

This Should Help Your Lazy Ass In Analysis B

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1 Normed Spaces

Proposition 1.1

A subset $C \subset X$ is a closed subset of $(X, \|\cdot\|)$ whenever the limit of any convergent sequence $(x_n)_n \subset C$ belongs to C .

In other words, C is closed if and only if once a sequence of elements of C is converging, the limit cannot escape C .

Proof

- 1 C closed and $(x_n)_n \subset C \Rightarrow \lim_n x_n = x \in C$.
We argue by contradiction that $x \notin C$; then x must lie in $X \setminus C$ which must be open. But if that is an open set then there must exist a ball of ϵ for some $\epsilon > 0$ such that $B(x, \epsilon) \subset X \setminus C$. But due to the definition of convergence there must exist a N such that $\|x_n - x\| < \epsilon$ ($n \geq N$), but then that x_n should be outside C which is a contradiction.
- 2 $\lim_n x_n = x \in C \Rightarrow C$ closed and $(x_n)_n \subset C$.
We argue by contradiction that C is not closed, so $X \setminus C$ must be not open. Not open sets are such that there exist $x \in X \setminus C$ such that for every $\epsilon > 0$ you can't find a ball that it is entirely in $X \setminus C$ which means $B(x, \epsilon) \cap C \neq \emptyset$. Then pick $x_n \in B(x, \frac{1}{n}) \cap C$ so that every x_n also belongs to C . Clearly this is a sequence of n such that for any $\epsilon > 0$ we have $\|x_n - x\| < \frac{1}{n}$ and this means that $\lim_n x_n = x$. We picked $x \in X \setminus C$ so we have found a limit of a sequence of elements of C that doesn't belong to C which is a contradiction. \square

Lemma 1.1

Young's inequality. Let $p > 1$ and let $q < 1$ be its conjugate exponent. Then, for any nonnegative $a, b \in \mathbb{R}$ it holds

$$ab \leq \frac{1}{p}a^p + \frac{1}{b}b^q.$$

Theorem 1.1

Holder Inequality. Let $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Assume that $f \in L^p(S, \mu)$ and $g \in L^q(S, \mu)$. Then $f \cdot g \in L^1$ and

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

Proof

The proof is trivial if $p = 1$ and $q = 1$. Remember Young's inequality

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q.$$

Now let's say $a = |f(s)|$ and $b = |g(s)|$. Now our inequality becomes

$$|f(s)g(s)| \leq \frac{1}{p}|f(s)|^p + \frac{1}{q}|g(s)|^q \quad \text{for } f, g \text{ are on } S.$$

Now we integrate over S and we get

$$\begin{aligned} \int_S |f(s)g(s)| &\leq \frac{1}{p} \int_S |f(s)|^p + \frac{1}{q} \int_S |g(s)|^q \\ &\Rightarrow \|fg\|_1 \leq \frac{1}{p} \|f\|_p^p + \frac{1}{q} \|g\|_q^q \end{aligned}$$

which means that $\|fg\|_1$ is finite and therefore $fg \in L^1(S, \mu)$. To end the proof let's substitute f with λf , $\forall \lambda > 0$. We get

$$\begin{aligned} \lambda \|fg\|_1 &\leq \frac{\lambda^p}{p} \|f\|_p^p + \frac{1}{q} \|g\|_q^q \quad \forall \lambda > 0 \\ &\Rightarrow \lambda \|fg\|_1 \leq \frac{\lambda^p}{p} \|f\|_p^p + \frac{1}{q} \|g\|_q^q \quad (\lambda > 0) \\ &\Rightarrow \|fg\|_1 \leq \frac{\lambda^{p-1}}{p} \|f\|_p^p + \frac{1}{\lambda q} \|g\|_q^q \quad \forall \lambda > 0 \end{aligned}$$

Now choose $\lambda = \frac{1}{\|f\|_p^{\frac{p}{p-1}}} \cdot \frac{q}{\|g\|_q^{\frac{1}{p-1}}}$. When we substitute this value for λ , we get:

$$\frac{q^p}{p} \|fg\|_1^p = \left(\frac{1}{\|f\|_p^{\frac{p}{p-1}}} \cdot \frac{q}{\|g\|_q^{\frac{1}{p-1}}} \right)^p \cdot \frac{1}{p} \|f\|_p^p$$

Expanding $\left(\frac{1}{\|f\|_p^{\frac{p}{p-1}}} \cdot \frac{q}{\|g\|_q^{\frac{1}{p-1}}} \right)^p$, we get:

$$\left(\frac{1}{\|f\|_p^{\frac{p}{p-1}}} \cdot \frac{q}{\|g\|_q^{\frac{1}{p-1}}} \right)^p = \frac{\|g\|_q^p}{\|f\|_p^p}$$

Substituting this, we have:

$$\frac{q^p}{p} \|fg\|_1^p = \frac{\|g\|_q^p}{\|f\|_p^p} \|fg\|_1^p = \frac{\|g\|_q^p}{p}$$

Substitute back into the inequality:

$$\lambda \|fg\|_1 \leq \frac{\|g\|_q^p}{p} + \frac{1}{\lambda q} \|g\|_q^q$$

Since $\frac{\|g\|_q^p}{p} + \frac{\|g\|_q^q}{q} = \|g\|_q^p \left(\frac{1}{p} + \frac{1}{q} \right)$ and $\frac{1}{p} + \frac{1}{q} = 1$, we get

$$\lambda \|fg\|_1 \leq \|g\|_q^p$$

Dividing by $\lambda = \frac{1}{\|f\|_p^{\frac{p}{p-1}}} \cdot \frac{q}{\|g\|_q^{\frac{1}{p-1}}}$:

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

This is Hölder's inequality:

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

□

Theorem 1.2

For any $1 \leq p \leq \infty$, $L^p(S, \mu)$ is a vector space and $\|\cdot\|_p$ is a norm.

Remark

For $1 < p < \infty$ the triangular inequality

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p \quad \forall f, g \in L^p(S, \mu)$$

is known as **Minkowski's Inequality**.

Proof

We already know that if $f \in L^p(S, \mu)$ then $\lambda f \in L^p(S, \mu)$. Homogeneity and uniqueness are also existent for $\|\cdot\|_p$ so in order to show that $L^p(S, \mu)$ is a vector space we only need to prove that if $f, g \in L^p(S, \mu)$ then $f + g \in L^p(S, \mu)$ and $\|\cdot\|_p$ is a norm. Fix $f, g \in L^p(S, \mu)$. We know that for any $x, y \in \mathbb{R}$ we get

$$\left| \frac{1}{2}x + \frac{1}{2}y \right|^p \leq \frac{1}{2}|x|^p + \frac{1}{2}|y|^p$$

since this mapping $r \rightarrow r^p$ is convex. This also means that

$$|x + y|^p \leq 2^{p-1}(|x|^p + |y|^p).$$

and this implies in particular that

$$|f(s) + g(s)|^p \leq 2^{p-1}(|f(s)|^p + |g(s)|^p) \quad \text{for } \mu\text{-a.e. on } s \in S.$$

If we integrate over S we get:

$$\int_S |f(s) + g(s)|^p \leq 2^{p-1} \left(\int_S |f(s)|^p + \int_S |g(s)|^p \right)$$

which means

$$\|f + g\|_p^p \leq 2^{p-1} (\|f\|_p^p + \|g\|_p^p)$$

which means that $f + g \in L^p(S, \mu)$.

We now must prove the Minkowski's inequality. We know that

$$\|f + g\|_p^p = \int_S |f + g|^p d\mu = \int_S |f + g| |f + g|^{p-1} d\mu$$

but since we know that $|f + g| \leq |f| + |g|$ then

$$\|f + g\|_p^p \leq \int_S |f| |f + g|^{p-1} d\mu + \int_S |g| |f + g|^{p-1} d\mu.$$

Call $\psi = |f + g|^{p-1}$. It clearly belongs to $L^q(S, \mu)$ because

$$|\psi|^q = (|f + g|^{p-1})^q = |f + g|^p \quad \text{since } q(p-1) = p$$

so

$$\|\psi\|_q = \left(\int_S |\psi|^q \right)^{\frac{1}{q}} = \left(\int_S |f + g|^p \right)^{\frac{1}{q}} = \|f + g\|_p^{\frac{p}{q}} < \infty$$

And this means that $|\psi|^q \in L^1(S, \mu) \implies \psi \in L^q(S, \mu)$. We also know that $|f| \in L^p(S, \mu)$ so we can apply Holder's inequality with f and ψ so that

$$\int_S |f| |f + g|^{p-1} d\mu = \|f\psi\|_1 \leq \|f\|_p \|\psi\|_q = \|f\|_p \|f + g\|_p^{\frac{p}{q}}$$

and

$$\int_S |g| |f + g|^{p-1} d\mu \leq \|g\|_p \|f + g\|_p^{\frac{p}{q}}$$

So that

$$\|f + g\|_p^p \leq \|f\|_p^p + \|g\|_p^p.$$

Dividing by $\|f + g\|_p^{\frac{p}{q}} \neq 0$ (otherwise the proof is trivial) we get

$$\|f + g\|_p^{p-\frac{p}{q}} \leq \|f\|_p + \|g\|_p.$$

□

Proposition 1.2

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability triplet. then the following holds:

$$L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \subset L^p(\Omega, \mathcal{F}, \mathbb{P}) \subset L^q(\Omega, \mathcal{F}, \mathbb{P}) \subset L^1(\Omega, \mathcal{F}, \mathbb{P}).$$

Of course, this result remains valid for every other measure space (S, \mathcal{S}, μ) as long as $\mu(S) < \infty$. In the special case in which

$$S = \mathbb{N} \quad \mathcal{F} = \mathcal{P}(\mathbb{N})$$

and $\mu(A)$ is the counting measure $\mu(A) = \sum_{k \in A} \delta_k(A)$, $A \in \mathbb{N}$ then knowing that sequences $n \mapsto f(n)$ can be identified as functions over \mathbb{N} of the type $f : \mathbb{N} \rightarrow \mathbb{R}$ we see that

$$L^1(S, \mu) = \underbrace{\mathcal{L}^1(S, \mu)}_{\text{actual functions, not equivalence classes}} = \ell^1(\mathbb{N}) = \left\{ \mathbf{x} = (x_n)_n; \|\mathbf{x}\|_1 := \sum_{n=1}^{\infty} |x_n| < \infty \right\}.$$

This means that $\ell^1(\mathbb{N})$ is a L^1 space for some special choice of S and μ . Since we chose our measure as the counting measure, we get

$$\int_{\mathbb{N}} |f(n)| d\mu(n) = \sum_{n=1}^{\infty} |f(n)| = \sum_{n=1}^{\infty} |x_n|.$$

Cool!

Proposition 1.3

Let $p \geq 1$ be given. We define the set

$$\ell^p(\mathbb{N}) = \left\{ \mathbf{x} = (x_n)_{n \in \mathbb{N}} \subset \mathbb{R} \text{ such that } \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}.$$

Then, $\ell^p(\mathbb{N})$ is a vector space. Moreover, if

$$\|\mathbf{x}\|_p := \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}} \quad \forall \mathbf{x} = (x_n)_n \in \ell^p(\mathbb{N})$$

then $(\ell^p(\mathbb{N}), \|\cdot\|_p)$ is a normed space.

1.1 The space of linear applications

Proposition 1.4

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed spaces and let $L; X \mapsto Y$ be a linear application. The following are equivalent:

- ① L is continuous on X ;
- ② L is continuous at $x = 0$;
- ③ there is a positive constant $C > 0$ such that

$$\|L(x)\|_Y \leq C \|x\|_X \quad \forall x \in X.$$

Proof

Of course $\textcircled{1} \Rightarrow \textcircled{2}$ but now let's prove $\textcircled{3}$. Consider the definition of continuity at 0 for the function L ; this means that for any $\varepsilon > 0$ there exists a constant $\delta > 0$ such that

$$\|x - 0_X\|_X \leq \delta \Rightarrow \|L(x) - L(0_Y)\|_Y \leq \varepsilon.$$

We know that L is linear, so $L(0_X) = 0_Y$ and therefore we get

$$\|x\|_X \leq \delta \Rightarrow \|L(x)\|_Y \leq \varepsilon.$$

Choose $\varepsilon = 1$ so that we get

$$\|x\|_X \leq \delta \Rightarrow \|L(x)\|_Y \leq 1.$$

Let $x \in X \setminus \{0_X\}$ and set $y = \frac{\delta}{\|x\|_X} x$. Now we have that

$$\|y\|_X = \left\| \frac{\delta}{\|x\|_X} x \right\|_X = \frac{\delta}{\|x\|_X} \|x\|_X = \delta$$

and, due to the linearity of L ,

$$\|L(y)\|_Y = \left\| L\left(\frac{\delta}{\|x\|_X} x\right) \right\|_Y = \frac{\delta}{\|x\|_X} \|L(x)\|_Y \leq \frac{\delta}{\|x\|_X} \|L(x)\|_Y \leq 1 \Rightarrow \|L(x)\|_Y \leq \frac{1}{\delta} \|x\|_X.$$

So, setting $C = \frac{1}{\delta}$, proves $\textcircled{3}$. We still need to prove $\textcircled{1} \Rightarrow \textcircled{3}$ is easy, since by linearity we have

$$\|L(x) - L(y)\|_Y = \|L(x - y)\|_Y \leq C \|x - y\|_X \quad \forall x, y \in X$$

and this clearly implies the continuity of L at any $x \in X$ (actually, the uniform continuity). \square

Definition 1.1

If $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are two normed spaces, we denote by $\mathcal{L}(X, Y)$ the space of continuous linear applications from X to Y . If $X = Y$ we simply denote $\mathcal{L}(X) = \mathcal{L}(X, X)$.

Proposition 1.5

If $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are two vector spaces and X is of finite dimension, any linear application $L : X \mapsto Y$ is continuous.

Remark

If $\dim(X) = n$ and $\dim(Y) = p$, the space $\mathcal{L}(X, Y)$ can be identified with the space $\mathcal{M}_{n \times p}(\mathbb{R})$ of matrices with n lines and p rows.

1.2 Compactness

Definition 1.2

Let $(X, \|\cdot\|_X)$ be a normed space and let $K \subset X$. We say that K is **compact** if every sequence $(x_n)_n$ contains a subsequence which converges to some $x \in K$.

Of course if K is compact then it is closed.

Lemma 1.2

If K is a compact subset of a normed space $(X, \|\cdot\|_X)$ then K is closed and there exists $M > 0$ such that $\sup_{x \in K} \|x\| \leq M$ which means that K is bounded.

Proposition 1.6

Product of compact spaces. Let $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$ be two compact normed spaces and let $X = X_1 \times X_2$. Then $(X, \|\cdot\|_+)$ and $(X, \|\cdot\|_{\max})$ are compact normed spaces.

Remember that

$$\|\mathbf{x}\|_+ = \|x_1\|_1 + \|x_2\|_2$$

and

$$\|\mathbf{x}\|_{\max} = \max(\|x_1\|_1, \|x_2\|_2)$$

Proof

Let $(\zeta_n)_n \subset X$ be a given sequence. This means, being in a product space, that there exist two sequences $(x_1)_n \subset X_1$ and $(y_n)_n \subset X_2$ such that $\zeta_n = (x_n, y_n)$ for any n . Consider the following subsequences:

- since $(X_1, \|\cdot\|_1)$ is compact, there exists a subsequence $(x_{\phi(n)})_n \subset X_1$ with a limit $x \in X_1$;
- take the analogous subsequence $(y_{\phi(n)})_n \subset X_2$. Since $(X_2, \|\cdot\|_2)$ is compact there is another (sub)subsequence $(y_{\phi(\psi(n))})_n$ of $(y_{\phi(n)})_n$ that converges to $y \in X_2$;
- now take the subsequence $(x_{\phi(\psi(n))})_n$ of $(x_{\phi(n)})_n$ which is still convergent in X_1 to x .

We can easily see that the subsequence

$$(\zeta_{\phi(\psi(n))})_n = ((x_{\phi(\psi(n))})_n, (y_{\phi(\psi(n))})_n)_n$$

is converging in X to $\mathbf{x} = (x, y)$. On every subsequence ζ_n contains a subsequence that converges to a point in X and this means that X is compact. \square

Of course, the above result readily extends to any finite product of compact normed spaces. On \mathbb{R} it is easy to describe a large class of compact sets:

Lemma 1.3

Let \mathbb{R} be endowed with the absolute value, $|\cdot|$. Any interval $[a, b] \subset \mathbb{R}$ is compact.

Proposition 1.7

Let $(X, \|\cdot\|)$ be a normed space and let K be a compact subset of X . If $A \subset K$ is a closed subset then A is compact.

Corollary

Heine-Borel theorem. A subset K of \mathbb{R}^N (where \mathbb{R}^N is endowed with, say, the usual Euclidean norm) is compact if and only if it is closed and bounded.

Proof

\Rightarrow We know that a compact subset of \mathbb{R}^N is closed and bounded.

\Leftarrow Let $K \subset \mathbb{R}^N$ be closed and bounded. Being bounded, there exists $R > 0$ such that

$$K \subset [-R, R]^N.$$

Since K is closed, from the previous proposition, it is sufficient to prove that $[-R, R]^N$ is a compact subset of \mathbb{R}^N , which is the same as checking that $[-R, R]$ is a compact subset of \mathbb{R} . Since every closed subset of a compact subset is compact. \square

This corollary can be reformulated as:

Every bounded sequence of \mathbb{R}^N has a convergent subsequence.

1.3 Compactness and continuous functions

Proposition 1.8

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed spaces and let $f : X \rightarrow Y$ be continuous. If $K \subset X$ is a compact subset of X then $f(K)$ is a compact subset of Y .

Proof

Let $(y_n)_n$ be a sequence in $f(K)$. It means that there is a sequence $(x_n)_n \subset K$ such that $y_n = f(x_n)$ for any n . Since K is compact, then there exists a subsequence $(x_{n_k})_k$ of $(x_n)_n$ which converges to $x \in K$. This means that $\lim_{k \rightarrow \infty} \|x_{n_k} - x\|_X = 0$, but since f is continuous then we know that

$$\lim_{k \rightarrow \infty} \|f(x_{n_k}) - f(x)\|_Y = 0 \Leftrightarrow \lim_{k \rightarrow \infty} \|y_{n_k} - y\|_Y = 0.$$

Since $y = f(x) \in f(K)$, we got that any sequence of $f(K)$ has a subsequence which converges to a limit in $f(K)$ which means that $f(K)$ is compact. \square

This has the following consequence:

Theorem 1.3

Let $(X, \|\cdot\|)$ be a normed space and let $K \subset X$ be compact. Let $f : K \rightarrow \mathbb{R}$ be continuous. Then, f assumes its maximum and minimum on K .

Proof

We know that $f(K)$ is a compact subset of \mathbb{R} and thanks to the Heine-Borel theorem we know that any compact subset of \mathbb{R} is closed and bounded (and vice versa...). So f is bounded and the fact that it reaches its maximum and minimum inside K is a simple consequence of the fact that $f(K)$ is closed. Let

$$M = \sup\{f(x), x \in K\}.$$

By definition there is a sequence $(x_n)_n \subset K$ such that $\lim_{n \rightarrow \infty} f(x_n) = M$. This sequence $(f(x_n))_n$ lies in $f(K)$ which is compact and is closed. But this means that also its limit M lies in $f(K)$ (i.e. there is $x \in K$ such that $f(x) = M$). This shows that M is the maximum value of f and by proceeding in an analogous manner we can show the same thing for the minimum. \square

Theorem 1.4

Heine Theorem. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed spaces and let $K \subset X$ be compact. Assume that $f : K \rightarrow Y$ is continuous. Then f is uniformly continuous on K .

Proof

Suppose that f is not uniformly continuous. This means that there exists $\varepsilon_0 > 0$ such that

$$\forall \delta > 0, \exists x, y \in K \text{ with } \|x - y\|_X \leq \delta \text{ and } \|f(x) - f(y)\|_Y \geq \varepsilon_0.$$

Now choosing $\delta = \frac{1}{n}$, $n \in \mathbb{N}$ this allows to build two sequences $(x_n)_n$ and $(y_n)_n$ such that

$$\|x_n - y_n\|_X \leq \frac{1}{n} \text{ and } \|f(x_n) - f(y_n)\|_Y \geq \varepsilon_0, \quad \forall n \in \mathbb{N}.$$

Since K is compact we can extract a subsequence of x_n that we call $(x_{n_k})_k$ and that converges to some $x_0 \in K$. It follows that $(y_{n_k})_k$ also converges to x_0 (why?). Since f is continuous we get that $\lim_{k \rightarrow \infty} f(x_{n_k}) = f(x_0) = \lim_{k \rightarrow \infty} f(y_{n_k})$ in Y , i.e.

$$\lim_{k \rightarrow \infty} \|f(x_{n_k}) - f(y_{n_k})\|_Y = 0.$$

But this contradicts the fact that $\|f(x_{n_k}) - f(y_{n_k})\|_Y \geq \varepsilon_0$ for each $n \in \mathbb{N}$. \square

So, in this case taking two sequences that get closer and closer does not correspond to the fact also their functions get closer and closer... and this is not possible.

1.4 Finite dimensional spaces

Proposition 1.9

Let $(X, \|\cdot\|)$ be a finite dimensional normed vector space with $\dim(X) = d$ and let $\{e_1, \dots, e_d\}$ be a basis for X . Then, there are positive constants $C_0, C_1 > 0$ such that

$$C_0 \sum_{i=1}^d |x_i| \leq \left\| \sum_{i=1}^d x_i e_i \right\| \leq C_1 \sum_{i=1}^d |x_i| \quad \forall (x_1, \dots, x_d) \in \mathbb{R}^d.$$

This proposition asserts that if $\dim(X) = d$ then *any* norm $\|\cdot\|$ is related to the $\|\cdot\|_1$ norm of \mathbb{R}^d . This translates in the following:

Proposition 1.10

If X is a finite dimensional vector space, all norms over X are equivalent.

So there is no weird norm, but everything is comparable to the simple $\|\cdot\|_1$ norm. This proposition also allows us to identify in a continuous way a finite dimensional space $(X, \|\cdot\|)$ and the space \mathbb{R}^d where d is the dimension of X . Indeed, introducing a basis $\{e_1, \dots, e_d\}$ of X , the mapping

$$\Phi : X \rightarrow \mathbb{R}^d$$

which associates $\Phi(\mathbf{x}) = (x_1, \dots, x_d)$ to some $\mathbf{x} = \sum_{i=1}^d x_i e_i \in X$, is a bijection from X to \mathbb{R}^d which is continuous whose inverse is also continuous. This results in the following:

Corollary

If $(X, \|\cdot\|)$ is a finite dimensional vector space and $K \subset X$ is closed and bounded then K is compact.

Again, this is very specific to finite dimensional spaces and, as we shall see, this actually characterizes finite dimensional spaces. Indeed, in infinite dimensional normed spaces, the closed unit ball cannot be compact. This shows that, in infinite dimensional spaces, the compact subsets do not coincide with closed and bounded subsets!! We first state the following technical lemma:

Lemma 1.4

Riesz Lemma. Let $(X, \|\cdot\|)$ be a normed vector space and let Y be a closed subspace of X (i.e. Y is closed in X and Y is a linear subspace of X). If $Y \neq X$ then for any $\varepsilon \in (0, 1)$ there exists $x \in X$ with $\|x\| = 1$ such that

$$\inf_{y \in Y} \|x - y\| \geq 1 - \varepsilon.$$

Remark

This lemma asserts that if $Y \neq X$ is a closed subspace then for any $\varepsilon \in (0, 1)$ there is some unit vector $x \in X$ such that $\text{dist}(x, Y) \geq 1 - \varepsilon$.

Proof

Let $z \in X \setminus Y$. Since Y is closed and $z \notin Y$, one has

$$\alpha = \text{dist}(z, Y) = \inf_{y \in Y} \|z - y\| > 0.$$

Pick $\varepsilon \in (0, 1)$. There exists $\bar{y} \in Y$ such that $\|\bar{y} - z\| \leq \frac{\alpha}{1-\varepsilon}$ (otherwise we would get $\text{dist}(z, Y) \geq \frac{\alpha}{1-\varepsilon} > \alpha$!). Notice that $\bar{y} \neq z$ so that $r := \frac{\alpha}{\|\bar{y} - z\|} > 0$. Set

$$x := \frac{1}{r}(z - \bar{y}).$$

Clearly, $\|x\| = 1$. Let $y \in Y$ be given. One can write

$$\|x - y\| = \frac{1}{r} \|z - \bar{y} - ry\|$$

and since Y is a linear subspace $\bar{y} + ry \in Y$ so that $\|z - \bar{y} - ry\| \geq \alpha$. Therefore $\|x - y\| \geq \frac{\alpha}{r} \geq 1 - \varepsilon$ by assumption on $r = \|z - \bar{y}\|$. Since this is true for any $y \in Y$, this proves the result. \square