MSO205: Introduction to Probability Theory
End Semester Examination (November 18, 2024)

8:00 am - 11:00 am (+60 minutes for DAP students),

Maximum Marks: 42, Minimum Marks: 0

Instructions:

a. Electronic devices (mobiles, calculators etc.) are prohibited.

b. Note the following p.d.fs of $\chi \sim \chi_n^2, T \sim t_n, F \sim F_{m,n}$ (m, n being positive integers):

$$f_\chi(x) = \begin{cases} \frac{1}{\Gamma(\frac{n}{2})} x^{\frac{n}{2}-1} 2^{-\frac{n}{2}} \exp(-\frac{x}{2}), \text{ if } x > 0, \\ 0, \text{ otherwise.} \end{cases}, \quad f_T(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{1}{2}\right)\sqrt{n}} \left(1 + \frac{x^2}{n}\right)^{-\frac{n+1}{2}}, \forall x \in \mathbb{R},$$

$$f_F(x) = \begin{cases} \frac{\Gamma(\frac{m+n}{2})}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})} \frac{m}{n} \left(\frac{m}{n}x\right)^{\frac{m}{2}-1} \left(1 + \frac{m}{n}x\right)^{-\frac{m+n}{2}}, \text{ if } x > 0, \\ 0, \text{ otherwise.} \end{cases}$$

1. Descriptive Type

Instructions: Provide all details while answering questions of this section.

Qn 1. (5 marks) It is known that the lifetime of a batch of electric bulbs follows the continuous probability distribution given by the p.d.f

$$f(x) = \begin{cases} \frac{1}{10} \exp(-\frac{x}{10}), \text{ if } x > 0, \\ 0, \text{ otherwise} \end{cases}.$$

In a box containing 100 electric bulbs from that batch, what is the probability (approximately) that 20 bulbs or more have a lifetime less or equal to $10 \ln 3$? Express the approximate value in terms of the DF of $N(0, 1)$ distribution.

Qn 2. (4 marks) Let X be a non-negative integer valued RV such that $\mathbb{E}X^2$ exists. Find the value of $\lim_{n\to\infty} n^2\, \mathbb{P}(X > n)$.

Qn 3. (1 + 5 marks) Let X be a 3-dimensional random vector. State and prove the non-decreasing property for the joint distribution function of X.

2. Short Answer Type

Instructions:

a. Answer all sub-questions in the same place. Only the final answer will be checked.

b. Not attempting carries 0 marks and incorrect answer carries negative (-2) marks in each sub-question and in each sub-question-parts, if any.

Qn 4 (i) (2 marks) Compute the mode for $Poisson(\sqrt{5})$ distribution.

Qn 4 (ii) (1 + 1 + 1 + 1 marks) An RV X has the MGF $M_X(t) = \frac{e^t}{4t}(e^{4t} - 1)$ for $t \neq 0$, and 1 if $t = 0$. Write the values of $\mathbb{P}(X > 2 | X < 3), \mathbb{E}X$ and $Var(X)$. Is $\mathbb{E}X > Var(X)$?

Qn 4 (iii) (3 marks) Fix $p_1 = p_2 = p_3 = \frac{1}{8}, p_4 = \frac{5}{8}$. Let the random vector $(X_1, X_2, X_3)^t$ have the joint p.m.f. given by

$$\mathbb{P}(X_1 = x_1, X_2 = x_2, X_3 = x_3) = \frac{20!}{x_1!x_2!x_3!(20 - \sum_{j=1}^3 x_j)!} p_1^{x_1} p_2^{x_2} p_3^{x_3} p_4^{20 - \sum_{j=1}^3 x_j},$$

if $x_1 + x_2 + x_3 \leq 20$ and 0 otherwise, where x_1, x_2, x_3 are non-negative integers. Write the value of correlation between X_2 & X_3.

Qn 4 (iv) (1 + 1 marks) There are 25 keys in a bunch and these are numbered $1, 2, \cdots, 25$. Except the numbers, the keys are identical. A person is told to open a lock, whose key is known to be there in the bunch. However, exactly one of the keys opens the lock and the person does not know the correct key number. The person tries the keys one by one by choosing one key at random, in each attempt, from one of the unused keys. Unsuccessful keys are not used for future attempts. Let X denote the number of attempts required to open the lock. Identify the distribution of X and write the value of $\mathbb{E}X$.

Qn 4 (v) (1.5 + 1.5 marks) Let $X \sim Poisson(2)$ and $Y \sim Poisson(3)$ be independent. Write the values of $\mathbb{E}[X \mid X+Y=3]$ and $Var[X \mid X+Y=3]$.

Qn 4 (vi) (2 + 1 marks) Let $X \sim N(1,4), Y \sim N(3,9), Z \sim N(-2,16)$ be independent RVs and set $W = 2Y - X$. Identify the distribution of W. Given that $\beta \frac{Z+2}{W-5} \sim t_1$ for some $\beta \in \mathbb{R}$. Write the value of $\beta^2 + 1$.

Qn 4 (vii) (1 + 1 + 2 marks) Let $\{X_n\}_n$ be a sequence of i.i.d. RVs defined on the same probability space.

(a) If $X_1 \sim Bernoulli(\frac{1}{4})$, then define $Y_n := \frac{1}{\sqrt{n}}\left(X_1 + X_2 + \cdots + X_n - \frac{n}{4}\right), \forall n$. It is known that $Y_n \xrightarrow[n\to\infty]{d} Z$. Identify the distribution of Z.

(b) If $X_1 \sim Beta(3,4)$, then define $Y_n := \frac{X_1+X_2+\cdots+X_n}{n}, \forall n$. It is known that $Y_n \xrightarrow[n\to\infty]{\mathbb{P}} V$. Identify the support of V.

(c) If $X_1 \sim N(-\sqrt{3}, 4)$, then define $Y_n := \sqrt{n}\left[\left(\frac{X_1+X_2+\cdots+X_n}{n}\right)^2 - 3\right], \forall n$. It is known that $Y_n \xrightarrow[n\to\infty]{d} W$. Write the value of $\mathbb{E}W^2$.

Qn 4 (viii) (3 marks) Let X be an RV with the distribution function F_X given by

$$F_X(x) = \begin{cases} 0, \text{ if } x < 0 \\ \alpha\, x + \frac{1}{5}, \text{ if } 0 \le x \le \frac{2}{3}, \\ \beta\, x^2, \text{ if } \frac{2}{3} < x \le \frac{2}{\sqrt{3}}, \\ 1, \text{ if } x > \frac{2}{\sqrt{3}}. \end{cases}$$

for some $\alpha, \beta \in \mathbb{R}$. Find the inter-quartile range. (Answers with α, β in the expression will not be accepted.)

Qn 4 (ix) (3 marks) Suppose the distribution of X is given by the p.d.f.

$$f_X(x) := \frac{1}{\pi(1+x^2)}, \forall x \in \mathbb{R}.$$

Then which of the following statement(s) is/are necessarily true? One or more than one options may be correct. Write down <u>ALL</u> correct option(s) to get full credit. Not selecting a correct option or selecting an incorrect option is taken as incorrect answer. No partial marking applies.

(a) $X \sim t_1$ (b) $X^2 \sim F_{1,1}$ (c) $\mathbb{E}|X|^{0.99} < \infty$ (d) $\mathbb{E}|X|^{1.01} < \infty$

(e) $\mathbb{E}|X|^{2.5} < \infty$ (f) The MGF of X exists. (g) $Var(X)$ does not exist.