Levy process: Markov process with stationary and independent increments

Characterized by the characteristic exponent at time 1 through

$$\psi_t(u) = t\,\psi_1(u)$$

Compound Poisson process:

$$X(t) = \sum_{i=1}^{N(t)} Z_i \qquad Z_i \overset{iid}{\sim} F$$

with no mass at 0

$$N(t) \sim \mathrm{Pois}(\lambda t)$$

C.E. $\psi(u) = \lambda \int_{\mathbb{R}} (1 - e^{iux})\, F(dx)$

Poisson rate for jump arrivals

distrib of jump sizes

$$\lambda = \int_{\mathbb{R}} \pi(dx) < \infty$$

by construction

If we add a drift to the CP process we obtain a Lévy process with C.E.

$$\lambda \int_{\mathbb{R}} (1-e^{iux}) F(dx) \quad - iub$$

where the drift is bt.

<u>Exercise</u>: show the triplet is (μ, σ, π):

- $\sigma = 0$
- $\pi = \lambda F$
- $\mu = -\left(b + \lambda \int_{-1}^{1} x F(dx) \right)$

<u>Example</u> We had shown $Y \sim Ga(\alpha, \beta)$ is I.D. with C.F.

$$(*) \quad \frac{1}{\left(1 - \frac{iu}{\beta}\right)^{\alpha}} = \exp\left\{ - \overbrace{\int_0^{\infty} (1-e^{iux}) \underbrace{\alpha x^{-1} e^{-\beta x} dx}_{\pi(dx)}}^{\psi(u)} \right\}$$

it can be shown (Frullani integral)

$+ iux \mathbb{1}_{(|x|<1)} - iux \mathbb{1}_{(|x|<1)}$

$\Rightarrow$ similarly to CP process we get

$$\psi(u) = \int_0^{\infty} \left(1 - e^{iux} - iux \mathbb{1}_{(|x|<1)}\right) \pi(dx) - iu \underbrace{\int_0^{1} x \, \pi(dx)}_{-\mu}$$

$\Rightarrow$ Lévy Triplet

$$\mu = -\int_0^1 x \, \pi(dx), \quad \sigma = 0, \quad \pi(dx) = \alpha x^{-1} e^{-\beta x} dx$$

we can use $\psi_t(u) = t\, \psi_1(u)$ to define a Gamma process as a Lévy process with Triplet as above, i.e. with exponent for $X(t)$

$$\psi_t(u) = \int_0^\infty (1 - e^{iux})\, \alpha t\, x^{-1} e^{-\beta x}\, dx$$

$\Rightarrow$ Through (*) we see That

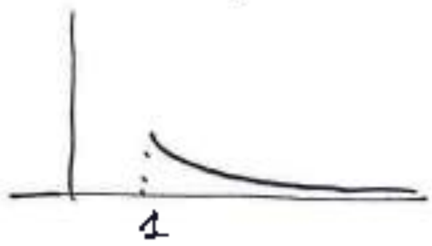
$$X(t) \stackrel{d}{=} X(s+t) - X(s) \sim Ga(\alpha t, \beta)$$

Note That:

- for $x \to \infty$, $\pi(dx) \approx e^{-\beta x}$ integrable

  $\Rightarrow$ $\pi((1,\infty)) < \infty$

  finite mass in Tail



- for $x \to 0^+$, $\pi(dx) \approx \frac{1}{x}$

  $\pi((0,1)) = \infty$

The requirement in LK formula

$$\int_{\mathbb{R}} \min(1, x^2)\, \pi(dx) < \infty$$

satisfied: around 0 we have $\frac{x^2}{x}$.

Since $\pi(\mathbb{R}_+) = \infty$, it <u>is not a CP</u>.

The Gamma process belongs to the following subclass of Lévy processes.

<u>Def.</u> A SUBORDINATOR is a Lévy process with a.s. non-decreasing sample paths, hence with triplet:

- $\mu \leq 0$ (so non negative drift $-\mu$)
- $\sigma = 0$ no Brownian component
- $\Pi((-\infty, 0]) = 0$, only select positive jumps

We want to understand path properties when Π has infinite total mass. Write a generic C.E. as follows

$$\psi(u) = \underbrace{i\mu u + \frac{1}{2}\sigma^2 u^2}_{\psi^{(1)}} \quad \leftarrow \text{exponent of } dx^{(1)}(t) = -\mu dt + \sigma dB(t)$$

$$+ \underbrace{\int_{(-1,1)^c} (1 - e^{iux}) \,\Pi(dx)}_{\psi^{(2)}}$$

$$+ \underbrace{\int_{(-1,1)} (1 - e^{iux} + iux \mathbb{1}_{(|x|<1)}) \,\Pi(dx)}_{\psi^{(3)}}$$

$\to \psi^{(1)}$ linear BM

$\to \psi^{(2)}$: outside $(-1,1)$ Π has finite mass

$\Rightarrow \psi^{(2)}$ can be written as

$$\lambda_0 \int_{(-1,1)^c} (1 - e^{iux}) F_0(dx)$$

$$\lambda_0 := \Pi\left((-1,1)^c\right)$$

$$F_0 := \lambda_0^{-1} \, \Pi\big|_{(-1,1)^c}$$

$\Rightarrow \psi^{(2)}$ corresponds to a CP process with λ_0 Poisson rate and F_0 distrib. for jumps of size $|x| \geq 1$

$\to \psi^{(3)}$: Two cases:

- $\Pi((-1,1)) < \infty$

$$\int_{-1}^{-1} (1 - e^{iux} + iux) \, \Pi(dx) =$$

$$= \underbrace{\int_{-1}^{+1} (1 - e^{iux}) \Pi(dx)}_{\text{C.P.}} + iu \underbrace{\int_{-1}^{1} x \, \Pi(dx)}_{\text{drift}}$$

- $\Pi((-1,1)) = \infty$

Jumps arrive at ∞ rate but we cannot normalize Π to get the jump distribution

So the condition $\int_{\mathbb{R}} \min(1, x^2)\, \pi(dx) < \infty$

implies:

- "big jumps" (size ≥ 1) arrive at finite rate $\pi((-1,1)^c) < \infty$

  hence they occur finitely often in every bounded interval

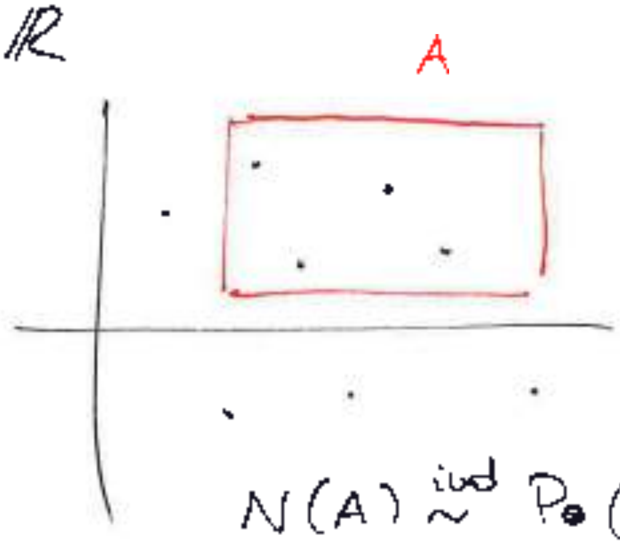
- if $\pi((-1,1)) = \infty$, "small jumps" arrive at infinite rate, hence they occur infinitely-often in every bounded interval.

  Such a process is said to have INFINITE ACTIVITY.

<u>Def.</u> A measure N on a σ-finite measure space $(\mathbb{X}, \mathcal{X}, \mu)$ is said to be a POISSON RANDOM MEASURE (PRM) with mean intensity measure μ if for mutually disjoint sets $A_1, \dots, A_k \in \mathcal{X}$

$$N(A_i) \overset{ind}{\sim} Po(\mu(A_i))$$

Ex. $\mathbb{X} = \mathbb{R}_+ \times \mathbb{R}$



$N(A) \overset{ind}{\sim} Po(\mu(A))$

if $A = [s,t] \times [a,b]$, $\mu = leb \times \pi$

$$\mu(A) = \lambda(t-s) \int_a^b F(dx)$$

$\pi = \lambda F$

<u>Proposition</u> Let N be a PRM on $[0,\infty) \times \mathbb{R}$ with mean intensity $\mu = Leb \times \pi$, with π finite on $\mathbb{R}$ s.t. $\pi(\{0\}) = 0$.

For any $B \in \mathcal{B}(\mathbb{R})$ ~~st~~

$$X_B(t) := \int_0^t \int_B x \, N(ds, dx)$$

is a CP process with rate $\lambda_B := \pi(B)$ and jump distrib. $F_B = \lambda_B^{-1} \pi|_B$

<u>Example</u> $B = \mathbb{R}$, π finite. Then we can show $N = \sum_{i \geq 1} \mathbb{1}_{(S_i, Z_i)}$ (random point configuration)

yields

$$\int_0^t \int_B x \sum_{i\geq 1} \mathbb{1}_{(S_i,Z_i)}(ds,dx) =$$

$$= \sum_{i\geq 1} Z_i \, \mathbb{1}(S_i \in [0,t], Z_i \in B)$$

i.e. sum of point height for points $(S_i, Z_i) \in [0,t]\times B$ (if $B = \mathbb{R}$ we sum all jumps/points sizes heights in $[0,t]$). See Fig 1 below.

If we now let, for general π (so in every $[0,t]\times[0,\varepsilon]$ there could be infinitely many points) See Fig 3-4 below

$$\varepsilon_m = \frac{1}{2^m} \quad m \geq 1$$

$$B_m = (-1, -\varepsilon_m] \cup [\varepsilon_m, 1)$$

Set

$$X^{(3,m)}(t) := \underbrace{\int_0^t \int_{B_m} x \, N(ds,dx)}_{\substack{CP \\ \text{since } \pi(B_m) < \infty}} - \underbrace{t \int_{B_m} x \, \pi(dx)}_{\text{drift}}$$

which has exponent

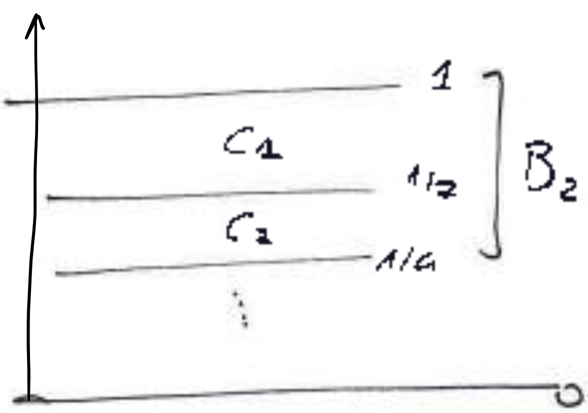
$$\psi^{(3,m)} = \int_{B_m} (1 - e^{iux})\,\pi(dx) + iu\int_{B_m} x\,\pi(dx)$$

<u>informally</u> as $m \to \infty$ we get

$$\psi^{(3)} = \int_{|x|<1} (1 - e^{iux} + iux)\,\pi(dx)$$

More formally:

<u>Proposition</u> Let $X^{(3,m)}$ be a Lévy process with exponent $\psi^{(3,m)}$ (CP + drift). As $m \to \infty$, $X^{(3,m)} \xrightarrow{a.s.} X^{(3)}$ uniformly over $[0,T]$, where $X^{(3)}$ is a Lévy process with exponent $\psi^{(3)}$ and with at most countably-many discontinuities in every bounded interval.



If $C_n = \left[\frac{1}{2^n}, \frac{1}{2^{n-1}}\right)$

$$B_m = \bigcup_{n=1}^{m} C_n$$

$$\psi^{(3,m)} = \int_{B_m} (1 - e^{iux} + iux)\,\pi(dx)$$

$$= \sum_{m=1}^{m} \Big[\underbrace{\lambda_m \int_{C_m} (1-e^{iux}) F_m(dx)}_{\text{c.p.}} + \underbrace{iu\lambda_m \int_{C_m} x F_m(dx)}_{\text{drift}} \Big]$$

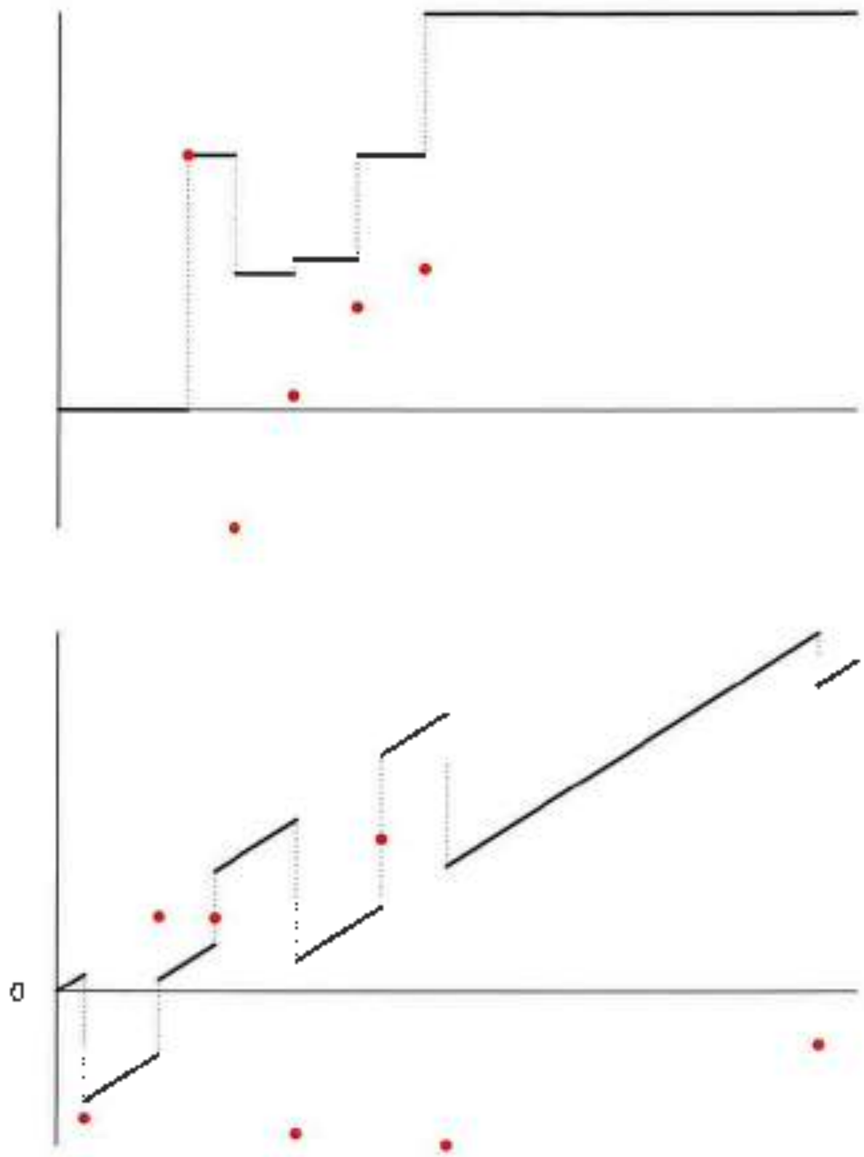
<u>Theorem</u> (LÉVY-ITO Decomposition)

Let (μ, σ, π) satisfy the conditions of the Lévy-Kintchine formula.

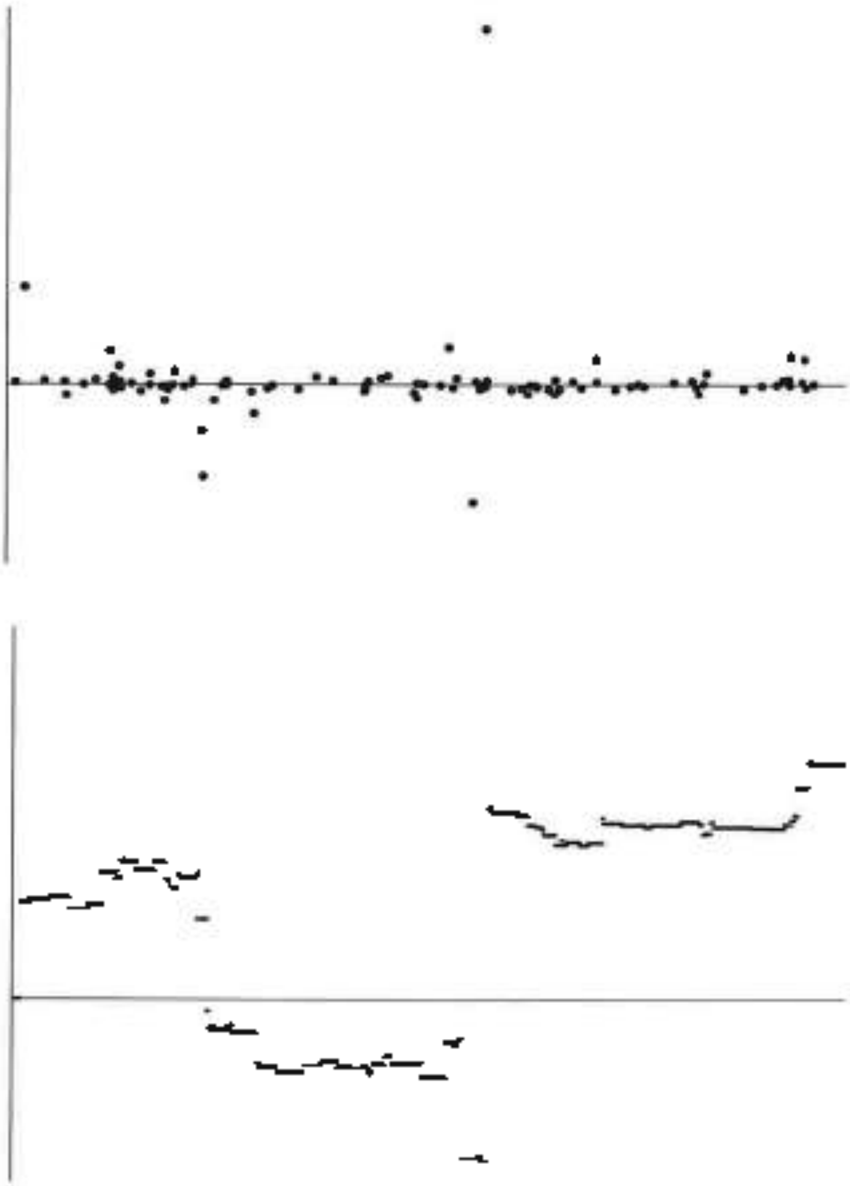
Any Lévy process X is the superposition of three independent Lévy processes, s.t. $X = X^{(1)} + X^{(2)} + X^{(3)}$ where:

- $X^{(1)}$ is a linear BM $dX^{(1)}(t) = -\mu dt + \sigma dB(t)$
- $X^{(2)}$ is a CP with jump law $F_0 := \lambda_0^{-1} \pi|_{(-1,1)^c}$, and rate $\lambda_0 := \pi((-1,1)^c)$
- $X^{(3)}$ is the sum of countably-many CP processes with drift and jumps in $(-1,1)$.

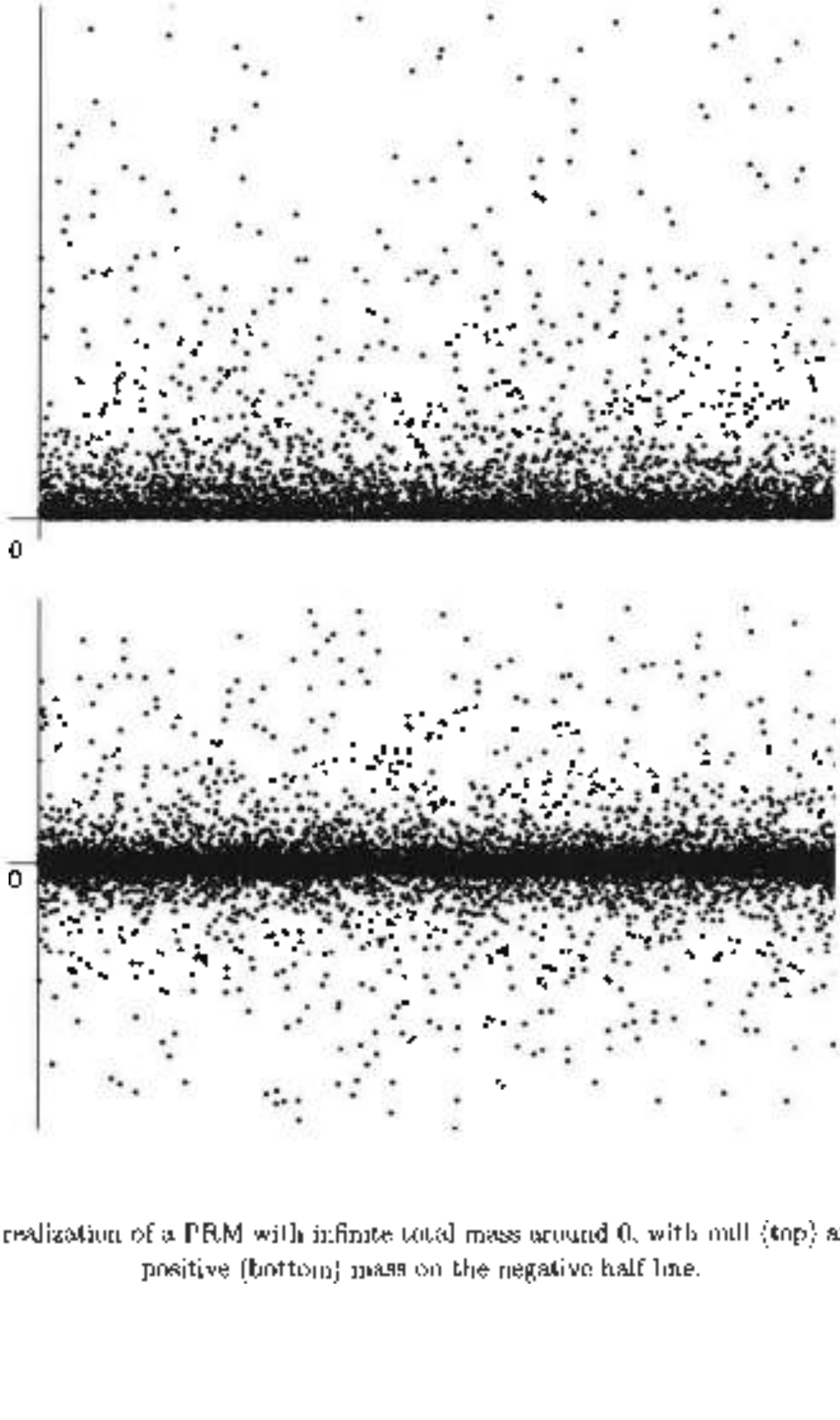
<u>Ex</u> verify superposition of CP and parameterization for Gamma process.



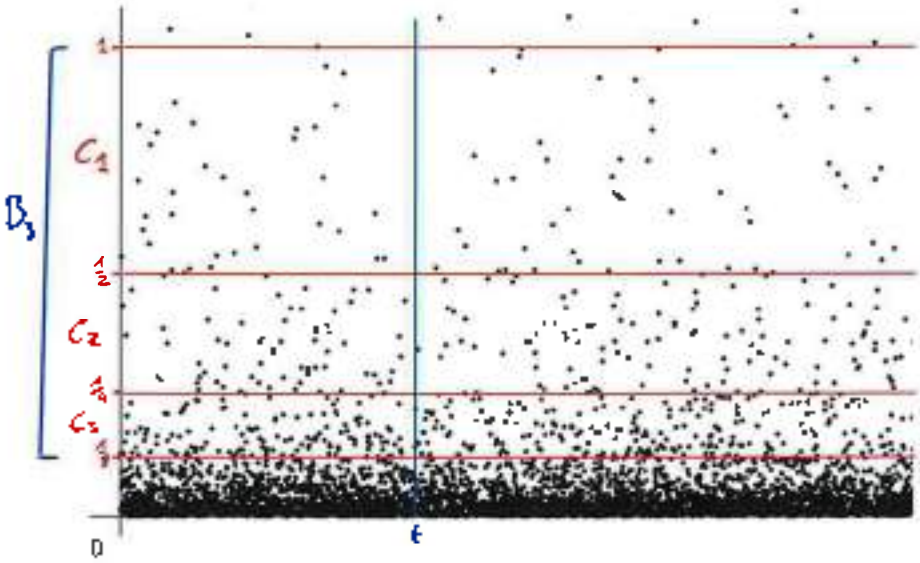
A realization of a PRM and the resulting trajectory of the CP process. Bottom: same with additional drift bt.



A realization of a PRM with Cauchy Lévy intensity (finite mass) and the resulting trajectory of the Cauchy CP process.



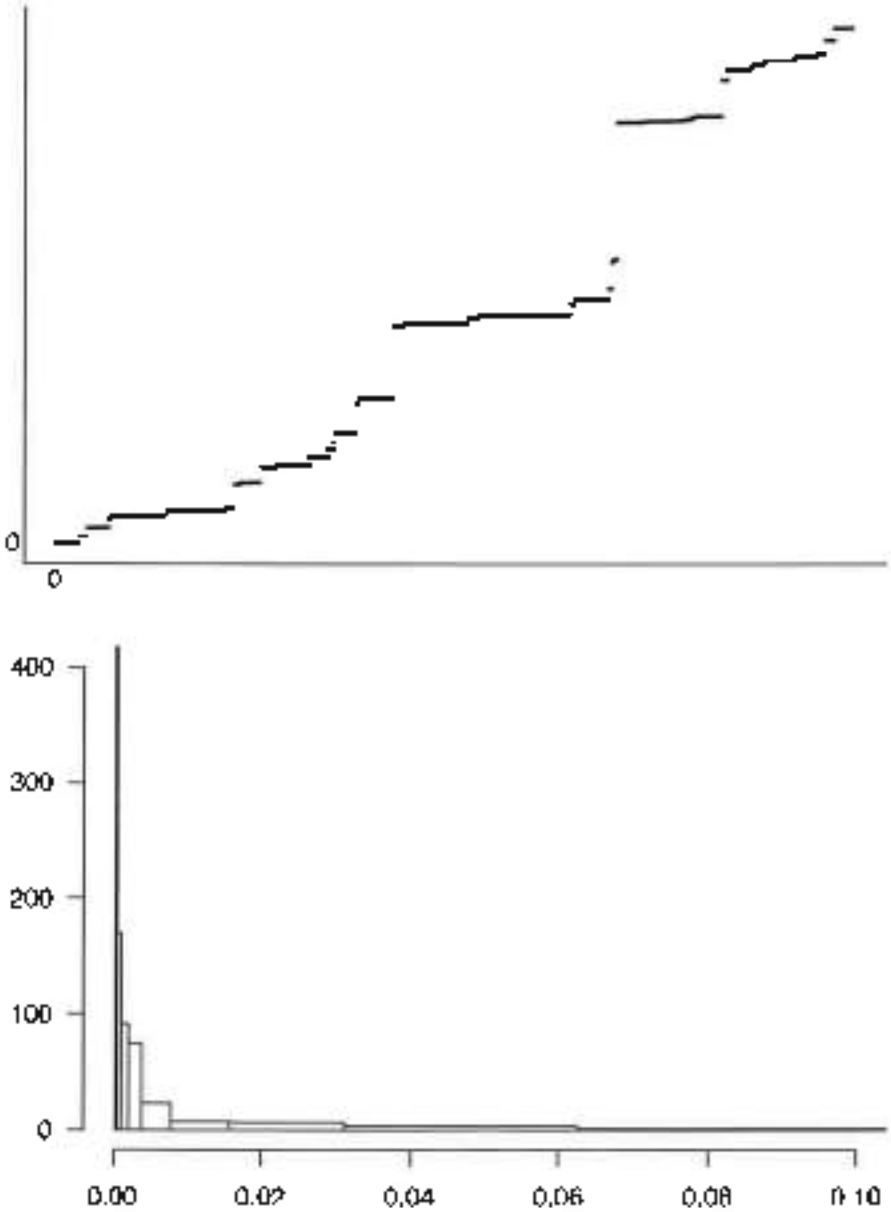
A realization of a PRM with infinite total mass around 0, with null (top) and positive (bottom) mass on the negative half line.



Slicing of the jump size space for given t into sets $C_m = [2^{-m}, 2^{-m+1})$. In each C_m every realization has almost surely finitely-many points, so

$$X_{C_m} = \int_0^t \int_{C_m} x N(\mathrm{d}s, \mathrm{d}x)$$

is a CP process. Then X_{B_m} is the superposition of finitely-many CP processes, and the a.s. limit of X_{B_m} yields a Lévy process with the desired exponent.



Trajectory of a Gamma process, and the histogram of the jumps ("empirical" π) with size in B_n, $1 \le n \le 14$.