MSO205 QUIZ 2 SOLUTIONS

Question 1 (1 mark) Let A denote the integer formed by the last two digits of your roll number. Let $X \sim Uniform(-2, 107 - A)$. Then

$\boxed{F_X(2) = \qquad\qquad}$ (answers involving the letter/character A will not be accepted)

Answer: General answer: $\frac{4}{109-A}$. Example of specific answer expected: if the roll number is 241234, then $A = 34$ and the answer is $\frac{4}{75}$.

Question 2 (3 marks) Suppose $X \sim N(\mu, \sigma^2)$ with unknown $\mu \in \mathbb{R}$ and $\sigma > 0$. If $EX^3 = 0$, what can you say about μ and σ? Justify your answer.

Answer: We have $\mathbb{E}X = \mu$ and $Var(X) = \mathbb{E}X^2 - \mu^2 = \sigma^2$. As, $X \sim N(\mu, \sigma^2)$ is symmetric about μ, $\mathbb{E}(X - \mu)^3 = 0$ and hence,

$$\mathbb{E}X^3 - 3\mu\mathbb{E}X^2 + 3\mu^2\mathbb{E}X - \mu^3 = 0,$$

which gives $\mathbb{E}X^3 = 3\mu(\mu^2 + \sigma^2) - 3\mu^3 + \mu^3 = \mu(\mu^2 + 3\sigma^2)$. (part marks: 1 marks). Since $\mu^2 + 3\sigma^2 \geq 3\sigma^2 > 0$, from $\mathbb{E}X^3 = 0$ we conclude $\mu = 0$. (part marks: 1 marks). As there is no further conditions on σ, it can be any positive real number, i.e. $\sigma > 0$.

Question 3

Set 1: Let X be an RV with the following Moment Generating function.

$$M_X(t) = \frac{1}{3} + \frac{1}{2}e^{2t} + \frac{1}{6}e^{-t}, \ \forall t \in \mathbb{R}.$$

Provide appropriate justification in all of the following sub-questions.

(i) (3 marks) Identify the distribution of X.

(ii) (1 + 1 marks) Does $E|X - 1|$ exist? Compute it.

(iii) (2 marks) Compute the Characteristic function of X.

Answer: Consider the discrete RV Y with support $S_Y = \{0, 2, -1\}$ and the probability mass function $f_Y : \mathbb{R} \to [0,1]$ defined by

$$f_Y(y) = \begin{cases} \frac{1}{3}, & \text{if } y = 0, \\ \frac{1}{2}, & \text{if } y = 2, \\ \frac{1}{6}, & \text{if } y = -1, \\ 0, & \text{otherwise.} \end{cases}$$

Then, as Y is a bounded RV, the MGF exists and $M_Y(t) = \sum_{y \in S_Y} e^{ty} f_Y(y) = \frac{1}{3} + \frac{1}{2}e^{2t} + \frac{1}{6}e^{-t}, \forall t \in \mathbb{R}$.

Since, $M_X(t) = M_Y(t), \forall t \in \mathbb{R}$ and since, the MGF, if it exists, uniquely identifies the distribution (this justification – part marks : 1), then X and Y are identically distributed. That is, X is a discrete RV with support $S_X = \{0, 2, -1\}$ (part marks : 0.5) and the probability mass function $f_X : \mathbb{R} \to [0,1]$ defined by (part marks : 1.5)

$$f_X(x) = \begin{cases} \frac{1}{3}, & \text{if } x = 0, \\ \frac{1}{2}, & \text{if } x = 2, \\ \frac{1}{6}, & \text{if } x = -1, \\ 0, & \text{otherwise.} \end{cases}$$

As X is a bounded RV, so is $|X - 1|$ and hence $\mathbb{E}|X - 1|$ exists (the justification – part marks : 1). Now, (part marks : 1)

$$\mathbb{E}|X - 1| = \sum_{x \in S_X} |x - 1| f_X(x) = \frac{1}{3} + \frac{1}{2} + \frac{1}{3} = \frac{2 + 3 + 2}{6} = \frac{7}{6}.$$

The Characteristic function $\Phi_X : \mathbb{R} \to \mathbb{C}$ of X is given by (part marks : 2)

$$\Phi_X(t) = \mathbb{E}e^{itx} = \sum_{x \in S_X} e^{itx} f_X(x) = \frac{1}{3} + \frac{1}{2}e^{2it} + \frac{1}{6}e^{-it}, \forall t \in \mathbb{R}.$$

Set 2: Let X be an RV with the following Moment Generating function.

$$M_X(t) = \frac{1}{3} + \frac{1}{6}e^{2t} + \frac{1}{2}e^{-t}, \forall t \in \mathbb{R}.$$

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Since, $M_X(t) = M_Y(t), \forall t \in \mathbb{R}$ and since, the MGF, if it exists, uniquely identifies the distribution (this justification – part marks : 1), then X and Y are identically distributed. That is, X is a discrete RV with support $S_X = \{0, 2, -1\}$ (part marks : 0.5) and the probability mass function $f_X : \mathbb{R} \to [0, 1]$ defined by (part marks : 1.5)

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As X is a bounded RV, so is $|X - 1|$ and hence $\mathbb{E}|X - 1|$ exists (the justification – part marks : 1). Now, (part marks : 1)

$$\mathbb{E}|X - 1| = \sum_{x \in S_X} |x - 1| f_X(x) = \frac{1}{3} + \frac{1}{6} + 1 = \frac{2 + 1 + 6}{6} = \frac{3}{2}.$$

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$$\Phi_X(t) = \mathbb{E}e^{itX} = \sum_{x \in S_X} e^{itx} f_X(x) = \frac{1}{3} + \frac{1}{6}e^{2it} + \frac{1}{2}e^{-it}, \ \forall t \in \mathbb{R}.$$

Set 3: Let X be an RV with the following Moment Generating function.

$$M_X(t) = \frac{1}{6} + \frac{1}{3}e^{2t} + \frac{1}{2}e^{-t}, \ \forall t \in \mathbb{R}.$$

Provide appropriate justification in all of the following sub-questions.

(i) (3 marks) Identify the distribution of X.

(ii) (1 + 1 marks) Does $E|X - 1|$ exist? Compute it.

(iii) (2 marks) Compute the Characteristic function of X.

Answer: Consider the discrete RV Y with support $S_Y = \{0, 2, -1\}$ and the probability mass function $f_Y : \mathbb{R} \to [0, 1]$ defined by

$$f_Y(y) = \begin{cases} \frac{1}{6}, & \text{if } y = 0, \\ \frac{1}{3}, & \text{if } y = 2, \\ \frac{1}{2}, & \text{if } y = -1, \\ 0, & \text{otherwise.} \end{cases}$$

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Since, $M_X(t) = M_Y(t), \forall t \in \mathbb{R}$ and since, the MGF, if it exists, uniquely identifies the distribution (this justification – part marks : 1), then X and Y are identically distributed. That is, X is a discrete RV with support $S_X = \{0, 2, -1\}$ (part marks : 0.5) and the probability mass function $f_X : \mathbb{R} \to [0, 1]$ defined by (part marks : 1.5)

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As X is a bounded RV, so is $|X - 1|$ and hence $\mathbb{E}|X - 1|$ exists (the justification – part marks : 1). Now, (part marks : 1)

$$\mathbb{E}|X - 1| = \sum_{x \in S_X} |x - 1| f_X(x) = \frac{1}{6} + \frac{1}{3} + 1 = \frac{1 + 2 + 6}{6} = \frac{3}{2}.$$

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$$\Phi_X(t) = \mathbb{E}e^{itx} = \sum_{x \in S_X} e^{itx} f_X(x) = \frac{1}{6} + \frac{1}{3}e^{2it} + \frac{1}{2}e^{-it}, \ \forall t \in \mathbb{R}.$$

Set 4: Let X be an RV with the following Moment Generating function.

$$M_X(t) = \frac{1}{6} + \frac{1}{2}e^{2t} + \frac{1}{3}e^{-t}, \ \forall t \in \mathbb{R}.$$

Provide appropriate justification in all of the following sub-questions.

(i) (3 marks) Identify the distribution of X.

(ii) (1 + 1 marks) Does $E|X - 1|$ exist? Compute it.

(iii) (2 marks) Compute the Characteristic function of X.

Answer: Consider the discrete RV Y with support $S_Y = \{0, 2, -1\}$ and the probability mass function $f_Y : \mathbb{R} \to [0,1]$ defined by

$$f_Y(y) = \begin{cases} \frac{1}{6}, & \text{if } y = 0, \\ \frac{1}{2}, & \text{if } y = 2, \\ \frac{1}{3}, & \text{if } y = -1, \\ 0, & \text{otherwise.} \end{cases}$$

Then, as Y is a bounded RV, the MGF exists and $M_Y(t) = \sum_{y \in S_Y} e^{ty} f_Y(y) = \frac{1}{6} + \frac{1}{2}e^{2t} + \frac{1}{3}e^{-t}, \forall t \in \mathbb{R}$.

Since, $M_X(t) = M_Y(t), \forall t \in \mathbb{R}$ and since, the MGF, if it exists, uniquely identifies the distribution (this justification – part marks : 1), then X and Y are identically distributed. That is, X is a discrete RV with support $S_X = \{0, 2, -1\}$ (part marks : 0.5) and the probability mass function $f_X : \mathbb{R} \to [0,1]$ defined by (part marks : 1.5)

$$f_X(x) = \begin{cases} \frac{1}{6}, & \text{if } x = 0, \\ \frac{1}{2}, & \text{if } x = 2, \\ \frac{1}{3}, & \text{if } x = -1, \\ 0, & \text{otherwise.} \end{cases}$$

As X is a bounded RV, so is $|X-1|$ and hence $\mathbb{E}|X-1|$ exists (the justification – part marks : 1). Now, (part marks : 1)

$$\mathbb{E}|X-1| = \sum_{x \in S_X} |x-1| f_X(x) = \frac{1}{6} + \frac{1}{2} + \frac{2}{3} = \frac{1+3+4}{6} = \frac{4}{3}.$$

The Characteristic function $\Phi_X : \mathbb{R} \to \mathbb{C}$ of X is given by (part marks : 2)

$$\Phi_X(t) = \mathbb{E}e^{itX} = \sum_{x \in S_X} e^{itx} f_X(x) = \frac{1}{6} + \frac{1}{2}e^{2it} + \frac{1}{3}e^{-it}, \forall t \in \mathbb{R}.$$

Question 4 (4 marks) Let X be a discrete RV supported on the set of non-negative integers $\{0, 1, 2, \cdots\}$. If

$$\mathbb{P}(X > n+1 | X > n) = \mathbb{P}(X \geq 1), \forall n = 0, 1, 2, \cdots,$$

then identify the distribution of X. Justify your answer.

Answer: Given that the support $S_X = \{0, 1, 2, \cdots\}$, we have $\sum_{n=0}^{\infty} \mathbb{P}(X = k) = 1$. We write $p = \mathbb{P}(X = 0)$. Then, $\mathbb{P}(X > 0) = \mathbb{P}(X \geq 1) = 1 - \mathbb{P}(X = 0) = 1 - p$.

Now, from the given relation, for $n = 0$, we have $1 - p = \mathbb{P}(X \geq 1) = \mathbb{P}(X > 1 | X > 0) =$

$\frac{\mathbb{P}(X>1\,\&\,X>0)}{\mathbb{P}(X>0)} = \frac{\mathbb{P}(X>1)}{\mathbb{P}(X>0)}$, which yields $\mathbb{P}(X > 1) = (1-p)\mathbb{P}(X > 0) = (1-p)^2$.
Then, $\mathbb{P}(X = 1) = \mathbb{P}(X \geq 1) - \mathbb{P}(X > 1) = (1-p) - (1-p)^2 = p(1-p)$.

Iterating this way, for $n = 1, 2, \cdots$, we have $1 - p = \mathbb{P}(X \geq 1) = \mathbb{P}(X > n+1 | X > n) = \frac{\mathbb{P}(X>n+1\,\&\,X>n)}{\mathbb{P}(X>n)} = \frac{\mathbb{P}(X>n+1)}{\mathbb{P}(X>n)}$, which yields $\mathbb{P}(X > n+1) = (1-p)\mathbb{P}(X > n) = (1-p)^{n+2}$.
Then, $\mathbb{P}(X = n+1) = \mathbb{P}(X \geq n+1) - \mathbb{P}(X > n+1) = (1-p)^{n+1} - (1-p)^{n+2} = p(1-p)^{n+1}$.
(derivation of the probability mass – part marks 2)

Hence, the p.m.f. of X is given by $f_X : \mathbb{R} \to [0,1]$ as (description of the p.m.f. – part marks 1)

$$f_X(x) = \begin{cases} p(1-p)^x, & \text{if } x \in S_X, \\ 0, & \text{otherwise,} \end{cases}$$

with $p = \mathbb{P}(X = 0)$. Thus, X follows the Geometric distribution with parameter p. (conclusion – part marks 1).

Note: Assuming X to be Geometric and verifying the property does not answer the question. If answered this way, part marks 1 shall be given.