2-quiz 3- Assignments / 4-08-2025

Stochastic process → collectⁿ of R.V. indexed by time.

System state → $\boxed{\text{state variable}}$

Process → induce a change in state of the system.
↓
indexed by time

→ Deterministic
↓
Random

1 - State variable
2 - Time index

		state variable	
		continuous	discrete
index	continuous	Brownian motion	
	discrete		

$X(t)$ ← functⁿ of time.
X_t ← indexed by time

$$E(X) = \sum x_i p_i$$

$$Var(X) = E(X_i - E(X))^2 = \sum p_i (X_i - E(X))^2$$
$$= E(X^2) - (E(X))^2$$

$$E(X^n) = \sum p_i X_i^n$$

↑ moment about mean zero

Stochastic Process — 7 Aug 2025

IID - independent identical distributⁿ

Bernoulli process:

betting $\rightarrow$ win : 1000₹
loss : 1000₹

↳ discrete time process

- $Y \rightarrow$ no. of success in 'n' trial in bernoulli process.

$$Y = \sum_{i=0}^{n} x_i$$

$$P(Y=k) = {}^{n}C_k \, q^{(n-k)} p^k$$

Z = no. of trial getting first success

$$E(Z) = \sum_{k=1}^{\infty} q^{k-1} p k$$

$$E(Z) = p \sum_{k=1}^{\infty} k (q)^{k-1}$$

$$K(1 - KP + P)$$

$$k(1-p)^{k-1}$$

$$k(1 - (k-1)(p))$$

$$\frac{k(1-p)^{k}(1-p)^{-1}}{k(1-p)^{k}(1+p)} \rightarrow \frac{k(1-p)^{k}}{(1-p)} \rightarrow \frac{k(k-kp)}{(1-p)}$$

$$p\sum_{k=1}^{\infty}(1-p)^{k-1}k \qquad \frac{d}{dq}q^{k}$$

$$= p\sum_{k=1}^{\infty}\frac{d}{dq}q^{k} \Rightarrow p\frac{d}{dq}\sum_{k=1}^{\infty}q^{k}$$

$$\hookrightarrow p\cdot\frac{d}{dq}\left(\frac{1}{1-q}\right) = p \times \frac{1}{(1-q)^{2}}$$

$$= \frac{1}{p}$$

W = no. of trial for getting k-success...

to get $(n-1)$ trial ... need $(k-1)$ success...

Stochastic process

Basic PnS → descriptive

→ Measure of central tendency
- Mean
- Median
- Mode

→ Measure of variability (spread)
- standard deviatⁿ
- Variance
- Range
- Kurtosis
- Skewness

→ Boole's inequality
for any n events:

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i)$$

↳ Bonferroni's inequality:

$$P\left(\bigcup_{i=1}^{n} A_i\right) \geq \sum_{i=1}^{n} P(A_i) - \sum_{1\leq i\leq j\leq n} P(A_i \cap A_j)$$

↳ Baye's theorem:
If $B_1, B_2, B_3, \ldots, B_n$ are mutually exclusive events contain in sample space S

$$\bigcup_{i=1}^{n} B_i = S \ \& \ P(B_i) > 0$$

The for any event A contained in S for which $P(A) > 0$

$$P(B_j/A) = \frac{P(A/B_j) \cdot P(B_j)}{\sum_{i=1}^{n} P(A/B_i) P(B_i)}$$

↳ Random variable
RV on a sample space S is a functⁿ $X: S \to \mathbb{R}$
s.t $X(s)$ be a real no., $s \in S$

Discrete RV

Pmf → collectⁿ pair of (x_i, p_i) is called PD of RV X

$X \to$ discrete RV
$P(X = x_i) = p_i$
↳ $p_i \geq 0 \ \forall i$
↳ $\sum_{i=1}^{\infty} p_i = 1$

Continuous RV

Pdf; X is a continuous RV

$$P\left(x - \frac{dx}{2} \leq X \leq x + \frac{dx}{2}\right) = f(x)\,dx$$

↳ $f(x) \geq 0 \ \forall x \in \mathbb{R}$
↳ $\int_{-\infty}^{\infty} f(x)\,dx = 1$

↳ $P(a \le X \le b)$ or $P(a < X < b) = \int_a^b f(x)\,dx$.

↳ $P(X=a) = \int_a^a f(x)\,dx = 0$

↳ impossible for a continuous RV to assume a particular value.

↳ Cummulative distribution function (cdf) or distribution function:

if X is RV (D or C) then

$P(X \le x)$ is called cdf

$F(x)$

↳ discreate.

$\bullet\ F(x) = \sum_{x_j \le x} P_j$

Continuous

$F(x) = \int_{-\infty}^{x} f(x)\,dx$.

→ Properties of distribution function

↳ $0 \le F(x) \le 1 \quad \forall x \in (-\infty, \infty)$

↳ $F(x)$ is a monotone ↑ function of x.

$F(x_1) \le F(x_2)$ for $x_1 \le x_2$.

Discrete RV

$P(X=x_i) = P(x_i) - P(x_{i-1})$

Continuous RV

$\frac{dF(x)}{dx} = f(x)$;

↓

Pdf of X

↳ visit & Practice Joint, marginal & conditional distribution...

Discrete distribution

↳ Binomial

→ Bernaulli trials.

↳ condition

→ each trial have 2-mutually exclusive outcome

→ Probability of success in each trial remain same

↳ outcome of successive trials are mutually exclusive..

Probability of x success in n trial.

$\binom{n}{x} p^x q^{n-x}$.

↳ Multinomial distribution

$E_1, E_2, E_3, \ldots, E_K$ are mutually exclusive event with

Probability $P_1, P_2, \ldots, P_K \Rightarrow P_1 + P_2 + \cdots + P_K = 1$.

X - R.V represent no. of time E_K occur.

$X_1 + X_2 + \cdots + X_K = n$

$\bullet\ P(X_1 = n_1, X_2 = n_2, \ldots, X_K = n_K) = \frac{n!}{n_1!\, n_2! \cdots n_K!} \times P_1^{n_1} \cdot P_2^{n_2} \cdots P_K^{n_K}$

$\bullet\ n_1 + n_2 + \cdots + n_K = n$

↳ Poisson distribut^n

X Discreate R.V

$$P(x) = P(X=x) = \frac{\lambda^x e^{-\lambda}}{x!} \quad ; x = 0,1,2\ldots$$

$\lambda \to$ +ve constant.

Poisson distribut^n → binomial

$n \to$ no. of trials are large

$p \to$ Probability of success is small

↳ $\lambda = np$; $P(x) = {}^nC_x \, p^x q^{n-x}$

moment generating function: $M_X(t) = (pe^t + q)^n$

about origin

characteristic:

$$\phi_X(t) = E(e^{ixt}) = (pe^{t} + q)^n$$

↳ hyper geometric distribut^n

- Probability of success changes on each trial.

Notation.

- 1000 - ball

700 → G & 300 R

sample 7.

Probability of 3G & 4R.

Populat^n	Sample
N	n
M → success	k ← success
$N-M$	$n-k$

N → M, $N-M$

n → k, $n-k$

- $$P(X=k) = \frac{\binom{M}{k}\binom{N-M}{n-k}}{\binom{N}{n}}$$

hyper geometric pmf ≈ the binomial pmf

no. of sample elements <<< Populat^n.

↳ Geometric distribution

↳ getting first success after x failure.

↳ $P(x) = pq^x$

- simple permutat^n & comb

↳ -ve Binomial distribution

use logic simple.

- occurance of success / failure in a sequence of independent trial before a specific no. of success occur

↳ trial repeat until a fixed no. of success say k occurs.

↳ assume k^{th} success occur at $(x+k)^{th}$ trial.

$$P(x) = \binom{x+k-1}{k-1} p^k q^x$$

Continuous probability distribution

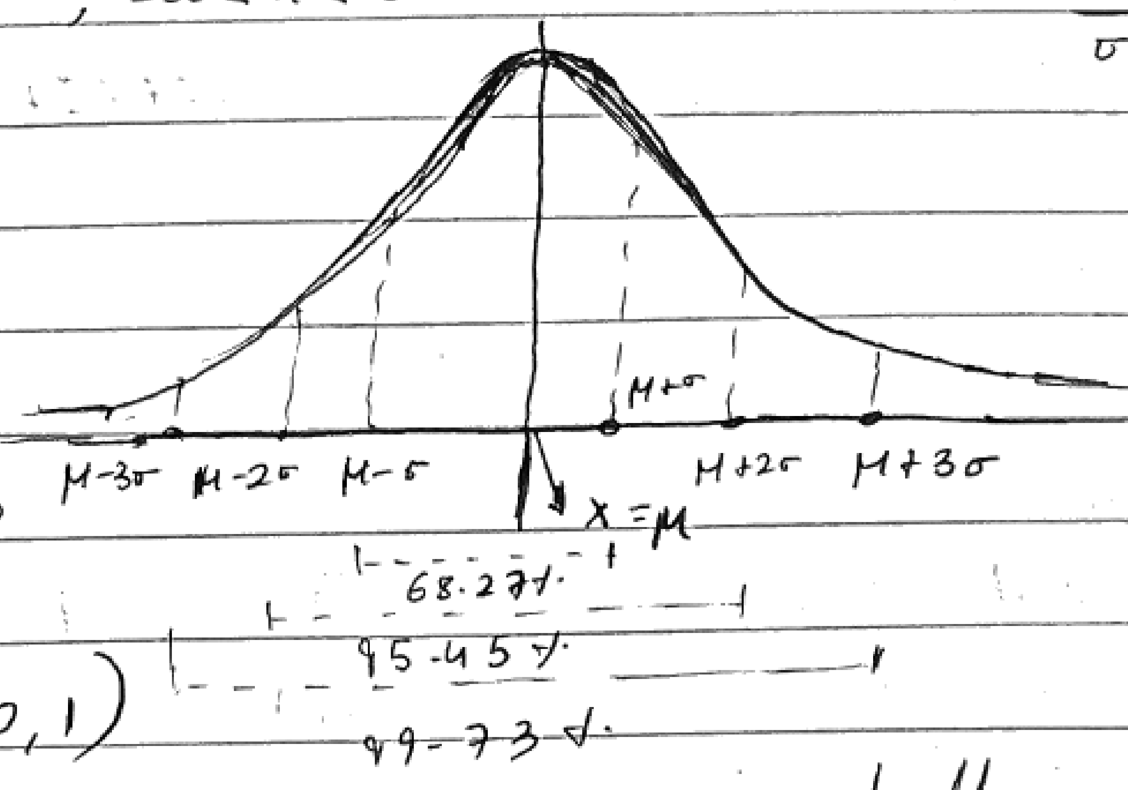
↳ Normal distribution:

pdf

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \times e^{-(x-\mu)^2/2\sigma^2} \; ; \; -\infty < x < \infty$$

$\circ \; Z = \frac{x - \mu}{\sigma}$

$X \sim N(\mu, \sigma)$



↳ Standard normal variate.

$$f(z) = \frac{1}{\sqrt{2\pi}} \times e^{-z^2/2} \qquad -\infty < z < \infty$$

$\mu_Z = 0 \; ; \; \sigma_Z = 1 \rightarrow Z \sim (0,1)$

↳ Z-score in general allow us to compare things that are not on the same scale, as long as they're normally distributed

↳ Exponential distribution

$$f(x) = \begin{cases} \alpha e^{-\alpha x} & ; \; x \geq 0 \\ 0 & ; \; x < 0 \end{cases} \qquad \alpha > 0$$

$E(x) = \frac{1}{\alpha} \; ; \quad Var(x) = \frac{1}{\alpha^2}$

Memory less property:

if X follow exponential distribution

$$P(X > m \mid x > n) = P(X > m-n)$$

for any $m, n > 0$

Proof:

$$P(X > s+t \mid X > t) = \frac{P(X > s+t \cap X > t)}{P(X > t)}$$

$$= \frac{P(X > s+t)}{P(X > t)} = \frac{\alpha \, e^{-\alpha(s+t)}}{\alpha \, e^{-\alpha(t)}} = e^{-\alpha s}$$

$$= P(X > s)$$

↳ memory less because future behaviour doesn't depend on the past ...

Moment Generating function

• discrete

↳ $M_X(t) = E(e^{tX})$

$= \sum_x e^{xt} f(x)$

• continuous

↳ $M_X(t) = \int_{-\infty}^{\infty} e^{xt} f(x)\,dx$

$$\bullet \; \mu_r' = \left[\frac{d^r M_X(t)}{dt^r}\right]_{t=0}$$

failure rate function

X cont.ous RV

hazard rate function

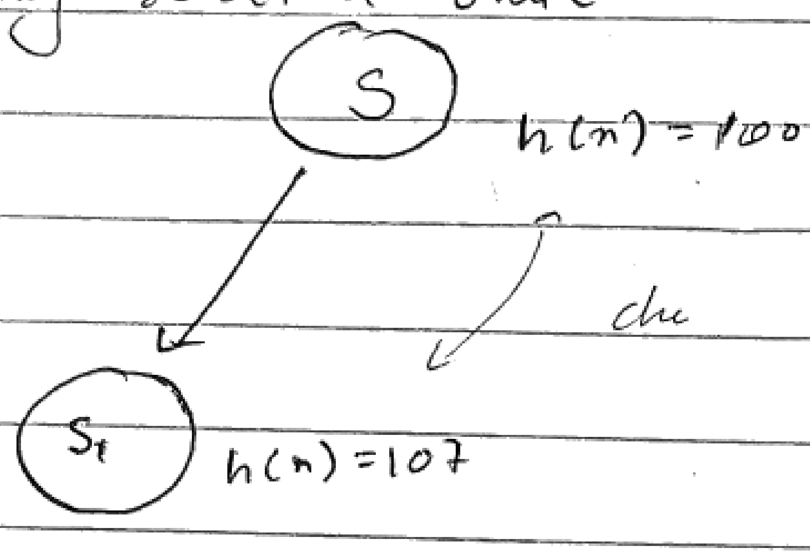
$$\bullet \; r(t) = \frac{f(t)}{1 - F(t)}$$

12-Aug 2025

stochastic hill Climbing

probability threshold = α

→ randomly select a state



probability threshold = 0.22

$$\Delta E = \text{Parent}(h(n)) - \text{child}(h(n))$$

→ to convert it in probability we use sigmoid.

$$a_n = \frac{1}{1 + e^{-\Delta E / T}}$$

↳ here T is constant

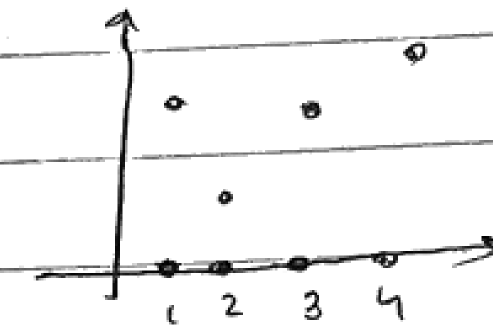
Simulated Anneling

↳ here we change value of T → start with a higher value & then decrease gradually at each level...

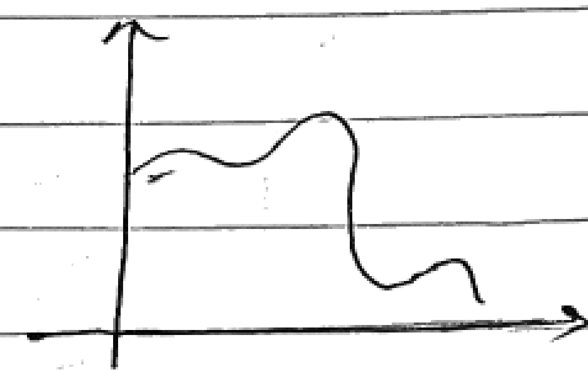
Stochastic process

↳ collection of random variables indexed by time.

$X_0, X_1, X_2, \ldots \rightarrow$ discrete time



$\{X_t\}_{t \geq 0} \rightarrow$ continuous



↳ Alternate definatⁿ:
probability distribution over a space of paths.

- uniform distributⁿ:

Nature: ↳ continuous b/w $[a,b]$; every value equally likely.

↳ Flat shape: maximum ignorance.

where it fits:

↳ when we only know bounds but nothing about Probabilities.

Question we study:

↳ what are the dependencies in the sequence of value?

↳ what is the long term behaviour?
(law of large no., central limit theron)

↳ what are boundary events.

ex: delivery time gauranteed b/w 2-5 days, but no idea where in b/w

$$\text{Pdf}: \quad f(x) = \begin{cases} \dfrac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise.} \end{cases}$$

$$\text{cdf}: \quad F(x) = \begin{cases} 0 & ; \ x < a \\ \dfrac{x-a}{b-a} & ; \ a \leq x < b \\ 1 & ; \ x \geq b. \end{cases}$$

$\bullet\ P(x_1 < X < x_2) = F(x_2) - F(x_1) = \dfrac{x_2 - x_1}{b-a}$; $a \le x_1 < x_2 \le b$.

$E(X) = \dfrac{a+b}{2}$; $V(X) = \dfrac{(b-a)^2}{12}$

ex: A bus arrives every 20 minute at a specified stop b/w 6:40 am & continuing until 8:40 AM. A certain passenger doesn't know the schedule but arrive randomly (uniform distr.) b/w 7:00 AM & 7:30 AM. Probability that passenger wait more the 5 minute.

sol^n: Buses come every 20 min,
relevant times around 7:00 - 7:30 are 7:00, 7:20, 7:40.
let X = arrival time in minutes past 7:00, since the passenger arrive b/w 7:00 - 7:30

$X \sim uniform\ (a, b)$ with $a = 0$ & $b = 30$.

PDF : $f_X(x) = \dfrac{1}{30}$ & CDF ; $F_X(x) = \dfrac{x-0}{30} = \dfrac{x}{30}$

Case 1 :
if $0 < x < 20$ next bus 7:20
wait time $= 20 - x$
$20 - x > 5 \rightarrow x < 15$
So, within $(0, 20)$ $0 < x < 15$

case 2 :
if $20 < x < 30 \rightarrow$ next bus 7:40.
wait time $\rightarrow x < 35$
so, within $(20, 30) \rightarrow$
$20 < x < 30$

$P(\text{wait} > 5) = \dfrac{\text{favourable length}}{\text{total length}}$

total favourable length
$15 + 10 = 25$

$= \dfrac{25}{30} \approx 0.8333$

"Exponential distribution" (only continuous distribution to follow memoryless property) --

Nature: ↳ continuous only +ve value ($t \geq 0$)
↳ memoryless ->
↳ shape: highest at 0, then decay exponentially ; mean = standard deviation.
↳ variability high

where it fits: model time b/w random arrivals or random service time when event occurs independently at a constant rate

ex; ↳ time b/w phone calls to a call center.
↳ time until the next data packet arrives at a router.

PdF :

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & ; x \geq 0 \\ 0 & ; \text{otherwise} \end{cases}$$

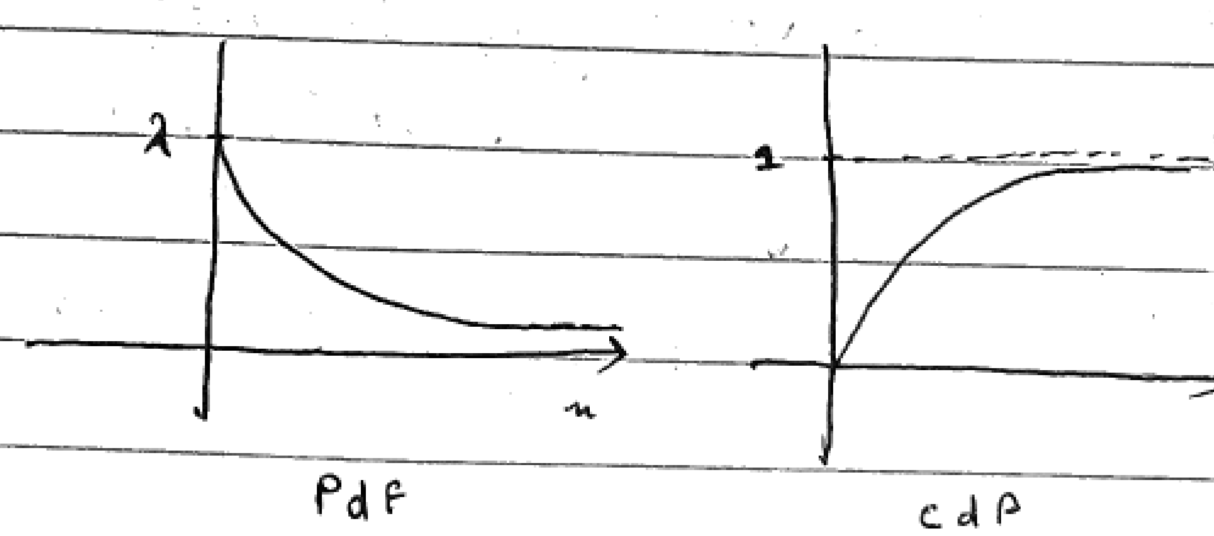
cdf:

$$F(x) = \begin{cases} 0 & ; x < 0 \\ \int_0^x \lambda e^{-\lambda x} dx = 1 - e^{-\lambda x} & ; x \geq 0 \end{cases}$$

$\lambda \rightarrow$ rate : arrival per hr or services per minute (example)

↳ also used to model the lifetime of a component that fails catastrophically (instantaneously), such as light bulb ; $\lambda \rightarrow$ rate of failure.

$E(X) = \frac{1}{\lambda}$; $V(X) = \frac{1}{\lambda^2}$



PdF   cdP

- memoryless condition

$X \rightarrow$ continuous R.V survival function $S(x) = P(X > x)$

$$P(X > s+t \mid X > s) = P(X > t)$$

$$\frac{S(s+t)}{S(s)} = S(t) \quad \forall s, t \geq 0$$

- Gamma Distributⁿ

Nature
- → continuous, +ve value only
- → A generalizatⁿ of the exponential: sum of several exponential stages.
- → Shape: skewed right (like exponential), but smoother & more peaked if the shape b/w parameter is larger.
- → Variability ↓ as the no. of stages ↑.

where it fits:
- → use when a process has multiple similar steps.
- → Time-to-failure when components go through stages before breaking...

ex:
- ↳ Time to finish airport security: check ID, scan luggage, body scan
- ↳ lead time in supply chains when many sub-activities are involve...

$$\text{PDF}: \quad f(x) = \begin{cases} \dfrac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} \cdot e^{-\beta x}, & x > 0 \\ 0 & ; \text{otherwise} \end{cases}$$

$\beta \rightarrow$ shape parameter.

$\theta \rightarrow$ scale parameter.

with θ.

$$f(x) = \begin{cases} \dfrac{\beta\theta}{\Gamma(\beta)} (\beta\theta x)^{\beta-1} e^{-\dots} \\ 0 & ; \text{otherwise} \end{cases}$$

$$E(X) = \frac{1}{\theta} \quad ; \quad V(X) = \frac{1}{\beta\theta}.$$

- Erlang → consider a series of k stations that must be passed through in order to complete the servicing of a customer.

pdf Erlang (k, λ)

$$f_X(x) = \frac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!} \quad x \geq 0$$

cdf

$$F_X(x) = P(X \leq x) = 1 - e^{-\lambda x} \sum_{n=0}^{k-1} \frac{(\lambda x)^n}{n!} ; x \geq 0$$

- Normal distributionⁿ

Nature:
- → continuous, symmetric bell curve
- → Defined by mean & standard deviatⁿ
- → Take value $(-\infty, \infty)$, so it needs truncation if modelling something non-negative.

where it fits:
- → R.V centered on an average with symmetric fluctuatⁿ
- → Not ideal for strongly skewed or bounded process.

Ex: Time for a machine operatⁿ that is very consistant

- Poisson distribution.

Nature:
- → Discrete count (0, 1, 2 ...)
- → describes no. of events in a fixed period when event occurs independently at a ~~random~~ rate .. → constant.
- → Mean = variance

where it fits: no. of arrivals or orders in a fixed window

ex: → call arriving per 10 minutes.
→ order per day in a grocery app.

• Weibull distribution:

Nature: ↳ Continous, +ve value only.
↳ Flexible hazard rate

• Shape (say): β ($\beta > 0$)

$\beta < 1 \rightarrow$ higher risk early (infant mortality)

$\beta = 1 \rightarrow$ become exponential

$\beta > 1 \rightarrow$ increase risk over time.. (wear-out)

where it fits:
↳ time to failure in machine & tools.
↳ service time when we need heavier or lighter tails than exponential..

Example: ↳ lifetime of a bearing; early defects.
↳ time customer spent in self-service kiosk..

Pdf:

$$f(x) = \begin{cases} \frac{\beta}{\alpha}\left(\frac{x-v}{\alpha}\right)^{\beta-1} e^{-\left(\frac{x-v}{\alpha}\right)^{\beta}} & ; x \geq v \\ 0 & ; \text{otherwise} \end{cases}$$

$v \rightarrow$ location parameter ($-\infty < v < \infty$)
α ($\alpha > 0$) $\rightarrow$ scale parameter
β ($\beta > 0$) $\rightarrow$ shape parameter.

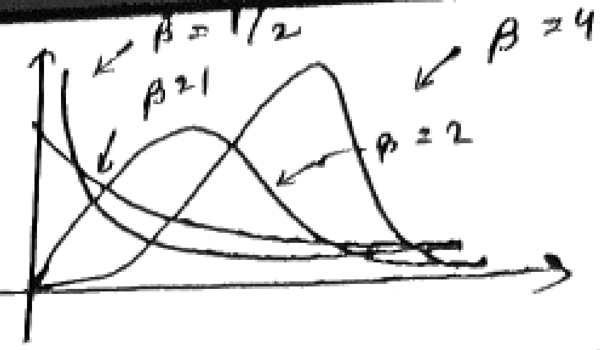
for $v = 0$ & $\beta = 1$.

$$f(x) = \begin{cases} \frac{1}{\alpha} e^{-x/\alpha} & ; x \geq 0 \\ 0 & ; \text{other} \end{cases}$$

$$E(X) = v + \alpha\,\Gamma\left(\frac{1}{\beta}+1\right)$$

$$V(X) = \alpha^2\left[\Gamma\left(\frac{2}{\beta}+1\right) - \Gamma\left(\frac{1}{\beta}+1\right)\right]^2$$

; $\lambda = 1/\alpha$ ← an exponential distributⁿ.



$$F(x) = \begin{cases} 0 & ; x < v \\ 1 - e^{-\left(\frac{x-v}{\alpha}\right)^{\beta}} & x \geq v \end{cases}$$

time to failure of a component screen weibull distribut^n
=0, $\beta = 1/3$ and $\alpha = 200$ hr

$E(x) = 200\ \Gamma(3+1) = 200\,(3!) = 1200$ hr.

robability that the unit fails before 2000 hr.

$$F(2000) = 1 - e^{-\left(\frac{2000}{200}\right)^{1/3}} = 0.884$$

gle distribut^n:

re:
- → continous defined by min, max, mode
- → shape : triangle → peak at mode
- → more realistic than uniform where we guess a typical value.

t fit: limited data but some intuition about "likely" value

rvice time 2 min, usually 5 min, never more than 8 in --

$$f(x) = \begin{cases} \dfrac{2(x-a)}{(b-a)(c-a)} & ; a \leq x \leq b \\[2ex] \dfrac{2(c-x)}{(c-b)(c-a)} & ; b < x \leq c \\[2ex] 0 & ; \text{elsewhere.} \end{cases}$$

here $a \leq b \leq c$, mode occur at $x = b$.

$$E(x) = \frac{a+b+c}{3}$$

cdf:

$$F(x) = \begin{cases} 0 & x \le a \\ \dfrac{(x-a)^2}{(b-a)(c-a)} & a \le x \le b \\ 1 - \dfrac{(c-x)^2}{(c-b)(c-a)} & b < x \le c \\ 1 & x > c \end{cases}$$

- Lognormal ~~normal~~ distribution:

Nature:
- continuous, +ve only
- skewed right (long tail)
- if we take log the data looks normal
- capture multiplicative randomness ; when random factors multiply together..

where it fits:

Task time or lifetime with occasional extreme large value..

Pdf :

$$f(x) = \begin{cases} \dfrac{1}{\sqrt{2\pi}\,\sigma_x} e^{-\left(\frac{(\ln x - \mu)^2}{2\sigma^2}\right)} & ; x > 0 \\ 0 & ; \text{otherwise..} \end{cases}$$

$E(X) = e^{\mu + \sigma^2/2}$

$V(X) = e^{2\mu + \sigma^2}(e^{\sigma^2} - 1)$; μ & σ are not mean & variance of lognormal.

$Y \sim N(\mu, \sigma^2)$

then $X = e^Y$ has lognormal distribution with parameter μ & σ^2

if mean & variance of the lognormal distribution are known to be μ_L & σ_L^2

$$\mu = \ln\left(\frac{\mu_L^2}{\sqrt{\mu_L^2+\sigma_L^2}}\right) \; ; \; \sigma^2 = \ln\left(\frac{\mu_L^2+\sigma_L^2}{\mu_L^2}\right)$$

~~Bernoulli process~~

- The range (possible values) of the R.V in a stochastic process is called the state space of the process.

  ↪ sample path is a collection of time-ordered data describing what happened to a dynamic process in one instance.

  ↪ A stochastic process is a probability model describing a collection of time-ordered R.V that represent the possible sample path.

Stochastic process:

Bernaulli process $\rightarrow$ seq of independent bernaulli trials, X_i

$\hookrightarrow$ at each trial, i:

$P(X_i = 1) = P(\text{success at the } i^{th} \text{ trial}) = p$

$P(X_i = 0) = P(\text{feullure } \quad " \quad ") = 1-p.$

assumptions.

$\hookrightarrow$ independent

$\hookrightarrow$ time - homogeneity

model of:

- sampling a deck of card with oreplacement

$\hookrightarrow$ seq of lottery wins/losses

$\hookrightarrow$ Arrival (each sec) to a bank.

- $E(X) = np$ & $Var(X) = npq$

$\hookrightarrow$ time until the first success/arrival:

$$T_1 = \min\{i : X_i = 1\}$$

$$P(T_1 = k) = P(\underbrace{0\,0\,0 \cdots 0}_{k-1}\ \underset{k}{1}) = (1-p)^{k-1} p$$

$\leftarrow$ geometric distribut^n.

$\{x_n\} \sim Ber(p)$

1 0 0 1 0 0 1

7

$Y_1 = X_7$

$Y_2 = X_8$

$\{Y_i\}$ $i = 1, 2, 3 \ldots$

(i) independent of $X_1, X_2, \ldots, X_7$

(ii) $Ber(p)$

$\hookrightarrow$ fresh start after time 7 (we can generalize it and say after time n)

The process $X_{N+1}, X_{N+2}, \ldots$ is

$\hookrightarrow$ a bernaulli process

$\hookrightarrow$ independent of $N, X_1, X_2, \ldots, X_N$

- as long as N is determined "casually"

- The distribut^n of busy period:-

$\hookrightarrow$ at each slot, a server is busy or ide (bernaulli process)

- First bysy period: $Geo(1-p)$

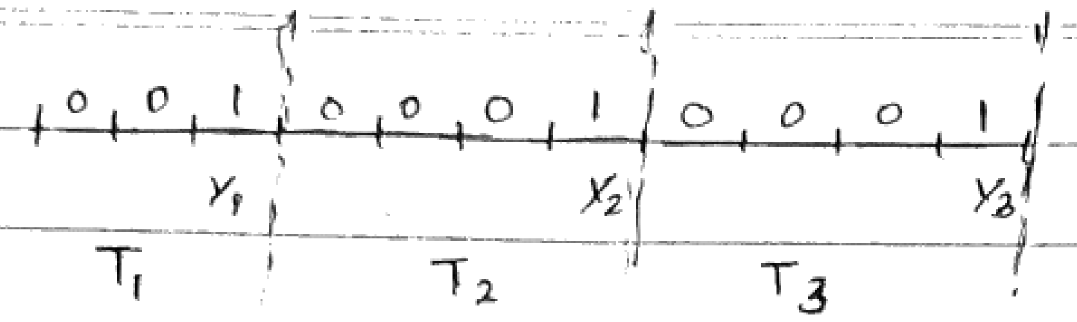
$\hookrightarrow$ start with first busy slot

$\hookrightarrow$ end just before the first subsequent ide slot

B B B I I I B B

I I I I B B B B I I

- Time of the k^{th} success/arrival.



$Y_k = T_1 + T_2 + \dots + T_k$

Y_k = time of k^{th} arrival

T_k = k^{th} inter-arrival time = $Y_k - Y_{k-1}$ $(k \geq 2)$

- The Process starts fresh after time T_1

T_2 is independent of T_1; Geometric (p); etc.

$E[Y_k] = \frac{k}{p}$; $Var(Y_k) = \frac{k(1-p)}{p^2}$

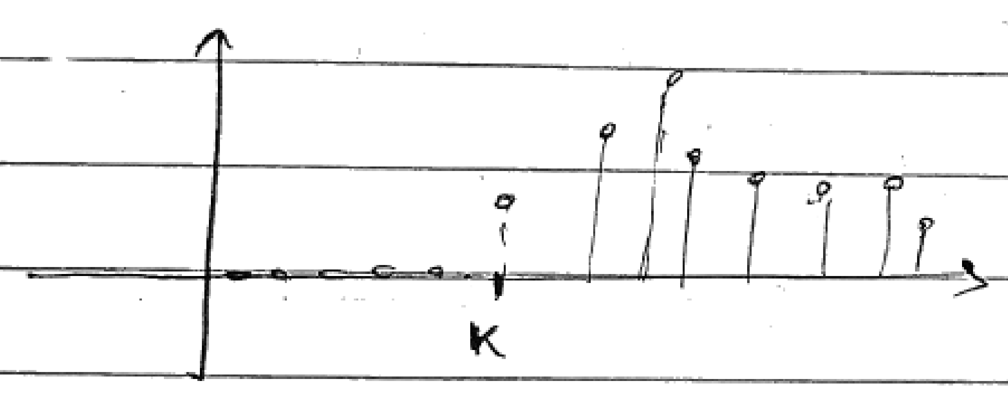
k^{th} arrival at time t

$P(Y_k = t)$

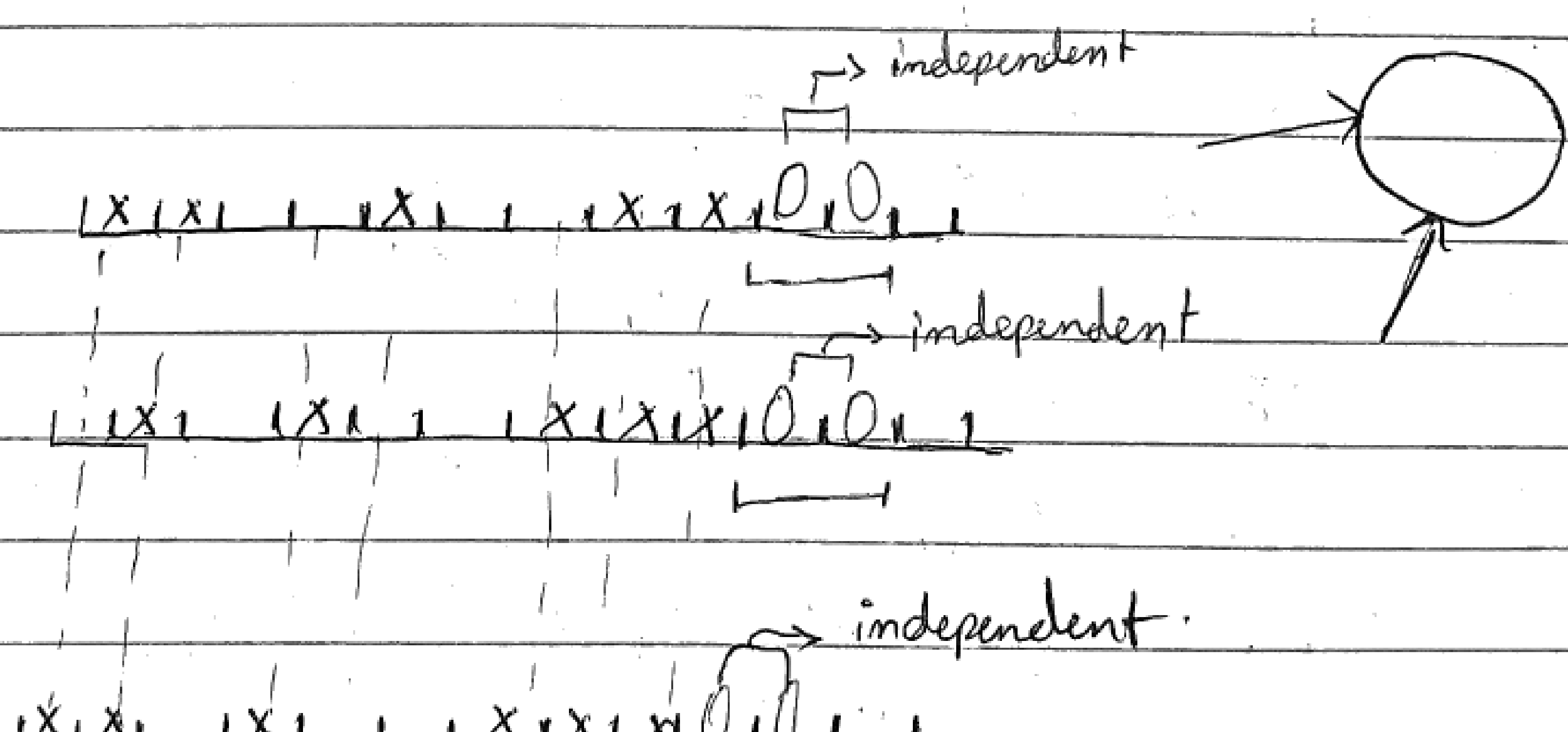
$= P($ k-1 arrival in t-1$)$

$= \binom{t-1}{k-1} p^{k-1} (1-p)^{t-k} \cdot p$

$t = k, k+1, \dots$



- Merging of independent Bernoulli processes:



X_t Bernoulli (p)

Y_t Bernoulli (q)

Z_t (merged process)

collision are counted as one arrival; Bernoulli $(p+q-pq)$

1	$(1-p)q$	pq
0	$(1-p)(1-q)$	$p(1-q)$
Y/X	0	1

$Z_t : g(X_t, Y_t)$

$Z_{t+1} : g(X_{t+1}, Y_{t+1})$

$\bullet \sum_{k=0}^{\infty} P(k, \tau) = 1$

Poisson process

↳ no. of arrivals in disjoint time intervals are independent

$P(k, \tau)$ = probability of k arrivals in interval of duration τ.

↳ small interval probabilities:

for very small δ:

$$P(k, \delta) \approx \begin{cases} 1 - \lambda\delta & ; k = 0 \\ \lambda\delta & ; k = 1 \\ 0 & ; k > 0 \end{cases}$$

λ: arrival rate

Poisson Pmf for the no. of arrivals

k arrival

$P(k, \delta)$ in interval of δ ← small no.

0 … τ time

divide in slots.

N_τ: arrivals in $[0, \tau]$ $\quad P(k, \tau) = P(N_\tau = k)$

$n = \tau/\delta$ intervals (slots of length δ)

↑ small

Bernoulli

no. of successes

$P_S(k) = \frac{n!}{(n-k)!\,k!} p^k q^{n-k}$

$q = 1 - p$

$k = 0, 1, \ldots, n$

P(some slots contain two or more arrival)

$\leq \left(\sum_i P(\text{slot } i \text{ has } \geq 2 \text{ arrivals})\right) = \underbrace{\frac{\tau}{\delta} O(\delta^2) \xrightarrow[\delta \to 0]{} 0}_{\text{negligible...}}$

$\bullet$ P (k arrivals in poisson) $\approx P$(k slots have arrival)

$N_\tau \approx$ binomial

$$p = \lambda\delta + O(\delta^2)$$

$$np = \lambda\tau + O(\delta) \approx \lambda\tau$$

so,

$$P(k,\tau) = \frac{(\lambda\tau)^k e^{-\lambda\tau}}{k!} \qquad k = 0, 1, 2 \ldots$$

Probability of k success in τ length of time interval.

$N_\tau \approx$ Binomial (n, p)

$n = \tau/\delta$, $p = \lambda\delta + O(\delta^2)$

$E(N_\tau) \approx \lambda\tau$; $Var(N_\tau) = \lambda\tau$

$\lambda = \frac{E(N_\tau)}{\tau}$ ← λ: Expected no. of arrival per unit time...

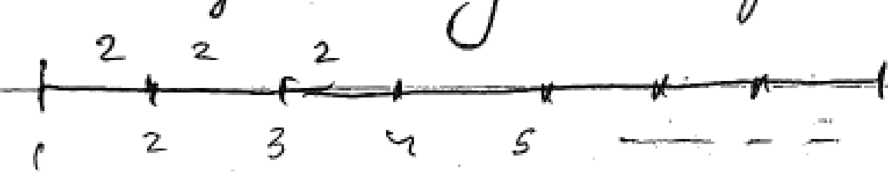
Example: you get email according to a poisson process, at a rate of $\lambda = 5$; message per hr..

$\mu = \lambda\tau = 5 \times 24 = \sigma$

• P(one message in next hr) $= P(1,1) = 5e^{-5}$

• P(exactly two message during each of the next 3 hr)

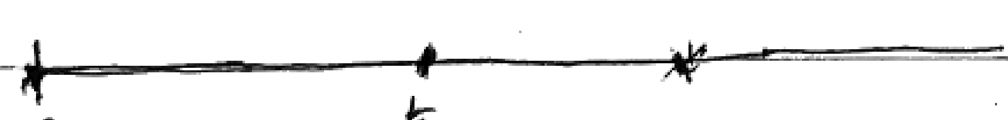
=



$$= (P(2,1))^3 = \left(\frac{5^2 \cdot e^{-5}}{2!}\right)^3$$

• The time T_1 until the first arrival

↑ R.V → continuous



CDF: $P(T_1 \le t) = 1 - P(T_1 > t)$

$= 1 - P(0,t)$

$= 1 - e^{-\lambda t}$

pdf: $f_{T_1}(t) = \lambda e^{-\lambda t}$ for $t \ge 0$

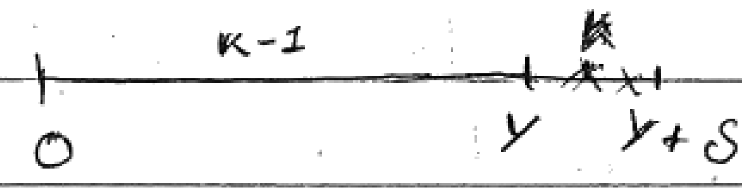
Exponential (λ)

memorylessness: conditⁿ on $T_1 > t$;
the PDF of $T_1 - t$ is again exponential.

• The time Y_K of the K^{th} arrival:-

$$P(Y_K \le y) = \sum_{n=K}^{\infty} P(n, y)$$

more intuitive argument:

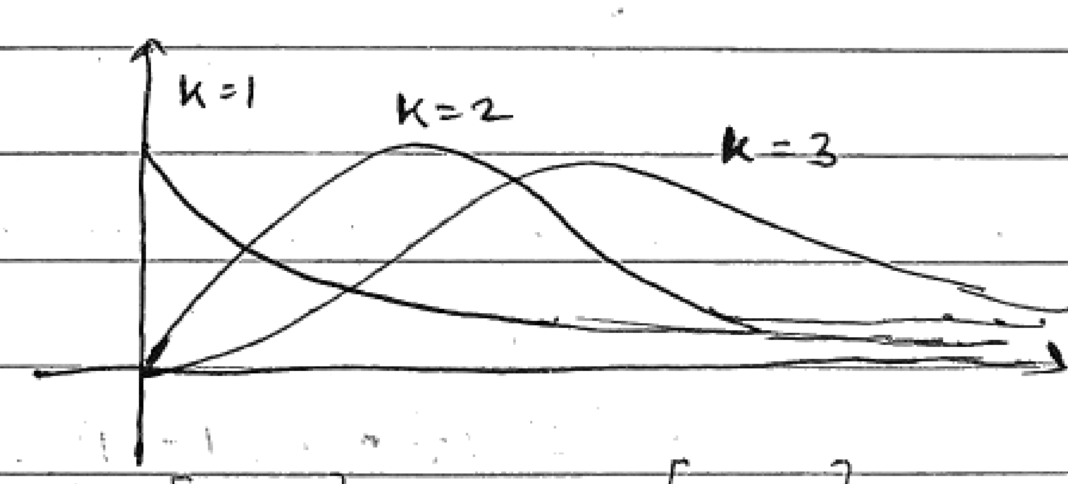


$$\underbrace{f_{Y_K}(y)\,\delta}_{\text{PDF}\times\delta \to \text{Probability}} \approx P(y \le Y_K \le y+\delta)$$

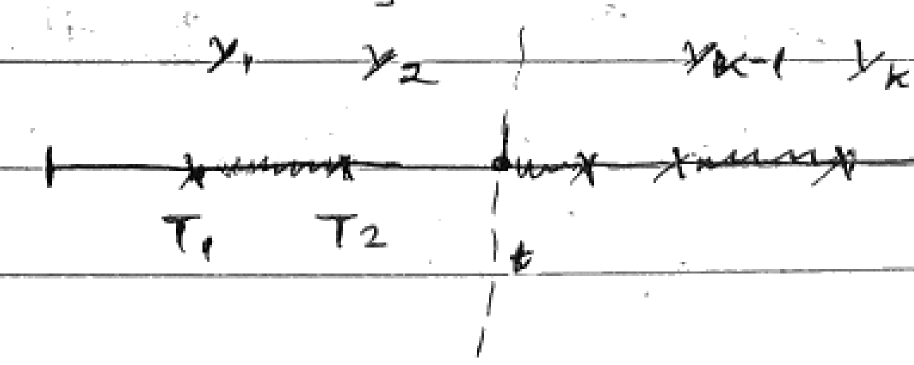
$$= P(K-1, y)\,\lambda\delta + \underbrace{P(K-2, y)\cdot O(\delta^2) + \cdots}_{\text{negligible}}$$

$$\circ\ f_{Y_K}(y)\,\cancel{\delta} \approx P(K-1, y)\,\lambda\,\cancel{\delta}$$

Erlang distributⁿ $\approx \dfrac{\lambda(\lambda y)^{K-1} e^{-\lambda y}}{(K-1)!}, \quad y \ge 0$



• memoryless & the fresh-start property:



If we start watch at time t, we see poisson process, independent of the history until time t.
↳ time until next interval: $Exp(\lambda)$, idependent of past

if we start watching at time T_1, we see poisson process, independent of the history until time T_1.

$T_K = Y_K - Y_{K-1}$; $K \ge 2$ • $Exp(\lambda)$
independent of T_{K-1}

$Y_K = T_1 + T_2 + \dots + T_K$ is sum of i.i.d exponentials.

↳ An equivalent defination

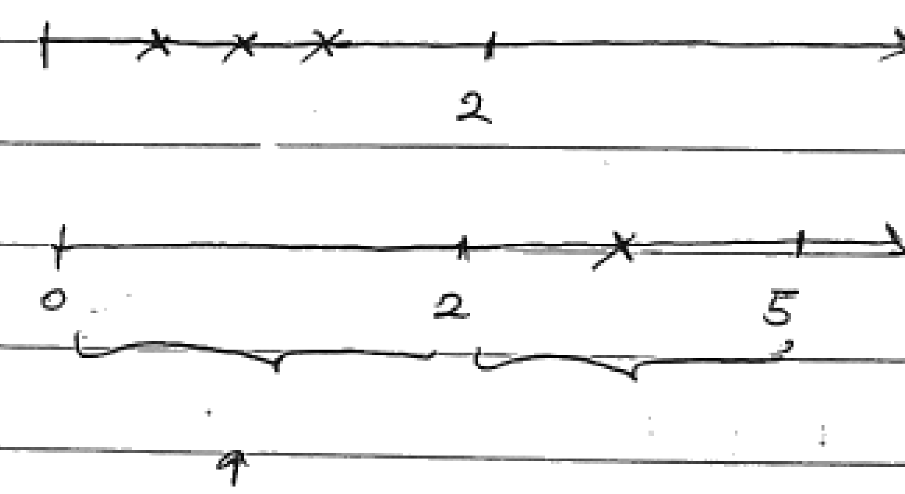
- $E[Y_K] = K/\lambda$ & $Var(Y_K) = K/\lambda^2$

	Poisson	Bernoulli
Time of arrival	continuous	discrete
Arrival rate	λ/unit time	P/per unit
PMF of # of arrivals	Poisson	Binomial
interarrival time	Expo	Geometric
Time to k-th arrival	Erlang	Pascal

Example: fish are caught as a poisson process, $\lambda = 0.6/hr$.
↳ fish for two hr..
↳ if you caught at least one fish, stop.
else continue until first fish is caught...

P(fish for more than 2 hr)

$= P(0,2) = \dfrac{(0.6\times2)^0 e^{-0.6\times2}}{0!}$



$=$

$P(T_1 > 2) = \int_2^\infty f_{T_1}(t)\,dt$

independent for disjoint time interval

P(fish more than 2 & less than 5 hr)

$P(0,2) \cdot (1 - P(0,3)) = \dfrac{(0.6\times2)^0 \cdot e^{-0.6\times2}}{0!} \times \left(1 - \dfrac{(0.6\times3)^0 \times e^{-0.6\times3}}{0!}\right)$

↑ atleast one success

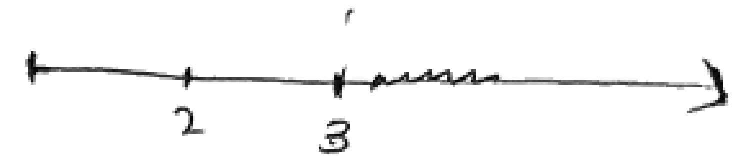
0, success in time interval of length 2...

$= e^{-1.2} - e^{-3.0}$

- P (catch at least 2 fish) $= \sum_{K=2}^{\infty} P(K,2) = 1 - P(0,2) - P(1,2)$

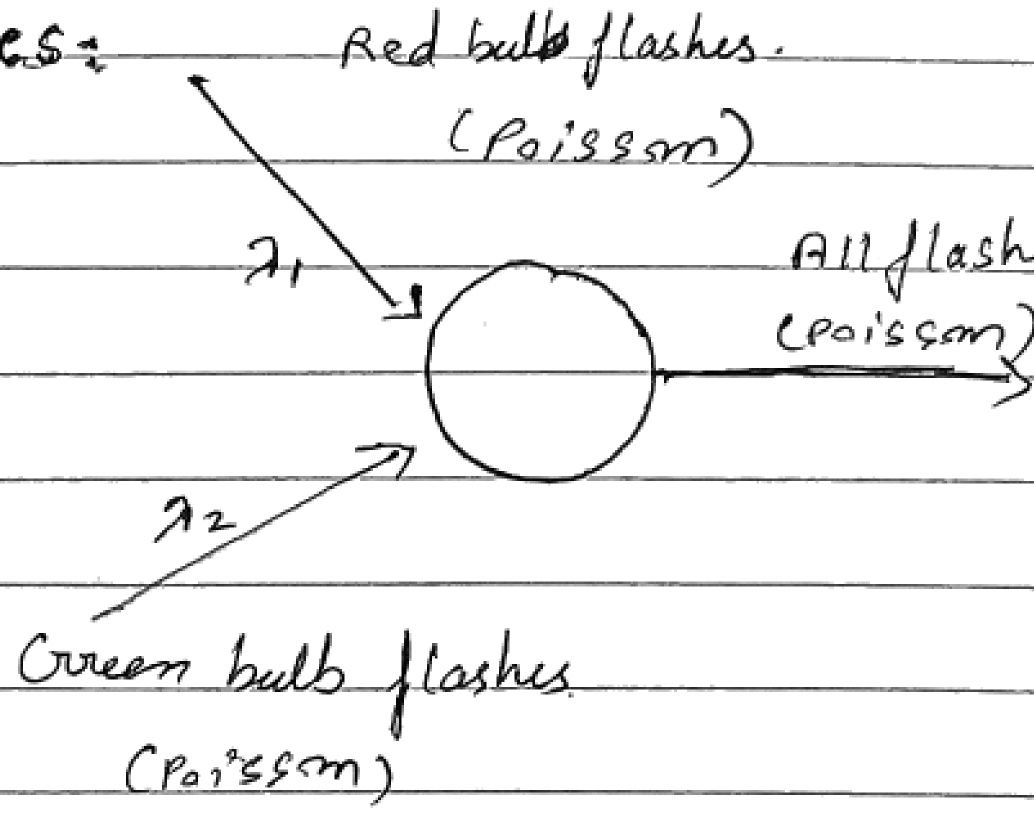
or $P(Y_2 \le 2) = \int_0^2 f_{Y_2}(y)\,dy = \dfrac{\lambda^K y^{K-1} e^{-\lambda y}}{(K-1)!}$

$K=2,\ y=2..$



$E(\text{Future Fishing} \mid \text{already fished for 3hr}) = \frac{1}{\lambda}$

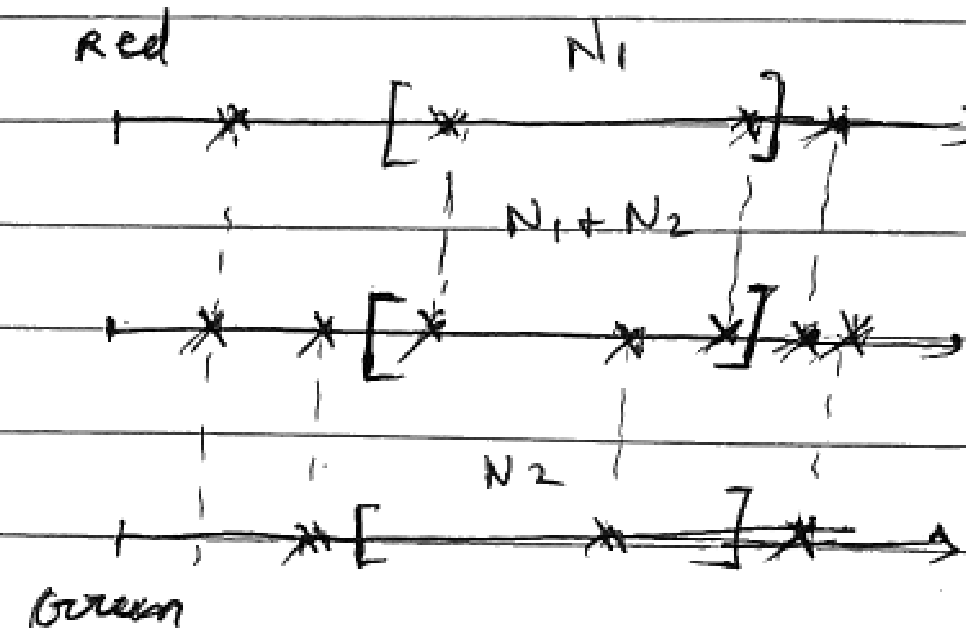
- Merging of independent poisson processes:



		$1-\lambda_1\delta$	$\lambda_1\delta$	$O(\delta^2)$
		0	1	≥ 2
$1-\lambda_2\delta$	0	$(1-\lambda_2\delta) \times (1-\lambda_1\delta)$	$(1-\lambda_2\delta) \times \lambda_1\delta$	·
$\lambda_2\delta$	1	$\lambda_2\delta(1-\lambda_1\delta)$	$\lambda_2\lambda_1\delta^2$	·
$O(\delta^2)$	≥ 2	·	·	·

$\geq 2 : O(\delta^2)$
ignoring δ^2 & higher term

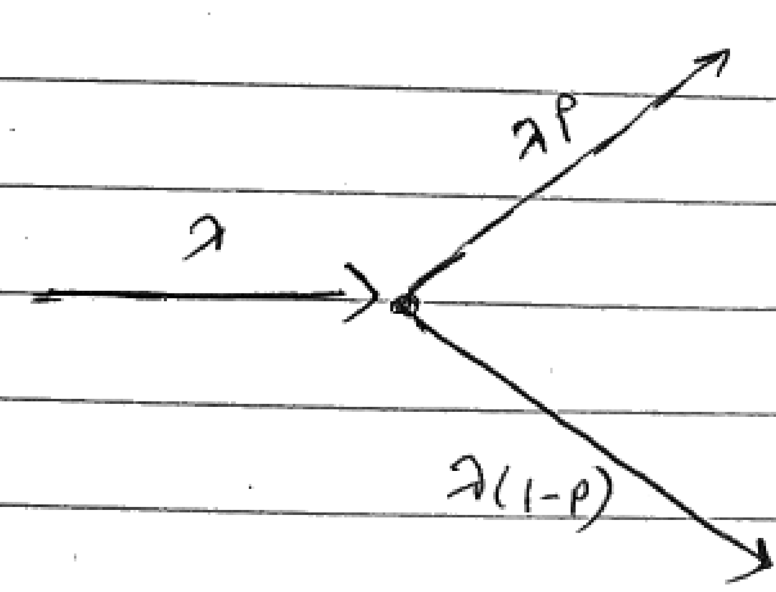
$0 : 1-(\lambda_1+\lambda_2)\delta$
$1 : (\lambda_1+\lambda_2)\delta$

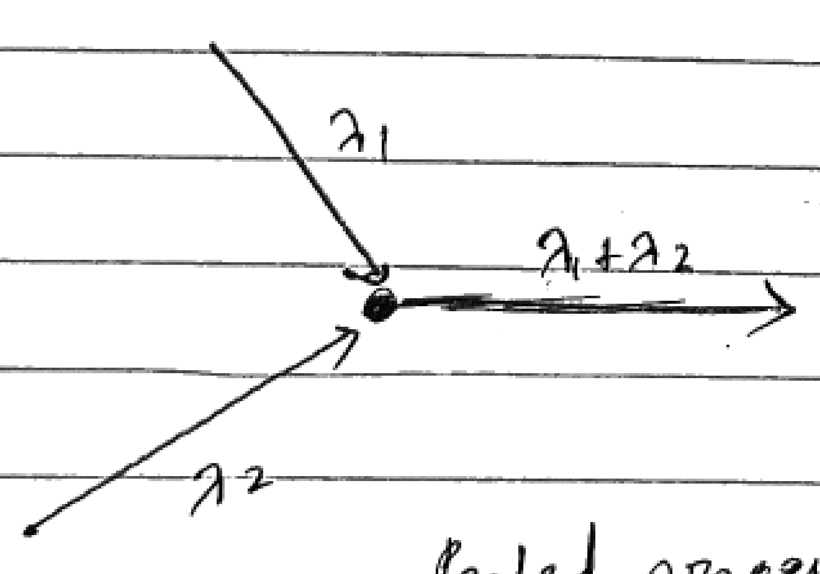


↳ idependent in disjoint interval.

$P(\text{4 out of first 10 arrival are Red})$

$$= \binom{10}{4}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^4\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^6$$





Random splitting

Pooled process

Stochastic

Normal distribution:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)^2}, \quad x \in \mathbb{R}$$

$$F(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy, \quad x \in \mathbb{R}$$

- Standard normal distribution,
  $\mu = 0;\ \sigma^2 = 1$ $\quad N(0,1)$

$$\varphi \overset{def}{=} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$\Theta \overset{def}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy$$

- $Z \sim N(0,1)$
  $\rightarrow X = \sigma Z + \mu$
  has $N(\mu, \sigma^2)$ distrib^{n}

CLT (central limit theorem).

If $\{X_i : i \geq 1\}$ are iid with finite mean $E(X) = \mu$ & finite non-zero variance $\sigma^2 = Var(X)$ then

$$Z_n \overset{def}{=} \frac{1}{\sigma\sqrt{n}} \left(\sum_{i=1}^{n} X_i - n\mu \right) \Rightarrow N(0,1), \quad n \to \infty, \text{ in distribut}^n$$

$$\lim_{n\to\infty} P(Z_n \leq x) = \Theta(x), \quad x \in \mathbb{R}$$

for any constant $c \neq 0$; since $cN(0,1) \sim N(0, c^2)$
$\mu = 0,\ \sigma^2 = 1$, $c \neq 0$, then the CLT becomes for any constant

$$\frac{c}{\sqrt{n}} \sum_{i=1}^{n} X_i \Rightarrow N(0, c^2)$$

Random walk

initial position of a particle..

$$X_n = X_0 + Z_1 + Z_2 + Z_3 + \cdots + Z_n$$

where $\{Z_i\}$ is a seq of iid R.V.

↳ a particle undergoes a general 1-D Random walk..

$$X_n = X_{n-1} + Z_n \quad (n = 1, 2, \ldots).$$

when the steps Z_i can only take value $-1, 0, 1$ with the distribution

$$prob(Z_i = 1) = p \; ; \; prob(Z_i = 0) = 1 - p - q$$

$$prob(Z_i = -1) = q$$

we shall call the process a simple Random walk...
(actual definatⁿ of simple Random walk can take -1 or 1 with $p+q=1$)

ex; Insuarance risk, ↳ start at period 0 with a fixed capital X_0, receive sum Y_i & pay out W_i

$$X_n = X_0 + (Y_1 - W_1) + (Y_2 - W_2) + \cdots + (Y_n - W_n)$$

and if any time n, $X_n < 0$, then the company is ruined & cannot continue in operatⁿ.. if we assume $\{Y_i\}$, $\{W_i\}$ are two seq of mutually iid R.V. then the capital X_n behaves like a Random walk start at X_0 with jump $Z_n = (Y_n - W_n)$

absorbing barrier at the origin..

$$X_n = \begin{cases} X_{n-1} + Z_n \; ; & (X_{n-1} > 0 \; ; \; X_{n-1} + Z_n > 0) \\ 0 & \text{otherwise..} \end{cases}$$

The content of dam, Gambler's ruin, The escape of comets from the solar system...

Unrestricted Random walk (no barrier):

Suppose, Random walk starts at origin & the particle is free to move indefinitely in either direction.

$$X_n = \sum_{r=1}^{n} Z_r$$

Possible position of the particle at time n

$$k = 0, \pm 1, \dots, \pm n.$$

to reach k at time n the particle has to make

$r_1 \rightarrow$ +ve jump ; $r_2 \rightarrow$ -ve jump ; $r_3 \rightarrow 0$ jump.

$$r_1 - r_2 = k \quad \& \quad r_3 = n - r_1 - r_2.$$

$$\text{Prob}(X_n = k) = \sum \frac{n!}{r_1!\, r_2!\, r_3!} \times p^{r_1} (1-p-q)^{r_3} q^{r_2}$$

$p+q=1$; Prob $(X_n = k)$ vanish for odd k when n is even, & for even k when n is odd.

$k = 2r_1 - n$

$$\Rightarrow \text{Prob}(X_n = k) = \sum \frac{n!}{r_1!\, r_2!} p^{r_1} q^{r_2}$$

Probability generating function:

$$G_X(z) = E(z^X) = \sum P(X=i)\, z^i$$
↑ dummy variable

{ only for discrete R.V, taking values in non-ve integer.

$M_X(t) = E(e^{tX})$ by taking $s = e^t$

$G_X(s) = E(e^{tX}) = M_X(t)$

Properties:

- $G_X(0) = P(X=0)$!
- $G_X(1) = 1$

$$G_X(0) = \sum_{x=0}^{\infty} 0^x P(X=x)$$

$$= 0^0 P(X=0) + 0^1 P(X=1) \ldots$$

$$= P(X=0)$$

$$G_X(1) = \sum_{x=0}^{\infty} 1^x P(X=x)$$

$$= \underbrace{\sum_{x=0}^{\infty} P(X=x)}_{pmf} \Rightarrow 1.$$

- $P(X=n) = \frac{1}{n!} G_X^{(n)}(0)$

$$= \frac{1}{n!} \frac{d^n}{ds^n} G_X(s) \Big|_{s=0}$$

↳ $P(X=0) = G_X(0)$

↳ $P(X=1) = G'_X(0)$

↳ $P(X=6) = \frac{1}{6!} G_X^{''''''}(0)$

- let $Y = X_1 + X_2 + \ldots + X_n$

$$G_Y(z) = E(z^Y) = E(z^{(X_1 + X_2 + \ldots + X_n)})$$

$$= E(z^{X_1}) \cdot E(z^{X_2}) \cdot \ldots \cdot E(z^{X_n})$$

as X_i's are iid $\quad = \{E(z^X)\}^n.$

now, back to our unrestricted Random walk...

the pgf of the jump Z_r is

$$G(z) = E(z^{Z_r}) = pz + (1-p-q) + \frac{q}{z}$$

hence for X_n

$$E(z^{X_n}) = \{G(z)\}^n.$$

since $X_0 = 0$ we define $G_0(z) = 1$

$$G(z,s) = \sum_{n=0}^{\infty} s^n \{G(z)\}^n = \frac{1}{1-sG(z)}$$

$$\{|sG(z)|<1\}$$

$$= \frac{z}{z - s(pz^2 + (1-q-p)z + q)}.$$

so, $G(z,s)$ contains all the informat^n about the process in the sense of prob $(X_n = k)$ is the coeff of $z^k s^n$ in $G(z,s)$...

$$E(X_n) = n\mu \quad \& \quad V(X_n) = n\sigma^2$$

where $\mu = p - q$ & $\sigma^2 = p+q - (p-q)^2$

probability that at time n, the particle is found in one of the state $j, j+1, ----, k$... , where j & k are possible value of X_n ($j<k$)...

by approximation provided by central limit theorem,

X_n will be approximate normally distributed with mean $n\mu$ & variance $n\sigma^2$ for large n...

$$\text{Prob}\,(j \le X_n \le k) \approx 2(\pi\sigma^2 n)^{-1/2} \int_j^k e^{-\left(\frac{(x-n\mu)^2}{2n\sigma^2}\right)} dx.$$

Better approximat^n could be get by employing continuety correction. i.e by using $j-c$ & $k+c$ as the limit..

$$\cancel{\text{prob}((j \le X_n \le k)) \simeq \phi\left(\frac{k+n\mu}{\sigma\sqrt{n}}\right)}$$

$$\text{Prob}\,(j \le X_n \le k) \simeq \phi\left(\frac{k+c-n\mu}{\sigma\sqrt{n}}\right) - \phi\left(\frac{j-c-n\mu}{\sigma\sqrt{n}}\right)$$

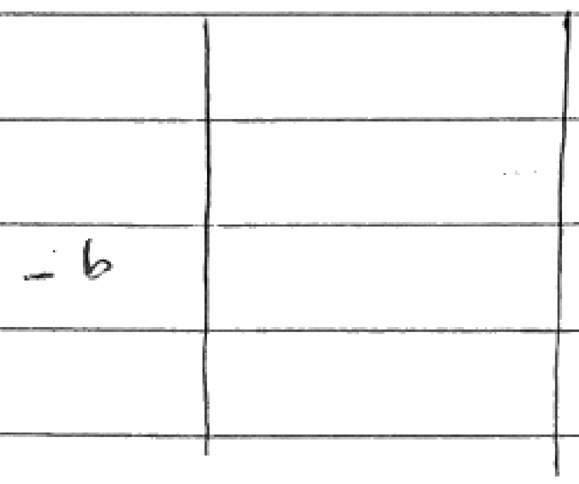
where

$$\phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{u^2}{2}} du$$

where $c = \frac{1}{2} \rightarrow p+q < 1$

$c = 1 \rightarrow p+q = 1$

Stochastic 8 september



↳

↳ probability of recurrance time

↳

$$Z = \begin{cases} +1 & p \\ 0 & 1-p-q \\ -1 & q \end{cases}$$

$P_{JK}^{(n)}$ be the probability proba of occupying the state k at time n standard form position at j

$$X_{n+1} = X_n + Z_{n+1} \begin{cases} 1 \\ 0 \\ -1 \end{cases}$$

$$P_{JK}^{(n)} = p \times P_{j,k-1}^{(n-1)} + (1-p-q) P_{jK}^{(n-1)} + q P_{j,k+1}^{(n-1)}$$

long term behaviour stabalised

↳ at time inf

$$P_{jK}^{(n)} = p \cdot P_{j,K-1}^{(n-1)} + (1-p-q) P_{j,K}^{(n-1)} + q \cdot P_{j,K+1}^{(n-1)} \quad ; 0 < K < a$$

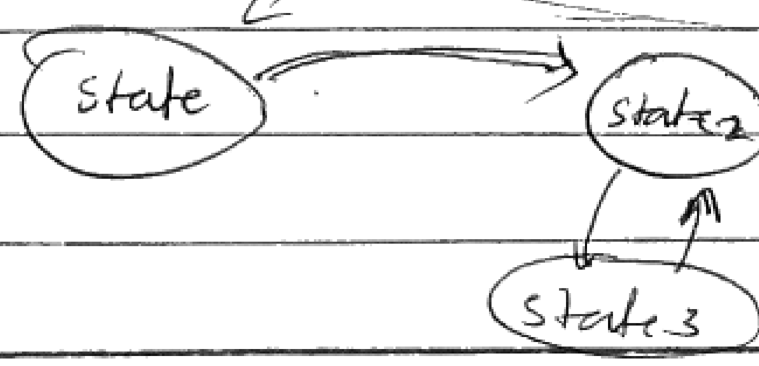
$$P_{Ja}^{(n)} = (1-q) \cdot P_{Ja}^{(n-1)} + p \cdot P_{ja-1}^{(n-1)} \quad + \quad P_{J0}^{(n-1)} = (1-p) P_{J0}^{(n-1)} + q P_{J1}^{(n-1)}$$

independent if n as $n \to \infty$

$P_{jK}^{(n)} \longrightarrow \Pi_K$ (steady state probability)

as $n \to \infty$

transient state



$$\pi_k = p\,\pi_{k-1} + (1-p-q)\,\pi_k + q\,\pi$$

$$\pi_a = (1-q)\,\pi_a + p\,\pi_{a-1}$$

$$\pi_k = \left(\frac{p}{q}\right)^k \pi_0$$

$$\sum_{k=0}^{a} \pi_k = 1 \longrightarrow \pi_0 \sum \left(\frac{p}{q}\right)^k = 1 \qquad \pi_0 = \frac{\frac{q-p}{q}}{\left(1-\left(\frac{p}{q}\right)^{k+1}\right)}$$

$$\hookrightarrow \quad \pi_k = \left[\frac{1-p/q}{1-(p/q)^{a+1}}\right]\left(\frac{p}{q}\right)^k$$

it's a process that develops probabilistically, meaning its future state is not fixed but is instead determined by a Random mechanism

$$\cancel{X_n = X_0 + Z_1} \qquad X_n = X_0 + \sum_{i=1}^{n} Z_i$$

$$X_n = X_{n-1} + Z_n$$

determined by the presence of barriers:

unrestricted

Absorbing Barriers

Reflecting Barriers

unrestricted Random walk

→ Central limit Theorem (CLT)

→ for a large no. of steps n, the position X_n is approx normally distributed with mean $n\mu$ & variance $n\sigma^2$

law of large numbers:

→ the strong law of large numbers state that if the mean jump μ is +ve, → the particle will drift off to $+\infty$ with high probability as $n \uparrow$...

$$E\left[e^{-\theta X_N}\{f^*(\theta)\}^{-N}\right] = 1$$

→ unrestricted random walk:

Absorbing barrier: