MSO205 QUIZ 1 SOLUTIONS

Question 1 Is the following statement 'True' or 'False'? Underline/Tick 'True' or 'False', as appropriate, in the space designated below. [1 mark]

Statement: The probability mass function of a discrete RV can be uniquely determined from the distribution function.

Answer: True

Question 2 Consider the random experiment of drawing a card at random from a standard deck of cards. Write down the sample space. [1 mark]

Answer: The sample space is

$$\Omega = \{(s, i) : s \in \{clubsuit, diamondsuit, heartsuit, spadesuit\}, i \in \{1, 2, \cdots, 13\}\}.$$

Question 3

Set 1 Suppose a standard six-sided die is rolled at random. If an even integer is observed, we choose box A containing 2 white balls and 4 black balls. Otherwise, we choose box B containing 3 white balls and 4 red balls. All balls of the same colour are identical. Next, a ball is drawn at random from the chosen box.

(a) What is the probability that the ball drawn is white? Justify your answer. [2 marks]

(b) If the ball drawn is white, what is the probability that box A was chosen? Justify your answer. [2 marks]

You may write your answer as a simple fraction.

Answer: We first find the probability that the ball drawn is white. Let E denote this event.

Suppose A and B denote the events that the box A and the box B is chosen respectively. Then the events A and B are mutually exclusive and exhaustive. (statement of mutually exclusive and exhaustive events: part marks: 1)

We need to find $\mathbb{P}(E)$ and $\mathbb{P}(A \mid E)$.

Now,

$$\mathbb{P}(A) = \text{probability of obtaining an even integer in rolling the die} = \frac{3}{6} = \frac{1}{2}.$$

and $\mathbb{P}(B) = 1 - \mathbb{P}(A) = \frac{1}{2}$. Further,

$$\mathbb{P}(E \mid A) = \frac{2}{2+4} = \frac{1}{3}$$

and

$$\mathbb{P}(E \mid B) = \frac{3}{3+4} = \frac{3}{7}.$$

By the Theorem of Total Probability we have

$$\mathbb{P}(E) = \mathbb{P}(A)\,\mathbb{P}(E \mid A) + \mathbb{P}(B)\,\mathbb{P}(E \mid B) \text{ (this statement: part marks: 0.5)}$$

Therefore,

$$\mathbb{P}(E) = \frac{1}{2}\frac{1}{3} + \frac{1}{2}\frac{3}{7} = \frac{7+9}{42} = \frac{8}{21}. \text{ (computation: part marks: 0.5)}$$

To find $\mathbb{P}(A \mid E)$. By Bayes Theorem,

$$\mathbb{P}(A \mid E) = \frac{\mathbb{P}(A)\,\mathbb{P}(E \mid A)}{\mathbb{P}(E)} \text{ (this statement: part marks: 1)}$$

Therefore,

$$\mathbb{P}(A \mid E) = \frac{1}{2}\frac{1}{3}\frac{21}{8} = \frac{7}{16}. \text{ (computation: part marks: 1)}$$

Set 2 Suppose a standard six-sided die is rolled at random. If an even integer is observed, we choose box A containing 3 white balls and 4 black balls. Otherwise, we choose box B containing 5 white balls and 3 red balls. All balls of the same colour are identical. Next, a ball is drawn at random from the chosen box.

(a) What is the probability that the ball drawn is white? Justify your answer. [2 marks]

(b) If the ball drawn is white, what is the probability that box A was chosen? Justify your answer. [2 marks]

You may write your answer as a simple fraction.

Answer: We first find the probability that the ball drawn is white. Let E denote this event.

Suppose A and B denote the events that the box A and the box B is chosen respectively. Then the events A and B are mutually exclusive and exhaustive. (statement of mutually exclusive and exhaustive events: part marks: 1)

We need to find $\mathbb{P}(E)$ and $\mathbb{P}(A \mid E)$.

Now,

$$\mathbb{P}(A) = \text{probability of obtaining an even integer in rolling the die} = \frac{3}{6} = \frac{1}{2}.$$

and $\mathbb{P}(B) = 1 - \mathbb{P}(A) = \frac{1}{2}$. Further,

$$\mathbb{P}(E \mid A) = \frac{3}{3+4} = \frac{3}{7}$$

and

$$\mathbb{P}(E \mid B) = \frac{5}{5+3} = \frac{5}{8}.$$

By the Theorem of Total Probability we have

$\mathbb{P}(E) = \mathbb{P}(A)\,\mathbb{P}(E \mid A) + \mathbb{P}(B)\,\mathbb{P}(E \mid B)$ (this statement: part marks: 0.5)

Therefore,

$$\mathbb{P}(E) = \frac{1}{2}\frac{3}{7} + \frac{1}{2}\frac{5}{8} = \frac{24+35}{112} = \frac{59}{112}.$$ (computation: part marks: 0.5)

To find $\mathbb{P}(A \mid E)$. By Bayes Theorem,

$$\mathbb{P}(A \mid E) = \frac{\mathbb{P}(A)\,\mathbb{P}(E \mid A)}{\mathbb{P}(E)}$$ (this statement: part marks: 1)

Therefore,

$$\mathbb{P}(A \mid E) = \frac{1}{2}\frac{3}{7}\frac{112}{59} = \frac{24}{59}.$$ (computation: part marks: 1)

Set 3 Suppose a standard six-sided die is rolled at random. If an even integer is observed, we choose box A containing 2 white balls and 5 black balls. Otherwise, we choose box B containing 4 white balls and 4 red balls. All balls of the same colour are identical. Next, a ball is drawn at random from the chosen box.

(a) What is the probability that the ball drawn is white? Justify your answer. [2 marks]

(b) If the ball drawn is white, what is the probability that box A was chosen? Justify your answer. [2 marks]

You may write your answer as a simple fraction.

Answer: We first find the probability that the ball drawn is white. Let E denote this event.

Suppose A and B denote the events that the box A and the box B is chosen respectively. Then the events A and B are mutually exclusive and exhaustive. (statement of mutually exclusive and exhaustive events: part marks: 1)

We need to find $\mathbb{P}(E)$ and $\mathbb{P}(A \mid E)$.

Now,

$$\mathbb{P}(A) = \text{probability of obtaining an even integer in rolling the die} = \frac{3}{6} = \frac{1}{2}.$$

and $\mathbb{P}(B) = 1 - \mathbb{P}(A) = \frac{1}{2}$. Further,

$$\mathbb{P}(E \mid A) = \frac{2}{2+5} = \frac{2}{7}$$

and

$$\mathbb{P}(E \mid B) = \frac{4}{4+4} = \frac{1}{2}.$$

By the Theorem of Total Probability we have

$$\mathbb{P}(E) = \mathbb{P}(A)\,\mathbb{P}(E \mid A) + \mathbb{P}(B)\,\mathbb{P}(E \mid B) \text{ (this statement: part marks: 0.5)}$$

Therefore,

$$\mathbb{P}(E) = \frac{1}{2}\frac{2}{7} + \frac{1}{2}\frac{1}{2} = \frac{4+7}{28} = \frac{11}{28}. \text{ (computation: part marks: 0.5)}$$

To find $\mathbb{P}(A \mid E)$. By Bayes Theorem,

$$\mathbb{P}(A \mid E) = \frac{\mathbb{P}(A)\,\mathbb{P}(E \mid A)}{\mathbb{P}(E)} \text{ (this statement: part marks: 1)}$$

Therefore,

$$\mathbb{P}(A \mid E) = \frac{1}{2}\frac{2}{7}\frac{28}{11} = \frac{4}{11}. \text{ (computation: part marks: 1)}$$

Set 4 Suppose a standard six-sided die is rolled at random. If an even integer is observed, we choose box A containing 4 white balls and 3 black balls. Otherwise, we choose box B containing 2 white balls and 4 red balls. All balls of the same colour are identical. Next, a ball is drawn at random from the chosen box.

(a) What is the probability that the ball drawn is white? Justify your answer. [2 marks]

(b) If the ball drawn is white, what is the probability that box A was chosen? Justify your answer. [2 marks]

You may write your answer as a simple fraction.

Answer: We first find the probability that the ball drawn is white. Let E denote this event.

Suppose A and B denote the events that the box A and the box B is chosen respectively. Then the events A and B are mutually exclusive and exhaustive. (statement of mutually exclusive and exhaustive events: part marks: 1)

We need to find $\mathbb{P}(E)$ and $\mathbb{P}(A \mid E)$.

Now,

$$\mathbb{P}(A) = \text{probability of obtaining an even integer in rolling the die} = \frac{3}{6} = \frac{1}{2}.$$

and $\mathbb{P}(B) = 1 - \mathbb{P}(A) = \frac{1}{2}$. Further,

$$\mathbb{P}(E \mid A) = \frac{4}{4+3} = \frac{4}{7}$$

and

$$\mathbb{P}(E \mid B) = \frac{2}{2+4} = \frac{1}{3}.$$

By the Theorem of Total Probability we have

$\mathbb{P}(E) = \mathbb{P}(A)\,\mathbb{P}(E \mid A) + \mathbb{P}(B)\,\mathbb{P}(E \mid B)$ (this statement: part marks: 0.5)

Therefore,

$$\mathbb{P}(E) = \frac{1}{2}\frac{4}{7} + \frac{1}{2}\frac{1}{3} = \frac{12+7}{42} = \frac{19}{42}.$$ (computation: part marks: 0.5)

To find $\mathbb{P}(A \mid E)$. By Bayes Theorem,

$$\mathbb{P}(A \mid E) = \frac{\mathbb{P}(A)\,\mathbb{P}(E \mid A)}{\mathbb{P}(E)}$$ (this statement: part marks: 1)

Therefore,

$$\mathbb{P}(A \mid E) = \frac{1}{2}\frac{4}{7}\frac{42}{19} = \frac{12}{19}.$$ (computation: part marks: 1)

Question 4

Set 1: Let X be an RV with the following distribution function $F_X : \mathbb{R} \to [0,1]$ given by

$$F_X(x) = \begin{cases} 0, \text{ if } x < 0, \\ \frac{1}{5} + \frac{x}{4}, \text{ if } 0 \le x \le 2, \\ \frac{1}{2} + \frac{x-1}{5}, \text{ if } 2 < x < \frac{7}{2}, \\ 1, \text{ if } x \ge \frac{7}{2}. \end{cases}$$

(a) Identify the points of discontinuity of F_X, if any. Also compute the jumps at these points, if any. [4 marks]

(b) Is X discrete or continuous? Justify your answer. [2 marks]

(c) Compute $\mathbb{P}(X > 0)$ and $\mathbb{P}(X > 1 \mid X \le \frac{5}{2})$. [1 + 2 marks]

Answer: Since F_X is continuous on the intervals $(-\infty, 0), (0, 2), (2, \frac{7}{2})$ and $(\frac{7}{2}, \infty)$, discontinuities may arise only at the points $0, 2, \frac{7}{2}$. F_X is also right-continuous at all points. (part marks: 1)

We have $F_X(0-) = \lim_{h \downarrow 0} F_X(0-h) = 0$ and $F_X(0) = \frac{1}{5}$. Therefore F_X is discontinuous at 0 with jump $F_X(0) - F_X(0-) = \frac{1}{5}$.(part marks: 1)

We have $F_X(2-) = \lim_{h \downarrow 0} F_X(2-h) = \lim_{h \downarrow 0}[\frac{1}{5} + \frac{2-h}{4}] = \frac{1}{5} + \frac{1}{2}$ and $F_X(2) = \frac{1}{5} + \frac{1}{2}$.

Therefore F_X is continuous at 2.(part marks: 1)

We have $F_X(\frac{7}{2}-) = \lim_{h\downarrow 0} F_X(\frac{7}{2}-h) = \lim_{h\downarrow 0}[\frac{1}{2}+\frac{\frac{7}{2}-h-1}{5}] = 1$ and $F(\frac{7}{2}) = 1$. Therefore F_X is continuous at $\frac{7}{2}$.(part marks: 1)

Only discontinuity of F_X is at the point 0. In particular, $\mathbb{P}(X = 0) = F_X(0) - F_X(0-) = \frac{1}{5}$. At all other points F_X is continuous and hence $\mathbb{P}(X = x) = 0, \forall x \neq 0$.

Here,

$$\sum_{x\in D} \mathbb{P}(X = x) = \mathbb{P}(X = 0) = \frac{1}{5} \neq 1,$$

with $D = \{0\}$ as the set of discontinuities of F. This RV X is not discrete.(part marks: 1)

Since, F_X has a discontinuity at the point 0. Therefore, an RV X with DF F_X is not a continuous RV.(part marks: 1)

Therefore, X is neither discrete nor continuous.

We have

$$\mathbb{P}(X > 0) = 1 - F_X(0) = 1 - \frac{1}{5} = \frac{4}{5},$$

$$\mathbb{P}(X \leq \frac{5}{2}) = F_X(\frac{5}{2}) = \frac{1}{2} + \frac{3}{10} = \frac{4}{5},$$

$$\mathbb{P}(X > 1 \mid X \leq \frac{5}{2}) = \frac{\mathbb{P}((X > 1) \cap (X \leq \frac{5}{2}))}{\mathbb{P}(X \leq \frac{5}{2})} = \frac{5}{4}\mathbb{P}(1 < X \leq \frac{5}{2}) = \frac{5}{4}[F_X(\frac{5}{2}) - F_X(1)] = \frac{5}{4}\frac{7}{20} = \frac{7}{16}.$$

Set 2: Let X be an RV with the following distribution function $F_X : \mathbb{R} \to [0,1]$ given by

$$F_X(x) = \begin{cases} 0, \text{ if } x < 0, \\ \frac{1}{4} + \frac{x}{5}, \text{ if } 0 \leq x \leq \frac{5}{2}, \\ \frac{1}{2} + \frac{2x-3}{8}, \text{ if } \frac{5}{2} < x < \frac{7}{2}, \\ 1, \text{ if } x \geq \frac{7}{2}. \end{cases}$$

(a) Identify the points of discontinuity of F_X, if any. Also compute the jumps at these points, if any. [4 marks]

(b) Is X discrete or continuous? Justify your answer. [2 marks]

(c) Compute $\mathbb{P}(X > 0)$ and $\mathbb{P}(X > 1 \mid X \leq \frac{5}{2})$. [1 + 2 marks]

Answer: Since F_X is continuous on the intervals $(-\infty, 0), (0, \frac{5}{2}), (\frac{5}{2}, \frac{7}{2})$ and $(\frac{7}{2}, \infty)$, discontinuities may arise only at the points $0, \frac{5}{2}, \frac{7}{2}$. F_X is also right-continuous at all points.(part marks: 1)

We have $F_X(0-) = \lim_{h\downarrow 0} F_X(0-h) = 0$ and $F_X(0) = \frac{1}{4}$. Therefore F_X is discontinuous at 0 with jump $F_X(0) - F_X(0-) = \frac{1}{4}$.(part marks: 1)

We have $F_X(\frac{5}{2}-) = \lim_{h\downarrow 0} F_X(\frac{5}{2}-h) = \lim_{h\downarrow 0}[\frac{1}{4} + \frac{\frac{5}{2}-h}{5}] = \frac{1}{4} + \frac{1}{2}$ and $F_X(\frac{5}{2}) = \frac{1}{4} + \frac{1}{2}$. Therefore F_X is continuous at $\frac{5}{2}$.(part marks: 1)

We have $F_X(\frac{7}{2}-) = \lim_{h\downarrow 0} F_X(\frac{7}{2}-h) = \lim_{h\downarrow 0}[\frac{1}{2} + \frac{7-2h-3}{8}] = 1$ and $F(\frac{7}{2}) = 1$. Therefore F_X is continuous at $\frac{7}{2}$.(part marks: 1)

Only discontinuity of F_X is at the point 0. In particular, $\mathbb{P}(X=0) = F_X(0) - F_X(0-) = \frac{1}{4}$. At all other points F_X is continuous and hence $\mathbb{P}(X=x) = 0, \forall x \neq 0$.

Here,

$$\sum_{x\in D} \mathbb{P}(X=x) = \mathbb{P}(X=0) = \frac{1}{4} \neq 1,$$

with $D = \{0\}$ as the set of discontinuities of F. This RV X is not discrete.(part marks: 1)

Since, F_X has a discontinuity at the point 0. Therefore, an RV X with DF F_X is not a continuous RV.(part marks: 1)

Therefore, X is neither discrete nor continuous.

We have

$$\mathbb{P}(X>0) = 1 - F_X(0) = 1 - \frac{1}{4} = \frac{3}{4},$$

$$\mathbb{P}(X \leq \frac{5}{2}) = F_X(\frac{5}{2}) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4},$$

$$\mathbb{P}(X>1 \mid X \leq \frac{5}{2}) = \frac{\mathbb{P}((X>1)\cap(X\leq\frac{5}{2}))}{\mathbb{P}(X\leq\frac{5}{2})} = \frac{4}{3}\mathbb{P}(1<X\leq\frac{5}{2}) = \frac{4}{3}[F_X(\frac{5}{2}) - F_X(1)] = \frac{4}{3}\frac{3}{10} = \frac{2}{5}.$$

Set 3: Let X be an RV with the following distribution function $F_X : \mathbb{R} \to [0,1]$ given by

$$F_X(x) = \begin{cases} 0, \text{ if } x<0, \\ \frac{2}{5} + \frac{x}{4}, \text{ if } 0 \leq x \leq 2, \\ \frac{1}{2} + \frac{x+2}{10}, \text{ if } 2<x<3, \\ 1, \text{ if } x \geq 3. \end{cases}$$

(a) Identify the points of discontinuity of F_X, if any. Also compute the jumps at these points, if any. [4 marks]

(b) Is X discrete or continuous? Justify your answer. [2 marks]

(c) Compute $\mathbb{P}(X>0)$ and $\mathbb{P}(X>1 \mid X \leq \frac{5}{2})$. [1 + 2 marks]

Answer: Since F_X is continuous on the intervals $(-\infty,0),(0,2),(2,3)$ and $(3,\infty)$, discontinuities may arise only at the points $0,2,3$. F_X is also right-continuous at all points.(part marks: 1)

We have $F_X(0-) = \lim_{h\downarrow 0} F_X(0-h) = 0$ and $F_X(0) = \frac{2}{5}$. Therefore F_X is discontinuous at 0 with jump $F_X(0) - F_X(0-) = \frac{2}{5}$.(part marks: 1)

We have $F_X(2-) = \lim_{h \downarrow 0} F_X(2-h) = \lim_{h \downarrow 0}[\frac{2}{5} + \frac{2-h}{4}] = \frac{2}{5} + \frac{1}{2}$ and $F_X(2) = \frac{2}{5} + \frac{1}{2}$. Therefore F_X is continuous at 2.(part marks: 1)

We have $F_X(3-) = \lim_{h \downarrow 0} F_X(3-h) = \lim_{h \downarrow 0}[\frac{1}{2} + \frac{3-h+2}{10}] = 1$ and $F(3) = 1$. Therefore F_X is continuous at 3.(part marks: 1)

Only discontinuity of F_X is at the point 0. In particular, $\mathbb{P}(X = 0) = F_X(0) - F_X(0-) = \frac{2}{5}$. At all other points F_X is continuous and hence $\mathbb{P}(X = x) = 0, \forall x \neq 0$.

Here,

$$\sum_{x \in D} \mathbb{P}(X = x) = \mathbb{P}(X = 0) = \frac{2}{5} \neq 1,$$

with $D = \{0\}$ as the set of discontinuities of F. This RV X is not discrete.(part marks: 1)

Since, F_X has a discontinuity at the point 0. Therefore, an RV X with DF F_X is not a continuous RV.(part marks: 1)

Therefore, X is neither discrete nor continuous.

We have

$$\mathbb{P}(X > 0) = 1 - F_X(0) = 1 - \frac{2}{5} = \frac{3}{5},$$

$$\mathbb{P}(X \leq \frac{5}{2}) = F_X(\frac{5}{2}) = \frac{1}{2} + \frac{9}{20} = \frac{19}{20},$$

$$\mathbb{P}(X > 1 \mid X \leq \frac{5}{2}) = \frac{\mathbb{P}((X > 1) \cap (X \leq \frac{5}{2}))}{\mathbb{P}(X \leq \frac{5}{2})} = \frac{20}{19}\mathbb{P}(1 < X \leq \frac{5}{2}) = \frac{20}{19}[F_X(\frac{5}{2}) - F_X(1)] = \frac{20}{19}\frac{6}{20} = \frac{6}{19}.$$

Set 4: Let X be an RV with the following distribution function $F_X : \mathbb{R} \to [0, 1]$ given by

$$F_X(x) = \begin{cases} 0, \text{ if } x < 0, \\ \frac{1}{5} + \frac{x}{4}, \text{ if } 0 \leq x \leq 1, \\ \frac{1}{4} + \frac{x}{5}, \text{ if } 1 < x < \frac{15}{4}, \\ 1, \text{ if } x \geq \frac{15}{4}. \end{cases}$$

(a) Identify the points of discontinuity of F_X, if any. Also compute the jumps at these points, if any. [4 marks]

(b) Is X discrete or continuous? Justify your answer. [2 marks]

(c) Compute $\mathbb{P}(X > 0)$ and $\mathbb{P}(X > 1 \mid X \leq \frac{5}{2})$. [1 + 2 marks]

Answer: Since F_X is continuous on the intervals $(-\infty, 0), (0, 1), (1, \frac{15}{4})$ and $(\frac{15}{4}, \infty)$, discontinuities may arise only at the points $0, 1, \frac{15}{4}$. F_X is also right-continuous at all points.(part marks: 1)

We have $F_X(0-) = \lim_{h\downarrow 0} F_X(0-h) = 0$ and $F_X(0) = \frac{1}{5}$. Therefore F_X is discontinuous at 0 with jump $F_X(0) - F_X(0-) = \frac{1}{5}$.(part marks: 1)
We have $F_X(1-) = \lim_{h\downarrow 0} F_X(1-h) = \lim_{h\downarrow 0}[\frac{1}{5} + \frac{1-h}{4}] = \frac{1}{5} + \frac{1}{4}$ and $F_X(1) = \frac{1}{5} + \frac{1}{4}$. Therefore F_X is continuous at 1.(part marks: 1)
We have $F_X(\frac{15}{4}-) = \lim_{h\downarrow 0} F_X(\frac{15}{4} - h) = \lim_{h\downarrow 0}[\frac{1}{4} + \frac{\frac{15}{4}-h}{5}] = 1$ and $F(\frac{15}{4}) = 1$. Therefore F_X is continuous at $\frac{15}{4}$.(part marks: 1)
Only discontinuity of F_X is at the point 0. In particular, $\mathbb{P}(X = 0) = F_X(0) - F_X(0-) = \frac{1}{5}$. At all other points F_X is continuous and hence $\mathbb{P}(X = x) = 0, \forall x \neq 0$.
Here,

$$\sum_{x\in D} \mathbb{P}(X = x) = \mathbb{P}(X = 0) = \frac{1}{5} \neq 1,$$

with $D = \{0\}$ as the set of discontinuities of F. This RV X is not discrete.(part marks: 1)
Since, F_X has a discontinuity at the point 0. Therefore, an RV X with DF F_X is not a continuous RV.(part marks: 1)
Therefore, X is neither discrete nor continuous.
We have

$$\mathbb{P}(X > 0) = 1 - F_X(0) = 1 - \frac{1}{5} = \frac{4}{5},$$

$$\mathbb{P}(X \leq \frac{5}{2}) = F_X(\frac{5}{2}) = \frac{1}{4} + \frac{1}{2} = \frac{3}{4},$$

$$\mathbb{P}(X > 1 \mid X \leq \frac{5}{2}) = \frac{\mathbb{P}((X > 1) \cap (X \leq \frac{5}{2}))}{\mathbb{P}(X \leq \frac{5}{2})} = \frac{4}{3}\mathbb{P}(1 < X \leq \frac{5}{2}) = \frac{4}{3}[F_X(\frac{5}{2}) - F_X(1)] = \frac{4}{3}\frac{3}{10} = \frac{2}{5}.$$