

# This Should Help Your Lazy Ass In Analysis B

Charles A. Mongus, world famous jazz composer and bassist

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## 1 Normed Spaces

### Proposition 1.1

A subset  $C \subset X$  is a closed subset of  $(X, \|\cdot\|)$  whenever the limit of any convergent sequence  $(x_n)_n \subset C$  belongs to  $C$ .

In other words,  $C$  is closed if and only if once a sequence of elements of  $C$  is converging, the limit cannot escape  $C$ .

### Proof

①  $C$  closed and  $(x_n)_n \subset C \implies \lim_n x_n = x \in C$ .

We argue by contradiction that  $x \notin C$ : then  $x$  must be in  $X \setminus C$  which must be open. But if that is an open set then there must exist a ball of  $x$  for some  $r > 0$  such that  $B(x, r) \subset X \setminus C$ . But due to the definition of convergence there must exist a  $N$  such that  $\|x_n - x\| < r \forall n \geq N$ , but then that  $x_n$  should be outside  $C$  which is a contradiction!

②  $\lim_n x_n = x \in C \implies C$  closed and  $(x_n)_n \subset C$ .

We argue by contradiction that  $C$  is not closed, so  $X \setminus C$  must be not open. Not open sets are such that there exist  $x \in X \setminus C$  such that for every  $r > 0$  you can't find a ball that it is entirely in  $X \setminus C$  which means  $B(x, r) \cap C \neq \emptyset$ . Then pick  $x_n \in B(x, \frac{1}{n}) \cap C$  so that every  $x_n$  also belongs to  $C$ . Clearly this is a sequence of  $n$  such that for any  $n \geq 1$  we have  $\|x_n - x\| < \frac{1}{n}$  and this means that  $\lim_n x_n = x$ . We picked  $x \in X \setminus C$  so we have found a limit of a sequence of elements of  $C$  that doesn't belong to  $C$  which is a contradiction!

**Lemma 1.1**

**Young's inequality.** Let  $p > 1$  and let  $q < 1$  be its conjugate exponent. Then, for any nonnegative  $a, b \in \mathbb{R}$  it holds

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q.$$

**Theorem 1.1**

**Holder Inequality.** Let  $1 \leq p \leq \infty$  and  $\frac{1}{p} + \frac{1}{q} = 1$ . Assume that  $f \in L^p(S, \mu)$  and  $g \in L^q(S, \mu)$ . Then  $f \cdot g \in L^1$  and

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

**Proof**

The proof is trivial if  $p = 1$  and  $q = 1$ . Remember Young's inequality

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q.$$

Now let's say  $a = |f(s)|$  and  $b = |g(s)|$ . Now our inequality becomes

$$|f(s)g(s)| \leq \frac{1}{p}|f(s)|^p + \frac{1}{q}|g(s)|^q \quad \text{for } \mu\text{-a.e. on } S.$$

Now we integrate over  $S$  and we get

$$\begin{aligned} \int_S |f(s)g(s)| &\leq \frac{1}{p} \int_S |f(s)|^p + \frac{1}{q} \int_S |g(s)|^q \\ \implies \|fg\|_1 &\leq \frac{1}{p} \|f\|_p^p + \frac{1}{q} \|g\|_q^q \end{aligned}$$

which means that  $\|fg\|_1$  is finite and therefore  $fg \in L^1(S, \mu)$ . To end the proof let's substitute  $f$  with  $\lambda f$ ,  $\forall \lambda > 0$ . We get

$$\begin{aligned} \lambda \|fg\|_1 &\leq \frac{\lambda^p}{p} \|f\|_p^p + \frac{1}{q} \|g\|_q^q \quad \forall \lambda > 0 \\ \implies \frac{\lambda}{\lambda} \|fg\|_1 &\leq \frac{\lambda^p}{p\lambda} \|f\|_p^p + \frac{1}{q\lambda} \|g\|_q^q \quad \forall \lambda > 0 \\ \implies \|fg\|_1 &\leq \frac{\lambda^{p-1}}{p} \|f\|_p^p + \frac{1}{\lambda q} \|g\|_q^q \quad \forall \lambda > 0. \end{aligned}$$

Now choose  $\lambda = \frac{1}{\|f\|_p} \cdot \|g\|_q^{\frac{p}{q}}$ . When we substitute this value for  $\lambda$ , we get:

$$\frac{\lambda^p}{p} \|f\|_p^p = \frac{\left(\frac{1}{\|f\|_p} \cdot \|g\|_q^{\frac{p}{q}}\right)^p}{p} \|f\|_p^p.$$

Expanding  $\left(\frac{1}{\|f\|_p} \cdot \|g\|_q^{\frac{p}{q}}\right)^p$ , we get:

$$\left(\frac{1}{\|f\|_p} \cdot \|g\|_q^{\frac{p}{q}}\right)^p = \frac{\|g\|_q^p}{\|f\|_p^p}.$$

Substituting this, we have:

$$\frac{\lambda^p}{p} \|f\|_p^p = \frac{\frac{\|g\|_q^p}{\|f\|_p^p}}{p} \|f\|_p^p = \frac{\|g\|_q^p}{p}.$$

Substitute back into the inequality:

$$\lambda \|fg\|_1 \leq \frac{\|g\|_q^p}{p} + \frac{1}{q} \|g\|_q^q.$$

Since  $\frac{\|g\|_q^p}{p} + \frac{\|g\|_q^q}{q} = \|g\|_q^p \left( \frac{1}{p} + \frac{1}{q} \right)$  and  $\frac{1}{p} + \frac{1}{q} = 1$ , we get:

$$\lambda \|fg\|_1 \leq \|g\|_q^p.$$

Dividing by  $\lambda = \frac{1}{\|f\|_p} \|g\|_q^{\frac{p}{q}}$ :

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

This is Hölder's inequality:

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

□

### Theorem 1.2

For any  $1 \leq p \leq \infty$ ,  $L^p(S, \mu)$  is a vector space and  $\|\cdot\|_p$  is a norm.

#### Remark

For  $1 < p < \infty$  the triangular inequality

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p \quad \forall f, g \in L^p(S, \mu)$$

is known as **Minkowski's Inequality**.

#### Proof

We already know that if  $f \in L^p(S, \mu)$  then  $\lambda f \in L^p(S, \mu)$ . Homogeneity and uniqueness are also existent for  $\|\cdot\|_p$  so in order to show that  $L^p(S, \mu)$  is a vector space we only need to prove that if  $f, g \in L^p(S, \mu)$  then  $f + g \in L^p(S, \mu)$  and  $\|\cdot\|_p$  is a norm.

Fix  $f, g \in L^p(S, \mu)$ . We know that for any  $x, y \in \mathbb{R}$  we get

$$\left| \frac{1}{2}x + \frac{1}{2}y \right|^p \leq \frac{1}{2}|x|^p + \frac{1}{2}|y|^p$$

since this mapping  $r \rightarrow r^p$  is convex. This also means that

$$|x + y|^p \leq 2^{p-1} (|x|^p + |y|^p).$$

and this implies in particular that

$$|f(s) + g(s)|^p \leq 2^{p-1} (|f(s)|^p + |g(s)|^p) \quad \text{for } \mu\text{-a.e. on } s \in S.$$

If we integrate over  $S$  we get:

$$\int_S |f(s) + g(s)|^p \leq 2^{p-1} \left( \int_S |f(s)|^p + \int_S |g(s)|^p \right)$$

which means

$$\|f + g\|_p^p \leq 2^{p-1} (\|f\|_p^p + \|g\|_p^p)$$

which means that  $f + g \in L^p(S, \mu)$ .

We now must prove the Minkowski's inequality. We know that

$$\|f + g\|_p^p = \int_S |f + g|^p d\mu = \int_S |f + g| |f + g|^{p-1} d\mu$$

but since we know that  $|f + g| \leq |f| + |g|$  then

$$\|f + g\|_p^p \leq \int_S |f| |f + g|^{p-1} d\mu + \int_S |g| |f + g|^{p-1} d\mu.$$

Call  $\psi = |f + g|^{p-1}$ . It clearly belongs to  $L^q(S, \mu)$  because

$$|\psi|^q = (|f + g|^{p-1})^q = |f + g|^p \quad \text{since } q(p-1) = p$$

so

$$\|\psi\|_q = \left( \int_S |\psi|^q \right)^{\frac{1}{q}} = \left( \int_S |f + g|^p \right)^{\frac{1}{q}} = \|f + g\|_p^{\frac{p}{q}} < \infty$$

And this means that  $|\psi|^q \in L^1(S, \mu) \implies \psi \in L^q(S, \mu)$ . We also know that  $|f| \in L^p(S, \mu)$  so we can apply Holder's inequality with  $f$  and  $\psi$  so that

$$\int_S |f| |f + g|^{p-1} d\mu = \|f\psi\|_1 \leq \|f\|_p \|\psi\|_q = \|f\|_p \|f + g\|_p^{\frac{p}{q}}$$

and

$$\int_S |g| |f + g|^{p-1} d\mu \leq \|g\|_p \|f + g\|_p^{\frac{p}{q}}$$

So that

$$\|f + g\|_p^p \leq \|f\|_p \|f + g\|_p^{\frac{p}{q}} + \|g\|_p \|f + g\|_p^{\frac{p}{q}}.$$

Dividing by  $\|f + g\|_p^{\frac{p}{q}} \neq 0$  (otherwise the proof is trivial) we get

$$\|f + g\|_p^{p-\frac{p}{q}} \leq \|f\|_p + \|g\|_p.$$

□

### Proposition 1.2

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability triplet. then the following holds:

$$L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \subset L^p(\Omega, \mathcal{F}, \mathbb{P}) \subset L^q(\Omega, \mathcal{F}, \mathbb{P}) \subset L^1(\Omega, \mathcal{F}, \mathbb{P}).$$

Of course, this result remains valid for every other measure space  $(S, \mathcal{S}, \mu)$  as long as  $\mu(S) < \infty$ . In the special case in which

$$S = \mathbb{N} \quad \mathcal{F} = \mathcal{P}(\mathbb{N})$$

and  $\mu(A)$  is the counting measure  $\mu(A) = \sum_{k \in A} \delta_k(A)$ ,  $A \in \mathbb{N}$  then knowing that sequences  $n \mapsto f(n)$  can be identified as functions over  $\mathbb{N}$  of the type  $f : \mathbb{N} \rightarrow \mathbb{R}$  we see that

$$L^1(S, \mu) = \underbrace{\mathcal{L}^1(S, \mu)}_{\text{actual functions, not equivalence classes}} = \ell^1(\mathbb{N}) = \left\{ \mathbf{x} = (x_n)_n; \|\mathbf{x}\|_1 := \sum_{n=1}^{\infty} |x_n| < \infty \right\}.$$

This means that  $\ell^1(\mathbb{N})$  is a  $L^1$  space for some special choice of  $S$  and  $\mu$ . Since we chose our measure as the counting measure, we get

$$\int_{\mathbb{N}} |f(n)| d\mu(n) = \sum_{n=1}^{\infty} |f(n)| = \sum_{n=1}^{\infty} |x_n|.$$

Cool!

### Proposition 1.3

Let  $p \geq 1$  be given. We define the set

$$\ell^p(\mathbb{N}) = \left\{ \mathbf{x} = (x_n)_{n \in \mathbb{N}} \subset \mathbb{R} \text{ such that } \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}.$$

Then,  $\ell^p(\mathbb{N})$  is a vector space. Moreover, if

$$\|\mathbf{x}\|_p := \left( \sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}} \quad \forall \mathbf{x} = (x_n)_n \in \ell^p(\mathbb{N})$$

then  $(\ell^p(\mathbb{N}), \|\cdot\|_p)$  is a normed space.

## 1.1 The space of linear applications

### Proposition 1.4

Let  $(X, \|\cdot\|_X)$  and  $(Y, \|\cdot\|_Y)$  be two normed spaces and let  $L: X \mapsto Y$  be a linear application. The following are equivalent:

- ①  $L$  is continuous on  $X$ ;
- ②  $L$  is continuous at  $x = 0$ ;
- ③ there is a positive constant  $C > 0$  such that

$$\|L(x)\|_Y \leq C \|x\|_X \quad \forall x \in X.$$

### Proof

Of course ①  $\Rightarrow$  ②. But now let's prove ③. Consider the definition of continuity at 0 for the function  $L$ : this means that for any  $\epsilon > 0$  there exists a constant  $\delta > 0$  such that

$$\|x - 0\|_X \leq \delta \Rightarrow \|L(x) - L(0_Y)\|_Y \leq \epsilon.$$

We know that  $L$  is linear, so  $L(0_X) = 0_Y$  and therefore we get

$$\|x\|_X \leq \delta \Rightarrow \|L(x)\|_Y \leq \epsilon.$$

Choose  $\epsilon = 1$  so that we get

$$\|x\|_X \leq \delta \Rightarrow \|L(x)\|_Y \leq 1.$$

Let  $x \in X \setminus \{0_X\}$  and set  $y = \frac{\delta}{\|x\|_X} x$ . Now we have that

$$\|y\|_X = \left\| \frac{\delta}{\|x\|_X} x \right\|_X = \frac{\delta}{\|x\|_X} \|x\|_X = \delta$$

and, due to the linearity of  $L$ ,

$$\|L(y)\|_Y = \left\| L\left(\frac{\delta}{\|x\|_X} x\right) \right\|_Y = \frac{\delta}{\|x\|_X} \|L(x)\|_Y = \frac{\delta}{\|x\|_X} \|L(x)\|_Y \leq 1 \Rightarrow \|L(x)\|_Y \leq \frac{1}{\delta} \|x\|_X.$$

So, setting  $C = \frac{1}{\delta}$  proves ③. We still need to prove ①  $\Rightarrow$  ③. It's easy. Since by linearity we have

$$\|L(x) - L(y)\|_Y = \|L(x - y)\|_Y \leq C \|x - y\|_X \quad \forall x, y \in X$$

and this clearly implies the continuity of  $L$  at any  $x \in X$  (actually, the uniform continuity).  $\square$

### Definition 1.1

If  $(X, \|\cdot\|_X)$  and  $(Y, \|\cdot\|_Y)$  are two normed spaces, we denote by  $\mathcal{L}(X, Y)$  the space of continuous linear applications from  $X$  to  $Y$ . If  $X = Y$  we simply denote  $\mathcal{L}(X) = \mathcal{L}(X, X)$ .

### Proposition 1.5

If  $(X, \|\cdot\|_X)$  and  $(Y, \|\cdot\|_Y)$  are two vector spaces and  $X$  is of finite dimension, any linear application  $L: X \mapsto Y$  is continuous.

### Remark

If  $\dim(X) = n$  and  $\dim(Y) = p$ , the space  $\mathcal{L}(X, Y)$  can be identified with the space  $\mathcal{M}_{n \times p}(\mathbb{R})$  of matrices with  $n$  lines and  $p$  rows.

## 1.2 Compactness

### Definition 1.2

Let  $(X, \|\cdot\|_X)$  be a normed space and let  $K \subset X$ . We say that  $K$  is **compact** if every sequence  $(x_n)_n$  contains a subsequence which converges to some  $x \in K$ .

Of course if  $K$  is compact then it is closed.

### Lemma 1.2

If  $K$  is a compact subset of a normed space  $(X, \|\cdot\|_X)$  then  $K$  is closed and there exists  $M > 0$  such that  $\sup_{x \in K} \|x\| \leq M$  which means that  $K$  is bounded.

### Proposition 1.6

**Product of compact spaces.** Let  $(X_1, \|\cdot\|_1)$  and  $(X_2, \|\cdot\|_2)$  be two compact normed spaces and let  $X = X_1 \times X_2$ . Then  $(X, \|\cdot\|_+)$  and  $(X, \|\cdot\|_{\max})$  are compact normed spaces.

Remember that

$$\|x\|_+ = \|x_1\|_1 + \|x_2\|_2$$

and

$$\|x\|_{\max} = \max(\|x_1\|_1, \|x_2\|_2)$$

### Proof

Let  $(\xi_n)_n \subset X$  be a given sequence. This means, being in a product space, that there exist two sequences  $(x_n)_n \subset X_1$  and  $(y_n)_n \subset X_2$  such that  $\xi_n = (x_n, y_n)$  for any  $n$ . Consider the following subsequences:

- since  $(X_1, \|\cdot\|_1)$  is compact, there exists a subsequence  $(x_{\varphi(n)})_n \subset X_1$  with a limit  $x \in X_1$ .
- take the analogous subsequence  $(y_{\varphi(n)})_n \subset X_2$ . Since  $(X_2, \|\cdot\|_2)$  is compact there is another (sub-)subsequence  $(y_{\psi(n)})_n$  of  $(y_{\varphi(n)})_n$  that converges to  $y \in X_2$ .
- now take the subsequence  $(x_{\psi(n)})_n$  of  $(x_{\varphi(n)})_n$  which is still convergent in  $X_1$  to  $x$ .

We can easily see that the subsequence

$$(\xi_{\psi(n)})_n = ((x_{\psi(n)}), (y_{\psi(n)}))_n$$

is converging in  $X$  to  $x = (x, y)$ . So every subsequence  $\xi_n$  contains a subsequence that converges to a point in  $X$  and this means that  $X$  is compact.  $\square$

Of course, the above result readily extends to any finite product of compact normed spaces. On  $\mathbb{R}$  it is easy to describe a large class of compact sets:

### Lemma 1.3

Let  $\mathbb{R}$  be endowed with the absolute value,  $|\cdot|$ . Any interval  $[a, b] \subset \mathbb{R}$  is compact.

### Proposition 1.7

Let  $(X, \|\cdot\|)$  be a normed space and let  $K$  be a compact subset of  $X$ . If  $A \subset K$  is a closed subset then  $A$  is compact.

### Corollary

**Heine-Borel theorem.** A subset  $K$  of  $\mathbb{R}^N$  (where  $\mathbb{R}^N$  is endowed with, say, the usual Euclidean norm) is compact if and only if it is closed and bounded.

### Proof

$\Leftarrow$  We know that a compact subset of  $\mathbb{R}^N$  is closed and bounded.

$\Rightarrow$  Let  $K \subset \mathbb{R}^N$  be closed and bounded. Being bounded, there exists  $R > 0$  such that

$$K \subset [-R, R]^N.$$

Since  $K$  is closed, from the previous proposition it is sufficient to prove that  $[-R, R]^N$  is a compact subset of  $\mathbb{R}^N$ , which is the same as checking that  $[-R, R]$  is a compact subset of  $\mathbb{R}$ , since every closed subset of a compact subset is compact.  $\square$

This corollary can be reformulated as:

Every bounded sequence of  $\mathbb{R}^N$  has a convergent subsequence.

## 1.3 Compactness and continuous functions

### Proposition 1.8

Let  $(X, \|\cdot\|_X)$  and  $(Y, \|\cdot\|_Y)$  be two normed spaces and let  $f : X \rightarrow Y$  be continuous. If  $K \subset X$  is a compact subset of  $X$  then  $f(K)$  is a compact subset of  $Y$ .

### Proof

Let  $(y_n)_n$  be a sequence of  $f(K)$ . It means that there is a sequence  $(x_n)_n \subset K$  such that  $y_n = f(x_n)$  for any  $n$ . Since  $K$  is compact, then there exists a subsequence  $(x_{n_k})_k$  of  $(x_n)_n$  which converges to  $x \in K$ . This means that  $\lim_k \|x_{n_k} - x\|_X = 0$ , but since  $f$  is continuous then we know that

$$\lim_k \|f(x_{n_k}) - f(x)\|_Y = 0 \Leftrightarrow \lim_k \|y_{n_k} - f(x)\|_Y = 0.$$

Since  $y = f(x) \in f(K)$ , we get that any sequence of  $f(K)$  has a subsequence which converges to a limit in  $f(K)$ , which means that  $f(K)$  is compact.  $\square$

This has the following consequence:

### Theorem 1.3

Let  $(X, \|\cdot\|)$  be a normed space and let  $K \subset X$  be compact. Let  $f : K \rightarrow \mathbb{R}$  be continuous. Then,  $f$  assumes its maximum and minimum on  $K$ .

### Proof

We know that  $f(K)$  is a compact subset of  $\mathbb{R}$  and thanks to the Heine-Borel theorem we know that any compact subset of  $\mathbb{R}$  is closed and bounded (and viceversa...). So  $f$  is bounded and the fact that it reaches its maximum and minimum inside  $K$  is a simple consequence of the fact that  $f(K)$  is closed. Let

$$M = \sup \{ f(x) : x \in K \}.$$

By definition, there is a sequence  $(x_n)_n \subset K$  such that  $\lim_n f(x_n) = M$ . The sequence  $(f(x_n))_n$  lies in  $f(K)$  which is compact and is closed. But this means that also its limit  $M$  lies in  $f(K)$  (i.e.) there is  $x \in K$  such that  $f(x) = M$ . This shows that  $M$  is the maximum value of  $f$  and by proceeding in an analogous manner we can show the same thing for the minimum.  $\square$

### Theorem 1.4

**Heine Theorem.** Let  $(X, \|\cdot\|_X)$  and  $(Y, \|\cdot\|_Y)$  be two normed spaces and let  $K \subset X$  be compact. Assume that  $f : K \rightarrow Y$  is continuous. Then  $f$  is uniformly continuous on  $K$ .

#### Proof

Suppose that  $f$  is not uniformly continuous. This means that there exists  $\epsilon_0 > 0$  such that

$$\forall \delta > 0, \exists x, y \in K \text{ with } \|x - y\|_X < \delta \text{ and } \|f(x) - f(y)\|_Y \geq \epsilon_0.$$

Now, choosing  $\delta = \frac{1}{n}, n \in \mathbb{N}$  this allows to build two sequences  $(x_n)_n$  and  $(y_n)_n$  such that

$$\|x_n - y_n\|_X < \frac{1}{n} \text{ and } \|f(x_n) - f(y_n)\|_Y \geq \epsilon_0, \quad \forall n \in \mathbb{N}.$$

Since  $K$  is compact we can extract a subsequence of  $x_n$  that we call  $(x_{\varphi(n)})_n$  and that converges to some  $x_0 \in K$ . It follows that  $(y_{\varphi(n)})_n$  also converges to  $x_0$  (why?). Since  $f$  is continuous we get that  $\lim_{n \rightarrow \infty} \|f(x_{\varphi(n)}) - f(y_{\varphi(n)})\|_Y = 0$ , i.e.

$$\lim_{n \rightarrow \infty} \|f(x_{\varphi(n)}) - f(y_{\varphi(n)})\|_Y = 0$$

but this contradicts the fact that  $\|f(x_{\varphi(n)}) - f(y_{\varphi(n)})\|_Y \geq \epsilon_0$  for each  $n \in \mathbb{N}$ .  $\square$

So, in this case taking two sequences that get closer and closer does not correspond to the fact also their functions get closer and closer... and this is not possible.

## 1.4 Finite dimensional spaces

### Proposition 1.9

Let  $(X, \|\cdot\|)$  be a finite dimensional normed vector space with  $\dim(X) = d$  and let  $\{e_1, \dots, e_d\}$  be a basis for  $X$ . Then, there are positive constants  $C_0, C_1 > 0$  such that

$$C_0 \sum_{i=1}^d |x_i| \leq \left\| \sum_{i=1}^d x_i e_i \right\| \leq C_1 \sum_{i=1}^d |x_i| \quad \forall (x_1, \dots, x_d) \in \mathbb{R}^d.$$

This proposition asserts that if  $\dim(X) = d$  then *any* norm  $\|\cdot\|$  is related to the  $\|\cdot\|_1$  norm of  $\mathbb{R}^d$ . This translates in the following:

### Proposition 1.10

If  $X$  is a finite dimensional vector space, all norms over  $X$  are equivalent.

So there is no weird norm, but everything is comparable to the simple  $\|\cdot\|_1$  norm. This proposition also allows us to identify in a continuous way a finite dimensional space  $(X, \|\cdot\|)$  and the space  $\mathbb{R}^d$  where  $d$  is the dimension of  $X$ . Indeed, introducing a basis  $\{e_1, \dots, e_d\}$  of  $X$ , the mapping

$$\Phi : X \rightarrow \mathbb{R}^d$$

which associates  $\Phi(\mathbf{x}) = (x_1, \dots, x_d)$  to some  $\mathbf{x} = \sum_{i=1}^d x_i e_i \in X$ , is a bijection from  $X$  to  $\mathbb{R}^d$  which is continuous whose inverse is also continuous. This results in the following:

### Corollary

If  $(X, \|\cdot\|)$  is a finite dimensional vector space and  $K \subset X$  is closed and bounded then  $K$  is compact.

Again, this is very specific to finite dimensional spaces and, as we shall see, this actually characterizes finite dimensional spaces. Indeed, in infinite dimensional normed spaces, the closed unit ball cannot be compact. This shows that, in infinite dimensional spaces, the compact subsets do not coincide with closed and bounded subsets!! We first state the following technical lemma:



### Lemma 1.4

**Riesz Lemma.** Let  $(X, \|\cdot\|)$  be a normed vector space and let  $Y$  be a closed subspace of  $X$  (i.e.  $Y$  is closed in  $X$  and  $Y$  is a linear subspace of  $X$ ). If  $Y \neq X$  then for any  $\varepsilon \in (0, 1)$  there exists  $x \in X$  with  $\|x\| = 1$  such that

$$\inf_{y \in Y} \|x - y\| \geq 1 - \varepsilon.$$

### Remark

This lemma asserts that if  $Y \neq X$  is a closed subspace then for any  $\varepsilon \in (0, 1)$  there is some unit vector  $x \in X$  such that  $\text{dist}(x, Y) \geq 1 - \varepsilon$ .

### Proof

Let  $z \in X \setminus Y$ . Since  $Y$  is closed and  $z \notin Y$ , one has

$$\alpha = \text{dist}(z, Y) = \inf_{y \in Y} \|z - y\| > 0.$$

Pick  $\varepsilon \in (0, 1)$ . Then, exists  $\bar{y} \in Y$  such that  $\|\bar{y} - z\| < \frac{\varepsilon}{1-\varepsilon}$  (otherwise, we would get  $\text{dist}(z, Y) \geq \frac{\varepsilon}{1-\varepsilon} > \alpha$ ). Notice that  $y \neq z$  so that  $y \neq 0$ ,  $\|y - z\| > 0$ . Set  $x = \frac{z - \bar{y}}{\|z - \bar{y}\|}$ .

$$x = \frac{z - \bar{y}}{\|z - \bar{y}\|}$$

Clearly,  $\|x\| = 1$ . Let  $y \in Y$  be given. One can write

$$\|x - y\| = \frac{1}{\alpha} \|z - \bar{y} - \alpha y\|$$

and, since  $Y$  is a linear subspace,  $y + \alpha y \in Y$  so that  $\|z - (y + \alpha y)\| \geq \alpha$ . Therefore  $\|x - y\| \geq \frac{\alpha}{\alpha} = 1 - \varepsilon$  by assumption on  $\bar{y} = \|z - \bar{y}\|$ . Since this is true for any  $y \in Y$ , this proves the result.  $\square$

### Lemma 1.5

If  $(X, \|\cdot\|)$  is a normed space, any linear subspace of finite dimension is closed.

### Theorem 1.5

**Riesz Theorem.** A normed space  $(X, \|\cdot\|)$  is finite dimensional if and only if the closed unit ball  $B_c(0, 1) = \{x \in X; \|x\| \leq 1\}$  of  $X$  is compact.

### Proof

We already know that in finite dimensional spaces the closed and bounded subsets are compact. Assume that  $B_c(0, 1)$  is compact. Argue by contradiction that  $X$  is infinite dimensional. Let us pick  $e_0 \in X$  with  $\|e_0\| = 1$  and let

$$E_0 = \text{Span}\{e_0\} = \{te_0; t \in \mathbb{R}\}.$$

According to the previous Lemma,  $E_0$  is a closed linear subspace of  $X$  (since it's of dimension 1). Being  $X$  infinite dimensional,  $E_0 \neq X$ . According to Riesz Lemma, there exists  $e_1 \notin E_0$  such that  $\|e_1\| = 1$  and  $\inf_{y \in E_0} \|e_1 - y\| \geq \frac{1}{2}$ . Set then  $E_1 = \text{Span}\{e_0, e_1\}$ . One constructs inductively a sequence  $(e_n)_{n \in \mathbb{N}} \subset X$  such that  $\|e_n\| = 1$  and such that the finite dimensional space  $E_n = \text{Span}\{e_0, \dots, e_n\}$  satisfies

$$e_n \notin E_{n-1} \quad \text{and} \quad \inf_{y \in E_{n-1}} \|e_n - y\| \geq \frac{1}{2}.$$

In particular,  $(e_n)_{n \in \mathbb{N}} \subset B_c(0, 1)$  which is compact, so that a subsequence of  $(e_n)_{n \in \mathbb{N}}$  should converge. But, for any  $n \neq m \in \mathbb{N}$ ,  $\|e_n - e_m\| \geq \frac{1}{2}$  (since  $e_n \notin E_{n-1}$ ). This is a contradiction.  $\square$

## 2 Banach spaces

### 2.1 Cauchy sequences & Banach spaces

#### Definition 2.1

Let  $(X, \|\cdot\|)$  be a normed space. A sequence  $(x_n)_n \subset X$  of elements in  $X$  is a Cauchy sequence if for any  $\varepsilon > 0$  there exists  $N = N(\varepsilon) \in \mathbb{N}$  such that

$$\|x_n - x_m\| \leq \varepsilon \quad \forall n, m \geq N.$$

One easily checks that every Cauchy sequence  $(x_n)_n \subset X$  is bounded. Indeed, by definition, for any  $\varepsilon > 0$  there is  $N \in \mathbb{N}$  such that

$$\|x_n - x_m\| < \varepsilon \quad \forall n, m \geq N.$$

For, say,  $\varepsilon = 1$  and  $m = N$  we see that for  $\forall n \geq N$

$$\|x_n\| = \|x_n - x_N + x_N\| \leq \|x_n - x_N\| + \|x_N\| \leq 1 + \|x_N\|.$$

Thus with  $C_1 = 1 + \|x_N\|$

$$\sup_{n \geq N} \|x_n\| \leq C_1.$$

Setting now  $C_2 : \max\{\|x_1\|, \dots, \|x_{N-1}\|\}$  one sees that  $C_2$  is finite since it is the maximum of only a finite number of real numbers. By definition

$$\|x_n\| \leq C_2 \quad \forall n < N.$$

Therefore

$$\sup_n \|x_n\| \leq C = \max\{C_1, C_2\} < \infty$$

which means  $\{(x_n)\}_n$  is bounded.

#### Lemma 2.1

Let  $(X, \|\cdot\|)$  be a normed space. Any convergent sequence is a Cauchy sequence.

#### Proof

Let  $(x_n)_n \subset X$  be a convergent sequence in  $X$  and let  $x$  be its limit. By definition, given  $\varepsilon > 0$ , there is  $N \in \mathbb{N}$  such that  $\|x_n - x\| < \varepsilon$  for any  $n \geq N$ . Pick then  $n, m \geq N$ . From the triangle inequality

$$\|x_n - x_m\| \leq \|x_n - x\| + \|x_m - x\| < 2\varepsilon.$$

Since  $\varepsilon > 0$  is arbitrary, we proved that the sequence is a Cauchy sequence (since we can replace  $2\varepsilon$  by  $\varepsilon$ ).  $\square$

The opposite is not true... We introduce the set  $c_{00}(\mathbb{N})$  of all sequences of real number that have only finitely many nonzero components, which means

$$c_{00} = \{x = (x_n)_n \subset X, \exists N \in \mathbb{N}, x_k = 0 \quad \forall k \geq N\}.$$

This is a subset of the space  $\ell^1(\mathbb{N})$  of finitely summable sequences and in particular it is a norm space with respect to the norm induced by the one in  $\ell^1(\mathbb{N})$  (which is  $\sum_{n=0}^{\infty} |x_n|$ ). Now for any  $n \in \mathbb{N}$  let  $x^{(n)}$  be the sequence given by

$$x^{(n)} = (2^{-1}, 2^{-2}, \dots, 2^{-n}, 0, 0, \dots)$$

so that  $x^{(n)} = (x_k^{(n)})_k$  with

$$x_k^{(n)} = \begin{cases} 2^{-k} & \text{if } k \leq n \\ 0 & \text{if } k > n. \end{cases}$$

Clearly,  $\mathbf{x}^{(n)} \in c_{00}(\mathbb{N})$  for any  $n \in \mathbb{N}$ . Therefore the family  $(\mathbf{x}^{(n)})_n$  is a sequence of elements of  $c_{00}(\mathbb{N})$  (it is a sequence of sequences...) and if  $m < n$

$$\|\mathbf{x}^{(n)} - \mathbf{x}^{(m)}\|_1 = \sum_{k=1}^{\infty} |x_k^{(n)} - x_k^{(m)}| = \sum_{k=m+1}^n 2^{-k}.$$

$(\mathbf{x}^{(n)})_n$  is a Cauchy sequence in  $(c_{00}(\mathbb{N}), \|\cdot\|_1)$ . However, it does not converge in  $(c_{00}(\mathbb{N}), \|\cdot\|_1)$ .

#### Proof

Assume the contrary: there exists

$$\mathbf{x} = (x_k)_k \in c_{00}(\mathbb{N}) \text{ such that } \lim \mathbf{x}^{(n)} = \mathbf{x}.$$

But then for any  $k \in \mathbb{N}$  it must hold

$$\lim_{n \rightarrow \infty} x_k^{(n)} = x_k,$$

and therefore it must hold that

$$x_k = 2^{-k} \quad (k \in \mathbb{N}).$$

But the sequence  $\mathbf{x} = (2^{-k})_k$  does not belong to  $c_{00}(\mathbb{N})$ ! □

#### Lemma 2.2

Let  $(X, \|\cdot\|)$  be a given normed space and let  $(x_n)_n$  be a Cauchy sequence in  $X$ . If  $(x_n)_n$  admits a limit point then it is convergent.

#### Definition 2.2

A normed space  $(X, \|\cdot\|)$  is said to be **complete** if any Cauchy sequence is convergent in  $X$ . A complete normed space  $(X, \|\cdot\|)$  is called a **Banach space**.

The most fundamental example of complete normed space is the set of real numbers.

#### Theorem 2.1

If  $X = \mathbb{R}$  is endowed with the usual norm  $|\cdot|$  for any  $x, y \in \mathbb{R}$  then  $(\mathbb{R}, |\cdot|)$  is a complete normed space. In other words, any Cauchy sequence in  $\mathbb{R}$  is convergent.

#### Definition 2.3

Let  $(X, \|\cdot\|)$  be a normed space and let  $(x_n)_n \in X$  be a given sequence. We define the sequence  $(s_N)_N \in X$  of partial sums:

$$s_N := \sum_{n=1}^N x_n, \quad N \in \mathbb{N}.$$

The series  $\sum_n x_n$  is said to be convergent if there exists  $x \in X$  such that

$$\lim_{N \rightarrow \infty} \|s_N - x\| = 0.$$

We write then

$$x = \sum_{n=1}^{\infty} x_n.$$

The series  $\sum_n x_n$  is said to be absolutely convergent if

$$\sum_n \|x_n\| < \infty.$$

### Theorem 2.2

Let  $(X, \|\cdot\|)$  be a normed space. The following are equivalent:

1.  $(X, \|\cdot\|)$  is a Banach space;
2. Every absolutely convergent series is convergent in  $(X, \|\cdot\|)$ , i.e. for any  $(x_n)_n \subset X$ ,

$$\sum_{n=1}^{\infty} \|x_n\| < \infty \implies \left( \sum_{n=1}^N x_n \right)_N \text{ converges in } (X, \|\cdot\|).$$

### Proof

(1  $\Rightarrow$  2) Let  $(x_n)_n$  be a sequence such that  $\sum_n \|x_n\| < \infty$  and let

$$S_n = \sum_{k=1}^n x_k, \quad n \in \mathbb{N}.$$

We need to prove that  $(S_n)_n$  converges in  $(X, \|\cdot\|)$ . Since, by assumption,  $(X, \|\cdot\|)$  is a Banach space, it is enough to show that  $(S_n)_n$  is a Cauchy sequence. Thus, for  $m \geq n \geq 1$ , we compute

$$\|S_m - S_n\| = \left\| \sum_{k=1}^m x_k - \sum_{k=1}^n x_k \right\| = \left\| \sum_{k=n+1}^m x_k \right\| \leq \sum_{k=n+1}^m \|x_k\|.$$

Thanks to the triangle inequality, we deduce that

$$\|S_m - S_n\| \leq \sum_{k=n+1}^m \|x_k\| \leq \sum_{k=n+1}^{\infty} \|x_k\| \rightarrow 0 \quad \text{as } m \rightarrow \infty,$$

since the numerical series  $\sum_k \|x_k\| < \infty$ . This proves that

$$\lim_{m \rightarrow \infty, n \rightarrow \infty} \|S_m - S_n\| = 0$$

which means that  $(S_n)_n$  is a Cauchy sequence. It is therefore convergent and the absolutely convergent series  $\sum_k x_k$  is convergent.

(2  $\Rightarrow$  1) Let us assume now that for any sequence  $(x_n)_n \subset X$  one has

$$\sum_n \|x_n\| < \infty \implies \lim_{n \rightarrow \infty} \sum_{k=1}^n x_k \text{ exists in } (X, \|\cdot\|) \quad \textcircled{1}$$

and let  $(x_n)_n$  be a Cauchy sequence in  $(X, \|\cdot\|)$ . Thus, for any  $\varepsilon > 0$  there is  $N_\varepsilon \in \mathbb{N}$  such that

$$\|x_m - x_n\| \leq \varepsilon \quad \forall m, n \geq N_\varepsilon.$$

Applying this with  $\varepsilon = \frac{1}{2}$ ,  $k \in \mathbb{N}$ , it is easy to check that one can construct a strictly increasing sequence  $(k_k)_k \subset \mathbb{N}$  of indices such that

$$\|x_{k_{k+1}} - x_{k_k}\| \leq \frac{1}{2^k} \quad k \in \mathbb{N}.$$

Therefore, setting  $y_k = x_{k_{k+1}} - x_{k_k}$ , one has  $\sum_k \|y_k\| < \sum_k \frac{1}{2^k} < \infty$ . Applying  $\textcircled{1}$  we deduce that there exists  $y \in X$  such that

$$\lim_{n \rightarrow \infty} \left\| \sum_{k=1}^n y_k - y \right\| = 0.$$

Observing that, for any  $k \in \mathbb{N}$

$$\sum_{j=1}^k x_{k_j} = \sum_{j=1}^k (x_{k_{j+1}} - x_{k_j}) = x_{k_{k+1}} - x_{k_1},$$

one sees that  $(x_{k_{k+1}})_k$  converges to  $x = x_{k_{k+1}} + y$  in  $X$ . Thus,  $(x_n)_n$  has a subsequence which converges. Since it is a Cauchy sequence, the whole sequence  $(x_n)_n$  actually converges and this proves the completeness of  $(X, \|\cdot\|)$ .  $\square$

## 2.2 Examples revisited

### Proposition 2.1

The space  $(\ell^1(\mathbb{N}), \|\cdot\|_1)$  is a Banach space.

#### Proof

Let  $(x^{(n)})_n$  be a Cauchy sequence in  $\ell^1(\mathbb{N})$ . Recall that, for any  $n \in \mathbb{N}$ ,  $x^{(n)}$  is a sequence given by

$$x^{(n)} = (x_k^{(n)})_k.$$

Since it is a Cauchy sequence, for any  $\varepsilon > 0$  there is  $N \in \mathbb{N}$  such that

$$\|x^{(n)} - x^{(m)}\|_1 = \sum_{k=1}^{\infty} |x_k^{(n)} - x_k^{(m)}| < \varepsilon \quad \forall n, m \geq N.$$

It is easy to see that for any  $k \in \mathbb{N}$  one has

$$|x_k^{(n)} - x_k^{(m)}| \leq \|x^{(n)} - x^{(m)}\|_1 < \varepsilon$$

i.e. for any  $k \in \mathbb{N}$  fixed the sequence  $(x_k^{(n)})_n \subset \mathbb{R}$  is a Cauchy sequence in  $\mathbb{R}$  endowed with its natural distance. Since  $\mathbb{R}$  is complete by the previous theorem, it must converge which means that for any  $k \in \mathbb{N}$

$$x_k = \lim_{n \rightarrow \infty} x_k^{(n)}$$

exists in  $\mathbb{R}$ . We introduce the sequence  $x = (x_k)_k$  and it is now that  $x \in \ell^1(\mathbb{N})$  and that  $x = \lim_{n \rightarrow \infty} x^{(n)}$  for the  $\ell^1(\mathbb{N})$  distance. To do so, we fix  $\varepsilon > 0$ . Because  $(x^{(n)})_n$  is a Cauchy sequence in  $\ell^1(\mathbb{N})$ , there exists  $N > 0$  such that

$$\|x^{(n)} - x^{(m)}\|_1 < \varepsilon \quad n, m \geq N.$$

Let us fix  $n \geq N$  and let us fix an integer  $k_0 > 0$ . Since  $\lim_{k \rightarrow \infty} x_k^{(n)} = x_k$  for any  $k \geq 1$  and since the  $k_0$  is finite one has

$$\sum_{k=1}^{k_0} |x_k^{(n)} - x_k| = \sum_{k=1}^{k_0} \lim_{m \rightarrow \infty} |x_k^{(n)} - x_k^{(m)}| \leq \lim_{m \rightarrow \infty} \sum_{k=1}^{k_0} |x_k^{(n)} - x_k^{(m)}| < \varepsilon.$$

Since this is true for any  $k_0 > 0$  and

$$\|x^{(n)} - x\|_1 = \lim_{k_0 \rightarrow \infty} \sum_{k=1}^{k_0} |x_k^{(n)} - x_k|$$

one deduces that  $\|x^{(n)} - x\|_1 < \varepsilon$  for any  $n \geq N$  which proves that the sequence is converging to  $x$  and in particular that  $x \in \ell^1(\mathbb{N})$ .  $\square$

### Proposition 2.2

For any  $p \in [1, \infty]$  the normed space  $(\ell^p(\mathbb{N}), \|\cdot\|_p)$  is a Banach space. Moreover,  $(\ell^\infty(\mathbb{N}), \|\cdot\|_\infty)$  is also a Banach space.

### Proposition 2.3

For any compact interval  $I \subset \mathbb{R}$ , the normed space  $(\mathcal{C}(I), \|\cdot\|_\infty)$  is a Banach space.

The norm  $\|\cdot\|_\infty$  in this case measures the max distance between two functions in  $\mathcal{C}(I)$ .

### Proposition 2.4

For any compact interval  $I \subset \mathbb{R}$ , the normed space  $(\mathcal{C}(I), \|\cdot\|_1)$  is not a Banach space.

This is because  $\|\cdot\|_\infty$  is stronger than  $\|\cdot\|_1$ . There are some functions that are continuous but converge (according to  $\ell^1$  norm  $\|f_n - f\|_1 \xrightarrow{n \rightarrow \infty} 0$ ) to a discontinuous function (not in  $\mathcal{C}(I)$ ). But if our criterion of convergence is  $\|f_n - f\|_\infty \xrightarrow{n \rightarrow \infty} 0$  then we cannot find such functions.

### Theorem 2.3

**Fischer-Riesz Theorem.** Let  $(S, \Sigma, \mu)$  be a given measure space. For any  $1 \leq p \leq \infty$ ,  $L^p(S, \Sigma, \mu)$  is a Banach space.

### Proof

To prove this we need this lemma:

#### Lemma 2.3

Let  $(S, \Sigma, \mu)$  be a given measure space and let  $1 \leq p < \infty$ . Assume that  $(f_n)_n \subset L^p(S, \Sigma, \mu)$  is a sequence such that

$$\sum_{n=1}^{\infty} \|f_n\|_p < \infty.$$

Then the series

$$\sum_{n=1}^{\infty} f_n$$

converges almost everywhere and in  $L^p(S, \Sigma, \mu)$  which means that there exists  $f \in L^p(S, \Sigma, \mu)$  such that

$$\lim_{N \rightarrow \infty} \left\| f - \sum_{n=1}^N f_n \right\|_p = 0.$$

From now on we simply write  $L^p(S, \mu)$  instead of  $L^p(S, \Sigma, \mu)$ . For the proof we distinguish between the cases  $1 \leq p < \infty$  and  $p = \infty$ .

- By virtue of the previous Lemma, in such a case any absolutely convergent series  $\sum_n f_n$  in  $L^p(S, \Sigma, \mu)$  is actually convergent. From this theorem we know that this proves that  $(L^p(S, \Sigma, \mu), \|\cdot\|_p)$  is a Banach Space.
- Assume that  $(f_n)_n$  is a Cauchy sequence in  $L^\infty(S, \mu)$ . Given an integer  $k \geq 1$  there is an integer  $N_k$  such that

$$\|f_m - f_n\|_\infty \leq \frac{1}{k} \quad \forall n, m \geq N_k.$$

Indeed, given  $m, n \geq N_k$  by definition there is a null set  $E_k(m, n)$  for which  $|f_m(x) - f_n(x)| \geq \frac{1}{k}$  for  $x \in E_k(m, n)$ . Taking then  $E_k = \bigcup_{n, m \geq N_k} E_k(m, n)$  one has  $E_k$  null set and the above property holds on  $S \setminus E_k$  for all  $n, m \geq N_k$ . Then we let

$$E = \bigcup_{k=1}^{\infty} E_k$$

so that  $\mu(E) = 0$  and get that, for any  $x \in S \setminus E$  the sequence  $(f_n(x))_n$  is a Cauchy sequence in  $\mathbb{R}$ . Thus for any  $x \in S \setminus E$ ,

$$\lim_n f_n(x)$$

exists and we denote it by  $f(x)$ . Passing to the limit in  $\square$  as  $m \rightarrow \infty$  we obtain then that

$$|f(x) - f_n(x)| \leq \frac{1}{k} \quad \forall x \in S \setminus E, \quad \forall n \geq N_k$$

from which we see that  $f \in L^\infty(S, \mu)$  with

$$\|f - f_n\|_\infty \leq \frac{1}{k} \quad \forall n \geq N_k.$$

Letting then  $n \rightarrow \infty$  and then  $k \rightarrow \infty$  we get that

$$\lim_n \|f - f_n\|_\infty = 0.$$

This proves that  $(L^\infty(S, \mu), \|\cdot\|_\infty)$  is complete. □

### Proposition 2.5

Let  $1 \leq p \leq \infty$  and let  $(f_n)_n$  be a Cauchy sequence in  $L^p(S, \mu)$  which converges to  $f$ . Then there is a subsequence  $(f_{n_k})$  which converges  $\mu$ -almost everywhere to  $f$ .

### Proposition 2.6

Let  $(X, \|\cdot\|_X)$  and  $(Y, \|\cdot\|_Y)$  be two normed spaces and assume that  $(Y, \|\cdot\|_Y)$  is a Banach space. Then the space  $(\mathcal{L}(X, Y), \|\cdot\|_{\mathcal{L}(X, Y)})$ .

Remember that

$$\|L\|_{\mathcal{L}(X, Y)} = \|L\|_{\text{op}} = \sup_{\|x\|_X \leq 1} \|L(x)\|_Y \quad \forall L \in \mathcal{L}(X, Y).$$

This measures how much the linear mapping “stretches” a small  $x \in X$  when it maps it to  $Y$ .

### Proof

Let  $(L_n)_n$  be a sequence of linear and continuous applications from  $X$  to  $Y$ . Assume that  $(L_n)_n$  is a Cauchy sequence for  $\|\cdot\|_{\mathcal{L}(X, Y)}$ , i.e. for any  $\varepsilon > 0$  there exist  $N \in \mathbb{N}$  such that

$$\|L_m - L_n\|_{\mathcal{L}(X, Y)} \leq \varepsilon \quad \forall m, n \geq N.$$

By definition of the norm, it holds

$$\|L_m(x) - L_n(x)\|_Y \leq \varepsilon \|x\|_X \quad \forall m, n \geq N, \forall x \in X.$$

In particular, for any  $x \in X$  the sequence  $(L_n(x))_n$  is a Cauchy sequence in  $Y$  and since  $Y$  is complete, the sequence converges to  $L(x)$ . This defines a mapping  $L : X \rightarrow Y$  and it is not difficult to prove that  $L$  is linear. Let us check then that  $L$  is continuous and that  $\|L_m - L\|_{\mathcal{L}(X, Y)} \rightarrow 0$  as  $m \rightarrow \infty$ . Taking the limit as  $n \rightarrow \infty$  in the previous inequality one gets that

$$\|L_m(x) - L(x)\|_Y \leq \varepsilon \|x\|_X \quad \forall m \geq N, \forall x \in X. \quad \square$$

In particular for any fixed  $m \geq N$  and any given  $x \in X$

$$\|L(x)\|_Y \leq \|L_m(x)\|_Y + \varepsilon \|x\|_X$$

and since  $L_m$  is continuous there exists  $C_m > 0$  such that

$$\|L_m(x)\|_Y \leq C_m \|x\|_X + \varepsilon \|x\|_X$$

This shows that  $L \in \mathcal{L}(X, Y)$ . Now,  $\square$  exactly means that  $\|L_m - L\|_{\mathcal{L}(X, Y)} \leq \varepsilon$  for any  $m \geq N$  which proves that  $(L_n)_n$  converges to  $L$  in  $(\mathcal{L}(X, Y), \|\cdot\|_{\mathcal{L}(X, Y)})$ . □

### Corollary

Let  $(X, \|\cdot\|_X)$  be a Banach space and let  $L \in \mathcal{L}(X)$  be such that

$$\|L\|_{\mathcal{L}(X)} < 1.$$

Then, the linear mapping  $I - L$  is invertible (i.e.  $I - L$  is bijective and its inverse  $(I - L)^{-1} \in \mathcal{L}(X)$ ) and

$$(I - L)^{-1} = \sum_{n=0}^{\infty} L^n$$

where  $I$  is the identity mapping of  $X$  and  $L^n$  is defined inductively by  $L^0 = I$  and  $L^{n+1} = L \circ L^n = L^n \circ L$  for any  $n \geq 0$ .

### Proof

For any  $n \in \mathbb{N}$  let

$$A_n := \sum_{k=0}^n L^k$$

Clearly  $A_n$  is a linear mapping as a composition and sum of linear mappings. For the same reason, it is continuous, i.e.  $A_n \in \mathcal{L}(X)$ . For any  $m > n$  it holds

$$\begin{aligned} \|A_m - A_n\|_{\mathcal{L}(X)} &= \left\| \sum_{k=n+1}^m L^k \right\|_{\mathcal{L}(X)} \\ &\leq \sum_{k=n+1}^m \|L^k\|_{\mathcal{L}(X)} \end{aligned}$$

One easily checks by induction that  $\|L^k\|_{\mathcal{L}(X)} \leq \|L\|_{\mathcal{L}(X)}^k$  for any  $k \geq 0$ . Thus

$$\|L^k\|_{\mathcal{L}(X)} \leq \|L\|_{\mathcal{L}(X)}^k \quad \forall m > n$$

Then  $(A_n)_n$  is a Cauchy sequence in  $\mathcal{L}(X)$ . According to the previous theorem,  $(A_n)_n$  converges to some linear mapping, say  $A \in \mathcal{L}(X)$ . Let us now prove that  $I - A$  is necessarily invertible and that  $A = (I - L)^{-1}$ . For any  $n \geq 0$ , it holds that

$$(I - L) \circ A_n = A_n \circ (I - L) = I - L^{n+1}$$

In particular

$$\|(I - L) \circ A_n - I\|_{\mathcal{L}(X)} \leq \|L\|_{\mathcal{L}(X)}^{n+1}$$

Letting  $n \rightarrow \infty$  one sees that  $(I - L) \circ A_n$  converges to  $I$  in  $\mathcal{L}(X)$ . By continuity, this means that

$$(I - L) \circ A = I$$

In the same way we have  $A \circ (I - L) = I$ . This shows that  $(I - L)$  is invertible and that  $A$  is its inverse.  $\square$

### Definition 2.4

If  $(X, \|\cdot\|)$  is a normed space, we define the dual space of  $X$  denoted by  $X^*$  the space of all continuous and linear mappings  $\Phi : X \rightarrow \mathbb{R}$ , i.e.

$$X^* = \mathcal{L}(X, \mathbb{R}).$$

We also denote

$$\|\Phi\|_* = \|\Phi\|_{\mathcal{L}(X, \mathbb{R})} = \sup_{\|x\|_X \leq 1} |\Phi(x)| \quad \forall \Phi \in X^*.$$



### Corollary

If  $(X, \|\cdot\|)$  is a normed vector space then  $(X^*, \|\cdot\|_*)$  is a Banach space.

### Theorem 2.4

**Riesz representation theorem.** Given  $1 < p < \infty$  and  $\Phi \in (L^p(S, \mu))^*$  there exists a unique  $g \in L^q(S, \mu)$  (with  $\frac{1}{p} + \frac{1}{q} = 1$ ) such that

$$\Phi(f) = \int_S fg \, d\mu \quad \forall f \in L^p(S, \mu).$$

Moreover,

$$\|\Phi\|_* = \|g\|_q.$$

If  $(S, \Sigma, \mu)$  is  $\sigma$ -finite, then the result is still true for  $p = 1$ .

## 2.3 Simple consequences of completeness

### Proposition 2.7

Let  $(X, \|\cdot\|)$  be a Banach space and let  $(x_n)_n \subset X$  be such that

$$\sum_{n=1}^{\infty} \|x_n\| < \infty.$$

Then the series  $\sum_{n=1}^{\infty} x_n$  converges in  $X$ , i.e. there exists  $x \in X$  such that

$$\lim_{N \rightarrow \infty} \left\| \sum_{n=1}^N x_n - x \right\| = 0.$$

### Definition 2.5

Let  $(X, \|\cdot\|_X)$  and  $(Y, \|\cdot\|_Y)$  be two normed spaces and  $k \in (0, 1)$ . A function  $f : X \rightarrow Y$  is said to be a  $k$ -contraction mapping if

$$\|f(x) - f(y)\|_Y \leq k \|x - y\|_X \quad \forall x, y \in X \times X.$$

It is obvious that any  $k$  contraction mapping is continuous. This is clear that a  $k$ -contraction mapping is an application which “contracts” distances: the images of  $x$  and  $y$  through  $f$  are closer to each other than  $x, y$ .

### Theorem 2.5

**Banach fixed point theorem.** Let  $(X, \|\cdot\|)$  be a complete normed space and let  $f : X \rightarrow X$  be a  $k$ -contraction mapping with  $k \in (0, 1)$ . Then there exists a unique fixed point  $a \in X$  for  $f$ , i.e. there is a unique  $a \in X$  such that

$$f(a) = a.$$

### Proof

We first prove that  $f$  can admit only one fixed point. Assume  $a, b$  are both fixed points of  $f$ . Since  $f$  is a  $k$  contraction mapping we have

$$\begin{aligned} \|f(a) - f(b)\| &\leq k \|a - b\| \\ &\Downarrow \\ \|a - b\| &\leq k \|a - b\|. \end{aligned}$$

Since  $k > 0$  and  $\|a, b\| \geq 0$  one sees necessarily that  $\|a - b\| = 0$  i.e.  $a = b$ .

To prove now the existence of some fixed point, one proves actually that the sequence defined in the above statement converges to some fixed-point of  $f$ . Let then  $x_0 \in X$  be given and define inductively

$$x_{n+1} = f(x_n).$$

Since  $f$  is a  $k$  contraction mapping we have

$$\|x_{n+1} - x_n\| \leq k \|x_n - x_{n-1}\|$$

for any  $n \geq 1$  and we easily deduce that

$$\|x_{n+1} - x_n\| \leq k^n \|x_1 - x_0\| \quad \forall n \in \mathbb{N}.$$

If  $m > n \geq 1$  are given, one deduces from the above inequality together with the triangle inequality that

$$\|x_n - x_m\| \leq \sum_{j=n}^{m-1} \|x_j - x_{j+1}\| \leq \|x_1 - x_0\| \sum_{j=n}^{m-1} k^j. \quad \square$$

Since the geometric series  $\sum_{j=1}^{\infty} k^j$  is convergent, one has

$$\lim_{n, m \rightarrow \infty} \sum_{j=n}^{m-1} k^j = 0$$

and therefore the sequence  $(x_n)_n$  is a Cauchy sequence in  $X$ . Since  $X$  is complete,  $(x_n)_n$  converges to some limit  $a \in X$ . Moreover, being  $k$ -contracting,  $f$  is continuous so that the sequence  $(f(x_n))_n$  converges to  $f(a)$ . Since  $x_{n+1} = f(x_n)$  one has  $a = f(a)$  and the result is proven.  $\square$

Notice that the above inequality  $\square$  provides the convergence rate of  $(x_n)_n$  to  $a$ . Indeed, taking the limit  $m \rightarrow \infty$  in  $\square$  we get

$$\|x_n - a\| \leq \|x_1 - x_0\| \sum_{j=n}^{\infty} k^j = \frac{k^n}{1-k} \|x_1 - x_0\| \quad \forall n \in \mathbb{N}.$$

## 2.4 Fundamental Properties of Banach spaces

### Definition 2.6

A normed space  $(X, \|\cdot\|)$  is said to have the Baire property if the intersection of any sequence of dense open sets of  $X$  is dense in  $X$ , i.e. for any  $(U_n)_n$  open subsets of  $X$  with  $\overline{U_n} = X$  for any  $n \in \mathbb{N}$  it holds

$$\bigcap_n \overline{U_n} = X.$$

The Baire property is clearly equivalent to the following property: the union of any sequence of closed subsets with empty interior has an empty interior. In particular, the theorem is often used in the following form: if  $(C_n)_n$  is a sequence of closed subsets such that

$$\text{Int} \left( \bigcup_{n \in \mathbb{N}} C_n \right) = X$$

Then there exists some  $n_0 \in \mathbb{N}$  such that  $\text{Int}(C_{n_0}) \neq \emptyset$ .

### Theorem 2.6

Any complete normed space  $(X, \|\cdot\|)$  has the Baire property.

### Proof

Let  $(U_n)_{n \in \mathbb{N}}$  be a sequence of open subsets of  $(X, \|\cdot\|)$  with  $\overline{U_n} = X$  for any  $n \in \mathbb{N}$  and let  $A = \bigcup_{n \in \mathbb{N}} U_n$ . To prove that  $A$  is dense in  $X$  we shall use the previous lemma and prove that

$$A \cap \overline{B(x_0, r_0)} \neq \emptyset$$

for any nonempty open subset  $O$  of  $X$ . Pick any  $x_0 \in O$  and let  $r_0 > 0$  be such that

$$B(x_0, r_0) \subset O.$$

Since  $U_1$  is dense, one has that  $B(x_0, r_0)$  is open and nonempty. So we can pick  $x_1 \in B(x_0, r_0) \cap U_1$  and  $r_1 > 0$  such that

$$B(x_1, r_1) \subset B(x_0, r_0) \cap U_1, \quad 0 < r_1 \leq \frac{r_0}{2}.$$

We iterate this reasoning and construct a sequence  $(x_n)_{n \in \mathbb{N}} \subset X$  and  $(r_n)_{n \in \mathbb{N}} \subset (0, \infty)$  such that

$$B(x_{n+1}, r_{n+1}) \subset B(x_n, r_n) \cap U_{n+1}, \quad \text{with } 0 < r_{n+1} \leq \frac{r_n}{2}, \quad \forall n \in \mathbb{N}.$$

In particular, one sees that  $r_n \leq \frac{r_0}{2^n}$  so that the sequence  $(x_n)_{n \in \mathbb{N}}$  is a Cauchy sequence. Since  $X$  is complete, there is  $x \in X$  such that  $\lim_{n \rightarrow \infty} x_n = x$ . Since  $x_{n+k} \in B(x_n, r_n)$  for any  $n \in \mathbb{N}$  and any  $k \in \mathbb{N}$  taking the limit as  $k \rightarrow \infty$  we get that

$$x \in \overline{B(x_n, r_n)} \subset B(x_n, r_n), \quad \forall n \in \mathbb{N}.$$

In particular,  $x \in U_1 \cap A$  which is non empty and this proves the result.  $\square$

### Theorem 2.7

**Banach-Steinhaus Theorem.** Let  $(X, \|\cdot\|_X)$  be a Banach space and let  $(Y, \|\cdot\|_Y)$  be a normed space. Let  $(T_i)_{i \in I} \subset \mathcal{L}(X, Y)$  be a given collection of continuous linear applications. Assume that, for any  $x \in X$ , there exists  $M_x > 0$  such that

$$\sup_{i \in I} \|T_i(x)\|_Y \leq M_x.$$

Then, there exists  $M > 0$  such that for any  $x \in X$  and any  $i \in I$  it holds

$$\|T_i(x)\|_Y \leq M \|x\|_X$$

i.e.  $\sup_{i \in I} \|T_i\|_{\mathcal{L}(X, Y)} \leq M < \infty$ .

The Uniform Boundedness Principle can be reformulated as follows: let  $(X, \|\cdot\|_X)$  be a Banach space and let  $(Y, \|\cdot\|_Y)$  be a normed space. Let  $\mathcal{F} \subset \mathcal{L}(X, Y)$  be a given collection of continuous linear applications (here above  $\mathcal{F} = (T_i)_{i \in I}$ ). Then the following are equivalent:

- Pointwise boundedness: for every  $x \in X$ , the set  $\{T(x); T \in \mathcal{F}\}$  is bounded in  $Y$ , i.e.

$$\sup_{T \in \mathcal{F}} \|T(x)\|_Y = M_x < \infty \quad \forall x \in X.$$

- Uniform boundedness: the operator norms  $\{\|T\|_{\mathcal{L}(X, Y)}; T \in \mathcal{F}\}$  are bounded, i.e.

$$\sup_{T \in \mathcal{F}} \|T\|_{\mathcal{L}(X, Y)} = M < \infty.$$

### Proof

For every  $n \in \mathbb{N}$  set

$$X_n = \{x \in X; \|T_i(x)\|_Y \leq n \forall i \in I\}.$$

Since for any  $i \in I$ ,  $T_i$  is continuous, one sees that  $X_n$  is closed as the intersection of the

closed subsets of  $X$ . Moreover, according to  $\boxtimes$ ,

$$X = \bigcup_n X_n.$$

It follows from Baire theorem that there exists  $n_0 \in \mathbb{N}$  such that  $\text{Int}(X_{n_0}) \neq \emptyset$ . Pick then  $x_0 \in X$  and  $r > 0$  so that  $B(x_0, r) \subset X_{n_0}$ . By definition it holds

$$\|T_1(x_0 + rz)\|_Y \leq n_0 \quad \forall i \in I; \forall z \in B(0, 1).$$

By linearity we get

$$\|T_i(z)\|_Y \leq \frac{1}{r}(\|T_i(x_0 + rz)\|_Y + \|T_i(x_0)\|_Y) \leq 2\frac{n_0}{r} \quad \forall i \in I; \forall z \in B(0, 1).$$

This clearly gives the result with  $M = 2\frac{n_0}{r}$ .  $\square$

### Theorem 2.8

**Open mapping theorem.** Let  $(X, \|\cdot\|_X)$  and  $(Y, \|\cdot\|_Y)$  be two Banach spaces and let

$$T : X \rightarrow Y$$

be a continuous linear application which is surjective (i.e.  $T \in \mathcal{L}(X, Y)$  is onto). Then  $T$  maps open sets of  $X$  into open sets of  $Y$ , i.e. if  $U \subset X$  is an open set then  $T(U)$  is an open set of  $Y$ .

### Corollary

Let  $(X_i, \|\cdot\|_i)$  with  $i = 1, 2$  be two Banach spaces and let  $T : X_1 \rightarrow X_2$  be a continuous linear application which is bijective. Then its inverse  $T_{-1}$  is also continuous, i.e.  $T^{-1} \in \mathcal{L}(X_2, X_1)$ .

### Proposition 2.8

Let  $(X, \|\cdot\|)$  be a Banach space and let  $\|\cdot\|_0$  be a norm on  $X$  such that

- $(X, \|\cdot\|_0)$  is a Banach space;
- there exists  $C_0 > 0$  such that

$$\|x\|_0 \leq C_0 \|x\| \quad \forall x \in X_0.$$

Then, the two norms  $\|\cdot\|$  and  $\|\cdot\|_0$  are equivalent.

So if a Banach norm dominates another Banach norm, then they are equivalent.

### Theorem 2.9

**Closed graph theorem.** Let  $(X_i, \|\cdot\|_i)$  with  $i = 1, 2$  be two Banach spaces and let  $T : X_1 \rightarrow X_2$  be a linear application. Assume that the graph of  $T$

$$\mathcal{G}(T) = \{(x_1, T(x_1)) \in X_1 \times X_2 \mid x_1 \in X_1\}$$

is closed in  $X_1 \times X_2$  (endowed with the norm  $\|(x_1, x_2)\|_{\max} = \max\{\|x_1\|_1, \|x_2\|_2\}$ ). Then,

$$T \in \mathcal{L}(X_1, X_2).$$

### Proof

Consider on  $X_1$  the norm

$$\|x\|_T = \|x\|_1 + \|T(x)\|_2 \quad x \in X_1.$$

One checks that  $\|\cdot\|_T$  is indeed a norm. Moreover, since  $\mathcal{G}(T)$  is closed in  $X_1 \times X_2$ , one can

check that  $(X_1, \|\cdot\|_T)$  is a Banach space. Moreover, one clearly has

$$\|x\|_1 \leq \|x\|_T \quad \forall x \in X_1.$$

Then, the previous proposition asserts that the norms  $\|\cdot\|_1$  and  $\|\cdot\|_T$  are equivalent norms on  $X_1$ , so there exists

$$c > 0 \quad \text{such that} \quad \|x\|_T \leq c \|x\|_1 \quad \forall x \in X_1.$$

This is enough to prove the continuity of  $T$ .  $\square$

## 2.5 Complements

### Definition 2.7

Let  $A \in \mathcal{L}(X)$ . We say that  $A$  is a compact operator if the image of  $B_c(0, 1)$  through  $A$  is contained in a compact set of  $X$ , i.e. for any sequence  $(x_n)_n \subset X$  with  $\|x_n\| \leq 1$ , the sequence  $(A(x_n))_n$  admits a subsequence which converges. We shall denote by  $\mathcal{K}(X)$  the collection of all compact operators in  $X$ .

### Definition 2.8

Let  $A \in \mathcal{L}(X)$ . We say that  $A$  is of finite rank if the image of  $A$ ,  $R(A) = \{Ax, x \in X\}$  is a linear subspace of finite dimension  $X$ .

### Lemma 2.4

If  $A \in \mathcal{K}(X)$  and  $B \in \mathcal{L}(X)$  then  $B \circ A$  and  $A \circ B$  belong to  $\mathcal{K}(X)$ .

### Proposition 2.9

Let  $(A_n)_n \subset \mathcal{K}(X)$  and let  $A \in \mathcal{L}(X)$  be given with

$$\lim_{n \rightarrow \infty} \|A_n - A\|_{\mathcal{L}(X)} = 0.$$

Then,  $A \in \mathcal{K}(X)$ .

## 2.6 More properties of the dual space

### Theorem 2.10

**Zorn's lemma.** Every non empty ordered set that is inductive has a maximal element.

### Theorem 2.11

**Helly, Hahn-Banach analytic form.** Let  $X$  be a vector space over  $\mathbb{R}$ . Let  $p : X \rightarrow \mathbb{R}$  be a function satisfying

$$1. \quad p(\lambda x) = \lambda p(x) \text{ for any } x \in X \text{ and any } \lambda > 0$$

$$2. \quad p(x + y) \leq p(x) + p(y) \text{ for any } x, y \in X.$$

Let  $Y \subset X$  be a linear subspace and  $g : Y \rightarrow \mathbb{R}$  a linear function such that

$$3. \quad g(x) \leq p(x) \text{ for any } x \in Y.$$

Then, there exists a linear function  $f : X \rightarrow \mathbb{R}$  such that

$$f(x) = g(x) \quad \forall x \in Y$$

(we say then that  $f$  extends  $g$  to  $X$ ) and

$$f(x) \leq p(x) \quad \forall x \in X.$$

### Corollary

Let  $(X, \|\cdot\|)$  be a normed space and let  $Y \subset X$  be a linear subspace. If  $g : Y \rightarrow \mathbb{R}$  is a continuous linear mapping, then there exists  $f \in X^*$  that extends  $g$  and such that

$$\|f\|_* = \|g\|_Y$$

### Corollary

Let  $(X, \|\cdot\|)$  be a normed space. For every  $x_0 \in X$  there exists  $\Phi_0 \in X^*$  such that

$$\|\Phi_0\|_* = \|x_0\|$$

and

$$\Phi_0(x_0) = \|x_0\|^2.$$

## 3 Inner product spaces and Hilbert spaces

### 3.1 General properties

#### Definition 3.1

Let  $H$  be a given vector space. An inner product  $\langle \cdot, \cdot \rangle$  is a mapping from  $H \times H$  with values in  $\mathbb{R}$  with the following properties:

1. **symmetry:**  $\langle x, y \rangle = \langle y, x \rangle$  for  $\forall x, y \in H$ ;
2. **bilinearity:**  $\langle x, \alpha y + \beta z \rangle = \alpha \langle x, y \rangle + \beta \langle x, z \rangle$  for  $\forall x, y, z \in H$  and  $\forall \alpha, \beta \in \mathbb{R}$ ;
3.  $\langle x, x \rangle \geq 0$  for  $\forall x \in H$ ;
4.  $\langle x, x \rangle = 0$  if and only if  $x = 0$ .

If  $H$  is endowed with an inner product, we say that  $(H, \langle \cdot, \cdot \rangle)$  is an inner product space.

#### Proposition 3.1

If  $(H, \langle \cdot, \cdot \rangle)$  is an inner product space then it is a norm space with respect to the norm given by

$$\|x\| = \sqrt{\langle x, x \rangle}.$$

The norm  $\|\cdot\|$  is called the norm induced by the inner product  $\langle \cdot, \cdot \rangle$  and it satisfies

$$\|x \pm y\|^2 = \|x\|^2 + \|y\|^2 \pm 2\langle x, y \rangle$$

and the parallelogram law

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

#### Proposition 3.2

**Cauchy-schwartz inequality.** If  $(H, \langle \cdot, \cdot \rangle)$  is an inner product space and  $\|\cdot\|$  defines the inner norm then it holds

$$|\langle x, y \rangle| \leq \|x\| \|y\|.$$

#### Proposition 3.3

Let  $(H, \langle \cdot, \cdot \rangle)$  be an inner product space and let  $(x_n)_n$  and  $(y_n)_n$  be two converging sequencer (in the inner norm) to  $x$  and  $y$ . Then

$$\lim_{n \rightarrow \infty} \langle x_n, y_n \rangle = \langle x, y \rangle.$$

## 3.2 Orthogonality

### Definition 3.2

Let  $(H, \langle \cdot, \cdot \rangle)$  be a given inner product space. Two vector  $x, y \in H$  are said to be **orthogonal** if  $\langle x, y \rangle = 0$ . Given two linear subspaces  $M, N \subset H$  we say that  $M \perp N$  if  $\langle x, y \rangle = 0$  for  $\forall x \in M, \forall y \in N$ .

### Definition 3.3

Let  $(H, \langle \cdot, \cdot \rangle)$  be a inner product space. A finite family  $\{e_1, \dots, e_N\} \subset H$  is said to be **orthonormal** if

$$\|e_k\| = 1 \quad \forall k = 1, \dots, N$$

and

$$\langle e_m, e_n \rangle = 0 \quad \forall n \neq m.$$

### Proposition 3.4

**Gram-Schmidt Procedure.** Let  $(H, \langle \cdot, \cdot \rangle)$  be a inner product space. Then

1. any orthonormal set  $\{e_1, \dots, e_N\} \subset H$  is linearly independent;
2. given a linearly independent subset  $\{v_1, \dots, v_N\}$  of  $H$  and given  $S = \text{span}(v_1, \dots, v_N)$  then there exists an orthonormal basis of  $S$   $\{e_1, \dots, e_N\}$ .

### Proof

1. We need to show that given  $\alpha_1, \dots, \alpha_N$  such that

$$\sum_{k=1}^N \alpha_k e_k = 0$$

then  $\alpha_j = 0$  for every  $j$ . Consider any  $e_j$  for  $j = 1, \dots, N$ . Then we have

$$\begin{aligned} 0 &= \left\langle e_j, \sum_{k=1}^N \alpha_k e_k \right\rangle \\ &= \sum_{k=1}^N \alpha_k \langle e_j, e_k \rangle \\ &= \alpha_j \text{ (taking into account } \langle e_j, e_j \rangle = \|e_j\|^2 = 1 \\ &= \alpha_j. \end{aligned}$$

2. We argue by induction over  $N$ . For  $N=1$  it is easy: we have  $\{v_1\}$  for which we can set  $e_1 = \frac{1}{\|v_1\|} v_1$ . We now know this is true for some integer  $N \geq 1$ . We need to find  $\{e_1, \dots, e_{N+1}\}$  which is an orthonormal basis for  $\text{span}(v_1, \dots, v_{N+1})$ . We know for sure that  $v_{N+1} \notin \text{span}(v_1, \dots, v_N)$  because they are linearly independent. Construct

$$h_{N+1} = v_{N+1} - \sum_{k=1}^N \langle v_{N+1}, e_k \rangle e_k$$

By construction we have  $\|h_{N+1}\| \neq 0$  and  $h_{N+1} \in \text{span}(v_1, \dots, v_{N+1})$ . Moreover  $\langle h_{N+1}, e_j \rangle = 0$  for  $j = 1, \dots, N$ . Now we just set

$$e_{N+1} = \frac{h_{N+1}}{\|h_{N+1}\|}$$

Now we have that  $\text{span}(e_1, \dots, e_{N+1}) \subset \text{span}(v_1, \dots, v_{N+1})$  but since we proved that  $\{e_1, \dots, e_N\}$  are linearly independent and since these two spans have the same dimension, we can say

$$\text{span}(v_1, \dots, v_{N+1}) = \text{span}(e_1, \dots, e_{N+1})$$

□

#### Definition 3.4

Let  $(H, \langle \cdot, \cdot \rangle)$  be an inner product space and let  $M$  be a subset of  $H$ . Then the **orthogonal complement** of  $M$  is defined as

$$M^\perp = \{x \in H; \langle x, u \rangle = 0 \quad \forall u \in M\}.$$

In particular, one sees that  $M^\perp \perp M$ .

#### Proposition 3.5

Let  $(H, \langle \cdot, \cdot \rangle)$  be an inner product space and let  $M \subset H$ . Then

1.  $0 \in M^\perp$ ;
2. if  $0 \in M$  then  $M \cap M^\perp = \{0\}$ , otherwise  $M \cap M^\perp = \emptyset$ ;
3.  $\{0\}^\perp = H$  and  $H^\perp = \{0\}$ ;
4. if  $M$  is a non-empty subset of  $H$  then  $M^\perp \neq \emptyset$ ;
5. if  $N \subset M$  then  $N^\perp \subset M^\perp$ ;
6.  $M^\perp$  is a closed linear subspace of  $H$ ;
7.  $M \subset (M^\perp)^\perp$ .

#### Proposition 3.6

Let  $M$  be a linear subspace of an inner product space  $(H, \langle \cdot, \cdot \rangle)$ . Then

$$x \in M^\perp \iff \|x - y\| \geq \|y\| \quad \forall y \in M.$$

### 3.3 Hilbert Spaces and Projection Theorem

#### Definition 3.5

Let  $(H, \langle \cdot, \cdot \rangle)$  be an inner product space and let  $\|\cdot\|$  be the associated inner norm. If  $(H, \|\cdot\|)$  is complete then  $(H, \langle \cdot, \cdot \rangle)$  is called a **Hilbert space**.

#### Definition 3.6

Let  $X$  be a given vector space. A subset  $C \subset X$  is convex if for any  $x, y \in C$  one has

$$tx + (1 - t)y \in C \quad \forall t \in [0, 1].$$

Linear subspaces of vector spaces are always convex.

#### Theorem 3.1

**Projection over closed subspaces.** Let  $(H, \langle \cdot, \cdot \rangle)$  be a given Hilbert space and let  $K \subset H$  be closed and convex. Then for every  $f \in H$  there exists a unique element  $u \in K$  such that

$$\|f - u\| = \min_{v \in K} \|f - v\| = \text{dist}(f, K).$$

Moreover,  $u$  is characterized by the property

$$u \in K$$

and

$$\langle f - u, w - u \rangle \leq 0 \quad \forall w \in K.$$



## Proof

Let  $(v_n)_n$  be a minimizing sequence for  $\|f - v\|$ , i.e.  $v_n \in K$  for any  $n$  and

$$\lim_{n \rightarrow \infty} \|f - v_n\| = \inf_{v \in K} \|f - v\|.$$

So call  $D = \inf_{v \in K} \|f - v\|$  and  $D_n = \|f - v_n\|$ . We claim that  $(v_n)_n$  is Cauchy. Call  $v = f - v_n$  and use the parallelogram rule to get

$$\begin{aligned} \|v - v_m\|^2 &\leq 2(\|f - v_n\|^2 + \|f - v_m\|^2) - \|2f - (v_n + v_m)\|^2 \\ &= 2(D_n^2 + D_m^2) - 4\left\|f - \frac{v_n + v_m}{2}\right\|^2. \end{aligned}$$

Since  $\mathcal{K}$  is convex then we know that  $\frac{v_n + v_m}{2} \in K$  and therefore  $\left\|f - \frac{v_n + v_m}{2}\right\|^2 \geq D^2$ . Therefore

$$\|v_n - v_m\| \leq 2(D_n^2 + D_m^2)^{1/2} - 4D^2.$$

Letting  $m, n \rightarrow \infty$  we get  $\lim_{m, n \rightarrow \infty} \|v_n - v_m\| = 0$  and so the sequence is Cauchy.

Since  $H$  is complete then it converges to some limit  $u \in H$  and since  $K$  is closed  $u \in K$  and in particular

$$\|f - u\| = D$$

which gives the existence of a  $u$  such that  $\|f - u\| = \text{dist}(f, K)$ .

Before proving uniqueness we show the equivalence between the two claims. Assume to have an  $u \in K$  that satisfies the first claim and let  $v \in K$ . We have

$$w = (1-t)u + tv \in K \quad \forall t \in (0, 1).$$

Therefore

$$\|f - u\| \leq \|f - v\| = \|(f - u) + t(v - u)\|$$

and taking the square and expanding the second norm we get

$$\|f - u\|^2 \leq \|f - v\|^2 + t^2\|v - u\|^2 + 2t\langle f - u, v - u \rangle$$

which means

$$\langle f - u, v - u \rangle \leq \frac{t}{2}\|v - u\|^2.$$

Letting  $t \rightarrow 0$  we see that  $u$  satisfies the second claim. Conversely, if  $u$  satisfies the second claim then for any  $v \in K$  we have

$$\|v - f\|^2 = \|v - u\|^2 = 2\langle f - u, v - u \rangle = \|v - u\|^2 \leq 0$$

which implies the first claim.

Let us now prove the uniqueness of  $u$ . Assume  $u_1$  and  $u_2$  satisfy the second claim. We have

$$\begin{aligned} \langle f - u_1, u - u_1 \rangle &\leq 0 & \forall u \in K \\ \langle f - u_2, u - u_2 \rangle &\leq 0 & \forall u \in K. \end{aligned}$$

Choosing  $u = u_2$  and  $u = u_1$  in the above inequalities yields

$$\begin{aligned} \langle f - u_1, u_2 - u_1 \rangle &\leq 0 \\ \langle f - u_2, u_1 - u_2 \rangle &\leq 0 \end{aligned}$$

and after adding both inequalities we get

$$\|u_1 - u_2\|^2 \leq 0 \Rightarrow u_1 = u_2.$$

□

### Definition 3.7

Given a closed convex subset  $K \subset H$  of a Hilbert space  $(H, \langle \cdot, \cdot \rangle)$  and  $f \in H$ , the unique  $u \in K$  satisfying the first claim is called the *projection* of  $f$  over  $K$  and is denoted by

$$u = \mathbf{P}_K f$$

### Proposition 3.7

Let  $(H, \langle \cdot, \cdot \rangle)$  be a Hilbert space and let  $K \subset H$  be a closed convex subset. Then

$$\|\mathbf{P}_K f - \mathbf{P}_K g\| \leq \|f - g\|$$

(the projection is a contraction).

### Proof

Let  $f, g \in H$  be given. Set  $u_1 = \mathbf{P}_K f$  and  $u_2 = \mathbf{P}_K g$ . According to the characterization of the second claim we have

$$\begin{aligned} \langle f - u_1, v - u_1 \rangle &\leq 0 \quad \forall v \in K \\ \langle g - u_2, w - u_2 \rangle &\leq 0 \quad \forall w \in K \end{aligned}$$

choosing  $v = u_2$  and  $w = u_1$  and adding the corresponding inequalities we obtain

$$\|u_1 - u_2\|^2 \leq \langle f - g, u_1 - u_2 \rangle$$

which together with the Cauchy-Schwarz inequality gives

$$\|u_1 - u_2\| \leq \|f - g\|.$$

□

### Corollary

Let  $(H, \langle \cdot, \cdot \rangle)$  be a Hilbert space. Assume that  $M \subset H$  is a closed linear subspace and let  $f \in H$ . Then  $u = \mathbf{P}_M f$  is characterized by

$$u \in M \quad \text{and} \quad \langle f - u, z \rangle = 0 \quad \forall z \in M.$$

Moreover,  $\mathbf{P}_M$  is a linear mapping,  $\mathbf{P}_M \in \mathcal{L}(H)$ , called the orthogonal projection over  $M$ .

### Proof

By the second claim of the projection theorem we know that

$$\langle f - u, w - u \rangle \leq 0 \quad \forall w \in M.$$

In particular, given  $z \in M$ , since  $w = z + u \in M$  one has

$$\langle f - u, z \rangle \leq 0.$$

Since  $-z \in M$  one has  $\langle f - u, z \rangle \geq 0$  and the claim in the corollary follows. The fact that the claim of the corollary implies the second claim of the theorem is obvious. Fuck you!

Let us now prove that  $\mathbf{P}_M$  is linear. To do so, let  $f_1, f_2 \in H$  with  $u_1 = \mathbf{P}_M f_1$  and  $u_2 = \mathbf{P}_M f_2$ . For  $\alpha_1, \alpha_2 \in \mathbb{R}$ , let  $f = \alpha_1 f_1 + \alpha_2 f_2$  and  $u = \mathbf{P}_M(f)$ . To show that  $\mathbf{P}_M$  is linear, we need to prove that  $u = \alpha_1 u_1 + \alpha_2 u_2$ . Recall that  $u$  is characterized by the claim in the corollary. Therefore, to prove the result, it is enough to show that  $\alpha_1 u_1 + \alpha_2 u_2$  also satisfies the claim of the corollary and then by uniqueness we will deduce  $\alpha_1 u_1 + \alpha_2 u_2 = u$ . First, since  $M$  is a linear

subspace and  $u_1, u_2 \in M$ ,  $\alpha_1 u_1 + \alpha_2 u_2 \in M$ . Moreover, for  $z \in M$ ,

$$\begin{aligned}\langle f - (\alpha_1 u_1 + \alpha_2 u_2), z \rangle &= \langle \alpha_1 f_1 + \alpha_2 f_2 - (\alpha_1 u_1 + \alpha_2 u_2), z \rangle \\ &= \alpha_1 \langle f_1 - u_1, z \rangle + \alpha_2 \langle f_2 - u_2, z \rangle \\ &= 0\end{aligned}$$

since, by the claim of the corollary, it is satisfied by  $(f_1, u_1)$  and  $(f_2, u_2)$  i.e.  $\langle f_1 - u_1, z \rangle = \langle f_2 - u_2, z \rangle = 0$ . Thus,  $\alpha_1 u_1 + \alpha_2 u_2$  satisfies the second claim and by uniqueness  $\alpha_1 u_1 + \alpha_2 u_2 = \mathbf{P}_M f$ . This shows the linearity of  $\mathbf{P}_M$ . The fact that then  $\mathbf{P}_M$  is continuous is an obvious consequence of the above proposition.  $\square$

### Corollary

Let  $(H, \langle \cdot, \cdot \rangle)$  be a Hilbert space. Assume that  $M \subset H$  is a closed linear subspace. For any  $f \in H$  there exists a unique  $u \in M$  and a unique  $v \in M^\perp$  such that  $f = u + v$ . Moreover

$$\|f\|^2 = \|u\|^2 + \|v\|^2$$

### Fundamental example: $L^2(S, \mu)$ and the construction of the conditional expectation

#### Theorem 3.2

The space  $L^2(S, \mu)$  endowed with the scalar product

$$\langle f, g \rangle = \int_S f g \, d\mu \quad \forall f, g \in L^2(S, \mu)$$

is an Hilbert space.

#### Theorem 3.3

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability triple and let  $X$  be a real random variable with  $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ . Let  $\mathcal{G}$  be a sub- $\sigma$ -algebra of  $\mathcal{F}$ . Then there exists a random variable  $Y$  such that

1.  $Y$  is  $\mathcal{G}$ -measurable;
2.  $\mathbb{E}(|Y|) < \infty$ ;
3. for any set  $A \in \mathcal{G}$  we have  $\mathbb{E}(Y \mathbb{1}_A) = \mathbb{E}(X \mathbb{1}_A)$  i.e.

$$\int_A Y \, d\mathbb{P} = \int_A X \, d\mathbb{P} \quad \forall A \in \mathcal{G}.$$

Moreover, if  $\tilde{Y}$  is another random variable with the above properties then  $\mathbb{P}[Y = \tilde{Y}] = 1$ . In such a case,  $Y$  is called a version of the conditional expectation of  $X$  given  $\mathcal{G}$  and we write

$$Y = \mathbb{E}(X|\mathcal{G}).$$

### 3.4 The dual of a Hilbert space

#### Theorem 3.4

**Riesz-Frechet representation theorem.** Let  $(H, \langle \cdot, \cdot \rangle)$  be a Hilbert space. Given any  $\Phi \in H^*$  (the dual space of  $H$ ) there exists an unique  $f \in H$  such that

$$\Phi(u) = \langle f, u \rangle \quad \forall u \in H \text{ and } \|f\| = \|\Phi\|_*.$$

### Proof

Let  $M = \Phi^{-1}(\{0\}) = \{y \in H; \Phi(y) = 0\}$ . Since  $\Phi$  is linear and continuous we know that  $M$  is a closed subspace of  $H$  and we assume  $M \neq H$  otherwise  $\Phi \equiv 0$  and the conclusion is obvious because  $f \equiv 0$ .

Imagine there exists an element  $y \in H$  such that

$$\|y_0\| = 1 \quad \langle y, v \rangle = 0 \quad \forall v \in M. \quad \blacksquare$$

Now pick  $y_1 \in H$  with  $y_0 \notin M$  and choose  $y_2 = P_M y_0$ . Since  $y_2 \in M$  and  $y_0 \notin M$  we have that  $\|y_0 - y_2\| \neq 0$ . Then choose

$$q = \frac{1}{\|y_0 - y_2\|} (y_0 - y_2) \in H$$

which implies  $q \in M^\perp$  which satisfies  $\blacksquare$  according to the projection theorem. Then pick  $v \in H$  and let

$$u = v - \lambda q \quad \lambda = \frac{\Phi(v)}{\Phi(q)}$$

which is fine because  $q \in M \Rightarrow \Phi(q) \neq 0$ . Now we have

$$\Phi(u) = \Phi(v) - \lambda \Phi(q) = \Phi(v) - \Phi(v) = 0$$

which means that  $u \in M$  and  $\langle q, u \rangle = 0$  because of  $\blacksquare$ . Moreover

$$\begin{aligned} \langle q, u \rangle &= \langle q, v - \lambda q \rangle \\ &= \langle q, v \rangle - \lambda \langle q, q \rangle \\ &= \lambda \langle q, q \rangle = \lambda \end{aligned}$$

and since  $\Phi(u) = \lambda \Phi(q)$  we have  $\Phi(u) = \Phi(v) \langle q, v \rangle$ . Setting  $f = q \langle q, v \rangle$  we finally have

$$\Phi(u) = \langle f, v \rangle$$

By CS we have  $|\Phi(u)| \leq \|f\| \|v\|$  for any  $v \in H$  so  $\|\Phi\|_* \leq \|f\|$ . Moreover

$$\|\Phi(f)\| = \|(f, f)\| = \|f\|^2 \Rightarrow \|\Phi\|_* = \|f\|.$$

$f$  is unique because  $\|f\|^2 = \|\Phi(f)\|$ . If we had  $f_1$  and  $f_2$  we would have  $\Phi(f_1 - f_2) = 0 \Rightarrow f_1 = f_2$ .  $\square$

## 3.5 Hilbert bases

### Proposition 3.8

Any infinite dimensional inner product space  $(H, \langle \cdot, \cdot \rangle)$  contains an orthonormal sequence, i.e. there is  $(e_n)_n$  such that

$$\langle e_n, e_m \rangle = \delta_{n,m} \quad \forall n, m \in \mathbb{N}.$$

### Lemma 3.1

**Bessel's inequality.** Let  $(H, \langle \cdot, \cdot \rangle)$  be an inner product space and let  $(e_n)_n$  be an orthonormal sequence of  $H$ . Then for any  $x \in H$  the sequence  $|\sum_{n=1}^{\infty} \langle x, e_n \rangle|^2$  converges in  $\mathbb{R}$  and