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23/MC/087
ASSIGNMENT-2
MC 207

Q1 $S = (1, 2, 3)$

P_1 = identity Permutation $= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$

$P_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ $\quad P_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$

$P_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ $\quad P_6 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$

$P_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$

- Closure

$P_2 \circ P_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$

$\in$ set of Permutation $\therefore$ closure satisfied

→ We know function Composition is always associative

$(P_1 \circ P_2) \circ P_3 = (P_1 \circ P_3) \circ P_2$

→ We We got P_1 = Identity such that $P_1 \circ$ (any Permutation) $=$

$P_1 \circ (P_n) = P_n$ $\forall$.

→ Inverse Let P_n' be inverse of P_n $\quad P \cdot P_n' = e$

$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ x & y & z \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ y & x & z \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$

$\Rightarrow$ $y = 1 \quad x = 2$

$z = 3$

Similarly inverse of each group exist $\therefore$ inverse exist $\therefore$ group.

Q2 Symmetric group S_n is group of all permutation of set $\{1, 2, n\}$
G is abelian if $g_1 \circ g_2 = g_2 \circ g_1$ $(g_1, g_2 \in G)$

$n = 3$

$(1, 2, 3)$

Let $P_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ $P_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$

$P_1 \cdot P_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ $P_1 \cdot P_2 \neq P_2 \cdot P_1$

$P_2 \cdot P_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \end{pmatrix}$ $\therefore$ from $n > 4$ also
Commutativity won't follow
$\therefore$ S_n is non abelian for $n \geq 4$

Q3 Lagranges' theorem let $H < G$ then $O(G) = k\,O(H)$

Let $O(G) = n$ and $O(H) = m$

We know $G = g_1H_1 \cup g_2H_2 \cup H_3 \cdots$

$O(G) = O(g_1H_1) + O(g_2H_2) + O(g_3H_3) \cdots$

We know order of $O(H) = O(g_nH)$

$O(G) = O(H) + O(H) \cdots O(H)$

$n = k(m)$

$\therefore$ Hence Proved

Q4 (i) Left coset = left coset of H in g wrt $g \in G$
is defined by gH

Right coset = Hg . def of H, $g \in G$

(ii) Let L be set of left coset and R be set of right coset then to Prove $n(L) = n(R)$ we should Prove ther exists a bijective function b/w L and R

Let $\phi_R : \to L$ $\quad \phi(Ha^{-1}) = aH$

Since each element of aH of L has a is image of R ~~a^{-1}~~ Ha^{-1} $\therefore$ onto of n

to Prove one - one

$\phi(Ha^{-1}) = \phi(Hb^{-1})$

$aH = bH$

$H = a^{-1}bH$

$a^{-1}b \in H \quad \therefore (a^{-1}b)^{-1} \in H$

$b^{-1}a \in H$

$H = Hb^{-1}a$

$Ha^{-1} = Hb^{-1} \quad \therefore$ one - one

$\therefore \phi =$ bijective $\therefore$ $n(L) = n(R)$

Q6 if G is non abelian and $aH = Ha$ then H is normal subgroup

$x \in G, h \in H$

$xhx^{-1} \in H$

Let S_4 be a group and A_4 be alternaly subgroup Centre e of $A_4 = \{e\}$

Let $a \in A_4$ and $x \in S_4$

xax^{-1} is also even $[xax^{-1} = ~~x~~ e)$

Because $\cdot\ xax^{-1} \in A$ $\quad = xa$

Q5 To Prove A_n is normal Subgroup of S_n
Let's Prove A_n is a Subgroup of S_n
Let $f, g \in A_n$
$f =$ even, $g =$ even
$\therefore f^{-1}, g^{-1} \Rightarrow$ even
$\therefore f \cdot g^{-1} = gf^{-1} \in A_n$ $\therefore$ Subgroup

Now to Prove $A_n \trianglelefteq S_n$
We need to Prove $f \circ g \circ f^{-1} \in A_n$
where $f \in S_n$ $g \in A_n$
f can be even and odd
Case 1 $f \to$ even $\Rightarrow f^{-1}$ even g is always even
$\therefore f \circ g \circ f^{-1} =$ even $\circ$ even $\circ$ even $=$ even $\in A_n$
Case 2 $f \to$ odd $\to f^{-1} =$ odd g is always even
$f \circ g \circ f^{-1} =$ odd $\circ$ even $\circ$ odd^{-1}
$\Rightarrow$ odd $\circ$ odd $\circ$ even $=$ even $\circ$ even
$=$ even $\in A_n$
$\therefore \forall f, g \in S_n, A_n$ $fgf^{-1} \in A_n$

Q7 To Prove every cyclic group abelian
Let ~~$S = (a\ b\ c)$~~ $\begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}$

Let $G = \langle x \rangle$ be cyclic group
$a, b \in G$
$a = x^n$ and $b = x^m$
$a \cdot b = x^{n+m} = x^{m+n} = x^m \times x^n = b \cdot a$
$a \cdot b = b \cdot a$ $\therefore$ abelian.

Q8 $Q_8 = \{\pm 1, \pm i, \pm J, \pm k\}$

$i^2 = J^2 = k^2 = i \cdot J \cdot k = -1$

$i \cdot j = k$
$j \cdot i = -k$
$j \cdot k = i$
$k \cdot j = -i$
$k \cdot i = j$
$i \cdot k = -j$

	1	-1	i	-i	j	-j	k	-k
1	1	-1	i	-i	j	-j	k	-k
-1	-1	1	-i	i	-j	j	-k	k
i	i	-i	-1	1	k	-k	-j	j
-i	-1	i	1	-1	-k	k	j	-j
j	j	-j	-k	k	-1	1	i	-i
-j	-j	j	k	-k	1	-1	-i	i
k	k	-k	j	-j	-i	i	-1	1
-k	-k	k	-j	j	i	-i	1	-1

Q. Find all left coset of $\langle 1, 11 \rangle$ in $U(30)$

$U(30) = \{1, \cancel{5}, 7, 11, 13, 17, 19, 23, 29\}$

$\cancel{H} = 1 \cdot H = \{1, 11\}$

$\cancel{5H \; \{5, 55\}}$

$7 \cdot H = \{7, 77\} \% 30 = \{7, 17\}$

$11 \cdot H = \{11, 121\} \% 30 \; \{1, 11\}$

$13 \cdot H = \{13, 143\} \% 30 = \{13, 23\}$

$17 \cdot H = \{17, 187\} \% 30 = \{17, 7\}$

$19 \cdot H = \{19, 209\} \% 30 = \{19, 29\}$

$23H = \{23, 253\} \% 30 = \{23, 13\}$

$29H = \{29, 319\} \% 30 = \{29, 23\}$

Distinct coset $= \{\{1, 11\}, \{7, 17\}, \{13, 23\}, \{19,$

Q10 $o(G) = n$ then $o(H) = m$ st $n = km$
60 = order of subgroup can be 1, 2, 3, 4, 5, 6
10, 12, 15, 20, 30, 60

Q11 $K_4 = \{e, a, b, c\}$
$a^2 = b^2 = c^2 = e^2 = e$
To Prove K_4 an abelian group
$\forall x, y \in K_4 \quad x \cdot y = y \cdot x$

$a \cdot e = e \cdot a = e$
$b \cdot e = e \cdot b = e$
$c \cdot e = e \cdot c = e$
$a \cdot b = b \cdot ab = b \cdot a$
$b \cdot c = bc = c \cdot b$
$c \cdot a = ca = a \cdot c$
$\therefore K_4$ abelian

Q12 For any group to be cyclic there must exist $x \in G$
such that $\forall y \in G = x^n$
also cyclic group of order 4 $g^4 = e$
and $g^k \neq e$ for $k < 4$
$k = 0, 1, 2, 3$
in K_4 we have $a^2 = e \quad b^2 = e \quad c^2 = e \quad e^2 = e$
each element has maximum order 2 not 4 $\therefore$ not cyclic

Q3 $K_4 = \{e, a, b, c\}$
subset Subgroup = $\{e\}$ $\{e, a, b, c\}$, $\{e, a\}$, $\{e, b\}$, $\{e, c\}$
$\{a, b\}$, $\{a, c\}$ $\{c, a\}$
~~Subgroups~~ subsets $\{e\}$, $\{e, a, b, c\}$, $\{c, e\}$ $\{e, b\}$
$\{e, c\}$ are subgroup

~~because don't have identity element ∴ every~~
~~$\{e\}$, $\{e, a, b, c\}$, $\{e, a\}$, $\{e, b\}$, $\{e, c\}$~~
~~$xHx^{-1} \in H$ $x \in G$ and~~
~~$\{e\}$ $e \in H$~~
~~$\{e, a, b, c\}$~~

K_4 = abelian group ∴ each subgroup is normal.

Q12 $S_3 = \{(e)(12), (13)(23), (123), (132)\}$

$H = \{e, 12\}$

$132 \cdot 12 (132^{-1}) = 13 \notin H$

∴ H is not normal in S_3

Q13 The cosets of subgroup of Q_8

$Q_8 = \{\pm 1, \pm i, \pm J, \pm k\}$

The cosets of subgroup of Q_8 are

(i) $\{1\} = \{1\}, \{i\} \{-1\}, \{-i\}, \{J\} \{-J\} \{k\} \{-k\}$

(ii) $\{1, -1\} = \{1, -1\} \{i, -i\} \{J, -J\}, \{k, -k\}$

(iii) $\{i\} = \{1, i, -1, -i\} \{J, J, k, -k\}$

(iv) For J $\{1, J, -1, -J\}, \{i, -1, J, -J\}, \{k, -k, i, -i\}$

(v) For k

$\{1, k, -1, -k\} \{i, -i, k, -k\} \{J, -J, k, -k\}$

Q14 ~~$N \triangleleft G$ let $x \in N$ let $n \in$~~
let $n \in (M \cap N)$
$\therefore \forall n \in (M \cap N)$ $n \in M$ and $n \in N$
since N is normal subgroup $\therefore x n x^{-1} \in N$ $[x \in G]$
also M is normal subgroup $x n x^{-1} \in M$
$\therefore x n x^{-1} \in (N \cap M)$
$\therefore N \cap M$ is normal subgroup of G

Q15 index = no. of distinct cosets (right or left)
index = 2 $\therefore$ cosets = H, gH

Let $h \in H$ then $x h x^{-1} = x x^{-1} h = h \in H$
$\therefore x h x^{-1} \in H$

$gH = gH \Rightarrow g h g^{-1} = g h^{-1}$ $h g^{-1} = h^{-1}$ or $h = g h^{-1} g$

in both case multiplying $\forall x \in G$ and it's inverse
with $h \in gH / h$ we get it in itself
$\therefore$ Hece Proved

Q16 (i) $(12)(3564)$ Order = lcm $(2,4) = 4$
(ii) $(1753)(264)$ order = lcm $(4,3) = 12$
(iii) $(124)(3578)$ order Lcm $(3,4) = 12$

Q17 Let $G/Z(G)$ be cyclic and form $\langle a \rangle$ of order n
$a = Z(G)$
$\therefore$
$G/Z(G) = \langle Z^1 \dots Z^n \rangle$ as for every $g \in G$ can be
equal to finite Product of number of $Z(G)$
hence g itself $\in Z(G)$ $\forall g \in G$ due to closure
Property

Q18 a) $g = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ $g^{-1} \begin{pmatrix} \frac{1}{a} & \frac{-b}{ac} \\ 0 & 1/c \end{pmatrix}$

$n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ $gng^{-1} = \begin{bmatrix} 1 & ax/c \\ 0 & 1 \end{bmatrix}$

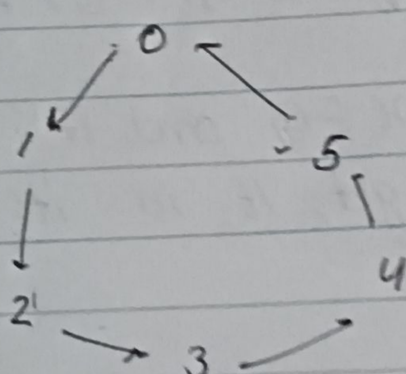
as $ac \neq 0 \Rightarrow c \neq 0$ hence $\frac{ax}{c}$ is defined
and $gng^{-1} \in N$ when $b = \frac{bx}{c}$

b) As element of G/N is of form xN. Let xN be and yN be cosets of G/N

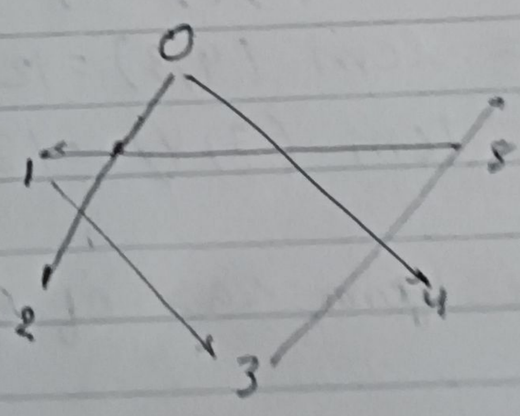
$xN = \begin{pmatrix} x & bx \\ 0 & x \end{pmatrix}$ $yN = \begin{pmatrix} y & by \\ 0 & y \end{pmatrix}$

$xN \cdot yN = yN \cdot xN = \begin{pmatrix} xy & abxy \\ 0 & xy \end{pmatrix}$ $\therefore \frac{G}{N}$ abelian

Q19



$\text{cay}(\{1\}, \mathbb{Z}_6)$



1 <2>

1 <3>

cay ({2,3}, 2)

Q20 $(1\ 3\ 5)(12) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 3 & 5 & 4 & 1 & 6 & 7 & 8 & 9 \end{pmatrix}$

$\bar{a} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 1 & 2 & 4 & 3 & 6 & 7 & 8 & 9 \end{pmatrix}$

6 $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 2 & 3 & 4 & 7 & 6 & 9 & 8 & 1 \end{pmatrix}$

$a^{-1}ba = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 1 & 3 & 4 & 5 & 6 & 9 & 8 & 2 \end{bmatrix}$

~~Q21 (i)~~ $\cancel{\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 1 & 3 & 4 & 5 & 6 & 9 & 8 & 2 \end{bmatrix}}$

Q21 (i) $(1\,2\,3\,4\,5)(8\,9)$

$= (15)(14)(13)(12)(89)$ odd

(ii) $(1\,2\,3\,4\,5)(1\,2\,3)$

$= (15)(14)(13)(12)(13)(12)$ even

Q22 $G \to$ abelian $N \triangleleft G$

$aN = \{an\}$ $n \in N$

$an \cdot bn = (a \cdot b)N = (ba)N$

$\therefore an \cdot bN = bN \cdot aN$ $\forall$ aN, bN G. G/N $\therefore G/N$

abelian group

a abelian
$\frac{a}{N}$

...ined

...t xN

$\begin{matrix} 9 \\ 9 \end{matrix}\Big)$

$)$