(Discrete Event System Stimulation pg 132)

Discrete Random variable → let X → random variable. If the no. of possible values of X is finite or countably infinite.

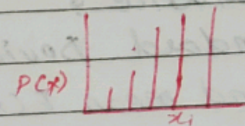
R_x → range space (possible values of X) $\quad x_i$ → possible outcome in R_x

$p(x_i) = P(X = x_i)$ → Probability of random variable equal to x_i ↳ Probability mass fn

The numbers $p(x_i)$ $i = 1, 2, \ldots$ must satisfy →

[Probability Distribution of X → $(x_i, p(x_i))$ ↳ $p(x_i)$ corresponding to x_i, Probability mass fn (pmf) of X ↳ $p(x_i)$]

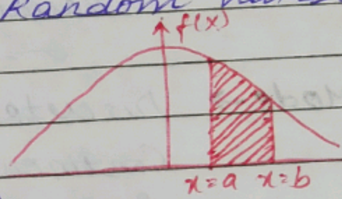
→ $p(x_i) \geq 0$ for all i $\qquad$ → $\sum_{i=1}^{\infty} p(x_i) = 1$



Continous Random Variable → X → random variable is an interval or a collection of intervals then X → Continous Random Variable.

Probability lies in $[a, b]$

$$P(a \leq X \leq b) = \int_a^b f(x)\,dx$$



R_x → range space

$f(x)$ → probability Density function (pdf)

↳ satisfies following properties →

a) $f(x) \geq 0$ for all x in R_x ↳ always true $\quad$ b) $\int_{R_x} f(x)\,dx = 1$ $\quad$ c) $f(x) = 0$, $x \notin R_x$

→ For any value x_0. $P(X = x_0) = 0$ because $\int_{x_0}^{x_0} f(x)\,dx = 0$ $\quad$ → $P(a \leq X \leq b) = P(a < X \leq b)$

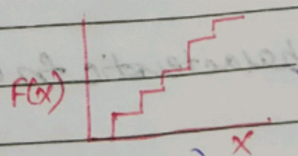
probability of being at $x_0 = 0$ $\qquad \therefore P(a \leq X < b) = P(a < X < b)$

Commulative Density fn → (denoted $F(x)$) (just like summation till a point)

$$F(x) = P(X \leq x) \begin{cases} \text{Discrete} & F(x) = \sum_{\substack{\text{all} \\ x_i \leq x}} p(x_i) \\ \text{Continous} & F(x) = \int_{-\infty}^{x} f(t)\,dt \end{cases}$$

↓ assumes a value $\leq x$

and $P(a \leq X \leq b) = F(b) - F(a)$ for all $a < b$.



Properties →

a) F is non-decreasing fn, If $a < b$ then $F(a) \leq F(b)$

b) $\lim_{x \to \infty} F(x) = 1$ $\qquad$ c) $\lim_{x \to -\infty} F(x) = 0$

Measure Central Tendency of a Random Variable

$E(x)$ Expected value

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Expectation → x = Discrete $\quad E(x) = \sum_{all\ i} x_i p(x_i)$

x = Continous : $E(x) = \int_{-\infty}^{\infty} x f(x) dx$

$E(x)$ is also called mean(μ), first moment of X

$E(x^n)$, $n \geq 1$ (n^{th} moment of x)

$E(x^n) = \sum_{all\ i} x_i^n p(x_i)$ Discrete $\qquad E(x^n) = \int_{-\infty}^{\infty} x^n f(x) dx$ Continous

Variance & Standard Deviation → Variance is defined as Spread/variation of possible values of X around mean $E(x)$.

Variance $\quad V(x) = E([X - E(x)]^2) = E(x^2) - (E(x))^2$

$\sigma = \sqrt{V(x)}$

Mode → Discrete = occurs most frequently

Continous = At which pdf ($f(x)$) is maximised.

2modal values → bimodal

Moment Generating fn → Expectation of a fn of the random variable

$\varphi(t) = E[e^{tx}] = \int e^{tx} dF(x)$

All moments of x, successively obtd by differenting φ & $t=0$

$\varphi'(t) = E[x e^{tx}]$

$\varphi''(t) = E[x^2 e^{tx}]$

$\vdots$

$\varphi^n(t) = E[x^n e^{tx}]$

Evaluating $t=0$ yields

$\varphi^n(0) = E[x^n] \quad n \geq 1$

Characteristic fn of X → $\phi(t) = E[e^{itx}] \quad -\infty < t < \infty$

↳ when mgf doesn't exist $\qquad i = \sqrt{-1}$

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Joint moment gen fn →

$$\psi(t_1, \ldots t_n) = E\left[\exp\left\{\sum_{j=1}^{n} t_j x_j\right\}\right]$$

Joint characteristic fn

$$\phi(t_1, \ldots t_n) = E\left[\exp\left\{i\sum_{j=1}^{n} t_j x_j\right\}\right]$$

Probability Generating fn → Power series representation of probability mass fn $(p(x_i))$ of random var.

fn of t for given random variable → $G_x(t) = E(t^x) = \sum_{x=0}^{\infty} t^x P(X=x)$ Discrete

Expected Value

$= \int_{-\infty}^{\infty} f(x) t^{x_i}$ Continous

→ If some random variable is in power of t then coeff of t^{x_i} will be probability of that fn.

for multivariables →

$$G_{x_1, x_2}(t_1, t_2) = E(t_1^{x_1} t_2^{x_2}) = \sum_{\forall x_1, x_j \in X_1, X_2 = 0}^{\infty} P(x_i\, x_j)\, t^{x_i} t^{x_j}$$

$$G(t_1, \ldots t_d) = E(t_1^{x_1} \ldots t_d^{x_d}) = \sum_{\substack{x_1, x_2 \ldots x_d = 0 \\ x_1, x_2 \ldots x_d \in X_1, X \ldots X_d}}^{\infty} P(x_1, \ldots x_d)\, t_1^{x_1} \ldots t_d^{x_d}$$

Power series converge absolutely at least for all complex vectors $z = (z_1, \ldots z_d) \in \mathbb{C}^d$

Discrete Distributions →

i) Bernoulli trials & Bernoulli Distribution

2 outcomes → for j^{th} trial $x_j = 1$ success
$x_j = 0$ failure

Bernoulli process → If the trials are independant, probabality of success remains constant from trial to trial have only 2 outcome (success/failure)

Independant so :- $p(x_1, x_2 \ldots x_n) = p_1(x_1) \cdot p_2(x_2) \ldots p_n(x_n)$

$$p_j(x_j) = p(x_j) = \begin{cases} p & x_j = 1 \quad j = 1, 2 \ldots n \\ 1-p = q & x_j = 0 \quad j = 1, 2 \ldots n \\ 0 & \text{otherwise} \end{cases}$$

↳ for one trial → called Bernoulli Distribution

(mean)

$E(x) = p$ $\quad V(x) = p(1-p) = pq$

$0(1-p) + 1p = p$ $\quad 0^2(1-p) + 1^2 p = pq = p(1-p)$

$E(X) = np$ $\quad V(X) = npq$

Probability Mass fn $(p(x))$ $= {}^nC_x\, p^x (1-p)^{n-x}$

$n \to$ total no. of trials

$P(\underbrace{S \ldots S}_{x} \underbrace{FF \ldots F}_{n-x}) = p^x q^{n-x}$

Moment Generating fn $(\phi(t))$ $= (pe^t + (1-p))^n$

2) Binomial Distribution $X \to$ no of successes in n Bernoulli trials

$$\text{Binomial Distribution } p(x) = \begin{cases} {}^nC_x\, p^x q^{n-x} & x = 0, 1, 2 \ldots n \\ 0 & \text{otherwise} \end{cases}$$

$X = X_1 + X_2 \ldots X_n$ (sum of n independant Bernoulli random variables)

Mean
$\hookrightarrow E(X) = p + p \ldots p = np$

Variance
$\hookrightarrow V(X) = pq + pq \ldots pq = npq$

3) Geometric & -ve binomial Distribution $\to$

$\hookrightarrow$ Sequence of bernoulli trials

$X \to$ no. of trials to achieve the first success

Distribution of X,

$$p(x) = \begin{cases} q^{x-1} p & x = 1, 2 \ldots \\ 0 & \text{otherwise} \end{cases}$$

* $P(X = x) = q^{x-1} p$

$\hookrightarrow$ When there are $x-1$ failures followed by a success $\to {}^{y-1}C_{k-1}$

$\to$ Mean $E(X) = 1/p$

Variance $V(X) = q/p^2$

$\to$ Distribution of no of trial until k^{th} success

$$p(y) = \begin{cases} \binom{y-1}{k-1} q^{y-k} p^k & y = k, k+1 \ldots \\ 0 & \text{otherwise} \end{cases}$$

$\hookrightarrow y$ has a -ve binomial distribution with parameters p & k

$k \to$ Until the k^{th} success

Good Write

$-12M$ $\quad -12M+1$ $\quad -14M+1$ $\quad -20M+1$ $\quad 0$

Mean = $E(Y) = \frac{k}{p}$ $V(X) = \frac{kp}{p^2}$ Cvariance

Moment Gen. fn $M_{S_n}(t) = (1-p+pt)^n$ (Binomial Dist^n)

$M_T(t) = \frac{tp}{1-t(1-p)}$ (Geometric Dist^n)

Properties of ~~Binomial~~ Bernoulli →

1) Binomial sums

No of 1's among $x_1, x_2 \dots x_n$ is Binomial(n, p) $S = \sum_{i=1}^{n} X_i$ $E(S) = np$ $Var(S) = np(1-p)$

2) Geometric first arrival time

T → time of 1st success

$T \geq 1$ $E(T) = \frac{1}{p}$, $Var(T) = \frac{1-p}{p^2}$ $p_T(t) = (1-p)^{t-1}p$

→ Don't depend on previous events, depends on trials remaining to achieve that

* Memorylessness & Geometric Inter arrivals →

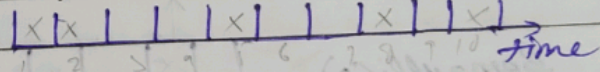
$i+T$ → time of 1st success after time i $(T \geq 1)$

$E(T) = \frac{1}{p}$ $Var(T) = \frac{1-p}{p^2} = \frac{q}{p^2}$ $p_T(t) = (1-p)^{t-1}p$ $t = 1, 2 \dots$

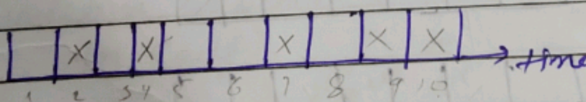
* ~~only~~ geo. posses Memorylessness → $(s, t \geq 0)$ $P(X > s+t \mid X > s) = P(X > t)$

Merging 2 Bernoulli Process → (independant) prob → p

X_t Bernoulli Process (BP) 1 →

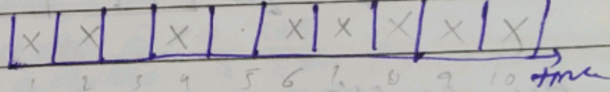


Y_t BP2 →



prob → q

Z_t merged process → $Z_t = g(X_t, Y_t)$



(merge all the arrivals) $Z_{t+1} = g(X_{t+1}, Y_{t+1})$

collision arrivals counted as 1

As $X_t, X_{t+1}, Y_t, Y_{t+1}$ are independant of each other Then $Z_t = g(X_t, Y_t)$ and $Z_{t+1} = g(X_{t+1}, Y_{t+1})$ are independant of each other.

→ Concludes the independance property $(Z_1, \dots Z_t)$ are independant random variables

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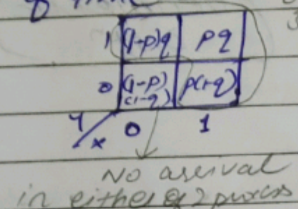
Probability of arrival at given pt of time $= (1-p)q + pq + p(1-q)$
$= 1-(1-p)(1-q)$
$= p+q-pq$ (Sum of these 3 probabs)

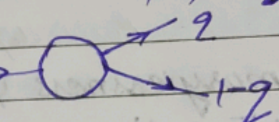


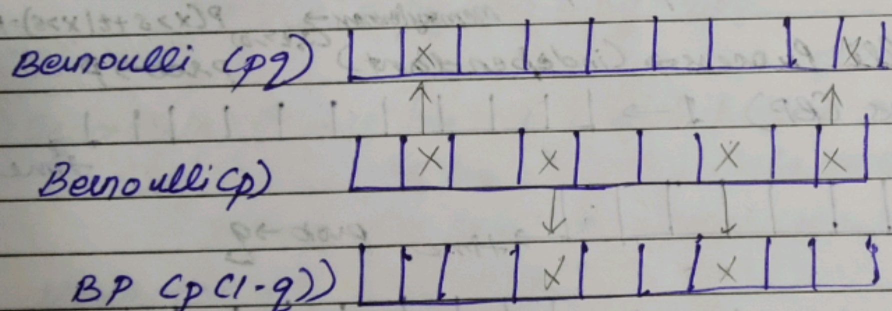
Probab of no arrival $= (1-p)(1-q)$

No arrival in either of 2 process

Probability of arrival in 1st process $= \dfrac{pq+p(1-q)}{p+q-pq}$ (total arrival) $= \dfrac{p}{p+q-pq}$

Splitting into two bernoulli ~~independent~~ processes →
Split sucesses into 2 streams, using independant flips of a coin with bias q
Assumption → Coin flips are independant from orignal Bernoulli proces.





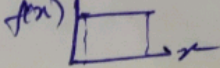
Are resulting streams independant? No
↳ output of 1 BP can be predicted by seeing other splitted BP

Describe random var that can be used to describe random phenomena in which variable can take any val in interval

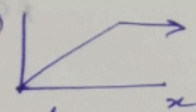
Continous Distributions

(1) Uniform Distribution →
Random var X is uniformely distributed in interval (a,b) if its probability density fn

$$f(x) = \begin{cases} \dfrac{1}{b-a} & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$



a,b → range of RV



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The commulative distribution fn

$$F(x) = \begin{cases} 0 & x<a \\ \frac{x-a}{b-a} & a\le x<b \\ 1 & x\ge b \end{cases}$$

f, pdf
$\left[\text{st} \int_a^b f(y)\,dy = 1 \right.$

as $P(x_1 \le X \le x_2) = F(x_2) - F(x_1) = \frac{x_2 - x_1}{b-a}$

Mean $E(X) = \frac{a+b}{2}$ Variance $V(X) = \frac{(b-a)^2}{12}$

Generating RV in range $y \in [a,b]$ from $x \in [0,1)$

$x_1 + x_2 = y$ $E(x_1) + E(x_2) = E(y)$ → 1/2

$y = a + x(b-a)$ $E(y) = a + E(x)(b-a) = \frac{a+b}{2}$

Density fn → $f(y) = \begin{cases} \frac{1}{b-a} & a \le y \le b \\ \text{otherwise} & 0 \end{cases}$

* in uniform RV → any possibility is likely

Poisson Distribution → (missed before continuous Dis topics)

Poisson probability mass fn $P(x) = \begin{cases} \frac{e^{-\alpha}\alpha^x}{x!} & x = 0, 1 \ldots \\ 0 & \text{otherwise} \end{cases}$

where $\alpha > 0$

$E(X) = Var(X) = \underset{\text{rate}}{\alpha}$

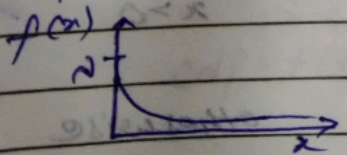
commulative Distribution fn (cdf) $- F(x) = \sum_{i=0}^{x} \frac{e^{-a}\alpha^i}{i!}$

→ with interarrival times

Exponential Distribution → parameter $\lambda > 0$ (rate)

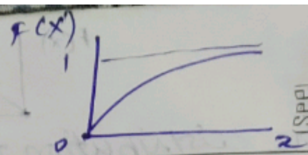
↳ Model interarrival times when arrivals are completely random
~~& to model service time~~ that are highly variable

probability Density fn (pdf) $f(x) = \begin{cases} \lambda e^{-\lambda x} & x \ge 0 \\ 0 & \text{otherwise} \end{cases}$



λ → arrivals per hour / service per min

$\underset{\text{mean}}{E(X)} = \frac{1}{\lambda}$, $Var(X) = \frac{1}{\lambda^2}$



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cdf $F(x) = \begin{cases} 0 & x<0 \\ \int_0^x \lambda e^{-\lambda t} dt = 1-e^{-\lambda x}, & x \geq 0 \end{cases}$

Memorylessness (property of exponential Dis^n)

$S \geq 0 \quad t \geq 0 \qquad P(X>t) = P(X>s+t \mid X>s) = \dfrac{P(X>s+t)}{P(X>s)}$

Component doesn't remember that its already for a time s. A used component is good as new.

$$\Rightarrow \frac{e^{-\lambda(s+t)}}{e^{-\lambda(s)}} = e^{-\lambda t}$$

* Geometric distribution is only discrete distribution that possess this property.

Gamma Distribution →

gamma fn → to define gamma distribution when $\beta > 0$

$$\Gamma(\beta) = \int_0^\infty x^{\beta-1} e^{-x} dx$$

$$\Gamma(\beta) = (\beta-1)\Gamma(\beta-1)$$

If β is an integer, using $\Gamma(1) = 1$ in above eq,

$$\Gamma(\beta) = (\beta-1)!$$

* Gamma fn ⇒ Generalization of factorial notion of all +ve nos

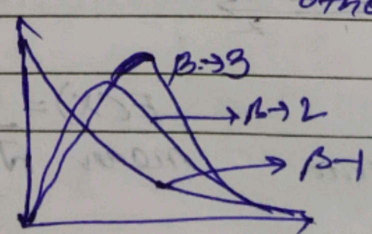
⇒ Random var X is gamma distributed

$\beta \to$ shape parameter

$\theta \to$ Scale parameter

$$f(x) = \begin{cases} \dfrac{\beta\theta}{\Gamma(\beta)} (\beta\theta x)^{\beta-1} e^{-\beta\theta x} & x>0 \\ 0 & \text{otherwise} \end{cases}$$

When $\theta = 1$ and β vary



Good Write

mean

$E(x) = \frac{1}{\theta}$

Variance

$V(x) = \frac{1}{\beta\theta^2}$

$$\text{cdf} \rightarrow F(x) = \begin{cases} 1 - \int_x^\infty \frac{\beta\theta}{\Gamma(\beta)} (\beta\theta t)^{\beta-1} e^{-\beta\theta t}\, dt & x > 0 \\ 0 & x \leq 0 \end{cases}$$

If $\beta \rightarrow$ integer, gamma dist rel^n to exp dist^n

RV $X \rightarrow$ sum of β independant, exp distributed rv each with parameters β, θ then X has a gamma dist^n with parameters β & θ.

if $X = X_1 + X_2 + \dots X_\beta$ (x_j are mutually independant)

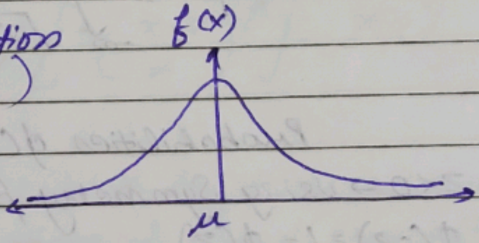
$$\text{pdf of } x_j \rightarrow g(x_j) = \begin{cases} (\beta\theta) e^{-\beta\theta x_j} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Representation

Normal Distribution $\rightarrow$ ($X \sim N(\mu, \sigma^2)$)

RV $X \Rightarrow$ mean $-\infty < \mu < \infty$

variance $\sigma^2 > 0$



$$\text{Pdf} = f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right], \quad -\infty < x < \infty$$

Properties $\rightarrow$

1) $\lim_{x \to -\infty} f(x) = 0$ & $\lim_{x \to \infty} f(x) = 0$

2) $f(\mu + x) = f(\mu - x)$ pdf is symmetric abt μ

3) Max value of pdf at $x = \mu$ (mean = mode)

$$\text{cdf} = F(x) = P(X \leq x) = \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} \exp\left[\frac{-1}{2}\left(\frac{t-\mu}{\sigma}\right)^2\right] dt$$

(Not possible to evaluate in this form)

$\therefore$ Let $Z = \frac{x-\mu}{\sigma}$ (allows eq to be independant of μ & σ)

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$$F(x) = P(X \le x) = P\left(Z \le \frac{x-\mu}{\sigma}\right)$$

$$= \int_{-\infty}^{\frac{x-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

$$= \int_{-\infty}^{\frac{x-\mu}{\sigma}} \phi(z) dz = \phi\left(\frac{x-\mu}{\sigma}\right)$$

pdf $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \quad -\infty < z < \infty$

Standard Normal Distribution $Z \sim N(0, 1)$
$(\mu \; \sigma^2) \rightarrow \mu = 1, \sigma^2 = 1$

$$\phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$$ (widely tabulated)

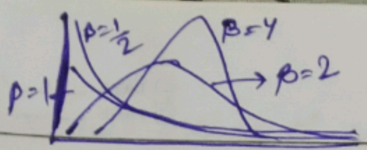
Probabilities $\phi(z)$ for $z \ge 0$ given in table
If $z < 0 \rightarrow$ using Symmetry property
$\phi(-z) = 1 - \phi(z)$

Weibull Distribution →
X has WD if

$$\text{pdf } f(x) = \begin{cases} \frac{\beta}{\alpha}\left(\frac{x-v}{\alpha}\right)^{\beta-1} \exp\left[-\left(\frac{x-v}{\alpha}\right)^{\beta}\right], & x \ge v \\ 0 & \text{otherwise} \end{cases}$$

$v \rightarrow$ location parameter $(-\infty < v < \infty)$
$\alpha \rightarrow$ scale parameter $(\alpha > 0)$
$\beta \rightarrow$ shape parameter $(\beta > 0)$
When $v = 0$,

$$f(x) = \begin{cases} \frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta-1} \exp\left[-\left(\frac{x}{\alpha}\right)^{\beta}\right] & x \ge 0 \\ 0 & \text{otherwise} \end{cases}$$



$\nu=0, \alpha=\frac{1}{2}, \beta=\frac{1}{2}, 1, 2, 4$

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When
$\nu=0, \alpha=1, \beta=1$ then

$$f(x)=\begin{cases}\frac{1}{\alpha}e^{-x/\alpha} & x\geq 0\\ 0 & \text{otherwise}\end{cases}$$

$\boxed{\lambda=\frac{1}{\alpha}}$ exponential Distribution with parameter λ

Mean $E(X)=\nu+\alpha\Gamma\left(\frac{1}{\beta}+1\right)$ Variance $V(X)=\alpha^2\left[\Gamma\left(\frac{2}{\beta}+1\right)-\left[\Gamma\left(\frac{1}{\beta}+1\right)\right]^2\right]$

$\nu\to$ has no effect on variance

where $\Gamma(\beta)=\int_0^\infty x^{\beta-1}e^{-x}dx$

$$\text{cdf } F(X)=\begin{cases}0 & x<\nu\\ 1-\exp\left[-\left(\frac{x-\nu}{\alpha}\right)^\beta\right] & x\geq\nu\end{cases}$$

Poisson Process $\rightarrow$

Counting Process $[N(t), t\geq 0]$ is said to be Poisson process, mean rate λ if following assumptions followed (given after def^n)

$N(t)\to$ counting fn for all $t\geq 0$ (Observation of a RV in given time)
↳ Represent no. of events that occur in $[0, t]$
$t=0\to$ observation began when

1. Arrivals occurs one at a time.
2. $\{N(t), t\geq 0\}$ has stationary increments: The distribution of number of arrivals b/w t & $t+s$ depends only on the length of interval s, not the starting pt t.
↳ Arrivals are completely at random without any rush / slack periods.
3. $\{N(t), t\geq 0\}$ has independent increments.
↳ No. of arrivals during non-overlapping time intervals are independent random variables.

Good Write

$\Rightarrow$ Future arrivals no correlation with past arrivals

If arrivals occurs acc to Poisson process $P[N(t)=n] = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$ for $t \geq 0$ & $n = 0, 1, 2 \ldots$

$N(t) \rightarrow$ has poisson distribution with $\alpha = \lambda t$ parameter

Mean $E(N(t)) = \alpha = \lambda t = V[N(t)]$ variance

No. of arrivals in intervals from s to t $= P[N(t)-N(s)=n] = \frac{e^{-\lambda(t-s)}[\lambda(t-s)]^n}{n!}$ $n=0,1,2\ldots$

$\hookrightarrow$ mean $= \lambda(t-s) = V[N(t)-N(s)]$

$E(N(t)-N(s))$

Obtaining Exponential Dist^n fn $\Rightarrow$

* Prob that 1st arrival occur in $[0,t]$ $= 1 - P(1^{st}$ arrival after time $t)$

$\hookrightarrow (A_1 > t) = \{N(t) = 0\}$

$P(A_1 > t) = P[N(t)=0] = e^{-\lambda t}$

$P(A_1 \leq t) = 1 - e^{-\lambda t}$

$\hookrightarrow$ distributed exponentialy $E(A_1) = \frac{1}{\lambda}$ (independantly dist^n interarrival times)

* The memoryless property is related to properties of independant & stationary increments of the Poisson process

$\Rightarrow$ No. of variables follows poisson process, inter arrival time follows exponential dist^n fn.

Conversion of Binomial Distribution fn to Poisson

$\hookrightarrow$ Poisson is limiting case of Binomial when n is very large, p - the probability of success is very small

Rule of Thumb $\rightarrow$

$n \geq 100$, $np \leq 10$ $(\lambda = np)$

$\hookrightarrow$ provide good approx

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Binomial probab of seeing x succes with n trials

$$P(x) = {}^nC_x p^x q^{n-x}$$

Let,

Expected value of binomial Dis^n be λ

$$\lambda = np \quad (E(x))$$

$$p = \frac{\lambda}{n} \quad \text{and} \quad q = 1 - \frac{\lambda}{n}$$

Rewriting $P(x)$ in terms of λ, n, x

$$P(x) = {}^nC_x \left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$P(x) = \frac{n(n-1)(n-2)\ldots(n-x+1)}{x!} \frac{\lambda^x}{n^x} \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$P(x) = \frac{n}{n} \cdot \frac{n-1}{n} \cdots \frac{n-x+1}{n} \cdot \frac{\lambda^x}{x!} \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$P(x) = \frac{n}{n} \cdot \frac{n-1}{n} \cdots \frac{n-x+1}{n} \cdot \frac{\lambda^x}{x!} \overbrace{\left(1-\frac{\lambda}{n}\right)^n}^{\text{tends towards } e^{-\lambda}} \left(1-\frac{\lambda}{n}\right)^{-x}$$

Applying limit to both sides, → tend towards 1 as $n \to \infty$

$$\lim_{n\to\infty} P(x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

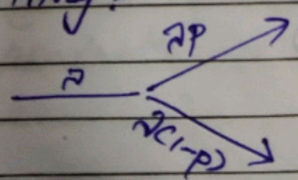
Properties of Poisson process

$N_1(t)$ and $N_2(t)$ → Number of type I and type II events occuring in $[0,t]$ interval.

$$N(t) = \underbrace{N_1(t) + N_2(t)}_{\text{poisson process that are independent}}$$

$$\text{Rates} \rightarrow \lambda p \qquad \lambda(1-p)$$

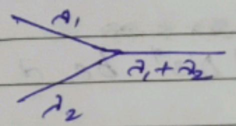
Random Splitting ⇒



* Joint pdf $= \lambda^n e^{-\lambda t_n}$

$\lambda \rightarrow$ rate

Pooled Poisson Process



$$N(t) = N_1(t) + N_2(t)$$

↳ Poisson pooled process with rate $\lambda_1+\lambda_2$

Non-Stationary Poisson Process (NSPP)

Non stationary so while computing the earlier values will also be considered

↳ with 1 & 3 assumptions but not 2nd assumption

characterised

$\text{NSPP} \overset{\text{||}}{\rightarrow} \lambda(t)$ arrival rate at time t

* Useful in situations where arrival rate varies during the period of interest

Expected no. of arrivals $\Lambda(t) = \int_0^t \lambda(s)\,ds$

↳ must be non-negative and integrable

No. of arrivals in $t \in [a,b]$ $= \int_a^b \lambda(t)\,dt$

For stationary process $\Lambda(t) = \lambda t \rightarrow$ (taking arrival rate as 1 in NSPP)

Let $T_1, T_2 \ldots$ arrival times of stationary Poisson process $N(t)$ with $\lambda = 1$, let $T_1, T_2, \ldots$ be arrival times for NSPP $N^{\prime}(t)$ with $\lambda(t)$

So $T_i = \Lambda(T_i^{\prime})$

$T_i^{\prime} = \Lambda^{-1}(T_i)$

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Stationary Random process (strict-sense)

Under what condition is ~~stationary~~ random process a stationary process :-

* Random process $\{X(t) : t \in T\}$ is stationary if its statistical properties don't change with time. (w.r.t to pdf & cdf)

i.e, for stationary process $X(t)$ and $X(t+\Delta t)$ have same probability distributions.

1st order statistical process $F_{X(t)}(x) = F_{X(t+\Delta t)}(x) \quad \forall\, t,\, t+\Delta \in T$ (cdf)

or $f_X(x, t) = f_X(x, t+\Delta)$ (pdf)

To conclude :- If a time shift doesn't change its stastical properties then the random process is a stationary process.

* If any process is not stationary then it is called evolutionary process.

Strict sense stationary process

A continous random process $\{X(t) : t \in T\}$ is strict sense stationary or simply stationary if all $t_1, t_2, \dots t_k \in T$ and all $\Delta \in R$

CDF

(k th order) $F_{X(t_1)X(t_2)\dots X(t_k)}(x_1, x_2 \dots x_k) = F_{X(t_1+\Delta)X(t_2+\Delta)\dots X(t_k+\Delta)}(x_1, x_2 \dots x_k)$

or PDF

$f_X(x_1, x_2 \dots x_k; t_1, t_2 \dots t_k) = f_X(x_1, x_2 \dots x_k; t_1+\Delta, t_2+\Delta \dots t_k+\Delta)$

Q. Consider discrete time ~~structure~~ random process $\{X(n), n \in Z\}$ in which $X(n)$'s are i.i.d with cdf $F_{X(n)}(x) = F(x)$. Show that this is (strict sense) stationary process.

i.i.d → independant so $f(x,y) = f(x)f(y)$

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$F_{X(n_1)X(n_2)\dots X(n_k)}(x_1, x_2 \dots x_k) = F_{X(n_1)}(x_1) F_{X(n_2)}(x_2) \dots F_{X(n_k)}(x_k)$

[using property given $\Rightarrow F_{X(n)}(x) = F(x)$]

$= F(x_1) F(x_2) \dots F(x_k)$

$F_{X(n_1+\Delta)X(n_2+\Delta)\dots X(n_k+\Delta)}(x_1, x_2 \dots x_k) = F_{X(n_1+\Delta)}(x_1) F_{X(n_2+\Delta)}(x_2) \dots F_{X(n_k+\Delta)}(x_k)$

(using property given $\rightarrow F_{X(n)}(x) = F(x)$)

$= F(x_1) F(x_2) \dots F(x_k)$

Hence, $\{X(n) : n \in \mathbb{Z}\}$ is a (strict sense) stationary process.

Remark $\rightarrow$
For a strict sense stationary process, all moments must be independant of time, i.e.,

$E[X(t)],\ E[X^2(t)],\ E[X^3(t)] \dots$
are all constant

If atleast one of them not constant $\rightarrow$ Non strict sense stationary

Theorem:- Mean & variance of first order stationary process are constant

Proof $\rightarrow$ Let $\{X(t)\}$ be first order stationary process

$\rightarrow f_X(x;t) = f_X(x; t+\Delta)$ —(1)

[To proof :- $E[X(t+\Delta)] = E(X(t))$
$Var(X(t+\Delta)) = Var(X(t))$

(considering it as continous so taking $\int$)

Taking LHS, $E[X(t+\Delta)] = \int_{-\infty}^{\infty} x f_X(x; t+\Delta)\, dx$

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using eq 1,

$$= \int_{-\infty}^{\infty} x f_x(x;t)\,dx$$

$$= E[X(t)]$$

Hence $E[X(t)]$ is constant.
Let $E[X(t)] = \mu_x$ is constant.

In case of variance, →

$$var(X(t+\Delta)) = E[(X(t+\Delta) - \mu_x)^2]$$

$$= \int_{-\infty}^{\infty} (x-\mu_x)^2 f_x(x; t+\Delta)\,dx$$

using eq(1)

$$= \int_{-\infty}^{\infty} (x-\mu_x)^2 f_x(x;t)\,dx$$

$$= var(X(t))$$

Thus, variance is constant.

Q If $\{X(t) = A\cos 2t + B\sin 2t\}$; $t \geq 0$ is a random process, where A and B are independant random vaiables each of which assumes the values -2 and $+1$ with probabilities $1/3$ and $2/3$ respectively. Show that $X(t)$ is not-strict sense stationary.

Sol^n → $E(X(t)) = E(A\cos 2t + B\sin 2t)$

always constants so taken out

$$= \cos 2t\, E(A) + \sin 2t\, E(B)$$

Since,

A/B	-2	1
P	1/3	2/3

So, $E(A) = (-2)\left(\frac{1}{3}\right) + (1)\left(\frac{2}{3}\right) = 0$

$E(B) = (-2)\left(\frac{1}{3}\right) + 1\left(\frac{2}{3}\right) = 0$

$\therefore E(X(t)) = 0$

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$$E(x^2(t)) = E\left(A^2\cos^2 2t + B^2\sin^2 2t + 2AB\cos 2t\sin 2t\right)$$

$$= \cos^2 2t\, E(A^2) + \sin^2 2t\, E(B^2) + \sin 2\,2t\, E(AB)$$

$$E(A^2) = E(B^2) = \frac{4}{3} + \frac{2}{3} = 2$$

* Since A, B are given independent $\rightarrow E(AB) = E(A)E(B) = 0$

Therefore,

→ only value of x squared & not its probability

$$E(x^2(t)) = (\cos^2 2t + \sin^2 2t)(2) + 0$$
$$= 2 \quad \text{constant } \forall t$$

and $E(x(t)) = 0$ constant $\forall t$

*Since, we have to show that it is not stationary, have to check for $E(x^3(t))$ as well

$$E[x^3(t)] = E[(A\cos 2t + B\sin 2t)^3]$$
$$= E[A^3\cos^3 2t + B^3\sin^3 2t + 3A^2B\cos^2 2t\sin 2t + 3AB^2\cos 2t\sin^2 2t]$$
$$= \cos^3 2t\, E(A^3) + 3\cos^2 2t\sin 2t\, E(A^2B) + 3\cos 2t\sin^2 2t\, E(AB^2) - \sin^3 2t\, E(B^3)$$

then,

$$E(A^3) = -8\left(\frac{1}{3}\right) + 1\left(\frac{2}{3}\right) = -2$$

$$E(B^3) = -8\left(\frac{1}{3}\right) + 1\left(\frac{2}{3}\right) = -2$$

Since A and B are independent,

$$E(A^2B) = E(A^2)E(B) = 2E(B) = 0 = E(AB^2)$$

In equation, $-2\cos^3 2t + 2\sin^3 2t$

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Which is not constant. Hence $\{X(t)\}$ is not strict sense stationary.

Q2 Consider a random process $X(t) = \cos(t+\phi)$, ϕ is random variable with density fn

$$f(\phi) = \frac{1}{\pi} ; \quad -\frac{\pi}{2} < \phi < \frac{\pi}{2}$$

check whether or not the process is stationary.

To check $E[X(t)] =$ costant or Not

$\Rightarrow E[X(t)] = E[\cos(t+\phi)]$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(t+\phi) f(\phi)\, d\phi$$

$$= \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \cos(t+\phi)\, d\phi$$

$$= \frac{1}{\pi} \left[\sin(t+\phi)\right]_{-\frac{\pi}{2}}^{\pi/2}$$

$$= \frac{1}{\pi} \left[\sin(t+\pi/2) - \sin(t-\pi/2)\right]$$

$$= \frac{1}{\pi}[2\cos t] = \frac{2}{\pi}\cos t$$ not constant

Therefore $E(X(t))$ is not constant, so $X(t)$ is not stationary.

Q3 Examine whether the poisson process $\{X(t)\}$ given by

$$P(X(t) = k) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}, \quad k = 0, 1, 2, 3 \dots$$ is stationary?

$\Rightarrow$ For poisson process,

$$P(X(t) = k) = \frac{e^{-\lambda t}(\lambda t)^k}{k!} \quad k = 0, 1, 2, 3 \dots$$

We know that mean of $X(t) = \lambda t$

i.e, $E[X(t)] = \lambda t$ which is not constant

Hence, $X(t)$ is not ~~covariance~~ stationary

Q. For the sine wave process $X(t) = Y\cos\omega_0 t$, $-\infty < t < \infty$, ω_0 is constant; the amplitude Y is a random variable with uniform distribution in the interval $0 < t < 1$. Check whether the process is stationary or not?

Ans → Y is a uniform random variable in $(0, 1)$

[Target :- check $E[X(t)] = \mu_X$ constant or not?]

pdf of uniform dist^n ⇒ $f(x) = \frac{1}{b-a}$ $a < x < b$

$E(X) = \frac{a+b}{2}$ $Var(X) = \frac{(b-a)^2}{12}$

So, $E(Y) = \frac{1}{2}$ $Var(Y) = \frac{1}{12}$

$$E[X(t)] = E[Y\cos\omega_0 t]$$
$$= \cos\omega_0 t \, E[Y]$$
$$= \frac{\cos\omega_0 t}{2}$$

Which is not constant. So, $X(t)$ is not stationary.

Independant Random Processes	Stationary Independent Increments
A random process $X(t)$ is called Independant if $\{X(t_i)\}$, $i = 1, 2, \dots n$ are independant random variables so that	A random process $\{X(t);\ t \geq 0\}$ is said to have independant inc^n if whenever $0 < t_1 < t_2 \dots < t_n$
$f_X(x_1, x_2, \dots x_n; t_1, t_2 \dots t_n)$	$X(0), X(t_1) - X(0), X(t_2) - X(t_1), \dots X(t_n) - X(t_{n-1})$ are
$= \prod_{i=1}^{n} f_X(x_i; t_i)$	independant random variables
	If $X(t)$ has independant increment & $X(t_2) - X(t_1)$ has same distribution as $X(t_2 + \Delta) - X(t_1 + \Delta)$ for all $t_1, t_2, \Delta \geq 0$, $t_1 < t_2$
$(t_3 - t_2 = t_2 - t_1)$	

then process $X(t)$ is said to have stationary independant increments.

Th:- If $\{X(t) : t \geq 0\}$ is a random process with stationary independant increments with

$$X(0) = 0, \quad \sigma_1^2 = Var[X(1)]$$

Prove following →

a) $Var[X(t)] = \sigma_1^2 t \quad \forall t \geq 0$

b) If $t_2 > t_1$, $Var[X(t_2) - X(t_1)] = \sigma_1^2 (t_2 - t_1)$

c) $C_{XX}(t_1, t_2) = Cov[X(t_1), X(t_2)] = \sigma_1^2 \min(t_1, t_2)$

Proof → (a)

Define $\phi(t) = Var[X(t)]$ assumed

$$\phi'(t) = Var[X(t) - X(0)]$$

Then for any t_1, t_2

$$\phi(t_1 + t_2) = Var[X(t_1 + t_2) - X(0)]$$
$$= Var[\underbrace{X(t_1 + t_2) - X(t_2)}_{x} + \underbrace{X(t_2) - X(0)}_{y}]$$

[If x and y independant then, $V(x+y) = V(x) + V(y)$]

$$= Var[X(t_1 + t_2) - X(t_2)] + Var[X(t_2) - X(0)]$$
$$= \phi(t_1) + \phi(t_2)$$

The only sol^n to the eq^n

$$\phi(t_1 + t_2) = \phi(t_1) + \phi(t_2) \text{ is}$$
$$\phi(t) = kt$$

where k is constant.

Since, $\phi(1) = k = Var[X(1)] = \sigma_1^2$

$$\Rightarrow \phi(t) = Var[X(t)] = \sigma_1^2 t, \quad t \geq 0$$

(Hence, proved)

(b) let $t_2 > t_1$. Then by property of stationary increments, we have,

$$Var[X(t_2)] = Var[X(t_2) - X(t_1) + X(t_1) - X(0)]$$
$$= Var[X(t_2) - X(t_1)] + Var[X(t_1) - X(0)]$$
$$= Var[X(t_2) - X(t_1)] + Var[X(t_1)]$$

By using result in part a,

$$Var[X(t_2) - X(t_1)] = Var[X(t_2)] - Var[X(t_1)]$$
$$= \sigma_1^2 t_2 - \sigma_1^2 t_1$$
$$= \sigma_1^2 (t_2 - t_1)$$

c) To proof :- $C_{XX}(t_1, t_2) = Cov[X(t_1), X(t_2)] = \sigma_1^2 \min(t_1, t_2)$

Prove → for any t_1, t_2 we have
$$Var[X(t_2) - X(t_1)] = Var[X(t_2)] + Var[X(t_1)] - 2cov[X(t_1)X(t_2)]$$

[using Result → $Var(aX+bY) = a^2 Var(X) + b^2 Var(Y) + 2ab\,cov(X, Y)$]

$\Rightarrow Cov[X(t_1)X(t_2)]$ (taken to left cov & var shifted)
$$= \frac{1}{2}\Big[\underbrace{Var[X(t_2)]}_{\text{part a}} + \underbrace{Var[X(t_1)]}_{\text{part a}} - \underbrace{Var[X(t_2) - X(t_1)]}_{\text{part b}}\Big]$$

$$= \frac{1}{2}\left[\sigma_1^2 t_2 + \sigma_1^2 t_1 - \sigma_1^2 (t_2 - t_1)\right] \quad \text{(By (a), (b) parts)}$$

$$= \frac{1}{2}\left[\sigma_1^2 t_2 + \sigma_1^2 t_1 - \sigma_1^2 (t_2 - t_1)\right]$$

Thus,
$$C_{XX}(t_1, t_2) = cov[X(t_1)X(t_2)]$$

$$= \begin{cases} \frac{1}{2}\sigma_1^2 [t_1 + t_2 - (t_2 - t_1)]; & t_2 \geq t_1 \\ \frac{1}{2}\sigma_1^2 [t_1 + t_2 - (t_1 - t_2)]; & t_1 > t_2 \end{cases}$$

$$= \begin{cases} \sigma_1^2 t_1 & ; t_2 \geq t_1 \\ \sigma_1^2 t_2 & t_1 > t_2 \end{cases}$$

$$= \sigma_1^2 \min(t_1, t_2)$$

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Major adv of stationary process is →
if a process is stationary, we can observe the past which will normally give a lot of info how process will behave in future. ($E(x) = const$)

However, many real life processes are not strict sense stationary. Even if it is, it is difficult to prove.
Hence WSS (wide sense stationary) or weak comes.

(from ppt)

Stationary Stochastic processes
All probabilities are invariant to time shifts i.e, for any s

$$P[X(t_1+s) \geq x_1, X(t_2+s) \geq x_2, \ldots X(t_k+s) \geq x_k] =$$

$$P[X(t_1) \geq x_1, X(t_2) \geq x_2, \ldots X(t_k) \geq x_k]$$

↳ If this rel^n true then called strict stationary

1st order → $P[X(t+s) \geq x] = P[X(t) \geq x]$ probs of single variable are shift invariant

2nd order → $P[X(t_1+s) \geq x_1, X(t_2+s) \geq x_2] = P[X(t_1) \geq x_1, X(t_2) \geq x_2]$

* for ss process, joint cdfs are shift invariant & pdfs are also.

$$f_{X(t+s)}(x) = f_{X(t)}(x) = f_{X(0)}(x) := f_X(x)$$

$f_{X(t_1)X(t_2)}(x_1, x_2) = f_{X(0)X(t_2-t_1)}(x_1, x_2)$ since it is stationary subtracting t_1 from both values

And, mean of ss process is constant

$$\mu(t) := E[X(t)] = \int_{-\infty}^{\infty} x f_{X(t)}(x) = \int_{-\infty}^{\infty} x f_X(x) = \mu$$

The variance of ss process is also constant

$$var[X(t)] = \int_{-\infty}^{\infty} (x-\mu)^2 f_{X(t)}(x) = \int_{-\infty}^{\infty} (x-\mu)^2 f_X(x) = \sigma^2$$

The power of ss process (second moment) is also constant

$$E[X^2(t)] = \int_{-\infty}^{\infty} x^2 f_{X(t)}(x) = \int_{-\infty}^{\infty} x^2 f_X(x) = \sigma^2 + \mu^2$$

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two values of
Joint pdfs of stationary process
$f_{X(t_1)X(t_2)}(x_1, x_2) = f_{X(0)X(t_2-t_1)}(x_1, x_2)$

Have use shift invariance fas t_1 shift $(t_1 - t_1 = 0,\ t_2 - t_1)$
→ This result is true for any pair t_1, t_2
⇒ Joint pdf depends only on time differences $s := t_2 - t_1$
writing $t_1 = t$ & $t_2 := t + s$ we equivalently have,
$f_{X(t)X(t+s)}(x_1, x_2) = f_{X(0)X(s)}(x_1, x_2) = f_X(x_1, x_2; s)$

⇒ Deterministic linear systems ⇒ transient + steady state behaviour

Guassian Random Process

Jointly random

Guassian process → $x(t)$ as guassian process
$y(t) = \int_0^T g(t)\, x(t)\, dt$ → any fn.

$y(t) = \int_0^T g(\tau)\, x(\tau)$

$y(t)$ and $g(t)$ are random process

Mean square value of a random variable Y is a guassian distributed random variable for every $g(t)$, then $x(t)$ is a Guassian process.

$f_Y(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-\mu)^2}{2\sigma^2}}$; μ = mean, σ = std deviation
σ^2 = Variance

Normalized guassian distribution :- $\mu = 0$, $\sigma^2 = 1$, $f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2} = N(0,1)$

Properties →
(i) When $x(t)$ is applied on a stable linear filter, $Y(t)$ obtained is guassian process.

(ii) If guassian process is wide sense stationary, then it is strict sense stationary as well.

$\mu = m(x)$ const for all x covariance $+ (x,x')$ for all x $K(x,x') = (x-x')$

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(iii) for $X(t_j)$

$$\mu = m(t_j) = E[X(t_j)], \; j = 1, 2 \ldots n$$
$$C_X(t_k, t_j) = E[X(t_k) - m(t_k)][X(t_j) - m(t_j)]$$
$$k, j = 1, 2 \ldots n$$

(iv) if $E[(X(t_k) - m(t_k))][X(t_j) - m(t_j)] = 0$

↳ Uncorrelated when $X(t_i)$ are independant

— (another video)

A random process $X(t)$ is a Guassian random process if the samples $X_1 = X(t_1), X_2 = X(t_2), \ldots X_k = X(t_k)$ are jointly random ~~process~~ variables for all K, and all choices of $t_1, t_2 \ldots t_K$.

- This definition applies for discrete-time & continous time processes.

- Joint pdf of jointly Guassian random variables is determined by the vector of means & by the covariance matrix.

$$f_{X_1, X_2, X_3 \ldots X_K}(x_1 \ldots x_K) = \frac{e^{-1/2 (x-m)^T C^{-1} (x-m)}}{(2\pi)^{K/2} |C|^{1/2}}$$

where

$$m = \begin{bmatrix} m_X(t_1) \\ m_X(t_2) \\ \vdots \\ m_X(t_K) \end{bmatrix} \quad C = \begin{bmatrix} C_X(t_1, t_1) & C_X(t_1, t_2) & \cdots & C_X(t_1, t_K) \\ C_X(t_2, t_1) & C_X(t_2, t_2) & \cdots & C_X(t_2, t_K) \\ \vdots & & & \vdots \\ C_X(t_K, t_1) & \cdots & \cdots & C_X(t_K, t_K) \end{bmatrix}$$

- A wide sense stationary Guassian process is also stationary in the strict sense.

Eg of Guassian Random process →

A Guassian Random process which is WSS $\bar{X} = 4$ & $R_{XX}(\tau) = 25 e^{-3|\tau|} + 16$. Specify the joint density fn for three rv $X(t_i)$ $i = 1, 2, 3$ $t_i = t_0 + \left[\frac{i-1}{2}\right]$, t_0 is constant

2 instances of time

$t_i = t_0 + \left[\frac{i-1}{2}\right]$ $\quad t_k = t_0 + \left[\frac{k-1}{2}\right]$

$\hookrightarrow t_k - t_i = \frac{k-i}{2}, \; i = 1, 2, 3$

$$R_{XX}(t_k - t_i) = 25e^{-\frac{3|k-i|}{2}} + 16 - (4)^2 = 25e^{-3|k-i|/2}$$

$$C_{XX}(t_k - t_i) = 25e^{-3\frac{|k-i|}{2}} + 16 - (4)^2 = 25e^{-\frac{3|k-i|}{2}}$$

$$[C_X] = 25\begin{bmatrix} 1 & e^{-3/2} & e^{-6/2} \\ e^{-3/2} & 1 & e^{-3/2} \\ e^{-6/2} & e^{-3/2} & 1 \end{bmatrix}$$

Special case → Independent Identical distributive Guassive sgⁿ

Let discrete time random process X_n be a sequence of independent Guassian random variables with mean m & variance σ^2. The covariance matrix for the times $t_1, \ldots t_k$ is

$$[C_X(t_i, t_j)] = [\sigma^2 \delta_{ij}] = \sigma^2 I,$$

where $\delta_{ij} = 1$ when $i = j$ and 0 otherwise, and I is identity matrix. Thus the corresponding joint pdf is →

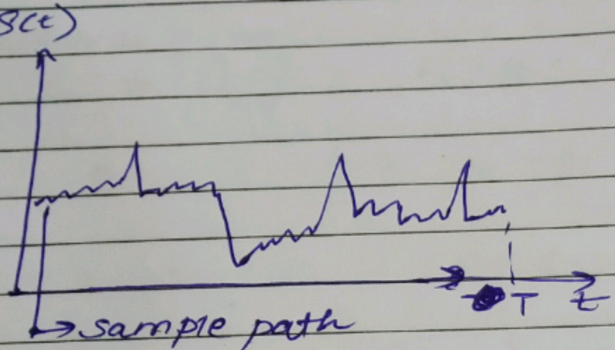
$$f_{X_1, X_2 \ldots X_k}(x_1, x_2 \ldots x_k) = \frac{1}{(2\pi\sigma^2)^{k/2}} \exp\left\{-\sum_{i=1}^{k}(x_i - m)^2/2\sigma^2\right\}$$

$$= f_X(x_1) f_X(x_2) \ldots f_X(x_k)$$

Joint pdf for all cases →

$$f_{X_1, X_2 \ldots X_k}(x_1, \ldots x_k) = \frac{e^{-\frac{1}{2}(x-m)^T C^{-1}(x-m)}}{(2\pi)^{k/2} |C|^{1/2}}$$

Brownian Distribution (process)



* Brownian process helps us to model it.

Symmetric random walk → (Drunkard's walk)

stoch. proc. generated by repeated tosses of a fair coin.

$\bar{\omega}$ head/tail $\omega_1 \quad \omega_2 \quad \omega_3 \quad \omega_4 \quad \omega_5$ ----

H H T T H H T T T

$\bar{\omega} = \bar{\omega}_1 \quad \bar{\omega}_2 \quad \bar{\omega}_3 \quad \bar{\omega}_4 \quad \bar{\omega}_5$ ----

T T H H T T T H H

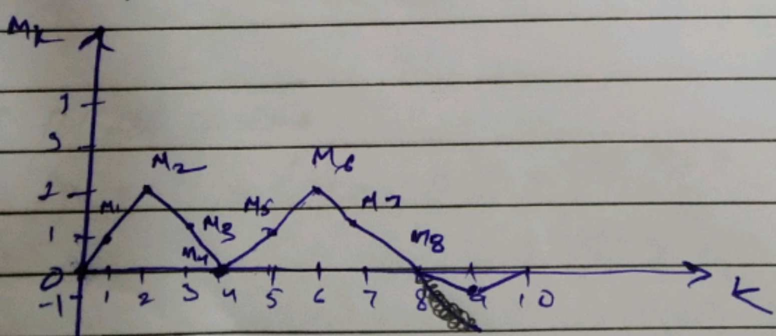
Two diff scenarios

Construct a Random variable

$$X_j := \begin{cases} 1 & \omega_j = H \\ -1 & \omega_j = T \end{cases}$$

Defining a new stochastic process $\{M_k\}_{k=0}^{\infty}$

$$M_0 = 0 \qquad M_k = \sum_{j=1}^{k} X_j$$



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This process have independant increments
$0 = k_0 < k_1 < k_2 < \dots < k_m$

$(M_{k_1} - M_{k_0}), (M_{k_2} - M_{k_1}), \dots (M_{k_m} - M_{k_{m-1}})$ are independant random var.
Non-overlapping intervals, this difference is independant

$$\left.\begin{aligned} E(x_j) &= 0 = 1\times\frac{1}{2} + (-1)\times\frac{1}{2} \\ Var(x_j) &= 1\times\frac{1}{2} + 1\times\frac{1}{2} = 1 \end{aligned}\right\} \forall j$$

$E[M_{k_{i+1}} - M_{k_i}] = 0$
$Var[M_{k_{i+1}} - M_{k_i}] = k_{i+1} - k_i$

* symmetric RW is a discrete Martingale.

$k < l$

$$E[M_l | F_k] = E[(M_l - M_k) + M_k | F_k]$$
$$= E[(M_l - M_k) | F_k] + E[M_k | F_k]$$

(since $l>k$) Independent of F_k, F_k doesn't have info abt anything before it happening

4at time k everything about M_k is known — constant RV

$$= E[(M_l - M_k)] + M_k E[1 | F_k]$$
$$= 0 + M_k \cdot 1 = M_k$$

Quadratic variation →

this value will always be 1

$[M,M]_k$ is defined as $\sum_{j=1}^{k} (M_j - M_{j-1})^2$

$[M,M]_k = k$ (independant of path)

$[M,M]_k = Var(M_k)$ averaging all path

Scaled Random walk (n^{th} level approximation of a Brownian motion)

$$W^{(n)}(t) = \frac{1}{\sqrt{n}} M_{nt}$$ (nt = integer)

Good Write

$0 \le t \le T$
$t \ge 0$

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When nt is not an integer →

take s, u near to t

$s \quad t \quad u$ st ns and nu. Then using linear interpolation to approximate the value of t.

* as $n \to \infty$ $W^{(n)}(t) \to W(t)$

Stochastical process — Symmetric RW in discrete setting

↳ Brownian motion

$\mu t + \sigma \Delta t$

Brownian motion → Continous analog of symmetric random walk. In continous setting, it behaves in the way a symmetric random walk behaves in the discrete setting.

Defined as $B(t; \sigma^2; \mu) = \underset{\text{drift}}{\mu t} + \underset{\text{voltality}}{\sigma} B(t)$ — std brownian motion

$(\Omega, \mathcal{F}, \mathbb{P})$

with $W(0) = 0$

A stochastic process $\{W(t)\}_{t \geq 0}$ is called Brownian motion if given at any time points

$0 = t_0 < t_1 < t_2 < \dots t_n$

the increments

$W(t_1) - W(t_0)$, $W(t_2) - W(t_1), \dots W(t_n) - W(t_{n-1})$ are independant random variables & $W(t_{i+1}) - W(t_i)$ follows normal distribution $\sim N(\underset{\text{or}}{0}, \underset{\text{variance}}{t_{i+1} - t_i})$

how much zigzaging taken place

$\{W(t)\}$ increasing normal

for any $i = 0, n-1$.

$m(t)$ mean fn $= 0$ (for al t) Cov fn $\underset{\text{autocorrelation}}{c(s,t)} = \min\{s,t\}$ $0 \leq s \leq t$

* To get non-negativity, always use exponential fn

Filtration associated with brownian motion

filtration $\{F(t)\}_{t \geq 0}$ is a collection of σ-algebras st

(i) $0 \leq s < t$ $F(s) \subset F(t)$

(ii) $\{W(t)\}$ must be adapted to filtration

(iii) $0 \leq t < u$ then $W(u) - W(t)$ is independant of $F(t)$

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Proof → Brownian motion is a martingale

$0 \le s \le t$

$E[W(t)/F(s)] = E[W(t) - W(s) + W(s) / F(s)]$

$= E[W(t) - W(s) / F(s)] + E[W(s) / F(s)]$

By using 3rd def, $W(t)-W(s)$ doesn't depend on $F(s)$.

$\therefore E[W(t)-W(s)] + W(s)$

$0 + W(s) = W(s)$ Hence proved as martingales

Probability of Brownian motion at a certain given time pt

$W(t) = W(t) - W(0) \sim N(0, t)$

$P(a \le W \le b) = \frac{1}{\sqrt{2\pi t}} \int_a^b e^{-\frac{x^2}{2t}} dt$. symmetric RV discrete $W(t)-W(t_0)$ indep ↳ Normal dist

$0 < t_1 < t_2 \ldots < t_n$

$P(a_1 \le W(t_1) \le b_1, \ldots a_n \le W(t_n) \le b_n)$

$P(A|B) = \frac{P(A \cap B)}{P(B)}$ expectation

$W(t_1) = x_1, \quad a_1 \le x_1 \le b_1$

$P(a_2 \le W(t_2) \le b_2 \mid W(t_1) = x_1) =$

$$\int_{a_2}^{b_2} \frac{1}{\sqrt{2\pi(t_2-t_1)}} e^{-\frac{(x_2-x_1)^2}{2(t_2-t_1)}} dx_2$$

$W(t_2) - x_1 = z$
$W(t_2) = z + x_1$
$W(t_2) \sim N(x_1, t_2 - t_1)$

↳ transitional prob or condition density fn

Marginal densities

Joint probability

$$P(a_1 \le W(t_1) \le b_1, a_2 \le W(t_2) \le b_2) = \int_{a_1}^{b_1}\int_{a_2}^{b_2} g(x_1, t_1 | 0)\, g(x_2, t_2 - t_1 | x_1)\, dx_2\, dx_1$$

Transition density

$$g(x, t | y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}}$$