}{L} \frac{L}{2}. \end{aligned}$$

Thus,

$$d_n = \frac{2}{L} \int_0^L g(x) \sin\frac{n\pi x}{L}\,dx, \qquad c_n = \frac{2}{n\pi c} \int_0^L h(x) \sin\frac{n\pi x}{L}\,dx. \tag{16.5}$$

The formulas (16.3), (16.4), and (16.5) define the solution.

Example (McOwen 3.1 #2). *Consider the initial/boundary value problem*

$$\begin{cases} u_{tt} - u_{xx} = 0 & 0 < x < \pi, \ t > 0 \\ u(x,0) = 1, \quad u_t(x,0) = 0 & 0 < x < \pi \\ u(0,t) = 0, \quad u(\pi,t) = 0 & t \geq 0. \end{cases} \tag{16.6}$$

Proof. Find $u(x,t)$ in the form

$$u(x,t) = \frac{a_0(t)}{2} + \sum_{n=1}^{\infty} a_n(t)\cos nx + b_n(t)\sin nx.$$

- Functions $a_n(t)$ and $b_n(t)$ are determined by the boundary conditions:

$$0 = u(0,t) = \frac{a_0(t)}{2} + \sum_{n=1}^{\infty} a_n(t) \quad \Rightarrow \quad a_n(t) = 0. \qquad \text{Thus,}$$

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\sin nx. \tag{16.7}$$

- If we substitute this into $u_{tt} - u_{xx} = 0$, we get

$$\sum_{n=1}^{\infty} b_n''(t)\sin nx + \sum_{n=1}^{\infty} b_n(t)n^2 \sin nx = 0, \qquad \text{or}$$
$$b_n''(t) + n^2 b_n(t) = 0,$$

whose general solution is

$$b_n(t) = c_n \sin nt + d_n \cos nt. \tag{16.8}$$

Also, $b_n'(t) = nc_n \cos nt - nd_n \sin nt$.

- The constants c_n and d_n are determined by the initial conditions:

$$\begin{aligned} 1 = u(x,0) &= \sum_{n=1}^{\infty} b_n(0)\sin nx = \sum_{n=1}^{\infty} d_n \sin nx, \\ 0 = u_t(x,0) &= \sum_{n=1}^{\infty} b_n'(0)\sin nx = \sum_{n=1}^{\infty} nc_n \sin nx. \end{aligned}$$

By orthogonality, we may multiply both equations by $\sin mx$ and integrate:

$$\int_0^\pi \sin mx\, dx = d_m \frac{\pi}{2},$$
$$\int_0^\pi 0\, dx = nc_n \frac{\pi}{2}.$$

Thus,

$$d_n = \frac{2}{n\pi}(1 - \cos n\pi) = \begin{cases} \frac{4}{n\pi}, & n \text{ odd}, \\ 0, & n \text{ even}, \end{cases} \qquad \text{and} \qquad c_n = 0. \tag{16.9}$$

Using this in (16.8) and (16.7), we get

$$b_n(t) = \begin{cases} \frac{4}{n\pi}\cos nt, & n \text{ odd}, \\ 0, & n \text{ even}, \end{cases}$$

$$\boxed{u(x,t) = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)t\,\sin(2n+1)x}{(2n+1)}.}$$

$\square$

We can **sum the series in regions bouded by characteristics**. We have

$$u(x,t) = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)t\,\sin(2n+1)x}{(2n+1)}, \qquad \text{or}$$

$$u(x,t) = \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{\sin[(2n+1)(x+t)]}{(2n+1)} + \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{\sin[(2n+1)(x-t)]}{(2n+1)}. \tag{16.10}$$

The initial condition may be written as

$$1 = u(x,0) = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)x}{(2n+1)} \qquad \text{for} \quad 0 < x < \pi. \tag{16.11}$$

We can use (16.11) to sum the series in (16.10).

$$\text{In } R_1, \qquad u(x,t) = \frac{1}{2} + \frac{1}{2} = 1.$$

Since $\sin[(2n+1)(x-t)] = -\sin[(2n+1)(-(x-t))]$, and $0 < -(x-t) < \pi$ in R_2,

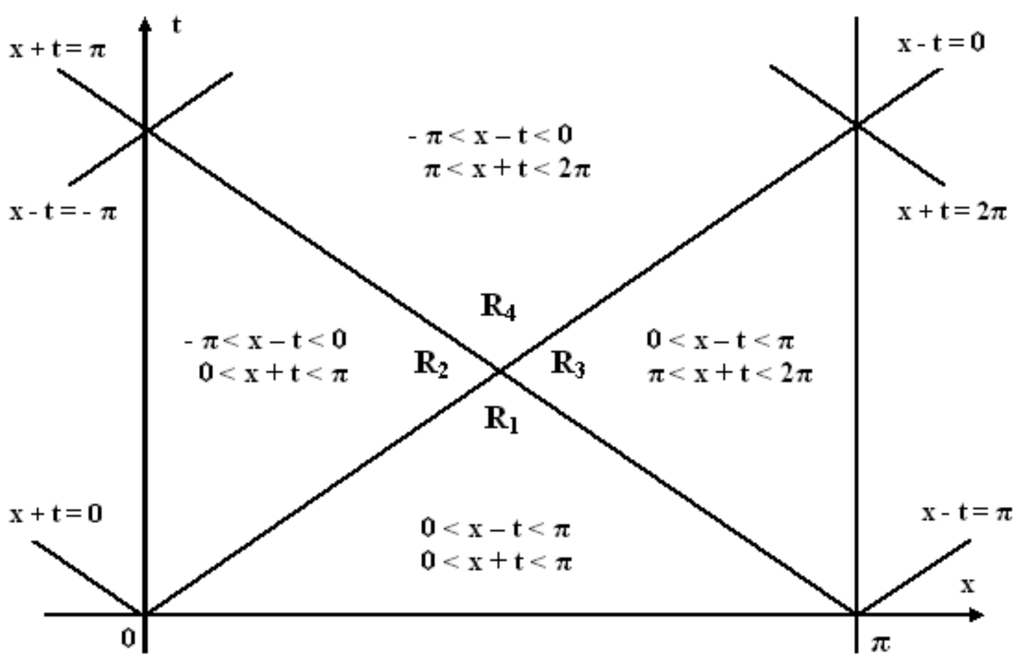
$$\text{in } R_2, \qquad u(x,t) = \frac{1}{2} - \frac{1}{2} = 0.$$

Since $\sin[(2n+1)(x+t)] = \sin[(2n+1)(x+t-2\pi)] = -\sin[(2n+1)(2\pi-(x+t))]$, and $0 < 2\pi-(x+t) < \pi$ in R_3,

$$\text{in } R_3, \qquad u(x,t) = -\frac{1}{2} + \frac{1}{2} = 0.$$

Since $0 < -(x-t) < \pi$ and $0 < 2\pi-(x+t) < \pi$ in R_4,

$$\text{in } R_4, \qquad u(x,t) = -\frac{1}{2} - \frac{1}{2} = -1.$$



□

Problem 2. *Consider the initial/boundary value problem*

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & 0 < x < L,\ t > 0 \\ u(x,0) = g(x), \quad u_t(x,0) = h(x) & 0 < x < L \\ u_x(0,t) = 0, \quad u_x(L,t) = 0 & t \geq 0. \end{cases} \tag{16.12}$$

Proof. Find $u(x,t)$ in the form

$$u(x,t) = \frac{a_0(t)}{2} + \sum_{n=1}^{\infty} a_n(t) \cos\frac{n\pi x}{L} + b_n(t) \sin\frac{n\pi x}{L}.$$

- Functions $a_n(t)$ and $b_n(t)$ are determined by the boundary conditions:

$$u_x(x,t) = \sum_{n=1}^{\infty} -a_n(t)\Big(\frac{n\pi}{L}\Big) \sin\frac{n\pi x}{L} + b_n(t)\Big(\frac{n\pi}{L}\Big)\cos\frac{n\pi x}{L},$$

$$0 = u_x(0,t) = \sum_{n=1}^{\infty} b_n(t)\Big(\frac{n\pi}{L}\Big) \quad \Rightarrow \quad b_n(t) = 0. \qquad \text{Thus,}$$

$$u(x,t) = \frac{a_0(t)}{2} + \sum_{n=1}^{\infty} a_n(t) \cos\frac{n\pi x}{L}. \tag{16.13}$$

- If we substitute (16.13) into the equation $u_{tt} - c^2 u_{xx} = 0$, we get

$$\frac{a_0''(t)}{2} + \sum_{n=1}^{\infty} a_n''(t) \cos\frac{n\pi x}{L} + c^2 \sum_{n=1}^{\infty} a_n(t)\Big(\frac{n\pi}{L}\Big)^2 \cos\frac{n\pi x}{L} = 0,$$

$$a_0''(t) = 0 \qquad \text{and} \qquad a_n''(t) + \Big(\frac{n\pi c}{L}\Big)^2 a_n(t) = 0,$$

whose general solutions are

$$a_0(t) = c_0 t + d_0 \qquad \text{and} \qquad a_n(t) = c_n \sin\frac{n\pi ct}{L} + d_n \cos\frac{n\pi ct}{L}. \tag{16.14}$$

Also, $a_0'(t) = c_0$ and $a_n'(t) = c_n(\frac{n\pi c}{L}) \cos\frac{n\pi ct}{L} - d_n(\frac{n\pi c}{L}) \sin\frac{n\pi ct}{L}$.

- The constants c_n and d_n are determined by the initial conditions:

$$\begin{aligned} g(x) = u(x,0) &= \frac{a_0(0)}{2} + \sum_{n=1}^{\infty} a_n(0) \cos\frac{n\pi x}{L} = \frac{d_0}{2} + \sum_{n=1}^{\infty} d_n \cos\frac{n\pi x}{L}, \\ h(x) = u_t(x,0) &= \frac{a_0'(0)}{2} + \sum_{n=1}^{\infty} a_n'(0) \cos\frac{n\pi x}{L} = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \frac{n\pi c}{L} \cos\frac{n\pi x}{L}. \end{aligned}$$

By orthogonality, we may multiply both equations by $\cos(m\pi x/L)$, including $m = 0$, and integrate:

$$\int_0^L g(x)\,dx = d_0 \frac{L}{2}, \qquad \int_0^L g(x) \cos\frac{m\pi x}{L}\,dx = d_m \frac{L}{2},$$

$$\int_0^L h(x)\,dx = c_0 \frac{L}{2}, \qquad \int_0^L h(x) \cos\frac{m\pi x}{L}\,dx = c_m \frac{m\pi c}{L}\frac{L}{2}.$$

Thus,

$$d_n = \frac{2}{L}\int_0^L g(x) \cos\frac{n\pi x}{L}\,dx, \qquad c_n = \frac{2}{n\pi c}\int_0^L h(x) \cos\frac{n\pi x}{L}\,dx, \qquad c_0 = \frac{2}{L}\int_0^L h(x)\,dx. \tag{16.15}$$

The formulas (16.13), (16.14), and (16.15) define the solution. $\square$

Example (McOwen 3.1 #3). *Consider the initial/boundary value problem*

$$\begin{cases} u_{tt} - u_{xx} = 0 & 0 < x < \pi,\ t > 0 \\ u(x,0) = x, \quad u_t(x,0) = 0 & 0 < x < \pi \\ u_x(0,t) = 0, \quad u_x(\pi,t) = 0 & t \geq 0. \end{cases} \tag{16.16}$$

Proof. Find $u(x,t)$ in the form

$$u(x,t) = \frac{a_0(t)}{2} + \sum_{n=1}^{\infty} a_n(t)\cos nx + b_n(t)\sin nx.$$

• Functions $a_n(t)$ and $b_n(t)$ are determined by the boundary conditions:

$$u_x(x,t) = \sum_{n=1}^{\infty} -a_n(t)n\sin nx + b_n(t)n\cos nx,$$

$$0 = u_x(0,t) = \sum_{n=1}^{\infty} b_n(t)n \quad \Rightarrow \quad b_n(t) = 0. \qquad \text{Thus,}$$

$$u(x,t) = \frac{a_0(t)}{2} + \sum_{n=1}^{\infty} a_n(t)\cos nx. \tag{16.17}$$

• If we substitute (16.17) into the equation $u_{tt} - u_{xx} = 0$, we get

$$\frac{a_0''(t)}{2} + \sum_{n=1}^{\infty} a_n''(t)\cos nx + \sum_{n=1}^{\infty} a_n(t)n^2\cos nx = 0,$$

$$a_0''(t) = 0 \qquad \text{and} \qquad a_n''(t) + n^2 a_n(t) = 0,$$

whose general solutions are

$$a_0(t) = c_0 t + d_0 \qquad \text{and} \qquad a_n(t) = c_n \sin nt + d_n \cos nt. \tag{16.18}$$

Also, $a_0'(t) = c_0$ and $a_n'(t) = c_n n\cos nt - d_n n\sin nt$.
• The constants c_n and d_n are determined by the initial conditions:

$$\begin{aligned} x = u(x,0) &= \frac{a_0(0)}{2} + \sum_{n=1}^{\infty} a_n(0)\cos nx = \frac{d_0}{2} + \sum_{n=1}^{\infty} d_n\cos nx, \\ 0 = u_t(x,0) &= \frac{a_0'(0)}{2} + \sum_{n=1}^{\infty} a_n'(0)\cos nx = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n n\cos nx. \end{aligned}$$

By orthogonality, we may multiply both equations by $\cos mx$, including $m = 0$, and integrate:

$$\int_0^\pi x\,dx = d_0\frac{\pi}{2}, \qquad \int_0^\pi x\cos mx\,dx = d_m\frac{\pi}{2},$$

$$\int_0^\pi 0\,dx = c_0\frac{\pi}{2}, \qquad \int_0^\pi 0\cos mx\,dx = c_m m\frac{\pi}{2}.$$

Thus,

$$d_0 = \pi, \qquad d_n = \frac{2}{\pi n^2}(\cos n\pi - 1), \qquad c_n = 0. \tag{16.19}$$

Using this in (16.18) and (16.17), we get

$$a_0(t) = d_0 = \pi, \qquad a_n(t) = \frac{2}{\pi n^2}(\cos n\pi - 1)\cos nt,$$

$$\boxed{u(x,t) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(\cos n\pi - 1)\cos nt \cos nx}{n^2}.}$$

$\square$

We can **sum the series in regions bouded by characteristics**. We have

$$u(x,t) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(\cos n\pi - 1)\cos nt \cos nx}{n^2}, \qquad \text{or}$$

$$u(x,t) = \frac{\pi}{2}+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(\cos n\pi - 1)\cos[n(x-t)]}{n^2}+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(\cos n\pi - 1)\cos[n(x+t)]}{n^2}. \quad (16.20)$$

The initial condition may be written as

$$u(x,0) = x = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(\cos n\pi - 1)\cos nx}{n^2} \qquad \text{for} \quad 0 < x < \pi,$$

which implies

$$\frac{x}{2} - \frac{\pi}{4} = \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(\cos n\pi - 1)\cos nx}{n^2} \qquad \text{for} \quad 0 < x < \pi, \quad (16.21)$$

We can use (16.21) to sum the series in (16.20).

$$\text{In } R_1, \qquad u(x,t) = \frac{\pi}{2} + \frac{x-t}{2} - \frac{\pi}{4} + \frac{x+t}{2} - \frac{\pi}{4} = x.$$

Since $\cos[n(x-t)] = \cos[n(-(x-t))]$, and $0 < -(x-t) < \pi$ in R_2,

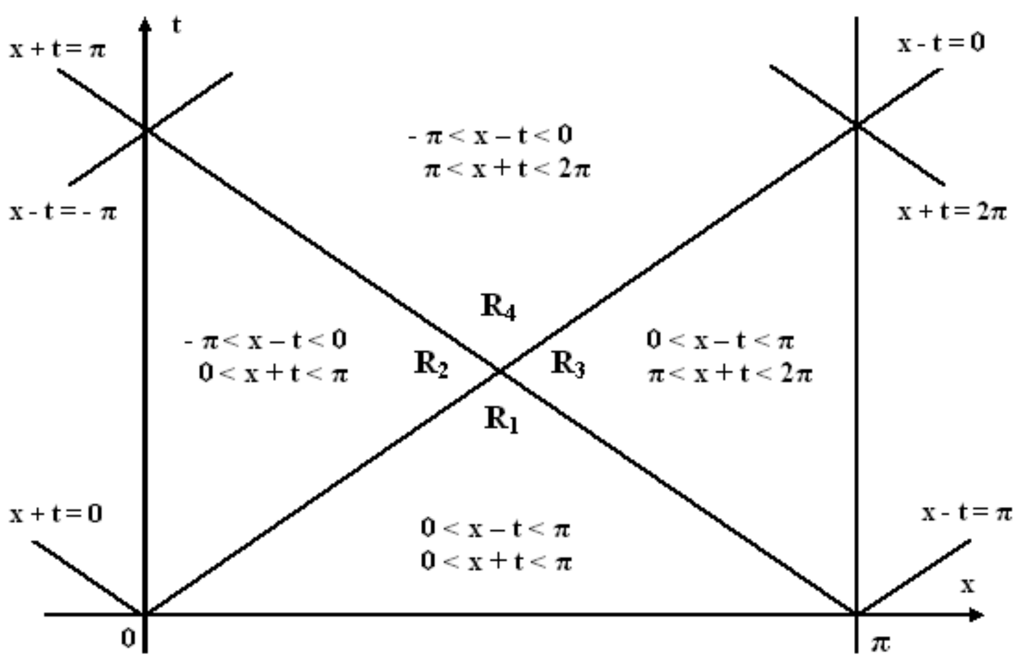
$$\text{in } R_2, \qquad u(x,t) = \frac{\pi}{2} + \frac{-(x-t)}{2} - \frac{\pi}{4} + \frac{x+t}{2} - \frac{\pi}{4} = t.$$

Since $\cos[n(x+t)] = \cos[n(x+t-2\pi)] = \cos[n(2\pi-(x+t))]$, and $0 < 2\pi-(x+t) < \pi$ in R_3,

$$\text{in } R_3, \qquad u(x,t) = \frac{\pi}{2} + \frac{x-t}{2} - \frac{\pi}{4} + \frac{2\pi-(x+t)}{2} - \frac{\pi}{4} = \pi - t.$$

Since $0 < -(x-t) < \pi$ and $0 < 2\pi-(x+t) < \pi$ in R_4

$$\text{in } R_4, \qquad u(x,t) = \frac{\pi}{2} + \frac{-(x-t)}{2} - \frac{\pi}{4} + \frac{2\pi-(x+t)}{2} - \frac{\pi}{4} = \pi - x.$$



Example (McOwen 3.1 #4). *Consider the initial boundary value problem*

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & \text{for} \quad x > 0,\, t > 0 \\ u(x,0) = g(x), \quad u_t(x,0) = h(x) & \text{for} \quad x > 0 \\ u(0,t) = 0 & \text{for } t \geq 0, \end{cases} \tag{16.22}$$

where $g(0) = 0 = h(0)$. *If we extend* g *and* h *as* **odd** *functions on* $-\infty < x < \infty$, *show that d'Alembert's formula gives the solution.*

Proof. Extend g and h as **odd** functions on $-\infty < x < \infty$:

$$\tilde{g}(x) = \begin{cases} g(x), & x \geq 0 \\ -g(-x), & x < 0 \end{cases} \qquad \tilde{h}(x) = \begin{cases} h(x), & x \geq 0 \\ -h(-x), & x < 0. \end{cases}$$

Then, we need to solve

$$\begin{cases} \tilde{u}_{tt} - c^2 \tilde{u}_{xx} = 0 & \text{for} \quad -\infty < x < \infty,\, t > 0 \\ \tilde{u}(x,0) = \tilde{g}(x), \quad \tilde{u}_t(x,0) = \tilde{h}(x) & \text{for} \quad -\infty < x < \infty. \end{cases} \tag{16.23}$$

To show that d'Alembert's formula gives the solution to (16.23), we need to show that the solution given by d'Alembert's formula satisfies the boundary condition $\tilde{u}(0,t) = 0$.

$$\begin{aligned} \tilde{u}(x,t) &= \frac{1}{2}(\tilde{g}(x+ct) + \tilde{g}(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \tilde{h}(\xi)\, d\xi, \\ \tilde{u}(0,t) &= \frac{1}{2}(\tilde{g}(ct) + \tilde{g}(-ct)) + \frac{1}{2c}\int_{-ct}^{ct} \tilde{h}(\xi)\, d\xi \\ &= \frac{1}{2}(\tilde{g}(ct) - \tilde{g}(ct)) + \frac{1}{2c}(H(ct) - H(-ct)) \\ &= 0 + \frac{1}{2c}(H(ct) - H(ct)) = 0, \end{aligned}$$

where we used $H(x) = \int_0^x \tilde{h}(\xi)\, d\xi$; and since $\tilde{h}$ is **odd**, then H is **even**. □

Example (McOwen 3.1 #5). *Find in closed form (similar to d'Alembet's formula) the solution* $u(x,t)$ *of*

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & \text{for} \quad x, t > 0 \\ u(x,0) = g(x), \quad u_t(x,0) = h(x) & \text{for} \quad x > 0 \\ u(0,t) = \alpha(t) & \text{for } t \geq 0, \end{cases} \tag{16.24}$$

where $g, h, \alpha \in C^2$ *satisfy* $\alpha(0) = g(0)$, $\alpha'(0) = h(0)$, *and* $\alpha''(0) = c^2 g''(0)$. *Verify that* $u \in C^2$, *even on the characteristic* $x = ct$.

Proof. As in (McOwen 3.1 #4), we can extend g and h to be odd functions. We want to transform the problem to have zero boundary conditions.
Consider the function:

$$U(x,t) = u(x,t) - \alpha(t). \tag{16.25}$$

Then (16.24) transforms to:

$$\begin{cases} U_{tt} - c^2 U_{xx} = \underbrace{-\alpha''(t)}_{f_U(x,t)} \\ U(x,0) = \underbrace{g(x) - \alpha(0)}_{g_U(x)}, \quad U_t(x,0) = \underbrace{h(x) - \alpha'(0)}_{h_U(x)} \\ U(0,t) = \underbrace{0}_{\alpha_u(t)}. \end{cases}$$

We use d'Alembert's formula and Duhamel's principle on U.
After getting U, we can get u from $u(x,t) = U(x,t) + \alpha(t)$. $\square$

Example (Zachmanoglou, Chapter 8, Example 7.2). *Find the solution of*

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & \textit{for} \;\; x > 0, t > 0 \\ u(x,0) = g(x), \quad u_t(x,0) = h(x) & \textit{for} \;\; x > 0 \\ u_x(0,t) = 0 & \textit{for } t > 0. \end{cases} \tag{16.26}$$

Proof. Extend g and h as **even** functions on $-\infty < x < \infty$:

$$\tilde{g}(x) = \begin{cases} g(x), & x \geq 0 \\ g(-x), & x < 0 \end{cases} \qquad \tilde{h}(x) = \begin{cases} h(x), & x \geq 0 \\ h(-x), & x < 0. \end{cases}$$

Then, we need to solve

$$\begin{cases} \tilde{u}_{tt} - c^2 \tilde{u}_{xx} = 0 & \text{for} \quad -\infty < x < \infty,\, t > 0 \\ \tilde{u}(x,0) = \tilde{g}(x), \quad \tilde{u}_t(x,0) = \tilde{h}(x) & \text{for} \quad -\infty < x < \infty. \end{cases} \tag{16.27}$$

To show that d'Alembert's formula gives the solution to (16.27), we need to show that the solution given by d'Alembert's formula satisfies the boundary condition $\tilde{u}_x(0,t) = 0$.

$$\begin{aligned} \tilde{u}(x,t) &= \frac{1}{2}(\tilde{g}(x+ct) + \tilde{g}(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \tilde{h}(\xi)\, d\xi. \\ \tilde{u}_x(x,t) &= \frac{1}{2}(\tilde{g}'(x+ct) + \tilde{g}'(x-ct)) + \frac{1}{2c}[\tilde{h}(x+ct) - \tilde{h}(x-ct)], \\ \tilde{u}_x(0,t) &= \frac{1}{2}(\tilde{g}'(ct) + \tilde{g}'(-ct)) + \frac{1}{2c}[\tilde{h}(ct) - \tilde{h}(-ct)] \;=\; 0. \end{aligned}$$

Since $\tilde{g}$ is **even**, then g' is **odd**. $\square$

Problem (F'89, #3). [36] *Let $\alpha \neq c$, constant. Find the solution of*

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & \textit{for} \;\; x > 0, t > 0 \\ u(x,0) = g(x), \quad u_t(x,0) = h(x) & \textit{for} \;\; x > 0 \\ u_t(0,t) = \alpha u_x(0,t) & \textit{for } t > 0, \end{cases} \tag{16.28}$$

where $g, h \in C^2$ for $x > 0$ and vanish near $x = 0$.
Hint: Use the fact that a general solution of (16.28) can be written as the sum of two traveling wave solutions.

Proof. D'Alembert's formula is derived by plugging in the following into the above equation and initial conditions:

$$u(x,t) = F(x+ct) + G(x-ct).$$

As in (Zachmanoglou 7.2), we can extend g and h to be even functions. $\square$

[36] Similar to McOwen 3.1 #5. The notation in this problem is changed to be consistent with McOwen.

Example (McOwen 3.1 #6). *Solve the initial/boundary value problem*

$$\begin{cases} u_{tt} - u_{xx} = 1 & \textit{for} \;\; 0 < x < \pi \;\; \textit{and} \;\; t > 0 \\ u(x,0) = 0, \quad u_t(x,0) = 0 & \textit{for} \;\; 0 < x < \pi \\ u(0,t) = 0, \quad u(\pi,t) = -\pi^2/2 & \textit{for} \; t \geq 0. \end{cases} \tag{16.29}$$

Proof. If we first find a particular solution of the nonhomogeneous equation, this reduces the problem to a boundary value problem for the homogeneous equation (as in (McOwen 3.1 #2) and (McOwen 3.1 #3)).
Hint: You should use a particular solution depending on x!

❶ First, find a particular solution. This is similar to the method of separation of variables. Assume

$$u_p(x,t) = X(x),$$

which gives

$$\begin{aligned} -X''(x) &= 1, \\ X''(x) &= -1. \end{aligned}$$

The solution to the above ODE is

$$X(x) = -\frac{x^2}{2} + ax + b.$$

The boundary conditions give

$$\begin{aligned} u_p(0,t) &= b = 0, \\ u_p(\pi,t) &= -\frac{\pi^2}{2} + a\pi + b = -\frac{\pi^2}{2}, \qquad \Rightarrow \qquad a = b = 0. \end{aligned}$$

Thus, the particular solution is

$$\boxed{u_p(x,t) = -\frac{x^2}{2}.}$$

This solution satisfies the following:

$$\begin{cases} u_{p_{tt}} - u_{p_{xx}} = 1 \\ u_p(x,0) = -\frac{x^2}{2}, \quad u_{p_t}(x,0) = 0 \\ u_p(0,t) = 0, \quad u_p(\pi,t) = -\frac{\pi^2}{2}. \end{cases}$$

❷ Second, we find a solution to a boundary value problem for the homogeneous equation:

$$\begin{cases} u_{tt} - u_{xx} = 0 \\ u(x,0) = \frac{x^2}{2}, \quad u_t(x,0) = 0 \\ u(0,t) = 0, \quad u(\pi,t) = 0. \end{cases}$$

This is solved by the method of Separation of Variables. See Separation of Variables subsection of "Problems: Separation of Variables: Wave Equation" McOwen 3.1 #2. The only difference there is that $u(x,0) = 1$.
We would find $u_h(x,t)$. Then,

$$\boxed{u(x,t) = u_h(x,t) + u_p(x,t).}$$

□

Problem (S'02, #2). ***a)*** *Given a continuous function f on $\mathbb{R}$ which vanishes for $|x| > R$, solve the initial value problem*

$$\begin{cases} u_{tt} - u_{xx} = f(x)\cos t, \\ u(x,0) = 0, \quad u_t(x,0) = 0, \qquad -\infty < x < \infty, \ 0 \le t < \infty \end{cases}$$

*by first finding a particular solution by **separation of variables** and then adding the appropriate solution of the homogeneous PDE.*
b) *Since the particular solution is not unique, it will not be obvious that the solution to the initial value problem that you have found in part (a) is unique. Prove that it is **unique**.*

Proof. **a)** ❶ First, find a particular solution by separation of variables. Assume

$$u_p(x,t) = X(x)\cos t,$$

which gives

$$\begin{aligned} -X(x)\cos t - X''(x)\cos t &= f(x)\cos t, \\ X'' + X &= -f(x). \end{aligned}$$

The solution to the above ODE is written as $X = X_h + X_p$. The homogeneous solution is

$$X_h(x) = a\cos x + b\sin x.$$

To find a particular solution, note that since f is continuous, $\exists G \in C^2(\mathbb{R})$, such that

$$G'' + G = -f(x).$$

Thus,

$$\begin{aligned} & X_p(x) = G(x). \\ \Rightarrow \quad & X(x) = X_h(x) + X_p(x) = a\cos x + b\sin x + G(x). \end{aligned}$$

$$\boxed{u_p(x,t) = \big[a\cos x + b\sin x + G(x)\big]\cos t.}$$

It can be verified that this solution satisfies the following:

$$\begin{cases} u_{p_{tt}} - u_{p_{xx}} = f(x)\cos t, \\ u_p(x,0) = a\cos x + b\sin x + G(x), \qquad u_{p_t}(x,0) = 0. \end{cases}$$

❷ Second, we find a solution of the homogeneous PDE:

$$\begin{cases} u_{tt} - u_{xx} = 0, \\ u(x,0) = \underbrace{-a\cos x - b\sin x - G(x)}_{g(x)}, \quad u_t(x,0) = \underbrace{0}_{h(x)}. \end{cases}$$

The solution is given by d'Alembert's formula (with $c = 1$):

$$\begin{aligned} u_h(x,t) &= u^A(x,t) = \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(\xi)\,d\xi \\ &= \frac{1}{2}\Big(\big(-a\cos(x+t) - b\sin(x+t) - G(x+t)\big) + \big(-a\cos(x-t) - b\sin(x-t) - G(x-t)\big)\Big) \\ &= -\frac{1}{2}\big(a\cos(x+t) + b\sin(x+t) + G(x+t)\big) - \frac{1}{2}\big(a\cos(x-t) + b\sin(x-t) + G(x-t)\big). \end{aligned}$$

It can be verified that the solution satisfies the above homogeneous PDE with the boundary conditions. Thus, the complete solution is:

$$\boxed{u(x,t) = u_h(x,t) + u_p(x,t).}$$

Alternatively, we could use Duhamel's principle to find the solution: [37]

$$\boxed{u(x,t) = \frac{1}{2}\int_0^t \left(\int_{x-(t-s)}^{x+(t-s)} f(\xi)\ \cos s\ d\xi \right) ds.}$$

However, this is not how it was suggested to do this problem.

b) The particular solution is **not** unique, since any constants a, b give the solution. However, we show that the solution to the initial value problem is unique.
Suppose u_1 and u_2 are two solutions. Then $w = u_1 - u_2$ satisfies:

$$\begin{cases} w_{tt} - w_{xx} = 0, \\ w(x,0) = 0, \quad w_t(x,0) = 0. \end{cases}$$

D'Alembert's formula gives

$$w(x,t) \;=\; \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(\xi)\,d\xi \;=\; 0.$$

Thus, the solution to the initial value problem is unique. $\square$

[37] Note the relationship: $x \leftrightarrow \xi,\ t \leftrightarrow s.$

16.3 Similarity Solutions

Problem (F'98, #7). *Look for a **similarity solution** of the form $v(x,t) = t^\alpha w(y = x/t^\beta)$ for the differential equation*

$$v_t = v_{xx} + (v^2)_x. \tag{16.30}$$

a) Find the parameters α and β.
b) Find a differential equation for $w(y)$ and show that this ODE can be reduced to first order.
c) Find a solution for the resulting first order ODE.

Proof. We can rewrite (16.30) as

$$v_t = v_{xx} + 2vv_x. \tag{16.31}$$

We look for a similarity solution of the form

$$v(x,t) = t^\alpha w(y), \qquad \Big(y = \frac{x}{t^\beta}\Big).$$

$$\begin{aligned}
v_t &= \alpha t^{\alpha-1} w + t^\alpha w' y_t = \alpha t^{\alpha-1} w + t^\alpha\Big(-\frac{\beta x}{t^{\beta+1}}\Big) w' = \alpha t^{\alpha-1} w - t^{\alpha-1}\beta y w', \\
v_x &= t^\alpha w' y_x = t^\alpha w' t^{-\beta} = t^{\alpha-\beta} w', \\
v_{xx} &= (t^{\alpha-\beta} w')_x = t^{\alpha-\beta} w'' y_x = t^{\alpha-\beta} w'' t^{-\beta} = t^{\alpha-2\beta} w''.
\end{aligned}$$

Plugging in the derivatives we calculated into (16.31), we obtain

$$\begin{aligned}
\alpha t^{\alpha-1} w - t^{\alpha-1}\beta y w' &= t^{\alpha-2\beta} w'' + 2(t^\alpha w)(t^{\alpha-\beta} w'), \\
\alpha w - \beta y w' &= t^{1-2\beta} w'' + 2t^{\alpha-\beta+1} w w'.
\end{aligned}$$

The parameters that would eliminate t from equation above are

$$\boxed{\beta = \frac{1}{2}, \quad \alpha = -\frac{1}{2}.}$$

With these parameters, we obtain the differential equation for $w(y)$:

$$-\frac{1}{2}w - \frac{1}{2}yw' = w'' + 2ww',$$

$$\boxed{w'' + 2ww' + \frac{1}{2}yw' + \frac{1}{2}w = 0.}$$

We can write the ODE as

$$w'' + 2ww' + \frac{1}{2}(yw)' = 0.$$

Integrating it with respect to y, we obtain the first order ODE:

$$\boxed{w' + w^2 + \frac{1}{2}yw = c.}$$

$\square$

16.4 Traveling Wave Solutions

Consider the *Korteweg-de Vries* (KdV) equation in the form [38]

$$u_t + 6uu_x + u_{xxx} = 0, \qquad -\infty < x < \infty,\ t > 0. \tag{16.32}$$

We look for a *traveling wave solution*

$$u(x,t) = f(x - ct). \tag{16.33}$$

We get the ODE

$$-cf' + 6ff' + f''' = 0. \tag{16.34}$$

We integrate (16.34) to get

$$-cf + 3f^2 + f'' = a, \tag{16.35}$$

where a is a constant. Multiplying this equality by f', we obtain

$$-cff' + 3f^2 f' + f'' f' = af'.$$

Integrating again, we get

$$-\frac{c}{2}f^2 + f^3 + \frac{(f')^2}{2} = af + b. \tag{16.36}$$

We are looking for solutions f which satisfy $f(x)$, $f'(x)$, $f''(x) \to 0$ as $x \to \pm\infty$. (In which case the function u having the form (16.33) is called a *solitary* wave.) Then (16.35) and (16.36) imply $a = b = 0$, so that

$$-\frac{c}{2}f^2 + f^3 + \frac{(f')^2}{2} = 0, \qquad \text{or} \quad f' = \pm f\sqrt{c - 2f}.$$

The solution of this ODE is

$$f(x) = \frac{c}{2}\text{sech}^2[\frac{\sqrt{c}}{2}(x - x_0)],$$

where x_0 is the constant of integration. A solution of this form is called a *soliton*.

[38] Evans, p. 174; Strauss, p. 367.

Problem (S'93, #6). *The generalized KdV equation is*

$$\frac{\partial u}{\partial t} = \frac{1}{2}(n+1)(n+2)u^n\frac{\partial u}{\partial x} - \frac{\partial^3 u}{\partial x^3},$$

where n is a positive integer. Solitary wave solutions are sought in which $u = f(\eta)$, where $\eta = x - ct$ and

$$f, f', f'' \to 0, \qquad as \ \ |\eta| \to \infty;$$

c, the wave speed, is constant.
Show that

$$f'^2 = f^{n+2} + cf^2.$$

Hence show that solitary waves do not exist if n is even.
Show also that, when $n = 1$, all conditions of the problem are satisfied provided $c > 0$ and

$$u = -c\, sech^2\Big[\frac{1}{2}\sqrt{c}(x - ct)\Big].$$

Proof. • We look for a *traveling wave solution*

$$u(x,t) = f(x - ct).$$

We get the ODE

$$-cf' = \frac{1}{2}(n+1)(n+2)f^n f' - f''',$$

Integrating this equation, we get

$$-cf = \frac{1}{2}(n+2)f^{n+1} - f'' + a, \tag{16.37}$$

where a is a constant. Multiplying this equality by f', we obtain

$$-cff' = \frac{1}{2}(n+2)f^{n+1}f' - f''f' + af'.$$

Integrating again, we get

$$-\frac{cf^2}{2} = \frac{1}{2}f^{n+2} - \frac{(f')^2}{2} + af + b. \tag{16.38}$$

We are looking for solutions f which satisfy $f, f', f'' \to 0$ as $x \to \pm\infty$. Then (16.37) and (16.38) imply $a = b = 0$, so that

$$\begin{aligned} -\frac{cf^2}{2} &= \frac{1}{2}f^{n+2} - \frac{(f')^2}{2}, \\ (f')^2 &= f^{n+2} + cf^2. \quad \checkmark \end{aligned}$$

• We show that solitary waves do not exist if n is even. We have

$$\begin{aligned} f' &= \pm\sqrt{f^{n+2} + cf^2} = \pm|f|\sqrt{f^n + c}, \\ \int_{-\infty}^{\infty} f'\, d\eta &= \pm\int_{-\infty}^{\infty} |f|\sqrt{f^n + c}\, d\eta, \\ f\Big|_{-\infty}^{\infty} &= \pm\int_{-\infty}^{\infty} |f|\sqrt{f^n + c}\, d\eta, \\ 0 &= \pm\int_{-\infty}^{\infty} |f|\sqrt{f^n + c}\, d\eta. \end{aligned}$$

Thus, either ① $|f| \equiv 0 \;\Rightarrow\; f = 0$, or
② $f^n + c = 0$. Since $f \to 0$ as $x \to \pm\infty$, we have $c = 0 \;\Rightarrow\; f = 0$.
Thus, solitary waves do not exist if n is even. ✓ $\square$

- When $n = 1$, we have

$$(f')^2 = f^3 + cf^2. \tag{16.39}$$

We show that all conditions of the problem are satisfied provided $c > 0$, including

$$\begin{aligned} u &= -c\,sech^2\Big[\frac{1}{2}\sqrt{c}(x-ct)\Big], \qquad \text{or} \\ f &= -c\,sech^2\Big[\frac{\eta\sqrt{c}}{2}\Big] = -\frac{c}{\cosh^2[\frac{\eta\sqrt{c}}{2}]} = -c\cosh\Big[\frac{\eta\sqrt{c}}{2}\Big]^{-2}. \end{aligned}$$

We have

$$\begin{aligned} f' &= 2c\cosh\Big[\frac{\eta\sqrt{c}}{2}\Big]^{-3}\cdot\sinh\Big[\frac{\eta\sqrt{c}}{2}\Big]\cdot\frac{\sqrt{c}}{2} = c\sqrt{c}\cosh\Big[\frac{\eta\sqrt{c}}{2}\Big]^{-3}\cdot\sinh\Big[\frac{\eta\sqrt{c}}{2}\Big], \\ (f')^2 &= \frac{c^3\ \sinh^2\Big[\frac{\eta\sqrt{c}}{2}\Big]}{\cosh^6\Big[\frac{\eta\sqrt{c}}{2}\Big]}, \\ f^3 &= -\frac{c^3}{\cosh^6\Big[\frac{\eta\sqrt{c}}{2}\Big]}, \\ cf^2 &= \frac{c^3}{\cosh^4\Big[\frac{\eta\sqrt{c}}{2}\Big]}. \end{aligned}$$

Plugging these into (16.39), we obtain: [39]

$$\begin{aligned} \frac{c^3\ \sinh^2\Big[\frac{\eta\sqrt{c}}{2}\Big]}{\cosh^6\Big[\frac{\eta\sqrt{c}}{2}\Big]} &= -\frac{c^3}{\cosh^6\Big[\frac{\eta\sqrt{c}}{2}\Big]} + \frac{c^3}{\cosh^4\Big[\frac{\eta\sqrt{c}}{2}\Big]}, \\ \frac{c^3\ \sinh^2\Big[\frac{\eta\sqrt{c}}{2}\Big]}{\cosh^6\Big[\frac{\eta\sqrt{c}}{2}\Big]} &= \frac{-c^3 + c^3\cosh^2\Big[\frac{\eta\sqrt{c}}{2}\Big]}{\cosh^6\Big[\frac{\eta\sqrt{c}}{2}\Big]}, \\ \frac{c^3\ \sinh^2\Big[\frac{\eta\sqrt{c}}{2}\Big]}{\cosh^6\Big[\frac{\eta\sqrt{c}}{2}\Big]} &= \frac{c^3\ \sinh^2\Big[\frac{\eta\sqrt{c}}{2}\Big]}{\cosh^6\Big[\frac{\eta\sqrt{c}}{2}\Big]}. \qquad \checkmark \end{aligned}$$

Also, $f, f', f'' \to 0$, as $|\eta| \to \infty$, since

$$f(\eta) = -c\,sech^2\Big[\frac{\eta\sqrt{c}}{2}\Big] = -\frac{c}{\cosh^2[\frac{\eta\sqrt{c}}{2}]} = -c\bigg(\frac{2}{e^{[\frac{\eta\sqrt{c}}{2}]} + e^{-[\frac{\eta\sqrt{c}}{2}]}}\bigg)^2 \to 0, \text{ as } |\eta| \to \infty.$$

Similarly, $f', f'' \to 0$, as $|\eta| \to \infty$. $\checkmark$

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[39] $\cosh^2 x - \sinh^2 x = 1$.

$$\cosh x = \frac{e^x + e^{-x}}{2}, \qquad \sinh x = \frac{e^x - e^{-x}}{2}$$

Problem (S'00, #5). *Look for a* ***traveling wave solution*** *of the PDE*

$$u_{tt} + (u^2)_{xx} = -u_{xxxx}$$

of the form $u(x,t) = v(x - ct)$. In particular, you should find an ODE for v. Under the assumption that v goes to a constant as $|x| \to \infty$, describe the form of the solution.

Proof. Since $(u^2)_x = 2uu_x$, and $(u^2)_{xx} = 2u_x^2 + 2uu_{xx}$, we have

$$u_{tt} + 2u_x^2 + 2uu_{xx} = -u_{xxxx}.$$

We look for a traveling wave solution

$$u(x,t) = v(x - ct).$$

We get the ODE

$$\begin{aligned}
c^2v'' + 2(v')^2 + 2vv'' &= -v'''', \\
c^2v'' + 2((v')^2 + vv'') &= -v'''', \\
c^2v'' + 2(vv')' &= -v'''', && \text{(exact differentials)} \\
c^2v' + 2vv' &= -v''' + a, && s = x - ct \\
c^2v + v^2 &= -v'' + as + b, && \circledast
\end{aligned}$$

$$\boxed{v'' + c^2v + v^2 = a(x - ct) + b.}$$

Since $v \to C = \text{const}$ as $|x| \to \infty$, we have v', $v'' \to 0$, as $|x| \to \infty$. Thus, $\circledast$ implies

$$c^2v + v^2 = as + b.$$

Since $|x| \to \infty$, but $v \to C$, we have $a = 0$:

$$v^2 + c^2v - b = 0.$$

$$v = \frac{-c^2 \pm \sqrt{c^4 + 4b}}{2}.$$

□

Problem (S'95, #2). *Consider the KdV-Burgers equation*

$$u_t + uu_x = \epsilon u_{xx} + \delta u_{xxx}$$

in which $\epsilon > 0$, $\delta > 0$.
a) *Find an ODE for* ***traveling wave solutions*** *of the form*

$$u(x,t) = \varphi(x - st)$$

with $s > 0$ *and*

$$\lim_{y\to-\infty} \varphi(y) = 0$$

and analyze the stationary points from this ODE.

b) *Find the possible (finite) values of*

$$\varphi_+ = \lim_{y\to\infty} \varphi(y).$$

Proof. **a)** We look for a traveling wave solution

$$u(x,t) = \varphi(x - st), \qquad y = x - st.$$

We get the ODE

$$\begin{aligned} -s\varphi' + \varphi\varphi' &= \epsilon\varphi'' + \delta\varphi''', \\ -s\varphi + \frac{1}{2}\varphi^2 &= \epsilon\varphi' + \delta\varphi'' + a. \end{aligned}$$

Since $\varphi \to 0$ as $y \to -\infty$, then φ', $\varphi'' \to 0$ as $y \to -\infty$. Therefore, at $y = -\infty$, $a = 0$.
We found the following ODE,

$$\boxed{\varphi'' + \frac{\epsilon}{\delta}\varphi' + \frac{s}{\delta}\varphi - \frac{1}{2\delta}\varphi^2 = 0.}$$

In order to find and analyze the stationary points of an ODE above, we write it as a first-order system.

$$\begin{aligned} \phi_1 &= \varphi, \\ \phi_2 &= \varphi'. \end{aligned}$$

$$\begin{aligned} \phi_1' &= \varphi' = \phi_2, \\ \phi_2' &= \varphi'' = -\frac{\epsilon}{\delta}\varphi' - \frac{s}{\delta}\varphi + \frac{1}{2\delta}\varphi^2 = -\frac{\epsilon}{\delta}\phi_2 - \frac{s}{\delta}\phi_1 + \frac{1}{2\delta}\phi_1^2. \end{aligned}$$

$$\begin{cases} \phi_1' = \phi_2 = 0, \\ \phi_2' = -\frac{\epsilon}{\delta}\phi_2 - \frac{s}{\delta}\phi_1 + \frac{1}{2\delta}\phi_1^2 = 0; \end{cases} \Rightarrow \begin{cases} \phi_1' = \phi_2 = 0, \\ \phi_2' = -\frac{s}{\delta}\phi_1 + \frac{1}{2\delta}\phi_1^2 = 0; \end{cases} \Rightarrow \begin{cases} \phi_1' = \phi_2 = 0, \\ \phi_2' = -\frac{1}{\delta}\phi_1(s - \frac{1}{2}\phi_1) = 0. \end{cases}$$

$$\boxed{\text{Stationary points: } (0,0),\ (2s,0), \quad s > 0.}$$

$$\begin{aligned} \phi_1' &= \phi_2 && = f(\phi_1, \phi_2), \\ \phi_2' &= -\frac{\epsilon}{\delta}\phi_2 - \frac{s}{\delta}\phi_1 + \frac{1}{2\delta}\phi_1^2 && = g(\phi_1, \phi_2). \end{aligned}$$

In order to classify a stationary point, need to find eigenvalues of a linearized system at that point.

$$J(f(\phi_1,\phi_2), g(\phi_1,\phi_2)) = \begin{bmatrix} \frac{\partial f}{\partial \phi_1} & \frac{\partial f}{\partial \phi_2} \\ \frac{\partial g}{\partial \phi_1} & \frac{\partial g}{\partial \phi_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{s}{\delta} + \frac{1}{\delta}\phi_1 & -\frac{\epsilon}{\delta} \end{bmatrix}.$$

- For $(\phi_1,\phi_2) = (0,0)$:

$\det(J|_{(0,0)} - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -\frac{s}{\delta} & -\frac{\epsilon}{\delta} - \lambda \end{vmatrix} = \lambda^2 + \frac{\epsilon}{\delta}\lambda + \frac{s}{\delta} = 0.$

$\lambda_\pm = -\frac{\epsilon}{2\delta} \pm \sqrt{\frac{\epsilon^2}{4\delta^2} - \frac{s}{\delta}}.$

If $\frac{\epsilon^2}{4\delta} > s \quad \Rightarrow \quad \lambda_\pm \in \mathbb{R}, \quad \lambda_\pm < 0.$

$\Rightarrow$ **(0,0)** is **Stable Improper Node**.

If $\frac{\epsilon^2}{4\delta} < s \quad \Rightarrow \quad \lambda_\pm \in \mathbb{C}, \quad \mathbb{R}e(\lambda_\pm) < 0.$

$\Rightarrow$ **(0,0)** is **Stable Spiral Point**.

- For $(\phi_1,\phi_2) = (2s,0)$:

$\det(J|_{(2s,0)} - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ \frac{s}{\delta} & -\frac{\epsilon}{\delta} - \lambda \end{vmatrix} = \lambda^2 + \frac{\epsilon}{\delta}\lambda - \frac{s}{\delta} = 0.$

$\lambda_\pm = -\frac{\epsilon}{2\delta} \pm \sqrt{\frac{\epsilon^2}{4\delta^2} + \frac{s}{\delta}}.$

$\Rightarrow \quad \lambda_+ > 0, \quad \lambda_- < 0.$

$\Rightarrow$ **(2s,0)** is **Untable Saddle Point**.

b) Since

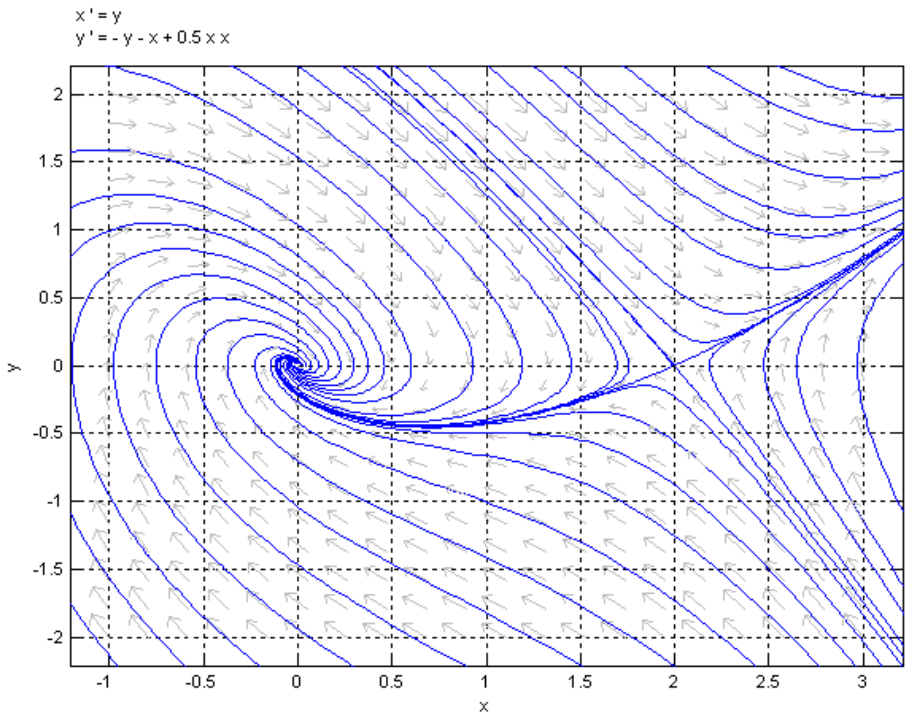
$$\lim_{y\to-\infty} \varphi(y) = 0 = \lim_{t\to\infty} \varphi(x - st),$$

we **may** have

$$\lim_{y\to+\infty} \varphi(y) = \lim_{t\to-\infty} \varphi(x - st) = 2s.$$

That is, a particle may start off at an unstable node $(2s,0)$ and as t increases, approach the stable node $(0,0)$.

A phase diagram with $(0,0)$ being a stable spiral point, is shown below.



$\square$

Problem (F'95, #8). *Consider the equation*

$$u_t + f(u)_x = \epsilon u_{xx}$$

where f is smooth and $\epsilon > 0$. We seek ***traveling wave solutions*** *to this equation, i.e., solutions of the form $u = \phi(x - st)$, under the boundary conditions*

$$u \to u_L \quad \text{and} \quad u_x \to 0 \quad \text{as} \quad x \to -\infty,$$
$$u \to u_R \quad \text{and} \quad u_x \to 0 \quad \text{as} \quad x \to +\infty.$$

Find a necessary and sufficient condition on f, u_L, u_R and s for such traveling waves to exist; in case this condition holds, write an equation which defines ϕ implicitly.

Proof. We look for traveling wave solutions

$$u(x,t) = \phi(x - st), \qquad y = x - st.$$

The boundary conditions become

$$\phi \to u_L \quad \text{and} \quad \phi' \to 0 \quad \text{as} \quad x \to -\infty,$$
$$\phi \to u_R \quad \text{and} \quad \phi' \to 0 \quad \text{as} \quad x \to +\infty. \qquad \circledast$$

Since $f(\phi(x - st))_x = f'(\phi)\phi'$, we get the ODE

$$-s\phi' + f'(\phi)\phi' = \epsilon\phi'',$$
$$-s\phi' + (f(\phi))' = \epsilon\phi'',$$
$$-s\phi + f(\phi) = \epsilon\phi' + a,$$
$$\phi' = \frac{-s\phi + f(\phi)}{\epsilon} + b.$$

We use boundary conditions to determine constant b:

$$\text{At} \quad x = -\infty, \qquad 0 = \phi' = \frac{-su_L + f(u_L)}{\epsilon} + b \qquad \Rightarrow \qquad b = \frac{su_L - f(u_L)}{\epsilon}.$$
$$\text{At} \quad x = +\infty, \qquad 0 = \phi' = \frac{-su_R + f(u_R)}{\epsilon} + b \qquad \Rightarrow \qquad b = \frac{su_R - f(u_R)}{\epsilon}.$$

$$\boxed{s = \frac{f(u_L) - f(u_R)}{u_L - u_R}.}$$

[40] □

[40] For the solution for the second part of the problem, refer to Chiu-Yen's solutions.

Problem (S'02, #5; F'90, #2). Fisher's Equation. *Consider*

$$u_t = u(1-u) + u_{xx}, \qquad -\infty < x < \infty, \ t > 0.$$

The solutions of physical interest satisfy $0 \le u \le 1$, *and*

$$\lim_{x\to-\infty} u(x,t) = 0, \qquad \lim_{x\to+\infty} u(x,t) = 1.$$

One class of solutions is the set of "wavefront" solutions. These have the form $u(x,t) = \phi(x+ct)$, $c \ge 0$.
Determine the ordinary differential equation and boundary conditions which ϕ *must satisfy (to be of physical interest). Carry out a phase plane analysis of this equation, and show that physically interesting wavefront solutions are possible if* $c \ge 2$, *but not if* $0 \le c < 2$.

Proof. We look for a traveling wave solution

$$u(x,t) = \phi(x+ct), \qquad s = x+ct.$$

We get the ODE

$$c\phi' = \phi(1-\phi) + \phi'',$$

$$\boxed{\phi'' - c\phi' + \phi - \phi^2 = 0,}$$

- ∘ $\phi(s) \to 0$, as $s \to -\infty$,
- ∘ $\phi(s) \to 1$, as $s \to +\infty$,
- ∘ $0 \le \phi \le 1$.

In order to find and analyze the stationary points of an ODE above, we write it as a first-order system.

$$y_1 = \phi,$$
$$y_2 = \phi'.$$

$$y_1' = \phi' = y_2,$$
$$y_2' = \phi'' = c\phi' - \phi + \phi^2 = cy_2 - y_1 + y_1^2.$$

$$\begin{cases} y_1' = y_2 = 0, \\ y_2' = cy_2 - y_1 + y_1^2 = 0; \end{cases} \quad \Rightarrow \quad \begin{cases} y_2 = 0, \\ y_1(y_1 - 1) = 0. \end{cases}$$

$$\boxed{\text{Stationary points: } (0,0),\ (1,0).}$$

$$y_1' = y_2 \qquad\qquad = f(y_1,y_2),$$
$$y_2' = cy_2 - y_1 + y_1^2 = g(y_1,y_2).$$

In order to classify a stationary point, need to find eigenvalues of a linearized system at that point.

$$J(f(y_1,y_2), g(y_1,y_2)) = \begin{bmatrix} \frac{\partial f}{\partial y_1} & \frac{\partial f}{\partial y_2} \\ \frac{\partial g}{\partial y_1} & \frac{\partial g}{\partial y_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2y_1 - 1 & c \end{bmatrix}.$$

- For $(y_1, y_2) = (0, 0)$:

$$\det(J|_{(0,0)} - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -1 & c - \lambda \end{vmatrix} = \lambda^2 - c\lambda + 1 = 0.$$

$$\lambda_\pm = \frac{c \pm \sqrt{c^2 - 4}}{2}.$$

If $c \geq 2 \quad \Rightarrow \quad \lambda_\pm \in \mathbb{R}, \quad \lambda_\pm > 0.$
(0,0) is **Unstable Improper** $(c > 2)$ **/ Proper** $(c = 2)$ **Node**.
If $0 \leq c < 2 \quad \Rightarrow \quad \lambda_\pm \in \mathbb{C}, \quad \mathbb{R}e(\lambda_\pm) \geq 0.$
(0,0) is **Unstable Spiral Node**.

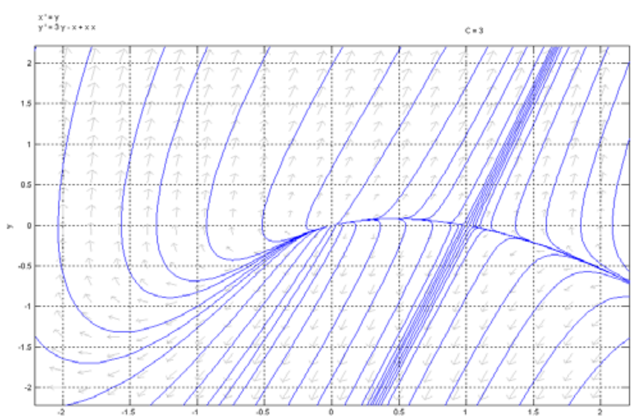
- For $(y_1, y_2) = (1, 0)$:

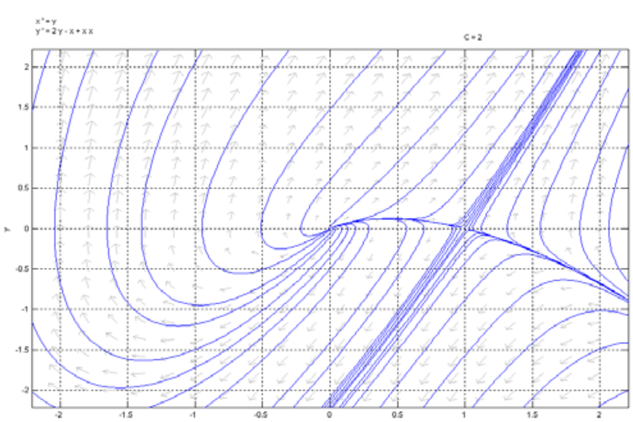
$$\det(J|_{(1,0)} - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ 1 & c - \lambda \end{vmatrix} = \lambda^2 - c\lambda - 1 = 0.$$

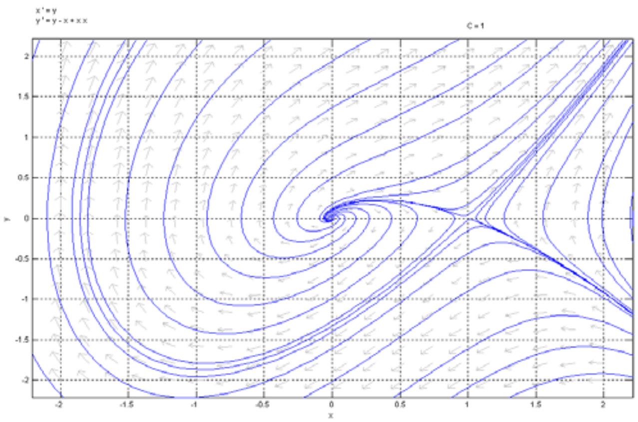
$$\lambda_\pm = \frac{c \pm \sqrt{c^2 + 4}}{2}.$$

If $c \geq 0 \quad \Rightarrow \quad \lambda_+ > 0, \quad \lambda_- < 0.$
(1,0) is **Unstable Saddle Point**.

By looking at the phase plot, a particle may start off at an unstable node $(0, 0)$ and as t increases, approach the unstable node $(1, 0)$.







$\square$

Problem (F'99, #6). *For the system*

$$\begin{aligned}\partial_t\rho + \partial_x u &= 0\\ \partial_t u + \partial_x(\rho u) &= \partial_x^2 u\end{aligned}$$

look for ***traveling wave solutions*** *of the form* $\rho(x,t) = \rho(y = x - st)$, $u(x,t) = u(y = x - st)$. *In particular*
a) *Find a first order ODE for* u.
b) *Show that this equation has solutions of the form*

$$u(y) = u_0 + u_1 \tanh(\alpha y + y_0),$$

for some constants u_0, u_1, α, y_0.

Proof. **a)** We rewrite the system:

$$\begin{aligned}\rho_t + u_x &= 0\\ u_t + \rho_x u + \rho u_x &= u_{xx}\end{aligned}$$

We look for traveling wave solutions

$$\rho(x,t) = \rho(x - st), \qquad u(x,t) = u(x - st), \qquad\qquad y = x - st.$$

We get the system of ODEs

$$\begin{cases}-s\rho' + u' = 0,\\ -su' + \rho' u + \rho u' = u''.\end{cases}$$

The first ODE gives

$$\begin{aligned}\rho' &= \frac{1}{s}u',\\ \rho &= \frac{1}{s}u + a,\end{aligned}$$

where a is a constant, and integration was done with respect to y. The second ODE gives

$$\begin{aligned}-su' + \frac{1}{s}u'u + \Big(\frac{1}{s}u + a\Big)u' &= u'',\\ -su' + \frac{2}{s}uu' + au' &= u''. \qquad \text{Integrating, we get}\\ -su + \frac{1}{s}u^2 + au &= u' + b.\end{aligned}$$

$$\boxed{u' = \frac{1}{s}u^2 + (a - s)u - b.}$$

b) Note that the ODE above may be written in the following form:

$$u' + Au^2 + Bu = C,$$

which is a nonlinear first order equation. $\square$

Problem (S'01, #7). *Consider the following system of PDEs:*

$$\begin{aligned} f_t + f_x &= g^2 - f^2 \\ g_t - g_x &= f^2 - g \end{aligned}$$

a) *Find a system of ODEs that describes* ***traveling wave solutions*** *of the PDE system; i.e. for solutions of the form $f(x,t) = f(x - st)$ and $g(x,t) = g(x - st)$.*
b) *Analyze the stationary points and draw the phase plane for this ODE system in the standing wave case $s = 0$.*

Proof. **a)** We look for traveling wave solutions

$$f(x,t) = f(x - st), \qquad g(x,t) = g(x - st).$$

We get the system of ODEs

$$\begin{aligned} -sf' + f' &= g^2 - f^2, \\ -sg' - g' &= f^2 - g. \end{aligned}$$

Thus,

$$\begin{aligned} f' &= \frac{g^2 - f^2}{1 - s}, \\ g' &= \frac{f^2 - g}{-1 - s}. \end{aligned}$$

b) If $s = 0$, the system becomes

$$\begin{cases} f' = g^2 - f^2, \\ g' = g - f^2. \end{cases}$$

Relabel the variables $f \to y_1, \quad g \to y_2$.

$$\begin{cases} y_1' = y_2^2 - y_1^2 = 0, \\ y_2' = y_2 - y_1^2 = 0. \end{cases}$$

$$\boxed{\text{Stationary points: } (0,0),\, (-1,1),\, (1,1).}$$

$$\begin{cases} y_1' = y_2^2 - y_1^2 = \phi(y_1, y_2), \\ y_2' = y_2 - y_1^2 = \psi(y_1, y_2). \end{cases}$$

In order to classify a stationary point, need to find eigenvalues of a linearized system at that point.

$$J(\phi(y_1,y_2), \psi(y_1,y_2)) = \begin{bmatrix} \frac{\partial \phi}{\partial y_1} & \frac{\partial \phi}{\partial y_2} \\ \frac{\partial \psi}{\partial y_1} & \frac{\partial \psi}{\partial y_2} \end{bmatrix} = \begin{bmatrix} -2y_1 & 2y_2 \\ -2y_1 & 1 \end{bmatrix}.$$

• For $(y_1, y_2) = (0,0)$:
$$\det(J|_{(0,0)} - \lambda I) = \begin{vmatrix} -\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = -\lambda(1 - \lambda) = 0.$$

$$\lambda_1 = 0,\ \lambda_2 = 1; \qquad \text{eigenvectors: } v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(0,0) is **Unstable Node**.

- For $(y_1, y_2) = (-1, 1)$:

$\det(J|_{(-1,1)} - \lambda I) = \begin{vmatrix} 2-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = \lambda^2 - 3\lambda - 2 = 0.$

$$\lambda_\pm = \frac{3}{2} \pm \frac{\sqrt{17}}{2}.$$

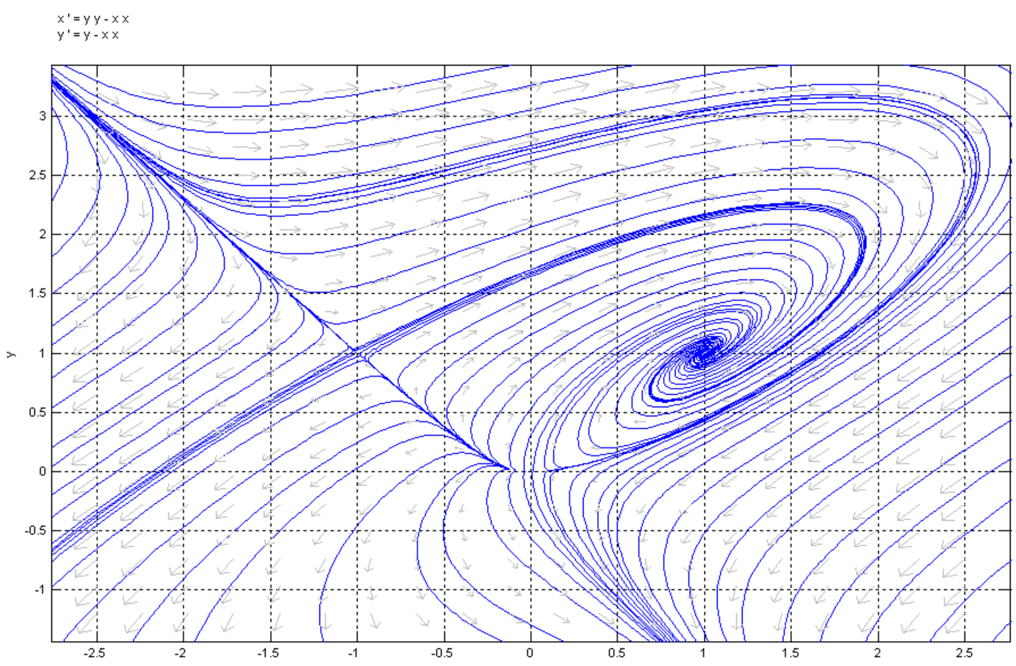
$\lambda_- < 0, \quad \lambda_+ > 0.$
(-1,1) is **Unstable Saddle Point**.

- For $(y_1, y_2) = (1, 1)$:

$\det(J|_{(1,1)} - \lambda I) = \begin{vmatrix} -2-\lambda & 2 \\ -2 & 1-\lambda \end{vmatrix} = \lambda^2 + \lambda + 2 = 0.$

$$\lambda_\pm = -\frac{1}{2} \pm i\frac{\sqrt{7}}{2}.$$

$\mathbb{R}e(\lambda_\pm) < 0.$
(1,1) is **Stable Spiral Point**.



$\square$

16.5 Dispersion

Problem (S'97, #8). *Consider the following equation*

$$u_t = (f(u_x))_x - \alpha u_{xxxx}, \qquad f(v) = v^2 - v, \tag{16.40}$$

with constant α.
a) *Linearize this equation around $u = 0$ and find the principal mode solution of the form $e^{\omega t + ikx}$. For which values of α are there unstable modes, i.e., modes with $\omega = 0$ for real k? For these values, find the maximally unstable mode, i.e., the value of k with the largest positive value of ω.*
b) *Consider the steady solution of the (fully nonlinear) problem. Show that the resulting equation can be written as a second order autonomous ODE for $v = u_x$ and draw the corresponding phase plane.*

Proof. **a)** We have

$$\begin{aligned} u_t &= (f(u_x))_x - \alpha u_{xxxx}, \\ u_t &= (u_x^2 - u_x)_x - \alpha u_{xxxx}, \\ u_t &= 2u_x u_{xx} - u_{xx} - \alpha u_{xxxx}. \quad \circledast \end{aligned}$$

However, we need to linearize (16.40) around $u = 0$. To do this, we need to linearize f.

$$f(u) = f(0) + uf'(0) + \frac{u^2}{2} f''(0) + \cdots = 0 + u(0-1) + \cdots = -u + \cdots .$$

Thus, we have

$$u_t = -u_{xx} - \alpha u_{xxxx}.$$

Consider $u(x,t) = e^{\omega t + ikx}$.

$$\begin{aligned} \omega e^{\omega t + ikx} &= (k^2 - \alpha k^4) e^{\omega t + ikx}, \\ \omega &= k^2 - \alpha k^4. \end{aligned}$$

To find unstable nodes, we set $\omega = 0$, to get

$$\boxed{\alpha = \frac{1}{k^2}.}$$

• To find the maximally unstable mode, i.e., the value of k with the largest positive value of ω, consider

$$\begin{aligned} \omega(k) &= k^2 - \alpha k^4, \\ \omega'(k) &= 2k - 4\alpha k^3. \end{aligned}$$

To find the extremas of ω, we set $\omega' = 0$. Thus,the extremas are at

$$k_1 = 0, \quad k_{2,3} = \pm\sqrt{\frac{1}{2\alpha}}.$$

To find if the extremas are maximums or minimums, we set $\omega'' = 0$:

$$\begin{aligned} \omega''(k) &= 2 - 12\alpha k^2 = 0, \\ \omega''(0) &= 2 > 0 \quad \Rightarrow \quad k = 0 \text{ is the minimum.} \\ \omega''\Big(\pm\sqrt{\frac{1}{2\alpha}}\Big) &= -4 < 0 \quad \Rightarrow \quad k = \pm\sqrt{\frac{1}{2\alpha}} \text{ is the } \textbf{maximum unstable mode}. \\ \omega\Big(\pm\sqrt{\frac{1}{2\alpha}}\Big) &= \frac{1}{4\alpha} \qquad \text{is the largest positive value of } \omega. \end{aligned}$$

b) Integrating $\circledast$, we get

$$u_x^2 - u_x - \alpha u_{xxx} = 0.$$

Let $v = u_x$. Then,

$$v^2 - v - \alpha v_{xx} = 0, \quad \text{or}$$
$$v'' = \frac{v^2 - v}{\alpha}.$$

In order to find and analyze the stationary points of an ODE above, we write it as a first-order system.

$$y_1 = v,$$
$$y_2 = v'.$$

$$y_1' = v' = y_2,$$
$$y_2' = v'' = \frac{v^2 - v}{\alpha} = \frac{y_1^2 - y_1}{\alpha}.$$

$$\begin{cases} y_1' = y_2 = 0, \\ y_2' = \frac{y_1^2 - y_1}{\alpha} = 0; \end{cases} \quad \Rightarrow \quad \begin{cases} y_2 = 0, \\ y_1(y_1 - 1) = 0. \end{cases}$$

$$\boxed{\text{Stationary points: } (0,0),\ (1,0).}$$

$$y_1' = y_2 \qquad\quad = f(y_1, y_2),$$
$$y_2' = \frac{y_1^2 - y_1}{\alpha} = g(y_1, y_2).$$

In order to classify a stationary point, need to find eigenvalues of a linearized system at that point.

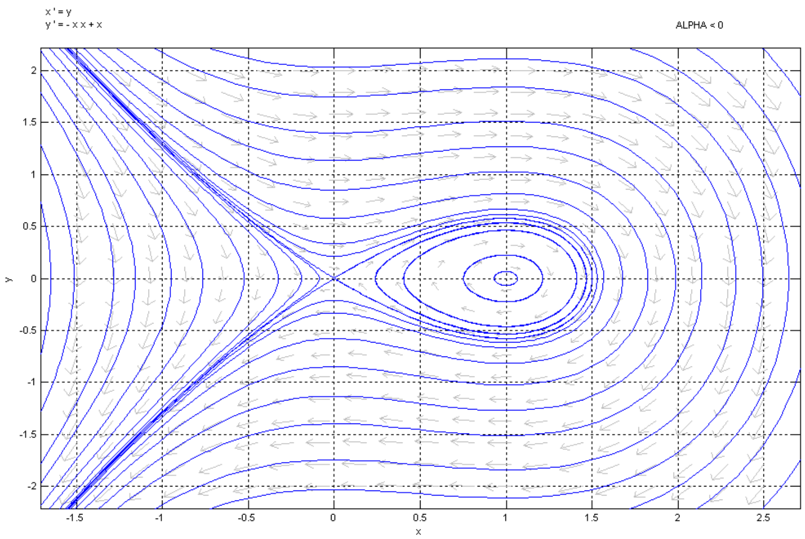
$$J(f(y_1,y_2), g(y_1,y_2)) = \begin{bmatrix} \frac{\partial f}{\partial y_1} & \frac{\partial f}{\partial y_2} \\ \frac{\partial g}{\partial y_1} & \frac{\partial g}{\partial y_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{2y_1 - 1}{\alpha} & 0 \end{bmatrix}.$$

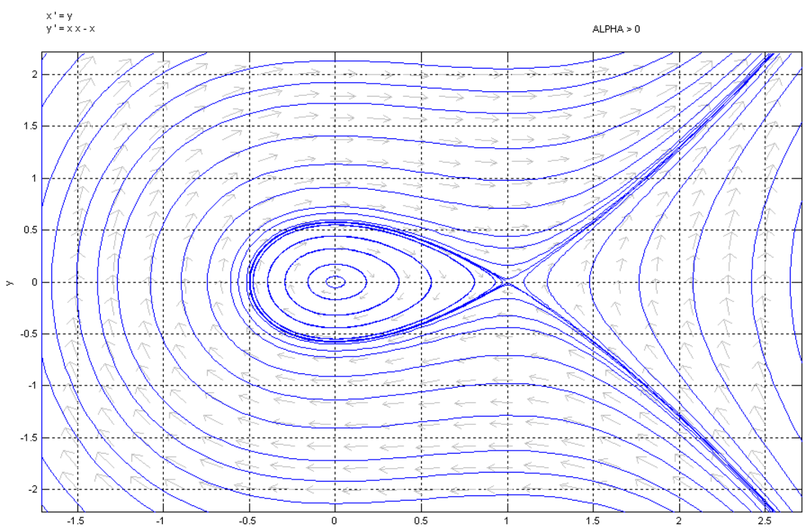
- For $(y_1, y_2) = (0,0)$, $\quad \lambda_\pm = \pm\sqrt{-\frac{1}{\alpha}}$.

If $\alpha < 0$, $\lambda_\pm \in \mathbb{R}$, $\lambda_+ > 0$, $\lambda_- < 0$. $\Rightarrow$ **(0,0)** is **Unstable Saddle Point**.
If $\alpha > 0$, $\lambda_\pm = \pm i\sqrt{\frac{1}{\alpha}} \in \mathbb{C}$, $\mathbb{R}e(\lambda_\pm) = 0$. $\Rightarrow$ **(0,0)** is **Spiral Point**.

- For $(y_1, y_2) = (1,0)$, $\quad \lambda_\pm = \pm\sqrt{\frac{1}{\alpha}}$.

If $\alpha < 0$, $\lambda_\pm = \pm i\sqrt{-\frac{1}{\alpha}} \in \mathbb{C}$, $\mathbb{R}e(\lambda_\pm) = 0$. $\Rightarrow$ **(1,0)** is **Spiral Point**.
If $\alpha > 0$, $\lambda_\pm \in \mathbb{R}$, $\lambda_+ > 0$, $\lambda_- < 0$. $\Rightarrow$ **(1,0)** is **Unstable Saddle Point**.





$\square$

16.6 Energy Methods

Problem (S'98, #9; S'96, #5). *Consider the following initial-boundary value problem for the multi-dimensional wave equation:*

$$\begin{aligned} u_{tt} &= \triangle u && \text{in } \Omega \times (0, \infty), \\ u(x, 0) = f(x), \quad \frac{\partial u}{\partial t}(x, 0) &= g(x) && \text{for } x \in \Omega, \\ \frac{\partial u}{\partial n} + a(x)\frac{\partial u}{\partial t} &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Here, Ω is a bounded domain in $\mathbb{R}^n$ and $a(x) \geq 0$. Define the Energy integral for this problem and use it in order to prove the uniqueness of the classical solution of the problem.

Proof.

$$\begin{aligned} \frac{d\tilde{E}}{dt} = 0 &= \int_\Omega (u_{tt} - \triangle u) u_t \, dx = \int_\Omega u_{tt} u_t \, dx - \int_{\partial\Omega} \frac{\partial u}{\partial n} u_t \, ds + \int_\Omega \nabla u \cdot \nabla u_t \, dx \\ &= \int_\Omega \frac{1}{2}\frac{\partial}{\partial t}(u_t^2) \, dx + \int_\Omega \frac{1}{2}\frac{\partial}{\partial t}|\nabla u|^2 \, dx + \int_{\partial\Omega} a(x) u_t^2 \, ds. \end{aligned}$$

Thus,

$$\underbrace{-\int_{\partial\Omega} a(x) u_t^2 \, dx}_{\leq 0} = \frac{1}{2}\frac{\partial}{\partial t}\int_\Omega u_t^2 + |\nabla u|^2 \, dx.$$

Let Energy integral be

$$\boxed{E(t) = \frac{1}{2}\int_\Omega u_t^2 + |\nabla u|^2 \, dx.}$$

In order to prove that the given $E(t) \leq 0$ from scratch, take its derivative with respect to t:

$$\begin{aligned} \frac{dE}{dt}(t) &= \int_\Omega \big(u_t u_{tt} + \nabla u \cdot \nabla u_t\big) \, dx \\ &= \int_\Omega u_t u_{tt} \, dx + \int_{\partial\Omega} u_t \frac{\partial u}{\partial n} \, ds - \int_\Omega u_t \triangle u \, dx \\ &= \underbrace{\int_\Omega u_t (u_{tt} - \triangle u) \, dx}_{=0} - \int_{\partial\Omega} a(x) u_t^2 \, dx \;\leq\; 0. \end{aligned}$$

Thus, $E(t) \leq E(0)$.
To prove the uniqueness of the classical solution, suppose u_1 and u_2 are two solutions of the initial boundary value problem. Let $w = u_1 - u_2$. Then, w satisfies

$$\begin{aligned} w_{tt} &= \triangle w && \text{in } \Omega \times (0, \infty), \\ w(x, 0) = 0, \quad w_t(x, 0) &= 0 && \text{for } x \in \Omega, \\ \frac{\partial w}{\partial n} + a(x)\frac{\partial w}{\partial t} &= 0 && \text{on } \partial\Omega. \end{aligned}$$

We have

$$E_w(0) = \frac{1}{2}\int_\Omega (w_t(x, 0)^2 + |\nabla w(x, 0)|^2) \, dx = 0.$$

$E_w(t) \le E_w(0) = 0 \;\Rightarrow\; E_w(t) = 0$. Thus, $w_t = 0$, $w_{x_i} = 0 \;\Rightarrow\; w(x,t) = \text{const} = 0$. Hence, $u_1 = u_2$.

$\square$

Problem (S'94, #7). *Consider the wave equation*

$$\begin{aligned} &\frac{1}{c^2(x)} u_{tt} = \triangle u && x \in \Omega \\ &\frac{\partial u}{\partial t} - \alpha(x)\frac{\partial u}{\partial n} = 0 && \textit{on} \;\; \partial\Omega \times \mathbb{R}. \end{aligned}$$

Assume that $\alpha(x)$ is of one sign for all x (i.e. α always positive or α always negative). For the energy

$$E(t) = \frac{1}{2}\int_\Omega \frac{1}{c^2(x)} u_t^2 + |\nabla u|^2 \, dx,$$

show that the sign of $\frac{dE}{dt}$ is determined by the sign of α.

Proof. We have

$$\begin{aligned} \frac{dE}{dt}(t) &= \int_\Omega \Big(\frac{1}{c^2(x)} u_t u_{tt} + \nabla u \cdot \nabla u_t\Big)\, dx \\ &= \int_\Omega \frac{1}{c^2(x)} u_t u_{tt}\, dx + \int_{\partial\Omega} u_t \frac{\partial u}{\partial n}\, ds - \int_\Omega u_t \triangle u \, dx \\ &= \underbrace{\int_\Omega u_t \Big(\frac{1}{c^2(x)} u_{tt} - \triangle u\Big)\, dx}_{=0} + \int_{\partial\Omega} \frac{1}{\alpha(x)} u_t^2 \, dx \\ &= \int_{\partial\Omega} \frac{1}{\alpha(x)} u_t^2 \, dx \;=\; \begin{cases} > 0, & \text{if } \;\alpha(x) > 0, \;\; \forall x \in \Omega, \\ < 0, & \text{if } \;\alpha(x) < 0, \;\; \forall x \in \Omega. \end{cases} \end{aligned}$$

$\square$

Problem (F'92, #2). *Let $\Omega \in \mathbb{R}^n$. Let $u(x,t)$ be a smooth solution of the following initial boundary value problem:*

$$\begin{aligned} u_{tt} - \triangle u + u^3 &= 0 \qquad \textit{for} \ \ (x,t) \in \Omega \times [0,T] \\ u(x,t) &= 0 \qquad \textit{for} \ \ (x,t) \in \partial\Omega \times [0,T]. \end{aligned}$$

a) *Derive an **energy equality** for u. (Hint: Multiply by u_t and integrate over $\Omega \times [0,T]$.)*
b) *Show that if $u|_{t=0} = u_t|_{t=0} = 0$ for $x \in \Omega$, then $u \equiv 0$.*

Proof. **a)** Multiply by u_t and integrate:

$$\begin{aligned} 0 &= \int_\Omega (u_{tt} - \triangle u + u^3) u_t \, dx = \int_\Omega u_{tt} u_t \, dx - \underbrace{\int_{\partial\Omega} \frac{\partial u}{\partial n} u_t \, ds}_{=0} + \int_\Omega \nabla u \cdot \nabla u_t \, dx + \int_\Omega u^3 u_t \, dx \\ &= \int_\Omega \frac{1}{2} \frac{\partial}{\partial t}(u_t^2) \, dx + \int_\Omega \frac{1}{2} \frac{\partial}{\partial t} |\nabla u|^2 \, dx + \int_\Omega \frac{1}{4} \frac{\partial}{\partial t}(u^4) \, dx = \frac{1}{2} \frac{d}{dt} \int_\Omega \big(u_t^2 + |\nabla u|^2 + \frac{1}{2} u^4\big) \, dx. \end{aligned}$$

Thus, the Energy integral is

$$\boxed{E(t) = \int_\Omega \big(u_t^2 + |\nabla u|^2 + \frac{1}{2} u^4\big) \, dx \ = \text{const} \ = \ E(0).}$$

b) Since $u(x,0) = 0$, $u_t(x,0) = 0$, we have

$$E(0) = \int_\Omega \big(u_t(x,0)^2 + |\nabla u(x,0)|^2 + \frac{1}{2} u(x,0)^4\big) \, dx = 0.$$

Since $E(t) = E(0) = 0$, we have

$$E(t) = \int_\Omega \big(u_t(x,t)^2 + |\nabla u(x,t)|^2 + \frac{1}{2} u(x,t)^4\big) \, dx = 0.$$

Thus, $u \equiv 0$. □

Problem (F'04, #3). *Consider a damped wave equation*

$$\begin{cases} u_{tt} - \triangle u + a(x)u_t = 0, & (x,t) \in \mathbb{R}^3 \times \mathbb{R}, \\ u|_{t=0} = u_0, \quad u_t|_{t=0} = u_1. \end{cases}$$

Here the damping coefficient $a \in C_0^\infty(\mathbb{R}^3)$ is a non-negative function and u_0, $u_1 \in C_0^\infty(\mathbb{R}^3)$. Show that the energy of the solution $u(x,t)$ at time t,

$$E(t) = \frac{1}{2} \int_{\mathbb{R}^3} \left(|\nabla_x u|^2 + |u_t|^2\right) dx$$

is a decreasing function of $t \geq 0$.

Proof. Take the derivative of $E(t)$ with respect to t. Note that the boundary integral is 0 by Huygen's principle.

$$\begin{aligned} \frac{dE}{dt}(t) &= \int_{\mathbb{R}^3} \left(\nabla u \cdot \nabla u_t + u_t u_{tt}\right) dx \\ &= \underbrace{\int_{\partial\mathbb{R}^3} u_t \frac{\partial u}{\partial n}\, ds}_{=0} - \int_{\mathbb{R}^3} u_t \triangle u \, dx + \int_{\mathbb{R}^3} u_t u_{tt} \, dx \\ &= \int_{\mathbb{R}^3} u_t(-\triangle u + u_{tt})\, dx = \int_{\mathbb{R}^3} u_t(-a(x)u_t)\, dx = \int_{\mathbb{R}^3} -a(x)u_t^2 \, dx \leq 0. \end{aligned}$$

Thus, $\frac{dE}{dt} \leq 0 \;\Rightarrow\; E(t) \leq E(0)$, i.e. $E(t)$ is a decreasing function of t. $\square$

Problem (W'03, #8). ***a)*** *Consider the damped wave equation for high-speed waves* $(0 < \epsilon << 1)$ *in a bounded region* D

$$\epsilon^2 u_{tt} + u_t = \triangle u \qquad \circledast$$

with the boundary condition $u(x,t) = 0$ *on* ∂D. *Show that the energy functional*

$$E(t) = \int_D \epsilon^2 u_t^2 + |\nabla u|^2 \, dx$$

is nonincreasing on solutions of the boundary value problem.

b) *Consider the solution to the boundary value problem in part (a) with initial data* $u^\epsilon(x,0) = 0$, $u_t^\epsilon(x,0) = \epsilon^{-\alpha} f(x)$, *where* f *does not depend on* ϵ *and* $\alpha < 1$. *Use part (a) to show that*

$$\int_D |\nabla u^\epsilon(x,t)|^2 \, dx \to 0$$

uniformly on $0 \le t \le T$ *for any* T *as* $\epsilon \to 0$.

c) *Show that the result in part (b) does not hold for* $\alpha = 1$. *To do this consider the case where* f *is an eigenfunction of the Laplacian, i.e.* $\triangle f + \lambda f = 0$ *in* D *and* $f = 0$ *on* ∂D, *and solve for* u^ϵ *explicitly.*

Proof. **a)**

$$\begin{aligned}
\frac{dE}{dt} &= \int_D 2\epsilon^2 u_t u_{tt} \, dx + \int_D 2\nabla u \cdot \nabla u_t \, dx \\
&= \int_D 2\epsilon^2 u_t u_{tt} \, dx + \underbrace{\int_{\partial D} 2\frac{\partial u}{\partial n} u_t \, ds}_{=0,\ (u=0 \text{ on } \partial D)} - \int_D 2\triangle u u_t \, dx \\
&= 2\int_D (\epsilon^2 u_{tt} - \triangle u) u_t \, dx \ = \ \circledast \ = \ -2\int_D |u_t|^2 \, dx \ \le \ 0.
\end{aligned}$$

Thus, $E(t) \le E(0)$, i.e. $E(t)$ is nonincreasing.

b) From (a), we know $\frac{dE}{dt} \le 0$. We also have

$$\begin{aligned}
E_\epsilon(0) &= \int_D \epsilon^2 (u_t^\epsilon(x,0))^2 + |\nabla u^\epsilon(x,0)|^2 \, dx \\
&= \int_D \epsilon^2 (\epsilon^{-\alpha} f(x))^2 + 0 \, dx \ = \ \int_D \epsilon^{2(1-\alpha)} f(x)^2 \, dx \to 0 \quad \text{as } \epsilon \to 0.
\end{aligned}$$

Since $E_\epsilon(0) \ge E_\epsilon(t) = \int_D \epsilon^2 (u_t^\epsilon)^2 + |\nabla u^\epsilon|^2 \, dx$, then $E_\epsilon(t) \to 0$ as $\epsilon \to 0$.
Thus, $\int_D |\nabla u^\epsilon|^2 \, dx \to 0$ as $\epsilon \to 0$.

c) If $\alpha = 1$,

$$E_\epsilon(0) = \int_D \epsilon^{2(1-\alpha)} f(x)^2 \, dx \ = \ \int_D f(x)^2 \, dx.$$

Since f is independent of ϵ, $E_\epsilon(0)$ does not approach 0 as $\epsilon \to 0$. We can not conclude that $\int_D |\nabla u^\epsilon(x,t)|^2 \, dx \to 0$. □

Problem (F'98, #6). *Let f solve the nonlinear wave equation*

$$f_{tt} - f_{xx} = -f(1+f^2)^{-1}$$

for $x \in [0,1]$, with $f(x=0,t) = f(x=1,t) = 0$ and with smooth initial data $f(x,t) = f_0(x)$.
a) *Find an energy integral $E(t)$ which is constant in time.*
b) *Show that $|f(x,t)| < c$ for all x and t, in which c is a constant.*
Hint: Note that

$$\frac{f}{1+f^2} = \frac{1}{2}\frac{d}{df}\log(1+f^2).$$

Proof. **a)** Since $f(0,t) = f(1,t) = 0$, $\forall t$, we have $f_t(0,t) = f_t(1,t) = 0$. Let

$$\begin{aligned}
\frac{dE}{dt} &= 0 = \int_0^1 \big(f_{tt} - f_{xx} + f(1+f^2)^{-1}\big) f_t \, dx \\
&= \int_0^1 f_{tt} f_t \, dx - \int_0^1 f_{xx} f_t \, dx + \int_0^1 \frac{f f_t}{1+f^2} \, dx \\
&= \int_0^1 f_{tt} f_t \, dx - [f_x \underbrace{f_t}_{=0}]_0^1 + \int_0^1 f_x f_{tx} \, dx + \int_0^1 \frac{f f_t}{1+f^2} \, dx \\
&= \int_0^1 \frac{1}{2}\frac{\partial}{\partial t}(f_t^2) \, dx + \int_0^1 \frac{1}{2}\frac{\partial}{\partial t}(f_x^2) \, dx + \int_0^1 \frac{1}{2}\frac{\partial}{\partial t}(ln(1+f^2)) \, dx \\
&= \frac{1}{2}\frac{d}{dt}\int_0^1 \big(f_t^2 + f_x^2 + ln(1+f^2)\big) \, dx.
\end{aligned}$$

Thus,

$$E(t) = \frac{1}{2}\int_0^1 \big(f_t^2 + f_x^2 + ln(1+f^2)\big) \, dx.$$

b) We want to show that f is bounded. For smooth $f(x,0) = f_0(x)$, we have

$$E(0) = \frac{1}{2}\int_0^1 \big(f_t(x,0)^2 + f_x(x,0)^2 + ln(1+f(x,0)^2)\big) \, dx < \infty.$$

Since $E(t)$ is constant in time, $E(t) = E(0) < \infty$. Thus,

$$\frac{1}{2}\int_0^1 ln(1+f^2) \, dx \ \le \ \frac{1}{2}\int_0^1 \big(f_t^2 + f_x^2 + ln(1+f^2)\big) \, dx = E(t) < \infty.$$

Hence, f is bounded. $\square$

Problem (F'97, #1). *Consider initial-boundary value problem*

$$u_{tt} + a^2(x,t)u_t - \triangle u(x,t) = 0 \qquad x \in \Omega \subset \mathbb{R}^n,\ 0 < t < +\infty$$
$$u(x) = 0 \qquad x \in \partial\Omega$$
$$u(x,0) = f(x), \qquad u_t(x,0) = g(x) \qquad x \in \Omega.$$

Prove that L_2-norm of the solution is bounded in t on $(0, +\infty)$.
Here Ω is a bounded domain, and $a(x,t)$, $f(x)$, $g(x)$ are smooth functions.

Proof. Multiply the equation by u_t and integrate over Ω:

$$u_t u_{tt} + a^2 u_t^2 - u_t \triangle u = 0,$$
$$\int_\Omega u_t u_{tt}\,dx + \int_\Omega a^2 u_t^2\,dx - \int_\Omega u_t \triangle u\,dx = 0,$$
$$\frac{1}{2}\frac{d}{dt}\int_\Omega u_t^2\,dx + \int_\Omega a^2 u_t^2\,dx - \underbrace{\int_{\partial\Omega} u_t \frac{\partial u}{\partial n}\,ds}_{=0,\ (u=0,\ x\in\partial\Omega)} + \int_\Omega \nabla u \cdot \nabla u_t\,dx = 0,$$
$$\frac{1}{2}\frac{d}{dt}\int_\Omega u_t^2\,dx + \int_\Omega a^2 u_t^2\,dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2\,dx = 0,$$
$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left(u_t^2 + |\nabla u|^2\right)dx = -\int_\Omega a^2 u_t^2\,dx \le 0.$$

Let Energy integral be

$$\boxed{E(t) = \int_\Omega \left(u_t^2 + |\nabla u|^2\right)dx.}$$

We have $\frac{dE}{dt} \le 0$, i.e. $E(t) \le E(0)$.

$$E(t) \le E(0) = \int_\Omega \left(u_t(x,0)^2 + |\nabla u(x,0)|^2\right)dx = \int_\Omega \left(g(x)^2 + |\nabla f(x)|^2\right)dx < \infty,$$

since f and g are smooth functions. Thus,

$$E(t) = \int_\Omega \left(u_t^2 + |\nabla u|^2\right)dx \quad < \quad \infty,$$
$$\int_\Omega |\nabla u|^2\,dx \quad < \quad \infty,$$
$$\int_\Omega u^2\,dx \le C\int_\Omega |\nabla u|^2\,dx \quad < \quad \infty, \qquad \text{by } \textbf{Poincare inequality}.$$

Thus, $||u||_2$ is bounded $\forall t$. $\square$

Problem (S'98, #4). ***a)*** *Let $u(x,y,z,t)$, $-\infty < x,y,z < \infty$ be a solution of the equation*

$$\begin{cases} u_{tt} + u_t = u_{xx} + u_{yy} + u_{zz} \\ u(x,y,z,0) = f(x,y,z), \\ u_t(x,y,z,0) = g(x,y,z). \end{cases} \tag{16.41}$$

*Here f, g are smooth functions which vanish if $\sqrt{x^2+y^2+z^2}$ is large enough. Prove that it is the **unique** solution for $t \geq 0$.*
b) *Suppose we want to solve the same equation (16.41) in the region $z \geq 0$, $-\infty < x,y < \infty$, with the additional conditions*

$$\begin{aligned} u(x,y,0,t) &= f(x,y,t) \\ u_z(x,y,0,t) &= g(x,y,t) \end{aligned}$$

with the same f, g as before in (16.41). What goes wrong?

Proof. **a)** Suppose u_1 and u_2 are two solutions. Let $w = u_1 - u_2$. Then,

$$\begin{cases} w_{tt} + w_t = \triangle w, \\ w(x,y,z,0) = 0, \\ w_t(x,y,z,0) = 0. \end{cases}$$

Multiply the equation by w_t and integrate:

$$\begin{aligned}
w_t w_{tt} + w_t^2 &= w_t \triangle w, \\
\int_{\mathbb{R}^3} w_t w_{tt}\,dx + \int_{\mathbb{R}^3} w_t^2\,dx &= \int_{\mathbb{R}^3} w_t \triangle w\,dx, \\
\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} w_t^2\,dx + \int_{\mathbb{R}^3} w_t^2\,dx &= \underbrace{\int_{\partial\mathbb{R}^3} w_t \frac{\partial w}{\partial n}\,dx}_{=0} - \int_{\mathbb{R}^3} \nabla w \cdot \nabla w_t\,dx, \\
\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} w_t^2\,dx + \int_{\mathbb{R}^3} w_t^2\,dx &= -\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} |\nabla w|^2\,dx, \\
\frac{d}{dt}\underbrace{\int_{\mathbb{R}^3} \left(w_t^2 + |\nabla w|^2\right)dx}_{E(t)} &= -2\int_{\mathbb{R}^3} w_t^2\,dx \;\leq\; 0, \\
\frac{dE}{dt} &\leq 0, \\
E(t) \;\leq\; E(0) &= \int_{\mathbb{R}^3} \left(w_t(x,0)^2 + |\nabla w(x,0)|^2\right)dx = 0, \\
\Rightarrow\; E(t) &= \int_{\mathbb{R}^3} \left(w_t^2 + |\nabla w|^2\right)dx = 0.
\end{aligned}$$

Thus, $w_t = 0$, $\nabla w = 0$, and $w =$ constant. Since $w(x,y,z,0) = 0$, we have $w \equiv 0$.

b) $\square$

Problem (F'94, #8). *The one-dimensional, isothermal fluid equations with viscosity and capillarity in Lagrangian variables are*

$$\begin{aligned} &v_t - u_x = 0 \\ &u_t + p(v)_x = \varepsilon u_{xx} - \delta v_{xxx} \end{aligned}$$

in which $v(= 1/\rho)$ *is specific volume,* u *is velocity, and* $p(v)$ *is pressure. The coefficients* ε *and* δ *are non-negative.*
Find an ***energy integral*** *which is non-increasing (as* t *increases) if* $\varepsilon > 0$ *and conserved if* $\varepsilon = 0$.
Hint: if $\delta = 0$, $E = \int u^2/2 - P(v)\, dx$ *where* $P'(v) = p(v)$.

Proof. Multiply the second equation by u and integrate over $\mathbb{R}$. We use $u_x = v_t$. Note that the boundary integrals are 0 due to finite speed of propagation.

$$\begin{aligned} &uu_t + up(v)_x = \varepsilon uu_{xx} - \delta uv_{xxx}, \\ &\int_{\mathbb{R}} uu_t\, dx + \int_{\mathbb{R}} up(v)_x\, dx = \varepsilon \int_{\mathbb{R}} uu_{xx}\, dx - \delta \int_{\mathbb{R}} uv_{xxx}\, dx, \\ &\frac{1}{2}\int_{\mathbb{R}} \frac{\partial}{\partial t}(u^2)\, dx + \underbrace{\int_{\partial\mathbb{R}} up(v)\, ds}_{=0} + \int_{\mathbb{R}} u_x p(v)\, dx \\ &\qquad\qquad = \varepsilon \underbrace{\int_{\partial\mathbb{R}} uu_x\, dx}_{=0} - \varepsilon \int_{\mathbb{R}} u_x^2\, dx - \delta \underbrace{\int_{\partial\mathbb{R}} uv_{xx}\, dx}_{=0} + \delta \int_{\mathbb{R}} u_x v_{xx}\, dx, \\ &\frac{1}{2}\int_{\mathbb{R}} \frac{\partial}{\partial t}(u^2)\, dx + \int_{\mathbb{R}} v_t p(v)\, dx = -\varepsilon \int_{\mathbb{R}} u_x^2\, dx + \delta \int_{\mathbb{R}} v_t v_{xx}\, dx, \\ &\frac{1}{2}\int_{\mathbb{R}} \frac{\partial}{\partial t}(u^2)\, dx + \int_{\mathbb{R}} \frac{\partial}{\partial t}P(v)\, dx = -\varepsilon \int_{\mathbb{R}} u_x^2\, dx + \delta \underbrace{\int_{\partial\mathbb{R}} v_t v_x\, dx}_{=0} - \delta \int_{\mathbb{R}} v_{xt} v_x\, dx, \\ &\frac{1}{2}\int_{\mathbb{R}} \frac{\partial}{\partial t}(u^2)\, dx + \int_{\mathbb{R}} \frac{\partial}{\partial t}P(v)\, dx + \frac{\delta}{2}\int_{\mathbb{R}} \frac{\partial}{\partial t}(v_x^2)\, dx = -\varepsilon \int_{\mathbb{R}} u_x^2\, dx, \\ &\frac{d}{dt}\int_{\mathbb{R}} \Big(\frac{u^2}{2} + P(v) + \frac{\delta}{2}v_x^2\Big)\, dx \;=\; -\varepsilon \int_{\mathbb{R}} u_x^2\, dx \;\le\; 0. \end{aligned}$$

$$\boxed{E(t) = \int_{\mathbb{R}} \Big(\frac{u^2}{2} + P(v) + \frac{\delta}{2}v_x^2\Big)\, dx}$$

is nonincreasing if $\varepsilon > 0$, and conserved if $\varepsilon = 0$. $\square$

Problem (S'99, #5). *Consider the equation*

$$u_{tt} = \frac{\partial}{\partial x}\sigma(u_x) \tag{16.42}$$

with $\sigma(z)$ a smooth function. This is to be solved for $t > 0$, $0 \le x \le 1$, with periodic boundary conditions and initial data $u(x,0) = u_0(x)$ and $u_t(x,0) = v_0(x)$.
a) *Multiply (16.42) by u_t and get an expression of the form*

$$\frac{d}{dt}\int_0^1 F(u_t, u_x) = 0$$

that is satisfied for an appropriate function $F(y,z)$ with $y = u_t$, $z = u_x$, where u is any smooth, periodic in space solution of (16.42).
b) *Under what conditions on $\sigma(z)$ is this function, $F(y,z)$,* ***convex*** *in its variables?*
c) *What à priori inequality is satisfied for smooth solutions when F is convex?*
d) *Discuss the special case $\sigma(z) = a^2z^3/3$, with $a > 0$ and constant.*

Proof. **a)** Multiply by u_t and integrate:

$$\begin{aligned}
u_t u_{tt} &= u_t \sigma(u_x)_x, \\
\int_0^1 u_t u_{tt}\, dx &= \int_0^1 u_t \sigma(u_x)_x\, dx, \\
\frac{d}{dt}\int_0^1 \frac{u_t^2}{2}\, dx &= \underbrace{u_t\sigma(u_x)\big|_0^1}_{=0,\ (2\pi\text{-periodic})} - \int_0^1 u_{tx}\sigma(u_x)\, dx \;=\; \circledast
\end{aligned}$$

Let $Q'(z) = \sigma(z)$, then $\frac{d}{dt}Q(u_x) = \sigma(u_x)u_{xt}$. Thus,

$$\circledast \;=\; -\int_0^1 u_{tx}\sigma(u_x)\, dx \;=\; -\frac{d}{dt}\int_0^1 Q(u_x)\, dx.$$

$$\boxed{\frac{d}{dt}\int_0^1 \Big(\frac{u_t^2}{2} + Q(u_x)\Big)\, dx \;=\; 0.}$$

b) We have

$$F(u_t, u_x) = \frac{u_t^2}{2} + Q(u_x).$$

[41] For F to be convex, the Hessian matrix of partial derivatives must be positive definite.

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[41] A function f is **convex** on a convex set S if it satisfies

$$f(\alpha x + (1-\alpha)y) \;\le\; \alpha f(x) + (1-\alpha)f(y)$$

for all $0 \le \alpha \le 1$ and for all $x, y \in S$.
If a one-dimensional function f has two continuous derivatives, then f is convex if and only if

$$f''(x) \ge 0.$$

In the multi-dimensional case the Hessian matrix of second derivatives must be positive semi-definite, that is, at every point $x \in S$

$$y^T\,\nabla^2 f(x)\, y \;\ge\; 0, \qquad \text{for all } \; y.$$

The **Hessian matrix** is the matrix with entries

$$[\nabla^2 f(x)]_{ij} \equiv \frac{\partial^2 f(x)}{\partial x_i \partial x_j}.$$

For functions with continuous second derivatives, it will always be symmetric matrix: $f_{x_i x_j} = f_{x_j x_i}$.

The Hessian matrix is

$$\nabla^2 F(u_t, u_x) = \begin{pmatrix} F_{u_t u_t} & F_{u_t u_x} \\ F_{u_x u_t} & F_{u_x u_x} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \sigma'(u_x) \end{pmatrix}.$$

$$y^T \nabla^2 F(x)\, y \;=\; \begin{pmatrix} y_1 & y_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \sigma'(u_x) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \;=\; y_1^2 + \sigma'(u_x) y_2^2 \underbrace{\geq}_{\text{need}} 0.$$

Thus, for a Hessian matrix to be positive definite, need $\sigma'(u_x) \geq 0$, so that the above inequality holds for all y.

c) We have

$$\begin{aligned}
\frac{d}{dt}\int_0^1 F(u_t, u_x)\, dx &= 0, \\
\int_0^1 F(u_t, u_x)\, dx &= \text{const}, \\
\int_0^1 F(u_t, u_x)\, dx &= \int_0^1 F(u_t(x,0), u_x(x,0))\, dx, \\
\int_0^1 \Big(\frac{u_t^2}{2} + Q(u_x)\Big)\, dx &= \int_0^1 \Big(\frac{v_0^2}{2} + Q(u_{0x})\Big)\, dx.
\end{aligned}$$

d) If $\sigma(z) = a^2 z^3/3$, we have

$$\begin{aligned}
F(u_t, u_x) = \frac{u_t^2}{2} + Q(u_x) &= \frac{u_t^2}{2} + \frac{a^2 u_x^4}{12}, \\
\frac{d}{dt}\int_0^1 \Big(\frac{u_t^2}{2} + \frac{a^2 u_x^4}{12}\Big)\, dx &= 0, \\
\int_0^1 \Big(\frac{u_t^2}{2} + \frac{a^2 u_x^4}{12}\Big)\, dx &= \text{const}, \\
\int_0^1 \Big(\frac{u_t^2}{2} + \frac{a^2 u_x^4}{12}\Big)\, dx &= \int_0^1 \Big(\frac{{v_0}^2}{2} + \frac{a^2 {u_{0x}}^4}{12}\Big)\, dx.
\end{aligned}$$

$\square$

Problem (S'96, #8). [42] *Let $u(x,t)$ be the solution of the Korteweg-de Vries equation*

$$u_t + uu_x = u_{xxx}, \qquad 0 \le x \le 2\pi,$$

with 2π-periodic boundary conditions and prescribed initial data

$$u(x, t = 0) = f(x).$$

a) *Prove that the energy integral*

$$I_1(u) = \int_0^{2\pi} u^2(x,t)\,dx$$

is independent of the time t.
b) *Prove that the second "energy integral",*

$$I_2(u) = \int_0^{2\pi} \Big(\frac{1}{2}u_x^2(x,t) + \frac{1}{6}u^3(x,t)\Big)\,dx$$

is also independent of the time t.
c) *Assume the initial data are such that $I_1(f) + I_2(f) < \infty$. Use (a) + (b) to prove that the **maximum norm** of the solution, $|u|_\infty = \sup_x |u(x,t)|$, is bounded in time. Hint: Use the following inequalities (here, $|u|_p$ is the L^p-norm of $u(x,t)$ at **fixed time** t):*

- $|u|_\infty^2 \le \dfrac{\pi}{6}(|u|_2^2 + |u_x|_2^2)$ *(one of Sobolev's inequalities);*
- $|u|_3^3 \le |u|_2^2\,|u|_\infty$ *(straightforward).*

Proof. **a)** Multiply the equation by u and integrate. Note that all boundary terms are 0 due to 2π-periodicity.

$$\begin{aligned}
uu_t + u^2u_x &= uu_{xxx},\\
\int_0^{2\pi} uu_t\,dx + \int_0^{2\pi} u^2u_x\,dx &= \int_0^{2\pi} uu_{xxx}\,dx,\\
\frac{1}{2}\frac{d}{dt}\int_0^{2\pi} u^2\,dx + \frac{1}{3}\int_0^{2\pi}(u^3)_x\,dx &= uu_{xx}\Big|_0^{2\pi} - \int_0^{2\pi} u_xu_{xx}\,dx,\\
\frac{1}{2}\frac{d}{dt}\int_0^{2\pi} u^2\,dx + \frac{1}{3}u^3\Big|_0^{2\pi} &= -\frac{1}{2}\int_0^{2\pi}(u_x^2)_x\,dx,\\
\frac{1}{2}\frac{d}{dt}\int_0^{2\pi} u^2\,dx &= -\frac{1}{2}u_x^2\Big|_0^{2\pi} = 0.\\
I_1(u) = \int_0^{2\pi} u^2\,dx &= C.
\end{aligned}$$

Thus, $I_1(u) = \int_0^{2\pi} u^2(x,t)\,dx$ is independent of the time t.
Alternatively, we may differentiate $I_1(u)$:

$$\begin{aligned}
\frac{dI_1}{dt}(u) &= \frac{d}{dt}\int_0^{2\pi} u^2\,dx = \int_0^{2\pi} 2uu_t\,dx = \int_0^{2\pi} 2u(-uu_x + u_{xxx})\,dx\\
&= \int_0^{2\pi} -2u^2u_x\,dx + \int_0^{2\pi} 2uu_{xxx}\,dx = \int_0^{2\pi} -\frac{2}{3}(u^3)_x\,dx + 2uu_{xx}\Big|_0^{2\pi} - \int_0^{2\pi} 2u_xu_{xx}\,dx\\
&= -\frac{2}{3}u^3\Big|_0^{2\pi} - \int_0^{2\pi}(u_x^2)_x\,dx = -u_x^2\Big|_0^{2\pi} = 0.
\end{aligned}$$

[42] Also, see S'92, #7.

b) Note that all boundary terms are 0 due to 2π-periodicity.

$$\frac{dI_2}{dt}(u) = \frac{d}{dt}\int_0^{2\pi}\Big(\frac{1}{2}u_x^2 + \frac{1}{6}u^3\Big)\,dx = \int_0^{2\pi}\Big(u_x u_{xt} + \frac{1}{2}u^2 u_t\Big)\,dx = \circledast$$

We differentiate the original equation with respect to x:

$$u_t = -uu_x + u_{xxx}$$
$$u_{tx} = -(uu_x)_x + u_{xxxx}.$$

$$\begin{aligned}
\circledast &= \int_0^{2\pi} u_x(-(uu_x)_x + u_{xxxx})\,dx + \frac{1}{2}\int_0^{2\pi} u^2(-uu_x + u_{xxx})\,dx \\
&= \int_0^{2\pi} -u_x(uu_x)_x\,dx + \int_0^{2\pi} u_x u_{xxxx}\,dx - \frac{1}{2}\int_0^{2\pi} u^3 u_x\,dx + \frac{1}{2}\int_0^{2\pi} u^2 u_{xxx}\,dx \\
&= -u_x uu_x\Big|_0^{2\pi} + \int_0^{2\pi} u_{xx}uu_x\,dx + u_x u_{xxx}\Big|_0^{2\pi} - \int_0^{2\pi} u_{xx}u_{xxx}\,dx \\
&\quad - \frac{1}{2}\int_0^{2\pi}\Big(\frac{u^4}{4}\Big)_x\,dx + \frac{1}{2}u^2 u_{xx}\Big|_0^{2\pi} - \frac{1}{2}\int_0^{2\pi} 2uu_x u_{xx}\,dx \\
&= \int_0^{2\pi} u_{xx}uu_x\,dx - \int_0^{2\pi} u_{xx}u_{xxx}\,dx - \frac{1}{2}\frac{u^4}{4}\Big|_0^{2\pi} - \int_0^{2\pi} uu_x u_{xx}\,dx \\
&= -\int_0^{2\pi} u_{xx}u_{xxx}\,dx = -u_{xx}^2\Big|_0^{2\pi} + \int_0^{2\pi} u_{xxx}u_{xx}\,dx = \int_0^{2\pi} u_{xxx}u_{xx}\,dx = 0,
\end{aligned}$$

since $-\int_0^{2\pi} u_{xx}u_{xxx}\,dx = +\int_0^{2\pi} u_{xx}u_{xxx}\,dx$. Thus,

$$I_2(u) = \int_0^{2\pi}\Big(\frac{1}{2}u_x^2(x,t) + \frac{1}{6}u^3(x,t)\Big)\,dx = C,$$

and $I_2(u)$ is independent of the time t.

c) From (a) and (b), we have

$$\begin{aligned}
I_1(u) &= \int_0^{2\pi} u^2\,dx = ||u||_2^2, \\
I_2(u) &= \int_0^{2\pi}\Big(\frac{1}{2}u_x^2 + \frac{1}{6}u^3\Big)\,dx = \frac{1}{2}||u_x||_2^2 + \frac{1}{6}||u||_3^3.
\end{aligned}$$

Using given inequalities, we have

$$\begin{aligned}
||u||_\infty^2 &\le \frac{\pi}{6}(||u||_2^2 + ||u_x||_2^2) \le \frac{\pi}{6}\Big(I_1(u) + 2I_2(u) - \frac{1}{3}||u||_3^3\Big) \\
&\le \frac{\pi}{6}I_1(u) + \frac{\pi}{3}I_2(u) + \frac{\pi}{18}||u||_2^2\,||u||_\infty \le \frac{\pi}{6}I_1(u) + \frac{\pi}{3}I_2(u) + \frac{\pi}{18}I_1(u)||u||_\infty \\
&= C + C_1||u||_\infty. \\
&\Rightarrow\ ||u||_\infty^2 \le C + C_1||u||_\infty, \\
&\Rightarrow\ ||u||_\infty \le C_2.
\end{aligned}$$

Thus, $||u||_\infty$ is bounded in time. $\square$

Also see Energy Methods problems for higher order equations (3rd and 4th) in the section on Gas Dynamics.

16.7 Wave Equation in 2D and 3D

Problem (F'97, #8); (McOwen 3.2 #90). *Solve*

$$u_{tt} = u_{xx} + u_{yy} + u_{zz}$$

with initial conditions

$$u(x, y, z, 0) = \underbrace{x^2 + y^2}_{g(x)}, \quad u_t(x, y, z, 0) = \underbrace{0}_{h(x)}.$$

Proof.
① We may use the **Kirchhoff's formula:**

$$\begin{aligned} u(x,t) &= \frac{1}{4\pi}\frac{\partial}{\partial t}\bigg(t\int_{|\xi|=1} g(x+ct\xi)\,dS_\xi\bigg) + \frac{t}{4\pi}\int_{|\xi|=1} h(x+ct\xi)\,dS_\xi \\ &= \frac{1}{4\pi}\frac{\partial}{\partial t}\bigg(t\int_{|\xi|=1} \big((x_1+ct\xi_1)^2+(x_2+ct\xi_2)^2\big)\,dS_\xi\bigg) + 0 = \end{aligned}$$

② We may solve the problem by Hadamard's **method of descent**, since initial conditions are independent of x_3. We need to convert surface integrals in $\mathbb{R}^3$ to domain integrals in $\mathbb{R}^2$. Specifically, we need to express the surface measure on the upper half of the unit sphere S_+^2 in terms of the two variables ξ_1 and ξ_2. To do this, consider

$$\begin{aligned} f(\xi_1, \xi_2) &= \sqrt{1-\xi_1^2-\xi_2^2} \qquad \text{over the unit disk } \xi_1^2+\xi_2^2<1. \\ dS_\xi &= \sqrt{1+(f_{\xi_1})^2+(f_{\xi_2})^2}\,d\xi_1 d\xi_2 = \frac{d\xi_1 d\xi_2}{\sqrt{1-\xi_1^2-\xi_2^2}}. \end{aligned}$$

$$
\begin{aligned}
u(x_1,x_2,t) &= \frac{1}{4\pi}\frac{\partial}{\partial t}\bigg(2t\int_{\xi_1^2+\xi_2^2<1}\frac{g(x_1+ct\xi_1,x_2+ct\xi_2)\,d\xi_1 d\xi_2}{\sqrt{1-\xi_1^2-\xi_2^2}}\bigg) \\
&+ \frac{t}{4\pi}\bigg(2\int_{\xi_1^2+\xi_2^2<1}\frac{h(x_1+ct\xi_1,x_2+ct\xi_2)\,d\xi_1 d\xi_2}{\sqrt{1-\xi_1^2-\xi_2^2}}\bigg) \\
&= \frac{1}{4\pi}\frac{\partial}{\partial t}\bigg(2t\int_{\xi_1^2+\xi_2^2<1}\frac{(x_1+t\xi_1)^2+(x_2+t\xi_2)^2}{\sqrt{1-\xi_1^2-\xi_2^2}}\,d\xi_1 d\xi_2\bigg) \;+\; 0, \\
&= \frac{1}{2\pi}\frac{\partial}{\partial t}\bigg(t\int_{\xi_1^2+\xi_2^2<1}\frac{x_1^2+2x_1t\xi_1+t^2\xi_1^2+x_2^2+2x_2t\xi_2+t^2\xi_2^2}{\sqrt{1-\xi_1^2-\xi_2^2}}\,d\xi_1 d\xi_2\bigg) \\
&= \frac{1}{2\pi}\frac{\partial}{\partial t}\bigg(\int_{\xi_1^2+\xi_2^2<1}\frac{tx_1^2+2x_1t^2\xi_1+t^3\xi_1^2+tx_2^2+2x_2t^2\xi_2+t^3\xi_2^2}{\sqrt{1-\xi_1^2-\xi_2^2}}\,d\xi_1 d\xi_2\bigg) \\
&= \frac{1}{2\pi}\bigg(\int_{\xi_1^2+\xi_2^2<1}\frac{x_1^2+4x_1t\xi_1+3t^2\xi_1^2+x_2^2+4x_2t\xi_2+3t^2\xi_2^2}{\sqrt{1-\xi_1^2-\xi_2^2}}\,d\xi_1 d\xi_2\bigg) \\
&= \frac{1}{2\pi}\bigg(\int_{\xi_1^2+\xi_2^2<1}\frac{(x_1^2+x_2^2)+4t(x_1\xi_1+x_2\xi_2)+3t^2(\xi_1^2+\xi_2^2)}{\sqrt{1-\xi_1^2-\xi_2^2}}\,d\xi_1 d\xi_2\bigg) \\
&= \underbrace{\frac{1}{2\pi}(x_1^2+x_2^2)\int_{\xi_1^2+\xi_2^2<1}\frac{d\xi_1 d\xi_2}{\sqrt{1-\xi_1^2-\xi_2^2}}}_{❶} + \underbrace{\frac{4t}{2\pi}\int_{\xi_1^2+\xi_2^2<1}\frac{x_1\xi_1+x_2\xi_2}{\sqrt{1-\xi_1^2-\xi_2^2}}\,d\xi_1 d\xi_2}_{❷} \\
&\quad + \underbrace{\frac{3t^2}{2\pi}\int_{\xi_1^2+\xi_2^2<1}\frac{\xi_1^2+\xi_2^2}{\sqrt{1-\xi_1^2-\xi_2^2}}\,d\xi_1 d\xi_2}_{❸} \;=\; \circledast
\end{aligned}
$$

$$
\begin{aligned}
❶ &= \frac{1}{2\pi}(x_1^2+x_2^2)\int_{\xi_1^2+\xi_2^2<1}\frac{d\xi_1 d\xi_2}{\sqrt{1-\xi_1^2-\xi_2^2}} = \frac{1}{2\pi}(x_1^2+x_2^2)\int_0^{2\pi}\int_0^1\frac{r\,dr\,d\theta}{\sqrt{1-r^2}} \\
&= \frac{1}{2\pi}(x_1^2+x_2^2)\int_0^{2\pi}-2\int_0^1\frac{-\frac{1}{2}\,du\,d\theta}{u^{\frac{1}{2}}} \qquad \big(u=1-r^2,\quad du=-2r\,dr\big) \\
&= \frac{1}{2\pi}(x_1^2+x_2^2)\int_0^{2\pi}1\,d\theta \;=\; x_1^2+x_2^2. \\
❷ &= \frac{4t}{2\pi}\int_{\xi_1^2+\xi_2^2<1}\frac{x_1\xi_1+x_2\xi_2}{\sqrt{1-\xi_1^2-\xi_2^2}}\,d\xi_1 d\xi_2 = \frac{4t}{2\pi}\int_{-1}^1\int_{-\sqrt{1-\xi_2^2}}^{\sqrt{1-\xi_2^2}}\frac{x_1\xi_1+x_2\xi_2}{\sqrt{1-\xi_1^2-\xi_2^2}}\,d\xi_1 d\xi_2 \\
&= 0. \\
❸ &= \frac{3t^2}{2\pi}\int_{\xi_1^2+\xi_2^2<1}\frac{\xi_1^2+\xi_2^2}{\sqrt{1-\xi_1^2-\xi_2^2}}\,d\xi_1 d\xi_2 = \frac{3t^2}{2\pi}\int_0^{2\pi}\int_0^1\frac{(r\cos\theta)^2+(r\sin\theta)^2}{\sqrt{1-r^2}}\,r\,drd\theta \\
&= \frac{3t^2}{2\pi}\int_0^{2\pi}\int_0^1\frac{r^3}{\sqrt{1-r^2}}\,drd\theta \qquad \big(u=1-r^2,\quad du=-2r\,dr\big) \\
&= \frac{3t^2}{2\pi}\int_0^{2\pi}\frac{2}{3}\,d\theta \;=\; \frac{t^2}{\pi}\int_0^{2\pi}d\theta \;=\; 2t^2.
\end{aligned}
$$

$$\circledast \quad\Rightarrow\quad u(x_1,x_2,t) \;=\; ❶+❷+❸ \;=\; x_1^2+x_2^2+2t^2.$$

③ We may guess what the solution is:

$$u(x,y,z,t) \;=\; \frac{1}{2}\big[(x+t)^2+(y+t)^2+(x-t)^2+(y-t)^2\big] \;=\; x^2+y^2+2t^2.$$

Check:

$$\begin{aligned}
u(x,y,z,0) &= x^2+y^2. \quad \checkmark \\
u_t(x,y,z,t) &= (x+t)+(y+t)-(x-t)-(y-t), \\
u_t(x,y,z,0) &= 0. \quad \checkmark \\
u_{tt}(x,y,z,t) &= 4, \\
u_x(x,y,z,t) &= (x+t)+(x-t), \\
u_{xx}(x,y,z,t) &= 2, \\
u_y(x,y,z,t) &= (y+t)+(y-t), \\
u_{yy}(x,y,z,t) &= 2, \\
u_{zz}(x,y,z,t) &= 0, \\
u_{tt} &= u_{xx}+u_{yy}+u_{zz}. \quad \checkmark
\end{aligned}$$

$\square$

Problem (S'98, #6).
Consider the two-dimensional wave equation $w_{tt} = a^2 \triangle w$, *with initial data which vanish for* x^2+y^2 *large enough. Prove that* $w(x, y, t)$ *satisfies the decay* $|w(x, y, t)| \leq C \cdot t^{-1}$. *(Note that the estimate is not uniform with respect to* x, y *since* C *may depend on* x, y*).*

Proof. Suppose we have the following problem with initial data:

$$u_{tt} = a^2 \triangle u \qquad x \in \mathbb{R}^2,\ t > 0,$$

$$u(x, 0) = g(x), \quad u_t(x, 0) = h(x) \qquad x \in \mathbb{R}^2.$$

The result is the consequence of the Huygens' principle and may be proved by Hadamard's **method of descent**: [43]

$$\begin{aligned}
u(x,t) &= \frac{1}{4\pi}\frac{\partial}{\partial t}\Big(2t\int_{\xi_1^2+\xi_2^2<1}\frac{g(x_1+ct\xi_1, x_2+ct\xi_2)\,d\xi_1 d\xi_2}{\sqrt{1-\xi_1^2-\xi_2^2}}\Big) \\
&+ \frac{t}{4\pi}\Big(2\int_{\xi_1^2+\xi_2^2<1}\frac{h(x_1+ct\xi_1, x_2+ct\xi_2)\,d\xi_1 d\xi_2}{\sqrt{1-\xi_1^2-\xi_2^2}}\Big) \\
&= \frac{1}{2\pi}\int_{|\xi|^2<c^2t^2}\frac{th(x+\xi)+g(x+\xi)}{\sqrt{1-\frac{|\xi|^2}{c^2t^2}}}\,\frac{d\xi_1 d\xi_2}{c^2t^2} \\
&+ \frac{t}{2\pi}\int_{|\xi|^2<c^2t^2}\frac{\nabla g(x+\xi)\cdot(ct, ct)}{\sqrt{1-\frac{|\xi|^2}{c^2t^2}}}\,\frac{d\xi_1 d\xi_2}{c^2t^2}.
\end{aligned}$$

For a given x, let $T(x)$ be so large that $T > 1$ and $\text{supp}(h+g) \subset B_T(x)$. Then for $t > 2T$ we have:

$$\begin{aligned}
|u(x,t)| &= \frac{1}{2\pi}\int_{|\xi|^2<c^2T^2}\frac{tM+M+2Mct}{\sqrt{1-\frac{c^2T^2}{c^2T^2 4}}}\,\frac{d\xi_1 d\xi_2}{c^2t^2} \\
&= \frac{\pi c^2T^2}{2\pi}\Big[\Big(\frac{M}{\sqrt{3/4}}\Big)\frac{1}{c^2t}+\Big(\frac{M}{\sqrt{3/4}}\Big)\frac{1}{c^2Tt}+\frac{2Mc}{c^2t}\Big].
\end{aligned}$$

$\Rightarrow \quad u(x,t) \leq C_1/t \quad$ for $t > 2T$.
For $t \leq 2T$:

$$\begin{aligned}
|u(x,t)| &= \frac{1}{2\pi}\int_{|\xi|^2<c^2t^2}\frac{2TM+M+4McT}{\sqrt{1-\frac{|\xi|^2}{c^2t^2}}}\,\frac{d\xi_1 d\xi_2}{c^2t^2} \\
&= \frac{1}{2\pi}(2TM+M+4Mct)2\pi\int_0^{ct}\frac{r\,dr/c^2t^2}{\sqrt{1-\frac{r^2}{c^2t^2}}} \\
&= \frac{M(2T+1+4cT)}{2}\int_0^1\frac{-du}{u^{1/2}} = \frac{M(2T+1+4cT)}{2}2 \leq \frac{M(2T+1+4cT)2T}{t}.
\end{aligned}$$

Letting $C = \max(C_1, M(2T+1+4cT)2T)$, we have $|u(x,t)| \leq C(x)/t$.

• For $n = 3$, suppose g, $h \in C_0^\infty(\mathbb{R}^3)$. The solution is given by the Kircchoff's formula. There is a constant C so that $u(x,t) \leq C/t$ for all $x \in \mathbb{R}^3$ and $t > 0$. As McOwen suggensts in Hints for Exercises, to prove the result, we need to estimate the

[43] Nick's solution follows.

area of intersection of the sphere of radius ct with the support of the functions g and h. $\square$

Problem (S'95, #6). *Spherical waves in 3-d are waves symmetric about the origin; i.e. $u = u(r,t)$ where r is the distance from the origin. The wave equation*

$$u_{tt} = c^2 \triangle u$$

then reduces to

$$\frac{1}{c^2} u_{tt} = u_{rr} + \frac{2}{r} u_r. \tag{16.43}$$

a) *Find the general solutions $u(r,t)$ by solving (16.43). Include both the incoming waves and outgoing waves in your solutions.*
b) *Consider only the outgoing waves and assume the finite out-flux condition*

$$0 < \lim_{r \to 0} r^2 u_r < \infty$$

for all t. The wavefront is defined as $r = ct$. How is the amplitude of the wavefront decaying in time?

Proof. **a)** We want to reduce (16.43) to the 1D wave equation. Let $v = ru$. Then

$$\begin{aligned} v_{tt} &= r u_{tt}, \\ v_r &= r u_r + u, \\ v_{rr} &= r u_{rr} + 2u_r. \end{aligned}$$

Thus, (16.43) becomes

$$\begin{aligned} \frac{1}{c^2}\frac{1}{r} v_{tt} &= \frac{1}{r} v_{rr}, \\ \frac{1}{c^2} v_{tt} &= v_{rr}, \\ v_{tt} &= c^2 v_{rr}, \end{aligned}$$

which has the solution

$$v(r,t) = f(r+ct) + g(r-ct).$$

Thus,

$$u(r,t) = \frac{1}{r} v(r,t) = \underbrace{\frac{1}{r} f(r+ct)}_{incoming,\ (c>0)} + \underbrace{\frac{1}{r} g(r-ct)}_{outgoing,\ (c>0)}.$$

b) We consider $u(r,t) = \frac{1}{r} g(r-ct)$:

$$\begin{aligned} 0 \ &< \ \lim_{r\to 0} r^2 u_r \ < \ \infty, \\ 0 \ &< \ \lim_{r\to 0} r^2 \Big(\frac{1}{r} g'(r-ct) - \frac{1}{r^2} g(r-ct)\Big) \ < \ \infty, \\ 0 \ &< \ \lim_{r\to 0} \big(r g'(r-ct) - g(r-ct)\big) \ < \ \infty, \\ 0 \ &< \ -g(-ct) \ < \ \infty, \\ 0 \ &< \ -g(-ct) \ = \ G(t) \ < \ \infty, \\ & g(t) \ = \ -G\Big(\frac{t}{-c}\Big). \end{aligned}$$

The wavefront is defined as $r = ct$. We have

$$
\begin{aligned}
u(r,t) &= \frac{1}{r}g(r-ct) = -\frac{1}{r}G\Big(\frac{r-ct}{-c}\Big) = -\frac{1}{ct}G(0). \\
|u(r,t)| &= \frac{1}{t}\Big| -\frac{1}{c}G(0)\Big|.
\end{aligned}
$$

The amplitude of the wavefront decays like $\frac{1}{t}$. $\square$

Problem (S'00, #8). ***a)*** *Show that for a smooth function F on the line, while $u(x,t) = F(ct+|x|)/|x|$ may look like a solution of the wave equation $u_{tt} = c^2\triangle u$ in $\mathbb{R}^3$, it actually is not. Do this by showing that for any smooth function $\phi(x,t)$ with compact support*

$$\int_{\mathbb{R}^3\times\mathbb{R}} u(x,t)(\phi_{tt} - \triangle\phi)\,dxdt = 4\pi\int_{\mathbb{R}}\phi(0,t)F(ct)\,dt.$$

Note that, setting $r = |x|$, for any function w which only depends on r one has
$\triangle w = r^{-2}(r^2 w_r)_r = w_{rr} + \frac{2}{r}w_r$.
b) *If $F(0) = F'(0) = 0$, what is the true solution to $u_{tt} = \triangle u$ with initial conditions $u(x,0) = F(|x|)/|x|$ and $u_t(x,0) = F'(|x|)/|x|$?*
c) ***(Ralston Hw)*** *Suppose $u(x,t)$ is a solution to the wave equation $u_{tt} = c^2\triangle u$ in $\mathbb{R}^3\times\mathbb{R}$ with $u(x,t) = w(|x|,t)$ and $u(x,0) = 0$. Show that*

$$u(x,t) = \frac{F(|x|+ct) - F(|x|-ct)}{|x|}$$

for a function F of one variable.

Proof. **a)** We have

$$\begin{aligned}\int_{\mathbb{R}^3}\int_{\mathbb{R}} u\,(\phi_{tt} - \triangle\phi)\,dxdt &= \lim_{\epsilon\to 0}\int_{\mathbb{R}} dt\int_{|x|>\epsilon} u\,(\phi_{tt} - \triangle\phi)\,dx \\ &= \lim_{\epsilon\to 0}\int_{\mathbb{R}} dt\Big[\int_{|x|>\epsilon}\phi\,(u_{tt} - \triangle u)\,dx + \int_{|x|=\epsilon}\frac{\partial u}{\partial n}\phi - u\frac{\partial\phi}{\partial n}\,dS\Big].\end{aligned}$$

The final equality is derived by integrating by parts twice in t, and using Green's theorem:

$$\int_\Omega (v\triangle u - u\triangle v)\,dx = \int_{\partial\Omega}(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n})ds.$$

Since $dS = \epsilon^2\sin\phi' d\phi' d\theta$ and $\frac{\partial}{\partial n} = -\frac{\partial}{\partial r}$, substituting $u(x,t) = F(|x|+ct)/|x|$ gives:

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}} u\,(\phi_{tt} - \triangle\phi)\,dxdt = \int_{\mathbb{R}} 4\pi\phi F(ct)\,dt.$$

Thus, u is not a weak solution to the wave equation.

b)
c) We want to show that $v(|x|,t) = |x|w(|x|,t)$ is a solution to the wave equation in one space dimension and hence must have the from $v = F(|x|+ct) + G(|x|-ct)$. Then we can argue that w will be undefined at $x = 0$ for some t unless $F(ct) + G(-ct) = 0$ for all t.
We work in spherical coordinates. Note that w and v are independent of ϕ and θ. We have:

$$\begin{aligned} v_{tt}(r,t) &= c^2\triangle w = c^2\frac{1}{r^2}(r^2 w_r)_r = c^2\frac{1}{r^2}(2rw_r + r^2 w_{rr}), \\ &\Rightarrow\ rw_{tt} = c^2 rw_{rr} + 2w_r.\end{aligned}$$

Thus we see that $v_{tt} = c^2 v_{rr}$, and we can conclude that

$$\begin{aligned} v(r,t) &= F(r+ct) + G(r-ct) \qquad \text{and} \\ w(r,t) &= \frac{F(r+ct) + G(r-ct)}{r}.\end{aligned}$$

$\lim_{r \to 0} w(r,t)$ does not exist unless $F(ct) + G(-ct) = 0$ for all t. Hence

$$\begin{aligned} w(r,t) &= \frac{F(ct+r) + G(ct-r)}{r}, \qquad \text{and} \\ u(x,t) &= \frac{F(ct+|x|) + G(ct-|x|)}{|x|}. \end{aligned}$$

$\square$

17 Problems: Laplace Equation

A fundamental solution $K(x)$ for the Laplace operator is a distribution satisfying [44]

$$\triangle K(x) = \delta(x)$$

The fundamental solution for the Laplace operator is

$$K(x) = \begin{cases} \frac{1}{2\pi}\log|x| & \text{if } n = 2 \\ \frac{1}{(2-n)\omega_n}|x|^{2-n} & \text{if } n \geq 3. \end{cases}$$

17.1 Green's Function and the Poisson Kernel

Green's function is a special fundamental solution satisfying [45]

$$\begin{cases} \triangle G(x,\xi) = \delta(x) & \text{for } \ x \in \Omega \\ G(x,\xi) = 0 & \text{for } \ x \in \partial\Omega, \end{cases} \tag{17.1}$$

To construct the Green's function,
① consider $w_\xi(x)$ with $\triangle w_\xi(x) = 0$ in Ω and $w_\xi(x) = -K(x-\xi)$ on $\partial\Omega$;
② consider $G(x,\xi) = K(x-\xi) + w_\xi(x)$, which is a fundamental solution satisfying (17.1).

Problem 1. *Given a particular distribution solution to the set of Dirichlet problems*

$$\begin{cases} \triangle u_\xi(x) = \delta_\xi(x) & \textit{for } \ x \in \Omega \\ u_\xi(x) = 0 & \textit{for } \ x \in \partial\Omega, \end{cases}$$

how would you use this to solve

$$\begin{cases} \triangle u = 0 & \textit{for } \ x \in \Omega \\ u(x) = g(x) & \textit{for } \ x \in \partial\Omega. \end{cases}$$

Proof. $u_\xi(x) = G(x,\xi)$, a Green's function. G is a fundamental solution to the Laplace operator, $G(x,\xi) = 0$, $x \in \partial\Omega$. In this problem, it is assumed that $G(x,\xi)$ is known for Ω. Then

$$u(\xi) = \int_\Omega G(x,\xi)\triangle u\,dx + \int_{\partial\Omega} u(x)\frac{\partial G(x,\xi)}{\partial n_x}dS_x$$

for every $u \in C^2(\overline{\Omega})$. In particular, if $\triangle u = 0$ in Ω and $u = g$ on $\partial\Omega$, then we obtain the *Poisson integral formula*

$$\boxed{u(\xi) = \int_{\partial\Omega} \frac{\partial G(x,\xi)}{\partial n_x} g(x)\,dS_x,}$$

[44] We know that $u(x) = \int_{\mathbb{R}^n} K(x-y)f(y)\,dy$ is a distribution solution of $\triangle u = f$ when f is integrable and has compact support. In particular, we have

$$u(x) = \int_{\mathbb{R}^n} K(x-y)\triangle u(y)\,dy \qquad \text{whenever } \ u \in C_0^\infty(\mathbb{R}^n).$$

The above result is a consequence of:

$$u(x) \ = \ \int_\Omega \delta(x-y)u(y)\,dy \ = \ (\triangle K) * u \ = \ K * (\triangle u) \ = \ \int_\Omega K(x-y)\,\triangle u(y)\,dy.$$

[45] Green's function is useful in satisfying Dirichlet boundary conditions.

where $H(x, \xi) = \frac{\partial G(x,\xi)}{\partial n_x}$ is the *Poisson kernel.*
Thus *if* we know that the Dirichlet problem has a solution $u \in C^2(\overline{\Omega})$, then we can calculate u from the Poisson integral formula (provided of course that we can compute $G(x, \xi)$). $\square$

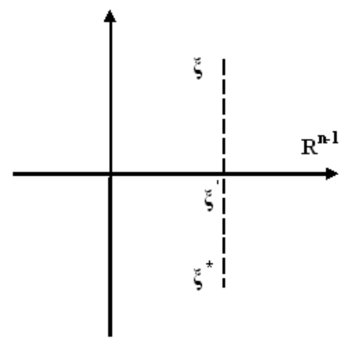
Dirichlet Problem on a Half-Space. *Solve the n-dimensional Laplace/Poisson equation on the half-space with Dirichlet boundary conditions.*

Proof. Use the **method of reflection** to construct Green's function. Let Ω be an upper half-space in $\mathbb{R}^n$. If $x = (x', x_n)$, where $x' \in \mathbb{R}^{n-1}$, we can see

$$|x' - \xi| = |x' - \xi^*|, \quad \text{and hence} \quad K(x' - \xi) = K(x' - \xi^*). \quad \text{Thus}$$

$$\boxed{G(x, \xi) = K(x - \xi) - K(x - \xi^*)}$$

is the Green's function on Ω. $G(x, \xi)$ is harmonic in Ω, and $G(x, \xi) = 0$ on $\partial\Omega$.
To compute the Poisson kernel, we must differentiate $G(x, \xi)$ in the *negative* x_n direction. For $n \geq 2$,



$$\frac{\partial}{\partial x_n} K(x - \xi) \;=\; \frac{x_n - \xi_n}{\omega_n} |x - \xi|^{-n},$$

so that the Poisson kernel is given by

$$-\frac{\partial}{\partial x_n} G(x, \xi)\Big|_{x_n = 0} \;=\; \frac{2\xi_n}{\omega_n} |x' - \xi|^{-n}, \quad \text{for} \;\; x' \in \mathbb{R}^{n-1}.$$

Thus, the solution is

$$\boxed{u(\xi) \;=\; \int_{\partial\Omega} \frac{\partial G(x, \xi)}{\partial n_x} g(x)\, dS_x \;=\; \frac{2\xi_n}{\omega_n} \int_{\mathbb{R}^{n-1}} \frac{g(x')}{|x' - \xi|^n}\, dx'.}$$

If $g(x')$ is bounded and continuous for $x' \in \mathbb{R}^{n-1}$, then $u(\xi)$ is C^∞ and harmonic in $\mathbb{R}^n_+$ and extends continuously to $\overline{\mathbb{R}^n_+}$ such that $u(\xi') = g(\xi')$.

□

Problem (F'95, #3): Neumann Problem on a Half-Space.
a) *Consider the Neumann problem in the upper half plane,*
$\Omega = \{x = (x_1, x_2) : -\infty < x_1 < \infty,\ x_2 > 0\}$*:*

$$\triangle u = u_{x_1x_1} + u_{x_2x_2} = 0 \qquad x \in \Omega,$$
$$u_{x_2}(x_1, 0) = f(x_1) \qquad -\infty < x_1 < \infty.$$

Find the corresponding ***Green's function*** *and conclude that*

$$u(\xi) = u(\xi_1, \xi_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty} ln\,[(x_1 - \xi_1)^2 + \xi_2^2] \cdot f(x_1)\, dx_1$$

is a solution of the problem.
b) *Show that this solution is bounded in Ω if and only if $\int_{-\infty}^{\infty} f(x_1)\, dx_1 = 0$.*

Proof. **a)** Notation: $x = (x, y)$, $\xi = (x_0, y_0)$. Since $K(x-\xi) = \frac{1}{2\pi}\log|x-\xi|$, $n = 2$.
① First, we find the Green's function. We have

$$K(x - \xi) = \frac{1}{2\pi}\log\sqrt{(x - x_0)^2 + (y - y_0)^2}.$$

Let $G(x, \xi) = K(x - \xi) + \omega(x)$.
Since the problem is Neumann, we need:

$$\begin{cases} \triangle G(x,\xi) \ = \ \delta(x-\xi), \\ \frac{\partial G}{\partial y}((x,0),\xi) \ = \ 0. \end{cases}$$

$$\begin{aligned}
G((x,y),\xi) &= \frac{1}{2\pi}\log\sqrt{(x - x_0)^2 + (y - y_0)^2} \ + \ \omega((x,y),\xi), \\
\frac{\partial G}{\partial y}((x,y),\xi) &= \frac{1}{2\pi}\frac{y - y_0}{(x - x_0)^2 + (y - y_0)^2} + \omega_y((x,y),\xi), \\
\frac{\partial G}{\partial y}((x,0),\xi) &= -\frac{1}{2\pi}\frac{y_0}{(x - x_0)^2 + y_0^2} + \omega_y((x,0),\xi) \ = \ 0.
\end{aligned}$$

Let

$$\begin{aligned}
\omega((x,y),\xi) &= \frac{a}{2\pi}\log\sqrt{(x - x_0)^2 + (y + y_0)^2}. \qquad \text{Then,} \\
\frac{\partial G}{\partial y}((x,0),\xi) &= -\frac{1}{2\pi}\frac{y_0}{(x - x_0)^2 + y_0^2} + \frac{a}{2\pi}\frac{y_0}{(x - x_0)^2 + y_0^2} \ = \ 0.
\end{aligned}$$

Thus, $a = 1$.

$$\boxed{G((x,y),\xi) \ = \ \frac{1}{2\pi}\log\sqrt{(x - x_0)^2 + (y - y_0)^2} \ + \ \frac{1}{2\pi}\log\sqrt{(x - x_0)^2 + (y + y_0)^2}.}$$

[46]

② Consider Green's identity (after cutting out $B_\epsilon(\xi)$ and having $\epsilon \to 0$):

$$\int_\Omega (u\triangle G - G\underbrace{\triangle u}_{=0})\, dx \ = \ \int_{\partial\Omega}\Big(u\underbrace{\frac{\partial G}{\partial n}}_{=0} - G\frac{\partial u}{\partial n}\Big)\, dS$$

[46] Note that for the Dirichlet problem, we would have gotten the "-" sign instead of "+" in front of ω.

Since $\frac{\partial u}{\partial n} = \frac{\partial u}{\partial(-y)} = -f(x)$, we have

$$\begin{aligned}
\int_\Omega u\,\delta(x-\xi)\,dx &= \int_{-\infty}^{\infty} G((x,y),\xi)\,f(x)\,dx, \\
u(\xi) &= \int_{-\infty}^{\infty} G((x,y),\xi)\,f(x)\,dx.
\end{aligned}$$

For $y = 0$, we have

$$\begin{aligned}
G((x,y),\xi) &= \frac{1}{2\pi}\log\sqrt{(x-x_0)^2+y_0^2} + \frac{1}{2\pi}\log\sqrt{(x-x_0)^2+y_0^2} \\
&= \frac{1}{2\pi}\,2\log\sqrt{(x-x_0)^2+y_0^2} \\
&= \frac{1}{2\pi}\log\left[(x-x_0)^2+y_0^2\right].
\end{aligned}$$

Thus,

$$u(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\log\left[(x-x_0)^2+y_0^2\right]f(x)\,dx. \quad \checkmark$$

b) *Show that this solution is bounded in Ω if and only if* $\int_{-\infty}^{\infty} f(x_1)\,dx_1 = 0$.

Consider the Green's identity:

$$\int_\Omega \triangle u\,dxdy = \int_{\partial\Omega}\frac{\partial u}{\partial n}\,dS = -\int_{-\infty}^{\infty}\frac{\partial u}{\partial y}\,dx = \int_{-\infty}^{\infty} f(x)\,dx = 0.$$

Note that the Green's identity applies to bounded domains Ω.

$$\int_{-R}^{R} f\,dx_1 + \int_0^{2\pi}\frac{\partial u}{\partial r}\,R\,d\theta = 0.$$

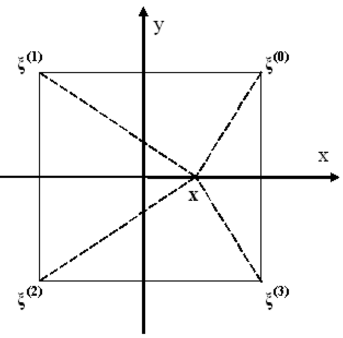
??? $\square$

McOwen 4.2 # 6. *For $n = 2$, use the method of reflections to find the Green's function for the **first quadrant** $\Omega = \{(x, y) : x, y > 0\}$.*

Proof. For $x \in \partial\Omega$,

$$|x - \xi^{(0)}| \cdot |x - \xi^{(2)}| = |x - \xi^{(1)}| \cdot |x - \xi^{(3)}|,$$

$$|x - \xi^{(0)}| = \frac{|x - \xi^{(1)}| \cdot |x - \xi^{(3)}|}{|x - \xi^{(2)}|}.$$



But $\xi^{(0)} = \xi$, so for $n = 2$,

$$\boxed{G(x, \xi) \;=\; \frac{1}{2\pi} \log|x - \xi| \;-\; \frac{1}{2\pi} \log \frac{|x - \xi^{(1)}| \cdot |x - \xi^{(3)}|}{|x - \xi^{(2)}|}}.$$

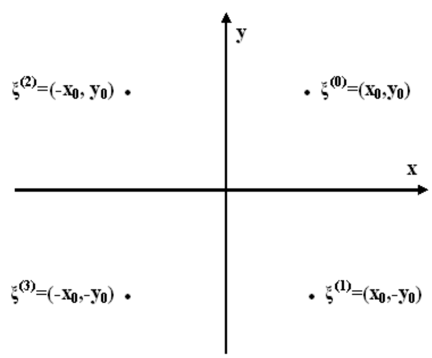
$G(x, \xi) = 0$, $x \in \partial\Omega$.

□

Problem. *Use the method of images to solve*

$$\triangle G = \delta(x - \xi)$$

in the first quadrant with $G = 0$ on the boundary.



Proof. To solve the problem in the first quadrant we take a reflection to the fourth quadrant and the two are reflected to the left half.

$$\begin{aligned}
\triangle G &= \delta(x - \xi^{(0)}) - \delta(x - \xi^{(1)}) - \delta(x - \xi^{(2)}) + \delta(x - \xi^{(3)}). \\
G &= \frac{1}{2\pi} \log \frac{|x - \xi^{(0)}|\,|x - \xi^{(3)}|}{|x - \xi^{(1)}|\,|x - \xi^{(2)}|} \\
&= \frac{1}{2\pi} \log \frac{\sqrt{(x - x_0)^2 + (y - y_0)^2}\,\sqrt{(x + x_0)^2 + (y + y_0)^2}}{\sqrt{(x - x_0)^2 + (y + y_0)^2}\,\sqrt{(x + x_0)^2 + (y - y_0)^2}}.
\end{aligned}$$

Note that on the axes $G = 0$. □

Problem (S'96, #3). *Construct a* ***Green's function*** *for the following* ***mixed*** *Dirichlet-Neumann problem in* $\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2 \, : \, x_1 > 0, \, x_2 > 0\}$*:*

$$\begin{aligned} \triangle u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} &= f, \qquad x \in \Omega, \\ u_{x_2}(x_1, 0) &= 0, \qquad x_1 > 0, \\ u(0, x_2) &= 0, \qquad x_2 > 0. \end{aligned}$$

Proof. Notation: $x = (x, y)$, $\xi = (x_0, y_0)$. Since $K(x - \xi) = \frac{1}{2\pi} \log |x - \xi|$, $n = 2$.

$$K(x - \xi) = \frac{1}{2\pi} \log \sqrt{(x - x_0)^2 + (y - y_0)^2}.$$

Let $G(x, \xi) = K(x - \xi) + \omega(x)$.
At $(0, y)$, $y > 0$,

$$G\big((0, y), \xi\big) = \frac{1}{2\pi} \log \sqrt{x_0^2 + (y - y_0)^2} + \omega(0, y) = 0.$$

Also,

$$\begin{aligned} G_y\big((x, y), \xi\big) &= \frac{1}{2\pi} \frac{\frac{1}{2} \cdot 2(y - y_0)}{(x - x_0)^2 + (y - y_0)^2} + w_y(x, y) \\ &= \frac{1}{2\pi} \frac{y - y_0}{(x - x_0)^2 + (y - y_0)^2} + w_y(x, y). \end{aligned}$$

At $(x, 0)$, $x > 0$,

$$G_y\big((x, 0), \xi\big) = -\frac{1}{2\pi} \frac{y_0}{(x - x_0)^2 + y_0^2} + w_y(x, 0) = 0.$$

We have

$$\begin{aligned} \omega((x, y), \xi) &= \frac{a}{2\pi} \log \sqrt{(x + x_0)^2 + (y - y_0)^2} \\ &+ \frac{b}{2\pi} \log \sqrt{(x - x_0)^2 + (y + y_0)^2} \\ &+ \frac{c}{2\pi} \log \sqrt{(x + x_0)^2 + (y + y_0)^2}. \end{aligned}$$

Using boundary conditions, we have

$$\begin{aligned} 0 &= G((0, y), \xi) = \frac{1}{2\pi} \log \sqrt{x_0^2 + (y - y_0)^2} + \omega(0, y) \\ &= \frac{1}{2\pi} \log \sqrt{x_0^2 + (y - y_0)^2} + \frac{a}{2\pi} \log \sqrt{x_0^2 + (y - y_0)^2} + \frac{b}{2\pi} \log \sqrt{x_0^2 + (y + y_0)^2} + \frac{c}{2\pi} \log \sqrt{x_0^2 + (y + y_0)^2}. \end{aligned}$$

Thus, $a = -1$, $c = -b$. Also,

$$\begin{aligned} 0 &= G_y((x, 0), \xi) = -\frac{1}{2\pi} \frac{y_0}{(x - x_0)^2 + y_0^2} + w_y(x, 0) \\ &= -\frac{1}{2\pi} \frac{y_0}{(x - x_0)^2 + y_0^2} - \frac{(-1)}{2\pi} \frac{y_0}{(x + x_0)^2 + y_0^2} + \frac{b}{2\pi} \frac{y_0}{(x - x_0)^2 + y_0^2} + \frac{(-b)}{2\pi} \frac{y_0}{(x + x_0)^2 + y_0^2}. \end{aligned}$$

Thus, $b = 1$, and

$$\boxed{G((x, y), \xi) = \frac{1}{2\pi} \log \sqrt{(x - x_0)^2 + (y - y_0)^2} + \omega(x) = \frac{1}{2\pi} \Big[\log \sqrt{(x - x_0)^2 + (y - y_0)^2}}$$

$$\boxed{\qquad - \log\sqrt{(x+x_0)^2+(y-y_0)^2} + \log\sqrt{(x-x_0)^2+(y+y_0)^2} - \log\sqrt{(x+x_0)^2+(y+y_0)^2}\Big].}$$

It can be seen that $G((x,y),\xi) = 0$ on $x = 0$, for example. $\square$

Dirichlet Problem on a Ball. *Solve the n-dimensional Laplace/Poisson equation on the ball with Dirichlet boundary conditions.*

Proof. Use the **method of reflection** to construct Green's function.
Let $\Omega = \{x \in \mathbb{R}^n : |x| < a\}$. For $\xi \in \Omega$, define $\xi^* = \frac{a^2\xi}{|\xi|^2}$ as its reflection in $\partial\Omega$; note $\xi^* \notin \Omega$.

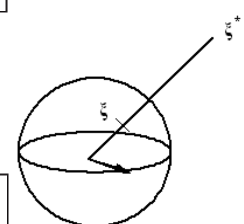
$$\frac{|x-\xi^*|}{|x-\xi|} = \frac{a}{|\xi|} \qquad \text{for} \quad |x| = a. \qquad \Rightarrow \quad |x-\xi| = \frac{|\xi|}{a}|x-\xi^*|. \tag{17.2}$$

From (17.2) we conclude that for $x \in \partial\Omega$ (i.e. $|x| = a$),

$$K(x-\xi) = \begin{cases} \frac{1}{2\pi}\log\left(\frac{|\xi|}{a}|x-\xi^*|\right) & \text{if } n = 2 \\ \left(\frac{a}{|\xi|}\right)^{n-2} K(x-\xi^*) & \text{if } n \geq 3. \end{cases} \tag{17.3}$$

Define for x, $\xi \in \Omega$:

$$\boxed{G(x,\xi) = \begin{cases} K(x-\xi) - \frac{1}{2\pi}\log\left(\frac{|\xi|}{a}|x-\xi^*|\right) & \text{if } n = 2 \\ K(x-\xi) - \left(\frac{a}{|\xi|}\right)^{n-2} K(x-\xi^*) & \text{if } n \geq 3. \end{cases}}$$



Since ξ^* is *not* in Ω, the second terms on the RHS are harmonic in $x \in \Omega$. Moreover, by (17.3) we have $G(x,\xi) = 0$ if $x \in \partial\Omega$. Thus, G is the Green's function for Ω.

$$\boxed{u(\xi) = \int_{\partial\Omega} \frac{\partial G(x,\xi)}{\partial n_x} g(x)\, dS_x = \frac{a^2 - |\xi|^2}{a\omega_n} \int_{|x|=a} \frac{g(x)}{|x-\xi|^n}\, dS_x.}$$

$\square$

17.2 The Fundamental Solution

Problem (F'99, #2). ① *Given that $K_a(x-y)$ and $K_b(x-y)$ are the kernels for the operators $(\triangle - aI)^{-1}$ and $(\triangle - bI)^{-1}$ on $L^2(\mathbb{R}^n)$, where $0 < a < b$, show that $(\triangle - aI)(\triangle - bI)$ has a* ***fundamental solution*** *of the form $c_1K_a + c_2K_b$.*
② *Use the preceding to find a fundamental solution for $\triangle^2 - \triangle$, when $n = 3$.*

Proof. **METHOD ❶**:
①

$$
\begin{array}{ccc}
(\triangle - aI)u = f & & (\triangle - bI)u = f \\
u = \underbrace{K_a}\star f & & u = \underbrace{K_b}\star f \\
\text{fundamental solution} & \Leftrightarrow & \text{kernel}
\end{array}
$$

$$
\begin{array}{ll}
\Rightarrow \quad \widehat{u} = \widehat{K_a}\widehat{f} & \widehat{u} = \widehat{K_b}\widehat{f} \qquad \text{if } \ u \in L^2, \\
(\widehat{\triangle - aI})u = (-|\xi|^2 - a)\widehat{u} = \widehat{f} & (\widehat{\triangle - bI})u = (-|\xi|^2 - b)\widehat{u} = \widehat{f} \\
\Rightarrow \quad \widehat{u} = -\dfrac{1}{(\xi^2+a)}\widehat{f}(\xi) & \widehat{u} = -\dfrac{1}{(\xi^2+b)}\widehat{f}(\xi) \\
\Rightarrow \quad \widehat{K_a} = -\dfrac{1}{\xi^2+a} & \widehat{K_b} = -\dfrac{1}{\xi^2+b}
\end{array}
$$

$$
\begin{aligned}
(\triangle - aI)(\triangle - bI)u &= f, \\
\big(\triangle^2 - (a+b)\triangle + abI\big)u &= f, \\
\widehat{u} = \frac{1}{(\xi^2+a)(\xi^2+b)}\widehat{f}(\xi) &= \widehat{K}_{new}\widehat{f}(\xi), \\
\widehat{K}_{new} = \frac{1}{(\xi^2+a)(\xi^2+b)} &= \frac{1}{b-a}\Big(-\frac{1}{\xi^2+b} + \frac{1}{\xi^2+a}\Big) = \frac{1}{b-a}(\widehat{K}_b - \widehat{K}_a), \\
K_{new} &= \frac{1}{b-a}(K_b - K_a), \\
c_1 = \frac{1}{b-a}, &\quad c_2 = -\frac{1}{b-a}.
\end{aligned}
$$

② $n = 3$ is not relevant (may be used to assume $K_a, K_b \in L^2$).
For $\triangle^2 - \triangle$, $a = 0$, $b = 1$ above, or more explicitly

$$
\begin{aligned}
(\triangle^2 - \triangle)u &= f, \\
(\xi^4 + \xi^2)\widehat{u} &= \widehat{f}, \\
\widehat{u} &= \frac{1}{(\xi^4+\xi^2)}\widehat{f}, \\
\widehat{K} &= \frac{1}{(\xi^4+\xi^2)} = \frac{1}{\xi^2(\xi^2+1)} = -\frac{1}{\xi^2+1} + \frac{1}{\xi^2} = \widehat{K}_1 - \widehat{K}_0.
\end{aligned}
$$

METHOD ❷:

- For $u \in C_0^\infty(\mathbb{R}^n)$ we have:

$$\begin{aligned} u(x) &= \int_{\mathbb{R}^n} K_a(x-y)\,(\triangle - aI)\,u(y)\,dy, \quad ① \\ u(x) &= \int_{\mathbb{R}^n} K_b(x-y)\,(\triangle - bI)\,u(y)\,dy. \quad ② \end{aligned}$$

Let

$$\begin{aligned} u(x) &= c_1(\triangle - bI)\,\phi(x), \qquad \text{for } ① \\ u(x) &= c_2(\triangle - aI)\,\phi(x), \qquad \text{for } ② \end{aligned}$$

for $\phi(x) \in C_0^\infty(\mathbb{R}^n)$. Then,

$$\begin{aligned} c_1(\triangle - bI)\phi(x) &= \int_{\mathbb{R}^n} K_a(x-y)\,(\triangle - aI)\,c_1(\triangle - bI)\phi(y)\,dy, \\ c_2(\triangle - aI)\phi(x) &= \int_{\mathbb{R}^n} K_b(x-y)\,(\triangle - bI)\,c_2(\triangle - aI)\phi(y)\,dy. \end{aligned}$$

We add two equations:

$$(c_1+c_2)\triangle\phi(x) - (c_1 b + c_2 a)\phi(x) = \int_{\mathbb{R}^n} (c_1K_a + c_2K_b)\,(\triangle - aI)\,(\triangle - bI)\,\phi(y)\,dy.$$

If $c_1 = -c_2$ and $-(c_1 b + c_2 a) = 1$, that is, $c_1 = \frac{1}{a-b}$, we have:

$$\phi(x) = \int_{\mathbb{R}^n} \frac{1}{a-b}(K_a - K_b)\,(\triangle - aI)\,(\triangle - bI)\,\phi(y)\,dy,$$

which means that $\frac{1}{a-b}(K_a - K_b)$ is a fundamental solution of $(\triangle - aI)(\triangle - bI)$. ✓

- $\triangle^2 - \triangle = \triangle(\triangle - 1) = (\triangle - 0I)(\triangle - 1I)$.

$(\triangle - 0I)$ has fundamental solution $K_0 = -\frac{1}{4\pi r}$ in $\mathbb{R}^3$.
To find K, a **fundamental solution for** $(\triangle - 1I)$, we need to solve for a radially symmetric solution of

$$(\triangle - 1I)K = \delta.$$

In spherical coordinates, in $\mathbb{R}^3$, the above expression may be written as:

$$K'' + \frac{2}{r}K' - K = 0. \qquad \circledast$$

Let

$$\begin{aligned} K &= \frac{1}{r}w(r), \\ K' &= \frac{1}{r}w' - \frac{1}{r^2}w, \\ K'' &= \frac{1}{r}w'' - \frac{2}{r^2}w' + \frac{2}{r^3}w. \end{aligned}$$

Plugging these into $\circledast$, we obtain:

$$\begin{aligned} \frac{1}{r}w'' - \frac{1}{r}w &= 0, \qquad \text{or} \\ w'' - w &= 0. \end{aligned}$$

Thus,

$$\begin{aligned} w &= c_1 e^r + c_2 e^{-r}, \\ K &= \frac{1}{r} w(r) \;=\; c_1 \frac{e^r}{r} + c_2 \frac{e^{-r}}{r}. \qquad \checkmark \end{aligned}$$

Suppose $v(x) \equiv 0$ for $|x| \geq R$ and let $\Omega = B_R(0)$; for small $\epsilon > 0$ let

$$\Omega_\epsilon = \Omega - B_\epsilon(0).$$

Note: $(\triangle - I)K(|x|) = 0$ in Ω_ϵ. Consider Green's identity ($\partial\Omega_\epsilon = \partial\Omega \cup \partial B_\epsilon(0)$):

$$\int_{\Omega_\epsilon} \Big(K(|x|)\triangle v - v\triangle K(|x|) \Big)\, dx = \underbrace{\int_{\partial\Omega} \Big(K(|x|)\frac{\partial v}{\partial n} - v\frac{\partial K(|x|)}{\partial n} \Big)\, dS}_{=0,\ since\ v\equiv 0\ for\ x\geq R} + \int_{\partial B_\epsilon(0)} \Big(K(|x|)\frac{\partial v}{\partial n} - v\frac{\partial K(|x|)}{\partial n} \Big)\, dS$$

We add $-\int_{\Omega_\epsilon} K(|x|)\, v\, dx + \int_{\Omega_\epsilon} v\, K(|x|)\, dx$ to LHS to get:

$$\int_{\Omega_\epsilon} \Big(K(|x|)(\triangle - I)v - v\underbrace{(\triangle - I)K(|x|)}_{=0,\ in\ \Omega_\epsilon} \Big)\, dx = \int_{\partial B_\epsilon(0)} \Big(K(|x|)\frac{\partial v}{\partial n} - v\frac{\partial K(|x|)}{\partial n} \Big)\, dS.$$

$$\lim_{\epsilon\to 0}\Big[\int_{\Omega_\epsilon} K(|x|)(\triangle - I)v\, dx\Big] = \int_{\Omega} K(|x|)(\triangle - I)v\, dx. \quad \Big(\text{Since } K(r) = c_1\frac{e^r}{r} + c_2\frac{e^{-r}}{r} \text{ is integrable at } x = 0.\Big)$$

On $\partial B_\epsilon(0)$, $K(|x|) = K(\epsilon)$. Thus, [47]

$$\Big|\int_{\partial B_\epsilon(0)} K(|x|)\frac{\partial v}{\partial n}\, dS\Big| = \big|K(\epsilon)\big|\int_{\partial B_\epsilon(0)} \Big|\frac{\partial v}{\partial n}\Big|\, dS \leq \Big|c_1\frac{e^\epsilon}{\epsilon} + c_2\frac{e^{-\epsilon}}{\epsilon}\Big| 4\pi\epsilon^2 \max|\nabla v| \;\to\; 0, \quad \text{as } \epsilon \to 0.$$

$$\begin{aligned} \int_{\partial B_\epsilon(0)} v(x)\frac{\partial K(|x|)}{\partial n}\, dS &= \int_{\partial B_\epsilon(0)} \Big[\frac{1}{\epsilon}\big(-c_1e^\epsilon + c_2e^{-\epsilon}\big) + \frac{1}{\epsilon^2}\big(c_1e^\epsilon + c_2e^{-\epsilon}\big)\Big] v(x)\, dS \\ &= \Big[\frac{1}{\epsilon}\big(-c_1e^\epsilon + c_2e^{-\epsilon}\big) + \frac{1}{\epsilon^2}\big(c_1e^\epsilon + c_2e^{-\epsilon}\big)\Big]\int_{\partial B_\epsilon(0)} v(x)\, dS \\ &= \Big[\frac{1}{\epsilon}\big(-c_1e^\epsilon + c_2e^{-\epsilon}\big) + \frac{1}{\epsilon^2}\big(c_1e^\epsilon + c_2e^{-\epsilon}\big)\Big]\int_{\partial B_\epsilon(0)} v(0)\, dS \\ &\quad + \Big[\frac{1}{\epsilon}\big(-c_1e^\epsilon + c_2e^{-\epsilon}\big) + \frac{1}{\epsilon^2}\big(c_1e^\epsilon + c_2e^{-\epsilon}\big)\Big]\int_{\partial B_\epsilon(0)} [v(x) - v(0)]\, dS \\ &\to \frac{1}{\epsilon^2}\big(c_1e^\epsilon + c_2e^{-\epsilon}\big)\, v(0)\, 4\pi\epsilon^2 \\ &\to 4\pi(c_1 + c_2)v(0) \;=\; -v(0). \end{aligned}$$

Thus, taking $c_1 = c_2$, we have $c_1 = c_2 = -\frac{1}{8\pi}$, which gives

$$\int_{\Omega} K(|x|)(\triangle - I)v\, dx = \lim_{\epsilon\to 0}\int_{\Omega_\epsilon} K(|x|)(\triangle - I)v\, dx = v(0),$$

[47] In $\mathbb{R}^3$, for $|x| = \epsilon$,

$$K(|x|) = K(\epsilon) = c_1\frac{e^\epsilon}{\epsilon} + c_2\frac{e^{-\epsilon}}{\epsilon}.$$

$$\frac{\partial K(|x|)}{\partial n} = -\frac{\partial K(\epsilon)}{\partial r} = -c_1\Big(\frac{e^\epsilon}{\epsilon} - \frac{e^\epsilon}{\epsilon^2}\Big) - c_2\Big(-\frac{e^{-\epsilon}}{\epsilon} - \frac{e^{-\epsilon}}{\epsilon^2}\Big) = \frac{1}{\epsilon}\big(-c_1e^\epsilon + c_2e^{-\epsilon}\big) + \frac{1}{\epsilon^2}\big(c_1e^\epsilon + c_2e^{-\epsilon}\big),$$

since n points inwards. n points *toward* 0 on the sphere $|x| = \epsilon$ (i.e., $n = -x/|x|$).

that is $K(r) = -\frac{1}{8\pi}\Big(\frac{e^r}{r} + \frac{e^{-r}}{r}\Big) = -\frac{1}{4\pi r}\cosh(r)$ is the fundamental solution of $(\triangle - I)$.
By part (a), $\frac{1}{a-b}(K_a - K_b)$ is a fundamental solution of $(\triangle - aI)(\triangle - bI)$.
Here, the fundamental solution of $(\triangle - 0I)(\triangle - 1I)$ is $\frac{1}{-1}(K_0 - K) = -\Big(-\frac{1}{4\pi r} + \frac{1}{4\pi r}\cosh(r)\Big) = \frac{1}{4\pi r}\big(1 - \cosh(r)\big)$. $\square$

Problem (F'91, #3). *Prove that*

$$-\frac{1}{4\pi}\frac{\cos k|x|}{|x|}$$

is a ***fundamental solution*** *for* $(\triangle + k^2)$ *in* $\mathbb{R}^3$ *where* $|x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$,
i.e. prove that for any smooth function $f(x)$ *with compact support*

$$u(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{\cos k|x-y|}{|x-y|}f(y)\,dy$$

is a solution to $(\triangle + k^2)u = f$.

Proof. For $v \in C_0^\infty(\mathbb{R}^n)$, we want to show that for $K(|x|) = -\frac{1}{4\pi}\frac{\cos k|x|}{|x|}$,
we have $(\triangle + k^2)K = \delta$, i.e.

$$\int_{\mathbb{R}^n} K(|x|)\,(\triangle + k^2)v(x)\,dx \;=\; v(0).$$

Suppose $v(x) \equiv 0$ for $|x| \geq R$ and let $\Omega = B_R(0)$; for small $\epsilon > 0$ let

$$\Omega_\epsilon = \Omega - B_\epsilon(0).$$

$(\triangle + k^2)K(|x|) = 0$ in Ω_ϵ. Consider Green's identity ($\partial\Omega_\epsilon = \partial\Omega \cup \partial B_\epsilon(0)$):

$$\int_{\Omega_\epsilon}\Big(K(|x|)\triangle v \;-\; v\triangle K(|x|)\Big)\,dx \;=\; \underbrace{\int_{\partial\Omega}\Big(K(|x|)\frac{\partial v}{\partial n} - v\frac{\partial K(|x|)}{\partial n}\Big)\,dS}_{=0,\ since\ v\equiv 0\ for\ x\geq R} \;+\; \int_{\partial B_\epsilon(0)}\Big(K(|x|)\frac{\partial v}{\partial n} - v\frac{\partial K(|x|)}{\partial n}\Big)\,dS$$

We add $\int_{\Omega_\epsilon} k^2\,K(|x|)\,v\,dx \;-\; \int_{\Omega_\epsilon} v\,k^2\,K(|x|)\,dx$ to LHS to get:

$$\int_{\Omega_\epsilon}\Big(K(|x|)(\triangle + k^2)v \;-\; v\,\underbrace{(\triangle + k^2)K(|x|)}_{=0,\ in\ \Omega_\epsilon}\Big)\,dx \;=\; \int_{\partial B_\epsilon(0)}\Big(K(|x|)\frac{\partial v}{\partial n} - v\frac{\partial K(|x|)}{\partial n}\Big)\,dS.$$

$$\lim_{\epsilon\to 0}\Big[\int_{\Omega_\epsilon} K(|x|)(\triangle + k^2)v\,dx\Big] \;=\; \int_\Omega K(|x|)(\triangle + k^2)v\,dx. \quad \Big(\text{Since } K(r) = -\frac{\cos kr}{4\pi r} \text{ is integrable at } x = 0.\Big)$$

On $\partial B_\epsilon(0)$, $K(|x|) = K(\epsilon)$. Thus, [48]

$$\Big|\int_{\partial B_\epsilon(0)} K(|x|)\frac{\partial v}{\partial n}\,dS\Big| \;=\; \big|K(\epsilon)\big|\int_{\partial B_\epsilon(0)}\Big|\frac{\partial v}{\partial n}\Big|\,dS \;\leq\; \Big|-\frac{\cos k\epsilon}{4\pi\epsilon}\Big|\,4\pi\epsilon^2\max\big|\nabla v\big| \;\to\; 0, \quad \text{as } \epsilon \to 0.$$

[48] In $\mathbb{R}^3$, for $|x| = \epsilon$,

$$K(|x|) \;=\; K(\epsilon) \;=\; -\frac{\cos k\epsilon}{4\pi\epsilon}.$$

$$\frac{\partial K(|x|)}{\partial n} \;=\; -\frac{\partial K(\epsilon)}{\partial r} \;=\; \frac{1}{4\pi}\Big(-\frac{k\sin k\epsilon}{\epsilon} - \frac{\cos k\epsilon}{\epsilon^2}\Big) \;=\; -\frac{1}{4\pi\epsilon}\Big(k\sin k\epsilon + \frac{\cos k\epsilon}{\epsilon}\Big),$$

since n points inwards. n points *toward* 0 on the sphere $|x| = \epsilon$ (i.e., $n = -x/|x|$).

$$\int_{\partial B_\epsilon(0)} v(x)\,\frac{\partial K(|x|)}{\partial n}\,dS \;=\; \int_{\partial B_\epsilon(0)} -\frac{1}{4\pi\epsilon}\Big(k\sin k\epsilon+\frac{\cos k\epsilon}{\epsilon}\Big)\,v(x)\,dS$$

$$= \;-\frac{1}{4\pi\epsilon}\Big(k\sin k\epsilon+\frac{\cos k\epsilon}{\epsilon}\Big)\int_{\partial B_\epsilon(0)} v(x)\,dS$$

$$= \;-\frac{1}{4\pi\epsilon}\Big(k\sin k\epsilon+\frac{\cos k\epsilon}{\epsilon}\Big)\int_{\partial B_\epsilon(0)} v(0)\,dS \;-\; \frac{1}{4\pi\epsilon}\Big(k\sin k\epsilon+\frac{\cos k\epsilon}{\epsilon}\Big)\int_{\partial B_\epsilon(0)}[v(x)-v(0)]\,dS$$

$$= \;-\frac{1}{4\pi\epsilon}\Big(k\sin k\epsilon+\frac{\cos k\epsilon}{\epsilon}\Big)\,v(0)\,4\pi\epsilon^2 \;-\; \underbrace{\frac{1}{4\pi\epsilon}\Big(k\sin k\epsilon+\frac{\cos k\epsilon}{\epsilon}\Big)\,[v(x)-v(0)]\,4\pi\epsilon^2}_{\to 0,\ (v\ is\ continuous)}$$

$$\to\; -\cos k\epsilon\; v(0) \;\to\; -v(0).$$

Thus,

$$\int_\Omega K(|x|)(\triangle+k^2)v\,dx \;=\; \lim_{\epsilon\to 0}\int_{\Omega_\epsilon} K(|x|)(\triangle+k^2)v\,dx \;=\; v(0),$$

that is, $K(r)=-\frac{1}{4\pi}\frac{\cos kr}{r}$ is the fundamental solution of $\triangle+k^2$. $\square$

Problem (F'97, #2). *Let $u(x)$ be a solution of the Helmholtz equation*

$$\triangle u+k^2u=0 \qquad x\in\mathbb{R}^3$$

satisfying the "radiation" conditions

$$u=O\Big(\frac{1}{r}\Big),\quad \frac{\partial u}{\partial r}-iku=O\Big(\frac{1}{r^2}\Big),\quad |x|=r\to\infty.$$

Prove that $u\equiv 0$.
*Hint: A **fundamental solution** to the Helmholtz equation is $\frac{1}{4\pi r}e^{ikr}$.*
Use the Green formula.

Proof. Denote $K(|x|)=\frac{1}{4\pi r}e^{ikr}$, a fundamental solution. Thus, $(\triangle+k^2)K=\delta$.
Let x_0 be any point and $\Omega=B_R(x_0)$; for small $\epsilon>0$ let

$$\Omega_\epsilon=\Omega-B_\epsilon(x_0).$$

$(\triangle+k^2)K(|x|)=0$ in Ω_ϵ. Consider Green's identity ($\partial\Omega_\epsilon=\partial\Omega\cup\partial B_\epsilon(x_0)$):

$$\underbrace{\int_{\Omega_\epsilon}\Big(u\,(\triangle+k^2)K \;-\; K(\triangle+k^2)u\Big)\,dx}_{=0} \;=\; \int_{\partial\Omega}\Big(u\,\frac{\partial K}{\partial n}-K\,\frac{\partial u}{\partial n}\Big)\,dS \;+\; \underbrace{\int_{\partial B_\epsilon(x_0)}\Big(u\,\frac{\partial K}{\partial n}-K\,\frac{\partial u}{\partial n}\Big)\,dS}_{\to u(x_0),\ as\ \epsilon\to 0}.$$

(It can be shown by the method previously used that the integral over $B_\epsilon(x_0)$ approaches $u(x_0)$ as $\epsilon\to 0$.) Taking the limit when $\epsilon\to 0$, we obtain

$$-u(x_0) \;=\; \int_{\partial\Omega}\Big(u\,\frac{\partial K}{\partial n}-K\,\frac{\partial u}{\partial n}\Big)\,dS \;=\; \int_{\partial\Omega}\Big(u\,\frac{\partial}{\partial r}\frac{e^{ik|x-x_0|}}{4\pi|x-x_0|}-\frac{e^{ik|x-x_0|}}{4\pi|x-x_0|}\,\frac{\partial u}{\partial r}\Big)\,dS$$

$$=\; \int_{\partial\Omega}\Bigg(u\,\underbrace{\Big[\frac{\partial}{\partial r}\frac{e^{ik|x-x_0|}}{4\pi|x-x_0|}-ik\frac{e^{ik|x-x_0|}}{4\pi|x-x_0|}\Big]}_{=O(\frac{1}{|x|^2});\ (can\ be\ shown)} - \frac{e^{ik|x-x_0|}}{4\pi|x-x_0|}\,\Big[\frac{\partial u}{\partial r}-iku\Big]\Bigg)\,dS$$

$$=\; O\Big(\frac{1}{R}\Big)\cdot O\Big(\frac{1}{R^2}\Big)\cdot 4\pi R^2 - O\Big(\frac{1}{R}\Big)\cdot O\Big(\frac{1}{R^2}\Big)\cdot 4\pi R^2 \;=\; 0.$$

Taking the limit when $R\to\infty$, we get $u(x_0)=0$. $\square$

Problem (S'02, #1). ***a)*** *Find a radially symmetric solution, u, to the equation in $\mathbb{R}^2$,*

$$\triangle u = \frac{1}{2\pi}\log|x|,$$

*and show that u is a **fundamental solution** for $\triangle^2$, i.e. show*

$$\phi(0) = \int_{\mathbb{R}^2} u\triangle^2\phi\,dx$$

for any smooth ϕ which vanishes for $|x|$ large.
b) *Explain how to construct the **Green's function** for the following boundary value in a bounded domain $D\subset\mathbb{R}^2$ with smooth boundary ∂D*

$$\begin{aligned}
&w = 0 \quad \text{and} \quad \frac{\partial w}{\partial n} = 0 \quad \text{on } \partial D,\\
&\triangle^2 w = f \quad \text{in } D.
\end{aligned}$$

Proof. **a)** Rewriting the equation in polar coordinates, we have

$$\triangle u \;=\; \frac{1}{r}\big(ru_r\big)_r + \frac{1}{r^2}u_{\theta\theta} \;=\; \frac{1}{2\pi}\log r.$$

For a radially symmetric solution $u(r)$, we have $u_{\theta\theta}=0$. Thus,

$$\begin{aligned}
\frac{1}{r}\big(ru_r\big)_r &= \frac{1}{2\pi}\log r,\\
\big(ru_r\big)_r &= \frac{1}{2\pi}r\log r,\\
ru_r &= \frac{1}{2\pi}\int r\log r\,dr \;=\; \frac{r^2\log r}{4\pi} - \frac{r^2}{8\pi},\\
u_r &= \frac{r\log r}{4\pi} - \frac{r}{8\pi},\\
u &= \frac{1}{4\pi}\int r\log r\,dr - \frac{1}{8\pi}\int r\,dr \;=\; \frac{1}{8\pi}r^2\big(\log r - 1\big).
\end{aligned}$$

$$\boxed{u(r) \;=\; \frac{1}{8\pi}r^2\big(\log r - 1\big).}$$

We want to show that u defined above is a fundamental solution of $\triangle^2$ for $n=2$. That is

$$\int_{\mathbb{R}^2} u\,\triangle^2 v\,dx \;=\; v(0), \qquad\qquad v\in C_0^\infty(\mathbb{R}^n).$$

See the next page that shows that u defined as $u(r) = \frac{1}{8\pi}r^2\log r$ is the Fundamental Solution of $\triangle^2$. (The $-\frac{1}{8\pi}r^2$ term does not play any role.)

In particular, the solution of

$$\triangle^2\omega = f(x),$$

if given by

$$\omega(x) \;=\; \int_{\mathbb{R}^2} u(x-y)\,\triangle^2\omega(y)\,dy \;=\; \frac{1}{8\pi}\int_{\mathbb{R}^2} |x-y|^2\big(\log|x-y| - 1\big)\,f(y)\,dy.$$

b) Let

$$K(x-\xi) = \frac{1}{8\pi}|x-\xi|^2\left(\log|x-\xi| - 1\right).$$

We use the method of images to construct the Green's function.
Let $G(x,\xi) = K(x-\xi) + \omega(x)$. We need $G(x,\xi) = 0$ and $\frac{\partial G}{\partial n}(x,\xi) = 0$ for $x \in \partial\Omega$.
Consider $w_\xi(x)$ with $\triangle^2 w_\xi(x) = 0$ in Ω, $w_\xi(x) = -K(x-\xi)$ and $\frac{\partial w_\xi}{\partial n}(x) = -\frac{\partial K}{\partial n}(x-\xi)$ on $\partial\Omega$. Note, we can find the Greens function for the upper-half plane, and then make a conformal map onto the domain. $\square$

Problem (S'97, #6). *Show that the* ***fundamental solution*** *of* $\triangle^2$ *in* $\mathbb{R}^2$ *is given by*

$$V(x_1, x_2) = \frac{1}{8\pi} r^2 ln(r), \qquad r = |x - \xi|,$$

and write the solution of

$$\triangle^2 w = F(x_1, x_2).$$

Hint: In polar coordinates, $\triangle = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$; *for example,* $\triangle V = \frac{1}{2\pi}(1 + ln(r))$.

Proof. Notation: $x = (x_1, x_2)$. We have

$$V(x) = \frac{1}{8\pi} r^2 \log(r),$$

In polar coordinates: (here, $V_{\theta\theta} = 0$)

$$\begin{aligned}
\triangle V &= \frac{1}{r}\big(rV_r\big)_r = \frac{1}{r}\bigg(r\Big(\frac{1}{8\pi}r^2\log(r)\Big)_r\bigg)_r = \frac{1}{8\pi}\frac{1}{r}\bigg(r\Big(2r\log(r) + r\Big)\bigg)_r \\
&= \frac{1}{8\pi}\frac{1}{r}\Big(2r^2\log(r) + r^2\Big)_r = \frac{1}{8\pi}\frac{1}{r}\big(4r + 4r\log r\big) \\
&= \frac{1}{2\pi}(1 + \log r).
\end{aligned}$$

The fundamental solution $V(x)$ for $\triangle^2$ is the distribution satisfying: $\triangle^2 V(r) = \delta(r)$.

$$\begin{aligned}
\triangle^2 V &= \triangle(\triangle V) = \triangle\Big(\frac{1}{2\pi}(1 + \log r)\Big) = \frac{1}{2\pi}\triangle(1 + \log r) = \frac{1}{2\pi}\frac{1}{r}\big(r(1 + \log r)_r\big)_r \\
&= \frac{1}{2\pi}\frac{1}{r}\Big(r\frac{1}{r}\Big)_r = \frac{1}{2\pi}\frac{1}{r}(1)_r = 0 \qquad \text{for } r \neq 0.
\end{aligned}$$

Thus, $\triangle^2 V(r) = \delta(r) \quad \Rightarrow \quad V$ is the fundamental solution. ✓

The approach above is not rigorous. See the next page that shows that V defined above is the Fundamental Solution of $\triangle^2$.

The solution of

$$\triangle^2 \omega = F(x),$$

if given by

$$\omega(x) = \int_{\mathbb{R}^2} V(x-y)\,\triangle^2\omega(y)\,dy = \frac{1}{8\pi}\int_{\mathbb{R}^2} |x-y|^2 \log|x-y|\,F(y)\,dy.$$

□

Show that the Fundamental Solution of $\triangle^2$ in $\mathbb{R}^2$ is given by:

$$K(x) = \frac{1}{8\pi} r^2 \ln(r), \qquad r = |x - \xi|, \tag{17.4}$$

Proof. For $v \in C_0^\infty(\mathbb{R}^n)$, we want to show

$$\int_{\mathbb{R}^n} K(|x|)\, \triangle^2 v(x)\, dx \;=\; v(0).$$

Suppose $v(x) \equiv 0$ for $|x| \ge R$ and let $\Omega = B_R(0)$; for small $\epsilon > 0$ let

$$\Omega_\epsilon = \Omega - B_\epsilon(0).$$

$K(|x|)$ is biharmonic ($\triangle^2 K(|x|) = 0$) in Ω_ϵ. Consider Green's identity ($\partial\Omega_\epsilon = \partial\Omega \cup \partial B_\epsilon(0)$):

$$\begin{aligned}
\int_{\Omega_\epsilon} K(|x|)\, \triangle^2 v\, dx \;&=\; \underbrace{\int_{\partial\Omega} \Big(K(|x|)\frac{\partial \triangle v}{\partial n} - v\frac{\partial \triangle K(|x|)}{\partial n}\Big)\, ds + \int_{\partial\Omega} \Big(\triangle K(|x|)\frac{\partial v}{\partial n} \;-\; \triangle v \frac{\partial K(|x|)}{\partial n}\Big)\, ds}_{=0,\ since\ v\equiv 0\ for\ x \ge R} \\
&+\; \int_{\partial B_\epsilon(0)} \Big(K(|x|)\frac{\partial \triangle v}{\partial n} - v\frac{\partial \triangle K(|x|)}{\partial n}\Big)\, ds + \int_{\partial B_\epsilon(0)} \Big(\triangle K(|x|)\frac{\partial v}{\partial n} \;-\; \triangle v \frac{\partial K(|x|)}{\partial n}\Big)\, ds.
\end{aligned}$$

$$\lim_{\epsilon \to 0}\left[\int_{\Omega_\epsilon} K(|x|)\triangle^2 v\, dx\right] \;=\; \int_\Omega K(|x|)\triangle v^2\, dx. \qquad \Big(\text{Since } K(r) \text{ is integrable at } x = 0.\Big)$$

On $\partial B_\epsilon(0)$, $K(|x|) = K(\epsilon)$. Thus, [49]

$$\begin{aligned}
\left|\int_{\partial B_\epsilon(0)} K(|x|)\, \frac{\partial \triangle v}{\partial n}\, dS\right| \;&=\; \big|K(\epsilon)\big| \int_{\partial B_\epsilon(0)} \left|\frac{\partial \triangle v}{\partial n}\right| dS \;\le\; \big|K(\epsilon)\big|\omega_n \epsilon^1 \max_{x\in\overline{\Omega}} \big|\nabla(\triangle v)\big| \\
&=\; \Big|\frac{1}{8\pi}\epsilon^2 \log(\epsilon)\Big|\omega_n \epsilon \max_{x\in\overline{\Omega}} \big|\nabla(\triangle v)\big| \to\; 0, \quad \text{as} \quad \epsilon \to 0.
\end{aligned}$$

$$\begin{aligned}
\int_{\partial B_\epsilon(0)} v(x)\, \frac{\partial \triangle K(|x|)}{\partial n}\, dS \;&=\; \int_{\partial B_\epsilon(0)} -\frac{1}{2\pi\epsilon} v(x)\, dS \\
&=\; \int_{\partial B_\epsilon(0)} -\frac{1}{2\pi\epsilon} v(0)\, dS + \int_{\partial B_\epsilon(0)} -\frac{1}{2\pi\epsilon}[v(x) - v(0)]\, dS \\
&=\; -\frac{1}{2\pi\epsilon}\, v(0)\, 2\pi\epsilon - \underbrace{\max_{x\in\partial B_\epsilon(0)} \big|v(x) - v(0)\big|}_{\to 0,\ (v\ is\ continuous)} \;=\; -v(0). \quad \checkmark
\end{aligned}$$

$$\left|\int_{\partial B_\epsilon(0)} \triangle K(|x|)\, \frac{\partial v}{\partial n}\, dS\right| \;=\; \big|\triangle K(\epsilon)\big| \int_{\partial B_\epsilon(0)} \left|\frac{\partial v}{\partial n}\right| dS \;\le\; \Big|\frac{1}{2\pi}(1 + \log\epsilon)\Big| 2\pi\epsilon \max_{x\in\overline{\Omega}} |\nabla v| \;\to 0, \quad \text{as} \quad \epsilon \to 0.$$

$$\begin{aligned}
\int_{\partial B_\epsilon(0)} \triangle v\, \frac{\partial K(|x|)}{\partial n}\, dS \;&=\; \int_{\partial B_\epsilon(0)} \Big(-\frac{1}{4\pi}\epsilon\log\epsilon - \frac{1}{8\pi}\epsilon\Big)\triangle v(x)\, dS \\
&\le\; \frac{\epsilon}{4\pi}\Big|\log\epsilon + \frac{1}{2}\Big| \cdot 2\pi\epsilon \max_{x\in\partial B_\epsilon(0)} |\triangle v| \;\to\; 0, \quad \text{as} \quad \epsilon \to 0.
\end{aligned}$$

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[49] Note that for $|x| = \epsilon$,

$$\begin{aligned}
K(|x|) \;&=\; K(\epsilon) \;=\; \frac{1}{8\pi}\epsilon^2 \log\epsilon, \qquad & \triangle K = \frac{1}{2\pi}(1 + \log\epsilon), \\
\frac{\partial K(|x|)}{\partial n} \;&=\; -\frac{\partial K(\epsilon)}{\partial r} \;= -\frac{1}{4\pi}\epsilon\log\epsilon - \frac{1}{8\pi}\epsilon, \qquad & \frac{\partial \triangle K}{\partial n} = -\frac{\partial \triangle K}{\partial r} \;=\; -\frac{1}{2\pi\epsilon}.
\end{aligned}$$

$$\Rightarrow \quad \int_{\Omega} K(|x|)\triangle^2 v\,dx \;=\; \lim_{\epsilon\to 0}\int_{\Omega_\epsilon} K(|x|)\triangle^2 v\,dx \;=\; v(0).$$

$\square$

17.3 Radial Variables

Problem (F'99, #8). *Let $u = u(x,t)$ solve the following PDE in* ***two*** *spatial dimensions*

$$-\triangle u = 1$$

for $r < R(t)$, in which $r = |x|$ is the ***radial variable****, with boundary condition*

$$u = 0$$

on $r = R(t)$. In addition assume that $R(t)$ satisfies

$$\frac{dR}{dt} = -\frac{\partial u}{\partial r}(r = R) \qquad \circledast$$

with initial condition $R(0) = R_0$.
a) *Find the solution $u(x,t)$.*
b) *Find an ODE for the outer radius $R(t)$, and solve for $R(t)$.*

Proof. **a)** Rewrite the equation in **polar coordinates**:

$$-\triangle u = -\Big(\frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta}\Big) = 1.$$

For a radially symmetric solution $u(r)$, we have $u_{\theta\theta} = 0$. Thus,

$$\begin{aligned}
\frac{1}{r}(ru_r)_r &= -1, \\
(ru_r)_r &= -r, \\
ru_r &= -\frac{r^2}{2} + c_1, \\
u_r &= -\frac{r}{2} + \frac{c_1}{r}, \\
u(r,t) &= -\frac{r^2}{4} + c_1 \log r + c_2.
\end{aligned}$$

Since we want u to be defined for $r = 0$, we have $c_1 = 0$. Thus,

$$u(r,t) = -\frac{r^2}{4} + c_2.$$

Using boundary conditions, we have

$$u(R(t),t) = -\frac{R(t)^2}{4} + c_2 = 0 \quad \Rightarrow \quad c_2 = \frac{R(t)^2}{4}. \qquad \text{Thus,}$$

$$\boxed{u(r,t) = -\frac{r^2}{4} + \frac{R(t)^2}{4}.}$$

b) We have

$$\begin{aligned}
u(r,t) &= -\frac{r^2}{4} + \frac{R(t)^2}{4}, \\
\frac{\partial u}{\partial r} &= -\frac{r}{2}, \\
\frac{dR}{dt} &= -\frac{\partial u}{\partial r}(r = R) \;=\; \frac{R}{2}, \qquad (\text{from } \circledast) \\
\frac{dR}{R} &= \frac{dt}{2}, \\
\log R &= \frac{t}{2}, \\
R(t) &= c_1 e^{\frac{t}{2}}, \qquad R(0) = c_1 = R_0. \qquad \text{Thus,}
\end{aligned}$$

$$\boxed{R(t) = R_0 e^{\frac{t}{2}}.}$$

$\square$

Problem (F'01, #3). *Let $u = u(x,t)$ solve the following PDE in* ***three*** *spatial dimensions*

$$\triangle u = 0$$

for $R_1 < r < R(t)$, in which $r = |x|$ is the ***radial variable****, with boundary conditions*

$$u(r = R(t), t) = 0, \quad \text{and} \quad u(r = R_1, t) = 1.$$

In addition assume that $R(t)$ satisfies

$$\frac{dR}{dt} = -\frac{\partial u}{\partial r}(r = R) \qquad \circledast$$

with initial condition $R(0) = R_0$ in which $R_0 > R_1$.
a) *Find the solution $u(x,t)$.*
b) *Find an ODE for the outer radius $R(t)$.*

Proof. **a)** Rewrite the equation in **spherical coordinates** ($n = 3$, radial functions):

$$\triangle u = \Big(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\Big) u = \frac{1}{r^2}(r^2 u_r)_r = 0.$$

$$\begin{aligned}
(r^2 u_r)_r &= 0, \\
r^2 u_r &= c_1, \\
u_r &= \frac{c_1}{r^2}, \\
u(r,t) &= -\frac{c_1}{r} + c_2.
\end{aligned}$$

Using boundary conditions, we have

$$\begin{aligned}
u(R(t),t) &= -\frac{c_1}{R(t)} + c_2 = 0 \qquad \Rightarrow \qquad c_2 = \frac{c_1}{R(t)}, \\
u(R_1,t) &= -\frac{c_1}{R_1} + c_2 = 1.
\end{aligned}$$

This gives

$$c_1 = \frac{R_1 R(t)}{R_1 - R(t)}, \qquad c_2 = \frac{R_1}{R_1 - R(t)}.$$

$$\boxed{u(r,t) = -\frac{R_1 R(t)}{R_1 - R(t)} \cdot \frac{1}{r} + \frac{R_1}{R_1 - R(t)}.}$$

b) We have

$$\begin{aligned}
u(r,t) &= -\frac{R_1 R(t)}{R_1 - R(t)} \cdot \frac{1}{r} + \frac{R_1}{R_1 - R(t)}, \\
\frac{\partial u}{\partial r} &= \frac{R_1 R(t)}{R_1 - R(t)} \cdot \frac{1}{r^2}, \\
\frac{dR}{dt} &= -\frac{\partial u}{\partial r}(r = R) = -\frac{R_1 R(t)}{R_1 - R(t)} \cdot \frac{1}{R(t)^2} = -\frac{R_1}{(R_1 - R(t))\,R(t)} \qquad (\text{from } \circledast)
\end{aligned}$$

Thus, an ODE for the outer radius $R(t)$ is

$$\boxed{\begin{cases} \frac{dR}{dt} = \frac{R_1}{(R(t) - R_1)\,R(t)}, \\ R(0) = R_0, \qquad R_0 > R_1. \end{cases}}$$

$\square$

Problem (S'02, #3). *Steady viscous flow in a cylindrical pipe is described by the equation*

$$(\vec{u}\cdot\nabla)\vec{u} + \frac{1}{\rho}\nabla p - \frac{\eta}{\rho}\triangle\vec{u} = 0$$

on the domain $-\infty < x_1 < \infty$, $x_2^2 + x_3^2 \le R^2$, *where* $\vec{u} = (u_1, u_2, u_3) = (U(x_2, x_3), 0, 0)$ *is the velocity vector,* $p(x_1, x_2, x_3)$ *is the pressure, and* η *and* ρ *are constants.*
a) *Show that* $\frac{\partial p}{\partial x_1}$ *is a constant* c, *and that* $\triangle U = c/\eta$.
b) *Assuming further that* U *is radially symmetric and* $U = 0$ *on the surface of the pipe, determine the mass* Q *of fluid passing through a cross-section of pipe per unit time in terms of* c, ρ, η, *and* R. *Note that*

$$Q = \rho\int_{\{x_2^2+x_3^2\le R^2\}} U\,dx_2dx_3.$$

Proof. **a)** Since $\vec{u} = (u_1, u_2, u_3) = (U(x_2, x_3), 0, 0)$, we have

$$(\vec{u}\cdot\nabla)\vec{u} = (u_1, u_2, u_3)\cdot\Big(\frac{\partial u_1}{\partial x_1}, \frac{\partial u_2}{\partial x_2}, \frac{\partial u_3}{\partial x_3}\Big) = (U(x_2, x_3), 0, 0)\cdot(0, 0, 0) = 0.$$

Thus,

$$\begin{aligned}
\frac{1}{\rho}\nabla p - \frac{\eta}{\rho}\triangle\vec{u} &= 0,\\
\nabla p &= \eta\triangle\vec{u},\\
\Big(\frac{\partial p}{\partial x_1}, \frac{\partial p}{\partial x_2}, \frac{\partial p}{\partial x_3}\Big) &= \eta(\triangle u_1, \triangle u_2, \triangle u_3),\\
\Big(\frac{\partial p}{\partial x_1}, \frac{\partial p}{\partial x_2}, \frac{\partial p}{\partial x_3}\Big) &= \eta(U_{x_2x_2} + U_{x_3x_3}, 0, 0).
\end{aligned}$$

We can make the following observations:

$$\begin{aligned}
\frac{\partial p}{\partial x_1} &= \eta\underbrace{(U_{x_2x_2} + U_{x_3x_3})}_{indep.\ of\ x_1},\\
\frac{\partial p}{\partial x_2} &= 0 \qquad\Rightarrow\qquad p = f(x_1, x_3),\\
\frac{\partial p}{\partial x_3} &= 0 \qquad\Rightarrow\qquad p = g(x_1, x_2).
\end{aligned}$$

Thus, $p = h(x_1)$. But $\frac{\partial p}{\partial x_1}$ is independent of x_1. Therefore, $\frac{\partial p}{\partial x_1} = c$.

$$\begin{aligned}
\frac{\partial p}{\partial x_1} &= \eta\triangle U,\\
\triangle U &= \frac{1}{\eta}\frac{\partial p}{\partial x_1} = \frac{c}{\eta}.
\end{aligned}$$

b) Cylindrical Laplacian in $\mathbb{R}^3$ for radial functions is

$$\begin{aligned}
\triangle U &= \frac{1}{r}\big(rU_r\big)_r, \\
\frac{1}{r}\big(rU_r\big)_r &= \frac{c}{\eta}, \\
\big(rU_r\big)_r &= \frac{cr}{\eta}, \\
rU_r &= \frac{cr^2}{2\eta} + c_1, \\
U_r &= \frac{cr}{2\eta} + \frac{c_1}{r}.
\end{aligned}$$

For U_r to stay bounded for $r = 0$, we need $c_1 = 0$. Thus,

$$\begin{aligned}
U_r &= \frac{cr}{2\eta}, \\
U &= \frac{cr^2}{4\eta} + c_2, \\
0 \ = \ U(R) &= \frac{cR^2}{4\eta} + c_2, \\
\Rightarrow \quad U &= \frac{cr^2}{4\eta} - \frac{cR^2}{4\eta} \ = \ \frac{c}{4\eta}(r^2 - R^2).
\end{aligned}$$

$$\begin{aligned}
Q &= \rho \int_{\{x_2^2 + x_3^2 \le R^2\}} U \, dx_2 dx_3 \ = \ \frac{c\rho}{4\eta} \int_0^{2\pi} \int_0^R (r^2 - R^2)\, r dr d\theta \ = \ -\frac{c\rho}{4\eta} \int_0^{2\pi} \frac{R^4}{4} d\theta \\
&= -\frac{c\rho R^4 \pi}{8\eta}.
\end{aligned}$$

It is not clear why Q is negative? $\square$

17.4 Weak Solutions

Problem (S'98, #2).
*A function $u \in H_0^2(\Omega)$ is a **weak solution** of the **biharmonic equation***

$$\begin{cases} \triangle^2 u = f & in \;\; \Omega \\ u = 0 & on \;\; \partial\Omega \\ \frac{\partial u}{\partial n} = 0 & on \;\; \partial\Omega \end{cases}$$

provided

$$\int_\Omega \triangle u \triangle v \, dx = \int_\Omega f v \, dx$$

for all test functions $v \in H_0^2(\Omega)$. Prove that for each $f \in L^2(\Omega)$, there exists a unique weak solution for this problem. Here, $H_0^2(\Omega)$ is the closure of all smooth functions in Ω which vanish on the boundary and with finite H^2 norm: $||u||_2^2 = \int_\Omega (u_{xx}^2 + u_{xy}^2 + u_{yy}^2) \, dxdy < \infty$.
*Hint: use **Lax-Milgram** lemma.*

Proof. Multiply the equation by $v \in H_0^2(\Omega)$ and integrate over Ω:

$$\begin{aligned} \triangle^2 u &= f, \\ \int_\Omega \triangle^2 u \, v \, dx &= \int_\Omega f \, v \, dx, \\ \underbrace{\int_{\partial\Omega} \frac{\partial \triangle u}{\partial n} v \, ds - \int_{\partial\Omega} \triangle u \frac{\partial v}{\partial n} \, ds}_{=0} + \int_\Omega \triangle u \triangle v \, dx &= \int_\Omega f \, v \, dx, \\ \underbrace{\int_\Omega \triangle u \triangle v \, dx}_{a(u,v)} &= \underbrace{\int_\Omega f \, v \, dx}_{L(v)}. \end{aligned}$$

Denote: $V = H_0^2(\Omega)$. Check the following conditions:

❶ $a(\cdot,\cdot)$ is continuous: $\exists \gamma > 0$, s.t. $|a(u,v)| \le \gamma ||u||_V ||v||_V, \quad \forall u, v \in V$;
❷ $a(\cdot,\cdot)$ is V-elliptic: $\exists \alpha > 0$, s.t. $a(v,v) \ge \alpha ||v||_V^2, \quad \forall v \in V$;
❸ $L(\cdot)$ is continuous: $\exists \Lambda > 0$, s.t. $|L(v)| \le \Lambda ||v||_V, \quad \forall v \in V$.

We have [50]

$$❶ \quad |a(u,v)|^2 = \Big|\int_\Omega \triangle u \triangle v \, dx\Big|^2 \le \Big(\int_\Omega (\triangle u)^2 \, dx\Big)\Big(\int_\Omega (\triangle v)^2 \, dx\Big) \le ||u||^2_{H^2_0(\Omega)} ||v||^2_{H^2_0(\Omega)}. \quad \checkmark$$

$$❷ \quad a(v,v) = \int_\Omega (\triangle v)^2 \, dx \ge ||v||_{H^2_0(\Omega)}. \quad \checkmark$$

$$
\begin{aligned}
❸ \quad |L(v)| &= \Big|\int_\Omega f\,v\,dx\Big| \le \int_\Omega |f|\,|v|\,dx \le \Big(\int_\Omega f^2\,dx\Big)^{\frac12}\Big(\int_\Omega v^2\,dx\Big)^{\frac12} \\
&= ||f||_{L^2(\Omega)}||v||_{L^2(\Omega)} \le \underbrace{||f||_{L^2(\Omega)}}_{\Lambda} ||v||_{H^2_0(\Omega)}. \quad \checkmark
\end{aligned}
$$

Thus, by Lax-Milgram theorem, there **exists a weak solution** $u \in H^2_0(\Omega)$.

Also, we can prove the stability result.

$$
\begin{aligned}
\alpha||u||^2_{H^2_0(\Omega)} \le a(u,u) = |L(u)| \le \Lambda||u||_{H^2_0(\Omega)}, \\
\Rightarrow \quad ||u||_{H^2_0(\Omega)} \le \frac{\Lambda}{\alpha}.
\end{aligned}
$$

Let u_1, u_2 be two solutions so that

$$
\begin{aligned}
a(u_1,v) &= L(v), \\
a(u_2,v) &= L(v)
\end{aligned}
$$

for all $v \in V$. Subtracting these two equations, we see that:

$$a(u_1-u_2,v) = 0 \qquad \forall v \in V.$$

Applying the stability estimate (with $L \equiv 0$, i.e. $\Lambda = 0$), we conclude that $||u_1-u_2||_{H^2_0(\Omega)} = 0$, i.e. $u_1 = u_2$. □

---

[50] **Cauchy-Schwarz Inequality:**

$$
\begin{aligned}
|(u,v)| &\le ||u||||v|| \text{ in any norm, for example } \int |uv|dx \le (\int u^2dx)^{\frac12}(\int v^2dx)^{\frac12}; \\
|a(u,v)| &\le a(u,u)^{\frac12}\, a(v,v)^{\frac12}; \\
\int |v|dx &= \int |v|\cdot 1\ dx = (\int |v|^2dx)^{\frac12}(\int 1^2dx)^{\frac12}.
\end{aligned}
$$

Poincare Inequality:

$$||v||_{H^2(\Omega)} \le C\int_\Omega (\triangle v)^2\,dx.$$

Green's formula:

$$\int_\Omega (\triangle u)^2\,dx = \int_\Omega (u^2_{xx}+u^2_{yy}+2u_{xx}u_{yy})\,dxdy = \int_\Omega (u^2_{xx}+u^2_{yy}-2u_{xxy}u_y)\,dxdy = \int_\Omega (u^2_{xx}+u^2_{yy}+2|u_{xy}|^2)\,dxdy \ge ||u||^2_{H^2_0(\Omega)}.$$

17.5 Uniqueness

Problem. *The solution of the* ***Robin problem***

$$\frac{\partial u}{\partial n} + \alpha u = \beta, \qquad x \in \partial\Omega$$

for the Laplace equation is ***unique*** *when* $\alpha > 0$ *is a constant.*

Proof. Let u_1 and u_2 be two solutions of the Robin problem. Let $w = u_1 - u_2$. Then

$$\begin{aligned} &\triangle w = 0 \qquad \text{in } \Omega, \\ &\frac{\partial w}{\partial n} + \alpha w = 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

Consider Green's formula:

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_{\partial\Omega} v \frac{\partial u}{\partial n} \, ds - \int_\Omega v \triangle u \, dx.$$

Setting $w = u = v$ gives

$$\int_\Omega |\nabla w|^2 \, dx = \int_{\partial\Omega} w \frac{\partial w}{\partial n} \, ds - \underbrace{\int_\Omega w \triangle w \, dx}_{=0}.$$

Boundary condition gives

$$\underbrace{\int_\Omega |\nabla w|^2 \, dx}_{\geq 0} = \underbrace{-\int_{\partial\Omega} \alpha w^2 \, ds}_{\leq 0}.$$

Thus, $w \equiv 0$, and $u_1 \equiv u_2$. Hence, the solution to the Robin problem is unique. ☐

Problem. *Suppose* $q(x) \geq 0$ *for* $x \in \Omega$ *and consider solutions* $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ *of*

$$\triangle u - q(x) u = 0 \qquad \text{in } \ \Omega.$$

Establish ***uniqueness*** *theorems for*
a) the ***Dirichlet*** *problem:* $u(x) = g(x), \quad x \in \partial\Omega;$
b) the ***Neumann*** *problem:* $\partial u/\partial n = h(x), \quad x \in \partial\Omega.$

Proof. Let u_1 and u_2 be two solutions of the Dirichlet or Neumann problem.
Let $w = u_1 - u_2$. Then

$$\begin{aligned} &\triangle w - q(x) w = 0 \ \text{ in } \Omega, \\ &w = 0 \quad \text{or} \quad \frac{\partial w}{\partial n} = 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

Consider Green's formula:

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_{\partial\Omega} v \frac{\partial u}{\partial n} \, ds - \int_\Omega v \triangle u \, dx.$$

Setting $w = u = v$ gives

$$\int_\Omega |\nabla w|^2 \, dx = \underbrace{\int_{\partial\Omega} w \frac{\partial w}{\partial n} \, ds}_{=0, \text{ Dirichlet or Neumann}} - \int_\Omega w \triangle w \, dx.$$

$$\underbrace{\int_\Omega |\nabla w|^2 \, dx}_{\geq 0} = \underbrace{-\int_\Omega q(x) w^2 \, dx}_{\leq 0}.$$

Thus, $w \equiv 0$, and $u_1 \equiv u_2$. Hence, the solution to the Dirichlet and Neumann problems are unique. □

Problem (F'02, #8; S'93, #5).
Let D be a bounded domain in $\mathbb{R}^3$. Show that a solution of the boundary value problem

$$\begin{aligned} &\triangle^2 u = f \quad \text{in } D, \\ &u = \triangle u = 0 \quad \text{on } \partial D \end{aligned}$$

*is **unique**.*

Proof. **Method I: Maximum Principle.** Let u_1, u_2 be two solutions of the boundary value problem. Define $w = u_1 - u_2$. Then w satisfies

$$\begin{aligned} &\triangle^2 w = 0 \quad \text{in } D, \\ &w = \triangle w = 0 \quad \text{on } \partial D. \end{aligned}$$

So $\triangle w$ is harmonic and thus achieves min and max on $\partial D \quad \Rightarrow \quad \triangle w \equiv 0$.
So w is harmonic, but $w \equiv 0$ on $\partial D \quad \Rightarrow \quad w \equiv 0$. Hence, $u_1 = u_2$.

Method II: Green's Identities. Multiply the equation by w and integrate:

$$\begin{aligned} w \triangle^2 w &= 0, \\ \int_\Omega w \triangle^2 w \; dx &= 0, \\ \underbrace{\int_{\partial\Omega} w \frac{\partial(\triangle w)}{\partial n} \, ds}_{=0} - \int_\Omega \nabla w \nabla(\triangle) w \; dx &= 0, \\ \underbrace{-\int_{\partial\Omega} \frac{\partial w}{\partial n} \triangle w \, ds}_{=0} + \int_\Omega (\triangle w)^2 \; dx &= 0. \end{aligned}$$

Thus, $\triangle w \equiv 0$. Now, multiply $\triangle w = 0$ by w. We get

$$\int_\Omega |\nabla w|^2 \; dx = 0.$$

Thus, $\nabla w = 0$ and w is a constant. Since $w = 0$ on $\partial\Omega$, we have $w \equiv 0$. □

Problem (F'97, #6).
a) *Let $u(x) \geq 0$ be continuous in closed bounded domain $\overline{\Omega} \subset \mathbb{R}^n$, $\triangle u$ is continuous in $\overline{\Omega}$,*

$$\triangle u = u^2 \qquad \text{and} \qquad u|_{\partial\Omega} = 0.$$

Prove that $u \equiv 0$.
b) *What can you say about $u(x)$ when the condition $u(x) \geq 0$ in $\overline{\Omega}$ is dropped?*

Proof. **a)** Multiply the equation by u and integrate:

$$\begin{aligned} u\triangle u &= u^3, \\ \int_\Omega u\triangle u \; dx &= \int_\Omega u^3 \; dx, \\ \underbrace{\int_{\partial\Omega} u\frac{\partial u}{\partial n} \; ds}_{=0} - \int_\Omega |\nabla u|^2 \; dx &= \int_\Omega u^3 \; dx, \\ \int_\Omega \big(u^3 + |\nabla u|^2\big) \; dx &= 0. \end{aligned}$$

Since $u(x) \geq 0$, we have $u \equiv 0$.

b) If we don't know that $u(x) \geq 0$, then u can not be nonnegative on the entire domain $\overline{\Omega}$. That is, $u(x) < 0$, on some (or all) parts of Ω. If u is nonnegative on Ω, then $u \equiv 0$. $\square$

Problem (W'02, #5). *Consider the boundary value problem*

$$\triangle u + \sum_{k=1}^{n} \alpha_k \frac{\partial u}{\partial x_k} - u^3 = 0 \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega,$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary. If the α_k's are constants, and $u(x)$ has continuous derivatives up to second order, prove that u must ***vanish identically****.*

Proof. Multiply the equation by u and integrate:

$$\begin{aligned}
u\triangle u + \sum_{k=1}^{n} \alpha_k \frac{\partial u}{\partial x_k} u - u^4 &= 0, \\
\int_\Omega u\triangle u\,dx + \int_\Omega \sum_{k=1}^{n} \alpha_k \frac{\partial u}{\partial x_k} u\,dx - \int_\Omega u^4\,dx &= 0, \\
\underbrace{\int_{\partial\Omega} u\frac{\partial u}{\partial n}\,ds}_{=0} - \int_\Omega |\nabla u|^2\,dx + \underbrace{\int_\Omega \sum_{k=1}^{n} \alpha_k \frac{\partial u}{\partial x_k} u\,dx}_{①} - \int_\Omega u^4\,dx &= 0.
\end{aligned}$$

We will show that ① $= 0$.

$$\begin{aligned}
\int_\Omega \alpha_k \frac{\partial u}{\partial x_k} u\,dx &= \underbrace{\int_{\partial\Omega} \alpha_k\, u^2\,ds}_{=0} - \int_\Omega \alpha_k\, u\frac{\partial u}{\partial x_k}\,dx, \\
\Rightarrow \quad 2\int_\Omega \alpha_k \frac{\partial u}{\partial x_k} u\,dx &= 0, \\
\Rightarrow \quad \int_\Omega \sum_{k=1}^{n} \alpha_k \frac{\partial u}{\partial x_k} u\,dx &= 0.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
-\int_\Omega |\nabla u|^2\,dx - \int_\Omega u^4\,dx &= 0, \\
\int_\Omega \big(|\nabla u|^2 + \int_\Omega u^4\big)\,dx &= 0.
\end{aligned}$$

Hence, $|\nabla u|^2 = 0$ and $u^4 = 0$. Thus, $u \equiv 0$. $\square$

Note that

$$\int_\Omega \sum_{k=1}^{n} \alpha_k \frac{\partial u}{\partial x_k} u\,dx = \int_\Omega \alpha\cdot\nabla u\,u\,dx = \underbrace{\int_{\partial\Omega} \alpha\cdot n u^2\,ds}_{=0} - \int_\Omega \alpha\cdot\nabla u\,u\,dx,$$

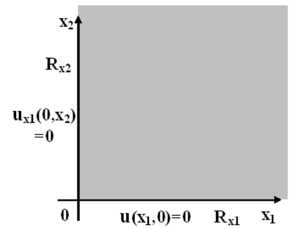
and thus,

$$\int_\Omega \alpha\cdot\nabla u\,u\,dx = 0.$$

Problem (W'02, #9). *Let $D = \{x \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0\}$, and assume that f is continuous on D and vanishes for $|x| > R$.*

a) *Show that the boundary value problem*

$$\triangle u = f \qquad in\ D,$$
$$u(x_1, 0) = \frac{\partial u}{\partial x_1}(0, x_2) = 0$$



can have only ***one*** *bounded solution.*

b) *Find an explicit* ***Green's function*** *for this boundary value problem.*

Proof. **a)** Let u_1, u_2 be two solutions of the boundary value problem. Define $w = u_1 - u_2$. Then w satisfies

$$\triangle w = 0 \qquad \text{in } D,$$
$$w(x_1, 0) = \frac{\partial w}{\partial x_1}(0, x_2) = 0.$$

Consider Green's formula:

$$\int_D \nabla u \cdot \nabla v \, dx = \int_{\partial D} v \frac{\partial u}{\partial n} \, ds - \int_D v \triangle u \, dx.$$

Setting $w = u = v$ gives

$$\begin{aligned}
\int_D |\nabla w|^2 \, dx &= \int_{\partial D} w \frac{\partial w}{\partial n} \, ds - \int_D w \triangle w \, dx, \\
\int_D |\nabla w|^2 \, dx &= \int_{\mathbb{R}_{x_1}} w \frac{\partial w}{\partial n} \, ds + \int_{\mathbb{R}_{x_2}} w \frac{\partial w}{\partial n} \, ds + \int_{|x|>R} w \frac{\partial w}{\partial n} \, ds - \int_D w \triangle w \, dx \\
&= \int_{\mathbb{R}_{x_1}} \underbrace{w(x_1, 0)}_{=0} \frac{\partial w}{\partial x_2} \, ds + \int_{\mathbb{R}_{x_2}} w(0, x_2) \underbrace{\frac{\partial w}{\partial x_1}}_{=0} \, ds + \int_{|x|>R} \underbrace{w}_{=0} \frac{\partial w}{\partial n} \, ds - \int_D w \underbrace{\triangle w}_{=0} \, dx, \\
\int_D |\nabla w|^2 \, dx &= 0 \quad \Rightarrow \quad |\nabla w|^2 = 0 \quad \Rightarrow \quad w = \text{const}.
\end{aligned}$$

Since $w(x_1, 0) = 0 \ \Rightarrow \ w \equiv 0$. Thus, $u_1 = u_2$.

b) The similar problem is solved in the appropriate section (S'96, #3).
Notice whenever you are on the boundary with variable x,

$$|x - \xi^{(0)}| = \frac{|x - \xi^{(1)}||x - \xi^{(3)}|}{|x - \xi^{(2)}|}.$$

$$\text{So,} \qquad G(x, \xi) = \frac{1}{2\pi} \left(\log|x - \xi| - \log \frac{|x - \xi^{(1)}||x - \xi^{(3)}|}{|x - \xi^{(2)}|} \right)$$

is the Green's function. ∎

Problem (F'98, #4). *In two dimensions* $\mathbf{x} = (x, y)$, *define the set* Ω_a *as*

$$\Omega_a = \Omega^+ \cup \Omega^-$$

in which

$$\begin{aligned}
\Omega^+ &= \{|\mathbf{x} - \mathbf{x}_0| \le a\} \cap \{x \ge 0\} \\
\Omega^- &= \{|\mathbf{x} + \mathbf{x}_0| \le a\} \cap \{x \le 0\} = -\Omega^+
\end{aligned}$$

and $\mathbf{x}_0 = (1, 0)$. *Note that* Ω_a *consists of two components when* $0 < a < 1$ *and a single component when* $a > 1$. *Consider the Neumann problem*

$$\begin{aligned}
\nabla^2 u &= f, & \mathbf{x} \in \Omega_a \\
\partial u/\partial n &= 0, & \mathbf{x} \in \partial\Omega_a
\end{aligned}$$

in which

$$\begin{aligned}
\int_{\Omega^+} f(\mathbf{x})\, d\mathbf{x} &= 1 \\
\int_{\Omega^-} f(\mathbf{x})\, d\mathbf{x} &= -1
\end{aligned}$$

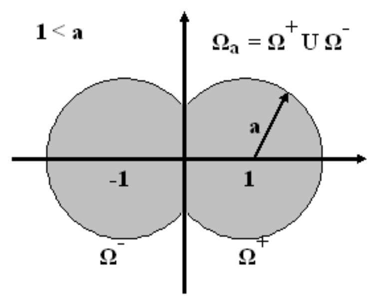
a) *Show that this problem has a solution for* $1 < a$, *but not for* $0 < a < 1$. *(You do not need to construct the solution, only demonstrate solveability.)*

b) *Show that* $\max_{\Omega_a} |\nabla u| \to \infty$ *as* $a \to 1$ *from above. (Hint: Denote* L *to be the line segment* $L = \Omega^+ \cap \Omega^-$, *and note that its length* $|L|$ *goes to* 0 *as* $a \to 1$.*)*

Proof. **a)** We use the Green's identity. For $1 < a$,

$$\begin{aligned}
0 &= \int_{\partial\Omega_a} \frac{\partial u}{\partial n}\, ds = \int_{\Omega_a} \triangle u\, dx = \int_{\Omega_a} f(x)\, dx \\
&= \int_{\Omega^+} f(x)\, dx + \int_{\Omega^-} f(x)\, dx = 1 - 1 = 0. \quad \checkmark
\end{aligned}$$

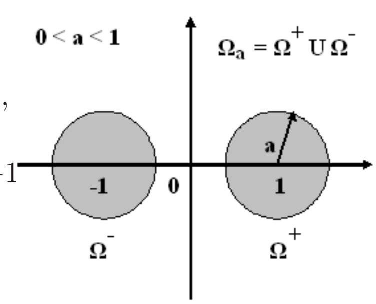
Thus, the problem has a solution for $1 < a$.



For $0 < a < 1$, Ω^+ and Ω^- are disjoint. Consider Ω^+:

$$\begin{aligned}
0 &= \int_{\partial\Omega^+} \frac{\partial u}{\partial n}\, ds = \int_{\Omega^+} \triangle u\, dx = \int_{\Omega^+} f(x)\, dx = 1, \\
0 &= \int_{\partial\Omega^-} \frac{\partial u}{\partial n}\, ds = \int_{\Omega^-} \triangle u\, dx = \int_{\Omega^-} f(x)\, dx = -1
\end{aligned}$$

We get *contradictions*.
Thus, the solution does *not* exist for $0 < a < 1$.



b) Using the Green's identity, we have: (n^+ is the unit normal to Ω^+)

$$\begin{aligned}
\int_{\Omega^+} \triangle u \, dx &= \int_{\partial\Omega^+} \frac{\partial u}{\partial n^+} \, ds = \int_L \frac{\partial u}{\partial n^+} \, ds, \\
\int_{\Omega^-} \triangle u \, dx &= \int_{\partial\Omega^-} \frac{\partial u}{\partial n^-} \, ds = \int_L \frac{\partial u}{\partial n^-} \, ds = -\int_L \frac{\partial u}{\partial n^+} \, ds.
\end{aligned}$$

$$\begin{aligned}
\int_{\Omega^+} \triangle u \, dx - \int_{\Omega^-} \triangle u \, dx &= 2\int_L \frac{\partial u}{\partial n^+} \, ds, \\
\int_{\Omega^+} f(x) \, dx - \int_{\Omega^-} f(x) \, dx &= 2\int_L \frac{\partial u}{\partial n^+} \, ds.
\end{aligned}$$

$$\begin{aligned}
2 &= 2\int_L \frac{\partial u}{\partial n^+} \, ds, \\
1 &= \int_L \frac{\partial u}{\partial n^+} \, ds \le \int_L \Big|\frac{\partial u}{\partial n^+}\Big| \, ds \le \int_L \sqrt{\Big(\frac{\partial u}{\partial n^+}\Big)^2 + \Big(\frac{\partial u}{\partial \tau}\Big)^2} \le |L| \max_L |\nabla u| \le |L| \max_{\Omega_a} |\nabla u|.
\end{aligned}$$

Thus,

$$\max_{\Omega_a} |\nabla u| \ge \frac{1}{|L|}.$$

As $a \to 1$ $(L \to 0)$ $\Rightarrow$ $\max_{\Omega_a} |\nabla u| \to \infty$. $\square$

Problem (F'00, #1). *Consider the Dirichlet problem in a bounded domain $D \subset \mathbb{R}^n$ with smooth boundary ∂D,*

$$\begin{aligned} \triangle u + a(x)u = f(x) \quad & \text{in } D, \\ u = 0 \quad & \text{on } \partial D. \end{aligned}$$

a) *Assuming that $|a(x)|$ is small enough, prove the uniqueness of the classical solution.*
b) *Prove the existence of the solution in the Sobolev space $H^1(D)$ assuming that $f \in L_2(D)$.*
Note: Use ***Poincare inequality****.*

Proof. **a)** By Poincare Inequality, for any $u \in C_0^1(D)$, we have $||u||_2^2 \leq C||\nabla u||_2^2$.
Consider two solutions of the Dirichlet problem above. Let $w = u_1 - u_2$. Then, w satisfies

$$\begin{cases} \triangle w + a(x)w = 0 & \text{in } D, \\ w = 0 & \text{on } \partial D. \end{cases}$$

$$\begin{aligned} w\triangle w + a(x)w^2 &= 0, \\ \int w\triangle w\,dx + \int a(x)w^2\,dx &= 0, \\ -\int |\nabla w|^2\,dx + \int a(x)w^2\,dx &= 0, \\ \int a(x)w^2\,dx \;=\; \int |\nabla w|^2\,dx &\geq \frac{1}{C}\int w^2\,dx, \qquad \text{(by Poincare inequality)} \\ \int a(x)w^2\,dx - \frac{1}{C}\int w^2\,dx &\geq 0, \\ |a(x)|\int w^2\,dx - \frac{1}{C}\int w^2\,dx &\geq 0, \\ \Big(|a(x)| - \frac{1}{C}\Big)\int w^2\,dx &\geq 0. \end{aligned}$$

If $|a(x)| < \frac{1}{C} \quad \Rightarrow \quad w \equiv 0$.
b) Consider

$$F(v,u) = -\int_\Omega (v\triangle u + a(x)vu)\,dx = -\int_\Omega vf(x)\,dx = F(v).$$

$F(v)$ is a bounded linear functional on $v \in H^{1,2}(D), \quad D = \Omega$.

$$|F(v)| \leq ||f||_2||v||_2 \leq ||f||_2 C||v||_{H^{1,2}(D)}$$

So by Riesz representation, there exists a solution $u \in H_0^{1,2}(D)$ of

$$-<u,v>= \int_\Omega v\triangle u + a(x)vu\,dx = \int_\Omega vf(x)\,dx = F(v) \qquad \forall v \in H_0^{1,2}(D).$$

$\square$

Problem (S'91, #8). *Define the operator*

$$Lu = u_{xx} + u_{yy} - 4(r^2+1)u$$

in which $r^2 = x^2 + y^2$.
a) *Show that* $\varphi = e^{r^2}$ *satisfies* $L\varphi = 0$.
b) *Use this to show that the equation*

$$\begin{aligned} Lu &= f \qquad in \;\; \Omega \\ \frac{\partial u}{\partial n} &= \gamma \qquad on \;\; \partial\Omega \end{aligned}$$

has a solution only if

$$\int_\Omega \varphi f\,dx = \int_{\partial\Omega} \varphi\gamma\,ds(x).$$

Proof. **a)** Expressing Laplacian in polar coordinates, we obtain:

$$\begin{aligned} Lu &= \frac{1}{r}(ru_r)_r - 4(r^2+1)u, \\ L\varphi &= \frac{1}{r}(r\varphi_r)_r - 4(r^2+1)\varphi = \frac{1}{r}(2r^2e^{r^2})_r - 4(r^2+1)e^{r^2} \\ &= \frac{1}{r}(4re^{r^2} + 2r^2\cdot 2re^{r^2}) - 4r^2e^{r^2} - 4e^{r^2} = 0. \quad \checkmark \end{aligned}$$

b) We have $\varphi = e^{r^2} = e^{x^2+y^2} = e^{x^2}e^{y^2}$. From part (a),

$$\begin{aligned} L\varphi &= 0, \\ \frac{\partial\varphi}{\partial n} &= \nabla\varphi\cdot n = (\varphi_x,\varphi_y)\cdot n = (2xe^{x^2}e^{y^2}, 2ye^{x^2}e^{y^2})\cdot n = 2e^{r^2}(x,y)\cdot(-y,x) = 0. \end{aligned}$$

[51] Consider two equations:

$$\begin{aligned} Lu &= \triangle u - 4(r^2+1)u, \\ L\varphi &= \triangle\varphi - 4(r^2+1)\varphi. \end{aligned}$$

Multiply the first equation by φ and the second by u and subtract the two equations:

$$\begin{aligned} \varphi Lu &= \varphi\triangle u - 4(r^2+1)u\varphi, \\ uL\varphi &= u\triangle\varphi - 4(r^2+1)u\varphi, \\ \varphi Lu - uL\varphi &= \varphi\triangle u - u\triangle\varphi. \end{aligned}$$

Then, we start from the LHS of the equality we need to prove and end up with RHS:

$$\begin{aligned} \int_\Omega \varphi f\,dx &= \int_\Omega \varphi Lu\,dx = \int_\Omega (\varphi Lu - uL\varphi)\,dx = \int_\Omega (\varphi\triangle u - u\triangle\varphi)\,dx \\ &= \int_\Omega \Big(\varphi\frac{\partial u}{\partial n} - u\frac{\partial\varphi}{\partial n}\Big)\,ds = \int_\Omega \varphi\frac{\partial u}{\partial n}\,ds = \int_\Omega \varphi\gamma\,ds. \quad \checkmark \end{aligned}$$

$\square$

[51] The only shortcoming in the above proof is that we assume $\vec{n} = (-y,x)$, without giving an explanation why it is so.

17.6 Self-Adjoint Operators

Consider an mth-order differential operator

$$Lu = \sum_{|\alpha|\le m} a_\alpha(x) D^\alpha u.$$

The integration by parts formula

$$\int_\Omega u_{x_k} v\, dx \;=\; \int_{\partial\Omega} uvn_k\, ds - \int_\Omega uv_{x_k}\, dx \qquad\qquad \vec{n} = (n_1, \ldots, n_n) \in \mathbb{R}^n,$$

with u or v vanishing near $\partial\Omega$ is:

$$\int_\Omega u_{x_k} v\, dx \;=\; -\int_\Omega uv_{x_k}\, dx.$$

We can repeat the integration by parts with any combination of derivatives $D^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n}$:

$$\int_\Omega (D^\alpha u) v\, dx \;=\; (-1)^m \int_\Omega u D^\alpha v\, dx, \qquad\qquad (m = |\alpha|).$$

We have

$$\begin{aligned}
\int_\Omega (Lu)\, v\; dx \;&=\; \int_\Omega \Big(\sum_{|\alpha|\le m} a_\alpha(x) D^\alpha u\Big) v\, dx \;=\; \sum_{|\alpha|\le m} \int_\Omega a_\alpha(x)\, v\, D^\alpha u\, dx \\
&=\; \sum_{|\alpha|\le m} (-1)^{|\alpha|} \int_\Omega D^\alpha(a_\alpha(x)\, v)\, u\, dx \;=\; \int_\Omega \underbrace{\sum_{|\alpha|\le m} (-1)^{|\alpha|} D^\alpha(a_\alpha(x)\, v)}_{L^*(v)}\, u\, dx \\
&=\; \int_\Omega L^*(v)\, u\; dx,
\end{aligned}$$

for all $u \in C^m(\Omega)$ and $v \in C_0^\infty$.
The operator

$$\boxed{L^*(v) \;=\; \sum_{|\alpha|\le m} (-1)^{|\alpha|} D^\alpha(a_\alpha(x)\, v)}$$

is called the **adjoint** of L.
The operator is **self-adjoint** if $L^* = L$.
Also, L is self-adjoint if [52]

$$\boxed{\int_\Omega vL(u)\, dx \;=\; \int_\Omega uL(v)\, dx.}$$

---

[52] $L = L^* \quad \Leftrightarrow \quad (Lu|v) = (u|L^*v) = (u|Lv)$.

Problem (F'92, #6).
Consider the Laplace operator $\triangle$ in the wedge $0 \le x \le y$ with boundary conditions

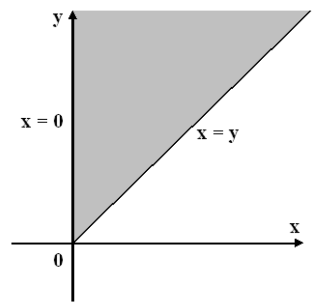
$$\frac{\partial f}{\partial x} = 0 \qquad on \quad x = 0$$
$$\frac{\partial f}{\partial x} - \alpha \frac{\partial f}{\partial y} = 0 \qquad on \quad x = y.$$

a) *For which values of α is this operator **self-adjoint**?*
b) *For such a value of α, suppose that*

$$\triangle f = e^{-r^2/2} \cos\theta$$

with these boundary conditions. Evaluate

$$\int_{C_R} \frac{\partial}{\partial r} f \, ds$$



in which C_R is the circular arc of radius R connecting the boundaries $x = 0$ and $x = y$.

Proof. **a)** We have

$$Lu = \triangle u = 0$$

$$\frac{\partial u}{\partial x} = 0 \qquad \text{on} \quad x = 0$$
$$\frac{\partial u}{\partial x} - \alpha \frac{\partial u}{\partial y} = 0 \qquad \text{on} \quad x = y.$$

The operator L is self-adjoint if:

$$\int_\Omega (u\, Lv - v\, Lu)\, dx = 0.$$

$$\begin{aligned}
\int_\Omega (u\, Lv - v\, Lu)\, dx &= \int_\Omega (u \triangle v - v \triangle u)\, dx = \int_{\partial\Omega} \Big(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \Big)\, ds \\
&= \int_{x=0} \Big(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \Big)\, ds + \int_{x=y} \Big(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \Big)\, ds \\
&= \int_{x=0} \Big(u\, (\nabla v \cdot n) - v\, (\nabla u \cdot n) \Big)\, ds + \int_{x=y} \Big(u\, (\nabla v \cdot n) - v\, (\nabla u \cdot n) \Big)\, ds \\
&= \int_{x=0} \Big(u\, \big((v_x, v_y) \cdot (-1, 0)\big) - v\, \big((u_x, u_y) \cdot (-1, 0)\big) \Big)\, ds \\
&\quad + \int_{x=y} \Big(u\, \big((v_x, v_y) \cdot (1/\sqrt{2}, -1/\sqrt{2})\big) - v\, \big((u_x, u_y) \cdot (1/\sqrt{2}, -1/\sqrt{2})\big) \Big)\, ds \\
&= \underbrace{\int_{x=0} \Big(u\, \big((0, v_y) \cdot (-1, 0)\big) - v\, \big((0, u_y) \cdot (-1, 0)\big) \Big)\, ds}_{=0} \\
&\quad + \int_{x=y} \Big(u\, \big((\alpha v_y, v_y) \cdot (1/\sqrt{2}, -1/\sqrt{2})\big) - v\, \big((\alpha u_y, u_y) \cdot (1/\sqrt{2}, -1/\sqrt{2})\big) \Big)\, ds \\
&= \int_{x=y} \Big(\frac{u v_y}{\sqrt{2}}\, (\alpha - 1) - \frac{v u_y}{\sqrt{2}}\, (\alpha - 1) \Big)\, ds \underbrace{=}_{need} 0.
\end{aligned}$$

Thus, we need $\alpha = 1$ so that L is self-adjoint.

b) We have $\alpha = 1$. Using Green's identity and results from part (a), ($\frac{\partial f}{\partial n} = 0$ on $x = 0$ and $x = y$):

$$\int_\Omega \triangle f \, dx = \int_{\partial\Omega} \frac{\partial f}{\partial n} \, ds = \int_{\partial C_R} \frac{\partial f}{\partial n} \, ds + \int_{x=0} \underbrace{\frac{\partial f}{\partial n}}_{=0} \, ds + \int_{x=y} \underbrace{\frac{\partial f}{\partial n}}_{=0} \, ds = \int_{\partial C_R} \frac{\partial f}{\partial r} \, ds.$$

Thus,

$$\begin{aligned} \int_{\partial C_R} \frac{\partial f}{\partial r} \, ds &= \int_\Omega \triangle f \, dx = \int_0^R \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} e^{-r^2/2} \cos\theta \; r \, dr d\theta \\ &= \Big(1 - \frac{1}{\sqrt{2}}\Big) \int_0^R e^{-r^2/2} \; r \, dr = \Big(1 - \frac{1}{\sqrt{2}}\Big)(1 - e^{-R^2/2}). \end{aligned}$$

$\square$

Problem (F'99, #1). *Suppose that $\triangle u = 0$ in the weak sense in $\mathbb{R}^n$ and that there is a constant C such that*

$$\int_{\{|x-y|<1\}} |u(y)|\,dy < C, \quad \forall x \in \mathbb{R}^n.$$

Show that u is constant.

Proof. Consider Green's formula:

$$\int_\Omega \nabla u \cdot \nabla v\,dx = \int_{\partial\Omega} v\frac{\partial u}{\partial n}\,ds - \int_\Omega v\,\triangle u\,dx$$

For $v = 1$, we have

$$\int_{\partial\Omega} \frac{\partial u}{\partial n}\,ds = \int_\Omega \triangle u\,dx.$$

Let $B_r(x_0)$ be a ball in $\mathbb{R}^n$. We have

$$\begin{aligned} 0 &= \int_{B_r(x_0)} \triangle u\,dx = \int_{\partial B_r(x_0)} \frac{\partial u}{\partial n}\,ds = r^{n-1}\int_{|x|=1}\frac{\partial u}{\partial r}(x_0+rx)\,ds \\ &= r^{n-1}\omega_n \frac{\partial}{\partial r}\frac{1}{\omega_n}\int_{|x|=1} u(x_0+rx)\,ds. \end{aligned}$$

Thus, $\frac{1}{\omega_n}\int_{|x|=1} u(x_0+rx)\,ds$ is independent of r. Hence, it is constant.
By continuity, as $r \to 0$, we obtain the Mean Value property:

$$\boxed{u(x_0) = \frac{1}{\omega_n}\int_{|x|=1} u(x_0+rx)\,ds.}$$

If $\int_{|x-y|<1} |u(y)|\,dy < C \quad \forall x \in \mathbb{R}^n$, we have $|u(x)| < C$ in $\mathbb{R}^n$.
Since u is harmonic and bounded in $\mathbb{R}^n$, u is constant by Liouville's theorem. [53] □

[53] **Liouville's Theorem:** A bounded harmonic function defined on $\mathbb{R}^n$ is a constant.

Problem (S'01, #1). *For bodies (bounded regions B in $\mathbb{R}^3$) which are not perfectly conducting one considers the boundary value problem*

$$0 \;=\; \nabla\cdot\gamma(x)\nabla u \;=\; \sum_{j=1}^{3}\frac{\partial}{\partial x_j}\Big(\gamma(x)\frac{\partial u}{\partial x_j}\Big)$$
$$u \;=\; f \qquad \text{on } \partial B.$$

The function $\gamma(x)$ is the "local conductivity" of B and u is the voltage. We define operator $\Lambda(f)$ mapping the boundary data f to the current density at the boundary by

$$\Lambda(f) = \gamma(x)\frac{\partial u}{\partial n},$$

and $\partial/\partial n$ is the inward normal derivative (this formula defines the current density).
a) *Show that Λ is a symmetric operator, i.e. prove*

$$\int_{\partial B} g\Lambda(f)\,dS \;=\; \int_{\partial B} f\Lambda(g)\,dS.$$

b) *Use the positivity of $\gamma(x) > 0$ to show that Λ is negative as an operator, i.e., prove*

$$\int_{\partial B} f\Lambda(f)\,dS \;\le\; 0.$$

Proof. **a)** Let

$$\begin{cases} \nabla\cdot\gamma(x)\nabla u = 0 & \text{on } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases} \qquad\qquad \begin{cases} \nabla\cdot\gamma(x)\nabla v = 0 & \text{on } \Omega, \\ v = g & \text{on } \partial\Omega. \end{cases}$$

$$\Lambda(f) = \gamma(x)\frac{\partial u}{\partial n}, \qquad\qquad \Lambda(g) = \gamma(x)\frac{\partial v}{\partial n}.$$

Since $\partial/\partial n$ is inward normal derivative, Green's formula is:

$$-\int_{\partial\Omega}\underbrace{v}_{=g}\gamma(x)\frac{\partial u}{\partial n}\,dS - \int_\Omega \nabla v\cdot\gamma(x)\nabla u\,dx = \int_\Omega v\nabla\cdot\gamma(x)\nabla u\,dx.$$

We have

$$\begin{aligned}
\int_{\partial\Omega} g\Lambda(f)\,dS &= \int_{\partial\Omega} g\gamma(x)\frac{\partial u}{\partial n}\,dS \;=\; -\int_\Omega \nabla v\cdot\gamma(x)\nabla u\,dx - \int_\Omega v\,\underbrace{\nabla\cdot\gamma(x)\nabla u}_{=0}\,dx \\
&= \int_{\partial\Omega} u\gamma(x)\frac{\partial v}{\partial n}\,dS \;+\; \int_\Omega u\,\underbrace{\nabla\cdot\gamma(x)\nabla v}_{=0}\,dx \\
&= \int_{\partial\Omega} f\gamma(x)\frac{\partial v}{\partial n}\,dS \;=\; \int_{\partial\Omega} f\Lambda(g)\,dS. \quad \checkmark
\end{aligned}$$

b) We have $\gamma(x) > 0$.

$$\begin{aligned}
\int_{\partial\Omega} f\Lambda(f)\,dS &= \int_{\partial\Omega} u\gamma(x)\frac{\partial u}{\partial n}\,dS \;=\; -\int_\Omega u\,\underbrace{\nabla\cdot\gamma(x)\nabla u}_{=0}\,dx - \int_\Omega \gamma(x)\nabla u\cdot\nabla u\,dx \\
&= -\int_\Omega \underbrace{\gamma(x)|\nabla u|^2}_{\ge 0} \;\le\; 0. \quad \checkmark
\end{aligned}$$

$\square$

Problem (S'01, #4). *The **Poincare Inequality** states that for any bounded domain Ω in $\mathbb{R}^n$ there is a constant C such that*

$$\int_\Omega |u|^2\,dx \le C\int_\Omega |\nabla u|^2\,dx$$

for all smooth functions u which vanish on the boundary of Ω.
a) *Find a formula for the "best" (smallest) constant for the domain Ω in terms of **the eigenvalues of the Laplacian** on Ω, and*
b) *give the best constant for the rectangular domain in $\mathbb{R}^2$*

$$\Omega = \{(x,y):\ 0\le x\le a,\ 0\le y\le b\}.$$

Proof. **a)** Consider Green's formula:

$$\int_\Omega \nabla u\cdot\nabla v\,dx \;=\; \int_{\partial\Omega} v\frac{\partial u}{\partial n}\,ds - \int_\Omega v\,\triangle u\,dx.$$

Setting $u=v$ and with u vanishing on $\partial\Omega$, Green's formula becomes:

$$\int_\Omega |\nabla u|^2\,dx \;=\; -\int_\Omega u\triangle u\,dx.$$

Expanding u in the eigenfunctions of the Laplacian, $u(x)=\sum a_n\phi_n(x)$, the formula above gives

$$\begin{aligned}\int_\Omega |\nabla u|^2\,dx \;&=\; -\int_\Omega \sum_{n=1}^\infty a_n\phi_n(x)\sum_{m=1}^\infty -\lambda_m a_m\phi_m(x)\,dx \;=\; \sum_{m,n=1}^\infty \lambda_m a_n a_m\int_\Omega \phi_n\phi_m\,dx\\ &=\; \sum_{n=1}^\infty \lambda_n|a_n|^2. \qquad \circledast\end{aligned}$$

Also,

$$\int_\Omega |u|^2\,dx \;=\; \int_\Omega\sum_{n=1}^\infty a_n\phi_n(x)\sum_{m=1}^\infty a_m\phi_m(x) \;=\; \sum_{n=1}^\infty |a_n|^2. \qquad \circledcirc$$

Comparing $\circledast$ and $\circledcirc$, and considering that λ_n increases as $n\to\infty$, we obtain

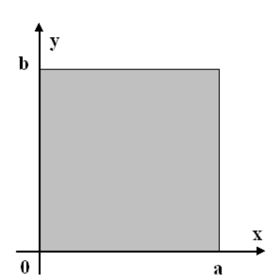
$$\lambda_1\int_\Omega |u|^2\,dx = \lambda_1\sum_{n=1}^\infty |a_n|^2 \le \sum_{n=1}^\infty \lambda_n|a_n|^2 = \int_\Omega |\nabla u|^2\,dx.$$

$$\boxed{\int_\Omega |u|^2\,dx \le \frac{1}{\lambda_1}\int_\Omega |\nabla u|^2\,dx,}$$

with $C = 1/\lambda_1$.

b) For the rectangular domain $\Omega = \{(x,y):\ 0\le x\le a,\ 0\le y\le b\}\subset\mathbb{R}^2$, the eigenvalues of the Laplacian are

$$\begin{aligned}\lambda_{mn} \;&=\; \pi^2\Big(\frac{m^2}{a^2}+\frac{n^2}{b^2}\Big), \qquad m,n=1,2,\ldots.\\ \lambda_1 \;&=\; \lambda_{11} = \pi^2\Big(\frac{1}{a^2}+\frac{1}{b^2}\Big),\\ \Rightarrow\ C \;&=\; \frac{1}{\lambda_{11}} = \frac{1}{\pi^2}\frac{1}{(\frac{1}{a^2}+\frac{1}{b^2})}.\end{aligned}$$



$\square$

Problem (S'01, #6). ***a)*** *Let B be a bounded region in $\mathbb{R}^3$ with smooth boundary ∂B. The "conductor" potential for the body B is the solution of Laplace's equation outside B*

$$\triangle V = 0 \quad in \quad \mathbb{R}^3/B$$

subject to the boundary conditions, $V = 1$ on ∂B and $V(x)$ tends to zero as $|x| \to \infty$. Assuming that the conductor potential exists, show that it is ***unique****.*
b) *The "capacity" $C(B)$ of B is defined to be the limit of $|x|V(x)$ as $|x| \to \infty$. Show that*

$$C(B) = -\frac{1}{4\pi}\int_{\partial B}\frac{\partial V}{\partial n}\,dS,$$

where ∂B is the boundary of B and n is the outer unit normal to it (i.e. the normal pointing "toward infinity").
c) *Suppose that $B' \subset B$. Show that $C(B') \le C(B)$.*

Proof. **a)** Let V_1, V_2 be two solutions of the boundary value problem. Define $W = V_1 - V_2$. Then W satisfies

$$\begin{cases} \triangle W = 0 & \text{in } \ \mathbb{R}^3/B \\ W = 0 & \text{on } \partial B \\ W \to 0 & \text{as } \ |x| \to \infty. \end{cases}$$

Consider Green's formula:

$$\int_B \nabla u \cdot \nabla v\,dx \ = \int_{\partial B} v\,\frac{\partial u}{\partial n}\,ds - \int_B v\,\triangle u\,dx.$$

Setting $W = u = v$ gives

$$\int_B |\nabla W|^2\,dx \quad = \quad \int_{\partial B} \underbrace{W}_{=0}\,\frac{\partial W}{\partial n}\,ds - \int_B W\,\underbrace{\triangle W}_{=0}\,dx = 0.$$

Thus, $|\nabla W|^2 = 0 \ \Rightarrow \ W = \text{const.}$ Since $W = 0$ on ∂B, $W \equiv 0$, and $V_1 = V_2$.

b & c) For $(b)\&(c)$, see the solutions from Ralston's homework (a few pages down). $\square$

Problem (W'03, #2). *Let L be the second order differential operator $L = \triangle - a(x)$ in which $x = (x_1, x_2, x_3)$ is in the three-dimensional cube $C = \{0 < x_i < 1,\ i = 1, 2, 3\}$. Suppose that $a > 0$ in C. Consider the eigenvalue problem*

$$\begin{aligned} Lu = \lambda u & \quad \text{for} \quad x \in C \\ u = 0 & \quad \text{for} \quad x \in \partial C. \end{aligned}$$

a) *Show that all eigenvalues are* ***negative***.
b) *If u and v are eigenfunctions for distinct eigenvalues λ and μ, show that u and v are* ***orthogonal*** *in the appropriate product.*
c) *If $a(x) = a_1(x_1) + a_2(x_2) + a_3(x_3)$ find an expression for the eigenvalues and eigenvectors of L in terms of the eigenvalues and eigenvectors of a set of one-dimensional problems.*

Proof. **a)** We have

$$\triangle u - a(x)u = \lambda u.$$

Multiply the equation by u and integrate:

$$\begin{aligned} u\triangle u - a(x)u^2 &= \lambda u^2, \\ \int_\Omega u\triangle u\,dx - \int_\Omega a(x)u^2\,dx &= \lambda\int_\Omega u^2\,dx, \\ \underbrace{\int_{\partial\Omega} u\frac{\partial u}{\partial n}\,ds}_{=0} - \int_\Omega |\nabla u|^2\,dx - \int_\Omega a(x)u^2\,dx &= \lambda\int_\Omega u^2\,dx, \\ \lambda = \frac{-\int_\Omega(|\nabla u|^2 + a(x)u^2)\,dx}{\int_\Omega u^2\,dx} &< 0. \end{aligned}$$

b) Let λ, μ, be the eigenvalues and u, v be the corresponding eigenfunctions. We have

$$\triangle u - a(x)u = \lambda u. \tag{17.5}$$

$$\triangle v - a(x)v = \mu v. \tag{17.6}$$

Multiply (17.5) by v and (17.6) by u and subtract equations from each other

$$\begin{aligned} v\triangle u - a(x)uv &= \lambda uv, \\ u\triangle v - a(x)uv &= \mu uv. \\ v\triangle u - u\triangle v &= (\lambda - \mu)uv. \end{aligned}$$

Integrating over Ω gives

$$\begin{aligned} \int_\Omega \big(v\triangle u - u\triangle v\big)\,dx &= (\lambda-\mu)\int_\Omega uv\,dx, \\ \int_{\partial\Omega}\Big(\underbrace{v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}}_{=0}\Big)\,dx &= (\lambda-\mu)\int_\Omega uv\,dx. \end{aligned}$$

Since $\lambda \neq \mu$, u and v are orthogonal on Ω.

c) The three one-dimensional eigenvalue problems are:

$$u_{1x_1x_1}(x_1) - a(x_1)u_1(x_1) = \lambda_1 u_1(x_1),$$
$$u_{2x_2x_2}(x_2) - a(x_2)u_2(x_2) = \lambda_2 u_2(x_2),$$
$$u_{3x_3x_3}(x_3) - a(x_3)u_3(x_3) = \lambda_3 u_3(x_3).$$

We need to derive how u_1, u_2, u_3 and $\lambda_1, \lambda_2, \lambda_3$ are related to u and λ. $\square$

17.7 Spherical Means

Problem (S'95, #4). *Consider the* ***biharmonic operator*** *in* $\mathbb{R}^3$

$$\triangle^2 u \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)^2 u.$$

a) *Show that* $\triangle^2$ *is* ***self-adjoint*** *on* $|x| < 1$ *with the following boundary conditions on* $|x| = 1$*:*

$$u = 0,$$
$$\triangle u = 0.$$

Proof. **a)** We have

$$\begin{aligned} Lu &= \triangle^2 u = 0 \\ \\ u &= 0 \qquad \text{on } |x| = 1 \\ \triangle u &= 0 \qquad \text{on } |x| = 1. \end{aligned}$$

The operator L is self-adjoint if:

$$\int_\Omega (u\,Lv - v\,Lu)\,dx = 0.$$

$$\begin{aligned} \int_\Omega (u\,Lv - v\,Lu)\,dx &= \int_\Omega (u\,\triangle^2 v - v\,\triangle^2 u)\,dx \\ &= \underbrace{\int_{\partial\Omega} u\frac{\partial \triangle v}{\partial n}\,ds}_{=0} - \int_\Omega \nabla u \cdot \nabla(\triangle v)\,dx - \underbrace{\int_{\partial\Omega} v\frac{\partial \triangle u}{\partial n}\,ds}_{=0} + \int_\Omega \nabla v \cdot \nabla(\triangle u)\,dx \\ &= -\underbrace{\int_{\partial\Omega} \triangle v\frac{\partial u}{\partial n}\,ds}_{=0} + \int_\Omega \triangle u \triangle v\,dx + \underbrace{\int_{\partial\Omega} \triangle u\frac{\partial v}{\partial n}\,ds}_{=0} - \int_\Omega \triangle v \triangle u\,dx = 0. \quad \checkmark \end{aligned}$$

b) *Denote* $|x| = r$ *and define the averages*

$$S(r) = (4\pi r^2)^{-1} \int_{|x|=r} u(x)\, ds,$$
$$V(r) = \left(\frac{4}{3}\pi r^3\right)^{-1} \int_{|x|\le r} \triangle u(x)\, dx.$$

Show that

$$\frac{d}{dr}S(r) = \frac{r}{3}V(r).$$

Hint: Rewrite $S(r)$ *as an integral over the unit sphere before differentiating; i.e.,*

$$S(r) = (4\pi)^{-1} \int_{|x'|=1} u(rx')\, dx'.$$

c) *Use the result of (b) to show that if* u *is biharmonic, i.e.* $\triangle^2 u = 0$*, then*

$$S(r) = u(0) + \frac{r^2}{6}\triangle u(0).$$

Hint: Use the mean value theorem for $\triangle u$*.*

b) Let $x' = x/|x|$. We have [54]

$$\begin{aligned} S(r) &= \frac{1}{4\pi r^2}\int_{|x|=r} u(x)\, dS_r = \frac{1}{4\pi r^2}\int_{|x'|=1} u(rx')\, r^2\, dS_1 = \frac{1}{4\pi}\int_{|x'|=1} u(rx')\, dS_1. \\ \frac{dS}{dr} &= \frac{1}{4\pi}\int_{|x'|=1} \frac{\partial u}{\partial r}(rx')\, dS_1 = \frac{1}{4\pi}\int_{|x'|=1} \frac{\partial u}{\partial n}(rx')\, dS_1 = \frac{1}{4\pi r^2}\int_{|x|=r} \frac{\partial u}{\partial n}(x)\, dS_r \\ &= \frac{1}{4\pi r^2}\int_{|x|\le r} \triangle u\, dx. \quad \checkmark \end{aligned}$$

where we have used Green's identity in the last equality. Also

$$\frac{r}{3}V(r) = \frac{1}{4\pi r^2}\int_{|x|\le r} \triangle u\, dx. \quad \checkmark$$

c) Since u is biharmonic (i.e. $\triangle u$ is harmonic), $\triangle u$ has a mean value property. We have

$$\begin{aligned} \frac{d}{dr}S(r) &= \frac{r}{3}V(r) = \frac{r}{3}\left(\frac{4}{3}\pi r^3\right)^{-1}\int_{|x|\le r} \triangle u(x)\, dx = \frac{r}{3}\triangle u(0), \\ S(r) &= \frac{r^2}{6}\triangle u(0) + S(0) = u(0) + \frac{r^2}{6}\triangle u(0). \end{aligned}$$

[54] **Change of variables:**
Surface integrals: $x = rx'$ in $\mathbb{R}^3$:

$$\int_{|x|=r} u(x)\, dS = \int_{|x'|=1} u(rx')\, r^2\, dS_1.$$

Volume integrals: $\xi' = r\xi$ in $\mathbb{R}^n$:

$$\int_{|\xi'|<r} h(x+\xi')\, d\xi' = \int_{|\xi|<1} h(x + r\xi)\, r^n\, d\xi.$$

$\square$

Problem (S'00, #7). *Suppose that $u = u(x)$ for $x \in \mathbb{R}^3$ is* ***biharmonic****; i.e. that $\triangle^2 u \equiv \triangle(\triangle u) = 0$. Show that*

$$(4\pi r^2)^{-1} \int_{|x|=r} u(x)\, ds(x) \;=\; u(0) + (r^2/6)\triangle u(0)$$

through the following steps:

a) *Show that for any smooth f,*

$$\frac{d}{dr}\int_{|x|\le r} f(x)\, dx \;=\; \int_{|x|=r} f(x)\, ds(x).$$

b) *Show that for any smooth f,*

$$\frac{d}{dr}(4\pi r^2)^{-1}\int_{|x|=r} f(x)\, ds(x) \;=\; (4\pi r^2)^{-1}\int_{|x|=r} n\cdot\nabla f(x,y)\, ds$$

in which n is the outward normal to the circle $|x| = r$.

c) *Use step (b) to show that*

$$\frac{d}{dr}(4\pi r^2)^{-1}\int_{|x|=r} f(x)\, ds(x) \;=\; (4\pi r^2)^{-1}\int_{|x|\le r} \triangle f(x)\, dx.$$

d) *Combine steps (a) and (c) to obtain the final result.*

Proof. **a)** We can express the integral in **Spherical Coordinates**: [55]

$$\begin{aligned}
\int_{|x|\le R} f(x)\, dx &= \int_0^R\int_0^{2\pi}\int_0^{\pi} f(\phi,\theta,r)\, r^2\, \sin\phi\; d\phi\; d\theta\; dr. \\
\frac{d}{dr}\int_{|x|\le R} f(x)\, dx &= \frac{d}{dr}\int_0^R\int_0^{2\pi}\int_0^{\pi} f(\phi,\theta,r)\, r^2\, \sin\phi\; d\phi\; d\theta\; dr \;= ??? \\
&= \int_0^{2\pi}\int_0^{\pi} f(\phi,\theta,r)\, R^2\, \sin\phi\; d\phi\; d\theta \\
&= \int_{|x|=R} f(x)\, dS.
\end{aligned}$$

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[55] Differential **Volume** in spherical coordinates:

$$\boxed{d^3\omega \;=\; \omega^2\, \sin\phi\; d\phi\; d\theta\; d\omega.}$$

Differential **Surface Area** on sphere:

$$\boxed{dS \;=\; \omega^2\, \sin\phi\; d\phi\; d\theta.}$$

b&c) We have

$$\begin{aligned}
\frac{d}{dr}\bigg(\frac{1}{4\pi r^2}\int_{|x|=r} f(x)\,dS\bigg) &= \frac{d}{dr}\bigg(\frac{1}{4\pi r^2}\int_{|x'|=1} f(rx')\,r^2\,dS_1\bigg) = \frac{1}{4\pi}\frac{d}{dr}\bigg(\int_{|x'|=1} f(rx')\,dS_1\bigg)\\
&= \frac{1}{4\pi}\int_{|x'|=1}\frac{\partial f}{\partial r}(rx')\,dS_1 = \frac{1}{4\pi}\int_{|x'|=1}\frac{\partial f}{\partial n}(rx')\,dS_1\\
&= \frac{1}{4\pi r^2}\int_{|x|=r}\frac{\partial f}{\partial n}(x)\,dS = \frac{1}{4\pi r^2}\int_{|x|=r}\nabla f\cdot n\,dS \quad \checkmark\\
&= \frac{1}{4\pi r^2}\int_{|x|\le r}\triangle f\,dx. \quad \checkmark
\end{aligned}$$

Green's formula was used in the last equality.

Alternatively,

$$\begin{aligned}
\frac{d}{dr}\bigg(\frac{1}{4\pi r^2}\int_{|x|=r} f(x)\,dS\bigg) &= \frac{d}{dr}\bigg(\frac{1}{4\pi r^2}\int_0^{2\pi}\int_0^{\pi} f(\phi,\theta,r)\,r^2\,\sin\phi\,d\phi\,d\theta\bigg)\\
&= \frac{d}{dr}\bigg(\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} f(\phi,\theta,r)\,\sin\phi\,d\phi\,d\theta\bigg)\\
&= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\frac{\partial f}{\partial r}(\phi,\theta,r)\,\sin\phi\,d\phi\,d\theta\\
&= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\nabla f\cdot n\,\sin\phi\,d\phi\,d\theta\\
&= \frac{1}{4\pi r^2}\int_0^{2\pi}\int_0^{\pi}\nabla f\cdot n\;r^2\,\sin\phi\,d\phi\,d\theta\\
&= \frac{1}{4\pi r^2}\int_{|x|=r}\nabla f\cdot n\,dS \quad \checkmark\\
&= \frac{1}{4\pi r^2}\int_{|x|=r}\triangle f\,dx. \quad \checkmark
\end{aligned}$$

d) Since f is biharmonic (i.e. $\triangle f$ is harmonic), $\triangle f$ has a mean value property. From (c), we have [56]

$$\begin{aligned}
\frac{d}{dr}\bigg(\frac{1}{4\pi r^2}\int_{|x|=r} f(x)\,ds(x)\bigg) &= \frac{1}{4\pi r^2}\int_{|x|\le r}\triangle f(x)\,dx = \frac{r}{3}\frac{1}{\frac{4}{3}\pi r^3}\int_{|x|\le r}\triangle f(x)\,dx\\
&= \frac{r}{3}\,\triangle f(0).\\
\frac{1}{4\pi r^2}\int_{|x|=r} f(x)\,ds(x) &= \frac{r^2}{6}\,\triangle f(0) + f(0).
\end{aligned}$$

$\square$

[56] Note that part (a) was not used. We use exactly the same derivation as we did in S'95 #4.

Problem (F'96, #4).
Consider smooth solutions of $\triangle u = k^2 u$ *in dimension* $d = 2$ *with* $k > 0$.
a) *Show that* u *satisfies the following 'mean value property':*

$$M_x''(r) + \frac{1}{r} M_x'(r) - k^2 M_x(r) = 0,$$

in which $M_x(r)$ *is defined by*

$$M_x(r) = \frac{1}{2\pi} \int_0^{2\pi} u(x + r\cos\theta,\ y + r\sin\theta)\, d\theta$$

and the derivatives (denoted by $'$*) are in* r *with* x *fixed.*
b) *For* $k = 1$*, this equation is the modified Bessel equation (of order 0)*

$$f'' + \frac{1}{r} f' - f = 0,$$

for which one solution (denoted as I_0*) is*

$$I_0(r) = \frac{1}{2\pi} \int_0^{2\pi} e^{r\sin\theta}\, d\theta.$$

Find an expression for $M_x(r)$ *in terms of* I_0.

Proof. **a)** Laplacian in polar coordinates written as:

$$\triangle u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}.$$

Thus, the equation may be written as

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = k^2 u.$$

$$\begin{aligned} M_x(r) &= \frac{1}{2\pi} \int_0^{2\pi} u\, d\theta, \\ M_x'(r) &= \frac{1}{2\pi} \int_0^{2\pi} u_r\, d\theta, \\ M_x''(r) &= \frac{1}{2\pi} \int_0^{2\pi} u_{rr}\, d\theta. \end{aligned}$$

$$\begin{aligned} M_x''(r) + \frac{1}{r} M_x'(r) - k^2 M_x(r) &= \frac{1}{2\pi} \int_0^{2\pi} \Big(u_{rr} + \frac{1}{r} u_r - k^2 u\Big)\, d\theta \\ &= -\frac{1}{2\pi r^2} \int_0^{2\pi} u_{\theta\theta}\, d\theta \ = \ -\frac{1}{2\pi r^2} \big[u_\theta\big]_0^{2\pi} \ = \ 0. \quad \checkmark \end{aligned}$$

b) Note that $w = e^{r\sin\theta}$ satisfies $\triangle w = w$, i.e.

$$\begin{aligned} \triangle w &= w_{rr} + \frac{1}{r} w_r + \frac{1}{r^2} w_{\theta\theta} \\ &= \sin^2\theta\, e^{r\sin\theta} + \frac{1}{r} \sin\theta\, e^{r\sin\theta} + \frac{1}{r^2}(-r\sin\theta\, e^{r\sin\theta} + r^2\cos^2\theta\, e^{r\sin\theta}) \ = \ e^{r\sin\theta} \ = \ w. \end{aligned}$$

Thus,

$$M_x(r) \ = \ e^y \frac{1}{2\pi} \int_0^{2\pi} e^{r\sin\theta}\, d\theta \ = \ e^y I_0.$$

[57] $\square$

[57] Check with someone about the last result.

17.8 Harmonic Extensions, Subharmonic Functions

Problem (S'94, #8). *Suppose that Ω is a bounded region in $\mathbb{R}^3$ and that $u = 1$ on $\partial\Omega$. If $\triangle u = 0$ in the exterior region $\mathbb{R}^3/\Omega$ and $u(x) \to 0$ as $|x| \to \infty$, prove the following:*

a) *$u > 0$ in $\mathbb{R}^3/\Omega$;*
b) *if $\rho(x)$ is a smooth function such that $\rho(x) = 1$ for $|x| > R$ and $\rho(x) = 0$ near $\partial\Omega$, then for $|x| > R$,*

$$u(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3/\Omega} \frac{(\triangle(\rho u))(y)}{|x-y|}\,dy.$$

c) $\lim_{|x|\to\infty} |x|u(x)$ *exists and is non-negative.*

Proof. **a)** Let $\overline{B}_r(0)$ denote the closed ball $\{x : |x| \geq r\}$.
Given $\varepsilon > 0$, we can find r large enough that $\Omega \in \overline{B}_{R_1}(0)$ and $\max_{x\in\partial\overline{B}_{R_1}(0)} |u(x)| < \varepsilon$, since $|u(x)| \to 0$ as $|x| \to \infty$.
Since u is harmonic in $\overline{B}_{R_1} - \Omega$, it takes its maximum and minimum on the boundary. Assume

$$\min_{x\in\partial\overline{B}_{R_1}(0)} u(x) \;=\; -a \;<\; 0 \qquad (\text{where } \; |a| < \varepsilon).$$

We can find an R_2 such that $\max_{x\in\overline{B}_{R_2}(0)} |u(x)| < \frac{a}{2}$; hence u takes a minimum *inside* $\overline{B}_{R_2}(0) - \Omega$, which is impossible; hence $u \geq 0$.
Now let $V = \{x : u(x) \neq 0\}$ and let $\alpha = \min_{x\in V} |x|$. Since u cannot take a minimum inside $\overline{B}_R(0)$ (where $R > \alpha$), it follows that $u \equiv C$ and $C = 0$, but this contradicts $u = 1$ on $\partial\Omega$. Hence $u > 0$ in $\mathbb{R}^3 - \Omega$.

b) For $n = 3$,

$$K(|x-y|) \;=\; \frac{1}{(2-n)\omega_n}|x-y|^{2-n} \;=\; -\frac{1}{4\pi}\frac{1}{|x-y|}.$$

Since $\rho(x) = 1$ for $|x| > R$, then for $x \notin B_R$, we have $\triangle(\rho u) = \triangle u = 0$. Thus,

$$\begin{aligned}
& -\frac{1}{4\pi}\int_{\mathbb{R}^3/\Omega} \frac{(\triangle(\rho u))(y)}{|x-y|}\,dy \\
=\;& -\frac{1}{4\pi}\int_{B_R/\Omega} \frac{(\triangle(\rho u))(y)}{|x-y|}\,dy \\
=\;& \frac{1}{4\pi}\int_{B_R/\Omega} \nabla_y\Big(\frac{1}{|x-y|}\Big)\cdot\nabla_y(\rho u)\,dy - \frac{1}{4\pi}\int_{\partial(B_R/\Omega)} \frac{\partial}{\partial n}(\rho u)\frac{1}{|x-y|}\,dS_y \\
=\;& -\frac{1}{4\pi}\int_{B_R/\Omega} \triangle\Big(\frac{1}{|x-y|}\Big)\rho u\,dy + \frac{1}{4\pi}\int_{\partial(B_R/\Omega)} \frac{\partial}{\partial n}\Big(\frac{1}{|x-y|}\Big)\rho u\,dS_y - \frac{1}{4\pi}\int_{\partial(B_R/\Omega)} \frac{\partial}{\partial n}(\rho u)\frac{1}{|x-y|}\,dS_y \\
=\;& \textbf{???} \quad = u(x) - \underbrace{\frac{1}{4\pi R^2}\int_{\partial B} u\,dS_y}_{\to 0,\ as\ R\to\infty} - \underbrace{\frac{1}{4\pi R}\int_{\partial B} \frac{\partial u}{\partial n}\,dS_y}_{\to 0,\ as\ R\to\infty} \\
=\;& u(x).
\end{aligned}$$

c) See the next problem. ◻

Ralston Hw. ***a)*** *Suppose that u is a smooth function on $\mathbb{R}^3$ and $\triangle u = 0$ for $|x| > R$. If $\lim_{x\to\infty} u(x) = 0$, show that you can write u as a convolution of $\triangle u$ with the $-\frac{1}{4\pi|x|}$ and prove that $\lim_{x\to\infty} |x|u(x) = 0$ exists.*
b) *The "conductor potential" for $\Omega \subset \mathbb{R}^3$ is the solution to the Dirichlet problem $\triangle v = 0$. The limit in part (a) is called the "capacity" of Ω. Show that if $\Omega_1 \subset \Omega_2$, then the capacity of Ω_2 is greater or equal the capacity of Ω_1.*

Proof. **a)** If we define

$$v(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\triangle u(y)}{|x-y|}\,dy,$$

then $\triangle(u-v) = 0$ in all $\mathbb{R}^3$, and, since $v(x) \to 0$ as $|x| \to \infty$, we have $\lim_{|x|\to\infty}(u(x) - v(x)) = 0$. Thus, $u - v$ must be bounded, and Liouville's theorem implies that it is identically zero. Since we now have

$$|x|u(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{|x|\,\triangle u(y)}{|x-y|}\,dy,$$

and $|x|/|x-y|$ converges uniformly to 1 on $\{|y| \le R\}$, it follows that

$$\lim_{|x|\to\infty} |x|u(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \triangle u(y)\,dy.$$

b) Note that part (a) implies that the limit $\lim_{|x|\to\infty} |x|v(x)$ exists, because we can apply (a) to $u(x) = \phi(x)v(x)$, where ϕ is smooth and vanishes on Ω, but $\phi(x) = 1$ for $|x| > R$.
Let v_1 be the conductor potential for Ω_1 and v_2 for Ω_2. Since $v_i \to \infty$ as $|x| \to \infty$ and $v_i = 1$ on $\partial\Omega_i$, the max principle says that $1 > v_i(x) > 0$ for $x \in \mathbb{R}^3 - \Omega_i$. Consider $v_2 - v_1$. Since $\Omega_1 \subset \Omega_2$, this is defined in $\mathbb{R}^3 - \Omega_2$, positive on $\partial\Omega_2$, and has limit 0 as $|x| \to \infty$. Thus, it must be positive in $\mathbb{R}^3 - \Omega_2$. Thus, $\lim_{|x|\to\infty} |x|(v_2 - v_1) \ge 0$. □

Problem (F'95, #4). [58] *Let Ω be a simply connected open domain in $\mathbb{R}^2$ and $u = u(x,y)$ be* ***subharmonic*** *there, i.e. $\triangle u \ge 0$ in Ω. Prove that if*

$$D_R = \{(x,y) : (x-x_0)^2 + (y-y_0)^2 \le R^2\} \subset \Omega$$

then

$$u(x_0,y_0) \le \frac{1}{2\pi}\int_0^{2\pi} u(x_0 + R\cos\theta,\, y_0 + R\sin\theta)\,d\theta.$$

Proof. Let

$$\begin{aligned} M(x_0,R) &= \frac{1}{2\pi}\int_0^{2\pi} u(x_0 + R\cos\theta,\, y_0 + R\sin\theta)\,d\theta, \\ w(r,\theta) &= u(x_0 + R\cos\theta,\, y_0 + R\sin\theta). \end{aligned}$$

Differentiate $M(x_0,R)$ with respect to R:

$$\frac{d}{dr}M(x_0,R) = \frac{1}{2\pi R}\int_0^{2\pi} w_r(R,\theta)R\,d\theta,$$

[58] See McOwen, Sec.4.3, p.131, #1.

[59] $\square$

[59]See ChiuYen's solutions and Sung Ha's solutions (in two places). Nick's solutions, as started above, have a very simplistic approach.

Ralston Hw (Maximum Principle).
Suppose that $u \in C(\Omega)$ satisfies the mean value property in the connected open set Ω.

a) *Show that u satisfies the maximum principle in Ω, i.e.*
either u is constant or $u(x) < \sup_\Omega u$ for all $x \in \Omega$.

b) *Show that, if v is a continuous function on a closed ball $B_r(\xi) \subset \Omega$ and has the mean value property in $B_r(\xi)$, then $u = v$ on $\partial B_r(\xi)$ implies $u = v$ in $B_r(\xi)$. Does this imply that u is harmonic in Ω?*

Proof. **a)** If $u(x)$ is not less than $\sup_\Omega u$ for all $x \in \Omega$, then the set

$$K = \{x \in \Omega : \ u(x) = \sup_\Omega u\}$$

is nonempty. This set is closed because u is continuous. We will show it is also open. This implies that $K = \Omega$ because Ω is connected. Thus u is constant on Ω.
Let $x_0 \in K$. Since Ω is open, $\exists \delta > 0$, s.t. $B_\delta(x_0) = \{x \in \mathbb{R}^n : \ |x - x_0| \le \delta\} \subset \Omega$. Let $\sup_\Omega u = M$. By the mean value property, for $0 \le r \le \delta$

$$M = u(x_0) = \frac{1}{A(S^{n-1})} \int_{|\xi|=1} u(x_0 + r\xi) dS_\xi, \text{ and } 0 = \frac{1}{A(S^{n-1})} \int_{|\xi|=1} (M - u(x_0 + r\xi)) dS_\xi.$$

Sinse $M - u(x_0 + r\xi)$ is a continuous nonnegative function on ξ, this implies $M - u(x_0 + r\xi) = 0$ for all $\xi \in S^{n-1}$. Thus $u = 0$ on $B_\delta(x_0)$.

b) Since $u - v$ has the mean value property in the open interior of $B_r(\xi)$, by part $a)$ it satisfies the maximum principle. Since it is continuous on $B_r(\xi)$, its supremum over the interior of $B_r(\xi)$ is its maximum on $B_r(\xi)$, and this maximum is assumed at a point x_0 in $B_r(\xi)$. If x_0 in the interior of $B_r(\xi)$, then $u - v$ is constant ant the constant must be zero, since this is the value of $u - v$ on the boundary. If x_0 is on the boundary, then $u - v$ must be nonpositive in the interior of $B_r(\xi)$.
Applying the same argument to $v - u$, one finds that it is either identically zero or nonpositive in the interior of $B_r(\xi)$. Thus, $u - v \equiv 0$ on $B_r(\xi)$.
Yes, it does follow that u harmonic in Ω. Take v in the preceding to be the harmonic function in the interior of $B_r(\xi)$ which agrees with u on the boundary. Since $u = v$ on $B_r(\xi)$, u is harmonic in the interior of $B_r(\xi)$. Since Ω is open we can do this for every $\xi \in \Omega$. Thus u is harmonic in Ω. $\square$

Ralston Hw. *Assume Ω is a bounded open set in $\mathbb{R}^n$ and the Green's function, $G(x, y)$, for Ω exists. Use the strong maximum principle, i.e. either $u(x) < \sup_\Omega u$ for all $x \in \Omega$, or u is constant, to prove that $G(x, y) < 0$ for $x, y \in \Omega$, $x \neq y$.*

Proof. $G(x, y) = K(x, y) + \omega(x, y)$. For each $x \in \Omega$, $f(y) = \omega(x, y)$ is continuous on $\overline{\Omega}$, thus, bounded. So $|\omega(x, y)| \leq M_x$ for all $y \in \overline{\Omega}$. $K(x - y) \to -\infty$ as $y \to x$. Thus, given M_x, there is $\delta > 0$, such that $K(x - y) < -M_x$ when $|x - y| = r$ and $0 < r \leq \delta$. So for $0 < r \leq \delta$ the Green's function with x fixed satisfies, $G(x, y)$ is harmonic on $\Omega - B_r(x)$, and $G(x, y) \leq 0$ on the boundary of $\Omega - B_r(x)$. Since we can choose r as small as we wish, we get $G(x, y) < 0$ for $y \in \Omega - \{x\}$. ☐

Problem (W'03, #6). *Assume that u is a* ***harmonic*** *function in the* ***half ball*** *$D = \{(x, y, z) : x^2 + y^2 + z^2 < 1,\ z \geq 0\}$ which is continuously differentiable, and satisfies $u(x, y, 0) = 0$. Show that u can be extended to be a harmonic function in the whole ball. If you propose and explicit extension for u, explain why the extension is harmonic.*

Proof. We can extend u to all of n-space by defining

$$u(x', x_n) = -u(x', -x_n)$$

for $x_n < 0$. Define

$$\omega(x) \;=\; \frac{1}{a\omega_n}\int_{|y|=1} \frac{a^2 - |x|^2}{|x - y|^n} v(y) dS_y$$

$\omega(x)$ is continuous on a closed ball B, harmonic in B.
Poisson kernel is symmetric in y at $x_n = 0$. $\Rightarrow$ $\omega(x) = 0$, $(x_n = 0)$.
ω is harmonic for $x \in B$, $x_n \geq 0$,with the same boundary values $\omega = u$.
ω is harmonic $\Rightarrow$ u can be extended to a harmonic function on the interior of B. ☐

Ralston Hw. *Show that a* ***bounded*** *solution to the* ***Dirichlet*** *problem in a* ***half space*** *is* ***unique****. (Note that one can show that a bounded solution* ***exists*** *for any given bounded continuous Dirichlet data by using the Poisson kernel for the half space.)*

Proof. We have to show that a function, u, which is harmonic in the half-space, continuous, equal to 0 when $x_n = 0$, and bounded, must be identically 0. We can extend u to all of n-space by defining

$$u(x', x_n) = -u(x', -x_n)$$

for $x_n < 0$. This extends u to a bounded harmonic function on all of n-space (by the problem above). Liouville's theorem says u must be constant, and since $u(x', 0) = 0$, the constant is 0. So the original u must be identically 0. ☐

Ralston Hw. *Suppose u is harmonic on the* ***ball minus the origin****, $B_0 = \{x \in \mathbb{R}^3 : 0 < |x| < a\}$. Show that $u(x)$ can be* ***extended*** *to a harmonic function on the ball $B = \{|x| < a\}$ iff $\lim_{|x|\to 0} |x|u(x) = 0$.*

Proof. The condition $\lim_{|x|\to 0} |x|u(x) = 0$ is necessary, because harmonic functions are continuous.
To prove the converse, let v be the function which is continuous on $\{|x| \leq a/2\}$, harmonic on $\{|x| < a/2\}$, and equals u on $\{|x| = a/2\}$. One can construct v using the Poisson kernel. Since v is continuous, it is bounded, and we can assume that $|v| \leq M$. Since $\lim_{|x|\to 0} |x|u(x) = 0$, given $\epsilon > 0$, we can choose δ, $0 < \delta < a/2$ such that $-\epsilon < |x|u(x) < \epsilon$ when $|x| < \delta$. Note that u, $v - 2\epsilon/|x|$, and $v + 2\epsilon/|x|$ are harmonic

on $\{0 < |x| < a/2\}$. Choose b, $0 < b < \min(\epsilon, a/2)$, so that $\epsilon/b > M$. Then on both $\{|x| = a/2\}$ and $\{|x| = b\}$ we have $v - 2\epsilon/|x| < u(x) < v + 2\epsilon/|x|$. Thus, by max principle these inequalities hold on $\{b \le |x| \le a/2\}$. Pick x with $0 < |x| \le a/2$. $u(x) = v(x)$. v is the extension of u on $\{|x| < a/2\}$, and u is extended on $\{|x| < a\}$. $\square$

18 Problems: Heat Equation

McOwen 5.2 #7(a). *Consider*

$$\begin{cases} u_t = u_{xx} & for\ \ x > 0,\, t > 0 \\ u(x,0) = g(x) & for\ \ x > 0 \\ u(0,t) = 0 & for\ \ t > 0, \end{cases}$$

where g is continuous and bounded for $x \geq 0$ and $g(0) = 0$.
Find a formula for the solution $u(x,t)$.

Proof. Extend g to be an **odd** function on all of $\mathbb{R}$:

$$\tilde{g}(x) = \begin{cases} g(x), & x \geq 0 \\ -g(-x), & x < 0. \end{cases}$$

Then, we need to solve

$$\begin{cases} \tilde{u}_t = \tilde{u}_{xx} & \text{for}\ \ x \in \mathbb{R},\, t > 0 \\ \tilde{u}(x,0) = \tilde{g}(x) & \text{for}\ \ x \in \mathbb{R}. \end{cases}$$

The solution is given by: [60]

$$\begin{aligned} \tilde{u}(x,t) &= \int_{\mathbb{R}} K(x,y,t) g(y)\, dy = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} \tilde{g}(y)\, dy \\ &= \frac{1}{\sqrt{4\pi t}} \left[\int_0^{\infty} e^{-\frac{(x-y)^2}{4t}} \tilde{g}(y)\, dy + \int_{-\infty}^{0} e^{-\frac{(x-y)^2}{4t}} \tilde{g}(y)\, dy \right] \\ &= \frac{1}{\sqrt{4\pi t}} \left[\int_0^{\infty} e^{-\frac{(x-y)^2}{4t}} g(y)\, dy - \int_0^{\infty} e^{-\frac{(x+y)^2}{4t}} g(y)\, dy \right] \\ &= \frac{1}{\sqrt{4\pi t}} \int_0^{\infty} \left(e^{\frac{-x^2+2xy-y^2}{4t}} - e^{\frac{-x^2-2xy-y^2}{4t}} \right) g(y)\, dy \\ &= \frac{1}{\sqrt{4\pi t}} \int_0^{\infty} e^{-\frac{(x^2+y^2)}{4t}} \left(e^{\frac{xy}{2t}} - e^{-\frac{xy}{2t}} \right) g(y)\, dy. \end{aligned}$$

$$\boxed{u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_0^{\infty} e^{-\frac{(x^2+y^2)}{4t}}\, 2\sinh\left(\frac{xy}{2t}\right) g(y)\, dy.}$$

Since $\sinh(0) = 0$, we can verify that $u(0,t) = 0$. ☐

[60] In calculations, we use: $\int_{-\infty}^{0} e^{y}\, dy = \int_0^{\infty} e^{-y}\, dy$, and $g(-y) = -g(y)$.

McOwen 5.2 #7(b). *Consider*

$$\begin{cases} u_t = u_{xx} & \textit{for } \; x > 0,\, t > 0 \\ u(x,0) = g(x) & \textit{for } \; x > 0 \\ u_x(0,t) = 0 & \textit{for } \; t > 0, \end{cases}$$

where g is continuous and bounded for $x \geq 0$.
Find a formula for the solution $u(x,t)$.

Proof. Extend g to be an **even** function [61] on all of $\mathbb{R}$:

$$\tilde{g}(x) = \begin{cases} g(x), & x \geq 0 \\ g(-x), & x < 0. \end{cases}$$

Then, we need to solve

$$\begin{cases} \tilde{u}_t = \tilde{u}_{xx} & \text{for } \; x \in \mathbb{R},\, t > 0 \\ \tilde{u}(x,0) = \tilde{g}(x) & \text{for } \; x \in \mathbb{R}. \end{cases}$$

The solution is given by: [62]

$$\begin{aligned} \tilde{u}(x,t) &= \int_{\mathbb{R}} K(x,y,t) g(y)\, dy = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} \tilde{g}(y)\, dy \\ &= \frac{1}{\sqrt{4\pi t}} \bigg[\int_0^{\infty} e^{-\frac{(x-y)^2}{4t}} \, \tilde{g}(y)\, dy + \int_{-\infty}^{0} e^{-\frac{(x-y)^2}{4t}} \, \tilde{g}(y)\, dy \bigg] \\ &= \frac{1}{\sqrt{4\pi t}} \bigg[\int_0^{\infty} e^{-\frac{(x-y)^2}{4t}} \, g(y)\, dy + \int_0^{\infty} e^{-\frac{(x+y)^2}{4t}} \, g(y)\, dy \bigg] \\ &= \frac{1}{\sqrt{4\pi t}} \int_0^{\infty} \Big(e^{\frac{-x^2+2xy-y^2}{4t}} + e^{\frac{-x^2-2xy-y^2}{4t}} \Big) g(y)\, dy \\ &= \frac{1}{\sqrt{4\pi t}} \int_0^{\infty} e^{-\frac{(x^2+y^2)}{4t}} \Big(e^{\frac{xy}{2t}} + e^{-\frac{xy}{2t}} \Big) g(y)\, dy. \end{aligned}$$

$$\boxed{u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_0^{\infty} e^{-\frac{(x^2+y^2)}{4t}} \, 2\cosh\Big(\frac{xy}{2t}\Big) \, g(y)\, dy.}$$

To check that the boundary condition holds, we perform the calculation:

$$\begin{aligned} u_x(x,t) &= \frac{1}{\sqrt{4\pi t}} \int_0^{\infty} \frac{d}{dx} \bigg[e^{-\frac{(x^2+y^2)}{4t}} \, 2\cosh\Big(\frac{xy}{2t}\Big) \bigg] g(y)\, dy \\ &= \frac{1}{\sqrt{4\pi t}} \int_0^{\infty} \bigg[-\frac{2x}{4t} e^{-\frac{(x^2+y^2)}{4t}} \, 2\cosh\Big(\frac{xy}{2t}\Big) + e^{-\frac{(x^2+y^2)}{4t}} \, 2\frac{y}{2t} \sinh\Big(\frac{xy}{2t}\Big) \bigg] g(y)\, dy, \\ u_x(0,t) &= \frac{1}{\sqrt{4\pi t}} \int_0^{\infty} \bigg[0 \cdot e^{-\frac{y^2}{4t}} \, 2\cosh 0 + e^{-\frac{y^2}{4t}} \, 2\frac{y}{2t} \sinh 0 \bigg] g(y)\, dy = 0. \end{aligned}$$

∎

[61] Even extensions are always continuous. Not true for odd extensions. g odd is continuous if $g(0) = 0$.

[62] In calculations, we use: $\int_{-\infty}^0 e^y\, dy = \int_0^\infty e^{-y}\, dy$, and $g(-y) = g(y)$.

Problem (F'90, #5).
The initial value problem for the heat equation on the whole real line is

$$f_t = f_{xx} \qquad t \geq 0$$
$$f(t=0,x) = f_0(x)$$

with f_0 smooth and bounded.
a) *Write down the* ***Green's function*** *$G(x,y,t)$ for this initial value problem.*
b) *Write the solution $f(x,t)$ as an integral involving G and f_0.*
c) *Show that the maximum values of $|f(x,t)|$ and $|f_x(x,t)|$ are non-increasing as t increases, i.e.*

$$\sup_x |f(x,t)| \leq \sup_x |f_0(x)| \qquad \sup_x |f_x(x,t)| \leq \sup_x |f_{0x}(x)|.$$

When are these inequalities actually equalities?

Proof. **a)** The fundamental solution

$$K(x,y,t) \;=\; \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x-y|^2}{4t}}.$$

The Green's function is: [63]

$$G(x,t;y,s) \;=\; \frac{1}{(2\pi)^n}\left[\frac{\pi}{k(t-s)}\right]^{\frac{n}{2}} e^{-\frac{(x-y)^2}{4k(t-s)}}.$$

b) The solution to the one-dimensional heat equation is

$$u(x,t) \;=\; \int_{\mathbb{R}} K(x,y,t)\, f_0(y)\, dy \;=\; \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{|x-y|^2}{4t}} f_0(y)\, dy.$$

c) We have

$$\begin{aligned}
\sup_x |u(x,t)| \;&=\; \left| \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} f_0(y)\, dy \right| \;\leq\; \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} \big|f_0(y)\big|\, dy \\
&=\; \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{y^2}{4t}} \big|f_0(x-y)\big|\, dy \\
&\leq\; \sup_x |f_0(x)|\, \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{y^2}{4t}}\, dy \qquad\qquad \left(\, z = \frac{y}{\sqrt{4t}}, \;\; dz = \frac{dy}{\sqrt{4t}} \,\right) \\
&\leq\; \sup_x |f_0(x)|\, \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-z^2} \sqrt{4t}\, dz \\
&=\; \sup_x |f_0(x)|\, \frac{1}{\sqrt{\pi}} \underbrace{\int_{\mathbb{R}} e^{-z^2}\, dz}_{=\sqrt{\pi}} \;=\; \sup_x |f_0(x)|. \quad \checkmark
\end{aligned}$$

[63] The Green's function for the heat equation on an infinite domain; derived in R. Haberman using the Fourier transform.

$$\begin{aligned}
u_x(x,t) &= \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} -\frac{2(x-y)}{4t} e^{-\frac{(x-y)^2}{4t}} f_0(y)\,dy \;=\; \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} -\frac{d}{dy}\Big[e^{-\frac{(x-y)^2}{4t}}\Big]\, f_0(y)\,dy \\
&= \underbrace{\frac{1}{\sqrt{4\pi t}}\Big[-e^{-\frac{(x-y)^2}{4t}}\, f_0(y)\Big]_{-\infty}^{\infty}}_{=0} + \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}}\, f_{0y}(y)\,dy, \\
\sup_x |u(x,t)| &\le \frac{1}{\sqrt{4\pi t}} \sup_x |f_{0x}(x)| \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}}\,dy \;=\; \frac{1}{\sqrt{4\pi t}} \sup_x |f_{0x}(x)| \int_{\mathbb{R}} e^{-z^2} \sqrt{4t}\,dz \\
&= \sup_x |f_{0x}(x)|. \quad \checkmark
\end{aligned}$$

These inequalities are equalities when $f_0(x)$ and $f_{0x}(x)$ are constants, respectively. $\square$

Problem (S'01, #5). ***a)*** *Show that the solution of the heat equation*

$$u_t = u_{xx}, \qquad -\infty < x < \infty$$

with square-integrable initial data $u(x,0) = f(x)$, decays in time, and there is a constant α independent of f and t such that for all $t > 0$

$$\max_x |u_x(x,t)| \le \alpha t^{-\frac{3}{4}} \Big(\int_x |f(x)|^2 \, dx \Big)^{\frac{1}{2}}.$$

b) *Consider the solution ρ of the transport equation $\rho_t + u\rho_x = 0$ with square-integrable initial data $\rho(x,0) = \rho_0(x)$ and the velocity u from part (a). Show that $\rho(x,t)$ remains square-integrable for all finite time*

$$\int_R |\rho(x,t)|^2 \, dx \le e^{Ct^{\frac{1}{4}}} \int_R |\rho_0(x)|^2 \, dx,$$

where C does not depend on ρ_0.

Proof. **a)** The solution to the one-dimensional homogeneous heat equation is

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} f(y) \, dy.$$

Take the derivative with respect to x, we get [64]

$$u_x(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} -\frac{2(x-y)}{4t} e^{-\frac{(x-y)^2}{4t}} f(y) \, dy = -\frac{1}{4t^{\frac{3}{2}}\sqrt{\pi}} \int_{\mathbb{R}} (x-y) e^{-\frac{(x-y)^2}{4t}} f(y) \, dy.$$

$$\begin{aligned}
|u_x(x,t)| &\le \frac{1}{4t^{\frac{3}{2}}\sqrt{\pi}} \int_{\mathbb{R}} \Big| (x-y) e^{-\frac{(x-y)^2}{4t}} f(y) \Big| \, dy && \text{(Cauchy-Schwarz)} \\
&\le \frac{1}{4t^{\frac{3}{2}}\sqrt{\pi}} \Big(\int_{\mathbb{R}} (x-y)^2 e^{-\frac{(x-y)^2}{2t}} \, dy \Big)^{\frac{1}{2}} ||f||_{L^2(\mathbb{R})} && \Big(z = \frac{x-y}{\sqrt{2t}}, \ \ dz = -\frac{dy}{\sqrt{2t}} \Big) \\
&= \frac{1}{4t^{\frac{3}{2}}\sqrt{\pi}} \Big(\int_{\mathbb{R}} \big| - z^2 (2t)^{\frac{3}{2}} e^{-z^2} \big| dz \Big)^{\frac{1}{2}} ||f||_{L^2(\mathbb{R})} \\
&= \frac{(2t)^{\frac{3}{4}}}{4t^{\frac{3}{2}}\sqrt{\pi}} \Big(\underbrace{\int_{\mathbb{R}} z^2 e^{-z^2} \, dz}_{M<\infty} \Big)^{\frac{1}{2}} ||f||_{L^2(\mathbb{R})} \\
&= Ct^{-\frac{3}{4}} M^{\frac{1}{2}} ||f||_{L^2(\mathbb{R})} \ = \ \alpha t^{-\frac{3}{4}} ||f||_{L^2(\mathbb{R})}.
\end{aligned}$$

b) Note:

$$\begin{aligned}
\max_x |u| &= \max_x \Big| \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} f(y) \, dy \Big| \ \le \ \frac{1}{\sqrt{4\pi t}} \Big(\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{2t}} \, dy \Big)^{\frac{1}{2}} ||f||_{L^2(\mathbb{R})} \\
&\le \frac{1}{\sqrt{4\pi t}} \Big(\int_{\mathbb{R}} \big| - e^{-z^2} \sqrt{2t} \big| \, dz \Big)^{\frac{1}{2}} ||f||_{L^2(\mathbb{R})} && \Big(z = \frac{x-y}{\sqrt{2t}}, \ \ dz = -\frac{dy}{\sqrt{2t}} \Big) \\
&= \frac{(2t)^{\frac{1}{4}}}{2\pi^{\frac{1}{2}} t^{\frac{1}{2}}} \Big(\underbrace{\int_{\mathbb{R}} e^{-z^2} \, dz}_{=\sqrt{\pi}} \Big)^{\frac{1}{2}} ||f||_{L^2(\mathbb{R})} \ = \ Ct^{-\frac{1}{4}} ||f||_{L^2(\mathbb{R})}.
\end{aligned}$$

[65]

$\square$

[64] Cauchy-Schwarz: $|(u,v)| \le ||u|| ||v||$ in any norm, for example $\int |uv| dx \le (\int u^2 dx)^{\frac{1}{2}} (\int v^2 dx)^{\frac{1}{2}}$

[65] See Yana's and Alan's solutions.

Problem (F'04, #2).
Let $u(x,t)$ be a bounded solution to the Cauchy problem for the heat equation

$$\begin{cases} u_t = a^2 u_{xx}, & t>0,\ x\in\mathbb{R},\ a>0, \\ u(x,0)=\varphi(x). \end{cases}$$

Here $\varphi(x)\in C(\mathbb{R})$ satisfies

$$\lim_{x\to+\infty}\varphi(x)=b, \qquad \lim_{x\to-\infty}\varphi(x)=c.$$

Compute the limit of $u(x,t)$ as $t\to+\infty$, $x\in\mathbb{R}$. Justify your argument carefully.

Proof. For $a=1$, the solution to the one-dimensional homogeneous heat equation is

$$u(x,t)=\frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}}\varphi(y)\,dy.$$

We want to transform the equation to $v_t = v_{xx}$. Make a change of variables: $x=ay$. $u(x,t)=u(x(y),t)=u(ay,t)=v(y,t)$. Then,

$$\begin{aligned} v_y &= u_x x_y = a u_x, \\ v_{yy} &= a u_{xx} x_y = a^2 u_{xx}, \\ v(y,0) &= u(ay,0)=\varphi(ay). \end{aligned}$$

Thus, the new problem is:

$$\begin{cases} v_t = v_{yy}, & t>0,\ y\in\mathbb{R}, \\ v(y,0)=\varphi(ay). \end{cases}$$

$$v(y,t)=\frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} e^{-\frac{(y-z)^2}{4t}}\varphi(az)\,dz. \qquad \circledast$$

Since φ is continuous, and $\lim_{x\to+\infty}\varphi(x)=b$, $\lim_{x\to-\infty}\varphi(x)=c$, we have

$|\varphi(x)|<M, \quad \forall x\in\mathbb{R}.$ Thus,

$$\begin{aligned} |v(y,t)| &\le \frac{M}{\sqrt{4\pi t}}\int_{\mathbb{R}} e^{-\frac{z^2}{4t}}\,dz \qquad \left(s=\frac{z}{\sqrt{4t}},\ ds=\frac{dz}{\sqrt{4t}}\right) \\ &= \frac{M}{\sqrt{4\pi t}}\int_{\mathbb{R}} e^{-s^2}\sqrt{4t}\,ds = \frac{M}{\sqrt{\pi}}\underbrace{\int_{\mathbb{R}} e^{-s^2}\,ds}_{\sqrt{\pi}} = M. \end{aligned}$$

Integral in $\circledast$ converges uniformly $\Rightarrow$ $\lim\int=\int\lim$. For $\psi=\varphi(a\cdot)$:

$$\begin{aligned} v(y,t) &= \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(y-z)^2}{4t}}\psi(z)\,dz = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{-\frac{z^2}{4t}}\psi(y-z)\,dz \\ &= \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{-s^2}\,\psi(y-s\sqrt{4t})\,\sqrt{4t}\,ds \\ &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-s^2}\,\psi(y-s\sqrt{4t})\,ds. \end{aligned}$$

$$\begin{aligned}
\lim_{t\to+\infty} v(y,t) &= \frac{1}{\sqrt{\pi}}\int_0^\infty e^{-s^2} \lim_{t\to+\infty}\psi(y-s\sqrt{4t})\,ds + \frac{1}{\sqrt{\pi}}\int_{-\infty}^0 e^{-s^2} \lim_{t\to+\infty}\psi(y-s\sqrt{4t})\,ds \\
&= \frac{1}{\sqrt{\pi}}\int_0^\infty e^{-s^2} c\,ds + \frac{1}{\sqrt{\pi}}\int_{-\infty}^0 e^{-s^2} b\,ds \;=\; c\frac{1}{\sqrt{\pi}}\frac{\sqrt{\pi}}{2} + b\frac{1}{\sqrt{\pi}}\frac{\sqrt{\pi}}{2} \\
&= \frac{c+b}{2}.
\end{aligned}$$

$\square$

Problem. *Consider*

$$\begin{aligned} u_t &= ku_{xx} + Q, \qquad 0 < x < 1 \\ u(0,t) &= 0, \\ u(1,t) &= 1. \end{aligned}$$

What is the steady state temperature?

Proof. Set $u_t = 0$, and integrate with respect to x twice:

$$\begin{aligned} ku_{xx} + Q = 0, \\ u_{xx} = -\frac{Q}{k}, \\ u_x = -\frac{Q}{k}x + a, \\ u = -\frac{Q}{k}\frac{x^2}{2} + ax + b. \end{aligned}$$

Boundary conditions give

$$u(x) = -\frac{Q}{2k}x^2 + \Big(1 + \frac{Q}{2k}\Big)x.$$

$\square$

18.1 Heat Equation with Lower Order Terms

McOwen 5.2 #11. *Find a formula for the solution of*

$$\begin{cases} u_t = \triangle u - cu & \text{in } \mathbb{R}^n \times (0,\infty) \\ u(x,0) = g(x) & \text{on } \mathbb{R}^n. \end{cases} \tag{18.1}$$

Show that such solutions, with initial data $g \in L^2(\mathbb{R}^n)$, are unique, even when c is negative.

Proof. **McOwen.** Consider $v(x,t) = e^{ct}u(x,t)$. The transformed problem is

$$\begin{cases} v_t = \triangle v & \text{in } \mathbb{R}^n \times (0,\infty) \\ v(x,0) = g(x) & \text{on } \mathbb{R}^n. \end{cases} \tag{18.2}$$

Since g is continuous and bounded in $\mathbb{R}^n$, we have

$$\begin{aligned} v(x,t) &= \int_{\mathbb{R}^n} K(x,y,t)\, g(y)\, dy = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\, dy, \\ u(x,t) &= e^{-ct}v(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t} - ct}\, g(y)\, dy. \end{aligned}$$

$u(x,t)$ is a **bounded** solution since $v(x,t)$ is.
To prove uniqueness, assume there is another solution v' of (18.2). $w = v - v'$ satisfies

$$\begin{cases} w_t = \triangle w & \text{in } \mathbb{R}^n \times (0,\infty) \\ w(x,0) = 0 & \text{on } \mathbb{R}^n. \end{cases} \tag{18.3}$$

Since bounded solutions of (18.3) are unique, and since w is a nontrivial solution, w is unbounded. Thus, v' is unbounded, and therefore, the bounded solution v is unique. ☐

18.1.1 Heat Equation Energy Estimates

Problem (F'94, #3). *Let $u(x, y, t)$ be a twice continuously differential solution of*

$$\begin{aligned} &u_t = \triangle u - u^3 && \text{in } \ \Omega \subset \mathbb{R}^2, \ t \geq 0 \\ &u(x, y, 0) = 0 && \text{in } \ \Omega \\ &u(x, y, t) = 0 && \text{in } \ \partial\Omega, \ t \geq 0. \end{aligned}$$

Prove that $\ u(x, y, t) \equiv 0 \ $ *in* $\Omega \times [0, T]$.

Proof. Multiply the equation by u and integrate:

$$\begin{aligned} uu_t &= u\triangle u - u^4, \\ \int_\Omega uu_t \, dx &= \int_\Omega u\triangle u \, dx - \int_\Omega u^4 \, dx, \\ \frac{1}{2}\frac{d}{dt}\int_\Omega u^2 \, dx &= \underbrace{\int_{\partial\Omega} u\frac{\partial u}{\partial n} \, ds}_{=0} - \int_\Omega |\nabla u|^2 \, dx - \int_\Omega u^4 \, dx, \\ \frac{1}{2}\frac{d}{dt}||u||_2^2 &= -\int_\Omega |\nabla u|^2 \, dx - \int_\Omega u^4 \, dx \ \leq \ 0. \end{aligned}$$

Thus,

$$||u(x, y, t)||_2 \ \leq \ ||u(x, y, 0)||_2 \ = \ 0.$$

Hence, $\ ||u(x, y, t)||_2 = 0$, and $\ u \equiv 0$. $\square$

Problem (F'98, #5). *Consider the heat equation*

$$u_t - \triangle u = 0$$

in a two dimensional region Ω. *Define the mass* M *as*

$$M(t) = \int_\Omega u(x,t)\,dx.$$

a) *For a fixed domain* Ω, *show* M *is a constant in time if the boundary conditions are* $\partial u/\partial n = 0$.
b) *Suppose that* $\Omega = \Omega(t)$ *is evolving in time, with a boundary that moves at velocity* v, *which may vary along the boundary. Find a modified boundary condition (in terms of local quantities only) for* u, *so that* M *is constant.*
Hint: You may use the fact that

$$\frac{d}{dt}\int_{\Omega(t)} f(x,t)\,dx = \int_{\Omega(t)} f_t(x,t)\,dx + \int_{\partial\Omega(t)} n\cdot v\, f(x,t)\,dl,$$

in which n *is a unit normal vector to the boundary* $\partial\Omega$.

Proof. **a)** We have

$$\begin{cases} u_t - \triangle u = 0, & \text{on } \Omega \\ \frac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega. \end{cases}$$

We want to show that $\frac{d}{dt}M(t) = 0$. We have [66]

$$\frac{d}{dt}M(t) = \frac{d}{dt}\int_\Omega u(x,t)\,dx = \int_\Omega u_t\,dx = \int_\Omega \triangle u\,dx = \int_{\partial\Omega}\frac{\partial u}{\partial n}\,ds = 0. \quad \checkmark$$

b) We need $\frac{d}{dt}M(t) = 0$.

$$\begin{aligned} 0 = \frac{d}{dt}M(t) &= \frac{d}{dt}\int_{\Omega(t)} u(x,t)\,dx = \int_{\Omega(t)} u_t\,dx + \int_{\partial\Omega(t)} n\cdot v\,u\,ds \\ &= \int_{\Omega(t)} \triangle u\,dx + \int_{\partial\Omega(t)} n\cdot v\,u\,ds = \int_{\partial\Omega(t)} \frac{\partial u}{\partial n}\,ds + \int_{\partial\Omega(t)} n\cdot v\,u\,ds \\ &= \int_{\partial\Omega(t)} \nabla u\cdot n\,ds + \int_{\partial\Omega(t)} n\cdot v\,u\,ds = \int_{\partial\Omega(t)} n\cdot(\nabla u + vu)\,ds. \end{aligned}$$

Thus, we need:

$$n\cdot(\nabla u + vu)\,ds = 0, \qquad \text{on } \partial\Omega.$$

$\square$

[66] The last equality below is obtained from the Green's formula:

$$\int_\Omega \triangle u\,dx = \int_\Omega \frac{\partial u}{\partial n}\,ds.$$

Problem (S'95, #3). *Write down an explicit formula for a function $u(x,t)$ solving*

$$\begin{cases} u_t + b\cdot\nabla u + cu = \triangle u & \text{in } \mathbb{R}^n\times(0,\infty) \\ u(x,0) = f(x) & \text{on } \mathbb{R}^n. \end{cases} \tag{18.4}$$

where $b\in\mathbb{R}^n$ and $c\in\mathbb{R}$ are constants.
Hint: First transform this to the ***heat equation*** *by a linear change of the dependent and independent variables. Then solve the heat equation using the fundamental solution.*

Proof. Consider

- $u(x,t) \ = \ e^{\alpha\cdot x+\beta t}v(x,t).$

$$\begin{aligned} u_t &= \beta e^{\alpha\cdot x+\beta t}v + e^{\alpha\cdot x+\beta t}v_t = (v_t+\beta v)e^{\alpha\cdot x+\beta t}, \\ \nabla u &= \alpha e^{\alpha\cdot x+\beta t}v + e^{\alpha\cdot x+\beta t}\nabla v = (\alpha v+\nabla v)e^{\alpha\cdot x+\beta t}, \\ \nabla\cdot(\nabla u) &= \nabla\cdot\big((\alpha v+\nabla v)e^{\alpha\cdot x+\beta t}\big) = (\alpha\cdot\nabla v+\triangle v)e^{\alpha\cdot x+\beta t} + (|\alpha|^2 v+\alpha\cdot\nabla v)e^{\alpha\cdot x+\beta t} \\ &= \big(\triangle v + 2\alpha\cdot\nabla v + |\alpha|^2 v\big)e^{\alpha\cdot x+\beta t}. \end{aligned}$$

Plugging this into (18.4), we obtain

$$\begin{aligned} &v_t + \beta v + b\cdot(\alpha v+\nabla v) + cv = \triangle v + 2\alpha\cdot\nabla v + |\alpha|^2 v, \\ &v_t + \big(b-2\alpha\big)\cdot\nabla v + \big(\beta + b\cdot\alpha + c - |\alpha|^2\big)v = \triangle v. \end{aligned}$$

In order to get homogeneous heat equation, we set

$$\alpha = \frac{b}{2}, \qquad \beta = -\frac{|b|^2}{4} - c,$$

which gives

$$\begin{cases} v_t = \triangle v & \text{in } \mathbb{R}^n\times(0,\infty) \\ v(x,0) = e^{-\frac{b}{2}\cdot x}f(x) & \text{on } \mathbb{R}^n. \end{cases}$$

The above PDE has the following solution:

$$v(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}}e^{-\frac{b}{2}\cdot y}f(y)\,dy.$$

Thus,

$$\boxed{u(x,t) = e^{\frac{b}{2}\cdot x-(\frac{|b|^2}{4}+c)t}v(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}}e^{\frac{b}{2}\cdot x-(\frac{|b|^2}{4}+c)t}\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}}e^{-\frac{b}{2}\cdot y}f(y)\,dy.}$$

$\square$

Problem (F'01, #7). *Consider the parabolic problem*

$$u_t = u_{xx} + c(x)u \tag{18.5}$$

for $-\infty < x < \infty$, *in which*

$$\begin{aligned} c(x) &= 0 \quad \text{for} \quad |x| > 1, \\ c(x) &= 1 \quad \text{for} \quad |x| < 1. \end{aligned}$$

Find solutions of the form $u(x,t) = e^{\lambda t}v(x)$ *in which* $\int_{-\infty}^{\infty} |u|^2\,dx < \infty$.
Hint: Look for v *to have the form*

$$\begin{aligned} v(x) &= ae^{-k|x|} \quad \text{for} \quad |x| > 1, \\ v(x) &= b\cos lx \quad \text{for} \quad |x| < 1, \end{aligned}$$

for some a, b, k, l.

Proof. Plug $u(x,t) = e^{\lambda t}v(x)$ into (18.5) to get:

$$\begin{aligned} \lambda e^{\lambda t}v(x) &= e^{\lambda t}v''(x) + ce^{\lambda t}v(x), \\ \lambda v(x) &= v''(x) + cv(x), \\ v''(x) - \lambda v(x) + cv(x) &= 0. \end{aligned}$$

- For $|x| > 1$, $c = 0$. We look for solutions of the form $v(x) = ae^{-k|x|}$.

$$\begin{aligned} v''(x) - \lambda v(x) &= 0, \\ ak^2e^{-k|x|} - a\lambda e^{-k|x|} &= 0, \\ k^2 - \lambda &= 0, \\ k^2 &= \lambda, \\ k &= \pm\sqrt{\lambda}. \end{aligned}$$

Thus, $v(x) = c_1e^{-\sqrt{\lambda}x} + c_2e^{\sqrt{\lambda}x}$. Since we want $\int_{-\infty}^{\infty} |u|^2\,dx < \infty$:

$$u(x,t) = ae^{\lambda t}e^{-\sqrt{\lambda}x}.$$

- For $|x| < 1$, $c = 1$. We look for solutions of the form $v(x) = b\cos lx$.

$$\begin{aligned} v''(x) - \lambda v(x) + v(x) &= 0, \\ -bl^2\cos lx + (1-\lambda)b\cos lx &= 0, \\ -l^2 + (1-\lambda) &= 0, \\ l^2 &= 1-\lambda, \\ l &= \pm\sqrt{1-\lambda}. \end{aligned}$$

Thus, (since $\cos(-x) = \cos x$)

$$u(x,t) = be^{\lambda t}\cos\sqrt{(1-\lambda)}x.$$

- We want $v(x)$ to be continuous on $\mathbb{R}$, and at $x = \pm 1$, in particular. Thus,

$$\begin{aligned} ae^{-\sqrt{\lambda}} &= b\cos\sqrt{(1-\lambda)}, \\ a &= be^{\sqrt{\lambda}}\cos\sqrt{(1-\lambda)}. \end{aligned}$$

- Also, $v(x)$ is symmetric:

$$\int_{-\infty}^{\infty} |u|^2\,dx = 2\int_0^{\infty} |u|^2\,dx = 2\left[\int_0^1 |u|^2\,dx + \int_1^{\infty} |u|^2\,dx\right] < \infty.$$

$\square$

Problem (F'03, #3). ❶ *The function*

$$h(X,T) = (4\pi T)^{-\frac{1}{2}} e^{-\frac{X^2}{4T}}$$

satisfies (you do not have to show this)

$$h_T = h_{XX}.$$

Using this result, verify that for any smooth function U

$$u(x,t) = e^{\frac{1}{3}t^3 - xt} \int_{-\infty}^{\infty} U(\xi)\, h(x - t^2 - \xi, t)\, d\xi$$

satisfies

$$u_t + xu = u_{xx}.$$

❷ *Given that $U(x)$ is bounded and continuous everywhere on $-\infty \le x \le \infty$, establish that*

$$\lim_{t\to 0} \int_{-\infty}^{\infty} U(\xi)\, h(x - \xi, t)\, d\xi = U(x)$$

❸ *and show that $u(x,t) \to U(x)$ as $t \to 0$. (You may use the fact that $\int_0^\infty e^{-\xi^2}\, d\xi = \frac{1}{2}\sqrt{\pi}$.)*

Proof. We change the notation: $h \to K$, $U \to g$, $\xi \to y$. We have

$$K(X,T) = \frac{1}{\sqrt{4\pi T}}\, e^{-\frac{X^2}{4T}}$$

❶ We want to verify that

$$u(x,t) = e^{\frac{1}{3}t^3 - xt} \int_{-\infty}^{\infty} K(x - y - t^2, t)\, g(y)\, dy.$$

satisfies

$$u_t + xu = u_{xx}. \qquad \circledast$$

We have

$$\begin{aligned}
u_t &= \int_{-\infty}^{\infty} \frac{d}{dt}\Big[e^{\frac{1}{3}t^3 - xt}\, K(x - y - t^2, t)\Big]\, g(y)\, dy \\
&= \int_{-\infty}^{\infty} \Big[(t^2 - x)\, e^{\frac{1}{3}t^3 - xt}\, K \,+\, e^{\frac{1}{3}t^3 - xt}\big(K_X \cdot (-2t) \,+\, K_T\big)\Big]\, g(y)\, dy, \\
xu &= \int_{-\infty}^{\infty} x\, e^{\frac{1}{3}t^3 - xt} K(x - y - t^2, t)\, g(y)\, dy, \\
u_x &= \int_{-\infty}^{\infty} \frac{d}{dx}\Big[e^{\frac{1}{3}t^3 - xt}\, K(x - y - t^2, t)\Big]\, g(y)\, dy \\
&= \int_{-\infty}^{\infty} \Big[-\, t\, e^{\frac{1}{3}t^3 - xt}\, K \,+\, e^{\frac{1}{3}t^3 - xt}\, K_X\Big]\, g(y)\, dy, \\
u_{xx} &= \int_{-\infty}^{\infty} \frac{d}{dx}\Big[-\, t\, e^{\frac{1}{3}t^3 - xt}\, K \,+\, e^{\frac{1}{3}t^3 - xt}\, K_X\Big]\, g(y)\, dy \\
&= \int_{-\infty}^{\infty} \Big[t^2\, e^{\frac{1}{3}t^3 - xt}\, K \,-\, t\, e^{\frac{1}{3}t^3 - xt}\, K_X - \, t\, e^{\frac{1}{3}t^3 - xt}\, K_X \,+\, e^{\frac{1}{3}t^3 - xt}\, K_{XX}\Big]\, g(y)\, dy.
\end{aligned}$$

Plugging these into $\circledast$, most of the terms cancel out. The remaining two terms cancel because $K_T = K_{XX}$.

❷ Given that $g(x)$ is bounded and continuous on $-\infty \le x \le \infty$, we establish that [67]

$$\lim_{t\to 0} \int_{-\infty}^{\infty} K(x-y,t)\, g(y)\, dy \;=\; g(x).$$

Fix $x_0 \in \mathbb{R}^n$, $\varepsilon > 0$. Choose $\delta > 0$ such that

$$|g(y) - g(x_0)| < \varepsilon \qquad \text{if } \; |y - x_0| < \delta,\; y \in \mathbb{R}^n.$$

Then if $|x - x_0| < \frac{\delta}{2}$, we have: $\quad (\int_{\mathbb{R}} K(x,t)\, dx = 1)$

$$\Big|\int_{\mathbb{R}} K(x-y,t)\, g(y)\, dy \;-\; g(x_0)\Big| \;\le\; \Big|\int_{\mathbb{R}} K(x-y,t)\, [g(y) - g(x_0)]\, dy\Big|$$

$$\le \underbrace{\int_{B_\delta(x_0)} K(x-y,t)\, \big|g(y) - g(x_0)\big|\, dy}_{\le\, \varepsilon \int_{\mathbb{R}} K(x-y,t)\, dy \;=\; \varepsilon} + \int_{\mathbb{R}-B_\delta(x_0)} K(x-y,t)\, \big|g(y) - g(x_0)\big|\, dy \;\circledast$$

Furthermore, if $|x - x_0| \;\le\; \frac{\delta}{2}$ and $|y - x_0| \ge \delta$, then

$$|y - x_0| \;\;\le\;\; |y - x| + \frac{\delta}{2} \;\le\; |y - x| + \frac{1}{2}|y - x_0|.$$

Thus, $|y - x| \ge \frac{1}{2}|y - x_0|$. Consequently,

$$\begin{aligned}
\circledast \;\;&=\;\; \varepsilon \,+\, 2||g||_{L^\infty} \int_{\mathbb{R}-B_\delta(x_0)} K(x-y,t)\, dy \\
&\le\;\; \varepsilon \,+\, \frac{C}{\sqrt{t}} \int_{\mathbb{R}-B_\delta(x_0)} e^{-\frac{|x-y|^2}{4t}}\, dy \\
&\le\;\; \varepsilon \,+\, \frac{C}{\sqrt{t}} \int_{\mathbb{R}-B_\delta(x_0)} e^{-\frac{|y-x_0|^2}{16t}}\, dy \\
&=\;\; \varepsilon \,+\, \frac{C}{\sqrt{t}} \int_{\delta}^{\infty} e^{-\frac{r^2}{16t}}\, r\, dr \;\;\to\;\; \varepsilon + 0 \qquad \text{as } \; t \to 0^+.
\end{aligned}$$

Hence, if $|x - x_0| < \frac{\delta}{2}$ and $t > 0$ is small enough, $|u(x,t) - g(x_0)| < 2\varepsilon$. $\square$

---

[67] Evans, p. 47, Theorem 1 (c).

Problem (S'93, #4). *The temperature $T(x,t)$ in a stationary medium, $x \geq 0$, is governed by the heat conduction equation*

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}. \tag{18.6}$$

Making the change of variable $(x,t) \to (u,t)$, where $u = x/2\sqrt{t}$, show that

$$4t\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial u^2} + 2u\frac{\partial T}{\partial u}. \tag{18.7}$$

*Solutions of (18.7) that depend on u alone are called **similarity solutions**.* [68]

Proof. We change notation: the change of variables is $(x,t) \to (u,\tau)$, where $t = \tau$. After the change of variables, we have $T = T(u(x,t),\tau(t))$.

$$\begin{array}{lll} u = \dfrac{x}{2\sqrt{t}} & \Rightarrow & u_t = -\dfrac{x}{4t^{\frac{3}{2}}}, \quad u_x = \dfrac{1}{2\sqrt{t}}, \quad u_{xx} = 0, \\ \tau = t & \Rightarrow & \tau_t = 1, \qquad \tau_x = 0. \end{array}$$

$$\begin{aligned} \frac{\partial T}{\partial t} &= \frac{\partial T}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial T}{\partial \tau}, \\ \frac{\partial T}{\partial x} &= \frac{\partial T}{\partial u}\frac{\partial u}{\partial x}, \\ \frac{\partial^2 T}{\partial x^2} &= \frac{\partial}{\partial x}\Big(\frac{\partial T}{\partial x}\Big) = \frac{\partial}{\partial x}\Big(\frac{\partial T}{\partial u}\frac{\partial u}{\partial x}\Big) = \Big(\frac{\partial^2 T}{\partial u^2}\frac{\partial u}{\partial x}\Big)\frac{\partial u}{\partial x} + \frac{\partial T}{\partial u}\underbrace{\frac{\partial^2 u}{\partial x^2}}_{=0} = \frac{\partial^2 T}{\partial u^2}\Big(\frac{\partial u}{\partial x}\Big)^2. \end{aligned}$$

Thus, (18.6) gives:

$$\begin{aligned} \frac{\partial T}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial T}{\partial \tau} &= \frac{\partial^2 T}{\partial u^2}\Big(\frac{\partial u}{\partial x}\Big)^2, \\ \frac{\partial T}{\partial u}\Big(-\frac{x}{4t^{\frac{3}{2}}}\Big) + \frac{\partial T}{\partial \tau} &= \frac{\partial^2 T}{\partial u^2}\Big(\frac{1}{2\sqrt{t}}\Big)^2, \\ \frac{\partial T}{\partial \tau} &= \frac{1}{4t}\frac{\partial^2 T}{\partial u^2} + \frac{x}{4t^{\frac{3}{2}}}\frac{\partial T}{\partial u}, \\ 4t\frac{\partial T}{\partial \tau} &= \frac{\partial^2 T}{\partial u^2} + \frac{x}{\sqrt{t}}\frac{\partial T}{\partial u}, \\ 4t\frac{\partial T}{\partial \tau} &= \frac{\partial^2 T}{\partial u^2} + 2u\frac{\partial T}{\partial u}. \quad \checkmark \end{aligned}$$

$\square$

[68] This is only the part of the qual problem.

19 Contraction Mapping and Uniqueness - Wave

Recall that the solution to

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t), \\ u(x,0) = g(x), \quad u_t(x,0) = h(x), \end{cases} \tag{19.1}$$

is given by adding together d'Alembert's formula and Duhamel's principle:

$$\boxed{u(x,t) = \frac{1}{2}(g(x+ct) + g(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} h(\xi)\,d\xi + \frac{1}{2c}\int_0^t \left(\int_{x-c(t-s)}^{x+c(t-s)} f(\xi,s)\,d\xi\right) ds.}$$

Problem (W'02, #8). ***a)*** *Find an explicit solution of the following Cauchy problem*

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = f(t,x), \\ u(0,x) = 0, \quad \frac{\partial u}{\partial x}(0,x) = 0. \end{cases} \tag{19.2}$$

b) *Use part (a) to prove the* ***uniqueness*** *of the solution of the Cauchy problem*

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + q(t,x)u = 0, \\ u(0,x) = 0, \quad \frac{\partial u}{\partial x}(0,x) = 0. \end{cases} \tag{19.3}$$

Here $f(t,x)$ and $q(t,x)$ are continuous functions.

Proof. **a)** It was probably meant to give the u_t initially. We rewrite (19.2) as

$$\begin{cases} u_{tt} - u_{xx} = f(x,t), \\ u(x,0) = 0, \quad u_t(x,0) = 0. \end{cases} \tag{19.4}$$

Duhamel's principle, with $c = 1$, gives the solution to (19.4):

$$u(x,t) = \frac{1}{2c}\int_0^t \left(\int_{x-c(t-s)}^{x+c(t-s)} f(\xi,s)\,d\xi\right) ds = \frac{1}{2}\int_0^t \left(\int_{x-(t-s)}^{x+(t-s)} f(\xi,s)\,d\xi\right) ds.$$

b) We use the Contraction Mapping Principle to prove uniqueness.
Define the operator

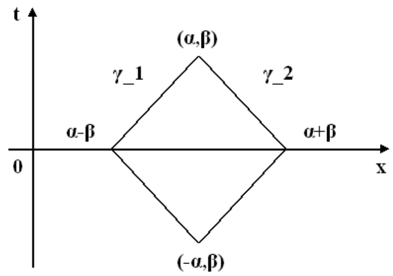
$$T(u) \ = \ \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} -q(\xi,s)\,u(\xi,s)\,d\xi\,ds.$$

on the Banach space $C^{2,2}$, $||\cdot||_\infty$.
We will show $|Tu_n - Tu_{n+1}| < \alpha||u_n - u_{n+1}||$ where $\alpha < 1$. Then $\{u_n\}_{n=1}^\infty$: $u_{n+1} = T(u_n)$ converges to a unique fixed point which is the unique solution of PDE.

$$\begin{aligned} |Tu_n - Tu_{n+1}| \ &= \ \left|\frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} -q(\xi,s)\big(u_n(\xi,s) - u_{n+1}(\xi,s)\big)\,d\xi\,ds\right| \\ &\le \ \frac{1}{2}\int_0^t ||q||_\infty ||u_n - u_{n+1}||_\infty\, 2(t-s)\,ds \\ &\le \ t^2||q||_\infty||u_n - u_{n+1}||_\infty \ \le \ \alpha||u_n - u_{n+1}||_\infty, \qquad \text{for small } t. \end{aligned}$$

Thus, T is a contraction $\Rightarrow$ $\exists$ a **unique** fixed point.
Since $Tu = u$, u is the solution to the PDE. $\square$

Problem (F'00, #3). *Consider the Goursat problem:*



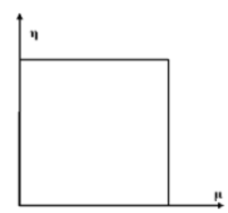
Find the solution of the equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + a(x,t)u = 0$$

in the square $\mathcal{D}$*, satisfying the boundary conditions*

$$u|_{\gamma_1} = \varphi, \quad u|_{\gamma_2} = \psi,$$

where γ_1*,* γ_2 *are two adjacent sides* $\mathcal{D}$*. Here* $a(x,t)$*,* φ *and* ψ *are continuous functions. Prove the* ***uniqueness*** *of the solution of this Goursat problem.*



Proof. The change of variable $\mu = x+t, \quad \eta = x-t$ transforms the equation to

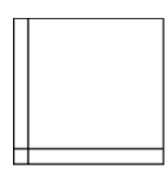
$$\tilde{u}_{\mu\eta} + \tilde{a}(\mu,\eta)\tilde{u} = 0.$$

We integrate the equation:

$$\begin{aligned}
\int_0^\eta \int_0^\mu \tilde{u}_{\mu\eta}(u,v)\,du\,dv &= -\int_0^\eta \int_0^\mu \tilde{a}(\mu,\eta)\,\tilde{u}\,du\,dv, \\
\int_0^\eta \big(\tilde{u}_\eta(\mu,v) - \tilde{u}_\eta(0,v)\big)\,dv &= -\int_0^\eta \int_0^\mu \tilde{a}(\mu,\eta)\,\tilde{u}\,du\,dv, \\
\tilde{u}(\mu,\eta) = \tilde{u}(\mu,0) + \tilde{u}(0,\eta) - u(0,0) &- \int_0^\eta \int_0^\mu \tilde{a}(\mu,\eta)\,\tilde{u}\,du\,dv.
\end{aligned}$$

We change the notation. In the new notation:

$$\begin{aligned}
f(x,y) &= \varphi(x,y) - \int_0^x \int_0^y a(u,v) f(u,v)\,du\,dv, \\
f &= \varphi + Kf, \\
f &= \varphi + K(\varphi + Kf), \\
&\dots \\
f &= \varphi + \sum_{n=1}^\infty K^n \varphi, \\
f &= Kf \qquad \Rightarrow \ f = 0, \\
\max_{0<x<\delta} |f| &\le \delta \max|a| \max|f|.
\end{aligned}$$



For small enough δ, the operator K is a contraction. Thus, there exists a unique fixed point of K, and $f = Kf$, where f is the unique solution. $\square$

20 Contraction Mapping and Uniqueness - Heat

The solution of the initial value problem

$$\begin{cases} u_t = \triangle u + f(x,t) & \text{for } \ t>0, \ x \in \mathbb{R}^n \\ u(x,0) = g(x) & \text{for } \ x \in \mathbb{R}^n. \end{cases} \tag{20.1}$$

is given by

$$\boxed{u(x,t) = \int_{\mathbb{R}^n} \tilde{K}(x-y,t)\, g(y)\, dy + \int_0^t \int_{\mathbb{R}^n} \tilde{K}(x-y,t-s)\, f(y,s)\, dy\, ds}$$

where

$$\tilde{K}(x,t) = \begin{cases} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} & \text{for } \ t>0, \\ 0 & \text{for } \ t \le 0. \end{cases}$$

Problem (F'00, #2). *Consider the Cauchy problem*

$$\begin{aligned} u_t - \triangle u + u^2(x,t) &= f(x,t), \qquad x \in \mathbb{R}^N, \quad 0<t<T \\ u(x,0) &= 0. \end{aligned}$$

Prove the ***uniqueness*** *of the* ***classical bounded solution*** *assuming that T is small enough.*

Proof. Let $\{u_n\}$ be a sequence of approximations to the solution, such that

$$S(u_n) \ = \ u_{n+1} \underbrace{=}_{use\ Duhamel's\ principle} \int_0^t \int_{\mathbb{R}^n} K(x-y,t-s)\big(f(y,s) - u_n^2(y,s)\big)\, dy\, ds.$$

We will show that S has a fixed point $\big(\ |S(u_n) - S(u_{n+1})| \le \alpha |u_n - u_{n+1}|, \ \ \alpha < 1 \ \big)$
$\Leftrightarrow \quad \{u_n\}$ converges to a uniques solution for small enough T.
Since u_n, $u_{n+1} \in C^2(\mathbb{R}^n) \cap C^1(t) \quad \Rightarrow \quad |u_{n+1} + u_n| \le M$.

$$\begin{aligned} |S(u_n) - S(u_{n+1})| &\le \int_0^t \int_{\mathbb{R}^n} \big|K(x-y,t-s)\big| \big|u_{n+1}^2 - u_n^2\big|\, dy\, ds \\ &= \int_0^t \int_{\mathbb{R}^n} \big|K(x-y,t-s)\big| \big|u_{n+1} - u_n\big| \big|u_{n+1} + u_n\big|\, dy\, ds \\ &\le M \int_0^t \int_{\mathbb{R}^n} \big|K(x-y,t-s)\big| \big|u_{n+1} - u_n\big|\, dy\, ds \\ &\le M M_1 \int_0^t \big|u_{n+1}(x,s) - u_n(x,s)\big|\, ds \\ &\le M M_1 T\, ||u_{n+1} - u_n||_\infty \ < \ ||u_{n+1} - u_n||_\infty \qquad \text{for small } \ T. \end{aligned}$$

Thus, S is a contraction $\quad \Rightarrow \quad \exists$ a **unique** fixed point $u \in C^2(\mathbb{R}^n) \cap C^1(t)$ such that $u = \lim_{n\to\infty} u_n$. $\ u$ is implicitly defined as

$$u(x,t) \ = \ \int_0^t \int_{\mathbb{R}^n} K(x-y,t-s)\big(f(y,s) - u^2(y,s)\big)\, dy\, ds.$$

$\square$

Problem (S'97, #3). ***a)*** *Let $Q(x) \geq 0$ such that $\int_{x=-\infty}^{\infty} Q(x)\,dx = 1$, and define $Q_\epsilon = \frac{1}{\epsilon}Q(\frac{x}{\epsilon})$. Show that (here $*$ denotes convolution)*

$$||Q_\epsilon(x) * w(x)||_{L^\infty} \leq ||w(x)||_{L^\infty}.$$

In particular, let $Q_t(x)$ denote the heat kernel (at time t), then

$$||Q_t(x) * w_1(x) - Q_t(x) * w_2(x)||_{L^\infty} \leq ||w_1(x) - w_2(x)||_{L^\infty}.$$

b) *Consider the parabolic equation $u_t = u_{xx} + u^2$ subject to initial conditions $u(x,0) = f(x)$. Show that the solution of this equation satisfies*

$$u(x,t) = Q_t(x) * f(x) + \int_0^t Q_{t-s}(x) * u^2(x,s)\,ds. \tag{20.2}$$

c) *Fix $t > 0$. Let $\{u_n(x,t)\}$, $n = 1, 2, \ldots$ the fixed point iterations for the solution of (20.2)*

$$u_{n+1}(x,t) = Q_t(x) * f(x) + \int_0^t Q_{t-s}(x) * u_n^2(x,s)\,ds. \tag{20.3}$$

Let $K_n(t) = \sup_{0\leq m\leq n} ||u_m(x,t)||_{L^\infty}$. Using (a) and (b) show that

$$||u_{n+1}(x,t) - u_n(x,t)||_{L^\infty} \leq 2 \sup_{0\leq\tau\leq t} K_n(\tau) \cdot \int_0^t ||u_n(x,s) - u_{n-1}(x,s)||_{L^\infty}\,ds.$$

Conclude that the fixed point iterations in (20.3) converge if t is sufficiently small.

Proof. **a)** We have

$$\begin{aligned}
||Q_\epsilon(x) * w(x)||_{L^\infty} &= \left| \int_{-\infty}^{\infty} Q_\epsilon(x-y)w(y)\,dy \right| \leq \int_{-\infty}^{\infty} \big|Q_\epsilon(x-y)w(y)\big|\,dy \\
&\leq ||w||_\infty \int_{-\infty}^{\infty} \big|Q_\epsilon(x-y)\big|\,dy = ||w||_\infty \int_{-\infty}^{\infty} \frac{1}{\epsilon}Q\Big(\frac{x-y}{\epsilon}\Big)\,dy \\
&= ||w||_\infty \int_{-\infty}^{\infty} \frac{1}{\epsilon}Q\Big(\frac{y}{\epsilon}\Big)\,dy \qquad \Big(z = \frac{y}{\epsilon},\ dz = \frac{dy}{\epsilon}\Big) \\
&= ||w||_\infty \int_{-\infty}^{\infty} Q(z)\,dz = ||w(x)||_\infty. \quad \checkmark
\end{aligned}$$

$Q_t(x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$, the heat kernel. We have [69]

$$\begin{aligned}
||Q_t(x) * w_1(x) - Q_t(x) * w_2(x)||_{L^\infty} &= \Big|\Big| \int_{-\infty}^{\infty} Q_t(x-y) w_1(y)\,dy - \int_{-\infty}^{\infty} Q_t(x-y) w_2(y)\,dy \Big|\Big|_\infty \\
&= \frac{1}{\sqrt{4\pi t}} \Big|\Big| \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} w_1(y)\,dy - \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} w_2(y)\,dy \Big|\Big|_\infty \\
&\le \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} \big| w_1(y) - w_2(y) \big|\,dy \\
&\le \big|\big| w_1(y) - w_2(y) \big|\big|_\infty \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}}\,dy \\
z = \frac{x-y}{\sqrt{4t}},\ dz = \frac{-dy}{\sqrt{4t}} \qquad &= \big|\big| w_1(y) - w_2(y) \big|\big|_\infty \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-z^2} \sqrt{4t}\,dz \\
&= \big|\big| w_1(y) - w_2(y) \big|\big|_\infty \frac{1}{\sqrt{\pi}} \underbrace{\int_{-\infty}^{\infty} e^{-z^2}\,dz}_{\sqrt{\pi}} \\
&= \big|\big| w_1(y) - w_2(y) \big|\big|_\infty. \quad \checkmark
\end{aligned}$$

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[69]Note:

$$\int_{-\infty}^{\infty} Q_t(x)\,dx \;=\; \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}}\,dy \;=\; \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-z^2} \sqrt{4t}\,dz \;=\; \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2}\,dz \;=\; 1.$$

b) Consider

$$\begin{cases} u_t = u_{xx} + u^2, \\ u(x,0) = f(x). \end{cases}$$

We will show that the solution of this equation satisfies

$$u(x,t) \;=\; Q_t(x) * f(x) + \int_0^t Q_{t-s}(x) * u^2(x,s)\,ds.$$

$$\begin{aligned}
\int_0^t Q_{t-s}(x) * u^2(x,s)\,ds \;&=\; \int_0^t \int_{\mathbb{R}} Q_{t-s}(x-y)\,u^2(y,s)\,dy\,ds \\
&= \int_0^t \int_{\mathbb{R}} Q_{t-s}(x-y)\,\Big(u_s(y,s) - u_{yy}(y,s)\Big)\,dy\,ds \\
&= \int_0^t \int_{\mathbb{R}} \frac{d}{ds}\big(Q_{t-s}(x-y)u(y,s)\big) - \frac{d}{ds}\big(Q_{t-s}(x-y)\big)u(y,s) - Q_{t-s}(x-y)u_{yy}(y,s)\,dy\,ds \\
&= \Big[\int_{\mathbb{R}} Q_0(x-y)u(y,t)\,dy - \int_{\mathbb{R}} Q_t(x-y)u(y,0)\,dy\Big] \\
&\quad - \int_0^t \int_{\mathbb{R}} \underbrace{\frac{d}{ds}\big(Q_{t-s}(x-y)\big)u(y,s) + \frac{d^2}{dy^2}Q_{t-s}(x-y)u(y,s)}_{=0,\quad since\ Q_t\ satisfies\ heat\,equation}\,dy\,ds \\
&= u(x,t) - \int_{\mathbb{R}} Q_t(x-y)f(y)\,dy \\
&= u(x,t) - Q_t(x) * f(x). \quad \checkmark
\end{aligned}$$

Note: $\lim_{t\to 0^+} Q(x,t) = \delta_0(x) = \delta(x)$.
$\lim_{t\to 0^+} \int_{\mathbb{R}} Q(x-y,t)v(y)\,dy = v(0)$.

Note that we used: $D^\alpha(f*g) \;=\; (D^\alpha f)*g \;=\; f*(D^\alpha g)$.

c) Let

$$u_{n+1}(x,t) \;=\; Q_t(x) * f(x) + \int_0^t Q_{t-s}(x) * u_n^2(x,s)\,ds.$$

$$\begin{aligned}
||u_{n+1}(x,t) - u_n(x,t)||_{L^\infty} \quad &= \quad \Big|\Big|\int_0^t Q_{t-s}(x) * \big(u_n^2(x,s) - u_{n-1}^2(x,s)\big)\,ds\Big|\Big|_\infty \\
&\le \quad \int_0^t \big|\big|Q_{t-s}(x) * \big(u_n^2(x,s) - u_{n-1}^2(x,s)\big)\big|\big|_\infty\,ds \\
&\underbrace{\le}_{(a)} \quad \int_0^t \big|\big|u_n^2(x,s) - u_{n-1}^2(x,s)\big|\big|_\infty\,ds \\
&\le \quad \int_0^t \big|\big|u_n(x,s) - u_{n-1}(x,s)\big|\big|_\infty \big|\big|u_n(x,s) + u_{n-1}(x,s)\big|\big|_\infty\,ds \\
&\le \quad \sup_{0\le\tau\le t} \big|\big|u_n(x,s) + u_{n-1}(x,s)\big|\big|_\infty \int_0^t \big|\big|u_n(x,s) - u_{n-1}(x,s)\big|\big|_\infty\,ds \\
&\le \quad 2\sup_{0\le\tau\le t} K_n(\tau)\cdot \int_0^t ||u_n(x,s) - u_{n-1}(x,s)||_{L^\infty}\,ds. \quad \checkmark
\end{aligned}$$

Also, $\quad ||u_{n+1}(x,t) - u_n(x,t)||_{L^\infty} \;\le\; 2t\sup_{0\le\tau\le t} K_n(\tau)\cdot ||u_n(x,s) - u_{n-1}(x,s)||_{L^\infty}.$

For t small enough, $2t\sup_{0\le\tau\le t} K_n(\tau) \le \alpha < 1$. Thus, T defined as

$$Tu = Q_t(x) * f(x) + \int_0^t Q_{t-s}(x) * u^2(x,s)\,ds$$

is a contraction, and has a **unique** fixed point $u = Tu$. $\square$

Problem (S'99, #3). *Consider the system of equations*

$$u_t = u_{xx} + f(u,v)$$
$$v_t = 2v_{xx} + g(u,v)$$

to be solved for $t > 0$, $-\infty < x < \infty$, *and smooth initial data with compact support:*

$$u(x,0) = u_0(x), \qquad v(x,0) = v_0(x).$$

If f and g are uniformly Lipschitz continuous, give a proof of ***existence*** *and* ***uniqueness*** *of the solution to this problem in the space of bounded continuous functions with* $||u(\cdot,t)|| = \sup_x |u(x,t)|$.

Proof. The space of continuous bounded functions forms a complete metric space so the contraction mapping principle applies.
First, let $v(x,t) = w\big(\frac{x}{\sqrt{2}}, t\big)$, then

$$u_t = u_{xx} + f(u,w)$$
$$w_t = w_{xx} + g(u,w).$$

These initial value problems have the following solutions (K is the heat kernel):

$$\begin{aligned}
u(x,t) &= \int_{\mathbb{R}^n} \tilde{K}(x-y,t)\, u_0(y)\, dy + \int_0^t \int_{\mathbb{R}^n} \tilde{K}(x-y,t-s)\, f(u,w)\, dy\, ds, \\
w(x,t) &= \int_{\mathbb{R}^n} \tilde{K}(x-y,t)\, w_0(y)\, dy + \int_0^t \int_{\mathbb{R}^n} \tilde{K}(x-y,t-s)\, g(u,w)\, dy\, ds.
\end{aligned}$$

By the Lipshitz conditions,

$$|f(u,w)| \le M_1 ||u||,$$
$$|g(u,w)| \le M_2 ||w||.$$

Now we can show the mappings, as defined below, are contractions:

$$\begin{aligned}
T_1 u &= \int_{\mathbb{R}^n} \tilde{K}(x-y,t)\, u_0(y)\, dy + \int_0^t \int_{\mathbb{R}^n} \tilde{K}(x-y,t-s)\, f(u,w)\, dy\, ds, \\
T_2 w &= \int_{\mathbb{R}^n} \tilde{K}(x-y,t)\, w_0(y)\, dy + \int_0^t \int_{\mathbb{R}^n} \tilde{K}(x-y,t-s)\, g(u,w)\, dy\, ds.
\end{aligned}$$

$$\begin{aligned}
|T_1(u_n) - T_1(u_{n+1})| &\le \int_0^t \int_{\mathbb{R}^n} \big|\tilde{K}(x-y,t-s)\big|\, \big|f(u_n,w) - f(u_{n+1},w)\big|\, dy\, ds \\
&\le M_1 \int_0^t \int_{\mathbb{R}^n} \big|\tilde{K}(x-y,t-s)\big|\, \big|u_n - u_{n+1}\big|\, dy\, ds \\
&\le M_1 \int_0^t \sup_x \big|u_n - u_{n+1}\big| \int_{\mathbb{R}^n} \tilde{K}(x-y,t-s) dy\, ds \\
&\le M_1 \int_0^t \sup_x \big|u_n - u_{n+1}\big|\, ds \ \le\ M_1 t \sup_x \big|u_n - u_{n+1}\big| \\
&< \sup_x \big|u_n - u_{n+1}\big| \qquad \text{for small } t.
\end{aligned}$$

We used the Lipshitz condition and $\int_{\mathbb{R}} \tilde{K}(x-y,t-s)\, dy = 1$.
Thus, for small t, T_1 is a contraction, and has a unique fixed point. Thus, the solution is defined as $u = T_1 u$.
Similarly, T_2 is a contraction and has a unique fixed point. The solution is defined as $w = T_2 w$. $\square$

21 Problems: Maximum Principle - Laplace and Heat

21.1 Heat Equation - Maximum Principle and Uniqueness

Let us introduce the "cylinder" $U = U_T = \Omega \times (0, T)$. We know that harmonic (and subharmonic) functions achieve their maximum on the boundary of the domain. For the heat equation, the result is improved in that the maximum is achieved on a certain part of the boundary, *parabolic boundary*:

$$\Gamma = \{(x, t) \in \overline{U} : x \in \partial\Omega \text{ or } t = 0\}.$$

Let us also denote by $C^{2;1}(U)$ functions satisfying $u_t,\ u_{x_i x_j} \in C(U)$.

Weak Maximum Principle. *Let $u \in C^{2;1}(U) \cap C(\overline{U})$ satisfy $\triangle u \geq u_t$ in U. Then u achieves its maximum on the parabolic boundary of U:*

$$\max_{\overline{U}} u(x, t) = \max_{\Gamma} u(x, t). \tag{21.1}$$

Proof. • First, assume $\triangle u > u_t$ in U. For $0 < \tau < T$ consider

$$U_\tau = \Omega \times (0, \tau), \qquad \Gamma_\tau = \{(x, t) \in \overline{U}_\tau : x \in \partial\Omega \quad \text{or} \quad t = 0\}.$$

If the maximum of u on $\overline{U}_\tau$ occurs at $x \in \Omega$ and $t = \tau$, then $u_t(x, \tau) \geq 0$ and $\triangle u(x, \tau) \leq 0$, violating our assumption; similarly, u cannot attain an interior maximum on U_τ. Hence (21.1) holds for U_τ: $\max_{\overline{U}_\tau} u = \max_{\Gamma_\tau} u$. But $\max_{\Gamma_\tau} u \leq \max_\Gamma u$ and by continuity of u, $\max_{\overline{U}} u = \lim_{\tau \to T} \max_{\overline{U}_\tau} u$. This establishes (21.1).
• Second, we consider the general case of $\triangle u \geq u_t$ in U. Let $u = v + \varepsilon t$ for $\varepsilon > 0$. Notice that $v \leq u$ on $\overline{U}$ and $\triangle v - v_t > 0$ in U. Thus we may apply (21.1) to v:

$$\max_{\overline{U}} u = \max_{\overline{U}} (v + \varepsilon t) \leq \max_{\overline{U}} v + \varepsilon T = \max_{\Gamma} v + \varepsilon T \leq \max_{\Gamma} u + \varepsilon T.$$

Letting $\varepsilon \to 0$ establishes (21.1) for u. $\square$

Problem (S'98, #7). *Prove that any smooth solution, $u(x, y, t)$ in the unit box $\Omega = \{(x, y) \mid -1 \le x, y \le 1\}$, of the following equation*

$$\begin{aligned} u_t &= uu_x + uu_y + \triangle u, && t \ge 0,\ (x, y) \in \Omega \\ u(x, y, 0) &= f(x, y), && (x, y) \in \Omega \end{aligned}$$

satisfies the ***weak maximum principle****,*

$$\max_{\Omega\times[0,T]} u(x, y, t) \le \max\{\max_{0\le t\le T} u(\pm 1, \pm 1, t), \max_{(x,y)\in\Omega} f(x, y)\}.$$

Proof. Suppose u satisfies given equation. Let $u = v + \varepsilon t$ for $\varepsilon > 0$. Then,

$$v_t + \varepsilon = vv_x + vv_y + \varepsilon t(v_x + v_y) + \triangle v.$$

Suppose v has a maximum at $(x_0, y_0, t_0) \in \Omega \times (0, T)$. Then

$$v_x = v_y = v_t = 0 \quad \Rightarrow \quad \varepsilon = \triangle v \quad \Rightarrow \quad \triangle v > 0$$

$\Rightarrow$ v has a minimum at (x_0, y_0, t_0), a contradiction.
Thus, the maximum of v is on the boundary of $\Omega \times (0, T)$.

Suppose v has a maximum at (x_0, y_0, T), $(x_0, y_0) \in \Omega$. Then

$$v_x = v_y = 0,\ v_t \ge 0 \quad \Rightarrow \quad \varepsilon \le \triangle v \quad \Rightarrow \quad \triangle v > 0$$

$\Rightarrow$ v has a minimum at (x_0, y_0, T), a contradiction. Thus,

$$\max_{\Omega\times[0,T]} v \le \max\{\max_{0\le t\le T} v(\pm 1, \pm 1, t), \max_{(x,y)\in\Omega} f(x, y)\}.$$

Now

$$\begin{aligned} \max_{\Omega\times[0,T]} u = \max_{\Omega\times[0,T]} (v + \varepsilon t) \le \max_{\Omega\times[0,T]} v + \varepsilon T &\le \max\{\max_{0\le t\le T} v(\pm 1, \pm 1, t), \max_{(x,y)\in\Omega} f(x, y)\} + \varepsilon T \\ &\le \max\{\max_{0\le t\le T} u(\pm 1, \pm 1, t), \max_{(x,y)\in\Omega} f(x, y)\} + \varepsilon T. \end{aligned}$$

Letting $\varepsilon \to 0$ establishes the result. $\square$

21.2 Laplace Equation - Maximum Principle

Problem (S'91, #6). *Suppose that u satisfies*

$$Lu = au_{xx} + bu_{yy} + cu_x + du_y - eu = 0$$

with $a > 0$, $b > 0$, $e > 0$, for $(x, y) \in \Omega$, with Ω a bounded open set in $\mathbb{R}^2$.

a) *Show that u cannot have a positive maximum or a negative minimum in the interior of Ω.*
b) *Use this to show that the only function u satisfying $Lu = 0$ in Ω, $u = 0$ on $\partial\Omega$ and u continuous on $\overline{\Omega}$ is $u = 0$.*

Proof. **a)** For an interior (local) maximum or minimum at an interior point (x, y), we have

$$u_x = 0, \quad u_y = 0.$$

- Suppose u has a positive maximum in the interior of Ω. Then

$$u > 0, \quad u_{xx} \le 0, \quad u_{yy} \le 0.$$

With these values, we have

$$\underbrace{au_{xx}}_{\le 0} + \underbrace{bu_{yy}}_{\le 0} + \underbrace{cu_x}_{=0} + \underbrace{du_y}_{=0} \underbrace{-eu}_{<0} = 0,$$

which leads to contradiction. Thus, u can not have a positive maximum in Ω.

- Suppose u has a negative minimum in the interior of Ω. Then

$$u < 0, \quad u_{xx} \ge 0, \quad u_{yy} \ge 0.$$

With these values, we have

$$\underbrace{au_{xx}}_{\ge 0} + \underbrace{bu_{yy}}_{\ge 0} + \underbrace{cu_x}_{=0} + \underbrace{du_y}_{=0} \underbrace{-eu}_{>0} = 0,$$

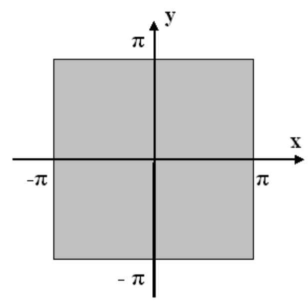
which leads to contradiction. Thus, u can not have a negative minimum in Ω.

b) Since u can not have positive maximum in the interior of Ω, then $\max u = 0$ on $\overline{\Omega}$.
Since u can not have negative minimum in the interior of Ω, then $\min u = 0$ on $\overline{\Omega}$.
Since u is continuous, $u \equiv 0$ on $\overline{\Omega}$. □

22 Problems: Separation of Variables - Laplace Equation

Problem 1: The 2D LAPLACE Equation on a Square.
Let $\Omega = (0,\pi) \times (0,\pi)$*, and use separation of variables to solve the boundary value problem*

$$\begin{cases} u_{xx} + u_{yy} = 0 & 0 < x, y < \pi \\ u(0,y) = 0 = u(\pi,y) & 0 \le y \le \pi \\ u(x,0) = 0, \quad u(x,\pi) = g(x) & 0 \le x \le \pi, \end{cases}$$



where g *is a continuous function satisfying* $g(0) = 0 = g(\pi)$*.*

Proof. Assume $u(x,y) = X(x)Y(y)$, then substitution in the PDE gives $X''Y + XY'' = 0$.

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda.$$

• From $X'' + \lambda X = 0$, we get $X_n(x) = a_n \cos nx + b_n \sin nx$. Boundary conditions give

$$\begin{cases} u(0,y) = X(0)Y(y) = 0 \\ u(\pi,y) = X(\pi)Y(y) = 0 \end{cases} \Rightarrow \quad X(0) = 0 = X(\pi).$$

Thus, $X_n(0) = a_n = 0$, and

$$X_n(x) = b_n \sin nx, \quad n = 1, 2, \ldots. \quad \checkmark$$
$$-n^2 b_n \sin nx + \lambda b_n \sin nx = 0,$$
$$\lambda_n = n^2, \quad n = 1, 2, \ldots. \quad \checkmark$$

• With these values of λ_n we solve $Y'' - n^2 Y = 0$ to find $Y_n(y) = c_n \cosh ny + d_n \sinh ny$.
Boundary conditions give

$$u(x,0) = X(x)Y(0) = 0 \quad \Rightarrow \quad Y(0) = 0 = c_n.$$
$$Y_n(x) = d_n \sinh ny. \quad \checkmark$$

• By superposition, we write

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} \tilde{a}_n \sin nx \sinh ny,}$$

which satifies the equation and the three homogeneous boundary conditions. The boundary condition at $y = \pi$ gives

$$u(x,\pi) = g(x) = \sum_{n=1}^{\infty} \tilde{a}_n \sin nx \sinh n\pi,$$
$$\int_0^\pi g(x) \sin mx \, dx = \sum_{n=1}^{\infty} \tilde{a}_n \sinh n\pi \int_0^\pi \sin nx \sin mx \, dx = \frac{\pi}{2} \tilde{a}_m \sinh m\pi.$$

$$\boxed{\tilde{a}_n \sinh n\pi = \frac{2}{\pi} \int_0^\pi g(x) \sin nx \, dx.}$$

$\square$

Problem 2: The 2D LAPLACE Equation on a Square. *Let* $\Omega = (0, \pi) \times (0, \pi)$, *and use separation of variables to solve the mixed boundary value problem*

$$\begin{cases} \triangle u = 0 & \text{in } \Omega \\ u_x(0, y) = 0 = u_x(\pi, y) & 0 < y < \pi \\ u(x, 0) = 0, \quad u(x, \pi) = g(x) & 0 < x < \pi. \end{cases}$$

Proof. Assume $u(x, y) = X(x)Y(y)$, then substitution in the PDE gives $X''Y + XY'' = 0$.

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda.$$

• Consider $X'' + \lambda X = 0$.
If $\lambda = 0$, $X_0(x) = a_0 x + b_0$.
If $\lambda > 0$, $X_n(x) = a_n \cos nx + b_n \sin nx$.
Boundary conditions give

$$\begin{cases} u_x(0, y) = X'(0)Y(y) = 0 \\ u_x(\pi, y) = X'(\pi)Y(y) = 0 \end{cases} \Rightarrow \quad X'(0) = 0 = X'(\pi).$$

Thus, $X_0'(0) = a_0 = 0$, and $X_n'(0) = nb_n = 0$.

$$X_0(x) = b_0, \quad X_n(x) = a_n \cos nx, \quad n = 1, 2, \ldots. \quad \checkmark$$
$$-n^2 a_n \cos nx + \lambda a_n \cos nx = 0,$$
$$\lambda_n = n^2, \quad n = 0, 1, 2, \ldots. \quad \checkmark$$

• With these values of λ_n we solve $Y'' - n^2 Y = 0$.
If $n = 0$, $Y_0(y) = c_0 y + d_0$.
If $n \neq 0$, $Y_n(y) = c_n \cosh ny + d_n \sinh ny$.
Boundary conditions give

$$u(x, 0) = X(x)Y(0) = 0 \quad \Rightarrow \quad Y(0) = 0.$$

Thus, $Y_0(0) = d_0 = 0$, and $Y_n(0) = c_n = 0$.

$$Y_0(y) = c_0 y, \quad Y_n(y) = d_n \sinh ny, \quad n = 1, 2, \ldots. \quad \checkmark$$

• We have

$$u_0(x, y) = X_0(x)Y_0(y) = b_0 c_0 y = \tilde{a}_0 y,$$
$$u_n(x, y) = X_n(x)Y_n(y) = (a_n \cos nx)(d_n \sinh ny) = \tilde{a}_n \cos nx \sinh ny.$$

By superposition, we write

$$\boxed{u(x, y) = \tilde{a}_0 y + \sum_{n=1}^{\infty} \tilde{a}_n \cos nx \sinh ny,}$$

which satifies the equation and the three homogeneous boundary conditions. The fourth boundary condition gives

$$u(x, \pi) = g(x) = \tilde{a}_0 \pi + \sum_{n=1}^{\infty} \tilde{a}_n \cos nx \sinh n\pi,$$

$$\begin{cases} \int_0^\pi g(x)\,dx = \int_0^\pi \left(\tilde{a}_0\pi + \sum_{n=1}^\infty \tilde{a}_n \cos nx \sinh n\pi\right) dx = \tilde{a}_0\pi^2, \\ \int_0^\pi g(x)\cos mx\,dx = \sum_{n=1}^\infty \tilde{a}_n \sinh n\pi \int_0^\pi \cos nx \cos mx\,dx = \frac{\pi}{2}\,\tilde{a}_m \sinh m\pi. \end{cases}$$

$$\boxed{\tilde{a}_0 = \frac{1}{\pi^2}\int_0^\pi g(x)\,dx,}$$

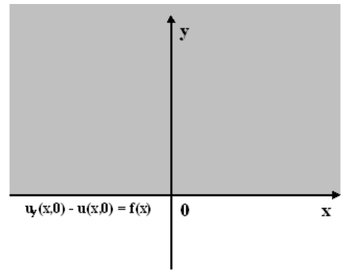
$$\boxed{\tilde{a}_n \sinh n\pi = \frac{2}{\pi}\int_0^\pi g(x)\cos nx\,dx.}$$

$\square$

Problem (W'04, #5) The 2D LAPLACE Equation in an Upper-Half Plane.
Consider the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad y > 0, \ -\infty < x < +\infty$$
$$\frac{\partial u(x,0)}{\partial y} - u(x,0) = f(x),$$

where $f(x) \in C_0^\infty(\mathbb{R}^1)$.



Find a bounded solution $u(x,y)$ *and show that* $u(x,y) \to 0$ *when* $|x| + y \to \infty$.

Proof. Assume $u(x,y) = X(x)Y(y)$, then substitution in the PDE gives $X''Y + XY'' = 0$.

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda. \qquad \circledast$$

- Consider $X'' + \lambda X = 0$.

If $\lambda = 0$, $X_0(x) = a_0 x + b_0$.
If $\lambda > 0$, $X_n(x) = a_n \cos\sqrt{\lambda_n}x + b_n \sin\sqrt{\lambda_n}x$.
Since we look for bounded solutions as $|x| \to \infty$, we have $a_0 = 0$.

- Consider $Y'' - \lambda_n Y = 0$.

If $\lambda_n = 0$, $Y_0(y) = c_0 y + d_0$.
If $\lambda_n > 0$, $Y_n(y) = c_n e^{-\sqrt{\lambda_n}y} + d_n e^{\sqrt{\lambda_n}y}$.
Since we look for bounded solutions as $y \to \infty$, we have $c_0 = 0$, $d_n = 0$. Thus,

$$u(x,y) = \tilde{a}_0 + \sum_{n=1}^{\infty} e^{-\sqrt{\lambda_n}y}\big(\tilde{a}_n \cos\sqrt{\lambda_n}x + \tilde{b}_n \sin\sqrt{\lambda_n}x\big).$$

Initial condition gives:

$$f(x) = u_y(x,0) - u(x,0) = -\tilde{a}_0 - \sum_{n=1}^{\infty}(\sqrt{\lambda_n}+1)\big(\tilde{a}_n \cos\sqrt{\lambda_n}x + \tilde{b}_n \sin\sqrt{\lambda_n}x\big).$$

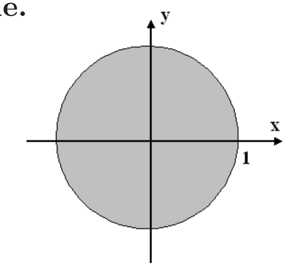
$f(x) \in C_0^\infty(\mathbb{R}^1)$, i.e. has compact support $[-L, L]$, for some $L > 0$. Thus the coefficients $\tilde{a}_n$, $\tilde{b}_n$ are given by

$$\int_{-L}^{L} f(x)\cos\sqrt{\lambda_n}x\,dx = -(\sqrt{\lambda_n}+1)\tilde{a}_n L.$$
$$\int_{-L}^{L} f(x)\sin\sqrt{\lambda_n}x\,dx = -(\sqrt{\lambda_n}+1)\tilde{b}_n L.$$

Thus, $u(x,y) \to 0$ when $|x| + y \to \infty$. [70] □

[70] Note that if we change the roles of X and Y in $\circledast$, the solution we get will be unbounded.

Problem 3: The 2D LAPLACE Equation on a Circle.
Let Ω be the unit disk in $\mathbb{R}^2$ and consider the problem



$$\begin{cases} \triangle u = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial n} = h & \text{on } \partial\Omega, \end{cases}$$

where h is a continuous function.

Proof. Use polar coordinates (r,θ)

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 & \text{for } 0 \le r < 1,\ 0 \le \theta < 2\pi \\ \frac{\partial u}{\partial r}(1,\theta) = h(\theta) & \text{for } 0 \le \theta < 2\pi. \end{cases}$$

$$r^2u_{rr} + ru_r + u_{\theta\theta} = 0.$$

Let $r = e^{-t}$, $u(r(t),\theta)$.

$$\begin{aligned} u_t &= u_r r_t = -e^{-t}u_r, \\ u_{tt} &= (-e^{-t}u_r)_t = e^{-t}u_r + e^{-2t}u_{rr} = ru_r + r^2u_{rr}. \end{aligned}$$

Thus, we have

$$u_{tt} + u_{\theta\theta} = 0.$$

Let $u(t,\theta) = X(t)Y(\theta)$, which gives $X''(t)Y(\theta) + X(t)Y''(\theta) = 0$.

$$\frac{X''(t)}{X(t)} = -\frac{Y''(\theta)}{Y(\theta)} = \lambda.$$

- From $Y''(\theta) + \lambda Y(\theta) = 0$, we get $Y_n(\theta) = a_n \cos n\theta + b_n \sin n\theta$.

$\lambda_n = n^2$, $n = 0,1,2,\ldots$.
- With these values of λ_n we solve $X''(t) - n^2X(t) = 0$.

If $n = 0$, $X_0(t) = c_0t + d_0$. $\Rightarrow$ $X_0(r) = -c_0 \log r + d_0$.
If $n \neq 0$, $X_n(t) = c_ne^{nt} + d_ne^{-nt}$ $\Rightarrow$ $X_n(r) = c_nr^{-n} + d_nr^n$.
- We have

$$\begin{aligned} u_0(r,\theta) &= X_0(r)Y_0(\theta) = (-c_0 \log r + d_0)a_0, \\ u_n(r,\theta) &= X_n(r)Y_n(\theta) = (c_nr^{-n} + d_nr^n)(a_n \cos n\theta + b_n \sin n\theta). \end{aligned}$$

But u must be finite at $r = 0$, so $c_n = 0$, $n = 0,1,2,\ldots$.

$$\begin{aligned} u_0(r,\theta) &= d_0a_0, \\ u_n(r,\theta) &= d_nr^n(a_n \cos n\theta + b_n \sin n\theta). \end{aligned}$$

By superposition, we write

$$\boxed{u(r,\theta) = \tilde{a}_0 + \sum_{n=1}^{\infty} r^n(\tilde{a}_n \cos n\theta + \tilde{b}_n \sin n\theta).}$$

Boundary condition gives

$$u_r(1,\theta) = \sum_{n=1}^{\infty} n(\tilde{a}_n \cos n\theta + \tilde{b}_n \sin n\theta) = h(\theta).$$

The coefficients a_n, b_n for $n \ge 1$ are determined from the Fourier series for $h(\theta)$.
a_0 is not determined by $h(\theta)$ and therefore may take an arbitrary value. Moreover,

the constant term in the Fourier series for $h(\theta)$ must be zero [i.e., $\int_0^{2\pi} h(\theta)d\theta = 0$]. Therefore, the problem is **not** solvable for an arbitrary function $h(\theta)$, and when it is solvable, the solution is **not** unique. $\square$

Problem 4: The 2D LAPLACE Equation on a Circle.
Let $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\} = \{(r, \theta) : 0 \le r < 1,\ 0 \le \theta < 2\pi\}$, *and use separation of variables* (r, θ) *to solve the Dirichlet problem*

$$\begin{cases} \triangle u = 0 & \textit{in } \Omega \\ u(1, \theta) = g(\theta) & \textit{for } 0 \le \theta < 2\pi. \end{cases}$$

Proof. Use polar coordinates (r, θ)

$$\begin{cases} u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0 & \text{for } 0 \le r < 1,\ 0 \le \theta < 2\pi \\ u(1, \theta) = g(\theta) & \text{for } 0 \le \theta < 2\pi. \end{cases}$$

$$r^2 u_{rr} + r u_r + u_{\theta\theta} = 0.$$

Let $r = e^{-t}$, $u(r(t), \theta)$.

$$\begin{aligned} u_t &= u_r r_t = -e^{-t} u_r, \\ u_{tt} &= (-e^{-t} u_r)_t = e^{-t} u_r + e^{-2t} u_{rr} = r u_r + r^2 u_{rr}. \end{aligned}$$

Thus, we have

$$u_{tt} + u_{\theta\theta} = 0.$$

Let $u(t, \theta) = X(t) Y(\theta)$, which gives $X''(t) Y(\theta) + X(t) Y''(\theta) = 0$.

$$\frac{X''(t)}{X(t)} = -\frac{Y''(\theta)}{Y(\theta)} = \lambda.$$

• From $Y''(\theta) + \lambda Y(\theta) = 0$, we get $Y_n(\theta) = a_n \cos n\theta + b_n \sin n\theta$.
$\lambda_n = n^2$, $n = 0, 1, 2, \ldots$.
• With these values of λ_n we solve $X''(t) - n^2 X(t) = 0$.
If $n = 0$, $X_0(t) = c_0 t + d_0$. $\Rightarrow$ $X_0(r) = -c_0 \log r + d_0$.
If $n \neq 0$, $X_n(t) = c_n e^{nt} + d_n e^{-nt}$ $\Rightarrow$ $X_n(r) = c_n r^{-n} + d_n r^n$.
• We have

$$\begin{aligned} u_0(r, \theta) &= X_0(r) Y_0(\theta) = (-c_0 \log r + d_0) a_0, \\ u_n(r, \theta) &= X_n(r) Y_n(\theta) = (c_n r^{-n} + d_n r^n)(a_n \cos n\theta + b_n \sin n\theta). \end{aligned}$$

But u must be finite at $r = 0$, so $c_n = 0$, $n = 0, 1, 2, \ldots$.

$$\begin{aligned} u_0(r, \theta) &= d_0 a_0, \\ u_n(r, \theta) &= d_n r^n (a_n \cos n\theta + b_n \sin n\theta). \end{aligned}$$

By superposition, we write

$$\boxed{u(r, \theta) = \tilde{a}_0 + \sum_{n=1}^{\infty} r^n (\tilde{a}_n \cos n\theta + \tilde{b}_n \sin n\theta).}$$

Boundary condition gives

$$u(1, \theta) = \tilde{a}_0 + \sum_{n=1}^{\infty} (\tilde{a}_n \cos n\theta + \tilde{b}_n \sin n\theta) = g(\theta).$$

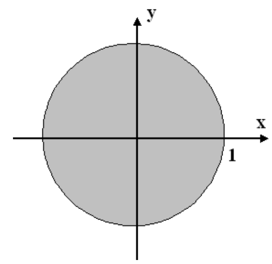
$$\begin{aligned}
\tilde{a}_0 &= \frac{1}{\pi}\int_0^{\pi} g(\theta)\,d\theta, \\
\tilde{a}_n &= \frac{2}{\pi}\int_0^{\pi} g(\theta)\cos n\theta\,d\theta, \\
\tilde{b}_n &= \frac{2}{\pi}\int_0^{\pi} g(\theta)\sin n\theta\,d\theta.
\end{aligned}$$

$\square$

Problem (F'94, #6): The 2D LAPLACE Equation on a Circle.
Find all solutions of the homogeneous equation

$$u_{xx} + u_{yy} = 0, \qquad x^2 + y^2 < 1,$$
$$\frac{\partial u}{\partial n} - u = 0, \qquad x^2 + y^2 = 1.$$

Hint: $\triangle = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$ *in polar coordinates.*



Proof. Use polar coordinates (r, θ):

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} \ = \ 0 & \text{for } \ 0 \le r < 1, \ 0 \le \theta < 2\pi \\ \frac{\partial u}{\partial r}(1,\theta) \ - \ u(1,\theta) \ = \ 0 & \text{for } \ 0 \le \theta < 2\pi. \end{cases}$$

Since we solve the equation on a circle, we have periodic conditions:

$$u(r,0) = u(r,2\pi) \quad \Rightarrow \quad X(r)Y(0) = X(r)Y(2\pi) \quad \Rightarrow \quad Y(0) = Y(2\pi),$$
$$u_\theta(r,0) = u_\theta(r,2\pi) \quad \Rightarrow \quad X(r)Y'(0) = X(r)Y'(2\pi) \quad \Rightarrow \quad Y'(0) = Y'(2\pi).$$

Also, we want the solution to be bounded. In particular, u is bounded for $r = 0$.

$$r^2 u_{rr} + r u_r + u_{\theta\theta} = 0.$$

Let $r = e^{-t}$, $u(r(t), \theta)$, we have

$$u_{tt} + u_{\theta\theta} = 0.$$

Let $u(t,\theta) = X(t)Y(\theta)$, which gives $X''(t)Y(\theta) + X(t)Y''(\theta) = 0$.

$$\frac{X''(t)}{X(t)} = -\frac{Y''(\theta)}{Y(\theta)} = \lambda.$$

• From $Y''(\theta) + \lambda Y(\theta) = 0$, we get $Y_n(\theta) = a_n \cos\sqrt{\lambda}\theta + b_n \sin\sqrt{\lambda}\theta$.
Using periodic condition: $Y_n(0) = a_n$,

$$Y_n(2\pi) = a_n \cos(\sqrt{\lambda_n}\,2\pi) + b_n \sin(\sqrt{\lambda_n}\,2\pi) = a_n \quad \Rightarrow \quad \sqrt{\lambda_n} = n \quad \Rightarrow \quad \lambda_n = n^2.$$

Thus, $Y_n(\theta) = a_n \cos n\theta + b_n \sin n\theta$.
• With these values of λ_n we solve $X''(t) - n^2 X(t) = 0$.
If $n = 0$, $X_0(t) = c_0 t + d_0$. $\Rightarrow$ $X_0(r) = -c_0 \log r + d_0$.
If $n \neq 0$, $X_n(t) = c_n e^{nt} + d_n e^{-nt}$ $\Rightarrow$ $X_n(r) = c_n r^{-n} + d_n r^n$.
u must be finite at $r = 0$ $\Rightarrow$ $c_n = 0$, $n = 0, 1, 2, \ldots$.

$$\boxed{u(r,\theta) = \tilde{a}_0 + \sum_{n=1}^{\infty} r^n(\tilde{a}_n \cos n\theta + \tilde{b}_n \sin n\theta).}$$

Boundary condition gives

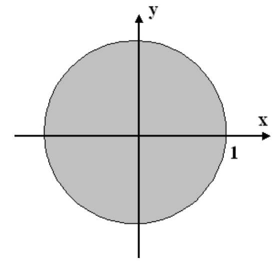
$$0 \ = \ u_r(1,\theta) \ - \ u(1,\theta) \ = \ -\tilde{a}_0 + \sum_{n=1}^{\infty}(n-1)(\tilde{a}_n \cos n\theta + \tilde{b}_n \sin n\theta).$$

Calculating Fourier coefficients gives $-2\pi\tilde{a}_0 = 0 \quad \Rightarrow \quad \tilde{a}_0 = 0$.
$\pi(n-1)a_n = 0 \quad \Rightarrow \quad \tilde{a}_n = 0, \ n = 2, 3, \ldots$.
a_1, b_1 are constants. Thus,

$$\boxed{u(r,\theta) = r(\tilde{a}_1 \cos\theta + \tilde{b}_1 \sin\theta).}$$

□

Problem (S'00, #4).
a) *Let (r,θ) be polar coordinates on the plane, i.e. $x_1 + ix_2 = re^{i\theta}$. Solve the boudary value problem*



$$\begin{aligned} \triangle u &= 0 \qquad \text{in } \ r<1 \\ \partial u/\partial r &= f(\theta) \qquad \text{on } \ r=1, \end{aligned}$$

*beginning with the **Fourier series** for f (you may assume that f is continuously differentiable). Give your answer as a power series in $x_1 + ix_2$ plus a power series in $x_1 - ix_2$. There is a necessary condition on f for this boundary value problem to be solvable that you will find in the course of doing this.*

b) *Sum the series in part (a) to get a representation of u in the form*

$$u(r,\theta) = \int_0^{2\pi} N(r,\theta-\theta')f(\theta')\,d\theta'.$$

Proof. **a)** Green's identity gives the necessary compatibility condition on f:

$$\int_0^{2\pi} f(\theta)\,d\theta \ = \ \int_{r=1} \frac{\partial u}{\partial r}\,d\theta \ = \ \int_{\partial\Omega} \frac{\partial u}{\partial n}\,ds \ = \ \int_\Omega \triangle u\,dx \ = \ 0.$$

Use polar coordinates (r,θ):

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} \ = \ 0 & \text{for } \ 0 \le r < 1, \ 0 \le \theta < 2\pi \\ \frac{\partial u}{\partial r}(1,\theta) \ = \ f(\theta) & \text{for } \ 0 \le \theta < 2\pi. \end{cases}$$

Since we solve the equation on a circle, we have periodic conditions:

$$\begin{aligned} u(r,0) = u(r,2\pi) \quad &\Rightarrow \quad X(r)Y(0) = X(r)Y(2\pi) \quad \Rightarrow \quad Y(0) = Y(2\pi), \\ u_\theta(r,0) = u_\theta(r,2\pi) \quad &\Rightarrow \quad X(r)Y'(0) = X(r)Y'(2\pi) \quad \Rightarrow \quad Y'(0) = Y'(2\pi). \end{aligned}$$

Also, we want the solution to be bounded. In particular, u is bounded for $r=0$.

$$r^2 u_{rr} + ru_r + u_{\theta\theta} = 0.$$

Let $r = e^{-t}$, $u(r(t),\theta)$, we have

$$u_{tt} + u_{\theta\theta} = 0.$$

Let $u(t,\theta) = X(t)Y(\theta)$, which gives $X''(t)Y(\theta) + X(t)Y''(\theta) = 0$.

$$\frac{X''(t)}{X(t)} = -\frac{Y''(\theta)}{Y(\theta)} = \lambda.$$

- From $Y''(\theta) + \lambda Y(\theta) = 0$, we get $Y_n(\theta) = a_n\cos\sqrt{\lambda}\theta + b_n\sin\sqrt{\lambda}\theta$.

Using periodic condition: $Y_n(0) = a_n$,

$$Y_n(2\pi) = a_n\cos(\sqrt{\lambda_n}\,2\pi) + b_n\sin(\sqrt{\lambda_n}\,2\pi) = a_n \quad \Rightarrow \quad \sqrt{\lambda_n} = n \quad \Rightarrow \quad \lambda_n = n^2.$$

Thus, $Y_n(\theta) = a_n\cos n\theta + b_n\sin n\theta$.

- With these values of λ_n we solve $X''(t) - n^2X(t) = 0$.

If $n=0$, $X_0(t) = c_0t + d_0$. $\quad\Rightarrow\quad X_0(r) = -c_0\log r + d_0$.

If $n \neq 0$, $\quad X_n(t) = c_n e^{nt} + d_n e^{-nt} \quad \Rightarrow \quad X_n(r) = c_n r^{-n} + d_n r^n$.
u must be finite at $r = 0 \;\Rightarrow\; c_n = 0$, $n = 0, 1, 2, \ldots$.

$$\boxed{u(r,\theta) = \tilde{a}_0 + \sum_{n=1}^{\infty} r^n (\tilde{a}_n \cos n\theta + \tilde{b}_n \sin n\theta).}$$

Since

$$u_r(r,\theta) \;=\; \sum_{n=1}^{\infty} n r^{n-1} (\tilde{a}_n \cos n\theta + \tilde{b}_n \sin n\theta),$$

the boundary condition gives

$$u_r(1,\theta) \;=\; \sum_{n=1}^{\infty} n\,(\tilde{a}_n \cos n\theta + \tilde{b}_n \sin n\theta) \;=\; f(\theta).$$

$$\begin{aligned}
\tilde{a}_n &= \frac{1}{n\pi} \int_0^{2\pi} f(\theta)\, \cos n\theta\, d\theta, \\
\tilde{b}_n &= \frac{1}{n\pi} \int_0^{2\pi} f(\theta)\, \sin n\theta\, d\theta.
\end{aligned}$$

$\tilde{a}_0$ is not determined by $f(\theta)$ (since $\int_0^{2\pi} f(\theta)\, d\theta = 0$). Therefore, it may take an arbitrary value. Moreover, the constant term in the Fourier series for $f(\theta)$ must be zero [i.e., $\int_0^{2\pi} f(\theta) d\theta = 0$]. Therefore, the problem is **not** solvable for an arbitrary function $f(\theta)$, and when it is solvable, the solution is **not** unique.

b) In part (a), we obtained the solution and the Fourier coefficients:

$$\begin{aligned}
\tilde{a}_n &= \frac{1}{n\pi} \int_0^{2\pi} f(\theta')\, \cos n\theta'\, d\theta', \\
\tilde{b}_n &= \frac{1}{n\pi} \int_0^{2\pi} f(\theta')\, \sin n\theta'\, d\theta'.
\end{aligned}$$

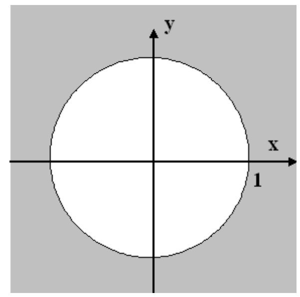
$$\begin{aligned}
u(r,\theta) &= \tilde{a}_0 + \sum_{n=1}^{\infty} r^n (\tilde{a}_n \cos n\theta + \tilde{b}_n \sin n\theta) \\
&= \tilde{a}_0 + \sum_{n=1}^{\infty} r^n \bigg(\Big[\frac{1}{n\pi} \int_0^{2\pi} f(\theta')\, \cos n\theta'\, d\theta' \Big] \cos n\theta + \Big[\frac{1}{n\pi} \int_0^{2\pi} f(\theta')\, \sin n\theta'\, d\theta' \Big] \sin n\theta \bigg) \\
&= \tilde{a}_0 + \sum_{n=1}^{\infty} \frac{r^n}{n\pi} \int_0^{2\pi} f(\theta') \big[\cos n\theta' \cos n\theta + \sin n\theta' \sin n\theta \big]\, d\theta' \\
&= \tilde{a}_0 + \sum_{n=1}^{\infty} \frac{r^n}{n\pi} \int_0^{2\pi} f(\theta')\, \cos n(\theta' - \theta)\, d\theta' \\
&= \tilde{a}_0 + \int_0^{2\pi} \underbrace{\sum_{n=1}^{\infty} \frac{r^n}{n\pi} \cos n(\theta - \theta')}_{N(r,\theta-\theta')} f(\theta')\, d\theta'.
\end{aligned}$$

$\square$

Problem (S'92, #6). *Consider the Laplace equation*

$$u_{xx} + u_{yy} = 0$$

for $x^2 + y^2 \geq 1$*. Denoting by* $x = r\cos\theta$*,* $y = r\sin\theta$ *polar coordinates, let* $f = f(\theta)$ *be a given smooth function of* θ*. Construct a uniformly bounded solution which satisfies boundary conditions*



$$u = f \qquad \text{for } x^2 + y^2 = 1.$$

What conditions has f *to satisfy such that*

$$\lim_{x^2+y^2\to\infty} (x^2 + y^2)u(x, y) = 0?$$

Proof. Use polar coordinates (r, θ):

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 & \text{for } r \geq 1 \\ u(1, \theta) = f(\theta) & \text{for } 0 \leq \theta < 2\pi. \end{cases}$$

Since we solve the equation on outside of a circle, we have periodic conditions:

$$u(r, 0) = u(r, 2\pi) \quad \Rightarrow \quad X(r)Y(0) = X(r)Y(2\pi) \quad \Rightarrow \quad Y(0) = Y(2\pi),$$
$$u_\theta(r, 0) = u(r, 2\pi) \quad \Rightarrow \quad X(r)Y'(0) = X(r)Y'(2\pi) \quad \Rightarrow \quad Y'(0) = Y'(2\pi).$$

Also, we want the solution to be bounded. In particular, u is bounded for $r = \infty$.

$$r^2 u_{rr} + r u_r + u_{\theta\theta} = 0.$$

Let $r = e^{-t}$, $u(r(t), \theta)$, we have

$$u_{tt} + u_{\theta\theta} = 0.$$

Let $u(t, \theta) = X(t)Y(\theta)$, which gives $X''(t)Y(\theta) + X(t)Y''(\theta) = 0$.

$$\frac{X''(t)}{X(t)} = -\frac{Y''(\theta)}{Y(\theta)} = \lambda.$$

- From $Y''(\theta) + \lambda Y(\theta) = 0$, we get $Y_n(\theta) = a_n \cos\sqrt{\lambda}\theta + b_n \sin\sqrt{\lambda}\theta$.

Using periodic condition: $Y_n(0) = a_n$,

$$Y_n(2\pi) = a_n \cos(\sqrt{\lambda_n}\, 2\pi) + b_n \sin(\sqrt{\lambda_n}\, 2\pi) = a_n \quad \Rightarrow \quad \sqrt{\lambda_n} = n \quad \Rightarrow \quad \lambda_n = n^2.$$

Thus, $Y_n(\theta) = a_n \cos n\theta + b_n \sin n\theta$.

- With these values of λ_n we solve $X''(t) - n^2 X(t) = 0$.

If $n = 0$, $X_0(t) = c_0 t + d_0$. $\Rightarrow$ $X_0(r) = -c_0 \log r + d_0$.
If $n \neq 0$, $X_n(t) = c_n e^{nt} + d_n e^{-nt}$ $\Rightarrow$ $X_n(r) = c_n r^{-n} + d_n r^n$.
u must be finite at $r = \infty$ $\Rightarrow$ $c_0 = 0$, $d_n = 0$, $n = 1, 2, \ldots$.

$$\boxed{u(r, \theta) = \tilde{a}_0 + \sum_{n=1}^{\infty} r^{-n}(\tilde{a}_n \cos n\theta + \tilde{b}_n \sin n\theta).}$$

Boundary condition gives

$$f(\theta) \;=\; u(1, \theta) \;=\; \tilde{a}_0 + \sum_{n=1}^{\infty}(\tilde{a}_n \cos n\theta + \tilde{b}_n \sin n\theta).$$

$$\begin{cases} 2\pi\tilde{a}_0 = \int_0^{2\pi} f(\theta)\,d\theta, \\ \pi\tilde{a}_n = \int_0^{2\pi} f(\theta)\,\cos n\theta\,d\theta, \\ \pi\tilde{b}_n = \int_0^{2\pi} f(\theta)\,\sin n\theta\,d\theta. \end{cases} \Rightarrow \begin{cases} f_0 = \tilde{a}_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,d\theta, \\ f_n = \tilde{a}_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\,\cos n\theta\,d\theta, \\ \tilde{f}_n = \tilde{b}_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\,\sin n\theta\,d\theta. \end{cases}$$

- We need to find conditions for f such that

$$\lim_{x^2+y^2\to\infty}(x^2+y^2)u(x,y) = 0, \qquad \text{or}$$

$$\lim_{r\to\infty} r^2 u(r,\theta) \underbrace{=}_{need} 0,$$

$$\lim_{r\to\infty} r^2\Big[f_0 + \sum_{n=1}^{\infty} r^{-n}(f_n\cos n\theta + \tilde{f}_n\sin n\theta)\Big] \underbrace{=}_{need} 0.$$

Since

$$\lim_{r\to\infty}\Big[\sum_{n>2}^{\infty} r^{2-n}(f_n\cos n\theta + \tilde{f}_n\sin n\theta)\Big] = 0,$$

we need

$$\lim_{r\to\infty}\Big[r^2 f_0 + \sum_{n=1}^{2} r^{2-n}(f_n\cos n\theta + \tilde{f}_n\sin n\theta)\Big] \underbrace{=}_{need} 0.$$

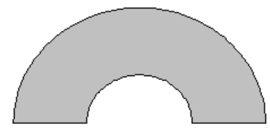
Thus, the conditions are

$$\boxed{f_n,\,\tilde{f}_n = 0, \quad n = 0,1,2.}$$

$\square$

Problem (F'96, #2): The 2D LAPLACE Equation on a Semi-Annulus.
Solve the Laplace equation in the semi-annulus

$$\begin{cases} \triangle u = 0, & 1 < r < 2,\ 0 < \theta < \pi, \\ u(r,0) = u(r,\pi) = 0, & 1 < r < 2, \\ u(1,\theta) = \sin\theta, & 0 < \theta < \pi, \\ u(2,\theta) = 0, & 0 < \theta < \pi. \end{cases}$$



Hint: Use the formula $\triangle = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$ *for the Laplacian in polar coordinates.*

Proof. Use polar coordinates (r,θ)

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \qquad 1 < r < 2,\ 0 < \theta < \pi,$$
$$r^2 u_{rr} + r u_r + u_{\theta\theta} = 0.$$

With $r = e^{-t}$, we have

$$u_{tt} + u_{\theta\theta} = 0.$$

Let $u(t,\theta) = X(t)Y(\theta)$, which gives $X''(t)Y(\theta) + X(t)Y''(\theta) = 0$.

$$\frac{X''(t)}{X(t)} = -\frac{Y''(\theta)}{Y(\theta)} = \lambda.$$

• From $Y''(\theta) + \lambda Y(\theta) = 0$, we get $Y_n(\theta) = a_n \cos\sqrt{\lambda}\theta + b_n \sin\sqrt{\lambda}\theta$.
Boundary conditions give

$$u_n(r,0) = 0 = X_n(r)Y_n(0) = 0, \quad \Rightarrow \quad Y_n(0) = 0,$$
$$u_n(r,\pi) = 0 = X_n(r)Y_n(\pi) = 0, \quad \Rightarrow \quad Y_n(\pi) = 0.$$

Thus, $0 = Y_n(0) = a_n$, and $Y_n(\pi) = b_n \sin\sqrt{\lambda}\pi = 0 \ \Rightarrow\ \sqrt{\lambda} = n \ \Rightarrow\ \lambda_n = n^2$.
Thus, $Y_n(\theta) = b_n \sin n\theta$, $n = 1, 2, \ldots$.
• With these values of λ_n we solve $X''(t) - n^2 X(t) = 0$.
If $n = 0$, $X_0(t) = c_0 t + d_0$. $\Rightarrow$ $X_0(r) = -c_0 \log r + d_0$.
If $n > 0$, $X_n(t) = c_n e^{nt} + d_n e^{-nt}$ $\Rightarrow$ $X_n(r) = c_n r^{-n} + d_n r^n$.
• We have,

$$\boxed{u(r,\theta) = \sum_{n=1}^{\infty} X_n(r)Y_n(\theta) = \sum_{n=1}^{\infty}(\tilde{c}_n r^{-n} + \tilde{d}_n r^n)\sin n\theta.}$$

Using the other two boundary conditions, we obtain

$$\sin\theta = u(1,\theta) = \sum_{n=1}^{\infty}(\tilde{c}_n + \tilde{d}_n)\sin n\theta \quad \Rightarrow \quad \begin{cases} \tilde{c}_1 + \tilde{d}_1 = 1, \\ \tilde{c}_n + \tilde{d}_n = 0, \quad n = 2, 3, \ldots. \end{cases}$$
$$0 = u(2,\theta) = \sum_{n=1}^{\infty}(\tilde{c}_n 2^{-n} + \tilde{d}_n 2^n)\sin n\theta \quad \Rightarrow \quad \tilde{c}_n 2^{-n} + \tilde{d}_n 2^n = 0, \quad n = 1, 2, \ldots.$$

Thus, the coefficients are given by

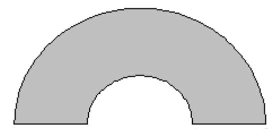
$$c_1 = \frac{4}{3}, \quad d_1 = -\frac{1}{3};$$
$$c_n = 0, \quad d_n = 0.$$

$$\boxed{u(r,\theta) = \left(\frac{4}{3r} - \frac{r}{3}\right)\sin\theta.}$$

$\square$

Problem (S'98, #8): The 2D LAPLACE Equation on a Semi-Annulus.
Solve

$$\begin{cases} \triangle u = 0, & 1 < r < 2,\ 0 < \theta < \pi, \\ u(r,0) = u(r,\pi) = 0, & 1 < r < 2, \\ u(1,\theta) = u(2,\theta) = 1, & 0 < \theta < \pi. \end{cases}$$



Proof. Use polar coordinates (r,θ)

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \qquad \text{for } 1 < r < 2,\ 0 < \theta < \pi,$$
$$r^2 u_{rr} + r u_r + u_{\theta\theta} = 0.$$

With $r = e^{-t}$, we have

$$u_{tt} + u_{\theta\theta} = 0.$$

Let $u(t,\theta) = X(t)Y(\theta)$, which gives $X''(t)Y(\theta) + X(t)Y''(\theta) = 0$.

$$\frac{X''(t)}{X(t)} = -\frac{Y''(\theta)}{Y(\theta)} = \lambda.$$

- From $Y''(\theta) + \lambda Y(\theta) = 0$, we get $Y_n(\theta) = a_n \cos n\theta + b_n \sin n\theta$.

Boundary conditions give

$$u_n(r,0) = 0 = X_n(r)Y_n(0) = 0, \quad \Rightarrow \quad Y_n(0) = 0,$$
$$u_n(r,\pi) = 0 = X_n(r)Y_n(\pi) = 0, \quad \Rightarrow \quad Y_n(\pi) = 0.$$

Thus, $0 = Y_n(0) = a_n$, and $Y_n(\theta) = b_n \sin n\theta$.
$\lambda_n = n^2, \ \ n = 1, 2, \ldots$.
- With these values of λ_n we solve $X''(t) - n^2 X(t) = 0$.

If $n = 0$, $X_0(t) = c_0 t + d_0$. $\quad\Rightarrow\quad X_0(r) = -c_0 \log r + d_0$.
If $n > 0$, $X_n(t) = c_n e^{nt} + d_n e^{-nt} \quad\Rightarrow\quad X_n(r) = c_n r^{-n} + d_n r^n$.
- We have,

$$\boxed{u(r,\theta) = \sum_{n=1}^{\infty} X_n(r)Y_n(\theta) = \sum_{n=1}^{\infty} (\tilde{c}_n r^{-n} + \tilde{d}_n r^n) \sin n\theta.}$$

Using the other two boundary conditions, we obtain

$$u(1,\theta) = 1 \ = \ \sum_{n=1}^{\infty} (\tilde{c}_n + \tilde{d}_n) \sin n\theta,$$
$$u(2,\theta) = 1 \ = \ \sum_{n=1}^{\infty} (\tilde{c}_n 2^{-n} + \tilde{d}_n 2^n) \sin n\theta,$$

which give the two equations for $\tilde{c}_n$ and $\tilde{d}_n$:

$$\int_0^{\pi} \sin n\theta \, d\theta \ = \ \frac{\pi}{2}(\tilde{c}_n + \tilde{d}_n),$$
$$\int_0^{\pi} \sin n\theta \, d\theta \ = \ \frac{\pi}{2}(\tilde{c}_n 2^{-n} + \tilde{d}_n 2^n),$$

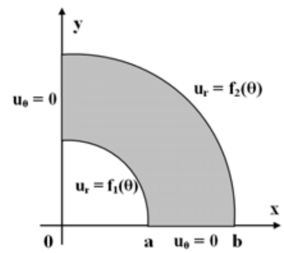
that can be solved. $\square$

Problem (F'89, #1). *Consider Laplace equation inside a* $90°$ *sector of a circular annulus*

$$\triangle u = 0 \qquad a < r < b, \ 0 < \theta < \frac{\pi}{2}$$

subject to the boundary conditions

$$\frac{\partial u}{\partial \theta}(r, 0) = 0, \qquad \frac{\partial u}{\partial \theta}(r, \frac{\pi}{2}) = 0,$$
$$\frac{\partial u}{\partial r}(a, \theta) = f_1(\theta), \quad \frac{\partial u}{\partial r}(b, \theta) = f_2(\theta),$$



where $f_1(\theta)$, $f_2(\theta)$ *are continuously differentiable.*

a) *Find the solution of this equation with the prescribed boundary conditions using separation of variables.*

Proof. **a)** Use polar coordinates (r, θ)

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \qquad \text{for} \ \ a < r < b, \ 0 < \theta < \frac{\pi}{2},$$
$$r^2 u_{rr} + r u_r + u_{\theta\theta} = 0.$$

With $r = e^{-t}$, we have

$$u_{tt} + u_{\theta\theta} = 0.$$

Let $u(t, \theta) = X(t)Y(\theta)$, which gives $X''(t)Y(\theta) + X(t)Y''(\theta) = 0$.

$$\frac{X''(t)}{X(t)} = -\frac{Y''(\theta)}{Y(\theta)} = \lambda.$$

- From $Y''(\theta) + \lambda Y(\theta) = 0$, we get $Y_n(\theta) = a_n \cos\sqrt{\lambda}\theta + b_n \sin\sqrt{\lambda}\theta$.

Boundary conditions give

$$u_{n\theta}(r, 0) \ = \ X_n(r)Y_n'(0) = 0 \ \Rightarrow \ Y_n'(0) = 0,$$
$$u_{n\theta}(r, \frac{\pi}{2}) \ = \ X_n(r)Y_n'(\frac{\pi}{2}) = 0 \ \Rightarrow \ Y_n'(\frac{\pi}{2}) = 0.$$

$Y_n'(\theta) = -a_n\sqrt{\lambda_n}\sin\sqrt{\lambda_n}\theta + b_n\sqrt{\lambda_n}\cos\sqrt{\lambda_n}\theta$. Thus, $Y_n'(0) = b_n\sqrt{\lambda_n} = 0 \ \Rightarrow \ b_n = 0$.
$Y_n'(\frac{\pi}{2}) = -a_n\sqrt{\lambda_n}\sin\sqrt{\lambda_n}\frac{\pi}{2} = 0 \ \Rightarrow \ \sqrt{\lambda_n}\frac{\pi}{2} = n\pi \ \Rightarrow \ \lambda_n = (2n)^2$.
Thus, $Y_n(\theta) = a_n \cos(2n\theta)$, $n = 0, 1, 2, \ldots$.
In particular, $Y_0(\theta) = a_0 t + b_0$. Boundary conditions give $Y_0(\theta) = b_0$.

- With these values of λ_n we solve $X''(t) - (2n)^2 X(t) = 0$.

If $n = 0$, $X_0(t) = c_0 t + d_0$. $\Rightarrow$ $X_0(r) = -c_0 \log r + d_0$.
If $n > 0$, $X_n(t) = c_n e^{2nt} + d_n e^{-2nt}$ $\Rightarrow$ $X_n(r) = c_n r^{-2n} + d_n r^{2n}$.

$$\boxed{u(r, \theta) \ = \ \tilde{c}_0 \log r + \tilde{d}_0 + \sum_{n=1}^{\infty}(\tilde{c}_n r^{-2n} + \tilde{d}_n r^{2n})\cos(2n\theta).}$$

Using the other two boundary conditions, we obtain

$$u_r(r, \theta) \ = \ \frac{\tilde{c}_0}{r} + \sum_{n=1}^{\infty}(-2n\tilde{c}_n r^{-2n-1} + 2n\tilde{d}_n r^{2n-1})\cos(2n\theta).$$

$$f_1(\theta) = u_r(a,\theta) = \frac{\tilde{c}_0}{a} + 2\sum_{n=1}^{\infty} n(-\tilde{c}_n a^{-2n-1} + \tilde{d}_n a^{2n-1})\cos(2n\theta),$$
$$f_2(\theta) = u_r(b,\theta) = \frac{\tilde{c}_0}{b} + 2\sum_{n=1}^{\infty} n(-\tilde{c}_n b^{-2n-1} + \tilde{d}_n b^{2n-1})\cos(2n\theta).$$

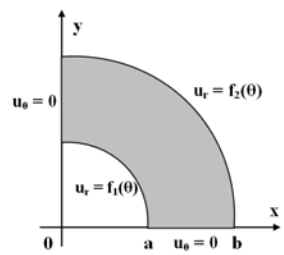
which give the two equations for $\tilde{c}_n$ and $\tilde{d}_n$:

$$\int_0^{\frac{\pi}{2}} f_1(\theta)\cos(2n\theta)\,d\theta = \frac{\pi}{2} n(-\tilde{c}_n a^{-2n-1} + \tilde{d}_n a^{2n-1}),$$
$$\int_0^{\frac{\pi}{2}} f_2(\theta)\sin(2n\theta)\,d\theta = \frac{\pi}{2} n(-\tilde{c}_n b^{-2n-1} + \tilde{d}_n b^{2n-1}).$$

□

b) *Show that the solution exists if and only if*

$$a\int_0^{\frac{\pi}{2}} f_1(\theta)\,d\theta \,-\, b\int_0^{\frac{\pi}{2}} f_2(\theta)\,d\theta \,=\, 0.$$



Proof. Using Green's identity, we obtain:

$$\begin{aligned}
0 &= \int_\Omega \triangle u\,dx \;=\; \int_{\partial\Omega} \frac{\partial u}{\partial n} \\
&= \int_0^{\frac{\pi}{2}} \frac{\partial u}{\partial r}(b,\theta)\,d\theta + \int_{\frac{\pi}{2}}^0 -\frac{\partial u}{\partial r}(a,\theta)\,d\theta + \int_a^b -\frac{\partial u}{\partial \theta}(r,0)\,dr + \int_b^a \frac{\partial u}{\partial \theta}\Big(r,\frac{\pi}{2}\Big)\,dr \\
&= \int_0^{\frac{\pi}{2}} f_2(\theta)\,d\theta + \int_0^{\frac{\pi}{2}} f_1(\theta)\,d\theta + 0 + 0 \\
&= \int_0^{\frac{\pi}{2}} f_1(\theta)\,d\theta + \int_0^{\frac{\pi}{2}} f_2(\theta)\,d\theta.
\end{aligned}$$

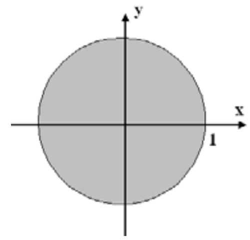
□

c) *Is the solution* ***unique****?*

Proof. No, since the boundary conditions are Neumann. The solution is unique only up to a constant. □

Problem (S'99, #4). *Let $u(x,y)$ be **harmonic** inside the unit disc, with boundary values along the unit circle*

$$u(x,y) = \begin{cases} 1, & y > 0 \\ 0, & y \le 0. \end{cases}$$



Compute $u(0,0)$ and $u(0,y)$.

Proof. Since u is harmonic, $\triangle u = 0$. Use polar coordinates (r,θ)

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 & 0 \le r < 1,\ 0 \le \theta < 2\pi \\ u(1,\theta) = \begin{cases} 1, & 0 < \theta < \pi \\ 0, & \pi \le \theta \le 2\pi. \end{cases} \end{cases}$$

$$r^2 u_{rr} + r u_r + u_{\theta\theta} = 0.$$

With $r = e^{-t}$, we have

$$u_{tt} + u_{\theta\theta} = 0.$$

Let $u(t,\theta) = X(t)Y(\theta)$, which gives $X''(t)Y(\theta) + X(t)Y''(\theta) = 0$.

$$\frac{X''(t)}{X(t)} = -\frac{Y''(\theta)}{Y(\theta)} = \lambda.$$

- From $Y''(\theta) + \lambda Y(\theta) = 0$, we get $Y_n(\theta) = a_n \cos n\theta + b_n \sin n\theta$.

$\lambda_n = n^2$, $n = 1, 2, \ldots$.

- With these values of λ_n we solve $X''(t) - n^2 X(t) = 0$.

If $n = 0$, $X_0(t) = c_0 t + d_0$. $\Rightarrow$ $X_0(r) = -c_0 \log r + d_0$.
If $n > 0$, $X_n(t) = c_n e^{nt} + d_n e^{-nt}$ $\Rightarrow$ $X_n(r) = c_n r^{-n} + d_n r^n$.

- We have

$$\begin{aligned} u_0(r,\theta) &= X_0(r)Y_0(\theta) = (-c_0 \log r + d_0)a_0, \\ u_n(r,\theta) &= X_n(r)Y_n(\theta) = (c_n r^{-n} + d_n r^n)(a_n \cos n\theta + b_n \sin n\theta). \end{aligned}$$

But u must be finite at $r = 0$, so $c_n = 0$, $n = 0, 1, 2, \ldots$.

$$\begin{aligned} u_0(r,\theta) &= \tilde{a}_0, \\ u_n(r,\theta) &= r^n(\tilde{a}_n \cos n\theta + \tilde{b}_n \sin n\theta). \end{aligned}$$

By superposition, we write

$$\boxed{u(r,\theta) = \tilde{a}_0 + \sum_{n=1}^{\infty} r^n(\tilde{a}_n \cos n\theta + \tilde{b}_n \sin n\theta).}$$

Boundary condition gives

$$u(1,\theta) = \tilde{a}_0 + \sum_{n=1}^{\infty}(\tilde{a}_n \cos n\theta + \tilde{b}_n \sin n\theta) = \begin{cases} 1, & 0 < \theta < \pi \\ 0, & \pi \le \theta \le 2\pi, \end{cases}$$

and the coefficients $\tilde{a}_n$ and $\tilde{b}_n$ are determined from the above equation.[71] ∎

[71] See Yana's solutions, where Green's function on a unit disk is constructed.

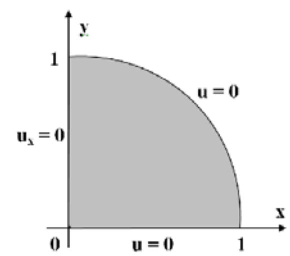
23 Problems: Separation of Variables - Poisson Equation

Problem (F'91, #2): The 2D POISSON Equation on a Quarter-Circle.
Solve explicitly the following boundary value problem

$$u_{xx} + u_{yy} = f(x, y)$$

in the domain $\Omega = \{(x, y),\, x > 0,\, y > 0,\, x^2 + y^2 < 1\}$
with boundary conditions

$$\begin{aligned} u &= 0 \qquad \text{for} \quad y = 0,\ 0 < x < 1, \\ \frac{\partial u}{\partial x} &= 0 \qquad \text{for} \quad x = 0,\ 0 < y < 1, \\ u &= 0 \qquad \text{for} \quad x > 0,\ y > 0,\ x^2 + y^2 = 1. \end{aligned}$$



Function $f(x, y)$ is known and is assumed to be continuous.

Proof. Use polar coordinates (r, θ):

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} \ = \ f(r, \theta) & 0 \leq r < 1,\ 0 \leq \theta < \frac{\pi}{2} \\ u(r, 0) \ = \ 0 & 0 \leq r < 1, \\ u_\theta(r, \frac{\pi}{2}) = 0 & 0 \leq r < 1, \\ u(1, \theta) \ = \ 0 & 0 \leq \theta \leq \frac{\pi}{2}. \end{cases}$$

We solve

$$r^2 u_{rr} + r u_r + u_{\theta\theta} = 0.$$

Let $r = e^{-t}$, $u(r(t), \theta)$, we have

$$u_{tt} + u_{\theta\theta} = 0. \qquad \circledast$$

Let $u(t, \theta) = X(t)Y(\theta)$, which gives $X''(t)Y(\theta) + X(t)Y''(\theta) = 0$.

$$\frac{X''(t)}{X(t)} = -\frac{Y''(\theta)}{Y(\theta)} = \lambda.$$

• From $Y''(\theta) + \lambda Y(\theta) = 0$, we get $Y_n(\theta) = a_n \cos \sqrt{\lambda}\theta + b_n \sin \sqrt{\lambda}\theta$. Boundary conditions:

$$\begin{cases} u(r, 0) = X(r)Y(0) = 0 \\ u_\theta(r, \frac{\pi}{2}) = X(r)Y'(\frac{\pi}{2}) = 0 \end{cases} \quad \Rightarrow \quad Y(0) = Y'\Big(\frac{\pi}{2}\Big) = 0.$$

Thus, $Y_n(0) = a_n = 0$, and $Y_n'(\frac{\pi}{2}) = \sqrt{\lambda_n} b_n \cos \sqrt{\lambda_n}\frac{\pi}{2} = 0$
$\Rightarrow \sqrt{\lambda_n}\frac{\pi}{2} = n\pi - \frac{\pi}{2},\ n = 1, 2, \ldots \ \Rightarrow \ \lambda_n = (2n-1)^2$.
Thus, $Y_n(\theta) = b_n \sin(2n-1)\theta$, $n = 1, 2, \ldots$. Thus, we have

$$u(r, \theta) = \sum_{n=1}^{\infty} X_n(r) \sin[(2n-1)\theta].$$

We now plug this equation into ⊛ with inhomogeneous term and obtain

$$\begin{aligned}
\sum_{n=1}^{\infty}\big(X_n''(t)\,\sin[(2n-1)\theta] \;-\; (2n-1)^2 X_n(t)\,\sin[(2n-1)\theta]\big) &= f(t,\theta),\\
\sum_{n=1}^{\infty}\big(X_n''(t)-(2n-1)^2X_n(t)\big)\sin[(2n-1)\theta] &= f(t,\theta),\\
\frac{\pi}{4}\big(X_n''(t)-(2n-1)^2X_n(t)\big) &= \int_0^{\frac{\pi}{2}} f(t,\theta)\,\sin[(2n-1)\theta]\,d\theta,\\
X_n''(t)-(2n-1)^2X_n(t) &= \frac{4}{\pi}\int_0^{\frac{\pi}{2}} f(t,\theta)\,\sin[(2n-1)\theta]\,d\theta.
\end{aligned}$$

The solution to this equation is

$$\begin{aligned}
X_n(t) &= c_n e^{(2n-1)t}+d_n e^{-(2n-1)t}+U_{np}(t), \qquad \text{or}\\
X_n(r) &= c_n r^{-(2n-1)}+d_n r^{(2n-1)}+u_{np}(r),
\end{aligned}$$

where u_{np} is the particular solution of inhomogeneous equation.
u must be finite at $r=0 \;\Rightarrow\; c_n=0,\ n=1,2,\ldots$. Thus,

$$\boxed{u(r,\theta)=\sum_{n=1}^{\infty}\big(d_n r^{(2n-1)}+u_{np}(r)\big)\;\sin[(2n-1)\theta].}$$

Using the last boundary condition, we have

$$\begin{aligned}
0=u(1,\theta) &= \sum_{n=1}^{\infty}\big(d_n+u_{np}(1)\big)\;\sin[(2n-1)\theta],\\
&\Rightarrow\;\; 0=\frac{\pi}{4}(d_n+u_{np}(1)),\\
&\Rightarrow\;\; d_n=-u_{np}(1).
\end{aligned}$$

$$\boxed{u(r,\theta)=\sum_{n=1}^{\infty}\big(-u_{np}(1)r^{(2n-1)}+u_{np}(r)\big)\;\sin[(2n-1)\theta].}$$

The method used to solve this problem is similar to section Problems: Eigenvalues of the Laplacian - Poisson Equation:
1) First, we find $Y_n(\theta)$ eigenfunctions.
2) Then, we plug in our guess $u(t,\theta)=X(t)Y(\theta)$ into the equation $u_{tt}+u_{\theta\theta}=f(t,\theta)$ and solve an ODE in $X(t)$.

Note the similar problem on 2D Poisson equation on a square domain. The problem is used by first finding the eigenvalues and eigenfunctions of the Laplacian, and then expanding $f(x,y)$ in eigenfunctions, and comparing coefficients of f with the general solution $u(x,y)$.
Here, however, this could not be done because of the circular geometry of the domain. In particular, the boundary conditions do not give enough information to find explicit representations for μ_m and ν_n. Also, the condition $u=0$ for $x>0,\ y>0,\ x^2+y^2=1$

can not be used.
[72] $\square$

[72] ChiuYen's solutions have attempts to solve this problem using Green's function.

24 Problems: Separation of Variables - Wave Equation

Example (McOwen 3.1 #2). *We considered the initial/boundary value problem and solved it using Fourier Series. We now solve it using the* ***Separation of Variables****.*

$$\begin{cases} u_{tt} - u_{xx} = 0 & 0 < x < \pi,\ t > 0 \\ u(x,0) = 1, \quad u_t(x,0) = 0 & 0 < x < \pi \\ u(0,t) = 0, \quad u(\pi,t) = 0 & t \geq 0. \end{cases} \tag{24.1}$$

Proof. Assume $u(x,t) = X(x)T(t)$, then substitution in the PDE gives $XT'' - X''T = 0$.

$$\frac{X''}{X} = \frac{T''}{T} = -\lambda.$$

• From $X'' + \lambda X = 0$, we get $X_n(x) = a_n \cos nx + b_n \sin nx$. Boundary conditions give

$$\begin{cases} u(0,t) = X(0)T(t) = 0 \\ u(\pi,t) = X(\pi)T(t) = 0 \end{cases} \Rightarrow \quad X(0) = X(\pi) = 0.$$

Thus, $X_n(0) = a_n = 0$, and $X_n(x) = b_n \sin nx$, $\lambda_n = n^2$, $n = 1, 2, \ldots$.
• With these values of λ_n, we solve $T'' + n^2 T = 0$ to find $T_n(t) = c_n \sin nt + d_n \cos nt$. Thus,

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} \big(\tilde{c}_n \sin nt + \tilde{d}_n \cos nt\big) \sin nx, \\ u_t(x,t) &= \sum_{n=1}^{\infty} \big(n\tilde{c}_n \cos nt - n\tilde{d}_n \sin nt\big) \sin nx. \end{aligned}$$

• Initial conditions give

$$\begin{aligned} 1 = u(x,0) &= \sum_{n=1}^{\infty} \tilde{d}_n \sin nx, \\ 0 = u_t(x,0) &= \sum_{n=1}^{\infty} n\tilde{c}_n \sin nx. \end{aligned}$$

By orthogonality, we may multiply both equations by $\sin mx$ and integrate:

$$\int_0^\pi \sin mx \, dx = \tilde{d}_m \frac{\pi}{2},$$
$$\int_0^\pi 0 \, dx = n\tilde{c}_n \frac{\pi}{2},$$

which gives the coefficients

$$\tilde{d}_n = \frac{2}{n\pi}(1 - \cos n\pi) = \begin{cases} \frac{4}{n\pi}, & n \text{ odd}, \\ 0, & n \text{ even}, \end{cases} \qquad \text{and} \qquad \tilde{c}_n = 0.$$

Plugging the coefficients into a formula for $u(x,t)$, we get

$$\boxed{u(x,t) = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\cos(2n+1)t \, \sin(2n+1)x}{(2n+1)}.}$$

$\square$

Example. *Use the method of separation of variables to find the solution to:*

$$\begin{cases} u_{tt} + 3u_t + u = u_{xx}, & 0 < x < 1 \\ u(0,t) = 0, \quad u(1,t) = 0, \\ u(x,0) = 0, \quad u_t(x,0) = x\sin(2\pi x). \end{cases}$$

Proof. Assume $u(x,t) = X(x)T(t)$, then substitution in the PDE gives

$$\begin{aligned} XT'' + 3XT' + XT &= X''T, \\ \frac{T''}{T} + 3\frac{T'}{T} + 1 &= \frac{X''}{X} = -\lambda. \end{aligned}$$

• From $X'' + \lambda X = 0$, $X_n(x) = a_n \cos\sqrt{\lambda_n}x + b_n \sin\sqrt{\lambda_n}x$. Boundary conditions give

$$\begin{cases} u(0,t) = X(0)T(t) = 0 \\ u(1,t) = X(1)T(t) = 0 \end{cases} \Rightarrow \quad X(0) = X(1) = 0.$$

Thus, $X_n(0) = a_n = 0$, and $X_n(x) = b_n \sin\sqrt{\lambda_n}x$.
$X_n(1) = b_n \sin\sqrt{\lambda_n} = 0$. Hence, $\sqrt{\lambda_n} = n\pi$, or $\lambda_n = (n\pi)^2$, $n = 1, 2, \ldots$.

$$\boxed{\lambda_n = (n\pi)^2, \qquad X_n(x) = b_n \sin n\pi x.}$$

• With these values of λ_n, we solve

$$\begin{aligned} T'' + 3T' + T &= -\lambda_n T, \\ T'' + 3T' + T &= -(n\pi)^2 T, \\ T'' + 3T' + (1 + (n\pi)^2)T &= 0. \end{aligned}$$

We can solve this 2nd-order ODE with the following guess, $T(t) = ce^{st}$ to obtain $s = -\frac{3}{2} \pm \sqrt{\frac{5}{4} - (n\pi)^2}$. For $n \geq 1$, $\frac{5}{4} - (n\pi)^2 < 0$. Thus, $s = -\frac{3}{2} \pm i\sqrt{(n\pi)^2 - \frac{5}{4}}$.

$$\boxed{T_n(t) = e^{-\frac{3}{2}t}\Big(c_n \cos\sqrt{(n\pi)^2 - \frac{5}{4}}t + d_n \sin\sqrt{(n\pi)^2 - \frac{5}{4}}t\Big).}$$

$$u(x,t) = X(x)T(t) = \sum_{n=1}^{\infty} e^{-\frac{3}{2}t}\Big(c_n \cos\sqrt{(n\pi)^2 - \frac{5}{4}}t + d_n \sin\sqrt{(n\pi)^2 - \frac{5}{4}}t\Big) \sin n\pi x.$$

• Initial conditions give

$$0 = u(x,0) = \sum_{n=1}^{\infty} c_n \sin n\pi x.$$

By orthogonality, we may multiply this equations by $\sin m\pi x$ and integrate:

$$\int_0^1 0\, dx = \frac{1}{2}c_m \qquad \Rightarrow \qquad c_m = 0.$$

Thus,

$$
\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} d_n e^{-\frac{3}{2}t}\Big(\sin\sqrt{(n\pi)^2-\frac{5}{4}}t\Big)\sin n\pi x. \\
u_t(x,t) &= \sum_{n=1}^{\infty}\Big[-\frac{3}{2}d_n e^{-\frac{3}{2}t}\Big(\sin\sqrt{(n\pi)^2-\frac{5}{4}}t\Big)+d_n e^{-\frac{3}{2}t}\Big(\sqrt{(n\pi)^2-\frac{5}{4}}\Big)\Big(\cos\sqrt{(n\pi)^2-\frac{5}{4}}t\Big)\Big]\sin n\pi x, \\
x\sin(2\pi x) &= u_t(x,0) = \sum_{n=1}^{\infty} d_n\Big(\sqrt{(n\pi)^2-\frac{5}{4}}\Big)\sin n\pi x.
\end{aligned}
$$

By orthogonality, we may multiply this equations by $\sin m\pi x$ and integrate:

$$
\int_0^1 x\sin(2\pi x)\,\sin(m\pi x)\,dx = d_m\frac{1}{2}\Big(\sqrt{(m\pi)^2-\frac{5}{4}}\Big),
$$

$$
d_n = \frac{2}{\sqrt{(n\pi)^2-\frac{5}{4}}}\int_0^1 x\sin(2\pi x)\,\sin(n\pi x)\,dx.
$$

$$
\boxed{u(x,t) \;=\; e^{-\frac{3}{2}t}\sum_{n=1}^{\infty} d_n\,\Big(\sin\sqrt{(n\pi)^2-\frac{5}{4}}t\Big)\,\sin n\pi x.}
$$

$\square$

Problem (F'04, #1). *Solve the following initial-boundary value problem for the wave equation with a potential term,*

$$
\begin{cases}
u_{tt}-u_{xx}+u=0 & 0<x<\pi,\ t<0 \\
u(0,t)=u(\pi,t)=0 & t>0 \\
u(x,0)=f(x), \quad u_t(x,0)=0 & 0<x<\pi,
\end{cases}
$$

where

$$
f(x)=\begin{cases} x & \text{if } x\in(0,\pi/2), \\ \pi-x & \text{if } x\in(\pi/2,\pi). \end{cases}
$$

The answer should be given in terms of an infinite series of explicitly given functions.

Proof. Assume $u(x,t)=X(x)T(t)$, then substitution in the PDE gives

$$
\begin{aligned}
XT''-X''T+XT &= 0, \\
\frac{T''}{T}+1 &= \frac{X''}{X} = -\lambda.
\end{aligned}
$$

- From $X''+\lambda X=0$, $X_n(x)=a_n\cos\sqrt{\lambda_n}x+b_n\sin\sqrt{\lambda_n}x$. Boundary conditions give

$$
\begin{cases}
u(0,t)=X(0)T(t)=0 \\
u(\pi,t)=X(\pi)T(t)=0
\end{cases}
\Rightarrow \quad X(0)=X(\pi)=0.
$$

Thus, $X_n(0)=a_n=0$, and $X_n(x)=b_n\sin\sqrt{\lambda_n}x$.
$X_n(\pi)=b_n\sin\sqrt{\lambda_n}\pi=0$. Hence, $\sqrt{\lambda_n}=n$, or $\lambda_n=n^2$, $n=1,2,\ldots$.

$$
\boxed{\lambda_n=n^2, \qquad X_n(x)=b_n\sin nx.}
$$

- With these values of λ_n, we solve

$$\begin{aligned} T'' + T &= -\lambda_n T, \\ T'' + T &= -n^2 T, \\ T_n'' + (1+n^2)T_n &= 0. \end{aligned}$$

The solution to this 2nd-order ODE is of the form:

$$\boxed{T_n(t) = c_n \cos\sqrt{1+n^2}\, t + d_n \sin\sqrt{1+n^2}\, t.}$$

$$\begin{aligned} u(x,t) &= X(x)T(t) = \sum_{n=1}^{\infty}\big(c_n \cos\sqrt{1+n^2}\, t + d_n \sin\sqrt{1+n^2}\, t\big)\sin nx. \\ u_t(x,t) &= \sum_{n=1}^{\infty}\big(-c_n(\sqrt{1+n^2})\sin\sqrt{1+n^2}\, t + d_n(\sqrt{1+n^2})\cos\sqrt{1+n^2}\, t\big)\sin nx. \end{aligned}$$

- Initial conditions give

$$\begin{aligned} f(x) &= u(x,0) = \sum_{n=1}^{\infty} c_n \sin nx. \\ 0 &= u_t(x,0) = \sum_{n=1}^{\infty} d_n(\sqrt{1+n^2})\,\sin nx. \end{aligned}$$

By orthogonality, we may multiply both equations by $\sin mx$ and integrate:

$$\begin{aligned} \int_0^{\pi} f(x)\,\sin mx\,dx &= c_m\frac{\pi}{2}, \\ \int_0^{\pi} 0\,dx &= d_m\frac{\pi}{2}\sqrt{1+m^2}, \end{aligned}$$

which gives the coefficients

$$\begin{aligned} c_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\,\sin nx\,dx \;=\; \frac{2}{\pi}\int_0^{\frac{\pi}{2}} x\,\sin nx\,dx + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi}(\pi - x)\,\sin nx\,dx \\ &= \frac{2}{\pi}\Big[-x\frac{1}{n}\cos nx\Big|_0^{\frac{\pi}{2}} + \frac{1}{n}\int_0^{\frac{\pi}{2}}\cos nx\,dx\Big] + \frac{2}{\pi}\Big[-\frac{\pi}{n}\cos nx\Big|_{\frac{\pi}{2}}^{\pi} + x\frac{1}{n}\cos nx\Big|_{\frac{\pi}{2}}^{\pi} - \frac{1}{n}\int_{\frac{\pi}{2}}^{\pi}\cos nx\,dx\Big] \\ &= \frac{2}{\pi}\Big[-\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2} - \frac{1}{n^2}\sin 0\Big] \\ &+ \frac{2}{\pi}\Big[-\frac{\pi}{n}\cos n\pi + \frac{\pi}{n}\cos\frac{n\pi}{2} + \frac{\pi}{n}\cos n\pi - \frac{\pi}{2n}\cos\frac{n\pi}{2} - \frac{1}{n^2}\sin n\pi + \frac{1}{n^2}\sin\frac{n\pi}{2}\Big] \\ &= \frac{2}{\pi}\Big[\frac{1}{n^2}\sin\frac{n\pi}{2}\Big] + \frac{2}{\pi}\Big[\frac{1}{n^2}\sin\frac{n\pi}{2}\Big] \;=\; \frac{4}{\pi n^2}\sin\frac{n\pi}{2} \\ &= \begin{cases} 0, & n = 2k \\ \frac{4}{\pi n^2}, & n = 4m+1 \\ -\frac{4}{\pi n^2}, & n = 4m+3 \end{cases} \;=\; \begin{cases} 0, & n = 2k \\ (-1)^{\frac{n-1}{2}}\frac{4}{\pi n^2}, & n = 2k+1. \end{cases} \\ d_n &= 0. \end{aligned}$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty}\big(c_n\cos\sqrt{1+n^2}\, t\big)\sin nx.}$$

$\square$

25 Problems: Separation of Variables - Heat Equation

Problem (F'94, #5).
Solve the initial-boundary value problem

$$\begin{cases} u_t = u_{xx} & 0 < x < 2, \ t > 0 \\ u(x,0) = x^2 - x + 1 & 0 \le x \le 2 \\ u(0,t) = 1, \quad u(2,t) = 3 & t > 0. \end{cases}$$

Find $\lim_{t \to +\infty} u(x,t)$.

Proof. ① First, we need to obtain function v that satisfies $v_t = v_{xx}$ and takes 0 boundary conditions. Let

$$\bullet \quad v(x,t) = u(x,t) + (ax+b), \tag{25.1}$$

where a and b are constants to be determined. Then,

$$\begin{aligned} v_t &= u_t, \\ v_{xx} &= u_{xx}. \end{aligned}$$

Thus,

$$v_t = v_{xx}.$$

We need equation (25.1) to take 0 boundary conditions for $v(0,t)$ and $v(2,t)$:

$$\begin{aligned} v(0,t) = 0 \ &= \ u(0,t) + b \ = \ 1 + b \qquad \Rightarrow \quad b = -1, \\ v(2,t) = 0 \ &= \ u(2,t) + 2a - 1 \ = \ 2a + 2 \quad \Rightarrow \quad a = -1. \end{aligned}$$

Thus, (25.1) becomes

$$v(x,t) = u(x,t) - x - 1. \tag{25.2}$$

The new problem is

$$\begin{cases} v_t = v_{xx}, \\ v(x,0) = (x^2 - x + 1) - x - 1 = x^2 - 2x, \\ v(0,t) = v(2,t) = 0. \end{cases}$$

② We solve the problem for v using the method of separation of variables.
Let $v(x,t) = X(x)T(t)$, which gives $XT' - X''T = 0$.

$$\frac{X''}{X} = \frac{T'}{T} = -\lambda.$$

From $X'' + \lambda X = 0$, we get $X_n(x) = a_n \cos\sqrt{\lambda}x + b_n \sin\sqrt{\lambda}x$.
Using boundary conditions, we have

$$\begin{cases} v(0,t) = X(0)T(t) = 0 \\ v(2,t) = X(2)T(t) = 0 \end{cases} \quad \Rightarrow \quad X(0) = X(2) = 0.$$

Hence, $X_n(0) = a_n = 0$, and $X_n(x) = b_n \sin\sqrt{\lambda}x$.
$X_n(2) = b_n \sin 2\sqrt{\lambda} = 0 \quad \Rightarrow \quad 2\sqrt{\lambda} = n\pi \quad \Rightarrow \quad \lambda_n = (\frac{n\pi}{2})^2$.

$$\boxed{X_n(x) = b_n \sin\frac{n\pi x}{2}, \qquad \lambda_n = \left(\frac{n\pi}{2}\right)^2.}$$

With these values of λ_n, we solve $T' + \left(\frac{n\pi}{2}\right)^2 T = 0$ to find

$$\boxed{T_n(t) = c_n e^{-(\frac{n\pi}{2})^2 t}.}$$

$$\boxed{v(x,t) = \sum_{n=1}^{\infty} X_n(x) T_n(t) = \sum_{n=1}^{\infty} \tilde{c}_n \, e^{-(\frac{n\pi}{2})^2 t} \sin\frac{n\pi x}{2}.}$$

Coefficients $\tilde{c}_n$ are obtained using the initial condition:

$$v(x,0) \;=\; \sum_{n=1}^{\infty} \tilde{c}_n \sin\frac{n\pi x}{2} \;=\; x^2 - 2x.$$

$$\tilde{c}_n = \int_0^2 (x^2 - 2x) \sin\frac{n\pi x}{2}\, dx \;=\; \begin{cases} 0 & n \textit{ is even}, \\ -\frac{32}{(n\pi)^3} & n \textit{ is odd}. \end{cases}$$

$$\Rightarrow \quad v(x,t) \;=\; \sum_{n=2k-1}^{\infty} -\frac{32}{(n\pi)^3}\, e^{-(\frac{n\pi}{2})^2 t} \sin\frac{n\pi x}{2}.$$

We now use equation (25.2) to convert back to function u:

$$u(x,t) = v(x,t) + x + 1.$$

$$\boxed{u(x,t) = \sum_{n=2k-1}^{\infty} -\frac{32}{(n\pi)^3}\, e^{-(\frac{n\pi}{2})^2 t} \sin\frac{n\pi x}{2} + x + 1.}$$

$$\lim_{t\to+\infty} u(x,t) \;=\; x + 1.$$

$\square$

Problem (S'96, #6).
Let $u(x,t)$ be the solution of the initial-boundary value problem for the heat equation

$$\begin{cases} u_t = u_{xx} & 0 < x < L, \ t > 0 \\ u(x,0) = f(x) & 0 \le x \le L \\ u_x(0,t) = u_x(L,t) = A & t > 0 \qquad (A = Const). \end{cases}$$

Find $v(x)$ - the limit of $u(x,t)$ when $t \to \infty$. Show that $v(x)$ is ***one*** *of the inifinitely many solutions of the stationary problem*

$$v_{xx} = 0 \qquad 0 < x < L$$
$$v_x(0) = v_x(L) = A.$$

Proof. ① First, we need to obtain function v that satisfies $v_t = v_{xx}$ and takes 0 boundary conditions. Let

- $$v(x,t) = u(x,t) + (ax + b), \tag{25.3}$$

where a and b are constants to be determined. Then,

$$\begin{aligned} v_t &= u_t, \\ v_{xx} &= u_{xx}. \end{aligned}$$

Thus,

$$v_t = v_{xx}.$$

We need equation (25.3) to take 0 boundary conditions for $v_x(0,t)$ and $v_x(L,t)$.
$v_x = u_x + a$.

$$\begin{aligned} v_x(0,t) = 0 &= u_x(0,t) + a = A + a \quad \Rightarrow \quad a = -A, \\ v_x(L,t) = 0 &= u_x(L,t) + a = A + a \quad \Rightarrow \quad a = -A. \end{aligned}$$

We may set $b = 0$ (infinitely many solutions are possible, one for each b).
Thus, (25.3) becomes

$$v(x,t) = u(x,t) - Ax. \tag{25.4}$$

The new problem is

$$\begin{cases} v_t = v_{xx}, \\ v(x,0) = f(x) - Ax, \\ v_x(0,t) = v_x(L,t) = 0. \end{cases}$$

② We solve the problem for v using the method of separation of variables.
Let $v(x,t) = X(x)T(t)$, which gives $XT' - X''T = 0$.

$$\frac{X''}{X} = \frac{T'}{T} = -\lambda.$$

From $X'' + \lambda X = 0$, we get $X_n(x) = a_n \cos\sqrt{\lambda}x + b_n \sin\sqrt{\lambda}x$.
Using boundary conditions, we have

$$\begin{cases} v_x(0,t) = X'(0)T(t) = 0 \\ v_x(L,t) = X'(L)T(t) = 0 \end{cases} \quad \Rightarrow \quad X'(0) = X'(L) = 0.$$

$X_n'(x) = -a_n\sqrt{\lambda}\sin\sqrt{\lambda}x + b_n\sqrt{\lambda}\cos\sqrt{\lambda}x$.
Hence, $X_n'(0) = b_n\sqrt{\lambda_n} = 0 \quad\Rightarrow\quad b_n = 0$; and $X_n(x) = a_n\cos\sqrt{\lambda}x$.
$X_n'(L) = -a_n\sqrt{\lambda}\sin L\sqrt{\lambda} = 0 \quad\Rightarrow\quad L\sqrt{\lambda} = n\pi \quad\Rightarrow\quad \lambda_n = (\frac{n\pi}{L})^2$.

$$\boxed{X_n(x) = a_n\cos\frac{n\pi x}{L}, \qquad \lambda_n = \Big(\frac{n\pi}{L}\Big)^2.}$$

With these values of λ_n, we solve $T' + \big(\frac{n\pi}{L}\big)^2 T = 0$ to find

$$\boxed{T_0(t) = c_0, \quad T_n(t) = c_n e^{-(\frac{n\pi}{L})^2 t}, \quad n = 1, 2, \ldots.}$$

$$\boxed{v(x,t) = \sum_{n=1}^{\infty} X_n(x)T_n(t) = \tilde{c}_0 + \sum_{n=1}^{\infty}\tilde{c}_n\, e^{-(\frac{n\pi}{L})^2 t}\,\cos\frac{n\pi x}{L}.}$$

Coefficients $\tilde{c}_n$ are obtained using the initial condition:

$$v(x,0) \;=\; \tilde{c}_0 + \sum_{n=1}^{\infty}\tilde{c}_n\,\cos\frac{n\pi x}{L} \;=\; f(x) - Ax.$$

$$\begin{aligned}
L\tilde{c}_0 &= \int_0^L (f(x) - Ax)\,dx = \int_0^L f(x)\,dx - \frac{AL^2}{2} \qquad\Rightarrow\quad \tilde{c}_0 = \frac{1}{L}\int_0^L f(x)\,dx - \frac{AL}{2},\\
\frac{L}{2}\tilde{c}_n &= \int_0^L (f(x) - Ax)\, cos\frac{n\pi x}{L}\,dx \qquad\Rightarrow\qquad \tilde{c}_n = \frac{1}{L}\int_0^L (f(x) - Ax)\, cos\frac{n\pi x}{L}\,dx.
\end{aligned}$$

$$\Rightarrow\quad v(x,t) \;=\; \frac{1}{L}\int_0^L f(x)\,dx - \frac{AL}{2} + \sum_{n}^{\infty}\tilde{c}_n\, e^{-(\frac{n\pi}{L})^2 t}\,\cos\frac{n\pi x}{L}.$$

We now use equation (25.4) to convert back to function u:

$$u(x,t) = v(x,t) + Ax.$$

$$\boxed{u(x,t) \;=\; \frac{1}{L}\int_0^L f(x)\,dx - \frac{AL}{2} + \sum_{n}^{\infty}\tilde{c}_n\, e^{-(\frac{n\pi}{L})^2 t}\,\cos\frac{n\pi x}{L} \;+\; Ax.}$$

$$\lim_{t\to+\infty} u(x,t) \;=\; Ax + b, \qquad b \text{ arbitrary.}$$

To show that $v(x)$ is **one** of the inifinitely many solutions of the stationary problem

$$\begin{aligned}
&v_{xx} = 0 \qquad 0 < x < L\\
&v_x(0) = v_x(L) = A,
\end{aligned}$$

we can solve the boundary value problem to obtain $v(x,t) = Ax+b$, where b is arbitrary.

$\square$

Heat Equation with Nonhomogeneous Time-Independent BC in N-dimensions.
The solution to this problem takes somewhat different approach than in the last few problems, but is similar.
Consider the following initial-boundary value problem,

$$\begin{cases} u_t = \triangle u, & x \in \Omega, \ t \geq 0 \\ u(x,0) = f(x), & x \in \Omega \\ u(x,t) = g(x), & x \in \partial\Omega, \ t > 0. \end{cases}$$

Proof. Let $w(x)$ be the solution of the Dirichlet problem:

$$\begin{cases} \triangle w = 0, & x \in \Omega \\ w(x) = g(x), & x \in \partial\Omega \end{cases}$$

and let $v(x,t)$ be the solution of the IBVP for the heat equation with homogeneous BC:

$$\begin{cases} v_t = \triangle v, & x \in \Omega, \ t \geq 0 \\ v(x,0) = f(x) - w(x), & x \in \Omega \\ v(x,t) = 0, & x \in \partial\Omega, \ t > 0. \end{cases}$$

Then $u(x,t)$ satisfies

$$u(x,t) = v(x,t) + w(x).$$

$$\lim_{t\to\infty} u(x,t) = w(x).$$

□

Nonhomogeneous Heat Equation with Nonhomogeneous Time-Independent BC in N dimensions.

Describe the method of solution of the problem

$$\begin{cases} u_t = \triangle u + F(x,t), & x \in \Omega, \ \ t \ge 0 \\ u(x,0) = f(x), & x \in \Omega \\ u(x,t) = g(x), & x \in \partial\Omega, \ \ t > 0. \end{cases}$$

Proof. ❶ We first find u_1, the solution to the homogeneous heat equation (no $F(x,t)$). Let $w(x)$ be the solution of the Dirichlet problem:

$$\begin{cases} \triangle w = 0, & x \in \Omega \\ w(x) = g(x), & x \in \partial\Omega \end{cases}$$

and let $v(x,t)$ be the solution of the IBVP for the heat equation with homogeneous BC:

$$\begin{cases} v_t = \triangle v, & x \in \Omega, \ \ t \ge 0 \\ v(x,0) = f(x) - w(x), & x \in \Omega \\ v(x,t) = 0, & x \in \partial\Omega, \ \ t > 0. \end{cases}$$

Then $u_1(x,t)$ satisfies

$$u_1(x,t) = v(x,t) + w(x).$$

$$\lim_{t\to\infty} u_1(x,t) = w(x).$$

❷ The solution to the homogeneous equation with 0 boundary conditions is given by Duhamel's principle.

$$\begin{cases} u_{2_t} = \triangle u_2 + F(x,t) & \text{for } \ t > 0, \ x \in \mathbb{R}^n \\ u_2(x,0) = 0 & \text{for } \ x \in \mathbb{R}^n. \end{cases} \tag{25.5}$$

Duhamel's principle gives the solution:

$$u_2(x,t) = \int_0^t \int_{\mathbb{R}^n} \tilde{K}(x-y, t-s)\, F(y,s)\, dy\, ds$$

Note: $u_2(x,t) = 0$ on $\partial\Omega$ may not be satisfied.

$$\boxed{u(x,t) = v(x,t) + w(x) + \int_0^t \int_{\mathbb{R}^n} \tilde{K}(x-y, t-s)\, F(y,s)\, dy\, ds.}$$

□

Problem (S'98, #5). *Find the solution of*

$$\begin{cases} u_t = u_{xx}, & t \geq 0, \ 0 < x < 1, \\ u(x,0) = 0, & 0 < x < 1, \\ u(0,t) = 1 - e^{-t}, \quad u_x(1,t) = e^{-t} - 1, & t > 0. \end{cases}$$

Prove that $\lim_{t\to\infty} u(x,t)$ *exists and find it.*

Proof. ① First, we need to obtain function v that satisfies $v_t = v_{xx}$ and takes 0 boundary conditions. Let

- $$v(x,t) = u(x,t) + (ax+b) + (c_1 \cos x + c_2 \sin x)e^{-t}, \tag{25.6}$$

where a, b, c_1, c_2 are constants to be determined. Then,

$$\begin{aligned} v_t &= u_t - (c_1 \cos x + c_2 \sin x)e^{-t}, \\ v_{xx} &= u_{xx} + (-c_1 \cos x - c_2 \sin x)e^{-t}. \end{aligned}$$

Thus,

$$v_t = v_{xx}.$$

We need equation (25.6) to take 0 boundary conditions for $v(0,t)$ and $v_x(1,t)$:

$$\begin{aligned} v(0,t) = 0 &= u(0,t) + b + c_1 e^{-t} \\ &= 1 - e^{-t} + b + c_1 e^{-t}. \end{aligned}$$

Thus, $b = -1$, $c_1 = 1$, and (25.6) becomes

$$v(x,t) = u(x,t) + (ax - 1) + (\cos x + c_2 \sin x)e^{-t}. \tag{25.7}$$

$$\begin{aligned} v_x(x,t) &= u_x(x,t) + a + (-\sin x + c_2 \cos x)e^{-t}, \\ v_x(1,t) = 0 &= u_x(1,t) + a + (-\sin 1 + c_2 \cos 1)e^{-t} \\ &= -1 + a + (1 - \sin 1 + c_2 \cos 1)e^{-t}. \end{aligned}$$

Thus, $a = 1$, $c_2 = \frac{\sin 1 - 1}{\cos 1}$, and equation (25.7) becomes

$$v(x,t) = u(x,t) + (x-1) + (\cos x + \frac{\sin 1 - 1}{\cos 1} \sin x)e^{-t}. \tag{25.8}$$

Initial condition tranforms to:

$$v(x,0) = u(x,0) + (x-1) + (\cos x + \frac{\sin 1 - 1}{\cos 1} \sin x) = (x-1) + (\cos x + \frac{\sin 1 - 1}{\cos 1} \sin x).$$

The new problem is

$$\begin{cases} v_t = v_{xx}, \\ v(x,0) = (x-1) + (\cos x + \frac{\sin 1 - 1}{\cos 1} \sin x), \\ v(0,t) = 0, \quad v_x(1,t) = 0. \end{cases}$$

② We solve the problem for v using the method of separation of variables.
Let $v(x,t) = X(x)T(t)$, which gives $XT' - X''T = 0$.

$$\frac{X''}{X} = \frac{T'}{T} = -\lambda.$$

From $X'' + \lambda X = 0$, we get $X_n(x) = a_n \cos\sqrt{\lambda}x + b_n \sin\sqrt{\lambda}x$.
Using the first boundary condition, we have

$$v(0,t) = X(0)T(t) = 0 \quad \Rightarrow \quad X(0) = 0.$$

Hence, $X_n(0) = a_n = 0$, and $X_n(x) = b_n \sin\sqrt{\lambda}x$. We also have

$$\begin{aligned}
v_x(1,t) &= X'(1)T(t) = 0 \quad \Rightarrow \quad X'(1) = 0. \\
X_n'(x) &= \sqrt{\lambda} b_n \cos\sqrt{\lambda}x, \\
X_n'(1) &= \sqrt{\lambda} b_n \cos\sqrt{\lambda} = 0, \\
& \cos\sqrt{\lambda} = 0, \\
& \sqrt{\lambda} = n\pi + \frac{\pi}{2}.
\end{aligned}$$

Thus,

$$\boxed{X_n(x) = b_n \sin\Big(n\pi + \frac{\pi}{2}\Big)x, \qquad \lambda_n = \Big(n\pi + \frac{\pi}{2}\Big)^2.}$$

With these values of λ_n, we solve $T' + \Big(n\pi + \frac{\pi}{2}\Big)^2 T = 0$ to find

$$\boxed{T_n(t) = c_n e^{-(n\pi+\frac{\pi}{2})^2 t}.}$$

$$\boxed{v(x,t) = \sum_{n=1}^{\infty} X_n(x)T_n(t) = \sum_{n=1}^{\infty} \tilde{b}_n \sin\Big(n\pi + \frac{\pi}{2}\Big)x \; e^{-(n\pi+\frac{\pi}{2})^2 t}.}$$

We now use equation (25.8) to convert back to function u:

$$u(x,t) = v(x,t) - (x-1) - (\cos x + \frac{\sin 1 - 1}{\cos 1}\sin x)e^{-t}.$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \tilde{b}_n \sin\Big(n\pi + \frac{\pi}{2}\Big)x \; e^{-(n\pi+\frac{\pi}{2})^2 t} - (x-1) - (\cos x + \frac{\sin 1 - 1}{\cos 1}\sin x)e^{-t}.}$$

Coefficients $\tilde{b}_n$ are obtained using the initial condition:

$$u(x,0) = \sum_{n=1}^{\infty} \tilde{b}_n \sin\Big(n\pi + \frac{\pi}{2}\Big)x \; - (x-1) - (\cos x + \frac{\sin 1 - 1}{\cos 1}\sin x).$$

③ Finally, we can check that the differential equation and the boundary conditions are satisfied:

$$\begin{aligned}
u(0,t) &= 1 - (1+0)e^{-t} = 1 - e^{-t}. \ \checkmark \\
u_x(x,t) &= \sum_{n=1}^{\infty} \tilde{b}_n \Big(n\pi + \frac{\pi}{2}\Big) \cos\Big(n\pi + \frac{\pi}{2}\Big)x \; e^{-(n\pi+\frac{\pi}{2})^2 t} - 1 + (\sin x - \frac{\sin 1 - 1}{\cos 1}\cos x)e^{-t}, \\
u_x(1,t) &= -1 + (\sin 1 - \frac{\sin 1 - 1}{\cos 1}\cos 1)e^{-t} = -1 + e^{-t}. \ \checkmark \\
u_t &= \sum_{n=1}^{\infty} -\tilde{b}_n \Big(n\pi + \frac{\pi}{2}\Big)^2 \sin\Big(n\pi + \frac{\pi}{2}\Big)x \; e^{-(n\pi+\frac{\pi}{2})^2 t} + (\cos x + \frac{\sin 1 - 1}{\cos 1}\sin x)e^{-t} = u_{xx}. \ \checkmark
\end{aligned}$$

$\square$

Problem (F'02, #6). *The temperature of a rod insulated at the ends with an exponentially decreasing heat source in it is a solution of the following boundary value problem:*

$$\begin{cases} u_t = u_{xx} + e^{-2t}g(x) & \textit{for} \quad (x,t) \in [0,1] \times \mathbb{R}_+ \\ u_x(0,t) = u_x(1,t) = 0 \\ u(x,0) = f(x). \end{cases}$$

Find the solution to this problem by writing u *as a cosine series,*

$$u(x,t) = \sum_{n=0}^{\infty} a_n(t) \cos n\pi x, \qquad \circledast$$

and determine $\lim_{t\to\infty} u(x,t)$.

Proof. Let g accept an expansion in eigenfunctions

$$g(x) = b_0 + \sum_{n=1}^{\infty} b_n \cos n\pi x \qquad \text{with} \qquad b_n = 2\int_0^1 g(x) \cos n\pi x \, dx.$$

Plugging $\circledast$ in the PDE gives:

$$a_0'(t) + \sum_{n=1}^{\infty} a_n'(t) \cos n\pi x = -\sum_{n=1}^{\infty} n^2\pi^2 a_n(t) \cos n\pi x + b_0 e^{-2t} + e^{-2t} \sum_{n=1}^{\infty} b_n \cos n\pi x,$$

which gives

$$\begin{cases} a_0'(t) = b_0 e^{-2t}, \\ a_n'(t) + n^2\pi^2 a_n(t) = b_n e^{-2t}, \quad n = 1, 2, \ldots. \end{cases}$$

Adding homogeneous and particular solutions of the above ODEs, we obtain the solutions

$$\begin{cases} a_0(t) = c_0 - \frac{b_0}{2} e^{-2t}, \\ a_n(t) = c_n e^{-n^2\pi^2 t} - \frac{b_n}{2-n^2\pi^2} e^{-2t}, \quad n = 1, 2, \ldots, \end{cases}$$

for some constants c_n, $n = 0, 1, 2, \ldots$. Thus,

$$\boxed{u(x,t) = \sum_{n=0}^{\infty} \Big(c_n e^{-n^2\pi^2 t} - \frac{b_n}{2 - n^2\pi^2} e^{-2t} \Big) \cos n\pi x.}$$

Initial condition gives

$$u(x,0) = \sum_{n=0}^{\infty} \Big(c_n - \frac{b_n}{2 - n^2\pi^2} \Big) \cos n\pi x = f(x),$$

As, $t \to \infty$, the only mode that survives is $n = 0$:

$$u(x,t) \ \to \ c_0 + \frac{b_0}{2} \qquad \text{as} \quad t \to \infty.$$

$\square$

Problem (F'93, #4). ***a)*** *Assume $f, g \in C^\infty$. Give the compatibility conditions which f and g must satisfy if the following problem is to possess a solution.*

$$\begin{aligned} &\triangle u = f(x) && x \in \Omega \\ &\frac{\partial u}{\partial n}(s) = g(s) && s \in \partial\Omega. \end{aligned}$$

Show that your condition is necessary for a solution to exist.

b) *Give an explicit solution to*

$$\begin{cases} u_t = u_{xx} + \cos x & x \in [0, 2\pi] \\ u_x(0,t) = u_x(2\pi, t) = 0 & t > 0 \\ u(x,0) = \cos x + \cos 2x & x \in [0, 2\pi]. \end{cases}$$

c) *Does there exist a steady state solution to the problem in (b) if*

$$u_x(0) = 1 \qquad u_x(2\pi) = 0 \quad ?$$

Explain your answer.

Proof. **a)** Integrating the equation and using Green's identity gives:

$$\int_\Omega f(x)\,dx \;=\; \int_\Omega \triangle u\,dx \;=\; \int_{\partial\Omega} \frac{\partial u}{\partial n}\,ds \;=\; \int_{\partial\Omega} g(s)\,ds.$$

b) With

- $v(x,t) = u(x,t) - \cos x$

the problem above transforms to

$$\begin{cases} v_t = v_{xx} \\ v_x(0,t) = v_x(2\pi,t) = 0 \\ v(x,0) = \cos 2x. \end{cases}$$

We solve this problem for v using the separation of variables. Let $v(x,t) = X(x)T(t)$, which gives $XT' = X''T$.

$$\frac{X''}{X} = \frac{T'}{T} = -\lambda.$$

From $X'' + \lambda X = 0$, we get $X_n(x) = a_n \cos\sqrt{\lambda}x + b_n \sin\sqrt{\lambda}x$.
$X_n'(x) = -\sqrt{\lambda_n}a_n \sin\sqrt{\lambda}x + \sqrt{\lambda_n}b_n \cos\sqrt{\lambda}x$.
Using boundary conditions, we have

$$\begin{cases} v_x(0,t) = X'(0)T(t) = 0 \\ v_x(2\pi,t) = X'(2\pi)T(t) = 0 \end{cases} \quad \Rightarrow \quad X'(0) = X'(2\pi) = 0.$$

Hence, $X_n'(0) = \sqrt{\lambda_n}b_n = 0$, and $X_n(x) = a_n \cos\sqrt{\lambda_n}x$.
$X_n'(2\pi) = -\sqrt{\lambda_n}a_n \sin\sqrt{\lambda_n}2\pi = 0 \quad \Rightarrow \quad \sqrt{\lambda_n} = \frac{n}{2} \quad \Rightarrow \quad \lambda_n = (\frac{n}{2})^2$. Thus,

$$\boxed{X_n(x) = a_n \cos\frac{nx}{2}, \qquad \lambda_n = \left(\frac{n}{2}\right)^2}$$

With these values of λ_n, we solve $T' + \left(\frac{n}{2}\right)^2 T = 0$ to find

$$\boxed{T_n(t) = c_n e^{-(\frac{n}{2})^2 t}.}$$

$$\boxed{v(x,t) = \sum_{n=0}^{\infty} X_n(x) T_n(t) = \sum_{n=0}^{\infty} \tilde{a}_n \, e^{-(\frac{n}{2})^2 t} \, \cos\frac{nx}{2}.}$$

Initial condition gives

$$v(x,0) \;=\; \sum_{n=0}^{\infty} \tilde{a}_n \cos\frac{nx}{2} \;=\; \cos 2x.$$

Thus, $\tilde{a}_4 = 1$, $\tilde{a}_n = 0$, $n \neq 4$. Hence,

$$v(x,t) \;=\; e^{-4t} \cos 2x.$$

$$\boxed{u(x,t) = v(x,t) + \cos x \;=\; e^{-4t} \cos 2x + \cos x.}$$

c) *Does there exist a steady state solution to the problem in (b) if*

$$u_x(0) = 1 \qquad u_x(2\pi) = 0 \quad ?$$

Explain your answer.

c) Set $u_t = 0$. We have

$$\begin{cases} u_{xx} + \cos x = 0 & x \in [0, 2\pi] \\ u_x(0) = 1, \quad u_x(2\pi) = 0. \end{cases}$$

$$\begin{aligned} u_{xx} &= -\cos x, \\ u_x &= -\sin x + C, \\ u(x) &= \cos x + Cx + D. \end{aligned}$$

Boundary conditions give:

$$\begin{aligned} 1 = u_x(0) &= C, \\ 0 = u_x(2\pi) &= C \qquad \Rightarrow \quad \text{contradiction} \end{aligned}$$

There exists no steady state solution.

We may use the result we obtained in part (a) with $u_{xx} = \cos x = f(x)$. We need

$$\begin{aligned} \int_\Omega f(x)\,dx &= \int_{\partial\Omega} \frac{\partial u}{\partial n}\,ds, \\ \underbrace{\int_0^{2\pi} \cos x\,dx}_{=0} &= u_x(2\pi) - u_x(0) = \underbrace{-1}_{given}. \end{aligned}$$

$\square$

Problem (F'96, #7). *Solve the parabolic problem*

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_{xx}, \qquad 0 \le x \le \pi, \ t > 0$$

$$\begin{aligned} u(x,0) &= \sin x, \qquad u(0,t) = u(\pi,t) = 0, \\ v(x,0) &= \sin x, \qquad v(0,t) = v(\pi,t) = 0. \end{aligned}$$

Prove the ***energy estimate*** *(for general initial data)*

$$\int_{x=0}^{\pi} [u^2(x,t) + v^2(x,t)]\, dx \le c \int_{x=0}^{\pi} [u^2(x,0) + v^2(x,0)]\, dx$$

for come constant c.

Proof. We can solve the second equation for v and then use the value of v to solve the first equation for u. [73]

① We have

$$\begin{cases} v_t = 2v_{xx}, & 0 \le x \le \pi, \ t > 0 \\ v(x,0) = \sin x, \\ v(0,t) = v(\pi,t) = 0. \end{cases}$$

Assume $v(x,t) = X(x)T(t)$, then substitution in the PDE gives $XT' = 2X''T$.

$$\frac{T'}{T} = 2\frac{X''}{X} = -\lambda.$$

From $X'' + \frac{\lambda}{2}X = 0$, we get $X_n(x) = a_n \cos\sqrt{\frac{\lambda}{2}}x + b_n \sin\sqrt{\frac{\lambda}{2}}x$.

Boundary conditions give

$$\begin{cases} v(0,t) = X(0)T(t) = 0 \\ v(\pi,t) = X(\pi)T(t) = 0 \end{cases} \qquad \Rightarrow \quad X(0) = X(\pi) = 0.$$

Thus, $X_n(0) = a_n = 0$, and $X_n(x) = b_n \sin\sqrt{\frac{\lambda}{2}}x$.

$X_n(\pi) = b_n \sin\sqrt{\frac{\lambda}{2}}\pi = 0$. Hence $\sqrt{\frac{\lambda}{2}} = n$, or $\lambda = 2n^2$.

$$\boxed{\lambda = 2n^2, \qquad X_n(x) = b_n \sin nx.}$$

With these values of λ_n, we solve $T' + 2n^2T = 0$ to get $T_n(t) = c_n e^{-2n^2t}$.

Thus, the solution may be written in the form

$$v(x,t) = \sum_{n=1}^{\infty} \tilde{a}_n e^{-2n^2t} \sin nx.$$

From initial condition, we get

$$v(x,0) = \sum_{n=1}^{\infty} \tilde{a}_n \sin nx = \sin x.$$

Thus, $\tilde{a}_1 = 1, \quad \tilde{a}_n = 0, \quad n = 2, 3, \ldots$.

$$\boxed{v(x,t) = e^{-2t} \sin x.}$$

[73]Note that if the matrix was fully inseparable, we would have to find eigenvalues and eigenvectors, just as we did for the hyperbolic systems.

② We have

$$\begin{cases} u_t = u_{xx} - \frac{1}{2}e^{-2t}\sin x, \qquad 0 \le x \le \pi,\ t > 0 \\ u(x,0) = \sin x, \\ u(0,t) = u(\pi,t) = 0. \end{cases}$$

Let $u(x,t) = \sum_{n=1}^{\infty} u_n(t)\sin nx$. Plugging this into the equation, we get

$$\sum_{n=1}^{\infty} u_n'(t)\sin nx + \sum_{n=1}^{\infty} n^2 u_n(t)\sin nx = -\frac{1}{2}e^{-2t}\sin x.$$

For $n = 1$:

$$u_1'(t) + u_1(t) = -\frac{1}{2}e^{-2t}.$$

Combining homogeneous and particular solution of the above equation, we obtain:

$$u_1(t) = \frac{1}{2}e^{-2t} + c_1 e^{-t}.$$

For $n = 2, 3, \ldots$:

$$u_n'(t) + n^2 u_n(t) = 0,$$
$$u_n(t) = c_n e^{-n^2 t}.$$

Thus,

$$u(x,t) = \Big(\frac{1}{2}e^{-2t} + c_1 e^{-t}\Big)\sin x + \sum_{n=2}^{\infty} c_n e^{-n^2 t}\sin nx = \frac{1}{2}e^{-2t}\sin x + \sum_{n=1}^{\infty} c_n e^{-n^2 t}\sin nx.$$

From initial condition, we get

$$u(x,0) = \frac{1}{2}\sin x + \sum_{n=1}^{\infty} c_n \sin nx = \sin x.$$

Thus, $c_1 = \frac{1}{2}$, $c_n = 0$, $n = 2, 3, \ldots$.

$$\boxed{u(x,t) = \frac{1}{2}\sin x\,(e^{-2t} + e^{-t}).}$$

To prove the **energy estimate** (for general initial data)

$$\int_{x=0}^{\pi} [u^2(x,t) + v^2(x,t)]\,dx \le c\int_{x=0}^{\pi} [u^2(x,0) + v^2(x,0)]\,dx$$

for come constant c, we assume that

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin nx, \qquad v(x,0) = \sum_{n=1}^{\infty} b_n \sin nx.$$

The general solutions are obtained by the same method as above

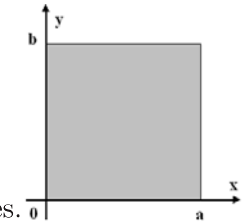
$$
\begin{aligned}
u(x,t) &= \frac{1}{2}e^{-2t}\sin x + \sum_{n=1}^{\infty} c_n e^{-n^2 t}\sin nx, \\
v(x,t) &= \sum_{n=1}^{\infty} b_n e^{-2n^2 t}\,\sin nx. \\
\int_{x=0}^{\pi} [u^2(x,t) + v^2(x,t)]\,dx &= \int_{x=0}^{\pi} \Big(\frac{1}{2}e^{-2t}\sin x + \sum_{n=1}^{\infty} c_n e^{-n^2 t}\sin nx\Big)^2 + \Big(\sum_{n=1}^{\infty} b_n e^{-2n^2 t}\,\sin nx\Big)^2\,dx \\
&\le \sum_{n=1}^{\infty}(b_n^2 + a_n^2)\int_{x=0}^{\pi}\sin^2 nx\,dx \;\le\; \int_{x=0}^{\pi}[u^2(x,0)+v^2(x,0)]\,dx.
\end{aligned}
$$

$\square$

26 Problems: Eigenvalues of the Laplacian - Laplace

The 2D LAPLACE Equation (eigenvalues/eigenfuctions of the Laplacian). *Consider*

$$\begin{cases} u_{xx} + u_{yy} + \lambda u = 0 & \textit{in } \Omega \\ u(0,y) = 0 = u(a,y) & \textit{for } 0 \le y \le b, \\ u(x,0) = 0 = u(x,b) & \textit{for } 0 \le x \le a. \end{cases} \tag{26.1}$$



Proof. We can solve this problem by separation of variables.
Let $u(x,y) = X(x)Y(y)$, then substitution in the PDE gives $X''Y + XY'' + \lambda XY = 0$.

$$\frac{X''}{X} + \frac{Y''}{Y} + \lambda = 0.$$

Letting $\lambda = \mu^2 + \nu^2$ and using boundary conditions, we find the equations for X and Y:

$$\begin{aligned} &X'' + \mu^2 X = 0 &\qquad& Y'' + \nu^2 Y = 0 \\ &X(0) = X(a) = 0 && Y(0) = Y(b) = 0. \end{aligned}$$

The solutions of these one-dimensional eigenvalue problems are

$$\begin{aligned} &\mu_m = \frac{m\pi}{a} &\qquad& \nu_n = \frac{n\pi}{b} \\ &X_m(x) = \sin\frac{m\pi x}{a} && Y_n(y) = \sin\frac{n\pi y}{b}, \end{aligned}$$

where $m, n = 1, 2, \ldots$. Thus we obtain solutions of (26.1) of the form

$$\lambda_{mn} = \pi^2\Big(\frac{m^2}{a^2} + \frac{n^2}{b^2}\Big) \qquad\qquad u_{mn}(x,y) = \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b},$$

where $m, n = 1, 2, \ldots$.

Observe that the eigenvalues $\{\lambda_{mn}\}_{m,n=1}^{\infty}$ are positive. The smallest eigenvalue λ_{11} has only one eigenfunction $u_{11}(x,y) = \sin(\pi x/a)\sin(\pi y/b)$; notice that u_{11} is positive in Ω. Other eigenvalues λ may correspond to more than one choice of m and n; for example, in the case $a = b$ we have $\lambda_{nm} = \lambda_{nm}$. For this λ, there are two linearly independent eigenfunctions. However, for a particular value of λ there are at most finitely many linearly independent eigenfunctions. Moreover,

$$\begin{aligned} \int_0^b\!\!\int_0^a u_{mn}(x,y)\,u_{m'n'}(x,y)\,dx\,dy &= \int_0^b\!\!\int_0^a \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\sin\frac{m'\pi x}{a}\sin\frac{n'\pi y}{b}\,dx\,dy \\ = \begin{cases} \frac{a}{2}\int_0^b \sin\frac{n\pi y}{b}\sin\frac{n'\pi y}{b}\,dy \\ 0 \end{cases} &= \begin{cases} \frac{ab}{4} & \text{if } m = m' \text{ and } n = n' \\ 0 & \text{if } m \ne m' \text{ or } n \ne n'. \end{cases} \end{aligned}$$

In particular, the $\{u_{mn}\}$ are pairwise orthogonal. We could *normalize* each u_{mn} by a scalar multiple (i.e. multiply by $\sqrt{4/ab}$) so that $ab/4$ above becomes 1. $\square$

Let us change the notation somewhat so that each eigenvalue λ_n corresponds to a particular eigenfunction $\phi_n(x)$. If we choose an orthonormal basis of eigenfunctions in each eigenspace, we may arrange that $\{\phi_n\}_{n=1}^{\infty}$ is pairwise orthonormal:

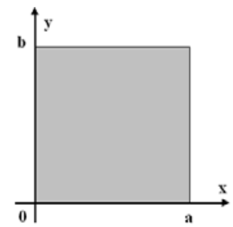
$$\int_\Omega \phi_n(x)\phi_m(x)\,dx = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \ne n. \end{cases}$$

In this notation, the eigenfunction expansion of $f(x)$ defined on Ω becomes

$$f(x) \sim \sum_{n=1}^{\infty} a_n \phi_n(x), \qquad \text{where} \qquad a_n = \int_{\Omega} f(x)\phi_n(x)\,dx.$$

Problem (S'96, #4). *Let D denote the rectangular*

$$D = \{(x,y) \in \mathbb{R}^2 : 0 < x < a, \ 0 < y < b\}.$$



*Find the **eigenvalues** of the following Dirichlet problem:*

$$(\triangle + \lambda)u = 0 \qquad in \ \ D$$
$$u = 0 \qquad on \ \ \partial D.$$

Proof. The problem may be rewritten as

$$\begin{cases} u_{xx} + u_{yy} + \lambda u = 0 & \text{in } \ \Omega \\ u(0,y) = 0 = u(a,y) & \text{for } \ 0 \le y \le b, \\ u(x,0) = 0 = u(x,b) & \text{for } \ 0 \le x \le a. \end{cases}$$

We may assume that the eigenvalues λ are positive, $\lambda = \mu^2 + \nu^2$. Then,

$$\boxed{\lambda_{mn} = \pi^2\Big(\frac{m^2}{a^2} + \frac{n^2}{b^2}\Big) \qquad u_{mn}(x,y) = \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}, \qquad m,n = 1,2,\ldots.}$$

□

Problem (W'04, #1). *Consider the differential equation:*

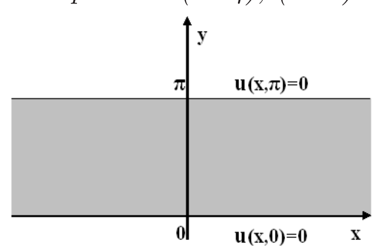
$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} + \lambda u(x,y) = 0 \tag{26.2}$$

in the strip $\{(x,y), 0 < y < \pi, -\infty < x < +\infty\}$ with boundary conditions

$$u(x,0) = 0, \quad u(x,\pi) = 0. \tag{26.3}$$

Find all bounded solutions of the boundary value problem (26.4), (26.5) when

a) $\lambda = 0$, ***b)*** $\lambda > 0$, ***c)*** $\lambda < 0$.



Proof. **a)** $\lambda = 0$. We have

$$u_{xx} + u_{yy} = 0.$$

Assume $u(x,y) = X(x)Y(y)$, then substitution in the PDE gives

$$X''Y + XY'' = 0.$$

Boundary conditions give

$$\begin{cases} u(x,0) = X(x)Y(0) = 0 \\ u(x,\pi) = X(x)Y(\pi) = 0 \end{cases} \Rightarrow \quad Y(0) = Y(\pi) = 0.$$

Method I: We have

$$\frac{X''}{X} = -\frac{Y''}{Y} = -c, \qquad c > 0.$$

From $X'' + cX = 0$, we have $X_n(x) = a_n \cos\sqrt{c}x + b_n \sin\sqrt{c}x$.
From $Y'' - cY = 0$, we have $Y_n(y) = c_n e^{-\sqrt{c}y} + d_n e^{\sqrt{c}y}$.
$Y(0) = c_n + d_n = 0 \quad \Rightarrow \quad c_n = -d_n$.

$Y(\pi) = c_n e^{-\sqrt{c}\pi} - c_n e^{\sqrt{c}\pi} = 0 \quad \Rightarrow \quad c_n = 0 \quad \Rightarrow \; Y_n(y) = 0.$
$\Rightarrow \quad u(x, y) = X(x)Y(y) = 0.$

Method II: We have

$$\frac{X''}{X} = -\frac{Y''}{Y} = c, \qquad c > 0.$$

From $X'' - cX = 0$, we have $X_n(x) = a_n e^{-\sqrt{c}x} + b_n e^{\sqrt{c}x}$.
Since we look for bounded solutions for $-\infty < x < \infty$, $a_n = b_n = 0 \quad \Rightarrow X_n(x) = 0$.
From $Y'' + cY = 0$, we have $Y_n(y) = c_n \cos\sqrt{c}y + d_n \sin\sqrt{c}y$.
$Y(0) = c_n = 0,$
$Y(\pi) = d_n \sin\sqrt{c}\pi = 0 \quad \Rightarrow \quad \sqrt{c} = n \quad \Rightarrow \quad c = n^2.$
$\Rightarrow \quad Y_n(y) = d_n \sin nx = 0.$
$\Rightarrow \quad u(x, y) = X(x)Y(y) = 0.$

b) $\lambda > 0$. We have

$$\frac{X''}{X} + \frac{Y''}{Y} + \lambda = 0.$$

Letting $\lambda = \mu^2 + \nu^2$, and using boundary conditions for Y, we find the equations:

$$X'' + \mu^2 X = 0 \qquad\qquad \begin{aligned} &Y'' + \nu^2 Y = 0 \\ &Y(0) = Y(\pi) = 0. \end{aligned}$$

The solutions of these one-dimensional eigenvalue problems are
$X_m(x) = a_m \cos\mu_m x + b_m \sin\mu_m x.$
$\nu_n = n, \; Y_n(y) = d_n \sin ny, \quad$ where $m, n = 1, 2, \ldots$.

$$\boxed{u(x, y) = \sum_{m,n=1}^{\infty} u_{mn}(x, y) = \sum_{m,n=1}^{\infty} (a_m \cos\mu_m x + b_m \sin\mu_m x) \sin ny.}$$

c) $\lambda < 0$. We have

$$\begin{aligned} u_{xx} + u_{yy} + \lambda u = 0&, \\ u(x, 0) = 0, \quad u(x, \pi) = 0&. \end{aligned}$$

$u \equiv 0$ is the solution to this equation. We will show that this solution is unique.
Let u_1 and u_2 be two solutions, and consider $w = u_1 - u_2$. Then,

$$\begin{aligned} \triangle w + \lambda w = 0&, \\ w(x, 0) = 0, \quad w(x, \pi) = 0&. \end{aligned}$$

Multiply the equation by w and integrate:

$$\begin{aligned} w\triangle w + \lambda w^2 &= 0, \\ \int_\Omega w\triangle w \, dx + \lambda \int_\Omega w^2 \, dx &= 0, \\ \underbrace{\int_{\partial\Omega} w \frac{\partial w}{\partial n} \, ds}_{=0} - \int_\Omega |\nabla w|^2 \, dx + \lambda \int_\Omega w^2 \, dx &= 0, \end{aligned}$$

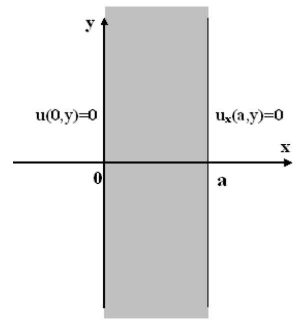
$$\underbrace{\int_\Omega |\nabla w|^2 \, dx}_{\ge 0} = \underbrace{\lambda \int_\Omega w^2 \, dx}_{\le 0}.$$

Thus, $w \equiv 0$ and the solution $u(x, y) \equiv 0$ is unique. $\square$

Problem (F'95, #5). *Find **all** bounded solutions for the following boundary value problem in the strip $0 < x < a$, $-\infty < y < \infty$,*

$$(\triangle + k^2)u = 0 \qquad (k = Const > 0),$$
$$u(0, y) = 0, \qquad u_x(a, y) = 0.$$

In particular, show that when $ak \le \pi$, the only bounded solution to this problem is $u \equiv 0$.



Proof. Let $u(x, y) = X(x)Y(y)$, then we have $X''Y + XY'' + k^2XY = 0$.

$$\frac{X''}{X} + \frac{Y''}{Y} + k^2 = 0.$$

Letting $k^2 = \mu^2 + \nu^2$ and using boundary conditions, we find:

$$X'' + \mu^2 X = 0, \qquad Y'' + \nu^2 Y = 0.$$
$$X(0) = X'(a) = 0.$$

The solutions of these one-dimensional eigenvalue problems are

$$\mu_m = \frac{(m - \frac{1}{2})\pi}{a},$$
$$X_m(x) = \sin\frac{(m - \frac{1}{2})\pi x}{a} \qquad Y_n(y) = c_n \cos\nu_n y + d_n \sin\nu_n y,$$

where $m, n = 1, 2, \ldots$. Thus we obtain solutions of the form

$$k_{mn}^2 = \left(\frac{(m - \frac{1}{2})\pi}{a}\right)^2 + \nu_n^2, \qquad u_{mn}(x, y) = \sin\frac{(m - \frac{1}{2})\pi x}{a}\bigg(c_n \cos\nu_n y + d_n \sin\nu_n y\bigg),$$

where $m, n = 1, 2, \ldots$.

$$\boxed{u(x, y) = \sum_{m,n=1}^{\infty} u_{mn}(x, y) = \sum_{m,n=1}^{\infty} \sin\frac{(m - \frac{1}{2})\pi x}{a}\bigg(c_n \cos\nu_n y + d_n \sin\nu_n y\bigg).}$$

• We can take an **alternate** approach and prove the second part of the question. We have

$$X''Y + XY'' + k^2XY = 0,$$
$$-\frac{Y''}{Y} = \frac{X''}{X} + k^2 = c^2.$$

We obtain $Y_n(y) = c_n \cos cy + d_n \sin cy$. The second equation gives

$$X'' + k^2 X = c^2 X,$$
$$X'' + (k^2 - c^2)X = 0,$$
$$X_m(x) = a_m e^{\sqrt{c^2-k^2}x} + b_m e^{\sqrt{c^2-k^2}x}.$$

Thus, $X_m(x)$ is bounded only if $k^2 - c^2 > 0$, (if $k^2 - c^2 = 0$, $X'' = 0$, and $X_m(x) = a_m x + b_m$, BC's give $X_m(x) = \pi x$, unbounded), in which case

$$X_m(x) = a_m \cos\sqrt{k^2 - c^2}\, x + b_m \sin\sqrt{k^2 - c^2}\, x.$$

Boundary conditions give $X_m(0) = a_m = 0$.

$$\begin{aligned}
X_m'(x) &= b_m\sqrt{k^2-c^2}\cos\sqrt{k^2-c^2}\,x,\\
X_m'(a) &= b_m\sqrt{k^2-c^2}\cos\sqrt{k^2-c^2}\,a \;=\; 0,\\
\sqrt{k^2-c^2}\,a &= m\pi - \frac{\pi}{2}, \qquad\qquad m = 1,2,\ldots,\\
k^2 - c^2 &= \Big(\frac{\pi}{a}\Big(m-\frac{1}{2}\Big)\Big)^2,\\
k^2 &= \Big(\frac{\pi}{a}\Big)^2\Big(m-\frac{1}{2}\Big)^2 + c^2,\\
a^2k^2 &> \pi^2\Big(m-\frac{1}{2}\Big)^2,\\
ak &> \pi\Big(m-\frac{1}{2}\Big), \qquad\qquad m = 1,2,\ldots.
\end{aligned}$$

Thus, bounded solutions exist only when $ak > \frac{\pi}{2}$. □

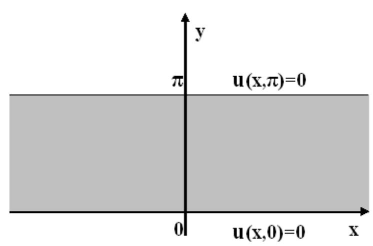
Problem (S'90, #2). *Show that the boundary value problem*

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} + k^2 u(x,y) = 0, \tag{26.4}$$

where $-\infty < x < +\infty$, $0 < y < \pi$, $k > 0$ *is a constant,*

$$u(x,0) = 0, \quad u(x,\pi) = 0 \tag{26.5}$$

has a bounded solution if and only if $k \geq 1$.



Proof. We have

$$\begin{aligned}
&u_{xx} + u_{yy} + k^2 u = 0,\\
&X''Y + XY'' + k^2XY \;=\; 0,\\
&-\frac{X''}{X} \;=\; \frac{Y''}{Y} + k^2 \;=\; c^2.
\end{aligned}$$

We obtain $X_m(x) = a_m\cos cx + b_m\sin cx$. The second equation gives

$$\begin{aligned}
Y'' + k^2Y &= c^2Y,\\
Y'' + (k^2-c^2)Y &= 0,\\
Y_n(y) &= c_n e^{\sqrt{c^2-k^2}y} + d_n e^{\sqrt{c^2-k^2}y}.
\end{aligned}$$

Thus, $Y_n(y)$ is bounded only if $k^2 - c^2 > 0$, (if $k^2-c^2 = 0$, $Y'' = 0$, and $Y_n(y) = c_ny + d_n$, BC's give $Y \equiv 0$), in which case

$$Y_n(y) \;=\; c_n\cos\sqrt{k^2-c^2}\,y + d_n\sin\sqrt{k^2-c^2}\,y.$$

Boundary conditions give $Y_n(0) = c_n = 0$.
$Y_n(\pi) = d_n\sin\sqrt{k^2-c^2}\,\pi = 0 \quad\Rightarrow\quad \sqrt{k^2-c^2} = n \quad\Rightarrow\quad k^2-c^2 = n^2 \quad\Rightarrow$
$k^2 = n^2 + c^2$, $n = 1,2,\ldots$. Hence, $k > n$, $n = 1,2,\ldots$.

Thus, bounded solutions exist if $k \geq 1$.
Note: If $k = 1$, then $c = 0$, which gives trivial solutions for $Y_n(y)$.

$$\boxed{u(x,y) = \sum_{m,n=1}^{\infty} X_m(x)Y_n(y) = \sum_{m,n=1}^{\infty} \sin ny\, X_m(x).}$$

$\square$

McOwen, 4.4 #7; 266B Ralston Hw. *Show that the boundary value problem*

$$\begin{cases} -\nabla \cdot a(x)\nabla u + b(x)u = \lambda u & \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{cases}$$

has only trivial solution with $\lambda \leq 0$, when $b(x) \geq 0$ and $a(x) > 0$ in Ω.

Proof. Multiplying the equation by u and integrating over Ω, we get

$$\int_\Omega -u\nabla \cdot a\nabla u\, dx + \int_\Omega bu^2\, dx = \lambda \int_\Omega u^2\, dx.$$

Since $\nabla \cdot (ua\nabla u) = u\nabla \cdot a\nabla u + a|\nabla u|^2$, we have

$$\int_\Omega -\nabla \cdot (ua\nabla u)\, dx + \int_\Omega a|\nabla u|^2\, dx + \int_\Omega bu^2\, dx = \lambda \int_\Omega u^2\, dx. \tag{26.6}$$

Using divergence theorem, we obtain

$$\int_{\partial\Omega} - \underbrace{u}_{=0} a\frac{\partial u}{\partial n}\, ds + \int_\Omega a|\nabla u|^2\, dx + \int_\Omega bu^2\, dx = \lambda \int_\Omega u^2\, dx,$$

$$\int_\Omega \underbrace{a}_{>0} |\nabla u|^2\, dx + \int_\Omega \underbrace{b}_{\geq 0} u^2\, dx = \underbrace{\lambda}_{\leq 0} \int_\Omega u^2\, dx,$$

Thus, $\nabla u = 0$ in Ω, and u is constant. Since $u = 0$ on $\partial\Omega$, $u \equiv 0$ on Ω.

Similar Problem I: Note that this argument also works with Neumann B.C.:

$$\begin{cases} -\nabla \cdot a(x)\nabla u + b(x)u = \lambda u & \text{in } \Omega \\ \partial u/\partial n = 0 \quad \text{on } \partial\Omega \end{cases}$$

Using divergence theorem, (26.6) becomes

$$\int_{\partial\Omega} -ua \underbrace{\frac{\partial u}{\partial n}}_{=0}\, ds + \int_\Omega a|\nabla u|^2\, dx + \int_\Omega bu^2\, dx = \lambda \int_\Omega u^2\, dx,$$

$$\int_\Omega \underbrace{a}_{>0} |\nabla u|^2\, dx + \int_\Omega \underbrace{b}_{\geq 0} u^2\, dx = \underbrace{\lambda}_{\leq 0} \int_\Omega u^2\, dx.$$

Thus, $\nabla u = 0$, and $u = \text{const}$ on Ω. Hence, we now have

$$\int_\Omega \underbrace{b}_{\geq 0} u^2\, dx = \underbrace{\lambda}_{\leq 0} \int_\Omega u^2\, dx,$$

which implies $\lambda = 0$. This gives the useful information that for the eigenvalue problem[74]

$$\begin{cases} -\nabla \cdot a(x)\nabla u + b(x)u = \lambda u \\ \partial u/\partial n = 0, \end{cases}$$

$\lambda = 0$ is an eigenvalue, its eigenspace is the set of constants, and all other λ's are **positive**.

[74] In Ralston's Hw#7 solutions, there is no '-' sign in front of $\nabla \cdot a(x)\nabla u$ below, which is probably a typo.

Similar Problem II: If $\lambda \leq 0$, we show that the only solution to the problem below is the trivial solution.

$$\begin{cases} \triangle u + \lambda u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$$\int_\Omega u \triangle u \, dx + \lambda \int_\Omega u^2 \, dx = 0,$$

$$\int_{\partial\Omega} \underbrace{u}_{=0} \frac{\partial u}{\partial n} \, ds - \int_\Omega |\nabla u|^2 \, dx + \underbrace{\lambda}_{\leq 0} \int_\Omega u^2 \, dx = 0.$$

Thus, $\nabla u = 0$ in Ω, and u is constant. Since $u = 0$ on $\partial\Omega$, $u \equiv 0$ on Ω. $\square$

27 Problems: Eigenvalues of the Laplacian - Poisson

The ND POISSON Equation (eigenvalues/eigenfunctions of the Laplacian).

Suppose we want to find the eigenfunction expansion of the solution of

$$\begin{aligned} \triangle u = f \qquad & \textit{in} \ \ \Omega \\ u = 0 \qquad & \textit{on} \ \ \partial\Omega, \end{aligned}$$

when f has the expansion in the orthonormal Dirichlet eigenfunctions ϕ_n:

$$f(x) \sim \sum_{n=1}^{\infty} a_n \phi_n(x), \qquad \textit{where} \qquad a_n = \int_\Omega f(x)\phi_n(x)\,dx.$$

Proof. Writing $u = \sum c_n \phi_n$ and inserting into $-\lambda u = f$, we get

$$\sum_{n=1}^{\infty} -\lambda_n c_n \phi_n = \sum_{n=1}^{\infty} a_n \phi_n(x).$$

Thus, $c_n = -a_n/\lambda_n$, and

$$\boxed{u(x) = -\sum_{n=1}^{\infty} \frac{a_n \phi_n(x)}{\lambda_n}.}$$

□

The 1D POISSON Equation (eigenvalues/eigenfunctions of the Laplacian).

For the boundary value problem

$$\begin{aligned} & u'' = f(x) \\ & u(0) = 0, \quad u(L) = 0, \end{aligned}$$

the related eigenvalue problem is

$$\begin{aligned} & \phi'' = -\lambda\phi \\ & \phi(0) = 0, \quad \phi(L) = 0. \end{aligned}$$

The eigenvalues are $\lambda_n = (n\pi/L)^2$, and the corresponding eigenfunctions are $\sin(n\pi x/L)$, $n = 1, 2, \ldots$.
Writing $u = \sum c_n \phi_n = \sum c_n \sin(n\pi x/L)$ and inserting into $-\lambda u = f$, we get

$$\begin{aligned} \sum_{n=1}^{\infty} -c_n \Big(\frac{n\pi}{L}\Big)^2 \sin\frac{n\pi x}{L} &= f(x), \\ \int_0^L \sum_{n=1}^{\infty} -c_n \Big(\frac{n\pi}{L}\Big)^2 \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L}\,dx &= \int_0^L f(x)\,\sin\frac{m\pi x}{L}\,dx, \\ -c_n \Big(\frac{n\pi}{L}\Big)^2 \frac{L}{2} &= \int_0^L f(x)\,\sin\frac{n\pi x}{L}\,dx, \\ c_n &= -\frac{2}{L}\frac{\int_0^L f(x)\,\sin(n\pi x/L)\,dx}{(n\pi/L)^2}. \end{aligned}$$

$$u(x) = \sum c_n \sin(n\pi x/L) = \sum_{n=1}^{\infty} -\frac{2}{L} \frac{\int_0^L f(\xi)\, \sin(n\pi x/L)\, \sin(n\pi\xi/L)\, d\xi}{(n\pi/L)^2},$$

$$\boxed{u = \int_0^L f(\xi) \underbrace{\Big[-\frac{2}{L} \sum_{n=1}^{\infty} \frac{\sin(n\pi x/L)\, \sin(n\pi\xi/L)}{(n\pi/L)^2} \Big]}_{=\,G(x,\xi)} d\xi.}$$

See similar, but more complicated, problem in Sturm-Liouville Problems (S'92, #2(c)).

Example: Eigenfunction Expansion of the GREEN's Function.
Suppose we fix x and attempt to expand the Green's function $G(x,y)$ in the orthonormal eigenfunctions $\phi_n(y)$:

$$G(x,y) \sim \sum_{n=1}^{\infty} a_n(x)\phi_n(y), \qquad \textit{where} \qquad a_n(x) = \int_\Omega G(x,z)\phi_n(z)\,dz.$$

Proof. We can rewrite $\triangle u + \lambda u = 0$ in Ω, $u = 0$ on $\partial\Omega$, as an integral equation [75]

$$u(x) + \lambda \int_\Omega G(x,y)u(y)\,dy = 0. \qquad \circledast$$

Suppose, $u(x) = \sum c_n\phi_n(x)$. Plugging this into $\circledast$, we get

$$\begin{aligned}
\sum_{m=1}^{\infty} c_m\phi_m(x) + \lambda \int_\Omega \sum_{n=1}^{\infty} a_n(x)\phi_n(y) \sum_{m=1}^{\infty} c_m\phi_m(y)\,dy &= 0,\\
\sum_{m=1}^{\infty} c_m\phi_m(x) + \lambda \sum_{n=1}^{\infty} a_n(x) \sum_{m=1}^{\infty} c_m \int_\Omega \phi_n(y)\phi_m(y)\,dy &= 0,\\
\sum_{n=1}^{\infty} c_n\phi_n(x) + \sum_{n=1}^{\infty} \lambda a_n(x)c_n &= 0,\\
\sum_{n=1}^{\infty} c_n\big(\phi_n(x) + \lambda a_n(x)\big) &= 0,\\
a_n(x) &= -\frac{\phi_n(x)}{\lambda_n}.
\end{aligned}$$

Thus,

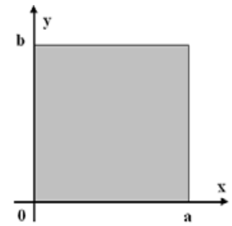
$$\boxed{G(x,y) \sim \sum_{n=1}^{\infty} -\frac{\phi_n(x)\phi_n(y)}{\lambda_n}.}$$

$\square$

[75] See the section: ODE - Integral Equations.

The 2D POISSON Equation (eigenvalues/eigenfunctions of the Laplacian).
Solve the boundary value problem

$$\begin{cases} u_{xx} + u_{yy} = f(x,y) & \textit{for } \ 0 < x < a,\, 0 < y < b \\ u(0,y) = 0 = u(a,y) & \textit{for } \ 0 \le y \le b, \\ u(x,0) = 0 = u(x,b) & \textit{for } \ 0 \le x \le a. \end{cases} \tag{27.1}$$



$f(x,y) \in C^2$, $\;f(x,y) = 0$ *if* $x = 0,\ x = a,\ y = 0,\ y = b,$

$$f(x,y) = \frac{2}{\sqrt{ab}} \sum_{m,n=1}^{\infty} c_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}.$$

Proof. ① First, we find eigenvalues/eigenfunctions of the Laplacian.

$$\begin{cases} u_{xx} + u_{yy} + \lambda u = 0 & \text{in } \ \Omega \\ u(0,y) = 0 = u(a,y) & \text{for } \ 0 \le y \le b, \\ u(x,0) = 0 = u(x,b) & \text{for } \ 0 \le x \le a. \end{cases}$$

Let $u(x,y) = X(x)Y(y)$, then substitution in the PDE gives $X''Y + XY'' + \lambda XY = 0$.

$$\frac{X''}{X} + \frac{Y''}{Y} + \lambda = 0.$$

Letting $\lambda = \mu^2 + \nu^2$ and using boundary conditions, we find the equations for X and Y:

$$\begin{array}{ll} X'' + \mu^2 X = 0 & \qquad Y'' + \nu^2 Y = 0 \\ X(0) = X(a) = 0 & \qquad Y(0) = Y(b) = 0. \end{array}$$

The solutions of these one-dimensional eigenvalue problems are

$$\begin{array}{ll} \mu_m = \dfrac{m\pi}{a} & \qquad \nu_n = \dfrac{n\pi}{b} \\ X_m(x) = \sin\dfrac{m\pi x}{a} & \qquad Y_n(y) = \sin\dfrac{n\pi y}{b}, \end{array}$$

where $m,\ n = 1, 2, \ldots$. Thus we obtain eigenvalues and normalized eigenfunctions of the Laplacian:

$$\boxed{\lambda_{mn} = \pi^2\Big(\frac{m^2}{a^2} + \frac{n^2}{b^2}\Big) \qquad \phi_{mn}(x,y) = \frac{2}{\sqrt{ab}} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b},}$$

where $m,\ n = 1, 2, \ldots$. Note that

$$f(x,y) = \sum_{m,n=1}^{\infty} c_{mn}\phi_{mn}.$$

② Second, writing $u(x,y) = \sum \tilde{c}_{mn}\phi_{mn}$ and inserting into $-\lambda u = f$, we get

$$-\sum_{m,n=1}^{\infty} \lambda_{mn}\tilde{c}_{mn}\phi_{mn}(x,y) = \sum_{m,n=1}^{\infty} c_{mn}\phi_{mn}(x,y).$$

Thus, $\quad \tilde{c}_{mn} = -\frac{c_{mn}}{\lambda_{mn}}$.

$$\boxed{u(x,y) = -\sum_{n=1}^{\infty} \frac{c_{mn}}{\lambda_{mn}}\phi_{mn}(x,y),}$$

with λ_{mn}, $\phi_{mn}(x)$ given above, and c_{mn} given by

$$\int_0^b \int_0^a f(x,y)\phi_{mn}\, dx\, dy = \int_0^b \int_0^a \sum_{m',n'=1}^{\infty} c_{m'n'}\phi_{m'n'}\phi_{mn}\, dx\, dy = c_{mn}.$$

$\square$

28 Problems: Eigenvalues of the Laplacian - Wave

In the section on the wave equation, we considered an initial boundary value problem for the one-dimensional wave equation on an interval, and we found that the solution could be obtained using Fourier series. If we replace the Fourier series by an expansion in eigenfunctions, we can consider an initial/boundary value problem for the n-dimensional wave equation.

The ND WAVE Equation (eigenvalues/eigenfunctions of the Laplacian). *Consider*

$$\begin{cases} u_{tt} = \triangle u & \textit{for } x \in \Omega, \ t > 0 \\ u(x,0) = g(x), \quad u_t(x,0) = h(x) & \textit{for } x \in \Omega \\ u(x,t) = 0 & \textit{for } x \in \partial\Omega, \ t > 0. \end{cases}$$

Proof. For $g, h \in C^2(\overline{\Omega})$ with $g = h = 0$ on $\partial\Omega$, we have eigenfunction expansions

$$g(x) = \sum_{n=1}^{\infty} a_n \phi_n(x) \qquad \text{and} \qquad h(x) = \sum_{n=1}^{\infty} b_n \phi_n(x). \qquad \circledast$$

Assume the solution $u(x,t)$ may be expanded in the eigenfunctions with coefficients depending on t: $u(x,t) = \sum_{n=1}^{\infty} u_n(t)\phi_n(x)$. This implies

$$\begin{aligned} \sum_{n=1}^{\infty} u_n''(t)\phi_n(x) &= -\sum_{n=1}^{\infty} \lambda_n u_n(t)\phi_n(x), \\ u_n''(t) + \lambda_n u_n(t) &= 0 \qquad \text{for each } n. \end{aligned}$$

Since $\lambda_n > 0$, this ordinary differential equation has general solution

$$\begin{aligned} u_n(t) &= A_n \cos\sqrt{\lambda_n}t + B_n \sin\sqrt{\lambda_n}t. \qquad \text{Thus,} \\ u(x,t) &= \sum_{n=1}^{\infty} \big(A_n \cos\sqrt{\lambda_n}t + B_n \sin\sqrt{\lambda_n}t\big)\phi_n(x), \\ u_t(x,t) &= \sum_{n=1}^{\infty} \big(-\sqrt{\lambda_n}A_n \sin\sqrt{\lambda_n}t + \sqrt{\lambda_n}B_n \cos\sqrt{\lambda_n}t\big)\phi_n(x), \\ u(x,0) &= \sum_{n=1}^{\infty} A_n\phi_n(x) = g(x), \\ u_t(x,0) &= \sum_{n=1}^{\infty} \sqrt{\lambda_n}B_n\phi_n(x) = h(x). \end{aligned}$$

Comparing with $\circledast$, we obtain

$$A_n = a_n, \qquad B_n = \frac{b_n}{\sqrt{\lambda_n}}.$$

Thus, the solution is given by

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \Big(a_n \cos\sqrt{\lambda_n}t + \frac{b_n}{\sqrt{\lambda_n}} \sin\sqrt{\lambda_n}t\Big)\phi_n(x),}$$

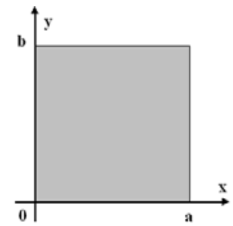
with

$$a_n = \int_\Omega g(x)\phi_n(x)\,dx,$$
$$b_n = \int_\Omega h(x)\phi_n(x)\,dx.$$

$\square$

The 2D WAVE Equation (eigenvalues/eigenfunctions of the Laplacian).
Let $\Omega = (0,a) \times (0,b)$ and consider

$$\begin{cases} u_{tt} = u_{xx} + u_{yy} & \text{for } x \in \Omega,\ t > 0 \\ u(x,0) = g(x), \quad u_t(x,0) = h(x) & \text{for } x \in \Omega \\ u(x,t) = 0 & \text{for } x \in \partial\Omega,\ t > 0. \end{cases} \tag{28.1}$$



Proof. ① First, we find eigenvalues/eigenfunctions of the Laplacian.

$$\begin{cases} u_{xx} + u_{yy} + \lambda u = 0 & \text{in } \Omega \\ u(0,y) = 0 = u(a,y) & \text{for } 0 \le y \le b, \\ u(x,0) = 0 = u(x,b) & \text{for } 0 \le x \le a. \end{cases}$$

Let $u(x,y) = X(x)Y(y)$, then substitution in the PDE gives $X''Y + XY'' + \lambda XY = 0$.

$$\frac{X''}{X} + \frac{Y''}{Y} + \lambda = 0.$$

Letting $\lambda = \mu^2 + \nu^2$ and using boundary conditions, we find the equations for X and Y:

$$\begin{aligned} &X'' + \mu^2 X = 0 && Y'' + \nu^2 Y = 0 \\ &X(0) = X(a) = 0 && Y(0) = Y(b) = 0. \end{aligned}$$

The solutions of these one-dimensional eigenvalue problems are

$$\begin{aligned} &\mu_m = \frac{m\pi}{a} && \nu_n = \frac{n\pi}{b} \\ &X_m(x) = \sin\frac{m\pi x}{a} && Y_n(y) = \sin\frac{n\pi y}{b}, \end{aligned}$$

where $m,\ n = 1, 2, \ldots$. Thus we obtain eigenvalues and normalized eigenfunctions of the Laplacian:

$$\boxed{\lambda_{mn} = \pi^2\Big(\frac{m^2}{a^2} + \frac{n^2}{b^2}\Big) \qquad \phi_{mn}(x,y) = \frac{2}{\sqrt{ab}} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b},}$$

where $m,\ n = 1, 2, \ldots$.
② Second, we solve the Wave Equation (28.1) using the "space" eigenfunctions.
For $g, h \in C^2(\overline{\Omega})$ with $g = h = 0$ on $\partial\Omega$, we have eigenfunction expansions [76]

$$g(x) = \sum_{n=1}^{\infty} a_n \phi_n(x) \qquad \text{and} \qquad h(x) = \sum_{n=1}^{\infty} b_n \phi_n(x). \qquad \circledast$$

Assume $u(x,t) = \sum_{n=1}^{\infty} u_n(t)\phi_n(x)$. This implies

$$u_n''(t) + \lambda_n u_n(t) = 0 \qquad \text{for each } n.$$

[76] In 2D, ϕ_n is really ϕ_{mn}, and x is (x,y).

Since $\lambda_n > 0$, this ordinary differential equation has general solution

$$
\begin{aligned}
u_n(t) &= A_n \cos\sqrt{\lambda_n}t + B_n \sin\sqrt{\lambda_n}t. \qquad \text{Thus,}\\
u(x,t) &= \sum_{n=1}^{\infty}\big(A_n \cos\sqrt{\lambda_n}t + B_n \sin\sqrt{\lambda_n}t\big)\phi_n(x),\\
u_t(x,t) &= \sum_{n=1}^{\infty}\big(-\sqrt{\lambda_n}A_n \sin\sqrt{\lambda_n}t + \sqrt{\lambda_n}B_n \cos\sqrt{\lambda_n}t\big)\phi_n(x),\\
u(x,0) &= \sum_{n=1}^{\infty} A_n\phi_n(x) = g(x),\\
u_t(x,0) &= \sum_{n=1}^{\infty} \sqrt{\lambda_n}B_n\phi_n(x) = h(x).
\end{aligned}
$$

Comparing with $\circledast$, we obtain

$$
A_n = a_n, \qquad B_n = \frac{b_n}{\sqrt{\lambda_n}}.
$$

Thus, the solution is given by

$$
\boxed{u(x,t) = \sum_{m,n=1}^{\infty}\Big(a_{mn}\cos\sqrt{\lambda_{mn}}t + \frac{b_{mn}}{\sqrt{\lambda_{mn}}}\sin\sqrt{\lambda_{mn}}t\Big)\phi_{mn}(x),}
$$

with λ_{mn}, $\phi_{mn}(x)$ given above, and

$$
\begin{aligned}
a_{mn} &= \int_\Omega g(x)\phi_{mn}(x)\,dx,\\
b_{mn} &= \int_\Omega h(x)\phi_{mn}(x)\,dx.
\end{aligned}
$$

$\square$

McOwen, 4.4 #3; 266B Ralston Hw. *Consider the initial-boundary value problem*

$$\begin{cases} u_{tt} = \triangle u + f(x,t) & \text{for } x \in \Omega,\ t > 0 \\ u(x,t) = 0 & \text{for } x \in \partial\Omega,\ t > 0 \\ u(x,0) = 0, \quad u_t(x,0) = 0 & \text{for } x \in \Omega. \end{cases}$$

Use Duhamel's principle and an expansion of f in eigenfunctions to obtain a (formal) solution.

Proof. **a)** We expand u in terms of the **Dirichlet eigenfunctions of Laplacian** in Ω.

$$\triangle\phi_n + \lambda_n\phi_n = 0 \quad \text{in } \Omega, \qquad \phi_n = 0 \quad \text{on } \partial\Omega.$$

Assume

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(x), \qquad a_n(t) = \int_\Omega \phi_n(x)u(x,t)\,dx.$$

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t)\phi_n(x), \qquad f_n(t) = \int_\Omega \phi_n(x)f(x,t)\,dx.$$

$$\begin{aligned} a_n''(t) &= \int_\Omega \phi_n(x)u_{tt}\,dx = \int_\Omega \phi_n(\triangle u + f)\,dx = \int_\Omega \phi_n\triangle u\,dx + \int_\Omega \phi_n f\,dx \\ &= \int_\Omega \triangle\phi_n u\,dx + \int_\Omega \phi_n f\,dx = -\lambda_n\int_\Omega \phi_n u\,dx + \underbrace{\int_\Omega \phi_n f\,dx}_{f_n} = -\lambda_n a_n(t) + f_n(t). \end{aligned}$$

$$\begin{aligned} a_n(0) &= \int_\Omega \phi_n(x)u(x,0)\,dx = 0. \\ a_n'(0) &= \int_\Omega \phi_n(x)u_t(x,0)\,dx = 0. \end{aligned}$$

[77] Thus, we have an ODE which is converted and solved by **Duhamel's principle**:

$$\begin{cases} a_n'' + \lambda_n a_n = f_n(t) \\ a_n(0) = 0 \\ a_n'(0) = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} \tilde{a}_n'' + \lambda_n\tilde{a}_n = 0 \\ \tilde{a}_n(0,s) = 0 \\ \tilde{a}_n'(0,s) = f_n(s) \end{cases} \qquad a_n(t) = \int_0^t \tilde{a}_n(t-s,s)\,ds.$$

With the anzats $\tilde{a}_n(t,s) = c_1\cos\sqrt{\lambda_n}t + c_2\sin\sqrt{\lambda_n}t$, we get $c_1 = 0$, $c_2 = f_n(s)/\sqrt{\lambda_n}$, or

$$\tilde{a}_n(t,s) = f_n(s)\frac{\sin\sqrt{\lambda_n}t}{\sqrt{\lambda_n}}.$$

Duhamel's principle gives

$$a_n(t) = \int_0^t \tilde{a}_n(t-s,s)\,ds = \int_0^t f_n(s)\frac{\sin(\sqrt{\lambda_n}(t-s))}{\sqrt{\lambda_n}}\,ds.$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty}\frac{\phi_n(x)}{\sqrt{\lambda_n}}\int_0^t f_n(s)\sin(\sqrt{\lambda_n}(t-s))\,ds.}$$

□

[77] We used Green's formula: $\int_{\partial\Omega}\left(\phi_n\frac{\partial u}{\partial n} - u\frac{\partial\phi_n}{\partial n}\right)ds = \int_\Omega(\phi_n\triangle u - \triangle\phi_n u)\,dx$.
On $\partial\Omega$, $u = 0$; $\phi_n = 0$ since eigenfunctions are Dirichlet.

Problem (F'90, #3). *Consider the initial-boundary value problem*

$$\begin{cases} u_{tt} = a(t)u_{xx} + f(x,t) & 0 \le x \le \pi,\ t \ge 0 \\ u(0,t) = u(\pi,t) = 0 & t \ge 0 \\ u(x,0) = g(x), \quad u_t(x,0) = h(x) & 0 \le x \le \pi, \end{cases}$$

where the coefficient $a(t) \neq 0$.
a) *Express (formally) the solution of this problem by the method of eigenfunction expansions.*
b) *Show that this problem is* ***not well-posed*** *if $a \equiv -1$.*
Hint: Take $f = 0$ and prove that the solution does not depend continuously on the initial data g, h.

Proof. **a)** We expand u in terms of the **Dirichlet eigenfunctions of Laplacian** in Ω.

$$\phi_{nxx} + \lambda_n \phi_n = 0 \quad \text{in } \Omega, \qquad \phi_n(0) = \phi_n(\pi) = 0.$$

That gives us the eigenvalues and eigenfunctions of the Laplacian: $\lambda_n = n^2$, $\phi_n(x) = \sin nx$.
Assume

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t)\phi_n(x), \qquad u_n(t) = \int_\Omega \phi_n(x)u(x,t)\,dx.$$

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t)\phi_n(x), \qquad f_n(t) = \int_\Omega \phi_n(x)f(x,t)\,dx.$$

$$g(x) = \sum_{n=1}^{\infty} g_n\phi_n(x), \qquad g_n = \int_\Omega \phi_n(x)g(x)\,dx.$$

$$h(x) = \sum_{n=1}^{\infty} h_n\phi_n(x), \qquad h_n = \int_\Omega \phi_n(x)h(x)\,dx.$$

$$\begin{aligned} u_n''(t) &= \int_\Omega \phi_n(x)u_{tt}\,dx = \int_\Omega \phi_n(a(t)u_{xx} + f)\,dx = a(t)\int_\Omega \phi_n u_{xx}\,dx + \int_\Omega \phi_n f\,dx \\ &= a(t)\int_\Omega \phi_{nxx}u\,dx + \int_\Omega \phi_n f\,dx = -\lambda_n a(t)\int_\Omega \phi_n u\,dx + \underbrace{\int_\Omega \phi_n f\,dx}_{f_n} \\ &= -\lambda_n a(t)u_n(t) + f_n(t). \\ u_n(0) &= \int_\Omega \phi_n(x)u(x,0)\,dx = \int_\Omega \phi_n(x)g(x)\,dx = g_n. \\ u_n'(0) &= \int_\Omega \phi_n(x)u_t(x,0)\,dx = \int_\Omega \phi_n(x)h(x)\,dx = h_n. \end{aligned}$$

Thus, we have an ODE which is converted and solved by **Duhamel's principle**:

$$\begin{cases} u_n'' + \lambda_n a(t)u_n = f_n(t) \\ u_n(0) = g_n \\ u_n'(0) = h_n. \end{cases} \qquad \circledast$$

Note: The initial data is not 0; therefore, the Duhamel's principle is not applicable. Also, the ODE is not linear in t, and it's solution is not obvious. Thus,

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} u_n(t)\phi_n(x),}$$

where $u_n(t)$ are solutions of $\circledast$.

b) Assume we have two solutions, u_1 and u_2, to the PDE:

$$\begin{cases} u_{1tt} + u_{1xx} = 0, \\ u_1(0,t) = u_1(\pi,t) = 0, \\ u_1(x,0) = g_1(x), \quad u_{1t}(x,0) = h_1(x); \end{cases} \qquad \begin{cases} u_{2tt} + u_{2xx} = 0, \\ u_2(0,t) = u_2(\pi,t) = 0, \\ u_2(x,0) = g_2(x), \quad u_{2t}(x,0) = h_2(x). \end{cases}$$

Note that the equation is **elliptic**, and therefore, the maximum principle holds.
In order to prove that the solution does not depend continuously on the initial data g, h, we need to show that one of the following conditions holds:

$$\begin{aligned} \max_{\overline{\Omega}} |u_1 - u_2| &> \max_{\partial\Omega} |g_1 - g_2|, \\ \max_{\overline{\Omega}} |u_{t1} - u_{t2}| &> \max_{\partial\Omega} |h_1 - h_2|. \end{aligned}$$

That is, the difference of the two solutions is not bounded by the difference of initial data.
By the method of separation of variables, we may obtain

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \sin nx, \\ u(x,0) &= \sum_{n=1}^{\infty} a_n \sin nx = g(x), \\ u_t(x,0) &= \sum_{n=1}^{\infty} n b_n \sin nx = h(x). \end{aligned}$$

Not complete.

We also know that for elliptic equations, and for Laplace equation in particular, the value of the function u has to be prescribed on the entire boundary, i.e. $u = g$ on $\partial\Omega$, which is not the case here, making the problem under-determined. Also, u_t is prescribed on one of the boundaries, making the problem overdetermined. $\square$

29 Problems: Eigenvalues of the Laplacian - Heat

The ND HEAT Equation (eigenvalues/eigenfunctions of the Laplacian).
Consider the initial value problem with homogeneous Dirichlet condition:

$$\begin{cases} u_t = \triangle u & \text{for } x \in \Omega,\ t > 0 \\ u(x,0) = g(x) & \text{for } x \in \overline{\Omega} \\ u(x,t) = 0 & \text{for } x \in \partial\Omega,\ t > 0. \end{cases}$$

Proof. For $g \in C^2(\overline{\Omega})$ with $g = 0$ on $\partial\Omega$, we have eigenfunction expansion

$$g(x) = \sum_{n=1}^{\infty} a_n \phi_n(x) \qquad \circledast$$

Assume the solution $u(x,t)$ may be expanded in the eigenfunctions with coefficients depending on t: $u(x,t) = \sum_{n=1}^{\infty} u_n(t)\phi_n(x)$. This implies

$$\sum_{n=1}^{\infty} u_n'(t)\phi_n(x) = -\lambda_n \sum_{n=1}^{\infty} u_n(t)\phi_n(x),$$

$$u_n'(t) + \lambda_n u_n(t) = 0, \quad \text{which has the general solution}$$

$$u_n(t) = A_n e^{-\lambda_n t}. \qquad \text{Thus,}$$

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\lambda_n t}\phi_n(x),$$

$$u(x,0) = \sum_{n=1}^{\infty} A_n \phi_n(x) = g(x).$$

Comparing with $\circledast$, we obtain $A_n = a_n$. Thus, the solution is given by

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n t}\phi_n(x),}$$

$$\text{with} \qquad a_n = \int_{\Omega} g(x)\phi_n(x)\,dx.$$

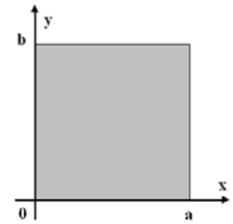
Also

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} a_n e^{-\lambda_n t}\phi_n(x) = \sum_{n=1}^{\infty} \Big(\int_{\Omega} g(y)\phi_n(y)\,dy\Big) e^{-\lambda_n t}\phi_n(x) \\ &= \int_{\Omega} \underbrace{\sum_{n=1}^{\infty} e^{-\lambda_n t}\phi_n(x)\phi_n(y)}_{K(x,y,t),\ \text{heat kernel}} g(y)\,dy \end{aligned}$$

$\square$

The 2D HEAT Equation (eigenvalues/eigenfunctions of the Laplacian).
Let $\Omega = (0,a) \times (0,b)$ *and consider*

$$\begin{cases} u_t = u_{xx} + u_{yy} & \textit{for } x \in \Omega,\ t > 0 \\ u(x,0) = g(x) & \textit{for } x \in \overline{\Omega} \\ u(x,t) = 0 & \textit{for } x \in \partial\Omega,\ t > 0. \end{cases} \tag{29.1}$$



Proof. ① First, we find eigenvalues/eigenfunctions of the Laplacian.

$$\begin{cases} u_{xx} + u_{yy} + \lambda u = 0 & \text{in } \Omega \\ u(0,y) = 0 = u(a,y) & \text{for } 0 \le y \le b, \\ u(x,0) = 0 = u(x,b) & \text{for } 0 \le x \le a. \end{cases}$$

Let $u(x,y) = X(x)Y(y)$, then substitution in the PDE gives $X''Y + XY'' + \lambda XY = 0$.

$$\frac{X''}{X} + \frac{Y''}{Y} + \lambda = 0.$$

Letting $\lambda = \mu^2 + \nu^2$ and using boundary conditions, we find the equations for X and Y:

$$\begin{aligned} &X'' + \mu^2 X = 0 && Y'' + \nu^2 Y = 0 \\ &X(0) = X(a) = 0 && Y(0) = Y(b) = 0. \end{aligned}$$

The solutions of these one-dimensional eigenvalue problems are

$$\begin{aligned} &\mu_m = \frac{m\pi}{a} && \nu_n = \frac{n\pi}{b} \\ &X_m(x) = \sin\frac{m\pi x}{a} && Y_n(y) = \sin\frac{n\pi y}{b}, \end{aligned}$$

where $m,\ n = 1, 2, \ldots$. Thus we obtain eigenvalues and normalized eigenfunctions of the Laplacian:

$$\boxed{\lambda_{mn} = \pi^2\Big(\frac{m^2}{a^2} + \frac{n^2}{b^2}\Big) \qquad \phi_{mn}(x,y) = \frac{2}{\sqrt{ab}}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b},}$$

where $m,\ n = 1, 2, \ldots$.
② Second, we solve the Heat Equation (29.1) using the "space" eigenfunctions.
For $g \in C^2(\overline{\Omega})$ with $g = 0$ on $\partial\Omega$, we have eigenfunction expansion

$$g(x) = \sum_{n=1}^{\infty} a_n \phi_n(x). \qquad \circledast$$

Assume $u(x,t) = \sum_{n=1}^{\infty} u_n(t)\phi_n(x)$. This implies

$u_n'(t) + \lambda_n u_n(t) = 0$, which has the general solution

$$u_n(t) = A_n e^{-\lambda_n t}. \qquad \text{Thus,}$$

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\lambda_n t}\phi_n(x),$$

$$u(x,0) = \sum_{n=1}^{\infty} A_n \phi_n(x) = g(x).$$

Comparing with $\circledast$, we obtain $A_n = a_n$. Thus, the solution is given by

$$\boxed{u(x,t) = \sum_{m,n=1}^{\infty} a_{mn} e^{-\lambda_{mn} t} \phi_{mn}(x),}$$

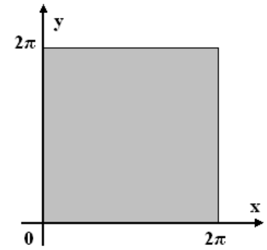
with λ_{mn}, ϕ_{mn} given above and $a_{mn} = \int_\Omega g(x)\phi_{mn}(x)\,dx$. $\square$

Problem (S'91, #2). *Consider the heat equation*

$$u_t = u_{xx} + u_{yy}$$

on the square $\Omega = \{0 \le x \le 2\pi,\ 0 \le y \le 2\pi\}$ *with* ***periodic boundary conditions*** *and with initial data*

$$u(0, x, y) = f(x, y).$$



a) *Find the solution using separation of variables.*

Proof. ① First, we find eigenvalues/eigenfunctions of the Laplacian.

$$\begin{cases} u_{xx} + u_{yy} + \lambda u = 0 & \text{in } \Omega \\ u(0, y) = u(2\pi, y) & \text{for } 0 \le y \le 2\pi, \\ u(x, 0) = u(x, 2\pi) & \text{for } 0 \le x \le 2\pi. \end{cases}$$

Let $u(x, y) = X(x)Y(y)$, then substitution in the PDE gives $X''Y + XY'' + \lambda XY = 0$.

$$\frac{X''}{X} + \frac{Y''}{Y} + \lambda = 0.$$

Letting $\lambda = \mu^2 + \nu^2$ and using periodic BC's, we find the equations for X and Y:

$$X'' + \mu^2 X = 0 \qquad\qquad Y'' + \nu^2 Y = 0$$
$$X(0) = X(2\pi) \qquad\qquad Y(0) = Y(2\pi).$$

The solutions of these one-dimensional eigenvalue problems are

$$\mu_m = m \qquad\qquad \nu_n = n$$
$$X_m(x) = e^{imx} \qquad\qquad Y_n(y) = e^{iny},$$

where $m, n = \dots, -2, -1, 0, 1, 2, \dots$. Thus we obtain eigenvalues and normalized eigenfunctions of the Laplacian:

$$\boxed{\lambda_{mn} = m^2 + n^2 \qquad\qquad \phi_{mn}(x, y) = e^{imx}e^{iny},}$$

where $m, n = \dots, -2, -1, 0, 1, 2, \dots$.
② Second, we solve the Heat Equation using the "space" eigenfunctions.
Assume $u(x, y, t) = \sum_{m,n=-\infty}^{\infty} u_{mn}(t)e^{imx}e^{iny}$. This implies

$$u'_{mn}(t) + (m^2 + n^2)u_{mn}(t) = 0, \quad \text{which has the general solution}$$

$$u_n(t) = c_{mn}e^{-(m^2+n^2)t}. \qquad\qquad \text{Thus,}$$

$$\boxed{u(x, y, t) = \sum_{m,n=-\infty}^{\infty} c_{mn}e^{-(m^2+n^2)t}e^{imx}e^{iny}.}$$

$$
\begin{aligned}
u(x,y,0) &= \sum_{m,n=-\infty}^{\infty} c_{mn} e^{imx} e^{iny} \;=\; f(x,y), \\
\int_0^{2\pi}\!\!\int_0^{2\pi} f(x,y)\, e^{\overline{imx}} e^{\overline{iny}}\, dxdy &= \int_0^{2\pi}\!\!\int_0^{2\pi} \sum_{m,n=-\infty}^{\infty} c_{mn} e^{imx} e^{iny} e^{\overline{im'x}} e^{\overline{in'y}}\, dxdy \\
&= 2\pi \int_0^{2\pi} \sum_{n=-\infty}^{\infty} c_{mn} e^{iny} e^{\overline{in'y}}\, dy \;=\; 4\pi^2 c_{mn}. \\
c_{mn} &= \frac{1}{4\pi^2} \int_0^{2\pi}\!\!\int_0^{2\pi} f(x,y)\, e^{-imx} e^{-iny}\, dxdy \;=\; f_{mn}.
\end{aligned}
$$

$\square$

b) *Show that the integral* $\int_\Omega u^2(x,y,t)\,dxdy$ *is decreasing in* t, *if* f *is not constant.*

Proof. We have

$$u_t = u_{xx} + u_{yy}$$

Multiply the equation by u and integrate:

$$\begin{aligned}
uu_t &= u\triangle u, \\
\frac{1}{2}\frac{d}{dt}u^2 &= u\triangle u, \\
\frac{1}{2}\frac{d}{dt}\int_\Omega u^2\,dxdy &= \int_\Omega u\triangle u\,dxdy \;=\; \underbrace{\int_{\partial\Omega} u\frac{\partial u}{\partial n}\,ds}_{=0,\ (\text{periodic BC})} - \int_\Omega |\nabla u|^2\,dxdy \\
&= -\int_\Omega |\nabla u|^2\,dxdy \;\le\; 0.
\end{aligned}$$

Equality is obtained only when $\nabla u = 0 \;\Rightarrow\; u = \text{constant} \;\Rightarrow\; f = \text{constant}$.
If f is not constant, $\int_\Omega u^2\,dxdy$ is decreasing in t. $\square$

Problem (F'98, #3). *Consider the eigenvalue problem*

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0,$$
$$\phi(0) - \frac{d\phi}{dx}(0) = 0, \qquad \phi(1) + \frac{d\phi}{dx}(1) = 0.$$

a) *Show that all eigenvalues are positive.*
b) *Show that there exist a sequence of eigenvalues $\lambda = \lambda_n$, each of which satisfies*

$$\tan\sqrt{\lambda} = \frac{2\sqrt{\lambda}}{\lambda - 1}.$$

c) *Solve the following initial-boundary value problem on $0 < x < 1$, $t > 0$*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},$$
$$u(0,t) - \frac{\partial u}{\partial x}(0,t) = 0, \qquad u(1,t) + \frac{\partial u}{\partial x}(1,t) = 0,$$
$$u(x,0) = f(x).$$

You may call the relevant eigenfunctions $\phi_n(x)$ and assume that they are known.

Proof. **a)** • If $\lambda = 0$, the ODE reduces to $\phi'' = 0$. Try $\phi(x) = Ax + B$.
From the first boundary condition,

$$\phi(0) - \phi'(0) = 0 = B - A \quad \Rightarrow \quad B = A.$$

Thus, the solution takes the form $\phi(x) = Ax + A$. The second boundary condition gives

$$\phi(1) + \phi'(1) = 0 = 3A \quad \Rightarrow \quad A = B = 0.$$

Thus the only solution is $\phi \equiv 0$, which is not an eigenfunction, and 0 not an eigenvalue. ✓
• If $\lambda < 0$, try $\phi(x) = e^{sx}$, which gives $s = \pm\sqrt{-\lambda} = \pm\beta \in \mathbb{R}$.
Hence, the family of solutions is $\phi(x) = Ae^{\beta x} + Be^{-\beta x}$. Also, $\phi'(x) = \beta Ae^{\beta x} - \beta Be^{-\beta x}$.
The boundary conditions give

$$\phi(0) - \phi'(0) = 0 = A + B - \beta A + \beta B = A(1-\beta) + B(1+\beta), \tag{29.2}$$

$$\phi(1) + \phi'(1) = 0 = Ae^{\beta} + Be^{-\beta} + \beta Ae^{\beta} - \beta Be^{-\beta} = Ae^{\beta}(1+\beta) + Be^{-\beta}(1-\beta). \tag{29.3}$$

From (29.2) and (29.3) we get

$$\frac{1+\beta}{1-\beta} = -\frac{A}{B} \quad \text{and} \quad \frac{1+\beta}{1-\beta} = -\frac{B}{A}e^{-2\beta}, \qquad \text{or} \qquad \frac{A}{B} = e^{-\beta}.$$

From (29.2), $\beta = \dfrac{A+B}{A-B}$ and thus, $\dfrac{A}{B} = e^{\frac{A+B}{B-A}}$, which has no solutions. ✓

b) Since $\lambda > 0$, the anzats $\phi = e^{sx}$ gives $s = \pm i\sqrt{\lambda}$ and the family of solutions takes the form

$$\phi(x) = A\sin(x\sqrt{\lambda}) + B\cos(x\sqrt{\lambda}).$$

Then, $\phi'(x) = A\sqrt{\lambda}\cos(x\sqrt{\lambda}) - B\sqrt{\lambda}\sin(x\sqrt{\lambda})$. The first boundary condition gives

$$\phi(0) - \phi'(0) \;=\; 0 = B - A\sqrt{\lambda} \qquad \Rightarrow \qquad B = A\sqrt{\lambda}.$$

Hence, $\phi(x) = A\sin(x\sqrt{\lambda}) + A\sqrt{\lambda}\cos(x\sqrt{\lambda})$. The second boundary condition gives

$$\begin{aligned}\phi(1)+\phi'(1)=0 \quad &= \quad A\sin(\sqrt{\lambda}) + A\sqrt{\lambda}\cos(\sqrt{\lambda}) + A\sqrt{\lambda}\cos(\sqrt{\lambda}) - A\lambda\sin(\sqrt{\lambda}) \\ &= \quad A\big[(1-\lambda)\sin(\sqrt{\lambda}) + 2\sqrt{\lambda}\cos(\sqrt{\lambda})\big]\end{aligned}$$

$A \neq 0$ (since $A = 0$ implies $B = 0$ and $\phi = 0$, which is not an eigenfunction). Therefore, $-(1-\lambda)\sin(\sqrt{\lambda}) = 2\sqrt{\lambda}\cos(\sqrt{\lambda})$, and thus $\tan(\sqrt{\lambda}) = \frac{2\sqrt{\lambda}}{\lambda-1}$.

c) We may assume that the eigenvalues/eigenfunctins of the Laplacian, λ_n and $\phi_n(x)$, are known. We solve the Heat Equation using the “space” eigenfunctions.

$$\begin{cases} u_t = u_{xx}, \\ u(0,t) - u_x(0,t) = 0, \qquad u(1,t) + u_x(1,t) = 0, \\ u(x,0) = f(x). \end{cases}$$

For f, we have an eigenfunction expansion

$$f(x) = \sum_{n=1}^{\infty} a_n \phi_n(x). \qquad \circledast$$

Assume $u(x,t) = \sum_{n=1}^{\infty} u_n(t)\phi_n(x)$. This implies

$u_n'(t) + \lambda_n u_n(t) = 0$, which has the general solution

$$\begin{aligned} u_n(t) &= A_n e^{-\lambda_n t}. \qquad \text{Thus,} \\ u(x,t) &= \sum_{n=1}^{\infty} A_n e^{-\lambda_n t}\phi_n(x), \\ u(x,0) &= \sum_{n=1}^{\infty} A_n \phi_n(x) = f(x). \end{aligned}$$

Comparing with $\circledast$, we have $A_n = a_n$. Thus, the solution is given by

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n t}\phi_n(x),}$$

with

$$a_n = \int_0^1 f(x)\phi_n(x)\,dx.$$

$\square$

Problem (W'03, #3); 266B Ralston Hw. *Let Ω be a smooth domain in three dimensions and consider the initial-boundary value problem for the heat equation*

$$\begin{cases} u_t = \triangle u + f(x) & \text{for } x \in \Omega,\ t > 0 \\ \partial u/\partial n = 0 & \text{for } x \in \partial\Omega,\ t > 0 \\ u(x,0) = g(x) & \text{for } x \in \Omega, \end{cases}$$

in which f and g are known smooth functions with

$$\partial g/\partial n = 0 \qquad \text{for } x \in \partial\Omega.$$

a) *Find an approximate formula for u as $t \to \infty$.*

Proof. We expand u in terms of the **Neumann eigenfunctions of Laplacian** in Ω.

$$\triangle\phi_n + \lambda_n\phi_n = 0 \quad \text{in } \Omega, \qquad \frac{\partial\phi_n}{\partial n} = 0 \quad \text{on } \partial\Omega.$$

Note that here $\lambda_1 = 0$ and ϕ_1 is the constant $V^{-1/2}$, where V is the volume of Ω. Assume

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(x), \qquad a_n(t) = \int_\Omega \phi_n(x)u(x,t)\,dx.$$

$$f(x) = \sum_{n=1}^{\infty} f_n\phi_n(x), \qquad f_n = \int_\Omega \phi_n(x)f(x)\,dx.$$

$$g(x) = \sum_{n=1}^{\infty} g_n\phi_n(x), \qquad g_n = \int_\Omega \phi_n(x)g(x)\,dx.$$

$$\begin{aligned} a_n'(t) &= \int_\Omega \phi_n(x)u_t\,dx = \int_\Omega \phi_n(\triangle u + f)\,dx = \int_\Omega \phi_n\triangle u\,dx + \int_\Omega \phi_n f\,dx \\ &= \int_\Omega \triangle\phi_n u\,dx + \int_\Omega \phi_n f\,dx = -\lambda_n\int_\Omega \phi_n u\,dx + \underbrace{\int_\Omega \phi_n f\,dx}_{f_n} = -\lambda_n a_n + f_n. \end{aligned}$$

$$a_n(0) = \int_\Omega \phi_n(x)u(x,0)\,dx = \int_\Omega \phi_n g\,dx = g_n.$$

[78] Thus, we solve the ODE:

$$\begin{cases} a_n' + \lambda_n a_n = f_n \\ a_n(0) = g_n. \end{cases}$$

For $n = 1$, $\lambda_1 = 0$, and we obtain $a_1(t) = f_1 t + g_1$.
For $n \geq 2$, the homogeneous solution is $a_{n_h} = ce^{-\lambda_n t}$. The anzats for a particular solution is $a_{n_p} = c_1 t + c_2$, which gives $c_1 = 0$ and $c_2 = f_n/\lambda_n$. Using the initial condition, we obtain

$$a_n(t) = \Big(g_n - \frac{f_n}{\lambda_n}\Big)e^{-\lambda_n t} + \frac{f_n}{\lambda_n}.$$

[78] We used Green's formula: $\int_{\partial\Omega}\big(\phi_n\frac{\partial u}{\partial n} - u\frac{\partial\phi_n}{\partial n}\big)ds = \int_\Omega(\phi_n\triangle u - \triangle\phi_n u)\,dx$.
On $\partial\Omega$, $\frac{\partial u}{\partial n} = 0$; $\frac{\partial\phi_n}{\partial n} = 0$ since eigenfunctions are Neumann.

$$\boxed{u(x,t) = (f_1 t + g_1)\phi_1(x) + \sum_{n=2}^{\infty}\Big[\Big(g_n - \frac{f_n}{\lambda_n}\Big)e^{-\lambda_n t} + \frac{f_n}{\lambda_n}\Big]\phi_n(x).}$$

If $f_1 = 0$ $\Big(\int_\Omega f(x)\,dx = 0\Big)$, $\quad \lim_{t\to\infty} u(x,t) = g_1\phi_1 + \sum_{n=2}^{\infty}\frac{f_n\phi_n}{\lambda_n}.$

If $f_1 \neq 0$ $\Big(\int_\Omega f(x)\,dx \neq 0\Big)$, $\quad \lim_{t\to\infty} u(x,t) \sim f_1\phi_1 t.$

b) *If* $g \geq 0$ *and* $f > 0$*, show that* $u > 0$ *for all* $t > 0$*.*

$\square$

Problem (S'97, #2). ***a)*** *Consider the eigenvalue problem for the Laplace operator* $\triangle$ *in* $\Omega \in \mathbb{R}^2$ *with zero Neumann boundary condition*

$$\begin{cases} u_{xx} + u_{yy} + \lambda u = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

Prove that $\lambda_0 = 0$ *is the lowest eigenvalue and that it is simple.*
b) *Assume that the eigenfunctions* $\phi_n(x, y)$ *of the problem in (a) form a complete orthogonal system, and that* $f(x, y)$ *has a uniformly convergent expansion*

$$f(x, y) = \sum_{n=0}^{\infty} f_n \phi_n(x, y).$$

Solve the initial value problem

$$u_t = \triangle u + f(x, y)$$

subject to initial and boundary conditions

$$u(x, y, 0) = 0, \qquad \frac{\partial u}{\partial n} u|_{\partial\Omega} = 0.$$

What is the behavior of $u(x, y, t)$ *as* $t \to \infty$*?*
c) *Consider the problem with Neumann boundary conditions*

$$\begin{cases} v_{xx} + v_{yy} + f(x, y) = 0 & \text{in } \Omega \\ \frac{\partial v}{\partial n} v = 0 & \text{on } \partial\Omega. \end{cases}$$

When does a solution exist? Find this solution, and find its relation with the behavior of $\lim u(x, y, t)$ *in (b) as* $t \to \infty$.

Proof. **a)** Suppose this eigenvalue problem did have a solution u with $\lambda \le 0$.
Multiplying $\triangle u + \lambda u = 0$ by u and integrating over Ω, we get

$$\int_\Omega u \triangle u \, dx + \lambda \int_\Omega u^2 \, dx = 0,$$

$$\int_{\partial\Omega} u \underbrace{\frac{\partial u}{\partial n}}_{=0} \, ds - \int_\Omega |\nabla u|^2 \, dx + \lambda \int_\Omega u^2 \, dx = 0,$$

$$\int_\Omega |\nabla u|^2 \, dx = \underbrace{\lambda}_{\le 0} \int_\Omega u^2 \, dx,$$

Thus, $\nabla u = 0$ in Ω, and u is constant in Ω. Hence, we now have

$$0 = \underbrace{\lambda}_{\le 0} \int_\Omega u^2 \, dx.$$

For nontrivial u, we have $\lambda = 0$. For this eigenvalue problem, $\lambda = 0$ is an eigenvalue, its eigenspace is the set of constants, and all other λ's are positive.

b) We expand u in terms of the **Neumann eigenfunctions of Laplacian** in Ω. [79]

$$\triangle\phi_n + \lambda_n\phi_n = 0 \quad \text{in} \quad \Omega, \qquad \frac{\partial\phi_n}{\partial n} = 0 \quad \text{on} \quad \partial\Omega.$$

$$u(x,y,t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(x,y), \qquad a_n(t) = \int_\Omega \phi_n(x,y)u(x,y,t)\,dx.$$

$$\begin{aligned} a_n'(t) &= \int_\Omega \phi_n(x,y)u_t\,dx = \int_\Omega \phi_n(\triangle u + f)\,dx = \int_\Omega \phi_n\triangle u\,dx + \int_\Omega \phi_n f\,dx \\ &= \int_\Omega \triangle\phi_n u\,dx + \int_\Omega \phi_n f\,dx = -\lambda_n \int_\Omega \phi_n u\,dx + \underbrace{\int_\Omega \phi_n f\,dx}_{f_n} = -\lambda_n a_n + f_n. \\ a_n(0) &= \int_\Omega \phi_n(x,y)u(x,y,0)\,dx = 0. \end{aligned}$$

[80] Thus, we solve the ODE:

$$\begin{cases} a_n' + \lambda_n a_n = f_n \\ a_n(0) = 0. \end{cases}$$

For $n = 1$, $\lambda_1 = 0$, and we obtain $a_1(t) = f_1 t$.
For $n \geq 2$, the homogeneous solution is $a_{n_h} = ce^{-\lambda_n t}$. The anzats for a particular solution is $a_{n_p} = c_1 t + c_2$, which gives $c_1 = 0$ and $c_2 = f_n/\lambda_n$. Using the initial condition, we obtain

$$a_n(t) = -\frac{f_n}{\lambda_n}e^{-\lambda_n t} + \frac{f_n}{\lambda_n}.$$

$$\boxed{u(x,t) = f_1\phi_1 t + \sum_{n=2}^{\infty}\Big(-\frac{f_n}{\lambda_n}e^{-\lambda_n t} + \frac{f_n}{\lambda_n}\Big)\phi_n(x).}$$

$$\text{If} \quad f_1 = 0 \quad \Big(\int_\Omega f(x)\,dx = 0\Big), \qquad \lim_{t\to\infty} u(x,t) = \sum_{n=2}^{\infty}\frac{f_n\phi_n}{\lambda_n}.$$

$$\text{If} \quad f_1 \neq 0 \quad \Big(\int_\Omega f(x)\,dx \neq 0\Big), \qquad \lim_{t\to\infty} u(x,t) \sim f_1\phi_1 t.$$

c) Integrate $\triangle v + f(x,y) = 0$ over Ω:

$$\int_\Omega f\,dx = -\int_\Omega \triangle v\,dx = -\int_\Omega \nabla\cdot\nabla v\,dx =^1 -\int_{\partial\Omega}\frac{\partial v}{\partial n}\,ds =^2 0,$$

where we used [1] divergence theorem and [2] Neumann boundary conditions. Thus, the solution exists only if

$$\int_\Omega f\,dx = 0.$$

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[79] We use $dx\,dy \to dx$.

[80] We used Green's formula: $\int_{\partial\Omega}\big(\phi_n\frac{\partial u}{\partial n} - u\frac{\partial\phi_n}{\partial n}\big)ds = \int_\Omega(\phi_n\triangle u - \triangle\phi_n u)\,dx$.
On $\partial\Omega$, $\frac{\partial u}{\partial n} = 0$; $\frac{\partial\phi_n}{\partial n} = 0$ since eigenfunctions are Neumann.

Assume $v(x,y) = \sum_{n=0}^{\infty} a_n \phi_n(x,y)$. Since we have $f(x,y) = \sum_{n=0}^{\infty} f_n \phi_n(x,y)$, we obtain

$$\begin{aligned}
-\sum_{n=0}^{\infty} \lambda_n a_n \phi_n + \sum_{n=0}^{\infty} f_n \phi_n &= 0, \\
-\lambda_n a_n \phi_n + f_n \phi_n &= 0, \\
a_n &= \frac{f_n}{\lambda_n}.
\end{aligned}$$

$v(x,y) = \sum_{n=0}^{\infty} (\frac{f_n}{\lambda_n}) \phi_n(x,y)$. $\square$

29.1 Heat Equation with Periodic Boundary Conditions in 2D (with extra terms)

Problem (F'99, #5). *In two spatial dimensions, consider the differential equation*

$$u_t = -\varepsilon \triangle u - \triangle^2 u$$

*with **periodic boundary conditions** on the unit square* $[0, 2\pi]^2$.
a) *If* $\varepsilon = 2$ *find a solution whose amplitude increases as* t *increases.*
b) *Find a value* ε_0*, so that the solution of this PDE stays bounded as* $t \to \infty$*, if* $\varepsilon < \varepsilon_0$*.*

Proof. **a) Eigenfunctions of the Laplacian.**
The periodic boundary conditions imply a Fourier Series solution of the form:

$$u(x,t) = \sum_{m,n} a_{mn}(t) e^{i(mx+ny)}.$$

$$\begin{aligned}
u_t &= \sum_{m,n} a'_{mn}(t) e^{i(mx+ny)}, \\
\triangle u &= u_{xx} + u_{yy} = -\sum_{m,n} (m^2+n^2)\, a_{mn}(t) e^{i(mx+ny)}, \\
\triangle^2 u &= u_{xxxx} + 2u_{xxyy} + u_{yyyy} = \sum_{m,n} (m^4 + 2m^2n^2 + n^4)\, a_{mn}(t) e^{i(mx+ny)} \\
&= \sum_{m,n} (m^2+n^2)^2\, a_{mn}(t) e^{i(mx+ny)}.
\end{aligned}$$

Plugging this into the PDE, we obtain

$$\begin{aligned}
&a'_{mn}(t) = \varepsilon(m^2+n^2) a_{mn}(t) - (m^2+n^2)^2 a_{mn}(t), \\
&a'_{mn}(t) - [\varepsilon(m^2+n^2) - (m^2+n^2)^2] a_{mn}(t) = 0, \\
&a'_{mn}(t) - (m^2+n^2)[\varepsilon - (m^2+n^2)] a_{mn}(t) = 0.
\end{aligned}$$

The solution to the ODE above is

$$a_{mn}(t) = \alpha_{mn}\, e^{(m^2+n^2)[\varepsilon-(m^2+n^2)]t}.$$

$$\boxed{u(x,t) = \sum_{m,n} \alpha_{mn}\, e^{(m^2+n^2)[\varepsilon-(m^2+n^2)]t} \underbrace{e^{i(mx+ny)}}_{oscillates}. \qquad \circledast}$$

When $\varepsilon = 2$, we have

$$u(x,t) = \sum_{m,n} \alpha_{mn}\, e^{(m^2+n^2)[2-(m^2+n^2)]t} e^{i(mx+ny)}.$$

We need a solution whose amplitude increases as t increases. Thus, we need those $\alpha_{mn} > 0$, with

$$\begin{aligned}
(m^2+n^2)[2-(m^2+n^2)] &> 0, \\
2-(m^2+n^2) &> 0, \\
2 &> m^2+n^2.
\end{aligned}$$

Hence, $\alpha_{mn} > 0$ for $(m,n) = (0,0)$, $(m,n) = (1,0)$, $(m,n) = (0,1)$.
Else, $\alpha_{mn} = 0$. Thus,

$$\begin{aligned}
u(x,t) &= \alpha_{00} + \alpha_{10} e^t e^{ix} + \alpha_{01} e^t e^{iy} = 1 + e^t e^{ix} + e^t e^{iy} \\
&= 1 + e^t(\cos x + i \sin x) + e^t(\cos y + i \sin y).
\end{aligned}$$

b) For $\varepsilon \leq \varepsilon_0 = 1$, the solution $\circledast$ stays bounded as $t \to \infty$. $\square$

Problem (F'93, #1).
Suppose that a and b are constants with $a \geq 0$, and consider the equation

$$u_t = u_{xx} + u_{yy} - au^3 + bu \tag{29.4}$$

*in which $u(x, y, t)$ is **2π-periodic** in x and y.*

a) *Let u be a solution of (29.4) with*

$$||u(t=0)|| = \int_0^{2\pi}\int_0^{2\pi} |u(x,y,t=0)|^2\, dxdy^{1/2} < \epsilon.$$

Derive an explicit bound on $||u(t)||$ and show that it stays finite for all t.
b) *If $a = 0$, construct the normal modes for (29.4); i.e. find all solutions of the form*

$$u(x,y,t) = e^{\lambda t + ikx + ily}.$$

c) *Use these normal modes to construct a solution of (29.4) with $a = 0$ for the initial data*

$$u(x,y,t=0) = \frac{1}{1-\frac{1}{2}e^{ix}} + \frac{1}{1-\frac{1}{2}e^{-ix}}.$$

Proof. **a)** Multiply the equation by u and integrate:

$$\begin{aligned}
u_t &= \triangle u - au^3 + bu, \\
uu_t &= u\triangle u - au^4 + bu^2, \\
\int_\Omega uu_t\, dx &= \int_\Omega u\triangle u\, dx - \int_\Omega au^4\, dx + \int_\Omega bu^2\, dx, \\
\frac{1}{2}\frac{d}{dt}\int_\Omega u^2\, dx &= \underbrace{\int_{\partial\Omega} u\frac{\partial u}{\partial n}\, ds}_{=0,\ u\ periodic\ on\ [0,2\pi]^2} \underbrace{- \int_\Omega |\nabla u|^2\, dx - \int_\Omega au^4\, dx}_{\leq 0} + \int_\Omega bu^2\, dx, \\
\frac{d}{dt}||u||_2^2 &\leq 2b\,||u||_2^2, \\
||u||_2^2 &\leq ||u(x,0)||_2^2\, e^{2bt}, \\
||u||_2 &\leq ||u(x,0)||_2\, e^{bt} \leq \varepsilon\, e^{bt}.
\end{aligned}$$

Thus, $||u||$ stays finite for all t.

b) Since $a = 0$, plugging $u = e^{\lambda t + ikx + ily}$ into the equation, we obtain:

$$\begin{aligned}
u_t &= u_{xx} + u_{yy} + bu, \\
\lambda\, e^{\lambda t + ikx + ily} &= (-k^2 - l^2 + b)\, e^{\lambda t + ikx + ily}, \\
\lambda &= -k^2 - l^2 + b.
\end{aligned}$$

Thus,

$$\begin{aligned}
u_{kl} &= e^{(-k^2 - l^2 + b)t + ikx + ily}, \\
u(x,y,t) &= \sum_{k,l} a_{kl}\, e^{(-k^2 - l^2 + b)t + ikx + ily}.
\end{aligned}$$

c) Using the initial condition, we obtain:

$$\begin{aligned}
u(x,y,0) = \sum_{k,l} a_{kl}\, e^{i(kx+ly)} &= \frac{1}{1-\frac{1}{2}e^{ix}} + \frac{1}{1-\frac{1}{2}e^{-ix}} \\
&= \sum_{k=0}^{\infty}\Big(\frac{1}{2}e^{ix}\Big)^k + \sum_{k=0}^{\infty}\Big(\frac{1}{2}e^{-ix}\Big)^k \\
&= \sum_{k=0}^{\infty}\frac{1}{2^k}e^{ikx} + \sum_{k=0}^{\infty}\frac{1}{2^k}e^{-ikx}, \\
&= 2 + \sum_{k=1}^{\infty}\frac{1}{2^k}e^{ikx} + \sum_{k=-1}^{-\infty}\frac{1}{2^{-k}}e^{ikx}.
\end{aligned}$$

Thus, $l = 0$, and we have

$$\sum_{k=-\infty}^{\infty} a_k\, e^{ikx} = 2 + \sum_{k=1}^{\infty}\frac{1}{2^k}e^{ikx} + \sum_{k=-1}^{-\infty}\frac{1}{2^{-k}}e^{ikx},$$

$$\Rightarrow \quad a_0 = 2; \quad a_k = \frac{1}{2^k}, \quad k>0; \quad a_k = \frac{1}{2^{-k}}, \quad k<0$$

$$\Rightarrow \quad a_0 = 2; \quad a_k = \frac{1}{2^{|k|}}, \quad k \neq 0.$$

$$\boxed{u(x,y,t) = 2e^{bt} + \sum_{k=-\infty,\, k\neq 0}^{+\infty} \frac{1}{2^{|k|}}\, e^{(-k^2+b)t+ikx}.}$$

[81] $\square$

[81]Note a similar question formulation in F'92 #3(b).

Problem (S'00, #3). *Consider the initial-boundary value problem for $u = u(x, y, t)$*

$$u_t = \triangle u - u$$

for $(x, y) \in [0, 2\pi]^2$, with ***periodic boundary conditions*** *and with*

$$u(x, y, 0) = u_0(x, y)$$

in which u_0 is periodic. Find an asymptotic expansion for u for t large with terms tending to zero increasingly rapidly as $t \to \infty$.

Proof. Since we have periodic boundary conditions, assume

$$u(x, y, t) \;=\; \sum_{m,n} u_{mn}(t)\, e^{i(mx+ny)}.$$

Plug this into the equation:

$$\begin{aligned}
\sum_{m,n} u'_{mn}(t)\, e^{i(mx+ny)} &= \sum_{m,n} (-m^2 - n^2 - 1)\, u_{mn}(t)\, e^{i(mx+ny)}, \\
u'_{mn}(t) &= (-m^2 - n^2 - 1)\, u_{mn}(t), \\
u_{mn}(t) &= a_{mn}\, e^{(-m^2-n^2-1)t}, \\
u(x, y, t) &= \sum_{m,n} a_{mn}\, e^{-(m^2+n^2+1)t}\, e^{i(mx+ny)}.
\end{aligned}$$

Since u_0 is periodic,

$$u_0(x, y) = \sum_{m,n} u_{0mn}\, e^{i(mx+ny)}, \qquad u_{0mn} = \frac{1}{4\pi^2}\int_0^{2\pi}\!\!\int_0^{2\pi} u_0(x, y)\, e^{-i(mx+ny)}\, dxdy.$$

Initial condition gives:

$$\begin{aligned}
u(x, y, 0) \;=\; \sum_{m,n} a_{mn}\, e^{i(mx+ny)} &= u_0(x, y), \\
\sum_{m,n} a_{mn}\, e^{i(mx+ny)} &= \sum_{m,n} u_{0mn}\, e^{i(mx+ny)}, \\
\Rightarrow \quad a_{mn} &= u_{0mn}.
\end{aligned}$$

$$\boxed{u(x, y, t) = \sum_{m,n} u_{0mn}\, e^{-(m^2+n^2+1)t}\, e^{i(mx+ny)}.}$$

$u_{0mn}\, e^{-(m^2+n^2+1)t}\, e^{i(mx+ny)} \to 0$ as $t \to \infty$, since $e^{-(m^2+n^2+1)t} \to 0$ as $t \to \infty$. □

30 Problems: Fourier Transform

Problem (S'01, #2b). *Write the solution of initial value problem*

$$U_t - \begin{pmatrix} 1 & 0 \\ 5 & 3 \end{pmatrix} U_x = 0,$$

for general initial data

$$\begin{pmatrix} u^{(1)}(x,0) \\ u^{(2)}(x,0) \end{pmatrix} = \begin{pmatrix} f(x) \\ 0 \end{pmatrix} \qquad \textit{as an } \boldsymbol{\textit{inverse Fourier transform.}}$$

You may assume that f is smooth and rapidly decreasing as $|x| \to \infty$.

Proof. Consider the original system:

$$\begin{aligned} u_t^{(1)} - u_x^{(1)} &= 0, \\ u_t^{(2)} - 5u_x^{(1)} - 3u_x^{(2)} &= 0. \end{aligned}$$

Take the Fourier transform in x. The transformed initial value problems are:

$$\begin{aligned} \widehat{u}_t^{(1)} - i\xi\widehat{u}^{(1)} &= 0, & \widehat{u}^{(1)}(\xi,0) &= \widehat{f}(\xi), \\ \widehat{u}_t^{(2)} - 5i\xi\widehat{u}^{(1)} - 3i\xi\widehat{u}^{(2)} &= 0, & \widehat{u}^{(2)}(\xi,0) &= 0. \end{aligned}$$

Solving the first ODE for $\widehat{u}^{(1)}$ gives:

$$\widehat{u}^{(1)}(\xi,t) = \widehat{f}(\xi)e^{i\xi t}. \qquad \checkmark$$

With this $\widehat{u}^{(1)}$, the second initial value problem becomes

$$\widehat{u}_t^{(2)} - 3i\xi\widehat{u}^{(2)} = 5i\xi\widehat{f}(\xi)e^{i\xi t}, \qquad \widehat{u}^{(2)}(\xi,0) = 0.$$

The homogeneous solution of the above ODE is:

$$\widehat{u}_h^{(2)}(\xi,t) = c_1 e^{3i\xi t}.$$

With $\widehat{u}_p^{(2)} = c_2 e^{i\xi t}$ as anzats for a particular solution, we obtain:

$$\begin{aligned} i\xi c_2 e^{i\xi t} - 3i\xi c_2 e^{i\xi t} &= 5i\xi\widehat{f}(\xi)e^{i\xi t}, \\ -2i\xi c_2 e^{i\xi t} &= 5i\xi\widehat{f}(\xi)e^{i\xi t}, \\ c_2 &= -\frac{5}{2}\widehat{f}(\xi). \\ \Rightarrow \quad \widehat{u}_p^{(2)}(\xi,t) &= -\frac{5}{2}\widehat{f}(\xi)e^{i\xi t}. \end{aligned}$$

$$\widehat{u}^{(2)}(\xi,t) = \widehat{u}_h^{(2)}(\xi,t) + \widehat{u}_p^{(2)}(\xi,t) = c_1 e^{3i\xi t} - \frac{5}{2}\widehat{f}(\xi)e^{i\xi t}.$$

We find c_1 using initial conditions:

$$\widehat{u}^{(2)}(\xi,0) = c_1 - \frac{5}{2}\widehat{f}(\xi) = 0 \qquad \Rightarrow \quad c_1 = \frac{5}{2}\widehat{f}(\xi).$$

Thus,

$$\widehat{u}^{(2)}(\xi,t) = \frac{5}{2}\widehat{f}(\xi)\big(e^{3i\xi t} - e^{i\xi t}\big). \qquad \checkmark$$

$u^{(1)}(x,t)$ and $u^{(2)}(x,t)$ are be obtained by taking **inverse Fourier transform**:

$$\begin{aligned} u^{(1)}(x,t) &= \big(\widehat{u}^{(1)}(\xi,t)\big)^{\vee} = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}^n} e^{ix\xi}\,\widehat{f}(\xi)\,e^{i\xi t}\,d\xi, \\ u^{(2)}(x,t) &= \big(\widehat{u}^{(2)}(\xi,t)\big)^{\vee} = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}^n} e^{ix\xi}\,\frac{5}{2}\widehat{f}(\xi)\big(e^{3i\xi t}-e^{i\xi t}\big)\,d\xi. \end{aligned}$$

$\square$

Problem (S'02, #4). *Use the* ***Fourier transform*** *on $L^2(\mathbb{R})$ to show that*

$$\frac{du}{dx} + cu(x) + u(x-1) = f \tag{30.1}$$

has a unique solution $u \in L^2(\mathbb{R})$ for each $f \in L^2(\mathbb{R})$ when $|c| > 1$ - you may assume that c is a real number.

Proof. $u \in L^2(\mathbb{R})$. Define its **Fourier transform** $\widehat{u}$ by

$$\begin{aligned}\widehat{u}(\xi) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ix\xi}u(x)\,dx \qquad \text{for } \xi \in \mathbb{R}.\\ \widehat{\frac{du}{dx}}(\xi) &= i\xi\widehat{u}(\xi).\end{aligned}$$

We can find $\widehat{u(x-1)}(\xi)$ in two ways.

- Let $u(\underbrace{x-1}_{y}) = v(x)$, and determinte $\widehat{v}(\xi)$:

$$\begin{aligned}\widehat{u(x-1)}(\xi) = \widehat{v}(\xi) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ix\xi}v(x)\,dx = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-i(y+1)\xi}u(y)\,dy\\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-iy\xi}e^{-i\xi}u(y)\,dy = e^{-i\xi}\widehat{u}(\xi). \qquad \circledast\end{aligned}$$

- We can also write the definition for $\widehat{u}(\xi)$ and substitute $x-1$ later in calculations:

$$\begin{aligned}\widehat{u}(\xi) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-iy\xi}u(y)\,dy = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-i(x-1)\xi}u(x-1)\,dx\\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ix\xi}e^{i\xi}u(x-1)\,dx = e^{i\xi}\widehat{u(x-1)}(\xi),\\ \Rightarrow \qquad & \widehat{u(x-1)}(\xi) = e^{-i\xi}\widehat{u}(\xi).\end{aligned}$$

Substituting into (30.1), we obtain

$$\begin{aligned} & i\xi\widehat{u}(\xi) + c\widehat{u}(\xi) + e^{-i\xi}\widehat{u}(\xi) = \widehat{f}(\xi),\\ \widehat{u}(\xi) &= \frac{\widehat{f}(\xi)}{i\xi + c + e^{-i\xi}}.\\ u(x) &= \left(\frac{\widehat{f}(\xi)}{i\xi + c + e^{-i\xi}}\right)^{\vee} = \big(\widehat{f}\,\widehat{B}\big)^{\vee} = \frac{1}{\sqrt{2\pi}}f * B,\\ \text{where } \widehat{B} &= \frac{1}{i\xi + c + e^{-i\xi}},\\ \Rightarrow \quad B &= \left(\frac{1}{i\xi + c + e^{-i\xi}}\right)^{\vee} = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \frac{e^{ix\xi}}{i\xi + c + e^{-i\xi}}\,d\xi.\end{aligned}$$

For $|c| > 1$, $\widehat{u}(\xi)$ exists for all $\xi \in \mathbb{R}$, so that $u(x) = (\widehat{u}(\xi))^{\vee}$ and this is unique by the Fourier Inversion Theorem. $\square$

Note that in $\mathbb{R}^n$, $\circledast$ becomes

$$\begin{aligned}\widehat{u(x-1)}(\xi) = \widehat{v}(\xi) &= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-ix\cdot\xi}v(x)\,dx = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-i(y+1)\cdot\xi}u(y)\,dy\\ &= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-iy\cdot\xi}e^{-i\vec{1}\cdot\xi}u(y)\,dy = e^{-i\vec{1}\cdot\xi}\widehat{u}(\xi) = e^{(-i\sum_j \xi_j)}\,\widehat{u}(\xi).\end{aligned}$$

Problem (F'96, #3). *Find the **fundamental solution** for the equation*

$$u_t = u_{xx} - xu_x. \tag{30.2}$$

*Hint: The **Fourier transform** converts this problem into a PDE which can be solved using the method of characteristics.*

Proof. $u \in L^2(\mathbb{R})$. Define its **Fourier transform** $\widehat{u}$ by

$$\begin{aligned}
\widehat{u}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} u(x)\, dx \qquad \text{for } \xi \in \mathbb{R}. \\
\widehat{u_x}(\xi) &= i\xi\, \widehat{u}(\xi), \\
\widehat{u_{xx}}(\xi) &= (i\xi)^2\, \widehat{u}(\xi) = -\xi^2\, \widehat{u}(\xi). \quad \checkmark
\end{aligned}$$

We find $\widehat{xu_x}(\xi)$ in two steps:
① Multiplication by x:

$$\begin{aligned}
\widehat{-ixu}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} \big(- ixu(x)\big)\, dx = \frac{d}{d\xi}\widehat{u}(\xi). \\
&\Rightarrow \widehat{xu(x)}(\xi) = i\frac{d}{d\xi}\widehat{u}(\xi).
\end{aligned}$$

② Using the previous result, we find:

$$\begin{aligned}
\widehat{xu_x(x)}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} \big(xu_x(x)\big)\, dx = \underbrace{\frac{1}{\sqrt{2\pi}} \Big[e^{-ix\xi} xu\Big]_{-\infty}^{\infty}}_{=0} - \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \Big((-i\xi)e^{-ix\xi}x + e^{-ix\xi}\Big)\, u\, dx \\
&= \frac{1}{\sqrt{2\pi}}\, i\xi \int_{\mathbb{R}} e^{-ix\xi}\, x\, u\, dx - \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi}\, u\, dx \\
&= i\xi\, \widehat{xu(x)}(\xi) - \widehat{u}(\xi) = i\xi\Big[i\frac{d}{d\xi}\widehat{u}(\xi)\Big] - \widehat{u}(\xi) = -\xi\frac{d}{d\xi}\widehat{u}(\xi) - \widehat{u}(\xi). \\
\Rightarrow \widehat{xu_x(x)}(\xi) &= -\xi\frac{d}{d\xi}\widehat{u}(\xi) - \widehat{u}(\xi). \quad \checkmark
\end{aligned}$$

Plugging these into (30.2), we get:

$$\begin{aligned}
\frac{\partial}{\partial t}\widehat{u}(\xi,t) &= -\xi^2\widehat{u}(\xi,t) - \Big(- \xi\frac{d}{d\xi}\widehat{u}(\xi,t) - \widehat{u}(\xi,t)\Big), \\
\widehat{u}_t &= -\xi^2\widehat{u} + \xi\widehat{u}_\xi + \widehat{u}, \\
\widehat{u}_t - \xi\widehat{u}_\xi &= -(\xi^2 - 1)\widehat{u}.
\end{aligned}$$

We now solve the above equation by characteristics.
We change the notation: $\widehat{u} \to u$, $t \to y$, $\xi \to x$. We have

$$u_y - xu_x = -(x^2 - 1)u.$$

$$\begin{aligned}
\frac{dx}{dt} &= -x \quad \Rightarrow \quad x = c_1e^{-t}, \qquad (c_1 = xe^t) \\
\frac{dy}{dt} &= 1 \quad \Rightarrow \quad y = t + c_2, \\
\frac{dz}{dt} &= -(x^2 - 1)z = -(c_1^2e^{-2t} - 1)z \quad \Rightarrow \quad \frac{dz}{z} = -(c_1^2e^{-2t} - 1)dt \\
&\Rightarrow \log z = \frac{1}{2}c_1^2e^{-2t} + t + c_3 = \frac{x^2}{2} + t + c_3 = \frac{x^2}{2} + y - c_2 + c_3 \quad \Rightarrow \quad z = ce^{\frac{x^2}{2}+y}.
\end{aligned}$$

Changing the notation back, we have

$$\widehat{u}(\xi,t) \;=\; ce^{\frac{\xi^2}{2}+t}.$$

Thus, we have

$$\widehat{u}(\xi,t) \;=\; ce^{\frac{\xi^2}{2}+t}.$$

We use Inverse Fourier Tranform to get $u(x,t)$: [82]

$$\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{ix\xi}\,\widehat{u}(\xi,t)\,d\xi \;=\; \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{ix\xi}\,ce^{\frac{\xi^2}{2}+t}\,d\xi \\
&= \frac{c}{\sqrt{2\pi}}\,e^t\int_{\mathbb{R}} e^{ix\xi}\,e^{\frac{\xi^2}{2}}\,d\xi \;=\; \frac{c}{\sqrt{2\pi}}\,e^t\int_{\mathbb{R}} e^{ix\xi+\frac{\xi^2}{2}}\,d\xi \\
&= \frac{c}{\sqrt{2\pi}}\,e^t\int_{\mathbb{R}} e^{\frac{2ix\xi+\xi^2}{2}}\,d\xi \;=\; \frac{c}{\sqrt{2\pi}}\,e^t\int_{\mathbb{R}} e^{\frac{(\xi+ix)^2}{2}}\,d\xi\; e^{\frac{x^2}{2}} \\
&= \frac{c}{\sqrt{2\pi}}\,e^te^{\frac{x^2}{2}}\int_{\mathbb{R}} e^{\frac{y^2}{2}}\,dy \;=\; \frac{c}{\sqrt{2\pi}}\,e^te^{\frac{x^2}{2}}\sqrt{2\pi} \;=\; c\,e^t\,e^{\frac{x^2}{2}}.
\end{aligned}$$

$$\boxed{u(x,t) \;=\; c\,e^t\,e^{\frac{x^2}{2}}.}$$

Check:

$$\begin{aligned}
u_t &= c\,e^t\,e^{\frac{x^2}{2}}, \\
u_x &= c\,e^t\,xe^{\frac{x^2}{2}}, \\
u_{xx} &= c\,e^t\left(e^{\frac{x^2}{2}} + x^2e^{\frac{x^2}{2}}\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
u_t &= u_{xx} - xu_x, \\
c\,e^t\,e^{\frac{x^2}{2}} &= c\,e^t\left(e^{\frac{x^2}{2}} + x^2e^{\frac{x^2}{2}}\right) - x\,c\,e^t\,xe^{\frac{x^2}{2}}. \quad \checkmark
\end{aligned}$$

$\square$

[82] We complete the square for powers of exponentials.

Problem (W'02, #4). ***a)*** *Solve the initial value problem*

$$\frac{\partial u}{\partial t} + \sum_{k=1}^{n} a_k(t)\frac{\partial u}{\partial x_k} + a_0(t)u = 0, \quad x \in \mathbb{R}^n,$$
$$u(0,x) = f(x)$$

where $a_k(t)$, $k = 1,\ldots,n$, and $a_0(t)$ are continuous functions, and f is a continuous function. You may assume f has compact support.
b) *Solve the initial value problem*

$$\frac{\partial u}{\partial t} + \sum_{k=1}^{n} a_k(t)\frac{\partial u}{\partial x_k} + a_0(t)u = f(x,t), \quad x \in \mathbb{R}^n,$$
$$u(0,x) = 0$$

where f is continuous in x and t.

Proof. **a)** Use the **Fourier transform** to solve this problem.

$$\widehat{u}(\xi,t) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-ix\cdot\xi}u(x,t)\,dx \qquad \text{for } \xi \in \mathbb{R}.$$
$$\widehat{\frac{\partial u}{\partial x_k}} = i\xi_k\widehat{u}.$$

Thus, the equation becomes:

$$\begin{cases} \widehat{u}_t + i\sum_{k=1}^{n} a_k(t)\xi_k\widehat{u} + a_0(t)\widehat{u} = 0, \\ \widehat{u}(\xi,0) = \widehat{f}(\xi), \end{cases}$$

or

$$\widehat{u}_t + i\,\vec{a}(t)\cdot\vec{\xi}\,\widehat{u} + a_0(t)\widehat{u} = 0,$$
$$\widehat{u}_t = -\big(i\,\vec{a}(t)\cdot\vec{\xi} + a_0(t)\big)\,\widehat{u}.$$

This is an ODE in $\widehat{u}$ with solution:

$$\widehat{u}(\xi,t) = ce^{-\int_0^t (i\,\vec{a}(s)\cdot\vec{\xi}+a_0(s))\,ds}, \qquad \widehat{u}(\xi,0) = c = \widehat{f}(\xi). \qquad \text{Thus,}$$
$$\widehat{u}(\xi,t) = \widehat{f}(\xi)\,e^{-\int_0^t (i\,\vec{a}(s)\cdot\vec{\xi}+a_0(s))\,ds}.$$

Use the Inverse Fourier transform to get $u(x,t)$:

$$u(x,t) = \widehat{u}(\xi,t)^{\vee} = \left[\widehat{f}(\xi)\,e^{-\int_0^t (i\,\vec{a}(s)\cdot\vec{\xi}+a_0(s))\,ds}\right]^{\vee} = \frac{(f*g)(x)}{(2\pi)^{\frac{n}{2}}},$$
$$\text{where} \quad \widehat{g}(\xi) = e^{-\int_0^t (i\,\vec{a}(s)\cdot\vec{\xi}+a_0(s))\,ds}.$$
$$g(x) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\,\widehat{g}(\xi)\,d\xi = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\left[e^{-\int_0^t (i\,\vec{a}(s)\cdot\vec{\xi}+a_0(s))\,ds}\right]d\xi.$$

$$\boxed{u(x,t) = \frac{(f*g)(x)}{(2\pi)^{\frac{n}{2}}} = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}\left[e^{-\int_0^t (i\,\vec{a}(s)\cdot\vec{\xi}+a_0(s))\,ds}\right]d\xi\,f(y)\,dy.}$$

b) Use **Duhamel's Principle** and the result from (a).

$$u(x,t) = \int_0^t U(x,t-s,s)\,ds, \qquad \text{where } U(x,t,s) \text{ solves}$$
$$\frac{\partial U}{\partial t} + \sum_{k=1}^{n} a_k(t)\frac{\partial U}{\partial x_k} + a_0(t)U = 0,$$
$$U(x,0,s) = f(x,s).$$

$$\boxed{u(x,t) \;=\; \int_0^t U(x,t-s,s)\,ds \;=\; \frac{1}{(2\pi)^n}\int_0^t \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}\left[e^{-\int_0^{\mathbf{t-s}}(i\,\vec{a}(s)\cdot\vec{\xi}+a_0(s))\,ds}\right] d\xi\, f(y,s)\,dy\,ds.}$$

$\square$

Problem (S'93, #2). ***a)*** *Define the Fourier transform* [83]

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} e^{ix\xi} f(x)\,dx.$$

State the inversion theorem. If

$$\widehat{f}(\xi) = \begin{cases} \pi, & |\xi| < a, \\ \frac{1}{2}\pi, & |\xi| = a, \\ 0, & |\xi| > a, \end{cases}$$

where a is a real constant, what $f(x)$ does the inversion theorem give?

b) *Show that*

$$\widehat{f(x-b)} = e^{i\xi b}\widehat{f(x)},$$

where b is a real constant. Hence, using part (a) and Parseval's theorem, show that

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin a(x+z)}{x+z}\,\frac{\sin a(x+\xi)}{x+\xi}\,dx = \frac{\sin a(z-\xi)}{z-\xi},$$

where z and ξ are real constants.

Proof. **a)** • The **inverse Fourier transform** for $f \in L^1(\mathbb{R}^n)$:

$$f^{\vee}(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ix\xi} f(x)\,dx \qquad \text{for } \ \xi \in \mathbb{R}.$$

Fourier Inversion Theorem: Assume $f \in L^2(\mathbb{R})$. Then

$$\boxed{f(x) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ix\xi}\widehat{f}(\xi)\,d\xi = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(y-x)\xi} f(y)\,dy\,d\xi \;=\; (\widehat{f})^{\vee}(x).}$$

• **Parseval's theorem (Plancherel's theorem)** (for this definition of the Fourier transform). Assume $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Then $\widehat{f}, f^{\vee} \in L^2(\mathbb{R}^n)$ and

$$\frac{1}{2\pi}||\widehat{f}||_{L^2(\mathbb{R}^n)} \;=\; ||f^{\vee}||_{L^2(\mathbb{R}^n)} \;=\; ||f||_{L^2(\mathbb{R}^n)}, \qquad \text{or}$$

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} |\widehat{f}(\xi)|^2\,d\xi.$$

Also,

$$\int_{-\infty}^{\infty} f(x)\,\overline{g(x)}\,dx \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(\xi)\,\overline{\widehat{g}(\xi)}\,d\xi.$$

• We can write

$$\widehat{f}(\xi) = \begin{cases} \pi, & |\xi| < a, \\ 0, & |\xi| > a. \end{cases}$$

[83] Note that the Fourier transform is defined incorrectly here. There should be '-' sign in $e^{-ix\xi}$. Need to be careful, since the consequences of this definition propagate throughout the solution.

$$
\begin{aligned}
f(x) &= (\widehat{f}(\xi))^{\vee} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ix\xi}\widehat{f}(\xi)\,d\xi = \frac{1}{2\pi}\int_{-\infty}^{-a} 0\,d\xi + \frac{1}{2\pi}\int_{-a}^{a} e^{-ix\xi}\pi\,d\xi + \frac{1}{2\pi}\int_{a}^{\infty} 0\,d\xi \\
&= \frac{1}{2}\int_{-a}^{a} e^{-ix\xi}\,d\xi = -\frac{1}{2ix}\Big[e^{-ix\xi}\Big]_{\xi=-a}^{\xi=a} = -\frac{1}{2ix}\Big[e^{-iax}-e^{iax}\Big] = \frac{\sin ax}{x}. \quad \checkmark
\end{aligned}
$$

b) • Let $f(\underbrace{x-b}_{y}) = g(x)$, and determinte $\widehat{g}(\xi)$:

$$
\begin{aligned}
\widehat{f(x-b)}(\xi) = \widehat{g}(\xi) &= \int_{\mathbb{R}} e^{ix\xi} g(x)\,dx = \int_{\mathbb{R}} e^{i(y+b)\xi} f(y)\,dy \\
&= \int_{\mathbb{R}} e^{iy\xi} e^{ib\xi} f(y)\,dy = e^{ib\xi}\widehat{f}(\xi). \quad \checkmark
\end{aligned}
$$

• With $f(x) = \frac{\sin ax}{x}$ (from (a)), we have

$$
\begin{aligned}
\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin a(x+z)}{x+z}\,\frac{\sin a(x+s)}{x+s}\,dx &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(x+z)f(x+s)\,dx \qquad (x' = x+s,\ \ dx' = dx) \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty} f(x'+z-s)f(x')\,dx' \qquad \text{(Parseval's)} \\
&= \frac{1}{\pi}\,\frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f(x'+z-s)}\,\overline{\widehat{f(x')}}\,d\xi \qquad \text{part (b)} \\
&= \frac{1}{2\pi^2}\int_{-\infty}^{\infty} \widehat{f(\xi)}\,e^{-i(z-s)\xi}\,\overline{\widehat{f(\xi)}}\,d\xi \\
&= \frac{1}{2\pi^2}\int_{-a}^{a} \widehat{f(\xi)}^{\,2}\,e^{-i(z-s)\xi}\,d\xi \\
&= \frac{1}{2\pi^2}\int_{-a}^{a} \pi^2\,e^{-i(z-s)\xi}\,d\xi \\
&= \frac{1}{2}\int_{-a}^{a} e^{-i(z-s)\xi}\,d\xi \\
&= \frac{1}{-2i(z-s)}\big[e^{-i(z-s)\xi}\big]_{\xi=-a}^{\xi=a} \\
&= \frac{e^{i(z-s)a}-e^{-i(z-s)a}}{2i(z-s)} = \frac{\sin a(z-s)}{z-s}. \quad \checkmark
\end{aligned}
$$

$\square$

Problem (F'03, #5). ❶ *State Parseval's relation for* ***Fourier transforms****.*

❷ *Find the Fourier transform $\hat{f}(\xi)$ of*

$$f(x) = \begin{cases} e^{i\alpha x}/2\sqrt{\pi y}, & |x| \le y \\ 0, & |x| > y, \end{cases}$$

in which y and α are constants.

❸ *Use this in Parseval's relation to show that*

$$\int_{-\infty}^{\infty} \frac{\sin^2(\alpha-\xi)y}{(\alpha-\xi)^2}\, d\xi = \pi y.$$

What does the transform $\hat{f}(\xi)$ become in the limit $y \to \infty$?

❹ *Use Parseval's relation to show that*

$$\frac{\sin(\alpha-\beta)y}{(\alpha-\beta)} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin(\alpha-\xi)y}{(\alpha-\xi)}\,\frac{\sin(\beta-\xi)y}{(\beta-\xi)}\, d\xi.$$

Proof. • $f \in L^2(\mathbb{R})$. Define its **Fourier transform** $\widehat{u}$ by

$$\widehat{f}(\xi) \;=\; \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ix\xi} f(x)\, dx \qquad \text{for } \ \xi \in \mathbb{R}.$$

❶ **Parseval's theorem (Plancherel's theorem):**
Assume $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Then $\widehat{f}, f^\vee \in L^2(\mathbb{R}^n)$ and

$$||\widehat{f}||_{L^2(\mathbb{R}^n)} \;=\; ||f^\vee||_{L^2(\mathbb{R}^n)} \;=\; ||f||_{L^2(\mathbb{R}^n)}, \qquad \text{or}$$

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx \;=\; \int_{-\infty}^{\infty} |\widehat{f}(\xi)|^2\, d\xi.$$

Also,

$$\int_{-\infty}^{\infty} f(x)\,\overline{g(x)}\, dx \;=\; \int_{-\infty}^{\infty} \widehat{f}(\xi)\,\overline{\widehat{g}(\xi)}\, d\xi.$$

❷ Find the Fourier transform of f:

$$\begin{aligned}
\widehat{f}(\xi) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ix\xi} f(x)\, dx \;=\; \frac{1}{\sqrt{2\pi}}\int_{-y}^{y} e^{-ix\xi}\frac{e^{i\alpha x}}{2\sqrt{\pi y}}\, dx \;=\; \frac{1}{2\pi\sqrt{2y}}\int_{-y}^{y} e^{i(\alpha-\xi)x}\, dx \\
&= \frac{1}{2\pi\sqrt{2y}}\,\frac{1}{i(\alpha-\xi)}\left[e^{i(\alpha-\xi)x}\right]_{x=-y}^{x=y} \;=\; \frac{1}{2i\pi\sqrt{2y}(\alpha-\xi)}\left[e^{i(\alpha-\xi)y} - e^{-i(\alpha-\xi)y}\right] \\
&= \frac{\sin y(\alpha-\xi)}{\pi\sqrt{2y}(\alpha-\xi)}. \qquad \checkmark
\end{aligned}$$

❸ Parseval's theorem gives:

$$\begin{aligned}
\int_{-\infty}^{\infty} |\widehat{f}(\xi)|^2\, d\xi &= \int_{-\infty}^{\infty} |f(x)|^2\, dx, \\
\int_{-\infty}^{\infty} \frac{\sin^2 y(\alpha-\xi)}{\pi^2 2y(\alpha-\xi)^2}\, d\xi &= \int_{-y}^{y} \frac{|e^{2i\alpha x}|}{4\pi y}\, dx, \\
\int_{-\infty}^{\infty} \frac{\sin^2 y(\alpha-\xi)}{(\alpha-\xi)^2}\, d\xi &= \frac{\pi}{2}\int_{-y}^{y} dx, \\
\int_{-\infty}^{\infty} \frac{\sin^2 y(\alpha-\xi)}{(\alpha-\xi)^2}\, d\xi &= \pi y. \qquad \checkmark
\end{aligned}$$

❹ We had

$$\widehat{f}(\xi) = \frac{\sin y(\alpha-\xi)}{\pi\sqrt{2y}(\alpha-\xi)}.$$

• We make change of variables: $\alpha-\xi=\beta-\xi'$. Then, $\xi=\xi'+\alpha-\beta$. We have

$$\widehat{f}(\xi) = \widehat{f}(\xi'+\alpha-\beta) = \frac{\sin y(\beta-\xi')}{(\beta-\xi')}, \qquad \text{or}$$

$$\widehat{f}(\xi+\alpha-\beta) = \frac{\sin y(\beta-\xi)}{(\beta-\xi)}.$$

• We will also use the following result.
Let $\widehat{f}(\underbrace{\xi+a}_{\xi'})=\widehat{g}(\xi)$, and determinte $\widehat{g}(\xi)^\vee$:

$$\begin{aligned}\widehat{f}(\xi+a)^\vee = \widehat{g}(\xi)^\vee &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{ix\xi}\widehat{g}(\xi)\,d\xi = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{ix(\xi'-a)}\widehat{f}(\xi')\,d\xi' \\ &= e^{-ixa}f(x).\end{aligned}$$

• Using these results, we have

$$\begin{aligned}\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin(\alpha-\xi)y}{(\alpha-\xi)}\,\frac{\sin(\beta-\xi)y}{(\beta-\xi)}\,d\xi &= \frac{1}{\pi}(\pi\sqrt{2y})^2\int_{-\infty}^{\infty}\widehat{f}(\xi)\,\widehat{f}(\xi+\alpha-\beta)\,d\xi \\ &= 2\pi y\int_{-\infty}^{\infty} f(x)\,e^{-(\alpha-\beta)ix}f(x)\,dx \\ &= 2\pi y\int_{-\infty}^{\infty} f(x)^2\,e^{-(\alpha-\beta)ix}\,dx \\ &= 2\pi y\int_{-y}^{y}\frac{\left|e^{2i\alpha x}\right|}{4\pi y}\,e^{-(\alpha-\beta)ix}\,dx \\ &= \frac{1}{2}\int_{-y}^{y} e^{-(\alpha-\beta)ix}\,dx \\ &= \frac{1}{-2i(\alpha-\beta)}\left[e^{-(\alpha-\beta)ix}\right]_{x=-y}^{x=y} \\ &= \frac{1}{-2i(\alpha-\beta)}\left[e^{-(\alpha-\beta)iy}-e^{(\alpha-\beta)iy}\right] \\ &= \frac{\sin(\alpha-\beta)y}{\alpha-\beta}. \qquad \checkmark\end{aligned}$$

$\square$

Problem (S'95, #5). *For the Laplace equation*

$$\triangle f \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f = 0 \tag{30.3}$$

in the upper half plane $y \geq 0$, consider
- *the Dirichlet problem $f(x,0) = g(x)$;*
- *the Neumann problem $\frac{\partial}{\partial y} f(x,0) = h(x)$.*

Assume that f, g and h are 2π periodic in x and that f is bounded at infinity.
Find the ***Fourier transform*** *N of the* ***Dirichlet-Neumann map****. In other words, find an operator N taking the Fourier transform of g to the Fourier transform of h; i.e.*

$$N\widehat{g}_k = \widehat{h}_k.$$

Proof. We solve the problem by two methods.
❶ **Fourier Series.**
Since f is 2π-periodic in x, we can write

$$f(x,y) = \sum_{n=-\infty}^{\infty} a_n(y)\, e^{inx}.$$

Plugging this into (30.3), we get the ODE:

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \big(-n^2 a_n(y) e^{inx} + a_n''(y) e^{inx}\big) &= 0, \\ a_n''(y) - n^2 a_n(y) &= 0. \end{aligned}$$

Initial conditions give: (g and h are 2π-periodic in x)

$$\begin{aligned} f(x,0) &= \sum_{n=-\infty}^{\infty} a_n(0) e^{inx} = g(x) = \sum_{n=-\infty}^{\infty} \widehat{g}_n e^{inx} \quad \Rightarrow \quad a_n(0) = \widehat{g}_n. \\ f_y(x,0) &= \sum_{n=-\infty}^{\infty} a_n'(0) e^{inx} = h(x) = \sum_{n=-\infty}^{\infty} \widehat{h}_n e^{inx} \quad \Rightarrow \quad a_n'(0) = \widehat{h}_n. \end{aligned}$$

Thus, the problems are:

$$\begin{aligned} a_n''(y) - n^2 a_n(y) &= 0, \\ a_n(0) &= \widehat{g}_n, \qquad \text{(Dirichlet)} \\ a_n'(0) &= \widehat{h}_n. \qquad \text{(Neumann)} \end{aligned}$$

$$\begin{aligned} \Rightarrow \qquad a_n(y) &= b_n e^{ny} + c_n e^{-ny}, \quad n = 1, 2, \ldots; \qquad a_0(y) = b_0 y + c_0. \\ a_n'(y) &= n b_n e^{ny} - n c_n e^{-ny}, \quad n = 1, 2, \ldots; \qquad a_0'(y) = b_0. \end{aligned}$$

Since f is bounded at $y = \pm\infty$, we have:

$$\begin{aligned} &b_n = 0 \qquad \text{for } n > 0, \\ &c_n = 0 \qquad \text{for } n < 0, \\ &b_0 = 0, \qquad c_0 \text{ arbitrary}. \end{aligned}$$

- $n > 0$:

$$\begin{aligned}
a_n(y) &= c_n e^{-ny}, \\
a_n(0) &= c_n \;=\; \widehat{g}_n, && \text{(Dirichlet)} \\
a_n'(0) &= -nc_n \;=\; \widehat{h}_n. && \text{(Neumann)} \\
&\Rightarrow\; -n\widehat{g}_n \;=\; \widehat{h}_n.
\end{aligned}$$

- $n < 0$:

$$\begin{aligned}
a_n(y) &= b_n e^{ny}, \\
a_n(0) &= b_n \;=\; \widehat{g}_n, && \text{(Dirichlet)} \\
a_n'(0) &= nb_n \;=\; \widehat{h}_n. && \text{(Neumann)} \\
&\Rightarrow\; n\widehat{g}_n \;=\; \widehat{h}_n.
\end{aligned}$$

$$\boxed{-|n|\widehat{g}_n \;=\; \widehat{h}_n, \qquad n \neq 0.}$$

- $n = 0:$

$$\begin{aligned}
a_0(y) &= c_0, \\
a_0(0) &= c_0 \;=\; \widehat{g}_0, && \text{(Dirichlet)} \\
a_0'(0) &= 0 \;=\; \widehat{h}_0. && \text{(Neumann)}
\end{aligned}$$

Note that solution $f(x,y)$ may be written as

$$\begin{aligned}
f(x,y) &= \sum_{n=-\infty}^{\infty} a_n(y)\, e^{inx} \;=\; a_0(y) + \sum_{n=-\infty}^{-1} a_n(y)\, e^{inx} + \sum_{n=1}^{\infty} a_n(y)\, e^{inx} \\
&= c_0 + \sum_{n=-\infty}^{-1} b_n e^{ny}\, e^{inx} + \sum_{n=1}^{\infty} c_n e^{-ny}\, e^{inx} \\
&= \begin{cases} \widehat{g}_0 + \sum_{n=-\infty}^{-1} \widehat{g}_n e^{ny}\, e^{inx} + \sum_{n=1}^{\infty} \widehat{g}_n e^{-ny}\, e^{inx}, & \text{(Dirichlet)} \\ c_0 + \sum_{n=-\infty}^{-1} \frac{\widehat{h}_n}{n} e^{ny}\, e^{inx} + \sum_{n=1}^{\infty} -\frac{\widehat{h}_n}{n} e^{-ny}\, e^{inx}. & \text{(Neumann)} \end{cases}
\end{aligned}$$

❷ **Fourier Transform.** The Fourier transform of $f(x,y)$ in x is:

$$\begin{aligned}
\widehat{f}(\xi,y) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi}\, f(x,y)\, dx, \\
f(x,y) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\xi}\, \widehat{f}(\xi,y)\, d\xi.
\end{aligned}$$

$$(i\xi)^2 \widehat{f}(\xi,y) + \widehat{f}_y y(\xi,y) \;=\; 0,$$

$\widehat{f}_{yy} \;-\; \xi^2 \widehat{f} \;=\; 0.$ The solution to this ODE is:

$$\widehat{f}(\xi,y) \;=\; c_1 e^{\xi y} + c_2 e^{-\xi y}.$$

For $\xi > 0$, $c_1 = 0$; for $\xi < 0$, $c_2 = 0$.

- $\xi > 0:$ $\widehat{f}(\xi,y) = c_2 e^{-\xi y}, \qquad \widehat{f}_y(\xi,y) = -\xi c_2 e^{-\xi y},$

$$\begin{aligned}
c_2 \;=\; \widehat{f}(\xi,0) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi}\, f(x,0)\, dx \;=\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi}\, g(x)\, dx \;=\; \widehat{g}(\xi), && \text{(Dirichlet)} \\
-\xi c_2 \;=\; \widehat{f}_y(\xi,0) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi}\, f_y(x,0)\, dx \;=\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi}\, h(x)\, dx \;=\; \widehat{h}(\xi). && \text{(Neumann)} \\
&\Rightarrow\; -\xi\widehat{g}(\xi) \;=\; \widehat{h}(\xi).
\end{aligned}$$

$$\bullet\ \xi < 0: \qquad \widehat{f}(\xi,y) = c_1 e^{\xi y}, \qquad \widehat{f}_y(\xi,y) = \xi c_1 e^{\xi y},$$

$$c_1 = \widehat{f}(\xi,0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ix\xi}\, f(x,0)\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ix\xi}\, g(x)\,dx = \widehat{g}(\xi), \qquad \text{(Dirichlet)}$$

$$\xi c_1 = \widehat{f}_y(\xi,0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ix\xi}\, f_y(x,0)\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ix\xi}\, h(x)\,dx = \widehat{h}(\xi). \qquad \text{(Neumann)}$$

$$\Rightarrow \quad \xi\widehat{g}(\xi) = \widehat{h}(\xi).$$

$$\boxed{-|\xi|\widehat{g}(\xi) = \widehat{h}(\xi).}$$

$\square$

Problem (F'97, #3). *Consider the Dirichlet problem in the half-space* $x_n > 0$, $n \geq 2$:

$$\triangle u + a\frac{\partial u}{\partial x_n} + k^2 u = 0, \qquad x_n > 0$$
$$u(x', 0) = f(x'), \qquad x' = (x_1, \ldots, x_{n-1}).$$

Here a *and* k *are constants.*
Use the ***Fourier transform*** *to show that for any* $f(x') \in L^2(\mathbb{R}^{n-1})$ *there exists a solution* $u(x', x_n)$ *of the Dirichlet problem such that*

$$\int_{\mathbb{R}^n} |u(x', x_n)|^2 \, dx' \leq C$$

for all $0 < x_n < +\infty$.

Proof. [84] Denote $\xi = (\xi', \xi_n)$. Transform in the first $n-1$ variables:

$$-|\xi'|^2 \, \widehat{u}(\xi', x_n) \; + \; \frac{\partial^2 \widehat{u}}{\partial x_n^2}(\xi', x_n) \; + \; a\frac{\partial \widehat{u}}{\partial x_n}(\xi', x_n) \; + \; k^2 \widehat{u}(\xi', x_n) \; = \; 0.$$

Thus, the ODE and initial conditions of the transformed problem become:

$$\begin{cases} \widehat{u}_{x_n x_n} + a\widehat{u}_{x_n} + (k^2 - |\xi'|^2)\widehat{u} \; = \; 0, \\ \widehat{u}(\xi', 0) \; = \; \widehat{f}(\xi'). \end{cases}$$

With the anzats $\widehat{u} = ce^{sx_n}$, we obtain $s^2 + as + (k^2 - |\xi'|^2) = 0$, and

$$s_{1,2} \; = \; \frac{-a \pm \sqrt{a^2 - 4(k^2 - |\xi'|^2)}}{2}.$$

Choosing only the negative root, we obtain the solution: [85]

$$\widehat{u}(\xi', x_n) \; = \; c(\xi') \, e^{\frac{-a - \sqrt{a^2 - 4(k^2 - |\xi'|^2)}}{2} x_n}. \qquad \widehat{u}(\xi', 0) \; = \; c \; = \; \widehat{f}(\xi'). \qquad \text{Thus,}$$
$$\widehat{u}(\xi', x_n) \; = \; \widehat{f}(\xi') \, e^{\frac{-a - \sqrt{a^2 - 4(k^2 - |\xi'|^2)}}{2} x_n}.$$

Parseval's theorem gives:

$$\begin{aligned} ||u||^2_{L^2(\mathbb{R}^{n-1})} \; &= \; ||\widehat{u}||^2_{L^2(\mathbb{R}^{n-1})} \; = \; \int_{\mathbb{R}^{n-1}} |\widehat{u}(\xi', x_n)|^2 \, d\xi' \\ &= \; \int_{\mathbb{R}^{n-1}} \Big| \widehat{f}(\xi') \, e^{\frac{-a - \sqrt{a^2 - 4(k^2 - |\xi'|^2)}}{2} x_n} \Big|^2 \, d\xi' \; \leq \; \int_{\mathbb{R}^{n-1}} \big| \widehat{f}(\xi') \big|^2 \, d\xi' \\ &= \; ||\widehat{f}||^2_{L^2(\mathbb{R}^{n-1})} \; = \; ||f||^2_{L^2(\mathbb{R}^{n-1})} \; \leq \; C, \end{aligned}$$

since $f(x') \in L^2(\mathbb{R}^{n-1})$. Thus, $u(x', x_n) \in L^2(\mathbb{R}^{n-1})$. □

[84] Note that the last element of $x = (x', x_n) = (x_1, \ldots, x_{n-1}, x_n)$, i.e. x_n, plays a role of time t. As such, the PDE may be written as

$$\triangle u + u_{tt} + au_t + k^2 u \; = \; 0.$$

[85] Note that $a > 0$ should have been provided by the statement of the problem.

Problem (F'89, #7). *Find the following* ***fundamental solutions***

$$\textbf{\textit{a)}} \qquad \begin{aligned} \frac{\partial G(x,y,t)}{\partial t} &= a(t)\frac{\partial^2 G(x,y,t)}{\partial x^2} + b(t)\frac{\partial G(x,y,t)}{\partial x} + c(t)G(x,y,t) \qquad \textit{for} \quad t>0 \\ G(x,y,0) &= \delta(x-y), \end{aligned}$$

where $a(t)$, $b(t)$, $c(t)$ *are continuous functions on* $[0,+\infty]$, $a(t)>0$ *for* $t>0$.

$$\textbf{\textit{b)}} \qquad \frac{\partial G}{\partial t}(x_1,\ldots,x_n,y_1,\ldots,y_n,t) = \sum_{k=1}^n a_k(t)\frac{\partial G}{\partial x_k} \qquad \textit{for} \quad t>0,$$
$$G(x_1,\ldots,x_n,y_1,\ldots,y_n,0) = \delta(x_1-y_1)\delta(x_2-y_2)\ldots\delta(x_n-y_n).$$

Proof. **a)** We use the **Fourier transform** to solve this problem.
Transform the equation in the first variable only. That is,

$$\widehat{G}(\xi,y,t) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ix\xi}\, G(x,y,t)\,dx.$$

The equation is transformed to an ODE, that can be solved:

$$\begin{aligned} \widehat{G}_t(\xi,y,t) &= -a(t)\,\xi^2\,\widehat{G}(\xi,y,t) \;+\; i\,b(t)\,\xi\,\widehat{G}(\xi,y,t) \;+\; c(t)\,\widehat{G}(\xi,y,t), \\ \widehat{G}_t(\xi,y,t) &= \big[-a(t)\,\xi^2 \;+\; i\,b(t)\,\xi \;+\; c(t)\big]\,\widehat{G}(\xi,y,t), \\ \widehat{G}(\xi,y,t) &= c\,e^{\int_0^t[-a(s)\xi^2+i\,b(s)\xi+c(s)]\,ds}. \end{aligned}$$

We can also transform the initial condition:

$$\widehat{G}(\xi,y,0) = \widehat{\delta(x-y)}(\xi) = e^{-iy\xi}\,\widehat{\delta}(\xi) = \frac{1}{\sqrt{2\pi}}\,e^{-iy\xi}.$$

Thus, the solution of the transformed problem is:

$$\widehat{G}(\xi,y,t) = \frac{1}{\sqrt{2\pi}}\,e^{-iy\xi}\,e^{\int_0^t[-a(s)\xi^2+i\,b(s)\xi+c(s)]\,ds}.$$

The inverse Fourier transform gives the solution to the original problem:

$$\begin{aligned} G(x,y,t) &= \big(\widehat{G}(\xi,y,t)\big)^\vee = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{ix\xi}\,\widehat{G}(\xi,y,t)\,d\xi \\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{ix\xi}\left[\frac{1}{\sqrt{2\pi}}\,e^{-iy\xi}\,e^{\int_0^t[-a(s)\xi^2+i\,b(s)\xi+c(s)]\,ds}\right]d\xi \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{i(x-y)\xi}\,e^{\int_0^t[-a(s)\xi^2+i\,b(s)\xi+c(s)]\,ds}\,d\xi. \qquad \checkmark \end{aligned}$$

b) Denote $\vec{x}=(x_1,\ldots,x_n)$, $\vec{y}=(y_1,\ldots,y_n)$. Transform in $\vec{x}$:

$$\widehat{G}(\vec{\xi},\vec{y},t) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-i\vec{x}\cdot\vec{\xi}}\,G(\vec{x},\vec{y},t)\,d\vec{x}.$$

The equation is transformed to an ODE, that can be solved:

$$\begin{aligned} \widehat{G}_t(\vec{\xi},\vec{y},t) &= \sum_{k=1}^n a_k(t)\,i\xi_k\,\widehat{G}(\vec{\xi},\vec{y},t), \\ \widehat{G}(\vec{\xi},\vec{y},t) &= c\,e^{i\int_0^t[\sum_{k=1}^n a_k(s)\,\xi_k]\,ds}. \end{aligned}$$

We can also transform the initial condition:

$$\widehat{G}(\vec{\xi},\vec{y},0) \;=\; \big[\delta(x_1-y_1)\delta(x_2-y_2)\dots\delta(x_n-y_n)\big]\widehat{\ }(\xi) \;=\; e^{-i\vec{y}\cdot\vec{\xi}}\,\widehat{\delta}(\vec{\xi}) \;=\; \frac{1}{(2\pi)^{\frac{n}{2}}}\,e^{-i\vec{y}\cdot\vec{\xi}}.$$

Thus, the solution of the transformed problem is:

$$\widehat{G}(\vec{\xi},\vec{y},t) \;=\; \frac{1}{(2\pi)^{\frac{n}{2}}}\,e^{-i\vec{y}\cdot\vec{\xi}}\,e^{i\int_0^t[\sum_{k=1}^n a_k(s)\,\xi_k]\,ds}.$$

The inverse Fourier transform gives the solution to the original problem:

$$\begin{aligned}
G(\vec{x},\vec{y},t) &= \big(\widehat{G}(\vec{\xi},\vec{y},t)\big)^{\vee} \;=\; \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{i\vec{x}\cdot\vec{\xi}}\,\widehat{G}(\vec{\xi},\vec{y},t)\,d\vec{\xi} \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{i\vec{x}\cdot\vec{\xi}}\left[\frac{1}{(2\pi)^{\frac{n}{2}}}\,e^{-i\vec{y}\cdot\vec{\xi}}\,e^{i\int_0^t[\sum_{k=1}^n a_k(s)\,\xi_k]\,ds}\right]d\vec{\xi} \\
&= \frac{1}{(2\pi)^{n}}\int_{\mathbb{R}^n} e^{i(\vec{x}-\vec{y})\cdot\vec{\xi}}\,e^{i\int_0^t[\sum_{k=1}^n a_k(s)\,\xi_k]\,ds}\,d\vec{\xi}. \qquad \checkmark
\end{aligned}$$

$\square$

Problem (W'02, #7). *Consider the equation*

$$\Big(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\Big) u = f \quad \textit{in } \mathbb{R}^n, \tag{30.4}$$

where f is an integrable function (i.e. $f \in L^1(\mathbb{R}^n)$), satisfying $f(x) = 0$ for $|x| \geq R$. Solve (30.4) by ***Fourier transform****, and prove the following results.*
a) *There is a solution of (30.4) belonging to $L^2(\mathbb{R}^n)$ if $n > 4$.*
b) *If $\int_{\mathbb{R}^n} f(x)\,dx = 0$, there is a solution of (30.4) belonging to $L^2(\mathbb{R}^n)$ if $n > 2$.*

Proof.

$$\begin{aligned}
\triangle u &= f, \\
-|\xi|^2 \widehat{u}(\xi) &= \widehat{f}(\xi), \\
\widehat{u}(\xi) &= -\frac{1}{|\xi|^2}\widehat{f}(\xi), \qquad \xi \in \mathbb{R}^n, \\
u(x) &= -\Big(\frac{\widehat{f}(\xi)}{|\xi|^2}\Big)^{\vee}.
\end{aligned}$$

a) Then

$$||\widehat{u}||_{L^2(\mathbb{R}^n)} = \Big(\int_{\mathbb{R}^n} \frac{|\widehat{f}(\xi)|^2}{|\xi|^4}\,d\xi\Big)^{\frac{1}{2}} \leq \Big(\underbrace{\int_{|\xi|<1} \frac{|\widehat{f}(\xi)|^2}{|\xi|^4}\,d\xi}_{A} + \underbrace{\int_{|\xi|\geq 1} \frac{|\widehat{f}(\xi)|^2}{|\xi|^4}\,d\xi}_{B}\Big)^{\frac{1}{2}}.$$

Notice, $||f||_2 = ||\widehat{f}||_2 \geq B$, so $B < \infty$.
Use **polar coordinates** on A.

$$A = \int_{|\xi|<1} \frac{|\widehat{f}(\xi)|^2}{|\xi|^4}\,d\xi = \int_0^1 \int_{S_{n-1}} \frac{|\widehat{f}|^2}{r^4} r^{n-1}\,dS_{n-1}\,dr = \int_0^1 \int_{S_{n-1}} |\widehat{f}|^2 r^{n-5}\,dS_{n-1}\,dr.$$

If $n > 4$,

$$A \leq \int_{S_{n-1}} |\widehat{f}|^2\,dS_{n-1} = ||\widehat{f}||_2^2 < \infty.$$

$$||u||_{L^2(\mathbb{R}^n)} = ||\widehat{u}||_{L^2(\mathbb{R}^n)} = (A+B)^{\frac{1}{2}} < \infty.$$

b) We have

$$
\begin{aligned}
u(x,t) &= -\Big(\frac{\widehat{f}(\xi)}{|\xi|^2}\Big)^{\vee} \;=\; -\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{-\infty}^{\infty} e^{ix\cdot\xi}\,\frac{\widehat{f}(\xi)}{|\xi|^2}\,d\xi \\
&= -\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{-\infty}^{\infty} \frac{e^{ix\cdot\xi}}{|\xi|^2}\Big(\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{-\infty}^{\infty} e^{-iy\cdot\xi}\,f(y)\,dy\Big)\,d\xi \\
&= -\frac{1}{(2\pi)^{n}}\int_{-\infty}^{\infty} f(y)\,\Big(\int_{-\infty}^{\infty} \frac{e^{i(x-y)\cdot\xi}}{|\xi|^2}\,d\xi\Big)\,dy \\
&= -\frac{1}{(2\pi)^{n}}\int_{-\infty}^{\infty} f(y)\,\Big(\int_0^1\int_{S_{n-1}} \frac{e^{i(x-y)r}}{r^2}\,r^{n-1}\,dS_{n-1}\,dr\Big)\,dy \\
&= -\frac{1}{(2\pi)^{n}}\int_{-\infty}^{\infty} f(y)\,\underbrace{\Big(\int_0^1\int_{S_{n-1}} e^{i(x-y)r}\,r^{n-3}\,dS_{n-1}\,dr\Big)}_{\le M<\infty,\quad if\ \ n>2.}\,dy. \\
|u(x,t)| &= \frac{1}{(2\pi)^{n}}\Big|\int_{-\infty}^{\infty} M\,f(y)\,dy\Big| \;<\; \infty.
\end{aligned}
$$

$\square$

Problem (F'02, #7). *For the right choice of the constant c, the function $F(x, y) = c(x + iy)^{-1}$ is a **fundamental solution** for the equation*

$$\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y} = f \qquad \text{in } \mathbb{R}^2.$$

*Find the right choice of c, and use your answer to compute the **Fourier transform** (in distribution sense) of $(x + iy)^{-1}$.*

Proof. [86]

$$\triangle = \Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big)\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big).$$

$F_1(x, y) = \frac{1}{2\pi}\log|z|$ is the fundamental solution of the Laplacian. $z = x + iy$.

$$\begin{aligned} \triangle F_1(x,y) &= \delta, \\ \Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big)\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big) F(x,y) &= \delta. \end{aligned}$$

$h_x + ih_y = e^{-i(x\xi_1+y\xi_2)}$.
Suppose $h = h(x\xi_1 + y\xi_2)$ or $h = ce^{-i(x\xi_1+y\xi_2)}$.

$$\begin{aligned} &\Rightarrow \quad c\big(-i\xi_1 e^{-i(x\xi_1+y\xi_2)} - i^2\xi_2 e^{-i(x\xi_1+y\xi_2)}\big) = -ic(\xi_1 - i\xi_2)\, e^{-i(x\xi_1+y\xi_2)} \equiv e^{-i(x\xi_1+y\xi_2)}, \\ &\Rightarrow \quad -ic(\xi_1 - i\xi_2) = 1, \\ &\Rightarrow \quad c = -\frac{1}{i(\xi_1 - i\xi_2)}, \\ &\Rightarrow \quad h(x,y) = -\frac{1}{i(\xi_1 - i\xi_2)}\, e^{-i(x\xi_1+y\xi_2)}. \end{aligned}$$

Integrate by parts:

$$\begin{aligned} \Big(\widehat{\frac{1}{x+iy}}\Big)(\xi) &= \int_{\mathbb{R}^2} e^{-i(x\xi_1+y\xi_2)}\frac{1}{i(\xi_1 - i\xi_2)}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big)\frac{1}{(x+iy)-0}\, dxdy \\ &= \frac{1}{i(\xi_1 - i\xi_2)} = \frac{1}{i(\xi_2 + i\xi_1)}. \end{aligned}$$

□

[86] Alan solved in this problem in class.

31 Laplace Transform

If $u \in L^1(\mathbb{R}_+)$, we define its **Laplace transform** to be

$$\boxed{\mathcal{L}[u(t)] \;=\; u^{\#}(s) \;=\; \int_0^\infty e^{-st}\, u(t)\, dt \qquad (s>0).}$$

In practice, for a PDE involving time, it may be useful to perform a Laplace transform in t, holding the space variables x fixed.
The **inversion formula** for the Laplace transform is:

$$\boxed{u(t) \;=\; \mathcal{L}^{-1}[u^{\#}(s)] \;=\; \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\, u^{\#}(s)\, ds.}$$

Example: $f(t)=1$.

$$\mathcal{L}[1] \;=\; \int_0^\infty e^{-st}\cdot 1\, dt \;=\; \Big[-\frac{1}{s}e^{-st}\Big]_{t=0}^{t=\infty} \;=\; \frac{1}{s} \qquad \text{for} \quad s>0.$$

Example: $f(t)=e^{at}$.

$$\mathcal{L}[e^{at}] \;=\; \int_0^\infty e^{-st}\, e^{at}\, dt \;=\; \int_0^\infty e^{(a-s)t}\, dt \;=\; \frac{1}{a-s}\Big[e^{(a-s)t}\Big]_{t=0}^{t=\infty} \;=\; \frac{1}{s-a} \qquad \text{for} \quad s>a.$$

Convolution: We want to find an inverse Laplace transform of $\frac{1}{s}\cdot\frac{1}{s^2+1}$.

$$\mathcal{L}^{-1}\Big[\underbrace{\frac{1}{s}}_{L[f]}\cdot\underbrace{\frac{1}{s^2+1}}_{L[g]}\Big] \;=\; f*g \;=\; \int_0^t 1\cdot\sin t'\, dt' \;=\; 1-\cos t.$$

Partial Derivatives: $u=u(x,t)$

$$\begin{aligned}
\mathcal{L}[u_t] &= \int_0^\infty e^{-st}\, u_t\, dt \;=\; \Big[e^{-st}u(x,t)\Big]_{t=0}^{t=\infty} + s\int_0^\infty e^{-st}\, u\, dt \;=\; s\mathcal{L}[u]-u(x,0),\\
\mathcal{L}[u_{tt}] &= \int_0^\infty e^{-st}\, u_{tt}\, dt \;=\; \Big[e^{-st}u_t\Big]_{t=0}^{t=\infty} + s\int_0^\infty e^{-st}\, u_t\, dt \;=\; -u_t(x,0)+s\mathcal{L}[u_t]\\
&= s^2\mathcal{L}[u]-su(x,0)-u_t(x,0),\\
\mathcal{L}[u_x] &= \int_0^\infty e^{-st}\, u_x\, dt \;=\; \frac{\partial}{\partial x}\mathcal{L}[u],\\
\mathcal{L}[u_{xx}] &= \int_0^\infty e^{-st}\, u_{xx}\, dt \;=\; \frac{\partial^2}{\partial x^2}\mathcal{L}[u].
\end{aligned}$$

Heat Equation: Consider

$$\begin{cases} u_t-\triangle u=0 & \text{in } U\times(0,\infty)\\ u=f & \text{on } U\times\{t=0\}, \end{cases}$$

and perform a Laplace transform with respect to time:

$$\begin{aligned}
\mathcal{L}[u_t] &= \int_0^\infty e^{-st}\, u_t\, dt \;=\; s\mathcal{L}[u]-u(x,0) \;=\; s\mathcal{L}[u]-f(x),\\
\mathcal{L}[\triangle u] &= \int_0^\infty e^{-st}\, \triangle u\, dt \;=\; \triangle\mathcal{L}[u].
\end{aligned}$$

Thus, the transformed problem is: $s\mathcal{L}[u] - f(x) \;=\; \triangle\mathcal{L}[u]$. Writing $v(x) = L[u]$, we have

$$-\triangle v + sv \;=\; f \qquad \text{in } U.$$

Thus, the solution of this equation with RHS f is the Laplace transform of the solution of the heat equation with initial data f.

Table of Laplace Transforms: $L[f] \;=\; f^{\#}(s)$

$$
\begin{aligned}
\mathcal{L}[\sin at] &= \frac{a}{s^2+a^2}, && s>0\\
\mathcal{L}[\cos at] &= \frac{s}{s^2+a^2}, && s>0\\
\mathcal{L}[\sinh at] &= \frac{a}{s^2-a^2}, && s>|a|\\
\mathcal{L}[\cosh at] &= \frac{s}{s^2-a^2}, && s>|a|\\
\mathcal{L}[e^{at}\sin bt] &= \frac{b}{(s-a)^2+b^2}, && s>a\\
\mathcal{L}[e^{at}\cos bt] &= \frac{s-a}{(s-a)^2+b^2}, && s>a\\
\mathcal{L}[t^n] &= \frac{n!}{s^{n+1}}, && s>0\\
\mathcal{L}[t^n e^{at}] &= \frac{n!}{(s-a)^{n+1}}, && s>a\\
\mathcal{L}[H(t-a)] &= \frac{e^{-as}}{s}, && s>0\\
\mathcal{L}[H(t-a)\,f(t-a)] &= e^{-as}\mathcal{L}[f],\\
\mathcal{L}[af(t)+bg(t)] &= a\mathcal{L}[f]+b\mathcal{L}[g],\\
\mathcal{L}[f(t)*g(t)] &= \mathcal{L}[f]\,\mathcal{L}[g],\\
\mathcal{L}\Big[\int_0^t g(t'-t)\,f(t')\,dt'\Big] &= \mathcal{L}[f]\,\mathcal{L}[g],\\
\mathcal{L}\Big[\frac{df}{dt}\Big] &= s\mathcal{L}[f]-f(0),\\
\mathcal{L}\Big[\frac{d^2f}{dt^2}\Big] &= s^2\mathcal{L}[f]-sf(0)-f'(0), && \Big(f'=\frac{df}{dt}\Big)\\
\mathcal{L}\Big[\frac{d^nf}{dt^n}\Big] &= s^n\mathcal{L}[f]-s^{n-1}f(0)-\ldots-f^{n-1}(0),\\
\mathcal{L}[f(at)] &= \frac{1}{a}f^{\#}\Big(\frac{s}{a}\Big),\\
\mathcal{L}[e^{bt}f(t)] &= f^{\#}(s-b),\\
\mathcal{L}[tf(t)] &= -\frac{d}{ds}\mathcal{L}[f],\\
\mathcal{L}\Big[\frac{f(t)}{t}\Big] &= \int_s^\infty f^{\#}(s')\,ds',\\
\mathcal{L}\Big[\int_0^t f(t')\,dt'\Big] &= \frac{1}{s}\mathcal{L}[f],\\
\mathcal{L}[J_0(at)] &= (s^2+a^2)^{-\frac{1}{2}},\\
\mathcal{L}[\delta(t-a)] &= e^{-sa}.
\end{aligned}
$$

Example: $f(t)=\sin t$. After integrating by parts twice, we obtain:

$$
\begin{aligned}
\mathcal{L}[\sin t] &= \int_0^\infty e^{-st}\,\sin t\,dt \;=1-s^2\int_0^\infty e^{-st}\,\sin t\,dt,\\
&\Rightarrow \int_0^\infty e^{-st}\,\sin t\,dt \;=\; \frac{1}{1+s^2}.
\end{aligned}
$$

Example: $f(t) = t^n$.

$$\begin{aligned}
\mathcal{L}[t^n] &= \int_0^\infty e^{-st}\, t^n \, dt \;=\; -\Big[\frac{t^n e^{-st}}{s}\Big]_0^\infty + \frac{n}{s}\int_0^\infty e^{-st}\, t^{n-1}\, dt \;=\; \frac{n}{s}\mathcal{L}[t^{n-1}] \\
&= \frac{n}{s}\Big(\frac{n-1}{s}\Big)\mathcal{L}[t^{n-2}] \;=\; \ldots \;=\; \frac{n!}{s^n}\mathcal{L}[1] \;=\; \frac{n!}{s^{n+1}}.
\end{aligned}$$

Problem (F'00, #6). *Consider the initial-boundary value problem*

$$\begin{aligned} u_t - u_{xx} + au &= 0, && t > 0,\ x > 0 \\ u(x,0) &= 0, && x > 0 \\ u(0,t) &= g(t), && t > 0, \end{aligned}$$

where $g(t)$ is continuous function with a compact support, and a is constant. Find the explicit solution of this problem.

Proof. We solve this problem using the **Laplace transform**.

$$\mathcal{L}[u(x,t)] \;=\; u^{\#}(x,s) \;=\; \int_0^\infty e^{-st}\, u(x,t)\, dt \qquad (s > 0).$$

$$\begin{aligned} \mathcal{L}[u_t] &= \int_0^\infty e^{-st}\, u_t\, dt \;=\; \Big[e^{-st} u(x,t)\Big]_{t=0}^{t=\infty} + s\int_0^\infty e^{-st}\, u\, dt \\ &= su^{\#}(x,s) - u(x,0) \;=\; su^{\#}(x,s), \qquad (\text{since } \ u(x,0) = 0) \\ \mathcal{L}[u_{xx}] &= \int_0^\infty e^{-st}\, u_{xx}\, dt \;=\; \frac{\partial^2}{\partial x^2} u^{\#}(x,s), \\ \mathcal{L}[u(0,t)] &= u^{\#}(0,s) \;=\; \int_0^\infty e^{-st}\, g(t)\, dt \;=\; g^{\#}(s). \end{aligned}$$

Plugging these into the equation, we obtain the ODE in $u^{\#}$:

$$su^{\#}(x,s) - \frac{\partial^2}{\partial x^2} u^{\#}(x,s) + au^{\#}(x,s) \;=\; 0.$$

$$\begin{cases} (u^{\#})_{xx} - (s+a)u^{\#} \;=\; 0, \\ u^{\#}(0,s) \;=\; g^{\#}(s). \end{cases}$$

This initial value problem has a solution:

$$u^{\#}(x,s) \;=\; c_1 e^{\sqrt{s+a}\,x} + c_2 e^{-\sqrt{s+a}\,x}.$$

Since we want u to be bounded as $x \to \infty$, we have $c_1 = 0$, so

$$u^{\#}(x,s) \;=\; c_2 e^{-\sqrt{s+a}\,x}. \qquad u^{\#}(0,s) \;=\; c_2 \;=\; g^{\#}(s), \quad \text{thus,}$$

$$\boxed{u^{\#}(x,s) \;=\; g^{\#}(s) e^{-\sqrt{s+a}\,x}.}$$

To obtain $u(x,t)$, we take the **inverse Laplace transform** of $u^{\#}(x,s)$:

$$\begin{aligned} u(x,t) &= L^{-1}[u^{\#}(x,s)] \;=\; L^{-1}\big[\underbrace{g^{\#}(s)}_{L[g]}\,\underbrace{e^{-\sqrt{s+a}\,x}}_{L[f]}\big] \;=\; g * f \\ &= g * L^{-1}\big[e^{-\sqrt{s+a}\,x}\big] \;=\; g * \Big[\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\, e^{-\sqrt{s+a}\,x}\, ds\Big], \end{aligned}$$

$$\boxed{u(x,t) \;=\; \int_0^t g(t-t')\Big[\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st'}\, e^{-\sqrt{s+a}\,x}\, ds\Big]\, dt'.}$$

∎

Problem (F'04, #8). *The function $y(x,t)$ satisfies the partial differential equation*

$$x\frac{\partial y}{\partial x}+\frac{\partial^2 y}{\partial x\partial t}+2y=0,$$

and the boundary conditions

$$y(x,0)=1, \qquad y(0,t)=e^{-at},$$

*where $a\geq 0$. Find the **Laplace transform**, $\overline{y}(x,s)$, of the solution, and hence derive an expression for $y(x,t)$ in the domain $x\geq 0$, $t\geq 0$.*

Proof. We change the notation: $y\to u$. We have

$$\begin{cases} xu_x+u_{xt}+2u=0,\\ u(x,0)=1, \quad u(0,t)=e^{-at}.\end{cases}$$

The **Laplace transform** is defined as:

$$\mathcal{L}[u(x,t)] \;=\; u^{\#}(x,s) \;=\; \int_0^\infty e^{-st}\,u(x,t)\,dt \qquad (s>0).$$

$$\begin{aligned}
\mathcal{L}[xu_x] &= \int_0^\infty e^{-st}\,xu_x\,dt \;=\; x\int_0^\infty e^{-st}\,u_x\,dt \;=\; x(u^{\#})_x,\\
\mathcal{L}[u_{xt}] &= \int_0^\infty e^{-st}\,u_{xt}\,dt \;=\; \Big[e^{-st}u_x(x,t)\Big]_{t=0}^{t=\infty} + s\int_0^\infty e^{-st}\,u_x\,dt\\
&= s(u^{\#})_x-u_x(x,0) \;=\; s(u^{\#})_x, \qquad \text{(since } u(x,0)=0)\\
\mathcal{L}[u(0,t)] &= u^{\#}(0,s) \;=\; \int_0^\infty e^{-st}\,e^{-at}\,dt \;=\; \int_0^\infty e^{-(s+a)t}\,dt \;=\; \Big[-\frac{1}{s+a}e^{-(s+a)t}\Big]_{t=0}^{t=\infty}\\
&= \frac{1}{s+a}.
\end{aligned}$$

Plugging these into the equation, we obtain the ODE in $u^{\#}$:

$$\begin{cases} (x+s)(u^{\#})_x+2u^{\#} \;=\; 0,\\ u^{\#}(0,s) \;=\; \frac{1}{s+a},\end{cases}$$

which can be solved:

$$\frac{(u^{\#})_x}{u^{\#}}=-\frac{2}{x+s} \quad\Rightarrow\quad \log u^{\#}=-2\log(x+s)+c_1 \quad\Rightarrow\quad u^{\#}=c_2e^{\log(x+s)^{-2}}=\frac{c_2}{(x+s)^2}.$$

From the initial conditions:

$$u^{\#}(0,s)=\frac{c_2}{s^2}=\frac{1}{s+a} \quad\Rightarrow\quad c_2=\frac{s^2}{s+a}.$$

$$\boxed{u^{\#}(x,s) \;=\; \frac{s^2}{(s+a)(x+s)^2}.}$$

To obtain $u(x,t)$, we take the **inverse Laplace transform** of $u^{\#}(x,s)$:

$$u(x,t) \;=\; L^{-1}[u^{\#}(x,s)] \;=\; L^{-1}\Big[\frac{s^2}{(s+a)(x+s)^2}\Big] \;=\; \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\Big[\frac{s^2}{(s+a)(x+s)^2}\Big]\,ds.$$

$$\boxed{u(x,t) \;=\; \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\Big[\frac{s^2}{(s+a)(x+s)^2}\Big]\,ds.}$$

□

Problem (F'90, #1). *Using the **Laplace transform**, or any other convenient method, solve the Volterra integral equation*

$$u(x) = \sin x + \int_0^x \sin(x-y)u(y)\,dy.$$

Proof. Rewrite the equation:

$$\begin{aligned} u(t) &= \sin t + \int_0^t \sin(t-t')u(t')\,dt', \\ u(t) &= \sin t + (\sin t) * u. \qquad \circledast \end{aligned}$$

Taking the Laplace transform of each of the elements in $\circledast$:

$$\begin{aligned} \mathcal{L}[u(t)] &= u^{\#}(s) = \int_0^\infty e^{-st}\,u(t)\,dt, \\ \mathcal{L}[\sin t] &= \frac{1}{1+s^2}, \\ \mathcal{L}[(\sin t) * u] &= \mathcal{L}[\sin t] * \mathcal{L}[u] = \frac{u^{\#}}{1+s^2}. \end{aligned}$$

Plugging these into the equation:

$$u^{\#} = \frac{1}{1+s^2} + \frac{u^{\#}}{1+s^2} = \frac{u^{\#}+1}{1+s^2}.$$

$$\boxed{u^{\#}(s) = \frac{1}{s^2}.}$$

To obtain $u(t)$, we take the **inverse Laplace transform** of $u^{\#}(s)$:

$$u(t) = L^{-1}[u^{\#}(s)] = L^{-1}\Big[\frac{1}{s^2}\Big] = t.$$

$$\boxed{u(t) = t.}$$

$\square$

Problem (F'91, #5). *In what follows, the* ***Laplace transform*** *of $x(t)$ is denoted either by $\overline{x}(s)$ or by $\mathcal{L}x(t)$.* ❶ *Show that, for integral $n \geq 0$,*

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}.$$

❷ *Hence show that*

$$\mathcal{L}\,J_0(2\sqrt{ut}) = \frac{1}{s}e^{-u/s},$$

where

$$J_0(z) = \sum_{n=0}^{\infty} \frac{(-1)^n(\frac{1}{2}z)^{2n}}{n!n!}$$

is a Bessel function. ❸ *Hence show that*

$$\mathcal{L}\left[\int_0^\infty J_0(2\sqrt{ut})x(u)\,du\right] = \frac{1}{s}\,\overline{x}\left(\frac{1}{s}\right). \tag{31.1}$$

❹ *Assuming that*

$$\mathcal{L}\,J_0(at) = \frac{1}{\sqrt{a^2+s^2}},$$

prove with the help of (31.1) that if $t \geq 0$

$$\int_0^\infty J_0(au)J_0(2\sqrt{ut})\,du = \frac{1}{a}J_0\left(\frac{t}{a}\right).$$

Hint: For the last part, use the uniqueness of the Laplace transform.

Proof.

❶
$$\mathcal{L}[t^n] = \int_0^\infty \underbrace{e^{-st}}_{g'}\,\underbrace{t^n}_{f}\,dt = \underbrace{-\left[\frac{t^n e^{-st}}{s}\right]_0^\infty}_{=0} + \frac{n}{s}\int_0^\infty e^{-st}\,t^{n-1}\,dt = \frac{n}{s}\mathcal{L}[t^{n-1}]$$
$$= \frac{n}{s}\left(\frac{n-1}{s}\right)\mathcal{L}[t^{n-2}] = \ldots = \frac{n!}{s^n}\mathcal{L}[1] = \frac{n!}{s^{n+1}}. \quad \checkmark$$

❷
$$\mathcal{L}\,J_0(2\sqrt{ut}) = \mathcal{L}\left[\sum_{n=0}^\infty \frac{(-1)^n u^n t^n}{n!n!}\right] = \sum_{n=0}^\infty \frac{(-1)^n u^n}{n!n!}\,\mathcal{L}[t^n] = \sum_{n=0}^\infty \frac{(-1)^n u^n}{n!s^{n+1}}$$
$$= \frac{1}{s}\sum_{n=0}^\infty \frac{(-1)^n}{n!}\left(\frac{u}{s}\right)^n = \frac{1}{s}e^{-\frac{u}{s}}. \quad \checkmark$$

❸
$$\mathcal{L}\left[\int_0^\infty J_0(2\sqrt{ut})\,x(u)\,du\right] = \int_0^\infty \mathcal{L}[J_0(2\sqrt{ut})]\,x(u)\,du = \frac{1}{s}\int_0^\infty e^{-\frac{u}{s}}\,x(u)\,du$$
$$= \frac{1}{s}x^{\#}\left(\frac{1}{s}\right), \quad \checkmark$$

where

$$x^{\#}(s) = \int_0^\infty e^{-us}\,x(u)\,du.$$

□

32 Linear Functional Analysis

32.1 Norms

$||\cdot||$ is a norm on a vector space X if
i) $||x|| = 0$ iff $x = 0$.
ii) $||\alpha x|| = |\alpha| \cdot ||x||$ for all scalars α.
iii) $||x+y|| \leq ||x|| + ||y||$ (the triangle inequality).
The norm induces the *distance function* $d(x,y) = ||x-y||$ so that X is a metric space, called a *normed vector space.*

32.2 Banach and Hilbert Spaces

A **Banach space** is a normed vector space that is complete in that norm's metric. I.e. a complete normed linear space is a Banach space.
A **Hilbert space** is an inner product space for which the corresponding normed space is complete. I.e. a complete inner product space is a Hilbert space.
Examples: 1) Let K be a compact set of R^n and let $C(K)$ denote the space of continuous functions on K. Since every $u \in C(K)$ achieves maximum and minimum values on K, we may define

$$||u||_\infty = \max_{x\in K} |u(x)|.$$

$||\cdot||_\infty$ is indeed a norm on $C(K)$ and since a uniform limit of continuous functions is continuous, $C(K)$ is a **Banach space**. However, this norm cannot be derived from an inner product, so $C(K)$ is **not a Hilbert space**.
2) $C(K)$ is **not** a Banach space with $||\cdot||_2$ norm. (Bell-shaped functions on $[0,1]$ may converge to a discontinuous δ-function). In general, the space of continuous functions on $[0,1]$, with the norm $||\cdot||_p$, $1 \leq p < \infty$, is **not** a Banach space, since it is not complete.
3) R^n and C^n are real and complex Banach spaces (with a Eucledian norm).
4) L^p are Banach spaces (with $||\cdot||_p$ norm).
5) The space of bounded real-valued functions on a set S, with the sup norm $||\cdot||_S$ are Banach spaces.
6) The space of bounded continuous real-valued functions on a metric space X is a Banach space.

32.3 Cauchy-Schwarz Inequality

$$\begin{aligned}
|(u,v)| &\leq ||u||||v|| \quad \text{in any norm, for example} \quad \textstyle\int |uv|dx \leq (\int u^2 dx)^{\frac{1}{2}} (\int v^2 dx)^{\frac{1}{2}} \\
|a(u,v)| &\leq a(u,u)^{\frac{1}{2}}\ a(v,v)^{\frac{1}{2}} \\
\textstyle\int |v| dx &= \textstyle\int |v| \cdot 1\ dx = (\int |v|^2 dx)^{\frac{1}{2}} (\int 1^2 dx)^{\frac{1}{2}}
\end{aligned}$$

32.4 Hölder Inequality

$$\int_\Omega |uv|\, dx \leq ||u||_p ||v||_q,$$

which holds for $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$, where $\frac{1}{p} + \frac{1}{q} = 1$. In particular, this shows $uv \in L^1(\Omega)$.

32.5 Minkowski Inequality

$$||u+v||_p \le ||u||_p + ||v||_p,$$

which holds for $u, v \in L^p(\Omega)$. In particular, it shows $u+v \in L^p(\Omega)$.
Using the Minkowski Inequality, we find that $||\cdot||_p$ is a norm on $L^p(\Omega)$.
The *Riesz-Fischer theorem* asserts that $L^p(\Omega)$ is complete in this norm, so $L^p(\Omega)$ is a Banach space under the norm $||\cdot||_p$.
If $p=2$, then $L^2(\Omega)$ is a Hilbert space with inner product

$$(u,v) = \int\limits_\Omega uv\,dx.$$

Example: $\Omega \in R^n$ bounded domain, $C^1(\bar{\Omega})$ denotes the functions that, along with their first-order derivatives, extend continuously to the compact set $\bar{\Omega}$. Then $C^1(\bar{\Omega})$ is a Banach space under the norm

$$||u||_{1,\infty} = \max_{x\in\bar{\Omega}}(|\nabla u(x)| + |u(x)|).$$

Note that $C^1(\Omega)$ is **not** a Banach space since $||u||_{1,\infty}$ need not be finite for $u \in C^1(\Omega)$.

32.6 Sobolev Spaces

A *Sobolev space* is a space of functions whose distributional derivatives (up to some fixed order) exist in an L^p-space.
Let Ω be a domain in R^n, and let us introduce

$$< u, v >_1 = \int\limits_\Omega (\nabla u \cdot \nabla v + uv)\,dx, \tag{32.1}$$

$$||u||_{1,2} = \sqrt{< u, u >_1} = \left(\int\limits_\Omega (|\nabla u|^2 + |u|^2)\,dx \right)^{\frac{1}{2}} \tag{32.2}$$

when these expressions are defined and finite. For example, (32.1) and (32.2) are defined for functions in $C_0^1(\Omega)$. However, $C_0^1(\Omega)$ is not complete under the norm (32.2), and so does not form a Hilbert space.

Divergence Theorem

$$\int_{\partial\Omega} \vec{A}\cdot n\,dS = \int_\Omega \operatorname{div}\vec{A}\,dx$$

Trace Theorem

$$\|u\|_{L_2(\partial\Omega)} \le C\|u\|_{H^1(\Omega)} \qquad \Omega \text{ smooth or square}$$

Poincare Inequality

$$\|u\|_p \le C\|\nabla u\|_p \qquad 1 \le p \le \infty$$

$$\int_\Omega |u(x)|^2\,dx \le C\int_\Omega |\nabla u(x)|^2\,dx \qquad u \in C_0^1(\Omega),\ H_0^{1,2}(\Omega) \qquad i.e.\ p=2$$

$$\|u - u_\Omega\|_p \le \|\nabla u\|_p \qquad u \in H_0^{1,p}(\Omega)$$

$$u_\Omega = \frac{1}{|\Omega|}\int_\Omega u(x)\,dx \qquad (\textit{Average value of } u \textit{ over } \Omega),\;\; |\Omega| \textit{ is the volume of } \Omega$$

Notes

$$\frac{\partial u}{\partial \vec{n}} = \nabla u \cdot \vec{n} = n_1 \frac{\partial u}{\partial x_1} + n_2 \frac{\partial u}{\partial x_2} \qquad\qquad |\nabla u|^2 = u_{x_1}^2 + u_{x_2}^2$$

$$\int_\Omega \nabla |u|\,dx = \int_\Omega \frac{|u|}{u}\nabla u\,dx$$

$$\sqrt{ab} \le \frac{a+b}{2} \quad\Rightarrow\quad ab \le \frac{a^2+b^2}{2} \quad\Rightarrow\quad ||\nabla u||\,||u|| \le \frac{||\nabla u||^2 + ||u||^2}{2}$$

$$u\nabla u = \nabla(\frac{u^2}{2})$$

$$\int_\Omega (u_{xy})^2\,dx = \int_\Omega u_{xx}u_{yy}\,dx \qquad \forall u \in H_0^2(\Omega) \qquad \Omega \;\text{ square}$$

Problem (F'04, #6). *Let $q \in C_0^1(\mathbb{R}^3)$. Prove that the vector field*

$$u(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{q(y)(x-y)}{|x-y|^3}\,dy$$

enjoys the following properties: [87]
a) *$u(x)$ is **conservative**;*
b) *div $u(x) = q(x)$ for all $x \in \mathbb{R}^3$;*
c) *$|u(x)| = O(|x|^{-2})$ for large x.*
Furthermore, prove that the proverties (1), (2), and (3) above determine the vector field $u(x)$ uniquely.

Proof. **a)** To show that $\vec{u}(x)$ is conservative, we need to show that curl $\vec{u} = 0$.
The curl of $\vec{V}$ is another vector field defined by

$$\text{curl}\,\vec{V} = \nabla \times \vec{V} = \det\begin{pmatrix} \mathbf{e_1} & \mathbf{e_2} & \mathbf{e_3} \\ \partial_1 & \partial_2 & \partial_3 \\ V_1 & V_2 & V_3 \end{pmatrix} = \Big(\frac{\partial V_3}{\partial x_2} - \frac{\partial V_2}{\partial x_3}, \frac{\partial V_1}{\partial x_3} - \frac{\partial V_3}{\partial x_1}, \frac{\partial V_2}{\partial x_1} - \frac{\partial V_1}{\partial x_2}\Big).$$

Consider

$$\vec{V}(x) = \frac{\vec{x}}{|\vec{x}|^3} = \frac{(x_1,x_2,x_3)}{(x_1^2+x_2^2+x_3^2)^{\frac{3}{2}}}.$$

Then,

$$\begin{aligned} \vec{u}(x) &= \frac{1}{4\pi}\int_{\mathbb{R}^3} q(y)\,V(x-y)\,dy, \\ \text{curl}\,\vec{u}(x) &= \frac{1}{4\pi}\int_{\mathbb{R}^3} q(y)\,\text{curl}_x\,V(x-y)\,dy. \end{aligned}$$

$$\begin{aligned} \text{curl}\,\vec{V}(x) &= \text{curl}\,\frac{(x_1,x_2,x_3)}{(x_1^2+x_2^2+x_3^2)^{\frac{3}{2}}} \\ &= \Big(\frac{-\frac{3}{2}\cdot 2x_2x_3}{(x_1^2+x_2^2+x_3^2)^{\frac{5}{2}}} - \frac{-\frac{3}{2}\cdot 2x_3x_2}{(x_1^2+x_2^2+x_3^2)^{\frac{5}{2}}}, \frac{-\frac{3}{2}\cdot 2x_3x_1}{(x_1^2+x_2^2+x_3^2)^{\frac{5}{2}}} - \frac{-\frac{3}{2}\cdot 2x_1x_3}{(x_1^2+x_2^2+x_3^2)^{\frac{5}{2}}}, \frac{-\frac{3}{2}\cdot 2x_1x_2}{(x_1^2+x_2^2+x_3^2)^{\frac{5}{2}}} - \cdot \\ &= (0,0,0). \end{aligned}$$

[87] McOwen, p. 138-140.

Thus, curl $\vec{u} = \frac{1}{4\pi}\int_{\mathbb{R}^3} q(y)\cdot 0\, dy = 0$, and $\vec{u}(x)$ is conservative. ✓
b) Note that the Laplace kernel in $\mathbb{R}^3$ is $-\frac{1}{4\pi r}$.

$$u(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{q(y)(x-y)}{|x-y|^3}\, dy = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{q(r)\, r}{r^3}\, r\, dr = \int_{\mathbb{R}^3} \frac{q(r)}{4\pi r}\, dr = q.$$

c) Consider

$$F(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{q(y)}{|x-y|}\, dy.$$

$F(x)$ is $O(|x|^{-1})$ as $|x| \to \infty$.
Note that $u = \nabla F$, which is clearly $O(|x|^{-2})$ as $|x| \to \infty$. ✓ □