

5 Renewal processes

5.1 Definition

Let the r.v. T denote the *failure time* or *lifetime* of a component, where the lifetime distribution is continuous with p.d.f. f and corresponding Laplace transform f^* . (The continuity assumption is convenient for avoiding certain complications and for ease of notation, but it is not strictly necessary for the development of the theory.)

Further, consider a sequence of such components. Assume that there is a new component at time 0 with failure time T_1 , replaced at time T_1 by a second component with failure time T_2 , replaced at time $T_1 + T_2$ by a third component The failure time of the n th component in the sequence is represented by T_n , where $\{T_n : n \geq 1\}$ is a sequence of i.i.d. r.v.s, each with p.d.f. f . The n -th failure/renewal occurs at time S_n , where

$$S_n = T_1 + T_2 + \dots + T_n \quad n \geq 1.$$

Define $S_0 = 0$ and

$$N(t) = \sup\{n \geq 0 : S_n \leq t\} \quad t \geq 0.$$

The r.v. $N(t)$ represents the number of renewals up to time t . The stochastic process $\{N(t) : t \geq 0\}$ is a *renewal process* with the given lifetime distribution.

5.2 The distribution of S_n and the distribution of $N(t)$

When the lifetime distribution is an exponential distribution with parameter λ , so that $f(t) = \lambda e^{-\lambda t}$, $t \geq 0$, then $\{N(t) : t \geq 0\}$ is a Poisson process with rate λ . In this case, as was shown in Section 1.3, S_n has a gamma distribution with parameters n and λ , and $N(t)$ has a Poisson distribution with parameter λt . Otherwise $\{N(t) : t \geq 0\}$ is not a Markov process — the exponential distributions are the only distributions with the “memoryless” property.

In general, if f_n denotes the p.d.f. of S_n and f_n^* its Laplace transform then

$$f_n^*(s) = f^*(s)^n \quad n \geq 0.$$

A relatively simple example occurs when the lifetime distribution is a gamma distribution with parameters ν and λ , in which case

$$f^*(s) = \left(\frac{\lambda}{\lambda + s} \right)^\nu$$

and

$$f_n^*(s) = \left(\frac{\lambda}{\lambda + s} \right)^{\nu n}.$$

Hence S_n has the gamma distribution with parameters νn and λ .

Furthermore, if the parameter ν of the gamma distribution is an integer then we may use the *method of stages* to obtain the distribution of $N(t)$. The gamma distribution may be thought of as the sum of ν i.i.d. exponential distributions with parameter λ . Hence the lifetime of each component may be thought of as the sum of ν independent stages, where each stage has an exponential distribution with parameter λ . The renewal process may be constructed from an underlying Poisson process with rate λ , where a renewal occurs at every ν -th arrival in the Poisson process. The event that exactly n renewals have occurred by time t is equivalent to the event that between νn and $\nu n + \nu - 1$ arrivals, inclusive, have occurred in the underlying Poisson process. Thus

$$\mathbb{P}(N(t) = n) = \sum_{i=\nu n}^{\nu n + \nu - 1} e^{-\lambda t} \frac{(\lambda t)^i}{i!} \quad n \geq 0.$$

In the general case, except for a few special cases, it will not be possible to obtain explicit expressions for the distributions of S_n and $N(t)$, but there are general results concerning the behaviour of S_n as $n \rightarrow \infty$ and $N(t)$ and $t \rightarrow \infty$. Assuming that f has a finite mean μ , where $\mu > 0$, by the Strong Law of Large Numbers, with probability 1,

$$\frac{S_n}{n} \rightarrow \mu$$

as $n \rightarrow \infty$. Now consider the behaviour of $N(t)$ as $t \rightarrow \infty$. Note that, by the definition of $N(t)$,

$$S_{N(t)} \leq t < S_{N(t)+1}.$$

Hence, for $N(t) > 0$,

$$\frac{S_{N(t)}}{N(t)} \leq \frac{t}{N(t)} < \frac{S_{N(t)+1}}{N(t)} = \frac{S_{N(t)+1}}{N(t)+1} \frac{N(t)+1}{N(t)}.$$

As $t \rightarrow \infty$, $N(t) \rightarrow \infty$ and, by the Strong Law of Large Numbers, with probability 1, both the left hand side and the right hand side in the above inequalities tends to μ . Hence with probability 1, as $t \rightarrow \infty$,

$$\frac{t}{N(t)} \rightarrow \mu$$

and

$$\frac{N(t)}{t} \rightarrow \frac{1}{\mu}. \quad (1)$$

So $1/\mu$ is the *rate* of the renewal process, the long-term average number of renewals per unit time.

5.3 The renewal function and the renewal density

Although it is difficult analytically to investigate the distribution of $N(t)$ for finite t , it is relatively easy to obtain general results about $E[N(t)]$.

Define the *renewal function* $H(t)$ by

$$H(t) = E[N(t)] \quad t \geq 0$$

and the *renewal density* $h(t)$ by

$$h(t) = H'(t) \quad t \geq 0.$$

The renewal function $H(t)$ is a finite non-decreasing function of t with $H(0) = 0$.

To obtain a formula for $H^*(s)$, first note the identity

$$\{N(t) \geq n\} = \{S_n \leq t\} \quad n \geq 1,$$

from which it follows that

$$\mathbb{P}(N(t) \geq n) = \mathbb{P}(S_n \leq t) = F_n(t) \quad n \geq 1,$$

where F_n is the d.f. of S_n . Note also that the Laplace transform F_n^* of F_n is given by

$$F_n^*(s) = \frac{f_n^*(s)}{s} = \frac{f^*(s)^n}{s} \quad n \geq 1.$$

Hence

$$H(t) = E[N(t)] = \sum_{n=1}^{\infty} \mathbb{P}(N(t) \geq n) = \sum_{n=1}^{\infty} F_n(t)$$

and, taking Laplace transforms,

$$H^*(s) = \sum_{n=1}^{\infty} \frac{f^*(s)^n}{s} ,$$

so that

$$H^*(s) = \frac{f^*(s)}{s[1 - f^*(s)]} . \quad (2)$$

The Laplace transform of the renewal density is given by

$$h^*(s) = sH^*(s) - H(0) = sH^*(s) ,$$

i.e.,

$$h^*(s) = \frac{f^*(s)}{1 - f^*(s)} . \quad (3)$$

We may also use what is known as the renewal argument to investigate $H(t)$. Conditioning on the time T_1 of the first renewal and noting that after each renewal the process starts anew,

$$E[N(t)|T_1 = u] = \begin{cases} 0 & u > t \\ 1 + H(t - u) & u \leq t \end{cases} .$$

Hence

$$\begin{aligned} H(t) = E[N(t)] &= \int_0^\infty E[N(t)|T_1 = u]f(u)du \\ &= \int_0^t [1 + H(t-u)]f(u)du . \end{aligned}$$

Thus

$$H(t) = F(t) + \int_0^t H(t-u)f(u)du \quad t \geq 0. \quad (4)$$

This is a version of the *renewal equation*, the integral equation of renewal theory.

Taking Laplace transforms in Equation (4),

$$H^*(s) = \frac{f^*(s)}{s} + H^*(s)f^*(s) ,$$

which is equivalent to Equation (2).

Equation (3) may be rewritten as

$$h^*(s) = f^*(s) + h^*(s)f^*(s) ,$$

which corresponds to the renewal equation

$$h(t) = f(t) + \int_0^t h(t-u)f(u)du \quad t \geq 0. \quad (5)$$

Alternatively, Equation (5) may be obtained by differentiating Equation (4).

- For the Poisson process with rate λ , since $N(t)$ has the Poisson distribution with parameter λt , $H(t) = \lambda t$ and $h(t) = \lambda$ for $t \geq 0$, so that $H^*(s) = \lambda/s^2$ and $h^*(s) = \lambda/s$. Here $f(u) = \lambda e^{-\lambda u}$, $u \geq 0$ so that $f^*(s) = \lambda/(\lambda + s)$, and it is easy to verify the truth of Equations (2)-(5).

Example Consider a renewal process with a lifetime distribution that is a mixture of two exponential distributions, so that its p.d.f. is given by

$$f(t) = \theta \lambda e^{-\lambda t} + (1 - \theta) \kappa e^{-\kappa t} \quad t \geq 0,$$

where $0 < \theta < 1$ and $\lambda \neq \kappa$.

$$f^*(s) = \frac{\theta \lambda}{\lambda + s} + \frac{(1 - \theta) \kappa}{\kappa + s} .$$

Substitution into Equation (3) leads to

$$\begin{aligned} h^*(s) &= \frac{[\theta \lambda + (1 - \theta) \kappa]s + \lambda \kappa}{s[(1 - \theta)\lambda + \theta \kappa + s]} \\ &= \frac{A}{s} + \frac{B}{(1 - \theta)\lambda + \theta \kappa + s} , \end{aligned}$$

where A and B must satisfy

$$[(1 - \theta)\lambda + \theta\kappa]A = \lambda\kappa$$

and

$$A + B = \theta\lambda + (1 - \theta)\kappa .$$

Thus

$$A = \frac{\lambda\kappa}{(1 - \theta)\lambda + \theta\kappa}$$

and

$$B = \frac{\theta(1 - \theta)(\lambda - \kappa)^2}{(1 - \theta)\lambda + \theta\kappa} .$$

Hence

$$h(t) = \frac{\lambda\kappa}{(1 - \theta)\lambda + \theta\kappa} + \frac{\theta(1 - \theta)(\lambda - \kappa)^2}{(1 - \theta)\lambda + \theta\kappa} e^{-[(1 - \theta)\lambda + \theta\kappa]t} .$$

Integrating and using the fact that $H(0) = 0$,

$$H(t) = \frac{\lambda\kappa t}{(1 - \theta)\lambda + \theta\kappa} + \frac{\theta(1 - \theta)(\lambda - \kappa)^2}{[(1 - \theta)\lambda + \theta\kappa]^2} [1 - e^{-[(1 - \theta)\lambda + \theta\kappa]t}] .$$

5.4 The limiting value of the renewal density

Given an arbitrary lifetime distribution, it is not in general possible to obtain explicit expressions for $H(t)$ or $h(t)$. However, there is a simple result for the limiting value of $h(t)$ as $t \rightarrow \infty$.

If the lifetime distribution has a finite mean μ then

$$f^*(s) = 1 - \mu s + o(s)$$

as $s \rightarrow 0$. Hence, substituting into Equation (3),

$$h^*(s) = \frac{1 - \mu s + o(s)}{\mu s + o(s)} \sim \frac{1}{\mu s}$$

as $s \rightarrow 0$. Using an asymptotic result about Laplace transforms from Section 3.3.2,

$$h(t) \rightarrow \frac{1}{\mu} \tag{6}$$

as $t \rightarrow \infty$, which is a version of the so-called *Renewal Theorem*.

For instance, in the example at the end of Section 5.3, as $t \rightarrow \infty$,

$$h(t) \rightarrow \frac{\lambda\kappa}{(1 - \theta)\lambda + \theta\kappa} = \frac{1}{\mu} .$$

5.5 Interpretation of the renewal density

$$\begin{aligned}
h(t) &= \lim_{\delta t \rightarrow 0} \frac{H(t + \delta t) - H(t)}{\delta t} \\
&= \lim_{\delta t \rightarrow 0} \frac{E[N(t + \delta t) - N(t)]}{\delta t}.
\end{aligned}$$

Now $N(t + \delta t) - N(t)$ represents the number of renewals in the time interval $(t, t + \delta t]$, and, as $\delta t \rightarrow 0$,

$$\mathbb{P}[N(t + \delta t) - N(t) \geq 2] = o(\delta t).$$

Hence

$$E[N(t + \delta t) - N(t)] = \mathbb{P}[N(t + \delta t) - N(t) = 1] + o(\delta t),$$

and

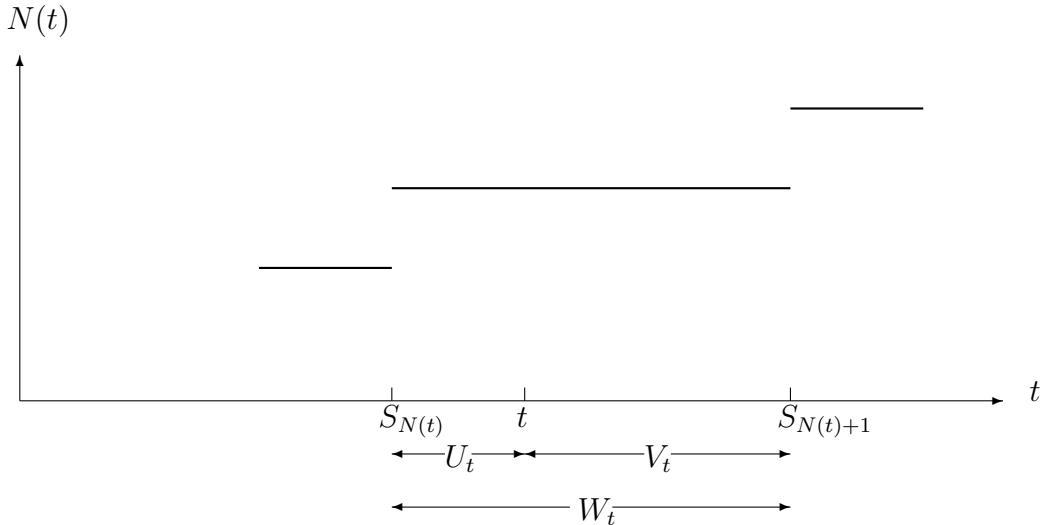
$$h(t) = \lim_{\delta t \rightarrow 0} \frac{\mathbb{P}(\text{a renewal occurs in } (t, t + \delta t])}{\delta t}.$$

From the result of Equation (6), $1/\mu$ is the asymptotic rate at which renewals occur. If the renewal process has been in operation for a long time then the probability of a renewal occurring in a given time interval $(t, t + \delta t]$ is

$$\frac{\delta t}{\mu} + o(\delta t).$$

- Note that the result of Equation (6) is similar to but not identical with the result of Equation (1).

5.6 Backward and forward recurrence times



The *backward recurrence time (current life)*, $U_t \equiv t - S_{N(t)}$, is the age of the component currently in use at a given time t . For $0 \leq u < t$,

$$\begin{aligned}\mathbb{P}(U_t \in (u, u + \delta u)) &= \mathbb{P}(\text{a renewal in } (t - u - \delta u, t - u), \\ &\quad \text{next failure time greater than } u) + o(\delta u) \\ &= h(t - u) \delta u Q(u) + o(\delta u),\end{aligned}$$

where $Q(t)$ is the survivor function of the lifetime distribution, as defined in Section 1.1. For any given u , as $t \rightarrow \infty$, by the result of Equation (6), $h(t - u) \rightarrow 1/\mu$. Hence, as $t \rightarrow \infty$, the distribution of U_t converges to a distribution with p.d.f. g , where

$$g(u) = \frac{Q(u)}{\mu} \quad u \geq 0. \quad (7)$$

(Note that the expression in Equation (7) really does specify a p.d.f., since, by Theorem 1.1.2 of Section 1.1, $\int_0^\infty Q(u)du = \mu$.)

The *forward recurrence time (excess life)*, $V_t \equiv S_{N(t)+1} - t$, is the length of time remaining until failure of the component currently in use at time t . Let T denote the lifetime of an arbitrary component.

$$\begin{aligned}\mathbb{P}(V_t > v | U_t = u) &= \mathbb{P}(T > u + v | T > u) \\ &= \frac{Q(u + v)}{Q(u)}.\end{aligned} \quad (8)$$

Differentiating the expression in Equation (8) with respect to v , we find that the conditional p.d.f. of V_t given $U_t = u$ is $g(v|u)$, where

$$g(v|u) = \frac{f(u + v)}{Q(u)} \quad v \geq 0. \quad (9)$$

Using Equations (7) and (9), we find that for large t the joint p.d.f. $g(u, v)$ of U_t and V_t is given by

$$\begin{aligned}g(u, v) &= g(v|u)g(u) \\ &= \frac{f(u + v)}{Q(u)} \frac{Q(u)}{\mu} \\ &= \frac{f(u + v)}{\mu} \quad u \geq 0, v \geq 0.\end{aligned} \quad (10)$$

Note the symmetry in u and v of the joint p.d.f. in Equation (10), so that the marginal p.d.f. of V_t is the same as the marginal p.d.f. of U_t . We may check this by integrating out u in Equation (10): the p.d.f. of V_t is given by

$$\int_0^\infty \frac{f(u + v)}{\mu} du = \frac{Q(v)}{\mu} \quad v \geq 0.$$

Let $W_t = U_t + V_t$, the lifetime (*total life*) of the component in use at time t . For large t , from Equation (10), the joint p.d.f. of U_t and W_t is given by

$$\frac{f(w)}{\mu} \quad 0 \leq u \leq w.$$

(The Jacobian of the corresponding transformation is 1.) Integrating out u , the marginal p.d.f. of W_t is

$$\int_0^w \frac{f(w)}{\mu} du = \frac{wf(w)}{\mu} \quad w \geq 0. \quad (11)$$

Assuming that the lifetime distribution has a finite variance σ^2 ,

$$\begin{aligned} E(W_t) &= \frac{1}{\mu} \int_0^\infty w^2 f(w) dw \\ &= \frac{1}{\mu} (\sigma^2 + \mu^2) \\ &= \mu(1 + C^2), \end{aligned} \quad (12)$$

where C is the *coefficient of variation*,

$$C = \frac{\sigma}{\mu}.$$

The inspection paradox

Equation (12) demonstrates the *inspection paradox*, that the component in use at a given time t tends to have a longer lifetime than a component chosen at random.

To obtain an intuitive feel for the inspection paradox, imagine the time-axis split up into intervals corresponding to the successive lifetimes of the components. If the point t is chosen at random on the time-axis, the probability of it falling in a given interval is proportional to the length of the interval, so that we have what is known as *length-biased sampling* of the intervals and hence of the lifetimes. In the long run, the proportion of components with failure times in $(t, t + dt)$ is $f(t)dt$, but the proportion of time for which components with failure times in $(t, t + dt)$ are in use is proportional to $tf(t)dt$. Hence the p.d.f. of Equation (11).

Since, in the limit as $t \rightarrow \infty$, U_t and V_t are identically distributed and $W_t = U_t + V_t$, it follows from Equation (12) that, in the limit,

$$E(U_t) = E(V_t) = \frac{1}{2}E(W_t) = \frac{1}{2}\mu(1 + C^2). \quad (13)$$

The waiting-time paradox

Perhaps an even more startling result than the inspection paradox is the *waiting-time paradox* that the expectation $E(V_t)$ of the forward recurrence time, i.e., the expected excess life, can be greater than the expectation μ of the lifetime distribution. From Equation (13) we see that this occurs if $C > 1$, i.e., the coefficient of variation of the lifetime distribution exceeds one.

For example, we may suppose that buses arrive at a bus-stop according to a renewal process. (This is an approximation, since inter-arrival times are unlikely to be independently distributed.) If the average inter-arrival time for buses is 10 minutes, say, it may be the case that, when we arrive at the bus-stop, the expected waiting time until the next arrival of a bus is greater than 10 minutes.

The equilibrium renewal process

We may consider what is known as an *equilibrium renewal process*, where $\{T_n : n \geq 2\}$ is a sequence of i.i.d. r.v.s, each with p.d.f. f , and T_1 is independently distributed of $\{T_n : n \geq 2\}$, but has p.d.f. $Q(t)/\mu$, $t \geq 0$, which corresponds to the limiting distribution of U_t as $t \rightarrow \infty$. Results similar to those for ordinary renewal processes, but with slight modifications, may be obtained. Some results are much simpler. For example, $H(t) = t/\mu$ and $h(t) = 1/\mu$, $t \geq 0$.