PAGE No
DATE 25 08 25

Chapter → 2 (Random Walk)

Ex ⇒ A particle is moving along a line.

- Consider a particle initially at x_0 on the x-axis and time = 1, the particle either ~~set~~ step or jump with some prob: distribution Z_1 at time $n = 1$.

- The particle takes another ~~I~~ jump Z_2 which is independent of Z_1
- At time 'n' the position of the particle will be

$$x_n = x_0 + Z_1 + Z_2 \dots + Z_n$$

Where Z_i is a sequence of mutually independent, identically distributed random variables.

⇒ $$\boxed{x_n = x_{n-1} + Z_n}$$

- The steps Z_i can take ~~a~~ values $1, 0, -1$.
- ~~$P(Z_i)$~~ $P(Z_i = 1) = p$
  $P(Z_i = -1) = 1 - p = q$
  $p(Z_i = 0) = 1 - p - q$.

- If the particle ~~continuous~~ continues to move indefinitly then the random walk is unrestricted.

- A random walk starting at $x_0 = 0$ may be restricted to within a distance $a - b$. The point a and $-b$ are called absorbing barriers.

Ex ⇒ Insurance risk.

Consider an insurance company which starts at $t = 0$ with fixed capital x_0 during periods 1, 2, ... so on. It receives some $Y_1, Y_2, \ldots$ in the form of premium, at it pays out some $W_1, W_2, \ldots$ so on.

⇒ $$X_n = x_0 + (Y_1 - W_1) + (Y_2 - W_2) + \ldots$$

$$X_n = x_0 + (Y_i - W_i)$$

$$X_n = x_0 + Z_1 + Z_2 + \ldots + Z_n.$$

Comparing $Z_1 = Y_1 - W_1$

$$\boxed{Z_i = Y_i - W_i}$$

- Here we have an absorbing barrier at 0.

~~In this~~

- **Insurance Company :-**

A company say XYZ has an initial capital of x_0, Rs at time $t=0$ years. In coming years $t=1,2,\dots$ the company gains capitals $Y_1, Y_2, \dots$ respectively in the form of premiums, interests etc. And it pays out $W_1, W_2, \dots$ in the form of insurance claims

$$\therefore \quad x_n = x_0 + (\underbrace{Y_1}_{\text{Gain of 1st yr}} - \overbrace{W_1}^{\text{Claim of 1st yr}}) + (Y_2 - W_2) + \dots (Y_n - W_n)$$

$$\boxed{x_n = Z_{n-1} + Z_n}$$

where $Z_{n-1} = x_0 + (Y_1 - W_1) + (Y_2 - W_2) + \dots (X_{n-1} - W_{n-1})$

and $Z_n = Y_n - W_n$ > 0 Company resumes

$= 0$ No loss, No profit

< 0 Profit.

- **Content of a Dam :-**

Let X_n be the amount of water at the end of n time, suppose that during day 'r', Y_r water flows into the dam in the form of rainfall and rivers, water is released from the dam at the beginning of each day. If the content of water at the end of day '$r-1$' is added to the inflow of day 'r' exceeds a quantity say 'A' then A unit of water is released during day r.

If B is the capacity of the dam. then,

$$X_{r-1} + Y_r - A > B$$

then ~~ovel~~ overflow will occur of amt. $X_{r-1} + Y_r - A - B$ on day r

- It can be seen as a random walk.

$$\begin{aligned} X_n &= X_{n-1} + Z_n \quad \text{where } X_{n-1}+Z_n < B \\ &= 0 \qquad\quad \text{if } X_{n+1} + Z_n \leq 0 \\ &= B \qquad\quad \text{if } X_{n-1} + Z_n > B. \end{aligned}$$

where $Z_n = Y_r - A$

- Also X_n is a random walk with reflecting barrier 0 and B

- A reflecting barrier is a state which when crossed in a given direction say downwards. hold the particle ~~tt~~ until a +ve. jump occurs and ~~att~~ allows the particle to move up and resume the random walk.

⇒ **Content of a Dam :-**

Let X_n be the amount of water at the end of n time, Suppose a day say 'k', Y_k is the water deposited in the ~~dar~~ dam on 'k' day 'k' and A be the water released during day 'k' then

$$X_{k-1} + Y_k - A > B$$

where B is the total capacity of the dam,

then overflow will occur of the amt. $X_{k-1} + Y_k - A - B$ on day k.

It is a random walk as.

$$\begin{aligned} X_n &= X_{n-1} + Z_n \quad \text{where } X_{n-1} + Z_n < B \\ &= 0 \qquad\qquad \text{if } X_{n-1} + Z_n \leq 0 \\ &= B \qquad\qquad \text{if } X_{n+1} + Z_n > B. \end{aligned}$$

The Game has independent turns.

- **Gambler's ruin:-**

Consider two gamblers MY & IJ both start with a certain amt. of money say 'a' & 'b' respectively. At each turn MY wins the 1 unit of IJ's capital at a prob. of 'q', Let X_n denote the gain of IJ at the end on 'n' turns., then

$$X_n = Z_0 + Z_1 + \dots + Z_n$$

$$\text{Where} \quad -b < X_n < a.$$

Now,
$$P(Z_i = 1) = P$$
$$P(Z_i = -1) = q$$

If at any stage $X_n = a$ then IJ will gain all of MY's capital then game will stop.

But if $X_n = -b$ then IJ will be ruined and game will stop.

- Therefore this is a random walk with absorbing barrier $-b$ & a.

- Consider a random walk with independent jump with prob.

$$P(Z_i = 1) = p$$
$$P(Z_i = -1) = q$$
$$P(Z_i = 0) = 1-p-q$$

- <u>Unrestricted R.W.</u> :- Consider the case where particle is free to move in either direction. indefinitely.

$$X_n = \sum_{r=1}^{n} Z_r$$

at time $t = n$, $K = 0, \pm 1, \pm 2, \dots \pm n$

for Ex ⇒ If a particle is to reach K at time $t = n$ then particle has to perform r_1 +ve jumps, r_2 -ve jumps, r_3 0 jumps. where $r_1, r_2, r_3 \geq 0$

- The probability that

$$P(X_r = K) = \frac{n!}{r_1!\, r_2!\, r_3!} \times p^{r_1} q^{r_2} (1-p-q)^{r_3}$$

The Prob. Generating function of jump Z_r is (PGF)

$$G(z) = pz + \frac{q}{z} + (1-p-q)z^0$$

Let μ and σ^2 be the mean and variance of the jump then,

$$E(X_n) = n\mu$$
$$V(X_n) = n\sigma^2$$

★ Central Limit Theorem, X_n will be normally distributed with mean $n\mu$ and variance $n\sigma^2$.

Imp: $p(j \leq X_n \leq k) \simeq$ Normal distribution

$$\frac{1}{\sqrt{2\pi\sigma^2 n}} \int_j^k e^{-\frac{(x-n\mu)^2}{n\sigma^2}} dx.$$

(I) —— $\simeq \phi\left(\frac{k+c-n\mu}{\sigma\sqrt{n}}\right) - \phi\left(\frac{j-(-n\mu)}{\sigma\sqrt{n}}\right)$

where $\phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-1/2 x^2} dx$

Two absorbing barriers.

- Suppose the particle starts at the origin and moves in the presence of two absorbing barriers at the point $-b$ and a the movement will stop when the particle enters either of the states $-b$ or a.

The Prob. that the particle is still in motion at time n, i.e. it occupies one of the non-absorbing states $-b+1$, $-b+2$, ..., $a-1$ cannot exceed prob. that an unrestricted particle occupies one of these states at time n.

from ① we can observe that Prob $\to 0$ as $n \to \infty$, this mean that the Prob. that the particle is not yet absorbed at time n also tends to zero as $n \to \infty$.

- Let $f_{ja}^{(n)}$ is the prob. that the particle is absorbed at exactly time n.

$\therefore f_{ja}^{(n)}$ is the prob. that an unrestricted particle reaches position a for the first time at time n without position $-b$ being occupied at any of the times $1, 2, \ldots, n-1$ all conditional on starting at j.

$$f_{ja}^{(n)} = P(-b < X_n < a,\ -b < X_{n-1} < a,\ -b < X_1 < a \mid X_0 = j)$$

for $n = 0$,

$$f_{ja}^{(0)} = \begin{cases} 1 & , j = a \\ 0 & , j \neq a \end{cases}$$

- **One Absorbing barrier :-**

Let the particle starts at $X_0 = 0$

- Our aim is to find the prob. that the particle will ever reach the absorbing state.

Let $f_{a}^{(n)}$ be the prob. that the particle will reach state a at time n.

Define the generating function.

$$F_a(s) = \sum_{n=1}^{\infty} f_a^{(n)} s^n.$$

In case of two absorbing barriers. the generating function is given by.

$$F_j(s) = F_{ja}(s) = \frac{(\lambda_1(s))^{j+b} - (\lambda_2(s))^{j+b}}{(\lambda_1(s))^{a+b} - (\lambda_2(s))^{a+b}}$$

where $\lambda_1(s) > \lambda_2(s)$

Letter $b \to \infty$
we obtain . F_{ja} for absorbing barriers.

- Where λ_1 and λ_2 are the sol.^{n} of the Eq^n

$$P(s)\lambda^2 - \lambda(1 - s(1-p-q)) + qs = 0$$

$$F_j(s) = \lim_{b \to \infty} \frac{1}{(\lambda f s)^a}$$

☆ Two Reflecting barriers :-

- Reflecting barrier is defined as suppose 'a' is the point above initial position if the particle reaches 'a' either it will either remain at a or it will return to the neighbouring state 'a - 1'.

Prob. of remaining at $a = q$
" " returning to $a-1 = 1-q$.

If the particle is initially at the state j and that 0 and a are reflecting barriers.

$$X_n = \begin{cases} X_{n-1} + Z_n & (0 \leq X_{n-1} + Z_n \leq a) \\ a & (X_{n-1} + Z_n > a), \\ 0 & (X_{n-1} + Z_n < 0). \end{cases}$$

- So there is no prob. of ceasing motion at any stage.

Let $P_{jk}^{(n)}$ be the probability of the particle k at time n initially at the state j, since the jumps are independent the position of the particle n depends only on its position at time $n-1$ and n^{th} jump.

$$P_{jk}^{(n)} = p \cdot P_{j,k-1}^{(n-1)} + (1-p-q) P_{jk}^{(n-1)} + q P_{j,k+1}^{(n-1)}$$