B.Tech V Semester (2025-26)
MC 303: Stochastic Processes

Assignment- I

Problem 1

A call center receives calls as a Poisson process with unknown rate λ. Each call generates a random revenue X_i (i.i.d.) with mean μ and variance σ^2. Let N(t) be number of calls in [0,t] and total revenue S(t)=∑_{i=1}^{N(t)} X_i.

a) Show that E[S(t)] = λ t μ and Var[S(t)] = λ t (σ^2 + μ^2). Carefully justify independence steps.
b) Suppose revenues follow an exponential distribution with mean μ. Derive the moment-generating function (MGF) of S(t) and use it to write the distribution of S(t) (name the distribution or express via compound Poisson series).
c) Estimation task (practical): You are given time-stamped call arrivals for 30 days and per-call revenue (CSV).
- (i) Propose an MLE for λ (per minute) assuming a homogeneous Poisson process; derive the log-likelihood and estimator.
- (ii) Implement the estimator in Python or MATLAB and report 95% CI for λ. Provide code and a short report (1 page) interpreting results and checking Poisson assumption (one statistical test or goodness-of-fit plot).

Problem 2

Consider two independent Poisson processes N1(t) and N2(t) with rates λ1 and λ2. Let N(t)=N1(t)+N2(t).

a) Prove N(t) is Poisson with rate λ1+λ2 (superposition). Provide the proof via PGFs or independent increments.
b) Suppose each arrival of N(t) is independently kept with probability p (random thinning), producing process M(t). Show that M(t) is Poisson with rate (λ1+λ2)p.
c) Application problem: In a sensor network two independent sensors generate event reports (rates λ1, λ2). Reports are transmitted to cloud but each packet is lost independently with probability q. Formulate the effective arrival process at cloud and compute its rate. If observed cloud arrivals appear non-Poisson, list three practical reasons (network or modeling) why the Poisson assumptions might fail.

Problem 3

Let {B(t): t≥ 0} be standard Brownian motion.

a) Prove that B(t) has stationary independent increments and that for fixed t, B(t)~N(0,t). Use these to show covariance Cov(B(s),B(t))=min(s,t).
b) Define X(t)=e^{μ t + σ B(t)}. Compute E[X(t)] and Var[X(t)]. Explain why X(t) is not a martingale unless drift corrected.

c) Estimation & model-check (practical): Given daily log-price series of a liquid stock for 1 year (CSV), fit a Brownian motion with drift model $\Delta \log S_t = \mu + \sigma \Delta B_t$.

- (i) Provide MLEs for μ and σ based on increments.
- (ii) Implement estimation, test residual normality (one test), and comment whether Brownian model is plausible. Deliver code, parameter estimates, diagnostic plot.

Problem 4

Consider a biased simple random walk on integers starting at 0: at each step move +1 with probability p and −1 with probability q=1−p. Let T_a be hitting time to +a (>0) and T_{-b} to −b (>0). Define absorption at ± boundaries (gambler's ruin).

a) Derive the probability u_0 that the walk eventually hits +a before −b. Solve the difference equation and present closed form for both cases $p \neq q$ and $p=q=1/2$.
b) Derive expected time to absorption E[T] starting from 0 for the fair case p=1/2 (show method using second difference equations). Discuss order of magnitude with a and b.
c) Challenging extension: For $p \neq q$, find E[T] (you may present expression in summation form or via generating functions). Discuss what happens when p>q and $b \to \infty$ (probability of eventual absorption at +a and expected time).

Problem 5

A particle on {0,1,…,N} moves with probabilities: from i ($1 \le i \le N-1$) go to i+1 with p and to i−1 with q=1−p. State 0 is reflecting (if at 0 it moves to 1 in next step with probability 1). State N is absorbing.

a) Write and solve for the stationary distribution (if it exists) for the chain when p<q and chain is irreducible/aperiodic (discuss conditions). If stationary does not exist due to absorption, explain.
b) Compute expected time to absorption at N starting from 0 for the symmetric case p=1/2 and compare asymptotically with N^2. Provide leading-order term.
c) Simulation task: Simulate 10,000 sample paths for N=200 and p=0.49 (slight drift down). Estimate probability of absorption at N and sample mean absorption time starting from 0. Provide code and brief analysis (are theoretical approximations matched?).

Problem 6

Consider scaling a simple symmetric random walk with step size Δx and time step Δt such that $\lim_{\Delta x, \Delta t \to 0}$ it converges to Brownian motion with variance parameter. Use Donsker's invariance principle heuristics.

a) Show how to choose Δx and Δt so that scaled walk converges to standard Brownian motion (match variance).
b) Using reflection principle for Brownian motion, derive probability that maximum of Brownian motion on [0,T] exceeds level a > 0. Compare this continuous result to discrete random walk approximation for large n.
c) Applied question: For barrier option pricing (up-and-out option) briefly explain how the discrete random walk approximation and continuous Brownian result differ in pricing and why barrier monitoring frequency matters.

Problem 7

Problem statement: Model customer arrivals (nonstationary) to a ride-hailing platform during a city festival. Data: timestamped rides for one festival day (CSV) and a separate "regular day" for control.

Deliverables:

a) Propose a stochastic process model for arrivals that can capture time-varying intensity (e.g., non-homogeneous Poisson process $\lambda(t)$ or a Cox process with stochastic $\lambda(t)$). Write down likelihood for observed arrivals under chosen model.

b) Implement both approaches and fit them to the festival-day data:

- (i) Fit a non-homogeneous Poisson process using e.g. piecewise-constant $\lambda(t)$ (hourly) or spline for $\lambda(t)$.
- (ii) Fit a Cox (doubly stochastic) model where $\lambda(t)$ is a Gaussian process (brief implementation using MCMC or an approximation such as discretized latent Gaussian).

c) Compare fits using AIC/BIC or predictive log-likelihood on hold-out data (the regular day). Produce arrival intensity plots, QQ plot of interarrival residuals, and a short report (max 3 pages) with interpretation (peak times, overdispersion, evidence of clustering beyond Poisson).

d) Submit code (well commented), plots, parameter estimates, and the report.