Derivatives

$D_x e^x = e^x$
$D_x \sin(x) = \cos(x)$
$D_x \cos(x) = -\sin(x)$
$D_x \tan(x) = \sec^2(x)$
$D_x \cot(x) = -\csc^2(x)$
$D_x \sec(x) = \sec(x)\tan(x)$
$D_x \csc(x) = -\csc(x)\cot(x)$
$D_x \sin^{-1} = \frac{1}{\sqrt{1-x^2}}, x \in [-1,1]$
$D_x \cos^{-1} = \frac{-1}{\sqrt{1-x^2}}, x \in [-1,1]$
$D_x \tan^{-1} = \frac{1}{1+x^2}, \frac{-\pi}{2} \le x \le \frac{\pi}{2}$
$D_x \sec^{-1} = \frac{1}{|x|\sqrt{x^2-1}}, |x| > 1$
$D_x \sinh(x) = \cosh(x)$
$D_x \cosh(x) = \sinh(x)$
$D_x \tanh(x) = sech^2(x)$
$D_x \coth(x) = -csch^2(x)$
$D_x sech(x) = -sech(x)\tanh(x)$
$D_x csch(x) = -csch(x)\coth(x)$
$D_x \sinh^{-1} = \frac{1}{\sqrt{x^2+1}}$
$D_x \cosh^{-1} = \frac{-1}{\sqrt{x^2-1}}, x > 1$
$D_x \tanh^{-1} = \frac{1}{1-x^2} -1 < x < 1$
$D_x sech^{-1} = \frac{-1}{x\sqrt{1-x^2}}, 0 < x < 1$
$D_x \ln(x) = \frac{1}{x}$

Integrals

$\int \frac{1}{x}dx = \ln|x| + c$
$\int e^x dx = e^x + c$
$\int a^x dx = \frac{1}{\ln a}a^x + c$
$\int e^{ax} dx = \frac{1}{a}e^{ax} + c$
$\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1}(x) + c$
$\int \frac{1}{1+x^2}dx = \tan^{-1}(x) + c$
$\int \frac{1}{x\sqrt{x^2-1}}dx = \sec^{-1}(x) + c$
$\int \sinh(x)dx = \cosh(x) + c$
$\int \cosh(x)dx = \sinh(x) + c$
$\int \tanh(x)dx = \ln|\cosh(x)| + c$
$\int \tanh(x)sech(x)dx = -sech(x) + c$
$\int sech^2(x)dx = \tanh(x) + c$
$\int csch(x)\coth(x)dx = -csch(x) + c$
$\int \tan(x)dx = -\ln|\cos(x)| + c$
$\int \cot(x)dx = \ln|\sin(x)| + c$
$\int \cos(x)dx = \sin(x) + c$
$\int \sin(x)dx = -\cos(x) + c$
$\int \frac{1}{\sqrt{a^2-u^2}}dx = \sin^{-1}(\frac{u}{a}) + c$
$\int \frac{1}{a^2+u^2}dx = \frac{1}{a}\tan^{-1}\frac{u}{a} + c$
$\int ln(x)dx = (xln(x)) - x + c$

U-Substitution
Let $u = f(x)$ (can be more than one variable).
Determine: $du = \frac{f(x)}{dx}dx$ and solve for dx.
Then, if a definite integral, substitute the bounds for $u = f(x)$ at each bounds
Solve the integral using u.

Integration by Parts
$\int udv = uv - \int vdu$

Fns and Identities

$\sin(\cos^{-1}(x)) = \sqrt{1-x^2}$
$\cos(\sin^{-1}(x)) = \sqrt{1-x^2}$
$\sec(\tan^{-1}(x)) = \sqrt{1+x^2}$
$\tan(\sec^{-1}(x))$
$= (\sqrt{x^2-1} \text{ if } x \ge 1)$
$= (-\sqrt{x^2-1} \text{ if } x \le -1)$
$\sinh^{-1}(x) = \ln x + \sqrt{x^2+1}$
$\sinh^{-1}(x) = \ln x + \sqrt{x^2-1},\ x \ge -1$
$\tanh^{-1}(x) = \frac{1}{2}\ln x + \frac{1+x}{1-x},\ 1 < x < -1$
$sech^{-1}(x) = \ln[\frac{1+\sqrt{1-x^2}}{x}],\ 0 < x \le -1$
$\sinh(x) = \frac{e^x - e^{-x}}{2}$
$\cosh(x) = \frac{e^x + e^{-x}}{2}$

Trig Identities

$\sin^2(x) + \cos^2(x) = 1$
$1 + \tan^2(x) = \sec^2(x)$
$1 + \cot^2(x) = \csc^2(x)$
$\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$
$\cos(x \pm y) = \cos(x)\cos(y) \pm \sin(x)\sin(y)$
$\tan(x \pm y) = \frac{\tan(x) \pm \tan(y)}{1 \mp \tan(x)\tan(y)}$
$\sin(2x) = 2\sin(x)\cos(x)$
$\cos(2x) = \cos^2(x) - \sin^2(x)$
$\cosh(n^2x) - \sinh^2 x = 1$
$1 + \tan^2(x) = \sec^2(x)$
$1 + \cot^2(x) = \csc^2(x)$
$\sin^2(x) = \frac{1-\cos(2x)}{2}$
$\cos^2(x) = \frac{1+\cos(2x)}{2}$
$\tan^2(x) = \frac{1-\cos(2x)}{1+\cos(2x)}$
$\sin(-x) = -\sin(x)$
$\cos(-x) = \cos(x)$
$\tan(-x) = -\tan(x)$

Calculus 3 Concepts

Cartesian coords in 3D

given two points:
(x_1, y_1, z_1) and (x_2, y_2, z_2),
Distance between them:
$\sqrt{(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2}$
Midpoint:
$(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2})$
Sphere with center (h,k,l) and radius r:
$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$

Vectors

Vector: $\vec{u}$
Unit Vector: $\hat{u}$
Magnitude: $||\vec{u}|| = \sqrt{u_1^2 + u_2^2 + u_3^2}$
Unit Vector: $\hat{u} = \frac{\vec{u}}{||\vec{u}||}$

Dot Product
$\vec{u} \cdot \vec{v}$
Produces a Scalar
(Geometrically, the dot product is a vector projection)
$\vec{u} =< u_1, u_2, u_3 >$
$\vec{v} =< v_1, v_2, v_3 >$
$\vec{u} \cdot \vec{v} = \vec{0}$ means the two vectors are Perpendicular θ is the angle between them.
$\vec{u} \cdot \vec{v} = ||\vec{u}||\ ||\vec{v}|| \cos(\theta)$
$\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3$
NOTE:
$\hat{u} \cdot \hat{v} = \cos(\theta)$
$||\vec{u}||^2 = \vec{u} \cdot \vec{u}$
$\vec{u} \cdot \vec{v} = 0$ when $\perp$
Angle Between $\vec{u}$ and $\vec{v}$:
$\theta = \cos^{-1}(\frac{\vec{u} \cdot \vec{v}}{||\vec{u}||\ ||\vec{v}||})$

Projection of $\vec{u}$ onto $\vec{v}$:
$pr_{\vec{v}}\vec{u} = (\frac{\vec{u} \cdot \vec{v}}{||\vec{v}||^2})\vec{v}$

Cross Product
$\vec{u} \times \vec{v}$
Produces a Vector
(Geometrically, the cross product is the area of a paralellogram with sides $||\vec{u}||$ and $||\vec{v}||$)
$\vec{u} =< u_1, u_2, u_3 >$
$\vec{v} =< v_1, v_2, v_3 >$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$\vec{u} \times \vec{v} = \vec{0}$ means the vectors are paralell

Lines and Planes

Equation of a Plane
(x_0, y_0, z_0) is a point on the plane and $< A, B, C >$ is a normal vector

$A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$
$< A, B, C > \cdot < x-x_0, y-y_0, z-z_0 >= 0$
$Ax + By + Cz = D$ where
$D = Ax_0 + By_0 + Cz_0$

Equation of a line
A line requires a Direction Vector $\vec{u} =< u_1, u_2, u_3 >$ and a point (x_1, y_1, z_1)
then,
a parameterization of a line could be:
$x = u_1t + x_1$
$y = u_2t + y_1$
$z = u_3t + z_1$

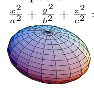
Distance from a Point to a Plane
The distance from a point (x_0, y_0, z_0) to a plane Ax+By+Cz=D can be expressed by the formula:
$d = \frac{|Ax_0 + By_0 + Cz_0 - D|}{\sqrt{A^2+B^2+C^2}}$

Coord Sys Conv

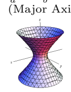
Cylindrical to Rectangular
$x = r\cos(\theta)$
$y = r\sin(\theta)$
$z = z$
Rectangular to Cylindrical
$r = \sqrt{x^2+y^2}$
$\tan(\theta) = \frac{y}{x}$
$z = z$
Spherical to Rectangular
$x = \rho\sin(\phi)\cos(\theta)$
$y = \rho\sin(\phi)\sin(\theta)$
$z = \rho\cos(\phi)$
Rectangular to Spherical
$\rho = \sqrt{x^2+y^2+z^2}$
$\tan(\theta) = \frac{y}{x}$
$\cos(\phi) = \frac{z}{\sqrt{x^2+y^2+z^2}}$
Spherical to Cylindrical
$r = \rho\sin(\phi)$
$\theta = \theta$
$z = \rho\cos(\phi)$
Cylindrical to Spherical
$\rho = \sqrt{r^2+z^2}$
$\theta = \theta$
$\cos(\phi) = \frac{z}{\sqrt{r^2+z^2}}$

Surfaces

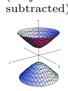
Ellipsoid
$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$



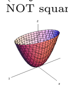
Hyperboloid of One Sheet
$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$
(Major Axis: z because it follows -)



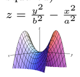
Hyperboloid of Two Sheets
$\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
(Major Axis: Z because it is the one not subtracted)



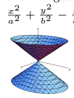
Elliptic Paraboloid
$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$
(Major Axis: z because it is the variable NOT squared)



Hyperbolic Paraboloid
(Major Axis: Z axis because it is not squared)
$z = \frac{y^2}{b^2} - \frac{x^2}{a^2}$



Elliptic Cone
(Major Axis: Z axis because it's the only one being subtracted)
$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$



Cylinder
1 of the variables is missing
OR
$(x-a)^2 + (y-b^2) = c$
(Major Axis is missing variable)

Partial Derivatives

Partial Derivatives are simply holding all other variables constant (and act like constants for the derivative) and only taking the derivative with respect to a given variable.

Given z=f(x,y), the partial derivative of z with respect to x is:
$f_x(x,y) = z_x = \frac{\partial z}{\partial x} = \frac{\partial f(x,y)}{\partial x}$
likewise for partial with respect to y:
$f_y(x,y) = z_y = \frac{\partial z}{\partial y} = \frac{\partial f(x,y)}{\partial y}$
Notation
For f_{xyy}, work "inside to outside" f_x then f_{xy}, then f_{xyy}
$f_{xyy} = \frac{\partial^3 f}{\partial x \partial^2 y}$,
For $\frac{\partial^3 f}{\partial x \partial^2 y}$, work right to left in the denominator

Gradients

The Gradient of a function in 2 variables is $\nabla f =< f_x, f_y >$
The Gradient of a function in 3 variables is $\nabla f =< f_x, f_y, f_z >$

Chain Rule(s)

Take the Partial derivative with respect to the first-order variables of the function times the partial (or normal) derivative of the first-order variable to the ultimate variable you are looking for summed with the same process for other first-order variables this makes sense for.
Example:
let x = x(s,t), y = y(t) and z = z(x,y).
z then has first partial derivative:
$\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$
x has the partial derivatives:
$\frac{\partial x}{\partial s}$ and $\frac{\partial x}{\partial t}$
and y has the derivative:
$\frac{dy}{dt}$
In this case (with z containing x and y as well as x and y both containing s and t), the chain rule for $\frac{\partial z}{\partial s}$ is $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s}$
The chain rule for $\frac{\partial z}{\partial t}$ is
$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{dy}{dt}$
Note: the use of "d" instead of "∂" with the function of only one independent variable

Limits and Continuity

Limits in 2 or more variables
Limits taken over a vectorized limit just evaluate separately for each component of the limit.
Strategies to show limit exists
1. Plug in Numbers, Everything is Fine
2. Algebraic Manipulation
-factoring/dividing out
-use trig identites
3. Change to polar coords
$if (x,y) \to (0,0) \Leftrightarrow r \to 0$
Strategies to show limit DNE
1. Show limit is different if approached from different paths
(x=y, $x = y^2$, etc.)
2. Switch to Polar coords and show the limit DNE.
Continuity
A fn, $z = f(x,y)$, is continuous at (a,b) if
$f(a,b) = \lim_{(x,y)\to(a,b)} f(x,y)$
Which means:
1. The limit exists
2. The fn value is defined
3. They are the same value

Directional Derivatives

Let z=f(x,y) be a fuction, (a,b) ap point in the domain (a valid input point) and $\hat{u}$ a unit vector (2D).
The Directional Derivative is then the derivative at the point (a,b) in the direction of $\hat{u}$ or:
$D_{\vec{u}} f(a,b) = \hat{u} \cdot \nabla f(a,b)$
This will return a *scalar*. 4-D version:
$D_{\vec{u}} f(a,b,c) = \hat{u} \cdot \nabla f(a,b,c)$

Tangent Planes

let F(x,y,z) = k be a surface and P = (x_0, y_0, z_0) be a point on that surface.
Equation of a Tangent Plane:
$\nabla F(x_0, y_0, z_0) \cdot < x-x_0, y-y_0, z-z_0 >$

Approximations

let $z = f(x,y)$ be a differentiable function total differential of f = dz
$dz = \nabla f \cdot < dx, dy >$
This is the *approximate* change in z
The actual change in z is the difference in z values:
$\Delta z = z - z_1$

Maxima and Minima

Internal Points
1. Take the Partial Derivatives with respect to X and Y (f_x and f_y) (Can use gradient)
2. Set derivatives equal to 0 and use to solve system of equations for x and y
3. Plug back into original equation for z. Use Second Derivative Test for whether points are local max, min, or saddle

Second Partial Derivative Test
1. Find all (x,y) points such that $\nabla f(x,y) = \vec{0}$
2. Let $D = f_{xx}(x,y)f_{yy}(x,y) - f_{xy}^2(x,y)$
IF (a) D > 0 AND $f_{xx} < 0$, f(x,y) is local max value
(b) D > 0 AND $f_{xx}(x,y) > 0$ f(x,y) is local min value
(c) D < 0, (x,y,f(x,y)) is a saddle point
(d) D = 0, test is inconclusive
3. Determine if any boundary point gives min or max. Typically, we have to parametrize boundary and then reduce to a Calc 1 type of min/max problem to solve.
The following only apply only if a boundary is given
1. check the corner points
2. Check each line ($0 \le x \le 5$ would give x=0 and x=5)
On Bounded Equations, this is the global min and max...second derivative test is not needed.

Lagrange Multipliers

Given a function f(x,y) with a constraint g(x,y), solve the following system of equations to find the max and min points on the constraint (NOTE: may need to also find internal points.):
$\nabla f = \lambda \nabla g$
$g(x,y) = 0 (orkifgiven)$

Double Integrals

With Respect to the xy-axis, if taking an integral,
$\int\int dydx$ is cutting in vertical rectangles,
$\int\int dxdy$ is cutting in horizontal rectangles

Polar Coordinates
When using polar coordinates,
$dA = rdrd\theta$

Surface Area of a Curve

let z = f(x,y) be continuous over S (a closed Region in 2D domain)
Then the surface area of z = f(x,y) over S is:
$SA = \int\int_S \sqrt{f_x^2 + f_y^2 + 1}dA$

Triple Integrals

$\int\int\int_s f(x,y,z)dv =$
$\int_{a_1}^{a_2}\int_{\phi_1(x)}^{\phi_2(x)}\int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x,y,z)dzdydx$
Note: dv can be exchanged for $dxdydz$ in any order, but you must then choose your limits of integration according to that order

Jacobian Method

$\int\int_G f(g(u,v), h(u,v))|J(u,v)|dudv =$
$\int\int_R f(x,y)dxdy$

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Common Jacobians:
Rect. to Cylindrical: r
Rect. to Spherical: $\rho^2\sin(\phi)$

Vector Fields

let $f(x,y,z)$ be a scalar field and
$\vec{F}(x,y,z) =$
$M(x,y,z)\hat{i} + N(x,y,z)\hat{j} + P(x,y,z)\hat{k}$ be a vector field,
Grandient of f = $\nabla f =< \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} >$
Divergence of $\vec{F}$:
$\nabla \cdot \vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$
Curl of $\vec{F}$:

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$$

Line Integrals

C given by $x = x(t), y = y(t), t \in [a,b]$
$\int_c f(x,y)ds = \int_a^b f(x(t), y(t))ds$
where $ds = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}dt$
or $\sqrt{1 + (\frac{dy}{dx})^2}dx$
or $\sqrt{1 + (\frac{dx}{dy})^2}dy$
To evaluate a Line Integral,
· get a paramaterized version of the line (usually in terms of t, though in exclusive terms of x or y is ok)
· evaluate for the derivatives needed (usually dy, dx, and/or dt)
· plug in to original equation to get in terms of the independant variable
· solve integral

Work
Let $\vec{F} = M\hat{i} + \hat{j} + \hat{k}$ (force)
$M = M(x,y,z), N = N(x,y,z), P = P(x,y,z)$
(Literally)$d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$
Work $w = \int_c \vec{F} \cdot d\vec{r}$
(Work done by moving a particle over curve C with force $\vec{F}$)

Independence of Path

Fund Thm of Line Integrals
C is curve given by $\vec{r}(t), t \in [a,b]$;
$\vec{r}\prime(t)$ exists. If $f(\vec{r})$ is continuously differentiable on an open set containing C, then $\int_c \nabla f(\vec{r}) \cdot d\vec{r} = f(\vec{b}) - f(\vec{a})$
Equivalent Conditions
$\vec{F}(\vec{r})$ continuous on open connected set D. Then,
$(a)\vec{F} = \nabla f$ for some fn f. (if $\vec{F}$ is conservative)
$\Leftrightarrow (b) \int_c \vec{F}(\vec{r}) \cdot d\vec{r} isindep.ofpathinD$
$\Leftrightarrow (c) \int_c \vec{F}(\vec{r}) \cdot d\vec{r} = 0$ for all closed paths in D.
Conservation Theorem
$\vec{F} = M\hat{i} + N\hat{j} + P\hat{k}$ continuously differentiable on open, simply connected set D.
$\vec{F}$ conservative $\Leftrightarrow \nabla \times \vec{F} = \vec{0}$
(in 2D $\nabla \times \vec{F} = \vec{0}$ iff $M_y = N_x$)

Green's Theorem

(method of changing line integral for double integral - Use for Flux and Circulation across 2D curve and line integrals over a closed boundary)
$\oint Mdy - Ndx = \int\int_R (M_x + N_y)dxdy$
$\oint Mdx + Ndy = \int\int_R (N_x - M_y)dxdy$
Let:
·R be a region in xy-plane
·C is simple, closed curve enclosing R (w/ paramerization $\vec{r}(t)$)
·$\vec{F}(x,y) = M(x,y)\hat{i} + N(x,y)\hat{j}$ be continuously differentiable over R∪C.
Form 1: Flux Across Boundary
$\vec{n}$ = unit normal vector to C
$\oint_c \vec{F} \cdot \vec{n} = \int\int_R \nabla \cdot \vec{F}dA$
$\Leftrightarrow \oint Mdy - Ndx = \int\int_R (M_x + N_y)dxdy$
Form 2: Circulation Along Boundary
$\oint_c \vec{F} \cdot d\vec{r} = \int\int_R \nabla \times \vec{F} \cdot \hat{u}dA$
$\Leftrightarrow \oint Mdx + Ndy = \int\int_R (N_x - M_y)dxdy$
Area of R
$A = \oint(\frac{-1}{2}ydx + \frac{1}{2}xdy)$

Gauss' Divergence Thm

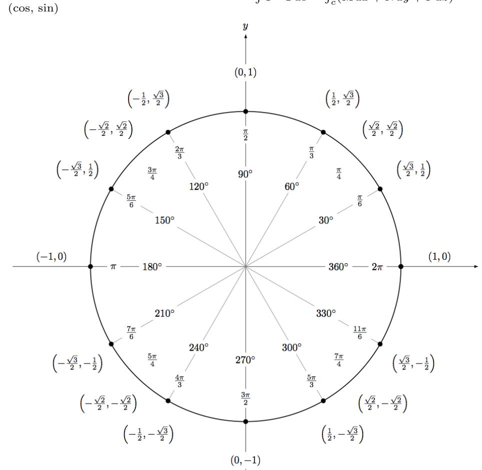
(3D Analog of Green's Theorem - Use for Flux over a 3D surface) Let:
·$\vec{F}(x,y,z)$ be vector field continuously differentiable in solid S
·S is a 3D solid ·∂S boundary of S (A Surface)
·$\hat{n}$unit outer normal to ∂S
Then,
$\int\int_{\partial S} \vec{F}(x,y,z) \cdot \hat{n}dS = \int\int\int_S \nabla \cdot \vec{F}dV$
(dV = dxdydz)

Surface Integrals

Let
·R be closed, bounded region in xy-plane
·f be a fn with first order partial derivatives on R
·G be a surface over R given by $z = f(x,y)$
·$g(x,y,z) = g(x,y,f(x,y))$ is cont. on R
Then,
$\int\int_G g(x,y,z)dS =$
$\int\int_R g(x,y,f(x,y))dS$
where $dS = \sqrt{f_x^2 + f_y^2 + 1}dydx$
Flux of $\vec{F}$ across G
$\int\int_G \vec{F} \cdot ndS =$
$\int\int_R [-Mf_x - Nf_y + P]dxdy$
where:
·$\vec{F}(x,y,z) =$
$M(x,y,z)\hat{i} + N(x,y,z)\hat{j} + P(x,y,z)\hat{k}$
·G is surface f(x,y)=z
·$\vec{n}$ is upward unit normal on G.
·f(x,y) has continuous 1^{st} order partial derivatives

Unit Circle

(cos, sin)



Other Information

$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$
Where a Cone is defined as
$z = \sqrt{a(x^2+y^2)}$,
In Spherical Coordinates,
$\phi = \cos^{-1}(\sqrt{\frac{a}{1+a}})$
Right Circular Cylinder:
$V = \pi r^2h, SA = \pi r^2 + 2\pi rh$
$\lim_{n\to\inf}(1 + \frac{m}{n})^{pn} = e^{mp}$
Law of Cosines:
$a^2 = b^2 + c^2 - 2bc(\cos(\theta))$

Stokes Theorem

Let:
·S be a 3D surface
·$\vec{F}(x,y,z) =$
$M(x,y,z)\hat{i} + N(x,y,z)\hat{j} + P(x,y,z)\hat{l}$
·M,N,P have continuous 1^{st} order partial derivatives
·C is piece-wise smooth, simple, closed, curve, positively oriented
·$\hat{T}$ is unit tangent vector to C.
Then,
$\oint \vec{F}_c \cdot \hat{T}dS = \int\int_s (\nabla \times \vec{F}) \cdot \hat{n}dS =$
$\int\int_R (\nabla \times \vec{F}) \cdot \vec{n}dxdy$
Remember:
$\oint \vec{F} \cdot \vec{T}ds = \int_c (Mdx + Ndy + Pdz)$

Originally Written By Daniel Kenner for MATH 2210 at the University of Utah.
Source code available at https://github.com/keytotime/Calc3_CheatShee
Thanks to Kelly Macarthur for Teaching and Providing Notes.

Harvard College

Math 21a: Multivariable Calculus

Formula and Theorem Review

Tommy MacWilliam, '13
tmacwilliam@college.harvard.edu

December 15, 2009

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9 Vectors and the Geometry of Space

9.1 Distance Formula in 3 Dimensions

The distance between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is given by:

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

9.2 Equation of a Sphere

The equation of a sphere with center (h, k, l) and radius r is given by:

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

9.3 Properties of Vectors

If $\vec{a}, \vec{b}$, and $\vec{c}$ are vectors and c and d are scalars:

$$\begin{array}{ll}
\vec{a} + \vec{b} = \vec{b} + \vec{a} & \vec{a} + 0 = \vec{a} \\
\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c} & \vec{a} + -\vec{a} = 0 \\
c(\vec{a} + \vec{b}) = c\vec{a} + c + \vec{b} & (c + d)\vec{a} = c\vec{a} + d\vec{a} \\
(cd)\vec{a} = c(d\vec{a}) &
\end{array}$$

9.4 Unit Vector

A unit vector is a vector whose length is 1. The unit vector $\vec{u}$ in the same direction as $\vec{a}$ is given by:

$$\vec{u} = \frac{\vec{a}}{|\vec{a}|}$$

9.5 Dot Product

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$$

$$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$$

9.6 Properties of the Dot Product

Two vectors are orthogonal if their dot product is 0.

$$\begin{array}{ll}
\vec{a} \cdot \vec{a} = |\vec{a}|^2 & \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a} \\
\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} & (c\vec{a}) \cdot \vec{b} = c(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (c\vec{b}) \\
0 \cdot \vec{a} = 0 &
\end{array}$$

9.7 Vector Projections

Scalar projection of $\vec{b}$ onto $\vec{a}$:

$$\text{comp}_{\vec{a}}\,\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$$

Vector projection of $\vec{b}$ onto $\vec{a}$:

$$\text{proj}_{\vec{a}}\,\vec{b} = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\right)\frac{\vec{a}}{|\vec{a}|}$$

9.8 Cross Product

$$\vec{a}\times\vec{b} = (|\vec{a}||\vec{b}|\sin\theta)\vec{n}$$

where $\vec{n}$ is the unit vector orthogonal to both $\vec{a}$ and $\vec{b}$.

$$\vec{a}\times\vec{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle$$

9.9 Properties of the Cross Product

Two vectors are parallel if their cross product is 0.

$$\vec{a}\times\vec{b} = -\vec{b}\times\vec{a} \qquad (c\vec{a})\times\vec{b} = c(\vec{a}\times\vec{b}) = \vec{a}\times(c\vec{b})$$

$$\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b}+\vec{a}\times\vec{c} \qquad (\vec{a}+\vec{b})\times\vec{c} = \vec{a}\times\vec{c}+\vec{b}\times\vec{c}$$

9.10 Scalar Triple Product

The volume of the parallelpiped determined by vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ is the magnitude of their scalar triple product:

$$V = |\vec{a}\cdot(\vec{b}\times\vec{c})|$$

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{c}\cdot(\vec{a}\times\vec{b})$$

9.11 Vector Equation of a Line

$$\vec{r} = \vec{r}_0 + t\vec{v}$$

9.12 Symmetric Equations of a Line

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

where the vector $\vec{c} = \langle a, b, c\rangle$ is the direction of the line.

The symmetric equations for a line passing through the points (x_0, y_0, z_0) and (x_1, y_1, z_1) are given by:

$$\frac{x-x_0}{x_1-x_0} = \frac{y-y_0}{y_1-y_0} = \frac{z-z_0}{z_1-z_0}$$

9.13 Segment of a Line

The line segment from $\vec{r}_0$ to $\vec{r}_1$ is given by:

$$\vec{r}(t) = (1-t)\vec{r}_0 + t\vec{r}_1 \quad \text{for} \quad 0 \le t \le 1$$

9.14 Vector Equation of a Plane

$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$$

where $\vec{n}$ is the vector orthogonal to every vector in the given plane and $\vec{r} - \vec{r}_0$ is the vector between any two points on the plane.

9.15 Scalar Equation of a Plane

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

where (x_0, y_0, z_0) is a point on the plane and $\langle a, b, c\rangle$ is the vector normal to the plane.

9.16 Distance Between Point and Plane

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

$$d(P, \Sigma) = \frac{|\vec{PQ} \cdot \vec{n}|}{|\vec{n}|}$$

where P is a point, Σ is a plane, Q is a point on plane Σ, and $\vec{n}$ is the vector orthogonal to the plane.

9.17 Distance Between Point and Line

$$d(P, L) = \frac{|\vec{PQ} \times \vec{u}|}{|\vec{u}|}$$

where P is a point in space, Q is a point on the line L, and $\vec{u}$ is the direction of line.

9.18 Distance Between Line and Line

$$d(L, M) = \frac{|(\vec{PQ}) \cdot (\vec{u} \times \vec{v})|}{|\vec{u} \times \vec{v}|}$$

where P is a point on line L, Q is a point on line M, $\vec{u}$ is the direction of line L, and $\vec{v}$ is the direction of line M.

9.19 Distance Between Plane and Plane

$$d = \frac{|e - d|}{|\vec{n}|}$$

where $\vec{n}$ is the vector orthogonal to both planes, e is the constant of one plane, and d is the constant of the other. The distance between non-parallel planes is 0.

9.20 Quadric Surfaces

$$\begin{aligned}
\text{Ellipsoid:} \quad & \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \\
\text{Elliptic Paraboloid:} \quad & \frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2} \\
\text{Hyperbolic Paraboloid:} \quad & \frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2} \\
\text{Cone:} \quad & \frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2} \\
\text{Hyperboloid of One Sheet:} \quad & \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \\
\text{Hyperboloid of Two Sheets:} \quad & -\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1
\end{aligned}$$

9.21 Cylindrical Coordinates

To convert from cylindrical to rectangular:

$$x = r\cos\theta \quad y = r\sin\theta \quad z = z$$

To convert from rectangular to cylindrical:

$$r^2 = x^2 + y^2 \quad \tan\theta = \frac{y}{x} \quad z = z$$

9.22 Spherical Coordinates

To convert from spherical to rectangular:

$$x = \rho\sin\phi\cos\theta \quad y = \rho\sin\phi\sin\theta \quad z = \rho\cos\phi$$

To convert from rectangular to spherical:

$$\rho^2 = x^2 + y^2 + z^2 \quad \tan\theta = \frac{y}{x} \quad \cos\phi = \frac{z}{\rho}$$

10 Vector Functions

10.1 Limit of a Vector Function

$$\lim_{t\to a} \vec{r}(t) = \left\langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) \right\rangle$$

10.2 Derivative of a Vector Function

$$\frac{d\vec{r}}{dt} = \vec{r}\,'(t) = \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

$$\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle$$

10.3 Unit Tangent Vector

$$T(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$$

10.4 Derivative Rules for Vector Functions

$$\frac{d}{dt}[\vec{u}(t) + \vec{v}(t)] = \vec{u}\,'(t) + \vec{v}\,'(t)$$

$$\frac{d}{dt}[c\vec{u}(t)] = c\vec{u}\,'(t)$$

$$\frac{d}{dt}[f(t)\vec{u}(t)] = f'(t)\ \vec{u}(t) + f(t)\vec{u}\,'(t)$$

$$\frac{d}{dt}[\vec{u}(t) \cdot \vec{v}(t)] = \vec{u}\,'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}\,'(t)$$

$$\frac{d}{dt}[\vec{u}(t) \times \vec{v}(t)] = \vec{u}\,'(t) \times \vec{v}(t) + \vec{u}(t) \times v\,'(t)$$

$$\frac{d}{dt}[\vec{u}(f(t))] = f'(t)\ \vec{u}\,'(f(t))$$

10.5 Integral of a Vector Function

$$\int_a^b \vec{r}(t)\ dt = \left\langle \int_a^b f(t)\ dt, \int_a^b g(t)\ dt, \int_a^b h(t)\ dt \right\rangle$$

10.6 Arc Length of a Vector Function

$$L = \int_a^b |r\,'(t)|\ dt$$

10.7 Curvature

$$\kappa = |\frac{d\vec{T}}{ds}| = \frac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|}$$

$$\kappa = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$$

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}$$

10.8 Normal and Binormal Vectors

$$\vec{N}(t) = \frac{\vec{T}\,'(t)}{|\vec{T}\,'(t)|}$$

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

10.9 Velocity and Acceleration

$$\vec{v}(t) = \vec{r}\,'(t)$$

$$\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$$

10.10 Parametric Equations of Trajectory

$$x = (v_0 \cos\alpha)t \quad y = (v_0 \sin\alpha)t - \frac{1}{2}gt^2$$

10.11 Tangential and Normal Components of Acceleration

$$\vec{a} = v'\vec{T} + \kappa v^2 \vec{N}$$

10.12 Equations of a Parametric Surface

$$x = x(u,v) \quad y = y(u,v) \quad z = z(u,v)$$

11 Partial Derivatives

11.1 Limit of $f(x, y)$

If $f(x,y) \to L_1$ as $(x,y) \to (a,b)$ along a path C_1 and $f(x,y) \to L_2$ as $(x,y) \to (a,b)$ along a path C_2, then $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

11.2 Strategy to Determine if Limit Exists

1. Substitute in for x and y. If point is defined, limit exists. If not, continue.
2. Approach (x, y) from the x-axis by setting $y = 0$ and taking $\lim_{x\to a}$. Compare this result to approaching (x, y) from the y-axis by setting $x = 0$ and taking $\lim_{y\to a}$. If these results are different, then the limit does not exist. If results are the same, continue.
3. Approach (x, y) from any nonvertical line by setting $y = mx$ and taking $\lim_{x\to a}$. If this limit depends on the value of m, then the limit of the function does not exist. If not, continue.
4. Rewrite the function in cylindrical coordinates and take $\lim_{r\to a}$. If this limit does not exist, then the limit of the function does not exist.

11.3 Continuity

A function is continuous at (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

11.4 Definition of Partial Derivative

$$f_x(a, b) = g'(a) \quad \text{where} \quad g(x) = f(x, b)$$

$$f_x(a, b) = \lim_{h\to 0} \frac{f(a + h, b) - f(a, b)}{h}$$

To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x.

11.5 Notation of Partial Derivative

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = D_x f$$

11.6 Clairaut's Theorem

If the functions f_{xy} and f_{yx} are both continuous, then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

11.7 Tangent Plane

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

11.8 The Chain Rule

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

11.9 Implicit Differentiation

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

11.10 Gradient

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle$$

11.11 Directional Derivative

$$D_{\vec{u}} f(x, y) = \nabla f(x, y) \cdot \vec{u}$$

where $\vec{u} = \langle a, b\rangle$ is a unit vector.

11.12 Maximizing the Directional Derivative

The maximum value of the directional derivative $D_{\vec{u}} f(x)$ is $|\nabla f(x)|$ and it occurs when $\vec{u}$ has the same direction as the gradient vector $\nabla f(x)$.

11.13 Second Derivative Test

Let $D = f_{xx}(a, b)f_{yy}(a, b) - (f_{xy}(a, b))^2$.

1. If $D > 0$ and $f_{xx}(a, b) > 0$ then $f(a, b)$ is a local minimum.
2. If $D > 0$ and $f_{xx}(a, b) < 0$ then $f(a, b)$ is a local maximum.
3. If $D < 0$ and $f_{xx}(a, b) > 0$ then $f(a, b)$ is a not a local maximum or minimum, but could be a saddle point.

11.14 Method of Lagrange Multipliers

To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$:

1. Find all values of x, y, z and λ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) \quad \text{and} \quad g(x, y, z) = k$$

2. Evaluate f at all of these points. The largest is the maximum value, and the smallest is the minimum value of f subject to the constraint g.

12 Multiple Integrals

12.1 Volume under a Surface

$$V = \iint_D f(x,y)\ dx\ dy$$

12.2 Average Value of a Function of Two Variables

$$f_{avg} = \frac{1}{A(R)} \iint_R f(x,y)\ dx\ dy$$

12.3 Fubini's Theorem

$$\iint_R f(x,y)\ dA = \int_a^b \int_c^d f(x,y)\ dy\ dx = \int_c^d \int_a^b f(x,y)\ dx\ dy$$

12.4 Splitting a Double Integral

$$\iint_R g(x)h(y)\ dA = \int_a^b g(x)\ dx\ \int_c^d h(y)\ dy$$

12.5 Double Integral in Polar Coordinates

$$\iint_R f(x,y)\ dA = \int_a^b \int_c^d f(r\cos\theta, r\sin\theta) r\ dr\ d\theta$$

12.6 Surface Area

$$A(S) = \iint_D |\vec{r}_u \times \vec{r}_v|\ dA$$

where a smooth parametric surface S is given by $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$.

12.7 Surface Area of a Graph

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$$

12.8 Triple Integrals in Spherical Coordinates

$$\iiint\limits_E f(x,y,z)\ dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin\phi\ d\rho\ d\theta\ d\phi$$

13 Vector Calculus

13.1 Line Integral

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)dt$$

13.2 Fundamental Theorem of Line Integrals

$$\int_C \nabla f\cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

13.3 Path Independence

$\int_C \vec{F}\cdot d\vec{r}$ is independent of path in D if and only if $\int_C \vec{F}\cdot d\vec{r} = 0$ for every closed path C in D.

13.4 Curl

$$\text{curl}(\vec{F}) = \nabla\times\vec{F}$$

13.5 Conservative Vector Field Test

$\vec{F}$ is conservative if curl $\vec{F} = 0$ and the domain is closed and simply connected.

13.6 Divergence

$$\text{div}(\vec{F}) = \nabla\cdot F$$

13.7 Green's Theorem

$$\int_C \vec{F}\cdot d\vec{r} = \iint\limits_R \text{curl}(\vec{F})\ dx\ dy$$

13.8 Surface Integral

$$\iint\limits_S f(x,y,z)\ dS = \iint\limits_D f(\vec{r}(u,v))|\vec{r}_u\times\vec{r}_v|\ dA$$

13.9 Flux

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v)\ dA$$

13.10 Stokes' Theorem

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}(\vec{F}) \cdot d\vec{S}$$

13.11 Divergence Theorem

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \text{div}(\vec{F})\ dV$$

14 Appendix A: Selected Surface Paramatrizations

14.1 Sphere of Radius ρ

$$\vec{r}(u,v) = \langle \rho \cos u \sin v, \rho \sin u \sin v, \rho \cos v \rangle$$

14.2 Graph of a Function $f(x,y)$

$$\vec{r}(u,v) = \langle u, v, f(u,v) \rangle$$

14.3 Graph of a Function $f(\phi, r)$

$$\vec{r}(u,v) = \langle v \cos u, v \sin u, f(u,v) \rangle$$

14.4 Plane Containing $P, \vec{u}$, and $\vec{v}$

$$\vec{r}(s,t) = \vec{OP} + s\vec{u} + t\vec{v}$$

14.5 Surface of Revolution

$$\vec{r}(u,v) = \langle g(v) \cos u, g(v) \sin u, v \rangle$$

where $g(z)$ gives the distance from the z-axis.

14.6 Cylinder

$$\vec{r}(u,v) = \langle \cos u, \sin u, v \rangle$$

14.7 Cone

$$\vec{r}(u,v) = \langle v\cos u, v\sin u, v\rangle$$

14.8 Paraboloid

$$\vec{r}(u,v) = \langle \sqrt{v}\cos u, \sqrt{v}\sin u, v\rangle$$

15 Appendix B: Selected Differential Equations

15.1 Heat Equation

$$f_t = f_{xx}$$

15.2 Wave Equation (Wavequation)

$$f_{tt} = f_{xx}$$

15.3 Transport (Advection) Equation

$$f_x = f_t$$

15.4 Laplace Equation

$$f_{xx} = -f_{yy}$$

15.5 Burgers Equation

$$f_{xx} = f_t + ff_x$$

Multivariable Calculus Study Guide: A $\LaTeX$ Version

Tyler Silber
University of Connecticut

December 11, 2011

1 Disclaimer

It is not guaranteed that I have every single bit of necessary information for the course. This happened to be some of what I needed to know this specific semester in my course. For example, Stokes' Theorem is not even mentioned.

2 Vectors Between Two Points

$$Given: \; P(x_1, y_1) \;\&\; Q(x_2, y_2)$$

$$\overrightarrow{PQ} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix}$$

3 Vectors in the Plane

$$let \; \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \& \; \mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

3.1 Simple Operations

$$c\mathbf{v} = \begin{pmatrix} cv_1 \\ cv_2 \end{pmatrix}$$

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2}$$

$$\mathbf{v} + \mathbf{u} = \begin{pmatrix} v_1 + u_1 \\ v_2 + u_2 \end{pmatrix}$$

3.2 Unit Vectors

$$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \;\&\; \mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j}$$

3.3 Vectors of a Specified Length

$$\left|\frac{c\mathbf{v}}{|\mathbf{v}|}\right| = |c|$$

$$\pm\frac{c\mathbf{v}}{|\mathbf{v}|} \parallel \mathbf{v}$$

4 Vectors in Three Dimensions

4.1 Notes

Everything in the above section can be expanded to three dimensions. Simply add another component.

$$\mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

4.2 Random Equations

$$xy\text{-plane } \{(x, y, z) : z = 0\}$$

$$xz\text{-plane } \{(x, y, z) : y = 0\}$$

$$yz\text{-plane } \{(x, y, z) : x = 0\}$$

$$\text{Sphere: } (x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$

5 Dot Product

5.1 Definitions

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3 = |\mathbf{u}||\mathbf{v}|\cos\theta$$

$$\mathbf{u} \perp \mathbf{v} \Leftrightarrow \mathbf{u} \cdot \mathbf{v} = 0$$

$$\mathbf{u} \parallel \mathbf{v} \Leftrightarrow \mathbf{u} \cdot \mathbf{v} = \pm|\mathbf{u}||\mathbf{v}|$$

5.2 Projections

The orthogonal projection of **u** onto **v** is denoted $\text{proj}_{\mathbf{v}}\mathbf{u}$ and the scalar component of **u** in the direction of **v** is denoted $\text{scal}_{\mathbf{v}}\mathbf{u}$.

$$\text{proj}_{\mathbf{v}}\mathbf{u} = |\mathbf{u}|\cos\theta\left(\frac{\mathbf{v}}{|\mathbf{v}|}\right) = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v}$$

$$\text{scal}_{\mathbf{v}}\mathbf{u} = |\mathbf{u}|\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}$$

6 Cross Product

$$|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta \tag{1}$$

$$\mathbf{u} \parallel \mathbf{v} \Leftrightarrow \mathbf{u}\times\mathbf{v} = \mathbf{0}$$

$$\mathbf{u}\times\mathbf{v} = \begin{pmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{pmatrix}$$

Note: $\mathbf{u}\times\mathbf{v}$ is orthogonal to both **u** and **v** and the direction is defined by the right-hand rule.

7 Lines and Curves in Space

7.1 Vector-Valued Functions

$$\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$$

7.2 Lines

$$\langle x, y, z\rangle = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle,\ \text{for } -\infty < t < \infty$$

7.3 Line Segments

$$Given:\ P_1(x_1, y_1, z_1)\ \&\ P_2(x_2, y_2, z_2)$$

$$\overrightarrow{P_1P_2} = \langle x_1, y_1, z_1\rangle + t\langle x_2 - x_1, y_2 - y_1, z_2 - z_1\rangle,\ \text{for } 0 \le t \le 1$$

7.4 Curves in Space

$$\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$$

Equation 1 is also equal to the area of the parallelogram created by the two vectors.

7.5 Limits

$$\lim_{t \to a} \mathbf{r}(t) = \left\langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \right\rangle$$

8 Calculus of Vector-Valued Functions

8.1 Derivative and Tangent Vector

$$\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$$

Note: $\mathbf{r}'(t)$ is the tangent vector to $\mathbf{r}(t)$ at the point $(f(t), g(t), h(t))$.

8.2 Indefinite Integral

$$\int \mathbf{r}(t)\, dt = \mathbf{R}(t) + \mathbf{C}$$

Note: $\mathbf{C}$ is an arbitrary constant vector and $\mathbf{R} = F\mathbf{i} + G\mathbf{j} + H\mathbf{k}$.

8.3 Definite Integral

$$\int_a^b \mathbf{r}(t)\, dt = \left[\int_a^b f(t)\, dt\right] \mathbf{i} + \left[\int_a^b g(t)\, dt\right] \mathbf{j} + \left[\int_a^b h(t)\, dt\right] \mathbf{k}$$

9 Motion in Space

9.1 Definitions

$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$$

$$Speed = |\mathbf{v}(t)|$$

9.2 Two-Dimensional Motion in a Gravitational Field

$$Given:\ \mathbf{v}(0) = \langle u_0, v_0 \rangle \ \&\ \mathbf{r}(0) = \langle x_0, y_0 \rangle$$

$$\mathbf{v}(t) = \langle x'(t), y'(t) \rangle = \langle u_0, -gt + v_0 \rangle$$

$$\mathbf{r}(t) = \langle x(t), y(t) \rangle = \left\langle u_0 t + x_0, -\frac{1}{2}gt^2 + v_0 t + y_0 \right\rangle$$

9.3 Two-Dimensional Motion

$$Given: \ \mathbf{v}(0) = \langle |\mathbf{v}_0| \cos\theta, |\mathbf{v}_0| \sin\theta \rangle \ \& \ \mathbf{r}(0) = \langle 0, 0 \rangle$$

$$Time = \frac{2|\mathbf{v}_0| \sin\theta}{g}$$

$$Range = \frac{|\mathbf{v}_0|^2 \sin 2\theta}{g}$$

$$Max\,Height = y\left(\frac{T}{2}\right) = \frac{(|\mathbf{v}_0| \sin\theta)^2}{2g}$$

10 Planes and Surfaces

10.1 Plane Equations

The plane passing through the point $P_0(x_0, y_0, z_0)$ with a normal vector $\mathbf{n} = \langle a, b, c, \rangle$ is described by the equations:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$ax + by + cz = d, \text{ where } d = ax_0 + by_0 + cz_0$$

In order to find the equation of a plane when given three points, simply create any two vectors out of the points and take the cross product to find the vector normal to the plane. Then use one of the above formulae.

10.2 Parallel and Orthogonal Planes

Two planes are parallel if their normal vectors are parallel. Two planes are orthogonal if their normal vectors are orthogonal.

10.3 Surfaces

10.3.1 Ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

10.3.2 Elliptic Paraboloid

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

It would be worth it to learn how to derive sections 9.2 and 9.3.

10.3.3 Hyperboloid of One Sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

10.3.4 Hyperboloid of Two Sheets

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

10.3.5 Elliptic Cone

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

10.3.6 Hyperbolic Paraboloid

$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

11 Graphs and Level Curves

11.1 Functions of Two Variables

$$\mathbb{R}^2 \to \mathbb{R}$$

$$z = f(x, y)$$

$$F(x, y, z) = 0$$

11.2 Functions of Three Variables

$$\mathbb{R}^3 \to \mathbb{R}$$

$$w = f(x, y, z)$$

$$F(w, x, y, z) = 0$$

11.3 Level Curves

Imagine stepping onto a surface and walking along a path with constant elevation. The path you walk on is known as the contour curve, while the projection of the path onto the xy-plane is known as a level curve.

12 Limits and Continuity

12.1 Limits

The function f has the limit L as $P(x, y)$ approaches $P_0(a, b)$.

$$\lim_{(x,y)\to(a,b)} f(x, y) = \lim_{P\to P_0} f(x, y) = L$$

If $f(x, y)$ approaches two different values as (x, y) approaches (a, b) along two different paths in the domain of f, then the limit does not exist.

12.2 Continuity

The function f if continuous at the point (a, b) provided:

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

13 Partial Derivatives

13.1 Definitions

$$f_x(a, b) = \lim_{h\to 0} \frac{f(a + h, b) - f(a, b)}{h}$$
$$f_y(a, b) = \lim_{h\to 0} \frac{f(a, b + h) - f(a, b)}{h}$$

So basically just take the derivative of one (the subscript) given that the other one is a constant.

13.2 Notation for Higher-Order Partial Derivatives

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = (f_x)_x = f_{xx}$$
$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = (f_y)_y = f_{yy}$$
$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = (f_y)_x = f_{yx}$$
$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = (f_x)_y = f_{xy}$$

Note: $f_{xy} = f_{yx}$ for nice functions.

13.3 Differentiability

Suppose the function f has partial derivatives f_x and f_y defined on an open region containing (a, b), with f_x and f_y continuous at (a, b). Then f is differentiable at (a, b). This also implies that it is continuous at (a, b).

14 Chain Rule

14.1 Examples

You can use a *tree diagram* to determine the equation for the chain rule. You can also just think about it. Refer to the following examples.

z is a function of x and y, while x and y are functions of t

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

w is a function of x, y, and z, while x, y, and z are functions of t

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

z is a function of x and y, while x and y are functions of s and t

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

w is a function of z, z is a function of x and y, x and y are functions of t

$$\frac{dw}{dt} = \frac{dw}{dz}\left(\frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}\right)$$

14.2 Implicit Differentiation

Let F be differentiable on its domain and suppose that $F(x, y) = 0$ defines y as a differentiable function of x. Provided $F_y \neq 0$,

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

15 Directional Derivatives and Gradient

15.1 Definitions

Let f be differentiable at (a, b) and let $\mathbf{u} = \langle u_1, u_2\rangle$ be a unit vector in the xy-plane. The directional derivative of f at (a, b) in the direction of $\mathbf{u}$ is

$$D_{\mathbf{u}}f(a, b) = \langle f_x(a, b), f_y(a, b)\rangle \cdot \langle u_1, u_2\rangle = \nabla f(a, b) \cdot \mathbf{u}$$

Gradient

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}$$

15.2 Directions of Change

- f has its maximum rate of increase at (a, b) in the direction of the gradient $\nabla f(a, b)$. The rate of increase in this direction is $|\nabla f(a, b)|$.
- f has its maximum rate of decrease at (a, b) in the direction of the gradient $-\nabla f(a, b)$. The rate of decrease in this direction is $-|\nabla f(a, b)|$.
- The directional derivative is zero in any direction orthogonal to $\nabla f(a, b)$.

15.3 Expanding to Three Dimensions

It's really intuitive how it expands into three dimensions. Just add another component or f_z where you think it should go.

16 Tangent Plane and Linear Approximation

16.1 Tangent Plane for $\mathbf{F(x, y, z) = 0}$

The tangent plane passes through the point $P_0(a, b, c)$.

$$F_x(a, b, c)(x - a) + F_y(a, b, c)(y - b) + F_z(a, b, c)(z - c) = 0$$

16.2 Tangent Plane for $\mathbf{z = f(x, y)}$

The tangent plane passes through the point $(a, b, f(a, b))$.

$$z = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b)$$

16.3 Linear Approximation

Firstly, calculate the equation of the tangent plane of a point near the point you wish to approximate. Then simply plug in the point and you're done.

16.4 The differential dz

The change in $z = f(x, y)$ as the independent variables change from (a, b) to $(a + dx, b + dy)$ is denoted Δz and is approximated by the differential dz:

$$\Delta z \approx dz = f_x(a, b)dx + f_y(a, b)dy$$

17 Max-Min Problems

17.1 Derivatives and Local Maximum/Minimum Values

If f has a local maximum or minimum value at (a, b) and the partial derivatives f_x and f_y exist at (a, b), then $f_x(a, b) = f_y(a, b) = 0$.

17.2 Critical Points

A critical point exists if either

- $f_x(a, b) = f_y(a, b) = 0$
- one (or both) of f_x or f_y does not exist at (a, b)

17.3 Second Derivative Test

Let $D(x, y) = f_{xx}f_{yy} - f_{xy}^2$

- If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then f has a local maximum at (a, b).
- If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then f has a local minimum at (a, b).
- If $D(a, b) < 0$, then f has a saddle point at (a, b).
- If $D(a, b) = 0$, then the test is inconclusive.

17.4 Absolute Maximum/Minimum Values

Let f be continuous on a closed bounded set R in $\mathbf{R}^2$. To find absolute maximum and minimum values of f on R:

1. Determine the values of f at all critical points in R.
2. Find the maximum and minimum values of f on the boundary of R.
3. The greatest function value found in Steps 1 and 2 is the absolute maximum value of f on R, and the least function value found in Steps 1 and 2 is the absolute minimum values of f on R.

18 Double Integrals

18.1 Double Integrals on Rectangular Regions

Let f be continuous on the rectangular region $R = \{(x, y) : a \leq x \leq b, c \leq y \leq d\}$. The double integral of f over R may be evaluated by either of two iterated integrals:

$$\iint\limits_R f(x, y)\, dA = \int_c^d \int_a^b f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx$$

18.2 Double Integrals over Nonrectangular Regions

Let R be a region bounded below and above by the graphs of the continuous functions $y = g(x)$ and $y = h(x)$, respectively, and by the lines $x = a$ and $x = b$. If f is continuous on R, then

$$\iint\limits_R f(x, y)\, dA = \int_a^b \int_{g(x)}^{h(x)} f(x, y)\, dy\, dx$$

Let R be a region bounded on the left and right by the graphs of the continuous functions $x = g(y)$ and $x = h(y)$, respectively, and by the lines $y = c$ and $y = d$. If f is continuous on R, then

$$\iint\limits_R f(x, y)\, dA = \int_c^d \int_{g(y)}^{h(y)} f(x, y)\, dx\, dy$$

18.3 Areas of Regions by Double Integrals

$$\text{area of } R = \iint\limits_R dA$$

19 Polar Double Integrals

19.1 Double Integrals over Polar Rectangular Regions

Let f be continuous on the region in the xy-plane $R = \{(r, \theta) : 0 \leq a \leq r \leq b, \alpha \leq \theta \leq \beta\}$, where $\beta - \alpha \leq 2\pi$. Then

$$\iint\limits_R f(r, \theta)\, dA = \int_\alpha^\beta \int_a^b f(r, \theta)\, r\, dr\, d\theta$$

19.2 Double Integrals over More General Polar Regions

Let f be continuous on the region in the xy-plane

$$R = \{(r, \theta) : 0 \leq g(\theta) \leq r \leq h(\theta), \alpha \leq \theta \leq \beta\}$$

where $\beta - \alpha \leq 2\pi$. Then.

$$\iint\limits_R f(r, \theta)\, dA = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} f(r, \theta)\, r\, dr\, d\theta$$

If f is nonnegative on R, the double integral gives the volume of the solid bounded by the surface $z = f(r, \theta)$ and R.

19.3 Area of Polar Regions

$$A = \iint\limits_R dA = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} r \, dr \, d\theta$$

20 Triple Integrals

Let $D = \{(x, y, z) : a \leq x \leq b, g(x) \leq y \leq h(x), G(x, y) \leq z \leq H(x, y)\}$, where g, h, G, H are continuous functions. The triple integral of a continuous function f on D is evaluated as the iterated integral

$$\iiint\limits_D f(x, y, z) \, dV = \int_a^b \int_{g(x)}^{h(x)} \int_{G(x,y)}^{H(x,y)} f(x, y, z) \, dz \, dy \, dx$$

21 Cylindrical and Spherical Coordinates

21.1 Definitions

21.1.1 Cylindrical Coordinates

(r, θ, z) An extension of polar coordinates into $\mathbf{R}^3$. Simply add a z component.

21.1.2 Spherical Coordinates

(ρ, φ, θ)

- ρ is the distance from the origin to a point P.
- φ is the angle between the positive z-axis and the line OP.
- θ is the same angle as in cylindrical coordinates; it measure rotation about the z-axis relative to the positive x-axis.

21.2 Rectangular to Cylindrical

$$r^2 = x^2 + y^2$$

$$\tan\theta = \frac{y}{x}$$

$$z = z$$

21.3 Cylindrical to Rectangular

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$z = z$$

21.4 Integration in Cylindrical Coordinates

$$\iiint\limits_D f(r,\theta,z)\,dV = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} \int_{G(r\cos\theta, r\sin\theta)}^{H(r\cos\theta, r\sin\theta)} f(r,\theta,z)\,dz\,r\,dr\,d\theta$$

21.5 Rectangular to Spherical

$$\rho^2 = x^2 + y^2 + z^2$$

You have to solve for φ and θ with trigonometry.

21.6 Spherical to Rectangular

$$x = \rho \sin\varphi \cos\theta$$
$$y = \rho \sin\varphi \sin\theta$$
$$z = \rho \cos\varphi$$

21.7 Integration in Spherical Coordinates

$$\iiint\limits_D f(\rho,\varphi,\theta)\,dV = \int_\alpha^\beta \int_a^b \int_{g(\varphi,\theta)}^{h(\varphi,\theta)} f(\rho,\varphi,\theta)\rho^2 \sin\varphi\,d\rho\,d\varphi\,d\theta$$

22 Change of Variables

22.1 Jacobian Determinant of a Transformation of Two Variables

Given a transformation $T : x = g(u,v), y = h(u,v)$, where g and h are differentiable on a region of the uv-plane, the Jacobian determinant of T is

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

22.2 Change of Variables for Double Integrals

$$\iint\limits_R f(x,y)\,dA = \iint\limits_S f(g(u,v),h(u,v))|J(u,v)|\,dA$$

22.3 Change of Variables for Triple Integrals

I am *SO* not typing out the expansion of the above into triple integrals. It's intuitive. Just add stuff where you think it should go.

22.4 YOU have to Choose the Transformation

Just cry.

23 Vector Fields

23.1 Vector Fields in Two Dimensions

$$\mathbf{F}(x, y) = \langle f(x, y), g(x, y) \rangle$$

23.2 Radial Vector Fields in $\mathbf{R}^2$

Let $\mathbf{r} = (x, y)$. A vector field of the form $\mathbf{F} = f(x, y)\mathbf{r}$, where f is a scalar-valued function, is a radial vector field.

$$\mathbf{F}(x, y) = \frac{\mathbf{r}}{|\mathbf{r}|^p} = \frac{\langle x, y \rangle}{|\mathbf{r}|^p}$$

p is a real number. At every point (sans origin), the vectors of this field are directed outward format he origin with a magnitude of $|\mathbf{F}| = \dfrac{1}{|\mathbf{r}|^{p-1}}$. You can also apply all of this to $\mathbf{R}^3$ by just adding a z component.

23.3 Gradient Fields and Potential Functions

Let $z = \varphi(x, y)$ and $w = \varphi(x, y, z)$ be differentiable functions on regions of $\mathbf{R}^2$ and $\mathbf{R}^3$, respectively. The vector field $\mathbf{F} = \nabla\varphi$ is a gradient field, and the function φ is a potential function for $\mathbf{F}$.

24 Line Integrals

24.1 Evaluating Scalar Line Integrals in $\mathbf{R}^2$

Let f be continuous on a region containing a smooth curve $C : \mathbf{r}(t) = \langle x(t), y(t) \rangle$, for $a \leq t \leq b$. Then

$$\int\limits_C f\, ds = \int_a^b f(x(t), y(t))|\mathbf{r}'(t)|\, dt = \int_a^b f(x(t), y(t))\sqrt{x'(t)^2 + y'(t)^2}\, dt$$

24.2 Evaluating Scalar Line Integrals in $\mathbf{R}^3$

Simply add a z component to the above where it obviously belongs.

24.3 Line Integrals of Vector Fields

24.3.1 Definition

Let $\mathbf{F}$ be a vector field that is continuous on a region containing a smooth oriented curve C parametrized by arc length. Let $\mathbf{T}$ be the unit tangent vector at each point of C consistent with the orientation. The line integral of $\mathbf{F}$ over C is $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$.

24.3.2 Different Forms

$\mathbf{F} = \langle f, g, h \rangle$ and C has a parametrization $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, for $a \leq t \leq b$

$$\int_a^b \mathbf{F} \cdot \mathbf{r}'(t)\, dt = \int_a^b (f x'(t) + g y'(t) + h z'(t))\, dt = \int\limits_C f\, dx + g\, dy + h\, dz = \int\limits_C \mathbf{F} \cdot d\mathbf{r}$$

For line integrals in the plane, we let $\mathbf{F} = \langle f, g \rangle$ and assume C is parametrized in the form $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, for $a \leq t \leq b$. Then

$$\int\limits_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_a^b (f x'(t) + g y'(t))\, dt = \int\limits_C f\, dx + g\, dy = \int\limits_C \mathbf{F} \cdot d\mathbf{r}$$

24.4 Work

$\mathbf{F}$ is a force field

$$W = \int\limits_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_a^b \mathbf{F} \cdot \mathbf{r}'(t)\, dt$$

24.5 Circulation

$\mathbf{F}$ is a vector field

$$Circulation = \int\limits_C \mathbf{F} \cdot \mathbf{T}\, ds$$

24.6 Flux

$$Flux = \int\limits_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_a^b (f y'(t) - g x'(t))\, dt$$

$\mathbf{n} = \mathbf{T} \times \mathbf{k}$, and a positive answer means a positive outward flux.

25 Conservative Vector Fields

25.1 Test for Conservative Vector Field

Let $\mathbf{F} = \langle f, g, h \rangle$ be a vector field defined on a connected and simply connected region D of $\mathbf{R}^3$, where f, g, and h have continuous first partial derivatives on

D. Then, $\mathbf{F}$ is a conservative vector field on D (there is a potential function φ such that $\mathbf{F} = \nabla\varphi$) if and only if

- $\dfrac{\partial f}{\partial y} = \dfrac{\partial g}{\partial x}$
- $\dfrac{\partial f}{\partial z} = \dfrac{\partial h}{\partial x}$
- $\dfrac{\partial g}{\partial z} = \dfrac{\partial h}{\partial y}$

For vector fields in $\mathbf{R}^2$, we have the single condition $\dfrac{\partial f}{\partial y} = \dfrac{\partial g}{\partial x}$.

25.2 Finding Potential Functions

Suppose $\mathbf{F} = \langle f, g, h \rangle$ is a conservative vector field. To find φ such that $\mathbf{F} = \nabla\varphi$, take the following steps:

1. Integrate $\varphi_x = f$ with respect to x to obtain φ, which includes an arbitrary function $c(y, z$.
2. Compute φ_y and equate it to g to obtain an expression for $c_y(y, z)$.
3. Integrate $c_y(y, z)$ with respect to y to obtain $c(y, z)$, including an arbitrary function $d(z)$.
4. Compute φ_z and equate it to h to get $d(z)$.

Beginning the procedure with $\varphi_y = g$ or $\varphi_z = h$ may be easier in some cases. This method can also be used to check if a vector field is conservative by seeing if there is a potential function.

25.3 Fundamental Theorem for Line Integrals

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(B) - \varphi(A)$$

25.4 Line Integrals on Closed Curves

Let R in $\mathbf{R}^2$ (or D in $\mathbf{R}^3$) be an open region. Then $\mathbf{F}$ is a conservative vector field on R if and only if $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ on all simple closed smooth oriented curves C in R.

26 Green's Theorem

26.1 Circulation Form

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C f\, dx + g\, dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA$$

26.2 Area of a Plane Region by Line Integrals

$$\oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C (x\,dy - y\,dx)$$

26.3 Flux Form

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \oint_C f\,dy - g\,dx = \iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right) dA$$

27 Divergence and Curl

27.1 Divergence of a Vector Field

$$\operatorname{div}(\mathbf{F}) = \nabla\cdot\mathbf{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

27.2 Divergence of Radial Vector Fields

$$\operatorname{div}(\mathbf{F}) = \frac{3-p}{|\mathbf{r}|^p}$$

$$\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|^p} = \frac{\langle x, y, z\rangle}{(x^2+y^2+z^2)^{p/2}}$$

27.3 Curl

$$\operatorname{curl}(\mathbf{F}) = \nabla\times\mathbf{F}$$

Just derive the curl by doing the cross product.

27.4 Divergence of the Curl

$$\nabla\cdot(\nabla\times\mathbf{F}) = 0$$

28 Surface Integrals

28.1 Parameterization

28.1.1 z is Explicitly Defined

Use $x = x$, $y = y$, and since z is explicitly defined, you already have what z equals.

28.1.2 Cylinder

Simply use cylindrical coordinates to parameterize the surface in terms of θ and z.

28.1.3 Sphere

Simply use spherical coordinates to parameterize the surface in terms of φ and θ.

28.1.4 Cone

Use:

- $x = v \cos u$
- $y = v \sin u$
- $z = v$

$0 \le u \le 2\pi$ and $0 \le v \le h$

28.2 Surface Integrals of Parameterized Surfaces

$$\iint\limits_{\Sigma} f(x, y, z)\, d\sigma = \iint\limits_{R} f(x(u, v), y(u, v), z(u, v)) \left| \frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t} \right| dA$$

29 Divergence Theorem

Let $\mathbf{F}$ be a vector field whose components have continuous first partial derivatives in a connected and simply connected region D enclosed by a smooth oriented surface S. Then

$$\iint\limits_{S} \mathbf{F} \cdot \mathbf{n}\, dS = \iiint\limits_{D} \nabla \cdot \mathbf{F}\, dV$$

where $\mathbf{n}$ is the outward normal vector on S.

Dr. Z's Math251 Handout #16.6 [Parametric Surfaces and Their Areas]

By Doron Zeilberger

Problem Type 16.6a: Find an equation of the tangent plane to the given parametric surface at the specified point.

$$x = x(u,v) \quad , \quad y = y(u,v) \quad , \quad z = z(u,v) \quad ; \quad u = 1, v = 1 \quad .$$

Example Problem 16.6a: Find an equation of the tangent plane to the given parametric surface at the specified point.

$$x = u^2 \quad , \quad y = v^2 \quad , \quad z = uv \quad ; \quad u = 1, v = 1 \quad .$$

Steps

1. Set-up

$$\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k} \quad ,$$

and compute the partial derivatives w.r.t. u and w.r.t. v :

$$\mathbf{r}_u = x_u\,\mathbf{i} + y_u\,\mathbf{j} + z_u\,\mathbf{k} \quad ,$$

$$\mathbf{r}_v = x_v\,\mathbf{i} + y_v\,\mathbf{j} + z_v\,\mathbf{k} \quad .$$

Then **plug-in** the given values of u and v.

Example

1. In this problem

$$\mathbf{r} = u^2\,\mathbf{i} + v^2\,\mathbf{j} + uv\,\mathbf{k} \quad ,$$

We have

$$\mathbf{r}_u = 2u\,\mathbf{i} + 0\,\mathbf{j} + v\,\mathbf{k} \quad ,$$

$$\mathbf{r}_v = 0\,\mathbf{i} + 2v\,\mathbf{j} + u\,\mathbf{k} \quad .$$

Now **plug-in** $u = 1, v = 1$ to get **numerical vectors**.

$$\mathbf{r}_u(1,1) = 2\,\mathbf{i} + 0\,\mathbf{j} + 1\,\mathbf{k} \quad ,$$

$$\mathbf{r}_v(1,1) = 0\,\mathbf{i} + 2\,\mathbf{j} + 1\,\mathbf{k} \quad .$$

So at this point, $r_u = \langle 2, 0, 1 \rangle$, $r_v = \langle 0, 2, 1 \rangle$.

2. Find the cross-product $\mathbf{r}_u \times \mathbf{r}_v$. This is a vector **normal** to the tangent plane.

2.

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & 1 \\ 0 & 2 & 1 \end{vmatrix}$$

$$= -2\,\mathbf{i} - 2\,\mathbf{j} + 4\,\mathbf{k} \quad .$$

Or in $\langle\rangle$ notation

$$\mathbf{N} = \langle -2, -2, 4 \rangle \quad .$$

3. Find the **point** (x_0, y_0, z_0) by plugging into x, y, z the specific values of u and v given in the problem The desired equation of the tangent plane is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \quad .$$

where $N = \langle a, b, c \rangle$ and the point is (x_0, y_0, z_0).

3. The point is $(1^2, 1^2, 1 \cdot 1) = (1, 1, 1)$. The desired equation of the tangent plane is

$$(-2)(x - 1) - 2(y - 1) + 4(z - 1) = 0 \quad .$$

Or, in expanded form

$$-2x - 2y + 4z = 0 \quad .$$

Dividing by -2 (to make it nicer), we get:

Ans.: $x + y - 2z = 0$.

Dr. Z's Math251 Handout #15.6 [Surface Area]

By Doron Zeilberger

Problem Type 15.6a:Find the area of the surface $z = F(x, y)$ that lies above a given region.

Example Problem 15.6a: Find the area of the surface $z = 1 + 3x + 2y^2$ that lies above the triangle with vertices $(0, 0)$,$(0, 1)$,$(2, 1)$.

Steps

1. Write the region either as a type I region

$$D = \{(x, y) \,|\, a \le x \le b \,, f_1(x) \le y \le f_2(x)\,\} \ ,$$

or type II region

$$D = \{(x, y) \,|\, c \le y \le d \,, g_1(y) \le x \le g_2(y)\,\} \ ,$$

whatever is convenient.

2. Write the surface as $z = F(x, y)$ (if not already in that form) and set-up the integral for the surface area

$$\int\int_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA \quad .$$

Compute $\frac{\partial z}{\partial x} = F_x(x, y)$ and $\frac{\partial z}{\partial y} = F_y(x, y)$ and plug them in. Then use the description of D to convert it into an iterated integral.

Example

1. The triangle with vertices $(0, 0)$,$(0, 1)$,$(2, 1)$ is most conveniently written as a type II region

$$D = \{(x, y) \,|\, 0 \le y \le 1 \,, 0 \le x \le 2y\,\} \quad .$$

2. In this problem, $F(x, y) = 1+3x+2y^2$. So $F_x = 3$ and $F_y = 4y$ and the integral is

$$\int\int_D \sqrt{1 + 3^2 + (4y)^2}\, dA$$

$$= \int_0^1 \int_0^{2y} \sqrt{10 + 16y^2}\, dx\, dy \quad .$$

3. Evaluate that iterated integral by first doing the inner integral and then the outer integral.

3. The inner integral is

$$\int_0^{2y} \sqrt{10+16y^2}\,dx = \sqrt{10+16y^2}\int_0^{2y} dx$$

$$= 2y\sqrt{10+16y^2} \quad ,$$

and the outer integral is

$$\int_0^1 \int_0^{2y} \sqrt{10+16y^2}\,dx\,dy$$

$$= \int_0^1 \left[\int_0^{2y} \sqrt{10+16y^2}\,dx\right] dy$$

$$= \int_0^1 2y(10{+}16y^2)^{1/2}\,dy = \frac{1}{16}\frac{(10+16y^2)^{3/2}}{3/2}\Big|_0^1$$

$$= \frac{1}{24}\left[(10+16\cdot 1^2)^{3/2} - (10+16\cdot 0^2)^{3/2}\right]$$

$$= \frac{26^{3/2}-10^{3/2}}{24} \quad .$$

Ans.: $\frac{26^{3/2}-10^{3/2}}{24}$.

Dr. Z's Math251 Handout #15.6 [Surface Area]

By Doron Zeilberger

Problem Type 15.6a:Find the area of the surface $z = F(x, y)$ that lies above a given region.

Example Problem 15.6a: Find the area of the surface $z = 1 + 3x + 2y^2$ that lies above the triangle with vertices $(0, 0)$,$(0, 1)$,$(2, 1)$.

Steps

1. Write the region either as a type I region

$$D = \{(x, y) \mid a \le x \le b \,, f_1(x) \le y \le f_2(x)\,\} \ ,$$

or type II region

$$D = \{(x, y) \mid c \le y \le d \,, g_1(y) \le x \le g_2(y)\,\} \ ,$$

whatever is convenient.

2. Write the surface as $z = F(x, y)$ (if not already in that form) and set-up the integral for the surface area

$$\int\int_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA \quad .$$

Compute $\frac{\partial z}{\partial x} = F_x(x, y)$ and $\frac{\partial z}{\partial y} = F_y(x, y)$ and plug them in. Then use the description of D to convert it into an iterated integral.

Example

1. The triangle with vertices $(0, 0)$,$(0, 1)$,$(2, 1)$ is most conveniently written as a type II region

$$D = \{(x, y) \mid 0 \le y \le 1 \,, 0 \le x \le 2y\,\} \quad .$$

2. In this problem, $F(x, y) = 1+3x+2y^2$. So $F_x = 3$ and $F_y = 4y$ and the integral is

$$\int\int_D \sqrt{1 + 3^2 + (4y)^2}\, dA$$

$$= \int_0^1 \int_0^{2y} \sqrt{10 + 16y^2}\, dx\, dy \quad .$$

Dr. Z's Math251 Handout #15.4 [Double Integrals in Polar Coordinates]

By Doron Zeilberger

Problem Type 15.4a: Evaluate the integral

$$\int\int_D F(x,y)\,dA \quad ,$$

where D is a region best described in polar coordinates,

$$D = \{\,(r,\theta)\,|\,\alpha \le \theta \le \beta\,,\, h_1(\theta) \le r \le h_2(\theta)\,\} \quad .$$

Example Problem 15.4a: Evaluate the integral

$$\int\int_D e^{-x^2-y^2}\,dA \quad ,$$

where D is the region bounded by the semi-circle $x = \sqrt{25-y^2}$ and the y-axis.

Steps

1. Draw the region and express it, if possible and convenient, as

$$D =$$

$$\{\,(r,\theta)\,|\,\alpha \le \theta \le \beta\,,\, h_1(\theta) \le r \le h_2(\theta)\,\} \quad .$$

Of course, in many problems, the $h_1(\theta)$ and/or $h_2(\theta)$ may be plain numbers (i.e. not involve θ).

Example

1. This is a **semi**-circle, i.e. **half** a circle, center origin, radius 5, and since it is bounded by the y-axis, and $x \ge 0$, it is the **right** half

[Had it been $x = -\sqrt{25-y^2}$ it would have been the left-half. Had it been $y = \sqrt{25-x^2}$ it would have been the upper-half. Had it been $y = -\sqrt{25-x^2}$ it would have been the lower-half.]

Since it is the right-half, θ ranges from $\theta = -\pi/2$ (the downwards direction) to $\theta = \pi/2$ (the upwards direction). For each ray $\theta = \theta_0$, r, the distance from the origin, ranges from $r = 0$ to $r = 5$ (and indeed does not depend on θ in this problem). So our region phrased in **polar coordinates** is:

$$D = \{\,(r,\theta)\,|\,-\pi/2 \le \theta \le \pi/2\,,\, 0 \le r \le 5\,\} \quad .$$

2. Rewrite the area integral

$$\int\int_D F(x,y)\,dA \quad ,$$

in **polar** coordinates by replacing

x by $r\cos\theta$, y by $r\sin\theta$, dA by $r\,dr\,d\theta$.

[**shortcut:** Whenever you see x^2+y^2 you can replace it by r^2.]

Write it as an iterated integral

$$\int\int_D F(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} F(r\cos\theta\,,\,r\sin\theta)\,r\,dr\,d\theta \quad ,$$

with the θ-integral being at the **outside** and the r-integral being in the **inside**.

2.

$$\int\int_D e^{-x^2-y^2}\,dA$$

$$= \int_{-\pi/2}^{\pi/2}\int_0^5 e^{-r^2}\,rdr\,d\theta \quad .$$

3. Evaluate this iterated integral by first doing the inner-integral (possibly getting an expression in θ, or just a number), and then the outer integral.

3. The inside integral is (do the change-of-variable $u=-r^2$):

$$\int_0^5 e^{-r^2}\,rdr \;=\; (-1/2)e^{-r^2}\Big|_0^5 = (1-e^{-25})/2 \quad ,$$

and the whole double-integral is

$$\int_{-\pi/2}^{\pi/2}\int_0^5 e^{-r^2}\,rdr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\left[\int_0^5 e^{-r^2}\,rdr\right]\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2}[(1-e^{-25})/2]\;\;d\theta = (1-e^{-25})/2\int_{-\pi/2}^{\pi/2} d\theta =$$

$$[(1-e^{-25})/2][\pi/2-(-\pi/2)] = \pi(1-e^{-25})/2 \quad .$$

Ans.: $\pi(1-e^{-25})/2$.

Problem Type 15.4b: Find the volume of the solid above the surface $z = f(x,y)$ and below the surface $z = g(x,y)$.

Example Problem 15.4b: Find the volume of the solid above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 2$.

Steps

1. Find the "floor", let's call it D, by setting $f(x,y) = g(x,y)$ (or if convenient already convert to polar coordinates).

Example

1. In polar coordinates, the two surfaces are $z = r$ and $z = \sqrt{2 - r^2}$. Setting them equal gives $r = \sqrt{2 - r^2}$. Squaring both sides gives $r^2 = 2 - r^2$, which gives $2r^2 = 2$, which gives $r^2 = 1$ and so $r = \pm 1$. But r is never negative, so $r = -1$ is nonsense. Hence the "floor", D, is the region bounded by the circle $r = 1$, or, if you wish, the disk $r \leq 1$.

So

$$D = \{(r,\theta) | 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\} \quad .$$

2. The volume is the area integral of TOP-BOTTOM

$$\int\int_D [f(x,y) - g(x,y)]\, dA \quad .$$

Set it up. Then convert it to polar-coordinates.

2. The bottom is $z = \sqrt{x^2 + y^2}$, and in polar $z = r$, and the top is $x^2 + y^2 + z^2 = 2$ which is $z = \sqrt{2 - x^2 - y^2}$ and in polar $z = \sqrt{2 - r^2}$. So the volume in polar coordinates is

$$\int_0^{2\pi} \int_0^1 [\sqrt{2 - r^2} - r]\, r\, dr\, d\theta \quad .$$

3. Evaluate the iterated integral. First do the inner integral (w.r.t. to r) getting an expression in θ (or just a number), and then do the outer integral.

3. The inner integral is

$$\int_0^1 [\sqrt{2-r^2} - r]\, r\, dr = \int_0^1 [r\sqrt{2-r^2} - r^2]\, dr$$

$$= \int_0^1 r(2-r^2)^{1/2}\, dr - \int_0^1 r^2\, dr$$

$$= -(1/3)(2-r^2)^{3/2}\Big|_0^1 - r^3/3\Big|_0^1$$

$$= -(1/3)(2-r^2)^{3/2}\Big|_0^1 - r^3/3\Big|_0^1$$

$$= -(1/3)[(2-1^2)^{3/2} - (2-0^2)^{3/2}] - 1/3$$

$$= [2^{3/2} - 2]/3 = (2\sqrt{2} - 1)/3 \quad .$$

The whole integral is thus:

$$\int_0^{2\pi} \int_0^1 [\sqrt{2-r^2} - r]\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\int_0^1 [\sqrt{2-r^2} - r]\, r\, dr \right] d\theta$$

$$= \int_0^{2\pi} (2\sqrt{2} - 1)/3\, d\theta$$

$$= 2\pi(2\sqrt{2} - 1)/3 \quad .$$

Ans.: The volume is $2\pi(2\sqrt{2} - 1)/3$.

Problem Type 15.4c: Evaluate the iterated integral by converting to polar coordinates.

$$\int_a^b \int_{f_1(y)}^{f_2(y)} F(x,y)\, dx\, dy$$

Example Problem 15.4c: Evaluate the iterated integral by converting to polar coordinates.

$$\int_0^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} x^2 y\, dx\, dy$$

Steps

Example

1. By looking at the limits of integration of the outer and inner integral signs, figure out the region D.

$$D = \{(x, y) \,|\, a \le y \le b\,,\ f_1(y) \le x \le f_2(y)\,\} \quad .$$

Draw this region, and express it in polar coordinates

$$D = \{(r, \theta) \,|\, \alpha \le \theta \le \beta\,,\ g_1(\theta) \le r \le g_2(\theta)\,\}$$

1. Our region is:

$$D = \{(x, y) \,|\, 0 \le y \le 3\,,\ -\sqrt{9-y^2} \le x \le \sqrt{9-y^2}\,\} \quad .$$

Drawing it (do it!), we see that this is the upper-half of the circle whose center is the origin and whose radius is 3. In polar coordiantes it is:

$$D = \{(r, \theta) \,|\, 0 \le \theta \le \pi\,,\ 0 \le r \le 3\,\} \quad .$$

2. Write the iterated integral as an area integral, then convert it to an iterated integral in polar coordinates. Use the "disc-tionary" $x = r\cos\theta$ $y = r\sin\theta$ $dx\,dy = r\,dr\,d\theta$.

2.

$$\int_0^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} x^2 y \, dx \, dy$$

$$= \int_0^{\pi} \int_0^3 (r\cos\theta)^2 (r\sin\theta) r \, dr \, d\theta$$

$$= \int_0^{\pi} \int_0^3 r^4 \sin\theta \cos^2\theta \, dr \, d\theta \quad .$$

3. Evaluate that iterated integral by doing the inner integral first, and then the outer integral.

3. The inner integral is

$$\int_0^3 r^4 \sin\theta \cos^2\theta \, dr = \sin\theta \cos^2\theta \int_0^3 r^4 \, dr$$

$$= \sin\theta \cos^2\theta \, \left[\frac{r^5}{5} \Big|_0^3 \right]$$

$$= \frac{243}{5} \sin\theta \cos^2\theta \quad .$$

The outer integral is:

$$\int_0^\pi \int_0^3 r^4 \sin\theta \cos^2\theta \, dr \, d\theta$$

$$= \int_0^\pi \left[\int_0^3 r^4 \sin\theta \cos^2\theta \, dr \right] d\theta$$

$$= \int_0^\pi \frac{243}{5} \cos^2\theta \sin\theta \, d\theta = \frac{243}{5} \int_0^\pi \cos^2\theta \sin\theta \, d\theta$$

$$= \frac{243}{5} \cdot \left[\frac{-\cos^3\theta}{3} \Big|_0^\pi \right] = \frac{81}{5} \cdot (-\cos^3(\pi) - -\cos^3(0)) = \frac{162}{5} \quad .$$

Ans.: $\frac{162}{5}$.

Dr. Z's Math251 Handout #15.7 [Triple Integrals]

By Doron Zeilberger

Problem Type 15.7a: Evaluate the iterated integral

$$\int_a^b \int_{f_1(x)}^{f_2(x)} \int_{g_1(x,y)}^{g_2(x,y)} F(x,y,z)\,dz\,dy\,dx \quad .$$

Example Problem 15.7a: Evaluate the iterated integral

$$\int_0^1 \int_x^{2x} \int_0^y 2xyz\,dz\,dy\,dx \quad .$$

Steps

1. You go from **inside to outside**. First isolate and compute the **inner** integral

$$\int_{g_1(x,y)}^{g_2(x,y)} F(x,y,z)\,dz \quad ,$$

which in this problem is a z-integral, since in $dz\,dy\,dx$, dz comes **first**. The answer should not have z in it, but in general depends on x and y.

Example

1.

$$\int_0^y 2xyz\,dz = xyz^2\Big|_0^y = xy \cdot y^2 - 0 = xy^3 \quad .$$

2. Compute the **middle integral**

$$\int_{f_1(x)}^{f_2(x)} \int_{g_1(x,y)}^{g_2(x,y)} F(x,y,z)\,dz\,dy \quad ,$$

using what you already know from Step 1. This should be an expression in x.

2.

$$\int_x^{2x} \int_0^y 2xyz\,dz\,dy$$

$$= \int_x^{2x} \left[\int_0^y 2xyz\,dz \right] dy$$

from step 1, this equals:

$$= \int_x^{2x} xy^3\,dy \quad .$$

This integral equals:

$$x \frac{y^4}{4} \Big|_x^{2x} = x\frac{(2x)^4}{4} - x\frac{(x)^4}{4} = \frac{15}{4} x^5 \quad .$$

3. Do the outside integral, by viewing it as

$$\int_a^b \left[\int_{f_1(x)}^{f_2(x)} \int_{g_1(x,y)}^{g_2(x,y)} F(x,y,z)\,dz\,dy \right] dx \quad .$$

and using the result from step 2. The final answer should not depend on anything involving x, y, z (i.e. is a pure number or a constant).

3.

$$\int_0^1 \int_x^{2x} \int_0^y 2xyz\,dz\,dy\,dx$$

$$= \int_0^1 \left[\int_x^{2x} \int_0^y 2xyz\,dz\,dy \right] dx \quad .$$

By step 2, this is

$$= \int_0^1 \frac{15}{4} x^5\,dx \quad .$$

This is a Calc I integral

$$= \frac{15}{4} \int_0^1 x^5\,dx \quad = \frac{15}{4} \frac{x^6}{6} \Big|_0^1 = \frac{5}{8} \quad .$$

Ans.: $\frac{5}{8}$.

Problem Type 15.7b: Evaluate the triple integral

$$\int\int\int_E F(x,y,z)\,dV \quad ,$$

where

$$E = \{(x,y,z) \,|\, a \le x \le b\,,\ f_1(x) \le y \le f_2(x)\,,\ g_1(x,y) \le z \le g_2(x,y)\,\} \quad .$$

Example Problem 15.7b: Evaluate the triple integral

$$\int\int\int_E yz\cos(x^5)\,dV \quad ,$$

where

$$E = \{(x,y,z)\,|\,0 \le x \le 1\,,\ 0 \le y \le x\,,\ x \le z \le 2x\,\} \quad .$$

Steps

1. Convert the volume integral into a triple iterated interal by transcribing the limits from left to right.

$$\int\int\int_E F(x,y,z)dV$$

$$\int_a^b \int_{f_1(x)}^{f_2(x)} \int_{g_1(x,y)}^{g_2(x,y)} F(x,y,z)\,dz\,dy\,dx \quad .$$

2. Compute the iterated integral like we did in 15.7a. First the inner intgeral, then the middle, then the outer.

Example

1. Our volume integral equals the iterated integral

$$\int_0^1 \int_0^x \int_x^{2x} yz\cos(x^5)\,dz\,dy\,dx \quad .$$

2. The inner integral is:

$$\int_x^{2x} yz\cos(x^5)\,dz = y\cos(x^5)\int_x^{2x} z\,dz$$

$$= y\cos(x^5)\frac{z^2}{2}\Big|_x^{2x} = y\cos(x^5)\frac{(2x)^2 - x^2}{2}$$

$$= \frac{3}{2}yx^2\cos(x^5) \quad .$$

3. Compute the middle integral

$$\int_{f_1(x)}^{f_2(x)} \int_{g_1(x,y)}^{g_2(x,y)} F(x,y,z)\,dz\,dy \quad ,$$

by writing it as

$$\int_{f_1(x)}^{f_2(x)} \left[\int_{g_1(x,y)}^{g_2(x,y)} F(x,y,z)\,dz \right] dy \quad ,$$

and using the result from the previous step.

3.

$$\int_0^x \int_x^{2x} yz\cos(x^5)\,dz\,dy = \int_0^x \left[\int_x^{2x} yz\cos(x^5)\,dz \right] dy$$

By step 2, this equals

$$\int_0^x \frac{3}{2}yx^2\cos(x^5)\,dy = \frac{3}{2}x^2\cos(x^5)\int_0^x y\,dy$$

$$= \frac{3}{2}x^2\cos(x^5)\frac{y^2}{2}\Big|_0^x$$

$$= \frac{3}{2}x^2\cos(x^5)\frac{x^2-0^2}{2} = \frac{3}{4}x^4\cos(x^5) \quad .$$

4. Compute the outside integral

$$\int_a^b \int_{f_1(x)}^{f_2(x)} \int_{g_1(x,y)}^{g_2(x,y)} F(x,y,z)\,dz\,dy\,dx \quad ,$$

by writing it as

$$\int_a^b \left[\int_{f_1(x)}^{f_2(x)} \int_{g_1(x,y)}^{g_2(x,y)} F(x,y,z)\,dz\,dy \right] dx \quad ,$$

and using the result from the previous step.

4.

$$\int_0^1 \int_0^x \int_x^{2x} yz\cos(x^5)\,dz\,dy\,dx$$

$$= \int_0^1 \left[\int_0^x \int_x^{2x} yz\cos(x^5)\,dz\,dy \right] dx \quad .$$

By step 3, this equals

$$= \int_0^1 \frac{3}{4}x^4\cos(x^5)\,dx = \frac{3}{4}\int_0^1 x^4\cos(x^5)\,dx \quad .$$

Making the u-substitution $u = x^5$, gives that this equals

$$= \frac{3}{4}\cdot\frac{1}{5}\sin(x^5)\Big|_0^1 = \frac{3}{20}(\sin(1)-\sin(0)) = \frac{3\sin(1)}{20} \quad .$$

Ans.: $\frac{3\sin(1)}{20}$.

Problem Type 15.7c: Evaluate the triple integral

$$\int\int\int_E F(x,y,z)\,dV \quad ,$$

where E is bounded by the surfaces $z = g_1(x,y)$ $z = g_2(x,y)$ and possibly some planes of the form $x = a$.

Example Problem 15.7c: Evaluate the triple integral

$$\int\int\int_E x^2y^2\,dV \quad ,$$

where E is bounded by the parabolic cylinder $z = 1 - y^2$ and the planes $z = 0$, $x = 1$ and $x = -1$.

Steps

1. Express the solid region E in the form

$$E = \{(x, y, z) | (x, y) \in D\,,$$

$$u_1(x, y) \leq z \leq u_2(x, y)\,\} \quad ,$$

where D is the projection of E on the xy-plane. To find D, set the two surfaces equal to each other i.e. solve $g_1(x, y) = g_2(x, y)$ and add to it the other restictions only involving x and/or y. After you figure out D, express E in the form

$$E = \{(x, y, z) | a \leq x \leq b\,, g_1(x) \leq y \leq g_2(x)\,,$$

$$u_1(x, y) \leq z \leq u_2(x, y)\,\} \quad .$$

Example

1. To get the projection D solve $1 - y^2 = 0$, which gives $y = -1$ and $y = 1$. Together with $x = 1$ and $x = -1$, the region D is

$$D = \{\,(x, y)\,| -1 \leq x \leq 1\,, -1 \leq y \leq 1\,\} \quad .$$

And so

$$E = \{(x, y, z) | (x, y) \in D\,,\, 0 \leq z \leq 1 - y^2\,\} \quad ,$$

and finally (combining it with D), we get

$$E = \{(x, y, z) | -1 \leq x \leq 1\,,\, -1 \leq y \leq 1\,,\, 0 \leq z \leq 1 - y^2\,\} \quad .$$

2. Convert the triple integral into an iterated integral.

$$\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} F(x,y,z)\,dz\,dy\,dx \quad .$$

Now evaluate that iterated integral by going from the inside to the outside.

2.

$$\int_{-1}^{1}\int_{-1}^{1}\int_{0}^{1-y^2} x^2y^2\,dz\,dy\,dx \quad .$$

The inner integral is:

$$\int_0^{1-y^2} x^2y^2\,dz = x^2y^2\int_0^{1-y^2} dz = x^2y^2(1-y^2) = x^2(y^2-y^4)$$

The middle integral is:

$$\int_{-1}^{1}\int_{0}^{1-y^2} x^2y^2\,dz\,dy = \int_{-1}^{1}\left[\int_0^{1-y^2} x^2y^2\,dz\right]dy$$

$$= \int_{-1}^{1} x^2(y^2-y^4)\,dy = x^2\left(\frac{y^3}{3}-\frac{y^5}{5}\right)\Big|_{-1}^{1} = \frac{4}{15}x^2 \quad .$$

The outer integral (i.e. the whole thing) is

$$\int_{-1}^{1}\int_{-1}^{1}\int_{0}^{1-y^2} x^2y^2\,dz\,dy\,dx$$

$$= \int_{-1}^{1}\left[\int_{-1}^{1}\int_0^{1-y^2} x^2y^2\,dz\,dy\right]dx \quad .$$

By the previous result, this is:

$$= \int_{-1}^{1}\frac{4}{15}x^2\,dx \quad = \frac{4}{15}\frac{x^3}{3}\Big|_{-1}^{1} = \frac{4}{15}\frac{2}{3} = \frac{8}{45} \quad .$$

Ans.: $\frac{8}{45}$.

Dr. Z's Math251 Handout #15.8 [Triple Integrals in Cylindrical and Spherical Coordinates]

By Doron Zeilberger

Problem Type 15.8a: Evaluate

$$\int\int\int_E F(x,y,z)\,dV \quad ,$$

where E is a solid region descibed in terms of cylinders and other stuff.

Example Problem 15.8a: Evaluate

$$\int\int\int_E x^2\,dV \quad ,$$

where E is the solid that lies within the cylinder $x^2+y^2=1$, above the plane $z=0$, and below the cone $z^2=4x^2+4y^2$.

Steps

1. Chances are that you are supposed to use *cylindrical coordinates*. Express E in the form

$$E = \{\,(r,\theta,z)\,|\,\alpha \le \theta \le \beta\,,$$

$$h_1(\theta) \le r \le h_2(\theta)\,,\, u_1(r,\theta) \le z \le u_2(r,\theta)\} \quad .$$

Example

1. Since $x^2+y^2=r^2$, the cone $z^2=4x^2+4y^2$ can be writen $z^2=4r^2$. The cylinder $x^2+y^2=1$ is really $r=1$, and this is the "base". $z^2=4r^2$ means that z ranges between $-2r$ and $2r$. **but** we are also told that our solid is **above** the plane $z=0$, so z ranges between 0 and $2r$. It turns out that

$$E = \{\,(r,\theta,z)\,|\,0 \le \theta \le 2\pi\,,\,0 \le r \le 1\,,\,0 \le z \le 2r\} \quad .$$

2. Express the integrand $F(x,y,z)$ in cylindrical coordinates, using the "dictionary"

$$x = r\cos\theta \quad , \quad y = r\sin\theta \quad , \quad x^2+y^2 = r^2 \quad .$$

Also $dV = r dr d\theta dz$. Then set up the volume integral as an iterated integral

$$\int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r,\theta)}^{u_2(r,\theta)} F(r\cos\theta, r\sin\theta, z) r\, dz\, dr\, d\theta \quad .$$

2.

$$\int_0^{2\pi} \int_0^1 \int_0^{2r} (r\cos\theta)^2 r\, dz\, dr\, d\theta \quad .$$

3. Evalute the integral from the inside to the outside.

3. The inside integral is:

$$\int_0^{2r} (r\cos\theta)^2 r\, dz = r^3\cos^2\theta \int_0^{2r} dz = 2r^4\cos^2\theta \quad .$$

The middle integral is

$$\int_0^1 \left[\int_{-2r}^{2r} (r\cos\theta)^2 r\, dz\right] dr$$

$$= \int_0^1 2r^4\cos^2\theta\, dr = \cos^2\theta \int_0^1 2r^4\, dr = \frac{2}{5}\cos^2\theta \quad .$$

The outer integral is

$$\int_0^{2\pi} \left[\int_0^1 \int_{-2r}^{2r} (r\cos\theta)^2 r\, dz\, dr\right] d\theta$$

$$= \int_0^{2\pi} \frac{2}{5}\cos^2\theta \quad d\theta$$

$$= \frac{2}{5}\int_0^{2\pi} \frac{1+\cos 2\theta}{2} \quad d\theta$$

$$= \frac{2}{5}\cdot[\frac{\theta + (1/2)\sin 2\theta}{2}]\Big|_0^{2\pi} = \frac{2\pi}{5} \quad .$$

Ans.: $\frac{2\pi}{5}$.

Problem Type 15.8b: Evaluate

$$\int\int\int_E F(x,y,z)\, dV \quad ,$$

where E is bounded by the xz-plane and the hemispheres $y = \sqrt{r_1^2 - x^2 - z^2}$ and $y = \sqrt{r_2^2 - x^2 - z^2}$.

Example Problem 15.8b: Evaluate

$$\int\int\int_E x^2 \, dV \quad ,$$

where E is bounded by the xz-plane and the hemispheres $y = \sqrt{1 - x^2 - z^2}$ and $y = \sqrt{4 - x^2 - z^2}$.

Steps

1. This is best handled with spherical coordinates. The hemispere $y = \sqrt{R^2 - x^2 - z^2}$ is half of the sphere $x^2 + y^2 + z^2 = R^2$ whose equation in spherical coordinates is really simple: $\rho = R$. Since $y > 0$ (the square-root is always positive), the range of θ is between 0 and π. ρ is between r_1 and r_2 and ϕ has its full range 0 to π. So

$$E = \{ (\rho, \theta, \phi) \, | r_1 \le \rho \le r_2 \, , \, 0 \le \theta \le \pi \, , \, 0 \le \phi \le \pi \} \quad .$$

2. Using the 'dictionary'

$$x = \rho \sin\phi \cos\theta \, , \; y = \rho \sin\phi \sin\theta \, ,$$

$$z = \rho \cos\phi \quad ; \quad dV = \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi \quad .$$

Convert the volume integral into an iterated spherical integral (using the description of E in step 1).

$$\int_0^\pi \int_0^\pi \int_{r_1}^{r_2} F(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi) \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi \quad .$$

Example

1. Here the radii are 1 and 2 so

$$E = \{ (\rho, \theta, \phi) \, | 1 \le \rho \le 2 \, , \, 0 \le \theta \le \pi \, , \, 0 \le \phi \le \pi \} \quad .$$

2.

$$\int_0^\pi \int_0^\pi \int_1^2 (\rho \sin\phi \cos\theta)^2 \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$= \int_0^\pi \int_0^\pi \int_1^2 \rho^4 \sin^3\phi \cos^2\theta \, d\rho \, d\theta \, d\phi \quad .$$

3. If you are lucky and all the limits of integration are numbers (do not involve ρ, ϕ or θ) and the integrand is a product of functions of a single variable, then the iterated integral is simply a product of three simple integrals. Express the big integral like that, and evaluate each single integral separately. Then multiply them together.

Warning: This is only possible if *all* the limits of integration are numbers and the integrand is *completely* separable as a product of functions of a single variable.

3.

$$\int_0^\pi \int_0^\pi \int_1^2 \rho^4 \sin^3\phi \cos^2\theta \, d\rho \, d\theta \, d\phi$$

$$= \int_0^\pi \sin^3\phi \, d\phi \int_0^\pi \cos^2\theta \, d\theta \int_1^2 \rho^4 \, d\rho \quad .$$

The first integral is ($u = \cos\phi$)

$$\int_0^\pi \sin^3\phi \, d\phi = \int_0^\pi \sin^2\phi \, d(-\cos\phi) =$$

$$\int_0^\pi (1-\cos^2\phi) \, d(-\cos\phi) = -\int_1^{-1} (1-u^2) \, du =$$

$$-u + \frac{u^3}{3}\Big|_1^{-1} = \frac{4}{3} \quad .$$

The second integral is

$$\int_0^\pi \cos^2\theta \, d\theta = \int_0^\pi \frac{1+\cos 2\theta}{2} \, d\theta =$$

$$= \frac{\theta}{2} + \frac{\sin 2\theta}{4}\Big|_0^\pi = \frac{\pi}{2} \quad .$$

The third integral is

$$\int_1^2 \rho^4 \, d\rho = \frac{\rho^5}{5}\Big|_1^2 = \frac{2^5 - 1^5}{5} = \frac{31}{5} \quad .$$

Multipluing these three single-integrals, we get that the original triple integral equals

$$\frac{4}{3} \cdot \frac{\pi}{2} \cdot \frac{31}{5} = \frac{62\pi}{15} \quad .$$

Ans.: $\frac{62\pi}{15}$.

Dr. Z's Math251 Handout #15.9 [Change of Variables in Multiple Integrals]

By Doron Zeilberger

Problem Type 15.9a: Find the Jacobian of the transformation

$$x = g(u,v,w) \quad , \quad y = h(u,v,w) \quad , \quad z = k(u,v,w) .$$

Example Problem 15.9a: Find the Jacobian of the transformation

$$x = u^2 v \quad , \quad y = v^2 w \quad , \quad z = w^2 u .$$

Steps

1. Compute all the entries in the Jacobian matrix

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \quad .$$

2. Evaluate the determinant:

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$= (\frac{\partial x}{\partial u}) \begin{vmatrix} \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} - (\frac{\partial x}{\partial v}) \begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$+ (\frac{\partial x}{\partial w}) \begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{vmatrix}$$

Example

1.

$$\begin{vmatrix} 2uv & u^2 & 0 \\ 0 & 2vw & v^2 \\ w^2 & 0 & 2uw \end{vmatrix} \quad .$$

2.

$$= 2uv \begin{vmatrix} 2vw & v^2 \\ 0 & 2wu \end{vmatrix} - u^2 \begin{vmatrix} 0 & v^2 \\ w^2 & 2wu \end{vmatrix}$$

$$+ 0 \cdot \begin{vmatrix} 0 & 2vw \\ w^2 & 0 \end{vmatrix}$$

$$= 2uv[(2vw)(2uw) - 0] - u^2[0 - (v^2)(w^2)] + 0$$

$$= 9u^2v^2w^2 \quad .$$

Ans.: $9u^2v^2w^2$.

Problem Type 15.9b: Use the given transformation to evaluate the integral

$$\int\int_R F(x,y)\, dA \quad ,$$

where R is the triangular region with vertices (p_1,p_2),(q_1,q_2), (r_1,r_2); $x = au + bv$, $y = cu + dv$.

Example Problem 15.9b: Use the given transformation to evaluate the integral

$$\int\int_R (x+y)\,dA \quad ,$$

where R is the triangular region with vertices $(0,0)$,$(2,1)$, $(1,2)$; $x = 2u + v$, $y = u + 2v$.

Steps

1. Figure out the region in the uv-plane that gets transformed. Since a triangle goes to a triangle, we need to find the 3 vertices. Solve for u, v in terms of x, y and find the three points. Call the new triangle R'.

2. Find the Jacobian of the transformation. In this case of a so-called linear transformation, the Jacobian is simply $ad-bc$. Also express $F(x,y)$ in terms of (u,v) using the transformation.

$$\int\int_R F(x,y)\,dA =$$

$$\int\int_{R'} F(au+bv, cu+dv)(ad-bc)\,dA \quad .$$

Example

1. Since $x = 2u + v$, $y = u + 2v$, when $(x,y) = (0,0)$ $u = 0, v = 0$ so the point $(0,0)$ goes to the point $(0,0)$. When $(x,y) = (1,2)$, we have to solve the system $1 = 2u+v, 2 = u+2v$ giving us $u = 0, v = 1$ so $(1,2)$ goes to $(0,1)$. Similarly, $(2,1)$ goes to $(1,0)$. So the region in the uv-plane is the far simpler triangle whose vertices are $(0,0), (1,0), (0,1)$. Let's call this region R'.

2. The Jacobian is $(2)(2) - (1)(1) = 3$, so

$$\int\int_R (x+y)\,dA = \int\int_{R'} (2u+v+u+2v)\cdot 3\,dA =$$

$$9\int\int_{R'} (u+v)\,dA \quad .$$

3. Draw the region (in this case triangle) in the uv- plane and express it as a type I (or type II) region. Then set-up the appropriate iterated integral, by deciding on the **main road** and the **side streets**.

3. The region is the triangle bounded by the axes and the line $u+v=1$. It can be written as

$$\{(u,v)\,|\,0 \le u \le 1\,,\; 0 \le v \le 1-u\,\} \quad .$$

Our area-integral is thus equal to the iterated integral

$$9\int_0^1 \int_0^{1-u} (u+v)\,dv\,du \quad .$$

The inner integral is

$$\int_0^{1-u} (u+v)\,dv = uv + \frac{v^2}{2}\Big|_0^{1-u}$$

$$= u(1-u) + \frac{(1-u)^2}{2} = (1-u^2)/2 \quad ,$$

and the whole integral is

$$\frac{9}{2}\int_0^1 (1-u^2)\,du = \frac{9}{2}\left[u - \frac{u^3}{3}\right]\Big|_0^1 = \frac{9}{2}\cdot\frac{2}{3} = 3 \quad .$$

Ans.: 3.