ASSIGNMENT-5

Q1.

Sol. An Integrad domain is a commutative ring D with no zero devisors.

D[x] is the ring of all polynomials with coefficient D.

1. ADDition and Multiplication in D[x]:
   - The set D[x] forms a ring under polynomial addition and multiplication.
   - polynomial multiplication is associative and distributive over addition, just as in any commutative ring.

2. Since D is Commutativity, so is D[x] under polynomial multiplication.

3. let $f(x), g(x) \in D[x]$ where, $f(x) \neq 0$ and $g(x) \neq 0$

$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_0$

- The degree of product of $f(x) g(x)$ is $n+m$ and the leading coefficient of $f(x) g(x)$ is $a_n b_m$.
- Since D is an integral domain $a_n b_m \neq 0$. Thus $f(x) g(x) \neq 0$.

Q2.

Sol. A prime ideal I in a commutative ring R is an ideal such that for any $a, b \in R$, if $ab \in I$, then either $a \in I$ or $b \in I$.

In Z, an ideal $\langle p \rangle$ is set of all multiples of P $\langle p \rangle = px \; x \in Z$

1. $\langle p \rangle$ is ideal

Let $a, b \in Z$ and $r \in \langle p \rangle$

$r = pk$ $k \in Z$

$m \in Z$ $r + ma \in \langle p \rangle$

Similarly, $n \cdot r \in \langle p \rangle$ for any $n \in Z$ because $n(pk) = p(nk)$

$\langle p \rangle$ is an ideal for Z.

2. $\langle p \rangle$ is prime:

$a, b \in Z$ such that $ab \in \langle p \rangle$

$ab = pk$ for some $k \in Z$

Since p is a prime number either $p|a$ or $p|b$.

if $p|a$, then $a \in \langle p \rangle$ because $a = pm$ for some $m \in Z$

Similarly if $p|b$, then $b \in \langle p \rangle$

Therefore $ab \in \langle p \rangle$ implies $a \in \langle p \rangle$ or $b \in \langle p \rangle$

Q3.

Sol. Ideals of Z_{12}.

$Z_{12} = \{0, 1, 2, 3, \ldots, 11\}$

$\langle 0 \rangle = 0$

$\langle 1 \rangle = \{0, 1, 2, \ldots, 11\}$

$\langle 2 \rangle = \{0, 2, 4, 6, 8, 10\}$

$\langle 3 \rangle = \{0, 3, 6, 9\}$

$\langle 4 \rangle = \{0, 4, 8\}$

$\langle 6 \rangle = \{0, 6\}$

Divisors of 12 are:

$1, 2, 3, 4, 6, 12$

For $\langle 2 \rangle$: $Z_{12} / \langle 2 \rangle \cong Z_2$ which is a field

Thus $\langle 2 \rangle$ is maximal

for $\langle 3 \rangle$: $Z_{12} / \langle 3 \rangle \cong Z_3$ field

Thus $\langle 3 \rangle$ is maximal.

Q4

Sol. $f(x) = 4x - 5 \quad g(x) = 2x^2 - 4x + 2$

$f(x) = 4x + 3$ $\qquad$ $f(x) + g(x) = 2x^2 + (4x + 4x) + (2 - 5)$

$g(x) = 2x^2 + 4x + 2$

$f(x) + g(x) = 2x^2 - 3 \mod 8$

$= 2x^2 + 5$

$f(x) \cdot g(x) = (4x - 5)(2x^2 - 4x + 2)$

$= 8x^3 - 16x^2 + 8x - 10x^2 + 20x - 10$

$= 6x^2 + 4x + 6.$

Q5.

Sol. Ring homomorphism $f: R \to F$

$f(a+b) = f(a) + f(b) \quad , f(ab) = f(a) \cdot f(b)$

$\ker f = \{x \in R : f(x) = 0_F$

Since f is a homomorphism onto F, every element of F has a pre image in R making f surjective.

First Isomorphism Theorem for ring states that

$R / \ker f \cong Im(f)$

$R / \ker f \cong F$, which implies $R / \ker f$ is a field.

Maximum Ideal :

$R / \ker f$ to be a field, $\ker f$ must be a maximum ideal of R.

If $\ker f$ were not maximal, there would exist an Ideal I such that $\ker f \subsetneq I \subsetneq R$

This would imply R/I is not a field contradicting the isomorphism $R/\ker f \cong F$..

Q6.

Sol: Let M be a maximal ideal of R;

$M \neq R$ and there is no ideal properly containing M other than R.

Consider the Quotient ring R/M

- In R/M, there are no zero divisors.
- For any $a, b \in R$, if $ab \in M$, then either $a \in M$ or $b \in M$. This is because in the field R/M the image of ab is zero only if the image of a or b is zero.

$ab \in M \Rightarrow a \in M$ or $b \in M$.

Since R/M being a field implies that M satisfies the condition for being a prime ideal, every maximal ideal of R is also a prime ideal.

Q7

Sol: A ideal M is maximal if R/M is a field.

An ideal P is prime if R/P is an integral domain.

eg: Consider the commutative ring $R = \mathbb{Z}_4$

$\mathbb{Z}_4 = \{0, 1, 2, 3\}$

1. Ideal of $\langle 2 \rangle = 0, 2$ is maximal, because $\mathbb{Z}_4/\langle 2 \rangle \cong \mathbb{Z}_2$ is a field.
2. The ideal $\langle 2 \rangle$ is not prime because $\mathbb{Z}_4/\langle 2 \rangle$ has zero divisors : $2 \cdot 2 = 4 \equiv 0 \pmod 4$.

In $R = \mathbb{Z}_4$ the ideal $\langle 2 \rangle$ is maximal but not prime.

Q8.

Sol. let P_1, P_2 be prime ideal of a ring R,

Assume $P_1 \cap P_2$ is a prime ideal of R.

1. suppose. $P_1 \subseteq P_2$: $a \in P_1$ such that $a \notin P_2$.

2. $P_2 \subseteq P_1$: $b \in P_2$ such that $b \in P_1$

3. Consider ab:

$a \in P_1$ and $b \in P_2$ we have

$ab \in P_1$ and $ab \in P_2$ (as P_1 and P_2 are ideal)

Therefore $ab \in P_1 \cap P_2$.

4. Using the prime property $P_1 \cap P_2$

Since $P_1 \cap P_2$ is prime and $ab \in P_1 \cap P_2$, it follow that either $a \in P_1 \cap P_2$ or $b \in P_1 \cap P_2$

C-1 $a \in P_1 \cap P_2$

This implies $a \in P_2$; which contradicts our assumption that $a \notin P_2$

C-2 $b \in P_1 \cap P_2$

This implies $b \in P_1$ which contradicts our assumption $b \notin P_1$.

One of our assumptions ($a \notin P_2$ or $b \notin P_1$) must be false.

Therefore either $P_1 \subseteq P_2$ or $P_2 \subseteq P_1$

Q9.

Sol. Commutative ring $R = Z$

Let $P_1 = \langle 2 \rangle$, $P_2 = \langle 3 \rangle$

Both P_1 and P_2 are prime ideals because

$Z/P_1 \cong Z_2$ (a field) and

$Z/P_2 \cong Z_3$ (a field.)

The intersection $P_1 \cap P_2$ consists of all integers

$P_1 \cap P_2 = \langle 2 \rangle \cap \langle 3 \rangle = \{0\}$

which is zero ideal in Z_6.

for any ideal I to be prime, it must satisfy the condition $ab \in I$, then either $a \in I$ or $b \in I$ for all $a, b \in R$

In this case $P_1 \cap P_2 = \{0\}$ is not a prime ideal because the zero ideal $\{0\}$ does not satisfy the prime ideal.

Thus intersection of two prime ideals $P_1 \cap P_2 = \{0\}$ is not a prime ideal in this case.

Q10

Sol. Consider the ring Z_6

Let $M_1 = \langle 2 \rangle = \{0, 2, 4\}$ which is a maximal ideal $(Z_6/\langle 2 \rangle \cong Z_2)$

Let $M_2 = \langle 3 \rangle = \{0, 3\}$ which is a maximal ideal $(Z_6/\langle 3 \rangle \cong Z_3)$

$M_1 \cap M_2 = \langle 2 \rangle \cap \langle 3 \rangle = \{0\}$

which is zero ideal of Z_6

for any ideal I to be maximal, the Quotient ring R/I must be a field. In this case $M_1 \cap M_2 = \{0\}$ which is not a field because it has zero divisor.

Since $\mathbb{Z}_6$ is not a field, the zero ideal $\{0\}$ is not a maximal ideal.

$\therefore$ The intersection of two maximal ideal, does not necessarily result in a maximal ideal.

Q.11.

Sol. Let R be a Boolean ring and let P be a prime ideal of R such that $P \neq R$.

consider the Quotient ring R/P. Since P is a prime ideal the quotient ring R/P inherits the boolean ring structure from R, means

$$(a+P)^2 = a+P \quad \text{for all } a \in R.$$

Thus R/P is also an boolean ring.

In a boolean ring, the Quotient ring by a prime ideal must be a field.

consider any non-zero element $a+P \in R/P$

we know that $a+P$ is idempotent and hence $a+P$ is its own inverse in the Quotient ring.

Every non-zero element in R/P is invertive means that R/P is field and thus it follows that P is maximal.

Q12.

Sol. (i) $\deg(f(x)+g(x)) < \max\{\deg f(x), \deg g(x)\}$

Let $f(x) = x^2 + x$

Let $g(x) = -x^2 + 2x$

$f(x) + g(x) = 3x$

Degree of $f(x)+g(x) = 1$

while $\max\{\deg f(x), \deg g(x)\} = \max\{2,2\}$
$= 2$

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(ii) $\deg(f(x) \cdot g(x)) \leq \deg(f(x)) + \deg(g(x))$

$f(x) \cdot g(x) = (x^2+x)(-x^2+2x)$

$= -x^4 + x^3 + 2x^2$

Degree of $f(x) \cdot g(x) = 4$

$\deg f(x) = 2$ $\deg g(x) = 2$

$4 \leq 4$.

So this satisfies the condition.