Real Analysis

• **Peano's Axioms :-**

The properties of the natural numbers are called the peano's Axioms.

• $1 \in N$, i.e; 1 is a natural number and 'N' is a non-empty set.

• for each element $n \in N$, there is a unique element $n_0 \in N$ called successor of small 'n' i.e; each element of N as a successor of N.

• For each $n \in N$ and $n_0 \neq 1$, 1 is not the successor of any element of N.

• For every $n, m \in N$, with $n \neq m$ and $m_0 \neq n_0$ distinct element in 'N' has distinct successor i.e each successor is unique.

• if 'A' is a subset or equal $A \subseteq N$ & $1 \in A$ and, Some $A \subseteq N$, $1 \in A$ and $P \in A \Rightarrow P_0 \in A$, then $A = N$.

• When we add 2 natural number we get a natural number but inverse operatⁿ (-) i.e substraction is not possible always if the domain is a set of Natural Number.

Thereafter we use these facts without notice,

$a + x = b$, $\forall a, b \in N$ for $a = b$ or $a > b$ there is no solution in domain N.

This equatⁿ gives solution within the domain N for only $a < b$, we consider the equatⁿ $x + 3 = 2$ and its solⁿ is not in domain 'N'.

- So, -ve integers $I = -1, -2, \ldots\ldots$
  & $a + (-a) = 0 \ \forall a \ N$, are included in domain I. which constitute the system of integer.

- The concept of fract^n for +ve & -ve numbers and 0 is called system of rational no. for eqt^n $ax = b \ \forall a \in I$ & $a \neq b$ there, system of rational number is denoted by "Q" is denoted by p/q, where $q \neq 0$. $p, q \in I$

- The Process of extracting the root, there is a no. Q whose square

- The system of rational no. is further extended by introduct^n of so called irrational number ex; $\sqrt{2}, \sqrt{3}, \pi$ --

- The union or combination of rational or irrational number is the Real number and the set of real no. is also called real line.

(i) $\sqrt{2}$ is irrational (Prove)

Sol^n: To Prove: $\sqrt{2}$ is a irrational number.

Sol^n: let us suppose the $\sqrt{2}$ is a rational number, then it can be written as p/q form, where p & q are relatively prime that are not both even no.'s.
In other ways, we can say that p & q are relative Prime.
i.e. the greatest common divisor is 1.

Thus, we have

$$\frac{p}{q} = \sqrt{2} \qquad (q \neq 0$$

$$q\sqrt{2} = p \Rightarrow \ 2q^2 = p^2$$

This shows p^2 is even $\Rightarrow$ p is also even

So that p^2 is divisible by 4. & q^2 is also even $\Rightarrow$ the q is even. Now both p & q are even it contradict our assumption

$\therefore$ $\sqrt{2}$ is a irrational number

$funct^n$.

let A & B be any two sets. A $funct^n$ f on A into B is a subset of $A \times B$ (and hence is a set of ordered pair $\langle a,b \rangle$) with the property that each $a \in A$ belongs to precisely one Pair $\langle a,b \rangle$.

Instead of $\langle x,y \rangle \in f$ we usually write $y = f(x)$, then y is called the image of x under f.

function

one-one

- $f(x_1) = f(x_2)$

iff $x_1 = x_2$

- ex; $f(x) = e^x$

$f(x) = \ln x$

many-one

ex; $f(x) = x^2$

$f(x) = |x|$

onto

Co-domain

$=$ ← equal to

Range

into

Co - domain

$\neq$ ← not equal to

Range

Thrm 1: The image of the union of two set is the union of images. or.

if $f: A \to B$ & $x \subset A$ and $y \subset B$.

then $f(x \cup y) = f(x) \cup f(y)$

Proof: (i) $f(x \cup y) \subseteq f(x) \cup f(y)$

(ii) $f(x) \cup f(y) \subseteq f(x \cup y)$

Part 1: $f(x \cup y) \subseteq f(x) \cup f(y)$

let b belongs to $f(x \cup y)$, then $b \in f(a)$ for some $a \in X \cup Y$.

$\therefore$ $a \in x$ or $a \in y$.

then $f(a) \in f(x)$ or $f(a) \in f(y)$

$b \in f(x)$ or $b \in f(y)$

$f(a) \in f(x) \cup f(y)$

$f(x \cup y) \in f(x) \cup f(y)$

$f(x \cup y) \subseteq f(x) \cup f(y)$ (i)

Part 2: $f(x) \cup f(y) \subseteq f(x \cup y)$

let $c \in f(x) \cup f(y)$

$\Rightarrow c \in f(x)$ or $c \in f(y)$

$\Rightarrow$ c is the image of some point in X

" " " " Y

" " " " $X \cup Y$

$\therefore$ $c \in f(X \cup X)$

$\Rightarrow \quad f(x) \cup f(y) \subseteq f(x \cup y)$ (ii)

from (i) & (ii) we conclude that.

$$f(x \cup y) = f(x) \cup f(y).$$

Thm 2: The inverse image of the union of two sets is the union of the inverse of images.

$x \subset A$ & $y \subset A$ then $f^{-1}(x \cup y) = f^{-1}(x) \cup f^{-1}(y)$

Proof $f^{-1}(x \cup y) = f^{-1}(x) \cup f^{-1}(y)$

$$\begin{aligned} a \in f^{-1}(x \cup y) &\Leftrightarrow f(a) \in x \cup y \\ &\Leftrightarrow f(a) \in x \text{ or } f(a) \in y \\ &\Leftrightarrow a \in f^{-1}(x) \text{ or } a \in f^{-1}(y) \\ &\Leftrightarrow a \in f^{-1}(x) \cup f^{-1}(y) \end{aligned}$$

Thm 3: If $f: A \to B$ if $x \subset B$, $y \in B$, then

$$f^{-1}(x \cap y) = f^{-1}(x) \cap f^{-1}(y)$$

Proof: $f^{-1}(x \cap y) = f^{-1}(x) \cap f^{-1}(y)$

$$\begin{aligned} \text{let } b \in f^{-1}(x \cap y) &\Leftrightarrow f(b) \in x \cap y \\ &\Leftrightarrow f(b) \in y \text{ \& } f(b) \in x \\ &\Leftrightarrow b \in f^{-1}(y) \text{ \& } b \in f^{-1}(x) \\ &\Leftrightarrow b \in f^{-1}(x \cap y) \end{aligned}$$

$$\therefore f^{-1}(x \cap y) = f^{-1}(x) \cap f^{-1}(y)$$

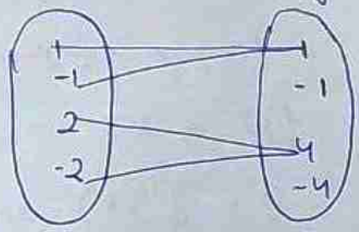
Thrm 4: It is nessarily true that the image of the intersection of two set is equal to the intersect^n of images or if $f: A \to B$ and $x \subset A$ & $y \subset A$ then.

$$f(x \cap y) \subset f(x) \cap f(y)$$

but the converse is not true . necessarily.

$$f(x) \cap f(y) \not\subset f(x \cap y)$$

~~Cont~~ Counter example : $f(n) = n^2$



Countable sets:

• A set 'A' is said to be countable if A is equivalent to $\mathbb{N}$, $A \equiv \mathbb{N}$

i.e if 'f' is one-one and onto from $\mathbb{N} \to A$, then the set A is countable.

ex; • the set of Powers of 3 $\{1, 3, 9, \ldots\ldots 3^n\}$.

• set of all +ve even or odd number is countable

• The set of all ordered pair of integer.

Countable set $f: \mathbb{N} \to A$ & bijective

Equivalent sets: Same number of elements.

Equal sets: Same elements.

Thrm 3: If $A_1, A_2, A_3, ----$ are countable sets, then $\bigcup_{n=1}^{\infty} A_n$ is countable i.e; the union of countable sets are countable.

Proof: Since, $A_1, A_2, A_3 ---$ are countable set and every set A_n is Assigned in a seq^n $\{a_{nk}\}$ $k = 1, 2, ---$.

we may write as follows.

$A_1 = \{a_{11} \quad a_{12} \quad a_{13} ---- a_{1n}\}$

$A_2 = \{a_{21} \quad a_{22} \quad a_{23} ---- a_{2n}\}$

$A_3 = \{a_{31} \quad a_{32} \quad a_{33} ---- a_{3n}\}$

|

|

$A_n = \{a_{n1}, a_{n2} \quad a_{n3} ----. a_{nn}\}$

now, we consider elements of the above sets. there is only one element a_{11} whose subscript sum is 2, there are 2 element a_{21}, a_{12} whose subscript sum is 3 ~~em~~, there are 3 elements whose subscript sum is 4 a_{31}, a_{22}, a_{13}

explain

and so on. we observe that for any +ve integer m (sum of subscript) ≥ 2, there are only $m-1$ elements. now, we arrange them in order of the sum of subscript

$a_{11} : a_{21}, a_{12} : a_{13}, a_{23} : a_{31} ----$

we have removed all the element a_{nk} which ~~are~~ have counted already.

$\therefore$ we established a one-one & onto mapping b/w natural no.'s and $\bigcup_{n=1}^{\infty} A_n$ which proves that.

$$N \sim \bigcup_{n=1}^{\infty} A_n$$

Thus $\bigcup_{n=1}^{\infty} A_n$ is countable.

Thrm 6: The set of all rational no.'s are countable

Proof: It is sufficient to show that the set of all positive rational no's is countable, then the collect^n of +ve & -ve & zero can be represented as a single list.

i.e $Q^+ = \left\{ \frac{m}{n}, m, n \in N \right\}$, where fract^n $\frac{n}{m}$ are written in the lowest terms arrange. all ratios in an infinite matrix $\left[\frac{n}{m}\right]_{n,m=1}^{\infty}$

$$\begin{bmatrix} 1/1 & 1/2 & 1/3 & 1/4 & 1/5 & \cdots \\ 2/1 & 2/2 & 2/3 & 2/4 & 2/5 & \cdots \\ 3/1 & 3/2 & 3/3 & 3/4 & 3/5 & \cdots \\ \vdots & \cdots & \cdots & \cdots & \cdots & \end{bmatrix}$$

and defined a map $f: N \to Q^+$ from the above sloping diagonals in successions.

now start at the lefthand corner and move along the north-east diagonal of these table

i.e; start with $1/1$ then move ~~along~~ ~~the~~ toward $2/1$ than $1/2$, again process to $3/1$, $2/2$ and move along $2/2$ and $1/3$ and so on.

when we hit a repitition [$2/2 = 1/1$] skip it.

$f(1) = 1/1 = 1$ $\quad f(5) = \frac{1}{3}$ $\quad f(9) = 1/4$

$f(2) = 2/1 = 2$ $\quad f(6) = 4/1 = 4$ $\quad f(10) = 5/1$

$f(3) = 1/2$. $\quad f(7) = 3/2$ $\quad f(11) = - - - -$

$f(4) = 3/1 = 3$ $\quad f(8) = 2/3$

skipped $\{ 4/2 = f(2), \; 3/3 = f(1), \; 2/6 = \frac{1}{3} = f(5), - - - \}$

clearly f is a bijective mapping thus $N \sim Q^+$

$\therefore$ The set of all rational numbers is countable.

Thrm 7: If N is a set of natural numbers then $N \times N$ is countable.

Proof: we consider the following collections of set is countable.

$A_1 = \{ (1,1) \quad (1,2) \quad (1,3) - - - - (1,n) - - - \}$

$A_2 = \{ (2,1) \quad (2,2) \quad (2,3) - - - - (2,n) - - - \}$

$A_3 = \{ (3,1) \quad (3,2) \quad (3,3) - - - - (3,n) - - - \}$

$\vdots$

$A_n = \{ (n,1) \quad (n,2) \quad (n,3) - - - (n,n) - - - - \}$

let funct^n $\boxed{f : N \to A}$ be obtained/defined by $f(n,m) = m$, clearly it is bijective.

Thus A_n is equivalent to N i.e; $N \sim A_n$.

~~By the~~ $\therefore A_n$ is countable

By the theorem " union of countable set is countable "

we have $N \times N = \cup \{A_n ; n \in N\}$.

$\therefore$ $N \times N$ itself is countable set.

Uncountable set: A set which is not countable is called uncountable set

ex; $x = \frac{1}{2} = \frac{5}{10} = 0.500000 = 0.499999\ldots$

" Shows that two decimals are distinct, that doesn't necessarily means that they expressed distinct real no.'s "

Thrm 8: The set $[0, 1]$ is uncountable.

Proof: By contradiction we prove $[0,1]$ is countable. let us suppose that $[0, 1]$ is a countable set i.e; all the real no.'s lying in the segment $[0, 1]$ can be written/ expressed in the form of an infinite decimal expansion and it can be arranged in the sequence as follows:.

$[0, 1] = \{x_1, x_2, x_3, \ldots\ldots, x_n\}$.

for ex; instead of 0.19999 we would use $0.2000\ldots$

$\therefore$ we can express x_i as follows.

$u_1 = 0 \cdot a_{11}\, a_{12}\, a_{13} \ldots\ldots$

$u_2 = 0 \cdot a_{21}\, a_{22}\, a_{23} \ldots\ldots$

$u_3 = 0 \cdot a_{31}\, a_{32}\, a_{33} \ldots\ldots$

$\vdots$

$u_n = 0 \cdot a_{n1}\, a_{n2}\, a_{n3} \ldots\ldots$

. where each a_{ij} is ~~of~~ one of the no.'s ~~$\{1,2,3,4,5,6,7,8,9\}$~~ $\{0,1,2,3,4,5,6,7,8,9\}$.

now, we'll construct a real no.'s

$y = 0 \cdot b_1\, b_2\, b_3 \ldots\ldots b_n \ldots$

. we choose each integer $b_n = \{n ; n \in \{0, 1, 2, \ldots 9\}\}$. Then choose another integer from 1 to 9 s.t $b_1 \neq a_{11}$. and proceeding like then we have $b_n \neq a_{nn}$ $b_2 \neq a_{22}$

. since $b_1 \neq a_{11}$ it follows the $y \neq u_1$ and $b_2 \neq a_{22}$ for each decimal places n. It means means $y \neq u_2$ the decimal expansion of y is different from the decimal expansion of $u_1, u_2, u_3, \ldots u_n, \ldots$ listed above.

$\therefore$ the decimal expansion of y is unique, since $y \in [0,1]$ this contradict the assumption that every (real no. / element) in $[0,1]$ can be listed as $u_1, u_2, u_3 \ldots u_n, \ldots$.

. Thus the assumpt^n that $[0,1]$ is countable is wrong.

$\therefore$ $[0,1]$ is not countable.

Thm 9: The set of all ~~real~~ Irrational no's are uncountable

Proof: Suppose, S is a set of irrational no. and countable. we know that the set $\mathbb{Q}$ of rational no.s is countable.

$\therefore$ $\mathbb{R} = S \cup \mathbb{Q}$ is also countable, but R is uncountable

Hence, $S = \mathbb{R} - \mathbb{Q}$ is also uncountable.

Thm 10: The set of all real no's are uncountable.

Proof: let us suppose, R is a countable set & $[0, 1]$ is an infinite subset of $\mathbb{R}$. Since, every infinite subset of a countable set is countable, then $[0, 1]$ is also countable. This is contradicting the theorem "8" mention the theorem statement
hence, the set $\mathbb{R}$ is uncountable.

Bounded below: A set $S \subset \mathbb{R}$ of real no's is said to bounded below if there exist a real no. k s.t.

$$k \le x \quad \forall x \in S.$$

- The real no. k is called a lower bound of S

ex; $\mathbb{N} = \{1, 2, 3, \ldots\}$ i.e $1 \le x \quad x \in \mathbb{N}$

Bounded Above: The set $S \subset \mathbb{R}$ of real no.s is said to bounded above if $\exists$ a real no. l s.t

$$x \le l \quad \forall x \in S$$

- The real no. l is called a ~~a~~ upper bound of S.

ex; $I = \{-1, -2, -3, \ldots\}$ i.e $x \le -1 \quad \forall x \in I$

Bounded set: A set is said to be bounded if it is bounded above as well as bounded below.

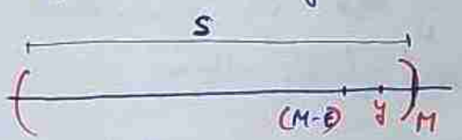
So. a set S is bounded if $\exists$ k, K $\in$ S s.t.

Least upper bound (sup): The smallest member in the set of all upper bound of a set is called the L.U.B or supremum (sup)

Properties: S = M equivalent to hold the next (i) & (ii)

(i) M is an upper bound of S i.e; $x \le M \;\forall\; x \in S$.

(ii) No number less than 'M' can be upper bound of S, i.e; $\epsilon > 0 \;\exists\; y \in S$ s.t $(M - \epsilon) < y \le M$



Necessity: For $\epsilon > 0$ $(M - \epsilon)$ is not an upper bound of S

$\Rightarrow \exists\; y \in S$ s.t $(M - \epsilon) < y \le M$ (by Property (i)).

Sufficient: by (i) M is upper bound of S.

So, by (ii) we see M is the LUB.

Greatest lower bound (Inf): The greatest member in the set of all lower bound of a set 'S' is called G.L.B or Infremum

Property: Inf S = m equivalent to hold the next (i) & (ii).

(i) m is the lower bound of set S.

(ii) no-no. greater than m can be lower bound of S i.e; $\epsilon > 0 \;\exists\; y \in S$ s.t $m \le y < (m + \epsilon)$

Necessity: for $\epsilon > 0$, $(m + \epsilon)$.

Sufficient: by (i) 'm' is lower bound of 'S' so by (ii) we say that 'm' is the G.L.B

ex; $N = \{1, 2, 3, 4, --- 10\}$. $Inf(N) = 1$.
$Sup(N) = 10$.

Thrm 10: Supremum of any set is unique, i.e; the set A cannot have more than one sup/Inf.

Proof: suppose that 'S' is ~~an~~ a non-empty set and we have to show that sup S is unique.

let m & m' be two sups of the set (S), that is both m & m' are also upper bound of 'S' then.

If m is the L.UB the m' is an upper bound then
$m' \geq m$ (i)

similarly if m' is the L.UB the m is an upper bound then $m \geq m$ (ii)

from (i) & (ii). we have $m = m'$
hence, the sup of the set is unique.

The completeness axiom:

1. every non-empty set of real no. which is bounded above, has the sup [L.U.B] in $\mathbb{R}$.
2. Every non-empty set of real no. which is bounded below has the inf [G.L.B] in $\mathbb{R}$.

say

The Archimedian Property of Real number.

Theorm: If $x, y \in \mathbb{R}$ and $x>0$ then ~~these~~ there is a +ve integer n s.t $nx > y$.

not

Proof: (i) If $y \le 0$ then the theorem is true.

Proving by contradictn

(ii) we need to show that the following.

If $x, y \in \mathbb{R}$ and $x, y > 0$ then $\exists\, n \in \mathbb{N}$ s.t $nx > y$.

Suppose this is false. then $nx < y$ $\forall n \in \mathbb{N}$

now, let $S = \{nx : n \in \mathbb{N}\} = \{x, 2x, 3x, \dots\} \le y$.

Thus the set 'S' bounded above and y will be a upper bound of 'S' { by completeness ~~theorem~~ Axioms

$\sup S = M$ i.e M is L.U.B of S.

$\Rightarrow$ M is the upper bound of 'S' i.e, $nx \le M$ $\forall x \in S$ & $n \in \mathbb{N}$

$\Rightarrow (n+1)x \le M$ $\quad (\forall n \in \mathbb{N} \Rightarrow n+1 \in \mathbb{N})$

$\Rightarrow nx \le M - x$ $\quad$ { $(M-x)$ is the upper bound of S

$\Rightarrow M - x < M$ $\quad (\because x > 0)$

Hence we have the upper bound $\underline{(M-x)}$ less than Sup S which is $\underline{M}$ which is a contradiction.

$\therefore$ $x, y \in \mathbb{R}$ and $x, y > 0$ then $\exists\, n \in \mathbb{N}$ s.t $nx > y$.