Chapter 2: Introduction to Quantum Mechanics

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1 Intro to Quantum Mechanics

1.1 Linear Algebra

Vector space

$$\begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$$

1.1.1 Bases and Linear Independence

A *spanning set* for a vector space is a set of vectors $|v_1\rangle, ..., |v_n\rangle$ such that any vector $|v\rangle$ in the vector space can be written as a linear combination $|v\rangle = \sum_i a_i |v_i\rangle$ in that set. The spanning set for the vector space $\mathbb{C}^2$ is the set

$$|v_1\rangle \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad |v_2\rangle \equiv \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

A second spanning set in $\mathbb{C}^2$ is

$$|v_1\rangle \equiv \tfrac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \quad |v_2\rangle \equiv \tfrac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

A set set of non-zero vectors $|v_1\rangle, ..., |v_n\rangle$ are *linear dependent* if there exists a set of complex numbers $a_1, ..., a_n$ with $a_i \neq 0$ for at least one value of i, such that

$$a_1 |v_1\rangle + a_2 |v_2\rangle + \ldots + a_n |v_n\rangle = 0.$$

1.1.2 Linear Operators and Matrices

A *linear operator* between vector spaces V and W is defined to be any function $A : V \to W$ which is linear in its inputs

$$A\left(\sum_i a_i |v_i\rangle\right) = \sum_i a_i A(|v_i\rangle)$$

1.1.3 Pauli Matrices

$$\sigma_0 \equiv I \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad \sigma_1 \equiv \sigma_x \equiv X \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\sigma_2 \equiv \sigma_y \equiv Y \equiv \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}; \quad \sigma_3 \equiv \sigma_z \equiv Z \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Preprint. Under review.

1.1.4 Inner Products

An *inner product* is a function which takes as input two vectors $|v\rangle$ and $|w\rangle$ form a vector space and produces a complex number as output.

A function from $V \times V \rightarrow \mathbb{C}$ is an inner product if it satisfies the requiments that:

(1) (., .) is linear in the second argument

$$\left(|v\rangle, \sum_i \lambda_i |w_i\rangle\right) = \sum_i \lambda_i (|v\rangle, |w_i\rangle)$$

(2) $(|v\rangle, |w\rangle) = (|w\rangle, |v\rangle)^*$

(3) $(|v\rangle, |v\rangle) \geq 0$ with equality if and only if $|v\rangle = 0$

For example, $\mathbb{C}^n$ has an inner product defined by

$$((y_1, ..., y_n), (z_1, ..., z_n)) \equiv \sum_i y_i^* z_i = [y_1^* \dots y_n^*] \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$$

Vectors $|v\rangle$ and $|v\rangle$ are *orthogonal* if their inner product is zero. We define the *norm* of a vector $|v\rangle$ by

$$\| \, |v\rangle \, \| \equiv \sqrt{\langle v|v\rangle}$$

A *unit vector* is a vector $|v\rangle$ such that $\| \, |v\rangle \, \| = 1$.

Suppose $|w_1\rangle, \dots, |w_d\rangle$ is a basis set for some vector space V with an inner product. There is a useful method, the *Gram-Schmidt* procedure, which can be used to produce an orthonormal basis set $|v_1\rangle, \dots, |v_d\rangle$ for a vector space V. Define $|v_1\rangle \equiv \frac{|w_1\rangle}{\||w_1\rangle\|}$ and for $1 \leq k \leq d-1$ define $|v_{k+1}\rangle$ inductively by

$$|v_{k+1}\rangle \equiv \frac{|w_{k+1}\rangle - \sum_{i=1}^{k} \langle v_i|w_{k+1}\rangle |v_i\rangle}{\||w_{k+1}\rangle - \sum_{i=1}^{k} \langle v_i|w_{k+1}\rangle |v_i\rangle\|}$$

1.1.5 Eigenvectors and Eigenvalues

An *eigenvector* of a linear operator A on a vector space is a non-zero vector $|v\rangle$ such that $A|v\rangle = v|v\rangle$, where $v \in \mathbb{C}$ and is known as the *eigenvalue* of A corresponding to $|v\rangle$. The *characteristic function* is defined as $c(\lambda) \equiv det|A - \lambda I|$ where det is the *determinant* function of matrices. The solutions to the characteristic equation, $c(\lambda) = 0$, are the eigenvalues of the operator A.

A *diagonal representation* for an operator A on a vector space V is a representation

$$A = \sum_i \lambda_i |i\rangle \langle i|$$

where the vectors $|i\rangle$ form an orthonormal set of eigenvectors for A. A diagonal representation of Pauli Z matrix:

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = |0\rangle \langle 0| - |1\rangle \langle 1|$$

1.1.6 Adjoints and Hermitian Operators

Suppose A is any linear operator on a Hilbert space, V, then $\exists A^\dagger$ on V such that $\forall |v\rangle, |w\rangle \in V$,

$$(|v\rangle, |w\rangle) = (A^\dagger |v\rangle, |w\rangle).$$

This linear operator is know ad the *adjoint* or *Hermitian conjugate* of the operator A.

Note: $(AB)^\dagger = B^\dagger A^\dagger$

($\dagger$) is called "dagger" and is equal to $A^\dagger \equiv (A^*)^T$, which is the transpose of the complex conjugate. Example:

$$\begin{bmatrix} 1+3i & 2i \\ 1+i & 1-4i \end{bmatrix}^\dagger = \begin{bmatrix} 1-3i & 1-i \\ -2i & 1+4i \end{bmatrix}$$

1.1.7 Tensor Products

The *tensor product* if a way of putting vector spaces together to form larger vector spaces. Suppose V and W are vector spaces of dimensions m and n respectively, then $V \otimes W$ is an nm dimensional vector space. The elements of $V \otimes W$ are linear combinations of tensor products $|v\rangle \otimes |w\rangle$ of elements $|v\rangle$ of V and $|w\rangle$ of W. A more intuitive way to represent this is through the *Kronecker product*. Suppose A is an m by n matrix and B is a p by q matrix then:

$$A \otimes B = \begin{bmatrix} A_{11}B & A_{12}B & \cdots & A_{1n}B \\ A_{21}B & A_{22}B & \cdots & A_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1}B & A_{m2}B & \cdots & A_{mn}B \end{bmatrix}$$

produces an $nq \times mp$ matrix.

Example:

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \otimes \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 6 \end{bmatrix}.$$

1.1.8 Operator Functions

Let $A = \sum_a a |a\rangle \langle a|$ be a spectral decomposition for a normal operator A. Define $f(A) \equiv \sum_a f(a) |a\rangle \langle a|$.
Example:

$$e^{\theta Z} = \begin{bmatrix} e^\theta & 0 \\ 0 & e^{-\theta} \end{bmatrix}$$

Since Z has eigenvectors $|0\rangle$ and $|1\rangle$.

The *trace* of A is defined to be the sum of its diagonal elements,

$$tr(A) \equiv \sum_i A_{ii}.$$

Some important properities:

The trace is *cyclic*:

$$tr(AB) = tr(BA)$$

The trace is *linear*:

$$tr(A+B) = tr(A) + tr(B) \text{ and } tr(zA) = ztr(A)$$

The trace is invariant under the *unitary similarity transform*:

$$A \to UAU^\dagger$$

as

$$tr(UAU^\dagger) = tr(U^\dagger UA) = tr(A)$$

Suppose $|\psi\rangle$ is a unit vector and A is an arbitrary operator. To evaluate $tr(A|\psi\rangle\langle\psi|)$ use the Gram-Schmidt procedure to extend $|\psi\rangle$ to an orthonormal basis $|i\rangle$ which includes $|\psi\rangle$ as the first element. Then we have

$$tr(A|\psi\rangle\langle\psi|) = \sum_i \langle i| A |\psi\rangle \langle\psi|i\rangle$$

$$= \langle\psi| A |\psi\rangle$$

1.1.9 The Commutator and Anti-Commutator

The *commutator* between two operators A and B is defined to be

$$[A, B] \equiv AB - BA.$$

If $[A, B] = 0$, that is, $AB = BA$, then we say that A *commutes* with B. The *anti-commutator* is defined as:

$$\{A, B\} \equiv AB + BA$$

And we say A *anti-commutes* with B if $\{A, B\} = 0$.

1.1.10 Polar and Singular Value Decomposition

Let A be a linear operator on a vector space V. $\exists$ a unitary U and positive operators J and K such that $A = UJ = KU$, defined by $J \equiv \sqrt{A^\dagger A}$ and $K \equiv \sqrt{AA^\dagger}$.

Let A be a square matrix. Then $\exists$ unitary matrics U and V, and a diagonal matrix D with non-negative entries such that $A = UDV$. The diagonal elements D are called the *singular values* of A.

1.2 Postulates of Quantum Mechanics

1.2.1 State Space

Postulate 1: Assocaited to any isolated physical system is a complex vector space with inner product (that is, Hilbert space) know as the *state space* of the system. The system is completely described by its *state vector*, which is a unit vector in the system's state space.

The simplest quantum mechanical system is the *qubit*. An arbitrary vector in the state space can be written $|\psi\rangle = a|0\rangle + b|1\rangle$ where $a, b \in \mathbb{C}$. The conditional that $|\psi\rangle$ be a unit vector, $\langle\psi|\psi\rangle = 1$, is equivalent to $|a|^2 + |b|^2 = 1$

1.2.2 Evolution

How does the state, $|\psi\rangle$, of a quantum mechanical system change with time?

Postulate 2.1: The evolution of a *closed* quantum system is described by a *unitary transformation*. That is, the state $|\psi\rangle$ of the system at time t_1 is related to the state $|\psi'\rangle$ of the system at time t_2 by a unitary operator U which depends only on the times t_1 and t_2:

$$|\psi'\rangle = U|\psi\rangle.$$

Postulate 2.2: The time evolution of the state of a closed quantum system is described by the *Schrödinger equation*.

$$i\hbar\frac{\partial|\psi\rangle}{\partial t} = H|\psi\rangle$$

1.2.3 Quantum Measurement

Postulate 3: Quantum measurements are described by a collection M_m of *measurement operators*.

If the state of the quantum system is $|\psi\rangle$ immediately before the measurement then the probability that the result m occurs is give by

$$p(m) = \langle\psi| M_m^\dagger M_m |\psi\rangle$$

and the state after the measurement is

$$\frac{M_m|\psi\rangle}{\sqrt{\langle\psi|M_m^\dagger M_m|\psi\rangle}}$$

1.2.4 Distinguishing Quantum States