MSO205 PRACTICE PROBLEMS SET 8

Question 1. Let X be a continuous RV with $\mathbb{P}(X > 0) = 1$ and such that $\mu_1' = \mathbb{E}X$ exists. Prove that $\mathbb{P}(X > 2\mu_1') \leq \frac{1}{2}$.

Question 2. Let $x_1, x_2, \cdots, x_k > 0$ be distinct real numbers and let n be a positive integer. Using Jensen's inequality discussed in the lecture notes, show that

$$\left(\frac{x_1 + x_2 + \cdots + x_k}{k}\right)^n \leq \frac{x_1^n + x_2^n + \cdots + x_k^n}{k}$$

Question 3. Let $x_1, x_2, \cdots, x_k, p_1, p_2, \cdots, p_k > 0$ be such that $\sum_{i=1}^k p_i = 1$. Prove the classical AM-GM-HM inequality using the AM-GM-HM inequality for RVs discussed in the lecture notes,

$$\sum_{i=1}^k x_i p_i \geq \prod_{i=1}^k x_i^{p_i} \geq \frac{1}{\sum_{i=1}^k \frac{p_i}{x_i}}$$

Question 4. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $X = (X_1, X_2, X_3) : \Omega \to \mathbb{R}^3$ be a 3-dimensional random vector. State and prove the non-decreasing property of the joint DF of X.

Question 5. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $A, B \in \mathcal{F}$. Define RVs $X, Y : \Omega \to \mathbb{R}$ by

$$X(\omega) = 1_A(\omega) = \begin{cases} 1, & \text{if } \omega \in A, \\ 0, & \text{otherwise} \end{cases}, \quad Y(\omega) = 1_B(\omega), \forall \omega \in \Omega.$$

Show that RVs X, Y are independent if and only if events A, B are independent.

Question 6. Let $X = (X_1, X_2, \cdots, X_p)$ be a discrete random vector with joint DF F_X, joint p.m.f. f_X and support S_X. Let f_{X_j} denote the marginal p.m.f. of X_j. If $X_1, X_2, \cdots, X_p$ are independent, then show that

$$f_{X_1, X_2, \cdots, X_p}(x_1, x_2, \cdots, x_p) = \prod_{j=1}^p f_{X_j}(x_j), \forall x_1, x_2, \cdots, x_p \in \mathbb{R}.$$