QUIZ 1 - MSO202M SOLUTIONS
IIT KANPUR - 2024–2025
DATE : 21 AUGUST, 2024, TIME : 60 MINS. (06:45–07:45 PM)
MAXIMUM MARKS: 40

Q 1. (a) Discuss continuity and differentiability of $f(z) = \sqrt{|xy|}$, where $z = x + iy$.

Q 1. (b) Compute $(1+i\sqrt{3})^{2024}$. **5 marks**

Ans : We have

$$1+i\sqrt{3}=re^{i\theta}$$

with

$$r=\sqrt{1^2+(\sqrt{3})^2}=2 \text{ and } \theta=\tan^{-1}(\sqrt{3})=\frac{\pi}{3}.$$

Thus,

$$1+i\sqrt{3}=2e^{\frac{i\pi}{3}}$$
$$(1+i\sqrt{3})^{2024}=2^{2024}(e^{\frac{i\pi}{3}})^{2024}. \tag{1}$$

Now,

$$2024=6\times 337+2$$
$$\frac{2024}{3}=2\times 337+\frac{2}{3}$$
$$\left(\frac{2024}{3}\right)i\pi=2\pi i\times 337+\frac{2\pi i}{3}$$
$$e^{\frac{2024}{3}i\pi}=e^{\frac{2\pi i}{3}}$$

Hence by (1),

$$(1+i\sqrt{3})^{2024}=2^{2024}e^{\frac{2\pi i}{3}}$$

Remark: Reduce 2-mark if it is not written in the reduced form.

Q 2. (a) Show that the function $u(x,y)=x+e^{-x}\cos y$ is harmonic and find its harmonic conjugate. **6 marks**

Ans: Given that

$$v(x,y)=x+e^{-x}\cos y.$$

Now,

$$v_x=1-e^{-x}\cos y$$
$$v_{xx}=e^{-x}\cos y$$

and

$$v_y=-e^{-x}\sin y$$
$$v_{yy}=-e^{-x}\cos y.$$

Hence,

$$v_{xx}+v_{yy}=0. \quad \textbf{(2-marks)}.$$

For Harmonic Conjugate,

$$v = \int_0^y u_x(x,t)dt - \int_0^x u_y(s,0)ds$$
$$v = \int_0^y (1 - e^{-x}\cos t)dt + \int_0^x e^{-s}\sin(0)ds$$
$$v = [t - e^{-x}\sin t]_0^y$$
$$v = y - e^{-x}\sin y.$$

Remark: 2 marks for showing u is harmonic, 2 marks for writing formula for harmonic conjugate and 2 marks for finding it.

Q 2. (b) Show that $f'(z)$ does not exist at any point if $f(z) = 2x + iy^2$. **4 marks**

Ans: We have, $u(x,y) = 2x$ and $v(x,y) = y^2$.

$$u_x = 2,\ \ v_x = 0$$
$$u_y = 0,\ \ v_y = 2y.$$

If it is analytic it satisfies the Cauchy Riemann (C-R) equation, $u_x = v_y, u_y = -v_x$. By C-R equation,

$$2 = 2y, \qquad 0 = 0.$$

- For writing C-R equation. **1 mark**
- C-R equation satisfies only at $y = 1$ i.e., at $z = x + i$. **1 mark**
- If you take any neighborhood of $x+i$, it contains a point of the form $x+iy$ with $y \neq 1$. Hence, C-R equation does not satisfy in a neighborhood and thus $f(z)$ is not analytic. **2 marks**

Q 3. Find radius and domain of convergence for the following series.

(a) **5 marks**

$$\sum_{n=1}^{\infty} \frac{(4n)!}{(n!)^4} z^{4n+3}.$$

Ans: Radius of convergence for power series with gaps formula

$$\frac{1}{R} = \lim_{n\to\infty} \left|\frac{a_n}{a_{n-1}}\right|^{\frac{1}{\lambda(n)-\lambda(n-1)}} \qquad \textbf{2 marks}$$

Here

$$a_n = \frac{(4n)!}{(n!)^4}, \quad \lambda(n) = 4n+3$$

Therefore,

$$\frac{a_n}{a_{n-1}} = \frac{(4n)!((n-1)!)^4}{(n!)^4(4(n-1)!)}$$
$$= \frac{4n(4n-1)(4n-2)(4n-3)(4n-4)!((n-1)!)^4}{n^4((n-1)!)^4(4n-4)!},$$

and

$$\lambda(n) - \lambda(n-1) = 4n+3 - (4(n-1)+3) = 4.$$

Hence, by formula we have $\frac{1}{R} = 4$ i.e., radius of convergence is $1/4$ and the domain of convergence is $|z| < 1/4$.

Remark: 2-mark for finding radius of convergence, 1-mark for domain of convergence.

Q 3. (b) Find radius and domain of convergence for the following series. **5 marks**

$$\sum_{n=1}^{\infty} a_n (z-5i)^n, \quad \text{where } a_n = \begin{cases} 2 & \text{if } n = 3k \\ 3 & \text{if } n = 3k+1 \\ 6 & \text{if } n = 3k+2. \end{cases}$$

Ans: Radius of convergence is given by

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n} \qquad \mathbf{1} \text{ mark}$$

$$\frac{1}{R} = \lim_{k\to\infty} 6^{\frac{1}{3k+2}} \qquad \mathbf{1} \text{ mark}$$

$$R = 1 \qquad \mathbf{1} \text{ mark}$$

Domain of convergence $|z-5i| < 1$ (**2 marks**).

Q 4. (a) Using ML inequality, show that **5 marks**

$$\left| \int_C \frac{dz}{\overline{z}^2 + z + 1} \right| \le \frac{3\pi}{10},$$

where C is the arc of the circle $|z| = 3$ from $z = 3$ to $z = 3i$, lying in first quadrant.

Ans: The ML inequality is given by

$$\left| \int_C f(z)dz \right| \le ML.$$

We have, on $|z| = 3$,

$$\begin{aligned} \overline{z}^2 + z + 1 &\ge |\overline{z}|^2 - |z| - 1 \\ &\ge 9 - 3 - 1 \\ \frac{1}{\overline{z}^2 + z + 1} &\le \frac{1}{5} (= M), \end{aligned}$$

and the length of the curve $z = 3e^{i\theta}$ is given by,

$$L = \int_0^{\pi/2} |dz| = \frac{3\pi}{2}.$$

So, $ML = \frac{3\pi}{10}$.

Remark 1-mark for formula, 2-mark for finding M, 2-mark for finding L

Q 4. (b) Let C be the circle $|z| = 3$, described in the positive sense. If **5 marks**

$$g(w) = \int_C \frac{z^2 - 5z + 3}{(z-w)^2} dz, \qquad (|w| \ne 3).$$

Find $g(2)$ and $g(5)$ using Cauchy integral formula.

Ans: Cauchy integral formula:

$$f^n(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}dz$$

where C is any curve enclosing a.(**1-mark for formula**). Since point 2 lies inside C, then

$$\int_C \frac{z^2-5z+3}{(z-2)^2}dz = f'(2)$$

where $f(z) = z^2 - 5z + 3$ this implies that $f'(2) = -1$. Therefore, $g(2) = -1$. (**2 marks**)

Since the point 5 lies outside C, then $\frac{f(z)}{(z-5)^2}$ is analytic and hence

$$\int_C \frac{f(z)}{(z-5)^2} = 0.$$

Hence, $g(5) = 0$. (**2 marks**)