Stochastic Processes notes

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vaffanculo

Preface

Let's have a fucking party!

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Chapter 1

Brownian Motion

1.1 Continuity of stochastic processes

The sample path in a discrete scenario is given by

$$\mathbb{P}(X(t) < x|\mathcal{F}_s) = \mathbb{P}(X(t) < x|X_s).$$

This gives us a sample path like this:

Figure 1.1: Fucking hell.

We want to think, though, about a continuous space and time process:

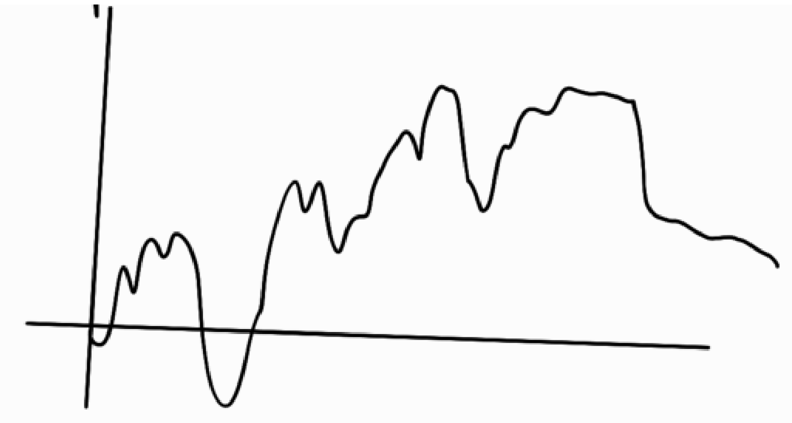

Figure 1.2: Leaning like my academic career.

How can we define the continuity of the sample paths? and how can we check this property? There are some possible definitions of continuity. To control for their robustness we check whether according to each of these definitions the Poisson process, a discrete process, is correctly classified as non continuous.

Definition 1.1.1

A stochastic process $\{M(t)\}$ is said to be a **counting process** if:

i. $M(t) > 0$;

ii. $M(t)$ is an integer;

iii. $M(t)$ is increasing, meaning that $s \leqslant t \implies M(s) \leqslant M(t)$.

In general, these processes count how many times an event happens.

Definition 1.1.2

A Poisson process is a counting process such that:

1. $N(0) = 0$;
2. for $t_1 < t_2 < t_3 < t_4$ we have
$$N(t_2) - N(t_1) \perp\!\!\!\perp N(t_4) - N(t_3)$$
meaning that the increments are independent;
3. for $\forall\, h > 0$ and $t > \tau$ it holds:
$$N(t) - N(\tau) \sim N(t + h) - N(\tau + h)$$
meaning that the increments are stationary (the origin start doesn't matter);
4. we have
$$\mathbb{P}(N(t) = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t} \qquad \text{for } k = 0, 1, \ldots$$

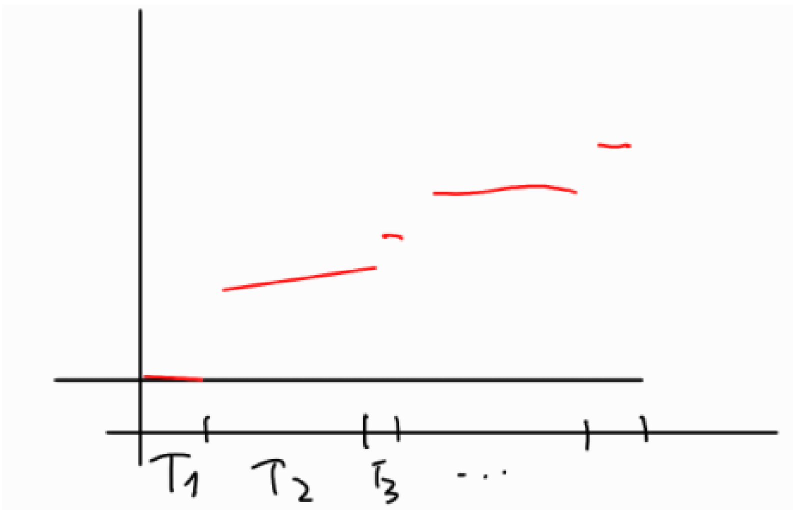

Figure 1.3: I am very good at drawing straight lines.

The inter-arrival times are i.i.d. random variables distributed as

$$T_i \sim \mathsf{Exp}\,(\lambda)\,.$$

The continuity of sample paths can be characterized in four different ways:

1. **Continuity in mean squares**: we have this if for $\forall\, t \geqslant 0$ we have
$$\lim_{s \to t} \mathbb{E}\left[|X(t) - X(s)|^2\right] = 0.$$
So according to this definition, the mean square of the distance goes to 0 if we go near s.

2. **Continuity in probability**: we have this if for $\forall\, t \geqslant 0$ and $\forall\, \varepsilon > 0$ we have

$$\lim_{s \to t} \mathbb{P}(|X(t) - X(s)| > \varepsilon) = 0.$$

This should be enough for all finite distributions, right?[1]

Theorem 1.1.1

Let $\{X(t)\}$ be a stochastic process such that $\mathbb{E}[X^2(t)] < \infty$ for all t. Then it is continuous in mean squaresif and only if:

(a) $m(t) = \mathbb{E}[X(t)]$ is continuous;

(b) the covariance function

$$\Gamma(s,t) = \mathbb{E}[(X(t) - m(t))(X(s) - m(s))]$$

is continuous on its diagonal set.

Proof

Consider the expectation

$$\mathbb{E}\left[|X(t) - X(s)|^2\right] = \mathbb{E}\left[|X^2(s) + X^2(t) - 2X(t)X(s)|\right] \qquad (◐)$$

but

$$\mathbb{E}\left[X^2(s)\right] = \underbrace{\mathbb{E}\left[(X(s) - m(s))^2\right]}_{\Gamma(s,s)} + 2m(s)\mathbb{E}\left[X(s)\right] - m^2(s)$$

and

$$\mathbb{E}\left[X^2(t)\right] = \Gamma(t,t) + 2m(t)\mathbb{E}[X(t)] - m^2(t)$$

and, moreover,

$$\mathbb{E}[X(s)X(t)] = \Gamma(s,t) - \cancel{m(t)m(s)} + m(t)\mathbb{E}[X(s)] + \cancel{m(s)\mathbb{E}[X(t)]}.$$

So ◐ becomes

$$\begin{aligned}
◐ &= \Gamma(s,s) + 2m^2(s) - m^2(s) + \Gamma(t,t) + 2m^2(t) - m^2(t) - 2\mathbb{E}\left[X(s)X(t)\right] \\
&= \Gamma(s,s) + \Gamma(t,t) - 2\Gamma(s,t) + m^2(t) + m^2(s) - 2m(t)m(s) \\
&= \Gamma(s,s) + \Gamma(t,t) - 2\Gamma(s,t) + [m(t) - m(s)]^2 .
\end{aligned}$$

Hence, if $m(t)$ is continuous and $\Gamma(s,t)$ is continuous for $s = t$ then the process is continuous because we have:

- $[m(t) - m(s)]^2 \to 0$ since it is a continuous function;
- $\Gamma(s,s) + \Gamma(t,t) - 2\Gamma(s,t)$ that becomes $\Gamma(t,t) + \Gamma(t,t) - 2\Gamma(t,t) = 0$.

I'd like to add that dear prof. Sacerdote didn't explain this last little point. Thank you! So now whe have

$$\mathbb{E}\left[|X(s) - X(t)|^2\right] \to 0.$$

If this holds for $m(t)$ and $\Gamma(t,t)$ then it is continuous in mean squares. □

Remark

A process continuous in mean □ (get it?) is continuous in probability (use Chebyshev[a]).

[a] Like use him? As a person? He is dead.

Is the Poisson process continuous in mean □ (and also in probability)? We know that

$$m(t) = \lambda t$$

[1] First, but not last, question without an answer.

and

$$
\begin{aligned}
\Gamma(s,t) &= \mathbb{E}[(N(t)-m(t))(N(s)-m(s))] \\
&= \mathbb{E}[N(t)N(s)] - 2m(t)m(s) - m(t)m(s) \\
&\underset{t>s}{=} \mathbb{E}[(N(t)-2N(s)+N(s))N(s)] - m(t)m(s) \\
&= \mathbb{E}[(N(t)-N(s))N(s)] + \mathbb{E}\left[N^2(s)\right] - m(t)m(s) \\
&= \underbrace{\mathbb{E}[N(t)-N(s)]}_{\lambda(t-s)}\underbrace{\mathbb{E}(N(s))}_{\lambda s} + \mathbb{E}\left[N^2(s)\right] - \underbrace{m(t)}_{\lambda t}\underbrace{m(s)}_{\lambda s} \\
&= \lambda^2(\cancel{t}-s)s - \cancel{\lambda^2 ts} + \mathbb{E}\left[N^2(s)\right] \\
&= -\lambda^2 s^2 + \underbrace{\operatorname{Var} N(s)}_{\lambda s} + \underbrace{[\mathbb{E}\left[N(s)\right]]^2}_{\lambda^2 s^2} \\
&= \lambda s \qquad \text{if } s<t.
\end{aligned}
$$

So $\Gamma(s,t) = \lambda \min(s,t)$ is continuous on diagonal and $m(t) = \lambda t$ is also continuous... but this would

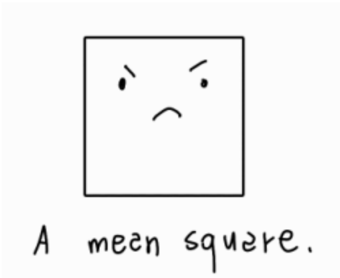

Figure 1.4: Sei un fallito.

mean that Poisson processes are continuous! Which they shouldn't be! Do I care? No!

3. **Almost sure continuity**: we can ask, as a requirement, that

$$\mathbb{P}\left(\lim_{s\to t} N(s) = N(t)\right) = 1.$$

Does this finally solve the problem with the Poisson processes? No, because it verifies the almost sure continuity (since it is discontinuous only in a countable number of instances).
It is not enought ot think point by point: we must think *uniformly*.

Definition 1.1.3

A stochastic process $\{X(t)\}$ has almost sure continuous sample paths if, with probability 1, $X(t)$ is a continuous function, that is:

$$\mathbb{P}(X(t) \text{ has continuous samples}) = 1.$$

Of course, the case in which you have exceptional points is not a problem since they have measure 0.

Figure 1.5: Me neither.

Remark

The set

$$\{\omega \in \Omega : t \to B_t(\omega) \text{ is continuous}\}$$

is not necessarily in the σ-algebra generated by the vectors

$$\left(B_{t_1}, B_{t_2}, \ldots, B_{t_n}\right) \qquad n \in \mathbb{N}.$$

List of Figures