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23/MC/087

ASSIGNMENT 4

Q1 $na = (na) \cdot b$

Base case $n=1$

$1 \cdot ab = ab$ $\quad (1 \cdot a) b = ab = a(1 \cdot b)$ $\quad [\because 1 \cdot x = x \ \forall x \in R]$

Inductive step

assume it hold for any k

$ka = (ka) b$

Now, For $k+1$

$(k+1) a = (k+1) a \cdot b$

$ka + a = k(a \cdot b) + a \cdot b$

$(ka) b + a = k(ab) + ab$

$\Rightarrow a(k+1) b = ab \cdot (n+1)$

Hence Proved

Q2 $(\Rightarrow)$ assume that $(a^2 - b^2) = (a+b)(a-b)$

$\cancel{a^2} - \cancel{b^2} = \cancel{a^2} - \cancel{b^2} + ab - ba$

$\boxed{ab = ba}$

$(\Leftarrow)$ assume $ab = ba$

$a^2 - b^2 - ab + ba$

$= \cancel{(a^2 - ab + ba) - b^2}$

$a^2 - ab + ba - b^2$

$a(a-b) + b(a-b)$

$(a+b)(a-b) \ \forall a, b \in R$

Q3 $S = \left\{ \begin{pmatrix} a & a \\ b & b \end{pmatrix} \middle| a, b \in \mathbb{Z} \right\} \subset M_2(\mathbb{Z})$ is

a subring of $M_2(\mathbb{Z})$ using subring test

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- ~~Ideal~~ Additive identity : The zero matrix $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is in S ~~so s Contains additive id~~ identity

- $\forall\ a, b \in S \quad a+b \in S$

- If $A, B \in S$ their ~~action~~ Product is not necessarily in S as it may not satisfy the form $\begin{pmatrix} a & a \\ b & b \end{pmatrix}$

Hence S is not closed under multiplication

~~So closure under multiplication fails~~ $\therefore$ S is not subring in R

Q4 Since A and B are ideal of Ring R, then ~~A+B = ...~~ they are closed under ring ~~multiplication~~ addition, contain 0 and are closed under ring addition.

- Additive ~~Cos~~ closure Let $x, y \in A+B$ with $x = a_1 + b_1$ and $y = a_2 + b_2$, where $a_1, a_2 \in A$ then $b_1, b_2 \in B$ then $x + y = (a_1 + b_1) + (a_2 + b_2)$
$= (a_1 + a_2) + (b_1 + b_2) \in A+B$

- Absorbtion for $r \in R$ and $x = a+b \in A+B$
$r \cdot x = r \cdot (a+b) = (r \cdot a) + (r \cdot b) \in A+B$ since $r \cdot a \in A$ and $r \cdot b \in B$ By the ideal Property of A+B

Thus A+B is an ideal of R

Q5 Let $R = Z$ and $A = 2Z$ and $B = 3Z$ as subring of R

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- Additive closure: elements in $A+B$ are of the form $2m+3n$ for $m, n \in Z$ which ~~are~~ do not closed under multiplication (eg $2+3=5 \notin A+B$)

This shows $A+B$ may not form Subring in general

Q16 Show that $Z(R)$ is Subring of R but may or may or may not be ideal.

$Z(R) = \{x \in R \mid xy = yx \ \forall y \in R\}$ is closed under addition also $0=0$ ∴ Contains 0 (additive identity)
$xy = yx$, $xy \in Z(R)$ ∴ closed under multiplication making it a subring.

- For $x \in Z(R)$ and $r \in R$ xr may not Commute with all elements in R, so $xr \notin Z(R)$ in General. Thus $Z(R)$ is not necessarily an ideal.

Q17 Consider $R = 2\mathbb{Z}$ (the set of even integers under multiplication) R does not have unity element as $\nexists$ any elements in $2\mathbb{Z}$ that act as multiplicative identity.

However $S = \mathbb{Z}$ as a subring of R, $1 \in \mathbb{Z}$
∴ S has unity but R doesn't.

Q8 No, Consider example $2\mathbb{Z}$ (subring of $\mathbb{Z}$) the subring $2\mathbb{Z}$ doesn't have unity but $\mathbb{Z}$ does have (1). In cases where subring does have subring is like $\mathbb{Z}$ is subring of $\mathbb{Q}$ and both have 1.

Q9 If $a+b\sqrt{d}$ and $c+\sqrt{d}$ are in $\mathbb{Q}[\sqrt{d}]$
then $a+b\sqrt{d}+c+e\sqrt{d}$ (closure under addition)
$= a+c+(b+e)\sqrt{d}$
$(a+b\sqrt{d})(c+e\sqrt{d}) = (ac+bde^2)+(ae+bc)\sqrt{d}$
[Closure under multiplication

multiplicative inverse $\frac{1}{a+b\sqrt{d}} = \frac{a-b\sqrt{d}}{a^2-ab^2} = \frac{a}{a^2-ab^2} - \frac{b}{a^2-ab^2}\sqrt{d}$
which belongs to $\mathbb{Q}[\sqrt{d}]$

Q10 Consider $M_2(\mathbb{Z}) \to \mathbb{Z}$ given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto a$
check homomorphis.

1. Addition $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$

$A+B = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$

multiplication $AB = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$ the

map gives only $ae+bg$, not preserving multiplication.

14 Let $\phi : R \to S$ be a ring homomorphism
Prove that the image of homomorphism is a subring S

- Closed under addition if $\phi(a), \phi(b) \in im(\phi)$ then $\phi(a) + \phi(b) = \phi(a+b) \in im(\phi)$

- Closed under multiplication if $\phi(a), \phi(b) \in Im(\phi)$ then $\phi(a)\phi(b) = \phi(ab) \in Im(\phi)$

- It contains zero as $\phi(0) = 0 \in Im(\phi)$
Thus $Im(\phi)$ is subring of S

Q16 Intersection if S_i be ~~any~~ subring ~~in~~ for $a, b \in \cap S_i$ $a, b \in$ ~~to every~~ subring S_i and also in $\cap S_i$ so since $\forall a, b \in \cap S_i$ S_i is subring so $\cap S_i$ is also subring as the elements are same.

Q17 $ab = ac$
$\Rightarrow ab - ac = 0$
$a(b-c) = 0$
Now $a \neq 0 \therefore b - c = 0 \Rightarrow b = c$

Hence Proved

Q18 Since R is a Commutative ring with zero

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divisor, if $n \neq 0$ and $n \cdot 1 = 0$ then n must be a prime ~~no~~ number. This is because if $n = a \cdot b$ for non zero $a, b \in \mathbb{Z}$, then we would have $(a \cdot 1) \cdot (b \cdot 1) = (a \cdot b) \cdot 1 = n \cdot 1 = 0$,

→ Which contradicts the assumption that R has zero divisors unless a or b is zero therefore n cannot be factored and must be prime.

→ Conclusion: Therefore characteristic of ~~Ring~~ R is either 0 (if no such n exists) ~~for~~ or n must be prime

Q19 Boolean ring (R) :- $\forall a \in R \; a^2 = a$

~~Additive inver we have,~~ ~~$a + a = 0 \Rightarrow \boxed{2a = 0}$~~

~~$\forall a \in R$. In boolean ring thus~~

$a^2 = a \Rightarrow a^2 - a = 0$

$a(a-1) = 0$

$\boxed{a = 1}$ $\boxed{a \neq 0}$

$2a = 0 \Rightarrow 2 \cdot 1 = 0$. ~~A~~ since

~~2 is the smallest element such that~~

$2a = 0$ $\therefore$ characteristic $= 2$

Q20 Assume $a, b \in R$ then $a^2 = a$ and $b^2 = b$

$(a+b)^2 = a^2 + b^2 + \cancel{2ab} + ba = a^2 + b^2 + ab + ba$

$\Rightarrow \cancel{a} + \cancel{b} + ab + ba = a + b$ $[\because (a+b)^2 = a+b]$

$\boxed{ab + ba = 0}$

$\boxed{ab = -ba}$ But since characteristic of Boolean Ring is 2 we can write $ab = ba$ $[\because -ba = ba]$

$\therefore$ R is commutative

Q11 $S = \{a+ib \mid a, b \in \mathbb{Z},\ b \text{ even}\}$ ~~show tha~~

→ S is closed closed under addition and multiplication
For $a+2bi,\ c+2di \in S$,
~~Sum $(a+c)+(2b+2di) \in S$~~
Product $(a+2bi)(c+2di) = (ac-4bd) + 2(ad+bc) \in S$
$\therefore$ S is subring

→ if $r \in \mathbb{Z}[i]$ $r \cdot S$ may not be in S since the imaginary part might not be even.
Hence : is not ideal.

Q12 ~~Find all idempote~~ An element x is idempotent if $x^2 = x \pmod n$

$\mathbb{Z}_{10}$: $x(x-1) = 0 \pmod{10}$ for $x = 0, 1$ satisfying this, so the idempotent elements are $\{0, 1\}$

$\mathbb{Z}_{12}$: the solution of $x(x-1) = 0 \pmod{12}$ are $x = 0, 1, 4, 9$ so idempotent elements are $\{0, 1, 4, 9\}$

Q13 in $\mathbb{Z}$ Let $A = \langle 4 \rangle$ and $B = \langle 8 \rangle$
~~Group isomorphism~~ since $B \subset A$ A/B is cyclic of order 4 generated by coset $A+B$
so $A/B \cong \mathbb{Z}_4$ as groups.

However A/B does not have a well defined multiplication that corresponds to $\mathbb{Z}_4$ since multiplication of cosets A/B may not be closed.