Stochastic Processes

Homework 2

M.S. in Stochastics and Data Science

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Exercise 1

Consider the one-dimensional diffusion process $\{X(t)\}$ defined on the interval $(0, \infty)$ with drift $\mu(x) = \frac{1}{2x}$ and infinitesimal variance $\sigma^2(x) = 1$. Let $a = 0$ and $b = \infty$ be the boundaries of the state space.

1. Compute the **scale function** $s(x)$ up to a constant.
2. Compute the **speed measure** $m(x)$ up to a constant.
3. Determine the **nature of boundaries** 0 and ∞. For each boundary decide whether it is:
   - regular;
   - exit;
   - entrance;
   - natural.
4. Interpret the classification in terms of the behavior of the process near 0 and ∞. Is the process **recurrent** or **transient**?

Solution

1. We know that the scale function (or measure) $S(x)$ for a process x is defined as

$$S(x) = \int^x s(\eta)\,\mathrm{d}\eta = \int^x \exp\left\{-\int^\eta \frac{2\mu(\xi)}{\sigma^2(\xi)}\,\mathrm{d}\xi\right\}\mathrm{d}\eta.$$

We start by computing the scale *density* $s(\eta)$:

$$
\begin{aligned}
s(\eta) &= \exp\left\{-\int^{\eta} \frac{2\mu(\xi)}{\sigma^2(\xi)}\,\mathrm{d}\xi\right\} \\
&= \exp\left\{\int^{\eta} \frac{2\cdot\frac{1}{2\eta}}{1}\,\mathrm{d}\xi\right\} \\
&= \exp\left\{-\int^{\eta} \frac{1}{\xi}\,\mathrm{d}\xi\right\} \\
&= \exp\left\{-\ln(\eta)\right\} = \frac{1}{\eta}.
\end{aligned}
$$

We can now plug this into the formula for $S(x)$ so that we get

$$
S(x) = \int^{x} \frac{1}{\eta}\,\mathrm{d}\eta = \ln|x| + c
$$

where $c \in \mathbb{R}$ is an additive constant.

2. We know that the speed measure $M(x)$ for a process x is

$$
M(x) = \int^{x} m(y)\,\mathrm{d}y = \int^{x} \frac{1}{\sigma^2(x)s(x)}\,\mathrm{d}y.
$$

We need to compute the speed *density* $m(y)$:

$$
\begin{aligned}
m(y) &= \frac{1}{\sigma^2(y)s(y)} \\
&= \frac{1}{1\cdot\frac{1}{y}} = y.
\end{aligned}
$$

So by plugging this result into the formula for $M(x)$ we get:

$$
M(x) = \int^{x} y\,\mathrm{d}y = \frac{1}{2}x^2 + c
$$

where $c \in \mathbb{R}$ is an additive constant.

3. Recall the scheme for the classification of boundaries (in this case, for the lower boundary a):

<table>
<tr><th colspan="4">Criteria</th><th colspan="4">Terminology</th></tr>
<tr><th>$S(a,x]$</th><th>$M(a,x]$</th><th>$\Sigma(a)$</th><th>$N(a)$</th><th>Feller</th><th colspan="3">Gikhman and Skorokhod</th></tr>
<tr><td>$< \infty$</td><td>$< \infty$</td><td>$< \infty$</td><td>$< \infty$</td><td colspan="2">Regular</td><td rowspan="3">Attracting</td><td rowspan="2">Attainable</td></tr>
<tr><td>$< \infty$</td><td>$= \infty$</td><td>$< \infty$</td><td>$= \infty$</td><td colspan="2">Exit-Trap-Absorbing</td></tr>
<tr><td>$< \infty$</td><td>$= \infty$</td><td>$= \infty$</td><td>$= \infty$</td><td rowspan="3">Natural
$(\Sigma(l) = \infty, N(l) = \infty)$</td><td>Attracting,
unattainable</td><td rowspan="4">Unattainable</td></tr>
<tr><td>$= \infty$</td><td>$< \infty$</td><td>$= \infty$</td><td>$= \infty$</td><td rowspan="3">Natural
$(S(l,x] = \infty$</td><td rowspan="3">Nonattracting</td></tr>
<tr><td>$= \infty$</td><td>$= \infty$</td><td>$= \infty$</td><td>$= \infty$</td></tr>
<tr><td>$= \infty$</td><td>$< \infty$</td><td>$= \infty$</td><td>$< \infty$</td><td>Entrance</td></tr>
</table>

In our case, $a = 0$ so we are left with the task of computing

$$S(0, x] \quad M(0, x] \quad \Sigma(0) \quad N(0).$$

- We can compute $M(0, x]$straight away:

$$M(0, x] = \int_0^x y \, \mathrm{d}y = \frac{1}{2} y^2 \bigg|_0^x = \frac{1}{2} x^2 < \infty.$$

- On the other hand, $S(0, x]$ presents itself as an improper integral which does not converge (since we cannot compute $\ln 0$ directly):

$$\begin{aligned} S(0, x] &= \int_0^x \frac{1}{\eta} \, \mathrm{d}\eta \\ &= \lim_{\varepsilon \to 0^+} \int_\varepsilon^x \frac{1}{\eta} \, \mathrm{d}\eta \\ &= \lim_{\varepsilon \to 0^+} (\ln x - \ln \varepsilon) = \infty. \end{aligned}$$

- We now have to compute $\Sigma(0)$. We defined the function $\Sigma(l)$ as

$$\begin{aligned} \Sigma(l) &= \lim_{a \searrow l} \int_a^x S[a, \xi] \, \mathrm{d}M(\xi) \\ &= \int_l^x M[\eta, x] \, \mathrm{d}S(\eta). \end{aligned}$$

  We know that

$$M[\eta, x] = \int_\eta^x y \, \mathrm{d}y = \frac{1}{2} y^2 \bigg|_\eta^x = \frac{1}{2} \left(x^2 - \eta^2 \right).$$

  We also know that $\mathrm{d}S(\eta) = s(\eta) \, \mathrm{d}\eta$ and since we already know that $s(\eta) = \frac{1}{\eta}$

$$\mathrm{d}S(\eta) = s(\eta) \, \mathrm{d}\eta = \frac{1}{\eta} \, \mathrm{d}\eta.$$

  Putting all together in the formula for $\Sigma(l)$ with $l = 0$ we get

$$\begin{aligned} \Sigma(0) &= \int_0^x \left(\frac{1}{2} \left(x^2 - \eta^2 \right) \right) \cdot \frac{1}{\eta} \, \mathrm{d}\eta \\ &= \frac{1}{2} \int_0^x \left(x^2 - \eta^2 \right) \cdot \frac{1}{\eta} \, \mathrm{d}\eta \\ &= \frac{1}{2} \Bigg[x^2 \cdot \underbrace{\int_0^x \frac{1}{\eta} \, \mathrm{d}\eta}_{=\infty} - \underbrace{\int_0^x \eta \, \mathrm{d}\eta}_{\frac{1}{2} x^2 < \infty} \Bigg] \end{aligned}$$

  Since the first term of the subtraction is infinite, we have that

$$\Sigma(0) = \infty.$$

- We defined $N(l)$ as

$$\begin{aligned} N(l) &= \int_l^x S[\eta, x]\,\mathrm{d}M(x) \\ &= \int_l^x M(l,\xi]\,\mathrm{d}S(\xi). \end{aligned}$$

  We already know that, for $l = 0$ then

$$M(0,\xi] = \int_0^\xi m(y)\,\mathrm{d}y = \int_0^\xi y\,\mathrm{d}y = \frac{\xi^2}{2}$$

  and

$$\mathrm{d}S(\xi) = \frac{1}{\xi}\,\mathrm{d}\xi.$$

  So putting all together we get

$$\begin{aligned} N(0) &= \int_0^x \frac{\xi^2}{2} \cdot \frac{1}{\xi}\,\mathrm{d}\xi \\ &= \frac{1}{2}\int_0^x \xi\,\mathrm{d}\xi \\ &= \frac{1}{2} \cdot \frac{x^2}{2} = \frac{x^2}{4} < \infty \qquad \forall\, x \in \mathbb{R}. \end{aligned}$$

So our situation for the boundary $a = 0$ is

$$S(0,x] = \infty \quad M(0,x] < \infty \quad \Sigma(0) = \infty \quad N(0) < \infty.$$

So, according to the Feller terminology this boundary is:

- an entrance boundary (or a “natural boundary” according to the Skorokhod terminology);
- non-attracting;
- unattainable.

We now have to do the same exact procedure for the upper boundary $b = \infty$ for which we have to compute

$$S(x,\infty] \quad M(x,\infty] \quad \Sigma(\infty) \quad N(\infty).$$

- We know that

$$\begin{aligned} S(x,\infty) &= \int_x^\infty \frac{1}{\eta}\,\mathrm{d}\eta \\ &= \lim_{\varepsilon\to\infty} \int_x^b \frac{1}{\eta}\,\mathrm{d}\eta \\ &= \lim_{\varepsilon\to\infty} (\ln b - \ln x) = \infty. \end{aligned}$$

- We know that
$$M(x,\infty] = \int_x^\infty y\,\mathrm{d}y = \frac{1}{2}y^2\Big|_x^\infty = \infty.$$
- We know that
$$\Sigma(\infty) = \int_x^\infty M[x,\xi]\,\mathrm{d}S(\xi)$$
and since $M[x,\xi] = \int_x^\xi \eta\,\mathrm{d}\eta = \frac{\xi^2-x^2}{2}$ and $\mathrm{d}S(\xi) = \frac{1}{\xi}\,\mathrm{d}\xi$ we get that
$$\begin{aligned}\Sigma(\infty) &= \int_x^\infty \frac{\xi^2-x^2}{2}\cdot\frac{1}{\xi}\,\mathrm{d}\xi\\ &= \frac{1}{2}\int_x^\infty\left(\xi-\frac{x^2}{\xi}\right)\mathrm{d}\xi\\ &= \frac{1}{2}\underbrace{\int_x^\infty \xi\,\mathrm{d}\xi}_{=\infty} - x^2\underbrace{\int_x^\infty \frac{1}{\xi}\,\mathrm{d}\xi}_{=\infty}.\end{aligned}$$
Since $\int_x^\infty \frac{1}{\xi}\,\mathrm{d}\xi$ grows logarithmically and this it is slower than $\int_x^\infty \xi\,\mathrm{d}\xi$ then we have
$$\Sigma(\infty) = \infty.$$
- For this problem it is easier to use the alternative definition of $N(l)$. We know that
$$N(\infty) = \int_x^\infty S[x,\xi]\,\mathrm{d}M(\xi).$$
Since $\mathrm{d}M(\xi) = m(\xi)\,\mathrm{d}\xi$ we can write
$$N(\infty) = \int_x^\infty S[x,\xi]\cdot m(\xi)\,\mathrm{d}\xi.$$
Moreover, we know that
$$S[x,\xi] = \int_x^\xi \frac{1}{\eta}\,\mathrm{d}\eta = \ln\xi - \ln x$$
and that $m(\xi) = \xi$ so we can write
$$\begin{aligned}N(\infty) &= \int_x^\infty \xi(\ln\xi - \ln x)\,\mathrm{d}\xi\\ &= \int_x^\infty \xi\ln\xi\,\mathrm{d}\xi - \ln x\int_x^\infty \xi\,\mathrm{d}\xi.\end{aligned}$$
We want to analyze the behavior of both $\int_x^\infty \xi\ln\xi\,\mathrm{d}\xi$ and $\int_x^\infty \xi\,\mathrm{d}\xi$.

- In $\int_x^\infty \xi \ln \xi \, \mathrm{d}\xi$ we have ξ that grows faster than $\ln \xi$, so $\xi \ln \xi \xrightarrow{\xi \to \infty} \infty$ so the integral diverges to ∞.
- In $\int_x^\infty \xi \, \mathrm{d}\xi$ we have

$$\int_x^\infty \xi \, \mathrm{d}\xi = \left. \frac{\xi^2}{2} \right|_x^\infty$$

but since $\frac{\xi^2}{2} \xrightarrow{\xi \to \infty} \infty$ then the integral diverges to ∞.

So we have

$$N(\infty) = \infty - \ln x \cdot \infty = \infty$$

since $\ln x$ is finite.

So our situation for the boundary $b = \infty$ is

$$S(x, \infty] = \infty \quad M(x, \infty] = \infty \quad \Sigma(\infty) = \infty \quad N(\infty) = \infty.$$

So, according to the Feller terminology this boundary is:

- a natural boundary (according to both terminologies);
- non-attracting;
- unattainable.

4. We know that 0 is an entrance boundary: this means that the process cannot "enter" 0 but if it starts from 0 then it can exit and drift away. On the other hand ∞ is a natural boundary, so the process can neither start from it or reach it in finite time from the interior of the diffusion interval. So this means that the process will drift away from 0 and will never return to it. This, together with the fact that ∞ is unattainable and non-attracting, makes the process **transient.**

Exercise 2

Show that:

1. the equilibrium distribution for the Wiener process between two reflecting barriers is a truncated exponential if the drift is non-zero and a uniform distribution if the drift is zero;

2. the equilibrium distribution for the Ornstein-Uhlenbeck process between two reflecting barriers is a truncated normal distribution;

3. $\{X(t)\}$ is not a Markov process when $\{X(t)\}$ is the displacement process of a particle whose velocity $U(t)$ follows the Ornstein-Uhlenbeck process.

Solution

1. If Brownian motion has a real drift and is locked between two reflective barriers, its overall conduct (the distribution of the equilibrium) depends on the drift.

   We know that the stationary distribution $\psi(x)$ satisfies the Fokker-Planck equation

   $$0 = \frac{1}{2}\frac{\partial^2}{\partial y^2}\left[\sigma^2(y)\psi(y)\right] - \frac{\partial}{\partial y}\left[\mu(y)\psi(y)\right]$$

   which can be solved to yield

   $$\psi(x) = m(x)\left[c_1 S(x) + c_2\right].$$

   Let's analyze the two different cases, when $\mu(x) = 0$ and when $\mu(x) \neq 0$.

   - **No drift**. We know that

     $$m(y) = \frac{1}{\sigma^2(y)s(y)}$$

     and in this case

     $$\begin{aligned} s(y) &= \exp\left\{-\int^y \frac{2\mu(\xi)}{\sigma^2(\xi)}\,\mathrm{d}\xi\right\} \\ &= \exp\left\{-\int^y \frac{2\cdot 0}{1}\,\mathrm{d}\xi\right\} \\ &= e^0 = 1 \end{aligned}$$

     so we get

     $$m(y) = 1.$$

     We can also calculate $S(x)$:

     $$\begin{aligned} S(x) &= \int^x s(\eta)\,\mathrm{d}\eta \\ &= \int^x 1\,\mathrm{d}\eta = x. \end{aligned}$$

     This means that the solution to the Fokker-Planck equation becomes

     $$\psi(x) = c_1 \cdot x + c_2.$$

     To determine the values of c_1 and c_2 we must take into account the conditions on the stationary distribution:

     $$\begin{cases} \psi(x) \geqslant 0 \\ \int_I \psi(x)\,\mathrm{d}x = 1. \end{cases}$$

For the conditions of non-negativity to hold, we try setting $c_1 = 0 \implies \psi(x) = c_2$. Knowing that the Wiener process between two reflecting boundaries a and b has diffusion interval $I = (a, b)$, the total probability condition becomes

$$\begin{aligned}
&\int_I \psi(x)\,\mathrm{d}x = 1 \\
\implies &\int_a^b c_2\,\mathrm{d}x = 1 \\
\implies &c_2(b-a) = 1 \\
\implies &c_2 = \frac{1}{b-a}.
\end{aligned}$$

So what we get is

$$\psi(x) = \frac{1}{b-a} \qquad x \in [a, b]$$

which is exactly the uniform distribution we were looking for.

- **With drift**. Since we are talking about a Wiener process, we have $\mu(x) = \mu$ constant. Here the values for $m(x)$ and $S(x)$ change and so does the solution to the Fokker-Planck equation. We have

$$\begin{aligned}
s(y) = \exp\left\{-\int^y \frac{2\mu}{1}\,\mathrm{d}\xi\right\} \\
\exp\left\{-2\mu\int^y \mathrm{d}\xi\right\} = e^{-2\mu x}
\end{aligned}$$

so our speed density becomes

$$m(y) = \frac{1}{1 \cdot e^{-2\mu x}} = e^{2\mu x}.$$

The scale measure $S(x)$ becomes

$$S(x) = \int^x e^{-2\mu\eta}\,\mathrm{d}\eta.$$

If we integrate from $-\infty$ then this integral is not well-defined for $\mu > 0$ but in this case we know that the lower bound is our reflecting boundary a. So we get

$$S(x) = \left.\frac{e^{-2\mu\eta}}{-2\mu}\right|_a^x = \frac{1}{2\mu}\left(e^{-2\mu x} - e^{-2\mu a}\right).$$

This means that the solution to the Fokker-Planck equation becomes

$$\begin{aligned}
\psi(x) &= e^{2\mu x}\left[c_1\frac{1}{2\mu}\left(e^{-2\mu x} - e^{-2\mu a}\right) + c_2\right] \\
&= e^{2\mu x}\left[\frac{c_1}{2\mu}\left(e^{-\mu x} - e^{-\mu a}\right) + c_2\right] \\
&= \frac{c_1}{2\mu}\left(e^{2\mu x}e^{-2\mu x} - e^{2\mu x}e^{2\mu a}\right) + c_2 e^{2\mu x} \\
&= \frac{c_1}{2\mu}\left(1 - e^{2\mu(x-a)}\right) + c_2 e^{2\mu x}.
\end{aligned}$$

As before, we try setting $c_1 = 0$ so that the equilibrium distribution becomes

$$\psi(x) = c_2 e^{2\mu x}$$

and so that the normalization condition becomes

$$\int_a^b \psi(x)\,\mathrm{d}x = \int_a^b c_2 e^{2\mu x}\,\mathrm{d}x = 1.$$

So we get

$$\begin{aligned}
c_2\int_a^b e^{2\mu x}\,\mathrm{d}x &= \frac{c_2}{2\mu}\left(e^{2\mu b} - e^{2\mu a}\right) = 1 \qquad \text{this is okay because } \mu \neq 0 \\
\implies c_2 &= \frac{2\mu}{e^{2\mu b} - e^{2\mu a}}.
\end{aligned}$$

This gives us

$$\psi(x) = \frac{2\mu \cdot e^{2\mu x}}{e^{2\mu b} - e^{2\mu a}}, \qquad x \in [a, b], \mu \neq 0$$

which is a truncated exponential distribution with parameter $\lambda = -2\mu$.

2. For the Ornstein-Uhlenbeck process we have

$$\begin{cases} \mu(x) = -\frac{x}{\vartheta} \\ \sigma^2(x) = \sigma^2. \end{cases}$$

We need to perform the calculations. We have that

$$\begin{aligned}
s(y) &= \exp\left\{-\int^y \frac{-2\frac{\xi}{\vartheta}}{\sigma^2}\,\mathrm{d}\xi\right\} \\
&= \exp\left\{\int^y \frac{2\xi}{\vartheta\sigma^2}\,\mathrm{d}\xi\right\} \\
&= \exp\left\{\frac{1}{\vartheta\sigma^2}\int^y 2\xi\,\mathrm{d}\xi\right\} \\
&= \exp\left\{\frac{y^2}{\vartheta\sigma^2}\right\}.
\end{aligned}$$

We get that

$$m(y) = \frac{1}{\sigma^2 \exp\left\{\frac{y^2}{\vartheta\sigma^2}\right\}} = \frac{1}{\sigma^2} \exp\left\{-\frac{y^2}{\vartheta\sigma^2}\right\}.$$

Moving on to $S(x)$:

$$S(x) = \int^x \exp\left\{\frac{\eta^2}{\vartheta\sigma^2}\right\} \mathrm{d}\eta.$$

This is not solvable in closed form... Anyway, we get that our stationary distribution satisfies

$$\begin{aligned}
\psi(x) &= m(x)\,[c_1 S(x) + c_2] \\
&= \frac{1}{\sigma^2} \exp\left\{-\frac{x^2}{\vartheta\sigma^2}\right\} \left[c_1 \int^x \exp\left\{\frac{\eta^2}{\vartheta\sigma^2}\right\} \mathrm{d}\eta + c_2\right] \\
&= c_1 \frac{1}{\sigma^2} \exp\left\{-\frac{x^2}{\vartheta\sigma^2}\right\} \int^x \exp\left\{\frac{\eta^2}{\vartheta\sigma^2}\right\} \mathrm{d}\eta + c_2 \frac{1}{\sigma} \exp\left\{-\frac{x^2}{\vartheta\sigma^2}\right\}.
\end{aligned}$$

We gladly set $c_1 = 0$ to get

$$\psi(x) = c_2 \frac{1}{\sigma^2} \exp\left\{-\frac{x^2}{\vartheta\sigma^2}\right\}.$$

We use the normalization constraint (with respect to the diffusion interval with boundaries $[a, b]$) to obtain

$$\begin{aligned}
&\int_a^b c_2 \frac{1}{\sigma^2} \exp\left\{-\frac{x^2}{\vartheta\sigma^2}\right\} \mathrm{d}x = 1 \\
\implies & c_2 \frac{1}{\sigma^2} \int_a^b \exp\left\{-\frac{x^2}{\vartheta\sigma^2}\right\} \mathrm{d}x = 1 \\
\implies & c_2 = \frac{\sigma^2}{\int_a^b \exp\left\{-\frac{x^2}{\vartheta\sigma^2}\right\} \mathrm{d}x}.
\end{aligned}$$

If we plug this back into the solution of the Fokker-Planck equation and we get

$$\psi(x) = \frac{\exp\left\{-\frac{x^2}{\vartheta\sigma^2}\right\}}{\int_a^b \exp\left\{-\frac{x^2}{\vartheta\sigma^2}\right\} \mathrm{d}x} \qquad x \in [a, b]$$

which is the density function of truncated Gaussian distribution at a and b.

3. If $X(t)$ is the displacement process with velocity $U(t)$ then we have that

$$X(t) = \int_0^t U(y)\,\mathrm{d}y.$$

We need to show that

$$\mathbb{P}(X(t+s) \in A | \mathcal{F}_t) \neq \mathbb{P}(X(t+s) \in A | X(t)).$$

We know that an alternative definition of Markov property is that any d-dimensional $\mathcal{F}_t$ adapted stochastic process $\{X_t\}_{t\geqslant 0}$ with right continuous paths is a Markov process if it satisfies

$$\mathbb{E}\left[u(X(t+s)) \mid \mathcal{F}_t\right] = \mathbb{E}\left[u(X(t+s) \mid X(t))\right]$$

for $\forall\, s, t \geqslant 0, u \in \mathcal{B}(\mathbb{R}^d)$. We can write $X(t+s)$ as

$$X(t+s) = X(t) + \int_t^{t+s} U(y)\,\mathrm{d}y.$$

This means that

$$\begin{aligned} \mathbb{E}\left[X(t+s)|\mathcal{F}_t\right] &= \mathbb{E}\left[X(t) + \int_t^{t+s} U(y)\,\mathrm{d}y \middle| \mathcal{F}_t\right] \\ &= \underbrace{X(t)}_{\in\mathcal{F}_t} + \mathbb{E}\left[\int_t^{t+s} U(y)\,\mathrm{d}y \middle| \mathcal{F}_t\right] \end{aligned}$$

and we know that Ornstein-Uhlenbeck processes are Markov, therefore

$$\mathbb{E}[U(y)|\mathcal{F}_t] = \mathbb{E}[U(y)|U(t)] \qquad \text{for } y > t.$$

So by linearity of expectation and Fubini's theorem we have

$$\mathbb{E}\left[X(t+s)|\mathcal{F}_t\right] = X(t) + \int_t^{t+s} \mathbb{E}\left[U(y)|U(t)\right]\mathrm{d}y.$$

We now want to compute $\mathbb{E}\left[u(X(t+s) \mid X(t))\right]$. Again, we start from the "split" version of $X(t+s)$.

$$\begin{aligned} \mathbb{E}\left[X(t+s) \mid X(s)\right] &= \mathbb{E}\left[X(t) + \int_t^{t+s} U(y)\,\mathrm{d}y \middle| X(t)\right] \\ &= X(t) + \mathbb{E}\left[\int_t^{t+s} U(y\,\mathrm{d}y) \middle| X(t)\right] \\ &= X(t) + \int_t^{t+s} \mathbb{E}\left[U(y)|X(t)\right]\mathrm{d}y. \end{aligned}$$

But now we cannot say that $\mathbb{E}\left[U(y)|X(t)\right] = \mathbb{E}\left[U(y)|U(t)\right]$ as before, so we got that

$$\mathbb{E}\left[u(X(t+s)) \mid \mathcal{F}_t\right] \neq \mathbb{E}\left[u(X(t+s) \mid X(t))\right]$$

And so the process $X(t)$ is *not* Markov. This makes sense intuitively because X, being the integral of U, "smooths out" the velocity. This makes the information generated by X at time t insufficient to know about the displacement at time $t+s$.

Exercise 3

Consider a variant $\{S_n\}_{n \leqslant 0}$ of the classical random walk such that $S_n = \sum_{i=0}^{n} X_i$ but $\{X_i\}$ is a sequence i.i.d. random variables with $\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = \frac{1}{4}$ and $\mathbb{P}(X_i = 0) = \frac{1}{2}$.

1. Determine the diffusion limit of this random walk.
2. Determine its diffusion interval and the nature of its boundaries.
3. Does this process admit stationary distribution? Justify the answer.

Solution

1. By diffusion limit here we mean investigating the distribution of $\{S_n\}_{n \leqslant 0}$ as the step size tends to 0. We want to apply the Central Limit Theorem and for this we need the mean and the variance of X_i. We know that

$$\mathbb{E}\left[X_i\right] = 1 \cdot \frac{1}{4} + (-1) \cdot \frac{1}{4} + 0 \cdot \frac{1}{2} = 0$$

and that

$$\begin{aligned}\mathbb{V}\mathrm{ar}\left(X_i\right) &= \mathbb{E}\left[X_i^2\right] - \mathbb{E}\left[X_i\right]^2 \\ &= \left(1^2 \cdot \frac{1}{4} + (-1)^2 \cdot \frac{1}{4} + 0^2 \cdot \frac{1}{2}\right) - 0^2 \\ &= \frac{1}{2}.\end{aligned}$$

If we turn to the variance of S_n we see that, due to the fact that the X_i's are i.i.d.,

$$\mathbb{V}\mathrm{ar}\left(S_n\right) = \sum_{i=0}^{n} \mathbb{V}\mathrm{ar}\left(X_i\right) = (n+1) \cdot \frac{1}{2}.$$

This is not something we should be surprised of: variances have an additive nature in sums, so we could say that, having a step in time (that in this case is 1 unit of time between each step of the random walk),

$$\text{variance} \propto n.$$

In diffusion processes we have that the time step is not fixed as it is here, but it is continuous. This is problematic because it causes the variance to explode as $n \to \infty$ if it is directly proportional to the number of time steps. The variance must be proportional to the time step so that changing the size of the step Δt doesn't change the behavior of the process.

$$\text{variance} \propto \text{time step}.$$

Each time step we will basically add a fixed variance σ^2 to the random step length: after n steps we will have $n\sigma^2$ cumulative variance; we want this variance to be the same when we take a single time step of length $n\Delta t$ and this corresponds to scaling by $\sqrt{n}$ so that the variance does not explode when letting $n \to \infty$. For this purpose we define the scaled random walk

$$S^{\star}_{(n)}(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{nt} X_i.$$

Here the random walk is rescaled by *space* by a factor of $\frac{1}{\sqrt{n}}$ and by *time* by "fitting" nt steps of the random walk in a time t. This means that instead of 1 unit of time per step, we make each step take time $\Delta t = \frac{1}{n}$. The effect of this is that as $n \to \infty$ the steps will become smaller (as effect of the space rescaling) and they will happen ever more often in a time t (by effect of the time rescaling).

We can now apply the CLT. We know that since $\mathbb{E}\left[X_i\right] = 0$ then

$$\mathbb{E}\left[S^{\star}_{(n)}(t)\right] = 0$$

and since $\mathbb{V}\mathrm{ar}\left(X_i\right) = \frac{1}{2}$ then

$$\begin{aligned}\mathbb{V}\mathrm{ar}\left(S^{\star}_{(n)}(t)\right) &= \frac{1}{n} \cdot \sum_{i=1}^{nt} \mathbb{V}\mathrm{ar}\left(X_i\right) \\ &= \frac{1}{\cancel{n}} \cdot \cancel{n} t \cdot \frac{1}{2} = \frac{t}{2}.\end{aligned}$$

This means that by means of the CLT

$$S^{\star}_{(n)}(t) \xrightarrow{\mathrm{d}} \mathsf{N}\left(0, \frac{t}{2}\right) \qquad \text{as } n \to \infty$$

and this hints at the fact that the limiting process may be a Brownian motion with diffusion coefficient $\sigma^2 = \frac{1}{2}$. We can check the 4 properties of Brownian motion:

1. initial condition:
$$S^{\star}_{(n)}(0) = 0$$
for every n, even when $n \to \infty$;
2. independent increments: this is true since the random walk had independent increments;
3. stationary and normally distributed increments: for $t \leqslant s$ we have
$$S^{\star}_{(n)}(t) - S^{\star}_{(n)}(s) = \frac{1}{\sqrt{(n)}} \sum_{i=ns+1}^{nt} X_i \xrightarrow[n\to\infty]{d} \mathsf{N}\left(0, \frac{t-s}{2}\right)$$
since this is always a sum of $(t-s)n$ i.i.d. random variables X_i with variance $\frac{1}{2}$;

④ continuity of paths: by Donsker's theorem[1] we know that, if the increments of the random walk have zero mean and finite variance, then

$$S^{\star}_{(n)}(t) \xrightarrow[n\to\infty]{d} B_t \qquad \forall\, t \in (0,1].$$

This convergence gives pointwise but not uniform results: to get the latter, we need the *invariance principle* by Donsker[2]: if $\Phi : \mathcal{C}([0,1],\mathbb{R}) \to \mathbb{R}$ (the space of continuous real-valued functions defined on $[0,1]$) is a uniformly continuous bounded functional then

$$\lim_{n\to\infty} \mathbb{E}\left[\Phi\left(S^{\star}_{(n)}(\cdot)\right)\right] = \mathbb{E}\left[\Phi(B(\cdot))\right].$$

This implies that the limit process lives in $\mathcal{C}([0,1],\mathbb{R})$ and therefore the limit (which is the Brownian motion) has continuous sample paths almost surely.

$$S^{\star}_{(n)}(\cdot) \Rightarrow B(\cdot) \quad \text{in } \mathcal{C}([0,1],\mathbb{R}).$$

2. The state space of the random walk is $\mathbb{Z}$ so the diffusion interval is $(-\infty,+\infty)$. To classify these borders we need to compute, as before, the four quantities of interest. In this case $\mu(x) = 0$ and $\sigma^2 = \frac{1}{2}$.

   i) *scale measure* $S(x)$:

$$\begin{aligned} S(x) &= \int_{-\infty}^{x} \exp\left\{-\int^{\eta} \frac{0}{\frac{1}{2}}\,\mathrm{d}\xi\right\}\mathrm{d}\eta \\ &= \int_{-\infty}^{x} \exp\{0\}\,\mathrm{d}\eta \\ &= \int_{-\infty}^{x} 1\,\mathrm{d}\eta = \infty. \end{aligned}$$

   ii) *speed measure* $M(x)$:

$$\begin{aligned} M(x) &= \int_{-\infty}^{x} \frac{1}{\sigma^2(y)\cdot s(y)}\,\mathrm{d}y \\ &= \int_{-\infty}^{x} \frac{1}{\frac{1}{2}\cdot 1}\,\mathrm{d}y \\ &= \int_{-\infty}^{x} 2\,\mathrm{d}y = \infty. \end{aligned}$$

We already have that $S(-\infty,x) = S(x,+\infty) = \infty$ and $M(-\infty,x) = M(x,+\infty) = \infty$. Since we defined

$$\Sigma(l) = \int_{l}^{x} M[\eta,x]\,\mathrm{d}S(\eta) \quad \text{and} \quad N(l) = \int_{l}^{x} M(l,\xi]\,\mathrm{d}S(\xi)$$

[1] SP12, p. 198.
[2] SP12, p. 199.

then we will have also that $\Sigma(-\infty) = \Sigma(\infty) = \infty$ and $N(-\infty) = N(\infty) = \infty$. According to our table, this means that both boundaries will be

- natural;
- non-attracting;
- unattainable.

3. A natural boundary is not accessible from the interior of the state space and it is neither reflecting or absorbing. We know that an hypothetical stationary distribution $\psi(x)$ should satisfy

$$\psi(x) = m(x)\,[c_1 S(x) + c_2]$$

and that if we cannot find the constants c_1 and c_2 such that $\psi(x)$ is a density then the stationary distribution doesn't exist. In our case we have $m(x) = 2$ and $[c_1 S(x) + c_2] = [c_1 \cdot \infty + c_2]$ so we will never find constants such that $\psi(x)$ integrates to 1! Even choosing $c_1 = 0$ we have

$$\int_I \psi(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} 2c_2\,\mathrm{d}x = \infty.$$

This makes sense because the process can move arbitrarily in any direction (there is no drift) and it will never return with certainty to any compact region.

References

[SP12] Renè L. Schilling and Lothar Partzsch. **Brownian Motion**. Ed. by De Gryter Graduate. 2012.