5. Week 5

Definition 5.1 (Support of a Continuous RV)**.** Let X be a continuous RV with DF F_X. The set

$$S := \{x \in \mathbb{R} : F_X(x+h) - F_X(x-h) > 0, \forall h > 0\}$$

is defined to be the support of X.

Remark 5.2. The support S of a continuous RV X can be expressed in terms of the law/distribution of X as follows.

$$S = \{x \in \mathbb{R} : \mathbb{P}(x-h < X \leq x+h) > 0, \forall h > 0\} = \{x \in \mathbb{R} : \mathbb{P}_X((x-h, x+h]) > 0, \forall h > 0\}.$$

Remark 5.3. The support S of a continuous RV X can be expressed in terms of the p.d.f. f_X as follows.

$$S = \{x \in \mathbb{R} : \int_{x-h}^{x+h} f_X(t)\,dt > 0, \forall h > 0\}.$$

Note 5.4. If $x \notin S$, where S is the support of a continuous RV X, then there exists $h > 0$ such that $F_X(x+h) = F_X(x-h)$. By the non-decreasing property of F_X, we conclude that F_X remains a constant on the interval $[x-h, x+h]$. In particular, $f_X(t) = F_X'(t) = 0, \forall t \in (x-h, x+h)$.

Example 5.5. Consider a continuous RV X with DF $F_X : \mathbb{R} \to [0,1]$ and p.d.f. $f_X : \mathbb{R} \to [0, \infty)$ given by

$$F_X(x) := \begin{cases} 0, & \text{if } x < 0, \\ x, & \text{if } 0 \leq x < 1, \\ 1, & \text{if } x \geq 1. \end{cases}, \qquad f_X(x) := \begin{cases} 1, & \text{if } 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

To identify the support S, we consider the following cases.

(a) Let $x \in (-\infty, 0)$. Then for all h with $-x > h > 0$, we have $x - h < x + h < 0$ and consequently, $F_X(x+h) - F_X(x-h) = 0 - 0 = 0$. Therefore $x \notin S$.

(b) Let $x \in (1, \infty)$. Then for all $0 < h < x - 1$, we have $1 < x - h < x + h$ and consequently, $F_X(x+h) - F_X(x-h) = 1 - 1 = 0$. Therefore $x \notin S$.

(c) Let $x \in (0, 1)$. For any $0 < h < \min\{x, 1-x\}$, we have $0 < x - h < x + h < 1$ and consequently, $F_X(x+h) - F_X(x-h) = (x+h) - (x-h) = 2h > 0$. For $h \geq \min\{x, 1-x\}$, at least one of $x-h, x+h$ is in $(0,1)^c$ and hence $F_X(x+h) - F_X(x-h) > 0$. Therefore $x \in S$.

(d) Let $x = 0$. Then for any $h > 0$, we have $F_X(0+h) - F_X(0-h) = F_X(0+h) > 0$. Then $0 \in S$. By a similar argument, $1 \in S$.

From the above discussion, we conclude that $S = [0,1]$.

Note 5.6. Cantor function (also known as the Devil's Staircase) is an example of a continuous distribution function, which is not absolutely continuous. In this case, the DF F is not representable as $\int_{-\infty}^{x} f(t)\,dt$ for any non-negative integrable function. In fact, in this example, $F'(x) = 0$ at 'almost all points x'. We do not discuss these types of examples in this course.

Definition 5.7 (Quantiles and Median for an RV)**.** Let X be an RV with DF F_X. For any $p \in (0,1)$, a number $x \in \mathbb{R}$ is called a quantile of order p if the following inequalities are satisfied, viz.

$$p \le F_X(x) \le p + \mathbb{P}(X = x).$$

A quantile of order $\frac{1}{2}$ is called a median.

Note 5.8. A quantile need not be unique.

Notation 5.9. We write $\mathfrak{z}_p(X)$ to denote a quantile of order p.

Notation 5.10. The quantiles of order $\frac{1}{4}$ and $\frac{3}{4}$ for an RV X are referred to as the lower and upper quartiles of X, respectively.

Note 5.11. The inequalities mentioned in Definition 5.7 can be restated as

$$\mathbb{P}(X \le x) \ge p, \quad \mathbb{P}(X \ge x) \ge 1 - p.$$

Note 5.12. Let X be a continuous RV with DF F_X. Then a quantile of order p is a solution to the equation $F_X(x) = p$, since $\mathbb{P}(X = x) = 0, \forall x \in \mathbb{R}$. Moreover, if F_X is strictly increasing, then $\mathfrak{z}_p(X)$ is unique for all $p \in (0,1)$.

Example 5.13. Consider a continuous RV X with DF $F_X : \mathbb{R} \to [0,1]$ given by

$$F_X(x) := \begin{cases} 0, & \text{if } x < 0, \\ x, & \text{if } 0 \leq x < 1, \\ 1, & \text{if } x \geq 1. \end{cases}$$

For any $p \in (0,1)$, the solution to $F_X(x) = p$ is given by $x = p$, i.e. $\mathfrak{z}_p(X) = p$. Moreover, the median is $\mathfrak{z}_{\frac{1}{2}}(X) = \frac{1}{2}$.

We now discuss functions of RVs and their law/distributions.

Remark 5.14 (Function of an RV is an RV). Let $h : \mathbb{R} \to \mathbb{R}$ be a function and let X be an RV defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Since $X : \Omega \to \mathbb{R}$ is a function, we can consider the composition of the functions h and X to obtain another function $h \circ X : \Omega \to \mathbb{R}$ defined by $(h \circ X)(\omega) := h(X(\omega)), \forall \omega \in \Omega$. Since $h \circ X$ is a real valued function defined on Ω with $(\Omega, \mathcal{F}, \mathbb{P})$, $h \circ X$ is an RV defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Notation 5.15. In the setting of the above remark, we shall write $h(X)$ to denote $h \circ X$.

Example 5.16. Let X be an RV defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and consider the function $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) = 3x^2 + \sin x + 1, \forall x \in \mathbb{R}$. Then $h(X) = h \circ X$ defined by $(h \circ X)(\omega) := 3X(\omega)^2 + \sin(X(\omega)) + 1, \forall \omega \in \Omega$ is an RV.

Remark 5.17 (DF of a function of an RV). We continue with the notations of Remark 5.14 and are interested in computing the law/distribution of $Y = h(X)$. Using Remark 3.20, we may equivalently, compute the DF of Y and that will identify the required law. Then for any $y \in \mathbb{R}$, we have

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(h(X) \leq y) = \mathbb{P}(h(X) \in (-\infty, y]) = \mathbb{P}(X \in h^{-1}((-\infty, y])),$$

where $h^{-1}((-\infty, y])$ denotes the pre-image of $(-\infty, y]$ under h (see Notation 2.39).

Example 5.18. Let X be a discrete RV with p.m.f.

$$f_X(x) := \begin{cases} \frac{|x|}{110} & \text{if } x \in \{\pm 1, \pm 2, \ldots, \pm 10\} \\ 0, & \text{otherwise} \end{cases}$$

and take $h : \mathbb{R} \to \mathbb{R}$ as $h(x) := |x|, \forall x \in \mathbb{R}$. Note that

$$h^{-1}((-\infty, y]) = \begin{cases} \emptyset, \text{ if } y < 0, \\ \{0\}, \text{ if } y = 0, \\ [-y, y], \text{ if } y > 0. \end{cases}$$

Then the DF of $Y = h(X) = |X|$ is given by

$$\begin{aligned} F_Y(y) &= \mathbb{P}(X \in h^{-1}((-\infty, y])) \\ &= \begin{cases} \mathbb{P}(X \in \emptyset), \text{ if } y < 0, \\ \mathbb{P}(X \in \{0\}), \text{ if } y = 0, \\ \mathbb{P}(X \in [-y, y]), \text{ if } y > 0. \end{cases} \\ &= \begin{cases} 0, \text{ if } y < 0, \\ \mathbb{P}(X = 0), \text{ if } y = 0, \\ \sum_{t \in [-y,y] \cap \{\pm 1, \pm 2, \ldots, \pm 10\}} f_X(t), \text{ if } y > 0. \end{cases} \\ &= \begin{cases} 0, \text{ if } y \leq 0, \\ \sum_{t \in [-y,y] \cap \{\pm 1, \pm 2, \ldots, \pm 10\}} \frac{|t|}{110}, \text{ if } y > 0. \end{cases} \end{aligned}$$

From the structure of the DF we conclude that the RV is discrete. The p.m.f. may be computed using the techniques discussed in earlier lectures.

Example 5.19. Let X be a continuous RV with p.d.f.

$$f_X(x) = \begin{cases} \frac{|x|}{2}, & \text{if } -1 < x < 1 \\ \frac{x}{3}, & \text{if } 1 \leq x < 2 \\ 0, & \text{otherwise} \end{cases}$$

and take $h : \mathbb{R} \to \mathbb{R}$ as $h(x) := x^2, \forall x \in \mathbb{R}$. Note that

$$h^{-1}((-\infty, y]) = \begin{cases} \emptyset, \text{ if } y < 0, \\ \{0\}, \text{ if } y = 0, \\ [-\sqrt{y}, \sqrt{y}], \text{ if } y > 0. \end{cases}$$

Then the DF of $Y = h(X) = X^2$ is given by

$$\begin{aligned} F_Y(y) &= \mathbb{P}(X \in h^{-1}((-\infty, y])) \\ &= \begin{cases} \mathbb{P}(X \in \emptyset), \text{ if } y < 0, \\ \mathbb{P}(X \in \{0\}), \text{ if } y = 0, \\ \mathbb{P}(X \in [-\sqrt{y}, \sqrt{y}]), \text{ if } y > 0. \end{cases} \\ &= \begin{cases} 0, \text{ if } y < 0, \\ \mathbb{P}(X = 0), \text{ if } y = 0, \\ \mathbb{P}(\{-\sqrt{y} \leq X \leq \sqrt{y}\}), \text{ if } y > 0. \end{cases} \\ &= \begin{cases} 0, \text{ if } y < 0, \\ 0, \text{ if } y = 0, \\ \int_{-\sqrt{y}}^{\sqrt{y}} f_X(x)\, dx, \text{ if } y > 0. \end{cases} \end{aligned}$$

$$= \begin{cases} 0, \text{ if } y < 0, \\ 0, \text{ if } y = 0, \\ \int_{-\sqrt{y}}^{\sqrt{y}} \frac{|x|}{2} dx, \text{ if } 0 \le y < 1 \\ \int_{-1}^{1} \frac{|x|}{2} \, dx + \int_{1}^{\sqrt{y}} \frac{x}{3} dx, \text{ if } 1 \le y < 4 \\ 1, \text{ if } y \ge 4 \end{cases}$$

$$= \begin{cases} 0, \text{ if } y \le 0, \\ \frac{y}{2}, \text{ if } 0 \le y < 1 \\ \frac{y+2}{6}, \text{ if } 1 \le y < 4 \\ 1, \text{ if } y \ge 4. \end{cases}$$

From the structure of the DF we conclude that the RV is continuous. The p.d.f. may be computed using the techniques discussed in earlier lectures.

Note 5.20. We continue the discussion in Remark 5.17. In general, we may not be able to reduce/simplify the expression $h^{-1}((-\infty, y])$ further, without additional information about h or X. In what follows, we shall consider the cases where X is discrete or continuous and then attempt to obtain the DF of $h(X)$.

Theorem 5.21. *Let X be a discrete RV with DF F_X, p.m.f. f_X and support S_X. Let $h : \mathbb{R} \to \mathbb{R}$ be a function. Then $Y = h(X)$ is a discrete RV with support $S_Y = h(S_X) := \{h(x) : x \in S_X\}$, p.m.f. f_Y given by*

$$f_Y(y) = \begin{cases} \sum_{x \in h^{-1}(\{y\})} f_X(x), & \text{if } y \in S_Y, \\ 0, & \text{otherwise} \end{cases}$$

and DF F_Y given by

$$F_Y(y) = \mathbb{P}(Y \le y) = \sum_{t \in S_Y \cap (-\infty, y]} f_Y(t) = \sum_{\substack{x \in S_X \\ h(x) \le y}} f_X(x) = \sum_{x \in S_X \cap h^{-1}((-\infty, y])} f_X(x).$$

Proof. Since S_X is a finite or a countably infinite set, the set $h(S_X)$ is also finite or countably infinite. Now,

$$\mathbb{P}(h(X) \in h(S_X)) = \mathbb{P}(X \in h^{-1}(h(S_X))) \geq \mathbb{P}(X \in S_X) = 1$$

and hence $\mathbb{P}(h(X) \in h(S_X)) = 1$. Here, we have used the fact that $h^{-1}(h(S_X)) \supseteq S_X$. Moreover, for any $x \in S_X$,

$$\mathbb{P}(h(X) = h(x)) = \mathbb{P}(X \in h^{-1}(\{h(x)\})) \geq \mathbb{P}(X \in \{x\}) = f_X(x) > 0$$

and hence $Y = h(X)$ is discrete with support $S_Y = h(S_X)$. The expressions for f_Y and F_Y follows from standard arguments. $\square$

Note 5.22. As a consequence of Theorem 5.21, we conclude that the functions of discrete RVs are also discrete RVs.

Note 5.23. In Theorem 5.21, the function h need not be one-to-one or onto and therefore need not have an inverse. This was the same problem encountered in Remark 5.17, which stops us in computing the DF of $h(X)$ for a general RV X.

As a special case of Remark 5.17, we get the next result. We do not give a separate proof, for brevity.

Corollary 5.24. *Continue with the notations of Theorem 5.21. Assume that $h : S_X \to \mathbb{R}$ is one-to-one. Then we have*

$$f_Y(y) = \begin{cases} f_X(h^{-1}(y)), & \textit{if } y \in S_Y, \\ 0, & \textit{otherwise} \end{cases}$$

where $h^{-1} : h(S_X) \to S_X$ denotes the inverse function of $h : S_X \to \mathbb{R}$.

Example 5.25. Let X be a discrete RV with p.m.f.

$$f_X(x) = \begin{cases} \frac{1}{7}, & \text{if } x \in \{-2, -1, 0, 1\} \\ \frac{3}{14}, & \text{if } x \in \{2, 3\} \\ 0, & \text{otherwise.} \end{cases}$$

Consider the RV $Y = X^2$. Here $S_X = \{-2, -1, 0, 1, 2, 3\}$ and $S_Y = \{0, 1, 4, 9\}$. Observe that,

$$\begin{aligned}
\mathbb{P}(Y = 0) &= \mathbb{P}\left(X^2 = 0\right) = \mathbb{P}(X = 0) = \tfrac{1}{7},\\
\mathbb{P}(Y = 1) &= \mathbb{P}\left(X^2 = 1\right) = \mathbb{P}(X \in \{-1, 1\}) = \tfrac{1}{7} + \tfrac{1}{7} = \tfrac{2}{7},\\
\mathbb{P}(Y = 4) &= \mathbb{P}\left(X^2 = 4\right) = \mathbb{P}(X \in \{-2, 2\}) = \tfrac{1}{7} + \tfrac{3}{14} = \tfrac{5}{14}\\
\mathbb{P}(Y = 9) &= \mathbb{P}\left(X^2 = 9\right) = \mathbb{P}(X \in \{-3, 3\}) = 0 + \tfrac{3}{14} = \tfrac{3}{14}.
\end{aligned}$$

Therefore, the p.m.f. of Y is

$$f_Y(y) = \begin{cases} \frac{1}{7}, & \text{if } y = 0\\ \frac{2}{7}, & \text{if } y = 1\\ \frac{5}{14}, & \text{if } y = 4\\ \frac{3}{14}, & \text{if } y = 9\\ 0, & \text{otherwise,} \end{cases}$$

and the DF of Y is

$$F_Y(y) = \begin{cases} 0, & \text{if } y < 0\\ \frac{1}{7}, & \text{if } 0 \le y < 1\\ \frac{3}{7}, & \text{if } 1 \le y < 4\\ \frac{11}{14}, & \text{if } 4 \le y < 9\\ 1, & \text{if } y \ge 9. \end{cases}$$

In fact, after identifying S_Y, we could have directly computed the DF F_Y as follows:

$$F_Y(y) = \mathbb{P}(Y \le y) = \begin{cases} 0, \text{ if } y < 0,\\ \mathbb{P}(Y = 0), \text{ if } 0 \le y < 1,\\ \mathbb{P}(Y = 0) + \mathbb{P}(Y = 1), \text{ if } 1 \le y < 4,\\ \mathbb{P}(Y = 0) + \mathbb{P}(Y = 1) + \mathbb{P}(Y = 4), \text{ if } 4 \le y < 9,\\ 1, \text{ if } y \ge 9. \end{cases}$$

and the p.m.f. f_Y from F_Y using standard techniques discussed in earlier lectures.

Example 5.26. Let X be a discrete RV with p.m.f.

$$f_X(x) = \begin{cases} \frac{x}{55} & \text{if } x \in \{1, 2, \ldots, 10\} \\ 0, & \text{otherwise.} \end{cases}$$

Now consider the RV $Y = X^2$. Note that the function $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) := x^2, \forall x \in \mathbb{R}$ is one-to-one on the support S_X. Here, Y is discrete with support $S_Y = \{1, 4, 9, \cdots, 100\}$ and by Corollary 5.24, the p.m.f. f_Y is given by

$$f_Y(y) = \begin{cases} f_X(\sqrt{y}), \text{ if } y \in S_Y, \\ 0, \text{ otherwise} \end{cases} = \begin{cases} \frac{\sqrt{y}}{55}, \text{ if } y \in S_Y, \\ 0, \text{ otherwise} \end{cases}.$$

The DF F_Y can now be computed from the p.m.f. f_Y using standard techniques.

Now we look at functions of continuous RVs.

Example 5.27. Let X be a continuous RV with p.d.f.

$$f_X(x) = \begin{cases} e^{-x}, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases}$$

and let $Y = [X]$, where $[x]$ denotes the largest integer not exceeding x for $x \in \mathbb{R}$. Note that $S_X = [0, \infty)$. Moreover,

$$\mathbb{P}(Y \in \{0, 1, 2, \ldots\}) = \mathbb{P}(X \in S_X) = 1$$

and hence Y is a discrete RV. Now, for $y \in \{0, 1, 2, \ldots\}$

$$f_Y(y) = \mathbb{P}(Y = y) = \mathbb{P}(y \le X < y + 1) = \int_y^{y+1} f_X(x)\,dx = \int_y^{y+1} e^{-x}\,dx = \left(1 - e^{-1}\right) e^{-y} > 0.$$

hence Y is a discrete RV with support $S_Y = \{0, 1, 2, \ldots\}$ and the above p.m.f. f_Y. Therefore, a function of a continuous RV need not be a continuous RV.

Remark 5.28. Given any continuous RV X and a constant function $h : \mathbb{R} \to \mathbb{R}$ given by $h(x) := c, \forall x \in \mathbb{R}$ for some $c \in \mathbb{R}$, the RV $h(X)$ is discrete. Together with the above example, we may conclude that additional information on h is required before we can conclude that $h(X)$ is continuous.