The next result is stated without proof.

Theorem 6.1. *Let X be a continuous RV with p.d.f. f_X and support S_X. Suppose $\{x \in \mathbb{R} : f_X(x) > 0\} = \cup_{i=1}^k (a_i, b_i)$ and f_X is continuous on each (a_i, b_i). We assume that the intervals (a_i, b_i) are pairwise disjoint.*

Let $h : \mathbb{R} \to \mathbb{R}$ be a function such that on each (a_i, b_i), $h : (a_i, b_i) \to \mathbb{R}$ is strictly monotone and continuously differentiable with inverse function h_i^{-1} for $i = 1, \dots, k$.

Then $Y = h(X)$ is a continuous RV with support $S_Y = \cup_{i=1}^k [c_i, d_i]$, where $c_i = \min\{h(a_i), h(b_i)\}$ and $d_i = \max\{h(a_i), h(b_i)\}$. The p.d.f. is given by

$$f_Y(y) = \sum_{i=1}^k f_X\left(h_i^{-1}(y)\right) \left|\frac{d}{dy} h_i^{-1}(y)\right| 1_{(c_i, d_i)}(y), y \in \mathbb{R}$$

where $1_{(c_i,d_i)}(y) = 1$ if $y \in (c_i, d_i)$ and 0 otherwise.

Note 6.2. In Theorem 6.1, the function h may be strictly monotone increasing in some (a_i, b_i) and strictly monotone decreasing in other intervals. Moreover, this monotonicity may be verified by looking at the sign of h'. If $h'(x) > 0, \forall x \in (a_i, b_i)$, then h is strictly monotone increasing on (a_i, b_i). If $h'(x) < 0, \forall x \in (a_i, b_i)$, then h is strictly monotone decreasing on (a_i, b_i).

Example 6.3. Let X be a continuous RV with p.d.f.

$$f_X(x) = \begin{cases} e^{-x}, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases}$$

and consider $Y = X^2$. Here, $S_X = [0, \infty)$ and the function $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) := x^2, \forall x \in \mathbb{R}$ is continuous differentiable on $(0, \infty)$. Moreover, $h'(x) = 2x > 0, \forall x \in (0, \infty)$ and hence h is strictly monotone increasing on $(0, \infty)$. The inverse function is given by $h^{-1}(y) = \sqrt{y}, \forall y \in (0, \infty)$.
The p.d.f. f_Y is given by

$$f_Y(y) = \begin{cases} \frac{e^{-\sqrt{y}}}{2\sqrt{y}}, & \text{if } y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

The DF F_Y can now be computed from the p.d.f. f_Y by standard techniques.

Example 6.4. Let X be a continuous RV with p.d.f.

$$f_X(x) = \begin{cases} \frac{|x|}{2}, & \text{if } -1 < x < 1 \\ \frac{x}{3}, & \text{if } 1 \le x < 2 \\ 0, & \text{otherwise} \end{cases}$$

and consider $Y = X^2$.

Observe that $\{x \in \mathbb{R} : f_X(x) > 0\} = (-1,0) \cup (0,2)$. Now, $h(x) = x^2$ is strictly decreasing on $(-1,0)$ with inverse function $h_1^{-1}(t) = -\sqrt{t}$; and $h(x) = x^2$ is strictly increasing on $(0,2)$ with inverse function $h_2^{-1}(t) = \sqrt{t}$. Note that $h((-1,0)) = (0,1)$ and $h((0,2)) = (0,4)$. Then, $Y = X^2$ has p.d.f. given by

$$\begin{aligned} f_Y(y) &= f_X(-\sqrt{y}) \left| \frac{d}{dy}(-\sqrt{y}) \right| 1_{(0,1)}(y) + f_X(\sqrt{y}) \left| \frac{d}{dy}(\sqrt{y}) \right| 1_{(0,4)}(y) \\ &= \begin{cases} \frac{1}{2}, & \text{if } 0 < y < 1 \\ \frac{1}{6}, & \text{if } 1 < y < 4 \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

We can compute the DF of Y and verify that this matches with our earlier computation in Example 5.19.

Let X be a discrete (or continuous) RV defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with DF F_X, support S_X and p.m.f. (or p.d.f.) f_X.

Definition 6.5 (Expectation/Expected value/Mean of the RV X)**.** The Expectation/Expected value/Mean of the RV X, denoted by $\mathbb{E}X$, is defined as the quantity

$$\mathbb{E}[X] := \begin{cases} \sum_{x \in S_X} x f_X(x), & \text{if } \sum_{x \in S_X} |x| f_X(x) < \infty \text{ for discrete } X, \\ \int_{-\infty}^{\infty} x f_X(x)\, dx, & \text{if } \int_{-\infty}^{\infty} |x| f_X(x)\, dx < \infty \text{ for continuous } X. \end{cases}$$

Remark 6.6. If the sum or the integral above converges absolutely, we say that the expectation $\mathbb{E}X$ exists or equivalently, $\mathbb{E}X$ is finite. Otherwise, we shall say that the expectation $\mathbb{E}X$ does not exist.

Note 6.7. Note that it is possible to define the expectation $\mathbb{E}X$ through the law/distribution $\mathbb{P}_X$ of X. However, this is beyond the scope of this course.

Example 6.8. Fix $c \in \mathbb{R}$. Let X be a discrete RV with p.m.f.

$$f_X(x) = \mathbb{P}(X = x) = \begin{cases} 1, & \text{if } x = c \\ 0, & \text{otherwise.} \end{cases}$$

Such RVs are called constant/degenerate RVs. Here, the support is a singleton set $S_X = \{c\}$ and $\sum_{x \in S_X} |x| f_X(x) = |c| < \infty$ and hence $\mathbb{E}X = \sum_{x \in S_X} x f_X(x) = c$.

Example 6.9. Let X be a discrete RV with p.m.f.

$$f_X(x) := \begin{cases} \frac{1}{6}, \forall x \in \{1,2,3,4,5,6\} \\ 0, \text{ otherwise.} \end{cases}$$

Here, the support is $S_X = \{1,2,3,4,5,6\}$, a finite set with all elements positive and hence $\sum_{x \in S_X} |x| f_X(x) = \sum_{x \in S_X} x f_X(x)$ is finite and

$$\mathbb{E}X = \sum_{x \in S_X} x f_X(x) = \frac{1}{6}(1+2+3+4+5+6) = \frac{7}{2}.$$

Example 6.10. Let X be a discrete RV with p.m.f.

$$f_X(x) := \begin{cases} \frac{1}{2^x}, \forall x \in \{1,2,3,\cdots\} \\ 0, \text{ otherwise.} \end{cases}$$

Here, the support is $S_X = \{1,2,3,\cdots\}$, the set of natural numbers. To check the existence of $\mathbb{E}X$, we need to check the convergence of the series $\sum_{x \in S_X} |x| f_X(x) = \sum_{x=1}^{\infty} x \frac{1}{2^x}$. Now, the x-th term is $\frac{x}{2^x}$ and

$$\lim_{x \to \infty} \frac{\frac{x+1}{2^{x+1}}}{\frac{x}{2^x}} = \frac{1}{2} < 1.$$

By ratio test, we have the required convergence and the existence of $\mathbb{E}X$ follows.
Observe that

$$\mathbb{E}X = \sum_{x=1}^{\infty} x\frac{1}{2^x} = \frac{1}{2} + \sum_{x=2}^{\infty} x\frac{1}{2^x} = \frac{1}{2} + \sum_{x=1}^{\infty}(x+1)\frac{1}{2^{x+1}} = \frac{1}{2} + \frac{1}{2}\sum_{x=1}^{\infty} x\frac{1}{2^x} + \frac{1}{2} = 1 + \frac{1}{2}\mathbb{E}X,$$

which gives $\mathbb{E}X = 2$.

Note 6.11. It is fact that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Example 6.12. Let X be a discrete RV with p.m.f.

$$f_X(x) := \begin{cases} \frac{3}{\pi^2 x^2}, \forall x \in \{\pm 1, \pm 2, \pm 3, \cdots\} \\ 0, \text{ otherwise.} \end{cases}$$

Here, the support is $S_X = \{\pm 1, \pm 2, \pm 3, \cdots\}$. To check the existence of $\mathbb{E}X$, we need to check the convergence of the series $\sum_{x \in S_X} |x| f_X(x) = 2\sum_{n=1}^{\infty} n\frac{3}{\pi^2 n^2} = \frac{6}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n}$. However, this series diverges and hence $\mathbb{E}X$ does not exist.

Example 6.13. Let X be a continuous RV with the p.d.f.

$$f_X(x) = \begin{cases} 1, \text{ if } 0 < x < 1, \\ 0, \text{ otherwise.} \end{cases}$$

To check the existence of $\mathbb{E}X$, we need to check the existence of $\int_{-\infty}^{\infty} |x| f_X(x)\, dx$. Now,

$$\int_{-\infty}^{\infty} |x| f_X(x)\, dx = \int_0^1 x\, dx = \frac{1}{2}$$

and hence $\mathbb{E}X = \frac{1}{2}$.

Example 6.14. Let X be a continuous RV with the p.d.f.

$$f_X(x) = \frac{1}{2}e^{-|x|}, \forall x \in \mathbb{R}.$$

To check the existence of $\mathbb{E}X$, we need to check the existence of $\int_{-\infty}^{\infty} |x| f_X(x)\, dx$. Now,

$$\int_{-\infty}^{\infty} |x| f_X(x)\, dx = \int_{-\infty}^{\infty} |x|\frac{1}{2}e^{-|x|}\, dx = \int_0^{\infty} xe^{-x}\, dx = 1 < \infty$$

and hence $\mathbb{E}X$ exists and

$$\mathbb{E}X = \int_{-\infty}^{\infty} x f_X(x)\, dx = \int_{-\infty}^{\infty} x \frac{1}{2} e^{-|x|}\, dx = 0.$$

Example 6.15. Let X be a continuous RV with the p.d.f.

$$f_X(x) = \frac{1}{\pi} \frac{1}{1+x^2}, \forall x \in \mathbb{R}.$$

To check the existence of $\mathbb{E}X$, we need to check the existence of $\int_{-\infty}^{\infty} |x| f_X(x)\, dx$. Now,

$$\int_{-\infty}^{\infty} |x| f_X(x)\, dx = \int_{-\infty}^{\infty} |x| \frac{1}{\pi} \frac{1}{1+x^2}\, dx = \frac{2}{\pi} \int_0^{\infty} \frac{x}{1+x^2}\, dx = \infty$$

and hence $\mathbb{E}X$ does not exist.

Proposition 6.16. *Let X be a discrete or continuous RV such that $\mathbb{E}X$ exists. Then,*

$$\mathbb{E}X = \int_0^{\infty} \mathbb{P}(X > x)\, dx - \int_{-\infty}^{0} \mathbb{P}(X < x)\, dx.$$

Proof. We prove the result when X is continuous. The case for discrete X can be proved in a similar manner. Observe that

$$\begin{aligned}
\mathbb{E}X &= \int_{-\infty}^{\infty} x f_X(x)\, dx = \int_{-\infty}^{0} x f_X(x)\, dx + \int_0^{\infty} x f_X(x)\, dx \\
&= -\int_{x=-\infty}^{0} \int_{y=x}^{0} f_X(x)\, dy dx + \int_{x=0}^{\infty} \int_{y=0}^{x} f_X(x)\, dy dx \\
&= -\int_{y=-\infty}^{0} \int_{x=-\infty}^{y} f_X(x)\, dx dy + \int_{y=0}^{\infty} \int_{x=y}^{\infty} f_X(x)\, dx dy \\
&= \int_0^{\infty} \mathbb{P}(X > y)\, dy - \int_{-\infty}^{0} \mathbb{P}(X < y)\, dy.
\end{aligned}$$

This completes the proof. $\square$

Remark 6.17. (a) Suppose X is discrete or continuous with $\mathbb{P}(X \geq 0) = 1$. Then $\mathbb{P}(X \leq x) = 0, \forall x < 0$ and hence $\mathbb{E}X = \int_0^{\infty} \mathbb{P}(X > x)\, dx$.

(b) Suppose that X is discrete with $\mathbb{P}(X \in \{0, 1, 2, \cdots\}) = 1$. Then $\mathbb{P}(X > x) = \mathbb{P}(X \geq n+1), \forall x \in [n, n+1), n \in \{0, 1, 2, \cdots\}$ and hence by part (a),

$$\mathbb{E}X = \int_0^\infty \mathbb{P}(X > x)\, dx = \sum_{n=0}^\infty \mathbb{P}(X \geq n+1) = \sum_{n=1}^\infty \mathbb{P}(X \geq n).$$

(c) For any RV X, recall that $\mathbb{P}(X = x) \neq 0$ for at most countably many points x and consequently, $\mathbb{P}(X < x) = \mathbb{P}(X \leq x)$ except those points x. Thus,

$$\begin{aligned}\mathbb{E}X &= \int_0^\infty \mathbb{P}(X > x)\, dx - \int_{-\infty}^0 \mathbb{P}(X < x)\, dx \\ &= \int_0^\infty \mathbb{P}(X > x)\, dx - \int_{-\infty}^0 \mathbb{P}(X \leq x)\, dx \\ &= \int_0^\infty (1 - F_X(x))\, dx - \int_{-\infty}^0 F_X(x)\, dx.\end{aligned}$$

Note 6.18 (Expectation of functions of RVs)**.** Given a function $h : \mathbb{R} \to \mathbb{R}$ and an RV X, we have already discussed about the distribution of $Y = h(X)$. If the p.m.f./p.d.f. f_Y is known, we can then consider the existence of $\mathbb{E}Y$ through f_Y, as per Definition 6.5. However, to do this, we first need to compute f_Y from X and then check the relevant existence. In what follows, we discuss the computation of $\mathbb{E}Y = \mathbb{E}h(X)$ directly from X, using the p.m.f./p.d.f. f_X.

Proposition 6.19. *(a) Let X be a discrete RV with p.m.f. f_X and support S_X and let $h : \mathbb{R} \to \mathbb{R}$ be a function. Consider the discrete RV $Y := h(X)$. Then $\mathbb{E}Y$ exists provided $\sum_{x \in S_X} |h(x)| f_X(x) < \infty$ and in this case,*

$$\mathbb{E}Y = \mathbb{E}h(X) = \sum_{x \in S_X} h(x) f_X(x).$$

(b) Let X be a continuous RV with p.d.f. f_X and support S_X and let $h : \mathbb{R} \to \mathbb{R}$ be a function. Consider the RV $Y := h(X)$. Then $\mathbb{E}Y$ exists provided $\int_{-\infty}^\infty |h(x)| f_X(x)\, dx < \infty$ and in this case,

$$\mathbb{E}Y = \mathbb{E}h(X) = \int_{-\infty}^\infty h(x) f_X(x)\, dx.$$

Proof. We consider the proof for the case when X is discrete. The other case can be proved by similar arguments.

By Theorem 5.21, $Y = h(X)$ is discrete with support $S_Y = h(S_X)$. Now,

$$\sum_{y \in S_Y} |y| f_Y(y) = \sum_{y \in S_Y} |y| \sum_{\{x \in S_X : h(x) = y\}} f_X(x) = \sum_{y \in S_Y} \sum_{\{x \in S_X : h(x) = y\}} |h(x)| f_X(x) = \sum_{x \in S_X} |h(x)| f_X(x).$$

Therefore, $\mathbb{E}Y$ exists provided $\sum_{x \in S_X} |h(x)| f_X(x) < \infty$ and in this case,

$$\mathbb{E}Y = \sum_{y \in S_Y} y f_Y(y) = \sum_{y \in S_Y} y \sum_{\{x \in S_X : h(x) = y\}} f_X(x) = \sum_{x \in S_X} h(x) f_X(x).$$

This completes the proof. □

Note 6.20. If X is discrete with p.m.f. f_X such that $\mathbb{E}X$ exists, then $\mathbb{E}|X| = \sum_{x \in S_X} |x| f_X(x) < \infty$. Similarly, if X is continuous with p.d.f. f_X such that $\mathbb{E}X$ exists, then $\mathbb{E}|X| = \int_{-\infty}^{\infty} |x| f_X(x)\, dx < \infty$. Therefore $\mathbb{E}X$ exists if and only if $\mathbb{E}|X| < \infty$. In other words, $\mathbb{E}X$ is finite if and only if $\mathbb{E}|X|$ is finite.

Note 6.21. Fix $a, b \in \mathbb{R}$ with $a \neq 0$. Let X be a discrete/continuous RV with p.m.f./p.d.f. f_X such that $\mathbb{E}X$ exists. Then $Y = aX + b$ is also a discrete/continuous RV. If X is discrete, then

$$\sum_{x \in S_X} |ax + b| f_X(x) \leq |a| \sum_{x \in S_X} |x| f_X(x) + |b| \sum_{x \in S_X} f_X(x) = |a| \mathbb{E}|X| + |b| < \infty$$

and hence $\mathbb{E}(aX + b)$ exists and equals

$$\mathbb{E}(aX + b) = \sum_{x \in S_X} (ax + b) f_X(x) = a \sum_{x \in S_X} x f_X(x) + b \sum_{x \in S_X} f_X(x) = a\,\mathbb{E}X + b.$$

If X is continuous, a similar argument shows $\mathbb{E}(aX + b) = a\,\mathbb{E}X + b$.

Using arguments similar to the above observations, we obtain the next result. We skip the details for brevity.

Proposition 6.22. *Let X be a discrete/continuous RV with p.m.f./p.d.f. f_X.*

(a) Let $h_i : \mathbb{R} \to \mathbb{R}$ be functions and let $a_i \in \mathbb{R}$ for $i = 1, 2, \cdots, n$. Then

$$\mathbb{E}\left(\sum_{i=1}^{n} a_i h_i(X)\right) = \sum_{i=1}^{n} a_i\, \mathbb{E}h_i(X),$$

provided all the expectations above exist.

(b) Let $h_1, h_2 : \mathbb{R} \to \mathbb{R}$ be functions such that $h_1(x) \le h_2(x), \forall x \in S_X$, where S_X denotes the support of X. Then,

$$\mathbb{E}h_1(X) \le \mathbb{E}h_2(X),$$

provided all the expectations above exist.

(c) Take $h_1(x) := -|x|, h_2(x) := x, h_3(x) := |x|, \forall x \in \mathbb{R}$. If $\mathbb{E}X$ exists, then

$$-\mathbb{E}|X| \le \mathbb{E}X \le \mathbb{E}|X|,$$

i.e. $|\mathbb{E}X| \le \mathbb{E}|X|$.

(d) If $\mathbb{P}(a \le X \le b) = 1$ for some $a, b \in \mathbb{R}$, then $\mathbb{E}X$ exists and $a \le \mathbb{E}X \le b$.

Note 6.23. Given an RV X, by choosing different functions $h : \mathbb{R} \to \mathbb{R}$, we obtain several quantities of interest of the form $\mathbb{E}h(X)$.

Definition 6.24 (Moments)**.** The quantity $\mu_r' := \mathbb{E}[X^r]$, if it exists, is called the r-th moment of RV X for $r > 0$.

Definition 6.25 (Absolute Moments)**.** The quantity $\mathbb{E}[|X|^r]$, if it exists, is called the r-th absolute moment of RV X for $r > 0$.

Definition 6.26 (Moments about a point)**.** Let $c \in \mathbb{R}$. The quantity $\mathbb{E}[(X-c)^r]$, if it exists, is called the r-th moment of RV X about c for $r > 0$.

Definition 6.27 (Absolute Moments about a point)**.** Let $c \in \mathbb{R}$. The quantity $\mathbb{E}[|X-c|^r]$, if it exists, is called the r-th absolute moment of RV X about c for $r > 0$.

Note 6.28. It is clear from the definitions above that the usual moments and absolute moments are moments and absolute moments about origin, respectively.

Proposition 6.29. *Let X be a discrete/continuous RV such that $\mathbb{E}|X|^r < \infty$ for some $r > 0$. Then $\mathbb{E}|X|^s < \infty$ for all $0 < s < r$.*

Proof. Observe that for all $x \in \mathbb{R}$, we have $|x|^s \le \max\{|x|^r, 1\} \le |x|^r + 1$ and hence

$$\mathbb{E}|X|^s \le \mathbb{E}|X|^r + 1 < \infty.$$

□

Remark 6.30. Suppose that the m-th moment $\mathbb{E}X^m$ of X exists for some positive integer m. Then we have $\mathbb{E}|X|^m < \infty$ (see Note 6.20). By Proposition 6.29, we have $\mathbb{E}|X|^n < \infty$ for all positive integers $n \leq m$ and hence the n-th moment $\mathbb{E}X^n$ exists for X. In particular, the existence of the second moment $\mathbb{E}X^2$ implies the existence of the first moment $\mathbb{E}X$, which is the expectation of X.

Definition 6.31 (Central Moments)**.** Let X be an RV such that $\mu_1' = \mathbb{E}X$ exists. The quantity $\mu_r := \mathbb{E}[(X - \mu_1')^r]$, if it exists, is called the r-th moment of RV X about the mean or r-th central moment of X for $r > 0$.

Remark 6.32.

(a) If $\mathbb{E}X$ exists, then $\mu_1 := \mathbb{E}[X - \mu_1'] = 0$.

(b) If $\mathbb{E}X^2$ exists, then so does $\mathbb{E}X$ and hence for any $c \in \mathbb{R}$,

$$\mathbb{E}(X - c)^2 = \mathbb{E}X^2 - 2c\mathbb{E}X + c^2$$

also exists. A similar argument shows that $\mathbb{E}(X - c)^2$ exists if and only if $\mathbb{E}(X - d)^2$ exists, for any $c, d \in \mathbb{R}$.

(c) If $\mathbb{E}X^2$ exists, then for any $c \in \mathbb{R}$,

$$\mathbb{E}(X-c)^2 = \mathbb{E}(X-\mu_1'+\mu_1'-c)^2 = \mathbb{E}(X-\mu_1')^2+(\mu_1'-c)^2-2(\mu_1'-c)\mathbb{E}(X-\mu_1') = \mathbb{E}(X-\mu_1')^2+(\mu_1'-c)^2$$

and hence,

$$\mathbb{E}(X - \mu_1')^2 = \inf_{c\in\mathbb{R}} \mathbb{E}(X - c)^2$$

Definition 6.33 (Variance)**.** The second central moment μ_2 of an RV X, if it exists, is called the variance of X and denoted by $Var(X)$. Note that $Var(X) = \mu_2 = \mathbb{E}\left[(X - \mu_1')^2\right]$.

Remark 6.34. The following are some simple observations about the variance of an RV X.

(a) We have

$$Var(X) = \mathbb{E}\left[(X - \mu_1')^2\right] = \mathbb{E}[X^2+(\mu_1')^2-2\mu_1'X] = \mu_2'-2(\mu_1')^2+(\mu_1')^2 = \mu_2'-(\mu_1')^2 = \mathbb{E}X^2-(\mathbb{E}X)^2.$$

(b) Since the RV $(X - \mu_1')^2$ takes non-negative values, we have $Var(X) = \mathbb{E}(X - \mu_1')^2 \geq 0$.

(c) We have $(\mathbb{E}X)^2 \leq \mathbb{E}X^2$.

(d) $Var(X) = 0$ if and only if $\mathbb{P}(X = \mu_1') = 1$. (see problem set 6).

(e) For any $a, b \in \mathbb{R}$, we have $Var(aX + b) = a^2 Var(X)$.

(f) Let $Var(X) > 0$. Then $Y := \frac{X - \mathbb{E}X}{\sqrt{Var(X)}}$ has the property that $\mathbb{E}Y = 0$ and $Var(Y) = 1$.

Definition 6.35 (Standard Deviation)**.** The quantity $\sigma(X) = \sqrt{Var(X)}$ is defined to be the standard deviation of X.

Example 6.36. Let X be a discrete RV with p.m.f.

$$f_X(x) := \begin{cases} \frac{1}{6}, \forall x \in \{1,2,3,4,5,6\} \\ 0, \text{ otherwise.} \end{cases}$$

Here, existence of $\mu_1' = \mathbb{E}X$ and $\mu_2' = \mathbb{E}X^2$ can be established by standard calculations. Moreover,

$$\mathbb{E}X = \sum_{x \in S_X} x f_X(x) = \frac{1}{6}(1+2+3+4+5+6) = \frac{7}{2}$$

and

$$\mathbb{E}X^2 = \sum_{x \in S_X} x^2 f_X(x) = \frac{1}{6}(1^2+2^2+3^2+4^2+5^2+6^2) = \frac{91}{6}.$$

Variance can now be computed using the relation $Var(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2$.

Example 6.37. In Example 6.13, we had shown $\mathbb{E}X = \frac{1}{2}$, where X is a continuous RV with the p.d.f.

$$f_X(x) = \begin{cases} 1, \text{ if } 0 < x < 1, \\ 0, \text{ otherwise.} \end{cases}$$

Now, $\mathbb{E}X^2 = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \int_0^1 x^2\,dx = \frac{1}{3}$. Then $Var(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$.

Note 6.38. We are familiar with the Laplace transform of a given real-valued function defined on $\mathbb{R}$. We also know that under certain conditions, the Laplace transform of a function determines the function almost uniquely. In probability theory, the Laplace transform of a p.m.f./p.d.f. of a random variable X plays an important role.

Let X be a discrete/continuous RV defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with DF F_X, p.m.f./p.d.f. f_X and support S_X.

Definition 6.39 (Moment Generating Function (MGF))**.** We say that the moment generating function (MGF) of X exists, denoted by M_X and equals $M_X(t) := \mathbb{E}e^{tX}$, provided $\mathbb{E}e^{tX}$ exists for all $t \in (-h, h)$, for some $h > 0$.

Note 6.40. Observe that $e^x > 0, \forall x \in \mathbb{R}$.

Note 6.41. If X is discrete/continuous with p.m.f./p.d.f. f_X, then following the definition of an expectation of an RV, we write

$$M_X(t) = \mathbb{E}e^{tX} = \begin{cases} \sum\limits_{x \in S_X} e^{tx} f_X(x), & \text{if } \sum\limits_{x \in S_X} e^{tx} f_X(x) < \infty \text{ for discrete } X, \forall t \in (-h, h) \text{ for some } h > 0 \\ \int_{-\infty}^{\infty} e^{tx} f_X(x)\, dx, & \text{if } \int_{-\infty}^{\infty} e^{tx} f_X(x)\, dx < \infty \text{ for continuous } X, t \in (-h, h) \text{ for some } h > 0. \end{cases}$$

In this case, we shall say that the MGF M_X exists on $(-h, h)$.

Remark 6.42. (a) $M_X(0) = 1$ and hence $A := \left\{t \in \mathbb{R} : \mathbb{E}[e^{tX}] \text{ is finite}\right\} \neq \emptyset$.

(b) $M_X(t) > 0\ \forall t \in A$, with A as above.

(c) For $c \in \mathbb{R}$, consider the constant/degenerate RV X given by the p.m.f. (see Example 6.8)

$$f_X(x) = \begin{cases} 1, & \text{if } x = c \\ 0, & \text{otherwise.} \end{cases}$$

Here, the support is $S_X = \{c\}$ and $M_X(t) = \mathbb{E}e^{tX} = \sum_{x \in S_X} e^{tx} f_X(x) = e^{tc}$ exists for all $t \in \mathbb{R}$.

(d) Suppose the MGF M_X exists on $(-h, h)$. Take constants $c, d \in \mathbb{R}$ with $c \neq 0$. Then, the RV $Y = cX + d$ is discrete/continuous, according to X being discrete/continuous and moreover,

$$M_Y(t) = \mathbb{E}e^{t(cX+d)} = e^{td} M_X(ct)$$

exists for all $t \in (-\frac{h}{|c|}, \frac{h}{|c|})$.

Note 6.43. The MGF can be used to compute the moments of an RV and this is the motivation behind the term 'Moment Generating Function'. This result is stated below. We skip the proof for brevity.

Theorem 6.44. *Let X be an RV with MGF M_X which exists on $(-h,h)$ for some $h>0$. Then, we have the following results.*

(a) $\mu_r' = \mathbb{E}[X^r]$ *is finite for each* $r \in \{1,2,\dots\}$.

(b) $\mu_r' = \mathbb{E}[X^r] = M_X^{(r)}(0)$, *where* $M_X^{(r)}(0) = \left[\frac{d^r}{dt^r} M_X(t)\right]_{t=0}$ *is the r-th derivative of* $M_X(t)$ *at the point* 0 *for each* $r \in \{1,2,\dots\}$.

(c) M_X *has the following Maclaurin's series expansion around* $t=0$ *of the following form* $M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r'$ *with* $t \in (-h,h)$.

Proposition 6.45. *Continue with the notations and assumptions of Theorem 6.44 and define* $\psi_X : (-h,h) \to \mathbb{R}$ *by* $\psi_X(t) := \ln M_X(t), t \in (-h,h)$. *Then*

$$\mu_1' = \mathbb{E}[X] = \psi_X^{(1)}(0) \quad \textit{and} \quad \mu_2 = Var(X) = \psi_X^{(2)}(0),$$

where $\psi_X^{(r)}$ *denotes the r-th* $(r \in \{1,2\})$ *derivative of* ψ_X.

Proof. We have, for $t \in (-h,h)$

$$\psi_X^{(1)}(t) = \frac{M_X^{(1)}(t)}{M_X(t)} \quad \text{and} \quad \psi_X^{(2)}(t) = \frac{M_X(t)M_X^{(2)}(t) - \left(M_X^{(1)}(t)\right)^2}{\left(M_X(t)\right)^2}.$$

Evaluating the above equalities at $t=0$ give the required results. □

Example 6.46. Let X be a discrete RV with p.m.f.

$$f_X(x) = \begin{cases} \frac{e^{-\lambda}\lambda^x}{x!}, & \text{if } x \in \{0,1,2,\dots\} \\ 0, & \text{otherwise,} \end{cases}$$

where $\lambda > 0$. We have

$$M_X(t) = \mathbb{E}\left[e^{tX}\right] = \sum_{x=0}^{\infty} e^{tx}\frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda}\sum_{x=0}^{\infty}\frac{\left(\lambda e^t\right)^x}{x!} = e^{-\lambda}e^{\lambda e^t} = e^{\lambda\left(e^t-1\right)} \;\forall\, t \in \mathbb{R}$$

since $A = \left\{t \in \mathbb{R} : \mathbb{E}\left(e^{tX}\right) < \infty\right\} = \mathbb{R}$. Now,

$$M_X^{(1)}(t) = \lambda e^t e^{\lambda\left(e^t - 1\right)} \quad \text{and} \quad M_X^{(2)}(t) = \lambda e^t e^{\lambda\left(e^t - 1\right)}\left(1 + \lambda e^t\right) \ \forall\, t \in \mathbb{R}.$$

Then,

$$\mu_1' = \mathbb{E}(X) = M_X^{(1)}(0) = \lambda,\ \mu_2' = \mathbb{E}(X^2) = M_X^{(2)}(0) = \lambda(1+\lambda),\ Var(X) = \mu_2 = \mu_2' - (\mu_1')^2 = \lambda.$$

Again, for $t \in \mathbb{R}$, $\psi_X(t) = \ln\left(M_X(t)\right) = \lambda\left(e^t - 1\right)$, which yields $\psi_X^{(1)}(t) = \psi_X^{(2)}(t) = \lambda e^t, \forall t \in \mathbb{R}$. Then, $\mu_1' = \mathbb{E}(X) = \lambda, \mu_2 = Var(X) = \lambda$. Higher order moments can be calculated by looking at higher order derivatives of M_X.

Example 6.47. Let X be a continuous RV with p.d.f.

$$f_X(x) = \begin{cases} e^{-x}, & \text{if } x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

We have

$$M_X(t) = \mathbb{E}\left(e^{tX}\right) = \int_0^\infty e^{tx} e^{-x}\, \mathrm{d}x = \int_0^\infty e^{-(1-t)x}\, \mathrm{d}x = (1-t)^{-1} < \infty, \text{ if } t < 1.$$

In particular, M_X exists on $(-1,1)$ and $A = \left\{t \in \mathbb{R} : \mathbb{E}\left(e^{tX}\right) < \infty\right\} = (-\infty, 1) \supset (-1,1)$. Now,

$$M_X^{(1)}(t) = (1-t)^{-2} \quad \text{and} \quad M_X^{(2)}(t) = 2(1-t)^{-3}, t < 1.$$

Then,

$$\mu_1' = \mathbb{E}(X) = M_X^{(1)}(0) = 1,\ \mu_2' = \mathbb{E}(X^2) = M_X^{(2)}(0) = 2,\ Var(X) = \mu_2 = \mu_2' - (\mu_1')^2 = 1.$$

Again, for $t < 1$, $\psi_X(t) = \ln\left(M_X(t)\right) = -\ln(1-t)$, which yields $\psi_X^{(1)}(t) = \frac{1}{1-t}, \psi_X^{(2)}(t) = \frac{1}{(1-t)^2}, \forall t < 1$. Then, $\mu_1' = \mathbb{E}(X) = 1, \mu_2 = Var(X) = 1$.

Now, consider the Maclaurin's series expansion for M_X around $t = 0$. We have

$$M_X(t) = (1-t)^{-1} = \sum_{r=0}^{\infty} t^r, \forall t \in (-1,1)$$

and hence $\mu_r' = r!$, which is the coefficient of $\frac{t^r}{r!}$ in the above power series.

Example 6.48. Let X be a continuous RV with p.d.f.

$$f_X(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2}, \forall x \in \mathbb{R}.$$

As observed earlier in Example 6.15, $\mathbb{E}X$ does not exist. Since the existence of moments is a necessary condition for the existence of MGF, we conclude that the MGF does not exist for this RV X.