MSO205 QUIZ 3 SOLUTIONS

Question 1 (3 marks) Let X be a continuous RV with DF F_X and p.d.f. f_X such that $\mathbb{P}(0 < X \le 2) = 1$. Is it necessary that the variance $Var(X) \le 1$? Justify.

Answer: Since X is bounded with probability 1, all moments exists. In particular, existence of $\mathbb{E}X^2$ yields that $Var(X)$ is finite. (part marks: 0.5)

As per the given condition, we have $F_X(2) - F_X(0) = \mathbb{P}(0 < X \le 2) = 1$. Since, $0 \le F_X(x) \le 1, \forall x \in \mathbb{R}$, we must have $F_X(2) = 1$ and $F_X(0) = 0$. Since, F_X is non-decreasing and $0 \le F_X(x) \le 1, \forall x \in \mathbb{R}$, this implies $F_X(x) = 0, \forall x \le 0$ and $F_X(x) = 1, \forall x \ge 2$. Consequently, $f_X(x)$ can be taken as 0 on $(-\infty, 0) \cup (2, \infty)$. (part marks: 0.5)

Now, $\mathbb{E}(X-1)^2 = \int_0^2 (x-1)^2 f_X(x)\,dx \le \int_0^2 f_X(x)\,dx = 1$. (part marks: 1)

But, $Var(X)$ is the minimum value of all second moments $\mathbb{E}(X-c)^2, c \in \mathbb{R}$ and hence, $Var(X) \le \mathbb{E}(X-1)^2 \le 1$. Hence, necessarily the inequality holds. (part marks: 1)

Question 2 (2 + 2 marks) A box contains 8 electric bulbs, out of which 4 are defective. All the bulbs look identical. Suppose 3 bulbs drawn at random and without replacement, and then tested. Let X denote the number of defective bulbs in this random experiment. Identify the distribution of X and compute $\mathbb{E}[X(X-1)(X-2)]$.

Answer: Since 4 bulbs are defective and only 3 bulbs in total are drawn, X is an non-negative-integer valued RV bounded above by 3. As per the description of the random experiment, where the draws are without replacement, we have X follows the Hypergeometric distribution with the parameters $N = 8, M = 4, n = 3$ (part marks: 0.5) . The p.m.f. is given by

$$f_X(x) = \begin{cases} \frac{\binom{4}{x}\binom{8-4}{3-x}}{\binom{8}{3}}, \text{ if } x \in \{0,1,2,3\}, (part\ marks: 1) \\ 0, \text{otherwise.} (part\ marks: 0.5^*) \end{cases}$$

Now, X is a bounded RV and so is $X(X-1)(X-2)$, and hence $\mathbb{E}[X(X-1)(X-2)]$ exists (part marks: 0.5). Now, (part marks: 1.5)

$$\mathbb{E}[X(X-1)(X-2)] = \sum_{x \in \{0,1,2,3\}} x(x-1)(x-2) f_X(x) = 3 \times 2 \times 1 \times \frac{\binom{4}{3}\binom{8-4}{3-3}}{\binom{8}{3}} = 3 \times 2 \times 1 \times \frac{4 \times 1}{\frac{8 \times 7 \times 6}{3 \times 2 \times 1}} = \frac{3}{7}.$$

* Note: If the support is clearly mentioned then this part marks shall be awarded

Question 3

Set 1: (3 + 1 marks) Let $Z = (X, Y)$ be a 2-dimensional continuous random vector with the joint p.d.f.

$$f_Z(x,y) = \begin{cases} 8xy, \text{ if } 0 < x < y < 1 \\ 0, \text{ otherwise.} \end{cases}$$

Find the conditional p.d.f. $f_{X|Y}(x \mid \frac{1}{3}), \forall x \in \mathbb{R}$ and compute $\mathbb{E}[X \mid Y = \frac{1}{3}]$.

Answer: First, we find the marginal p.d.f. of Y. We have, (part marks: 1)

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx = \begin{cases} \int_0^y 8xy\,dx, & \text{if } y \in (0,1) \\ 0, & \text{otherwise.} \end{cases} = \begin{cases} 4y^3, & \text{if } y \in (0,1) \\ 0, & \text{otherwise.} \end{cases}$$

Then the conditional p.d.f $f_{X|Y}(x \mid \frac{1}{3}), \forall x \in \mathbb{R}$ is given by (part marks: 2)

$$f_{X|Y}(x \mid \frac{1}{3}) = \begin{cases} \frac{f_{X,Y}(x,\frac{1}{3})}{f_Y(\frac{1}{3})}, & \text{if } x \in (0,\frac{1}{3}) \\ 0, & \text{otherwise.} \end{cases} = \begin{cases} \frac{\frac{8}{3}x}{\frac{4}{3^3}}, & \text{if } x \in (0,\frac{1}{3}) \\ 0, & \text{otherwise.} \end{cases} = \begin{cases} 18x, & \text{if } x \in (0,\frac{1}{3}) \\ 0, & \text{otherwise.} \end{cases}$$

and (part marks: 1)

$$\mathbb{E}[X \mid Y = \frac{1}{3}] = \int_{-\infty}^{\infty} x f_{X|Y}(x \mid \frac{1}{3})dx = \int_0^{\frac{1}{3}} 18x^2\,dx = \frac{2}{9}.$$

Set 2: (3 + 1 marks) Let $Z = (X,Y)$ be a 2-dimensional continuous random vector with the joint p.d.f.

$$f_Z(x,y) = \begin{cases} 8xy, & \text{if } 0 < x < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Find the conditional p.d.f. $f_{X|Y}(x \mid \frac{1}{4}), \forall x \in \mathbb{R}$ and compute $\mathbb{E}[X \mid Y = \frac{1}{4}]$.

Answer: First, we find the marginal p.d.f. of Y. We have, (part marks: 1)

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx = \begin{cases} \int_0^y 8xy\,dx, & \text{if } y \in (0,1) \\ 0, & \text{otherwise.} \end{cases} = \begin{cases} 4y^3, & \text{if } y \in (0,1) \\ 0, & \text{otherwise.} \end{cases}$$

Then the conditional p.d.f $f_{X|Y}(x \mid \frac{1}{4}), \forall x \in \mathbb{R}$ is given by (part marks: 2)

$$f_{X|Y}(x \mid \frac{1}{4}) = \begin{cases} \frac{f_{X,Y}(x,\frac{1}{4})}{f_Y(\frac{1}{4})}, & \text{if } x \in (0,\frac{1}{4}) \\ 0, & \text{otherwise.} \end{cases} = \begin{cases} \frac{\frac{8}{4}x}{\frac{4}{4^3}}, & \text{if } x \in (0,\frac{1}{4}) \\ 0, & \text{otherwise.} \end{cases} = \begin{cases} 32x, & \text{if } x \in (0,\frac{1}{4}) \\ 0, & \text{otherwise.} \end{cases}$$

and (part marks: 1)

$$\mathbb{E}[X \mid Y = \frac{1}{4}] = \int_{-\infty}^{\infty} x f_{X|Y}(x \mid \frac{1}{4})dx = \int_0^{\frac{1}{4}} 32x^2\,dx = \frac{1}{6}.$$

Set 3: (3 + 1 marks) Let $Z = (X,Y)$ be a 2-dimensional continuous random vector with the joint p.d.f.

$$f_Z(x,y) = \begin{cases} 8xy, & \text{if } 0 < y < x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Find the conditional p.d.f. $f_{Y|X}(y \mid \frac{1}{3}), \forall y \in \mathbb{R}$ and compute $\mathbb{E}[Y \mid X = \frac{1}{3}]$.

Answer: First, we find the marginal p.d.f. of X. We have, (part marks: 1)

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy = \begin{cases} \int_0^x 8xy\,dy, & \text{if } x \in (0,1) \\ 0, & \text{otherwise.} \end{cases} = \begin{cases} 4x^3, & \text{if } x \in (0,1) \\ 0, & \text{otherwise.} \end{cases}$$

Then the conditional p.d.f $f_{Y|X}(y \mid \frac{1}{3}), \forall y \in \mathbb{R}$ is given by (part marks: 2)

$$f_{Y|X}(y \mid \frac{1}{3}) = \begin{cases} \frac{f_{X,Y}(\frac{1}{3},y)}{f_X(\frac{1}{3})}, \text{ if } y \in (0,\frac{1}{3}) \\ 0, \text{ otherwise.} \end{cases} = \begin{cases} \frac{\frac{8}{3}y}{\frac{4}{3^3}}, \text{ if } y \in (0,\frac{1}{3}) \\ 0, \text{ otherwise.} \end{cases} = \begin{cases} 18y, \text{ if } y \in (0,\frac{1}{3}) \\ 0, \text{ otherwise.} \end{cases}$$

and (part marks: 1)

$$\mathbb{E}[Y \mid X = \frac{1}{3}] = \int_{-\infty}^{\infty} y f_{X|Y}(y \mid \frac{1}{3}) dy = \int_0^{\frac{1}{3}} 18y^2\, dy = \frac{2}{9}.$$

Set 4: (3 + 1 marks) Let $Z = (X, Y)$ be a 2-dimensional continuous random vector with the joint p.d.f.

$$f_Z(x,y) = \begin{cases} 8xy, \text{ if } 0 < y < x < 1 \\ 0, \text{ otherwise.} \end{cases}$$

Find the conditional p.d.f. $f_{Y|X}(y \mid \frac{1}{4}), \forall y \in \mathbb{R}$ and compute $\mathbb{E}[Y \mid X = \frac{1}{4}]$.

Answer: First, we find the marginal p.d.f. of X. We have, (part marks: 1)

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy = \begin{cases} \int_0^x 8xy\, dy, \text{ if } x \in (0,1) \\ 0, \text{ otherwise.} \end{cases} = \begin{cases} 4x^3, \text{ if } x \in (0,1) \\ 0, \text{ otherwise.} \end{cases}$$

Then the conditional p.d.f $f_{Y|X}(y \mid \frac{1}{4}), \forall y \in \mathbb{R}$ is given by (part marks: 2)

$$f_{Y|X}(y \mid \frac{1}{4}) = \begin{cases} \frac{f_{X,Y}(\frac{1}{4},y)}{f_X(\frac{1}{4})}, \text{ if } y \in (0,\frac{1}{4}) \\ 0, \text{ otherwise.} \end{cases} = \begin{cases} \frac{\frac{8}{4}y}{\frac{4}{4^3}}, \text{ if } y \in (0,\frac{1}{4}) \\ 0, \text{ otherwise.} \end{cases} = \begin{cases} 32y, \text{ if } y \in (0,\frac{1}{4}) \\ 0, \text{ otherwise.} \end{cases}$$

and (part marks: 1)

$$\mathbb{E}[Y \mid X = \frac{1}{4}] = \int_{-\infty}^{\infty} y f_{X|Y}(y \mid \frac{1}{4}) dy = \int_0^{\frac{1}{4}} 32y^2\, dy = \frac{1}{6}.$$

Question 4 (3 + 1 marks) Fix $p \in (0,1)$. Let X_1, X_2, X_3 be a random sample of size 3 from the *Bernoulli*(p) distribution. Find the joint distribution of $X_{(2)}$ and $X_{(3)}$. Are $X_{(2)}$ and $X_{(3)}$ independent? Justify your answer.

Answer: By definition, $X_{(2)} \leq X_{(3)}$. Since, X_1, X_2, X_3 are $\{0,1\}$ valued, so are $X_{(2)}$ and $X_{(3)}$. Consequently, using independence of X_1, X_2, X_3, the joint distribution of $X_{(2)}$ and $X_{(3)}$ is described by

(i) $\mathbb{P}(X_{(2)} = 0, X_{(3)} = 0) = \mathbb{P}(X_1 = 0, X_2 = 0, X_3 = 0) = (1-p)^3$.

(ii) $\mathbb{P}(X_{(2)} = 1, X_{(3)} = 0) = 0$.

(iii) $\mathbb{P}(X_{(2)} = 0, X_{(3)} = 1) = \mathbb{P}(\text{exactly two of } X_1, X_2, X_3 \text{ are } 0 \text{ and the other is } 1) = \binom{3}{2}(1-p)^2 \times p = 3p(1-p)^2$.

(iv) $\mathbb{P}(X_{(2)} = 1, X_{(3)} = 1) = \mathbb{P}(\text{exactly one of } X_1, X_2, X_3 \text{ is } 0 \text{ and the other two are } 1) + \mathbb{P}(X_1 = 1, X_2 = 1, X_3 = 1) = \binom{3}{1}(1-p) \times p^2 + p^3 = 3p^2(1-p) + p^3$.

We have,

$$\mathbb{P}(X_{(2)} = 1) \geq \mathbb{P}(X_{(2)} = 1, X_{(3)} = 1) > 0$$

and

$$\mathbb{P}(X_{(3)} = 0) \geq \mathbb{P}(X_{(2)} = 0, X_{(3)} = 0) > 0.$$

Hence, $\mathbb{P}(X_{(2)} = 1)\mathbb{P}(X_{(3)} = 0) \neq \mathbb{P}(X_{(2)} = 1, X_{(3)} = 0)$. Therefore, $X_{(2)}$ and $X_{(3)}$ are not independent.