This Should Help Your Lazy Ass In Analisys B

Charles A. Mongus, world famous jazz composer and bassist

August 23, 2025

Contents

1 Normed Spaces

Proposition 1.1

A subset $C \subset X$ is a closed subset of $(X, \|\cdot\|)$ whenever the limit of any convergent sequence $(x_n)_n \subset C$ belongs to C.

In other words, C is closed if and only if once a sequence of elements of C is converging, the limit cannot escape C.

Proof

[redacted]

Lemma 1.1

Young's inequality. Let $p > 1$ and let $q < 1$ be its conjugate exponent. Then, for any nonnegative $a, b \in \mathbb{R}$ it holds

$$ab \leqslant \frac{1}{p}a^p + \frac{1}{b}b^q.$$

Theorem 1.1

Holder Inequality. Let $1 \leqslant p \leqslant \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Assume that $f \in L^p(S, \mu)$ and $g \in L^q(S, \mu)$. Then $f \cdot g \in L^1$ and

$$\|fg\|_1 \leqslant \|f\|_p \|g\|_q .$$

Proof

The proof is trivial if $p = 1$ and $q = 1$. Remember Young's inequality

$$ab \leqslant \frac{1}{p}a^p + \frac{1}{q}b^q.$$

Now let's say $a = |f(s)|$ and $b = |g(s)|$. Now our inequality becomes

$$|f(s)g(s)| \leqslant \frac{1}{p}|f(s)|^p + \frac{1}{q}|g(s)|^q \qquad \text{for } \mu\text{-a.e. on } S.$$

Now we integrate over S and we get

$$\begin{aligned}
&\int_S |f(s)g(s)| \leqslant \frac{1}{p}\int_S |f(s)|^p + \frac{1}{q}\int_S |g(s)|^q \\
\implies &\|fg\|_1 \leqslant \frac{1}{p}\|f\|_p^p + \frac{1}{q}\|g\|_q^q
\end{aligned}$$

which means that $\|fg\|_1$ is finite and therefore $fg \in L^1(S, \mu)$. To end the proof let's substitute f with λf, $\forall\, \lambda > 0$. We get

$$\begin{aligned}
&\lambda \|fg\|_1 \leqslant \frac{\lambda^p}{p}\|f\|_p^p + \frac{1}{q}\|g\|_q^q \qquad \forall\, \lambda > 0 \\
\implies &\frac{\lambda}{\lambda}\|fg\|_1 \leqslant \frac{\lambda^p}{p\lambda}\|f\|_p^p + \frac{1}{q\lambda}\|g\|_q^q \qquad \forall\, \lambda > 0 \\
\implies &\|fg\|_1 \leqslant \frac{\lambda^{p-1}}{p}\|f\|_p^p + \frac{1}{\lambda q}\|g\|_q^q \qquad \forall\, \lambda > 0.
\end{aligned}$$

Now choose $\lambda = \frac{1}{\|f\|_p} \cdot \|g\|_q^{\frac{q}{p}}$. When we substitute this value for λ, we get:

$$\frac{\lambda^p}{p}\|f\|_p^p = \frac{\left(\frac{1}{\|f\|_p} \cdot \|g\|_q^{\frac{p}{q}}\right)^p}{p}\|f\|_p^p.$$

Expanding $\left(\frac{1}{\|f\|_p} \cdot \|g\|_q^{\frac{p}{q}}\right)^p$, we get:

$$\left(\frac{1}{\|f\|_p} \cdot \|g\|_q^{\frac{p}{q}}\right)^p = \frac{\|g\|_q^p}{\|f\|_p^p}.$$

Substituting this, we have:

$$\frac{\lambda^p}{p}\|f\|_p^p = \frac{\frac{\|g\|_q^p}{\cancel{\|f\|_p^p}}}{p}\cancel{\|f\|_p^p} = \frac{\|g\|_q^p}{p}.$$

Substitute back into the inequality:

$$\lambda\|fg\|_1 \leqslant \frac{\|g\|_q^p}{p} + \frac{1}{q}\|g\|_q^q.$$

Since $\frac{\|g\|_q^p}{p} + \frac{\|g\|_q^q}{q} = \|g\|_q^p \left(\frac{1}{p} + \frac{1}{q}\right)$ and $\frac{1}{p} + \frac{1}{q} = 1$, we get:

$$\lambda \|fg\|_1 \leqslant \|g\|_q^p.$$

Dividing by $\lambda = \frac{1}{\|f\|_p} \|g\|_q^{\frac{p}{q}}$:

$$\|fg\|_1 \leqslant \|f\|_p \|g\|_q.$$

This is Hölder's inequality:

$$\|fg\|_1 \leqslant \|f\|_p \|g\|_q.$$

□

Theorem 1.2

For any $1 \leqslant p \leqslant \infty$, $L^p(S, \mu)$ is a vector space and $\|\cdot\|_p$ is a norm.

Remark

For $1 < p < \infty$ the triangular inequality

$$\|f+g\|_p \leqslant \|f\|_p + \|g\|_p \qquad \forall\, f, g \in L^p(S, \mu)$$

is known as **Minkowski's Inequality**.

Proof

We already know that if $f \in L^p(S, \mu)$ then $\lambda f \in L^p(S, \mu)$. Homogeneity and uniqueness are also existent for $\|\cdot\|_p$ so in order to show that $L^p(S, \mu)$ is a vector space we only need to prove that if $f, g \in L^p(S, \mu)$ then $f + g \in L^p(S, \mu)$ and $\|\cdot\|$ is a norm.
Fix $f, g \in L^p(S, \mu)$. We know that for any $x, y \in \mathbb{R}$ we get

$$\left|\frac{1}{2}x + \frac{1}{2}y\right|^p \leqslant \frac{1}{2}|x|^p + \frac{1}{2}|y|^p$$

since this mapping $r \to r^p$ is convex. This also means that

$$|x+y|^p \leqslant 2^{p-1}\left(|x|^p + |y|^p\right).$$

and this implies in particular that

$$|f(s) + g(s)|^p \leqslant 2^{p-1}\left(|f(s)|^p + |g(s)|^p\right) \qquad \text{for } \mu\text{-a.e. on } s \in S.$$

If we integrate over S we get:

$$\int_S |f(s) + g(s)|^p \leqslant 2^{p-1}\left(\int_S |f(s)|^p + \int_S |g(s)|^p\right)$$

which means

$$\|f+g\|_p^p \leqslant 2^{p-1}\left(\|f\|_p^p + \|g\|_p^p\right)$$

which means that $f + g \in L^p(S, \mu)$.
We now must prove the Minkowski's inequality. We know that

$$\|f+g\|_p^p = \int_S |f+g|^p \, \mathrm{d}\mu = \int_S |f+g|\, |f+g|^{p-1} \, \mathrm{d}\mu$$

but since we know that $|f+g| \leqslant |f| + |g|$ then

$$\|f+g\|_p^p \leqslant \int_S |f| |f+g|^{p-1} \, \mathrm{d}\mu + \int_S |g| |f+g|^{p-1} \, \mathrm{d}\mu.$$

Call $\psi = |f+g|^{p-1}$. It clearly belongs to $L^q(S, \mu)$ because

$$|\psi|^q = \left(|f+g|^{p-1}\right) q = |f+g|^p \qquad \text{since } q(p-1) = p$$

so

$$\|\psi\|_q = \left(\int_S |\psi|^q\right)^{\frac{1}{q}} = \left(\int_S |f+g|^p\right)^{\frac{1}{q}} = \|f+g\|_p^{\frac{p}{q}} < \infty$$

And this means that $|\psi|^q \in L^1(S,\mu) \implies \psi \in L^q(S,\mu)$. We also know that $|f| \in L^p(S,\mu)$ so we can apply Holder's inequality with f and ψ so that

$$\int_S |f||f+g|^{p-1}\,\mathrm{d}\mu = \|f\psi\|_1 \leqslant \|f\|_p \|\psi\|_q = \|f\|_p \|f+g\|_p^{\frac{p}{q}}$$

and

$$\int_S |g||f+g|^{p-1}\,\mathrm{d}\mu \leqslant \|g\|_p \|f+g\|_p^{\frac{p}{q}}$$

So that

$$\|f+g\|_p^p \leqslant \|f\|_p \|f+g\|_p^{\frac{p}{q}} + \|g\|_p \|f+g\|_p^{\frac{p}{q}}.$$

Dividing by $\|f+g\|_p^{\frac{p}{q}} \neq 0$ (otherwise the proof is trivial) we get

$$\|f+g\|_p^{p-\frac{p}{q}} \leqslant \|f\|_p + \|g\|_p.$$

□

Proposition 1.2

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability triplet. then the following holds:

$$L^\infty(\Omega,\mathcal{F},\mathbb{P}) \subset L^p(\Omega,\mathcal{F},\mathbb{P}) \subset L^q(\Omega,\mathcal{F},\mathbb{P}) \subset L^1(\Omega,\mathcal{F},\mathbb{P}).$$

Of course, this result remains valid for every other measure space $(S, \mathscr{S}, \mu)$ as long as $\mu(S) < \infty$. In the special case in which

$$S = \mathbb{N} \qquad \mathcal{F} = \mathscr{P}(\mathbb{N})$$

and $\mu(A)$ is the counting measure $\mu(A) = \sum_{k\in A} \delta_k(A), A \in \mathbb{N}$ then knowing that sequences $n \mapsto f(n)$ can be identified as functions over $\mathbb{N}$ of the type $f : \mathbb{N} \to \mathbb{R}$ we see that

$$L^1(S,\mu) = \underbrace{\mathscr{L}^1(S,\mu)}_{\text{actual functions, not equivalence classes}} = \ell^1(\mathbb{N}) = \left\{\mathbf{x} = (x_n)_n; \|\mathbf{x}\|_1 := \sum_{n=1}^{\infty} |x_n| < \infty\right\}.$$

This means that $\ell^1(\mathbb{N})$ is a L^1 space for some special choice of S and μ. Since we chose our measure as the counting measure, we get

$$\int_{\mathbb{N}} |f(n)|\,\mathrm{d}\mu(n) = \sum_{n=1}^{\infty} |f(n)| = \sum_{n=1}^{\infty} |x_n|.$$

Cool!

Proposition 1.3

Let $p \geqslant 1$ be given. We define the set

$$\ell^p(\mathbb{N}) = \left\{\mathbf{x} = (x_n)_{n\in\mathbb{N}} \subset \mathbb{R} \text{ such that } \sum_{n=1}^{\infty} |x_n|^p < \infty\right\}.$$

Then, $\ell^p(\mathbb{N})$ is a vector space. Moreover, if

$$\|\mathbf{x}\|_p := \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}} \qquad \forall\, \mathbf{x} = (x_n)_n \in \ell^p(\mathbb{N})$$

then $\left(\ell^p(\mathbb{N}), \|\cdot\|_p\right)$ is a normed space.

1.1 The space of linear applications

Proposition 1.4

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed spaces and let $L; X \mapsto Y$ be a linear application. The following are equivalent:

1. L is continuous on X;
2. L is continuous at $x = 0$;
3. there is a positive constant $C > 0$ such that
$$\|L(x)\|_Y \leqslant C \|x\|_X \qquad \forall x \in X.$$

Proof

Of course 1 $\implies$ 2, but now let's prove 3. Consider the definition of continuity at 0 for the function L: this means that for any $\varepsilon > 0$ there exists a constant $\delta > 0$ such that

$$\|x - 0_X\|_X < \delta \implies \|L(x) - L(0_X)\|_Y < \varepsilon.$$

We know that L is linear, so $L(0_X) = 0_Y$ and therefore we get

$$\|x\|_X < \delta \implies \|L(x)\|_Y < \varepsilon.$$

Choose $\varepsilon = 1$ so that we get

$$\|x\|_X < \delta \implies \|L(x)\|_Y < 1.$$

Let $x \in X \setminus \{0_X\}$ and set $y = \frac{\delta}{\|x\|_X} x$. Now we have that

$$\|y\|_X = \left\| \frac{\delta}{\|x\|_X} x \right\|_X = \frac{\delta}{\|x\|_X} \|x\|_X = \delta$$

and, due to the linearity of L

$$\|L(y)\|_Y = \left\| L\left(\frac{\delta}{\|x\|_X} x \right) \right\|_Y = \frac{\delta}{\|x\|_X} \|L(x)\|_Y = \frac{\delta}{\|x\|_X} \|L(x)\|_Y < 1 \implies \|L(x)\|_Y < \frac{1}{\delta} \|x\|_X$$

So setting $C = \frac{1}{\delta}$ proves 3. We still need to prove 1 $\implies$ 3: it is easy, since by linearity we have

$$\|L(x) - L(y)\|_Y = \|L(x - y)\|_Y \leqslant C \|x - y\|_X \qquad \forall x, y \in X$$

and this clearly implies the continuity of L at any $x \in X$ (actually, the uniform continuity). $\square$

Definition 1.1

If $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are two normed spaces, we denote by $\mathscr{L}(X, Y)$ the space of continuous linear applications from X to Y. If $X = Y$ we simply denote $\mathscr{L}(X) = \mathscr{L}(X, X)$.

Proposition 1.5

If $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are two vector spaces and X is of finite dimension, any linear application $L : X \mapsto Y$ is continuous.

Remark

If $\dim(X) = n$ and $\dim(Y) = p$, the space $\mathscr{L}(X, Y)$ can be identified with the space $\mathscr{M}_{n \times p}(\mathbb{R})$ of matrices with n lines and p rows.

1.2 Compactness

Definition 1.2

Let $(X, \|\cdot\|_X)$ be a normed space and let $K \subset X$. We say that K is **compact** if every sequence $(x_n)_n$ contains a subsequence which converges to some $x \in K$.

Of course if K is compact then it is closed.

Lemma 1.2

If K is a compact subset of a normed space $(X, \|\cdot\|_X)$ then K is closed and there exists $M > 0$ such that $\sup_{x \in K} \|x\| \leqslant M$ which means that K is bounded.

Proposition 1.6

Product of compact spaces. Let $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$ be two compact normed spaces and let $X = X_1 \times X_2$. Then $(X, \|\cdot\|_+)$ and $(X, \|\cdot\|_{\max})$ are compact normed spaces.

Remember that

$$\|\mathbf{x}\|_+ = \|x_1\|_1 + \|x_2\|_2$$

and

$$\|\mathbf{x}\|_{\max} = \max(\|x_1\|_1, \|x_2\|_2)$$

Proof

[redacted]

Of course, the above result readily extends to any finite product of compact normed spaces. On $\mathbb{R}$ it is easy to describe a large class of compact sets:

Lemma 1.3

Let $\mathbb{R}$ be endowed with the absolute value, $|\cdot|$. Any interval $[a, b] \subset \mathbb{R}$ is compact.

Proposition 1.7

Let $(X, \|\cdot\|)$ be a normed space and let K be a compact subset of X. If $A \subset K$ is a closed subset then A is compact.

Corollary

Heine-Borel theorem. A subset K of $\mathbb{R}^N$ (where $\mathbb{R}^N$ is endowed with, say, the usual Euclidean norm) is compact if and only if it is closed and bounded.

Proof

[redacted]

This corollary can be reformulated as:

Every bounded sequence of $\mathbb{R}^N$ has a convergent subsequence.

1.3 Compactness and continuous functions

Proposition 1.8

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed spaces and let $f : X \to Y$ be continuous. If $K \subset X$ is a compact subset of X then $f(K)$ is a compact subset of Y.

Proof

[redacted]

This has the following consequence:

Theorem 1.3

Let $(X, \|\cdot\|)$ be a normed space and let $K \subset X$ be compact. Let $f : K \to \mathbb{R}$ be continuous. Then, f assumes its maximum and minimum on K.

Proof

[redacted]

Theorem 1.4

Heine Theorem. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed spaces and let $K \subset X$ be compact. Assume that $f : K \to Y$ is continuous. Then f is uniformly continuous on K.

Proof

[redacted]

So, in this case taking two sequences that get closer and closer does not correspond to the fact also their functions get closer and closer... and this is not possible.

1.4 Finite dimensional spaces

Proposition 1.9

Let $(X, \|\cdot\|)$ be a finite dimensional normed vector space with $\dim(X) = d$ and let $\{e_1, \ldots, e_d\}$ be a basis for X. Then, there are positive constants $C_0, C_1 > 0$ such that

$$C_0 \sum_{i=1}^{d} |x_i| \leqslant \left\| \sum_{i=1}^{d} x_i e_i \right\| \leqslant C_1 \sum_{i=1}^{d} |x_i| \qquad \forall (x_1, \ldots, x_d) \in \mathbb{R}^d.$$

This proposition asserts that if $\dim(X) = d$ then *any* norm $\|\cdot\|$ is related to the $\|\cdot\|_1$ norm of $\mathbb{R}^d$. This translates in the following:

Proposition 1.10

If X is a finite dimensional vector space, all norms over X are equivalent.

So there is no weird norm, but everything is comparable to the simple $\|\cdot\|_1$ norm. This proposition also allows us to identify in a continuous way a finite dimensional space $(X, \|\cdot\|)$ and the space $\mathbb{R}^d$ where d is the dimension of X. Indeed, introducing a basis $\{e_1, \ldots, e_d\}$ of X, the mapping

$$\Phi : X \to \mathbb{R}^d$$

which associates $\Phi(\mathbf{x}) = (x_1, \ldots, x_d)$ to some $\mathbf{x} = \sum_{i=1}^{d} x_i e_i \in X$, is a bijection from X to $\mathbb{R}^d$ which is continuous whose inverse is also continuous. This results in the following:

Corollary

If $(X, \|\cdot\|)$ is a finite dimensional vector space and $K \subset X$ is closed and bounded then K is compact.

Again, this is very specific to finite dimensional spaces and, as we shall see, this actually characterizes finite dimensional spaces. Indeed, in infinite dimensional normed spaces, the closed unit ball cannot be compact. This shows that, in infinite dimensional spaces, the compact subsets do not coincide with closed and bounded subsets!! We first state the following technical lemma:

Lemma 1.4

Riesz Lemma. Let $(X, \|\cdot\|)$ be a normed vector space and let Y be a closed subspace of X (i.e. Y is closed in X and Y is a linear subspace of X). If $Y \neq X$ then for any $\varepsilon \in (0, 1)$ there exists

$x \in X$ with $\|x\| = 1$ such that

$$\inf_{y \in Y} \|x - y\| \geqslant 1 - \varepsilon.$$

Remark

This lemma asserts that if $Y \neq X$ is a closed subspace then for any $\varepsilon \in (0,1)$ there is some unit vector $x \in X$ such that $\text{dist}(x, Y) \geqslant 1 - \varepsilon$.

Proof

[redacted]

Lemma 1.5

If $(X, \|\cdot\|)$ is a normed space, any linear subspace of finite dimension is closed.

Theorem 1.5

Riesz Theorem. A normed space $(X, \|\cdot\|)$ is finite dimensional if and only if the closed unit ball $B_c(0,1) = \{x \in X; \|x\| \leqslant 1\}$ of X is compact.

Proof

[redacted]

2 Banach spaces

2.1 Cauchy sequences & Banach spaces

Definition 2.1

Let $(X, \|\cdot\|)$ be a normed space. A sequence $(x_n)_n \subset X$ of elements in X is a Cauchy sequence if for any $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \mathbb{N}$ such that

$$\|x_n - x_m\| \leqslant \varepsilon \qquad \forall\, n, m \geqslant \mathbb{N}.$$

One easily checks that every Cauchy sequence $(x_n)_n \subset X$ is bounded. Indeed, by definition, for any $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that

$$\|x_n - x_m\| < \varepsilon \qquad \forall\, n, m \geqslant N.$$

For, say, $\varepsilon = 1$ and $m = N$ we see that for $\forall\, n \geqslant N$

$$\|x_n\| = \|x_n - x_N + x_N\| \leqslant \|x_n - x_N\| + \|x_N\| \leqslant 1 + \|x_N\|.$$

Thus with $C_1 = 1 + \|x_N\|$

$$\sup_{n \geqslant N} \|x_n\| \leqslant C_1.$$

Setting now $C_2 : \max\{\|x_1\|, \ldots, \|x_{N-1}\|\}$ one sees that C_2 is finite since it is the maximum of only a finite number of real numbers. By definition

$$\|x_n\| \leqslant C_2 \qquad \forall\, n < N.$$

Therefore

$$\sup_n \|x_n\| \leqslant C = \max\{C_1, C_2\} < \infty$$

which means $\{(x_n)\}_n$ is bounded.

Lemma 2.1

Let $(X, \|\cdot\|)$ be a normed space. Any convergent sequence is a Cauchy sequence.

Proof

[redacted]

The opposite is not true... We introduce the set $c_{00}(\mathbb{N})$ of all sequences of real number that have only finitely many nonzero components, which means

$$c_{00} = \{x = (x_n)_n \subset X, \exists N \in \mathbb{N}, x_k = 0 \quad \forall\, k \geqslant N\}.$$

This is a subset of the space $\ell^1(\mathbb{N})$ of finitely summable sequences and in particular it is a norm space with respect to the norm induced by the one in $\ell^1(\mathbb{N})$ (which is $\sum_{n=0}^{\infty} |x_n|$). Now for any $n \in \mathbb{N}$ let $\boldsymbol{x}^{(n)}$ be the sequence given by

$$\boldsymbol{x}^{(n)} = (2^{-1}, 2^{-2}, \ldots, 2^{-n}, 0, 0, \ldots)$$

so that $x^{(n)} = \left(x_k^{(n)}\right)_k$ with

$$x_k^{(n)} = \begin{cases} 2^{-k} & \text{if } k \leqslant n \\ 0 & \text{if } k > n. \end{cases}$$

Clearly, $\boldsymbol{x}^{(n)} \in c_{00}(\mathbb{N})$ for any $n \in \mathbb{N}$. Therefore the family $\left(\boldsymbol{x}^{(n)}\right)_n$ is a sequence of elements of $c_{00}(\mathbb{N})$ (it is a sequence of sequences...) and if $m < n$

$$\left\|\boldsymbol{x}^{(n)} - \boldsymbol{x}^{(m)}\right\|_1 = \sum_{k=1}^{\infty} \left|x_k^{(n)} - x_k^{(m)}\right| = \sum_{k=m+1}^{n} 2^{-k}.$$

$\left(\boldsymbol{x}^{(n)}\right)_n$ is a Cauchy sequence in $(c_{00}(\mathbb{N}), \|\cdot\|_1)$. However, it does not converge in $(c_{00}(\mathbb{N}), \|\cdot\|_1)$.

Proof

[redacted]

Lemma 2.2

Let $(X, \|\cdot\|)$ be a given normed space and let $(x_n)_n$ be a Cauchy sequence in X. If $(x_n)_n$ admits a limit point then it is convergent.

Definition 2.2

A normed space $(X, \|\cdot\|)$ is said to be **complete** if any Cauchy sequence is convergent in X. A complete normed space $(X, \|\cdot\|)$ is called a **Banach space**.

The most fundamental example of complete normed space is the set of real numbers.

Theorem 2.1

If $X = \mathbb{R}$ is endowed with the usual norm $|\cdot|$ for any $x, y \in \mathbb{R}$ then $(\mathbb{R}, |\cdot|)$ is a complete normed space. In other words, any Cauchy sequence in $\mathbb{R}$ is convergent.

Definition 2.3

Let $(X, \|\cdot\|)$ be a normed space and let $(x_n)_n \in X$ be a given sequence. We define the sequence $(s_N)_N \in X$ of partial sums:

$$s_N := \sum_{n=1}^{N} x_n, \qquad N \in \mathbb{N}.$$

The series $\sum_n x_n$ is said to be convergent if there exists $x \in X$ such that

$$\lim_{N \to \infty} \|s_N - x\| = 0.$$

We write then

$$x = \sum_{n=1}^{\infty} x_n.$$

The series $\sum_n x_n$ is said to be absolutely convergent if

$$\sum_n \|x_n\| < \infty.$$

Theorem 2.2

Let $(X, \|\cdot\|)$ be a normed space. The following are equivalent:

1. $(X, \|\cdot\|)$ is a Banach space;
2. Every absolutely convergent series is convergent in $(X, \|\cdot\|)$, i.e. for any $(x_n)_n \subset X$,

$$\sum_{n=1}^{\infty} \|x_n\| < \infty \implies \left(\sum_{n=1}^{N} x_n \right)_N \quad \text{converges in } (X, \|\cdot\|).$$

Proof

$(1 \Rightarrow 2)$ Let $(x_n)_n$ be a sequence such that $\sum_n \|x_n\| < \infty$ and let

$$S_n = \sum_{k=1}^{n} x_k, \qquad n \in \mathbb{N}.$$

We need to prove that $(S_N)_N$ converges in $(X, \|\cdot\|)$. Since, by assumption, $(X, \|\cdot\|)$ is a Banach space, it is enough to show that $(S_n)_n$ is a Cauchy sequence. Thus, for $m > n \geq 1$, we compute

$$\|S_m - S_n\| = \left\| \sum_{k=1}^{m} x_k - \sum_{k=1}^{n} x_k \right\| = \left\| \sum_{k=n+1}^{m} x_k \right\|.$$

Thanks to the triangle inequality, we deduce that

$$\|S_m - S_n\| \leq \sum_{k=n+1}^{m} \|x_k\| \leq \sum_{k=n+1}^{\infty} \|x_k\| \to 0 \qquad \text{as } n \to \infty.$$

since the numerical series $\sum_k \|x_k\| < \infty$. This proves that

$$\lim_{m \to \infty, n \to \infty} \|S_m - S_n\| = 0$$

which means that $(S_n)_n$ is a Cauchy sequence. It is therefore convergent and the absolutely convergent series $\sum_k x_k$ is convergent.

$(2 \Rightarrow 1)$ Let us assume now that for any sequence $(y_n)_n \subset X$ one has

$$\sum_k \|y_k\| < \infty \implies \lim_n \sum_{k=1}^{n} y_k \quad \text{exists in } (X, \|\cdot\|) \qquad (\varnothing)$$

and let $(x_n)_n$ be a Cauchy sequence in $(X, \|\cdot\|)$. Thus, for any $\varepsilon > 0$ there is $N_\varepsilon \in \mathbb{N}$ such that

$$\|x_n - x_m\| \leq \varepsilon \qquad \forall\, n, m \geq N_\varepsilon.$$

Applying this with $\varepsilon = \frac{1}{2^k}$, $k \in \mathbb{N}$, it is easy to check that one can construct a strictly increasing sequence $(n_k)_k \subset \mathbb{N}$ of indices such that

$$\|x_{n_{k+1}} - x_{n_k}\| \leq \frac{1}{2^k} \qquad k \in \mathbb{N}.$$

Therefore, setting $y_k = x_{n_{k+1}} - x_{n_k}$, one has $\sum_k \|y_k\| \leq \sum_k \frac{1}{2^k} < \infty$. Applying $(\varnothing)$ we deduce that there exists $y \in X$ such that

$$\lim_{n \to \infty} \left\| \sum_{j=1}^{k} y_j - y \right\| = 0.$$

Observing that, for any $k \in \mathbb{N}$

$$\sum_{j=1}^{k} y_j = \sum_{j=1}^{k} (x_{n_{j+1}} - x_{n_j}) = x_{n_{k+1}} - x_{n_1},$$

one sees that $(x_{n_{k+1}})_k$ converges to $x = x_{n_1} + y$ in X. Thus, $(x_n)_n$ has a subsequence which converges. Since it is a Cauchy sequence, the whole sequence $(x)_n$ actually converges and this proves the completeness of $(X, \|\cdot\|)$. ∎

2.2 Examples revisited

Proposition 2.1

The space $\left(\ell^1(\mathbb{N}), \|\cdot\|_1\right)$ is a Banach space.

Proof

Let $\left(\boldsymbol{x}^{(n)}\right)_n$ be a Cauchy sequence in $\ell^1(\mathbb{N})$. Recall that, for any $n \in \mathbb{N}$, $\boldsymbol{x}^{(n)}$ is a sequence given by

$$\boldsymbol{x}^{(n)} = \left(\boldsymbol{x}_k^{(n)}\right)_k.$$

Since it is a Cauchy sequence, for any $\varepsilon > 0$ there is $N > 0$ such that

$$\left\|\boldsymbol{x}^{(n)} - \boldsymbol{x}^{(m)}\right\|_1 = \sum_{k=1}^{\infty} \left|\boldsymbol{x}_k^{(n)} - \boldsymbol{x}_k^{(m)}\right| < \varepsilon \qquad \forall n, m > N.$$

It is easy to see that for any $k \in \mathbb{N}$ one has

$$\left|\boldsymbol{x}_k^{(n)} - \boldsymbol{x}_k^{(m)}\right| \leq \left\|\boldsymbol{x}^{(n)} - \boldsymbol{x}^{(m)}\right\|_1 < \varepsilon$$

i.e. for any $k \in \mathbb{N}$ fixed the sequence $\left(\boldsymbol{x}_k^{(n)}\right)_n \subset \mathbb{R}$ is a Cauchy sequence in $\mathbb{R}$ endowed with its natural distance. Since $\mathbb{R}$ is complete by the previous theorem, it must converge which means that for any $k \in \mathbb{N}$

$$x_k = \lim_{n \to \infty} \boldsymbol{x}_k^{(n)}$$

exists in $\mathbb{R}$. We introduce the sequence $\boldsymbol{x} = (x_k)_k$ and let us show that $\boldsymbol{x} \in \ell^1(\mathbb{N})$ and that $\boldsymbol{x} = \lim_n \boldsymbol{x}^{(n)}$ for the $\ell^1(\mathbb{N})$ distance. To do so, we fix $\varepsilon > 0$. Because $\left(\boldsymbol{x}^{(n)}\right)_n$ is a Cauchy sequence in $\ell^1(\mathbb{N})$, there exists $N > 0$ such that

$$\left\|\boldsymbol{x}^{(n)} - \boldsymbol{x}^{(m)}\right\|_1 < \varepsilon \qquad n, m > N.$$

Let us fix $n > N$ and let us fix an integer $k_0 > 0$. Since $\lim_{m \to \infty} \boldsymbol{x}^{(m)}_k = \boldsymbol{x}_k$ for any $k \geq 1$ and since the k_0 is finite one has

$$\sum_{k=1}^{k_0} \left|\boldsymbol{x}_k^{(n)} - \boldsymbol{x}_k\right| = \sum_{k=1}^{k_0} \lim_{m \to \infty} \left|\boldsymbol{x}_k^{(n)} - \boldsymbol{x}_k^{(m)}\right| \leq \lim_{m \to \infty} \left\|\boldsymbol{x}^{(n)} - \boldsymbol{x}^{(m)}\right\|_1 < \varepsilon.$$

Since this is true for any $k_0 > 0$ and

$$\left\|\boldsymbol{x}^{(n)} - \boldsymbol{x}\right\|_1 = \lim_{k_0 \to \infty} \sum_{k=1}^{k_0} \left|\boldsymbol{x}_k^{(n)} - \boldsymbol{x}_k\right|$$

one deduces that $\|\boldsymbol{x}^{(n)} - \boldsymbol{x}\|_1 < \varepsilon$ for any $n > N$ which proves that the sequence is converging to $\boldsymbol{x}$ and in particular that $\boldsymbol{x} \in \ell^1(\mathbb{N})$. $\square$

Proposition 2.2

For any $p \in [1, \infty]$ the normed space $\left(\ell^p(\mathbb{N}), \|\cdot\|_p\right)$ is a Banach space. Moreover, $(\ell^\infty(\mathbb{N}), \|\cdot\|_\infty)$ is also a Banach space.

Proposition 2.3

For any compact interval $I \subset \mathbb{R}$, the normed space $(\mathcal{C}(I), \|\cdot\|_\infty)$ is a Banach space.

The norm $\|\cdot\|_\infty$ in this case measures the max distance between two functions in $\mathcal{C}(I)$.

Proposition 2.4

For any compact interval $I \subset \mathbb{R}$, the normed space $(\mathcal{C}(I), \|\cdot\|_1)$ is not a Banach space.

This is because $\|\cdot\|_\infty$ is stronger than $\|\cdot\|_1$. There are some functions that are continuous but converge (according to ℓ^1 norm $\|f_n - f\|_1 \xrightarrow{n\to\infty} 0$) to a discontinuous function (not in $\mathcal{C}(I)$). But if our criterion of convergence is $\|f_n - f\|_\infty \xrightarrow{n\to\infty} 0$ then we cannot find such functions.

Theorem 2.3

Fischer-Riesz Theorem. Let (S, Σ, μ) be a given measure space. For any $1 \leqslant p \leqslant \infty$, $L^p(S, \Sigma, \mu)$ is a Banach space.

Proof

To prove this we need this lemma:

Lemma 2.3

Let (S, Σ, μ) be a given measure space and let $1 \leqslant p < \infty$. Assume that $(f_n)_n \subset L^p(S, \Sigma, \mu)$ is a sequence such that

$$\sum_{n=1}^{\infty} \|f_n\|_p < \infty.$$

Then the series

$$\sum_{n=1}^{\infty} f_n$$

converges almost everywhere and in $L^p(S, \Sigma, \mu)$ which means that there exists $f \in L^p(S, \Sigma, \mu)$ such that

$$\lim_{N\to\infty} \left\| f - \sum_{n=1}^{\infty} f_n \right\|_p = 0.$$

From now on we simply write $L^p(S, \mu)$ instead of $L^p(S, \Sigma, \mu)$. For the proof we distinguish between the cases $1 \leqslant p < \infty$ and $p = \infty$.

- By virtue of the previous Lemma, in such a case any absolutely convergent series $\sum_n f_n$ in $L^p(S, \Sigma, \mu)$ is actually convergent. From this theorem we know that this proves that $\left(L^p(S, \Sigma, \mu) \, \|\cdot\|_p\right)$ is a Banach Space.

- Assume that $(f_n)_n$ is a Cauchy sequence in $L^\infty(S, \mu)$. Given an integer $k \geqslant 1$ there is an integer N_k such that
$$\|f_m - f_n\|_\infty \leqslant \frac{1}{k} \qquad \forall n, m \geqslant N_k.$$
Indeed, given $m, n \geqslant N_k$ by definition there is a null set $E_k(m, n)$ for which $|f_m(k) - f_n(x)| \geqslant \frac{1}{k}$ for $x \in S \setminus E_k(m, n)$. Taking then $E_k = \bigcup_{n,m\geqslant N_k} E_k(m, n)$ one has E_k null set and the above property holds on $S \setminus E_k$ for all $n, m \geqslant N_k$. Then we let
$$E = \bigcup_{k=1}^{\infty} E_k$$
so that $\mu(E) = 0$ and get that, for any $x \in S \setminus E$ the sequence $(f_n(x))_n$ is a Cauchy sequence in $\mathbb{R}$. Thus for any $x \in S \setminus E$,
$$\lim_n f_n(x)$$
exists and we denote it by $f(x)$. Passing to the limit in as $m \to \infty$ we obtain then that
$$|f(x) - f_n(x)| \leqslant \frac{1}{k} \qquad \forall x \in S \setminus E, \, \forall n \geqslant N_k$$

from which we see that $f \in L^\infty(S,\mu)$ with

$$\|f - f_n\|_\infty \leqslant \frac{1}{k} \qquad \forall\, n \geqslant N_k.$$

Letting then $n \to \infty$ and then $k \to \infty$ we get that

$$\lim_n \|f - f_n\|_\infty = 0.$$

This proves that $(L^\infty(S,\mu), \|\cdot\|_\infty)$ is complete.

$\square$

Proposition 2.5

Let $1 \leqslant p \leqslant \infty$ and let $(f_n)_n$ be a Cauchy sequence in $L^p(S,\mu)$ which converges to f. Then there is a subsequence (f_{n_k}) which converges μ-almost everywhere to f.

Proposition 2.6

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed spaces and assume that $(Y, \|\cdot\|_Y)$ is a Banach space. Then the space $\left(\mathscr{L}(X,Y), \|\cdot\|_{\mathscr{L}(X,Y)}\right)$.

Remember that

$$\|L\|_{\mathscr{L}(X,Y)} = \|L\|_{\mathrm{op}} = \sup_{\|x\|_X \leqslant 1} \|L(x)\|_Y \qquad \forall\, L \in \mathscr{L}(X,Y).$$

This measures how much the linear mapping "stretches" a small $x \in X$ when it maps it to Y.

Proof

[redacted]

Corollary

Let $(X, \|\cdot\|_X)$ be a Banach space and let $L \in \mathscr{L}(X)$ be such that

$$\|L\|_{\mathscr{L}(X)} < 1.$$

Then, the linear mapping $I - L$ is invertible (i.e. $I - L$ is bijective and its inverse $(I-L)^{-1} \in \mathscr{L}(X)$) and

$$(I - L)^{-1} = \sum_{n=0}^{\infty} L^n$$

where I is the identity mapping of X and L^n is defined inductively by $L^0 = I$ and $L^{n+1} = L \circ L^n = L^n \circ L$ for any $n \geqslant 0$.

Proof

[redacted]

Definition 2.4

If $(X, \|\cdot\|)$ is a normed space, we define the dual space of X denoted by $X^\star$ the space of all continuous and linear mappings $\Phi : X \to \mathbb{R}$, i.e.

$$X^\star = \mathscr{L}(X, \mathbb{R}).$$

We also denote

$$\|\Phi\|_\star = \|\Phi\|_{\mathscr{L}(X,\mathbb{R})} = \sup_{\|x\|_X \leqslant 1} |\Phi(x)| \qquad \forall\, \Phi \in X^\star.$$

Corollary

If $(X, \|\cdot\|)$ is a normed vector space then $(X^\star, \|\cdot\|_\star)$ is a Banach space.

Theorem 2.4

Riesz representation theorem. Given $1 < p < \infty$ and $\Phi \in (L^p(S,\mu))^\star$ there exists a unique $g \in L^q(S,\mu)$ (with $\frac{1}{p} + \frac{1}{q} = 1$) such that

$$\Phi(f) = \int_S fg \, \mathrm{d}\mu \qquad \forall f \in L^p(S,\mu).$$

Moreover,

$$\|\Phi\|_\star = \|g\|_q .$$

If (S, Σ, μ) is σ-finite, then the result is still true for $p = 1$.

2.3 Simple consequences of completeness

Proposition 2.7

Let $(X, \|\cdot\|)$ be a Banach space and let $(x_n)_n \subset X$ be such that

$$\sum_{n=1}^{\infty} \|x_n\| < \infty.$$

Then the series $\sum_{n=1}^{\infty} x_n$ converges in X, i.e. there exists $x \in X$ such that

$$\lim_{N\to\infty} \left\| \sum_{n=1}^{N} x_n - x \right\| = 0.$$

Definition 2.5

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed spaces and $k \in (0,1)$. A function $f : X \to Y$ is said to be a k-contraction mapping if

$$\|f(x) - f(y)\|_y \leqslant k \|x - y\|_X \qquad \forall x, y \in X \times X.$$

It is obvious that any k contraction mapping is continuous. This is clear that a k-contraction mapping is an application which "contracts" distances: the images of x and y through f are closer to each other than x, y.

Theorem 2.5

Banach fixed point theorem. Let $(X, \|\cdot\|)$ be a complete normed space and let $f : X \to X$ be a k-contraction mapping with $k \in (0,1)$. Then there exists a unique fixed point $\boldsymbol{a} \in X$ for f, i.e. there is a unique $\boldsymbol{a} \in X$ such that

$$f(\boldsymbol{a}) = \boldsymbol{a}.$$

Proof

We first prove that f can admit only one fixed point. Assume $\boldsymbol{a}, \boldsymbol{b}$ are both fixed points of f. Since f is a k contraction mapping we have

$$\begin{gathered} \|f(\boldsymbol{a}) - f(\boldsymbol{b})\| \leqslant k \|\boldsymbol{a} - \boldsymbol{b}\| \\ \Downarrow \\ \|\boldsymbol{a} - \boldsymbol{b}\| \leqslant k \|\boldsymbol{a} - \boldsymbol{b}\| . \end{gathered}$$

Since $k > 0$ and $\|\boldsymbol{a}, \boldsymbol{b}\| \geqslant 0$ one sees necessarily that $\|\boldsymbol{a} - \boldsymbol{b}\| = 0$ i.e. $\boldsymbol{a} = \boldsymbol{b}$.

To prove now the existence of some fixed point, one proves actually that the sequence defined in the above statement converges to some fixed-point of f. Let then $x_0 \in X$ be given and define inductively

$$x_{n+1} = f(x_n).$$

Since f is a k contraction mapping we have

$$\|x_{n+1} - x_n\| \leqslant k \, \|x_n - x_{n-1}\|$$

for any $n \geqslant 1$ and we easily deduce that

$$\|x_{n+1} - x_n\| \leqslant k^n \, \|x_1 - x_0\| \qquad \forall \, n \in \mathbb{N}.$$

If $m > n \geqslant 1$ are given, one deduces from the above inequality together with the triangle inequality that

$$\|x_n - x_m\| \leqslant \sum_{j=n}^{m-1} \|x_j - x_{j+1}\| \leqslant \|x_1 - x_0\| \sum_{j=n}^{m-1} k^j.$$

Since the geometric series $\sum_{j=1}^{\infty} k^j$ is convergent, one has

$$\lim_{n,m \to \infty} \sum_{j=n}^{m-1} k^j = 0$$

and therefore the sequence $(x_n)_n$ is a Cauchy sequence in X. Since X is complete, $(x_n)_n$ converges to some limit $\boldsymbol{a} \in X$. Moreover, being k-contracting, f is continuous so that the sequence $(f(x)_n)_n$ converges to $f(\boldsymbol{a})$. Since $x_{n+1} = f(x_n)$ one has $\boldsymbol{a} = f(\boldsymbol{a})$ and the result is proven. □

Notice tat the above inequality provides the convergence rate of $(x_n)_n$ to $\boldsymbol{a}$. Indeed, taking the limit $m \to \infty$ in we get

$$\|x_n - \boldsymbol{a}\| \leqslant \|x_1 - x_0\| \sum_{j=n}^{\infty} k^j = \frac{k^n}{1-k} \|x_1 - x_0\| \, s \qquad \forall \, n \in \mathbb{N}.$$

2.4 Fundamental Properties of Banach spaces

Definition 2.6

A normed space $(X, \|\cdot\|)$ is said to have the Baire property if the intersection of any sequence of dense open sets of X is dense in X, i.e. for any $(U_n)_n$ open subsets of X with $\overline{U_n} = X$ for any $n \in \mathbb{N}$ it holds

$$\overline{\bigcap_n U_n} = X.$$

The Baire property is clearly equivalent to the following property: the union of any sequence of closed substes with empty interior has an empty interior. In particular, the theorem is often used in the following form: if $(C_n)_n$ is a sequence of closed subsets such that

$$\text{Int}\left(\bigcup_{n \in \mathbb{N}} C_n\right) = X$$

Then there exists some $n_0 \in \mathbb{N}$ such that $\text{Int}(C_{n_0}) \neq \varnothing$.

Theorem 2.6

Any complete normed space $(X, \|\cdot\|)$ has the Baire property.

Proof

Let $(U_n)_n$ be a sequence of open subsets of $(X, \|\|)$ with $\overline{U_n} = X$ for any $n \in \mathbb{N}$ and let $A = \bigcup_{n\in\mathbb{N}} U_n$. To prove that A is dense in X we shall use the previous lemma and prove that

$$A \cap \mathcal{O} \neq \varnothing$$

for any non-empty open subset $\mathcal{O}$ of X. Pick any $x_0 \in \mathcal{O}$ and let $r_0 > 0$ be such that

$$B_c(x_0, r_0) \subset \mathcal{O}$$

Since U_1 is dense, one has that $B(x_0, r_0)$ is open and non-empty. So we can pick $x_1 \in B(x_0, r_0) \cap U_1$ and $r_1 > 0$ such that

$$B_c(x_1, r_1) \subset B(x_0, r_0) \cap U_1 \qquad 0 < r_1 < \frac{r_0}{2}.$$

We iterate this reasoning and construct a sequence $(x_n)_n \subset X$ and $(r_n)_n \subset (0, \infty)$ such that

$$B_c(x_{n+1}, r_{n+1}) \subset B(x_n, r_n) \cap U_{n+1} \qquad \text{with } 0 < r_{n+1} < \frac{r_n}{2} \qquad \forall n \in \mathbb{N}$$

In particular, one sees that $r_n < \frac{1}{2^n}$ so that the sequence $(x_n)_n$ is a Cauchy sequence. Since X is complete, there is $x \in X$ such that $\lim_n x_n = x$. Since $x_{n+k} \in B(x_n, r_n)$ for any $n \in \mathbb{N}$ and any $k \in \mathbb{N}$ taking the limit as $k \to \infty$ we get that

$$x \in \overline{B(x_n, r_n)} \subset B_c(x_n, r_n) \qquad \forall n \in \mathbb{N}.$$

In particular, $x \in \mathcal{O} \cap A$ which is non empty and this proves the result. $\square$

Theorem 2.7

Banach-Steinhaus Theorem. Let $(X, \|\cdot\|_X)$ be a Banach space and let $(Y, \|\cdot\|_Y)$ be a normed space. Let $(T_i)_{i\in I} \subset \mathscr{L}(X, Y)$ be a given collection of continuous linear applications. Assume that, for any $x \in X$, there exists $M_x > 0$ such that

$$\sup_{i\in I} \|T_i(x)\|_Y \leqslant M_x.$$

Then, there exists $M > 0$ such that for any $x \in X$ and any $i \in I$ it holds

$$\|T_i(x)\|_Y \leqslant M \|x\|_X$$

i.e. $\sup_{i\in I} \|T_i\|_{\mathscr{L}(X,Y)} \leqslant M \leqslant \infty$.

The Uniform Boundedness Principle can be reformulated as follows: let $(X, \|\cdot\|_X)$ be a Banach space and let $(Y, \|\cdot\|_Y)$ be a normed space. Let $\mathscr{F} \subset \mathscr{L}(X, Y)$ be a given collection of continuous linear applications (here above $\mathscr{F} = (T_i)_{i\in I}$). Then the following are equivalent:

- Pointwise boundedness: for every $x \in X$, the set $\{T(x); T \in \mathscr{F}\}$ is bounded in Y, i.e.

$$\sup_{T\in\mathscr{F}} \|T(x)\|_Y = M_x < \infty \qquad \forall x \in X.$$

- Uniform boundedness: the operator norms $\left\{\|T\|_{\mathscr{L}(X,Y)}; T \in \mathscr{F}\right\}$ are bounded, i.e.

$$\sup_{T\in\mathscr{F}} \|T\|_{\mathscr{L}(X,Y)} = M < \infty.$$

Proof

For every $n \in \mathbb{N}$ set

$$X_n = \{x \in X; \|T_i(x)\|_Y \leqslant n \forall i \in I\}.$$

Since for any $i \in I, T$, T_i is continuous, one sees that X_n is closed as the intersection of the closed subsets of X. Moreover, according to ✖,

$$X = \bigcup_n X_n.$$

It follows from Baire theorem that there exists $n_0 \in \mathbb{N}$ such that Int $(X_{n_0}) \neq \varnothing$. Pick then $x_0 \in X$ and $r > 0$ so that $B(x_0, r) \subset X_{n_0}$. By definition it holds

$$\|T_1(x_0 + rz)\|_Y \leqslant n_0 \qquad \forall\, i \in I; \forall\, z \in B(0,1).$$

By linearity we get

$$\|T_i(z)\|_Y \leqslant \frac{1}{r}(\|T_i(x_0 + r_z)\|_Y + \|T_i(x_0)\|_Y) \leqslant 2\frac{n_0}{r} \qquad \forall\, i \in I; \forall\, z \in B(0,1).$$

This clearly gives the result with $M = 2\frac{n_0}{r}$. □

Theorem 2.8

Open mapping theorem. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two Banach spaces and let

$$T : X \to Y$$

be a continuous linear application which is surjective (i.e. $T \in \mathscr{L}(X, Y)$ is onto). Then T maps open sets of X into open sets of Y, i.e. if $U \subset X$ is an open set then $T(U)$ is an open set of Y.

Corollary

Let $(X_i, \|\cdot\|_i)$ with $i = 1, 2$ be two Banach spaces and let $T : X_1 \to X_2$ be a continuous linear application which is bijective. Then its inverse T_{-1} is also continuous, i.e. $T^{-1} \in \mathscr{L}(X_2, X_1)$.

Proposition 2.8

Let $(X, \|\cdot\|)$ be a Banach space and let $\|\cdot\|_0$ be a norm on X such that

- $(X, \|\cdot\|_0)$ is a Banach space;
- there exists $C_0 > 0$ such that

$$\|x\|_0 \leqslant C_0 \|x\| \qquad \forall\, x \in X_0.$$

Then, the two norms $\|\cdot\|$ and $\|\cdot\|_0$ are equivalent.

So if a Banach norm dominates another Banach norm, then they are equivalent.

Theorem 2.9

Closed graph theorem. Let $(X_i, \|\cdot\|_i)$ with $i = 1, 2$ be two Banach spaces and let $T : X_1 \to X_2$ be a linear application. Assume that the graph of T

$$\mathscr{G}(T) = \{(x_1, T(x_1)) \in X_1 \times X_1\ x_1 \in X_1\}$$

is closed in $X_1 \times X_2$ (endowed with the norm $\|(x_1, x_s)\|_{\max} = \max\{\|x_1\|_1, \|x_2\|_2\}$). Then,

$$T \in \mathscr{L}(X_1, X_2).$$

Proof

Consider on X_1 the norm

$$\|x\|_T = \|x\|_1 + \|T(x)\|_2 \qquad x \in X_1.$$

One checks that $\|\cdot\|_T$ is indeed a norm. Moreover, since $\mathscr{G}(T)$ is closed in $X_1 \times X_2$, one can check that $(X_1, \|\cdot\|_T)$ is a Banach space. Moreover, one clearly has

$$\|x\|_1 \leqslant \|x\|_T \qquad \forall x \in X_1.$$

Then, the previous proposition asserts that the norms $\|\cdot\|_1$ and $\|\cdot\|_T$ are equivalent norms on X_1, so there exists

$$c > 0 \quad \text{such that } \|x\|_T \leqslant x \|x\|_1 \qquad \forall x \in X_1.$$

This is enough to prove the continuity of T. □

2.5 Complements

Definition 2.7

Let $A \in \mathscr{L}(X)$. We say that A is a compact operator if the image of $B_c(0,1)$ through A is contained in a compact set of X, i.e. for any sequence $(\boldsymbol{x}_n)_n \subset X$ with $\|x_n\| \leqslant 1$, the sequence $(A(\boldsymbol{x}_n))_n$ admits a subsequence which converges. We shall denote by $\mathscr{K}(X)$ the collection of all compact operators in X.

Definition 2.8

Let $A \in \mathscr{L}(X)$. We say that A is of finite rank if the image of A, $R(A) = \{A\boldsymbol{x}, \boldsymbol{x} \in X\}$ is a linear subspace of finite dimension X.

Lemma 2.4

If $A \in \mathscr{K}(X)$ and $B \in \mathscr{L}(X)$ then $B \circ A$ and $A \circ B$ belong to $\mathscr{K}(X)$.

Proposition 2.9

Let $(A_n)_n \subset \mathscr{K}(X)$ and let $A \in \mathscr{L}(X)$ be given with

$$\lim_{n\to\infty} \|A_n - A\|_{\mathscr{L}(X)} = 0.$$

Then, $A \in \mathscr{K}(X)$.

2.6 More properties of the dual space

Theorem 2.10

Zorn's lemma. Every non empty ordered set that is inductive has a maximal element.

Theorem 2.11

Helly, Hahn-Banach analytic form. Let X be a vector space over $\mathbb{R}$. Let $p : X \to \mathbb{R}$ be a function satisfying

1. $p(\lambda x) = \lambda p(x)$ for any $x \in X$ and any $\lambda > 0$

2. $p(x + y) \leqslant p(x) + p(y)$ for any $x, y \in X$.

Let $Y \subset X$ be a linear subspace and $g : Y \to \mathbb{R}$ a linear function such that

3 $g(x) \leqslant p(x)$ for any $x \in Y$.

Then, there exists a linear function $f : X \to \mathbb{R}$ such that

$$f(x) = g(x) \qquad \forall x \in Y$$

(we say then that f extends g to X) and

$$f(x) \leqslant p(x) \qquad \forall x \in X.$$

Corollary

Let $(X, \|\cdot\|)$ be a normed space and let $Y \subset X$ be a linear subspace. If $g : Y \to \mathbb{R}$ is a continuous linear mapping, then there exsts $f \in X^\star$ that extends g and such that

$$\|f\|_\star = \|g\|_Y$$

Corollary

Let $(X, \|\cdot\|)$ be a normed space. For every $x_0 \in X$ there exists $\Phi_0 \in X^\star$ such that

$$\|\Phi_0\|_\star = \|x_0\|$$

and

$$\Phi_0(x_0) = \|x_0\|^2 .$$

3 Inner product spaces and Hilbert spaces

3.1 General properties

Definition 3.1

Let H be a given vector space. An inner product $\langle \cdot, \cdot \rangle$ is a mapping from $H \times H$ with values in $\mathbb{R}$ with the following properties:

1. **symmetry**: $\langle x, y \rangle = \langle y, x \rangle$ for $\forall\, x, y \in H$;
2. **bilinearity**: $\langle x, \alpha y + \beta z \rangle = \alpha \langle x, y \rangle + \beta \langle x, z \rangle$ for $\forall\, x, y, z \in H$ and $\forall\, \alpha, \beta \in \mathbb{R}$;
3. $\langle x, x \rangle \geqslant 0$ for $\forall\, x \in H$;
4. $\langle x, y \rangle = 0$ if and only if $x = 0$.

If H is endowed with an inner product, we say that $(H, \langle \cdot, \cdot \rangle)$ is an inner product space.

Proposition 3.1

If $(H, \langle \cdot, \cdot \rangle)$ is an inner product space then it is a norm space with respect to the norm given by

$$\|x\| = \sqrt{\langle x, x \rangle}.$$

The norm $\|\cdot\|$ is called the norm induced by the inner product $\langle \cdot, \cdot \rangle$ and it satisfies

$$\|x \pm y\|^2 = \|x\|^2 + \|y\|^2 \pm 2\langle x, y \rangle$$

and the parallelogram law

$$\|x + y\|^2 + \|x - y\|^2 = 2\,\|x\|^2 + 2\,\|y\|^2 .$$

Proposition 3.2

Cauchy-schwartz inequality. If $(H, \langle \cdot, \cdot \rangle)$ is an inner product space and $\|\cdot\|$ defines the inner norm then it holds

$$|\langle x, y \rangle| \leqslant \|x\|\,\|y\| .$$

Proposition 3.3

Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space and let $(x_n)_n$ and $(y_n)_n$ be two converging sequencer (in the inner norm) to x and y. Then

$$\lim_{n\to\infty} \langle x_n, y_n \rangle = \langle x, y \rangle.$$

3.2 Orthogonality

Definition 3.2

Let $(H, \langle\cdot,\cdot\rangle)$ be a given inner product space. Two vector $x, y \in H$ are said to be **orthogonal** if $\langle x, y\rangle = 0$. Given two linear subspaces $M, N \subset H$ we say that $M \perp N$ if $\langle x, y\rangle = 0$ for $\forall\, x \in M, \forall\, y \in N$.

Definition 3.3

Let $(H, \langle\cdot,\cdot\rangle)$ be a inner product space. A finite family $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_N\} \subset H$ is said to be **orthonormal** if

$$\|\boldsymbol{e}_k\| = 1 \qquad \forall\, k = 1, \ldots, N$$

and

$$\langle \boldsymbol{e}_m, \boldsymbol{e}_n\rangle = 0 \qquad \forall\, n \neq m.$$

Proposition 3.4

Gram-Schmidt Procedure. Let $(H, \langle\cdot,\cdot\rangle)$ be a inner product space. Then

1. any orthonormal set $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_N\} \subset H$ is linearly independent;
2. given a linearly independent subset $\{v_1, \ldots, v_N\}$ of H and given $S = \mathrm{span}(v_1, \ldots, v_N)$ then there exists an orthonormal basis of S $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_N\}$.

Proof

[redacted]

Definition 3.4

Let $(H, \langle\cdot,\cdot\rangle)$ be an inner product space and let M be a subset of H. Then the **orthogonal complement** of M is defined as

$$M^\perp = \{x \in H; \langle x, u\rangle = 0 \quad \forall u \in M\}.$$

In particular, one sees that $M^\perp \perp M$.

Proposition 3.5

Let $(H, \langle\cdot,\cdot\rangle)$ be an inner product space and let $M \subset H$. Then

1. $0 \in M^\perp$;
2. if $0 \in M$ then $M \cap M^\perp = \{0\}$, otherwise $M \cap M^\perp = \varnothing$;
3. $\{0\}^\perp = H$ and $H^\perp = \{0\}$;
4. if M is a non-empty subset of H then $M^\perp = 0$;
5. if $N \subset M$ then $N^\perp \subset M^\perp$;
6. $M^\perp$ is a closed linear subspace of H;
7. $M \subset \left(M^\perp\right)^\perp$.

Proposition 3.6

Let M be a linear subspace of an inner product space $(H, \langle\cdot,\cdot\rangle)$. Then

$$x \in M^\perp \iff \|x - y\| \geqslant \|y\| \qquad \forall y \in M.$$

3.3 Hilbert Spaces and Projection Theorem

Definition 3.5

Let $(H, \langle\cdot,\cdot\rangle)$ be an inner product space and let $\|\cdot\|$ be the associated inner norm. If $(H, \|\cdot\|)$ is complete then $(H, \langle\cdot,\cdot\rangle)$ is called a **Hilbert space**.

Definition 3.6

Let X be a given vector space. A subset $C \subset X$ is convex if for any $x, y \in C$ one has

$$tx + (1-t)y \in C \qquad \forall t \in [0,1].$$

Linear subspaces of vector spaces are always convex.

Theorem 3.1

Projection over closed subspaces. Let $(H, \langle\cdot,\cdot\rangle)$ be a given Hilbert space and let $K \subset H$ be closed and convex.