MSO205 PRACTICE PROBLEMS SET 4

Question 1. Which of the following functions are distribution functions?

(a) $F_1(x) := \begin{cases} 0, \text{ if } x \leq 1, \\ 1, \text{ if } x > 1. \end{cases}$

(b) $F_2(x) := \begin{cases} 1, \text{ if } x < 1, \\ 0, \text{ if } x \geq 1. \end{cases}$

(c) $F_3(x) := \begin{cases} 0, \text{ if } x < 1, \\ 1, \text{ if } x \geq 1. \end{cases}$

(d) $F_4(x) := \begin{cases} 0, \text{ if } x < 0, \\ \frac{1}{4} + \frac{x}{2}, \text{ if } 0 \leq x \leq 1, \\ \frac{1}{2} + \frac{x}{4}, \text{ if } 1 < x < 2, \\ 1, \text{ if } x \geq 2. \end{cases}$.

Question 2. Let X be an RV defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with law $\mathbb{P}_X$ and DF F_X. Consider the set $D := \{x \in \mathbb{R} : F_X \text{ is discontinuous at } x\}$. Show that it is either finite or countably infinite. (Hint: for each $n = 1, 2, \cdots$, consider the set $D_n := \{x \in \mathbb{R} : F_X(x+) - F_X(x-) > \frac{1}{n}\} = \{x \in \mathbb{R} : F_X(x) - F_X(x-) > \frac{1}{n}\} = \{x \in \mathbb{R} : \mathbb{P}(X = x) > \frac{1}{n}\}$. Then $D = \cup_n D_n$. What can you say about D_n?)

Question 3. Is it possible that the following function $f : \mathbb{R} \to \mathbb{R}$ is a p.m.f. of an RV? If yes, also compute the corresponding DF. Take f as follows: for some constant $c \in \mathbb{R}$,

$$f(x) = \begin{cases} \frac{c}{(2x-1)(2x+1)}, \text{ if } x \in \{1, 2, 3, \cdots\}, \\ 0, \text{ otherwise.} \end{cases}$$

Question 4. Let X be a continuous RV with p.d.f. $f_X : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} -\frac{4}{3}x, \text{ if } x \in [-1, 0), \\ \frac{x^2}{8}, \text{ if } x \in [0, 2], \\ 0, \text{ otherwise.} \end{cases}$$

Write down the DF F_X. Compute $\mathbb{P}(X > 0), \mathbb{P}(X \leq 1)$ and $\mathbb{P}(X > 0 \mid X \leq 1)$.

Question 5. Suppose that an RV X has the DF F_X given by any of the following functions. In each case, check if X is discrete or continuous. If so, also find the corresponding p.m.f./p.d.f. (as appropriate) of X.

(a)

$$F_X(x) := \begin{cases} 0, \text{ if } x < 0, \\ \frac{x}{4}, \text{ if } 0 \leq x < 1, \\ \frac{x}{3}, \text{ if } 1 \leq x < 2, \\ \frac{3x}{8}, \text{ if } 2 \leq x < \frac{5}{2}, \\ 1, \text{ if } x \geq \frac{5}{2}. \end{cases}$$

(b)

$$F_X(x) = \begin{cases} 0, \text{ if } x < 0, \\ \frac{1}{2}, \text{ if } 0 \leq x < 2, \\ \frac{3}{4}, \text{ if } 2 \leq x < 3, \\ 1, \text{ if } x \geq 3. \end{cases}$$

(c)

$$F_X(x) := \begin{cases} 0, \text{ if } x < 0, \\ \frac{x^2}{2}, \text{ if } 0 \leq x < 1, \\ \frac{x}{2}, \text{ if } 1 \leq x < 2, \\ 1, \text{ if } x \geq 2. \end{cases}$$

Question 6. Verify that the following distribution function $F : \mathbb{R} \to \mathbb{R}$ corresponds to a discrete RV. Take

$$F(x) := \begin{cases} 0, \text{ if } x < 0, \\ \frac{1}{2}, \text{ if } 0 \leq x < 1, \\ 1, \text{ if } x \geq 1. \end{cases}$$

Compute the p.m.f. f_X.

Question 7. Verify that the following distribution function $F : \mathbb{R} \to \mathbb{R}$ corresponds to a continuous RV. Take

$$F(x) := \begin{cases} 0, \text{ if } x < 0, \\ \frac{x}{2}, \text{ if } 0 \leq x < 1, \\ \frac{1}{2}, \text{ if } 1 \leq x < 2, \\ \frac{x-1}{2}, \text{ if } 2 \leq x < 3, \\ 1, \text{ if } x \geq 3. \end{cases}$$

Compute the p.d.f. f_X.