This Should Help Your Lazy Ass In Analisys B

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1 Normed Spaces

Proposition 1.1

A subset $C \subset X$ is a closed subset of $(X, \|\cdot\|)$ whenever the limit of any convergent sequence $(x_n)_n \subset C$ belongs to C.

In other words, C is closed if and only if once a sequence of elements of C is converging, the limit cannot escape C.

Proof

[redacted]

Lemma 1.1

Young's inequality. Let $p > 1$ and let $q < 1$ be its conjugate exponent. Then, for any nonnegative $a, b \in \mathbb{R}$ it holds

$$ab \leqslant \frac{1}{p}a^p + \frac{1}{b}b^q.$$

Theorem 1.1

Holder Inequality. Let $1 \leqslant p \leqslant \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Assume that $f \in L^p(S, \mu)$ and $g \in L^q(S, \mu)$. Then $f \cdot g \in L^1$ and

$$\|fg\|_1 \leqslant \|f\|_p \|g\|_q .$$

Proof

The proof is trivial if $p = 1$ and $q = 1$. Remember Young's inequality

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q.$$

Now let's say $a = |f(s)|$ and $b = |g(s)|$. Now our inequality becomes

$$|f(s)g(s)| \leq \frac{1}{p}|f(s)|^p + \frac{1}{q}|g(s)|^q \quad \text{for } \mu\text{-a.e. on } S.$$

Now we integrate over S and we get

$$\int_S |f(s)g(s)| \leq \frac{1}{p}\int_S |f(s)|^p + \frac{1}{q}\int_S |g(s)|^q$$

$$\implies \|fg\|_1 \leq \frac{1}{p}\|f\|_p^p + \frac{1}{q}\|g\|_q^q$$

which means that $\|fg\|_1$ is finite and therefore $fg \in L^1(S, \mu)$. To end the proof let's substitute f with λf, $\forall \lambda > 0$. We get

$$\lambda\|fg\|_1 \leq \frac{\lambda^p}{p}\|f\|_p^p + \frac{1}{q}\|g\|_q^q \qquad \forall \lambda > 0$$

$$\implies \frac{\lambda}{\lambda}\|fg\|_1 \leq \frac{\lambda^p}{p\lambda}\|f\|_p^p + \frac{1}{q\lambda}\|g\|_q^q \qquad \forall \lambda > 0$$

$$\implies \|fg\|_1 \leq \frac{\lambda^{p-1}}{p}\|f\|_p^p + \frac{1}{\lambda q}\|g\|_q^q \qquad \forall \lambda > 0$$

Now choose $\lambda = \frac{1}{\|f\|_p} \cdot \|g\|_q^{\frac{q}{p}}$. When we substitute this value for λ, we get:

$$\frac{\lambda^p}{p}\|f\|_p^p = \frac{\left(\frac{1}{\|f\|_p} \cdot \|g\|_q^{\frac{p}{q}}\right)^p}{p}\|f\|_p^p$$

Expanding $\left(\frac{1}{\|f\|_p} \cdot \|g\|_q^{\frac{p}{q}}\right)^p$, we get:

$$\left(\frac{1}{\|f\|_p} \cdot \|g\|_q^{\frac{p}{q}}\right)^p = \frac{\|g\|_q^p}{\|f\|_p^p}.$$

Substituting this, we have:

$$\frac{\lambda^p}{p}\|f\|_p^p = \frac{\frac{\|g\|_q^p}{\cancel{\|f\|_p^p}}}{p}\cancel{\|f\|_p^p} = \frac{\|g\|_q^p}{p}$$

Substitute back into the inequality:

$$\lambda\|fg\|_1 \leq \frac{\|g\|_q^p}{p} + \frac{1}{q}\|g\|_q^q.$$

Since $\frac{\|g\|_q^p}{p} + \frac{\|g\|_q^q}{q} = \|g\|_q^p\left(\frac{1}{p} + \frac{1}{q}\right)$ and $\frac{1}{p} + \frac{1}{q} = 1$, we get:

$$\lambda\|fg\|_1 \leq \|g\|_q^p.$$

Dividing by $\lambda = \frac{1}{\|f\|_p}\|g\|_q^{\frac{p}{q}}$:

$$\|fg\|_1 \leq \|f\|_p\|g\|_q.$$

This is Hölder's inequality:

$$\|fg\|_1 \leq \|f\|_p\|g\|_q$$

□

Theorem 1.2

For any $1 \leqslant p \leqslant \infty$, $L^p(S, \mu)$ is a vector space and $\|\cdot\|_p$ is a norm.

Remark

For $1 < p < \infty$ the triangular inequality

$$\|f+g\|_p \leqslant \|f\|_p + \|g\|_p \qquad \forall\, f, g \in L^p(S, \mu)$$

is known as **Minkowski's Inequality**.

Proof

We already know that if $f \in L^p(S, \mu)$ then $\lambda f \in L^p(S, \mu)$. Homogeneity and uniqueness are also existent for $\|\cdot\|_p$ so in order to show that $L^p(S, \mu)$ is a vector space we only need to prove that if $f, g \in L^p(S, \mu)$ then $f + g \in L^p(S, \mu)$ and $\|\cdot\|$ is a norm.
Fix $f, g \in L^p(S, \mu)$. We know that for any $x, y \in \mathbb{R}$ we get

$$\left|\frac{1}{2}x + \frac{1}{2}y\right|^p \leqslant \frac{1}{2}|x|^p + \frac{1}{2}|y|^p$$

since this mapping $r \to r^p$ is convex. This also means that

$$|x+y|^p \leqslant 2^{p-1}\left(|x|^p + |y|^p\right).$$

and this implies in particular that

$$|f(s) + g(s)|^p \leqslant 2^{p-1}\left(|f(s)|^p + |g(s)|^p\right) \qquad \text{for } \mu\text{-a.e. on } s \in S.$$

If we integrate over S we get:

$$\int_S |f(s) + g(s)|^p \leqslant 2^{p-1}\left(\int_S |f(s)|^p + \int_S |g(s)|^p\right)$$

which means

$$\|f+g\|_p^p \leqslant 2^{p-1}\left(\|f\|_p^p + \|g\|_p^p\right)$$

which means that $f + g \in L^p(S, \mu)$.
We now must prove the Minkowski's inequality. We know that

$$\|f+g\|_p^p = \int_S |f+g|^p \,\mathrm{d}\mu = \int_S |f+g|\,|f+g|^{p-1} \,\mathrm{d}\mu$$

but since we know that $|f+g| \leqslant |f| + |g|$ then

$$\|f+g\|_p^p \leqslant \int_S |f||f+g|^{p-1} \,\mathrm{d}\mu + \int_S |g||f+g|^{p-1} \,\mathrm{d}\mu.$$

Call $\psi = |f+g|^{p-1}$. It clearly belongs to $L^q(S, \mu)$ because

$$|\psi|^q = \left(|f+g|^{p-1}\right)q = |f+g|^p \qquad \text{since } q(p-1) = p$$

so

$$\|\psi\|_q = \left(\int_S |\psi|^q\right)^{\frac{1}{q}} = \left(\int_S |f+g|^p\right)^{\frac{1}{q}} = \|f+g\|_p^{\frac{p}{q}} < \infty$$

And this means that $|\psi|^q \in L^1(S, \mu) \implies \psi \in L^q(S, \mu)$. We also know that $|f| \in L^p(S, \mu)$ so we can apply Holder's inequality with f and ψ so that

$$\int_S |f||f+g|^{p-1} \,\mathrm{d}\mu = \|f\psi\|_1 \leqslant \|f\|_p \|\psi\|_q = \|f\|_p \|f+g\|_p^{\frac{p}{q}}$$

and

$$\int_S |g||f+g|^{p-1} \,\mathrm{d}\mu \leqslant \|g\|_p \|f+g\|_p^{\frac{p}{q}}$$

So that

$$\|f+g\|_p^p \leqslant \|f\|_p \|f+g\|_p^{\frac{p}{q}} + \|g\|_p \|f+g\|_p^{\frac{p}{q}}.$$

Dividing by $\|f+g\|_p^{\frac{p}{q}} \neq 0$ (otherwise the proof is trivial) we get

$$\|f+g\|_p^{p-\frac{p}{q}} \leqslant \|f\|_p + \|g\|_p.$$

□

Proposition 1.2

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability triplet. then the following holds:

$$L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \subset L^p(\Omega, \mathcal{F}, \mathbb{P}) \subset L^q(\Omega, \mathcal{F}, \mathbb{P}) \subset L^1(\Omega, \mathcal{F}, \mathbb{P}).$$

Of course, this result remains valid for every other measure space $(S, \mathcal{S}, \mu)$ as long as $\mu(S) < \infty$. In the special case in which

$$S = \mathbb{N} \qquad \mathcal{F} = \mathcal{P}(\mathbb{N})$$

and $\mu(A)$ is the counting measure $\mu(A) = \sum_{k\in A} \delta_k(A), A \in \mathbb{N}$ then knowing that sequences $n \mapsto f(n)$ can be identified as functions over $\mathbb{N}$ of the type $f : \mathbb{N} \to \mathbb{R}$ we see that

$$L^1(S,\mu) = \underbrace{\mathcal{L}^1(S,\mu)}_{\text{actual functions, not equivalence classes}} = \ell^1(\mathbb{N}) = \left\{ \mathbf{x} = (x_n)_n ; \|\mathbf{x}\|_1 := \sum_{n=1}^{\infty} |x_n| < \infty \right\}.$$

This means that $\ell^1(\mathbb{N})$ is a L^1 space for some special choice of S and μ. Since we chose our measure as the counting measure, we get

$$\int_{\mathbb{N}} |f(n)| \, \mathrm{d}\mu(n) = \sum_{n=1}^{\infty} |f(n)| = \sum_{n=1}^{\infty} |x_n|.$$

Cool!

Proposition 1.3

Let $p \geqslant 1$ be given. We define the set

$$\ell^p(\mathbb{N}) = \left\{ \mathbf{x} = (x_n)_{n\in\mathbb{N}} \subset \mathbb{R} \text{ such that } \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}.$$

Then, $\ell^p(\mathbb{N})$ is a vector space. Moreover, if

$$\|\mathbf{x}\|_p := \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}} \qquad \forall \mathbf{x} = (x_n)_n \in \ell^p(\mathbb{N})$$

then $\left(\ell^p(\mathbb{N}), \|\cdot\|_p\right)$ is a normed space.

1.1 The space of linear applications

Proposition 1.4

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed spaces and let $L; X \mapsto Y$ be a linear application. The following are equivalent:

1. L is continuous on X;
2. L is continuous at $x = 0$;
3. there is a positive constant $C > 0$ such that

$$\|L(x)\|_Y \leqslant C \|x\|_X \qquad \forall x \in X.$$

Proof

Of course ① $\implies$ ②, but now let's prove ③. Consider the definition of continuity at 0 for the function L: this means that for any $\varepsilon > 0$ there exists a constant $\delta > 0$ such that

$$\|x - 0_X\|_X \leqslant \delta \implies \|L(x) - L(0_Y)\|_Y \leqslant \varepsilon.$$

We know that L is linear, so $L(0_X) = 0_Y$ and therefore we get

$$\|x\|_X \leqslant \delta \implies \|L(x)\|_Y \leqslant \varepsilon.$$

Choose $\varepsilon = 1$ so that we get

$$\|x\|_X \leqslant \delta \implies \|L(x)\|_Y \leqslant 1.$$

Let $x \in X \setminus \{0_X\}$ and set $y = \frac{\delta}{\|x\|_X} x$. Now we have that

$$\|y\|_X = \left\| \frac{\delta}{\|x\|_X} x \right\|_X = \frac{\delta}{\|x\|_X} \|x\|_X = \delta$$

and, due to the linearity of L,

$$\|L(y)\|_Y = \left\| L\left(\frac{\delta}{\|x\|_X} x \right) \right\|_Y = \frac{\delta}{\|x\|_X} \|L(x)\|_Y = \frac{\delta}{\|x\|_X} \|L(x)\|_Y \leqslant 1 \implies \|L(x)\|_Y \leqslant \frac{1}{\delta} \|x\|_X$$

So, setting $C = \frac{1}{\delta}$ proves ③. We still need to prove ① $\implies$ ③: it is easy, since by linearity we have

$$\|L(x) - L(y)\|_X = \|L(x-y)\|_Y \leqslant C \|x - y\|_Y \quad \forall x, y \in X$$

and this clearly implies the continuity of L at any $x \in X$ (actually, the uniform continuity). □

Definition 1.1

If $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are two normed spaces, we denote by $\mathcal{L}(X, Y)$ the space of continuous linear applications from X to Y. If $X = Y$ we simply denote $\mathcal{L}(X) = \mathcal{L}(X, X)$.

Proposition 1.5

If $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are two vector spaces and X is of finite dimension, any linear application $L : X \mapsto Y$ is continuous.

Remark

If $\dim(X) = n$ and $\dim(Y) = p$, the space $\mathcal{L}(X, Y)$ can be identified with the space $\mathcal{M}_{n \times p}(\mathbb{R})$ of matrices with n lines and p rows.

1.2 Compactness

Definition 1.2

Let $(X, \|\cdot\|_X)$ be a normed space and let $K \subset X$. We say that K is **compact** if every sequence $(x_n)_n$ contains a subsequence which converges to some $x \in K$.

Of course if K is compact then it is closed.

Lemma 1.2

If K is a compact subset of a normed space $(X, \|\cdot\|_X)$ then K is closed and there exists $M > 0$ such that $\sup_{x \in K} \|x\| \leqslant M$ which means that K is bounded.

Proposition 1.6

Product of compact spaces. Let $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$ be two compact normed spaces and let $X = X_1 \times X_2$. Then $(X, \|\cdot\|_+)$ and $(X, \|\cdot\|_{\max})$ are compact normed spaces.

Remember that

$$\|\mathbf{x}\|_+ = \|x_1\|_1 + \|x_2\|_2$$

and

$$\|\mathbf{x}\|_{\max} = \max(\|x_1\|_1, \|x_2\|_2)$$

Proof

[redacted]

Of course, the above result readily extends to any finite product of compact normed spaces. On $\mathbb{R}$ it is easy to describe a large class of compact sets:

Lemma 1.3

Let $\mathbb{R}$ be endowed with the absolute value, $|\cdot|$. Any interval $[a, b] \subset \mathbb{R}$ is compact.

Proposition 1.7

Let $(X, \|\cdot\|)$ be a normed space and let K be a compact subset of X. If $A \subset K$ is a closed subset then A is compact.

Corollary

Heine-Borel theorem. A subset K of $\mathbb{R}^N$ (where $\mathbb{R}^N$ is endowed with, say, the usual Euclidean norm) is compact if and only if it is closed and bounded.

Proof

[redacted]

This corollary can be reformulated as:

Every bounded sequence of $\mathbb{R}^N$ has a convergent subsequence.

1.3 Compactness and continuous functions

Proposition 1.8

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed spaces and let $f : X \to Y$ be continuous. If $K \subset X$ is a compact subset of X then $f(K)$ is a compact subset of Y.

Proof

[redacted]

This has the following consequence:

Theorem 1.3

Let $(X, \|\cdot\|)$ be a normed space and let $K \subset X$ be compact. Let $f : K \to \mathbb{R}$ be continuous. Then, f assumes its maximum and minimum on K.

Proof

[redacted]

Theorem 1.4

Heine Theorem. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed spaces and let $K \subset X$ be compact. Assume that $f : K \to Y$ is continuous. Then f is uniformly continuous on K.

Proof

[redacted]

So, in this case taking two sequences that get closer and closer does not correspond to the fact also their functions get closer and closer... and this is not possible.

1.4 Finite dimensional spaces

Proposition 1.9

Let $(X, \|\cdot\|)$ be a finite dimensional normed vector space with $\dim(X) = d$ and let $\{e_1, \ldots, e_d\}$ be a basis for X. Then, there are positive constants $C_0, C_1 > 0$ such that

$$C_0 \sum_{i=1}^{d} |x_i| \leqslant \left\| \sum_{i=1}^{d} x_i e_i \right\| \leqslant C_1 \sum_{i=1}^{d} |x_i| \qquad \forall (x_1, \ldots, x_d) \in \mathbb{R}^d.$$

This proposition asserts that if $\dim(X) = d$ then *any* norm $\|\cdot\|$ is related to the $\|\cdot\|_1$ norm of $\mathbb{R}^d$. This translates in the following:

Proposition 1.10

If X is a finite dimensional vector space, all norms over X are equivalent.

So there is no weird norm, but everything is comparable to the simple $\|\cdot\|_1$ norm. This proposition also allows us to identify in a continuous way a finite dimensional space $(X, \|\cdot\|)$ and the space $\mathbb{R}^d$ where d is the dimension of X. Indeed, introducing a basis $\{e_1, \ldots, e_d\}$ of X, the mapping

$$\Phi : X \to \mathbb{R}^d$$

which associates $\Phi(\mathbf{x}) = (x_1, \ldots, x_d)$ to some $\mathbf{x} = \sum_{i=1}^{d} x_i e_i \in X$, is a bijection from X to $\mathbb{R}^d$ which is continuous whose inverse is also continuous. This results in the following:

Corollary

If $(X, \|\cdot\|)$ is a finite dimensional vector space and $K \subset X$ is closed and bounded then K is compact.

Again, this is very specific to finite dimensional spaces and, as we shall see, this actually characterizes finite dimensional spaces. Indeed, in infinite dimensional normed spaces, the closed unit ball cannot be compact. This shows that, in infinite dimensional spaces, the compact subsets do not coincide with closed and bounded subsets!! We first state the following technical lemma:

Lemma 1.4

Riesz Lemma. Let $(X, \|\cdot\|)$ be a normed vector space and let Y be a closed subspace of X (i.e. Y is closed in X and Y is a linear subspace of X). If $Y \neq X$ then for any $\varepsilon \in (0,1)$ there exists $x \in X$ with $\|x\| = 1$ such that

$$\inf_{y \in Y} \|x - y\| \geqslant 1 - \varepsilon.$$

Remark

This lemma asserts that if $Y \neq X$ is a closed subspace then for any $\varepsilon \in (0,1)$ there is some unit vector $x \in X$ such that $\text{dist}(x, Y) \geqslant 1 - \varepsilon$.

Proof

Let $z \in X \setminus Y$. Since Y is closed and $z \notin Y$, one has

$$\alpha = \text{dist}(z, Y) = \inf_{y \in Y} \|z - y\| > 0.$$

Pick $\varepsilon \in (0,1)$. There exists $\overline{y} \in Y$ such that $\|\overline{y} - z\| \leqslant \frac{\alpha}{1-\varepsilon}$ (otherwise we would get $\text{dist}(z, Y) \geqslant \frac{\alpha}{1-\varepsilon} > \alpha$!). Notice that $\overline{y} \neq z$ so that $r : +\|\overline{y} - z\| > 0$. Set

$$x := \frac{1}{r}(z - \overline{y}).$$

Clearly, $\|x\| = 1$. Let $y \in Y$ be given. One can write

$$\|x - y\| = \frac{1}{r} \|z - \overline{y} - ry\|$$

and since Y is a linear subspace $\overline{y} + ry \in Y$ so that $\|z - \overline{y} - ry\| \geqslant \alpha$. Therefore $\|x - y\| \geqslant \frac{\alpha}{r} \geqslant 1 - \varepsilon$ by assumption on $r = \|z - \overline{y}\|$. Since this is true for any $y \in Y$, this proves the result. $\square$