AP CALCULUS AB and BC

Final Notes

Trigonometric Formulas

1. $\sin^2\theta + \cos^2\theta = 1$
2. $1 + \tan^2\theta = \sec^2\theta$
3. $1 + \cot^2\theta = \csc^2\theta$
4. $\sin(-\theta) = -\sin\theta$
5. $\cos(-\theta) = \cos\theta$
6. $\tan(-\theta) = -\tan\theta$
7. $\sin(A+B) = \sin A\cos B + \sin B\cos A$
8. $\sin(A-B) = \sin A\cos B - \sin B\cos A$
9. $\cos(A+B) = \cos A\cos B - \sin A\sin B$
10. $\cos(A-B) = \cos A\cos B + \sin A\sin B$
11. $\sin 2\theta = 2\sin\theta\cos\theta$
12. $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$
13. $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{1}{\cot\theta}$
14. $\cot\theta = \dfrac{\cos\theta}{\sin\theta} = \dfrac{1}{\tan\theta}$
15. $\sec\theta = \dfrac{1}{\cos\theta}$
16. $\csc\theta = \dfrac{1}{\sin\theta}$
17. $\cos^2\theta = \dfrac{1}{2}\left(1 + \cos 2\theta\right)$
18. $\sin^2\theta = \dfrac{1}{2}\left(1 - \cos 2\theta\right)$

Differentiation Formulas

1. $\dfrac{d}{dx}(x^n) = nx^{n-1}$
2. $\dfrac{d}{dx}(fg) = fg' + gf'$ *Product rule*
3. $\dfrac{d}{dx}\left(\dfrac{f}{g}\right) = \dfrac{gf' - fg'}{g^2}$ *Quotient rule*
4. $\dfrac{d}{dx}f(g(x)) = f'(g(x))g'(x)$ *Chain rule*
5. $\dfrac{d}{dx}(\sin x) = \cos x$
6. $\dfrac{d}{dx}(\cos x) = -\sin x$
7. $\dfrac{d}{dx}(\tan x) = \sec^2 x$
8. $\dfrac{d}{dx}(\cot x) = -\csc^2 x$
9. $\dfrac{d}{dx}(\sec x) = \sec x\tan x$
10. $\dfrac{d}{dx}(\csc x) = -\csc x\cot x$
11. $\dfrac{d}{dx}(e^x) = e^x$
12. $\dfrac{d}{dx}(a^x) = a^x \ln a$
13. $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$
14. $\dfrac{d}{dx}(Arc\sin x) = \dfrac{1}{\sqrt{1-x^2}}$
15. $\dfrac{d}{dx}(Arc\tan x) = \dfrac{1}{1+x^2}$
16. $\dfrac{d}{dx}(Arc\sec x) = \dfrac{1}{|x|\sqrt{x^2-1}}$
17. $\dfrac{d}{dx}\left[c\right] = 0$
18. $\dfrac{d}{dx}\left[cf(x)\right] = cf'(x)$

Integration Formulas

1. $\int a\, dx = ax + C$

2. $\int x^n\, dx = \dfrac{x^{n+1}}{n+1} + C, \quad n \neq -1$

3. $\int \dfrac{1}{x} dx = \ln|x| + C$

4. $\int e^x\, dx = e^x + C$

5. $\int a^x dx = \dfrac{a^x}{\ln a} + C$

6. $\int \ln x\, dx = x\ln x - x + C$

7. $\int \sin x\, dx = -\cos x + C$

8. $\int \cos x\, dx = \sin x + C$

9. $\int \tan x\, dx = \ln|\sec x| + C$ or $-\ln|\cos x| + C$

10. $\int \cot x\, dx = \ln|\sin x| + C$

11. $\int \sec x\, dx = \ln|\sec x + \tan x| + C$

12. $\int \csc \mathrm{x}\ \mathrm{dx} = -\ln|\csc \mathrm{x} + \cot \mathrm{x}| + C$

13. $\int \sec^2 x\, dx = \tan x + C$

14. $\int \sec x \tan x\, dx = \sec x + C$

15. $\int \csc^2 x\, dx = -\cot x + C$

16. $\int \csc x \cot x\, dx = -\csc x + C$

17. $\int \tan^2 x\, dx = \tan x - x + C$

18. $\int \dfrac{dx}{a^2 + x^2} = \dfrac{1}{a} Arc\tan\left(\dfrac{x}{a}\right) + C$

19. $\int \dfrac{dx}{\sqrt{a^2 - x^2}} = Arc\sin\left(\dfrac{x}{a}\right) + C$

20. $\int \dfrac{dx}{x\sqrt{x^2 - a^2}} = \dfrac{1}{a} Arc\sec\dfrac{|x|}{a} + C = \dfrac{1}{a} Arc\cos\left|\dfrac{a}{x}\right| + C$

Formulas and Theorems

1. Limits and Continuity:

A function $y = f(x)$ is continuous at $x = a$ if

i). $f(a)$ exists

ii). $\lim_{x \to a} f(x)$ exists

iii). $\lim_{x \to a} f(x) = f(a)$

Otherwise, f is discontinuous at x = a.

The limit $\lim_{x \to a} f(x)$ exists if and only if both corresponding one-sided limits exist and are equal – that is,

$$\lim_{x \to a} f(x) = L \to \lim_{x \to a^+} f(x) = L = \lim_{x \to a^-} f(x)$$

2. Even and Odd Functions

1. A function $y = f(x)$ is even if $f(-x) = f(x)$ for every x in the function's domain. Every even function is symmetric about the y-axis.

2. A function $y = f(x)$ is odd if $f(-x) = -f(x)$ for every x in the function's domain. Every odd function is symmetric about the origin.

3. Periodicity

A function $f(x)$ is periodic with period $p\ (p > 0)$ if $f(x + p) = f(x)$ for every value of x.

Note: The period of the function $y = A\sin(Bx + C)$ or $y = A\cos(Bx + C)$ is $\dfrac{2\pi}{|B|}$.

The amplitude is $|A|$. The period of $y = \tan x$ is π.

4. Intermediate-Value Theorem

A function $y = f(x)$ that is continuous on a closed interval $[a,b]$ takes on every value between $f(a)$ and $f(b)$.

Note: If f is continuous on $[a,b]$ and $f(a)$ and $f(b)$ differ in sign, then the equation $f(x) = 0$ has at least one solution in the open interval (a,b).

5. Limits of Rational Functions as $x \to \pm\infty$

i). $\lim_{x \to \pm\infty} \dfrac{f(x)}{g(x)} = 0$ if the degree of $f(x) <$ the degree of $g(x)$

Example: $\lim_{x \to \infty} \dfrac{x^2 - 2x}{x^3 + 3} = 0$

ii). $\lim_{x \to \pm\infty} \dfrac{f(x)}{g(x)}$ is infinite if the degrees of $f(x) >$ the degree of $g(x)$

Example: $\lim_{x \to \infty} \dfrac{x^3 + 2x}{x^2 - 8} = \infty$

iii). $\lim_{x \to \pm\infty} \dfrac{f(x)}{g(x)}$ is finite if the degree of $f(x) =$ the degree of $g(x)$

Example: $\lim_{x \to \infty} \dfrac{2x^2 - 3x + 2}{10x - 5x^2} = -\dfrac{2}{5}$

6. Horizontal and Vertical Asymptotes

1. A line $y = b$ is a horizontal asymptote of the graph $y = f(x)$ if either $\lim_{x\to\infty} f(x) = b$ or $\lim_{x\to-\infty} f(x) = b$.(Compare degrees of functions in fraction)

2. A line $x = a$ is a vertical asymptote of the graph $y = f(x)$ if either $\lim_{x\to a^+} f(x) = \pm\infty$ or $\lim_{x\to a^-} f(x) = \pm\infty$ (Values that make the denominator 0 but not numerator)

7. Average and Instantaneous Rate of Change

i). *Average Rate of Change*: If (x_0, y_0) and (x_1, y_1) are points on the graph of $y = f(x)$, then the average rate of change of y with respect to x over the interval $[x_0, x_1]$ is $\dfrac{f(x_1) - f(x_0)}{x_1 - x_0} = \dfrac{y_1 - y_0}{x_1 - x_0} = \dfrac{\Delta y}{\Delta x}$.

ii). *Instantaneous Rate of Change*: If (x_0, y_0) is a point on the graph of $y = f(x)$, then the instantaneous rate of change of y with respect to x at x_0 is $f'(x_0)$.

8. Definition of Derivative

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} \quad \text{or} \quad f'(\mathrm{a}) = \lim_{\mathrm{x}\to\mathrm{a}}\frac{f(\mathrm{x}) - f(\mathrm{a})}{\mathrm{x} - \mathrm{a}}$$

The latter definition of the derivative is the instantaneous rate of change of $f(\mathrm{x})$ with respect to x at x = a.

Geometrically, the derivative of a function at a point is the slope of the tangent line to the graph of the function at that point.

9. The Number ***e*** as a limit

i). $\lim_{n\to\infty}\left(1 + \dfrac{1}{n}\right)^n = e$

ii). $\lim_{n\to 0}(1+n)^{1/n} = e$

10. Rolle's Theorem (this is a weak version of the MVT)

If f is continuous on $[a,b]$ and differentiable on (a,b) such that $f(a) = f(b)$, then there is at least one number c in the open interval (a,b) such that $f'(c) = 0$.

11. Mean Value Theorem

If f is continuous on $[a,b]$ and differentiable on (a,b), then there is at least one number c in (a,b) such that $\dfrac{f(b) - f(a)}{b - a} = f'(c)$.

12. Extreme-Value Theorem

If f is continuous on a closed interval $[a,b]$, then $f(x)$ has both a maximum and minimum on $[a,b]$.

13. ***Absolute Mins and Maxs***: To find the maximum and minimum values of a function $y = f(x)$, locate

1. the points where $f'(x)$ is zero or where $f'(x)$ fails to exist.
2. the end points, if any, on the domain of $f(x)$.
3. Plug those values into $f(x)$ to see which gives you the max and which gives you this min values (the x-value is where that value occurs)

Note: These are the only candidates for the value of x where $f(x)$ may have a maximum or a minimum.

14. **<u>Increasing and Decreasing</u>**: Let f be differentiable for $a < x < b$ and continuous for a $a \le x \le b$,

1. If $f'(x) > 0$ for every x in (a,b), then f is increasing on $[a,b]$.
2. If $f'(x) < 0$ for every x in (a,b), then f is decreasing on $[a,b]$.

15. **<u>Concavity</u>:** Suppose that $f''(x)$ exists on the interval (a,b)

1. If $f''(x) > 0$ in (a,b), then f is concave upward in (a,b).
2. If $f''(x) < 0$ in (a,b), then f is concave downward in (a,b).

To locate the ***<u>points of inflection</u>*** of $y = f(x)$, find the points where $f''(x) = 0$ <u>or</u> where $f''(x)$ fails to exist. These are the only candidates where $f(x)$ may have a point of inflection. Then test these points to make sure that $f''(x) < 0$ on one side and $f''(x) > 0$ on the other.

16a. If a function is differentiable at point $x = a$, it is continuous at that point. The converse is false, in other words, continuity does <u>not</u> imply differentiability.

16b. <u>Local Linearity and Linear Approximations</u>
The linear approximation to $f(x)$ near $x = x_0$ is given by $y = f(x_0) + f'(x_0)(x - x_0)$ for x sufficiently close to x_0. In other words, find the equation of the tangent line at $(x_0, f(x_0))$ and use that equation to approximate the value at the value you need an estimate for.

17. ***<u>Dominance and Comparison of Rates of Change</u> **<u>(BC topic only)</u>**
Logarithm functions grow slower than any power function (x^n).

Among power functions, those with higher powers grow faster than those with lower powers.

All power functions grow slower than any exponential function $(a^x, a > 1)$.

Among exponential functions, those with larger bases grow faster than those with smaller bases.

We say, that as $x \to \infty$:

1. $f(x)$ grows <u>faster</u> than $g(x)$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$ or if $\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0$.

   If $f(x)$ grows faster than $g(x)$ as $x \to \infty$, then $g(x)$ grows <u>slower</u> than $f(x)$ as $x \to \infty$.

2. $f(x)$ and $g(x)$ grow at the <u>same</u> rate as $x \to \infty$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L \ne 0$ (L is finite and nonzero).

For example,

1. e^x grows faster than x^3 as $x \to \infty$ since $\lim_{x\to\infty} \frac{e^x}{x^3} = \infty$
2. x^4 grows faster than $\ln x$ as $x \to \infty$ since $\lim_{x\to\infty} \frac{x^4}{\ln x} = \infty$
3. $x^2 + 2x$ grows at the same rate as x^2 as $x \to \infty$ since $\lim_{x\to\infty} \frac{x^2 + 2x}{x^2} = 1$

To find some of these limits as $x \to \infty$, you may use the graphing calculator. Make sure that an appropriate viewing window is used.

18. ****L'Hôpital's Rule (BC topic, but useful for AB)***

If $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}$ is of the form $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$, and if $\lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}$ exists, then $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}=\lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}$.

19. Inverse function

1. If f and g are two functions such that $f(g(x))=x$ for every x in the domain of g and $g(f(x))=x$ for every x in the domain of f, then f and g are inverse functions of each other.
2. A function f has an inverse if and only if no horizontal line intersects its graph more than once.
3. If f is strictly either increasing or decreasing in an interval, then f has an inverse.
4. If f is differentiable at every point on an interval I, and $f'(x)\neq 0$ on I, then $g=f^{-1}(x)$ is differentiable at every point of the interior of the interval $f(I)$ and if the point (a,b) is on $f(x)$, then the point (b,a) is on $g=f^{-1}(x)$; furthermore $g'(b)=\dfrac{1}{f'(a)}$.

20. Properties of $y=e^x$

1. The exponential function $y=e^x$ is the inverse function of $y=\ln x$.
2. The domain is the set of all real numbers, $-\infty<x<\infty$.
3. The range is the set of all positive numbers, $y>0$.
4. $\dfrac{d}{dx}(e^x)=e^x$ and $\dfrac{d}{dx}\left(e^{f(x)}\right)=f'(x)e^{f(x)}$
5. $e^{x_1}\cdot e^{x_2}=e^{x_1+x_2}$
6. $y=e^x$ is continuous, increasing, and concave up for all x.
7. $\lim\limits_{x\to\infty}e^x=+\infty$ and $\lim\limits_{x\to-\infty}e^x=0$.
8. $e^{\ln x}=x$, for $x>0$; $\ln(e^x)=x$ for all x.

21. Properties of $y=\ln x$

1. The domain of $y=\ln x$ is the set of all positive numbers, $x>0$.
2. The range of $y=\ln x$ is the set of all real numbers, $-\infty<y<\infty$.
3. $y=\ln x$ is continuous and increasing everywhere on its domain.
4. $\ln(ab)=\ln a+\ln b$.
5. $\ln\left(\dfrac{a}{b}\right)=\ln a-\ln b$.
6. $\ln a^r=r\ln a$.
7. $y=\ln x<0$ if $0<x<1$.
8. $\lim\limits_{x\to+\infty}\ln x=+\infty$ and $\lim\limits_{x\to 0^+}\ln x=-\infty$.
9. $\log_a x=\dfrac{\ln x}{\ln a}$

10. $\frac{d}{dx}\left(\ln f(x)\right)=\frac{f'(x)}{f(x)}$ and $\frac{d}{dx}\left(\ln(x)\right)=\frac{1}{x}$

22. Trapezoidal Rule

If a function f is continuous on the closed interval $[a,b]$ where $[a,b]$ has been ***equally*** partitioned into n subintervals $[x_0,x_1],[x_1,x_2],\ldots[x_{n-1},x_n]$, each length $\frac{b-a}{n}$, then $\int_a^b f(x)\,dx \approx \frac{b-a}{2n}\left[f(x_0)+2f(x_1)+2f(x_2)+\ldots+2f(x_{n-1})+f(x_n)\right]$, which is equivalent to $\frac{1}{2}\left(Leftsum+Rightsum\right)$

23a. Definition of Definite Integral as the Limit of a Sum

Suppose that a function $f(x)$ is continuous on the closed interval $[a,b]$. Divide the interval into n equal subintervals, of length $\Delta x=\frac{b-a}{n}$. Choose one number in each subinterval, in other words, x_1 in the first, x_2 in the second, …, x_k in the k th,…, and x_n in the n th. Then $\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x=\int_a^b f(x)\,dx=F(b)-F(a)$.

23b. Properties of the Definite Integral

Let $f(x)$ and $g(x)$ be continuous on $[a,b]$.

i). $\int_a^b c\cdot f(x)\,dx = c\int_a^b f(x)\,dx$ for any constant c.

ii). $\int_a^a f(x)\,dx = 0$

iii). $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

iv). $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, where f is continuous on an interval containing the numbers a, b, and c.

v). If $f(x)$ is an odd function, then $\int_{-a}^{a} f(x)\,dx = 0$

vi). If $f(x)$ is an even function, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$

vii). If $f(x)\ge 0$ on $[a,b]$, then $\int_a^b f(x)\,dx \ge 0$

viii). If $g(x)\ge f(x)$ on $[a,b]$, then $\int_a^b g(x)\,dx \ge \int_a^b f(x)\,dx$

24. Fundamental Theorem of Calculus:

$$\int_a^b f(x)\,dx = F(b) - F(a), \text{ where } F'(x) = f(x), \text{ or } \frac{d}{dx}\int_a^b f(x)\,dx = f(x).$$

25. Second Fundamental Theorem of Calculus (Steve's Theorem):

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x) \quad \text{or} \quad \frac{d}{dx}\int_{h(x)}^{g(x)} f(t)\,dt = g'(x)f(g(x)) - h'(x)f(h(x))$$

26. Velocity, Speed, and Acceleration

1. The velocity of an object tells how fast it is going and in which direction. Velocity is an instantaneous rate of change. If velocity is positive (graphically above the "x"-axis), then the object is moving away from its point of origin. If velocity is negative (graphically below the "x"-axis), then the object is moving back towards its point of origin. If velocity is 0 (graphically the point(s) where it hits the "x"-axis), then the object is not moving at that time.
2. The speed of an object is the absolute value of the velocity, $|v(t)|$. It tells how fast it is going disregarding its direction.
   The speed of a particle increases (speeds up) when the velocity and acceleration have the same signs. The speed decreases (slows down) when the velocity and acceleration have opposite signs.
3. The acceleration is the instantaneous rate of change of velocity – it is the derivative of the velocity – that is, $a(t) = v'(t)$. Negative acceleration (deceleration) means that the velocity is decreasing (i.e. the velocity graph would be going down at that time), and vice-versa for acceleration increasing. The acceleration gives the rate at which the velocity is changing.

Therefore, if x is the displacement of a moving object and t is time, then:

i) velocity = $v(t) = x'(t) = \dfrac{dx}{dt}$

ii) acceleration = $a(t) = x''(t) = v'(t) = \dfrac{dv}{dt} = \dfrac{d^2x}{dt^2}$

iii) $v(t) = \int a(t)\,dt$

iv) $x(t) = \int v(t)\,dt$

Note: The average velocity of a particle over the time interval from t_0 to another time t, is

Average Velocity = $\dfrac{\text{Change in position}}{\text{Length of time}} = \dfrac{s(t) - s(t_0)}{t - t_0}$, where $s(t)$ is the position of the particle at time t or $\dfrac{1}{b-a}\displaystyle\int_a^b v(t)\,dt$ if given the velocity function.

27. The average value of $f(x)$ on $[a,b]$ is $\dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$.

28. Area Between Curves

If f and g are continuous functions such that $f(x) \geq g(x)$ on $[a,b]$, then area between the curves is $\displaystyle\int_a^b [f(x) - g(x)]\,dx$ or $\displaystyle\int_a^b [top - bottom]\,dx$ or $\displaystyle\int_c^d [right - left]\,dy$.

29. ***Integration By "Parts"

If $u = f(x)$ and $v = g(x)$ and if $f'(x)$ and $g'(x)$ are continuous, then

$\int u\,dx = uv - \int v\,du$.

Note: The goal of the procedure is to choose u and dv so that $\int v\,du$ is easier to solve than the original problem.

Suggestion:

When "choosing" u, remember **L.I.A.T.E**, where **L** is the logarithmic function, **I** is an inverse trigonometric function, **A** is an algebraic function, **T** is a trigonometric function, and **E** is the exponential function. Just choose u as the first expression in **L.I.A.T.E** (and dv will be the remaining part of the integrand). For example, when integrating $\int x \ln x\,dx$, choose $u = \ln x$ since **L** comes first in **L.I.A.T.E**, and $dv = x\,dx$. When integrating $\int xe^{x}\,dx$, choose $u = x$, since x is an algebraic function, and **A** comes before **E** in **L.I.A.T.E**, and $dv = e^{x}dx$. One more example, when integrating $\int x\,Arc\tan(x)\,dx$, let $u = Arc\tan(x)$, since **I** comes before **A** in **L.I.A.T.E**, and $dv = x\,dx$.

30. Volume of Solids of Revolution (rectangles drawn perpendicular to the axis of revolution)

- Revolving around a horizontal line (y=# or x-axis) where $a \le x \le b$:

  Axis of Revolution and the region being revolved:

$$V = \pi\int_a^b (\textit{furthest from a.r.} - a.r.)^2 - (\textit{closest to a.r.} - a.r.)^2\,dx$$

- Revolving around a vertical line (x=# or y-axis) where $c \le y \le d$ (or use Shell Method):

  Axis of Revolution and the region being revolved:

$$V = \pi\int_c^d (\textit{furthest from a.r.} - a.r.)^2 - (\textit{closest to a.r.} - a.r.)^2\,dy$$

30b. Volume of Solids with Known Cross Sections

1. For cross sections of area $A(x)$, taken perpendicular to the x-axis, volume = $\int_a^b A(x)\,dx$.

Cross-sections {if only one function is used then just use that function, if it is between two functions use *top-bottom if perpendicular to the x-axis or right-left if perpendicular to the y-axis*} mostly all the same only varying by a constant, with the only exception being the rectangular cross-sections:

- Square cross-sections:

$$V = \int_a^b (\textit{top function} - \textit{bottom function})^2\,dx$$

- Equilateral cross-sections:

$$V = \frac{\sqrt{3}}{4}\int_a^b (\textit{top function} - \textit{bottom function})^2\,dx$$

- Isosceles Right Triangle cross-sections (hypotenuse in the *xy* plane):

$$V = \frac{1}{4}\int_a^b (\textit{top function} - \textit{bottom function})^2\,dx$$

- Isosceles Right Triangle cross-sections (leg in the *xy* plane):

$$V = \frac{1}{2}\int_a^b \left(top\ function - bottom\ function\right)^2\, dx$$

- Semi-circular cross-sections:

$$V = \frac{\pi}{8}\int_a^b \left(top\ function - bottom\ function\right)^2\, dx$$

- Rectangular cross-sections (height function or value must be given or articulated somehow – notice no "square" on the {top – bottom} part):

$$V = \int_a^b \left(top\ function - bottom\ function\right) \bullet \left(height\ function / value\right) dx$$

- Circular cross-sections with the diameter in the *xy* plane:

$$V = \frac{\pi}{4}\int_a^b \left(top\ function - bottom\ function\right)^2\, dx$$

- Square cross-sections with the diagonal in the *xy* plane:

$$V = \frac{1}{2}\int_a^b \left(top\ function - bottom\ function\right)^2\, dx$$

2. For cross sections of area $A(y)$, taken perpendicular to the y-axis, volume = $\int_a^b A(y)\, dy$.

30c. ***Shell Method (used if function is in terms of x and revolving around a vertical line) where $a \le x \le b$:

$$V = 2\pi\int_a^b r(x)h(x)dx$$

$r(x) = x$ if a.r. is y-axis ($x = 0$)

$r(x) = (x - a.r.)$ if a.r. is to the left of the region

$r(x) = (a.r. - x)$ if a.r. is to the right of the region

$h(x) = f(x)$ if only revolving with one function

$h(x) = \left(top - bottom\right)$ if revolving the region between two functions

31. Solving Differential Equations: Graphically and Numerically

Slope Fields

At every point $(\mathrm{x},\ \mathrm{y})$ a differential equation of the form $\frac{\mathrm{dy}}{\mathrm{dx}} = f(\mathrm{x},\ \mathrm{y})$ gives the slope of the member of the family of solutions that contains that point. A slope field is a graphical representation of this family of curves. At each point in the plane, a short segment is drawn whose slope is equal to the value of the derivative at that point. These segments are tangent to the solution's graph at the point.

The slope field allows you to sketch the graph of the solution curve even though you do not have its equation. This is done by starting at any point (usually the point given by the initial condition), and moving from one point to the next in the direction indicated by the segments of the slope field.

Some calculators have built in operations for drawing slope fields; for calculators without this feature there are programs available for drawing them.

***Euler's Method (BC topic)

Euler's Method is a way of approximating points on the solution of a differential equation $\frac{dy}{dx} = f(x, y)$. The calculation uses the tangent line approximation to move from one point to the next. That is, starting with the given point (x_1, y_1) – the initial condition, the point $(x_1 + \Delta x, y_1 + f'(x_1, y_1)\Delta x)$ approximates a nearby point on the solution graph. This aproximation may then be used as the starting point to calculate a third point and so on. The accuracy of the method decreases with large values of Δx. The error increases as each successive point is used to find the next.

$(x, y): given$	$\frac{dy}{dx}: given$	$\Delta x: given$	$\Delta y = \frac{dy}{dx}\Delta x$	$(x+\Delta x, y+\Delta y)$
Start again				

32. ***Logistics (BC topic)

1. Rate is jointly proportional to its size and the difference between a fixed positive number (L) and its size.

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right) \text{ OR } \frac{dy}{dt} = ky(M - y) \text{ which yields}$$

$$y = \frac{L}{1 + Ce^{-kt}} \quad \text{through separation of variables}$$

2. $\lim_{t \to \infty} y = L$; L = carrying capacity (Maximum); horizontal asymptote

3. y-coordinate of inflection point is $\frac{L}{2}$, i.e. when it is growing the fastest (or max rate).

32(a). ***Decomposition:

Steps:

1. Use Long Division first if the degree of the Numerator is equal or more than the Denominator to get $\int \frac{N(x)}{D(x)}dx = \int q(x)dx + \int \frac{r(x)}{D(x)}dx$
2. For the second integral, factor $D(x)$ completely into Linear factors to get $\frac{r(x)}{D(x)} = \frac{A}{linearfactor\#1} + \frac{B}{linearfactor\#2} + \ldots$
3. Multiply both sides by $D(x)$ to eliminate the fractions
4. Choose your x-values wisely so that you can easily solve for A, B, C, etc
5. Rewrite your integral that has been decomposed and integrate everything.

33. ***Definition of Arc Length

If the function given by $y = f(x)$ represents a smooth curve on the interval $[a,b]$, then the arc length of f between a and b is given by $s = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$.

34. ***<u>Improper Integral</u>

$\int_a^b f(x)\,dx$ is an improper integral if

1. f becomes infinite at one or more points of the interval of integration, or
2. one or both of the limits of integration is infinite, or
3. both (1) and (2) hold.

35. ***<u>Parametric Form of the Derivative</u>

If a smooth curve **C** is given by the parametric equations $x = f(x)$ and $y = g(t)$, then the slope of the curve **C** at (x, y) is $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt},\ \frac{dx}{dt} \neq 0$.

<u>Note</u>: The second derivative, $\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dt}\left[\frac{dy}{dx}\right] \div \frac{dx}{dt}$.

36. ***<u>Arc Length in Parametric Form</u>

If a smooth curve **C** is given by $x = f(t)$ and $y = g(t)$ and these functions have continuous first derivatives with respect to t for $a \leq t \leq b$, and if the point $P(x, y)$ traces the curve exactly once as t moves from $t = a$ to $t = b$, then the length of the curve is given by

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_a^b \sqrt{\left(f'(t)\right)^2 + \left(g'(t)\right)^2}\,dt.$$

$$\text{speed} = \sqrt{\left(\mathrm{f}'(\mathrm{t})\right)^2 + \left(\mathrm{g}'(\mathrm{t})\right)^2}$$

37. ***<u>**Vectors**</u>

Velocity, speed, acceleration, and direction of motion in Vector form

- position vector is $r(t) = \langle x(t), y(t) \rangle$
- velocity vector is $v(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle$
- speed is the magnitude of velocity because $speed = |v(t)| = \sqrt{\left(\frac{dx}{dx}\right)^2 + \left(\frac{dy}{dt}\right)^2}$
- acceleration vector is $a(t) = \left\langle \frac{d^2x}{dt^2}, \frac{d^2y}{dt^2} \right\rangle$
- the direction of motion is based on the velocity vector and the signs on its components

Displacement and distance travelled in vector form

- Displacement in vector form $\left\langle \int_a^b v_1(t)\,dt, \int_a^b v_2(t)\,dt \right\rangle$
- Final position in vector form $\left(x_1 + \int_a^b v_1(t)\,dt, x_2 + \int_a^b v_2(t)\,dt \right)$
- Distance travelled from

  $t = a$ to $t = b$ is given by $\int_a^b |v(t)|\,dt = \int_a^b \sqrt{\left(v_1(t)\right)^2 + \left(v_2(t)\right)^2}\,dt$

38. ***<u>Polar Coordinates</u>

1. Cartesian vs. Polar Coordinates. The polar coordinates (r,θ) are related to the Cartesian coordinates (x,y) as follows:

$$x = r\cos\theta \text{ and } y = r\sin\theta$$

$$\tan\theta = \frac{y}{x} \text{ and } x^2 + y^2 = r^2$$

2. To find the points of intersection of two polar curves, find (r,θ) satisfying the first equation for which some points $(r,\theta+2n\pi)$ or $(-r,\theta+\pi+2n\pi)$ satisfy the second equation. Check separately to see if the origin lies on both curves, i.e. if r can be 0. Sketch the curves.

3. Area in Polar Coordinates: If f is continuous and nonnegative on the interval $[\alpha,\beta]$, then the area of the region bounded by the graph of $r = f(\theta)$ between the radial lines $\theta = \alpha$ and $\theta = \beta$ is given by

$$A = \frac{1}{2}\int_{\alpha}^{\beta}\left(f(\theta)\right)^2 d\theta = \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta$$

4. Derivative of Polar function: Given $r = f(\theta)$, to find the derivative, use parametric equations.

$$x = r\cos\theta = f(\theta)\cos\theta \quad \text{and} \quad y = r\sin\theta = f(\theta)\sin\theta.$$

Then $$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta}$$

5. Arc Length in Polar Form: $s = \int_{\alpha}^{\beta}\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

39. ***<u>Sequences and Series</u>

1. If a sequence $\{a_n\}$ has a limit L, that is, $\lim_{n\to\infty} a_n = L$, then the sequence is said to converge to L. If there is no limit, the series diverges. If the sequence $\{a_n\}$ converges, then its limit is unique. Keep in mind that

$\lim_{n\to\infty}\frac{\ln n}{n} = 0$; $\lim_{n\to\infty} x^{\left(\frac{1}{n}\right)} = 1$; $\lim_{n\to\infty}\sqrt[n]{n} = 1$; $\lim_{n\to\infty}\frac{x^n}{n!} = 0$. These limits are useful and arise frequently.

2. The harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges; the geometric series $\sum_{n=0}^{\infty} ar^n$ converges to $\frac{a}{1-r}$ if $|r| < 1$ and diverges if $|r| \geq 1$ and $a \neq 0$.

3. The p-series $\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

4. Limit Comparison Test: Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be a series of nonnegative terms, with $a_n \neq 0$ for all sufficiently large n, and suppose that $\lim_{n\to\infty} \frac{b_n}{a_n} = c > 0$. Then the two series either both converge or both diverge.

5. Alternating Series: Let $\sum_{n=1}^{\infty} a_n$ be a series such that
   - i) the series is alternating
   - ii) $\left| a_{n+1} \right| \leq \left| a_n \right|$ for all n, and
   - iii) $\lim_{n\to\infty} a_n = 0$

   Then the series *converges*.

   Alternating Series Remainder: The remainder R_N is less than (or equal to) the first neglected term

$$\left| R_N \right| \leq a_{N+1}$$

6. The *n*-th Term Test for Divergence: If $\lim_{n\to\infty} a_n \neq 0$, then the series diverges. Note that the converse is *false*, that is, if $\lim_{n\to\infty} a_n = 0$, the series may or may not converge.

7. A series $\sum a_n$ is absolutely convergent if the series $\sum |a_n|$ converges. If $\sum a_n$ converges, but $\sum |a_n|$ does not converge, then the series is conditionally convergent. Keep in mind that if $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

8. Comparison Test: If $0 \leq a_n \leq b_n$ for all sufficiently large n, and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

9. Integral Test: If $f(x)$ is a positive, continuous, and decreasing function on $[1,\infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ will converge if the improper integral $\int_1^{\infty} f(x)\,dx$ converges. If the improper integral $\int_1^{\infty} f(x)\,dx$ diverges, then the infinite series $\sum_{n=1}^{\infty} a_n$ diverges.

10. Ratio Test: Let $\sum a_n$ be a series with nonzero terms.

i) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$, then the series converges absolutely.

ii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$, then the series is divergent.

iii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, then the test is inconclusive (and another test must be used).

11. Power Series: A power series is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n + \ldots$$ or

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots + c_n(x-a)^n + \ldots$$ in which the center a and the coefficients $c_0, c_1, c_2, \ldots, c_n, \ldots$ are constants. The set of all numbers x for which the power series converges is called the interval of convergence.

12. Taylor Series: Let f be a function with derivatives of all orders throughout some intervale containing a as an interior point. Then the Taylor series generated by f at a is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots$$

The remaining terms after the term containing the nth derivative can be expressed as a remainder to Taylor's Theorem:

$$f(x) = f(a) + \sum_{1}^{n} f^{(n)}(a)(x-a)^n + R_n(x) \text{ where } R_n(x) = \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\,dt$$

Lagrange's form of the remainder: $|f(x) - P_n(x)| = |R_n x| = \frac{f^{(n+1)}(c)\,|(x-a)|^{n+1}}{(n+1)!}$

, where $a < c < x$.

The series will converge for all values of x for which the remainder approaches zero as $x \to \infty$.

13. Frequently Used Series and their Interval of Convergence

$$\frac{1}{1-x} = 1 + x + x^2 + \ldots + x^n + \ldots = \sum_{n=0}^{\infty} x^n \text{ , } |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ , } |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \text{ , } |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)x^{2n}}{(2n)!} \text{ , } |x| < \infty$$

Limits

Definitions

Precise Definition : We say $\lim_{x\to a} f(x) = L$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that whenever $0 < |x-a| < \delta$ then $|f(x) - L| < \varepsilon$.

"Working" Definition : We say $\lim_{x\to a} f(x) = L$ if we can make $f(x)$ as close to L as we want by taking x sufficiently close to a (on either side of a) without letting $x = a$.

Right hand limit : $\lim_{x\to a^+} f(x) = L$. This has the same definition as the limit except it requires $x > a$.

Left hand limit : $\lim_{x\to a^-} f(x) = L$. This has the same definition as the limit except it requires $x < a$.

Limit at Infinity : We say $\lim_{x\to\infty} f(x) = L$ if we can make $f(x)$ as close to L as we want by taking x large enough and positive.

There is a similar definition for $\lim_{x\to-\infty} f(x) = L$ except we require x large and negative.

Infinite Limit : We say $\lim_{x\to a} f(x) = \infty$ if we can make $f(x)$ arbitrarily large (and positive) by taking x sufficiently close to a (on either side of a) without letting $x = a$.

There is a similar definition for $\lim_{x\to a} f(x) = -\infty$ except we make $f(x)$ arbitrarily large and negative.

Relationship between the limit and one-sided limits

$\lim_{x\to a} f(x) = L \;\Rightarrow\; \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L$ $\qquad$ $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L \;\Rightarrow\; \lim_{x\to a} f(x) = L$

$\lim_{x\to a^+} f(x) \neq \lim_{x\to a^-} f(x) \;\Rightarrow\; \lim_{x\to a} f(x)$ Does Not Exist

Properties

Assume $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist and c is any number then,

1. $\lim_{x\to a}\left[c f(x)\right] = c\lim_{x\to a} f(x)$
2. $\lim_{x\to a}\left[f(x) \pm g(x)\right] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
3. $\lim_{x\to a}\left[f(x) g(x)\right] = \lim_{x\to a} f(x)\,\lim_{x\to a} g(x)$
4. $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ provided $\lim_{x\to a} g(x) \neq 0$
5. $\lim_{x\to a}\left[f(x)\right]^n = \left[\lim_{x\to a} f(x)\right]^n$
6. $\lim_{x\to a}\left[\sqrt[n]{f(x)}\right] = \sqrt[n]{\lim_{x\to a} f(x)}$

Basic Limit Evaluations at $\pm\infty$

Note : $\operatorname{sgn}(a) = 1$ if $a > 0$ and $\operatorname{sgn}(a) = -1$ if $a < 0$.

1. $\lim_{x\to\infty} \mathbf{e}^x = \infty$ & $\lim_{x\to-\infty} \mathbf{e}^x = 0$
2. $\lim_{x\to\infty} \ln(x) = \infty$ & $\lim_{x\to 0^+} \ln(x) = -\infty$
3. If $r > 0$ then $\lim_{x\to\infty} \frac{b}{x^r} = 0$
4. If $r > 0$ and x^r is real for negative x then $\lim_{x\to-\infty} \frac{b}{x^r} = 0$
5. n even : $\lim_{x\to\pm\infty} x^n = \infty$
6. n odd : $\lim_{x\to\infty} x^n = \infty$ & $\lim_{x\to-\infty} x^n = -\infty$
7. n even : $\lim_{x\to\pm\infty} a\,x^n + \cdots + b\,x + c = \operatorname{sgn}(a)\infty$
8. n odd : $\lim_{x\to\infty} a\,x^n + \cdots + b\,x + c = \operatorname{sgn}(a)\infty$
9. n odd : $\lim_{x\to-\infty} a\,x^n + \cdots + c\,x + d = -\operatorname{sgn}(a)\infty$

Evaluation Techniques

Continuous Functions

If $f(x)$ is continuous at a then $\lim_{x \to a} f(x) = f(a)$

Continuous Functions and Composition

$f(x)$ is continuous at b and $\lim_{x \to a} g(x) = b$ then

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right) = f(b)$$

Factor and Cancel

$$\lim_{x \to 2} \frac{x^2 + 4x - 12}{x^2 - 2x} = \lim_{x \to 2} \frac{(x-2)(x+6)}{x(x-2)}$$

$$= \lim_{x \to 2} \frac{x+6}{x} = \frac{8}{2} = 4$$

Rationalize Numerator/Denominator

$$\lim_{x \to 9} \frac{3 - \sqrt{x}}{x^2 - 81} = \lim_{x \to 9} \frac{3 - \sqrt{x}}{x^2 - 81} \frac{3 + \sqrt{x}}{3 + \sqrt{x}}$$

$$= \lim_{x \to 9} \frac{9 - x}{(x^2 - 81)(3 + \sqrt{x})} = \lim_{x \to 9} \frac{-1}{(x+9)(3 + \sqrt{x})}$$

$$= \frac{-1}{(18)(6)} = -\frac{1}{108}$$

Combine Rational Expressions

$$\lim_{h \to 0} \frac{1}{h}\left(\frac{1}{x+h} - \frac{1}{x}\right) = \lim_{h \to 0} \frac{1}{h}\left(\frac{x - (x+h)}{x(x+h)}\right)$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{-h}{x(x+h)}\right) = \lim_{h \to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2}$$

L'Hospital's Rule

If $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0}$ or $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\pm\infty}{\pm\infty}$ then,

$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$ a is a number, ∞ or $-\infty$

Polynomials at Infinity

$p(x)$ and $q(x)$ are polynomials. To compute

$\lim_{x \to \pm\infty} \frac{p(x)}{q(x)}$ factor largest power of x in $q(x)$ out

of both $p(x)$ and $q(x)$ then compute limit.

$$\lim_{x \to -\infty} \frac{3x^2 - 4}{5x - 2x^2} = \lim_{x \to -\infty} \frac{x^2\left(3 - \frac{4}{x^2}\right)}{x^2\left(\frac{5}{x} - 2\right)} = \lim_{x \to -\infty} \frac{3 - \frac{4}{x^2}}{\frac{5}{x} - 2} = -\frac{3}{2}$$

Piecewise Function

$\lim_{x \to -2} g(x)$ where $g(x) = \begin{cases} x^2 + 5 & \text{if } x < -2 \\ 1 - 3x & \text{if } x \ge -2 \end{cases}$

Compute two one sided limits,

$$\lim_{x \to -2^-} g(x) = \lim_{x \to -2^-} x^2 + 5 = 9$$

$$\lim_{x \to -2^+} g(x) = \lim_{x \to -2^+} 1 - 3x = 7$$

One sided limits are different so $\lim_{x \to -2} g(x)$ doesn't exist. If the two one sided limits had been equal then $\lim_{x \to -2} g(x)$ would have existed and had the same value.

Some Continuous Functions

Partial list of continuous functions and the values of x for which they are continuous.

1. Polynomials for all x.
2. Rational function, except for x's that give division by zero.
3. $\sqrt[n]{x}$ (n odd) for all x.
4. $\sqrt[n]{x}$ (n even) for all $x \ge 0$.
5. $\mathbf{e}^x$ for all x.
6. $\ln x$ for $x > 0$.
7. $\cos(x)$ and $\sin(x)$ for all x.
8. $\tan(x)$ and $\sec(x)$ provided $x \ne \cdots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \cdots$
9. $\cot(x)$ and $\csc(x)$ provided $x \ne \cdots, -2\pi, -\pi, 0, \pi, 2\pi, \cdots$

Intermediate Value Theorem

Suppose that $f(x)$ is continuous on $[a, b]$ and let M be any number between $f(a)$ and $f(b)$.

Then there exists a number c such that $a < c < b$ and $f(c) = M$.

Derivatives

Definition and Notation

If $y = f(x)$ then the derivative is defined to be $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

If $y = f(x)$ then all of the following are equivalent notations for the derivative.

$$f'(x) = y' = \frac{df}{dx} = \frac{dy}{dx} = \frac{d}{dx}(f(x)) = Df(x)$$

If $y = f(x)$ all of the following are equivalent notations for derivative evaluated at $x = a$.

$$f'(a) = y'\big|_{x=a} = \frac{df}{dx}\bigg|_{x=a} = \frac{dy}{dx}\bigg|_{x=a} = Df(a)$$

Interpretation of the Derivative

If $y = f(x)$ then,

1. $m = f'(a)$ is the slope of the tangent line to $y = f(x)$ at $x = a$ and the equation of the tangent line at $x = a$ is given by $y = f(a) + f'(a)(x-a)$.
2. $f'(a)$ is the instantaneous rate of change of $f(x)$ at $x = a$.
3. If $f(x)$ is the position of an object at time x then $f'(a)$ is the velocity of the object at $x = a$.

Basic Properties and Formulas

If $f(x)$ and $g(x)$ are differentiable functions (the derivative exists), c and n are any real numbers,

1. $(c\,f)' = c\,f'(x)$
2. $(f \pm g)' = f'(x) \pm g'(x)$
3. $(f\,g)' = f'\,g + f\,g'$ – **Product Rule**
4. $\left(\frac{f}{g}\right)' = \frac{f'\,g - f\,g'}{g^2}$ – **Quotient Rule**
5. $\frac{d}{dx}(c) = 0$
6. $\frac{d}{dx}(x^n) = n\,x^{n-1}$ – **Power Rule**
7. $\frac{d}{dx}(f(g(x))) = f'(g(x))\,g'(x)$

   This is the **Chain Rule**

Common Derivatives

$\frac{d}{dx}(x) = 1$

$\frac{d}{dx}(\sin x) = \cos x$

$\frac{d}{dx}(\cos x) = -\sin x$

$\frac{d}{dx}(\tan x) = \sec^2 x$

$\frac{d}{dx}(\sec x) = \sec x \tan x$

$\frac{d}{dx}(\csc x) = -\csc x \cot x$

$\frac{d}{dx}(\cot x) = -\csc^2 x$

$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

$\frac{d}{dx}(a^x) = a^x \ln(a)$

$\frac{d}{dx}(\mathbf{e}^x) = \mathbf{e}^x$

$\frac{d}{dx}(\ln(x)) = \frac{1}{x}, \quad x > 0$

$\frac{d}{dx}(\ln|x|) = \frac{1}{x}, \quad x \neq 0$

$\frac{d}{dx}(\log_a(x)) = \frac{1}{x \ln a}, \quad x > 0$

Chain Rule Variants

The chain rule applied to some specific functions.

1. $\frac{d}{dx}\left(\left[f(x)\right]^n\right) = n\left[f(x)\right]^{n-1} f'(x)$
2. $\frac{d}{dx}\left(\mathbf{e}^{f(x)}\right) = f'(x)\mathbf{e}^{f(x)}$
3. $\frac{d}{dx}\left(\ln\left[f(x)\right]\right) = \frac{f'(x)}{f(x)}$
4. $\frac{d}{dx}\left(\sin\left[f(x)\right]\right) = f'(x)\cos\left[f(x)\right]$
5. $\frac{d}{dx}\left(\cos\left[f(x)\right]\right) = -f'(x)\sin\left[f(x)\right]$
6. $\frac{d}{dx}\left(\tan\left[f(x)\right]\right) = f'(x)\sec^2\left[f(x)\right]$
7. $\frac{d}{dx}\left(\sec\left[f(x)\right]\right) = f'(x)\sec\left[f(x)\right]\tan\left[f(x)\right]$
8. $\frac{d}{dx}\left(\tan^{-1}\left[f(x)\right]\right) = \frac{f'(x)}{1+\left[f(x)\right]^2}$

Higher Order Derivatives

The Second Derivative is denoted as

$f''(x) = f^{(2)}(x) = \frac{d^2 f}{dx^2}$ and is defined as

$f''(x) = \left(f'(x)\right)'$, *i.e.* the derivative of the first derivative, $f'(x)$.

The nth Derivative is denoted as

$f^{(n)}(x) = \frac{d^n f}{dx^n}$ and is defined as

$f^{(n)}(x) = \left(f^{(n-1)}(x)\right)'$, *i.e.* the derivative of the (n-1)st derivative, $f^{(n-1)}(x)$.

Implicit Differentiation

Find y' if $\mathbf{e}^{2x-9y} + x^3y^2 = \sin(y) + 11x$. Remember $y = y(x)$ here, so products/quotients of x and y will use the product/quotient rule and derivatives of y will use the chain rule. The "trick" is to differentiate as normal and every time you differentiate a y you tack on a y' (from the chain rule). After differentiating solve for y'.

$$\mathbf{e}^{2x-9y}\left(2-9y'\right) + 3x^2y^2 + 2x^3y\,y' = \cos(y)\,y' + 11$$
$$2\mathbf{e}^{2x-9y} - 9y'\mathbf{e}^{2x-9y} + 3x^2y^2 + 2x^3y\,y' = \cos(y)\,y' + 11 \qquad \Rightarrow \qquad y' = \frac{11-2\mathbf{e}^{2x-9y}-3x^2y^2}{2x^3y-9\mathbf{e}^{2x-9y}-\cos(y)}$$
$$\left(2x^3y - 9\mathbf{e}^{2x-9y} - \cos(y)\right)y' = 11 - 2\mathbf{e}^{2x-9y} - 3x^2y^2$$

Increasing/Decreasing – Concave Up/Concave Down

Critical Points

$x = c$ is a critical point of $f(x)$ provided either

1. $f'(c) = 0$ or **2.** $f'(c)$ doesn't exist.

Increasing/Decreasing

1. If $f'(x) > 0$ for all x in an interval I then $f(x)$ is increasing on the interval I.
2. If $f'(x) < 0$ for all x in an interval I then $f(x)$ is decreasing on the interval I.
3. If $f'(x) = 0$ for all x in an interval I then $f(x)$ is constant on the interval I.

Concave Up/Concave Down

1. If $f''(x) > 0$ for all x in an interval I then $f(x)$ is concave up on the interval I.
2. If $f''(x) < 0$ for all x in an interval I then $f(x)$ is concave down on the interval I.

Inflection Points

$x = c$ is a inflection point of $f(x)$ if the concavity changes at $x = c$.

Extrema

Absolute Extrema

1. $x = c$ is an absolute maximum of $f(x)$ if $f(c) \ge f(x)$ for all x in the domain.
2. $x = c$ is an absolute minimum of $f(x)$ if $f(c) \le f(x)$ for all x in the domain.

Fermat's Theorem

If $f(x)$ has a relative (or local) extrema at $x = c$, then $x = c$ is a critical point of $f(x)$.

Extreme Value Theorem

If $f(x)$ is continuous on the closed interval $[a,b]$ then there exist numbers c and d so that, **1.** $a \le c, d \le b$, **2.** $f(c)$ is the abs. max. in $[a,b]$, **3.** $f(d)$ is the abs. min. in $[a,b]$.

Finding Absolute Extrema

To find the absolute extrema of the continuous function $f(x)$ on the interval $[a,b]$ use the following process.

1. Find all critical points of $f(x)$ in $[a,b]$.
2. Evaluate $f(x)$ at all points found in Step 1.
3. Evaluate $f(a)$ and $f(b)$.
4. Identify the abs. max. (largest function value) and the abs. min.(smallest function value) from the evaluations in Steps 2 & 3.

Relative (local) Extrema

1. $x = c$ is a relative (or local) maximum of $f(x)$ if $f(c) \ge f(x)$ for all x near c.
2. $x = c$ is a relative (or local) minimum of $f(x)$ if $f(c) \le f(x)$ for all x near c.

1st Derivative Test

If $x = c$ is a critical point of $f(x)$ then $x = c$ is

1. a rel. max. of $f(x)$ if $f'(x) > 0$ to the left of $x = c$ and $f'(x) < 0$ to the right of $x = c$.
2. a rel. min. of $f(x)$ if $f'(x) < 0$ to the left of $x = c$ and $f'(x) > 0$ to the right of $x = c$.
3. not a relative extrema of $f(x)$ if $f'(x)$ is the same sign on both sides of $x = c$.

2nd Derivative Test

If $x = c$ is a critical point of $f(x)$ such that $f'(c) = 0$ then $x = c$

1. is a relative maximum of $f(x)$ if $f''(c) < 0$.
2. is a relative minimum of $f(x)$ if $f''(c) > 0$.
3. may be a relative maximum, relative minimum, or neither if $f''(c) = 0$.

Finding Relative Extrema and/or Classify Critical Points

1. Find all critical points of $f(x)$.
2. Use the 1st derivative test or the 2nd derivative test on each critical point.

Mean Value Theorem

If $f(x)$ is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b) then there is a number $a < c < b$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

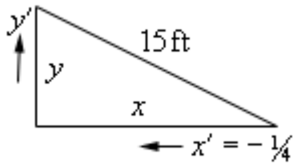
Newton's Method

If x_n is the n^{th} guess for the root/solution of $f(x) = 0$ then $(n+1)^{\text{st}}$ guess is $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ provided $f'(x_n)$ exists.

Related Rates

Sketch picture and identify known/unknown quantities. Write down equation relating quantities and differentiate with respect to t using implicit differentiation (*i.e.* add on a derivative every time you differentiate a function of t). Plug in known quantities and solve for the unknown quantity.

Ex. A 15 foot ladder is resting against a wall. The bottom is initially 10 ft away and is being pushed towards the wall at $\frac{1}{4}$ ft/sec. How fast is the top moving after 12 sec?



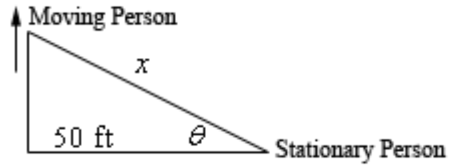
x' is negative because x is decreasing. Using Pythagorean Theorem and differentiating,

$$x^2 + y^2 = 15^2 \quad \Rightarrow \quad 2x\,x' + 2y\,y' = 0$$

After 12 sec we have $x = 10 - 12\left(\frac{1}{4}\right) = 7$ and so $y = \sqrt{15^2 - 7^2} = \sqrt{176}$. Plug in and solve for y'.

$$7\left(-\tfrac{1}{4}\right) + \sqrt{176}\; y' = 0 \;\Rightarrow\; y' = \frac{7}{4\sqrt{176}} \text{ ft/sec}$$

Ex. Two people are 50 ft apart when one starts walking north. The angle θ changes at 0.01 rad/min. At what rate is the distance between them changing when $\theta = 0.5$ rad?



We have $\theta' = 0.01$ rad/min. and want to find x'. We can use various trig fcns but easiest is,

$$\sec\theta = \frac{x}{50} \quad \Rightarrow \quad \sec\theta\tan\theta\;\theta' = \frac{x'}{50}$$

We know $\theta = 0.5$ so plug in θ' and solve.

$$\sec(0.5)\tan(0.5)(0.01) = \frac{x'}{50}$$

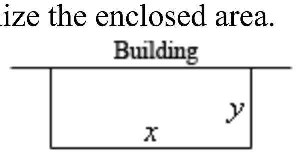
$$x' = 0.3112 \text{ ft/sec}$$

Remember to have calculator in radians!

Optimization

Sketch picture if needed, write down equation to be optimized and constraint. Solve constraint for one of the two variables and plug into first equation. Find critical points of equation in range of variables and verify that they are min/max as needed.

Ex. We're enclosing a rectangular field with 500 ft of fence material and one side of the field is a building. Determine dimensions that will maximize the enclosed area.



Maximize $A = xy$ subject to constraint of $x + 2y = 500$. Solve constraint for x and plug into area.

$$x = 500 - 2y \quad \Rightarrow \quad \begin{aligned} A &= y(500 - 2y) \\ &= 500y - 2y^2 \end{aligned}$$

Differentiate and find critical point(s).

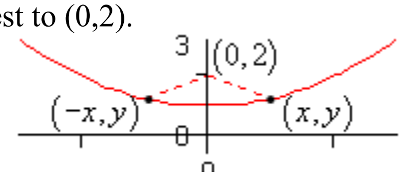
$$A' = 500 - 4y \quad \Rightarrow \quad y = 125$$

By 2nd deriv. test this is a rel. max. and so is the answer we're after. Finally, find x.

$$x = 500 - 2(125) = 250$$

The dimensions are then 250 x 125.

Ex. Determine point(s) on $y = x^2 + 1$ that are closest to (0,2).



Minimize $f = d^2 = (x-0)^2 + (y-2)^2$ and the constraint is $y = x^2 + 1$. Solve constraint for x^2 and plug into the function.

$$x^2 = y - 1 \;\Rightarrow\; f = x^2 + (y-2)^2$$

$$= y - 1 + (y-2)^2 = y^2 - 3y + 3$$

Differentiate and find critical point(s).

$$f' = 2y - 3 \quad \Rightarrow \quad y = \tfrac{3}{2}$$

By the 2nd derivative test this is a rel. min. and so all we need to do is find x value(s).

$$x^2 = \tfrac{3}{2} - 1 = \tfrac{1}{2} \quad \Rightarrow \quad x = \pm\tfrac{1}{\sqrt{2}}$$

The 2 points are then $\left(\frac{1}{\sqrt{2}}, \frac{3}{2}\right)$ and $\left(-\frac{1}{\sqrt{2}}, \frac{3}{2}\right)$.

Integrals

Definitions

Definite Integral: Suppose $f(x)$ is continuous on $[a,b]$. Divide $[a,b]$ into n subintervals of width Δx and choose x_i^* from each interval.

Then $\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$.

Anti-Derivative : An anti-derivative of $f(x)$ is a function, $F(x)$, such that $F'(x) = f(x)$.

Indefinite Integral : $\int f(x)\,dx = F(x) + c$ where $F(x)$ is an anti-derivative of $f(x)$.

Fundamental Theorem of Calculus

Part I : If $f(x)$ is continuous on $[a,b]$ then $g(x) = \int_a^x f(t)\,dt$ is also continuous on $[a,b]$ and $g'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$.

Part II : $f(x)$ is continuous on $[a,b]$, $F(x)$ is an anti-derivative of $f(x)$ (*i.e.* $F(x) = \int f(x)\,dx$) then $\int_a^b f(x)\,dx = F(b) - F(a)$.

Variants of Part I :

$$\frac{d}{dx}\int_a^{u(x)} f(t)\,dt = u'(x)\,f[u(x)]$$

$$\frac{d}{dx}\int_{v(x)}^{b} f(t)\,dt = -v'(x)\,f[v(x)]$$

$$\frac{d}{dx}\int_{v(x)}^{u(x)} f(t)\,dt = u'(x)\,f[u(x)] - v'(x)\,f[v(x)]$$

Properties

$\int f(x) \pm g(x)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

$\int cf(x)\,dx = c\int f(x)\,dx$, c is a constant

$\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, c is a constant

$\int_a^a f(x)\,dx = 0$

$\int_a^b c\,dx = c(b-a)$

$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$

$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ for any value of c.

If $f(x) \ge g(x)$ on $a \le x \le b$ then $\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

If $f(x) \ge 0$ on $a \le x \le b$ then $\int_a^b f(x)\,dx \ge 0$

If $m \le f(x) \le M$ on $a \le x \le b$ then $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$

Common Integrals

$\int k\,dx = k\,x + c$

$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c,\ n \ne -1$

$\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln|x| + c$

$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + c$

$\int \ln u\,du = u\ln(u) - u + c$

$\int \mathbf{e}^u\,du = \mathbf{e}^u + c$

$\int \cos u\,du = \sin u + c$

$\int \sin u\,du = -\cos u + c$

$\int \sec^2 u\,du = \tan u + c$

$\int \sec u \tan u\,du = \sec u + c$

$\int \csc u \cot u\,du = -\csc u + c$

$\int \csc^2 u\,du = -\cot u + c$

$\int \tan u\,du = \ln|\sec u| + c$

$\int \sec u\,du = \ln|\sec u + \tan u| + c$

$\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + c$

$\int \frac{1}{\sqrt{a^2-u^2}}\,du = \sin^{-1}\left(\frac{u}{a}\right) + c$

Standard Integration Techniques

Note that at many schools all but the Substitution Rule tend to be taught in a Calculus II class.

***u* Substitution :** The substitution $u = g(x)$ will convert $\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$ using $du = g'(x)\,dx$. For indefinite integrals drop the limits of integration.

Ex. $\int_1^2 5x^2 \cos(x^3)\,dx$

$u = x^3 \;\Rightarrow\; du = 3x^2 dx \;\Rightarrow\; x^2 dx = \frac{1}{3} du$

$x = 1 \;\Rightarrow\; u = 1^3 = 1 \;::\; x = 2 \;\Rightarrow\; u = 2^3 = 8$

$$\int_1^2 5x^2 \cos(x^3)\,dx = \int_1^8 \tfrac{5}{3}\cos(u)\,du$$
$$= \tfrac{5}{3}\sin(u)\Big|_1^8 = \tfrac{5}{3}\left(\sin(8) - \sin(1)\right)$$

Integration by Parts : $\int u\,dv = uv - \int v\,du$ and $\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$. Choose u and dv from integral and compute du by differentiating u and compute v using $v = \int dv$.

Ex. $\int x\mathbf{e}^{-x}\,dx$

$u = x \quad dv = \mathbf{e}^{-x} \quad \Rightarrow \quad du = dx \quad v = -\mathbf{e}^{-x}$

$$\int x\mathbf{e}^{-x}\,dx = -x\mathbf{e}^{-x} + \int \mathbf{e}^{-x}\,dx = -x\mathbf{e}^{-x} - \mathbf{e}^{-x} + c$$

Ex. $\int_3^5 \ln x\,dx$

$u = \ln x \quad dv = dx \quad \Rightarrow \quad du = \frac{1}{x} dx \quad v = x$

$$\int_3^5 \ln x\,dx = x\ln x\Big|_3^5 - \int_3^5 dx = \left(x\ln(x) - x\right)\Big|_3^5$$
$$= 5\ln(5) - 3\ln(3) - 2$$

Products and (some) Quotients of Trig Functions

For $\int \sin^n x \cos^m x\,dx$ we have the following :

1. ***n* odd.** Strip 1 sine out and convert rest to cosines using $\sin^2 x = 1 - \cos^2 x$, then use the substitution $u = \cos x$.
2. ***m* odd.** Strip 1 cosine out and convert rest to sines using $\cos^2 x = 1 - \sin^2 x$, then use the substitution $u = \sin x$.
3. ***n* and *m* both odd.** Use either 1. or 2.
4. ***n* and *m* both even.** Use double angle and/or half angle formulas to reduce the integral into a form that can be integrated.

For $\int \tan^n x \sec^m x\,dx$ we have the following :

1. ***n* odd.** Strip 1 tangent and 1 secant out and convert the rest to secants using $\tan^2 x = \sec^2 x - 1$, then use the substitution $u = \sec x$.
2. ***m* even.** Strip 2 secants out and convert rest to tangents using $\sec^2 x = 1 + \tan^2 x$, then use the substitution $u = \tan x$.
3. ***n* odd and *m* even.** Use either 1. or 2.
4. ***n* even and *m* odd.** Each integral will be dealt with differently.

Trig Formulas **:** $\sin(2x) = 2\sin(x)\cos(x)$, $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$, $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$

Ex. $\int \tan^3 x \sec^5 x\,dx$

$$\int \tan^3 x \sec^5 x\,dx = \int \tan^2 x \sec^4 x \tan x \sec x\,dx$$
$$= \int \left(\sec^2 x - 1\right)\sec^4 x \tan x \sec x\,dx$$
$$= \int \left(u^2 - 1\right)u^4\,du \qquad (u = \sec x)$$
$$= \tfrac{1}{7}\sec^7 x - \tfrac{1}{5}\sec^5 x + c$$

Ex. $\int \frac{\sin^5 x}{\cos^3 x}\,dx$

$$\int \frac{\sin^5 x}{\cos^3 x}\,dx = \int \frac{\sin^4 x \sin x}{\cos^3 x}\,dx = \int \frac{(\sin^2 x)^2 \sin x}{\cos^3 x}\,dx$$
$$= \int \frac{(1-\cos^2 x)^2 \sin x}{\cos^3 x}\,dx \qquad (u = \cos x)$$
$$= -\int \frac{(1-u^2)^2}{u^3}\,du = -\int \frac{1-2u^2+u^4}{u^3}\,du$$
$$= \tfrac{1}{2}\sec^2 x + 2\ln\left|\cos x\right| - \tfrac{1}{2}\cos^2 x + c$$

Trig Substitutions : If the integral contains the following root use the given substitution and formula to convert into an integral involving trig functions.

$\sqrt{a^2-b^2x^2} \Rightarrow x=\frac{a}{b}\sin\theta$	$\sqrt{b^2x^2-a^2} \Rightarrow x=\frac{a}{b}\sec\theta$	$\sqrt{a^2+b^2x^2} \Rightarrow x=\frac{a}{b}\tan\theta$
$\cos^2\theta=1-\sin^2\theta$	$\tan^2\theta=\sec^2\theta-1$	$\sec^2\theta=1+\tan^2\theta$

Ex. $\int \frac{16}{x^2\sqrt{4-9x^2}}\,dx$

$x=\frac{2}{3}\sin\theta \quad \Rightarrow \quad dx=\frac{2}{3}\cos\theta\,d\theta$

$\sqrt{4-9x^2}=\sqrt{4-4\sin^2\theta}=\sqrt{4\cos^2\theta}=2\left|\cos\theta\right|$

Recall $\sqrt{x^2}=\left|x\right|$. Because we have an indefinite integral we'll assume positive and drop absolute value bars. If we had a definite integral we'd need to compute θ's and remove absolute value bars based on that and,

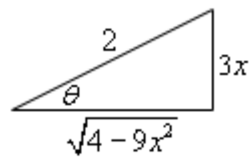
$$\left|x\right|=\begin{cases} x & \text{if } x\geq 0 \\ -x & \text{if } x<0 \end{cases}$$

In this case we have $\sqrt{4-9x^2}=2\cos\theta$.

$$\int \frac{16}{\frac{4}{9}\sin^2\theta(2\cos\theta)}\left(\tfrac{2}{3}\cos\theta\right)d\theta=\int \frac{12}{\sin^2\theta}\,d\theta$$

$$=\int 12\csc^2 d\theta=-12\cot\theta+c$$

Use Right Triangle Trig to go back to x's. From substitution we have $\sin\theta=\frac{3x}{2}$ so,



From this we see that $\cot\theta=\frac{\sqrt{4-9x^2}}{3x}$. So,

$$\int \frac{16}{x^2\sqrt{4-9x^2}}\,dx=-\frac{4\sqrt{4-9x^2}}{x}+c$$

Partial Fractions : If integrating $\int \frac{P(x)}{Q(x)}\,dx$ where the degree of $P(x)$ is smaller than the degree of $Q(x)$. Factor denominator as completely as possible and find the partial fraction decomposition of the rational expression. Integrate the partial fraction decomposition (P.F.D.). For each factor in the denominator we get term(s) in the decomposition according to the following table.

Factor in $Q(x)$	Term in P.F.D	Factor in $Q(x)$	Term in P.F.D
$ax+b$	$\frac{A}{ax+b}$	$(ax+b)^k$	$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\cdots+\frac{A_k}{(ax+b)^k}$
ax^2+bx+c	$\frac{Ax+B}{ax^2+bx+c}$	$(ax^2+bx+c)^k$	$\frac{A_1x+B_1}{ax^2+bx+c}+\cdots+\frac{A_kx+B_k}{(ax^2+bx+c)^k}$

Ex. $\int \frac{7x^2+13x}{(x-1)(x^2+4)}\,dx$

$$\int \frac{7x^2+13x}{(x-1)(x^2+4)}\,dx=\int \frac{4}{x-1}+\frac{3x+16}{x^2+4}\,dx$$

$$=\int \frac{4}{x-1}+\frac{3x}{x^2+4}+\frac{16}{x^2+4}\,dx$$

$$=4\ln\left|x-1\right|+\tfrac{3}{2}\ln\left(x^2+4\right)+8\tan^{-1}\left(\tfrac{x}{2}\right)$$

Here is partial fraction form and recombined.

$$\frac{7x^2+13x}{(x-1)(x^2+4)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+4}=\frac{A(x^2+4)+(Bx+C)(x-1)}{(x-1)(x^2+4)}$$

Set numerators equal and collect like terms.

$$7x^2+13x=\left(A+B\right)x^2+\left(C-B\right)x+4A-C$$

Set coefficients equal to get a system and solve to get constants.

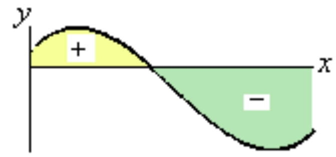
$$A+B=7 \qquad C-B=13 \qquad 4A-C=0$$

$$A=4 \qquad B=3 \qquad C=16$$

An alternate method that *sometimes* works to find constants. Start with setting numerators equal in previous example : $7x^2+13x=A\left(x^2+4\right)+\left(Bx+C\right)\left(x-1\right)$. Chose *nice* values of x and plug in. For example if $x=1$ we get $20=5A$ which gives $A=4$. This won't always work easily.

Applications of Integrals

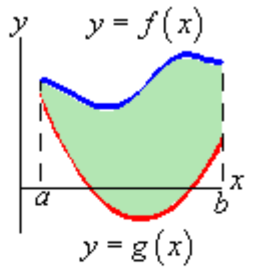
Net Area : $\int_a^b f(x)\,dx$ represents the net area between $f(x)$ and the x-axis with area above x-axis positive and area below x-axis negative.



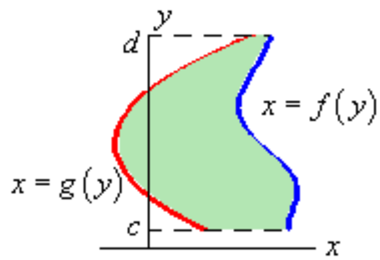
Area Between Curves : The general formulas for the two main cases for each are,

$$y = f(x) \Rightarrow A = \int_a^b [\text{upper function}] - [\text{lower function}]\,dx \quad \& \quad x = f(y) \Rightarrow A = \int_c^d [\text{right function}] - [\text{left function}]\,dy$$

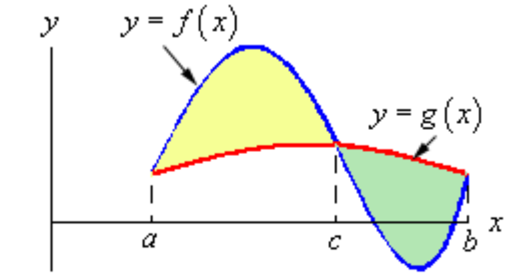
If the curves intersect then the area of each portion must be found individually. Here are some sketches of a couple possible situations and formulas for a couple of possible cases.



$$A = \int_a^b f(x) - g(x)\,dx$$



$$A = \int_c^d f(y) - g(y)\,dy$$



$$A = \int_a^c f(x) - g(x)\,dx + \int_c^b g(x) - f(x)\,dx$$

Volumes of Revolution : The two main formulas are $V = \int A(x)\,dx$ and $V = \int A(y)\,dy$. Here is some general information about each method of computing and some examples.

Rings

$$A = \pi\left((\text{outer radius})^2 - (\text{inner radius})^2\right)$$

Limits: x/y of right/bot ring to x/y of left/top ring

Horz. Axis use $f(x)$, $g(x)$, $A(x)$ and dx.

Vert. Axis use $f(y)$, $g(y)$, $A(y)$ and dy.

Cylinders

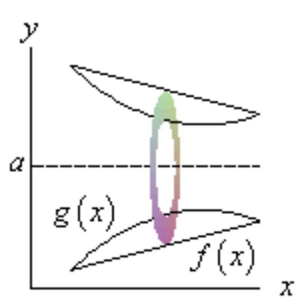
$$A = 2\pi\,(\text{radius})(\text{width / height})$$

Limits : x/y of inner cyl. to x/y of outer cyl.

Horz. Axis use $f(y)$, $g(y)$, $A(y)$ and dy.

Vert. Axis use $f(x)$, $g(x)$, $A(x)$ and dx.

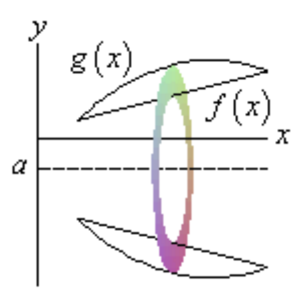
Ex. Axis : $y = a > 0$



outer radius : $a - f(x)$

inner radius : $a - g(x)$

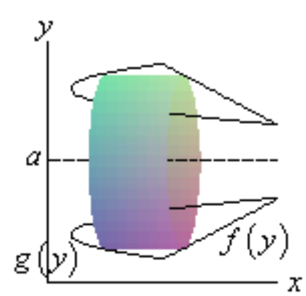
Ex. Axis : $y = a \le 0$



outer radius: $|a| + g(x)$

inner radius: $|a| + f(x)$

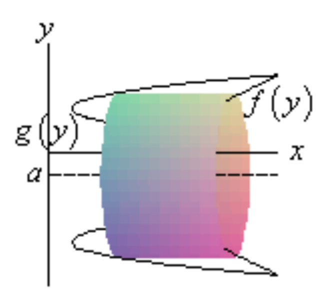
Ex. Axis : $y = a > 0$



radius : $a - y$

width : $f(y) - g(y)$

Ex. Axis : $y = a \le 0$



radius : $|a| + y$

width : $f(y) - g(y)$

These are only a few cases for horizontal axis of rotation. If axis of rotation is the x-axis use the $y = a \le 0$ case with $a = 0$. For vertical axis of rotation ($x = a > 0$ and $x = a \le 0$) interchange x and y to get appropriate formulas.

Work : If a force of $F(x)$ moves an object in $a \le x \le b$, the work done is $W = \int_a^b F(x)\,dx$

Average Function Value : The average value of $f(x)$ on $a \le x \le b$ is $f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$

Arc Length Surface Area : Note that this is often a Calc II topic. The three basic formulas are,

$L = \int_a^b ds$ $\quad$ $SA = \int_a^b 2\pi y\,ds$ (rotate about x-axis) $\quad$ $SA = \int_a^b 2\pi x\,ds$ (rotate about y-axis)

where ds is dependent upon the form of the function being worked with as follows.

$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ if $y = f(x),\ a \le x \le b$ $\quad$ $ds = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$ if $x = f(t), y = g(t),\ a \le t \le b$

$ds = \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$ if $x = f(y),\ a \le y \le b$ $\quad$ $ds = \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$ if $r = f(\theta),\ a \le \theta \le b$

With surface area you *may* have to substitute in for the x or y depending on your choice of ds to match the differential in the ds. With parametric and polar you will always need to substitute.

Improper Integral

An improper integral is an integral with one or more infinite limits and/or discontinuous integrands. Integral is called convergent if the limit exists and has a finite value and divergent if the limit doesn't exist or has infinite value. This is typically a Calc II topic.

Infinite Limit

1. $\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$
2. $\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx$
3. $\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$ provided BOTH integrals are convergent.

Discontinuous Integrand

1. Discont. at a: $\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx$
2. Discont. at b : $\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx$
3. Discontinuity at $a < c < b$: $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ provided both are convergent.

Comparison Test for Improper Integrals : If $f(x) \ge g(x) \ge 0$ on $[a,\infty)$ then,

1. If $\int_a^\infty f(x)\,dx$ conv. then $\int_a^\infty g(x)\,dx$ conv.
2. If $\int_a^\infty g(x)\,dx$ divg. then $\int_a^\infty f(x)\,dx$ divg.

Useful fact : If $a > 0$ then $\int_a^\infty \frac{1}{x^p}\,dx$ converges if $p > 1$ and diverges for $p \le 1$.

Approximating Definite Integrals

For given integral $\int_a^b f(x)\,dx$ and a n (must be even for Simpson's Rule) define $\Delta x = \frac{b-a}{n}$ and divide $[a,b]$ into n subintervals $[x_0,x_1]$, $[x_1,x_2]$, … , $[x_{n-1},x_n]$ with $x_0 = a$ and $x_n = b$ then,

Midpoint Rule : $\int_a^b f(x)\,dx \approx \Delta x\left[f(x_1^*)+f(x_2^*)+\cdots+f(x_n^*)\right]$, x_i^* is midpoint $[x_{i-1},x_i]$

Trapezoid Rule : $\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}\left[f(x_0)+2f(x_1)++2f(x_2)+\cdots+2f(x_{n-1})+f(x_n)\right]$

Simpson's Rule : $\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\left[f(x_0)+4f(x_1)+2f(x_2)+\cdots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)\right]$

AP CALCULUS BC
Stuff you MUST Know Cold

l'Hopital's Rule

If $\frac{f(a)}{g(a)} = \frac{0}{0}$ or $= \frac{\infty}{\infty}$,

then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

Average Rate of Change (slope of the secant line)

If the points $(a, f(a))$ and $(b, f(b))$ are on the graph of $f(x)$ the average rate of change of $f(x)$ on the interval $[a,b]$ is

$$\frac{f(b)-f(a)}{b-a}$$

Definition of Derivative (slope of the tangent line)

$$f'(x) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

Derivatives

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \tan x \sec x$$

$$\frac{d}{dx}(\csc x) = -\cot x \csc x$$

$$\frac{d}{dx}(\ln u) = \frac{1}{u}du$$

$$\frac{d}{dx}\left(e^u\right) = e^u du$$

$$\frac{d}{dx}(\log_a x) = \frac{1}{x\ln a}$$

$$\frac{d}{dx}\left(a^u\right) = a^x(\ln a)du$$

Properties of Log and Ln

1. $\ln 1 = 0$
2. $\ln e^a = a$
3. $e^{\ln x} = x$
4. $\ln x^n = n\ln x$
5. $\ln(ab) = \ln a + \ln b$
6. $\ln\left(a/b\right) = \ln a - \ln b$

Differentiation Rules

Chain Rule

$$\frac{d}{dx}\left[f(u)\right] = f'(u)\frac{du}{dx}$$

Product Rule

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Quotient Rule

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Mean Value & Rolle's Theorem

If the function $f(x)$ is continuous on $[a, b]$ and the first derivative exists on the interval (a, b), then there exists a number x = c on (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

if f(a) = f(b), then $f'(c) = 0$.

Curve sketching and analysis

y = $f(x)$ must be continuous at each:

critical point: $\frac{dy}{dx} = 0$ or undefined.

local minimum: $\frac{dy}{dx}$ goes (-,0,+) or (-,und,+) or $\frac{d^2y}{dx^2} > 0$

local maximum: $\frac{dy}{dx}$ goes (+,0,-) or (+,und,-) or $\frac{d^2y}{dx^2} < 0$

Absolute Max/Min.: Compare local extreme values to values at endpoints.

pt of inflection : concavity changes.

$\frac{d^2y}{dx^2}$ goes (+,0,-),(-,0,+), (+,und,-), or (-,und,+)

"PLUS A CONSTANT"

The Fundamental Theorem of Calculus

$$\int_a^b f(x)dx = F(b) - F(a)$$

where $F'(x) = f(x)$

2nd Fundamental Theorem of Calculus

$$\frac{d}{dx}\int_{\#}^{g(x)} f(x)dx = f(g(x))\cdot g'(x)$$

Average Value

If the function $f(x)$ is continuous on $[a, b]$ and the first derivative exist on the interval (a, b), then there exists a number x = c on (a, b) such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$

$f(c)$ is the average value

Euler's Method

If given that $\frac{dy}{dx} = f(x, y)$ and that the solution passes through (x_0, y_0), then

$$x_{new} = x_{old} + \Delta x$$

$$y_{new} = y_{old} + \left.\frac{dy}{dx}\right|_{(x_{old}, y_{old})}\cdot\Delta x$$

Logistics Curves

$$P(t) = \frac{L}{1+Ce^{-(Lk)t}},$$

where L is carrying capacity

Maximum growth rate occurs when P = ½ L

$$\frac{dP}{dt} = kP(L-P) \text{ or}$$

$$\frac{dP}{dt} = (Lk)P\left(1-\frac{P}{L}\right)$$

Integrals

$\int kf(u)du = k\int f(u)du$

$\int du = u + C$

$\int u^n du = \frac{u^{n+1}}{n+1} + C, n \neq -1$

$\int \frac{1}{u} du = \ln|u| + C$

$\int e^u du = e^u + C$

$\int a^u du = \left(\frac{1}{\ln a}\right) a^u + C$

$\int \cos u du = \sin u + C$

$\int \sin u du = -\cos u + C$

$\int \tan u du = -\ln|\cos u| + C$

$\int \cot u du = \ln|\sin u| + C$

$\int \sec u du = \ln|\sec u + \tan u| + C$

$\int \csc u du = -\ln|\csc u + \cot u| + C$

$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} arc\sec\left(\frac{|u|}{a}\right) + C$

$\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin\left(\frac{u}{a}\right) + C$

$\int \frac{du}{a^2 + u^2} = \frac{1}{a}\arctan\left(\frac{u}{a}\right) + C$

Integration by Parts

$$\int udv = uv - \int vdu$$

Arc Length

For a function, $f(x)$

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

For a polar graph, r(θ)

$$L = \int_{\theta 1}^{\theta 2} \sqrt{[r(\theta)]^2 + [r'(\theta)]^2}\, d\theta$$

Lagrange Error Bound

If $P_n(x)$ is the nth degree Taylor polynomial of $f(x)$ about c, then

$$\left|f(x) - P_n(x)\right| \le \frac{\max\left|f^{(n+1)}(z)\right|}{(n+1)!}\left|x - c\right|^{n+1}$$

for all z between x and c.

Distance, velocity and Acceleration

Velocity = $\frac{d}{dt}(position)$

Acceleration = $\frac{d}{dt}(velocity)$

Velocity Vector = $\left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle$

Speed = $|v(t)| = \sqrt{(x')^2 + (y')^2}$.

Distance Traveled =

$$\int_{initial\ time}^{final\ time} |v(t)|\, dt = \int_{initial\ time}^{final\ time} \sqrt{(x')^2 + (y')^2}\, dt$$

$x(b) = x(a) + \int_a^b x'(t)dt$

$y(b) = y(a) + \int_a^b y'(t)dt$

Polar Curves

For a polar curve r(θ), the

Area inside a "leaf" is $\frac{1}{2}\int_{\theta 1}^{\theta 2} [r(\theta)]^2\, d\theta$

where θ1 and θ2 are the "first" two times that $r = 0$.

The slope of $r(\theta)$ at a given θ is

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{d}{d\theta}[r(\theta)\sin\theta]}{\frac{d}{d\theta}[r(\theta)\cos\theta]}$$

Ratio Test
(use for interval of convergence)

The series $\sum_{n=0}^{\infty} a_n$ converges if

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$ **CHECK ENDPOINTS**

Alternating Series Error Bound

If $S_N = \sum_{n=1}^{N} (-1)^n a_n$ is the Nth partial sum of a convergent alternating series, then

$$\left|S_\infty - S_N\right| \le \left|a_{N+1}\right|$$

Volume
Solids of Revolution

Disk Method: $V = \pi\int_a^b [R(x)]^2\, dx$

Washer Method:

$$V = \pi\int_a^b \left([R(x)]^2 - [r(x)]^2\right) dx$$

Shell Method: $V = 2\pi\int_a^b r(x)h(x)dx$

Volume of Known Cross Sections

Perpendicular to

x-axis: $V = \int_a^b A(x)dx$

y-axis: $V = \int_c^d A(y)dy$

Taylor Series

If the function f is "smooth"at x = c, then it can be approximated by the nth degree polynomial

$$f(x) \approx f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \ldots + \frac{f'''(c)}{3!}(x - c)^3 + \ldots + \frac{f^{(n)}(c)}{n!}(x - c)^n$$

Elementary Functions

Centered at x = 0

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$

$\frac{1}{1 - x} = 1 + x + x^2 + x^3 + \ldots$

$\ln(x + 1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$

Most Common Series

$\sum_{n=1}^{\infty}\frac{1}{n}$ diverges $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ converges $\sum_{n=0}^{\infty} A(r)^n$ converges to $\frac{A}{1 - r}$ if $|r| < 1$