MC 303

Stochastic Processes

Random ⟵ (Stochastic)

CWS	PRE	MTE	ETE
15	25	20	40
↓ 5 assign + Attendance	↓ exp		

→ Series of action to achieve some goal
→ Set of steps which progresses
→ set of instructions

change of variables
↳ represent change of $\{V_1, V_2, V_3 \ldots V_n\}$ states

State 1 $\xrightarrow{P_1}$ State 2

Randomness in variability
↳ observing variance (could also indicate error)

HW
* 14 test → NIST Randomness Test

Diff b/w randomness & Deterministic
↓
having state or time variable

(Environmental) EF_1, EF_2, EF_3, EF_4, EF_5 → state 1: V_1, V_2, V_3, V_4 → state 2: V_1', V_2', V_3', V_4'

Random process: Set of Random variables in "changing time"

Random events → in terms of random variables

$X \to$ discrete
Probability mass fn
↳ $\sum_{i=1}^{n} P(x_i) = 1$

Cummulative mass fn
$\sum_{i=1} P(x_i) = 1$

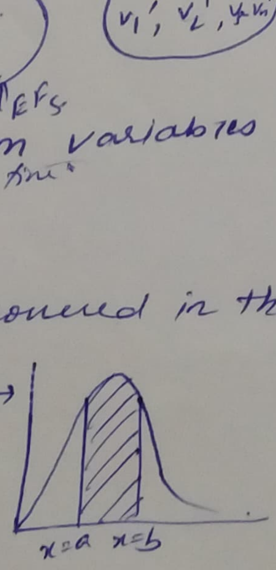
$\boxed{F(x) = P(X \le x) = \sum_{\text{all } x_i \le x} P(x_i)}$

Distribution fn
$P(a \le X \le b)$
↳ $\int_a^b f(x)\,dx$

X continous var
Probability - area covered in that interval

$\boxed{F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt}$

$x \in X$ $f(x) \to$



$P(a \le X \le b)$
↓
$\int_a^b f(x)\,dx$

$\boxed{\int_{-\infty}^{\infty} f(x)\,dx = 1}$

Expected value $E(x) = \sum_{\forall x_i \in X} P(x_i) x_i$

$$\boxed{E(x) = \int_{-\infty}^{\infty} x f(x)\, dx}$$

Variance of x $\quad$ var(x) = $E[(x-E(x))^2] = E(x^2) - [E(x)]^2$

$$\left[Var(x) = \sum_{\forall x_i \in X} P(x_i) x_i^2 - \left(\sum_{\forall x_i \in X} P(x_i) x_i\right)^2 \right]$$

*Imp

(1) Moment Generating function $\quad$ 2) characteristic function

(3) Probability Generating fn. $\quad$ Analysis $\to$ Joint Distribution system

(1) 1st $E(x)$ $\sum P_i x_i$

2nd $E(x^2)$ $\sum P_i x_i^2$

3rd $E(x^3)$ $\sum P_i x_i^3$

MGF $\to$

(1) $\psi_x(t) = E[e^{tx}]$ $\quad$ 1st moment $\to$ diff wrt to time

$\psi'_x(t) = E[x \cdot e^{tx}]$ (for 1st moment $t=0$)

so $\psi'_x(t) = E_{at\, t=0}[x \cdot e^{tx}] = E(x)$ $\to$ expected value

2nd moment $= \psi''_x(t) = E[x^2 \cdot e^{tx}]_{at\, t=0} = E(x^2)$

[* MGF doesn't exist for all random variable so we use charac. fn for all random var]

$$\boxed{\int_{-\infty}^{\infty} e^{tu} f(u)\, du}$$

(2) Characteristic fn $\to$

$\phi(t) = E(e^{itX})$ $\quad -\infty < t < \infty \quad i = \sqrt{-1}$

*Imp

3) Probability Gen fn $\to$

$G_x(t) = \underbrace{E(t^x)}_{\text{Expected value}}$ $\quad \sum_{\forall x_i} P(x_i) t^{x_i}$ — discrete

fn of t for given random var

$\int_{-\infty} t^u f(u)\, du$ $\quad$ continous

$\to$ to find this $\to$ deal in inv of fn

$\to$ If some random var is in power of t then coeff of t^{x_i} will be the probability of that fn

multivariable

$\to G_{x_1, x_2}(t_1, t_2) = E(t_1^{x_1}, t_2^{x_2}) = \sum_{\forall x_i x_j \in X_1, X_2} P(x_i, x_j) t_1^{x_i} t_2^{x_j}$

Some Thm

* At large pbs $\to$ probability of practical = theory val

* Bernoulli $\to$ success & failure of prev val event doesn't affect the next event $\to$ Independant & Identical (follow same distribution fn)

Discrete RV $\quad$ Continous RV

Bernoulli Trials,

Bernoulli Dis. fn

Bern n trials $\to P(x_1, x_2 \ldots x_n)$
$\quad t_1 \quad p^{t_2} \ldots t_n$

$x_j < \begin{matrix} 1 & S \\ 0 & F \end{matrix}$ $\quad p \quad 1-p$

Independant so $= P_{t_1}(x_1) P_{t_2}(x_2) P_{t_3}(x_3) \ldots P_{t_n}(x_n)$

IID = RV

$\to$ Independantly Dis. Var

$E(x) = p \quad var(x) = pq$

IID → Independant Identical Distribution

$$E(x) = 0 \times (1-p) + 1 \cdot p = p$$
$$var(x) = 0^2(1-p) + 1^2 p = p(1-p) = pq$$
$$p(x_1, x_2, x_3 \dots x_n) = p(x_1)\, p(x_2) \dots p(x_n)$$

Let, x denote the no. of successes in n Bernoulli Trial

$x = t_1 + t_2 + t_3 + \dots t_n \qquad x \in \{0, 1, \dots n\}$

↳ give no. of 1's trials

out of 10 trial $P(x=3)$

$P(F\,S\,F\,F\,S\,F\,F\,F\,S\,F) = p^3 q^7$

↳ multiply since they're independant to diffs

ways to find success $^{10}C_3$

↳ all comb. to be added (exclusive event)

Probability $= {}^{10}C_3\, p^3 (1-p)^7$

Probability for Bernoulli distribution

$$\boxed{P(x) = {}^nC_x\, p^x q^{n-x}}$$

$E(x) = \sum x\, p(x)$

or use identity

$E(x) = E(x_1) + E(x_2) + E(x_3)$

(No. of trial = random var) until you get 1st success

$p(x) = q^{x-1} p \quad \forall\, x = \{1, 2, \dots \infty\}$

Geometric Distribution fn

$var(x) = \dfrac{q}{p^2}$

If 10 RV added for GV

↳ No of trials until 1st success

No. of trials until k^{th} success)

$x = \{k, k+1, \dots \infty\}$

so $\; ^{x-1}C_{k-1} \times p^k q^{x-k}$ (-ve binomial dist^n)

↳ last one fixed

$\dfrac{k}{p}$

(Basics of RV & probab)

$X = \{0, 1\}$ $\quad E(X) = p$

$X = \{1, 2, 3 \dots n\}$ $\quad E(x) = P/2$

$X = \{0, 1, 2 \dots \infty\}$ $\quad E(x) = kP/\sim$

$X \to \{k, k+1, \quad \infty\}$

$p^n \approx 0$ (n = very large) $\quad$ (In GM $X \to$ total no of trials to get 1st success

(memory less → events b/w 2 events are indepn.)
↳ Don't depend on prev events & depends on trials sem. to achieve that

$P(X > s+t \mid X > s)$
↳ to get 1st success

Merging 2 bernoulli processes

BP1 → p $\quad X = \{0, 1\}$
$P(X=1) = p$

BP2 → q → $Y = \{0, 1\}$
$P(Y=1) = q$

Binomial process

$p \to x_1 \; x_2 \; x_3 \; x_4 \dots \infty$
$q \to y_1 \; y_2 \; y_3 \; y_4 \dots \infty$
BP3 → $z_1 \; z_2 \; z_3 \; z_4 \dots \infty$
↳ $x_1 = 1 \quad y_1 = 1 \quad z_1 = 1$
$0 \quad 1 \quad 1$
$1 \quad 0 \quad 1$
$0 \quad 0 \quad 0$

$P(Z=1) =$ ~~[illegible]~~ $p \cdot q$

$P(Z=0) =$ ~~[illegible]~~ Prob$(X=0, Y=0)$ = Prob$(Z=0)$ $= (1-p)(1-q)$
both are independant so multiply

Splitting into 2 BP. →

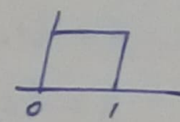
$\theta_i \to q$ $\quad X_1 \quad X_2 \quad X_3 \dots X_n \to p$
$=1$
$=0 \to 1-q$ $\quad Y_1 \quad Y_2 \quad Y_3 \dots Y_n$
$Z_1 \quad Z_2 \quad Z_3 \dots Z_n$

→ Prob mass density fn → pq (if 1 occurs assigned to this)
(if 0 ,,)
→ PMD = $p(1-q)$

* Y & Z dependant as they depend on outcome of θ_i

In random var X can take any value
$X = [0 \; 1]$
$X = [a \; b]$
$X = (0 \; \infty)$
$X = (-\infty, \infty)$

(1) Uniform Distribution fn

(Eg: Density fn) $f(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$ graph 

$\int f(x)\,dx = 1$

$E(x) = \int_{-\infty}^{\infty} x f(x)\,dx \rightarrow \int_0^1 x \cdot dx = \frac{1}{2}$

Q ⇒ Gen RV $\cdot\, y \in [a, b]$ using $X(0,1)$

Using property ⇒ $X_1 + X_2 = Y$ $\quad E(X_1) + E(X_2) = E(Y)$

→ forming a relⁿ b/w X & Y $\quad Y = a + X(b-a)$

$E(Y) = E(a) + E[X(b-a)]$

$= a + E(X)(b-a) = a + \frac{1}{2}(b-a) = \frac{1}{2}(a+b)$

Density fn ⇒ $f(y) = \begin{cases} \frac{1}{b-a} & a \le y \le b \\ \text{otherwise} & 0 \end{cases}$ st $\int_a^b f(y)\,dx = 1$

Generating no.s in continous fns

$x_1, x_2, x_3 \ldots y_n \quad \cdot x_i \in (0,1)$

(*) Exponential Distribution fn →

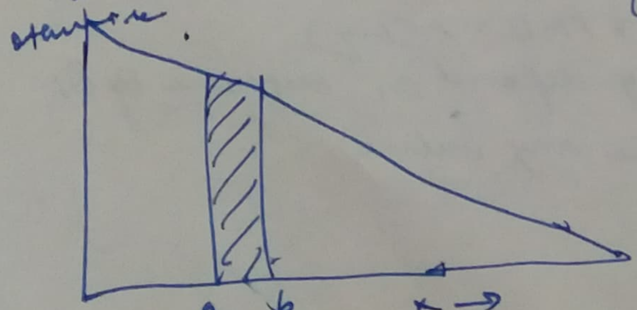
$f(y) = P(X \le y) \begin{cases} 0 & y < a \\ \frac{y-a}{a-b} & a \le y \le b \\ 1 & y > b \end{cases}$

Q λ → Arrival time of someone visiting

Let $\lambda = 3$ yrs

Avg waiting time $= 12/3 = 4$ months $\left(\frac{1}{\lambda}\right)$

$\int_{-\infty}^{\infty} f(x)\,dx = 1 \leftarrow f(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$



$E(x) = \int_{-\infty}^{\infty} x f(x)\,dx = \frac{1}{\lambda}$

$\int_{-\infty}^{\infty} x \lambda e^{-\lambda x}\,dx$

$= \cancel{\lambda} e^{\lambda x} + x$

$$\text{var}(X) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_{-\infty}^{\infty} x^2 \lambda e^{-\lambda x}\,dx = \frac{1}{\lambda^2}$$

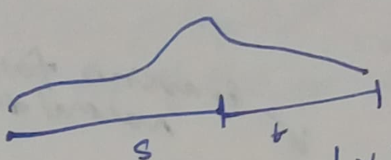
Gamma fn

$$E(X) = \int_{-\infty}^{\infty} x \lambda e^{-\lambda x}\,dx$$

$$= \frac{1}{\lambda}$$

Distribution fn →

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dx = \int_{0}^{x} f(t)\,dx = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & x \ge 0 \end{cases}$$



$P(X > t+s \mid X > s)$

* Independant events gen are not memory less but here are.

$$\frac{P(X > t+s,\ X > s)}{P(X > s)} = \frac{P(X > t+s)}{P(X > s)}$$

$$\frac{1-(1-e^{-(t+s)\lambda})}{1-(1-e^{-\lambda s})} = \frac{\cancel{x} - \cancel{x} + e^{-(t+s)\lambda}}{\cancel{x} - \cancel{x} + e^{-\lambda s}} = e^{-\lambda t}$$

$$= 1 - (1 - e^{-\lambda t}) = 1 - P(X \le t) = P(X > t)$$

fn with gamma r^{v}

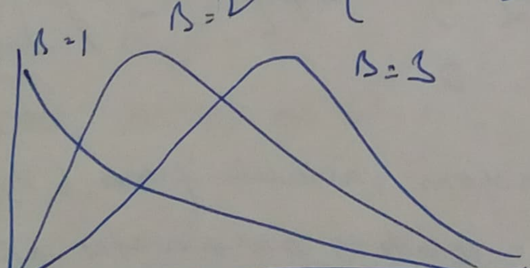
$x \in [0, \infty)$

$\beta \to$ shape pose

$\theta \to$ scale pose

$$\Gamma(\beta) = \int_0^{\infty} x^{\beta-1} e^{-x}\,dx$$

$$f(x) = \begin{cases} \frac{\beta\theta}{\Gamma(\beta)} (\beta\theta x)^{\beta-1} e^{-\beta\theta x} & x > 0 \\ 0 & \text{otherwis} \end{cases}$$



$x e^{-t} dx$

$\frac{-1}{\lambda} \cdot t e^{-t} dt$

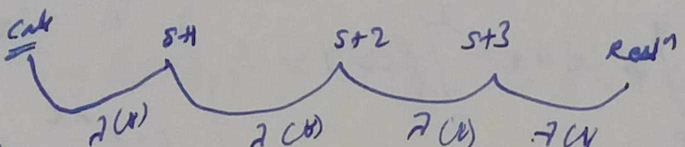
e^{-ax}

$\beta \to$ shape parameter $\theta \to$ scale parameter

$$E(X) = \frac{1}{\theta}$$

$$Var(X) = \frac{1}{\beta\theta^2}$$

$$f(x) = \begin{cases} \frac{\beta\theta}{\Gamma(\beta)} (\beta\theta x)^{\beta-1} e^{-\beta\theta x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$



$$X = X_1 + X_2 + X_3 + X_4$$

$$E(X) = E(X_1) + E(X_2) + E(X_3) + E(X_4)$$

Guassian for Normal dist.

Normal distribution $\to$

$$1 = \int_{-\infty}^{\infty} f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) \quad -\infty < x < \infty$$

↳ This fn is symmetric in nature

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\,dx$$

It willn't have a close form

Let $\frac{x-\mu}{\sigma} = z$ Since $x = z\sigma + \mu$

$$F(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}z^2\right) = \int_{-\infty}^{(x-\mu)\sigma} \phi(z)\,dz$$

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

(normal $\mu = 0$, $\sigma^2 = 1$)

Weibull Distribution $\to$

$$f(x) = \begin{cases} \frac{\beta}{\alpha}\left(\frac{x-\nu}{\alpha}\right)^{\beta-1} \exp\left[-\left(\frac{x-\nu}{\alpha}\right)^{\beta}\right] & x \ge \nu \\ 0 & \text{otherwise} \end{cases}$$

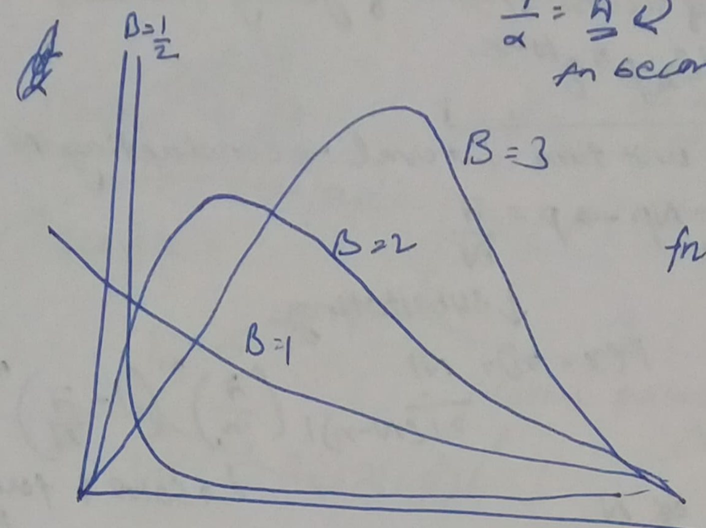
↳ give random variable from (ν, ∞)

ν = location parameter $\to (-\infty, \infty)$

$\alpha\ (>0) \to$ scale parameter

$\beta > 0 \to$ shape

if $\beta = 1$, $\gamma = 0$ exp distribution fn with parameter
$\frac{1}{\alpha} = \lambda$ &
fn becoms exp



approximates fn = gaussian fn ($\beta = 3$)

→ Eg of bernoulli process

flip of bias coin — for n^{th} step

1 step fwd — $H \to p \to np$

1 step bwd — $T \to 1-p \to n(1-p)$

Poisson process

$N(t)$ $t \geq 0$
(no of arrival)
counting process is said to be poisson process with max rate λ if

(1) arrival occur one at a time
(2) $(N(t), t \geq 0)$ stationary increment
(3) $\{N(t), t \geq 0\}$ has independently increment

$$P[N(t) = n] = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

using Binomal dist^n (No. of succes of N trials)

Stationary bec^n in time interval t & $t+s$, time depends on s only.

counting process $N(t_1)$ $N(t_2)$ ....
time of arrival in
time → $(0 - t_i)$

Stationary increment → $N[t_{i+1} - t_i] = N(t_{i+1}) - N(t_i)$
$N(t, t+s) \Rightarrow N(s)$

Binomial Distribution fn

N = trials x = no. of success (probab of getting 1 success)

$$\underbrace{P(X=x) = {}^{N}C_{x}\, p^{x} q^{N-x}}$$

* Happening in 1 unit time interval → conducting N Bernoulli Trials

No. of success while performing all trials $\lambda = Np \rightarrow p = \frac{\lambda}{N}$

↓ subsituting

$$P(X=x) = \frac{N!}{x!(N-x)!}\left(\frac{\lambda}{n}\right)^{n}\left(1-\frac{\lambda}{N}\right)^{N-n}$$

↳ solve & form poisson process

* For large value of N

$$P[N(t)=n] = \frac{e^{-\lambda}(\lambda)^{n}}{n!}$$ No. of arrival in $[0,t)$

Identity $n \to \infty$ $\left(1+\frac{1}{n}\right)^{n} = e$

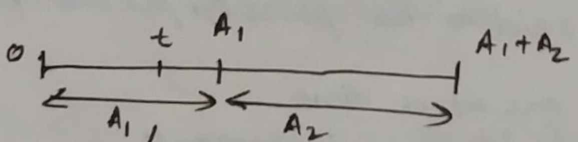
<u>Obtaining exponential Dist^n process</u> →

$P(X > t) = P[N(t)=0] = e^{-\lambda t}$

X ↓ Inter arrival time (of any succes)

no arrival in time t

X → interval arrival time (bet^n 2 successive arrivals)



t → std Arrival time

↳ no arrival then $A_1 > t$

↳ person take more arrival time than t

subtract eq by 1 (arrival in time 0-t)

$P(A \leq t) = 1 - P[N(t)=0] = 1-e^{-\lambda t}$

⇓

$1 - P(A_1 > t)$

↳ commulative dist^n fn from exp distribution fn

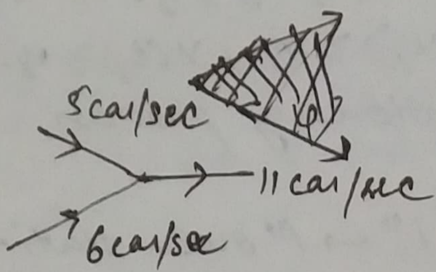
$f(x) = \lambda e^{-\lambda x}$

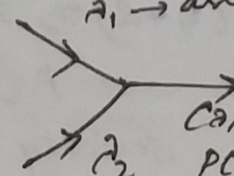
$P(X \leq x) = F(x) = 1 - e^{-\lambda x}$ $f(x) = \lambda e^{-\lambda x}$

v.imp * No. of variables follows poisson process then inter arrival time follows the exponential distribution fn.

Pooled poisson process

eg



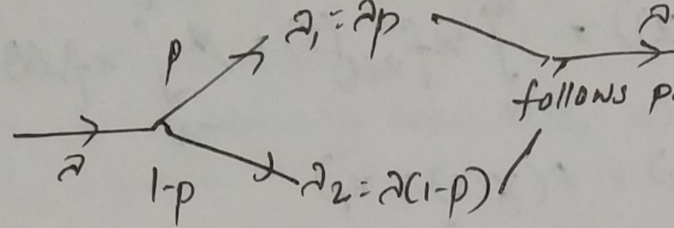


parallel process

P(t) will be higher

$$P(N(t)=10) = \frac{\lambda^{10} e^{-\lambda t}}{10!}$$

$\Rightarrow$



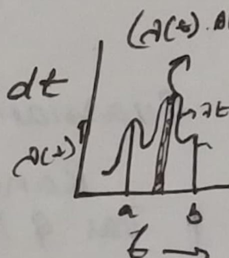
follows poisson process (become same)

Non-stationary poisson process

$\to$ arrival rate isn't fixed $\quad \lambda \to$ fn of time t

Arrival rate $\lambda(t)$

No. of arrivals in $t \in [a,b) = \int_a^b \lambda(t)\,dt$



[Expected value/Avg arrival $P(X=n) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$

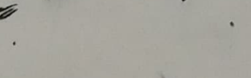
$E(X) = \sum_{x=0}^{\infty} x_i P(x=i) = \lambda t$

$\hookrightarrow$ in stationary process

$$P(\{N(t) - N(S)\} = n) = \frac{[\lambda(t-s)]^n e^{-\lambda(t-s)}}{n!}$$

N(t) $\to$ no. of arrival in 0-t

N(S) $\to$ No of arrival in (0-s)



$$E(X) = \int_{t_1}^{t_2} \lambda(t)\,dt$$

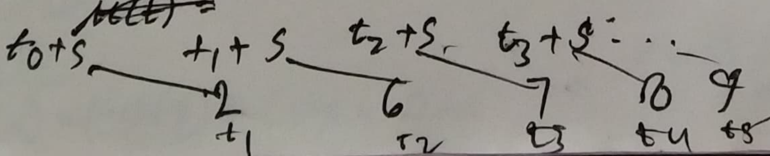
(Replacing in eq instead of $\lambda(t)$)

Q Rel^n b/w Poisson, exponential & binomial

how binomal $\xrightarrow{\text{reduces}}$ Poisson $\xrightarrow{\text{reduces}}$ exponential

Stationary Process

parameters like variance, mean, pdf etc wouldn't change due to shifting (as only parameters are changed) $\hookrightarrow$ index independent parameters



$$x_i \frac{(\lambda t)^{x_i} e^{-\lambda t}}{x_i!}$$

Joint Probab Density fn $P(X(t_1+s) \geq x_1, X(t_2+s) \geq x_2 \dots X(t_k+s) \geq x_k]$

$= P[X(t_1) \geq x_1, X(t_2) \geq x_2 \dots P(t_k) \geq x_k]$

strictly stationary process

$\rightarrow$ k^{th} order stationality (1st $\rightarrow$ 1st order stationary)

$$\mu(t) = E(X(t)) = \int_{-\infty}^{\infty} x f_{X(t)}(x) = \int_{-\infty}^{\infty} x f_X(x) = \mu$$

$P(X(t) \geq x_1) = P(X(0) \geq x_1)$

Probab Density fn $\rightarrow f_{X(t_1)X(t_2)}(x_1, x_2) = f_{X(0)X(t_2-t_1)}(x_1, x_2)$

since it is stationary $\rightarrow$ subtract t_1 from both values

Guassian Process

Random var & Independent

	t_1	t_2	t_3	...	t_n
o/p of	X_1	X_2	X_3	...	X_n
Expected value $\rightarrow$	μ_1	μ_2	μ_3		μ_n $\rightarrow$ Guassian process

Joint density fn

Dependant $\rightarrow$ $P(a < X_1 < b, c < X_2 < d) = \int_a^b \int_c^d f_{X_1X_2}(x_1, x_2)\, dx_2\, dx_1$

$P(a \leq X_1 < b) * P(c < X_2 < d)$

Two theorm

- Law of Large no.
- central no theorm

Stationary guassian brownian Renewal

Unit-2

Barrier

→ Reflecting: The process will never stop but will reflect

Absorbing: The process & the process ends.

RW

Z_i — +ve p Prob(Z_i=1); 0 $1-p-q$ Prob(Z_i=0); -ve q Prob(Z_i=-1) ⇐ Generalised Random walk ⇒ 1+ve p; 1-ve $1-p$

* Either move fwd/bwd

IID - Independently Identically Distributed

$z_0, z_1 \ldots z_i$ are IID

$X_n = X_0 + Z_1 + Z_2 \ldots Z_n$ (position of a particle at n^{th} step)

Probability Generating fn → $G_Z(z) = E(z^Z) = p z^{(1)} + (1-p-q) z^{(0)} + q z^{(-1)}$

z → Dummy variable

Possible values of X_n →

$X_n = \{-n, \ldots -1, 0, 1, \ldots n\}$

To find $P(X_n = k)$??

Ans Let r_1 → no. of +ve step, r_2 → no. of -ve step, r_3 → no. of 0 step

$[r_1 + r_2 + r_3 = n]$ n → no. of jumps

$k = r_1 - 1 \times r_2 + 0 r_3$

Probability of getting $= \frac{n!}{r_1! r_2! r_3!} p^{r_1} q^{r_2} (1-p-q)^{r_3}$

r_1 success, r_2 back, $=0$

eg $n=3$ $r_1=2$ $r_2=0$ $r_3=1$

0 1 1

1 0 1

1 1 0

No. of ways to arrange $= \frac{n!}{r_1! r_2! r_3!}$

How to find out r_1, r_2 acc to k

$$P(X_n = k) = \sum_{\text{for which } \{r_1 - r_2 = k\}} \frac{n!}{r_1! r_2! r_3!} p^{r_1} q^{r_2} (1-p-q)^{n-r_1-r_2}$$

r_3 or $(n-r_1-r_2)$

* Not sure

r_1, r_2, r_3 are mutually exclusive

Central unit theorem → If no. of obs ↑, rv follows normal distribution.

for Z_i

$E(X) = (-1)(q) + (1)p = p - q$

$Var(z) = E(X^2) - (E(X))^2$

$= p + q - (p-q)^2$

finding mean, variance for X_n

as $X_n = X_0 + Z_1 + Z_2 \dots Z_n$ → independant
↳ constant so $=0$

$E(X_n) = E(Z_1) + E(Z_2) + \dots E(Z_n) = n(p-q)$

$Var(X_n) = n\sigma^2 = n[p+q-(p-q)^2]$

PGF X_n -??

$= Z_1 + Z_2 \dots Z_n$ and $X_0 = 0$

$G_{X_n}(z) = E(z^{X_n}) = E(z^{Z_1+Z_2+\dots Z_n})$

dummy val

$= E(z^{Z_1} \cdot z^{Z_2} \dots z^{Z_n}] = \{G_Z(z)\}^n$

↳ same & independant

If value of n is anything $(\pm\infty)$
↳ without barrier

To give the value of n (a dummy var s added)

$$G_X(z,s) = \sum_{n=0}^{\infty} s^n \{G_Z(z)\}^n$$

↓ using GM property $s = \frac{a}{1-r}$

$$= \frac{1}{1 - sG_Z(z)}$$

→ $(= pz^1 + (1-p-q)z^0 + qz^{-1})$

$$= \frac{1}{1 - s(pz + (1-p-q) + \frac{q}{z})} = \frac{z}{z - pz^2 s + (1-p-q)zs + qs}$$

$$= \cancel{\frac{z}{z - s(pz^2 - (p+q)z + q)}}$$

$$= \frac{z}{z - s(pz^2 + (1-p+q)z + q)}$$

$P(X_7 = 3)$ is eq to coeff of $s^7 z^3$

How to approximate this wrt to Normal distribution for X_n?

Probability Density fn (Gaussian in normal) $f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \approx \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$

when $\mu = 0$, $\sigma^2 = 1$

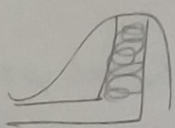
~~$P(X \le x)$~~

Commulative Distribution fn

$$\text{Prob}(j \le X_n \le k) = \frac{1}{n\sigma\sqrt{2\pi}} \int_j^k \exp\left[-\left(\frac{-x - n\mu}{2n\sigma^2}\right)^2\right] dx$$

* If have to solve in 0,1 → assume any value

so $P(a \le X \le b) = F(b) - F(a)$ so,



$P(j \le X_n \le k) = P(X_n \le k) - P(X_n \le j)$

$\hookrightarrow \phi(y) \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{1}{2}x^2} dx$

when $y = k / j$

(subsitute)

can be solved using table and,

$$P(j \le X_n \le k) = \phi\left(\frac{k - c + n\mu}{\sigma\sqrt{n}}\right) - \phi\left(\frac{j - c - n\mu}{\sigma\sqrt{n}}\right)$$

Value of c — $\frac{1}{2}$ if $p+q<1$; 1 $p+q=1$

while normalizing

How to convert $Y = (0,1)$ into $X = (a,b)$

$Y = \frac{x-a}{b-a}$ for $x = a$ $y = 0$, $x = b$ $y = 1$

$$\text{Prob}(J \le X_n \le K) \approx (2\pi\sigma^2 n)^{-1/2} \int_J^K \exp\left(-\frac{(x - n\mu)^2}{2\sigma^2}\right) dx$$

Find value st $\sigma = 1$ $\mu = 0$

(to bring it int $N(0,1)$)

converting into a form using it $N(0,1)$ form

Unrestricted Random Walk → n can be anything

Probability that the drift higher is always?

Eg $p = 0.7$ $q = 0.2$ $n = 500$

$\text{Prob}(400 \le X_{500} \le 450)$

(higher; as moving fwd is high)

$\text{Prob}(10 \le X_{800} \le 30)$

mean $= n(p-q) = 500 \times 0.5 = 250$

variance $= -(p-q)^2 + (p+q) = 0.9 - 0.25$

$= 0.65 = 500 \times 0.65$

$= 325$

$= \phi\left(\frac{450 - 1/2 + 250}{\sqrt{325}}\right) - \phi\left(\frac{400 - 1/2 - 250}{\sqrt{350}}\right)$

$= \phi\left(\frac{700 - 1/2}{\sqrt{325}}\right) - \phi\left(\frac{150 - 1/2}{\sqrt{325}}\right) = \frac{1399}{2\sqrt{325}}$ $\frac{299}{2\sqrt{325}}$

$P(-450 \le X_{800} \le 400)$

↳ very low

* Study:- If RV are not Independant — Eg Insurance, Gambling

1) ~~undistributed~~ Unrestricted Random walk

$X_n \to (-\infty, \infty)$

2) Restricted Random walk

$X_n \to [0, \infty)$

both end absorbing — one end absorbing — upper limit / lower limit

Absorbing barrier

At some pt the process has to stop

Barrier → Reflecting barrier (same) — upper limit / lower limit

Initial position

$X_n = \{j\}$

* P that it lies in bet?

prob of → $=0$ (positive drift is higher)

$X_n = a$ — $(a-1)$ — a (absorbing barrier)

$Prob(-b+1 \le X_n \le a-1 \mid X_0 = j) = 0$

$X_n = -b$ — $-b$

(will absorb as $n \to \infty$)

Possible state space. $X_n = \{-b, -b+1, \ldots 0, \ldots a-1, a\}$

* Reflecting barrier can go upto ∞

→ property! get one upward thrust then start moving upwards

3 cases in both absorbing & reflective →

$X_n = \{-b \ldots a\}$ $X_n = \{-\infty \ldots a\}$ $X_n = \{-b \ldots +\infty\}$

1/2 absorbing barrier

coeff of s^n $\lambda_1(s), \lambda_2(s) = \dfrac{1 - s(1-p-q) \pm [\{1-(1-p-q)\}^2 - 4pq s^2]^{1/2}}{2ps}$

$$F_j(s) = F_{ja}(s) = \frac{\{\lambda_1(s)\}^{j+b} - \{\lambda_2(s)\}^{j+b}}{\{\lambda_1(s)\}^{a+b} - \{\lambda_2(s)\}^{a+b}}$$

$$F_{0a}^{(n)} = (2ps)^a \sum_{v=1}^{a+b-1} \frac{\alpha_v}{1 - s/s_v}$$

$$s_v = \frac{1}{1-p-q+2(pq)^{1/2}\cos\left(\frac{v\pi}{a+b}\right)} \quad v = \{1, \dots a+b-1\}$$

$$\lambda(s_v) = \left(\frac{2}{p}\right)^{1/2} \exp\left(\pm \frac{v\pi i}{a+b}\right) \quad v = \{1, 2, \dots a+b-1\}$$

$$\alpha_v = \lim_{s \to s_v} \frac{f_{0a}(s)}{(2ps)^a}\left(1 - \frac{s}{s_v}\right) \to \frac{(-1)^{v+1}\sin\left(\frac{bv\pi}{a+b}\right)\sin\left(\frac{v\pi}{a+b}\right)}{2(a+b)(4pq)^{\frac{a}{2}} s_v^{a-b}}$$

$$\left[F_{0a}(s) = F_{0a}(1)\, E(s^N \mid X_n = a) \right]$$

$\hookrightarrow$ Generating fn

$$f_{0a}^{(n)} = (2p)^a \sum_{v=1}^{a+b-1} \frac{\alpha_v}{s_v^{(n-a)}}$$

Q Prob. for $n = 5$, $p = 0.8$, $q = 0.2$

$$\cancel{\alpha_v = (-1)^{v+1} \sin\left(\frac{bv\pi}{a+b}\right) \sin\left(\frac{v\pi}{a+b}\right)}$$

$a = 5$ $b = 3$

$$\cancel{f_{0a}^{(n)} = (2p)^a \sum_{v=1}^{a+b-1} \frac{\alpha_v}{s_v^{n-a}}}$$

$$f_{0a}^{(n)} = (1.6)^a \sum_{v=1}^{a+b-1} \cdot (-1)^{v+1} \cdot$$

$$\text{Prob}(N = n) = f_{0a}^{(n)} + f_{0b}^{(n)}$$

$$E(s^N) = F_{0a}^{(s)} + F_{0b}^{(s)}$$