18.01 Single Variable Calculus
Fall 2006

Lecture 3
Derivatives of Products, Quotients, Sine, and Cosine

Derivative Formulas

There are two kinds of derivative formulas:

1. Specific Examples: $\frac{d}{dx}x^n$ or $\frac{d}{dx}\left(\frac{1}{x}\right)$
2. General Examples: $(u+v)' = u' + v'$ and $(cu) = cu'$ (where c is a constant)

A notational convention we will use today is:

$$(u+v)(x) = u(x) + v(x); \quad uv(x) = u(x)v(x)$$

Proof of $(u+v) = u' + v'$. (General)

Start by using the definition of the derivative.

$$\begin{aligned}
(u+v)'(x) &= \lim_{\Delta x \to 0} \frac{(u+v)(x+\Delta x) - (u+v)(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{u(x+\Delta x) + v(x+\Delta x) - u(x) - v(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left\{ \frac{u(x+\Delta x) - u(x)}{\Delta x} + \frac{v(x+\Delta x) - v(x)}{\Delta x} \right\} \\
(u+v)'(x) &= u'(x) + v'(x)
\end{aligned}$$

Follow the same procedure to prove that $(cu)' = cu'$.

Derivatives of $\sin x$ and $\cos x$. (Specific)

Last time, we computed

$$\begin{aligned}
\lim_{x \to 0} \frac{\sin x}{x} &= 1 \\
\frac{d}{dx}(\sin x)|_{x=0} &= \lim_{\Delta x \to 0} \frac{\sin(0+\Delta x) - \sin(0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sin(\Delta x)}{\Delta x} = 1 \\
\frac{d}{dx}(\cos x)|_{x=0} &= \lim_{\Delta x \to 0} \frac{\cos(0+\Delta x) - \cos(0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\cos(\Delta x) - 1}{\Delta x} = 0
\end{aligned}$$

So, we know the value of $\frac{d}{dx}\sin x$ and of $\frac{d}{dx}\cos x$ at $x = 0$. Let us find these for arbitrary x.

$$\frac{d}{dx}\sin x = \lim_{\Delta x \to 0} \frac{\sin(x+\Delta x) - \sin(x)}{\Delta x}$$

Recall:

$$\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a)$$

So,

$$\begin{aligned}
\frac{d}{dx}\sin x &= \lim_{\Delta x \to 0} \frac{\sin x \cos \Delta x + \cos x \sin \Delta x - \sin(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left[\frac{\sin x(\cos \Delta x - 1)}{\Delta x} + \frac{\cos x \sin \Delta x}{\Delta x}\right] \\
&= \lim_{\Delta x \to 0} \sin x \left(\frac{\cos \Delta x - 1}{\Delta x}\right) + \lim_{\Delta x \to 0} \cos x \left(\frac{\sin \Delta x}{\Delta x}\right)
\end{aligned}$$

Since $\dfrac{\cos \Delta x - 1}{\Delta x} \to 0$ and that $\dfrac{\sin \Delta x}{\Delta x} \to 1$, the equation above simplifies to

$$\frac{d}{dx}\sin x = \cos x$$

A similar calculation gives

$$\frac{d}{dx}\cos x = -\sin x$$

Product formula (General)

$$(uv)' = u'v + uv'$$

Proof:

$$(uv)' = \lim_{\Delta x \to 0} \frac{(uv)(x+\Delta x) - (uv)(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{u(x+\Delta x)v(x+\Delta x) - u(x)v(x)}{\Delta x}$$

对于product的证明：△y = △（uv），uv同时随x变化，无法证。所以，尝试一次只变化一个变量，固定u变化v，另一个同理。因而如下面加减一个内容，重新分组处理，即可以。而后面的几何证明，本质上，也体现了固定一个，变化另一个，避免uv同时变化

Now obviously,

$$u(x+\Delta x)v(x) - u(x+\Delta x)v(x) = 0$$

so adding that to the numerator won't change anything.

$$(uv)' = \lim_{\Delta x \to 0} \frac{u(x+\Delta x)v(x) - u(x)v(x) + u(x+\Delta x)v(x+\Delta x) - u(x+\Delta x)v(x)}{\Delta x}$$

We can re-arrange that expression to get

$$(uv)' = \lim_{\Delta x \to 0} \left(\frac{u(x+\Delta x) - u(x)}{\Delta x}\right) v(x) + u(x+\Delta x)\left(\frac{v(x+\Delta x) - v(x)}{\Delta x}\right)$$

Remember, the limit of a sum is the sum of the limits.

$$\left[\lim_{\Delta x \to 0} \frac{u(x+\Delta x) - u(x)}{\Delta x}\right] v(x) + \lim_{\Delta x \to 0} \left(u(x+\Delta x)\left[\frac{v(x+\Delta x) - v(x)}{\Delta x}\right]\right)$$

$$(uv)' = u'(x)v(x) + u(x)v'(x)$$

Note: we also used the fact that

$$\lim_{\Delta x \to 0} u(x+\Delta x) = u(x) \qquad \text{(true because } u \text{ is continuous)}$$

This proof of the product rule assumes that u and v have derivatives, which implies both functions are continuous.

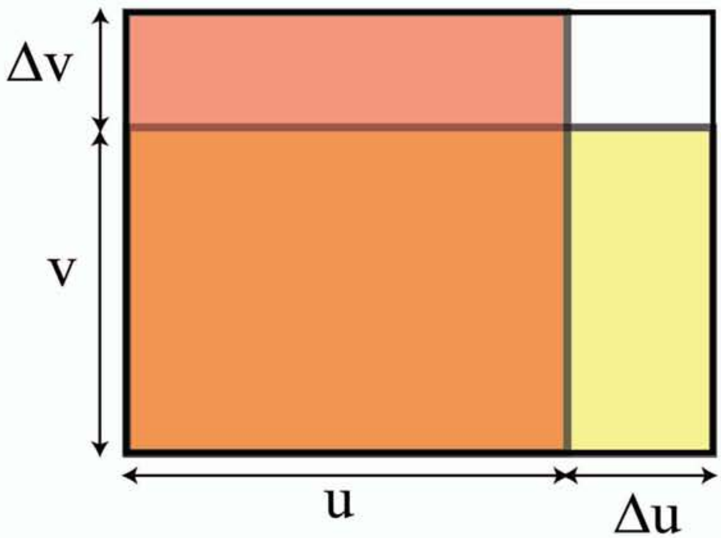

Figure 1: A graphical "proof" of the product rule

An intuitive justification:

We want to find the difference in area between the large rectangle and the smaller, inner rectangle. The inner (orange) rectangle has area uv. Define Δu, the change in u, by

$$\Delta u = u(x + \Delta x) - u(x)$$

We also abbreviate $u = u(x)$, so that $u(x + \Delta x) = u + \Delta u$, and, similarly, $v(x + \Delta x) = v + \Delta v$. Therefore the area of the largest rectangle is $(u + \Delta u)(v + \Delta v)$.

If you let v increase and keep u constant, you add the area shaded in red. If you let u increase and keep v constant, you add the area shaded in yellow. The sum of areas of the red and yellow rectangles is:

$$[u(v + \Delta v) - uv] + [v(u + \Delta u) - uv] = u\Delta v + v\Delta u$$

If Δu and Δv are small, then $(\Delta u)(\Delta v) \approx 0$, that is, the area of the white rectangle is very small. Therefore the difference in area between the largest rectangle and the orange rectangle is approximately the same as the sum of areas of the red and yellow rectangles. Thus we have:

u+△u体现的是u随x变化下，u变化后的值。v同理。所以，(u+△u) (v+△v)是uv在x变化下的值，这个很直观。求△uv，即用其减去uv。就是上面剩下的三部分，这三部分可以理解为：如果u不变，v变化对uv的影响；v不变同理；以及△u△v，这个我认为可以理解为，前面两部分分别假设一部分不动，这与实际有偏差，这一部分弥补偏差，值为△u△v，在这里因为二级无穷小为0，所以得证

$$[(u + \Delta u)(v + \Delta v) - uv] \approx u\Delta v + v\Delta u$$

(Divide by Δx and let $\Delta x \to 0$ to finish the argument.)

由上一个证明，△u可以表示u随x变化，△x对u的影响，这个肯定没问题。

因而我们可以用u+△u表示u(x+△x)

Quotient formula (General)

To calculate the derivative of u/v, we use the notations Δu and Δv above. Thus,

$$\begin{aligned}
\frac{u(x+\Delta x)}{v(x+\Delta x)} - \frac{u(x)}{v(x)} &= \frac{u+\Delta u}{v+\Delta v} - \frac{u}{v} \\
&= \frac{(u+\Delta u)v - u(v+\Delta v)}{(v+\Delta v)v} \quad \text{(common denominator)} \\
&= \frac{(\Delta u)v - u(\Delta v)}{(v+\Delta v)v} \quad (\text{cancel } uv - uv)
\end{aligned}$$

Hence,

$$\frac{1}{\Delta x}\left(\frac{u+\Delta u}{v+\Delta v} - \frac{u}{v}\right) = \frac{(\frac{\Delta u}{\Delta x})v - u(\frac{\Delta v}{\Delta x})}{(v+\Delta v)v} \longrightarrow \frac{v(\frac{du}{dx}) - u(\frac{dv}{dx})}{v^2} \quad \text{as } \Delta x \to 0$$

Therefore,

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

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