Total No. of Pages: 02 *Roll No.:……………*

Answer Key

V SEMESTER

B.Tech.

MID TERM EXAMINATION **September-2023**

MC 303 Stochastic Process

Time: 1 Hour 30 Minutes **Max. Marks: 20**

Note: Answer all questions. (Assume suitable missing data, if any.)

1) How can you determine that a process is a stochastic process? Stochastic processes can be categorized based on nature of index parameter (Discrete/Continuous) and state space (Discrete/Continuous). Categorize Bernoulli process, Poisson process, Renewal process, counting process, Gaussian Process, Brownian Motion based on state space and index set with proper justification.

Answer:

A stochastic process, also known as a random process, is a mathematical model used to describe a system that evolves over time where the outcome at any given moment is not completely determined by the previous outcomes. Instead, it incorporates an element of randomness or uncertainty.

There are several ways to determine whether a process is a stochastic process:

Randomness or Uncertainty: Stochastic processes inherently involve randomness or uncertainty. If the outcome of a process at any given time is not entirely predictable based on the information available up to that point, it suggests the process is stochastic.

Probabilistic Description: Stochastic processes are typically described using probability distributions. If you can describe the future behavior of the process in terms of probabilities and likelihoods, it is likely a stochastic process.

Lack of Deterministic Rules: In contrast to deterministic processes, which follow fixed rules or equations, stochastic processes lack such strict determinism. Instead, they often involve probabilistic rules or equations that govern the system's evolution.

Historical Data Analysis: Examining historical data or observations of a process can provide insights into whether it exhibits random behavior. If the data show variability that cannot be attributed solely to deterministic factors, stochasticity may be involved. (2 Marks)

Process	Category
Bernoulli process	Discrete Index and Discrete State
Poisson process	Continuous Index and Discrete State
Renewal process	Continuous Index and Discrete State
counting process	Continuous Index and Discrete State
Gaussian Process	Continuous Index and Continuous State
Brownian Motion	Continuous Index and Continuous State

(3 Marks)

2) Explain the counting process and give definition of Poisson process accordingly. A random variable X has a Poisson distribution with parameter $\lambda > 0$ and for positive integer $k \geq 0, P[X = k] = \frac{e^{-\lambda}\lambda^k}{k!}$. Show that the moment generating function $M(t) = e^{\lambda(e^t - 1)}$ for Poisson(λ). Let two independent random variable X and Y follows Poisson(λ_1) and Poisson(λ_2), respectively where $\lambda_1$and λ_2 are parameter representing the arrival rates. Then derive the distribution of the random variable X+Y.

Answer:

A stochastic process $(N(t))_{t\geq 0}$ is said to be a *counting process* if $N(t)$ counts the total number of 'events' that have occurred up to time t. Hence, it must satisfy:

(i) $N(t) \geq 0$ for all $t \geq 0$.

(ii) $N(t)$ is integer-valued.

(iii) If $s < t$, then $N(s) \leq N(t)$.

(iv) For $s < t$, the increment $N((s,t]) \stackrel{\text{def}}{=} N(t) - N(s)$ equals the number of events that have occurred in the interval $(s,t]$.

(1 Mark)

Definition 1.1. [The Axiomatic Way]. A counting process $(N(t))_{t\geq 0}$ is said to be a *Poisson process with rate (or intensity)* λ, $\lambda > 0$, if:

(PP1) $N(0) = 0$.

(PP2) The process has independent increments.

(PP3) The number of events in any time interval of length t is Poisson distributed with mean λt. That is, $N((s,t]) \stackrel{d}{=} Poi(\lambda t)$ for all $s, t \geq 0$:

$$\mathbb{P}(N((s,t]) = n) \;=\; e^{-\lambda t}\frac{(\lambda t)^n}{n!}, \quad n \in \mathbb{N}_0.$$

If $\lambda = 1$, then $(N(t))_{t\geq 0}$ is also called *standard Poisson process*.

(1 Mark)

Theorem 5.7. *Let X be a r.v. with a Poisson distribution with parameter $\lambda > 0$, then $E[X] = \lambda$, $\mathrm{Var}(X) = \lambda$ and $M(t) = e^{\lambda(e^t-1)}$.*

Proof. Using that $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$,

$$M(t) = E[e^{tX}] = \sum_{k=0}^{\infty} e^{tk}\frac{e^{-\lambda}\lambda^k}{k!} = e^{-\lambda}\sum_{k=0}^{\infty}\frac{(e^t\lambda)^k}{k!} = e^{-\lambda} + e^{e^t\lambda} = e^{\lambda(e^t-1)}.$$

Hence,

$$M(t) = e^{-\lambda}e^{\lambda e^t},$$
$$M'(t) = e^{-\lambda}e^{\lambda e^t}\lambda e^t, \;\; E[X] = M'(0) = \lambda$$
$$M''(t) = e^{-\lambda}e^{\lambda e^t}(\lambda e^t)^2 + e^{-\lambda}e^{\lambda e^t}\lambda e^t, \;\; E[X^2] = M''(0) = \lambda^2 + \lambda$$
$$\mathrm{Var}(X) = \lambda$$

Q.E.D. (1 Mark)

Theorem 5.8. *Let $X \sim$ Poisson(λ_1) and let $Y \sim$ Poisson(λ_2). Suppose that X and Y are independent. Then, $X + Y \sim$ Poisson$(\lambda_1 + \lambda_2)$.*

Proof. The moment generating function of X is $M_X(t) = e^{\lambda_1(e^t-1)}$. The moment generating function of X is $M_X(t) = e^{\lambda_2(e^t-1)}$. Since X and Y are independent r.v.'s, the moment generating function of $X + Y$ is

$$M_{X+Y}(t) = M_X(t)M_Y(t) = e^{\lambda_1(e^t-1)}e^{\lambda_2(e^t-1)} = e^{(\lambda_1+\lambda_2)(e^t-1)},$$

which is the moment generating function of a Poisson$(\lambda_1 + \lambda_2)$. So, $X + Y$ has a Poisson$(\lambda_1 + \lambda_2)$ distribution. Q.E.D. (2 Marks)

3) Discuss random walk problem using any real world scenario of your choice and explain the advantages of such random walk simulations on those real world scenario. Describe unrestricted random walk along with deriving the probability of being at state k after n steps in terms of probability generating function.

Answer: Any Suitable Example of Students choice (2 Marks)

(i) UNRESTRICTED

Suppose that the random walk starts at the origin and that the particle is free to move indefinitely in either direction. Then we have

$$X_n = \sum_{r=1}^{n} Z_r.$$

The possible positions of the particle at time n are $k = 0, \pm 1, \ldots, \pm n$. In order to reach the point k at time n the particle has to make r_1 positive jumps, r_2 negative jumps and r_3 zero jumps where r_1, r_2, r_3 may be any non-negative integers satisfying the simultaneous equalities

$$r_1 - r_2 = k, \qquad r_3 = n - r_1 - r_2. \tag{8}$$

Hence the probability that $X_r = k$ is given by the summation of multinomial probabilities,

$$\text{prob}(X_n = k) = \sum \frac{n!}{r_1!\,r_2!\,r_3!} p^{r_1}(1-p-q)^{r_3} q^{r_2}$$

over values of r_1, r_2 and r_3 satisfying (8). It should be noted that if $p + q = 1$ then prob$(X_n = k)$ vanishes for odd k when n is even and for even k when n is odd. The probability generating function (p.g.f.) of the jump Z_r is

$$G(z) = E(z^{Z_r}) = pz + (1-p-q) + qz^{-1}$$

and hence that of X_n is

$$E(z^{X_n}) = \{G(z)\}^n.$$

Since $X_0 = 0$ we define $G_0(z) = 1$ and introduce a generating function

$$G(z,s) = \sum_{n=0}^{\infty} s^n \{G(z)\}^n = \frac{1}{1-sG(z)} \quad (|sG(z)| < 1)$$

$$= \frac{z}{-spz^2 + z\{1-s(1-p-q)\} - sq}.$$

Then $G(z,s)$ contains all the information about the process in the sense that prob$(X_n = k)$ is the coefficient of $z^k s^n$ in $G(z,s)$.

(3 Marks)

4) The limiting equilibrium distribution of the state occupation probabilities $p_{jk}^n \to \pi_k \forall\, k \in \{0,1,2,3,\dots,a\} as\, n \to \infty$ for the random walk with two reflecting barrier (0 and a) satisfy following conditions

$$\pi_k = p\pi_{k-1} + (1-p-q)\pi_k + q\pi_{k+1}\; k \in \{1,2,3,\dots,a-1\}$$

$$\pi_0 = (1-p)\pi_0 + q\pi_1$$

$$\pi_a = p\pi_{a-1} + (1-q)\pi_a$$

Where p and q are denoting the probability of forward and backward steps in random walk. Solve the above for π_k. Discuss scenario of the steady state probability for (i) $p > q$ (ii) $p < q$ and (iii) $p = q$

Answer:

We may solve for π_1 in terms of π_0, obtaining

$$\pi_1 = \left(\frac{p}{q}\right)\pi_0.$$

Solving (45) recursively, we have that

$$\pi_2 = \{(p+q)\pi_1 - p\pi_0\}/q$$

$$= \left(\frac{p}{q}\right)^2 \pi_0$$

and in general

$$\pi_k = \left(\frac{p}{q}\right)^k \pi_0 \quad (k = 0,\dots,a). \tag{46}$$

We require the solution to be a probability distribution, i.e. $\sum \pi_k = 1$, and this enables us to find π_0. Hence we obtain the truncated geometric distribution

$$\pi_k = \frac{1-\frac{p}{q}}{1-\left(\frac{p}{q}\right)^{a+1}}\left(\frac{p}{q}\right)^k \quad (k = 0,\dots,a) \tag{47}$$

(2 Marks)

as the equilibrium set of state occupation probabilities. If $p > q$ then π_k decreases geometrically away from the upper barrier whereas if $p < q$, π_k decreases geometrically away from the lower barrier. If $p = q$ then from (46) we see that $\pi_k = \pi_0 = 1/(a+1)$ for all k, so that in the equilibrium situation all states are equally likely to be occupied by the particle.

In the following example we describe a model of a finite queue in discrete time and we shall see that it has a representation like a random walk with reflecting barriers. It will serve to illustrate both the method of solving these problems and the distinctive property of independent increments possessed by the random walk. (3 Marks)