Chapter 4: Quantum Circuits

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1 Quantum Circuits

1.1 Exercise

1.2 Exercise

Let $x \in \mathbb{R}$ and A a matrix such that $A^2 = I$. Show that

$$e^{iAx} = I \cos x + iA \sin x$$

Solution

From the definition of the *rotational operator* $\hat{y}$,

$$R_a(-2\theta) = \cos\theta I + i \sin\theta A = e^{iA\theta}$$

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1.3 Exercise

Show that, up to global phase, the $\frac{\pi}{8}$ gate satisfies $T = R_z(\frac{\pi}{4})$

Solution

$R_z(\frac{\pi}{4}) = e^{-i\pi Z/8} = \begin{bmatrix} e^{-i\pi/8} & 0 \\ 0 & e^{i\pi/8} \end{bmatrix} = e^{-i\pi/8} \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix} = T$

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1.4 Exercise

Express Hadamard gate H as a product of R_x and R_z rotations and $e^{i\phi}$ for some ϕ

Solution

Let $c = \frac{1}{\sqrt{2}} e^{i\phi/2}$.

Then $cR_x(\phi)R_z(\phi) = c(e^{-i\phi Y/2}e^{-i\phi Z/2}) = c(e^{-i\frac{\phi}{2}(X+Z)}) = c(e^{-i\frac{\phi}{\sqrt{2}}H}) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & e^{i\phi} \end{bmatrix}$

$\therefore \quad \phi = (2k+1)\pi; \quad \forall k \in \mathbb{N}$

$\implies \frac{1}{\sqrt{2}} e^{-(2k+1)\pi/2} R_x((2k+1)\pi) R_z((2k+1)\pi) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & e^{i(2k+1)\pi} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = H.$

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1.5 Exercise

Prove that $(\hat{n}, \vec{\sigma})^2 = I$, and use this to verify that

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$$R_{\hat{n}}(\theta) \equiv e^{-i\theta\hat{n}\vec{\sigma}/2} = \cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}(n_xX + n_yY + n_zZ)$$

where $\vec{\sigma}$ denotes the three component vector (X, Y, Z) of Pauli matrices and where $\hat{n}$ is a real unit vector.

Solution

$\hat{n} = (n_x, n_y, n_z) \in \mathbb{R}^3$

$\vec{\sigma} = (X, Y, Z)$

$A^2 = (\hat{n}\vec{\sigma})^2 = (n_xX + n_yY + n_zZ)^2 = \begin{bmatrix} n_z & n_x - in_y \\ n_x + in_y & -n_z \end{bmatrix}^2$

$= \begin{bmatrix} n_x^2 + n_y^2 + n_z^2 & 0 \\ 0 & n_x^2 + n_y^2 + n_z^2 \end{bmatrix}$

We are given $\hat{n}$ is a unit vector $\implies n_x^2 + n_y^2 + n_z^2 = 1$

$\therefore \quad A^2 = I$

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From **Exercise 1.4**, we know $e^{iAx} = \cos(x)I + i\sin(x)A$ and using $A^2 = I$

$A = \hat{n}\vec{\sigma}$ and $x = -\theta/2$ so

$R_{\hat{n}}(\theta) = \cos(\theta/2) - i\sin(\theta/2)\hat{n}\vec{\sigma}$

$\implies \cos(\frac{\theta}{2}) - i\sin(\frac{\theta}{2})(n_xX + n_yY + n_zZ)$

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1.6 Exercise-Block Sphere Interpretation of Rotations

$R_{\hat{n}}(\theta)$ operators are referred to as rotation operators is the following fact, which you are to prove. Suppose a single qubit has a state represented by the Block vector $\vec{\lambda}$. Then the effect of the rotation $R_{\hat{n}}(\theta)$ on the state is to rotate it by an angle θ about the $\hat{n}$ axis of the Block sphere. This fact explains the rather mysterious looking factor of two in the definition of the rotation matrices.

1.7 Exercise

Show that $XYX = -Y$ and use this to prove that $XR_y(\theta)X = R_y(-\theta)$

Solution

$XYX = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} = -Y$

$XR_y(\theta)X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \cos(\frac{\theta}{2}) & -\sin(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -\sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \\ \cos(\frac{\theta}{2}) & \sin(\frac{\theta}{2}) \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \cos(\frac{\theta}{2}) & \sin(\frac{\theta}{2}) \\ -\sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{bmatrix} = R_y(-\theta)$

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1.8 Exercise

An arbitrary single qubit unitary operator can be written in the form

$$U = e^{\alpha i}R_{\hat{n}}(\theta)$$

for some $\alpha, \theta \in \mathbb{R}$ and a three-dimensional unit vector $\hat{n}$.

1. Prove this fact

2. Find values for α, θ, and $\hat{n}$ giving the Hadamard gate H.

3. Find values for α, θ, and $\hat{n}$ giving the phase gate

$$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}.$$

Solution

1. A operator will be unitary if $U^\dagger U = I$ holds.

$$U = \cos{(\theta/2)} I - i \sin{(\theta/2)} A$$

and

$$U^\dagger = \cos{(\theta/2)} I + i \sin{(\theta/2)} A^\dagger$$

where

$$A = (\hat{n}, \vec{\sigma}) = (n_x X + n_y Y + n_z Z)$$

Then

$$U^\dagger U = \cos^2{(\theta/2)} I + \sin^2{(\theta/2)} A^\dagger A$$

From **Exercise 5**, we know $A = A^\dagger \implies A^\dagger A = I$

$\therefore$

$$U^\dagger U = I(\cos^2{(\theta/2)} + \sin^2{(\theta/2)}) = I$$

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1.9 Exercise

1.10 Exercise

1.11 Exercise

1.12 Exercise

1.13 Exercise

1.14 Exercise

1.15 Exercise

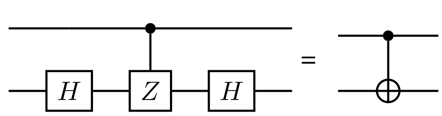
1.16 Exercise

1.17 Exercise

Construct a $CNOT$ gate from one controlled-Z gate and two Hadamard gates, where

$$Z = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Solution



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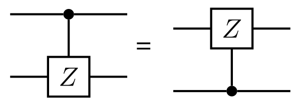
1.18 Exercise

Show that



Solution

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