LÉVY PROCESSES

Introduced in the 1930's by Paul Lévy and Bruno de Finetti.

Important applications to math finance, Bayesian nonparametrics, math biology among other fields.

<u>Def.</u> A R.V. Y is said to be INFINITELY DIVISIBLE (I.D.) if $\forall n \in \mathbb{N}$ there are $Y_i^{(n)}$ iid s.t.

$$Y \overset{d}{=} Y_1^{(n)} + \dots + Y_n^{(n)}$$

i.e., Y can be written as sum of an arbitrary no. of iid RV's.

<u>Examples</u>

- $Y \sim Po(\lambda)$: $\quad Y_i^{(n)} \overset{iid}{\sim} Po\left(\frac{\lambda}{n}\right)$

$$\Rightarrow \sum_{i=1}^{n} Y_i^{(n)} \sim Po\left(\sum_{i=1}^{n} \frac{\lambda}{n}\right) = Po(\lambda)$$

- $Y \sim N(\mu, s^2)$: $\quad Y_i^{(n)} \overset{iid}{\sim} N\left(\frac{\mu}{n}, \frac{s^2}{n}\right)$

$$\Rightarrow \sum_{i=1}^{n} Y_i^{(n)} \sim N(\mu, s^2)$$

- $Y \sim \text{Gamma}(\alpha, \beta)$:

$$Y_i^{(m)} \overset{iid}{\sim} \text{Gamma}\left(\frac{\alpha}{m}, \beta\right)$$

$$\Rightarrow \sum_{i=1}^{m} Y_i^{(m)} \sim \text{Gamma}(\alpha, \beta)$$

A simple way to establish I.D. is using $E(e^{iuY})$

- $Y \sim Po(\lambda)$

$$E(e^{iuY}) = \sum_{k \geq 0} e^{iuk} \frac{\lambda^k e^{-\lambda}}{k!}$$

$$= e^{-\lambda} \sum_{k \geq 0} \frac{(\lambda e^{iu})^k}{k!} = e^{-\lambda} e^{\lambda e^{iu}}$$

$$= e^{-\lambda(1-e^{iu})}$$

$$Y_i^{(m)} \overset{iid}{\sim} Po\left(\frac{\lambda}{m}\right)$$

$$E\left(e^{iu\sum_1^m Y_i^{(m)}}\right) = \left[e^{-\frac{\lambda}{m}(1-e^{iu})}\right]^m \quad \text{using iid}$$

$$= e^{-\lambda(1-e^{iu})}$$

This suggests a strategy: define the CHARACTERISTIC EXPONENT (C.E.) of Y to be

$$\psi(u) = -\log E(e^{iuY})$$

Then Y is I.D. if $\exists$ a R.V. with C.E. $\psi^{(n)}$ s.t.

$$\psi(u) = n\,\psi^{(n)}(u)$$

Check for $N(\mu, s^2)$

Example $Y \sim \text{Gamma}(\alpha, \beta)$

$$E(e^{iuY}) = \frac{1}{\left(1 - \frac{iu}{\beta}\right)^{\alpha}} = \left(\frac{1}{\left(1 - \frac{iu}{\beta}\right)^{\alpha/n}}\right)^{n}$$

$$\Rightarrow Y_i^{(n)} \overset{iid}{\sim} \text{Gamma}\left(\frac{\alpha}{n}, \beta\right)$$

<u>Def.</u>

Let $N \sim Po(\lambda)$ and $Z_i \overset{iid}{\sim} F$ on $\mathbb{R}$ with no mass at 0. The R.V.

$$Y = \sum_{i=1}^{N} Z_i \qquad (Y = 0 \text{ if } N = 0)$$

is said to have COMPOUND POISSON distribution. Its C.F. is

$$E\left(e^{iu\sum_{i=1}^{N} Z_i}\right) = E\left[E\left(e^{iu\sum_{i=1}^{k} Z_i} \,\middle|\, N = k\right)\right]$$

$$= \sum_{k \geq 0} \frac{\lambda^k e^{-\lambda}}{k!} E\left(e^{iu \sum_{i=1}^{k} Z_i}\right)$$

$$= \sum_{k \geq 0} \frac{\lambda^k e^{-\lambda}}{k!} \Big[\underbrace{E\left(e^{iu Z_1}\right)}_{\int_{\mathbb{R}} e^{iux} F(dx)} \Big]^k$$

$$= e^{-\lambda} e^{\lambda \int_{\mathbb{R}} e^{iux} F(dx)}$$

$$= e^{-\lambda \int_{\mathbb{R}} (1 - e^{iux}) F(dx)}$$

Now

$$\psi(u) = \lambda \int_{\mathbb{R}} (1 - e^{iux}) F(dx) \cdot \frac{m}{m}$$

$$= m \underbrace{\frac{\lambda}{m} \int_{\mathbb{R}} (1 - e^{iux}) F(dx)}_{\psi^{(m)}(u)}$$

so Y is I.D. r.v.

$$Y = \sum_{i=1}^{N} Z_i \stackrel{d}{=} \sum_{j=1}^{m} Y_j^{(m)}$$

still CP with Poisson rate/mean $\frac{\lambda}{m}$

$$\left[\quad Y_j^{(m)} = \sum_{i=1}^{N_j^{(m)}} Z_i \, , \quad Z_i \stackrel{iid}{\sim} F, \quad N_j^{(m)} \stackrel{iid}{\sim} Po\left(\frac{\lambda}{m}\right) \right.$$

Theorem (Lévy-Khintchine formula)

A R.V. on $\mathbb{R}$ with C.E. $\psi(u)$ is I.D.
if and only if there exist:
$\mu \in \mathbb{R}$, $\sigma \geq 0$ and a
measure π on $\mathbb{R} \setminus \{0\}$ satisfying

$$\int_{\mathbb{R}} \underbrace{(1 \wedge x^2)}_{\min(1, x^2)} \pi(dx) < \infty$$

s.t., $\forall u \in \mathbb{R}$,

$$\psi(u) = i\mu u + \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} \left(1 - e^{iux} + iux \mathbb{1}_{(|x|<1)}\right) \pi(dx).$$

This characterizes I.D. distributions.

- π is called Lévy intensity measure
- (μ, σ, π) is called characteristic TRIPLET.

Examples:

- $(\mu, \sigma, \pi) = (-m, s, 0)$ i.e. $\pi \equiv 0$

$$\psi(u) = -imu + \frac{1}{2} s^2 u^2$$

C.E. of $N(m, s^2)$

- $(\mu, \sigma, \pi) = (0, 0, \lambda \delta_1)$

$$\psi(u) = \lambda \int_{\mathbb{R}} (1 - e^{iux} + iux \underbrace{\mathbb{1}_{(|x|<1)}}_{=0}) \delta_1(dx)$$

$$= \lambda \int_{\mathbb{R}} (1 - e^{iux}) \delta_1(dx) = \lambda(1 - e^{iu})$$

$$\Rightarrow Y \sim Po(\lambda)$$

The requirement

$$\int_{\mathbb{R}} \min(1, x^2) \pi(dx) < \infty$$

implies

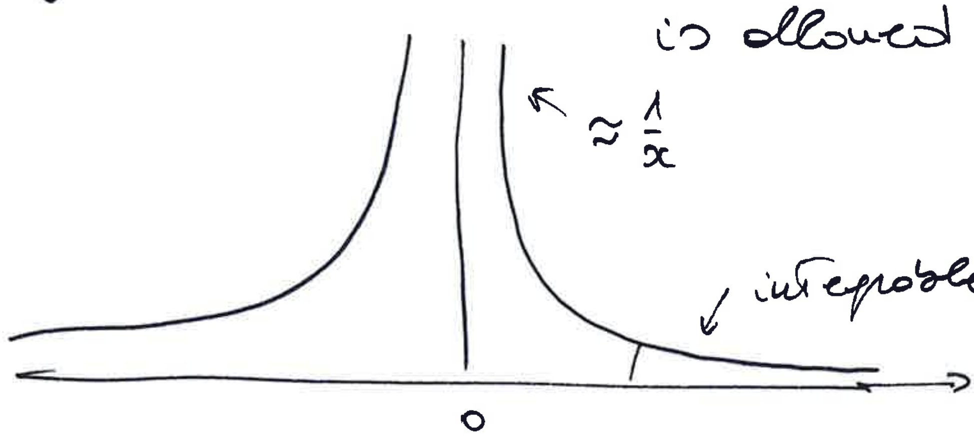
- $\int_{|x| \geq 1} \pi(dx) < \infty$ finite mass in both tails of π

- $\int_{|x| < 1} x^2 \pi(dx) < \infty$

$$\Rightarrow \pi((-1, 1)) = \infty$$ ~~only~~ allowed

as long as $x^2\, \Pi(dx)$ is integrable around 0.

E.g. $\Pi(dx) \approx \frac{1}{x}\, dx$ around 0 is allowed



Def.

A Lévy process on $\mathbb{R}$ is a continuous-time càdlàg process $\{X(t), t\geq 0\}$ s.t.:

- $X(0)=0$ a.s.
- $X(s+t)-X(s) \overset{d}{=} X(u+t)-X(u) \quad \forall s,u,t\geq 0$

  stationary increments
- $X(s+t)-X(s) \perp\!\!\!\perp \{X(u), u\leq s\}$

  independent increments.

Example Poisson process

- $X(0)=0$ a.s.
- $\underbrace{X(s+t)-X(s)}_{\perp\!\!\!\perp\ s,\, X(s)} \sim Po(\lambda t)$

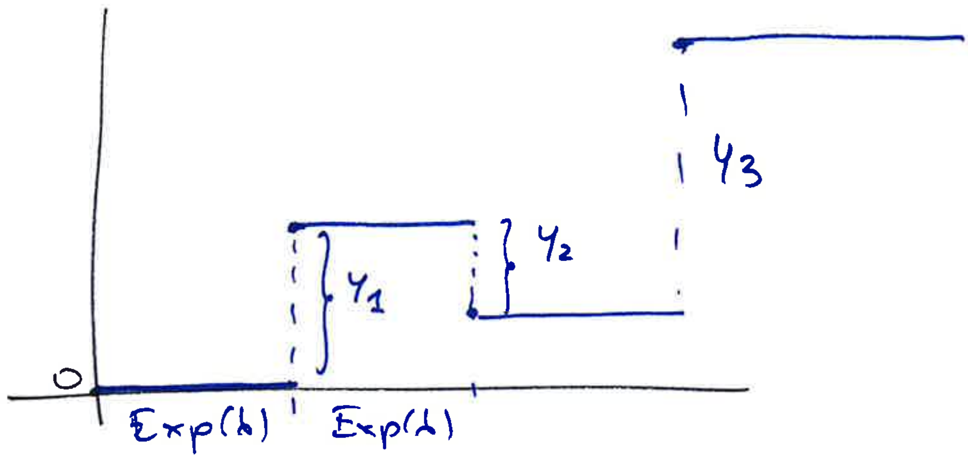
Example Brownian motion

- $X(0)=0$ a.s.
- $X(s+t)-X(s) \sim N(0,t) \perp\!\!\!\perp s, X(s)$

Def.
Let $N(t)$ be a rate λ Poisson process and let $Y_i \overset{iid}{\sim} F$, indep. of $N(t)$.
Then $X(t) := \sum_{i=1}^{N(t)} Y_i \quad t \geq 0$
is called COMPOUND POISSON PROCESS.

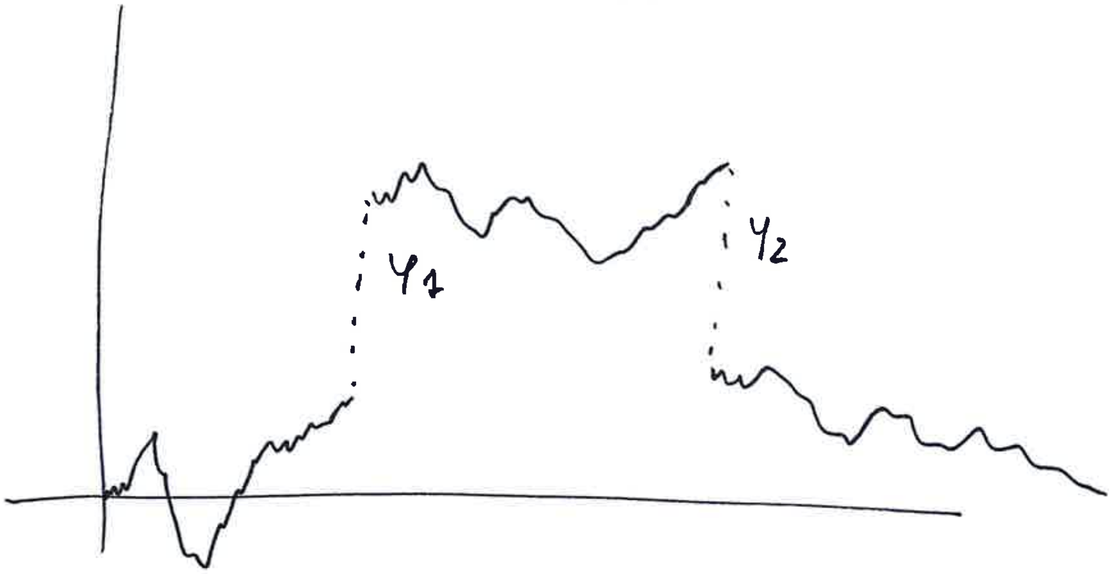
Exercise: - show it is a Lévy process.

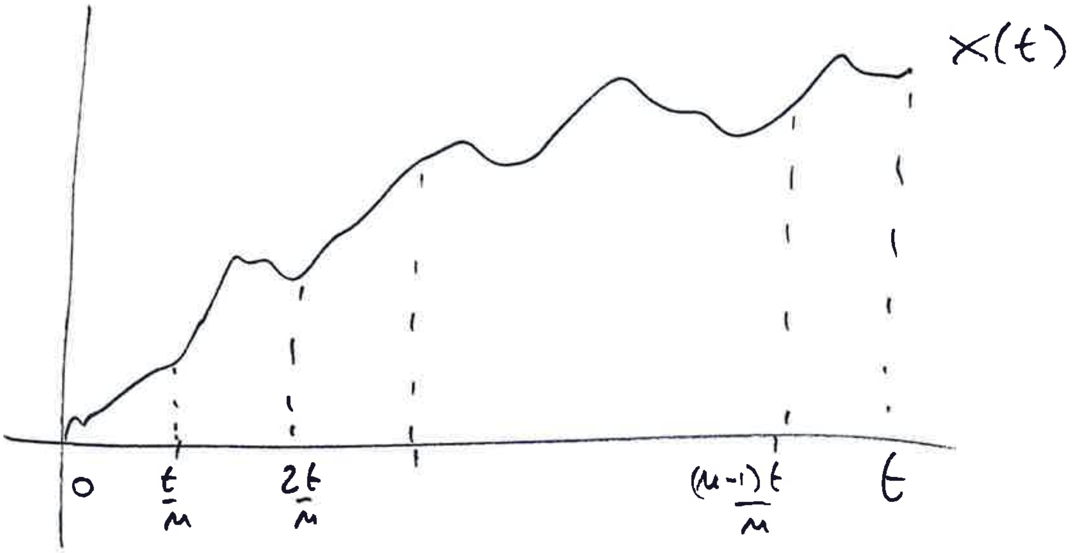
- $F = \delta_1 \Rightarrow$ Poisson process



Exercise: A finite sum of Lévy process is a Lévy process.

Example $\quad X(t) = \underbrace{X^{(1)}(t)}_{CP} + \underbrace{X^{(2)}(t)}_{BM}$





Let $X(t)$ be a Lévy process and write

$$X(t) = \underbrace{X\left(\tfrac{t}{n}\right) - X(0)}_{Y_1^{(n)}} + \underbrace{X\left(\tfrac{2t}{n}\right) - X\left(\tfrac{t}{n}\right)}_{Y_2^{(n)}} + \dots + \underbrace{X(t) - X\left(\tfrac{(n-1)t}{n}\right)}_{Y_n^{(n)}}$$

$\Rightarrow Y_i^{(n)} \overset{iid}{\sim}$ from indep. + station. of increments

$$= \sum_{i=1}^{n} Y_i^{(n)} \quad \Rightarrow \quad X(t) \text{ is I.D.}$$

Define $\psi_t(u) := -\log E(e^{iuX(t)})$

The C.E. of $X(t)$, from the sum:

if $t = m \in \mathbb{N}$

$$\begin{cases} \psi_m = n\,\psi_{\frac{m}{n}} & n \in \mathbb{N} \\ \psi_m = m\,\psi_1 & n = m \end{cases}$$

$$\Rightarrow \quad \psi_{\frac{m}{n}} = \frac{m}{n}\psi_1$$

i.e. $\forall t$ rational $\psi_t(u) = t\,\psi_1(u)$

can be extended to $t \geq 0$.

So
$$\boxed{\forall t \geq 0 \quad \psi_t(u) = t\,\psi_1(u)}$$

so any Lévy process is s.t.

$$E(e^{iuX(t)}) = e^{-t\psi_1(u)}$$

and so it is characterized by its C.E. at $t=1$; therefore the Lévy-Khintchine formula provides a characterization of all Lévy processes.

<u>Example</u> Linear brownian motion

$$dX(t) = m\,dt + s\,dB(t)$$

$$\Rightarrow X(t) \sim N(mt, s^2 t)$$

$$\Rightarrow E\left(e^{iuX(t)}\right) = e^{i(mt)u - \frac{1}{2}(s^2 t)u^2}$$

$$= \exp\Big(\underbrace{-t\,[\underbrace{-imu + \tfrac{1}{2}s^2u^2}_{\psi_1(u)}]}_{t\,\psi_1(u)}\Big)$$

so at $t=1$ we find the triplet $(\mu, \sigma, \pi) = (-m, s, 0)$. This suggests:

- μ describes a constant drift with slope $-\mu$
- σ describes a Brownian component with diffusion coefficient σ^2

<u>Example</u> $X(t) \sim Po(\lambda t)$ Poisson process

$\psi_t(u) = t\,\psi_1(u)$

at $t=1$ $Po(\lambda)$ which has triplet $(0, 0, \lambda\delta_1)$

so it seems π describes the jump structure: rate λ and size 1 a.s.

Example CP: $X(t) = \sum_{i=1}^{N(t)} Z_i$

$N(t) \sim Po(\lambda t)$, $Z_i \overset{iid}{\sim} F$

$\psi_t(u) = t\,\psi_1(u) \Rightarrow$ at $t=1$ we get

a CP R.V. $X(1) = \sum_{i=1}^{N(1)} Z_i$

$N(1) \sim Po(\lambda)$

$$\Rightarrow \psi_1(u) = \lambda \int_{\mathbb{R}} (1 - e^{iux}) F(dx)$$

$$\psi_1(u) = \lambda \int_{\mathbb{R}} \left(1 - e^{iux} + iux\,\mathbb{1}_{(|x|<1)} - iux\,\mathbb{1}_{(|x|<1)}\right) F(dx)$$

$$= \int_{\mathbb{R}} \left(1 - e^{iux} + iux\,\mathbb{1}_{(|x|<1)}\right) \underbrace{\lambda F(dx)}_{\Pi(dx)} - \underbrace{iu\lambda \int_{-1}^{1} x F(dx)}_{-\mu}$$

$\leadsto$ The Triplet is

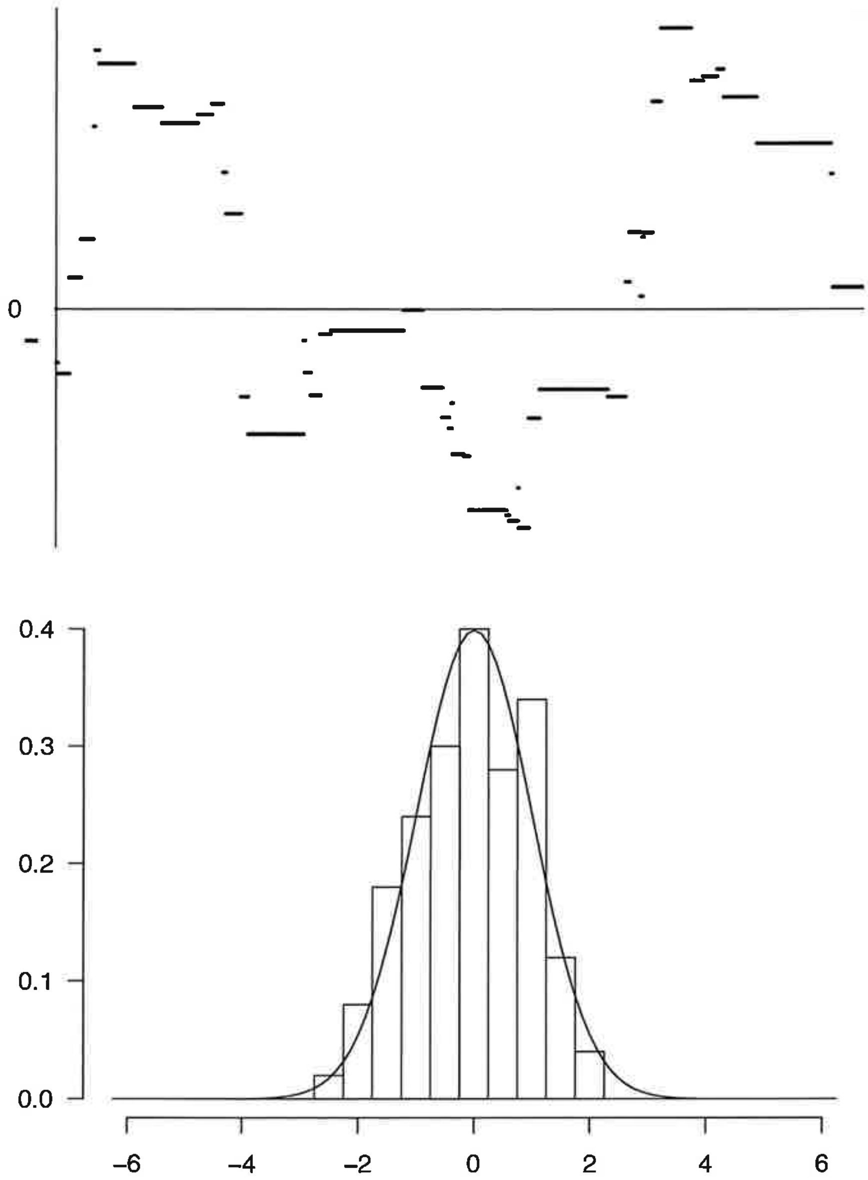
$$\mu = -\lambda \int_{-1}^{1} x F(dx), \quad \sigma = 0, \quad \Pi = \lambda F$$

$\leadsto$: $\lambda = \int_{\mathbb{R}} \Pi(dx)$ called TOTAL MASS of Π, is the Poisson rate for jump arrivals

The Lévy intensity of a CP has finite total mass by construction

The normalized Π yields $F = \lambda^{-1}\Pi$

which gives the distribution of the jumps.



A CP trajectory and empirical distribution of the jumps vs. a N(0, 1).