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18.01 Single Variable Calculus  
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# Lecture 6: Exponential and Log, Logarithmic Differentiation, Hyperbolic Functions

## Taking the derivatives of exponentials and logarithms

### Background

We always assume the base,  $a$ , is greater than 1.

$$a^0 = 1; \quad a^1 = a; \quad a^2 = a \cdot a; \quad \dots$$

$$\begin{aligned} a^{x_1+x_2} &= a^{x_1} a^{x_2} \\ (a^{x_1})^{x_2} &= a^{x_1 x_2} \\ a^{\frac{p}{q}} &= \sqrt[q]{a^p} \quad (\text{where } p \text{ and } q \text{ are integers}) \end{aligned}$$

To define  $a^r$  for real numbers  $r$ , fill in by continuity.

**Today's main task: find  $\frac{d}{dx} a^x$**

We can write

$$\frac{d}{dx} a^x = \lim_{\Delta x \rightarrow 0} \frac{a^{x+\Delta x} - a^x}{\Delta x}$$

We can factor out the  $a^x$ :

$$\lim_{\Delta x \rightarrow 0} \frac{a^{x+\Delta x} - a^x}{\Delta x} = \lim_{\Delta x \rightarrow 0} a^x \frac{a^{\Delta x} - 1}{\Delta x} = a^x \lim_{\Delta x \rightarrow 0} \frac{a^{\Delta x} - 1}{\Delta x}$$

Let's call

$$M(a) \equiv \lim_{\Delta x \rightarrow 0} \frac{a^{\Delta x} - 1}{\Delta x}$$

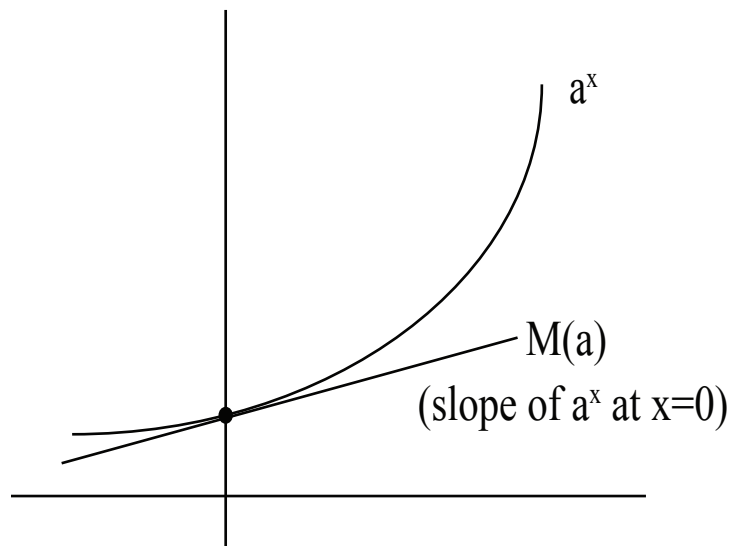
We don't yet know what  $M(a)$  is, but we can say

$$\frac{d}{dx} a^x = M(a) a^x$$

Here are two ways to describe  $M(a)$ :

1. Analytically  $M(a) = \frac{d}{dx} a^x$  at  $x = 0$ .

$$\text{Indeed, } M(a) = \lim_{\Delta x \rightarrow 0} \frac{a^{0+\Delta x} - a^0}{\Delta x} = \left. \frac{d}{dx} a^x \right|_{x=0}$$

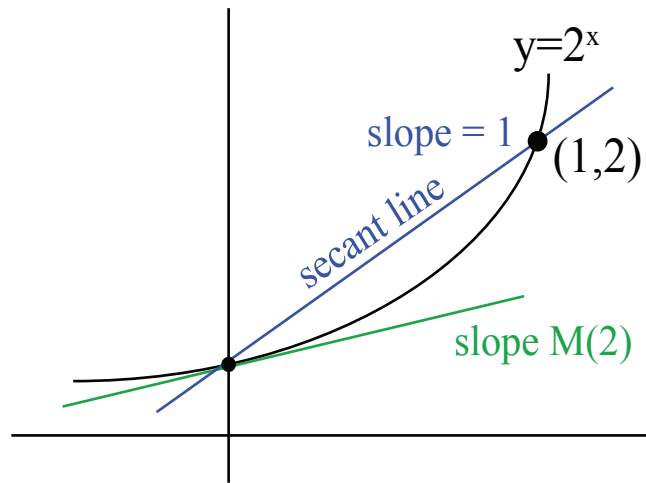
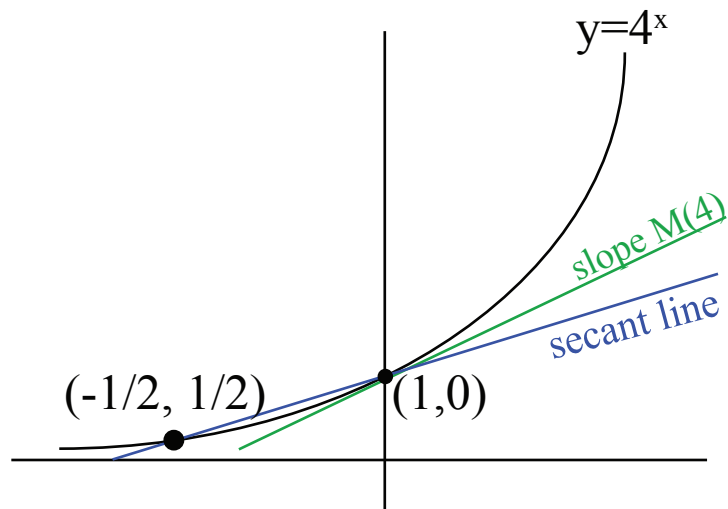
Figure 1: Geometric definition of  $M(a)$ 

2. Geometrically,  $M(a)$  is the slope of the graph  $y = a^x$  at  $x = 0$ .

The trick to figuring out what  $M(a)$  is is to beg the question and define  $e$  as the number such that  $M(e) = 1$ . Now can we be sure there is such a number  $e$ ? First notice that as the base  $a$  increases, the graph  $a^x$  gets steeper. Next, we will estimate the slope  $M(a)$  for  $a = 2$  and  $a = 4$  geometrically. Look at the graph of  $2^x$  in Fig. 2. The secant line from  $(0, 1)$  to  $(1, 2)$  of the graph  $y = 2^x$  has slope 1. Therefore, the slope of  $y = 2^x$  at  $x = 0$  is less:  $M(2) < 1$  (see Fig. 2).

Next, look at the graph of  $4^x$  in Fig. 3. The secant line from  $(-\frac{1}{2}, \frac{1}{2})$  to  $(1, 4)$  on the graph of  $y = 4^x$  has slope 1. Therefore, the slope of  $y = 4^x$  at  $x = 0$  is greater than  $M(4) > 1$  (see Fig. 3).

Somewhere in between 2 and 4 there is a base whose slope at  $x = 0$  is 1.

Figure 2: Slope  $M(2) < 1$ Figure 3: Slope  $M(4) > 1$

Thus we can *define*  $e$  to be the unique number such that

$$M(e) = 1$$

or, to put it another way,

$$\lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1$$

or, to put it still another way,

$$\frac{d}{dx}(e^x) = 1 \quad \text{at } x = 0$$

What is  $\frac{d}{dx}(e^x)$ ? We just defined  $M(e) = 1$ , and  $\frac{d}{dx}(e^x) = M(e)e^x$ . So

$$\boxed{\frac{d}{dx}(e^x) = e^x}$$

## Natural log (inverse function of $e^x$ )

To understand  $M(a)$  better, we study the natural log function  $\ln(x)$ . This function is defined as follows:

$$\boxed{\text{If } y = e^x, \text{ then } \ln(y) = x}$$

(or)

$$\boxed{\text{If } w = \ln(x), \text{ then } e^w = x}$$

Note that  $e^x$  is always positive, even if  $x$  is negative.

Recall that  $\ln(1) = 0$ ;  $\ln(x) < 0$  for  $0 < x < 1$ ;  $\ln(x) > 0$  for  $x > 1$ . Recall also that

$$\ln(x_1 x_2) = \ln x_1 + \ln x_2$$

Let us use implicit differentiation to find  $\frac{d}{dx} \ln(x)$ .  $w = \ln(x)$ . We want to find  $\frac{dw}{dx}$ .

$$\begin{aligned} e^w &= x \\ \frac{d}{dx}(e^w) &= \frac{d}{dx}(x) \\ \frac{d}{dw}(e^w) \frac{dw}{dx} &= 1 \\ e^w \frac{dw}{dx} &= 1 \\ \frac{dw}{dx} &= \frac{1}{e^w} = \frac{1}{x} \end{aligned}$$

$$\boxed{\frac{d}{dx}(\ln(x)) = \frac{1}{x}}$$

Finally, what about  $\frac{d}{dx}(a^x)$ ?

There are two methods we can use:



**Method 1: Write base e and use chain rule.**

Rewrite  $a$  as  $e^{\ln(a)}$ . Then,

$$a^x = \left(e^{\ln(a)}\right)^x = e^{x \ln(a)}$$

That looks like it might be tricky to differentiate. Let's work up to it:

$$\begin{aligned} \frac{d}{dx} e^x &= e^x \\ \text{and by the chain rule,} \\ \frac{d}{dx} e^{3x} &= 3e^{3x} \end{aligned}$$

Remember,  $\ln(a)$  is just a constant number— not a variable! Therefore,

$$\frac{d}{dx} e^{(\ln a)x} = (\ln a) e^{(\ln a)x}$$

or

$$\frac{d}{dx}(a^x) = \ln(a) \cdot a^x$$

Recall that

$$\frac{d}{dx}(a^x) = M(a) \cdot a^x$$

So now we know the value of  $M(a)$ :  $M(a) = \ln(a)$ .

Even if we insist on starting with another base, like 10, the natural logarithm appears:

$$\frac{d}{dx} 10^x = (\ln 10) 10^x$$

The base  $e$  may seem strange at first. But, it comes up everywhere. After a while, you'll learn to appreciate just how natural it is.



**Method 2: Logarithmic Differentiation.**

The idea is to find  $\frac{d}{dx} f(x)$  by finding  $\frac{d}{dx} \ln(f(x))$  instead. Sometimes this approach is easier. Let  $u = f(x)$ .

$$\frac{d}{dx} \ln(u) = \frac{d \ln(u)}{du} \frac{du}{dx} = \frac{1}{u} \left( \frac{du}{dx} \right)$$

Since  $u = f$  and  $\frac{du}{dx} = f'$ , we can also write

$$(\ln f)' = \frac{f'}{f} \quad \text{or} \quad f' = f(\ln f)'$$

Apply this to  $f(x) = a^x$ .

$$\ln f(x) = x \ln a \implies \frac{d}{dx} \ln(f) = \frac{d}{dx} \ln(a^x) = \frac{d}{dx} (x \ln(a)) = \ln(a).$$

(Remember,  $\ln(a)$  is a constant, *not* a variable.) Hence,

$$\frac{d}{dx}(\ln f) = \ln(a) \implies \frac{f'}{f} = \ln(a) \implies f' = \ln(a)f \implies \frac{d}{dx} a^x = \ln(a) a^x$$

**Example 1.**  $\frac{d}{dx}(x^x) = ?$

With variable (“moving”) exponents, you should use either base  $e$  or logarithmic differentiation. In this example, we will use the latter.

$$\begin{aligned} f &= x^x \\ \ln f &= x \ln x \\ (\ln f)' &= 1 \cdot (\ln x) + x \left( \frac{1}{x} \right) = \ln(x) + 1 \\ (\ln f)' &= \frac{f'}{f} \end{aligned}$$

Therefore,

$$f' = f(\ln f)' = x^x (\ln(x) + 1)$$

If you wanted to solve this using the base  $e$  approach, you would say  $f = e^{x \ln x}$  and differentiate it using the chain rule. It gets you the same answer, but requires a little more writing.

**Example 2.** Use logs to evaluate  $\lim_{k \rightarrow \infty} \left(1 + \frac{1}{k}\right)^k$ .

Because the exponent  $k$  changes, it is better to find the limit of the logarithm.

$$\lim_{k \rightarrow \infty} \ln \left[ \left(1 + \frac{1}{k}\right)^k \right]$$

We know that

$$\ln \left[ \left(1 + \frac{1}{k}\right)^k \right] = k \ln \left(1 + \frac{1}{k}\right)$$

This expression has two competing parts, which balance:  $k \rightarrow \infty$  while  $\ln \left(1 + \frac{1}{k}\right) \rightarrow 0$ .

$$\ln \left[ \left(1 + \frac{1}{k}\right)^k \right] = k \ln \left(1 + \frac{1}{k}\right) = \frac{\ln \left(1 + \frac{1}{k}\right)}{\frac{1}{k}} = \frac{\ln(1+h)}{h} \quad (\text{with } h = \frac{1}{k})$$

Next, because  $\ln 1 = 0$

$$\ln \left[ \left(1 + \frac{1}{k}\right)^k \right] = \frac{\ln(1+h) - \ln(1)}{h}$$

Take the limit:  $h = \frac{1}{k} \rightarrow 0$  as  $k \rightarrow \infty$ , so that

$$\lim_{h \rightarrow 0} \frac{\ln(1+h) - \ln(1)}{h} = \left. \frac{d}{dx} \ln(x) \right|_{x=1} = 1$$

In all,

$$\lim_{k \rightarrow \infty} \ln \left( 1 + \frac{1}{k} \right)^k = 1.$$

We have just found that  $a_k = \ln \left[ \left( 1 + \frac{1}{k} \right)^k \right] \rightarrow 1$  as  $k \rightarrow \infty$ .

If  $b_k = \left( 1 + \frac{1}{k} \right)^k$ , then  $b_k = e^{a_k} \rightarrow e^1$  as  $k \rightarrow \infty$ . In other words, we have evaluated the limit we wanted:

$$\lim_{k \rightarrow \infty} \left( 1 + \frac{1}{k} \right)^k = e$$

**Remark 1.** We never figured out what the exact numerical value of  $e$  was. Now we can use this limit formula;  $k = 10$  gives a pretty good approximation to the actual value of  $e$ .

**Remark 2.** Logs are used in all sciences and even in finance. Think about the stock market. If I say the market fell 50 points today, you'd need to know whether the market average before the drop was 300 points or 10,000. In other words, you care about the percent change, or the ratio of the change to the starting value:

$$\frac{f'(t)}{f(t)} = \frac{d}{dt} \ln(f(t))$$