SCALING LIMITS

We are interested in

$$X^{(N)} \xrightarrow{d} X \qquad \text{as } N \to \infty$$

where

- $X^{(N)}$ is a CTMC $\{X^{(N)}(t), t \geq 0\}$ indexed by $N \geq 1$

  e.g. N = • pop. size
  • parameter in the transition prob.

- $\{X^{(N)}, N \geq 1\}$ sequence of CTMC's

- X is a limit process

Let X be any stoch. proc. on $S \subseteq \mathbb{R}$ indexed by $T \subseteq [0, \infty)$

Def. We call

- PROJECTIONS of X the vectors $(X(t_1), \dots, X(t_n))$ $\quad t_i \in T, n \geq 1$

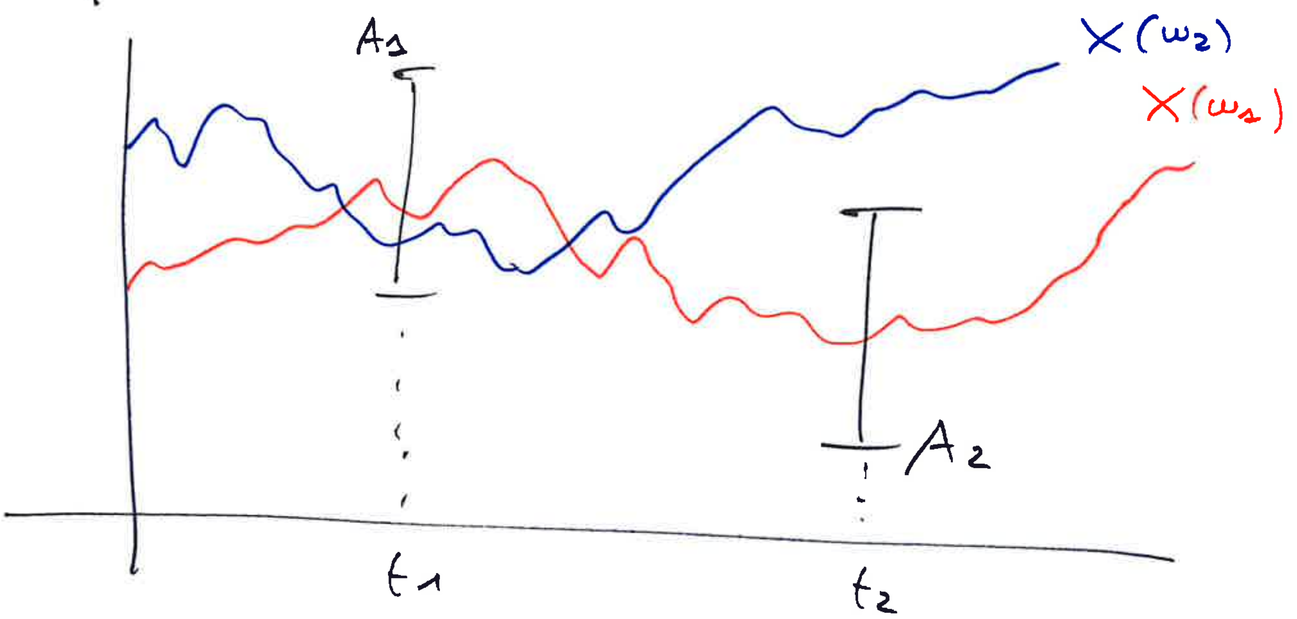
- FINITE-DIMENSIONAL DISTRIBUTIONS of X the laws of its projections, i.e. the family $\mathcal{M} = \{Q_{t_1, \dots, t_n} : t_i \in T, n \geq 1\}$ s.t.

$$Q_{t_1,\dots,t_m}(A_1 \times \dots \times A_m) = P(X(t_1) \in A_1, \dots, X(t_m) \in A_m)$$

for $A_i \in \mathcal{B}(S)$.

The FDDs give the prob. assigned to CYLINDER SETS

$$C = \{ x \in \{ f : [0,\infty) \to \mathbb{R} \} : x(t_1) \in A_1, \dots, x(t_m) \in A_m \}$$



$X(\omega_2) \notin C$ $\qquad$ $X(\omega_1) \in C$

The FDDs of a stoch. process satisfy two properties, called KOLMOGOROV CONSISTENCY CONDITIONS:

C1) $Q_{t_1,\dots,t_{m-1},t_m}(A_1 \times \dots \times A_{m-1} \times \mathbb{R}) =$

$$= Q_{t_1,\dots,t_{m-1}}(A_1 \times \dots \times A_{m-1})$$

marginalization

C2) for all permutations π of $\{1,2,\dots,m\}$
with $\pi(i)$ the new position of i

$$Q_{t_1,\dots,t_m}(A_1 \times \dots \times A_m) =$$

$$= Q_{t_{\pi(1)},\dots,t_{\pi(m)}}(A_{\pi(1)} \times \dots \times A_{\pi(m)})$$

joint permutation of indices and arguments

Converse non trivial:

<u>Thm</u> (Kolmogorov's extension Thm)

Let $\mathcal{M}$ be a family (as above) of probability measures that satisfy C1-C2. Then there exists a prob. space $(\Omega, \mathcal{F}, P)$ and a stoch proc. X on such space s.t. the elements of $\mathcal{M}$ are the FDDs of X.

Remarks

— $(\Omega, \mathcal{F}, P)$ always $\exists$
$\Rightarrow$ irrelevant in its specifics

— To construct a SP, enough to specify the law of the projections

(satisfying C1 - C2) at arbitrary
finitely - many coordinates
(X is ∞-dimensional)

Weak convergence of CADLAG processes

R.v.'s $Z^{(N)} \sim \nu_N$, $Z \sim \nu$

$Z^{(N)} \xrightarrow{d} Z$ as $N \to \infty$ iff

$\nu_N \Rightarrow \nu$ i.e. $\int f \, d\nu_N \to \int f \, d\nu$
$f \in B(S)$

$Z^{(N)}$, Z

- can be defined on different prob. spaces $(\Omega_N, \mathcal{F}_N, P_N)$
- Take value on different spaces S_N
- can have different continuity structure e.g. ν_N discrete $\forall N$
ν continuous

For SPs we also need to take care of the nature of the trajectories.

Z is a CÀDLÀG process if it trajectories are right-continuous with left limits.

E.g. a CTMC.

Denote D_S the space of càdlàg functions from $[0,\infty)$ to S.

Take

- $X, X^{(N)}$ càdlàg processes with values in S, S_N respectively assuming $\lim_N S_N$ dense in S.

so these are r.v.'s taking values in D_S and D_{S_N} respectively.

If $X^{(N)} \xrightarrow{d} X$ then this implies convergence of all FDDs, i.e.

$$\left[X^{(N)}(t_1), \dots, X^{(N)}(t_n)\right] \xrightarrow{d} \left[X(t_1), \dots, X(t_n)\right]$$

The converse is true with additional requirements (tightness).

Denote by C_S space of continuous functions from $[0,\infty)$ to S.

It is allowed to have, all $X^{(N)} \in D_S$ and $X \in C_S$

limit process has continuous
trajectories, but none of the
other $X^{(N)}$.

Let $X(t)$ be cadlag on $S \subseteq \mathbb{R}$.
Define its increments

$$\Delta_h X(t) := X(t+h) - X(t)$$

and

$$E_x\left[\Delta_h X(t)\right] := E\left[\Delta_h X(t) \mid X(t) = x\right]$$

X is a diffusion if three conditions hold:

- $E_x[\Delta_h X(t)] = \underbrace{\mu(x)}_{\substack{\text{drift of } X \\ \text{infinitesimal mean}}} h + o(h)$
- $E_x[(\Delta_h X(t))^2] = \underbrace{\sigma^2(x)}_{\substack{\text{diffusion coefficient} \\ \text{infinitesimal variance}}} h + o(h)$
- $E_x[|\Delta_h X(t)|^p] = o(h)$ some $p > 2$

$\Rightarrow$ continuity of trajectories via Dynkin's condition

If these hold X is the solution of the stoch. differential equation

(SDE)

$$dX(t) = \mu(X(t))\,dt + \sigma(X(t))\,dB(t)$$

standard BM.

Interpretation: if $X(t) = x$

$$\Delta_h X(t) \approx \mu(x)\,h + \sigma(x)\,\underbrace{\Delta_h B(t)}_{\substack{\text{increment of SBM} \\ \sim N(0,h)}}$$

cf. EULER - MARUYAMA scheme for approximating numerically an SDE.

Let now $\{Y^{(N)}\}_{N \geq 1}$ be a sequence of DTMCs on S_N countable subset of S with limit dense in S;

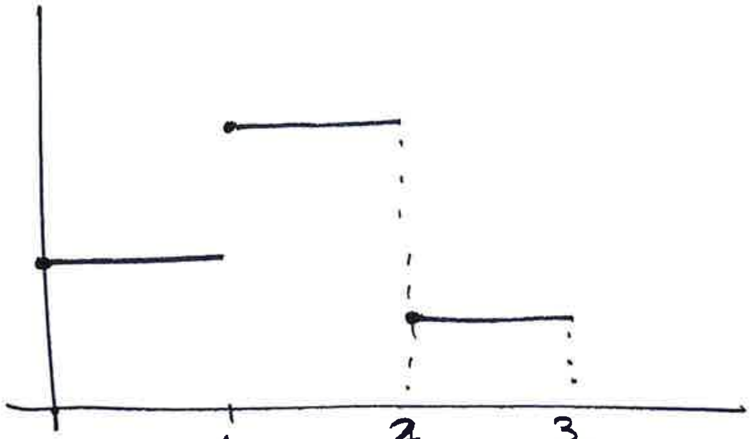
let $\{h_N, N \geq 1\} \subset \mathbb{R}_+$ s.t. $h_N \to 0$.

Define the continuous-time process

$$X^{(N)}(t) := Y^{(N)}_{\lfloor t/h_N \rfloor}$$

$\lfloor z \rfloor = \sup\{n \in \mathbb{N} : n \leq z\}$ floor function

E.g. $h_N = \frac{1}{N}$



$$X^{(N)}(t) = \begin{cases} Y_0^{(N)} & 0 \le t < \frac{1}{N} \\ Y_1^{(N)} & \frac{1}{N} \le t < \frac{2}{N} \\ \vdots & \end{cases}$$

a unit interval for $X^{(N)}$ corresponds to N steps of $Y^{(N)}$

Define $\Delta X^{(N)}(t) = \Delta_{h_N} X^{(N)}(t) = X^{(N)}(t+h_N) - X^{(N)}(t)$

and denote $E_x[\cdot] := E(\cdot \mid X^{(N)}(t) = x)$

If we can show:

- $E_x[\Delta X^{(N)}(t)] = \mu(x) h_N + o(h_N)$

$$\Rightarrow \lim_{N \to \infty} \frac{1}{h_N} E_x[\quad] = \mu(x)$$

- $E_x[(\Delta X^{(N)}(t))^2] = \sigma^2(x) h_N + o(h_N)$

$$\Rightarrow \lim_{N \to \infty} \frac{1}{h_N} E_x[(\quad)^2] = \sigma^2(x)$$

- $E_x[(\Delta X^{(N)}(t))^4] = o(h_N)$

Then (with some additional technical conditions) we can claim

$$X^{(N)} \xrightarrow{d} X \quad \text{as } N \to \infty$$

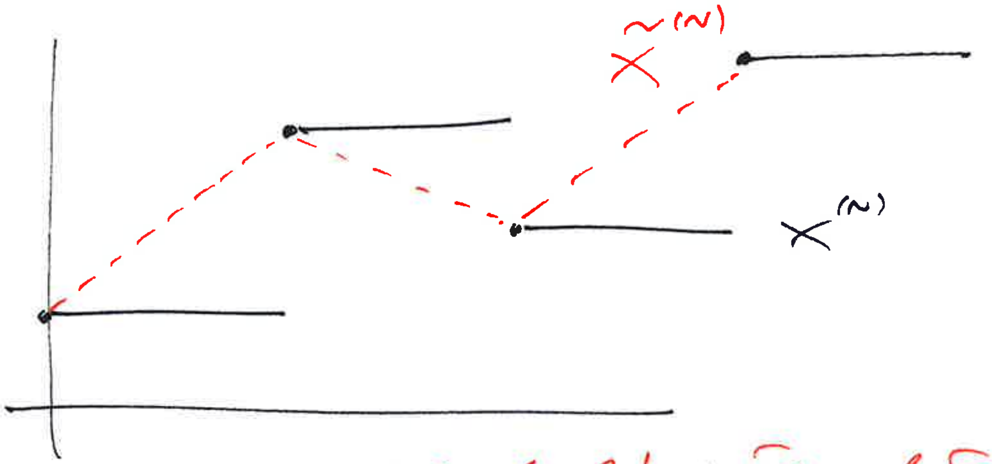
with X the solution of the SDE

$$dx(t) = \mu(x(t))\,dt + \sigma(x(t))\,dB(t)$$

so X approximates $X^{(N)}$ for large N.

Comments:

- The above $X^{(N)}$ are cadlag and discontinuous



we could interpolate to have each $\tilde{X}^{(N)}$ continuous (not necessary)

- Time rescaling:

we have used deterministic h_N intervals.

otherwise we can define a

uniform chain with jump chain $Y^{(N)}$
and h_N^{-1}-rate Poisson process

$$\Rightarrow \quad T^{(i)} \overset{iid}{\sim} Exp(h_N^{-1})$$

$$h_N = \frac{1}{N} \qquad Exp(N)$$

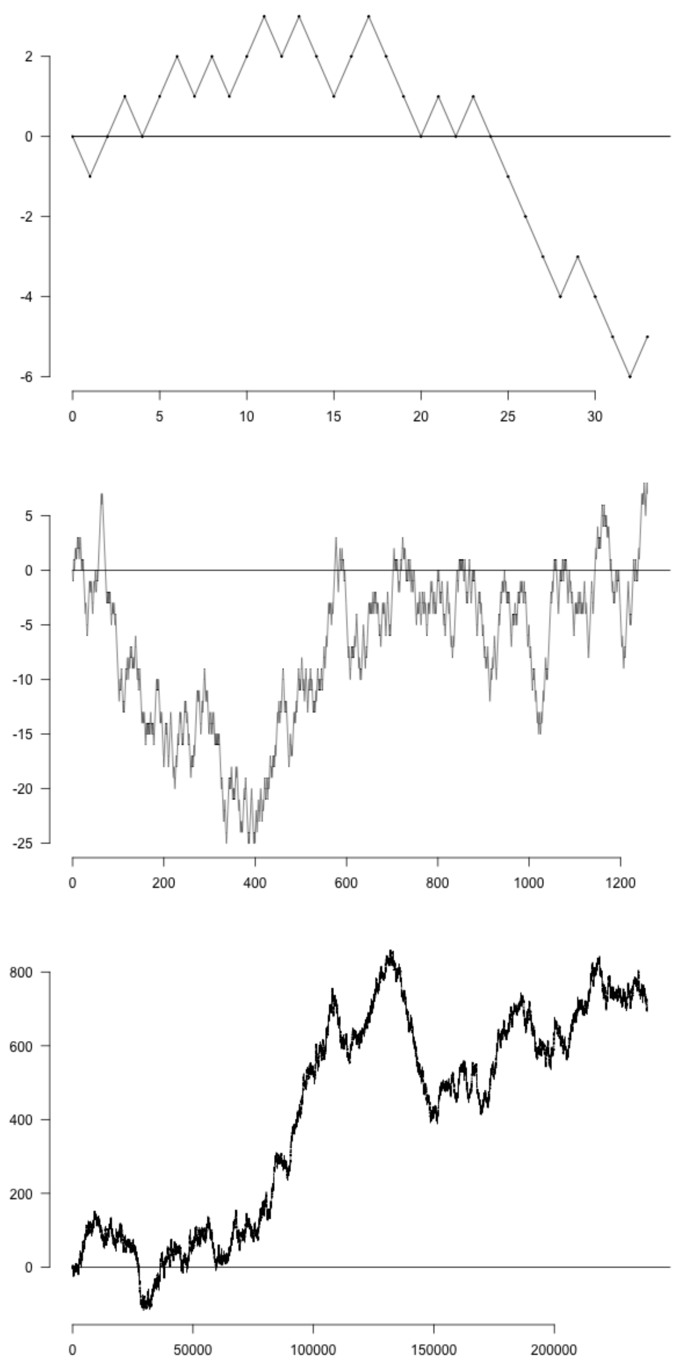
$$T^{(i)} \xrightarrow{m.s.} 0$$

- Space rescaling : above we have implicitly assumed the states of $Y^{(N)}$ get closer and closer as N increases.
  In general one may need to rescale space too
  e.g. Take $\frac{\mathbb{Z}}{N} = \{0, \pm\frac{1}{N}, \pm\frac{2}{N}, \dots\}$

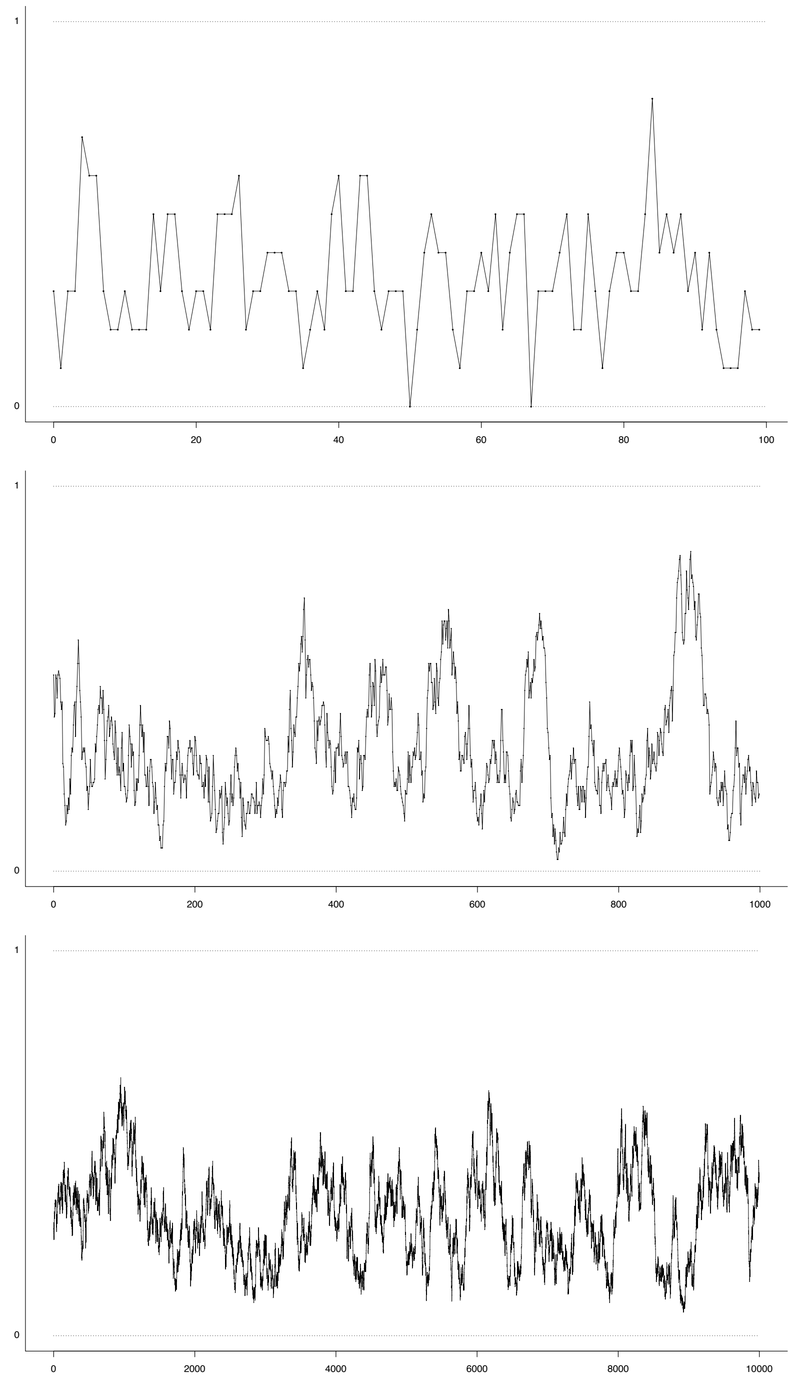
So in general we are interested in conditions for convergence of

$$X^{(N)}(t) = \frac{Y^{(N)}_{\lfloor t/h_N \rfloor} - a_N}{b_N} \xrightarrow{d} X(t)$$

- centering a_N
- Time rescaling h_N
- space rescaling b_N



RWs $X^{(N)}$ for three values of N.



WF chains $X^{(N)}$ for three values of N.