MSO205 PRACTICE PROBLEMS SET 2

Question 1 (Matching problem). A secretary types 3 letters and prepares 3 corresponding envelopes. In a hurry, she places one letter in each envelope at random. What is the probability that at least one letter is in the correct envelope?

Question 2. Draw 3 cards successively at random without replacement from a standard deck of 52 cards. Find the probability that exactly 2 cards are King and one card is a Queen.

Question 3 (Monty Hall Problem). There are 3 doors with one door having an expensive car behind it and each of the other 2 doors having a goat behind them. Monty Hall, being a host of the game, knows what it behind each door. A contestant is asked to select one of the doors and he wins the item (car or goat) behind the selected door. The contestant selects one of the doors at random, and then Monty Hall opens one of the other two doors to reveal a goat behind it (Monty Hall knows the doors behind which there are goats). Monty Hall offers to trade the door the contestant has chosen for the other door that is closed. Should the contestant switch doors if the goal is to win the car? (The problem is based on the American Television game show 'Let's make a deal' hosted by Monty Hall).

Question 4. Suppose that a population comprises of 45% females and 55% males. Suppose that 60% of the females and 80% of the males in the population have jobs. Choose a person at random from the population.

(a) Find the probability that the person chosen has a job.
(b) Given that the selected person has a job, find the probability that the person is female.

Question 5. Let A, B be two independent events in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Show that

(1) A and B^c are independent.
(2) A^c and B are independent.
(3) A^c and B^c are independent.

Question 6. (1) Let $\{E_1, E_2, \cdots, E_n\}$ be a finite collection of mutually independent events in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Construct a collection $\{A_1, A_2, \cdots, A_n\}$ of events in the following way. For each $1 \leq i \leq n$, make a choice for A_i between E_i or E_i^c. For every

$2 \leq k \leq n$ and indices $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, show that $\{A_{i_1}, A_{i_2}, \cdots, A_{i_k}\}$ is a collection of mutually independent events.

(2) Let $\mathcal{I}$ be an infinite indexing set and let $\{E_i : i \in \mathcal{I}\}$ be a collection of mutually independent events in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Construct a collection $\{A_i : i \in \mathcal{I}\}$ of events in the following way. For each $i \in \mathcal{I}$, make a choice for A_i between E_i or E_i^c. For positive integers $k \geq 2$ and distinct indices $i_1, i_2, \cdots, i_k$, show that $\{A_{i_1}, A_{i_2}, \cdots, A_{i_k}\}$ is a collection of mutually independent events.

Question 7. A student appears in the examinations of four subjects Biology, Chemistry, Physics and Mathematics. Suppose that the performances of the student in four subjects are independent and that the probabilities of the student obtaining a passing grade in these subjects are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ and $\frac{1}{5}$ respectively. Find the probability that the student will obtain a passing grade in (a) all the subjects, (b) no subject, (c) at least one subject.

Question 8. Let Ω be a non-empty set and let $X : \Omega \to \mathbb{R}$ be a function. Given any subset A of $\mathbb{R}$, we consider the subset $X^{-1}(A)$ of Ω defined by

$$X^{-1}(A) := \{\omega \in \Omega : X(\omega) \in A\}.$$

The set $X^{-1}(A)$ shall be referred to as the pre-image of A under the function X. Verify the following properties of the pre-images under X, which follow from the fact that X is a function.

(i) $X^{-1}(\mathbb{R}) = \Omega$.

(ii) $X^{-1}(\emptyset_{\mathbb{R}}) = \emptyset_{\Omega}$, where $\emptyset_{\mathbb{R}}$ and $\emptyset_{\Omega}$ denote the empty sets under $\mathbb{R}$ and Ω, respectively. (Note: When there is no chance of confusion, we simply write $X^{-1}(\emptyset) = \emptyset$.)

(iii) For any two subsets A, B of $\mathbb{R}$ with $A \cap B = \emptyset$, we have $X^{-1}(A) \cap X^{-1}(B) = \emptyset$.

(iv) For any subset A of $\mathbb{R}$, we have $X^{-1}(A^c) = \left(X^{-1}(A)\right)^c$.

(v) Let $\mathcal{I}$ be an indexing set. For any collection $\{A_i : i \in \mathcal{I}\}$ of subsets of $\mathbb{R}$, we have

$$X^{-1}\left(\bigcup_{i \in \mathcal{I}} A_i\right) = \bigcup_{i \in \mathcal{I}} X^{-1}(A_i), \quad X^{-1}\left(\bigcap_{i \in \mathcal{I}} A_i\right) = \bigcap_{i \in \mathcal{I}} X^{-1}(A_i).$$