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2K21/MC/90

MC-205

Assignment -1

Q1.)

No. of cars	Mid pt. (x_i)	No. of cycles (f)	$u = \frac{x_i - 62.5}{5}$	fu	fu^2
45-50	47.5	5	-3	-15	45
50-55	52.5	10	-2	-20	40
55-60	57.5	6	-1	-6	6
60-65	62.5	13	0	0	0
65-70	67.5	9	1	9	9
70-75	72.5	4	2	8	16
75-80	77.5	3	3	9	27
		50		$\Sigma f_i u_i = -15$	143

$a = 62.5 \qquad h = 5$

$$\text{Mean} = a + \left(\frac{\Sigma f_i u_i}{\Sigma f_i}\right) \times h = 62.5 + \frac{(-15)}{50} \times 5 = 61$$

$\frac{N}{2} = 25$, $cf = 21$, $f = 13$, $h = 5$, $l = 60$

$$\text{Median} = l + \frac{(N/2 - cf)}{f} \times h = 60 + \frac{(25-21)}{13} \times 5 = 61.53$$

~~l = 60~~

for mode

$l = 60 \qquad f_1 = 13 \qquad f_0 = 6 \qquad f_2 = 9 \qquad h = 5$

$$\text{Mode} = l + \left(\frac{f_1 - f_0}{2f_1 - f_0 - f_2}\right) \times h = 60 + \left(\frac{16-6}{26-6-9}\right) \times 5 = 63.18$$

$$\text{Variance}_u = \sigma_u^2 = \frac{1}{N} \Sigma f_i u_i^2 - \left(\frac{1}{N} \Sigma f_i u_i\right)^2$$

$$= \frac{143}{50} - \left(\frac{-15}{50}\right)^2$$

$$= 2.86 - 0.09 = 2.77$$

$\sigma_x^2 = 25\,\sigma_u^2 = 25 \times 2.77 = 69.25$

Q2.

Range	Mid pt.	Frequency	$u = \frac{x - 38.75}{1.5}$	fu	fu^2
32 – 33.5	32.75	12	−4	−48	192
33.5 – 35	34.25	10	−3	−30	90
35 – 36.5	35.75	8	−2	−16	32
36.5 – 38	37.25	5	−1	−5	5
38 – 39.5	38.75	7	0	0	0
39.5 – 41	40.25	8	1	8	8
41 – 42.5	41.75	6	2	12	24
42.5 – 44	43.25	6	3	18	54
		$\Sigma f_i = 62$		$\Sigma f_i u_i = -61$	$\Sigma f_i u_i^2 = 505$

$a = 38.75 \qquad h = 1.5$

$$\text{Mean} = 38.75 + \frac{(-61)}{65} \times 1.5 = 38.75 - 1.47 = 37.28$$

$l = 36.5 \qquad N/2 = 31 \qquad cf = 30 \qquad f = 5 \qquad h = 1.5$

$$\text{Median} = 36.5 + \frac{(31 - 30)}{5} \times 1.5 = 36.8$$

For mode

$l = 32 \qquad f_1 = 12 \qquad f_0 = 0 \qquad f_2 = 10 \qquad h = 1.5$

$$\text{Mode} = 32 + \frac{(12 - 0)}{(24 - 10 - 0)} \times 1.5 = 33.28$$

$$\text{Standard Deviation}_u = \sigma_u = \sqrt{\frac{505}{62} - \left(\frac{-61}{62}\right)^2}$$

$$= \sqrt{8.1 - 1}$$

$$= \sqrt{7.1}$$

$$\sigma_x = 1.5 \times \sqrt{7.1}$$

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Q3. To prove $\rightarrow \sum_{i=1}^{n} f_i (x_i - a)^2$ is minimum at $a = \bar{x}$

Let $S = \sum_{i=1}^{n} f_i (x_i - a)^2$

$$\frac{dS}{da} = 2 \sum_{i=1}^{n} f_i (x_i - a)(-1) = 0$$

$$a = \frac{1}{N} \sum_{i=1}^{N} f_i x_i = \bar{x}$$

Also, $\frac{d^2 S}{da^2} = 2N > 0$

Therefore, the minima of S exists at $a = \bar{x}$.

Q4. Mean $= \frac{1+2+3+4+\dots\dots n}{n} = \frac{n(n+1)}{2 \times n} = \frac{n+1}{2}$

Weights $(w_i) = i$

Weighted mean $= \frac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} w_i} = \frac{n(n+1)(2n+1) \times 2}{6 \times n(n+1)} = \frac{2n+1}{3}$

Q5. $f_1 = 900$ $\bar{x}_1 = 60$; $f_2 = 3000$ $\bar{x}_2 = 25$ $f_3 = 400$ $\bar{x}_3 = 350$
$f_4 = 15$ $\bar{x}_4 = 25$

$$\bar{x} = \frac{f_1 \bar{x}_1 + f_2 \bar{x}_2 + f_3 \bar{x}_3 + f_4 \bar{x}_4}{f_1 + f_2 + f_3 + f_4}$$

$$= \frac{900 \times 60 + 3000 \times 25 + 400 \times 350 + 15 \times 25}{900 + 3000 + 400 + 15}$$

$$= \frac{54000 + 75000 + 14000 + 375}{4315}$$

$$= 33.22 \text{ Km/hr}$$

Q6. $P(\text{sum } 6) = \frac{5}{36}$

$P(\text{sum } 7) = \frac{6}{36} = \frac{1}{6}$

$$P(A \text{ wins}) = \frac{5}{36} + \frac{31}{36} \times \frac{5}{6} \times \frac{5}{36} + \left(\frac{31}{6} \times \frac{5}{6}\right)^2 \times \frac{5}{36} \cdots\cdots \infty$$

$$= \frac{\frac{5}{36}}{1 - \frac{31}{36} \times \frac{5}{6}} = \frac{30}{61}$$

$$P(B \text{ wins}) = \frac{31}{36} \times \frac{1}{6} + \left(\frac{31}{36} \times \frac{1}{6}\right)^2 + \cdots\cdots \infty$$

$$= \frac{\frac{31}{36} \times \frac{1}{6}}{1 - \frac{31}{36} \times \frac{1}{6}} = \frac{1}{5}$$

Q7. $P(9^{th} \text{ is last defective}) = \frac{{}^{11}C_5 \times {}^{4}C_3}{{}^{15}C_8} \times \frac{1}{7}$

$$= \frac{\cancel{11} \times \cancel{10} \times \cancel{9} \times 8 \times \cancel{7}}{\cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times 1} \times \frac{4 \times \cancel{7} \times \cancel{6} \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2}}{15 \times \cancel{14}_{2} \times 13 \times \cancel{12}_{2} \times \cancel{11} \times \cancel{10} \times \cancel{9}} \times \frac{1}{\cancel{7}}$$

$$= \frac{8}{195}$$

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Q8. $P(\text{know} \cap \text{answered correctly}) = P(\text{know}) = p$

~~P(answered correctly) =~~

$P(\text{guess}) = (1-p)$

$P(\text{know} \mid \text{answered correctly}) = \dfrac{P(\text{know} \cap \text{answered correctly})}{P(\text{answered correctly})}$

$= \dfrac{p}{p + \frac{1-p}{5}} = \dfrac{5p}{4p+1}$

Q9. A → produces a defective item
B → follows instructions
C → Not follow

$P(B) = 0.9$

$P(C) = 0.1$

$P(A \mid B) = 0.01$

$P(A \mid C) = 0.03$

$P(A) = P(B) \times P(A \mid B) + P(C) \times P(A \mid C)$

$= 0.9 \times 0.01 + 0.1 \times 0.03$

$= 0.009 + 0.003 = 0.012$

Q10. $P(\text{critical}) = P(C) = 0.2$

$P(\text{serious}) = P(S) = 0.3$

$P(\text{stable}) = P(T) = 0.5$

D → die

$P(D \mid C) = 0.3$

$P(D \mid S) = 0.1$

$P(D \mid T) = 0.01$

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$$P(C|D) = \frac{0.2 \times 0.3}{0.2\times0.3 + 0.3\times0.1 + 0.5\times0.01} = \frac{0.06}{0.06+0.03+0.005}$$

$$= \frac{0.06}{0.095} = \frac{60}{95} = \frac{12}{19}$$

Q11.

$$f(x) = \begin{cases} 2x & 0 < x \leq 1 \\ 4-2x & 1 < x < 2 \\ 0 & \text{elsewhere} \end{cases}$$

$$\int_{-\infty}^{\infty} f(x)\,dx \cancel{=1}$$

$$\int_0^1 2x\,dx + \int_1^2 (4-2x)\,dx = 1 + (4x - x^2)$$

$$= 1 + (8-4-4+1)$$

$$= 2 \neq 1$$

$\therefore$ $f(x)$ is not a p.d.f.

Q12.

c.d.f $\rightarrow$ $F(x) = \begin{cases} 0 & x < -a \\ \frac{1}{2}\left(\frac{x}{a}+1\right) & -a \leq x \leq a \\ 1 & x > -a \end{cases}$

$$\text{p.d.f.} = f(x) = \frac{d\,F(x)}{dx}$$

$$\therefore f(x) = \begin{cases} 0 & x < -a \\ \frac{1}{2a} & -a \leq x \leq a \\ 0 & x > -a \end{cases}$$

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Q13. $p(x_1) + p(x_2) + p(x_3) + p(x_4) = 1$ for discrete random variable X

$2p(x_1) = 3p(x_2) = p(x_3) = 5p(x_4)$

$p(x_1) + \frac{2}{3}p(x_1) + 2p(x_1) + \frac{2}{5}p(x_1) = 1$

$\left(\frac{15+10+30+6}{15}\right) p(x_1) = 1$

$p(x_1) = \frac{15}{61}$, $p(x_2) = \frac{10}{61}$, $p(x_3) = \frac{30}{61}$, $p(x_3) = \frac{6}{61}$

Random variable X	:	x_1	x_2	x_3	x_4
Probability distribution $p(x)$	:	$15/61$	$10/61$	$30/61$	$6/61$
Cumulative distribution $F(x)$	:	$15/61$	$25/61$	$55/61$	1

Q14. $\sum_{x=0}^{6} f(x) = 1$ $f(x) = k\left(\frac{1}{2}\right)^x$

$k + \frac{k}{2} + \frac{k}{4} + \frac{k}{8} + \frac{k}{16} + \frac{k}{32} + \frac{k}{64} = 1$

$\frac{127}{64}k = 1 \Rightarrow k = \frac{64}{127}$

Expression for cumulative probabilities $F(x) = \sum_{x=0}^{n} \frac{64}{127}\left(\frac{1}{2}\right)^x$

$= \frac{64}{127}\left(\frac{1-\left(\frac{1}{2}\right)^{x+1}}{1/2}\right) = \frac{128}{127}\left[1 - \left(\frac{1}{2}\right)^x\right]$

Q15. $f(x) = 3x^2$ $\quad 0 \le x \le 1$

(i) $P(X \le a) = P(X > a)$

$$\int_0^a 3x^2 dx = \int_a^1 3x^2 dx$$

$$a^3 = 1 - a^3$$

$$2a^3 = 1$$

$$a^3 = \frac{1}{2} \Rightarrow a = \frac{1}{\sqrt[3]{2}}$$

(ii) $P(X > b) = 0.05$

$$\int_b^1 3x^2 dx = 0.05$$

$$1 - b^3 = 0.05$$

$$b = \sqrt[3]{0.95}$$

Q16. $f(x,y) = \dfrac{9(1+x+y)}{2(1+x)^4(1+x)^4}$ $\quad 0 \le x < \infty, \; 0 < y < \infty$

Marginal distribution function of x

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$

$$= \int_0^{\infty} \frac{9(1+x+y)}{2(1+x)^4(1+y)^4} dy = \frac{9}{2(1+x)^4} \int_0^{\infty} \frac{x+y+1}{(1+y)^4} dy$$

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$$= \frac{9}{2(1+x)^4}\left(\frac{-1}{2(1+y)^2} - \frac{x}{3(1+y)^3}\right)_0^\infty$$

$$= \frac{9}{2(1+x)^4}\left(\frac{1}{2} + \frac{x}{3}\right)$$

$$= \frac{3(3+2x)}{4(1+x)^4},$$

$$f_Y(y) = \frac{9}{2(1+y)^4}\int_0^\infty \frac{y+x+1}{(1+x)^4}\,dx$$

$$= \frac{3}{4}\left(\frac{3+2y}{(1+y)^4}\right)$$

$$P(Y=y \mid X=x) = \frac{f(x,y)}{f_X(x)} = \frac{\frac{\cancel{9}^{3}}{\cancel{2}} \times \frac{1}{\cancel{(1+x)^4}} \times \frac{x+y+1}{(1+y)^4}}{\frac{\cancel{3}}{\cancel{2}}\left(\frac{3+2x}{\cancel{(1+x)^4}}\right)}$$

$$= \frac{3(x+y+1)}{(1+y)^4(3+2x)}$$

Q17. The joint p.d.f. of X and Y is

$$f(x,y) = 4xy\,e^{-(x^2+y^2)} \quad ; \; x, y \geq 0$$

The marginal density function of X is

$$f_X(x) = \int_0^\infty f(x,y)\,dy = \int_0^\infty 4xy\,e^{-(x^2+y^2)}\,dy$$

$$= 4xe^{-x^2}\int_0^\infty ye^{-y^2}\,dy$$

$$= 2xe^{-x^2}\left[e^{-y^2}\right]_0^\infty = 2xe^{-x^2}, \quad x \geq 0$$

Similarly, the marginal density fn. of Y is

$$f_Y(y) = \int_0^{\infty} f(x,y)\,dx = 2ye^{-y^2}, \quad y \geq 0$$

Since $f(x,y) = f_X(x)\, f_Y(y)$, thus X and Y are independent.

Q18.

$$f(x,y) = \begin{cases} \frac{1}{8}(6-x-y) & , 0 < x < 2, \ 2 < y < 4 \\ 0 & , \text{otherwise} \end{cases}$$

(1) $$P((X<1)\cap(Y<3)) = \int_0^1 \int_2^3 \frac{1}{8}(6-x-y)\,dy\,dx$$

$$= \int_0^1 \left(\frac{1}{8}\right)\left((6-x)y - \frac{y^2}{2}\right)_2^3 dx$$

$$= \int_0^1 \frac{1}{8}\left((6-x) - \frac{9}{2} + \frac{4}{2}\right) dx$$

$$= \int_0^1 \frac{1}{8}\left((6-x) - \frac{5}{2}\right) dx$$

$$= \frac{1}{8}\left(6x - \frac{x^2}{2} + \frac{5}{2}x\right)_0^1$$

$$= \frac{1}{8}\left(6 - \frac{1}{2} - \frac{5}{2}\right) = \frac{1}{8} \times 3 = \frac{3}{8}$$

(ii) $$\cancel{f_{X/Y}(X=x \mid Y=y) = \frac{f(x,y)}{f_Y(y)}}$$

$$\cancel{P(X+Y<3) = \int_2^4 \int_0^{3-y} \frac{1}{8}(6-x-y)\,dx\,dy}$$

$$\cancel{= \int_2^4 \frac{1}{8}\left[(6-y)x - \frac{x^2}{2}\right]_0^{3-y} dy}$$

(ii) $P(X+Y<3) = \int_2^3 \int_0^{3-y} \frac{1}{8}(6-x-y)\,dx\,dy$

$= \int_2^3 \frac{1}{8}\left[(6-y)x - \frac{x^2}{2}\right]_0^{3-y} dy$

$= \int_2^3 \frac{1}{8}\left[18 - 9y + y^2 - \frac{(3-y)^2}{2}\right] dy$

$= \frac{1}{8}\int_2^3 \left[18 - 9y + y^2 - \frac{9}{2} - \frac{y^2}{2} + 3y\right] dy$

$= \frac{1}{8}\int_2^3 \left[\frac{y^2}{2} - 6y + \frac{27}{2}\right] dy$

$= \frac{1}{8}\left[\frac{y^3}{6} - 3y^2 + \frac{27y}{2}\right]_2^3$

$= \frac{1}{8}\left[\frac{27}{6} - 27 + \frac{81}{2} - \frac{8}{6} + 12 - \frac{54}{2}\right]$

$= \frac{1}{8}\left[\frac{19}{6} - 15 + \frac{27}{2}\right] = \frac{1}{8}\left[\frac{19-9}{6}\right] = \frac{10}{6\times 8} = \frac{5}{24}$

(iii) $P(X<1 \mid Y<3) = \frac{P(X<1 \cap Y<3)}{P(Y<3)}$

$= \cancel{\frac{\int_0^1\int_2^3 \frac{1}{8}(6-x-y)\,dy\,dx}{\int_0^2 \frac{1}{8}\int (6-x-y)\,dx}}$

$= \frac{3/8}{\int_2^3\int_0^2 \frac{1}{8}(6-x-y)\,dx\,dy}$

$= \frac{3/8}{5/8} = \frac{3}{5}$