Modern algebra (MC-207)
Assignment - 3
2024 (ODD semester).

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given: $(\mathbb{Z}, +)$, $(E, +)$, group of integers & even integers respectively under addition operation.

To determine: if $\phi: \mathbb{Z} \to E$ defined by $\phi(x) = 2x \quad \forall x \in \mathbb{Z}$ is an isomorphism.

Sol^n: we need to show following.

1. Homomorphism: ϕ preserve the grp operat^n.
2. Bijective: ϕ should by bijective.

1- Homomorphism:

for any $x, y \in \mathbb{Z}$

$$\phi(x+y) = 2(x+y) = 2x + 2y = \phi(x) + \phi(y)$$

Thus, ϕ preserves the group operation, so it's a homomorphism.

2- To check if ϕ is injective, assume $\phi(x) = \phi(y)$:

$$\phi(x) = \phi(y)$$
$$2x = 2y$$
$$x = y.$$

$\Big\{$ · $\phi(x) = 2x$
· left cancellation

$\therefore$ ϕ is injective.

- surjective: let $e \in E$, since E is the set of all even integers, we can write $e = 2k$, for some $k \in \mathbb{Z}$

$$\phi(k) = 2k = e.$$

Thus $\forall$ $\phi(x) \in E$ $\exists$ $x \in \mathbb{Z}$ s.t $\phi(x) = 2x$.

$\therefore$ ϕ is surjective (onto).

$\square$

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2. given: $(\mathbb{Z}, +)$ & $G = \{-1, 1\}$ under multiplication.

$\phi : \mathbb{Z} \to G$.

$$\phi(x) = \begin{cases} 1 & ; \text{ if } x \text{ is even} \\ -1 & ; \text{ if } x \text{ is odd} \end{cases}$$

To check: ϕ is homomorphism or isomorphism.

Soln: Check for homomorphism;-

let $x, y \in \mathbb{Z}$

case 1: x, y both are even.

$$\phi(x+y) = 1 = 1 \cdot 1 = \phi(x) \cdot \phi(y).$$

- even + even = even.
- odd + odd = even.
- odd + even = odd.

case 2: x, y both are odd

$$\phi(x+y) = 1 = -1 \cdot -1 = \phi(x) \cdot \phi(y)$$

Case 3: x, is odd & y is even

$$\phi(x+y) = -1 = -1 \cdot 1 = \phi(x) \cdot \phi(y)$$

In all three case $\phi(x)$ preserve the group operation.

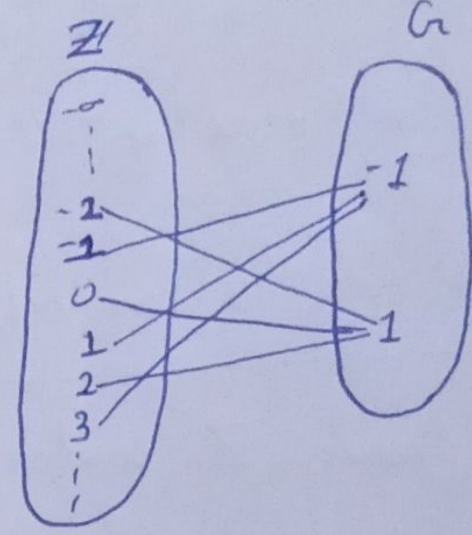
$\therefore$ $\phi(x)$ is group homomorphism.

2. Check $\phi(x)$ is one-one :-

let $\phi(x) = \phi(y)$ where $x, y \in \mathbb{Z}$

if x is even & y is odd

$\phi(x) = 1$ & $\phi(y) = -1$.



also all even integer map to 1.

& all odd integers map to -1

$\therefore$ $\phi(x)$ is not injective (one-one)

hence $\phi(x)$ is not an isomorphism.

3. given: $\phi : G \to G'$ is a homomorphism

To show: $\phi(x^{-1}) = (\phi(x))^{-1}$

Solⁿ: $\phi(e) = \phi(x x^{-1})$

$e' = \phi(x) . \phi(x^{-1})$

$\Rightarrow (\phi(x))^{-1} = \phi(x^{-1})$

{ $e \to$ identity of grp G
x, x^{-1} elements of G.
• $\phi(x)$ is homomorphism
• $\phi(e) = e'$

$\square$

4. given: $\phi : G \to G$, $\phi(x) = x^{-1}$ is a homomorphism.

To show: G is abelian.

Sol^n: let $x, y \in G \Rightarrow xy \in G$

$\phi(xy) = \phi(x).\phi(y)$ $\quad \phi(x), \phi(y), \phi(x.y) \in G.$

$\phi(xy) = \phi(x)\,\phi(y)$

$(xy)^{-1} = x^{-1} y^{-1}$

$y^{-1}x^{-1} = x^{-1}y^{-1}$ $\quad \left\{ \begin{array}{l} (xy)^{-1} = y^{-1}x^{-1} \\ x^{-1}, y^{-1} \in G. \end{array} \right.$

$\Rightarrow$ G is an abelian group.

$\left\{ \begin{array}{l} \cdot\ G \text{ is a group.} \\ \cdot\ \phi(x) \text{ is a homomorphism} \\ \cdot\ \phi(x) = x^{-1} \quad \forall x \in G \end{array} \right.$

$\to A\}$

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5. To show: Z_4 under addition modulo 4 is isomorphic to the grp U_5 under multiplication modulo 5.

Sol^n: $Z_4 = \{0, 1, 2, 3\}$ $\quad U_5 = \{1, 2, 3, 4\}.$

now, finding order of elements of Z_4 & U_5

In Z_4:

- 0 has order 1
- 1 has order 4
- 2 has order 2
- 3 has order 4

In U_5:

- 1 has order 1
- 2 has order 4
- 3 has order 4
- 4 has order 2

based on the order we can match elements between Z_4 & U_5 as follow.

map $0 \in Z_4$ to $1 \in U_5$, since both have order 1

map $1 \in Z_4$ to $2 \in U_5$, since both have order 4

map $2 \in Z_4$ to $4 \in U_5$, since both have order 2.

map $3 \in Z_4$ to $3 \in U_5$, since both have order 4.

Thus, define the mapping:

$$\phi : Z_4 \to U_5 \text{ by } \phi(0) = 1,\ \phi(1) = 2,\ \phi(2) = 4,\ \phi(3) = 3$$

Clearly ϕ is a bijection mapping as each member of Z_4 uniquely mapped with the element of U_5, with order matching, covering all elements of U_5.

now; Check that ϕ Preserve group operation; i.e;

$$\phi(a \oplus_4 b) = \phi(a) \odot_5 \phi(b) \qquad a, b \in Z_4$$

- For $a = 1$ & $b = 1$.

$$\phi(1 \oplus_4 1) = \phi(2) = 4 = 2 \cdot 2 = \phi(1) \cdot \phi(1).$$

for $a = 2$ & $b = 3$

$$\phi(2 \oplus_4 3) = \cancel{\ } 2 : \quad \& \quad \phi(2) \odot_5 \phi(3) = 4 \odot_5 3 = 2.$$

So,
$$\phi(2 \oplus_4 3) = \phi(2) \odot_5 \phi(3).$$

in each case $\phi(a \oplus_4 b) = \phi(a) \odot_5 \phi(b)$, so, ϕ is a homomorphism.

$\square$

6. To Prove: Cayley's theorem

Sol^n: Cayley's theorem states that: every group is isomorphic to some permutation group

$$G \cong P_A$$

let G be a group, ~~then~~ and corresponding to every element say $a \in G$, we define a map f_a as follow.

$$f_a(x) = ax \qquad x \in G.$$

$\therefore$ $a \in G$, $x \in G \Rightarrow ax \in G$ {closer law hold in group.

$$f_a : G \to G.$$

further, for any $x, y \in G$.

$$f_a(x) = f_a(y) \Rightarrow ax = ay$$
$$x = y.$$
{by left cancellation

$\therefore$ f_a is one-one

and for every $x \in G$. $\exists$ $a^{-1}x \in G$ s.t.

$$f_a(a^{-1}x) = a(a^{-1}x) = (aa^{-1})x = ex = x.$$

$\therefore$ f_a is onto

hence f_a is one-one mapping of G onto G itself.

$\Rightarrow$ f_a is a permutation of G.

now, let $G' = \{ f_a \mid a \in G \}$.

Clearly $G' \subset S_G$ {$S_G \to$ set of all permutation of G

Let us now consider a mapping ϕ from G to S_G, defined

$$\phi : G \to S_G \quad , \quad \phi(x) = f_x \quad \forall x \in G.$$

now, for every $x, y \in G$.

$$\phi(xy) = f_{xy} = f_x f_y = \phi(x)\phi(y)$$

$\therefore$ ϕ is a homomorphism from a group G to S_G.

Consequently G' is a subgroup of the permutation group S_G. and ϕ is an epimorphism from G onto G'.

also for any $a, b \in G$.

$\phi(a) = \phi(b) \Rightarrow f_a = f_b.$

$\Rightarrow f_a(x) = f_b(x)$

$ax = bx$ $\quad$ [right cancellation

$a = b.$

$\therefore$ ϕ is one-one.

hence, ϕ is an isomorphism from a group G onto permutation.

$\boxed{}$

7. To show: homomorphic image of an abelian group is abelian.

Soln: let G be an abelian group & H be some another group.

and $\phi : G \to H$ be a homomorphism from G to H.

Since G is abelian, for any $a, b \in G$, we have.

$$ab = ba.$$

$$\phi(ab) = \phi(ba)$$ {taking ϕ both side.

$$\phi(a)\phi(b) = \phi(b)\phi(a)$$ {$\because \phi$ is an homomorphism $\phi(a), \phi(b) \in H$.

$\Rightarrow$ H is abelian.

$\therefore$ the homomorphic image of an abelian group is abelian.

but, Converse doesn't hold.

Counter example: Consider S_3 which is non-abelian and Z_2 cyclic group of order 2.

Define a mapping $\phi: S_3 \rightarrow \mathbb{Z}_2$, by mapping even permutation to 0 & odd permutation to 1.

- Since Z_2 is abelian, the image of ϕ, which is Z_2, is abelian. However S_3 is not abelian, demonstrating that the converse doesn't hold.

10:-

given: G is a finite abelian group & $o(G)$ & n are co-prime.

To show: $\phi: G \rightarrow G$ s.t $\phi(x) = x^n$ is an isomorphism.

Soln: we have G, finite abelian group, so for every $a, b \in G$ $ab = ba$.

now; $\phi(a) = a^n$ & $\phi(b) = b^n$, $a^n, b^n \in G$.

$$\phi(ab) = (ab)^n = \underbrace{ab \cdot ab \cdot ab \cdots\cdots ab}_{n\text{-times}}$$

$$\phi(ab) = (\underbrace{a \cdot a \cdot a \cdots\cdots a}_{n\text{-times}})(\underbrace{b \cdot b \cdot b \cdots\cdots b}_{n\text{-times}})$$ $\{ ab = ba.$

$$= a^n \cdot b^n = \phi(a) \cdot \phi(b)$$

Thus, ϕ preserve group operation and is a homomorphism.

now; to show ϕ is injective (one-one).

let us assume $\phi(a) = \phi(b)$

$$\Rightarrow \quad a^n = b^n$$

$$a^n b^{-n} = b^n \cdot b^{-n}.$$

$$(ab^{-1})^n = e$$

$\{$ Post multiply by b^{-n} $\cdot\, b^n \cdot b^{-n} = e$, where e is the identity of G.

$\therefore$ The element $z = xy^{-1}$ satisfy $z^n = e$.

since, G is a finite abelian group, order of any element of G. divides order of G. let order of z be d, so $d \mid o(G)$.

since $z^n = e$, $\Rightarrow$ $d \mid n$.

but $o(G)$ & n are co-prime, so the only integer divides both $o(G)$ & n is 1.

$$\Rightarrow z = e \Rightarrow xy^{-1} = e$$

$$xy^{-1} \cdot y = ey$$

$$\Rightarrow x = y.$$

$\{$ Post multiply by y.

$\therefore$ $x = y$. $\Rightarrow$ ϕ is injective.

since, G is finite & ϕ is an injective homomorphism, it must also be surjective (by Pigeonhole principle).

11. To Proof: If $\phi : G \to G'$ is homomorphism then kernel of ϕ is a normal subgroup of G.

Sol^n: $\ker\phi = \{x \mid \phi(x) = e', \text{ where } x \in G\}$

{ $e' \to$ identity element of G'

we need to show following:

(i) $\ker\phi \neq \{\}$.
(ii) if $a, b \in \ker\phi$ then $ab^{-1} \in \ker\phi$] — subgroup condition.
(iii) $g \ker\phi\, g^{-1} \subset \ker\phi$] — normal subgroup condition.

(i) – $\ker\phi \neq \{\}$:

we have e, identity element of G.

$\phi(e) = e'$ { ϕ is a homomorphism

$\Rightarrow e \in \ker\phi$

$\Rightarrow \ker\phi \neq \{\}$.

(ii) – If $a, b \in \ker\phi$ then $ab^{-1} \in \ker\phi$:

let $a, b \in \ker\phi \Rightarrow \phi(a) = e'$ & $\phi(b) = e'$

$$\phi(ab^{-1}) = \phi(a)\phi(b^{-1})$$
$$= \phi(a)(\phi(b))^{-1}$$
$$= e'.e'^{-1} = e'$$

{ ϕ is a homomorphism, $\phi(b^{-1}) = (\phi(b))^{-1}$

$\Rightarrow ab^{-1} \in \ker\phi$

$\therefore \ker\phi \leq G$.

now let $k \in \ker\phi$ & $g \in G$.

$$\phi(gkg^{-1}) = \phi(g)\,\phi(kg^{-1})$$
$$= \phi(g)\,\phi(k)\,\phi(g^{-1})$$
$$= \phi(g)\cdot e'\cdot(\phi(g))^{-1}$$
$$= \phi(g)\cdot(\phi(g))^{-1}$$
$$= e'$$

$\{\ \phi(k) = e'$
$\ \ \phi(g^{-1}) = (\phi(g))^{-1}$

$\Rightarrow\ gkg^{-1} \in \ker\phi$

$\Rightarrow\ g\ker\phi\, g^{-1} \subseteq \ker\phi$ hence $\ker\phi \trianglelefteq G$ $\square$

13: To Proof: Group G is abelian iff the mapping $\phi : G \to G$, given by $\phi(x) = x^2$, is a homomorphism.

Soln: (i) To Prove: If G is abelian then the mapping $\phi : G \to G$ given by $\phi(x) = x^2$ is homomorphism.

let $a, b \in G \Rightarrow ab = ba$ $\{\because G$ is abelian

$\Rightarrow \phi(a) = a^2$ & $\phi(b) = b^2$.

now; $\phi(ab) = (ab)^2$
$= ab\cdot ab$.
$= (aa)(bb) = a^2\cdot b^2 = \phi(a)\cdot\phi(b)$

thus, ϕ Preserve the group operation.

$\therefore$ ϕ is a homomorphism.

now; (ii) if $\phi : G \to G$; $\phi(x) = x^2$ is a homomorphism, then G is abelian.

let $a, b \in G$, $\Rightarrow ab \in G$ {G is a group.

$\phi(ab) = \phi(a)\,\phi(b)$ {∵ ϕ is a homomorphism. $\phi(x) = x^2$; $x \in G$.

$(ab)^2 = a^2 \cdot b^2$

$abab = a \cdot a \cdot b \cdot b$ {by left & right cancellation

$ab = ba$.

$\Rightarrow G$ is an abelian group.

hence Proved $\square$

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14. To Prove: If $\phi: G \to G'$ is a homomorphism, then $\mathrm{Im}\,\phi$, the image of ϕ, is indeed a subgroup of G'

soln: To Prove that $\mathrm{Im}\,\phi \leq G'$, we need to prove follows:

(i) $\mathrm{Im}\,\phi$ is non-empty.

(ii) if $a, b \in \mathrm{Im}\,\phi \Rightarrow ab^{-1} \in \mathrm{Im}\,\phi$

(i) Since ϕ is a homomorphism, $\phi(e) = e'$, where e & e' is the identity of G & G' respectively.

$\Rightarrow e' \in \mathrm{Im}\,\phi$

so, $\mathrm{Im}\,\phi$ is non-empty.

(ii) let $a, b \in \mathrm{Im}\,\phi$, then $\exists\, g_1, g_2$ s.t $\phi(g_1) = a$ & $\phi(g_2) = b$

$ab^{-1} = \phi(g_1)(\phi(g_2))^{-1} = \phi(g_1)\,\phi(g_2^{-1}) = \phi(g_1 g_2^{-1})$. {∵ ϕ is a homomorphism

$g_1 g_2^{-1} \in G \Rightarrow ab^{-1} \in \mathrm{Im}\,\phi$

$\therefore \mathrm{Im}\,\phi \leq G'$ $\square$

To Prove; If $\phi: G \to G'$ is a homomorphism, then $\ker \phi = \{e\}$ iff ϕ is one-one.

Soln:- 1st we take if ϕ is one-one then $\ker \phi = \{e\}$.

If ϕ is one-one then the only element in G that maps to the identity element e' in G', because, otherwise ϕ would not be injective

$\therefore \ker \phi = \{g \in G : \phi(g) = e'\} = \{e\}$. (i)

now; ~~If ker {φ} ≠ e'~~
if $\ker \phi = \{e\}$ then ϕ is one-one.

Let us assume $\phi(g_1) = \phi(g_2)$ for some $g_1, g_2 \in G$.

$$\phi(g_1) = \phi(g_2)$$

{ Post multiply by $(\phi(g_2))^{-1}$

$$\phi(g_1) . (\phi(g_2))^{-1} = \underbrace{\phi(g_2) . (\phi(g_2))^{-1}}_{e'}$$

$$\phi(g_1) . (\phi(g_2))^{-1} = e'$$

$$\phi(g_1 g_2^{-1}) = e'$$

{ $\because \phi$ is a homomorphism

$$\Rightarrow g_1 g_2^{-1} \in \ker \phi$$

so, $g_1 g_2^{-1} = e$

$$\Rightarrow g_1 = g_2.$$

$\therefore \phi$ is one-one (ii)

from (i) & (ii).

$\ker \phi = \{e\} \Longleftrightarrow \phi$ is one-one $\square$

9\. To Prove: $G/\{e\} \cong G$ & $G/G \cong \{e\}$.

Proof: Consider the quotient group $G/\{e\}$., where $\{e\}$ is the trivial subgroup containing only the identity element of G.

Thus the coset of $\{e\}$ in G of the form $g.\{e\} = \{g\}$ $\forall$ $g \in G$.
So every coset is simply the element itself: $\{g\} = g$.

So, we define a map $\phi : G \to G/\{e\}$ by $\phi(g) = g.\{e\}$.
This map is:

well defined: for any $g \in G$, $g.\{e\}$ is a unique coset.

- Bijective:
- Injective: if $\phi(g_1) = \phi(g_2)$ then $g_1\{e\} = g_2\{e\}$ $\Big\{$ right cancellat-
$\Rightarrow g_1 = g_2$.

- Surjective: for any $g.\{e\} \in G/\{e\}$ there is a corresponding element $g \in G$.

- Homomorphism: for $g_1, g_2 \in G$ $\phi(g_1 g_2) = (g_1 g_2)\{e\} = (g_1.\{e\})(g_2.\{e\})$
$= \phi(g_1).\phi(g_2)$

Hence, ϕ is an isomorphism so $G/\{e\} \cong G$.

now; consider the quotient group G/G, so the only coset in G/G is G itself: $G = g.G$ for any $g \in G$.

Thus, G/G consist of a single element, which is the identity element in the quotient group.

- Define a map $\phi : G/G \to \{e\}$ by mapping the only coset

G to e

- This map is clearly:
    - well defined: There is only one element in G/G, which ~~is~~ maps to e.
    - It is both injective & surjective as there is only one element in each set.
    - Homomorphism: Trivially held as the group operation in G/G maps directly to the operation in $\{e\}$.

Thus $G/G \cong \{e\}$.

$\square$

8. given: G : group of 2x2 matrices over reals of the type

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ s.t } ad-bc \neq 0 \text{ under matrix multiplicat}^n$$

G' : group of non-zero real numbers under multiplication.

To show: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \to ad-bc$ is an onto homomorphism.

Sol^n: let ϕ be the given map s.t.

$$\phi : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \to ad-bc \quad \text{or} \quad \phi : G \to G'$$

$$\phi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad-bc \qquad \begin{cases} ad-bc \in G' \\ \& \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G. \end{cases}$$

now, let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ & $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \in G'$

$$\text{so} \quad \phi(AB) = \phi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix}\right)$$

$$A \cdot B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}.$$

So, $\phi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix}\right) = \phi\left(\begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}\right).$

$$\Rightarrow \phi\left(\begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}\right) = (ae+bg)(cf+dh) - (af+bh)(ce+dg)$$

$$= a\cancel{ecf} + aedh + bgcf + bg\cancel{dh} - a\cancel{fce} - afdg - bhce - b\cancel{hdg}$$

$$= ad(eh-fg) - bc(eh-fg)$$

$$= (ad-bc)(eh-fg)$$

$$= \phi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)\phi\left(\begin{bmatrix} e & f \\ g & h \end{bmatrix}\right)$$

$$= \phi(A)\,\phi(B)$$

$$\Rightarrow \phi(AB) = \phi(A)\,\phi(B)$$

$\Rightarrow$ ϕ preserve the group operation hence ϕ is a homomorphism.

also for every $ad-bc \in G'$ $\exists$ a 2x2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G$ s.t $\phi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad-bc$.

hence ϕ is an epimorphisam or we say ϕ is an onto homomorphism.

To verify: Fundamental theorem of Group homomorphism. ϕ

sol^n:-

$$\phi(x) = \begin{cases} 1 & ; \text{ if } x \text{ is even} \\ -1 & ; \text{ if } x \text{ is odd} \end{cases} \qquad \phi: \mathbb{Z} \to G.$$

where $\mathbb{Z}$ is the group of integers under addition & $G = \{1, -1\}$ is a group under multiplication.

- verify that ϕ is a homomorphism

  Let $a, b \in \mathbb{Z}$

  To show: $\phi(a+b) = \phi(a)\phi(b)$

Case 1: both a & b are even ; $a = 2n$ & $b = 2m$.

$\phi(a) = 1$ & $\phi(b) = 1$.

$\phi(a+b) = \phi(2n+2m) = \phi(2(m+n)) = 1 = 1 \cdot 1 = \cancel{2m \cdot 2n}$

$= \phi(2n)\phi(2m)$

$= \phi(a)\phi(b)$.

Case 2: both a & b are odd; $a = 2n+1$ & $b = 2m+1$

$\phi(a) = \phi(2n+1) = -1$

$\phi(b) = \phi(2m+1) = -1$

so; $\Rightarrow \phi(a+b) = \phi(2n+1+2m+1)$

$\Rightarrow \phi(a+b) = \phi(2(m+n+1)) = 1 = -1 \cdot -1$

$= \phi(2n+1)\phi(2m+1)$

$= \phi(a)\phi(b)$

$\Rightarrow \phi(a+b) = \phi(a)\phi(b)$

Case 3: either of a or b is odd and other is even.

Let b is odd s.t $b = 2n+1$ & a is even s.t $a = 2m$.

$\phi(a) = \phi(2m) = 1$

$\phi(b) = \phi(2n+1) = -1$

so; $\phi(a+b) = \phi(2m+2n+1) = -1$

$= 1 \cdot (-1)$

$= \phi(2m)\phi(2n+1)$

$= \phi(a)\phi(b)$

$\Rightarrow \phi(a+b) = \phi(a)\phi(b)$

now; kernal of ϕ:

$\ker\phi = \{n \in \mathbb{Z} \mid \phi(n) = e'\}$

{ e' is the identity element of group $\{-1, 1\}$ • $e' = 1$

$\Rightarrow$ for $\phi(n) = e' = 1$, n must be even

Thus $\ker\phi$ is the set of all even integers

$\therefore \ker\phi = 2\mathbb{Z}$

now; Image of ϕ:

The image of ϕ is:

$$Im(\phi) = \{\phi(n) \mid n \in \mathbb{Z}\}.$$

- The possible value of $\phi(n)$ are 1 & -1.

Thus $Im(\phi) = G' = \{-1, 1\}$

So, The fundamental theorem of Group homomorphism states:

$$\mathbb{Z}/\ker\phi \cong Im(\phi)$$

we have $\ker\phi = 2\mathbb{Z}$. & The quotient grp $\mathbb{Z}/2\mathbb{Z}$ consists of two cosets:

- The coset of even integers: $2\mathbb{Z}$
- The coset of odd integers: $1 + 2\mathbb{Z}$
  - The group $\mathbb{Z}/2\mathbb{Z}$ has two elements: $[0]$ & $[1]$.

now; consider the map $\varphi : \mathbb{Z}/2\mathbb{Z} \to G$ defined by,

$$\varphi([n]) = \phi(n)$$

- If n is even, then $[n] = 0$ & $\varphi([0]) = \phi(0) = 1$.
- If n is odd, then $[n] = [1]$ & $\varphi([1]) = \phi(1) = -1$.

This map is well defined isomorphism between $\mathbb{Z}/2\mathbb{Z}$ & $Im\ \phi$

{ $2\mathbb{Z} = \ker\phi$ & $Im\ \phi = \{-1, 1\}$

hence $\mathbb{Z}/\ker\phi \cong \{-1, 1\}$

$\square$