Chapter 1: Introduction and Overview

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1 Introduction

1.1 Global Perspective

n/a

1.2 Quantum Bits

What is a qubit? Just as a classical bit has a *state* - either 0 or 1 - a qubit also has a state. Two possible states for a qubit are the states $|0\rangle$ and $|1\rangle$, which as you might guess corresponds to the states 0 and 1 for a classical bit. The difference between bits and qubits is that a qubit can be in a state *other* than $|0\rangle$ or $|1\rangle$. It is also possible to form *a linear combination* of states, often called *superpositions*:

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$$

Where $\alpha, \beta \in \mathbb{C}$. Unfortunately, we cannot examine a qubit to determine its quantum state, that is, the values of α and β. Instead, quantum mechanics tells us that we can only aquire much more restricted information about the quantum state. When we measure a qubit we get either the result 0, with probability $|\alpha|^2$, or the result 1, with probability $|\beta|^2$ with $|\alpha|^2 + |\beta|^2 = 1$, since probabilities must sum to 1.

This equation can also be represented as

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right) |0\rangle + e^{i\phi} \sin\left(\frac{\theta}{2}\right) |1\rangle$$

1.2.1 Multiple Qubits

A two qubit system will have four *computational basis states* denoted $|00\rangle, |01\rangle, |10\rangle, |11\rangle$. The state vector describing two qubits is

$$|\psi\rangle = \alpha_{00} |00\rangle + \alpha_{01} |01\rangle + \alpha_{10} |10\rangle + \alpha_{11} |11\rangle$$

And again, the probabilities must sum to one, satisfying the condition $\sum_{x \in \{0,1\}^2} \left|\alpha_x^2\right| = 1$, where the notation '$x \in \{0,1\}^2$' means 'the set of strings of length two with each letter being either zero or one'. Subsets of multi-qubit systems can be measured. Measuring the first qubit of the above two qubit system gives 0 with probability $\left|\alpha_{00}^2\right| + \left|\alpha_{01}^2\right|$. The post-measurement state would then be

$$|\psi'\rangle = \frac{\alpha_{00}|00\rangle + \alpha_{01}|01\rangle}{\sqrt{\left|\alpha_{00}^2\right| + \left|\alpha_{01}^2\right|}}$$

An important two qubit state is the *Bell state* or *EPR pair*,

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

Bell states are important because their measurement outcomes are *correlated*.

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1.3 Quantum Computation

1.3.1 Single Qubit Gates

The quantum $\mathbf{NOT}$ gate acts *linearly*.

$$X \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

If we write the quantum state $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$ in vector notation

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

then the $\mathbf{NOT}$ gate operation can be represented as

$$X \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \beta \\ \alpha \end{bmatrix}$$

There is a restriction on the the type of matrix that can be a quantum gate and that is that the operation must preserve the normalization condition (i.e. $|\alpha|^2 + |\beta|^2 = 1$). The appropriate condition on the matrix representing the gate is that the matrix U describing the single qubit gate must be *unitary*, that is $U^\dagger U = I$ (where $X^\dagger$ is the transpose of the complex conjugate of X).

Other important single qubit quantum gates are the Z gate

$$Z \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

which leaves $|0\rangle$ unchanged and flips the sign of $|1\rangle$, and the *Hadamard* gate

$$H \equiv \tfrac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

The Hadamard operation is just a rotation of the sphere about the $\hat{y}$ axis by $90°$, followed by a rotation about the $\hat{x}$ axis by $180°$.

1.3.2 Multiple Qubit Gates

The prototypical multi-qubit logic gate is the *controlled*-$\mathbf{NOT}$ or $\mathbf{CNOT}$ gate.

$$U_{CN} \equiv \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The gate has two inputs, a *control* qubit and a *target* qubit. If the control qubit is set to 0, then the target qubit is left alone. If the control qubit is set to 1, then the target qubit is flipped.

$$|00\rangle \to |00\rangle;\, |01\rangle \to |01\rangle;\, |10\rangle \to |11\rangle;\, |11\rangle \to |10\rangle;$$

1.3.3 Measurement in non-standard basis

The states $|0\rangle$ represent just one of many possible choices of basis states for a qubit. Another possible choice is

$$|+\rangle \equiv \tfrac{(|0\rangle + |1\rangle)}{\sqrt{2}} \text{ and } |-\rangle \equiv \tfrac{(|0\rangle - |1\rangle)}{\sqrt{2}}$$

1.3.4 Quantum Circuits?

1.3.5 Copying Circuit?

1.3.6 Example: Bell States

$$|\beta_{00}\rangle = \tfrac{|00\rangle+|11\rangle}{\sqrt{2}};$$

$$|\beta_{01}\rangle = \tfrac{|01\rangle+|10\rangle}{\sqrt{2}};$$

$$|\beta_{10}\rangle = \tfrac{|00\rangle-|11\rangle}{\sqrt{2}};$$

$$|\beta_{11}\rangle = \tfrac{|01\rangle-|10\rangle}{\sqrt{2}};$$

These are known as the *Bell states*, or *EPR states or pairs*.

1.4 Quantum Algorithms

1.4.1 Classical Computations on a Quantum Computer?

1.4.2 Quantum Parallelism

Suppose $f(x) : \{0,1\} \to \{0,1\}$ is a function with a one-bit domain and range. A convenient way of computing this function on a quantum computer is to consider a two qubit quantum computer which starts in the state $|x, y\rangle$. It is possible with a sequence of gates to turn this into the state $U_f : |x, y\rangle \to |x, y \oplus f(x)\rangle$. In particular, if we apply the Hadamard gate to $H\,|0\rangle$ we obtain $\frac{|0\rangle+|1\rangle}{\sqrt{2}}$. Now applying U_f we get $\frac{|y,y\oplus f(0)\rangle+|y,y\oplus f(1)\rangle}{\sqrt{2}}$. And now letting $y = |0\rangle$, we get $\frac{|0,f(0)\rangle+|0,f(1)\rangle}{\sqrt{2}}$. This is a remarkable state, in that we have information about $f(0)$ and $f(1)$ simultaneously!

This procedure can me generalized to functions on an arbitrary number of bits, by using the *Walsh-Hadamard transform*. For $n = 2$ qubits we get

$$\tfrac{|0\rangle+|1\rangle}{\sqrt{2}}\tfrac{|0\rangle+|1\rangle}{\sqrt{2}} = \tfrac{|00\rangle+|01\rangle+|10\rangle+|11\rangle}{2}$$

We can write $H^{\otimes 2}$ to denote the parallel action of two Hadamard gates. More generally, the result of performing the Walsh-Hadamard transform on n qubits initially in the $|0\rangle$ state is

$$\tfrac{1}{\sqrt{2^n}}\sum_x |x\rangle$$

And we write $H^{\otimes n}$ to denote this operation.