Lectures on Symplectic Geometry

Ana Cannas da Silva[1]

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[1]E-mail: acannas@math.ist.utl.pt or acannas@math.princeton.edu

Foreword

These notes approximately transcribe a 15-week course on symplectic geometry I taught at UC Berkeley in the Fall of 1997.

The course at Berkeley was greatly inspired in content and style by Victor Guillemin, whose masterly teaching of beautiful courses on topics related to symplectic geometry at MIT, I was lucky enough to experience as a graduate student. I am very thankful to him!

That course also borrowed from the 1997 Park City summer courses on symplectic geometry and topology, and from many talks and discussions of the symplectic geometry group at MIT. Among the regular participants in the MIT informal symplectic seminar 93-96, I would like to acknowledge the contributions of Allen Knutson, Chris Woodward, David Metzler, Eckhard Meinrenken, Elisa Prato, Eugene Lerman, Jonathan Weitsman, Lisa Jeffrey, Reyer Sjamaar, Shaun Martin, Stephanie Singer, Sue Tolman and, last but not least, Yael Karshon.

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Ana Cannas da Silva

Berkeley, November 1998
and Lisbon, September 2000

Contents

Introduction

The goal of these notes is to provide a fast introduction to symplectic geometry.

A symplectic form is a closed nondegenerate 2-form. A symplectic manifold is a manifold equipped with a symplectic form. Symplectic geometry is the geometry of symplectic manifolds. Symplectic manifolds are necessarily even-dimensional and orientable, since nondegeneracy says that the top exterior power of a symplectic form is a volume form. The closedness condition is a natural differential equation, which forces all symplectic manifolds to being locally indistinguishable. (These assertions will be explained in Lecture 1 and Homework 2.)

The list of questions on symplectic forms begins with those of existence and uniqueness on a given manifold. For specific symplectic manifolds, one would like to understand the geometry and the topology of special submanifolds, the dynamics of certain vector fields or systems of differential equations, the symmetries and extra structure, etc.

Two centuries ago, symplectic geometry provided a language for classical mechanics. Through its recent huge development, it conquered an independent and rich territory, as a central branch of differential geometry and topology. To mention just a few key landmarks, one may say that symplectic geometry began to take its modern shape with the formulation of the Arnold conjectures in the 60's and with the foundational work of Weinstein in the 70's. A paper of Gromov [49] in the 80's gave the subject a whole new set of tools: pseudo-holomorphic curves. Gromov also first showed that important results from complex Kähler geometry remain true in the more general symplectic category, and this direction was continued rather dramatically in the 90's in the work of Donaldson on the topology of symplectic manifolds and their symplectic submanifolds, and in the work of Taubes in the context of the Seiberg-Witten invariants. Symplectic geometry is significantly stimulated by important interactions with global analysis, mathematical physics, low-dimensional topology, dynamical systems, algebraic geometry, integrable systems, microlocal analysis, partial differential equations, representation theory, quantization, equivariant cohomology, geometric combinatorics, etc.

As a curiosity, note that two centuries ago the name *symplectic geometry* did not exist. If you consult a major English dictionary, you are likely to find that *symplectic* is the name for a bone in a fish's head. However, as clarified in [105], the word *symplectic* in mathematics was coined by Weyl [110, p.165] who substituted the Latin root in *complex* by the corresponding Greek root, in order to label the symplectic group. Weyl thus avoided that this group connote the complex numbers, and also spared us from much confusion that would have arisen, had the name remained the former one in honor of Abel: *abelian linear group*.

This text is essentially the set of notes of a 15-week course on symplectic geometry with 2 hour-and-a-half lectures per week. The course targeted second-year graduate students in mathematics, though the audience was more diverse, including advanced undergraduates, post-docs and graduate students from other departments. The present text should hence still be appropriate for a second-year graduate course or for an independent study project.

There are scattered short exercises throughout the text. At the end of most lectures, some longer guided problems, called homework, were designed to complement the exposition or extend the reader's understanding.

Geometry of manifolds was the basic prerequisite for the original course, so the same holds now for the notes. In particular, some familiarity with de Rham theory and classical Lie groups is expected.

As for conventions: unless otherwise indicated, all vector spaces are real and finite-dimensional, all maps are smooth (i.e., C^∞) and all manifolds are smooth, Hausdorff and second countable.

Here is a brief summary of the contents of this book. Parts I-III explain classical topics, including cotangent bundles, symplectomorphisms, lagrangian submanifolds and local forms. Parts IV-VI concentrate on important related areas, such as contact geometry and Kähler geometry. Classical hamiltonian theory enters in Parts VII-VIII, starting the second half of this book, which is devoted to a selection of themes from hamiltonian dynamical systems and symmetry. Parts IX-XI discuss the moment map whose preponderance has been growing steadily for the past twenty years.

There are by now excellent references on symplectic geometry, a subset of which is in the bibliography. However, the most efficient introduction to a subject is often a short elementary treatment, and these notes attempt to serve that purpose. The author hopes that these notes provide a taste of areas of current research, and will prepare the reader to explore recent papers and extensive books in symplectic geometry, where the pace is much faster.

Part I
Symplectic Manifolds

A symplectic form is a 2-form satisfying an algebraic condition – nondegeneracy – and an analytical condition – closedness. In Lectures 1 and 2 we define symplectic forms, describe some of their basic properties, introduce the first examples, namely even-dimensional euclidean spaces and cotangent bundles.

1 Symplectic Forms

1.1 Skew-Symmetric Bilinear Maps

Let V be an m-dimensional vector space over $\mathbb{R}$, and let $\Omega : V \times V \to \mathbb{R}$ be a bilinear map. The map Ω is **skew-symmetric** if $\Omega(u, v) = -\Omega(v, u)$, for all $u, v \in V$.

Theorem 1.1 (Standard Form for Skew-symmetric Bilinear Maps)
Let Ω be a skew-symmetric bilinear map on V. Then there is a basis $u_1, \ldots, u_k, e_1, \ldots, e_n, f_1, \ldots, f_n$ of V such that

$$\begin{array}{ll} \Omega(u_i, v) = 0 \ , & \textit{for all } i \textit{ and all } v \in V, \\ \Omega(e_i, e_j) = 0 = \Omega(f_i, f_j) \ , & \textit{for all } i, j, \textit{ and} \\ \Omega(e_i, f_j) = \delta_{ij} \ , & \textit{for all } i, j. \end{array}$$

Remarks.

1. The basis in Theorem 1.1 is not unique, though it is traditionally also called a "canonical" basis.

2. In matrix notation with respect to such basis, we have

$$\Omega(u, v) = [\text{—}u\text{—}] \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathrm{Id} \\ 0 & -\mathrm{Id} & 0 \end{bmatrix} \begin{bmatrix} | \\ v \\ | \end{bmatrix} .$$

$\diamondsuit$

Proof. This induction proof is a skew-symmetric version of the Gram-Schmidt process.

Let $U := \{u \in V \mid \Omega(u, v) = 0 \text{ for all } v \in V\}$. Choose a basis $u_1, \ldots, u_k$ of U, and choose a complementary space W to U in V,

$$V = U \oplus W \ .$$

Take any nonzero $e_1 \in W$. Then there is $f_1 \in W$ such that $\Omega(e_1, f_1) \neq 0$. Assume that $\Omega(e_1, f_1) = 1$. Let

$$\begin{array}{rcl} W_1 & = & \text{span of } e_1, f_1 \\ W_1^\Omega & = & \{w \in W \mid \Omega(w, v) = 0 \text{ for all } v \in W_1\} \ . \end{array}$$

Claim. $W_1 \cap W_1^\Omega = \{0\}$.

Suppose that $v = ae_1 + bf_1 \in W_1 \cap W_1^\Omega$.

$$\left. \begin{array}{l} 0 = \Omega(v, e_1) = -b \\ 0 = \Omega(v, f_1) = \quad a \end{array} \right\} \quad \Longrightarrow \quad v = 0 \ .$$

Claim. $W = W_1 \oplus W_1^\Omega$.

Suppose that $v \in W$ has $\Omega(v, e_1) = c$ and $\Omega(v, f_1) = d$. Then

$$v = \underbrace{(-cf_1 + de_1)}_{\in W_1} + \underbrace{(v + cf_1 - de_1)}_{\in W_1^\Omega} \ .$$

Go on: let $e_2 \in W_1^\Omega$, $e_2 \neq 0$. There is $f_2 \in W_1^\Omega$ such that $\Omega(e_2, f_2) \neq 0$. Assume that $\Omega(e_2, f_2) = 1$. Let $W_2 =$ span of e_2, f_2. Etc.

This process eventually stops because $\dim V < \infty$. We hence obtain

$$V = U \oplus W_1 \oplus W_2 \oplus \ldots \oplus W_n$$

where all summands are orthogonal with respect to Ω, and where W_i has basis e_i, f_i with $\Omega(e_i, f_i) = 1$. $\square$

The dimension of the subspace $U = \{u \in V \mid \Omega(u, v) = 0, \text{ for all } v \in V\}$ does not depend on the choice of basis.

$\Longrightarrow \quad k := \dim U$ is an invariant of (V, Ω) .

Since $k + 2n = m = \dim V$,

$\Longrightarrow \quad n$ is an invariant of (V, Ω); $2n$ is called the **rank** of Ω.

1.2 Symplectic Vector Spaces

Let V be an m-dimensional vector space over $\mathbb{R}$, and let $\Omega : V \times V \to \mathbb{R}$ be a bilinear map.

Definition 1.2 *The map $\widetilde{\Omega} : V \to V^*$ is the linear map defined by $\widetilde{\Omega}(v)(u) = \Omega(v, u)$.*

The kernel of $\widetilde{\Omega}$ is the subspace U above.

Definition 1.3 *A skew-symmetric bilinear map Ω is* **symplectic** *(or* **nondegenerate***) if $\widetilde{\Omega}$ is bijective, i.e., $U = \{0\}$. The map Ω is then called a* **linear symplectic structure** *on V, and (V, Ω) is called a* **symplectic vector space***.*

The following are immediate properties of a linear symplectic structure Ω:

- **Duality**: the map $\widetilde{\Omega} : V \overset{\simeq}{\to} V^*$ is a bijection.
- By the standard form theorem, $k = \dim U = 0$, so $\dim V = 2n$ is **even**.
- By Theorem 1.1, a symplectic vector space (V, Ω) has a basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ satisfying

$$\Omega(e_i, f_j) = \delta_{ij} \quad \text{and} \quad \Omega(e_i, e_j) = 0 = \Omega(f_i, f_j) \ .$$

  Such a basis is called a **symplectic basis** of (V, Ω). We have

$$\Omega(u, v) = [\text{—}u\text{—}] \begin{bmatrix} 0 & \mathsf{Id} \\ -\mathsf{Id} & 0 \end{bmatrix} \begin{bmatrix} | \\ v \\ | \end{bmatrix},$$

where the symbol $\begin{bmatrix} | \\ v \\ | \end{bmatrix}$ represents the column of coordinates of the vector v with respect to a symplectic basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ whereas $[\text{—}v\text{—}]$ represents its transpose line.

Not all subspaces W of a symplectic vector space (V, Ω) look the same:

- A subspace W is called **symplectic** if $\Omega|_W$ is nondegenerate. For instance, the span of e_1, f_1 is symplectic.
- A subspace W is called **isotropic** if $\Omega|_W \equiv 0$. For instance, the span of e_1, e_2 is isotropic.

Homework 1 describes subspaces W of (V, Ω) in terms of the relation between W and W^Ω.

The **prototype of a symplectic vector space** is $(\mathbb{R}^{2n}, \Omega_0)$ with Ω_0 such that the basis

$$\begin{array}{lll} e_1 = (1, 0, \ldots, 0), & \ldots, & e_n = (0, \ldots, 0, \overbrace{1}^{n}, 0, \ldots, 0), \\ f_1 = (0, \ldots, 0, \underbrace{1}_{n+1}, 0, \ldots, 0), & \ldots, & f_n = (0, \ldots, 0, 1) \ , \end{array}$$

is a symplectic basis. The map Ω_0 on other vectors is determined by its values on a basis and bilinearity.

Definition 1.4 *A* **symplectomorphism** φ *between symplectic vector spaces* (V, Ω) *and* (V', Ω') *is a linear isomorphism* $\varphi : V \overset{\simeq}{\to} V'$ *such that* $\varphi^*\Omega' = \Omega$. *(By definition,* $(\varphi^*\Omega')(u, v) = \Omega'(\varphi(u), \varphi(v))$*.) If a symplectomorphism exists,* (V, Ω) *and* (V', Ω') *are said to be* **symplectomorphic**.

The relation of being symplectomorphic is clearly an equivalence relation in the set of all even-dimensional vector spaces. Furthermore, by Theorem 1.1, every $2n$-dimensional symplectic vector space (V,Ω) is symplectomorphic to the prototype $(\mathbb{R}^{2n},\Omega_0)$; a choice of a symplectic basis for (V,Ω) yields a symplectomorphism to $(\mathbb{R}^{2n},\Omega_0)$. Hence, nonnegative even integers classify equivalence classes for the relation of being symplectomorphic.

1.3 Symplectic Manifolds

Let ω be a de Rham 2-form on a manifold M, that is, for each $p \in M$, the map $\omega_p : T_pM \times T_pM \to \mathbb{R}$ is skew-symmetric bilinear on the tangent space to M at p, and ω_p varies smoothly in p. We say that ω is closed if it satisfies the differential equation $d\omega = 0$, where d is the de Rham differential (i.e., exterior derivative).

Definition 1.5 *The 2-form ω is* **symplectic** *if ω is closed and ω_p is symplectic for all $p \in M$.*

If ω is symplectic, then $\dim T_pM = \dim M$ must be even.

Definition 1.6 *A* **symplectic manifold** *is a pair (M,ω) where M is a manifold and ω is a symplectic form.*

Example. Let $M = \mathbb{R}^{2n}$ with linear coordinates $x_1,\ldots,x_n,y_1,\ldots,y_n$. The form

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i$$

is symplectic as can be easily checked, and the set

$$\left\{ \left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_n}\right)_p, \left(\frac{\partial}{\partial y_1}\right)_p, \ldots, \left(\frac{\partial}{\partial y_n}\right)_p \right\}$$

is a symplectic basis of T_pM. $\diamondsuit$

Example. Let $M = \mathbb{C}^n$ with linear coordinates $z_1,\ldots,z_n$. The form

$$\omega_0 = \frac{i}{2}\sum_{k=1}^{n} dz_k \wedge d\bar{z}_k$$

is symplectic. In fact, this form equals that of the previous example under the identification $\mathbb{C}^n \simeq \mathbb{R}^{2n}$, $z_k = x_k + iy_k$. $\diamondsuit$

Example. Let $M = S^2$ regarded as the set of unit vectors in $\mathbb{R}^3$. Tangent vectors to S^2 at p may then be identified with vectors orthogonal to p. The standard symplectic form on S^2 is induced by the inner and exterior products:

$$\omega_p(u,v) := \langle p, u \times v\rangle\ , \qquad \text{for } u,v \in T_pS^2 = \{p\}^{\perp}\ .$$

This form is closed because it is of top degree; it is nondegenerate because $\langle p, u \times v\rangle \neq 0$ when $u \neq 0$ and we take, for instance, $v = u \times p$. $\diamondsuit$

1.4 Symplectomorphisms

Definition 1.7 *Let (M_1, ω_1) and (M_2, ω_2) be $2n$-dimensional symplectic manifolds, and let $\varphi : M_1 \to M_2$ be a diffeomorphism. Then φ is a* **symplectomorphism** *if $\varphi^* \omega_2 = \omega_1$.*[1]

We would like to classify symplectic manifolds up to symplectomorphism. The Darboux theorem (proved in Lecture 8 and stated below) takes care of this classification locally: the dimension is the only local invariant of symplectic manifolds up to symplectomorphisms. Just as any n-dimensional manifold looks locally like $\mathbb{R}^n$, any $2n$-dimensional *symplectic* manifold looks locally like $(\mathbb{R}^{2n}, \omega_0)$. More precisely, any symplectic manifold (M^{2n}, ω) is locally symplectomorphic to $(\mathbb{R}^{2n}, \omega_0)$.

Theorem 8.1 (Darboux) *Let (M, ω) be a $2n$-dimensional symplectic manifold, and let p be any point in M.*
Then there is a coordinate chart $(\mathcal{U}, x_1, \ldots, x_n, y_1, \ldots, y_n)$ centered at p such that on $\mathcal{U}$

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i \ .$$

A chart $(\mathcal{U}, x_1, \ldots, x_n, y_1, \ldots, y_n)$ as in Theorem 8.1 is called a **Darboux chart**.

By Theorem 8.1, the **prototype of a local piece of a $2n$-dimensional symplectic manifold** is $M = \mathbb{R}^{2n}$, with linear coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$, and with symplectic form

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i \ .$$

[1] Recall that, by definition of pullback, at tangent vectors $u, v \in T_pM_1$, we have $(\varphi^*\omega_2)_p(u, v) = (\omega_2)_{\varphi(p)}(d\varphi_p(u), d\varphi_p(v))$.

Homework 1: Symplectic Linear Algebra

Given a linear subspace Y of a symplectic vector space (V,Ω), its **symplectic orthogonal** Y^Ω is the linear subspace defined by

$$Y^\Omega := \{v \in V \mid \Omega(v,u) = 0 \text{ for all } u \in Y\} \; .$$

1. Show that $\quad \dim Y + \dim Y^\Omega = \dim V$.

   Hint: What is the kernel and image of the map

   $$\begin{array}{ccc} V & \longrightarrow & Y^* = \mathrm{Hom}(Y,\mathbb{R}) \quad ? \\ v & \longmapsto & \Omega(v,\cdot)|_Y \end{array}$$

2. Show that $(Y^\Omega)^\Omega = Y$.

3. Show that, if Y and W are subspaces, then

   $$Y \subseteq W \iff W^\Omega \subseteq Y^\Omega \; .$$

4. Show that:

   Y is **symplectic** (i.e., $\Omega|_{Y\times Y}$ is nondegenerate) $\iff Y \cap Y^\Omega = \{0\} \iff V = Y \oplus Y^\Omega$.

5. We call Y **isotropic** when $Y \subseteq Y^\Omega$ (i.e., $\Omega|_{Y\times Y} \equiv 0$).

   Show that, if Y is isotropic, then $\dim Y \leq \frac{1}{2}\dim V$.

6. We call Y **coisotropic** when $Y^\Omega \subseteq Y$.

   Check that every codimension 1 subspace Y is coisotropic.

7. An isotropic subspace Y of (V,Ω) is called **lagrangian** when $\dim Y = \frac{1}{2}\dim V$.

   Check that:

   $$Y \text{ is lagrangian} \iff Y \text{ is isotropic and coisotropic} \iff Y = Y^\Omega \; .$$

8. Show that, if Y is a lagrangian subspace of (V,Ω), then any basis $e_1,\ldots,e_n$ of Y can be extended to a symplectic basis $e_1,\ldots,e_n,f_1,\ldots,f_n$ of (V,Ω).

   Hint: Choose f_1 in W^Ω, where W is the linear span of $\{e_2,\ldots,e_n\}$.

9. Show that, if Y is a lagrangian subspace, (V,Ω) is symplectomorphic to the space $(Y \oplus Y^*, \Omega_0)$, where Ω_0 is determined by the formula

   $$\Omega_0(u \oplus \alpha, v \oplus \beta) = \beta(u) - \alpha(v) \; .$$

   In fact, for any vector space E, the direct sum $V = E \oplus E^*$ has a canonical symplectic structure determined by the formula above. If $e_1,\ldots,e_n$ is a basis of E, and $f_1,\ldots,f_n$ is the dual basis, then $e_1\oplus 0,\ldots,e_n\oplus 0, 0\oplus f_1,\ldots,0\oplus f_n$ is a symplectic basis for V.

2 Symplectic Form on the Cotangent Bundle

2.1 Cotangent Bundle

Let X be any n-dimensional manifold and $M = T^*X$ its cotangent bundle. If the manifold structure on X is described by coordinate charts $(\mathcal{U}, x_1, \ldots, x_n)$ with $x_i : \mathcal{U} \to \mathbb{R}$, then at any $x \in \mathcal{U}$, the differentials $(dx_1)_x, \ldots (dx_n)_x$ form a basis of T_x^*X. Namely, if $\xi \in T_x^*X$, then $\xi = \sum_{i=1}^n \xi_i (dx_i)_x$ for some real coefficients $\xi_1, \ldots, \xi_n$. This induces a map

$$\begin{array}{ccc} T^*\mathcal{U} & \longrightarrow & \mathbb{R}^{2n} \\ (x, \xi) & \longmapsto & (x_1, \ldots, x_n, \xi_1, \ldots, \xi_n) \ . \end{array}$$

The chart $(T^*\mathcal{U}, x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ is a coordinate chart for T^*X; the coordinates $x_1, \ldots, x_n, \xi_1, \ldots, \xi_n$ are the **cotangent coordinates** associated to the coordinates $x_1, \ldots, x_n$ on $\mathcal{U}$. The transition functions on the overlaps are smooth: given two charts $(\mathcal{U}, x_1, \ldots, x_n)$, $(\mathcal{U}', x_1', \ldots, x_n')$, and $x \in \mathcal{U} \cap \mathcal{U}'$, if $\xi \in T_x^*X$, then

$$\xi = \sum_{i=1}^n \xi_i \, (dx_i)_x = \sum_{i,j} \xi_i \left(\frac{\partial x_i}{\partial x_j'} \right) (dx_j')_x = \sum_{j=1}^n \xi_j' \, (dx_j')_x$$

where $\xi_j' = \sum_i \xi_i \left(\frac{\partial x_i}{\partial x_j'} \right)$ is smooth. Hence, T^*X is a $2n$-dimensional manifold.

We will now construct a major class of examples of symplectic forms. The *canonical forms* on cotangent bundles are relevant for several branches, including analysis of differential operators, dynamical systems and classical mechanics.

2.2 Tautological and Canonical Forms in Coordinates

Let $(\mathcal{U}, x_1, \ldots, x_n)$ be a coordinate chart for X, with associated cotangent coordinates $(T^*\mathcal{U}, x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$. Define a 2-form ω on $T^*\mathcal{U}$ by

$$\omega = \sum_{i=1}^n dx_i \wedge d\xi_i \ .$$

In order to check that this definition is coordinate-independent, consider the 1-form on $T^*\mathcal{U}$

$$\alpha = \sum_{i=1}^n \xi_i \ dx_i \ .$$

Clearly, $\omega = -d\alpha$.

Claim. The form α is intrinsically defined (and hence the form ω is also intrinsically defined) .

Proof. Let $(\mathcal{U}, x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ and $(\mathcal{U}', x'_1, \ldots, x'_n, \xi'_1, \ldots, \xi'_n)$ be two cotangent coordinate charts. On $\mathcal{U} \cap \mathcal{U}'$, the two sets of coordinates are related by $\xi'_j = \sum_i \xi_i \left(\frac{\partial x_i}{\partial x'_j}\right)$. Since $dx'_j = \sum_i \left(\frac{\partial x'_j}{\partial x_i}\right) dx_i$, we have

$$\alpha = \sum_i \xi_i dx_i = \sum_j \xi'_j dx'_j = \alpha' \ .$$

□

The 1-form α is the **tautological form** or **Liouville 1-form** and the 2-form ω is the **canonical symplectic form**. The following section provides an alternative proof of the intrinsic character of these forms.

2.3 Coordinate-Free Definitions

Let

$$\begin{array}{ccccc} M = T^*X & \qquad & p = (x,\xi) & \qquad & \xi \in T^*_x X \\ \downarrow \pi & & \downarrow & & \\ X & & x & & \end{array}$$

be the natural projection. The **tautological 1-form** α may be defined pointwise by

$$\alpha_p = (d\pi_p)^* \xi \quad \in T^*_p M \ ,$$

where $(d\pi_p)^*$ is the transpose of $d\pi_p$, that is, $(d\pi_p)^*\xi = \xi \circ d\pi_p$:

$$\begin{array}{ccccc} p = (x,\xi) & \qquad & T_pM & \qquad & T^*_pM \\ \downarrow \pi & & \downarrow d\pi_p & & \uparrow (d\pi_p)^* \\ x & & T_xX & & T^*_xX \end{array}$$

Equivalently,

$$\alpha_p(v) = \xi\Big((d\pi_p)v\Big) \ , \qquad \text{for } v \in T_pM \ .$$

Exercise. Let $(\mathcal{U}, x_1, \ldots, x_n)$ be a chart on X with associated cotangent coordinates $x_1, \ldots, x_n, \xi_1, \ldots, \xi_n$. Show that on $T^*\mathcal{U}$, $\alpha = \sum\limits_{i=1}^{n} \xi_i \ dx_i$. ♢

The **canonical symplectic 2-form** ω on T^*X is defined as

$$\omega = -d\alpha \ .$$

Locally, $\omega = \sum_{i=1}^n dx_i \wedge d\xi_i$.

Exercise. Show that the tautological 1-form α is uniquely characterized by the property that, for every 1-form $\mu : X \to T^*X$, $\mu^*\alpha = \mu$. (See Lecture 3.) ♢

2.4 Naturality of the Tautological and Canonical Forms

Let X_1 and X_2 be n-dimensional manifolds with cotangent bundles $M_1 = T^*X_1$ and $M_2 = T^*X_2$, and tautological 1-forms α_1 and α_2. Suppose that $f : X_1 \to X_2$ is a diffeomorphism. Then there is a natural diffeomorphism

$$f_\sharp : M_1 \to M_2$$

which **lifts** f; namely, if $p_1 = (x_1, \xi_1) \in M_1$ for $x_1 \in X_1$ and $\xi_1 \in T^*_{x_1}X_1$, then we define

$$f_\sharp(p_1) = p_2 = (x_2, \xi_2) \ , \quad \text{with } \begin{cases} x_2 = f(x_1) \in X_2 \text{ and} \\ \xi_1 = (df_{x_1})^*\xi_2 \ , \end{cases}$$

where $(df_{x_1})^* : T^*_{x_2}X_2 \xrightarrow{\simeq} T^*_{x_1}X_1$, so $f_\sharp|_{T^*_{x_1}}$ is the inverse map of $(df_{x_1})^*$.

Exercise. Check that $f_\sharp$ is a diffeomorphism. Here are some hints:

1. $$\begin{array}{ccc} M_1 & \xrightarrow{f_\sharp} & M_2 \\ \pi_1 \downarrow & & \downarrow \pi_2 \\ X_1 & \xrightarrow{f} & X_2 \end{array}$$ commutes;
2. $f_\sharp : M_1 \to M_2$ is bijective;
3. $f_\sharp$ and $f_\sharp^{-1}$ are smooth.

$\diamondsuit$

Proposition 2.1 *The lift $f_\sharp$ of a diffeomorphism $f : X_1 \to X_2$ pulls the tautological form on T^*X_2 back to the tautological form on T^*X_1, i.e.,*

$$(f_\sharp)^*\alpha_2 = \alpha_1 \ .$$

Proof. At $p_1 = (x_1, \xi_1) \in M_1$, this identity says

$$(df_\sharp)^*_{p_1} (\alpha_2)_{p_2} = (\alpha_1)_{p_1} \qquad (\star)$$

where $p_2 = f_\sharp(p_1)$.

Using the following facts,

- Definition of $f_\sharp$:

  $p_2 = f_\sharp(p_1) \iff p_2 = (x_2, \xi_2)$ where $x_2 = f(x_1)$ and $(df_{x_1})^*\xi_2 = \xi_1$.

- Definition of tautological 1-form:

  $(\alpha_1)_{p_1} = (d\pi_1)^*_{p_1}\xi_1$ and $(\alpha_2)_{p_2} = (d\pi_2)^*_{p_2}\xi_2$.

- The diagram $$\begin{array}{ccc} M_1 & \xrightarrow{f_\sharp} & M_2 \\ \pi_1 \downarrow & & \downarrow \pi_2 \\ X_1 & \xrightarrow{f} & X_2 \end{array}$$ commutes.

the proof of $(\star)$ is:

$$\begin{array}{rclcl}
(df_\sharp)^*_{p_1}(\alpha_2)_{p_2} & = & (df_\sharp)^*_{p_1}(d\pi_2)^*_{p_2}\xi_2 & = & (d(\pi_2 \circ f_\sharp))^*_{p_1}\xi_2 \\
& = & (d(f \circ \pi_1))^*_{p_1}\xi_2 & = & (d\pi_1)^*_{p_1}(df)^*_{x_1}\xi_2 \\
& = & (d\pi_1)^*_{p_1}\xi_1 & = & (\alpha_1)_{p_1} \; .
\end{array}$$

$\square$

Corollary 2.2 *The lift $f_\sharp$ of a diffeomorphism $f : X_1 \to X_2$ is a symplectomorphism, i.e.,*

$$(f_\sharp)^*\omega_2 = \omega_1 \; ,$$

where ω_1, ω_2 are the canonical symplectic forms.

In summary, a diffeomorphism of manifolds induces a canonical symplectomorphism of cotangent bundles:

$$\begin{array}{rccc}
f_\sharp : & T^*X_1 & \longrightarrow & T^*X_2 \\
& & \uparrow & \\
f : & X_1 & \longrightarrow & X_2
\end{array}$$

Example. Let $X_1 = X_2 = S^1$. Then T^*S^1 is an infinite cylinder $S^1 \times \mathbb{R}$. The canonical 2-form ω is the area form $\omega = d\theta \wedge d\xi$. If $f : S^1 \to S^1$ is any diffeomorphism, then $f_\sharp : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$ is a symplectomorphism, i.e., is an area-preserving diffeomorphism of the cylinder. $\diamondsuit$

If $f : X_1 \to X_2$ and $g : X_2 \to X_3$ are diffeomorphisms, then $(g \circ f)_\sharp = g_\sharp \circ f_\sharp$. In terms of the group $\mathrm{Diff}(X)$ of diffeomorphisms of X and the group $\mathrm{Sympl}(M, \omega)$ of symplectomorphisms of (M, ω), we say that the map

$$\begin{array}{rcl}
\mathrm{Diff}(X) & \longrightarrow & \mathrm{Sympl}(M, \omega) \\
f & \longmapsto & f_\sharp
\end{array}$$

is a group homomorphism. This map is clearly injective. Is it surjective? Do all symplectomorphisms $T^*X \to T^*X$ come from diffeomorphisms $X \to X$? No: for instance, translation along cotangent fibers is not induced by a diffeomorphism of the base manifold. A criterion for which symplectomorphisms arise as lifts of diffeomorphisms is discussed in Homework 3.

Homework 2: Symplectic Volume

1. Given a vector space V, the exterior algebra of its dual space is

$$\wedge^*(V^*) = \bigoplus_{k=0}^{\dim V} \wedge^k(V^*) \ ,$$

where $\wedge^k(V^*)$ is the set of maps $\alpha : \overbrace{V \times \cdots \times V}^{k} \to \mathbb{R}$ which are linear in each entry, and for any permutation π, $\alpha(v_{\pi_1}, \ldots, v_{\pi_k}) = (\operatorname{sign} \pi) \cdot \alpha(v_1, \ldots, v_k)$. The elements of $\wedge^k(V^*)$ are known as **skew-symmetric k-linear maps** or **k-forms** on V.

   (a) Show that any $\Omega \in \wedge^2(V^*)$ is of the form $\Omega = e_1^* \wedge f_1^* + \ldots + e_n^* \wedge f_n^*$, where $u_1^*, \ldots, u_k^*, e_1^*, \ldots, e_n^*, f_1^*, \ldots, f_n^*$ is a basis of V^* dual to the standard basis ($k + 2n = \dim V$).

   (b) In this language, a symplectic map $\Omega : V \times V \to \mathbb{R}$ is just a nondegenerate 2-form $\Omega \in \wedge^2(V^*)$, called a **symplectic form** on V.

   Show that, if Ω is any symplectic form on a vector space V of dimension $2n$, then the nth exterior power $\Omega^n = \underbrace{\Omega \wedge \ldots \wedge \Omega}_{n}$ does not vanish.

   (c) Deduce that the nth exterior power ω^n of any symplectic form ω on a $2n$-dimensional manifold M is a volume form.[2]

   Hence, any symplectic manifold (M, ω) is canonically oriented by the symplectic structure. The form $\frac{\omega^n}{n!}$ is called the **symplectic volume** or the **Liouville volume** of (M, ω).

   Does the Möbius strip support a symplectic structure?

   (d) Conversely, given a 2-form $\Omega \in \wedge^2(V^*)$, show that, if $\Omega^n \neq 0$, then Ω is symplectic.

   Hint: Standard form.

2. Let (M, ω) be a $2n$-dimensional symplectic manifold, and let ω^n be the volume form obtained by wedging ω with itself n times.

   (a) Show that, if M is compact, the de Rham cohomology class $[\omega^n] \in H^{2n}(M; \mathbb{R})$ is non-zero.

   Hint: Stokes' theorem.

   (b) Conclude that $[\omega]$ itself is non-zero (in other words, that ω is not exact).

   (c) Show that if $n > 1$ there are no symplectic structures on the sphere S^{2n}.

[2] A **volume form** is a nonvanishing form of top degree.

Part II

Symplectomorphisms

Equivalence between symplectic manifolds is expressed by a *symplectomorphism*. By Weinstein's lagrangian creed [105], everything is a lagrangian manifold! We will study symplectomorphisms according to the creed.

3 Lagrangian Submanifolds

3.1 Submanifolds

Let M and X be manifolds with $\dim X < \dim M$.

Definition 3.1 *A map $i : X \to M$ is an* **immersion** *if $di_p : T_pX \to T_{i(p)}M$ is injective for any point $p \in X$.*

An **embedding** *is an immersion which is a homeomorphism onto its image.*[3]

A **closed embedding** *is a proper*[4] *injective immersion.*

Exercise. Show that a map $i : X \to M$ is a closed embedding if and only if i is an embedding and its image $i(X)$ is closed in M.

Hints:

- If i is injective and proper, then for any neighborhood $\mathcal{U}$ of $p \in X$, there is a neighborhood $\mathcal{V}$ of $i(p)$ such that $f^{-1}(\mathcal{V}) \subseteq \mathcal{U}$.
- On a Hausdorff space, any compact set is closed. On any topological space, a closed subset of a compact set is compact.
- An embedding is proper if and only if its image is closed.

$\diamondsuit$

Definition 3.2 *A* **submanifold** *of M is a manifold X with a closed embedding $i : X \hookrightarrow M$.*[5]

Notation. Given a submanifold, we regard the embedding $i : X \hookrightarrow M$ as an inclusion, in order to identify points and tangent vectors:

$$p = i(p) \quad \text{and} \quad T_pX = di_p(T_pX) \subset T_pM \ .$$

[3] The image has the topology induced by the target manifold.

[4] A map is **proper** if the preimage of any compact set is compact.

[5] When X is an open subset of a manifold M, we refer to it as an *open* submanifold.

3.2 Lagrangian Submanifolds of T^*X

Definition 3.3 *Let (M,ω) be a $2n$-dimensional symplectic manifold. A submanifold Y of M is a* **lagrangian submanifold** *if, at each $p \in Y$, T_pY is a lagrangian subspace of T_pM, i.e., $\omega_p|_{T_pY} \equiv 0$ and $\dim T_pY = \frac{1}{2}\dim T_pM$. Equivalently, if $i : Y \hookrightarrow M$ is the inclusion map, then Y is* **lagrangian** *if and only if $i^*\omega = 0$ and $\dim Y = \frac{1}{2}\dim M$.*

Let X be an n-dimensional manifold, with $M = T^*X$ its cotangent bundle. If $x_1, \ldots, x_n$ are coordinates on $U \subseteq X$, with associated cotangent coordinates $x_1, \ldots, x_n, \xi_1, \ldots, \xi_n$ on T^*U, then the tautological 1-form on T^*X is

$$\alpha = \sum \xi_i dx_i$$

and the canonical 2-form on T^*X is

$$\omega = -d\alpha = \sum dx_i \wedge d\xi_i \ .$$

The **zero section** of T^*X

$$X_0 := \{(x,\xi) \in T^*X \mid \xi = 0 \text{ in } T_x^*X\}$$

is an n-dimensional submanifold of T^*X whose intersection with T^*U is given by the equations $\xi_1 = \ldots = \xi_n = 0$. Clearly $\alpha = \sum \xi_i dx_i$ vanishes on $X_0 \cap T^*U$. In particular, if $i_0 : X_0 \hookrightarrow T^*X$ is the inclusion map, we have $i_0^*\alpha = 0$. Hence, $i_0^*\omega = i_0^* d\alpha = 0$, and X_0 is lagrangian.

What are all the lagrangian submanifolds of T^*X which are "C^1-close to X_0"?

Let X_μ be (the image of) another section, that is, an n-dimensional submanifold of T^*X of the form

$$X_\mu = \{(x,\mu_x) \mid x \in X,\ \mu_x \in T_x^*X\} \qquad (\star)$$

where the covector μ_x depends smoothly on x, and $\mu : X \to T^*X$ is a de Rham 1-form. Relative to the inclusion $i : X_\mu \hookrightarrow T^*X$ and the cotangent projection $\pi : T^*X \to X$, X_μ is of the form $(\star)$ if and only if $\pi \circ i : X_\mu \to X$ is a diffeomorphism.

When is such an X_μ lagrangian?

Proposition 3.4 *Let X_μ be of the form $(\star)$, and let μ be the associated de Rham 1-form. Denote by $s_\mu : X \to T^*X$, $x \mapsto (x,\mu_x)$, the 1-form μ regarded exclusively as a map. Notice that the image of s_μ is X_μ. Let α be the tautological 1-form on T^*X. Then*

$$s_\mu^*\alpha = \mu \ .$$

Proof. By definition of α (previous lecture), $\alpha_p = (d\pi_p)^*\xi$ at $p = (x,\xi) \in M$. For $p = s_\mu(x) = (x,\mu_x)$, we have $\alpha_p = (d\pi_p)^*\mu_x$. Then

$$(s_\mu^*\alpha)_x = (ds_\mu)_x^*\alpha_p = (ds_\mu)_x^*(d\pi_p)^*\mu_x = (d\underbrace{(\pi \circ s_\mu)}_{\mathrm{id}_X})_x^*\mu_x = \mu_x \ .$$

$\square$

Suppose that X_μ is an n-dimensional submanifold of T^*X of the form $(\star)$, with associated de Rham 1-form μ. Then $s_\mu : X \to T^*X$ is an embedding with image X_μ, and there is a diffeomorphism $\tau : X \to X_\mu$, $\tau(x) := (x, \mu_x)$, such that the following diagram commutes.

$$\begin{tikzcd} X \arrow[rr, "s_\mu"] \arrow[dr, "\tau"', "\simeq"] & & T^*X \\ & X_\mu \arrow[ur, "i"'] & \end{tikzcd}$$

We want to express the condition of X_μ being lagrangian in terms of the form μ:

$$\begin{array}{rcl} X_\mu \text{ is lagrangian} & \iff & i^*d\alpha = 0 \\ & \iff & \tau^*i^*d\alpha = 0 \\ & \iff & (i\circ\tau)^*d\alpha = 0 \\ & \iff & s_\mu^*d\alpha = 0 \\ & \iff & ds_\mu^*\alpha = 0 \\ & \iff & d\mu = 0 \\ & \iff & \mu \text{ is closed} . \end{array}$$

Therefore, there is a one-to-one correspondence between the set of lagrangian submanifolds of T^*X of the form $(\star)$ and the set of closed 1-forms on X.

When X is simply connected, $H^1_{\mathrm{deRham}}(X) = 0$, so every closed 1-form μ is equal to df for some $f \in C^\infty(X)$. Any such primitive f is then called a **generating function** for the lagrangian submanifold X_μ associated to μ. (Two functions generate the same lagrangian submanifold if and only if they differ by a locally constant function.) On arbitrary manifolds X, functions $f \in C^\infty(X)$ originate lagrangian submanifolds as images of df.

Exercise. Check that, if X is compact (and not just one point) and $f \in C^\infty(X)$, then $\#(X_{df} \cap X_0) \geq 2$. $\diamondsuit$

There are lots of lagrangian submanifolds of T^*X not covered by the description in terms of closed 1-forms, starting with the cotangent fibers.

3.3 Conormal Bundles

Let S be any k-dimensional submanifold of an n-dimensional manifold X.

Definition 3.5 *The* **conormal space** *at* $x \in S$ *is*

$$N_x^*S = \{\xi \in T_x^*X \mid \xi(v) = 0 \ , \ \textit{for all } v \in T_xS\} \ .$$

The **conormal bundle** *of* S *is*

$$N^*S = \{(x,\xi) \in T^*X \mid x \in S, \ \xi \in N_x^*S\} \ .$$

Exercise. The conormal bundle N^*S is an n-dimensional submanifold of T^*X.
Hint: Use coordinates on X adapted[6] to S. $\diamondsuit$

Proposition 3.6 *Let $i : N^*S \hookrightarrow T^*X$ be the inclusion, and let α be the tautological 1-form on T^*X. Then*

$$i^*\alpha = 0 \ .$$

Proof. Let $(\mathcal{U}, x_1, \ldots, x_n)$ be a coordinate system on X centered at $x \in S$ and adapted to S, so that $\mathcal{U} \cap S$ is described by $x_{k+1} = \ldots = x_n = 0$. Let $(T^*\mathcal{U}, x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ be the associated cotangent coordinate system. The submanifold $N^*S \cap T^*\mathcal{U}$ is then described by

$$x_{k+1} = \ldots = x_n = 0 \qquad \text{and} \qquad \xi_1 = \ldots = \xi_k = 0 \ .$$

Since $\alpha = \sum \xi_i dx_i$ on $T^*\mathcal{U}$, we conclude that, at $p \in N^*S$,

$$(i^*\alpha)_p = \alpha_p|_{T_p(N^*S)} = \sum_{i>k} \xi_i dx_i \Big|_{\operatorname{span}\{\frac{\partial}{\partial x_i}, i \le k\}} = 0 \ .$$

$\square$

Corollary 3.7 *For any submanifold $S \subset X$, the conormal bundle N^*S is a lagrangian submanifold of T^*X.*

Taking $S = \{x\}$ to be one point, the conormal bundle $L = N^*S = T_x^*X$ is a cotangent fiber. Taking $S = X$, the conormal bundle $L = X_0$ is the zero section of T^*X.

3.4 Application to Symplectomorphisms

Let (M_1, ω_1) and (M_2, ω_2) be two $2n$-dimensional symplectic manifolds. Given a diffeomorphism $\varphi : M_1 \xrightarrow{\simeq} M_2$, when is it a symplectomorphism? (I.e., when is $\varphi^*\omega_2 = \omega_1$?)

Consider the two projection maps

$$\begin{tikzcd}
(p_1, p_2) \arrow[dd] & & M_1 \times M_2 \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & & (p_1, p_2) \arrow[dd] \\
& & & & \\
p_1 & M_1 & & M_2 & p_2
\end{tikzcd}$$

[6] A coordinate chart $(\mathcal{U}, x_1, \ldots, x_n)$ on X is adapted to a k-dimensional submanifold S if $S \cap \mathcal{U}$ is described by $x_{k+1} = \ldots = x_n = 0$.

Then $\omega = (\mathrm{pr}_1)^*\omega_1 + (\mathrm{pr}_2)^*\omega_2$ is a 2-form on $M_1 \times M_2$ which is closed,

$$d\omega = (\mathrm{pr}_1)^* \underbrace{d\omega_1}_{0} + (\mathrm{pr}_2)^* \underbrace{d\omega_2}_{0} = 0 \ ,$$

and symplectic,

$$\omega^{2n} = \left(\begin{array}{c} 2n \\ n \end{array} \right) \Big((\mathrm{pr}_1)^*\omega_1 \Big)^n \wedge \Big((\mathrm{pr}_2)^*\omega_2 \Big)^n \neq 0 \ .$$

More generally, if $\lambda_1, \lambda_2 \in \mathbb{R}\backslash\{0\}$, then $\lambda_1(\mathrm{pr}_1)^*\omega_1 + \lambda_2(\mathrm{pr}_2)^*\omega_2$ is also a symplectic form on $M_1 \times M_2$. Take $\lambda_1 = 1$, $\lambda_2 = -1$ to obtain the **twisted product form** on $M_1 \times M_2$:

$$\widetilde{\omega} = (\mathrm{pr}_1)^*\omega_1 - (\mathrm{pr}_2)^*\omega_2 \ .$$

The graph of a diffeomorphism $\varphi : M_1 \xrightarrow{\simeq} M_2$ is the $2n$-dimensional submanifold of $M_1 \times M_2$:

$$\Gamma_\varphi := \mathrm{Graph}\, \varphi = \{(p, \varphi(p)) \mid p \in M_1\} \ .$$

The submanifold Γ_φ is an embedded image of M_1 in $M_1 \times M_2$, the embedding being the map

$$\begin{array}{rccc} \gamma : & M_1 & \longrightarrow & M_1 \times M_2 \\ & p & \longmapsto & (p, \varphi(p)) \ . \end{array}$$

Proposition 3.8 *A diffeomorphism φ is a symplectomorphism if and only if Γ_φ is a lagrangian submanifold of $(M_1 \times M_2, \widetilde{\omega})$.*

Proof. The graph Γ_φ is lagrangian if and only if $\gamma^*\widetilde{\omega} = 0$. But

$$\begin{array}{rcl} \gamma^*\widetilde{\omega} & = & \gamma^*\ \mathrm{pr}_1^*\ \omega_1 - \gamma^*\ \mathrm{pr}_2^*\ \omega_2 \\ & = & (\mathrm{pr}_1 \circ \gamma)^*\omega_1 - (\mathrm{pr}_2 \circ \gamma)^*\omega_2 \end{array}$$

and $\mathrm{pr}_1 \circ \gamma$ is the identity map on M_1 whereas $\mathrm{pr}_2 \circ \gamma = \varphi$. Therefore,

$$\gamma^*\widetilde{\omega} = 0 \quad \Longleftrightarrow \quad \varphi^*\omega_2 = \omega_1 \ .$$

$\square$

Homework 3:
Tautological Form and Symplectomorphisms

This set of problems is from [53].

1. Let (M, ω) be a symplectic manifold, and let α be a 1-form such that
$$\omega = -d\alpha \ .$$
   Show that there exists a unique vector field v such that its interior product with ω is α, i.e., $\imath_v \omega = -\alpha$.

   Prove that, if g is a symplectomorphism which preserves α (that is, $g^*\alpha = \alpha$), then g commutes with the one-parameter group of diffeomorphisms generated by v, i.e.,
$$(\exp tv) \circ g = g \circ (\exp tv) \ .$$
   Hint: Recall that, for $p \in M$, $(\exp tv)(p)$ is the *unique* curve in M solving the ordinary differential equation
$$\begin{cases} \frac{d}{dt}(\exp tv(p)) = v(\exp tv(p)) \\ (\exp tv)(p)|_{t=0} = p \end{cases}$$
   for t in some neighborhood of 0. Show that $g \circ (\exp tv) \circ g^{-1}$ is the one-parameter group of diffeomorphisms generated by $g_* v$. (The push-forward of v by g is defined by $(g_* v)_{g(p)} = dg_p(v_p)$.) Finally check that g preserves v (that is, $g_* v = v$).

2. Let X be an arbitrary n-dimensional manifold, and let $M = T^*X$. Let $(\mathcal{U}, x_1, \ldots, x_n)$ be a coordinate system on X, and let $x_1, \ldots, x_n, \xi_1, \ldots, \xi_n$ be the corresponding coordinates on $T^*\mathcal{U}$.

   Show that, when α is the tautological 1-form on M (which, in these coordinates, is $\sum \xi_i \, dx_i$), the vector field v in the previous exercise is just the vector field $\sum \xi_i \frac{\partial}{\partial \xi_i}$.

   Let $\exp tv$, $-\infty < t < \infty$, be the one-parameter group of diffeomorphisms generated by v.

   Show that, for every point $p = (x, \xi)$ in M,
$$(\exp tv)(p) = p_t \quad \text{where} \quad p_t = (x, e^t \xi) \ .$$

3. Let M be as in exercise 2.

   Show that, if g is a symplectomorphism of M which preserves α, then

   $$g(x,\xi) = (y,\eta) \quad \Longrightarrow \quad g(x,\lambda\xi) = (y,\lambda\eta)$$

   for all $(x,\xi) \in M$ and $\lambda \in \mathbb{R}$.

   Conclude that g has to preserve the cotangent fibration, i.e., show that there exists a diffeomorphism $f : X \to X$ such that $\pi \circ g = f \circ \pi$, where $\pi : M \to X$ is the projection map $\pi(x,\xi) = x$.

   Finally prove that $g = f_\#$, the map $f_\#$ being the symplectomorphism of M lifting f.

   Hint: Suppose that $g(p) = q$ where $p = (x,\xi)$ and $q = (y,\eta)$.
   Combine the identity
   $$(dg_p)^*\alpha_q = \alpha_p$$
   with the identity
   $$d\pi_q \circ dg_p = df_x \circ d\pi_p \ .$$
   (The first identity expresses the fact that $g^*\alpha = \alpha$, and the second identity is obtained by differentiating both sides of the equation $\pi \circ g = f \circ \pi$ at p.)

4. Let M be as in exercise 2, and let h be a smooth function on X. Define $\tau_h : M \to M$ by setting

   $$\tau_h(x,\xi) = (x, \xi + dh_x) \ .$$

   Prove that
   $$\tau_h^*\alpha = \alpha + \pi^* dh$$
   where π is the projection map

   $$\begin{array}{ccc} M & & (x,\xi) \\ \downarrow \pi & & \downarrow \\ X & & x \end{array}$$

   Deduce that
   $$\tau_h^*\omega = \omega \ ,$$
   i.e., that τ_h is a symplectomorphism.

4 Generating Functions

4.1 Constructing Symplectomorphisms

Let X_1, X_2 be n-dimensional manifolds, with cotangent bundles $M_1 = T^*X_1$, $M_2 = T^*X_2$, tautological 1-forms α_1, α_2, and canonical 2-forms ω_1, ω_2.

Under the natural identification

$$M_1 \times M_2 = T^*X_1 \times T^*X_2 \simeq T^*(X_1 \times X_2) \ ,$$

the tautological 1-form on $T^*(X_1 \times X_2)$ is

$$\alpha = (\mathrm{pr}_1)^*\alpha_1 + (\mathrm{pr}_2)^*\alpha_2 \ ,$$

where $\mathrm{pr}_i : M_1 \times M_2 \to M_i$, $i = 1, 2$ are the two projections. The canonical 2-form on $T^*(X_1 \times X_2)$ is

$$\omega = -d\alpha = -d\mathrm{pr}_1^*\alpha_1 - d\mathrm{pr}_2^*\alpha_2 = \mathrm{pr}_1^*\omega_1 + \mathrm{pr}_2^*\omega_2 \ .$$

In order to describe the twisted form $\widetilde{\omega} = \mathrm{pr}_1^*\omega_1 - \mathrm{pr}_2^*\omega_2$, we define an involution of $M_2 = T^*X_2$ by

$$\begin{array}{rccc} \sigma_2 : & M_2 & \longrightarrow & M_2 \\ & (x_2, \xi_2) & \longmapsto & (x_2, -\xi_2) \end{array}$$

which yields $\sigma_2^*\alpha_2 = -\alpha_2$. Let $\sigma = \mathrm{id}_{M_1} \times \sigma_2 : M_1 \times M_2 \to M_1 \times M_2$. Then

$$\sigma^*\widetilde{\omega} = \mathrm{pr}_1^*\omega_1 + \mathrm{pr}_2^*\omega_2 = \omega \ .$$

If Y is a lagrangian submanifold of $(M_1 \times M_2, \omega)$, then its "twist" $Y^\sigma := \sigma(Y)$ is a lagrangian submanifold of $(M_1 \times M_2, \widetilde{\omega})$.

Recipe for producing symplectomorphisms $M_1 = T^*X_1 \to M_2 = T^*X_2$:

1. Start with a lagrangian submanifold Y of $(M_1 \times M_2, \omega)$.
2. Twist it to obtain a lagrangian submanifold Y^σ of $(M_1 \times M_2, \widetilde{\omega})$.
3. Check whether Y^σ is the graph of some diffeomorphism $\varphi : M_1 \to M_2$.
4. If it is, then φ is a symplectomorphism (by Proposition 3.8).

Let $i : Y^\sigma \hookrightarrow M_1 \times M_2$ be the inclusion map

$$\begin{tikzcd} & Y^\sigma \arrow[dl, "\mathrm{pr}_1 \circ i"'] \arrow[dr, "\mathrm{pr}_2 \circ i"] & \\ M_1 \arrow[rr, "\varphi ?"] & & M_2 \end{tikzcd}$$

Step 3 amounts to checking whether $\mathrm{pr}_1 \circ i$ and $\mathrm{pr}_2 \circ i$ are diffeomorphisms. If yes, then $\varphi := (\mathrm{pr}_2 \circ i) \circ (\mathrm{pr}_1 \circ i)^{-1}$ is a diffeomorphism.

In order to obtain lagrangian submanifolds of $M_1 \times M_2 \simeq T^*(X_1 \times X_2)$, we can use the *method of generating functions*.

4.2 Method of Generating Functions

For any $f \in C^\infty(X_1 \times X_2)$, df is a closed 1-form on $X_1 \times X_2$. The **lagrangian submanifold generated by** f is

$$Y_f := \{((x,y),(df)_{(x,y)}) \mid (x,y) \in X_1 \times X_2\} \ .$$

We adopt the notation

$$\begin{array}{rcl} d_x f & := & (df)_{(x,y)} \text{ projected to } T_x^* X_1 \times \{0\}, \\ d_y f & := & (df)_{(x,y)} \text{ projected to } \{0\} \times T_y^* X_2 \ , \end{array}$$

which enables us to write

$$Y_f = \{(x,y,d_x f, d_y f) \mid (x,y) \in X_1 \times X_2\}$$

and

$$Y_f^\sigma = \{(x,y,d_x f, -d_y f) \mid (x,y) \in X_1 \times X_2\} \ .$$

When Y_f^σ is in fact the graph of a diffeomorphism $\varphi : M_1 \to M_2$, we call φ the **symplectomorphism generated by** f, and call f the **generating function**, of $\varphi : M_1 \to M_2$.

So when is Y_f^σ the graph of a diffeomorphism $\varphi : M_1 \to M_2$?

Let $(\mathcal{U}_1, x_1, \ldots, x_n), (\mathcal{U}_2, y_1, \ldots, y_n)$ be coordinate charts for X_1, X_2, with associated charts $(T^*\mathcal{U}_1, x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$, $(T^*\mathcal{U}_2, y_1, \ldots, y_n, \eta_1, \ldots, \eta_n)$ for M_1, M_2. The set Y_f^σ is the graph of $\varphi : M_1 \to M_2$ if and only if, for any $(x,\xi) \in M_1$ and $(y,\eta) \in M_2$, we have

$$\varphi(x,\xi) = (y,\eta) \iff \xi = d_x f \text{ and } \eta = -d_y f \ .$$

Therefore, given a point $(x,\xi) \in M_1$, to find its image $(y,\eta) = \varphi(x,\xi)$ we must solve the “Hamilton” equations

$$\left\{ \begin{array}{rcll} \xi_i & = & \dfrac{\partial f}{\partial x_i}(x,y) & \qquad (\star) \\ \eta_i & = & -\dfrac{\partial f}{\partial y_i}(x,y) \ . & \qquad (\star\star) \end{array} \right.$$

If there is a solution $y = \varphi_1(x,\xi)$ of $(\star)$, we may feed it to $(\star\star)$ thus obtaining $\eta = \varphi_2(x,\xi)$, so that $\varphi(x,\xi) = (\varphi_1(x,\xi), \varphi_2(x,\xi))$. Now by the implicit function theorem, in order to solve $(\star)$ locally for y in terms of x and ξ, we need the condition

$$\det \left[\frac{\partial}{\partial y_j} \left(\frac{\partial f}{\partial x_i} \right) \right]_{i,j=1}^n \neq 0 \ .$$

This is a necessary local condition for f to generate a symplectomorphism φ. Locally this is also sufficient, but globally there is the usual bijectivity issue.

Example. Let $X_1 = \mathcal{U}_1 \simeq \mathbb{R}^n$, $X_2 = \mathcal{U}_2 \simeq \mathbb{R}^n$, and $f(x,y) = -\frac{|x-y|^2}{2}$, the square of euclidean distance up to a constant.

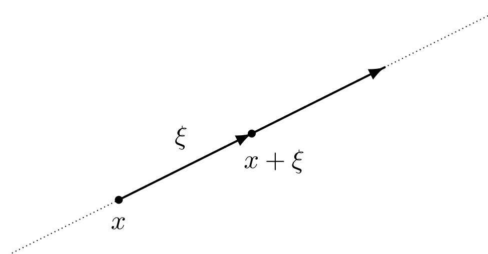
The "Hamilton" equations are

$$\begin{cases} \xi_i = \dfrac{\partial f}{\partial x_i} = y_i - x_i \\ \eta_i = -\dfrac{\partial f}{\partial y_i} = y_i - x_i \end{cases} \iff \begin{cases} y_i = x_i + \xi_i \\ \eta_i = \xi_i \ . \end{cases}$$

The symplectomorphism generated by f is

$$\varphi(x,\xi) = (x + \xi, \xi) \ .$$

If we use the euclidean inner product to identify $T^*\mathbb{R}^n$ with $T\mathbb{R}^n$, and hence regard φ as $\widetilde{\varphi} : T\mathbb{R}^n \to T\mathbb{R}^n$ and interpret ξ as the velocity vector, then the symplectomorphism φ corresponds to free translational motion in euclidean space.



$\diamondsuit$

4.3 Application to Geodesic Flow

Let V be an n-dimensional vector space. A **positive inner product** G on V is a bilinear map $G : V \times V \to \mathbb{R}$ which is

symmetric :	$G(v,w) = G(w,v)$,	and
positive-definite :	$G(v,v) > 0$	when $v \neq 0$.

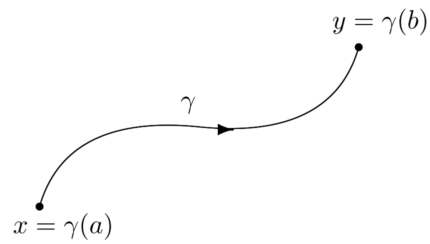
Definition 4.1 *A* **riemannian metric** *on a manifold X is a function g which assigns to each point $x \in X$ a positive inner product g_x on T_xX.*

A riemannian metric g is **smooth** *if for every smooth vector field $v : X \to TX$ the real-valued function $x \mapsto g_x(v_x, v_x)$ is a smooth function on X.*

Definition 4.2 *A* **riemannian manifold** (X, g) *is a manifold X equipped with a smooth riemannian metric g.*

The **arc-length** of a piecewise smooth curve $\gamma : [a,b] \to X$ on a riemannian manifold (X,g) is

$$\int_a^b \sqrt{g_{\gamma(t)}\left(\frac{d\gamma}{dt},\frac{d\gamma}{dt}\right)}dt \ .$$



Definition 4.3 *The* **riemannian distance** *between two points x and y of a connected riemannian manifold (X,g) is the infimum $d(x,y)$ of the set of all arc-lengths for piecewise smooth curves joining x to y.*

A smooth curve joining x to y is a **minimizing geodesic**[7] *if its arc-length is the riemannian distance $d(x,y)$.*

A riemannian manifold (X,g) is **geodesically convex** *if every point x is joined to every other point y by a unique minimizing geodesic.*

Example. On $X = \mathbb{R}^n$ with $TX \simeq \mathbb{R}^n \times \mathbb{R}^n$, let $g_x(v,w) = \langle v,w\rangle$, $g_x(v,v) = |v|^2$, where $\langle\cdot,\cdot\rangle$ is the euclidean inner product, and $|\cdot|$ is the euclidean norm. Then $(\mathbb{R}^n,\langle\cdot,\cdot\rangle)$ is a geodesically convex riemannian manifold, and the riemannian distance is the usual euclidean distance $d(x,y) = |x-y|$. $\diamondsuit$

Suppose that (X,g) is a geodesically convex riemannian manifold. Consider the function

$$f : X \times X \longrightarrow \mathbb{R} \ , \qquad f(x,y) = -\frac{d(x,y)^2}{2} \ .$$

What is the symplectomorphism $\varphi : T^*X \to T^*X$ generated by f?

The metric $g_x : T_xX \times T_xX \to \mathbb{R}$ induces an identification

$$\begin{array}{rccc} \widetilde{g}_x : & T_xX & \xrightarrow{\simeq} & T_x^*X \\ & v & \longmapsto & g_x(v,\cdot) \end{array}$$

Use $\widetilde{g}$ to translate φ into a map $\widetilde{\varphi} : TX \to TX$.

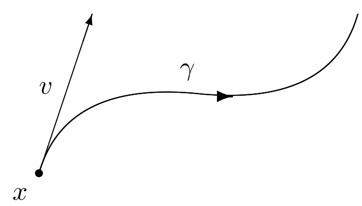
We need to solve

$$\left\{ \begin{array}{lclcr} \widetilde{g}_x(v) & = & \xi_i & = & d_xf(x,y) \\ \widetilde{g}_y(w) & = & \eta_i & = & -d_yf(x,y) \end{array} \right.$$

[7]In riemannian geometry, a **geodesic** is a curve which locally minimizes distance and whose velocity is constant.

for (y, η) in terms of (x, ξ) in order to find φ, or, equivalently, for (y, w) in terms (x, v) in order to find $\widetilde{\varphi}$.

Let γ be the geodesic with initial conditions $\gamma(0) = x$ and $\frac{d\gamma}{dt}(0) = v$.



Then the symplectomorphism φ corresponds to the map

$$\begin{array}{rccc} \widetilde{\varphi}: & TX & \longrightarrow & TX \\ & (x, v) & \longmapsto & (\gamma(1), \frac{d\gamma}{dt}(1)) \ . \end{array}$$

This is called the **geodesic flow** on X (see Homework 4).

Homework 4: Geodesic Flow

This set of problems is adapted from [53].

Let (X, g) be a riemannian manifold. The arc-length of a smooth curve $\gamma : [a, b] \to X$ is

$$\text{arc-length of } \gamma \ := \int_a^b \left|\frac{d\gamma}{dt}\right| dt \ , \qquad \text{where} \qquad \left|\frac{d\gamma}{dt}\right| \ := \ \sqrt{g_{\gamma(t)}\left(\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\right)} \ .$$

1. Show that the arc-length of γ is independent of the parametrization of γ, i.e., show that, if we reparametrize γ by $\tau : [a', b'] \to [a, b]$, the new curve $\gamma' = \gamma \circ \tau : [a', b'] \to X$ has the same arc-length. A curve γ is called a curve of *constant velocity* when $\left|\frac{d\gamma}{dt}\right|$ is independent of t. Show that, given any curve $\gamma : [a, b] \to X$ (with $\frac{d\gamma}{dt}$ never vanishing), there is a reparametrization $\tau : [a, b] \to [a, b]$ such that $\gamma \circ \tau : [a, b] \to X$ is of constant velocity.

2. Given a smooth curve $\gamma : [a, b] \to X$, the *action* of γ is $\mathcal{A}(\gamma) \ := \displaystyle\int_a^b \left|\frac{d\gamma}{dt}\right|^2 dt$. Show that, among all curves joining x to y, γ minimizes the action if and only if γ is of constant velocity and γ minimizes arc-length.

   Hint: Suppose that γ is of constant velocity, and let $\tau : [a, b] \to [a, b]$ be a reparametrization. Show that $\mathcal{A}(\gamma \circ \tau) \geq \mathcal{A}(\gamma)$, with equality only when $\tau = \text{identity}$.

3. Assume that (X, g) is geodesically convex, that is, any two points $x, y \in X$ are joined by a unique (up to reparametrization) minimizing geodesic; its arc-length $d(x, y)$ is called the riemannian distance between x and y.

   Assume also that (X, g) is *geodesically complete*, that is, every geodesic can be extended indefinitely. Given $(x, v) \in TX$, let $\exp(x, v) : \mathbb{R} \to X$ be the unique minimizing geodesic of constant velocity with initial conditions $\exp(x, v)(0) = x$ and $\frac{d \exp(x,v)}{dt}(0) = v$.

   Consider the function $f : X \times X \to \mathbb{R}$ given by $f(x, y) = -\frac{1}{2} \cdot d(x, y)^2$. Let $d_x f$ and $d_y f$ be the components of $df_{(x,y)}$ with respect to $T^*_{(x,y)}(X \times X) \simeq T^*_x X \times T^*_y X$. Recall that, if

   $$\Gamma^\sigma_f = \{(x, y, d_x f, -d_y f) \mid (x, y) \in X \times X\}$$

   is the graph of a diffeomorphism $f : T^*X \to T^*X$, then f is the symplectomorphism generated by f. In this case, $f(x, \xi) = (y, \eta)$ if and only if $\xi = d_x f$ and $\eta = -d_y f$.

   Show that, under the identification of TX with T^*X by g, the symplectomorphism generated by f coincides with the map $TX \to TX$, $(x, v) \mapsto \exp(x, v)(1)$.

Hint: The metric g provides the identifications $T_xXv \simeq \xi(\cdot) = g_x(v,\cdot) \in T^*_xX$. We need to show that, given $(x,v) \in TX$, the unique solution of $(\star)$ $\begin{cases} g_x(v,\cdot) = d_xf(\cdot) \\ g_y(w,\cdot) = -d_yf(\cdot) \end{cases}$ is $(y,w) = (\exp(x,v)(1), d\frac{\exp(x,v)}{dt}(1))$.
Look up the Gauss lemma in a book on riemannian geometry. It asserts that geodesics are orthogonal to the level sets of the distance function.
To solve the first line in $(\star)$ for y, evaluate both sides at $v = \frac{d\exp(x,v)}{dt}(0)$. Conclude that $y = \exp(x,v)(1)$. Check that $d_xf(v') = 0$ for vectors $v' \in T_xX$ orthogonal to v (that is, $g_x(v,v') = 0$); this is a consequence of $f(x,y)$ being the arc-length of a *minimizing* geodesic, and it suffices to check locally.
The vector w is obtained from the second line of $(\star)$. Compute $-d_yf(\frac{d\exp(x,v)}{dt}(1))$. Then evaluate $-d_yf$ at vectors $w' \in T_yX$ orthogonal to $\frac{d\exp(x,v)}{dt}(1)$; this pairing is again 0 because $f(x,y)$ is the arc-length of a minimizing geodesic. Conclude, using the nondegeneracy of g, that $w = \frac{d\exp(x,v)}{dt}(1)$.
For both steps, it might be useful to recall that, given a function $\varphi : X \to \mathbb{R}$ and a tangent vector $v \in T_xX$, we have $d\varphi_x(v) = \dfrac{d}{du}\left[\varphi(\exp(x,v)(u))\right]_{u=0}$.

5 Recurrence

5.1 Periodic Points

Let X be an n-dimensional manifold. Let $M = T^*X$ be its cotangent bundle with canonical symplectic form ω.

Suppose that we are given a smooth function $f : X \times X \to \mathbb{R}$ which generates a symplectomorphism $\varphi : M \to M$, $\varphi(x, d_x f) = (y, -d_y f)$, by the recipe of the previous lecture.

What are the fixed points of φ?

Define $\psi : X \to \mathbb{R}$ by $\psi(x) = f(x, x)$.

Proposition 5.1 *There is a one-to-one correspondence between the fixed points of φ and the critical points of ψ.*

Proof. At $x_0 \in X$, $d_{x_0}\psi = (d_x f + d_y f)|_{(x,y)=(x_0,x_0)}$. Let $\xi = d_x f|_{(x,y)=(x_0,x_0)}$.

$$x_0 \text{ is a critical point of } \psi \iff d_{x_0}\psi = 0 \iff d_y f|_{(x,y)=(x_0,x_0)} = -\xi \ .$$

Hence, the point in Γ_f^σ corresponding to $(x, y) = (x_0, x_0)$ is (x_0, x_0, ξ, ξ). But Γ_f^σ is the graph of φ, so $\varphi(x_0, \xi) = (x_0, \xi)$ is a fixed point. This argument also works backwards. $\square$

Consider the iterates of φ,

$$\varphi^{(N)} = \underbrace{\varphi \circ \varphi \circ \ldots \circ \varphi}_{N} : M \longrightarrow M \ , \qquad\qquad N = 1, 2, \ldots \ ,$$

each of which is a symplectomorphism of M. According to the previous proposition, if $\varphi^{(N)} : M \to M$ is generated by $f^{(N)}$, then there is a correspondence

$$\left\{ \text{fixed points of } \varphi^{(N)} \right\} \stackrel{1-1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{critical points of} \\ \psi^{(N)} : X \to \mathbb{R} \ , \ \psi^{(N)}(x) = f^{(N)}(x, x) \end{array} \right\}$$

Knowing that φ is generated by f, does $\varphi^{(2)}$ have a generating function? The answer is a partial yes:

Fix $x, y \in X$. Define a map

$$\begin{array}{rcl} X & \longrightarrow & \mathbb{R} \\ z & \longmapsto & f(x, z) + f(z, y) \ . \end{array}$$

Suppose that this map has a unique critical point z_0, and that z_0 is nondegenerate. Let

$$f^{(2)}(x, y) := f(x, z_0) + f(z_0, y) \ .$$

Proposition 5.2 *The function $f^{(2)} : X \times X \to \mathbb{R}$ is smooth and is a generating function for $\varphi^{(2)}$ if we assume that, for each $\xi \in T_x^* X$, there is a unique $y \in X$ for which $d_x f^{(2)} = \xi$.*

Proof. The point z_0 is given implicitly by $d_y f(x, z_0) + d_x f(z_0, y) = 0$. The nondegeneracy condition is

$$\det\left[\frac{\partial}{\partial z_i}\left(\frac{\partial f}{\partial y_j}(x,z) + \frac{\partial f}{\partial x_j}(z,y)\right)\right] \neq 0 \ .$$

By the implicit function theorem, $z_0 = z_0(x,y)$ is smooth.

As for the second assertion, $f^{(2)}(x,y)$ is a generating function for $\varphi^{(2)}$ if and only if

$$\varphi^{(2)}(x, d_x f^{(2)}) = (y, -d_y f^{(2)})$$

(assuming that, for each $\xi \in T_x^* X$, there is a unique $y \in X$ for which $d_x f^{(2)} = \xi$). Since φ is generated by f, and z_0 is critical, we obtain

$$\begin{aligned}
\varphi^{(2)}(x, d_x f^{(2)}(x,y)) &= \varphi(\varphi(x, \underbrace{d_x f^{(2)}(x,y)}_{=d_x f(x,z_0)}) &&= \varphi(z_0, -d_y f(x, z_0)) \\
&= \varphi(z_0, d_x f(z_0, y)) &&= (y, \underbrace{-d_y f(z_0,y)}_{=-d_y f^{(2)}(x,y)}) \ .
\end{aligned}$$

□

Exercise. What is a generating function for $\varphi^{(3)}$?

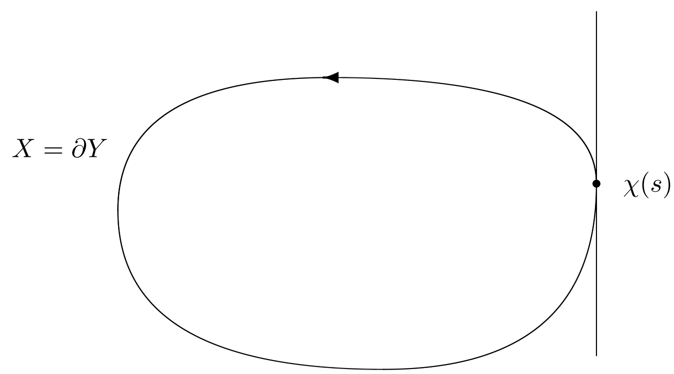
Hint: Suppose that the function

$$\begin{array}{rcl} X \times X & \longrightarrow & \mathbb{R} \\ (z,u) & \longmapsto & f(x,z) + f(z,u) + f(u,y) \end{array}$$

has a unique critical point (z_0, u_0), and that it is a nondegenerate critical point. Let $f^{(3)}(x,y) = f(x,z_0) + f(z_0,u_0) + f(u_0,y)$. ♢

5.2 Billiards

Let $\chi : \mathbb{R} \to \mathbb{R}^2$ be a smooth plane curve which is 1-periodic, i.e., $\chi(s+1) = \chi(s)$, and parametrized by arc-length, i.e., $\left|\frac{d\chi}{ds}\right| = 1$. Assume that the region Y enclosed by χ is *convex*, i.e., for any $s \in \mathbb{R}$, the tangent line $\{\chi(s) + t\frac{d\chi}{ds} \mid t \in \mathbb{R}\}$ intersects $X := \partial Y$ (= the image of χ) at only the point $\chi(s)$.

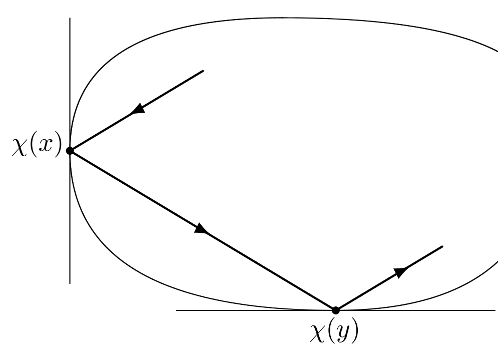


Suppose that we throw a ball into Y rolling with constant velocity and bouncing off the boundary with the usual law of reflection. This determines a map

$$\begin{array}{rrcl} \varphi : & \mathbb{R}/\mathbb{Z} \times (-1,1) & \longrightarrow & \mathbb{R}/\mathbb{Z} \times (-1,1) \\ & (x,v) & \longmapsto & (y,w) \end{array}$$

by the rule

when the ball bounces off $\chi(x)$ with angle $\theta = \arccos v$, it will next collide with $\chi(y)$ and bounce off with angle $\nu = \arccos w$.



Let $f : \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ be defined by $f(x,y) = -|\chi(x) - \chi(y)|$; f is smooth off the diagonal. Use χ to identify $\mathbb{R}/\mathbb{Z}$ with the image curve X.

Suppose that $\varphi(x,v) = (y,w)$, i.e., (x,v) and (y,w) are successive points on the orbit described by the ball. Then

$$\begin{cases} \dfrac{df}{dx} = -\dfrac{x-y}{|x-y|} \text{ projected onto } T_xX = v \\[2ex] \dfrac{df}{dy} = -\dfrac{y-x}{|x-y|} \text{ projected onto } T_yX = -w \end{cases}$$

or, equivalently,

$$\begin{cases} \dfrac{d}{ds}f(\chi(s),y) = \dfrac{y-x}{|x-y|}\cdot\dfrac{d\chi}{ds} = \cos\theta = v \\[2ex] \dfrac{d}{ds}f(x,\chi(s)) = \dfrac{x-y}{|x-y|}\cdot\dfrac{d\chi}{ds} = -\cos\nu = -w\ . \end{cases}$$

We conclude that f is a generating function for φ. Similar approaches work for higher dimensional billiards problems.

Periodic points are obtained by finding critical points of

$$\begin{array}{rcll} \underbrace{X\times\ldots\times X}_{N} & \longrightarrow & \mathbb{R}\ , & N>1 \\ (x_1,\ldots,x_N) & \longmapsto & f(x_1,x_2)+f(x_2,x_3)+\ldots+f(x_{N-1},x_N)+f(x_N,x_1) \\ & & =|x_1-x_2|+\ldots+|x_{N-1}-x_N|+|x_N-x_1|\ , \end{array}$$

that is, by finding the N-sided (generalized) polygons inscribed in X of critical perimeter.

Notice that

$$\mathbb{R}/\mathbb{Z}\times(-1,1)\simeq\{(x,v)\mid x\in X, v\in T_xX, |v|<1\}\simeq A$$

is the open unit tangent ball bundle of a circle X, that is, an open annulus A. The map $\varphi : A\to A$ is area-preserving.

5.3 Poincaré Recurrence

Theorem 5.3 (Poincaré Recurrence Theorem) *Suppose that $\varphi : A\to A$ is an area-preserving diffeomorphism of a finite-area manifold A. Let $p\in A$, and let $\mathcal{U}$ be a neighborhood of p. Then there is $q\in\mathcal{U}$ and a positive integer N such that $\varphi^{(N)}(q)\in\mathcal{U}$.*

Proof. Let $\mathcal{U}_0=\mathcal{U},\mathcal{U}_1=\varphi(\mathcal{U}),\mathcal{U}_2=\varphi^{(2)}(\mathcal{U}),\ldots$. If all of these sets were disjoint, then, since Area $(\mathcal{U}_i)=$ Area $(\mathcal{U})>0$ for all i, we would have

$$\text{Area } A \geq \text{ Area } (\mathcal{U}_0\cup\mathcal{U}_1\cup\mathcal{U}_2\cup\ldots)=\sum_i \text{ Area } (\mathcal{U}_i)=\infty\ .$$

To avoid this contradiction we must have $\varphi^{(k)}(\mathcal{U}) \cap \varphi^{(l)}(\mathcal{U}) \neq \emptyset$ for some $k > l$, which implies $\varphi^{(k-l)}(\mathcal{U}) \cap \mathcal{U} \neq \emptyset$. $\square$

Hence, eternal return applies to billiards...

Remark. Theorem 5.3 clearly generalizes to volume-preserving diffeomorphisms in higher dimensions. $\diamondsuit$

Theorem 5.4 (Poincaré's Last Geometric Theorem) *Suppose $\varphi : A \to A$ is an area-preserving diffeomorphism of the closed annulus $A = \mathbb{R}/\mathbb{Z} \times [-1, 1]$ which preserves the two components of the boundary, and twists them in opposite directions. Then φ has at least two fixed points.*

This theorem was proved in 1913 by Birkhoff, and hence is also called the **Poincaré-Birkhoff theorem**. It has important applications to dynamical systems and celestial mechanics. The Arnold conjecture (1966) on the existence of fixed points for symplectomorphisms of compact manifolds (see Lecture 9) may be regarded as a generalization of the Poincaré-Birkhoff theorem. This conjecture has motivated a significant amount of recent research involving a more general notion of generating function; see, for instance, [34, 45].

Part III
Local Forms

Inspired by the elementary normal form in symplectic linear algebra (Theorem 1.1), we will go on to describe normal neighborhoods of a point (the Darboux theorem) and of a lagrangian submanifold (the Weinstein theorems), inside a symplectic manifold. The main tool is the Moser trick, explained in Lecture 7, which leads to the crucial Moser theorems and which is at the heart of many arguments in symplectic geometry.

In order to prove the normal forms, we need the (non-symplectic) ingredients discussed in Lecture 6; for more on these topics, see, for instance, [18, 55, 96].

6 Preparation for the Local Theory

6.1 Isotopies and Vector Fields

Let M be a manifold, and $\rho : M \times \mathbb{R} \to M$ a map, where we set $\rho_t(p) := \rho(p,t)$.

Definition 6.1 *The map ρ is an* **isotopy** *if each $\rho_t : M \to M$ is a diffeomorphism, and $\rho_0 = \mathrm{id}_M$.*

Given an isotopy ρ, we obtain a **time-dependent vector field**, that is, a family of vector fields v_t, $t \in \mathbb{R}$, which at $p \in M$ satisfy

$$v_t(p) = \left. \frac{d}{ds}\rho_s(q)\right|_{s=t} \qquad \text{where} \qquad q = \rho_t^{-1}(p) \ ,$$

i.e.,

$$\frac{d\rho_t}{dt} = v_t \circ \rho_t \ .$$

Conversely, given a time-dependent vector field v_t, if M is compact or if the v_t's are compactly supported, there exists an isotopy ρ satisfying the previous ordinary differential equation.

Suppose that M is compact. Then we have a one-to-one correspondence

$$\begin{array}{ccc} \{\text{isotopies of } M\} & \stackrel{1-1}{\longleftrightarrow} & \{\text{time-dependent vector fields on } M\} \\ \rho_t,\ t \in \mathbb{R} & \longleftrightarrow & v_t,\ t \in \mathbb{R} \end{array}$$

Definition 6.2 *When $v_t = v$ is independent of t, the associated isotopy is called the* **exponential map** *or the* **flow** *of v and is denoted $\exp tv$; i.e., $\{\exp tv : M \to M \mid t \in \mathbb{R}\}$ is the unique smooth family of diffeomorphisms satisfying*

$$\exp tv|_{t=0} = \mathrm{id}_M \quad \text{and} \quad \frac{d}{dt}(\exp tv)(p) = v(\exp tv(p)) \ .$$

Definition 6.3 *The* **Lie derivative** *is the operator*

$$\mathcal{L}_v : \Omega^k(M) \longrightarrow \Omega^k(M) \qquad \textit{defined by} \qquad \mathcal{L}_v\omega := \frac{d}{dt}(\exp tv)^*\omega|_{t=0} \ .$$

When a vector field v_t is time-dependent, its flow, that is, the corresponding isotopy ρ, still locally exists by Picard's theorem. More precisely, in the neighborhood of any point p and for sufficiently small time t, there is a one-parameter family of local diffeomorphisms ρ_t satisfying

$$\frac{d\rho_t}{dt} = v_t \circ \rho_t \qquad \text{and} \qquad \rho_0 = \mathrm{id} \ .$$

Hence, we say that the **Lie derivative** by v_t is

$$\mathcal{L}_{v_t} : \Omega^k(M) \longrightarrow \Omega^k(M) \qquad \text{defined by} \qquad \mathcal{L}_{v_t}\omega := \frac{d}{dt}(\rho_t)^*\omega|_{t=0} \ .$$

Exercise. Prove the **Cartan magic formula**,

$$\mathcal{L}_v\omega = \imath_v d\omega + d\imath_v\omega \ ,$$

and the formula

$$\frac{d}{dt}\rho_t^*\omega = \rho_t^*\mathcal{L}_{v_t}\omega \ , \qquad (\star)$$

where ρ is the (local) isotopy generated by v_t. A good strategy for each formula is to follow the steps:

1. Check the formula for 0-forms $\omega \in \Omega^0(M) = C^\infty(M)$.
2. Check that both sides commute with d.
3. Check that both sides are derivations of the algebra $(\Omega^*(M), \wedge)$. For instance, check that
$$\mathcal{L}_v(\omega \wedge \alpha) = (\mathcal{L}_v\omega) \wedge \alpha + \omega \wedge (\mathcal{L}_v\alpha) \ .$$
4. Notice that, if $\mathcal{U}$ is the domain of a coordinate system, then $\Omega^\bullet(\mathcal{U})$ is generated as an algebra by $\Omega^0(\mathcal{U})$ and $d\Omega^0(\mathcal{U})$, i.e., every element in $\Omega^\bullet(\mathcal{U})$ is a linear combination of wedge products of elements in $\Omega^0(\mathcal{U})$ and elements in $d\Omega^0(\mathcal{U})$.

$\diamondsuit$

We will need the following improved version of formula $(\star)$.

Proposition 6.4 *For a smooth family ω_t, $t \in \mathbb{R}$, of d-forms, we have*

$$\frac{d}{dt}\rho_t^*\omega_t = \rho_t^*\left(\mathcal{L}_{v_t}\omega_t + \frac{d\omega_t}{dt}\right) \ .$$

Proof. If $f(x,y)$ is a real function of two variables, by the chain rule we have

$$\frac{d}{dt}f(t,t) = \left.\frac{d}{dx}f(x,t)\right|_{x=t} + \left.\frac{d}{dy}f(t,y)\right|_{y=t} .$$

Therefore,

$$\begin{aligned}\frac{d}{dt}\rho_t^*\omega_t &= \underbrace{\left.\frac{d}{dx}\rho_x^*\omega_t\right|_{x=t}}_{\left.\rho_x^*\mathcal{L}_{v_x}\omega_t\right|_{x=t} \text{ by } (\star)} + \underbrace{\left.\frac{d}{dy}\rho_t^*\omega_y\right|_{y=t}}_{\left.\rho_t^*\frac{d\omega_y}{dy}\right|_{y=t}} \\ &= \rho_t^*\left(\mathcal{L}_{v_t}\omega_t + \frac{d\omega_t}{dt}\right) .\end{aligned}$$

$\square$

6.2 Tubular Neighborhood Theorem

Let M be an n-dimensional manifold, and let X be a k-dimensional submanifold where $k < n$ and with inclusion map

$$i : X \hookrightarrow M .$$

At each $x \in X$, the tangent space to X is viewed as a subspace of the tangent space to M via the linear inclusion $di_x : T_xX \hookrightarrow T_xM$, where we denote $x = i(x)$. The quotient $N_xX := T_xM/T_xX$ is an $(n-k)$-dimensional vector space, known as the **normal space** to X at x. The **normal bundle** of X is

$$NX = \{(x,v) \mid x \in X \ , \ v \in N_xX\} .$$

The set NX has the structure of a vector bundle over X of rank $n-k$ under the natural projection, hence as a manifold NX is n-dimensional. The zero section of NX,

$$i_0 : X \hookrightarrow NX \ , \qquad x \mapsto (x,0) \ ,$$

embeds X as a closed submanifold of NX. A neighborhood $\mathcal{U}_0$ of the zero section X in NX is called **convex** if the intersection $\mathcal{U}_0 \cap N_xX$ with each fiber is convex.

Theorem 6.5 (Tubular Neighborhood Theorem) *There exist a convex neighborhood $\mathcal{U}_0$ of X in NX, a neighborhood $\mathcal{U}$ of X in M, and a diffeomorphism $\varphi : \mathcal{U}_0 \to \mathcal{U}$ such that*

$$\begin{tikzcd} NX \supseteq \mathcal{U}_0 \arrow[rr, "\varphi", "\simeq"'] & & \mathcal{U} \subseteq M \\ & X \arrow[ul, "i_0"] \arrow[ur, "i"'] & \end{tikzcd}$$

commutes.

Outline of the proof.

- *Case of $M = \mathbb{R}^n$, and X is a compact submanifold of $\mathbb{R}^n$.*

Theorem 6.6 (ε-Neighborhood Theorem)
Let $\mathcal{U}^\varepsilon = \{p \in \mathbb{R}^n : \ |p-q| < \varepsilon$ for some $q \in X\}$ be the set of points at a distance less than ε from X. Then, for ε sufficiently small, each $p \in \mathcal{U}^\varepsilon$ has a unique nearest point $q \in X$ (i.e., a unique $q \in X$ minimizing $|q-x|$).

Moreover, setting $q = \pi(p)$, the map $\mathcal{U}^\varepsilon \xrightarrow{\pi} X$ is a (smooth) submersion with the property that, for all $p \in \mathcal{U}^\varepsilon$, the line segment $(1-t)p + tq$, $0 \le t \le 1$, is in $\mathcal{U}^\varepsilon$.

The proof is part of Homework 5. Here are some hints.

At any $x \in X$, the *normal* space $N_x X$ may be regarded as an $(n-k)$-dimensional subspace of $\mathbb{R}^n$, namely the orthogonal complement in $\mathbb{R}^n$ of the tangent space to X at x:

$$N_x X \simeq \{v \in \mathbb{R}^n : \ v \perp w \ , \text{ for all } w \in T_x X\} \ .$$

We define the following open neighborhood of X in NX:

$$NX^\varepsilon = \{(x,v) \in NX : \ |v| < \varepsilon\} \ .$$

Let

$$\begin{array}{rccc} \exp : & NX & \longrightarrow & \mathbb{R}^n \\ & (x,v) & \longmapsto & x+v \ . \end{array}$$

Restricted to the zero section, $\exp$ is the identity map on X.

Prove that, for ε sufficiently small, $\exp$ maps NX^ε diffeomorphically onto $\mathcal{U}^\varepsilon$, and show also that the diagram

$$\begin{tikzcd} NX^\varepsilon \arrow[rr, "\exp"] \arrow[dr, "\pi_0"'] & & \mathcal{U}^\varepsilon \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$$

commutes.

- *Case where X is a compact submanifold of an arbitrary manifold M.*

Put a riemannian metric g on M, and let $d(p,q)$ be the riemannian distance between $p, q \in M$. The ε-neighborhood of a compact submanifold X is

$$\mathcal{U}^\varepsilon = \{p \in M \mid d(p,q) < \varepsilon \text{ for some } q \in X\} \ .$$

Prove the ε-neighborhood theorem in this setting: for ε small enough, the following assertions hold.

– Any $p \in \mathcal{U}^\varepsilon$ has a unique point $q \in X$ with minimal $d(p,q)$. Set $q = \pi(p)$.

– The map $\mathcal{U}^\varepsilon \xrightarrow{\pi} X$ is a submersion and, for all $p \in \mathcal{U}^\varepsilon$, there is a unique geodesic curve γ joining p to $q = \pi(p)$.

– The normal space to X at $x \in X$ is naturally identified with a subspace of T_xM:

$$N_xX \simeq \{v \in T_xM \mid g_x(v,w) = 0 \ , \text{ for any } w \in T_xX\} \ .$$

Let $NX^\varepsilon = \{(x,v) \in NX \mid \sqrt{g_x(v,v)} < \varepsilon\}$.

– Define $\exp : NX^\varepsilon \to M$ by $\exp(x,v) = \gamma(1)$, where $\gamma : [0,1] \to M$ is the geodesic with $\gamma(0) = x$ and $\frac{d\gamma}{dt}(0) = v$. Then $\exp$ maps NX^ε diffeomorphically to $\mathcal{U}^\varepsilon$.

- *General case.*

  When X is not compact, adapt the previous argument by replacing ε by an appropriate continuous function $\varepsilon : X \to \mathbb{R}^+$ which tends to zero fast enough as x tends to infinity.

$\square$

Restricting to the subset $\mathcal{U}^0 \subseteq NX$ from the tubular neighborhood theorem, we obtain a submersion $\mathcal{U}_0 \xrightarrow{\pi_0} X$ with all fibers $\pi_0^{-1}(x)$ convex. We can carry this fibration to $\mathcal{U}$ by setting $\pi = \pi_0 \circ \varphi^{-1}$:

$$\begin{array}{lll} \mathcal{U}_0 & \subseteq NX & \text{is a fibration} \\ \pi_0 \downarrow & & \\ X & & \end{array} \quad \Longrightarrow \quad \begin{array}{lll} \mathcal{U} & \subseteq M & \text{is a fibration} \\ \pi \downarrow & & \\ X & & \end{array}$$

This is called the **tubular neighborhood fibration**.

6.3 Homotopy Formula

Let $\mathcal{U}$ be a tubular neighborhood of a submanifold X in M. The restriction $i^* : H^d_{\text{deRham}}(\mathcal{U}) \to H^d_{\text{deRham}}(X)$ by the inclusion map is surjective. As a corollary of the tubular neighborhood fibration, i^* is also injective: this follows from the homotopy-invariance of de Rham cohomology.

Corollary 6.7 *For any degree ℓ, $H^\ell_{\text{deRham}}(\mathcal{U}) \simeq H^\ell_{\text{deRham}}(X)$.*

At the level of forms, this means that, if ω is a closed ℓ-form on $\mathcal{U}$ and $i^*\omega$ is exact on X, then ω is exact. We will need the following related result.

Proposition 6.8 *If a closed ℓ-form ω on $\mathcal{U}$ has restriction $i^*\omega = 0$, then ω is exact, i.e., $\omega = d\mu$ for some $\mu \in \Omega^{d-1}(\mathcal{U})$. Moreover, we can choose μ such that $\mu_x = 0$ at all $x \in X$.*

Proof. Via $\varphi : \mathcal{U}_0 \xrightarrow{\simeq} \mathcal{U}$, it is equivalent to work over $\mathcal{U}_0$. Define for every $0 \le t \le 1$ a map

$$\begin{array}{rccl} \rho_t : & \mathcal{U}_0 & \longrightarrow & \mathcal{U}_0 \\ & (x,v) & \longmapsto & (x,tv) \ . \end{array}$$

This is well-defined since $\mathcal{U}_0$ is convex. The map ρ_1 is the identity, $\rho_0 = i_0 \circ \pi_0$, and each ρ_t fixes X, that is, $\rho_t \circ i_0 = i_0$. We hence say that the family $\{\rho_t \mid 0 \le t \le 1\}$ is a **homotopy** from $i_0 \circ \pi_0$ to the identity fixing X. The map $\pi_0 : \mathcal{U}_0 \to X$ is called a **retraction** because $\pi_0 \circ i_0$ is the identity. The submanifold X is then called a **deformation retract** of $\mathcal{U}$.

A (de Rham) **homotopy operator** between $\rho_0 = i_0 \circ \pi_0$ and $\rho_1 = \mathrm{id}$ is a linear map

$$Q : \Omega^d(\mathcal{U}_0) \longrightarrow \Omega^{d-1}(\mathcal{U}_0)$$

satisfying the **homotopy formula**

$$\mathrm{Id} - (i_0 \circ \pi_0)^* = dQ + Qd \ .$$

When $d\omega = 0$ and $i_0^*\omega = 0$, the operator Q gives $\omega = dQ\omega$, so that we can take $\mu = Q\omega$. A concrete operator Q is given by the formula:

$$Q\omega = \int_0^1 \rho_t^*(\imath_{v_t}\omega)\ dt \ ,$$

where v_t, at the point $q = \rho_t(p)$, is the vector tangent to the curve $\rho_s(p)$ at $s = t$. The proof that Q satisfies the homotopy formula is below.

In our case, for $x \in X$, $\rho_t(x) = x$ (all t) is the constant curve, so v_t vanishes at all x for all t, hence $\mu_x = 0$. $\square$

To check that Q above satisfies the homotopy formula, we compute

$$\begin{aligned} Qd\omega + dQ\omega &= \int_0^1 \rho_t^*(\imath_{v_t} d\omega)dt + d\int_0^1 \rho_t^*(\imath_{v_t}\omega)dt \\ &= \int_0^1 \rho_t^*(\underbrace{\imath_{v_t} d\omega + d\imath_{v_t}\omega}_{\mathcal{L}_{v_t}\omega})dt \ , \end{aligned}$$

where $\mathcal{L}_v$ denotes the Lie derivative along v (reviewed in the next section), and we used the Cartan magic formula: $\mathcal{L}_v\omega = \imath_v d\omega + d\imath_v\omega$. The result now follows from

$$\frac{d}{dt}\rho_t^*\omega = \rho_t^*\mathcal{L}_{v_t}\omega$$

and from the fundamental theorem of calculus:

$$Qd\omega + dQ\omega = \int_0^1 \frac{d}{dt}\rho_t^*\omega\ dt = \rho_1^*\omega - \rho_0^*\omega \ .$$

Homework 5: Tubular Neighborhoods in $\mathbb{R}^n$

1. Let X be a k-dimensional submanifold of an n-dimensional manifold M. Let x be a point in X. The **normal space** to X at x is the quotient space

$$N_x X = T_x M / T_x X \ ,$$

and the **normal bundle** of X in M is the vector bundle NX over X whose fiber at x is $N_x X$.

   (a) Prove that NX is indeed a vector bundle.

   (b) If M is $\mathbb{R}^n$, show that $N_x X$ can be identified with the usual "normal space" to X in $\mathbb{R}^n$, that is, the orthogonal complement in $\mathbb{R}^n$ of the tangent space to X at x.

2. Let X be a k-dimensional compact submanifold of $\mathbb{R}^n$. Prove the **tubular neighborhood theorem** in the following form.

   (a) Given $\varepsilon > 0$ let $\mathcal{U}_\varepsilon$ be the set of all points in $\mathbb{R}^n$ which are at a distance less than ε from X. Show that, for ε sufficiently small, every point $p \in \mathcal{U}_\varepsilon$ has a *unique* nearest point $\pi(p) \in X$.

   (b) Let $\pi : \mathcal{U}_\varepsilon \to X$ be the map defined in (a) for ε sufficiently small. Show that, if $p \in \mathcal{U}_\varepsilon$, then the line segment $(1-t) \cdot p + t \cdot \pi(p)$, $0 \leq t \leq 1$, joining p to $\pi(p)$ lies in $\mathcal{U}_\varepsilon$.

   (c) Let $NX_\varepsilon = \{(x, v) \in NX \text{ such that } |v| < \varepsilon\}$. Let $\exp : NX \to \mathbb{R}^n$ be the map $(x, v) \mapsto x + v$, and let $\nu : NX_\varepsilon \to X$ be the map $(x, v) \mapsto x$. Show that, for ε sufficiently small, $\exp$ maps NX_ε diffeomorphically onto $\mathcal{U}_\varepsilon$, and show also that the following diagram commutes:

$$\begin{tikzcd}
NX_\varepsilon \arrow[rr, "\exp"] \arrow[dr, "\nu"'] & & \mathcal{U}_\varepsilon \arrow[dl, "\pi"] \\
& X &
\end{tikzcd}$$

3. Suppose that the manifold X in the previous exercise is not compact.

   Prove that the assertion about $\exp$ is still true provided we replace ε by a continuous function

$$\varepsilon : X \to \mathbb{R}^+$$

   which tends to zero fast enough as x tends to infinity.

7 Moser Theorems

7.1 Notions of Equivalence for Symplectic Structures

Let M be a $2n$-dimensional manifold with two symplectic forms ω_0 and ω_1, so that (M, ω_0) and (M, ω_1) are two symplectic manifolds.

Definition 7.1 *We say that*

- (M, ω_0) *and* (M, ω_1) *are* **symplectomorphic** *if there is a diffeomorphism* $\varphi : M \to M$ *with* $\varphi^*\omega_1 = \omega_0$*;*
- (M, ω_0) *and* (M, ω_1) *are* **strongly isotopic** *if there is an isotopy* $\rho_t : M \to M$ *such that* $\rho_1^*\omega_1 = \omega_0$*;*
- (M, ω_0) *and* (M, ω_1) *are* **deformation-equivalent** *if there is a smooth family* ω_t *of symplectic forms joining* ω_0 *to* ω_1*;*
- (M, ω_0) *and* (M, ω_1) *are* **isotopic** *if they are deformation-equivalent with* $[\omega_t]$ *independent of* t*.*

Clearly, we have

$$\text{strongly isotopic} \quad \Longrightarrow \quad \text{symplectomorphic}\ , \qquad \text{and}$$

$$\text{isotopic} \quad \Longrightarrow \quad \text{deformation-equivalent}\ .$$

We also have

$$\text{strongly isotopic} \quad \Longrightarrow \quad \text{isotopic}$$

because, if $\rho_t : M \to M$ is an isotopy such that $\rho_1^*\omega_1 = \omega_0$, then the set $\omega_t := \rho_t^*\omega_1$ is a smooth family of symplectic forms joining ω_1 to ω_0 and $[\omega_t] = [\omega_1]$, $\forall t$, by the homotopy invariance of de Rham cohomology. As we will see below, the Moser theorem states that, on a compact manifold,

$$\text{isotopic} \quad \Longrightarrow \quad \text{strongly isotopic}\ .$$

7.2 Moser Trick

Problem. Given a $2n$-dimensional manifold M, a k-dimensional submanifold X, neighborhoods $\mathcal{U}_0, \mathcal{U}_1$ of X, and symplectic forms ω_0, ω_1 on $\mathcal{U}_0, \mathcal{U}_1$, does there exist a symplectomorphism preserving X? More precisely, does there exist a diffeomorphism $\varphi : \mathcal{U}_0 \to \mathcal{U}_1$ with $\varphi^*\omega_1 = \omega_0$ and $\varphi(X) = X$?

At the two extremes, we have:
Case $X =$ *point*: Darboux theorem – see Lecture 8.
Case $X = M$: Moser theorem – discussed here:

Let M be a *compact* manifold with symplectic forms ω_0 and ω_1.

– Are (M, ω_0) and (M, ω_1) symplectomorphic?
I.e., does there exist a diffeomorphism $\varphi : M \to M$ such that $\varphi_1^* \omega_0 = \omega_1$?

Moser asked whether we can find such an φ which is homotopic to id_M. A necessary condition is $[\omega_0] = [\omega_1] \in H^2(M; \mathbb{R})$ because: if $\varphi \sim \mathrm{id}_M$, then, by the homotopy formula, there exists a homotopy operator Q such that

$$
\begin{array}{rl}
 & \mathrm{id}_M^* \omega_1 - \varphi^* \omega_1 = dQ\omega_1 + Q \underbrace{d\omega_1}_{0} \\
\Longrightarrow & \omega_1 = \varphi^* \omega_1 + d(Q\omega_1) \\
\Longrightarrow & [\omega_1] = [\varphi^* \omega_1] = [\omega_0] \ .
\end{array}
$$

– If $[\omega_0] = [\omega_1]$, does there exist a diffeomorphism φ homotopic to id_M such that $\varphi^* \omega_1 = \omega_0$?

Moser [87] proved that the answer is yes, with a further hypothesis as in Theorem 7.2. McDuff showed that, in general, the answer is no; for a counterexample, see Example 7.23 in [83].

Theorem 7.2 (Moser Theorem – Version I) *Suppose that M is compact, $[\omega_0] = [\omega_1]$ and that the 2-form $\omega_t = (1-t)\omega_0 + t\omega_1$ is symplectic for each $t \in [0,1]$. Then there exists an isotopy $\rho : M \times \mathbb{R} \to M$ such that $\rho_t^* \omega_t = \omega_0$ for all $t \in [0,1]$.*

In particular, $\varphi = \rho_1 : M \xrightarrow{\simeq} M$, satisfies $\varphi^* \omega_1 = \omega_0$.

The following argument, due to Moser, is extremely useful; it is known as the **Moser trick**.

Proof. Suppose that there exists an isotopy $\rho : M \times \mathbb{R} \to M$ such that $\rho_t^* \omega_t = \omega_0$, $0 \le t \le 1$. Let

$$
v_t = \frac{d\rho_t}{dt} \circ \rho_t^{-1} \ , \qquad t \in \mathbb{R} \ .
$$

Then

$$
\begin{array}{rl}
 & 0 = \frac{d}{dt}(\rho_t^* \omega_t) = \rho_t^* \big(\mathcal{L}_{v_t} \omega_t + \frac{d\omega_t}{dt} \big) \\
\Longleftrightarrow & \mathcal{L}_{v_t} \omega_t + \frac{d\omega_t}{dt} = 0 \ . \qquad (\star)
\end{array}
$$

Suppose conversely that we can find a smooth time-dependent vector field v_t, $t \in \mathbb{R}$, such that $(\star)$ holds for $0 \le t \le 1$. Since M is compact, we can integrate v_t to an isotopy $\rho : M \times \mathbb{R} \to M$ with

$$
\frac{d}{dt}(\rho_t^* \omega_t) = 0 \quad \Longrightarrow \quad \rho_t^* \omega_t = \rho_0^* \omega_0 = \omega_0 \ .
$$

So everything boils down to solving $(\star)$ for v_t.

First, from $\omega_t = (1-t)\omega_0 + t\omega_1$, we conclude that

$$
\frac{d\omega_t}{dt} = \omega_1 - \omega_0 \ .
$$

Second, since $[\omega_0] = [\omega_1]$, there exists a 1-form μ such that

$$\omega_1 - \omega_0 = d\mu \ .$$

Third, by the Cartan magic formula, we have

$$\mathcal{L}_{v_t}\omega_t = d\imath_{v_t}\omega_t + \imath_{v_t}\underbrace{d\omega_t}_{0} \ .$$

Putting everything together, we must find v_t such that

$$d\imath_{v_t}\omega_t + d\mu = 0 \ .$$

It is sufficient to solve $\imath_{v_t}\omega_t + \mu = 0$. By the nondegeneracy of ω_t, we can solve this pointwise, to obtain a unique (smooth) v_t. □

Theorem 7.3 (Moser Theorem – Version II) *Let M be a compact manifold with symplectic forms ω_0 and ω_1. Suppose that ω_t, $0 \le t \le 1$, is a smooth family of closed 2-forms joining ω_0 to ω_1 and satisfying:*

(1) cohomology assumption: $[\omega_t]$ *is independent of* t, *i.e.,* $\frac{d}{dt}[\omega_t] = \left[\frac{d}{dt}\omega_t\right] = 0$,

(2) nondegeneracy assumption: ω_t *is nondegenerate for* $0 \le t \le 1$.

Then there exists an isotopy $\rho : M \times \mathbb{R} \to M$ such that $\rho_t^*\omega_t = \omega_0$, $0 \le t \le 1$.

Proof. *(Moser trick)* We have the following implications from the hypotheses:

(1) $\implies \exists$ family of 1-forms μ_t such that

$$\frac{d\omega_t}{dt} = d\mu_t \ , \quad 0 \le t \le 1 \ .$$

We can indeed find a *smooth* family of 1-forms μ_t such that $\frac{d\omega_t}{dt} = d\mu_t$. The argument involves the Poincaré lemma for compactly-supported forms, together with the Mayer-Vietoris sequence in order to use induction on the number of charts in a good cover of M. For a sketch of the argument, see page 95 in [83].

(2) $\implies \exists$ unique family of vector fields v_t such that

$$\imath_{v_t}\omega_t + \mu_t = 0 \qquad \textbf{(Moser equation)} \ .$$

Extend v_t to all $t \in \mathbb{R}$. Let ρ be the isotopy generated by v_t (ρ exists by compactness of M). Then we indeed have

$$\frac{d}{dt}(\rho_t^*\omega_t) = \rho_t^*(\mathcal{L}_{v_t}\omega_t + \frac{d\omega_t}{dt}) = \rho_t^*(d\imath_{v_t}\omega_t + d\mu_t) = 0 \ .$$

□

The compactness of M was used to be able to integrate v_t for all $t \in \mathbb{R}$. If M is *not* compact, we need to check the existence of a solution ρ_t for the differential equation $\frac{d\rho_t}{dt} = v_t \circ \rho_t$ for $0 \leq t \leq 1$.

Picture. Fix $c \in H^2(M)$. Define $S_c = \{\text{symplectic forms } \omega \text{ in } M \text{ with } [\omega] = c\}$. The Moser theorem implies that, on a compact manifold, all symplectic forms on the same path-connected component of S_c are symplectomorphic.

7.3 Moser Relative Theorem

Theorem 7.4 (Moser Theorem – Relative Version) *Let M be a manifold, X a compact submanifold of M, $i : X \hookrightarrow M$ the inclusion map, ω_0 and ω_1 symplectic forms in M.*

Hypothesis: $\omega_0|_p = \omega_1|_p \ , \quad \forall p \in X \ .$
Conclusion: *There exist neighborhoods $\mathcal{U}_0, \mathcal{U}_1$ of X in M, and a diffeomorphism $\varphi : \mathcal{U}_0 \to \mathcal{U}_1$ such that*

$$\begin{tikzcd} \mathcal{U}_0 \arrow[rr, "\varphi"] & & \mathcal{U}_1 \\ & X \arrow[ul, "i"] \arrow[ur, "i"'] & \end{tikzcd}$$

commutes

and $\varphi^*\omega_1 = \omega_0$.

Proof.

1. Pick a tubular neighborhood $\mathcal{U}_0$ of X. The 2-form $\omega_1 - \omega_0$ is closed on $\mathcal{U}_0$, and $(\omega_1 - \omega_0)_p = 0$ at all $p \in X$. By the homotopy formula on the tubular neighborhood, there exists a 1-form μ on $\mathcal{U}_0$ such that $\omega_1 - \omega_0 = d\mu$ and $\mu_p = 0$ at all $p \in X$.

2. Consider the family $\omega_t = (1-t)\omega_0 + t\omega_1 = \omega_0 + td\mu$ of closed 2-forms on $\mathcal{U}_0$. Shrinking $\mathcal{U}_0$ if necessary, we can assume that ω_t is symplectic for $0 \leq t \leq 1$.

3. Solve the Moser equation: $\imath_{v_t}\omega_t = -\mu$. Notice that $v_t = 0$ on X.

4. Integrate v_t. Shrinking $\mathcal{U}_0$ again if necessary, there exists an isotopy $\rho : \mathcal{U}_0 \times [0,1] \to M$ with $\rho_t^*\omega_t = \omega_0$, for all $t \in [0,1]$. Since $v_t|_X = 0$, we have $\rho_t|_X = \mathrm{id}_X$.

Set $\varphi = \rho_1$, $\mathcal{U}_1 = \rho_1(\mathcal{U}_0)$. □

Exercise. Prove the Darboux theorem. (Hint: apply the relative version of the Moser theorem to $X = \{p\}$, as in the next lecture.) ◊

8 Darboux-Moser-Weinstein Theory

8.1 Darboux Theorem

Theorem 8.1 (Darboux) *Let (M, ω) be a symplectic manifold, and let p be any point in M. Then we can find a coordinate system $(\mathcal{U}, x_1, \ldots, x_n, y_1, \ldots y_n)$ centered at p such that on $\mathcal{U}$*

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i \ .$$

As a consequence of Theorem 8.1, if we prove for $(\mathbb{R}^{2n}, \sum dx_i \wedge dy_i)$ a local assertion which is invariant under symplectomorphisms, then that assertion holds for any symplectic manifold.

Proof. Apply the Moser relative theorem (Theorem 7.4) to $X = \{p\}$:

Use any symplectic basis for T_pM to construct coordinates $(x'_1, \ldots, x'_n, y'_1, \ldots y'_n)$ centered at p and valid on some neighborhood $\mathcal{U}'$, so that

$$\omega_p = \sum dx'_i \wedge dy'_i \Big|_p \ .$$

There are two symplectic forms on $\mathcal{U}'$: the given $\omega_0 = \omega$ and $\omega_1 = \sum dx'_i \wedge dy'_i$. By the Moser theorem, there are neighborhoods $\mathcal{U}_0$ and $\mathcal{U}_1$ of p, and a diffeomorphism $\varphi : \mathcal{U}_0 \to \mathcal{U}_1$ such that

$$\varphi(p) = p \quad \text{and} \quad \varphi^*(\sum dx'_i \wedge dy'_i) = \omega \ .$$

Since $\varphi^*(\sum dx'_i \wedge dy'_i) = \sum d(x'_i \circ \varphi) \wedge d(y'_i \circ \varphi)$, we only need to set new coordinates $x_i = x'_i \circ \varphi$ and $y_i = y'_i \circ \varphi$. $\square$

If in the Moser relative theorem (Theorem 7.4) we assume instead

Hypothesis: X is an n-dimensional submanifold with $i^*\omega_0 = i^*\omega_1 = 0$ where $i : X \hookrightarrow M$ is inclusion, i.e., X is a submanifold lagrangian for ω_0 and ω_1 ,

then Weinstein [104] proved that the conclusion still holds. We need some algebra for the Weinstein theorem.

8.2 Lagrangian Subspaces

Suppose that U, W are n-dimensional vector spaces, and $\Omega : U \times W \to \mathbb{R}$ is a bilinear pairing; the map Ω gives rise to a linear map $\widetilde{\Omega} : U \to W^*$, $\widetilde{\Omega}(u) = \Omega(u, \cdot)$. Then Ω is nondegenerate if and only if $\widetilde{\Omega}$ is bijective.

Proposition 8.2 *Suppose that (V,Ω) is a $2n$-dimensional symplectic vector space. Let U be a lagrangian subspace of (V,Ω) (i.e., $\Omega|_{U\times U}=0$ and U is n-dimensional). Let W be any vector space complement to U, not necessarily lagrangian.*

Then from W we can canonically *build a* lagrangian *complement to U.*

Proof. The pairing Ω gives a nondegenerate pairing $U \times W \overset{\Omega'}{\to} \mathbb{R}$. Therefore, $\widetilde{\Omega}' : U \to W^*$ is bijective. We look for a lagrangian complement to U of the form

$$W' = \{w + Aw \mid w \in W\} ,$$

the map $A : W \to U$ being linear. For W' to be lagrangian we need

$$\forall\ w_1, w_2 \in W\ ,\quad \Omega(w_1 + Aw_1, w_2 + Aw_2) = 0$$

$$\Longrightarrow\ \Omega(w_1,w_2) + \Omega(w_1, Aw_2) + \Omega(Aw_1, w_2) + \underbrace{\Omega(\underbrace{Aw_1, Aw_2}_{\in U})}_{0} = 0$$

$$\begin{aligned} \Longrightarrow\ \Omega(w_1,w_2) &= \Omega(Aw_2, w_1) - \Omega(Aw_1, w_2) \\ &= \widetilde{\Omega}'(Aw_2)(w_1) - \widetilde{\Omega}'(Aw_1)(w_2)\ . \end{aligned}$$

Let $A' = \widetilde{\Omega}' \circ A : W \to W^*$, and look for A' such that

$$\forall\ w_1, w_2 \in W\ ,\quad \Omega(w_1,w_2) = A'(w_2)(w_1) - A'(w_1)(w_2)\ .$$

The canonical choice is $A'(w) = -\frac{1}{2}\Omega(w,\cdot)$. Then set $A = (\widetilde{\Omega}')^{-1} \circ A'$. $\square$

Proposition 8.3 *Let V be a $2n$-dimensional vector space, let Ω_0 and Ω_1 be symplectic forms in V, let U be a subspace of V lagrangian for Ω_0 and Ω_1, and let W be any complement to U in V. Then from W we can canonically construct a linear isomorphism $L : V \overset{\simeq}{\to} V$ such that $L|_U = \mathrm{Id}_U$ and $L^*\Omega_1 = \Omega_0$.*

Proof. From W we canonically obtain complements W_0 and W_1 to U in V such that W_0 is lagrangian for Ω_0 and W_1 is lagrangian for Ω_1. The nondegenerate bilinear pairings

$$\begin{array}{l} W_0 \times U \xrightarrow{\Omega_0} \mathbb{R} \\ W_1 \times U \xrightarrow{\Omega_1} \mathbb{R} \end{array} \quad \text{give isomorphisms} \quad \begin{array}{l} \widetilde{\Omega}_0 : W_0 \xrightarrow{\simeq} U^* \\ \widetilde{\Omega}_1 : W_1 \xrightarrow{\simeq} U^*\ . \end{array}$$

Consider the diagram

$$\begin{array}{ccc} W_0 & \xrightarrow{\widetilde{\Omega}_0} & U^* \\ B \downarrow & & \downarrow \mathrm{id} \\ W_1 & \xrightarrow{\widetilde{\Omega}_1} & U^* \end{array}$$

where the linear map B satisfies $\widetilde{\Omega}_1 \circ B = \widetilde{\Omega}_0$, i.e., $\Omega_0(w_0,u) = \Omega_1(Bw_0,u)$, $\forall w_0 \in W_0$, $\forall u \in U$. Extend B to the rest of V by setting it to be the identity on U:

$$L := \mathrm{Id}_U \oplus B : U \oplus W_0 \longrightarrow U \oplus W_1\ .$$

Finally, we check that $L^*\Omega_1 = \Omega_0$.

$$\begin{aligned}(L^*\Omega_1)(u \oplus w_0, u' \oplus w_0') &= \Omega_1(u \oplus Bw_0, u' \oplus B\omega_0') \\ &= \Omega_1(u, Bw_0') + \Omega_1(Bw_0, u') \\ &= \Omega_0(u, w_0') + \Omega_0(w_0, u') \\ &= \Omega_0(u \oplus w_0, u' \oplus w_0') \ .\end{aligned}$$

□

8.3 Weinstein Lagrangian Neighborhood Theorem

Theorem 8.4 (Weinstein Lagrangian Neighborhood Theorem [104]) *Let M be a $2n$-dimensional manifold, X a compact n-dimensional submanifold, $i : X \hookrightarrow M$ the inclusion map, and ω_0 and ω_1 symplectic forms on M such that $i^*\omega_0 = i^*\omega_1 = 0$, i.e., X is a lagrangian submanifold of both (M, ω_0) and (M, ω_1). Then there exist neighborhoods $\mathcal{U}_0$ and $\mathcal{U}_1$ of X in M and a diffeomorphism $\varphi : \mathcal{U}_0 \to \mathcal{U}_1$ such that*

$$\begin{tikzcd} \mathcal{U}_0 \arrow[rr, "\varphi"] & & \mathcal{U}_1 \\ & X \arrow[ul, "i"] \arrow[ur, "i"'] & \end{tikzcd} \qquad \textit{commutes} \qquad \textit{and} \qquad \varphi^*\omega_1 = \omega_0 \ .$$

The proof of the Weinstein theorem uses the Whitney extension theorem.

Theorem 8.5 (Whitney Extension Theorem) *Let M be an n-dimensional manifold and X a k-dimensional submanifold with $k < n$. Suppose that at each $p \in X$ we are given a linear isomorphism $L_p : T_pM \xrightarrow{\simeq} T_pM$ such that $L_p|_{T_pX} = \mathrm{Id}_{T_pX}$ and L_p depends smoothly on p. Then there exists an embedding $h : \mathcal{N} \to M$ of some neighborhood $\mathcal{N}$ of X in M such that $h|_X = \mathrm{id}_X$ and $dh_p = L_p$ for all $p \in X$.*

The linear maps L serve as "germs" for the embedding.

Proof of the Weinstein theorem. Put a riemannian metric g on M; at each $p \in M$, $g_p(\cdot, \cdot)$ is a positive-definite inner product. Fix $p \in X$, and let $V = T_pM$, $U = T_pX$ and $W = U^\perp$ the orthocomplement of U in V relative to $g_p(\cdot, \cdot)$.

Since $i^*\omega_0 = i^*\omega_1 = 0$, the space U is a lagrangian subspace of both $(V, \omega_0|_p)$ and $(V, \omega_1|_p)$. By symplectic linear algebra, we canonically get from $U^\perp$ a linear isomorphism $L_p : T_pM \to T_pM$, such that $L_p|_{T_pX} = \mathrm{Id}_{T_pX}$ and $L_p^*\omega_1|_p = \omega_0|_p$. L_p varies smoothly with respect to p since our recipe is canonical!

By the Whitney theorem, there are a neighborhood $\mathcal{N}$ of X and an embedding $h : \mathcal{N} \hookrightarrow M$ with $h|_X = \mathrm{id}_X$ and $dh_p = L_p$ for $p \in X$. Hence, at any $p \in X$,

$$(h^*\omega_1)_p = (dh_p)^*\omega_1|_p = L_p^*\omega_1|_p = \omega_0|_p \ .$$

Applying the Moser relative theorem (Theorem 7.4) to ω_0 and $h^*\omega_1$, we find a neighborhood $\mathcal{U}_0$ of X and an embedding $f : \mathcal{U}_0 \to \mathcal{N}$ such that $f|_X = \mathrm{id}_X$ and $f^*(h^*\omega_1) = \omega_0$ on $\mathcal{U}_o$. Set $\varphi = h \circ f$. $\square$

Sketch of proof for the Whitney theorem.

Case $M = \mathbb{R}^n$:

For a compact k-dimensional submanifold X, take a neighborhood of the form

$$\mathcal{U}^\varepsilon = \{p \in M \mid \text{ distance } (p, X) \leq \varepsilon\} .$$

For ε sufficiently small so that any $p \in \mathcal{U}^\varepsilon$ has a unique nearest point in X, define a projection $\pi : \mathcal{U}^\varepsilon \to X$, $p \mapsto$ point on X closest to p. If $\pi(p) = q$, then $p = q + v$ for some $v \in N_qX$ where $N_qX = (T_qX)^\perp$ is the normal space at q; see Homework 5. Let

$$\begin{array}{rccc} h: & \mathcal{U}^\varepsilon & \longrightarrow & \mathbb{R}^n \\ & p & \longmapsto & q + L_q v \ , \end{array}$$

where $q = \pi(p)$ and $v = p - \pi(p) \in N_qX$. Then $h_X = \mathrm{id}_X$ and $dh_p = L_p$ for $p \in X$. If X is not compact, replace ε by a continuous function $\varepsilon : X \to \mathbb{R}^+$ which tends to zero fast enough as x tends to infinity.

General case:

Choose a riemannian metric on M. Replace distance by riemannian distance, replace straight lines $q + tv$ by geodesics $\exp(q, v)(t)$ and replace $q + L_qv$ by the value at $t = 1$ of the geodesic with initial value q and initial velocity L_qv. $\square$

In Lecture 30 we will need the following generalization of Theorem 8.4. For a proof see, for instance, either of [47, 58, 107].

Theorem 8.6 (Coisotropic Embedding Theorem) *Let M be a manifold of dimension $2n$, X a submanifold of dimension $k \geq n$, $i : X \hookrightarrow M$ the inclusion map, and ω_0 and ω_1 symplectic forms on M, such that $i^*\omega_0 = i^*\omega_1$ and X is coisotropic for both (M, ω_0) and (M, ω_1). Then there exist neighborhoods $\mathcal{U}_0$ and $\mathcal{U}_1$ of X in M and a diffeomorphism $\varphi : \mathcal{U}_0 \to \mathcal{U}_1$ such that*

$$\begin{tikzcd} \mathcal{U}_0 \arrow[rr, "\varphi"] & & \mathcal{U}_1 \\ & X \arrow[ul, "i"] \arrow[ur, "i"'] & \end{tikzcd}$$

commutes and $\varphi^*\omega_1 = \omega_0$.

Homework 6: Oriented Surfaces

1. The standard symplectic form on the 2-sphere is the standard area form: If we think of S^2 as the unit sphere in 3-space

$$S^2 = \{u \in \mathbb{R}^3 \text{ such that } |u| = 1\} ,$$

then the induced area form is given by

$$\omega_u(v, w) = \langle u, v \times w \rangle$$

where $u \in S^2$, $v, w \in T_u S^2$ are vectors in $\mathbb{R}^3$, $\times$ is the exterior product, and $\langle \cdot, \cdot \rangle$ is the standard inner product. With this form, the total area of S^2 is 4π.

Consider cylindrical polar coordinates (θ, z) on S^2 away from its poles, where $0 \leq \theta < 2\pi$ and $-1 \leq z \leq 1$.

Show that, in these coordinates,

$$\omega = d\theta \wedge dz \ .$$

2. Prove the Darboux theorem in the 2-dimensional case, using the fact that every nonvanishing 1-form on a surface can be written locally as $f\, dg$ for suitable functions f, g.

Hint: $\omega = df \wedge dg$ is nondegenerate $\iff$ (f, g) is a local diffeomorphism.

3. Any oriented 2-dimensional manifold with an area form is a symplectic manifold.

   (a) Show that convex combinations of two area forms ω_0, ω_1 that induce the same orientation are symplectic.

   This is wrong in dimension 4: find two symplectic forms on the vector space $\mathbb{R}^4$ that induce the same orientation, yet some convex combination of which is degenerate. Find a path of symplectic forms that connect them.

   (b) Suppose that we have two area forms ω_0, ω_1 on a compact 2-dimensional manifold M representing the same de Rham cohomology class, i.e., $[\omega_0] = [\omega_1] \in H^2_{\text{deRham}}(M)$.

   Prove that there is a 1-parameter family of diffeomorphisms $\varphi_t : M \to M$ such that $\varphi_1^* \omega_0 = \omega_1$, $\varphi_0 = \text{id}$, and $\varphi_t^* \omega_0$ is symplectic for all $t \in [0, 1]$.

   Hint: Exercise (a) and the Moser trick.

   Such a 1-parameter family φ_t is called a *strong isotopy* between ω_0 and ω_1. In this language, this exercise shows that, up to strong isotopy, there is a unique symplectic representative in each non-zero 2-cohomology class of M.

9 Weinstein Tubular Neighborhood Theorem

9.1 Observation from Linear Algebra

Let (V, Ω) be a symplectic linear space, and let U be a lagrangian subspace.

Claim. There is a canonical nondegenerate bilinear pairing $\Omega' : V/U \times U \to \mathbb{R}$.

Proof. Define $\Omega'([v], u) = \Omega(v, u)$ where $[v]$ is the equivalence class of v in V/U.

Exercise. Check that Ω' is well-defined and nondegenerate. $\diamondsuit$ $\square$

Consequently, we get
$\Longrightarrow \widetilde{\Omega}' : V/U \to U^*$ defined by $\widetilde{\Omega}'([v]) = \Omega'([v], \cdot)$ is an isomorphism.
$\Longrightarrow V/U \simeq U^*$ are canonically identified.

In particular, if (M, ω) is a symplectic manifold, and X is a lagrangian submanifold, then T_xX is a lagrangian subspace of (T_xM, ω_x) for each $x \in X$.
The space $N_xX := T_xM/T_xX$ is called the **normal space** of X at x.

$\Longrightarrow$ There is a canonical identification $N_xX \simeq T_x^*X$.
$\Longrightarrow$

Theorem 9.1 *The vector bundles NX and T^*X are canonically identified.*

9.2 Tubular Neighborhoods

Theorem 9.2 (Standard Tubular Neighborhood Theorem) *Let M be an n-dimensional manifold, X a k-dimensional submanifold, NX the normal bundle of X in M, $i_0 : X \hookrightarrow NX$ the zero section, and $i : X \hookrightarrow M$ inclusion. Then there are neighborhoods $\mathcal{U}_0$ of X in NX, $\mathcal{U}$ of X in M and a diffeomorphism $\psi : \mathcal{U}_0 \to \mathcal{U}$ such that*

$$\begin{tikzcd} \mathcal{U}_0 \arrow[rr, "\psi"] & & \mathcal{U} \\ & X \arrow[ul, "i_0"] \arrow[ur, "i"'] & \end{tikzcd}$$

commutes .

For the proof, see Lecture 6.

Theorem 9.3 (Weinstein Tubular Neighborhood Theorem) *Let (M, ω) be a symplectic manifold, X a compact lagrangian submanifold, ω_0 the canonical symplectic form on T^*X, $i_0 : X \hookrightarrow T^*X$ the lagrangian embedding as the zero section, and $i : X \hookrightarrow M$ the lagrangian embedding given by inclusion.*

Then there are neighborhoods $\mathcal{U}_0$ of X in T^*X, $\mathcal{U}$ of X in M, and a diffeomorphism $\varphi : \mathcal{U}_0 \to \mathcal{U}$ such that

$$\begin{tikzcd} \mathcal{U}_0 \arrow[rr, "\varphi"] & & \mathcal{U} \\ & X \arrow[ul, "i_0"] \arrow[ur, "i"'] & \end{tikzcd} \quad \textit{commutes} \quad \textit{and} \quad \varphi^*\omega = \omega_0 \ .$$

Proof. This proof relies on (1) the standard tubular neighborhood theorem, and (2) the Weinstein lagrangian neighborhood theorem.

(1) Since $NX \simeq T^*X$, we can find a neighborhood $\mathcal{N}_0$ of X in T^*X, a neighborhood $\mathcal{N}$ of X in M, and a diffeomorphism $\psi : \mathcal{N}_0 \to \mathcal{N}$ such that

$$\begin{tikzcd} \mathcal{N}_0 \arrow[rr, "\psi"] & & \mathcal{N} \\ & X \arrow[ul, "i_0"] \arrow[ur, "i"'] & \end{tikzcd} \quad \text{commutes .}$$

Let $\left.\begin{array}{l} \omega_0 = \text{canonical form on } T^*X \\ \omega_1 = \psi^*\omega \end{array}\right\}$ symplectic forms on $\mathcal{N}_0$.

The submanifold X is lagrangian for both ω_0 and ω_1.

(2) There exist neighborhoods $\mathcal{U}_0$ and $\mathcal{U}_1$ of X in $\mathcal{N}_0$ and a diffeomorphism $\theta : \mathcal{U}_0 \to \mathcal{U}_1$ such that

$$\begin{tikzcd} \mathcal{U}_0 \arrow[rr, "\theta"] & & \mathcal{U}_1 \\ & X \arrow[ul, "i_0"] \arrow[ur, "i_0"'] & \end{tikzcd} \quad \text{commutes} \quad \text{and} \quad \theta^*\omega_1 = \omega_0 \ .$$

Take $\varphi = \psi \circ \theta$ and $\mathcal{U} = \varphi(\mathcal{U}_0)$. Check that $\varphi^*\omega = \theta^*\underbrace{\psi^*\omega}_{\omega_1} = \omega_0$.

$\square$

Remark. Theorem 9.3 classifies lagrangian embeddings: up to local symplectomorphism, the set of lagrangian embeddings is the set of embeddings of manifolds into their cotangent bundles as zero sections.

The classification of *isotropic* embeddings was also carried out by Weinstein in [105, 107]. An **isotropic embedding** of a manifold X into a symplectic manifold (M,ω) is a closed embedding $i : X \hookrightarrow M$ such that $i^*\omega = 0$. Weinstein showed that neighbourhood equivalence of isotropic embeddings is in one-to-one correspondence with isomorphism classes of symplectic vector bundles.

The classification of *coisotropic embeddings* is due to Gotay [47]. A **coisotropic embedding** of a manifold X carrying a closed 2-form α of constant rank into a symplectic manifold (M,ω) is an embedding $i : X \hookrightarrow M$ such that $i^*\omega = \alpha$ and $i(X)$ is coisotropic as a submanifold of M. Let E be the **characteristic distribution** of a closed form α of constant rank on X, i.e., E_p is the kernel of α_p at $p \in X$. Gotay showed that then E^* carries a symplectic structure in a neighbourhood of the zero section, such that X embeds coisotropically onto this zero section, and, moreover every coisotropic embedding is equivalent to this in some neighbourhood of the zero section. $\diamondsuit$

9.3 Application 1: Tangent Space to the Group of Symplectomorphisms

The symplectomorphisms of a symplectic manifold (M,ω) form the group

$$\mathrm{Sympl}(M,\omega) = \{f : M \xrightarrow{\simeq} M \mid f^*\omega = \omega\} \ .$$

– What is $T_{\mathrm{id}}(\mathrm{Sympl}(M,\omega))$?
(What is the "Lie algebra" of the group of symplectomorphisms?)
– What does a neighborhood of id in $\mathrm{Sympl}(M,\omega)$ look like?

We use notions from the C^1-topology:

C^1-topology.
Let X and Y be manifolds.

Definition 9.4 *A sequence of maps $f_i : X \to Y$* **converges in the C^0-topology** *to $f : X \to Y$ if and only if f_i converges uniformly on compact sets.*

Definition 9.5 *A sequence of C^1 maps $f_i : X \to Y$* **converges in the C^1-topology** *to $f : X \to Y$ if and only if it and the sequence of derivatives $df_i : TX \to TY$ converge uniformly on compact sets.*

Let (M,ω) be a compact symplectic manifold and $f \in \mathrm{Sympl}(M,\omega)$. Then

$$\left.\begin{array}{l}\text{Graph } f\\ \text{Graph id} = \Delta\end{array}\right\}\ \text{are lagrangian submanifolds of } (M\times M, \mathrm{pr}_1^*\omega - \mathrm{pr}_2^*\omega).$$

($\mathrm{pr}_i : M \times M \to M$, $i = 1, 2$, are the projections to each factor.)

By the Weinstein tubular neighborhood theorem, there exists a neighborhood $\mathcal{U}$ of Δ ($\simeq M$) in $(M \times M, \mathrm{pr}_1^*\omega - \mathrm{pr}_2^*\omega)$ which is symplectomorphic to a neighborhood $\mathcal{U}_0$ of M in (T^*M, ω_0). Let $\varphi : \mathcal{U} \to \mathcal{U}_0$ be the symplectomorphism satisfying $\varphi(p,p) = (p,0)$, $\forall p \in M$.

Suppose that f is sufficiently $\boldsymbol{C^1}$**-close** to id, i.e., f is in some sufficiently small neighborhood of id in the C^1-topology. Then:

1. We can assume that $\mathrm{Graph}\ f \subseteq \mathcal{U}$.

   Let $j : M \hookrightarrow \mathcal{U}$ be the embedding as $\mathrm{Graph}\ f$,
   $i : M \hookrightarrow \mathcal{U}$ be the embedding as $\mathrm{Graph\ id} = \Delta$.

2. The map j is sufficiently C^1-close to i.

3. By the Weinstein theorem, $\mathcal{U} \simeq \mathcal{U}_0 \subseteq T^*M$, so the above j and i induce

   $j_0 : M \hookrightarrow \mathcal{U}_0$ embedding, where $j_0 = \varphi \circ j$,
   $i_0 : M \hookrightarrow \mathcal{U}_0$ embedding as 0-section .

   Hence, we have

$$\begin{tikzcd} \mathcal{U} \arrow[rr, "\varphi"] & & \mathcal{U}_0 \\ & M \arrow[ul, "i"] \arrow[ur, "i_0"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{U} \arrow[rr, "\varphi"] & & \mathcal{U}_0 \\ & M \arrow[ul, "j"] \arrow[ur, "j_0"'] & \end{tikzcd}$$

   where $i(p) = (p, p)$, $i_0(p) = (p, 0)$, $j(p) = (p, f(p))$ and $j_0(p) = \varphi(p, f(p))$ for $p \in M$.

4. The map j_0 is sufficiently C^1-close to i_0.

   $\Downarrow$

   The image set $j_0(M)$ intersects each T_p^*M at one point μ_p depending smoothly on p.

5. The image of j_0 is the image of a smooth section $\mu : M \to T^*M$, that is, a 1-form $\mu = j_0 \circ (\pi \circ j_0)^{-1}$.

Therefore, $\mathrm{Graph}\ f \simeq \{(p, \mu_p) \mid p \in M,\ \mu_p \in T_p^*M\}$.

Exercise. Vice-versa: if μ is a 1-form sufficiently C^1-close to the zero 1-form, then

$$\{(p, \mu_p) \mid p \in M,\ \mu_p \in T_p^*M\} \ \simeq\ \mathrm{Graph}\ f \ ,$$

for some diffeomorphism $f : M \to M$. By Lecture 3, we have

$\mathrm{Graph}\ f$ is lagrangian $\iff$ μ is closed. $\diamondsuit$

Conclusion. A small C^1-neighborhood of id in $\mathrm{Sympl}(M, \omega)$ is homeomorphic to a C^1-neighborhood of zero in the vector space of closed 1-forms on M. So:

$$T_{\mathrm{id}}(\mathrm{Sympl}(M, \omega)) \simeq \{\mu \in \Omega^1(M) \mid d\mu = 0\} \ .$$

In particular, $T_{\mathrm{id}}(\mathrm{Sympl}(M, \omega))$ contains the space of exact 1-forms

$$\{\mu = dh \mid h \in C^\infty(M)\} \ \simeq\ C^\infty(M)/ \text{ locally constant functions} \ .$$

9.4 Application 2: Fixed Points of Symplectomorphisms

Theorem 9.6 *Let (M, ω) be a compact symplectic manifold with $H^1_{\text{deRham}}(M) = 0$. Then any symplectomorphism of M which is sufficiently C^1-close to the identity has at least two fixed points.*

Proof. Suppose that $f \in \text{Sympl}(M, \omega)$ is sufficiently C^1-close to id.

Then $\text{Graph } f \simeq$ closed 1-form μ on M.

$$\left.\begin{array}{l} d\mu = 0 \\ H^1_{\text{deRham}}(M) = 0 \end{array}\right\} \Longrightarrow \mu = dh \text{ for some } h \in C^\infty(M) \ .$$

Since M is compact, h has at least 2 critical points.

$$\begin{array}{ccc} \text{Fixed points of f} & = & \text{critical points of } h \\ \| & & \| \\ \text{Graph } f \cap \Delta & = & \{p : \mu_p = dh_p = 0\} \ . \end{array}$$

□

Lagrangian intersection problem:

A submanifold Y of M is **C^1-close** to X when there is a diffeomorphism $X \to Y$ which is, as a map into M, C^1-close to the inclusion $X \hookrightarrow M$.

Theorem 9.7 *Let (M, ω) be a symplectic manifold. Suppose that X is a compact lagrangian submanifold of M with $H^1_{\text{deRham}}(X) = 0$. Then every lagrangian submanifold of M which is C^1-close to X intersects X in at least two points.*

Proof. Exercise. □

Arnold conjecture:

Let (M, ω) be a compact symplectic manifold, and $f : M \to M$ a symplectomorphism which is "exactly homotopic to the identity" (see below). Then

$$\#\{\text{fixed points of } f\} \geq \begin{array}{l} \text{minimal } \# \text{ of critical points} \\ \text{a smooth function on } M \text{ can have .} \end{array}$$

Together with Morse theory,[8] we obtain[9]

$$\begin{array}{rcl} \#\{\text{nondegenerate fixed points of } f\} & \geq & \begin{array}{l} \text{minimal } \# \text{ of critical points} \\ \text{a Morse function on } M \text{ can have} \end{array} \\ & \geq & \displaystyle\sum_{i=0}^{2n} \dim H^i(M; \mathbb{R}) \ . \end{array}$$

[8] A **Morse function** on M is a function $h : M \to \mathbb{R}$ whose critical points (i.e., points p where $dh_p = 0$) are all nondegenerate (i.e., the hessian at those points is nonsingular: $\det \left(\frac{\partial^2 h}{\partial x_i \partial x_j} \right)_p \neq 0$).

[9] A fixed point p of $f : M \to M$ is **nondegenerate** if $df_p : T_pM \to T_pM$ is nonsingular.

The Arnold conjecture was proved by Conley-Zehnder, Floer, Hofer-Salamon, Ono, Fukaya-Ono, Liu-Tian using Floer homology (which is an ∞-dimensional analogue of Morse theory). There are open conjectures for sharper bounds on the number of fixed points.

Meaning of "f is exactly homotopic to the identity:"

Suppose that $h_t : M \to \mathbb{R}$ is a smooth family of functions which is 1-periodic, i.e., $h_t = h_{t+1}$. Let $\rho : M \times \mathbb{R} \to M$ be the isotopy generated by the time-dependent vector field v_t defined by $\omega(v_t, \cdot) = dh_t$. Then "$f$ being exactly homotopic to the identity" means $f = \rho_1$ for some such h_t.

In other words, f is **exactly homotopic to the identity** when f is the time-1 map of an isotopy generated by some smooth time-dependent 1-periodic hamiltonian function.

There is a one-to-one correspondence

$$\text{fixed points of } f \quad \overset{1-1}{\longleftrightarrow} \quad \text{period-1 orbits of } \rho : M \times \mathbb{R} \to M$$

because $f(p) = p$ if and only if $\{\rho(t,p)\ ,\ t \in [0,1]\}$ is a closed orbit.

Proof of the Arnold conjecture in the case when $h : M \to \mathbb{R}$ is independent of t:
p is a critical point of $h \iff dh_p = 0 \iff v_p = 0$
$\Longrightarrow \rho(t,p) = p\ ,\ \forall t \in \mathbb{R} \Longrightarrow p$ is a fixed point of ρ_1. $\square$

Exercise. Compute these estimates for the number of fixed points on some compact symplectic manifolds (for instance, S^2, $S^2 \times S^2$ and $T^2 = S^1 \times S^1$). $\diamondsuit$

Part IV
Contact Manifolds

Contact geometry is also known as "the odd-dimensional analogue of symplectic geometry." We will browse through the basics of contact manifolds and their relation to symplectic manifolds.

10 Contact Forms

10.1 Contact Structures

Definition 10.1 *A* **contact element** *on a manifold M is a point $p \in M$, called the* **contact point***, together with a tangent hyperplane at p, $H_p \subset T_pM$, that is, a codimension-1 subspace of T_pM.*

A hyperplane $H_p \subset T_pM$ determines a covector $\alpha_p \in T_p^*M \setminus \{0\}$, up to multiplication by a nonzero scalar:

$$(p, H_p) \text{ is a contact element} \longleftrightarrow H_p = \ker \alpha_p \text{ with } \alpha_p : T_pM \longrightarrow \mathbb{R} \text{ linear }, \neq 0$$

$$\ker \alpha_p = \ker \alpha'_p \iff \alpha_p = \lambda \alpha'_p \text{ for some } \lambda \in \mathbb{R} \setminus \{0\} \ .$$

Suppose that H is a smooth field of contact elements (i.e., of tangent hyperplanes) on M:

$$H : p \longmapsto H_p \subset T_pM \ .$$

Locally, $H = \ker \alpha$ for some 1-form α, called a **locally defining 1-form** for H. (α is not unique: $\ker \alpha = \ker(f\alpha)$, for any nowhere vanishing $f : M \to \mathbb{R}$.)

Definition 10.2 *A* **contact structure** *on M is a smooth field of tangent hyperplanes $H \subset TM$, such that, for any locally defining 1-form α, we have $d\alpha|_H$ nondegenerate (i.e., symplectic). The pair (M, H) is then called a* **contact manifold** *and α is called a* **local contact form**.

At each $p \in M$,

$$T_pM = \underbrace{\ker \alpha_p}_{H_p} \oplus \underbrace{\ker d\alpha_p}_{1-\text{dimensional}} \ .$$

The $\ker d\alpha_p$ summand in this splitting depends on the choice of α.

$$\begin{array}{ll} d\alpha_p|_{H_p} \text{ nondegenerate} & \Longrightarrow \begin{cases} \dim H_p = 2n & \text{is even} \\ (d\alpha_p)^n|_{H_p} \neq 0 & \text{is a volume form on } H_p \end{cases} \\ \alpha_p|_{\ker d\alpha_p} \text{ nondegenerate} & \end{array}$$

Therefore,

- any contact manifold (M, H) has $\dim M = 2n + 1$ *odd*, and

- if α is a (global) contact form, then $\alpha \wedge (d\alpha)^n$ is a volume form on M.

Remark. Let (M, H) be a contact manifold. A *global* contact form exists if and only if the quotient line bundle TM/H is orientable. Since H is also orientable, this implies that M is orientable. ♢

Proposition 10.3 *Let H be a field of tangent hyperplanes on M. Then*

$$H \text{ is a contact structure} \iff \alpha \wedge (d\alpha)^n \neq 0 \text{ for every locally defining 1-form } \alpha \ .$$

Proof.
$\Longrightarrow$ Done above.
$\Longleftarrow$ Suppose that $H = \ker \alpha$ locally. We need to show:

$$d\alpha|_H \text{ nondegenerate} \iff \alpha \wedge (d\alpha)^n \neq 0 \ .$$

Take a local trivialization $\{e_1, f_1, \ldots, e_n, f_n, r\}$ of $TM = \ker \alpha \oplus$ rest , such that $\ker \alpha = \mathrm{span}\{e_1, f_1, \ldots, e_n, f_n\}$ and rest $= \mathrm{span}\{r\}$.

$$(\alpha \wedge (d\alpha)^n)(e_1, f_1, \ldots, e_n, f_n, r) = \underbrace{\alpha(r)}_{\neq 0} \cdot (d\alpha)^n(e_1, f_1, \ldots, e_n, f_n)$$

and hence $\alpha \wedge (d\alpha)^n \neq 0 \iff (d\alpha)^n|_H \neq 0 \iff d\alpha|_H$ is nondegenerate . □

10.2 Examples

1. On $\mathbb{R}^3$ with coordinates (x, y, z), consider $\alpha = xdy + dz$. Since

   $$\alpha \wedge d\alpha = (xdy + dz) \wedge (dx \wedge dy) = dx \wedge dy \wedge dz \neq 0 \ ,$$

   α is a contact form on $\mathbb{R}^3$.

   The corresponding field of hyperplanes $H = \ker \alpha$ at $(x, y, z) \in \mathbb{R}^3$ is

   $$H_{(x,y,z)} = \{v = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z} \mid \alpha(v) = bx + c = 0\} \ .$$

   Exercise. Picture these hyperplanes. ♢

2. **(Martinet [80], 1971)** Any compact orientable 3-manifold admits a contact structure.

   Open Problem, 2000. The classification of compact orientable contact 3-manifolds is still not known. There is by now a huge collection of results in contact topology related to the classification of contact manifolds. For a review of the state of the knowledge and interesting questions on contact 3-manifolds, see [33, 43, 100].

3. Let X be a manifold and T^*X its cotangent bundle. There are two canonical contact manifolds associated to X (see Homework 7):

$$\begin{array}{rcl} \mathbb{P}(T^*X) & = & \text{the projectivization of } T^*X \text{ , and} \\ S(T^*X) & = & \text{the cotangent sphere bundle .} \end{array}$$

4. On $\mathbb{R}^{2n+1}$ with coordinates $(x_1, y_1, \ldots, x_n, y_n, z)$, $\alpha = \sum_i x_i dy_i + dz$ is contact.

10.3 First Properties

There is a local normal form theorem for contact manifolds analogous to the Darboux theorem for symplectic manifolds.

Theorem 10.4 *Let (M, H) be a contact manifold and $p \in M$. Then there exists a coordinate system $(\mathcal{U}, x_1, y_1, \ldots, x_n, y_n, z)$ centered at p such that on $\mathcal{U}$*

$$\alpha = \sum x_i dy_i + dz \textit{ is a local contact form for } H \text{ .}$$

The idea behind the proof is sketched in the next lecture.

There is also a Moser-type theorem for contact forms.

Theorem 10.5 (Gray) *Let M be a compact manifold. Suppose that α_t, $t \in [0,1]$, is a smooth family of (global) contact forms on M. Let $H_t = \ker \alpha_t$. Then there exists an isotopy $\rho : M \times \mathbb{R} \longrightarrow M$ such that $H_t = \rho_{t*} H_0$, for all $0 \le t \le 1$.*

Exercise. Show that $H_t = \rho_{t*} H_0 \iff \rho_t^* \alpha_t = u_t \cdot \alpha_0$ for some family $u_t : M \longrightarrow \mathbb{R}$, $0 \le t \le 1$, of nowhere vanishing functions. $\diamondsuit$

Proof. *(À la Moser)*

We need to find ρ_t such that $\left\{ \begin{array}{l} \rho_0 = \mathrm{id} \\ \frac{d}{dt}(\rho_t^* \alpha_t) = \frac{d}{dt}(u_t \alpha_0) \text{ .} \end{array} \right.$ For any isotopy ρ,

$$\frac{d}{dt}(\rho_t^* \alpha_t) = \rho_t^* \left(\mathcal{L}_{v_t} \alpha_t + \frac{d\alpha_t}{dt} \right) \text{ ,}$$

where $v_t = \frac{d\rho_t}{dt} \circ \rho_t^{-1}$ is the vector field generated by ρ_t. By the Moser trick, it suffices to find v_t and then integrate it to ρ_t. We will search for v_t in $H_t = \ker \alpha_t$; this unnecessary assumption simplifies the proof.

We need to solve

$$
\begin{array}{rrcl}
 & \rho_t^*(\underbrace{\mathcal{L}_{v_t}\alpha_t}_{d\imath_{v_t}\alpha_t+\imath_{v_t}d\alpha_t}+\frac{d\alpha_t}{dt}) & = & \frac{du_t}{dt}\underbrace{\alpha_0}_{\frac{1}{u_t}\rho_t^*\alpha_t} \\
\Longrightarrow & \rho_t^*\left(\imath_{v_t}d\alpha_t+\frac{d\alpha_t}{dt}\right) & = & \frac{du_t}{dt}\cdot\frac{1}{u_t}\cdot\rho_t^*\alpha_t \\
\Longleftrightarrow & \imath_{v_t}d\alpha_t+\frac{d\alpha_t}{dt} & = & (\rho_t^*)^{-1}\left(\frac{du_t}{dt}\cdot\frac{1}{u_t}\right)\alpha_t\ . \qquad (\star)
\end{array}
$$

Restricting to the hyperplane $H_t=\ker\alpha_t$, equation $(\star)$ reads

$$
\imath_{v_t}d\alpha_t|_{H_t}=-\left.\frac{d\alpha_t}{dt}\right|_{H_t}
$$

which determines v_t uniquely, since $d\alpha_t|_{H_t}$ is nondegenerate. After integrating v_t to ρ_t, the factor u_t is determined by the relation $\rho_t^*\alpha_t=u_t\cdot\alpha_0$. Check that this indeed gives a solution. $\square$

Homework 7: Manifolds of Contact Elements

Given any manifold X of dimension n, there is a canonical symplectic manifold of dimension $2n$ attached to it, namely its cotangent bundle with the standard symplectic structure. The exercises below show that there is also a canonical *contact* manifold of dimension $2n-1$ attached to X.

The **manifold of contact elements** of an n-dimensional manifold X is

$$\mathcal{C} = \{(x, \chi_x) \,|\, x \in X \text{ and } \chi_x \text{ is a hyperplane in } T_xX\} \ .$$

On the other hand, the projectivization of the cotangent bundle of X is

$$\mathbb{P}^*X = (T^*X \setminus \text{zero section})/ \sim$$

where $(x,\xi) \sim (x,\xi')$ whenever $\xi = \lambda\xi'$ for some $\lambda \in \mathbb{R} \setminus \{0\}$ (here $x \in X$ and $\xi, \xi' \in T_x^*X \setminus \{0\}$). We will denote elements of $\mathbb{P}^*X$ by $(x, [\xi])$, $[\xi]$ being the $\sim$ equivalence class of ξ.

1. Show that $\mathcal{C}$ is naturally isomorphic to $\mathbb{P}^*X$ as a bundle over X, i.e., exhibit a diffeomorphism $\varphi : \mathcal{C} \to \mathbb{P}^*X$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\varphi} & \mathbb{P}^*X \\ \pi \downarrow & & \downarrow \pi \\ X & = & X \end{array}$$

   where the vertical maps are the natural projections $(x, \chi_x) \mapsto x$ and $(x,\xi) \mapsto x$.

   Hint: The kernel of a non-zero $\xi \in T_x^*X$ is a hyperplane $\chi_x \subset T_xX$. What is the relation between ξ and ξ' if $\ker \xi = \ker \xi'$?

2. There is on $\mathcal{C}$ a canonical field of hyperplanes $\mathcal{H}$ (that is, a smooth map attaching to each point in $\mathcal{C}$ a hyperplane in the tangent space to $\mathcal{C}$ at that point): $\mathcal{H}$ at the point $p = (x, \chi_x) \in \mathcal{C}$ is the hyperplane

$$\mathcal{H}_p = (d\pi_p)^{-1}\chi_x \ \subset T_p\mathcal{C} \ ,$$

   where

$$\begin{array}{ccccc} \mathcal{C} & \qquad & p = (x, \chi_x) & \qquad & T_p\mathcal{C} \\ \downarrow \pi & & \downarrow & & \downarrow d\pi_p \\ X & & x & & T_xX \end{array}$$

   are the natural projections, and $(d\pi_p)^{-1}\chi_x$ is the preimage of $\chi_x \subset T_xX$ by $d\pi_p$.

   Under the isomorphism $\mathcal{C} \simeq \mathbb{P}^*X$ from exercise 1, $\mathcal{H}$ induces a field of hyperplanes $\mathbb{H}$ on $\mathbb{P}^*X$. Describe $\mathbb{H}$.

   Hint: If $\xi \in T_x^*X \setminus \{0\}$ has kernel χ_x, what is the kernel of the canonical 1-form $\alpha_{(x,\xi)} = (d\pi_{(x,\xi)})^*\xi$?

3. Check that $(\mathbb{P}^*X, \mathbb{H})$ is a contact manifold, and therefore $(\mathcal{C}, \mathcal{H})$ is a contact manifold.

   Hint: Let $(x, [\xi]) \in \mathbb{P}^*X$. For any ξ representing the class $[\xi]$, we have
   $$\mathbb{H}_{(x,[\xi])} = \ker\left((d\pi_{(x,[\xi])})^*\xi\right) .$$
   Let $x_1, \ldots, x_n$ be local coordinates on X, and let $x_1, \ldots, x_n, \xi_1, \ldots, \xi_n$ be the associated local coordinates on T^*X. In these coordinates, $(x, [\xi])$ is given by $(x_1, \ldots, x_n, [\xi_1, \ldots, \xi_n])$. Since at least one of the ξ_i's is nonzero, without loss of generality we may assume that $\xi_1 \neq 0$ so that we may divide ξ by ξ_1 to obtain a representative with coordinates $(1, \xi_2, \ldots, \xi_n)$. Hence, by choosing always the representative of $[\xi]$ with $\xi_1 = 1$, the set $x_1, \ldots, x_n, \xi_2, \ldots, \xi_n$ defines coordinates on some neighborhood $\mathcal{U}$ of $(x, [\xi])$ in $\mathbb{P}^*X$. On $\mathcal{U}$, consider the 1-form
   $$\alpha = dx_1 + \sum_{i \geq 2} \xi_i dx_i .$$
   Show that α is a contact form on $\mathcal{U}$, i.e., show that $\ker \alpha_{(x,[\xi])} = \mathbb{H}_{(x,[\xi])}$, and that $d\alpha_{(x,[\xi])}$ is nondegenerate on $\mathbb{H}_{(x,[\xi])}$.

4. What is the symplectization of $\mathcal{C}$?

   What is the manifold $\mathcal{C}$ when $X = \mathbb{R}^3$ and when $X = S^1 \times S^1$?

Remark. Similarly, we could have defined the **manifold of oriented contact elements** of X to be
$$\mathcal{C}^o = \left\{ (x, \chi_x^o) \;\middle|\; x \in X \text{ and } \begin{array}{c} \chi_x^o \text{ is a hyperplane in } T_xX \\ \text{equipped with an orientation} \end{array} \right\} .$$
The manifold $\mathcal{C}^o$ is isomorphic to the cotangent sphere bundle of X
$$S^*X := (T^*X \setminus \text{zero section})/ \approx$$
where $(x, \xi) \approx (x, \xi')$ whenever $\xi = \lambda\xi'$ for some $\lambda \in \mathbb{R}^+$.

A construction analogous to the above produces a canonical contact structure on $\mathcal{C}^o$. See [3, Appendix 4].

$\diamondsuit$

11 Contact Dynamics

11.1 Reeb Vector Fields

Let (M,H) be a contact manifold with a contact form α.

Claim. There exists a unique vector field R on M such that $\begin{cases} \imath_R d\alpha &= 0 \\ \imath_R \alpha &= 1 \end{cases}$

Proof. $\begin{cases} \imath_R d\alpha = 0 & \Longrightarrow \quad R \in \ker d\alpha \text{ , which is a line bundle, and} \\ \imath_R \alpha = 1 & \Longrightarrow \quad \text{normalizes } R \text{ .} \end{cases}$ □

The vector field R is called the **Reeb vector field** determined by α.

Claim. The flow of R preserves the contact form, i.e., if $\rho_t = \exp tR$ is the isotopy generated by R, then $\rho_t^* \alpha = \alpha$, $\forall t \in \mathbb{R}$.

Proof. We have $\frac{d}{dt}(\rho_t^*\alpha) = \rho_t^*(\mathcal{L}_R \alpha) = \rho_t^*(d \underbrace{\imath_R \alpha}_{1} + \underbrace{\imath_R d\alpha}_{0}) = 0$.

Hence, $\rho_t^*\alpha = \rho_0^*\alpha = \alpha$, $\forall t \in \mathbb{R}$. □

Definition 11.1 *A* **contactomorphism** *is a diffeomorphism f of a contact manifold (M,H) which preserves the contact structure (i.e., $f_* H = H$).*

Examples.

1. Euclidean space $\mathbb{R}^{2n+1}$ with $\alpha = \sum_i x_i dy_i + dz$.

$$\left.\begin{array}{rcl} \imath_R \sum dx_i \wedge dy_i &=& 0 \\ \imath_R \sum x_i dy_i + dz &=& 1 \end{array}\right\} \Longrightarrow R = \frac{\partial}{\partial z} \text{ is the Reeb vector field.}$$

   The contactomorphisms generated by R are translations

$$\rho_t(x_1, y_1, \ldots, x_n, y_n, z) = (x_1, y_1, \ldots, x_n, y_n, z + t) \ .$$

2. Regard the odd sphere $S^{2n-1} \overset{i}{\hookrightarrow} \mathbb{R}^{2n}$ as the set of unit vectors

$$\{(x_1, y_1, \ldots, x_n, y_n) \mid \sum (x_i^2 + y_i^2) = 1\} \ .$$

   Consider the 1-form on $\mathbb{R}^{2n}$, $\sigma = \frac{1}{2}\sum(x_i dy_i - y_i dx_i)$.

   Claim. The form $\alpha = i^*\sigma$ is a contact form on S^{2n-1}.

   Proof. We need to show that $\alpha \wedge (d\alpha)^{n-1} \neq 0$. The 1-form on $\mathbb{R}^{2n}$ $\nu = d\sum(x_i^2 + y_i^2) = 2\sum(x_i dx_i + y_i dy_i)$ satisfies $T_p S^{2n-1} = \ker \nu_p$, at $p \in S^{2n-1}$. Check that $\nu \wedge \sigma \wedge (d\sigma)^{n-1} \neq 0$. □

The distribution $H = \ker\alpha$ is called the **standard contact structure** on S^{2n-1}. The Reeb vector field is $R = 2\sum\left(x_i\frac{\partial}{\partial y_i} - y_i\frac{\partial}{\partial x_i}\right)$, and is also known as the **Hopf vector field** on S^{2n-1}, as the orbits of its flow are the circles of the Hopf fibration.

$\diamondsuit$

11.2 Symplectization

Example. Let $\widetilde{M} = S^{2n-1}\times\mathbb{R}$, with coordinate τ in the $\mathbb{R}$-factor, and projection $\pi : \widetilde{M} \to S^{2n-1}$, $(p,\tau)\mapsto p$. Under the identification $\widetilde{M}\simeq\mathbb{R}^{2n}\backslash\{0\}$, where the $\mathbb{R}$-factor represents the logarithm of the square of the radius, the projection π becomes

$$\begin{array}{cccc} \pi: & \mathbb{R}^{2n}\backslash\{0\} & \longrightarrow & S^{2n-1} \\ & (X_1,Y_1,\ldots,X_n,Y_n) & \longmapsto & \left(\frac{X_1}{\sqrt{e^\tau}},\frac{Y_1}{\sqrt{e^\tau}},\ldots,\frac{X_n}{\sqrt{e^\tau}},\frac{Y_n}{\sqrt{e^\tau}}\right) \end{array}$$

where $e^\tau = \sum(X_i^2+Y_i^2)$. Let $\alpha = i^*\sigma$ be the standard contact form on S^{2n-1} (see the previous example). Then $\omega = d(e^\tau\pi^*\alpha)$ is a closed 2-form on $\mathbb{R}^{2n}\backslash\{0\}$. Since $\pi^*i^*x_i = \frac{X_i}{\sqrt{e^\tau}}$, $\pi^*i^*y_i = \frac{Y_i}{\sqrt{e^\tau}}$, we have

$$\begin{array}{rcl} \pi^*\alpha = \pi^*i^*\sigma & = & \frac{1}{2}\sum\left(\frac{X_i}{\sqrt{e^\tau}}d(\frac{Y_i}{\sqrt{e^\tau}}) - \frac{Y_i}{\sqrt{e^\tau}}d(\frac{X_i}{\sqrt{e^\tau}})\right) \\ & = & \frac{1}{2e^\tau}\sum(X_i dY_i - Y_i dX_i)\ . \end{array}$$

Therefore, $\omega = \sum dX_i\wedge dY_i$ is the standard symplectic form on $\mathbb{R}^{2n}\backslash\{0\}\subset\mathbb{R}^{2n}$. $(\widetilde{M},\omega)$ is called the *symplectization* of (S^{2n-1},α). $\diamondsuit$

Proposition 11.2 *Let (M,H) be a contact manifold with a contact form α. Let $\widetilde{M} = M\times\mathbb{R}$, and let $\pi : \widetilde{M}\to M$, $(p,\tau)\mapsto p$, be the projection. Then $\omega = d(e^\tau\pi^*\alpha)$ is a symplectic form on $\widetilde{M}$, where τ is a coordinate on $\mathbb{R}$.*

Proof. Exercise. $\square$

Hence, $\widetilde{M}$ has a symplectic form ω canonically determined by a contact form α on M and a coordinate function on $\mathbb{R}$; $(\widetilde{M},\omega)$ is called the **symplectization** of (M,α).

Remarks.

1. The contact version of the Darboux theorem can now be derived by applying the symplectic theorem to the symplectization of the contact manifold (with appropriate choice of coordinates); see [3, Appendix 4].

2. There is a coordinate-free description of $\widetilde{M}$ as

$$\widetilde{M} = \{(p, \xi) \mid p \in M,\ \xi \in T_p^* M,\ \text{such that} \ \ \ker \xi = H_p\} \ .$$

The group $\mathbb{R} \setminus \{0\}$ acts on $\widetilde{M}$ by multiplication on the cotangent vector:

$$\lambda \cdot (p, \xi) = (p, \lambda \xi) \ , \quad \lambda \in \mathbb{R} \setminus \{0\} \ .$$

The quotient $\widetilde{M}/(\mathbb{R} \setminus \{0\})$ is diffeomorphic to M. $\widetilde{M}$ has a canonical 1-form $\widetilde{\alpha}$ defined at $v \in T_{(p,\xi)}\widetilde{M}$ by

$$\widetilde{\alpha}_{(p,\xi)}(v) = \xi((d\,\mathrm{pr})_{(p,\xi)} v) \ ,$$

where $\mathrm{pr} : \widetilde{M} \to M$ is the bundle projection.

$\diamondsuit$

11.3 Conjectures of Seifert and Weinstein

Question. **(Seifert, 1948)** Let v be a nowhere vanishing vector field on the 3-sphere. Does the flow of v have any periodic orbits?

Counterexamples.

- **(Schweitzer, 1974)** $\exists\, C^1$ vector field without periodic orbits.
- **(Kristina Kuperberg, 1994)** $\exists\, C^\infty$ vector field without periodic orbits.

Question. How about volume-preserving vector fields?

- **(Greg Kuperberg, 1997)** $\exists\, C^1$ counterexample.
- C^∞ counterexamples are not known.

Natural generalization of this problem:

Let $M = S^3$ be the 3-sphere, and let γ be a volume form on M. Suppose that v is a nowhere vanishing vector field, and suppose that v is volume-preserving, i.e.,

$$\mathcal{L}_v \gamma = 0 \iff d\imath_v \gamma = 0 \iff \imath_v \gamma = d\alpha$$

for some 1-form α, since $H^2(S^3) = 0$.

Given a 1-form α, we would like to study vector fields v such that

$$\begin{cases} \imath_v \gamma = d\alpha \\ \imath_v \alpha > 0 \ . \end{cases}$$

A vector field v satisfying $\imath_v\alpha > 0$ is called **positive**. For instance, vector fields in a neighborhood of the Hopf vector field are positive relative to the standard contact form on S^3.

Renormalizing as $R := \frac{v}{\imath_v\alpha}$, we should study instead

$$\begin{cases} \imath_R d\alpha = 0 \\ \imath_\alpha = 1 \\ \alpha \wedge d\alpha \text{ is a volume form,} \end{cases}$$

that is, study pairs (α, R) where

$$\begin{cases} \alpha \text{ is a } \textbf{contact} \text{ form, and} \\ R \text{ is its } \textbf{Reeb} \text{ vector field.} \end{cases}$$

Conjecture. (Weinstein, 1978 [106]) *Suppose that M is a 3-dimensional manifold with a (global) contact form α. Let v be the Reeb vector field for α. Then v has a periodic orbit.*

Theorem 11.3 (Viterbo and Hofer, 1993 [63, 64, 103]) *The Weinstein conjecture is true when*

1) $M = S^3$, or

2) $\pi_2(M) \neq 0$, or

3) the contact structure is overtwisted.[10]

Open questions.

- How many periodic orbits are there?
- What do they look like?
- Is there always an unknotted one?
- What about the linking behavior?

[10] A surface S inside a contact 3-manifold determines a singular foliation on S, called the **characteristic foliation** of S, by the intersection of the contact planes with the tangent spaces to S. A contact structure on a 3-manifold M is called **overtwisted** if there exists an embedded 2-disk whose characteristic foliation contains one closed leaf C and exactly one singular point inside C; otherwise, the contact structure is called **tight**. Eliashberg [32] showed that the isotopy classification of overtwisted contact structures on closed 3-manifolds coincides with their homotopy classification as tangent plane fields. The classification of tight contact structures is still open.

Part V

Compatible Almost Complex Structures

The fact that any symplectic manifold possesses almost complex structures, and even so in a *compatible* sense, establishes a link from symplectic geometry to complex geometry, and is the point of departure for the modern technique of counting pseudo-holomorphic curves, as first proposed by Gromov [49].

12 Almost Complex Structures

12.1 Three Geometries

1. Symplectic geometry:
   geometry of a closed nondegenerate skew-symmetric bilinear form.

2. Riemannian geometry:
   geometry of a positive-definite symmetric bilinear map.

3. Complex geometry:
   geometry of a linear map with square -1.

Example. The euclidean space $\mathbb{R}^{2n}$ with the standard linear coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ has standard structures:

$$\begin{array}{rcll} \omega_0 & = & \sum dx_j \wedge dy_j \ , & \text{standard symplectic structure;} \\ g_0 & = & \langle \cdot, \cdot \rangle \ , & \text{standard inner product; and} \end{array}$$

if we identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ with coordinates $z_j = x_j + \sqrt{-1}\, y_j$, then multiplication by $\sqrt{-1}$ induces a constant linear map J_0 on the tangent spaces of $\mathbb{R}^{2n}$:

$$J_0(\frac{\partial}{\partial x_j}) = \frac{\partial}{\partial y_j} \ , \qquad J_0(\frac{\partial}{\partial y_j}) = -\frac{\partial}{\partial x_j} \ ,$$

with $J_0^2 = -\mathrm{Id}$. Relative to the basis $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}$, the maps J_0, ω_0 and g_0 are represented by

$$\begin{array}{rcl} J_0(u) & = & \left(\begin{array}{cc} 0 & -\mathrm{Id} \\ \mathrm{Id} & 0 \end{array} \right) u \\ \omega_0(u,v) & = & v^t \left(\begin{array}{cc} 0 & -\mathrm{Id} \\ \mathrm{Id} & 0 \end{array} \right) u \\ g_0(u,v) & = & v^t u \end{array}$$

where $u, v \in \mathbb{R}^{2n}$ and v^t is the transpose of v. The following compatibility relation holds:

$$\omega_0(u,v) = g_0(J_0(u), v) \ .$$

$\diamondsuit$

12.2 Complex Structures on Vector Spaces

Definition 12.1 *Let V be a vector space. A* **complex structure** *on V is a linear map:*

$$J : V \to V \qquad \text{with} \qquad J^2 = -\mathrm{Id} \ .$$

The pair (V, J) is called a **complex vector space**.

A complex structure J is equivalent to a structure of vector space over $\mathbb{C}$ if we identify the map J with multiplication by $\sqrt{-1}$.

Definition 12.2 *Let (V, Ω) be a symplectic vector space. A complex structure J on V is said to be* **compatible** *(with Ω, or Ω-compatible) if*

$$G_J(u,v) := \Omega(u, Jv) \ , \quad \forall u, v \in V \ , \ \text{is a positive inner product on } V \ .$$

That is,

$$J \text{ is } \Omega\text{-compatible} \iff \begin{cases} \Omega(Ju, Jv) = \Omega(u,v) & \text{[symplectomorphism]} \\ \Omega(u, Ju) > 0, \quad \forall u \neq 0 & \text{[taming condition]} \end{cases}$$

Compatible complex structures always exist on symplectic vector spaces:

Proposition 12.3 *Let (V, Ω) be a symplectic vector space. Then there is a compatible complex structure J on V.*

Proof. Choose a positive inner product G on V. Since Ω and G are nondegenerate,

$$\left.\begin{array}{rcl} u \in V & \longmapsto & \Omega(u, \cdot) \in V^* \\ w \in V & \longmapsto & G(w, \cdot) \in V^* \end{array}\right\} \text{ are isomorphisms between } V \text{ and } V^*.$$

Hence, $\Omega(u,v) = G(Au, v)$ for some linear map $A : V \to V$. This map A is skew-symmetric because

$$\begin{aligned} G(A^*u, v) &= G(u, Av) = G(Av, u) \\ &= \Omega(v, u) = -\Omega(u,v) = G(-Au, v) \ . \end{aligned}$$

Also:

- AA^* is symmetric: $(AA^*)^* = AA^*$.
- AA^* is positive: $G(AA^*u, u) = G(A^*u, A^*u) > 0$, for $u \neq 0$.

These properties imply that AA^* diagonalizes with positive eigenvalues λ_i,

$$AA^* = B \text{ diag}\,(\lambda_1, \ldots, \lambda_{2n})\ B^{-1}\ .$$

We may hence define an arbitrary real power of AA^* by rescaling the eigenspaces, in particular,

$$\sqrt{AA^*} := B \,\text{diag}\,(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_{2n}})\ B^{-1}\ .$$

Then $\sqrt{AA^*}$ is symmetric and positive-definite. Let

$$J = (\sqrt{AA^*})^{-1} A\ .$$

The factorization $A = \sqrt{AA^*}\, J$ is called the **polar decomposition** of A. Since A commutes with $\sqrt{AA^*}$, J commutes with $\sqrt{AA^*}$. Check that J is orthogonal, $JJ^* = \mathrm{Id}$, as well as skew-adjoint, $J^* = -J$, and hence it is a complex structure on V:

$$J^2 = -JJ^* = -\mathrm{Id}\ .$$

Compatibility:

$$\begin{array}{rcl} \Omega(Ju, Jv) & = & G(AJu, Jv) = G(JAu, Jv) = G(Au, v) \\ & = & \Omega(u, v) \\ \Omega(u, Ju) & = & G(Au, Ju) = G(-JAu, u) \\ & = & G(\sqrt{AA^*}\, u, u) > 0\ , \quad \text{for } u \neq 0\ . \end{array}$$

Therefore, J is a compatible complex structure on V. □

As indicated in the proof, in general, the positive inner product defined by

$$\Omega(u, Jv) = G(\sqrt{AA^*}\, u, v) \ \text{ is different from } \ G(u, v)\ .$$

Remarks.

1. This construction is canonical after an initial choice of G. To see this, notice that $\sqrt{AA^*}$ does not depend on the choice of B nor of the ordering of the eigenvalues in diag $(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_{2n}})$. The linear transformation $\sqrt{AA^*}$ is completely determined by its effect on each eigenspace of AA^*: on the eigenspace corresponding to the eigenvalue λ_k, the map $\sqrt{AA^*}$ is defined to be multiplication by $\sqrt{\lambda_k}$.

2. If (V_t, Ω_t) is a family of symplectic vector spaces with a family G_t of positive inner products, all depending smoothly on a real parameter t, then, adapting the proof of the previous proposition, we can show that there is a smooth family J_t of compatible complex structures on V_t.

3. To check just the existence of compatible complex structures on a symplectic vector space (V, Ω), we could also proceed as follows. Given a symplectic basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ (i.e., $\Omega(e_i, e_j) = \Omega(f_i, f_j) = 0$ and $\Omega(e_i, f_j) = \delta_{ij}$), one can define $Je_j = f_j$ and $Jf_j = -e_j$. This is a compatible complex

structure on (V, Ω). Moreover, given Ω and J compatible on V, there exists a symplectic basis of V of the form:

$$e_1, \ldots, e_n, f_1 = Je_1, \ldots, f_n = Je_n \ .$$

The proof is part of Homework 8.

4. Conversely, given (V, J), there is always a symplectic structure Ω such that J is Ω-compatible: pick any positive inner product G such that $J^* = -J$ and take $\Omega(u, v) = G(Ju, v)$.

$\diamondsuit$

12.3 Compatible Structures

Definition 12.4 *An* **almost complex structure** *on a manifold M is a smooth field of complex structures on the tangent spaces:*

$$x \longmapsto J_x : T_xM \to T_xM \quad \textit{linear,} \quad \textit{and} \quad J_x^2 = -\mathrm{Id} \ .$$

The pair (M, J) is then called an **almost complex manifold**.

Definition 12.5 *Let (M, ω) be a symplectic manifold. An almost complex structure J on M is called* **compatible** *(with ω or ω-compatible) if the assignment*

$$\begin{aligned} x \ \longmapsto \ & g_x : T_xM \times T_xM \to \mathbb{R} \\ & g_x(u, v) := \omega_x(u, J_xv) \end{aligned}$$

is a riemannian metric on M.

For a manifold M,

ω is a symplectic form	$\Longrightarrow$	$x \longmapsto$	$\omega_x : T_xM \times T_xM \to \mathbb{R}$ is bilinear, nondegenerate, skew-symmetric;
g is a riemannian metric	$\Longrightarrow$	$x \longmapsto$	$g_x : T_xM \times T_xM \to \mathbb{R}$ is a positive inner product;
J almost complex structure	$\Longrightarrow$	$x \longmapsto$	$J_x : T_xM \to T_xM$ is linear and $J^2 = -\mathrm{Id}$.

The triple (ω, g, J) is called a **compatible triple** when $g(\cdot, \cdot) = \omega(\cdot, J\cdot)$.

Proposition 12.6 *Let (M, ω) be a symplectic manifold, and g a riemannian metric on M. Then there exists a canonical almost complex structure J on M which is compatible.*

Proof. The polar decomposition is *canonical* (after a choice of metric), hence this construction of J on M is *smooth*; cf. Remark 2 of the previous section. $\square$

Remark. In general, $g_J(\cdot, \cdot) := \omega(\cdot, J\cdot) \neq g(\cdot, \cdot)$. $\diamondsuit$

Since riemannian metrics always exist, we conclude:

Corollary 12.7 *Any symplectic manifold has compatible almost complex structures.*

– How different can compatible almost complex structures be?

Proposition 12.8 *Let (M, ω) be a symplectic manifold, and J_0, J_1 two almost complex structures compatible with ω. Then there is a smooth family $J_t, 0 \leq t \leq 1$, of compatible almost complex structures joining J_0 to J_1.*

Proof. By compatibility, we get

$$\left.\begin{array}{lcl} \omega, J_0 & \rightsquigarrow & g_0(\cdot,\cdot) = \omega(\cdot, J_0 \cdot) \\ \omega, J_1 & \rightsquigarrow & g_1(\cdot,\cdot) = \omega(\cdot, J_1 \cdot) \end{array}\right\} \quad \text{two riemannian metrics on } M \ .$$

Their convex combinations

$$g_t(\cdot,\cdot) = (1-t)g_0(\cdot,\cdot) + tg_1(\cdot,\cdot) \ , \qquad 0 \leq t \leq 1 \ ,$$

form a smooth family of riemannian metrics. Apply the polar decomposition to (ω, g_t) to obtain a smooth family of J_t's joining J_0 to J_1. $\square$

Corollary 12.9 *The set of all compatible almost complex structures on a symplectic manifold is path-connected.*

Homework 8: Compatible Linear Structures

1. Let $\Omega(V)$ and $J(V)$ be the spaces of symplectic forms and complex structures on the vector space V, respectively. Take $\Omega \in \Omega(V)$ and $J \in J(V)$. Let $\mathrm{GL}(V)$ be the group of all isomorphisms of V, let $\mathrm{Sp}(V,\Omega)$ be the group of symplectomorphisms of (V,Ω), and let $\mathrm{GL}(V,J)$ be the group of complex isomorphisms of (V,J).

   Show that

   $$\Omega(V) \simeq \mathrm{GL}(V)/\mathrm{Sp}(V,\Omega) \quad \text{and} \quad J(V) \simeq \mathrm{GL}(V)/\mathrm{GL}(V,J) .$$

   Hint: The group $\mathrm{GL}(V)$ acts on $\Omega(V)$ by pullback. What is the stabilizer of a given Ω?

2. Let $(\mathbb{R}^{2n},\Omega_0)$ be the standard $2n$-dimensional symplectic euclidean space.

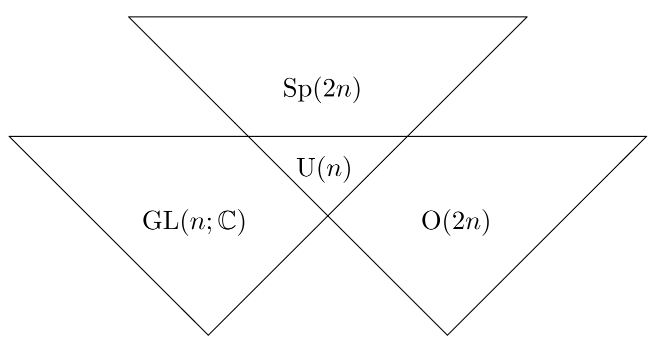
   The **symplectic linear group** is the group of all linear transformations of $\mathbb{R}^{2n}$ which preserve the symplectic structure:

   $$\mathrm{Sp}(2n) := \{A \in \mathrm{GL}(2n;\mathbb{R}) \,|\, \Omega_0(Au,Av) = \Omega_0(u,v) \text{ for all } u,v \in \mathbb{R}^{2n}\} .$$

   Identifying the complex $n \times n$ matrix $X + iY$ with the real $2n \times 2n$ matrix $\begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}$, consider the following subgroups of $\mathrm{GL}(2n;\mathbb{R})$:

   $$\mathrm{Sp}(2n) \ , \ \mathrm{O}(2n) \ , \ \mathrm{GL}(n;\mathbb{C}) \ \text{ and } \ \mathrm{U}(n) \ .$$

   Show that the intersection of any two of them is $\mathrm{U}(n)$. (From [83, p.41].)



3. Let (V, Ω) be a symplectic vector space of dimension $2n$, and let $J : V \to V$, $J^2 = -\mathrm{Id}$, be a complex structure on V.

   (a) Prove that, if J is Ω-compatible and L is a lagrangian subspace of (V, Ω), then JL is also lagrangian and $JL = L^\perp$, where $\perp$ denotes orthogonality with respect to the positive inner product $G_J(u, v) = \Omega(u, Jv)$.

   (b) Deduce that J is Ω-compatible if and only if there exists a symplectic basis for V of the form

   $$e_1, e_2, \ldots, e_n, f_1 = Je_1, f_2 = Je_2, \ldots, f_n = Je_n$$

   where $\Omega(e_i, e_j) = \Omega(f_i, f_j) = 0$ and $\Omega(e_i, f_j) = \delta_{ij}$.

13 Compatible Triples

13.1 Compatibility

Let (M, ω) be a symplectic manifold. As shown in the previous lecture, compatible almost complex structures always exist on (M, ω). We also showed that the set of all compatible almost complex structures on (M, ω) is path-connected. In fact, the set of all compatible almost complex structures is even contractible. (This is important for defining invariants.) Let $\mathcal{J}(T_xM, \omega_x)$ be the set of all compatible complex structures on (T_xM, ω_x) for $x \in M$.

Proposition 13.1 *The set $\mathcal{J}(T_xM, \omega_x)$ is contractible, i.e., there exists a homotopy*

$$h_t : \mathcal{J}(T_xM, \omega_x) \longrightarrow \mathcal{J}(T_xM, \omega_x) \ , \quad 0 \leq t \leq 1 \ ,$$

starting at the identity $h_0 = \mathrm{Id}$,
finishing at a trivial map $h_1 : \mathcal{J}(T_xM, \omega_x) \to \{J_0\}$,
and fixing J_0 (i.e., $h_t(J_0) = J_0$, $\forall t$) for some $J_0 \in \mathcal{J}(T_xM, \omega_x)$.

Proof. Homework 9. $\square$

Consider the fiber bundle $\mathcal{J} \to M$ with fiber

$$\mathcal{J}_x := \mathcal{J}(T_xM, \omega_x) \quad \text{over } x \in M \ .$$

A compatible almost complex structure J on (M, ω) is a section of $\mathcal{J}$. The space of sections of $\mathcal{J}$ is contractible because the fibers are contractible.

Remarks.

- We never used the closedness of ω to construct compatible almost complex structures. The construction holds for an **almost symplectic manifold** (M, ω), that is, a pair of a manifold M and a nondegenerate 2-form ω, not necessarily closed.

- Similarly, we could define a **symplectic vector bundle** to be a vector bundle $E \to M$ over a manifold M equipped with a smooth field ω of fiberwise nondegenerate skew-symmetric bilinear maps

  $$\omega_x : E_x \times E_x \longrightarrow \mathbb{R} \ .$$

  The existence of such a field ω is equivalent to being able to reduce the structure group of the bundle from the general linear group to the linear symplectic group. As a consequence of our discussion, a symplectic vector bundle is always a complex vector bundle, and vice-versa.

$\diamondsuit$

13.2 Triple of Structures

If (ω, J, g) is a **compatible triple**, then any one of ω, J or g can be written in terms of the other two:

$$\begin{aligned} g(u,v) &= \omega(u,Jv) \\ \omega(u,v) &= g(Ju,v) \\ J(u) &= \widetilde{g}^{-1}(\widetilde{\omega}(u)) \end{aligned}$$

where

$$\begin{array}{lllllll} \widetilde{\omega}: & TM & \longrightarrow & T^*M & \qquad u & \longmapsto & \omega(u,\cdot) \\ \widetilde{g}: & TM & \longrightarrow & T^*M & \qquad u & \longmapsto & g(u,\cdot) \end{array}$$

are the linear isomorphisms induced by the bilinear forms ω and g.

The relations among ω, J and g can be summarized in the following table. The last column lists differential equations these structures are usually asked to satisfy.

Data	**Condition/Technique**	**Consequence**	**Question**
ω, J	$\omega(Ju,Jv) = \omega(u,v)$ $\omega(u,Ju) > 0, u \neq 0$	$g(u,v) := \omega(u,Jv)$ is positive inner product	(g flat?)
g, J	$g(Ju,Jv) = g(u,v)$ (i.e., J is orthogonal)	$\omega(u,v) := g(Ju,v)$ is nondeg., skew-symm.	ω closed?
ω, g	polar decomposition $\rightsquigarrow$	J almost complex str.	J integrable?

An almost complex structure J on a manifold M is called **integrable** if and only if J is induced by a structure of complex manifold on M. In Lecture 15 we will discuss tests to check whether a given J is integrable.

13.3 First Consequences

Proposition 13.2 *Let (M, J) be an almost complex manifold. Suppose that J is compatible with two symplectic structures ω_0, ω_1 Then ω_0, ω_1 are deformation-equivalent, that is, there exists a smooth family ω_t, $0 \le t \le 1$, of symplectic forms joining ω_0 to ω_1.*

Proof. Take $\omega_t = (1-t)\omega_0 + t\omega_1$, $0 \le t \le 1$. Then:

- ω_t is closed.
- ω_t is nondegenerate, since

$$g_t(\cdot,\cdot) := \omega_t(\cdot, J\cdot) = (1-t)g_0(\cdot,\cdot) + tg_1(\cdot,\cdot)$$

is positive, hence nondegenerate.

□

Remark. The converse of this proposition is not true. A counterexample is provided by the following family in $\mathbb{R}^4$:

$$\omega_t = \cos \pi t \ dx_1 dy_1 + \sin \pi t \ dx_1 dy_2 + \sin \pi t \ dy_1 dx_2 + \cos \pi t \ dx_2 dy_2 \ , 0 \leq t \leq 1 \ .$$

There is no J in $\mathbb{R}^4$ compatible with both ω_0 and ω_1. ♢

Definition 13.3 *A submanifold X of an almost complex manifold (M, J) is an* **almost complex submanifold** *when $J(TX) \subseteq TX$, i.e., for all $x \in X, v \in T_x X$, we have $J_x v \in T_x X$.*

Proposition 13.4 *Let (M, ω) be a symplectic manifold equipped with a compatible almost complex structure J. Then any almost complex submanifold X of (M, J) is a symplectic submanifold of (M, ω).*

Proof. Let $i : X \hookrightarrow M$ be the inclusion. Then $i^*\omega$ is a closed 2-form on X.
Nondegeneracy:

$$\omega_x(u, v) = g_x(J_x u, v) \ , \qquad \forall x \in X \ , \ \forall u, v \in T_x X \ .$$

Since $g_x|_{T_x X}$ is nondegenerate, so is $\omega_x|_{T_x X}$. Hence, $i^*\omega$ is symplectic. □

– When is an almost complex manifold a complex manifold? See Lecture 15.

Examples.

S^2 is an almost complex manifold and it is a complex manifold.

S^4 is not an almost complex manifold (proved by Ehresmann and Hopf).

S^6 is almost complex and it is not yet known whether it is complex.

S^8 and higher spheres are not almost complex manifolds.

♢

Homework 9: Contractibility

The following proof illustrates in a geometric way the relation between lagrangian subspaces, complex structures and inner products; from [11, p.45].

Let (V,Ω) be a symplectic vector space, and let $\mathcal{J}(V,\Omega)$ be the set of all complex structures on (V,Ω) which are Ω-compatible; i.e., given a complex structure J on V we have

$$J \in \mathcal{J}(V,\Omega) \iff G_J(\cdot,\cdot) := \Omega(\cdot, J\cdot) \text{ is a positive inner product on } V \ .$$

Fix a lagrangian subspace L_0 of (V,Ω). Let $\mathcal{L}(V,\Omega,L_0)$ be the space of all lagrangian subspaces of (V,Ω) which intersect L_0 transversally. Let $\mathcal{G}(L_0)$ be the space of all positive inner products on L_0.

Consider the map

$$\begin{array}{rrcl} \Psi : & \mathcal{J}(V,\Omega) & \to & \mathcal{L}(V,\Omega,L_0) \times \mathcal{G}(L_0) \\ & J & \mapsto & (JL_0, G_J|_{L_0}) \end{array}$$

Show that:

1. Ψ is well-defined.

2. Ψ is a bijection.

   Hint: Given $(L,G) \in \mathcal{L}(V,\Omega,L_0) \times \mathcal{G}(L_0)$, define J in the following manner: For $v \in L_0$, $v^\perp = \{u \in L_0 \mid G(u,v) = 0\}$ is a $(n-1)$-dimensional space of L_0; its symplectic orthogonal $(v^\perp)^\Omega$ is $(n+1)$-dimensional. Check that $(v^\perp)^\Omega \cap L$ is 1-dimensional. Let Jv be the unique vector in this line such that $\Omega(v, Jv) = 1$. Check that, if we take v's in some G-orthonormal basis of L_0, this defines the required element of $\mathcal{J}(V,\Omega)$.

3. $\mathcal{L}(V,\Omega,L_0)$ is contractible.

   Hint: Prove that $\mathcal{L}(V,\Omega,L_0)$ can be identified with the vector space of all symmetric $n \times n$ matrices. Notice that any n-dimensional subspace L of V which is transversal to L_0 is the graph of a linear map $S : JL_0 \to L_0$, i.e.,

   $$\begin{array}{rrcl} & L & = & \text{span of } \{Je_1 + SJe_1, \ldots, Je_n + SJe_n\} \\ \text{when} & L_0 & = & \text{span of } \{e_1, \ldots, e_n\} \ . \end{array}$$

4. $\mathcal{G}(L_0)$ is contractible.

   Hint: $\mathcal{G}(L_0)$ is even convex.

Conclude that $\mathcal{J}(V,\Omega)$ is contractible.

14 Dolbeault Theory

14.1 Splittings

Let (M, J) be an almost complex manifold. The complexified tangent bundle of M is the bundle

$$\begin{array}{c} TM \otimes \mathbb{C} \\ \downarrow \\ M \end{array}$$

with fiber $(TM \otimes \mathbb{C})_p = T_pM \otimes \mathbb{C}$ at $p \in M$. If

$$\begin{array}{ll} T_pM & \text{is a } 2n\text{-dimensional vector space over } \mathbb{R}\ , \text{ then} \\ T_pM \otimes \mathbb{C} & \text{is a } 2n\text{-dimensional vector space over } \mathbb{C}\ . \end{array}$$

We may extend J linearly to $TM \otimes \mathbb{C}$:

$$J(v \otimes c) = Jv \otimes c\ , \quad v \in TM\ , \quad c \in \mathbb{C}\ .$$

Since $J^2 = -\mathrm{Id}$, on the complex vector space $(TM \otimes \mathbb{C})_p$, the linear map J_p has eigenvalues $\pm i$. Let

$$\begin{array}{rcl} T_{1,0} & = & \{v \in TM \otimes \mathbb{C} \mid Jv = +iv\} = (+i)\text{-eigenspace of } J \\ & = & \{v \otimes 1 - Jv \otimes i \mid v \in TM\} \\ & = & \textbf{(}\boldsymbol{J}\textbf{-)holomorphic tangent vectors}\ ; \\ \\ T_{0,1} & = & \{v \in TM \otimes \mathbb{C} \mid Jv = -iv\} = (-i)\text{-eigenspace of } J \\ & = & \{v \otimes 1 + Jv \otimes i \mid v \in TM\} \\ & = & \textbf{(}\boldsymbol{J}\textbf{-)anti-holomorphic tangent vectors}\ . \end{array}$$

Since

$$\begin{array}{rcl} \pi_{1,0} : TM & \longrightarrow & T_{1,0} \\ v & \longmapsto & \frac{1}{2}(v \otimes 1 - Jv \otimes i) \end{array}$$

is a (real) bundle isomorphism such that $\pi_{1,0} \circ J = i\pi_{1,0}$, and

$$\begin{array}{rcl} \pi_{0,1} : TM & \longrightarrow & T_{0,1} \\ v & \longmapsto & \frac{1}{2}(v \otimes 1 + Jv \otimes i) \end{array}$$

is also a (real) bundle isomorphism such that $\pi_{0,1} \circ J = -i\pi_{0,1}$, we conclude that we have isomorphisms of complex vector bundles

$$(TM, J) \simeq T_{1,0} \simeq \overline{T_{0,1}}\ ,$$

where $\overline{T_{0,1}}$ denotes the complex conjugate bundle of $T_{0,1}$. Extending $\pi_{1,0}$ and $\pi_{0,1}$ to projections of $TM \otimes \mathbb{C}$, we obtain an isomorphism

$$(\pi_{1,0}, \pi_{0,1}) : TM \otimes \mathbb{C} \xrightarrow{\simeq} T_{1,0} \oplus T_{0,1}\ .$$

Similarly, the complexified cotangent bundle splits as

$$(\pi^{1,0}, \pi^{0,1}) : T^*M \otimes \mathbb{C} \xrightarrow{\simeq} T^{1,0} \oplus T^{0,1}$$

where

$$\begin{aligned} T^{1,0} &= (T_{1,0})^* = \{\eta \in T^* \otimes \mathbb{C} \mid \eta(J\omega) = i\eta(\omega) \ , \forall \omega \in TM \otimes \mathbb{C}\} \\ &= \{\xi \otimes 1 - (\xi \circ J) \otimes i \mid \xi \in T^*M\} \\ &= \textbf{complex-linear cotangent vectors} \ , \end{aligned}$$

$$\begin{aligned} T^{0,1} &= (T_{0,1})^* = \{\eta \in T^* \otimes \mathbb{C} \mid \eta(J\omega) = -i\eta(\omega) \ , \forall \omega \in TM \otimes \mathbb{C}\} \\ &= \{\xi \otimes 1 + (\xi \circ J) \otimes i \mid \xi \in T^*M\} \\ &= \textbf{complex-antilinear cotangent vectors} \ , \end{aligned}$$

and $\pi^{1,0}, \pi^{0,1}$ are the two natural projections

$$\begin{array}{rcl} \pi^{1,0} : T^*M \otimes \mathbb{C} & \longrightarrow & T^{1,0} \\ \eta & \longmapsto & \eta^{1,0} := \frac{1}{2}(\eta - i\eta \circ J) \ ; \end{array}$$

$$\begin{array}{rcl} \pi^{0,1} : T^*M \otimes \mathbb{C} & \longrightarrow & T^{0,1} \\ \eta & \longmapsto & \eta^{0,1} := \frac{1}{2}(\eta + i\eta \circ J) \ . \end{array}$$

14.2 Forms of Type (ℓ, m)

For an almost complex manifold (M, J), let

$$\begin{aligned} \Omega^k(M; \mathbb{C}) &:= \text{sections of } \Lambda^k(T^*M \otimes \mathbb{C}) \\ &= \textbf{complex-valued k-forms on } M, where \end{aligned}$$

$$\begin{aligned} \Lambda^k(T^*M \otimes \mathbb{C}) &:= \Lambda^k(T^{1,0} \oplus T^{0,1}) \\ &= \oplus_{\ell+m=k} \underbrace{(\Lambda^\ell T^{1,0}) \wedge (\Lambda^m T^{0,1})}_{\Lambda^{\ell,m}(\text{definition})} \\ &= \oplus_{\ell+m=k} \Lambda^{\ell,m} \ . \end{aligned}$$

In particular, $\Lambda^{1,0} = T^{1,0}$ and $\Lambda^{0,1} = T^{0,1}$.

Definition 14.1 *The* **differential forms of type (ℓ, m)** *on (M, J) are the sections of $\Lambda^{\ell,m}$:*

$$\Omega^{\ell,m} := \textit{ sections of } \Lambda^{\ell,m} \ .$$

Then

$$\Omega^k(M; \mathbb{C}) = \oplus_{\ell+m=k} \Omega^{\ell,m} \ .$$

Let $\pi^{\ell,m} : \Lambda^k(T^*M \otimes \mathbb{C}) \to \Lambda^{\ell,m}$ be the projection map, where $\ell + m = k$. The usual exterior derivative d composed with two of these projections induces differential operators ∂ and $\bar{\partial}$ on forms of type (ℓ, m):

$$\begin{aligned} \partial &:= \pi^{\ell+1,m} \circ d : \Omega^{\ell,m}(M) \longrightarrow \Omega^{\ell+1,m}(M) \\ \bar{\partial} &:= \pi^{\ell,m+1} \circ d : \Omega^{\ell,m}(M) \longrightarrow \Omega^{\ell,m+1}(M) \ . \end{aligned}$$

If $\beta \in \Omega^{\ell,m}(M)$, with $k = \ell + m$, then $d\beta \in \Omega^{k+1}(M;\mathbb{C})$:

$$d\beta = \sum_{r+s=k+1} \pi^{r,s} d\beta = \pi^{k+1,0} d\beta + \cdots + \partial\beta + \bar{\partial}\beta + \cdots + \pi^{0,k+1} d\beta \ .$$

14.3 J-Holomorphic Functions

Let $f : M \to \mathbb{C}$ be a smooth complex-valued function on M. The exterior derivative d extends linearly to $\mathbb{C}$-valued functions as $df = d(\mathrm{Re} f) + i\, d(\mathrm{Im} f)$.

Definition 14.2 *A function f is* **(J-)holomorphic at** $x \in M$ *if df_p is complex linear, i.e., $df_p \circ J = i\, df_p$. A function f is* **(J-)holomorphic** *if it is holomorphic at all $p \in M$.*

Exercise. Show that

$$df_p \circ J = i\, df_p \quad \Longleftrightarrow \quad df_p \in T_p^{1,0} \quad \Longleftrightarrow \quad \pi_p^{0,1} df_p = 0 \ .$$

$\diamondsuit$

Definition 14.3 *A function f is* **(J-)anti-holomorphic at** $p \in M$ *if df_p is complex antilinear, i.e., $df_p \circ J = -i\, df_p$.*

Exercise.

$$\begin{aligned} df_p \circ J = -i\, df_p \quad &\Longleftrightarrow \quad df_p \in T_p^{0,1} \quad \Longleftrightarrow \quad \pi_p^{1,0} df_p = 0 \\ &\Longleftrightarrow \quad d\bar{f}_p \in T_p^{1,0} \quad \Longleftrightarrow \quad \pi_p^{0,1} d\bar{f}_p = 0 \\ &\Longleftrightarrow \quad \bar{f} \text{ is holomorphic at } p \in M \ . \end{aligned}$$

$\diamondsuit$

Definition 14.4 *On functions, $d = \partial + \bar{\partial}$, where*

$$\partial := \pi^{1,0} \circ d \quad \text{and} \quad \bar{\partial} := \pi^{0,1} \circ d \ .$$

Then

$$\begin{aligned} f \text{ is holomorphic} \quad &\Longleftrightarrow \quad \bar{\partial} f = 0 \ , \\ f \text{ is anti-holomorphic} \quad &\Longleftrightarrow \quad \partial f = 0 \ . \end{aligned}$$

– What about higher differential forms?

14.4 Dolbeault Cohomology

Suppose that $d = \partial + \bar{\partial}$, i.e.,

$$d\beta = \underbrace{\partial\beta}_{\in\Omega^{\ell+1,m}} + \underbrace{\bar{\partial}\beta}_{\in\Omega^{\ell,m+1}} \ , \quad \forall \beta \in \Omega^{\ell,m} \ .$$

Then, for any form $\beta \in \Omega^{\ell,m}$,

$$0 = d^2\beta = \underbrace{\partial^2\beta}_{\in\Omega^{\ell+2,m}} + \underbrace{\partial\bar{\partial}\beta + \bar{\partial}\partial\beta}_{\in\Omega^{\ell+1,m+1}} + \underbrace{\bar{\partial}^2\beta}_{\in\Omega^{\ell,m+2}} \ ,$$

which implies

$$\begin{cases} \bar{\partial}^2 = 0 \\ \partial\bar{\partial} + \bar{\partial}\partial = 0 \\ \partial^2 = 0 \end{cases}$$

Since $\bar{\partial}^2 = 0$, the chain

$$0 \longrightarrow \Omega^{\ell,0} \xrightarrow{\bar{\partial}} \Omega^{\ell,1} \xrightarrow{\bar{\partial}} \Omega^{\ell,2} \xrightarrow{\bar{\partial}} \cdots$$

is a differential complex; its cohomology groups

$$H^{\ell,m}_{\text{Dolbeault}}(M) := \frac{\ker\ \bar{\partial} : \Omega^{\ell,m} \longrightarrow \Omega^{\ell,m+1}}{\text{im}\ \bar{\partial} : \Omega^{\ell,m-1} \longrightarrow \Omega^{\ell,m}}$$

are called the **Dolbeault cohomology** groups.

$$\begin{tikzcd}
 & & \Omega^{0,0} \arrow[dl, "\partial"'] \arrow[dr, "\bar{\partial}"] & & \\
 & \Omega^{1,0} \arrow[dl, "\partial"'] \arrow[dr, "\bar{\partial}"] & & \Omega^{0,1} \arrow[dl, "\partial"'] \arrow[dr, "\bar{\partial}"] & \\
\Omega^{2,0} & & \Omega^{1,1} & & \Omega^{0,2} \\
\vdots & & \vdots & & \vdots
\end{tikzcd}$$

– When is $d = \partial + \bar{\partial}$? See the next lecture.

Homework 10: Integrability

This set of problems is from [11, p.46-47].

1. Let (M, J) be an almost complex manifold. Its **Nijenhuis tensor** $\mathcal{N}$ is:

$$\mathcal{N}(v, w) := [Jv, Jw] - J[v, Jw] - J[Jv, w] - [v, w] \ ,$$

where v and w are vector fields on M, $[\cdot, \cdot]$ is the usual bracket

$$[v, w] \cdot f := v \cdot (w \cdot f) - w \cdot (v \cdot f) \ , \ \text{ for } f \in C^\infty(M) \ ,$$

and $v \cdot f = df(v)$.

   (a) Check that, if the map $v \mapsto [v, w]$ is complex linear (in the sense that it commutes with J), then $\mathcal{N} \equiv 0$.

   (b) Show that $\mathcal{N}$ is actually a tensor, that is: $\mathcal{N}(v, w)$ at $x \in M$ depends only on the values $v_x, w_x \in T_xM$ and not really on the vector fields v and w.

   (c) Compute $\mathcal{N}(v, Jv)$. Deduce that, if M is a surface, then $\mathcal{N} \equiv 0$.

A theorem of Newlander and Nirenberg [89] states that an almost complex manifold (M, J) is a complex (analytic) manifold if and only if $\mathcal{N} \equiv 0$. Combining (c) with the fact that any orientable surface is symplectic, we conclude that any orientable surface is a complex manifold, a result already known to Gauss.

2. Let $\mathcal{N}$ be as above. For any map $f : \mathbb{R}^{2n} \to \mathbb{C}$ and any vector field v on $\mathbb{R}^{2n}$, we have $v \cdot f = v \cdot (f_1 + if_2) = v \cdot f_1 + i\, v \cdot f_2$, so that $f \mapsto v \cdot f$ is a complex linear map.

   (a) Let $\mathbb{R}^{2n}$ be endowed with an almost complex structure J, and suppose that f is a J-**holomorphic function**, that is,

$$df \circ J = i\, df \ .$$

   Show that $df(\mathcal{N}(v, w)) = 0$ for all vector fields v, w.

   (b) Suppose that there exist n J-holomorphic functions, $f_1, \ldots, f_n$, on $\mathbb{R}^{2n}$, which are independent at some point p, i.e., the real and imaginary parts of $(df_1)_p, \ldots, (df_n)_p$ form a basis of $T_p^*\mathbb{R}^{2n}$. Show that $\mathcal{N}$ vanishes identically at p.

   (c) Assume that M is a complex manifold and J is its complex structure. Show that $\mathcal{N}$ vanishes identically everywhere on M.

In general, an almost complex manifold has *no* J-holomorphic functions at all. On the other hand, it has *plenty* of J-**holomorphic curves**: maps $f : \mathbb{C} \to M$ such that $df \circ i = J \circ df$. J-holomorphic curves, also known as **pseudo-holomorphic curves**, provide a main tool in symplectic topology, as first realized by Gromov [49].

Part VI
Kähler Manifolds

Kähler geometry lies at the intersection of complex, riemannian and symplectic geometries, and plays a central role in all of these fields. We will start by reviewing complex manifolds. After describing the local normal form for Kähler manifolds (Lecture 16), we conclude with a summary of Hodge theory for compact Kähler manifolds (Lecture 17).

15 Complex Manifolds

15.1 Complex Charts

Definition 15.1 *A* **complex manifold** *of (complex) dimension n is a set M with a complete complex atlas*

$$\mathcal{A} = \{(\mathcal{U}_\alpha, \mathcal{V}_\alpha, \varphi_\alpha)\ , \alpha \in \text{ index set } I\}$$

where $M = \cup_\alpha \mathcal{U}_\alpha$, the $\mathcal{V}_\alpha$'s are open subsets of $\mathbb{C}^n$, and the maps $\varphi_\alpha : \mathcal{U}_\alpha \to \mathcal{V}_\alpha$ are such that the transition maps $\psi_{\alpha\beta}$ are biholomorphic *as maps on open subsets of $\mathbb{C}^n$:*

$$\begin{tikzcd}
 & \mathcal{U}_\alpha \cap \mathcal{U}_\beta \arrow[dl, "\varphi_\alpha"', "\simeq"] \arrow[dr, "\varphi_\beta", "\simeq"'] & \\
\mathcal{V}_{\alpha\beta} \arrow[rr, "\psi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1}"] & & \mathcal{V}_{\beta\alpha}
\end{tikzcd}$$

where $\mathcal{V}_{\alpha\beta} = \varphi_\alpha(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \subseteq \mathbb{C}^n$ and $\mathcal{V}_{\beta\alpha} = \varphi_\beta(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \subseteq \mathbb{C}^n$. $\psi_{\alpha\beta}$ being biholomorphic means that $\psi_{\alpha\beta}$ is a bijection and that $\psi_{\alpha\beta}$ and $\psi_{\alpha\beta}^{-1}$ are both holomorphic.

Proposition 15.2 *Any complex manifold has a canonical almost complex structure.*

Proof.

1) *Local definition of J*:

   Let $(\mathcal{U}, \mathcal{V}, \varphi : \mathcal{U} \to \mathcal{V})$ be a complex chart for a complex manifold M with $\varphi = (z_1, \ldots, z_n)$ written in components relative to complex coordinates $z_j = x_j + iy_j$. At $p \in \mathcal{U}$

$$T_pM = \mathbb{R}\text{-span of } \left\{ \left.\frac{\partial}{\partial x_j}\right|_p , \left.\frac{\partial}{\partial y_j}\right|_p \ :\ j = 1, \ldots, n \right\} \ .$$

Define J over $\mathcal{U}$ by

$$\left.\begin{array}{rcl} J_p\left(\left.\frac{\partial}{\partial x_j}\right|_p\right) & = & \left.\frac{\partial}{\partial y_j}\right|_p \\ J_p\left(\left.\frac{\partial}{\partial y_j}\right|_p\right) & = & -\left.\frac{\partial}{\partial x_j}\right|_p \end{array}\right. \qquad j = 1, \ldots, n .$$

2) *This J is well-defined globally*:

If $(\mathcal{U}, \mathcal{V}, \varphi)$ and $(\mathcal{U}', \mathcal{V}', \varphi')$ are two charts, we need to show that $J = J'$ on their overlap.

On $\mathcal{U} \cap \mathcal{U}'$, $\psi \circ \varphi = \varphi'$. If $z_j = x_j + iy_j$ and $w_j = u_j + iv_j$ are coordinates on $\mathcal{U}$ and $\mathcal{U}'$, respectively, so that φ and φ' can be written in components $\varphi = (z_1, \ldots, z_n)$, $\varphi' = (w_1, \ldots, w_n)$, then $\psi(z_1, \ldots, z_n) = (w_1, \ldots, w_n)$. Taking the derivative of a composition

$$\left\{\begin{array}{rcl} \frac{\partial}{\partial x_k} & = & \sum_j \left(\frac{\partial u_j}{\partial x_k}\frac{\partial}{\partial u_j} + \frac{\partial v_j}{\partial x_k}\frac{\partial}{\partial v_j}\right) \\ \frac{\partial}{\partial y_k} & = & \sum_j \left(\frac{\partial u_j}{\partial y_k}\frac{\partial}{\partial u_j} + \frac{\partial v_j}{\partial y_k}\frac{\partial}{\partial v_j}\right) \end{array}\right.$$

Since ψ is biholomorphic, each component of ψ satisfies the **Cauchy-Riemann equations**:

$$\left\{\begin{array}{rcl} \frac{\partial u_j}{\partial x_k} & = & \frac{\partial v_j}{\partial y_k} \\ \frac{\partial u_j}{\partial y_k} & = & -\frac{\partial v_j}{\partial x_k} \end{array}\right. \qquad j, k = 1, \ldots, n .$$

These equations imply

$$\underbrace{J' \sum_j \left(\frac{\partial u_j}{\partial x_k}\frac{\partial}{\partial u_j} + \frac{\partial v_j}{\partial x_k}\frac{\partial}{\partial v_j}\right)}_{\sum_j \left(\underbrace{\frac{\partial u_j}{\partial x_k}}_{\frac{\partial v_j}{\partial y_k}}\frac{\partial}{\partial v_j} - \underbrace{\frac{\partial v_j}{\partial x_j}}_{-\frac{\partial u_j}{\partial y_k}}\frac{\partial}{\partial u_j}\right)} = \sum_j \left(\frac{\partial u_j}{\partial y_k}\frac{\partial}{\partial u_j} + \frac{\partial v_j}{\partial y_k}\frac{\partial}{\partial v_j}\right)$$

which matches the equation

$$J\frac{\partial}{\partial x_k} = \frac{\partial}{\partial y_k} .$$

$\square$

15.2 Forms on Complex Manifolds

Suppose that M is a complex manifold and J is its canonical almost complex structure. What does the splitting $\Omega^k(M;\mathbb{C}) = \oplus_{\ell+m=k}\Omega^{\ell,m}$ look like? ([22, 48, 66, 109] are good references for this material.)

Let $\mathcal{U} \subseteq M$ be a coordinate neighborhood with complex coordinates $z_1,\ldots,z_n$, $z_j = x_j + iy_j$, and real coordinates $x_1, y_1, \ldots, x_n, y_n$. At $p \in \mathcal{U}$,

$$\begin{aligned} T_pM &= \mathbb{R}\text{-span}\left\{ \left.\frac{\partial}{\partial x_j}\right|_p, \left.\frac{\partial}{\partial y_j}\right|_p \right\} \\ T_pM\otimes\mathbb{C} &= \mathbb{C}\text{-span}\left\{ \left.\frac{\partial}{\partial x_j}\right|_p, \left.\frac{\partial}{\partial y_j}\right|_p \right\} \end{aligned}$$

$$= \underbrace{\mathbb{C}\text{-span}\left\{\frac{1}{2}\left(\left.\frac{\partial}{\partial x_j}\right|_p - i\left.\frac{\partial}{\partial y_j}\right|_p\right)\right\}}_{\substack{T_{1,0} = (+i)\text{-eigenspace of } J \\ J\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right) = i\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right)}} \oplus \underbrace{\mathbb{C}\text{-span}\left\{\frac{1}{2}\left(\left.\frac{\partial}{\partial x_j}\right|_p + i\left.\frac{\partial}{\partial y_j}\right|_p\right)\right\}}_{\substack{T_{0,1} = (-i)\text{-eigenspace of } J \\ J\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right) = -i\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)}}$$

This can be written more concisely using:

Definition 15.3

$$\frac{\partial}{\partial z_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right) \quad \textit{and} \quad \frac{\partial}{\partial \bar{z}_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right) .$$

Hence,

$$(T_{1,0})_p = \mathbb{C}\text{-span}\left\{ \left.\frac{\partial}{\partial z_j}\right|_p : j = 1,\ldots,n \right\},\ (T_{0,1})_p = \mathbb{C}\text{-span}\left\{ \left.\frac{\partial}{\partial \bar{z}_j}\right|_p : j = 1,\ldots,n \right\} .$$

Similarly,

$$T^*M\otimes\mathbb{C} = \mathbb{C}\text{-span}\{dx_j, dy_j : j = 1,\ldots,n\}$$

$$= \underbrace{\mathbb{C}\text{-span}\{dx_j + idy_j : j = 1,\ldots,n\}}_{\substack{T^{1,0} \\ (dx_j + idy_j)\circ J = i(dx_j + idy_j)}} \oplus \underbrace{\mathbb{C}\text{-span}\{dx_j - idy_j : j = 1,\ldots,n\}}_{\substack{T^{0,1} \\ (dx_j - idy_j)\circ J = -i(dx_j - idy_j)}}$$

Putting

$$dz_j = dx_j + idy_j \qquad \text{and} \qquad d\bar{z}_j = dx_j - idy_j ,$$

we obtain

$$T^{1,0} = \mathbb{C}\text{-span}\{dz_j : j = 1,\ldots,n\} , \quad T^{0,1} = \mathbb{C}\text{-span}\{d\bar{z}_j : j = 1,\ldots,n\} .$$

On the coordinate neighborhood $\mathcal{U}$,

$$\begin{array}{rcl}
(1,0)\text{-forms} & = & \left\{\sum_j b_j dz_j \mid b_j \in C^\infty(\mathcal{U};\mathbb{C})\right\} \\
(0,1)\text{-forms} & = & \left\{\sum_j b_j d\bar{z}_j \mid b_j \in C^\infty(\mathcal{U};\mathbb{C})\right\} \\
(2,0)\text{-forms} & = & \left\{\sum_{j_1<j_2} \; b_{j_1,j_2} dz_{j_1} \wedge dz_{j_2} \mid b_{j_1,j_2} \in C^\infty(\mathcal{U};\mathbb{C})\right\} \\
(1,1)\text{-forms} & = & \left\{\sum_{j_1,j_2} \; b_{j_1,j_2} dz_{j_1} \wedge d\bar{z}_{j_2} \mid b_{j_1,j_2} \in C^\infty(\mathcal{U};\mathbb{C})\right\} \\
(0,2)\text{-forms} & = & \left\{\sum_{j_1<j_2} \; b_{j_1,j_2} d\bar{z}_{j_1} \wedge d\bar{z}_{j_2} \mid b_{j_1,j_2} \in C^\infty(\mathcal{U};\mathbb{C})\right\}
\end{array}$$

If we use multi-index notation:

$$\begin{array}{rcll}
J & = & (j_1,\ldots,j_m) & \qquad 1 \le j_1 < \ldots < j_m \le n \\
|J| & = & m & \\
dz_{_J} & = & dz_{j_1} \wedge dz_{j_2} \wedge \ldots \wedge dz_{j_m} &
\end{array}$$

then

$$\Omega^{\ell,m} = (\ell,m)\text{-forms} = \left\{ \sum_{|J|=\ell,|K|=m} b_{_{J,K}} dz_{_J} \wedge d\bar{z}_{_K} \mid b_{_{J,K}} \in C^\infty(\mathcal{U};\mathbb{C}) \right\} \ .$$

15.3 Differentials

On a coordinate neighborhood $\mathcal{U}$, a form $\beta \in \Omega^k(M;\mathbb{C})$ may be written as

$$\beta = \sum_{|J|+|K|=k} a_{_{J,K}} dx_{_J} \wedge dy_{_K} \ , \qquad \text{with } a_{_{J,K}} \in C^\infty(\mathcal{U};\mathbb{C}) \ .$$

We would like to know whether the following equality holds:

$$d\beta = \sum (\partial a_{_{J,K}} + \bar{\partial} a_{_{J,K}}) dx_{_J} \wedge dy_{_K} \stackrel{?}{=} (\partial + \bar{\partial}) \sum a_{_{J,K}} dx_{_J} \wedge dy_{_K} \ .$$

If we use the identities

$$\left\{ \begin{array}{l} dx_j + i dy_j = dz_j \\ dx_j - i dy_j = d\bar{z}_j \end{array} \right. \iff \left\{ \begin{array}{l} dx_j = \frac{1}{2}(dz_j + d\bar{z}_j) \\ dy_j = \frac{1}{2i}(dz_j - d\bar{z}_j) \end{array} \right.$$

after substituting and reshuffling, we obtain

$$\begin{aligned}
\beta &= \sum_{|J|+|K|=k} b_{J,K}\, dz_J \wedge d\bar{z}_K \\
&= \sum_{\ell+m=k} \underbrace{\left(\sum_{|J|=\ell,|K|=m} b_{J,K}\, dz_J \wedge d\bar{z}_K \right)}_{\in \Omega^{\ell,m}} ,
\end{aligned}$$

$$\begin{aligned}
d\beta &= \sum_{\ell+m=k} \left(\sum_{|J|=\ell,|K|=m} db_{J,K} \wedge dz_J \wedge d\bar{z}_K \right) \\
&= \sum_{\ell+m=k} \sum_{|J|=\ell,|K|=m} \left(\partial b_{J,K} + \bar{\partial} b_{J,K} \right) \wedge dz_J \wedge d\bar{z}_K \\
&\qquad\qquad\qquad\qquad \text{(because } d = \partial + \bar{\partial} \text{ on functions)} \\
&= \sum_{\ell+m=k} \left(\underbrace{\sum_{|J|=\ell,|K|=m} \partial b_{J,K} \wedge dz_J \wedge d\bar{z}_K}_{\in \Omega^{\ell+1,m}} + \underbrace{\sum_{|J|=\ell,|K|=m} \bar{\partial} b_{J,K} \wedge dz_J \wedge d\bar{z}_K}_{\in \Omega^{\ell,m+1}} \right) \\
&= \partial\beta + \bar{\partial}\beta \ .
\end{aligned}$$

Therefore, $d = \partial + \bar{\partial}$ on forms of any degree for a *complex* manifold.

Conclusion. If M is a complex manifold, then $d = \partial + \bar{\partial}$. (For an almost complex manifold this fails because there are no coordinate functions z_j to give a suitable basis of 1-forms.)

Remark. If $b \in C^\infty(\mathcal{U}; \mathbb{C})$, in terms of z and $\bar{z}$, we obtain the following formulas:

$$\begin{aligned}
db &= \sum_j \left(\frac{\partial b}{\partial x_j} dx_j + \frac{\partial b}{\partial y_j} dy_j \right) \\
&= \sum_j \left[\frac{1}{2} \left(\frac{\partial b}{\partial x_j} - i \frac{\partial b}{\partial y_j} \right) (dx_j + i dy_j) + \frac{1}{2} \left(\frac{\partial b}{\partial x_j} + i \frac{\partial b}{\partial y_j} \right) (dx_j - i dy_j) \right] \\
&= \sum_j \left(\frac{\partial b}{\partial z_j} dz_j + \frac{\partial b}{\partial \bar{z}_j} d\bar{z}_j \right) .
\end{aligned}$$

Hence:

$$\begin{cases} \partial b = \pi^{1,0} db = \sum_j \frac{\partial b}{\partial z_j} dz_j \\ \bar{\partial} b = \pi^{0,1} db = \sum_j \frac{\partial b}{\partial \bar{z}_j} d\bar{z}_j \end{cases}$$

$\diamondsuit$

In the case where $\beta \in \Omega^{\ell,m}$, we have

$$\begin{array}{rcl} d\beta & = & \partial\beta + \bar{\partial}\beta = (\ell+1,m)\text{-form} + (\ell,m+1)\text{-form} \\ 0 = d^2\beta & = & (\ell+2,m)\text{-form} + (\ell+1,m+1)\text{-form} + (\ell,m+2)\text{-form} \\ & = & \underbrace{\partial^2\beta}_{0} + \underbrace{(\partial\bar{\partial}+\bar{\partial}\partial)\beta}_{0} + \underbrace{\bar{\partial}^2\beta}_{0} \ . \end{array}$$

Hence, $\bar{\partial}^2 = 0$.

The Dolbeault theorem states that for complex manifolds

$$H^{\ell,m}_{\text{Dolbeault}}(M) = H^m(M; \mathcal{O}(\Omega^{(\ell,0)})) \ ,$$

where $\mathcal{O}(\Omega^{(\ell,0)})$ is the sheaf of forms of type $(\ell,0)$ over M.

Theorem 15.4 (Newlander-Nirenberg, 1957 [89])
Let (M,J) be an almost complex manifold. Let $\mathcal{N}$ be the **Nijenhuis tensor** *(defined in Homework 10). Then:*

$$\begin{array}{rcl} M \textit{ is a complex manifold} & \Longleftrightarrow & J \textit{ is integrable} \\ & \Longleftrightarrow & \mathcal{N} \equiv 0 \\ & \Longleftrightarrow & d = \partial + \bar{\partial} \\ & \Longleftrightarrow & \bar{\partial}^2 = 0 \\ & \Longleftrightarrow & \pi^{2,0} d|_{\Omega^{0,1}} = 0 \ . \end{array}$$

For the proof of this theorem, besides the original reference, see also [22, 30, 48, 66, 109]. Naturally most almost complex manifolds have $d \neq \partial + \bar{\partial}$.

Homework 11: Complex Projective Space

The complex projective space $\mathbb{CP}^n$ is the space of complex lines in $\mathbb{C}^{n+1}$:

$\mathbb{CP}^n$ is obtained from $\mathbb{C}^{n+1} \setminus \{0\}$ by making the identifications $(z_0, \ldots, z_n) \sim (\lambda z_0, \ldots, \lambda z_n)$ for all $\lambda \in \mathbb{C} \setminus \{0\}$. One denotes by $[z_0, \ldots, z_n]$ the equivalence class of $(z_0, \ldots, z_n)$, and calls $z_0, \ldots, z_n$ the homogeneous coordinates of the point $p = [z_0, \ldots, z_n]$. (The homogeneous coordinates are, of course, only determined up to multiplication by a non-zero complex number λ.)

Let $\mathcal{U}_i$ be the subset of $\mathbb{CP}^n$ consisting of all points $p = [z_0, \ldots, z_n]$ for which $z_i \neq 0$. Let $\varphi_i : \mathcal{U}_i \to \mathbb{C}^n$ be the map

$$\varphi_i([z_0, \ldots, z_n]) = \left(\tfrac{z_0}{z_i}, \ldots, \tfrac{z_{i-1}}{z_i}, \tfrac{z_{i+1}}{z_i}, \ldots, \tfrac{z_n}{z_i} \right) .$$

1. Show that the collection

   $$\{(\mathcal{U}_i, \mathbb{C}^n, \varphi_i), i = 0, \ldots, n\}$$

   is an atlas in the *complex* sense, i.e., the transition maps are biholomorphic. Conclude that $\mathbb{CP}^n$ is a complex manifold.

   Hint: Work out the transition maps associated with $(\mathcal{U}_0, \mathbb{C}^n, \varphi_0)$ and $(\mathcal{U}_1, \mathbb{C}^n, \varphi_1)$. Show that the transition diagram has the form

   $$\begin{tikzcd} & \mathcal{U}_0 \cap \mathcal{U}_1 \arrow[dl, "\varphi_0"'] \arrow[dr, "\varphi_1"] & \\ \mathcal{V}_{0,1} \arrow[rr, "\varphi_{0,1}"] & & \mathcal{V}_{1,0} \end{tikzcd}$$

   where $\mathcal{V}_{0,1} = \mathcal{V}_{1,0} = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_1 \neq 0\}$ and

   $$\varphi_{0,1}(z_1, \ldots, z_n) = (\tfrac{1}{z_1}, \tfrac{z_2}{z_1}, \ldots, \tfrac{z_n}{z_1}) .$$

2. Show that the 1-dimensional complex manifold $\mathbb{CP}^1$ is diffeomorphic, as a real 2-dimensional manifold, to S^2.

   Hint: Stereographic projection.

16 Kähler Forms

16.1 Kähler Forms

Definition 16.1 *A* **Kähler manifold** *is a symplectic manifold (M, ω) equipped with an integrable compatible almost complex structure. The symplectic form ω is then called a* **Kähler form***.*

It follows immediately from the previous definition that

$$\begin{aligned} (M,\omega) \text{ is Kähler} \quad &\Longrightarrow \quad M \text{ is a complex manifold} \\ &\Longrightarrow \quad \begin{cases} \Omega^k(M;\mathbb{C}) = \oplus_{\ell+m=k}\Omega^{\ell,m} \\ d = \partial + \bar\partial \end{cases} \end{aligned}$$

where

$$\begin{array}{rcl} \partial & = & \pi^{\ell+1,m} \circ d : \Omega^{\ell,m} \to \Omega^{\ell+1,m} \\ \bar\partial & = & \pi^{\ell,m+1} \circ d : \Omega^{\ell,m} \to \Omega^{\ell,m+1} \ . \end{array}$$

On a complex chart $(\mathcal{U}, z_1, \ldots, z_n)$, $n = \dim_{\mathbb{C}} M$,

$$\Omega^{\ell,m} = \left\{ \sum_{|J|=\ell,|K|=m} b_{JK}\, dz_J \wedge d\bar z_K \mid b_{JK} \in C^\infty(\mathcal{U};\mathbb{C}) \right\} \ ,$$

where

$$\begin{array}{rclcccrcl} J & = & (j_1,\ldots,j_\ell) \ , & & j_1 < \ldots < j_\ell \ , & & dz_J & = & dz_{j_1} \wedge \ldots \wedge dz_{j_\ell} \ , \\ K & = & (k_1,\ldots,k_m) \ , & & k_1 < \ldots < k_m \ , & & d\bar z_K & = & d\bar z_{k_1} \wedge \ldots \wedge d\bar z_{k_m} \ . \end{array}$$

On the other hand,

$$(M,\omega) \text{ is Kähler} \quad \Longrightarrow \quad \omega \text{ is a symplectic form} \ .$$

– Where does ω fit with respect to the above decomposition?
A Kähler form ω is

1. a 2-form,
2. compatible with the complex structure,
3. closed,
4. real-valued, and
5. nondegenerate.

These properties translate into:

1. $\Omega^2(M;\mathbb{C}) = \Omega^{2,0} \oplus \Omega^{1,1} \oplus \Omega^{0,2}$.

   On a local complex chart $(\mathcal{U}, z_1, \ldots, z_n)$,

$$\omega = \sum a_{jk}\ dz_j \wedge dz_k + \sum b_{jk}\ dz_j \wedge d\bar{z}_k + \sum c_{jk}\ d\bar{z}_j \wedge d\bar{z}_k$$

   for some $a_{jk}, b_{jk}, c_{jk} \in C^\infty(\mathcal{U};\mathbb{C})$.

2. J is a symplectomorphism, that is, $J^*\omega = \omega$ where $(J^*\omega)(u,v) := \omega(Ju, Jv)$.

$$\begin{array}{rcl} J^*dz_j & = & dz_j \circ J = \quad i dz_j \\ J^*d\bar{z}_j & = & d\bar{z}_j \circ J = -i d\bar{z}_j \end{array}$$

$$J^*\omega = \sum \underset{\overset{\|}{-1}}{(i \cdot i)}\ a_{jk}\ dz_j \wedge dz_k + \underset{\overset{\|}{1}}{i(-i)} \sum b_{jk}\ dz_j \wedge d\bar{z}_k + \underset{\overset{\|}{-1}}{(-i)^2} \sum c_{jk} d\ \bar{z}_j \wedge d\bar{z}_k$$

$$J^*\omega = \omega \quad \iff \quad a_{jk} = 0 = c_{jk}\ ,\ \text{all } j,k \quad \iff \quad \omega \in \Omega^{1,1}\ .$$

3. $$0 = d\omega = \underbrace{\partial\omega}_{(2,1)-\text{form}} + \underbrace{\bar{\partial}\omega}_{(1,2)-\text{form}} \implies \begin{cases} \partial\omega = 0 & \omega \text{ is } \partial\text{-closed} \\ \bar{\partial}\omega = 0 & \omega \text{ is } \bar{\partial}\text{-closed} \end{cases}$$

   Hence, ω defines a Dolbeault $(1,1)$ cohomology class,

$$[\omega] \in H^{1,1}_{\text{Dolbeault}}(M)\ .$$

   Putting $b_{jk} = \frac{i}{2} h_{jk}$,

$$\omega = \frac{i}{2} \sum_{j,k=1}^{n} h_{jk}\ dz_j \wedge d\bar{z}_k\ , \qquad h_{jk} \in C^\infty(\mathcal{U};\mathbb{C}).$$

4. ω real-valued $\iff$ $\omega = \overline{\omega}$.

$$\overline{\omega} = -\frac{i}{2} \sum \overline{h_{jk}}\ d\bar{z}_j \wedge dz_k = \frac{i}{2} \sum \overline{h_{jk}}\ dz_k \wedge d\bar{z}_j = \frac{i}{2} \sum \overline{h_{kj}}\ dz_j \wedge d\bar{z}_k$$

$$\omega \text{ real} \quad \iff \quad h_{jk} = \overline{h_{kj}}\ ,$$

   i.e., at every point $p \in \mathcal{U}$, the $n \times n$ matrix $(h_{jk}(p))$ is hermitian.

5. nondegeneracy: $\omega^n = \underbrace{\omega \wedge \ldots \wedge \omega}_{n} \neq 0$.

   Exercise. Check that

$$\omega^n = n! \left(\frac{i}{2}\right)^n \det(h_{jk})\, dz_1 \wedge d\bar{z}_1 \wedge \ldots \wedge dz_n \wedge d\bar{z}_n\ .$$

$\diamondsuit$

Now

$$\omega \text{ nondegenerate} \quad \iff \quad \det{}_{\mathbb{C}}(h_{jk}) \neq 0\ ,$$

i.e., at every $p \in M$, $(h_{jk}(p))$ is a nonsingular matrix.

2. Again the positivity condition: $\omega(v, Jv) > 0$, $\forall v \neq 0$.

   Exercise. Show that $(h_{jk}(p))$ is positive-definite. $\diamondsuit$

$$\omega \text{ positive} \iff (h_{jk}) \gg 0 \ ,$$

   i.e., at each $p \in \mathcal{U}$, $(h_{jk}(p))$ is positive-definite.

Conclusion. Kähler forms are ∂- and $\bar{\partial}$-closed $(1,1)$-forms, which are given on a local chart $(\mathcal{U}, z_1, \ldots, z_n)$ by

$$\omega = \frac{i}{2} \sum_{j,k=1}^{n} h_{jk} \ dz_j \wedge d\bar{z}_k$$

where, at every point $p \in \mathcal{U}$, $(h_{jk}(p))$ is a positive-definite hermitian matrix.

16.2 An Application

Theorem 16.2 (Banyaga) *Let M be a compact complex manifold. Let ω_0 and ω_1 be Kähler forms on M. If $[\omega_0] = [\omega_1] \in H^2_{\mathrm{deRham}}(M)$, then (M, ω_0) and (M, ω_1) are symplectomorphic.*

Proof. Any combination $\omega_t = (1-t)\omega_0 + t\omega_1$ is symplectic for $0 \leq t \leq 1$, because, on a complex chart $(\mathcal{U}, z_1, \ldots, z_n)$, where $n = \dim_{\mathbb{C}} M$, we have

$$\begin{aligned} \omega_0 &= \tfrac{i}{2} \sum h^0_{jk} dz_j \wedge d\bar{z}_k \\ \omega_1 &= \tfrac{i}{2} \sum h^1_{jk} dz_j \wedge d\bar{z}_k \\ \omega_t &= \tfrac{i}{2} \sum h^t_{jk} dz_j \wedge d\bar{z}_k \ , \qquad \text{where } h^t_{jk} = (1-t)h^0_{jk} + t h^1_{jk} \ . \end{aligned}$$

$$(h^0_{jk}) \gg 0 \ , (h^1_{jk}) \gg 0 \implies (h^t_{jk}) \gg 0 \ .$$

Apply the Moser theorem (Theorem 7.2). $\square$

16.3 Recipe to Obtain Kähler Forms

Definition 16.3 *Let M be a complex manifold. A function $\rho \in C^\infty(M; \mathbb{R})$ is* **strictly plurisubharmonic** *(s.p.s.h.) if, on each local complex chart $(\mathcal{U}, z_1, \ldots, z_n)$, where $n = \dim_{\mathbb{C}} M$, the matrix $\left(\frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(p)\right)$ is positive-definite at all $p \in \mathcal{U}$.*

Proposition 16.4 *Let M be a complex manifold and let $\rho \in C^\infty(M; \mathbb{R})$ be s.p.s.h.. Then*

$$\omega = \frac{i}{2} \partial \bar{\partial} \rho \qquad \text{is Kähler} \ .$$

A function ρ as in the previous proposition is called a (global) **Kähler potential**.

Proof. Simply observe that:

$$\begin{cases} \partial\omega & = & \frac{i}{2}\overbrace{\partial^2}^{0}\bar{\partial}\rho = 0 \\ \bar{\partial}\omega & = & \frac{i}{2}\underbrace{\bar{\partial}\partial}_{-\partial\bar{\partial}}\bar{\partial}\rho = -\frac{i}{2}\partial\underbrace{\bar{\partial}^2}_{0}\rho = 0 \end{cases}$$

$$d\omega = \partial\omega + \bar{\partial}\omega = 0 \implies \omega \text{ is closed .}$$

$$\overline{\omega} = -\tfrac{i}{2}\bar{\partial}\partial\rho = \tfrac{i}{2}\partial\bar{\partial}\rho = \omega \implies \omega \text{ is real .}$$

$$\omega \in \Omega^{1,1} \implies J^*\omega = \omega \implies \omega(\cdot, J\cdot) \text{ is symmetric .}$$

Exercise. Show that, for $f \in C^\infty(\mathcal{U};\mathbb{C})$,

$$\partial f = \sum \frac{\partial f}{\partial z_j} dz_j \quad \text{and} \quad \bar{\partial} f = \sum \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j \ .$$

Since the right-hand sides are in $\Omega^{1,0}$ and $\Omega^{0,1}$, respectively, it suffices to show that the sum of the two expressions is df. ♢

$$\omega = \frac{i}{2}\partial\bar{\partial}\rho = \frac{i}{2}\sum \frac{\partial}{\partial z_j}\left(\frac{\partial\rho}{\partial\bar{z}_k}\right) dz_j \wedge d\bar{z}_k = \frac{i}{2}\sum \underbrace{\left(\frac{\partial^2\rho}{\partial z_j \partial \bar{z}_k}\right)}_{h_{jk}} dz_j \wedge d\bar{z}_k \ .$$

$$\rho \text{ is s.p.s.h} \implies (h_{jk}) \gg 0 \implies \omega(\cdot, J\cdot) \text{ is positive .}$$

In particular, ω is nondegenerate. □

Example. Let $M = \mathbb{C}^n \simeq \mathbb{R}^{2n}$, with complex coordinates $(z_1, \ldots, z_n)$ and corresponding real coordinates $(x_1, y_1, \ldots, x_n, y_n)$ via $z_j = x_j + iy_j$. Let

$$\rho(x_1, y_1, \ldots, x_n, y_n) = \sum_{j=1}^{n}(x_j^2 + y_j^2) = \sum |z_j|^2 = \sum z_j \bar{z}_j \ .$$

Then

$$\frac{\partial}{\partial z_j}\frac{\partial\rho}{\partial\bar{z}_k} = \frac{\partial}{\partial z_j} z_k = \delta_{jk} \ ,$$

so

$$(h_{jk}) = \left(\frac{\partial^2\rho}{\partial z_j \partial\bar{z}_k}\right) = (\delta_{jk}) = \mathrm{Id} \gg 0 \implies \rho \text{ is s.p.s.h. .}$$

The corresponding Kähler form

$$\begin{aligned}\omega &= \tfrac{i}{2}\partial\bar\partial\rho = \tfrac{i}{2}\sum_{j,k}\delta_{jk}\ dz_j\wedge d\bar z_k\\ &= \tfrac{i}{2}\sum_j dz_j\wedge d\bar z_j = \sum_j dx_j\wedge dy_j \quad \text{is the standard form .}\end{aligned}$$

$\diamondsuit$

16.4 Local Canonical Form for Kähler Forms

There is a local converse to the previous construction of Kähler forms.

Theorem 16.5 *Let ω be a closed real-valued $(1,1)$-form on a complex manifold M and let $p \in M$. Then there exist a neighborhood $\mathcal{U}$ of p and $\rho \in C^\infty(\mathcal{U};\mathbb{R})$ such that, on $\mathcal{U}$,*

$$\omega = \frac{i}{2}\partial\bar\partial\rho\ .$$

The function ρ is then called a (local) **Kähler potential**.

The proof requires holomorphic versions of Poincaré's lemma, namely, the local triviality of Dolbeault groups:

$$\forall p\in M \quad \exists \text{ neighborhood } \mathcal{U} \text{ of } p \text{ such that} \quad H^{\ell,m}_{\text{Dolbeault}}(\mathcal{U}) = 0\ ,\ m>0\ ,$$

and the local triviality of the holomorphic de Rham groups; see [48].

Proposition 16.6 *Let M be a complex manifold, $\rho \in C^\infty(M;\mathbb{R})$ s.p.s.h., X a complex submanifold, and $i : X \hookrightarrow M$ the inclusion map. Then $i^*\rho$ is s.p.s.h..*

Proof. Let $\dim_{\mathbb{C}} M = n$ and $\dim_{\mathbb{C}} X = n-m$. For $p\in X$, choose a chart $(\mathcal{U}, z_1,\dots,z_n)$ for M centered at p and adapted to X, i.e., $X\cap\mathcal{U}$ is given by $z_1=\ldots=z_m=0$. In this chart, $i^*\rho = \rho(0,0,\dots,0,z_{m+1},\dots,z_n)$.

$$i^*p \text{ is s.p.s.h.} \iff \left(\frac{\partial^2\rho}{\partial z_{m+j}\partial\bar z_{m+k}}(0,\dots,0,z_{m+1},\dots,z_n)\right) \text{ is positive-definite ,}$$

which holds since this is a minor of $\left(\frac{\partial^2}{\partial z_j\partial\bar z_k}(0,\dots,0,z_{m+1},\dots,z_n)\right)$. $\square$

Corollary 16.7 *Any complex submanifold of a Kähler manifold is also Kähler.*

Definition 16.8 *Let (M,ω) be a Kähler manifold, X a complex submanifold, and $i: X\hookrightarrow M$ the inclusion. Then $(X, i^*\omega)$ is called a* **Kähler submanifold**.

Example. Complex vector space $(\mathbb{C}^n, \omega_0)$ where $\omega_0 = \frac{i}{2} \sum dz_j \wedge d\bar{z}_j$ is Kähler. Every complex submanifold of $\mathbb{C}^n$ is Kähler. $\diamondsuit$

Example. The complex projective space is

$$\mathbb{CP}^n = \mathbb{C}^{n+1} \backslash \{0\} / \sim$$

where

$$(z_0, \ldots, z_n) \sim (\lambda z_0, \ldots, \lambda z_n) \ , \quad \lambda \in \mathbb{C} \backslash \{0\} \ .$$

The Fubini-Study form (see Homework 12) is Kähler. Therefore, every **non-singular projective variety** is a Kähler submanifold. Here we mean

non-singular	$=$	smooth
projective variety	$=$	zero locus of a collection of homogeneous polynomials .

$\diamondsuit$

Homework 12: The Fubini-Study Structure

The purpose of the following exercises is to describe the natural Kähler structure on complex projective space, $\mathbb{CP}^n$.

1. Show that the function on $\mathbb{C}^n$

$$z \longmapsto \log(|z|^2 + 1)$$

is strictly plurisubharmonic. Conclude that the 2-form

$$\omega_{\mathrm{FS}} = \tfrac{i}{2}\partial\bar{\partial}\log(|z|^2 + 1)$$

is a Kähler form. (It is usually called the **Fubini-Study form** on $\mathbb{C}^n$.)

Hint: A hermitian $n \times n$ matrix H is positive definite if and only if $v^* H v > 0$ for any $v \in \mathbb{C}^n \setminus \{0\}$, where v^* is the transpose of the vector $\bar{v}$. To prove positive-definiteness, either apply the Cauchy-Schwarz inequality, or use the following symmetry observation: $\mathrm{U}(n)$ acts transitively on S^{2n-1} and ω_{FS} is $\mathrm{U}(n)$-invariant, thus it suffices to show positive-definiteness along *one* direction.

2. Let $\mathcal{U}$ be the open subset of $\mathbb{C}^n$ defined by the inequality $z_1 \neq 0$, and let $\varphi : \mathcal{U} \to \mathcal{U}$ be the map

$$\varphi(z_1, \ldots, z_n) = \tfrac{1}{z_1}(1, z_2, \ldots, z_n) \ .$$

Show that φ maps $\mathcal{U}$ biholomorphically onto $\mathcal{U}$ and that

$$\varphi^* \log(|z|^2 + 1) = \log(|z|^2 + 1) + \log \tfrac{1}{|z_1|^2} \ . \qquad (\star)$$

3. Notice that, for every point $p \in \mathcal{U}$, we can write the second term in $(\star)$ as the sum of a holomorphic and an anti-holomorphic function:

$$-\log z_1 - \log \overline{z_1}$$

on a neighborhood of p. Conclude that

$$\partial\bar{\partial}\varphi^* \log(|z|^2 + 1) = \partial\bar{\partial}\log(|z|^2 + 1)$$

and hence that $\varphi^*\omega_{\mathrm{FS}} = \omega_{\mathrm{FS}}$.

Hint: You need to use the fact that the pullback by a holomorphic map φ^* commutes with the ∂ and $\bar{\partial}$ operators. This is a consequence of φ^* preserving form type, $\varphi^*(\Omega^{p,q}) \subseteq \Omega^{p,q}$, which in turn is implied by $\varphi^* dz_j = \partial\varphi_j \subseteq \Omega^{1,0}$ and $\varphi^* d\overline{z_j} = \bar{\partial}\overline{\varphi_j} \subseteq \Omega^{0,1}$, where φ_j is the jth component of φ with respect to local complex coordinates $(z_1, \ldots, z_n)$.

4. Recall that $\mathbb{CP}^n$ is obtained from $\mathbb{C}^{n+1} \setminus \{0\}$ by making the identifications $(z_0, \ldots, z_n) \sim (\lambda z_0, \ldots, \lambda z_n)$ for all $\lambda \in \mathbb{C} \setminus \{0\}$; $[z_0, \ldots, z_n]$ is the equivalence class of $(z_0, \ldots, z_n)$.

For $i = 0, 1, \ldots, n$, let

$$\begin{aligned} &\mathcal{U}_i = \{[z_0, \ldots, z_n] \in \mathbb{CP}^n \,|\, z_i \neq 0\} \\ &\varphi_i : \mathcal{U}_i \to \mathbb{C}^n \qquad \varphi_i([z_0, \ldots, z_n]) = \left(\tfrac{z_0}{z_i}, \ldots, \tfrac{z_{i-1}}{z_i}, \tfrac{z_{i+1}}{z_i}, \ldots, \tfrac{z_n}{z_i}\right) . \end{aligned}$$

Homework 11 showed that the collection $\{(\mathcal{U}_i, \mathbb{C}^n, \varphi_i), i = 0, \ldots, n\}$ is a complex atlas (i.e., the transition maps are biholomorphic). In particular, it was shown that the transition diagram associated with $(\mathcal{U}_0, \mathbb{C}^n, \varphi_0)$ and $(\mathcal{U}_1, \mathbb{C}^n, \varphi_1)$ has the form

$$\begin{tikzcd} & \mathcal{U}_0 \cap \mathcal{U}_1 \arrow[dl, "\varphi_0"'] \arrow[dr, "\varphi_1"] & \\ \mathcal{V}_{0,1} \arrow[rr, "\varphi_{0,1}"] & & \mathcal{V}_{1,0} \end{tikzcd}$$

where $\mathcal{V}_{0,1} = \mathcal{V}_{1,0} = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \,|\, z_1 \neq 0\}$ and $\varphi_{0,1}(z_1, \ldots, z_n) = (\frac{1}{z_1}, \frac{z_2}{z_1}, \ldots, \frac{z_n}{z_1})$. Now the set $\mathcal{U}$ in exercise 2 is equal to the sets $\mathcal{V}_{0,1}$ and $\mathcal{V}_{1,0}$, and the map φ coincides with $\varphi_{0,1}$.

Show that $\varphi_0^* \omega_{\text{FS}}$ and $\varphi_1^* \omega_{\text{FS}}$ are identical on the overlap $\mathcal{U}_0 \cap \mathcal{U}_1$.

More generally, show that the Kähler forms $\varphi_i^* \omega_{\text{FS}}$ "glue together" to define a Kähler structure on $\mathbb{CP}^n$. This is called the **Fubini-Study form** on complex projective space.

5. Prove that for $\mathbb{CP}^1$ the Fubini-Study form on the chart $\mathcal{U}_0 = \{[z_0, z_1] \in \mathbb{CP}^1 \,|\, z_0 \neq 0\}$ is given by the formula

$$\omega_{\text{FS}} = \frac{dx \wedge dy}{(x^2 + y^2 + 1)^2}$$

where $\frac{z_1}{z_0} = z = x + iy$ is the usual coordinate on $\mathbb{C}$.

6. Compute the total area of $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ with respect to ω_{FS}:

$$\int_{\mathbb{CP}^1} \omega_{\text{FS}} = \int_{\mathbb{R}^2} \frac{dx \wedge dy}{(x^2 + y^2 + 1)^2} .$$

7. Recall that $\mathbb{CP}^1 \simeq S^2$ as real 2-dimensional manifolds (Homework 11). On S^2 there is the standard area form ω_{std} induced by regarding it as the unit sphere in $\mathbb{R}^3$ (Homework 6): in cylindrical polar coordinates (θ, h) on S^2 away from its poles ($0 \leq \theta < 2\pi$ and $-1 \leq h \leq 1$), we have

$$\omega_{\text{std}} = d\theta \wedge dh .$$

Using stereographic projection, show that

$$\omega_{\text{FS}} = \frac{1}{4} \omega_{\text{std}} .$$

17 Compact Kähler Manifolds

17.1 Hodge Theory

Let M be a complex manifold. A Kähler form ω on M is a symplectic form which is compatible with the complex structure. Equivalently, a Kähler form ω is a ∂- and $\bar{\partial}$-closed form of type $(1,1)$ which, on a local chart $(\mathcal{U}, z_1, \ldots, z_n)$ is given by $\omega = \frac{i}{2}\sum_{j,k=1}^{n} h_{jk} dz_j \wedge d\bar{z}_k$, where, at each $x \in \mathcal{U}$, $(h_{jk}(x))$ is a positive-definite hermitian matrix. The pair (M, ω) is then called a Kähler manifold.

Theorem 17.1 (Hodge) *On a compact Kähler manifold (M, ω) the Dolbeault cohomology groups satisfy*

$$H^k_{\mathrm{deRham}}(M;\mathbb{C}) \simeq \bigoplus_{\ell+m=k} H^{\ell,m}_{\mathrm{Dolbeault}}(M) \qquad \textbf{(Hodge decomposition)}$$

with $H^{\ell,m} \simeq \overline{H^{m,\ell}}$. In particular, the spaces $H^{\ell,m}_{\mathrm{Dolbeault}}$ are finite-dimensional.

Hodge identified the spaces of cohomology classes of forms with spaces of actual forms, by picking *the* representative from each class which solves a certain differential equation, namely the *harmonic* representative.

(1) The **Hodge $*$-operator**.

Each tangent space $V = T_xM$ has a positive inner product $\langle \cdot, \cdot \rangle$, part of the riemannian metric in a compatible triple; we forget about the complex and symplectic structures until part (4).

Let $e_1, \ldots, e_n$ be a positively oriented orthonormal basis of V.

The star operator is a linear operator $* : \Lambda(V) \to \Lambda(V)$ defined by

$$\begin{array}{rcl} *(1) & = & e_1 \wedge \ldots \wedge e_n \\ *(e_1 \wedge \ldots \wedge e_n) & = & 1 \\ *(e_1 \wedge \ldots \wedge e_k) & = & e_{k+1} \wedge \ldots \wedge e_n \ . \end{array}$$

We see that $* : \Lambda^k(V) \to \Lambda^{n-k}(V)$ and satisfies $** = (-1)^{k(n-k)}$.

(2) The **codifferential** and the **laplacian** are the operators defined by:

$$\begin{array}{rclcl} \delta & = & (-1)^{n(k+1)+1} * d * & : & \Omega^k(M) \to \Omega^{k-1}(M) \\ \Delta & = & d\delta + \delta d & : & \Omega^k(M) \to \Omega^k(M) \ . \end{array}$$

The operator Δ is also called the **Laplace-Beltrami operator**.

Exercise. Check that, on $\Omega^0(\mathbb{R}^n) = C^\infty(\mathbb{R}^n)$, $\Delta = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$. ♢

Exercise. Check that $\Delta * = * \Delta$. ♢

Suppose that M is compact. Define an inner product on forms by

$$\langle \cdot, \cdot \rangle : \Omega^k \times \Omega^k \to \mathbb{R} \ , \qquad \langle \alpha, \beta \rangle = \int_M \alpha \wedge *\beta \ .$$

Exercise. Check that this is symmetric, positive-definite and satisfies $\langle d\alpha, \beta\rangle = \langle \alpha, \delta\beta\rangle$. $\diamondsuit$

Therefore, δ is often denoted by d^* and called the adjoint of d. (When M is not compact, we still have a formal adjoint of d with respect to the nondegenerate bilinear pairing $\langle \cdot, \cdot \rangle : \Omega^k \times \Omega^k_c \to \mathbb{R}$ defined by a similar formula, where Ω^k_c is the space of compactly supported k-forms.) Also, Δ is self-adjoint:

Exercise. Check that $\langle \Delta\alpha, \beta\rangle = \langle \alpha, \Delta\beta\rangle$, and that $\langle \Delta\alpha, \alpha\rangle = |d\alpha|^2 + |\delta\alpha|^2 \geq 0$, where $|\cdot|$ is the norm with respect to this inner product. $\diamondsuit$

(3) The **harmonic k-forms** are the elements of $\mathcal{H}^k := \{\alpha \in \Omega^k \mid \Delta\alpha = 0\}$.

Note that $\Delta\alpha = 0 \iff d\alpha = \delta\alpha = 0$. Since a harmonic form is d-closed, it defines a de Rham cohomology class.

Theorem 17.2 (Hodge) *Every de Rham cohomology class on a compact oriented riemannian manifold (M, g) possesses a unique harmonic representative, i.e.,*

$$\mathcal{H}^k \simeq H^k_{\mathrm{deRham}}(M; \mathbb{R}) \ .$$

In particular, the spaces $\mathcal{H}^k$ are finite-dimensional. We also have the following orthogonal decomposition with respect to $\langle \cdot, \cdot \rangle$:

$$\begin{array}{rcl} \Omega^k & \simeq & \mathcal{H}^k \oplus \Delta(\Omega^k(M)) \\ & \simeq & \mathcal{H}^k \oplus d\Omega^{k-1} \oplus \delta\Omega^{k+1} \end{array} \qquad \textbf{(Hodge decomposition on forms)} \ .$$

The proof involves functional analysis, elliptic differential operators, pseudodifferential operators and Fourier analysis; see [48, 109].

So far, this was ordinary Hodge theory, considering only the metric and not the complex structure.

(4) **Complex Hodge Theory.**

When M is Kähler, the laplacian satisfies $\Delta = 2(\bar\partial\bar\partial^* + \bar\partial^*\bar\partial)$ (see, for example, [48]) and preserves the decomposition according to type, $\Delta : \Omega^{\ell,m} \to \Omega^{\ell,m}$. Hence, harmonic forms are also bigraded

$$\mathcal{H}^k = \bigoplus_{\ell+m=k} \mathcal{H}^{\ell,m} \ .$$

Theorem 17.3 (Hodge) *Every Dolbeault cohomology class on a compact Kähler manifold (M, ω) possesses a unique harmonic representative, i.e.,*

$$\mathcal{H}^{\ell,m} \simeq H^{\ell,m}_{\mathrm{Dolbeault}}(M)$$

and the spaces $\mathcal{H}^{\ell,m}$ are finite-dimensional. Hence, we have the following isomorphisms:

$$H^k_{\mathrm{deRham}}(M) \simeq \mathcal{H}^k \simeq \bigoplus_{\ell+m=k} \mathcal{H}^{\ell,m} \simeq \bigoplus_{\ell+m=k} H^{\ell,m}_{\mathrm{Dolbeault}}(M) \ .$$

For the proof, see for instance [48, 109].

17.2 Immediate Topological Consequences

Let $b^k(M) := \dim H^k_{\mathrm{deRham}}(M)$ be the usual **Betti numbers** of M, and let $h^{\ell,m}(M) := \dim H^{\ell,m}_{\mathrm{Dolbeault}}(M)$ be the so-called **Hodge numbers** of M.

$$\text{Hodge Theorem} \quad \Longrightarrow \quad \left\{ \begin{array}{rcl} b^k & = & \sum_{\ell+m=k} h^{\ell,m} \\ h^{\ell,m} & = & h^{m,\ell} \end{array} \right.$$

Some immediate topological consequences are:

1. On compact Kähler manifolds "the odd Betti numbers are even:"

$$b^{2k+1} = \sum_{\ell+m=2k+1} h^{\ell,m} = 2\sum_{\ell=0}^{k} h^{\ell,(2k+1-\ell)} \quad \text{is even} \ .$$

2. On compact Kähler manifolds, $h^{1,0} = \frac{1}{2} b^1$ is a topological invariant.

3. On compact symplectic manifolds, "even Betti numbers are positive," because ω^k is closed but not exact $(k = 0, 1, \ldots, n)$.

   Proof. If $\omega^k = d\alpha$, by Stokes' theorem, $\displaystyle\int_M \omega^n = \int_M d(\alpha \wedge \omega^{n-k}) = 0$. This cannot happen since ω^n is a volume form. □

4. On compact Kähler manifolds, the $h^{\ell,\ell}$ are positive.

   Claim. $0 \neq [\omega^\ell] \in H^{\ell,\ell}_{\mathrm{Dolbeault}}(M)$.

   Proof.

$$\begin{array}{rcl} \omega \in \Omega^{1,1} & \Longrightarrow & \omega^\ell \in \Omega^{\ell,\ell} \\ d\omega = 0 & \Longrightarrow & 0 = d\omega^\ell = \underbrace{\partial\omega^\ell}_{(\ell+1,\ell)} + \underbrace{\bar\partial\omega^\ell}_{(\ell,\ell+1)} \\ & \Longrightarrow & \bar\partial\omega^\ell = 0 \ , \end{array}$$

   so $[\omega^\ell]$ defines an element of $H^{\ell,\ell}_{\mathrm{Dolbeault}}$. Why is ω^ℓ not $\bar\partial$-exact? If $\omega^\ell = \bar\partial\beta$ for some $\beta \in \Omega^{\ell-1,\ell}$, then

$$\omega^n = \omega^\ell \wedge \omega^{n-\ell} = \bar\partial(\beta \wedge \omega^{n-\ell}) \implies 0 = [\omega^n] \in H^{n,n}_{\mathrm{Dolbeault}}(M) \ .$$

   But $[\omega^n] \neq 0$ in $H^{2n}_{\mathrm{deRham}}(M;\mathbb{C}) \simeq H^{n,n}_{\mathrm{Dolbeault}}(M)$ since it is a volume form. □

There are other constraints on the Hodge numbers of compact Kähler manifolds, and ongoing research on how to compute $H^{\ell,m}_{\text{Dolbeault}}$. A popular picture to describe the relations is the **Hodge diamond**:

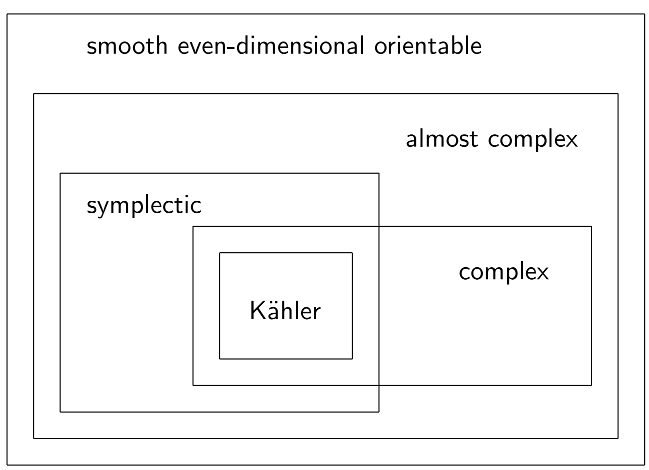
$$
\begin{array}{ccccc}
 & & h^{n,n} & & \\
 & h^{n,n-1} & & h^{n-1,n} & \\
h^{n,n-2} & & h^{n-1,n-1} & & h^{n-2,n} \\
 & & \vdots & & \\
h^{2,0} & & h^{1,1} & & h^{0,2} \\
 & h^{1,0} & & h^{0,1} & \\
 & & h^{0,0} & &
\end{array}
$$

Complex conjugation gives symmetry with respect to the middle vertical, whereas the Hodge $*$ induces symmetry about the center of the diamond. The middle vertical axis is all non-zero. There are further symmetries induced by isomorphisms given by wedging with ω.

The **Hodge conjecture** relates $H^{\ell,\ell}_{\text{Dolbeault}}(M) \cap H^{2\ell}(M;\mathbb{Z})$ for projective manifolds M (i.e., submanifolds of complex projective space) to codim$_{\mathbb{C}} = \ell$ complex submanifolds of M.

17.3 Compact Examples and Counterexamples

$$
\begin{array}{ccc}
\text{symplectic} & \Longleftarrow & \text{Kähler} \\
\Downarrow & & \Downarrow \\
\text{almost complex} & \Longleftarrow & \text{complex}
\end{array}
$$



Is each of these regions nonempty? Can we even find representatives of each region which are simply connected or have any specified fundamental group?

- Not all smooth even-dimensional manifolds are almost complex. For example, S^4, S^8, S^{10}, etc., are not almost complex.

- If M is both symplectic and complex, is it necessarily Kähler?

  No. For some time, it had been suspected that every compact symplectic manifold might have an underlying Kähler structure, or, at least, that a compact symplectic manifold might have to satisfy the Hodge relations on its Betti numbers [52]. The following example first demonstrated otherwise.

  The **Kodaira-Thurston example (Thurston, 1976 [101]):**

  Take $\mathbb{R}^4$ with $dx_1 \wedge dy_1 + dx_2 \wedge dy_2$, and Γ the discrete group generated by the following symplectomorphisms:

$$\begin{array}{rcl} \gamma_1 & : & (x_1, x_2, y_1, y_2) \longmapsto (x_1, x_2 + 1, y_1, y_2) \\ \gamma_2 & : & (x_1, x_2, y_1, y_2) \longmapsto (x_1, x_2, y_1, y_2 + 1) \\ \gamma_3 & : & (x_1, x_2, y_1, y_2) \longmapsto (x_1 + 1, x_2, y_1, y_2) \\ \gamma_4 & : & (x_1, x_2, y_1, y_2) \longmapsto (x_1, x_2 + y_2, y_1 + 1, y_2) \end{array}$$

  Then $M = \mathbb{R}^4/\Gamma$ is a flat 2-torus bundle over a 2-torus. Kodaira [70] had shown that M has a complex structure. However, $\pi_1(M) = \Gamma$, hence $H^1(\mathbb{R}^4/\Gamma; \mathbb{Z}) = \Gamma/[\Gamma, \Gamma]$ has rank 3, $b^1 = 3$ is *odd*, so M is *not* Kähler [101].

- Does any symplectic manifold admit some complex structure (not necessarily compatible)?

  No.

  (Fernández-Gotay-Gray, 1988 [37]): There are symplectic manifolds which do *not* admit any complex structure [37]. Their examples are circle bundles over circle bundles over a 2-torus.

$$\begin{array}{ccc} S^1 & \hookrightarrow & M \\ & & \downarrow \\ S^1 & \hookrightarrow & P \\ & & \downarrow \\ & & \mathbb{T}^2 \end{array} \qquad \text{tower of circle fibrations}$$

- Given a complex structure on M, is there always a symplectic structure (not necessarily compatible)?

  No.

  The **Hopf surface** $S^1 \times S^3$ is not symplectic because $H^2(S^1 \times S^3) = 0$. But it is complex since $S^1 \times S^3 \simeq \mathbb{C}^2 \backslash \{0\}/\Gamma$ where $\Gamma = \{2^n \mathrm{Id} \mid n \in \mathbb{Z}\}$ is a group of *complex* transformations, i.e., we factor $\mathbb{C}^2 \backslash \{0\}$ by the equivalence relation $(z_1, z_2) \sim (2z_1, 2z_2)$.

- Is any almost complex manifold either complex or symplectic?

  No.

  $\mathbb{CP}^2 \# \mathbb{CP}^2 \# \mathbb{CP}^2$ is almost complex (proved by a computation with characteristic classes), but is neither complex (since it does not fit Kodaira's classification of complex surfaces), nor symplectic (as shown by **Taubes** [97] in 1995 using Seiberg-Witten invariants).

- In 1993 **Gompf** [46] provided a construction that yields a compact symplectic 4-manifold with fundamental group equal to any given finitely-presented group. In particular, we can find simply connected examples. His construction can be adapted to produce *non*Kähler examples.

17.4 Main Kähler Manifolds

- **Compact Riemann surfaces**

  As real manifolds, these are the 2-dimensional compact orientable manifolds classified by genus. An area form is a symplectic form. Any compatible almost complex structure is always integrable for dimension reasons (see Homework 10).

- **Stein manifolds**

  Definition 17.4 *A* **Stein manifold** *is a Kähler manifold* (M, ω) *which admits a global proper Kähler potential, i.e.,* $\omega = \frac{i}{2}\partial\bar{\partial}\rho$ *for some proper function* $\rho : M \to \mathbb{R}$.

  Proper means that the preimage by ρ of a compact set is compact, i.e., "$\rho(p) \to \infty$ as $p \to \infty$."

  Stein manifolds can be also characterized as the properly embedded analytic submanifolds of $\mathbb{C}^n$.

- **Complex tori**

  Complex tori look like $M = \mathbb{C}^n / \mathbb{Z}^n$ where $\mathbb{Z}^n$ is a lattice in $\mathbb{C}^n$. The form $\omega = \sum dz_j \wedge d\bar{z}_j$ induced by the euclidean structure is Kähler.

- **Complex projective spaces**

  The standard Kähler form on $\mathbb{CP}^n$ is the Fubini-Study form (see Homework 12). (In 1995, Taubes showed that $\mathbb{CP}^2$ has a unique symplectic structure up to symplectomorphism.)

- **Products of Kähler manifolds**

- **Complex submanifolds of Kähler manifolds**

Part VII
Hamiltonian Mechanics

The equations of motion in classical mechanics arise as solutions of variational problems. For a general mechanical system of n particles in $\mathbb{R}^3$, the physical path satisfies Newton's second law. On the other hand, the physical path minimizes the mean value of kinetic minus potential energy. This quantity is called the action. For a system with constraints, the physical path is the path which minimizes the action among all paths satisfying the constraint.

The Legendre transform (Lecture 20) gives the relation between the variational (Euler-Lagrange) and the symplectic (Hamilton-Jacobi) formulations of the equations of motion.

18 Hamiltonian Vector Fields

18.1 Hamiltonian and Symplectic Vector Fields

– What does a symplectic geometer do with a real function?...

Let (M, ω) be a symplectic manifold and let $H : M \to \mathbb{R}$ be a smooth function. Its differential dH is a 1-form. By nondegeneracy, there is a unique vector field X_H on M such that $\imath_{X_H}\omega = dH$. Integrate X_H. Supposing that M is compact, or at least that X_H is complete, let $\rho_t : M \to M$, $t \in \mathbb{R}$, be the one-parameter family of diffeomorphisms generated by X_H:

$$\begin{cases} \rho_0 = \mathrm{id}_M \\[2ex] \dfrac{d\rho_t}{dt} \circ \rho_t^{-1} = X_H \ . \end{cases}$$

Claim. Each diffeomorphism ρ_t preserves ω, i.e., $\rho_t^*\omega = \omega$, $\forall t$.

Proof. We have $\frac{d}{dt}\rho_t^*\omega = \rho_t^*\mathcal{L}_{X_H}\omega = \rho_t^*(d\underbrace{\imath_{X_H}\omega}_{dH} + \imath_{X_H}\underbrace{d\omega}_{0}) = 0$. $\square$

Therefore, every function on (M, ω) gives a family of symplectomorphisms. Notice how the proof involved both the *nondegeneracy* and the *closedness* of ω.

Definition 18.1 *A vector field X_H as above is called the* **hamiltonian vector field** *with* **hamiltonian function** *H.*

Example. The height function $H(\theta, h) = h$ on the sphere $(M, \omega) = (S^2, d\theta \wedge dh)$ has

$$\imath_{X_H}(d\theta \wedge dh) = dh \qquad \Longleftrightarrow \qquad X_H = \frac{\partial}{\partial\theta} \ .$$

Thus, $\rho_t(\theta, h) = (\theta + t, h)$, which is rotation about the vertical axis; the height function H is preserved by this motion. $\diamondsuit$

Exercise. Let X be a vector field on an abstract manifold W. There is a unique vector field $X_\sharp$ on the cotangent bundle T^*W, whose flow is the lift of the flow of X; cf. Lecture 2. Let α be the tautological 1-form on T^*W and let $\omega = -d\alpha$ be the canonical symplectic form on T^*W. Show that $X_\sharp$ is a hamiltonian vector field with hamiltonian function $H := \imath_{X_\sharp}\alpha$. $\diamondsuit$

Remark. If X_H is hamiltonian, then

$$\mathcal{L}_{X_H} H = \imath_{X_H} dH = \imath_{X_H} \imath_{X_H} \omega = 0 \ .$$

Therefore, hamiltonian vector fields preserve their hamiltonian functions, and each integral curve $\{\rho_t(x) \mid t \in \mathbb{R}\}$ of X_H must be contained in a level set of H:

$$H(x) = (\rho_t^* H)(x) = H(\rho_t(x)) \ , \quad \forall t \ .$$

$\diamondsuit$

Definition 18.2 *A vector field X on M preserving ω (i.e., such that $\mathcal{L}_X \omega = 0$) is called a* **symplectic vector field**.

$$\begin{cases} X \text{ is symplectic} & \iff \quad \imath_X\omega \text{ is closed ,} \\ X \text{ is hamiltonian} & \iff \quad \imath_X\omega \text{ is exact .} \end{cases}$$

Locally, on every contractible open set, every symplectic vector field is hamiltonian. If $H^1_{\mathrm{deRham}}(M) = 0$, then globally every symplectic vector field is hamiltonian. In general, $H^1_{\mathrm{deRham}}(M)$ measures the obstruction for symplectic vector fields to be hamiltonian.

Example. On the 2-torus $(M, \omega) = (\mathbb{T}^2, d\theta_1 \wedge d\theta_2)$, the vector fields $X_1 = \frac{\partial}{\partial\theta_1}$ and $X_2 = \frac{\partial}{\partial\theta_2}$ are symplectic but not hamiltonian. $\diamondsuit$

To summarize, vector fields on a symplectic manifold (M, ω) which preserve ω are called **symplectic**. The following are equivalent:

- X is a symplectic vector field;
- the flow ρ_t of X preserves ω, i.e., $\rho_t^*\omega = \omega$, for all t;
- $\mathcal{L}_X\omega = 0$;
- $\imath_X\omega$ is closed.

A **hamiltonian** vector field is a vector field X for which

- $\imath_X\omega$ is exact,

i.e., $\imath_X\omega = dH$ for some $H \in C^\infty(M)$. A primitive H of $\imath_X\omega$ is then called a **hamiltonian function** of X.

18.2 Classical Mechanics

Consider euclidean space $\mathbb{R}^{2n}$ with coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ and $\omega_0 = \sum dq_j \wedge dp_j$. The curve $\rho_t = (q(t), p(t))$ is an integral curve for X_H exactly if

$$\begin{cases} \dfrac{dq_i}{dt}(t) = \dfrac{\partial H}{\partial p_i} \\[2ex] \dfrac{dp_i}{dt}(t) = -\dfrac{\partial H}{\partial q_i} \end{cases} \qquad \textbf{(Hamilton equations)}$$

Indeed, let $X_H = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i} \right)$. Then,

$$\begin{aligned} \imath_{X_H}\omega &= \sum_{j=1}^{n} \imath_{X_H}(dq_j \wedge dp_j) = \sum_{j=1}^{n} [(\imath_{X_H} dq_j) \wedge dp_j - dq_j \wedge (\imath_{X_H} dp_j)] \\ &= \sum_{j=1}^{n} \left(\frac{\partial H}{\partial p_j} dp_j + \frac{\partial H}{\partial q_j} dq_j \right) = dH \ . \end{aligned}$$

Remark. The gradient vector field of H relative to the euclidean metric is

$$\nabla H := \sum_{i=1}^{n} \left(\frac{\partial H}{\partial q_i}\frac{\partial}{\partial q_i} + \frac{\partial H}{\partial p_i}\frac{\partial}{\partial p_i} \right) \ .$$

If J is the standard (almost) complex structure so that $J(\frac{\partial}{\partial q_i}) = \frac{\partial}{\partial p_i}$ and $J(\frac{\partial}{\partial p_i}) = -\frac{\partial}{\partial q_i}$, we have $JX_H = \nabla H$. $\diamondsuit$

The case where $n = 3$ has a simple physical illustration. Newton's second law states that a particle of mass m moving in **configuration space** $\mathbb{R}^3$ with coordinates $q = (q_1, q_2, q_3)$ under a potential $V(q)$ moves along a curve $q(t)$ such that

$$m\frac{d^2q}{dt^2} = -\nabla V(q) \ .$$

Introduce the **momenta** $p_i = m\frac{dq_i}{dt}$ for $i = 1, 2, 3$, and **energy** function $H(p, q) = \frac{1}{2m}|p|^2 + V(q)$. Let $\mathbb{R}^6 = T^*\mathbb{R}^3$ be the corresponding **phase space**, with coordinates $(q_1, q_2, q_3, p_1, p_2, p_3)$. Newton's second law in $\mathbb{R}^3$ is equivalent to the Hamilton equations in $\mathbb{R}^6$:

$$\begin{cases} \dfrac{dq_i}{dt} = \dfrac{1}{m}p_i = \dfrac{\partial H}{\partial p_i} \\[2ex] \dfrac{dp_i}{dt} = m\dfrac{d^2q_i}{dt^2} = -\dfrac{\partial V}{\partial q_i} = -\dfrac{\partial H}{\partial q_i} \ . \end{cases}$$

The energy H is conserved by the physical motion.

18.3 Brackets

Vector fields are differential operators on functions: if X is a vector field and $f \in C^\infty(M)$, df being the corresponding 1-form, then

$$X \cdot f := df(X) = \mathcal{L}_X f \ .$$

Given two vector fields X, Y, there is a unique vector field W such that

$$\mathcal{L}_W f = \mathcal{L}_X(\mathcal{L}_Y f) - \mathcal{L}_Y(\mathcal{L}_X f) \ .$$

The vector field W is called the **Lie bracket** of the vector fields X and Y and denoted $W = [X,Y]$, since $\mathcal{L}_W = [\mathcal{L}_X, \mathcal{L}_Y]$ is the commutator.

Exercise. Check that, for any form α,

$$\imath_{[X,Y]}\alpha = \mathcal{L}_X \imath_Y \alpha - \imath_Y \mathcal{L}_X \alpha = [\mathcal{L}_X, \imath_Y]\alpha \ .$$

Since each side is an anti-derivation with respect to the wedge product, it suffices to check this formula on local generators of the exterior algebra of forms, namely functions and exact 1-forms. ♢

Proposition 18.3 *If X and Y are symplectic vector fields on a symplectic manifold (M,ω), then $[X,Y]$ is hamiltonian with hamiltonian function $\omega(Y,X)$.*

Proof.

$$\begin{aligned} \imath_{[X,Y]}\omega &= \mathcal{L}_X \imath_Y \omega - \imath_Y \mathcal{L}_X \omega \\ &= d\imath_X \imath_Y \omega + \imath_X \underbrace{d\imath_Y \omega}_{0} - \imath_Y \underbrace{d\imath_X \omega}_{0} - \imath_Y \imath_X \underbrace{d\omega}_{0} \\ &= d(\omega(Y,X)) \ . \end{aligned}$$

□

A (real) **Lie algebra** is a (real) vector space $\mathfrak{g}$ together with a **Lie bracket** $[\cdot,\cdot]$, i.e., a bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying:

(a) $[x,y] = -[y,x]$, $\quad \forall x,y \in \mathfrak{g}$, **(antisymmetry)**

(b) $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$, $\quad \forall x,y,z \in \mathfrak{g}$. **(Jacobi identity)**

Let

$$\begin{aligned} \chi(M) &= \{ \text{ vector fields on } M \ \} \\ \chi^{\text{sympl}}(M) &= \{ \text{ symplectic vector fields on } M \ \} \\ \chi^{\text{ham}}(M) &= \{ \text{ hamiltonian vector fields on } M \ \} \ . \end{aligned}$$

Corollary 18.4 *The inclusions $(\chi^{\text{ham}}(M),[\cdot,\cdot]) \subseteq (\chi^{\text{sympl}}(M),[\cdot,\cdot]) \subseteq (\chi(M),[\cdot,\cdot])$ are inclusions of Lie algebras.*

Definition 18.5 *The* **Poisson bracket** *of two functions $f,g \in C^\infty(M;\mathbb{R})$ is*

$$\{f,g\} := \omega(X_f, X_g) \ .$$

We have $X_{\{f,g\}} = -[X_f, X_g]$ because $X_{\omega(X_f,X_g)} = [X_g, X_f]$.

Theorem 18.6 *The bracket $\{\cdot,\cdot\}$ satisfies the Jacobi identity, i.e.,*

$$\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0 \ .$$

Proof. Exercise. □

Definition 18.7 *A* **Poisson algebra** $(\mathcal{P},\{\cdot,\cdot\})$ *is a commutative associative algebra $\mathcal{P}$ with a Lie bracket $\{\cdot,\cdot\}$ satisfying the* **Leibniz rule***:*

$$\{f,gh\} = \{f,g\}h + g\{f,h\} \ .$$

Exercise. Check that the Poisson bracket $\{\cdot,\cdot\}$ defined above satisfies the Leibniz rule. ♢

We conclude that, if (M,ω) is a symplectic manifold, then $(C^\infty(M),\{\cdot,\cdot\})$ is a Poisson algebra. Furthermore, we have a Lie algebra anti-homomorphism

$$\begin{array}{ccc} C^\infty(M) & \longrightarrow & \chi(M) \\ H & \longmapsto & X_{_H} \\ \{\cdot,\cdot\} & \leadsto & -[\cdot,\cdot] \ . \end{array}$$

18.4 Integrable Systems

Definition 18.8 *A* **hamiltonian system** *is a triple (M,ω,H), where (M,ω) is a symplectic manifold and $H \in C^\infty(M;\mathbb{R})$ is a function, called the* **hamiltonian function**.

Theorem 18.9 *We have $\{f,H\} = 0$ if and only if f is constant along integral curves of $X_{_H}$.*

Proof. Let ρ_t be the flow of $X_{_H}$. Then

$$\begin{array}{rcl} \frac{d}{dt}(f\circ\rho_t) & = & \rho_t^*\mathcal{L}_{X_H} f = \rho_t^*\imath_{X_H} df = \rho_t^*\imath_{X_H}\imath_{X_f}\omega \\ & = & \rho_t^*\omega(X_f, X_{_H}) = \rho_t^*\{f,H\} \ . \end{array}$$

□

A function f as in Theorem 18.9 is called an **integral of motion** (or a **first integral** or a **constant of motion**). In general, hamiltonian systems do not admit integrals of motion which are *independent* of the hamiltonian function. Functions $f_1,\ldots,f_n$ on M are said to be **independent** if their differentials $(df_1)_p,\ldots,(df_n)_p$ are linearly independent at all points p in some open dense subset of M. Loosely speaking, a hamiltonian system is *(completely) integrable* if it has as many commuting integrals of motion as possible. Commutativity is with respect to the Poisson bracket. Notice that, if $f_1, \ldots, f_n$ are commuting integrals of motion for a hamiltonian system (M, ω, H), then, at each $p \in M$, their hamiltonian vector fields generate an isotropic subspace of T_pM:

$$\omega(X_{f_i}, X_{f_j}) = \{f_i, f_j\} = 0 \ .$$

If $f_1, \ldots, f_n$ are independent, then, by symplectic linear algebra, n can be at most half the dimension of M.

Definition 18.10 *A hamiltonian system (M, ω, H) is* **(completely) integrable** *if it possesses $n = \frac{1}{2} \dim M$ independent integrals of motion, $f_1 = H, f_2, \ldots, f_n$, which are pairwise in involution with respect to the Poisson bracket, i.e., $\{f_i, f_j\} = 0$, for all i, j.*

Example. The simple pendulum (Homework 13) and the harmonic oscillator are trivially integrable systems – any 2-dimensional hamiltonian system (where the set of non-fixed points is dense) is integrable. $\diamondsuit$

Example. A hamiltonian system (M, ω, H) where M is 4-dimensional is integrable if there is an integral of motion independent of H (the commutativity condition is automatically satisfied). Homework 18 shows that the spherical pendulum is integrable. $\diamondsuit$

For sophisticated examples of integrable systems, see [10, 62].

Let (M, ω, H) be an integrable system of dimension $2n$ with integrals of motion $f_1 = H, f_2, \ldots, f_n$. Let $c \in \mathbb{R}^n$ be a regular value of $f := (f_1, \ldots, f_n)$. The corresponding level set, $f^{-1}(c)$, is a lagrangian submanifold, because it is n-dimensional and its tangent bundle is isotropic.

Lemma 18.11 *If the hamiltonian vector fields $X_{f_1}, \ldots, X_{f_n}$ are complete on the level $f^{-1}(c)$, then the connected components of $f^{-1}(c)$ are homogeneous spaces for $\mathbb{R}^n$, i.e., are of the form $\mathbb{R}^{n-k} \times \mathbb{T}^k$ for some k, $0 \leq k \leq n$, where $\mathbb{T}^k$ is a k-dimensional torus.*

Proof. Exercise (just follow the flows to obtain coordinates). $\square$

Any compact component of $f^{-1}(c)$ must hence be a torus. These components, when they exist, are called **Liouville tori**. (The easiest way to ensure that compact components exist is to have one of the f_i's proper.)

Theorem 18.12 (Arnold-Liouville [3]) *Let (M, ω, H) be an integrable system of dimension $2n$ with integrals of motion $f_1 = H, f_2, \ldots, f_n$. Let $c \in \mathbb{R}^n$ be a regular value of $f := (f_1, \ldots, f_n)$. The corresponding level $f^{-1}(c)$ is a lagrangian submanifold of M.*

(a) If the flows of $X_{f_1}, \ldots, X_{f_n}$ starting at a point $p \in f^{-1}(c)$ are complete, then the connected component of $f^{-1}(c)$ containing p is a homogeneous space for $\mathbb{R}^n$. With respect to this affine structure, that component has coordinates $\varphi_1, \ldots, \varphi_n$, known as **angle coordinates**, *in which the flows of the vector fields $X_{f_1}, \ldots, X_{f_n}$ are linear.*

(b) *There are coordinates $\psi_1, \ldots, \psi_n$, known as* **action coordinates***, complementary to the angle coordinates such that the ψ_i's are integrals of motion and $\varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_n$ form a Darboux chart.*

Therefore, the dynamics of an integrable system is extremely simple and the system has an explicit solution in action-angle coordinates. The proof of part (a) – the easy part – of the Arnold-Liouville theorem is sketched above. For the proof of part (b), see [3, 28].

Geometrically, regular levels being lagrangian submanifolds implies that, in a neighborhood of a regular value, the map $f : M \to \mathbb{R}^n$ collecting the given integrals of motion is a **lagrangian fibration**, i.e., it is locally trivial and its fibers are lagrangian submanifolds. Part (a) of the Arnold-Liouville theorem states that there are coordinates along the fibers, the angle coordinates φ_i,[11] in which the flows of $X_{f_1}, \ldots, X_{f_n}$ are linear. Part (b) of the theorem guarantees the existence of coordinates on $\mathbb{R}^n$, the action coordinates ψ_i, which (Poisson) commute among themselves and satisfy $\{\varphi_i, \psi_j\} = \delta_{ij}$ with respect to the angle coordinates. Notice that, in general, the action coordinates are not the given integrals of motion because $\varphi_1, \ldots, \varphi_n, f_1, \ldots, f_n$ do not form a Darboux chart.

[11] The name "angle coordinates" is used even if the fibers are not tori.

Homework 13: Simple Pendulum

This problem is adapted from [53].

The **simple pendulum** is a mechanical system consisting of a massless rigid rod of length l, fixed at one end, whereas the other end has a plumb bob of mass m, which may oscillate in the vertical plane. Assume that the force of gravity is constant pointing vertically downwards, and that this is the only external force acting on this system.

(a) Let θ be the oriented angle between the rod (regarded as a line segment) and the vertical direction. Let ξ be the coordinate along the fibers of T^*S^1 induced by the standard angle coordinate on S^1. Show that the function $H : T^*S^1 \to \mathbb{R}$ given by

$$H(\theta, \xi) = \underbrace{\frac{\xi^2}{2ml^2}}_{K} + \underbrace{ml(1 - \cos\theta)}_{V} ,$$

is an appropriate hamiltonian function to describe the simple pendulum. More precisely, check that gravity corresponds to the potential energy $V(\theta) = ml(1 - \cos\theta)$ (we omit universal constants), and that the kinetic energy is given by $K(\theta, \xi) = \frac{1}{2ml^2}\xi^2$.

(b) For simplicity assume that $m = l = 1$.
Plot the level curves of H in the (θ, ξ) plane.

Show that there exists a number c such that for $0 < h < c$ the level curve $H = h$ is a disjoint union of closed curves. Show that the projection of each of these curves onto the θ-axis is an interval of length less than π.

Show that neither of these assertions is true if $h > c$.

What types of motion are described by these two types of curves?
What about the case $H = c$?

(c) Compute the critical points of the function H. Show that, modulo 2π in θ, there are exactly two critical points: a critical point s where H vanishes, and a critical point u where H equals c. These points are called the **stable** and **unstable** points of H, respectively. Justify this terminology, i.e., show that a trajectory of the hamiltonian vector field of H whose initial point is close to s stays close to s forever, and show that this is not the case for u. What is happening physically?

19 Variational Principles

19.1 Equations of Motion

The equations of motion in classical mechanics arise as solutions of variational problems:

> A general mechanical system possesses both kinetic and potential energy. The quantity that is minimized is the mean value of kinetic minus potential energy.

Example. Suppose that a point-particle of mass m moves in $\mathbb{R}^3$ under a force field F; let $x(t)$, $a \le t \le b$, be its path of motion in $\mathbb{R}^3$. Newton's second law states that

$$m\frac{d^2x}{dt^2}(t) = F(x(t)) \ .$$

Define the **work** of a path $\gamma : [a,b] \longrightarrow \mathbb{R}^3$, with $\gamma(a) = p$ and $\gamma(b) = q$, to be

$$W_\gamma = \int_a^b F(\gamma(t)) \cdot \frac{d\gamma}{dt}(t)dt \ .$$

Suppose that F is **conservative**, i.e., W_γ depends only on p and q. Then we can define the **potential energy** $V : \mathbb{R}^3 \longrightarrow \mathbb{R}$ of the system as

$$V(q) := W_\gamma$$

where γ is a path joining a fixed base point $p_0 \in \mathbb{R}^3$ (the "origin") to q. Newton's second law can now be written

$$m\frac{d^2x}{dt^2}(t) = -\frac{\partial V}{\partial x}(x(t)) \ .$$

In the previous lecture we saw that

Newton's second law in $\mathbb{R}^3 = \{(q_1,q_2,q_3)\}$	$\Longleftrightarrow$	Hamilton equations in $T^*\mathbb{R}^3 = \{(q_1,q_2,q_3,p_1,p_2,p_3)\}$

where $p_i = m\frac{dq_i}{dt}$ and the hamiltonian is $H(p,q) = \frac{1}{2m}|p|^2 + V(q)$. Hence, solving Newton's second law in **configuration space** $\mathbb{R}^3$ is equivalent to solving in **phase space** $T^*\mathbb{R}^3$ for the integral curve of the hamiltonian vector field with hamiltonian function H. ♢

Example. The motion of earth about the sun, both regarded as point-masses and assuming that the sun to be stationary at the origin, obeys the **inverse square law**

$$m\frac{d^2x}{dt^2} = -\frac{\partial V}{\partial x} \ ,$$

where $x(t)$ is the position of earth at time t, and $V(x) = \frac{\text{const.}}{|x|}$ is the **gravitational potential**. ♢

19.2 Principle of Least Action

When we need to deal with systems with constraints, such as the simple pendulum, or two point masses attached by a rigid rod, or a rigid body, the language of variational principles becomes more appropriate than the explicit analogues of Newton's second laws. Variational principles are due mostly to D'Alembert, Maupertius, Euler and Lagrange.

Example. (The $\mathbf{n}$-particle system.) Suppose that we have n point-particles of masses $m_1, \ldots, m_n$ moving in 3-space. At any time t, the configuration of this system is described by a vector in configuration space $\mathbb{R}^{3n}$

$$x = (x_1, \ldots, x_n) \in \mathbb{R}^{3n}$$

with $x_i \in \mathbb{R}^3$ describing the position of the ith particle. If $V \in C^\infty(\mathbb{R}^{3n})$ is the potential energy, then a path of motion $x(t)$, $a \leq t \leq b$, satisfies

$$m_i \frac{d^2 x_i}{dt^2}(t) = -\frac{\partial V}{\partial x_i}(x_1(t), \ldots, x_n(t)) \ .$$

Consider this path in configuration space as a map $\gamma_0 : [a, b] \to \mathbb{R}^{3n}$ with $\gamma_0(a) = p$ and $\gamma_0(b) = q$, and let

$$\mathcal{P} = \{\gamma : [a, b] \longrightarrow \mathbb{R}^{3n} \mid \gamma(a) = p \text{ and } \gamma(b) = q\}$$

be the set of all paths going from p to q over time $t \in [a, b]$. $\diamondsuit$

Definition 19.1 *The* **action** *of a path $\gamma \in \mathcal{P}$ is*

$$\mathcal{A}_\gamma := \int_a^b \left(\sum_{i=1}^n \frac{m_i}{2} \left| \frac{d\gamma_i}{dt}(t) \right|^2 - V(\gamma(t)) \right) dt \ .$$

Principle of least action.
The physical path γ_0 is the path for which $\mathcal{A}_\gamma$ is minimal.

Newton's second law for a constrained system.
Suppose that the n point-masses are restricted to move on a submanifold M of $\mathbb{R}^{3n}$ called the **constraint set**. We can now single out the actual physical path $\gamma_0 : [a, b] \to M$, with $\gamma_0(a) = p$ and $\gamma_0(b) = q$, as being "the" path which minimizes $\mathcal{A}_\gamma$ among all those hypothetical paths $\gamma : [a, b] \to \mathbb{R}^{3n}$ with $\gamma(a) = p$, $\gamma(b) = q$ and satisfying the rigid constraints $\gamma(t) \in M$ for all t.

19.3 Variational Problems

Let M be an n-dimensional manifold. Its tangent bundle TM is a $2n$-dimensional manifold. Let $F : TM \to \mathbb{R}$ be a smooth function.

If $\gamma : [a,b] \to M$ is a smooth curve on M, define the **lift of γ to TM** to be the smooth curve on TM given by

$$\begin{array}{rcl} \tilde{\gamma} : [a,b] & \longrightarrow & TM \\ t & \longmapsto & \left(\gamma(t), \frac{d\gamma}{dt}(t)\right) \ . \end{array}$$

The **action** of γ is

$$\mathcal{A}_\gamma := \int_a^b (\tilde{\gamma}^* F)(t) dt = \int_a^b F\left(\gamma(t), \frac{d\gamma}{dt}(t)\right) dt \ .$$

For fixed $p, q \in M$, let

$$\mathcal{P}(a,b,p,q) := \{\gamma : [a,b] \longrightarrow M \mid \gamma(a) = p,\ \gamma(b) = q\} \ .$$

Problem.
Find, among all $\gamma \in \mathcal{P}(a,b,p,q)$, the curve γ_0 which "minimizes" $\mathcal{A}_\gamma$.

First observe that minimizing curves are always locally minimizing:

Lemma 19.2 *Suppose that $\gamma_0 : [a,b] \to M$ is minimizing. Let $[a_1,b_1]$ be a subinterval of $[a,b]$ and let $p_1 = \gamma_0(a_1)$, $q_1 = \gamma_0(b_1)$. Then $\gamma_0|_{[a_1,b_1]}$ is minimizing among the curves in $\mathcal{P}(a_1,b_1,p_1,q_1)$.*

Proof. Exercise:
Argue by contradiction. Suppose that there were $\gamma_1 \in \mathcal{P}(a_1,b_1,p_1,q_1)$ for which $\mathcal{A}_{\gamma_1} < \mathcal{A}_{\gamma_0|_{[a_1,b_1]}}$. Consider a broken path obtained from γ_0 by replacing the segment $\gamma_0|_{[a_1,b_1]}$ by γ_1. Construct a smooth curve $\gamma_2 \in \mathcal{P}(a,b,p,q)$ for which $\mathcal{A}_{\gamma_2} < \mathcal{A}_{\gamma_0}$ by rounding off the corners of the broken path. □

We will now assume that p, q and γ_0 lie in a coordinate neighborhood $(\mathcal{U}, x_1, \ldots, x_n)$. On $T\mathcal{U}$ we have coordinates $(x_1, \ldots, x_n, v_1, \ldots, v_n)$ associated with a trivialization of $T\mathcal{U}$ by $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$. Using this trivialization, the curve

$$\gamma : [a,b] \longrightarrow \mathcal{U} \ , \qquad \gamma(t) = (\gamma_1(t), \ldots, \gamma_n(t))$$

lifts to

$$\tilde{\gamma} : [a,b] \longrightarrow T\mathcal{U} \ , \qquad \tilde{\gamma}(t) = \left(\gamma_1(t), \ldots, \gamma_n(t), \frac{d\gamma_1}{dt}(t), \ldots, \frac{d\gamma_n}{dt}(t)\right) \ .$$

Necessary condition for $\gamma_0 \in \mathcal{P}(a,b,p,q)$ to minimize the action.
Let $c_1, \ldots, c_n \in C^\infty([a,b])$ be such that $c_i(a) = c_i(b) = 0$. Let $\gamma_\varepsilon : [a,b] \longrightarrow \mathcal{U}$ be the curve

$$\gamma_\varepsilon(t) = (\gamma_1(t) + \varepsilon c_1(t), \ldots, \gamma_n(t) + \varepsilon c_n(t)) \ .$$

For ε small, γ_ε is well-defined and in $\mathcal{P}(a,b,p,q)$.

Let $\mathcal{A}_\varepsilon = \mathcal{A}_{\gamma_\varepsilon} = \int_a^b F\left(\gamma_\varepsilon(t), \frac{d\gamma_\varepsilon}{dt}(t)\right) dt$. If γ_0 minimizes $\mathcal{A}$, then

$$\frac{d\mathcal{A}_\varepsilon}{d\varepsilon}(0) = 0 \ .$$

$$\begin{aligned}
\frac{d\mathcal{A}_\varepsilon}{d\varepsilon}(0) &= \int_a^b \sum_i \left[\frac{\partial F}{\partial x_i}\left(\gamma_0(t), \frac{d\gamma_0}{dt}(t)\right) c_i(t) + \frac{\partial F}{\partial v_i}\left(\gamma_0(t), \frac{d\gamma_0}{dt}(t)\right)\frac{dc_i}{dt}(t)\right] dt \\
&= \int_a^b \sum_i \left[\frac{\partial F}{\partial x_i}(\ldots) - \frac{d}{dt}\frac{\partial F}{\partial v_i}(\ldots)\right] c_i(t) dt = 0
\end{aligned}$$

where the first equality follows from the Leibniz rule and the second equality follows from integration by parts. Since this is true for all c_i's satisfying the boundary conditions $c_i(a) = c_i(b) = 0$, we conclude that

$$\frac{\partial F}{\partial x_i}\left(\gamma_0(t), \frac{d\gamma_0}{dt}(t)\right) = \frac{d}{dt}\frac{\partial F}{\partial v_i}\left(\gamma_0(t), \frac{d\gamma_0}{dt}(t)\right) \ . \qquad \textbf{E-L}$$

These are the **Euler-Lagrange equations**.

19.4 Solving the Euler-Lagrange Equations

Case 1: Suppose that $F(x, v)$ does not depend on v.

The Euler-Lagrange equations become

$$\frac{\partial F}{\partial x_i}\left(\gamma_0(t), \frac{d\gamma_0}{dt}(t)\right) = 0 \iff \text{the curve } \gamma_0 \text{ sits on the critical set of } F \ .$$

For generic F, the critical points are isolated, hence $\gamma_0(t)$ must be a constant curve.

Case 2: Suppose that $F(x, v)$ depends affinely on v:

$$F(x, v) = F_0(x) + \sum_{j=1}^n F_j(x) v_j \ .$$

$$\text{LHS of } \textbf{E-L}: \qquad \frac{\partial F_0}{\partial x_i}(\gamma(t)) + \sum_{j=1}^n \frac{\partial F_j}{\partial x_i}(\gamma(t))\frac{d\gamma_j}{dt}(t)$$

$$\text{RHS of } \textbf{E-L}: \qquad \frac{d}{dt}F_i(\gamma(t)) = \sum_{j=1}^n \frac{\partial F_i}{\partial x_j}(\gamma(t))\frac{d\gamma_j}{dt}(t)$$

The Euler-Lagrange equations become

$$\frac{\partial F_0}{\partial x_i}(\gamma(t)) = \sum_{j=1}^n \underbrace{\left(\frac{\partial F_i}{\partial x_j} - \frac{\partial F_j}{\partial x_i}\right)}_{n\times n \text{ matrix}}(\gamma(t))\frac{d\gamma_j}{dt}(t) \ .$$

If the $n \times n$ matrix $\left(\frac{\partial F_i}{\partial x_j} - \frac{\partial F_j}{\partial x_i}\right)$ has an inverse $G_{ij}(x)$, then

$$\frac{d\gamma_j}{dt}(t) = \sum_{i=1}^{n} G_{ji}(\gamma(t)) \frac{\partial F_0}{\partial x_i}(\gamma(t))$$

is a system of first order ordinary differential equations. Locally it has a unique solution through each point p. If q is not on this curve, there is no solution at all to the Euler-Lagrange equations belonging to $\mathcal{P}(a,b,p,q)$.

Therefore, we need non-linear dependence of F on the v variables in order to have appropriate solutions. From now on, assume that the

Legendre condition: $$\det\left(\frac{\partial^2 F}{\partial v_i \partial v_j}\right) \neq 0 \ .$$

Letting $G_{ij}(x,v) = \left(\frac{\partial^2 F}{\partial v_i \partial v_j}(x,v)\right)^{-1}$, the Euler-Lagrange equations become

$$\frac{d^2\gamma_j}{dt^2} = \sum_i G_{ji} \frac{\partial F}{\partial x_i}\left(\gamma, \frac{d\gamma}{dt}\right) - \sum_{i,k} G_{ji} \frac{\partial^2 F}{\partial v_i \partial x_k}\left(\gamma, \frac{d\gamma}{dt}\right) \frac{d\gamma_k}{dt} \ .$$

This second order ordinary differential equation has a unique solution given initial conditions

$$\gamma(a) = p \qquad \text{and} \qquad \frac{d\gamma}{dt}(a) = v \ .$$

19.5 Minimizing Properties

Is the above solution locally minimizing?

Assume that $\left(\frac{\partial^2 F}{\partial v_i \partial v_j}(x,v)\right) \gg 0$, $\forall (x,v)$, i.e., with the x variable frozen, the function $v \mapsto F(x,v)$ is **strictly convex**.

Suppose that $\gamma_0 \in \mathcal{P}(a,b,p,q)$ satisfies **E-L**. Does γ_0 minimize $\mathcal{A}_\gamma$? Locally, yes, according to the following theorem. (Globally it is only critical.)

Theorem 19.3 *For every sufficiently small subinterval* $[a_1,b_1]$ *of* $[a,b]$*,* $\gamma_0|_{[a_1,b_1]}$ *is locally minimizing in* $\mathcal{P}(a_1,b_1,p_1,q_1)$ *where* $p_1 = \gamma_0(a_1)$*,* $q_1 = \gamma_0(b_1)$*.*

Proof. As an exercise in Fourier series, show the **Wirtinger inequality**: for $f \in C^1([a,b])$ with $f(a) = f(b) = 0$, we have

$$\int_a^b \left|\frac{df}{dt}\right|^2 dt \geq \frac{\pi^2}{(b-a)^2} \int_a^b |f|^2 dt \ .$$

Suppose that $\gamma_0 : [a,b] \to \mathcal{U}$ satisfies **E-L**. Take $c_i \in C^\infty([a,b])$, $c_i(a) = c_i(b) = 0$. Let $c = (c_1,\ldots,c_n)$. Let $\gamma_\varepsilon = \gamma_0 + \varepsilon c \in \mathcal{P}(a,b,p,q)$, and let $\mathcal{A}_\varepsilon = \mathcal{A}_{\gamma_\varepsilon}$.

E-L $\iff \frac{d\mathcal{A}_\varepsilon}{d\varepsilon}(0) = 0.$

$$\begin{aligned}
\frac{d^2\mathcal{A}_\varepsilon}{d\varepsilon^2}(0) &= \int_a^b \sum_{i,j} \frac{\partial^2 F}{\partial x_i \partial x_j}\left(\gamma_0, \frac{d\gamma_0}{dt}\right) c_i \; c_j \; dt && \text{(I)} \\
&+ \; 2\int_a^b \sum_{i,j} \frac{\partial^2 F}{\partial x_i \partial v_j}\left(\gamma_0, \frac{d\gamma_0}{dt}\right) c_i \; \frac{dc_j}{dt} \; dt && \text{(II)} \\
&+ \; \int_a^b \sum_{i,j} \frac{\partial^2 F}{\partial v_i \partial v_j}\left(\gamma_0, \frac{d\gamma_0}{dt}\right) \frac{dc_i}{dt} \; \frac{dc_j}{dt} \; dt && \text{(III)} \; .
\end{aligned}$$

Since $\left(\frac{\partial^2 F}{\partial v_i \partial v_j}(x,v)\right) \gg 0$ at all x, v,

$$\begin{aligned}
\text{III} &\geq K_{\text{III}} \left|\frac{dc}{dt}\right|^2_{L^2[a,b]} \\
|\text{I}| &\leq K_{\text{I}} |c|^2_{L^2[a,b]} \\
|\text{II}| &\leq K_{\text{II}} |c|_{L^2[a,b]} \left|\frac{dc}{dt}\right|_{L^2[a,b]}
\end{aligned}$$

where $K_{\text{I}}, K_{\text{II}}, K_{\text{III}} > 0$. By the Wirtinger inequality, if $b - a$ is very small, then III $> |\text{I}| + |\text{II}|$ when $c \neq 0$. Hence, γ_0 is a local minimum. $\square$

Homework 14: Minimizing Geodesics

This set of problems is adapted from [53].

Let (M, g) be a riemannian manifold. From the riemannian metric, we get a function $F : TM \to \mathbb{R}$, whose restriction to each tangent space T_pM is the quadratic form defined by the metric.

Let p and q be points on M, and let $\gamma : [a,b] \to M$ be a smooth curve joining p to q. Let $\tilde{\gamma} : [a,b] \to TM$, $\tilde{\gamma}(t) = (\gamma(t), \frac{d\gamma}{dt}(t))$ be the lift of γ to TM. The **action** of γ is

$$\mathcal{A}(\gamma) = \int_a^b (\tilde{\gamma}^* F)\ dt = \int_a^b \left| \frac{d\gamma}{dt} \right|^2 dt\ .$$

1. Let $\gamma : [a,b] \to M$ be a smooth curve joining p to q. Show that the arc-length of γ is independent of the parametrization of γ, i.e., show that if we reparametrize γ by $\tau : [a', b'] \to [a,b]$, the new curve $\gamma' = \gamma \circ \tau : [a', b'] \to M$ has the same arc-length.

2. Show that, given any curve $\gamma : [a,b] \to M$ (with $\frac{d\gamma}{dt}$ never vanishing), there is a reparametrization $\tau : [a,b] \to [a,b]$ such that $\gamma \circ \tau : [a,b] \to M$ is of constant velocity, that is, $|\frac{d\gamma}{dt}|$ is independent of t.

3. Let $\tau : [a,b] \to [a,b]$ be a smooth monotone map taking the endpoints of $[a,b]$ to the endpoints of $[a,b]$. Prove that
$$\int_a^b \left(\frac{d\tau}{dt} \right)^2 dt \geq b - a\ ,$$
with equality holding if and only if $\frac{d\tau}{dt} = 1$.

4. Let $\gamma : [a,b] \to M$ be a smooth curve joining p to q. Suppose that, as s goes from a to b, its image $\gamma(s)$ moves at constant velocity, i.e., suppose that $|\frac{d\gamma}{ds}|$ is constant as a function of s. Let $\gamma' = \gamma \circ \tau : [a,b] \to M$ be a reparametrization of γ. Show that $\mathcal{A}(\gamma') \geq \mathcal{A}(\gamma)$, with equality holding if and only if $\tau(t) \equiv t$.

5. Let $\gamma_0 : [a,b] \to M$ be a curve joining p to q. Suppose that γ_0 is **action-minimizing**, i.e., suppose that

$$\mathcal{A}(\gamma_0) \leq \mathcal{A}(\gamma)$$

for any other curve $\gamma : [a,b] \to M$ joining p to q. Prove that γ_0 is also **arc-length-minimizing**, i.e., show that γ_0 is the shortest geodesic joining p to q.

6. Show that, among all curves joining p to q, γ_0 minimizes the action if and only if γ_0 is of constant velocity and γ_0 minimizes arc-length.

7. On a coordinate chart $(\mathcal{U}, x^1, \ldots, x^n)$ on M, we have

$$F(x,v) = \sum g_{ij}(x) v^i v^j \ .$$

Show that the Euler-Lagrange equations associated to the action reduce to the **Christoffel equations** for a geodesic

$$\frac{d^2\gamma^k}{dt^2} + \sum (\Gamma^k_{ij} \circ \gamma) \frac{d\gamma^i}{dt} \frac{d\gamma^j}{dt} = 0 \ ,$$

where the Γ^k_{ij}'s (called the **Christoffel symbols**) are defined in terms of the coefficients of the riemannian metric by

$$\Gamma^k_{ij} = \frac{1}{2} \sum_{\ell} g^{\ell k} \left(\frac{\partial g_{\ell i}}{\partial x_j} + \frac{\partial g_{\ell j}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_\ell} \right) \ ,$$

(g^{ij}) being the matrix inverse to (g_{ij}).

8. Let p and q be two non-antipodal points on S^n. Show that the geodesic joining p to q is an arc of a great circle, the great circle in question being the intersection of S^n with the two-dimensional subspace of $\mathbb{R}^{n+1}$ spanned by p and q.

   Hint: No calculations are needed: Show that an isometry of a riemannian manifold has to carry geodesics into geodesics, and show that there is an isometry of $\mathbb{R}^{n+1}$ whose fixed point set is the plane spanned by p and q, and show that this isometry induces on S^n an isometry whose fixed point set is the great circle containing p and q.

20 Legendre Transform

20.1 Strict Convexity

Let V be an n-dimensional vector space, with $e_1, \ldots, e_n$ a basis of V and $v_1, \ldots, v_n$ the associated coordinates. Let $F : V \to \mathbb{R}$, $F = F(v_1, \ldots, v_n)$, be a smooth function. Let $p \in V$, $u = \sum_{i=1}^n u_i e_i \in V$. The **hessian** of F is the quadratic function on V defined by

$$(d^2F)_p(u) := \sum_{i,j} \frac{\partial^2 F}{\partial v_i \partial v_j}(p) u_i u_j \ .$$

Exercise. Show that $(d^2F)_p(u) = \frac{d^2}{dt^2} F(p + tu)|_{t=0}$. $\diamondsuit$

Definition 20.1 *The function F is* **strictly convex** *if $(d^2F)_p \gg 0$, $\forall p \in V$.*

Proposition 20.2 *For a strictly convex function F on V, the following are equivalent:*

(a) F has a critical point, i.e., a point where $dF_p = 0$;

(b) F has a local minimum at some point;

(c) F has a unique critical point (global minimum); and

(d) F is proper, that is, $F(p) \to +\infty$ as $p \to \infty$ in V.

Proof. Homework 15. $\square$

Definition 20.3 *A strictly convex function F is* **stable** *when it satisfies conditions (a)-(d) in Proposition 20.2.*

Example. The function $e^x + ax$ is strictly convex for any $a \in \mathbb{R}$, but it is stable only for $a < 0$. The function $x^2 + ax$ is strictly convex and stable for any $a \in \mathbb{R}$. $\diamondsuit$

20.2 Legendre Transform

Let F be any strictly convex function on V. Given $\ell \in V^*$, let

$$F_\ell : V \longrightarrow \mathbb{R} \ , \qquad F_\ell(v) = F(v) - \ell(v) \ .$$

Since $(d^2F)_p = (d^2F_\ell)_p$,

$$F \text{ is strictly convex} \quad \Longleftrightarrow \quad F_\ell \text{ is strictly convex.}$$

Definition 20.4 *The* **stability set** *of a strictly convex function F is*

$$S_F = \{\ell \in V^* \mid F_\ell \text{ is stable}\} \ .$$

Proposition 20.5 *The set S_F is an open and convex subset of V^*.*

Proof. Homework 15. $\square$

Homework 15 also describes a sufficient condition for $S_F = V^*$.

Definition 20.6 *The* **Legendre transform** *associated to $F \in C^\infty(V;\mathbb{R})$ is the map*

$$\begin{array}{rccl} L_F : & V & \longrightarrow & V^* \\ & p & \longmapsto & dF_p \in T_p^*V \simeq V^* \ . \end{array}$$

Proposition 20.7 *Suppose that F is strictly convex. Then*

$$L_F : V \xrightarrow{\simeq} S_F \ ,$$

i.e., L_F is a diffeomorphism onto S_F.

The inverse map $L_F^{-1} : S_F \to V$ is described as follows: for $\ell \in S_F$, the value $L_F^{-1}(\ell)$ is the unique minimum point $p_\ell \in V$ of $F_\ell = F - \ell$.

Exercise. Check that p is the minimum of $F(v) - dF_p(v)$. $\diamondsuit$

Definition 20.8 *The* **dual function** *F^* to F is*

$$F^* : S_F \longrightarrow \mathbb{R} \ , \quad F^*(\ell) = -\min_{p \in V} F_\ell(p) \ .$$

Theorem 20.9 *We have that* $\quad L_F^{-1} = L_{F^*} \ .$

Proof. Homework 15. $\square$

20.3 Application to Variational Problems

Let M be a manifold and $F : TM \to \mathbb{R}$ a function on TM.

Problem. Minimize $\mathcal{A}_\gamma = \int \tilde{\gamma}^* F$.

At $p \in M$, let

$$F_p := F|_{T_pM} : T_pM \longrightarrow \mathbb{R} \ .$$

Assume that F_p is strictly convex for all $p \in M$. To simplify notation, assume also that $S_{F_p} = T_p^*M$. The Legendre transform on each tangent space

$$L_{F_p} : T_pM \xrightarrow{\simeq} T_p^*M$$

is essentially given by the first derivatives of F in the v directions. The dual function to F_p is $F_p^* : T_p^*M \longrightarrow \mathbb{R}$. Collect these fiberwise maps into

$$\begin{array}{llllll} \mathcal{L}: & TM & \longrightarrow & T^*M \ , & \mathcal{L}|_{T_pM} & = & L_{F_p} \ , & \text{and} \\ H: & T^*M & \longrightarrow & \mathbb{R} \ , & H|_{T_p^*M} & = & F_p^* \ . \end{array}$$

Exercise. The maps H and $\mathcal{L}$ are smooth, and $\mathcal{L}$ is a diffeomorphism. $\diamondsuit$

Let

$$\begin{array}{ll} \gamma : [a,b] \longrightarrow M & \text{be a curve, and} \\ \tilde{\gamma} : [a,b] \longrightarrow TM & \text{its lift.} \end{array}$$

Theorem 20.10 *The curve γ satisfies the Euler-Lagrange equations on every coordinate chart if and only if $\mathcal{L} \circ \tilde{\gamma} : [a,b] \to T^*M$ is an integral curve of the hamiltonian vector field X_H.*

Proof. Let

$$\begin{array}{ll} (\mathcal{U}, x_1, \ldots, x_n) & \text{coordinate neighborhood in } M \ , \\ (T\mathcal{U}, x_1, \ldots, x_n, v_1, \ldots, v_n) & \text{coordinates in } TM \ , \\ (T^*\mathcal{U}, x_1, \ldots, x_n, \xi_1, \ldots, \xi_n) & \text{coordinates in } T^*M \ . \end{array}$$

On $T\mathcal{U}$ we have $F = F(x,v)$.
On $T^*\mathcal{U}$ we have $H = H(u,\xi)$.

$$\begin{array}{llll} \mathcal{L}: & T\mathcal{U} & \longrightarrow & T^*\mathcal{U} \\ & (x,v) & \longmapsto & (x,\xi) \qquad \text{where} \qquad \xi = L_{F_x}(v) = \frac{\partial F}{\partial v}(x,v) \ . \end{array}$$

(This is the definition of **momentum** ξ.)

$$H(x,\xi) = F_x^*(\xi) = \xi \cdot v - F(x,v) \quad \text{where} \quad \mathcal{L}(x,v) = (x,\xi) \ .$$

Integral curves $(x(t),\xi(t))$ of X_H satisfy the Hamilton equations:

$$\textbf{H} \qquad \begin{cases} \dfrac{dx}{dt} & = \quad \dfrac{\partial H}{\partial \xi}(x,\xi) \\[2ex] \dfrac{d\xi}{dt} & = \ -\dfrac{\partial H}{\partial x}(x,\xi) \ , \end{cases}$$

whereas the physical path $x(t)$ satisfies the Euler-Lagrange equations:

$$\textbf{E-L} \qquad \frac{\partial F}{\partial x}\left(x, \frac{dx}{dt}\right) = \frac{d}{dt}\frac{\partial F}{\partial v}\left(x, \frac{dx}{dt}\right) \ .$$

Let $(x(t),\xi(t)) = \mathcal{L}\left(x(t), \frac{dx}{dt}(t)\right)$. We want to prove:

$$t \mapsto (x(t),\xi(t)) \ \text{ satisfies } \textbf{H} \qquad \Longleftrightarrow \qquad t \mapsto \left(x(t), \frac{dx}{dt}(t)\right) \ \text{ satisfies } \textbf{E-L} \ .$$

The first line of **H** is automatically satisfied:

$$\frac{dx}{dt} = \frac{\partial H}{\partial \xi}(x,\xi) = L_{F_x^*}(\xi) = L_{F_x}^{-1}(\xi) \qquad \Longleftrightarrow \qquad \xi = L_{F_x}\left(\frac{dx}{dt}\right)$$

Claim. If $(x,\xi) = \mathcal{L}(x,v)$, then $\frac{\partial F}{\partial x}(x,v) = -\frac{\partial H}{\partial x}(x,\xi)$.

This follows from differentiating both sides of $H(x,\xi) = \xi \cdot v - F(x,v)$ with respect to x, where $\xi = L_{F_x}(v) = \xi(x,v)$.

$$\frac{\partial H}{\partial x} + \underbrace{\frac{\partial H}{\partial \xi}}_{v}\frac{\partial \xi}{\partial x} = \frac{\partial \xi}{\partial x} \cdot v - \frac{\partial F}{\partial x} \ .$$

Now the second line of **H** becomes

$$\underbrace{\frac{d}{dt}\frac{\partial F}{\partial v}(x,v) = \frac{d\xi}{dt}}_{\text{since } \xi = L_{F_x}(v)} = \underbrace{-\frac{\partial H}{\partial x}(x,\xi) = \frac{\partial F}{\partial x}(x,v)}_{\text{by the claim}} \qquad \Longleftrightarrow \qquad \textbf{E-L} \ .$$

$\square$

Homework 15: Legendre Transform

This set of problems is adapted from [54].

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function. f is called **strictly convex** if $f''(x) > 0$ for all $x \in \mathbb{R}$. Assuming that f is strictly convex, prove that the following four conditions are equivalent:

   (a) $f'(x) = 0$ for some point x_0,
   (b) f has a local minimum at some point x_0,
   (c) f has a unique (global) minimum at some point x_0,
   (d) $f(x) \to +\infty$ as $x \to \pm\infty$.

   The function f is **stable** if it satisfies one (and hence all) of these conditions.

   For what values of a is the function $e^x + ax$ stable? For those values of a for which it is not stable, what does the graph look like?

2. Let V be an n-dimensional vector space and $F : V \to \mathbb{R}$ a smooth function. The function F is said to be **strictly convex** if for every pair of elements $p, v \in V$, $v \neq 0$, the restriction of F to the line $\{p + xv \,|\, x \in \mathbb{R}\}$ is strictly convex.

   The **hessian** of F at p is the quadratic form

$$d^2F_p : v \longmapsto \frac{d^2}{dx^2} F(p + xv)|_{x=0} \ .$$

   Show that F is strictly convex if and only if d^2F_p is positive definite for all $p \in V$.

   Prove the n-dimensional analogue of the result you proved in (1). Namely, assuming that F is strictly convex, show that the four following assertions are equivalent:

   (a) $dF_p = 0$ at some point p_0,
   (b) F has a local minimum at some point p_0,
   (c) F has a unique (global) minimum at some point p_0,
   (d) $F(p) \to +\infty$ as $p \to \infty$.

3. As in exercise 2, let V be an n-dimensional vector space and $F : V \to \mathbb{R}$ a smooth function. Since V is a vector space, there is a canonical identification $T_p^*V \simeq V^*$, for every $p \in V$. Therefore, we can define a map

$$L_F : V \longrightarrow V^* \qquad \textbf{(Legendre transform)}$$

   by setting

$$L_F(p) = dF_p \in T_p^*V \simeq V^* \ .$$

   Show that, if F is strictly convex, then, for every point $p \in V$, L_F maps a neighborhood of p diffeomorphically onto a neighborhood of $L_F(p)$.

4. A strictly convex function $F : V \to \mathbb{R}$ is **stable** if it satisfies the four equivalent conditions of exercise 2. Given any strictly convex function F, we will denote by S_F the set of $l \in V^*$ for which the function $F_l : V \to \mathbb{R}$, $p \mapsto F(p) - l(p)$, is stable. Prove that:

   (a) The set S_F is open and convex.

   (b) L_F maps V diffeomorphically onto S_F.

   (c) If $\ell \in S_F$ and $p_0 = L_F^{-1}(\ell)$, then p_0 is the unique minimum point of the function F_ℓ.

   Let $F^* : S_F \to \mathbb{R}$ be the function whose value at l is the quantity $-\min_{p \in V} F_l(p)$. Show that F^* is a smooth function.

   The function F^* is called the **dual** of the function F.

5. Let F be a strictly convex function. F is said to have **quadratic growth at infinity** if there exists a positive-definite quadratic form Q on V and a constant K such that $F(p) \geq Q(p) - K$, for all p. Show that, if F has quadratic growth at infinity, then $S_F = V^*$ and hence L_F maps V diffeomorphically onto V^*.

6. Let $F : V \to \mathbb{R}$ be strictly convex and let $F^* : S_F \to \mathbb{R}$ be the dual function. Prove that for all $p \in V$ and all $\ell \in S_F$,

$$F(p) + F^*(\ell) \geq \ell(p) \qquad \textbf{(Young inequality)} \ .$$

7. On one hand we have $V \times V^* \simeq T^*V$, and on the other hand, since $V = V^{**}$, we have $V \times V^* \simeq V^* \times V \simeq T^*V^*$.

   Let α_1 be the canonical 1-form on T^*V and α_2 be the canonical 1-form on T^*V^*. Via the identifications above, we can think of both of these forms as living on $V \times V^*$. Show that $\alpha_1 = d\beta - \alpha_2$, where $\beta : V \times V^* \to \mathbb{R}$ is the function $\beta(p, \ell) = \ell(p)$.

   Conclude that the forms $\omega_1 = d\alpha_1$ and $\omega_2 = d\alpha_2$ satisfy $\omega_1 = -\omega_2$.

8. Let $F : V \to \mathbb{R}$ be strictly convex. Assume that F has quadratic growth at infinity so that $S_F = V^*$. Let Λ_F be the graph of the Legendre transform L_F. The graph Λ_F is a lagrangian submanifold of $V \times V^*$ with respect to the symplectic form ω_1; why? Hence, Λ_F is also lagrangian for ω_2.

   Let $\mathrm{pr}_1 : \Lambda_F \to V$ and $\mathrm{pr}_2 : \Lambda_F \to V^*$ be the restrictions of the projection maps $V \times V^* \to V$ and $V \times V^* \to V^*$, and let $i : \Lambda_F \hookrightarrow V \times V^*$ be the inclusion map. Show that

$$i^*\alpha_1 = d(\mathrm{pr}_1)^*F \ .$$

   Conclude that

$$i^*\alpha_2 = d(i^*\beta - (\mathrm{pr}_1)^*F) = d(\mathrm{pr}_2)^*F^* \ ,$$

   and from this conclude that the inverse of the Legendre transform associated with F is the Legendre transform associated with F^*.

Part VIII

Moment Maps

The concept of a *moment map*[12] is a generalization of that of a hamiltonian function. The notion of a moment map associated to a group action on a symplectic manifold formalizes the Noether principle, which states that to every symmetry (such as a group action) in a mechanical system, there corresponds a conserved quantity.

21 Actions

21.1 One-Parameter Groups of Diffeomorphisms

Let M be a manifold and X a complete vector field on M. Let $\rho_t : M \to M$, $t \in \mathbb{R}$, be the family of diffeomorphisms generated by X. For each $p \in M$, $\rho_t(p)$, $t \in \mathbb{R}$, is by definition the unique integral curve of X passing through p at time 0, i.e., $\rho_t(p)$ satisfies

$$\begin{cases} \rho_0(p) &= p \\ \dfrac{d\rho_t(p)}{dt} &= X(\rho_t(p)) \ . \end{cases}$$

Claim. We have that $\rho_t \circ \rho_s = \rho_{t+s}$.

Proof. Let $\rho_s(q) = p$. We need to show that $(\rho_t \circ \rho_s)(q) = \rho_{t+s}(q)$, for all $t \in \mathbb{R}$. Reparametrize as $\tilde{\rho}_t(q) := \rho_{t+s}(q)$. Then

$$\begin{cases} \tilde{\rho}_0(q) &= \rho_s(q) = p \\ \dfrac{d\tilde{\rho}_t(q)}{dt} &= \dfrac{d\rho_{t+s}(q)}{dt} = X(\rho_{t+s}(q)) = X(\tilde{\rho}_t(q)) \ , \end{cases}$$

i.e., $\tilde{\rho}_t(q)$ is an integral curve of X through p. By uniqueness we must have $\tilde{\rho}_t(q) = \rho_t(p)$, that is, $\rho_{t+s}(q) = \rho_t(\rho_s(q))$. $\square$

Consequence. We have that $\rho_t^{-1} = \rho_{-t}$.

In terms of the group $(\mathbb{R}, +)$ and the group $(\mathrm{Diff}(M), \circ)$ of all diffeomorphisms of M, these results can be summarized as:

Corollary 21.1 *The map $\mathbb{R} \to \mathrm{Diff}(M)$, $t \mapsto \rho_t$, is a group homomorphism.*

The family $\{\rho_t \mid t \in \mathbb{R}\}$ is then called a **one-parameter group of diffeomorphisms** of M and denoted

$$\rho_t = \exp tX \ .$$

[12] Souriau [95] invented the french name "application moment." In the US, East and West coasts could be distinguished by the choice of translation: *moment map* and *momentum map*, respectively. We will stick to the more economical version.

21.2 Lie Groups

Definition 21.2 *A* **Lie group** *is a manifold G equipped with a group structure where the group operations*

$$\begin{array}{rcl} G \times G & \longrightarrow & G \\ (a,b) & \longmapsto & a \cdot b \end{array} \qquad \text{and} \qquad \begin{array}{rcl} G & \longrightarrow & G \\ a & \longmapsto & a^{-1} \end{array}$$

are smooth maps.

Examples.

- $\mathbb{R}$ (with addition[13]).
- S^1 regarded as unit complex numbers with multiplication, represents rotations of the plane: $S^1 = \mathrm{U}(1) = \mathrm{SO}(2)$.
- $\mathrm{U}(n)$, unitary linear transformations of $\mathbb{C}^n$.
- $\mathrm{SU}(n)$, unitary linear transformations of $\mathbb{C}^n$ with $\det = 1$.
- $\mathrm{O}(n)$, orthogonal linear transformations of $\mathbb{R}^n$.
- $\mathrm{SO}(n)$, elements of $\mathrm{O}(n)$ with $\det = 1$.
- $\mathrm{GL}(V)$, invertible linear transformations of a vector space V.

$\diamondsuit$

Definition 21.3 *A* **representation** *of a Lie group G on a vector space V is a group homomorphism $G \to \mathrm{GL}(V)$.*

21.3 Smooth Actions

Let M be a manifold.

Definition 21.4 *An* **action** *of a Lie group G on M is a group homomorphism*

$$\begin{array}{rrcl} \psi : & G & \longrightarrow & \mathrm{Diff}(M) \\ & g & \longmapsto & \psi_g \ . \end{array}$$

(We will only consider left actions where ψ is a homomorphism. A **right action** *is defined with ψ being an anti-homomorphism.) The* **evaluation map** *associated with an action $\psi : G \to \mathrm{Diff}(M)$ is*

$$\begin{array}{rrcl} \mathrm{ev}_\psi : & M \times G & \longrightarrow & M \\ & (p,g) & \longmapsto & \psi_g(p) \ . \end{array}$$

The action ψ is **smooth** *if ev_ψ is a smooth map.*

[13] The operation will be omitted when it is clear from the context.

Example. If X is a complete vector field on M, then

$$\begin{array}{rccl} \rho: & \mathbb{R} & \longrightarrow & \mathrm{Diff}(M) \\ & t & \longmapsto & \rho_t = \exp tX \end{array}$$

is a smooth action of $\mathbb{R}$ on M. ♢

Every complete vector field gives rise to a smooth action of $\mathbb{R}$ on M. Conversely, every smooth action of $\mathbb{R}$ on M is defined by a complete vector field.

$$\begin{array}{rcl} \{\text{complete vector fields on } M\} & \stackrel{1-1}{\longleftrightarrow} & \{\text{smooth actions of } \mathbb{R} \text{ on } M\} \\ X & \longmapsto & \exp tX \\ X_p = \left.\dfrac{d\psi_t(p)}{dt}\right|_{t=0} & \longleftarrow & \psi \end{array}$$

21.4 Symplectic and Hamiltonian Actions

Let (M, ω) be a symplectic manifold, and G a Lie group. Let $\psi : G \longrightarrow \mathrm{Diff}(M)$ be a (smooth) action.

Definition 21.5 *The action ψ is a* **symplectic action** *if*

$$\psi : G \longrightarrow \mathrm{Sympl}(M, \omega) \subset \mathrm{Diff}(M) \ ,$$

i.e., G "acts by symplectomorphisms."

$$\{\text{complete symplectic vector fields on } M\} \stackrel{1-1}{\longleftrightarrow} \{\text{symplectic actions of } \mathbb{R} \text{ on } M\}$$

Example. On $\mathbb{R}^{2n}$ with $\omega = \sum dx_i \wedge dy_i$, let $X = -\frac{\partial}{\partial y_1}$. The orbits of the action generated by X are lines parallel to the y_1-axis,

$$\{(x_1, y_1 - t, x_2, y_2, \ldots, x_n, y_n) \mid t \in \mathbb{R}\} \ .$$

Since $X = X_{x_1}$ is hamiltonian (with hamiltonian function $H = x_1$), this is actually an example of a *hamiltonian action* of $\mathbb{R}$. ♢

Example. On S^2 with $\omega = d\theta \wedge dh$ (cylindrical coordinates), let $X = \frac{\partial}{\partial \theta}$. Each orbit is a horizontal circle (called a "parallel") $\{(\theta + t, h) \mid t \in \mathbb{R}\}$. Notice that all orbits of this $\mathbb{R}$-action close up after time 2π, so that this is an action of S^1:

$$\begin{array}{rccl} \psi: & S^1 & \longrightarrow & \mathrm{Sympl}(S^2, \omega) \\ & t & \longmapsto & \text{rotation by angle } t \text{ around } h\text{-axis} \ . \end{array}$$

Since $X = X_h$ is hamiltonian (with hamiltonian function $H = h$), this is an example of a *hamiltonian action* of S^1. ♢

Definition 21.6 *A symplectic action ψ of S^1 or $\mathbb{R}$ on (M,ω) is* **hamiltonian** *if the vector field generated by ψ is hamiltonian. Equivalently, an action ψ of S^1 or $\mathbb{R}$ on (M,ω) is* **hamiltonian** *if there is $H : M \to \mathbb{R}$ with $dH = \imath_X\omega$, where X is the vector field generated by ψ.*

What is a "hamiltonian action" of an arbitrary Lie group?

For the case where $G = \mathbb{T}^n = S^1 \times \ldots \times S^1$ is an n-torus, an action $\psi : G \to \mathrm{Sympl}(M,\omega)$ should be called *hamiltonian* when each restriction

$$\psi^i := \psi|_{i\text{th } S^1 \text{ factor}} : S^1 \longrightarrow \mathrm{Sympl}(M,\omega)$$

is hamiltonian in the previous sense with hamiltonian function preserved by the action of the rest of G.

When G is not a product of S^1's or $\mathbb{R}$'s, the solution is to use an upgraded hamiltonian function, known as a *moment map*. Before its definition though (in Lecture 22), we need a little Lie theory.

21.5 Adjoint and Coadjoint Representations

Let G be a Lie group. Given $g \in G$ let

$$\begin{array}{rrcl} L_g : & G & \longrightarrow & G \\ & a & \longmapsto & g \cdot a \end{array}$$

be **left multiplication** by g. A vector field X on G is called **left-invariant** if $(L_g)_*X = X$ for every $g \in G$. (There are similar *right* notions.)

Let $\mathfrak{g}$ be the vector space of all left-invariant vector fields on G. Together with the Lie bracket $[\cdot,\cdot]$ of vector fields, $\mathfrak{g}$ forms a Lie algebra, called the **Lie algebra of the Lie group** G.

Exercise. Show that the map

$$\begin{array}{rcl} \mathfrak{g} & \longrightarrow & T_eG \\ X & \longmapsto & X_e \end{array}$$

where e is the identity element in G, is an isomorphism of vector spaces. ♢

Any Lie group G acts on itself by **conjugation**:

$$\begin{array}{rcl} G & \longrightarrow & \mathrm{Diff}(G) \\ g & \longmapsto & \psi_g \ , \end{array} \qquad \psi_g(a) = g \cdot a \cdot g^{-1} \ .$$

The derivative at the identity of

$$\begin{array}{rrcl} \psi_g : & G & \longrightarrow & G \\ & a & \longmapsto & g \cdot a \cdot g^{-1} \end{array}$$

is an invertible linear map $\mathrm{Ad}_g : \mathfrak{g} \longrightarrow \mathfrak{g}$. Here we identify the Lie algebra $\mathfrak{g}$ with the tangent space T_eG. Letting g vary, we obtain the **adjoint representation** (or **adjoint action**) of G on $\mathfrak{g}$:

$$\begin{array}{rccl} \mathrm{Ad}: & G & \longrightarrow & \mathrm{GL}(\mathfrak{g}) \\ & g & \longmapsto & \mathrm{Ad}_g \ . \end{array}$$

Exercise. Check for matrix groups that

$$\left.\frac{d}{dt}\mathrm{Ad}_{\exp tX}Y\right|_{t=0} = [X,Y] \ , \qquad \forall X,Y \in \mathfrak{g} \ .$$

Hint: For a matrix group G (i.e., a subgroup of $\mathrm{GL}(n;\mathbb{R})$ for some n), we have

$$\mathrm{Ad}_g(Y) = gYg^{-1} \ , \qquad \forall g \in G \ , \ \forall Y \in \mathfrak{g}$$

and

$$[X,Y] = XY - YX \ , \qquad \forall X,Y \in \mathfrak{g} \ .$$

$\diamondsuit$

Let $\langle \cdot, \cdot \rangle$ be the natural pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$:

$$\begin{array}{rccl} \langle \cdot,\cdot \rangle : & \mathfrak{g}^* \times \mathfrak{g} & \longrightarrow & \mathbb{R} \\ & (\xi, X) & \longmapsto & \langle \xi, X\rangle = \xi(X) \ . \end{array}$$

Given $\xi \in \mathfrak{g}^*$, we define $\mathrm{Ad}_g^*\xi$ by

$$\langle \mathrm{Ad}_g^*\xi, X\rangle = \langle \xi, \mathrm{Ad}_{g^{-1}}X\rangle \ , \qquad \text{for any } X \in \mathfrak{g} \ .$$

The collection of maps Ad_g^* forms the **coadjoint representation** (or **coadjoint action**) of G on $\mathfrak{g}^*$:

$$\begin{array}{rccl} \mathrm{Ad}^*: & G & \longrightarrow & \mathrm{GL}(\mathfrak{g}^*) \\ & g & \longmapsto & \mathrm{Ad}_g^* \ . \end{array}$$

We take g^{-1} in the definition of $\mathrm{Ad}_g^*\xi$ in order to obtain a (left) representation, i.e., a group homomorphism, instead of a "right" representation, i.e., a group anti-homomorphism.

Exercise. Show that $\quad \mathrm{Ad}_g \circ \mathrm{Ad}_h = \mathrm{Ad}_{gh} \quad$ and $\quad \mathrm{Ad}_g^* \circ \mathrm{Ad}_h^* = \mathrm{Ad}_{gh}^*$. $\qquad \diamondsuit$

Homework 16: Hermitian Matrices

Let $\mathcal{H}$ be the vector space of $n \times n$ complex hermitian matrices.

The unitary group $\mathrm{U}(n)$ acts on $\mathcal{H}$ by conjugation: $A \cdot \xi = A\xi A^{-1}$, for $A \in \mathrm{U}(n)$, $\xi \in \mathcal{H}$.

For each $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$, let $\mathcal{H}_\lambda$ be the set of all $n \times n$ complex hermitian matrices whose spectrum is λ.

1. Show that the orbits of the $\mathrm{U}(n)$-action are the manifolds $\mathcal{H}_\lambda$.
   For a fixed $\lambda \in \mathbb{R}^n$, what is the stabilizer of a point in $\mathcal{H}_\lambda$?

   Hint: If $\lambda_1, \ldots, \lambda_n$ are all distinct, the stabilizer of the diagonal matrix is the torus $\mathbb{T}^n$ of all diagonal unitary matrices.

2. Show that the symmetric bilinear form on $\mathcal{H}$, $(X, Y) \mapsto \mathrm{trace}\ (XY)$, is nondegenerate.
   For $\xi \in \mathcal{H}$, define a skew-symmetric bilinear form ω_ξ on $\mathfrak{u}(n) = T_1\mathrm{U}(n) = i\mathcal{H}$ (space of skew-hermitian matrices) by

   $$\omega_\xi(X, Y) = i\,\mathrm{trace}\ ([X, Y]\xi) \ , \qquad X, Y \in i\mathcal{H} \ .$$

   Check that $\omega_\xi(X, Y) = i\,\mathrm{trace}\ (X(Y\xi - \xi Y))$ and $Y\xi - \xi Y \in \mathcal{H}$.
   Show that the kernel of ω_ξ is $K_\xi := \{Y \in \mathfrak{u}(n) \mid [Y, \xi] = 0\}$.

3. Show that K_ξ is the Lie algebra of the stabilizer of ξ.

   Hint: Differentiate the relation $A\xi A^{-1} = \xi$.

   Show that the ω_ξ's induce nondegenerate 2-forms on the orbits $\mathcal{H}_\lambda$.
   Show that these 2-forms are closed.
   Conclude that all the orbits $\mathcal{H}_\lambda$ are compact symplectic manifolds.

4. Describe the manifolds $\mathcal{H}_\lambda$.
   When all eigenvalues are equal, there is only one point in the orbit.
   Suppose that $\lambda_1 \neq \lambda_2 = \ldots = \lambda_n$. Then the eigenspace associated with λ_1 is a line, and the one associated with λ_2 is the orthogonal hyperplane. Show that there is a diffeomorphism $\mathcal{H}_\lambda \simeq \mathbb{CP}^{n-1}$. We have thus exhibited a lot of symplectic forms on $\mathbb{CP}^{n-1}$, on for each pair of distinct real numbers.
   What about the other cases?

   Hint: When the eigenvalues $\lambda_1 < \ldots < \lambda_n$ are all distinct, any element in $\mathcal{H}_\lambda$ defines a family of pairwise orthogonal lines in $\mathbb{C}^n$: its eigenspaces.

5. Show that, for any skew-hermitian matrix $X \in \mathfrak{u}(n)$, the vector field on $\mathcal{H}$ generated by $X \in \mathfrak{u}(n)$ for the $\mathrm{U}(n)$-action by conjugation is $X^\#_\xi = [X, \xi]$.

22 Hamiltonian Actions

22.1 Moment and Comoment Maps

Let

$$\begin{array}{rl} (M,\omega) & \text{be a symplectic manifold,} \\ G & \text{a Lie group, and} \\ \psi : G \to \mathrm{Sympl}(M,\omega) & \text{a (smooth) symplectic action, i.e., a group homomorphism} \\ & \text{such that the evaluation map } \mathrm{ev}_\psi(g,p) := \psi_g(p) \text{ is smooth.} \end{array}$$

Case $G = \mathbb{R}$:

We have the following bijective correspondence:

$$\begin{array}{rcl} \{\text{symplectic actions of } \mathbb{R} \text{ on } M\} & \stackrel{1-1}{\longleftrightarrow} & \{\text{complete symplectic vector fields on } M\} \\ \psi & \longmapsto & X_p = \frac{d\psi_t(p)}{dt} \\ \psi = \exp tX & \longleftarrow & X \\ \text{"flow of } X\text{"} & & \text{"vector field generated by } \psi\text{"} \end{array}$$

The action ψ is **hamiltonian** if there exists a function $H : M \to \mathbb{R}$ such that $dH = \imath_X \omega$ where X is the vector field on M generated by ψ.

Case $G = S^1$:

An action of S^1 is an action of $\mathbb{R}$ which is 2π-periodic: $\psi_{2\pi} = \psi_0$. The S^1-action is called **hamiltonian** if the underlying $\mathbb{R}$-action is hamiltonian.

General case:

Let

$$\begin{array}{rl} (M,\omega) & \text{be a symplectic manifold,} \\ G & \text{a Lie group,} \\ \mathfrak{g} & \text{the Lie algebra of } G, \\ \mathfrak{g}^* & \text{the dual vector space of } \mathfrak{g}, \text{ and} \\ & \\ \psi : G \longrightarrow \mathrm{Sympl}(M,\omega) & \text{a symplectic action.} \end{array}$$

Definition 22.1 *The action ψ is a* **hamiltonian action** *if there exists a map*

$$\mu : M \longrightarrow \mathfrak{g}^*$$

satisfying:

1. *For each $X \in \mathfrak{g}$, let*

- *$\mu^X : M \to \mathbb{R}$, $\mu^X(p) := \langle \mu(p), X \rangle$, be the component of μ along X,*
- *$X^\#$ be the vector field on M generated by the one-parameter subgroup $\{\exp tX \mid t \in \mathbb{R}\} \subseteq G$.*

Then

$$d\mu^X = \imath_{X^\#}\omega$$

i.e., μ^X is a hamiltonian function for the vector field $X^\#$.

2. *μ is* equivariant *with respect to the given action ψ of G on M and the coadjoint action Ad^* of G on $\mathfrak{g}^*$:*

$$\mu \circ \psi_g = \mathrm{Ad}_g^* \circ \mu \ , \qquad \text{for all } g \in G \ .$$

The vector (M, ω, G, μ) is then called a **hamiltonian G-space** *and μ is a* **moment map**.

For connected Lie groups, hamiltonian actions can be equivalently defined in terms of a **comoment map**

$$\mu^* : \mathfrak{g} \longrightarrow C^\infty(M) \ ,$$

with the two conditions rephrased as:

1. $\mu^*(X) := \mu^X$ is a hamiltonian function for the vector field $X^\#$,
2. μ^* is a Lie algebra homomorphism:

$$\mu^*[X, Y] = \{\mu^*(X), \mu^*(Y)\}$$

where $\{\cdot, \cdot\}$ is the Poisson bracket on $C^\infty(M)$.

These definitions match the previous ones for the cases $G = \mathbb{R}$, S^1, torus, where equivariance becomes invariance since the coadjoint action is trivial.

Case $G = S^1$ (or $\mathbb{R}$):

Here $\mathfrak{g} \simeq \mathbb{R}$, $\mathfrak{g}^* \simeq \mathbb{R}$. A moment map $\mu : M \longrightarrow \mathbb{R}$ satisfies:

1. For the generator $X = 1$ of $\mathfrak{g}$, we have $\mu^X(p) = \mu(p) \cdot 1$, i.e., $\mu^X = \mu$, and $X^\#$ is the standard vector field on M generated by S^1. Then $d\mu = \imath_{X^\#}\omega$.
2. μ is invariant: $\mathcal{L}_{X^\#}\mu = \imath_{X^\#} d\mu = 0$.

Case $G = \mathbb{T}^n = n$-torus:

Here $\mathfrak{g} \simeq \mathbb{R}^n$, $\mathfrak{g}^* \simeq \mathbb{R}^n$. A moment map $\mu : M \longrightarrow \mathbb{R}^n$ satisfies:

1. For each basis vector X_i of $\mathbb{R}^n$, μ^{X_i} is a hamiltonian function for $X_i^\#$.
2. μ is invariant.

22.2 Orbit Spaces

Let $\psi : G \to \mathrm{Diff}(M)$ be any action.

Definition 22.2 *The* **orbit** *of G through $p \in M$ is $\{\psi_g(p) \mid g \in G\}$.*

The **stabilizer** *(or* **isotropy***) of $p \in M$ is the subgroup $G_p := \{g \in G \mid \psi_g(p) = p\}$.*

Exercise. If q is in the orbit of p, then G_q and G_p are conjugate subgroups. $\diamondsuit$

Definition 22.3 *We say that the action of G on M is ...*

- **transitive** *if there is just one orbit,*
- **free** *if all stabilizers are trivial $\{e\}$,*
- **locally free** *if all stabilizers are discrete.*

Let $\sim$ be the orbit equivalence relation; for $p, q \in M$,

$$p \sim q \quad \Longleftrightarrow \quad p \text{ and } q \text{ are on the same orbit.}$$

The space of orbits $M/\sim\, = M/G$ is called the **orbit space**. Let

$$\begin{array}{rrcl} \pi: & M & \longrightarrow & M/G \\ & p & \longmapsto & \text{orbit through } p \end{array}$$

be the **point-orbit projection**.

Topology of the orbit space:

We equip M/G with the weakest topology for which π is continuous, i.e., $\mathcal{U} \subseteq M/G$ is open if and only if $\pi^{-1}(\mathcal{U})$ is open in M. This is called the **quotient topology**. This topology can be "bad." For instance:

Example. Let $G = \mathbb{R}$ act on $M = \mathbb{R}$ by

$$t \longmapsto \psi_t = \text{multiplication by } e^t.$$

There are three orbits $\mathbb{R}^+$, $\mathbb{R}^-$ and $\{0\}$. The point in the three-point orbit space corresponding to the orbit $\{0\}$ is not open, so the orbit space with the quotient topology is *not* Hausdorff. $\diamondsuit$

Example. Let $G = \mathbb{C}\backslash\{0\}$ act on $M = \mathbb{C}^n$ by

$$\lambda \longmapsto \psi_\lambda = \text{multiplication by } \lambda \ .$$

The orbits are the punctured complex lines (through non-zero vectors $z \in \mathbb{C}^n$), plus one "unstable" orbit through 0, which has a single point. The orbit space is

$$M/G = \mathbb{CP}^{n-1} \sqcup \{\text{point}\} \ .$$

The quotient topology restricts to the usual topology on $\mathbb{CP}^{n-1}$. The only open set containing $\{\text{point}\}$ in the quotient topology is the full space. Again the quotient topology in M/G is *not* Hausdorff.

However, it suffices to remove 0 from $\mathbb{C}^n$ to obtain a Hausdorff orbit space: $\mathbb{CP}^{n-1}$. Then there is also a compact (yet not complex) description of the orbit space by taking only unit vectors:

$$\mathbb{CP}^{n-1} = \Big(\mathbb{C}^n \backslash \{0\}\Big) \Big/ \Big(\mathbb{C} \backslash \{0\}\Big) = S^{2n-1}/S^1 \ .$$

$\diamondsuit$

22.3 Preview of Reduction

Let $\omega = \frac{i}{2}\sum dz_i \wedge d\bar{z}_i = \sum dx_i \wedge dy_i = \sum r_i dr_i \wedge d\theta_i$ be the standard symplectic form on $\mathbb{C}^n$. Consider the following S^1-action on $(\mathbb{C}^n, \omega)$:

$$t \in S^1 \longmapsto \psi_t = \text{multiplication by } t \ .$$

The action ψ is hamiltonian with moment map

$$\begin{array}{rccl} \mu: & \mathbb{C}^n & \longrightarrow & \mathbb{R} \\ & z & \longmapsto & -\frac{|z|^2}{2} + \text{constant} \end{array}$$

since

$$\begin{array}{rcl} d\mu & = & -\frac{1}{2}d(\sum r_i^2) \\ X^\# & = & \dfrac{\partial}{\partial\theta_1} + \dfrac{\partial}{\partial\theta_2} + \ldots + \dfrac{\partial}{\partial\theta_n} \\ \imath_{X^\#}\omega & = & -\sum r_i dr_i = -\frac{1}{2}\sum dr_i^2 \ . \end{array}$$

If we choose the constant to be $\frac{1}{2}$, then $\mu^{-1}(0) = S^{2n-1}$ is the unit sphere. The orbit space of the zero level of the moment map is

$$\mu^{-1}(0)/S^1 = S^{2n-1}/S^1 = \mathbb{CP}^{n-1} \ .$$

$\mathbb{CP}^{n-1}$ is thus called a **reduced space**. Notice also that the image of the moment map is half-space.

These particular observations are related to major theorems:

Under assumptions (explained in Lectures 23-29),

- [Marsden-Weinstein-Meyer] reduced spaces are symplectic manifolds;
- [Atiyah-Guillemin-Sternberg] the image of the moment map is a convex polytope;
- [Delzant] hamiltonian $\mathbb{T}^n$-spaces are classified by the image of the moment map.

22.4 Classical Examples

Example.

Let $G = \mathrm{SO}(3) = \{A \in \mathrm{GL}(3;\mathbb{R}) \mid A\ {}^tA = \mathrm{Id} \text{ and } \det A = 1\}$. Then $\mathfrak{g} = \{A \in \mathfrak{gl}(3;\mathbb{R}) \mid A + A^t = 0\}$ is the space of 3×3 skew-symmetric matrices and can be identified with $\mathbb{R}^3$. The Lie bracket on $\mathfrak{g}$ can be identified with the exterior product via

$$A = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \quad \longmapsto \quad \vec{a} = (a_1, a_2, a_3)$$

$$[A, B] = AB - BA \quad \longmapsto \quad \vec{a} \times \vec{b}.$$

Exercise. Under the identifications $\mathfrak{g}, \mathfrak{g}^* \simeq \mathbb{R}^3$, the adjoint and coadjoint actions are the usual $\mathrm{SO}(3)$-action on $\mathbb{R}^3$ by rotations. $\diamondsuit$

Therefore, the coadjoint orbits are the spheres in $\mathbb{R}^3$ centered at the origin. Homework 17 shows that coadjoint orbits are symplectic. $\diamondsuit$

The name "moment map" comes from being the generalization of linear and angular momenta in classical mechanics.

Translation: Consider $\mathbb{R}^6$ with coordinates $x_1, x_2, x_3, y_1, y_2, y_3$ and symplectic form $\omega = \sum dx_i \wedge dy_i$. Let $\mathbb{R}^3$ act on $\mathbb{R}^6$ by translations:

$$\vec{a} \in \mathbb{R}^3 \longmapsto \quad \psi_{\vec{a}} \in \mathrm{Sympl}(\mathbb{R}^6, \omega)$$

$$\psi_{\vec{a}}(\vec{x}, \vec{y}) = (\vec{x} + \vec{a}, \vec{y})\ .$$

Then $X^\# = a_1 \frac{\partial}{\partial x_1} + a_2 \frac{\partial}{\partial x_2} + a_3 \frac{\partial}{\partial x_3}$ for $X = \vec{a}$, and

$$\mu : \mathbb{R}^6 \longrightarrow \mathbb{R}^3\ , \quad \mu(\vec{x}, \vec{y}) = \vec{y}$$

is a moment map, with

$$\mu^{\vec{a}}(\vec{x}, \vec{y}) = \langle \mu(\vec{x}, \vec{y}), \vec{a} \rangle = \vec{y} \cdot \vec{a}\ .$$

Classically, $\vec{y}$ is called the **momentum vector** corresponding to the **position vector** $\vec{x}$, and the map μ is called the **linear momentum**.

Rotation: The $\mathrm{SO}(3)$-action on $\mathbb{R}^3$ by rotations lifts to a symplectic action ψ on the cotangent bundle $\mathbb{R}^6$. The infinitesimal version of this action is

$$\vec{a} \in \mathbb{R}^3 \quad \longmapsto \quad d\psi(\vec{a}) \in \chi^{\mathrm{sympl}}(\mathbb{R}^6)$$

$$d\psi(\vec{a})(\vec{x}, \vec{y}) = (\vec{a} \times \vec{x}, \vec{a} \times \vec{y})\ .$$

Then

$$\mu : \mathbb{R}^6 \longrightarrow \mathbb{R}^3 \ , \quad \mu(\overrightarrow{x}, \overrightarrow{y}) = \overrightarrow{x} \times \overrightarrow{y}$$

is a moment map, with

$$\mu^{\overrightarrow{a}}(\overrightarrow{x}, \overrightarrow{y}) = \langle \mu(\overrightarrow{x}, \overrightarrow{y}), \overrightarrow{a} \rangle = (\overrightarrow{x} \times \overrightarrow{y}) \cdot \overrightarrow{a} .$$

The map μ is called the **angular momentum**.

Homework 17: Coadjoint Orbits

Let G be a Lie group, $\mathfrak{g}$ its Lie algebra and $\mathfrak{g}^*$ the dual vector space of $\mathfrak{g}$.

1. Let ${}^{\mathfrak{g}}X^\#$ be the vector field generated by $X \in \mathfrak{g}$ for the adjoint representation of G on $\mathfrak{g}$. Show that
$$ {}^{\mathfrak{g}}X^\#_Y = [X, Y] \qquad \forall\, Y \in \mathfrak{g} \ .$$

2. Let $X^\#$ be the vector field generated by $X \in \mathfrak{g}$ for the coadjoint representation of G on $\mathfrak{g}^*$. Show that
$$\langle X^\#_\xi, Y\rangle = \langle \xi, [Y, X]\rangle \qquad \forall\, Y \in \mathfrak{g} \ .$$

3. For any $\xi \in \mathfrak{g}^*$, define a skew-symmetric bilinear form on $\mathfrak{g}$ by
$$\omega_\xi(X, Y) := \langle \xi, [X, Y]\rangle \ .$$
Show that the kernel of ω_ξ is the Lie algebra $\mathfrak{g}_\xi$ of the stabilizer of ξ for the coadjoint representation.

4. Show that ω_ξ defines a nondegenerate 2-form on the tangent space at ξ to the coadjoint orbit through ξ.

5. Show that ω_ξ defines a closed 2-form on the orbit of ξ in $\mathfrak{g}^*$.

   Hint: The tangent space to the orbit being generated by the vector fields $X^\#$, this is a consequence of the Jacobi identity in $\mathfrak{g}$.

   This **canonical symplectic form** on the coadjoint orbits in $\mathfrak{g}^*$ is also known as the **Lie-Poisson** or **Kostant-Kirillov symplectic structure**.

6. The Lie algebra structure of $\mathfrak{g}$ defines a canonical Poisson structure on $\mathfrak{g}^*$:
$$\{f, g\}(\xi) := \langle \xi, [df_\xi, dg_\xi]\rangle$$
for $f, g \in C^\infty(\mathfrak{g}^*)$ and $\xi \in \mathfrak{g}^*$. Notice that $df_\xi : T_\xi\mathfrak{g}^* \simeq \mathfrak{g}^* \to \mathbb{R}$ is identified with an element of $\mathfrak{g} \simeq \mathfrak{g}^{**}$.
Check that $\{\cdot, \cdot\}$ satisfies the Leibniz rule:
$$\{f, gh\} = g\{f, h\} + h\{f, g\} \ .$$

7. Show that the **jacobiator**
$$J(f, g, h) := \{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\}$$
is a trivector field, i.e., J is a skew-symmetric trilinear map $C^\infty(\mathfrak{g}^*)\times C^\infty(\mathfrak{g}^*)\times C^\infty(\mathfrak{g}^*) \to C^\infty(\mathfrak{g}^*)$, which is a derivation in each argument.

   Hint: Being a derivation amounts to the Leibniz rule from exercise 6.

8. Show that $J \equiv 0$, i.e., $\{\cdot, \cdot\}$ satisfies the Jacobi identity.

   Hint: Follows from the Jacobi identity for $[\cdot, \cdot]$ in $\mathfrak{g}$. It is enough to check on coordinate functions.

Part IX

Symplectic Reduction

The phase space of a system of n particles is the space parametrizing the position and momenta of the particles. The mathematical model for the phase space is a symplectic manifold. Classical physicists realized that, whenever there is a symmetry group of dimension k acting on a mechanical system, then the number of degrees of freedom for the position and momenta of the particles may be reduced by $2k$. Symplectic reduction formulates this feature mathematically.

23 The Marsden-Weinstein-Meyer Theorem

23.1 Statement

Theorem 23.1 (Marsden-Weinstein-Meyer [77, 85]) *Let (M, ω, G, μ) be a hamiltonian G-space for a compact Lie group G. Let $i : \mu^{-1}(0) \hookrightarrow M$ be the inclusion map. Assume that G acts freely on $\mu^{-1}(0)$. Then*

- *the orbit space $M_{\mathrm{red}} = \mu^{-1}(0)/G$ is a manifold,*
- *$\pi : \mu^{-1}(0) \to M_{\mathrm{red}}$ is a principal G-bundle, and*
- *there is a symplectic form ω_{red} on M_{red} satisfying $i^*\omega = \pi^*\omega_{\mathrm{red}}$.*

Definition 23.2 *The pair $(M_{\mathrm{red}}, \omega_{\mathrm{red}})$ is called the* **reduction** *of (M, ω) with respect to G, μ, or the* **reduced space***, or the* **symplectic quotient***, or the* **Marsden-Weinstein-Meyer quotient***, etc.*

Low-brow proof for the case $G = S^1$ and $\dim M = 4$.

In this case the moment map is $\mu : M \to \mathbb{R}$. Let $p \in \mu^{-1}(0)$. Choose local coordinates:

- θ along the orbit through p,
- μ given by the moment map, and
- η_1, η_2 pullback of coordinates on $\mu^{-1}(0)/S^1$.

Then the symplectic form can be written

$$\omega = A\ d\theta \wedge d\mu + B_j\ d\theta \wedge d\eta_j + C_j\ d\mu \wedge d\eta_j + D\ d\eta_1 \wedge d\eta_2\ .$$

Since $d\mu = \imath\left(\frac{\partial}{\partial\theta}\right)\omega$, we must have $A = 1$, $B_j = 0$. Hence,

$$\omega = d\theta \wedge d\mu + C_j\ d\mu \wedge d\eta_j + D\ d\eta_1 \wedge d\eta_2\ .$$

Since ω is symplectic, we must have $D \neq 0$. Therefore, $i^*\omega = D\ d\eta_1 \wedge d\eta_2$ is the pullback of a symplectic form on M_{red}. $\square$

The actual proof of the Marsden-Weinstein-Meyer theorem requires the following ingredients.

23.2 Ingredients

1. Let $\mathfrak{g}_p$ be the Lie algebra of the stabilizer of $p \in M$. Then $d\mu_p : T_pM \to \mathfrak{g}^*$ has
$$\begin{array}{rcl} \ker d\mu_p & = & (T_p\mathcal{O}_p)^{\omega_p} \\ \operatorname{im} d\mu_p & = & \mathfrak{g}_p^0 \end{array}$$
where $\mathcal{O}_p$ is the G-orbit through p, and $\mathfrak{g}_p^0 = \{\xi \in \mathfrak{g}^* \mid \langle \xi, X\rangle = 0,\ \forall X \in \mathfrak{g}_p\}$ is the annihilator of $\mathfrak{g}_p$.

   Proof. Stare at the expression $\omega_p(X_p^\#, v) = \langle d\mu_p(v), X\rangle$, for all $v \in T_pM$ and all $X \in \mathfrak{g}$, and count dimensions. $\square$

   Consequences:

   - The action is locally free at p
     $\iff$ $\mathfrak{g}_p = \{0\}$
     $\iff$ $d\mu_p$ is surjective
     $\iff$ p is a regular point of μ.
   - G acts freely on $\mu^{-1}(0)$
     $\Longrightarrow$ 0 is a regular value of μ
     $\Longrightarrow$ $\mu^{-1}(0)$ is a closed submanifold of M of codimension equal to $\dim G$.
   - G acts freely on $\mu^{-1}(0)$
     $\Longrightarrow$ $T_p\mu^{-1}(0) = \ker d\mu_p$ (for $p \in \mu^{-1}(0)$)
     $\Longrightarrow$ $T_p\mu^{-1}(0)$ and $T_p\mathcal{O}_p$ are symplectic orthocomplements in T_pM.
     In particular, the tangent space to the orbit through $p \in \mu^{-1}(0)$ is an isotropic subspace of T_pM. Hence, orbits in $\mu^{-1}(0)$ are isotropic.

   Since any tangent vector to the orbit is the value of a vector field generated by the group, we can confirm that orbits are isotropic directly by computing, for any $X, Y \in \mathfrak{g}$ and any $p \in \mu^{-1}(0)$,
$$\begin{array}{rcl} \omega_p(X_p^\#, Y_p^\#) & = & \text{hamiltonian function for } [Y^\#, X^\#] \text{ at } p \\ & = & \text{hamiltonian function for } [Y, X]^\# \text{ at } p \\ & = & \mu^{[Y,X]}(p) = 0\ . \end{array}$$

2. **Lemma 23.3** *Let (V, ω) be a symplectic vector space. Suppose that I is an isotropic subspace, that is, $\omega|_I \equiv 0$. Then ω induces a canonical symplectic form Ω on I^ω/I.*

   Proof. Let $u, v \in I^\omega$, and $[u], [v] \in I^\omega/I$. Define $\Omega([u], [v]) = \omega(u, v)$.

   - Ω is well-defined:
$$\omega(u + i, v + j) = \omega(u, v) + \underbrace{\omega(u, j)}_{0} + \underbrace{\omega(i, v)}_{0} + \underbrace{\omega(i, j)}_{0}\ , \quad \forall i, j \in I\ .$$

- Ω is nondegenerate:
  Suppose that $u \in I^\omega$ has $\omega(u,v) = 0$, for all $v \in I^\omega$.
  Then $u \in (I^\omega)^\omega = I$, i.e., $[u] = 0$.

$\square$

3. **Theorem 23.4** *If a compact Lie group G acts freely on a manifold M, then M/G is a manifold and the map $\pi : M \to M/G$ is a principal G-bundle.*

   Proof. We will first show that, for any $p \in M$, the G-orbit through p is a compact embedded submanifold of M diffeomorphic to G.

   Since the action is smooth, the evaluation map $\mathrm{ev} : G \times M \to M$, $\mathrm{ev}(g,p) = g \cdot p$, is smooth. Let $\mathrm{ev}_p : G \to M$ be defined by $\mathrm{ev}_p(g) = g \cdot p$. The map ev_p provides the embedding we seek:

   The image of ev_p is the G-orbit through p. Injectivity of ev_p follows from the action of G being free. The map ev_p is proper because, if A is a compact, hence closed, subset of M, then its inverse image $(\mathrm{ev}_p)^{-1}(A)$, being a closed subset of the compact Lie group G, is also compact. It remains to show that ev_p is an immersion. For $X \in \mathfrak{g} \simeq T_eG$, we have

   $$d(\mathrm{ev}_p)_e(X) = 0 \iff X_p^\# = 0 \iff X = 0 \ ,$$

   as the action is free. We conclude that $d(\mathrm{ev}_p)_e$ is injective. At any other point $g \in G$, for $X \in T_gG$, we have

   $$d(\mathrm{ev}_p)_g(X) = 0 \iff d(\mathrm{ev}_p \circ R_g)_e \circ (dR_{g^{-1}})_g(X) = 0 \ ,$$

   where $R_g : G \to G$ is right multiplication by g. But $\mathrm{ev}_p \circ R_g = \mathrm{ev}_{g \cdot p}$ has an injective differential at e, and $(dR_{g^{-1}})_g$ is an isomorphism. It follows that $d(\mathrm{ev}_p)_g$ is always injective.

   Exercise. Show that, even if the action is not free, the G-orbit through p is a compact embedded submanifold of M. In that case, the orbit is diffeomorphic to the quotient of G by the isotropy of p: $\mathcal{O}_p \simeq G/G_p$. $\diamondsuit$

   Let S be a transverse section to $\mathcal{O}_p$ at p; this is called a **slice**. Choose a coordinate system $x_1, \ldots, x_n$ centered at p such that

   $$\begin{array}{lccccccc} \mathcal{O}_p \simeq G & : & x_1 & = & \ldots & = & x_k & = & 0 \\ S & : & x_{k+1} & = & \ldots & = & x_n & = & 0 \ . \end{array}$$

   Let $S_\varepsilon = S \cap B_\varepsilon(0, \mathbb{R}^n)$ where $B_\varepsilon(0, \mathbb{R}^n)$ is the ball of radius ε centered at 0 in $\mathbb{R}^n$. Let $\eta : G \times S \to M$, $\eta(g,s) = g \cdot s$. Apply the following equivariant tubular neighborhood theorem.

Theorem 23.5 (Slice Theorem) *Let G be a compact Lie group acting on a manifold M such that G acts freely at $p \in M$. For sufficiently small ε, $\eta : G \times S_\varepsilon \to M$ maps $G \times S_\varepsilon$ diffeomorphically onto a G-invariant neighborhood $\mathcal{U}$ of the G-orbit through p.*

The proof of this slice theorem is sketched further below.

Corollary 23.6 *If the action of G is free at p, then the action is free on $\mathcal{U}$.*

Corollary 23.7 *The set of points where G acts freely is open.*

Corollary 23.8 *The set $G \times S_\varepsilon \simeq \mathcal{U}$ is G-invariant. Hence, the quotient $\mathcal{U}/G \simeq S_\varepsilon$ is smooth.*

Conclusion of the proof that M/G is a manifold and $\pi : M \to M/G$ is a smooth fiber map.

For $p \in M$, let $q = \pi(p) \in M/G$. Choose a G-invariant neighborhood $\mathcal{U}$ of p as in the slice theorem: $\mathcal{U} \simeq G \times S$ (where $S = S_\varepsilon$ for an appropriate ε). Then $\pi(\mathcal{U}) = \mathcal{U}/G =: \mathcal{V}$ is an open neighborhood of q in M/G. By the slice theorem, $S \xrightarrow{\simeq} \mathcal{V}$ is a homeomorphism. We will use such neighborhoods $\mathcal{V}$ as charts on M/G. To show that the transition functions associated with these charts are smooth, consider two G-invariant open sets $\mathcal{U}_1, \mathcal{U}_2$ in M and corresponding slices S_1, S_2 of the G-action. Then $S_{12} = S_1 \cap \mathcal{U}_2$, $S_{21} = S_2 \cap \mathcal{U}_1$ are both slices for the G-action on $\mathcal{U}_1 \cap \mathcal{U}_2$. To compute the transition map $S_{12} \to S_{21}$, consider the diagram

$$\begin{array}{ccccccc} S_{12} & \xrightarrow{\simeq} & \mathrm{id} \times S_{12} & \hookrightarrow & G \times S_{12} & & \\ & & & & & \searrow^{\simeq} & \\ & & & & & & \mathcal{U}_1 \cap \mathcal{U}_2 \ . \\ & & & & & \nearrow_{\simeq} & \\ S_{21} & \xrightarrow{\simeq} & \mathrm{id} \times S_{21} & \hookrightarrow & G \times S_{21} & & \end{array}$$

Then the composition

$$S_{12} \hookrightarrow \mathcal{U}_1 \cap \mathcal{U}_2 \xrightarrow{\simeq} G \times S_{21} \xrightarrow{pr} S_{21}$$

is smooth.

Finally, we need to show that $\pi : M \to M/G$ is a smooth fiber map. For $p \in M$, $q = \pi(p)$, choose a G-invariant neighborhood $\mathcal{U}$ of the G-orbit through p of the form $\eta : G \times S \xrightarrow{\simeq} \mathcal{U}$. Then $\mathcal{V} = \mathcal{U}/G \simeq S$ is the corresponding neighborhood of q in M/G:

$$\begin{array}{ccccccc} M \supseteq & \mathcal{U} & \overset{\eta}{\simeq} & G \times S & \simeq & G \times \mathcal{V} \\ & \downarrow \pi & & & & \downarrow \\ M/G \supseteq & \mathcal{V} & & = & & \mathcal{V} \end{array}$$

Since the projection on the right is smooth, π is smooth.

Exercise. Check that the transition functions for the bundle defined by π are smooth. $\diamondsuit$

$\square$

Sketch for the proof of the slice theorem. We need to show that, for ε sufficiently small, $\eta : G \times S_\varepsilon \to \mathcal{U}$ is a diffeomorphism where $\mathcal{U} \subseteq M$ is a G-invariant neighborhood of the G-orbit through p. Show that:

(a) $d\eta_{(\mathrm{id},p)}$ is bijective.

(b) Let G act on $G \times S$ by the product of its left action on G and trivial action on S. Then $\eta : G \times S \to M$ is G-equivariant.

(c) $d\eta$ is bijective at all points of $G \times \{p\}$. This follows from (a) and (b).

(d) The set $G \times \{p\}$ is compact, and $\eta : G \times S \to M$ is injective on $G \times \{p\}$ with $d\eta$ bijective at all these points. By the implicit function theorem, there is a neighborhood $\mathcal{U}_0$ of $G \times \{p\}$ in $G \times S$ such that η maps $\mathcal{U}_0$ diffeomorphically onto a neighborhood $\mathcal{U}$ of the G-orbit through p.

(e) The sets $G \times S_\varepsilon$, varying ε, form a neighborhood base for $G \times \{p\}$ in $G \times S$. So in (d) we may take $\mathcal{U}_0 = G \times S_\varepsilon$.

$\square$

23.3 Proof of the Marsden-Weinstein-Meyer Theorem

Since

$$\begin{array}{rcl} G \text{ acts freely on } \mu^{-1}(0) & \Longrightarrow & d\mu_p \text{ is surjective for all } p \in \mu^{-1}(0) \\ & \Longrightarrow & 0 \text{ is a regular value} \\ & \Longrightarrow & \mu^{-1}(0) \text{ is a submanifold of codimension} = \dim G \end{array}$$

for the first two parts of the Marsden-Weinstein-Meyer theorem it is enough to apply the third ingredient from Section 23.2 to the free action of G on $\mu^{-1}(0)$.

At $p \in \mu^{-1}(0)$ the tangent space to the orbit $T_p\mathcal{O}_p$ is an isotropic subspace of the symplectic vector space (T_pM, ω_p), i.e., $T_p\mathcal{O}_p \subseteq (T_p\mathcal{O}_p)^\omega$.

$$(T_p\mathcal{O}_p)^\omega = \ker d\mu_p = T_p\mu^{-1}(0) \ .$$

The lemma (second ingredient) gives a canonical symplectic structure on the quotient $T_p\mu^{-1}(0)/T_p\mathcal{O}_p$. The point $[p] \in M_{\mathrm{red}} = \mu^{-1}(0)/G$ has tangent space $T_{[p]}M_{\mathrm{red}} \simeq T_p\mu^{-1}(0)/T_p\mathcal{O}_p$. Thus the lemma defines a nondegenerate 2-form ω_{red} on M_{red}. This is well-defined because ω is G-invariant.

By construction $i^*\omega = \pi^*\omega_{\mathrm{red}}$ where

$$\begin{array}{ccc} \mu^{-1}(0) & \stackrel{i}{\hookrightarrow} & M \\ \downarrow \pi & & \\ M_{\mathrm{red}} & & \end{array}$$

Hence, $\pi^* d\omega_{\rm red} = d\pi^*\omega_{\rm red} = d\imath^*\omega = \imath^* d\omega = 0$. The closedness of $\omega_{\rm red}$ follows from the injectivity of π^*. $\square$

Remark. Suppose that another Lie group H acts on (M, ω) in a hamiltonian way with moment map $\phi : M \to \mathfrak{h}^*$. If the H-action commutes with the G-action, and if ϕ is G-invariant, then $M_{\rm red}$ inherits a hamiltonian action of H, with moment map $\phi_{\rm red} : M_{\rm red} \to \mathfrak{h}^*$ satisfying $\phi_{\rm red} \circ \pi = \phi \circ i$. $\diamondsuit$

24 Reduction

24.1 Noether Principle

Let (M, ω, G, μ) be a hamiltonian G-space.

Theorem 24.1 (Noether) *A function $f : M \to \mathbb{R}$ is G-invariant if and only if μ is constant on the trajectories of the hamiltonian vector field of f.*

Proof. Let v_f be the hamiltonian vector field of f. Let $X \in \mathfrak{g}$ and $\mu^X = \langle \mu, X \rangle : M \to \mathbb{R}$. We have

$$\begin{aligned}
\mathcal{L}_{v_f} \mu^X &= \imath_{v_f} d\mu^X = \imath_{v_f} \imath_{X^\#} \omega \\
&= -\imath_{X^\#} \imath_{v_f} \omega = -\imath_{X^\#} df \\
&= -\mathcal{L}_{X^\#} f = 0
\end{aligned}$$

because f is G-invariant. □

Definition 24.2 *A G-invariant function $f : M \to \mathbb{R}$ is called an* **integral of motion** *of (M, ω, G, μ). If μ is constant on the trajectories of a hamiltonian vector field v_f, then the corresponding one-parameter group of diffeomorphisms $\{\exp tv_f \mid t \in \mathbb{R}\}$ is called a* **symmetry** *of (M, ω, G, μ).*

The **Noether principle** asserts that there is a one-to-one correspondence between symmetries and integrals of motion.

24.2 Elementary Theory of Reduction

Finding a symmetry for a $2n$-dimensional mechanical problem may reduce it to a $(2n-2)$-dimensional problem as follows: an integral of motion f for a $2n$-dimensional hamiltonian system (M, ω, H) may enable us to understand the trajectories of this system in terms of the trajectories of a $(2n-2)$-dimensional hamiltonian system $(M_{\mathrm{red}}, \omega_{\mathrm{red}}, H_{\mathrm{red}})$. To make this precise, we will describe this process locally. Suppose that $\mathcal{U}$ is an open set in M with Darboux coordinates $x_1, \ldots, x_n, \xi_1, \ldots, \xi_n$ such that $f = \xi_n$ for this chart, and write H in these coordinates: $H = H(x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$. Then

$$\xi_n \text{ is an integral of motion } \Longrightarrow \left\{ \begin{array}{l} \text{the trajectories of } v_H \text{ lie on the} \\ \quad \text{hyperplane } \xi_n = \text{constant} \\ \{\xi_n, H\} = 0 = -\frac{\partial H}{\partial x_n} \\ \quad \Longrightarrow H = H(x_1, \ldots, x_{n-1}, \xi_1, \ldots, \xi_n) \ . \end{array} \right.$$

If we set $\xi_n = c$, the motion of the system on this hyperplane is described by the following Hamilton equations:

$$\begin{cases} \dfrac{dx_1}{dt} & = & \dfrac{\partial H}{\partial \xi_1} & (x_1, \ldots, x_{n-1}, \xi_1, \ldots, \xi_{n-1}, c) \\ \vdots & & & \\ \dfrac{dx_{n-1}}{dt} & = & \dfrac{\partial H}{\partial \xi_{n-1}} & (x_1, \ldots, x_{n-1}, \xi_1, \ldots, \xi_{n-1}, c) \\ \dfrac{d\xi_1}{dt} & = & -\dfrac{\partial H}{\partial x_1} & (x_1, \ldots, x_{n-1}, \xi_1, \ldots, \xi_{n-1}, c) \\ \vdots & & & \\ \dfrac{d\xi_{n-1}}{dt} & = & -\dfrac{\partial H}{\partial x_{n-1}} & (x_1, \ldots, x_{n-1}, \xi_1, \ldots, \xi_{n-1}, c) \end{cases}$$

$$\begin{aligned} \frac{dx_n}{dt} &= \frac{\partial H}{\partial \xi_n} \\ \frac{d\xi_n}{dt} &= -\frac{\partial H}{\partial x_n} = 0 \ . \end{aligned}$$

The **reduced phase space** is

$$\begin{aligned} \mathcal{U}_{\mathrm{red}} = \ & \{(x_1, \ldots, x_{n-1}, \xi_1, \ldots, \xi_{n-1}) \in \mathbb{R}^{2n-2} \mid \\ & (x_1, \ldots, x_{n-1}, a, \xi_1, \ldots, \xi_{n-1}, c) \in \mathcal{U} \text{ for some } a\} \ . \end{aligned}$$

The **reduced hamiltonian** is

$$\begin{aligned} & H_{\mathrm{red}} : \mathcal{U}_{\mathrm{red}} \longrightarrow \mathbb{R} \ , \\ & H_{\mathrm{red}}(x_1, \ldots, x_{n-1}, \xi_1, \ldots, \xi_{n-1}) = H(x_1, \ldots, x_{n-1}, \xi_1, \ldots, \xi_{n-1}, c) \ . \end{aligned}$$

In order to find the trajectories of the original system on the hypersurface $\xi_n = c$, we look for the trajectories

$$x_1(t), \ldots, x_{n-1}(t), \xi_1(t), \ldots, \xi_{n-1}(t)$$

of the reduced system on $\mathcal{U}_{\mathrm{red}}$. We integrate the equation

$$\frac{dx_n}{dt}(t) = \frac{\partial H}{\partial \xi_n}(x_1(t), \ldots, x_{n-1}(t), \xi_1(t), \ldots, \xi_{n-1}(t), c)$$

to obtain the original trajectories

$$\begin{cases} x_n(t) & = & x_n(0) + \int_0^t \frac{\partial H}{\partial \xi_n}(\ldots) dt \\ \xi_n(t) & = & c \ . \end{cases}$$

24.3 Reduction for Product Groups

Let G_1 and G_2 be compact connected Lie groups and let $G = G_1 \times G_2$. Then

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \qquad \text{and} \qquad \mathfrak{g}^* = \mathfrak{g}_1^* \oplus \mathfrak{g}_2^* \ .$$

Suppose that (M, ω, G, ψ) is a hamiltonian G-space with moment map

$$\psi : M \longrightarrow \mathfrak{g}_1^* \oplus \mathfrak{g}_2^* \ .$$

Write $\psi = (\psi_1, \psi_2)$ where $\psi_i : M \to \mathfrak{g}_i^*$ for $i = 1, 2$. The fact that ψ is equivariant implies that ψ_1 is invariant under G_2 and ψ_2 is invariant under G_1. Now reduce (M, ω) with respect to the G_1-action. Let

$$Z_1 = \psi_1^{-1}(0) \ .$$

Assume that G_1 acts freely on Z_1. Let $M_1 = Z_1/G_1$ be the reduced space and let ω_1 be the corresponding reduced symplectic form. The action of G_2 on Z_1 commutes with the G_1-action. Since G_2 preserves ω, it follows that G_2 acts symplectically on (M_1, ω_1). Since G_1 preserves ψ_2, G_1 also preserves $\psi_2 \circ \iota_1 : Z_1 \to \mathfrak{g}_2^*$, where $\iota_1 : Z_1 \hookrightarrow M$ is inclusion. Thus $\psi_2 \circ \iota_1$ is constant on fibers of $Z_1 \xrightarrow{p_1} M_1$. We conclude that there exists a smooth map $\mu_2 : M_1 \to \mathfrak{g}_2^*$ such that $\mu_2 \circ p_1 = \psi_2 \circ \iota_1$.

Exercise. Show that:

(a) the map μ_2 is a moment map for the action of G_2 on (M_1, ω_1), and

(b) if G acts freely on $\psi^{-1}(0,0)$, then G_2 acts freely on $\mu_2^{-1}(0)$, and there is a natural symplectomorphism

$$\mu_2^{-1}(0)/G_2 \ \simeq \ \psi^{-1}(0,0)/G \ .$$

$\diamondsuit$

This technique of performing reduction with respect to one factor of a product group at a time is called **reduction in stages**. It may be extended to reduction by a normal subgroup $H \subset G$ and by the corresponding quotient group G/H.

24.4 Reduction at Other Levels

Suppose that a compact Lie group G acts on a symplectic manifold (M, ω) in a hamiltonian way with moment map $\mu : M \to \mathfrak{g}^*$. Let $\xi \in \mathfrak{g}^*$.

To reduce at the level ξ of μ, we need $\mu^{-1}(\xi)$ to be preserved by G, or else take the G-orbit of $\mu^{-1}(\xi)$, or else take the quotient by the maximal subgroup of G which preserves $\mu^{-1}(\xi)$.

Since μ is equivariant,

$$\begin{aligned} G \text{ preserves } \mu^{-1}(\xi) \quad &\Longleftrightarrow \quad G \text{ preserves } \xi \\ &\Longleftrightarrow \quad \mathrm{Ad}_g^* \xi = \xi, \ \forall g \in G \ . \end{aligned}$$

Of course the level 0 is always preserved. Also, when G is a torus, any level is preserved and reduction at ξ for the moment map μ, is equivalent to reduction at 0 for a shifted moment map $\phi : M \to \mathfrak{g}^*$, $\phi(p) := \mu(p) - \xi$.

Let $\mathcal{O}$ be a coadjoint orbit in $\mathfrak{g}^*$ equipped with the **canonical symplectic form** (also know as the **Kostant-Kirillov symplectic form** or the **Lie-Poisson symplectic form**) $\omega_{\mathcal{O}}$ defined in Homework 17. Let $\mathcal{O}^-$ be the orbit $\mathcal{O}$ equipped with $-\omega_{\mathcal{O}}$. The natural product action of G on $M \times \mathcal{O}^-$ is hamiltonian with moment map $\mu_{\mathcal{O}}(p, \xi) = \mu(p) - \xi$. If the Marsden-Weinstein-Meyer hypothesis is satisfied for $M \times \mathcal{O}^-$, then one obtains a **reduced space with respect to the coadjoint orbit** $\mathcal{O}$.

24.5 Orbifolds

Example. Let $G = \mathbb{T}^n$ be an n-torus. For any $\xi \in (\mathfrak{t}^n)^*$, $\mu^{-1}(\xi)$ is preserved by the $\mathbb{T}^n$-action. Suppose that ξ is a regular value of μ. (By Sard's theorem, the singular values of μ form a set of measure zero.) Then $\mu^{-1}(\xi)$ is a submanifold of codimension n. Note that

$$\begin{array}{rcl} \xi \text{ regular} & \Longrightarrow & d\mu_p \text{ is surjective at all } p \in \mu^{-1}(\xi) \\ & \Longrightarrow & \mathfrak{g}_p = 0 \;\text{ for all } p \in \mu^{-1}(\xi) \\ & \Longrightarrow & \text{the stabilizers on } \mu^{-1}(\xi) \text{ are finite} \\ & \Longrightarrow & \mu^{-1}(\xi)/G \text{ is an } \textbf{orbifold} \text{ [91, 92]} . \end{array}$$

Let G_p be the stabilizer of p. By the slice theorem (Lecture 23), $\mu^{-1}(\xi)/G$ is modeled by S/G_p, where S is a G_p-invariant disk in $\mu^{-1}(\xi)$ through p and transverse to $\mathcal{O}_p$. Hence, locally $\mu^{-1}(\xi)/G$ looks indeed like $\mathbb{R}^n$ divided by a finite group action. ♢

Example. Consider the S^1-action on $\mathbb{C}^2$ given by $e^{i\theta} \cdot (z_1, z_2) = (e^{ik\theta} z_1, e^{i\theta} z_2)$ for some fixed integer $k \geq 2$. This is hamiltonian with moment map

$$\begin{array}{rccc} \mu : & \mathbb{C}^2 & \longrightarrow & \mathbb{R} \\ & (z_1, z_2) & \longmapsto & -\frac{1}{2}(k|z_1|^2 + |z_2|^2) \ . \end{array}$$

Any $\xi < 0$ is a regular value and $\mu^{-1}(\xi)$ is a 3-dimensional ellipsoid. The stabilizer of $(z_1, z_2) \in \mu^{-1}(\xi)$ is $\{1\}$ if $z_2 \neq 0$, and is $\mathbb{Z}_k = \left\{ e^{i\frac{2\pi\ell}{k}} \mid \ell = 0, 1, \ldots, k-1 \right\}$ if $z_2 = 0$. The reduced space $\mu^{-1}(\xi)/S^1$ is called a **teardrop** orbifold or **conehead**; it has one **cone** (also known as a **dunce cap**) singularity of type k (with cone angle $\frac{2\pi}{k}$). ♢

Example. Let S^1 act on $\mathbb{C}^2$ by $e^{i\theta} \cdot (z_1, z_2) = (e^{ik\theta} z_1, e^{i\ell\theta} z_2)$ for some integers $k, \ell \geq 2$. Suppose that k and ℓ are relatively prime. Then

$$\begin{array}{rll} (z_1, 0) & \text{has stabilizer } \mathbb{Z}_k & (\text{for } z_1 \neq 0) \ , \\ (0, z_2) & \text{has stabilizer } \mathbb{Z}_\ell & (\text{for } z_2 \neq 0) \ , \\ (z_1, z_2) & \text{has stabilizer } \{1\} & (\text{for } z_1, z_2 \neq 0) \ . \end{array}$$

The quotient $\mu^{-1}(\xi)/S^1$ is called a **football** orbifold. It has two cone singularities, one of type k and another of type ℓ. $\diamondsuit$

Example. More generally, the reduced spaces of S^1 acting on $\mathbb{C}^n$ by

$$e^{i\theta} \cdot (z_1, \ldots, z_n) = (e^{ik_1\theta} z_1, \ldots, e^{ik_n\theta} z_n) \ ,$$

are called **weighted** (or **twisted**) **projective spaces**. $\diamondsuit$

Homework 18: Spherical Pendulum

This set of problems is from [53].

The **spherical pendulum** is a mechanical system consisting of a massless rigid rod of length l, fixed at one end, whereas the other end has a plumb bob of mass m, which may oscillate freely in all directions. Assume that the force of gravity is constant pointing vertically downwards, and that this is the only external force acting on this system.

Let φ, θ ($0 < \varphi < \pi$, $0 < \theta < 2\pi$) be spherical coordinates for the bob. For simplicity assume that $m = l = 1$.

1. Let η, ξ be the coordinates along the fibers of T^*S^2 induced by the spherical coordinates φ, θ on S^2. Show that the function $H : T^*S^2 \to \mathbb{R}$ given by

$$H(\varphi, \theta, \eta, \xi) = \frac{1}{2}\left(\eta^2 + \frac{\xi^2}{(\sin\varphi)^2}\right) + \cos\varphi \ ,$$

   is an appropriate hamiltonian function to describe the spherical pendulum.

2. Compute the critical points of the function H. Show that, on S^2, there are exactly two critical points: s (where H has a minimum) and u. These points are called the **stable** and **unstable** points of H, respectively. Justify this terminology, i.e., show that a trajectory whose initial point is close to s stays close to s forever, and show that this is not the case for u. What is happening physically?

3. Show that the group of rotations about the vertical axis is a group of symmetries of the spherical pendulum.

   Show that, in the coordinates above, the integral of motion associated with these symmetries is the function

$$J(\varphi, \theta, \eta, \xi) = \xi \ .$$

   Give a more coordinate-independent description of J, one that makes sense also on the cotangent fibers above the North and South poles.

4. Locate all points $p \in T^*S^2$ where dH_p and dJ_p are linearly dependent:

   (a) Clearly, the two critical points s and u belong to this set. Show that these are the only two points where $dH_p = dJ_p = 0$.

   (b) Show that, if $x \in S^2$ is in the southern hemisphere ($x_3 < 0$), then there exist exactly two points, $p_+ = (x, \eta, \xi)$ and $p_- = (x, -\eta, -\xi)$, in the cotangent fiber above x where dH_p and dJ_p are linearly dependent.

   (c) Show that dH_p and dJ_p are linearly dependent along the trajectory of the hamiltonian vector field of H through p_+.

   Conclude that this trajectory is also a trajectory of the hamiltonian vector field of J, and, hence, that its projection onto S^2 is a latitudinal circle (of the form $x_3 =$ constant).

   Show that the projection of the trajectory through p_- is the same latitudinal circle traced in the opposite direction.

5. Show that any nonzero value j is a regular value of J, and that S^1 acts freely on the level set $J = j$. What happens on the cotangent fibers above the North and South poles?

6. For $j \neq 0$ describe the reduced system and sketch the level curves of the reduced hamiltonian.

7. Show that the integral curves of the original system on the level set $J = j$ can be obtained from those of the reduced system by "quadrature", in other words, by a simple integration.

8. Show that the reduced system for $j \neq 0$ has exactly one equilibrium point. Show that the corresponding relative equilibrium for the original system is one of the horizontal curves in exercise 4.

9. The **energy-momentum map** is the map $(H, J) : T^*S^2 \to \mathbb{R}^2$. Show that, if $j \neq 0$, the level set $(H, J) = (h, j)$ of the energy-momentum map is either a circle (in which case it is one of the horizontal curves in exercise 4), or a two-torus. Show that the projection onto the configuration space of the two-torus is an annular region on S^2.

Part X
Moment Maps Revisited

Moment maps and symplectic reduction have been finding infinite-dimensional incarnations with amazing consequences for differential geometry. Lecture 25 sketches the symplectic approach of Atiyah and Bott to Yang-Mills theory.

Lecture 27 describes the convexity of the image of a torus moment map, one of the most striking geometric characteristics of moment maps.

25 Moment Map in Gauge Theory

25.1 Connections on a Principal Bundle

Let G be a Lie group and B a manifold.

Definition 25.1 *A* **principal** G**-bundle over** B *is a manifold* P *with a smooth map* $\pi : P \to B$ *satisfying the following conditions:*

(a) G *acts freely on* P *(on the left),*

(b) B *is the orbit space for this action and* π *is the point-orbit projection, and*

(c) there is an open covering of B*, such that, to each set* $\mathcal{U}$ *in that covering corresponds a map* $\varphi_{\mathcal{U}} : \pi^{-1}(\mathcal{U}) \to \mathcal{U} \times G$ *with*

$$\varphi_{\mathcal{U}}(p) = (\pi(p), s_{\mathcal{U}}(p)) \quad \text{and} \quad s_{\mathcal{U}}(g \cdot p) = g \cdot s_{\mathcal{U}}(p) \ , \quad \forall p \in \pi^{-1}(\mathcal{U}) \ .$$

The G*-valued maps* $s_{\mathcal{U}}$ *are determined by the corresponding* $\varphi_{\mathcal{U}}$*. Condition (c) is called the property of being* **locally trivial***.*

If P with map $\pi : P \to B$ is a principal G-bundle over B, then the manifold B is called the **base**, the manifold P is called the **total space**, the Lie group G is called the **structure group**, and the map π is called the **projection**. This principal bundle is also represented by the following diagram:

$$\begin{tikzcd} G \arrow[r, hook] & P \arrow[d, "\pi"] \\ & B \end{tikzcd}$$

Example. Let P be the 3-sphere regarded as unit vectors in $\mathbb{C}^2$:

$$P = S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\} \ .$$

Let G be the circle group, where $e^{i\theta} \in S^1$ acts on S^3 by complex multiplication,

$$(z_1, z_2) \longmapsto (e^{i\theta} z_1, e^{i\theta} z_2) \ .$$

Then the quotient space B is the first complex projective space, that is, the two-sphere. This data forms a principal S^1-bundle, known as the **Hopf fibration**:

$$\begin{tikzcd} S^1 \arrow[r, hook] & S^3 \arrow[d, "\pi"] \\ & S^2 \end{tikzcd}$$

$\diamondsuit$

An action $\psi : G \to \mathrm{Diff}(P)$ induces an infinitesimal action

$$\begin{array}{rccll} d\psi : & \mathfrak{g} & \longrightarrow & \chi(P) & \\ & X & \longmapsto & X^\# = & \text{vector field generated by the} \\ & & & & \text{one-parameter group } \{\exp tX(e) \mid t \in \mathbb{R}\} \ . \end{array}$$

From now on, fix a basis $X_1, \ldots, X_k$ of $\mathfrak{g}$.

Let P be a principal G-bundle over B. Since the G-action is free, the vector fields $X_1^\#, \ldots, X_k^\#$ are linearly independent at each $p \in P$. The **vertical bundle** V is the rank k subbundle of TP generated by $X_1^\#, \ldots, X_k^\#$.

Exercise. Check that the vertical bundle V is the set of vectors tangent to P which lie in the kernel of the derivative of the bundle projection π. (This shows that V is independent of the choice of basis for $\mathfrak{g}$.) $\diamondsuit$

Definition 25.2 *A* **(Ehresmann) connection** *on a principal bundle P is a choice of a splitting*

$$TP = V \oplus H \ ,$$

where H is a G-invariant subbundle of TP complementary to the vertical bundle V. The bundle H is called the **horizontal bundle**.

25.2 Connection and Curvature Forms

A connection on a principal bundle P may be equivalently described in terms of 1-forms.

Definition 25.3 *A* **connection form** *on a principal bundle P is a Lie-algebra-valued 1-form*

$$A = \sum_{i=1}^{k} A_i \otimes X_i \qquad \in \ \Omega^1(P) \otimes \mathfrak{g}$$

such that:

(a) *A is G-invariant, with respect to the product action of G on $\Omega^1(P)$ (induced by the action on P) and on $\mathfrak{g}$ (the adjoint representation), and*

(b) *A is vertical, in the sense that $\imath_{X^\#}A = X$ for any $X \in \mathfrak{g}$.*

Exercise. Show that a connection $TP = V \oplus H$ determines a connection form A and vice-versa by the formula

$$H = \ker A = \{v \in TP \mid \imath_v A = 0\} \ .$$

$\diamondsuit$

Given a connection on P, the splitting $TP = V \oplus H$ induces the following splittings for bundles:

$$\begin{array}{rcl} T^*P &=& V^* \oplus H^* \\ \wedge^2 T^*P &=& (\wedge^2 V^*) \ \oplus \ (V^* \wedge H^*) \ \oplus \ (\wedge^2 H^*) \\ &\vdots& \end{array}$$

and for their sections:

$$\begin{array}{rcl} \Omega^1(P) &=& \Omega^1_{\rm vert}(P) \oplus \Omega^1_{\rm horiz}(P) \\ \Omega^2(P) &=& \Omega^2_{\rm vert}(P) \oplus \Omega^2_{\rm mix}(P) \oplus \Omega^2_{\rm horiz}(P) \\ &\vdots& \end{array}$$

The corresponding connection form A is in $\Omega^1_{\rm vert} \otimes \mathfrak{g}$. Its exterior derivative dA is in

$$\Omega^2(P) \otimes \mathfrak{g} = \left(\Omega^2_{\rm vert} \oplus \Omega^2_{\rm mix} \oplus \Omega^2_{\rm horiz}\right) \otimes \mathfrak{g} \ ,$$

and thus decomposes into three components,

$$dA = (dA)_{\rm vert} + (dA)_{\rm mix} + (dA)_{\rm horiz} \ .$$

Exercise. Check that:

(a) $(dA)_{\rm vert}(X,Y) = [X,Y]$, i.e., $(dA)_{\rm vert} = \frac{1}{2} \sum\limits_{i,\ell,m} c^i_{\ell m} A_\ell \wedge A_m \otimes X_i$, where the $c^i_{\ell m}$'s are the **structure constants** of the Lie algebra with respect to the chosen basis, and defined by $[X_\ell, X_m] = \sum\limits_{i,\ell,m} c^i_{\ell m} X_i$;

(b) $(dA)_{\rm mix} = 0$.

$\diamondsuit$

According to the previous exercise, the relevance of dA may come only from its horizontal component.

Definition 25.4 *The* **curvature form** *of a connection is the horizontal component of its connection form. I.e., if A is the connection form, then*

$$\mathrm{curv}\ A = (dA)_{\rm horiz} \qquad \in \ \Omega^2_{\rm horiz} \otimes \mathfrak{g} \ .$$

Definition 25.5 *A connection is called* **flat** *if its curvature is zero.*

25.3 Symplectic Structure on the Space of Connections

Let P be a principal G-bundle over B. If A is a connection form on P, and if $a \in \Omega^1_{\text{horiz}} \otimes \mathfrak{g}$ is G-invariant for the product action, then it is easy to check that $A + a$ is also a connection form on P. Reciprocally, any two connection forms on P differ by an $a \in (\Omega^1_{\text{horiz}} \otimes \mathfrak{g})^G$. We conclude that the set $\mathcal{A}$ of all connections on the principal G-bundle P is an affine space modeled on the linear space

$$\mathfrak{a} = (\Omega^1_{\text{horiz}} \otimes \mathfrak{g})^G \ .$$

Now let P be a principal G-bundle over a compact oriented 2-dimensional riemannian manifold B (for instance, B is a Riemann surface). Suppose that the group G is compact or semisimple. Atiyah and Bott [7] noticed that the corresponding space $\mathcal{A}$ of all connections may be treated as an *infinite-dimensional symplectic manifold*. This will require choosing a G-invariant inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$, which always exists, either by averaging any inner product when G is compact, or by using the Killing form on semisimple groups.

Since $\mathcal{A}$ is an affine space, its tangent space at any point A is identified with the model linear space $\mathfrak{a}$. With respect to a basis $X_1, \ldots, X_k$ for the Lie algebra $\mathfrak{g}$, elements $a, b \in \mathfrak{a}$[14] are written

$$a = \sum a_i \otimes X_i \quad \text{and} \quad b = \sum b_i \otimes X_i \ .$$

If we wedge a and b, and then integrate over B using the riemannian volume, we obtain a real number:

$$\begin{array}{rccccc} \omega : & \mathfrak{a} \times \mathfrak{a} & \longrightarrow & \left(\Omega^2_{\text{horiz}}(P)\right)^G \simeq \Omega^2(B) & \longrightarrow & \mathbb{R} \\ & (a,b) & \longmapsto & \sum\limits_{i,j} a_i \wedge b_j \langle X_i, X_j \rangle & \longmapsto & \int\limits_B \sum\limits_{i,j} a_i \wedge b_j \langle X_i, X_j \rangle \ . \end{array}$$

We have used that the pullback $\pi^* : \Omega^2(B) \to \Omega^2(P)$ is an isomorphism onto its image $\left(\Omega^2_{\text{horiz}}(P)\right)^G$.

Exercise. Show that if $\omega(a, b) = 0$ for all $b \in \mathfrak{a}$, then a must be zero. $\diamondsuit$

The map ω is nondegenerate, skew-symmetric, bilinear and constant in the sense that it does not depend on the base point A. Therefore, it has the right to be called a symplectic form on $\mathcal{A}$, so the pair $(\mathcal{A}, \omega)$ is an infinite-dimensional symplectic manifold.

25.4 Action of the Gauge Group

Let P be a principal G-bundle over B. A diffeomorphism $f : P \to P$ commuting with the G-action determines a diffeomorphism $f_{\text{basic}} : B \to B$ by projection.

[14] The choice of symbols is in honor of Atiyah and Bott!

Definition 25.6 *A diffeomorphism $f : P \to P$ commuting with the G-action is a* **gauge transformation** *if the induced f_{basic} is the identity. The* **gauge group** *of P is the group $\mathcal{G}$ of all gauge transformations of P.*

The derivative of an $f \in \mathcal{G}$ takes a connection $TP = V \oplus H$ to another connection $TP = V \oplus H_f$, and thus induces an action of $\mathcal{G}$ in the space $\mathcal{A}$ of all connections. Recall that $\mathcal{A}$ has a symplectic form ω. Atiyah and Bott [7] noticed that the action of $\mathcal{G}$ on $(\mathcal{A}, \omega)$ is hamiltonian, where the moment map (appropriately interpreted) is the map

$$\begin{array}{rccc} \mu : & \mathcal{A} & \longrightarrow & \left(\Omega^2(P) \otimes \mathfrak{g}\right)^G \\ & A & \longmapsto & \operatorname{curv} A \ , \end{array}$$

i.e., the moment map "is" the curvature! We will describe this construction in detail for the case of circle bundles in the next section.

Remark. The reduced space at level zero

$$\mathcal{M} = \mu^{-1}(0)/\mathcal{G}$$

is the space of flat connections modulo gauge equivalence, known as the **moduli space of flat connections**. It turns out that $\mathcal{M}$ is a finite-dimensional symplectic orbifold. ♢

25.5 Case of Circle Bundles

What does the Atiyah-Bott construction of the previous section look like for the case when $G = S^1$?

$$\begin{array}{ccc} S^1 & \hookrightarrow & P \\ & & \downarrow \pi \\ & & B \end{array}$$

Let v be the generator of the S^1-action on P, corresponding to the basis 1 of $\mathfrak{g} \simeq \mathbb{R}$. A connection form on P is a usual 1-form $A \in \Omega^1(P)$ such that

$$\mathcal{L}_v A = 0 \quad \text{and} \quad \imath_v A = 1 \ .$$

If we fix one particular connection A_0, then any other connection is of the form $A = A_0 + a$ for some $a \in \mathfrak{a} = \left(\Omega^1_{\text{horiz}}(P)\right)^G = \Omega^1(B)$. The symplectic form on $\mathfrak{a} = \Omega^1(B)$ is simply

$$\begin{array}{rccc} \omega : & \mathfrak{a} \times \mathfrak{a} & \longrightarrow & \mathbb{R} \\ & (a, b) & \longmapsto & \displaystyle\int_B \underbrace{a \wedge b}_{\in \Omega^2(B)} \ . \end{array}$$

The gauge group is $\mathcal{G} = \mathrm{Maps}(B, S^1)$, because a gauge transformation is multiplication by some element of S^1 over each point in B:

$$\begin{array}{rcll} \phi : \quad \mathcal{G} & \longrightarrow & \mathrm{Diff}(P) & \\ h : B \to S^1 & \longmapsto & \phi_h : & P \to P \\ & & & p \to h(\pi(p)) \cdot p \end{array}$$

The Lie algebra of $\mathcal{G}$ is

$$\text{Lie } \mathcal{G} = \mathrm{Maps}(B, \mathbb{R}) = C^\infty(B) \ .$$

Its dual space is

$$(\text{Lie } \mathcal{G})^* = \Omega^2(B) \ ,$$

where the duality is provided by integration over B

$$\begin{array}{rcl} C^\infty(B) \times \Omega^2(B) & \longrightarrow & \mathbb{R} \\ (h, \beta) & \longmapsto & \displaystyle\int_B h\beta \ . \end{array}$$

(it is topological or smooth duality, as opposed to algebraic duality) .

The gauge group acts on the space of all connections by

$$\begin{array}{rcl} \mathcal{G} & \longrightarrow & \mathrm{Diff}(\mathcal{A}) \\ h(x) = e^{i\theta(x)} & \longmapsto & (A \mapsto A \underbrace{-\pi^* d\theta}_{\in \mathfrak{a}}) \end{array}$$

Exercise. Check the previous assertion about the action on connections.

> **Hint:** First deal with the case where $P = S^1 \times B$ is a trivial bundle, in which case $h \in \mathcal{G}$ acts on P by
>
> $$\phi_h : (t, x) \longmapsto (t + \theta(x), x) \ ,$$
>
> and where every connection can be written $A = dt + \beta$, with $\beta \in \Omega^1(B)$. A gauge transformation $h \in \mathcal{G}$ acts on $\mathcal{A}$ by
>
> $$A \longmapsto \phi^*_{h^{-1}}(A) \ .$$

$\diamondsuit$

The infinitesimal action of $\mathcal{G}$ on $\mathcal{A}$ is

$$\begin{array}{rcl} d\phi : \quad \text{Lie } \mathcal{G} & \longrightarrow & \chi(\mathcal{A}) \\ X & \longmapsto & X^\# = \text{ vector field described by the transformation} \\ & & (A \ \mapsto \ A \underbrace{-dX}_{\in \Omega^1(B) = \mathfrak{a}}) \end{array}$$

so that $X^\# = -dX$.

Finally, we will check that

$$\begin{array}{rccl} \mu : & \mathcal{A} & \longrightarrow & (\mathrm{Lie}\ \mathcal{G})^* = \Omega^2(B) \\ & A & \longmapsto & \mathrm{curv}\ A \end{array}$$

is indeed a moment map for the action of the gauge group on $\mathcal{A}$.

Exercise. Check that in this case:

(a) $\mathrm{curv}\ A = dA \quad \in \quad \left(\Omega^2_{\mathrm{horiz}}(P)\right)^G = \Omega^2(B)$,

(b) μ is $\mathcal{G}$-invariant.

◇

The previous exercise takes care of the equivariance condition, since the action of $\mathcal{G}$ on $\Omega^2(B)$ is trivial.

Take any $X \in \mathrm{Lie}\ \mathcal{G} = C^\infty(B)$. We need to check that

$$d\mu^X(a) = \omega(X^\#, a) \ , \qquad \forall a \in \Omega^1(B) \ . \qquad (\star)$$

As for the left-hand side of $(\star)$, the map μ^X,

$$\begin{array}{rccl} \mu^X : & \mathcal{A} & \longrightarrow & \mathbb{R} \\ & A & \longmapsto & \langle \underbrace{X}_{\in C^\infty(B)} , \underbrace{dA}_{\in \Omega^2(B)} \rangle = \int_B X \cdot dA \ , \end{array}$$

is linear in A. Consequently,

$$\begin{array}{rccl} d\mu^X : & \mathfrak{a} & \longrightarrow & \mathbb{R} \\ & a & \longmapsto & \int_B X \cdot da \ . \end{array}$$

As for the right-hand side of $(\star)$, by definition of ω, we have

$$\omega(X^\#, a) = \int_B X^\# \cdot a = -\int_B dX \cdot a \ .$$

But, by Stokes theorem, the last integral is

$$-\int_B dX \cdot a = \int_B X \cdot da \ ,$$

so we are done in proving that μ is the moment map.

Homework 19: Examples of Moment Maps

1. Suppose that a Lie group G acts in a hamiltonian way on two symplectic manifolds (M_j, ω_j), $j = 1, 2$, with moment maps $\mu_j : M_j \to \mathfrak{g}^*$. Prove that the diagonal action of G on $M_1 \times M_2$ is hamiltonian with moment map $\mu : M_1 \times M_2 \to \mathfrak{g}^*$ given by

$$\mu(p_1, p_2) = \mu_1(p_1) + \mu_2(p_2) \ , \quad \text{for } p_j \in M_j \ .$$

2. Let $\mathbb{T}^n = \{(t_1, \ldots, t_n) \in \mathbb{C}^n \ : \ |t_j| = 1, \ \text{for all } j\}$ be a torus acting on $\mathbb{C}^n$ by

$$(t_1, \ldots, t_n) \cdot (z_1, \ldots, z_n) = (t_1^{k_1} z_1, \ldots, t_n^{k_n} z_n) \ ,$$

where $k_1, \ldots, k_n \in \mathbb{Z}$ are fixed. Check that this action is hamiltonian with moment map $\mu : \mathbb{C}^n \to (\mathfrak{t}^n)^* \simeq \mathbb{R}^n$ given by

$$\mu(z_1, \ldots, z_n) = -\tfrac{1}{2}(k_1|z_1|^2, \ldots, k_n|z_n|^2) \ (+ \text{constant} \ ) \ .$$

3. The vector field $X^\#$ generated by $X \in \mathfrak{g}$ for the coadjoint representation of a Lie group G on $\mathfrak{g}^*$ satisfies $\langle X^\#_\xi, Y \rangle = \langle \xi, [Y, X] \rangle$, for any $Y \in \mathfrak{g}$. Equip the coadjoint orbits with the canonical symplectic forms. Show that, for each $\xi \in \mathfrak{g}^*$, the coadjoint action on the orbit $G \cdot \xi$ is hamiltonian with moment map the inclusion map:

$$\mu : G \cdot \xi \hookrightarrow \mathfrak{g}^* \ .$$

4. Consider the natural action of $\mathrm{U}(n)$ on $(\mathbb{C}^n, \omega_0)$. Show that this action is hamiltonian with moment map $\mu : \mathbb{C}^n \to \mathfrak{u}(n)$ given by

$$\mu(z) = \tfrac{i}{2} z z^* \ ,$$

where we identify the Lie algebra $\mathfrak{u}(n)$ with its dual via the inner product $(A, B) = \mathrm{trace}(A^* B)$.

Hint: Denote the elements of $\mathrm{U}(n)$ in terms of real and imaginary parts $g = h + i\,k$. Then g acts on $\mathbb{R}^{2n}$ by the linear symplectomorphism $\begin{pmatrix} h & -k \\ k & h \end{pmatrix}$. The Lie algebra $\mathfrak{u}(n)$ is the set of skew-hermitian matrices $X = V + i\,W$ where $V = -V^t \in \mathbb{R}^{n \times n}$ and $W = W^t \in \mathbb{R}^{n \times n}$. Show that the infinitesimal action is generated by the hamiltonian functions

$$\mu^X(z) = -\tfrac{1}{2}(x, Wx) + (y, Vx) - \tfrac{1}{2}(y, Wy)$$

where $z = x + i\,y$, $x, y \in \mathbb{R}^n$ and $(\cdot, \cdot)$ is the standard inner product. Show that

$$\mu^X(z) = \tfrac{1}{2} i\, z^* X z = \tfrac{1}{2} i\, \mathrm{trace}(z z^* X) \ .$$

Check that μ is equivariant.

5. Consider the natural action of $\mathrm{U}(k)$ on the space $(\mathbb{C}^{k\times n}, \omega_0)$ of complex $(k \times n)$-matrices. Identify the Lie algebra $\mathfrak{u}(k)$ with its dual via the inner product $(A, B) = \mathrm{trace}(A^*B)$. Prove that a moment map for this action is given by
$$\mu(A) = \tfrac{i}{2}AA^* + \tfrac{\mathrm{Id}}{2i}\ , \quad \text{for } A \in \mathbb{C}^{k\times n}\ .$$
(The choice of the constant $\frac{\mathrm{Id}}{2i}$ is for convenience in Homework 20.)

   Hint: Exercises 1 and 4.

6. Consider the $\mathrm{U}(n)$-action by conjugation on the space $(\mathbb{C}^{n^2}, \omega_0)$ of complex $(n \times n)$-matrices. Show that a moment map for this action is given by
$$\mu(A) = \tfrac{i}{2}[A, A^*]\ .$$

   Hint: Previous exercise and its "transpose" version.

26 Existence and Uniqueness of Moment Maps

26.1 Lie Algebras of Vector Fields

Let (M, ω) be a symplectic manifold and $v \in \chi(M)$ a vector field on M.

$$\begin{array}{rcl} v \text{ is symplectic} & \Longleftrightarrow & \imath_v \omega \text{ is closed },\\ v \text{ is hamiltonian} & \Longleftrightarrow & \imath_v \omega \text{ is exact }. \end{array}$$

The spaces

$$\begin{array}{rcl} \chi^{\mathrm{sympl}}(M) & = & \text{symplectic vector fields on } M \ ,\\ \chi^{\mathrm{ham}}(M) & = & \text{hamiltonian vector fields on } M \ . \end{array}$$

are Lie algebras for the Lie bracket of vector fields. $C^\infty(M)$ is a Lie algebra for the Poisson bracket, $\{f, g\} = \omega(v_f, v_g)$. $H^1(M; \mathbb{R})$ and $\mathbb{R}$ are regarded as Lie algebras for the trivial bracket. We have two exact sequences of Lie algebras:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \chi^{\mathrm{ham}}(M) & \hookrightarrow & \chi^{\mathrm{sympl}}(M) & \longrightarrow & H^1(M;\mathbb{R}) & \longrightarrow & 0\\ & & & & v & \longmapsto & [\imath_v \omega] & & \\ & & & & & & & & \\ 0 & \longrightarrow & \mathbb{R} & \hookrightarrow & C^\infty(M) & \longrightarrow & \chi^{\mathrm{ham}}(M) & \longrightarrow & 0\\ & & & & f & \longmapsto & v_f \ . & & \end{array}$$

In particular, if $H^1(M; \mathbb{R}) = 0$, then $\chi^{\mathrm{ham}}(M) = \chi^{\mathrm{sympl}}(M)$.

Let G be a connected Lie group. A symplectic action $\psi : G \to \mathrm{Sympl}(M, \omega)$ induces an infinitesimal action

$$\begin{array}{rccl} d\psi : & \mathfrak{g} & \longrightarrow & \chi^{\mathrm{sympl}}(M)\\ & X & \longmapsto & X^\# = \text{ vector field generated by the}\\ & & & \qquad\text{one-parameter group } \{\exp tX(e) \mid t \in \mathbb{R}\} \ . \end{array}$$

Exercise. Check that the map $d\psi$ is a Lie algebra anti-homomorphism. $\diamondsuit$

The action ψ is **hamiltonian** if and only if there is a Lie algebra homomorphism $\mu^* : \mathfrak{g} \to C^\infty(M)$ lifting $d\psi$, i.e., making the following diagram commute.

$$\begin{tikzcd} C^\infty(M) \arrow[rr] & & \chi^{\mathrm{sympl}}(M) \\ & \mathfrak{g} \arrow[ul, "\mu^*"] \arrow[ur, "d\psi"'] & \end{tikzcd}$$

The map μ^* is then called a **comoment map** (defined in Lecture 22).

$$\begin{array}{ccc} \text{Existence of } \mu^* & \Longleftrightarrow & \text{Existence of } \mu\\ \text{comoment map} & & \text{moment map}\\ & & \\ \text{Lie algebra homomorphism} & \longleftrightarrow & \text{equivariance} \end{array}$$

26.2 Lie Algebra Cohomology

Let $\mathfrak{g}$ be a Lie algebra, and

$$\begin{array}{rcl} C^k & := & \Lambda^k\mathfrak{g}^* = k\text{-cochains on } \mathfrak{g} \\ & = & \text{alternating } k\text{-linear maps } \underbrace{\mathfrak{g}\times\ldots\times\mathfrak{g}}_{k} \longrightarrow \mathbb{R} \ . \end{array}$$

Define a linear operator $\delta : C^k \to C^{k+1}$ by

$$\delta c(X_0,\ldots,X_k) = \sum_{i<j}(-1)^{i+j}c([X_i,X_j],X_0,\ldots,\widehat{X}_i,\ldots,\widehat{X}_j,\ldots,X_k) \ .$$

Exercise. Check that $\delta^2 = 0$. $\diamondsuit$

The **Lie algebra cohomology groups** (or **Chevalley cohomology groups**) of $\mathfrak{g}$ are the cohomology groups of the complex $0 \xrightarrow{\delta} C^0 \xrightarrow{\delta} C^1 \xrightarrow{\delta} \ldots$:

$$H^k(\mathfrak{g};\mathbb{R}) := \frac{\ker\delta : C^k \longrightarrow C^{k+1}}{\mathrm{im}\ \delta : C^{k-1} \longrightarrow C^k} \ .$$

Theorem 26.1 *If $\mathfrak{g}$ is the Lie algebra of a compact connected Lie group G, then*

$$H^k(\mathfrak{g};\mathbb{R}) = H^k_{\mathrm{deRham}}(G) \ .$$

Proof. Exercise. Hint: by averaging show that the de Rham cohomology can be computed from the subcomplex of G-invariant forms. $\square$

Meaning of $H^1(\mathfrak{g};\mathbb{R})$ and $H^2(\mathfrak{g};\mathbb{R})$:

- An element of $C^1 = \mathfrak{g}^*$ is a linear functional on $\mathfrak{g}$. If $c \in \mathfrak{g}^*$, then $\delta c(X_0,X_1) = -c([X_0,X_1])$. The **commutator ideal** of $\mathfrak{g}$ is

  $$[\mathfrak{g},\mathfrak{g}] := \{\text{linear combinations of } [X,Y] \text{ for any } X,Y \in \mathfrak{g}\} \ .$$

  Since $\delta c = 0$ if and only if c vanishes on $[\mathfrak{g},\mathfrak{g}]$, we conclude that

  $$H^1(\mathfrak{g};\mathbb{R}) = [\mathfrak{g},\mathfrak{g}]^0$$

  where $[\mathfrak{g},\mathfrak{g}]^0 \subseteq \mathfrak{g}^*$ is the annihilator of $[\mathfrak{g},\mathfrak{g}]$.

- An element of C^2 is an alternating bilinear map $c : \mathfrak{g}\times\mathfrak{g} \to \mathbb{R}$.

  $$\delta c(X_0,X_1,X_2) = -c([X_0,X_1],X_2) + c([X_0,X_2],X_1) - c([X_1,X_2],X_0) \ .$$

  If $c = \delta b$ for some $b \in C^1$, then

  $$c(X_0,X_1) = (\delta b)(X_0,X_1) = -b([X_0,X_1]\ ).$$

26.3 Existence of Moment Maps

Theorem 26.2 *If $H^1(\mathfrak{g};\mathbb{R}) = H^2(\mathfrak{g},\mathbb{R}) = 0$, then any symplectic G-action is hamiltonian.*

Proof. Let $\psi : G \to \mathrm{Sympl}(M,\omega)$ be a symplectic action of G on a symplectic manifold (M,ω). Since

$$H^1(\mathfrak{g};\mathbb{R}) = 0 \iff [\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$$

and since commutators of symplectic vector fields are hamiltonian, we have

$$d\psi : \mathfrak{g} = [\mathfrak{g},\mathfrak{g}] \longrightarrow \chi^{\mathrm{ham}}(M).$$

The action ψ is hamiltonian if and only if there is a Lie algebra homomorphism $\mu^* : \mathfrak{g} \to C^\infty(M)$ such that the following diagram commutes.

$$\begin{tikzcd}
\mathbb{R} \arrow[r] & C^\infty(M) \arrow[rr] & & \chi^{\mathrm{ham}}(M) \\
& & \mathfrak{g} \arrow[ul, "?"] \arrow[ur, "d\psi"'] &
\end{tikzcd}$$

We first take an arbitrary vector space lift $\tau : \mathfrak{g} \to C^\infty(M)$ making the diagram commute, i.e., for each basis vector $X \in \mathfrak{g}$, we choose

$$\tau(X) = \tau^X \in C^\infty(M) \qquad \text{such that} \qquad v_{(\tau^X)} = d\psi(X)\ .$$

The map $X \mapsto \tau^X$ may not be a Lie algebra homomorphism. By construction, $\tau^{[X,Y]}$ is a hamiltonian function for $[X,Y]^\#$, and (as computed in Lecture 16) $\{\tau^X,\tau^Y\}$ is a hamiltonian function for $-[X^\#,Y^\#]$. Since $[X,Y]^\# = -[X^\#,Y^\#]$, the corresponding hamiltonian functions must differ by a constant:

$$\tau^{[X,Y]} - \{\tau^X,\tau^Y\} = c(X,Y) \in \mathbb{R}\ .$$

By the Jacobi identity, $\delta c = 0$. Since $H^2(\mathfrak{g};\mathbb{R}) = 0$, there is $b \in \mathfrak{g}^*$ satisfying $c = \delta b$, $c(X,Y) = -b([X,Y])$. We define

$$\begin{array}{rccl}
\mu^* : & \mathfrak{g} & \longrightarrow & C^\infty(M) \\
& X & \longmapsto & \mu^*(X) = \tau^X + b(X) = \mu^X\ .
\end{array}$$

Now μ^* is a Lie algebra homomorphism:

$$\mu^*([X,Y]) = \tau^{[X,Y]} + b([X,Y]) = \{\tau^X,\tau^Y\} = \{\mu^X,\mu^Y\}\ .$$

$\square$

So when is $H^1(\mathfrak{g};\mathbb{R}) = H^2(\mathfrak{g};\mathbb{R}) = 0$?

A compact Lie group G is **semisimple** if $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$.

Examples. The unitary group $\mathrm{U}(n)$ is not semisimple because the multiples of the identity, $S^1 \cdot \mathrm{Id}$, form a nontrivial center; at the level of the Lie algebra, this corresponds to the 1-dimensional subspace $\mathbb{R} \cdot \mathrm{Id}$ of scalar matrices which are not commutators since they are not traceless.

Any direct product of the other compact classical groups $\mathrm{SU}(n)$, $\mathrm{SO}(n)$ and $\mathrm{Sp}(n)$ is semisimple $(n > 1)$. Any commutative Lie group is not semisimple. $\diamondsuit$

Theorem 26.3 (Whitehead Lemmas) *Let G be a compact Lie group.*

$$G \text{ is semisimple} \qquad \Longleftrightarrow \qquad H^1(\mathfrak{g}; \mathbb{R}) = H^2(\mathfrak{g}; \mathbb{R}) = 0 \ .$$

A proof can be found in [67, pages 93-95].

Corollary 26.4 *If G is semisimple, then any symplectic G-action is hamiltonian.*

26.4 Uniqueness of Moment Maps

Let G be a compact connected Lie group.

Theorem 26.5 *If $H^1(\mathfrak{g}; \mathbb{R}) = 0$, then moment maps for hamiltonian G-actions are unique.*

Proof. Suppose that μ_1^* and μ_2^* are two comoment maps for an action ψ:

$$\begin{tikzcd}
C^\infty(M) \arrow[rr] & & \chi^{\mathrm{ham}}(M) \\
& \mathfrak{g} \arrow[ul, "\mu_2^*", "\mu_1^*"'] \arrow[ur, "d\psi"'] &
\end{tikzcd}$$

For each $X \in \mathfrak{g}$, μ_1^X and μ_2^X are both hamiltonian functions for $X^\#$, thus $\mu_1^X - \mu_2^X = c(X)$ is locally constant. This defines $c \in \mathfrak{g}^*$, $X \mapsto c(X)$.

Since μ_1^*, μ_2^* are Lie algebra homomorphisms, we have $c([X, Y]) = 0$, $\forall X, Y \in \mathfrak{g}$, i.e., $c \in [\mathfrak{g}, \mathfrak{g}]^0 = \{0\}$. Hence, $\mu_1^* = \mu_2^*$. $\square$

Corollary of this proof. *In general, if $\mu : M \to \mathfrak{g}^*$ is a moment map, then given any $c \in [\mathfrak{g}, \mathfrak{g}]^0$, $\mu_1 = \mu + c$ is another moment map.*

In other words, moment maps are unique up to elements of the dual of the Lie algebra which annihilate the commutator ideal.

The two extreme cases are:

G semisimple:	any symplectic action is hamiltonian ,
	moment maps are unique .
G commutative:	symplectic actions may not be hamiltonian ,
	moment maps are unique up to any constant $c \in \mathfrak{g}^*$.

Example. The circle action on $(\mathbb{T}^2, \omega = d\theta_1 \wedge d\theta_2)$ by rotations in the θ_1 direction has vector field $X^\# = \frac{\partial}{\partial \theta_1}$; this is a symplectic action but is not hamiltonian. $\diamondsuit$

Homework 20: Examples of Reduction

1. For the natural action of $\mathrm{U}(k)$ on $\mathbb{C}^{k\times n}$ with moment map computed in exercise 5 of Homework 19, we have $\mu^{-1}(0) = \{A \in \mathbb{C}^{k\times n} \mid AA^* = \mathrm{Id}\}$. Show that the quotient

$$\mu^{-1}(0)/\mathrm{U}(k) = \mathbb{G}(k,n)$$

is the grassmannian of k-planes in $\mathbb{C}^n$.

2. Consider the S^1-action on $(\mathbb{R}^{2n+2}, \omega_0)$ which, under the usual identification of $\mathbb{R}^{2n+2}$ with $\mathbb{C}^{n+1}$, corresponds to multiplication by e^{it}. This action is hamiltonian with a moment map $\mu : \mathbb{C}^{n+1} \to \mathbb{R}$ given by

$$\mu(z) = -\tfrac{1}{2}|z|^2 + \tfrac{1}{2} \ .$$

Prove that the reduction $\mu^{-1}(0)/S^1$ is $\mathbb{CP}^n$ with the Fubini-Study symplectic form $\omega_{\mathrm{red}} = \omega_{\mathrm{FS}}$.

Hint: Let $\mathrm{pr} : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ denote the standard projection. Check that

$$\mathrm{pr}^*\omega_{\mathrm{FS}} = \tfrac{i}{2}\partial\bar{\partial}\log(|z|^2) \ .$$

Prove that this form has the same restriction to S^{2n+1} as ω_0.

3. Show that the natural actions of $\mathbb{T}^{n+1}$ and $\mathrm{U}(n+1)$ on $(\mathbb{CP}^n, \omega_{\mathrm{FS}})$ are hamiltonian, and find formulas for their moment maps.

Hint: Previous exercise and exercises 2 and 4 of Homework 19.

27 Convexity

27.1 Convexity Theorem

From now on, we will concentrate on actions of a torus $G = \mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m$.

Theorem 27.1 (Atiyah [6], Guillemin-Sternberg [57])
Let (M, ω) be a compact connected symplectic manifold, and let $\mathbb{T}^m$ be an m-torus. Suppose that $\psi : \mathbb{T}^m \to \mathrm{Sympl}(M, \omega)$ is a hamiltonian action with moment map $\mu : M \to \mathbb{R}^m$. Then:

1. *the levels of μ are connected;*
2. *the image of μ is convex;*
3. *the image of μ is the convex hull of the images of the fixed points of the action.*

The image $\mu(M)$ of the moment map is hence called the **moment polytope**.

Proof. This proof (due to Atiyah) involves induction over $m = \dim \mathbb{T}^m$. Consider the statements:

A_m: "the levels of μ are connected, for any $\mathbb{T}^m$-action;"

B_m: "the image of μ is convex, for any $\mathbb{T}^m$-action."

Then

$$\begin{array}{ccl} (1) & \iff & A_m \text{ holds for all } m \ , \\ (2) & \iff & B_m \text{ holds for all } m \ . \end{array}$$

- A_1 is a non-trivial result in Morse theory.
- $A_{m-1} \Longrightarrow A_m$ (induction step) is in Homework 21.
- B_1 is trivial because in $\mathbb{R}$ connectedness is convexity.
- $A_{m-1} \Longrightarrow B_m$ is proved below.

Choose an injective matrix $A \in \mathbb{Z}^{m \times (m-1)}$. Consider the action of an $(m-1)$-subtorus

$$\begin{array}{rccl} \psi_A : & \mathbb{T}^{m-1} & \longrightarrow & \mathrm{Sympl}(M, \omega) \\ & \theta & \longmapsto & \psi_{A\theta} \ . \end{array}$$

Exercise. The action ψ_A is hamiltonian with moment map $\mu_A = A^t \mu : M \to \mathbb{R}^{m-1}$. $\diamondsuit$

Given any $p_0 \in \mu_A^{-1}(\xi)$,

$$p \in \mu_A^{-1}(\xi) \iff A^t\mu(p) = \xi = A^t\mu(p_0)$$

so that

$$\mu_A^{-1}(\xi) = \{p \in M \mid \mu(p) - \mu(p_0) \in \ker A^t\} \ .$$

By the first part (statement A_{m-1}), $\mu_A^{-1}(\xi)$ is connected. Therefore, if we connect p_0 to p_1 by a path p_t in $\mu_A^{-1}(\xi)$, we obtain a path $\mu(p_t) - \mu(p_0)$ in $\ker A^t$. But $\ker A^t$ is 1-dimensional. Hence, $\mu(p_t)$ must go through any convex combination of $\mu(p_0)$ and $\mu(p_1)$, which shows that any point on the line segment from $\mu(p_0)$ to $\mu(p_1)$ must be in $\mu(M)$:

$$(1-t)\mu(p_0) + t\mu(p_1) \in \mu(M) \ , \quad 0 \le t \le 1 \ .$$

Any $p_0, p_1 \in M$ can be approximated arbitrarily closely by points p_0' and p_1' with $\mu(p_1') - \mu(p_0') \in \ker A^t$ for some injective matrix $A \in \mathbb{Z}^{m \times (m-1)}$. Taking limits $p_0' \to p_0$, $p_1' \to p_1$, we obtain that $\mu(M)$ is convex.[15]

To prove part *3*, consider the fixed point set C of ψ. Homework 21 shows that C is a finite union of connected symplectic submanifolds, $C = C_1 \cup \ldots \cup C_N$. The moment map is constant on each C_j, $\mu(C_j) = \eta_j \in \mathbb{R}^m$, $j = 1, \ldots, N$. By the second part, the convex hull of $\{\eta_1, \ldots, \eta_N\}$ is contained in $\mu(M)$.

For the converse, suppose that $\xi \in \mathbb{R}^m$ and $\xi \notin$ convex hull of $\{\eta_1, \ldots, \eta_N\}$. Choose $X \in \mathbb{R}^m$ with rationally independent components and satisfying

$$\langle \xi, X \rangle > \langle \eta_j, X \rangle, \text{ for all } j \ .$$

By the irrationality of X, the set $\{\exp tX(e) \mid t \in \mathbb{R}\}$ is dense in $\mathbb{T}^m$, hence the zeros of the vector field $X^\#$ on M are the fixed points of the $\mathbb{T}^m$-action. Since $\mu^X = \langle \mu, X \rangle$ attains its maximum on one of the sets C_j, this implies

$$\langle \xi, X \rangle > \sup_{p \in M} \langle \mu(p), X \rangle \ ,$$

hence $\xi \notin \mu(M)$. Therefore,

$$\mu(M) = \text{ convex hull of } \{\eta_1, \ldots, \eta_N\} \ .$$

□

27.2 Effective Actions

An action of a group G on a manifold M is called **effective** if each group element $g \neq e$ moves at least one $p \in M$, that is,

$$\bigcap_{p \in M} G_p = \{e\} \ ,$$

where $G_p = \{g \in G \mid g \cdot p = p\}$ is the stabilizer of p.

[15] Clearly $\mu(M)$ is closed because it is compact.

Corollary 27.2 *Under the conditions of the convexity theorem, if the $\mathbb{T}^m$-action is effective, then there must be at least $m+1$ fixed points.*

Proof. If the $\mathbb{T}^m$-action is effective, there must be a point p where the moment map is a submersion, i.e., $(d\mu_1)_p, \ldots, (d\mu_m)_p$ are linearly independent. Hence, $\mu(p)$ is an interior point of $\mu(M)$, and $\mu(M)$ is a nondegenerate convex polytope. Any nondegenerate convex polytope in $\mathbb{R}^m$ must have at least $m+1$ vertices. The vertices of $\mu(M)$ are images of fixed points. $\square$

Theorem 27.3 *Let $(M, \omega, \mathbb{T}^m, \mu)$ be a hamiltonian $\mathbb{T}^m$-space. If the $\mathbb{T}^m$-action is effective, then $\dim M \geq 2m$.*

Proof. On an orbit $\mathcal{O}$, the moment map $\mu(\mathcal{O}) = \xi$ is constant. For $p \in \mathcal{O}$, the exterior derivative

$$d\mu_p : T_pM \longrightarrow \mathfrak{g}^*$$

maps $T_p\mathcal{O}$ to 0. Thus

$$T_p\mathcal{O} \subseteq \ker d\mu_p = (T_p\mathcal{O})^\omega ,$$

which shows that *orbits $\mathcal{O}$ of a hamiltonian torus action are always isotropic submanifolds of M*. In particular, $\dim \mathcal{O} \leq \frac{1}{2} \dim M$.

Fact: If $\psi : \mathbb{T}^m \to \mathrm{Diff}(M)$ is an effective action, then it has orbits of dimension m; a proof may be found in [17]. $\square$

Definition 27.4 *A* **(symplectic) toric manifold**[16] *is a compact connected symplectic manifold (M, ω) equipped with an effective hamiltonian action of a torus $\mathbb{T}$ of dimension equal to half the dimension of the manifold:*

$$\dim \mathbb{T} = \frac{1}{2} \dim M$$

and with a choice of a corresponding moment map μ.

Exercise. Show that an effective hamiltonian action of a torus $\mathbb{T}^n$ on a $2n$-dimensional symplectic manifold gives rise to an integrable system.

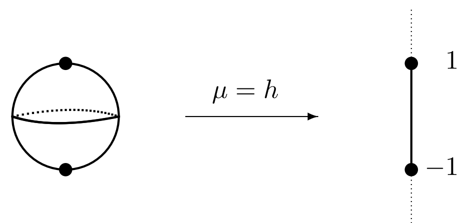
Hint: The coordinates of the moment map are commuting integrals of motion.

$\diamondsuit$

[16] In these notes, a toric manifold is always a *symplectic* toric manifold.

27.3 Examples

1. The circle $\mathbf{S}^1$ **acts on the** 2**-sphere** $(S^2, \omega_{\text{standard}} = d\theta \wedge dh)$ by rotations with moment map $\mu = h$ equal to the height function and moment polytope $[-1, 1]$.



1.' The circle $\mathbf{S}^1$ **acts on** $\mathbb{CP}^1 = \mathbb{C}^2 - 0/\sim$ with the Fubini-Study form $\omega_{\text{FS}} = \frac{1}{4}\omega_{\text{standard}}$, by $e^{i\theta} \cdot [z_0, z_1] = [z_0, e^{i\theta} z_1]$. This is hamiltonian with moment map $\mu[z_0, z_1] = -\frac{1}{2} \cdot \frac{|z_1|^2}{|z_0|^2 + |z_1|^2}$, and moment polytope $\left[-\frac{1}{2}, 0\right]$.

2. The $\mathbb{T}^2$**-action on** $\mathbb{CP}^2$ by

$$(e^{i\theta_1}, e^{i\theta_2}) \cdot [z_0, z_1, z_2] = [z_0, e^{i\theta_1} z_1, e^{i\theta_2} z_2]$$

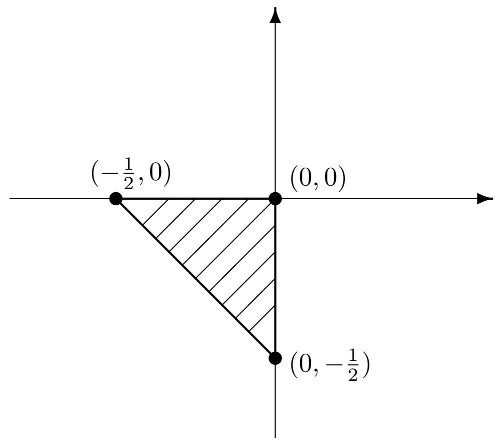
has moment map

$$\mu[z_0, z_1, z_2] = -\frac{1}{2}\left(\frac{|z_1|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2}, \frac{|z_2|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2}\right) .$$

The fixed points get mapped as

$$\begin{array}{lcl} [1,0,0] & \longmapsto & (0,0) \\ [0,1,0] & \longmapsto & \left(-\frac{1}{2}, 0\right) \\ [0,0,1] & \longmapsto & \left(0, -\frac{1}{2}\right) . \end{array}$$

Notice that the stabilizer of a preimage of the edges is S^1, while the action is free at preimages of interior points of the moment polytope.



Exercise. What is the moment polytope for the $\mathbb{T}^3$-action on $\mathbb{CP}^3$ as

$$(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}) \cdot [z_0, z_1, z_2, z_3] = [z_0, e^{i\theta_1} z_1, e^{i\theta_2} z_2, e^{i\theta_3} z_3] \ ?$$

$\diamondsuit$

Exercise. What is the moment polytope for the $\mathbb{T}^2$-action on $\mathbb{CP}^1 \times \mathbb{CP}^1$ as

$$(e^{i\theta}, e^{i\eta}) \cdot ([z_0, z_1], [w_0, w_1]) = ([z_0, e^{i\theta} z_1], [w_0, e^{i\eta} w_1]) \ ?$$

$\diamondsuit$

Homework 21: Connectedness

Consider a hamiltonian action $\psi : \mathbb{T}^m \to \mathrm{Sympl}\,(M,\omega)$, $\theta \mapsto \psi_\theta$, of an m-dimensional torus on a $2n$-dimensional compact connected symplectic manifold (M,ω). If we identify the Lie algebra of $\mathbb{T}^m$ with $\mathbb{R}^m$ by viewing $\mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m$, and we identify the Lie algebra with its dual via the standard inner product, then the moment map for ψ is $\mu : M \to \mathbb{R}^m$.

1. Show that there exists a compatible almost complex structure J on (M,ω) which is invariant under the $\mathbb{T}^m$-action, that is, $\psi_\theta^* J = J\psi_\theta^*$, for all $\theta \in \mathbb{T}^m$.

   Hint: We cannot average almost complex structures, but we can average riemannian metrics (why?). Given a riemannian metric g_0 on M, its $\mathbb{T}^m$-average $g = \int_{\mathbb{T}^m} \psi_\theta^* g_0 d\theta$ is $\mathbb{T}^m$-invariant.

2. Show that, for any subgroup $G \subseteq \mathbb{T}^m$, the fixed-point set for G ,
$$\mathrm{Fix}\,(G) = \bigcap_{\theta\in G} \mathrm{Fix}\,(\psi_\theta)\ ,$$
is a symplectic submanifold of M.

   Hint: For each $p \in \mathrm{Fix}\,(G)$ and each $\theta \in G$, the differential of ψ_θ at p,
$$d\psi_\theta(p) : T_pM \longrightarrow T_pM\ ,$$
preserves the complex structure J_p on T_pM. Consider the exponential map $\exp_p : T_pM \to M$ with respect to the invariant riemannian metric $g(\cdot,\cdot) = \omega(\cdot, J\cdot)$. Show that, by uniqueness of geodesics, $\exp_p$ is equivariant, i.e.,
$$\exp_p(d\psi_\theta(p)v) = \psi_\theta(\exp_p v)$$
for any $\theta \in G$, $v \in T_pM$. Conclude that the fixed points of ψ_θ near p correspond to the fixed points of $d\psi_\theta(p)$ on T_pM, that is
$$T_p\mathrm{Fix}\,(G) = \bigcap_{\theta\in G} \ker(\mathrm{Id} - d\psi_\theta(p))\ .$$
Since $d\psi_\theta(p) \circ J_p = J_p \circ d\psi_\theta(p)$, the eigenspace with eigenvalue 1 is invariant under J_p, and is therefore a symplectic subspace.

3. A smooth function $f : M \to \mathbb{R}$ on a compact riemannian manifold M is called a **Morse-Bott function** if its critical set $\mathrm{Crit}\,(f) = \{p \in M \,|\, df(p) = 0\}$ is a submanifold of M and for every $p \in \mathrm{Crit}\,(f)$, $T_p\mathrm{Crit}\,(f) = \ker \nabla^2 f(p)$ where $\nabla^2 f(p) : T_pM \to T_pM$ denotes the linear operator obtained from the hessian via the riemannian metric. This is the natural generalization of the notion of Morse function to the case where the critical set is not just isolated points. If f is a Morse-Bott function, then $\mathrm{Crit}\,(f)$ decomposes into finitely many connected critical manifolds C. The tangent space T_pM at $p \in C$ decomposes as a direct sum
$$T_pM = T_pC \oplus E_p^+ \oplus E_p^-$$
where E_p^+ and E_p^- are spanned by the positive and negative eigenspaces of $\nabla^2 f(p)$. The *index* of a connected critical submanifold C is $n_C^- = \dim E_p^-$, for any $p \in C$, whereas the *coindex* of C is $n_C^+ = \dim E_p^+$.

For each $X \in \mathbb{R}^m$, let $\mu^X = \langle \mu, X \rangle : M \to \mathbb{R}$ be the component of μ along X. Show that μ^X is a Morse-Bott function with even-dimensional critical manifolds of even index. Moreover, show that the critical set

$$\mathrm{Crit}\ (\mu^X) = \bigcap_{\theta \in \mathbb{T}^X} \mathrm{Fix}\ (\psi_\theta)$$

is a symplectic manifold, where $\mathbb{T}^X$ is the closure of the subgroup of $\mathbb{T}^m$ generated by X.

Hint: Assume first that X has components independent over $\mathbb{Q}$, so that $\mathbb{T}^X = \mathbb{T}^m$ and $\mathrm{Crit}\ (\mu^X) = \mathrm{Fix}\ (\mathbb{T}^m)$. Apply exercise 2. To prove that $T_p\mathrm{Crit}\ (\mu^X) = \ker \nabla^2 \mu^X(p)$, show that $\ker \nabla^2 \mu^X(p) = \cap_{\theta \in \mathbb{T}^m} \ker(\mathrm{Id} - d\psi_\theta(p))$. To see this, notice that the 1-parameter group of matrices $(d\psi_{\exp tX})_p$ coincides with $\exp(tv_p)$, where $v_p = -J_p \nabla^2 \mu^X(p) : T_pM \to T_pM$ is a vector field on T_pM. The kernel of $\nabla^2 \mu^X(p)$ corresponds to the fixed points of $d\psi_{tX}(p)$, and since X has rationally independent components, these are the common fixed points of all $d\psi_\theta(p)$, $\theta \in \mathbb{T}^m$. The eigenspaces of $\nabla^2 \mu^X(p)$ are even-dimensional because they are invariant under J_p.

4. The moment map $\mu = (\mu_1, \ldots, \mu_m)$ is called **effective** if the 1-forms $d\mu_1, \ldots, d\mu_m$ of its components are linearly independent. Show that, if μ is not effective, then the action reduces to that of an $(m-1)$-subtorus.

   Hint: If μ is not effective, then the function $\mu^X = \langle \mu, X \rangle$ is constant for some nonzero $X \in \mathbb{R}^m$. Show that we can neglect the direction of X.

5. Prove that the level set $\mu^{-1}(\xi)$ is connected for every regular value $\xi \in \mathbb{R}^m$.

   Hint: Prove by induction over $m = \dim \mathbb{T}^m$. For the case $m = 1$, use the lemma that all level sets $f^{-1}(c)$ of a Morse-Bott function $f : M \to \mathbb{R}$ on a compact manifold M are necessarily connected, if the critical manifolds all have index and coindex $\neq 1$ (see [83, p.178-179]). For the induction step, you can assume that ψ is effective. Then, for every $0 \neq X \in \mathbb{R}^m$, the function $\mu^X : M \to \mathbb{R}$ is not constant. Show that $\mathcal{C} := \cup_{X \neq 0} \mathrm{Crit}\ \mu^X = \cup_{0 \neq X \in \mathbb{Z}^m} \mathrm{Crit}\ \mu^X$ where each $\mathrm{Crit}\ \mu^X$ is an even-dimensional proper submanifold, so the complement $M \setminus \mathcal{C}$ must be dense in M. Show that $M \setminus \mathcal{C}$ is open. Hence, by continuity, to show that $\mu^{-1}(\xi)$ is connected for every regular value $\xi = (\xi_1, \ldots, \xi_m) \in \mathbb{R}^m$, it suffices to show that $\mu^{-1}(\xi)$ is connected whenever $(\xi_1, \ldots, \xi_{m-1})$ is a regular value for a reduced moment map $(\mu_1, \ldots, \mu_{m-1})$. By the induction hypothesis, the manifold $Q = \cap_{j=1}^{m-1} \mu_j^{-1}(\xi_j)$ is connected whenever $(\xi_1, \ldots, \xi_{m-1})$ is a regular value for $(\mu_1, \ldots, \mu_{m-1})$. It suffices to show that the function $\mu_m : Q \to \mathbb{R}$ has only critical manifolds of even index and coindex (see [83, p.183]), because then, by the lemma, the level sets $\mu^{-1}(\xi) = Q \cap \mu_m^{-1}(\xi_m)$ are connected for every ξ_m.

Part XI

Symplectic Toric Manifolds

Native to algebraic geometry, toric manifolds have been studied by symplectic geometers as examples of extremely symmetric hamiltonian spaces, and as guinea pigs for new theorems. Delzant showed that symplectic toric manifolds are classified (as hamiltonian spaces) by a set of special polytopes.

28 Classification of Symplectic Toric Manifolds

28.1 Delzant Polytopes

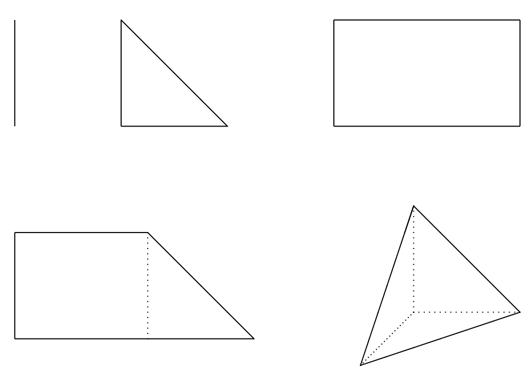
A $2n$-dimensional **(symplectic) toric manifold** is a compact connected symplectic manifold (M^{2n}, ω) equipped with an effective hamiltonian action of an n-torus $\mathbb{T}^n$ and with a corresponding moment map $\mu : M \to \mathbb{R}^n$.

Definition 28.1 *A* **Delzant polytope** Δ *in* $\mathbb{R}^n$ *is a convex polytope satisfying:*

- *it is* **simple**, *i.e., there are* n *edges meeting at each vertex;*
- *it is* **rational**, *i.e., the edges meeting at the vertex* p *are rational in the sense that each edge is of the form* $p + tu_i$, $t \geq 0$, *where* $u_i \in \mathbb{Z}^n$*;*
- *it is* **smooth**, *i.e., for each vertex, the corresponding* $u_1, \ldots, u_n$ *can be chosen to be a* $\mathbb{Z}$*-basis of* $\mathbb{Z}^n$*.*

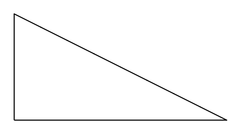
Remark. The Delzant polytopes are the simple rational smooth polytopes. These are closely related to the **Newton polytopes** (which are the nonsingular n-valent polytopes), except that the vertices of a Newton polytope are required to lie on the integer lattice and for a Delzant polytope they are not. $\diamondsuit$

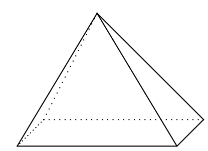
Examples of Delzant polytopes:



The dotted vertical line in the trapezoidal example means nothing, except that it is a picture of a rectangle plus an isosceles triangle. For "taller" triangles, smoothness would be violated. "Wider" triangles (with integral slope) may still be Delzant. The family of the Delzant trapezoids of this type, starting with the rectangle, correspond, under the Delzant construction, to **Hirzebruch surfaces**; see Homework 22.

Examples of polytopes which are **not Delzant**:





The picture on the left fails the smoothness condition, whereas the picture on the right fails the simplicity condition.

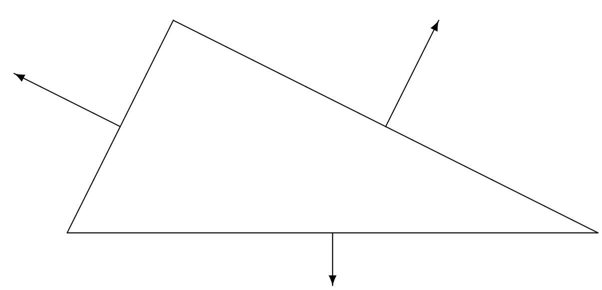
Algebraic description of Delzant polytopes:

A **facet** of a polytope is a $(n-1)$-dimensional face.

Let Δ be a Delzant polytope with $n = \dim \Delta$ and $d =$ number of facets.

A lattice vector $v \in \mathbb{Z}^n$ is **primitive** if it cannot be written as $v = ku$ with $u \in \mathbb{Z}^n$, $k \in \mathbb{Z}$ and $|k| > 1$; for instance, $(1,1)$, $(4,3)$, $(1,0)$ are primitive, but $(2,2)$, $(3,6)$ are not.

Let $v_i \in \mathbb{Z}^n$, $i = 1, \ldots, d$, be the primitive outward-pointing normal vectors to the facets.



$n = 2$

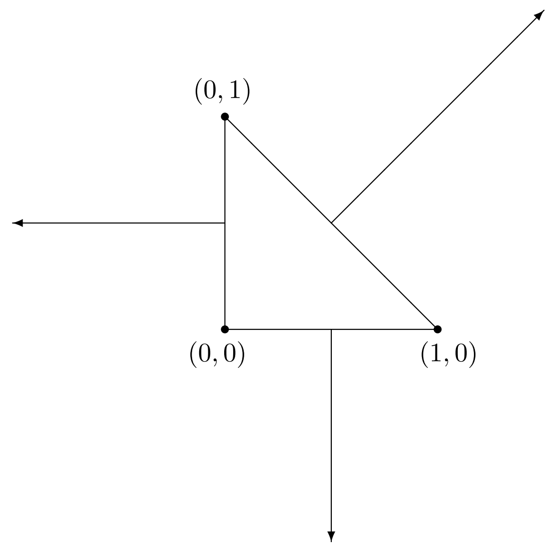
$d = 3$

Then we can describe Δ as an intersection of halfspaces

$$\Delta = \{x \in (\mathbb{R}^n)^* \mid \langle x, v_i \rangle \leq \lambda_i,\ i = 1, \ldots, d\} \quad \text{for some } \lambda_i \in \mathbb{R} .$$

Example. For the picture below, we have

$$\begin{aligned}\Delta &= \{x \in (\mathbb{R}^2)^* \mid x_1 \geq 0,\ x_2 \geq 0,\ x_1 + x_2 \leq 1\} \\ &= \{x \in (\mathbb{R}^2)^* \mid \langle x, (-1,0)\rangle \leq 0\ ,\ \langle x, (0,-1)\rangle \leq 0\ ,\ \langle x, (1,1)\rangle \leq 1\}\ .\end{aligned}$$



◊

28.2 Delzant Theorem

We do not have a classification of symplectic manifolds, but we do have a classification of toric manifolds in terms of combinatorial data. This is the content of the Delzant theorem.

Theorem 28.2 (Delzant [23]) *Toric manifolds are classified by Delzant polytopes. More specifically, there is the following one-to-one correspondence*

$$\begin{array}{ccc}\{\textit{toric manifolds}\} & \xrightarrow{1-1} & \{\textit{Delzant polytopes}\} \\ (M^{2n}, \omega, \mathbb{T}^n, \mu) & \longmapsto & \mu(M).\end{array}$$

We will prove the existence part (or surjectivity) in the Delzant theorem following [54]. Given a Delzant polytope, what is the corresponding toric manifold?

$$(M_\Delta, \omega_\Delta, \mathbb{T}^n, \mu) \quad \xleftarrow{?} \quad \Delta^n$$

28.3 Sketch of Delzant Construction

Let Δ be a Delzant polytope with d facets. Let $v_i \in \mathbb{Z}^n$, $i = 1, \ldots, d$, be the primitive outward-pointing normal vectors to the facets. For some $\lambda_i \in \mathbb{R}$,

$$\Delta = \{x \in (\mathbb{R}^n)^* \mid \langle x, v_i \rangle \leq \lambda_i,\ i = 1, \ldots, d\} \ .$$

Let $e_1 = (1, 0, \ldots, 0), \ldots, e_d = (0, \ldots, 0, 1)$ be the standard basis of $\mathbb{R}^d$. Consider

$$\begin{array}{rccc} \pi : & \mathbb{R}^d & \longrightarrow & \mathbb{R}^n \\ & e_i & \longmapsto & v_i \ . \end{array}$$

Claim. The map π is onto and maps $\mathbb{Z}^d$ onto $\mathbb{Z}^n$.

Proof. The set $\{e_1, \ldots, e_d\}$ is a basis of $\mathbb{Z}^d$. The set $\{v_1, \ldots, v_d\}$ spans $\mathbb{Z}^n$ for the following reason. At a vertex p, the edge vectors $u_1, \ldots, u_n \in (\mathbb{R}^n)^*$, form a basis for $(\mathbb{Z}^n)^*$ which, without loss of generality, we may assume is the standard basis. Then the corresponding primitive normal vectors to the facets meeting at p are symmetric (in the sense of multiplication by -1) to the u_i's, hence form a basis of $\mathbb{Z}^n$. $\square$

Therefore, π induces a surjective map, still called π, between tori:

$$\begin{array}{ccccc} \mathbb{R}^d/\mathbb{Z}^d & \stackrel{\pi}{\longrightarrow} & \mathbb{R}^n/\mathbb{Z}^n & & \\ \| & & \| & & \\ \mathbb{T}^d & \longrightarrow & \mathbb{T}^n & \longrightarrow & 0 \ . \end{array}$$

Let

$$\begin{array}{rcl} N & = & \text{kernel of } \pi \ \ (N \text{ is a Lie subgroup of } \mathbb{T}^d) \\ \mathfrak{n} & = & \text{Lie algebra of } N \\ \mathbb{R}^d & = & \text{Lie algebra of } \mathbb{T}^d \\ \mathbb{R}^n & = & \text{Lie algebra of } \mathbb{T}^n. \end{array}$$

The exact sequence of tori

$$0 \longrightarrow N \stackrel{i}{\longrightarrow} \mathbb{T}^d \stackrel{\pi}{\longrightarrow} \mathbb{T}^n \longrightarrow 0$$

induces an exact sequence of Lie algebras

$$0 \longrightarrow \mathfrak{n} \stackrel{i}{\longrightarrow} \mathbb{R}^d \stackrel{\pi}{\longrightarrow} \mathbb{R}^n \longrightarrow 0$$

with dual exact sequence

$$0 \longrightarrow (\mathbb{R}^n)^* \stackrel{\pi^*}{\longrightarrow} (\mathbb{R}^d)^* \stackrel{i^*}{\longrightarrow} \mathfrak{n}^* \longrightarrow 0 \ .$$

Now consider $\mathbb{C}^d$ with symplectic form $\omega_0 = \frac{i}{2} \sum dz_k \wedge d\bar{z}_k$, and standard hamiltonian action of $\mathbb{T}^d$

$$(e^{2\pi i t_1}, \ldots, e^{2\pi i t_d}) \cdot (z_1, \ldots, z_d) = (e^{2\pi i t_1} z_1, \ldots, e^{2\pi i t_d} z_d) \ .$$

The moment map is $\phi : \mathbb{C}^d \longrightarrow (\mathbb{R}^d)^*$

$$\phi(z_1, \ldots, z_d) = -\pi(|z_1|^2, \ldots, |z_d|^2) + \text{ constant },$$

where we choose the constant to be $(\lambda_1, \ldots, \lambda_d)$. What is the moment map for the action restricted to the subgroup N?

Exercise. Let G be any compact Lie group and H a closed subgroup of G, with $\mathfrak{g}$ and $\mathfrak{h}$ the respective Lie algebras. The inclusion $i : \mathfrak{h} \hookrightarrow \mathfrak{g}$ is dual to the projection $i^* : \mathfrak{g}^* \to \mathfrak{h}^*$. Suppose that (M, ω, G, ϕ) is a hamiltonian G-space. Show that the restriction of the G-action to H is hamiltonian with moment map

$$i^* \circ \phi : M \longrightarrow \mathfrak{h}^* .$$

$\diamondsuit$

The subtorus N acts on $\mathbb{C}^d$ in a hamiltonian way with moment map

$$i^* \circ \phi : \mathbb{C}^d \longrightarrow \mathfrak{n}^* .$$

Let $Z = (i^* \circ \phi)^{-1}(0)$ be the zero-level set.

Claim. The set Z is compact and N acts freely on Z.

This claim will be proved in the next lecture.

By the first claim, $0 \in \mathfrak{n}^*$ is a regular value of $i^* \circ \phi$. Hence, Z is a compact submanifold of $\mathbb{C}^d$ of dimension

$$\dim_{\mathbb{R}} Z = 2d - \underbrace{(d-n)}_{\dim \mathfrak{n}^*} = d + n .$$

The orbit space $M_\Delta = Z/N$ is a compact manifold of dimension

$$\dim_{\mathbb{R}} M_\Delta = d + n - \underbrace{(d-n)}_{\dim N} = 2n .$$

The point-orbit map $p : Z \to M_\Delta$ is a principal N-bundle over M_Δ.
Consider the diagram

$$\begin{array}{ccc} Z & \stackrel{j}{\hookrightarrow} & \mathbb{C}^d \\ {\scriptstyle p} \downarrow & & \\ M_\Delta & & \end{array}$$

where $j : Z \hookrightarrow \mathbb{C}^d$ is inclusion. The Marsden-Weinstein-Meyer theorem guarantees the existence of a symplectic form ω_Δ on M_Δ satisfying

$$p^* \omega_\Delta = j^* \omega_0 .$$

Exercise. Work out all details in the following simple example.

Let $\Delta = [0, a] \subset \mathbb{R}^*$ $(n = 1, d = 2)$. Let $v(= 1)$ be the standard basis vector in $\mathbb{R}$. Then

$$\begin{array}{lll} \Delta : & \langle x, v_1 \rangle \leq 0 & \qquad v_1 = -v \\ & \langle x, v_2 \rangle \leq a & \qquad v_2 = v . \end{array}$$

The projection

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{\pi} & \mathbb{R} \\ e_1 & \longmapsto & -v \\ e_2 & \longmapsto & v \end{array}$$

has kernel equal to the span of $(e_1 + e_2)$, so that N is the diagonal subgroup of $\mathbb{T}^2 = S^1 \times S^1$. The exact sequences become

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \xrightarrow{i} & \mathbb{T}^2 & \xrightarrow{\pi} & S^1 & \longrightarrow & 0 \\ 0 & \longrightarrow & \mathbb{R}^* & \xrightarrow{\pi^*} & (\mathbb{R}^2)^* & \xrightarrow{i^*} & \mathfrak{n}^* & \longrightarrow & 0 \\ & & & & (x_1, x_2) & \longmapsto & x_1 + x_2 \ . & & \end{array}$$

The action of the diagonal subgroup $N = \{(e^{2\pi it}, e^{2\pi it}) \in S^1 \times S^1\}$ on $\mathbb{C}^2$,

$$(e^{2\pi it}, e^{2\pi it}) \cdot (z_1, z_2) = (e^{2\pi it} z_1, e^{2\pi it} z_2) \ ,$$

has moment map

$$(i^* \circ \phi)(z_1, z_2) = -\pi(|z_1|^2 + |z_2|^2) + a \ ,$$

with zero-level set

$$(i^* \circ \phi)^{-1}(0) = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = \frac{a}{\pi}\} \ .$$

Hence, the reduced space is

$$(i^* \circ \phi)^{-1}(0)/N = \mathbb{CP}^1 \qquad \text{projective space!}$$

$\diamondsuit$

29 Delzant Construction

29.1 Algebraic Set-Up

Let Δ be a Delzant polytope with d facets. We can write Δ as

$$\Delta = \{x \in (\mathbb{R}^n)^* \mid \langle x, v_i \rangle \leq \lambda_i \ , \ i = 1, \dots, d\} \ ,$$

for some $\lambda_i \in \mathbb{R}$. Recall the exact sequences from the previous lecture

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \stackrel{i}{\longrightarrow} & \mathbb{T}^d & \stackrel{\pi}{\longrightarrow} & \mathbb{T}^n & \longrightarrow & 0 \\ 0 & \longrightarrow & \mathfrak{n} & \stackrel{i}{\longrightarrow} & \mathbb{R}^d & \stackrel{\pi}{\longrightarrow} & \mathbb{R}^n & \longrightarrow & 0 \\ & & & & e_i & \longmapsto & v_i & & \end{array}$$

and the dual sequence

$$0 \longrightarrow (\mathbb{R}^n)^* \stackrel{\pi^*}{\longrightarrow} (\mathbb{R}^d)^* \stackrel{i^*}{\longrightarrow} \mathfrak{n}^* \longrightarrow 0 \ .$$

The standard hamiltonian action of $\mathbb{T}^d$ on $\mathbb{C}^d$

$$(e^{2\pi i t_1}, \dots, e^{2\pi i t_d}) \cdot (z_1, \dots, z_d) = (e^{2\pi i t_1} z_1, \dots, e^{2\pi i t_d} z_d)$$

has moment map $\phi : \mathbb{C}^d \to (\mathbb{R}^d)^*$ given by

$$\phi(z_1, \dots, z_d) = -\pi(|z_1|^2, \dots, |z_d|^2) + (\lambda_1, \dots, \lambda_d) \ .$$

The restriction of this action to N has moment map

$$i^* \circ \phi : \mathbb{C}^d \longrightarrow \mathfrak{n}^* \ .$$

29.2 The Zero-Level

Let $Z = (i^* \circ \phi)^{-1}(0)$.

Proposition 29.1 *The level Z is compact and N acts freely on Z.*

Proof. Let Δ' be the image of Δ by π^*. We will show that $\phi(Z) = \Delta'$. Since ϕ is a proper map and Δ' is compact, it will follow that Z is compact.

Lemma 29.2 *Let $y \in (\mathbb{R}^d)^*$. Then:*

$$y \in \Delta' \iff y \text{ is in the image of } Z \text{ by } \phi \ .$$

Proof of the lemma. The value y is in the image of Z by ϕ if and only if both of the following conditions hold:

1. y is in the image of ϕ;

2. $i^* y = 0$.

Using the expression for ϕ and the third exact sequence, we see that these conditions are equivalent to:

1. $\langle y, e_i \rangle \leq \lambda_i$ for $i = 1, \ldots, d$.
2. $y = \pi^*(x)$ for some $x \in (\mathbb{R}^n)^*$.

Suppose that the second condition holds, so that $y = \pi^*(x)$. Then

$$\begin{aligned} \langle y, e_i \rangle \leq \lambda_i, \forall i &\iff \langle \pi^*(x), e_i \rangle \leq \lambda_i, \forall i \\ &\iff \langle x, \pi(e_i) \rangle \leq \lambda_i, \forall i \\ &\iff x \in \Delta. \end{aligned}$$

Thus, $y \in \phi(Z) \iff y \in \pi^*(\Delta) = \Delta'$. □

Hence, we have a surjective proper map $\phi : Z \to \Delta'$. Since Δ' is compact, we conclude that Z is compact. It remains to show that N acts freely on Z.

We define a stratification of Z with three equivalent descriptions:

- Define a stratification on Δ' whose ith stratum is the closure of the union of the i-dimensional faces of Δ'. Pull this stratification back to Z by ϕ.

  We can obtain a more explicit description of the stratification on Z:

- Let F be a face of Δ' with $\dim F = n - r$. Then F is characterized (as a subset of Δ') by r equations

  $$\langle y, e_i \rangle = \lambda_i \ , \quad i = i_1, \ldots, i_r \ .$$

  We write $F = F_I$ where $I = (i_1, \ldots, i_r)$ has $1 \leq i_1 < i_2 \ldots < i_r \leq d$.
  Let $z = (z_1, \ldots, z_d) \in Z$.

  $$\begin{aligned} z \in \phi^{-1}(F_I) &\iff \phi(z) \in F_I \\ &\iff \langle \phi(z), e_i \rangle = \lambda_i \ , \quad \forall i \in I \\ &\iff -\pi |z_i|^2 + \lambda_i = \lambda_i \ , \quad \forall i \in I \\ &\iff z_i = 0 \ , \quad \forall i \in I \ . \end{aligned}$$

- The $\mathbb{T}^d$-action on $\mathbb{C}^d$ preserves ϕ, so the $\mathbb{T}^d$-action takes $Z = \phi^{-1}(\Delta')$ onto itself, so $\mathbb{T}^d$ acts on Z.

  Exercise. The stratification of Z is just the stratification of Z into $\mathbb{T}^d$ orbit types. More specifically, if $z \in Z$ and $\phi(z) \in F_I$ then the stabilizer of z in $\mathbb{T}^d$ is $(\mathbb{T}^d)_I$ where

  $$I = (i_1, \ldots, i_r) \ ,$$
  $$F_I = \{ y \in \Delta' \mid \langle y, e_i \rangle = \lambda_i, \forall i \in I \} \ ,$$

  and

  $$(\mathbb{T}^d)_I = \{ (e^{2\pi i t_1}, \ldots, e^{2\pi i t_d}) \mid e^{2\pi i t_s} = 1, \forall s \notin I \}$$

Hint: Suppose that $z = (z_1, \ldots, z_d) \in \mathbb{C}^d$. Then

$$(e^{2\pi i t_1} z_1, \ldots, e^{2\pi i t_d} z_d) = (z_1, \ldots, z_d)$$

if and only if $e^{2\pi i t_s} = 1$ whenever $z_s \neq 0$.

$\diamondsuit$

In order to show that N acts freely on Z, consider the worst case scenario of points $z \in Z$ whose stabilizer under the action of $\mathbb{T}^d$ is a large as possible. Now $(\mathbb{T}^d)_I$ is largest when $F_I = \{y\}$ is a vertex of Δ'. Then y satisfies n equations

$$\langle y, e_i \rangle = \lambda_i \ , \quad i \in I = \{i_1, \ldots, i_n\} \ .$$

Lemma 29.3 *Let $z \in Z$ be such that $\phi(z)$ is a vertex of Δ'. Let $(\mathbb{T}^d)_I$ be the stabilizer of z. Then the map $\pi : \mathbb{T}^d \to \mathbb{T}^n$ maps $(\mathbb{T}^d)_I$ bijectively onto $\mathbb{T}^n$.*

Since $N = \ker \pi$, this lemma shows that in the worst case, the stabilizer of z intersects N in the trivial group. It will follow that N acts freely at this point and hence on Z.

Proof of the lemma. Suppose that $\phi(z) = y$ is a vertex of Δ'. Renumber the indices so that

$$I = (1, 2, \ldots, n) \ .$$

Then

$$(\mathbb{T}^d)_I = \{(e^{2\pi i t_1}, \ldots, e^{2\pi i t_n}, 1, \ldots, 1) \mid t_i \in \mathbb{R}\} \ .$$

The hyperplanes meeting at y are

$$\langle y', e_i \rangle = \lambda_i \ , \quad i = 1, \ldots, n \ .$$

By definition of Delzant polytope, the set $\pi(e_1), \ldots, \pi(e_n)$ is a basis of $\mathbb{Z}^n$. Thus, $\pi : (\mathbb{T}^d)_I \to \mathbb{T}^n$ is bijective. $\square$

This proves the theorem in the worst case scenario, and hence in general. $\square$

29.3 Conclusion of the Delzant Construction

We continue the construction of $(M_\Delta, \omega_\Delta)$ from Δ. We already have that

$$M_\Delta = Z/N$$

is a compact $2n$-dimensional manifold. Let ω_Δ be the reduced symplectic form.

Claim. The manifold $(M_\Delta, \omega_\Delta)$ is a hamiltonian $\mathbb{T}^n$-space with a moment map μ having image $\mu(M_\Delta) = \Delta$.

Suppose that $z \in Z$. The stabilizer of z with respect to the $\mathbb{T}^d$-action is $(\mathbb{T}^d)_I$, and

$$(\mathbb{T}^d)_I \cap N = \{e\} \ .$$

In the worst case scenario, F_I is a vertex of Δ' and $(\mathbb{T}^d)_I$ is an n-dimensional subgroup of $\mathbb{T}^d$. In any case, there is a right inverse map $\pi^{-1} : \mathbb{T}^n \to (\mathbb{T}^d)_I$. Thus, the exact sequence

$$0 \longrightarrow N \longrightarrow \mathbb{T}^d \longrightarrow \mathbb{T}^n \longrightarrow 0$$

splits, and $\mathbb{T}^d = N \times \mathbb{T}^n$.

Apply the results on reduction for product groups (Section 24.3) to our situation of $\mathbb{T}^d = N \times \mathbb{T}^n$ acting on $(M_\Delta, \omega_\Delta)$. The moment map is

$$\phi : \mathbb{C}^d \longrightarrow (\mathbb{R}^d)^* = \mathfrak{n}^* \oplus (\mathbb{R}^n)^* \ .$$

Let $j : Z \hookrightarrow \mathbb{C}^d$ be the inclusion map, and let

$$\mathrm{pr}_1 : (\mathbb{R}^d)^* \longrightarrow \mathfrak{n}^* \qquad \text{and} \qquad \mathrm{pr}_2 : (\mathbb{R}^d)^* \longrightarrow (\mathbb{R}^n)^*$$

be the projection maps. The map

$$\mathrm{pr}_2 \circ \phi \circ j : Z \longrightarrow (\mathbb{R}^n)^*$$

is constant on N-orbits. Thus there exists a map

$$\mu : M_\Delta \longrightarrow (\mathbb{R}^n)^*$$

such that

$$\mu \circ p = \mathrm{pr}_2 \circ \phi \circ j \ .$$

The image of μ is equal to the image of $\mathrm{pr}_2 \circ \phi \circ j$. We showed earlier that $\phi(Z) = \Delta'$. Thus

$$\text{Image of } \mu = \mathrm{pr}_2(\Delta') = \underbrace{\mathrm{pr}_2 \circ \pi^*}_{\mathrm{id}}(\Delta) = \Delta \ .$$

Thus $(M_\Delta, \omega_\Delta)$ is the required toric manifold corresponding to Δ.

29.4 Idea Behind the Delzant Construction

We use the idea that $\mathbb{R}^d$ is "universal" in the sense that any n-dimensional polytope Δ with d facets can be obtained by intersecting the negative orthant $\mathbb{R}^d_-$ with an affine plane A. Given Δ, to construct A first write Δ as:

$$\Delta = \{x \in \mathbb{R}^n \mid \langle x, v_i \rangle \le \lambda_i,\ i = 1, \ldots, d\} \ .$$

Define

$$\begin{array}{rccc} \pi : & \mathbb{R}^d & \longrightarrow & \mathbb{R}^n \\ & e_i & \longmapsto & v_i \end{array} \qquad \text{with dual map} \qquad \pi^* : \ \mathbb{R}^n \longrightarrow \mathbb{R}^d \ .$$

Then

$$\pi^* - \lambda : \mathbb{R}^n \longrightarrow \mathbb{R}^d$$

is an affine map, where $\lambda = (\lambda_1, \ldots, \lambda_d)$. Let A be the image of $\pi^* - \lambda$. Then A is an n-dimensional affine plane.

Claim. We have the equality $(\pi^* - \lambda)(\Delta) = \mathbb{R}^d_- \cap A$.

Proof. Let $x \in \mathbb{R}^n$. Then

$$\begin{array}{rcl}(\pi^* - \lambda)(x) \in \mathbb{R}^d_- & \iff & \langle \pi^*(x) - \lambda, e_i \rangle \leq 0, \forall i \\ & \iff & \langle x, \pi(e_i) \rangle - \lambda_i \leq 0, \forall i \\ & \iff & \langle x, v_i \rangle \leq \lambda_i, \forall i \\ & \iff & x \in \Delta \ .\end{array}$$

$\square$

We conclude that $\Delta \simeq \mathbb{R}^d_- \cap A$. Now $\mathbb{R}^d_-$ is the image of the moment map for the standard hamiltonian action of $\mathbb{T}^d$ on $\mathbb{C}^d$

$$\begin{array}{rcl}\phi : \mathbb{C}^d & \longrightarrow & \mathbb{R}^d \\ (z_1, \ldots, z_d) & \longmapsto & -\pi(|z_1|^2, \ldots, |z_d|^2) \ .\end{array}$$

Facts.

- The set $\phi^{-1}(A) \subset \mathbb{C}^d$ is a compact submanifold. Let $i : \phi^{-1}(A) \hookrightarrow \mathbb{C}^d$ denote inclusion. Then $i^*\omega_0$ is a closed 2-form which is degenerate. Its kernel is an integrable distribution. The corresponding foliation is called the **null foliation**.
- The null foliation of $i^*\omega_0$ is a principal fibration, so we take the quotient:

$$\begin{array}{ccccc} N & \hookrightarrow & \phi^{-1}(A) & & \\ & & \downarrow & & \\ & & M_\Delta & = & \phi^{-1}(A)/N \end{array}$$

  Let ω_Δ be the reduced symplectic form.

- The (non-effective) action of $\mathbb{T}^d = N \times \mathbb{T}^n$ on $\phi^{-1}(A)$ has a "moment map" with image $\phi(\phi^{-1}(A)) = \Delta$. (By "moment map" we mean a map satisfying the usual definition even though the closed 2-form is not symplectic.)

Theorem 29.4 *For any $x \in \Delta$, we have that $\mu^{-1}(x)$ is a single $\mathbb{T}^n$-orbit.*

Proof. Exercise.

First consider the standard $\mathbb{T}^d$-action on $\mathbb{C}^d$ with moment map $\phi : \mathbb{C}^d \to \mathbb{R}^d$. Show that $\phi^{-1}(y)$ is a single $\mathbb{T}^d$-orbit for any $y \in \phi(\mathbb{C}^d)$. Now observe that

$$y \in \Delta' = \pi^*(\Delta) \iff \phi^{-1}(y) \subseteq Z \ .$$

Suppose that $y = \pi^*(x)$. Show that $\mu^{-1}(x) = \phi^{-1}(y)/N$. But $\phi^{-1}(y)$ is a single $\mathbb{T}^d$-orbit where $\mathbb{T}^d = N \times \mathbb{T}^n$, hence $\mu^{-1}(x)$ is a single $\mathbb{T}^n$-orbit. $\square$

Therefore, for toric manifolds, Δ is the orbit space.

Now Δ is a *manifold with corners*. At every point p in a face F, the tangent space $T_p\Delta$ is the subspace of $\mathbb{R}^n$ tangent to F. We can visualize $(M_\Delta, \omega_\Delta, \mathbb{T}^n, \mu)$ from Δ as follows. First take the product $\mathbb{T}^n \times \Delta$. Let p lie in the interior of $\mathbb{T}^n \times \Delta$. The tangent space at p is $\mathbb{R}^n \times (\mathbb{R}^n)^*$. Define ω_p by:

$$\omega_p(v, \xi) = \xi(v) = -\omega_p(\xi, v) \quad \text{and} \quad \omega_p(v, v') = \omega(\xi, \xi') = 0 \ .$$

for all $v, v' \in \mathbb{R}^n$ and $\xi, \xi' \in (\mathbb{R}^n)^*$. Then ω is a closed nondegenerate 2-form on the interior of $\mathbb{T}^n \times \Delta$. At the corner there are directions missing in $(\mathbb{R}^n)^*$, so ω is a degenerate pairing. Hence, we need to eliminate the corresponding directions in $\mathbb{R}^n$. To do this, we collapse the orbits corresponding to subgroups of $\mathbb{T}^n$ generated by directions orthogonal to the annihilator of that face.

Example. Consider

$$(S^2, \omega = d\theta \wedge dh, S^1, \mu = h) \ ,$$

where S^1 acts on S^2 by rotation. The image of μ is the line segment $I = [-1, 1]$. The product $S^1 \times I$ is an open-ended cylinder. By collapsing each end of the cylinder to a point, we recover the 2-sphere. $\diamondsuit$

Exercise. Build $\mathbb{CP}^2$ from $\mathbb{T}^2 \times \Delta$ where Δ is a right-angled isosceles triangle. $\diamondsuit$

Finally, $\mathbb{T}^n$ acts on $\mathbb{T}^n \times \Delta$ by multiplication on the $\mathbb{T}^n$ factor. The moment map for this action is projection onto the Δ factor.

Homework 22: Delzant Theorem

1. (a) Consider the standard $(S^1)^3$-action on $\mathbb{CP}^3$:

$$(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}) \cdot [z_0, z_1, z_2, z_3] = [z_0, e^{i\theta_1} z_1, e^{i\theta_2} z_2, e^{i\theta_3} z_3] \ .$$

   Exhibit explicitly the subsets of $\mathbb{CP}^3$ for which the stabilizer under this action is $\{1\}$, S^1, $(S^1)^2$ and $(S^1)^3$. Show that the images of these subsets under the moment map are the interior, the facets, the edges and the vertices, respectively.

   (b) Classify all 2-dimensional Delzant polytopes with 4 vertices, up to translation and the action of $\mathrm{SL}(2;\mathbb{Z})$.

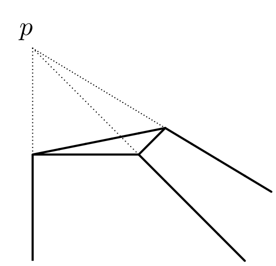
   Hint: By a linear transformation in $\mathrm{SL}(2;\mathbb{Z})$, you can make one of the angles in the polytope into a square angle. Check that automatically another angle also becomes 90^o.

   (c) What are all the 4-dimensional symplectic toric manifolds that have four fixed points?

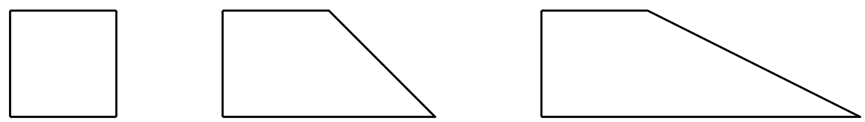
2. Take a Delzant polytope in $\mathbb{R}^n$ with a vertex p and with primitive (inward-pointing) edge vectors $u_1, \ldots, u_n$ at p. Chop off the corner to obtain a new polytope with the same vertices except p, and with p replaced by n new vertices:

$$p + \varepsilon u_j \ , \quad j = 1, \ldots, n \ ,$$

   where ε is a small positive real number. Show that this new polytope is also Delzant. The corresponding toric manifold is the ε-**symplectic blowup** of the original one.

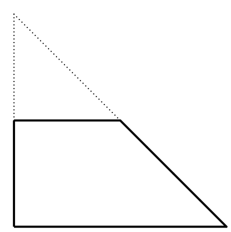


3. The toric 4-manifold $\mathcal{H}_n$ corresponding to the polygon with vertices $(0,0)$, $(n+1,0)$, $(0,1)$ and $(1,1)$, for n a nonnegative integer, is called a **Hirzebruch surface**.



   (a) What is the manifold $\mathcal{H}_0$? What is the manifold $\mathcal{H}_1$?

   Hint:



   (b) Construct the manifold $\mathcal{H}_n$ by symplectic reduction of $\mathbb{C}^4$ with respect to an action of $(S^1)^2$.

   (c) Exhibit $\mathcal{H}_n$ as a $\mathbb{CP}^1$-bundle over $\mathbb{CP}^1$.

4. Which $2n$-dimensional toric manifolds have exactly $n+1$ fixed points?

30 Duistermaat-Heckman Theorems

30.1 Duistermaat-Heckman Polynomial

Let (M^{2n}, ω) be a symplectic manifold. Then $\frac{\omega^n}{n!}$ is the symplectic volume form.

Definition 30.1 *The* **Liouville measure** *(or* **symplectic measure***) of a Borel subset*[17] $\mathcal{U}$ *of* M *is*

$$m_\omega(\mathcal{U}) = \int_\mathcal{U} \frac{\omega^n}{n!} \ .$$

Let G be a torus. Suppose that (M, ω, G, μ) is a hamiltonian G-space, and that the moment map μ is proper.

Definition 30.2 *The* **Duistermaat-Heckman measure**, m_{DH}*, on* $\mathfrak{g}^*$ *is the push-forward of* m_ω *by* $\mu : M \to \mathfrak{g}^*$*. That is,*

$$m_{DH}(U) = (\mu_* m_\omega)(U) = \int_{\mu^{-1}(U)} \frac{\omega^n}{n!}$$

for any Borel subset U *of* $\mathfrak{g}^*$*.*

For a compactly-supported function $h \in C^\infty(\mathfrak{g}^*)$, we define its integral with respect to the Duistermaat-Heckman measure to be

$$\int_{\mathfrak{g}^*} h \ dm_{DH} = \int_M (h \circ \mu) \frac{\omega^n}{n!} \ .$$

On $\mathfrak{g}^*$ regarded as a vector space, say $\mathbb{R}^n$, there is also the Lebesgue (or euclidean) measure, m_0. The relation between m_{DH} and m_0 is governed by the Radon-Nikodym derivative, denoted by $\frac{dm_{DH}}{dm_0}$, which is a generalized function satisfying

$$\int_{\mathfrak{g}^*} h \ dm_{DH} = \int_{\mathfrak{g}^*} h \ \frac{dm_{DH}}{dm_0} \ dm_0 \ .$$

Theorem 30.3 (Duistermaat-Heckman, 1982 [31]) *The Duistermaat-Heckman measure is a piecewise polynomial multiple of Lebesgue (or euclidean) measure* m_0 *on* $\mathfrak{g}^* \simeq \mathbb{R}^n$*, that is, the Radon-Nikodym derivative*

$$f = \frac{dm_{DH}}{dm_0}$$

is piecewise polynomial. More specifically, for any Borel subset U *of* $\mathfrak{g}^*$*,*

$$m_{DH}(U) = \int_U f(x) \, dx \ ,$$

where $dx = dm_0$ *is the Lebesgue volume form on* U *and* $f : \mathfrak{g}^* \simeq \mathbb{R}^n \to \mathbb{R}$ *is polynomial on any region consisting of regular values of* μ*.*

[17] The set $\mathcal{B}$ of **Borel subsets** is the σ-*ring* generated by the set of compact subsets, i.e., if $A, B \in \mathcal{B}$, then $A \setminus B \in \mathcal{B}$, and if $A_i \in \mathcal{B}$, $i = 1, 2, \ldots$, then $\cup_{i=1}^\infty A_i \in \mathcal{B}$.

The proof of Theorem 30.3 for the case $G = S^1$ is in Section 30.3. The proof for the general case, which follows along similar lines, can be found in, for instance, [54], besides the original articles.

The Radon-Nikodym derivative f is called the **Duistermaat-Heckman polynomial**. In the case of a toric manifold, the Duistermaat-Heckman polynomial is a universal constant equal to $(2\pi)^n$ when Δ is n-dimensional. Thus the symplectic volume of $(M_\Delta, \omega_\Delta)$ is $(2\pi)^n$ times the euclidean volume of Δ.

Example. Consider $(S^2, \omega = d\theta \wedge dh, S^1, \mu = h)$. The image of μ is the interval $[-1, 1]$. The Lebesgue measure of $[a, b] \subseteq [-1, 1]$ is

$$m_0([a,b]) = b - a \ .$$

The Duistermaat-Heckman measure of $[a, b]$ is

$$m_{DH}([a,b]) = \int_{\{(\theta,h)\in S^2 | a \le h \le b\}} d\theta \ dh = 2\pi(b-a) \ .$$

Consequently, the spherical area between two horizontal circles depends only on the vertical distance between them, a result which was known to Archimedes around 230 BC.

Corollary 30.4 *For the standard hamiltonian action of S^1 on (S^2, ω), we have*

$$m_{DH} = 2\pi \ m_0 \ .$$

$\diamondsuit$

30.2 Local Form for Reduced Spaces

Let (M, ω, G, μ) be a hamiltonian G-space, where G is an n-torus.[18] Assume that μ is proper. If G acts freely on $\mu^{-1}(0)$, it also acts freely on nearby levels $\mu^{-1}(t)$, $t \in \mathfrak{g}^*$ and $t \approx 0$. Consider the reduced spaces

$$M_{\text{red}} = \mu^{-1}(0)/G \qquad \text{and} \qquad M_t = \mu^{-1}(t)/G$$

with reduced symplectic forms ω_{red} and ω_t. What is the relation between these reduced spaces as symplectic manifolds?

For simplicity, we will assume G to be the circle S^1. Let $Z = \mu^{-1}(0)$ and let $i : Z \hookrightarrow M$ be the inclusion map. We fix a connection form $\alpha \in \Omega^1(Z)$ for the principal bundle

$$\begin{array}{ccc} S^1 & \hookrightarrow & Z \\ & & \downarrow \pi \\ & & M_{\text{red}} \end{array}$$

[18] The discussion in this section may be extended to hamiltonian actions of other compact Lie groups, not necessarily tori; see [54, Exercises 2.1-2.10].

that is, $\mathcal{L}_{X^\#}\alpha = 0$ and $\imath_{X^\#}\alpha = 1$, where $X^\#$ is the infinitesimal generator for the S^1-action. From α we construct a 2-form on the product manifold $Z \times (-\varepsilon, \varepsilon)$ by the recipe

$$\sigma = \pi^*\omega_{\text{red}} - d(x\alpha) \ ,$$

x being a linear coordinate on the interval $(-\varepsilon, \varepsilon) \subset \mathbb{R} \simeq \mathfrak{g}^*$. (By abuse of notation, we shorten the symbols for forms on $Z \times (-\varepsilon, \varepsilon)$ which arise by pullback via projection onto each factor.)

Lemma 30.5 *The 2-form σ is symplectic for ε small enough.*

Proof. The form σ is clearly closed. At points where $x = 0$, we have

$$\sigma|_{x=0} = \pi^*\omega_{\text{red}} + \alpha \wedge dx \ ,$$

which satisfies

$$\sigma|_{x=0}\left(X^\#, \frac{\partial}{\partial x}\right) = 1 \ ,$$

so σ is nondegenerate along $Z \times \{0\}$. Since nondegeneracy is an open condition, we conclude that σ is nondegenerate for x in a sufficiently small neighborhood of 0. □

Notice that σ is invariant with respect to the S^1-action on the first factor of $Z \times (-\varepsilon, \varepsilon)$. In fact, this S^1-action is hamiltonian with moment map given by projection onto the second factor,

$$x : Z \times (-\varepsilon, \varepsilon) \longrightarrow (-\varepsilon, \varepsilon) \ ,$$

as is easily verified:

$$\imath_{X^\#}\sigma = -\imath_{X^\#}d(x\alpha) = -\underbrace{\mathcal{L}_{X^\#}(x\alpha)}_{0} + d\underbrace{\imath_{X^\#}(x\alpha)}_{x} = dx \ .$$

Lemma 30.6 *There exists an equivariant symplectomorphism between a neighborhood of Z in M and a neighborhood of $Z \times \{0\}$ in $Z \times (-\varepsilon, \varepsilon)$, intertwining the two moment maps, for ε small enough.*

Proof. The inclusion $i_0 : Z \hookrightarrow Z \times (-\varepsilon, \varepsilon)$ as $Z \times \{0\}$ and the natural inclusion $i : Z \hookrightarrow M$ are S^1-equivariant coisotropic embeddings. Moreover, they satisfy $i_0^*\sigma = i^*\omega$ since both sides are equal to $\pi^*\omega_{\text{red}}$, and the moment maps coincide on Z because $i_0^*x = 0 = i^*\mu$. Replacing ε by a smaller positive number if necessary, the result follows from the equivariant version of the coisotropic embedding theorem stated in Section 8.3.[19] □

[19] The equivariant version of Theorem 8.6 needed for this purpose may be phrased as follows: *Let (M_0, ω_0), (M_1, ω_1) be symplectic manifolds of dimension $2n$, G a compact Lie group acting on (M_i, ω_i), $i = 0, 1$, in a hamiltonian way with moment maps μ_0 and μ_1, respectively, Z a manifold of dimension $k \geq n$ with a G-action, and $\iota_i : Z \hookrightarrow M_i$, $i = 0, 1$, G-equivariant coisotropic embeddings. Suppose that $\iota_0^*\omega_0 = \iota_1^*\omega_1$ and $\iota_0^*\mu_0 = \iota_1^*\mu_1$. Then there exist G-invariant neighborhoods $\mathcal{U}_0$ and $\mathcal{U}_1$ of $\iota_0(Z)$ and $\iota_1(Z)$ in M_0 and M_1, respectively, and a G-equivariant symplectomorphism $\varphi : \mathcal{U}_0 \to \mathcal{U}_1$ such that $\varphi \circ \iota_0 = \iota_1$ and $\mu_0 = \varphi^*\mu_1$.*

Therefore, in order to compare the reduced spaces

$$M_t = \mu^{-1}(t)/S^1 \ , \qquad t \approx 0 \ ,$$

we can work in $Z \times (-\varepsilon, \varepsilon)$ and compare instead the reduced spaces

$$x^{-1}(t)/S^1 \ , \qquad t \approx 0 \ .$$

Proposition 30.7 *The reduced space (M_t, ω_t) is symplectomorphic to*

$$(M_{\rm red}, \omega_{\rm red} - t\beta) \ ,$$

where β is the curvature form of the connection α.

Proof. By Lemma 30.6, (M_t, ω_t) is symplectomorphic to the reduced space at level t for the hamiltonian space $(Z \times (-\varepsilon, \varepsilon), \sigma, S^1, x)$. Since $x^{-1}(t) = Z \times \{t\}$, where S^1 acts on the first factor, all the manifolds $x^{-1}(t)/S^1$ are diffeomorphic to $Z/S^1 = M_{\rm red}$. As for the symplectic forms, let $\iota_t : Z \times \{t\} \hookrightarrow Z \times (-\varepsilon, \varepsilon)$ be the inclusion map. The restriction of σ to $Z \times \{t\}$ is

$$\iota_t^* \sigma = \pi^* \omega_{\rm red} - t d\alpha \ .$$

By definition of curvature, $d\alpha = \pi^*\beta$. Hence, the reduced symplectic form on $x^{-1}(t)/S^1$ is

$$\omega_{\rm red} - t\beta \ .$$

$\square$

In loose terms, Proposition 30.7 says that the reduced forms ω_t vary linearly in t, for t close enough to 0. However, the identification of M_t with $M_{\rm red}$ as abstract manifolds is not natural. Nonetheless, any two such identifications are isotopic. By the homotopy invariance of de Rham classes, we obtain:

Theorem 30.8 (Duistermaat-Heckman, 1982 [31]) *The cohomology class of the reduced symplectic form $[\omega_t]$ varies linearly in t. More specifically,*

$$[\omega_t] = [\omega_{\rm red}] + tc \ ,$$

where $c = [-\beta] \in H^2_{\rm deRham}(M_{\rm red})$ is the first Chern class of the S^1-bundle $Z \to M_{\rm red}$.

Remark on conventions. Connections on principal bundles are Lie algebra-valued 1-forms; cf. Section 25.2. Often the Lie algebra of S^1 is identified with $2\pi i\mathbb{R}$ under the exponential map $\exp : \mathfrak{g} \simeq 2\pi i\mathbb{R} \to S^1$, $\xi \mapsto e^\xi$. Given a principal S^1-bundle, by this identification the infinitesimal action maps the generator $2\pi i$ of $2\pi i\mathbb{R}$ to the generating vector field $X^\#$. A connection form A is then an imaginary-valued 1-form on the total space satisfying $\mathcal{L}_{X^\#} A = 0$ and $\imath_{X^\#} A = 2\pi i$. Its curvature form B is an imaginary-valued 2-form on the base satisfying $\pi^* B = dA$. By the Chern-Weil isomorphism, the **first Chern class** of the principal S^1-bundle is $c = [\frac{i}{2\pi} B]$.

In this lecture, we identify the Lie algebra of S^1 with $\mathbb{R}$ and implicitly use the exponential map $\exp : \mathfrak{g} \simeq \mathbb{R} \to S^1$, $t \mapsto e^{2\pi i t}$. Hence, given a principal S^1-bundle, the infinitesimal action maps the generator 1 of $\mathbb{R}$ to $X^\#$, and here a connection form α is an ordinary 1-form on the total space satisfying $\mathcal{L}_{X^\#}\alpha = 0$ and $\imath_{X^\#}\alpha = 1$. The curvature form β is an ordinary 2-form on the base satisfying $\pi^*\beta = d\alpha$. Consequently, we have $A = 2\pi i\alpha$, $B = 2\pi i\beta$ and the first Chern class is given by $c = [-\beta]$. $\diamondsuit$

30.3 Variation of the Symplectic Volume

Let (M, ω, S^1, μ) be a hamiltonian S^1-space of dimension $2n$ and let (M_x, ω_x) be its reduced space at level x. Proposition 30.7 or Theorem 30.8 imply that, for x in a sufficiently narrow neighborhood of 0, the symplectic volume of M_x,

$$\mathrm{vol}(M_x) = \int_{M_x} \frac{\omega_x^{n-1}}{(n-1)!} = \int_{M_{\mathrm{red}}} \frac{(\omega_{\mathrm{red}} - x\beta)^{n-1}}{(n-1)!} \ ,$$

is a polynomial in x of degree $n-1$. This volume can be also expressed as

$$\mathrm{vol}(M_x) = \int_Z \frac{\pi^*(\omega_{\mathrm{red}} - x\beta)^{n-1}}{(n-1)!} \wedge \alpha \ .$$

Recall that α is a chosen connection form for the S^1-bundle $Z \to M_{\mathrm{red}}$ and β is its curvature form.

Now we go back to the computation of the Duistermaat-Heckman measure. For a Borel subset U of $(-\varepsilon, \varepsilon)$, the Duistermaat-Heckman measure is, by definition,

$$m_{DH}(U) = \int_{\mu^{-1}(U)} \frac{\omega^n}{n!} \ .$$

Using the fact that $(\mu^{-1}(-\varepsilon, \varepsilon), \omega)$ is symplectomorphic to $(Z \times (-\varepsilon, \varepsilon), \sigma)$ and, moreover, they are isomorphic as hamiltonian S^1-spaces, we obtain

$$m_{DH}(U) = \int_{Z \times U} \frac{\sigma^n}{n!} \ .$$

Since $\sigma = \pi^*\omega_{\mathrm{red}} - d(x\alpha)$, its power is

$$\sigma^n = n(\pi^*\omega_{\mathrm{red}} - x d\alpha)^{n-1} \wedge \alpha \wedge dx \ .$$

By the Fubini theorem, we then have

$$m_{DH}(U) = \int_U \left[\int_Z \frac{\pi^*(\omega_{\mathrm{red}} - x\beta)^{n-1}}{(n-1)!} \wedge \alpha \right] \wedge dx \ .$$

Therefore, the Radon-Nikodym derivative of m_{DH} with respect to the Lebesgue measure, dx, is

$$f(x) = \int_Z \frac{\pi^*(\omega_{\mathrm{red}} - x\beta)^{n-1}}{(n-1)!} \wedge \alpha = \mathrm{vol}(M_x) \ .$$

The previous discussion proves that, for $x \approx 0$, $f(x)$ is a polynomial in x. The same holds for a neighborhood of any other regular value of μ, because we may change the moment map μ by an arbitrary additive constant.

Homework 23: S^1-Equivariant Cohomology

1. Let M be a manifold with a circle action and $X^\#$ the vector field on M generated by S^1. The algebra of S^1-**equivariant forms on** M is the algebra of S^1-invariant forms on M tensored with complex polynomials in x,

$$\Omega^\bullet_{S^1}(M) := (\Omega^\bullet(M))^{S^1} \otimes_{\mathbb{R}} \mathbb{C}[x] \ .$$

The product $\wedge$ on $\Omega^\bullet_{S^1}(M)$ combines the wedge product on $\Omega^\bullet(M)$ with the product of polynomials on $\mathbb{C}[x]$.

   (a) We grade $\Omega^\bullet_{S^1}(M)$ by adding the usual grading on $\Omega^\bullet(M)$ to a grading on $\mathbb{C}[x]$ where the monomial x has degree 2. Check that $(\Omega^\bullet_{S^1}(M), \wedge)$ is then a supercommutative graded algebra, i.e.,

$$\underline{\alpha} \wedge \underline{\beta} = (-1)^{\deg \underline{\alpha} \cdot \deg \underline{\beta}} \underline{\beta} \wedge \underline{\alpha}$$

   for elements of pure degree $\underline{\alpha}, \underline{\beta} \in \Omega^\bullet_{S^1}(M)$.

   (b) On $\Omega^\bullet_{S^1}(M)$ we define an operator

$$d_{S^1} := d \otimes 1 - \imath_{X^\#} \otimes x \ .$$

   In other words, for an elementary form $\underline{\alpha} = \alpha \otimes p(x)$,

$$d_{S^1} \underline{\alpha} = d\alpha \otimes p(x) - \imath_{X^\#}\alpha \otimes x p(x) \ .$$

   The operator d_{S^1} is called the **Cartan differentiation**. Show that d_{S^1} is a superderivation of degree 1, i.e., check that it increases degree by 1 and that it satisfies the *super* Leibniz rule:

$$d_{S^1}(\underline{\alpha} \wedge \underline{\beta}) = (d_{S^1}\underline{\alpha}) \wedge \underline{\beta} + (-1)^{\deg \underline{\alpha}}\ \underline{\alpha} \wedge d_{S^1}\underline{\beta} \ .$$

   (c) Show that $d^2_{S^1} = 0$.

   Hint: Cartan magic formula.

2. The previous exercise shows that the sequence

$$0 \longrightarrow \Omega^0_{S^1}(M) \xrightarrow{d_{S^1}} \Omega^1_{S^1}(M) \xrightarrow{d_{S^1}} \Omega^2_{S^1}(M) \xrightarrow{d_{S^1}} \dots$$

forms a graded complex whose cohomology is called the **equivariant cohomology**[20] of M for the given action of S^1. The kth equivariant cohomology group of M is

$$H^k_{S^1}(M) := \frac{\ker\ d_{S^1} : \Omega^k_{S^1} \longrightarrow \Omega^{k+1}_{S^1}}{\mathrm{im}\ d_{S^1} : \Omega^{k-1}_{S^1} \longrightarrow \Omega^k_{S^1}} \ .$$

[20] The **equivariant cohomology** of a topological space M endowed with a continuous action of a topological group G is, by definition, the cohomology of the diagonal quotient $(M \times EG)/G$, where EG is the *universal bundle* of G, i.e., EG is a contractible space where G acts freely. H. Cartan [21, 59] showed that, for the action of a compact Lie group G on a manifold M, the de Rham model $(\Omega^\bullet_G(M), d_G)$ computes the equivariant cohomology, where $\Omega^\bullet_G(M)$ are the *G-equivariant forms* on M. [8, 9, 29, 54] explain equivariant cohomology in the symplectic context and [59] discusses equivariant de Rham theory and many applications.

(a) What is the equivariant cohomology of a point?

(b) What is the equivariant cohomology of S^1 with its multiplication action on itself?

(c) Show that the equivariant cohomology of a manifold M with a free S^1-action is isomorphic to the ordinary cohomology of the quotient space M/S^1.

Hint: Let $\pi : M \to M/S^1$ be projection. Show that

$$\begin{array}{rccc} \pi^* : & H^\bullet(M/S^1) & \longrightarrow & H^\bullet_{S^1}(M) \\ & [\alpha] & \longmapsto & [\pi^*\alpha \otimes 1] \end{array}$$

is a well-defined isomorphism. It helps to choose a connection on the principal S^1-bundle $M \to M/S^1$, that is, a 1-form θ on M such that $\mathcal{L}_{X^\#}\theta = 0$ and $\imath_{X^\#}\theta = 1$. Keep in mind that a form β on M is of type $\pi^*\alpha$ for some α if and only if it is *basic*, that is $\mathcal{L}_{X^\#}\beta = 0$ and $\imath_{X^\#}\beta = 0$.

3. Suppose that (M, ω) is a symplectic manifold with an S^1-action. Let $\mu \in C^\infty(M)$ be a real function. Consider the equivariant form

$$\underline{\omega} := \omega \otimes 1 + \mu \otimes x \ .$$

Show that $\underline{\omega}$ is **equivariantly closed**, i.e., $d_{S^1}\underline{\omega} = 0$ if and only if μ is a moment map. The equivariant form $\underline{\omega}$ is called the **equivariant symplectic form**.

4. Let M^{2n} be a compact oriented manifold, not necessarily symplectic, acted upon by S^1. Suppose that the set M^{S^1} of fixed points for this action is finite. Let $\alpha^{(2n)}$ be an S^1-invariant form which is the top degree part of an equivariantly closed form of even degree, that is, $\alpha^{(2n)} \in \Omega^{2n}(M)^{S^1}$ is such that there exists $\underline{\alpha} \in \Omega^\bullet_{S^1}(M)$ with

$$\underline{\alpha} = \alpha^{(2n)} + \alpha^{(2n-2)} + \ldots + \alpha^{(0)}$$

where $\alpha^{(2k)} \in (\Omega^{2k}(M))^{S^1} \otimes \mathbb{C}[x]$ and $d_{S^1}\underline{\alpha} = 0$.

(a) Show that the restriction of $\alpha^{(2n)}$ to $M \setminus M^{S^1}$ is exact.

Hint: The generator $X^\#$ of the S^1-action does not vanish on $M \setminus M^{S^1}$. Hence, we can define a *connection* on $M \setminus M^{S^1}$ by $\theta(Y) = \frac{\langle Y, X^\# \rangle}{\langle X^\#, X^\# \rangle}$, where $\langle \cdot, \cdot \rangle$ is some S^1-invariant metric on M. Use $\theta \in \Omega^1(M \setminus M^{S^1})$ to chase the primitive of $\alpha^{(2n)}$ all the way up from $\alpha^{(0)}$.

(b) Compute the integral of $\alpha^{(2n)}$ over M.

Hint: Stokes' theorem allows to *localize* the answer near the fixed points.

This exercise is a very special case of the Atiyah-Bott-Berline-Vergne localization theorem for equivariant cohomology [8, 14].

5. What is the integral of the symplectic form ω on a surface with a hamiltonian S^1-action, knowing that the S^1-action is free outside a finite set of fixed points?

Hint: Exercises 3 and 4.

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Lectures on Symplectic Topology* Park City 1997

Dusa McDuff

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Instead of trying to give a comprehensive overview of the subject, I will concentrate on explaining a few key concepts and their implications, notably "Moser's argument" (or the homotopy method) in Lecture 2, capacity in Lecture 4 and Gromov's proof of the nonsqueezing theorem in Lecture 5. The first exhibits the flexibility of symplectic geometry while the latter two show its rigidity. Quite a lot of time is spent on the linear theory since this is the basis of everything else. The last lecture sketches the bare outlines of the theory of J-holomorphic spheres, to give an introduction to a fascinating and powerful technique.

Throughout the notation is consistent with that used in [MS1] and [MS2]. Readers may consult those books for more details on almost every topic mentioned here, as well as for a much fuller list of references.

I wish to thank Jenn Slimowitz for taking the notes and making useful comments on an earlier version of this manuscript.

1 Lecture 1: Basics

Symplectic geometry is the geometry of a skew-symmetric form. Let M be a manifold of dimension $2n$. A symplectic form (or symplectic structure) on M is a closed nondegenerate 2-form ω. Nondegeneracy means that $\omega(v,w) = 0$ for all $w \in TM$ only when $v = 0$. Therefore the map

$$I_\omega : T_pM \to T_p^*M : \quad v \mapsto \iota(v)\omega = \omega(v,\cdot)$$

is injective and hence an isomorphism. The basic example is

$$\omega_0 = dx_1 \wedge dy_1 + \ldots + dx_n \wedge dy_n$$

on $\mathbf{R}^{2n}$.

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Here are some fundamental questions.

• Can one get a geometric understanding of the structure defined by a symplectic form?

• Which manifolds admit symplectic forms?

• When are two symplectic manifolds (eg two open sets in $(\mathbf{R}^{2n}, \omega_0)$) symplectomorphic?

Definition 1.1 A diffeomorphism $\phi : (M, \omega) \to (M'\omega')$ is called a symplectomorphism if $\phi^*(\omega') = \omega$. The group of all symplectomorphisms is written $\mathrm{Symp}(M)$.

Existence of many symplectomorphisms

Given a function $H : M \to \mathbf{R}$ – often called the *energy* function or *Hamiltonian* – let X_H be the vector field defined by

$$\iota(X_H)\omega = dH.$$

(Observe that $X_H = (I_\omega)^{-1}(dH)$ is well defined because of the nondegeneracy of ω. Also, many authors put a minus sign in the above equation.) When M is compact, X_H integrates to a flow ϕ_t^H that preserves ω because

$$\mathcal{L}_{X_H}\omega = \iota(X_H)d\omega + d(\iota(X_H)\omega) = ddH = 0.$$

Here we have used both that ω is closed and that it is nondegenerate. The calculation

$$dH(X_H) = (\iota(X_H))\,\omega(X_H) = \omega(X_H, X_H) = 0$$

shows that X_H is tangent to the level sets of H. Thus the flow of ϕ_t^H preserves the function H.

Example 1.2 With $H : \mathbf{R}^2 \to \mathbf{R}$ given by $H(x, y) = y$ and with $\omega = dx \wedge dy$, we get

$$X_H = \frac{\partial}{\partial x} \quad \text{and} \quad \phi_t^H(x, y) = (x + t, y).$$

If $H : \mathbf{R}^{2n} \to R$ and $\omega = \omega_0$, we get

$$X_H = \sum_i \frac{\partial H}{\partial y_i}\frac{\partial}{\partial x_i} - \frac{\partial H}{\partial x_i}\frac{\partial}{\partial y_i}.$$

The solution curves $(x_i(t), y_i(t)) = \phi_t(x(0), y(0))$ satisfy Hamilton's equations

$$\dot{x}_i = \frac{\partial H}{\partial y_i}, \quad \dot{y}_i = -\frac{\partial H}{\partial x_i}.$$

With $H = \frac{1}{2}\sum x_j^2 + y_j^2$, the orbits of this action are circles. In complex coordinates $z_j = x_j + iy_j$ we have

$$\phi_t^H(z_1, \ldots, z_n) = (e^{-it}z_1, \ldots, e^{-it}z_n).$$

Thus we get a circle action on $\mathbf{R}^{2n} = \mathbf{C}^n$. The function H that generates it is called the *moment map* of this action.

Exercise 1.3 Often it is useful to consider Hamiltonian functions that depend on time: viz:

$$H : M \times [0,1] \to \mathbf{R}, \quad H(p,t) = H_t(p).$$

Then one defines X_{H_t} as before, and gets a smooth family ϕ_t^H of symplectomorphisms with $\phi_0^H = id$ which at time t are tangent to X_{H_t}:

$$\frac{d}{dt}(\phi_t^H(p)) = X_{H_t}(\phi_t^H(p)), \quad p \in M, t \in [0,1].$$

Such a family is called a *Hamiltonian isotopy.* Show that the set of all time-1 maps ϕ_1^H forms a subgroup of $\mathrm{Symp}(M)$. This is called the group of *Hamiltonian symplectomorphisms* $\mathrm{Ham}(M)$. Its elements are also often called *exact* symplectomorphisms.

Linear symplectic geometry

To get a better understanding of what is going on, let's now look at what happens at a point. As we shall see, linear symplectic geometry contains a surprising amount of structure. Moreover, most of this structure at a point corresponds very clearly to nonlinear phenomena. One example of this is Darboux's theorem. We shall see in a minute that there is only one symplectic structure on a given (finite-dimensional) vector space, up to isomorphism. Darboux's theorem says that, locally, there is only one symplectic form on a smooth manifold. In other words, every symplectic form ω on M is locally symplectomorphic to the standard form ω_0 on $\mathbf{R}^{2n}$. One might think that this implies there is no interesting local structure (just as if one were in the category of smooth manifolds.) But this is false, since, as we shall see, the standard structure ω_0 on $\mathbf{R}^{2n}$ is itself very interesting.

So let V be a vector space (over $\mathbf{R}$) with a nondegenerate skew bilinear form ω. Thus

$$\omega(v,w) = -\omega(w,v), \quad \omega(v,w) = 0 \text{ for all } v \in V \text{ implies } w = 0.$$

The basic example is $\mathbf{R}^{2n}$ with the form ω_0 considered as a bilinear form.

Given a subspace W define its *symplectic orthogonal* W^ω by:

$$W^\omega = \{v : \omega(v,w) = 0 \text{ for all } w \in W\}.$$

Lemma 1.4 $\dim W + \dim W^\omega = \dim V$.

Proof: Check that the map

$$I : V \to W^* : \quad v \mapsto \omega(v, \cdot)|_W$$

is surjective with kernel W^ω. $\square$

A subspace W is said to be *symplectic* if $\omega|_W$ is nondegenerate. It is easy to check that:

Lemma 1.5 *W is symplectic $\iff W \cap W^\omega = \{0\} \iff V \cong W \oplus W^\omega$.*

Proof: Exercise. $\square$

Further we say that W is *isotropic* iff $W \subset W^\omega$ and that W is *Lagrangian* iff $W = W^\omega$. In the latter case $\dim W = n$ by Lemma 1.4.

Proposition 1.6 *Every symplectic vector space is isomorphic to* $(\mathbf{R}^{2n}, \omega_0)$.

Proof: A basis $u_1, v_1, \ldots, u_n, v_n$ of (V, ω) is said to be standard if, for all i, j,

$$\omega(u_i, u_j) = \omega(v_i, v_j) = 0, \quad \omega(u_i, v_j) = \delta_{ij}.$$

Clearly $(\mathbf{R}^{2n}, \omega_0)$ posses such a basis. Further any linear map that takes one such basis into another preserves the symplectic form. Hence we just have to construct a standard basis for (V, ω).

To do this, start with any $u_1 \neq 0$. Choose v so that $\omega(u_1, v) = \lambda \neq 0$ and set $v_1 = v/\lambda$. Let W be the span of u_1, v_1. Then W is symplectic, so $V = W \oplus W^\omega$ by Lemma 1.5. By induction, we may assume that W^ω has a standard basis $u_2, v_2, \ldots, u_n, v_n$. It is easy to check that adding u_1, v_1 to this makes a standard basis for (V, ω). $\square$

Exercise 1.7 (i) Show that if L is a Lagrangian subspace of the symplectic vector space (V, ω), any basis $u_1, \ldots, u_n$ for V can be extended to a standard basis $u_1, v_1, \ldots, u_n, v_n$ for (V, ω). (Hint: choose $v_1 \in W^\omega$ where W is the span of $u_2, \ldots, u_n$.)
(ii) Show that (V, ω) is symplectomorphic to the space $(L \oplus L^*, \tau)$ where

$$\tau((\ell, v^*), (\ell', v'^*)) = v'^*(\ell) - v^*(\ell').$$

The next exercise connects the linear theory with the Hamiltonian flows we were considering earlier.

Exercise 1.8 (i) Check that every codimension 1 subspace W is *coisotropic* in the sense that $W^\omega \subset W$. Note that W^ω is 1-dimensional. For obvious reasons its direction is called the *null* direction in W.
(ii) Given $H : M \to \mathbf{R}$ let $Q = H^{-1}(c)$ be a regular level set. Show that $X_H(p) \in (T_pQ)^\omega$ for all $p \in Q$. Thus the direction of X_H is determined by the level set Q. (Its size is determined by H.)

Exercise 1.9 Show that if ω is any symplectic form on a vector space of dimension $2n$ then the nth exterior power ω^n does not vanish. Deduce that the nth exterior power $\Omega = \omega^n$ of any symplectic form ω on a $2n$-dimensional manifold M is a volume form. Further every symplectomorphism of M preserves this volume form.

The cotangent bundle

This is another basic example of a symplectic manifold. The cotangent bundle TX carries a canonical 1-form λ_{can} defined by

$$(\lambda_{can})_{(x,v^*)}(w) = v^*(\pi_*(w)), \quad \text{for} \quad w \in T_{(x,v^*)}(T^*X),$$

where $\pi : TX \to X$ is the projection. (Here x is a point in X and $v^* \in T_x^*X$.) Then $\Omega_{can} = -d\lambda_{can}$ is a symplectic form. Clearly the fibers of $\pi : T^*X \to X$ are Lagrangian with respect to Ω_{can}, as is the zero section. Moreover, it is not hard to see that:

Lemma 1.10 *Let $\sigma_\alpha : X \to T^*X$ be the section determined by the 1-form α on X. Then $\sigma_\alpha^*(\lambda_{can}) = \alpha$. Hence the manifold $\sigma_\alpha(X)$ is Lagrangian iff α is closed.*

Exercise 1.11 (i) Take a function H on X and let $\widetilde{H} = H \circ \pi$. Describe the resulting flow on T^*X.
(ii) Every diffeomorphism ϕ of X lifts to a diffeomorphism $\widetilde{\phi}$ of T^*X by

$$\widetilde{\phi}(x, v^*) = (\phi(x), (\phi^{-1})^* v^*).$$

Show that $\widetilde{\phi}^*(\lambda_{can}) = \lambda_{can}$.
(iii) Let ϕ_t be the flow on X generated by a vector field Y. If $\widetilde{\phi}_t$ is the lift of this flow to T^*X show that the Hamiltonian $H : T^*X \to \mathbf{R}$ that generates this flow has the form

$$H(x, v^*) = v^*(Y(x)).$$

Hint: use (ii) and write down the defining equation for $\widetilde{Y} = X_H$ in terms of λ_{can}.

2 Lecture 2: Moser's argument

In this lecture I will show you a powerful argument due to Moser [M] which exhibits the "flabbiness" or lack of local structure in symplectic geometry. Here is the basic argument.

Lemma 2.1 *Suppose that ω_t is a family of symplectic forms on a closed manifold M whose time derivative is exact. Thus*

$$\dot{\omega}_t = d\sigma_t,$$

where σ_t is a smooth family of 1-forms. Then there is a smooth family of diffeomorphisms ϕ_t with $\phi_0 = id$ such that

$$\phi_t^*(\omega_t) = \omega_0.$$

Proof: We construct ϕ_t as the flow of a time-dependent vector field X_t. We know

$$\begin{aligned}
\phi_t^*(\omega_t) = \omega \quad &\Longleftrightarrow \quad \frac{d}{dt}(\phi_t^* \omega_t) = 0 \\
&\Longleftrightarrow \quad \phi_t^*\left(\dot{\omega}_t + \mathcal{L}_{X_t}\omega_t\right) = 0 \\
&\Longleftrightarrow \quad \dot{\omega}_t + \iota(X_t)d\omega_t + d(\iota(X_t)\omega_t) = 0 \\
&\Longleftrightarrow \quad d(\sigma_t + \iota(X_t)\omega_t) = 0.
\end{aligned}$$

This last equation will hold if $\sigma_t + \iota(X_t)\omega_t = 0$. Observe that for any choice of 1-forms σ_t the latter equation can always be solved for X_t because of the nondegeneracy of the ω_t. Therefore, reading this backwards, we see that we can always find an X_t and hence a family ϕ_t that will do what we want. $\square$

Remark 2.2 (i) The condition $\dot{\omega}_t = d\sigma_t$ is equivalent to requiring that the cohomology class $[\omega_t]$ be constant. For if this class is constant the derivative $\dot{\omega}_t$ is exact for each t so that for each t there is a form σ_t with $\dot{\omega}_t = d\sigma_t$. Thus the problem is to construct these σ_t so that they depend smoothly on t. This can be accomplished in various ways (eg by using Hodge theory, or see Bott–Tu [BT].)

(ii) The previous lemma uses the fact that the forms ω_t are closed and the fact that the equation $\sigma_t + \iota(X_t)\omega_t$ can always be solved. This last is possible only for nondegenerate 2-forms and for nonvanishing top dimensional forms. In particular the argument does apply to volume forms. Note that this case is very different from the symplectic case because there is never any problem in constructing homotopies of volume forms. Indeed, the set of volume forms in a given cohomology class is convex: if ω_0, ω_1 are volume forms with the same orientation the forms $(1-t)\omega_0 + t\omega_1, 0 \le t \le 1$ are also volume forms. Thus all such forms are diffeomorphic. This is not true for symplectic forms. (Exercise: find an example.)

The previous remarks show that one cannot get interesting new symplectic structures by deforming a given structure within its cohomology class, ie:

Corollary 2.3 (Moser's stability theorem) *If $\omega_t, 0 \le t \le 1$, is a family of cohomologous symplectic forms on a closed manifold M then there is an isotopy ϕ_t with $\phi_0 = id$ such that $\phi_t^*(\omega_t) = \omega_0$ for all t.*

Other corollaries apply Moser's argument to noncompact manifolds M. In this case, to be able to define the flow of the vector field X_t one must be very careful to control its support. Since $X_t = 0 \Longleftrightarrow \sigma_t = 0$ the problem becomes that of controlling the support of the forms σ_t. We illustrate what is involved by proving Darboux's theorem.

Theorem 2.1 (Darboux) *Every symplectic form on M is locally diffeomorphic to the standard form ω_0 on $\mathbf{R}^{2n}$.*

Proof: Given a point p on M let $\psi : nbhd(p) \to \mathbf{R}^{2n}$ be a local chart that takes p to the origin 0. We have to show that the form ω' obtained by pushing ω forward by ψ is diffeomorphic to the standard form ω_0 near 0. By Proposition 1.6 we can choose ψ so that $\omega' = \omega_0$ at the point 0. Now consider the family

$$\omega_t = (1-t)\omega_0 + t\omega'.$$

Since $\omega_t = \omega_0$ at 0 by construction and nondegeneracy is an open condition, there is some open ball U containing 0 on which all these forms are nondegenerate. Observe that $\dot{\omega}_t = \omega' - \omega_0$. Since U is contractible there is a 1-form σ such that $d\sigma = \omega' - \omega_0$. Moreover, by subtracting the constant form $\sigma(0)$ we can arrange that $\sigma = 0$ at the point 0. Thus the corresponding family of vector fields X_t vanishes at 0. Let ϕ_t be the partially defined flow of X_t. Since 0 is a fixed point, it is easy to see that there is a very small neighborhood V of 0 such that the orbits $\phi_t(p), 0 \le t \le 1$, of the points p in V remain inside U. Thus the ϕ_t are defined on V and $\phi_1^*(\omega') = \omega_0$. □

For another proof of Darboux's theorem (together with much else) see Arnold [A]. The next applications apply this idea to neighborhoods of submanifolds of M. The basic proposition is:

Proposition 2.4 *Let ω_0, ω_1 be two symplectic forms on M whose restrictions to the full tangent bundle of M along some submanifold Q of M agree: ie*

$$\omega_0|_{T_pM} = \omega_1|_{T_pM} \quad for \quad p \in Q.$$

Then there is a diffeomorphism ϕ of M such that

$$\phi(p) = p, \quad for \quad p \in Q, \qquad \phi^*(\omega_1) = \omega_0 \quad near\ Q.$$

Proof: Again look at the forms $\omega_t = (1-t)\omega_0 + t\omega_1$. As before these are nondegenerate in some neighborhood of Q. Moreover

$$\dot{\omega}_t = \omega_1 - \omega_0$$

is exact near Q. If we find a 1-form σ that vanishes at all points of Q and is such that $d\sigma = \omega_1 - \omega_0$, then the corresponding vector fields X_t will also vanish along Q and will integrate to give the required diffeomorphisms ϕ_t near Q. Such a form σ can be constructed by suitably adapting the usual proof of Poincaré's lemma: see [BT], for example. $\square$

We can get better results by considering special submanifolds Q. Consider for example a symplectic submanifold Q of (M,ω).[1] Then by Lemma 1.5 the normal bundle $\nu_Q = TM/TQ$ of Q may be identified with the symplectic orthogonal $(TQ)^\omega$ to TQ. Moreover ω restricts to give a symplectic structure on ν_Q: this means that each fiber has a natural symplectic structure that is preserved by the transition functions of the bundle. (See Lecture 3.)

Corollary 2.5 (Symplectic neighborhood theorem) *If ω_0, ω_1 are symplectic forms on M that restrict to the same symplectic form ω_Q on the submanifold Q, then there is a diffeomorphism ϕ of M that fixes the points of Q and is such that $\phi^*(\omega_1) = \omega_0$ near Q provided that ω_0 and ω_1 induce isomorphic symplectic structures on the normal bundle ν_Q.*

Proof: The hypothesis implies that there is a linear isomorphism

$$L : TM|_Q \to TM|_Q$$

that is the identity on the subbundle TQ and is such that $L^*(\omega_1) = \omega_0$. It is not hard to see that L may be realised by a diffeomorphism ψ of M that fixes the points of Q. In other words there is a diffeomorphism with $d\psi_p = L_p$ at each point of Q. Then

$$\omega_0|_{T_pM} = \psi^*\omega_1|_{T_pM}, \quad p \in Q,$$

and so the result follows from Proposition 2.4. $\square$

We will see in the next lecture that giving an isomorphism class of symplectic structures on a bundle is equivalent to giving an isomorphism class of complex structures on it. Hence the normal data needed to make ω_0 and ω_1 agree near Q is quite weak. For example, if Q has codimension 2, all we need to check is that the two forms induce the same orientation on the normal bundle since the Euler class (or first Chern class) of ν_Q is determined up to sign by its topology.

Another important case is when Q is Lagrangian. In this case one can check that the normal bundle ν_Q is canonically isomorphic to the dual bundle TQ^*. Moreover this dual bundle is also Lagrangian. (Cf Exercise 1.7.) Thus when Q is Lagrangian with respect to both ω_0 and ω_1 there always is a linear isomorphism

$$L : TM|_Q \to TM|_Q$$

that is the identity on the subbundle TQ and is such that $L^*(\omega_1) = \omega_0$. Moreover, just as in the case of Darboux's theorem there is a standard model for Q, namely the zero section in the cotangent bundle (T^*Q, Ω_{can}). Thus we have:

[1] A submanifold Q of M is called symplectic if ω restricts to a symplectic form on Q, or, equivalently, if all its tangent spaces $T_pQ, p \in Q$, are symplectic subspaces. Similarly, Q is Lagrangian if $\omega|_Q \equiv 0$ and $\dim Q = n$.

Corollary 2.6 (Weinstein's Lagrangian neighborhood theorem) *If Q is a Lagrangian submanifold of (M, ω) there is a neighborhood of Q that is symplectomorphic to a neighborhood of the zero section in the cotangent bundle (T^*Q, Ω_{can}).*

Exercise 2.7 Given any two diffeomorphic closed smooth domains U, V in $\mathbf{R}^n$ that have the same total volume, show that there is a diffeomorphism $\phi : U \to V$ that preserves volume. Hint: first choose any diffeomorphism $\psi : U \to V$ and look at the forms $\omega_0, \omega_1 = \psi^*(\omega_0)$ on U. Adjust ψ by hand near the boundary ∂U so that $\omega_0 = \omega_1$ at all points on ∂U. Then use a Moser type argument to make the forms agree in the interior.

The last important result of this kind is the symplectic isotopy extension theorem due to Banyaga. The proof is left as an exercise.

Proposition 2.8 (Isotopy extension) *Let Q be a compact submanifold of (M, ω) and suppose that $\phi_t : M \to M$ is a family of diffeomorphisms of M starting at $\phi_0 = id$ such that $\phi_t^*(\omega) = \omega$ near Q. Then, if for every relative cycle $Z \in H_2(M, Q)$*

$$\int_Z \phi_t^*(\omega) = \int_Z \omega,$$

there is a family of symplectic diffeomorphisms ψ_t and a neighborhood U of Q such that $\psi_t(p) = \phi_t(p)$ for all $p \in U$.

3 Lecture 3: The linear theory

We will consider the vector space $\mathbf{R}^{2n}$ with its standard symplectic form ω_0. This may be written in vector notation as

$$\omega_0(v, w) = w^T J_0 v,$$

where w^T is the transpose of the column vector w and J_0 is the block diagonal matrix

$$J_0 = \operatorname{diag}\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right).$$

The symplectic linear group $\mathrm{Sp}(2n, \mathbf{R})$ (sometimes written $\mathrm{Sp}(2n)$) consists of all matrices A such that

$$\omega_0(Av, Aw) = \omega_0(v, w),$$

or equivalently of all A such that

$$A^T J_0 A = J_0.$$

Clearly $\mathrm{Sp}(2n, \mathbf{R})$ is a group. The identity $J_0^T = -J_0 = J_0^{-1}$ gives rise to interesting algebraic properties of this group. Firstly, it is closed under transpose, and secondly every symplectic matrix is conjugate to its inverse transpose $(A^{-1})^T$. The former statement is proved by inverting the identity

$$(A^{-1})^T J_0 A^{-1} = J_0,$$

and the second by multiplying the defining equation $A^T J_0 A = J_0$ on the left by $(J_0)^{-1}(A^T)^{-1}$.

Exercise 3.1 Show that if $\lambda \in \mathbf{C}$ is an the eigenvalue of a symplectic matrix A then so are $1/\lambda, \bar{\lambda}, 1/\bar{\lambda}$. What happens when $\lambda \in \mathbf{R}$, or $|\lambda| = 1$?

Recall that we are identifying $\mathbf{C}^n$ with $\mathbf{R}^{2n}$ by setting $z_j = x_j + iy_j$. Under this identification, J_0 corresponds to multiplication by i. Hence we may consider $\mathrm{GL}(n, \mathbf{C})$ to be the subgroup of $\mathrm{GL}(2n, \mathbf{R})$ consisting of all matrices A such that $AJ_0 = J_0 A$.

Exercise 3.2 Given an $n \times n$ matrix A with complex entries, find a formula for the corresponding real $2n \times 2n$ matrix.

Lemma 3.3

$$\mathrm{Sp}(2n, \mathbf{R}) \cap \mathrm{O}(2n) = \mathrm{Sp}(2n, \mathbf{R}) \cap \mathrm{GL}(n, \mathbf{C}) = \mathrm{O}(2n) \cap \mathrm{GL}(n, \mathbf{C}) = \mathrm{U}(n).$$

Proof: Exercise. □

Our first main result is that $\mathrm{U}(n)$ is a maximal compact subgroup of $\mathrm{Sp}(2n)$ and hence, by the general theory of Lie groups, the quotient space $\mathrm{Sp}(2n)/\mathrm{U}(n)$ is contractible. The first statement above means that any compact subgroup G of $\mathrm{Sp}(2n)$ is conjugate to a subgroup of $\mathrm{U}(n)$. We won't prove this here since we will not use it. However, we will give an independent proof of the second.

To begin, recall the usual proof that $\mathrm{GL}(n, \mathbf{R})/\mathrm{O}(n)$ is contractible. One looks at the polar decomposition

$$A = (AA^T)^{\frac{1}{2}} O$$

of A. Here $P = AA^T$ is a symmetric, positive definite[2] matrix and hence diagonalises with real positive eigenvalues. In other words P may be written $X \Lambda X^{-1}$ where Λ is a diagonal matrix with positive entries. One can therefore define an arbitrary real power P^α of P by

$$P^\alpha = X \Lambda^\alpha X^{-1}.$$

[2]Usually a positive definite matrix is assumed to be symmetric (ie $P = P^T$). However, in symplectic goemetry one does come across matrices that satisfy the positivity condition $v^T P v > 0$ for all nonzero v but that are not symmetric. Hence it is better to mention the symmetry explicitly.

It is easy to check that

$$O = (AA^T)^{-\frac{1}{2}}A$$

is orthogonal. Hence one can define a deformation retraction of $\mathrm{GL}(2n, \mathbf{R})$ onto $\mathrm{O}(2n)$ by

$$A \mapsto (AA^T)^{\frac{1-t}{2}}O, \quad 0 \le t \le 1.$$

The claim is that this argument carries over to the symplectic context. To see this we need to show:

Lemma 3.4 *If $P \in \mathrm{Sp}(2n)$ is positive and symmetric then all its powers P^α, $\alpha \in \mathbf{R}$ are also symplectic.*

Proof: Let V_λ be the eigenspace of P corresponding to the eigenvalue λ. Then, if $v_\lambda \in V_\lambda, v_{\lambda'} \in V_{\lambda'}$,

$$\omega_0(v_\lambda, v_{\lambda'}) = \omega_0(Pv_\lambda, Pv_{\lambda'}) = \omega_0(\lambda v_\lambda, \lambda' v_{\lambda'}) = \lambda\lambda'\omega_0(v_\lambda, v_{\lambda'}).$$

Hence $\omega_0(v_\lambda, v_{\lambda'}) = 0$ unless $\lambda' = 1/\lambda$. In other words the eigenspaces $V_\lambda, V_{\lambda'}$ are symplectically orthogonal unless $\lambda' = 1/\lambda$. To check that P^α is symplectic we just need to know that

$$\omega_0(P^\alpha v_\lambda, P^\alpha v_{\lambda'}) = \omega_0(v_\lambda, v_{\lambda'}),$$

for all eigenvectors $v_\lambda, v_{\lambda'}$. But this holds since

$$\omega_0(P^\alpha v_\lambda, P^\alpha v_{\lambda'}) = \omega_0(\lambda^\alpha v_\lambda, (\lambda')^\alpha v_{\lambda'}) = (\lambda\lambda')^\alpha \omega_0(v_\lambda, v_{\lambda'}) = \omega_0(v_\lambda, v_{\lambda'}).$$

(Observe that everything vanishes when $\lambda\lambda' \neq 1$!) $\square$

Thus the argument given above in the real context extends to the symplectic context, and we have:

Proposition 3.5 *The subgroup $\mathrm{U}(n)$ is a deformation retract of $\mathrm{Sp}(2n)$.*

ω-compatible almost complex structures

An almost complex structure on a vector space V is a linear automorphism $J : V \to V$ with J^2 equal to $-\mathbb{1}$. Thus one can define an action of $\mathbf{C}$ on V by

$$(a + ib)v = a + Jb,$$

so that (V, J) is a complex vector space. If V also has a symplectic form ω we say that ω and J are compatible if for all nonzero v, w,

$$\omega(Jv, Jw) = \omega(v, w), \quad \omega(v, Jv) > 0.$$

The basic example is the pair (ω_0, J_0) on $\mathbf{R}^{2n}$.

Any such pair (J, ω) defines a corresponding metric (inner product) g_J by

$$g_J(v, w) = \omega(v, Jw).$$

This is symmetric because

$$g_J(w, v) = \omega(w, Jv) = \omega(Jw, J^2v) = \omega(Jw, -v) = \omega(v, Jw) = g_J(v, w).$$

Exercise 3.6 Show that J is ω-compatible iff there is a standard basis of the form

$$u_1, v_1 = Ju_1, \ldots, u_n, v_n = Ju_n.$$

Deduce that there is a linear symplectomorphism $\Phi : (\mathbf{R}^{2n}, \omega_0) \to (V, \omega)$ such that $J = \Phi J_0 \Phi^{-1}$.

Proposition 3.7 *The space of ω-compatible almost complex structures J on V is contractible.*

Proof: Without loss of generality we may suppose that (V, ω) is standard Euclidean space $(\mathbf{R}^{2n}, \omega_0)$. Clearly, $\mathrm{Sp}(2n)$ acts on the space $\mathcal{J}(\omega_0)$ of ω_0-compatible almost complex structures on $\mathbf{R}^{2n}$ by

$$A \cdot J = AJA^{-1}.$$

The preceding exercise shows that this action is transitive, since every J may be written as $J = AJ_0A^{-1}$ and so is in the orbit of J_0. The kernel of the action consists of elements that commute with J_0, in other words of unitary transformations. (See Lemma 3.3.) Thus $\mathcal{J}(\omega_0)$ is isomorphic to the homogeneous space $\mathrm{Sp}(2n)/\mathrm{U}(n)$ and so is contractible by Proposition 3.5. $\square$

Exercise 3.8 Define the form ω_B by

$$\omega_B(v, w) = w^T BJ_0v.$$

Under what conditions on B is ω_B compatible with J_0? Deduce that for each fixed almost complex structure J on V the space of compatible ω is contractible.

Vector bundles

A (real) $2n$-dimensional vector bundle $\pi : E \to B$ is said to be *symplectic* if it has an atlas of local trivializations $\tau_\alpha : \pi^{-1}U_\alpha \to \mathbf{R}^{2n} \times U_\alpha$ such that for all $p \in U_\alpha \cap U_\beta$ the corresponding transition map

$$\phi_{\alpha,\beta}(p) = \tau_\alpha \circ (\tau_\beta)^{-1} : \mathbf{R}^{2n} \times p \to \mathbf{R}^{2n} \times p$$

is a linear symplectomorphism. Using a parametized version of Proposition 1.6, one can easily show that this is equivalent to requiring that there is a bilinear skew form σ on E that is nondegenerate on each fiber. For, given such σ, one can use Proposition 1.6 to choose the trivializations τ_α so that at each point $p \in B$ they pull back the standard form on $\mathbf{R}^{2n} \times p$ to the given form $\sigma(p)$. Then the transition maps have to be symplectic. Conversely, if the transition maps are symplectic the pull backs of the standard form by the τ_α agree on the overlaps to give a well-defined global form σ.

A σ-compatible almost complex structure J on E is an automorphism of E that at each point $p \in B$ is a $\sigma(p)$-compatible almost complex structure on the fiber.

Proposition 3.9 *Every symplectic vector bundle (E, σ) admits a contractible family of compatible almost complex structures, and hence gives rise to a complex structure on E that is unique up to isomorphism. Conversely, any complex vector bundle admits a contractible family of compatible symplectic forms, and hence has a symplectic structure that is unique up to isomorphism. Thus classifying isomorphism classes of symplectic bundles is the same as classifying isomorphism classes of complex bundles.*

Proof: (Sketch) One way of proving the first statement is to note that the space of compatible almost complex structures on E forms a fiber bundle over B that, by Proposition 3.7, has contractible fibers. Another way is to start from the contractible space of inner products on E and to show that each such inner product gives rise to a unique almost complex structure. (The details of this second argument can be found in 2.5,6 of [MS2].) The second statement follows by similar arguments, using Exercise 3.8. $\Box$

Clearly the tangent bundle TM of every symplectic manifold (M, ω) is a symplectic vector bundle with symplectic structure given by ω. The previous proposition shows that TM has a well-defined complex structure, and so, in particular, has Chern classes $c_i(TM)$. The first Chern class $c_1(TM) \in H^2(M, \mathbf{Z})$ is a particularly useful class as it enters into the dimension formula for moduli spaces of J-holomorphic curves. (See Lecture 5.)

The Lagrangian Grassmannian

Another interesting piece of linear structure concerns the space $\mathcal{L}(n)$ of all Lagrangian subspaces of a $2n$-dimensional symplectic vector space (V, ω). This is also known as the Lagrangian Grassmannian. Here we will consider the space of unoriented Lagrangian subspaces, but it is easy to adapt our remarks to the oriented case.

Lemma 3.10 *Let J be any ω-compatible almost complex structure on the symplectic vector space (V, ω). Then the subspace $L \subset V$ is Lagrangian if and only if there is a standard basis $u_1, v_1, \dots, u_n, v_n$, for (V, ω) such that $u_1, \dots, u_n$ span L and $v_j = Ju_j$ for all j.*

Proof: Let g_J be the associated metric and choose a g_J-orthonormal basis for L. Then

$$\omega(u_i, Ju_j) = -g_J(u_i, u_j) = \delta_{ij}.$$

Hence we get a standard basis by setting $v_j = Ju_j$ for all j. The converse is clear. $\square$

Corollary 3.11 $\mathcal{L}(n) \cong \mathrm{U}(n)/\mathrm{O}(n)$.

Proof: We may take $(V, \omega, J) = (\mathbf{R}^{2n}, \omega_0, J_0)$. Let L_0 be the Lagrangian spanned by the vectors $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$. The previous lemma shows that every Lagrangian subspace L is the image $A(L_0)$ of L_0 under the unitary transformation A that takes $\frac{\partial}{\partial x_j}$ to u_j and $\frac{\partial}{\partial y_j}$ to v_j. Moreover, $A(L_0) = L_0$ exactly when A belongs to the orthogonal subgroup $\mathrm{O}(n) \subset \mathrm{U}(n)$. $\square$

Exercise 3.12 Show that the group $\mathrm{Sp}(2n)$ acts transitively on pairs of transversally intersecting Lagrangians.

Lemma 3.13 $\pi_1(\mathcal{L}(n)) = \mathbf{Z}$.

Proof: The long exact homotopy sequence of the fibration $\mathrm{O}(n) \to \mathrm{U}(n) \to \mathcal{L}(n)$ contains the terms

$$\pi_1(\mathrm{O}(n)) \to \pi_1(\mathrm{U}(n)) \to \pi_1(\mathcal{L}(n)) \to \pi_0(\mathrm{O}(n)) = \mathbf{Z}/2\mathbf{Z}.$$

It is easy to check that the map

$$\mathrm{U}(n) \to S^1 : \quad A \mapsto \det(A)$$

induces an isomorphism on π_1 (where det denotes the determinant over $\mathbf{C}$.) On the other hand $\pi_1(\mathrm{O}(n))$ is generated by a loop in $\mathrm{O}(2)$ and the inclusion $\mathrm{O}(2) \hookrightarrow \mathrm{U}(2)$ takes its image in $\mathrm{SU}(2)$. Hence the map $\pi_1(\mathrm{O}(n)) \to \pi_1(\mathrm{U}(n))$ is trivial. $\square$

It is not hard to check that a generating loop of $\pi_1(\mathcal{L}(n)$ is

$$t \mapsto (e^{\pi i t}\mathbf{R}) \oplus \mathbf{R} \oplus \cdot \oplus \mathbf{R} \subset \mathbf{C}^n, \quad 0 \le t \le 1.$$

The Maslov index

There are several ways to use the structure of the Lagrangian Grassmannian to get invariants. Typically the resulting invariants are called the "Maslov index". Here is one way that is relevant when considering the Lagrangian intersection problem. Suppose we are given two Lagrangian submanifolds Q_0, Q_1 in (M, ω) that intersect transversally. For example Q_0 might be the zero section of the cotangent bundle T^*Q and Q_1 might be the graph of an exact 1-form df

with nondegenerate zeros. In this case one can assign an index to each transversal intersection point $x \in Q_0 \cap Q_1$ by using the usual Morse index for critical points of the function f. Although this is not possible in the general situation, we will now explain how it is possible to define a relative index of pairs x_+, x_- of intersection points. If one has chosen a homotopy class of connecting trajectories u, this index takes values in $\mathbf{Z}$. Here, a connecting trajectory means a map $u : D^2 \to M$ such that

$$\begin{aligned} u(1) &= x_+, & u(-1) &= x_-, \\ u(e^{\pi it}) &\in Q_0, & 0 &\le t \le 1, \\ u(e^{\pi it}) &\in Q_1, & 1 &\le t \le 2. \end{aligned}$$

Let us first see how to use this data to define a closed loop $L(t), 0 \le t \le 4$, in $\mathcal{L}(n)$. Note that $u^*(TM)$ is a symplectic bundle over the disc and so is symplectically trivial. Choose a trivialization $\phi : u^*(TM) \to D^2 \times \mathbf{R}^{2n}$. Then, define

$$L(t) = \begin{cases} \phi(u^*(T_{u(e^{\pi it})}Q_0)), & 0 \le t \le 1, \\ \phi(u^*(T_{u(e^{\pi it})}Q_1)), & 2 \le t \le 3. \end{cases}$$

For $t \in [1, 2]$ choose any path in $\mathcal{L}(n)$ from $L(1)$ to $L(2)$. To complete the loop, observe that by Exercise 3.12 there is $A \in \mathrm{Sp}(n)$ such that

$$A(L(0)) = L(1), \quad A(L(3)) = L(2).$$

Therefore, we may set

$$L(3+s) = A(L(2-s)), \quad 0 \le s \le 1.$$

The Maslov index $\mu_u(x_-, x_+)$ is now defined to be the element in $\pi_1(\mathcal{L}(n) \cong \mathbf{Z}$ represented by this path.

Exercise 3.14 Check that this index is independent of choices.

4 Lecture 4: The Nonsqueezing theorem

In Lecture 2, I explained various results that showed how flexible symplectomorphisms are and how little local structure a symplectic manifold has. Now I want to show you the phenomenon of symplectic rigidity that is encapsulated in Gromov's nonsqueezing theorem [G]. We will consider the cylinder

$$Z(r) = B^2(r) \times \mathbf{R}^{2n-2} = \{(x, y) \in \mathbf{R}^{2n} : x_1^2 + y_1^2 \le r\}$$

with the restriction of the usual symplectic form ω_0.

Theorem 4.1 (Gromov) *If there is a symplectomorphism that maps the the unit ball $B^{2n}(1)$ in $(\mathbf{R}^{2n}, \omega_0)$ into the cylinder $Z(r)$ then $r \ge 1$.*

This deceptively simple result is, as we shall see, enough to characterise symplectomorphisms among all diffeomorphisms. It clearly shows that symplectomorphisms are different from volume-preserving diffeomorphisms since it is easy to construct a volume-preserving diffeomorphism that squeezes the unit ball into an arbitrarily thin cylinder. We will begin discussing the proof at the end of this lecture. For now, let's look at its implications.

The clearest way to understand the force of Theorem 4.1 is to use the Ekeland–Hofer idea of capacity. A *symplectic capacity* is a function c that assigns an element in $[0, \infty]$ to each symplectic manifold of dimension $2n$ and satisfies the following axioms:

(i) (*monotonicity*) if there is a symplectic embedding $\phi : (U, \omega) \to (U', \omega')$ then $c(U, \omega) \leq c(U', \omega')$.
(ii) (*conformal invariance*) $c(U, \lambda\omega) = \lambda^2 c(U, \omega)$.
(iii) (*nontriviality*)

$$0 < c(B^{2n}(1), \omega_0) = c(Z(1), \omega_0) < \infty.$$

It is the last property $c(Z(1), \omega_0) < \infty$ that implies that capacity is an essentially 2-dimensional invariant, for example that it cannot be a power of the total volume. Sometimes one considers capacities that satisfy a less stringent version of (iii): namely
(iii′)

$$0 < c(B^{2n}(1), \omega_0), \quad c(Z(1), \omega_0) < \infty.$$

However, below we will use the strong form (iii).

The interesting question is: do symplectic capacities exist? A moment's reflection shows that the fact that they do is essentially equivalent to the nonsqueezing theorem. Let us define the Gromov capacity c_G by

$$c_G(U, \omega) = \sup\{\pi r^2 : B^{2n}(r) \text{ embeds symplectically in } U\}.$$

Then c_G clearly satisfies the conditions (i), (ii), and also $c_G(B^{2n}(r)) = \pi r^2$. The only difficult thing to check is that $c_G(Z(1)) < \infty$, but in fact

$$c_G(Z(r)) = \pi r^2$$

by the nonsqueezing theorem. Thus c_G is a capacity. There are now several other known capacities, (cf work by Ekeland–Hofer [EH], Hofer–Zehnder [HZ], Viterbo [V]) mostly defined by looking at properties of the periodic flows of certain Hamiltonian functions H that are associated to U.

The main result is

Theorem 4.2 (Ekeland–Hofer) *A (local) orientation-preserving diffeomorphism ϕ of $(\mathbf{R}^{2n}, \omega_0)$ is symplectic iff it preserves the capacity of all open subsets of $\mathbf{R}^{2n}$, ie iff there is a capacity c such that $c(\phi(U)) = c(U)$ for all open U.*

The proof is based on the corresponding result at the linear level.

Proposition 4.1 *A linear map L that preserves the capacity of ellipsoids is either symplectic or antisymplectic, ie $L^*(\omega_0) = \pm\omega_0$.*

Proof: If L is neither symplectic nor antisymplectic the same can be said of its transpose L^T. Therefore there are vectors v, w so that

$$\omega_0(v, w) \neq \pm\, \omega_0(L^T v, L^T w).$$

By perturbing v, w and using the openness of the above condition we can suppose that both $\omega_0(v, w)$ and $\omega_0(L^T v, L^T w)$ are nonzero. Then, replacing L^T by its inverse if necessary, we can arrange that

$$0 < \lambda^2 = |\omega_0(L^T v, L^T w)| < \omega_0(v, w) = 1.$$

Now construct two standard bases of $\mathbf{R}^{2n}$, the first starting as

$$u_1 = v, \quad v_1 = w, \quad u_2, \ldots,$$

and the second starting as

$$u_1' = \frac{L^T v}{\lambda}, \quad v_1' = \pm\frac{L^T w}{\lambda}, \quad u_2', \ldots.$$

Let A, resp A', be the symplectic linear map that takes the standard basis $e_1, e_2, e_3, \ldots$ of $\mathbf{R}^{2n}$ to $u_1, v_1, u_2, \ldots$, resp $u_1', v_1', u_2', \ldots$. Then, setting $C = (A')^{-1} L^T A$, we have

$$C(e_1) = \lambda e_1, \quad C(e_2) = \lambda e_2.$$

Thus the matrices for C and C^T have the form

$$C = \left(\begin{array}{cc|ccc} \lambda & 0 & * & \ldots & * \\ 0 & \lambda & * & \ldots & * \\ \hline 0 & 0 & * & \ldots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & * & \ldots & * \end{array}\right), \qquad C^T = \left(\begin{array}{cc|ccc} \lambda & 0 & 0 & \ldots & 0 \\ 0 & \lambda & 0 & \ldots & 0 \\ \hline * & * & * & \ldots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ & * & * & \ldots & * \end{array}\right).$$

It is now easy to check that C^T maps the unit ball into the cylinder $Z(\lambda)$. But because A, L and A' preserve capacity, so does $C^T = A^T L (A'^T)^{-1}$. This contradiction proves the result. Note that we have only needed the fact that C^T preserves the capacity of the unit ball. Hence we only need to know that L preserves the capacity of all sets that are images of the ball by symplectic linear maps, ie the ellipsoids. □

Proof of Theorem 4.2 We want to show that the derivative $d\phi_p$ of ϕ at every point p in its domain is symplectic. By pre- and post-composing with suitable translations, it is easy to see that it suffices to consider the case when $p = 0$ and $\phi(0) = 0$. Then the derivative $d\phi_0$ is the limit in the compact open topology of the diffeomorphisms ϕ_t given by

$$\phi_t(v) = \frac{\phi(tv)}{t}.$$

Because ϕ preserves capacity, and capacity behaves well under rescaling (see condition (ii)), the diffeomorphisms ϕ_t also preserve capacity. Moreover, by the exercise below, the capacity of convex sets is continuous with respect to the Hausdorff topology on sets. Thus the uniform limit $d\phi_0$ of the ϕ_t preserves capacity and so must be either symplectic or anti-symplectic.

To complete the proof, we must show that $d\phi_0$ is symplectic rather than anti-symplectic. If n is odd this follows immediately from the fact that $d\phi_0$ preserves orientation. If n is even, repeat the previous argument replacing ϕ by $id_{R^2} \times \phi$. □

Exercise 4.2 Recall that the Hausdorff distance $d(U, V)$ between two subsets U, V of $\mathbf{R}^{2n}$ is defined to be

$$d(U, V) = \max_{x \in U} \left(\min_{y \in V} \|x - y\| \right) + \max_{y \in V} \left(\min_{x \in U} \|x - y\| \right).$$

Suppose that U is a convex set containing the origin. Show that for all $\varepsilon > 0$ there is $\delta > 0$ such that

$$(1 - \varepsilon)U \subset V \subset (1 + \varepsilon)U, \quad \text{whenever} \quad d(U, V) < \delta.$$

Using this, prove the claim in the previous proof that $d\phi_0$ preserves capacity.

Corollary 4.3 (Eliashberg, Ekeland–Hofer) *The group* $\mathrm{Symp}(M, \omega)$ *is* C^0*-closed in the group of all diffeomorphisms.*

Proof: We must show that if ϕ_n is a sequence of symplectomorphisms that converge uniformly to a diffeomorphism ϕ_0 then ϕ_0 is itself symplectic. But ϕ_0 preserves the capacity of ellipsoids because capacity is continuous with respect to the Hausdorff topology on convex sets. Hence result. □

Note that these results give us a way of defining symplectic homeomorphisms. In fact, there are two possibly different definitions. One says that a (local) homeomorphism of $\mathbf{R}^{2n}$ is symplectic if it preserves the capacity of all open sets, the other that it is symplectic if it preserves the capacity of all sufficiently small ellipsoids. Very little is known about the properties of such homeomorphisms. In particular, it is unknown whether these two definitions agree and the extent to which they depend on the particular choice of capacity.

Theorem 4.2 makes clear that symplectic capacity is the basic symplectic invariant from which all others are derived. The fact that capacity is C^0-continuous shows the robustness of the property of being symplectic, and is really the reason why there is an interesting theory of symplectic topology. There is much recent work that develops the ideas presented here. Here is a short list of key references: Floer–Hofer [FH] on the theory of symplectic homology, Cieliebak–Floer–Hofer–Wysocki [CFHW] on its appplications, Hofer [H] and Lalonde–McDuff [LM] on the Hofer norm on the group $\mathrm{Ham}(M, \omega)$, and Polterovich [P] on its applications.

There are now many known proofs of the nonsqueezing theorem that are based on the different notions of capacity that have been developed: see for example Ekeland–Hofer [EH] and Viterbo [V]. We shall follow the original proof of Gromov [G] that uses J-holomorphic curves.

Preliminaries on J-holomorphic curves

A J-holomorphic curve (of genus 0) in an almost complex manifold (M, J) is a map

$$u : (S^2, j) \to (M, J) : \quad J \circ du = du \circ j,$$

where j is the usual almost complex structure on S^2. This equation may be rewritten as

$$\overline{\partial}_J u = \frac{1}{2}(du + J \circ du \circ j) = 0.$$

In local holomorphic coordinates $z = s + it$ on S^2, j acts by $j(\frac{\partial}{\partial s}) = \frac{\partial}{\partial t}$ and so this translates to the pair of equations:

$$\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} = 0, \quad \frac{\partial u}{\partial t} - J(u)\frac{\partial u}{\partial s} = 0.$$

Note that the second of these follows from the first by multiplying by J. Further, if J were constant in local coordinates on M (which is equivalent to requiring that J be integrable) these would reduce to the usual Cauchy–Riemann equations. As it is, these are quasi-linear equations that agree with the Cauchy–Riemann equations up to terms of order zero. Hence they are elliptic.

There is one very important point about J-holomorphic curves in the case when J is compatible with a symplectic form ω. We then have an associated metric g_J and we find (in obvious but rather inexact notation)

$$\begin{aligned}
\int_{S^2} u^*(\omega) &= \int_{S^2} \omega(\frac{\partial u}{\partial s}, \frac{\partial u}{\partial t}) \\
&= \int_{S^2} \omega(\frac{\partial u}{\partial s}, J\frac{\partial u}{\partial s}) \\
&= \int_{S^2} g_J(\frac{\partial u}{\partial s}, \frac{\partial u}{\partial s})\, ds\, dt
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\int_{S^2}\left(\left|\frac{\partial u}{\partial s}\right|^2 + \left|\frac{\partial u}{\partial t}\right|^2\right) ds\,dt \\
&= g_J\text{-area of } \operatorname{Im} u.
\end{aligned}
$$

Thus the g_J-area of a J-holomorphic curve is determined entirely by the homology class A that it represents. Note that $\omega(A)$ is always strictly positive unless $A = 0$: indeed the restriction of ω to a J-holomorphic curve is nondegenerate at all nonsingular points. Further the next exercise implies that such curves are g_J-minimal surfaces. It is possible to develop much of the theory of J-holomorphic curves using this fact. (This is Gromov's original approach. More details can be found in some of the articles in Audin–Lafontaine [AL].) In the next lecture we will sketch the outlines of a rather different approach using standard elliptic analysis.

Exercise 4.4 Let $(V, \omega J)$ be a symplectic vector space with compatible almost complex structure J and associated inner product g_J. Given two vectors v, w denote by $P(v, w)$ the parallelogram they span. Show that

$$
\omega(v, w) \le g_J\text{-area of } P(v, w)
$$

with equality if and only if $w = Jv$. Deduce that J-hol curves in (M, ω, J) are g_J-minimal surfaces.

5 Lecture 5

Sketch of the proof of the nonsqueezing theorem.

Suppose that $\phi : B^{2n}(1) \to Z(r)$ is a symplectic embedding. Its image lies in some compact subset $B^2(r) \times K$ of $Z(r)$ that can be considered as a subset of the compact manifold $(S^2 \times T^{2n-2}, \Omega)$, where Ω is the sum $\sigma \oplus \kappa\omega_0$ of a symplectic form σ on S^2 with total area $\pi r^2 + \varepsilon$ and a suitable multiple of the standard form ω_0 on T^{2n-2}. Let J_0 be the usual almost complex structure on $\mathbf{R}^{2n}$ and let J be an Ω-compatible almost complex structure on $S^2 \times T^{2n-2}$ that restricts to $\phi_*(J_0)$ on the image of the ball. (It is easy to construct such J using the methods of proof of Proposition 3.9.) As we will see below, the theory of J-holomorphic curves ensures that there is at least one J-holomorphic curve through each point of $S^2 \times T^{2n-2}$ in the class $A = [S^2 \times pt]$. Let C be such a curve through the image $\phi(0)$ of the origin, and let S be the component of the inverse image $\phi^{-1}(C)$ that goes through the origin. Then S is a proper[3] J_0-holomorphic curve in the ball $B^{2n}(1)$ through 0. Since J_0 is the usual complex structure, this means that S is a g_0-minimal surface (where g_0 is the usual metric

[3]This means that the intersection of S with any compact subset of the ball is compact. Thus it goes all the way out to the boundary.

on $\mathbf{R}^{2n}$.) But it is well-known that the proper surface of smallest area through the center of a ball of radius 1 is a flat disc with area π. Hence

$$\pi \le g_0\text{-area of } S = \int_S \omega_0 = \int_{\phi(S)} \Omega < \int_C \Omega = \int_{S^2 \times pt} \Omega = \pi r^2 + \varepsilon.$$

Since this is true for all $\varepsilon > 0$ we must have $r \ge 1$. $\square$

What we have used here from the theory of J-holomorphic curves is the existence of a curve in class $A = [S^2 \times pt]$ through an arbitrary point in $S^2 \times T^{2n-2}$. It is easy to check that when J equals the product almost complex structure J_{split} there is exactly one such curve through every point. For in this case the two projections are holomorphic so that every J_{split}-holomorphic curve in $S^2 \times \mathbf{R}^{2n-2}$ is the product of curves in each factor. But the curve in T^{2n-2} represents the zero homology class and so must be constant. Now, the basic theory of J-holomorphic curves is really a deformation theory: if you know that curves exist for one J you can often prove they exist for all other J.[4] That is exactly what we need here. Here is an outline of how this works. For more details see [MS1] as well as the Park City lectures by Salamon.

Fredholm theory

Let $\mathcal{M}(A, \mathcal{J})$ be the space of all pairs (u, J), where $u : (S^2, j) \to (M, J)$ is J-holomorphic , $u_*([S^2]) = A \in H_2(M)$, and $J \in \mathcal{J}(\omega)$. One shows that a suitable completion of $\mathcal{M}(A, \mathcal{J})$ is a Banach manifold and that the projection

$$\pi : \mathcal{M}(A, \mathcal{J}) \to \mathcal{J}$$

is Fredholm of index

$$2(c_1(A) + n),$$

where $c_1 = c_1(TM, J)$. In this situation one can apply an infinite dimensional version of Sard's theorem (due to Smale) that states that there is a set $\mathcal{J}_{reg}$ of second category in $\mathcal{J}$ consisting of regular values of π. Moreover by the implicit function theorem for Banach manifolds the inverse image of a regular value is a smooth manifold of dimension equal to the index of the Fredholm operator. Thus one finds that for almost every J

$$\pi^{-1}(J) = \mathcal{M}(A, J)$$

is a smooth manifold of dimension $2(c_1(A) + n)$. Moreover, by a transversality theorem for paths, given any two elements $J_0, J_1 \in \mathcal{J}_{reg}$ there is a path $J_t, 0 \le t \le 1$, such that the union

$$W = \cup_t \mathcal{M}(A, J_t) = \pi^{-1}(\cup_t J_t)$$

[4] An *existence* theory for J-holomorphic curves had to wait until the recent work of Donaldson and Taubes.

is a smooth (and also oriented) manifold with boundary

$$\partial W = \mathcal{M}(A, J_1) \cup -\mathcal{M}(A, J_0).$$

It follows that the evaluation map

$$ev_J : \mathcal{M}(A, J) \times_G S^2 \to M, \quad (u, z) \mapsto u(z),$$

is independent of the choice of (regular) J up to oriented bordism.[5] (Here $G = PSL(2, \mathbf{C})$ is the reparametrization group and has dimension 6.) In particular, *if* we could ensure that everything is compact and *if* we arrange that ev maps between manifolds of the same dimension then the degree of this map would be independent of J.

In the case of the nonsqueezing theorem we are interested in looking at curves in the class $A = [S^2 \times pt]$ in the cylinder $S^2 \times T^{2n-2}$. Thus $c_1(A) = 2$ since the normal bundle to $S^2 \times pt$ is trivial (as a complex vector bundle with the induced structure from J_{split}.) Thus

$$\begin{aligned}\dim(\mathcal{M}(A, J) \times_G S^2) &= 2(c_1(A) + n) - 6 + 2 \\ &= 4 + 2n - 6 + 2 = 2n = \dim(M).\end{aligned}$$

Further when $J = J_{split}$ the unparametrized moduli space $\mathcal{M}(A, J)/G)$ is compact (it is diffeomorphic to T^{2n-2}) and ev_J has degree 1. It is also possible to check that J_{split} is regular. So the problem is to check that compactness holds for all J. If so, we would know that ev_J has degree 1 for all regular J, ie there is at least one J-holomorphic curve through every point.

Compactness

This is the most interesting part of the theory and leads to all sorts of new developments such as the connection with stable maps and Deligne–Mumford compactifications.

We proved the following lemma at the end of Lecture 4. It is the basic reason why spaces of J-holomorphic curves can be compactified.

Lemma 5.1 *If u is J-holomorphic for some ω-compatible J then*

$$\|u\|_{1,2} = \int_{S^2} u^*(\omega) = g_J\text{-}area\ of\ (\operatorname{Im} u).$$

[5] Two maps, $e_i : M_i \to X$ for $i = 1, 2$, are said to be *oriented bordant* if there is an oriented manifold W with boundary $\partial W = M_1 \cup (-M_2)$ and a map $e : W \to X$ that restricts to e_i on the boundary component M_i. Often the compactness that is needed to get any results from this notion is built into the definition. For example, if all manifolds M_1, M_2, W are compact and if M_1, M_2 have no boundary then bordant maps e_i represent the same homology class.

Here $\|u\|_{1,2}$ denotes the L^2-norm of the first derivative of u. If we just knew a little more we would have compactness by the following basic regularity theorem for solutions of elliptic differential equations.

Lemma 5.2 *If $u_n : S^2 \to M$ are J-holomorphic curves such that for some $p > 2$ and $K < \infty$*

$$\|u_n\|_{1,p} \leq K,$$

then a subsequence of the u_n converges uniformly with all derivatives to a J-holomorphic map u_∞.

It follows from the above two lemmas that if $u_n \in \mathcal{M}(A, J)$ is a sequence with no convergent subsequence then the size of the derivatives du_n must tend to infinity. In other words

$$c_n = \max_{z \in S^2} |du_n(z)| \to \infty.$$

By reparametrizing by suitable rotations we can assume that this maximum is always assumed at the point $0 \in \mathbf{C} \subset \mathbf{C} \cup \infty = S^2$. The claim is that as $n \to \infty$ a "bubble" is forming at 0, i.e. the image curve is breaking up into two or more spheres. To see this analytically consider the reparametrized maps $v_n : \mathbf{C} \to M$ defined by

$$v_n(z) = u_n(z/c_n).$$

Then

$$|dv_n(0)| = 1 \quad \text{and} \quad |dv_n(z)| \leq 1, z \in \mathbf{C}.$$

Therefore by Lemma 5.2 a suitable subsequence of the v_n converge to a map $v_\infty : \mathbf{C} \to M$. Moreover because the energy (or symplectic area) of the image of the limit v_∞ is bounded (by $\omega(A)$), the image points $v_\infty(z)$ converge as $z \to \infty$. In other words v_∞can be extended to a map $v_\infty : S^2 \to M$. (Here we are applying a removable singularity theorem for J-holomorphic maps $v : D^2 - \{0\} \to M$ that have finite area.)

Usually the image curve $C_\infty = v_\infty(S^2)$ will be just a part of the limit of the set-theoretic limit of the curves $C_n = u_n(S^2)$. What we have done in constructing C_∞ is focus on the part of C_n that is the image of a very small neighborhood of 0, and there usually are other parts of C_n (separated by a "neck".) Thus typically the the curves C_n converge (as point sets) to a union of several spheres, and the bubble C_∞ is just one of them. (Such a union of spheres is often called a "cusp-curve" or reducible curve.) It can happen that the bubble C_∞ is the whole limit of the C_n. But in this case one can show that it is possible to reparametrize the original maps u_n so that they converge. In other words, the u_n converge in the space $\mathcal{M}(A, J)/\mathrm{G}$ of *unparametrized* curves. For example, if we started with a sequence of the form $u \circ \gamma_n$ where γ_n is a nonconvergent sequence in the reparametrization group G, then the effect of the reparametrizations v_n of u_n is essentially to undo the γ_n. More precisely, v_n would have the form $u \circ \gamma_n'$ where the γ_n' do converge in G.

This argument (when made somewhat more precise) shows that the only way the unparametrized moduli space $\mathcal{M}(A, J)/G$ can be noncompact is if there is a reducible J-holomorphic curve in class A consisting of several nontrivial spheres that represent classes $A_1, A_2, \ldots$. Since each $\omega(A_i) > 0$, this is possible only when $\omega(A)$ is not minimal (among all positive values of ω on spheres). In particular, when $M = S^2 \times T^{2n-2}$, $\pi_2(M)$ is generated by A. Therefore there is no reducible J-holomorphic curve in class A, and $\mathcal{M}(A, J)/G$ is always compact. Hence the space $\mathcal{M}(A, J) \times_G S^2$ is also always compact. This completes our sketch of the proof of the nonsqueezing theorem.

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An Introduction to Symplectic Topology through Sheaf theory

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C. Viterbo

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Contents

CHAPTER 1

Introduction

This are the notes of graduate lectures given in the fall semester 2010 at Princeton University, and then as the Eilenberg lectures at Columbia in the spring 2011. The first part of the symplectic part of the course (chapter 2 to 4) corresponds to a course given at Beijing Unversity on 2007 and 2009, with notes by Hao Yin (Shanghai Jiaotong University). The aim of this course is to present the recent work connecting sheaf theory and symplectic topology, due to several authors, Nadler ([**Nad, Nad-Z**], [**Tam**], Guillermou-Kashiwara-Schapira [**G-K-S**]. This is completed by the approach of [**F-S-S**], and the paper [**F-S-S2**] really helped us to understand the content of these works.

Even though the goal of the paper is to present the proof of the classical Arnold conjecture on intersection of Lagrangians, and the more recent work of [**F-S-S**] and [**Nad**] on the topology of exact Lagrangians in T^*X, we tried to explore new connections between objects. We also tried to keep to the minium the requirements in category theory and sheaf theory necessary for proving our result. Even though the appendices contain some material that will be useful for those interested in pursuing the sheaf theoretical approach, much more has been omitted, or restricted to the setting we actually use[1] The experts will certainly find that our approach is "not the right one", as we take advantage of many special features of the category of sheafs, and base our approach of derived categories on the Cartan-Eilenberg resolution. We can only refer to the papers and books in the bibliography for a much more complete account of the theory.

The starting point is the idea of Kontsevich, about the homological interpretation of Mirror symmetry. This should be an equivalence between the derived category of the $D^b(\mathbf{Fuk}(\mathbf{M},\omega))$, the derived cateogory of the category having objects the (exact) Lagrangians in (M,ω) and morphisms the elements in the Floer cohomology (i.e. $\mathrm{Mor}(L_1,L_2)=FH^*(L_1,L_2)$) the derived category of coherent sheafs on the Mirror, $D^b(\mathbf{Coh}(\check{\mathbf{M}},\mathbf{J}))$. Our situation is a toy model, in which $(M,\omega)=(T^*X,d(pdq))$, and $D^b(\mathbf{Coh}(\check{\mathbf{M}},\mathbf{J}))$ is then replaced by $D^b(\mathbf{Sheaf_{cstr}}(\mathbf{X}\times\mathbb{R}))$ the category of constructible sheafs (with possibly more restrictions) on $X\times\mathbb{R}$.

There is a functor

$$SS: D^b(\mathbf{Sheaf_{cstr}}(\mathbf{X}\times\mathbb{R}))\longrightarrow D^b(\mathbf{Fuk}(\mathbf{T^*X},\omega))$$

[1]For example since the spaces on which our sheafs are defined are manifolds, we only rarely discuss assumptions of finite cohomological dimension.

determined by the singular support functor. The image does not really fall in $D^b(\mathbf{Fuk}(\mathbf{T^*X},\omega))$, since we must add the singular Lagrangians, but this a more a feature than a bug. Moreover we show that there is an inverse map, called " Quantization" obtained by associating to a smooth Lagrangian L, a sheaf over X, $\mathscr{F}_L$ with fiber $(\mathscr{F}_L)_x = (CF_*(L,V_x),\partial_x)$ where V_x is the Lagrangian fiber over x and $CF_*(L,V_x),\partial_x)$ is the Floer complex of the intersection of L and V_x. This is the **Floer quantization** of L. This proves in particular that the functor SS is essentially an equivalence of categories. We are also able to explain the condition for the **Floer quantization** of L to be an actual quantization (i.e. to be well defined and provide an inverse to SS). Due to this equivalence, for complexes of sheafs $\mathscr{F}^\bullet,\mathscr{G}^\bullet$ on X, we are able to define $H^*(\mathscr{F}^\bullet,\mathscr{G}^\bullet) = H^*(\mathscr{F}^\bullet\otimes(\mathscr{G}^\bullet)^*)$ as well as $FH^*(SS(\mathscr{F}),SS(\mathscr{G}))$ and these two objects coincide. We may also define $FH^*(L,\mathscr{G})$ as $H^*(\mathscr{F}_L,\mathscr{G})$.

I thank Hao Yin for allowing me to use his lecture notes from Beijing. I am very grateful to the authors of [**Tam**], [**Nad**], [**F-S-S**] and [**F-S-S2**] and [**G-K-S**] from where theses notes drew much inspiration, and in particular to Stéphane Guillermou for a talk he gave at Symplect'X seminar, which led me to presomptuously believe I could understand this beautiful theory, and to Pierre Schapira for patiently explaining me many ideas of his theory and dispelling some naive preconceptions, to Paul Seidel and Mohammed Abouzaid for discussions relevant to the General quantization theorem. Finally I thank the University of Princeton, the Institute for Advanced Study and Columbia University for hospitality during the preparation of this course. A warm thanks to Helmut Hofer for many discussions and for encouraging me to turn these notes into book form.

New-York, Spring 2011

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Part 1

Elementary symplectic geometry

CHAPTER 2

Symplectic linear algebra

1. Basic facts

Let V be a finite dimensional real vector space.

DEFINITION 2.1. A symplectic form on V is a skew-symmetric bilinear nondegenerate form, i.e. a two-form satisfying:

(1)
$$\forall x, y \in V \ \ \omega(x, y) = -\omega(y, x)$$
($\Longrightarrow \ \forall x \in V \ \ \omega(x, x) = 0$);

(2) $\forall x, \exists y$ such that $\omega(x, y) \neq 0$.

For a general 2-form ω on a vector space, V, we denote by $\mathrm{Ker}(\omega)$ the subspace given by
$$\mathrm{Ker}(\omega) = \{\mathrm{v} \in \mathrm{V} \mid \forall \mathrm{w} \in \mathrm{V}\, \omega(\mathrm{v}, \mathrm{w}) = 0\}$$
The second condition implies that $\mathrm{Ker}(\omega)$ reduces to zero, so when ω is symplectic, there are no "preferred directions" in V.

There are special types of subspaces in symplectic manifolds. For a vector subspace F, we denote by
$$F^{\omega} = \{v \in V \mid \forall\, w \in F\, ,\ \omega(v, w) = 0\}$$
the symplectic orthogonal From Grassmann's formula applied to the surjective map $\varphi_F : V \to F^*$ given by $\varphi_F(v) = \omega(v, \bullet)$, it follows that $\dim(F^{\omega}) = \dim(\mathrm{Ker}(\varphi_{\mathrm{F}})) = \mathrm{codim}(\mathrm{F}) = \dim(\mathrm{V}) - \dim(\mathrm{F})$. Moreover the proof of the following is left to the reader

PROPOSITION 2.2.
$$(F^{\omega})^{\omega} = F$$
$$(F_1 + F_2)^{\omega} = F_1^{\omega} \cap F_2^{\omega}$$

DEFINITION 2.3. A map $\varphi : (V_1, \omega_1) \to (V_2, \omega_2)$ is a **symplectic map** if $\varphi^*(\omega_2) = \omega_1)$ that is $\forall x, y \in V_1, \omega_2(\varphi(x), \varphi(y)) = \omega_1(x, y)$. It is a **symplectomorphism** if and only if it is invertible- its inverse is then necessarily symplectic. A subspace F of (V, ω) is

- isotropic if $F \subset F^{\omega}$ ($\Longleftrightarrow \omega|_F = 0$);
- coisotropic if $F^{\omega} \subset F$
- Lagrangian if $F^{\omega} = F$.

PROPOSITION 2.4. *(1) Any symplectic vector space has even dimension.*

(2) *Any isotropic subspace is contained in a Lagrangian subspace and Lagrangians have dimension equal to half the dimension of the total space.*
(3) *If (V_1,ω_1), (V_2,ω_2) are symplectic vector spaces with L_1, L_2 Lagrangian subspaces, and if $\dim(V_1) = \dim(V_2)$, then there is a linear isomorphism $\varphi : V_1 \to V_2$ such that $\varphi^*\omega_2 = \omega_1$ and $\varphi(L_1) = L_2$. As a consequence, any two symplectic vector spaces of the same dimension are symplectomorphic.*

PROOF. We first prove that if I is an isotropic subspace it is contained in a Lagrangian subspace. Indeed, I is contained in a maximal isotropic subspace. We denote it again by I and we just have to prove $2\dim(I) = \dim(V)$.

Since $I \subset I^\omega$ we have $\dim(I) \le \dim(I^\omega) = \dim(V) - \dim(I)$ so that $2\dim(I) \le \dim(V)$. Now assume the inequality is strict. Then there exist a non zero vector, e, in $I^\omega \setminus I$, and $I \oplus \mathbb{R}e$ is isotropic and contains I. Therefore I was not maximal, a contradiction.

We thus proved that a maximal isotropic subspace I satisfies $I = I^\omega$ hence $2\dim(I) = \dim(V)$, and $\dim(V)$ is even.

Since $\{0\}$ is an isotropic subspace, maximal isotropic subspaces exist[1], and we conclude that we may always find a Lagrangian subspace, hence V is always even-dimensional.

This proves (1) and (2).

Let us now prove (3).

We shall consider a standard symplectic vector space $(\mathbb{R}^2,\sigma)$ with canonical base e_x, e_y and the symplectic form given by

$$\sigma(x_1e_x + y_1e_y, x_2e_x + y_2e_y) = x_1y_2 - y_1x_2.$$

Similarly by orthogonal direct sum, we get the symplectic space $(\mathbb{R}^{2n},\sigma_n)$

$$\sigma((x_1,\dots,x_n,y_1,\dots,y_n),(x'_1,\dots,x'_n,y'_1,\dots,y'_n)) = \sum_{j=1}^{n} x_j y'_j - x'_j y_j$$

It contains an obvious Lagrangian subspace,

$$Z_n = \mathbb{R}^n \oplus 0 = \{(x_1,\dots,x_n,y_1,\dots,y_n) \mid \forall j,\ 1 \le j \le n, y_j = 0\}$$

Let (V,ω) be a symplectic vector space and L a Lagrangian. We are going to prove by induction on $n = \dim(L) = \frac{1}{2}\dim(V)$ that there exists a symplectic map φ_n sending Z_n to L.

Assume this has been proved in dimension less or equal than $n-1$, and let us prove it in dimension n.

Pick any $e_1 \in L$. Since ω is nondegenerate, there exists an $f_1 \in V$ such that $\omega(e_1, f_1) = 1$. Then $f_1 \notin L$. Define

$$V' = Vect(e_1, f_1)^\omega = \{x \in V | \omega(x,e_1) = \omega(x, f_1) = 0\}.$$

[1]no need to invoke Zorn's lemma, a dimension argument is sufficient.

It is easy to see that $(V', \omega_{|V'})$ is symplectic since only non-degeneracy is an issue, which follows from the fact that

$$\mathrm{Ker}(\omega_{|\mathrm{V}'}) = \mathrm{V}' \cap (\mathrm{V}')^{\omega} = \{0\}$$

We now claim that $L' = L \cap V'$ is a Lagrangian in V' and $L = L' \oplus \mathbb{R}e_1$. First, since $\omega_{|L'}$ is the restriction of $\omega_{|L}$, we see that L' is isotropic. It is maximal isotropic, since otherwise, there would be an isotropic W such that $V' \supset W \supsetneq L'$, and then $W \oplus \mathbb{R}e_1$ would be a strictly larger isotropic subspace than L, which is impossible. Since $L \subset L' \oplus \mathbb{R}e_1$ our second claim follows by comparing dimensions.

Now the induction assumption implies that there is a symplectic map, φ_{n-1} from $(\mathbb{R}^{2n-2}, \sigma)$ to (V_2, ω) sending Z_{n-1} to L'. Then the map

$$\begin{aligned} \varphi_n : (\mathbb{R}^2, \sigma_2) \oplus (\mathbb{R}^{2n-2}, \sigma) &\longrightarrow (V, \omega) \\ (x_1, y_1; z) &\longrightarrow x_1 e_1 + y_1 f_1 + \varphi_{n-1}(z) \end{aligned}$$

is symplectic and sends Z_n to L.

Now the last statement of our theorem easily follows from the above: given two symplectic manifolds, $(V_1, \omega_1), (V_2, \omega_2)$ of dimension $2n$, and two lagrangians L_1, L_2, we get two symplectic maps

$$\psi_j : (\mathbb{R}^{2n}, \sigma_n) \longrightarrow (V_j, \omega_j)$$

sending Z_n to L_j. Then the map $\psi_2 \circ \psi_1^{-1}$ is a symplectic map from (V_1, ω_1) to (V_2, ω_2) sending L_1 to L_2.

$\square$

REMARK 2.5. As we shall see, the map φ is not unique.

Since any symplectic vector space is isomorphic to $(\mathbb{R}^{2n}, \sigma)$, the group of symplectic automorphisms of (V, ω) denoted by $Sp(V, \omega) = \{\varphi \in GL(V) | \varphi^* \omega = \omega\}$ is isomorphic to $Sp(n) = Sp(\mathbb{R}^{2n}, \omega)$.

We now give a better description of the set of lagrangian subspaces of (V, ω).

PROPOSITION 2.6. *(1) There is a homeomorphism between the set*

$$\{\Lambda \mid \Lambda \textit{ is Lagrangian and } \Lambda \cap L = \{0\}\}$$

and the set of quadratic forms on L^*. As a result, $\Lambda(n)$ is a smooth manifold of dimension $\frac{n(n+1)}{2}$.

(2) The action of $Sp(n) = \{\varphi \in GL(V) | \varphi^* \omega = \omega\}$ on the set of pairs of transverse Lagrangians is transitive.

PROOF. For (1), we notice that $W = L \oplus L^*$ with the symplectic form

$$\sigma((e, f), (e', f')) = \langle e', f \rangle - \langle e, f' \rangle$$

is a symplectic vector space and that $L \oplus 0$ is a Lagrangian subspace.

According to the previous proposition there is a symplectic map $\psi : V \longrightarrow W$ such that $\psi(L) = L \oplus 0$, so we can work in W.

Let Λ be a Lagrangian in W with $\Lambda \cap L = \{0\}$. Then Λ is the graph of a linear map $A : L^* \to L$, more precisely

$$\Lambda = \{(Ay^*, y^*) | y^* \in L^*\}.$$

The subspace Λ is Lagrangian if and only if

$$\sigma((Ay_1^*, y_1^*), (Ay_2^*, y_2)) = 0, \text{ for all } y_1, y_2$$

i.e. if and only if

$$\langle y_1^*, Ay_2^* \rangle = \langle y_2^*, Ay_1^* \rangle$$

that is if $\langle \cdot, A \cdot \rangle$ is a bilinear symmetric form on L^*. But such bilinear form are in 1-1 correspondence with quadratic forms. The second statement immediately follows from the fact that the set of quadratic forms on an n-dimensional vector space is a vector space of dimension $\frac{n(n+1)}{2}$, and the fact that to any Lagrangian L_0 we may associate a transverse Lagrangian L_0', and L_0 is contained in the open set of Lagrangians transverse to L_0' (Well we still have to check the change of charts maps are smooth, this is left as an exercise).

To prove (2) let (L_1, L_2) and $L_1', L_2')$ be two pairs of transverse Lagrangians. By the previous proposition, we may assume $V = (L \oplus L^*, \sigma)$ and $L_1 = L_1' = L$. It is enough to find $\varphi \in Sp(V, \omega)$ such that $\varphi(L) = L, \varphi(L^*) = \Lambda$. The map $(x, y) \longrightarrow (x + Ay^*, y^*)$ is symplectic provided A is symmetric and sends $L \oplus 0$ to $L \oplus 0$ and L^* to $\Lambda = \{(Ay^*, y^*) \mid y^* \in L^*\}$. □

EXERCICES 1. (1) Prove that if K is a coisotropic subspace, K/K^ω is symplectic.

(2) Compute the dimension of the space of Lagrangians containing a given isotropic subspace I. Hint: show that it is the space of Lagrangians in I^ω/I.
(3) (Witt's Theorem) Let V_1 and V_2 be two symplectic vector spaces with the same dimension and $F_i \subset (V_i, \omega_i), i = 1,2$. Assume that there exists a linear isomorphism $\varphi : F_1 \cong F_2$, i.e. $\varphi^*(\omega_2)_{|F_2} = (\omega_1)_{|F_1}$. Then φ extends to a symplectic map $\widetilde{\varphi} : (V_1, \omega_1) \to (V_2, \omega_2)$. Hint: show that symplectic maps are the same thing as Lagrangians in $(V_1 \oplus V_2, \omega_1 - \omega_2)$ which are transverse to V_1 and V_2, and the map we are looking for, correspond to Lagrangians transverse to V_1, V_2 containing $I = \{(x, \varphi(x)) \mid x \in F_1\}$. Compute the dimension of the non transverse ones.
(4) The action of $Sp(n)$ is not transitive on the triples of pairwise transverse Lagrangian spaces. Using the notion of index of a quadratic form prove that this has at least (in fact exactly) $n+1$ connected components. This is responsible for the existence of the Maslov index.
(5) Prove that the above results are valid over any field of any characteristic, except in characteristic 2 because quadratic forms and bilinear symmetric forms are not equivalent.

2. Complex structure

Let h be a hermitian form on a complex vector space V in the sense:
1) $h(z,z') = \overline{h(z',z)}$;
2) $h(\lambda z, z') = \lambda h(z,z')$ for $\lambda \in \mathbb{C}$;
3) $h(z, \lambda z') = \bar{\lambda} h(z,z')$ for $\lambda \in \mathbb{C}$;
4) $h(z,z) > 0$ for all $z \neq 0$.
Then

$$h(z,z') = g(z,z') + i\omega(z,z'),$$

where g is a scalar product and ω is symplectic, since $\omega(iz,z) > 0$ for $z \neq 0$.

Example: On $\mathbb{C}^n$, define

$$h((z_1,\cdots,z_n),(z'_1,\cdots,z'_n)) = \sum_{j=1}^{n} z_j \bar{z'_j} \in \mathbb{C}.$$

Then the symmetric part is the usual scalar product on $\mathbb{R}^{2n}$ while ω is the standard symplectic form.

Denote by J the multiplication by $i = \sqrt{-1}$.

PROPOSITION 2.7.

$$\begin{cases} g(Jz,z') = -\omega(z,z') \\ \omega(z,Jz') = -g(z,z') \end{cases}$$

REMARK 2.8. ω is nondegenerate because $\omega(z,Jz) = -g(z,z) < 0$ for all $z \neq 0$.

Conclusion: Any hermitian space V has a canonical symplectic form.

We will now answer the following question: can a symplectic vector space be made into a hermitian space? In how many ways?

PROPOSITION 2.9. *Let (V,ω) be a symplectic vector space. Then there is a complex structure on V such that $\omega(J\xi,\eta)$ is a scalar product. Moreover, the set $\mathscr{J}(\omega)$ of such J is contractible.*

PROOF. Let $(\cdot,\cdot)$ be any fixed scalar product on V. Then there exists A such that

$$\omega(x,y) = (Ax,y).$$

Since ω is skew-symmetric, $A^* = -A$ where A^* is the adjoint of A with respect to $(\cdot,\cdot)$. Since any other scalar product can be given by a positive definite symmetric matrix M, we look for J such that $J^2 = -I$ and M such that $M^* = M$ and setting $(x,y)_M = (Mx,y)$ we have $\omega(Jx,y) = (x,y)_M$. The last equality can be rewritten as

$$(AJx,y) = (Mx,y) \text{ for all } x,y.$$

This is equivalent to finding a symmetric M such that $M = AJ$. It's easy to check that there is a unique solution given by $M = (AA^*)^{1/2}$ and $J = A^{-1}M$ solves $AJ = M$, $J^2 = -I$ and $M^* = M$.

In summary, for any fixed scalar product $(\cdot,\cdot)$, we can find a pair (J_0, M_0) such that $\omega(J_0 x, y)$ is the scalar product $(M_0\cdot,\cdot)$. If we know (J_0, M_0) is such a pair and we start from the scalar product $(M_0\cdot,\cdot)$, then we get the pair (J_0, id).

Define $\mathcal{J}(\omega)$ to be the set of all J's such that $\omega(J\cdot,\cdot)$ is a scalar product. Define $\mathcal{S}$ to be the set of all scalar products on V. By previous discussion, there is continuous map

$$\Psi : \mathcal{S} \to \mathcal{J}(\omega).$$

Moreover, if J is in $\mathcal{J}(\omega)$, Ψ maps $\omega(J\cdot,\cdot)$ to J. On the other hand, we have a continuous embedding i from $\mathcal{J}(\omega)$ to $\mathcal{S}$ which maps J to $\omega(J\cdot,\cdot)$. Clearly, $\Psi \circ i = \mathrm{id}_{\mathcal{S}}$.

Let now $M_p \in \mathcal{S}$ be in the image. Since we know $\mathcal{S}$ is contractible, there is a continuous family

$$F_t : \mathcal{S} \to \mathcal{S}$$

such that $F_0 = id$ and $F_1(\mathcal{S}) = M_p$. Consider

$$\tilde{F}_t : \mathcal{J}(\omega) \to \mathcal{J}(\omega)$$

given by

$$\tilde{F}_t = \Psi \circ F_t \circ i.$$

By the definition of Ψ, we know $\tilde{F}_0 = id$ and $\tilde{F}_1 = J_p$. This shows that $\mathcal{J}(\omega)$ is contractible. $\square$

EXERCICE 2. Let L be a Lagrangian subspace, show that JL is also a Lagrangian and $L \cap JL = \{0\}$.

We finally study the structure of the symplectic group,

PROPOSITION 2.10. *The group $Sp(n)$ of linear symplectic maps of (V,ω) is connected, has fundamental group isomorphic to $\mathbb{Z}$ and the homotopy type of $U(n)$.*

PROOF. Let $\langle Jx, y\rangle = \sigma(x, y)$ with $J^2 = -\mathrm{Id}$ and $J^* = -J$ Let $R \in Sp(n)$, then $\sigma(Rx, Ry) = \sigma(x, y)$ i.e.

$$\langle JRx, Ry\rangle = \langle x, y\rangle$$

Thus $R \in Sp(n)$ is equivalent to $R^* JR = J$.

Thus, if R is symplectic, so is R^*, since $(R^*)JRJ = J^2 = -\mathrm{Id}$ we may conclude that $(R^*)^{-1}[(R^*)JRJ]R^* = -\mathrm{Id}$, that is $JRJR^* = -\mathrm{Id}$, so that $RJR^* = J$.

Now decompose R as $R = PQ$ with P symmetric and Q orthogonal, by setting $P = (RR^*)^{1/2}$ and $Q = P^{-1}R$. Since R, R^* are symplectic so is P and hence Q. Now

$$Q^{-1}JQ = R^{-1}PJP^{-1}R = R^{-1}(PJP^{-1})R =$$

$$\hat{\mathrm{E}} - R^{-1}JP^{-2}R = R^{-1}J(RR^*)^{-1}R = R^{-1}JR^* = J$$

Thus Q is symplectic and complex, that is unitary. Then since P is also positive definite, the map $t \longrightarrow P^t$ is well defined (as $\exp(t\log(P))$ and $\log(P)$ is well defined for a positive symmetric matrix) for $t \in \mathbb{R}$ and the path $PQ \longrightarrow P^t Q$ yields a retraction form $Sp(n)$ to $U(n)$. $\square$

EXERCICE 3. Prove that $Sp(n)$ acts transitively on the set of isotropic subspaces (resp. coisotropic subspaces) of given dimension (use Witt's theorem).

EXERCICE 4. Prove that the set $\tilde{\mathscr{J}}(\omega)$ made of complex structures J such that $\omega(J\xi,\xi) > 0$ for all $\xi \neq 0$ is also contractible (of course it contains $\mathscr{J}(\omega)$. Elements of $\mathscr{J}(\omega)$ are called **compatible almost complex structures** while those in $\tilde{\mathscr{J}}(\omega)$ are called **tame almost complex structures**.

CHAPTER 3

Symplectic differential geometry

1. Moser's lemma and local triviality of symplectic differential geometry

DEFINITION 3.1. A two form ω on a manifold M is symplectic if and only if
1) $\forall x \in M$, $\omega(x)$ is symplectic on T_xM;
2) $d\omega = 0$ (ω is closed).

Examples:
1) $(\mathbb{R}^{2n}, \sigma)$ is symplectic manifold.
2) If N is a manifold, then

$$T^*N = \{(q,p) | p \text{ linear form on } T_qM\}$$

is a symplectic manifold. Let $q_1, \cdots, q_n$ be local coordinates on N and let $p^1, \cdots, p^n$ be the dual coordinates. Then the symplectic form is defined by

$$\omega = \sum_{i=1}^{n} dp^i \wedge dq_i.$$

One can check that ω does not depend on the choice of coordinates and is a symplectic form. We can also define a one form, called the Liouville form

$$\lambda = p\,dq = \sum_{i=1}^{n} p^i dq_i.$$

It is well defined and $d\lambda = \omega$.

3) Projective algebraic manifolds (See also Kähler manifolds)

$\mathbb{C}P^n$ has a canonical symplectic structure σ and is also a complex manifold. The restriction to the tangent space at any point of the complex structure J and the symplectic form σ are compatible. The manifold $\mathbb{C}P^n$ has a hermitian metric h, called the Fubini-Study metric. For any $z \in \mathbb{C}P^n$, $h(z)$ is a hermitian inner product on $T_z\mathbb{C}P^n$. $h = g + i\sigma$, where g is a Riemannian metric and $\sigma(J\xi, \xi) = g(\xi, \xi)$.

Claim: A complex submanifold M of $\mathbb{C}P^n$ carries a natural symplectic structure.

Indeed, consider $\sigma|_M$. It's obviously skew-symmetric and closed. We must prove that $\sigma|_M$ is non-degenerate. This is true because if $\xi \in T_xM_{\{0\}}$ and $J\xi \in T_xM$, then $\omega(x)(\xi, J\xi) \neq 0$

DEFINITION 3.2. A submanifold in symplectic manifold (M,ω) is Lagrangian if and only if $\omega|_{T_xL} = 0$ for all $x \in M$ and $\dim L = \frac{1}{2} \dim M$. In other words T_xL is a Lagrangian subspace of $(T_xM, \omega(x))$.

We are going to prove that locally symplectic manifolds "have no geometry". A crucial lemma is

LEMMA 3.3 (Moser). *Let N be a compact submanifold in M. Let ω_t be a family of symplectic forms such that $\omega_t|_{T_N M}$ is constant. Then there is a diffeomorphism φ defined near N such that $\varphi^*\omega_1 = \omega_0$ and $\varphi|_N = id|_N$.*

PROOF. We will construct a vector field $X(t,x) = X_t(x)$ whose flow φ^t satisfies $\varphi^0 = id$ and $(\varphi^t)^*\omega_t = \omega_0$. Differentiate the last equality

$$(\frac{d}{dt}(\varphi^t)^*)\omega_t + (\varphi^t)^*(\frac{d}{dt}\omega_t) = 0.$$

Then

$$(\varphi^t)^* L_{X_t}\omega_t + (\varphi^t)^*(\frac{d}{dt}\omega_t) = 0.$$

Since φ^t is diffeomorphism, this is equivalent to

$$L_{X_t}\omega_t + \frac{d}{dt}\omega_t = 0.$$

Using Cartan's formula

$$L_X = d \circ i_X + i_X \circ d,$$

we get

$$d(i_{X_t}\omega_t) + \frac{d}{dt}\omega_t = 0.$$

Since ω_t is nondegenerate, the map $T_xM \to (T_xM)^*$ which maps X to $\omega(X,\cdot)$ is an isomorphism. Therefore, for any one form β, the equation $i_X\omega = \beta$ has a unique solution X_β. It suffices to solve for β_t,

$$d\beta_t = -\frac{d}{dt}\omega_t.$$

with the requirement that $\beta_t = 0$ on $T_N M$ for all t, because we want $\varphi_{|N} = \mathrm{Id}_{|N}$, that is $X_t 0$ on N. On the other hand, the assumption that $\omega_t = \omega_0$ on $T_N M$ implies $(\frac{d}{dt}\omega_t) \equiv 0$ on $T_N M$. Denote the right hand side of the above equation by α, then α is defined in a neighborhood U of N. The solution of β_t is given by Poincaré's Lemma on the tubular neighborhood of N. Here by a tubular neighborhood we mean a neighborhood of N in M diffeomorphic to the unit disc bundle $D\nu_N M$ of $\nu_M N$ the normal bundle of N in M (i.e. $\nu_M N = \{(x,\xi) \in T_N M \mid \xi \perp TN\}$).

LEMMA 3.4. *(Poincaré) If α is a p-form on U, closed and vanishing on N, then there exists β such that $\alpha = d\beta$ and β vanishes on $T_N M$.*

PROOF. [1]

[1]The proof is easier if one is willing to admit that the set of exact forms is closed for the C^∞ topology, i.e. if $\alpha = d\beta_\varepsilon + \gamma_\varepsilon$ and $\lim_{\varepsilon\to 0}\gamma_\varepsilon = 0$ then α is exact. This follows for example from the fact that exactness of a closed form can be checked by verifying that its integral over a finite number of cycles vanishes.

This means that for a tubular neighborhood $H^*(U,N)=0$.

Indeed, let r_t be the map on $\nu_N M$ defined by $r_t(x,\xi)=(x,t\xi)$ and V the vector field $V_t(x,\xi)=-\frac{\xi}{t}$, well defined for $t\neq 0$. This vector field satisfies $\frac{d}{dt}r_t(x,\xi)=V_t(r_t(x,\xi))$. Since r_0 sends $\nu_N M$ to its zero section, N, we have $r_0^*\alpha=0$ and $r_1=\mathrm{Id}$.

Then

$$\frac{d}{dt}(r_t)^*(\alpha)=r_t^*(L_{V_t}\alpha)=d\left(r_t^*(i_{V_t}\alpha)\right)$$

Note that $r_t^*(i_{V_t}\alpha)$ is well defined, continuous and bounded as t goes to zero, since writing (locally) (u,η) for a tangent vector to $T_{(x,\xi)}\nu_N M$

$$(r_t^*(i_{V_t}\alpha))(x,\xi)((u_2,\eta_2)....(u_p,\eta_p))=\alpha(x,t\xi)((0,\xi),(u_2,t\eta_2)...(u_p,t\eta_p))$$

remains C^1 bounded as t goes to zero. Let us denote by β_t the above form. We can write for ε positive

$$r_1^*(\alpha)-r_\varepsilon^*(\alpha)=\int_\varepsilon^1\frac{d}{dt}[(r_t)^*(\alpha)]dt=d\left(\int_\varepsilon^1(r_t)^*(i_{V_t}\alpha)dt\right)$$

Since as t goes to zero, $d(r_t^*(i_{V_t}\alpha))$ remains bounded, thus $\lim_{\varepsilon\to 0}\int_0^\varepsilon d(r_t^*(i_{V_t}\alpha))=0$ and we have that

$$\alpha=r_1^*(\alpha)-r_0^*(\alpha)=\lim_{\varepsilon\to 0}[r_1^*(\alpha)-r_\varepsilon^*(\alpha)]=$$
$$\lim_{\varepsilon\to 0}d\left(\int_\varepsilon^1(r_t)^*(i_{V_t}\alpha)dt\right)=d\left(\lim_{\varepsilon\to 0}\int_0^1(r_t)^*(i_{V_t}\alpha)dt\right)=d\beta$$

where

$$\beta=\int_0^1(r_t)^*(i_{V_t}\alpha)dt=\int_0^1\beta_t dt$$

but β_t vanishes on N, since

$$\beta_t(x,0)((u_2,\eta_2)...(u_p,\eta_p)=\alpha(x,0)((0,0),(u_2,t\eta_2)...(u_p,t\eta_p)=0$$

This proves our lemma. □

□

EXERCICE 1. Prove using the above lemma that if N is a submanifold of M, $H^*(M,N)$ can either be defined as the set of closed forms vanishing on TN modulo the differential forms vanishing on TN or as the set of closed form vanishing in a neighborhood of N modulo the differential of forms vanishing near N.

As an application, we have

PROPOSITION 3.5 (Darboux). *Let (M,ω) be a symplectic manifold. Then for each $z\in M$, there is a local diffeomorphism φ from a neighborhood of z to a neighborhood of o in $\mathbb{R}^{2n}$ such that $\varphi^*\sigma=\omega$.*

PROOF. According to Lecture 1, there exists a linear map $L : T_zM \to \mathbb{R}^{2n}$ such that $L^*\sigma = \omega(z)$. Hence, using a local diffeomorphism $\varphi_0 : U \to W$ such that $d\varphi_0(z) = L$, where U and W are neighborhoods of $z \in M$ and $o \in \mathbb{R}^{2n}$ respectively, we are reduced to considering the case where $\varphi_0^*\sigma$ is a symplectic form defined in U and $\omega(z) = (\varphi_0^*)\sigma$.

Define $\omega_t = (1-t)\varphi_0^*\sigma + t\omega$ in U. It's easy to check ω_t satisfies the assumptions of Moser's Lemma, therefore, there exists ψ such that $\psi^*\omega_1 = \omega_0$, i.e.

$$\psi^*\omega = \varphi_0^*\sigma.$$

Then $\varphi = \varphi_0 \circ \psi^{-1}$ is the required diffeomorphism. $\square$

EXERCICES 2. (1) Show the analogue of Moser's Lemma for volume forms.
(2) Let ω_1, ω_2 be symplectic forms on a compact surface without boundary. Then there exists a diffeomorphism φ such that $\varphi^*\omega_1 = \omega_2$ if and only if $\int\omega_1 = \int\omega_2$.

PROPOSITION 3.6. *(Weinstein) Let L be a closed Lagrangian submanifold in* (M,ω). *Then L has a neighborhood symplectomorphic to a neighborhood of* $O_L \subset T^*L$. *(Here,* $O_L = \{(q,0)|q \in L\}$ *is the zero section.)*

PROOF. The idea of the proof is the same as that of Darboux Lemma.

First, for any $x \in L$, find a subspace $V(x)$ in T_xM such that
1) $V(x) \subset T_xM$ is Lagrangian subspace;
2) $V(x) \cap T_xL = \{0\}$;
3) $x \to V(x)$ is smooth.

According to our discussion in linear symplectic space, we can find such $V(x)$ at least pointwise. To see 3), note that at each point $x \in L$ the set of all Lagrangian subspaces in T_xM transverse to T_xL may be identified with quadratic forms on $(T_xL)^*$. It's then possible to find a smooth section of such an "affine bundle".

Abusing notations a little, we write L for the zero section in T^*L. Denote by $T_L(T^*L)$ the restriction of the tangent bundle T^*L to L. Denote by T_LM the restriction of the bundle TM to L. Both bundles are over L. For $x \in L$, their fibers are

$$T_x(T^*L) = T_xL \oplus T_x(T_x^*L)$$

and

$$T_xM = T_xL \oplus V(x).$$

Construct a bundle map $L_0 : T_L(T^*L) \to T_LM$ which restricts to identity on factor T_xL and sends $T_x(T_x^*L)$ to $V(x)$. Moreover, we require

$$\omega(L_0u, L_0v) = \sigma(u,v),$$

where $u \in T_x(T_x^*L) = T_x^*L$ and $v \in T_xL$. This defines L_0 uniquely. Again, we can find φ_0 from a neighborhood of L in T^*L to a neighborhood of L in M such that $d\varphi_0|_{T_L(T^*L)} = L_0$. By the construction of L_0, one may check that

$$\varphi_0^*\omega = \sigma \text{ on } T_L(T^*L).$$

Define

$$\omega_t = (1-t)\varphi_0^*\omega + t\sigma, \quad t \in [0,1].$$

ω_t is a family of symplectic forms in a neighborhood of O_L. Moreover, $\omega_t \equiv \omega_0$ on $T_L(T^*L)$. By Moser's Lemma, there exists Ψ defined near O_L such that $\Psi^*\omega_1 = \omega_0$, i.e. $\Psi^*\sigma = \varphi_0^*\omega$. Then $\varphi_0 \circ \Psi^{-1}$ is the diffeomorphism we need. □

EXERCICE 3. Let I_1, I_2 be two diffeomorphic isotropic submanifold in (M_1,ω_1), (M_2,ω_2). Let $E_1 = (TI_1)^{\omega_1}/(TI_1)$ and $E_2 = (TI_2)^{\omega_2}/(TI_2)$. E_1, E_2 are symplectic vector bundles over I_1 and I_2. Show that I_1 and I_2 have symplectomorphic neighborhoods if and only if $E_1 \cong E_2$ as symplectic vector bundles.

EXERCICE 4. Same exercise in the coisotropic situation.

2. The groups Ham and $Diff_\omega$

Since Klein's Erlangen's program, geometry has meant the study of symmetry groups. The group playing the first role here is $Diff_\omega(M)$. Let (M,ω) be a symplectic manifold. Define

$$Diff_\omega(M) = \{\varphi \in Diff(M) | \varphi^*\omega = \omega\}.$$

This is a very large group since it contains $Ham(M,\omega)$, which we will now define.

Let $H(t,x)$ be any smooth function and X_H the unique vector field such that

$$\omega(X_H(t,x),\xi) = d_x H(t,x)\xi, \quad \forall \xi \in T_x M.$$

Here d_x means exterior derivative with respect to x only.

Claim: The flow of X_H is in $Diff_\omega(M)$.

To see this,

$$\begin{aligned} \frac{d}{dt}(\varphi^t)^*\omega &= (\varphi^t)^*(L_{X_H}\omega) \\ &= (\varphi^t)^*(d \circ i_{X_H}\omega + i_{X_H} \circ d\omega) \\ &= (\varphi^t)^*(d(dH)) = 0. \end{aligned}$$

DEFINITION 3.7. The set of all diffeomorphism φ that can be obtained as the flow of some H is a subgroup $Diff(M,\omega))$ called $Ham(M,\omega)$.

To prove that $Ham(M,\omega)$ is a subgroup, we proceed as follows: first notice that the Hamiltonian isotopy can be reparametrized, and still yields a Hamiltonian isotopy $\varphi_{s(t)}$ satisfying

$$(\frac{d}{dt}\varphi_{s(t)})_{t=t_0} = s'(t)(\frac{d}{ds}\varphi_s)_{s=s(t_0)} = s'(t)X_H(s(t_0),\varphi_{s(t_0)})$$

which is the Hamiltonian flow of

$$s'(t)H(s(t),z)$$

Therefore we may use a function $s(t)$ on $[0,1]$ such that $s(0)=0, s(1/2)=1, s'(t)=0$ for t close to $1/2$ and we find a Hamiltonian flow ending at φ_1 in time $1/2$ and such that

H vanishes near $t=1/2$. Similarly if ψ_t is the flow associated to $K(t,z)$ we may modify it in a similar way using $r(t)$ so that $K\equiv 0$ for t in a neighborhood of $[0,1/2]$. We can then consider the flow associated to $H(t,z)+K(t,z)=L(t,z)$ it will be $\varphi_{s(t)}\circ\psi_{r(t)}$ and for $t=1$ we get $\varphi_1\circ\psi_1$.

That φ_1^{-1} is also Hamiltonian follows from the fact that $-H(t,\varphi_t(z))$ has flow φ_t^{-1}.

EXERCICE 5. Show that $(\varphi^t)^{-1}\psi^t$ is the Hamiltonian flow of

$$L(t,z)=K(t,\varphi_t(z))-H(t,\varphi_t(z))$$

This immediately proves that $Ham(M,\omega)$ is a group.

REMARK 3.8. Denote by $Diff_{\omega,0}$ the component of $Diff_\omega(M)$ in which the identity lies. It's obvious that $Ham(M,\omega)\subset(Diff_{\omega,0}(M)$.

REMARK 3.9. If $H(t,x)=H(x)$, then $H\circ\varphi^t=H$. This is what physicists call conservation of energy. Indeed H is the energy of the system, and for time-independent conservative systems, energy is preserved. This is not the case in time-dependent situations.

REMARK 3.10. If we choose local coordinates $q_1,...,q_n$ and their dual $p_1,...,p_n$ in the cotangent space, p_i,q_i, the flow is given by the ODE

$$\begin{cases} \dot q_i = \frac{\partial H}{\partial p_i}(t,q,p) \\ \dot p_i = -\frac{\partial H}{\partial q_i}(t,q,p) \end{cases}$$

Question: How big is the quotient $Diff_{\omega_0}/Ham(M,\omega)$?

Given $\varphi\in Diff_{\omega_0}$, there is an obvious obstruction for φ to belong to $Ham(M,\omega)$. Assume $\omega=d\lambda$. Then $\varphi^*\lambda-\lambda$ is closed for all $\varphi\in Diff_\omega$, since

$$d(\varphi^*\lambda-\lambda)=\varphi^*\omega-\omega=0.$$

If φ^t is the flow of X_H,

$$\begin{array}{rcl} \dfrac{d}{dt}((\varphi^t)^*\lambda) & = & (\varphi^t)^*(L_{H_X}\lambda) \\ & = & (\varphi^t)^*(d(i_{X_H}\lambda)+i_{X_H}d\lambda) \\ & = & (\varphi^t)^*d(i_{X_H}\lambda+H) \\ & = & d((\varphi^t)^*(i_{X_H}\lambda+H)). \end{array}$$

This implies that $\varphi^*\lambda-\lambda$ is exact.

In summary, we can define map

$$\begin{array}{rrcl} \text{Flux}: & (Diff_\omega)_0(M) & \to & H^1(M,\mathbb{R}) \\ & \varphi & \mapsto & [\varphi^*\lambda-\lambda] \end{array}$$

We know

$$Ham_\omega(M)=\ker(Flux).$$

Examples:

(1) On T^*T^1 the translation $\varphi : (x,p) \longrightarrow (x, p+p_0)$ is symplectic, but $\text{Flux}(\varphi) = p_0$.
(2) Similarly if $M = T^2$ and $\sigma = dx \wedge dy$, the map $(x,y) \longrightarrow (x, y+y_0)$ is not in $Ham(T^2,\sigma)$ for $y_0 \not\equiv 0 \mod 1$.

Indeed, since the projection $\pi : T^*T^1 \longrightarrow T^2$ is a symplectic covering, any Hamiltonian isotopy on T^2 ending in φ would lift to a Hamiltonian isotopy on T^*T^1 (if $H(t,z)$ is the Hamiltonian on T^2, $H(t,\pi(z))$ is the Hamiltonian on T^*T^1) ending to some lift of φ. But the lifts of φ are given by $(x,y) \longrightarrow (x+m, y+y_0+n)$ for $(m,n) \in \mathbb{Z}^2$, with Flux given by $y_0 + n \neq 0$.

EXERCICES 6. (1) Prove the Darboux-Weinstein-Givental theorem: Let S_1, S_2 be two submanifolds in $(M_1,\omega_1),(M_2,\omega_2)$. Assume there is a map $\varphi : S_1 \longrightarrow S_2$ which lifts to bundle map

$$\Phi : T_{S_1}M_1 \longrightarrow T_{S_2}M_2$$

coinciding with $d\varphi$ on the subbundle TS_1, and preserving the symplectic structures, i.e. $\Phi^*(\omega_2) = \omega_1$.

Then there is a symplectic diffeomorphism between a neighborhood U_1 of S_1 and a neighborhood U_2 of S_2.

(2) Use the Darboux-Weinstein-Givental theorem to prove that all closed curves have symplectomorphic neighborhoods. Hint: Show that all symplectic vector bundle on the circle are trivial.

(3) (a) Prove that the Flux homomorphism can be defined on (M,ω) as follows. Let φ_t be a symplectic isotopy. Then $\frac{d}{dt}\varphi_t(z) = X(t,\varphi_t(z))$ and $\omega(X(t,z)) = \alpha_t$ is a closed form. Then

$$\widetilde{\text{Flux}}(\varphi) = \int_0^1 \alpha_t dt \in H^1(M,\mathbb{R})$$

depends only on the homotopy class of the path φ_t. If Γ is the image by Flux of the set of closed loops, we get a well defined map

$$\text{Flux} : Diff(M,\omega)_0 \longrightarrow H^1(M,\mathbb{R})/\Gamma$$

(b) Prove that when ω is exact, Γ vanishes and the new definition coincides with the old one.

CHAPTER 4

More Symplectic differential Geometry: Reduction and Generating functions

Philosophical Principle: Everything important is a Lagrangian submanifold.
Examples:

(1) If $(M_i, \omega_i), i = 1, 2$ are symplectic manifolds and φ a symplectomorphism between them, that is a map from M_1 to M_2 such that $\varphi^*\omega_2 = \omega_1$. Consider the graph of φ,
$$\Gamma(\varphi) = \{(x, \varphi(x))\} \subset M_1 \times M_2.$$
This is a Lagrangian submanifold of $M_1 \times \overline{M_2}$ if we define M_2 as the manifold M_2 with the symplectic form $-\omega_2$ and the symplectic form on $M_1 \times \overline{M_2}$ is given by
$$(\omega_1 \ominus \omega_2)((\xi_1, \xi_2), (\eta_1, \eta_2)) = \omega_1(\xi_1, \eta_1) - \omega_2(\xi_2, \eta_2).$$
In fact, it's easy to see $\Gamma(\varphi)$ is a Lagrangian submanifold if and only if $\varphi^*\omega_2 = \omega_1$. Note that if $M_1 = M_2$, then $\Gamma(\varphi) \cap \triangle_M = \mathrm{Fix}(\varphi)$.

(2) Let (M, J, ω) be a smooth projective manifold, i.e. a smooth manifold given by
$$M = \{P_1(z_0, \cdots, z_N) = \cdots = P_i(z_0, \cdots, z_N) = 0\}$$
where P_j are homogeneous polynomials. We shall assume the map from $\mathbb{C}^n \setminus \{0\}$ to $\mathbb{C}^r$
$$(z_0, ..., z_n) \mapsto (P_1(z_0, ..., z_n), ..., P_r(z_0, ..., z_n))$$
has zero as a regular value, so that M is a smooth manifold.

If P_j's have real coefficients, then real algebraic geometry is concerned with
$$\begin{aligned} M_{\mathbb{R}} &= \{[x_0, \cdots, x_N] \in \mathbb{R}P^N | P_j(x_0, \cdots, x_N) = 0\} \\ &= M \cap \mathbb{R}P^N. \end{aligned}$$
The problem is to "determine the relation" between M and $M_{\mathbb{R}}$". It is easy to see that $M_{\mathbb{R}}$ is a Lagrangian of (M, ω) (of course, possibly empty).

1. Symplectic Reduction

Let (M, ω) be a symplectic manifold and K a submanifold. K is said to be *coisotropic* if $\forall x \in K$, we have $T_xK \supset (T_xK)^\omega$. As x varies in K, $(T_xK)^\omega$ gives a distribution in T_xK.

LEMMA 4.1. *This distribution is integrable.*

PROOF. According to Frobenius theorem, it suffices to check that for all vector field $X, Y \in (T_xK)^\omega$, η in T_xK,

$$\omega([X,Y],\eta) = 0$$

where X and Y are vector fields in $(T_xK)^\omega$.

$d\omega(X,Y,\eta)$ vanishes, but on the other hand is a sum of terms of the form:

$X \cdot \omega(Y,\eta)$ but since $\omega(Y,\eta)$ is identically zero these terms vanishes. The same holds if we exchange X and Y.

$\eta \cdot \omega(X,Y)$ vanishes for the same reason.

$\omega(X,[Y,\eta])$ and $\omega(Y,[X,\eta])$ vanish since $[X,\eta],[Y,\eta]$ are tangent to K.

$\omega([X,Y],\eta)$ is the only remaining term. But since the sum of all terms must vanish, this must also vanish, hence $[X,Y] \in (T_xK)^\omega$ □

This integrable distribution gives a foliation of K, denoted by $\mathscr{C}_K$. We can check that ω induces a symplectic form (we only need to check it is nondegenerate) on the quotient space $(T_xK)/(T_xK)^\omega$. One might expect $K/\mathscr{C}_K$ to be a a "symplectic something".

Unfortunately, due to global topological difficulties, there is no nice manifold structure on the quotient. However, in certain special cases, as will be illustrated by examples in the end of this section, $K/\mathscr{C}_K$ is a manifold, and therefore a symplectic manifold.

Let us now see the effect of the above operation on symplectic manifolds.

LEMMA 4.2. *(Automatic Transversality) If L is a Lagrangian in M and L intersects K transversally, i.e. $T_xL + T_xK = T_xM$ for $x \in K \cap L$, then L intersects the leaves of C_K transversally, $T_xL \cap T_x\mathscr{C}_K = \{0\}$, for $x \in K \cap L$.*

PROOF. Recall from symplectic linear algebra that if F_i are subspaces of a symplectic vector space, then

$$(F_1 + F_2)^\omega = F_1^\omega \cap F_2^\omega.$$

We know $(T_xL)^\omega = T_xL$ and $(T_xM)^\omega = \{0\}$, then the lemma follows from $T_xL + T_xK = T_xM$. □

Now, let's pretend $K/\mathscr{C}_K$ is a manifold and denote the projection by $\pi : K \to K/\mathscr{C}_K$.

1) K and L intersect transversally, so in particular $L \cap K$ is a manifold.

2) The projection $\pi : (L \cap K) \to K/\mathscr{C}_K$ is an immersion.

$$\ker d\pi(x) = T_x\mathscr{C}_K = (T_xK)^\omega.$$

$$\begin{aligned} \ker d\pi(x)|_{T_x(L\cap K)} &\subset \ker d\pi(x) \cap T_xL \\ &\subset (T_xK)^\omega \cap TxL = \{0\}. \end{aligned}$$

Therefore $d\pi(x)|_{L\cap K}$ is injective and $\pi|_{L\cap K}$ is immersion.

To summarize our findings, given a symplectic manifold (M,ω) and a coisotropic submanifold K, let L be a Lagrangian of M intersecting K transversally. Define L_K to be the image of the above immersion. Then it is a Lagrangian in $K/\mathscr{C}_K$. This operation is called *symplectic reduction*.

The only thing left to check is that L_K is Lagrangian. Let $\tilde{\omega}$ be the induced symplectic form on $K/\mathscr{C}_K$ and $\tilde{v}$ is a tangent vector of L_K. Assume the preimage of $\tilde{v}$ is v, a tangent vector of L. Since L is Lagrangian and $\tilde{\omega}$ is induced from ω, we know L_K is isotropic. It's Lagrangian by a dimension count. The same argument shows that the reduction of an isotropic submanifold (resp. coisotropic submanifold) is isotropic (resp. coisotropic).

Example 1: Let N be a symplectic manifold, and V be any smooth submanifold. Define

$$K = T^*_V N = \{(x,p) | x \in V, p \in T^*_x N\}.$$

This is a coisotropic submanifold, and its coisotropic foliation $\mathscr{C}_K$ is given by specifying the leaf through $(x,p) \in K$ to be

$$\mathscr{C}_K(x,p) = \{(x,\tilde{p}) \in K \mid \tilde{p} - p \text{ vanishes on } T_x V\}.$$

It is natural to identify $K/\mathscr{C}_K$ with T^*V.

Symplectic reduction in this case, sends Lagrangian in T^*N to Lagrangian in T^*V.

Example 2: Let N_1, N_2 are smooth manifolds and $N = N_1 \times N_2$. Suppose we choose local coordinates near a point in T^*N is written as

$$(x_1, p_1, x_2, p_2).$$

where $(x_1,p_1) \in T^*N_1, (x_2,p_2) \in T^*N_2$. Define $K = \{(x_1,p_1,x_2,p_2) | p_2 = 0\}$. The tangent space of K at a point $z = (x_1,p_1,x_2,p_2)$ is given by

$$(v_1, w_1, v_2, 0),$$

$$(T_z K)^\omega = \{(0,0,0,w_2)\}.$$

Then we can identify $K/\mathscr{C}_K$ with T^*N_1.

Symplectic reduction sends a Lagrangian in T^*N to a Lagrangian in T^*N_1.

1.1. Lagrangian correspondences. Let Λ be a Lagrangian submanifold in $\overline{T^*X} \times T^*Y$. Then it induces a correspondence from T^*X to T^*Y as follows: consider a set $C \subset T^*X$, and $C \times \Lambda \subset T^*X \times \overline{T^*X} \times T^*Y$. Now, denote by Δ_{T^*X} the diagonal in $T^*X \times \overline{T^*X}$. The submanifold $K = \Delta_{T^*X} \times T^*Y$ is coisotropic, and we define $\Lambda \circ C$ as $C \times \Lambda \cap K/\mathscr{K} \subset K/\mathscr{K} = T^*Y$. When C is a submanifold, then $\Lambda \circ C$ is a submanifold provided $C \times T^*Y$ is transverse to Λ.

If C is isotropic or coisotropic, it is easy to check that the same will hold for $\Lambda \circ C$. In particular if L is a Lagrangian submanifold, the correspondence maps $\mathscr{L}(T^*X)$ to $\mathscr{L}(T^*Y)$ (well, not everywhere defined) can alternatively be defined as follows : take the symplectic reduction of Λ by $L \times \overline{T^*Y}$. This is well defined at least when L is generic. We denote it by $\Lambda \circ L$.

Note that Λ^a (sometimes denoted as Λ^{-1}) is defined as $\Lambda^a = \{(x,\xi,y,\eta) \mid (y,\eta,x,\xi)Ê \in \Lambda\}$. This is a Lagrangian correspondence from T^*Y to T^*X. The composition $\Lambda \circ \Lambda^a \subset$

$T^*X \times \overline{T^*X}$ is, in general, not equal to the identity (i.e. Δ_{T^*X}, the diagonal in T^*X), even though this is the case if Λ is the graph of a symplectomorphism.

EXERCICE 1. Compute $\Lambda \circ \Lambda^a$ for $\Lambda = V_x \times V_y$, where V_x is the cotangent fiber over x.

2. Generating functions

Our goal is to describe Lagrangian submanifolds in T^*N. Let $\lambda = pdx$ be the Liouville form of T^*N. Given any 1-form α on N, we can define a smooth manifold

$$L_\alpha = \{(x, \alpha(x)) | x \in N, \alpha(x) \in T_x^*N\} \subset T^*N.$$

LEMMA 4.3. *L_α is Lagrangian if and only if α is closed.*

PROOF. Let $i : N \to TN$ be the embedding map $i(x) = (x, \alpha(x))$. Notice that

$$\lambda|_{L_\alpha} = \alpha$$

i.e.

$$i^*(\lambda) = \alpha.$$

Lagrangian condition is $(d\lambda)|_{L_\alpha} = 0$, i.e. $d\alpha = 0$. □

DEFINITION 4.4. If $\lambda|_L$ is exact, we say L is exact Lagrangian.

In particular, L_α is exact if and only if $\alpha = df$ for some function f on N. In this case,

$$L_\alpha \cap O_N = \{x | \alpha(x) = df(x) = 0\} = Crit(f),$$

where O_N is the zero section of TN.

REMARK 4.5. 1) If L is C^1 close to O_N, then $L = L_\alpha$ for some α. To see this, L_α is 'graph' of α in TN and a C^1 perturbation of a graph is a graph.

2) If L is exact, C^1 close to O_N, then $L = L_{df}$. Therefore, $\#(L \cap O_N) \geq 2$, if we assume N is compact. (f has at least two critical points, corresponding to maximum and minimum, and we may find more with more sophisticated tools.)

Arnold Conjecture: If $\varphi \in Ham_\omega(T^*N)$ and $L = \varphi(O_N)$, then $\#(L \cap O_N) \geq \mathrm{cat}_{\mathrm{LS}}(N)$, where $\mathrm{cat}_{\mathrm{LS}}(N)$ is the minimal number of critical points for a function on N.

DEFINITION 4.6. A generating function for L is a smooth function $S : N \times \mathbb{R}^k \to \mathbb{R}$ such that

1) The map

$$(x, \xi) \quad \mapsto \quad \frac{\partial S}{\partial \xi}(x, \xi)$$

has zero as a regular value. As a result $\Sigma_S = \{(x, \xi) | \frac{\partial S}{\partial \xi}(x, \xi) = 0\}$ is a submanifold. (Note that $\partial S / \partial \xi$ is a vector of dimension k, so Σ_S is a manifold with the same dimension as N, but may have a different topology.)

2)

$$\begin{array}{rccc} i_S: & \Sigma_S & \to & T^*N \\ & (x,\xi) & \mapsto & (x, \frac{\partial S}{\partial x}(x,\xi)) \end{array}$$

has image $L = L_S$.

LEMMA 4.7. *If for some given S satisfying 1) of the definition and L_S is given by 2), then L_S is an immersed Lagrangian in T^*N.*

PROOF. Since S is a function from $N \times \mathbb{R}^k$ to $\mathbb{R}$, the graph of dS in $T^*(N \times \mathbb{R}^k)$ is a Lagrangian in $T^*(N \times \mathbb{R}^k)$. We will use the symplectic reduction as in the Example 2 in the last section. Define K as a submanifold in $T^*(N \times \mathbb{R}^k)$,

$$K = T^*N \times \mathbb{R}^k \times \{0\}.$$

K is coisotropic as shown in Example 2. Locally, the graph of dS is given by

$$gr(dS) = \{(x, \xi, \frac{\partial S}{\partial x}(x,\xi), \frac{\partial S}{\partial \xi}(x,\xi))\}.$$

Then

$$\Sigma_S = gr(dS) \cap K.$$

The regular value condition in 1) ensures that $gr(dS)$ intersects K transversally. By symplectic reduction, we know i_S is an immersion and L_S is a Lagrangian in T^*N because $gr(dS)$ is Lagrangian in $T^*(N \times \mathbb{R}^k)$. $\square$

REMARK 4.8. If L_S is embedded, we have

$$L_S \cap O_N \simeq Crit(S).$$

Question: Which L have a generating function?
Answer: (Giroux) It is given by conditions on the tangent bundle TL.

DEFINITION 4.9. Let S be a generating function on $N \times \mathbb{R}^k$. We say that S is quadratic at infinity if there exists a nondegenerate quadratic form Q on $\mathbb{R}^k$ such that

$$S(x,\xi) = Q(\xi) \quad \text{for } |\xi| >> 0.$$

For simplicity, we will use GFQI to mean generating function quadratic at infinity.

PROPOSITION 4.10. *Let S be a generating function of L_S such that*
(1) $\|\nabla(S-Q)\|_{C^0} \le C$,
(2) $\|S-Q\|_{C^0(B(0,r))} \le Cr$,
then there exists $\tilde{S}$ GFQI for L_S.

PROOF. (sketch) Let $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ be a nonincreasing function such that $\rho \equiv 1$ on $[0, A]$, $\rho \equiv 0$ on $[B, +\infty)$ and $-\varepsilon \le \rho' \le 0$. Define

$$S_1(x,\xi) = \rho(|\xi|)S(x,\xi) + (1 - \rho(|\xi|))Q(\xi)$$

We are going to prove that

$$\frac{\partial}{\partial \xi} S_1(x,\xi) = 0 \Longleftrightarrow \frac{\partial}{\partial \xi} S_0(x,\xi) = 0$$

Indeed,

$$\frac{\partial}{\partial \xi} S_1(x,\xi) = \frac{\partial}{\partial \xi}\left(\rho(|\xi|)\,(S(x,\xi) - Q(\xi)) + Q(\xi)\right)$$

$$= \rho'(|\xi|)\frac{\xi}{|\xi|}(S(x,\xi) - Q(\xi)) + \rho(|\xi|)\frac{\partial}{\partial \xi}(S-Q)(x,\xi) + A_Q\xi = 0$$

For this one must have, if $|A\xi| \geq k|\xi|$

$$c|\xi| \leq \varepsilon \|S - Q\|_{C^0} + \|\nabla(S-Q)\|_{C^0} \leq \varepsilon C|\xi| + C$$

therefore for ε small enough, this implies

$$|\xi| \leq \frac{C}{c - \varepsilon C}$$

and this remains bounded for ε small enough. If we choose A large enough so that it is larger than $\frac{C}{c-\varepsilon C}$, then $S_1 = S_0$ and therefore Σ_{S_1} and Σ_{S_0} coincide, and also i_{S_1} and i_{S_0}. □

THEOREM 4.11. *(Sikorav) N is compact. Let $L = \varphi(O_N)$ and $\varphi \in Ham(T^*N)$. Then L has a GFQI.*

PROOF. (Brunella) Consider a "special" case $N = \mathbb{R}^N$ and $\varphi \in Ham^0(\mathbb{R}^N)$. By superscript 0, we mean compactly supported.

There is a ' 'correspondence" between function $h : N \times N \to \mathbb{R}$ and maps $\varphi_h : T^*N \to T^*N$ given by

$$\varphi_h(x_1,p_1) = (x_2,p_2) \quad\Longleftrightarrow\quad \begin{cases} p_1 = \frac{\partial}{\partial x_1} h(x_1,x_2) \\ p_2 = -\frac{\partial}{\partial x_2} h(x_1,x_2) \end{cases}$$

The graph of φ_h is a submanifold in $T^*\mathbb{R}^n \times T^*\mathbb{R}^n$ with symplectic form given by $\omega = dp_1 \wedge dx_1 - dp_2 \wedge dx_2$. It's a Lagrangian if and only if φ_h is a symplectic diffeomorphism.

The graph of dh is a submanifold in $T^*(\mathbb{R}^n \times \mathbb{R}^n)$ with the natural symplectic structure and it's Lagrangian.

Note that the first is a graph of a map T^*N to T^*N while the second is the graph of a map $N \times N$ to $\mathbb{R}^l \times \mathbb{R}^l$ (in particular the first is transverse to $\{0\}(T^*N)$, while the second is transverse to $\{0\} \times \mathbb{R}^l$).

There is a symplectic isomorphism between $T^*\mathbb{R}^n \times T^*\mathbb{R}^n$ and $T^*(\mathbb{R}^n \times \mathbb{R}^n)$, given by

$$(x_1,p_1,x_2,p_2) \longmapsto (x_1,x_2,p_1,-p_2).$$

and this maps the graph of dh to the graph of φ_h.

Set $h_0(x_1, x_2) = \frac{1}{2}|x_1 - x_2|^2$, then

$$\varphi_{h_0}(x_1, p_1) = (x_1 - p_1, p_1).$$

If h is C^2 close to h_0, then $gr(dh)$ is C^1 close to $gr(dh_0)$, under isomorphism, $\Gamma(\varphi_{h_0})$, since C^1 perturbation of a graph is a graph, we know (up to isomorphism) $gr(dh) = \Gamma(\varphi_h)$. Since $gr(dh)$ is always Lagrangian, φ_h is symplectic isomorphism.

REMARK 4.12. We can do the same with $-h_0$.

$$\varphi_{-h_0} = (\varphi_{h_0})^{-1}.$$

REMARK 4.13. We can do the inverse. Any φ C^1 close to φ_{h_0} is of the form φ_h.

PROPOSITION 4.14 (Chekanov's composition formula). *Let L be a Lagrangian in $T^*\mathbb{R}^n$. L coincides with O_N outside a compact set and has a GFQI $S(x,\xi)$. If $h = h_0$ near infinity, then $\varphi_h(L)$ has GFQI*

$$\tilde{S}(x, \xi, y) = h(x, y) + S(y, \xi).$$

REMARK 4.15. $\tilde{S}$ is only approximately quadratic at infinity. We use the last proposition to make it real GFQI.

For the proof of the claim, check that $L_{\tilde{S}}$ is $\varphi_h(L)$.

$$\frac{\partial \tilde{S}}{\partial \xi}(x, \xi, y) = 0 \Longleftrightarrow \frac{\partial S}{\partial \xi}(y, \xi) = 0.$$

$$\frac{\partial \tilde{S}}{\partial y}(x, \xi, y) = 0 \Longleftrightarrow \frac{\partial h}{\partial y}(x, y) + \frac{\partial S}{\partial y}(y, \xi) = 0.$$

A point in $L_{\tilde{S}}$ is

$$\begin{aligned}
(x, \frac{\partial \tilde{S}}{\partial x}(x, \xi, y)) &= (x, \frac{\partial h}{\partial x}(x, y)) \\
&= \varphi_h(y, -\frac{\partial h}{\partial y}(x, y)) \\
&= \varphi_h(y, \frac{\partial S}{\partial y}(y, \xi)).
\end{aligned}$$

$(y, \frac{\partial S}{\partial y}(y, \xi))$ is a point in L_S.

If k is close to $-h_0$, $\varphi_k \circ \varphi_h(L)$ has GFQI. If $k = -h_0$, then $(\varphi_{h_0}^{-1} \circ \varphi_h)(L)$ has GFQI.

Any C^1 small symplectic map ψ can be given as

$$\varphi_h = \varphi_{h_0} \circ \psi.$$

So the conclusion is for any ψ C^1 close to the identity, if L has GFQI, then $\psi(L)$ has GFQI.

Now take $\varphi^t \in Ham(T^*N)$.

$$\varphi^1 = \varphi^1_{\frac{N-1}{N}} \circ \varphi^{\frac{N-1}{N}}_{\frac{N-2}{N}} \cdots \varphi_0^{\frac{1}{N}}.$$

Each factor is C^1 small. Then If L has GFQI, then $\varphi^1(L)$ has GFQI.

□

3. The Maslov class

The Maslov or Arnold-Maslov class is a topological invariant of a Lagrangian submanifold, measuring how much its tangent space "turns" with respect to a given Lagrangian distribution.

4. Contact and homogeneous symplectic geometry

4.1. Contact geometry, symplectization and contactization. Let (N,ξ) be a pair constituted of a manifold N, and a hyperplane field ξ on N. This means that locally, there is a non-vanishing 1-form α such that $\xi = \mathrm{Ker}(\alpha)$.

DEFINITION 4.16. The pair (N,ξ) is a **contact manifold** if integral submanifolds of ξ (i.e. submanifolds everywhere tangent to ξ) have the minimal possible dimension, i.e. $\frac{\dim(N)-1}{2}$. Such an integral manifold is called a **Legendrian submanifold**.

It is easy to check that if locally $\xi = \mathrm{Ker}(\alpha)$, the contact type condition is equivalent to requiring that $\alpha \wedge (d\alpha)^{n-1}$ is nowhere vanishing. Note also that the global existence if α is equivalent to the co-orientability of ξ. Sometimes we assume the existence of α. This is always possible, at the cost of going to a double cover.

Examples:

(1) the standard example is $\mathbb{R}^{2n+1}$, with coordinates $q_1,...,q_n,p^1,...,p^n,z$ and $\xi = \ker(\alpha)$ with $\alpha = dz - p^1 dq_1 - ... - p^n dq_n$.
(2) A slightly more general case is $J^1(N)$ for any manifold N. This is the set of (q,p,z) where $z \in N$, $p \in T_q^*N$ and $z \in \mathbb{R}$, the contact form being $dz - pdq$. Note that for any smooth function f on N, the set $j^1f = \{(q,df(q),f(q)0 \mid q \in N\}$ is Legendrian. Moreover any Legendrian graph is of this form.
(3) The manifold $ST^*N = \{(q,p) \in T^*N \mid |p| = 1\}$, where $|\bullet|$ is induced by any riemannian metric on N, endowed with the restriction of the Liouville form. The same holds for $PT^*N = ST^*N/\simeq$ where $(q,p_1) \simeq (q,p_2)$ if and only if $p_1 = \pm p_2$.

EXERCICE 2. Prove that $PT^*\mathbb{R}^n$ is contactomorphic to J^1S^{n-1}. There is a natural contactomorphism called Euler coordinates: a point $(q,p) \in PT^*(\mathbb{R}^n)$ corresponds in a unique way to a to a point in $\mathbb{R}^n$ and a linear hyperplane (i.e. the pair $(q,\ker(p))$), that may be replaced by the parallel linear hyperplane through this point. In other words we identify $PT^*\mathbb{R}^n$ to the set of pairs constituted of an affine hyperplanes and a point on the hyperplane. The hyperplane may be associated to its normal vector, q, in S^{n-1}, the distance from the origin to the hyperplane, a real number z, and a vector in the hyperplane, connecting the orthogonal projection of the origin on the hyperplane and the point, p. Now (q,p,z) are in $J^1(S^{n-1})$ because p is orthogonal to q , provided we use the canonical metric in $\mathbb{R}^n$ to identify vectors and covectors.

There are two constructions relating symplectic and contact manifolds.

DEFINITION 4.17 (Symplectization of a contact manifold). Let (N,ξ) be a contact manifold, with contact form α. Then $(N\times\mathbb{R}_+^*, d(t\alpha))$ is a symplectic manifold called the symplectization of (N,ξ).

PROPOSITION 4.18 (Uniqueness of the Symplectization). *If* $\mathrm{Ker}(\alpha)=\mathrm{Ker}(\beta)=\xi$ *we have a symplectomorphism between* $(N\times\mathbb{R}_+^*, d(t\alpha))$ *and* $(N\times\mathbb{R}_+^*, d(t\beta))$. *Indeed, we have* $\beta = f\alpha$ *where* f *is a non-vanishing function on* N. *Then the map* $F:(z,t)\mapsto(z,f(z)\cdot t)$ *satisfies* $F^*(t\alpha)=tf(z)\alpha=t\beta$, *so realizes a symplectomorphism* $F:(N\times\mathbb{R}_+^*, d(t\beta))\to(N\times\mathbb{R}_+^*, d(t\alpha))$

Let (M,ω) be a symplectic manifold. Assume $\omega=d\lambda$. Then $(M\times\mathbb{R}, dz-\lambda)$ is a contact manifold. If we only know that ω is an integral class, and P is the circle bundle over M with first Chern class ω, then the canonical $U(1)$-connection, θ on P with curvature ω makes (P,θ) into a contact manifold[1].

EXERCICE 3. State and prove the analogue of Darboux and Weinstein's theorem in the contact setting.

PROPOSITION 4.19 (Symplectization of a Legendrian submanifold). *Let L be a Legendrian submanifold in* (N,ξ). *Then* $L\times\mathbb{R}$ *is a Lagrangian in the symplectization of* (N,ξ). *Let L be a Lagrangian in* (M,ω) *with* ω *exact. Assume L is exact, that is* λ_L *is an exact form (it is automatically closed, since* ω *vanishes on L). Then L has a lift to a Legendrian* Λ *in* $(M\times\mathbb{R}, dz-\lambda)$, *unique up to a translation in* z. *Similarly if* ω *is integral, and the holonomy of* θ *along L is integral, we have a Legendrian lift* Λ *of L, unique up to a rotation in* $U(1)$.

The proof is left as an exercise.

4.2. Homogeneous symplectic geometry. We now show that contact structures are equivalent to homogeneous symplectic structures. Indeed,

DEFINITION 4.20. A homogeneous symplectic manifold is a symplectic manifold (M,ω) endowed with a smooth proper and free action of $\mathbb{R}_+^*$, such that denoting by $\frac{\partial}{\partial\lambda}$ the vector field associated to the action, we have $L_{\frac{1}{\lambda}\frac{\partial}{\partial\lambda}}\omega=\omega$.

Clearly the symplectization of a contact manifold is a homogeneous symplectic manifold. We now prove the converse.

[1]The 1-form θ is the unique S^1 invariant form such that $d\theta=\pi^*(\omega)$. In both cases, we call the manifold the contactization or the prequantization of (M,ω).

Example: Let M be a smooth manifold. We denote by $\mathring{T}^*M$ the manifold $T^*M\setminus 0_M$ endowed with the obvious action $\lambda\cdot(q,p)=(q,\lambda,p)$. This is the symplectization of ST^*M.

PROPOSITION 4.21 (Homogeneous symplectic geometry is contact geometry). *Let (M,ω) be a homogeneous symplectic manifold. Then (M,ω) is symplectomorphic (by a homogeneous map) to the symplectization of $(M/\mathbb{R}_+^*, i_X\omega)$*

PROOF. Let $X=\frac{1}{\lambda}\frac{\partial}{\partial\lambda}$, and consider the form $\alpha(\xi)=\omega(X,\xi)$ which is well defined on the quotient $C=M/\mathbb{R}_+^*$. this is a contact form on C, since $i_X\omega\wedge(d(i_X\omega)^{n-1}=i_X\omega\wedge(L_X\omega)^n=i_X\omega\wedge\omega^{n-1}=\frac{1}{n}i_X(\omega^n)$, and since tangent vectors to C are identified to tangent vectors to M transverse to C, this does not vanish. Let t be a coordinate on M such that $dt(X)=1$, and $\widetilde{\omega}=d(t\pi^*(\alpha))$, then (M,ω) is equal to $(M,d(t\alpha))$. Indeed, let us consider two vectors, first of all in the case where one is X and the other is in $dt(Y)=0$. Then $\widetilde{\omega}(X,Y)=(dt\wedge\alpha+td\alpha)(X,Y)=dt(X)\alpha(Y)=(i_X\omega)(Y)=\omega(X,Y)$. Now assume Y,Z are bot in $\ker(dt)$. Then $\widetilde{\omega}(Y,Z)=dt\wedge t\alpha(Y,Z)+td\alpha(Y,Z)$ but $d\alpha=di_X\omega=\omega$ so that $\widetilde{\omega}(Y,Z)=\omega(Y,Z)$. □

EXERCICE 4. Prove that $\mathring{T}^*(M\times\mathbb{R})$ is symplectomorphic to $T^*M\times\mathbb{R}\times\mathbb{R}_+^*$, the symplectization of $J^1(M)$. Hint: prove that the contact manifold $J^1(M)$ is contactomorphic to an open set of $ST^*(M\times\mathbb{R})$.

PROPOSITION 4.22 (Symplectization of a contact map). *Let $\Phi:(N,\xi)\to(P,\eta)$ be a contact transformation, that is a diffeomorphism such that $d\Phi$ sends ξ to η. Then there exists a homogeneous lift of Φ*

$$\widetilde{\Phi}:(\widetilde{N},\omega_\xi)\to(\widetilde{P},\omega_\eta).$$

Conversely any homogeneous symplectomorphism from $(\widetilde{N},\omega_\xi)\to(\widetilde{P},\omega_\eta)$ is obtained in this way.

PROOF. Assume that $\Phi^*(\beta)=\alpha$ where $\mathrm{Ker}(\alpha)=\xi,\mathrm{Ker}(\beta)=\eta$. Then this induces a symplectic map $\widetilde{\Phi}$ between $(N\times\mathbb{R}_+^*,d(t\alpha))$ and $(P\times\mathbb{R}_+^*,d(t\beta))$ and by uniqueness of the symplectization (or rather the fact that it does not depend on the choice of the contact form) we are done. Conversely if $\Psi^*\omega_\eta=\omega_\xi$ that is $\Psi^*d(t\beta)=dt\alpha$, in other words, $d\big(\Psi^*(t\beta)-t\alpha\big)=0$. If the map is exact, this means, $\Psi^*\beta=\alpha+df$

□

EXERCICES 5. (1) Prove that the above lift is functorial, that is the lift of $\Phi\circ\Psi$ is $\widetilde{\Phi}\circ\widetilde{\Psi}$

(2) Let $\varphi:T^*M\to T^*M$ be an exact symplectic map, that is a map such that $\varphi^*(\lambda)-\lambda$ is exact. Prove that there is a lift of φ to a contact map $\widetilde{\varphi}:J^1M\to J^1M$. Prove that if (N,α) is a contact manifold and ψ a diffeomorphism of N such that $\psi^*(\alpha)=\alpha$ (note that this is stronger than requiring that ψ is a contact diffeomorphism, that is $\psi^*(\alpha)=f\cdot\alpha$ for some nonzero function f) then ψ lifts in turn to a homogeneous symplectic map $(N\times\mathbb{R}_+^*,d(t\alpha))$ to itself.

(3) Prove that the symplectization of $J^1(M)$ is $T^*(M)\times\mathbb{R}\times\mathbb{R}_+^*$ and explicit the symplectomorphism obtained from the above $\widetilde{\varphi}$ by symplectization. Thus to any symplectomorphism $\varphi : T^*M \to T^*M$ we may associate a homogeneous symplectomorphism

$$\Phi : T^*(M)\times\mathbb{R}\times\mathbb{R}_+^* \to T^*(M)\times\mathbb{R}\times\mathbb{R}_+^*$$

Prove that the lift is functorial. That is the lift of $\varphi\circ\psi$ is $\Phi\circ\Psi$.

As a result of Proposition 4.21 we have

COROLLARY 4.23. *An exact Lagrangian submanifold L in $(M,\omega = d\lambda)$ has a unique lift $\widehat{L}$ to the (homogeneous) symplectization of its contactization,* $(\widehat{M},\Omega) = (M\times\mathbb{R}_*^+\times\mathbb{R}, dt\wedge d\tau - dt\wedge\lambda)$.

PROOF. Indeed, let $f(z)$ be a primitive of λ on L. Set $\widehat{L} = \{(z,t,\tau) \mid z\in L, \tau = f(z)\}$. Then, $d(td\tau - t\lambda)$ restricted to $\widehat{L}$ equals zero. □

PROPOSITION 4.24. *Let L be an exact Lagrangian. Then L is a conical (or homogeneous) Lagrangian in T^*X if and only if $\lambda_L = 0$.*

PROOF. Let X be the homogeneous vector field, that is the vector filed such that $i_X\omega = \lambda$. Then since for every vector $Y\in TL$ we have $\lambda(Y) = \omega(X,Y) = 0$ since both X and Y are tangent to L, we have $\lambda_L = 0$. □

Locally, L is given by a homogeneous generating function, that is a generating function $S(q,\xi)$ such that $S(q,\lambda\cdot\xi) = \lambda\cdot S(q,\xi)$.

PROPOSITION 4.25 (See [**Duis**], page 83.). *Let L be a germ of homogeneous Lagrangian. Then L is locally defined by a homogeneous generating function.*

EXERCICE 6. Let $S(q,\xi)$ be a (local) generating function for L. What is the generating function for $\widehat{L}$?

PROPOSITION 4.26. *Let Σ be a germ of hypersurface near z in a homogeneous symplectic manifold. Then after a homogenous symplectic diffeomorphism we may assume Σ is either in $\{q_1 = 0\}$ or $\{p_1 = 0\}$.*

PROOF. Let us consider a transverse germ, V, to X.Then V is transverse to Σ, and denote $\Sigma_0 = V\cap\Sigma$. By a linear change of variable, we may assume the tangent space $T_z\Sigma$ □

CHAPTER 5

Generating functions for general Hamiltonians.

In the previous lecture, we proved that if $L_0 = O_{\mathbb{R}^n}$ outside a compact set and has GFQI, and φ is compactly supported Hamiltonian map of $T^*\mathbb{R}^n$, then $\varphi(L)$ has a GFQI.

Let us return to the general case: let N be a compact manifold. For l large enough, there exists an embedding $i : N \hookrightarrow \mathbb{R}^l$. It gives rise to an embedding $\tilde{i}$ of T^*N into $T^*\mathbb{R}^l$, obtained by choosing a metric on $\mathbb{R}^l$. This can be defined as

$$\begin{array}{ccc} T^*N & \hookrightarrow & T^*\mathbb{R}^l \\ (x,p) & \mapsto & (\tilde{x}(x,p), \tilde{p}(x,p)) \end{array}$$

where $\tilde{(}x)(x,p) = i(x)$ and $\tilde{p}(x,p) = p \circ \pi(x)$. $\pi(x)$ is the orthogonal projection $T\mathbb{R}^l \to T_xN$.

It's easy to check that $\tilde{i}^* \tilde{p} d\tilde{x} = p dx$, i.e. $\tilde{i}$ is a symplectic map(embedding). Moreover, if we denote by $N \times (\mathbb{R}^l)^*$ the restriction of $T^*\mathbb{R}^l$ to N, then it's coisotropic as in Example 1 of symplectic reduction. To any Lagrangian in $T^*\mathbb{R}^l$(transversal to $N \times (\mathbb{R}^l)^*$), we may associate the reduction, that is a Lagrangian of T^*N.

Let $\tilde{L} \subset T^*\mathbb{R}^l$ be a Lagrangian. Assume $\tilde{L}$ coincides with $O_{\mathbb{R}^l}$ outside a compact set and $\tilde{L}$ is transverse to $N \times (\mathbb{R}^l)^*$. Denote its symplectic reduction by $\tilde{L}_N = \tilde{L}_{N\times(\mathbb{R}^l)^*} = \tilde{L} \cap (N \times (\mathbb{R}^l)^*)/\sim$.

Claim: For $\varphi \in Ham(T^*N)$, if $\tilde{L}$ has GFQI, then $\varphi(\tilde{L}_N)$ has GFQI.

REMARK 5.1. If $\tilde{L}$ has $\tilde{S} : \mathbb{R}^l \times \mathbb{R}^k \to \mathbb{R}$ as GFQI, then $\tilde{L}_N$ has $\tilde{S}|_{N\times\mathbb{R}^k}$ as GFQI.

For the proof of the claim, we will construct $\tilde{\varphi}$ with compact support such that

$$(\tilde{\varphi}(\tilde{L}))_N = \varphi(\tilde{L}_N).$$

Then, the claim follows from the last remark and first part of the proof. Assume φ is the time one map of φ^t associated to $H(t,x,p)$, where (x,p) is coordinates for T^*N. Locally, we can write (x,u,p,v) for points in $\mathbb{R}^l$ so that $N = \{u = 0\}$. We define

$$\tilde{H}(t,x,u,p,v) = \chi(u)H(t,x,p),$$

where χ is some bump function which is 1 on N and 0 outside a neighborhood of N. By the construction, $X_{\tilde{H}} = X_H$ on $N \times (\mathbb{R}^l)^*$. $\tilde{\varphi} = \tilde{\varphi}^1$, the time one flow of $\tilde{H}$, is the map we need.

The theorem follows by noticing that if we take $\tilde{L} = O_{\mathbb{R}^l}$, which is the same as zero section outside compact set and has GFQI, then $\tilde{L}_N = O_N$. □

Exercice: Show that if L has a GFQI, then $\varphi(L)$ has GFQI for $\varphi \in Ham(T^*N)$.

Hint. If $S : N \times \mathbb{R}^k \to \mathbb{R}$ is a GFQI for L, then L is the reduction of $gr(dS)$.

REMARK 5.2. 1) O_N is generated by

$$\begin{array}{rccc} S: & N \times \mathbb{R} & \to & \mathbb{R} \\ & (x, \xi) & \mapsto & \xi^2 \end{array}$$

2) There is no general upper bound on k (the minimal number of parameter of a generating functions needed to produce all Lagrangian.)

Reason: Consider a curve in T^*S^1

1. Applications

We first need to show that GFQI has critical points. Let us consider a smooth function f on noncompact manifold M satisfying (PS) condition.

(PS): If a sequence (x_n) satisfying $df(x_n) \to 0$ and $f(x_n) \to c$, then (x_n) has a converging subsequence.

REMARK 5.3. Clearly, the limit of the subsequence is a critical point at level c.

REMARK 5.4. A GFQI satisfies (PS). It suffices to check this for a nondegenerate quadratic form Q. Let $Q(x) = \frac{1}{2}(A_Q x, x)$, then $dQ(x) = A_Q(x)$. Since Q is nondegenerate, we know A_Q is invertible and

$$dQ(x_n) \to 0 \Longrightarrow A_Q x_n \to 0 \Longrightarrow x_n \to 0.$$

PROPOSITION 5.5. *If f satisfies (PS) and $H^*(f^b, f^a) \neq 0$, then f has a critical point in $f^{-1}([a,b])$, where $f^\lambda = \{x \in M | f(x) \leq \lambda\}$.*

PROPOSITION 5.6. *For $b >> 0$ and $a << 0$ we have*

$$H^*(S^b, S^a) \cong H^{*-i}(N).$$

PROOF. One can replace S by Q since $S = Q$ at infinity. Define

$$Q^\lambda = \{\xi | Q(\xi) \leq \lambda\}.$$

$$\begin{aligned} H^*(S^b, S^a) &= H^*(N \times Q^b, N \times Q^a) \\ &= H^*(N) \times H^*(Q^b, Q^a). \end{aligned}$$

Since Q is a quadratic form, it's easy to see $H^*(Q^b, Q^a)$ is the same as $H^*(D^-, \partial D^-)$ where D^- is the disk in the negative eigenspace of Q (hence has dimension $index(Q)$, the number of negative eigenvalues). ☐

Conjecture:(Arnold) Let $L \subset T^*N$ be an exact Lagrangian. Is there $\varphi \in Ham(T^*N)$ such that $L = \varphi(O_N)$?

REMARK 5.7. L_S is always exact since $\lambda|_{L_S} = dS|_{\Sigma_S}$.

$$L_S = \{(x, \frac{\partial S}{\partial x}(x,\xi)) | \frac{\partial S}{\partial \xi}(x,\xi) = 0\}.$$

$$\lambda|_{L_S} = p dx = \frac{\partial S}{\partial x}(x,\xi) dx = dS,$$

since for points on L_S, $\frac{\partial S}{\partial \xi} = 0$.

A recent result by Fukaya, Seidel and Smith ([**F-S-S**]) grants that under quite general assumptions, the degree of the projection $\deg(\pi : L \to N) = \pm 1$ and $H^*(L) = H^*(N)$.

Ex: Prove that if L has GFQI S, then $\deg(\pi : L \to N) = \pm 1$.

Indication: Choose a generic point $x_0 \in N$. The degree is the multiplicity with sign of the intersection of L and the fiber over x_0. That is counting the number of ξ with $\frac{\partial S}{\partial \xi}(x_0, \xi) = 0$, i.e. the number of critical points of function $\xi \mapsto S(x_0, \xi)$ with sign

$$(-1)^{index(\frac{d^2 S}{d\xi^2}(x_0,\xi))}.$$

Therefore

$$\deg(\pi : L \to N) = \sum_{\xi_j} (-1)^{index(\frac{d^2 S}{d\xi^2}(x_0,x_j))}$$

where the summation is over all ξ_j with $\frac{\partial S}{\partial \xi}(x_0, \xi_j) = 0$. The summation is finite since S has quadratic infinity and the sum is the euler number of the pair (S^b, S^a) for large b and small a. Finally, check that for all quadratic form Q, the euler number of (Q^b, Q^a) is ± 1.

By the previous claim, for large b and small a

$$H^*(S^b, S^a) \cong H^{*-i}(N).$$

Since N is compact, we know $H^*(N) \neq 0$. This implies that S has at least one critical point and $(L_S \cap O_N) \neq \emptyset$.

THEOREM 5.8 (Hofer). *Let N be a compact manifold and $L = \varphi(O_N)$ for some $\varphi \in Ham(T^*N)$, then*

$$\#(L \cap O_N) \geq cl(N).$$

If all intersection points are transverse, then

$$\#(L \cap O_N) \geq \sum b_j(N).$$

Here

$$cl(N) = \max\{k | \exists \alpha_1, \cdots, \alpha_{k-1} \in H^*(N) \setminus H^0(N) \text{ such that } \alpha_1 \cup \cdots \cup \alpha_{k-1} \neq 0\}$$

and

$$b_j(N) = \dim H^j(N).$$

COROLLARY 5.9.

$$\#(L \cap O_N) \geq 1.$$

We shall postpone the proof of the theorem. However we may prove the corollary: since by Theorem of Sikorav, L has GFQI, and by proposition 1.4 and 1.3 it must have a critical point. Some calculus of critical levels as in the next lectures will allow us to recover the full strength of Hofer's theorem.

THEOREM 5.10. *(Conley-Zehnder) Let $\varphi \in Ham(T^{2n})$, then*

$$\#Fix(\varphi) \geq 2n+1.$$

If all fixed points are nondegenerate, then

$$\#Fix(\varphi) \geq 2^{2n}.$$

REMARK 5.11. $2n+1$ is the cup product length of T^{2n} and 2^{2n} is the sum of Betti numbers of T^{2n}.

PROOF. Let (x_i, y_i) be coordinates of T^{2n}. We will write (x, y) for simplicity. The symplectic form is given by $\omega = dy \wedge dx$. Consider $T^{2n} \times \overline{T^{2n}}$ with coordinates (x, y, X, Y), whose symplectic form is given by

$$\omega = dy \wedge dx - dY \wedge dX.$$

With this ω, the graph of φ, $\Gamma(\varphi)$ is a Lagrangian. Consider another symplectic manifold T^*T^{2n}, denote the coordinates by (a, b, A, B). Note that x, y, X, Y, a, b take value in $T^n = \mathbb{R}^n/\mathbb{Z}^n$ and A, B takes value in $\mathbb{R}^n$.

It has the natural symplectic form as a cotangent bundle

$$\omega = dA \wedge da + dB \wedge db.$$

Define a map $F : T^*T^{2n} \to T^{2n} \times \overline{T^{2n}}$

$$F(a, b, A, B) = (\frac{2a-B}{2}, \frac{2b+A}{2}, \frac{2a+B}{2}, \frac{2b-A}{2}) \bmod \mathbb{Z}^n.$$

It's straightforward to check that F is a symplectic covering.

Let $\triangle_{T^{2n}}$ be the diagonal in $T^{2n} \times \overline{T^{2n}}$. It lifts to $O_{T^{2n}} \subset T^*T^{2n}$ and the projection π induces a bijection between $O_{T^{2n}}$ and $\triangle_{T^{2n}}$. Of course $O_{T^{2n}}$ is only one component in the preimage of $\triangle_{T^{2n}}$ corresponding to $A = B = 0$ (other components are given by $A = A_0, B = B_0$ where $A_0, B_0 \in \mathbb{Z}^n$. Now assume φ is the time one map of $\varphi^t \in Ham(T^{2n})$.

$$\Gamma(\varphi^t) = (id \times \varphi^t)(\triangle_{T^{2n}}).$$

This Hamiltonian isotopy lifts to a Hamiltonian isotopy Φ^t of T^*T^{2n} such that

$$\pi \circ \Phi^t = \phi^t \circ \pi.$$

Then the restriction of the projection to $\Phi^t(O_{T^{2n}})$ remains injective, since

$$\pi(\Phi^t(u)) = \pi(\Phi^t(v))$$

implies

$$\phi^t(\pi(u)) = \phi^t(\pi(v))$$

but since π is injective on $O_{T^{2n}}$ and ϕ^t is injective, this implies $u = v$.

Therefore to distinct points in $\Phi^t(O_{T^{2n}}) \cap O_{T^{2n}}$ correspond distinct points in $\Gamma(\varphi) \cap \triangle_{T^{2n}} = Fix(\varphi)$.

According to Hofer's theorem, the first set has at least $2n+1$ points, so the same holds for the latter. □

REMARK 5.12. The theorem doesn't include all fixed point φ. Indeed, we could have done the same with any other component of $\pi^{-1}(\triangle_{T^{2n}})$ (remember, they are parametrized by pairs of vectors $(A_0, B_0) \in \mathbb{Z}^n \times \mathbb{Z}^n$), and possibly obtained other fixed points. What is so special about those we obtained ? It is not hard to check that they correspond to periodic contractible trajectories on the torus. Indeed, a closed curve on the torus is contractible if and only if it lifts to a closed curve on $\mathbb{R}^{2n}$. Now, our curve is $\Phi^t(a,b,0,0)$ and projects on $(id \times \varphi^t)(x,y,x,y) = (x,y,\phi^t(x,y))$. Since $\Phi^1(a,b,0,0) \in O_{T^{2n}}$, we may denote $\Phi^1(a,b,0,0) = (a',b',0,0)$, and since $\phi^1(x,y)) = (x,y)$, we have $a' = x = a, b' = y = b$. Thus $\Phi^t(a,b,0,0)$ is a closed loop projecting on $(id \times \varphi^t)(x,y,x,y)$, this last loop is therefore contractible, hence the loop $\varphi^t(x,y)$ is also contractible.

Historical comment: Conley-Zehnder proof of the Arnold conjecture for the torus came before Hofer's theorem. It was the first result in higher dimensional symplectic topology, followed shortly after by Gromov's non-squeezing.

THEOREM 5.13. *(Poincaré and Birkhoff) Let φ be an area preserving map of the annulus, shifting each circle (boundary) in opposite direction, then $\#Fix(\varphi) \geq 2$.*

PROOF. Assume φ is the time one map of a Hamiltonian flow φ^t associated to $H = H(t,r,\theta)$, where (r,θ) is the polar coordinates of the annulus$(1 \leq r \leq 2)$. Assume without loss of generality
$$\frac{\partial H}{\partial r} > 0 \text{ for } r = 2$$
and
$$\frac{\partial H}{\partial r} < 0 \text{ for } r = 1.$$
One can extend H to $[\frac{1}{2}, \frac{5}{2}] \times S^1$ such that
$$H(t,r,\theta) = -r \text{ on } [\frac{1}{2}, \frac{2}{3}]$$
and
$$H(t,r,\theta) = r \text{ on } [\frac{7}{3}, \frac{5}{2}].$$
Take two copies of this enlarged annulus and glue them together to make a torus. Then $\#Fix(\varphi) \geq 3$. At least one copy has two fixed points. □

2. The calculus of critical values and first proof of the Arnold Conjecture

Let N be a compact manifold and $\varphi \in Ham(T^*N)$, then $L = \varphi(O_N)$ is a Lagrangian. We have proved the following

THEOREM 5.14. *L has a GFQI.*

There are several consequences

- Hofer's theorem: $\#(\varphi(O_N)\cap O_N)\geq 1$; (In fact Hofer's theorem says more.)
- Conley-Zehnder theorem: $\#Fix(\varphi)\geq 2n+1$ for $\varphi\in Ham(T^{2n})$;
- Poincaré-Birkhoff Theorem.

Today, we are going to talk about 1) Uniqueness of GFQI of L and 2) Calculus of critical levels.

REMARK 5.15. Theorem 5.14 extends to continuous family, i.e. if φ_λ is a continuous family of Hamiltonian diffeomorphisms and $L_\lambda=\varphi_\lambda(O_N)$, then there exists a continuous family of GFQI S_λ.

REMARK 5.16. The Theorem 1.1 (you mean 5.14? Yes (Claude) holds also for Legendrian isotopies(Chekanov). Let $J^1(N,\mathbb{R})\equiv T^*N\times\mathbb{R}$ and define

$$\alpha=dz-pdq.$$

DEFINITION 5.17. Λ is called a Legendrian if and only if $\alpha|_\Lambda=0$.

Example: Given a smooth function $f\in C^\infty(N,\mathbb{R})$, the submanifold defined by

$$z=f(x),p=df,q=x$$

is a Legendrian. One similarly associates to a generating function, $S:N\times\mathbb{R}^k\longrightarrow\mathbb{R}$ a legendrian submanifold (under the same transversality assumptions as for the Legendrian case)

$$\Lambda_S=\{(x,\frac{\partial S}{\partial x}(x,\xi),S(x,\xi))\mid\frac{\partial S}{\partial\xi}=0\}$$

Denote the projection from $T^*N\times\mathbb{R}$ to T^*N by π. Then any Legendrian submanifold projects down to an (exact) Lagrangian. Moreover, any exact Lagrangian can be lifted to a Legendrian. Note however that there are legendrian isotopies that do not project to Lagrangian ones. So Chekanov's theorem is in fact stronger than Sikorav's theorem, even though the proof is the same.

2.1. Uniqueness of GFQI. Let $\varphi\in Ham(T^*N)$ and $L=\varphi(O_N)$. Denote a GFQI for L by S. We will show that we can obtain different GFQI by the following three operations.

Operation 1:(Conjugation) If smooth map $\xi:N\times\mathbb{R}^k\to\mathbb{R}^k$ satisfies that for each $x\in N$, $\xi(x,\cdot):\mathbb{R}^k\to\mathbb{R}^k$ is a diffeomorphism, then we claim:

$$\tilde{S}(x,\eta)=S(x,\xi(x,\eta))$$

is again GFQI for L.

Recall from the definition of generating function

$$L_{\tilde{S}}=\{(x,\frac{\partial\tilde{S}}{\partial x}(x,\eta))|\frac{\partial\tilde{S}}{\partial\eta}(x,\eta)=0\}$$

and

$$L_S = \{(x, \frac{\partial S}{\partial x}(x,\xi)) | \frac{\partial S}{\partial \xi}(x,\xi) = 0\}.$$

Since $\frac{\partial \xi}{\partial \eta}$ is invertible, the chain rule says $\frac{\partial \tilde{S}}{\partial \eta}(x,\eta)$ and $\frac{\partial S}{\partial \xi}(x,\xi(x,\eta))$ simultaneously. On such points,

$$\frac{\partial \tilde{S}}{\partial x}(x,\eta) = \frac{\partial S}{\partial x}(x,\xi(x,\eta)) + \frac{\partial S}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} = \frac{\partial S}{\partial x}(x,\xi(x,\eta)).$$

Operation 2: (Stabilization) If q is a nondegenerate quadratic form, then

$$\tilde{S}(x,\xi,\eta) = S(x,\xi) + q(\eta)$$

is a GFQI for L.

The reason is

$$\frac{\partial \tilde{S}}{\partial x}(x,\xi,\eta) = \frac{\partial S}{\partial x}(x,\xi)$$

and

$$\tfrac{\partial \tilde{S}}{\partial \xi} = \tfrac{\partial \tilde{S}}{\partial \eta} = 0 \quad \Longleftrightarrow \quad \begin{cases} A_q \eta = 0 \Longrightarrow \eta = 0 \\ \frac{\partial S}{\partial \xi}(x,\xi) = 0 \end{cases}$$

where A_q is given by $(A_q\eta, \eta) = q(\eta)$ for all η and is invertible since q is nondegenerate.

Operation 3: (Shift) By adding constant,

$$\tilde{S}(x,\xi) = S(x,\xi) + c.$$

The GFQI is unique up to the above operations in the sense that

THEOREM 5.18 (Uniqueness theorem for GFQI). *If S_1, S_2 are GFQI for $L = \varphi(O_N)$, then there exists $\tilde{S}_1$, $\tilde{S}_2$ obtained from S_1 and S_2 by a sequence of operations 1,2,3 such that $\tilde{S}_1 = \tilde{S}_2$.*

For the proof, see [**Theret**].

The main consequence of this theorem is that given $L = \varphi(O_N)$, for different choices of GFQI, we know the relation between $H^*(S^b, S^a)$. It suffices to trace how $H^*(S^b, S^a)$ changes by operation 1,2,3.

It's easy to see that $H^*(S^b, S^a)$ is left invariant by operation 1, because the pair (S^b, S^a) is diffeomorphic to $(\tilde{S}^b, \tilde{S}^a)$.

For operation 3,

$$H^*(\tilde{S}^b, \tilde{S}^a) = H^*(S^{b-c}, S^{a-c}).$$

For operation 2, we claim without proof for $b > a$

$$H^*(\tilde{S}^b, \tilde{S}^a) = H^{*-i}(S^b, S^a)$$

where i is the index of q.

REMARK 5.19. The theorem holds for $L = \varphi(O_N)$ only, no result is known for general L). Moreover, the theorem holds for families.

2.2. Calculus of critical levels. In this section, we assume M is a manifold and $f \in C^\infty(M,\mathbb{R})$ is a smooth function satisfying (PS) condition. Given $a < b < c$, there is natural embedding map

$$(f^b, f^a) \hookrightarrow (f^c, f^a).$$

It induces

$$H^*(f^c, f^a) \to H^*(f^b, f^a).$$

DEFINITION 5.20. Let $\alpha \in H^*(f^c, f^a)$. Define

$$c(\alpha, f) = \inf\{b | \text{ image of } \alpha \text{ in } H^*(f^b, f^a) \text{ is not zero}\}.$$

Since the embedding also induces

$$H_*(f^b, f^a) \hookrightarrow H_*(f^c, f^a),$$

the same can be done for $\omega \in H_*(f^c, f^a) \setminus \{0\}$.

DEFINITION 5.21. For $\omega \in H_*(f^c, f^a) \setminus \{0\}$, define

$$c(\omega, f) = \inf\{b | \omega \text{ is in the image of } H_*(f^b, f^a)\}.$$

PROPOSITION 5.22. *$c(\alpha, f)$ and $c(\omega, f)$ are critical values of f.*

PROOF. Prove the first one only. Proof for the other is similar. Let $\gamma = c(\alpha, f)$, assume γ is not a critical value. Since f satisfies (PS) condition, we have

$$H^*(f^{\gamma+\varepsilon}, f^{\gamma-\varepsilon}) = 0.$$

Study the long exact sequence for the triple $(f^{\gamma+\varepsilon}, f^{\gamma-\varepsilon}, f^a)$,

$$H^*(f^{\gamma+\varepsilon}, f^{\gamma-\varepsilon}) \to H^*(f^{\gamma-\varepsilon}, f^a) \to H^(f^{\gamma+\varepsilon}, f^a) \to H^{*+1}(f^{\gamma+\varepsilon}, f^{\gamma-\varepsilon})$$

Since the first and the last space are $\{0\}$, we know

$$H^*(f^{\gamma-\varepsilon}, f^a) \cong H^*(f^{\gamma+\varepsilon}, f^a).$$

By the definition of γ, the image of α in $H^*(f^{\gamma-\varepsilon}, f^a)$ is zero, but the image of α in $H^*(f^{\gamma+\varepsilon}, f^a)$ is not zero. This is a contradiction. $\square$

Recall Alexander duality:

$$AD: \quad H^*(f^c, f^a) \quad \to \quad H_{n-*}(X - f^a, X - f^c) = H_{n-*}((-f)^{-a}, (-f)^{-c}).$$

PROPOSITION 5.23. *Assume that M is a compact, connected and oriented manifold, then for $\alpha \in H^*(f^c, f^a) \setminus \{0\}$,*

1) $c(\alpha, f) = -c(AD(\alpha), -f)$;

2) $c(1, f) = -c(\mu, -f)$ where $1 \in H^0(M)$ and $\mu \in H^n(M)$ are generators. (In fact, any nonzero element will do since they are all proportional. Here we assumed $a = -\infty$ and $c = +\infty$.)

PROOF. 1) Diagram chasing on the following diagram, using the fact that $X \setminus f^a = (-f)^{-a}$.

$$\begin{array}{ccc}
H_*(f^c, f^a) & \xrightarrow{AD} & H^{n-*}(X \setminus f^a, X \setminus f^c) \\
\downarrow & & \downarrow \\
H_*(f^b, f^a) & \xrightarrow{AD} & H^{n-*}(X \setminus f^a, X \setminus f^b) \\
\downarrow & & \downarrow \\
H_*(f^b, f^c) & \xrightarrow{AD} & H^{n-*}(X \setminus f^c, X \setminus f^b)
\end{array}$$

2) It suffices to show

$$c(1, f) = \min(f) \text{ and } c(\mu, f) = \max(f).$$

□

THEOREM 5.24. *(Lusternik-Schnirelmann) Assume $\alpha \in H^*(f^c, f^a) \setminus \{0\}$ and $\beta \in H^*(M) \setminus H^0(M)$, then*

$$c(\alpha \cap \beta, f) \geq c(\alpha, f) \tag{5.1}$$

If equality holds in equation 5.1 with common value γ, then for any neighborhood U of $K_\gamma = \{x | f(x) = \gamma, df(x) = 0\}$, we have $\beta \neq 0$ in $H^*(U)$.

REMARK 5.25. If $\beta \notin H^0(M)$ and equality in (5.1) holds, then $H^p(U) \neq 0$ for all U and some $p \neq 0$. This implies K_γ is infinite. Otherwise, take U to be disjoint union of balls then $H^p(U) = 0$ for all $p \neq 0$, which is a contradiction. One can even show that K_γ is uncountable by the same argument.

COROLLARY 5.26. *Let $f \in C^\infty(M, \mathbb{R})$ with compact M, then*

$$\#Crit(f) \geq cl(M).$$

PROOF. Inequality 5.1 is obvious because $\alpha = 0$ in $H^*(f^b, f^a)$ implies $\alpha \cap \beta = 0$ in $H^*(f^b, f^a)$.

If equality in (5.1) holds, for any given U, take ε sufficiently small so that

1) There exists a saturated neighborhood $V \subset U$ of K_γ for the negative gradient flow of f between $\gamma + \varepsilon$ and $\gamma - \varepsilon$, in the sense that any flow line coming into V will either go to K_γ for all later time or go into $f^{\gamma-\varepsilon}$.(Never come out of V between $\gamma + \varepsilon$ and $\gamma - \varepsilon$). Moreover by (PS) we may assume V contains all critical points in $f^{\gamma+\varepsilon} \setminus f^{\gamma-\varepsilon}$.

2) (PS) condition ensures a lower bound for $|\nabla f|$ for all $x \in f^{\gamma+\varepsilon} \setminus (V \cup f^{\gamma-\varepsilon})$.

Let $X = -\nabla f$ and consider its flow φ^t.

$$\frac{d}{dt} f(\varphi^t(x)) = -|\nabla f|^2 (\varphi^t(x)).$$

Therefore, we have

- If $x \in f^{\gamma-\varepsilon}$, then $\varphi^t(x) \in f^{\gamma-\varepsilon}$.

- (V is saturated) $x \in V$ implies $\varphi^t(x) \in V \cup f^{\gamma-\varepsilon}$.
- For $x \notin V$ and $x \in f^{\gamma+\varepsilon}$. By 1), $\varphi^t(x) \notin V$ and as long as $f(\varphi^t(x)) \geq \gamma-\varepsilon$, we have (due to 2))
$$\left|\nabla f\right|(\varphi^t(x)) \geq \delta_0.$$
This implies that there exists $T > 0$ such that for $x \in f^{\gamma+\varepsilon} \setminus V$,
$$f(\varphi^T(x)) < \gamma - \varepsilon.$$

In conclusion, we get an isotopy $\varphi^T : f^{\gamma+\varepsilon} \to f^{\gamma-\varepsilon} \cup V \subset f^{\gamma-\varepsilon} \cup U$.

Assume $\beta = 0$ in $H^*(U)$. By definition $\alpha = 0$ in $H^*(f^{\gamma-\varepsilon}, f^a)$, then $\alpha \cup \beta = 0$ in $H^*(f^{\gamma-\varepsilon} \cup U, f^a)$. But φ^T is an isotopy, we know
$$\alpha \cup \beta = 0 \text{ in } H^*(f^{\gamma+\varepsilon}, f^a).$$
This is a contradiction to $c(\alpha \cup \beta, f) = \gamma$. □

2.3. The case of GFQI. If S is a GFQI for L, we know
$$H^*(S^\infty, S^{-\infty}) \cong H^{*-i}(N)$$
where i is the index of the nondegenerate quadratic form associated with S.

Due to this isomorphism, to each $\alpha \in H^*(N)$, we associate $\tilde{\alpha} \in H^*(S^\infty, S^{-\infty})$. Define
$$c(\alpha, S) = c(\tilde{\alpha}, S).$$

We claim the next result but omit the proof.

PROPOSITION 5.27. *For $\alpha_1, \alpha_2 \in H^*(N)$,*
$$c(\alpha_1 \cup \alpha_2, S_1 \oplus S_2) \geq c(\alpha_1, S_1) + c(\alpha_2, S_2),$$
where
$$(S_1 \oplus S_2)(x, \xi_1, \xi_2) = S_1(x, \xi_1) + S_2(x, \xi_2).$$

REMARK 5.28. The isomorphism mentioned above is precisely
$$\begin{array}{ccc} H^*(N) \otimes H^*(D^-, \partial D^-) & = & H^*(S^\infty, S^{-\infty}) \\ \alpha \otimes T & \mapsto & \tilde{T} \cup p^*\alpha \end{array}$$
where $p : N \times \mathbb{R}^k \to N$ is the projection.

$$\begin{array}{ccccccc} H^*((S_1 \oplus S_2)^\infty, (S_1 \oplus S_2)^{-\infty}) & \cong & H^*(N) & \otimes & H^*(D_1^-, \partial D_1^-) & \otimes & H^*(D_2^-, \partial D_2^-) \\ \tilde{T} \cup p^*\alpha & & \alpha & & T_1 & & T_2 \end{array}$$

$$\begin{aligned} \tilde{T} \cup p^*\alpha &= \tilde{T}_1 \cup \tilde{T}_2 \cup p^*(\alpha_1 \cup \alpha_2) \\ &= (\tilde{T}_1 \cup p^*\alpha_1) \cup (\tilde{T}_2 \cup p^*\alpha_2) \end{aligned}$$

Part 2

Sheaf theory and derived categories

CHAPTER 6

Categories and Sheaves

1. The language of categories

DEFINITION 6.1. A category $\mathscr{C}$ is a pair $(\mathrm{Ob}(\mathscr{C}), \mathrm{Mor}_{\mathscr{C}})$ where

- $\mathrm{Ob}(\mathscr{C})$ is a class of Objects [1]
- Mor is a map from $\mathscr{C} \times \mathscr{C}$ to a class, together with a composition map

$$\mathrm{Mor}(A,B) \times \mathrm{Mor}(B,C) \longrightarrow \mathrm{Mor}(A,C)$$
$$(f,g) \mapsto g \circ f$$

The composition is :

(1) associative
(2) has an identity element, $\mathrm{id}_A \in \mathrm{Mor}(A,A)$ such that $\mathrm{id}_B \circ f = f \circ \mathrm{id}_A = f$ for all $f \in \mathrm{Mor}(A,B)$.

The category is said to be small if $\mathrm{Ob}(\mathscr{C})$ and $\mathrm{Mor}_{\mathscr{C}}$ are actually sets. It is locally small if $\mathrm{Mor}_{\mathscr{C}}(A,B)$ is a set for any A,B in $\mathrm{Ob}(\mathscr{C})$.

Examples:

(1) The category **Sets** of sets, where objects are sets and morphisms are maps. The subcategory **Top** where objects are topological spaces and morphisms are continuous maps.
(2) The category **Group** of groups, where objects are groups and morphisms are group morphisms. It has a subcategory, **Ab** with objects the abelian groups and morphisms the group morphisms. This is a **full subcategory**, which means that $\mathrm{Mor}_{\mathbf{Group}}(A,B) = \mathrm{Mor}_{\mathbf{Ab}}(A,B)$ for any pair A,B of abelian groups; the set of morphisms between two abelian groups does not depend on whether you consider them as abelian groups or just groups. An example of a subcategory which is **not** a full subcategory is given by the subcategory **Top** of **Sets**.
(3) The category **R-mod** of R-modules, where objects are left R-modules and morphisms are R-modules morphisms.
(4) The category **k-vect** of k-vector spaces, where objects are k-vector spaces and morphisms are k-linear maps.
(5) The category **Man** of smooth manifolds, where objects are smooth manifolds and morphisms are smooth maps.

[1] The class of Objects can be and often is a “set of sets”. There is clean set-theoretic approach to this, using “Grothendieck Universe”, but we will not worry about these questions here (nor elsewhere...).

(6) Given a manifold M, the category **Vect(M)** of smooth vector bundles over M and morphisms are smooth linear fiber maps.
(7) If P is a partially ordered set (a poset), **Ord(P)** is a category with objects the element of P, and morphisms $\mathrm{Mor}(x, y) = \emptyset$ unless $x \leq y$ in which case $\mathrm{Mor}(x, y) = \{*\}$.
(8) The category **Pos** of partially ordered sets (i.e. posets), where morphisms are monotone maps.
(9) If G is a group, the category **Group (G)** has objets the one element set $\{*\}$ and $\mathrm{Mor}_{\mathscr{C}}(*, *) = G$, where composition corresponds to multiplication.
(10) The simplicial category **Simplicial** whose objects are sets $[i] = \{0, 1, ..., i\}$ for $i \geq -1$ ($[-1] = \emptyset$) and morphisms are the monotone maps.
(11) if X is a topological space, **Open(X)** is the category where objects are open sets, and morphisms are such that $\mathrm{Mor}(U, V) = \{*\}$ if $U \subset V$, and $\mathrm{Mor}(U, V) = \emptyset$ otherwise. (since the set of open sets in X is a set ordered by inclusion, this is related to **Pos**).
(12) Given a category $\mathscr{C}$, the opposite category is the category denoted $\mathscr{C}^{op}$ having the same objects as $\mathscr{C}$, but such that $\mathrm{Mor}_{\mathscr{C}^{\mathbf{op}}}(A, B) = \mathrm{Mor}_{\mathscr{C}^{\mathbf{op}}}(B, A)$ with the obvious composition map: if we denote by $f^* \in \mathrm{Mor}_{\mathscr{C}^{\mathbf{op}}}(A, B)$ the image of $f \in \mathrm{Mor}_{\mathscr{C}}(B, A)$, we have $f^* \circ g^* = (g \circ f)^*$. In some cases there is a simple identification of $\mathscr{C}^{op}$ with a natural category (example: the opposite category of **k-vect** is the category with objects the space of linear forms on a vector space).
(13) Given a category $\mathscr{C}$, we can consider the quotient category by isomorphism. The standard construction, at least if the category is not too large, is to choose for each isomorphism class of objects a given object (using the axiom of choice), and consider the subcategory $\mathscr{C}'$ of $\mathscr{C}$ generated by these objects.

An initial object in a category is an element I such that $\mathrm{Mor}(I, A)$ ha exactly one element. A terminal object T is an object such that $\mathrm{Mor}(A, T)$ is a singleton for each A. Equivalently T is an initial object in the opposite category.

Examples: $\emptyset$ in **Sets**, $\{e\}$ in **Group**, $\{0\}$ in **R-mod** or **K-Vect**, the smallest object in **P** if it exists, $[-1]$ in **Simplicial**.

DEFINITION 6.2. A **functor** between the categories $\mathscr{C}$ and $\mathscr{D}$ is a "pair of maps", one from $\mathrm{Ob}(\mathscr{C})$ to $\mathrm{Ob}(\mathscr{D})$ the second one sending $\mathrm{Mor}_{\mathscr{C}}(A, B)$ to $\mathrm{Mor}_{\mathscr{D}}(F(A), F(B))$ such that $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$ and $F(f \circ g) = F(f) \circ F(g)$.

Examples: A functor from **Group(G)** to **Group(H)** is a morphism from G to H. There are lots of **forgetful functors**, like **Group** to **Sets**. There is also a functor from **Top** to **Ord** sending X to the set of its open subsets ordered by inclusion.

DEFINITION 6.3. A functor is **fully faithful** if for any pair X, Y the map $F_{X,Y} : Mor(X, Y) \to \mathrm{Mor}(F(X), F(Y))$ is bijective. We say that F is an **equivalence of categories** if it is fully faithful, and moreover for any $X' \in \mathscr{D}$ there is X such that $F(X)$ is isomorphic to X'.

Note that for an equivalence of categories, we only require that F is a bijection between equivalence classes of isomorphic objects.

There is also a notion of transformation of functors. If $F : \mathbf{A} \to \mathbf{B}, G : \mathbf{A} \to \mathbf{B}$ are functors, a transformation of functors is a family of maps parametrized by X, $T_X \in \mathrm{Mor}(F(X), G(X))$ making the following diagram commutative for every f in $\mathrm{Mor}(X, Y)$

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "T_X"] & F(Y) \arrow[d, "T_Y"] \\ G(X) \arrow[r, "G(f)"] & G(Y) \end{tikzcd}$$

Notice that some categories are categories of categories, the morphisms being the functors. This is the case for **Group** with objects the set of categories of the type **Group(G)**, or of **Pos** whose objects are the **Ord(P)**. We may also, given two categories, **A**, **B** define the category with objects the functors from **A** to **B**, and morphisms the transformations of these functors. We shall see for example that **presheaves over** X (see the next section) are nothing but functors defined on the category **Open(X)**. And so on, and so forth....

Finally as in maps, we have the notion of monomorphism and epimorphisms

DEFINITION 6.4. An element $f \in \mathrm{Mor}(B, C)$ is a **monomorphism** if for any $g_1, g_2 \in \mathrm{Mor}(A, B)$ we have $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$. An element $f \in \mathrm{Mor}(A, B)$ is an **epimorphism** if for any $g_1, g_2 \in \mathrm{Mor}(B, C)$ we have $g_1 \circ f = g_2 \circ f$ implies $g_1 = g_2$. An isomorphism is a morphism $f \in \mathrm{Mor}(A, B)$ such that there exists g such that $f \circ g = \mathrm{Id}_B$ and $g \circ f = \mathrm{id}_A$.

EXERCICES 1. (1) Is being an isomorphism equivalent to being both a monomorphism and an epimorphism ?

(2) Prove that in the category **Sets** monomorphisms and epimorphisms are just injective and surjective maps. What are monomorphisms and epimorphisms in the other categories. In which of the above categories the following statement holds: "a morphism is an isomorphism if and only if it is both an epimorphism and a monomorphism" (such a category is said to be "balanced") ?

(3) In the category **Groups**: prove that the cokernel of f is $G/N(\mathrm{Im}(f))$, where $N(H)$ is the normalizer[2] of H in G, but epimorphisms are surjective morphisms, In particular, to have cokernel 0 is not equivalent to being an epimorphism. Prove that the category
CatGroups is balanced.

Hint to prove that an epimorphism is onto: prove that for any proper subgroup H of G (not necessarily normal), there is a group K and two different morphisms g_1, g_2 in $\mathrm{Mor}(G, K)$ such that $g_1 = g_2$ on H. For this use the action

[2] i.e. the largest subgroup such that H is a normal subgroup of $N(H)$.

of G on the classes of H/G to reduce the problem to $\mathfrak{S}_{q-1} \subset \mathfrak{S}_q$ and prove that there are two different morphisms $\mathfrak{S}_q \to \mathfrak{S}_{q+1}$ equal to the inclusion on $\mathfrak{S}_{q-1}$.

(4) Prove that the injection $\mathbb{Z} \to \mathbb{Q}$ is an epimorphism in the category **Rng** of commutative rings with unit. Hint: use the fact that if a map $g : \mathbb{Q} \to R$ is non-zero then R has characteristic zero[3].

(5) Prove that in the category **Top** an epimorphism is surjective. Is the category balanced ? Give an example of a balanced subcategory. Find also a subcategory of **Top** such that any continuous map with dense image is an epimorphism.

2. Additive and Abelian categories

DEFINITION 6.5. An additive category is a category such that

(1) It has a 0 object which is both initial and terminal. The zero map is defined as the unique composition $A \to 0 \to B$.
(2) $\mathrm{Mor}(A,B)$ is an abelian group, 0 is the zero map, composition is bilinear.
(3) It has finite biproducts (see below for the definition).

A category has finite products if for any A_1, A_2 there exists an object denoted $A_1 \times A_2$ and maps $p_k : A_1 \times A_2 \longrightarrow A_k$ such that

$$\mathrm{Mor}(Y, A_1) \times \mathrm{Mor}(Y, A_2) = \mathrm{Mor}(Y, A)$$

by the map $f \to (p_1 \circ f, p_2 \circ f)$ and which are universal in the following sense[4]. For any maps $f_1 : Y \to A_1$ and $f_2 : Y \to A_2$ there is a **unique** map $f : Y \to A_1 \times A_2$ making the following diagram commutative

$$\begin{tikzcd}
 & Y \arrow[dl, "f_1"'] \arrow[d, dashed, "f"] \arrow[dr, "f_2"] & \\
A_1 & A_1 \times A_2 \arrow[l, "p_1"] \arrow[r, "p_2"'] & A_2
\end{tikzcd}$$

It has finite coproducts if given any A_1, A_2 there exists an object denoted $A_1 + A_2$ and maps $i_k : A_k \longrightarrow A_1 + A_2$ such that

$$\mathrm{Mor}(A_1, Y) \times \mathrm{Mor}(A_2, Y) = Mor(A_1 + A_2, Y)$$

and this is given by $g \to (g \circ i_1, g \circ i_2)$. In other words for any $g_1 : A_1 \to Y$ and $g_2 : A_2 \to Y$ there exists a **unique** map $g : A_1 + A_2 \to Y$ making the following diagram commutative

[3]There are in fact two possible definitions for a ring morphism: either it is just a map such that $f(x+y) = f(x) + f(y), f(xy) = f(x)f(y)$ or we also impose $f(1) = 1$. In the latter case $\mathrm{Mor}(\mathbb{Q}, R) = \emptyset$ unless R has zero characteristic.

[4]The maps (p_1, p_2) correspond to Id_A under the identification of $\mathrm{Mor}(A, A)$ and $\mathrm{Mor}(A, A_1) \times \mathrm{Mor}(A, A_2)$. The maps i_1, i_2 mentioned later are obtained similarly.

$$\begin{tikzcd}
A_1 \arrow[r, "i_1"] \arrow[dr, "g_1"'] & A_1 + A_2 \arrow[d, dashed, "f"] & A_2 \arrow[l, "i_2"'] \arrow[dl, "g_2"] \\
& Y &
\end{tikzcd}$$

the category has **biproducts** if it has both products and coproducts, these are equal and moreover

(1) $p_j \circ i_k$ is id_{A_j} if $j = k$ and 0 for $j \neq k$,
(2) $i_1 \circ p_1 + i_2 \circ p_2 = \mathrm{id}_{A_1 \oplus A_2}$.

We then denote the biproduct of A_1 and A_2 by $A_1 \oplus A_2$. According to exercice 5 on the following page, if biproducts exist, they are **unique** up to a **unique isomorphism**.

DEFINITION 6.6. A **kernel** for a morphism $f \in \mathrm{Mor}(A, B)$ is a pair (K, k) where $K \xrightarrow{k} A \xrightarrow{f} B$ such that $f \circ k = 0$ and if $g \in \mathrm{Mor}(P, A)$ and $f \circ g = 0$ there is a **unique** map $h \in \mathrm{Mor}(P, K)$ such that $g = k \circ h$.

$$\begin{tikzcd}
K \arrow[r, "k"] & A \arrow[r, "f"] & B \\
& P \arrow[u, "g"'] \arrow[ul, dashed, "h"] &
\end{tikzcd}$$

A **cokernel** is a pair (C, c) such that $c \circ f = 0$ and if $g \in \mathrm{Mor}(B, Q)$ is such that $g \circ f = 0$ there is a **unique** $d \in \mathrm{Mor}(C, Q)$ such that $d \circ c = g$.

$$\begin{tikzcd}
A \arrow[r, "f"] & B \arrow[r, "c"] \arrow[dr, "g"'] & C \arrow[d, dashed, "d"] \\
& & Q
\end{tikzcd}$$

A **Coimage** is the kernel of the cokernel. An **Image** is the cokernel of the kernel.

DEFINITION 6.7 (Abelian category). An **abelian category** is an additive category such that

(1) It has both kernels and cokernels
(2) The natural map from the coimage to the image (see the map σ in Exercice 2, (7)) is an isomorphism. To prove our statement, use the fact that the direct limit is exact.

The second statement can be replaced by the more intuitive one: every morphism $f : A \to B$ has a factorization

(2')
$$\begin{tikzcd} K \arrow[r, "i"] & A \arrow[r, "u"] \arrow[rr, bend left, "f"] & \mathrm{Im}(f) \arrow[r, "v"] & B \arrow[r] & \mathrm{Coker}(f) \end{tikzcd}$$

where u and v are the natural maps (see Exercice 2, (7)).

EXERCICES 2. (1) Identify Kernel and Cokernel in the category of R-modules and in the category of groups.

(2) Which one of the categories from the list of examples starting on page 49 are abelian ?

(3) Prove that a kernel is a monomorphism, that is if (K, k) is the kernel of $A \xrightarrow{f} B$, then $k : K \to A$ is a monomorphism. Prove that a cokernel is an epimorphism (use the uniqueness of the maps), and that in (2'), u is an epimorphism and v a monomorphism.

(4) Prove that the composition of two monomorphisms (resp. epimorphism) is a monomorphism (resp. epimorphism)

(5) It is a general fact that solutions to **universal problems**, if they exists, are unique up to isomorphisms. Prove this for products, coproducts, Kernels and Cokernels.

(6) Prove that the kernel of f is zero if and only if f is mono. Prove that $\mathrm{Coker}(f) = 0$ if and only if $\mathrm{Im}(f)$ is isomorphic to B and this in turn means f is an epimorphism. If a map (in a non-abelian category) is both mono and epi, is it an isomorphism (f is an isomorphism if and only if there exists g such that $f \circ g = g \circ f = \mathrm{id}$) ? Consider the case of a group morphism for example.

(7) Assuming property (1) holds prove the factorization of morphisms (2') is equivalent to property (2). Use the following diagram, justifying the existence of the dotted arrows

(6.1)
$$\begin{tikzcd} \mathrm{Ker(f)} \arrow[r, "i"] & A \arrow[r, "f"] \arrow[d, "u"] & B \arrow[r, "p"] & \mathrm{Coker}(f) \\ & \mathrm{Coim}(f) = \mathrm{Coker}(i) \arrow[ur, dashed, "\psi"] & \mathrm{Ker(p)} = \mathrm{Im(f)} \arrow[u, "v"'] & \end{tikzcd}$$

Then $p \circ \psi = 0$, since u is an epimorphism according to Exercise 2 (3), and $p \circ \psi \circ u = p \circ f = 0$, and this implies $p \circ \psi = 0$ hence ψ factors through Ker(p) and we now have the diagram with the unique map σ

(6.2)
$$\begin{tikzcd}
\mathrm{Ker(f)} \arrow[r, "i"] & A \arrow[r, "f"] \arrow[d, "u"] & B \arrow[r, "p"] & \mathrm{Coker}(f) \\
& \mathrm{Coim}(f) = \mathrm{Coker}(i) \arrow[ur, "\psi"] \arrow[r, dashed, "\sigma"] & \mathrm{Ker(p)} = \mathrm{Im(f)} \arrow[u, "v"] &
\end{tikzcd}$$

by assumption we are in an abelian category if and only if the map σ is an isomorphism.

PROPOSITION 6.8. *Let $\mathscr{C}$ be an abelian category. Then a morphism which is both a monomorphism and an epimorphism is an isomorphism.*

PROOF. Notice first that $0 \to A$ has cokernel equal to (A, Id). Similarly the kernel of $B \to 0$ is (B, Id). Assuming f is both an epimorphism and a monomorphism, we get the commutative diagram

(6.3)
$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\mathrm{Id}"] & B \arrow[r] & 0 \\
& \mathrm{Coim}(f) = \mathrm{Coker}(i) = A \arrow[r, dashed, "\sigma"] & \mathrm{Ker(p)} = \mathrm{Im(f)} = \mathrm{B} \arrow[u, "\mathrm{Id}"] &
\end{tikzcd}$$

and the result follows from the invertibility of σ. □

DEFINITION 6.9. In an abelian category, the notion of **exact sequence** is defined as follows. A sequence of maps $A \xrightarrow{f} B \xrightarrow{g} C$ is exact if and only if $g \circ f = 0$ and the map from $\mathrm{Im}(f)$ to Ker(g) is an isomorphism. The exact sequence is said to be **split** if there is a map $h : C \to B$ such that $g \circ h = \mathrm{Id}_C$.

The map from $\mathrm{Im}(f)$ to Ker(g) is obtained from the following diagram

(6.4)
$$\begin{tikzcd}
\mathrm{Im}(f) = \mathrm{Ker(p)} \arrow[rr, dashed, "w"] \arrow[dr, "v"] & & \mathrm{Ker(g)} \arrow[dl, "i"'] \\
A \arrow[u, "u"] \arrow[r, "f"] & B \arrow[r, "g"] \arrow[d, "p"] & C \\
& \mathrm{Coker}(f) &
\end{tikzcd}$$

Here u, v come from the canonical factorization of f. We claim that $g \circ v = 0$ since $g \circ v \circ u = g \circ f = 0$ and u is an epimorphism according to Exercice 2, (3). As a result v factors through a map $w : \mathrm{Im}(f) \to \mathrm{Ker(g)}$.

EXERCICE 3. Prove that if an exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is split, that is there is a map $h : C \to B$ such that $g \circ h = \mathrm{Id}_C$, then $B \simeq A \oplus C$. Prove that the same conclusion holds if there exists k such that $k \circ f = \mathrm{Id}_A$.

Hint: prove that there exists a map $k: B \to A$ such that $\mathrm{Id}_B = f \circ k + h \circ g$. Indeed, $g \circ h \circ g = g$ and since g is an epimorphism, and $g \circ (\mathrm{Id}_B - h \circ g) = 0$, we get that since $f: A \to B$ is the kernel of g, that $(\mathrm{Id}_B - h \circ g) = f \circ k$ for some map $k: B \to A$.

Now $f \oplus h: A \oplus C \to B$ is an isomorphism, with inverse $k \oplus g: B \to A \oplus C$.

Note that Property 2' can be replaced by either of the following conditions:

(2") any monomorphism is a kernel, and any epimorphism is a cokernel. In other words, monomorphism $0 \to A \xrightarrow{f} B$ can be completed to an exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C$, and any $B \xrightarrow{g} C \to 0$ can be completed to an exact sequence $A \xrightarrow{f} B \xrightarrow{g} C \to 0$

(2"') If $A \xrightarrow{f} B \xrightarrow{g} C$ is an exact sequence, then we have a factorization

$$A \xrightarrow{f} B \xrightarrow{g_1} \mathrm{Coker}(g) \xrightarrow{g_2} C$$

where the last map is monomorphism.

Note that $0 \to A \xrightarrow{f} B$ is exact if and only if f is a monomorphism, and $A \xrightarrow{f} B \to 0$ is exact if and only if f is an epimorphism. Moreover

PROPOSITION 6.10. *If*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is an exact sequence, then $(A, f) = \mathrm{Ker(g)}$ *and* $(C, g) = \mathrm{Coker}(f)$.

PROOF. Consider $0 \to A \xrightarrow{f} B$. We claim the map $A \xrightarrow{u} \mathrm{Im}(f)$ is an isomorphism. It is a monomorphism, because the factorization (2') of f is written $0 \to A \xrightarrow{u} \mathrm{Im}(f) \xrightarrow{v} B$. Moreover it is an epimorphism according to Exercice 2, (3).

Since the map w from (*) is an isomorphism (due to the exactness of the sequence), we have the commutative diagram

$$\begin{tikzcd}
& \mathrm{Im}(f) = \mathrm{Ker(p)} \arrow[rr, dashed, "\simeq", "w"'] \arrow[dr, "v"] & & \mathrm{Ker(g)} \arrow[dl, "i"'] \\
0 \arrow[r] & A \arrow[u, "\simeq", "u"'] \arrow[r, "f"] & B \arrow[r, "g"] & C
\end{tikzcd}$$

and thus (A, f) is isomorphic to $(\mathrm{Ker(g)}, \mathrm{i})$. We leave the proof of the dual statement to the reader. □

DEFINITION 6.11. Let F be a functor between additive categories. We say that F is additive if the associated map from $\mathrm{Hom}(A, B)$ to $\mathrm{Hom}(F(A), F(B))$ is a morphism of abelian groups. Let F be a functor between abelian categories. We say that the functor F is **exact** if it transforms an exact sequence in an exact sequence. It is **left-exact** if it transforms an exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C$ to an exact sequence $0 \to F(A) \xrightarrow{F(f)} B \xrightarrow{F(g)} F(C)$. It is **right-exact**, if it transforms an exact sequence $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ to an exact sequence $A \xrightarrow{F(f)} B \xrightarrow{F(g)} C \to 0$.

Example:

(1) The functor $X \to \mathrm{Mor}(X,A)$ (from $\mathscr{C}$ to **Ab**) is left-exact. Indeed, consider an exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C$, and the corresponding sequence $0 \to \mathrm{Mor}(X,A) \xrightarrow{f_*} \mathrm{Mor}(X,B) \xrightarrow{g_*} \mathrm{Mor}(X,C)$ is exact, since the fact that f is a monomorphism is equivalent to the fact that f_* is injective, while the fact that $\mathrm{Im}(f_*) = \mathrm{Ker}(g_*)$ follows from the fact that $A \xrightarrow{f} B$ is the kernel of g (according to Prop. 6.10), so that for any X and $u \in \mathrm{Mor}(X,B)$ such that $g \circ u = 0$, there exists a unique v making the following diagram commutative:

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, "g"] & C \\ & X \arrow[u, "u"'] \arrow[ul, dashed, "v"] & \end{tikzcd}$$

(2) The contravariant functor $M \to \mathrm{Mor}(M,X)$ is right-exact. This means that it transforms $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ to $\mathrm{Mor}(C,X) \xrightarrow{g^*} \mathrm{Mor}(B,X) \xrightarrow{f^*} \mathrm{Mor}(A,X) \to 0$.

(3) In the category R-**mod**, the functor $M \to M \otimes_R N$ is right-exact.

(4) If a functor has a **right-adjoint** it is right-exact, if it has a **left-adjoint**, it is left-exact (see Lemma 7.15, for the meaning and proof).

EXERCICES 4. (1) Let $\mathscr{C}$ be a small category and $\mathscr{A}$ an abelian category. Prove that the category $\mathscr{C}^{\mathscr{A}}$ of functors from $\mathscr{C}$ to $\mathscr{A}$ is an abelian category.

3. The category of Chain complexes

To any abelian category $\mathscr{C}$ we may associate the category **Chain($\mathscr{C}$)** of **chain complexes**. Its objects are sequences

$$\dots \xrightarrow{d_{m-1}} I_m \xrightarrow{d_m} I_{m+1} \xrightarrow{d_{m+1}} I_{m+2} \xrightarrow{d_{m+2}} I_{m+3} \dots.$$

where the **boundary maps** d_m satisfy the condition $d_m \circ d_{m-1} = 0$. Its morphisms are commutative diagrams

$$\begin{tikzcd} \cdots \arrow[r, "d_{m-1}"] & I_m \arrow[r, "d_m"] \arrow[d, "u_m"] & I_{m+1} \arrow[r, "d_{m+1}"] \arrow[d, "u_{m+1}"] & I_{m+2} \arrow[r, "d_{m+2}"] \arrow[d, "u_{m+2}"] & I_{m+3} \arrow[r, "d_{m+3}"] \arrow[d, "u_{m+3}"] & \cdots . \\ \cdots \arrow[r] & J_m \arrow[r, "\partial_m"] & J_{m+1} \arrow[r, "\partial_{m+1}"] & J_{m+2} \arrow[r, "\partial_{m+2}"] & I_{m+3} \arrow[r, "\partial_{m+3}"] & \cdots \end{tikzcd}$$

It has several natural subcategories, in particular the subcategory of bounded complexes $\mathbf{Chain}^b(\mathscr{C})$, complexes bounded from below $\mathbf{Chain}^+(\mathscr{C})$, complexes bounded from above $\mathbf{Chain}^-(\mathscr{C})$. The cohomology $\mathscr{H}^m(A^\bullet)$ of the chain complex $A^\bullet$ is given by $\ker(d_m)/\mathrm{Im}(d_{m-1})$. We may consider $\mathscr{H}^m(A^\bullet)$ as a chain complex with boundary maps equal to zero.

PROPOSITION 6.12. *Let $\mathscr{C}$ be an abelian category. Then* $\mathbf{Chain}^b(\mathscr{C})$, $\mathbf{Chain}^+(\mathscr{C})$, $\mathbf{Chain}^-(\mathscr{C})$ *are abelian categories.*

The map from **Chain** $(\mathscr{C})$ *to* **Chain** $(\mathscr{C})$ *induced by taking homology is a functor. In particular any morphism $u = (u_m)_{m\in\mathbb{N}}$ from the complex $A^\bullet$ to the complex $B^\bullet$ induces a map $u_* : \mathscr{H}(A^\bullet) \to \mathscr{H}(B^\bullet)$. If moreover u, v are* **chain homotopic**, *that is there exists a map $s = (s_m)_{m\in\mathbb{N}}$ such that $s_m : I_m \to J_{m-1}$ and $u - v = \partial_{m-1} \circ s_m + s_{m+1} \circ d_m$ then $\mathscr{H}(u) = \mathscr{H}(v)$.*

PROOF. The proof is left to the reader or referred for example to [**Weib**]. $\square$

EXERCICES 5. (1) Show that the definition of $\mathscr{H}(A^\bullet)$ indeed makes sense in an abstract category: one must prove that there is a mapping $\mathrm{Im}(d_{m-1}) \to \mathrm{Ker}(d_m)$ (see the map w after definition 6.9) and $\mathscr{H}^m(C^\bullet)$ is the cokernel of this map.
(2) Determine the kernel and cokernel in the category **Chain**$(\mathscr{C})$.

The abelian category $\mathscr{C}$ is a subcategory of **Chain**$(\mathscr{C})$ by identifying A to $0 \to A \to 0$ and it is then a full subcategory.

DEFINITION 6.13. A map $u : A^\bullet \to B^\bullet$ is a quasi-isomorphism if the induced map $\mathscr{H}(u)$ is an isomorphism from $\mathscr{H}(A^\bullet)$ to $\mathscr{H}(B^\bullet)$.

Note that a chain map $u : A^\bullet \to B^\bullet$ is a **chain homotopy equivalence** if and only if there exists a chain map $v : B^\bullet \to A^\bullet$ such that $u \circ v$ and $v \circ u$ are chain homotopic to the Identity. A chain homotopy equivalence is a quasi-isomorphism, but the converse is not true. A fundamental result in homological algebra is the existence of long exact sequences associated to a short exact sequence.

PROPOSITION 6.14. *To a short exact sequence of chain complexes,*

$$0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$$

corresponds a long exact sequence

$$.. \to \mathscr{H}^m(A^\bullet) \to \mathscr{H}^m(B^\bullet) \to \mathscr{H}^m(C^\bullet) \xrightarrow{\delta} \mathscr{H}^{m+1}(A^\bullet) \to ...$$

PROOF. See any book on Algebraic topology or [**Weib**] page 10. $\square$

REMARK 6.15. If the exact sequence is split (i.e. there exists $h : C^\bullet \to B^\bullet$ such that $g \circ h = \mathrm{Id}_C$, then we can construct a **sequence of chain maps**,

$$... \to A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \xrightarrow{\delta} A^\bullet[1] \xrightarrow{f[1]} B^\bullet \xrightarrow{g[1]} ...$$

where we set $(A^\bullet[k])^n = A^{n+k}$ and $\partial_{A^\bullet[k]} = (-1)^k\partial$, and such that the long exact sequence is obtained by taking the cohomology of the above sequence.

This does not hold in general, but these **distinguished triangles** play an important role in triangulated categories (of which the Derived category is the main example), where exact sequences do not make much sense.

Finally, the Freyd-Mitchell theorem tells us that if $\mathscr{C}$ is a small abelian category[5], then there exists a ring R and a **fully faithful and exact**[6] functor $F : \mathscr{C} \to \mathbf{R-mod}$ for some R. The functor F identifies A with a subcategory of R-Mod : F yields an equivalence between $\mathscr{C}$ and a subcategory of R-Mod in such a way that kernels and cokernels computed in $\mathscr{C}$ correspond to the ordinary kernels and cokernels computed in R-Mod. We can thus, whenever this simplifies the proofs, assume that an abelian category is a subcategory of the category of R-modules. As a result, all diagram theorems in an abelian categories, can be proved by assuming the objects are R-modules, and the maps are R-modules morphisms, and in particular maps between sets[7].

We refer to [**Weib**] for the sketch of a proof, but it let us mention here Yoneda's lemma, that is a crucial ingredient in the proof of Freyd-Mitchell theorem.

LEMMA 6.16. *Given two objects A, A' in $\mathscr{C}$, and assume for all C there is a bijection $i_C : \mathrm{Mor}(A,C) \to \mathrm{Mor}(A',C)$, commuting with the maps $f^* : \mathrm{Mor}(C,A) \to \mathrm{Mor}(B,A)$ induced by $f : B \to C$. Then A and A' are isomorphic.*

As a consequence of the Freyd-Mitchell theorem, we see that all results of homological algebra obtained by diagram chasing are valid in any abelian category. For example we have :

LEMMA 6.17 (Snake Lemma). *In an abelian category, consider a commutative diagram:*

$$\begin{tikzcd} & A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' & \end{tikzcd}$$

where the rows are exact sequences and 0 is the zero object. Then there is an exact sequence relating the kernels and cokernels of a, b, and c:

$$\mathrm{Ker(a)} \longrightarrow \mathrm{Ker(b)} \longrightarrow \mathrm{Ker(c)} \xrightarrow{d} \mathrm{Coker}(a) \longrightarrow \mathrm{Coker}(b) \longrightarrow \mathrm{Coker}(c)$$

Furthermore, if the morphism f is a monomorphism, then so is the morphism $\mathrm{Ker(a)} \longrightarrow \mathrm{Ker(b)}$*, and if g' is an epimorphism, then so is* $\mathrm{Coker}(b) \longrightarrow \mathrm{Coker}(c)$.

PROOF. First we may work in the abelian category generated by the objects and maps of the diagram. This will be a small abelian category. According to the Freyd-Mitchell theorem, we may assume the objects are R-modules and the morphisms are R-modules morphisms. Note that apart from the map d, whose existence we need to prove, the other maps are induced by f, g, f', g'. Note also that the existence of d in the general abelian category follows from the R-module case and the Freyd-Mitchell

[5]Remember that this means that objects and morphism are in fact sets.

[6]A functor F is fully faithful if $F_{X,Y} : \mathrm{Mor}(X,Y) \to \mathrm{Mor}(F(X),F(Y))$ is bijective.

[7]see http://unapologetic.wordpress.com/2007/09/28/diagram-chases-done-right/ for an alternative approach to this specific problem.

theorem, since the functor provided by the theorem is fully-faithful. Let us construct d. Let $z \in \mathrm{Ker(c)}$, then $z = g(y)$ because g is onto, and $g'b(y) = 0$, hence $b(y) = f'(x')$ and we set $x' = d(z)$. We must prove that x' is well defined in $\mathrm{Coker}(f') = A'/a(A)$. For this it is enough to see that if $z = 0$, $y \in \mathrm{Ker(g)} = \mathrm{Im(f)}$ that is $y = f(x)$, and so if $bf(x) = b(y) = f'(x')$, we have $f'(x') = f'(a(x))$ and since f' is monomorphism, we get $x' = a(x)$.

Let us now prove the maps are exact at Ker(b). Let $v \in \mathrm{Ker(b)}$ (i.e. $b(v) = 0$) such that $g(v) = 0$. Then by exactness of the top sequence, $v = f(u)$ with $u \in A$. We have $f'a(u) = b(f(u)) = b(v) = 0$, and since f' is injective, $a(u) = 0$ that is $u \in \mathrm{Ker(a)}$. □

4. Presheaves and sheaves

Let X be a topological space, $\mathscr{C}$ a category.

DEFINITION 6.18. A $\mathscr{C}$-**presheaf** on X is a functor from the category **Open(X)**^{op} to an other category.

DEFINITION 6.19. A presheaf $\mathscr{F}$ of R-modules on X is defined by associating to each open set U in X an R-module, $\mathscr{F}(U)$, such that If $V \subset U$ there is a unique module morphism $r_{V,U} : \mathscr{F}(U) \longrightarrow \mathscr{F}(V)$ such that $r_{W,V} \circ r_{V,U} = r_{W,U}$ and $r_{U,U} = \mathrm{id}$. Equivalently, a presheaf is a functor from the category **Open(X)**^{op} to the category R-**mod**.

Notation: if $s \in \mathscr{F}(U)$ we often denote by $s_{|V}$ the element $r_{V,U}(s) \in \mathscr{F}(V)$. From now on we shall, unless otherwise mentioned, only deal with presheaves in the category R-**mod**. Our results extend to sheaves in any abelian category. The reader can either check this for himself (most proofs translate verbatim to a general abelian category), or use the Freyd-Mitchell theorem (see page 59).

DEFINITION 6.20. A presheaf $\mathscr{F}$ on X is a **sheaf** if whenever $(U_j)_{j\in I}$ are open sets in X covering U (i.e. $\bigcup_{j\in I} U_j = U$, the map

$$\mathscr{F}(U) \longrightarrow \{(s_j)_{j\in I} \mid \prod_{j\in I} \mathscr{F}(U_j),\ r_{U_j,U_j\cap U_k}(s_j) = r_{U_k,U_j\cap U_k}(s_k)\}$$

is bijective.

This means that elements of $\mathscr{F}(U)$ are defined by **local properties**, and that we may check whether they are equal to zero by local considerations. We denote by R-**Presheaf(X)** and R-**Sheaf(X)** the category of R-modules presheaves or sheafs.

EXERCICE 6. Does the above definition imply that for a sheaf, $\mathscr{F}(\emptyset)$ is the terminal object in the category ? One usually adds this condition to the definition of a sheaf, and we stick to these tradition.

Examples:

(1) The skyscraper R-sheaf over x, denoted R_x is given by $R_x(U) = 0$ if $x \notin U$ and $R_x(U) = R$ for $x \in U$.

(2) Let $f : E \to X$ be a continuous map, and $\mathscr{F}(U)$ be the sheaf of continuous sections of f defined over U, that is the set of maps $s : U \to E$ such that $f \circ s = \mathrm{id}_U$.
(3) Let $E \to X$ be a map between manifolds, and Π be a subbundle of T_zE. Consider $\mathscr{F}(U)$ to be the set of sections $s : X \to E$ such that $ds(x) \subset \Pi(s(x))$.
(4) If $f : Y \to X$ is a map, then we define a sheaf as $\mathscr{F}_f(U) = f^{-1}(U)$. This is a sheaf of **Open(Y)** on X but can also be considered as a sheaf of sets, or a sheaf of topological spaces.
(5) Set $\mathscr{F}(U)$ to be the set of constant functions on U. This is a presheaf. It is not a sheaf, because local considerations can only tell whether a function is **locally constant**. On the other hand the sheaf of locally constant functions is indeed a sheaf. It is called **the constant sheaf**, and denoted R_X. It can also be defined by setting $\mathscr{F}(U)$ to be the set of locally constant functions from U to the discrete set R.
(6) Let $\mathscr{F}$ be a sheaf. We say that $\mathscr{F}$ is **locally constant** if and only if $\mathscr{F}$ every point is contained in an open set U such that the sheaf $\mathscr{F}_U$ defined on U by $\mathscr{F}_U(V) = \mathscr{F}(V)$ for $V \subset U$ is a constant sheaf. There are non-constant locally constant sheafs, for example the set of locally constant sections of the $\mathbb{Z}/2$ Möbius band, defined by $M = [0,1] \times \{\pm 1\}/\{(0,1) = (1,-1)\}$.
(7) If A is a closed subset of X, then k_A, **the constant sheaf over** A is the sheaf such that $k_A(U)$ is the set of locally constant functions from $A \cap U$ to k.
(8) If U is an open set in X, then k_U, **the constant sheaf over** U is defined by $k_U(V)$ is the subset of $k(U \cap V)$ made of sections of the constant sheaf with support closed in V. This means that $k(U \cap V) = k^{\pi_0(U \cap V)}$, where $\pi_0(U \cap V)$ is the number of connected component of $U \cap V$ such that $\overline{U \cap V} \subset U$.
(9) The sheaf $C^0(U)$ of continuous functions on U is a sheaf. The same holds for $C^p(U)$ on a C^p manifold, or $\Omega^p(U)$ the space of smooth p-forms on a smooth manifold, or $\mathscr{D}(U)$ the space of distributions on U, or $\mathscr{T}^p(U)$ the set of p-currents on U.
(10) If X is a complex manifold, the sheaf of holomorphic functions $\mathscr{O}_X$ is a sheaf. Similarly if E is a holomorphic vector bundle over X, then $\mathscr{O}_X(E)$ the set of holomorphic sections of the bundle E.
(11) The functor **Top** $\to$ **Chains** associating to a topological space M its singular cochain complex $(C^*(M,R),\partial)$ yields a sheaf of R-modules by associating to U, the R-module of singular cochains on U, $C^*(U,R)$. It is obviously a presheaf, and one proves it is a sheaf by using the exact sequence

$$0 \to C^*(U \cup V) \to C^*(U) \oplus C^*(V) \to C^*(U \cap V) \to 0$$

On the other hand using the functor **Top** $\longrightarrow$ **R – mod** given by $U \to H^*(U)$, we get a presheaf of R-modules by $\mathscr{H}(U) = H^*(U)$. This is not a sheaf, because

Mayer-Vietoris is a long exact sequence

$$\dots \to H^{*-1}(U\cup V)\to H^*(U\cup V)\to H^*(U)\oplus H^*(V)\to H^*(U\cap V)\to H^{*+1}(U\cup V)\to\dots$$

not a short exact sequence, so two elements in $H^*(U)$ and $H^*(V)$ with same image in $H^*(U\cap V)$ do come from an element in $H^*(U\cup V)$, but this element is not unique: the indeterminacy is given by the image of the coboundary map $\delta : H^{*-1}(U\cap V)\to H^*(U\cup V)$. The stalk of this presheaf is $\lim_{U\ni x} H^*(U)$, the local cohomology of X at x. If X is a manifold, the Poincaré lemma tells us that this is R in degree zero and 0 otherwise.

EXERCICE 7. Prove that a locally constant sheaf is the same as local coefficients. In particular prove that on a simply connected manifold, all locally constant sheaf are of the form $k_X\otimes V$ for some vector space V.

Because a sheaf is defined by local considerations, it makes sense to define the germ of $\mathscr{F}$ at x. The following definition makes sense if the category has direct limits. This means that given a family $(A_\alpha)_{\alpha\in J}$ of objects indexed by a totally ordered set, J, and morphisms $f_{\alpha,\beta}: A_\alpha\to A_\beta$ defined for $\alpha\le\beta$, we define the direct limit of the sequence as an object A and maps $f_\alpha : A_\alpha\to A$ with the universal property: for each family of maps $g_\alpha : A_\alpha\to B$ such that $f_{\alpha,\beta}\circ g_\beta = g_\alpha$, we have a map $\varphi : B\to A$ making the following diagram commutative :

$$\begin{tikzcd}
A \arrow[r, "\varphi"] & B \\
A_\alpha \arrow[u, "f_\alpha"] \arrow[r, "f_{\alpha,\beta}"'] \arrow[ur, "g_\alpha"] & A_\beta \arrow[u, "g_\beta"'] \arrow[ul, "f_\beta"]
\end{tikzcd}$$

Note that if we restrict ourselves to metric spaces, for example manifolds, we only need this concept for $J=\mathbb{N}$.

DEFINITION 6.21. Let $\mathscr{F}$ be a presheaf on X and assume that direct limits exists in the category where the sheaf takes its values. The **stalk** of $\mathscr{F}$ at x, denoted $\mathscr{F}_x$ is defined as the **direct limit**

$$\varinjlim_{U\ni x}\mathscr{F}(U)$$

An element in $\mathscr{F}_x$ is just an element $s\in\mathscr{F}_U$ for some $U\ni x$, but two such objects are identified if they coincide in a neighborhood of x: they are "germs of sections". For example if $\mathbb{C}_X$ is the constant sheaf, $(\mathbb{C}_X)_x=\mathbb{C}$. If $\mathscr{F}$ is the skyscraper sheaf at x, we have $\mathscr{F}_y=0$ for $y\neq x$ and $\mathscr{F}_x=R$.

REMARK 6.22. (1) Be careful, the data of $\mathscr{F}_x$ for each x, does not in general, define an element in $\mathscr{F}(X)$. On the other hand if it does, the element is then unique.

(2) For any closed F, we denote by $\mathscr{F}(F) = \lim_{U\supset F}\mathscr{F}(U)$. Be careful, for V open, it is not true that $\mathscr{F}(V) = \lim_{U\supsetneq V}\mathscr{F}(U)$, since some sections on V may not extend to any neighborhood (e.g. continuous functions on V going to infinity near ∂V do not extend). But of course, replacing $\supsetneq$ by $\supset$ we do have equality.

(3) Using the stalk, we see that any sheaf can be identified with the sheaf of continuous sections of the map $\bigcup_{x\in X}\mathscr{F}_x \to X$ sending $\mathscr{F}_x$ to x. The main point is to endow $\bigcup_{x\in X}\mathscr{F}_x$ with a suitable topology, and this topology is rather strange, for example the fibers are always totally disconnected. Indeed, the topology is given as follows: open sets in $\bigcup_{x\in X}\mathscr{F}_x$ are generated by $U_s = \{s(x) \mid x \in U, s \in \mathscr{F}(U)\}$.

(4) For a section $s \in \mathscr{F}(X)$ define the support $\mathrm{supp}(s)$ of s as the set of x such that $s(x) \in \mathscr{F}_x$ is nonzero. Note that this set is closed, or equivalently the set of x such that $s(x) = 0$ is open, contrary to what one would expect, before a moment's reflexion shows that the stalk is a set of germs, and if a germ of a function is zero, the germ at nearby points are also zero.

First we set

Definition 6.23. Let $\mathscr{F},\mathscr{G}$ be presheaves. A morphism f from $\mathscr{F}$ to $\mathscr{G}$ is a family of maps $f_U : \mathscr{F}(U) \to \mathscr{G}(U)$ such that $r_{V,U}\circ f_U = f_V \circ r_{V,U}$. Such a morphism induces a map $f_x : \mathscr{F}_x \to \mathscr{G}_x$.

4.1. Sheafification. The notion of stalk will allow us to associate to each presheaf a sheaf. Let $\mathscr{F}$ be a presheaf.

Definition 6.24. The sheaf $\widetilde{\mathscr{F}}$ is defined as follows. Define $\widetilde{\mathscr{F}}(U)$to be the subset of $\prod_{x\in U}\mathscr{F}_x$ made of families $(s_x)_{x\in U}$ such that for each $x \in U$, there is $W \ni x$ and $t \in \mathscr{F}(W)$ such that for all y in W $s_y = t_y$ in $\mathscr{F}_y$.

Clearly we made the property of belonging to $\widetilde{\mathscr{F}}$ local, so this is a sheaf (Check !). Contrary to what one may think, even if we are only interested in sheafs, we cannot avoid presheaves or sheafification.

Proposition 6.25. *Let $\mathscr{F}$ be a presheaf, $\widetilde{\mathscr{F}}$ the associated sheaf. Then $\widetilde{\mathscr{F}}$ is characterized by the following universal property: there is a natural morphism $i : \mathscr{F} \to \widetilde{\mathscr{F}}$ inducing an isomorphism $i_x : \mathscr{F}_x \to \widetilde{\mathscr{F}}_x$, and such that for any $f : \mathscr{F} \to \mathscr{G}$ morphisms of presheaves such that $\mathscr{G}$ is a sheaf, there is a unique $\widetilde{f} : \widetilde{\mathscr{F}} \to \mathscr{G}$ making the following diagram commutative*

$$\begin{tikzcd} \mathscr{F} \arrow[r, "i"] \arrow[dr, "f"'] & \widetilde{\mathscr{F}} \arrow[d, "\widetilde{f}"] \\ & \mathscr{G} \end{tikzcd}$$

Proposition 6.26. *Let $f : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves. Then*

(1) If for all x we have $f_x = 0$, then $f = 0$

(2) *If for all x we have f_x is injective, then f_U is injective*
(3) *If for all x, the map f_x is an isomorphism, then so is f_U*

PROOF. Let $s \in \mathscr{F}(U)$. Then $f_x = 0$ implies that for all $x \in U$ there is a neighborhood U_x such that $f_U(s)_x = 0$. This implies that $f_U(s) = 0$, hence $f = 0$ Let us now assume that $f_U(s) = 0$, and let us prove $s = 0$. Indeed, since $f_x s_x = 0$ we have $s_x = 0$ for all $x \in U$. But this implies $s = 0$ in $\mathscr{F}(U)$ by the locality property of sheafs. Finally, if f_x is bijective, it is injective and so is f_U. We have to prove that if moreover f_x is surjective, so is f_U. Indeed, let $t \in \mathscr{G}(U)$. By assumption, for each x, there exists s_x defined on a neighborhood V_x of x, such that $f_{V_x}(s_x) = t_x$ on $W_x \subset V_x$ containing x. We may of course replace V_x by W_x. By injectivity, such a s_x is unique. If s_x is defined over W_x, and s_y over W_y then on $W_x \cap W_y$ we have $f_{V_x}(s_x) = f_{W_x}(s_y) = t_{W_x \cap W_y}$, hence $s_x = s_y$ on $W_x \cap W_y$. As a result, according to the definition of a sheaf, there exists s equal to s_x on each W_x and $f(s) = t$. As a result the map f_U has a unique inverse, g_U for each open et U and we may check that g_U is a sheaf morphism, and $g \circ f = \mathrm{Id}_{\mathscr{F}}, f \circ g = \mathrm{Id}_{\mathscr{G}}$. □

Of course we do not have a surjectivity analogue of the above, because it does not hold in general.

In terms of categories, R-**Presheaf(X)** being the category of presheaves, and R-**Sheaf(X)** the category of sheaves of R-modules, these are abelian categories. The 0 object is the sheaf associating the R-module 0 to any open set. This is equivalent to $\mathscr{F}_x = 0$ for all x. The biproduct of $\mathscr{F}_1, ..., \mathscr{F}_2$ is the sheaf associating to U the R-module $\mathscr{F}_1(U) \oplus \mathscr{F}_2(U)$. Clearly $\mathrm{Mor}(\mathscr{F}, \mathscr{G})$ is abelian and makes R-**Sheaf(X)** into an additive category. We also have that $\mathrm{Ker(f)(U)} = \mathrm{Ker(f_U)}$. Indeed, this defines a sheaf on X, since if s_j satisfies $f_{U_j}(s_j) = 0$ and $s_{U_j} = s_{U_k}$ on $U_j \cap U_k$, then$f_U(s) = 0$. On the other hand $\mathrm{Im}(f)(U)$ is not defined as $\mathrm{Im}(f_U)$, since this is not a sheaf. Indeed, $t_{U_j} = f_{U_j}(s_j)$ and $t_j = t_k$ on $U_j \cap U_k$ does not imply that $t_j = t_k$ on $U_j \cap U_k$, so here is no way to guarantee that there exists s such that $t = f(s)$. However $\mathrm{Im}(f_U)$ defines a presheaf. Then the Image in the category of Sheaves, denoted by $\mathrm{Im}(f)$ is the sheafification of $\mathrm{Im}(f_U)$. The same holds for $\mathrm{Coker}(f)$. Indeed, the universal property of sheafification means that if $f : \mathscr{F} \to \mathscr{G}$ is a morphism, and $\mathscr{F} \xrightarrow{f} \mathscr{G} \xrightarrow{p} \mathscr{H}$ is the cokernel in the category of presheaves, so that for any sheaf $\mathscr{L}$ such that

$$\mathscr{F} \xrightarrow{f} \mathscr{G} \xrightarrow{g} \mathscr{L}$$

satisfies $g \circ f = 0$, we have a pair $(\mathscr{C}, q)$ such that there exits a unique h making this diagram commutative

$$\begin{tikzcd}
\mathscr{F} \arrow[r, "f"] & \mathscr{G} \arrow[r, "g"] \arrow[dr, "q"] & \mathscr{L} & \\
 & & \mathscr{H} \arrow[u, dashed, "h"] \arrow[r, "i_{\mathscr{H}}"] & \widetilde{\mathscr{H}}
\end{tikzcd}$$

But if $\mathscr{L}$ is a sheaf, the map $\mathscr{H} \stackrel{h}{\to} \mathscr{L}$ lifts to a map $\widetilde{\mathscr{H}} \stackrel{\tilde{h}}{\to} \mathscr{L}$. Now set $\tilde{q} = i_{\mathscr{H}} \circ q$, it is easy to check that $(\widetilde{\mathscr{H}}, \tilde{q})$ has the universal property we are looking for, hence this is the cokernel of f in the category R-**Sheaf (X)**. Because $(i_{\mathscr{H}})_x$ is an isomorphism, we see that $\mathrm{Coker}(f)_x = \mathrm{Coker}(f_x)$.

To conclude, we have an inclusion functor from R-**Presheaf(X)** to R-**Sheaf(X)**, and the sheafification functor: $Sh : R$-**Presheaf(X)** $\to R$-**Sheaf(X)**.

⚠ CAUTION: It follows from the above that the Image in the category of presehaves does not coincide with the Image in the category of sheaves. Since we mostly work with sheaves, $\mathrm{Im}(f)$ will designate the Image in the category of sheaves, unless otherwise mentioned.

Now the definition of an exact sequence in the abelian category of sheaves translates as follows.

DEFINITION 6.27. A sequence of sheaves over X, $\mathscr{F} \stackrel{f}{\to} \mathscr{G} \stackrel{g}{\to} \mathscr{H}$ is exact, if and only if for all $x \in X$, $\mathscr{F}_x \stackrel{f_x}{\to} \mathscr{G}_x \stackrel{g_x}{\to} \mathscr{H}_x$ is exact.

Example:

(1) Let $U = X \setminus A$ where A is a closed subset of X. Then we have an exact sequence
$$0 \to k_{X\setminus A} \to k_X \to k_A \to 0$$
obtained from the obvious maps.

(2) Given a sheaf $\mathscr{F}$ and a closed subset A of X, we have as above an exact sequence
$$0 \to \mathscr{F}_{X\setminus A} \to \mathscr{F}_X \to \mathscr{F}_A \to 0$$
were $\mathscr{F}_A(U) = \mathscr{F}(U \cap A)$ while $\mathscr{F}_{X\setminus A}(U)$ is the set of sections of $\mathscr{F}(U \cap (X \setminus A))$ with closed support in $X \setminus A$.

Now consider the functor Γ_U from $\mathbf{R-Sheaf(X)} \longrightarrow \mathbf{R-mod}$ given by $\Gamma_U(\mathscr{F}) = \mathscr{F}(U)$. We have that a short exact sequence, i.e. a sequence $0 \to \mathscr{A} \stackrel{f}{\to} \mathscr{B} \stackrel{g}{\to} \mathscr{C} \to 0$ such that for each x $0 \to \mathscr{A}_x \stackrel{f_x}{\to} \mathscr{B}_x \stackrel{g_x}{\to} \mathscr{C}_x \to 0$ is exact, then
$$0 \to \mathscr{A}(U) \stackrel{f_U}{\to} \mathscr{B}(U) \stackrel{g_U}{\to} \mathscr{C}(U)$$
is exact, and f_U is injective by proposition 6.25, but the map g_U is not necessarily surjective. Indeed, we wish to prove that $\mathrm{Im}(f_U) = \mathrm{Ker(g_U)}$. Because $g_x \circ f_x = 0$ we have $g_U \circ f_U = 0$, so that $\mathrm{Im}(f_U) \subset \mathrm{Ker(g_U)}$. Let us prove the reverse inclusion. Let $t \in \mathrm{Ker(g_U)}$. Then for each $x \in U$, there exists s_x such that on some neighborhood U_x we have $t_x = f_x(s_x)$, and by injectivity of f_x, s_x is unique. This implies that on $U_x \cap U_y$, $s_x = s_y$. But this implies that the s_x are restrictions of an element in $\mathscr{A}(U)$.

We just proved

PROPOSITION 6.28. *For any open set, U, the functor $\Gamma_U : \mathbf{Sheaf}(\mathbf{X}) \to R-mod$ is left exact.*

5. Appendix: Freyd-Mitchell without Freyd-Mitchell

If the only application of the Freyd-Mitchell theorem was to allow us to prove theorems on abelian categories as if the objects were modules and the maps module morphisms, there would be the following simpler approach. Let us first prove that pull-back exist in any abelian category.

Consider the diagram:

(6.5)
$$\begin{tikzcd} & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

The above diagram has a pull-back (P, i, j) where $i \in \mathrm{Mor}(P, X)$, $j \in \mathrm{Mor}(P, Y)$ if for any Q and maps $u \in \mathrm{Mor}(Q, X)$, $v \in \mathrm{Mor}(Q, Y)$ such that $f \circ u = g \circ v$ there is a unique map $\rho \in \mathrm{Mor}(Q, P)$ such that $i \circ \rho = u$, $j \circ \rho = v$.

(6.6)
$$\begin{tikzcd} Q \arrow[dr, dashed, "\rho"] \arrow[drr, bend left, "u"] \arrow[ddr, bend right, "v"'] & & \\ & P \arrow[r, "i"] \arrow[d, "j"] & X \arrow[d, "f"] \\ & Y \arrow[r, "g"] & Z \end{tikzcd}$$

We can construct a pull-back in any abelian category by taking for (P, i, j) the kernel of the map $f - g : X \oplus Y \to Z$. Then $(f - g) \circ (u, v) = 0$ and the existence and uniqueness of ρ follows form existence and uniqueness of the dotted map in the definition fo the kernel.

Let us define the relation $x \in_m A$ to mean $x \in \mathrm{Mor}(B, A)$ for some B, and identify x and y if and only if there are epimorphisms u, v such that $x \circ u = y \circ v$. This is obviously a reflexive and symmetric relation. We need to prove it is transitive through the following diagram

(6.7)
$$\begin{tikzcd} \bullet \arrow[r, "u'"] \arrow[d, "v'"] & \bullet \arrow[r, "t"] \arrow[d, "u"] & \bullet \arrow[d, "x"] \\ \bullet \arrow[r, "v"] \arrow[d, "w"] & \bullet \arrow[r, "y"] \arrow[d, "y"] & A \\ \bullet \arrow[r, "z"] & A & \end{tikzcd}$$

The existence of u', v' follows from pull-back from the other diagrams. Moreover u', v' are epimorphisms, so $x \equiv z$ since $x \circ (t \circ u') = z \circ (w \circ v')$. Let us denote by $\overline{A}$ the set of $x \in_m A$ modulo the equivalence relation. $\overline{A}$ is an abelian group:

(1) 0 is represented by the zero map, and any zero map in $\mathrm{Mor}(B, A)$ is equivalent to it.
(2) if $x \in_m A$, then $-x \in_m A$.
(3) If $f \in \mathrm{Mor}(A, A')$ and $x \in_m A$ then $f \circ x \in_m A'$. We denote this by $f(x)$.

Now

(1) if f is a monomorphism, if and only if $f \circ x = 0$ implies $x = 0$. This is also equivalent to $f(x) = f(x')$ implies $x = x'$.
(2) the sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is exact if and only if $g \circ f = 0$ and for any y such that $g(y) = 0$ we have $y = f(x)$

Indeed, $f(x) = 0$ means there is an epimorphism u such that $f \circ x \circ u = 0$. Since f is a monomorphism this implies $x \circ u = 0$ that is $x \equiv 0$. The second statement follows from the fact that $f(x) = f(x')$ is equivalent to $f(x - x') = 0$.

We thus constructed a functor from $\mathscr{C}$ to **Sets**. Its image is an abelian subcategory of the category of sets, and Freyd-Mitchell tells us that this is a category of R-modules, for some R, but the first embedding is enough for “diagram chasing with elements”.

CHAPTER 7

More on categories and sheaves.

1. Injective objects and resolutions

Let I be an object in a category.

DEFINITION 7.1. The object I is said to be injective, if for any maps h, f such that f is a monomorphism, there exists g making the following diagram commutative

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "h"'] & B \arrow[d, dashed, "g"] \\ & I \end{tikzcd} .$$

This is equivalent to saying that $A \to \mathrm{Mor}(A, I)$ sends monomorphisms to epimorphisms. Note that g is by no means unique ! An injective sheaf is an injective object in R-**Sheaf (X)**.

PROPOSITION 7.2. *If I is injective in an abelian category $\mathscr{C}$, the functor $A \to \mathrm{Mor}(A, I)$ from $\mathscr{C}$ to* **Ab** *is exact.*

DEFINITION 7.3. A category has enough injectives, if any object A has a monomorphism into an injective object.

EXERCICE 1. Prove that in the category **Ab** of abelian groups, the group $\mathbb{Q}/\mathbb{Z}$ is injective. Prove that **Ab** has enough injectives (prove that a sum of injectives is injective).

In a category with enough injectives, we have the notion of **injective resolution**.

PROPOSITION 7.4 ([**Iv**], p.15). *Assume $\mathscr{C}$ has enough injectives, and let B be an object in $\mathscr{C}$. Then there is an exact sequence*

$$0 \to B \xrightarrow{i_B} J_0 \xrightarrow{d_0} J_1 \xrightarrow{d_1} J_2 \to ....$$

where the J_k are injectives.This is called an **injective resolution** *of B. Moreover given an object A in $\mathscr{C}$ and a map $f : A \to B$ and a resolution of A (not necessarily injective), that is an exact sequence*

$$0 \to A \xrightarrow{i_A} L_0 \xrightarrow{d_0} L_1 \xrightarrow{d_1} L_2 ....$$

and an injective resolution of B as above, then there is a morphism (i.e. a family of maps $u_k : L_k \to J_k$) such that the following diagram is commutative

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i_A"] \arrow[d, "f"] & L_0 \arrow[r, "d_0"] \arrow[d, "u_0"] & L_1 \arrow[r, "d_1"] \arrow[d, "u_1"] & L_2 \arrow[r, "d_2"] \arrow[d, "u_2"] & \cdots . \\
0 \arrow[r] & B \arrow[r, "i_B"] & J_0 \arrow[r, "\partial_0"] & J_1 \arrow[r, "\partial_1"] & I_2 \arrow[r, "\partial_2"] & \cdots
\end{tikzcd}$$

Moreover any two such maps are homotopic (i.e. $u_k - v_k = \partial_{k-1} s_k + s_{k+1}\delta_k$, *where* $s^k : I_k \to J_{k-1}$*).*

PROOF. The existence of a resolution is proved as follows: existence of J_0 is by definition of having enough injectives. Then let $M_1 = \mathrm{Coker}(i_B)$ so that $0 \to B \overset{d_0}{\to} J_0 \overset{f_0}{\to} M_1 \to 0$ is exact. A map $0 \to M_1 \to J_1$ induces a map $0 \to B \overset{i_B}{\to} J_0 \overset{d_0}{\to} J_1$, exact at J_0. Continuing this procedure we get the injective resolution of B. Now let $f : A \to B$ and consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & A \arrow[r, "i_A"] \arrow[d, "f"] & L_0 \\
0 \arrow[r] & B \arrow[r, "i_B"] & J_0
\end{tikzcd}$$

Since J_0 is injective, i_B is monomorphism and $i_A \circ f$ lifts to a map $u_0 : L_0 \to J_0$. Let us now assume inductively that the map u_k is defined, and let us define u_{k+1}. We decompose using property (2) of Definition 6.7:

$$\begin{tikzcd}
L_{k-1} \arrow[r, "d_{k-1}"] \arrow[d, "u_{k-1}"] & L_k \arrow[r, "d_k"] \arrow[d, "u_k"] & L_{k+1} \\
J_{k-1} \arrow[r, "\partial_{k-1}"] & J_k \arrow[r, "\partial_k"] & J_{k+1}
\end{tikzcd}$$

as

$$\begin{tikzcd}
L_{k-1} \arrow[r, "d_{k-1}"] \arrow[d, "u_{k-1}"] & L_k \arrow[r, "d_k"] \arrow[d, "u_k"] & \mathrm{Coker}(d_{k-1}) \arrow[r, "i_k"] \arrow[d, dashed, "v_{k+1}"] & L_{k+1} \\
J_{k-1} \arrow[r, "\partial_{k-1}"] & J_k \arrow[r, "\partial_k"] & \mathrm{Coker}(\partial_{k-1}) \arrow[r, "j_k"] & J_{k+1}
\end{tikzcd}$$

Since $(\partial_k \circ u_k) \circ d_{k-1} = 0$, there exists by definition of the cokernel a map $v_{k+1} : \mathrm{Coker}(d_{k-1}) \to \mathrm{Coker}(\partial_{k-1})$, making the above diagram commutative. Then since i_k is monomorphism (due to exactness at L_k) and J_{k+1} is injective, the map $j_k \circ v_{k+1}$ factors through i_k so that there exists $u_{k+1} : L_{k+1} \to J_{k+1}$ making the above diagram commutative. The construction of the homotopy is left to the reader. □

PROPOSITION 7.5. *The category* $\mathbf{R-Sheaf(X)}$ *has enough injectives.*

PROOF. The proposition is proved as follows.

Step 1: One proves that for each x there is an injective $\mathscr{D}(x)$ such that $\mathscr{F}_x$ injects into $D(x)$. In other words we need to show that R-mod has enough injectives. We omit this step since it is trivial for $\mathbb{C}$-sheaves (any vector space is injective).

Step 2: Construction of $\mathscr{D}$. The category **R-mod** has enough injectives, so choose for each x a map $\tilde{\jmath}_x : \mathscr{F}_x \to \mathscr{D}(x)$ where $\mathscr{D}(x)$ is injective, and consider the sheaf $\mathscr{D}(U) = \prod_{x\in U}\mathscr{D}(x)$. Thus a section is the choice for each x of an element $\mathscr{D}(x)$ (without any "continuity condition"). One should be careful. The sheaf $\mathscr{D}$ does not have $\mathscr{D}(x)$ as its stalk: the stalk of $\mathscr{D}$ is the set of germs of functions (without continuity condition) $x \mapsto \mathscr{D}(x)$ for x in a neighborhood of x_0. Obviously, $\mathscr{D}_{x_0}$ surjects on $\mathscr{D}(x_0)$. However, for each $\mathscr{F}$ we have $\mathrm{Mor}(\mathscr{F},\mathscr{D}) = \prod_{x\in X} Hom(\mathscr{F}_x,\mathscr{D}(x))$: indeed, an element $(f_x)_{x\in X}$ in the right hand side will define a morphism f by $s \to f_x(s_x)$, and vice-versa, an element f in the left hand side, defines a family $(f_x)_{x\in X}$ by taking the value $f_x(s_x) = f(s)_x$. So $\tilde{j}_x$ defines an element j in $\mathrm{Mor}(\mathscr{F},\mathscr{D})$. Clearly $\mathscr{D}$ is injective since for each x, there exists a lifting g_x

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{F}_x \arrow[r, "f"] \arrow[dr, "h"'] & \mathscr{G}_x \arrow[d, dashed, "g_x"] \\ & & \mathscr{D}(x) \end{tikzcd} .$$

and the family (g_x) defines a morphism $g : \mathscr{G} \to \mathscr{D}$ (one may need the axiom of choice to choose g_x for each x).

Step 3: Let $\mathscr{F}$ an object in R-**Sheaf(X)** and $\mathscr{D}$ be the above associated sheaf. Then the obvious map $i : \mathscr{F} \to \mathscr{D}$ induces an injection $i_x : \mathscr{F}_x \to \mathscr{D}(x)$ hence is a monomorphism.

□

When R is a field, there is a unique injective sheaf with $\mathscr{D}(x) = R^q$. It is called the canonical injective R^q-sheaf. Let us now define

DEFINITION 7.6. Let $\mathscr{F}$ be a sheaf, and consider an injective resolution of $\mathscr{F}$

$$0 \to \mathscr{F} \xrightarrow{d_0} \mathscr{J}_0 \xrightarrow{d_1} \mathscr{J}_1 \xrightarrow{d_2} \mathscr{J}_2 \ldots.$$

Then the cohomology $\mathscr{H}^*(X,\mathscr{F})$ (also denoted $R\Gamma(X,\mathscr{F})$) is the (co)homology of the sequence

$$0 \to \mathscr{J}_0(X) \xrightarrow{d_{0,X}} \mathscr{J}_2(X) \xrightarrow{d_{1,X}} \mathscr{J}_2(X) \ldots.$$

In other words $\mathscr{H}^m(X,\mathscr{F}) = \mathrm{Ker}(\mathrm{d_{m,X}})/\mathrm{Im}(\mathrm{d_{m-1,X}})$

Check that $\mathscr{H}^0(X,\mathscr{F}) = \mathscr{F}(X)$. Note that the second sequence is not an exact sequence of R-modules, because exactness of a sequence of sheafs means exactness of the sequence of R-modules obtained by taking the stalk at x (for each x). In other words, the functor from **Sheaf(X)** to R-**mod** defined by $\Gamma_x : \mathscr{F} \to \mathscr{F}_x$ is exact, but the functor $\Gamma_U : \mathscr{F} \to \mathscr{F}(U)$ is not.

This is a general construction that can be applied to any left-exact functor: take an injective resolution of an object, apply the functor to the resolution after having removed the object, and compute the cohomology. According to Proposition 7.4, this does not depend on the choice of the resolution, since two resolutions are chain homotopy equivalent, and F sends chain homotopic maps to chain homotopic maps, hence preserves chain homotopy equivalences . This is the idea of derived functors, that we are going to explain in full generality (i.e. applied to chain complexes). It is here applied to the functor Γ_X. It is a way of measuring how this left exact functor fails to be exact: if the functor is exact, then $\mathscr{H}^0(X,\mathscr{F}) = \mathscr{F}(X)$ and $\mathscr{H}^m(X,\mathscr{F}) = 0$ for $m \geq 1$.

For the moment we set

DEFINITION 7.7. Let $\mathscr{C}$ be a category with enough injectives, and F be a left-exact functor. Then $R^jF(A)$ is obtained as follows: take an injective resolution of A,

$$0 \to A \xrightarrow{i_A} I_0 \xrightarrow{d_0} I_1 \xrightarrow{d_1} I_2 \to ....$$

then $R^jF(A)$ is the j-th cohomology of the complex

$$0 \to F(I_0) \xrightarrow{d_0} F(I_1) \xrightarrow{d_1} F(I_2) \to ....$$

We say that A is F-acyclic, if $R^jF(A) = 0$ for $j \geq 1$.

Note that the left-exactness of F implies that we always have $R^0F(A) = A$. Since according to Proposition 7.4, the $R^jF(A) = 0$ do not depend on the choice of the resolution, an injective object is acyclic: take $0 \to I \to I \to 0$ as an injective resolution, and notice that the cohomology of $0 \to I \to 0$ vanishes in degree greater than 0.

However, as we saw in the case of sheafs, injective objects do not appear naturally. So we would like to be able to use resolutions with a wider class of objects

DEFINITION 7.8. A **flabby** sheaf is a sheaf such that the map $\mathscr{F}(U) \to \mathscr{F}(V)$ is onto for any $V \subset U$.

Notice that by composing the restriction maps, $\mathscr{F}$ is flabby if and only if $\mathscr{F}(X) \to \mathscr{F}(V)$ is onto for any $V \subset X$. This clearly implies that the restriction of a flabby sheaf is flabby.

PROPOSITION 7.9. *An injective sheaf is flabby. A flabby sheaf is* Γ_X*-acyclic.*

PROOF. First note that the sheaf we constructed to prove that **Sheaf(X)** has enough injectives is clearly flabby. Therefore any injective sheaf $\mathscr{I}$ injects into a flabby sheaf, $\mathscr{D}$. Moreover there is a map $p : \mathscr{D} \to \mathscr{I}$ such that $p \circ i = \mathrm{id}$, since the following diagram yields the arrow p

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{I} \arrow[r, "i"] \arrow[dr, "\mathrm{id}"'] & \mathscr{D} \arrow[d, dashed, "p"] \\ & & \mathscr{I} \end{tikzcd}$$

As a result, we have diagrams

$$\begin{tikzcd}
\mathscr{D}(U) \arrow[r, "p_U"] \arrow[d, "s_{V,U}"] & \mathscr{I}(U) \arrow[d, "r_{V,U}"] \\
\mathscr{D}(V) \arrow[r, "p_V"] & \mathscr{I}(V)
\end{tikzcd}$$

Since $p_U \circ i_U = \mathrm{id}$, we have that p_U is onto, hence $r_{V,U}$ is onto.

We now want to prove the following: let $0 \to \mathscr{E} \xrightarrow{u} \mathscr{F} \xrightarrow{v} \mathscr{G} \to 0$ be an exact sequence, where $\mathscr{E}, \mathscr{F}$ are flabby. Then $\mathscr{G}$ is flabby.

Let us first consider an exact sequence $0 \to \mathscr{E} \xrightarrow{u} \mathscr{F} \xrightarrow{v} \mathscr{G} \to 0$ with $\mathscr{E}$ flabby. We want to prove that the map $v(X) : \mathscr{F}(X) \to \mathscr{G}(X)$ is onto. Indeed, let $s \in \Gamma(X, \mathscr{G})$, and a maximal set for inclusion, U, such that there exists a section $t \in \Gamma(U, \mathscr{F})$ such that $v(t) = s$ on U. We claim $U = X$ otherwise there exists $x \in X \setminus U$, a section t_x defined in a neighborhood V of x such that $v(t_x) = s$ on V. Then $t - t_x$ is defined in $\Gamma(U \cap V, \mathscr{F})$, but since $v(t - t_x) = 0$, we have, by left-exactness of $\Gamma(U \cap V, -)$, $t - t_x = u(z)$ for $z \in \Gamma(U \cap V, \mathscr{E})$. Since $\mathscr{E}$ is flabby, we may extend z to X, and then $t = t_x + u(z)$ on $U \cap V$. We may the find a section $\tilde{t} \in \Gamma(U \cup V, \mathscr{F})$ such that $\tilde{t} = t$ on U and $\tilde{t} = t_x + u(z)$ on V. Clearly $v(\tilde{t})_U = s_{|U}$ and $v(\tilde{t})_V = v(t_x) + vu(z) = v(t_x) = s_{|V}$, hence $v(\tilde{t}) = s$ on $U \cup V$. This contradicts the maximality of U.

As a result, we have the following diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{E}(X) \arrow[r, "u(X)"] \arrow[d, "\rho_{X,U}"] & \mathscr{F}(X) \arrow[r, "v(X)"] \arrow[d, "\sigma_{X,U}"] & \mathscr{G}(X) \arrow[r] \arrow[d, "\tau_{X,U}"] & 0 \\
0 \arrow[r] & \mathscr{E}(U) \arrow[r, "u(U)"] & \mathscr{F}(U) \arrow[r, "v(U)"] & \mathscr{G}(U) \arrow[r] & 0
\end{tikzcd}$$

and $\rho_{U,X}, \sigma_{U,X}$ are onto. This immediately implies that $\tau_{X,U}$ is onto. Finally, let us prove that a flabby sheaf $\mathscr{F}$ is acyclic. We consider the exact map $0 \to \mathscr{F} \to \mathscr{I}$ where $\mathscr{I}$ is injective. Using the existence of the cokernel, this yields an exact sequence $0 \to \mathscr{F} \to \mathscr{I} \to \mathscr{K} \to 0$. By the above remark, $\mathscr{K}$ is flabby. Consider then the long exact sequence associated to the short exact sequence of sheaves:

$$0 \to H^0(X, \mathscr{F}) \to H^0(X, \mathscr{I}) \to H^0(X, \mathscr{K}) \to H^1(X, \mathscr{F}) \to H^1(X, \mathscr{I}) \to H^1(X, \mathscr{K}) \to \dots$$

We prove by induction on n that for any $n \geq 1$ and any flabby sheaf, $H^n(X, \mathscr{F}) = 0$. Indeed, we just proved that $H^0(X, \mathscr{I}) \to H^0(X, \mathscr{K})$ is onto, and we know that $H^1(X, \mathscr{I}) = 0$. this implies $H^1(X, \mathscr{F}) = 0$. Assume now, that for any flabby sheaf and $j \leq n$, H^j vanishes. Then the long exact sequence yields

$$.. \to H^n(X, \mathscr{K}) \to H^{n+1}(X, \mathscr{F}) \to H^{n+1}(X, \mathscr{I}) \to \dots$$

Since $\mathscr{I}$ is injective, $H^{n+1}(X, \mathscr{I}) = 0$ and since $\mathscr{K}$ is flabby $H^n(X, \mathscr{K}) = 0$ hence $H^{n+1}(X, \mathscr{F})$ vanishes. $\square$

Example: Flabby sheafs are much more natural than injective ones, and we shall see they are just as useful. The sheaf of distributions, that is $\mathscr{D}_X(U)$ is the dual of $C_0^\infty(U)$, the sheaf of differential forms with distribution coefficients, the set of singular cochain defined on X (see Exemple **??**)... are all flabby.

A related notion is the notion of soft sheaves. A soft sheaf is a sheaf such that the map $\mathscr{F}(X) \to \mathscr{F}(K)$ is surjective for any closed set K. Of course, we define $\mathscr{F}(K) = \lim_{K\subset U}\mathscr{F}(U)$. In other words, an element defined in a neighborhood of K has an extension (maybe after reducing the neighborhood) to all of X. The sheafs of smooth functions, smooth forms, continuous functions... are all soft.

We refer to subsection 3.1 for applications of these notions.

EXERCICE 2. (1) Prove that for a locally contractible space, the sheaf of singular cochains is flabby. Prove that the singular cohomology of a locally contractible space X is isomorphic to the sheaf cohomology $H^*(X, k_X)$.
(2) Prove that soft sheaves are acyclic.

2. Operations on sheaves. Sheaves in mathematics.

First of all, if $\mathscr{F}$ is sheaf over X, and U an open subset of X, we denote by $\mathscr{F}_{|U}$ the sheaf on U defined by $\mathscr{F}_{|U}(V) = \mathscr{F}(V)$ for all $V \subset U$. For clarity, we define $\Gamma(U, \bullet)$ as the functor $\mathscr{F} \to \Gamma(U, \mathscr{F}) = \mathscr{F}(U)$.

DEFINITION 7.10. Let $\mathscr{F}, \mathscr{G}$ be sheafs over X. We define $\mathscr{H}om(\mathscr{F}, \mathscr{G})$ as the sheaf associated to the presheaf $\mathrm{Mor}(\mathscr{F}_{|U}, \mathscr{G}_{|U})$. We define $\mathscr{F} \otimes \mathscr{G}$ to be the sheafification of the presheaf $U \mapsto \mathscr{F}(U) \otimes \mathscr{G}(U)$. The same constructions hold for sheafs of modules over a sheaf of rings $\mathscr{R}$, and we then write $\mathscr{H}om_{\mathscr{R}}(\mathscr{F}, \mathscr{G})$ and $\mathscr{F} \otimes_{\mathscr{R}} \mathscr{G}$.

REMARK 7.11. (1) Note that $\mathrm{Mor}(\mathscr{F}_{|U}, \mathscr{G}_{|U}) \neq Hom(\mathscr{F}(U), \mathscr{G}(U))$ in general, since an element f in the left hand side defines compatible $f_V \in \mathscr{H}om(\mathscr{F}(V), \mathscr{G}(V))$ for all open sets V in U, while the right-hand side does not. There is however a connection between the two definitions: $\mathrm{Mor}(\mathscr{F}, \mathscr{G}) = \Gamma(X, \mathscr{H}om(\mathscr{F}, \mathscr{G})$.
(2) Note that tensor products commute with direct limits, so $(\mathscr{F} \otimes \mathscr{G})_x = \mathscr{F}_x \otimes \mathscr{G}_x$. On the other hand Mor does not commute with direct limits, so $\mathscr{H}om(\mathscr{F}, \mathscr{G})_x$ is generally different from $\mathscr{H}om(\mathscr{F}_x, \mathscr{G}_x)$.

Let $f : X \to Y$ be a continuous map. We define a number of functors associated to f as follows.

DEFINITION 7.12. Let $f : X \to Y$ be a continuous map, $\mathscr{F} \in \mathbf{Sheaf(X)}, \mathscr{G} \in \mathbf{Sheaf(Y)}$ The sheaf $f_*\mathscr{F}$ is defined by

$$f_*(\mathscr{F})(U) = \mathscr{F}(f^{-1}(U))$$

The sheaf $f^{-1}(\mathscr{G})(U)$ is the sheaf associated to the presheaf $Pf^{-1}(\mathscr{F}) : U \mapsto \lim_{V\supset f(U)} \mathscr{G}(V)$. We also define $\mathscr{F} \boxtimes \mathscr{G}$ as follows. If p_X, p_Y are the projections of $X \times Y$ on the respective factors, we have $\mathscr{F} \boxtimes \mathscr{G} = p_X^{-1}\mathscr{F} \otimes p_Y^{-1}(\mathscr{G})$. When $X = Y$ and d is the diagonal map, we define $d^{-1}(\mathscr{F} \boxtimes G) = \mathscr{F} \otimes \mathscr{G}$. This is the sheaf associated to the presheaf $U \to \mathscr{F}(U) \otimes \mathscr{G}(U)$.

It is also useful to have the definition of

PROPOSITION 7.13. *The functors f_*, f^{-1} are respectively left-exact and exact. Moreover , let f, g be continuous maps, then $(f \circ g)_* = f_* \circ g_*$ and $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.*

PROOF. For the first statement, let us prove that f^{-1} is exact. We use the fact that $f^{-1}(\mathscr{G})_x = \mathscr{G}_{f(x)}$. Thus an exact sequence $0 \to \mathscr{F} \xrightarrow{u} \mathscr{G} \xrightarrow{v} \mathscr{H} \to 0$ is transformed into the sequence $0 \to f^{-1}(\mathscr{F}) \xrightarrow{u\circ f} f^{-1}(\mathscr{G}) \xrightarrow{v\circ f} f^{-1}(\mathscr{H}) \to 0$ which has germs

$$0 \to (f^{-1}(\mathscr{F}))_x \xrightarrow{u(f(x))} (f^{-1}(\mathscr{G}))_x \xrightarrow{v(f(x))} (f^{-1}(\mathscr{H}))_x \to 0$$

equal to

$$0 \to \mathscr{F}_{f(x))} \xrightarrow{u(f(x))} \mathscr{G}_{f(x)} \xrightarrow{v(f(x))} \mathscr{H}_{f(x)} \to 0$$

which is exact. Now we prove that f_* is left-exact. Indeed, consider an exact sequence $0 \to \mathscr{E} \xrightarrow{u} \mathscr{F} \xrightarrow{v} G$. By left-exactness of Γ_U, the sequence

$$0 \to \mathscr{E}(U) \xrightarrow{u(U)} \mathscr{F}(U) \xrightarrow{v(U)} G(U)$$

is exact, hence for any $V \subset Y$, the sequence

$$0 \to \mathscr{E}(f^{-1}(V)) \xrightarrow{v(f^{-1}(V))} \mathscr{F}(f^{-1}(V)) \xrightarrow{v(f^{-1}(V))} G(f^{-1}(V))$$

is exact, which by taking limits on $V \ni x$ implies the exactness of

$$0 \to (f_*\mathscr{E})_x \xrightarrow{(f_* u)_x} (f_*\mathscr{F})_x \xrightarrow{(f_* v)_x} (f_*\mathscr{G})_x.$$

□

PROPOSITION 7.14. *We have* $\mathrm{Mor}(\mathscr{G}, f_*\mathscr{F}) = \mathrm{Mor}(f^{-1}(\mathscr{G}), \mathscr{F})$. *We say that f_* is* **right-adjoint** *to f^{-1} or that f^{-1} is left adjoint to f_*.*

PROOF. We claim that an element in either space, is defined by the following data, called a f-homomorphism: consider for each x a morphism $k_x : \mathscr{G}_{f(x)} \to \mathscr{F}_x$ such that for any section s of $\mathscr{G}(U)$, $k_x \circ s(f(x))$ is a (continuous) section of $\mathscr{F}(U)$. Notice that there are in general many x such that $f(x) = y$ is given, and also that a f-homomorphism is the way one defines morphisms in the category **Sheaves** of sheaves over all manifold (so that we must be able to define a morphism between a sheaf over X and a sheaf over Y). Now, we claim that an element in $\mathrm{Mor}(f^{-1}(\mathscr{G}), \mathscr{F})$ defines k_x, since $(f^{-1}(\mathscr{G}))_x = \mathscr{G}_{f(x)}$, so a map sending elements of $f^{-1}(\mathscr{G})(U)$ to elements of $\mathscr{F}(U)$ localizes to a map k_x having the above property. Conversely, given a map k_x as above, let $s \in f^{-1}(\mathscr{G})(U)$. By definition, for each point $x \in U$ there exists a section $t_{f(x)}$ defined near $f(x)$ such that $s = t_{f(x)}$ near x. Now define $s'_x = k_x t_{f(x)}$. We have that $s'_x \in F_x$, and by varying x in U, this defines a section of $\mathscr{F}(U)$. So k_x defines a morphism from $f^{-1}(\mathscr{G})$ to $\mathscr{F}$.

Now an element in $\mathrm{Mor}(\mathscr{G}, f_*\mathscr{F})$ sends for each U, $\mathscr{G}(U)$ to $\mathscr{F}(f^{-1}(U))$, hence an element in $\mathscr{G}_{f(y)}$ to an element in some $\mathscr{F}(f^{-1}(V_{f(y)}))$, where $V_{f(y)}$ is a neighborhood of $f(y)$, which induces by restriction an element in $\mathscr{F}_y$, hence defines k_x. Vice-versa, let $s \in \mathscr{G}(V)$ then for $y \in V$ and $x \in f^{-1}(y)$, we define $s'_x = k_x s_y$. The section s'_x is defined on V_x a neighborhood of x, and by assumption $k_x s_{f(x)}$ is continuous, so s' is continuous.

We thus identified the set of f-homomorphism both with $\mathrm{Mor}(\mathscr{G}, f_*\mathscr{F})$ and with $\mathrm{Mor}(f^{-1}(\mathscr{G}), \mathscr{F})$, which are thus isomorphic.

EXERCICE 3. Prove that $f_*\mathscr{H}om(f^{-1}(\mathscr{G}), \mathscr{F}) = \mathscr{H}om(\mathscr{G}, f_*\mathscr{F})$.

$\square$

The notion of adjointness is important in view of the following.

PROPOSITION 7.15. *Any right-adjoint functor is left exact. Any left-adjoint functor is right-exact.*

PROOF. Let F be right-adjoint to G, that is $\mathrm{Mor}(A, F(B)) = \mathrm{Mor}(G(A), B)$. We wish to prove that F is left-exact. The exactness of the sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C$ is equivalent to

$$0 \to \mathrm{Mor}(X, A) \xrightarrow{f^*} \mathrm{Mor}(X, B) \xrightarrow{g^*} \mathrm{Mor}(X, A) \tag{7.1}$$

Indeed, exactness of the sequence is equivalent to the fact that $A \xrightarrow{f} B$ is the kernel of g, or else that for any X, and $u : X \to A$ such that $g \circ u = 0$, there exists a unique $v : X \to B$ such that the following diagram commutes

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, "g"] & C \\ & X \arrow[u, "u"'] \arrow[ul, dashed, "v"] & \end{tikzcd}$$

The existence of v implies exactness of 7.1 at $\mathrm{Mor}(X, B)$, while uniqueness yields exactness at $\mathrm{Mor}(X, A)$.

As a result, left-exactness of F is equivalent to the fact that for each X, and each exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$, the induced sequence

$$0 \to \mathrm{Mor}(X, F(A)) \xrightarrow{F(f)^*} \mathrm{Mor}(X, F(B)) \xrightarrow{F(g)^*} \mathrm{Mor}(X, F(A))$$

is exact. But this sequence coincides with

$$0 \to \mathrm{Mor}(G(X), A) \xrightarrow{f^*} \mathrm{Mor}(G(X), B) \xrightarrow{g^*} \mathrm{Mor}(G(X), C)$$

its exactness follows from the left-exactness of $M \to \mathrm{Mor}(X, M)$.

□

Note that in the literature, f^{-1} is sometimes denoted f^*. Note also that if f is the constant map, then $f_*\mathscr{F} = \Gamma(X,\mathscr{F})$, so that $Rf_* = R\Gamma(X,\bullet)$.

EXERCICE 4. Show that Sheafification is the right adjoint functor to the inclusion of sheaves onto presheaves. Conclude that Sheafification is a left-exact functor.

COROLLARY 7.16. *The functor f_* maps injective sheafs to injective sheafs. The same holds for Γ_X.*

PROOF. Indeed, we have to check that $\mathscr{F} \to \mathscr{H}om(\mathscr{F}, f_*(\mathscr{I}))$ is an exact functor. But this is the same as checking that $\mathscr{F} \to \mathscr{H}om(f^{-1}\mathscr{F},\mathscr{I})$ is exact. Now $F \to f^{-1}(\mathscr{F})$ is exact, and since $\mathscr{I}$ is injective, $\mathscr{G} \to \mathscr{H}om(\mathscr{G},\mathscr{I})$ is exact. Thus $\mathscr{F} \to \mathscr{H}om(\mathscr{F}, f_*(\mathscr{I}))$ is the composition of two exact functors, hence is exact. The second statement is a special case of the first by taking f to be the constant map.

□

There is at least another simple functor: $f_!$ given by

DEFINITION 7.17. $f_!(\mathscr{F})(U) = \{s \in \mathscr{F}(f^{-1}(U)) \mid f : \mathrm{supp}(s) \to U \text{ is a proper map}\}$.

If f is proper, then $f_!$ and f_* coincide. Even though $f_!$ has a right-adjoint $f^!$, we shall not construct this as it requires a slightly complicated construction, extending Poincaré duality, the so-called Poincaré-Verdier duality (see [**Iv**] chapter V).

Example:

(1) Let A be a closed subset of X, and k_A be the constant sheaf on A, and $i : A \to X$ be the inclusion of A in X. Then $i_! = i_*$ and $i_*(k_A) = k_A$ and $i^{-1}(k_A) = k_X$. Thus if $i : A \to X$ is the inclusion of the closed set A in X, and $\mathscr{F}$ a sheaf on X, then $\mathscr{F}_A = i_* i^{-1}(\mathscr{F})$. This does not hold for A open, as we shall see in a moment.
(2) Let U be an open set in X and j the inclusion. Then $\mathscr{F}_U = j_! j^{-1}(\mathscr{F})$. This formula in fact holds for U locally closed (i.e. the intersection of a closed set and an open set).
(3) We have, with the above notations,
$$j^{-1} \circ j_* = j^{-1} \circ j_! = i^! \circ i_* = i^{-1} \circ i_* = \mathrm{id}$$

Note that the above operations extend to complexes of sheaves:

DEFINITION 7.18. Let $A^\bullet, B^\bullet$ be two bounded complexes. Then we define $(A^\bullet \otimes B^\bullet)^m = \sum_j A^j \otimes B^{m-j}$ with boundary map $d_m(u_j \otimes v_{m-j}) = \partial_j u_j \otimes v_{m-j} + u_j \otimes \partial_{m-j} v_{m-j}$. and $\mathscr{H}om(A^\bullet, B^\bullet)^m = \sum_j Hom(A^j \otimes B^{m+j})$, with boundary map $d_m f = \sum_p \partial_{m+p} f^p + (-1)^{m+1} f^{p+1}\partial_p$.

Finally we define the functor $\Gamma_Z : \mathbf{Sheaf(X)} \to \mathbf{Sheaf(X)}$ defined by

DEFINITION 7.19. Let Z be a locally closed set. Let $\mathscr{F} \in \mathbf{Sheaf(X)}$. Then the sheaf $\Gamma_Z\mathscr{F}$ is defined by $\Gamma_Z\mathscr{F}(U) = \ker(\mathscr{F}(U) \to \mathscr{F}(U \setminus Z))$.

EXERCICE 5. Here Z is a closed subset of X. Check the following statements:

(1) Show that the support of Γ_Z is contained in Z.
(2) Show that Γ_Z is a left exact functor from **Sheaf(X)** to **Sheaf(X)**.
(3) Show that Γ_Z maps injectives to injectives.
(4) Show that $\mathscr{F}_Z = k_Z \otimes \mathscr{F}$ and $\Gamma_Z(\mathscr{F}) = \mathscr{H}om(k_Z, \mathscr{F})$.

PROPOSITION 7.20. *The functor Γ_Z is left-exact. It sends flabby sheafs to flabby sheafs (an in particular injective sheafs to acyclic sheafs).*

PROOF. One checks that Γ_Z is left-exact from the left-exactness of the functor $\mathscr{F} \to \mathscr{F}_{|X\setminus Z}$. Applying the Snake lemma (Lemma 6.17) to the following diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{F} \arrow[r, "f"] \arrow[d, "a"] & \mathscr{G} \arrow[r, "g"] \arrow[d, "b"] & \mathscr{H} \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & \mathscr{F}_{X\setminus Z} \arrow[r, "f'"] & \mathscr{G}_{X\setminus Z} \arrow[r, "g'"] & \mathscr{H}_{X\setminus Z} &
\end{tikzcd}$$

yields exactness of the sequence $0 \to \mathrm{Ker(a)} \to \mathrm{Ker(b)} \to \mathrm{Ker(c)}$ that is exactness of $0 \to \Gamma_Z(\mathscr{F}) \to \Gamma_Z(\mathscr{G}) \to \Gamma_Z(\mathscr{H})$.

We must now prove that if $\mathscr{F}$ is flabby, $\Gamma_Z(X, \mathscr{F}) \to \Gamma_Z(U, \mathscr{F})$ is onto. Let $s \in \Gamma_Z(U, \mathscr{F})$, that is an element in $\mathscr{F}(U)$ vanishing on $U \setminus Z$. We may thus first extend s by 0 on $X \setminus Z$ to the open set $(X \setminus Z) \cup U$. By flabbiness of $\mathscr{F}$ we then extend s to X. $\square$

2.1. Sheaves and D-modules. Note that the rings we shall consider in this subsection are non-commutative, a situation we had not explicitly considered above. A D-module is a module over the ring D_X of algebraic differentials operators over an algebraic manifold X. Let O_X be the ring of holomorphic functions, Θ_X the ring of linear operators on O_X (i.e. holomorphic vector fields), and D_X the noncommutative ring generated by O_X and Θ_X, that is the sheaf of holomorphic differential operators on X. A D-module is a module over the ring D_X. More generally, given a sheaf of rings $\mathscr{R}$, we can consider $\mathscr{R}$-modules, that is for each open U, $\mathscr{F}(U)$ is an $\mathscr{R}(U)$-module and the restriction morphism is compatible with the $\mathscr{R}$-module structure. What we did for R-modules also hold for $\mathscr{R}$-modules.

Let us show how D-modules appear naturally. Let P be a general differential operator, that is, locally, $Pu = (\sum_{j=1}^m P_{1,j} u_j, \ldots, \sum_{j=1}^m P_{q,j} u_j)$ or else $\sum_{j=1}^m P_{i,j} u_i = v_j$, and let us start with $u = 0$. The operator P yields a linear map $D_X^p \to D_X^q$ and we may consider the map

$$\begin{array}{rcl}
\Phi(u) : D_X^p & \longrightarrow & O_X \\
(Q_j)_{1 \le j \le p} & \longrightarrow & \sum_{j=1}^{p} Q_j u_j
\end{array}$$

so that if $(u_1,...,u_p)$ is a solution of our equation, then $\Phi(u)$ vanishes on $D_X\cdot P_1+...+D_XP_q$ where

$$P_j=\begin{pmatrix}P_{1,j}\\ \vdots\\ P_{q,j}\end{pmatrix}$$

Conversely, a map $\Phi: D_X^p \longrightarrow O_X$ vanishing on $D_X\cdot P_1+...+D_XP_q$ yields a solution of our equation, setting $u_j=\Phi(0,..,1,0...0)$.

Then, let $\mathscr{M}$ be the D-module $D_X/(D_X\cdot P)$, the set of solutions of the equation corresponds to $\mathrm{Mor}(\mathscr{M},O_X)$.

3. Injective and acyclic resolutions

One of the goals of this section, is to show why the injective complexes can be used to define the derived category. One of the main reasons, is that on those complexes, quasi-isomorphism coincides with chain homotopy equivalence. We also explain why acyclic resolutions are enough to compute the derived functors, and finally work out the examples of the deRham and Čech complexes, proving that they both compute the cohomology of X with coefficients in the constant sheaf.

We start with the following

PROPOSITION 7.21. *Let $f: C^\bullet\to I^\bullet$ be a quasi-isomorphism where the I^p are injective. Then there exists $g: I^\bullet\to C^\bullet$ such that $g\circ f$ is homotopic to* id.

PROOF. We first construct the mapping cone of a map. Let $f^\bullet: A^\bullet\to B^\bullet$ be morphism of chain complexes, and $C(f)^\bullet = A^\bullet[1]\oplus B^\bullet$ with boundary map

$$d=\begin{pmatrix}-\partial_A & 0\\ -f & \partial_B\end{pmatrix}$$

Then there is a short exact sequence of chain complexes

$$0\longrightarrow B^\bullet \xrightarrow{u=\binom{0}{1}} C(f)^\bullet \xrightarrow{v=\binom{1}{0}} A^\bullet[1]\longrightarrow 0$$

The above exact sequence (or distinguished triangle) yields a long exact sequence in homology:

$$\longrightarrow H^n(A^\bullet,\partial_A)\xrightarrow{H^n(f_*)} H^n(B^\bullet,\partial_B)\xrightarrow{H^n(u)} H^n(C(f)^\bullet,d)\xrightarrow{\delta_f^*} H^{n+1}(A^\bullet,\partial_A)\longrightarrow\cdots$$

where the connecting map can be identified with $H^\bullet(f)$ and $\delta_f^*=H^*(f)$ coincides with the connecting map defined in the long exact sequence of Proposition 6.14. Note that $H^n(A^\bullet[1],\partial_A)=H^{n+1}(A^\bullet,\partial_A)$. Now we see that if $H^n(f)$ is an isomorphism then $H^n(C(f)^\bullet,d)=0$ for all n, we have an acyclic complex $(C(f)^\bullet,d)$, and a map $C(f)^\bullet\to$

$A^\bullet[1]$. We claim that it is sufficient to prove that this map is homotopic to zero. Indeed, let s be such a homotopy. It induces a map $s^\bullet : C(f)^\bullet \to A^\bullet$ such that $-\partial_A s(a,b) + sd(a,b) = a$ or else

$$-\partial_A s(a,b) + s(-\partial_A(a), -f(a) + \partial_B(b)) = a$$

so setting $g(b) = s(0,b)$ and $t(a) = s(-a,0)$ we get (apply successively to $(0,-b)$ and $(a,0)$),

$$\partial_A g(b) - g(\partial_B b) = 0$$

so g is a chain map, and

$$\partial_A t(a) + gf(a) + t\partial(a) = a$$

so gf is homotopic to Id_A.

The proposition thus follows from the following lemma.

LEMMA 7.22. *Any morphism from an acyclic complex $C^\bullet$ to an injective complex $I^\bullet$ is homotopic to* 0.

Let f be the morphism. We will construct the map s such that $f = \partial s + sd$ by induction using the injectivity. Assume we have constructed the solid maps and we wish to construct the dotted one in the following (non commutative !) diagram, such that $f_{m-1} = \partial_{m-2}s_{m-1} + s_m d_{m-1}$.

$$\begin{tikzcd}
\cdots & {} \arrow[r] & C^{m-2} \arrow[r, "d_{m-2}"] \arrow[dl, "s_{m-2}"'] \arrow[d, "f_{m-2}"] & C^{m-1} \arrow[r, "d_{m-1}"] \arrow[dl, "s_{m-1}"] \arrow[d, "f_{m-1}"] & C^m \arrow[r, "d_m"] \arrow[dl, dashed] \arrow[d, "f_m"] & \cdots \\
\cdots & {} \arrow[r, "\partial_{m-3}"] & I^{m-2} \arrow[r, "\partial_{m-2}"] & I^{m-1} \arrow[r, "\partial_{m-1}"] & I^m \arrow[r, "\partial_m"] & \cdots
\end{tikzcd}$$

The horizontal maps are not injective, but we may replace them by the following commutative diagram

$$\begin{tikzcd}
& {} \arrow[d, "d_{m-2}"] & \\
& C^{m-1} \arrow[d] \arrow[dr, "d_{m-1}"] \arrow[dd, bend right=60, "f_{m-1}-\partial_{m-2}s_{m-1}"'] & \\
0 \arrow[r] & \mathrm{Im}(d_{m-1}) = \mathrm{Ker}(\mathrm{d_m}) \arrow[r, "d'_m"] \arrow[d, dashed, "w"'] & C^m \arrow[dl, dashed, "s_m"] \\
& I^{m-1} &
\end{tikzcd}$$

where we first prove the existence of w and then the existence of s_m. The existence of w follows from the fact that $\mathrm{Ker(d_{m-1})} \to \mathrm{ker(f_{m-1} - \partial_{m-2}s_{m-1})}$ since $\mathrm{Ker(d_{m-1})} = \mathrm{Im(d_{m-2})}$ and we just have to check that $(f_{m-1} - \partial_{m-2}s_{m-1}) \circ d_{m-2} = 0$ which is obvious from the diagram and the induction assumption, since

$$f_{m-1} \circ d_{m-2} = \partial_{m-2} \circ f_{m-2} =$$
$$\partial_{m-2} \circ (\partial_{m-3}s_{m-2} + s_{m-1}d_{m-2}) = \partial_{m-2}s_{m-1}d_{m-2}$$

The injectivity of d'_m follows from the exactness of the sequence, and the existence of s_m follows from the injectivity of I^{m-1}. □

Notice that proposition 7.21 implies

COROLLARY 7.23. *Let $I^\bullet$ be an acyclic chain complex of injective elements, and F is any left-exact functor, then $F(I^\bullet)$ is also acyclic.*

PROOF OF THE COROLLARY. Indeed, since the 0 map from $I^\bullet$ to itself is a quasi-isomorphism, we get a homotopy between $\mathrm{id}_{I^\bullet}$ and 0. In other words $\mathrm{id}_{I^\bullet} = ds + sd$. As a result $F(\mathrm{id}_{I^\bullet}) = F(d)F(s) + F(s)F(d) = dF(s) + F(s)d$ and this implies that $F(\mathrm{id}_{I^\bullet}) : F(I^\bullet) \to F(I^\bullet)$ is homotopic to zero, which is equivalent to the acyclicity of $F(I^\bullet)$. □

Note that this implies that to compute the right-derived functor, we may replace the injective resolution by any F-acyclic resolution, that is resolution by objects L_m such that $H^j(L_m) = 0$ for all $j \neq 0$:

COROLLARY 7.24. *Let $0 \to A \to L_0 \to L_1 \to ...$ be a resolution of A such that the L_j are F-acyclic, that is $R^mF(L_j) = 0$ for any $m \geq 1$. Then $RF(A)$ is quasi-isomorphic to the chain complex $0 \to F(L_0) \to F(L_1) \to .....$ In particular $R^mF(A)$ can be computed as the cohomology of this last chain complex.*

PROOF. Let $I^\bullet$ be an injective resolution of A. There is according to 7.4 a morphism $f : L^\bullet \to I^\bullet$ extending the identity map. Because the map f is a quasi-isomorphism (there is no homology except in degree zero, and then by assumption f_* induces the identity), according to the previous result there exists $g : I^\bullet \to L^\bullet$ such that $g \circ f$ is homotopic to the identity. But then $F(g) \circ F(f)$ is homotopic to the identity, and $F(f)$ is an isomorphism between the cohomology of $F(I^\bullet)$, that is $RF^*(A)$ and that of $F(L^\bullet)$. □

Note that the above corollary will be proved again using spectral sequences in Proposition 8.13 on page 93.

Note that if $\mathscr{I}$ is injective, $0 \to \mathscr{I} \to \mathscr{I} \to 0$ is an injective resolution, and then clearly $H^0(X, \mathscr{I}) = \Gamma(X, \mathscr{I})$ and $H^j(X, \mathscr{I}) = 0$ for $j \geq 1$. A sheaf such that $H^j(X, \mathscr{F}) = 0$ for $j \geq 1$ is said to be Γ_X**-acyclic** (or **acyclic** for short).

3.1. Complements: DeRham, singular and Čech cohomology. We shall prove here that DeRham or Čech cohomology compute the usual cohomology.

Let $\mathbb{R}_X$ be the constant sheaf on X. Let Ω^j be the sheaf of differential forms on X, that is $\Omega^j(U)$ is the set of differential forms defined on U. This is clearly a soft sheaf, and we claim that we have a resolution

$$0 \to \mathbb{R}_X \xrightarrow{i} \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \Omega^3 \xrightarrow{d} .... \xrightarrow{d} \Omega^n \to 0$$

where d is the exterior differential. The fact that it is a resolution is checked by the exactness of

$$0 \to \mathbb{R}_X \xrightarrow{i} \Omega^0_x \xrightarrow{d} \Omega^1_x \xrightarrow{d} \Omega^2_x \xrightarrow{d} \Omega^3_x \xrightarrow{d} .... \xrightarrow{d} \Omega^n_x \to 0$$

which in turn follows from the Poincaré lemma, since for U contractible, we already have the exactness of

$$0 \to \mathbb{R}_X \xrightarrow{i} \Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \Omega^2(U) \xrightarrow{d} \Omega^3(U) \xrightarrow{d} .... \xrightarrow{d} \Omega^n(U) \to 0$$

and x has a fundamental basis of contractible neighborhoods. Since soft sheafs are acyclic, we may compute $H^*(X, \mathbb{R}_X)$ by applying $\Gamma(X, \bullet)$ to the above resolution. That is the cohomology of

$$0 \to \Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d} \Omega^2(X) \xrightarrow{d} \Omega^3(X) \xrightarrow{d} .... \xrightarrow{d} \Omega^n(X) \to 0$$

or else the DeRham cohomology.

3.2. Singular cohomology. Let $f : X \to Y$ be a continuous map between topological spaces, and $\mathscr{C}^*_f$ be the complex of singular cochains over f, that is $\mathscr{C}^q_f(U)$ is the set of singular q-cochains over $f^{-1}(U)$. There is of course a boundary map $\delta : \mathscr{C}^q_f(U) \to \mathscr{C}^{q+1}_f(U)$. For $X = Y$ and $f = \mathrm{Id}$ this is just the sheaf of singular cochains on X. If moreover the space X is locally contractible, the sequence

$$0 \to k_X \to \mathscr{C}^0 \xrightarrow{\delta} \mathscr{C}^1 \xrightarrow{\delta} \mathscr{C}^2 \to ....$$

yields a resolution of the constant sheaf, the exactness of the sequence at the stalk level follows from its exactness on any contractible open set U. Thus, since the $\mathscr{C}^q$ are flabby, the cohomology $H^*(X, k_X)$ is computed as the cohomology of the complex

$$0 \to \mathscr{C}^0(X) \xrightarrow{\delta} \mathscr{C}^1(X) \xrightarrow{\delta} \mathscr{C}^2(X) \to$$

3.3. Čech cohomology. Let $\mathscr{F}$ be a sheaf of R-modules on X.

DEFINITION 7.25. Given a covering $\mathfrak{U}$ of X by open sets U_j, an element of $C^q(\mathfrak{U}, \mathscr{F})$ consists in defining for each $(q+1)$-uple $(U_{i_0}, ..., U_{i_q})$ an element $s(i_0, ..., i_q) \in \mathscr{F}(U_{i_0} \cap ... \cap U_{i_q})$ such that $s(i_{\sigma(0)}, i_{\sigma(1)}, ..., i_{\sigma(q)}) = \varepsilon(\sigma) s(i_0, ..., i_q)$.

If $s \in \check{C}^q(\mathfrak{U}, \mathscr{F})$ we define $(\delta s)(i_0, i_1, ..., i_{q+1}) = \sum_j (-1)^j s(i_0, i_1, ., \widehat{i_j} .., i_{q+1})$. This construction defines a sheaf on X as follows: to an open set V we associate the covering of V by the $U_j \cap V$, and there is a natural map induced by restriction of the sections of $\mathscr{F}$,

$\check{C}^q(\mathfrak{U},\mathscr{F}) \to \check{C}^q(\mathfrak{U}\cap V,\mathscr{F})$ obtained by replacing U_j by $U_j \cup V$. Thus the Čech complex associated to a covering is a sheaf over X. We may consider the sheaf of complexes

$$0 \to \mathscr{F} \xrightarrow{i} \check{C}^0(\mathfrak{U},\mathscr{F}) \xrightarrow{\delta} \dots \xrightarrow{\delta} \check{C}^q(\mathfrak{U},\mathscr{F}) \xrightarrow{\delta} \check{C}^{q+1}(\mathfrak{U},\mathscr{F}) \xrightarrow{\delta} \dots$$

However when the $H^j(U_{i_0}\cap\dots\cap U_{i_q},\mathscr{F})$ are zero for $j \geq 1$, we say we have an acyclic cover, and the cohomology of $\check{C}^q(\mathfrak{U},\mathscr{F})$ computes the cohomology of the sheaf $\mathscr{F}$. This will follow from a spectral sequence argument.

3.4. Exercices.

(1) Let $\mathscr{A}$ be a sheaf over $\mathbb{N}$, $\mathbb{N}$ being endowed with the topology for which the open sets are $\{1,2,\dots,n\}$, $\mathbb{N}$ and $\emptyset$. Prove that a sheaf over $\mathbb{N}$ is equivalent to a sequence of R-modules, A_n and maps

$$\dots \to A_n \to A_{n-1} \to \dots . \to A_0$$

and that $H^0(\mathbb{N},\mathscr{A}) = \lim_n A_n$. Describe $\lim^1(A_n)_{n\geq 1} \overset{def}{=} H^1(\mathbb{N},\mathscr{A})$

(2) Show that the above sheaf is flabby if and only if the maps $A_n \to A_{n-1}$ are onto, and that the sheaf is acyclic if and only if the sequence satisfies the Mittag-Leffler condition: the image of A_k in A_j is stationary as k goes to infinity.

4. Appendix: More on injective objects

Let us first show that the functor $A \to \mathrm{Mor}(A,L)$ is left exact, regardless of whether L is injective or not. Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an exact sequence. Since f is a monomorphism $\mathrm{Mor}(f) : \mathrm{Mor}(B,L) \to \mathrm{Mor}(A,L)$ is the map $u \to u\circ f$. By definition of monomorphisms, this is injective, and we only have to prove $\mathrm{Im}(\mathrm{Mor}(g)) = \mathrm{Ker}(\mathrm{Mor}(\mathrm{f}))$. Assume $u \in \mathrm{Ker}(\mathrm{Mor}(\mathrm{f}))$ so that $u\circ f = 0$. According to proposition 6.10, $(C,g) = \mathrm{Coker}(f)$, so by definition of the cokernel we get the factorization $u = v\circ g$.

LEMMA 7.26. *Let* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *be an exact sequence such that A is injective. Then there exists $w : B \to A$ such that $w\circ f = \mathrm{id}_A$. As a result there exists of $u : C \to B$ and $v : B \to A$ such that* $\mathrm{id}_B = f\circ v + u\circ g$, *and the sequence splits.*

PROOF. The existence of w follows from the definition of injectivity applied to $h = \mathrm{id}_A$. The map w is then given as the dotted map. Now since $f = f\circ w\circ f$ we get $(\mathrm{id}_A - f\circ w)\circ f = 0$, hence by definition of the Cokernel, and the fact that $C = \mathrm{Coker}(g)$, there is a map $u : C \to B$ such that $(\mathrm{id}_A - f\circ w) = u\circ g$. This proves the formula $\mathrm{id}_B = f\circ v + u\circ g$ with $v = w$. As a result, $g = g\circ \mathrm{Id}_B = g\circ f\circ v + g\circ u\circ g$, and $g\circ f = 0$, and since g is an epimorphism and $g = g\circ u\circ g$ we have $\mathrm{Id}_C = g\circ u$ and the sequence is split according to Definition 6.9 and Exercice 3. □

LEMMA 7.27. *Let* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *be an exact sequence with A, B injective. Then C is injective.*

PROOF. Indeed, the above lemma implies that the sequence splits, $B \simeq A \oplus C$, but the sum of two objects is injective if and only if they are both injectives: as injectivity is a lifting property, to lift a map to a direct sum, we must be able to lift to each factor. $\square$

As a consequence any additive functor F will send a short exact sequences of injectives to a short exact sequences of injectives, since the image by F will be split, and a split sequence is exact. The same holds for a general exact sequence since it decomposes as $0 \to I_0 \to I_1 \to \mathrm{Ker}(\mathrm{d}_2) = \mathrm{Im}(\mathrm{d}_1) \to 0$. Since I_0, I_1 are injectives, so is $\mathrm{Ker}(\mathrm{d}_2) = \mathrm{Im}(\mathrm{d}_1)$. Now we use the exact sequence $0 \to \mathrm{Im}(d_1) \to I_2 \to \mathrm{Ker}(\mathrm{d}_3) = \mathrm{Im}(\mathrm{d}_2) \to 0$ to show that $\mathrm{Ker}(\mathrm{d}_3) = \mathrm{Im}(\mathrm{d}_2)$ is injective. Finally all the $\mathrm{Ker}(\mathrm{d_j})$ and $\mathrm{Im}(d_j)$ are injective. But this implies that the sequences $0 \to \mathrm{Im}(d_{m-1}) \to I_m \to \mathrm{Ker}(\mathrm{d_{m+1}}) = \mathrm{Im}(\mathrm{d_m}) \to 0$ are split, hence $0 \to F(\mathrm{Im}(d_{m-1})) \to F(I_m) \to F(\mathrm{Ker}(\mathrm{d_{m+1}})) = \mathrm{F}(\mathrm{Im}(\mathrm{d_m})) \to 0$ is split hence exact. This implies (Check !) that the sequence $0 \to F(I_0) \to F(I_1) \to F(I_2) \to F(I_3) \to$ is exact.

LEMMA 7.28 (Horseshoe lemma). *Let $0 \to A \to B \to C \to 0$ be an exact sequence, and, $I_A^\bullet$, $I_C^\bullet$ be injective resolutions of A and C. Then there exists an injective resolution of B, $I_B^\bullet$, such that $0 \to I_A^\bullet \to I_B^\bullet \to I_C^\bullet \to 0$ is an exact sequence of complexes. Moreover, we can take $I_B^\bullet = I_A^\bullet \oplus I_C^\bullet$.*

PROOF. See [**Weib**] page 37. One can also use the Freyd-Mitchell theorem. $\square$

PROPOSITION 7.29. *Let $\mathscr{C}$ be an abelian category with enough injectives. Let $f : A \to B$ be a morphism. Assume for any injective object I, the induced map $f^* : \mathrm{Mor}(B, I) \to \mathrm{Mor}(A, I)$ is an isomorphism, then f is an isomorphism.*

PROOF. Assume f is not a monomorphism. Then there exists a non-zero $u : K \to A$ such that $f \circ u = 0$. We first assume u is a monomorphism. Let $\pi : K \to I$ be a monomorphism into an injective I. Then there exists $v : A \to I$ such that $v \circ u = \pi$. Let $h : B \to I$ be such that $v = h \circ f$. We have $h \circ f \circ u = v \circ u = \pi$ but also $f \circ u = 0$ hence $h \circ f \circ u = 0$ which implies $\pi = 0$ a contradiction. Now we still have to prove that u may be supposed to be injective. But the map u can be factored as $t \circ s$ where $s : K \to \mathrm{Im}(u)$ and $t : \mathrm{Im}(u) \to A$ and t is mono and s is epi. Thus since $f \circ u = 0$, we have $f \circ t \circ s = 0$, but since s is epimorphisms, we have $f \circ t = 0$ with t mono. Assume now f is not an epimorphism; Then there exists a nonzero map $v : B \to C$ such that $v \circ f = 0$. We now send C to an injective I by a monomorphism π. Then $(\pi \circ v) \circ f = 0$, and $\pi \circ v$ is nonzero, since π is a monomorphism. We thus get a non zero map $\pi \circ v \in \mathrm{Mor}(B, I)$ such that its image by f^* in $\mathrm{Mor}(A, I)$ is zero. $\square$

As an example we consider the case of sheaves. Let $\mathscr{F}, \mathscr{G}$ be sheaves over X, and $f : \mathscr{F} \to \mathscr{G}$ a morphism of sheaves. We consider an injective sheaf, $\mathscr{I}$, then $\mathrm{Mor}(\mathscr{F}, \mathscr{I}) = \bigcup_x \mathrm{Mor}(\mathscr{F}_x, \mathscr{I}(x))$, so that the map f^* on each component will give $f_x : \mathscr{F}_x \to \mathscr{G}_x$. If this map is an isomorphism, then f is an isomorphism.

One should be careful: the map f must be given, and the fact that $\mathscr{F}_x$ and $\mathscr{G}_x$ are isomorphic for all x does not imply the isomorphism of $\mathscr{F}$ and $\mathscr{G}$.

4.1. Appendix: Poincaré-Verdier Duality. Let $f : X \to Y$ be a continuous map between manifolds. We want to define the map $f^!$, and then of course $Rf^!$, adjoint of $f_!$ and $Rf_!$. This is the sheaf theoretic version of Poincaré duality.

EXERCICES 6. (1) Prove that the inverse limit functor $\lim_{\leftarrow}$ is left-exact, while the direct limit functor $\lim_{\rightarrow}$ is exact. Prove that

$$H^*(\varinjlim C_\alpha) = \varinjlim H^*(C_\alpha)$$

(2) Use the above to prove that for $\mathscr{F}^\bullet$ a complex of sheaves over X, we have $\mathscr{H}(\mathscr{F}_x^\bullet) = \lim_U H^*(U, \mathscr{F}^\bullet)$. In other words the presheaf $U \mapsto H^*(U, \mathscr{F}^\bullet)$ has stalk $\mathscr{H}(\mathscr{F}_x^\bullet)$, and of course the same holds for the associated sheaf. So the stalk of the sheaf associated to the presheaf $U \mapsto H^*(U, \mathscr{F}^\bullet)$ is the homology of the stalk complex $\mathscr{F}_x^\bullet$.

CHAPTER 8

Derived categories of Sheaves, and spectral sequences

One of the main reasons to introduce derived categories is to do without spectral sequences. It may then seem ironic to base our presentation of derived categories on spectral sequences, via Cartan-Eilenberg resolutions. We coudl then rephrase our point of view: the goal of spectral sequences is to actually do computations. The derived category allows us to make this computation simpler hence more efficient by applying the spectral sequence only once at the end of our categorical reasoning. This is a common method in mathematics: we keep all information in an algebraic object, and only make explicit computations after performing all the algebraic operations.

1. The categories of chain complexes

As we mentioned in the prevous lecture, one can consider the different categories of chain complexes, $\mathbf{Ch^b}(\mathscr{C})$, $\mathbf{Ch}^+(\mathscr{C})$, $\mathbf{Ch}^-(\mathscr{C})$ respectively of chain complexes bounded, bounded from below, and bounded from above. We denote by $A^\bullet$ an object in $\mathbf{Ch}^+(\mathscr{C})$, we write it as

$$\dots \xrightarrow{d_{m-1}} A^m \xrightarrow{d_m} A^{m+1} \xrightarrow{d_{m+1}} A^{m+2} \xrightarrow{d_{m+2}} \dots$$

The functor $\mathscr{H}(A^\bullet)$ denotes the cohomology of this chain complex, that is $\mathscr{H}^m(A^\bullet) = \mathrm{Ker(d_m)/Im(d_{m-1})}$. We can see this is a complex with zero differential, so that $\mathscr{H}$ is a functor from $\mathbf{Ch}(\mathscr{C})$ to itself. When $\mathscr{F}^\bullet$ is a complex of sheaves, one should be careful not to confuse this with $H^*(X,\mathscr{F}^m)$ obtained by looking at the sheaf cohomology of each term, nor is it equal to something we have not defined yet, $H^*(X,\mathscr{F}^\bullet)$ that is computed from a spectral sequence involving both $\mathscr{H}$ and H^* as we shall se later.

Because we are interested in cohomologies, we will identify two chain homotopic chain complexes, but replacing chain complexes by their cohomology loses too much information. There are two notions which are relevant. The first is chain homotopy. The second is quasi-isomorphism.

DEFINITION 8.1. A chain map $f^\bullet$ is a **quasi-isomorphism**, if the induced map $\mathscr{H}(f^\bullet):\mathscr{H}(A^\bullet)\to\mathscr{H}(B^\bullet)$ is an isomorphism.

It is easy to construct two chain complexes with the same cohomology, but not chain homotopic.

The following definition shall not be used in these notes, but we give it for the sake of completeness

DEFINITION 8.2. Two chain complexes $A^\bullet, B^\bullet$ are quasi-isomorphic if and only if there exists $C^\bullet$ and chain maps $f^\bullet : C^\bullet \to A^\bullet$ and $g^\bullet : C^\bullet \to B^\bullet$ such that $f^\bullet, g^\bullet$ are quasi-isomorphisms (i.e. induce an isomorphism in cohomology).

We shall restrict ourselves to derived categories of bounded complexes. The derived category is philosophically the category of chain complexes quotiented by the relation of quasi-isomorphisms. This is usually acheived in two steps. We first quotient out by chain-homotopies, because it is easy to prove that homotopy between maps is a transitive relation, and only afterwords by quasi-isomorphism, for which transitivity is more complicated.

Note that if

$$0 \to A \to B^1 \to B^2 \to B^3 \to ...$$

is a resolution of A, then $0 \to A \to 0$ is quasi-isomorphic to $0 \to B^1 \to B^2 \to B^3 \to ....$ Indeed the map $i : A \to B^1$ induces obviously a chain map and a quasi-isomorphism

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "i_0"] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & B^1 \arrow[r, "\partial_1"] & B^2 \arrow[r, "\partial_2"] & B^3 \arrow[r, "\partial_3"] & \cdots
\end{tikzcd}$$

The idea of the derived category, is that it is a universal category such that any functor sending quasi-isomorphisms to isomorphisms, factors through the derived category. Because we do not use this property, we shall give here a particular construction, in a case sufficiently general for our purposes: the case when the category $\mathscr{C}$ is a **category having enough injectives**. We refer to the bibliography for the general construction.

DEFINITION 8.3. Let $\mathscr{C}$ be an abelian category. The homotopy category, $\mathbf{K^b}(\mathscr{C})$ is the category having the same objects as $\mathbf{Chain^b}(\mathscr{C})$ and morphisms are equivalence classes of chain maps for the cahin homotopy equivalence relation : $\mathrm{Mor}_{\mathbf{K^b}(\mathscr{C})}(A,B) = \mathrm{Mor}_{\mathscr{C}}(A,B)/\simeq$ where $f \simeq g$ means that f is chain homotopic to g.

Note that $\mathbf{K^b}(\mathscr{C})$ is not an abelian category: by moding out by the chain homotopies, we lost the notion of kernels and cokernels. As a result there is no good notion of exact sequence. However $\mathbf{K^b}(\mathscr{C})$ is a **triangulated category**. We shall not go into the details of this notion here, but to remark that this is related to the property that short exact sequences of complexes only yield long exact sequences in homology. Before taking homology, a long exact sequence is a sequence of complexes

$$.. \to A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1] \to ..$$

usually only homotopy exact. Let **Inj(**$\mathscr{C}$**)** be the category of injective objects. This is a full subcategory of $\mathscr{C}$. Let $\mathbf{K^b}(\mathbf{Inj}(\mathscr{C}))$ be the same category constructed on injective objects. To each chain complex, we can associate a chain complex of injective objects as follows:

Let

$$\cdots \xrightarrow{d_{m-1}} A^m \xrightarrow{d_m} A^{m+1} \xrightarrow{d_{m+1}} A^{m+2} \xrightarrow{d_{m+2}} \cdots$$

be the chain complex, and for each A^m an injective resolution

$$0 \longrightarrow A^m \xrightarrow{i_m} I_0^m \xrightarrow{d_0^m} I_1^m \xrightarrow{d_1^m} I_2^m \xrightarrow{d_2^m} \cdots$$

By slightly refining this construction, we get the notion of Cartan-Eilenberg resolution:

DEFINITION 8.4. A Cartan-Eilenberg resolution of $A^\bullet$ is a commutative diagram, where the lines are injective resolutions:

$$\begin{tikzcd}
 & {} \arrow[d, "\partial^{m-2}"] & {} \arrow[d, "\partial_0^{m-2}"] & {} \arrow[d, "\partial_1^{m-2}"] & {} \arrow[d, "\partial_2^{m-2}"] & \\
0 \arrow[r] & A^{m-1} \arrow[r, "i_{m-1}"] \arrow[d, "\partial^{m-1}"] & I_0^{m-1} \arrow[r, "\delta_0^{m-1}"] \arrow[d, "\partial_0^{m-1}"] & I_1^{m-1} \arrow[r, "\delta_1^{m-1}"] \arrow[d, "\partial_1^{m-1}"] & I_2^{m-1} \arrow[r, "\delta_2^{m-1}"] \arrow[d, "\partial_2^{m-1}"] & \cdots \\
0 \arrow[r] & A^{m} \arrow[r, "i_{m}"] \arrow[d, "\partial^{m}"] & I_0^{m} \arrow[r, "\delta_0^{m}"] \arrow[d, "\partial_0^{m}"] & I_1^{m} \arrow[r, "\delta_1^{m}"] \arrow[d, "\partial_1^{m}"] & I_2^{m} \arrow[r, "\delta_2^{m}"] \arrow[d, "\partial_2^{m}"] & \cdots \\
0 \arrow[r] & A^{m+1} \arrow[r, "i_{m+1}"] \arrow[d, "\partial^{m+1}"] & I_0^{m+1} \arrow[r, "\delta_0^{m+1}"] \arrow[d, "\partial_0^{m+1}"] & I_1^{m+1} \arrow[r, "\delta_1^{m+1}"] \arrow[d, "\partial_1^{m+1}"] & I_2^{m+1} \arrow[r, "\delta_2^{m+1}"] \arrow[d, "\partial_2^{m+1}"] & \cdots \\
 & {} & {} & {} & {} &
\end{tikzcd}$$

Moreover

(1) If $A^m = 0$, then for all j, the I_j^m are zero.
(2) The lines yield injective resolutions of $\mathrm{Ker}(\partial^m)$, $\mathrm{Im}(\partial^m)$ and $\mathscr{H}^m(A^*)$. In other words, the $\mathrm{Im}(\partial_j^m)$ are an injective resolution of $\mathrm{Im}(\partial^m)$, the $\mathrm{Ker}(\partial_j^m)/\mathrm{Im}(\partial_j^{m-1})$ are an injective resolution of $\mathrm{Ker}(\partial^m)/\mathrm{Im}(\partial^{m-1}) = \mathscr{H}^m(A^\bullet, \partial)$. This implies that the $\mathrm{Ker}(\partial_j^m)$ are an injective resolution of $\mathrm{Ker}(\partial^m)$.

REMARK 8.5. We decided to work in categories of finite complexes. This raises a question: are Cartan-Eilenberg resolutions of such complexes themselves finite. Clearly this is equivalent to asking whether an object has a finite resolution. The answer is positive over manifolds: they have cohomological dimension n, so we can always find resolutions of length at most n (see [**Br**] chap 2, thm 16.4 and 16.28). If we want to work with bounded from below complexes, we do not need this result, but then we shall need to be slightly more careful about convergence results for spectral sequences, even though there is no real difficulty. The case of complexes unbounded from above and below is more complicated- because of spectral sequence convergence issues- and we shall not deal with it.

Now we claim

PROPOSITION 8.6. *(1) Every chain complex has a Cartan-Eilenberg resolution.*

(2) Let $A^\bullet, B^\bullet$ be two complexes, $I^{\bullet,\bullet}$ and $J^{\bullet,\bullet}$ be Cartan-Eilenberg resolutions of $A^\bullet, B^\bullet$, and $f : A^\bullet \to B^\bullet$ be a chain map. Then f lifts to a chain map $\tilde{f} : I^{\bullet,\bullet} \to J^{\bullet,\bullet}$. Moreover two such lifts are chain homotopic.

PROOF. (see [**Weib**]) Set $B^m(A^\bullet) = \mathrm{Im}(\partial^m)$, $Z^m(A^\bullet) = \mathrm{Ker}(\partial^{\mathrm{m}})$ and $H^m(A^\bullet)$, and consider the exact sequence $0 \to B^m(A^\bullet) \to Z^m(A^\bullet) \to H^m(A^\bullet) \to 0$. Starting from injective resolutions $I^\bullet_{B^m}$ of $B^m(A)$ and $I^\bullet_{H^m}$ of $H^m(A^\bullet)$, the Horseshoe lemma (lemma 7.28 on page 84) yields an exact sequence of injective resolutions $0 \to I^\bullet_{B^m} \to I^\bullet_{Z^m} \to I^\bullet_{H^m} \to 0$. Applying the Horseshoe lemma again to $0 \to Z^m(A^\bullet) \to A^m \to B^{m+1}(A^\bullet) \to 0$ we get an injective resolution $I^\bullet_{A^m}$ of A^m and exact sequence $0 \to I^\bullet_{Z^m} \to I^\bullet_{A^m} \to I^\bullet_{B^{m+1}} \to 0$. Then $I^\bullet_{A^m} \xrightarrow{\partial^\bullet_m} I^\bullet_{A^{m+1}}$ is the composition of $I^\bullet_{A^m} \to I^\bullet_{B^{m+1}} \to I^\bullet_{Z^{m+1}} \to I^\bullet_{A^{m+1}}$. This proves (1). Property (2) is left to the reader. ☐

Note: a chain homotopy between $f, g : I^{\bullet,\bullet} \to J^{\bullet,\bullet}$ is a pair of maps $s^h_{p,q} : I^{p,q} \to J^{p+1,q}$ and $s^v_{p,q} : I^{p,q} \to J^{p,q+1}$ such that $g - f = (\delta s^h + s^h \delta) + (\partial s^v + s^v \partial)$. This is equivalent to requiring that $s^h + s^v$ is a chain homotopy between $Tot(I^{\bullet,\bullet})$ and $Tot(J^{\bullet,\bullet})$.

PROPOSITION 8.7. *Let I^m_j be the double complex as above, and $Tot(I^{\bullet,\bullet})$ be the chain complex given by $T^q = \oplus_{j+m=q} I^m_j$ and $d = \partial + (-1)^m \delta$, in other words $d_{|I^m_j} = d^m_j + (-1)^m \delta^m_j$. Then $A^\bullet$ is quasi-isomorphic to $T^\bullet$.*

LEMMA 8.8 (Tic-Tac-Toe). *Consider the following bi-complex*

$$
\begin{tikzcd}
 & {} \arrow[d, "\partial^{m-2}"] & {} \arrow[d, "\partial_0^{m-2}"] & {} \arrow[d, "\partial_1^{m-2}"] & {} \arrow[d, "\partial_2^{m-2}"] & \\
0 \arrow[r] & A^{m-1} \arrow[r, "i_{m-1}"] \arrow[d, "\partial^{m-1}"] & I_0^{m-1} \arrow[r, "\delta_0^{m-1}"] \arrow[d, "\partial_0^{m-1}"] & I_1^{m-1} \arrow[r, "\delta_1^{m-1}"] \arrow[d, "\partial_1^{m-1}"] & I_2^{m-1} \arrow[r, "\delta_2^{m-1}"] \arrow[d, "\partial_2^{m-1}"] & \cdots \\
0 \arrow[r] & A^{m} \arrow[r, "i_{m}"] \arrow[d, "\partial^{m}"] & I_0^{m} \arrow[r, "\delta_0^{m}"] \arrow[d, "\partial_0^{m}"] & I_1^{m} \arrow[r, "\delta_1^{m}"] \arrow[d, "\partial_1^{m}"] & I_2^{m} \arrow[r, "\delta_2^{m}"] \arrow[d, "\partial_2^{m}"] & \cdots \\
0 \arrow[r] & A^{m+1} \arrow[r, "i_{m+1}"] \arrow[d, "\partial^{m+1}"] & I_0^{m+1} \arrow[r, "\delta_0^{m+1}"] \arrow[d, "\partial_0^{m+1}"] & I_1^{m+1} \arrow[r, "\delta_1^{m+1}"] \arrow[d, "\partial_1^{m+1}"] & I_2^{m+1} \arrow[r, "\delta_2^{m+1}"] \arrow[d, "\partial_2^{m+1}"] & \cdots \\
 & {} & {} & {} & {} &
\end{tikzcd}
$$

Assume the lines are exact (i.e. i_m is injective and $\mathrm{Im}(i_m) = \ker(\delta_0^m)$ and $\mathrm{Im}(\delta_j^m) = \mathrm{Ker}(\delta^{\mathrm{m}}_{\mathrm{j+1}})$). Then the maps i_m induce a quasi-isomorphism between the total complex $T^q = \oplus_{j+m=q} I^m_j$ endowed with $d = \partial + (-1)^m \delta$ and the chain complex $A^\bullet$.

PROOF. The proof is the same as the proof of the spectral sequence computing the cohomology of a bicomplex, except that here we get an exact result. Let us write for convenience $\overline{\delta} = (-1)^m\delta$. Then notice that the maps i_m yield a chain map between $A^\bullet$ and $T^\bullet$. Indeed, if $u_m \in A^m$, $(\partial+\overline{\delta})(i_m(u_m)) = \partial_0^m i_m(u_m)$ since $\overline{\delta}_0^m \circ i_m = 0$. But $\partial_0^m i_m(u_m) = i_{m+1}\partial_m(u_m) = 0$ since u_m is ∂_m-closed. Similarly if u_m is exact, $i_m(u_m)$ is exact, so that i_m induces a map in cohomology. We must now prove that this induces an isomorphism in cohomology. Injectivity is easy: suppose $i_m(u_m) = (\partial+\overline{\delta})(y)$. Because there is no element left of I_0^m, we must have $y = y_0^{m-1}$ hence $i_m(u_m) = \partial_0^{m-1}(y_0^{m-1})$ and $\overline{\delta}_0^{m-1}(y_0^{m-1}) = 0$. This implies by exactness of the lines that $y_0^{m-1} = i_{m-1}(u_{m-1})$, and

$$i_m(u_m) = \partial_0^{m-1}(y_0^{m-1}) = \partial_0^{m-1}(i_{m-1}(u_{m-1}) = i_m\partial_{m-1}(u_{m-1})$$

injectivity of i_m implies that $u_m = \partial_m u_{m-1}$, so u_m was zero in the cohomology of $A^\bullet$. We finally prove surjectivity of the map induced by i_m in cohomology.

Indeed, let $x = \sum_{j+m=q} x_j^m$ such that $(\partial+\overline{\delta})(x) = 0$. Looking at the component of $(\partial+\overline{\delta})(x)$ in I_j^m we see that this is equivalent to $\partial x_{j-1}^{m-1} + \overline{\delta} x_j^{m-1} = 0$. Since the complexes are bounded, there is a smallest $j = j_0$ such that $x_j^m \neq 0$. Then we have $\overline{\delta} x_{j_0}^{m_0-1} = 0$ (since $x_{j_0-1}^{m_0} = 0$), and by exactness of $\overline{\delta}$, we have $x_{j_0}^{m_0-1} = \overline{\delta} y_{j_0}^{m_0}$. Then $x - (\partial+\overline{\delta})(y_{j_0}^{m_0})$ has for all components in I_j^m vanishing for $j \geq j_0 - 1$. By induction, we see that we can replace x by a $(\partial+\overline{\delta})$ cohomologous element with a single component w_0^m in I_0^m and since $(\partial+\overline{\delta})(w_0^m) = 0$, we have $w_0^m = i_m(u_m)$ and we easily check $\partial(u_m) = 0$. $\square$

If we are talking about an element in $\mathscr{C}$ identified with the chain complex $0 \to A \to 0$ the total complex above is quasi-isomorphic to an injective resolution of A. Then if F is a left-exact functor, we denote $RF(A)$ to be the element

$$0 \to F(I^0) \to F(I_1) \to ....$$

in $\mathbf{K^b}(\mathbf{Inj}(\mathscr{C}))$. And $R^jF(A)$ is the j-th homology of the above. sequence[1]. But if we want to work in the category of chain complexes, we must give a a meaning to $RF(A^\bullet)$ for a complex $A^\bullet$.

REMARK 8.9. The idea of the total complex of a double complex has an important consequence: we will never have to consider triple, quadruple or more complicated complexes, since these can all eventually be reduced to usual complexes.

DEFINITION 8.10. Assume $\mathscr{C}$ is a category with enough injectives. The derived category of $\mathscr{C}$, denoted $D^b(\mathscr{C})$ is defined as $\mathbf{K^b}(\mathbf{Inj}(\mathscr{C}))$. The functor $D : \mathbf{Chain^b}(\mathscr{C}) \to D^b(\mathscr{C})$ is the map associating to $\mathscr{F}^\bullet$ the total complex of a Cartan-Eilenberg resolution of $\mathscr{F}^\bullet$.

[1]The notation does not convey the idea that information is lost from $RF(A)$ to $R^jF(A)$, as always when taking homology.

REMARKS 8.11. (1) The category $D^b(\mathscr{C})$ has the following fundamental property. Let F be a functor from $\mathbf{Chain^b}(\mathscr{C})$ to a category $\mathscr{D}$, which sends quasi-isomorphisms to isomorphisms, then F can be factored through $D^b(\mathscr{C})$: there is a functor $G: D^b(\mathscr{C}) \to \mathscr{D}$ such that $F = G \circ D$.

(2) We need to choose for each complex, a Cartan-Eilenberg resolution of it, and the functor $D: \mathbf{Chain^b}(\mathscr{C}) \to D^b(\mathscr{C})$ depends on this choice. However, chosing for each complex a resolution yields a functor, and any two functors obtained in such a way are isomorphic (I would hope...).

DEFINITION 8.12. Assume $\mathscr{C}$ is a category with enough injectives, and $D^b(\mathscr{C}) = \mathbf{K^b}(\mathbf{Inj}(\mathscr{C}))$ its derived category. Let F be a left-exact functor. Then the **right-derived functor of** F, $RF: D^b(\mathscr{C}) \to D^b(\mathscr{D})$ is obtained by associating to $\mathscr{F}^\bullet$ the image by F of the total complex of a Cartan-Eilenberg resolution of $\mathscr{F}^\bullet$.

Note that Proposition 8.6 (2) shows that $RF(A^\bullet)$ does not depend on the choice of the Cartan-Eilenberg resolution Most of the time, we only compute $RF(A)$ for an element A in $\mathscr{C}$. For this take an injective resolution of A

Examples:

(1) Let $\mathscr{F}^\bullet$ be a complex of sheaves. Then, $H^m(X,\mathscr{F}^\bullet)$ is defined as follows: apply Γ_X to a Cartan-Eilenberg resolution of $F^\bullet$, and take the cohomology. In other words, $H^m(X,\mathscr{F}^\bullet) = (R^m\Gamma_X)(\mathscr{F}^\bullet)$. As we pointed out before, this is different from $H^m(X,\mathscr{F}^q)$. But we shall see that there is a spectral sequence with $E_2 = H^p(X,\mathscr{F}^q)$ (resp. $E_2^{p,q} = H^p(X,\mathscr{H}^p(\mathscr{F}^\bullet)))$ converging to $H^{p+q}(X,\mathscr{F}^\bullet)$.

(2) Computing Tor. Let M be an R-module, and $0 \to M \to I_1 \to I_2 \to ....$ be an injective resolution. Let F be the $\otimes_R N$ functor, then $R^jF(M) = Tor^j(M,N)$ is the j-th cohomology of $RF(M)$ given by $0 \to F(I_1) \to F(I_2) \to F(I_3) \to ....$ For example the $\mathbb{Z}$-module $\mathbb{Z}/2\mathbb{Z}$ has the resolution

$$0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{f} \mathbb{Q}/\mathbb{Z} \xrightarrow{g} \mathbb{Q}/\mathbb{Z} \to 0$$

where the map f sends 1 to $\frac{1}{2}$ and $g(x) = 2x$. Then $Tor(\mathbb{Z}/2\mathbb{Z},\mathbb{Z}/2\mathbb{Z})$ is the complex $0 \to \mathbb{Q}/\mathbb{Z}\otimes\mathbb{Z}/2\mathbb{Z} \xrightarrow{\bar{g}} \mathbb{Q}/\mathbb{Z}\otimes\mathbb{Z}/2\mathbb{Z} \to 0$. This is isomorphic to $0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{2} \mathbb{Z}/2\mathbb{Z} \to 0$, so that $Tor^0(\mathbb{Z}/2\mathbb{Z},\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ and $Tor^1(\mathbb{Z}/2\mathbb{Z},\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$, while $Tor^k(\mathbb{Z}/2\mathbb{Z},\mathbb{Z}/2\mathbb{Z}) = 0$ for $k \geq 2$. However this is usually done using projective resolutions, which cannot be done for sheaves, since they do not have enough projectives:

we start from

$$0 \to \mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

which yields

$$0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z}/2\mathbb{Z} \to 0.$$

Finally the notion of spectral sequence allows us to replace injective resolutions by acyclic ones, as we already proved in corollary 7.23:

PROPOSITION 8.13. *Let $0 \to A \to B^1 \to B^2 \to B^3 \to \ldots$ be a resolution by F-acyclic objects, that is $R^jF(B^m) = 0$ for all $j \geq 1$ and all m. Then $RF(A)$ is quasi-isomorphic to $0 \to F(B^1) \to F(B^2) \to F(B^3) \to \ldots$*

PROOF. The proposition tells us that injective resolutions are not necessary to compute derived functors: F-acyclic ones are sufficient. Indeed we saw that $0 \to A \to 0$ is quasi-isomorphic to

$$0 \longrightarrow B^1 \xrightarrow{\partial_1} B^2 \xrightarrow{\partial_2} B^3 \xrightarrow{\partial_3} \cdots$$

To compute the image by RF of this last complex, we use again the Cartan-Eilenberg resolution of the above exact sequence.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & B^1 \arrow[r, "i_1"] \arrow[d, "\partial_1"] & I_1^0 \arrow[r, "\delta_0^1"] \arrow[d, "\partial_1^0"] & I_1^1 \arrow[r, "\delta_1^1"] \arrow[d, "\partial_1^1"] & I_2^1 \arrow[r, "\delta_2^1"] \arrow[d, "\partial_1^2"] & \cdots \\
0 \arrow[r] & B^2 \arrow[r, "i_2"] \arrow[d, "\partial_2"] & I_0^2 \arrow[r, "\delta_0^2"] \arrow[d, "\partial_2^0"] & I_1^2 \arrow[r, "\delta_1^2"] \arrow[d, "\partial_2^1"] & I_2^2 \arrow[r, "\delta_2^2"] \arrow[d, "\partial_2^2"] & \cdots \\
 & {} & {} & {} & {} &
\end{tikzcd}$$

We must then apply F to the above diagram, and we must compute the cohomology of the total complex obtained by removing the column containing the B^j. But by assumptions the horizontal lines remain exact, since the B^j are F-acyclic, while

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & F(B^1) \arrow[r, "i_1"] \arrow[d, "\partial_1"] & F(I_1^0) \arrow[r, "\delta_0^1"] \arrow[d, "\partial_1^0"] & F(I_1^1) \arrow[r, "\delta_1^1"] \arrow[d, "\partial_1^1"] & F(I_2^1) \arrow[r, "\delta_2^1"] \arrow[d, "\partial_1^2"] & \cdots \\
0 \arrow[r] & F(B^2) \arrow[r, "i_2"] \arrow[d, "\partial_2"] & F(I_0^2) \arrow[r, "\delta_0^2"] \arrow[d, "\partial_2^0"] & F(I_1^2) \arrow[r, "\delta_1^2"] \arrow[d, "\partial_2^1"] & F(I_2^2) \arrow[r, "\delta_2^2"] \arrow[d, "\partial_2^2"] & \cdots \\
 & {} & {} & {} & {} &
\end{tikzcd}$$

Since the horizontal lines remain exact by assumption, using Tic-Tac-Toe, we can represent any cohomology class of the total complex $F(Tot(I^{p,q}))$ by a closed element in $F(B^{p+q})$. $\square$

2. Spectral sequences of a bicomplex. Grothendieck and Leray-Serre spectral sequences

Apart from simple situations, we cannot apply the Tic-Tac-Toe lemma to a general bicomplex. However one should hope to recover at least SOME information on total cohomology, from the homology of lines and columns.

Let us start with algebraic study. Let $(K^{p,q},\partial,\delta)$ be a double (or bigraded)complex. In other words, $\delta_q^p : K^{p,q} \to K^{p,q+1}$ and $\partial_q^p : K^{q,p} \to K^{p+1,q}$ each define a complex. We moreover assume that ∂ and δ commute. This yields a third chain complex, called the total complex, given by $Tot(K^{\bullet,\bullet})^m = \oplus_{p+q=m} K^{q,p}$ and $d_m = \sum_{p+q=m} \partial_q^p + (-1)^p \delta_q^p$.

DEFINITION 8.14. A spectral sequence is a sequence of bigraded complexes $(E_r^{p,q}, d_r^{p,q})$, such that $d_r^2 = 0$, $d_r^{p,q} : E_r^{p,q} \to E_r^{p-r+1,q+r}$, such that $E_{r+1}^{p,q} = H(E_r^{p,q}, d_r^{p,q})$. The spectral sequence is said to converge to a graded complex F^p endowed with a homogeneous increasing filtration F_m, if for r large enough, $F_m^p / F_{m-1}^p = E_r^{p,m-p}$.

This is not the most general definition, since convergence could be reached in infinite time. This will not happen in our situation, as long as we stick with bounded complexes (and bounded resolutions). Note that the map ∂ obviously induces a boundary map on $H_\delta^{p,q}(K^{\bullet,\bullet}) = H^{p,q}(K^{\bullet,\bullet},\delta) \to H_\delta^{p+1,q}(K^{\bullet,\bullet})$.

THEOREM 8.15 (Spectral sequence of a total complex). *There is a spectral sequence from $H_\partial H_\delta(K^{\bullet,\bullet})$ converging to $H^{p+q}(Tot(K^{\bullet,\bullet}))$.*

PROOF. For simplicity we assume $K^{p,q} = 0$ for p or q nonpositive.

Then a cohomology class in $H^m(Tot(K^{\bullet,\bullet}), d = \partial + \overline{\delta})$ is just a sequence $\mathbf{x} = (x_0, \dots, x_m)$ of elements in $K^{p,m-p}$ such that $\partial x_0 = 0$ and $\overline{\delta} x_j + \partial x_{j+1} = 0$ for $j \geq 1$ and finally $\overline{\delta} x_m = 0$. This is represented by the **zig-zag**

$$\begin{tikzcd}
0 & & & & \\
x_0 \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & 0 & & & \\
& x_1 \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & 0 & & \\
& & \ddots \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & 0 & \\
& & & x_m \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & 0
\end{tikzcd}$$

FIGURE 1. $\mathbf{x} = x_0 + \dots + x_m$ a cocycle in $Tot(K^{\bullet,\bullet})^m$

where the zeros indicate that the sum of the images of the arrows abutting there is zero. This is well defined modulo addition of coboundaries, that correspond to sequences $(y_0, ..., y_{m-1})$, such that $x_0 = \partial y_0, x_j = \overline{\delta} y_j + \partial y_{j+1}, \overline{\delta} y_{m-1} = x_m$, that is represented as follows

$$\begin{tikzcd}
x_0 & & & & \\
y_0 \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & x_1 & & & \\
& y_1 \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & x_2 & & \\
& & \ddots \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & x_m & \\
& & & y_{m-1} \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & x_m
\end{tikzcd}$$

FIGURE 2. $\mathbf{y} = y_0 + ... + y_{m-1}$ and $\mathbf{x} = x_0 + ... + x_m$ is the coboundary of $\mathbf{y}$

The idea of the spectral sequence, is that a zig-zag as in Figure 1 can be approximated by zig-zags of length at most r. Replace the $K^{p,q}$ by $E_r^{p,q}$ as follows:

the space $E_r^{p,q}$ is a quotient of $Z_r^{p,q}$, the set of sequences $\mathbf{x} = x_0 + ... + x_{r-2}$ such that

(1) $x_j \in K^{p-j,q+j}$
(2) $\partial x_0 = 0$ and $\overline{\delta} x_j + \partial x_{j+1} = 0$ for $j \geq 1$
(3) there exists x_{r-1} satisfying $-\overline{\delta} x_{r-2} = \partial x_{r-1}$

It will be convenient to use the notation $\mathbf{x} = x_0 + ... + x_{r-2} + (x_{r-1})$, where only the existence of x_{r-1} matters and not its value, which explains why we put parenthesis around x_{r-1}. Another possible notation would be to replace x_{r-1} by $x_{r-1} + \ker(\partial) \cap K^{p-r+1,q+r-1}$ (so that (x_{r-1}) designates an element in $K^{p-r+1,q+r-1}/\mathrm{Ker}(\partial)$). An element of $Z_r^{p,q}$ is thus represented by the **zig-zag**

$$\begin{tikzcd}
0 & & & \\
x_0 \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & 0 & & \\
 & x_1 \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & 0 & \\
 & & \ddots \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & 0 \\
 & & & (x_{r-1}) \arrow[u, "\partial"]
\end{tikzcd}$$

FIGURE 3. An element $\mathbf{x} = x_0 + \dots + x_{r-2} + (x_{r-1})$ in $Z_r^{p,q}$

Note that one or more of the x_j could be taken equal to 0 (and that all unwritten elements are assumed to be zeros).

Then $E_r^{p,q}$ is defined as the quotient of $Z_r^{p,q}$ by the subgroup $B_r^{p,q}$ of $Z_r^{p,q}$ of elements of the type $D(y_0 + \dots + y_{r-1})$ represented as

$$\begin{tikzcd}
0 & & & & & \\
y_0 \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & x_0 & & & & \\
 & y_1 \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & x_1 & & & \\
 & & y_2 \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & x_2 & & \\
 & & & \ddots \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & x_{r-2} & \\
 & & & & y_{r-1} \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & (x_{r-1})
\end{tikzcd}$$

FIGURE 4. The element $\mathbf{x} = x_0 + \dots + x_{r-2}$ is in $B_r^{p,q}$ as it is the d_{r-1}-boundary of $\mathbf{y}$.

Again we do not worry about the value of $\overline{\delta} y_{r-1}$. We denote by $E_r^{p,q}$ the set of such equivalence classes of objects obtained with $x_0 \in K^{p,q}$.

Clearly a cohomology class of the total complex, yields by truncation, a class in $E_r^{p,q}$, and it is clear that for r large enough (namely $r \geq \min\{p,q\}$), an element of $E_r^{p,q}$ is nothing else than a cohomology class.

Our claim is that there is a differential $d_r : E_r^{p,q} \to E_r^{p-r+1,q+r}$ such that $E_{r+1}^{p,q}$ is the cohomology of $(E_r^{p,q}, d_r)$. Let us first study the space $E_r^{p,q}$ for small values of r. Clearly, $E_0^{p,q} = K^{p,q}$ and for $r = 1$, $E_1^{p,q} = H^{p,q}(K^{\bullet,\bullet}, \partial)$. Then to x_0 such that $\partial x_0 = 0$ we associate $-\overline{\delta} x_0$. This yields a map $\delta : H^{p,q}(K^{\bullet,\bullet}, \partial) \to H^{p,q+1}(K^{\bullet,\bullet}, \partial)$, and for the class of x_0 to be in the kernel of this map, means that $\overline{\delta} x_1 \in \mathrm{Im}(\partial)$ so there exists x_1 such that $\partial x_1 = -\overline{\delta} x_0$, and so we may associate to it the element

$$\begin{tikzcd}
0 & \\
x_0 \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & 0 \\
 & (x_1) \arrow[u, "\partial"]
\end{tikzcd}$$

FIGURE 5. An element in $Z_2^{p,q}$.

the parenthesis around x_1 means, as usual, that the choice of x_1 is not part of the data defining the element, only its existence matters.

Then for any choice of x_1 as above, the element $\mathbf{x} = x_0 + (x_1)$ vanishes in $E_2^{p,q}$ if there exists $\mathbf{y} = y_0 + y_1$ such that $\partial y_0 = 0, x_0 = \overline{\delta} y_0 + \partial y_1$, and $\overline{\delta} y_1 = x_1$ (note that this last equality can be taken as the choice of x_1 which automatically satisfies $\partial x_1 = -\overline{\delta} x_0$). This is clearly the definition of an element in $E_2^{p,q}$, so we may indeed identify E_2 with the cohomology of (E_1, d_1), that is $H_\delta^q H_\partial^p (K^{\bullet,\bullet})$. The map d_2 is then defined as the class of $\overline{\delta} x_1$.

In the general case, we define the map d_r as follows. For the sequence $(x_0, ..., x_r)$ we define its image by d_r to be the class of $-\overline{\delta} x_r$. Note that x_r is only defined up to an element z in the kernel of ∂, but $\overline{\delta} x_r$ is well defined, since $\overline{\delta} z = D(z)$.

Clearly $\overline{\delta} x_r \in K^{p-r,q+r}$. We have to prove on one hand that if $\overline{\delta} x_r$ is zero (in the quotient space $E_r^{p,q}$) we may associate to $\mathbf{x}$ an element in $E_{r+1}^{p,q}$, and that this map is an isomorphism. Clearly if $\overline{\delta} x_r = 0$ in the quotient space, this means we have the following two diagrams

$$\begin{tikzcd}
0 & & & & \\
x_1 \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & 0 & & & \\
& x_2 \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & 0 & & \\
& & \ddots \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & 0 & \\
& & & (x_{r-1}) \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & u
\end{tikzcd}$$

FIGURE 6. The class u represents $d_r(\mathbf{x})$

Now claiming that u vanishes in the quotient E_r, means that we have a diagram of the following type

$$\begin{tikzcd}
0 & & & & & \\
y_1 \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & 0 & & & & \\
& \ddots \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & 0 & & & \\
& & y_j \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & u & & \\
& & & y_{j+1} \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & 0 & \\
& & & & \ddots \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & 0 \\
& & & & & y_r \arrow[u, "\partial"]
\end{tikzcd}$$

FIGURE 7. Representing the vanishing of $d_r\mathbf{x}$ in $E_r^{p,q}$

In particular in the above case, u is not in the image of ∂, but of the form $\overline{\delta} y_j + \partial y_{j+1}$ with $\partial y_1 = 0$. Then the fllowing sequence represents an element in $Z_{r+1}^{p,q}$:

$$
\begin{tikzcd}
0 \\
x_1 \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & 0 \\
& x_2 \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & 0 \\
& & \ddots \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & 0 \\
& & & x_{r-j} - y_1 \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & 0 \\
& & & & x_{r-j+1} - y_2 \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & 0 \\
& & & & & \ddots \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & 0 \\
& & & & & & -y_j \arrow[u, "\partial"] \arrow[r, "\overline{\delta}"] & 0
\end{tikzcd}
$$

FIGURE 8. How to make $\mathbf{x}$ into an element of $E_{r+1}^{p,q}$ assuming $d_r \mathbf{x} = 0$ in $E_r^{p,q}$.

However by substracting from x the above coboundary, we can make u to vanish, and then we get an element of $E_{r+1}^{p,q}$. Conversely, it is easy to see that an element in $E_{r+1}^{p,q}$ corresponds by truncation to an element $\mathbf{x}$ in $E_r^{p,q}$ with $d_r(\mathbf{x}) = 0$. □

REMARK 8.16. Because ∂ and δ play symmetric roles, there is also a spectral sequence from $H_\delta H_\partial(K^{\bullet,\bullet})$ converging to $H^{p+q}(Tot(K^{\bullet,\bullet}))$. This is often very useful in applications.

PROPOSITION 8.17 (The canonical spectral sequence of a derived functor). *Let $A^\bullet \in$ **Chain**$(\mathscr{C})$, and F a left-exact functor. Then there are two spectral sequences with respectively $E_2^{p,q} = H^p(R^q F(A))$ and $E_2^{p,q} = R^p F(H^q(A))$, converging to $R^{p+q}F(A)$.*

PROOF. Consider a Cartan-Eilenberg resolution of $A^\bullet$, and denote it by $(I^{p,q},\partial,\delta)$. Then, consider the complex $(F(I^{p,q},F(\partial),F(\delta))$. By definition $R^mF(A^\bullet)$ is the cohomology of $(Tot(F(I^{p,q}),F(d))$. Now $H^q_\delta(F(I^{p,q})) = R^qF(A^\bullet)$, since the lines are injective resolutions of A^p, and so the cohomology of each line is $R^qF(A^\bullet)$. Thus the first spectral sequence has $E_2^{p,q} = H^p_\partial H^q_\delta(F(I^{\bullet,\bullet})) = H^p_\partial(R^qF(A^\bullet))$. Now consider the other spectral sequence. We must first compute $H_\partial(F(I^{p,q}))$. But by our assumptions the columns are injective, and have ∂ homology giving an injective resolution of $H^q(A^\bullet)$, so applying F and taking the δ cohomology, we get $R^pF(H^q(A))$. □

COROLLARY 8.18. *There is a spectral sequence with E_2 term $H^p(X,\mathcal{H}^q(\mathcal{F}^\bullet))$ and converging to $H^{p+q}(X,\mathcal{F}^\bullet)$. Similarly there is a spectral sequence from $E_2^{p,q} = H^p(X,\mathcal{F}^q)$ converging to $H^{p+q}(X,\mathcal{F}^\bullet)$.*

PROOF. Apply the above to the left-exact functor on **Sheaf (X)**, $F(\mathcal{F}) = \Gamma(X,\bullet)$. □

The following result is often useful:

PROPOSITION 8.19 (Comparison theorem for spectral sequences). *Let $A^\bullet, B^\bullet$ be two objects in* **Chain**$(\mathcal{C})$*, $f^\bullet : A^\bullet \to B^\bullet$ a chain morphism. Let F be a left-exact functor, and assume that the induced map from $H^p(R^qF(A))$ to $H^p(R^qF(B))$ is an isomorphism. Then the induced map $RF(A) \to RF(B)$ is also an isomorphism.*

PROOF. □

Besides the above canonical spectral sequence, the simplest exampe of a spectral sequence is the following topological theorem, constructing the cohomology of the total space of a fibre bundle from the cohomology of the base and fiber. Indeed,

THEOREM 8.20 (Leray-Serre spectral sequence). *Let $\pi : E \to B$ be a smooth fibre bundle. Then there exists a spectral sequence with E_2 term $H^*(B,\mathcal{H}^q(F_x))$ and converging to $H^{p+q}(E)$.*

For the proof see 102. Note that $\mathcal{H}^q(F_x)$ is a locally constant sheaf, i.e. local coefficients, with stalk $H^*(F)$, since $\mathcal{H}^q(\pi^{-1}(U)) \simeq H^q(U\times F) = H^q(F)$ for U small enough and contractible. In particular when B is simply connected, and we take coefficients in a field, $H^*(B,\mathcal{H}^q(F_x)) = H^*(B)\otimes H^*(F)$.

At the level of derived categories, this is even simpler. Let G be a left-exact functors, from $\mathcal{C}$ to $\mathcal{D}$ and F a left-exact functor from $\mathcal{D}$ to $\mathcal{E}$. We are interested in the derived functor $R(G\circ F)$

THEOREM 8.21 (Grothendieck's spectral sequence). *Assume the category $\mathcal{C}$ has enough injectives, and G transforms injectives into F-acyclic objects (i.e. such that $R^jF(A) = 0$ for $j \ge 1$). Then*

$$R(F\circ G) = RF\circ RG$$

PROOF. Let $I^\bullet$ be an injective resolution of A. Then $G(I^\bullet)$ is a complex representing $RG(A)$. Since this is F-acyclic, it can be used to compute $RF(RG(A))$, and this is then represented by $FG(I^\bullet)$. But obviously this represents $R(G\circ F)(A)$. □

Note that this theorem could not be formulated if we only have the R^jF without derived categories, as was the case before Grothendieck and Verdier. Indeed, if we only know the R^jF there is no way of composing derived functors. This has the following important application:

THEOREM 8.22 (Cohomological Fubini theorem). *Let $f : X \to Y$ be a continuous map between compact spaces. Then , we have $R\Gamma(X,\mathscr{F}) = R\Gamma(Y, Rf_*(\mathscr{F}))$ hence, taking cohomology, $H^*(X,\mathscr{F}) = H^*(Y, Rf_*\mathscr{F})$.*

PROOF. Apply Grothendieck's theorem to $G = f_*$ and $F = \Gamma(Y,\bullet)$, use the fact that $\Gamma(X,\bullet) = \Gamma(Y,\bullet)\circ f_*$, and remember that $H^j(X,\mathscr{F}) = R^j\Gamma(X,\mathscr{F})$. We still have to check that f_* sends injective sheafs to $\Gamma(Y,\bullet)$ acyclic objects, but this is a consequence of corollary 7.16. The second statement follows from the first by taking homology. □

REMARKS 8.23.

(1) The Grothendieck spectral sequence looks like "three card monty" trick: there is no apparent spectral sequence, and the proof is essentially trivial. So what ? See the next theorem for an explanantion.
(2) Note that a priori we have not defined the cohomology of a an object in the derived category of sheafs. This does not even fall in the framework of sheafs with values in an abelian category, since the derived category is not abelian. However, $R\Gamma(X,) : D^b(\mathbf{Sheaf(X)}) \to D^b(\mathbf{Ab})$. Now taking homology does not lose anything, because any complex of abelian groups is quasi-isomorphic to its homology, since the category of abelian groups has homological dimension 1 ([**?**]). This fails for general modules, so in general, $R\Gamma(X, Rf_*(\mathscr{F}))$ is only defined in $D^b(\mathbf{R}-\mathbf{mod})$, which is not well understood, except that any element has a well defined homology, so $R^p\Gamma(X,(Rf_*))$ is well defined.
(3) If c is the constant map, we get $H^*(X,\mathscr{F}^\bullet) = H^*(\{pt\},(Rc)_*(\mathscr{F}^\bullet))$, but $(Rc)_*(\mathscr{F}^\bullet)$ is a complex of sheaves over a point, that is just an ordinary complex. We thus associate a complex in $D^b(\mathbf{R}-\mathbf{mod})$ to the cohomology of X with coefficients in $\mathscr{F}^\bullet$.

Example: Let us consider the functor Γ_Z, then by Grothendieck's theorem, $\Gamma_Z(X,\mathscr{F}) = \Gamma(X,\Gamma_Z(\mathscr{F}))$ so that $R\Gamma_Z(X,\mathscr{F}) = R\Gamma(X, R\Gamma_Z(\mathscr{F}))$.

THEOREM 8.24 (Grothendieck's spectral sequence-cohomological version). *Under the assumptions of theorem 8.21, there is a spectral sequence from $E_2^{p,q} = R^pF\circ R^qG$ to $R^{p+q}(F\circ G)$.*

PROOF. Let $I^\bullet$ be an injective resolution of A, and consider $C^\bullet = G(I^\bullet)$.

Then one of the canonical spectral sequence of theorem 8.17 applied to RF and $C^\bullet$, has $E_2^{p,q}$ given by $R^pF(H^q(C^\bullet))$ and converges to $R^{p+q}F(C^\bullet)$. But since $H^q(C^\bullet) = R^qG(A)$ by definition, we get that this spectral sequence is $R^pF(R^qG(A))$, and converges to $R^{p+q}F(G(I^\bullet))$ that is the $p+q$ cohomology of $RF(G(I^\bullet)) = RF\circ RG(A)$. But we saw that $RF\circ RG(A) = R(F\circ G)(A)$, so the spectral sequence converges to $R^{p+q}(F\circ G)(A)$. □

Ideally, one should never have to construct a spectral sequence directly, any spectral sequence should be obtained from the Grothendieck's spectral sequence fro some suitable pair fo functors F, G.

EXERCICE 1. Let F_1, F_2 be functors such that we have an isomorphism $RF_1 = RF_2$ on elements of $\mathscr{C}$. Then $RF_1 = RF_2$ on the derived category.

PROOF OF LERAY-SERRE. Let us see how this implies the Leray spectral sequence: take $\mathscr{C}$ =**Sheaves(X)**, $\mathscr{D}$ =**Sheaves(Y)**, $\mathscr{E}$ =**Ab** ad $F = f_*$, $G = \Gamma_Y$. Since $\Gamma_Y \circ f_* = \Gamma_X$, we get $R\Gamma_X = R\Gamma_Y \circ Rf_*$, since f_* sends injectives to injectives (because f_* has an adjoint f^{-1}). So we get a spectral sequence $E_2^{p,q} = H^p(Y, R^q f_*(\mathscr{F}))$ to $H^{p+q}(X, \mathscr{F})$ is the sheaf associated to the presheaf $H^q(f^{-1}(U))$. If f is a fibration, this is a constant sheaf. Moreover the sheaf $R^q f_*(\mathscr{F})$ has stalk $\lim_{x\in U} H^q(f^{-1}(U))$ which is equal to $H^q(f^{-1}(x))$ if f is a fibration such that the $f^{-1}(U)$ form a fundamental basis of neighbourhoods of $f^{-1}(x)$. □

EXERCICE 2. Prove that if $\mathscr{U}$ is a covering of X such that for all q and all sequences $(i_0, i_1, ..., i_q)$, we have $H^j(U_{i_0} \cap .... \cap U_{i_q}) = 0$ for $j \geq 1$, then the cohomology of the Čech complex, $\mathscr{C}(\mathscr{U}, \mathscr{F})$ coincides with $H^*(X, \mathscr{F})$. Hint: consider an injective resolution of $\mathscr{F}$, $0 \to \mathscr{F} \to \mathscr{I}^0 \to \mathscr{I}^1 \to ...$ and the double complex having as rows the Čech resolution of $\mathscr{I}^p$.

3. Complements on functors and useful properties on the Derived category

3.1. Derived functors of operations and some useful properties of Derived functors. Consider the operations $\mathscr{H}om, \otimes, f_*, f^{-1}$. The operations f^{-1} is exact, so it is its own derived functor. The functor f_* is left exact, hence has a right-derived functor, Rf_*. The operation $\mathscr{H}om$ is covariant in the second variable and contravariant in the first. Considering it as a functor of the second variable it is left exact, so has a right-derived functor, $R\mathscr{H}om$. Finally the tensor product is right-exact,hence has a left derived functor denoted $\otimes^L$. Note that in the case of $\mathscr{H}om$ and $\otimes$, the symmetry of the functor is not really reflected, since for the moment one of the two factors must be a sheaf and not a chain complex of sheaves. For a satisfactory theory one would have to work with bifunctors, which we shall avoid (see [**K-S**], page 56). In particular we have as a complex of sheaves, $(\mathscr{F}^\bullet \otimes \mathscr{G}^\bullet)^r = \sum_{p+q=r} \mathscr{F}^p \otimes \mathscr{G}^q$ and $\mathscr{H}om(\mathscr{F}^\bullet, \mathscr{G}^\bullet)^r = \sum_{p+q=r} \mathscr{H}om(\mathscr{F}^p, \mathscr{G}^q)$.

Again acording to [**K-S**], under suitable assumptions, whether we consider $\mathscr{H}om$ as a bifunctor, or we consider the functor $\mathscr{F} \to \mathscr{H}om(\mathscr{F}, \mathscr{G})$ (resp. $\mathscr{G} \to \mathscr{H}om(\mathscr{F}, \mathscr{G})$), their derived functors coincide.

REMARK 8.25. (1) Let $\Gamma(\{x\}, \mathscr{F}) = \mathscr{F}_x$. This is an exact functor, since by definition a sequence is exact, if and only if the induced sequence at the stalk level is exact. So $R\Gamma(\{x\}, \mathscr{F}) = \mathscr{F}_x$.
(2) Be careful: there is no equality $\Gamma(\{x\}, \Gamma_Z(\mathscr{F})) = \Gamma(\{x\}, \mathscr{F})$, so we cannot use Grothendieck's theorem 8.21.

(3) As long as we are working over fields, and finite dimensional vector spaces, the tensor product and $\mathscr{H}om$ functors on the category k-**vect** are exact, so they coincide with their derived functors. We shall make this assumption whenever necessary.

3.2. More on Derived categories and functors and triangulated categories. There is no good notion of exact sequence in a derived category. Of course, the exact sequence of sheaves has a corresponding exact sequence of complexes of their injective resolution as the following extension of the Horseshoe lemma (Lemma 7.28) proves:

PROPOSITION 8.26. *Let* $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ *be an exact sequence of complexes. There is an exact sequence of injective resolutions* $0 \to I_A^\bullet \to I_B^\bullet \to I_C^\bullet \to 0$ *and chain maps which are quasi-isomorphisms*

$$\begin{tikzcd}
0 \arrow[r] & A^\bullet \arrow[r] \arrow[d, "a"] & B^\bullet \arrow[r] \arrow[d, "b"] & C^\bullet \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & I_A^\bullet \arrow[r] & I_B^\bullet \arrow[r] & I_C^\bullet \arrow[r] & 0
\end{tikzcd}$$

PROOF. Indeed, if the complexes are reduced to single objects, this is just the Horseshoe lemma 7.28 applied to $0 \to A \to B \to C \to 0$. The general case follows from the theorem 8.6, (2), by replacing the double complexes by their total complex. $\square$

However, since the derived category does not have kernels or cokernels, the notion of exact sequence is not well defined. It is replaced by the notion of distinguished triangle, defined as follows.

DEFINITION 8.27. A distinguished triangle is a triangle

$$\begin{tikzcd}
& A^\bullet \arrow[dl, squiggly, "v"'] & \\
B^\bullet \arrow[rr, "f"] & & C^\bullet \arrow[ul, "u"']
\end{tikzcd}$$

isomorphic to a triangle of the form

$$\begin{tikzcd}
& C(f)^\bullet \arrow[dl, squiggly, "v"'] & \\
M^\bullet \arrow[rr, "f"] & & N^\bullet \arrow[ul, "u"']
\end{tikzcd}$$

associated to a map $f : M \to N$.

We now claim that to an exact sequence in $\mathbf{Chain^b}(\mathscr{C})$, we may associate a distinguished triangle in the derived category

Indeed, an exact sequence of injective sheaves $0 \to I_A^\bullet \to I_B^\bullet \to I_C^\bullet \to 0$ is split, so is isomorphic to $0 \to I_A^\bullet \to I_A^\bullet \oplus I_C \to I_C^\bullet \to 0$ and hence isomorphic to the above exact sequence for $M^\bullet = I_C^\bullet[-1]$, $N^\bullet = I_A$ and $f = 0$.

$0 \to I_A^\bullet \to I_A^\bullet \oplus I_C \to I_C^\bullet \to 0$ and this is isomorphic to $0 \to I_A^\bullet \to I_B^\bullet \to I_C^\bullet \to 0$
The following property will be useful in the proof of Proposition 9.3.

PROPOSITION 8.28 ([**Iv**], p.58). *Let F be a left exact functor from $\mathscr{C}$ to $\mathscr{D}$, where $\mathscr{C},\mathscr{D}$ are categories having enough injectives. Then the functor $RF : D^b(\mathscr{C}) \to D^b(\mathscr{D})$ preserves distinguished triangles.*

PROOF. □

Let now F, G, H be left-exact functors, and λ, μ be transformations of functors from F to G and G to H respectively.

PROPOSITION 8.29 ([**K-S**] prop. 1.8.8, page 52). *Assume for each injective I we have an exact sequence $0 \to F(I) \xrightarrow{\lambda} G(I) \xrightarrow{\lambda} H(I) \to 0$. Then there is a transformation of functors ν and a distinguished triangle*

$$\to RF(A) \xrightarrow{R\lambda} RG(A) \xrightarrow{R\lambda} RH(A) \xrightarrow{\nu} RF(A)[1] \xrightarrow{R\lambda[1]} \ldots$$

PROOF. □

Example: We have an exact sequence $0 \to \Gamma_Z(\mathscr{F}) \to \mathscr{F} \to \mathscr{F}_{X-Z}$ that extends for $\mathscr{F}$ flabby to an exact sequence

$$0 \to \Gamma_Z(\mathscr{F}) \to \mathscr{F} \to \mathscr{F}_{X-Z} \to 0$$

therefore

COROLLARY 8.30. *There is a distinguished triangle*

$$R\Gamma_Z(\mathscr{F}) \to R\Gamma(\mathscr{F}) \to R\Gamma(\mathscr{F}_{X-Z}) \xrightarrow{[+1]} R\Gamma_Z(\mathscr{F})[1] \ldots$$

yielding a cohomology long exact sequence

$$\ldots \to H^j\Gamma_Z(\mathscr{F}) \to H^j(X,\mathscr{F}) \to H^j(X \setminus Z;\mathscr{F}) \to H_Z^{j+1}(\mathscr{F}) \to \ldots$$

REMARK 8.31. For each open U, we may consider $R\Gamma(U,\mathscr{F}^\bullet)$ that is an element in $D^b(\mathrm{R}-mod)$. We would like to put these toghether to make a sheaf. The only obstruction is that this would not be a sheaf in an abelian category, but only in a triangulated category. However, consider an injective resolution of $\mathscr{F}^\bullet$, $\mathscr{I}^\bullet$. Then $\mathscr{I}^\bullet(U)$ represents $R(\Gamma(U,\mathscr{F}^\bullet)$, so that $R\Gamma$ is just the functor associating to $\mathscr{F}^\bullet$ the injective resolution, which is the map so that we may define $R\Gamma(\mathscr{F}^\bullet) = \mathscr{I}^\bullet$ in the derived category, i.e. this is the functor D of Definiton 8.10. Then $R\Gamma(U,\mathscr{F}^\bullet) = R\Gamma(\mathscr{F}^\bullet)(U)$.

Part 3

Applications of sheaf theory to symplectic topology

CHAPTER 9

Singular support in the Derived category of Sheaves.

1. Singular support

1.1. Definition and first properties. From now on, we shall denote by $D^b(X)$ the derived category of (bounded) sheaves over X, that is $D^b(\mathbf{Sheaf(X)})$.

Let U be an open set. The functor $\Gamma(U;\bullet)$ sends sheaves on X to R-modules, and has a derived functor $R\Gamma(U;\bullet)$. Its cohomology $R^j\Gamma(U;\mathscr{F}) = H^j(U,\mathscr{F})$. Now if Z is a closed set, we defined the functor Γ_Z as the set of sections supported in Z, that is $\Gamma_Z(U,\mathscr{F})$ is the kernel of $\mathscr{F}(U) \longrightarrow \mathscr{F}(U\setminus Z)$. This is a sheaf, so Γ_Z is a functor from **Sheaf(X)** to **Sheaf(X)**. One checks that this is left-exact, as follows from the left-exactness of the functor $\mathscr{F} \to \mathscr{F}_{|X\setminus Z}$, where $\mathscr{F}_{|X\setminus Z}(U) = \mathscr{F}(U\setminus(Z\cap U))$. Hence we may define $R\Gamma_Z : D^b(X) \longrightarrow D^b(X)$. This is defined for example for a sheaf $\mathscr{F}$ as follows: construct an injective resolution $\mathscr{F}$, that is $0\to\mathscr{F}\to\mathscr{I}_0\to\mathscr{I}_1\to\mathscr{I}_2\to\mathscr{I}_3\to ....$

Then the complex of sheaves

$$0\to\Gamma_Z\mathscr{I}_0\to\Gamma_Z\mathscr{I}_1\to\Gamma_Z\mathscr{I}_2\to\Gamma_Z\mathscr{I}_3\to\Gamma_Z\mathscr{I}_4\to ...$$

represents $R\Gamma_Z(\mathscr{F})$. The cohomology space $\mathscr{H}^j(R\Gamma_Z(\mathscr{F}))$ is an element in $D^b(X)$, often denoted $H^j_Z(\mathscr{F})$. Moreover we denote by $H^j_Z(X,\mathscr{F}) = H^j(R\Gamma_Z(X,\mathscr{F}))$.

often denoted $H^j_Z(\mathscr{F})$.

DEFINITION 9.1. Let $\mathscr{F}^\bullet$ be an element in $D^b(X)$. The singular support of $\mathscr{F}^\bullet$, $SS(\mathscr{F}^\bullet)$ is the closure of the set of (x,p) such that **there exists** a real function $\varphi : M\to\mathbb{R}$ such that $d\varphi(x) = p$, and we have

$$R\Gamma_{\{x\mid\varphi(x)\geq 0\}}(\mathscr{F}^\bullet)_x \neq 0$$

Note that this is equivalent to the existence of j such that $R^j\Gamma_{\{x\mid\varphi(x)\geq 0\}}(\mathscr{F}^\bullet)_x = \mathscr{H}^j(R\Gamma_Z(\mathscr{F}^\bullet))_x = H^j_Z(\mathscr{F}^\bullet)_x \neq 0$.

REMARK 9.2. (1) The set $SS(\mathscr{F})$ is a homogeneous subset in T^*X. Note that $SS(\mathscr{F})$ is in T^*X not $\mathring{T}^*X$.

(2) It is easy to see that $SS(\mathscr{F}^\bullet)\cap 0_X = \mathrm{supp}(\mathscr{F}^\bullet)$ where $\mathrm{supp}(\mathscr{F}^\bullet) = \overline{\{x\in X \mid \mathscr{H}^j(\mathscr{F}^\bullet)_x = 0\}}$. Take $\varphi = 0$, then $R\Gamma_{\{x\mid\varphi(x)\geq 0\}}(\mathscr{F}) = R\Gamma(\mathscr{F})$, and $R^j\Gamma(\mathscr{F})_x = \mathscr{H}^j(\mathscr{F}_x)$.

(3) Clearly $(x,p)\in SS(\mathscr{F}^\bullet)$ only depends on $\mathscr{F}^\bullet$ near x. In other words if $\mathscr{F}^\bullet = \mathscr{G}^\bullet$ in a neighbourhood V of X, then

$$(x,p)\in SS(\mathscr{F}^\bullet) \Leftrightarrow (x,p)\in SS(\mathscr{G}^\bullet)$$

(4) Assume for simplicity that we are dealing with a single sheaf $\mathscr{F}$, rather than with a complex. The above vanishing can be restated by asking that the natural restriction morphism

$$\lim_{U\ni x} H^j(U;\mathscr{F}) \longrightarrow \lim_{U\ni x} H^j(U\cap\{\varphi<0\};\mathscr{F})$$

is an isomorphism for any $j\in Z$. This implies in particular ($j=0$) that "sections" of $\mathscr{F}$ defined on $U\cap\{\varphi<0\}$ uniquely extend to a neighborhood of x.

Indeed, let $\mathscr{I}^\bullet$ be a complex of injective sheafs quasi-isomorphic to $\mathscr{F}^\bullet$. Then we have an exact sequence

$$0\to\Gamma_Z\mathscr{I}^\bullet\to\mathscr{I}^\bullet\to\mathscr{I}^\bullet_{X\setminus Z}\to 0$$

where the surjectivity of the last map follows from the flabbiness of injective sheafs. This yields the long exact sequence

$$\to H^j_Z(U,\mathscr{F}^\bullet)\to H^j(U,\mathscr{F}^\bullet)\to H^j(U\setminus Z,\mathscr{F}^\bullet)\to H^{j+1}_Z(U,\mathscr{F}^\bullet)\to\ldots$$

so that the vanishing of $H^j_Z(U,\mathscr{F}^\bullet)=R\Gamma^j_Z(\mathscr{F}^\bullet)$ for all j is equivalent to the fact that $H^j(U,\mathscr{F}^\bullet)\to H^j(U\setminus Z,\mathscr{F}^\bullet)$ is an isomorphism.

(5) Using proposition 8.22, one can reformulate the condition of the definition as

$$R\Gamma_{\{t\geq 0\}}(\mathbb{R},R\varphi_*(\mathscr{F}^\bullet))_{\{t=0\}}=0.$$

The main properties of $SS(\mathscr{F})$ are given by the following proposition

PROPOSITION 9.3. *The singular support has the following properties*

(1) $SS(\mathscr{F}^\bullet)$ is a conical subset of T^*X.

(2) If $\mathscr{F}^\bullet_1\to\mathscr{F}^\bullet_2\to\mathscr{F}^\bullet_3\xrightarrow{+1}\mathscr{F}^\bullet_1[1]$ is a distinguished triangle in $\mathscr{D}^b(X)$, then $SS(\mathscr{F}^\bullet_i)\subset SS(\mathscr{F}^\bullet_j)\cup SS(\mathscr{F}^\bullet_k)$ and $(SS(\mathscr{F}^\bullet_i)\setminus SS(\mathscr{F}^\bullet_j))\cup(SS(\mathscr{F}^\bullet_j)\setminus SS(\mathscr{F}^\bullet_i))\subset SS(\mathscr{F}^\bullet_k))$ for any i,j,k such that $\{i,j,k\}=\{1,2,3\}$.

(3) $SS(\mathscr{F}^\bullet)\subset\bigcup_j SS(\mathscr{H}^j(\mathscr{F}^\bullet))$.

PROOF. The first statement is obvious. For the second, we first notice that $SS(\mathscr{F}^\bullet)=SS(\mathscr{F}^\bullet[1])$. Now according to Proposition 8.28, $R\Gamma_Z$ maps a triangle as in (2) to a similar triangle, so that we get the following distinguished triangle $R\Gamma_Z(\mathscr{F}^\bullet_1)\to R\Gamma_Z(\mathscr{F}^\bullet_2)\to R\Gamma_Z(\mathscr{F}^\bullet_3)\xrightarrow{+1}R\Gamma_Z(\mathscr{F}^\bullet_1)[1]\to\ldots$

which yields

$\ldots\to R\Gamma_Z(\mathscr{F}^\bullet_1)_x\to R\Gamma_Z(\mathscr{F}^\bullet_2)_x\to R\Gamma_Z(\mathscr{F}^\bullet_3)_x\xrightarrow{+1}R\Gamma_Z(\mathscr{F}^\bullet_1)_x[1]\to\ldots$

and in particular, taking $Z=\{y\mid\psi(y)\geq 0\}$ where $\psi(x)=0$ and $d\psi(x)=p$, if two of the above vanish, so does the third. This implies the first part of (2). Moreover if one of the above cohomologies vanish, for example $R\Gamma_Z(\mathscr{F}^\bullet_1)_x\simeq 0$, then the other two are isomorphic, hence vanish simultaneously. Thus $(x,p)\notin SS(\mathscr{F}^\bullet_1)$ implies that $(x,p)\notin$

$SS(\mathscr{F}_2^\bullet)\Delta SS(\mathscr{F}_3^\bullet)$, where Δ is the symmetric difference. This implies the second part of (2).

Consider the canonical spectral sequence of Proposition 8.17 applied to $F = \Gamma_Z$. This yields a spectral sequence from $R^p\Gamma_Z(H^q(\mathscr{F}^\bullet))$, converging to $R^{p+q}\Gamma_Z(\mathscr{F}^\bullet)$. So if $\left(R^p\Gamma_Z(H^q(\mathscr{F}^\bullet))\right)_x$ vanishes we also have that $\left(R^{p+q}\Gamma_Z(\mathscr{F}^\bullet)\right)_x$ vanishes. □

Examples:

(1) An exact sequence of complexes of sheaves $0 \to \mathscr{F}_1^\bullet \to \mathscr{F}_2^\bullet \to \mathscr{F}_3^\bullet \to 0$ is a special case of a distinguished triangle (or rather its image in the derived category is a distinguished triangle). So in this case, we have the inclusions $SS(\mathscr{F}_i^\bullet) \subset SS(\mathscr{F}_j^\bullet) \cup SS(\mathscr{F}_k^\bullet)$ and $(SS(\mathscr{F}_i^\bullet) \setminus SS(\mathscr{F}_j^\bullet)) \cup (SS(\mathscr{F}_j^\bullet) \setminus SS(\mathscr{F}_i^\bullet)) \subset SS(\mathscr{F}_k^\bullet))$ for any i, j, k such that $\{i, j, k\} = \{1, 2, 3\}$.

(2) If $\mathscr{F}$ is the 0-sheaf that is $\mathscr{F}_x = 0$ for all x (hence $\mathscr{F}(U) = 0$ for all U), we have $SS(\mathscr{F}) = \emptyset$. Indeed, for all x and ψ, $R\Gamma_{\{\psi(x)\geq 0\}}(X, \mathscr{F})_x = 0$, hence the result. It is easy to check that this if $SS(\mathscr{F}) = \emptyset$, then $\mathscr{F}$ is equivalent to the zero sheaf (in $D^b(X)$), that is $\mathscr{F}$ is a complex of sheaves with exact stalks.

(3) Let k_X be the constant sheaf on X. Then $SS(k_X) = 0_X$. Indeed, consider the deRham resolution of k_X,

$$0 \to k_U \to \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \Omega^3 \xrightarrow{d} \dots$$

and apply Γ_Z. We obtain

$$0 \to \Gamma_Z\Omega^0 \xrightarrow{d} \Gamma_Z\Omega^1 \xrightarrow{d} \Gamma_Z\Omega^2 \xrightarrow{d} \Gamma_Z\Omega^3 \xrightarrow{d} \dots$$

where $\Gamma_Z\Omega^j$ is the set of j-forms vanishing on Z, and the cohomology of the above complex is obtained by considering closed forms, vanihing on Z, modulo differential of forms vanishing on Z.

But if Z is the set $\{y\varphi(y) \geq 0\}$ where $p = d\varphi(y) \neq 0$, a chart reduces this to the case where Z is a half space. Then, Poincaré's lemma tells us that any closed form on a small ball, vanishing on the half ball is the differential of a form vanishing on the half ball. Thus $SS(k_X)$ does not intersect the complement of 0_X, and since the support of k_X is X, we get $SS(k_X) = 0_X$.

Since SS is defined by a local property, $SS(F) = 0_X$ for any locally constant sheaf on X.

(4) We have

$$SS(\mathscr{F}^\bullet \oplus \mathscr{G}^\bullet) = SS(\mathscr{F}^\bullet) \cup SS(\mathscr{G}^\bullet)$$

since $R\Gamma_Z(\mathscr{F}^\bullet \oplus \mathscr{G}^\bullet) = R\Gamma_Z(\mathscr{F}^\bullet) \oplus R\Gamma_Z(\mathscr{G}^\bullet)$.

(5) Let U be an open set with smooth boundary, ∂U and k_U be the constant sheaf over U. Then $SS(k_U) = \{(x, p) \mid x \in U, p = 0, \text{ or } x \in \partial U, p = \lambda\nu(x), \lambda < 0\}$ where $\nu(x)$ is the exterior normal.

Indeed, in a point outside ∂U the sheaf is locally constant, and the statement is obvious. If x is a point in U, then the singular support over T_x^*X is computed as in the case of the constant sheaf (since k_U is locally isomorphic to the constant sheaf) and we get that $SS(k_U)\cap T^*_x X = 0_x$. For x in $X\setminus\overline{U}$, the same argument, but comparing to the zero sheaf, shows that $SS(k_U)\cap T^*_x X = \emptyset$. We must then consider the case $x\in\overline{U}\setminus U$.

Now let Ω^j_U be the sheaf defined by $\Omega^j_U(V)$ is the set of j-forms in $\Omega^j(U\cap V)$ supported in a closed subset of V. We then have an acyclic resolution

$$0\to k_U\to\Omega^0_U\xrightarrow{d}\Omega^1_U\xrightarrow{d}\Omega^2_U\xrightarrow{d}\Omega^3_U\xrightarrow{d}\dots$$

so that $R\Gamma_Z(k_U)$ is defined by

$$0\to\Gamma_Z\Omega^0_U\xrightarrow{d}\Gamma_Z\Omega^1_U\xrightarrow{d}\Gamma_Z\Omega^2_U\xrightarrow{d}\Gamma_Z\Omega^3_U\xrightarrow{d}\dots$$

where $Z=\{\varphi(x)\geq 0\}$ and $\Gamma_Z\Omega^j_U$ means the space of j-forms vanishing on the complement of Z. Now assume U and Z are half-spaces (respectively open and closed). Consider the closed forms in $(\Gamma_Z\Omega^k_U)$ modulo differentials of forms in $(\Gamma_Z\Omega^{k-1}_U)$. But any closed form vanishing in a sector is the differential of a form vanishing in the same sector (by the proof of Poincaré's lemma 3.4). There is an exception, of course, if the sector is empty and $k=0$, in which case the constant function is not exact. So at a point x of ∂U, $(R^j\Gamma_Z\Omega_U)_x=0$ unless $Z\cap U=\emptyset$, in which case $(R^0\Gamma_Z\Omega_U)_x=k_x=k$ and $d\varphi(x)$ is a positive multiple of the interior normal.

We may reduce to the above case by a chart of U, and using the locality of singular support.

(6) For U as above and $F=\overline{U}$, we have

$$SS(k_F)=\{(x,p)\mid x\in U, p=0,\text{ or } x\in\partial U, p=\lambda\nu(x),\lambda>0\}$$

This follows from (2) of the above proposition applied to the exact sequence (which is a special case of a distinguished triangle) $0\to k_{X\setminus F}\to k_X\to k_F\to 0$.

(7) Let k_Z be the constant sheaf on the closed submanifold Z. Then $SS(k_Z)=\nu_Z=\{(x,p)\mid x\in Z, p_{|T_xZ}=0\}$. This is the conormal bundle to Z.

EXERCICE 1. Compute $SS(\mathscr{F})$ for $\mathscr{F}$ an injective sheaf defined by $\mathscr{F}(U)=\{(s_x)_{x\in U}\mid s_x\in\mathbb{C}\}$. What about the sheaf $\mathscr{F}_W(U)=\{(s_x)_{x\in U}\mid s_x\in\mathbb{C}\text{ for }x\in W, s_x=0\text{ for }x\notin W\}$

Let us now see how our operations on sheaves act on $SS(\mathscr{F}^\bullet)$.

PROPOSITION 9.4. *Let $f: X\to Y$ be a proper map on* $\operatorname{supp}(\mathscr{F}^\bullet)$. *Then*

$$SS(Rf_*(\mathscr{F}^\bullet))\subset\pi_Y(T^*f)^{-1}(SS(\mathscr{F}^\bullet))=\Lambda_f\circ SS(\mathscr{F}^\bullet)$$

and this is an equality if f is a closed embedding. We also have

$$SS(Rf_!(\mathscr{F}^\bullet))\subset\pi_Y(T^*f)^{-1}(SS(\mathscr{F}^\bullet))=\Lambda_f\circ SS(\mathscr{F}^\bullet)$$

If f is any submersive map,

$$SS(f^{-1}\mathscr{G}^\bullet)=T^*f(\pi_Y^{-1}(SS(\mathscr{G}^\bullet))=\Lambda_f^{-1}\circ SS(\mathscr{G}^\bullet)$$

Note that the maps π_Y ad T^*f are defined as follows: $\pi_Y : T^*X \times \overline{T^*Y} \to T^*Y$ is the projection, while $T^*f : T^*X \to T^*Y$ is the map $(x,\xi) \mapsto (f(x), df(x)\xi)$.

For L a Lagrangian, $\pi_Y(T^*f)^{-1}(L)$ is obtained as follows: consider $T^*X \times \overline{T^*Y}$ and the Lagrangian $\Lambda_f = \{(x,\xi,y,\eta) \mid y = f(x), \xi = \eta \circ df(x)\}$. This is a conical Lagrangian submanifold. Let $K_L = L \times \overline{T^*Y}$. This is a coisotropic submanifold, $\mathscr{K}_L^\omega(x,\xi,y,\eta) = L \times \{(y,\eta)\}$, so $K_L/\mathscr{K}_L^\omega \simeq \overline{T^*Y}$, and $\pi_Y(T^*f)^{-1}(L) = (\Lambda_f \cap K_L)/\mathscr{K}_L^\omega$. In other words, if Λ_f is the Lagrangian relation associated to f, we have $\pi_Y(T^*f)^{-1}(L) = \Lambda_f(L)$.

PROOF. Let ψ be a smooth function on Y such that $\psi(f(x)) = 0$ and $p = d\psi(f(x))df(x)$. Assume we have $(x,p) \notin SS(\mathscr{F}^\bullet)$ for all $x \in f^{-1}(y)$. Then we have

$$R\Gamma_{\{\psi\circ f \geq 0\}}(\mathscr{F}^\bullet)_{|f^{-1}(y)} = 0$$

But

$$R\Gamma_{\{\psi \geq 0\}}(Rf_*(\mathscr{F}^\bullet))_y = Rf_*(R\Gamma_{\{\psi\circ f\geq 0\}}(\mathscr{F}^\bullet))_y = R\Gamma(f^{-1}(y), R\Gamma_{\{\psi\circ f\geq 0\}}(\mathscr{F}^\bullet)) = 0$$

Here the first equality follows from the fact that $\Gamma_Z \circ f_* = f_* \circ \Gamma_{f^{-1}(Z)}$ so the same holds for the corresponding derived functors. The second equality follows from the fact that if f is proper on $\mathrm{supp}(\mathscr{F}^\bullet)$, we have $(f_*\mathscr{F}^\bullet)_y = \Gamma(f^{-1}(y), \mathscr{F}^\bullet_{|f^{-1}(y)})$.

Indeed, let $j : Z \to X$ be the inclusion of a closed set. We define $\Gamma(Z,\mathscr{F}^\bullet)$ as $\Gamma(Z, j^{-1}(\mathscr{F}^\bullet))$. We also have $\Gamma(Z,\mathscr{F}^\bullet) = \lim_{Z\subset U}\Gamma(U,\mathscr{F}^\bullet)$. Then $(f_*\mathscr{F}^\bullet)(U) = \mathscr{F}^\bullet(f^{-1}(U))$, so $(f_*\mathscr{F}^\bullet)_y = \lim_{U\ni y}\mathscr{F}^\bullet(f^{-1}(U))$ and since f is proper, $f^{-1}(U)$ is a cofinal family of neighbourhoods of $f^{-1}(y)$. This implies $(f_*\mathscr{F}^\bullet)_y \overset{def}{=} \Gamma(y, f_*\mathscr{F}^\bullet) = \Gamma(f^{-1}(y),\mathscr{F}^\bullet)$, hence taking the derived functors $(Rf_*\mathscr{F}^\bullet)_y = R\Gamma(y, Rf_*\mathscr{F}^\bullet) = R\Gamma(f^{-1}(y),\mathscr{F}^\bullet)$. Clearly if for all $x \in f^{-1}(y)$ we have $R\Gamma(x,\mathscr{F}^\bullet) = 0$, whe will have $(Rf_*\mathscr{F}^\bullet)_y = 0$. We thus proved that $(x, p\circ df(x)) \notin SS(\mathscr{F}^\bullet)$ implies $(f(x),p) \notin SS(Rf_*(\mathscr{F}^\bullet))$. If f is a closed embedding, $f^{-1}(y)$ is a discrete set of points, $R\Gamma(f^{-1}(y),\mathscr{F}^\bullet)$ vanishes if and only if for all x in $f^{-1}(y)$, the stalks $R\Gamma(\mathscr{F}^\bullet)_x$ vanish. □

The following continuity result is sometiems useful. Let $(\mathscr{F}^\bullet_\nu)_{\nu\geq 1}$ be a directed system of sheaves, i.e. there are maps $f_{\mu,\nu} : \mathscr{F}^\bullet_\mu \to \mathscr{F}^\bullet_\nu$ satisfying the obvious compatibility conditions, and let $\mathscr{F}^\bullet = \lim_{\nu\to+\infty}\mathscr{F}^\bullet_\nu$ (we will assume the limit is a bounded complex, so the $\mathscr{F}^\bullet_\nu$ are uniformly bounded.

Now let S_ν be a sequence of closed sets in a metric space M. Then $\lim_{\nu\to+\infty}S_\nu = S$ means that each point x in S is the accumulation point of some sequence of points x_ν) in S_ν. With these notions at hand, we may now state

LEMMA 9.5. *Let $(\mathscr{F}^\bullet_\nu)_{\nu\geq 1}$ be a directed system of sheaves. Then we have*

$$SS(\lim_{\nu\to+\infty}\mathscr{F}^\bullet_\nu) \subset \lim_{\nu\to+\infty}SS(\mathscr{F}^\bullet_\nu)$$

PROOF. Indeed, we must compute $R\Gamma_Z(\lim_{\nu\to+\infty}\mathscr{F}^\bullet_\nu)_x = \lim_{\nu\to+\infty}R\Gamma_Z(\mathscr{F}^\bullet_\nu)_x$ the equality follows from the fact that the direct limit is an exact functor, and thus commutes with Γ_Z (since it commutes with $\Gamma(U,\bullet)$). Set $Z = \{y \mid \psi(y) \geq 0\}$, where ψ is a function such that $\psi(x) = 0, d\psi(x) = p$. As a result $(x_0,p_0) \notin SS(\lim_{\nu\to+\infty}\mathscr{F}^\bullet_\nu)$ if and only $R\Gamma_Z(\lim_{\nu\to+\infty}\mathscr{F}_\nu^\bullet)_x=0$ for all (x,p) in a neighbourhood of (x_0,p_0), and this implies our statement. $\square$

1.2. The sheaf associated to a Generating function. Let $S(x,\xi)$ be a GFQI for a Lagrangian L, that is
$L=\{(x,\frac{\partial}{\partial x}S(x,\xi))\mid \frac{\partial}{\partial\xi}S(x,\xi)=0\}$. We set $\Sigma_S=\{(x,\xi)\mid \frac{\partial}{\partial\xi}S(x,\xi)=0\}$, $\widehat{\Sigma}_S=\{(x,\xi,\lambda)\mid \frac{\partial S}{\partial\xi}(x,\xi)=0,\lambda=S(x,\xi)\}$, and $\widehat{L}=\{(x,\tau p,\lambda,\tau)\mid p=\frac{\partial S}{\partial x}(x,\xi),\frac{\partial S}{\partial\xi}(x,\xi)=0,\lambda=S(x,\xi)\}$. We moreover assume the sets $\pi^{-1}(x,\lambda)\cap\widehat{\Sigma}_S$ are discrete sets.

Set $U_S=\{(x,\xi,\lambda)\mid S(x,\xi)\le\lambda\}\subset M\times\mathbb{R}^q\times\mathbb{R}$. Let $\mathscr{F}_S=R\pi_*(k_{U_S})$, where π is the projection $\pi: M\times\mathbb{R}^q\times\mathbb{R}\to M\times\mathbb{R}$.

We claim that $SS(\mathscr{F}_S)=\widehat{L}$. It is easy to prove that $SS(\mathscr{F}_S)\subset\widehat{L}$, since $\Lambda_\pi\circ SS(k_S)=\widehat{L}$. Indeed, the correspondence Λ_π corresponds to symplectic reduction by $p_\xi=0$, i.e. sends A to $\Lambda_\pi\circ A=A\cap\{p_\xi=0\}/(\xi)$.

To prove equality, we use the formula from the proof of the above proposition

$$R\Gamma_{\{\psi\ge0\}}(R\pi_*(k_{U_S}))_{(x,\lambda)}=R\pi_*(R\Gamma_{\{\psi\circ\pi\ge0\}}(k_{U_S}))_{(x,\lambda)}=$$
$$R\Gamma(\pi^{-1}(x,\lambda),R\Gamma_{\{\psi\circ\pi\ge0\}}(k_{U_S}))=0$$

But $R\Gamma_{\{\psi\circ\pi\ge0\}}(k_{U_S})_{(x,\xi,\lambda)}$ is non zero if and only if $(x,\xi,\lambda,d\psi(\pi(x,\xi,\lambda))d\pi(x,\xi,\lambda))\in SS(k_{U_S})$ that is $(x,d\psi(x,\lambda))\in\widehat{L}$. This is a discrete set by assumption (for (x,λ) fixed), thus $R\Gamma_{\{\psi\circ\pi\ge0\}}(k_{U_S})_{|\pi^{-1}(x,\lambda)}$ has vanishing stalk except over the discrete set of points of $\widehat{\Sigma}_S\cap\pi^{-1}(x,\lambda)$. Note that such a sheaf is zero if and only if each of the stalks is zero. So we have that

$$R\Gamma(\pi^{-1}(x,\lambda),R\Gamma_{\{\psi\circ\pi\ge0\}}(k_{U_S}))=0$$

if and only if for all $(x,\xi,\lambda)\in M\times\mathbb{R}^q\times\mathbb{R}$ we have $(x,\tau p,\lambda,\tau)\in\widehat{L}=\Lambda_\pi\circ SS(k_{U_S})$.

REMARKS 9.6. (1) With the notations of the previous remark, note that if $\lim_{\nu\to+\infty}S_\nu=S$, where the limit is for the uniform C^0 convergence, we have $\lim_{\nu\to+\infty}U_{S_\nu}=U_S$, and thus $\lim_{\nu\to+\infty}(k_{S_\nu})=k_S$ (where we wrote k_S for k_{U_S}). Thus $SS(k_S)\subset\lim_{\nu\to+\infty}SS(k_{S_\nu})$.

1.3. Uniqueness of the quantization sheaf of the zero section. The following plays the role of the uniqueness result for GFQI (see Theorem 5.18).

PROPOSITION 9.7. *Let $\mathscr{F}^\bullet$ in $D^b(X)$, be such that $SS(\mathscr{F}^\bullet)\subset 0_X$. Then $\mathscr{F}^\bullet$ is equivalent in $D^b(X)$ to a locally constant sheaf.*

PROOF. We start by proving the proposition for the case $X=\mathbb{R}$ (see [**K-S**] page 118, proposition 2.7.2 and lemma 2.7.3). First, since the support of $\Gamma_Z(\mathscr{F})$ is contained in Z, we have that $\Gamma_{\{t\ge s\}}\mathscr{F}(]-\infty,s+\varepsilon[)=\Gamma_{\{t\ge s\}}\mathscr{F}(]s-\varepsilon,s+\varepsilon[)$. Moreover this last space is the kernel of the map

$$\mathscr{F}(]-\infty,s+\varepsilon[)\to\mathscr{F}(]-\infty,s+\varepsilon[\setminus\{t\ge s\})=\mathscr{F}(]-\infty,s[)$$

so we have an exact sequence

$$0 \to \Gamma_{\{t\geq s\}}\mathscr{F}(]s-\varepsilon, s+\varepsilon[) \to \mathscr{F}(]-\infty, s+\varepsilon[) \to \mathscr{F}(]-\infty, s[)$$

which in the case of a flabby (and in particular for an injective) sheaf, $\mathscr{I}$ extends to

$$0 \to \Gamma_{\{t\geq s\}}\mathscr{I}(]s-\varepsilon, s+\varepsilon[) \to \mathscr{I}(]-\infty, s+\varepsilon[) \to \mathscr{I}(]-\infty, s[) \to 0$$

since the last map is surjective by flabbiness.

Thus, given an injective resolution $0 \to \mathscr{F} \to \mathscr{I}^0 \to \mathscr{I}^1 \to \mathscr{I}^2 \to \dots$ we get a sequence

$$0 \to \Gamma_{\{t\geq s\}}\mathscr{I}^\bullet(]s-\varepsilon, s+\varepsilon[) \to \mathscr{I}^\bullet(]-\infty, s+\varepsilon[) \to \mathscr{I}^\bullet(]-\infty, s[) \to 0$$

By definition, the complex $\Gamma_{\{t\geq s\}}\mathscr{I}^\bullet(]s-\varepsilon, s+\varepsilon[)$ represents $R\Gamma_{\{t\geq s\}}\mathscr{F}(]s-\varepsilon, s+\varepsilon[)$ which converges as ε goes to zero to $R\Gamma_{\{t\geq s\}}(\mathscr{F})_s$, which vanishes by assumption. Thus using the exactness of the direct limit, and this exact sequence we get an isomorphism

$$\varinjlim_{\varepsilon\to 0} R\Gamma(]-\infty, s+\varepsilon[, \mathscr{F}) \to R\Gamma(]-\infty, s[, \mathscr{F}) \tag{9.1}$$

We claim that this implies that the map

$$R\Gamma(]-\infty, s_1[, \mathscr{F}) \to R\Gamma(]-\infty, s_0[, \mathscr{F})$$

is an isomorphism for any $s_1 > s_0$. Indeed, the map 9.1 must be surjective for any ε small enough, thus for s_1 close enough to s_0, the above map is onto. On the other hand if the map was not injective, consider $u \in R\Gamma(]-\infty, s_1[, \mathscr{F})$, and s_0 be the least upper bound of the set of real numbers such that the map

$$R\Gamma(]-\infty, s_1[, \mathscr{F}) \to R\Gamma(]-\infty, s[, \mathscr{F})$$

sends u to 0. Consider now the fact that the map

$$\varinjlim_{\varepsilon\to 0} R\Gamma(]-\infty, s_0+\varepsilon[, \mathscr{F}) \to R\Gamma(]-\infty, s_0[, \mathscr{F})$$

is injective. If the image of u vanishes in $R\Gamma(]-\infty, s_0[, \mathscr{F})$, this implies that u already vanishes[1] $R\Gamma(]-\infty, s_0+\varepsilon[, \mathscr{F})$, but this contradicts the defintion of s_0. We thus proved that there is an element $u \neq 0$ in $R\Gamma(]-\infty, s_1[, \mathscr{F})$ vanishing in all $R\Gamma(]-\infty, s_1-\varepsilon[, \mathscr{F})$ for $\varepsilon > 0$.

To complete the proof, it is sufficient to prove that such a u vanishes in $R\Gamma(]-\infty, s_1[, \mathscr{F})$ which would follow from the definition of s_0 and the equality

$$R\Gamma(]-\infty, s_1[, \mathscr{F}) = \varprojlim_{\varepsilon\to 0} R\Gamma(]-\infty, s_1-\varepsilon[, \mathscr{F}).$$

which holds for any complex of sheaves[2].

We thus proved that $R\Gamma(]-\infty, s[, \mathscr{F})$ is constant. Now in the general case, we have to prove that if $\mathscr{F}^\bullet$ is in $D^b(X)$, it is locally constant. Let $B(x_0, R)$ be a small ball in

[1] Since an element $(a_n)_{n\geq 1}$ in the direct limit $\lim_n A_n$ is zero if and only if a_n is zero for n large enough.

[2] Because $\bigcup_{\bar{V}\subset U} V = U$ implies $\lim_{\bar{V}\subset U}\mathscr{F}(V) = \mathscr{F}(U)$.

X, that is of radius smaller than the injectivity radius of the manifold. Consider the function $r(x) = d(x, x_0)$. Then $SS(r_*\mathscr{F}) \subset \Lambda_r \circ SS(\mathscr{F}^\bullet)$, but since $SS(\mathscr{F}) \subset 0_X$, and r has no positive critical value, we get $\Lambda_r \circ 0_X \subset 0_{\mathbb{R}}$, so that $R\Gamma(]-\infty, R[, Rf_*(\mathscr{F}^\bullet)) \longrightarrow R\Gamma(]-\infty, \varepsilon[, Rf_*(\mathscr{F}^\bullet))$ is an isomorphism. In other words, $R\Gamma(B(x_0,R),\mathscr{F}^\bullet) \longrightarrow R\Gamma(B(x_0,\varepsilon),\mathscr{F}^\bullet)$ is an isomorphism, and by going to the limit as ε goes to zero, we get $R\Gamma(B(x_0,R),\mathscr{F}^\bullet) \simeq R\Gamma(\mathscr{F}^\bullet)_{x_0}$, hence $R\Gamma(\mathscr{F}^\bullet)$ is locally constant, i.e. $\mathscr{F}^\bullet$ is locally constant in $D^b(X)$.

□

REMARK 9.8. One should not imagine that sheafs on contractible spaces have vanishing cohomology.

EXERCICE 2. Compute the cohomology of the skyscraper sheaf at 0 in $\mathbb{R}$. Then compute its singular support.

2. The sheaf theoretic Morse lemma and applications

The last paragraph in the proof of Proposition 9.7 can be generalized as follows.

PROPOSITION 9.9. *Let us consider a function $f : M \to \mathbb{R}$ proper on* supp$(\mathscr{F})$. *Assume that $\{(x, df(x)) \mid x \in f^{-1}([a,b])\} \cap SS(\mathscr{F})$ is empty. Then for $t \in [a,b]$ the natural maps $R\Gamma(\{x \mid f(x) \le t\},\mathscr{F}) \longrightarrow R\Gamma(\{x \mid f(x) \le a\},\mathscr{F})$ are isomorphisms. In particular $H^*(f^{-1}(a),\mathscr{F}) \simeq H^*(f^{-1}(b),\mathscr{F})$.*

PROOF. The proposition is equivalent to proving that the $R\Gamma(]-\infty,t], Rf_*(\mathscr{F}))$ are all canonically isomorphic for $t \in [a,b]$. But this follows from Proposition 9.7, since $SS(Rf_*\mathscr{F}) \cap T^*([a,b]) \subset \Lambda_f \circ SS(\mathscr{F}) = \{(x, \tau df(x)) \mid x \in f^{-1}([a,b]), \tau \in \mathbb{R}_+\} \cap SS(\mathscr{F})$ and this is contained in the zero section by our assumption. □

Note that the standard Morse lemma corresponds to the case $\mathscr{F} = k_M$.

LEMMA 9.10. *Let φ be a smooth function on X such that* 0 *is a regular level. Let $x \in \varphi^{-1}(0)$ and assume there is a neighbourhood U of x such that*

$$R\Gamma(U \cap \{\varphi(z) \le t\},\mathscr{F}^\bullet) \longrightarrow R\Gamma(U \cap \{\varphi(z) \le 0\},\mathscr{F}^\bullet)$$

is an isomorphism for all positive t small enough. Then $R\Gamma_{\{\varphi \ge 0\}}(\mathscr{F}^\bullet)_x = 0$.

PROOF. Again, we have $R\Gamma(]-\infty,t[, R\varphi_*(\mathscr{F}^\bullet)) \to R\Gamma(]-\infty,0[, R\varphi_*(\mathscr{F}^\bullet))$ is an isomorphism. So if $\mathscr{G}^\bullet$ is a sheaf over $\mathbb{R}$, the fact that $R\Gamma(]-\infty,t[,\mathscr{G}^\bullet) \to R\Gamma(]-\infty,0[,\mathscr{G}^\bullet)$ is an isomorphism implies $R\Gamma_{\{t\ge 0\}}(\mathscr{G}^\bullet)_{t=0} = 0$ since for $\mathscr{I}^\bullet$ an injective resolution of $\mathscr{G}^\bullet$ we have

$$0 \to \Gamma_{\{t \ge s\}}\mathscr{I}^\bullet(]s-\varepsilon, s+\varepsilon[) \to \mathscr{I}^\bullet(]-\infty, s+\varepsilon[) \to \mathscr{I}^\bullet(]-\infty, s[) \to 0$$

□

Let C, D be two conic subsets in T^*M.

DEFINITION 9.11. Let C, D be two closed cones. Then $C\hat{+}D$ is defined as follows: $(z,\zeta) \in C\hat{+}D$ if and only if there are sequences $(x_n,\xi_n),(y_n,\eta_n)$ such that $\lim_n x_n = \lim_n y_n = z$, $\lim_n(\xi_n+\eta_n) = \zeta$ and $\lim_n |x_n - y_n||\xi_n| = 0$. We write $C\hat{+}D = (C+D) + C\underset{\infty}{\hat{+}}D$

PROPOSITION 9.12. *We have*

$$SS(\mathscr{F} \boxtimes^L \mathscr{G}) \subset SS(\mathscr{F}) \times SS(\mathscr{G})$$

$$SS(\mathscr{F} \otimes^L \mathscr{G}) \subset SS(\mathscr{F}) \hat{+} SS(\mathscr{G})$$

PROOF. Again, we limit ourselves to the situation of complexes of $\mathbb{C}$-modules sheaves, so that $\boxtimes^L, \otimes^L, RHom$ coincide with $\boxtimes, \otimes, \mathscr{H}om$, since vector spaces are always projective and injective. Note that the second equality follows from the first, since if $d : X \to X \times X$ is the diagonal map, we have $\mathscr{F} \otimes \mathscr{G} = d^{-1}(\mathscr{F} \boxtimes \mathscr{G})$, and

$$SS(d^{-1}\mathscr{F}) = (\Lambda_d^{-1})^{\#}(SS(\mathscr{F}) \times SS(\mathscr{G}))$$

but

$$\Lambda_d^{-1} = \{(x_1,\xi_1,x_2,\xi_2,x_3,\xi_3) \mid x_1 = x_2 = x_3, \xi_3 = \xi_1 + \xi_2\}$$

therefore $(\Lambda_d^{-1})^{\#}(SS(\mathscr{F}) \times SS(\mathscr{G}))$ is equal to $SS(\mathscr{F}) \hat{+} SS(\mathscr{G})$.

Assume now $(x_0,\xi_0) \notin SS(\mathscr{F})$. This implies that if $U \subset X$ is an smooth codimension zero submanifold, and $U_t = \varphi^{-1}(]-\infty, t])$, with $x_0 \in U_0$ and $d\varphi(x_0) = \xi_0$, then $R\Gamma(U_t, \mathscr{F}^\bullet) \longrightarrow R\Gamma(U_0, \mathscr{F}^\bullet)$ is an isomorphism, and also $R\Gamma(U_t \times V, \mathscr{F}^\bullet) \longrightarrow R\Gamma(U_0 \times V, \mathscr{F}^\bullet)$. Now let $H_t = \{(x,y) \mid \psi(x,y) \geq t\}$ where $d\psi(x_0,y_0) = (\xi_0,\eta_0)$,

$$\begin{tikzcd}
R\Gamma(H_0,\mathscr{F}^\bullet) \arrow[r] & R\Gamma(U_0 \times V, \mathscr{F}^\bullet \boxtimes \mathscr{G}^\bullet) & R\Gamma(U_\varepsilon \times V, \mathscr{F}^\bullet \boxtimes \mathscr{G}^\bullet) \arrow[l, "\simeq"'] \arrow[d] \\
& R\Gamma(U_{\eta/2} \times V, \mathscr{F}^\bullet \boxtimes \mathscr{G}^\bullet) \arrow[ul] \arrow[u, "\simeq"] & R\Gamma(H_\varepsilon, \mathscr{F}^\bullet \boxtimes \mathscr{G}^\bullet) \arrow[l]
\end{tikzcd}$$

Here U_η is determined so that $U_\eta \times V \subset H_\varepsilon$ and $U_\varepsilon \times V \subset H_\varepsilon$. This clearly implies that $R\Gamma(H_\varepsilon, \mathscr{F}^\bullet)_x \longrightarrow R\Gamma(H_0, \mathscr{F}^\bullet)_x$ is an isomorphism for ε small enough, for x close to x_0.

LEMMA 9.13 ([**K-S**], 2.6.6, p. 112, [**Iv**], p.320). *Let $f : X \to Y$ be a continuous map, and $\mathscr{F} \in D^b(X), G \in D^b(Y)$. Then*

$$Rf_!(\mathscr{F}^\bullet \otimes^L f^{-1}\mathscr{G}^\bullet) = Rf_!(\mathscr{F}^\bullet) \otimes^L \mathscr{G}^\bullet$$

PROOF. Again, we do not consider the derived tensor products, since we are dealing with $\mathbb{C}$-vector spaces. Then, there is a natural isomorphism from

$$f_!(F) \otimes G \simeq f_!(F \otimes f^{-1}(G))$$

□

□

LEMMA 9.14 (Base change theorem ([**Iv**], p. 322). *Let us consider the following* **cartesian square** *of maps,*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"] & B \arrow[d, "g"] \\ C \arrow[r, "v"] & D \end{tikzcd}$$

that is the square is commutative, and A is isomorphic to the fiber product $B\times_D C$. Then $Ru_! \circ f^{-1} = v^{-1} \circ Rg_!$

2.1. Resolutions of constant sheafs, the DeRham and Morse complexes. Let $W(f) = \{(x,\lambda) \mid f(x) \le \lambda\}$. We consider k_f the constant sheaf over $W(f)$, and we saw we have a quasi-isomorphism 3.1, between k_f and $\Omega_f^\bullet$ the set of differential forms on $W(f)$. Moreover according to LePeutrec-Nier-Viterbo ([**LePeutrec-Nier-V**], there is a quasi-isomorphism from $\Omega_f^\bullet$ to $BM_f^\bullet$ the Barannikov-Morse complex of f.

3. Quantization of symplectic maps

We assume in this section that X, Y, Z are manifolds. Now we want to quantize symplectic maps in T^*X, that is to a homogeneous Hamiltonian symplectomorphism $\Phi : T^*X \to T^*Y$ we want to associate a map $\widehat{\Phi} : D^b(X) \to D^b(Y)$. There are (at least) two posibilites to do that, and one should not be surprised. In microlocal analysis, there are several possible quantizations from symbols to operators: pseudodifferential, Weyl, coherent state, etc...

Define $q_X : X\times Y \to X$ (resp. $q_Y : X\times Y \to Y$) and $q_{XY} : X\times Y\times Z \to X\times Y$ (resp. $q_{XZ} : X\times Y\times Z \to X\times Z$, $q_{XY} : X\times Y\times Z \to Y\times Z$) be the projections.

DEFINITION 9.15. Let $\mathscr{K} \in D^b(X\times Y)$. We then define the following operators: for $\mathscr{F} \in D^b(X)$ and $\mathscr{G} \in D^b(Y)$ define

$$\Psi_{\mathscr{K}}(\mathscr{F}) = (Rq_{Y*})(RHom(\mathscr{K}, q_X^!(\mathscr{F}))$$

$$\Phi_{\mathscr{K}}(\mathscr{G}) = (Rq_{X!})(\mathscr{K} \otimes^L q_Y^{-1}(\mathscr{G}))$$

Then $\Psi_{\mathscr{K}}, \Phi_{\mathscr{K}}$ are operators from $\mathscr{D}^b(X)$ to $D^b(Y)$ and $\mathscr{D}^b(Y)$ to $D^b(X)$ respectively.

REMARK 9.16. (1) The method is reminiscent of the definition of operators on the space of C^k functions using kernels.

(2) For the sake of completeness, we have used the derived functor language in all cases. However, for sheafs in the category of finite dimensional vector spaces, $R\mathscr{H}om = \mathscr{H}om$ and $\otimes^L = \otimes$. Also, if the projections are proper, i.e. if X, Y are compact, $R(q_{X!}) = R(q_{X*})$

(3) In the category of coherent sheaves over a projective algebraic manifold, the above definition extends to the Fourier-Mukai transform. Indeed if $\mathscr{K} \in D^b_{Coh}(X\times$

Y) is an element in the derived category of the coherent sheafs on the product of two algebraic varieties, we define the Fourier-Mukai transform from $D^b_{Coh}(X)$ to $D^b_{Coh}(Y)$ as

$$\Phi_{\mathcal{K}}(\mathcal{G}) = (Rq_{X*})(K \otimes^L q_Y^{-1}(\mathcal{G}))$$

Consider Mirror symmetry as an equivalence of categories $\mathcal{M} : Fuk(T^*X) \longrightarrow D^b(X)$ sending $\mathrm{Mor}(L_1, L_2) = FH^*(L_1, L_2)$ to $\mathrm{Mor}_{D^b}(\mathcal{M}(L_1), \mathcal{M}(L_1))$. Moreover, let us consider the functor $SS : D^b(X) \longrightarrow Fuk(T^*X)$. This should send the element $\Phi_{\mathcal{K}} \in \mathrm{Mor}(D^b(X), D^b(Y))$ to the Lagrangian correspondence, $\Lambda_{SS(\mathcal{K})} : T^*X \longrightarrow T^*Y$. Vice-versa any such Lagrangian correspondence can be quantized, for example for each exact embedded Lagrangian L we can find $\mathcal{F}$ such that $SS(\mathcal{F}) = L$. We shall see this can be done using Floer homology. Can one use other methods, for example the theory of Fourier integral operators: ?

(4) According to [**K-S**] proposition 7.1.8, the two functors $\Phi_{\mathcal{K}}, \Psi_{\mathcal{K}}$ are adoint functors.

The sheaf $\mathcal{K}$ is called the **kernel** of the transform (or functor).We say that $\mathcal{K} \in D^b(X \times Y)$ is a **good kernel** if the map

$$SS(K) \longrightarrow T^*X$$

is proper. We denote by $N(X, Y)$ the set of good kernels. Note that any sheaf $\mathcal{F} \in D^b(X)$ can be considered as a kernel in $D^b(X) = D^b(X \times \{pt\})$, and it automatically belongs to $N(X, \{pt\})$, because $SS(\mathcal{F}) \to T^*X$ is trivially proper. We shall see that transforms defined by kernels can be composed, and, in the case of good kernels, act on the singular support in the way we expect. Let X, Y, Z three manifolds, and q_X (resp. q_Y, q_Z) be the projection of $X \times Y \times Z$ on X (resp. Y, Z) and q_{XY} (resp. q_{YZ}, q_{XZ}) be the projections on $X \times Y$ (resp. $Y \times Z, X \times Z$). Similarly π_{XY} etc... are the projections $T^*X \times T^*Y \times T^*Z \to T^*X \times T^*Y$.

We may now state

PROPOSITION 9.17. *Let $\mathcal{K}_1 \in D^b(X \times Y)$ and $\mathcal{K}_2 \in D^b(Y \times Z)$. Set*

$$\mathcal{K} = (Rq_{XZ})_!(q_{XY}^{-1}(\mathcal{K}_1) \otimes^L q_{YZ}^{-1}(\mathcal{K}_2))$$

Then $\mathcal{K} \in D^b(X \times Z)$, and $\Psi_{\mathcal{K}} = \Psi_{\mathcal{K}_1} \circ \Psi_{\mathcal{K}_2}$ and $\Phi_{\mathcal{K}} = \Phi_{\mathcal{K}_1} \circ \Phi_{\mathcal{K}_2}$. We will denote $\mathcal{K} = \mathcal{K}_1 \circ \mathcal{K}_2$.

PROOF. Consider the following diagram

$$\begin{tikzcd}
 & X \times Y \times Z \arrow[dl, "q_{XY}"'] \arrow[d, "q_{XZ}"] \arrow[dr, "q_{YZ}"] & \\
X \times Y \arrow[d, "q_X^{XY}"'] \arrow[dr, "q_Y^{XY}"] & X \times Z \arrow[dl, "q_X^{XZ}"'] \arrow[dr, "q_Z^{XZ}"] & Y \times Z \arrow[dl, "q_Y^{YZ}"'] \arrow[d, "q_Z^{YZ}"] \\
X & Y & Z
\end{tikzcd}$$

Let $\mathscr{G} \in D^b(Z)$. We first claim that

$$(\star) \qquad (Rq_X^{XY})_!(\mathscr{K}_1 \otimes (q_Y^{XY})^{-1}((Rq_Y^{YZ})_!(\mathscr{K}_2 \otimes (q_Z^{YZ})^{-1}(\mathscr{G})))) = (Rq_X)_!(q_{XY}^{-1}(\mathscr{K}_1) \otimes q_{YZ}^{-1}(\mathscr{K}_2) \otimes q_Z^{-1}(\mathscr{G}))$$

The cartesian square with vertices $X \times Y \times Z, X \times Y, Y \times Z, Y$ and lemma 9.14 yields an isomorphism between the image of $\mathscr{K}_2 \otimes (q_Z^{YZ})^{-1}(\mathscr{G})$ by $(Rq_{XY})_! q_{YZ}^{-1}$ and its image by $(q_Y^{XY})^{-1}(Rq_Y^{YZ})_!$. The first image is

$$(Rq_{XY})_! q_{YZ}^{-1}(\mathscr{K}_2 \otimes (q_Z^{YZ})^{-1}(\mathscr{G})) = (Rq_{XY})_!(q_{YZ}^{-1}(\mathscr{K}_2) \otimes q_{YZ}^{-1} \circ (q_Z^{YZ})^{-1}(\mathscr{G})) = (Rq_{XY})_!(q_{YZ}^{-1}(\mathscr{K}_2) \otimes q_Z^{-1}(\mathscr{G}))$$

using for the last equality that $q_Z^{YZ} \circ q_{YZ} = q_Z$.

This is thus equal to

$$(q_Y^{XY})^{-1}(Rq_Y^{YZ})_!(\mathscr{K}_2 \otimes (q_Z^{YZ})^{-1}(\mathscr{G})))$$

Apply now $\otimes\mathscr{K}_2$ and then $(Rq_X^{XY})_!$, we get

$$(Rq_X^{XY})_!(\mathscr{K}_1 \otimes (q_Y^{XY})^{-1}(Rq_Y^{YZ})_!(\mathscr{K}_2 \otimes (q_Z^{YZ})^{-1}(\mathscr{G})))$$

for the first term and

$$(Rq_X^{XY})_!(\mathscr{K}_1 \otimes (Rq_{XY})_!(q_{YZ}^{-1}(\mathscr{K}_2) \otimes q_Z^{-1}(\mathscr{G}))))$$

for the second term.

Using lemma 9.13 applied to $f = q_{XY}$, we get

$$\mathscr{F} \otimes (Rq_{XY})!\mathscr{G} = (Rq_{XY})!(q_{XY}^{-1}(\mathscr{F}) \otimes \mathscr{G})$$

hence applying $(Rq_X^{XY})_!$ and using the composition formula $(Rq_X^{XY})_! \circ (Rq_{XY})_! = (Rq_X)_!$, we get

$$(Rq_X^{XY})_!(\mathscr{K}_1 \otimes (Rq_{XY})_!(q_{YZ}^{-1}(\mathscr{K}_2) \otimes q_Z^{-1}(\mathscr{G})) = (Rq_X)_!(q_{XY}^{-1}(\mathscr{K}_1) \otimes q_{YZ}^{-1}(\mathscr{K}_2) \otimes q_Z^{-1}(\mathscr{G})))$$

This proves our equality.

We must prove the right hand side above is equal to

$$(Rq_X^{XZ})!((Rq_{XZ})_!(q_{XY}^{-1}(\mathscr{K}_1)\otimes^L q_{YZ}^{-1}(\mathscr{K}_2))\otimes(q_Z^{XZ})^{-1}(\mathscr{G}))$$

But

$$(Rq_{XZ})_!(\mathscr{F}\otimes(q_{XZ})^{-1}(\mathscr{G}))=(Rq_{XZ})_!(\mathscr{F})\otimes\mathscr{G}$$

and $(Rq_X^{XZ})_!\circ(Rq_{XZ})_!=(Rq_X)_!$, so

$$(Rq_X^{XZ})!((Rq_{XZ})_!(q_{XY}^{-1}(\mathscr{K}_1)\otimes^L q_{YZ}^{-1}(\mathscr{K}_2))\otimes(q_Z^{XZ})^{-1}(\mathscr{G}))=$$
$$(Rq_X^{XZ})!(Rq_{XZ})_!(q_{XY}^{-1}(\mathscr{K}_1)\otimes^L q_{YZ}^{-1}(\mathscr{K}_2)\otimes q_{XZ}^{-1}(q_Z^{XZ})^{-1}(\mathscr{G}))=$$
$$(Rq_X)_!(q_{XY}^{-1}(\mathscr{K}_1)\otimes^L q_{YZ}^{-1}(\mathscr{K}_2))\otimes q_Z^{-1}(\mathscr{G}))$$

□

The next proposition tells us that Φ_K,Ψ_K act as expected on $SS(\mathscr{F})$.

PROPOSITION 9.18 ([**K-S**], Proposition 7.12). *We assume $\mathscr{K}\in D^b(X\times Y)$ and $\mathscr{L}\in D^b(Y\times Z)$ are good kernels. Then $\mathscr{K}\circ\mathscr{L}$ is a good kernel and*

$$SS(\mathscr{K}\circ\mathscr{L})=SS(\mathscr{K})\circ SS(\mathscr{L})$$

In particular,

$$SS(\Psi_{\mathscr{K}}(\mathscr{F}))\subset\pi_Y^a(SS(\mathscr{K})\cap\pi_X^{-1}(SS(\mathscr{F}))=SS(\mathscr{K})\circ SS(\mathscr{F})$$

$$SS(\Phi_{\mathscr{K}}(\mathscr{G}))\subset\pi_{XZ}(SS(\mathscr{K})\times_{T^*Y}SS(\mathscr{G}))=SS(\mathscr{K})^{-1}\circ SS(\mathscr{G})$$

PROOF. We first notice that the properness assumption for good kernels implies that

$$\pi_{XY}^{-1}(SS(\mathscr{K}))\underset{\infty}{\hat{+}}\pi_{YZ}^{-1}(SS(\mathscr{L}))=\emptyset \tag{$*$}$$

Indeed, a sequence (x_n,y_n,ξ_n,η_n) and $(y_n',z_n,\eta_n',\zeta_n)$ respectively in $SS(\mathscr{K})$ and $SS(\mathscr{L})$ such that

$$\lim_n x_n=x_\infty,\ \lim_n y_n=\lim_n y_n'=y_\infty,\ \lim_n z_n=z_\infty,\ \lim_n\xi_n=\xi_\infty,\ \lim_n(\eta_n+\eta_n')=\eta_\infty \tag{9.2}$$

By properness of the projection $SS(K)\longrightarrow T^*X$, we have that the sequence η_n is bounded, hence η_n' is also bounded, and this proves $(*)$. Now we have

$$SS(q_{XY}^{-1}(\mathscr{K})\otimes^L q_{YZ}^{-1}(\mathscr{L}))\subset\pi_{XY}^{-1}(SS(\mathscr{K}))+\pi_{XY}^{-1}(SS(\mathscr{L}))$$

Then

$$SS(R_{q_{XZ}!}(q_{XY}^{-1}(\mathscr{K})\otimes^L q_{YZ}^{-1}(\mathscr{L})))\subset\Lambda_{q_{XZ}}(SS((q_{XY}^{-1}(\mathscr{K})\otimes^L q_{YZ}^{-1}(\mathscr{L})))=$$
$$\Lambda_{q_{XZ}}(\pi_{XY}^{-1}(SS(\mathscr{K}))+\pi_{YZ}^{-1}(SS(\mathscr{L})))=SS(\mathscr{K})\circ SS(\mathscr{L})$$

□

REMARK 9.19. Assume $X = Y$ and $SS(\mathscr{K})$ be the graph of a symplectomorphism, then set $\mathscr{K}^a \in D^b(Y \times X)$ to be the direct image by $\sigma(x,y) = (y,x)$ of $\mathscr{K}$ (i.e. $\mathscr{K}^a = \sigma_*\mathscr{K}$). Then set for a Lagrangian in $T^*X \times \overline{T^*X}$, $L^a = \{(y,\eta,x,\xi) \mid (x,\xi,y,\eta) \in L\}$. Then $SS(\mathscr{K}^a) = SS(\mathscr{K})^a \subset T^*X \times \overline{T^*X}$, and $\Psi_{\mathscr{K}} \circ \Psi_{\mathscr{K}^a} = \Psi_{\mathscr{L}}$ where $SS(\mathscr{L}) = SS(\mathscr{K}) \circ SS(\mathscr{K}^a) = SS(\mathscr{K}) \circ SS(K)^a = SS(\mathrm{Id}) = \Delta_{T^*X}$.

From this we can prove the following result. Even though we technically do not use it in concrete questions (our singular support will be Lagrangian by construction), the following is an essential result, due to Kashiwara-Schapira ([**K-S**], theorem 6.5.4), Gabber [**Ga**] (for the general algebraic case)

PROPOSITION 9.20 (Involutivity theorem). *Let $\mathscr{F}^\bullet$ be an element in $D^b(X)$. Then $SS(\mathscr{F}^\bullet)$ is a coisotropic submanifold.*

Some remarks are however in order. Proving that $C = SS(\mathscr{F}^\bullet)$ is coisotropic is equivalent to proving that given any hypersurface Σ such that $C \subset \Sigma$, the characteristic vector field X_Σ of Σ is tangent to C. Besides, this is a local property, so we may asume we are in a neighbourhood of $0 \in \mathbb{R}^n$. Now consider the example $C \subset \Sigma = \{(q,p) \mid \langle v, q\rangle = 0\}$. Then $X_\Sigma = \mathbb{R}(0,v)$. Now remember that $C \cap 0_{\mathbb{R}^n} = \mathrm{supp}(\mathscr{F})$. Thus if $\mathscr{F}$ is nonzero near 0, since our assumption implies that $\mathrm{supp}(\mathscr{F}) \subset \{q \mid \langle v,q\rangle\}$, whenever we move in the v direction, we certainly change $\Gamma\mathscr{F}_x$, hence $\mathbb{R}(0,v) \subset C$.

Let us start with the case $M = \mathbb{R}^n$. We wish to prove that for a sheaf $\mathscr{F}$, $SS(\mathscr{F})$ cannot be contained in $\{q_1 = p_1 = 0\}$. Indeed, let $f : \mathbb{R}^n \to \mathbb{R}$ be the projection on q_1. Then $\Lambda_f \circ SS(\mathscr{F}) \subset \{0\} \subset T^*\mathbb{R}$. Thus $Rf_*\mathscr{F}$ is a sheaf on $\mathbb{R}$ with singular support contained in $\{0\}$.

Here we should rather consider the embedding $j : \mathbb{R} \to \mathbb{R}^n$ given by $j(x) = (x,0,...0)$, and $j^{-1}(SS(\mathscr{F}))$ has singular support $\Lambda_j^{-1}(SS(\mathscr{F})) \subset \{(0,0)\}$ and now use the fact that $SS(f^{-1}(\mathscr{F})) = \Lambda_f^{-1}(SS(\mathscr{F}))$. Assume we could find such a sheaf. Then $SS(\mathscr{F})$ being conic, locally, it either contains vertical lines, or is contained in a singleton. We may thus assume $SS(\mathscr{F}) = \{0\}$ and find a contradiction. But locally $SS(\mathscr{F}) \subset \{0\}$ implies $\mathrm{supp}(\mathscr{F}) \subset \{0\}$ hence $F = \mathscr{F}_x$ is a sky-scraper sheaf at points of $f^{-1}(y)$, and $SS(\mathscr{F}) = T_0^*\mathbb{R}$ a contradiction. A way to rephrase this is that the singular support can not be too small. In fact the proof can be reduced to the above.

LEMMA 9.21. *Let C_0 be a homogeneous submanifold of T^*X and $(x_0,p_0) \in C_0$. There is a homogeneous Lagrangian correspondence Λ, such that $C = \Lambda \circ C_0$ sends $T_{(x_0,p_0)}C_0$ to $T_{(x,p)}C$. If we moreover assume C_0 is not coisotropic, we may find local homogeneous coordinates $T_{(x,p)}C \subset \{(x,p) \mid x_1 = p_1 = 0\}$*

PROOF. A space is coisotropic if andf ony if it is contained in no proper symplectic subspace. Let H be a hyperplane, ξ a vector transverse to H, C a subspace containing ξ. Assume □

PROOF OF THE INVOLUTIVITY THEOREM. Let us coinsider $C_0 = SS(\mathscr{F})$ and assume we are at a smooth point (x_0, p_0) which is not coisotropic. Because the result is local, we may always assume we are working on $T^*\mathbb{R}^n$. Then there exists a local symplectic diffeomorphism, sending (x_0, p_0) to $(0, p_0)$ sending C_0 to C, such that $T_{(0,p_0)}C \subset \{x_1 = p_1 = 0\}$. By applying a further C^1 small symplectic ap, we may assume Now let Λ be the correspondence in $T^*\mathbb{R}^n \times T^*\mathbb{R}^2$ given by $\{x_1 = t_1, ..., x_n = t_n, x_{n+1} = t_{n+1}, p_1 = t_{n+1}, p_2 = t_2, ..., p_{n+1} = t_{n+1}\}$, and $\mathscr{K}$ the corresponding kernel. Then $\Lambda \circ C$ is obtained by projecting on $T^*\mathbb{R}$ the intersection $\Lambda \cap (C \times T^*\mathbb{R})$. We have near (x, p) that $\Lambda \cap C \times T^*\mathbb{R} \subset \{x_1 = x_{n+1} = p_1 = p_{n+1} = 0\}$ so that the projection on $T^*\mathbb{R}^2$ is contained in $\{0, 0\}$. But we proved that this is impossible, since this would mean that $\mathscr{K} \circ \mathscr{F}$ satisfies locally $SS(\mathscr{K} \circ \mathscr{F}) \subset \{0\}$. $\square$

PROOF OF THE LEMMA. Clearly if V is a proper symplectic subspace and $C \subset V$ be isotropic, we have $V^\omega \subset C^\omega$, but C^ω is isotropic, a contradiction.

$\square$

DEFINITION 9.22. A sheaf is **constructible** if and only if there is a stratification of X, such that $\mathscr{F}^\bullet$ is locally constant on each strata.

PROPOSITION 9.23. *If $\mathscr{F}^\bullet$ is constructible, then it is Lagrangian.*

PROOF. We refer to the existing literature, since we will not really use this proposition: our singular suports will be Lagrangian by construction. We can actually take this as the definition of constructible.

$\square$

However the following turns out to be useful.

DEFINITION 9.24. We shall say that a sheaf on a metric space is **locally stable** if for any x there is a positive δ such that $H^*(B(x,\delta), \mathscr{F}^\bullet) \to H^*(\mathscr{F}^\bullet_x)$ is an isomorphism.

PROPOSITION 9.25. *Constructible sheafs are locally stable*

4. Appendix: More on sheafs and singular support

4.1. The Mittag-Leffler property. The question we are dealing with here, is to whether $R\Gamma(U, \mathscr{F}) = \varprojlim_{V \subset U} R\Gamma(U, \mathscr{F})$. Notice that by defintion of sheaves, we have

$$\Gamma(U, \mathscr{F}) = \varprojlim_{V \subset U} \Gamma(U, \mathscr{F})$$

so our question deals with the commutation of inverse limit and cohomology.

4.2. Appendix: More on singular supports of $f^{-1}(\mathscr{F}^\bullet)$.

DEFINITION 9.26. We shall say that the map $f : X \longrightarrow Y$ is non-characteristic for $A \subset T^*Y$ if

$$\eta \circ df(x) = 0 \text{ and } (f(x), \eta) \in A \Longrightarrow \eta = 0$$

We say that f is non-characterisitic for $\mathscr{F}$ if it is non-characterisitic for $SS(\mathscr{F})$.

REMARK 9.27. Let $d: X \to X \times X$ be the diagonal map. Then $A_1 \times A_2 \subset T^*X \times T^*X$ is non characteristic for d if and only if

$$(x,\eta_1) \in A_1, (x,\eta_2) \in A_2, \eta_1 + \eta_2 = 0 \Longrightarrow \eta_1 = \eta_2 = 0$$

Or in other words $A_1 \cap A_2^a \subset 0_X$

PROPOSITION 9.28. *Assume f is an embedding. Then if f is non-characteristic for $SS(\mathscr{F})$, and we have*

$$SS(f^{-1}(\mathscr{F})) \subset \Lambda_f^{-1} \circ SS(\mathscr{F})$$

PROOF. Saying that $f: X \to Y$ is non-characteristic, means

□

PROPOSITION 9.29. *Let f be non-characteristic for $SS(\mathscr{F})$. Then*

$$SS(f^{-1}(\mathscr{F})) \subset \Lambda_f^{-1} \circ SS(\mathscr{F})$$

PROOF. This follows from the fact that f can be written as the composition of a non-characteristic embedding $X \to X \times Y$ and a submersion $X \times Y \to Y$ □

PROPOSITION 9.30 ([**K-S**] page 235, Corollary 5.4.11 and Prop. 5.4.13). *Let us consider an embedding of V in X. Then*

$$SS(\mathscr{F}_V) \subset SS(F) + \nu_V^*$$

Let f be a smooth map such that $f_\pi^{-1}(A) \cap \nu_V^* \subset Y \times_X 0_X$. Then

$$SS(f^{-1}(\mathscr{F})) \subset \Lambda_f^{-1} \circ SS(\mathscr{F})$$

EXERCICE 3. Prove that if $\mathscr{F}$ is a sheaf over X and Z a smooth submanifold,

$$SS(\mathscr{F}_{|Z}) = (SS(\mathscr{F}) \cap \nu^* Z)/\sim \subset (\nu^* Z/\sim) = T^* Z$$

This is the symplectic reduction of $SS(\mathscr{F})$.

A VERIFIER

Let f be an open map, that is such that the image of an open set is an open set. Examples f such maps are embeddings, or submersions. We want to prove

LEMMA 9.31. *For f an open map, we have*

$$\Gamma_{f^{-1}(Z)} f^{-1}(\mathscr{F}) = f^{-1}\Gamma_Z(\mathscr{F})$$

therefore

$$R\Gamma_{f^{-1}(Z)} f^{-1}(\mathscr{F}) = f^{-1} R\Gamma_Z(\mathscr{F})$$

PROOF. Consider the functor Pf^{-1} defined on presheaves by $Pf^{-1}(\mathscr{F})(V) = \lim_{U \supset f(V)} \mathscr{F}(U)$ which for f open is given by $Pf^{-1}(\mathscr{F})(V) = \mathscr{F}(f(V))$. We claim that the functor Γ_Z, which is also well-defined and left-exact on presehaves, given by $\Gamma_Z(\mathscr{F})(V) = \{s \in$

$\mathscr{F}(V) \mid s(y) = 0 \forall y \in Y \setminus Z\}$ (one should be careful for presheaves, there maybe nonzero sections over V which are pointwise zero). This satisfies $\Gamma_{f^{-1}(Z)} \circ Pf^{-1} = Pf^{-1}\Gamma_Z$ since

$$\Gamma_{f^{-1}(Z)} \circ Pf^{-1}(\mathscr{F})(V) = \{t \in Pf^{-1}(\mathscr{F})(V) \mid t(x) = 0,\ \forall x \in X \setminus f^{-1}(Z)\} =$$
$$\{t \in \mathscr{F}(f(V)) \mid t(x) = 0,\ \forall x \in X \setminus f^{-1}(Z)\} =$$
$$\{t \in \mathscr{F}(f(V)) \mid t(x) = 0\ \forall x \text{ such that } f(x) \in Y \setminus Z\} = Pf^{-1}\Gamma_Z(\mathscr{F})$$

Moreover if Sh is the sheafification functor, we have $Sh \circ \Gamma_Z = \Gamma_Z \circ Sh$. As a result,

$$f^{-1}\Gamma_Z = Sh \circ Pf^{-1} \circ \Gamma_Z = Sh \circ \Gamma_{f^{-1}(Z)} \circ Pf^{-1} = \Gamma_{f^{-1}(Z)} \circ Sh \circ Pf^{-1} = \Gamma_{f^{-1}(Z)} \circ f^{-1}$$

This implies $f^{-1}R\Gamma_Z = R\Gamma_{f^{-1}(Z)} \circ f^{-1}$ in the derived category. □

As a result, if for some φ such that $\varphi(y) = 0, d\varphi(y) = \eta$, we have $\left(R\Gamma_{\{\varphi(v)\geq 0\}}(\mathscr{F})\right)_y \neq 0$ that is $\left(f^{-1}R\Gamma_{\{\varphi(u)\geq 0\}}(\mathscr{F})\right)_x \neq 0$ for all $x \in f^{-1}(y)$, then we have $\left(R\Gamma_{\{\varphi\circ f(u)\geq 0\}}(f^{-1}\mathscr{F})\right)_x \neq 0$ for all x in $f^1(y)$. Note that $d(\varphi \circ f)(x) = d\varphi(y) \circ df(x) = \eta \circ df(x)$.

As a result, there exists (x, p) and ψ such that $\psi(x) = 0, d\psi(x) = \eta \circ df(x)$ and $\left(R\Gamma_{\{\varphi\circ f(u)\geq 0\}}(f^{-1}\mathscr{F})\right)_x \neq 0$.

Now $SS(f^{-1}(\mathscr{F}))$ is in the closure of this set. So we proved

LEMMA 9.32. *We have the inclusion*

$$SS(f^{-1}(\mathscr{F})) \subset \overline{\Lambda_f^{-1}(SS(\mathscr{F}))} \overset{def}{=} (\Lambda_f^{-1})^b(SS(\mathscr{F}))$$

Note that $(x,\xi) \in (\Lambda_f^{-1})^b(SS(\mathscr{F}))$ is defined as the existence of a sequence (x_n, y_n, η_n) such that $(y_n, \eta_n) \in SS(\mathscr{F})$, and

$$f(x_n) = y_n, x_n \to x, \eta_n \circ df(x_n) \to \xi$$

There are a priori two kind of points $(x, p) \in (\Lambda_f^{-1})^b(SS(\mathscr{F}))$. Those obtained by using a bounded sequence η_n, but then taking a subsequence, we get $\eta_n \to \eta$, and thus $\eta \circ df(x) = \xi$ that is $(x, p) \in \Lambda_f^{-1}(SS(F))$, and the set obtained by taking an unbounded sequence, denoted $\Lambda_{f,\infty}^{-1}(SS(F))$.

Note that if the map f is non-characteristic, the set $\Lambda_{f,\infty}^{-1}(SS(F))$ is empty. Indeed, considering the sequence $\frac{\eta_n}{|\eta_n|}$ which has a subsequence converging to some η_∞ of norm one, we get $\eta_\infty \circ df(x) = 0$, i.e. f is characteristic.

4.3. Convolution of sheaves. Let $s(u, v) = u + v$. then $\Lambda_s \in T^*(\mathbb{R} \times \mathbb{R} \times \mathbb{R})$ is given by

$$\Lambda_s = \{(u, \xi, v, \eta, w, \zeta) \mid w = u + v, \zeta = \eta = \xi\}$$

DEFINITION 9.33 (Convolution). Let E be a real vector space, and $s : E \times E \to E$ be the map $s(u, v) = u + v$. We similarly denote by s the map $s : (X \times E) \times (Y \times E) \to (X \times Y) \times E$ given by $s(x, u, y, v) = (x, y, u + v)$. Let $\mathscr{F}, \mathscr{G}$ be sheafs on $X \times E$ and $Y \times E$. We set

$$\mathscr{F} * \mathscr{G} = Rs_!(q_X^{-1}\mathscr{F} \boxtimes^L q_Y^{-1}\mathscr{G})$$

This is a sheaf on $D^b(X\times Y\times E)$. where $q_X : X\times Y\times E \to X\times E$ and $q_Y X\times Y\times E \to Y\times E$ are the projections.

EXERCICE 4 ([**K-S**] page 135-exercice II.20)). (1) Prove that the operation $*$ is commutative and associative.
(2) Prove that $k_{X\times\{0\}} * \mathscr{G} = \mathscr{G}$.
(3) Let $U(f) = \{(x,u) \in X\times\mathbb{R} \mid f(x)\le u\}$, $V(g) = \{(y,v)\in Y\times\mathbb{R} \mid g(y)\le v\}$, and $W(h) = \{(x,y,w)\in X\times\mathbb{R} \mid h(x,y)\le w\}$. Then $k_{U(f)} * k_{V(g)} = k_{W(f\oplus g)}$ where $(f\oplus g)(x,y) = f(x)+g(y)$.
(4)

$$SS(\mathscr{F} * \mathscr{G}) = \Lambda_s \circ (SS(\mathscr{F})\times SS(\mathscr{G})) =$$

$$\{(x,p_x,y,p_y,w,\eta) \mid \exists (x,p_x,u,\eta)\in SS(\mathscr{F}), \exists (x,p_x,v,\eta)\in SS(\mathscr{G}), w=u+v\}$$

As a consequence

$$SS(k_{U(f)} * k_{V(g)}) = SS(k_{W(f\oplus g)})$$

(5) Let us consider a function $g(u,v)$ on $E\times E$ and assume $(u,\frac{\partial g}{\partial u}(u,v)) \to (v,-\frac{\partial g}{\partial v}(u,v))$ define a (necessarily Hamiltonian) map φ_g. Then, let Φ_g be the operator $\mathscr{F}\to k_{W(g)} * \mathscr{F}$. Prove that $SS(\Phi_g(\mathscr{F}))\subset \varphi_g(SS(\mathscr{F}))$.

Note that one can define the adjoint functor of the convolution, $RHom^*$ satisfying $\mathrm{Mor}(\mathscr{F} * \mathscr{G},\mathscr{H}) = \mathrm{Mor}(F, RHom^*(\mathscr{G},\mathscr{H}))$.

DEFINITION 9.34. We set

$$R\mathscr{H}om^*(\mathscr{F},\mathscr{G}) = (Rq_X)_* R\mathscr{H}om(q_Y^{-1}\mathscr{F}, s^!\mathscr{G})$$

PROPOSITION 9.35. *We have*

$$SS(\mathscr{F} * \mathscr{G}) \subset SS(\mathscr{F}) \hat{*} SS(\mathscr{G})$$

where $A\hat{*}B = s_\# j^\#(A\times B)$

CHAPTER 10

The proof of Arnold's conjecture using sheafs.

1. Statement of the Main theorem

Here is the theorem we wish to prove

THEOREM 10.1 (Guillermou-Kashiwara-Schapira). *Let M be a (non-compact manifold) and N be a compact submanifold. Let Φ^t be a homogenous Hamiltonian flow on $T^*M \setminus 0_M$ and ψ be a function without critical point in M. Then for all t we have*

$$\Phi^t(\nu^* N) \cap \{(x, d\psi(x)) \mid x \in M\} \neq \emptyset$$

Of course, Φ^t can be identified with a contact flow $\hat{\Phi}^t$ on ST^*M, $\nu^*N \cap ST^*M = S\nu^*(N)$ is Legendrian, the set $L_\psi = \{(x, \frac{d\psi(x)}{|d\psi(x)|} \mid x \in M\}$ is co-Legendrian, and we get

COROLLARY 10.2. *Under the assumptions of the theorem, we have*

$$\hat{\Phi}^t(S\nu^*(N)) \cap L_\psi \neq \emptyset$$

Let us prove how this implies the Arnold conjecture, first proved on T^*T^n by Chaperon ([**Cha**]), using the methods of Conley and Zehnder ([**Co-Z**]), then in general cotangent bundles of compact manifolds by Hofer ([**Hofer**]) and simplified by Laudenbach and Sikorav ([**Lau-Sik**]), who established the estimate of the number of fixed points in the non-degenerate case (this was done in the general case in terms of cup-length in [**Hofer**]).

THEOREM 10.3. *Let φ^t be a Hamiltonian isotopy of T^*N, the cotangent bundle of a compact manifold.Then $\varphi^1(0_N) \cap 0_N \neq \emptyset$. If moroever the intersection points are transverse, there are at least $\sum_j \dim(H^j(N))$ of them.*

PROOF OF THEOREM 10.3 ASSUMING THEOREM 10.1. Consider $M = N \times \mathbb{R}$ and $\psi(z,t) = t$. Then $\varphi^s : T^*N \to T^*N$ can be assumed to be supported in a compact region containing $\bigcup_{s\in[0,1]} \varphi^s(L)$, so we may set $\Phi^s(q,p,t,\tau) = (x_s(x,\tau^{-1}p), \tau p_s(x,\tau^{-1}p), f_s(t,x,p,\tau),\tau)$ where $\varphi^s(x,p) = (x_s(x,p), p_s(x,p))$, and this is now a homogeneous flow on T^*M. We identify N to $N \times \{0\}$, and apply the main theorem: $\nu^*N = 0_N \times \{0\} \times \mathbb{R}$ and $L_\psi = \{(x,0,t,1) \mid (x,t) \in N \times \mathbb{R}\}$, so that $\Phi^s(\nu^*N) = \{(x_s(x,0), \tau p_s(x,0), f_s(0,x,0,\tau),\tau) \mid x \in N, \tau \in \mathbb{R}\}$ so that $\Phi^s(\nu^*N) \cap L_\psi = \{(x_s(x,0), p_s(x,0), f_s(0,x,0,\tau),\tau) \mid p_s(x,0) = 0, \tau = 1\} = \varphi^s(0_N) \cap 0_N$. According to the main theorem this is not empty, and this concludes the proof. □

2. The proof

PROOF OF THE MAIN THEOREM. We start with the sheaf $\mathbb{C}_N$, which satisfies $SS(\mathbb{C}_N) = \nu^* N$. We first consider a lift of Φ^t to $\widetilde{\Phi} : T^*(M \times I) \to T^*(M \times I)$ given by the formula

$$\widetilde{\Phi} : (q,p,t,\tau) \longrightarrow (\Phi^t(q,p), t, \tau + F(t,q,p))$$

where $\Phi^t(q,p) = (Q_t(q,p), P_t(q,p))$ and $F(t,q,p) = -P_t(q,p)\frac{\partial}{\partial t} Q_t(q,p)$ because denoting $\Phi^t(q,p) = (Q_t(q,p), P_t(q,p))$ the homogeneity of Φ^t and Proposition 4.24 imply that $P_t dQ_t = pdq$ and $F(t,q,p)$ is homogeneous in p. Let $\mathscr{K}$ be a kernel in $D^b(M \times I \times M \times I)$ such that $SS(\mathscr{K}) = graph(\widetilde{\Phi})$. The existence of such a kernel will be proved in Proposition 10.4. Then consider the sheaf $\mathscr{K}(\mathbb{C}_{N\times I}) \in D^b(M \times I)$. It has singular support given by

$$SS(\mathscr{K}(\mathbb{C}_{N\times I})) \subset \widetilde{\Phi}(SS(\mathbb{C}_{N\times I})) \subset \widetilde{\Phi}(\nu^* N \times 0_I)$$

Now consider the function $f(q,t) = t$ on $M \times I$. It satisfies $L_f = \{(q,t,0,1) \mid q \in M, t \in I\} \notin SS(\mathscr{K}(\mathbb{C}_{N\times I}))$ since this last set is contained in

$$\widetilde{\Phi}(\nu^* N \times 0_I) = \{(Q_t(q,p), P_t(q,p), t, F(t,q,p)) \mid (q,t) \in N \times I, p = 0 \text{ on } T_q N\}$$

If we had a point in $L_f \cap SS(\mathscr{K}(\mathbb{C}_{N\times I}))$ it should then satisfy $P_t(q,p) = 0$, but then we would have $F(t,q,p) = -P_t(q,p)\frac{\partial}{\partial t} Q_t(q,p) = 0$ which contradicts $\tau = 1$. We now denote by $\mathscr{K}_t \in D^b(M)$ the sheaf obtained by restricting $\mathscr{K}$ to $M \times \{t\} \times M \times \{t\}$.

The Morse lemma (cf. lemma 9.9) then implies that $H^*(M \times [0,t], \mathscr{K}(\mathbb{C}_{N\times I})) \longrightarrow H^*(M \times [0,s], \mathscr{K}(\mathbb{C}_{N\times I}))$ is an isomorphism for all $s < t$, and also that

$$H^*(M \times \{0\}, \mathscr{K}(\mathbb{C}_{N\times I})) \simeq H^*(M \times \{t\}, \mathscr{K}(\mathbb{C}_{N\times I}))$$

for all t. But on one hand

$$H^*(M \times \{0\}, \mathscr{K}(\mathbb{C}_{N\times I})) \simeq H^*(M, \mathscr{K}_0(\mathbb{C}_N)) = H^*(M, \mathbb{C}_N) \simeq H^*(N, \mathbb{R})$$

on the other hand,

$$H^*(M, \mathscr{K}_1(\mathbb{C}_{N\times I})) = H^*(\mathbb{R}, \psi_*(\mathscr{K}_1(\mathbb{C}_{N\times I})) = 0$$

the last equality follows from the fact that

$$SS(\psi_*(\mathscr{K}_1(\mathbb{C}_{N\times I})) \subset \Lambda_\psi \cap SS((\mathscr{K}_1(\mathbb{C}_{N\times I})) = \Lambda_\psi \cap \Phi(\nu^* N) = \emptyset,$$

$\psi_*(\mathscr{K}_1(\mathbb{C}_{N\times I})$ is compact supported and Proposition 9.7. This is a contradiction and concludes the proof modulo the next Proposition. □

PROPOSITION 10.4. *Let $\Phi : T^*X \to T^*X$ be a compact supported symplectic diffeomorphism C^1-close to the identity, and $\widetilde{\Phi}$ its homogeneous lift to $\mathring{T}^*(X \times \mathbb{R}) \to \mathring{T}^*(X \times \mathbb{R})$, given by $\widetilde{\Phi}(q,p,t,\tau) = (Q(q,\tau^{-1}p), \tau P(q,\tau^{-1}p), F(q,p,t,\tau), \tau)$. Then there is a kernel $\mathscr{K} \in D^b(X \times \mathbb{R} \times X \times \mathbb{R})$ such that $SS(\mathscr{K}) = \Gamma_{\widetilde{\Phi}}$.*

PROOF ("TRANSLATED" FROM [**Bru**]). Because any Hamiltonian symplectomorphism is the product of C^1-small symplectomorphisms, thanks to the decomposition formula

$$\Phi_0^1 = \prod_{j=1}^{n} \Phi_{\frac{j-1}{N}}^{\frac{j}{N}}$$

we can restrict ourselves to the case where Φ is C^∞-small. Note also that $\widetilde{\Phi}$ is well defined by the compact support assumption: for τ close to zero,

$$(Q(q,\tau^{-1}p),\tau P(q,\tau^{-1}p)) = (q,\tau\tau^{-1}p) = (q,p)$$

Let us start with the case $X = \mathbb{R}^n$. Let $f(q,Q)$ be a generating function for Φ so that $p = \frac{\partial f}{\partial q}(q,Q), P = -\frac{\partial f}{\partial Q}(q,Q)$ defines the map Φ. Let $W = \{(q,t,Q,T) \mid f(q,Q) \le t-T\}$ and $\mathscr{F}_f = k_W \in D^b(X\times\mathbb{R})$. Then $SS(\mathscr{F}_f) = \Gamma_{\widetilde{\Phi}}$. Let us start with $X = Y = \mathbb{R}^n$, and $f_0(q,Q) = |q-Q|^2$. Then we get $\mathscr{K}_0$ with $SS(\mathscr{K}_0) = \Gamma_{\widetilde{\Phi}_0}$. Now if f is C^2 close to f_0, we will get any possible $\widetilde{\Phi}_f$, C^1-close to the map $(q,p)\to(q+p,p)$. Then $\widetilde{\Phi}_{f_1}\circ\widetilde{\Phi}_{f_2}$ where f_1 is close to f_0 and f_2 close to $-f_0$ will be C^1-close to the identity. Now since any time one of a Hamiltonian isotopy can be written as the decomposition of C^1-small symplectomorphisms, we get the general case.

Now let $i : N \to \mathbb{R}^n$ be an embedding. Then the standard Riemannian metric on $\mathbb{R}^l$ induces a symplectic embedding $\tilde{i} : T^*N \to T^*\mathbb{R}^n$ given by $(x,p)\mapsto(i(x),\widetilde{p}(i(x)))$ where $\widetilde{p}(i(x))$ is the linear form on $\mathbb{R}^l$ that equals p on T_xN and zero on $(T_xN)^\perp$. Now let Φ^t be a Hamiltonian isotopy of T^*N. We claim that it can be extended to $\widetilde{\Phi}^t$ such that

(1) $\widetilde{\Phi}^t$ preserves $\nu^*N = N\times(\mathbb{R}^l)^*$, and thus the leaves of this coisotropic submanifold. This implies that $\widetilde{\Phi}^t$ induces a map from the reduction of $N\times(\mathbb{R}^l)^*$ to itself, that is T^{*N}.
(2) we require that this map equals Φ^t.

The existence of $\widetilde{\Phi}^t$ follows from the following construction:

Assume Φ is the time one map of Φ^t associated to $H(t,x,p)$, where (x,p) is coordinates for T^*N. Locally, we can write (x,u,p,v) for points in $\mathbb{R}^l$ so that $N = \{u=0\}$. We define

$$\widetilde{H}(t,x,u,p,v) = \chi(u)H(t,x,p),$$

where χ is some bump function whichequals is 1 on N (i.e. $\{u=0\}$) and 0 outside a neighborhood of N. By the construction, $X_{\widetilde{H}} = X_H$ on $N\times(\mathbb{R}^l)^*$. Then $\widetilde{\Phi} = \widetilde{\Phi}^1$, the time one flow of $\widetilde{H}$, is the map we need.

The theorem follows by noticing that if we take $\widetilde{L} = O_{\mathbb{R}^l}$, coinciding with the zero section outside compact set and has GFQI, then $\widetilde{L}_N = O_N$. □

Exercice: Show that if L has a GFQI, then $\varphi(L)$ has a GFQI for $\varphi\in Ham(T^*N)$. **Hint.** If $S : N\times\mathbb{R}^k\to\mathbb{R}$ is a GFQI for L, then L is the reduction of $gr(dS)$.

REMARK 10.5. (1) We could have used directly that the graph of Φ has a GFQI.

(2) 0_N is generated by the zero function over the zero bundle over N, or less formally

$$\begin{array}{rccc} S: & N\times\mathbb{R} & \to & \mathbb{R} \\ & (x,\xi) & \mapsto & \xi^2 \end{array}$$

(3) There is no general upper bound on k (the minimal number of parameter of a generating functions needed to produce all Lagrangian.)

Reason: Consider a curve in T^*S^1

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