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Assignment-1
Real Analysis

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Ans 1. (a) given, $a \in \mathbb{R}$ and $a \cdot a = a$

Case 1) $a = 0$, as $0 \cdot 0 = 0$ hence a can be 0.

Case 2) $a \neq 0$, hence $\exists\, b$ such that $ab = ba = 1$
(Existence of Inverse of multiplication)

$a \cdot a = a$

$(a \cdot a) \cdot b = a \cdot b$ [multiplying both sides by b]

$a \cdot (a \cdot b) = 1$ [associative law]

$a \cdot 1 = a = 1$ [$x \cdot 1 = x$, Identity]

hence we conclude that $a \in \{0, 1\}$.

(b) as $a, b \neq 0$ hence $ab \neq 0 \Rightarrow$ Inverse of ab exists, that is $\exists\, x \in \mathbb{R}$ such that

$(ab)x = x(ab) = 1$ [multiplicative Inverse]

as multiplication over $\mathbb{R}$ is an abelian Group

hence, $(ab)x = x(ab)$ [Commutative law]

$(ab)x = 1$

$\frac{1}{a} \cdot (ab)x = \frac{1}{a} \cdot 1$ [multiplying both sides by multiplicative Inverse of a]

$\left(\frac{1}{a} \cdot a\right) \cdot b \cdot x = \frac{1}{a}$ [associative and Identity law]

$1 \cdot b \cdot x = \frac{1}{a} \Rightarrow b \cdot x = 1/a$

pot $(b \cdot x) \cdot \frac{1}{b} = \frac{1}{a} \cdot \frac{1}{b}$ (post multiply by $\frac{1}{b}$)

$(x \cdot b) \cdot \frac{1}{b} = \frac{1}{a} \cdot \frac{1}{b}$ (Commutative law)

$x \cdot \left(b \cdot \frac{1}{b}\right) = \frac{1}{a} \cdot \frac{1}{b}$ (associative law)

$x \cdot 1 = \frac{1}{a} \cdot \frac{1}{b}$ $\left(x \cdot \frac{1}{x} = 1\right)$

$x = \frac{1}{a} \cdot \frac{1}{b}$

as x is inverse of ab, hence, $\frac{1}{(ab)} = \frac{1}{a} \cdot \frac{1}{b}$

Ans 2. To Prove $a^{m+n} = a^m \cdot a^n$, where $a \in \mathbb{R}$ and $m, n \in \mathbb{N}$

Base Step (for $m=1$); $a^{1+n} = \underbrace{a \cdot \ldots \cdot a}_{n+1 \text{ times}}$

$= a \cdot \underbrace{a \cdot \ldots \cdot a}_{n \text{ times}} = a \cdot a^n$ hence proved.

Induction Step: let $a^{m+n} = a^m \cdot a^n$ $\forall\, m \in \{1, 2, 3, \ldots k\}$.

hence, $a^{k+n} = a^k \cdot a^n$

$a \cdot a^{k+n} = a(a^k \cdot a^n)$ (premultiply by a)

$a^{k+1+n} = (a \cdot a^k) \cdot a^n$ (associative law)

$a^{(k+1)+n} = a^{k+1} \cdot a^n$

hence, we conclude that $T(k) \rightarrow T(k+1)$ $\forall k \in \mathbb{N}$
hence, proved.

$(a^m)^n$ from definition of exponents, can be rewritten as

$$\underbrace{a^m \cdot a^m \cdot a^m \cdots a^m}_{n \text{ times}} \quad \text{— (1)}$$

from previous results $a^m \cdot a^m = a^{m+m}$ $(m=n)$

or, $\underbrace{a^m \cdot a^m}_{2 \text{ times}} = a^{2m}$

Similarly, let $\underbrace{a^m \cdot a^m \cdots a^m}_{k} = a^{km}$ $\forall k \in \{1, \ldots t\}$

Induction step: To check for $k = t+1$,

$$\underbrace{a^m \cdots a^m}_{t} = a^{tm}$$

$$a^m \cdot \underbrace{(a^m \cdots a^m)}_{t \text{ times}} = a^m \cdot a^{tm} \quad \left(\text{pre multiply by } a^m\right)$$

$$\underbrace{a^m \cdot a^m \cdots a^m}_{(t+1) \text{ times}} = a^{m+tm} \quad (\text{from previous result})$$

$$(a^m)^{(t+1)} = a^{m(1+t)} \quad [\text{distributive law}]$$

hence, $T(k) \to T(k+1)$ hence, $(a^m)^n = a^{mn}$ is true for all $m, n \in \mathbb{N}$ and $a \in \mathbb{R}$.

Ans 3. (a) by definition we know that

$$x^2 \leq y^2 \iff |x| \leq |y| \quad — (1)$$

$|a| \geq a$ and $|b| \geq b \quad \forall a, b \in \mathbb{R}$

$|a| \cdot |b| \geq a \cdot b$

$2 |a| \cdot |b| \geq 2 \cdot a \cdot b$ (as $2 > 0$)

$(-1) \cdot 2 |a| \cdot |b| \leq (-1) 2 \cdot a \cdot b$ (as $(-1) < 0$)

$-2 |a| |b| \leq -2 \cdot a \cdot b$

$|a|^2 + |b|^2 - 2 |a| |b| \leq |a|^2 + |b|^2 - 2 \cdot a \cdot b$

(adding $|a|^2 + |b|^2$ in both sides)

$(|a| - |b|)^2 \leq a^2 + b^2 - 2ab$ ($|a|^2 = a^2$)

$(|a| - |b|)^2 \leq (a-b)^2$ [$(x-y)^2 = x^2 + y^2 - 2xy$] — (2)

from (1) and (2)

$||a| - |b|| \leq |a-b|$ hence proved.

(b). for $a, b \in \mathbb{R}$

$a \cdot b \leq |a| |b| \quad — (1)$

$-2 \leq 0 \leq 2$ (as $2 = 1+1 > 0+0 \geq 0$) — (2)

$-2 \cdot (a \cdot b) \leq 2 |a| |b|$ (multiplying both (1) & (2))

$a^2 + b^2 - 2ab \leq a^2 + b^2 + 2 |a| |b|$ (adding $a^2 + b^2$ both sides)

$(a-b)^2 \leq |a|^2 + |b|^2 + 2|a||b|$ $(|x|^2 = x^2)$ (5)

$(a-b)^2 \leq (|a| + |b|)^2$ (Identity $(x+y)^2 = x^2+y^2+2xy$)

$|a-b| \leq ||a| + |b||$

$|a-b| \leq ||a| + |b|| \leq |a| + |b|$ (as $|a| \geq 0$, $|b| \geq 0$)

$|a-b| \leq |a| + |b|$ hence proved.

Ans. let $\varepsilon > 0$, and defined neighborhood of a as $V_\varepsilon(a) = (a-\varepsilon, a+\varepsilon)$.

let $x \in (a-\varepsilon, a+\varepsilon)$, $\forall \varepsilon_i \in \mathbb{R}$ and $\varepsilon > 0$

$\Rightarrow a - \varepsilon < x < a + \varepsilon$, $\forall \varepsilon > 0$

$\Rightarrow -\varepsilon < x - a < \varepsilon$, $\forall \varepsilon > 0$

$\Rightarrow |x-a| < \varepsilon$, $\forall \varepsilon > 0$ — (1)

$\not\Rightarrow$ like let $x \neq a \Rightarrow x = a + t$, for some $t \in \mathbb{R}$, where $t \neq 0$. — (2)

Substitute (2) in (1),

$|x-a| < \varepsilon \Rightarrow |a+t-a| < \varepsilon \Rightarrow |t| < \varepsilon$ $\forall \varepsilon > 0$

let $\varepsilon = |t|/2 \Rightarrow |t| < |t|/2 \Rightarrow |t| < 0$, contradicting the definition of $|t|$ hence,

$t = 0 \Rightarrow$ ~~$x \neq a$~~ $x = a$. hence proved.

Ans/. Completeness property of $\mathbb{R}$: Given a non-empty subset $X \subseteq \mathbb{R}$, such that it contains atleast one $y \in X$, such that $y \geq x, \forall x \in X$ then Supremum (X) exists in $\mathbb{R}$. [Similar definition for inf form: Infimum (X)]

a) $S_1 = \{ 1 - \frac{(-1)^n}{n} \mid n \in \mathbb{N} \}$

as $n > 0 \Rightarrow \frac{1}{n} > 0$ — ①

as $n \geq 1 \Rightarrow \frac{1}{n} \leq 1 \Rightarrow -\frac{1}{n} \geq -1$ — ②

as $\frac{-1}{n} \leq \frac{(-1)^n}{n} \leq \frac{1}{n}$ [as $-1 \leq (-1)^n \leq 1$]

$\Rightarrow \frac{-1}{n} \leq -\frac{(-1)^n}{n} \leq \frac{1}{n}$ — ③

Combing ② and ③

$-1 \leq \frac{-1}{n} \leq -\frac{(-1)^n}{n} \leq \frac{1}{n} \Rightarrow$

$1 - 1 \leq 1 - \frac{(-1)^n}{n} \leq 1 + 1/n$ (adding 1)

$\boxed{0 \leq t_n \leq 1 + 1/n, \ \forall n \in \mathbb{N}}$ — ④

Combing ② and ④

$0 \leq t_n \leq 1 + 1/n \leq 2 \Rightarrow \boxed{0 \leq t_n \leq 2}$

(b) $S_2 = \{\frac{1}{n} - \frac{1}{m} \mid n, m \in \mathbb{N}\}$

as $n \geq 1 \Rightarrow 0 < 1/n \leq 1 \Rightarrow 1/n - 1/m \leq 1 - 1/m$ — (1)

Similarly $0 < \frac{1}{m} \leq 1 \Rightarrow 0 > -\frac{1}{m} \geq -1$ — (2)

Combining (1) and (2)

$$0 - \frac{1}{m} \leq \frac{1}{n} - \frac{1}{m} \leq \frac{1}{n} \leq 1 \Rightarrow -1 \leq -\frac{1}{m} \leq \frac{1}{n} - \frac{1}{m} \leq 1$$

or $\boxed{-1 \leq \frac{1}{n} - \frac{1}{m} \leq 1}$ $\forall m, n \in \mathbb{N}$ — (3)

from (3) we conclude that, $\sup(S_2) \leq 1$ and $\inf(S_1) \geq -1$ — (4)

~~as for~~

let $\sup(S_2) = 1 - \varepsilon$, $\varepsilon > 0 \Rightarrow$

$\frac{1}{n} - \frac{1}{m} \leq 1 - \varepsilon$, $\forall n, m \in \mathbb{N}$

$\Rightarrow 1 - 1/m \leq 1 - \varepsilon$, $\forall m \in \mathbb{N}$

$\Rightarrow \frac{1}{m} \geq \varepsilon \ \forall m \in \mathbb{N}$, contradicting the Archimedean Property hence, $\sup(S_2) \geq 1$ — (5)

Combining (4) and (5) $\boxed{\sup(S_2) = 1}$

let's consider $-S_2 = \{\frac{1}{m} - \frac{1}{n} \mid m, n \in \mathbb{N}\}$

from observation $S_1 = -S_2$ — (6)

hence, $S_1 \subseteq \mathbb{R}$ contains lower bound as 0 and upper bound as 2, hence $\exists\ l$ and u such that $l, u \in \mathbb{R}$ and <u>$l = \inf(S_1)$</u> and <u>$u = \sup(S_1)$</u>

as for $n = 1$, $t_1 = 1 - \frac{(-1)^1}{1} = 2$.

hence, $\sup(S_1) \geq 2$. and 2 is an upper bound hence, 2 is least upper bound i.e.

$\boxed{\sup(S_1) = 2.}$

let's focus on $t_{2n} = 1 - \frac{1}{2n} \leq 1$

as $\frac{1}{n} \leq 1 \Rightarrow \frac{1}{2n} \leq \frac{1}{2} \Rightarrow \frac{-1}{2n} \geq \frac{-1}{2}$

$1 - \frac{1}{2n} \geq \frac{1}{2}$. (hence $\frac{1}{2}$ is a lower bound of sequence t_{2n}) — 5

also, note that $t_{2n-1} = 1 + \frac{1}{2n-1} \geq 1 > \frac{1}{2}$

hence $\frac{1}{2}$ is a lower bound for all $t_i \in S_1$,

$\Rightarrow \inf(S_1) \geq \frac{1}{2}$. — 6

as $t_2 = 1 - \frac{1}{2} = \frac{1}{2} \Rightarrow \inf(S_1) \leq \frac{1}{2}$ — 7

Combining 6 and 7

$\boxed{\inf(S_1) = \frac{1}{2}.}$

in other words, $\sup(S_2) = \sup(-S_2)$ (9)

$\sup(S_2) = -\inf(S_2)$ $\{\sup(X) = \inf(-X)\}$

$\Rightarrow \boxed{\inf(S_2) = -\sup(S_2) = -1}$

hence, proved.

(c) $S_3 = \{x \in \mathbb{R} \mid x > 0\}$.

as, $\forall x \in \mathbb{R}$, $\exists n \in \mathbb{N}$ such that $n > x$,
hence S_3 does not contain an upper bound
i.e, $\sup(S_3)$ does not exist.

as $0 < x$, hence 0 is a lower bound of $S_3 \subseteq \mathbb{R}$
and $1 \in S_3$ hence S_3 is non empty $\Rightarrow$ S_3 contains
a maximum lower bound.

let $\varepsilon > 0$ be the $\inf(S_3) \Rightarrow 0 < \varepsilon \leq x, \forall x > 0$

let $x = 1/n$, when $n \in \mathbb{N}$

$\Rightarrow 0 < \varepsilon \leq 1/n, \forall n \in \mathbb{N}$, which contradicts
with Archimedean Property hence, $\inf(S_3) \leq 0$.

Also, as $0 < x$, $\forall x \in S_3 \Rightarrow \boxed{\inf(S_3) = 0}$

Ans 7. let $X = \{x \in \mathbb{R} \mid x^2 < 2\}$, as $1 \in X$, and
$x < 2, \forall x \in X \Rightarrow X$ contains a Supremum(X)
$\in \mathbb{R}$, by Order Completeness Property.

let $y = \sup(X)$

Case 1) $y^2 < 2$.

consider $y+\varepsilon$, $\varepsilon > 0$, let

$(y+\varepsilon)^2 < 2$ iff

$y^2 + 2y\varepsilon + \varepsilon^2 < 2$ if

$2y\varepsilon + \varepsilon^2 < 2 - y^2$ if

$2y\varepsilon + \varepsilon^2 < 2y\varepsilon + \varepsilon < 2 - y^2$ if

$\varepsilon < \dfrac{2-y^2}{2y+1}$, we conclude that $\exists \varepsilon > 0$

such that $y < y+\varepsilon$ and $(y+\varepsilon)^2 < 2$, which means y^2 ~~isn't an~~ < 2 a so can't be ~~to~~ the supremum of set X.

Case 2). let $y^2 > 2$

Consider $y-\varepsilon$, $\varepsilon > 0$, let

$(y-\varepsilon)^2 > 2$ if

$y^2 - 2 > -\varepsilon^2 + 2y\varepsilon$ if

$y^2 - 2 > 2y\varepsilon > 2y\varepsilon - \varepsilon^2$ if

$\varepsilon < \dfrac{y^2-2}{2y}$, we conclude that $\exists \varepsilon > 0$, such

that $y-\varepsilon < y$ and $(y-\varepsilon)^2 > 2 \Rightarrow y^2 > 2$, satisfying $y^2 > 2$, can't be the Supremum(X), that is $\exists y \in \mathbb{R}$ such that $y^2 = 2$. Hence proved.

b) let $\exists$ a rational number $\frac{p}{q}$, where $p, q \in \mathbb{Z}$ and $\left(\frac{p}{q}\right)^2 = 2$.

let's choose p and q such that hcf $(p, q) = 1$

$\left(\frac{p}{q}\right)^2 = 2 \Rightarrow \frac{p^2}{q^2} = 2 \Rightarrow \frac{p^2}{q^2} \cdot q^2 = 2 \cdot q^2$ (multiplying both sides by q^2)

$p^2 = 2q^2$ ① $\Rightarrow 2 | p^2 \Rightarrow 2 | p \Rightarrow p$ is even

that is $\underline{p = 2x}$, for $\underline{\text{some } x \in \mathbb{Z}}$.

Substitute p in equation ①

$(2x)^2 = 2q^2 \Rightarrow 4x^2 = 2q^2 \Rightarrow \frac{1}{2} \cdot 4x^2 = q^2$

$q^2 = 2x^2 \Rightarrow 2 | q^2 \Rightarrow 2 | q \Rightarrow q$ is divisible by 2 that is $\underline{q = 2m, \text{ for some } m \in \mathbb{Z}}$.

as, $2 | p$ and $2 | q \Rightarrow 2 \mid$ hcf $(p, q) \Rightarrow$

$2 | 1$, contradiction arises because, our assumption that $r^2 = 2$ where r is rational was wrong.

Ans 8. Bernoulli's Inequality: for $x \geq -1$ and $r \geq 1$, $(1+x)^r \geq 1+xr$, where $r \in \mathbb{N}$

for $r=1$. $(1+x)^1 \geq 1+x$

for $r>1$, let $r = 1+\varepsilon$ $(\varepsilon > 0)$

$(1+x)^{1+\varepsilon} \geq 1+x(1+\varepsilon)$ if

$(1+x)(1+x)^{\varepsilon} \geq (1+x)+x\varepsilon$ if

$(1+x)^{\varepsilon} \geq 1+x\varepsilon$ $\{$for $x>-1$ and $\varepsilon>0\}$

let $(1+x)^r \geq 1+xr$, $\forall r \in \{1,2,..k\}$

$(1+x)^k \geq 1+xk$ $[$as, $1+x \geq 0]$

$(1+x)(1+x)^k \geq (1+x)(1+xk)$ [premultiplying by $(1+x)$]

$(1+x)^{1+k} \geq 1+x(1+k)+x^2k \geq 1+x(1+k)$

$(1+x)^{1+k} \geq 1+x(k+1)$

hence, $T(k) \rightarrow T(k+1)$ and $T(1)$ is True,
So by Mathematical Induction, $(1+x)^r \geq 1+xr$
$\forall r \in \mathbb{N}$.

Ans/. Countable Set: A set X is called countable if $\exists$ f, a bijective map from X to $Y \subseteq \mathbb{N}$. (13)

let $\mathbb{R}$ be a countable set, as $\mathbb{N} \subseteq \mathbb{R}$ hence, if $\mathbb{R}$ is countable then $f: \mathbb{R} \to \mathbb{N}$ must be bijective, let f be chosen such that,

$x < y \iff f(x) < f(y) \quad \forall x, y > 0$ and $x, y \in \mathbb{R}$

x_1 0. 45783... $\to$ 1234009... n_1

x_2 10. 983... $\to$ 8376400... n_2

20. 678... $\to$ 983100...

we conclude that given any two x_1, x_2 $\exists\, n_1, n_2 \in \mathbb{N}$.

we conclude that $\exists$ exactly $(n_2 - n_1 - 1)$ terms between x_1 and x_2, let that be

$X = \{q_1, \dots q_{n_2-n_1-1}\}$

~~let define $y = \frac{q + \dots q_{n_2-n_1-1}}{n_2 - n_1 - 1}$~~

we conclude that $f(\sup(X)) = \sup\{n_1+1, \dots n_2-1\}$

Similarly $f[\sup(X - x_2)] = n_2 - 2$

that is $q_{n_2-n_1-2} < q_{n_2-n_1-1}$.

but $\bar{x} = \dfrac{q_{n_2-n_1-2} + q_{n_2-n_1-1}}{2} \in X$, don't have it's image in $\mathbb{N}$, hence such mapping does not exist.

An 10. Between any two real number x and y $\exists$ a rational r, such that $x < r < y$.

Similarly, $\exists\, r'$ between $\sqrt{2}x$ and $\sqrt{2}y$

$$\sqrt{2}x < r' < \sqrt{2}y \Rightarrow x < \frac{r'}{\sqrt{2}} < y$$

now, $\frac{r'}{\sqrt{2}} \notin \mathbb{Q}$ [as let $\frac{r'}{\sqrt{2}} = q \in \mathbb{Q} \Rightarrow \sqrt{2} \in \mathbb{Q}$, which is not the case]

hence, between any 2 ~~ratio~~ reals there exist an irrational number q, $x < q < y$.