CALCULUS WITH ANALYTIC GEOMETRY

SECOND EDITION

GEORGE F. SIMMONS
Colorado College, Colorado Springs

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ABOUT THE AUTHOR

George F. Simmons has the usual academic degrees (CalTech, Chicago, Yale) and taught at several colleges and universities before joining the faculty of Colorado College in 1962. He is also the author of *Introduction to Topology and Modern Analysis* (McGraw-Hill, 1963), *Differential Equations with Applications and Historical Notes* (McGraw-Hill, 1972, 2nd edition 1991), *Precalculus Mathematics in a Nutshell* (Janson Publications, 1981), and *Calculus Gems: Brief Lives and Memorable Mathematics* (McGraw-Hill, 1992).

When not working or talking or eating or drinking or cooking, Professor Simmons is likely to be traveling (Western and Southern Europe, Turkey, Israel, Egypt, Russia, China, Southeast Asia), trout fishing (Rocky Mountain states), playing pocket billiards, or reading (literature, history, biography and autobiography, science, and enough thrillers to achieve enjoyment without guilt). One of his personal heroes is the older friend who once said to him, "I should probably spend about an hour a week revising my opinions."

For My Grandson Nicky—
without young people to continue
to wonder and care and study and learn,
it's all over.

With all humility, I think "Whatsoever thy hand findeth to do, do it with thy might" infinitely more important than the vain attempt to love one's neighbor as oneself. If you want to hit a bird on the wing, you must have all your will in a focus; you must not be thinking about yourself, and, equally, you must not be thinking about your neighbor; you must be living in your eye on that bird. Every achievement is a bird on the wing.

Oliver Wendell Holmes

If you bring forth what is within you, what you bring forth will save you. If you do not bring forth what is within you, what you do not bring forth will destroy you.

Jesus, *The Gospel of Thomas*
in the Nag Hammadi manuscripts

The more I work and practice, the luckier I seem to get.

Gary Player
(professional golfer)

A witty chess master once said that the difference between a master and a beginning chess player is that the beginner has everything clearly fixed in mind, while to the master everything is a mystery.

N. Ia. Vilenkin

Marshall's Generalized Iceberg Theorem: Seven-eighths of *everything* can't be seen.

Everything should be made as simple as possible, but not simpler.

Albert Einstein

CONTENTS

Preface to the Instructor xiii
To the Student xxiii

PART I

1 NUMBERS, FUNCTIONS, AND GRAPHS

1.1 Introduction 1
1.2 The Real Line and Coordinate Plane. Pythagoras 2
1.3 Slopes and Equations of Straight Lines 11
1.4 Circles and Parabolas. Descartes and Fermat 15
1.5 The Concept of a Function 22
1.6 Graphs of Functions 30
1.7 Introductory Trigonometry. The Functions sin θ and cos θ 37
Review: Definitions, Concepts, Methods 46
Additional Problems 47

2 THE DERIVATIVE OF A FUNCTION

2.1 What is Calculus? The Problem of Tangents 51
2.2 How to Calculate the Slope of the Tangent 53
2.3 The Definition of the Derivative 58
2.4 Velocity and Rates of Change. Newton and Leibniz 62
2.5 The Concept of a Limit. Two Trigonometric Limits 68
2.6 Continuous Functions. The Mean Value Theorem and Other Theorems 74
Review: Definitions, Concepts, Methods 81
Additional Problems 81

3 THE COMPUTATION OF DERIVATIVES

3.1 Derivatives of Polynomials 83
3.2 The Product and Quotient Rules 88
3.3 Composite Functions and the Chain Rule 92
3.4 Some Trigonometric Derivatives 98
3.5 Implicit Functions and Fractional Exponents 102
3.6 Derivatives of Higher Order 107
Review: Concepts, Formulas, Methods 111
Additional Problems 111

4 APPLICATIONS OF DERIVATIVES

4.1 Increasing and Decreasing Functions. Maxima and Minima 115
4.2 Concavity and Points of Inflection 120
4.3 Applied Maximum and Minimum Problems 123
4.4 More Maximum-Minimum Problems. Reflection and Refraction 131
4.5 Related Rates 139

	4.6	Newton's Method for Solving Equations	143
	4.7	(Optional) Applications to Economics. Marginal Analysis	146
		Review: Concepts, Methods	156
		Additional Problems	156
5 INDEFINITE INTEGRALS AND DIFFERENTIAL EQUATIONS	5.1	Introduction	163
	5.2	Differentials and Tangent Line Approximations	163
	5.3	Indefinite Integrals. Integration by Substitution	170
	5.4	Differential Equations. Separation of Variables	178
	5.5	Motion under Gravity. Escape Velocity and Black Holes	181
		Review: Concepts, Methods	188
		Additional Problems	188
6 DEFINITE INTEGRALS	6.1	Introduction	190
	6.2	The Problem of Areas	191
	6.3	The Sigma Notation and Certain Special Sums	194
	6.4	The Area under a Curve. Definite Integrals. Riemann	197
	6.5	The Computation of Areas as Limits	203
	6.6	The Fundamental Theorem of Calculus	206
	6.7	Properties of Definite Integrals	213
		Review: Concepts, Methods	217
		Additional Problems	217
		Appendix: The Lunes of Hippocrates	218
7 APPLICATIONS OF INTEGRATION	7.1	Introduction. The Intuitive Meaning of Integration	221
	7.2	The Area between Two Curves	222
	7.3	Volumes: The Disk Method	225
	7.4	Volumes: The Method of Cylindrical Shells	231
	7.5	Arc Length	236
	7.6	The Area of a Surface of Revolution	240
	7.7	Work and Energy	244
	7.8	Hydrostatic Force	252
		Review: Concepts, Methods	254
		Additional Problems	254
		Appendix: Archimedes and the Volume of a Sphere	257

PART II

8 EXPONENTIAL AND LOGARITHM FUNCTIONS	8.1	Introduction	260
	8.2	Review of Exponents and Logarithms	261
	8.3	The Number e and the Function $y = e^x$	264
	8.4	The Natural Logarithm Function $y = \ln x$. Euler	269
	8.5	Applications. Population Growth and Radioactive Decay	277
	8.6	More Applications. Inhibited Population Growth, etc.	283
		Review: Concepts, Formulas	287
		Additional Problems	288
9 TRIGONOMETRIC FUNCTIONS	9.1	Review of Trigonometry	292
	9.2	The Derivatives of the Sine and Cosine	301
	9.3	The Integrals of the Sine and Cosine. The Needle Problem	306

	9.4	The Derivatives of the Other Four Functions	310
	9.5	The Inverse Trigonometric Functions	313
	9.6	Simple Harmonic Motion. The Pendulum	319
	9.7	(Optional) Hyperbolic Functions	324
		Review: Definitions, Formulas	330
		Additional Problems	330
10 METHODS OF INTEGRATION	10.1	Introduction. The Basic Formulas	334
	10.2	The Method of Substitution	337
	10.3	Certain Trigonometric Integrals	340
	10.4	Trigonometric Substitutions	344
	10.5	Completing the Square	348
	10.6	The Method of Partial Fractions	351
	10.7	Integration by Parts	357
	10.8	A Mixed Bag. Strategy for Dealing with Integrals of Miscellaneous Types	362
	10.9	Numerical Integration. Simpson's Rule	369
		Review: Formulas, Methods	375
		Additional Problems	375
		Appendix 1: The Catenary, or Curve of a Hanging Chain	378
		Appendix 2: Wallis's Product $\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots$	380
		Appendix 3: How Leibniz Discovered His Formula $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$	382
11 FURTHER APPLICATIONS OF INTEGRATION	11.1	The Center of Mass of a Discrete System	384
	11.2	Centroids	386
	11.3	The Theorems of Pappus	391
	11.4	Moment of Inertia	393
		Review: Definitions, Concepts	396
		Additional Problems	396
12 INDETERMINATE FORMS AND IMPROPER INTEGRALS	12.1	Introduction. The Mean Value Theorem Revisited	398
	12.2	The Indeterminate Form 0/0. L'Hospital's Rule	400
	12.3	Other Indeterminate Forms	404
	12.4	Improper Integrals	409
	12.5	The Normal Distribution. Gauss	414
		Review: Definitions, Concepts	424
		Additional Problems	424
13 INFINITE SERIES OF CONSTANTS	13.1	What Is an Infinite Series?	427
	13.2	Convergent Sequences	432
	13.3	Convergent and Divergent Series	439
	13.4	General Properties of Convergent Series	445
	13.5	Series of Nonnegative Terms. Comparison Tests	451
	13.6	The Integral Test. Euler's Constant	455
	13.7	The Ratio Test and Root Test	461
	13.8	The Alternating Series Test. Absolute Convergence	465
		Review: Definitions, Concepts, Tests	470
		Additional Problems	470

Appendix 1: Euler's Discovery of the Formula $\Sigma_1^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$ 476
Appendix 2: More about Irrational Numbers. π *Is Irrational* 478
Appendix 3: The Series $\Sigma\, 1/p_n$ *of the Reciprocals of the Primes* 480

14 POWER SERIES
14.1 Introduction 483
14.2 The Interval of Convergence 484
14.3 Differentiation and Integration of Power Series 489
14.4 Taylor Series and Taylor's Formula 494
14.5 Computations Using Taylor's Formula 504
14.6 Applications to Differential Equations 509
14.7 (Optional) Operations on Power Series 514
14.8 (Optional) Complex Numbers and Euler's Formula 521
Review: Concepts, Formulas, Methods 523
Additional Problems 523
Appendix: The Bernoulli Numbers and Some Wonderful Discoveries of Euler 525

PART III

15 CONIC SECTIONS
15.1 Introduction. Sections of a Cone 529
15.2 Another Look at Circles and Parabolas 531
15.3 Ellipses 535
15.4 Hyperbolas 543
15.5 The Focus-Directrix-Eccentricity Definitions 550
15.6 (Optional) Second-Degree Equations. Rotation of Axes 552
Review: Definitions, Properties 557
Additional Problems 558

16 POLAR COORDINATES
16.1 The Polar Coordinate System 560
16.2 More Graphs of Polar Equations 564
16.3 Polar Equations of Circles, Conics, and Spirals 569
16.4 Arc Length and Tangent Lines 575
16.5 Areas in Polar Coordinates 580
Review: Concepts, Formulas 583
Additional Problems 583

17 PARAMETRIC EQUATIONS. VECTORS IN THE PLANE
17.1 Parametric Equations of Curves 586
17.2 The Cycloid and Other Similar Curves 592
17.3 Vector Algebra. The Unit Vectors **i** and **j** 600
17.4 Derivatives of Vector Functions. Velocity and Acceleration 605
17.5 Curvature and the Unit Normal Vector 611
17.6 Tangential and Normal Components of Acceleration 615
17.7 Kepler's Laws and Newton's Law of Gravitation 619
Review: Concepts, Formulas 627
Additional Problems 627
Appendix: Bernoulli's Solution of the Brachistochrone Problem 629

18 VECTORS IN THREE-DIMENSIONAL SPACE. SURFACES

18.1 Coordinates and Vectors in Three-Dimensional Space 632
18.2 The Dot Product of Two Vectors 636
18.3 The Cross Product of Two Vectors 640
18.4 Lines and Planes 646
18.5 Cylinders and Surfaces of Revolution 653
18.6 Quadric Surfaces 656
18.7 Cylindrical and Spherical Coordinates 661
Review: Definitions, Equations 664

19 PARTIAL DERIVATIVES

19.1 Functions of Several Variables 665
19.2 Partial Derivatives 669
19.3 The Tangent Plane to a Surface 675
19.4 Increments and Differentials. The Fundamental Lemma 679
19.5 Directional Derivatives and the Gradient 681
19.6 The Chain Rule for Partial Derivatives 686
19.7 Maximum and Minimum Problems 692
19.8 Constrained Maxima and Minima. Lagrange Multipliers 696
19.9 (Optional) Laplace's Equation, the Heat Equation, and the Wave Equation. Laplace and Fourier 702
19.10 (Optional) Implicit Functions 708
Review: Definitions, Methods 713

20 MULTIPLE INTEGRALS

20.1 Volumes as Iterated Integrals 714
20.2 Double Integrals and Iterated Integrals 718
20.3 Physical Applications of Double Integrals 722
20.4 Double Integrals in Polar Coordinates 726
20.5 Triple Integrals 731
20.6 Cylindrical Coordinates 736
20.7 Spherical Coordinates. Gravitational Attraction 739
20.8 Areas of Curved Surfaces. Legendre's Formula 744
Review: Methods, Formulas 748

Appendix: Euler's Formula $\Sigma_1^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$ *by Double Integration* 748

21 LINE AND SURFACE INTEGRALS. GREEN'S THEOREM, GAUSS'S THEOREM, AND STOKES' THEOREM

21.1 Line Integrals in the Plane 751
21.2 Independence of Path. Conservative Fields 758
21.3 Green's Theorem 764
21.4 Surface Integrals and Gauss's Theorem 771
21.5 Stokes' Theorem 778
21.6 Maxwell's Equations. A Final Thought 784
Review: Concepts, Theorems 786

APPENDICES

A The Theory of Calculus 787

A.1 The Real Number System 787
A.2 Theorems about Limits 791
A.3 Some Deeper Properties of Continuous Functions 796
A.4 The Mean Value Theorem 800
A.5 The Integrability of Continuous Functions 804
A.6 Another Proof of the Fundamental Theorem of Calculus 808
A.7 Continuous Curves with No Length 808
A.8 The Existence of $e = \lim_{h\to 0}(1 + h)^{1/h}$ 811

A.9 *Functions That Cannot Be Integrated* 812
A.10 *The Validity of Integration by Inverse Substitution* 817
A.11 *Proof of the Partial Fractions Theorem* 818
A.12 *The Extended Ratio Tests of Raabe and Gauss* 821
A.13 *Absolute vs. Conditional Convergence* 825
A.14 *Dirichlet's Test. Dirichlet* 830
A.15 *Uniform Convergence for Power Series* 834
A.16 *Division of Power Series* 836
A.17 *The Equality of Mixed Partial Derivatives* 837
A.18 *Differentiation under the Integral Sign* 838
A.19 *A Proof of the Fundamental Lemma* 839
A.20 *A Proof of the Implicit Function Theorem* 840
A.21 *Change of Variables in Multiple Integrals. Jacobians* 841

B A Few Review Topics 845

B.1 *The Binomial Theorem* 845
B.2 *Mathematical Induction* 849

Answers 856
Index 877

PREFACE TO THE INSTRUCTOR

It is a curious fact that people who write thousand-page textbooks still seem to find it necessary to write prefaces to explain their purposes. Enough is enough, one would think. However, every textbook — and this one is no exception — is both an expression of dissatisfaction with existing books and a statement by the author of what he thinks such a book ought to contain, and a preface offers one last chance to be heard and understood. Furthermore, anyone who adds to the glut of introductory calculus books should be called upon to justify his action (or perhaps apologize for it) to his colleagues in the mathematics community.

THE CALCULUS TURMOIL

I borrow this phrase from my old friend Paul Halmos as a handy label for the noise and confusion that have agitated the calculus community for the past dozen years or so. Regardless of one's attitude toward these debates and manifestoes, it seems reasonably clear that two opinions lie at the center of it all: first, too many students fail calculus; and second, our calculus textbooks are so bad that it's natural for these students to fail.

About the books, I completely—or almost completely—disagree. By and large, our calculus textbooks are written by excellent teachers who love their subject and write clear expository English. Naturally, each author has a personal agenda, and this is what separates their books from one another and provides diversity and choice for a healthy marketplace. Some writers prefer to emphasize the theoretical parts of calculus. Others are technology buffs. Yet others (like myself) want a modest amount of biography and history, and believe that interesting and substantial applications from other parts of mathematics and other sciences are highly desirable.

But let there be no misunderstanding: textbooks are servants of teachers, and not their masters. Any group of ten calculus teachers gathered together in a room will have ten very different views of what should be in their courses and how it should be taught. They will differ on the proper amount of theory; on how much numerical calculation is desirable; on whether or not to make regular use of graphing calculators or computer software; on whether some of the more elaborate applications to science are too difficult; on whether biography and history are interesting or boring for their students; and so on. But the bottom line is that only the teachers themselves are in a position to decide what goes on in their own classrooms—and certainly not textbook writers who are completely ignorant of local conditions.

Those of us who write these books try to provide everything we can think of that a teacher might want or need, in full awareness that some parts of what we offer have no place in the course plans of many teachers. Every teacher omits some sections (and even some chapters) and amplifies others, in accordance with individual judgment and personal taste. It is my hope that this book will be useful and agreeable for many diverse tastes and interests. I want it to be a convenient tool for teachers that offers help when help is wanted, and gets out of the way when it is not wanted.

As for the fact that too many of our students fail—if indeed it is a fact—what are the reasons for this? To understand these reasons, let us consider for a moment what is needed for success in calculus. There are clearly three main requirements: a decent background in high school algebra and geometry, some of which is remembered and understood; the ability to read closely and carefully; and tenacity of purpose.

In the matter of preparation in algebra and geometry, our students are in deep trouble. This is suggested by the fact that a few years ago the United States ranked last among the thirteen industrialized nations for the mathematics achievement of its high school graduates. As for reading skills and tenacity of purpose, some of our young people have these qualities, but the great majority do not. Unfortunately, tenacity of purpose is especially important for genuine success in calculus, because this is a subject in which almost every stage depends on having a reasonable command of all that went before, and which therefore requires steady application day after day, week after week, for many months.

We know from our own experience as teachers that calculus is very difficult for most students, and we fully understand the reasons why this is so. But improving our high school mathematics education, and arresting the decline of serious reading and instilling tenacity of purpose among the majority of our young people, are only remote possibilities. Obviously help from outside is not coming, so we must look within ourselves for better ways of doing our jobs.

Most of these ways are familiar to us. Regular class meetings over periods of many months, with frequent quizzes, are intended to encourage steady application to the task of learning. We praise (whenever possible), plead, cajole, and warn. We constantly review the elementary mathematics our students either never learned or have forgotten. We do today's homework problems for them in class, continually thinking out loud and welcoming questions, in the hope that some of the useful ways of thought will rub off to smooth the path for their efforts on tomorrow's homework. However, there is one big thing we can do but rarely do.

Most calculus courses concentrate on the technical details, on developing in students the ability to differentiate and integrate lots of functions. We turn out many students who can perform these somewhat routine tasks. However, if we regularly pause to ask these successful differentiators and integrators just what derivatives and integrals actually are, and what they are for, we rarely get a satisfactory answer—by which I mean an answer that reveals genuine understanding on the part of the student. Many can give the standard limit definitions, but we should expect more than parroted formal definitions. I believe we ought to do a better job of conveying a solid sense of what calculus is really about, what its purpose is, why we need the elaborate machinery of methods for computing derivatives and integrals, and why the Fundamental Theorem of Calculus is truly "fundamental." In a word, we need to communicate what calculus is *for*. More generally, we ought to do more toward encouraging students to learn *why* things are true, rather than merely memorizing ways of solving a few problems to pass examinations. It is clear to us, but not to them, that the only way to learn calculus is to understand it—it is much too massive and complex for mere memorizing to be more than a temporary stopgap—and we have an obligation to help students get this message.

If we can give more attention to these matters, we have a good chance of making calculus less frightening and more relevant for many more students than we have in the past. One of the main purposes of this book is to help us move our teaching in this direction, to convey more light to our students—and less mystery.

CHANGES FROM THE FIRST EDITION

1. **Early Trig.** In the First Edition, I thought it preferable to place trigonometry just before methods of integration. I still agree with myself, but most users think otherwise. I have therefore inserted an account of sines and cosines in Chapter 1, with the calculus of these functions at appropriate places in the following chapters. Since a solid command of trigonometry is so essential for methods of integration, a full review is still given just before the chapter on these methods (Chapter 10).

2. **Homework Problems.** I have added many new problems, mostly of the routine drill type, raising the total to well over 7,000. This is an increase of more than 15 percent and provides about four times as many as most instructors will want to use for their class assignments.

3. **Chapter Summaries.** It seems to help students in their efforts to review and pull things together if they have the ideas and methods of each chapter boiled down to a few pregnant phrases. I have tried to provide this assistance in the summaries at the ends of the chapters.

4. **Appendices.** The first edition had several massive appendices totaling hundreds of pages and containing enrichment material that I thought was so interesting that others would be interested, too. Many were, but I failed to realize that students barely keeping their heads above water in the regular work of the course would take a dim view of any unnecessary burdens. The first two of these long appendices were a collection of material that I thought of as "miscellaneous fun stuff," and a biographical history of calculus. These have been removed, augmented, and published separately in a little paperback book called *Calculus Gems: Brief Lives and Memorable Mathematics* (McGraw-Hill, 1992). However, I have retained some of this material in greatly abbreviated form and placed it in unobtrusive locations throughout the present book.

5. **Theory.** The third of the long appendices in the first edition was on the theory of calculus. I have retained this appendix with a few additions because many colleges and universities offer honors sections that use this material to provide greater theoretical depth than is appropriate for regular sections. Most instructors seem to agree with me in my desire to avoid cluttering our regular courses with any more theory than is absolutely necessary. This approach says: Do not try to prove what no one doubts. However, a number of people have asked me to expand my very condensed discussion of limits and continuous functions and also to give an informal descriptive treatment of the Mean Value Theorem, pointing out its practical uses as they arise. This new material can be found at the end of Chapter 2.

6. **Infinite Series.** My idea for handling this subject in the first edition was not a good one. Most students moving from the first chapter of informal overview into the second of detailed systematic treatment were impatient because they thought they were wasting their time by studying the same concepts all over again. I have therefore completely reorganized these two chapters into a traditional treatment, with series of constants developed first, and then power series.

7. **Vector Analysis.** In the first edition I closed my discussion of vector analysis with Green's Theorem. However, there seems to be general agreement these days that multivariable calculus should go a bit further, and include Gauss's Theorem (the divergence theorem) and Stokes' Theorem. I have rewritten Chapter 21 accordingly.

8. **The Workman Logo.** I thought it would be useful for students if there were some way to signal passages in the text that always cause trouble, because most students are not accustomed to the very slow and careful reading these passages require. The logo I chose for this purpose is copied from a European road sign:

It suggests that hard work is necessary to get through the adjoining passage. I have tried to use it sparingly.

9. **Simplify, Simplify!** When writing this book the first time, I thought I was aiming at the middle of my target, but many users thought I aimed too high. During the preparation of this revision, I kept a poster with these words on it directly in my line of sight as I sat at my work, and of course I looked at this message thousands of times. I hope it worked.

GRAPHING CALCULATORS

These marvelous tools are great fun to use and can make many contributions to the teaching and learning of calculus. But like all tools they should be used wisely, and this means very different things to different people. A scythe can harvest grain or cut off a foot, depending on the skill and judgment of the user.

Some of those in the calculus reform movement believe that the role of numbers and numerical computations should be greatly increased to reach a parity with symbolic (algebraic) and geometric ways of thinking. But I believe we should stop far short of this. In my opinion, there are five subject areas of calculus in which calculators are clearly of great value:

graphing;
calculation of limits;
Newton's method;
numerical integration;
computations using Taylor's formula.

In the last four of these areas, our calculators do heavy computational labor for us, and we are all grateful. But there are dangers, and one of these is an increasing tendency to replace mathematical thinking and learning by button-pushing.

The most surprising examples of this that I've seen involve teachers whose students use graphing calculators—*instead of* factoring or the quadratic formula—to solve quadratic equations as simple as $x^2 - 2x - 3 = 0$. The procedure is to "plot" the function $y = x^2 - 2x - 3$ on the calculator by pushing suitable buttons and then look at the graph the calculator produces to see where it crosses the x-axis. These students are enthusiastic about their calculators and enjoy experimenting with them, and I applaud the teachers who take advantage of this natural interest. But unfortunately, in many cases these students *do not know* how to sketch simple graphs, or how to factor or use the quadratic formula, and are not learning these basic methods of elementary algebra. More generally, sketching the graphs of functions by *thinking* is a fundamental part of learning mathematics. Let us use calculators in our classes to supplement this thinking—but not to replace it. Let us remember that the action that matters takes place in the mind of the student.

These wonderful graphing calculators are superb instruments when used in the right way. It is sobering to reflect that Leibniz himself would perhaps have given a year of his life to possess one—Leibniz who not only (along with Newton) created calculus, but also invented the first calculating machine that could multiply and divide as well as add and subtract.

The many problems in this book that require the use of a calculator are signaled by the standard symbol 🖩.

THE PURPOSE OF THIS BOOK

This book is intended to be a mainstream calculus text that is suitable for every kind of course at every level. It is designed particularly for the standard course of three semesters for students of science, engineering, or mathematics. Students are expected to have a background of high school algebra and geometry, and hopefully, some trigonometry as well.

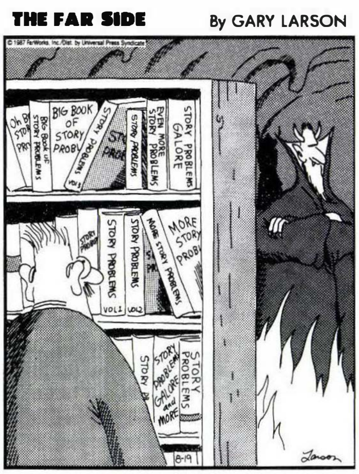
The text itself—that is, the 21 chapters without considering Appendix A—is traditional in subject matter and organization. I have placed great emphasis on *motivation* and *intuitive understanding,* and the refinements of theory are downplayed. Most students are impatient with the theory of the subject, and justifiably so, because the essence of calculus does not lie in theorems and how to prove them, but rather in tools and how to use them. My overriding purpose has been to present calculus as a problem-solving art of immense power that is indispensable in all the quantitative sciences. Naturally, I wish to convince students that the standard tools of calculus are reasonable and legitimate, but not at the expense of turning the subject into a stuffy logical discipline dominated by extra-careful definitions, formal statements of theorems, and meticulous proofs. It is my hope that every mathematical explanation in these chapters will seem to the thoughtful student to be as natural and inevitable as the fact that water flows downhill (rather than uphill) along a canyon floor. The main theme of our work is what calculus is good for—what it enables us to do and understand—and not what its logical nature is as seen from the specialized (and limited) point of view of the modern pure mathematician.

There are several additional features of the book that it might be useful for me to comment on.

Precalculus Material Because of the great amount of calculus that must be covered, it is desirable to get off to a fast start and introduce the derivative quickly, and to spend as little time as possible reviewing precalculus material. However, college freshmen constitute a very diverse group, with widely different levels of mathematical preparation. For this reason I have included a first chapter on precalculus material, which I urge teachers to skim over as lightly as they think advisable for their particular students. This chapter is written in enough detail so that individual students who need to spend more time on the preliminaries should be able to absorb most of it on their own with a little extra effort.*

Problems For students, the most important parts of their calculus book may well be the problem sets, because this is where they spend most of their time and energy. There are more than 7,000 problems in this book, including many old standbys familiar to all calculus teachers and dating back to the time of Euler and even earlier. I have tried to repay my debt to the past by inventing new problems whenever possible. The problem sets are carefully constructed, beginning with routine drill exercises and building up to more complex problems requiring higher levels of thought and skill. The most challenging problems are marked with an asterisk (*). In general, each set contains approximately twice as many problems as most teachers will want to assign for homework, so that a large number will be left over for students to use as review material.

Most of the chapters conclude with long lists of additional problems. Many of these are intended only to provide further scope and variety to the problems sets at the ends of the sections. However, teachers and students alike should treat these additional problems with special care, because a few are quite subtle and difficult and should be attacked only by students with ample reserves of drive and tenacity.



Hell's library

I should also mention that there are several sections scattered throughout the book with no corresponding problems at all. Sometimes these sections occur in small groups and are merely convenient subdivisions of what I consider a single topic and intend as a single assignment, as with Sections 6.1, 6.2, 6.3, and 6.4, 6.5. In other cases (e.g., Sections 15.5 and 19.4), the absence of problems is a tacit suggestion that the subject matter of these sections should be touched upon only lightly and briefly.

There are a great many so-called "story problems" spread through the entire book. All teachers know that students shudder at these problems, because they usually require nonroutine thinking. However, the usefulness of mathematics in the various sciences demands that we try to teach our students how to penetrate into the meaning of a story problem, how to judge what is relevant to it, and how to translate it from words into sketches and equations. Without these skills—which are equally valuable for students who will become doctors, lawyers, financial analysts, or thinkers of any kind—there is no mathematics education worthy of the name.†

*A more complete exposition of high school mathematics that is still respectably concise can be found in my little book, *Precalculus Mathematics in a Nutshell* (Janson Publications, Dedham, MA, 1981), 119 pages.

†I cannot let the opportunity pass without quoting a classic story problem that appeared in *The New Yorker* magazine many years ago. "You know those terrible arithmetic problems about how many peaches some people buy, and so forth? Well, here's one we *like,* made up by a third-grader who was asked to think up a problem similar to the ones in his book: "My father is forty-four years old. My dog is eight. If my dog was a human being, he would be fifty-six years old. How old would my father plus my dog be if they were both human beings?"

Differential Equations and Vector Analysis Each of these subjects is an important branch of mathematics in its own right. They should be taught in separate courses, after calculus, with ample time to explore their distinctive methods and applications. One of the main responsibilities of a calculus course is to prepare the way for these more advanced subjects and take a few preliminary steps in their direction, but just how far one should go is a debatable question. Some writers on calculus try to include mini-courses on these subjects in large chapters at the ends of their books. I disagree with this practice and believe that few teachers make much use of these chapters. Instead, in the case of differential equations I prefer to introduce the subject as early as possible (Section 5.4) and return to it in a low-key way whenever the opportunity arises (Sections 5.5, 7.7, 8.5, 9.6, 17.7, 19.9); and in vector analysis I have responded to reviewers by including a discussion of Gauss's Theorem and Stokes' Theorem in Chapter 21.

Appendix A One of the major ways in which this book is unique and different from all its competitors can be understood by examining Appendix A, which I will now comment on very briefly. Before doing so, I emphasize that this material is entirely separate from the main text and can be carefully studied, dipped into occasionally, or completely ignored, as each individual student or instructor desires.

In the main text, the level of mathematical rigor rises and falls in accordance with the nature of the subject under discussion. It is rather low in the geometrical chapters, where for the most part I rely on common sense together with intuition aided by illustrations; and it is rather high in the chapters on infinite series, where the substance of the subject cannot really be understood without careful thought. I have constantly kept in mind the fact that most students have very little interest in purely mathematical reasoning for its own sake, and I have tried to prevent this type of material from intruding any more than is absolutely necessary. Some students, however, have a natural taste for theory, and some instructors feel as a matter of principle that all students should be exposed to a certain amount of theory for the good of their souls. This appendix contains virtually all of the theoretical material that by any stretch of the imagination might be considered appropriate for the study of calculus. From the purely mathematical point of view, it is possible for instructors to teach courses at many different levels of sophistication by using—or not using—material selected from this appendix.

Supplements The following supplements have been developed to accompany this Second Edition of *Calculus with Analytic Geometry.*

A *Student Solutions Manual* is available for students and contains detailed solutions to the odd-numbered problems. An *Instructor's Solutions Manual* is available for instructors and contains detailed solutions to the even-numbered problems. Also available to instructors adopting the text are a Print Test Bank and an algorithmic Computerized Test Bank.

There are a variety of texts available from McGraw-Hill that support the use of specific graphing calculators and mathematical software programs for calculus. Please contact your local McGraw-Hill representative for more information on these titles.

ACKNOWLEDGMENTS

Every project of this magnitude obviously depends on the cooperative efforts of many people.

For this second edition, the editor Jack Shira provided friendly encouragement and smoothed my way throughout. I am profoundly grateful to my friend Maggie Lanzillo, the associate editor, who was a source of skilled support, assistance, and guidance on innumerable occasions—extending even to restaurant suggestions for dining in Italy. Thanks, Maggie. I owe you more than I can express. And as another piece of extraordinary good luck, this second edition was designed by Joan O'Connor, who designed the first edition, and whose inspired artistic taste seems to work miracles on a daily basis.

Also, I offer my sincere thanks to the publisher's reviewers. These astute people shared their knowledge and judgment with me in many important ways.

FOR THE FIRST EDITION:

Joe Browne, *Onondaga Community College*
Carol Crawford, *United States Naval Academy*
Bruce Edwards, *University of Florida*
Susan L. Friedman, *Baruch College*
Melvin Hausner, *New York University*
Louis Hoelzle, *Bucks County Community College*
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Maurice Oleson, *Memorial University*
David Pengelley, *New Mexico State University*
Mark Schwartz, *Ohio Wesleyan University*
Robert Smith, *Millersville University*

As to the flaws and errors that undoubtedly remain—for there are always a pesky few that manage to hide no matter how fervently we try to find them—there is no one to blame but myself. I will consider it a great kindness if colleagues and student users will take the trouble to inform me of any blemishes they detect, for correction in future printings and editions. As Confucius said, "A man who makes a mistake and doesn't correct it is making two mistakes."

George F. Simmons

TO THE STUDENT

Appearances to the contrary, no writer deliberately sets out to produce an unreadable book; we all do what we can and hope for the best. Naturally, I hope that my language will be clear and helpful to students, and in the end only they are qualified to judge. However, it would be a great advantage to all of us—teachers and students alike—if student users of mathematics textbooks could somehow be given a few hints on the art of reading mathematics, which is a very different thing from reading novels or magazines or newspapers.

In high school mathematics courses, most students are accustomed to tackling their homework problems first, out of impatience to have the whole burdensome task over and done with as soon as possible. These students read the explanations in the text only as a last resort, if at all. This is a grotesque reversal of reasonable procedure, and makes about as much sense as trying to put on one's shoes before one's socks. I suggest that students should read the text first, and when this has been thoroughly assimilated, *then and only then* turn to the homework problems. After all, the purpose of these problems is to nail down the ideas and methods described and illustrated in the text.

How should a student read the text in a book like this? Slowly and carefully, and in full awareness that a great many details have been deliberately omitted. If this book contained every detail of every discussion, it would be five times as long, which God forbid! There is a saying of Voltaire: "The secret of being a bore is to tell everything." Every writer of a book of this kind tries to walk a narrow path between saying too much and saying too little.

The words "clearly," "it is easy to see," and similar expressions are not intended to be taken literally, and should never be interpreted by any student as a putdown on his or her abilities. These are code-phrases that have been used in mathematical writing for hundreds of years. Their purpose is to give a signal to the careful reader that in this particular place, the exposition is somewhat condensed, and perhaps a few details of calculations have been omitted. Any phrase like this amounts to a friendly hint to the student that it might be a good idea to read even more carefully and thoughtfully in order to fill in omissions in the exposition, or perhaps get out a piece of scratch paper to verify omitted details of calculations. Or better yet, make full use of the margins of this book to emphasize points, raise questions, perform little computations, and correct misprints.

George F. Simmons

CALCULUS
WITH ANALYTIC
GEOMETRY

1 NUMBERS, FUNCTIONS, AND GRAPHS

1.1 INTRODUCTION

Everyone knows that the world in which we live is dominated by motion and change. The earth moves in its orbit around the sun; a rock thrown upward slows and stops, and then falls back to earth with increasing speed; the population of India grows each year at an increasing rate; and radioactive elements decay. These are merely a few items in the endless array of phenomena for which mathematics is the most natural medium of communication and understanding. As Galileo said more than 300 years ago, "The Great Book of Nature is written in mathematical symbols."

Calculus is that branch of mathematics whose primary purpose is the study of motion and change. It is an indispensable tool of thought in almost every field of pure and applied science—in physics, chemistry, biology, astronomy, geology, engineering, and even some of the social sciences. It also has many important uses in other parts of mathematics, especially geometry. By any standard, the methods and applications of calculus constitute one of the greatest intellectual achievements of civilization, and to become acquainted with these ideas is to open many doors that lead to a broader and richer life of the mind.

The main objects of study in calculus are functions. But what is a function? Roughly speaking, it is a rule or law that tells us how one variable quantity depends upon another. This is the master concept of the exact sciences. It offers us the prospect of understanding and correlating natural phenomena by means of mathematical machinery of great and sometimes mysterious power. The concept of a function is so vitally important for all our work that we must strive to clarify it beyond any possibility of confusion. This purpose is the theme of the present chapter.

The following sections contain a good deal of material that many readers have studied before. Some will welcome the opportunity to review and refresh their ideas. Those who already understand this material and find it irksome to tread the same path over again may discover some interesting sidelights and stimulating challenges among the Additional Problems at the end of the chapter. This chapter is intended solely for purposes of review. It can be studied carefully, or lightly, or even skipped altogether, depending on the reader's level of preparation. The actual subject matter of this course begins in Chapter 2, and it would be very unfortunate if even a single student should come to feel that this preliminary chapter is more of an obstacle than a source of assistance—for its only purpose is to smooth the way.

1.2 THE REAL LINE AND COORDINATE PLANE. PYTHAGORAS

Most of the variable quantities we study—such as length, area, volume, position, time, and velocity—are measured by means of real numbers, and in this sense calculus is based on the real number system. It is true that there are other important and useful number systems—for instance, the complex numbers. It is also true that two- and three-dimensional treatments of position and velocity require the use of vectors. These ideas will be examined in due course, but for a long time to come the only numbers we shall be working with are the real numbers.*

It is assumed in this book that students are familiar with the elementary algebra of the real number system. Nevertheless, in this section we give a brief descriptive survey that may be helpful. For our purposes this is sufficient, but any reader who wishes to probe more deeply into the nature of real numbers will find a more precise discussion in Appendix A.1 at the back of the book.

The real number system contains several types of numbers that deserve special mention: the *positive integers* (or *natural numbers*)

$$1, 2, 3, 4, 5, \ldots;$$

the *integers*

$$\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots;$$

and the *rational numbers,* which are those real numbers that can be represented as fractions (or quotients of integers), such as

$$\tfrac{2}{3}, -\tfrac{7}{4}, 4, 0, -5, 3.87, 2\tfrac{1}{4}.$$

A real number that is not rational is said to be *irrational;* for example,

$$\sqrt{2}, \sqrt{3}, \sqrt{2}+\sqrt{3}, \sqrt{5}, \sqrt[3]{5}, \qquad \text{and} \qquad \pi$$

are irrational numbers.†

We take this opportunity to remind the reader that for any positive number a, the symbol $\sqrt{a}$ always means its positive square root. Thus, $\sqrt{4}$ is equal to 2 and not -2, even though $(-2)^2 = 4$. If we wish to designate both square roots of 4, we must write $\pm\sqrt{4}$. Similarly, $\sqrt[n]{a}$ always means the positive nth root of a.

THE REAL LINE

The use of the real numbers for measurement is reflected in the very convenient custom of representing these numbers graphically by points on a horizontal straight line (Fig. 1.1).

This representation begins with the choice of an arbitrary point as the origin or zero point, and another arbitrary point to the right of it as the point 1. The dis-

*The adjective "real" was originally used to distinguish these numbers from numbers like $\sqrt{-1}$, which were once thought to be "unreal" or "imaginary."

†Our aims in the present book are almost entirely practical. Nevertheless, our discussions often give rise to certain "impractical" questions that some readers may find interesting and appealing. As an example, how do we know that the number $\sqrt{2}$ is irrational? For readers with the time and inclination to pursue such questions—and also because we consider the answers to be worth knowing about for their own sake—we offer food for further thought in a little paperback book entitled *Calculus Gems: Brief Lives and Memorable Mathematics* (McGraw-Hill, 1992). Some of the facts about irrational numbers, with proofs, are discussed in Section B.2 of this book.

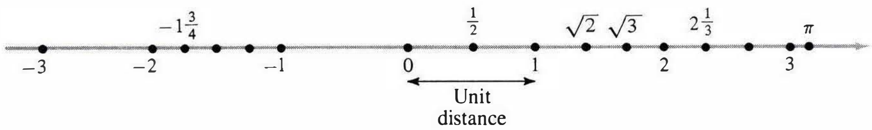


Figure 1.1 The real line.

tance between these two points (the unit distance) then serves as a scale by means of which we can assign a point on the line to every positive and negative integer, as illustrated in the figure, and also to every rational number. Notice that all positive numbers lie to the right of 0, in the "positive direction," and all negative numbers lie to the left.* The method of assigning a point to a rational number is shown in the figure for the number $\frac{7}{3} = 2\frac{1}{3}$: the segment between 2 and 3 is subdivided by two points into three equal segments, and the first of these points is labeled $2\frac{1}{3}$. This procedure of using equal subdivisions clearly serves to determine the point on the line which corresponds to any rational number whatever. Furthermore, this correspondence between rational numbers and points can be extended to irrational numbers; for the decimal expansion of an irrational number, such as

$$\sqrt{2} = 1.414\ldots, \qquad \sqrt{3} = 1.732\ldots, \qquad \pi = 3.14159\ldots,$$

can be interpreted as a set of instructions specifying the exact position of the corresponding point. For example, by looking at the expansion we see that the point corresponding to $\sqrt{2}$ lies between 1 and 2, between 1.4 and 1.5, between 1.41 and 1.42, and so on, and these requirements uniquely determine the position of the corresponding point.

We have described a one-to-one correspondence between all real numbers and all points on the line which establishes these numbers as a coordinate system for the line. This coordinatized line is called the *real line*. It is convenient and customary to merge the logically distinct concepts of the real number system and the real line, and we shall freely speak of points on the line as if they were numbers and of numbers as if they were points on the line. Thus, such mixed expressions as "irrational point" and "the segment between 2 and 3" are quite natural and will be used without further comment.

INEQUALITIES

The left-to-right linear succession of points on the real line corresponds to an important part of the algebra of the real number system, that dealing with inequalities. These ideas play a larger role in calculus than in earlier mathematics courses, so we briefly recall the essential points.

The geometric meaning of the inequality $a < b$ (read "a is less than b") is simply that a lies to the left of b; the equivalent inequality $b > a$ ("b is greater than a") means that b lies to the right of a. A number a is positive or negative according as $a > 0$ or $a < 0$. The main rules used in working with inequalities are the following:

*The arrowhead on the right end of the real line indicates the positive direction and nothing more.

1. If $a > 0$ and $b < c$, then $ab < ac$.
2. If $a < 0$ and $b < c$, then $ab > ac$.
3. If $a < b$, then $a + c < b + c$ for any number c.

Rules 1 and 2 are usually expressed by saying that an inequality is preserved on multiplication by a positive number, and reversed on multiplication by a negative number; and rule 3 says that an inequality is preserved when any number (positive or negative) is added to both sides. It is often desirable to replace an inequality $a > b$ by the equivalent inequality $a - b > 0$, with rule 3 being used to establish the equivalence.

If we wish to say that a is positive or equal to 0, we write $a \geq 0$ and read this "a is greater than or equal to zero." Similarly, $a \geq b$ means that $a > b$ or $a = b$. Thus, $3 \geq 2$ and $3 \geq 3$ are both true inequalities.

We also recall that a product of two or more numbers is zero if and only if one of its factors is zero. If none of its factors are zero, it is positive or negative according as it has an even or an odd number of negative factors.

ABSOLUTE VALUES

The *absolute value* of a number a is denoted by $|a|$ and defined by

$$|a| = \begin{cases} a & \text{if } a \geq 0, \\ -a & \text{if } a < 0. \end{cases}$$

For example, $|3| = 3$, $|-2| = -(-2) = 2$, and $|0| = 0$. It is clear that the operation of forming the absolute value leaves positive numbers unchanged and replaces each negative number by the corresponding positive number. The main properties of this operation are

$$|ab| = |a||b| \qquad \text{and} \qquad |a + b| \leq |a| + |b|.$$

In geometric language, the absolute value of a number a is simply the distance from the point a to the origin. Similarly, the distance from a to b is $|a - b|$.

To solve an equation such as $|x + 2| = 3$, we can write it in the form $|x - (-2)| = 3$ and think of it as saying that "the distance from x to -2 is 3." With Fig. 1.1 in mind, it is evident that the solutions are $x = 1$ and $x = -5$. We can also solve this equation by using the fact that $|x + 2| = 3$ means that $x + 2 = 3$ or $x + 2 = -3$; the solutions are $x = 1$ and $x = -5$, as before.

INTERVALS

The sets of real numbers we shall be dealing with most frequently are intervals. An *interval* is simply a segment on the real line. If its endpoints are the numbers a and b, then the interval consists of all numbers that lie between a and b. However, we may or may not want to include the endpoints themselves as part of the interval.

To be more precise, suppose that a and b are numbers, with $a < b$. The *closed interval* from a to b, denoted by $[a, b]$—using brackets—includes its endpoints, and therefore consists of all real numbers x such that $a \leq x \leq b$. Parentheses are used to indicate excluded endpoints. The interval (a, b), with both endpoints excluded, is called the *open interval* from a to b; it consists of all x such that

$a < x < b$. Sometimes we wish to include only one endpoint in an interval. Thus, the intervals denoted by $[a, b)$ and $(a, b]$ are defined by the inequalities $a \le x < b$ and $a < x \le b$, respectively. In each of these cases, any number c such that $a < c < b$ is called an *interior point* of the interval (Fig. 1.2).

Strictly speaking, the notations $a \le x \le b$ and $[a, b]$ have different meanings —the first represents a restriction imposed on x, while the second denotes a set —but both designate the same interval. We will therefore consider them to be equivalent and use them interchangeably, and the reader should become familiar with both. However, the geometric meaning of the notation $a \le x \le b$ is more easily grasped by the eye, and for this reason we usually prefer it to the other.

A half-line is often considered to be an interval extending to infinity in one direction. The symbol ∞ (read "infinity") is frequently used in designating such an interval. Thus, for any real number a the intervals defined by the inequalities $a < x$ and $x \le a$ can be written as $a < x < \infty$ and $-\infty < x \le a$, or equivalently as (a, ∞) and $(-\infty, a]$. Remember, however, that the symbols ∞ and $-\infty$ do not denote real numbers; they are used in this manner only as a convenient way of emphasizing that x is allowed to be arbitrarily large (in either the positive or negative direction). As an aid in keeping the notation clear in one's mind, it may be helpful to think of $-\infty$ and ∞ as "fictitious numbers" located at the left and right "ends" of the real line, as suggested in Fig. 1.3. Also, it is sometimes convenient to think of the entire real line itself as an interval, $-\infty < x < \infty$ or $(-\infty, \infty)$.

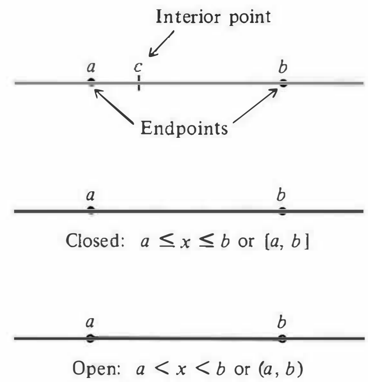


Figure 1.2 Intervals.

Figure 1.3

Sets of numbers described by means of inequalities and absolute values are often intervals. It is clear, for instance, that the set of all x such that $|x| < 2$ is the interval $-2 < x < 2$ or $(-2, 2)$.

Example 1 Solve the inequality $x^2 - 2 < x$.

Solution To "solve" an inequality like this means to find all numbers x for which the inequality is true. We begin by writing it as $x^2 - x - 2 < 0$, and then we write it in the factored form

$$(x + 1)(x - 2) < 0.$$

For this to be true, the two factors must have opposite signs: $x + 1 > 0$ and $x - 2 < 0$, or $x + 1 < 0$ and $x - 2 > 0$. These conditions are equivalent to $x > -1$ and $x < 2$, or $x < -1$ and $x > 2$. The second pair of conditions is easily seen to be impossible. The first pair of conditions means that x lies in the open interval $-1 < x < 2$, and these x's constitute the solution of the given inequality.

THE COORDINATE PLANE

Just as real numbers are used as coordinates for points on a line, pairs of real numbers can be used as coordinates for points in a plane. For this purpose we establish a *rectangular coordinate system* in the plane, as follows.

Draw two perpendicular straight lines in the plane, one horizontal and the other vertical, as shown in Fig. 1.4. These lines are called the *x-axis* and *y-axis*, respectively, and their point of intersection is called the *origin*. Coordinates are assigned to these axes in the manner described earlier, with the origin as the zero point on both and the same unit of distance measurement on both. The positive

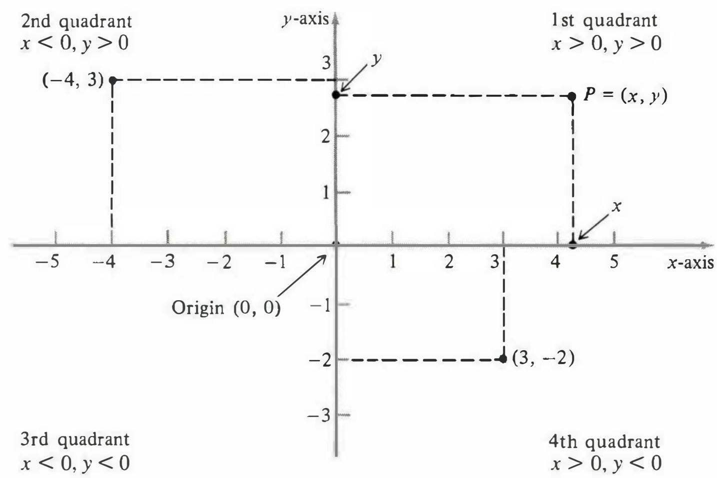


Figure 1.4 The coordinate plane or *xy*-plane.

x-axis is to the right of the origin and the negative x-axis to the left, as before; and the positive y-axis is above the origin and the negative y-axis below.

Now consider a point P anywhere in the plane. Draw a line through P parallel to the y-axis, and let x be the coordinate of the point where this line crosses the x-axis. Similarly, draw a line through P parallel to the x-axis, and let y be the coordinate of the point where this line crosses the y-axis. The numbers x and y determined in this way are called the *x-coordinate* and *y-coordinate* of P. In referring to the coordinates of P, it is customary to write them as an ordered pair (x, y) with the x-coordinate written first; we say that P has coordinates (x, y).* This correspondence between P and its coordinates establishes a one-to-one correspondence between all points in the plane and all ordered pairs of real numbers; for P determines its coordinates uniquely, and by reversing the process we see that each ordered pair of real numbers uniquely determines a point P with these numbers as its coordinates. As in the case of the real line, it is customary to drop the distinction between a point and its coordinates, and to speak of "the point (x, y)" instead of "the point with coordinates (x, y)." The coordinates x and y of the point P are sometimes called the *abscissa* and *ordinate* of P. Notice particularly that points $(x, 0)$ lie on the x-axis, that points $(0, y)$ lie on the y-axis, and that $(0, 0)$ is the origin. Also, the axes divide the plane into four quadrants, as shown in Fig. 1.4, and these quadrants are characterized as follows by the signs of x and y: first quadrant, $x > 0$ and $y > 0$; second quadrant, $x < 0$ and $y > 0$; third quadrant, $x < 0$ and $y < 0$; fourth quadrant, $x > 0$ and $y < 0$.

When the plane is equipped with the coordinate system described here, it is usually called the *coordinate plane* or the *xy-plane.*

THE DISTANCE FORMULA

Much of our work involves geometric ideas—right triangles, similar triangles, circles, spheres, cones, etc.—and we assume that students have acquired a reasonable grasp of elementary geometry from earlier mathematics courses. A ma-

*In practice, the use of the same notation for ordered pairs as for open intervals never leads to confusion, because in any specific context it is always clear which is meant.

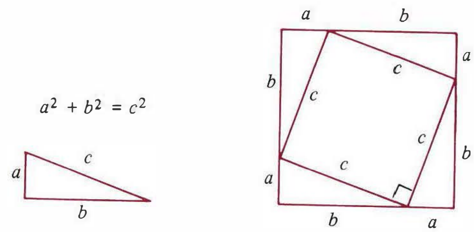


Figure 1.5 The Pythagorean theorem and a proof.

jor fact of particular importance is the Pythagorean theorem: In any right triangle, the sum of the squares of the legs equals the square of the hypotenuse (Fig. 1.5). There are many proofs of this theorem, but the following is probably simpler than most. Let the legs be a and b and the hypotenuse c, and arrange four replicas of the triangle in the corners of a square of side $a + b$, as shown on the right in Fig. 1.5. Then the area of the large square equals 4 times the area of the triangle plus the area of the small square; that is,

$$(a + b)^2 = 4(\tfrac{1}{2}ab) + c^2.$$

This simplifies at once to $a^2 + b^2 = c^2$, which is the Pythagorean theorem.

As the first of many applications of this fact, we obtain the formula for the distance d between any two points in the coordinate plane. If the points are $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$, then the segment joining them is the hypotenuse of a right triangle (Fig. 1.6) with legs $|x_1 - x_2|$ and $|y_1 - y_2|$. By the Pythagorean theorem,

$$\begin{aligned} d^2 &= |x_1 - x_2|^2 + |y_1 - y_2|^2 \\ &= (x_1 - x_2)^2 + (y_1 - y_2)^2, \end{aligned}$$

so

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}. \tag{1}$$

This is the *distance formula*.

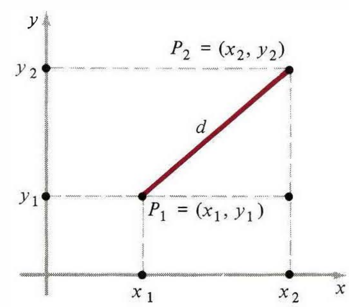


Figure 1.6

Example 2 The distance d between the points $(-4, 3)$ and $(3, -2)$ in Fig. 1.4 is

$$d = \sqrt{(-4 - 3)^2 + (3 + 2)^2} = \sqrt{74}.$$

Notice that in applying formula (1) it does not matter in which order the points are taken.

Example 3 Find the lengths of the sides of the triangle whose vertices are $P_1 = (-1, -3)$, $P_2 = (5, -1)$, $P_3 = (-2, 10)$.

By (1), these lengths are

$$P_1P_2 = \sqrt{(-1 - 5)^2 + (-3 + 1)^2} = \sqrt{40} = 2\sqrt{10},$$

$$P_1P_3 = \sqrt{(-1 + 2)^2 + (-3 - 10)^2} = \sqrt{170},$$

$$P_2P_3 = \sqrt{(5 + 2)^2 + (-1 - 10)^2} = \sqrt{170}.$$

These calculations reveal that the triangle is isosceles, with P_1P_3 and P_2P_3 as the equal sides.

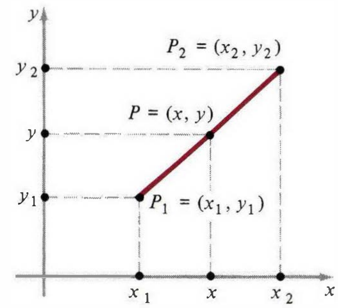


Figure 1.7

THE MIDPOINT FORMULAS

It is often useful to know the coordinates of the midpoint of the segment joining two given distinct points. If the given points are $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$, and if $P = (x, y)$ is the midpoint, then it is clear from Fig. 1.7 that x is the midpoint of the projection of the segment on the x-axis, and similarly for y. This tells us (examine the figure—and think!) that $x = x_1 + \frac{1}{2}(x_2 - x_1)$ and $y = y_1 + \frac{1}{2}(y_2 - y_1)$, so

$$x = \tfrac{1}{2}(x_1 + x_2) \qquad \text{and} \qquad y = \tfrac{1}{2}(y_1 + y_2).$$

Another way of obtaining these formulas is to notice from Fig. 1.7 that $x - x_1 = x_2 - x$, so $2x = x_1 + x_2$ or $x = \frac{1}{2}(x_1 + x_2)$, with the same argument applying to y. Similarly, if P is a trisection point of the segment joining P_1 and P_2, its coordinates can be found from the fact that x and y are trisection points of the corresponding segments on the x-axis and y-axis.

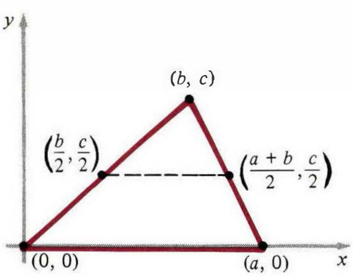


Figure 1.8

Example 4 In any triangle, the segment joining the midpoints of two sides is parallel to the third side and half its length. We know this from elementary geometry; but to prove it by our methods, we begin by noticing that the triangle can always be placed in the position shown in Fig. 1.8, with its third side along the positive x-axis and the left endpoint of this side at the origin. We then insert the midpoints of the other two sides, as shown, and observe that since they have the same y-coordinate, the segment joining them is parallel to the third side lying on the x-axis. The length of this segment is simply the difference between the x-coordinates of its endpoints,

$$\frac{a+b}{2} - \frac{b}{2} = \frac{a}{2},$$

which is half the length of the third side.

This example illustrates the way in which coordinates can often be used to give algebraic proofs of geometric theorems. The device employed here, of placing the figure in a convenient position relative to the coordinate system, has the purpose of simplifying the algebra.

NOTE ON PYTHAGORAS

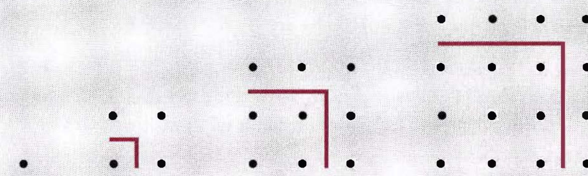
Who was this Pythagoras, whose name is attached to the great theorem of geometry we have just been using? And why should we care?

The pre-Socratic philosophers of ancient Greece—that is, those who lived before the time of Socrates (470?–399 B.C.)—were one of the most remarkable and influential groups of people in human history. The best known of these was Pythagoras of Samos (580?–500? B.C.), a mathematician, scientist, and mystic whose ideas live on today as part of the bone and flesh of our modern civilization.

Greek geometry was certainly one of the half-dozen supreme intellectual achievements of all time. Pythagoras' master Thales (625?–547? B.C.) had created geometry as the contemplation of abstract patterns of lines and figures and constructed the first proofs of the first theorems. But Pythagoras was the first person to see geometry as an organized system of thought held together by deductive proof, with one theorem depending on another in a tightly woven fabric of logic. Also, tradition tells us that he himself discovered many theorems, most notably, the fact that the sum of the angles in any triangle equals two right angles, and the famous Pythagorean theorem discussed above.

Pythagoras was born on the beautiful island of Samos, a mile or two off the Aegean coast of Turkey and a good day's walk along the shore from Thales' home town of Miletus. At the age of about 50, he migrated from Samos to the Greek colony of Crotona in southern Italy, where he established the famous Pythagorean school, a quasi-religious society with a solid claim to the honor of being the world's first university. The Pythagoreans were best known for two teachings: the doctrine of transmigration of souls at death from one body into another, and the theory that numbers constitute the true essence of all things. Believers performed rites of purification and followed strict moral and dietary rules (no sex, no meat) to enable their souls to rise to higher levels of spirituality in subsequent lives. Their beliefs also led them to consider the sexes as equal and to treat animals and slaves humanely. For who knows? In a subsequent life one might return as a slave, or one's soul might take up residence in an animal's body, or even—alas!—an insect's.

As a way of achieving purification of the mind, the Pythagoreans studied geometry, arithmetic, music, and astronomy—arithmetic not in the sense of useful computational skills but rather as the abstract theory of numbers. They were particularly fond of the "figurate numbers," which arise by arranging dots or points in regular geometric patterns. For example, there are the square numbers 1, 4, 9, 16, . . . :



As indicated, each square number can be obtained from its predecessor by adding an L-shaped border called a *gnomon,* meaning a carpenter's square. Since the successive gnomons are the successive odd numbers, it is immediately clear from the square arrays that the sum of the first n odd numbers equals n^2:

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$

Who would have believed that the common odd numbers and the relatively rare perfect squares are related in such a simple yet remarkable way? The Pythagoreans were fascinated, and rightly so, by the grave and beautiful games that numbers play with each other—games that seemed to them to take place outside of space and time and to be quite independent of the human mind itself.

Further, Pythagoras performed the first deliberate scientific experiment, on the relation between positive whole numbers and the musical notes emitted by a plucked lyre string. Also, he was the first person to conceive the supremely daring conjecture that the world is an ordered, understandable whole, and he applied the word *kosmos*—which previously meant order or harmony—to this whole.

In these and other ways Pythagoras was one of the prime creators of the Western civilization that sustains us all—as fish are sustained by the water in which they swim.

PROBLEMS

1 Among the words "integer," "rational," and "irrational," state the ones that apply to

(a) $-\frac{2}{3}$; (b) 0;
(c) $\frac{45}{9}$; (d) 0.75;
(e) $-\sqrt{49}$; (f) $1/\pi$;
(g) 9.000 . . . ; (h) $3^{1/2}$;
(i) $-\frac{20}{7}$; (j) $\frac{94}{7}$.

2 Every integer is either even or odd. The even integers are those that are divisible by 2, so n is even if and only if it has the form $n = 2k$ for some integer k. The odd integers are those that have the form $n = 2k + 1$ for some integer k.

(a) If n is even, prove that n^2 is also even.
(b) If n is odd, prove that n^2 is also odd.

In Problems 3–12, rewrite the given expression without using the absolute value symbol.

3 $|7 - 18|$. **4** $|7| - |-18|$.
5 $|\pi - 3|$. **6** $|3 - \pi|$.
7 $|x - 5|$ if $x < 5$. **8** $|x - 5|$ if $x > 5$.
9 $|x^2 + 10|$. **10** $|-11| - |-10|$.
11 $|1 - 3x^2|$ if $x \geq 1$. **12** $|\sqrt{10} - 10|$.

13 Solve the following inequalities:
(a) $x(x - 1) > 0$;
(b) $(x - 1)(x + 2) < 0$;

(c) $x^2 + 4x - 21 > 0$;
(d) $2x^2 + x < 3$;
(e) $4x^2 + 10x - 6 < 0$;
(f) $x^2 + 2x + 4 > 0$.

14 Recall that $\sqrt{a}$ is a real number if and only if $a \geq 0$, and find the values of x for which each of the following is a real number:
(a) $\sqrt{4 - x^2}$; (b) $\sqrt{x^2 - 9}$;
(c) $\dfrac{1}{\sqrt{4 - 3x}}$; (d) $\dfrac{1}{\sqrt{x^2 - x - 12}}$.

15 Find the values of x for which each of the following is positive:
(a) $\dfrac{x}{x^2 + 4}$; (b) $\dfrac{x}{x^2 - 4}$;
(c) $\dfrac{x + 1}{x - 3}$; (d) $\dfrac{x^2 - 1}{x^2 - 3x}$.

16 State the values of a for which the following inequalities are valid:
(a) $a \leq a$; (b) $a < a$.

17 If $a \leq b$ and $b \leq a$, what conclusion can be drawn about a and b?

18 (a) If $a < b$ is true, is it also necessarily true that $a \leq b$?
(b) If $a \leq b$ is true, is it also necessarily true that $a < b$?

19 State whether each pair of points lies on a horizontal or a vertical line:
(a) $(-2, -5), (-2, 3)$; (b) $(-2, -5), (7, -5)$;
(c) $(-3, 4), (6, 4)$; (d) $(2, -11), (2, 5)$;
(e) $(2, 2), (-13, 2)$; (f) $(-7, -7), (-7, 7)$;
(g) $(3, 5), (3, -2)$; (h) $(-1, -2), (2, -2)$.

20 Three vertices of a rectangle are $(-1, 2)$, $(3, -5)$, $(-1, -5)$. What is the fourth vertex?

21 Find the distance between each pair of points:
(a) $(1, 2), (6, 7)$; (b) $(2, 5), (-1, 3)$;
(c) $(-7, 3), (1, -2)$; (d) $(a, b), (b, a)$.

22 In Problem 21 find the midpoint of the segment joining each pair of points.

23 Draw a sketch indicating the points (x, y) in the plane for which
(a) $x < 2$;
(b) $-1 < y \leq 2$;
(c) $0 \leq x \leq 1$ and $0 \leq y \leq 1$;
(d) $x = -1$;
(e) $y = 3$;
(f) $x = y$.

24 Use the distance formula to show that the points $(-2, 1)$, $(2, 2)$, and $(10, 4)$ lie on a straight line.

25 Show that the point $(6, 5)$ lies on the perpendicular bisector of the segment joining the points $(-2, 1)$ and $(2, -3)$.

26 Show that the triangle whose vertices are $(3, -3)$, $(-3, 3)$, and $(3\sqrt{3}, 3\sqrt{3})$ is equilateral.

27 The two points $(2, -2)$ and $(-6, 5)$ are the endpoints of a diameter of a circle. Find the center and radius of the circle.

28 Find every point whose distance from each of the two coordinate axes equals its distance from the point $(4, 2)$.

29 Find the point equidistant from the three points $(-9, 0)$, $(6, 3)$, and $(-5, 6)$.

30 If a and b are any two numbers, convince yourself that:
(a) the points (a, b) and $(a, -b)$ are symmetric with respect to the x-axis;
(b) (a, b) and $(-a, b)$ are symmetric with respect to the y-axis;
(c) (a, b) and $(-a, -b)$ are symmetric with respect to the origin.

31 What symmetry statement can be made about the points (a, b) and (b, a)?

32 In each case, place the figure in a convenient position relative to the coordinate system and prove the statement algebraically:
(a) The diagonals of a parallelogram bisect each other.
(b) The sum of the squares of the diagonals of a parallelogram equals the sum of the squares of the sides.
(c) The midpoint of the hypotenuse of a right triangle is equidistant from the three vertices.

Use the fact stated in (c) to show that when the acute angles of a right triangle are 30° and 60°, the side opposite the 30° angle is half the hypotenuse.

33 In an isosceles right triangle, both acute angles are 45°. If the hypotenuse is h, what is the length of each of the other sides?

34 Let $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ be distinct points. If $P = (x, y)$ is on the segment joining P_1 and P_2 and one-third of the way from P_1 to P_2, show that

$$x = \tfrac{1}{3}(2x_1 + x_2) \qquad \text{and} \qquad y = \tfrac{1}{3}(2y_1 + y_2).$$

Find the corresponding formulas if P is two-thirds of the way from P_1 to P_2.

35 Consider an arbitrary triangle with vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3). Find the point on each median which is two-thirds of the way from the vertex to the midpoint of the opposite side.* Perform the calculations separately for each median and verify that these three points are all the same, with coordinates

$$\tfrac{1}{3}(x_1 + x_2 + x_3) \qquad \text{and} \qquad \tfrac{1}{3}(y_1 + y_2 + y_3).$$

This proves that the medians of any triangle intersect at a point which is two-thirds of the way from each vertex to the midpoint of the opposite side.

*A *median* of a triangle is a segment joining a vertex to the midpoint of the opposite side.

1.3 SLOPES AND EQUATIONS OF STRAIGHT LINES

In this section we use the language of algebra to describe the set of all points that lie on a given straight line. This algebraic description is called the *equation of the line.* First, however, it is necessary to discuss an important preliminary concept: the slope of a line.

THE SLOPE OF A LINE

Any nonvertical straight line has a number associated with it that specifies its direction, called its *slope*. This number is defined as follows (Fig. 1.9 illustrates the definition). Choose any two distinct points on the line, say $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$. Then the slope is denoted by m and defined to be the ratio

$$m = \frac{y_2 - y_1}{x_2 - x_1}. \tag{1}$$

If we reverse the order of subtraction in both numerator and denominator, then the sign of each is changed, so m is unchanged:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}.$$

This shows that the slope can be computed as the difference of the y-coordinates divided by the difference of the x-coordinates — in either order, as long as both differences are formed in the same order. In Fig. 1.9, where P_2 is placed to the right of P_1 and the line rises to the right, it is clear that the slope as defined by (1) is simply the ratio of the height to the base in the indicated right triangle. It is necessary to know that the value of m depends only on the line itself and is the same no matter where the points P_1 and P_2 happen to be located on the line. This is easy to see by visualizing the effect of moving P_1 and P_2 to different positions on the line; this change gives rise to a similar right triangle and therefore leaves the ratio in (1) unaltered.

If we choose the position of P_2 so that $x_2 - x_1 = 1$, that is, if we place P_2 1 unit to the right of P_1, then $m = y_2 - y_1$. This tells us that the slope is simply the change in y as a point (x, y) moves along the line in such a way that x increases by 1 unit. This change in y can be positive, negative, or zero, depending on the direction of the line. We therefore have the following important correlations between the sign of m and the indicated directions:

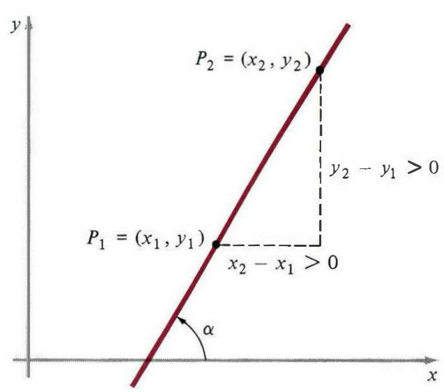


Figure 1.9

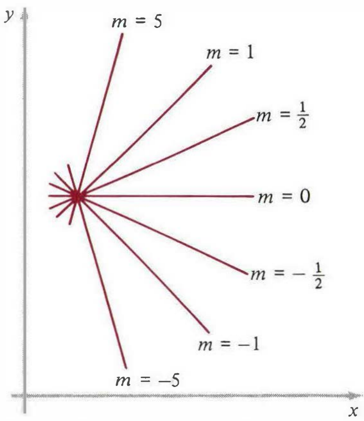


Figure 1.10 A variety of slopes.

$m > 0$, line rises to the right;

$m < 0$, line falls to the right;

$m = 0$, line horizontal.

Further, the absolute value of m is a measure of the steepness of the line (Fig. 1.10). It is evident from (1) why a vertical line has no slope, for in this case the two points have equal x-coordinates and the denominator in (1) is 0—and we know that division by 0 is undefined.

If the line under discussion crosses the x-axis, then the angle α from the positive x-direction to the line, measured counterclockwise, is called the *inclination* —or sometimes the *angle of inclination*—of the line. Students who have studied trigonometry will see from Fig. 1.9 that the slope is the tangent of this angle, $m = \tan \alpha$.

EQUATIONS OF A LINE

A vertical line is characterized by the fact that all points on it have the same x-coordinate. If the line crosses the x-axis at the point $(a, 0)$, then a point (x, y) lies on the line if and only if

$$x = a, \tag{2}$$

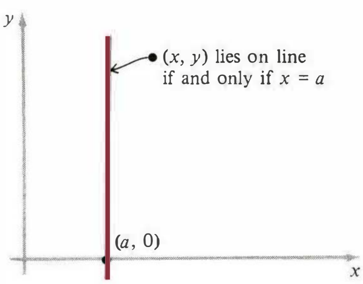


Figure 1.11

as illustrated in Fig. 1.11. The statement that (2) is the equation of the line means precisely this: A point (x, y) lies on the line if and only if condition (2) is satisfied.

Next consider a nonvertical line, and let it be "given" in the sense that we know a point (x_0, y_0) on it and its slope m (Fig. 1.12). If (x, y) is a point in the plane that does not lie on the vertical line through (x_0, y_0), then it is easy to see that this point lies on the given line if and only if the line determined by (x_0, y_0) and (x, y) has the same slope as the given line:

$$\frac{y - y_0}{x - x_0} = m. \tag{3}$$

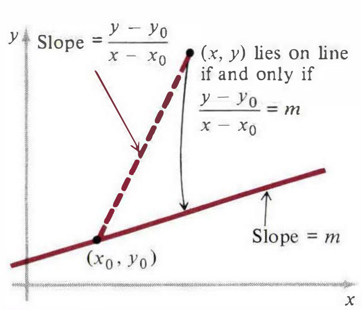


Figure 1.12

This would be the equation of our line except for the minor flaw that the coordinates of the point (x_0, y_0)—which is certainly on the line—do not satisfy the equation (they reduce the left side to the meaningless expression 0/0). This flaw is easily removed by writing equation (3) in the form

$$y - y_0 = m(x - x_0). \tag{4}$$

Nevertheless, we usually prefer the form (3), because its direct connection with the geometric idea illustrated in Fig. 1.12 makes it easy to remember. Either equation (or both) is called the *point-slope equation* of a line, since the line is initially specified by means of a known point on it and its known slope. To grasp more firmly the meaning of equation (4), imagine a point (x, y) moving along the given line. As this point moves, its coordinates x and y change; but even though they change, they are bound together by the fixed relationship expressed by equation (4).

If the known point on the line happens to be the point where the line crosses the y-axis, and if this point is denoted by $(0, b)$, then equation (4) becomes $y - b = mx$ or

$$y = mx + b. \tag{5}$$

The number b is called the *y-intercept* of the line, and (5) is called the *slope-intercept equation* of a line. This form is especially convenient because it tells at a glance the location and direction of a line. For example, if the equation

$$6x - 2y - 4 = 0 \tag{6}$$

is solved for y, we see that

$$y = 3x - 2. \tag{7}$$

Comparing (7) with (5) shows at once that $m = 3$ and $b = -2$, and so (6) and (7) both represent the line that passes through $(0, -2)$ with slope 3. This information makes it very easy to sketch the line. It may seem that (6) and (7) are different equations, so that (6) should be referred to as "an" equation of the line and (7) as "another" equation of the line, but we prefer to regard them as merely different forms of a single equation. Many other forms are possible, for instance,

$$y + 2 = 3x, \qquad x = \tfrac{1}{3}y + \tfrac{2}{3}, \qquad 3x - y = 2.$$

It is reasonable to cut through appearances and speak of any one of these as "the" equation of the line.

More generally, every equation of the form

$$Ax + By + C = 0, \tag{8}$$

where the constants A and B are not both zero, represents a straight line. For if $B = 0$, then $A \neq 0$, and the equation can be written as

$$x = -\frac{C}{A},$$

which is clearly the equation of a vertical line. On the other hand, if $B \neq 0$, then

$$y = -\frac{A}{B}x - \frac{C}{B},$$

and this equation has the form (5) with $m = -A/B$ and $b = -C/B$. Equation (8) is rather inconvenient for most purposes because its constants are not directly related to the geometry of the line. Its main merit is that it is capable of representing all lines, without any need for distinguishing between the vertical and nonvertical cases. For this reason it is called the *general linear equation.*

PARALLEL AND PERPENDICULAR LINES

Two distinct nonvertical straight lines with slopes m_1 and m_2 are evidently parallel if and only if their slopes are equal:

$$m_1 = m_2.$$

The criterion for perpendicularity is the relation

$$m_1 m_2 = -1. \tag{9}$$

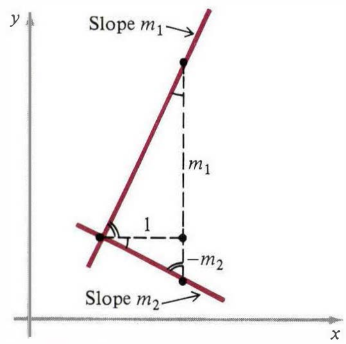


Figure 1.13

This is not obvious, but can be established quite easily by using similar triangles, as follows (Fig. 1.13). Suppose that the lines are perpendicular, as shown in Fig. 1.13. Draw a segment of length 1 to the right from their point of intersection, and from its right endpoint draw vertical segments up and down to the two lines. From the meaning of the slopes, the two right triangles formed in this way have sides of the indicated lengths. Since the lines are perpendicular, the indicated angles are equal and the triangles are similar. This similarity implies that the following ratios of corresponding sides are equal:

$$\frac{m_1}{1} = \frac{1}{-m_2}.$$

This is equivalent to (9), so (9) is true when the lines are perpendicular. The reasoning given here is easily reversed, telling us that if (9) is true, then the lines are perpendicular. Since equation (9) is equivalent to

$$m_1 = -\frac{1}{m_2} \quad \text{and} \quad m_2 = -\frac{1}{m_1},$$

we see that two nonvertical lines are perpendicular if and only if their slopes are negative reciprocals of one another.

The ideas of this section enlarge our supply of tools for proving geometric theorems by algebraic methods.

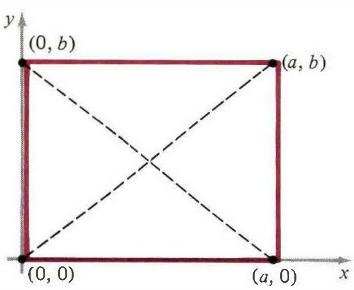


Figure 1.14

Example If the diagonals of a rectangle are perpendicular, then the rectangle is a square. To establish this, we place the rectangle in the convenient position shown in Fig. 1.14. The slopes of the diagonals are clearly b/a and $-b/a$. If these diagonals are perpendicular, then

$$\frac{b}{a} = \frac{a}{b}, \qquad a^2 = b^2, \qquad a^2 - b^2 = 0, \qquad \text{and} \qquad (a+b)(a-b) = 0.$$

The last equation implies that $a = b$, so the rectangle is a square.

PROBLEMS

1 Plot each pair of points, draw the line they determine, and compute the slope of this line:
(a) $(-3, 1), (4, -1)$; (b) $(2, 7), (-1, -1)$;
(c) $(-4, 0), (2, 1)$; (d) $(-4, 3), (5, -6)$;
(e) $(-5, 2), (7, 2)$; (f) $(0, -4), (1, 6)$.

2 Plot each of the following sets of three points, and use slopes to determine in each case whether all three points lie on a single straight line:
(a) $(5, -1), (2, 2), (-4, 6)$;
(b) $(1, 1), (-5, -2), (5, 3)$;
(c) $(4, 3), (10, 14), (-2, -8)$;
(d) $(-1, 3), (6, -1), (-9, 7)$.

3 Plot the points $(-1, -1), (9, 1), (8, 6)$, and $(-2, 4)$, and show that they are the vertices of a rectangle.

4 Plot the points $(-3, 8), (3, 5), (0, -1)$, and $(-6, 2)$, and show that they are the vertices of a square.

5 Plot each of the following sets of three points, and use slopes to determine in each case whether the points form a right triangle:
(a) $(2, -3), (5, 2), (0, 5)$;
(b) $(10, -5), (5, 4), (-7, -2)$;
(c) $(8, 2), (-1, -1), (2, -7)$;
(d) $(-2, 6), (3, -4), (8, 11)$.

6 Write the equation of each line in Problem 1 using the point-slope form; then rewrite each of these equations in the form $y = mx + b$ and find the y-intercept.

7 Find the equation of the line:
(a) through $(2, -3)$ with slope -4;
(b) through $(-4, 2)$ and $(3, -1)$;
(c) with slope $\frac{2}{3}$ and y-intercept -4;
(d) through $(2, -4)$ and parallel to the x-axis;
(e) through $(1, 6)$ and parallel to the y-axis;

(f) through $(4, -2)$ and parallel to $x + 3y = 7$;
(g) through $(5, 3)$ and perpendicular to $y + 7 = 2x$;
(h) through $(-4, 3)$ and parallel to the line determined by $(-2, -2)$ and $(1, 0)$;
(i) that is the perpendicular bisector of the segment joining $(1, -1)$ and $(5, 7)$;
(j) through $(-2, 3)$ with inclination 135°.

8 If a line crosses the x-axis at the point $(a, 0)$, the number a is called the *x-intercept* of the line. If a line has x-intercept $a \neq 0$ and y-intercept $b \neq 0$, show that its equation can be written as

$$\frac{x}{a} + \frac{y}{b} = 1.$$

This is called the *intercept form* of the equation of a line. Notice that it is easy to put $y = 0$ and see that the line crosses the x-axis at $x = a$, and to put $x = 0$ and see that the line crosses the y-axis at $y = b$.

9 Put each equation in intercept form and sketch the corresponding line:
(a) $5x + 3y + 15 = 0$; (b) $3x = 8y - 24$;
(c) $y = 6 - 6x$; (d) $2x - 3y = 9$.

10 The set of all points (x, y) that are equally distant from the points $P_1 = (-1, -3)$ and $P_2 = (5, -1)$ is the perpendicular bisector of the segment joining these points. Find its equation
(a) by equating the distances from (x, y) to P_1 and P_2, and simplifying the resulting equation;
(b) by finding the midpoint of the given segment and using a suitable slope.

11 Sketch the lines $3x + 4y = 7$ and $x - 2y = 6$, and find their point of intersection. Hint: Their point of intersection is that point (x, y) whose coordinates satisfy both equations simultaneously.

12 Find the point of intersection of each of the following pairs of lines:
(a) $2x + 2y = 2$, $y = x - 1$;
(b) $10x + 7y = 24$, $15x - 4y = 7$;
(c) $3x - 5y = 7$, $15y + 25 = 9x$.

13 Let F and C denote temperature in degrees Fahrenheit and degrees Celsius. Find the equation connecting F and C, given that it is linear and that $F = 32$ when $C = 0$, $F = 212$ when $C = 100$.

14 Find the values of the constant k for which the line $(k - 3)x - (4 - k^2)y + k^2 - 7k + 6 = 0$ is
(a) parallel to the x-axis;
(b) parallel to the y-axis;
(c) through the origin.

15 Show that the segments joining the midpoints of adjacent sides of any quadrilateral form a parallelogram.

16 Show that the lines from any vertex of a parallelogram to the midpoints of the opposite sides trisect a diagonal.

17 Let $(0, 0)$, $(a, 0)$, and (b, c) be the vertices of an arbitrary triangle placed so that one side lies along the positive x-axis with its left endpoint at the origin. If the square of this side equals the sum of the squares of the other two sides, use slopes to show that the triangle is a right triangle. Thus, the converse of the Pythagorean theorem is also true.

1.4 CIRCLES AND PARABOLAS. DESCARTES AND FERMAT

The coordinate plane or xy-plane is often called the *Cartesian plane,* and x and y are frequently referred to as the *Cartesian coordinates* of the point $P = (x, y)$. The word "Cartesian" comes from Cartesius, the Latinized name of the French philosopher-mathematician Descartes, who is considered one of the two principal founders of analytic geometry.* The basic idea of this subject is quite simple: Exploit the correspondence between points and their coordinates to study geometric problems—especially the properties of curves—with the tools of algebra. The reader will see this idea in action throughout this book. Generally speaking, geometry is visual and intuitive, while algebra is rich in computational machinery, and each can serve the other in many fruitful ways.

Most people who have had a course in algebra have learned that an equation

$$F(x, y) = 0 \tag{1}$$

usually determines a curve (its *graph*) which consists of all points $P = (x, y)$ whose coordinates satisfy the given equation. Conversely, a curve defined by some geometric condition can usually be described algebraically by an equation

*The other (also French) was Fermat, a less well known figure than Descartes but a much greater mathematician. The names of these two men are pronounced "Fair-MA" and "Day-CART."

of the form (1). It is intuitively clear that straight lines are the simplest curves, and our work in Section 1.3 demonstrated that straight lines in the coordinate plane correspond to linear equations in x and y. We now develop algebraic descriptions of several other curves that will be useful as illustrative examples in the next few chapters.

CIRCLES

The distance formula of Section 1.2 is often useful in finding the equation of a curve whose geometric definition depends on one or more distances.

One of the simplest curves of this kind is a *circle,* which can be defined as the set of all points at a given distance (the radius) from a given point (the center). If the center is the point (h, k) and the radius is the positive number r (Fig. 1.15), and if (x, y) is an arbitrary point on the circle, then the defining condition says that

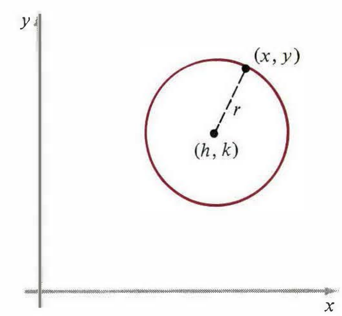


Figure 1.15 Circle.

$$\sqrt{(x-h)^2+(y-k)^2}=r.$$

It is convenient to eliminate the radical sign by squaring, which yields

$$(x-h)^2+(y-k)^2=r^2. \tag{2}$$

This is therefore the equation of the circle with center (h, k) and radius r. In particular, if the center happens to be the origin, so that $h = k = 0$, then

$$x^2+y^2=r^2$$

is the equation of the circle.

Example 1 If the radius of a circle is $\sqrt{10}$ and its center is $(-3, 4)$, then its equation is

$$(x+3)^2+(y-4)^2=10.$$

Notice that the coordinates of the center are the numbers *subtracted* from x and y in the parentheses.

Example 2 An angle inscribed in a semicircle is necessarily a right angle.* To prove this algebraically, let the semicircle have radius r and center at the origin (Fig. 1.16), so that its equation is $x^2 + y^2 = r^2$ with $y \geq 0$. The inscribed angle is a right angle if and only if the product of the slopes of its sides is -1, that is,

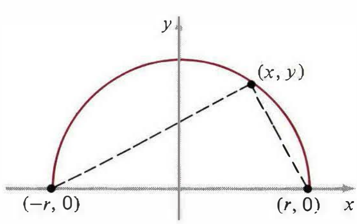


Figure 1.16

$$\frac{y}{x-r}\cdot\frac{y}{x+r}=-1. \tag{3}$$

This is easily seen to be equivalent to $x^2 + y^2 = r^2$, which is certainly true for any point (x, y) on the semicircle, so (3) is true and the angle is a right angle.

It is clear that any equation of the form (2) is easy to interpret geometrically. For instance,

*According to tradition, this is one of the theorems discovered and proved by Thales.

$$(x-5)^2+(y+2)^2=16 \tag{4}$$

is immediately recognizable as the equation of the circle with center $(5, -2)$ and radius 4, and this information enables us to sketch the graph without difficulty. However, if the equation has been roughly treated by someone who likes to "simplify" things algebraically, then it might have the form

$$x^2+y^2-10x+4y+13=0. \tag{5}$$

This is an equivalent but scrambled version of (4), and its constants tell us nothing directly about the nature of the graph. To find out what the graph is, we must "unscramble" by *completing the square.** To do this, we begin by rewriting equation (5) as

$$(x^2-10x+\quad)+(y^2+4y+\quad)=-13,$$

with the constant term moved to the right and blank spaces provided for the insertion of suitable constants. When the square of half the coefficient of x is added in the first blank space and the square of half the coefficient of y in the second, and the same constants are added to the right side to maintain the balance of the equation, we get

$$(x^2-10x+25)+(y^2+4y+4)=-13+25+4$$

or

$$(x-5)^2+(y+2)^2=16. \tag{6}$$

Exactly the same process can be applied to the general equation of the form (5), namely,

$$x^2+y^2+Ax+By+C=0, \tag{7}$$

but there is little to be gained by writing out the details in this general case. However, it is important to notice that if the constant term 13 in (5) is replaced by 29, then (6) becomes

$$(x-5)^2+(y+2)^2=0,$$

whose graph is the single point $(5, -2)$. Similarly, if this constant term is replaced by any number greater than 29, then the right-hand side of (6) becomes negative and the graph is empty, in the sense that there are no points (x, y) in the plane whose coordinates satisfy the equation. We therefore see that the graph of (7) is sometimes a circle, sometimes a single point, and sometimes empty—depending entirely on the constants A, B, and C.

PARABOLAS

The definition we use for a *parabola* is the following (Fig. 1.17*a*): It is the curve consisting of all points that are equally distant from a fixed point F (called the *focus*) and a fixed line d (called the *directrix*). The distance from a point to a line is always understood to mean the perpendicular distance.

*The form of the equation $(x+a)^2=x^2+2ax+a^2$ is the key to the process of completing the square. Notice that the right side is a perfect square—the square of $x+a$—precisely because its constant term is the square of half the coefficient of x.

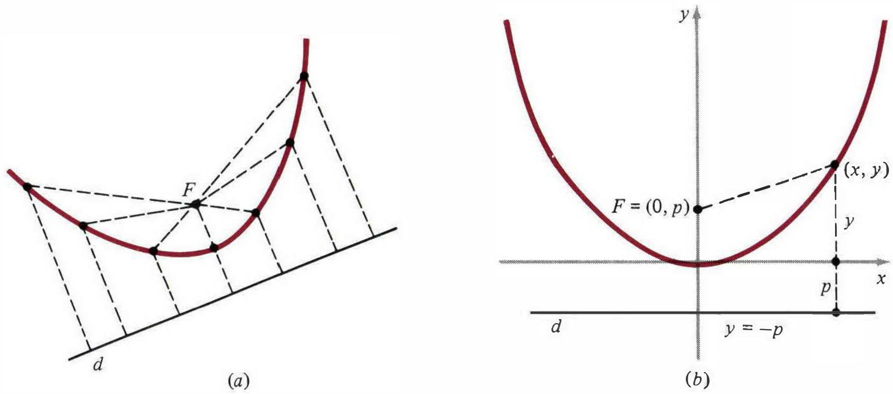


Figure 1.17 Parabola.

To find a simple equation for a parabola, we place it in the coordinate system as shown in Fig. 1.17*b*, with the focus and directrix equally far above and below the x-axis. The line through the focus perpendicular to the directrix is called the *axis* of the parabola; this is the axis of symmetry of the curve, and is the y-axis in the figure. The point on the axis halfway between the focus and the directrix is called the *vertex* of the parabola; in the figure this point is the origin. If (x, y) is an arbitrary point on the parabola, the condition expressed in the definition is stated algebraically by the equation

$$\sqrt{x^2 + (y - p)^2} = y + p. \tag{8}$$

On squaring both sides and simplifying, we obtain

$$x^2 + y^2 - 2py + p^2 = y^2 + 2py + p^2$$

or

$$x^2 = 4py. \tag{9}$$

These steps are reversible, so (8) and (9) are equivalent and (9) is the equation of the parabola whose focus and directrix are located as shown in Fig. 1.17*b*. Notice particularly that the positive constant p in (9) is the distance from the focus to the vertex, and also from the vertex to the directrix.

If we change the position of the parabola relative to the coordinate axes, we naturally change its equation. Three other positions are shown in Fig. 1.18, each with its corresponding equation and with $p > 0$ in each case. Students should

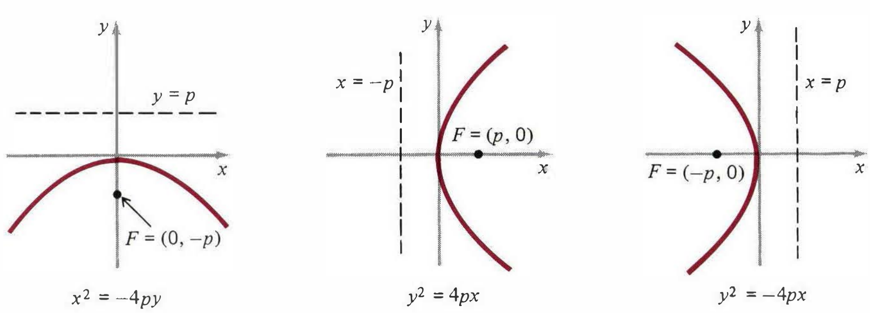


Figure 1.18 Various parabolas.

verify the correctness of all three equations. We also point out that each of these four equations can be put in the form

$$y = ax^2 \tag{10}$$

or

$$x = ay^2.$$

These forms conceal the constant p, with its geometric significance, but as compensation they are more useful in visualizing the overall appearance of the graph. For instance, in (10) the variable x is squared but y is not. This tells us that as a point (x, y) moves out along the curve, y increases much faster than x, and so the curve opens in the y-direction—upward or downward, according as a is positive or negative. It also tells us that the graph is symmetric with respect to the y-axis, because x is squared, and therefore we get the same number y for any number x and its negative.

Example 3 What is the graph of the equation $12x + y^2 = 0$? If this is put in the form $y^2 = -12x$ and compared with the equation on the right in Fig. 1.18, it is clear that the graph is a parabola with vertex at the origin and opening to the left. Since $4p = 12$ and therefore $p = 3$, the point $(-3, 0)$ is the focus and $x = 3$ is the directrix.

Example 4 The graph of $y = 2x^2$ is evidently a parabola with vertex at the origin and opening upward. To find its focus and directrix, the equation must be rewritten as $x^2 = \frac{1}{2}y$ and compared with equation (9). This yields $4p = \frac{1}{2}$, so $p = \frac{1}{8}$. The focus is therefore $(0, \frac{1}{8})$, and the directrix is $y = -\frac{1}{8}$.

We illustrate one last point about parabolas by examining the equation

$$y = x^2 - 4x + 5. \tag{11}$$

If this is written as

$$y - 5 = x^2 - 4x,$$

and if we complete the square on the terms involving x, then the result is

$$y - 1 = (x - 2)^2. \tag{12}$$

If we now introduce the new variables

$$X = x - 2, \quad Y = y - 1, \tag{13}$$

then equation (12) becomes

$$Y = X^2.$$

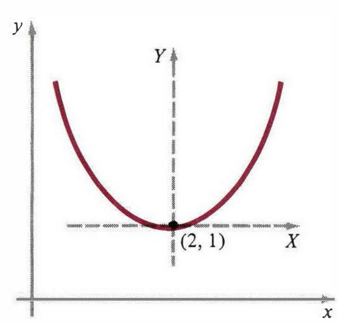


Figure 1.19

The graph of this equation is clearly a parabola opening upward with vertex at the origin of the XY coordinate system. By equations (13), the origin in the XY system is the point (2, 1) in the xy system, as shown in Fig. 1.19. What has happened here is that the coordinate system has been shifted or translated to a new position in the plane, and the axes renamed, and equations (13) express the relation between the coordinates of an arbitrary point with respect to each of the two coordinate systems. In exactly the same way, any equation of the form

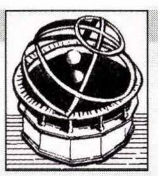
$$y = ax^2 + bx + c, \qquad a \neq 0,$$

represents a parabola with vertical axis which is congruent to $y = ax^2$ and opens up or down according as the number a is positive or negative. Similarly, the equation

$$x = ay^2 + by + c, \qquad a \neq 0,$$

represents a parabola with horizontal axis which opens to the right or left according as $a > 0$ or $a < 0$.

In our work up to this stage we have used the static concept of a curve as a certain set of points or geometric figure. It is often possible to adopt the dynamic point of view, in which a curve is thought of as the path of a moving point. For instance, a circle is the path of a point that moves in such a way that it maintains a fixed distance from a given point. When this mode of thought is used—with its advantage of greater intuitive vividness—a curve is often called a *locus*. Thus, a parabola is the locus of a point that moves in such a way that it maintains equal distances from a given point and a given line.



NOTE ON DESCARTES AND FERMAT

Are there some people among us who feel that what passes for "knowledge" in our time is an uncritical mishmash of sense and nonsense, fact and guesswork, gossip and hearsay and clumsy propaganda—mostly acquired from wishful thinking, lazy reasoning, inadequate senses, credulous parents, overworked teachers, and self-serving institutions? This was also the opinion of the 23-year-old Frenchman René Descartes (1596–1650) on Nov. 10, 1619. For this was the day above all others when the modern world began, our world of victorious rationality and triumphant science.

On this day—a famous day in the history of thought—in a state of exhaustion and feverish excitement, Descartes found the method he sought for extending the certainty of mathematics to all other fields of knowledge:

> The long chains of simple reasoning which geometers use to arrive at their most difficult conclusions made me believe that all things which are the objects of human knowledge are similarly interdependent; and that if we will only abstain from assuming something to be true which is not, and always follow the necessary order in deducing one thing from another, there is nothing so remote that we cannot reach it, nor so hidden that we cannot discover it.

This is a quotation from Part 2 of his *Discourse on Method*, a short and highly readable book published in 1637 which is commonly considered to mark the birth of modern philosophy. In this work he rejected the sterile scholasticism prevailing at the time and set himself the task of rebuilding knowledge from the ground up, on a foundation of reason and science instead of authority and faith. He provided the fresh points of view needed for the vigorous development of the Scientific Revolution, whose influence has been the dominant fact of modern history. Further, in an appendix to the *Discourse* on his ideas about geometry, he foreshadowed the new forms of mathematics—analytic geometry and calculus—without which this Revolution would have died in infancy. It was no exaggeration for the great American jurist Oliver Wendell Holmes to write: "Descartes commanded the future from his study more than Napoleon from his throne."

Descartes was a brilliant man—and enormously influential with a corresponding ego—but he was not quite as brilliant as he thought. His contemporary Pierre Fermat (1601–1665) was a man of genius and perhaps the greatest mathematician of the seventeenth century; and when the two men collided on issues of science or mathematics, it was always Descartes's nose that was bloodied.

By profession Fermat was a lawyer and a member of the provincial supreme court in Toulouse, a city in southwestern France. However, his hobby and private passion was mathematics, and his casual creativity was one of the wonders of the age to the few who knew about it. His letters suggest that he was a shy and retiring man, courteous and affable but slightly remote. His outward life was as quiet and orderly as one would expect of a provincial judge with a sense of responsibility toward his work. Fortunately this work was not too demanding, and left ample leisure for the extraordinary inner life that flourished by lamplight in the silence of his study at night.

He invented analytic geometry in 1629 and described his ideas in a short work that circulated in manuscript from early 1637 on, but was not published in his lifetime. The credit for this achievement has usually been given to Descartes on the basis of his *Geometry*, which was published late in 1637 as an appendix to his *Discourse on Method*. However, nothing that we would recognize as analytic geometry can be found in Descartes's essay, except perhaps the idea of using algebra as a language for discussing geometric problems. Fermat had the same idea but did something important with it: He introduced perpendicular axes and found the general equations of straight lines and circles and the simplest equations of parabolas, ellipses, and hyperbolas; and he further showed in a fairly complete and systematic way that every first- or second-degree equation can be reduced to one of these types. Descartes certainly knew some analytic geometry by the late 1630s; but since he had possession of the original manuscript of Fermat's short essay (of which Fermat himself did not bother to keep a copy) several months before the publication of his own *Geometry*, it is likely that much of what he knew he learned from Fermat.

The invention of calculus is usually credited to Newton and Leibniz, whose ideas and methods were not published until about 20 years after Fermat's death. However, if differential calculus is considered to be the mathematics of finding maxima and minima of functions and drawing tangents to curves, then Fermat was the true creator of this subject as early as 1629, more than a decade before either Newton or Leibniz was born. With his usual honesty in such matters, Newton stated—in a letter that was discovered only in 1934—that his own early ideas about calculus came directly from "Fermat's way of drawing tangents."

Fermat was also the founder of mathematical optics and the joint founder (in correspondence with Blaise Pascal) of the theory of probability. But to him all these activities were of minor importance compared with the consuming passion of his life, the theory of numbers. It was here that his genius shone most brilliantly, for his insight into the properties of the familiar but mysterious positive integers has perhaps never been equaled. He was the sole and undisputed founder of the modern era in this important branch of pure mathematics, without any rivals and with few followers until the next century.

To illustrate the nature of his achievement in number theory, we mention his profound and beautiful *four squares theorem:* Every positive integer is either a square or the sum of two, three, or four squares. Like many of his discoveries, this was jotted down in the margin of one of his books, and his proof went unrecorded and was lost forever when he died. A proof was found at last in 1772—more than a century after Fermat's death—as the culmination of 40 years of effort by one of the greatest mathematicians of the eighteenth century. As we see, mathematicians are people who are not only irresistibly attracted by truths of this kind but also cannot rest until they know *why* they are true.

Without visibly trying, and as naturally as a hawk sustains itself on the wind, Fermat attained immortal fame among mathematicians. There are many reasons for this immortality, one of the most interesting being the legacy of what is now known as *Fermat's last theorem:* If $n > 2$, then the equation $x^n + y^n = z^n$ has no positive integer solutions x, y, z. Again, he wrote this statement in the margin of a book he was studying, near a passage dealing with the fact that $x^2 + y^2 = z^2$ has many solutions—3, 4, 5 and 5, 12, 13, among others. He then added the tantalizing remark, "I have found a truly wonderful proof which this margin is too narrow to contain." Unfortunately no proof has ever been discovered by anyone else, and Fermat's last theorem remains to this day one of the most baffling unsolved problems of mathematics.*

*Late report from the cutting edge: It appears that Fermat's last theorem may have been proved by Andrew Wiles of Princeton University. This was announced on June 23, 1993, in the last of three lectures Wiles gave at Cambridge University, in England. The proof is about 200 pages long and follows a tortuous, roundabout path through many tangled jungles of sophisticated pure mathematics. The careful checking of every line of this proof may take years to carry out. It is estimated that perhaps a tenth of 1 percent of mathematicians could understand all details of the proof—and this definitely does not include the present writer. If Wiles's proof checks out, the challenge will still remain of discovering a one- or two-page (or even a three- or four-page) proof of Fermat's one-sentence theorem. For further details, see *Newsweek*, July 5, 1993, or *Scientific American*, September 1993.

PROBLEMS

1 Find the equation of the circle with the given point as center and the given number as radius:
(a) (4, 6), 3; (b) $(-3, 7)$, $\sqrt{5}$;
(c) $(-5, -9)$, 7; (d) $(1, -6)$, $\sqrt{2}$;
(e) $(a, 0)$, a; (f) $(0, a)$, a.

2 In each case find the equation of the circle determined by the given conditions:
(a) Center (2, 3) and passes through $(-1, -2)$.
(b) The ends of a diameter are $(-3, 2)$ and $(5, -8)$.
(c) Center (4, 5) and tangent to the x-axis.
(d) Center $(-4, 1)$ and tangent to the line $x = 3$.
(e) Center $(-2, 3)$ and tangent to the line $4y - 3x + 2 = 0$.
(f) Center on the line $x + y = 1$, passes through $(-2, 1)$ and $(-4, 3)$.
(g) Center on the line $y = 3x$ and tangent to the line $x = 2y$ at the point (2, 1).

3 In each of the following, determine the nature of the graph of the given equation by completing the square:
(a) $x^2 + y^2 - 4x - 4y = 0$.
(b) $x^2 + y^2 - 18x - 14y + 130 = 0$.
(c) $x^2 + y^2 + 8x + 10y + 40 = 0$.
(d) $4x^2 + 4y^2 + 12x - 32y + 37 = 0$.
(e) $x^2 + y^2 - 8x + 12y + 53 = 0$.
(f) $x^2 + y^2 - \sqrt{2}x + \sqrt{2}y + 1 = 0$.
(g) $x^2 + y^2 - 16x + 6y - 48 = 0$.

4 Find the equation of the locus of a point $P = (x, y)$ that moves in accordance with each of the following conditions, and sketch the graphs:
(a) The sum of the squares of the distances from P to the points $(a, 0)$ and $(-a, 0)$ is $4b^2$, where $b \geq a/\sqrt{2} > 0$.
(b) The distance of P from the point (8, 0) is twice its distance from the point (0, 4).

5 The *quadratic formula* for the roots of the quadratic equation $ax^2 + bx + c = 0$ is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Derive this formula from the equation by dividing through by a, moving the constant term to the right side, and completing the square. Under what circumstances does the equation have distinct real roots, equal real roots, and no real roots?

6 At what points does the circle $x^2 + y^2 - 8x - 6y - 11 = 0$ intersect
(a) the x-axis? (b) the y-axis?
(c) the line $x + y = 1$?
Sketch the figure, and use this picture to judge whether your answers are reasonable or not.

7 Find the equations of all lines that are tangent to the circle $x^2 + y^2 = 2y$ and pass through the point (0, 4). Hint: The line $y = mx + 4$ is tangent to the circle if it intersects the circle at only one point.

8 Find the focus and directrix of each of the following parabolas, and sketch the curves:
(a) $y^2 = 12x$; (b) $y = 4x^2$;
(c) $2x^2 + 5y = 0$; (d) $4x + 9y^2 = 0$;
(e) $x = -2y^2$; (f) $12y = -x^2$;
(g) $16y^2 = x$; (h) $24x^2 = y$;
(i) $y^2 + 8y - 16x = 16$; (j) $x^2 + 2x + 29 = 7y$.

9 Sketch the parabola and find its equation if it has
(a) vertex (0, 0) and focus $(-3, 0)$;
(b) vertex (0, 0) and directrix $y = -1$;
(c) vertex (0, 0) and directrix $x = -2$;
(d) vertex (0, 0) and focus $(0, -\frac{1}{3})$;
(e) directrix $x = 2$ and focus $(-4, 0)$;
(f) focus (3, 3) and directrix $y = -1$.

10 Find the focus and directrix of each of the following parabolas, and sketch the curves:
(a) $y = x^2 + 1$; (b) $y = (x - 1)^2$;
(c) $y = (x - 1)^2 + 1$; (d) $y = x^2 - x$.

11 Water squirting out of a horizontal nozzle held 4 ft above the ground describes a parabolic curve with the vertex at the nozzle. If the stream of water drops 1 ft in the first 10 ft of horizontal motion, at what horizontal distance from the nozzle will it strike the ground?

12 Show that there is exactly one line with given slope m which is tangent to the parabola $x^2 = 4py$, and find its equation.

13 Prove that the two tangents to a parabola from any point on the directrix are perpendicular.

1.5 THE CONCEPT OF A FUNCTION

The most important concept in all of mathematics is that of a function. No matter what branch of the subject we consider—algebra, geometry, number theory, probability, or any other—it almost always turns out that functions are the primary objects of investigation. This is particularly true of calculus, in which most of our work will be concerned with constructing machinery for the study of functions and applying this machinery to problems in science and geometry.

What is a function? Briefly—and we expand on this below—if x and y are two variables that are related in such a way that whenever a permissible numerical value is assigned to x, there is determined one and only one corresponding numerical value for y, then y is called *a function of* x.

Example 1 (a) If a rock is dropped from the edge of a cliff, and it falls s feet in t seconds, then s is a function of t. It is known from experiment that (approximately) $s = 16t^2$.

(b) The area A of a circle is a function of its radius r. It is known from geometry that $A = \pi r^2$.

(c) If the manager of a bookstore buys n books from a publisher at \$12 per copy and the shipping charges are \$35, then his cost C for these books is a function of n given by the formula $C = 12n + 35$.

We continue building our understanding of the concept of a function by considering an example directly related to our work in the preceding section.

Example 2 We examine the equation

$$y = x^2$$

and its corresponding graph, which we know is a parabola that opens upward and has its vertex at the origin (Fig. 1.20). In Section 1.4 we thought of this equation as a relation between the variable coordinates of a point (x, y) moving along the curve. We now shift our point of view, and instead think of it as a formula that provides a mechanism for calculating the numerical value of y when the numerical value of x is given. Thus, $y = 1$ when $x = 1$, $y = 4$ when $x = 2$, $y = \frac{1}{4}$ when $x = \frac{1}{2}$, $y = 1$ when $x = -1$, and so on. The value of y is therefore said to *depend on*, or to be *a function of*, the value of x. This dependence can be expressed in functional notation by writing

$$y = f(x) \qquad \text{where} \qquad f(x) = x^2.$$

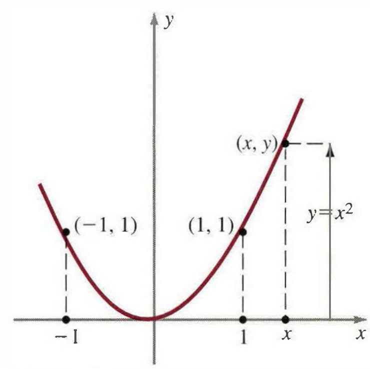


Figure 1.20

The symbol $f(x)$ is read "f of x," and the letter f represents the rule or process—squaring, in this particular case—which is applied to any number x to yield the corresponding number y. The numerical examples just given can therefore be written as $f(1) = 1, f(2) = 4, f(\frac{1}{2}) = \frac{1}{4}$, and $f(-1) = 1$. The meaning of this notation can perhaps be further clarified by observing that

$$f(x + 1) = (x + 1)^2 = x^2 + 2x + 1 \qquad \text{and} \qquad f(x^3) = (x^3)^2 = x^6;$$

that is, the rule f simply produces the square of whatever quantity follows it in parentheses.

This example suggests the general concept of a function as we shall use it in most of our work. We formulate this concept as follows.

Let D be a given set of real numbers. A *function* f defined on D is a formula, or rule, or law of correspondence that assigns a single real number y to each number x in D. The set D of allowed values of x is called the *domain* (or *domain of definition*) of the function, and the set of corresponding values of y is called its *range*. The number y that is assigned to x by the function f is usually written $f(x)$—so that $y = f(x)$—and is called the *value* of f at x. It is customary to call x the *independent variable* because it is free to assume any value in the domain, and to call y the *dependent variable* because its numerical value depends on the choice of x.

There is nothing illegal or immoral about using other letters than x and y to denote the variables. In Example 1, for instance, the independent variables are t, r, and n, and the dependent variables are s, A, and C. Also, as we see in the next example, there is nothing sacred about the letter f, and other letters can be used to designate functions.

Example 3 (a) If a function $f(x)$ is defined by the formula $f(x) = x^3 - 3x^2 + 5$, then $f(2) = 2^3 - 3 \cdot 2^2 + 5 = 1$, $f(0) = 5$, and $f(-2) = (-2)^3 - 3(-2)^2 + 5 = -15$.

(b) If a function $g(x)$ is defined by the formula $g(x) = \sqrt{x}$, then $g(1) = \sqrt{1} = 1$, $g(4) = \sqrt{4} = 2$, and a calculator tells us that $g(10) = \sqrt{10} = 3.16227766017$, approximately. In this case the only allowed values of x are those for which $x \geq 0$, because square roots of negative numbers are not real numbers.

(c) If a function $h(x)$ is defined by the formula $h(x) = 1/(4 - x)$, then $h(1) = 1/(4 - 1) = \frac{1}{3}$, $h(2) = 1/(4 - 2) = \frac{1}{2}$, and $h(4) = 1/(4 - 4) = \frac{1}{0}$ does not exist, because division by zero is not permitted in algebra. Thus, $x = 4$ is the only value of x that is not allowed.

We point out that a function is not fully known until we know precisely which real numbers are permissible values for the independent variable x. The domain is therefore an indispensable part of the concept of a function. In practice, however, most of the specific functions we deal with are defined only by formulas like the ones in Example 3, and nothing is said about the domain. Unless we state otherwise, the domain of such a function is understood to be the set of all real numbers x for which the formula makes sense. In part (a) of Example 3, this means all real numbers; in (b), all real numbers $x \geq 0$; and in (c), all real numbers except $x = 4$.

The reader is undoubtedly acquainted with the idea of the *graph* of a function f: If we imagine the domain D spread out on the x-axis in the coordinate plane (Fig. 1.21*a*), then to each number x in D there corresponds a number $y = f(x)$, and the set of all the resulting points (x, y) in the plane is the graph. Graphs are pictures of functions that enable us to see these functions in their entirety, and we will examine many in the next section.

Many people find it helpful to visualize a function by means of a *machine diagram,* as shown in Fig. 1.21*b*. Here a number x in the domain is fed into the machine, where it is acted upon by the specific instructions built into the function f, and this action produces the resulting number $f(x)$. The domain is the set of all permissible inputs x, and the range is the set of all outputs $f(x)$.

Another way to picture a function is by an *arrow diagram,* in which the domain is thought of as a certain set of points on the page and the range as another set of points (Fig. 1.21*c*). The arrow shows that x has $f(x)$ corresponding to it, and the function f is the complete collection of all these correspondences thought of as a mapping of the first set onto the second.

We mention machine diagrams and arrow diagrams *only* to help students who may be having difficulty grasping the concept of a function. The basic tool for visualizing functions throughout our work will always be graphs. Also, we will

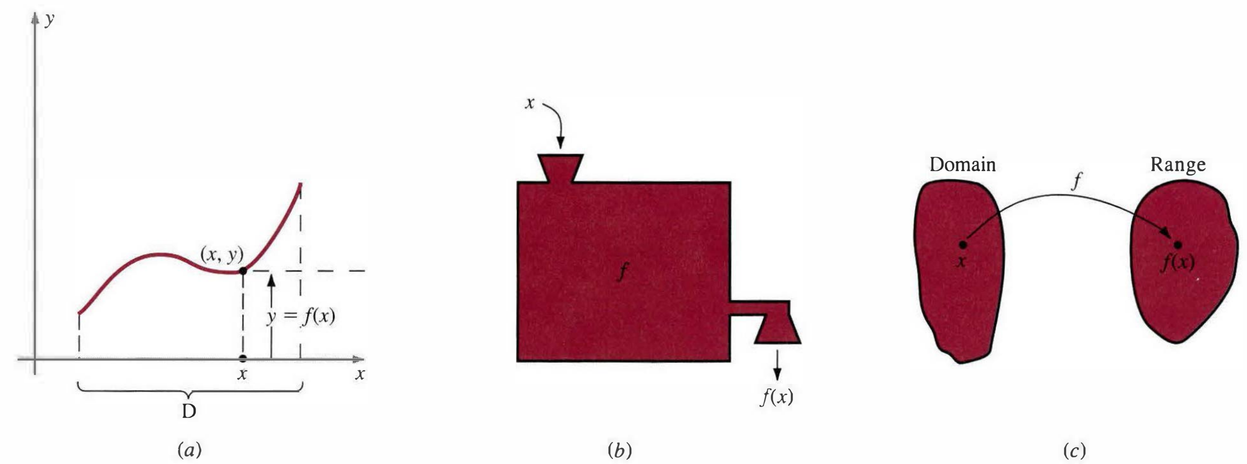


Figure 1.21

see in Section 2.1 that graphs are essential for formulating the main purposes of calculus.

Originally, the only functions mathematicians considered were those defined by formulas. This led to the useful intuitive idea that a function f "does something" to each number x in its domain to "produce" the corresponding number $y = f(x)$. Thus, if

$$y = f(x) = (x^3 + 4)^2,$$

then y is the result of applying certain specific operations to x: Cube it, add 4, and square the sum. On the other hand, the following is also a perfectly legitimate function which is defined by a verbal prescription instead of a formula:

$$y = f(x) = \begin{cases} 1 & \text{if } x \text{ is a rational number,} \\ 0 & \text{if } x \text{ is an irrational number.} \end{cases}$$

All that is really required of a function is that y be uniquely determined—in any manner whatever—when x is specified; beyond this, nothing is said about the nature of the rule f. In discussions that focus on ideas instead of specific functions, such broad generality is often an advantage. We will understand this better in Chapter 6, where one of our problems is to discover what conditions must be imposed on an arbitrary function to guarantee that its integral exists.

An additional remark on usage is perhaps in order. Strictly speaking, the word "function" refers to the rule of correspondence f that assigns a unique number $y = f(x)$ to each number x in the domain. Purists are fond of emphasizing the distinction between the function f and its value $f(x)$ at x. However, once this distinction is clearly understood, most people who work with mathematics prefer to use the word loosely and speak of "the function $y = f(x)$," or even "the function $f(x)$."

The functions we work with in calculus are often composite (or compound) functions built up out of simpler ones. As an illustration of this idea, consider the two functions

$$f(x) = x^2 + 3x \qquad \text{and} \qquad g(x) = x^2 - 1.$$

The single function that results from first applying g to x and then applying f to $g(x)$ is

$$f(g(x)) = f(x^2 - 1) = (x^2 - 1)^2 + 3(x^2 - 1)$$
$$= x^4 + x^2 - 2.$$

Notice that $f(x^2 - 1)$ is obtained by replacing x by the entire quantity $x^2 - 1$ in the formula $f(x) = x^2 + 3x$. The symbol $f(g(x))$ is read "f of g of x" and is called a *function of a function.* If we apply the functions in the other order (first f, then g), we have

$$g(f(x)) = g(x^2 + 3x) = (x^2 + 3x)^2 - 1$$
$$= x^4 + 6x^3 + 9x^2 - 1,$$

so $f(g(x))$ and $g(f(x))$ are different. In special cases it can happen that $f(g(x))$ and $g(f(x))$ are the same function of x; for example, if $f(x) = 2x - 3$ and $g(x) = -x + 6$:

$$f(g(x)) = f(-x + 6) = 2(-x + 6) - 3 = -2x + 9,$$
$$g(f(x)) = g(2x - 3) = -(2x - 3) + 6 = -2x + 9.$$

In each of these examples two given functions are combined into a single composite function. In most practical work we proceed in the other direction, and dissect composite functions into their simpler constituents. For example, if

$$y = (x^3 + 1)^7,$$

we can introduce an auxiliary variable u by writing $u = x^3 + 1$ and decompose the above function into the two simpler functions

$$y = u^7 \qquad \text{and} \qquad u = x^3 + 1.$$

We shall see that decompositions of this kind are often useful in the problems of calculus.

In practice, functions often arise from algebraic relations between variables. Thus, an equation involving x and y determines y as a function of x if the equation is equivalent to one that expresses y *uniquely* in terms of x. For example, the equation $4x + 2y = 6$ can be solved for y, $y = 3 - 2x$, and this second equation defines y as a function of x. However, in some cases it happens that the process of solving for y leads to more than one value of y. For example, if the equation is $y^2 = x$, we get $y = \pm\sqrt{x}$. Since this gives two values of y for each positive value of x, the equation $y^2 = x$ does not by itself determine y as a function of x. If we wish, we can split the formula $y = \pm\sqrt{x}$ into two separate formulas, $y = \sqrt{x}$ and $y = -\sqrt{x}$. Each of these formulas defines y as a function of x, so that out of one equation we obtain two functions.

The number of distinct individual functions is clearly unlimited. However, most of those appearing in this book are relatively simple and can be classified into a few convenient categories. It may help students to orient themselves if we give a rough description of these categories in order of increasing complexity.

POLYNOMIALS

The simplest functions are the powers of x with nonnegative integer exponents,

$$1, x, x^2, x^3, \ldots, x^n, \ldots .$$

If a finite number of these are multiplied by constants and the results are added, we obtain a general polynomial,

$$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n.$$

The *degree* of a polynomial is the largest exponent that occurs in it; if $a_n \neq 0$, the degree of $p(x)$ is n. The following are polynomials of degrees 1, 2, and 3:

$$y = 3x - 2, \qquad y = 1 - 2x + x^2, \qquad y = x - x^3.$$

Polynomials can evidently be multiplied by constants, added, subtracted, and multiplied together, and the results are again polynomials.

RATIONAL FUNCTIONS

If division is also allowed, we pass beyond the polynomials into the more inclusive class of rational functions, such as

$$\frac{x}{x^2+1}, \qquad \frac{x+2}{x-2}, \qquad \frac{x^3-4x^2+x+6}{x^2+x+1}, \qquad x + \frac{1}{x}.$$

The general rational function is a quotient of polynomials,

$$\frac{a_0 + a_1x + a_2x^2 + \cdots + a_nx^n}{b_0 + b_1x + b_2x^2 + \cdots + b_mx^m},$$

and a specific function is rational if it is (or can be expressed as) such a quotient. If the denominator here is a nonzero constant, this quotient is itself a polynomial. Thus, the polynomials are included among the rational functions.

ALGEBRAIC FUNCTIONS

If root extractions are also allowed, we pass beyond the rational functions into the larger class of algebraic functions, which will be properly defined in a later chapter. Some simple examples are

$$y = \sqrt{x}, \qquad y = x + \sqrt[3]{x^2+1}, \qquad y = \frac{1}{\sqrt{1-x}}, \qquad y = \sqrt[4]{\frac{x+1}{x-1}}.$$

If we replace the root symbols by fractional exponents in accordance with the rules of algebra, then these functions can be written

$$y = x^{1/2}, \qquad y = x + (x^2+1)^{1/3}, \qquad y = (1-x)^{-1/2}, \qquad y = \left(\frac{x+1}{x-1}\right)^{1/4}.$$

TRANSCENDENTAL FUNCTIONS

Any function that is not algebraic is called *transcendental.* The transcendental functions studied in calculus are the trigonometric, inverse trigonometric, exponential, and logarithm functions. We do not assume that students have any previous knowledge of these functions. All will be carefully explained later.

We conclude this section with a brief review of some important functions arising in geometry. A ready grasp of the geometric formulas given in Fig. 1.22 is

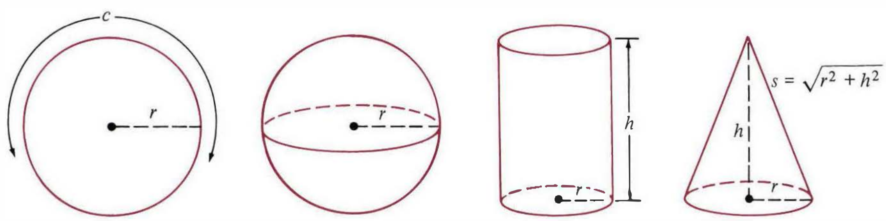


Figure 1.22 Geometric formulas.

essential for coping with many examples and problems in the following chapters. These formulas—for the area and circumference of a circle, the volume and total surface area of a sphere, and the volume and lateral surface area of a cylinder and a cone—should be understood if possible, but remembered in any event. Each of the first four formulas, those for the circle and the sphere, defines a function of the independent variable r, in which a given positive value of r determines the corresponding value of the dependent variable.

Most of our attention in this book will be directed at functions of a single independent variable, as previously defined and discussed. Nevertheless, we point out that each of the last four formulas in Fig. 1.22 defines a function of the two variables r and h; these variables are called *independent* (of each other) because the value assigned to either need not be related to the value assigned to the other. In special circumstances a function of this kind can be expressed as a function of one variable alone. For example, if the height of a cone is known to be twice the radius of its base so that $h = 2r$, then the formula for its volume can be written as a function of r or as a function of h:

$$V = \tfrac{1}{3}\pi r^2(2r) = \tfrac{2}{3}\pi r^3 \qquad \text{or} \qquad V = \tfrac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \tfrac{1}{12}\pi h^3.$$

The formulas in Fig. 1.22 also illustrate the custom of choosing letters for variables that suggest the quantities under discussion, such as A for area, V for volume, r for radius, h for height, and so on.

PROBLEMS

1 If $f(x) = 5x^2 - 3$, find:
(a) $f(-3)$; (b) $f(2)$;
(c) $f(0)$; (d) $f(-\sqrt{7})$;
(e) $f(a + 3)$; (f) $f(5t)$.

2 If $g(x) = \dfrac{x-1}{x+1}$, find:
(a) $g(3)$; (b) $g(-3)$;
(c) $g(\frac{1}{3})$; (d) $g\left(\dfrac{1}{a}\right)$;
(e) $g(a + 1)$; (f) $g(t - 1)$.

In each of Problems 3–8, compute and simplify the quantity $\dfrac{f(x+h) - f(x)}{h}$.

3 $f(x) = 5x - 3$. **4** $f(x) = 3 - 2x$.
5 $f(x) = x^2$. **6** $f(x) = 2x^2 + x$.
7 $f(x) = \dfrac{1}{x}$. **8** $f(x) = \dfrac{3}{1 - x}$.

9 If $f(x) = x^3 - 3x^2 + 4x - 2$, compute $f(1)$, $f(2)$, $f(3)$, $f(0)$, $f(-1)$, and $f(-2)$.

10 If $f(x) = 2^x$, compute $f(1)$, $f(3)$, $f(5)$, $f(0)$, and $f(-2)$.

11 If $f(x) = 4x - 3$, show that $f(2x) = 2f(x) + 3$.

12 What are the domains of $f(x) = 1/(x - 8)$ and $g(x) = x^3$? What is $h(x) = f(g(x))$? What is the domain of $h(x)$?

13 Find the domain of each of the following functions:
(a) $\sqrt{x}$; (b) $\sqrt{-x}$;
(c) $\sqrt{x^2}$; (d) $\sqrt{x^2 - 4}$;
(e) $\dfrac{1}{x^2 - 4}$; (f) $\dfrac{1}{x^2 + 4}$;
(g) $\sqrt{(x - 1)(x + 2)}$; (h) $\dfrac{1}{\sqrt{(x - 1)(x + 2)}}$;
(i) $\sqrt{3 - 2x - x^2}$; (j) $\sqrt{\dfrac{x}{x - 2}}$.

14 If $f(x) = 1 - x$, show that $f(f(x)) = x$.

15 If $f(x) = x/(x - 1)$, compute $f(0)$, $f(1)$, $f(2)$, $f(3)$, and $f(f(3))$. Show that $f(f(x)) = x$.

16 If $f(x) = (ax + b)/(x - a)$, show that $f(f(x)) = x$.

17 If $f(x) = 1/(1 - x)$, compute $f(0), f(1), f(2), f(f(2))$, and $f(f(f(2)))$. Show that $f(f(f(x))) = x$.

18 If $f(x) = ax$, show that $f(x) + f(1 - x) = f(1)$. Also verify that $f(x_1 + x_2) = f(x_1) + f(x_2)$ for all x_1 and x_2.

19 If $f(x) = 2^x$, use functional notation to express the fact that $2^{x_1} \cdot 2^{x_2} = 2^{x_1 + x_2}$.

20 Find $f(x)$ if $f(x + 1) = x^2 - 5x + 3$. Hint: Let $u = x + 1$ and find $f(u)$.

21 A *linear* function is one that has the form $f(x) = ax + b$, where a and b are constants. If $g(x) = cx + d$ is also linear, is it always true that $f(g(x)) = g(f(x))$?

22 If $f(x) = ax + b$ is a linear function with $a \neq 0$, show that there exists a linear function $g(x) = \alpha x + \beta$ such that $f(g(x)) = x$.* Also show that for these two functions it is true that $f(g(x)) = g(f(x))$.

23 A *quadratic* function is one that has the form $f(x) = ax^2 + bx + c$, where a, b, c, are constants and $a \neq 0$.
(a) Find the values of the coefficients a, b, c if $f(0) = 3$, $f(1) = 2, f(2) = 9$.
(b) Show that, no matter what values may be given to the coefficients, a, b, c, the range of a quadratic function cannot be the set of all real numbers.

24 In each case, decide whether or not the equation determines y as a function of x, and if it does, find a formula for the function:
(a) $3x^2 + y^2 = 1$; (b) $3x^2 + y = 1$;
(c) $\dfrac{y + 1}{y - 1} = x$; (d) $x = y - \dfrac{1}{y}$.

25 Split the equation $2x^2 + 2xy + y^2 = 3$ into two equations, each of which determines y as a function of x.

The following problems all involve geometry. In working on such a problem, always draw a sketch and use this sketch as a source of ideas.

26 If an equilateral triangle has side x, express its area as a function of x.

27 The equal sides of an isosceles triangle are 2. If x is the base, express the area as a function of x.

28 If the edge of a cube is x, express its volume, its surface area, and its diagonal as functions of x.

29 A rectangle whose base has length x is inscribed in a fixed circle of radius a. Express the area of the rectangle as a function of x.

30 A string of length L is cut into two pieces, and these pieces are shaped into a circle and a square. If x is the side of the square, express the total enclosed area as a function of x.

31 (a) Is the area of a circle a function of its circumference? If so, what function?
(b) Is the area of a square a function of its perimeter? If so, what function?
(c) Is the area of a triangle a function of its perimeter? If so, what function?

32 The volume of a sphere is a function of its surface area. Find a formula for this function.

33 A cylinder is inscribed in a sphere with fixed radius a. If h is the height and r is the radius of the base of the cylinder, express its volume and total surface area as functions of r, and also as functions of h.

34 A cylinder is circumscribed about a sphere. If their volumes are denoted by C and S, find C as a function of S.

35 A cylinder has fixed volume V. Express its total surface area as a function of the radius r of its base.

36 A fixed cone has height H and base radius R. If a cylinder with base radius r is inscribed in the cone, express the volume of the cylinder as a function of r.

37 (a) A farmer has 100 ft of fencing with which to build a rectangular chicken pen. If x is the length of one side of the pen, show that the enclosed area is

$$A = 50x - x^2 = 625 - (x - 25)^2.$$

Use this result to find the largest possible area and the lengths of the sides that yield this largest area.
(b) Suppose the farmer in part (a) decides to build the pen against a side of the barn so that he will have to fence only three sides of it. If x is the length of a side perpendicular to the barn wall, find the enclosed area as a function of x. Also find the largest possible area and the lengths of the sides that yield this largest area.

*The symbols α and β are letters of the Greek alphabet whose names are "alpha" and "beta." The letters of this alphabet (see the front endpaper) are used so frequently in mathematics and science that serious students should learn them at the earliest opportunity. Among other benefits, this will avoid the annoyance of reading printed matter containing symbols we don't know how to pronounce.

1.6 GRAPHS OF FUNCTIONS

In the previous section we discussed the concept of a function at some length. This discussion can be summarized in a few sentences, as follows.

If x and y are two variables that are related in such a way that whenever a suitable numerical value is assigned to x there is determined a single corresponding numerical value for y, then y is called *a function of* x and this is expressed by writing $y = f(x)$. The letter f symbolizes the function itself, which is the operation or rule of correspondence that yields y when applied to x. However, for practical reasons we prefer to speak of "the function $y = f(x)$" instead of "the function f." As a matter of principle, students should clearly understand that a function is not a formula and need not be specified by a formula—even though most of ours are.

Now for graphs.

The Chinese have a well-known proverb that can be interpreted as expressing a basic truth about the study of mathematics: One picture is worth a thousand words.* For us, in our study of functions, this means *draw graphs! Even more, cultivate the habit of thinking graphically, to the point where it becomes almost second nature.*

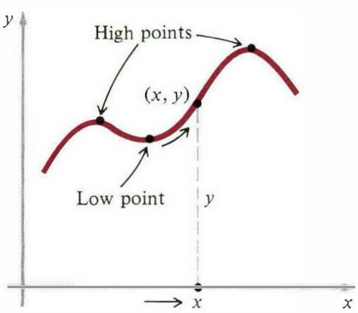


Figure 1.23

Before getting down to the details of specific functions, we emphasize that it is often possible to think of the graph of a function $y = f(x)$ very concretely, as the path of a moving point (Fig. 1.23). The independent variable x can be visualized as a point moving along the x-axis from left to right; each x determines a value of the dependent variable y, which is the height of the point (x, y) above the x-axis. The graph of the function is simply the path of the point (x, y) as it moves across the coordinate plane, sometimes rising and sometimes falling, and in general varying in height according to the nature of the particular function under consideration. The graph as a whole is intended to provide a clear overall picture of this variation. The graph shown in Fig. 1.23 happens to be a smooth curve with two high points and one low point, but many diverse phenomena are possible.

We now discuss the graphs of a few representative examples of the types of functions described in Section 1.5.

POLYNOMIALS

We have seen that the simplest polynomials are the powers of x with nonnegative integral exponents,

$$y = 1, x, x^2, x^3, \dots, x^n, \dots .$$

As we know, the graph of $y = 1$ is the horizontal straight line through the point (0, 1), and the graph of $y = x$ is the straight line through the origin with slope 1 (Fig. 1.24*a*). For larger values of the exponent n, the graphs of $y = x^n$ are of two distinct types, depending on whether n is even or odd:

$$y = x^2, x^4, x^6, \dots$$

and

$$y = x^3, x^5, x^7, \dots .$$

*See *Bartlett's Familiar Quotations,* 16th ed. (Little, Brown and Co., 1992), fn. 8, p. 782.

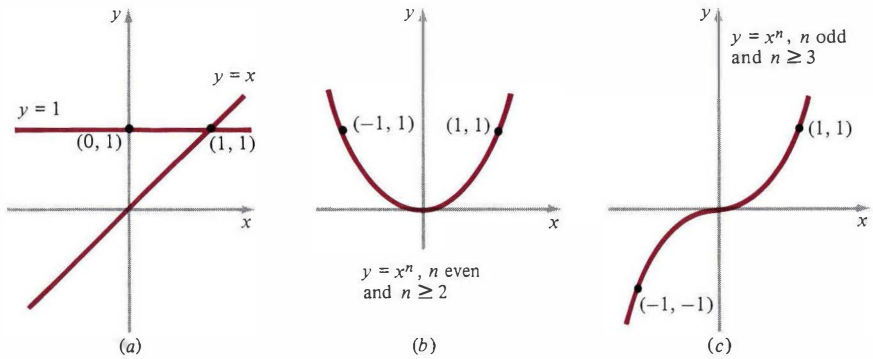


Figure 1.24 Graphs of $y = x^n$.

These types are shown in parts *b* and *c* of Fig. 1.24. As *n* increases, these curves become flatter near the origin and steeper outside the interval $[-1, 1]$.

We already know that the graphs of all first- and second-degree polynomials, such as

$$y = 2x - 1$$

and

$$y = 3x^2 - 2x + 1,$$

are straight lines and parabolas. These graphs are easy to draw—without plotting points—on the basis of the ideas in Sections 1.3 and 1.4.

For our next remark we need a bit of new terminology. A *zero* of a function $y = f(x)$ is a root of the corresponding equation $f(x) = 0$. Geometrically, the zeros of this function (if it has any) are the values of x at which its graph crosses or touches the x-axis; they are the x-intercepts of this graph.

Now consider the general second-degree polynomial

$$y = ax^2 + bx + c, \qquad a \neq 0. \tag{1}$$

As we know, the graph of this function is a parabola for all values of the coefficients. If we assume that $a > 0$, so that the parabola opens upward, then there are three possibilities for the zeros of (1), and these are shown in Fig. 1.25. Since the roots of the quadratic equation $ax^2 + bx + c = 0$ are given by the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

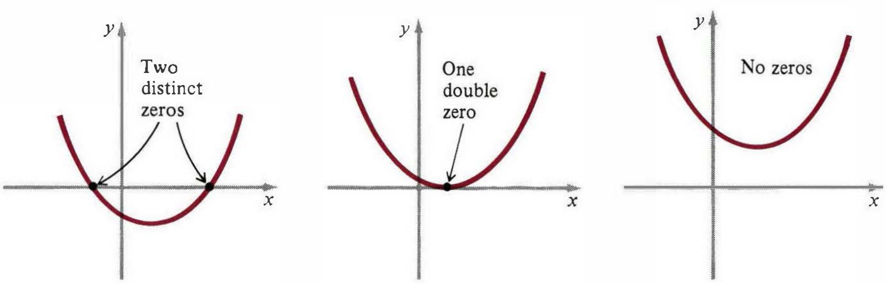


Figure 1.25

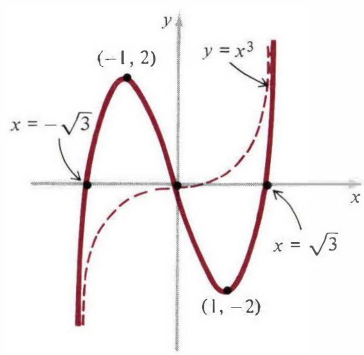


Figure 1.26

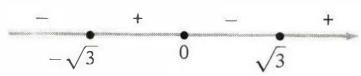


Figure 1.27

it is clear that the three possibilities in Fig. 1.25 correspond to the algebraic conditions $b^2 - 4ac > 0$, $b^2 - 4ac = 0$, $b^2 - 4ac < 0$.

The problem of graphing polynomials of degree $n \geq 3$ is not easy. Our discussion of the following example suggests several useful ideas.

Example 1 The graph of

$$y = x^3 - 3x \tag{2}$$

is shown in Fig. 1.26. At present we have no methods available for discovering such important features of this curve as the precise location of the indicated high and low points. This will come later. Nevertheless, a few observations can be made, and these provide at least some details and a good enough impression of the shape of the graph so that students should be able to sketch it for themselves.

We begin by pointing out that if (2) is written in factored form, as

$$y = x(x^2 - 3) = x(x + \sqrt{3})(x - \sqrt{3}), \tag{3}$$

then its zeros are obviously $0, -\sqrt{3}, \sqrt{3}$. These three numbers divide the x-axis into four intervals, as shown in Fig. 1.27, and a careful inspection of the factors of (3) tells us that in each interval y has the sign given in this figure. The details of this determination of the sign of y are important to understand, so we pause and carefully think it through, as follows:

for $x < -\sqrt{3}$, x is negative,
$x + \sqrt{3}$ is negative, and
$x - \sqrt{3}$ is negative,
so their product y is negative;

for $-\sqrt{3} < x < 0$, x is negative,
$x + \sqrt{3}$ is positive, and
$x - \sqrt{3}$ is negative,
so their product y is positive;

for $0 < x < \sqrt{3}$, x is positive,
$x + \sqrt{3}$ is positive, and
$x - \sqrt{3}$ is negative,
so their product y is negative;

for $x > \sqrt{3}$, x is positive,
$x + \sqrt{3}$ is positive, and
$x - \sqrt{3}$ is positive,
so their product y is positive.

We therefore know, for each interval, whether the graph of (2) lies above or below the x-axis (see Fig. 1.26). We have described this method of analysis in detail because it will often be useful in other problems of curve sketching.

Our second observation relates to the behavior of the graph of (2) when x is numerically large, that is, far to the right or far to the left in Fig. 1.26. If (2) is written in the form

$$y = x^3\left(1 - \frac{3}{x^2}\right), \qquad x \neq 0,$$

then for large positive or negative values of x the expression in parentheses is nearly 1, so y is close to x^3. In geometric language, when x is large, the graph of (2) is close to the graph of $y = x^3$, as Fig. 1.26 suggests. In particular, the graph of (2) rises on the far right and falls on the far left.

Students will notice that they can always sketch a graph by laboriously plotting many points and joining these points by a reasonable curve. Nevertheless, this rather clumsy procedure should be adopted only as a last resort, when more imaginative methods fail. The important features of functions and their graphs are much more clearly revealed by the qualitative approach to curve sketching that we have tried to suggest in Example 1 and will continue to emphasize.

RATIONAL FUNCTIONS

Example 2 The simplest rational function that is not a polynomial is

$$y = \frac{1}{x}. \tag{4}$$

On examining (4), we notice the following facts: y is undefined when $x = 0$; y is positive when x is positive, and is small when x is large and large when x is near 0 on the right; y is negative when x is negative, and is small when x is large and large when x is near 0 on the left. The graph of (4) given in Fig. 1.28 is a direct pictorial version of these statements. In this particular case the graph is also easy to sketch by plotting a few points, as shown in the figure. However, students will profit much more from simply visualizing the behavior of such a function on the various parts of its domain and drawing what they see in the mind's eye.

A straight line is called an *asymptote* of a curve if, as a point moves out along an extremity of the curve, the distance from this point to the line approaches 0. It is clear that both the x-axis and the y-axis are asymptotes of the graph shown

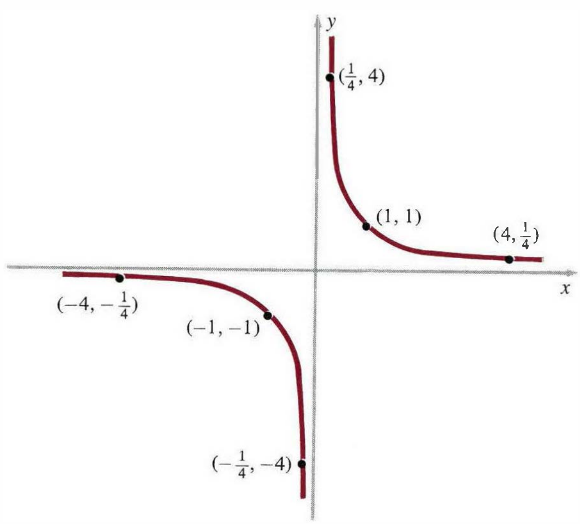


Figure 1.28

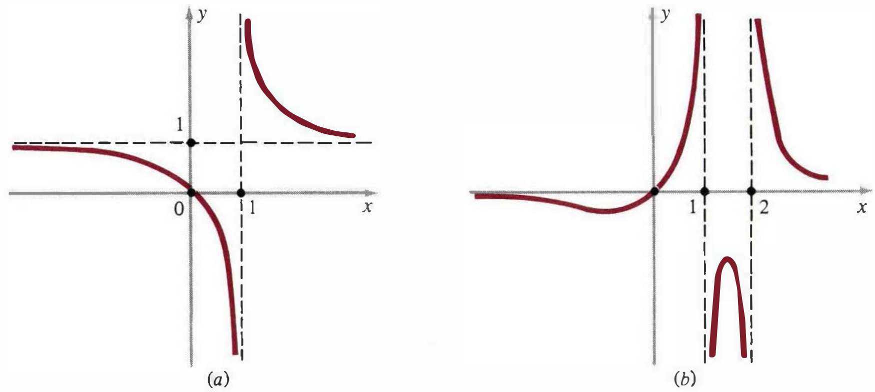


Figure 1.29

in Fig. 1.28. The behavior of the function (4) at and near the point $x = 0$, that is, the fact that y is undefined at $x = 0$ and "becomes infinite" near $x = 0$, is described by calling this point an *infinite discontinuity* of the function.

Example 3 In the case of the function

$$y = \frac{x}{x-1}, \tag{5}$$

it is clear that the point $x = 1$ is particularly significant, since y is undefined at $x = 1$ and is large in absolute value when x is near 1 ($x = 1$ is an infinite discontinuity). Also, y is near 1 and slightly greater than 1 when x is large and positive, and is near 1 and slightly less than 1 when x is large and negative.* These observations suggest drawing the vertical and horizontal guidelines shown in Fig. 1.29*a*. If we notice that $y = 0$ when $x = 0$, and use the method of Example 1 to find the sign of y in each of the intervals $-\infty < x < 0$, $0 < x < 1$, and $1 < x$, then the graph as given in Fig. 1.29*a* is quite easy to sketch. The lines $x = 1$ and $y = 1$ are both asymptotes.

Example 4 The function

$$y = \frac{x}{x^2 - 3x + 2} = \frac{x}{(x-1)(x-2)} \tag{6}$$

is similar to (5) but somewhat more complicated. Here the factored form of the denominator reveals two infinite discontinuities, $x = 1$ and $x = 2$. Again, $y = 0$ when $x = 0$, but this time y is small when x is large, since the degree of the denominator is greater than that of the numerator. If we combine these facts with the observable sign of y in each of the intervals $-\infty < x < 0$, $0 < x < 1$, $1 < x < 2$, and $2 < x$ (think it through in the manner of Example 1 for each interval!), then it is fairly straightforward to sketch the graph as shown in Fig. 1.29*b*. There is evidently a high point between 1 and 2, and a low point to the left of 0, but at present we are unable to determine the precise location of these points (we shall see later that they occur at $x = \sqrt{2}$ and $x = -\sqrt{2}$).

*To see this, test with convenient specific values of x; thus, for example, $y = \frac{10}{9}$ when $x = 10$ and $y = \frac{10}{11}$ when $x = -10$.

Example 5 The function

$$y = x + \frac{1}{x} \tag{7}$$

has an infinite discontinuity at $x = 0$, and is positive or negative according as x is positive or negative. For small positive x's, the first term on the right of (7) is negligible and the second term is large; and for large positive x's, the second term is negligible and y is approximately equal to x. We therefore sketch the part of the graph in the right half-plane as follows: Draw the guideline $y = x$ (Fig. 1.30); insert the two extremities of the curve, approaching this guideline and the positive y-axis, as suggested by the behavior previously stated; and connect these extremities in a reasonable way in the middle, where this part of the graph has an obvious low point. The function behaves similarly—with a corresponding high point—for negative values of x. The y-axis and the line $y = x$ are both asymptotes.

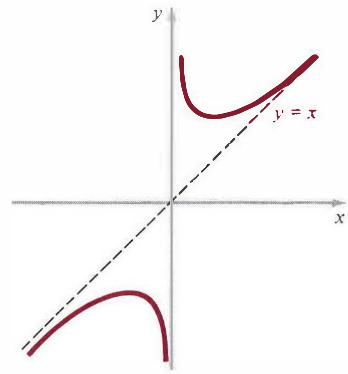


Figure 1.30

Example 6 The denominator of

$$y = \frac{x}{x^2 + 1} \tag{8}$$

is positive (in fact ≥ 1) for all x, so $y = 0$ when $x = 0$, y is positive when x is positive, and y is negative when x is negative. Also, y is small when x is large, because the degree of the denominator is greater than that of the numerator.* These properties of the function force the graph to have the shape shown in Fig. 1.31, with one high point and one low point.

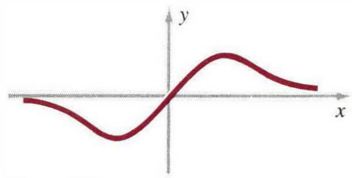


Figure 1.31

Example 7 In considering the function

$$y = \frac{x^2 - 1}{x - 1}, \tag{9}$$

it is natural to factor the numerator, obtaining

$$y = \frac{(x + 1)(x - 1)}{x - 1},$$

and then to cancel the common factor, which yields

$$y = x + 1.^\dagger \tag{10}$$

*Notice that when the numerator x is large, the denominator $x^2 + 1$ is enormous, so y is small.

†A word of warning about a point of algebra. To "cancel" a common *factor*, as in the text, is OK:

$$\frac{a\cancel{c}}{b\cancel{c}} = \frac{a}{b} \qquad \text{if } c \neq 0.$$

But "canceling" a common *term*, as in

$$\frac{a + \cancel{c}}{b + \cancel{c}} = \frac{a}{b},$$

is WRONG. Try it: Is

$$\frac{1 + 2}{2 + 2} = \frac{1}{2}?$$

Of course not.

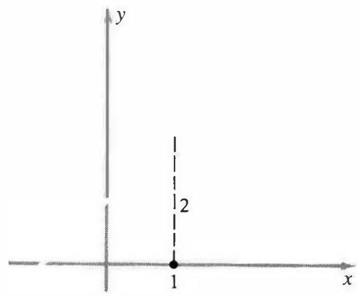


Figure 1.32

This cancellation is valid *except when* $x = 1$. At this point the value of (10) is 2, but (9) has no value ($y = 0/0$, which is meaningless). To graph (9), we therefore draw the straight line (10) and delete the single point (1, 2), as shown in Fig. 1.32.

Two functions $y = f(x)$ and $y = g(x)$ are said to be *equal* if they have the same domain and if $f(x) = g(x)$ for every x in their common domain. Accordingly, the functions (9) and (10) are not equal, because they have different domains—the point $x = 1$ is in the domain of (10) but is not in the domain of (9). The fact that the graph of (9) has a gap (or hole) corresponding to $x = 1$ is expressed by saying that (9) is *discontinuous* at $x = 1$, or has a *discontinuity* at this point.

ALGEBRAIC FUNCTIONS

Example 8 The functions

$$y = \sqrt{x} \qquad \text{and} \qquad y = \sqrt{25 - x^2} \tag{11}$$

can be obtained by solving the equations

$$y^2 = x \qquad \text{and} \qquad x^2 + y^2 = 25 \tag{12}$$

for y and choosing the positive square roots. We know that the graphs of equations (12) are a parabola and a circle, as shown in Fig. 1.33, so the graphs of (11) are the parts of these curves that lie on or above the x-axis.

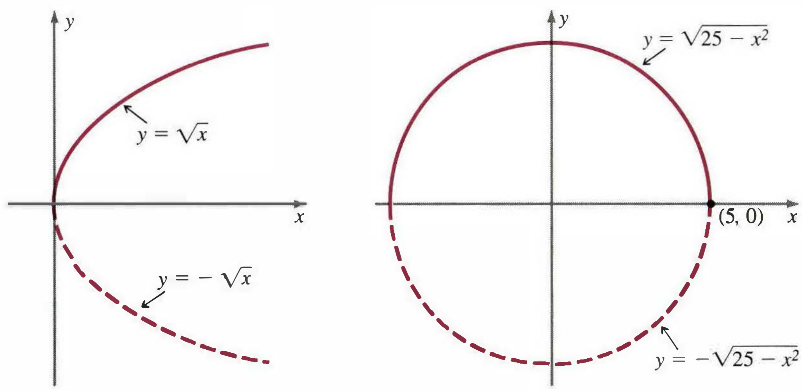


Figure 1.33

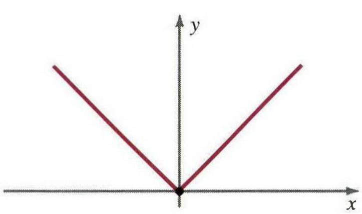


Figure 1.34

Example 9 The graph of the absolute value function

$$y = |x|$$

is easy to draw (Fig. 1.34). To see that this function is algebraic, we have only to notice the fact that $|x| = \sqrt{x^2}$ for every value of x.

As these examples show, many of the basic features of a function are made transparently clear by sketching its graph. We are interested less in sketches of high accuracy than in those that display broad general features: where the graph is rising and where falling, the presence of gaps, the presence of high points and low points, and what its approximate shape is. Formulas are obviously important in the study of functions—indeed, they are indispensable whenever our purposes require exact calculations yielding quantitative results. But we should never forget that *the primary aim of mathematics is insight*, and graphs are invaluable aids for gaining visual insight into the individual characteristics of functions.

PROBLEMS

1 Sketch the graphs of the following polynomials, paying special attention to the location of their zeros and their behavior for large values of x:
(a) $y = x^2 + x - 2$;
(b) $y = x^3 - 3x^2 + 2x$;
(c) $y = (1 - x)(2 - x)(3 - x)$;
(d) $y = x^4 - x^2$;
(e) $y = x^4 - 5x^2 + 4$.

2 Sketch the graphs of the following rational functions:
(a) $y = \dfrac{1}{x^2}$; (b) $y = \dfrac{1}{x^3}$;
(c) $y = x^2 + \dfrac{1}{x}$; (d) $y = x^2 + \dfrac{1}{x^2}$;
(e) $y = \dfrac{1}{x^2 + 1}$; (f) $y = \dfrac{x^2}{x^2 + 1}$;
(g) $y = \dfrac{1}{x^2 - 1}$; (h) $y = \dfrac{x}{x^2 - 1}$;
(i) $y = \dfrac{x^2}{x^2 - 1}$; (j) $y = \dfrac{x^2 - 3x + 2}{2 - x}$;
(k) $y = \dfrac{x^3 - x^2}{x - 1}$;
(l) $y = \dfrac{(x + 2)(x - 5)(x^2 + 2x - 8)}{(x - 2)(x^2 - 3x - 10)}$.

3 Sketch the graphs of the following algebraic functions:
(a) $y = \sqrt{(x - 1)(3 - x)}$;
(b) $y = \dfrac{1}{\sqrt{(x - 1)(3 - x)}}$;
(c) $y = \dfrac{1}{\sqrt{x - 1}}$; (d) $y = \sqrt{\dfrac{x}{3 - x}}$;
(e) $y = \sqrt{\dfrac{4 - x}{x - 2}}$; (f) $y = \sqrt{\dfrac{x - 4}{x - 2}}$.

4 In each of the following, sketch the graphs of all three functions on a single coordinate system:
(a) $y = |x|$, $y = |x| + 1$, $y = |x| - 1$;
(b) $y = |x|$, $y = |x + 1|$, $y = |x - 1|$;
(c) $y = |x|$, $y = 2|x|$, $y = \frac{1}{2}|x|$.

5 Sketch the graphs of the following functions:
(a) $y = \dfrac{|x|}{x}$; (b) $y = |2x + 3|$;
(c) $y = x + |x|$; (d) $y = 2x + |x|$;
(e) $y = x - |x|$; (f) $y = 1 + x - |x|$;
(g) $y = |x^2 - 1|$.

6 Considering only positive values of x, show that

$$y = \frac{|x + 1| - |x - 1|}{x} = \begin{cases} 2 & 0 < x < 1, \\ \dfrac{2}{x} & x \geq 1, \end{cases}$$

and sketch the graph.

7 Are any of the following pairs of functions equal?
(a) $f(x) = \dfrac{x}{x}$, $g(x) = 1$.
(b) $f(x) = x^2 - 1$, $g(x) = (x + 1)(x - 1)$.
(c) $f(x) = x$, $g(x) = \sqrt{x^2}$.
(d) $f(x) = x$, $g(x) = (\sqrt{x})^2$.

1.7 INTRODUCTORY TRIGONOMETRY. THE FUNCTIONS $\sin\theta$ AND $\cos\theta$

Periodic phenomena are found everywhere in the world around us—vibrating springs, alternating currents, swinging pendulums, revolving planets, etc.—and scientists describe these phenomena by using trigonometric functions. For this and other reasons, students beginning the study of calculus are often expected to know something about trigonometry.

Although most users of this book have some familiarity with basic trigonometry, we nevertheless review a few of the fundamental ideas, especially the radian measure of angles and the definitions and simpler properties of the very important functions $\sin\theta$ and $\cos\theta$.* This review is continued in Section 9.1, where the discussion is broadened to include the other four trigonometric functions $\tan\theta$, $\cot\theta$, $\sec\theta$, $\csc\theta$—all of which are indispensable in Chapter 10 but will not be needed until then.

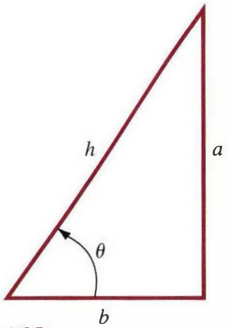


Figure 1.35

In high school trigonometry courses the sine and cosine of an acute angle θ are first defined as ratios of sides in a right triangle, as follows (see Fig. 1.35):

$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{a}{h},$$

$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{b}{h}.$$

Because similar triangles have proportional sides, the values of $\sin\theta$ and $\cos\theta$ depend only on the size of the acute angle θ, and not at all on the size of the right triangle whose sides are used to compute these values.

Example 1 We know from geometry that in a 30°–60° right triangle, the side opposite the 30° angle is half the hypotenuse (see Problem 32 in Section 1.2). This enables us to draw the familiar right triangles shown in Fig. 1.36, and from these triangles we see that

$$\sin 30^\circ = \frac{1}{2}, \qquad \sin 60^\circ = \frac{\sqrt{3}}{2}, \qquad \sin 45^\circ = \frac{1}{\sqrt{2}},$$

$$\cos 30^\circ = \frac{\sqrt{3}}{2}, \qquad \cos 60^\circ = \frac{1}{2}, \qquad \cos 45^\circ = \frac{1}{\sqrt{2}}.$$

It is customary to rationalize the denominators on the right by writing

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{1}{2}\sqrt{2},$$

but for the moment we leave these values as they stand in order to emphasize the defining ratios.

*The Greek letter θ is pronounced "theta."

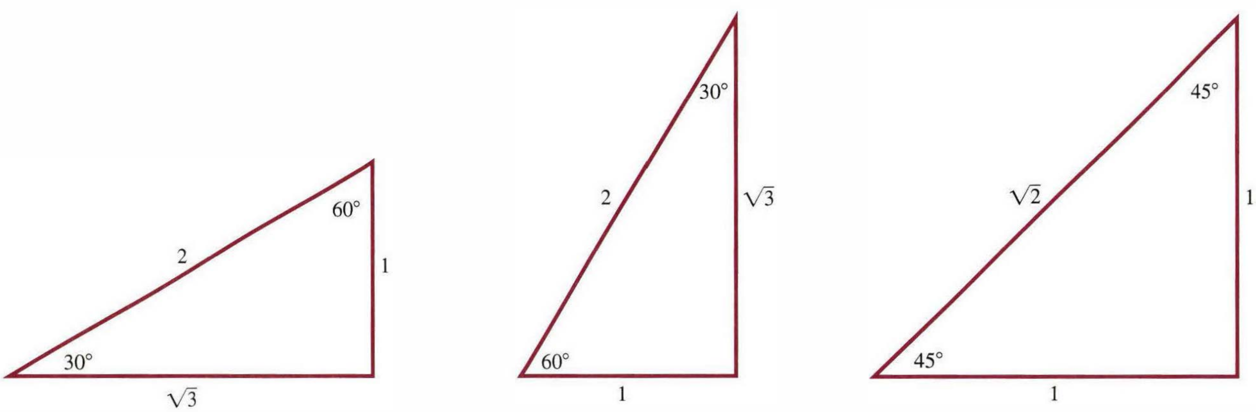


Figure 1.36

The ideas described here are part of what is called *right triangle trigonometry,* in which angles are measured in degrees and sines and cosines are defined only for acute angles of right triangles. In the equivalent forms

$$a = h\sin\theta \qquad \text{and} \qquad b = h\cos\theta,$$

these definitions have a number of applications in geometry and physics. This is all right as far as it goes. However, for the purposes of calculus the limitations of this approach are crippling. We therefore start all over again at the beginning and give a capsule development of *analytic trigonometry,* in which the trigonometric functions are freed from their dependence on right triangles and are defined as real-valued functions of a real variable. As an example of what we mean by analytic trigonometry, let us consider the motion of an object oscillating up and down at the end of a spring (Fig. 1.37). If this motion is described by the position function

$$s = f(t) = \cos t,$$

which gives the position s as a function of the time t, then it makes little sense to think of t as an angle and measure its values in degrees. We must consider what $\cos t$ means when t is not an angle but a *number*—the number of seconds that have elapsed since the motion began when $t = 0$.

Our treatment below is self-contained. Even a student who knows nothing of the subject will be able to learn everything that matters by reading with close attention and working through the problems at the end of the section.

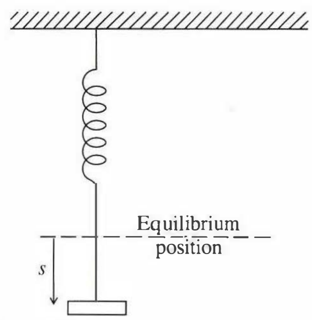
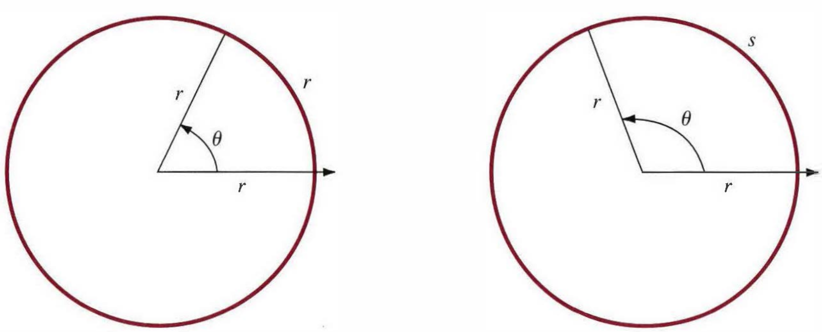


Figure 1.37

RADIAN MEASURE

In elementary mathematics and daily life, angles are measured in degrees, with 90° measuring a right angle. But the degree is an arbitrary measure inherited from the ancient Babylonian astronomers, and its use in calculus would make many of our formulas intolerably messy. In calculus we use a much more natural and convenient system called *radian measure,* which is defined in terms of how much arc an angle cuts off on a circle.

In this system the unit of angle measurement is called the *radian.* One radian is the angle which, placed at the center of a circle, subtends (cuts off) an arc whose length equals the radius (Fig. 1.38, left). More generally, the number of radians θ in an arbitrary central angle (Fig. 1.38, right) is defined to be the ratio of the length s of the subtended arc to the radius r, $\theta = s/r$, so that $s = r\theta$. We



$\theta = 1$ radian $\qquad$ $\theta = \frac{s}{r}$ radians

Figure 1.38

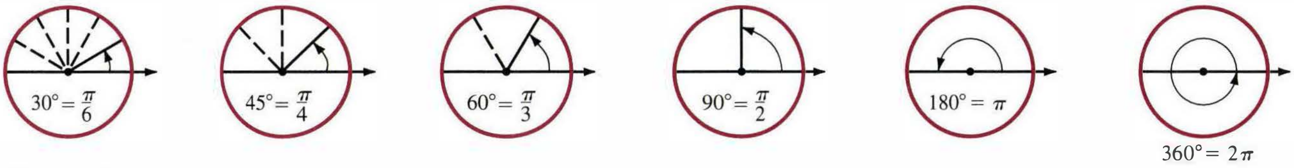


Figure 1.39

note especially that in a unit circle ($r = 1$), a central angle of θ radians subtends an arc of length $s = \theta$. Since the circumference of a circle is $c = 2\pi r$, a complete central angle of 360° is equivalent to $2\pi r/r = 2\pi$ radians. Thus,

$$2\pi \text{ radians} = 360° \qquad \text{or} \qquad \pi \text{ radians} = 180°;$$

and it follows from this that

$$1 \text{ radian} = \frac{180}{\pi} \cong 57.296°, \qquad 1° = \frac{\pi}{180} \cong 0.0175 \text{ radian}.$$

Further, $90° = \pi/2$, $60° = \pi/3$, $45° = \pi/4$, and $30° = \pi/6$, where we follow the convention of omitting the word "radian" in using radian measure. It is a good idea for students to memorize these common conversions with the aid of the circle diagrams in Fig. 1.39. In addition to *knowing* the conversions in these diagrams, it will help students feel more comfortable with radians if they also think through and verify the additional conversions in the following table.

Degrees	30	45	60	90	120	135	150	180	210	225	240	270	300	315	330	360
Radians	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π

The specific reason why radian measure for angles is preferred in calculus will appear in Section 3.4. In most of our work we will use radian measure routinely and mention degrees only in passing.

DEFINITIONS OF sin θ AND cos θ

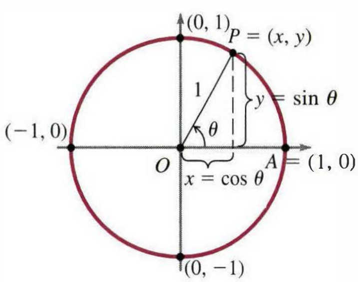


Figure 1.40

We approach trigonometry by way of analytic geometry. Consider the unit circle $x^2 + y^2 = 1$ in the xy-plane (Fig. 1.40), and let θ be an arbitrary real number. If θ is positive, let the radius OP start in the position OA and revolve counterclockwise through θ radians. Thus, $\theta = \pi$ produces half a revolution and $\theta = 2\pi$ produces a complete revolution, both counterclockwise. If θ is negative, we form the positive number $-\theta$ and let OP revolve clockwise through $-\theta$ radians. See Fig. 1.41. In this way, each real number θ (positive, negative, or zero) determines a unique position of the radius OP in Fig. 1.40, and therefore a unique point $P = (x, y)$ with the property that $x^2 + y^2 = 1$.

The *sine* and *cosine* of θ are now defined by

$$\sin\theta = y \qquad \text{and} \qquad \cos\theta = x.$$

The word "sine," *sinus* in Latin, is a corruption of an Arabic word meaning "chord" or "bowstring." Since sin and cos are the names of functions, the proper notation should be $\sin(\theta)$ and $\cos(\theta)$, just as we write $f(\theta)$ when the function is

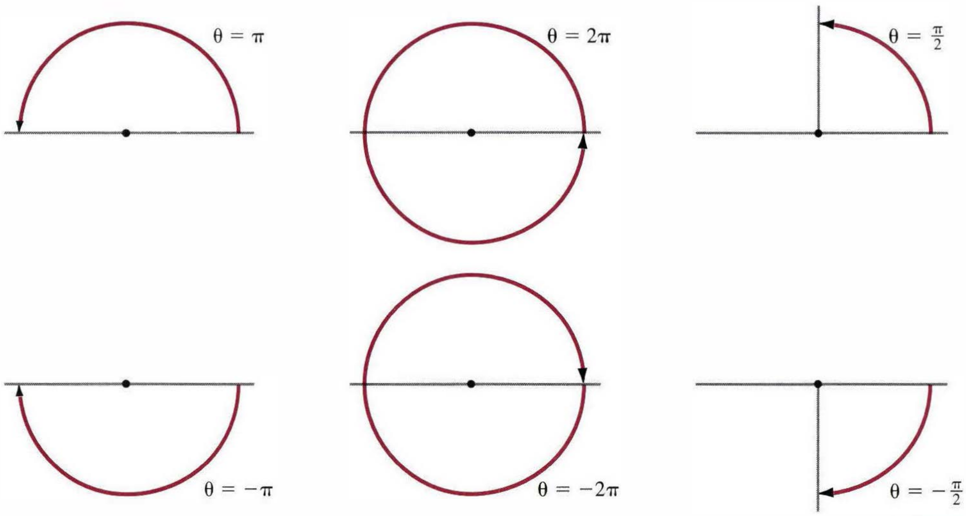


Figure 1.41

f. However, in the case of trigonometric functions it is customary to omit the parentheses. It is evident from the definition that $-1 \le \sin\theta \le 1$, and similarly for $\cos\theta$. The algebraic signs of these quantities depend on which quadrant of the plane the point P happens to lie in (Fig. 1.42). For values of θ such that $0 < \theta < \pi/2$, these definitions agree with the right triangle definitions given above, because in the triangle in Fig. 1.40 we have $\sin\theta = y = y/1 =$ (opposite side)/(hypotenuse) and $\cos\theta = x = x/1 =$ (adjacent side)/(hypotenuse).

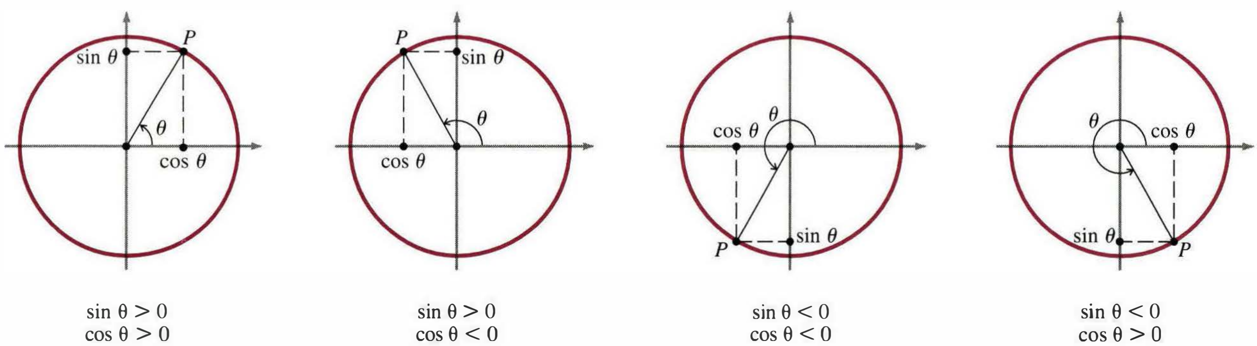


Figure 1.42

IDENTITIES

If we compare the angles θ and $-\theta$ in Fig. 1.43, we see at once that

$$\sin(-\theta) = -\sin\theta \quad \text{and} \quad \cos(-\theta) = \cos\theta. \tag{1}$$

The equation $x^2 + y^2 = 1$, or equivalently $y^2 + x^2 = 1$, translates immediately into the important identity

$$\sin^2\theta + \cos^2\theta = 1. \tag{2}$$

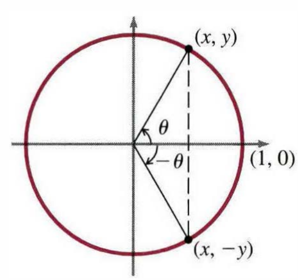


Figure 1.43

[The somewhat strange notation $\sin^2\theta$ is the standard way of writing the square of the number $\sin \theta$, that is, $(\sin \theta)^2$; and similarly for $\cos^2 \theta$.] Problem 10 in Section 9.1 outlines a general proof of the *addition formulas*

$$\sin(\theta + \phi) = \sin \theta \cos \phi + \cos \theta \sin \phi, \tag{3}$$

$$\cos(\theta + \phi) = \cos \theta \cos \phi - \sin \theta \sin \phi. \tag{4}$$

We give a proof of (3) below, in connection with Fig. 1.44, for the restricted case in which θ and ϕ are both positive angles whose sum is less than $\pi/2$. First, however, we point out that if we put $\phi = \theta$ in (3) and (4), we obtain the *double-angle formulas*

$$\sin 2\theta = 2 \sin \theta \cos \theta, \tag{5}$$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta. \tag{6}$$

And finally, if we write (2) and (6) together as

$$\cos^2 \theta + \sin^2 \theta = 1,$$

$$\cos^2 \theta - \sin^2 \theta = \cos 2\theta,$$

then by adding and subtracting we get the *half-angle formulas*

$$\cos^2 \theta = \tfrac{1}{2}(1 + \cos 2\theta), \tag{7}$$

$$\sin^2 \theta = \tfrac{1}{2}(1 - \cos 2\theta). \tag{8}$$

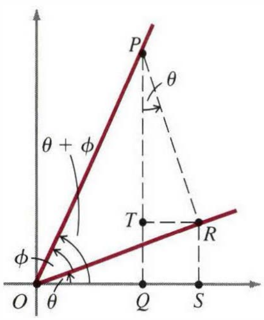


Figure 1.44

Now, to prove (3) for the restricted case mentioned above, we consult Fig. 1.44 and write

$$\begin{aligned}\sin(\theta + \phi) &= \frac{PQ}{OP} = \frac{PT + TQ}{OP} \\ &= \frac{PT + RS}{OP} = \frac{PT}{OP} + \frac{RS}{OP} \\ &= \frac{PT}{PR} \cdot \frac{PR}{OP} + \frac{RS}{OR} \cdot \frac{OR}{OP} \\ &= \cos \theta \sin \phi + \sin \theta \cos \phi.\end{aligned}$$

A similar argument can be given for formula (4).

VALUES AND GRAPHS

Example 1 provides several first-quadrant θ's for which exact values of $\sin \theta$ and $\cos \theta$ are easy to find. These facts can also be obtained by looking carefully at the three parts of Fig. 1.45 and remembering the Pythagorean theorem:

$$\sin \frac{\pi}{6} = \frac{1}{2}, \qquad \sin \frac{\pi}{4} = \frac{1}{2}\sqrt{2}, \qquad \sin \frac{\pi}{3} = \frac{1}{2}\sqrt{3},$$

$$\cos \frac{\pi}{6} = \frac{1}{2}\sqrt{3}, \qquad \cos \frac{\pi}{4} = \frac{1}{2}\sqrt{2}, \qquad \cos \frac{\pi}{3} = \frac{1}{2}.$$

Also, an inspection of Fig. 1.40 with OP in various positions gives us similar information for the cases $\theta = 0, \pi/2, \pi, 3\pi/2, 2\pi$:

$$\sin 0 = 0, \qquad \sin \frac{\pi}{2} = 1, \qquad \sin \pi = 0, \qquad \sin \frac{3\pi}{2} = -1, \qquad \sin 2\pi = 0,$$

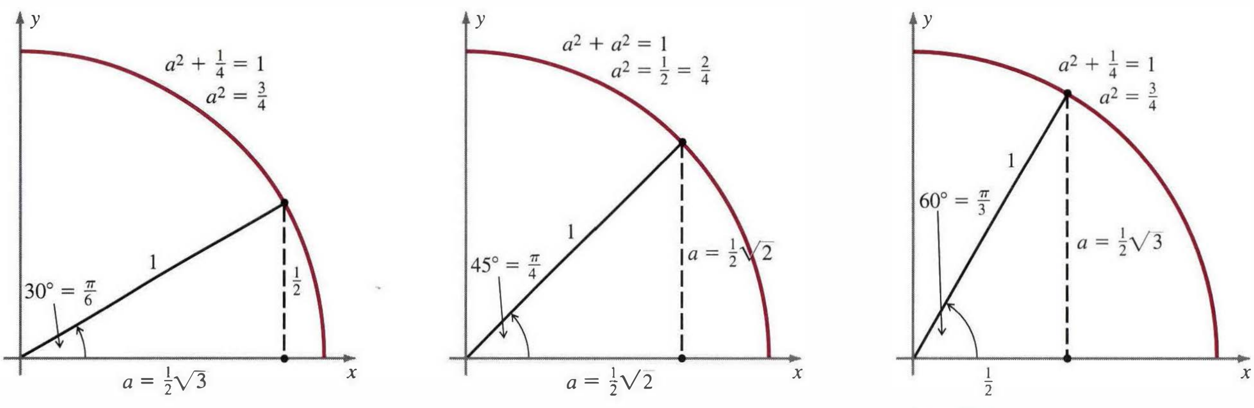


Figure 1.45

$$\cos 0 = 1, \qquad \cos\frac{\pi}{2} = 0, \qquad \cos\pi = -1, \qquad \cos\frac{3\pi}{2} = 0, \qquad \cos 2\pi = 1.$$

Further, by drawing pictures and using the ideas in Fig. 1.45 we can find the exact values of sin θ and cos θ for any value of θ that represents an angle one-third, one-half, or two-thirds of the way through any quadrant.

Example 2 To illustrate this remark, we point out (Fig. 1.46) that $135° = 3\pi/4$ is halfway from $\pi/2$ to π, so the point P is in the second quadrant with coordinates $(-\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2})$, and consequently we have

$$\sin\frac{3\pi}{4} = \frac{1}{2}\sqrt{2}, \qquad \cos\frac{3\pi}{4} = -\frac{1}{2}\sqrt{2}.$$

Similarly, $300° = 5\pi/3$ is one-third of the way from $3\pi/2$ to 2π, so P is in the fourth quadrant with coordinates $(\frac{1}{2}, -\frac{1}{2}\sqrt{3})$, and we have

$$\sin\frac{5\pi}{3} = -\frac{1}{2}\sqrt{3}, \qquad \cos\frac{5\pi}{3} = \frac{1}{2}.$$

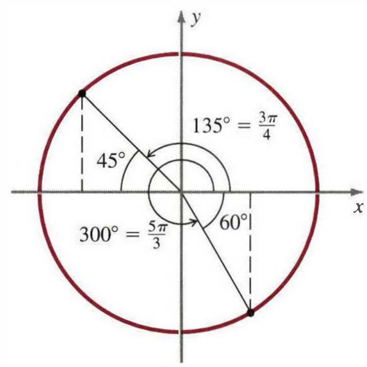


Figure 1.46

Of course, most θ's are beyond the scope of these methods, and in these cases the values of sin θ and cos θ can be found from trigonometric tables or a calculator. The problem of how these values themselves are calculated is more difficult, and will be discussed in Chapter 14.

For every θ, the numbers θ and $\theta + 2\pi$ clearly determine the same point P, so

$$\sin(\theta + 2\pi) = \sin\theta \qquad \text{and} \qquad \cos(\theta + 2\pi) = \cos\theta.$$

This says that the values of sin θ and cos θ repeat when θ increases by 2π. We express these properties of sin θ and cos θ by saying that these functions are *periodic* with *period* 2π.

The graph of sin θ is easy to sketch by looking at Fig. 1.40 and using imagination to follow the way y varies as θ increases from 0 to 2π, that is, as the radius swings around through one complete counterclockwise revolution. It is clear that sin θ starts at 0, increases to 1, decreases to 0, decreases further to -1, and increases to 0. This gives one complete cycle of sin θ on the interval $0 \leq \theta \leq 2\pi$, as shown on the left in Fig. 1.47. By using the periodicity of sin θ, we see

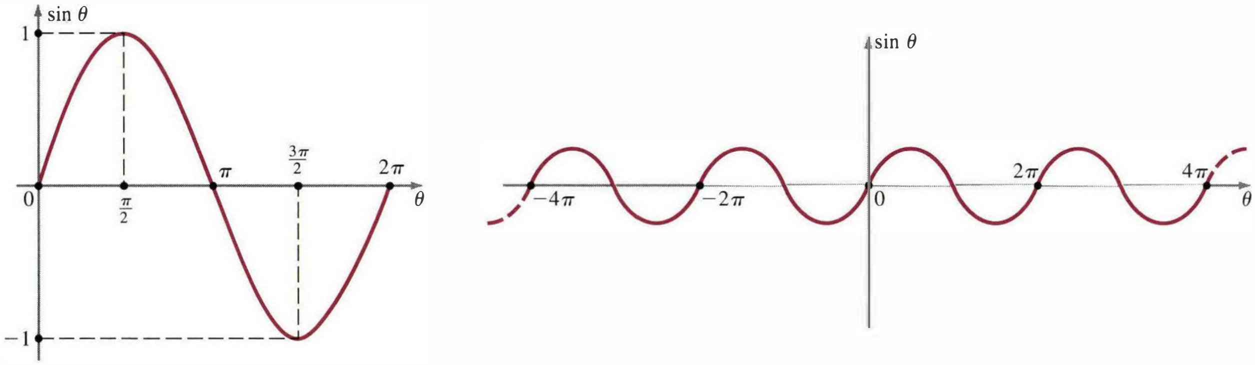


Figure 1.47

that the complete graph (on the right in the figure) consists of infinitely many repetitions of this cycle, to the right and to the left. The graph of cos θ can be sketched in essentially the same way (Fig. 1.48). The main difference is that cos θ starts at 1 when $\theta = 0$, decreases to 0, decreases further to -1, increases to 0, and increases further to 1.

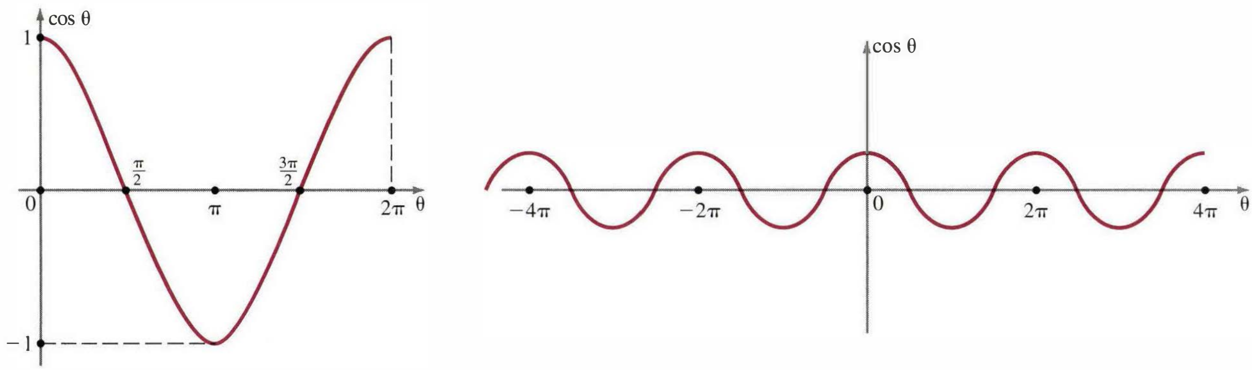


Figure 1.48

On the other hand, the graph of sin 2θ makes one complete cycle on the interval $0 \le \theta \le \pi$, because 2θ increases from 0 to 2π as θ increases from 0 to π (Fig. 1.49, left). This says that sin 2θ oscillates twice as fast as sin θ. In the same

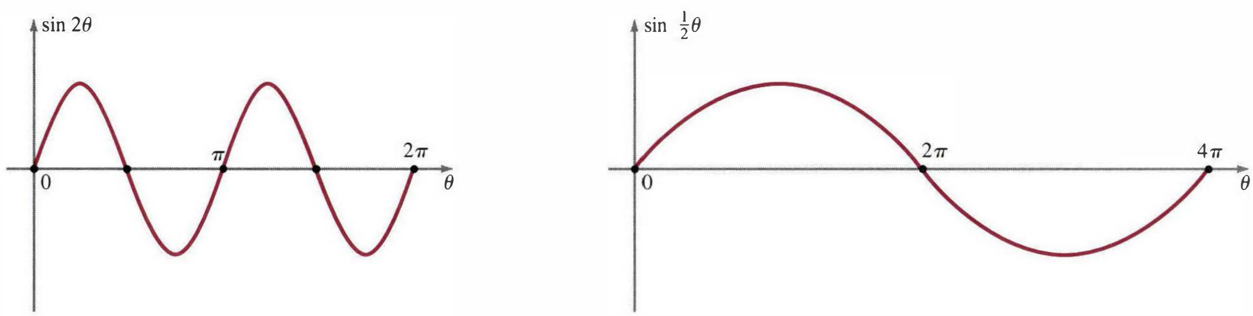


Figure 1.49

way we see that $\sin\frac{1}{2}\theta$ oscillates half as fast as $\sin\theta$ (Fig. 1.49, right). In general, both $\sin k\theta$ and $\cos k\theta$ make one complete cycle for $0 \le k\theta \le 2\pi$, or equivalently, on the interval $0 \le \theta \le 2\pi/k$.

Notice that degrees are almost entirely banished from this way of thinking about trigonometry. Trigonometric *values* can be written using degree measure or radian measure: either $\sin 30°$ or $\sin\pi/6$; either $\cos 90°$ or $\cos\pi/2$. But whenever we think of trigonometric *functions*, as in writing $y = \sin\theta$ or $f(\theta) = \cos\theta$, the independent variable θ is always understood to be in radians.

The functions $\sin\theta$ and $\cos\theta$ are the basic trigonometric functions, but there are four others that are also important though less fundamental: the tangent, cotangent, secant, and cosecant. These can be defined as follows:

$$\tan\theta = \frac{\sin\theta}{\cos\theta},$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta},$$

$$\sec\theta = \frac{1}{\cos\theta},$$

$$\csc\theta = \frac{1}{\sin\theta}.$$

Even though we mention them here, these four functions will not be essential for our work until we reach Chapter 10. At that time we will review them thoroughly.

PROBLEMS

1 Convert the given angle from degrees to radians:
(a) 15°; (b) 150°;
(c) 1500°; (d) −36°;
(e) −110°; (f) 7°.

2 Convert the given angle from radians to degrees:
(a) $\pi/15$; (b) $\pi/45$;
(c) $-\pi/36$; (d) -3;
(e) π^2; (f) 30.

3 Find the value of the given expression without using tables or a calculator:
(a) $\cos(-120°)$; (b) $\sin 780°$;
(c) $\sin\dfrac{17\pi}{3}$; (d) $\cos\left(-\dfrac{15\pi}{4}\right)$;
(e) $\sin\dfrac{19\pi}{6}$; (f) $\cos\dfrac{99\pi}{4}$.

4 Is the given number positive, negative, or zero?
(a) $\sin 500\pi$; (b) $\cos 7$;
(c) $\sin 901°$; (d) $\cos 2^4$.

5 Verify the given identities:
(a) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$ (Hint: $3\theta = 2\theta + \theta$);
(b) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

6 By sketching the angles in a unit circle and using the facts that $\sin\pi/6 = \frac{1}{2}$, $\cos\pi/6 = \sqrt{3}/2$, find
(a) $\sin\left(-\dfrac{\pi}{6}\right)$; (b) $\sin\dfrac{7\pi}{6}$;
(c) $\sin\dfrac{13\pi}{6}$; (d) $\cos\left(-\dfrac{\pi}{6}\right)$;
(e) $\cos\dfrac{7\pi}{6}$; (f) $\cos\dfrac{13\pi}{6}$.

7 Express each trigonometric function as a corresponding function of an angle in the first quadrant ($0 \le \theta \le \pi/2$) preceded by a + or − sign:
(a) $\sin\dfrac{9\pi}{2}$; (b) $\sin 7\pi$;
(c) $\sin\left(-\dfrac{7\pi}{3}\right)$; (d) $\sin\left(-\dfrac{8\pi}{3}\right)$;
(e) $\cos 10\pi$; (f) $\cos\dfrac{9\pi}{4}$;
(g) $\cos\left(-\dfrac{6\pi}{5}\right)$; (h) $\sin\left(-\dfrac{11\pi}{2}\right)$;
(i) $\cos\dfrac{11\pi}{3}$.

8 Replace ϕ by $-\phi$ in the addition formulas (3) and (4), and use the identities (1), to obtain the *subtraction formulas*:

$$\sin(\theta - \phi) = \sin\theta\cos\phi - \cos\theta\sin\phi,$$

$$\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi.$$

9 By examining Fig. 1.50, obtain the following identities:
(a) $\sin(\pi - \theta) = \sin\theta$, $\cos(\pi - \theta) = -\cos\theta$;
(b) $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$, $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$.
Use similar arguments—based on appropriate pictures—to obtain identities (c) and (d):
(c) $\sin(\theta + \pi) = -\sin\theta$, $\cos(\theta + \pi) = -\cos\theta$;
(d) $\sin\left(\theta + \frac{\pi}{2}\right) = \cos\theta$, $\cos\left(\theta + \frac{\pi}{2}\right) = -\sin\theta$.

10 Derive the identities in Problem 9 as special cases of the addition and subtraction formulas.

11 The half-angle formulas (7) and (8) are called by this name because if we set $2\theta = \alpha$, they can be written as

$$\sin\tfrac{1}{2}\alpha = \pm\sqrt{\frac{1 - \cos\alpha}{2}},$$

$$\cos\tfrac{1}{2}\alpha = \pm\sqrt{\frac{1 + \cos\alpha}{2}}.$$

Use these formulas to find the values of $\sin 15°$ and $\cos 15°$.

12 Apply the formulas in Problem 11 to find the values of $\sin 30°$ and $\cos 30°$ from the fact that $\cos 60° = \frac{1}{2}$.

13 Apply the half-angle formula for the cosine to find
(a) $\cos\frac{\pi}{4}$; (b) $\cos\frac{3\pi}{4}$.

14 Apply the half-angle formula for the sine to find
(a) $\sin\frac{\pi}{4}$; (b) $\sin\left(-\frac{\pi}{2}\right)$.

15 Use the appropriate addition or subtraction formula to find
(a) $\sin\frac{2\pi}{3}$ from $\frac{2\pi}{3} = \pi - \frac{\pi}{3}$;
(b) $\cos\frac{5\pi}{4}$ from $\frac{5\pi}{4} = \pi + \frac{\pi}{4}$;
(c) $\sin\frac{17\pi}{6}$ from $\frac{17\pi}{6} = 3\pi - \frac{\pi}{6}$.

16 Use the method of the preceding problem to find
(a) $\cos\frac{19\pi}{6}$; (b) $\cos\frac{10\pi}{3}$; (c) $\sin\frac{11\pi}{6}$.

17 Check the identity for $\sin(\theta + \phi)$ when
(a) $\theta = \frac{\pi}{6}$ and $\phi = \frac{\pi}{3}$;
(b) $\theta = \frac{\pi}{4}$ and $\phi = \frac{\pi}{4}$.

18 Check the identity for $\cos(\theta + \phi)$ when
(a) $\theta = \frac{\pi}{6}$ and $\phi = \frac{\pi}{3}$;
(b) $\theta = \frac{\pi}{4}$ and $\phi = \frac{\pi}{4}$.

19 Find $\sin 5\pi/12$ by using the fact that $5\pi/12 = \pi/4 + \pi/6$.

20 Find $\sin \pi/12$ by using the fact that $\pi/12 = \pi/4 - \pi/6$. Reconcile your answer here with the first answer in Problem 11.

21 Establish the addition formula (4) for the cosine by the method suggested in the text, that is, by using Fig. 1.44.

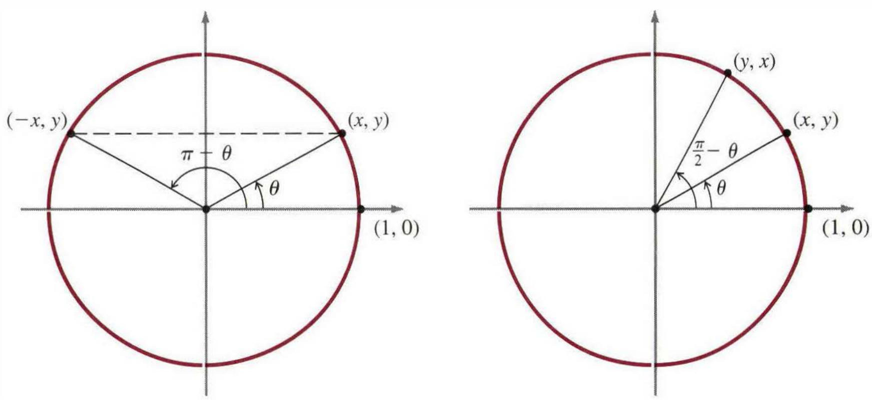


Figure 1.50

CHAPTER 1 REVIEW: DEFINITIONS, CONCEPTS, METHODS

Define, state, or think through the following.

1 Rational and irrational numbers.
2 Real line.
3 Rules for inequalities.
4 Absolute value of a number.
5 Closed and open intervals.
6 Coordinate plane.
7 Pythagorean theorem.

8 Distance formula.
9 Midpoint formulas.
10 Slope of a straight line.
11 Point-slope equation of a line.
12 Slope-intercept equation of a line.
13 Slope criterion for parallel lines.
14 Slope criterion for perpendicular lines.
15 Equation of a circle.
16 Completing the square.
17 Definition of a parabola.
18 Equations of parabolas.
19 Function.
20 Domain and range of a function.
21 Independent and dependent variables.
22 Polynomials.
23 Rational functions.
24 Algebraic functions.
25 Transcendental functions.
26 Graph of a function.
27 Zero of a function.
28 Asymptote of a curve.
29 Infinite discontinuity of a function.
30 Radian measure.
31 Sine and cosine of θ.
32 Addition and subtraction formulas.
33 Values and graphs of $\sin\theta$ and $\cos\theta$.
34 Double-angle formulas.
35 Half-angle formulas.

ADDITIONAL PROBLEMS FOR CHAPTER 1

SECTION 1.2

1 If a and b are positive numbers, prove the inequality $\sqrt{ab} \le \frac{1}{2}(a+b)$ as Euclid did, by considering a right triangle inscribed in a semicircle (Fig. 1.51).

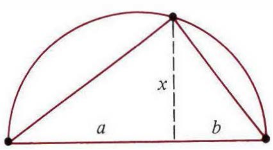


Figure 1.51

2 If a and b are any two numbers, denote the larger by max (a, b) and the smaller by min (a, b). Show that

$$\max(a, b) = \tfrac{1}{2}(a + b + |a - b|),$$

and find a similar expression for min (a, b).

3 Show that if $a \le b$ and $c \le d$, then $a + c \le b + d$. Use this fact to prove that $|a + b| \le |a| + |b|$. Hint: Begin by noticing that $-|a| \le a \le |a|$ and $-|b| \le b \le |b|$.

4 If a is a positive rational number, explain why the following method for calculating the square root of a works. First, choose a rational number which is a reasonable guess at the value of $\sqrt{a}$, and call this initial approximation x_1. Next, divide a by x_1 and average the result with x_1, thereby obtaining a second approximation x_2. Next, divide a by x_2 and average the result with x_2, obtaining a third approximation x_3. This procedure is expressed by the formula

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right), \qquad n = 1, 2, 3, \ldots .$$

Hint: If x_1 is reasonably close to $\sqrt{a}$ but different from it, then $\sqrt{a}$ lies between x_1 and a/x_1 (why?), and so the average of x_1 and a/x_1 is likely to be even closer to $\sqrt{a}$; also note that

$$x_{n+1} - \sqrt{a} = \frac{1}{2}\left(x_n - 2\sqrt{a} + \frac{a}{x_n}\right) = \frac{1}{2x_n}(x_n - \sqrt{a})^2.$$

5 Use the method of Problem 4 to calculate $\sqrt{2}$, first with $x_1 = 1$ and then with $x_1 = \frac{3}{2}$.

6 Use the method of Problem 4 to calculate $\sqrt{3}$, first with $x_1 = 2$ and then with $x_1 = \frac{3}{2}$.

7 If a and b are real numbers with $a < b$, show that there exists at least one rational number c such that $a < c < b$, and hence infinitely many. In particular, between any two irrationals there exist an infinite number of rationals.

8 If a is a nonzero rational number and b is irrational, show that $a + b$, $a - b$, ab, a/b, and b/a are all irrational.

9 If a and b are irrational, is $a + b$ necessarily irrational? Is ab?

10 If a and b are real numbers with $a < b$, show that there exists at least one irrational number c such that $a < c < b$, and hence infinitely many. In particular, between any two rationals there exist an infinite number of irrationals.

11 Give another proof of the Pythagorean theorem by using the equations

$$\frac{a}{c} = \frac{e}{a} \quad \text{and} \quad \frac{b}{c} = \frac{d}{b},$$

obtained from similar triangles in Fig. 1.52.*

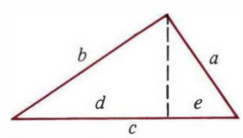


Figure 1.52

*Further proofs can be found in Section B.1 of Simmons, *Calculus Gems* (McGraw-Hill, 1992).

12 In each case place the figure in a convenient position relative to the coordinate system and prove the statement algebraically:
(a) The sum of the squares of the distances of any point from two opposite vertices of a rectangle equals the sum of the squares of its distances from the other two vertices.
(b) In any triangle, 4 times the sum of the squares of the medians equals 3 times the sum of the squares of the sides.

13 If $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ are distinct points, and if $P = (x, y)$ is located on the segment joining them in such a position that the ratio of its distance from P_1 to its distance from P_2 is q/p, show that

$$x = \frac{px_1 + qx_2}{p+q} \quad \text{and} \quad y = \frac{py_1 + qy_2}{p+q}.$$

14 Find the point on the segment joining (1, 2) and (5, 9) that is $\frac{11}{17}$ of the way from the first point to the second.

SECTION 1.3

15 If the line determined by two distinct points (x_1, y_1) and (x_2, y_2) is not vertical, and therefore has slope $(y_2 - y_1)/(x_2 - x_1)$, show that the point-slope form of its equation is the same regardless of which point is used as the given point.

16 Determine what each of the following statements implies about the constants A, B, C in the equation $Ax + By + C = 0$:
(a) The line goes through the origin.
(b) The line is parallel to the y-axis.
(c) The line is perpendicular to the y-axis.
(d) The line goes through (1, 1).
(e) The line is parallel to $5x + 3y = 2$.
(f) The line is perpendicular to $x + 10y = 3$.

17 If the lines $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$ are not parallel and k is any constant, show that

$$(A_1x + B_1y + C_1) + k(A_2x + B_2y + C_2) = 0$$

is a line through the point of intersection of the given lines. When k is assigned various values, this equation represents various members of the family of all lines through the point of intersection.

18 Given the lines $x + 3y - 2 = 0$ and $2x - y + 4 = 0$, use Problem 17 to find the equation of the line through their point of intersection which
(a) passes through $(-2, 1)$;
(b) is perpendicular to the line $3y + x = 21$;
(c) passes through the origin.

19 The points (0, 0), $(a, 0)$, and (b, c) are the vertices of an arbitrary triangle which is placed in a convenient position relative to the coordinate system.
(a) Find the equation of the line through each vertex perpendicular to the opposite side, and show algebraically that these three lines intersect at a single point.
(b) Find the equation of the perpendicular bisector of each side, and show algebraically that these three lines intersect at a single point. Why is this fact geometrically obvious?
(c) Find the equation of the line through each vertex and the midpoint of the opposite side, and show algebraically that these three lines intersect at a single point. Also, verify that this point is two-thirds of the way from each vertex to the midpoint of the opposite side.

20 Show that each of the following is the equation of a straight line:
(a) $x^3 - x^2y - 2x^2 + 3x - 3y - 6 = 0$.
(b) $3xy^2 + 5y^2 - y^3 - 4y + 12x + 20 = 0$.

21 Show that the distance from a point (x_0, y_0) to a line $Ax + By + C = 0$ is given by

$$\frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}.$$

22 Find the distance between the parallel lines $4x + 3y + 12 = 0$ and $4x + 3y - 38 = 0$.

23 If two intersecting straight lines are given, then it is easy to see that the bisectors of the angles formed by these lines are two other straight lines whose points are equidistant from the given lines. Use this fact to find the equations of the bisectors of the angles formed by the lines
(a) $3x + 4y - 10 = 0$ and $4x - 3y - 5 = 0$;
(b) $y = 0$ and $y = x$.

24 Why is it geometrically obvious (without calculation) that the bisectors of the angles of any triangle intersect at a single point?

SECTION 1.4

25 Find the values of b for which the line $y = 3x + b$ intersects the circle $x^2 + y^2 = 4$.

26 If the line $y = mx + b$ is tangent to the circle $x^2 + y^2 = r^2$, find an equation relating m, b, and r.

27 Find the equation of the locus of a point $P = (x, y)$ that moves in such a way that
(a) its distance from (0, 0) is twice its distance from $(a, 0)$;
(b) the product of its distances from $(a, 0)$ and $(-a, 0)$ is a^2 (this curve is called a *lemniscate*).
In each case, sketch the graph.

28 A line segment of length 6 moves in such a way that its endpoints remain on the x-axis and y-axis. What is the equation of the locus of its midpoint?

29 A point moves in such a way that the ratio of its distances from two fixed points is a constant $k \neq 1$. Show that the locus is a circle.

30 Find the equation of the line which is tangent to the circle $x^2 + y^2 + 8x + 6y + 8 = 0$ at the point $(-8, -2)$.

31 Find the equations of the lines that pass through the point $(1, 3)$ and are tangent to the circle $x^2 + y^2 = 2$.

32 If two circles

$$x^2 + y^2 + A_1x + B_1y + C_1 = 0$$

and

$$x^2 + y^2 + A_2x + B_2y + C_2 = 0$$

intersect in two points, and if k is a constant $\neq -1$, explain why

$$(x^2 + y^2 + A_1x + B_1y + C_1) + k(x^2 + y^2 + A_2x + B_2y + C_2) = 0$$

is the equation of a circle through the points of intersection. If $k = -1$, what does the equation represent?

33 Use Problem 32 to find the equation of the line joining the points of intersection of the circles $x^2 + y^2 = 4x + 4y - 4$ and $x^2 + y^2 = 2y$. Also find these points of intersection.

34 Show that a parabola with focus at the origin, axis the x-axis, and opening to the right has an equation of the form $y^2 = 4p(x + p)$, where $p > 0$.

35 Find the equation of the parabola with focus $(1, 1)$ and directrix $x + y = 0$, and simplify this equation to a form without radicals. Hint: See Problem 21.

36 Let the vertex of the parabola $x^2 = 4py$ be joined to every other point of the parabola. Show that the midpoints of the resulting chords lie on another parabola. Find the focus and directrix of this second parabola.

37 Consider all chords with given slope m that have endpoints on the parabola $x^2 = 4py$. Prove that the locus of the midpoints of these chords is a straight line parallel to the y-axis.

38 A *focal chord* of a parabola is the segment cut by the parabola from a straight line through the focus.

(a) If A and B are the endpoints of a focal chord, and if the line through A and the vertex intersects the directrix at a point C, show that the line through B and C is parallel to the axis of the parabola.

(b) Show that the length of a focal chord is twice the distance from its midpoint to the directrix.

(c) Show that if the two tangents to a parabola are drawn from any point on the directrix, then the points of tangency are the endpoints of a focal chord.

39 Given the two points $A = (4p, 0)$ and $B = (4p, 4p)$, divide the segments OA and AB into equal numbers of equal parts, number the points of division as shown in Fig. 1.53, and join the points of division on AB to the origin by straight lines. Show that the points of intersection of each of these lines with the corresponding vertical lines lie on the parabola $x^2 = 4py$.

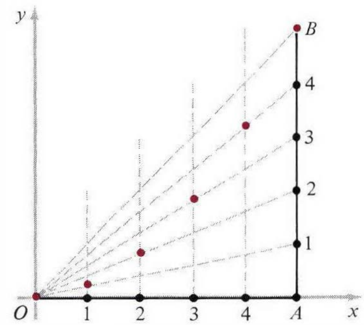


Figure 1.53

SECTION 1.5

40 Find the domain of each of the following functions:

(a) $5 - x$; (b) $\dfrac{x}{2x-3}$;

(c) $\sqrt{3x-2}$; (d) $\sqrt{5-3x}$;

(e) $\dfrac{x+7}{x^2-9}$; (f) $\sqrt[3]{x}$;

(g) $\sqrt{9-4x^2}$; (h) $\dfrac{1}{\sqrt{x+3}}$;

(i) $\sqrt{7x^2+5}$.

41 If $f(x) = ax + b$, show that

$$f\left(\frac{x_1 + x_2}{2}\right) = \frac{f(x_1) + f(x_2)}{2}.$$

Is this true for $f(x) = x^2$?

42 If $f(x) = (1 + x)/(1 - x)$, find

(a) $f(-x)$; (b) $f\left(\dfrac{1}{x}\right)$;

(c) $f\left(\dfrac{1}{1-x}\right)$; (d) $f(f(x))$.

43 If $f(x) = \sqrt[3]{x}$, what function $g(x)$ has the property that $g(f(x)) = x$?

44 The perimeter of a right triangle is 6 and its hypotenuse is x. Express the area as a function of x.

45 A cylinder has fixed total surface area A. Express its volume as a function of the radius r of its base.

46 A cone is inscribed in a sphere with fixed radius a. If r is the radius of the base of the cone, express its volume as a function of r.

47 A cone is circumscribed about a sphere with fixed radius a. If r is the radius of the base of the cone, express its volume as a function of r.

48 If $f(x) = (x - 3)/(x + 1)$, show that $f(f(f(x))) = x$.

49 Let a, b, c, d be given constants with the property that $ad - bc \neq 0$. If $f(x) = (ax + b)/(cx + d)$, show that there exists a function $g(x) = (\alpha x + \beta)/(\gamma x + \delta)$ such that $f(g(x)) = x$. Also show that for these two functions it is true that $f(g(x)) = g(f(x))$.

SECTION 1.6

50 Let $p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ be a polynomial of degree $n \geq 1$, and prove the following statements:

(a) If $p(0) = 0$, then $p(x) = xq(x)$, where $q(x)$ is a polynomial of degree $n - 1$.

(b) If a is any real number, the function $f(x)$ defined by $f(x) = p(x + a)$ is a polynomial of degree n.

(c) If a is a real number for which $p(a) = 0$, that is, if a is a zero of $p(x)$, then $p(x) = (x - a)r(x)$, where $r(x)$ is a polynomial of degree $n - 1$. Hint: Consider $f(x) = p(x + a)$.

(d) $p(x)$ has at most n zeros.

51 If n is any integer ≥ 1, show that there exists a polynomial of degree n with n zeros. If n is even, find a polynomial of degree n with no zeros; and if n is odd, find one with only one zero.

52 Let $p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ be a polynomial of degree $n \geq 1$. If $p(x)$ has n zeros $x_1, x_2, \ldots, x_n$, and is therefore expressible in the form

$$p(x) = a_n(x - x_1)(x - x_2) \cdots (x - x_n),$$

show that

(a) $x_1x_2 \cdots x_n = (-1)^n \dfrac{a_0}{a_n}$;

(b) $x_1 + x_2 + \cdots + x_n = -\dfrac{a_{n-1}}{a_n}$.

53 A function f is said to be *even* if $f(-x) = f(x)$ for every x in its domain, and it is said to be *odd* if $f(-x) = -f(x)$ for every x in its domain (in each case, it is understood that $-x$ is in the domain of f whenever x is). Determine whether each of the following functions is even, odd, or neither:

(a) $f(x) = x^3$; (b) $f(x) = x(x^3 + x)$;

(c) $f(x) = |x|$; (d) $f(x) = x + \dfrac{1}{x}$;

(e) $f(x) = x^2 + \dfrac{1}{x}$; (f) $f(x) = \dfrac{x^3 + x}{x^2 + 1}$;

(g) $f(x) = x^5 + 1$; (h) $f(x) = x(x + 1)$.

54 What is the distinguishing feature of the graph of an even function? Of an odd function?

55 What can be said about

(a) the product of two even functions?

(b) the product of two odd functions?

(c) the product of an even function and an odd function?

56 If $f(x)$ is an arbitrary function defined on an interval of the form $[-a, a]$, show that $f(x)$ is expressible in one and only one way as the sum of an even function $g(x)$ and an odd function $h(x)$, $f(x) = g(x) + h(x)$. Hint: $f(-x) = g(x) - h(x)$.

57 Write down a second-degree polynomial whose values at 1, 2, and 3 are π, $\sqrt{3}$, and 550.

58 If a and b are positive constants, sketch the graph of

$$y = \frac{b}{2a}(|x + a| + |x - a| - 2|x|).$$

59 The symbol $[x]$ (read "bracket x") is often used to denote the greatest integer $\leq$ a real number x. For example, $[1] = 1$, $[2.1] = 2$, $[\pi] = 3$, and $[-1.7] = -2$. Sketch the graphs of the following functions:

(a) $y = [x]$;

(b) $y = x - [x]$;

(c) $y = \sqrt{x - [x]}$;

(d) $y = [x] + \sqrt{x - [x]}$;

(e) $y = \sqrt{x} - [\sqrt{x}]$, $0 \leq x \leq 9$.

60 Express the number of squares $\leq$ a positive number x in terms of the bracket function defined in Problem 59. Do the same for the number of cubes $\leq x$.

61 If the symbol $\{x\}$ (read "brace x") denotes the distance from a real number x to the nearest integer, graph the following functions:

(a) $y = \{x\}$; (b) $y = \{2x\}$;

(c) $y = \{4x\}$; (d) $y = \frac{1}{4}\{4x\}$.

SECTION 1.7

62 Verify the given identities:

(a) $\sin 4\theta = 8 \sin\theta \cos^3\theta - 4 \sin\theta \cos\theta$;

(b) $\cos 4\theta = 8 \cos^4\theta - 8 \cos^2\theta + 1$.

63 Prove that

$$\sin m\theta \sin n\theta = \tfrac{1}{2}[\cos(m - n)\theta - \cos(m + n)\theta],$$

$$\sin m\theta \cos n\theta = \tfrac{1}{2}[\sin(m + n)\theta + \sin(m - n)\theta],$$

$$\cos m\theta \cos n\theta = \tfrac{1}{2}[\cos(m + n)\theta + \cos(m - n)\theta].$$

64 Given a triangle with angles A, B, C and sides a, b, c opposite these angles, prove the *law of sines*:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

65 Show that the area of the triangle in the preceding problem is

$$\tfrac{1}{2}ab \sin C = \tfrac{1}{2}ac \sin B = \tfrac{1}{2}bc \sin A.$$

66 What is the area of an equilateral triangle whose side has length s?

67 What is the area of an isosceles right triangle whose hypotenuse has length h?

68 Prove that the sine of an angle inscribed in a circle of unit diameter equals the length of the chord of the subtended arc. Hint: First consider the case in which one side of the angle is a diameter and use the fact that the resulting triangle is a right triangle; then use the fact that all inscribed angles subtending the same arc are equal. Deduce both of these facts from the theorem that any inscribed angle is half the corresponding central angle, and prove this theorem.

2 THE DERIVATIVE OF A FUNCTION

2.1 WHAT IS CALCULUS? THE PROBLEM OF TANGENTS

We begin our study of calculus with a brief statement of what the subject is about and why it is important. Such a bird's-eye view of the road that lies ahead can help us attain a clarity of purpose and sense of direction that will serve us well among the many technical details that constitute the bulk of our work.

Calculus is usually divided into two main parts, called *differential calculus* and *integral calculus*. Each of these parts has its own unfamiliar terminology, puzzling notation, and specialized computational methods. Getting accustomed to all this takes time and practice, much like the process of learning a new language. Nevertheless, this fact should not prevent us from seeing at the beginning that the central problems of the subject are really quite simple and clear, with nothing strange or mysterious about them.

Almost all the ideas and applications of calculus revolve around two geometric problems that are very easy to understand. Both problems refer to the graph of a function $y = f(x)$. We avoid complications by assuming that this graph lies entirely above the x-axis, as shown in Fig. 2.1.

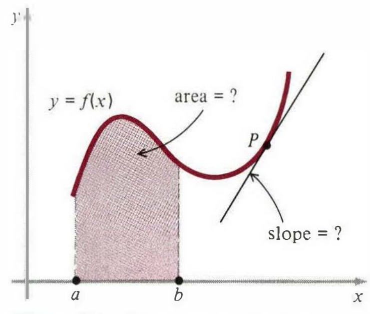


Figure 2.1 The essence of calculus.

PROBLEM 1 The basic problem of differential calculus is the *problem of tangents:* Calculate the slope of the tangent line to the graph at a given point P.

PROBLEM 2 The basic problem of integral calculus is the *problem of areas:* Calculate the area under the graph between the points $x = a$ and $x = b$.

Our work in the rest of this book will be focused on these two problems, on the ideas and techniques that have been developed for solving them, and on the applications that arise from them.*

At first sight these problems seem rather limited in scope. We expect them to shed significant light on geometry, and they do. What is very surprising is to find

*For readers who are interested in the origins of words, a *calculus* in ancient Rome was a small stone or pebble used in counting and gambling, and the Latin verb *calculare* came to mean "to figure out," "to compute," "to calculate." Today a calculus is a method or system of methods for solving quantitative problems of a particular kind, as in calculus of probabilities, calculus of finite differences, tensor calculus, calculus of variations, calculus of residues, etc. Our calculus—the branch of mathematics consisting of differential and integral calculus taken together—is sometimes called *the* calculus to distinguish it from all these other subordinate calculuses.

that they also have many profound and far-reaching applications to the various sciences. Calculus pays its way in the great world outside of mathematics through these scientific applications, and one of our major purposes is to introduce the student to as wide a variety of them as possible. At the same time we will continue to emphasize geometry and geometric applications, for this is the context in which the ideas of calculus are most easily understood.

It is sometimes said that calculus was "invented" by those two great geniuses of the late seventeenth century, Newton and Leibniz.* In reality, calculus is the product of a long evolutionary process that began in ancient Greece and continued into the nineteenth century. Newton and Leibniz were indeed remarkable men, and their contributions were of decisive importance, but the subject neither started nor ended with them. The problems stated above were much on the minds of many European scientists of the middle seventeenth century—most notably Fermat—and considerable progress was made on each of them by ingenious special methods. It was the great achievement of Newton and Leibniz to recognize and exploit the close connection between these problems, which no one else had fully understood. Specifically, they were the first to grasp the significance of the *Fundamental Theorem of Calculus,* which says, in effect, that the solution of the tangent problem can be used to solve the area problem. This theorem—certainly the most important in the whole of mathematics—was discovered by each man independently of the other, and they and their successors used it to weld the two halves of the subject together into a problem-solving art of astonishing power and versatility.

As these remarks suggest, we begin our work by undertaking a fairly thorough study of the tangent problem in the next four chapters. Then, in Chapters 6 and 7, we turn to the area problem. From there we push outward in a number of directions, extending our basic concepts and tools to broader classes of functions with a greater variety of significant applications.

Before attempting to calculate the slope of a tangent line, we must first decide what a tangent line *is,* and this is not as easy as it seems.

In the case of a circle there is no difficulty. A tangent to a circle (Fig. 2.2, left) is a line that intersects the circle at only one point, called the point of tangency; lines that are not tangents either intersect the circle at two different points or miss it altogether. This situation reflects the clear intuitive idea most of us have that a tangent to a curve at a given point is a line that "just touches" the curve at that

*The Latin spelling "Leibnitz" is sometimes used in order to suggest the correct pronunciation, which is "LIBE-nits."

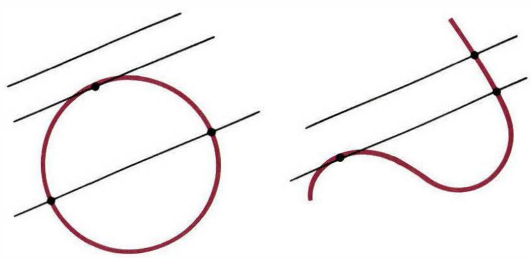


Figure 2.2

point.* It also suggests the possibility of defining a tangent to a curve as a line that intersects the curve at only one point. This definition was used successfully by the Greeks in dealing with circles and a few other special curves, but for curves in general it is wholly unsatisfactory. To understand why, consider the curve shown on the right in Fig. 2.2: It has a perfectly acceptable tangent (the lower line) that this definition would reject, and an obvious nontangent (the upper line) that this definition would accept.

The modern concept of a tangent line was originated by Fermat around 1630. As students will come to see, this concept is not only a reasonable statement about the geometric nature of tangents, it is also the key to a practical process for the construction of tangents.

Briefly, the idea is this: Consider a curve $y = f(x)$, and let P be a given fixed point on this curve (Fig. 2.3). Let Q be a second nearby point on the curve, and draw the secant line PQ. The tangent line at P can now be thought of as the limiting position of the variable secant as Q slides along the curve toward P. We shall see in Section 2.2 how this qualitative idea leads at once to a quantitative method for calculating the exact slope of the tangent in terms of the given function $f(x)$.

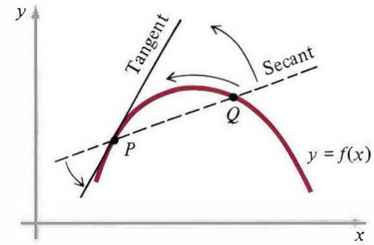


Figure 2.3 Fermat's idea.

Let there be no misunderstanding. This way of thinking about tangents is not a minor technical point in the geometry of curves. On the contrary, it is one of the three or four most fruitful ideas that any mathematician has ever had, for without it there would have been no concept of velocity or acceleration or force in physics, no Newtonian dynamics or astronomy, no physical science of any kind except as the mere verbal description of phenomena, and certainly no modern age of engineering and technology.

2.2 HOW TO CALCULATE THE SLOPE OF THE TANGENT

General discussions have their place, but the time has come to get down to details.

Let $P = (x_0, y_0)$ be an arbitrary fixed point on the parabola $y = x^2$, as shown in Fig. 2.4. As our first illustration of the basic idea of this chapter, we calculate the slope of the tangent to this parabola at the given point P. To begin the process, we choose a second nearby point $Q = (x_1, y_1)$ on the curve. Next, we draw the

*The Latin word *tangere* means "to touch," as in the English word "tangible."

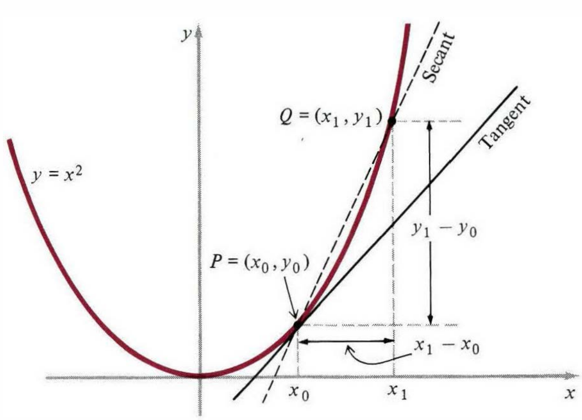


Figure 2.4

secant line PQ which is determined by these two points. The slope of this secant is evidently

$$m_{\text{sec}} = \text{slope of } PQ = \frac{y_1 - y_0}{x_1 - x_0}. \tag{1}$$

Now for the crucial step: We let x_1 approach x_0, so that the variable point Q approaches the fixed point P by sliding along the curve—much like a bead sliding along a curved wire. As this happens, the secant changes direction and visibly approaches the tangent at P as its limiting position. Also, it is intuitively clear that the slope m of the tangent is the limiting value approached by the slope m_{sec} of the secant. If we use the standard symbol $\rightarrow$ to mean "approaches," then the last statement can be expressed in the concise and convenient form

$$m = \lim_{Q \to P} m_{\text{sec}} = \lim_{x_1 \to x_0} \frac{y_1 - y_0}{x_1 - x_0}. \tag{2}$$

The abbreviation "lim," with "$x_1 \rightarrow x_0$" written below it, is read "the limit, as x_1 approaches x_0, of . . . ."

We cannot calculate the limiting value m in (2) by simply setting $x_1 = x_0$, because then $y_1 = y_0$ and this would give the meaningless result

$$m = \frac{y_0 - y_0}{x_0 - x_0} = \frac{0}{0}.$$

We must think of x_1 as coming very close to x_0 *but remaining distinct from it*. However, as this happens, both $y_1 - y_0$ and $x_1 - x_0$ become arbitrarily small, and it isn't at all clear what limiting value their quotient approaches.

The way out of this difficulty is to use the equation of the curve. Since P and Q both lie on the curve, we have $y_0 = {x_0}^2$ and $y_1 = {x_1}^2$, so (1) can be written

$$m_{\text{sec}} = \frac{y_1 - y_0}{x_1 - x_0} = \frac{{x_1}^2 - {x_0}^2}{x_1 - x_0}. \tag{3}$$

The reason this numerator becomes small is that it contains the denominator $x_1 - x_0$ as a factor. If this common factor is canceled, we obtain

$$m_{\text{sec}} = \frac{y_1 - y_0}{x_1 - x_0} = \frac{{x_1}^2 - {x_0}^2}{x_1 - x_0} = \frac{(x_1 - x_0)(x_1 + x_0)}{x_1 - x_0} = x_1 + x_0,$$

and (2) becomes

$$m = \lim_{x_1 \to x_0} \frac{y_1 - y_0}{x_1 - x_0} = \lim_{x_1 \to x_0} (x_1 + x_0).$$

It is now very easy to see what is happening: As x_1 gets closer and closer to x_0, $x_1 + x_0$ becomes more and more nearly equal to $x_0 + x_0 = 2x_0$. Accordingly,

$$m = 2x_0 \tag{4}$$

is the slope of the tangent to the curve $y = x^2$ at the point (x_0, y_0).

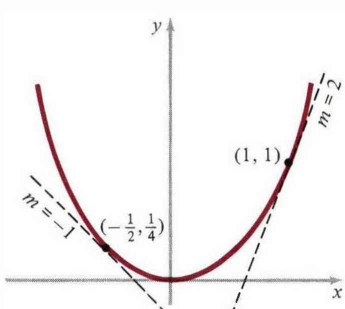


Figure 2.5

Example 1 The points $(1, 1)$ and $(-\frac{1}{2}, \frac{1}{4})$ lie on the parabola $y = x^2$ (Fig. 2.5). By formula (4), the slopes of the tangents at these points are $m = 2$ and $m = -1$. Using the point-slope form of the equation of a line, our two tangent lines clearly have equations

$$\frac{y-1}{x-1} = 2 \quad \text{and} \quad \frac{y-\frac{1}{4}}{x+\frac{1}{2}} = -1.$$

In exactly the same way,

$$\frac{y-x_0^2}{x-x_0} = 2x_0$$

is the equation of the tangent at a general point (x_0, x_0^2) on the curve.

We now introduce a widely used piece of symbolism called the *delta notation.* The procedure just described begins by changing the independent variable x from a first value x_0 to a second value x_1. The standard notation for the amount of such a change is Δx (read "delta x"), so that

$$\Delta x = x_1 - x_0 \tag{5}$$

is the change in x in going from the first value to the second. We can also think of the second value as being obtained from the first by adding the change:

$$x_1 = x_0 + \Delta x. \tag{6}$$

It is essential to understand that Δx is not the product of a number Δ and a number x, but a single number called an *increment* of x. An increment Δx can be either positive or negative. Thus, if $x_0 = 1$ and $x_1 = 3$, then $\Delta x = 3 - 1 = 2$; and if $x_0 = 1$ and $x_1 = -2$, then $\Delta x = -2 - 1 = -3$.

The letter Δ is the Greek d; when it is written in front of a variable, it signifies the difference between two values of that variable. This simple notational device turns out to be extremely convenient and has spread into almost every part of mathematics and science. We illustrate its role in our present work by using it to reformulate the above calculations.

In view of (5) and (6), formula (3) for the slope of the secant can be written in the form

$$m_{\text{sec}} = \frac{x_1^2 - x_0^2}{x_1 - x_0} = \frac{(x_0 + \Delta x)^2 - x_0^2}{\Delta x}. \tag{7}$$

This time, instead of factoring the numerator, we expand its first term and simplify the result, obtaining

$$\begin{aligned}(x_0 + \Delta x)^2 - x_0^2 &= x_0^2 + 2x_0\,\Delta x + (\Delta x)^2 - x_0^2 \\ &= 2x_0\,\Delta x + (\Delta x)^2 \\ &= \Delta x(2x_0 + \Delta x),\end{aligned}$$

so (7) becomes

$$m_{\text{sec}} = 2x_0 + \Delta x.$$

If we insert this in (2) and use the fact that $x_1 \to x_0$ is equivalent to $\Delta x \to 0$, we find that

$$m = \lim_{\Delta x \to 0} (2x_0 + \Delta x) = 2x_0,$$

as before. Again it is very easy to see what is happening in the indicated limit process: as Δx gets closer and closer to zero, $2x_0 + \Delta x$ becomes more and more nearly equal to $2x_0$.

This second method, using the delta notation, depends on expanding the square $(x_0 + \Delta x)^2$, whereas the first depends on factoring the expression $x_1^2 - x_0^2$. In this particular case neither calculation is noticeably harder than the other. In general, however, expanding is easier than factoring, and for this reason we adopt the method of increments as our standard procedure.

The calculation that we have just carried out for the parabola $y = x^2$ can be described in principle for the graph of any function $y = f(x)$ (Fig. 2.6). We first compute the slope of the secant through the two points P and Q corresponding to x_0 and $x_0 + \Delta x$,

$$m_{\text{sec}} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.$$

We then calculate the limit of m_{sec} as Δx approaches zero, obtaining a number m that we interpret geometrically as the slope of the tangent to the curve at the point P:

$$m = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.$$

The value of this limit is usually denoted by the symbol $f'(x_0)$, read "f prime of x_0," in order to emphasize its dependence on both the point x_0 and the function $f(x)$. Thus, by definition we have

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}. \tag{8}$$

In this notation, the result of the calculation given above can be expressed as follows: If $f(x) = x^2$, then $f'(x_0) = 2x_0$.

Example 2 Calculate $f'(x_0)$ if $f(x) = 2x^2 - 3x$.

Solution For this function, the numerator of the quotient in (8) is

$$\begin{aligned} f(x_0 + \Delta x) - f(x_0) &= [2(x_0 + \Delta x)^2 - 3(x_0 + \Delta x)] - [2x_0^2 - 3x_0] \\ &= 2x_0^2 + 4x_0\,\Delta x + 2(\Delta x)^2 - 3x_0 - 3\Delta x - 2x_0^2 + 3x_0 \end{aligned}$$

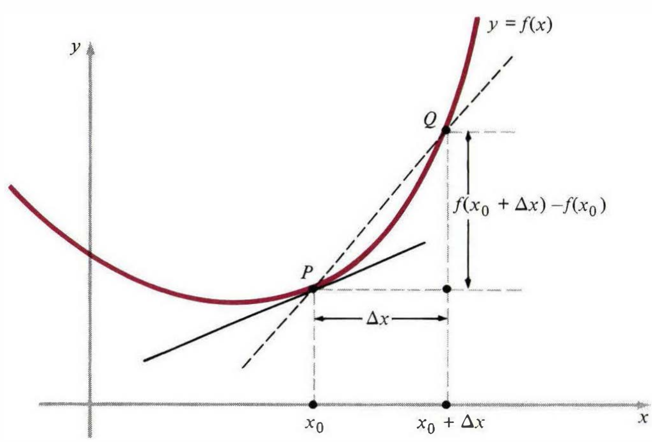


Figure 2.6

$$= 4x_0\,\Delta x + 2(\Delta x)^2 - 3\Delta x$$
$$= \Delta x(4x_0 + 2\Delta x - 3).$$

The quotient in (8) is therefore

$$\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = 4x_0 + 2\Delta x - 3,$$

and

$$f'(x_0) = \lim_{\Delta x \to 0} (4x_0 + 2\Delta x - 3)$$
$$= 4x_0 - 3.$$

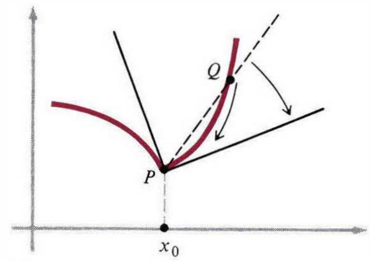


Figure 2.7

We have assumed in the remarks leading to (8) that the curve under discussion actually has a single definite tangent at the point P. This is a genuine assumption, because some curves do not have such a tangent at every point (Fig. 2.7). However, when a tangent exists, it is clearly necessary for the secant PQ to approach the same limiting position whether Q approaches P from the right or from the left. These two modes of approach correspond, respectively, to Δx approaching zero through only positive or only negative values. It is therefore part of the meaning of (8) that for this limit to exist we must have the same limiting value for both directions of approach.

PROBLEMS

1 Find the equation of the tangent to the parabola $y = x^2$
(a) at the point $(-2, 4)$;
(b) at the point where the slope is 8;
(c) if the x-intercept of the tangent is 2.

2 Show that the tangent to the parabola $y = x^2$ at a point (x_0, y_0) other than the vertex always has x-intercept $\frac{1}{2}x_0$.

3 A straight line $y = mx + b$ is presumably its own tangent line at any point. Verify this by using formula (8) to show that $f'(x_0) = m$ if $f(x) = mx + b$.

4 Sketch the graph of $y = x - x^2$ on the interval $-2 \le x \le 3$.
(a) Use the method of increments to compute the slope of the tangent line at an arbitrary point (x_0, y_0) on the curve.
(b) What are the slopes of the tangent lines at the points $(-1, -2)$, $(0, 0)$, $(1, 0)$, and $(2, -2)$ on the curve? Use these slopes to draw the tangents at these points in your sketch.
(c) At what point on the curve is the tangent horizontal?

5 Use formula (8) to calculate $f'(x_0)$ if $f(x)$ is equal to
(a) $x^2 - 4x - 5$; (b) $x^2 - 2x + 1$;
(c) $2x^2 + 1$; (d) $x^2 - 4$.
The results of these calculations will be needed in Problems 6–9.

6 Sketch the given curve and the tangent line at the given point, and find the equation of this tangent line:
(a) $y = x^2 - 4x - 5$, $(4, -5)$.
(b) $y = x^2 - 2x + 1$, $(-1, 4)$.

7 Find the equation of the tangent line to the curve $y = 2x^2 + 1$ that is parallel to the line $8x + y - 2 = 0$.

8 Find the equations of the two lines through the point $(3, 1)$ that are tangent to the curve $y = x^2 - 4$. Hint: Draw the graph, let $(a, a^2 - 4)$ be the point of tangency, and find a.

9 Prove analytically (that is, without appealing to geometric reasoning) that there is no line through the point $(1, -2)$ that is tangent to the curve $y = x^2 - 4$.

10 One of the two lines that pass through the point $(2, 0)$ and are tangent to the parabola $y = x^2$ is the x-axis. Find the equation of the other line.

11 Find equations for the two lines through the point $(3, 13)$ that are tangent to the parabola $y = 6x - x^2$.

12 Draw the graph of $y = f(x) = |x - 1|$.
(a) Is there any point on the graph at which there is no tangent line?
(b) Find $f'(x_0)$ if $x_0 > 1$. If $x_0 < 1$. What can be said about $f'(x_0)$ if $x_0 = 1$?

2.3 THE DEFINITION OF THE DERIVATIVE

If we separate formula (8) in Section 2.2 from its geometric motivation, and also drop the subscript on x_0, then we arrive at our basic definition: Given any function $f(x)$, its *derivative* $f'(x)$ is the new function whose value at a point x is defined by

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}. \tag{1}$$

In calculating this limit, x is held fixed while Δx varies and approaches zero. The indicated limit may exist for some values of x and fail to exist for other values. If the limit exists for $x = a$, then the function is said to be *differentiable at a*. A *differentiable function* is one that is differentiable at each point of its domain. Most of the specific functions considered in this book have this property.

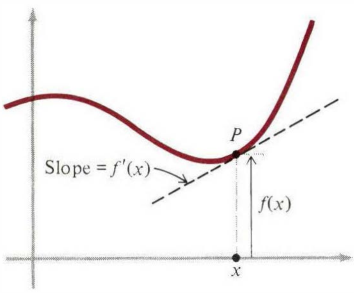


Figure 2.8

We know that the derivative $f'(x)$ can be visualized in the way suggested by Fig. 2.8, in which $f(x)$ is the variable height of a point P moving along the curve and $f'(x)$ is the variable slope of the tangent at P. Strictly speaking, however, the above definition of the derivative does not depend in any way on geometric ideas. Our thoughts about Fig. 2.8 constitute a *geometric interpretation,* and important as this may be as an aid to understanding, it is not an essential part of the concept of the derivative. In the next section we will meet other equally important interpretations that have nothing to do with geometry. We must therefore be prepared to consider $f'(x)$ purely as a function, and to recognize that it has several interpretations but no necessary connection with any one of them.

The process of actually forming the derivative $f'(x)$ is called the *differentiation* of the given function $f(x)$. This is the fundamental operation of calculus, upon which everything else depends. In principle, we merely follow the computational instructions specified in (1). These instructions can be arranged into a systematic procedure called the *three-step rule.*

STEP 1 Write down the difference $f(x + \Delta x) - f(x)$ for the particular function under consideration, and if possible simplify it to the point where Δx is a factor.

STEP 2 Divide by Δx to form the *difference quotient*

$$\frac{f(x + \Delta x) - f(x)}{\Delta x},$$

and manipulate this to prepare the way for evaluating its limit as $\Delta x \to 0$. In most of the examples and problems of the present chapter, this manipulation involves nothing more than canceling Δx from the numerator and denominator.

STEP 3 Evaluate the limit of the difference quotient as $\Delta x \to 0$. If Step 2 has accomplished its purpose, a simple inspection is usually all that is needed here.

If we remember that the innocent-looking notation $f(x)$ encompasses all conceivable functions, then we will understand that these steps are sometimes easy to carry out and sometimes very difficult. The following examples depend only on elementary algebra, but even this requires a little knowledge and ingenuity.

Example 1 Find $f'(x)$ if $f(x) = x^3$.

STEP 1:

$$\begin{aligned} f(x + \Delta x) - f(x) &= (x + \Delta x)^3 - x^3 \\ &= x^3 + 3x^2 \Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - x^3 \\ &= 3x^2 \Delta x + 3x(\Delta x)^2 + (\Delta x)^3 \\ &= \Delta x[3x^2 + 3x \Delta x + (\Delta x)^2]. \end{aligned}$$

STEP 2:

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} = 3x^2 + 3x \Delta x + (\Delta x)^2.$$

STEP 3:

$$f'(x) = \lim_{\Delta x \to 0} [3x^2 + 3x \Delta x + (\Delta x)^2] = 3x^2.$$

Example 2 Find $f'(x)$ if $f(x) = 1/x$.

STEP 1:

$$\begin{aligned} f(x + \Delta x) - f(x) &= \frac{1}{x + \Delta x} - \frac{1}{x} \\ &= \frac{x - (x + \Delta x)}{x(x + \Delta x)} = \frac{-\Delta x}{x(x + \Delta x)}. \end{aligned}$$

STEP 2:

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{-1}{x(x + \Delta x)}.$$

STEP 3:

$$f'(x) = \lim_{\Delta x \to 0} \frac{-1}{x(x + \Delta x)} = -\frac{1}{x^2}.$$

Let us briefly consider what the result of Example 2 can tell us about the graph of the function $y = f(x) = 1/x$. First, $f'(x) = -1/x^2$ is clearly negative for all $x \neq 0$, and since this is the slope of the tangent, all tangent lines point down to the right. Further, when x is near 0, $f'(x)$ is very large, which means that these tangent lines are steep; and when x is large, $f'(x)$ is small, so these tangent lines are nearly horizontal. It is instructive to verify our observations by examining Fig. 1.28. Generally speaking, derivatives are capable of telling us a great deal about the behavior of functions and the properties of their graphs, since the derivative at a point gives the slope of the tangent at that point. We explore this topic more fully in Chapter 4.

Example 3 Find $f'(x)$ if $f(x) = \sqrt{x}$.

STEP 1:

$$f(x + \Delta x) - f(x) = \sqrt{x + \Delta x} - \sqrt{x}.$$

STEP 2:

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x}.$$

This is not in a form that is convenient for canceling the Δx's, so we use an ingenious algebraic trick to remove the square roots from the numerator. We multiply both numerator and denominator of the last fraction by $\sqrt{x + \Delta x} + \sqrt{x}$, which amounts to multiplying this fraction by 1, and then we simplify the numerator by using the fact expressed in the algebraic identity $(a - b)(a + b) = a^2 - b^2$:

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} \cdot \frac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}}$$

$$= \frac{(x + \Delta x) - x}{\Delta x(\sqrt{x + \Delta x} + \sqrt{x})} = \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}}.$$

Now the next step is easy.

STEP 3:

$$f'(x) = \lim_{\Delta x \to 0} \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

REMARKS ON NOTATION

There is a slightly disconcerting feature of calculus that we might as well confront here. It is the fact that several different notations are in common use for derivatives, with preference shifting from one to another according to the circumstances in which the symbols are being used. Some may ask, What does it matter which symbols are used? The fact is that it matters a great deal, for good notations can smooth the way and do much of our work for us, while bad ones create a swamp under our feet through which easy movement is difficult.

The derivative of a function $f(x)$ has been denoted above by $f'(x)$. This notation has the merit of emphasizing that the derivative of $f(x)$ is another function of x which is associated in a certain way with the given function. If our function is given in the form $y = f(x)$, with the dependent variable displayed, then the shorter symbol y' is often used in place of $f'(x)$.

The main disadvantage of this prime notation for derivatives is that it doesn't suggest the nature of the process by which $f'(x)$ is obtained from $f(x)$. The notation devised by Leibniz for his version of calculus is better in this respect, and in other ways as well.

To explain Leibniz's notation, we begin with a function $y = f(x)$ and write the difference quotient

$$\frac{f(x + \Delta x) - f(x)}{\Delta x}$$

in the form

$$\frac{\Delta y}{\Delta x},$$

where $\Delta y = f(x + \Delta x) - f(x)$. Here Δy is not just any change in y; it is the specific change that results when the independent variable is changed from x to $x + \Delta x$. As we know, the difference quotient $\Delta y/\Delta x$ can be interpreted as the ratio of the change in y to the change in x along the curve $y = f(x)$, and this is the slope of the secant (Fig. 2.9). Leibniz wrote the limit of this difference quotient, which of course is the derivative $f'(x)$, in the form dy/dx (read "dy over dx"). In this notation, the definition of the derivative becomes

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}, \tag{2}$$

and this is the slope of the tangent in Fig. 2.9. Two slightly different equivalent forms of dy/dx are

$$\frac{df(x)}{dx} \quad \text{and} \quad \frac{d}{dx}f(x).$$

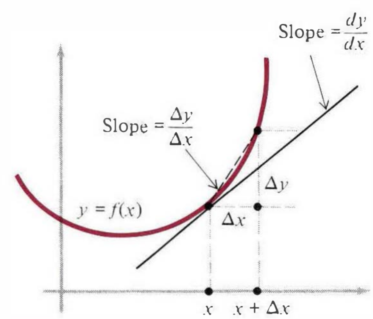


Figure 2.9

In the second of these, the symbol d/dx should be thought of as an operation which can be applied to the function $f(x)$ to yield its derivative $f'(x)$, as suggested by the equation

$$\frac{d}{dx}f(x) = f'(x).$$

The symbol d/dx can be read "the derivative with respect to x of . . ." whatever function of x follows it.

It is important to understand that dy/dx in (2) is a single indivisible symbol. In spite of the way it is written, it is *not* the quotient of two quantities dy and dx, because dy and dx have not been defined and have no independent existence. In Leibniz's notation, the formation of the limit on the right of (2) is symbolically expressed by replacing the letter Δ by the letter d. From this point of view, the symbol dy/dx for the derivative has the psychological advantage that it quickly reminds us of the whole process of forming the difference quotient $\Delta y/\Delta x$ and calculating its limit as $\Delta x \to 0$. There is also a practical advantage, for certain fundamental formulas developed in the next chapter are easier to remember and use when derivatives are written in the Leibniz notation.

But good though it is, this notation is not perfect. For instance, suppose we wish to write down the numerical value of the derivative at a specific point, say $x = 3$. Since dy/dx doesn't display the variable x in the convenient way that $f'(x)$ does, we are forced into using some such clumsy notation as

$$\left(\frac{dy}{dx}\right)_{x=3} \quad \text{or} \quad \left.\frac{dy}{dx}\right|_{x=3}.$$

The clear and concise symbol $f'(3)$ is obviously superior to these awkward expressions.

As we have seen, each of the notations described above is good in its own way. All are widely used in the literature of science and mathematics, and to help the student become thoroughly familiar with them, we shall use them freely and interchangeably from now on.

PROBLEMS

1 Use the three-step rule to show that if $f(x) = ax^2 + bx + c$, then $f'(x) = 2ax + b$.

Use the general rule in Problem 1 to write down the indicated derivatives in Problems 2–16.

2 $f(x) = 5x - 3$; find $f'(x)$.
3 $g(x) = 50 - 8x^2$; find $g'(x)$.
4 $h(x) = x(50 - x)$; find $h'(x)$.
5 $F(x) = 20 - 72x$; find $F'(x)$.
6 $G(x) = 3x^2 + 5x - 7$; find $G'(x)$.
7 $H(x) = 5 - 10x + 15x^2$; find $H'(x)$.
8 $y = 5x^2 - 9x + 15$; find dy/dx.
9 $x = 3y^2 + 7y - 6$; find dx/dy.
10 $u = 7t^2 - 11t + 109$; find du/dt.
11 $v = 7x(500 - x)$; find dv/dx.
12 $w = -7z^2 + 22z - 71$; find dw/dz.
13 $f(x) = 5x(x + 5)$; find $f'(x)$.
14 $y = 5x - (x/10)^2$; find dy/dx.
15 $g(x) = 3 - (4x + 5)^2$; find $g'(x)$.
16 $h(x) = (3x - 2)^2 - 5x$; find $h'(x)$.

In Problems 17–22, find all points on the curve $y = f(x)$ at which the tangent is horizontal.

17 $y = 6 - x^2$.
18 $y = 6x - x^2$.
19 $y = x^2 - 6x + 9$.
20 $y = x^2 + x - 5$.
21 $y = x(20 - x)$.
22 $y = x - (x/20)^2$.

In Problems 23–38, use the three-step rule to calculate $f'(x)$ if $f(x)$ is equal to the given expression.

23 $5x - x^3$. **24** $x^3 + 2x^2 - 5x$.
25 $2x^3 - 3x^2 + 6x - 5$. **26** x^4.
27 $x - \frac{1}{x}$. **28** $\frac{1}{3x + 2}$.
29 $\frac{x}{x + 1}$. **30** $\frac{5 - 2x}{13 + x}$.
31 $\frac{1}{x^2}$. **32** $\frac{1}{x^3}$.
33 $\frac{1}{x^2 + 1}$. **34** $\frac{3}{2 + x^2}$.
35 $\frac{2x}{x^2 - 1}$. **36** $\sqrt{2x}$.
37 $\sqrt{x - 1}$. **38** $2\sqrt{5 - x}$.

39 Consider the part of the curve $y = 1/x$ that lies in the first quadrant, and draw the tangent at an arbitrary point (x_0, y_0) on this curve.
(a) Show that the portion of the tangent which is cut off by the axes is bisected by the point of tangency.
(b) Find the area of the triangle formed by the axes and the tangent, and verify that this area is independent of the location of the point of tangency.

40 Find $f'(x)$ if $f(x) = x^3 - 3x$. Use this result to verify the positions of the high and low points on the curve $y = x^3 - 3x$ that are shown in Fig. 1.26. Hint: At the high and low points the tangent is horizontal.

41 Graph the function $y = f(x) = |x| + x$, and prove that this function is not differentiable at $x = 0$. Hint: In formula (1), first take Δx positive, obtaining one limiting value; then take Δx negative, obtaining a different limiting value. In a situation of this kind we say that the function has a *right derivative* and a *left derivative*, but not a derivative.

42 If $f(x) = 2x^2 - 5$, find $f'(2)$ and use it to write the equation of the tangent line to the parabola $y = 2x^2 - 5$ at the point $(2, 3)$.

43 If $g(x) = 3 - 2x^3$, find $g'(0)$ and use it to write the equation of the tangent line to the curve $y = 3 - 2x^3$ at the point $(0, 3)$.

44 If $h(x) = \sqrt{x + 5}$, find $h'(4)$ and use it to write the equation of the tangent line to the curve $y = \sqrt{x + 5}$ at the point $(4, 3)$.

45 Find the point on the graph of $y = x^2$ that is closest to the point $(0, 3)$. Hint: Draw the graph, let (a, a^2) be the point, and find a as a root of a certain cubic equation that can be solved by inspection.

2.4 VELOCITY AND RATES OF CHANGE. NEWTON AND LEIBNIZ

The concept of the derivative is closely related to the problem of computing the velocity of a moving object. It was this fact that made calculus an essential tool of thought for Newton in his efforts to uncover the principles of dynamics and understand the motions of the planets. It might appear that only students of physics would find it worthwhile to concern themselves with precise ideas about velocity. However, we shall see that these ideas provide a fairly easy introduction to the general concept of rate of change, and this concept is important in many other fields of study, including the biological and social sciences.

In this section we consider a special case of the general velocity problem: that in which the object in question can be thought of as a point moving along a straight line, so that the position of the point is determined by a single coordinate s (Fig. 2.10). The motion is fully known if we know where the moving point is at each moment, that is, if we know the position s as a function of the time t,

$$s = f(t). \tag{1}$$

The time is usually measured from some convenient initial moment $t = 0$.

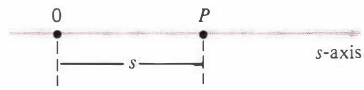


Figure 2.10

Example 1 Consider a freely falling object, say a rock dropped from the edge of a cliff 400 ft high (Fig. 2.11). It is known from many experiments that this rock falls

$$s = 16t^2 \tag{2}$$

feet in t seconds. We see that when $t = 5$, $s = 400$. The rock therefore hits the ground 5 seconds after it starts to fall, and formula (2) is valid only for $0 \leq t \leq 5$.

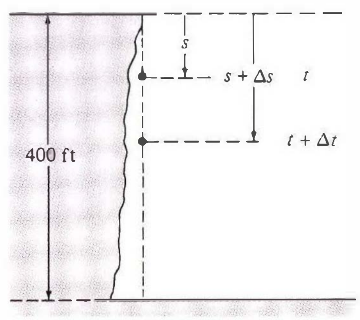


Figure 2.11

Two basic questions can be asked about the motion described in this example. First, what is meant by the velocity of the falling rock at a given instant? And second, how can this velocity be computed from (2)?

We are all familiar with the idea of velocity in its everyday sense, as a number measuring the rate at which distance is being traversed. We speak of walking 3 miles per hour (mi/h), driving 55 mi/h, and so on. We also speak of *average* velocities, and these are the numbers we usually compute. If we drive a distance of 200 mi in 5 hours, then our average velocity is 40 mi/h, because

$$\frac{\text{distance traveled}}{\text{elapsed time}} = \frac{200 \text{ mi}}{5 \text{ h}} = 40 \text{ mi/h}.$$

In general,

$$\text{average velocity} = \frac{\text{distance traveled}}{\text{elapsed time}},$$

and this is a formula almost everyone knows.

Example 1 (continued) The position function for the falling rock, $f(t) = 16t^2$, tells us that in the first second after the rock is released it falls $f(1) = 16$ ft, in the first 2 seconds $f(2) = 64$ ft, in the first 3 seconds $f(3) = 144$ ft, and so on. The average velocities during each of the first 3 seconds of fall are therefore

$$\frac{16}{1} = 16 \text{ ft/s}, \qquad \frac{64 - 16}{1} = 48 \text{ ft/s}, \qquad \text{and} \qquad \frac{144 - 64}{1} = 80 \text{ ft/s}.$$

The rock is clearly falling faster and faster from moment to moment, but we still do not know exactly how fast it is falling at any given instant.

To find the velocity v of the rock at a given instant t, we proceed as follows. In the time interval of length Δt between t and a slightly later instant $t + \Delta t$, the rock falls a distance Δs (see Fig. 2.11). The average velocity during this interval is the quotient $\Delta s/\Delta t$. When Δt is small, this average velocity is close to the exact velocity v at the beginning of the interval; that is,

$$v \cong \frac{\Delta s}{\Delta t},$$

where the symbol $\cong$ is read "is approximately equal to." Further, as Δt is made smaller and smaller, this approximation gets better and better, so we have

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}. \tag{3}$$

Our point of view here is that the velocity v is a direct intuitive concept, and (3) shows us how to compute it. However, it is also possible to regard (3) as the *definition* of the velocity, with the preceding remarks serving as motivation. The limit in (3) is clearly the derivative ds/dt, and carrying out the details we have

$$\begin{aligned} v = \frac{ds}{dt} &= \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{16(t + \Delta t)^2 - 16t^2}{\Delta t} \\ &= \lim_{\Delta t \to 0} (32t + 16\Delta t) = 32t. \end{aligned}$$

This formula tells us that the velocity of the rock after 1, 2, and 3 seconds of fall is 32, 64, and 96 ft/s, and also that the rock hits the ground at 160 ft/s. We notice that the velocity increases by 32 ft/s during each second of fall. This fact is usually expressed by saying that the acceleration of the rock is 32 feet per second per second (ft/s^2), or, in the metric system, 9.80 meters per second per second (m/s^2).

The reasoning used in this example is valid for any motion along a straight line. For the general motion (1), we therefore calculate the velocity v at time t in exactly the same way; that is, we approximate v more and more closely by the average velocity over a shorter and shorter interval of time beginning at the instant t:

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

We recognize this as the derivative of the function $s = f(t)$, so the *velocity* of a point moving on a straight line is simply the derivative of its position function,

$$v = \frac{ds}{dt} = f'(t).$$

Sometimes this is called the *instantaneous* velocity, in order to emphasize that it is calculated at an instant t. However, once this point has been made, it is customary to omit the adjective. The words "velocity" and "speed" are used interchangeably in everyday speech, but in mathematics and physics it is useful to distinguish them from one another. The *speed* of our point is defined to be the absolute value of the velocity,

$$\text{speed} = |v| = \left|\frac{ds}{dt}\right| = |f'(t)|.$$

The velocity may be positive or negative, depending on whether the point is moving along the line in the positive or negative direction; but the speed, being the magnitude of the velocity, is always positive or zero. The concept of speed is particularly useful in studies of motion along curved paths, for it tells us how fast the point is moving regardless of its direction. In our everyday experience, we learn the speed of a car at any moment by looking at the speedometer.

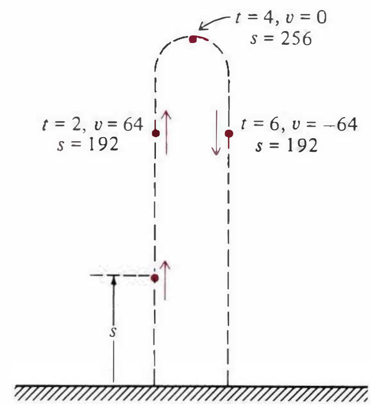


Figure 2.12

Example 2 Consider a projectile fired straight up from the ground with an initial velocity of 128 ft/s. This projectile moves up and down along a straight line. However, the two parts of its path are shown slightly separated in Fig. 2.12, for the sake of visual clarity. Let $s = f(t)$ be the height in feet of the projectile t seconds after firing. If the force of gravity were absent, the projectile would continue moving upward with a constant velocity of 128 ft/s, and we would have $s = f(t) = 128t$. However, the action of gravity causes it to slow down, stop momentarily at the top of its flight, and then fall back to earth with increasing speed. Experimental evidence suggests that the height of the projectile during its flight is given by the formula

$$s = f(t) = 128t - 16t^2. \tag{4}$$

If we write this in the factored form $s = 16t(8 - t)$, we see that $s = 0$ when $t = 0$ and when $t = 8$. The projectile therefore returns to earth 8 seconds after it starts up, and (4) is valid only for $0 \le t \le 8$.

To learn more about the nature of this motion, it is necessary to know the velocity. If the general rule for computing derivatives of second-degree polynomials is applied to (4), we find that the velocity at time t is

$$v = \frac{ds}{dt} = 128 - 32t. \tag{5}$$

At the top of its flight the projectile is momentarily at rest, and therefore $v = 0$. By (5), $t = 4$ when $v = 0$; and by (4), $s = 256$ when $t = 4$. In this way we find the maximum height reached by the projectile and the time required to reach this height (see Fig. 2.12). As t increases from 0 to 8, it is clear from (5) that v decreases from 128 ft/s to -128 ft/s; in fact, v decreases by 32 ft/s during each second of flight, and this is expressed by saying that the acceleration is -32 feet per second per second (ft/s^2). We notice explicitly that the velocity is positive from $t = 0$ to $t = 4$, when s is increasing; and it is negative from $t = 4$ to $t = 8$, when s is decreasing. In particular, it is easy to see from (5) that $v = 64$ ft/s when $t = 2$ and $v = -64$ ft/s when $t = 6$ (the speed is 64 ft/s at both times).

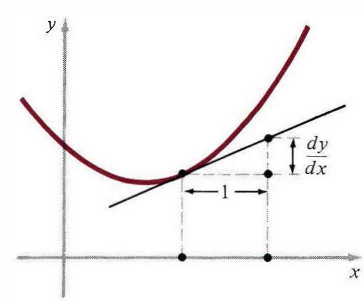


Figure 2.13

Velocity is an example of the concept of rate of change, which is basic for all the sciences. For any function $y = f(x)$, the derivative dy/dx is called the *rate of change* of y with respect to x. Intuitively, this is the change in y that would be produced by an increase of one unit in x if the rate of change remained constant (Fig. 2.13). In this terminology, velocity is simply the rate of change of position with respect to time. When time is the independent variable, we often omit the phrase "with respect to time" and speak only of the "rate of change."

Example 3 (a) We know that velocity is important in studying the motion of a point along a straight line, but the way the velocity changes is also important. By definition, the *acceleration* of a moving point is the rate of change of its velocity v,

$$a = \frac{dv}{dt}.$$

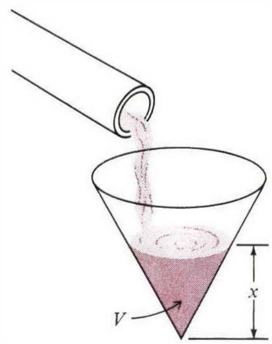


Figure 2.14

(b) Suppose that water is being pumped into the conical tank shown in Fig. 2.14 at the rate of 5 ft^3/min. If V denotes the volume of water in the tank at time t, then

$$\frac{dV}{dt} = 5.$$

The rate of change of the depth x is the derivative dx/dt, and this is not constant. It is intuitively clear that this rate of change is large when the area of the surface of the water is small, and becomes smaller as this area increases.

(c) In economics, the rate of change of a quantity Q with respect to a suitable independent variable is usually called *marginal* Q. Thus we have marginal cost, marginal revenue, marginal profit, etc. If $C(x)$ is the cost of manufacturing x pieces of a product, then the marginal cost is dC/dx. In most cases x is a large number, so 1 is small compared with x and dC/dx is approximately $C(x+1) - C(x)$. For this reason, many economists describe marginal cost as "the cost of producing one more piece." In Section 4.7 we discuss in some detail the applications of calculus to economics.

(d) We know that the area A of a circle in terms of its radius r is given by the formula $A = \pi r^2$, and the derivative of this function is easy to compute by the three-step rule:

$$\frac{dA}{dr} = 2\pi r. \tag{6}$$

This says that the rate of change of the area of a circle with respect to its radius equals its circumference. To understand the geometric reason for this remarkable fact, let Δr be an increment of the radius and ΔA the corresponding increment of the area (Fig. 2.15). It is clear that ΔA is the area of the narrow band around the circle, and this is approximately the product of the circumference $2\pi r$ and the width Δr of the band. The difference quotient $\Delta A/\Delta r$ is therefore close to $2\pi r$, and by letting $\Delta r \to 0$ we obtain (6) by geometric reasoning.

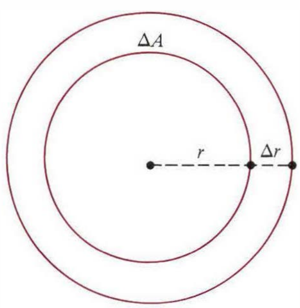
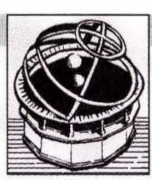


Figure 2.15

We have introduced two topics in this section: velocity, which is the rate of change of the position of a moving object, and rates of change in general, including acceleration, which is the rate of change of velocity. These are themes of major importance for calculus, and we shall return to them repeatedly throughout the rest of this book.



NOTE ON NEWTON AND LEIBNIZ

As we suggested earlier, the historical roots of calculus lie deep in antiquity—in the mind of Archimedes (287–212 B.C.)—and also in several profound ideas of Fermat in the early seventeenth century. Nevertheless, the actual discovery of calculus can be credited jointly to Isaac Newton (1642–1727) in England and Gottfried Wilhelm Leibniz (1646–1716) in Germany.

Newton's discovery is perhaps the only benefit that the Great Plague of London conferred upon humanity. This plague—which killed more than 75,000 people out of a total population of about 500,000—spread out from London into other parts of the country and forced the closing of Cambridge University in 1665. The young Newton left Trinity College and returned to his family farm in northern England for two years of rustic solitary meditation. During this enforced long vacation his genius burst into flame: He discovered the binomial series for negative and fractional exponents; differential and integral calculus; universal gravitation as the key to the mechanism of the solar system; and the resolution of sunlight into the visual spectrum by means of a prism, with its implications for understanding the colors of the rainbow and the nature of light in general. At the age of 22 to 23, in this miraculous period of his youth, he began the lifework in which he virtually created modern physical science. From the perspective of more than 300 years we see that he had a deeper influence on the direction of civilized life than the rise and fall of nations. When Newton died, England went into national mourning, and the poet Alexander Pope wrote his epitaph as follows: "Nature and Nature's laws lay hid in night: God said, *Let Newton be!* and all was light."

But our concern here is with his mathematics. Newton usually thought of functions in terms of motion, and what we refer to as "differential and integral calculus" he called "the direct and inverse method of fluxions." His treatise on these subjects was written in 1671 but remained unpublished until 1736, nine years after his death. His ideas became known in a limited way through correspondence and conversations with his friends, and were mentioned in vague general terms in his great treatise of 1687 on the laws of motion and the astronomy of the solar system.* Newton was pathologically secretive—as private as a snail in its shell. He had none of the itch to publish that afflicts the modern world, and most of his great works had to be dragged out of him by the cajolery and persistence of his friends. He was especially unwilling to publish anything about his mathematical discoveries, and it is not surprising that Leibniz and other mathematicians on the Continent, though starting a few years later, soon caught up with Newton and passed beyond him.

Mathematics and physics were only a small part of Newton's intellectual life, and he left these occupations behind him at the age of about 45. Throughout his life he spent thousands of hours of thought on theology and alchemy, and wrote bushels of manuscripts recording his studies of these subjects. In his later years he moved to London and became Master of the Mint. He was knighted for his success in stabilizing the British currency by catching and executing many counterfeiters with relentless and ferocious efficiency. A very strange man, unlike any other, and the more we learn about him the stranger he seems.

Newton's great rival Leibniz was a man of transcendent genius who made creative contributions across the entire spectrum of human knowledge. He is equally famous as mathematician and philosopher, and the graduate department of philosophy in every respectable university offers a course on Leibniz. He was also a lawyer, diplomat, historian, librarian, physicist, geologist, logician, theologian, landscape architect, economist, and much else. He spent most of his life in the service of the successive Dukes of Brunswick at Hanover in northern Germany, working as court historian and librarian. Without his researches as historian and genealogist, his employer the Elector George Louis of Hanover could never have become George I, the first German King of England; and George's descendants, including Queen Victoria and the present royal family of Britain (known as the House of Windsor since 1917) would never have been heard of. His ideas about the purposes and organization of scholarly libraries were so farsighted that the Director and Principal Librarian of the British Museum from 1959 to 1968 called him "the greatest librarian of his age."

He founded the Berlin Academy of Sciences and also *Acta Eruditorum,* the most influential European journal of the time in science and mathematics, and he was its editor-in-chief for many years. In this journal he published the first accounts of his version of calculus, in 1684 and 1686. He had started his mathematical work in 1673, eight years after Newton, and in 1675 he invented the basic notations dy/dx and $\int y\,dx$.† His early publications had little effect in Germany or England, but in Switzerland the Bernoulli brothers eagerly absorbed Leibniz's ideas and methods and contributed many of their own. Calculus grew rapidly from 1690 on and reached roughly its present state around 1800. However, subtle difficulties in the theory of calculus were not fully settled until the twentieth century.

***Philosophiae Naturalis Principia Mathematica* (The Mathematical Principles of Natural Philosophy), usually called the *Principia.* When this work was published, it was immediately recognized as one of the supreme achievements of the human mind. It is still universally considered to be the greatest contribution to science ever made by one man.

†The latter notation will be introduced in Section 5.3.

PROBLEMS

According to Problem 1 of Section 2.3, the general quadratic function

$$s = f(t) = at^2 + bt + c$$

has derivative

$$\frac{ds}{dt} = f'(t) = 2at + b.$$

Each of the formulas in Problems 1–7 below describes the motion of a point along a horizontal line whose positive direction is to the right. In each case use the result stated here to write down the velocity $v = ds/dt$ by inspection. Also, find (*a*) the times when the velocity is zero, so that the point is momentarily at rest; and (*b*) the times when the point is moving to the right.

1 $s = 3t^2 - 12t + 7$.
2 $s = 1 - 6t - t^2$.
3 $s = 2t^2 + 28t - 6$.
4 $s = -19 + 10t - 5t^2$.
5 $s = 7t^2 + 2$.
6 $s = 2 + 7t$.
7 $s = (2t - 6)^2$.

8 Two points start from the origin on the s-axis at time $t = 0$ and move along this axis in accordance with the formulas

$$s_1 = t^2 - 6t \quad \text{and} \quad s_2 = 9t - 2t^2,$$

where s_1 and s_2 are measured in feet and t in seconds.
(a) When will the two points have the same speed?
(b) What are the velocities of the two points at the times when they have the same position?

9 A camera is accidentally knocked off a ledge on the World Trade Center in New York City and falls to the ground below. The ledge is 784 ft above the ground. The camera falls a distance of $s = 16t^2$ feet in t seconds.
(a) How long does the camera fall before it hits the ground?
(b) What is the average velocity at which the camera falls during the first 3 seconds?
(c) What is the average velocity at which the camera falls during the last 3 seconds?
(d) What is the instantaneous velocity of the camera when it hits the ground?

10 A point moves along a straight line in such a way that after t seconds its distance from the origin is $s = 6t^2 + 2t$ feet.
(a) Find the average velocity between $t = 3$ and $t = 6$.
(b) Find the instantaneous velocity when $t = 3$.
(c) Find the instantaneous velocity when $t = 6$.

11 Consider the function $y = 3x^2 + 4$.
(a) Find the average rate of change of y with respect to x between the points $x = 1$ and $x = 3$.
(b) Find the instantaneous rate of change of y with respect to x at the point $x = 1$.
(c) Find the instantaneous rate of change of y with respect to x at the point $x = 3$.

12 Consider the function $y = \dfrac{1}{x^2 + 1}$ (see Problem 33 in Section 2.3).
(a) Find the average rate of change of y with respect to x between the points $x = -1$ and $x = 1$.
(b) Find the instantaneous rate of change of y with respect to x at the point $x = -1$.
(c) Find the instantaneous rate of change of y with respect to x at the point $x = 1$.

13 Starting from rest, a certain car moves s feet in t seconds, where $s = 4.4t^2$. How long does it take the car to reach the velocity of 60 mi/h (= 88 ft/s)?

14 Assume that a projectile fired straight up from the ground with an initial velocity of v_0 ft/s reaches a height of s feet in t seconds, where

$$s = v_0 t - 16t^2.$$

(a) Find the velocity v at time t.
(b) How much time is required for the projectile to reach its maximum height?
(c) What is the maximum height?
(d) What is the velocity when the projectile hits the ground?
(e) What must the initial velocity be for the projectile to hit the ground 15 seconds after firing?

15 An oil tank is to be drained for cleaning. If there are V gallons of oil left in the tank t minutes after the draining begins, where $V = 40(50 - t)^2$, find
(a) the average rate at which oil drains out of the tank during the first 20 minutes;
(b) the rate at which oil is flowing out of the tank 20 minutes after the draining begins.

16 Consider a square of area A and side s, so that $A = s^2$. If $x = \frac{1}{2}s$, use the idea of Example 3*d* to make a conjecture about the value of dA/dx. Verify your conjecture by calculation.

17 Suppose a balloon of volume V and radius r is being inflated, so that V and r are both functions of the time t. If dV/dt is constant, what can be said (without calculation) about the behavior of dr/dt as r increases?

2.5 THE CONCEPT OF A LIMIT. TWO TRIGONOMETRIC LIMITS

It is evident from the preceding sections that the definition of the derivative rests on the concept of the limit of a function, which we have freely used with only the briefest explanation. Now that we understand the purpose of this concept, the time has come to examine its meaning with somewhat more care and attention.

Let us consider a function $f(x)$ that is defined for all values of x near a point a on the x-axis but not necessarily at a itself. Suppose there exists a real number L with the property that $f(x)$ gets closer and closer to L as x gets closer and closer to a (Fig. 2.16). Under these circumstances we say that L is the *limit* of $f(x)$ as x approaches a, and we express this symbolically by writing

$$\lim_{x \to a} f(x) = L. \tag{1}$$

If there is no real number L with this property, we say that $f(x)$ *has no limit* as x approaches a, or that $\lim_{x \to a} f(x)$ *does not exist*. Another widely used notation equivalent to (1) is

$$f(x) \to L \quad \text{as} \quad x \to a,$$

which is read "$f(x)$ approaches L as x approaches a." In thinking about the meaning of (1), it is essential to understand that it does not matter what happens to $f(x)$ when x *equals* a; all that matters is the behavior of $f(x)$ for x's that are *near* a.

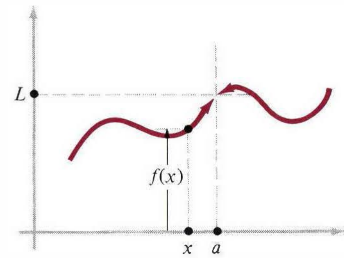


Figure 2.16

These informal descriptions of the meaning of (1) are helpful to the intuition and are adequate for most practical purposes. Nevertheless, they are too loose and imprecise to be acceptable as definitions, because of the vagueness of such expressions as "closer and closer" and "approaches." The exact meaning of (1) is too important to be left mainly to the student's imagination, and at the risk of being overly technical, we will try to give a satisfactory definition as briefly and clearly as possible. For the next few paragraphs we ask students to read even more carefully and thoughtfully than usual, and to suspend their natural impatience with what appears to be excessive, nit-picking precision.

We begin by analyzing a specific example with the hope of extracting the essence of the general situation:

$$\lim_{x \to 0} \frac{2x^2 + x}{x} = 1.$$

Here the function we must examine is

$$y = f(x) = \frac{2x^2 + x}{x}.$$

This function is not defined for $x = 0$, and for $x \neq 0$ its values are given by the simpler expression

$$f(x) = \frac{x(2x + 1)}{x} = 2x + 1.$$

If we examine the graph (Fig. 2.17), it is clear that $f(x)$ is close to 1 when x is close to 0. In order to give a quantitative description of this qualitative behavior, we need a formula for the difference between $f(x)$ and the limiting value 1:

$$f(x) - 1 = (2x + 1) - 1 = 2x.$$

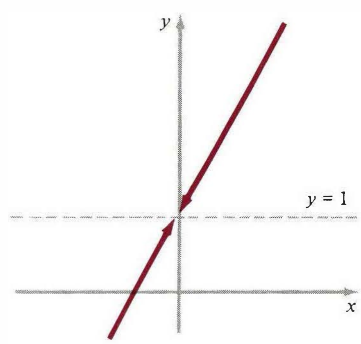


Figure 2.17

We see from this formula that $f(x)$ can be made *as close as we please* to 1, that is, this difference can be made *as small as we please*, by taking x *sufficiently close* to 0. Thus,

$$f(x) - 1 = \tfrac{1}{100} \quad \text{when} \quad x = \tfrac{1}{200},$$
$$f(x) - 1 = \tfrac{1}{1000} \quad \text{when} \quad x = \tfrac{1}{2000},$$

and so on. More generally, let ϵ (epsilon) be any positive number given in advance, no matter how small, and define δ (delta) by $\delta = \frac{1}{2}\epsilon$. Then the distance from $f(x)$ to 1 will be smaller than ϵ, provided only that the distance from x to 0 is smaller than δ; that is,

$$\text{if} \quad |x| < \delta = \tfrac{1}{2}\epsilon \quad \text{then} \quad |f(x) - 1| = 2|x| < \epsilon.$$

This assertion is much more precise than the vague statement that $f(x)$ is "close" to 1 when x is "close" to 0. It tells us exactly how close x must be to 0 in order to guarantee that $f(x)$ will attain a previously specified degree of closeness to 1. Of course, x is not permitted to equal 0 here, because $f(x)$ has no meaning for $x = 0$.

The so-called *epsilon-delta definition* of the meaning of (1) should now be easy to grasp. The defining condition is this:

> For each positive number ϵ there exists a positive number δ with the property that
> $$|f(x) - L| < \epsilon$$
> for any number $x \neq a$ that satisfies the inequality
> $$|x - a| < \delta.$$

In words: If $\epsilon > 0$ is given, then $\delta > 0$ can be found with the property that $f(x)$ will be "ϵ-close" to L whenever x is "δ-close" to a. As usual, we are concerned only with the behavior of $f(x)$ *near* the point $x = a$, and not at all with what happens *at* $x = a$.

It may be helpful to students if we interpret these ideas in terms of the graph of the function $y = f(x)$, as shown in Fig. 2.18. In this figure 2ϵ is the width of the horizontal strip centered on the line $y = L$, 2δ is the width of the vertical strip centered on the line $x = a$, and the defining condition stated above can be expressed this way:

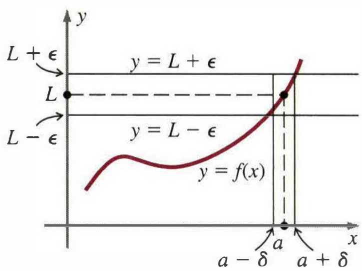


Figure 2.18 The epsilon-delta definition.

> For each horizontal strip, no matter how narrow, there exists a vertical strip such that if $x \neq a$ is confined to the vertical strip, then the corresponding part of the graph will be confined to the horizontal strip.

Students should read the precise definition of the meaning of (1) very carefully and be aware of its crucial role in the theory of calculus. However, an intuitive understanding of limits is quite enough for our purposes, and from this point of view the following examples should present no difficulties.

Example 1 First,

$$\lim_{x \to 2} (3x + 4) = 10.$$

Here it is clear that as x approaches 2, $3x$ approaches 6 and $3x + 4$ approaches $6 + 4 = 10$. Next,

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x + 1)(x - 1)}{x - 1} = \lim_{x \to 1} (x + 1) = 2.$$

The first thing we notice here is that the function $(x^2 - 1)/(x - 1)$ is undefined at $x = 1$, since both numerator and denominator equal 0. But this fact is irrelevant, since all that matters is the behavior of the function for x's that are near 1 but different from 1, and for all such x's the indicated cancellation is valid, the function equals $x + 1$, and this is near 2.

Example 2 It is illuminating to consider a few limits that do not exist, for instance,

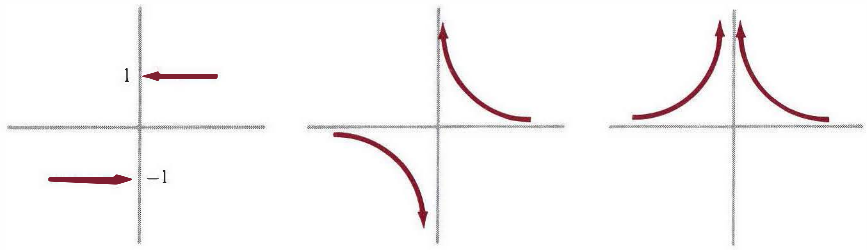


Figure 2.19

$$\lim_{x\to 0}\frac{x}{|x|},\qquad \lim_{x\to 0}\frac{1}{x},\qquad \text{and}\qquad \lim_{x\to 0}\frac{1}{x^2}.$$

The behavior of these limits is most easily understood by looking at the graphs of the functions $x/|x|$, $1/x$, and $1/x^2$ (Fig. 2.19). In the first case the function equals 1 when x is positive and -1 when x is negative (and is undefined for $x = 0$), so there is no single number that the values of the function approach as x approaches 0 without regard to sign. We can be a bit more specific about the way this limit fails to exist, by writing

$$\lim_{x\to 0+}\frac{x}{|x|} = 1 \qquad \text{and} \qquad \lim_{x\to 0-}\frac{x}{|x|} = -1.$$

The notations $x \to 0+$ and $x \to 0-$ are intended to suggest that the variable x approaches 0 from the positive side (the right) and from the negative side (the left), respectively. The other two limits fail to exist because in each case the values of the function become arbitrarily large in absolute value as x approaches 0. In symbols,

$$\lim_{x\to 0+}\frac{1}{x} = \infty,\qquad \lim_{x\to 0-}\frac{1}{x} = -\infty,\qquad \text{and}\qquad \lim_{x\to 0}\frac{1}{x^2} = \infty.$$

Remember: The number L in (1) must be a *real* number; $L = \infty$ does not qualify.

The main rules for calculating with limits are exactly what we would expect. For instance,

$$\lim_{x\to a} x = a;$$

and if c is a constant, then

$$\lim_{x\to a} c = c.$$

Also, if $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then

$$\lim_{x\to a}\,[f(x) + g(x)] = L + M,$$

$$\lim_{x\to a}\,[f(x) - g(x)] = L - M,$$

$$\lim_{x\to a} f(x)g(x) = LM,$$

and

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{L}{M} \qquad (\text{if } M \neq 0).$$

In words, the limit of a sum is the sum of the limits, with similar statements for differences, products, and quotients. These are called the *limit laws,* or *limit theorems.*

We remarked earlier that calculus is a problem-solving art and not a branch of logic. It has more to do with insight nourished by intuitive understanding than it does with careful deductive reasoning. Naturally, we will try to convince the reader of the truth of our statements and the legitimacy of our procedures. However, these efforts will be brief and rather informal, in order to avoid clogging the text with massive indigestible chunks of theoretical material. Those who wish to devote more attention to the purely mathematical side of the subject will find logically rigorous proofs of the major theorems in Appendix A at the back of the book. In particular, the properties of limits stated here are proved in Appendix A.2.

Before leaving these topics, we discuss two specific trigonometric limits that will be of crucial importance in the next chapter. The first is

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} \qquad (\theta \text{ in radians}). \tag{2}$$

We cannot simply set $\theta = 0$ here, because the result is the meaningless quotient 0/0. We notice how different this is from an algebraic limit like

$$\lim_{x\to 0}\frac{3x^2+2x}{x} = \lim_{x\to 0}\frac{x(3x+2)}{x} = \lim_{x\to 0}(3x+2) = 2,$$

θ	$\dfrac{\sin\theta}{\theta}$
1	0.84147098
0.5	0.95885108
0.4	0.97354586
0.3	0.98506736
0.2	0.99334665
0.1	0.99833417
0.05	0.99958339
0.01	0.99998333
0.005	0.99999583
0.001	0.99999983

because there is no apparent way to cancel θ from $\sin\theta$. To get an impression of what is happening in (2), let us calculate the numerical value of the ratio for several small values of θ. We begin by observing that if we replace θ by $-\theta$ in the ratio, then we have

$$\frac{\sin(-\theta)}{-\theta} = \frac{-\sin\theta}{-\theta} = \frac{\sin\theta}{\theta},$$

so we can restrict our attention to positive θ's. Using a calculator set to the radians mode, we can easily construct the adjoining table of values correct to eight decimal places. This numerical evidence strongly suggests (but does not prove!) that

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1. \tag{3}$$

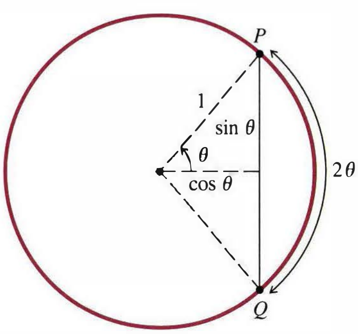


Figure 2.20

We now establish (3) by a simple geometric argument. Let P and Q be two nearby points on a unit circle (Fig. 2.20), and let $\overline{PQ}$ and $\widehat{PQ}$ denote the lengths of the chord and the arc connecting these points. Then the ratio of the chord length to the arc length evidently approaches 1 as the two points move together:

$$\frac{\text{chord length } \overline{PQ}}{\text{arc length } \widehat{PQ}} \to 1 \qquad \text{as} \qquad \widehat{PQ} \to 0.$$

With the notation in the figure, this geometric statement is equivalent to

$$\frac{2\sin\theta}{2\theta} = \frac{\sin\theta}{\theta} \to 1 \qquad \text{as} \qquad 2\theta \to 0 \qquad \text{or} \qquad \theta \to 0,$$

and this is (3).

The second limit necessary for our work in the next chapter is

$$\lim_{\theta \to 0} \frac{1 - \cos \theta}{\theta} = 0. \tag{4}$$

This follows from (3) by an ingenious use of the trigonometric identity $\sin^2 \theta + \cos^2 \theta = 1$:

$$\begin{aligned}
\lim_{\theta \to 0} \frac{1 - \cos \theta}{\theta} &= \lim_{\theta \to 0} \left(\frac{1 - \cos \theta}{\theta} \cdot \frac{1 + \cos \theta}{1 + \cos \theta} \right) \\
&= \lim_{\theta \to 0} \frac{1 - \cos^2 \theta}{\theta(1 + \cos \theta)} \\
&= \lim_{\theta \to 0} \frac{\sin^2 \theta}{\theta(1 + \cos \theta)} \\
&= \lim_{\theta \to 0} \left(\frac{\sin \theta}{\theta} \cdot \frac{\sin \theta}{1 + \cos \theta} \right) \\
&= \left(\lim_{\theta \to 0} \frac{\sin \theta}{\theta} \right) \left(\lim_{\theta \to 0} \frac{\sin \theta}{1 + \cos \theta} \right) \\
&= 1 \cdot \frac{0}{1 + 1} = 0.
\end{aligned}$$

The end of this calculation uses the facts that $\sin \theta \to 0$ and $\cos \theta \to 1$ as $\theta \to 0$, which are easily verified by examining the geometric meaning of $\sin \theta$ and $\cos \theta$ in Fig. 2.20.

PROBLEMS

Some of the following limits exist, and others do not. Evaluate those that do.

1 $\displaystyle\lim_{x \to 3} (7x - 6)$.

2 $\displaystyle\lim_{x \to 2} \frac{10}{3 + x}$.

3 $\displaystyle\lim_{x \to 0} \frac{5}{x - 1}$.

4 $\displaystyle\lim_{x \to 2} \frac{6}{2x - 4}$.

5 $\displaystyle\lim_{x \to 3} \frac{3x - 9}{x - 3}$.

6 $\displaystyle\lim_{x \to 3} \frac{x^2 + 3x}{x^2 - x + 3}$.

7 $\displaystyle\lim_{x \to 5} \frac{x - 3 - 2x^2}{1 + 3x}$.

8 $\displaystyle\lim_{x \to -3} \frac{4x}{x + 3}$.

9 $\displaystyle\lim_{x \to -3} \left(\frac{4x}{x + 3} + \frac{12}{x + 3} \right)$.

10 $\displaystyle\lim_{x \to 0.001} \frac{x}{|x|}$.

11 $\displaystyle\lim_{x \to 7} \frac{x^2 + x - 56}{x^2 - 11x + 28}$.

12 $\displaystyle\lim_{x \to -2} \frac{(x + 2)(x^2 - x + 3)}{x^2 + x - 2}$.

13 $\displaystyle\lim_{x \to 0} \frac{x^2}{|x|}$.

14 $\displaystyle\lim_{x \to 4} \frac{x - 4}{\sqrt{x} - 2}$.

15 $\displaystyle\lim_{x \to 4} \frac{x - 4}{x - \sqrt{x} - 2}$.

16 $\displaystyle\lim_{x \to 3} \frac{\sqrt{x^2 + 16} - 5}{x^2 - 3x}$.

17 If $\lim_{x \to a} f(x) = 4$, $\lim_{x \to a} g(x) = -2$, and $\lim_{x \to a} h(x) = 0$, evaluate the following limits:
(a) $\displaystyle\lim_{x \to a} [f(x) - g(x)]$; (b) $\displaystyle\lim_{x \to a} [g(x)]^2$;
(c) $\displaystyle\lim_{x \to a} \frac{f(x)}{g(x)}$; (d) $\displaystyle\lim_{x \to a} \frac{h(x)}{f(x)}$;
(e) $\displaystyle\lim_{x \to a} \frac{f(x)}{h(x)}$; (f) $\displaystyle\lim_{x \to a} \frac{1}{[f(x) + g(x)]^2}$.

18 In many situations we are interested in the behavior of $f(x)$ when x is large and positive. If there exists a real number L with the property that $f(x)$ gets closer and closer to L as x gets larger and larger (Fig. 2.21), then

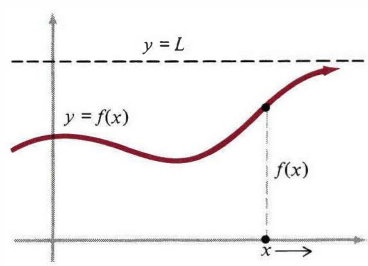


Figure 2.21

we say that L is the *limit* of $f(x)$ as x approaches infinity, and we symbolize this by writing $\lim_{x\to\infty} f(x) = L$. Evaluate the following limits:

(a) $\lim\limits_{x\to\infty} \frac{1}{x}$; (b) $\lim\limits_{x\to\infty} \left(2 + \frac{100}{x}\right)$;

(c) $\lim\limits_{x\to\infty} \frac{5x+3}{2x-7}$;* (d) $\lim\limits_{x\to\infty} \frac{2x^2+x-5}{3x^2-7x+2}$;

(e) $\lim\limits_{x\to\infty} \frac{x}{x^2+1}$; (f) $\lim\limits_{x\to\infty} \frac{x^2-2x+5}{x^3+7x^2+2x-1}$.

19 Evaluate the following limits:

(a) $\lim\limits_{\theta\to 0} \frac{\sin 5\theta}{\theta}$; (b) $\lim\limits_{\theta\to 0} \frac{\sin \theta}{2\theta}$;

(c) $\lim\limits_{x\to\infty} \sin \frac{1}{x}$; (d) $\lim\limits_{x\to\infty} x \sin \frac{1}{x}$;

(e) $\lim\limits_{x\to\infty} \cos \frac{500}{x}$; (f) $\lim\limits_{x\to 0} \frac{\sin^2 x}{3x^2}$;

(g) $\lim\limits_{x\to 0} \frac{\sin 2x}{\sin 3x}$.

20 Evaluate the following limits:

(a) $\lim\limits_{x\to 0} \frac{\sin x}{3\sqrt{x}}$; (b) $\lim\limits_{x\to 0} \frac{\sin^2 x}{x}$;

(c) $\lim\limits_{x\to 0} \frac{x^2}{1-\cos^2 x}$; (d) $\lim\limits_{\theta\to 0} \frac{\theta}{\cos \theta}$;

(e) $\lim\limits_{x\to 0} \frac{3x+\sin x}{x}$; (f) $\lim\limits_{\theta\to 0} \frac{\theta^2 - 2\sin \theta}{\theta}$;

(g) $\lim\limits_{x\to 0} \frac{3x^2+4x}{\sin 2x}$.

Each of the following problems requires the use of a calculator. Hereafter, problems of this kind will be signaled by the symbol.

21 Verify the limit (4) numerically by using a calculator to construct a table of values of $(1 - \cos \theta)/\theta$ corresponding to the same θ's used in the text.

*Hint: Notice that dividing both numerator and denominator of this quotient by x gives

$$\frac{5x+3}{2x-7} = \frac{5 + \frac{3}{x}}{2 - \frac{7}{x}}.$$

What becomes of the expression on the right as $x \to \infty$?

22 Consider the limit

$$\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta^2}.$$

(a) Use a calculator to construct a table of values of the function for small θ's, and thereby form a conjecture about the value of this limit.

(b) Prove your conjecture.

23 The limit (3) says that $\frac{\sin\theta}{\theta} \cong 1$ or $\sin\theta \cong \theta$ for small θ. Test this approximation by using a calculator to find the value of $\sin\theta$ for

(a) $\theta = 0.1$; (b) $\theta = 0.01$; (c) $\theta = 0.001$.

Give a geometric explanation for the fact that each $\sin\theta$ is slightly less than its corresponding θ.

24 Using the trigonometric identity $\cos 2\alpha = 1 - 2\sin^2\alpha$ with $2\alpha = \theta$, and the approximation $\sin\theta \cong \theta$ for small θ, show that for these θ we have

$$\cos\theta \cong 1 - \tfrac{1}{2}\theta^2.$$

Use a calculator to test this approximation for

(a) $\theta = 0.1$; (b) $\theta = 0.01$; (c) $\theta = 0.001$.

25 Consider the limit

$$\lim_{x\to 0+} x^x.$$

(a) Use a calculator to construct a table of values of x^x for $x = 1$, 0.9, 0.8, 0.7, 0.6, 0.5, 0.4, 0.3, 0.2, 0.1, 0.05, 0.01, 0.005, 0.001. Use this evidence to form a conjecture about the value of the limit.

(b) Use the information in (a) to sketch the graph of $y = x^x$ for $0 < x < 1$. Estimate the location of the lowest point.

26 The existence of the limit

$$\lim_{x\to 0} (1+x)^{1/x}$$

will be established later, in Chapter 8 and Appendix A.8.† Estimate the value of this limit to five decimal places by using a calculator to find the value of the function for $x = 1$, 0.1, 0.01, 0.001, 0.0001, 0.00001, 0.000001, 0.0000001, 0.00000001, 0.000000001.

†This number defines the constant e, which is the most important constant in mathematics after π.

2.6 CONTINUOUS FUNCTIONS. THE MEAN VALUE THEOREM AND OTHER THEOREMS

As we penetrate further into our subject, it will often be important for us to know what is meant by a *continuous function*. In everyday speech a "continuous" process is one that proceeds without gaps or interruptions or sudden changes. Roughly speaking, a function $y = f(x)$ is continuous if it displays similar behavior, that is, if a small change in x produces a small change in the corresponding value $f(x)$. The function shown in Fig. 2.22 is continuous at the point a because $f(x)$ is close to $f(a)$ when x is close to a, or more precisely, because $f(x)$ can be made as close as we please to $f(a)$ by taking x sufficiently close to a. In the language of limits this says that

$$\lim_{x \to a} f(x) = f(a). \tag{1}$$

Up to this stage our remarks about continuity have been rather loose and intuitive, and intended more to explain than to define. We now adopt equation (1) as the definition of the statement that $f(x)$ is *continuous at a*. The reader should observe that the continuity of $f(x)$ at a requires three things to happen: a must be in the domain of $f(x)$, so that $f(a)$ exists; $f(x)$ must have a limit as x approaches a; and this limit must equal $f(a)$. We can understand these ideas more clearly by examining Fig. 2.22, in which the function is discontinuous in different ways at the points b, c, and d: At the point b, $\lim_{x\to b} f(x)$ exists but $f(b)$ does not; at c, $f(c)$ exists but $\lim_{x\to c} f(x)$ does not; and at d, $f(d)$ and $\lim_{x\to d} f(x)$ both exist but have different values. The graph of this function therefore has "gaps" or "holes" of three different kinds.

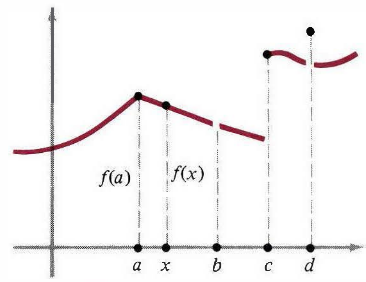


Figure 2.22

The definition given here tells us what it means for a function to be continuous at a particular point in its domain. A function is called a *continuous function* if it is continuous at every point in its domain. In particular, by the properties of limits this is easily seen to be true for all polynomials and rational functions; and by looking at their graphs, we see that the functions $\sqrt{x}$, $\sin x$, and $\cos x$ are also continuous. We will be especially interested in functions that are continuous on closed intervals. These functions are often described as those whose graphs can be drawn without lifting the pencil from the paper.

With a slight change of notation, we can express the continuity of our function $f(x)$ at a point x (instead of a) in either of the equivalent forms

$$\lim_{\Delta x\to 0} f(x + \Delta x) = f(x) \qquad \text{or} \qquad \lim_{\Delta x\to 0} [f(x + \Delta x) - f(x)] = 0;$$

and if we write $\Delta y = f(x + \Delta x) - f(x)$, then this condition becomes

$$\lim_{\Delta x\to 0} \Delta y = 0.$$

The purpose of this reformulation is to make it possible to give a very short proof of a fact that we will need in the next chapter, namely, that *a function which is differentiable at a point is continuous at that point*. The proof occupies only a single line:

$$\lim_{\Delta x\to 0} \Delta y = \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x} \cdot \Delta x = \left[\lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x}\right]\left[\lim_{\Delta x\to 0} \Delta x\right] = \frac{dy}{dx} \cdot 0 = 0.$$

The converse of this statement is not true, since a function can easily be continuous at a point without being differentiable there (for example, see the point a in Fig. 2.22).

There are several other facts in the theory of calculus that will be needed in the next few chapters to convert plausible reasoning into solid proof. Students should be aware of these facts but not smothered by them. We state them in the form of three basic theorems, which we present without proof but with a few comments on each that we hope will illuminate their meaning. Proofs are not given here because a careful examination of the theoretical foundations of calculus does not belong in a first course. All three theorems are extremely plausible—what some might call "intuitively obvious." Part of the difficulty they cause for beginning students lies in the effort required to doubt them in the face of their compelling believability:

The Mean Value Theorem Let $y = f(x)$ be a function with the following two properties:

$f(x)$ is continuous on the closed interval $[a, b]$; and

$f(x)$ is differentiable on the open interval (a, b).

Then there exists at least one point c in the open interval (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}, \tag{2}$$

or equivalently,

$$f(b) - f(a) = f'(c)(b - a). \tag{3}$$

COMMENTS ON THE MEAN VALUE THEOREM (MVT)

We see that the statement is reasonable by looking at its geometric meaning as shown in Fig. 2.23. The right side of equation (2) is the slope of the chord joining the endpoints A and B of the graph, and the left side is the slope of the tangent line at the point on the graph corresponding to $x = c$; and the MVT says that for at least one intermediate point on the graph the tangent is parallel to the chord. In Fig. 2.24 there are two such points, corresponding to $x = c_1$ and $x = c_2$. But this is perfectly all right, because the phrase "at least one point c" allows for the possibility of two such points, or three, or any number whatever.

The conclusion of the MVT is crucially dependent on its hypotheses, because this conclusion does not follow if the hypotheses are weakened ever so slightly. We see this by considering the example of the function $y = |x|$ defined on the closed interval $[-1, 1]$. This function (Fig. 2.25) is continuous on the closed interval $[-1, 1]$ and is differentiable on the open interval $(-1, 1)$, *except at the single point* $x = 0$, where the derivative does not exist. The conclusion fails, because the chord joining A and B is horizontal and clearly the graph has no horizontal tangent.

To understand the significance of the Mean Value Theorem, we briefly and informally consider three simple consequences that we will need in Chapters 4 and

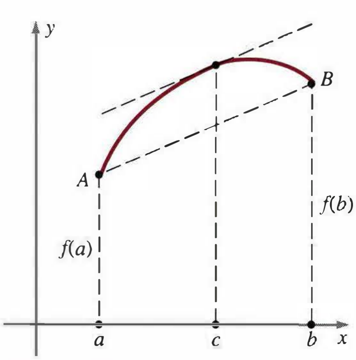


Figure 2.23

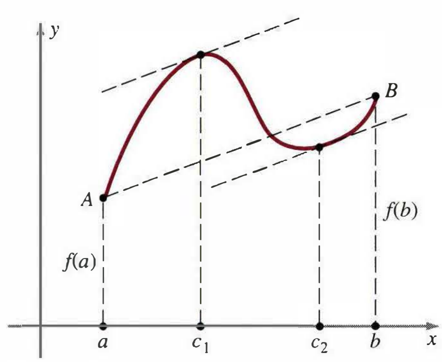


Figure 2.24

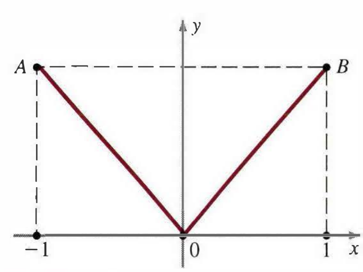


Figure 2.25

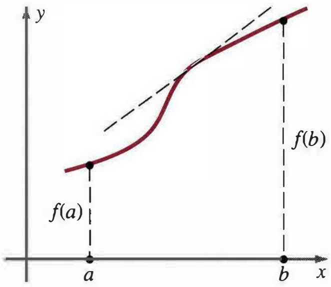


Figure 2.26

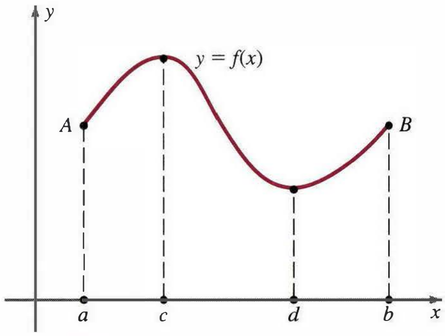


Figure 2.27

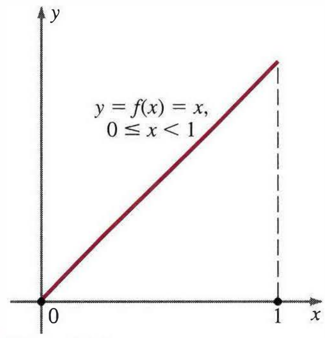


Figure 2.28

5. In each case we have a property of the derivative that implies a property of the function, and the MVT is the link between the two properties.

1. *If* $f'(x) > 0$ *on an interval, then* $f(x)$ *is increasing on that interval* ["increasing" means that $a < b$ implies $f(a) < f(b)$]. Our geometric intuition assures us that this is true, because $f'(x) > 0$ means that the tangent points up to the right everywhere (Fig. 2.26). For a more explicit argument based on the MVT, we point out that in this situation the right side of (3) is positive, so the left side is also positive, and this means that $f(a) < f(b)$.
2. Similarly, *if* $f'(x) < 0$ *on an interval, then* $f(x)$ *is decreasing on that interval* ["decreasing" means that $a < b$ implies $f(a) > f(b)$].
3. *If* $f'(x) = 0$ *on an interval, then* $f(x)$ *is constant on that interval.* To show this we assume the contrary, namely, that the function is not constant. Then there exist two points a and b with $a < b$ at which the function has different values $f(a)$ and $f(b)$. But this implies that the left side of (3) is not equal to 0, whereas the right side must equal 0. This contradiction shows that our assumption—that the function is not constant—cannot be true.

The Extreme Value Theorem If $y = f(x)$ is a function that is defined and continuous on a closed interval $[a, b]$, then this function attains both a maximum value and a minimum value at points of the interval; that is, there exist points c and d in $[a, b]$ such that $f(c) \geq f(x) \geq f(d)$ for all x in $[a, b]$.*

COMMENTS ON THE EXTREME VALUE THEOREM (EVT)

Informally, this theorem asserts that the graph of a continuous function on a closed interval always has both a high point and a low point. If we think of the graph as drawn by moving a pencil across the paper from the point A to the point B (see Fig. 2.27), then the statement is so visibly true that we wonder how anyone could doubt it. However, it is difficult to prove in a fully rigorous manner, because it depends on a subtle property of the real line (completeness, meaning that no points are "missing" from the line) that is normally discussed only in advanced courses.

Also, just as in the case of the Mean Value Theorem, the conclusion here is crucially dependent on the hypotheses that the function is *continuous* and the interval is *closed*. For example, the function in Fig. 2.28 is continuous on the in-

*Maximum values and minimum values are known collectively as *extreme values*.

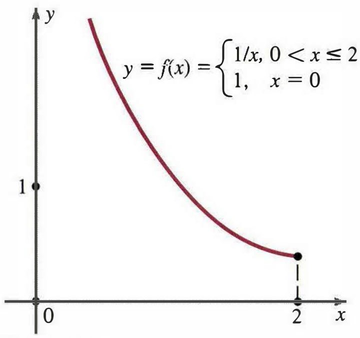


Figure 2.29

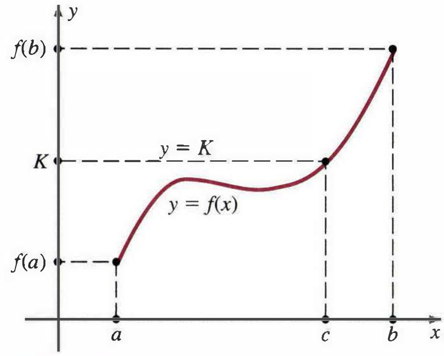


Figure 2.30

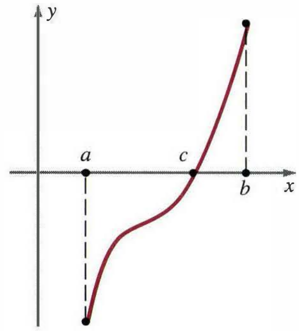


Figure 2.31

terval $[0, 1)$, but this interval is not closed because it lacks the right endpoint. We see that this function attains *no* maximum value at any point of $[0, 1)$, because the only possible candidate for a maximum value is 1 at $x = 1$, but $f(1)$ is not defined. On the other hand, in Fig. 2.29 the interval $[0, 2]$ is closed and the function is continuous at every point of this interval *except for the single point* $x = 0$, and again the function attains no maximum value at any point of the interval.

There is a further important fact about extreme values that is known as *Fermat's theorem: If a continuous function $f(x)$ on a closed interval $[a, b]$ attains its maximum or minimum value at an interior point c of $[a, b]$, and if $f(x)$ is differentiable at c, then $f'(c) = 0$.* In later chapters we will often be trying to locate extreme values of continuous functions on closed intervals. Fermat's theorem tells us that we must seek such points either at the endpoints of the interval or at those interior points where $f'(x) = 0$ or $f'(x)$ does not exist.

The Intermediate Value Theorem If $y = f(x)$ is a function that is defined and continuous on a closed interval $[a, b]$, then this function assumes every value between $f(a)$ and $f(b)$; that is, if K is any number strictly between $f(a)$ and $f(b)$, then there exists at least one point c in (a, b) such that $f(c) = K$.

COMMENTS ON THE INTERMEDIATE VALUE THEOREM (IVT)

In the language of graphs (Fig. 2.30), every horizontal line of height K intersects the graph of $y = f(x)$ if K is between $f(a)$ and $f(b)$.

The most vivid form of the IVT says that if $y = f(x)$ is continuous on $[a, b]$ and $f(a)$ and $f(b)$ have opposite signs, then $f(c) = 0$ for at least one point c in (a, b). In other words, the graph cannot get from one side of the x-axis to the other without actually crossing this axis (Fig. 2.31). This may seem to be very obvious indeed, but the statement can be false if the function fails to be continuous at even a single point. We see this by considering the function defined on $[0, 2]$ by

$$y = f(x) = \begin{cases} -1 & \text{if } 0 \le x < 1, \\ 1 & \text{if } 1 \le x \le 2. \end{cases}$$

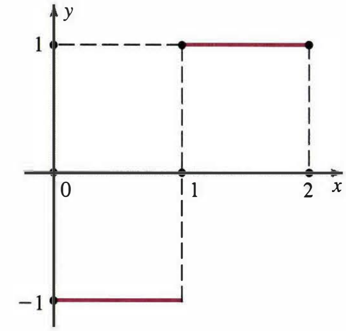


Figure 2.32

It is clear that for this function (Fig. 2.32) we have $f(0) < 0$ and $f(2) > 0$, and yet—because of the discontinuity at the single point $x = 1$—there does not exist any point c in (0, 2) for which $f(c) = 0$.

The practical significance of the IVT can best be understood by means of an example. We observe that the equation

$$x^3 + 2x - 4 = 0 \tag{4}$$

is not easy to solve by factoring, because the left side has no obvious factors. However, the continuous function $f(x) = x^3 + 2x - 4$ is negative at $x = 1$ and positive at $x = 2$ [$f(1) = -1$ and $f(2) = 8$]. The IVT therefore guarantees that $f(x)$ has a zero at some point in (1, 2), so equation (4) has a solution in this interval. Further, $f'(x) = 3x^2 + 2 > 0$ for all x, so $f(x)$ has only one zero and (4) has only one solution. This follows from the fact that if there were two zeros, then the Mean Value Theorem would imply that $f'(c) = 0$ for some intermediate point c, which cannot happen. In Section 4.6 we develop a method for calculating this solution of (4) to any desired degree of accuracy.

Remark Most students of calculus are impatient with the theory of the subject, and rightly so, because the essence of calculus does not lie in theorems and how to prove them but rather in tools and how to use them. A moderate dose of rigorous thinking in mathematics—rigidly correct, leakproof thinking that can withstand the closest skeptical scrutiny—is a good thing; but like virtue, it can be overdone, and in calculus courses it often is. This preoccupation with the techniques and fine points of proof, rather than with the central ideas of the subject, can make a forbidding mystery out of concepts that are essentially simple and clear. We mention these issues to point out that even though most of our discussions in this book involve plausibility arguments intended to be reasonably convincing to reasonable people, full and rigorous proofs of all theorems are available in Appendix A for those who may wish to examine them.

PROBLEMS

1 Find the points of discontinuity of the following functions:

(a) $\dfrac{x}{x^2+1}$; (b) $\dfrac{x}{x^2-1}$;

(c) $\dfrac{x^2-1}{x-1}$; (d) $\sqrt{x}$;

(e) $\dfrac{1}{\sqrt{x}}$; (f) $\sqrt{x^2}$;

(g) $\dfrac{1}{x^2+x-12}$; (h) $\dfrac{1}{x^2+4x+5}$.

In Problems 2–9, verify that the function $f(x)$ satisfies the hypotheses of the Mean Value Theorem on the given interval, and find all points c whose existence is guaranteed by the theorem.

2 $f(x) = \sin x$, $[0, 2\pi]$.

3 $f(x) = x^2 + 1$, $[1, 2]$.

4 $f(x) = x^3 + 1$, $[1, 2]$.

5 $f(x) = x^2 - 4x + 6$, $[2, 4]$.

6 $f(x) = x^2 + x$, $[-2, 8]$.

7 $f(x) = \sqrt{x+1}$, $[0, 3]$.

8 $f(x) = \sqrt{25 - x^2}$, $[-5, 5]$.

9 $f(x) = 1/x$, $[\frac{1}{2}, 2]$.

10 If $f(x)$ is an arbitrary quadratic polynomial, that is, if $f(x) = Ax^2 + Bx + C$ $(A \neq 0)$, show that the point c whose existence is guaranteed by the Mean Value Theorem is the midpoint of the interval $[a, b]$.

11 If $f(x) = 1/x$ and $g(x) = 1/x + x/|x|$, show that these functions have identical derivatives, so that $[f(x) - g(x)]' = 0$. However, their difference is not constant. Explain how this is possible in view of consequence 3 of the Mean Value Theorem.

12 A car starts from rest and travels 4 mi along a straight road in 6 minutes. Use the Mean Value Theorem to show

that at some moment during the trip its speed was exactly 40 mi/h.

13 If $f'(x) = c$, a constant, for all x, show that $f(x) = cx + d$ for some constant d.

14 If $f(x)$ and $g(x)$ are two functions with equal derivatives on an interval, what can be said about their difference $f(x) - g(x)$?

15 For each of the given intervals, find the maximum value of $\sin x$ on that interval, and also find the value of x at which it occurs:

(a) $\left[0, \frac{\pi}{6}\right]$; (b) $\left[0, \frac{\pi}{4}\right]$; (c) $[0, \pi]$.

16 For each of the given intervals, find the maximum value of $\cos x$ on that interval, and also find the value of x at which it occurs:

(a) $\left[0, \frac{\pi}{2}\right]$; (b) $\left[\frac{\pi}{3}, \pi\right]$; (c) $[0, 2\pi]$.

17 Does the function $\dfrac{x^3 - x^5}{1 + 9x^4 + 5x^6}$ have

(a) a maximum value on $[5, 8]$?
(b) a minimum value on $[5, 8]$?

18 Does the function x^3 have a maximum value on (a) $[1, 5]$; (b) $[-5, 2]$; (c) $(3, 4)$? If so, where?

19 Does the function x^4 have a minimum value on

(a) $[-3, 5]$; (b) $(-4, 2)$;
(c) $(2, 3)$; (d) $(-1, 5)$?

If so, where?

20 Does the function $4 - x^2$ have

(a) a maximum value on $(-2, 2)$?
(b) a minimum value on $(-2, 2)$?

If so, where?

21 Does the function $4 + x^2$ have

(a) a maximum value on $(-2, 2)$?
(b) a minimum value on $(-2, 2)$?

If so, where?

In Problems 22–29, find the maximum and minimum values attained by the given function on the given interval.

22 $1/x$, $(0, 1)$.
23 $1/x$, $(0, 1]$.
24 $1 - x^2$, $(0, 1)$.
25 $1 - x^2$, $[0, 1)$.
26 $1 - x^2$, $[0, 1]$.
27 $1 - x^2$, $[-3, -2]$.
28 $|3x - 4|$, $[1, 2]$.
29 $2 + |2x - 3|$, $(0, 2)$.

In each of Problems 30–33, apply the Intermediate Value Theorem to show that the given equation has a solution in the given interval.

30 $x^3 + 2x + 5 = 0$, $[-2, -1]$.
31 $x^4 + 3x - 5 = 0$, $[1, 2]$.
32 $x^5 - 4x^3 + 127 = 0$, $[-3, -2]$.
33 $x^6 - 3x + 1 = 0$, $[-1, 1]$.

34 Show that the equation $x^3 - 5x + 1 = 0$ has three distinct roots by calculating the value of $f(x) = x^3 - 5x + 1$ at the points $x = -3, -2, -1, 0, 1, 2, 3$. State the intervals in which the roots lie.

35 If $p(x)$ is a polynomial of odd degree, show that the equation $p(x) = 0$ has at least one solution.

36 If A and B are positive constants, show that the equation

$$\frac{A}{x-1} + \frac{B}{x-2} = 0$$

has a solution in the interval $(1, 2)$.

37 If $f(x)$ and $g(x)$ are continuous on $[a, b]$, and if $f(a) < g(a)$ and $f(b) > g(b)$, show that the equation $f(x) = g(x)$ has at least one solution in (a, b).

38 Assume for a moment that the rational numbers are the only numbers that exist. Under this assumption, show that the Intermediate Value Theorem is false by considering the function $y = f(x) = x^2 - 2$ on the interval $[1, 2]$. This example shows that the truth of the Intermediate Value Theorem depends on the profound fact that no points are "missing" from the real line.

39 Let $y = f(x)$ be a continuous function defined on the closed interval $[0, b]$ with the property that $0 < f(x) < b$ for all x in $[0, b]$. Show that there exists a point c in $(0, b)$ with the property that $f(c) = c$. Hint: Consider the function $g(x) = f(x) - x$.

40 The rectangle in Fig. 2.33 represents the floor of a room, and AB a straight piece of string lying on the floor whose ends touch the opposite walls W_1 and W_2. The tangle represents the same piece of string wadded up and thrown back down on the floor. Show that there is at least one point of the wadded string whose distances from the two walls are exactly the same as they were before. Hint: See the preceding problem.

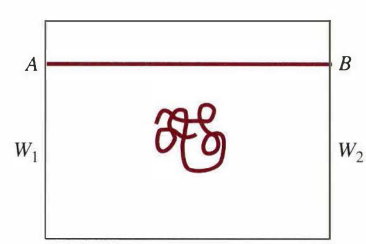


Figure 2.33

CHAPTER 2 REVIEW: DEFINITIONS, CONCEPTS, METHODS

Define, state, or think through the following.

1 Tangent line according to Fermat.
2 Delta notation.
3 Derivative of a function.
4 Differentiable function.
5 Three-step rule (or process).
6 Leibniz notation.
7 Derivative of $f(x) = ax^2 + bx + c$.
8 Average and instantaneous velocity.
9 Speed.
10 Rate of change.
11 Acceleration.
12 $\lim_{x \to a} f(x) = L$.
13 Limit laws.
14 $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$.
15 $\lim_{\theta \to 0} \frac{1 - \cos \theta}{\theta} = 0$.
16 Continuity of $f(x)$ at $x = a$.
17 Continuous function.
18 Differentiability implies continuity.
19 Mean Value Theorem.
20 Extreme Value Theorem.
21 Intermediate Value Theorem.

ADDITIONAL PROBLEMS FOR CHAPTER 2

SECTION 2.2

1 For what value of b does the graph of $y = x^2 + bx + 1$ have a horizontal tangent at $x = 3$?

2 Find the two points on the curve $y = x - \frac{1}{4}x^2$ at which the tangent passes through the point $(\frac{9}{2}, 0)$.

3 Let $P = (x_0, y_0)$ be a point on the parabola $y = x^2$. Show that a nonvertical line passing through P which does not intersect the curve at any other point is necessarily the tangent at P; that is, show that if the line

$$y - y_0 = m(x - x_0)$$

intersects $y = x^2$ only at (x_0, y_0), then $m = 2x_0$.

4 If (x_1, y_1) and (x_2, y_2) are distinct points on the parabola $y = x^2$, at what point on the curve is the tangent parallel to the chord joining these two given points?

5 The curve $y = x^2$ is a particular parabola, but if a is an unspecified positive constant, $y = f(x) = ax^2$ is a completely general parabola located in a convenient position.
(a) Show that $f'(x_0) = 2ax_0$.
(b) Show that the tangent at a point $P = (x_0, y_0)$ other than the vertex has y-intercept $-y_0$, and use this fact to formulate a geometric method for constructing the tangent at P.

SECTION 2.3

6 Use the three-step rule to calculate $f'(x)$ if $f(x)$ is equal to
(a) $\dfrac{x + 1}{x}$; (b) $\dfrac{3 - 2x}{x - 2}$;
(c) $\sqrt{3x + 2}$; (d) $\sqrt{x^2 + 1}$.

7 Sketch the graph of each of the following functions and state where it is not differentiable:
(a) $\sqrt{|x|}$; (b) $|x^2 - 4|$;
(c) $|2x - 3|$; (d) $x|x|$.

8 Let $f(x)$ be a function with the property that $f(x_1 + x_2) = f(x_1)f(x_2)$ for all x_1 and x_2. If $f(0) = 1$ and $f'(0) = 1$, show that $f'(x) = f(x)$ for all x.

9 If the derivative $f'(x)$ exists, then it can be calculated from the formula

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x - \Delta x)}{2\Delta x}.$$

Verify this statement for the special case $f(x) = x^2$, and then prove it in general. [To understand the statement, let P, Q, R be the points on the curve $y = f(x)$ that correspond to x, $x + \Delta x$, $x - \Delta x$, and write the slope of the secant through Q and R; and to prove it, notice that $f(x + \Delta x) - f(x - \Delta x) = f(x + \Delta x) - f(x) + f(x) - f(x - \Delta x)$.]

10 Show that the following function is differentiable at $x = 0$:

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

11 Show that the following function is not differentiable at $x = 0$:

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

12 If $f(x)$ is a function with the property that $|f(x)| \le x^2$ for all x, prove that $f(x)$ is differentiable at $x = 0$.

13 Consider the function $f(x)$ defined by

$$f(x) = \begin{cases} x^2 & \text{if } x \le a, \\ mx + b & \text{if } x > a, \end{cases}$$

where a, b, m are constants. Find what values m and b must have (in terms of a) in order for this function to be differentiable at all points.

SECTION 2.4

14 On a certain bicycle trip, the first half of the distance was covered at 30 mi/h and the second half at 20 mi/h. What was the average velocity? (It was not 25 mi/h.)

15 A silver dollar is thrown straight up from the roof of a 200-ft building. After t seconds, it is

$$s = 200 + 24t - 16t^2$$

feet above the ground. When does the dollar begin to fall? What is its speed when it has fallen 1 ft?

16 A capacitor (or condenser) in an electric circuit is a device for storing electric charge. If the amount of charge on a given capacitor at time t is $Q = 3t^2 + 5t + 2$ coulombs, find the current $I = dQ/dt$ in the circuit when $t = 3$.

17 Use the three-step rule to show that the rate of change of the volume of a sphere with respect to its radius equals the surface area.

SECTION 2.5

Evaluate the following limits.

18 $\lim_{x\to 2} \dfrac{2x - x^2}{2 - x}$.

19 $\lim_{x\to 0} \left(x + \dfrac{5}{x}\right)$.

20 $\lim_{x\to 3} \dfrac{x^2 - 6x + 9}{x - 3}$.

21 $\lim_{x\to 0} \dfrac{4x^2 - 5x}{x}$.

22 $\lim_{x\to 0} \dfrac{x^2(1 - x)}{3x}$.

23 $\lim_{x\to 0} \dfrac{x(1 - x)}{3x^2}$.

24 $\lim_{x\to 1} \dfrac{x + 2}{x^2 - 4}$.

25 $\lim_{x\to 2} \dfrac{x + 2}{x^2 - 4}$.

26 $\lim_{x\to 3} \dfrac{2x^2 + 3}{x + 4}$.

27 $\lim_{x\to 0} \dfrac{2 - 3\sqrt{x}}{1 + 9\sqrt{x}}$.

28 $\lim_{x\to 2} \dfrac{x^2 - 6x + 8}{x^2 - 5x + 6}$.

29 $\lim_{x\to -1} \dfrac{x^2 - 2x - 3}{x^2 - 1}$.

30 $\lim_{x\to 1} \dfrac{(x^2 + 3x - 4)^2}{x^2 - 7x + 6}$.

31 $\lim_{x\to 0} \dfrac{2x^2 + x - 6}{x + 2}$.

32 $\lim_{x\to -2} \dfrac{2x^2 + x - 6}{x + 2}$.

33 $\lim_{x\to 3} \dfrac{x - 3}{x^2 + x - 12}$.

34 $\lim_{x\to -4} \dfrac{x - 3}{x^2 + x - 12}$.

35 $\lim_{x\to 3} \dfrac{x^2 - x - 6}{x^2 - 7x + 12}$.

36 $\lim_{x\to 4} \dfrac{x^2 - x - 6}{x^2 - 7x + 12}$.

37 $\lim_{x\to 1} \dfrac{x + \sqrt{x} - 2}{x^3 - 1}$.

38 $\lim_{x\to 1} \dfrac{x^3 - 6x^2 + 3x + 2}{x^3 + x^2 - 3x + 1}$.*

39 $\lim_{x\to 2} \dfrac{x^3 - 4x}{x^3 - 3x^2 + 2x}$.

40 $\lim_{x\to 4} \dfrac{x^3 - 64}{x - 4}$.

41 $\lim_{x\to a} \dfrac{x^3 - a^3}{x^2 - a^2}$.

*If $x = a$ is a zero of a polynomial $p(x)$, then $x - a$ is a factor of $p(x)$ and the other factor can be found by long division (see Additional Problem 50 at the end of Chapter 1).

42 $\lim_{x\to -a} \dfrac{x^4 - a^4}{x^3 + a^3}$.

43 $\lim_{x\to 0} 2^{x^2}$.

44 $\lim_{x\to 0} 2^{-x^2}$.

45 $\lim_{x\to 0} 2^{-1/x^2}$.

46 $\lim_{x\to 0} \dfrac{2^{1/x^2} + 1}{2^{1/x^2} - 1}$.

47 $\lim_{x\to 0} \dfrac{2^{1/x} + 1}{2^{1/x} - 1}$.

48 $\lim_{x\to\infty} \dfrac{x}{\sqrt{x^2 + 1}}$.

49 $\lim_{x\to\infty} \dfrac{x}{\sqrt{x + 1}}$.

50 $\lim_{x\to\infty} \dfrac{2x^3 - x^2 + 7x - 3}{2 - x + 5x^2 - 4x^3}$.

51 $\lim_{x\to\infty} \dfrac{9x^{45} - x^9 + 2}{3x^{45} + x^{29} - 19}$.

52 $\lim_{x\to\infty} 2^x$.

53 $\lim_{x\to\infty} 2^{-x}$.

54 $\lim_{x\to\infty} 2^{1/x}$.

55 $\lim_{x\to\infty} (\sqrt{x + 1} - \sqrt{x})$.

56 $\lim_{x\to\infty} \dfrac{\sqrt{x + 1}}{\sqrt{9x + 1}}$.

57 $\lim_{x\to\infty} \dfrac{2^x - 2^{-x}}{2^x + 2^{-x}}$.

58 Consider the function $f(x)$ defined for $x \neq 0$ by $f(x) = [1/x]$, where $[1/x]$ denotes the greatest integer $\leq 1/x$, as in Additional Problem 59 at the end of Chapter 1. Sketch the graph of this function for $\frac{1}{4} \leq x \leq 2$, and also for $-2 \leq x \leq -\frac{1}{4}$. How does $f(x)$ behave as x approaches 0 from the positive side? From the negative side? Does $\lim_{x\to 0} f(x)$ exist?

59 Follow the directions in Problem 58 for the function $f(x) = (-1)^{[1/x]}$.

60 Follow the directions in Problem 58 for the function $f(x) = |x|(-1)^{[1/x]}$.

61 Consider the function $f(x)$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational,} \\ 1 & \text{if } x \text{ is irrational.} \end{cases}$$

For every a, $\lim_{x\to a} f(x)$ does not exist. Why?

62 Define a function $f(x)$ by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational,} \\ \dfrac{1}{n} & \text{if } x \text{ is a rational number } m/n \\ & \text{in lowest terms with } n > 0. \end{cases}$$

Show that $f(x)$ is continuous at irrational points and discontinuous at rational points.

63 The slope of the tangent line to the graph of the exponential function $y = 2^x$ at the point $(0, 1)$ is

$$\lim_{x\to 0} \frac{2^x - 1}{x}.$$

Estimate this slope to three decimal places by using a calculator to find the value of $(2^x - 1)/x$ for $x = 1$, 0.5, 0.1, 0.05, 0.01, 0.005, 0.001, 0.0005, 0.0001.

64 Same as Problem 63 for the function $y = 3^x$ and its tangent line at the point $(0, 1)$.

3 THE COMPUTATION OF DERIVATIVES

3.1 DERIVATIVES OF POLYNOMIALS

Differential calculus—the calculus of derivatives—takes its special flavor and importance from its many applications to the physical, biological, and social sciences. It would be pleasant to plunge into these applications immediately and get to the heart of the matter without any further delay. However, from the point of view of overall efficiency it is better to postpone this to the next chapter, and instead take a little time now to learn how to calculate derivatives with speed and accuracy.

As we know, the process of finding the derivative of a function is called *differentiation.* In Chapter 2 this process was based directly on the limit definition of the derivative,

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x},$$

or equivalently,

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

We have seen that this approach is rather slow and clumsy. Our purpose in the present chapter is to develop a small number of formal rules that will enable us to differentiate large classes of functions quickly, by purely mechanical procedures. In this section we learn how to write down the derivative of any polynomial by inspection, without having to think about limits at all; and by the end of the chapter we will be able to cope quite easily with messy algebraic functions like

$$\frac{x}{\sqrt{x^2 + 1}}, \qquad \left[\frac{x + \sqrt{x + 1}}{x - \sqrt{x + 1}}\right]^{1/3}, \qquad \text{and} \qquad \sqrt{1 + \sqrt{1 + \sqrt{1 + x}}}.$$

We will also learn how to differentiate many trigonometric functions. *Our goal in this phase of our work is computational skill, and needless to say, such skill comes only with practice. No one learns how to spell, or ski, or play a musical instrument, without constant practice accompanied by constant self-correction, and differentiation is no exception to this rule.*

Students will recall that a polynomial in x is a sum of constant multiples of powers of x in which each exponent is zero or a positive integer:

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

The way a polynomial is put together out of simpler pieces suggests the differentiation rules that we now discuss.

1 *The derivative of a constant is zero,*

$$\frac{d}{dx} c = 0.$$

The geometric meaning of this statement is that a horizontal straight line $y = f(x) = c$ has zero slope. To prove the statement from the definition, we notice that $\Delta y = f(x + \Delta x) - f(x) = c - c = 0$, so

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{0}{\Delta x} = \lim_{\Delta x \to 0} 0 = 0.$$

2 *If n is a positive integer, then*

$$\frac{d}{dx} x^n = nx^{n-1}.$$

In words, the derivative of x^n is obtained by bringing the exponent n down in front as a coefficient, then subtracting 1 from it to form the new exponent $n - 1$. We already know three special cases of this rule from Chapter 2:

$$\frac{d}{dx} x^2 = 2x, \qquad \frac{d}{dx} x^3 = 3x^2, \qquad \text{and} \qquad \frac{d}{dx} x^4 = 4x^3.$$

To prove this rule in general, we write $y = f(x) = x^n$ and use the *binomial theorem** to obtain

$$\begin{aligned} \Delta y &= f(x + \Delta x) - f(x) = (x + \Delta x)^n - x^n \\ &= \left[x^n + nx^{n-1} \Delta x + \frac{n(n-1)}{2} x^{n-2} (\Delta x)^2 + \cdots + (\Delta x)^n \right] - x^n \\ &= nx^{n-1} \Delta x + \frac{n(n-1)}{2} x^{n-2} (\Delta x)^2 + \cdots + (\Delta x)^n. \end{aligned}$$

This yields

$$\begin{aligned} \frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \\ &= \lim_{\Delta x \to 0} \left[nx^{n-1} + \frac{n(n-1)}{2} x^{n-2} \Delta x + \cdots + (\Delta x)^{n-1} \right] \\ &= nx^{n-1}, \end{aligned}$$

because Δx is a factor of each term in brackets beyond the first.

*For students who have forgotten the details of the binomial theorem, we state it as follows: If n is a positive integer, then

$$(a + b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2} a^{n-2}b^2 + \cdots + \frac{n(n-1) \cdots (n - k + 1)}{1 \cdot 2 \cdots k} a^{n-k}b^k + \cdots + b^n.$$

The precise form of this expansion can be understood without too much difficulty by simply thinking about the n-factor product

$$(a + b)^n = (a + b)(a + b) \cdots (a + b).$$

To multiply these factors out, we begin by choosing a from each factor, which gives the term a^n. If we next choose b from one factor and a from all the others, this can be done in n ways, so we get ba^{n-1} n times, or $na^{n-1}b$. Similarly, $n(n-1)/2$ is the number of ways b can be chosen from two factors and a from all the others, etc. The "etc." is explained more fully in Appendix B.1.

Our rule remains true when the exponent is a negative integer or a fraction. However, it is convenient to postpone giving a proof of this to a later part of the chapter.

3 *If c is a constant and $u = f(x)$ is a differentiable function of x, then*

$$\frac{d}{dx}(cu) = c\,\frac{du}{dx}.$$

That is, the derivative of a constant times a function equals the constant times the derivative of the function.* To prove this, we write $y = cu = cf(x)$ and observe that $\Delta y = cf(x + \Delta x) - cf(x) = c[f(x + \Delta x) - f(x)] = c\,\Delta u$, so

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{c\,\Delta u}{\Delta x} = c \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = c\,\frac{du}{dx}.$$

Combining rules 2 and 3, we see that

$$\frac{d}{dx}\,cx^n = cnx^{n-1}$$

for any constant c and any positive integer n.

Example 1 We are now in a position to calculate the following derivatives as fast as we can write:

$$\frac{d}{dx}\,3x^7 = 21x^6, \qquad \frac{d}{dx}\left(-\frac{1}{2}x^{12}\right) = -6x^{11}, \qquad \frac{d}{dx}\,22x^{101} = 2222x^{100},$$

$$\frac{d}{dx}\,55x = 55x^0 = 55, \qquad \frac{d}{dx}\left(\frac{10^{\sqrt{2}} + \log_{10}\pi}{\sqrt{19} + 1024}\right)^{999} = 0.$$

4 *If $u = f(x)$ and $v = g(x)$ are functions of x, then*

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}.$$

That is, the derivative of the sum of two functions equals the sum of the individual derivatives. The proof is routine: If we write $y = u + v = f(x) + g(x)$, then $\Delta y = [f(x + \Delta x) + g(x + \Delta x)] - [f(x) + g(x)] = [f(x + \Delta x) - f(x)] + [g(x + \Delta x) - g(x)] = \Delta u + \Delta v$, and therefore

$$\begin{aligned}\frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta u + \Delta v}{\Delta x} = \lim_{\Delta x \to 0}\left[\frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x}\right] \\ &= \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} + \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} = \frac{du}{dx} + \frac{dv}{dx}.\end{aligned}$$

In essentially the same way we can show that the derivative of a difference equals the difference of the derivatives,

$$\frac{d}{dx}(u - v) = \frac{du}{dx} - \frac{dv}{dx}.$$

*From now on we assume that every function we deal with is differentiable unless a specific statement is made to the contrary.

Further, these results can be extended without difficulty to any finite number of terms, as in

$$\frac{d}{dx}(u - v + w) = \frac{du}{dx} - \frac{dv}{dx} + \frac{dw}{dx}.$$

Example 2 It is now easy to differentiate any polynomial. For instance,

$$\begin{aligned}\frac{d}{dx}(15x^4 + 9x^3 - 7x^2 - 3x + 5) &= \frac{d}{dx}15x^4 + \frac{d}{dx}9x^3 - \frac{d}{dx}7x^2 - \frac{d}{dx}3x + \frac{d}{dx}5 \\ &= 60x^3 + 27x^2 - 14x - 3.\end{aligned}$$

With a little practice we can omit the middle step and write down the final result immediately by inspection.

Example 3 The function $y = (3x - 2)^4$ is a polynomial but is not in standard polynomial form. None of the rules established so far apply to this function directly, though later we will prove a formula that can be used here. Meanwhile we must first expand by the binomial theorem. This gives

$$\begin{aligned}y &= (3x - 2)^4 = [3x + (-2)]^4 \\ &= (3x)^4 + 4(3x)^3(-2) + \frac{4 \cdot 3}{2}(3x)^2(-2)^2 + \frac{4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3}(3x)(-2)^3 + (-2)^4 \\ &= 81x^4 - 216x^3 + 216x^2 - 96x + 16,\end{aligned}$$

so

$$\frac{dy}{dx} = 324x^3 - 648x^2 + 432x - 96.$$

Example 4 Even though the letters x and y are often used for the independent and dependent variables, there is obviously nothing to prevent us from using any letters we please, and the calculations work in just the same way. Thus,

$$s = 13t^3 - 11t^2 + 25$$

is a polynomial in t; and by the rules developed in this section, its derivative is clearly

$$\frac{ds}{dt} = 39t^2 - 22t.$$

Example 5 An object moves on a straight line in such a way that its position s at time t is given by

$$s = t^3 + 5t^2 - 8t.$$

What is its acceleration when it is at rest?

The velocity v and acceleration a are

$$v = \frac{ds}{dt} = 3t^2 + 10t - 8 \qquad \text{and} \qquad a = \frac{dv}{dt} = 6t + 10.$$

The object is at rest when $v = 0$ or

$$3t^2 + 10t - 8 = (3t - 2)(t + 4) = 0,$$

that is, when $t = \frac{2}{3}, -4$. The corresponding values of the acceleration are $a = 14, -14$.

PROBLEMS

1 Find the derivative of each function:
(a) $6x^9$;
(b) π^5;
(c) $-15x^4$;
(d) $3x^{500} + 15x^{100}$;
(e) $(x - 3)^2$;
(f) $\frac{1}{5}x^5 + \frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 + x$;
(g) $x^4 + x^3 + x^2 + x + 1$;
(h) $(x - 2)^5$;
(i) $x^{12} + 2x^6 - 4x^3 - 6x^2$;
(j) $(2x - 1)(3x^2 + 2)$.

2 Differentiate each of the following functions:
(a) $f(x) = x^{200} - 100x + 50$;
(b) $g(x) = (10x)^4$;
(c) $h(t) = t^{10} + 7t^8 - 9t^3 + 5t$;
(d) $F(y) = (y^2 - 1)(3y - 5)$;
(e) $G(x) = (x^2 + 3)(x^2 + x + 1)$;
(f) $H(t) = (1 + t^2)(1 + t^2 + t^3)$;
(g) $f(x) = (3x^3 - 2x^2)(15x^4 - 2x + 5)$;
(h) $g(x) = x(2x + 1)(2x - 1)$;
(i) $h(t) = (3t - 5)^2$;
(j) $y = 2x(3x^2 + 1)(x^2 - x + 2)$;
(k) $y = (3x + 2)(2x - 3)$;
(l) $y = (2x^2 + 3)(3x^3 - 4)$.

3 If s is the position at time t of an object moving on a straight line, find the velocity v and the acceleration a:
(a) $s = 12 - 6t + 3t^2$;
(b) $s = 13 - 9t + 6t^3$;
(c) $s = (3t - 2)^2$.

4 Find a function of x whose derivative is the given function:
(a) $3x^2$;
(b) $4x^2$;
(c) $3x^2 + 2x - 5$.

5 Find the line tangent to the curve $y = 3x^2 - 5x + 2$ at the point $(2, 4)$.

6 Find the points on the curve $y = 4x^3 + 6x^2 - 24x + 10$ at which the tangent is horizontal.

7 At what points on the curve $y = x^3 - x^2 - x - 1$ is the tangent horizontal?

8 At what points on the curve $y = 2x^3 - 3x^2 + 6x - 39$ is the tangent horizontal?

9 Show that the curve $y = 4x^3 + 4x - 2$ has no tangent line with slope 3.

10 Show that the curve $y = 2x^5 + 3x$ has no horizontal tangent line. What is the smallest slope a tangent line can have?

11 The line $x = a$ intersects the curve $y = \frac{1}{3}x^3 + 4x + 2$ at a point P and the curve $y = 2x^2 + x$ at a point Q. For what value (or values) of a are the tangents to these curves at P and Q parallel?

12 Find the vertex of the parabola $y = x^2 - 8x + 18$. Hint: The tangent at the vertex is horizontal.

13 Find the vertex of the parabola $y = ax^2 + bx + c$ by the method of Problem 12.

14 What values must the constants a, b, c have if the two curves $y = x^2 + ax + b$ and $y = cx - x^2$ have the same tangent at the point $(3, 3)$?

15 State conditions on the coefficients a, b, c, d so that the graph of the polynomial $y = ax^3 + bx^2 + cx + d$ has precisely
(a) two horizontal tangents;
(b) one horizontal tangent;
(c) no horizontal tangent.

16 Show that any two tangent lines to the parabola $y = ax^2$ intersect at a point that lies on the vertical line halfway between the points of tangency.

17 Find the equation of the tangent to the curve $y = x^3$ at the point (a, a^3). For what values of a does this tangent intersect the curve at another point?

18 Find the tangent to the curve $y = x^3$ that passes through the point $(0, 2)$.

19 There are two lines through the point $(2, 8)$ that are tangent to the curve $y = x^3$. Find them.

20 Sketch the curves $y = x^2$ and $y = -x^2 + 2x - 2$ on a single coordinate system, and use the sketch to decide whether there are any lines that are simultaneously tangent to both curves. If there are any, find their equations.

21 Let p be a positive constant and consider the parabola $x^2 = 4py$ with vertex at the origin and focus at the point $(0, p)$, as shown on the left in Fig. 3.1. Let (x_0, y_0) be a point on this parabola other than the vertex.
(a) Show that the tangent at (x_0, y_0) has y-intercept $-y_0$.
(b) Show that the triangle with vertices (x_0, y_0), $(0, -y_0)$,

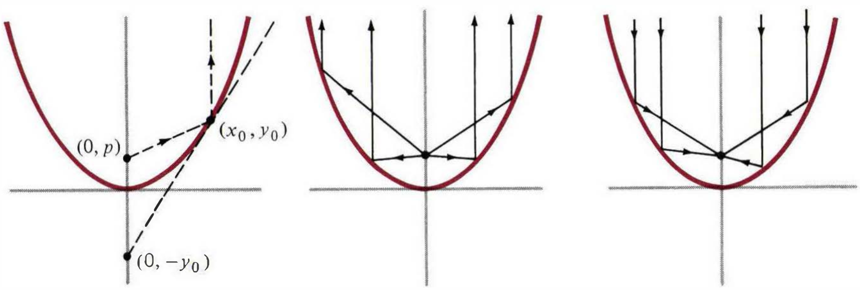


Figure 3.1 A parabolic reflector.

and (0, p) is isosceles. Hint: Use the distance formula.

(c) Suppose that a source of light is placed at the focus, and assume that each ray of light leaving the focus is reflected off the parabola in such a way that it makes equal angles with the tangent line at the point of reflection (the angle of incidence equals the angle of reflection). Use (b) to show that after reflection each ray points vertically upward, parallel to the axis (Fig. 3.1, center).*

22 The line through a point on a curve which is perpendicular to the tangent at that point is called the *normal* to the curve at that point. Find the normal to the curve $4y + x^2 = 5$ at the point (1, 1).

23 Consider the normal to the curve $y = x - x^2$ at the point (1, 0). Where does this line intersect the curve a second time?

24 Find the normal to the curve $y = 1 - x^2$ at the point $(3, -8)$.

*This is called the *reflection property* of parabolas. To form a three-dimensional idea of the way this property is used in the design of searchlights and automobile headlights, we have only to imagine a mirror constructed by rotating a parabola about its axis and silvering the inside of the resulting surface. Such a parabolic reflector can also be used in reverse (Fig. 3.1, right), to gather faint incoming rays parallel to the axis and concentrate them at the focus. This is the basic principle of radar antennas, radio telescopes, and reflecting optical telescopes. The great telescope on Palomar Mountain in California has a 15-ton glass reflector that is 200 in (almost 17 ft) in diameter, and the accurate grinding of this enormous mirror required 11 years of work.

3.2 THE PRODUCT AND QUOTIENT RULES

In Section 3.1 we learned how to differentiate sums, differences, and constant multiples of functions. We now consider

$$\text{products } uv \qquad \text{and} \qquad \text{quotients } \frac{u}{v},$$

where u and v are understood to be differentiable functions of x.

Since the derivative of a sum is the sum of the derivatives, it is natural to guess that the derivative of a product might equal the product of the derivatives. However, it is quite easy to construct examples showing that this is not true. For instance, the product of x^3 and x^4 is x^7, so the derivative of the product is $7x^6$, but the product of the individual derivatives is $3x^2 \cdot 4x^3 = 12x^5$. This shows that our preliminary guess about the derivative of a product is incorrect. The correct formula for differentiating products is rather surprising.

5* *The product rule:*

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}. \tag{1}$$

Students may wish to keep in mind the following verbal statement of this rule: *The derivative of the product of two functions is the first times the derivative of*

*We continue the numbering started in Section 3.1.

the second plus the second times the derivative of the first. To prove this, we write $y = uv$ and let the independent variable x be changed by an amount Δx, to $x + \Delta x$. This produces corresponding changes Δu, Δv, Δy in the variables u, v, y, and we have

$$y + \Delta y = (u + \Delta u)(v + \Delta v) = uv + u\,\Delta v + v\,\Delta u + \Delta u\,\Delta v,$$

$$\Delta y = (y + \Delta y) - y = u\,\Delta v + v\,\Delta u + \Delta u\,\Delta v,$$

$$\frac{\Delta y}{\Delta x} = u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \Delta u\frac{\Delta v}{\Delta x}.$$

Taking limits as $\Delta x \to 0$ yields

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} + 0 \cdot \frac{dv}{dx},$$

which is equivalent to (1). We have used the fact that $\Delta u \to 0$ as $\Delta x \to 0$. This is due to the continuity of u, which follows from the differentiability by the argument given in Section 2.6.

Example 1 We first test formula (1) on the factors x^3 and x^4, whose product we already know has derivative $7x^6$. We get

$$\begin{aligned}\frac{d}{dx}(x^3 \cdot x^4) &= x^3\frac{d}{dx}x^4 + x^4\frac{d}{dx}x^3\\ &= x^3 \cdot 4x^3 + x^4 \cdot 3x^2 = 7x^6,\end{aligned}$$

as we should. As a more complicated example, we apply our formula to the function $y = (x^3 - 4x)(3x^4 + 2)$:

$$\begin{aligned}\frac{dy}{dx} &= (x^3 - 4x)\frac{d}{dx}(3x^4 + 2) + (3x^4 + 2)\frac{d}{dx}(x^3 - 4x)\\ &= (x^3 - 4x)(12x^3) + (3x^4 + 2)(3x^2 - 4)\\ &= 12x^6 - 48x^4 + 9x^6 - 12x^4 + 6x^2 - 8\\ &= 21x^6 - 60x^4 + 6x^2 - 8.\end{aligned}$$

Notice that we can also proceed by multiplying the two factors at the beginning and then differentiating. This gives

$$y = 3x^7 - 12x^5 + 2x^3 - 8x,$$

so

$$\frac{dy}{dx} = 21x^6 - 60x^4 + 6x^2 - 8,$$

as we expect. Since we can solve this problem without using the product rule, it may appear that this rule is unnecessary. This is indeed true when both factors are polynomials, because the product of two polynomials is also a polynomial. However, in the more complex situations that lie ahead—in which the factors are often different types of functions—it will be clear that the product rule is indispensable.

6 *The quotient rule:*

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\,du/dx - u\,dv/dx}{v^2} \tag{2}$$

at all values of x where $v \neq 0$.

Most people find it easier to remember the working instructions given by (2) in words rather than symbols: *The derivative of a quotient is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the denominator squared.* To prove this, we write $y = u/v$ and let x change by an amount Δx. As before, this produces changes Δu, Δv, Δy in the variables u, v, y, and we have

$$y + \Delta y = \frac{u + \Delta u}{v + \Delta v}, \qquad \Delta y = \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v},$$

$$\Delta y = \frac{uv + v\,\Delta u - uv - u\,\Delta v}{v(v + \Delta v)} = \frac{v\,\Delta u - u\,\Delta v}{v(v + \Delta v)},$$

$$\frac{\Delta y}{\Delta x} = \frac{v\,\Delta u/\Delta x - u\,\Delta v/\Delta x}{v(v + \Delta v)}.$$

If we now take limits as $\Delta x \to 0$ we obtain formula (2),

$$\frac{dy}{dx} = \frac{v\,du/dx - u\,dv/dx}{v^2},$$

since $\Delta v \to 0$ as $\Delta x \to 0$, by the continuity of v (recall that v is continuous *because* it is differentiable).

Example 2 To differentiate the quotient $y = (3x^2 - 2)/(x^2 + 1)$, we follow the verbal prescription,

$$\frac{dy}{dx} = \frac{(x^2 + 1)(d/dx)(3x^2 - 2) - (3x^2 - 2)(d/dx)(x^2 + 1)}{(x^2 + 1)^2}$$

$$= \frac{(x^2 + 1)(6x) - (3x^2 - 2)(2x)}{(x^2 + 1)^2}$$

$$= \frac{6x^3 + 6x - 6x^3 + 4x}{(x^2 + 1)^2} = \frac{10x}{(x^2 + 1)^2}.$$

With practice, calculations like this can be performed very quickly. For instance,

$$\frac{d}{dx}\frac{1}{x^2 + 1} = \frac{(x^2 + 1)(0) - 1(2x)}{(x^2 + 1)^2} = \frac{-2x}{(x^2 + 1)^2},$$

$$\frac{d}{dx}\frac{3x}{x^2 + 1} = \frac{(x^2 + 1)(3) - 3x(2x)}{(x^2 + 1)^2} = \frac{3 - 3x^2}{(x^2 + 1)^2},$$

$$\frac{d}{dx}\frac{2x + 1}{3x - 1} = \frac{(3x - 1)(2) - (2x + 1)(3)}{(3x - 1)^2} = \frac{-5}{(3x - 1)^2}.$$

The quotient rule enables us to extend rule 2 of Section 3.1,

$$\frac{d}{dx}x^n = nx^{n-1}, \tag{3}$$

to the case in which n is a negative integer. To make the negative character of n more visible, we write $n = -m$, where m is a positive integer. Now, using (2) and the fact that (3) is known to be valid for positive integer exponents, we have

$$\frac{d}{dx}x^n = \frac{d}{dx}x^{-m} = \frac{d}{dx}\frac{1}{x^m} = \frac{x^m(0) - 1(mx^{m-1})}{(x^m)^2}$$

$$= \frac{-mx^{m-1}}{x^{2m}} = -mx^{-m-1} = nx^{n-1},$$

which proves our statement. Thus, for example,

$$\frac{d}{dx}x^{-1} = (-1)x^{-2} = -x^{-2}, \qquad \frac{d}{dx}x^{-2} = (-2)x^{-3} = -2x^{-3}, \qquad \text{etc.}$$

Since (3) is clearly true for $n = 0$, we now know that it is valid for all integer exponents.

Example 3 To differentiate

$$y = 3x^2 - \frac{2}{x^3},$$

we write it as

$$y = 3x^2 - 2x^{-3}.$$

Then

$$\frac{dy}{dx} = 6x + 6x^{-4},$$

which can be rewritten as

$$\frac{dy}{dx} = 6x + \frac{6}{x^4}$$

if positive exponents are preferred.

We urge students to memorize the product and quotient rules, and to anchor them in their minds by conscientious practice.

PROBLEMS

In Problems 1–8, differentiate each function two ways, and verify that your answers agree.

1 $(x - 1)(x + 1)$.
2 $(2x - 6)(3x^2 + 9)$.
3 $(3x^2 + 1)(x^3 + 6x)$.
4 $(x - 1)(x^4 + x^3 + x^2 + x + 1)$.
5 $(3x - 1)(2x^2 + x)$.
6 $(x^3 - 3x)(x^2 + 5)$.
7 $(4x^5 + x)(3x + 1)$.
8 $(x^4 + 1)(x^4 - 1)$.

In Problems 9–28, differentiate each function and simplify your answer as much as possible.

9 $\dfrac{x + 1}{x - 1}$.

10 $\dfrac{1}{x^2 + 2}$.

11 $\dfrac{2x^3 + 1}{x + 2}$.

12 $\dfrac{3x + 4}{7x + 8}$.

13 $\dfrac{3x}{1 + 2x^2}$.

14 $\dfrac{4x - x^4}{x^3 + 2}$.

15 $\dfrac{1 - x^2}{1 + x^2}$.

16 $\dfrac{2x + 1}{1 - x^2}$.

17 $\dfrac{1}{x-1} - \dfrac{1}{x+1}$.

18 $4x^5 - \dfrac{1}{x^3}$.

19 $\dfrac{1}{2 - \dfrac{3}{x}}$.

20 $\dfrac{1}{3x} - \dfrac{1}{4x^2}$.

21 $\dfrac{x-1}{x^2+2x+1}$.

22 $\dfrac{3x^3+2x^2-3x+7}{2x-3}$.

23 $\dfrac{\dfrac{1}{x} - \dfrac{3}{x^2}}{\dfrac{5}{x^3} - \dfrac{7}{x^4}}$.

24 $\dfrac{1}{1-2x^{-2}}$.

25 $\dfrac{2x^2}{4x - \dfrac{5}{6x^4}}$.

26 $\dfrac{4x}{x^3+5x-3}$.

27 $\dfrac{2x}{x-1} - \dfrac{x+2}{2x}$.

28 $x^4 - \dfrac{1}{x^2-1}$.

In Problems 29–34, find the derivative in two ways, first by dividing and then by using the quotient rule, and show that your answers agree.

29 $\dfrac{4x+4}{x}$.

30 $\dfrac{2x+6x^4-2x^6}{x^5}$.

31 $\dfrac{1+x^4}{x^2}$.

32 $\dfrac{3x^4-4x^3+5x-4}{x^2}$.

33 $\dfrac{x+15}{x^2}$.

34 $\dfrac{6-3x^2+x^6}{x^4}$.

35 Find all points on the curve $y = 6/x$ where the tangent is parallel to the line $2x + 3y + 1 = 0$.

36 Find the equations of the tangent lines to the curve $y = (x-1)/(x+1)$ that are parallel to the line $x - 2y = 1$.

37 Sketch the graph of the curve $y = x/(x+1)$. How many tangent lines pass through the point $(1, 3)$? Find the x-coordinates of the points of tangency of these lines.

In Problems 38–43, find the equations of the stated lines.

38 The tangent and normal to $y = \dfrac{6}{x+2}$ at $(1, 2)$.

39 The tangent and normal to $y = \dfrac{5}{x^2+1}$ at $x = 2$.

40 The tangent to $y = \dfrac{x^3+x}{x-1}$ at $(2, 10)$.

41 The normal to $y = \dfrac{1-2x+3x^2}{1+x^2}$ at $(0, 1)$.

42 The tangent and normal to $y = \dfrac{1}{2x-1}$ at $(2, \frac{1}{3})$.

43 The tangent and normal to $y = \dfrac{x-2}{x+1}$ at $(2, 0)$.

44 Show that the tangents to the curves $y = (x^2+45)/x^2$ and $y = (x^2-4)/(x^2+1)$ at $x = 3$ are perpendicular to each other.

45 Let P be a point on the first-quadrant part of the curve $y = 1/x$. Show that the triangle determined by the x-axis, the tangent at P, and the line from P to the origin is isosceles, and find its area.

46 Use the product rule to verify rule 3 of Section 3.1: If c is a constant and u is a function of x, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}.$$

47 Sketch the curve $y = 2/(1+x^2)$, and find the points on it at which the normal passes through the origin.

48 Verify the location of the high and low points on the graph of

$$y = \frac{x}{x^2-3x+2}$$

as stated in Example 4 of Section 1.6.

3.3 COMPOSITE FUNCTIONS AND THE CHAIN RULE

Let us consider the problem of differentiating the function

$$y = (x^3+2)^5. \tag{1}$$

We can do this with the tools we now have by using the binomial theorem to expand the function into the polynomial

$$y = x^{15} + 10x^{12} + 40x^9 + 80x^6 + 80x^3 + 32. \tag{2}$$

It now follows at once that

$$\frac{dy}{dx} = 15x^{14} + 120x^{11} + 360x^8 + 480x^5 + 240x^2. \tag{3}$$

In this case the work of expansion is bothersome but not too difficult. However, few of us would willingly attempt to carry out the same procedure for the function $y = (x^3 + 2)^{100}$. It is much better to develop the chain rule, which enables us to differentiate both functions with equal ease—and a host of others as well.

For this purpose it is important to understand the structure of the function (1). We accomplish this by introducing an auxiliary variable $u = x^3 + 2$, so that (1) can be decomposed into simpler pieces as follows:

$$y = u^5 \qquad \text{where} \qquad u = x^3 + 2. \tag{4}$$

Working in the other direction, we can reconstruct (1) out of these pieces by substituting the expression for u into $y = u^5$. Such a function is called a *composite function*, or often a *function of a function*. We have already encountered this idea in Section 1.5. In general, suppose that y is a function of u, where u in turn is a function of x, say

$$y = f(u) \qquad \text{where} \qquad u = g(x). \tag{5}$$

The corresponding composite function is the single function

$$y = f(g(x)), \tag{6}$$

obtained by substituting $u = g(x)$ into $y = f(u)$.

Our position now is this. We assume we are confronted by the composite function (6), and we wish to learn how to differentiate it by decomposing it into the simpler functions (5) and using the presumably simpler derivatives of these functions. This is what the chain rule is all about.

7 *The chain rule: Under the circumstances described above, we have*

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}. \tag{7}$$

As we see, in this form the chain rule has the appearance of a trivial algebraic identity; it is easily remembered because the Leibniz fractional notation for derivatives suggests that du can be canceled from the two "fractions" on the right. Its intuitive content is easy to grasp if we think of derivatives as rates of change:

If y changes a times as fast as u
and u changes b times as fast as x,
then y changes ab times as fast as x.

Or, in everyday terms, if a car travels twice as fast as a bicycle and the bicycle is four times as fast as a walking man, then the car travels $2 \cdot 4 = 8$ times as fast as the man.

Before looking into the proof of the chain rule, let us see how it applies to the problem just discussed, in which (1) is the given function and (4) is its decomposition. Formula (7) gives

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 5u^4 \cdot 3x^2 = 15x^2(x^3 + 2)^4, \tag{8}$$

where the auxiliary variable u is replaced by $x^3 + 2$ in the last step. It is not immediately obvious that this result is the same as (3), but the equivalence is easy to establish.* Further, the derivative of $y = (x^3 + 2)^{100}$ can be computed just as easily in just the same way: We write

$$y = u^{100} \qquad \text{where} \qquad u = x^3 + 2$$

and use (7) to obtain

$$\begin{aligned}\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} &= 100u^{99} \cdot 3x^2 \\ &= 300x^2(x^3 + 2)^{99}.\end{aligned}$$

As these examples show, the chain rule is a very powerful tool.

We begin the proof of (7) with the usual change Δx in the independent variable x; this produces a change Δu in the variable u, and this in turn produces a change Δy in the variable y. We know that differentiability implies continuity, so $\Delta u \to 0$ as $\Delta x \to 0$. If we look at the definitions of the three derivatives we are trying to link together,

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}, \qquad \frac{dy}{du} = \lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u}, \qquad \frac{du}{dx} = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}, \tag{9}$$

then it is natural to try to complete the proof as follows: By simple algebra we have

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}, \tag{10}$$

so

$$\begin{aligned}\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x} = \left[\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u}\right]\left[\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}\right] \\ &= \left[\lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u}\right]\left[\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}\right] = \frac{dy}{du} \cdot \frac{du}{dx}\end{aligned} \tag{11}$$

This reasoning is almost correct, but not quite. The difficulty centers on a possible division by zero. In computing dy/dx from the definition in (9), we know that it is part of the meaning of this formula that the increment Δx is small and approaches zero *but is never equal to zero.* On the other hand, it can happen that Δx induces no actual change in u, so that $\Delta u = 0$, and this possibility invalidates (10) and (11). This flaw can be patched up by an ingenious bit of mathematical trickery. We give the argument in the footnote below for students who wish to examine it.†

It will become clear as we proceed that the chain rule is indispensable for almost all differentiations above the simplest level. An important special case

*We hope students did not accept the expansion in (2)—and similarly will not accept the stated equivalence of (8) and (3)—without checking the details for themselves. Total skepticism is the recommended state of mind for studying this (or any similar) book: Take nothing on faith; verify all omitted calculations; believe nothing unless you have seen and understood it for yourself.

†We begin with the definition of the derivative dy/du, which is

$$\frac{dy}{du} = \lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u}.$$

This is equivalent to

has been illustrated in connection with finding the derivatives of $(x^3 + 2)^5$ and $(x^3 + 2)^{100}$. The general principle here is expressed by the formula

$$\frac{d}{dx}(\ )^n = n(\ )^{n-1}\frac{d}{dx}(\ ),$$

where any differentiable function of x can be inserted in the parentheses. If we denote the function by u, the formula can be written as follows.

8 *The power rule:*

$$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}. \tag{12}$$

At this stage of our work we know that the exponent n is allowed to be any positive or negative integer (or zero). In Section 3.5 we will see that (12) is also valid for all fractional exponents.

Example 1 To differentiate $y = (3x^4 + 1)^7$, we make a routine application of (12):

$$\frac{dy}{dx} = 7(3x^4 + 1)^6\frac{d}{dx}(3x^4 + 1) = 7(3x^4 + 1)^6 \cdot 12x^3.$$

But to differentiate $y = [(3x^4 + 1)^7 + 1]^5$, we apply (12) twice in succession:

$$\begin{aligned}\frac{dy}{dx} &= 5[(3x^4 + 1)^7 + 1]^4\frac{d}{dx}[(3x^4 + 1)^7 + 1]\\ &= 5[(3x^4 + 1)^7 + 1]^4 \cdot 7(3x^4 + 1)^6\frac{d}{dx}(3x^4 + 1)\\ &= 5[(3x^4 + 1)^7 + 1]^4 \cdot 7(3x^4 + 1)^6 \cdot 12x^3.\end{aligned}$$

After this procedure becomes familiar and more or less automatic, it is often possible to skip the intermediate steps and write down the answer at once. For the sake of clarity, in these calculations we have left the various factors in exactly the positions where they appear in the successive steps of the work. Normally, of course, we tidy things up a bit, and write the first answer, for example, in the more compact form $84x^3(3x^4 + 1)^6$.

Example 2 If $y = [(1 - 2x)/(1 + 2x)]^4$, then by (12) and the quotient rule we have

$$\frac{\Delta y}{\Delta u} = \frac{dy}{du} + \epsilon$$

or

$$\Delta y = \frac{dy}{du}\Delta u + \epsilon\,\Delta u,$$

where $\epsilon \to 0$ as $\Delta u \to 0$. It is assumed in these equations that Δu is a nonzero increment in u, but the last equation is valid even when $\Delta u = 0$. Dividing this by a nonzero increment Δx yields

$$\frac{\Delta y}{\Delta x} = \frac{dy}{du}\frac{\Delta u}{\Delta x} + \epsilon\frac{\Delta u}{\Delta x},$$

and on letting $\Delta x \to 0$ we obtain the chain rule (7), since $\epsilon \to 0$.

$$\begin{aligned}\frac{dy}{dx} &= 4\left(\frac{1-2x}{1+2x}\right)^3 \frac{d}{dx}\left(\frac{1-2x}{1+2x}\right)\\ &= 4\left(\frac{1-2x}{1+2x}\right)^3 \cdot \frac{(1+2x)(-2)-(1-2x)(2)}{(1+2x)^2}\\ &= \frac{-16(1-2x)^3}{(1+2x)^5}.\end{aligned}$$

Example 3 If $y = (x^2 - 1)^3(x^2+1)^{-2}$, then by combining (12) with the product rule we have

$$\begin{aligned}\frac{dy}{dx} &= (x^2-1)^3 \frac{d}{dx}(x^2+1)^{-2} + (x^2+1)^{-2}\frac{d}{dx}(x^2-1)^3\\ &= (x^2-1)^3 \cdot (-2)(x^2+1)^{-3}(2x) + (x^2+1)^{-2}\cdot 3(x^2-1)^2(2x).\end{aligned}$$

To simplify this, we take out the factor $2x(x^2-1)^2$, get rid of the negative exponents, and reduce to a common denominator:

$$\begin{aligned}\frac{dy}{dx} &= 2x(x^2-1)^2\left[\frac{-2(x^2-1)}{(x^2+1)^3} + \frac{3}{(x^2+1)^2}\right]\\ &= 2x(x^2-1)^2\left[\frac{-2(x^2-1)+3(x^2+1)}{(x^2+1)^3}\right] = \frac{2x(x^2-1)^2(x^2+5)}{(x^2+1)^3}.\end{aligned}$$

In Chapter 4 we will be using derivatives as tools in many concrete problems of science and geometry, and it will then be clear that it is worth a little extra effort to put the derivatives we calculate into their simplest possible forms.

There are a few concluding remarks that ought to be made. We have not yet explained why the expression "chain rule" is appropriate. The reason is this. In (7) we are dealing with three variables y, u, and x that are linked together step by step in a chain in such a way that each is dependent on the next. We can suggest this relation by writing

$$y \quad \text{depends on} \quad u \quad \text{depends on} \quad x.$$

The formula

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

tells us how to differentiate the first variable with respect to the last by taking into account each individual link in the chain. This formula can easily be extended to more variables. For instance, if x depends in turn on z, then

$$\frac{dy}{dz} = \frac{dy}{du}\cdot\frac{du}{dx}\cdot\frac{dx}{dz};$$

if z depends on w, then

$$\frac{dy}{dw} = \frac{dy}{du}\cdot\frac{du}{dx}\cdot\frac{dx}{dz}\cdot\frac{dz}{dw};$$

and so on. Each new variable adds a new link to the chain and a new derivative to the formula.

PROBLEMS

Find dy/dx in Problems 1–10.

1 $y = \dfrac{1}{(2-5x)^2}$.

2 $y = (4+5x)^4$.

3 $y = (x^2+4x-1)^3$.

4 $y = \dfrac{x-2}{(3x+5)^3}$.

5 $y = (5-x)^3(4+x)^5$.

6 $y = [1+(1+x)^5]^6$.

7 $y = \dfrac{1}{(3x+1)^4}$.

8 $y = (5-7x^4)^{-5}$.

9 $y = (1-6x)^6$.

10 $y = \dfrac{2-x}{(3-x)^4}$.

In Problems 11–16, express dy/dx in terms of x.

11 $y = (1-u)^4$, $u = \dfrac{1}{x^3}$.

12 $y = \dfrac{1}{u^2} - \dfrac{1}{u^3}$, $u = 3x-1$.

13 $y = (1+u^2)^2$, $u = (2x+1)^2$.

14 $y = u^7$, $u = \dfrac{1}{3-4x}$.

15 $y = u(1-u)^4$, $u = \dfrac{1}{x^5}$.

16 $y = \dfrac{u}{1+u}$, $u = \dfrac{x}{1+x}$.

In Problems 17–32, find dy/dx.

17 $y = (x^5-3x)^4$.

18 $y = (x^2-2)^{500}$.

19 $y = (x+x^2-2x^5)^6$.

20 $y = (1-3x)^{-1}$.

21 $y = (12-x^2)^{-2}$.

22 $y = [1-(3x-2)^3]^4$.

23 $y = (x^2+3x-5)^7$.

24 $y = (x^3-7x)^5$.

25 $y = (3x^2-5x+2)^{-6}$.

26 $y = \dfrac{1}{(x^3-5x+1)^5}$.

27 $y = (5x+3)^4(4x-3)^7$.

28 $y = (x^2-2)^5(x^2+2)^{10}$.

29 $y = x^2(9-x^2)^{-2}$.

30 $y = (1-2x)^{-4}(x^2-x)^2$.

31 $y = (2x-3)^8(3x^2-x+2)^{10}$.

32 $y = (5x^2+6)^3(x^3-3)^4$.

In Problems 33–36, find ds/dt.

33 $s = \dfrac{(2t-1)^3}{(t^2+3)^2}$.

34 $s = \dfrac{1}{(2t-1)^2}$.

35 $s = \dfrac{6}{(5-4t)^3}$.

36 $s = \dfrac{t^4-10t^2}{(t^2-6)^2}$.

In Problems 37–39, find dy/dx by two methods, and verify that your answers agree.

37 $y = (2x-1)^5(x+3)^5 = (2x^2+5x-3)^5$.

38 $y = \dfrac{1}{(1-2x^2)^3} = (1-2x^2)^{-3}$.

39 $y = \dfrac{(3x+1)^4}{(1-2x)^4} = \left(\dfrac{3x+1}{1-2x}\right)^4$.

In Problems 40–42, find dy/dx as a function of x in two ways, first without the power rule and then using the power rule.

40 $y = u^2$, $u = x^2-3x+2$.

41 $y = u^2-3u+2$, $u = 7-5x$.

42 $y = u^3$, $u = x - \dfrac{1}{x}$.

In Problems 43 and 44, find the equation of the tangent line to the given curve at the given point.

43 $y = (x^3-x^2+x)^8$, $(1, 1)$.

44 $y = \dfrac{x}{(8-x^2)^5}$, $(3, -3)$.

45 If u is a function of x, express each of the following in terms of u and du/dx:

(a) $\dfrac{d}{dx}u^3$;

(b) $\dfrac{d}{dx}(2u-1)^2$;

(c) $\dfrac{d}{dx}(u^2-2)^2$.

46 Find a function $y = f(x)$ for which

(a) $\dfrac{dy}{dx} = 2(x^2-1)\cdot 2x$;

(b) $\dfrac{dy}{dx} = 4(x^2-1)^2\cdot 2x$;

(c) $\dfrac{dy}{dx} = 2(x^3-2)\cdot 3x^2$;

(d) $\dfrac{dy}{dx} = 3(x^3-2)^2\cdot 3x^2$.

3.4 SOME TRIGONOMETRIC DERIVATIVES

So far, the only truly fundamental functions we have learned how to differentiate are the simple power functions x^n:

$$\frac{d}{dx} x^n = nx^{n-1}.$$

All other functions have been constructed from these by addition, subtraction, multiplication, division and by forming a function of a function, and our general rules have allowed us to find the derivatives of these combinations. We now expand our kit of tools beyond elementary algebra by learning how to differentiate the basic trigonometric functions $\sin x$ and $\cos x$:

$$\frac{d}{dx} \sin x = \cos x \tag{1}$$

and

$$\frac{d}{dx} \cos x = -\sin x. \tag{2}$$

To obtain these formulas we go back to the definition of the derivative of an arbitrary function $f(x)$,

$$\frac{d}{dx} f(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

When we apply this definition to the function $f(x) = \sin x$, and use the addition formula for the sine [identity (3) in Section 1.7], we obtain

$$\begin{aligned} \frac{d}{dx} \sin x &= \lim_{\Delta x \to 0} \frac{\sin (x + \Delta x) - \sin x}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\sin x \cos \Delta x + \cos x \sin \Delta x - \sin x}{\Delta x}. \end{aligned} \tag{3}$$

An algebraic rearrangement of (3) gives

$$\begin{aligned} \frac{d}{dx} \sin x &= \lim_{\Delta x \to 0} \left[\cos x \left(\frac{\sin \Delta x}{\Delta x} \right) - \sin x \left(\frac{1 - \cos \Delta x}{\Delta x} \right) \right] \\ &= \cos x \left[\lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x} \right] - \sin x \left[\lim_{\Delta x \to 0} \frac{1 - \cos \Delta x}{\Delta x} \right], \end{aligned} \tag{4}$$

since $\cos x$ and $\sin x$ are constants with respect to this limit operation. The limits in brackets here are precisely the limits (3) and (4) in Section 2.5, with θ replaced by Δx:

$$\lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x} = 1 \qquad \text{and} \qquad \lim_{\Delta x \to 0} \frac{1 - \cos \Delta x}{\Delta x} = 0.$$

These facts enable us to write (4) as

$$\begin{aligned} \frac{d}{dx} \sin x &= \cos x \cdot 1 - \sin x \cdot 0 \\ &= \cos x, \end{aligned}$$

which is (1).

The proof of (2) is similar, with the difference that we use the addition formula for the cosine [identity (4) in Section 1.7]:

$$
\begin{aligned}
\frac{d}{dx}\cos x &= \lim_{\Delta x\to 0}\frac{\cos(x+\Delta x)-\cos x}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{\cos x\cos\Delta x-\sin x\sin\Delta x-\cos x}{\Delta x}\\
&= \lim_{\Delta x\to 0}\left[-\sin x\left(\frac{\sin\Delta x}{\Delta x}\right)-\cos x\left(\frac{1-\cos\Delta x}{\Delta x}\right)\right]\\
&= -\sin x\left[\lim_{\Delta x\to 0}\frac{\sin\Delta x}{\Delta x}\right]-\cos x\left[\lim_{\Delta x\to 0}\frac{1-\cos\Delta x}{\Delta x}\right]\\
&= -\sin x\cdot 1-\cos x\cdot 0=-\sin x.
\end{aligned}
$$

This establishes (2).

When (1) and (2) are combined with the chain rule, we have the main tools of this section,

$$\frac{d}{dx}\sin u=\cos u\,\frac{du}{dx} \tag{5}$$

and

$$\frac{d}{dx}\cos u=-\sin u\,\frac{du}{dx}, \tag{6}$$

where $u = u(x)$ is understood to be any differentiable function of x. Students must learn to use these formulas in combination with all previous rules of differentiation.

Example 1 Find dy/dx if $y = \sin(5 + 4x^3)$. Here $u = 5 + 4x^3$, so by (5),

$$\frac{dy}{dx}=\cos(5+4x^3)\,\frac{d}{dx}(5+4x^3)=12x^2\cos(5+4x^3).$$

Example 2 Find dy/dx if $y = \cos(\sin x)$. Here $u = \sin x$, so by (6) and (1),

$$\frac{dy}{dx}=-\sin(\sin x)\,\frac{d}{dx}(\sin x)=-\cos x\cdot\sin(\sin x).$$

Example 3 Find dy/dx if $y = \sin[(1 - x^2)/(1 + x^2)]$. Here $u = (1 - x^2)/(1 + x^2)$, so by (5) and the quotient rule,

$$
\begin{aligned}
\frac{dy}{dx}&=\cos\left(\frac{1-x^2}{1+x^2}\right)\frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right)\\
&=\cos\left(\frac{1-x^2}{1+x^2}\right)\cdot\frac{(1+x^2)(-2x)-(1-x^2)2x}{(1+x^2)^2}\\
&=\frac{-4x}{(1+x^2)^2}\cos\left(\frac{1-x^2}{1+x^2}\right).
\end{aligned}
$$

Example 4 Find dy/dx if $y = \cos(1 + \sin 5x)$. Here $u = 1 + \sin 5x$, so finding du/dx requires an application of the chain rule, and we have

$$\frac{dy}{dx} = -\sin(1 + \sin 5x)\frac{d}{dx}(1 + \sin 5x)$$

$$= -\sin(1 + \sin 5x) \cdot \cos 5x \cdot \frac{d}{dx}(5x)$$

$$= -5 \cos 5x \cdot \sin(1 + \sin 5x).$$

In these examples the chain rule acquires additional scope by being used in ways not covered by the ideas of the preceding section.

We remind the reader of the standard notational convention for powers of the trigonometric functions: In general $\sin^n x$ means $(\sin x)^n$. However, it must not be forgotten that $(\sin x)^{-1}$ is *never* written $\sin^{-1}x$. The reason for this is that the latter notation is reserved for the *inverse sine* function, which will not play any part in our work until Chapter 9 but will be in regular use from that point on.

Example 5 Find dy/dx if $y = \sin^5 7x^2$. Here we let $w = \sin 7x^2$. Then $y = w^5$ and

$$\frac{dy}{dx} = 5w^4\frac{dw}{dx} = 5w^4 \cdot \cos 7x^2 \cdot \frac{d}{dx}(7x^2)$$

$$= 5w^4 \cdot \cos 7x^2 \cdot 14x$$

$$= 70x \sin^4 7x^2 \cdot \cos 7x^2.$$

In Section 1.7 we stated that in calculus it is preferred to use radian measure for angles instead of degree measure. We are now able to explain the reason for this. Let $\sin x°$ and $\cos x°$ denote the sine and cosine of an angle of x degrees. Since an angle of x degrees has radian measure $\pi x/180$, we have

$$\sin x° = \sin\frac{\pi x}{180}.$$

Then

$$\frac{d}{dx}\sin x° = \cos\frac{\pi x}{180}\frac{d}{dx}\left(\frac{\pi x}{180}\right) = \frac{\pi}{180}\cos\frac{\pi x}{180},$$

so

$$\frac{d}{dx}\sin x° = \frac{\pi}{180}\cos x°.$$

If we insist on using degrees to measure angles, then we are forced to use this formula instead of the simpler formula (1). We therefore use radian measure to prevent our calculations from being cluttered up by the repeated occurrence of the unwelcome factor $\pi/180$.

The remaining four trigonometric functions can be defined in terms of $\sin x$ and $\cos x$, as at the end of Section 1.7, and their derivatives are calculated from these definitions. Here are the definitions again:

$$\tan x = \frac{\sin x}{\cos x}, \tag{7}$$

$$\cot x = \frac{\cos x}{\sin x}\left(= \frac{1}{\tan x}\right),$$

$$\sec x = \frac{1}{\cos x},$$

$$\csc x = \frac{1}{\sin x}.$$

These are the tangent, cotangent, secant, and cosecant functions. All four of these functions will be thoroughly developed in Chapter 9 and used extensively thereafter, but for the present we confine our attention to the tangent function and its derivative,

$$\frac{d}{dx}\tan x = \sec^2 x. \tag{8}$$

To establish this formula we refer to (7) and use the quotient rule:

$$\begin{aligned}\frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x} &= \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.\end{aligned}$$

The chain rule extension of (8) is

$$\frac{d}{dx}\tan u = \sec^2 u \frac{du}{dx}. \tag{9}$$

Example 6 Find dy/dx if $y = \tan^5 (3x^2 + 1)$. If we put $w = \tan (3x^2 + 1)$, then $y = w^5$ and by (9),

$$\begin{aligned}\frac{dy}{dx} = 5w^4 \frac{dw}{dx} &= 5w^4 \cdot \sec^2 (3x^2 + 1) \cdot \frac{d}{dx}(3x^2 + 1) \\ &= 5w^4 \cdot \sec^2 (3x^2 + 1) \cdot 6x \\ &= 30x \cdot \tan^4 (3x^2 + 1) \cdot \sec^2 (3x^2 + 1).\end{aligned}$$

PROBLEMS

In Problems 1–32, find dy/dx.

1 $y = \sin (5x - 2)$.
2 $y = \cos (x^5 + 1)$.
3 $y = \sin (\cos x)$.
4 $y = \sin [\sin (\sin x)]$.
5 $y = \sin^3 x$.
6 $y = \sin^2 (4x - 1)^3$.
7 $y = \dfrac{\sin x}{1 + \cos x}$.
8 $y = \cos^3 4x$.
9 $y = \sin x^3$.
10 $y = (1 + \sin^2 x)^4$.
11 $y = \tan 5x$.
12 $y = \tan^2 3x$.
13 $y = \tan (\sin x)$.
14 $y = \tan^3 x^4$.
15 $y = (1 + \tan^2 x^2)^2$.
16 $y = 1 - \cos 3x$.
17 $y = 5 \sin 3x - 3 \cos 5x$.
18 $y = 3 \sin (4x - 5)$.
19 $y = \cos 3(5x - 3)^3$.
20 $y = \cos^7 2x$.
21 $y = \dfrac{\sin x}{1 - \sin x}$.
22 $y = \sin x \cos^2 x$.
23 $y = \dfrac{1 - \cos 3x}{\sin 3x}$.
24 $y = x \sin x$.
25 $y = \dfrac{\cos x}{x}$.
26 $y = \sin x - x \cos x$.
27 $y = x^3 \sin \dfrac{1}{x^2}$.
28 $y = \sin 2x \cos 3x$.
29 $y = (2 - \cos^2 x)^3$.
30 $y = \sin^2 x + \cos^2 x$.
31 $y = \sin (\tan x)$.
32 $y = \tan^2 (1 - \sin^3 x)$.

33 Derive formula (2) in another way, by using the identities

$$\cos x = \sin\left(\frac{\pi}{2} - x\right) \quad \text{and} \quad \sin x = \cos\left(\frac{\pi}{2} - x\right).$$

34 Find the values of x for which the graph of $y = x + 2 \sin x$ has a horizontal tangent.

35 Same as Problem 34 for $y = (\cos x)/(2 + \sin x)$.

36 By differentiating the first of the following double-angle formulas, obtain the second:

$$\sin 2x = 2 \sin x \cos x, \qquad \cos 2x = \cos^2 x - \sin^2 x.$$

37 An object moving on a straight line has position $s = A \cos kt$ at time t, where A and k are constants.
(a) Describe the motion, giving physical meaning to the constants A and k.
(b) Find the velocity v.
(c) Show that $v^2 + k^2s^2$ has the same value at all times. Give a physical interpretation.
(d) Find the acceleration a and show that a is proportional to s but oppositely directed.

Trigonometric identities are often useful in simplifying the form of functions or their derivatives. We mention particularly the double-angle formulas in Problem 36 and the half-angle formulas in Problem 11 of Section 1.7. Show that each of the following derivatives can be expressed in the given form.

38 $\dfrac{d}{dx}\left(\dfrac{1}{2}x - \dfrac{1}{4a}\sin 2ax\right) = \sin^2 ax.$

39 $\dfrac{d}{dx}\left(\dfrac{1}{2}x + \dfrac{1}{4a}\sin 2ax\right) = \cos^2 ax.$

40 $\dfrac{d}{dx}\left(\dfrac{1}{a}\sin ax - \dfrac{1}{3a}\sin^3 ax\right) = \cos^3 ax.$

41 $\dfrac{d}{dx}\left(-\dfrac{2}{3a}\cos ax - \dfrac{1}{6a}\sin 2ax \sin ax\right) = \sin^3 ax.$

3.5 IMPLICIT FUNCTIONS AND FRACTIONAL EXPONENTS

Most of the functions we have met so far have been of the form $y = f(x)$, in which y is expressed directly—or explicitly—in terms of x. In contrast to this, it often happens that y is defined as a function of x by means of an equation

$$F(x, y) = 0, \tag{1}$$

which is not solved for y but in which x and y are more or less entangled with each other. When x is given a suitable numerical value, the resulting equation usually determines one or more corresponding values of y. In such a case we say that equation (1) determines y as one or more *implicit functions* of x.

Example 1 (a) The very simple equation $xy = 1$ determines one implicit function of x, which can be written explicitly as

$$y = \frac{1}{x}.$$

(b) The equation $x^2 + y^2 = 25$ determines two implicit functions of x, which can be written explicitly as

$$y = \sqrt{25 - x^2} \qquad \text{and} \qquad y = -\sqrt{25 - x^2}.$$

As we know, the graphs of these two functions are the upper and lower halves of the circle of radius 5 shown in Fig. 3.2.

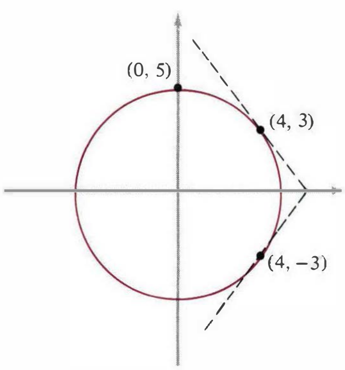


Figure 3.2

(c) The equation $2x^2 - 2xy = 5 - y^2$ also determines two implicit functions. If we use the quadratic formula to solve for y, we find that these functions are

$$y = x + \sqrt{5 - x^2} \qquad \text{and} \qquad y = x - \sqrt{5 - x^2}.$$

(d) The equation $x^3 + y^3 = 3axy$ $(a > 0)$ determines several implicit functions, but the problem of solving this equation for y is so forbidding that we might as well forget it.

It is rather surprising that we can often calculate the derivative dy/dx of an implicit function without first solving the given equation for y. We start the process by differentiating the given equation through with respect to x, using the chain rule (or power rule) and consciously thinking of y as a function of x wherever it appears. Thus, for example, y^3 is treated as the cube of a function of x and its derivative is

$$\frac{d}{dx} y^3 = 3y^2 \frac{dy}{dx};$$

and x^3y^4 is thought of as the product of two functions of x and its derivative is

$$\frac{d}{dx}(x^3y^4) = x^3 \cdot 4y^3 \frac{dy}{dx} + y^4 \cdot 3x^2.$$

To complete the process, we solve the resulting equation for dy/dx as the unknown. This method is called *implicit differentiation*. We show how it works by applying it to the equations in Example 1.

Example 2 (a) We can think of the equation $xy = 1$ as stating that two functions of x (namely, xy and 1) are equal. It follows that the derivatives of these functions are equal, so

$$x\frac{dy}{dx} + y = 0 \qquad \text{or} \qquad \frac{dy}{dx} = -\frac{y}{x}.$$

In this case it is possible to solve the original equation for y and check our result: Since $y = 1/x$, the formula we have just obtained becomes

$$\frac{dy}{dx} = -\frac{y}{x} = -\frac{1}{x} \cdot y = -\frac{1}{x} \cdot \frac{1}{x} = -\frac{1}{x^2};$$

and differentiating $y = 1/x$ directly also yields

$$\frac{dy}{dx} = -\frac{1}{x^2}.$$

(b) From the equation $x^2 + y^2 = 25$ we get

$$2x + 2y\frac{dy}{dx} = 0 \qquad \text{or} \qquad \frac{dy}{dx} = -\frac{x}{y}.$$

This gives the correct result whichever of the two implicit functions we are thinking about. Thus, at the point (4, 3) on the upper curve in Fig. 3.2, the value of dy/dx is $-\frac{4}{3}$, and at $(4, -3)$ on the lower curve, its value is $\frac{4}{3}$.

(c) If we apply this process of implicit differentiation to the equation $2x^2 - 2xy = 5 - y^2$, we obtain

$$4x - 2x\frac{dy}{dx} - 2y = -2y\frac{dy}{dx} \qquad \text{or} \qquad \frac{dy}{dx} = \frac{2x - y}{x - y}.$$

(d) In Example 1(d) the derivative dy/dx is clearly beyond direct calculation. However, it is easily found by our present method: Since $x^3 + y^3 = 3axy$, we have

$$3x^2 + 3y^2\frac{dy}{dx} = 3ax\frac{dy}{dx} + 3ay \qquad \text{or} \qquad \frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}.$$

It is apparent that implicit differentiation usually gives an expression for dy/dx in terms of both x and y, instead of in terms of x alone. However, in many cases this is not a real disadvantage. For instance, if we want the slope of the tangent to the graph of the equation at a point (x_0, y_0), all we have to do is substitute x_0 and y_0 for x and y in the formula for dy/dx. This is illustrated in Example 2(b) for the points (4, 3) and $(4, -3)$.

We now use implicit differentiation to show that the vital formula

$$\frac{d}{dx}x^n = nx^{n-1} \tag{2}$$

is valid for all fractional exponents $n = p/q$.*

For the sake of convenience, we begin the proof of (2) for fractional exponents by introducing y as the dependent variable,

$$y = x^{p/q}.$$

Raising both sides of this to the qth power yields

$$y^q = x^p;$$

and by differentiating implicitly with respect to x and using the known validity of the power rule for integral exponents, we obtain

$$qy^{q-1}\frac{dy}{dx} = px^{p-1}$$

or

$$\frac{dy}{dx} = \frac{p}{q}\frac{x^{p-1}}{y^{q-1}}.$$

But $y^{q-1} = y^q/y = x^p/x^{p/q}$, so

$$\frac{dy}{dx} = \frac{p}{q}\frac{x^{p-1}}{y^{q-1}} = \frac{p}{q}\frac{x^{p-1}}{x^p}\cdot x^{p/q} = \frac{p}{q}x^{p/q-1},$$

and the proof is complete.

Example 3 We immediately have

$$\frac{d}{dx}x^{1/2} = \frac{1}{2}x^{-1/2}, \qquad \frac{d}{dx}x^{-2/3} = -\frac{2}{3}x^{-5/3}, \qquad \frac{d}{dx}x^{5/4} = \frac{5}{4}x^{1/4}.$$

The first of these derivatives is often used in the form

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}.$$

This formula was established directly from the definition of the derivative in Example 3 of Section 2.3.

Example 4 By the chain rule, the power rule of Section 3.3 is now known to be valid for all fractional exponents. Accordingly,

$$\begin{aligned}\frac{d}{dx}(4-x^2)^{-5/2} &= -\frac{5}{2}(4-x^2)^{-7/2}\frac{d}{dx}(4-x^2)\\ &= -\frac{5}{2}(4-x^2)^{-7/2}(-2x) = \frac{5x}{(4-x^2)^{7/2}}.\end{aligned}$$

*Students who are comfortable with fractional exponents should ignore this footnote. However, for those who have forgotten the meaning of these exponents, we offer a brief review. We begin by recalling that the square root $\sqrt{x}$, the cube root $\sqrt[3]{x}$, and more generally the qth root $\sqrt[q]{x}$, where q is any positive integer, are all defined for $x \geq 0$; if q is odd, $\sqrt[q]{x}$ is also defined for $x < 0$. The definition of fractional exponents now proceeds in two stages: First, $x^{1/q}$ is defined for $q > 0$ by $x^{1/q} = \sqrt[q]{x}$; and second, if p/q is in lowest terms and $q > 0$, $x^{p/q}$ is defined by $x^{p/q} = (x^{1/q})^p$. It is sometimes

Example 5 In differentiating expressions containing radicals, it is necessary to begin by replacing all radical signs by fractional exponents. Thus,

$$\frac{d}{dx}\frac{x}{\sqrt{x^2-1}} = \frac{d}{dx}x(x^2-1)^{-1/2} = x\left(-\frac{1}{2}\right)(x^2-1)^{-3/2}(2x) + (x^2-1)^{-1/2}$$

$$= \frac{-x^2}{(x^2-1)^{3/2}} + \frac{1}{(x^2-1)^{1/2}} = \frac{-x^2+(x^2-1)}{(x^2-1)^{3/2}} = \frac{-1}{(x^2-1)^{3/2}}.$$

For convenience of reference, we list together all the differentiation rules developed in this chapter.

1 $\frac{d}{dx}c = 0.$

2 $\frac{d}{dx}x^n = nx^{n-1}$ (n any integer or fraction).

3 $\frac{d}{dx}(cu) = c\frac{du}{dx}.$

4 $\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}.$

5 *The product rule*: $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$

6 *The quotient rule*: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\,du/dx - u\,dv/dx}{v^2}.$

7 *The chain rule*: $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}.$

8 *The power rule*: $\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}$ (n any integer or fraction).

9 $\frac{d}{dx}\sin u = \cos u\frac{du}{dx}.$

10 $\frac{d}{dx}\cos u = -\sin u\frac{du}{dx}.$

These rules will be used in many ways in almost everything we do from this point on. We therefore urge students who have not already done so to commit them to memory and practice them until their use becomes almost automatic. The eminent philosopher A. N. Whitehead might well have had these rules in mind when he said, "Civilization advances by extending the number of important operations which we can perform without thinking about them."

It is worth pointing out that most mistakes in differentiation come from misusing the power rule or the quotient rule. For instance, in applying the power rule it is easy to forget the essential final factor du/dx:

Common mistake	Right answer
$\frac{d}{dx}(1+6x^2)^4 = 4(1+6x^2)^3$	$4(1+6x^2)^3\cdot 12x$
$\frac{d}{dx}(1+2x)^{1/3} = \frac{1}{3}(1+2x)^{-2/3}$	$\frac{1}{3}(1+2x)^{-2/3}\cdot 2$

useful to know (and it is not difficult to prove) that $(x^p)^{1/q} = (x^{1/q})^p$ if $x > 0$. For example, $8^{2/3}$ is easy to evaluate both ways, since $8^{2/3} = (8^2)^{1/3} = 64^{1/3} = 4$ and $8^{2/3} = (8^{1/3})^2 = 2^2 = 4$; but $32^{3/5} = (32^3)^{1/5}$ is hard, while $32^{3/5} = (32^{1/5})^3 = 2^3 = 8$ is easy.

The difficulty with the quotient rule lies in remembering the order of subtraction in the numerator. If we forget, one way of quickly recalling the correct order is to use the product rule as follows:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{d}{dx}(uv^{-1}) = u \cdot (-1)v^{-2}\frac{dv}{dx} + v^{-1}\frac{du}{dx}$$

$$= \frac{1}{v}\frac{du}{dx} - \frac{u}{v^2}\frac{dv}{dx} = \frac{v\, du/dx - u\, dv/dx}{v^2}.$$

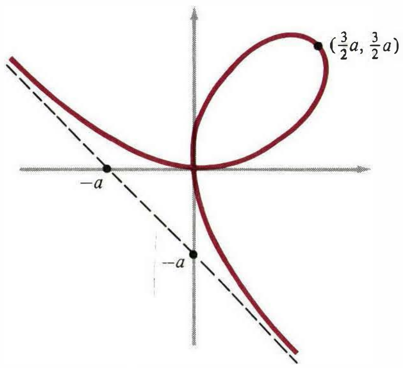


Figure 3.3 The folium of Descartes.

Remark The equation in Example 1(d) has a long history and deserves a bit of further comment. Its graph is called the *folium of Descartes* and is shown in Fig. 3.3. If we consider the simplest case by putting $a = 1$, the equation becomes

$$x^3 + y^3 = 3xy, \tag{3}$$

and the problem of solving this for y in terms of x—which we airily dismissed above—is not absolutely out of the question. And thereby hangs a tale of considerable historical interest.

In 1545 the boisterous Italian physician-mathematician-astrologer Girolamo Cardano (1501–1576) discovered a formula for solving any cubic equation by means of radicals.* This formula resembles the familiar quadratic formula but is much more complicated. If Cardano's formula is used to solve equation (3) for y, the three solution functions that arise are†

$$y_1 = \sqrt[3]{-\frac{x^3}{2} + \sqrt{\frac{x^6}{4} - x^3}} + \sqrt[3]{-\frac{x^3}{2} - \sqrt{\frac{x^6}{4} - x^3}}$$

and

$$y_2, y_3 = -\tfrac{1}{2}y_1 \pm \tfrac{1}{2}\sqrt{-3}\left(\sqrt[3]{-\frac{x^3}{2} + \sqrt{\frac{x^6}{4} - x^3}} - \sqrt[3]{-\frac{x^3}{2} - \sqrt{\frac{x^6}{4} - x^3}}\right).$$

The method of implicit differentiation as carried out in Example 2(d) is clearly preferable to the task of directly differentiating horrors like these. Furthermore, implicit differentiation works just as easily for equations such as

$$x^5 + 5x^4y^2 + 3xy^3 + y^5 = 1,$$

which are actually *impossible* to solve for y in terms of x.‡

*At one point in his turbulent life Cardano was imprisoned for heresy: his offense was that he published a horoscope for Jesus.

†These formulas can be obtained from the ideas in Chapter X of H. Tietze, *Famous Problems of Mathematics* (Graylock Press, 1965).

‡In 1824 the Norwegian mathematician Niels Henrik Abel (1802–1829) proved that no general formula exists for solving a fifth-degree equation by means of radicals, as is possible for equations of lower degree. This young man made many profound discoveries in his short life, and it has been said that he was the greatest genius produced by the Scandinavian countries.

PROBLEMS

In Problems 1–10, find dy/dx by implicit differentiation.

1 $3x^3 + 4y^3 + 8 = 0$.
2 $xy^2 - x^2y + x^2 + 2y^2 = 0$.
3 $x = y - y^7$.
4 $x^4y^3 - 3xy = 60$.
5 $x^3 - y^3 = 4xy$.
6 $\dfrac{1}{x} + \dfrac{1}{y} = 1$.
7 $\sqrt{x} + \sqrt{y} = 6$.
8 $\dfrac{x + y}{x - y} = x^2$.
9 $x^{2/3} + y^{2/3} = 1$.
10 $\sqrt{xy} + 4 = y$.

In Problems 11–18, find dy/dx by implicit differentiation and also by solving for y and then differentiating, and verify that your two answers are equivalent.

11 $3xy + 2 = 0$.
12 $x^2 + y^2 = 9$.
13 $y^2 = 3x - 1$.
14 $2x^2 + 3x + y^2 = 12$.
15 $\dfrac{1 - y}{1 + y} = x$.
16 $x^2 + 5x + xy = 3$.
17 $9x^2 + 4y^2 = 36$.
18 $x^2 + y - y^2 = 5$.

In Problems 19–28, find the derivative of each function.

19 $x^{4/5}$.
20 $x^{5/6}$.
21 $x^{-3/4}$.
22 $x^{-7/11}$.
23 $3\sqrt[5]{x^2}$.
24 $(1 + x^{2/3})^{3/2}$.
25 $\left(\dfrac{x^3 + 8}{x^2}\right)^{3/4}$.
26 $\sqrt{1 + \sqrt{1 + x}}$.
27 $\left(\dfrac{x + 2}{x - 1}\right)^{3/2}$.
28 $\sqrt{\dfrac{x^2 + 3}{x^2 - 3}}$.

29 Find the equation of
(a) the tangent to $y = (5 - 3x)^{1/3}$ at $(-1, 2)$;
(b) the tangent to $x^4 + 16y^4 = 32$ at $(2, 1)$;
(c) the normal to $y = x\sqrt{9 + x^2}$ at the origin;
(d) the normal to $y^2 - 4xy = 12$ at $(1, 6)$.

30 Show that the curves $x^2 + 3y^2 = 12$ and $3x^2 - y^2 = 6$ intersect at right angles at the point $(\sqrt{3}, \sqrt{3})$.

31 Show that for the "curve" $x(x + 6) + y^2 - 4y + 15 = 0$, implicit differentiation gives

$$\frac{dy}{dx} = \frac{x + 3}{2 - y}.$$

Show further that this result is completely meaningless, because there are no points on this "curve."

32 Verify that the normal at any point (x_0, y_0) on the circle $x^2 + y^2 = a^2$ passes through the center.

33 Find a function $y = f(x)$ for which
(a) $\dfrac{dy}{dx} = 3\sqrt{x}$; (b) $\dfrac{dy}{dx} = 5x\sqrt{x}$.

34 Show that the curve $xy^3 + x^3y = 4$ has no horizontal tangent.

35 Find the highest point on the loop of the folium of Descartes (Fig. 3.3) whose equation is (3).

In Problems 36–40, find dy/dx by implicit differentiation.

36 $\tan y = x$.
37 $y^3 + y^2 = \sin x$.
38 $\cos y = x$.
39 $\sin y = xy$.
40 $\cos xy = x^2 + y^2$.

In Problems 41–48, find the derivative of each function.

41 $\sin \sqrt{x}$.
42 $\dfrac{\cos x}{\sqrt{x}}$.
43 $\tan^2 \sqrt{x}$.
44 $\sin^3 (1 - 5x)^{4/3}$.
45 $\sqrt{6 - 5 \cos 2x}$.
46 $\sqrt{\dfrac{\sin^2 x}{1 + \cos x}}$.
47 $\tan (3x - 1)^{-1/2}$.
48 $[\tan (3x - 1)]^{-1/2}$.

3.6 DERIVATIVES OF HIGHER ORDER

The derivative of $y = x^4$ is clearly $y' = 4x^3$. But $4x^3$ can also be differentiated, yielding $12x^2$. It is natural to denote this function by y'' and call it the *second derivative* of the original function. By differentiating the second derivative $y'' = 12x^2$ we obtain the *third derivative* $y''' = 24x$, and so on indefinitely. Several notations are in common use for these higher-order derivatives, and students should become familiar with all of them. The successive derivatives of a function $y = f(x)$ can be written as follows:

First derivative	$f'(x)$	y'	$\dfrac{dy}{dx}$	$\dfrac{d}{dx} f(x)$
Second derivative	$f''(x)$	y''	$\dfrac{d^2y}{dx^2}$	$\dfrac{d^2}{dx^2} f(x)$

Third derivative	$f'''(x)$	y'''	$\dfrac{d^3y}{dx^3}$	$\dfrac{d^3}{dx^3}f(x)$
nth derivative	$f^{(n)}(x)$	$y^{(n)}$	$\dfrac{d^ny}{dx^n}$	$\dfrac{d^n}{dx^n}f(x)$

A few remarks about these notations are perhaps in order. The entries in the first column are read "f prime of x," "f double prime of x," "f triple prime of x," "f upper n of x"; similarly, those in the second column are read "y prime," "y double prime," and so on. The prime notation quickly becomes unwieldy and is not often used beyond the third order. It is sometimes convenient to think of the original function as the zeroth-order derivative and to write $f(x) = f^{(0)}(x)$. The seemingly strange position of the superscripts in the third column can be understood if we remember that the second derivative is the derivative of the first derivative,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right).$$

On the left side of this, the superscript 2 is attached to the d on top and to the dx on the bottom, and this is consistent with the way these symbols are written on the right.

What are the uses of these higher derivatives? In geometry, as we will see in Chapter 4, the sign of $f''(x)$ tells us whether the curve $y = f(x)$ is concave up or concave down. Also, in a later chapter this qualitative interpretation of the second derivative will be refined into a quantitative formula for the curvature of the curve.

In physics, second derivatives are of very great importance. If $s = f(t)$ gives the position of a moving body at time t, then we know that the first and second derivatives of this position function,

$$v = \frac{ds}{dt} \qquad \text{and} \qquad a = \frac{dv}{dt} = \frac{d^2s}{dt^2},$$

are the velocity and acceleration of the body at time t. The central role of acceleration arises from Newton's second law of motion, which states that the acceleration of a moving body is proportional to the force acting on it. The basic problem of Newtonian dynamics is to use calculus to deduce the nature of the motion from the given force. We shall begin examining problems of this kind in Chapter 5.

Derivatives of higher order than the second do not have any such fundamental geometric or physical interpretations. However, as we shall see later, these derivatives have their uses too, mainly in connection with expanding functions into infinite series.

All these applications will be discussed in detail at the proper time. Meanwhile, our present task is to develop proficiency at performing the calculations.

Example 1 It is easy to find all the derivatives of $y = x^5$:

$$y' = 5x^4, \qquad y'' = 20x^3, \qquad y''' = 60x^2,$$
$$y^{(4)} = 120x, \qquad y^{(5)} = 120, \qquad y^{(n)} = 0 \qquad \text{for } n > 5.$$

The following notation will often be useful. For any positive integer n, the symbol $n!$ (read "n factorial") is defined to be the product of all the positive integers from 1 up to n:

$$n! = 1 \cdot 2 \cdot 3 \cdots n.$$

Thus, $1! = 1$, $2! = 1 \cdot 2 = 2$, $3! = 1 \cdot 2 \cdot 3 = 6$, $4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$, etc. If we differentiate $y = x^n$ repeatedly we clearly get

$$\begin{aligned} y' &= nx^{n-1}, \\ y'' &= n(n-1)x^{n-2}, \\ y''' &= n(n-1)(n-2)x^{n-3}, \ldots, \\ y^{(n)} &= n(n-1)(n-2)\cdots 2 \cdot 1 = n!, \\ y^{(k)} &= 0 \qquad \text{for } k > n. \end{aligned}$$

Example 2 To discover a formula for the nth derivative of $y = 1/x = x^{-1}$, we compute until a pattern emerges:

$$\begin{aligned} y' &= -x^{-2}, \\ y'' &= 2x^{-3}, \\ y''' &= -2 \cdot 3x^{-4} = -3!x^{-4}, \\ y^{(4)} &= 2 \cdot 3 \cdot 4x^{-5} = 4!x^{-5}, \\ y^{(5)} &= -2 \cdot 3 \cdot 4 \cdot 5x^{-6} = -5!x^{-6}. \end{aligned}$$

From the evidence so far and the way the process of differentiation works, it is clear that except for sign $y^{(n)}$ is $n!x^{-(n+1)}$. A convenient way of expressing the alternating sign is provided by the number $(-1)^n$, which equals -1 if n is odd and 1 if n is even. We therefore have

$$y^{(n)} = (-1)^n n! x^{-(n+1)}$$

for every positive integer n.

Example 3 Implicit differentiation can be used to find a simple formula for y'' on the circle $x^2 + y^2 = a^2$. To begin the process, we differentiate and obtain

$$2x + 2yy' = 0 \qquad \text{or} \qquad y' = -\frac{x}{y}. \tag{1}$$

Differentiating again by the quotient rule and remembering that y is a function of x, we get

$$y'' = -\frac{y - xy'}{y^2}.$$

When (1) is substituted into this, the formula becomes

$$y'' = -\frac{y - x(-x/y)}{y^2} = -\frac{y^2 + x^2}{y^3} = -\frac{a^2}{y^3},$$

which should be simple enough for anyone.

Example 4 Repeated differentiation enables us to give a relatively easy proof of the binomial theorem. For any positive integer n, we consider the function

$$(1+x)^n = (1+x)(1+x)\cdots(1+x).$$

It is obvious that this function is a polynomial of degree n, that is,

$$(1+x)^n = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n, \tag{2}$$

and our problem is to find out what the coefficients are. If we put $x = 0$, we immediately obtain $a_0 = 1$. Next, differentiating both sides of (2) repeatedly yields

$$n(1+x)^{n-1} = a_1 + 2a_2x + 3a_3x^2 + \cdots + na_nx^{n-1},$$

$$n(n-1)(1+x)^{n-2} = 2a_2 + 3\cdot 2a_3x + \cdots + n(n-1)a_nx^{n-2},$$

$$n(n-1)(n-2)(1+x)^{n-3} = 3\cdot 2a_3 + \cdots + n(n-1)(n-2)a_nx^{n-3},$$

and so on. These equations hold for all values of x, so we can put $x = 0$ in each of them. This procedure gives the following expressions for the coefficients a_1, a_2, a_3, . . . :

$$a_1 = n, \qquad a_2 = \frac{n(n-1)}{2}, \qquad a_3 = \frac{n(n-1)(n-2)}{2\cdot 3}, \qquad \ldots,$$

$$a_k = \frac{n(n-1)(n-2)\cdots(n-k+1)}{1\cdot 2\cdot 3\cdots k}, \qquad \ldots, \qquad a_n = 1.$$

With these coefficients, equation (2) takes the form

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1\cdot 2}x^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3 + \cdots$$
$$+ \frac{n(n-1)(n-2)\cdots(n-k+1)}{1\cdot 2\cdot 3\cdots k}x^k + \cdots + x^n, \tag{3}$$

and this is the binomial theorem.*

*To obtain the equivalent version given in the footnote of Section 3.1, substitute $x = b/a$ in equation (3) and then multiply by a^n.

PROBLEMS

1 Find the first four derivatives of
(a) $8x - 3$;
(b) $8x^2 - 11x + 2$;
(c) $8x^3 + 7x^2 - x + 9$;
(d) $x^4 - 13x^3 + 5x^2 + 3x - 2$;
(e) $x^{5/2}$.

2 Calculate the indicated derivative in each case:
(a) y'' if $y = \dfrac{x}{1-x}$; (b) y'' if $y = x^2 - \dfrac{1}{x^2}$;
(c) $\dfrac{d^2}{dx^2}\left(\dfrac{1-x}{1+x}\right)$; (d) $\dfrac{d^2}{dx^2}\left(x^3 + \dfrac{1}{x^3}\right)$;
(e) $\dfrac{d^{500}}{dx^{500}}(x^{131} - 3x^{79} + 4)$.

3 Find a general formula for $y^{(n)}$ in each case:
(a) $y = \dfrac{1}{1-x}$; (b) $y = \dfrac{1}{1+3x}$;
(c) $y = \dfrac{x}{1+x}$.

4 Use implicit differentiation to find a simple formula for y'' in each case:
(a) $b^2x^2 + a^2y^2 = a^2b^2$; (b) $y^2 = 4px$;
(c) $x^{1/2} + y^{1/2} = a^{1/2}$; (d) $x^3 + y^3 = a^3$;
(e) $x^4 + y^4 = a^4$.

5 Find a simple formula for y'' on the curve $x^n + y^n = a^n$ and show that your results in parts (c), (d), and (e) of Problem 4 are all special cases of this formula.

6 Find the values of y', y'', and y''' at the point (4, 3) on the circle $x^2 + y^2 = 25$.

7 If s is the position of a moving body at time t, find the time, position, and velocity at each moment when the acceleration is zero:
(a) $s = 8t^2 - \frac{1}{t}$ $(t > 0)$;
(b) $s = 12t^{1/2} + t^{3/2}$ $(t > 0)$;
(c) $s = \frac{24}{3 + t^2}$ $(t \geq 0)$.

8 (a) What is the 23rd derivative of

$$x^{22} - 501x^{17} + \tfrac{19}{35}x^6 - \pi^3x^2?$$

(b) What is the 22nd derivative?

9 If $f(x) = x^3 - 2x^2 - x$, for what values of x is $f'(x) = f''(x)$?

10 Show the following:
(a) if y' is proportional to x^2, then y'' is proportional to x;
(b) if y' is proportional to y^2, then y'' is proportional to y^3.

11 It is natural to expect from the chain rule that the formula

$$\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} \cdot \frac{d^2u}{dx^2}$$

might be true. Disprove this guess by considering $y = \sqrt{u}$ where $u = x^2 + 1$. Prove that in fact

$$\frac{d^2y}{dx^2} = \frac{d^2y}{du^2}\left(\frac{du}{dx}\right)^2 + \frac{d^2u}{dx^2}\frac{dy}{du},$$

and verify this for the given functions.

12 If u and v are functions of x, and $y = uv$, show that

$$y'' = u''v + 2u'v' + uv''.$$

Find a similar formula for y'''.

13 For each of the functions $\sin x$ and $\cos x$, obtain
(a) the 4th derivative;
(b) the 10th derivative;
(c) the 100th derivative;
(d) the 159th derivative.

CHAPTER 3 REVIEW: CONCEPTS, FORMULAS, METHODS

Memorize and learn to use, or think through the following.

1 Derivative of a function.
2 Binomial theorem.
3 Derivative of a polynomial.
4 Reflection property of parabolas.
5 Normal line to a curve.
6 Product rule.
7 Quotient rule.
8 Composite function.
9 Chain rule.
10 Power rule.
11 Derivatives of trigonometric functions.
12 Implicit function.
13 Implicit differentiation.
14 Second derivative and higher derivatives.

ADDITIONAL PROBLEMS FOR CHAPTER 3

SECTION 3.1

1 Find the points on the curve $y = x^3 - 3x^2 - 9x + 5$ at which the tangent is horizontal.

2 Find the points on the curve $y = x^3 - x^2$ at which the tangent has slope 1.

3 Find the points on the curve $y = x^3 + x$ at which the tangent has slope 4. What is the smallest value the slope of the tangent to this curve can have, and where on the curve does the slope of the tangent have this smallest value?

4 At what points on the curve $y = x^3 - x^2 + x$ is the tangent parallel to the line $2x - y - 7 = 0$?

5 Find the slope of the tangent to the curve $y = x^4 - 2x^2 + 2$ at any point. For what values of x is the tangent horizontal? For what values of x does the tangent point upward to the right?

6 The curve $y = ax^2 + bx + 2$ is tangent to the line $8x + y = 14$ at the point $(2, -2)$. Find a and b.

7 Find the constants a, b, and c if the curve $y = ax^2 + bx + c$ passes through the point $(-1, 0)$ and is tangent to the line $y = x$ at the origin.

8 If the curve $y = ax^2 + bx + c$ passes through the point $(-1, 0)$ and has the line $3x + y = 5$ as its tangent at the point $(1, 2)$, what values must the constants a, b, and c have?

9 The curves $y = x^2 + ax + b$ and $y = x^3 - c$ have the same tangent at the point $(1, 2)$. What are the values of a, b, and c?

10 Find the equations of the tangents to the curve $y = x^2 - 4x$ that pass through the point $(1, -4)$.

11 If $a \neq 0$, show that the tangent to the curve $y = x^3$ at

(a, a^3) intersects the curve a second time at the point where $x = -2a$.

12 Show that the tangents to the curve $y = x^2$ at the points (a, a^2) and $(a + 2, (a + 2)^2)$ intersect on the curve $y = x^2 - 1$.

13 Find the values of a, b, c, and d if the curve $y = ax^3 + bx^2 + cx + d$ is tangent to the line $y = x - 1$ at the point $(1, 0)$ and is tangent to the line $y = 6x - 9$ at the point $(2, 3)$.

14 Use the reflection property of parabolas to show that the two tangents to a parabola at the ends of a chord through the focus are perpendicular to each other.

15 Show that the tangent to the curve $y = x^3 - 2x^2 - 3x + 8$ at the point $(2, 2)$ is one of the normals of $y = x^2 - 3x + 3$.

16 There is only one normal to the parabola $x^2 = 2y$ that passes through the point $(4, 1)$. Find its equation.

17 The point $P = (6, 9)$ lies on the parabola $x^2 = 4y$. Find all points Q on this parabola with the property that the normal at Q passes through P.

SECTION 3.2

18 Differentiate each of the following functions two ways and verify that your answers agree:
(a) $(x^2 - 1)(x^3 - 1)$; (b) $3x^4(x^2 + 2x)$;
(c) $(x^2 - 3)(x - 1)$; (d) $(x + 1)(x^2 - 2x - 3)$.

19 Differentiate each of the following functions and simplify your answer as much as possible:
(a) $\dfrac{x + x^{-1}}{x - x^{-1}}$; (b) $\dfrac{x^2 + 2x + 1}{x^2 - 2x + 1}$;
(c) $\dfrac{x^2}{x^3 + 2}$; (d) $\dfrac{2x + 3}{x^2 + x - 4}$;
(e) $\dfrac{x^3}{1 - x^2}$; (f) $\dfrac{1 - x}{1 + x}$;
(g) $\dfrac{6x^4 + 9}{x - 1}$; (h) $\dfrac{x^2 + 6x + 9}{x^2 - 4x + 4}$.

20 Find dy/dx two ways, first by dividing and then by using the quotient rule, and show that your answers agree:
(a) $\dfrac{9 - x^3}{x^2}$; (b) $\dfrac{5 - 3x}{x^4}$; (c) $\dfrac{x^3 - 6x}{x^4}$.

21 Prove the quotient rule from the product rule as follows: Write $y = u/v$ in the form $yv = u$, differentiate this with respect to x by the product rule, and solve the resulting equation for dy/dx.

22 Extend the product rule to a product of three functions by showing that

$$\frac{d}{dx}(uvw) = vw\frac{du}{dx} + uw\frac{dv}{dx} + uv\frac{dw}{dx}.$$

Hint: Treat uvw as a product $(uv)w$ of two factors. (Notice that the right-hand side of this extended product rule is the sum of all terms in which the derivative of one factor is multiplied by the other factors unchanged. This pattern persists for products of more than three factors.)

23 Use Problem 22 to differentiate
(a) $(x + 1)(x + 2)(x + 3)$;
(b) $(x^2 + 2x)(x^3 + 3x^2)(x^4 + 4)$.

24 Use Problem 22 to show that $(d/dx)\, u^3 = 3u^2\, du/dx$, and apply this formula to calculate

$$\frac{d}{dx}(6x^{11} + 9x^5 - 3)^3.$$

25 Sketch the curve $y = 10\sqrt{5}/(1 + x^2)$ and find the points on it at which the normal passes through the origin.

26 Consider the curve $y = a/(1 + x^2)$, where a is a positive constant. For what values of a does there exist a point $P = (x_0, y_0)$ on the first-quadrant part of the curve at which the normal passes through the origin? If the normal at the point for which $x_0 = 2$ passes through the origin, what must be the value of a?

27 There are two points on the curve $y = (x + 4)/(x - 5)$ at which the tangent passes through the origin. Sketch the curve and find these points.

SECTION 3.3

28 Find dy/dx in each case:
(a) $y = (4x^2 - 2)^{12}$; (b) $y = (x^4 + 1)^{125}$;
(c) $y = (x^4 - x^8)^{16}$; (d) $y = (x^{-1} - x^{-2})^{-3}$;
(e) $y = (4x^2 + 5)^{-1}$; (f) $y = (x + x^2 + x^3 + x^4)^5$.

29 Find dy/dx in each case:
(a) $y = (1 + 2x)^3(4 - 5x)^6$;
(b) $y = (x^2 + 1)^{10}(x^2 - 1)^{15}$;
(c) $y = (x^2 - 1)(16 + x^2)^{-3}$;
(d) $y = (4x^3 - 9x^2)^2(3x - 2x^2)^3$.

30 Find dx/dt in each case:
(a) $s = \dfrac{(t + 3t^2)^2}{t + 1}$; (b) $s = \dfrac{1}{(t^3 - 1)^5}$;
(c) $s = \dfrac{(t^2 + 1)^4}{(t^2 - 1)^3}$; (d) $s = \dfrac{(1 + 2t^2)^5}{(1 - 3t^3)^4}$.

31 Find a function $y = f(x)$ for which
(a) $\dfrac{dy}{dx} = 12x^3(x^4 + 1)^2$;
(b) $\dfrac{dy}{dx} = 72x^5(x^6 + 1)^5$.

32 Prove the power rule for positive integral exponents n by writing $y = u^n$, expanding $\Delta y = (u + \Delta u)^n - u^n$ by the binomial theorem, and then dividing by Δx. Use the quotient rule to extend this result to negative integral exponents.

SECTION 3.4

33 Find dy/dx for each of the following functions:
(a) $y = \cos(1 - 3x)$; (b) $y = \sin(1 - x^7)$;
(c) $y = \cos(\cos x)$; (d) $y = \cos[\sin(\cos x)]$;
(e) $y = \cos^4 x$; (f) $y = \cos^5(1 - 3x^2)^3$;

(g) $y = \dfrac{\cos x}{1 - \sin x}$; (h) $y = \sin^5 3x$;
(i) $y = \cos x^4$; (j) $y = (1 - \cos^5 x)^3$;
(k) $y = \tan(1 - 3x)$; (l) $y = \tan^4(1 - 2x^3)$;
(m) $y = \cos(\tan x)$; (n) $y = \sin[\cos(\tan x)]$;
(o) $y = \tan^4 x^5$.

34 Differentiate each of the given functions:
(a) $y = \sin(10x - 1)$; (b) $y = \cos^2 x$;
(c) $y = 5\cos(7 - 2x)$; (d) $y = \sin^5 x^5$;
(e) $y = 4\cos^4(1 - x)$; (f) $y = \dfrac{1 - \cos x}{1 + \cos x}$;
(g) $y = \cos^3 x \sin^2 x$; (h) $y = \dfrac{1}{5 - 3\cos 2x}$;
(i) $y = x\cos x$; (j) $y = \dfrac{x}{\sin x}$;
(k) $y = x^4 \sin\dfrac{1}{x}$; (l) $y = \dfrac{1}{\sin x + \cos x}$;
(m) $y = (1 + \sin x)^4$; (n) $y = \sin^3 x - \cos^3 x$;
(o) $y = \cos^2 x - \sin^2 x$; (p) $y = \tan(\sin 5x)$;
(q) $y = \sin^2(\tan^2 x)$.

35 Find the values of x for which the graph of $y = 2\sin x + \sin^2 x$ has a horizontal tangent.

36 Same as Problem 35 for $y = \sin 2x - 2\sin x$.

37 Assuming that the derivative of $\sin x$ is known to be $\cos x$, find the derivative of $\cos x$ by differentiating the identity $\sin^2 x + \cos^2 x = 1$.

38 Consider the function defined by

$$f(x) = \begin{cases} x \sin \dfrac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

(a) The graph of this function is shown in Fig. 3.4. Make a careful verification of the correctness of the general features of this graph. What happens to y when x is large?
(b) Show that $f(x)$ is continuous at $x = 0$.
(c) Find $f'(x)$ for $x \neq 0$.
(d) Show that $f'(0)$ does not exist.

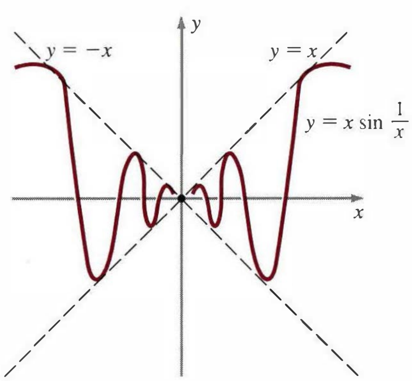


Figure 3.4

39 Consider the function defined by

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

(a) The graph of this function is shown in Fig. 3.5. Examine it carefully and make sure that it correctly reflects the main characteristics of the function. What happens to y when x is large?
(b) Show that $f(x)$ is continuous at $x = 0$.
(c) Find $f'(x)$ for $x \neq 0$.
(d) Find $f'(0)$.
(e) Show that $f'(x)$ is not continuous at $x = 0$.

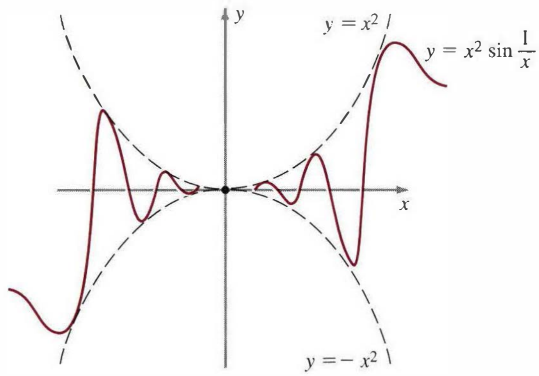


Figure 3.5

SECTION 3.5

40 Find dy/dx by implicit differentiation:
(a) $x^4 + 2xy^3 + 2y^4 = 4$; (b) $\dfrac{y}{x} - 2x = y$;
(c) $y^2 = \dfrac{x^2 + 2}{x^2 - 2}$; (d) $x^4y^4 = x^4 + y^4$;
(e) $\sqrt{xy} + 2y = \sqrt{x}$.

41 Find dy/dx by implicit differentiation and also by solving for y and then differentiating, and verify that your two answers are equivalent:
(a) $y^3 = 3x^2 + 5x - 1$; (b) $y^5 = x^2$;
(c) $4y^2 = 3xy + x^2$; (d) $x^{3/2} + y^{3/2} = 8$.

42 Find the derivative in each case:
(a) $x^{5/2} - x^{3/2}$; (b) $(x^2 + 2)^{4/9}$;
(c) $\sqrt[3]{x + \sqrt{x^5}}$; (d) $\dfrac{x^2}{\sqrt{1 - x^2}}$;
(e) $\sqrt{x} + \dfrac{1}{\sqrt{x}}$; (f) $\sqrt[4]{2x^2 - 1}$;
(g) $\sqrt{\dfrac{x^2 - 1}{x^2 + 1}}$; (h) $\sqrt{2 + \sqrt{2 - x}}$.

43 Find the equation of
(a) the tangent to $x^3 + y^3 = 2xy + 5$ at $(2, 1)$;

(b) the tangent to $y = \dfrac{2x}{\sqrt[3]{x^2 - 1}}$ at (3, 3);

(c) the normal to $x^3 + 3xy^3 - xy^2 = xy + 10$ at (2, 1);

(d) the normal to $x^{2/3} + y^{2/3} = 5$ at (−8, 1).

44 Show that the sum of the x- and y-intercepts of any line tangent to the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ is equal to a.

45 The curve $x^{2/3} + y^{2/3} = a^{2/3}$ is called a *hypocycloid of four cusps*. Sketch it and show that the tangent at (x_0, y_0) is $x_0^{-1/3}x + y_0^{-1/3}y = a^{2/3}$. Use this equation to show that the segment cut from the tangent by the axes has constant length a, so that a segment of length a with its ends sliding along the axes always touches the curve.

46 Find the derivatives of the given functions:

(a) $\cos \sqrt{x}$; (b) $\sqrt{x} \sin \sqrt{x}$;

(c) $\sqrt[3]{\sin\sqrt{x}}$; (d) $\cos^3 \sqrt[3]{x^4 + 1}$;

(e) $\sqrt{1 + \sin\sqrt{x}}$; (f) $\tan^{1/3} (1 + 3x)$;

(g) $\dfrac{1}{\sqrt{\cos x^3}}$; (h) $\sqrt{\tan(\sqrt{\sin x})}$.

SECTION 3.6

47 Calculate y'' if

(a) $y = (1 + 3x)^{1/3}$; (b) $y = \dfrac{x}{\sqrt{x + 1}}$;

(c) $y = x^{4/5}$; (d) $y = x^3\sqrt{x} - 7x$;

(e) $y = \sqrt{x} + \dfrac{1}{\sqrt{x}}$; (f) $y = (x^2 + 4)^{5/2}$.

48 Find a general formula for $y^{(n)}$ if

(a) $y = \dfrac{1}{1 - 2x}$; (b) $y = \dfrac{1}{a + bx}$.

49 Show that

$$\frac{d^n}{dx^n}\left[\frac{1}{x(1-x)}\right] = n!\left[\frac{(-1)^n}{x^{n+1}} + \frac{1}{(1-x)^{n+1}}\right].$$

50 Consider the function $f(x)$ defined by

$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0, \\ -x^2 & \text{if } x < 0. \end{cases}$$

Sketch the graph, show that $f'(x) = 2|x|$, and conclude that $f''(0)$ does not exist.

51 For each of the following functions, find $f''(x)$ and then calculate the limit

$$\lim_{\Delta x \to 0} \frac{f(x + 2\Delta x) - 2f(x + \Delta x) + f(x)}{(\Delta x)^2},$$

and notice that they are equal:

(a) $f(x) = x^3$; (b) $f(x) = 1/x$.

52 Solve Problem 51 after replacing the limit given there by

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - 2f(x) + f(x - \Delta x)}{(\Delta x)^2}.$$

53 Find the given derivatives by calculating the first few derivatives and noticing a pattern:

(a) the 20th derivative of $x \sin x$;

(b) the 62nd derivative of $\sin 3x$.

4 APPLICATIONS OF DERIVATIVES

4.1 INCREASING AND DECREASING FUNCTIONS. MAXIMA AND MINIMA

In this chapter we begin to justify the effort we have spent on learning how to calculate derivatives.

Our first applications are based on the interpretation of the derivative as the slope of the tangent line to a curve. The purpose of this work is to enable us to use the derivative as a tool for quickly discovering the most important features of a function and sketching its graph. This art of curve sketching is essential in the physical sciences. It is also one of the most useful skills that calculus can provide for those who need to use mathematics in their study of economics or biology or psychology.

A function $f(x)$ is said to be *increasing* on a certain interval of the x-axis if on this interval $x_1 < x_2$ implies $f(x_1) < f(x_2)$. In geometric language, this means that the graph is rising as the point that traces it moves from left to right. Similarly, the function is said to be *decreasing* (the graph is falling) if $x_1 < x_2$ implies $f(x_1) > f(x_2)$. These concepts are illustrated in Fig. 4.1.

In sketching the graph of a function, it is important to know the intervals on which it is increasing and those on which it is decreasing. The sign of the derivative gives us this information:

> *A function $f(x)$ is increasing on any interval in which $f'(x) > 0$, and it is decreasing on any interval in which $f'(x) < 0$.*

This is geometrically evident if we keep in mind the fact that a straight line points

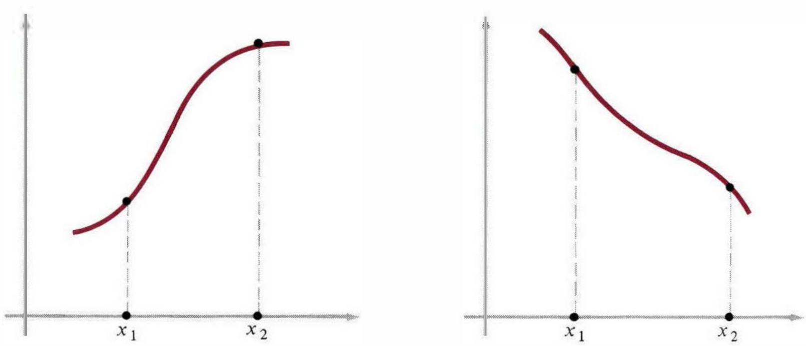


Figure 4.1 Increasing and decreasing functions.

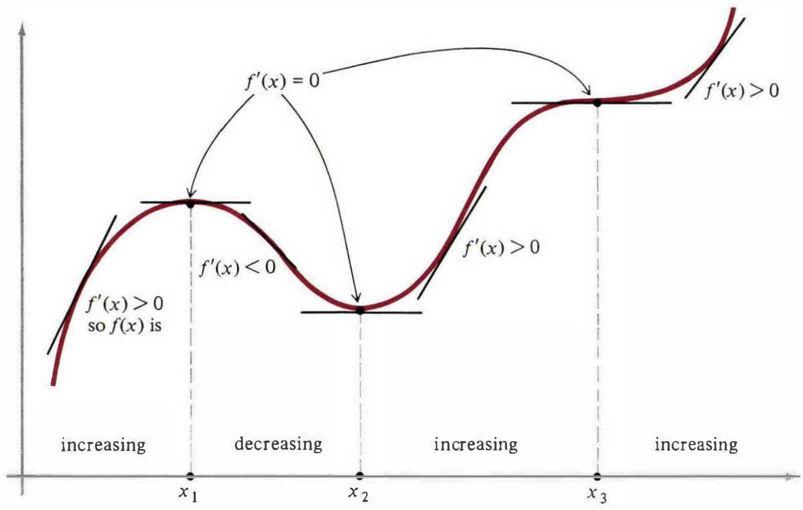


Figure 4.2

upward to the right if its slope is positive and downward to the right if its slope is negative (Fig. 4.2).

It is clear that a smooth curve can make the transition from rising to falling only by passing over a peak where the slope is zero. Similarly, it can change from falling to rising only by going through a trough where the slope is zero. At such points we have a *maximum* or *minimum value* of the function. We locate these values by finding the *critical points* of the function, which are the solutions of the equation $f'(x) = 0$; that is, we force the tangent to be horizontal by equating the derivative to zero, and we then solve the equation $f'(x) = 0$ to discover where this happens. In Fig. 4.2 the critical points are x_1, x_2, x_3, and the corresponding *critical values* are the values of the function at these points, that is, $f(x_1), f(x_2), f(x_3)$.

It is important to understand that a critical value is not necessarily either a maximum or a minimum. This is shown by $f(x_3)$ in Fig. 4.2; at the critical point x_3 the graph does not pass either over a peak or through a trough, but instead merely flattens out momentarily between two intervals on each of which the derivative is positive.

It should also be pointed out that we are discussing the so-called *relative* (or *local*) maximum or minimum values. These are values that are maximal or minimal compared only with nearby points on the curve. In Fig. 4.2, for instance, $f(x_1)$ is a maximum even though there are many higher points on the curve, off to the right. If we seek the *absolute* maximum of a function, we must compare its relative maxima with one another to determine which (if any) is larger than any other value assumed by the function.

Example 1 To sketch the graph of the polynomial

$$y = f(x) = 2x^3 - 3x^2 - 12x + 12,$$

we begin by computing the derivative and factoring this derivative as completely as possible:

$$f'(x) = 6x^2 - 6x - 12 = 6(x + 1)(x - 2).$$

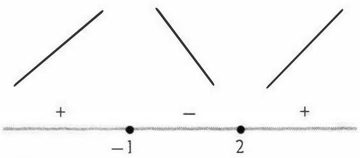


Figure 4.3

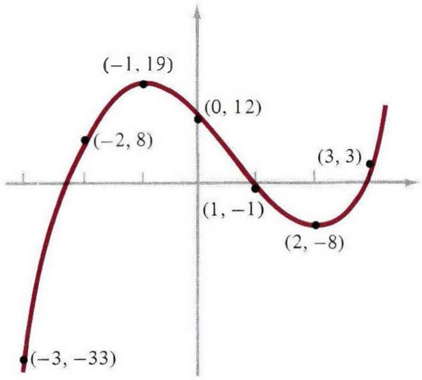


Figure 4.4

The critical points are evidently $x = -1$ and $x = 2$, and by substituting -1 and 2 in $f(x)$, we see that the corresponding critical values are $y = 19$ and $y = -8$. We now examine the three intervals into which the critical points divide the x-axis, for on each of these intervals $f'(x)$ has constant sign. When $x < -1$, $x + 1$ and $x - 2$ are both negative, so their product is positive and $f'(x) > 0$. When $-1 < x < 2$, $x + 1$ is positive and $x - 2$ is negative, so their product is negative and $f'(x) < 0$. When $x > 2$, $x + 1$ and $x - 2$ are both positive, so their product is positive and $f'(x) > 0$. These results are displayed in Fig. 4.3, where the slanted lines give a schematic suggestion of the direction of the graph in each interval. In Fig. 4.4 we now plot the points $(-1, 19)$ and $(2, -8)$ and sketch a smooth curve through these points, using the information in Fig. 4.3 provided by the sign of the derivative; that is, $f(x)$ is increasing when $x < -1$, decreasing when $-1 < x < 2$, and increasing when $x > 2$. Notice that in Fig. 4.4 we use different units of length on the two axes, as a matter of convenience in drawing a picture of reasonable size.* It is clear that our function has a maximum at $x = -1$ and a minimum at $x = 2$, and also that no absolute maximum or minimum exists.

The zeros of a function are always valuable aids in curve sketching when they can be found, but finding them can be quite difficult. We have plotted a few additional points in Fig. 4.4 to suggest that the zeros of this particular function are approximately -2.2, 0.9, and 2.9. As a matter of fact, we sometimes sketch the graph of a function to help us estimate the approximate location of its zeros, just as we have done here, as a first step toward the numerical calculation of these zeros to any desired degree of accuracy. In Section 4.6 we describe a standard method for carrying out such calculations.†

Example 2 The rational function

$$y = \frac{x}{x^2 + 1}$$

*The basic idea of a graph as a visual aid displaying the qualitative nature of the function does not require the use of equal units on the two axes. It is only when we work with certain quantitative aspects of the geometry of the plane, such as distances between points, areas of regions, or angles between lines, that it is necessary to use equal units on both axes.

†Those students who have a graphing calculator will enjoy plotting the graph of the polynomial in this example, with the vertical scale changed by a factor of 10 or more.

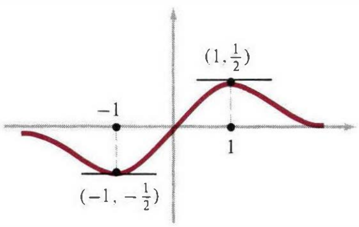


Figure 4.5

was discussed in Example 6 of Section 1.6, and we explained there why the graph has the shape it does (Fig. 4.5). To find the precise location of the indicated maximum and minimum, we calculate the derivative and equate it to zero:

$$y' = \frac{(x^2+1)\cdot 1 - x\cdot 2x}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} = 0.$$

The roots of this equation (the critical points) are $x = 1$ and $x = -1$, so the maximum and minimum occur at $x = 1$ and $x = -1$, respectively. The actual maximum and minimum values are $y = \frac{1}{2}$ and $y = -\frac{1}{2}$. With these facts and our initial awareness of the overall shape of the graph, it is obvious that this function increases on the interval $-1 < x < 1$ and decreases for $x < -1$ and $x > 1$. However, these conclusions can also be drawn directly from the sign of the derivative, which is clearly positive for $-1 < x < 1$ and negative for $x < -1$ and $x > 1$.

These examples, as well as our past experience, suggest a few informal rules that are useful in sketching the graph of a function $f(x)$. If possible and convenient, we should determine

1 The critical points of $f(x)$.
2 The critical values of $f(x)$.
3 The sign of $f'(x)$ between critical points.
4 The zeros of $f(x)$.
5 The behavior of $f(x)$ as $x \to \infty$ and as $x \to -\infty$.
6 The behavior of $f(x)$ near points at which the function is not defined.

However, perhaps the most important rule of all is this: *Don't be a slave to any rule, be flexible, use common sense.* Remember the old Hungarian proverb: "All fixed ideas are wrong, including this one."

Remark 1 Maxima and minima can occur in three ways not covered by the preceding discussion: at *endpoints, cusps,* and *corners.* As examples we consider the three functions

$$x = \sqrt{1-x^2}, \qquad y = x^{2/3}, \qquad y = 1 - \sqrt{x^2} = 1 - |x|.$$

Their graphs are shown in Fig. 4.6. The first function has the closed interval $-1 \le x \le 1$ as its domain, and at the endpoints it has minima that are not revealed by equating the derivative to zero. The second function has a minimum at $x = 0$ that is a cusp, because its derivative

$$y' = \tfrac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$$

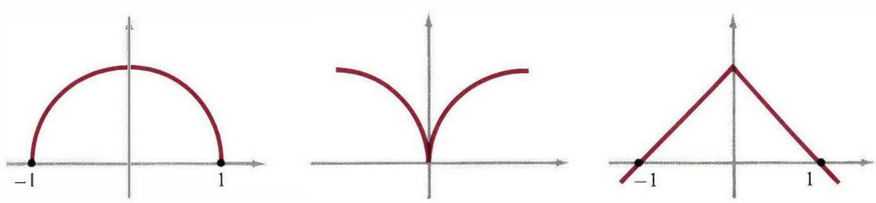


Figure 4.6 Endpoints, cusp, and corner.

is negative to the left of 0 and positive to the right of 0, and becomes infinite near 0. The third function has a maximum at $x = 0$, and this maximum is called a corner for obvious reasons. In seeking the maxima and minima of functions, equate the derivative to zero by all means, but do so carefully, keeping these three possibilities in mind as well.

Remark 2 Among other things, mathematicians are professional skeptics. On one side of their nature they have trained themselves to attack loose arguments and to accept only those statements that they find impossible to doubt, in the hope that Ultimate Certainty will reward their efforts. Our statements about increasing and decreasing functions and maxima and minima are supported only by geometric plausibility arguments. The statements themselves are true, and when examined more carefully they fall into a field of study called *Analysis*, where the foundations of calculus are fully investigated. However, this book is for students, not mathematicians, and our main concern is with the use of the tools rather than the tools themselves. We gave a few preliminary discussions in Section 2.6, and we remind any reader who might be interested that fully rigorous proofs can be found in Appendices A.3 and A.4 at the back of the book.

PROBLEMS

Sketch the graphs of the following functions by using the first derivative and the methods of this section; in particular, find the intervals on which each function is increasing and those on which it is decreasing, and locate any maximum or minimum values it may have.

1 $y = x^2 - 2x$.
2 $y = 2 + x - x^2$.
3 $y = x^2 - 6x + 9$.
4 $y = x^2 - 4x + 5$.
5 $y = 2x^3 - 3x^2 + 1$.
6 $y = x^3 - 3x^2 + 3x - 1$.
7 $y = x^3 - x$.
8 $y = x^4 - 2x^2 + 1$.
9 $y = 3x^4 + 4x^3$.
10 $y = 3x^5 - 20x^3$.
11 $y = x + \dfrac{1}{x}$.
12 $y = 2x + \dfrac{1}{x^2}$.
13 $y = \dfrac{1}{x^2 + x}$.
14 $y = \dfrac{x}{(x-1)^2}$.
15 $y = x\sqrt{3 - x}$.
***16** $y = 5x^{2/3} - x^{5/3}$.

17 The function $f(x) = x^3 + x - 1$, being a third-degree polynomial, obviously crosses the x-axis (why?) and therefore has at least one zero. By examining $f'(x)$, show that this function has only one zero. Show similarly that $f(x) = 7x^{131} + 11x^{73} + x - 500$ has one and only one zero.

18 Consider the function $y = x^m(1 - x)^n$, where m and n are positive integers, and show that
(a) if m is even, y has a minimum at $x = 0$;
(b) if n is even, y has a minimum at $x = 1$;
(c) y has a maximum at $x = m/(m + n)$ regardless of whether m and n are even or not.

19 Sketch the graph of a function $f(x)$ defined for $x > 0$ and having the properties
$$f(1) = 0 \qquad \text{and} \qquad f'(x) = \frac{1}{x} \text{ (all } x > 0\text{)}.$$

20 Sketch the graph of a function $f(x)$ with the properties
$$f'(x) < 0 \text{ for } x < 2 \qquad \text{and} \qquad f'(x) > 0 \text{ for } x > 2$$
(a) if $f'(x)$ is continuous at $x = 2$;
(b) if $f'(x) \to -1$ as $x \to 2-$ and $f'(x) \to 1$ as $x \to 2+$.

21 In each case, sketch the graph of a function with all the stated properties:
(a) $f(1) = 1$, $f'(x) > 0$ for $x < 1$, $f'(x) < 0$ for $x > 1$;
(b) $f(-1) = 2$ and $f(2) = -1$, $f'(x) > 0$ for $x < -1$ and $x > 2$, $f'(x) < 0$ for $-1 < x < 2$;
(c) $f(-1) = 1$ and $f'(-1) = 0$, $f'(x) < 0$ for $x < -1$ and $-1 < x < 2$, $f'(x) > 0$ for $x > 2$;
(d) $f'(x) < 0$ for $-2 < x < 0$ and $x > 1$, $f'(x) > 0$ for $x < -2$ and $0 < x < 1$, $f'(-2) = f'(0) = 0$, $f'(1)$ does not exist.

22 Construct a formula for a function $f(x)$ with a maximum at $x = -2$ and a minimum at $x = 1$.

23 Find the critical points and corresponding critical values for the function $y = \cos 2x + 2\cos x$ on the interval $[0, 2\pi]$. Sketch the graph.

24 Find $a > 0$ so that the curves $y = \sin ax$ and $y = \cos ax$ intersect at right angles.

25 Show that the largest possible value of $y = \sin x - \cos x$ is $\sqrt{2}$.

26 Find the largest possible value of each of the following functions:

(a) $y = \sin x + \cos x$;
(b) $y = \sin x + \cos^2 x$;
(c) $y = \sin^2 x + \cos^2 x$.

27 Find the maximum and minimum values of $y = 2 \sin 2x + \sin 4x$ on the interval $[0, \pi]$, and state where these values occur. Sketch the graph.

28 Find the maximum and minimum values of each of the following functions on the interval $[0, 2\pi]$, and sketch the graphs:
(a) $y = \sin(\cos x)$; (b) $y = \cos(\sin x)$.

29 Show that $y = 27/(\sin x) + 64/(\cos x)$ has a minimum value but no maximum value on the interval $0 < x < \pi/2$, and find this minimum value.

30 Find all critical points for each of the following functions, use the first derivative to decide whether each of the corresponding critical values is a maximum, a minimum, or neither, and sketch the graphs:
(a) $y = x + \sin x$; (b) $y = \sin^2 2x$.

4.2 CONCAVITY AND POINTS OF INFLECTION

One of the most distinctive features of a graph is the direction in which it curves or bends. The graph on the left in Fig. 4.7 curves upward as the point that traces it moves from left to right, and the graph on the right curves downward. The sign of the second derivative gives us this information, as follows.

A positive second derivative, $f''(x) > 0$, tells us that the slope $f'(x)$ is an increasing function of x. This means that the tangent turns counterclockwise as we move along the curve from left to right, as shown on the left side of Fig. 4.8. The curve is said to be *concave up* (the concave side of a curve is its hollow side). Such a curve lies above its tangent except at the point of tangency. Similarly, if the second derivative is negative, $f''(x) < 0$, then the slope $f'(x)$ is a decreasing function, and the tangent turns clockwise as we move to the right (see the right side of Fig. 4.8). Under these circumstances the curve is *concave down;* it lies below its tangent except at the point of tangency.

Most curves are concave up on some intervals and concave down on other intervals. A point like P in Fig. 4.8, across which the direction of concavity changes, is called a *point of inflection.** If $f''(x)$ is continuous and has opposite signs on each side of P, then it must have a zero at P itself. The search for points of inflection is mainly a matter of solving the equation $f''(x) = 0$ and checking the direction of concavity on both sides of each root.

Example 1 To investigate the function

$$y = f(x) = 2x^3 - 12x^2 + 18x - 2$$

*The word "inflection" comes from the Latin *inflectere,* meaning "to bend."

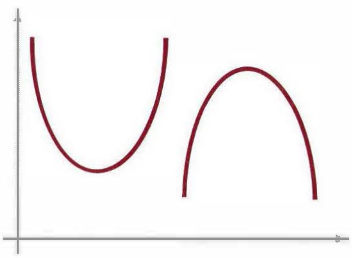


Figure 4.7

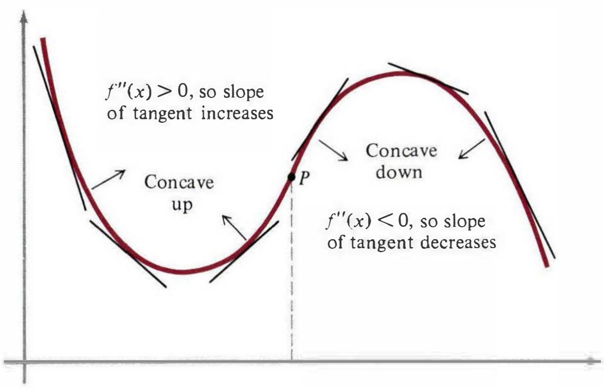


Figure 4.8

for concavity and points of inflection, we calculate

$$f'(x) = 6x^2 - 24x + 18 = 6(x-1)(x-3)$$

and

$$f''(x) = 12x - 24 = 12(x-2).$$

The critical points [the roots of $f'(x) = 0$] are clearly $x = 1$ and $x = 3$, and the corresponding critical values are $y = 6$ and $y = -2$. We have a possible point of inflection at $x = 2$, since this is the only root of $f''(x) = 0$. It is evident that $f''(x)$ is negative for $x < 2$ and positive for $x > 2$, so the graph is concave down on the left of $x = 2$ and concave up on the right. This tells us that we really have a point of inflection at $x = 2$, as indicated in Fig. 4.9.

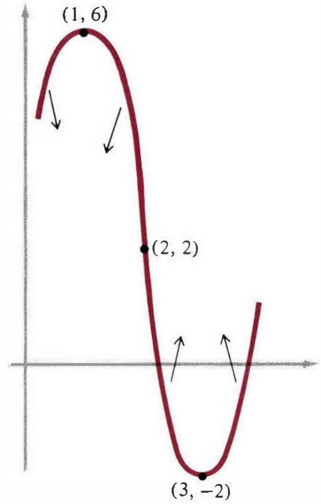


Figure 4.9

Example 2 The rational function

$$y = \frac{1}{x^2+1}$$

is very easy to sketch by inspection if we notice the following clues: it is symmetric about the y-axis because the exponent is an even number, its values are all positive, it has a maximum at $x = 0$ because this yields the smallest denominator, and $y \to 0$ as $|x| \to \infty$. It is therefore intuitively clear that the graph has the shape shown in Fig. 4.10. There are evidently two points of inflection, and the only question is, What are their precise locations? To discover this, we compute

$$y' = \frac{-2x}{(x^2+1)^2}$$

and

$$y'' = \frac{(x^2+1)^2 \cdot (-2) + 2x \cdot 2(x^2+1) \cdot 2x}{(x^2+1)^4}$$

$$= \frac{(x^2+1) \cdot (-2) + 8x^2}{(x^2+1)^3} = \frac{2(3x^2-1)}{(x^2+1)^3}.$$

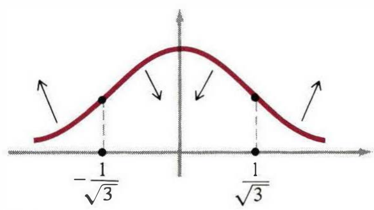


Figure 4.10

Equating y'' to zero and solving gives $x = \pm 1/\sqrt{3}$, which locates the points of inflection. If we wish, we can verify our first impression about the direction of concavity on various parts of the curve, as shown in Fig. 4.10, by observing that $y'' < 0$ when $x^2 < \frac{1}{3}$ and $y'' > 0$ when $x^2 > \frac{1}{3}$. These facts tell us that the graph is concave down for $-1/\sqrt{3} < x < 1/\sqrt{3}$ and concave up for $x < -1/\sqrt{3}$ and $x > 1/\sqrt{3}$.

Remark 1 As we have tried to suggest in these examples, knowing that $f''(x_0) = 0$ is not enough to guarantee that $x = x_0$ furnishes a point of inflection. We must also know that the graph is concave up on one side of x_0 and concave down on the other. The simplest function that shows this difficulty is $y = f(x) = x^4$ (Fig. 4.11). Here $f'(x) = 4x^3$ and $f''(x) = 12x^2$, so $f''(x) = 0$ at $x = 0$. However, $f''(x)$ is clearly positive on both sides of the point $x = 0$, and therefore—as we already know from the graph—this point corresponds to a minimum, not a point of inflection. The function $y = x^5 - 5x^4$ provides a more complicated example of the same phenomenon. Here

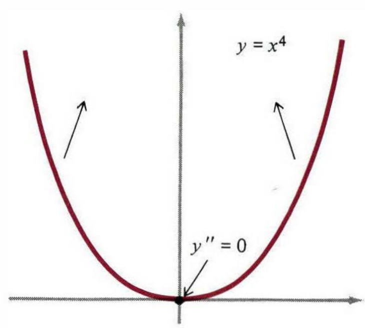


Figure 4.11

$$y' = 5x^4 - 20x^3 \qquad \text{and} \qquad y'' = 20x^3 - 60x^2 = 20x^2(x - 3).$$

The roots of $y'' = 0$ are $x = 0$ and $x = 3$. However, y'' does not change sign at $x = 0$, so the only point of inflection is at $x = 3$. The graph is concave down on the left of this point and concave up on the right.

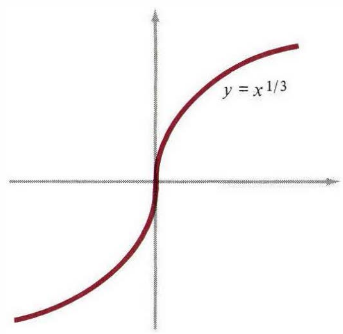


Figure 4.12

Remark 2 The graph of $y = x^{1/3} = \sqrt[3]{x}$ is easy to sketch and has an obvious point of inflection at $x = 0$ (Fig. 4.12). We can also discover this by inspecting the second derivative. We have

$$y' = \tfrac{1}{3}x^{-2/3}$$

and

$$y'' = -\tfrac{2}{9}x^{-5/3} = \frac{-2}{9\sqrt[3]{x^5}},$$

so y'' is positive if $x < 0$ and negative if $x > 0$. However, y'' does not exist at $x = 0$. In searching for points of inflection, we must therefore consider not only points at which $y'' = 0$, but also points (if there are any) at which y'' does not exist.

Remark 3 In the so-called *second derivative test*—which we state informally in Fig. 4.13—the sign of the second derivative is used to decide whether a critical point furnishes a maximum or a minimum value. This test is sometimes useful, but its importance is often exaggerated. We will see in the next two sections that in most applied problems it is clear from the context whether we have a maximum or minimum value, and no further testing is necessary.

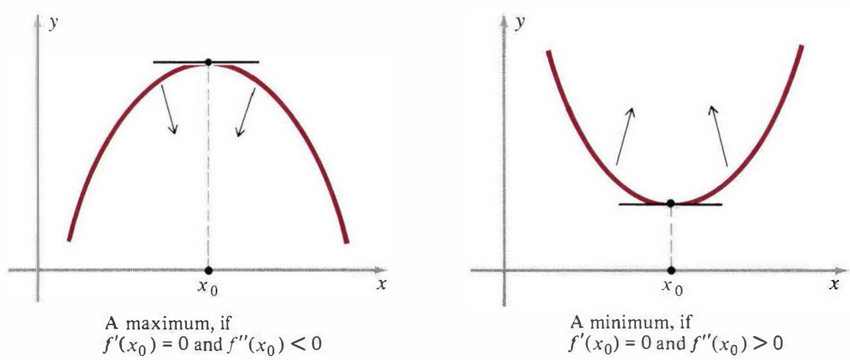


Figure 4.13 The second derivative test.

PROBLEMS

For each of the following, locate the points of inflection, find the intervals on which the curve is concave up and those on which it is concave down, and sketch.

1 $y = (x - a)^3 + b$.

2 $y = x^3 - 6x^2$.

3 $y = x^3 + 3x^2 + 4$.

4 $y = 2x^3 + 3x^2 - 12x$.

5 $y = x^4 + 2x^3 + 1$.

6 $y = x^4 - 6x^2$.

7 $y = x^4 - 2x^3$.

8 $y = 3x^5 - 5x^4$.

9 $y = \dfrac{9}{x^2 + 9}$.

10 $y = \dfrac{ax}{x^2 + b^2}$ $(a, b > 0)$.

11 $y = \dfrac{4x^2}{x^2 + 3}$.

12 $y = \dfrac{12}{x^2} - \dfrac{12}{x}$.

13 $y = x - \dfrac{1}{x}$.

14 In each part of this problem, use the given formula for the second derivative of a function to locate the points

of inflection, the intervals on which the graph is concave up, and the intervals on which the graph is concave down:
(a) $y'' = 8x^2 + 32x$; (b) $y'' = 15x^3 + 39x$;
(c) $y'' = 3x^4 - 27x^2$; (d) $y'' = (x + 2)(x^2 - 4)$.

15 Sketch the graph of a function $f(x)$ defined for all x such that
(a) $f(x) > 0, f'(x) > 0$, and $f''(x) > 0$;
(b) $f'(x) < 0$ and $f''(x) < 0$.

16 Is it possible for a function $f(x)$ defined for all x to have the three properties $f(x) > 0, f'(x) < 0$, and $f''(x) < 0$? Explain.

17 (a) By sketching, show that $y = x^2 + a/x$ has a minimum but no maximum for every value of the constant a. Also, verify this by calculation.
(b) Find the point of inflection of $y = x^2 - 8/x$.

18 Starting from $x^2 + y^2 = a^2$, calculate d^2y/dx^2 by implicit differentiation and state why its sign should be opposite to the sign of y.

19 Find the value of a that makes $y = x^3 - ax^2 + 1$ have a point of inflection at $x = 1$.

20 Find a and b such that $y = a\sqrt{x} + b/\sqrt{x}$ has $(1, 4)$ as a point of inflection.

21 If k is a positive number $\neq 1$, show that the first quadrant part of the curve $y = x^k$ is
(a) concave up if $k > 1$;
(b) concave down if $k < 1$.

22 If k is a positive number $\neq 1$ and $y = x^k - kx$, show that
(a) if $k < 1$, y has a maximum at $x = 1$;
(b) if $k > 1$, y has a minimum at $x = 1$.

23 Show that the graph of a quadratic function $y = ax^2 + bx + c$ has no points of inflection. Give a condition under which the graph is (a) concave up; (b) concave down.

24 Show that the general cubic curve $y = ax^3 + bx^2 + cx + d$ has a single point of inflection and three possible shapes depending on whether $b^2 > 3ac$, $b^2 = 3ac$, or $b^2 < 3ac$. Sketch these shapes.

25 In each of the following, sketch the graph of a function with all the stated properties:
(a) $f(0) = 2, f(2) = 0, f'(0) = f'(2) = 0, f'(x) > 0$ for $|x - 1| > 1, f'(x) < 0$ for $|x - 1| < 1, f''(x) < 0$ for $x < 1, f''(x) > 0$ for $x > 1$;
(b) $f(-2) = 6,\ f(1) = 2,\ f(3) = 4,\ f'(1) = f'(3) = 0,$ $f''(x) < 0$ for $|x - 2| > 1, f'(x) > 0$ for $|x - 2| < 1, f''(x) < 0$ for $x > 2$ or $|x + 1| < 1, f''(x) > 0$ for $|x - 1| < 1$ or $x < -2$;
(c) $f(0) = 0,\ f(2) = f(-2) = 1,\ f'(0) = 0,\ f'(x) > 0$ for $x > 0,\ f'(x) < 0$ for $x < 0,\ f''(x) > 0$ for $|x| < 2,\ f''(x) < 0$ for $|x| > 2,\ \lim_{x\to\infty} f(x) = 2,$ $\lim_{x\to-\infty} f(x) = 2$;
(d) $f(2) = 4, f'(x) > 0$ for $x < 2, f'(x) < 0$ for $x > 2, f''(x) > 0$ for $x \neq 2, \lim_{x\to 2} |f'(x)| = \infty, \lim_{x\to\infty} f(x) = 2, \lim_{x\to-\infty} f(x) = 2$.

26 Show that the graph of $y = \sin x$ is concave down when it is above the x-axis and concave up when it is below the x-axis.

27 For each of the following functions defined on the interval $0 \leq x \leq 2\pi$, find all points of inflection and determine the intervals on which the graph is concave up and those on which the graph is concave down, and sketch the graph:
(a) $y = x - 2\sin x$;
(b) $y = \sin^2 x$;
(c) $y = x + \sin x$.

4.3 APPLIED MAXIMUM AND MINIMUM PROBLEMS

Among the most striking applications of calculus are those that depend on finding the maximum or minimum values of functions.

Practical everyday life is filled with such problems, and it is natural that mathematicians and others should find them interesting and important. A businessperson seeks to maximize profits and minimize costs. An engineer designing a new automobile wishes to maximize its efficiency. An airline pilot tries to minimize flight times and fuel consumption. In science, we often find that nature acts in a way that maximizes or minimizes a certain quantity. For example, a ray of light traverses a system of lenses along a path that minimizes its total time of travel, and a flexible hanging chain assumes a shape that minimizes its potential energy due to gravity.

Whenever we use such words as "largest," "smallest," "most," "least," "best," and so on, it is a reasonable guess that some kind of maximum or minimum problem is lurking nearby. If this problem can be expressed in terms of variables and functions—which is not always possible by any means—then the methods of calculus stand ready to help us understand it and solve it.

Many of our examples and problems deal with geometric ideas, because maximum and minimum values often appear with particular vividness in geometric settings. In order to be ready for this work, students should make sure they know the formulas for areas and volumes given in Fig. 1.22 of Chapter 1.

We begin with a fairly simple example about numbers.

Example 1 Find two positive numbers whose sum is 16 and whose product is as large as possible.

Solution If x and y are two variable positive numbers whose sum is 16, so that

$$x + y = 16, \tag{1}$$

then we are asked to find the particular values of x and y that maximize their product

$$P = xy. \tag{2}$$

Our initial difficulty is that P depends on two variables, whereas our calculus of derivatives works only for functions of a single independent variable. Equation (1) gets us over this difficulty. It enables us to express y in terms of x, $y = 16 - x$, and thereby to express P as a function of x alone,

$$P = x(16 - x) = 16x - x^2. \tag{3}$$

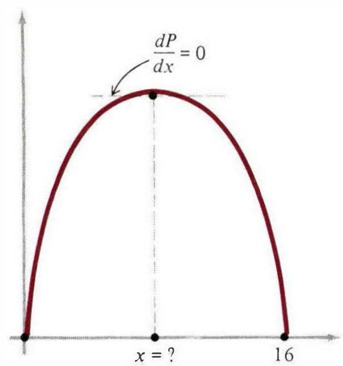


Figure 4.14

In Fig. 4.14 we give a rough sketch of the graph of (3). Our only purpose here is to provide visual emphasis for the following obvious facts about this function: that $P = 0$ for $x = 0$ and $x = 16$, that $P > 0$ for $0 < x < 16$, and that therefore the highest point (where P has its largest value) is characterized by the condition $dP/dx = 0$, since this condition means that the tangent is horizontal. To solve the problem we compute this derivative from (3),

$$\frac{dP}{dx} = 16 - 2x;$$

we equate this derivative to zero,

$$16 - 2x = 0;$$

and we see that $x = 8$ is the solution of this equation. This is the value of x that maximizes P, and by (1) the corresponding value of y is also 8. It is quite clear from Fig. 4.14 that $x = 8$ actually does maximize P; but if we wish to verify this, we can do so by computing the second derivative,

$$\frac{d^2P}{dx^2} = -2,$$

and by recalling that a negative second derivative implies that the curve is concave down and therefore we have a maximum—which we already knew from Fig. 4.14. The related problem of making the product P as small as possible within the stated restrictions has no solution, because the restriction that x and y must be positive numbers means that x must belong to the *open* interval $0 < x < 16$, and this part of the graph has no lowest point.

Example 2 A rectangular garden 450 ft^2 in area is to be fenced off against rabbits. If one side of the garden is already protected by a barn wall, what dimensions will require the shortest length of fence?

Solution We begin by drawing a sketch and introducing notation that will make it convenient to deal with the area of the garden and the total length of the fence (Fig. 4.15). If L denotes the length of the fence, we are to minimize

$$L = 2x + y \tag{4}$$

subject to the restriction that

$$xy = 450. \tag{5}$$

By using (5), L can be written as a function of x alone,

$$L = 2x + \frac{450}{x}. \tag{6}$$

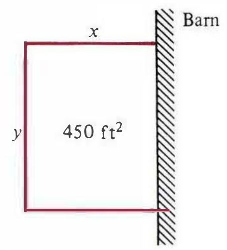


Figure 4.15

A quick sketch (Fig. 4.16) helps us to visualize this function and feel comfortable with its properties, especially the fact that it has a minimum and no maximum (we are only interested in positive values of x). Our next steps are to compute the derivative of (6),

$$\frac{dL}{dx} = 2 - \frac{450}{x^2},$$

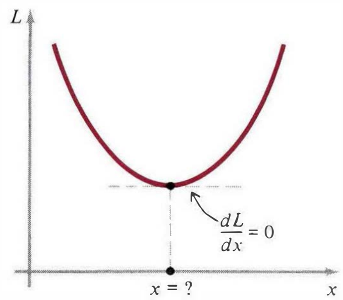


Figure 4.16

and then to equate this derivative to zero and solve the resulting equation,

$$2 - \frac{450}{x^2} = 0, \qquad x^2 = 225, \qquad x = 15.$$

(We ignore the root $x = -15$ for the reason stated.) By (5), the corresponding value of y is $y = 30$, so the garden with the shortest fence is 15 by 30, or twice as long as it is wide.

Example 3 Find the dimensions of the rectangle of greatest area that can be inscribed in a semicircle of radius a.

Solution If we take our semicircle to be the top half of the circle $x^2 + y^2 = a^2$ (Fig. 4.17, left), then our notation is ready and waiting: We must maximize

$$A = 2xy \tag{7}$$

with the restriction that

$$x^2 + y^2 = a^2. \tag{8}$$

Since (8) yields $y = \sqrt{a^2 - x^2} = (a^2 - x^2)^{1/2}$, (7) becomes

$$A = 2x(a^2 - x^2)^{1/2}. \tag{9}$$

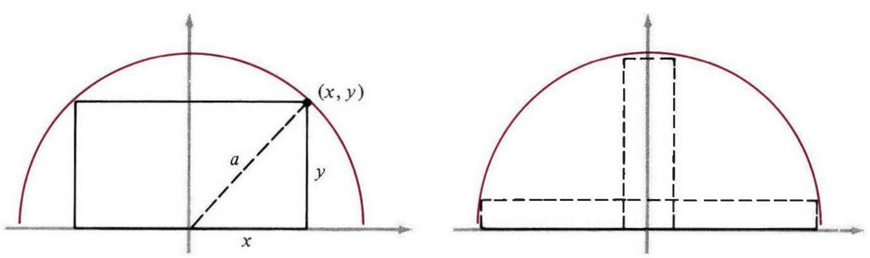


Figure 4.17

It is clear that x lies in the interval $0 < x < a$. On the right in Fig. 4.17 we imagine the extreme cases: When x is close to 0, the rectangle is tall and thin, and when x is close to a, it is short and wide—and in each case the area is small, so somewhere in between we have a maximum area. To locate this maximum, we compute dA/dx from (9), equate it to zero, and solve:

$$2x \cdot \tfrac{1}{2}(a^2 - x^2)^{-1/2} \cdot (-2x) + 2(a^2 - x^2)^{1/2} = 0, \qquad \frac{x^2}{\sqrt{a^2 - x^2}} = \sqrt{a^2 - x^2},$$

$$x^2 = a^2 - x^2, \qquad 2x^2 = a^2, \qquad x = \frac{a}{\sqrt{2}} = \tfrac{1}{2}\sqrt{2}a.$$

Since $y = \sqrt{a^2 - x^2}$, we see that the corresponding value of y is also $\frac{1}{2}\sqrt{2}a$, so the dimensions of the largest inscribed rectangle are $2x = \sqrt{2}a$ and $y = \frac{1}{2}\sqrt{2}a$, and this rectangle is twice as long as it is wide.

There is a more efficient way of solving this problem if we don't care about the actual dimensions of the largest rectangle but only about its shape. The first step is to notice that (8) determines y as an implicit function of x, so implicit differentiation with respect to x yields

$$2x + 2y\frac{dy}{dx} = 0 \qquad \text{or} \qquad \frac{dy}{dx} = -\frac{x}{y}. \tag{10}$$

Next, by differentiating (7) with respect to x and using the fact that $dA/dx = 0$ at the maximum, we obtain

$$2x\frac{dy}{dx} + 2y = 0 \qquad \text{or} \qquad x\frac{dy}{dx} + y = 0. \tag{11}$$

When (10) is inserted in (11), the result is

$$x\left(-\frac{x}{y}\right) + y = 0, \qquad -x^2 + y^2 = 0, \qquad y^2 = x^2, \qquad \text{or} \qquad y = x,$$

where the last equation expresses the shape of the rectangle with the largest area. We can also describe this shape by saying that the ratio of the height of the rectangle to its base is

$$\frac{y}{2x} = \frac{x}{2x} = \frac{1}{2}.$$

Example 4 A wire of length L is to be cut into two pieces, one being bent to form a square and the other to form a circle. How should the wire be cut if the sum of the areas enclosed by the two pieces is to be (a) a maximum? (b) a minimum?

Solution If x denotes the side of the square and r the radius of the circle, as shown on the left in Fig. 4.18, then the sum of the areas is

$$A = x^2 + \pi r^2 \tag{12}$$

where x and r are related by

$$4x + 2\pi r = L. \tag{13}$$

We solve (13) for r in terms of x,

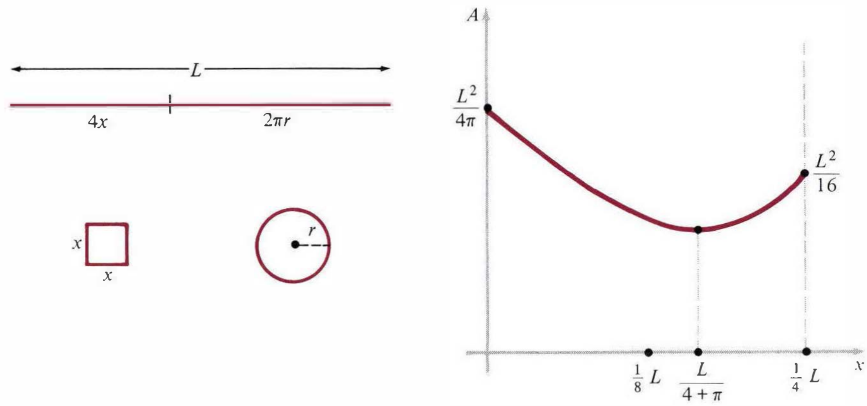


Figure 4.18

$$r = \frac{1}{2\pi}(L - 4x),$$

and use this to express A in terms of x alone,

$$\begin{aligned} A &= x^2 + \pi \cdot \frac{1}{4\pi^2}(L - 4x)^2 \\ &= x^2 + \frac{1}{4\pi}(L - 4x)^2. \end{aligned} \tag{14}$$

Now we notice that when $x = 0$ all the wire is used for the circle, and when $x = \frac{1}{4}L$ all the wire is used for the square. To solve the problem, we must fully understand the behavior of the function (14) on the interval $0 \leq x \leq \frac{1}{4}L$. Its values at $x = 0$ and $x = \frac{1}{4}L$ are clearly $L^2/4\pi$ and $L^2/16$, and the first of these values is the larger because $16 > 4\pi$. This is indicated on the right in Fig. 4.18. Since (14) shows that the graph is a parabola that opens up (x^2 has a positive coefficient), all that is needed to verify the shape of the graph shown in the figure is to find the location of the low point. For this we calculate the derivative of (14),

$$\begin{aligned} \frac{dA}{dx} &= 2x + \frac{1}{4\pi} \cdot 2(L - 4x) \cdot (-4) \\ &= 2x - \frac{2}{\pi}(L - 4x). \end{aligned}$$

On setting this equal to zero and solving the resulting equation, we get

$$x - \frac{1}{\pi}(L - 4x) = 0, \qquad \pi x = L - 4x, \qquad x = \frac{L}{4 + \pi}.$$

This number lies between $\frac{1}{8}L$ and $\frac{1}{4}L$, so the graph shown in the figure is correct and we complete the solution of the problem as follows.

The highest point of the graph is at the left end, and therefore to maximize A we must choose $x = 0$ and use all the wire for the circle. If we insist that the wire must actually be cut, then (*a*) has no answer; for no matter how little of the wire is used for the square, we can always increase the total area by using still less.

For (b), the total area is minimized when $x = L/(4 + \pi)$. Accordingly, the length of wire used for the square is $4x = 4L/(4 + \pi)$ and the length used for the circle is

$$L - 4x = L - \frac{4L}{4+\pi} = \frac{\pi L}{4+\pi}.$$

We also notice that the minimal area is attained when the diameter of the circle equals the side of the square, since

$$2r = \frac{1}{\pi}(L - 4x) = \frac{1}{\pi} \cdot \frac{\pi L}{4+\pi} = \frac{L}{4+\pi}.$$

Example 5 At a price of \$1.50, a door-to-door salesperson can sell 500 potato peelers that cost 70 cents each. For every cent that the salesperson lowers the price, the number sold can be increased by 25. What selling price will maximize the total profit?

Solution If x denotes the number of cents the salesperson lowers the price, then the profit on each peeler is $80 - x$ cents and the number sold is $500 + 25x$. The total profit (in cents) is therefore

$$P = (80 - x)(500 + 25x) = 40{,}000 + 1500x - 25x^2.$$

We maximize this function by setting the derivative equal to zero and solving the resulting equation,

$$\frac{dP}{dx} = 1500 - 50x, \qquad 1500 - 50x = 0 \qquad 50x = 1500, \qquad x = 30.$$

The most advantageous selling price is therefore \$1.20.

As these examples show, the mathematical techniques required in most maximum-minimum problems are relatively simple. The hardest part of such a problem is usually "setting it up" in a convenient form. This is the analytical, thinking part of the problem, as opposed to the computational part. We emphasize this because it is clear that calculus is unlikely to be of much value as a tool in the sciences unless one learns how to understand what a problem is about and how to translate its words into appropriate mathematical language. This is what "word problems" or "story problems" are for—to help students learn these critically important skills.

No set of rules for problem solving really works, because the essential ingredients are imagination and intelligence, which cannot be taught. However, the following general suggestions may be helpful. They don't guarantee success, but without them progress is unlikely.

STRATEGY FOR SOLVING MAXIMUM-MINIMUM PROBLEMS

1 *Understand the problem.* Begin by reading the problem carefully, several times if necessary, until it is fully understood. It is a sad fact of life that students often seem driven to start working on a problem before they have any clear idea of what it is about. Take your time and make your efforts count.

2 If geometry is involved—as it often is—*make a fairly careful sketch of reasonable size.* Show the general configuration. For instance, if a problem is about a general triangle, don't mislead yourself by drawing one that looks like a right triangle or an isosceles triangle. Don't be hasty or sloppy. You hope your sketch will be a source of fruitful ideas, so treat it with respect.

3 *Label your figure carefully,* making sure you understand which quantities are constant and which are allowed to vary. If convenient, use initial letters to suggest the quantities they represent, as A for area, V for volume, h for height. Be aware of geometric relations among the quantities in your figure, especially those involving right triangles and similar triangles. Write these relations down in the form of equations and be prepared to use them if needed.

4 If Q is the quantity to be maximized or minimized, write it down in terms of the quantities in the figure, and try to *use the relations in Step 3 to express Q as a function of a single variable.* Draw a quick informal graph of this function on a suitable interval, perform little thought experiments in which you visualize the extreme cases, and use derivatives to discover details and thereby solve your problem.*

*Serious mathematical problem solving is mental activity on the highest level. Even exceptionally able students, who are confident they have the necessary imagination and intelligence, may derive additional comfort from the possession of a method. Such a method was distilled by George Polya (1887–1985) out of his own vast experience as an eminent creative mathematician (250 research papers and 10 books) and the foremost mathematics teacher of his generation. Polya's method consists of four simple principles that will be recognized at once as only common sense: (1) understand the problem; (2) devise a plan; (3) carry out your plan; (4) look back on your work and learn. The strategy given above is nothing but Polya's principles (1), (2), and (3) adapted to the special circumstances of maximum-minimum problems. Polya developed his ideas in a series of great and delightful books that should be required reading for every undergraduate mathematics major: *How To Solve It* (Princeton Press, 1945, 2nd ed., 1957); *Mathematics and Plausible Reasoning* (2 vols., Princeton Press, 1954); and *Mathematical Discovery* (2 vols., Wiley, 1962). His discussion of the following problem in *How To Solve It,* under the heading Working Backwards, is only one of many unforgettable gems: How can you bring up from the river exactly 6 quarts of water when you have only two containers, a 4-quart pail and a 9-quart pail (Fig. 4.19), to measure with?

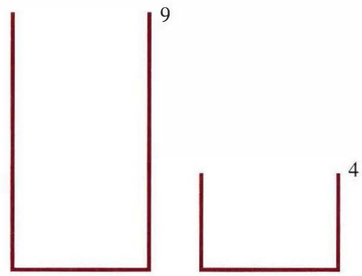


Figure 4.19

PROBLEMS

1 Find the positive number that exceeds its square by the largest amount. Why would you expect this number to be in the open interval (0, 1)?

2 Express the number 18 as the sum of two positive numbers in such a way that the product of the first and the square of the second is as large as possible.

3 Show that the rectangle with maximum area for a given perimeter is a square.†

4 Show that the rectangle with minimum perimeter for a given area is a square.

5 Show that the square has the largest area among all rectangles inscribed in a given fixed circle $x^2 + y^2 = a^2$.

6 If we maximize the perimeter of the rectangle instead of the area in Problem 5, show that the solution is still a square.

7 An east-west and a north-south road intersect at a point O. A diagonal road is to be constructed from a point A east of O to a point B north of O, passing through a town C which is a miles east and b miles north of O. Find the ratio of OA to OB if the triangular area OAB is as small as possible. Show that this minimal area is attained when C bisects the segment AB.

8 A certain poster requires 96 in^2 for the printed message and must have 3-in margins at the top and bottom and a 2-in margin on each side. Find the overall dimensions of the poster if the amount of paper used is a minimum.

9 A university bookstore can get the book *Courtship Rituals of the American College Student* at a cost of \$4 a copy from the publisher. The bookstore manager estimates that she can sell 180 copies at a price of \$10 and that each 50-cent reduction in the price will increase the sales by 30 copies. What should be the price of the book to maximize the bookstore's total profit?

10 A new branch bank is to have a floor area of 3500 ft^2. It is to be a rectangle with three solid brick walls and a decorative glass front. The glass costs 1.8 times as much as the brick wall per linear foot. What dimensions of the building will minimize the cost of materials for the walls and front?

11 At noon a ship A is 50 mi north of a ship B and is steaming south at 16 mi/h. Ship B is headed west at 12 mi/h. At what time are they closest together, and what is the minimal distance between them?

12 Express the number 8 as the sum of two nonnegative

†This was the earliest maximum-minimum problem solved by the methods of calculus (by Fermat, about 1629).

numbers in such a way that the sum of the square of the first and the cube of the second is as small as possible. Also solve the problem if this sum is to be as large as possible.

13 Find two positive numbers whose product is 16 and whose sum is as small as possible.

14 A triangle of base b and height h has acute base angles. A rectangle is inscribed in the triangle with one side on the base of the triangle. Show that the largest such rectangle has base $b/2$ and height $h/2$, so that its area is one-half the area of the triangle.

15 Find the area of the largest rectangle with lower base on the x-axis and upper vertices on the parabola $y = 27 - x^2$.

16 An isosceles triangle has its vertex at the origin, its base parallel to and above the x-axis, and the vertices of its base on the parabola $9y = 27 - x^2$. Find the area of the largest such triangle.

17 If a rectangle has an area of 32 in^2, what are its dimensions if the distance from one corner to the midpoint of a nonadjacent side is as small as possible?

18 If the cost per hour of running a small riverboat is proportional to the cube of its speed through the water, find the speed at which it should be run against a current of a miles per hour to minimize the cost of an upstream journey over a distance of b miles.

19 A Norman window has the shape of a rectangle surmounted by a semicircle. If the total perimeter is fixed, find the proportions of the window (i.e., the ratio of the height of the window to its base) that will admit the most light.

20 Solve the Norman window problem in Problem 19 if the semicircular part is made of stained glass that transmits only half as much light per unit area as does the clear glass in the rectangular part.

21 A trough is to be made from three planks, each 12 in wide. If the cross section has the shape of a trapezoid, how far apart should the tops of the sides be placed to give the trough maximum carrying capacity?

22 Solve Problem 21 if there is one 12-in plank and two 6-in planks.

23 The strength of a rectangular beam is jointly proportional to its width and the cube of its depth.[†] Find the proportions (ratio of depth to width) of the strongest beam that can be cut from a given cylindrical log.

24 Among all isosceles triangles with fixed perimeter, show that the triangle of greatest area is equilateral.

25 An isosceles triangle is inscribed in the circle $x^2 + y^2 = a^2$ with its base parallel to the x-axis and one vertex at the point $(0, a)$. Find the height of the triangle with maximum area and show that this triangle is equilateral. (Can you show by geometric reasoning alone that the largest triangle inscribed in the circle is necessarily equilateral?)

26 A wire of length L is to be cut into two pieces, one bent to form a square and the other to form an equilateral triangle. How should the wire be cut if the sum of the areas enclosed by the two pieces is to be (a) a maximum? (b) a minimum? Show that case (b) occurs when the side of the square is $\frac{2}{3}$ the height of the triangle.

*27 A man 6 ft tall wants to construct a greenhouse of length L and width 18 ft against the outer wall of his house by building a sloping glass roof of slant height y from the ground to the wall, as shown in Fig. 4.20. He considers space in the greenhouse to be *usable* if he can stand upright without bumping his head. If the cost of building the roof is proportional to y, find the slope of the roof that minimizes the cost per square foot of usable space. Hint: Notice that this amounts to minimizing y/x.

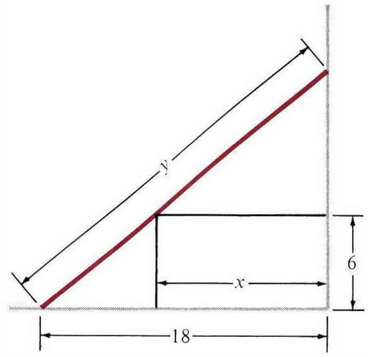


Figure 4.20

*28 (The fireman's problem) A fence a feet high is b feet from a high burning building. Find the length of the shortest ladder that will reach from the ground across the top of the fence to the building.

*29 A corridor of width a is at right angles to a second corridor of width b. A long, thin, heavy rod is to be pushed along the floor from the first corridor into the second. What is the length of the longest rod that can get around the corner?

*30 A long sheet of paper is a units wide. One corner of the paper is folded over as shown in Fig. 4.21. Find the value of x that minimizes (a) the area of the triangle ABC; (b) the length of the crease AC.

31 The speed v of a wave on the surface of a still body of liquid depends as follows on its wavelength λ:

$$v = \sqrt{\frac{g}{2\pi}\lambda + \frac{2\pi\sigma}{\delta\lambda}},$$

[†]This means that if x is the width and y is the depth, then the strength S is given by the formula $S = cxy^3$, where c is a constant of proportionality.

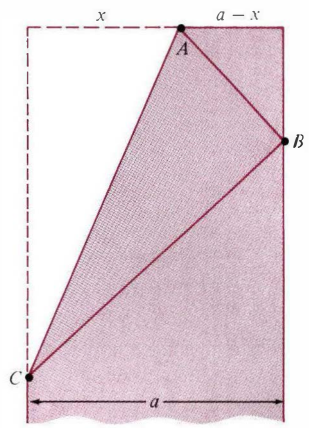


Figure 4.21

where the constants are the acceleration g due to gravity, the surface tension σ of the liquid, and the density δ of the liquid. Find the minimum speed of a wave and the corresponding wavelength.

32 The illumination provided on a flat surface by a light source is inversely proportional to the square of the distance from the source and directly proportional to the sine of the angle of incidence. How high should a light be placed on a pole to maximize the illumination on the ground 50 ft from the pole?

33 Find the maximum area of a rectangle that can be circumscribed about a given rectangle with base B and height H (Fig. 4.22).

34 Water has the very unusual property among common liquids of having a temperature above its freezing point at which its density is a maximum. (This property is important for the survival of life in ponds and lakes: the denser water sinks and prevents the bottom water from freezing.) If 1 liter of water at 0°C occupies a volume of

$$V = 1 - \frac{6.42}{10^5}T + \frac{8.51}{10^6}T^2 - \frac{6.79}{10^8}T^3$$

liters at T°C, find the temperature at which the density is greatest.

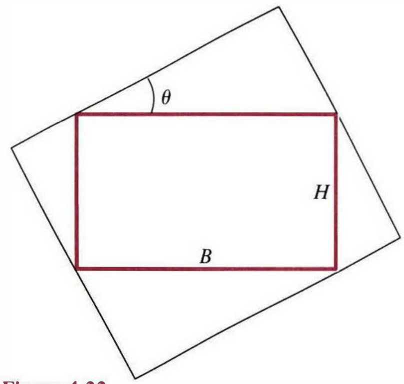


Figure 4.22

4.4 MORE MAXIMUM-MINIMUM PROBLEMS. REFLECTION AND REFRACTION

We continue to develop the basic ideas of Section 4.3 by means of additional examples.

Example 1 A manufacturer of cylindrical soup cans receives a very large order for cans of a specified volume V_0. What dimensions will minimize the total surface area of such a can, and therefore the amount of metal needed to manufacture it?

Solution If r and h are the radius of the base and the height of a cylindrical can (Fig. 4.23, left), then the volume is

$$V_0 = \pi r^2 h \tag{1}$$

and the total surface area is

$$A = 2\pi r^2 + 2\pi rh. \tag{2}$$

We must minimize A, which is a function of two variables, by using the fact that equation (1) relates these variables to one another. We therefore solve (1) for h, $h = V_0/\pi r^2$, and use this to express A as a function of r alone,

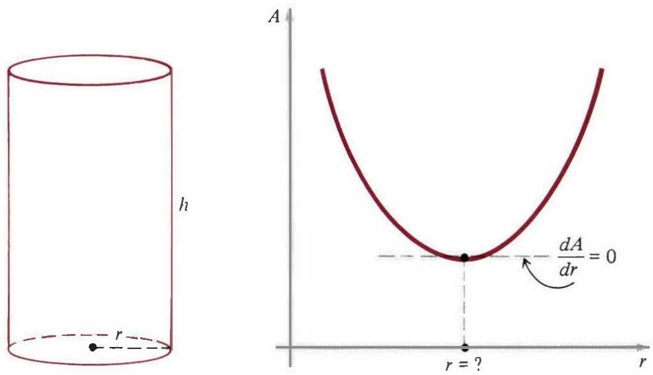


Figure 4.23

$$A = 2\pi r^2 + 2\pi r \cdot \frac{V_0}{\pi r^2}$$

$$= 2\pi r^2 + \frac{2V_0}{r}. \tag{3}$$

The graph of this function (Fig. 4.23, right) shows that A is large when r is small and also when r is large, with a minimum somewhere in between. As usual, to discover the precise location of this minimum, we differentiate (3), equate the derivative to zero, and solve,

$$\frac{dA}{dr} = 4\pi r - \frac{2V_0}{r^2}, \qquad 4\pi r - \frac{2V_0}{r^2} = 0, \qquad 4\pi r^3 = 2V_0,$$

$$2\pi r^3 = V_0. \tag{4}$$

If we want the actual dimensions of the most efficient can, we can solve equation (4) for r and then use this to calculate h,

$$r = \sqrt[3]{\frac{V_0}{2\pi}}, \qquad h = \frac{V_0}{\pi r^2} = \frac{V_0}{\pi}\left(\frac{2\pi}{V_0}\right)^{2/3} = 2\sqrt[3]{\frac{V_0}{2\pi}},$$

from which we observe that $h = 2r$. Or, if we are interested primarily in the shape, we can replace V_0 in (4) by $\pi r^2 h$ and immediately obtain

$$2\pi r^3 = \pi r^2 h \qquad \text{or} \qquad 2r = h.$$

From the point of view of lowering costs for raw materials—an extremely serious matter for manufacturers—this remarkable result tells us that the "best" shape for a cylindrical can is that in which the height equals the diameter of the base.

Example 2 Find the ratio of the height to the diameter of the base for the cylinder of maximum volume that can be inscribed in a sphere of radius R.

Solution If we sketch a cylinder inscribed in the sphere and label it as shown on the left in Fig. 4.24, then we see that

$$V = 2\pi x^2 y \tag{5}$$

where

$$x^2 + y^2 = R^2. \tag{6}$$

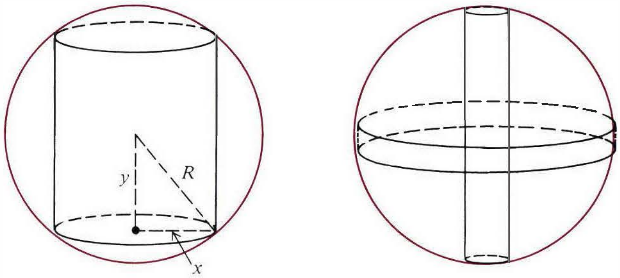


Figure 4.24

Visualizing the extreme cases (Fig. 4.24, right) tells us that V is small when x is near zero and also when x is near R, so between these extremes there is a shape of maximum volume. To find it, we use (6) to write (5) as

$$V = 2\pi y(R^2 - y^2) = 2\pi(R^2y - y^3),$$

from which we obtain

$$\frac{dV}{dy} = 2\pi(R^2 - 3y^2).$$

Setting this equal to zero to find y and then using (6) to find x gives

$$y = \frac{R}{\sqrt{3}} \qquad \text{and} \qquad x = \sqrt{R^2 - \tfrac{1}{3}R^2} = \frac{\sqrt{2}}{\sqrt{3}}R.$$

The ratio of the height to the diameter of the base for the largest cylinder is therefore

$$\frac{2y}{2x} = \frac{y}{x} = \frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2} \cong 0.707.$$

This result can be obtained more efficiently by the method of implicit differentiation. If x is taken as the independent variable and y is thought of as a function of x, then (6) yields

$$2x + 2y\frac{dy}{dx} = 0 \qquad \text{or} \qquad \frac{dy}{dx} = -\frac{x}{y}.$$

From (5) we find that

$$\frac{dV}{dx} = 2\pi\left(x^2\frac{dy}{dx} + 2xy\right) = 2\pi\left[x^2\left(-\frac{x}{y}\right) + 2xy\right]$$
$$= 2\pi\left(\frac{-x^3 + 2xy^2}{y}\right) = \frac{2\pi x}{y}(2y^2 - x^2).$$

It therefore follows that $dV/dx = 0$ when

$$2y^2 = x^2 \qquad \text{or} \qquad \frac{y}{x} = \frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2},$$

as before.

Example 3 If a ray of light travels from a point A to a point P on a flat mirror and is then reflected to a point B, as shown in Fig. 4.25, then the most careful measurements show that the incident ray and the reflected ray make equal an-

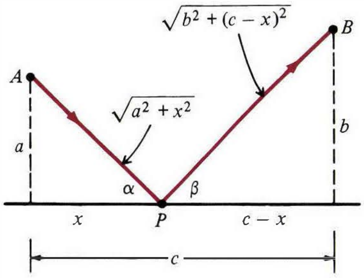


Figure 4.25 Reflection of light.

gles with the mirror: $\alpha = \beta$. Assume that the ray of light takes the shortest path from A to B by way of the mirror, and prove this law of reflection by showing that the path APB is shortest when $\alpha = \beta$.

Solution If we think of the point P as assuming various positions on the mirror, with each position determined by a value of x, then we wish to consider the length L of the path as a function of x. From Fig. 4.25 this function is clearly

$$L = \sqrt{a^2 + x^2} + \sqrt{b^2 + (c-x)^2}$$
$$= (a^2 + x^2)^{1/2} + [b^2 + (c-x)^2]^{1/2},$$

and differentiation yields

$$\frac{dL}{dx} = \tfrac{1}{2}(a^2 + x^2)^{-1/2} \cdot (2x) + \tfrac{1}{2}[b^2 + (c-x)^2]^{-1/2} \cdot 2(c-x) \cdot (-1)$$
$$= \frac{x}{\sqrt{a^2 + x^2}} - \frac{c-x}{\sqrt{b^2 + (c-x)^2}}. \tag{7}$$

If we minimize L by equating this derivative to zero, we get

$$\frac{x}{\sqrt{a^2 + x^2}} = \frac{c-x}{\sqrt{b^2 + (c-x)^2}}, \tag{8}$$

and this equation can be changed in form as follows:

$$\frac{\sqrt{a^2 + x^2}}{x} = \frac{\sqrt{b^2 + (c-x)^2}}{c-x}, \qquad \sqrt{\left(\frac{a}{x}\right)^2 + 1} = \sqrt{\left(\frac{b}{c-x}\right)^2 + 1},$$
$$\frac{a}{x} = \frac{b}{c-x}.$$

The equation last written can easily be solved for x. However, there is no need to do this, because the equation as it stands tells us what we want to know: For the angles α and β in the two right triangles shown in the figure, the ratios of the opposite side to the adjacent side are equal, so α and β are equal.

It is fairly clear on intuitive grounds that we have minimized L. If we wish to verify this by the second derivative test, we can use (7) to compute

$$\frac{d^2L}{dx^2} = \frac{a^2}{(a^2 + x^2)^{3/2}} + \frac{b^2}{[b^2 + (c-x)^2]^{3/2}}$$

(we omit the details of the computation), and all that remains is to notice that this quantity is positive.

Remark 1 The reasoning in Example 3 can be made simpler if we recall from Section 1.7 the definition of the cosine of a positive acute angle A. If we think of A as one of the acute angles of a right triangle (Fig. 4.26), then by definition

$$\cos A = \frac{b}{c} = \frac{\text{adjacent side}}{\text{hypotenuse}}.$$

Using this, the minimizing condition (8) can be written as

$$\cos \alpha = \cos \beta,$$

so $\alpha = \beta$. For use in the next example, we also recall the definition of the sine of A,

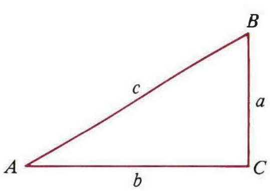


Figure 4.26

$$\sin A = \frac{a}{c} = \frac{\text{opposite side}}{\text{hypotenuse}}.$$

Remark 2 The law of reflection discussed in Example 3 was known to the ancient Greeks. However, the fact that a reflected ray of light follows the shortest path was discovered much later, by Heron of Alexandria in the first century A.D. Heron's geometric proof is simple but ingenious. The argument goes as follows. Let A and B be the same points as before (Fig. 4.27), and let B' be the mirror image of B, so that the surface of the mirror is the perpendicular bisector of BB'. The segment AB' intersects the mirror at a point P, and this is the point where a ray of light is reflected in passing from A to B; for $\alpha = \gamma$ and $\gamma = \beta$, so $\alpha = \beta$. The total length of the path is $AP + PB = AP + PB' = AB'$. For any other point P' on the mirror the total length of the path is $AP' + P'B = AP' + P'B'$, and this is greater than AB' because the sum of two sides of a triangle is greater than the third side. This shows that the actual path of our reflected ray of light is the shortest possible path from A to B by way of the mirror.

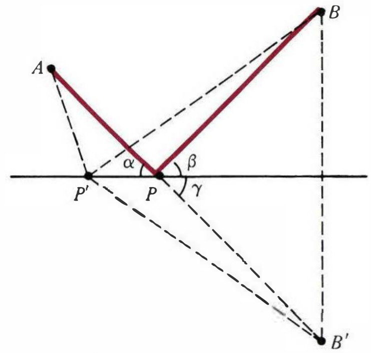


Figure 4.27

Example 4 The reflected ray of light previously discussed travels in a single medium at a constant speed. However, in different media (air, water, glass) light travels at different speeds. If a ray of light passes from air into water as shown in Fig. 4.28, it is refracted (bent) toward the perpendicular at the interface. The path APB is clearly no longer the shortest path from A to B. What law determines it? In 1621 the Dutch scientist Snell discovered empirically that the actual path of the ray is that for which

$$\frac{\sin \alpha}{\sin \beta} = \text{a constant}, \tag{9}$$

where this constant is independent of the positions of A and B. This fact is now called *Snell's law of refraction.** Prove Snell's law by assuming that the ray takes the path from A to B that minimizes the total time of travel.

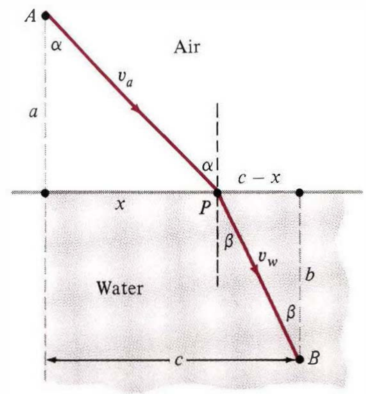


Figure 4.28 Refraction of light.

Solution If the speed of light in air is v_a and in water is v_w, then the total time of travel T is the time in air plus the time in water,

$$T = \frac{\sqrt{a^2 + x^2}}{v_a} + \frac{\sqrt{b^2 + (c - x)^2}}{v_w}$$
$$= \frac{1}{v_a}(a^2 + x^2)^{1/2} + \frac{1}{v_w}[b^2 + (c - x)^2]^{1/2}.$$

If we compute the derivative of this function and notice its meaning in terms of Fig. 4.28, we obtain

$$\frac{dT}{dx} = \frac{1}{v_a}\frac{x}{\sqrt{a^2 + x^2}} - \frac{1}{v_w}\frac{c - x}{\sqrt{b^2 + (c - x)^2}}$$
$$= \frac{\sin \alpha}{v_a} - \frac{\sin \beta}{v_w}. \tag{10}$$

If we now minimize T by equating this to zero, the result is

*Willebrord Snell (1591–1626) was an astronomer and mathematician. At the age of 22 he succeeded his father as professor of mathematics at Leiden. His fame rests mainly on his discovery of the law of refraction (9).

$$\frac{\sin \alpha}{v_a} = \frac{\sin \beta}{v_w} \quad \text{or} \quad \frac{\sin \alpha}{\sin \beta} = \frac{v_a}{v_w}. \tag{11}$$

This is a more revealing form of Snell's law, because it tells us the physical meaning of the constant on the right side of (9): It is the ratio of the speed of light in air to the (smaller) speed of light in water. This constant is called the *index of refraction* of water. If the water in this experiment is replaced by any other translucent medium, such as alcohol, glycerin, or glass, then this constant has a different numerical value, which is the index of refraction of the medium in question.

As in Example 3, we can verify that the configuration (11) actually minimizes T by computing the second derivative and noting that this quantity is positive:

$$\frac{d^2T}{dx^2} = \frac{1}{v_a}\frac{a^2}{(a^2 + x^2)^{3/2}} + \frac{1}{v_w}\frac{b^2}{[b^2 + (c - x)^2]^{3/2}} > 0.$$

But there is another method that is worth mentioning. We begin by observing that dT/dx as given by (10) is a difference of two terms. As x increases from 0 to c, the first term, $(\sin \alpha)/v_a$, increases from 0 to some positive value. The second term, $(\sin \beta)/v_w$, decreases from some positive value to 0. This shows that dT/dx is negative at $x = 0$ and increases to a positive value at $x = c$. The minimum value of T therefore occurs at the only x for which $dT/dx = 0$, and this is precisely the configuration described by (11).

Remark 3 The ideas of Example 4 were discovered in 1657 by Fermat (Section 1.4), and for this reason the statement that a ray of light traverses an optical system along the path that minimizes its total time of travel is called *Fermat's principle of least time.* (It should be noticed that when a ray of light travels in a single uniform medium, "shortest path" is equivalent to "least time," so Example 3 falls under the same principle.) During the next two centuries Fermat's ideas stimulated a broad development of the general theory of maxima and minima, leading first to Euler's creation of the calculus of variations and then to Hamilton's principle of least action, which has turned out to be one of the deepest unifying principles of physical science. Euler expressed his enthusiasm in the following memorable words: "Since the fabric of the world is the most perfect and was established by the wisest Creator, nothing happens in this world in which some reason of maximum or minimum would not come to light."*

Remark 4 Snell's sine law (9) was first published by Descartes in 1637 (without any mention of Snell), and he purported to prove it in an incorrect form equivalent to

$$\frac{\sin \alpha}{\sin \beta} = \frac{v_w}{v_a}.$$

Descartes based his argument on a fanciful model and on the metaphysically based opinion that light travels faster in a denser medium. Fermat rejected both the opinion ("shocking to common sense") and the argument ("demonstrations which do not force belief cannot bear this name"). After many years of passive but exasperated skepticism he at last actively confronted the problem and proved the correct law himself in 1657, creating the necessary calculus techniques as he went along.

*For a brief account of the great Swiss mathematician Euler (pronounced "OIL-er") see Section 8.4.

PROBLEMS

1 A closed rectangular box with a square base is to be made out of plywood. If the volume is given, find the shape (ratio of height to side of base) that minimizes the total number of square feet of plywood that are needed.

2 Solve Problem 1 if the box is open on top.

3 Find the radius of the cylinder of maximum volume that can be inscribed in a cone of height H and radius of base R.

4 Find the height of the cone of maximum volume that can be inscribed in a sphere of radius R.

5 A square piece of tin 24 in on each side is to be made into an open-top box by cutting a small square from each corner and bending up the flaps to form the sides. How large a square should be cut from each corner to make the volume of the box as large as possible?

6 Solve Problem 5 if the given piece of tin is a rectangle 15 by 24 in.

7 A cylindrical can without a top is to be made from a specified weight of sheet metal. Find the ratio of the height to the diameter of the base when the volume of the can is greatest.

8 A cylindrical tank without a top is to have a specified volume. If the cost of the material used for the bottom is three times the cost of that used for the curved lateral part, find the ratio of the height to the diameter of the base for which the total cost is least.

9 Draw a reasonably good sketch of $y = \sqrt{x}$ and mark the point on this graph that seems to be closest to the point $(\frac{3}{2}, 0)$. Then calculate the coordinates of this closest point. Hint: Minimize the square of the distance from the point $(\frac{3}{2}, 0)$ to the point $(x, \sqrt{x})$.

10 Generalize Problem 9 by finding the point on the graph of $y = \sqrt{x}$ that is closest to the point $(a, 0)$ for any $a > 0$.

11 A spy climbs out of a submarine into a rubber boat 2 mi east of a point P on a straight north-south shoreline. He wants to get to a house on the shore 6 mi north of P. He can row 3 mi/h and walk 5 mi/h, and he intends to row directly to a point somewhere north of P and then walk the rest of the way.

(a) How far north of P should he land in order to get to the house in the shortest possible time?

(b) How long does the trip take?

(c) How much longer will it take if he rows directly to P and then walks to the house?

12 Show that the answer to part (a) of Problem 11 does not change if the house is 8 mi north of P.

13 If the rubber boat in Problem 11 has a small outboard motor and can go 5 mi/h, then it is obvious by common sense that the fastest route is entirely by boat. What is the slowest speed for which the fastest route is still entirely by boat?

*14 The intensity of illumination at a point P due to a light source is directly proportional to the strength of the source and inversely proportional to the square of the distance from P to the source. Two light sources of strengths a and b are a distance L apart. What point on the line segment joining these sources receives the least total illumination? If a is 8 times as large as b, where is this point? (Assume that the intensity at any point is the sum of the intensities from the two sources.)

*15 Two towns, A and B, lie on the same side of a straight highway. Their distance apart is c, and their distances from the highway are a and b. Show that the length of the shortest road that goes from A to the highway and then on to B is $\sqrt{c^2 + 4ab}$

(a) by using calculus;

(b) without calculus. Hint: Introduce the "mirror image" of B on the other side of the highway.

16 Find the minimum vertical distance between the curves $y = 16x^2$ and $y = -1/x^2$.

17 An isosceles triangle is circumscribed about a circle of radius R. If x is the height of the triangle, show that its area A is least when $x = 3R$. Hint: Minimize A^2.

18 If the figure in Problem 17 is revolved about the altitude of the triangle, the result is a cone circumscribed about a sphere of radius R. Show that the volume of the cone is least when $x = 4R$, and that this least volume is twice the volume of the sphere.

19 A silo has cylindrical walls, a flat circular floor, and a hemispherical top. For a given volume, find the ratio of the total height to the diameter of the base that minimizes the total surface area.

20 In Problem 19, if the cost of construction per square foot is twice as great for the hemispherical top as for the walls and the floor, find the ratio of the total height to the diameter of the base that minimizes the total cost of construction.

21 What is the smallest value of the constant a for which the inequality $ax + 1/x \geq 2\sqrt{2}$ is valid for all positive numbers x?

*22 There is a refinery at a point A on a straight highway and an oil well at a point B which can be reached by traveling 5 mi along the highway to a point C and then 12 mi across country perpendicular to the highway. If a pipeline is built from A to B, it costs k times as much per mile to build it across country as along the highway, because of the difficult terrain. The line will be built either directly from A to B or along the highway to a point P part of the way toward C and then across country to B, whichever is cheaper. Decide on the cheapest route (a) if $k = 3$; (b) if $k = 2$. (c) What is the largest value of k for which it is cheapest to build the pipeline directly from A to B?

23 A circular ring of radius a is uniformly charged with electricity, the total charge being Q. The force exerted by this charge on a unit charge located at a distance x from the center of the ring, in a direction perpendicular to the plane of the ring, is given by $F = Qx(x^2 + a^2)^{-3/2}$. Sketch the graph of this function and find the value of x that maximizes F.

24 A cylindrical hole of radius x is bored through a sphere of radius R in such a way that the axis of the hole passes through the center of the sphere. Find the value of x that maximizes the complete surface area of the remaining solid. Hint: The area of a segment of height h on a sphere of radius R is $2\pi Rh$.

***25** The sum of the surface areas of a cube and a sphere is given. What is the ratio of the edge of the cube to the diameter of the sphere when (a) the sum of their volumes is a maximum? (b) the sum of their volumes is a minimum?

***26** Consider two spheres of radii 1 and 2 whose centers are 6 units apart. At what point on the line joining their centers will an observer be able to see the most total surface area? (See the hint for Problem 24.)

***27** Find the point on the parabola $y = x^2$ that is closest to the point (6, 3).

28 A man at point A on the shore of a circular lake of radius 1 mi wants to reach the opposite point C as soon as possible (Fig. 4.29). He can walk 6 mi/h and row his boat 3 mi/h. At what angle θ to the diameter AC should he row?

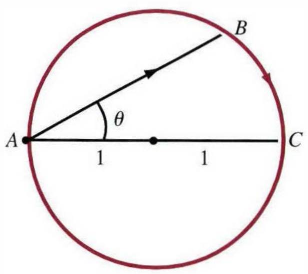


Figure 4.29

29 Find the maximum possible area A of a trapezoid inscribed in a semicircle of radius 1 (Fig. 4.30)
(a) by expressing A as a function of θ;
(b) by expressing A as a function of ϕ.

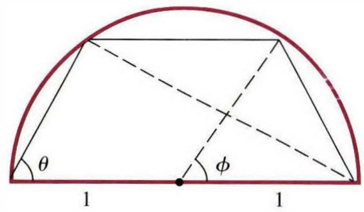


Figure 4.30

30 A laboratory scientist performs an experiment n times to measure a physical quantity x and obtains the n results $x_1, x_2, \ldots, x_n$. These measurements deviate from the "true value" of x because of unavoidable environmental factors of many kinds, such as fluctuations in temperature or air pressure. She decides to use an estimate $\bar{x}$ for x based on the *method of least squares*. This means that $\bar{x}$ is chosen to minimize the quantity

$$s = (\bar{x} - x_1)^2 + (\bar{x} - x_2)^2 + \cdots + (\bar{x} - x_n)^2,$$

which is the sum of the squares of the deviations of the estimate $\bar{x}$ from the measured values. Show that this estimate $\bar{x}$ is the average of the measured values:

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n}.$$

31 A spider at a corner S of the ceiling of a cubic room 8 ft on each side wishes to catch a bug at the opposite corner B of the floor (Fig. 4.31). The spider, who must walk on the ceiling, the walls, or the floor, wishes to find the shortest path to the bug. Find a shortest path
(a) by using calculus;
(b) without calculus, by merely thinking.

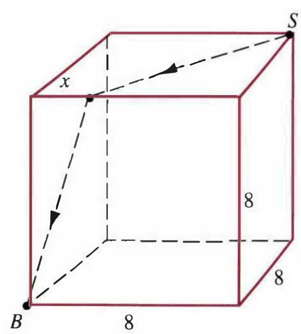


Figure 4.31

4.5 RELATED RATES

If a tank is being filled with water, then the water level rises. To describe how fast the water level rises, we speak of the rate of change of the water level or, equivalently, the rate of change of the depth. If the depth is denoted by h, and t is the time measured from some convenient moment, then the derivative dh/dt is the rate of change of the depth. Further, the volume V of water in the tank is also changing, and dV/dt is the rate of change of this volume.

Similarly, any geometric or physical quantity Q that changes with time is *a function of time,* say $Q = Q(t)$, and its derivative dQ/dt is the *rate of change of the quantity.* The problems that we now consider are based on the fact that if two changing quantities are related to one another, then their rates of change are also related.

Example 1 Gas is being pumped into a large spherical rubber balloon at the constant rate of 8 ft^3/min. Find how fast the radius r of the balloon is increasing (a) when $r = 2$ ft; (b) when $r = 4$ ft.

Solution The volume of the balloon (Fig. 4.32) is given by the formula for the volume of a sphere,

$$V = \tfrac{4}{3}\pi r^3. \qquad (1)$$

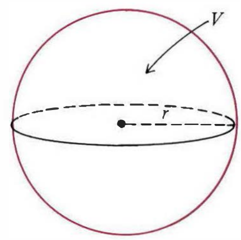


Figure 4.32

From the statement of the problem we know that $dV/dt = 8$, and we must find dr/dt for two specific values of r. It is essential to understand the background of this situation, namely, the fact that V and r are both dependent variables with the time t as the underlying independent variable. With this in mind, it is natural to introduce the rates of change of V and r by differentiating (1) with respect to t,

$$\frac{dV}{dt} = \tfrac{4}{3}\pi \cdot 3r^2 \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}, \qquad (2)$$

where the chain rule is needed in the calculation. It follows from (2) that

$$\frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt} = \frac{2}{\pi r^2},$$

since $dV/dt = 8$. In case (a) we therefore have

$$\frac{dr}{dt} = \frac{1}{2\pi} \cong 0.16 \text{ ft/min},$$

and in case (b),

$$\frac{dr}{dt} = \frac{1}{8\pi} \cong 0.04 \text{ ft/min}.$$

These conclusions confirm our commonsense awareness that since the volume of the balloon is increasing at a constant rate, the radius increases more and more slowly as the volume grows larger.

Example 2 A ladder 13 ft long is leaning against a wall. The bottom of the ladder is being pulled away from the wall at the constant rate of 6 ft/min. How fast is the top of the ladder moving down the wall when the bottom of the ladder is 5 ft from the wall?

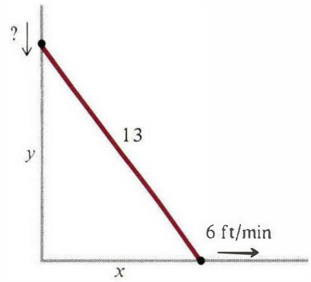


Figure 4.33

Solution The first thing we do is draw a diagram of the situation and label it, being careful to use letters to represent quantities that are changing (Fig. 4.33). In terms of this figure, we can clarify our thoughts by stating what is known and what we are trying to find:

$$\frac{dx}{dt} = 6, \qquad -\frac{dy}{dt} = ? \text{ when } x = 5.$$

(The use of the minus sign here can best be understood by thinking of dy/dt as the rate at which y is increasing and $-dy/dt$ as the rate at which y is decreasing. The problem asks for the latter.) Roughly speaking, we know one time derivative and we want to find the other. We therefore seek an equation connecting x and y which we can differentiate with respect to t to obtain a second equation connecting their rates of change. It is clear from the figure that our starting point must be the Pythagorean theorem,

$$x^2 + y^2 = 169. \tag{3}$$

When this is differentiated with respect to t, we get

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \quad \text{or} \quad \frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt} \quad \text{or} \quad -\frac{dy}{dt} = \frac{x}{y}\frac{dx}{dt},$$

and therefore

$$-\frac{dy}{dt} = \frac{6x}{y}, \tag{4}$$

since $dx/dt = 6$. Finally, equation (3) tells us that $y = 12$ when $x = 5$, so (4) yields our conclusion,

$$-\frac{dy}{dt} = \frac{6 \cdot 5}{12} = 2\tfrac{1}{2} \text{ ft/min when } x = 5.$$

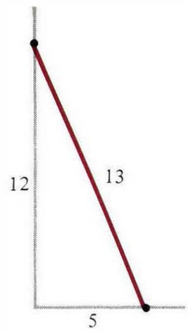


Figure 4.34

Warning: Don't substitute the values $x = 5$, $y = 12$ prematurely. The essence of the problem is the fact that x and y are variables; if we pin them down to specific values too soon, as is done in Fig. 4.34, then this makes it impossible to understand or solve the problem. In other words, preserve the fluidity of the situation until the last possible moment.

Example 3 A conical tank with its vertex down is 12 ft high and 12 ft in diameter at the top. Water is being pumped in at the rate of 4 ft^3/min. Find the rate at which the water level is rising (a) when the water is 2 ft deep; (b) when the water is 8 ft deep.

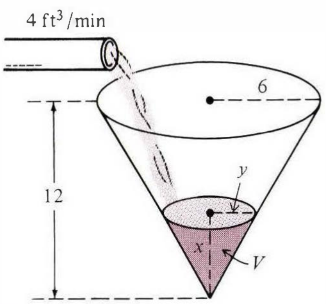


Figure 4.35

Solution As before, we begin by drawing and labeling a diagram (Fig. 4.35), with the purpose of visualizing the situation and establishing notation. Our next step is to use this notation to state as follows what is given and what we are trying to find:

$$\frac{dV}{dt} = 4, \qquad \frac{dx}{dt} = ? \text{ when } x = 2 \text{ and } x = 8.$$

The changing volume V of water in the tank has the shape of a cone, so our starting point is the formula for the volume of a cone,

$$V = \tfrac{1}{3}\pi y^2 x. \tag{5}$$

The only dependent variables we care about are V and x, so we wish to eliminate the superfluous variable y. By examining Fig. 4.35 and using similar triangles, we see that

$$\frac{y}{x} = \frac{6}{12} = \frac{1}{2} \quad \text{or} \quad y = \tfrac{1}{2}x, \tag{6}$$

and substituting this in (5) gives

$$V = \frac{\pi}{12}x^3. \tag{7}$$

We are now in a position to introduce the rates of change by differentiating (7) with respect to t, which yields

$$\frac{dV}{dt} = \frac{\pi}{4}x^2\frac{dx}{dt} \tag{8}$$

or

$$\frac{dx}{dt} = \frac{4}{\pi x^2}\frac{dV}{dt} = \frac{16}{\pi x^2},$$

since $dV/dt = 4$. This formula tells us that when $x = 2$,

$$\frac{dx}{dt} = \frac{4}{\pi} \cong 1.27 \text{ ft/min},$$

and when $x = 8$,

$$\frac{dx}{dt} = \frac{1}{4\pi} \cong 0.08 \text{ ft/min},$$

and the solution is complete.

It may be helpful to students if we now summarize the method that emerges from these examples:

STRATEGY FOR SOLVING RELATED RATE PROBLEMS

1 Read the problem carefully, several times if necessary, until it is fully understood.

2 Draw a careful sketch of the situation being considered. Add to the sketch all numerical quantities that remain constant throughout the problem. Now add as letters all quantities—the dependent variables—that are functions of time.

3 Write down the given rate of change and the required rate of change in terms of derivatives.

4 Find an equation that connects the two dependent variables in Step 3, using geometry if necessary to eliminate any superfluous dependent variable. Use the chain rule to differentiate both sides of this equation with respect to t.

5 Substitute the given rate of change from Step 3 into the differentiated equation obtained in Step 4, and solve for the required rate of change.

Warning: Don't give the dependent variables numerical values too soon. This should be done only *after* the differentiation in Step 4.

PROBLEMS

1 A stone dropped into a pond sends out a series of concentric ripples. If the radius r of the outer ripple increases steadily at the rate of 6 ft/s, find the rate at which the area of disturbed water is increasing (a) when $r = 10$ ft, and (b) when $r = 20$ ft.

2 A large spherical snowball is melting at the rate of 2π ft^3/h. At the moment when it is 30 inches in diameter, determine (a) how fast the radius is changing, and (b) how fast the surface area is changing.

3 Sand is being poured onto a conical pile at the constant rate of 50 ft^3/min. Frictional forces in the sand are such that the height of the pile is always equal to the radius of its base. How fast is the height of the pile increasing when the sand is 5 ft deep?

4 A girl 5 ft tall is running at the rate of 12 ft/s and passes under a street light 20 ft above the ground. Find how rapidly the tip of her shadow is moving when she is (a) 20 ft past the street light, and (b) 50 ft past the street light.

5 In Problem 4, find how rapidly the length of the girl's shadow is increasing at each of the stated moments.

6 A light is at the top of a pole 80 ft high. A ball is dropped from the same height from a point 20 ft away from the light. Find how fast the shadow of the ball is moving along the ground (a) 1 second later; (b) 2 seconds later. (Assume that the ball falls $s = 16t^2$ feet in t seconds.)

7 A woman raises a bucket of cement to a platform 40 ft above her head by means of a rope 80 ft long that passes over a pulley on the platform. If she holds her end of the rope firmly at head level and walks away at 5 ft/s, how fast is the bucket rising when she is 30 ft away from the spot directly below the pulley?

8 A boy is flying a kite at a height of 80 ft, and the wind is blowing the kite horizontally away from the boy at the rate of 20 ft/s. How fast is the boy paying out string when the kite is 100 ft away from him?

9 A boat is being pulled in to a dock by means of a rope with one end tied to the bow of the boat and the other end passing through a ring attached to the dock at a point 5 ft higher than the bow of the boat. If the rope is being pulled in at the rate of 4 ft/s, how fast is the boat moving through the water when 13 ft of rope are out?

10 A trough is 10 ft long and has a cross section in the shape of an equilateral triangle 2 ft on each side. If water is being pumped in at the rate of 20 ft^3/min, how fast is the water level rising when the water is 1 ft deep?

11 A spherical meteorite enters the earth's atmosphere and burns up at a rate proportional to its surface area. Show that its radius decreases at a constant rate.

12 A point moves around the circle $x^2 + y^2 = a^2$ in such a way that the x-component of its velocity is given by $dx/dt = -y$. Find dy/dt and decide whether the direction of the motion is clockwise or counterclockwise.

13 A car moving at 60 mi/h along a straight road passes under a weather balloon rising vertically at 20 mi/h. If the balloon is 1 mi up when the car is directly beneath it, how fast is the distance between the car and the balloon increasing 1 minute later?

14 Most gases obey Boyle's law: If a sample of the gas is held at a constant temperature while being compressed by a piston in a cylinder, then its pressure p and volume V are related by the equation $pV = c$, where c is a constant. Find dp/dt in terms of p and dV/dt.

15 At a certain moment a sample of gas obeying Boyle's law (Problem 14) occupies a volume of 1000 in^3 at a pressure of 10 lb/in^2. If this gas is being compressed isothermally at the rate of 12 in^3/min, find the rate at which the pressure is increasing at the instant when the volume is 600 in^3.

*16 A ladder 20 ft long is leaning against a wall 12 ft high, with its top projecting over the wall. Its bottom is being pulled away from the wall at the constant rate of 5 ft/min. Find how rapidly the top of the ladder is approaching the ground (a) when 5 ft of the ladder projects over the wall; (b) when the top of the ladder reaches the top of the wall.

17 A conical party hat made of cardboard has a radius of 4 in and a height of 12 in. When filled with beer, it leaks at the rate of 4 in^3/min. At what rate is the level of beer falling (a) when the beer is 6 in deep? (b) when the hat is half empty?

18 A hemispherical bowl of radius 8 in is being filled with water at a constant rate. If the water level is rising at the rate of $\frac{1}{3}$ in/s at the instant when the water is 6 in deep, find how fast the water is flowing in

(a) by using the fact that a segment of a sphere has volume

$$V = \pi h^2 \left(a - \frac{h}{3} \right)$$

where a is the radius of the sphere and h is the height of the segment;

(b) by using the fact that if V is the volume of the water at time t, then

$$\frac{dV}{dt} = \pi r^2 \frac{dh}{dt}$$

where r is the radius of the surface and h is the depth.

19 Water is being poured into a hemispherical bowl of radius 3 in at the rate of 1 in^3/s. How fast is the water level rising when the water is 1 in deep?

*20 In Problem 19, suppose that the bowl contains a lead ball 2 inches in diameter, and find how fast the water level is rising when the ball is half submerged.

21 Assume that a snowball melts in such a way that its volume decreases at a rate proportional to its surface area. If half the original snowball has melted away after 2 hours, how much longer will it take for the snowball to disappear completely?

22 A man in a hot air balloon is rising at the rate of 20 ft/s. How fast is the distance to the horizon increasing when the balloon is 2000 ft high? Assume that the earth is a sphere of radius 4000 mi.

23 A drawbridge with two 20-ft spans is being raised at the rate of 2 radians/min (Fig. 4.36). How fast is the distance between the ends of the spans increasing when they are elevated $\pi/4$ radians?

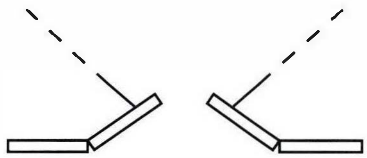


Figure 4.36

4.6 NEWTON'S METHOD FOR SOLVING EQUATIONS

Consider the cubic equation

$$x^3 - 3x - 5 = 0. \tag{1}$$

It is possible to solve this equation by exact methods, that is, by formulas yielding a solution in terms of radicals in the same sense that the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

provides an exact solution of the quadratic equation $ax^2 + bx + c = 0$. However, if what we need is a numerical solution of (1) that is accurate to a reasonable number of decimal places, then it is more convenient to find this solution by the approximation method to be described here than to try to use the exact solution. Furthermore, while formulas that yield exact solutions in terms of radicals for equations of degree 2, 3, and 4 do exist, it is known to be impossible to solve the general equation of degree 5 or more in terms of radicals. Therefore, in order to solve a fifth-degree equation like $x^5 - 3x^2 + 9x - 11 = 0$, we would be forced to use an approximation method, since no other method is available.

Returning to equation (1), if we denote $x^3 - 3x - 5$ by $f(x)$, then we can easily calculate the following values:

$$f(-2) = -7, \quad f(-1) = -3, \quad f(0) = -5,$$
$$f(1) = -7, \quad f(2) = -3, \quad f(3) = 13.$$

The pair of values $f(2) = -3$ and $f(3) = 13$ suggests that as x varies continuously from $x = 2$ to $x = 3$, $f(x)$ varies continuously from -3 to 13, and that consequently there is some intermediate value of x where $f(x) = 0$. This is true, but even though it is intuitively obvious, it is quite difficult to give a rigorous proof. We do not attempt such a proof here, but instead take it for granted that if a continuous function $f(x)$ has values $f(a)$ and $f(b)$ with opposite signs, then there is at least one root of the equation $f(x) = 0$ between a and b.* This tells us that (1) has a root between $x = 2$ and $x = 3$, and we can take either of these numbers as a first approximation to this root. The approximation $x = 2$ would seem to be the better choice, since -3 is closer to 0 than 13 is.

*This property of continuous functions is discussed in Section 2.6 under the heading The Intermediate Value Theorem.

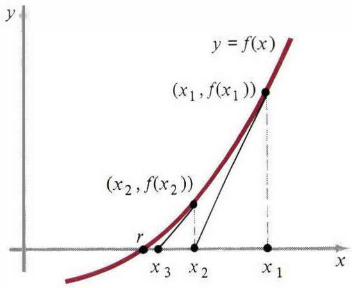


Figure 4.37

In general, suppose we have a first approximation $x = x_1$ to a root r of an equation $f(x) = 0$. This root is a point where the curve $y = f(x)$ crosses the x-axis, as shown in Fig. 4.37; and the idea of Newton's method is to use the tangent line to the curve at the point where $x = x_1$ as a stepping-stone to a better approximation $x = x_2$. Beginning with the approximation $x = x_1$, we draw the tangent line to the curve at the point $(x_1, f(x_1))$. This line intersects the x-axis at the point $x = x_2$, which seems to be a better approximation than x_1 (see Fig. 4.37). Repeating the process, we use the tangent line at $(x_2, f(x_2))$ to get to the point $x = x_3$, which is a still better approximation. Figure 4.37 illustrates the idea as a geometric procedure, but to apply it in calculations we need a formula. This formula is easily derived as follows.

The slope of the first tangent line is $f'(x_1)$. If we consider this line to be determined by the points $(x_2, 0)$ and $(x_1, f(x_1))$, then the slope is also

$$\frac{0 - f(x_1)}{x_2 - x_1}, \qquad \text{so} \qquad \frac{0 - f(x_1)}{x_2 - x_1} = f'(x_1).$$

This equation yields

$$-f(x_1) = (x_2 - x_1)f'(x_1) \qquad \text{or} \qquad x_2 - x_1 = -\frac{f(x_1)}{f'(x_1)},$$

so

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}. \tag{2}$$

In this way our first approximation x_1 leads to a second approximation x_2 given by (2); this in turn leads to a third approximation x_3, given by

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)};$$

and so on indefinitely.

Example 1 On applying this method to equation (1), we have

$$f(x) = x^3 - 3x - 5, \qquad f'(x) = 3x^2 - 3, \qquad x_1 = 2,$$

$$f(x_1) = -3, \qquad f'(x_1) = 9, \qquad x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2 - \frac{-3}{9} = 2\tfrac{1}{3}.$$

By writing x_2 in decimal form as $x_2 = 2.333333$, correct to six decimal places, and continuing with a good calculator, we get

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.333333 - \frac{0.703699}{13.333329} = 2.280556,$$

rounding off to six decimal places. Since the burden of computation is on our calculator—and calculators are cheap labor—we shall continue working to six decimal places. When two successive approximations are equal in their first six decimal places, we shall consider this as evidence of accuracy. Thus, in the case of equation (1) we obtained $x_3 = 2.280556$ after two applications of the procedure. Repeating this procedure yields

$$x_4 = 2.279020,$$

$$x_5 = 2.279019,$$

$$x_6 = 2.279019.$$

We therefore conclude that $x = 2.279019$ is the desired solution of equation (1) accurate to six decimal places.

Newton's method is not restricted to the solution of polynomial equations like (1), but can also be applied to any equation containing functions whose derivatives we can calculate.

Example 2 To illustrate this remark we consider the equation

$$x = \cos x, \tag{3}$$

where x on the right side is understood to be measured in radians. The best way to begin thinking is to graph the two functions $y = x$ and $y = \cos x$ on the same set of axes, as shown in Fig. 4.38. It is then easy to understand that these curves intersect at only one point and the x-coordinate of this point is the solution of (3), because at the point of intersection the two y's are equal. By inspecting Fig. 4.38 we are able to give a good first approximation to this solution:

$$x_1 = 0.7.$$

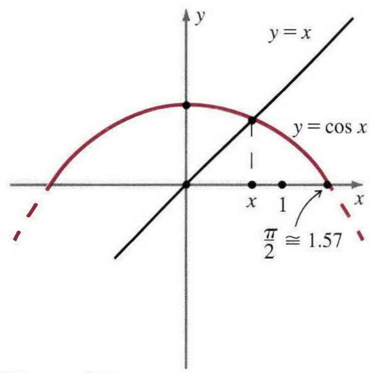


Figure 4.38

To apply Newton's method we write (3) in the form

$$x - \cos x = 0$$

and put $f(x) = x - \cos x$ so that $f'(x) = 1 + \sin x$. Now, setting our calculator to the radians mode, we find that

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.739436,$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.739085,$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 0.739085.$$

This finishes the calculation and gives us the desired solution $x = 0.739085$ correct to six decimal places.

Remark In some cases, the sequence of approximations produced by Newton's method may fail to converge to the desired root. For example, Fig. 4.39 shows a function for which the approximation x_1 leads to x_2 and x_2 leads back to x_1, so repetitions of the process do not bring us any closer to the root r than our initial guess. Specific examples of this behavior—and worse—are given in the problems. The mathematical theory providing conditions under which Newton's method is guaranteed to succeed can be found in books on numerical analysis.

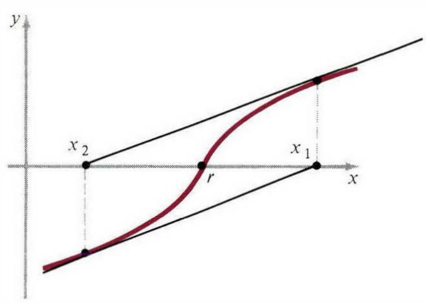


Figure 4.39

PROBLEMS

1 By sketching the graph of $y = f(x) = x^3 - 3x - 5$, show that equation (1) has only one real root. Hint: Use the derivative $f'(x) = 3x^2 - 3 = 3(x^2 - 1)$ to locate the maxima and minima of the function and to learn where it is increasing and decreasing.

2 (a) Show that $x^3 + 3x^2 - 6 = 0$ has only one real root, and calculate it to six decimal places of accuracy.
(b) Show that $x^3 + 3x = 8$ has only one real root, and calculate it to six decimal places of accuracy.

3 Use Newton's method to calculate the positive root of $x^2 + x - 1 = 0$ to six decimal places of accuracy.

4 Calculate $\sqrt{5}$ to six decimal places of accuracy by solving the equation $x^2 - 5 = 0$, and use this result in the quadratic formula to check the answer to Problem 3.

5 Use Newton's method to calculate $\sqrt[3]{10}$ to six decimal places of accuracy.

6 Consider a spherical shell 1 ft thick whose volume equals the volume of the hollow space inside it. Use Newton's method to calculate the shell's outer radius to six decimal places of accuracy.

7 A hollow spherical buoy of radius 2 ft has specific gravity $\frac{1}{4}$, so it floats on water in such a way as to displace $\frac{1}{4}$ its own volume. Show that the depth x to which it is submerged is a root of the equation $x^3 - 6x^2 + 8 = 0$, and use Newton's method to calculate this root to six decimal places of accuracy. Hint: The volume of a spherical segment of height h cut from a sphere of radius a is $\pi h^2(a - h/3)$.

8 Suppose that by good luck our first approximation x_1 happens to be the root of $f(x) = 0$ that we are seeking. What does this imply about x_2, x_3, etc.?

9 Show that the function $y = f(x)$ defined by

$$f(x) = \begin{cases} \sqrt{x - r} & x \ge r, \\ -\sqrt{r - x} & x \le r, \end{cases}$$

has the property illustrated in Fig. 4.39; that is, for any positive number a, if $x_1 = r + a$, then $x_2 = r - a$; and if $x_1 = r - a$, then $x_2 = r + a$.

10 Show that Newton's method applied to the function $y = f(x) = \sqrt[3]{x}$ leads to $x_2 = -2x_1$, and is therefore useless for finding where $f(x) = 0$. Sketch the situation.

11 In Example 1 of Section 4.1 we saw from its graph that the function $y = f(x) = 2x^3 - 3x^2 - 12x + 12$ has positive zeros close to $x = 0.9$ and $x = 2.9$. Use Newton's method to calculate these zeros to six decimal places of accuracy.

12 Find a solution of $2x = \cos x$ correct to six decimal places.

13 Find the smallest positive solution of each of the following equations, correct to six decimal places:
(a) $4(x - 1) = \sin x$;
(b) $x^2 = \sin x$.

14 How many solutions does the equation

$$x = \sin x$$

have? Why?

4.7 (OPTIONAL) APPLICATIONS TO ECONOMICS. MARGINAL ANALYSIS

Ever since its beginning, calculus has served primarily as a tool for the physical sciences. The uses of mathematics in the social sciences have arisen more recently. In this section we discuss several applications of calculus to *microeconomics*, the branch of economics that studies the economic decisions of individual businesses or industries. More precisely, we focus our attention on the production and marketing of a single commodity by a single firm.

The most important management decisions in a particular firm usually depend on the costs and revenues involved. We shall examine applications of derivatives to the cost and revenue functions.

COST, MARGINAL COST, AND AVERAGE COST

The total cost to a firm of producing x units of a given commodity is a certain function of x called the *cost function* and denoted by $C(x)$. Here x can be the number of pieces produced, or the number of pounds, or the number of bushels, and so on. The cost $C(x)$ can be measured in dollars, in thousands of dollars, in French francs, or in any other monetary unit.

To determine the cost function $C(x)$ is a difficult task for experts in bookkeeping and accounting. Here, however, we take this function as given. We shall

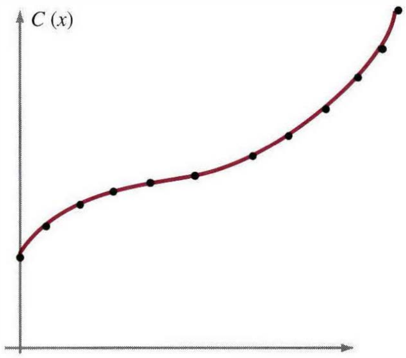


Figure 4.40

assume, for the sake of definiteness, that x is the number of pieces or units produced, and therefore a nonnegative integer, and also that the cost $C(x)$ is measured in dollars. For most commodities, such as TV sets or calculators, x can only be a nonnegative integer, so the graph of $C(x)$ might look like the sequence of dots in Fig. 4.40. However, economists usually assume that these dots are connected by a smooth curve as shown in the figure. Accordingly, $C(x)$ is understood to be defined for all nonnegative values of x, not just for nonnegative integers.

Many components make up the total cost. Some, like capital expenditures for buildings and machinery, are fixed and do not depend on x. Others, like wages and the cost of raw materials, are roughly proportional to the amount x produced. If this were all, then the cost function would have the very simple form

$$C(x) = a + bx,$$

where a is the fixed cost and b is the constant running cost per unit.

But this is not all, and most cost functions are not as simple as this. The essential point is the fact that a time restriction is present, and that $C(x)$ is the cost of producing x units of the product *in a given time interval*, say 1 week. There will then be a fixed cost of a dollars per week, as before, but the variable part of the cost will probably increase more than proportionally to x as the weekly production x increases, because of overtime wages, the need to use older machinery that breaks down more frequently, and other inefficiencies that arise from forcing production to higher and higher levels. The cost function $C(x)$ might then have the form

$$a + bx + cx^2 \qquad \text{or} \qquad a + bx + cx^2 + dx^3,$$

or it might be a function even more complicated than these. The general nature of such a cost function is suggested in Fig. 4.40.

The derivative $C'(x)$ of the cost function is called the *marginal cost*. This derivative is, or course, the rate of change of cost with respect to the production level x. The economic meaning of this important concept will become clearer as we proceed.

As a first step in this direction we point out that it is a good approximation to think of the marginal cost $C'(x)$, at a given production level x, as the extra cost of producing one more unit. To see this we recall the definition of the derivative,

$$C'(x) = \lim_{\Delta x \to 0} \frac{C(x + \Delta x) - C(x)}{\Delta x}.$$

We therefore have the approximation

$$C'(x) \cong \frac{C(x + \Delta x) - C(x)}{\Delta x},$$

where this approximation is good if Δx is "suitably small." It is customary in economics to assume that $\Delta x = 1$ meets the requirement of being suitably small. Therefore we have

$$C'(x) = C(x + 1) - C(x),$$

approximately. In words, the marginal cost at each level of production x is the extra cost required to produce the next unit of output [the $(x + 1)$st unit].

Example 1 Suppose a company has estimated that the cost (in dollars) of producing x units is

$$C(x) = 5000 + 7x + 0.02x^2.$$

Then the marginal cost is

$$C'(x) = 7 + 0.04x.$$

The marginal cost at the production level of 1000 units is

$$C'(1000) = 7 + 0.04(1000) = \$47/\text{unit}.$$

The exact cost of producing the 1001st unit is

$$\begin{aligned} C(1001) - C(1000) &= [5000 + 7(1001) + 0.02(1001)^2] \\ &\qquad - [5000 + 7(1000) + 0.02(1000)^2] \\ &= \$47.02. \end{aligned}$$

The difference between the marginal cost for $x = 1000$ and the exact cost of producing the 1001st unit is clearly negligible.

The graph of a typical cost function is shown in Fig. 4.41. This cost function is increasing because it costs more to produce more. The marginal cost $C'(x)$ is the slope of the tangent to the cost curve. The cost curve is initially concave down (the marginal cost is decreasing), because it costs more to produce the first piece than to produce one more piece when many are being produced; this reflects the more efficient use of the fixed costs of production. At a certain production level x_0 there is a point of inflection P_0 and the cost curve becomes concave up (the marginal cost is increasing), because when we produce almost as much as we can, it becomes more expensive to increase production by even a small amount. As we suggested earlier, reasons for this might include greater overtime costs or more frequent breakdowns of the equipment as we strain our productive capacity.

It is a reasonable view that the most efficient production level for a manufacturer is that which minimizes the *average cost*

$$\frac{C(x)}{x},$$

which, of course, is the cost per unit when x units are produced. We sketch a typical average cost curve in Fig. 4.42 by noticing that $C(x)/x$ is the slope of the line joining the origin to the point P in Fig. 4.41. We know that some cost is un-

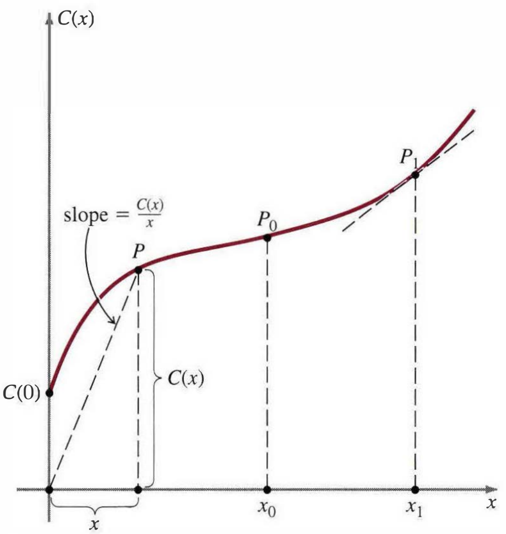


Figure 4.41 A cost function.

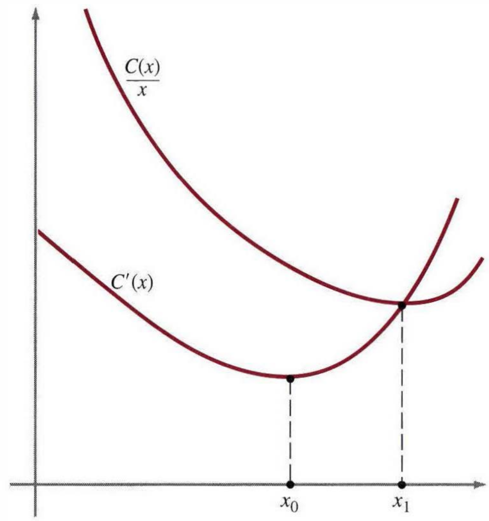


Figure 4.42 Marginal cost and average cost.

avoidable even before a single unit is produced—for instance, the capital expenditures mentioned earlier, utilities, insurance, and so on—so $C(0) > 0$. This shows that $C(x)/x$ has the limit $+\infty$ as $x \to 0+$. We are particularly interested in the minimum that $C(x)/x$ appears to have. To locate this minimum we find the critical point x_1 of the function $C(x)/x$ by calculating the derivative by means of the quotient rule,

$$\frac{d}{dx}\left(\frac{C(x)}{x}\right) = \frac{xC'(x) - C(x)}{x^2}.$$

This derivative must be zero at the critical point, so $xC'(x) - C(x) = 0$ or

$$C'(x) = \frac{C(x)}{x}. \qquad (1)$$

We therefore have the following basic law of economics:

> If the average cost is a minimum, then
> marginal cost = average cost.

In other words, *at the peak of operating efficiency the marginal cost equals the average cost.* We see from this that the graphs of marginal cost and average cost intersect at the point of minimum average cost, as shown in Fig. 4.42. Like other principles of economics, this is usually established by extensive verbal discussions supported by tables and graphs. However, the calculus derivation is brief and clear.

Equation (1) has an interesting geometric interpretation for the cost function shown in Fig. 4.41: At the production level $x = x_1$ where $C(x)/x$ has a minimum, the line from the origin to the point P_1 is tangent to the graph. We can see the reason for this by noticing that the average cost $C(x)/x$ decreases as P moves to the right along the curve toward P_1, and then increases as P moves beyond P_1.

Example 2 A firm estimates that the cost (in dollars) of producing x units is $C(x) = 3400 + 4x + 0.002x^2$.

(a) Find the cost, marginal cost, and average cost of producing 500 units, 1000 units, 1500 units, and 2000 units.

(b) What is the minimum average cost, and at what production level is this achieved?

Solution (a) The marginal cost is

$$C'(x) = 4 + 0.004x.$$

The average cost is

$$\frac{C(x)}{x} = \frac{3400}{x} + 4 + 0.002x.$$

We use these formulas to calculate the entries in the following table, giving all amounts in dollars (or dollars per unit) rounded to the nearest cent.

x	$C(x)$	$C'(x)$	$C(x)/x$
500	5,900	6	11.80
1000	9,400	8	9.40
1500	13,900	10	9.27
2000	19,400	12	9.70

(b) When average cost is a minimum, we must have

$$\text{marginal cost} = \text{average cost},$$

$$C'(x) = \frac{C(x)}{x},$$

$$4 + 0.004x = \frac{3400}{x} + 4 + 0.002x.$$

This equation simplifies to

$$0.002x = \frac{3400}{x},$$

so

$$x^2 = 1{,}700{,}000$$

and

$$x = \sqrt{1{,}700{,}000} = 1304.$$

To verify that this production level actually gives a minimum for $C(x)/x$, we observe that the second derivative $(C(x)/x)'' = 6800/x^3 > 0$, so the graph of $C(x)/x$ is concave up for all $x > 0$ and we have a minimum. Finally, the minimum average cost is

$$\frac{C(1304)}{1304} = \frac{3400}{1304} + 4 + 0.002(1304) = \$9.22.$$

REVENUE, PROFIT, AND DEMAND

It is clearly important for a manager to know all about the cost function, but this is not enough. The overall purpose of the firm is to make a profit, and for this it is essential to consider the income from sales, or the *revenue,* as economists call it. And this requires us to bring the consumers (buyers) of the product into the picture.

The *revenue function* $R(x)$ is the total revenue (or income) derived from producing and selling x units of the product. The *marginal revenue* is the derivative $R'(x)$ of this function. By the same type of interpretation as used above, the marginal revenue can be thought of as the extra revenue received from the sale of one more unit,

$$R'(x) = R(x + 1) - R(x),$$

approximately.

Example 3 Many business decisions are based on an analysis of the costs and revenues "at the margin," or at the edge—hence the expression *marginal analysis* for this kind of thinking.

To understand this, let us suppose we are running a taxi company in New York City and are trying to decide whether to add one more cab to our large fleet. If it will make money for the company then we add it, otherwise not. Clearly we need to consider the costs and revenues involved. Since the choice is between adding this cab and leaving the fleet the same size, the crucial question is whether the *additional revenue* generated by one more cab is greater or smaller than the *additional cost* incurred. This additional revenue and cost are precisely the *marginal revenue* and *marginal cost.* Therefore, if the marginal revenue is greater than the marginal cost, then we should clearly add the cab and increase our profit. This is nothing but simple common sense expressed in the economists' language of "marginal this" and "marginal that."

The total profit derived from producing and selling x units is

$$P(x) = R(x) - C(x).$$

This is called the *profit function;* it is what is left over from the revenue after the cost is deducted. A firm will lose money when production is too low, because of fixed costs, and also when production is too high, because of high marginal costs. Unless the firm can operate profitably at some in-between level of production, the business will fail, so we can assume that the profit curve looks something like Fig. 4.43.

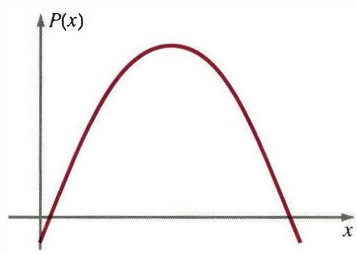


Figure 4.43 A profit function.

The *marginal profit* is the derivative $P'(x)$ of the profit function. In order to maximize profit we look for the critical points of $P(x)$, that is, the points where the marginal profit is zero. But if

$$P'(x) = R'(x) - C'(x) = 0$$

then

$$R'(x) = C'(x).$$

This gives another basic law of economics:

> If the profit is a maximum, then
> marginal revenue = marginal cost.

To satisfy ourselves that this condition gives a maximum and not a minimum, we can use the second derivative test,

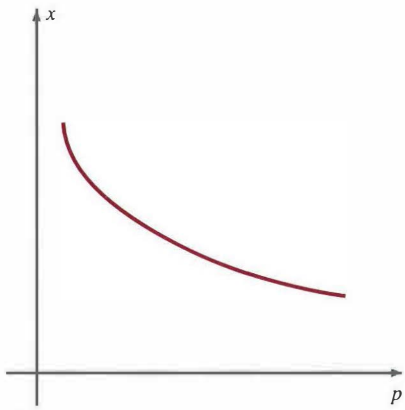


Figure 4.44 Demand curve.

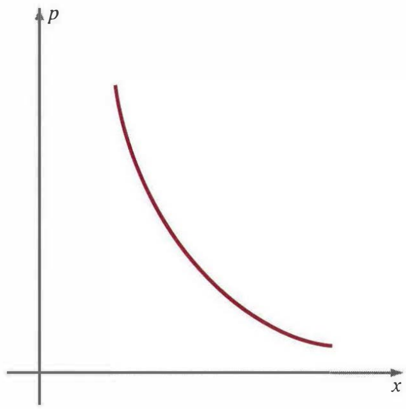


Figure 4.45 Price function.

$$P''(x) = R''(x) - C''(x) < 0$$

or

$$R''(x) < C''(x).$$

Thus the profit will be a maximum if

$$R'(x) = C'(x) \qquad \text{and} \qquad R''(x) < C''(x).$$

In order to put teeth into these generalities, we must consider the nature of the consumers who constitute the market. Normally, the higher the price of a commodity, the smaller the number x that will be sold. Thus x, the number "demanded," is a decreasing function of the price p of a unit, and this function is usually determined by market research. The *demand curve* (Fig. 4.44) displays this dependence, and under these circumstances the variable x is called the *demand* and the function $x = x(p)$ is the *demand function.* For the sake of convenience in comparing the demand curve with the cost function, economists usually interchange the axes and consider p as a function of x, $p = p(x)$, as shown in Fig. 4.45. This function is called the *price function.*

When x units of a commodity are sold at a price of $p(x)$ dollars per unit, then the revenue $R(x)$ is evidently the product of the price per unit and the number of units sold, $R(x) = xp(x)$, and the profit is

$$P(x) = xp(x) - C(x).$$

If both the price function $p(x)$ and the cost function $C(x)$ are known, then the law stated above can be used to find the value of x that maximizes profit. It is clear that this value of x need not be the one that minimizes the average cost, for the latter depends only on the cost function $C(x)$. That is, profit depends on the whims of the marketplace, while efficiency is an internal matter.

Example 4 What production level will maximize profit for a firm with cost function

$$C(x) = 2400 + 9x + 0.002x^2$$

and demand function $x = 12{,}000 - 500p$?

Solution First, we point out that the economic meaning of the demand function is that no units will be sold ($x = 0$) at a price of \$24 per unit, but for every dollar decrease in price, 500 more units will be sold. If we solve for p, then we obtain the price function

$$p(x) = 24 - \frac{1}{500}x.$$

The revenue function is therefore

$$R(x) = xp(x) = 24x - \frac{1}{500}x^2,$$

so the marginal revenue is

$$R'(x) = 24 - \frac{1}{250}x$$

and the marginal cost is

$$C'(x) = 9 + 0.004x = 9 + \frac{1}{250}x.$$

When the profit is a maximum, then marginal revenue equals marginal cost, that is,

$$24 - \frac{1}{250}x = 9 + \frac{1}{250}x,$$

and solving yields

$$15 = \frac{1}{125}x \qquad \text{or} \qquad x = 1875.$$

To verify that this gives a maximum we calculate the second derivatives,

$$R''(x) = -\frac{1}{250}, \qquad C''(x) = \frac{1}{250}.$$

Since $R''(x) < C''(x)$ for all x, the production level $x = 1875$ maximizes profit. The corresponding price is $p(1875) = \$20.25$.

ELASTICITY OF DEMAND

The nature of the demand curve in Fig. 4.44 depends on the particular product under consideration. It is relatively flat (or *inelastic*) for bread and motor oil, since people tend to buy what they need without much regard for the price, and relatively steep (or *elastic*) for candy, since no one really needs it but more people buy more of it when the price is low.

The elasticity of demand is an important concept of quantitative economics. To introduce it in a precise way, let (p, x) be an arbitrary point on the demand curve in Fig. 4.44. If p increases by a small amount Δp and $-\Delta x$ is the corresponding decrease in x, then the ratio of the percentage decrease in x to the percentage increase in p is

$$\frac{100(-\Delta x/x)}{100(\Delta p/p)} = -\frac{p}{x}\frac{\Delta x}{\Delta p}.$$

The *elasticity of demand* $E(p)$ at the price level p is now defined by

$$E(p) = \lim_{\Delta p \to 0}\left(-\frac{p}{x}\frac{\Delta x}{\Delta p}\right) = -\frac{p}{x}\frac{dx}{dp}.$$

The demand is said to be *elastic* if $E(p) > 1$ and *inelastic* if $E(p) < 1$. The positive function $E(p)$ is a useful tool of economic analysis because it measures the responsiveness of the demand to changes in the price p: it is small when the demand curve is relatively flat, so that changes in p induce relatively smaller changes in x, and large when this curve is relatively steep. It also has the merit of being independent of the units of measurement used for p and x. This is a great convenience in many economic and business situations. For example, changing the units of p from dollars to French francs, say, and the units of x from pounds to kilograms would leave the value of the elasticity $E(p)$ unchanged, because this quantity involves only the percentage changes in p and x.

Example 5 In Example 4 the demand function is $x = 12{,}000 - 500p$. Find $E(p)$. At what price p is the demand elastic? Inelastic?

Solution From the definition we have

$$E(p) = -\frac{p}{x}\frac{dx}{dp} = -\frac{p}{12{,}000 - 500p}\cdot(-500)$$
$$= \frac{500p}{12{,}000 - 500p} = \frac{p}{24 - p}.$$

From Example 4 we understand that $0 < p < 24$, so the condition $E(p) > 1$ is equivalent to

$$\frac{p}{24 - p} > 1 \quad \text{or} \quad p > 24 - p \quad \text{or} \quad 2p > 24 \quad \text{or} \quad p > 12,$$

so the demand is elastic for $p > 12$. Similarly, the demand is inelastic for $p < 12$. To understand what this means, we observe that when the revenue is expressed as a function of p (instead of x) we have $R(p) = p(12{,}000 - 500p) = 500p \cdot (24 - p)$. It is easy to see from this that revenue is maximized for $p = 12$. Therefore to maximize revenue, the price must be lowered if the demand is elastic and raised if the demand is inelastic. These conclusions are valid for any decreasing demand function, whether it is linear or not (see Problem 26).

The commonsense interpretation of all this is clear: If the demand is elastic at a given price, then a price decrease by a certain percentage causes a proportionally larger increase in sales, so the revenue, which is the product of price and sales, is increased. Similarly, if the demand is inelastic, then a price increase by a certain percentage causes a proportionally smaller decrease in sales, so again the revenue is increased.

The discussions of this section suggest several ways in which derivatives can be used in economics. The most influential contribution to this subject in the twentieth century was perhaps Keynes's *General Theory of Employment, Interest and Money,* which has been characterized as "an endless desert of economics, algebra and abstraction, with trackless wastes of differential calculus, and only an oasis here and there of delightfully refreshing prose."* This may be some-

*Chapter IX of *The Wordly Philosophers,* by Robert L. Heilbroner.

what exaggerated for the sake of its juicy phrases, but nevertheless the general impression is valid—that modern economics makes extensive use of many kinds of mathematics, especially calculus.

PROBLEMS

In Problems 1–4 a cost function (in dollars) is given for producing x units of a certain commodity. In each case find the marginal cost at the production level of 500 units and also the actual cost of producing the 501st unit.

1 $C(x) = 15{,}000 + 13x + 0.03x^2$.

2 $C(x) = 400 + \dfrac{x}{10} + \dfrac{x^2}{100}$.

3 $C(x) = 5000 + 15x - 0.01x^2 + 0.0001x^3$.

4 $C(x) = 3000 + 100\sqrt{x}$.

For each of the cost functions in Problems 5–10, find the minimum average cost and the production level at which this is achieved.

5 $C(x) = 8000 + 15x + x^2$.

6 $C(x) = 2400 + 3x + 0.02x^2$.

7 $C(x) = 60 + \frac{1}{3}x + \frac{1}{900}x^2$.

8 $C(x) = 5000 + 2x + 0.001x^3$.

9 $C(x) = 2\sqrt{x} + \dfrac{x^2}{900}$.

10 $C(x) = 10{,}000 + 8x + 4x^{3/2}$.

For each of the cost and price functions in Problems 11–16, find the production level that maximizes profit.

11 $C(x) = 1240 + 8x + 0.02x^2$, $p(x) = 16$.

12 $C(x) = 1240 + 8x + 0.02x^2$, $p(x) = 16 - \frac{1}{50}x$.

13 $C(x) = 900 + 35x + 0.001x^2$, $p(x) = 65 - \frac{1}{500}x$.

14 $C(x) = 750 + 140x - 0.2x^2 + \frac{1}{30}x^3$,
$p(x) = 300 - \frac{1}{5}x$.

15 $C(x) = 4500 + 50x - x^2 + 0.002x^3$,
$p(x) = 80 - 0.01x$.

16 $C(x) = 6000 + 15x - \dfrac{x^2}{200} + 0.001x^3$,
$p(x) = 120 - 0.05x$.

17 A Broadway theater has seats for 2000 playgoers. With the ticket price at \$50, the average attendance at a moderately successful play has been 1200. When the ticket price was lowered to \$40, the average attendance rose to 1400.
(a) Find the price function, assuming that it is linear.
(b) What should the ticket price be to maximize revenue?

In Problems 18–21 use the given demand function to find the selling price p that maximizes the revenue.

18 $x = 1200 - 20p$.

19 $x = 800 - 2.5p$.

20 $x = 160 - p^{3/2}$.

21 $x = 768 - p^2$.

22 A *perfect competitor* is an enterprise that has such a small share of the market that it cannot influence the price of its product and can sell as much as it produces at the prevailing market price p. Figure 4.46 shows the cost and revenue curves of a certain perfect competitor. Sketch the profit curve.

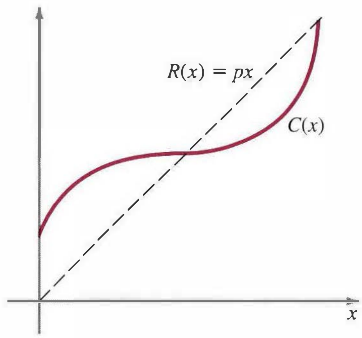


Figure 4.46

23 The daily cost to a small company of producing x hand calculators is $C(x) = 1560 + 50x - 8x^2 + \frac{1}{3}x^3$ dollars. The market price of this calculator is \$130. What is the maximum daily profit, and what is the daily output x that yields this profit?

24 A small company with fixed costs (overhead) of a dollars produces x units of a commodity which it sells at a fixed price of p dollars per unit. If it costs b dollars to produce each unit, where $b < p$, at what output level does the company break even, and what is the graphical interpretation of this *break-even point?*

25 Suppose the company in Problem 24 produces trout fishing instructional videotapes for \$8 that it sells for \$30. If the overhead is \$14,000, how many tapes must be sold to break even?

26 Consider a demand curve $x = x(p)$, where $x(p)$ is any decreasing function.
(a) If $E(p) > 1$, show that the revenue $R = px$ is increased by lowering the price.
(b) If $E(p) < 1$, show that the revenue is increased by raising the price.
(c) Establish the formula

$$\frac{dR}{dp} = x[1 - E(p)],$$

and use this to deduce that $E(p) = 1$ at a point on the demand curve where the revenue is a maximum.

27 Verify the conclusion in part (c) of Problem 26 for each of the demand curves in Problems 18–21:

(a) $x = 1200 - 20p$;
(b) $x = 800 - 2.5p$;
(c) $x = 160 - p^{3/2}$;
(d) $x = 768 - p^2$.

CHAPTER 4 REVIEW: CONCEPTS, METHODS

Define, state, or think through the following.

1 Increasing and decreasing functions.
2 Critical points and critical values.
3 Relative (or local) maximum and minimum values.
4 Absolute maximum and minimum values.
5 Procedure for curve sketching.
6 Endpoints, cusps, and corners.
7 Concave up and concave down.
8 Point of inflection.
9 Second derivative test.
10 Strategy for applied maximum-minimum problems.
11 Law of reflection.
12 Snell's law of refraction.
13 Fermat's principle of least time.
14 Rate of change.
15 Strategy for related rate problems.
16 Newton's method.

ADDITIONAL PROBLEMS FOR CHAPTER 4

SECTION 4.1

Sketch the graphs of the following functions by using the first derivative and the methods of Section 4.1; in particular, find the intervals on which each function is increasing and those on which it is decreasing, and locate any maximum or minimum values it may have.

1 $y = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x + \frac{4}{3}$.
2 $y = x^3 + 6x^2 + 12x + 8$.
3 $y = -x^3 + 3x + 2$.
4 $y = x^3 + 3x - 2$.
5 $y = x^4 - 6x^2 + 8x$.
6 $y = (x + 2)^3(x - 4)^3$.
7 $y = x^4 - 4x^3 + 16$.
8 $y = 3x^5 - 10x^3 + 15x + 3$.
9 $y = x^2(x + 1)^2$.
10 $y = x^3(x - 1)^2$.
11 $y = x^2(4 - x^2)$.
*12 $y = \dfrac{x^3}{x + 1}$.
13 $y = \dfrac{x}{(x + 1)^2}$.
14 $y = \frac{16}{3}x^3 + \dfrac{1}{x}$.
15 $y = \dfrac{4(x^2 - 1)}{x^4}$.
16 $y = \dfrac{4(x - 1)}{x^2}$.
17 $y = x^2 + \dfrac{16}{x^2}$.
18 $y = \dfrac{4 - 2x}{1 - x}$.
19 $y = \dfrac{5x^2 + 2}{x^2 + 1}$.
20 $y = \dfrac{5x^2 - 20x + 21}{x^2 - 4x + 5}$.
*21 $y = x^2(x - 4)^{2/3}$.
22 $y = \sqrt{x} + \dfrac{2}{\sqrt{x}} - 2\sqrt{2}$.

SECTION 4.2

For each of the following, locate the points of inflection, find the intervals on which the curve is concave up and those on which it is concave down, and sketch.

23 $y = x^3 + x$.
24 $y = x^3 + 3x^2 + 6x + 7$.
25 $y = x^3 - 12x + 2$.
26 $y = x^4 - 2x^2$.
27 $y = x^4 + 4x^3$.
28 $y = (x + 2)(x - 2)^3$.
29 $y = x^4 - 4x^3 - 2x^2 + 12x - 1$.
30 $y = \dfrac{x}{\sqrt{x^2 + 1}}$.
*31 $y = \dfrac{x^3}{x^2 + 3a^2}$ $(a > 0)$.
*32 $y = \dfrac{1}{x^3 + 1}$.
33 $y = \dfrac{5}{3x^4 + 5}$.
*34 $y = \dfrac{x^3}{(x - 1)^2}$.
35 $y = \dfrac{8}{x^3} - \dfrac{2}{x}$.
36 $y = \dfrac{6}{x} + \dfrac{6}{x^2}$.
37 In each part of this problem, use the given formula for the second derivative of a function to locate the points of inflection, the intervals on which the graph is concave up, and the intervals on which the graph is concave down:
(a) $y'' = x^2(x - 1)(x - 2)^2$;
(b) $y'' = (x^2 + 2)(x + 2)^2(x - 1)(x - 2)$;
(c) $y'' = x(x - 1)(x^2 - 4)(x - 3)$.
38 If $f(x) = (x - a)(x - b)(x - c)$, find the x-coordinate of the point of inflection. Hint: See Additional Problem 22 in Chapter 3.

39 Find the value of a that makes $f(x) = ax^2 + 1/x^2$ have a point of inflection at $x = 1$.

***40** Consider the general cubic curve $y = ax^3 + bx^2 + cx + d$.

(a) Show that the curve has one and only one point of inflection,

$$I = \left(-\frac{b}{3a}, k\right), \qquad \text{where } k = \frac{2b^3}{27a^2} - \frac{bc}{3a} + d.$$

(b) Show that the curve has one maximum point and one minimum point if and only if $b^2 - 3ac > 0$.

(c) When the curve has a maximum point P and a minimum point Q, show that the abscissa (x-coordinate) of I is the average of the abscissas of P and Q. Hint: Recall how to find the sum of the roots of a quadratic equation from its coefficients.

(d) Part (c) suggests that our general cubic curve might be symmetric with respect to its point of inflection I. Prove this by (1) introducing a new X-axis and Y-axis by means of

$$X = x + \frac{b}{3a} \qquad \text{and} \qquad Y = y - k,$$

so that the origin of the XY-system is the point I; (2) showing that the equation of our curve in the XY-system is

$$Y = aX\left(X^2 - \frac{b^2 - 3ac}{3a^2}\right);$$

and (3) observing that this transformed equation is symmetric with respect to the origin of the XY-system.

SECTION 4.3

41 Find the positive number that exceeds its cube by the largest amount.

42 Find two positive numbers x and y such that their sum is 30 and the product xy^4 is a maximum.

43 Find two positive numbers x and y such that their sum is 56 and the product x^3y^5 is a maximum.

44 (Generalization of the preceding problems) Let m and n be given positive integers. If x and y are positive numbers such that $x + y = S$, where S is a constant, show that the maximum value of the product $P = x^m y^n$ is attained when

$$x = \frac{mS}{m + n} \qquad \text{and} \qquad y = \frac{nS}{m + n}.$$

***45** Express the number 18 as the sum of two positive numbers in such a way that the sum of the square of the first and the fourth power of the second is as small as possible.

46 Find the positive number such that the sum of its cube and 48 times the reciprocal of its square is as small as possible.

47 The sum of three positive numbers is 15. Twice the first plus three times the second plus four times the third is 45. Choose the numbers so that the product of all three is as large as possible.

***48** (A generalization of Problem 6 in Section 4.3) Consider a rectangle with sides $2x$ and $2y$ inscribed in a given fixed circle $x^2 + y^2 = a^2$, and let n be a positive number. We wish to find the rectangle that maximizes the quantity $z = x^n + y^n$. If $n = 2$, it is clear that z has the constant value a^2 for all rectangles. If $n < 2$, show that the square maximizes z; and if $n > 2$, show that z is maximized by a degenerate rectangle in which x or y is zero.

49 Show that of all triangles with given base and given perimeter, the one with the greatest area is isosceles. Hint: Use Heron's formula for the area,

$$A = \sqrt{s(s - a)(s - b)(s - c)},$$

where a, b, c are the sides and s is the semiperimeter (half the perimeter).

50 Show that of all triangles with given base and given area, the one with the least perimeter is isosceles. Hint: If the base lies on the x-axis and is bisected by the origin, and if the third vertex (x, h) has a fixed height above the x-axis, then the triangle is isosceles if $x = 0$.

51 If a and b are positive constants, the region between the parabola $a^2y = a^2b - 4bx^2$ and the x-axis is a parabolic segment of base a and height b. Find the base and height of the largest rectangle with lower base on the x-axis and upper vertices on the parabola.

52 A circle of radius a is divided into two segments by a line L at a distance b from the center. The rectangle of greatest possible area is inscribed in the smaller of these segments. How far from the center is the side of this rectangle that is opposite to the line L?

***53** Two straight fences meet at a point, but not necessarily at right angles. A post stands in the angle between them. If a triangular corral is constructed by building a new straight fence containing this post, show that the fenced-off triangle has minimal area when the old post is in the center of the new fence. (Notice that this generalizes the result of Problem 7 in Section 4.3.)

***54** A line through a fixed point (a, b) in the first quadrant intersects the x-axis at A and the y-axis at B. Show that the minimum values of AB and $OA + OB$ are

$$(a^{2/3} + b^{2/3})^{3/2} \qquad \text{and} \qquad (\sqrt{a} + \sqrt{b})^2.$$

***55** (A generalization of Example 4 and Problem 26 in Section 4.3) First, notice that areas of similar figures are proportional to the squares of corresponding lengths, as in Fig. 4.47, where

$$A = c_1p^2 = c_2d^2 = c_3x^2 = c_4y^2$$

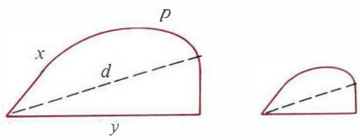


Figure 4.47

for suitable constants c_1, c_2, c_3, c_4. Here p is the perimeter, d is the diameter—the length of the longest chord—and x and y are the indicated lengths. The constants c_1, c_2, c_3, c_4 are evidently the areas when $p = 1$, $d = 1$, $x = 1$, $y = 1$. Now, cut a wire of length L into two pieces and use these pieces as the perimeters p and P of figures of two specified shapes (Fig. 4.48), so that $p + P = L$. Then the sum of the areas is

$$A = A_1 + A_2 = ap^2 + bP^2 = ap^2 + b(L - p)^2,$$

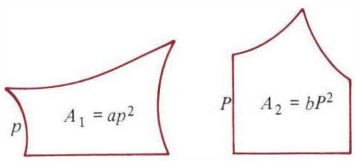


Figure 4.48

where $0 \le p \le L$ (we allow either perimeter to be zero). Show that

(a) the minimum combined area is $abL^2/(a + b)$, corresponding to

$$p = \frac{b}{a+b}L \quad \text{and} \quad P = \frac{a}{a+b}L;$$

(b) the maximum combined area is the larger of the one-figure areas aL^2 and bL^2, corresponding to $p = L$ and $p = 0$.

Also, verify that these conclusions contain as special cases the results of Example 4 and Problem 26 in Section 4.3.

56 A printed page must have A square inches of printed matter and is required to have side margins of width a inches and top and bottom margins of width b inches. Find the length of the printed lines if the page is designed to use the least paper.

57 For a certain printed page, the widths of the four margins (possibly all different) and the area of the printed matter are specified. Show that the least paper is required if the full page is similar in shape to the rectangle of printed matter.

58 A dormer window has the shape of a rectangle surmounted by an equilateral triangle. If the total perimeter is fixed, find the proportions of the window (i.e., the ratio of the height of the window to its base) that will admit the most light.

59 A long strip of sheet metal 8 in wide is to be made into a rain gutter by turning up two sides at right angles to the bottom. If the gutter is to have maximum capacity, how many inches should be turned up on the sides?

60 A playing field is to be built in the shape of a rectangle with a semicircular part at each end, and the perimeter is to be a race track of specified length. Find the proportions of the field that will give the rectangular part as large an area as possible.

61 A farmer wishes to use 5 acres of land along a straight river to construct 6 small pens by means of a fence parallel to the river and 7 fences perpendicular to it. Show that if the total amount of fencing is to be minimized, the parallel fence should be as long as all the others combined.

62 An automobile manufacturer estimates that he can sell 5000 cars a month at \$10,000 each and that he can sell 500 more cars per month for each \$200 decrease in price.

(a) What price per car will bring the largest gross income?

(b) If each car costs \$4000 to make, what price will bring the largest total profit?

63 A manufacturer of electric knives estimates that her weekly production costs are given by the formula $C = 9500 + 8x + 0.00025x^2$, where x is the number of knives manufactured in a week.[†] The sales department estimates that if the selling price is set at y, then $x = 13{,}000 - 500y$ knives can be sold.[‡] How many knives should be manufactured each week, and what should their selling price be, in order to achieve maximum profit?

*64 The cost for fuel of running a large paddlewheel steamboat at a speed of v miles per hour through still water is $\$v^3/24$ per hour. Other costs—wages, insurance, etc.—are \$108 per hour. What is the most economical speed for a certain trip upstream against a current of 2 mi/h?

65 A feedlot operator has a herd of 200 cows in his pens, each weighing 600 lb. The cost of keeping one cow for a day is 80 cents. The cows are gaining weight at the rate of 8 lb/day. The market price for cows is now \$1.25/lb, but is dropping 1 cent a day. How many days should the operator wait in order to sell his cows for the largest profit?

[†]The overhead is \$9500 per week; the cost of labor and materials is \$8 per knife; and the term $0.00025x^2$, which is small unless x is very large, says—in effect—that the factory has a fixed size and loses efficiency if too much production is attempted.

[‡]This formula says that sales are expected to be 5000 at a selling price of \$16, with a loss of 500 sales for each \$1 increase in price.

66 An estimate of the numerical value of a certain quantity is to be determined from n measurements $x_1, x_2, \ldots, x_n$. The *least squares* estimate is the number x that minimizes the sum of the squares,

$$S = (x - x_1)^2 + (x - x_2)^2 + \cdots + (x - x_n)^2.$$

Show that this least squares estimate is the arithmetic mean of the measurements,

$$x = \frac{x_1 + x_2 + \cdots + x_n}{n}.$$

67 As a woman starts jogging across a 300-ft bridge, a man in a canoe passes directly below the center of the bridge. The woman is moving at the rate of 9 ft/s and the man at the rate of 12 ft/s.

(a) What is the shortest *horizontal distance* between the woman and the man?

(b) If the bridge is 288 ft high, what is the shortest *distance* between the woman and the man?

SECTION 4.4

68 Find the height of the cylinder of maximum lateral area that can be inscribed in a sphere of radius R. Show that this maximum lateral area is half the surface area of the sphere.

69 A cylinder is generated by revolving a rectangle of given perimeter about one of its sides. What is the shape (ratio of height to diameter of base) of the cylinder of maximum volume?

*70 The cone of smallest possible volume is circumscribed about a given hemisphere. What is the ratio of its height to the diameter of its base?

71 If the volume of a cone is fixed, what shape (ratio of height to diameter of base) minimizes its total surface area?

72 A pyramid has a square base and four equal sloping triangular faces. If the total area of the bottom and faces is given, show that the volume is greatest when the height is $\sqrt{2}$ times the edge of the base.

73 A cylinder is generated by revolving a rectangle about the x-axis, where the base of the rectangle lies on the x-axis and its upper vertices lie on the curve $y = x/(x^2 + 1)$. What is the largest volume such a cylinder can have?

74 (A problem of Kepler) Consider a cylinder with a given fixed distance D from the center of a generator to the farthest point of the cylinder. If this cylinder has the largest possible volume, what is the ratio of its height to the diameter of its base?

75 A solid is formed by cutting hemispherical cavities in the ends of a cylinder. If the total surface area of this solid is given, find the shape of the cylinder (ratio of height to diameter of base) that maximizes the volume of the solid.

76 A given cone has height H and radius of base R. A second cone is inscribed in the first with its vertex at the center of the base of the given cone and its base parallel to the base of the given cone. Find the height of the second cone if its volume is as large as possible.

77 Closed cylindrical cans are to be made with a specified volume. There is no waste involved in cutting the sheet metal that goes into the curved lateral part, but each end is to be cut from a square piece of metal and the scraps discarded. Find the ratio of the height to the diameter of the base that minimizes the cost of sheet metal.

78 A certain tank consists of a cylinder with hemispherical ends. For a given surface area, describe the shape of the tank with maximum volume.

79 A rectangle of tin whose sides are a and b is to be made into an open-top box by cutting a square from each corner and bending up the flaps to form the sides. How large a square should be cut from each corner to make the volume of the box as large as possible?

80 An aquarium is to be 4 ft high and is to have a volume of 88 ft^3. The base, ends, and back are to be made of slate, and the front is to be made of special reinforced glass that costs 1.75 times as much as the slate per square foot. What should the dimensions be to minimize the cost of materials?

81 A circular filter paper of radius a is to be formed into a conical filter by folding under a circular sector. Find the ratio of the radius to the depth for the filter of greatest capacity.

82 A frame for a cylindrical lampshade is to be made from a piece of wire 20 ft long. The frame consists of two equal circles, four wires from the upper circle to the lower circle, and two diametrical wires in the upper circle. Find the height and radius that will maximize the volume of the cylinder.

83 A box with a lid is to be made from a square sheet of cardboard 18 in on a side by cutting along the dotted lines as shown in Fig. 4.49. The cardboard is then

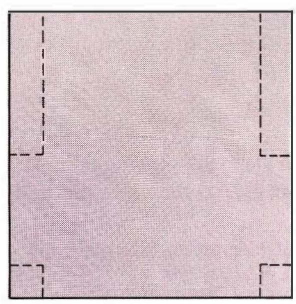


Figure 4.49

folded up to form the ends and sides, and the flap is folded over to form the lid. What are the dimensions of the box of greatest volume?

84 On a calm day the atmospheric pollution spreading out from a city is directly proportional to the population of the city and inversely proportional to the distance from the city. A retired forester wishes to start a tree nursery somewhere on a straight highway between two cities 60 km apart. If the first city is four times as large as the second, where should the forester locate his nursery to minimize the effect of pollution on his young trees?

85 The x-axis is the southern shore of a lake containing a small island at the point (a, b) in the first quadrant. A woman at the origin can run r meters per second along the shore and can swim s meters per second, where $r > s$. If she wants to reach the island as quickly as possible, how far should she run before she starts to swim?

86 Two towers 30 m apart are 30 and 70 m high, respectively. A taut wire fastened to the top of each tower is anchored to the ground between the towers. How far from the shorter tower will the wire touch the ground if its total length is a minimum? (Can you solve this problem without calculus?)

87 Find the equation of the circle with center at the origin that is internally tangent to the parabola $8y = 48 - x^2$.

88 Sketch the curve $y = \sqrt{x^2 + 16}$ and find the point on it that is closest to the point $(6, 0)$.

*89 Find the point on the parabola $y^2 = 3x$ that is closest to the point $(4, 1)$.

90 What points on the curve $x^2y = 16$ are closest to the origin?

91 For what points on the circle $x^2 + y^2 = 25$ is the sum of the distances from $(2, 0)$ and $(-2, 0)$ a minimum?

92 Let $P = (x, y)$ be a variable point on the line $ax + by + c = 0$ and let $P_0 = (x_0, y_0)$ be a fixed point not on this line.

(a) If s is the distance from P_0 to P, use the methods of calculus to show that s^2 (and therefore s) is a minimum when PP_0 is perpendicular to the given line.

(b) Show that the minimum distance is

$$\frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}.$$

93 A smooth graph not passing through the origin always has a point (x_0, y_0) that is closest to the origin.† If this point is not an endpoint, show that the line from the origin to (x_0, y_0) is perpendicular to the graph.

94 If a, b, c are positive constants, show that $ax + b/x \geq c$ for all positive numbers x if and only if $4ab \geq c^2$.

95 If a, b, c are positive constants, show that $ax^2 + b/x \geq c$ for all positive numbers x if and only if $27ab^2 \geq 4c^3$.

96 Consider the general quadratic function $f(x) = ax^2 + bx + c$ with $a > 0$. By calculating the minimum value of this function, show that $f(x) \geq 0$ for all x if and only if $b^2 - 4ac \leq 0$.

97 By applying the idea of Problem 96 to the function

$$f(x) = (a_1x + b_1)^2 + (a_2x + b_2)^2 + \cdots + (a_nx + b_n)^2,$$

establish *Schwarz's inequality*:

$$|a_1b_1 + \cdots + a_nb_n| \leq (a_1{}^2 + \cdots + a_n{}^2)^{1/2}(b_1{}^2 + \cdots + b_n{}^2)^{1/2}.$$

Also show that equality holds here if and only if there exists a number x such that $b_i = -a_ix$ for every $i = 1, 2, \ldots, n$.

SECTION 4.5

98 A cubic block of ice is melting at the rate of 6 in³/min. How fast is its surface area changing when its edge is 12 in long?

99 A light is on the ground 50 ft from a building. A man 6 ft tall walks from the light toward the building at 4 ft/s. Find how rapidly the length of his shadow on the building is decreasing (a) when he is 40 ft from the building; (b) when he is 30 ft from the building.

100 Two airplanes are flying westward on parallel courses 9 mi apart. One flies at 425 mi/h and the other at 500 mi/h. How fast is the distance between the planes decreasing when the slower plane is 12 mi farther west than the faster plane?

101 A conical tank with its vertex down is 8 ft high and 4 ft in diameter at the top. It is full of water, but the water is leaking out through a hole in the bottom at the rate of 1 ft³/min. Find the rate at which the water level is falling when the tank is $\frac{7}{8}$ empty.

102 Assume that water squirts out a hole in the bottom of a tank at a speed proportional to the square root of the depth y of the water. If the tank has the shape of a cone with its vertex down, show that the rate of change of the depth is

$$\frac{dy}{dt} = -\frac{c}{y^{3/2}}$$

where c is a positive constant.

103 Water is being pumped into an open-top cylindrical tank of radius 5 ft at the rate of 6 ft³/min. At the same

†For the purposes of this problem, interpret the phrase "smooth graph" to mean the graph of a function $y = f(x)$ defined for all x or on a closed interval $a \leq x \leq b$, whose derivative $f'(x)$ exists in the interior of its domain.

time, water is squirting out a hole in the bottom of the tank at the rate of $2\sqrt{y}$ ft^3/min, where y is the depth of the water. How high must the tank be for the water level to stabilize before it overflows?

104 A long rectangular tank has a sliding panel that divides it into two adjustable tanks of width 4 ft (see Fig. 4.50). Water is pumped into the left compartment at the rate of 12 ft^3/min. At the same time the panel is moved steadily to the right at the rate of 1 ft/min. In each of the following situations determine whether the water level is rising or falling, and how fast: (a) the left compartment contains 144 ft^3 of water and is 9 ft long; (b) the left compartment contains 144 ft^3 of water and is 18 ft long.

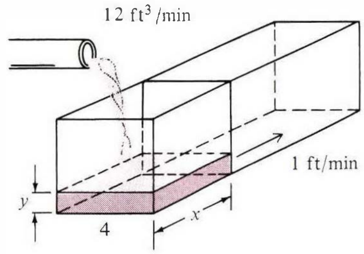


Figure 4.50

105 A large volume V of oil is spilled into a calm sea from a broken tanker. After the initial turbulence has died down, the oil spreads in a circular pattern of radius r and uniform thickness h, where r increases and h decreases in a manner determined by the viscosity and buoyancy of the oil. Laboratory experiments suggest that the thickness is inversely proportional to the square root of the elapsed time, $h = c/\sqrt{t}$. Show that the rate dr/dt at which the oil spreads is inversely proportional to $t^{3/4}$.

106 String of radius $\frac{1}{10}$ in is being wound onto a ball at the rate of 32 in/s. If the ball is assumed to remain spherical and to consist entirely of string with no empty space, find the rate at which its radius is increasing when the radius is 2 in.

107 Thread is being unwound at the rate of a inches per second from a spool of radius r inches. The unwound part of the thread has length x inches and is stretched taut into a segment PT tangent to the spool at the point T. Find the rate of increase of the distance y from the axis of the spool to the point P at the end of the thread.

108 Meteorologists are interested in the adiabatic expansion or compression of large masses of air, in which temperatures may change but no heat is added or subtracted. The adiabatic gas law for air is $pV^{1.4} = c$, where p is pressure, V is volume, and c is a constant. The volume of a certain insulated chamber of air is decreasing steadily at the rate of 1 ft^3/min. Find how rapidly the pressure is increasing at an instant when the pressure is 65 lb/in^2 and the volume is 13 ft^3.

109 If a rocket weighs 1000 lb on the surface of the earth, then it weighs

$$W = \frac{1000}{(1 + r/4000)^2}$$

pounds when it is r miles above the surface of the earth. If the rocket is rising at the rate of 1.25 mi/min, how fast is it losing weight when its altitude is 1000 mi?

110 Wheat is being poured into a pile at the constant rate of 36 ft^3/min. If the pile always has the shape of a cone whose height is half the radius of the base, at what rate is the height increasing when the diameter of the pile is 12 ft?

111 Gravel is being poured onto a pile, forming a cone. If the radius of the base is increasing at the rate of 3 m/min and the height is increasing at the rate of 1 m/min, how rapidly is the volume increasing when the height is 4 m?

112 A chord moves across a circle of radius 5 ft at the rate of 4 ft/min. How fast is the length of the chord decreasing when it is $\frac{4}{5}$ of the way across the circle?

113 A point moves along the parabola $x^2 = 4py$ in such a way that its projection on the x-axis has constant velocity. Show that its projection on the y-axis has constant acceleration.

114 Two points A and B are moving along the x-axis and y-axis, respectively, in such a way that the perpendicular distance k from the origin O to the segment AB remains constant. If A is moving away from O at the rate of $4k$ units per minute, find how fast OB is changing, and whether it is increasing or decreasing, at the moment when $OA = 3k$.

115 One side of a rectangle is increasing at the rate of 7 in/min and the other side is decreasing at the rate of 5 in/min. At a certain moment the lengths of these two sides are 10 and 7 in, respectively. Is the area of the rectangle increasing or decreasing at that moment? How fast?

116 Two concentric circles are expanding. At a certain moment, designated by $t = 0$, the inner radius is 2 ft and the outer radius is 10 ft; and for $t > 0$, these radii are increasing at the steady rates of 4 ft/min and 3 ft/min, respectively. If A is the area between the circles, when will A have its largest value?

***117** Two concentric spheres are expanding. At time $t = 0$, the inner and outer radii r and R have the values r_0 and R_0 feet, respectively. For $t > 0$, these radii are increasing at the steady rates of a and b feet per minute, where $a > b > ar_0^2/R_0^2$. If V is the volume between the spheres, when will V have its largest value?

SECTION 4.6

118 Show that each of the following equations has only one real root, and calculate it to six decimal places: (a) $x^3 + 5x - 2 = 0$; (b) $x^3 + 2x - 4 = 0$.

119 Calculate each of the following to six decimal places of accuracy: (a) $\sqrt{11}$; (b) $\sqrt[3]{6.9}$; (c) $\sqrt[4]{19}$.

120 Let a be a given positive number and x_1 a positive number that approximates $\sqrt{a}$.

(a) Show that Newton's method applied to the equation $x^2 - a = 0$ gives $x_2 = \frac{1}{2}(x_1 + a/x_1)$ as the next approximation.†

(b) If $x_1 \neq \sqrt{a}$, show that the approximation $\frac{1}{2}(x_1 + a/x_1)$ is greater than $\sqrt{a}$, regardless of whether x_1 is greater than $\sqrt{a}$ or less than $\sqrt{a}$. Hint: Show that the inequality $\frac{1}{2}(x_1 + a/x_1) > \sqrt{a}$ is equivalent to $(\sqrt{x_1} - \sqrt{a/x_1})^2 > 0$.

(c) If the approximation x_1 is too large, i.e., if $x_1 > \sqrt{a}$, show that $\frac{1}{2}(x_1 + a/x_1)$ is a better approximation in the sense that

$$\frac{1}{2}\left(x_1 + \frac{a}{x_1}\right) - \sqrt{a} < x_1 - \sqrt{a}.$$

(d) Assume that the approximation x_1 is too small, i.e., $x_1 < \sqrt{a}$, but is large enough so that $x_1 > \frac{1}{3}\sqrt{a}$. Show that $\frac{1}{2}(x_1 + a/x_1)$ is a better approximation in the sense that

$$\frac{1}{2}\left(x_1 + \frac{a}{x_1}\right) - \sqrt{a} < \sqrt{a} - x_1.$$

Hint: Show that this inequality is equivalent to $x_1 + a/x_1 - 2\sqrt{a} < 2\sqrt{a} - 2x_1$, $3x_1 - 4\sqrt{a} + a/x_1 < 0$, and

$$\frac{(3x_1 - \sqrt{a})(x_1 - \sqrt{a})}{x_1} < 0.$$

121 If a is a given positive number and $\sqrt[3]{a}$ is calculated by applying Newton's method to the equation $x^3 - a = 0$, show that

$$x_2 = \frac{1}{3}\left(2x_1 + \frac{a}{x_1^2}\right).$$

122 Consider a spherical shell 1 ft thick whose volume is twice the volume of the hollow space inside it. Use Newton's method to calculate the shell's outer radius to six decimal places of accuracy.

123 A conical paper cup is 4 in deep and 4 inches in diameter. Its vertex is pushed up inside, as shown in Fig. 4.51. How far does its tip penetrate the space inside the cup if the new volume is four-fifths of the original volume?

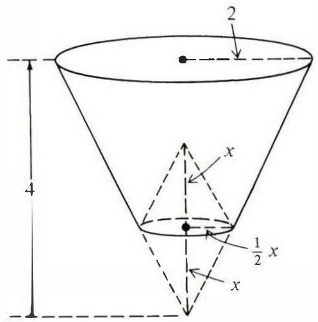


Figure 4.51

124 The formula in Problem 7 of Section 4.6 was discovered by Archimedes. Use it to show that if a plane at distance x from the center of a sphere of radius 1 cuts off $\frac{1}{3}$ the volume of the sphere, then x is a solution of the equation

$$3x^3 - 9x + 2 = 0.$$

Use Newton's method to calculate x correct to six decimal places.

SECTION 4.7

125 An economist studying a certain appliance business finds that the overhead and wholesale cost involved in handling x electric mixers a week is $56 + 24x$ dollars, and that each week $x = 30 - \frac{1}{2}p$ mixers are sold at a retail price of p dollars apiece. What retail price should she advise the owner to charge in order to earn the greatest profit?

126 (a) Suppose a manufacturer can sell x bicycles per year at a price of $p = 300 - 0.01x$ dollars apiece, and that it costs him $C(x) = 60{,}000 + 75x$ dollars to produce the x bicycles. For maximum profit, what should his production be and what price should he charge?

(b) If the government imposes on the manufacturer a tax of \$25 for each bicycle and the other features of the situation are unchanged, how much of the tax should he absorb himself and how much should he pass on to his customers if he wishes to continue making the maximum profit?

127 If the marginal revenue from producing x units of a certain commodity is $40 - \frac{1}{60}x^2$ dollars/unit and the marginal cost is $10 + \frac{1}{60}x^2$ dollars/unit, how many units should be produced to maximize the profit?

†See Additional Problem 4 at the end of Chapter 1.

5 INDEFINITE INTEGRALS AND DIFFERENTIAL EQUATIONS

5.1 INTRODUCTION

Our work in the preceding chapters was concerned with the *problem of tangents* as described in Section 2.1—given a curve, find the slope of its tangent; or equivalently, given a function, find its derivative.

In addition to launching the full-scale study of derivatives, Newton and Leibniz also discovered that many problems in geometry and physics depend on "backwards differentiation," or "antidifferentiation." This is sometimes called the *inverse problem of tangents*: Given the derivative of a function, find the function itself.

In this chapter we work with the same derivative rules as in Chapter 3. Here, however, these rules are read backwards, and lead in particular to the "integration" of polynomials. Even these relatively simple procedures have some remarkable applications, which we discuss in Section 5.5.

5.2 DIFFERENTIALS AND TANGENT LINE APPROXIMATIONS

As we know, the definition of the derivative $f'(x)$ of a function $y = f(x)$ can be stated as follows:

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}. \tag{1}$$

It is understood here that Δx is a nonzero change in the independent variable x, and that $\Delta y = f(x + \Delta x) - f(x)$ is the corresponding change in y. In Section 2.3 we introduced the equivalent notation

$$\frac{dy}{dx} \tag{2}$$

for this derivative, and we emphasized there that (2) is a single symbol and not a fraction. However, it is certainly true that (2) looks like a fraction, and in some circumstances it even acts like one. The most important example of this is the chain rule,

$$\frac{dy}{du}\frac{du}{dx} = \frac{dy}{\cancel{du}}\frac{\cancel{du}}{dx} = \frac{dy}{dx},$$

where the correct formula for the derivative of a composite function is obtained by canceling as if the derivatives were fractions.

Our present purpose is to give individual meanings to the pieces of (2), namely, to dy and dx, in such a way that their quotient is indeed the derivative $f'(x)$. Our reasons for doing this are difficult to explain in advance. Suffice it to say that this notational device is a necessary prelude to the powerful computational methods introduced in this chapter—integration by substitution, and the solution of certain differential equations by separating the variables.

We begin by considering the special case in which y is a linear function of x,

$$y = mx + b. \tag{3}$$

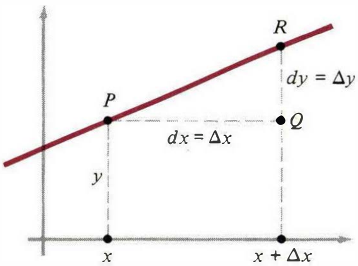


Figure 5.1

Let $P = (x, y)$ be a point on this line (Fig. 5.1). If x is given an increment Δx and if the corresponding increment in y is Δy, then the slope of the line (3) is

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x},$$

so

$$\Delta y = m\,\Delta x. \tag{4}$$

When working in this way with increments along a straight line, we denote these increments by the symbols dx and dy, so that by definition

$$dx = \Delta x \qquad \text{and} \qquad dy = \Delta y,$$

and we call them *differentials*. With this notation, (4) becomes

$$dy = m\,dx. \tag{5}$$

Now consider an arbitrary function

$$y = f(x), \tag{6}$$

and assume that this function has a derivative at x. If P is the corresponding point on the graph (Fig. 5.2), then the tangent at P is the straight line PR with slope $m = f'(x)$. By the *differentials* dx and dy arising from (6), we mean the increments in the variables x and y that are associated with this tangent line. To state this more precisely, the differential dx of the independent variable x is any increment Δx in x, as shown,

$$dx = \Delta x; \tag{7}$$

and the differential dy of the dependent variable y is the corresponding increment in y *along the tangent line*, namely,

$$dy = f'(x)\,dx. \tag{8}$$

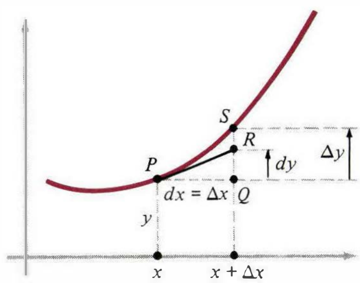


Figure 5.2

Thus, as Fig. 5.2 shows, if $dx = \Delta x = PQ$ is any change in x, then $\Delta y = QS$ and $dy = QR$ are the corresponding changes in y along the curve and along the tangent line, respectively. We observe that (8) reduces to (5) when $f(x) = mx + b$.

If $dx \neq 0$, then we can divide (8) by it and obtain

$$\frac{dy}{dx} = f'(x). \tag{9}$$

Up to this point equation (9) has been trivially true because its two sides have been merely two different ways of writing the same thing, namely, the derivative of the function $y = f(x)$. The new feature of (9) in our present discussion is that now the Leibniz symbol on the left not only looks like a fraction but *is* a fraction,

$$\frac{dy}{dx} = \frac{\text{differential of } y}{\text{differential of } x}.$$

The Leibniz notation for derivatives makes it particularly easy to produce the differential formula (8) when the function $y = f(x)$ is given, by computing the derivative and multiplying by dx. The calculation in the first column gives the general pattern,

$$\begin{array}{ll} y = f(x) & y = x^2 \\ \dfrac{dy}{dx} = f'(x) & \dfrac{dy}{dx} = 2x \\ dy = f'(x)\, dx & dy = 2x\, dx, \end{array}$$

and the calculation in the second column shows how it works for the special case $y = x^2$. A little experience with the use of this notation makes us realize that we can proceed directly from $y = x^2$ to the formula $dy = 2x\,dx$ without bothering to write the intermediate step $dy/dx = 2x$. We emphasize that a differential on the left side of an equation requires that the right side must also contain a differential. Thus, we never write $dy = 2x$, but instead $dy = 2x\,dx$.

It is often convenient to write $df(x)$ instead of dy.

Example 1 As illustrations of this remark we have

$$d(x^2) = 2x\,dx, \qquad d(5x^4) = 20x^3\,dx, \qquad d\left(\frac{1}{x}\right) = \left(-\frac{1}{x^2}\right) dx = -\frac{dx}{x^2},$$

$$d(x^4 + 7x^2 + 6) = (4x^3 + 14x)\,dx,$$

and

$$d(x \sin x) = (x \cos x + \sin x)\,dx.$$

Our familiar formulas for calculating derivatives can now be given useful equivalent formulations in the notation of differentials. Suppose $y = f(u)$, so that $dy = f'(u)\,du$. Then for various choices of the function $f(u)$ we get the formulas

$$d(u^n) = nu^{n-1}\,du, \qquad d(\sin u) = \cos u\,du, \tag{10}$$

and so on. When the differential notation is used in this way, it allows us to write derivative formulas without any need to mention the independent variable. In this spirit, if we multiply the product and quotient rules by dx, then they take the form

$$d(uv) = u\,dv + v\,du \qquad \text{and} \qquad d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}.$$

Further, if we have $y = f(u)$, and u in turn is a function of another variable x, say $u = g(x)$, then we can substitute $du = g'(x)\,dx$ in the formula $dy = f'(u)\,du$ and obtain

$$dy = f'(u)g'(x)\,dx. \tag{11}$$

This is the differential version of the chain rule

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \tag{12}$$

that we mentioned earlier. In both versions (11) and (12), the chain rule appears to be the result of simple algebraic manipulations of differentials. It is this perfect key-in-the-lock fit with the chain rule that makes the differential notation such an indispensable tool in calculus, as we shall see in the next section and thereafter.

Example 2 In the following specific applications of formulas (10) we see the differential chain rule in action:

$$\begin{aligned} d(x^2+1)^4 &= 4(x^2+1)^3\, d(x^2+1) \\ &= 4(x^2+1)^3 \cdot 2x\, dx \\ &= 8x(x^2+1)^3\, dx, \end{aligned}$$

and

$$\begin{aligned} d(\sin 4x^3) &= \cos 4x^3\, d(4x^3) \\ &= \cos 4x^3 \cdot 12x^2\, dx \\ &= 12x^2 \cos 4x^3\, dx. \end{aligned}$$

Most people who use calculus routinely as a tool in their work think of differentials as very small quantities, even though the definitions contain no such requirement. There are several good reasons for this. One such reason can be seen in Fig. 5.2, which shows that the tangent to a curve hugs the curve closely near the point of tangency. This means that when dx is small, the curve is virtually indistinguishable from its tangent, and therefore the differential dy, which is comparatively easy to calculate, provides a very good approximation to the exact increment Δy, which may be harder to calculate. We express this as a practical procedure in the following way (Fig. 5.3):

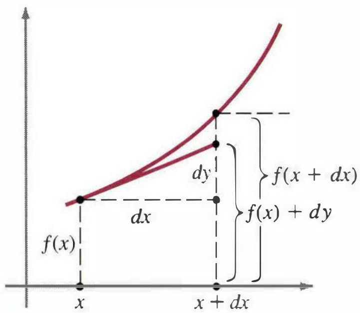


Figure 5.3

When $f(x)$ and $f'(x)$ have been found for a particular value of x so that $dy = f'(x)\, dx$ is known, then we can use the formula

$$f(x+dx) \cong f(x) + dy \tag{13}$$

to compute approximate values of the function at nearby values of x.

Formula (13) is called a *tangent line approximation,* or sometimes a *linear approximation.*

Example 3 Use differentials to find an approximate value for $\sqrt[3]{28}$.

Solution The evaluation of $\sqrt[3]{x}$ is easy for $x = 27$, so we take $y = f(x) = \sqrt[3]{x}$ and $dx = 1$. Since $dy = \frac{1}{3}x^{-2/3}\, dx$, we have

$$dy = \tfrac{1}{3}(27)^{-2/3} \cdot 1 = \frac{1}{3 \cdot 9} = \frac{1}{27},$$

and therefore

$$\begin{aligned} \sqrt[3]{28} &= f(28) \cong f(27) + dy \\ &= 3 + \tfrac{1}{27} \cong 3.037. \end{aligned}$$

The actual value of $\sqrt[3]{28}$ (by calculator) is 3.036588972. . . . Our approximation by differentials is therefore accurate to three decimal places even when $dx = 1$.

Of course, the calculation in this example has little practical value, because calculators can easily find cube roots to great accuracy. The real purpose of the example is to emphasize that differentials provide good *linear* approximations to the increments of more complicated functions. We will understand the great importance of this idea more clearly in a later chapter, when we work with differentials of functions of several variables.

Perhaps a few more examples would not be amiss.

Example 4 Calculate the actual and approximate volumes of a 4.01-ft cube.

Solution If x is an edge, then the volume is $V(x) = x^3$. The actual volume is $V(4.01) = (4.01)^3 = 64.481201$ ft^3. Now $dV = 3x^2\,dx$, and by putting $x = 4$ and $dx = 0.01$ we have the approximate volume

$$V(4.01) \cong V(4) + dV = 4^3 + 3\cdot 4^2(0.01) = 64.48 \text{ ft}^3,$$

which is not bad.

Example 5 If the earth's radius were increased by 1 ft, approximately how much would its surface area increase?

Solution The surface area of a sphere of radius r is $A = 4\pi r^2$, and the earth's radius is about 4000 mi.* If we approximate the actual increment ΔA of the surface area by the differential dA evaluated at $r = 4000$ with $dr = 1$ ft, we get

$$\Delta A \cong dA = 8\pi r\,dr = 8\pi(4000)\cdot\frac{1}{5280}\text{ mi}^2,$$

since 1 ft = 1/5280 mi. By doing the arithmetic we find that $\Delta A \cong 19.04$ mi^2. This is close to the area of Manhattan Island, which is about 22 mi^2.

Example 6 If the earth's radius were to shrink by 1 in, approximately how much would its volume decrease?

Solution The volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$, so $dV = 4\pi r^2\,dr$ and

$$\begin{aligned}\Delta V \cong dV &= 4\pi(4000)^2\cdot\left(\frac{-1}{12\cdot 5280}\right)\\ &\cong -3173.32 \text{ mi}^3.\end{aligned}$$

The minus sign appears here because r decreases, so the answer is that the volume decreases about 3173.32 mi^3.

Remark 1 One of our standard notations for the second derivative of a function $y = f(x)$ is d^2y/dx^2. In view of our work in this section it is desirable to point out that the numerator, d^2y, and the denominator, dx^2, *have absolutely no meaning*

*In this example and the next we ignore minor irregularities like mountain ranges and deep ocean trenches, and the ellipsoidal shape of the earth, and assume that the earth is a perfect sphere of radius 4000 mi.

by themselves and will never be given such a meaning. The expression d^2y/dx^2 is an inseparable symbol representing the second derivative and is written this way for reasons explained in the second paragraph of Section 3.6.

Remark 2 *The Leibnizian myths about curves and differentials.* The modern concept of limit did not arise until the early nineteenth century, so no definition of the derivative resembling equation (1) was possible for Leibniz or his immediate successors. What were the early ideas about the nature of derivatives and differentials?

Most of the fruitful mathematical thinking of the period was based on one form or another of the notion of the "infinitely small." Leibniz's attitude toward the equation

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

would have been essentially as follows: As Δx approaches zero, both Δy and Δx become "infinitely small" or "infinitesimal" together. It is therefore reasonable to think of the limit dy/dx as the quotient of two infinitesimal quantities denoted by dy and dx and called "differentials." In Leibniz's imagination, an *infinitesimal* was a special kind of number that is not zero and yet is smaller than any other number.

There was also a geometric version of these ideas, in which a curve was thought of as consisting of an infinite number of infinitely small straight line segments (Fig. 5.4). A tangent was a line containing one of these tiny segments. To find the slope of the tangent at a point (x, y), we move an infinitesimal distance along the curve to a point $(x + dx, y + dy)$ and observe that the slope of the infinitesimal segment joining these two points is the quotient of two infinitesimals, dy/dx.

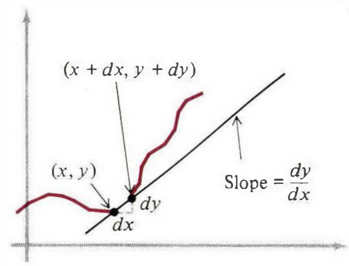


Figure 5.4 The Leibniz myth.

We have suggested that Leibniz introduced his differentials dx and dy to denote corresponding infinitesimal changes in the variables x and y. To get an idea of how these differentials were used, let us suppose that the variables x and y are related by the equation

$$y = x^2. \tag{14}$$

Leibniz would then replace x and y by $x + dx$ and $y + dy$ to obtain

$$y + dy = (x + dx)^2 = x^2 + 2x\,dx + dx^2,$$

which in view of (14) yields

$$dy = 2x\,dx + dx^2. \tag{15}$$

At this stage Leibniz would simply discard the term dx^2 and arrive at our familiar formula

$$dy = 2x\,dx, \tag{16}$$

which after division by dx takes its fractional form

$$\frac{dy}{dx} = 2x. \tag{17}$$

He would justify this step by claiming that any square of an infinitely small number is "infinitely infinitely small," or "an infinitesimal of higher order," and therefore entirely negligible. For Leibniz the derivative was a genuine quotient, a quotient of infinitesimals as calculated in formula (17) and illustrated in Fig. 5.4, and his form of calculus came to be widely known as "infinitesimal calculus."

It may be instructive to compare this Leibnizian use of infinitesimals with the modern approach based on limits. Thus, with the function $y = x^2$, if Δx is a given nonzero change in x and Δy is the corresponding change in y, then by essentially the same calculation as above we obtain

$$\Delta y = 2x\,\Delta x + \Delta x^2.$$

Instead of discarding the term Δx^2 as Leibniz would have done, in the modern approach we divide through by Δx to obtain the quotient $\Delta y/\Delta x$ and then define the derivative to be the limit of this quotient as Δx approaches zero,

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x) = 2x.$$

This produces formula (17) in a way that replaces the use of infinitesimals by a limit calculation.

The ideas of Leibniz worked with almost miraculous ease and effectiveness, and dominated the historical development of calculus and the physical sciences for almost 150 years. However, these ideas were flawed by the fact that infinitesimals in the sense described above clearly do not exist, for there is no such thing as a positive number that is smaller than all other positive numbers. Throughout this period of more than a century the enormous success of calculus as a problem-solving tool was obvious to all, and yet no one was able to give a logically acceptable explanation of what calculus *is*. The fog that obscured its fundamental concepts was at last dispelled in the early nineteenth century by the classical theory of limits. Fortunately the early mathematicians of the modern period—Leibniz himself, the Bernoullis, Euler, Lagrange—had sound intuitive feelings for what was reasonable and correct in the problems they studied. Even though their arguments often lacked rigor from the modern point of view, these pioneers rarely went astray in their conclusions.

If a myth is a veiled, condensed, symbolic expression of a more complicated and perhaps partially hidden truth, then mathematics has its myths just as history and literature do. Leibniz's differentials were banished from "official calculus" by the theory of limits, but nevertheless they remain a living part of the mythology of the subject.*

*It should be added that a logically acceptable concept of infinitesimals was constructed in the 1960s by the American mathematician Abraham Robinson [see his book *Non-Standard Analysis* (North-Holland Publishing Co., 1966), especially Sections 1.1 and 10.1]. While Robinson's achievement is of great interest to logicians and mathematicians, his ideas depend on mathematical logic and abstract set theory and are not likely to have much influence on the teaching or learning of calculus.

PROBLEMS

Calculate each of the following.

1 $d(7x^9 - 3x^5 + 34)$.

2 $d(\sqrt{1 - x^2})$.

3 $d(x^2\sqrt{1 - x^2})$.

4 $d\left(\dfrac{x-2}{x+3}\right)$.

5 $d(\sqrt{4x - x^2})$.

6 $d\left(\dfrac{x}{\sqrt{a^2 + x^2}}\right)$.

7 $d(3x^{2/3} + 10x^{1/5} - 17x)$.

8 $d\left[\dfrac{(1-2x)^3}{3-4x}\right]$.

9 $d(x^2\sqrt{3x+2})$. **10** $d(\sqrt{x+\sqrt{x+1}})$.

11 Use differentials to find dy/dx, given that

$$y = \frac{3u-1}{u^2-u} \quad \text{and} \quad u = (x^3+2)^5.$$

12 Use differentials to find dy/dx, given that

$$y = \frac{u+1}{u-1}, \quad u = \frac{v^3+6v-2}{\sqrt{v-1}}, \quad v = x^4+5x^2-3.$$

13 Consider a circle of radius r and area $A = \pi r^2$. If the radius is increased by a small amount Δr, find the increment ΔA and the differential dA. Draw a sketch, and observe that ΔA is the area of the thin circular ring between two concentric circles. Use the fact that the inner circle has circumference $2\pi r$ to understand geometrically why dA is a good approximation to ΔA.

14 A sphere of radius r has volume $V = \frac{4}{3}\pi r^3$ and surface area $A = 4\pi r^2$. If the radius is increased by a small amount Δr, find ΔV and dV. In the spirit of Problem 13, understand geometrically why dV is a good approximation to ΔV.

15 A coat of paint of thickness 0.02 in is applied to the faces of a cube whose edge is 10 in, thereby producing a slightly larger cube. Use differentials to find approximately the number of cubic inches of paint used. Also find the exact amount used by computing volumes before and after painting.

In Problems 16–23, use differentials to find approximate values for the given quantities. In each case use a calculator to find the value correct to six decimal places, and compare.

16 $\sqrt[4]{83}$. **17** $65^{2/3}$.
18 $\sqrt{102}$. **19** $80^{3/4}$.
20 $\sqrt{119}$. **21** $\sqrt[3]{218}$.
22 sin 59°. (Remember: First translate into radians.)
23 cos 32°.

In Problems 24–26, find the approximate amount by which the radius of the earth would have to be increased to produce additional surface area the size of each of the given states.

24 Rhode Island ($\cong$ 1215 mi^2).
25 Colorado ($\cong$ 104,250 mi^2).
26 Alaska ($\cong$ 580,400 mi^2).

27 Suppose a red ribbon is wrapped tightly around the earth at the equator. Approximately how much must the ribbon be lengthened if it is to be strung on poles 20 ft above the ground all the way around the earth?

5.3 INDEFINITE INTEGRALS. INTEGRATION BY SUBSTITUTION

If $y = F(x)$ is a function whose derivative is known, say, for example,

$$\frac{d}{dx}F(x) = 2x, \tag{1}$$

can we discover what the function $F(x)$ is? It doesn't take much imagination to write down one function with this property, namely, $F(x) = x^2$. Moreover, adding a constant term doesn't change the derivative, so each of the functions

$$x^2+1, \qquad x^2-\sqrt{3}, \qquad x^2+5\pi,$$

and more generally

$$x^2+c$$

where c is any constant, also has the property (1). Are there any others? The answer is *no*.

The justification for this answer lies in the following principle:

If $F(x)$ and $G(x)$ are two functions having the same derivative $f(x)$ on a certain interval, then $G(x)$ differs from $F(x)$ by a constant, that is, there exists a constant c with the property that

$$G(x) = F(x) + c$$

for all x in the interval.

To see why this statement is true, we notice that the derivative of the difference $G(x) - F(x)$ is zero on the interval,

$$\frac{d}{dx}[G(x) - F(x)] = \frac{d}{dx}G(x) - \frac{d}{dx}F(x) = f(x) - f(x) = 0.$$

It now follows that this difference itself must have a constant value c, so

$$G(x) - F(x) = c \quad \text{or} \quad G(x) = F(x) + c,$$

which is what we wanted to establish.*

This principle allows us to conclude that every solution of equation (1) must have the form $x^2 + c$ for some constant c.

The problem just discussed involved finding an unknown function whose derivative is known. If $f(x)$ is given, then a function $F(x)$ such that

$$\frac{d}{dx}F(x) = f(x) \tag{2}$$

is called an *antiderivative* of $f(x)$, and the process of finding $F(x)$ from $f(x)$ is *antidifferentiation.* We have seen that $f(x)$ does not have a single, uniquely determined antiderivative, but if we can find one antiderivative $F(x)$, then all others have the form

$$F(x) + c$$

for various values of the constant c. For example, $\frac{1}{3}x^3$ is one antiderivative of x^2, and the formula

$$\tfrac{1}{3}x^3 + c$$

comprises all possible antiderivatives of x^2.

For historical reasons, an antiderivative of $f(x)$ is usually called an *integral* of $f(x)$, and antidifferentiation is called *integration.* The standard notation for an integral of $f(x)$ is

$$\int f(x)\,dx, \tag{3}$$

which is read "the integral of $f(x)\,dx$." The equation

$$\int f(x)\,dx = F(x)$$

is therefore completely equivalent to (2). The function $f(x)$ is called the *integrand.* The "elongated S" symbol in (3) is called the *integral sign,* and was introduced by Leibniz in the earliest days of calculus. Its origin will become clear in the next chapter.

To illustrate a point of usage, we remark that the formulas

$$\int x^2\,dx = \tfrac{1}{3}x^3 \qquad \text{and} \qquad \int x^2\,dx = \tfrac{1}{3}x^3 + c \tag{4}$$

are both correct, but the first provides one integral while the second provides all possible integrals. For this reason the integral (3) is usually called the *indefinite*

*The crucial step in this reasoning can be expressed in several different ways: for instance, if the rate of change of a function is always zero, then the function cannot change and therefore must be constant; or equivalently, if every tangent line to a graph is horizontal, then the graph can neither rise nor fall and therefore must be a horizontal straight line. The theoretical basis for this inference is called the Mean Value Theorem and is examined more closely in Section 2.6.

integral, in contrast to the definite integrals discussed in the next chapter. The constant c in the second formula of (4) is called the *constant of integration* and is often referred to as an "arbitrary" constant. Our previous discussion shows that to find all integrals of a given function $f(x)$, it suffices to find one integral by any method that works—calculation, intelligent guessing, or asking a knowledgeable friend—and then to add an arbitrary constant at the end.

Every derivative that we have ever calculated can be reversed and rewritten as an integral. In particular, the power rule

$$\frac{d}{dx}x^n = nx^{n-1} \qquad \text{becomes} \qquad \int nx^{n-1}\,dx = x^n.$$

For our present purposes the formula

$$\frac{d}{dx}\frac{x^{n+1}}{n+1} = x^n$$

is a more convenient version of the power rule. This gives the form of the integral that we shall memorize and use,

$$\int x^n\,dx = \frac{x^{n+1}}{n+1}, \qquad n \neq -1. \tag{5}$$

In words: *To integrate a power, add 1 to the exponent and divide by the new exponent.*

Example 1 The following integrals are all special cases of (5):

$$\int x^3\,dx = \frac{x^4}{4} = \frac{1}{4}x^4, \qquad \int x^{572}\,dx = \frac{x^{573}}{573} = \frac{1}{573}x^{573},$$

$$\int \frac{dx}{x^5} = \int x^{-5}\,dx = \frac{x^{-4}}{-4} = -\frac{1}{4x^4},$$

$$\int \sqrt{x}\,dx = \int x^{1/2}\,dx = \frac{x^{3/2}}{\frac{3}{2}} = \frac{2}{3}x^{3/2}.$$

The reader will notice that when $n = -1$, the right side of (5) has zero denominator and is therefore meaningless. The treatment of this case, that is, the determination of the integral

$$\int \frac{dx}{x},$$

is one of the most important and fascinating parts of calculus, with a wide variety of applications. We return to this problem in Chapter 8.

The following additional integration rules are also slightly disguised versions of familiar facts about derivatives:

$$\int cf(x)\,dx = c\int f(x)\,dx \tag{6}$$

and

$$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx. \tag{7}$$

The first says that a constant factor can be moved from one side of the integral sign to the other. It is important to understand that this does not apply to variable factors, as can be seen from the fact that

$$\int x^2\,dx \neq x\int x\,dx,$$

since the left and right sides are, respectively, $\frac{1}{3}x^3$ and $x\cdot\frac{1}{2}x^2=\frac{1}{2}x^3$. Formula (7) says that the integral of a sum is the sum of the separate integrals. This applies to any finite number of terms.

To verify (6) and (7), it is enough to notice that they are equivalent to the differentiation formulas

$$\frac{d}{dx}\,cF(x)=c\,\frac{d}{dx}\,F(x)$$

and

$$\frac{d}{dx}\,[F(x)+G(x)]=\frac{d}{dx}\,F(x)+\frac{d}{dx}\,G(x),$$

where $(d/dx)F(x)=f(x)$ and $(d/dx)G(x)=g(x)$.

Example 2 When rules (5), (6), and (7) are combined, they enable us to integrate any polynomial. For instance,

$$\begin{aligned}\int(3x^4+6x^2)\,dx&=3\int x^4\,dx+6\int x^2\,dx\\&=\tfrac{3}{5}x^5+2x^3+c\end{aligned}$$

and

$$\begin{aligned}\int(5-2x^5+3x^{11})\,dx&=5\int dx-2\int x^5\,dx+3\int x^{11}\,dx\\&=5x-\tfrac{1}{3}x^6+\tfrac{1}{4}x^{12}+c.\end{aligned}$$

Observe that $\int dx=\int 1\,dx=x$. In each of these calculations an arbitrary constant is added at the end so that all possible integrals are included.

Example 3 We can also integrate many nonpolynomials that are expressible as linear combinations of powers:

$$\int\sqrt[3]{x^2}\,dx=\int x^{2/3}\,dx=\tfrac{3}{5}x^{5/3}+c;$$

$$\begin{aligned}\int\frac{2x^3-x^2-2}{x^2}\,dx&=\int(2x-1-2x^{-2})\,dx\\&=x^2-x+\frac{2}{x}+c;\end{aligned}$$

$$\begin{aligned}\int\frac{5x^{1/3}-2x^{-1/3}}{\sqrt{x}}\,dx&=\int(5x^{-1/6}-2x^{-5/6})\,dx\\&=6x^{5/6}-12x^{1/6}+c.\end{aligned}$$

The formula

$$\int u^n\,du = \frac{u^{n+1}}{n+1}, \qquad n \neq -1, \tag{8}$$

appears to be a trivial variation of (5) in which the letter x is replaced by u. However, let us think of u as some function $f(x)$ of x and take du seriously as the differential of u, so that

$$u = f(x)$$

and

$$du = f'(x)\,dx.$$

Then (8) becomes

$$\int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1}, \qquad n \neq -1, \tag{9}$$

which is a far-reaching generalization of (5).

Example 4 In practice, we usually exploit this idea by explicitly changing the variable in order to reduce a given complicated integral to an integral of the simple form (8). For instance, in the case of

$$\int (3x^2 - 1)^{1/3} 4x\,dx,$$

we notice that the differential of the expression in parentheses is $6x\,dx$, which differs from $4x\,dx$ only by a constant factor, so we write

$$u = 3x^2 - 1,$$
$$du = 6x\,dx,$$
$$x\,dx = \tfrac{1}{6}du.$$

These equations constitute a small "dictionary" that enables us to translate the given integral from the x-notation to the u-notation, as follows:

$$\int (3x^2 - 1)^{1/3} 4x\,dx = \int u^{1/3} \cdot 4 \cdot \tfrac{1}{6}\,du = \tfrac{2}{3}\int u^{1/3}\,du$$
$$= \tfrac{2}{3} \cdot \tfrac{3}{4}u^{4/3} + c = \tfrac{1}{2}u^{4/3} + c;$$

and by returning to the x-notation we obtain our result,

$$\int (3x^2 - 1)^{1/3} 4x\,dx = \tfrac{1}{2}(3x^2 - 1)^{4/3} + c.$$

This method is called *integration by substitution,* because it depends on a substitution or change of variable to simplify the problem. As formula (9) suggests, the success of the method depends on having an integral in which one part of the integrand is essentially the derivative of another part—where "essentially" means "except for a constant factor."

Remark 1 The integral in Example 4 was deliberately constructed so that the method of substitution works. To emphasize this point, we observe that the similar integral

$$\int (3x^2 - 1)^{1/3}\, dx \tag{10}$$

seems to be "simpler" than the one in Example 4 but is actually much more difficult because the integrand lacks the important factor x. If we try the substitution that worked before, we get

$$\int (3x^2 - 1)^{1/3}\, dx = \int u^{1/3} \cdot \frac{du}{6x},$$

and there is no practical way to get rid of the x in the denominator. In a later chapter we will study deeper methods that succeed in this type of problem, but just now there is nothing further we can do.

Remark 2 Many students are tempted to try to integrate (10) by writing

$$\int (3x^2 - 1)^{1/3}\, dx = \frac{(3x^2 - 1)^{4/3}}{4/3} = \frac{3}{4}(3x^2 - 1)^{4/3} + c, \tag{11}$$

which is incorrect. To understand why this is incorrect, recall that in calculating integrals we can always check our work quite easily, for if we have a suspected integral of a function $f(x)$, we can test it by computing its derivative to see if the result really equals $f(x)$. It is clear that (11) fails this test, because the derivative of the right side is

$$\tfrac{3}{4} \cdot \tfrac{4}{3}(3x^2 - 1)^{1/3} \cdot 6x = (3x^2 - 1)^{1/3} 6x,$$

which is certainly not the integrand of (10).

Finally, our derivative formulas for the sine and cosine yield the following important integration formulas:

$$\int \cos u\, du = \sin u + c \tag{12}$$

and

$$\int \sin u\, du = -\cos u + c. \tag{13}$$

These are tools with innumerable applications, ranging from the theory of probability to the propagation of sound waves.

Example 5 (a) To integrate

$$\int \cos 3x\, dx,$$

we look at (12) and see that we must put $u = 3x$ so that $du = 3\, dx$ and $dx = \frac{1}{3}\, du$. We then write

$$\int \cos 3x\, dx = \int \cos u \cdot \tfrac{1}{3}\, du = \tfrac{1}{3} \int \cos u\, du$$
$$= \tfrac{1}{3} \sin u + c = \tfrac{1}{3} \sin 3x + c.$$

(b) To integrate

$$\int x \sin (1 - x^2)\, dx,$$

we put $u = 1 - x^2$ so that $du = -2x\,dx$ and $x\,dx = -\frac{1}{2}\,du$, and then use (13):

$$\int x \sin(1 - x^2)\,dx = \int \sin u \cdot (-\tfrac{1}{2}\,du) = -\tfrac{1}{2}\int \sin u\,du$$

$$= \tfrac{1}{2}\cos u + c = \tfrac{1}{2}\cos(1 - x^2) + c.$$

Remark 3 It is clear from Examples 4 and 5 that the notation of differentials is extremely useful for calculating indefinite integrals by the method of substitution. This method strikes many students as a kind of magic. To understand why it is legitimate (magic is not allowed in mathematics!), let us consider the form of integral to which the method applies:

$$\int f[g(x)]g'(x)\,dx. \tag{14}$$

What we have done above is put $u = g(x)$ and then write $du = g'(x)\,dx$. The integral (14) now takes the new form

$$\int f[g(x)]g'(x)\,dx = \int f(u)\,du.$$

If we can integrate this, so that

$$\int f(u)\,du = F(u) + c$$

or

$$F'(u) = f(u),$$

then since $u = g(x)$ we want to be able to integrate (14) by writing

$$\int f[g(x)]g'(x)\,dx = \int f(u)\,du = F(u) + c = F[g(x)] + c. \tag{15}$$

All that is needed to justify this procedure is to observe that (15) is a correct result, because

$$\frac{d}{dx}F[g(x)] = F'[g(x)]g'(x) = f[g(x)]g'(x)$$

by the chain rule. It is therefore the chain rule that allows us to work with the symbols dx and du after the integral signs as if they were differentials. This smooth compatibility with the chain rule is the main reason for the extraordinary value of the differential notation in calculus.

Finally, it may be helpful to students if we give a formal outline of the process of *integration by substitution*:

1 Make a careful choice of u, say $u = g(x)$.
2 Compute $du = g'(x)\,dx$.
3 Substitute $g(x) = u$ and $g'(x)\,dx = du$ in the given integral. At this point the integral must be wholly in terms of u, and no x's should be present. If this is not the case, try another choice of u.
4 Calculate the integral obtained in Step 3.
5 Replace u by $g(x)$ to express the final result wholly in terms of x.

PROBLEMS

In Problems 1–36, compute the integrals. Be sure to include the constant of integration in each answer.

1 $\int (x+1)\,dx.$

2 $\int (3x-2)\,dx.$

3 $\int (x^2+x^3+x^4)\,dx.$

4 $\int x^7\,dx.$

5 $\int \frac{dx}{\sqrt{x}}.$

6 $\int (3x^2+2x+1)\,dx.$

7 $\int x^{3/4}\,dx.$

8 $\int x^2(x^2-1)\,dx.$

9 $\int \frac{dx}{\sqrt[3]{x}}.$

10 $\int (600x-6x^5)\,dx.$

11 $\int \left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)dx.$

12 $\int (2x-7)\,dx.$

13 $\int \frac{3+2x}{\sqrt{x}}\,dx.$

14 $\int x^2\sqrt{x}\,dx.$

15 $\int x^2(1+x^3)\,dx.$

16 $\int x^{7/5}\,dx.$

17 $\int (7+x)\,dx.$

18 $\int \frac{1}{x^6}\,dx.$

19 $\int (4x^5+6x-5)\,dx.$

20 $\int (1-2x^2-3x^3)\,dx.$

21 $\int \frac{4}{x^3}\,dx.$

22 $\int \left(\frac{6}{x^4}+2x^{3/2}-5\right)dx.$

23 $\int \left(\sqrt{x}-14x^{5/2}+\frac{3}{x^2}\right)dx.$

24 $\int (\frac{3}{2}x^{1/2}-5)\,dx.$

25 $\int \left(\frac{3}{x^{2/3}}-\frac{4}{x^{3/4}}\right)dx.$

26 $\int \left(\sqrt[3]{x^2}-\frac{6}{\sqrt[4]{x^5}}\right)dx.$

27 $\int \left(\frac{1}{\sqrt{x}}-\frac{1}{3}x^{7/3}\right)dx.$

28 $\int 10\,dx.$

29 $\int (4x^3-8x+17)\,dx.$

30 $\int (12x^5-3x^{-2})\,dx.$

31 $\int x^{1/3}(x+2)^2\,dx.$

32 $\int \frac{(2x+3)^2}{\sqrt{x}}\,dx.$

33 $\int \sqrt{x}(2-3x^2)^2\,dx.$

34 $\int x^4(5-12x^{55})\,dx.$

35 $\int 100x^{499}\,dx.$

36 $\int \sqrt[3]{\sqrt{x}}\,dx.$

In Problems 37–44, compute the integrals by using the given substitutions.

37 $\int \sqrt{3+4x}\,dx,\quad u=3+4x.$

38 $\int \sqrt{3x^2+1}\,x\,dx,\quad u=3x^2+1.$

39 $\int \frac{dx}{(2x-3)^2},\quad u=2x-3.$

40 $\int x^2(1-4x^3)^{1/5}\,dx,\quad u=1-4x^3.$

41 $\int \frac{x\,dx}{\sqrt{5-4x^2}},\quad u=5-4x^2.$

42 $\int x^{2/3}(2-x^{5/3})^{-5}\,dx,\quad u=2-x^{5/3}.$

43 $\int \frac{(1+\sqrt{x})^{1/4}}{\sqrt{x}}\,dx,\quad u=1+\sqrt{x}.$

44 $\int \frac{(2+3x)\,dx}{\sqrt{1+4x+3x^2}},\quad u=1+4x+3x^2.$

In Problems 45–58, compute the integrals by using substitutions of your own devising.

45 $\int \sqrt{x^2+x^4}\,dx.$

46 $\int \frac{dx}{(x-7)^7}.$

47 $\int \frac{dx}{(7-x)^7}.$

48 $\int \frac{4\,dx}{\sqrt{(x-1)^3}}.$

49 $\int x\sqrt{2-x^2}\,dx.$

50 $\int 24x(4x^2-1)^9\,dx.$

51 $\int \frac{3x^2\,dx}{\sqrt{x^3-5}}.$

52 $\int \frac{40\,dx}{(4x+5)^6}.$

53 $\int (10x+10)^{10}\,dx.$

54 $\int x^6(x^7+8)^9\,dx.$

55 $\int x\sqrt{(3x^2+4)^3}\,dx.$

56 $\int x^2\sqrt[3]{x^3+1}\,dx.$

57 $\int \frac{(18x^2-2)\,dx}{\sqrt{3x^3-x+2}}.$

58 $\int \frac{(4+12x)\,dx}{\sqrt[3]{1-2x-3x^2}}.$

59 Integrate $\int (x^3)^4\cdot 3x^2\,dx$ as $\int u^4\,du$ and also as $\int 3x^{14}\,dx$, and compare your results.

60 Integrate $\int (x^3+1)^2\cdot 3x^2\,dx$ as $\int u^2\,du$ and also by multiplying out, and compare your results.

61 Find the integral of $3x^2$ that has the value 10 when $x=2$. Hint: Since every integral of $3x^2$ has the form $y=x^3+c$, find the value of c that makes $y=10$ when $x=2$.

62 Find the integral $F(x)$ of $\sqrt{x}$ with the property that $F(9)=9$.

63 Find the following integrals:
(a) $\int \cos 2x\,dx$;
(b) $\int \sin 5x\,dx$;
(c) $\int (4\cos 2x+15\sin 5x)\,dx$;
(d) $\int (\sin 2x+\cos 5x)\,dx$.

64 Construct an example to show that

$$\int f(x)g(x)\,dx=\left(\int f(x)\,dx\right)\left(\int g(x)\,dx\right)$$

is *not* a valid integration formula.

65 (a) Show that

$$\int \sin^2 x\,dx=\tfrac{1}{3}\sin^3 x$$

is not true.
(b) Calculate

$$\int \sin^2 x\,dx$$

by using the half-angle formula $\sin^2 x=\frac{1}{2}(1-\cos 2x)$.
(c) Use differentiation to verify the correctness of your answer in (b).

66 Find the following integrals:
(a) $\int \sin(2-x)\,dx$;
(b) $\int x \cos x^2\,dx$;
(c) $\int x^3 \sin x^4\,dx$.

67 Find the following integrals:
(a) $\int \sin^4 x \cos x\,dx$;
(b) $\int \cos^5 x \sin x\,dx$;
(c) $\int \cos x \cos(\sin x)\,dx$.

68 Show that both of the following integrals are correct:

$$\int \frac{dx}{(1-x)^2} = \frac{1}{1-x} \qquad \text{and} \qquad \int \frac{dx}{(1-x)^2} = \frac{x}{1-x}.$$

Explain.

69 Calculate $\int \sin x \cos x\,dx$
(a) by using the substitution $u = \sin x$;
(b) by using $u = \cos x$.
Reconcile your answers in (a) and (b).

5.4 DIFFERENTIAL EQUATIONS. SEPARATION OF VARIABLES

We have seen that the equation

$$\int f(x)\,dx = F(x) \tag{1}$$

is equivalent to

$$\frac{d}{dx} F(x) = f(x). \tag{2}$$

This statement can be interpreted in two ways.

(a) In accordance with the explanation in Section 5.3, we can think of the symbol

$$\int \cdots dx$$

as operating on the function $f(x)$ to produce its integral, or antiderivative, $F(x)$. From this point of view the integral sign and the dx go together as parts of a single symbol; the integral sign specifies the operation of integration, and the only role of the dx is to tell us that x is the "variable of integration."

(b) A second interpretation is suggested by our treatment of Examples 4 and 5 in Section 5.3. Let us write (2) in the form

$$dF(x) = f(x)\,dx,$$

so that $f(x)\,dx$ is explicitly seen to be the differential of $F(x)$. If we now take dx in (1) at its face value, as the differential of x, then the integral sign in (1) acts on the differential of a function $F(x)$, namely, on $f(x)\,dx$, and produces the function itself. Thus, the symbol $\int$ for integration (without considering the dx as part of the symbol) stands for the operation which is the inverse of the operation denoted by the symbol d.

We shall use both interpretations. However, the second is particularly convenient, not only for the actual procedures used in computing integrals, but also for solving certain simple differential equations.

A *differential equation* is an equation involving an unknown function and one or more of its derivatives. The *order* of such an equation is the order of the highest derivative that occurs in it.

In the process of integration we have been solving first-order differential equations of the form

$$\frac{dy}{dx} = f(x),$$

where $f(x)$ is a given function. Thus, the equation

$$\frac{dy}{dx} = 3x^2 \qquad \text{is equivalent to} \qquad dy = 3x^2\,dx, \tag{3}$$

and we integrate to obtain the solution,

$$\int dy = \int 3x^2\,dx \qquad \text{or} \qquad y = x^3 + c. \tag{4}$$

Notice that a constant of integration arises on both sides here,

$$y + c_1 = x^3 + c_2,$$

but this can be written as $y = x^3 + (c_2 - c_1)$, and no generality is lost by replacing $c_2 - c_1$ by c. Accordingly, it suffices to add a constant of integration to one side only, as we have done in (4).

We can also handle more complicated differential equations. Let us find y if

$$\frac{dy}{dx} = -2xy^2. \tag{5}$$

If we set aside the obvious trivial solution $y = 0$, this can be written as

$$-\frac{dy}{y^2} = 2x\,dx.$$

Integration now yields

$$\frac{1}{y} = x^2 + c$$

or

$$y = \frac{1}{x^2 + c}. \tag{6}$$

This is called the *general solution* of (5), and different choices of c give different *particular solutions.*

We were able to solve equation (5) by the method of *separation of variables,* that is, by isolating the y's from the x's and integrating. In general, if a first-order differential equation can be written in the form

$$g(y)\,dy = f(x)\,dx,$$

with its variables separated, and if we can carry out the integrations, then we have the solution

$$\int g(y)\,dy = \int f(x)\,dx + c. \tag{7}$$

It should be noted that only in very special cases can the variables be separated in this way. For instance, the differential equation

$$\frac{dy}{dx} = \frac{x + y}{x - y} \tag{8}$$

cannot be solved by this method.

Each of the solutions (4) and (6) of equations (3) and (5) consists of a family of curves corresponding to various values of the constant c. These families are shown in Figs. 5.5 and 5.6. The arbitrary constant that appears in the general solution of a first-order equation is given a specific numerical value by prescrib-

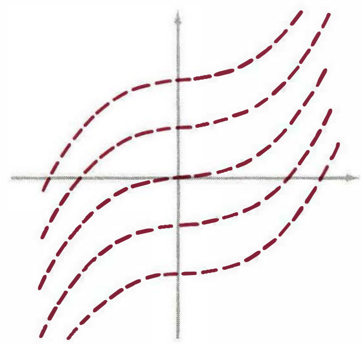


Figure 5.5 $y = x^3 + c$

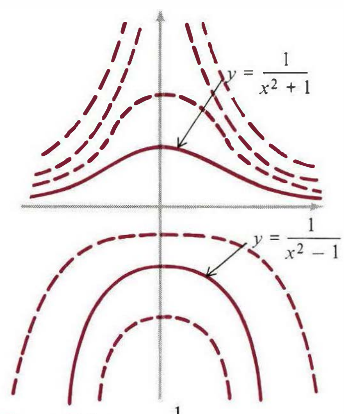


Figure 5.6 $y = \dfrac{1}{x^2 + c}$

ing, as an *initial condition*, the value of the unknown function $y = y(x)$ at a single value of x, say $y = y_0$ when $x = x_0$. In geometric language, an initial condition means that the solution curve is required to pass through a specific point in the plane. Thus, in Fig. 5.6 the upper and lower solid curves correspond to the initial conditions

$$y = 1 \text{ when } x = 0 \qquad \text{and} \qquad y = -1 \text{ when } x = 0,$$

respectively. We shall see in the next section that this terminology is particularly suitable for mechanical problems, where time is the independent variable and the initial positions or initial velocities of moving bodies are specified.

In the problems just discussed, equation (7) was easily solved for y to yield the solution of the given differential equation as a family of *functions*. It is often convenient not to press this point, and to accept a family of *equations* as the general solution, without demanding explicitly displayed functions.

We illustrate by finding the most general curve whose normal at each point passes through the origin O, and also the particular curve with this property through the point (2, 3). The normal OP has slope y/x (see Fig. 5.7), and the slope of the tangent is the negative reciprocal of this, so our differential equation is

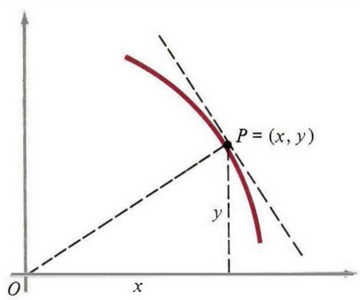


Figure 5.7

$$\frac{dy}{dx} = -\frac{x}{y}. \tag{9}$$

Separating variables gives $y\,dy = -x\,dx$, and by integrating we get

$$\tfrac{1}{2}y^2 = -\tfrac{1}{2}x^2 + c.$$

If we put $r^2 = 2c$, our general solution of (9) takes the neater form

$$x^2 + y^2 = r^2.$$

This is the family of all circles with center at the origin, as the reader has probably foreseen. By setting $x = 2$ and $y = 3$, we find that $r^2 = 13$, so

$$x^2 + y^2 = 13$$

is the particular solution of (9) passing through the point (2, 3). It is clearly more reasonable to leave this solution as it is than to insist that it be solved for y.

Remark 1 By rights, differential equations should perhaps be called *derivative equations*. However, as we saw in Section 5.2, in the early days of calculus differentials were the primary concepts and derivatives were secondary, so the term arose in a natural way. In any case, it has been in standard use for hundreds of years and no one dreams of changing it now.

Remark 2 The mathematical description of a physical (or biological or chemical) process is usually given in terms of functions that show how the quantities involved change as time goes on. When we know such a function, we can find its rate of change by calculating the derivative. Often, however, we are faced with the reverse problem of finding an unknown function from given information about its rate of change. This information is usually expressed in the form of an equation involving derivatives of the unknown function. These differential equations arise so frequently in scientific problems that their study constitutes one of the main branches of mathematics. We continue with some important applications of this subject in the next section, and return to it from time to time throughout the rest of our work.

PROBLEMS

Find the general solution of each of the following differential equations.

1 $\dfrac{dy}{dx} = 6x^2 + 4x - 5.$

2 $\dfrac{dy}{dx} = (3x + 1)^3.$

3 $\dfrac{dy}{dx} = 24x^3 + 18x^2 - 8x + 3.$

4 $\dfrac{dy}{dx} = 2\sqrt{y}.$

5 $\dfrac{dy}{dx} = \dfrac{x + \sqrt{x}}{y - \sqrt{y}}.$

6 $\dfrac{dy}{dx} = \sqrt[3]{\dfrac{y}{x}}.$

7 $\dfrac{dy}{dx} = \dfrac{1}{x^2} + x.$

Find the particular solution of each of the following differential equations that satisfies the given initial condition.

8 $\dfrac{dy}{dx} = 10x + 5, \quad y = 15$ when $x = 0.$

9 $\dfrac{dy}{dx} = 2xy^2, \quad y = 1$ when $x = 2.$

10 $\dfrac{dy}{dx} = \dfrac{x}{y}, \quad y = 3$ when $x = 2.$

11 $y\dfrac{dy}{dx} = x(y^4 + 2y^2 + 1), \quad y = 1$ when $x = 4.$

12 $\dfrac{dy}{dx} = \dfrac{5 + 3x^2}{2 + 2y}, \quad y = 2$ when $x = -2.$

13 $\dfrac{dy}{dx} = \sqrt{xy}, \quad y = 64$ when $x = 9.$

In Problems 14–19, verify that the given function is a solution of the given differential equation for all choices of the constants A and B.

14 $y = x + Ax^2, \quad x\dfrac{dy}{dx} = 2y - x.$

15 $y = Ax + x^3, \quad x\dfrac{dy}{dx} = y + 2x^3.$

16 $y = x + A\sqrt{x^2 + 1}, \quad (x^2 + 1)\dfrac{dy}{dx} = xy + 1.$

17 $y = Ax + \sqrt{x^2 + 1}, \quad x\dfrac{dy}{dx} = y - \dfrac{1}{\sqrt{x^2 + 1}}.$

18 $y = Ax + \dfrac{B}{x}, \quad x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - y = 0.$

19 $y = Ax + Bx^2, \quad x^2\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + 2y = 0.$

20 In a certain barbarous land, two neighboring tribes have hated one another from time immemorial. Being barbarous peoples, their powers of belief are strong, and a solemn curse pronounced by the medicine man of the first tribe deranges the members of the second tribe and drives them to murder and suicide. If the rate of change of the population P of the second tribe is $-\sqrt{P}$ people per week, and if the population is 676 when the curse is uttered, when will they all be dead?

5.5 MOTION UNDER GRAVITY. ESCAPE VELOCITY AND BLACK HOLES

Much of the original inspiration for the development of calculus came from the science of mechanics, and these two subjects have continued to be inseparably connected down to the present day. Mechanics rests on certain basic principles that were first laid down by Newton. The statement of these principles requires the concept of the derivative, and we shall see in this section that their applications depend on integration and the solution of differential equations.

Rectilinear motion is motion along a straight line. In contrast, motion along a curved path is sometimes called *curvilinear motion.* Our present purpose is to study the rectilinear motion of a single *particle,* that is, of a point at which a body of mass m is imagined to be concentrated. In discussing the motion of physical objects, such as cars, bullets, falling rocks, etc., we often ignore the size and shape of the object and think of it as if it were a particle.

The position of our particle is completely determined by its coordinate s with respect to a conveniently chosen coordinate system on the line (Fig. 5.8). Since the particle moves, s is a function of the time t, as measured from some convenient initial instant $t = 0$. We symbolize this by writing $s = s(t)$. As we know

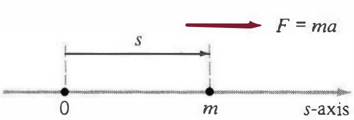


Figure 5.8

from the discussion in Section 2.4, the *velocity* v of the particle is the rate of change of its position,

$$v = \frac{ds}{dt},$$

and the *speed* is the absolute value of the velocity.* In general, the velocity of a moving particle changes with time, and the *acceleration* a is the rate of change of velocity,

$$a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2s}{dt^2}.$$

This is positive or negative according as v is increasing or decreasing.

The basic assumption of Newtonian mechanics is that force is required in order to change velocity; that is, acceleration is caused by force. The concept of force originates in the subjective feeling of effort that we experience when we change the velocity of a physical object, for instance, when we push a stalled car or throw a rock. In the case of rectilinear motion, we assume that a force can be expressed by a number, which is positive or negative according as the force acts in the positive or negative direction.

Newton's second law of motion states that the acceleration of a particle is directly proportional to the force F acting on it and inversely proportional to its mass m,

$$a = \frac{F}{m}, \tag{1}$$

or equivalently,

$$F = ma. \tag{2}$$

[The units of measurement for these quantities are always chosen so that the constant of proportionality in equation (1) has the value 1, as shown.] Thus, if the force is doubled, then by (1) the resulting acceleration is also doubled; and if the mass is doubled, the acceleration is cut in half. In this context, the mass of a body can be interpreted as its capacity to resist acceleration.†

From one point of view, equation (2) can be considered as nothing more than a definition of force, because the right side is a quantity that can be calculated by measuring the mass and observing the motion, and this determines the force. On the other hand, the force F is often known in advance from fairly simple physical considerations. The innocent-looking equation $F = ma$ then becomes the second-order differential equation

$$m\frac{d^2s}{dt^2} = F. \tag{3}$$

This equation has profound consequences, for in principle we can find the par-

*We have pointed out before that even though the words "velocity" and "speed" are more or less synonymous in ordinary usage, in physics (and mathematics) they have different meanings. The distinction lies in the fact that the velocity v is sometimes positive and sometimes negative, depending on whether s is increasing or decreasing. On the other hand, the speed is $|v|$, and hence is never negative.

†*Newton's first law of motion* asserts that if no force acts on a particle, then its velocity does not change, that is, its acceleration is zero. This is clearly a special case of (1).

ticle's position s at any time t by solving (3) with appropriate initial conditions.*

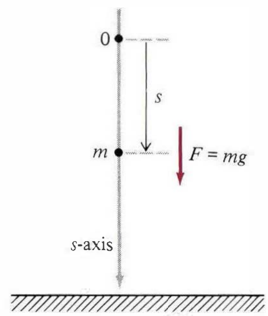


Figure 5.9

Example 1 Find the motion of a stone of mass m which is dropped from a point above the surface of the earth.

Solution The most important example of a known force is the familiar "force of gravity." From direct experimental evidence, we know that the force of gravity acting on the stone (this is the *weight* of the stone) is directed downward and has magnitude $F = mg$, where g is the constant acceleration due to gravity near the surface of the earth ($g = 32$ ft/s^2 or 9.80 m/s^2, approximately). If s is the stone's position as measured along a vertical axis, with the positive direction pointing downward and the origin at the initial position of the stone (Fig. 5.9), then equation (3) is

$$m\frac{d^2s}{dt^2} = mg \qquad \text{or} \qquad \frac{d^2s}{dt^2} = g.$$

Integrating this equation twice gives

$$v = \frac{ds}{dt} = gt + c_1, \tag{4}$$

$$s = \tfrac{1}{2}gt^2 + c_1t + c_2, \tag{5}$$

where c_1 and c_2 are constants of integration. Since the stone is "dropped" (that is, released with no initial velocity) at time $t = 0$ from the point chosen as the origin, the initial conditions are

$$v = 0 \qquad \text{and} \qquad s = 0 \qquad \text{when} \qquad t = 0.$$

The condition $v = 0$ when $t = 0$ gives $c_1 = 0$, and $s = 0$ when $t = 0$ gives $c_2 = 0$. We therefore have

$$v = gt, \tag{6}$$

$$s = \tfrac{1}{2}gt^2, \tag{7}$$

at least until the stone hits the ground. If we change the situation and require that the stone be thrown downward with an initial velocity v_0 from the initial position $s = s_0$ at time $t = 0$, then the initial conditions are

$$v = v_0 \qquad \text{and} \qquad s = s_0 \qquad \text{when} \qquad t = 0,$$

and (4) and (5) become

$$v = gt + v_0,$$
$$s = \tfrac{1}{2}gt^2 + v_0t + s_0.$$

It should be pointed out that in this discussion we have ignored the effect of air resistance, and have assumed that the only force acting on the falling stone

*The intellectual impact of Newton's $F = ma$ on the seventeenth and eighteenth centuries was even greater than that of Einstein's $E = mc^2$ on the twentieth century.

is the force of gravity. It is possible to take the air resistance into account, but in this case equation (3) is too complicated for us to cope with here. We return to this topic in Chapter 8.

We also remark that if distance is measured in feet and time in seconds, so that g has the numerical value 32, then (6) and (7) become

$$v = 32t \qquad \text{and} \qquad s = 16t^2.$$

It is clear from the first of these equations that the velocity of the stone increases by 32 ft/s during each second of fall, and of course this is what is meant by the statement that the acceleration due to gravity is 32 feet per second per second (ft/s^2).

Example 2 A stone is thrown upward with an initial velocity of 128 ft/s from the roof of a building 320 ft high. Express its height above the ground as a function of time. Find the maximum height the stone attains. Assuming that the stone misses the building on its way down, how long does it take to hit the ground? What are the velocity and speed of the stone at the moment it hits the ground?

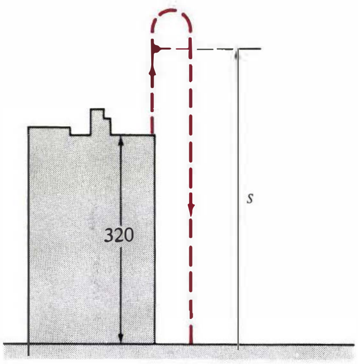


Figure 5.10

Solution We place the s-axis with its origin on the ground and the positive direction pointing upward (Fig. 5.10). Since the force of gravity is directed downward, and by equation (2) the force and acceleration have the same sign, the acceleration of the stone is given by

$$a = \frac{d^2s}{dt^2} = -32. \tag{8}$$

Integrating this equation yields

$$v = \frac{ds}{dt} = -32t + c_1,$$

and by using the initial condition $v = 128$ when $t = 0$, we get

$$v = \frac{ds}{dt} = -32t + 128. \tag{9}$$

A second integration gives

$$s = -16t^2 + 128t + c_2,$$

and since $s = 320$ when $t = 0$, we obtain

$$s = -16t^2 + 128t + 320 \tag{10}$$

as the height of the stone above the ground at any time t.

To find the maximum height attained by the stone, we write (9) in the form

$$v = -32(t - 4).$$

This tells us that for $t < 4$, the velocity is positive, so the stone is moving upward. When $t = 4$, the velocity is zero and the stone is motionless for an instant. For $t > 4$, the velocity is negative and the stone is falling. We therefore find the maximum height by putting $t = 4$ into equation (10). This gives $s = -16 \cdot 16 + 128 \cdot 4 + 320 = -256 + 512 + 320 = 576$ as the maximum height.

The stone hits the ground when $s = 0$. By using equation (10) we see that this leads us to the sequence of equivalent equations

$$-16t^2 + 128t + 320 = 0,$$

$$-16(t^2 - 8t - 20) = 0,$$

$$(t - 10)(t + 2) = 0.$$

Thus $s = 0$ when $t = 10$ or $t = -2$. The second answer is meaningless in the circumstances, and can be discarded. Therefore the stone hits the ground 10 s after being thrown.

To find the velocity of the stone at the moment it hits the ground, we put $t = 10$ into equation (9): $v = -32 \cdot 10 + 128 = -320 + 128 = -192$. The velocity at that moment is therefore -192 ft/s, and the minus sign tells us that the stone is moving downward. The speed at that moment is $|-192| = 192$ ft/s.

In these examples we have treated the acceleration due to gravity as if it were a constant. This is almost true for moving bodies that stay fairly close to the surface of the earth. However, to study the motion of a body that moves away from the earth into space, we must take account of the fact that the force of gravity varies inversely as the square of the distance from the center of the earth.

Example 3 Suppose a rocket is fired vertically upward with initial velocity v_0 and thereafter coasts with no further expenditure of energy. For larger values of v_0 it rises higher before coming to rest and falling back to earth. What must v_0 be in order for the rocket never to come to rest, and thereby to escape completely from the earth's gravitational attraction?

Solution According to *Newton's law of gravitation,* any two particles of matter in the universe attract each other with a force that is jointly proportional to their masses and inversely proportional to the square of the distance between them. In the present situation (see Fig. 5.11), this means that the force F attracting the rocket back to earth is given by the inverse square law

$$F = -G\frac{Mm}{s^2},$$

where G is a positive constant, M and m are the masses of the earth and the rocket, and s is the distance from the center of the earth to the rocket.*

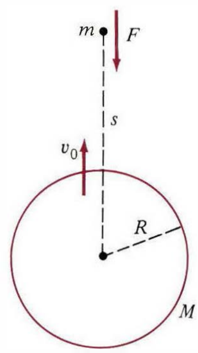


Figure 5.11

We begin our detailed analysis of the problem by observing that in this case Newton's second law of motion $F = ma$ becomes

$$m\frac{d^2s}{dt^2} = -G\frac{Mm}{s^2},$$

so

$$\frac{d^2s}{dt^2} = -\frac{GM}{s^2}. \tag{11}$$

*It can be proved—and will be proved in a later chapter—that the gravitational attraction exerted on the rocket by the earth as a whole is the same as that which would be exerted by a particle of mass M located at the center of the earth. In other words, the entire mass of the earth can be treated as if it were concentrated at its center.

This tells us at the outset that the motion of the rocket does not depend on the rocket's own mass. We can put the constants here into a more convenient form by noticing that the acceleration d^2s/dt^2 has the value $-g$ when $s = R$, where R is the radius of the earth. This gives

$$-g = -\frac{GM}{R^2} \qquad \text{or} \qquad GM = gR^2;$$

and since $d^2s/dt^2 = dv/dt$, we can write (11) as

$$\frac{dv}{dt} = -\frac{gR^2}{s^2}. \tag{12}$$

Our next step is to eliminate t from this equation by using the chain rule to write

$$\frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = \frac{dv}{ds}v.$$

Equation (12) now becomes

$$v\frac{dv}{ds} = -\frac{gR^2}{s^2}.$$

By separating variables and integrating, we obtain

$$\int v\,dv = gR^2\int -\frac{ds}{s^2}$$

or

$$\tfrac{1}{2}v^2 = \frac{gR^2}{s} + c. \tag{13}$$

To evaluate the constant of integration c, we use the initial condition that $v = v_0$ when $s = R$, so

$$\tfrac{1}{2}v_0^2 = gR + c$$

and

$$c = \tfrac{1}{2}v_0^2 - gR.$$

With this value of c, equation (13) becomes

$$\tfrac{1}{2}v^2 = \frac{gR^2}{s} + (\tfrac{1}{2}v_0^2 - gR). \tag{14}$$

Our final conclusion emerges from (14) as follows: For the rocket to escape from the earth, it must move in such a way that $\frac{1}{2}v^2$ is always positive, for if $\frac{1}{2}v^2$ vanishes, the rocket stops moving and then falls back to earth. But the first term on the right of (14) evidently approaches zero as s increases. Therefore, in order to guarantee that $\frac{1}{2}v^2$ is positive no matter how large s is, we must have $\frac{1}{2}v_0^2 - gR \geq 0$. This is equivalent to $v_0^2 \geq 2gR$ or $v_0 \geq \sqrt{2gR}$. The quantity $\sqrt{2gR}$ is usually called the *escape velocity* for the earth. We can easily estimate its value by using the approximations $g \cong 32$ ft/s^2 and $R \cong 4000$ mi:

$$\begin{aligned}\sqrt{2gR} &\cong \sqrt{2 \cdot 32 \text{ ft/s}^2 \cdot 4000 \text{ mi}} \\ &\cong \sqrt{2 \cdot 32 \cdot \tfrac{1}{5280} \text{ mi/s}^2 \cdot 4000 \text{ mi}} \\ &\cong 7 \text{ mi/s} \cong 25{,}000 \text{ mi/h}.\end{aligned}$$

Remark 1 In just the same way as in this example, the quantity $\sqrt{2g'R'}$ is the escape velocity for any planet, satellite, or star, where R' and g' are understood to be the radius and the acceleration due to gravity at the surface. If the radius of such a body is decreased while the mass is unchanged, the escape velocity at the surface increases. Why?

Remark 2 Most normal stars are maintained in their gaseous, puffed-up state by radiation pressure from within, which is generated by the burning of nuclear fuel. When the nuclear fuel gives out, the star undergoes gravitational collapse into a very much smaller sphere of essentially the same mass. The crushed, degenerate matter of these collapsed stars can sustain two types of equilibrium, depending on the mass of the star. *White dwarfs* are those that result when the mass is less than about 1.3 solar masses, and *neutron stars* arise when the mass is between 1.3 and 2 solar masses. For heavier stars no equilibrium is possible, and collapse continues until the escape velocity at the surface reaches the speed of light. Collapsed stars of this type are completely invisible, since no radiation can ever escape. These are the so-called *black holes*.

PROBLEMS

1 In Example 2, how long after the stone is thrown does it pass the roof of the building on its way down? What are the velocity and speed at that moment?

2 In Example 2, if the stone were simply dropped from the roof, what would s be as a function of time? How long would the stone fall?

3 In Example 2, the origin of the s-axis is at ground level. If the origin is placed at the top of the building, what are the formulas for v and s that correspond to (9) and (10)?

4 A ball is thrown upward from the top of a cliff 96 ft high with an initial velocity of 64 ft/s. Find the maximum height of the ball above the ground below. Assuming that the ball misses the cliff on its way down, how long does it take to hit the ground?

5 A bag of ballast is accidentally dropped from a balloon which is stationary at an altitude of 4900 m. How long does it take for the bag to hit the ground?

6 With what velocity must an arrow be shot upward in order to fall back to its starting point 10 seconds later? How high will it rise?

7 A boy at the top of a cliff 299 ft high throws a rock straight down, and it hits the ground $3\frac{1}{4}$ seconds later. With what velocity does the boy throw the rock?

8 A woman standing on a bridge throws a stone straight up. Exactly 5 seconds later the stone passes the woman on the way down, and 1 second after that it hits the water below. Find the initial velocity of the stone and the height of the bridge above the water.

9 A stone is dropped from the roof of a building 256 ft high. Two seconds later a second stone is thrown downward from the roof of the same building with an initial velocity of v_0 feet per second. If both stones hit the ground at the same time, what is v_0?

10 How much time does a train traveling 144 km/h take to stop if it has a constant negative acceleration of 4 m/s^2? How far does the train travel in this time?

11 A man standing on the ground throws a stone straight up. Neglecting the height of the man, find the maximum height of the stone in terms of initial velocity v_0. What is the smallest value of v_0 that will make it possible for the stone to land on top of a 144-ft building?

12 On the surface of the moon the acceleration due to gravity is approximately $\frac{1}{6}$ that at the surface of the earth, and on the surface of the sun it is approximately 29 times as great as at the surface of the earth. If a person on earth can jump with enough initial velocity to rise 5 ft, how high will the same initial velocity carry that person (a) on the moon? (b) on the sun?

13 Newton's law of gravitation implies that the acceleration due to gravity at the surface of a planet (or the moon or the sun) is directly proportional to the mass of the planet and inversely proportional to the square of the radius.

(a) If g_m denotes the acceleration due to gravity at the surface of the moon, use the fact that the moon has approximately $\frac{3}{11}$ the radius and $\frac{1}{81}$ the mass of the earth to show that g_m is approximately $g/6$.

(b) Use part (a) to show that the escape velocity for the moon is approximately 1.5 mi/s.

14 Show that the point between the earth and the moon where the two exert equal but opposite gravitational forces on a particle is $\frac{9}{10}$ of the way from the center of the earth to the center of the moon.

CHAPTER 5 REVIEW: CONCEPTS, METHODS

Define, state, or think through the following.

1 Differentials dx and dy.
2 Tangent line approximation.
3 Indefinite integral (or antiderivative).
4 Integrand and constant of integration.
5 Integration formulas.
6 Integration by substitution.
7 Differential equation.
8 Order of a differential equation.
9 General solution, particular solution.
10 Separation of variables.
11 Initial condition.
12 Rectilinear motion.
13 Velocity, speed, acceleration.
14 Newton's second law of motion.
15 Newton's law of gravitation.
16 Escape velocity.

ADDITIONAL PROBLEMS FOR CHAPTER 5

SECTION 5.3

Compute the following integrals. Be sure to include the constant of integration in each answer.

1 $\int (3x^4 - 7x^3 + 10)\,dx.$ 2 $\int \frac{dx}{\sqrt[3]{x^4}}.$

3 $\int \frac{x^3 - 3x^2 + x - 2\sqrt{x}}{x}\,dx.$

4 $\int \left(x + \frac{1}{x}\right)^2 dx.$

5 $\int x(x+1)^2\,dx.$ 6 $\int (x+3)(x^2-1)\,dx.$

7 $\int (51x^2 - 108x^3)\,dx.$ 8 $\int \frac{x^3+2}{x^2}\,dx.$

9 $\int (2-\sqrt{x})(3+\sqrt{x})\,dx.$ 10 $\int \sqrt{x}(7x^2 - 5x + 3)\,dx.$

11 $\int \sqrt{2-3x}\,dx.$ 12 $\int (3+7x^2)^9 5x\,dx.$

13 $\int (5x+2)^{164}\,dx.$ 14 $\int (3-4x)^{3/4}\,dx.$

15 $\int \frac{5x\,dx}{\sqrt{1+x^2}}.$ 16 $\int \sqrt{3x^2-2}\;x\,dx.$

17 $\int \frac{x^2\,dx}{\sqrt{2x^3-1}}.$ 18 $\int \frac{dx}{\sqrt[3]{(7x+3)^2}}.$

19 $\int \frac{(x-1)\,dx}{\sqrt[3]{x^2-2x+3}}.$ 20 $\int \frac{dx}{x\sqrt{3x}}.$

21 $\int \frac{x\,dx}{\sqrt[3]{(2-x^2)^2}}.$ 22 $\int \frac{x\,dx}{\sqrt{(x^2-4)^3}}.$

23 $\int \left(1+\frac{1}{x}\right)^2 \frac{dx}{x^2}.$ 24 $\int \frac{x^2\,dx}{(2+3x^3)^3}.$

25 $\int (x^2+2x+1)^{2/3}\,dx.$ 26 $\int x\sqrt[3]{1+x^2}\,dx.$

27 $\int x\sqrt[3]{1+x}\,dx.$ 28 $\int \frac{\sqrt{2x^6+x^4}}{x}\,dx.$

29 $\int (x^3+x+32)^{9/2}(3x^2+1)\,dx.$

30 $\int (x^2+1)^7 x^3\,dx.$

31 $\int (x^3-1)^{1/3} x^5\,dx.$

SECTION 5.4

32 Find the general solution of each of the following differential equations:

(a) $\frac{dy}{dx} = 2y^2(4x^3 + 4x^{-3});$

(b) $\frac{dy}{dx} = \sqrt{(x^2 - x^{-2})^2 + 4}.$

33 Find the indicated particular solution of each of the following differential equations:

(a) $\frac{dy}{dx} = \frac{x(1+y^2)^2}{y(1+x^2)^2},\quad y = 1$ when $x = 2;$

(b) $\frac{dy}{dx} = \sqrt{xy - 4x - y + 4},\quad y = 8$ when $x = 5.$

34 The equation $x^2 = 4py$ represents the family of all parabolas with vertex at the origin and axis the y-axis. Find the family of curves that intersect the curves of this given family at right angles. Hint: Show first that the slope of the tangent at every point (x, y) $(y \neq 0)$ on each curve of the given family is $2y/x$.

35 Solve Problem 34 if the given family is $xy = c$.

36 Find y as a function of x if $dy/dx + y/x = 0$.

37 Equation (8) in Section 5.4 can be written as

$$\frac{dy}{dx} = \frac{1 + y/x}{1 - y/x},$$

and this suggests the substitution $z = y/x$. Use this idea to replace y by z as the dependent variable, and show that the variables can be separated in the resulting differential equation. Notice that the necessary integrations are beyond our capacity at the present stage, so in spite of our progress we have reached a temporary dead end.

SECTION 5.5

38 A ball is thrown vertically upward with an initial velocity of 78 ft/s from the roof of a building 400 ft high. Find

the distance s from the ground up to the ball t seconds later. If the ball misses the building on the way down, how long does it take to hit the ground?

39 (a) A bullet is fired downward with a velocity of 400 ft/s from an airplane 20,000 ft above the ocean. Neglecting air resistance, how long does it take the bullet to reach the water, and what is its velocity at the moment of impact?
(b) If the bullet is merely dropped from the airplane, how long does it take to fall, and what is its velocity on impact?

40 Show that a rock thrown straight up from the ground takes just as long to rise to its highest point as it does to fall back to its initial position. How is the velocity with which it hits the ground related to its initial velocity? Answer the same question for its speed.

41 A ball is dropped out of a window 19.6 m above the ground. At the same time another ball is thrown straight down from a window 79.6 m above the ground. If both balls reach the ground at the same moment, find the initial velocity of the second ball.

42 An automobile is traveling in a straight line at a velocity of v_0 feet per second. The driver suddenly applies the brakes, and the car stops in T seconds after traveling S feet. If the brakes produce a constant negative acceleration of $-a_0$ ft/s^2, find formulas for T and S in terms of v_0 and a_0.

43 An astronaut stands on the edge of a cliff and drops a stone. She observes that it takes 4 seconds for the stone to fall to the ground at the bottom. On earth, this would mean that the cliff is 256 ft high. How high is the cliff (a) if the astronaut is on the moon, where the acceleration due to gravity is approximately 5.5 ft/s^2? (b) if she is on Jupiter, where the acceleration due to gravity is approximately 85 ft/s^2?

44 The results of Problem 13 in Section 5.5 are given in the second column of the following table:

	Earth	*Moon*	*Jupiter*	*Saturn*	*Sun*
Mass (earth = 1)	1	$\frac{1}{81}$	317	95	332,000
Radius (mi)	4000	1100	43,000	36,000	432,000
Acceleration of gravity	g	$g/6$	$2.6g$	$1.2g$	$29g$
Escape velocity (mi/s)	7	1.5	38	23	400

Verify the rough approximations given in the third and fourth rows for Jupiter, Saturn, and the sun.

45 If the sun could be crushed into a smaller sphere with the same mass, estimate what its new radius would have to be in order to increase the escape velocity at its surface to the speed of light (approximately 186,000 mi/s or 300,000 km/s). What would the new radius have to be in the case of the earth?

46 Estimate the escape velocity from the surface of a *white dwarf,* a type of star in which a mass about equal to that of the sun is compressed into a volume about equal to that of the earth.

47 Estimate the escape velocity from the surface of a *neutron star,* a type of star in which a mass about equal to two solar masses is compressed into a sphere of radius about 4 mi.

48 According to currently accepted ideas among astronomers, the universe came into existence about 15 billion years ago in an explosion called the *Big Bang.* Ever since that time the universe has been expanding in such a way that the velocity v of a galaxy at distance R from our galaxy (the Milky Way) is given by *Hubble's law* $v = HR$, where H is *Hubble's constant,* about 16 km/s per million light-years (a light-year is about 9.47×10^{12} km). No one knows whether this expansion of the universe will continue indefinitely. If the universe contains enough matter, then the gravitational forces exerted by this matter on itself will ultimately slow down and stop the expansion. Then there will be a period of contraction ending in a complete gravitational collapse called the *Big Crunch,* in which the universe as we know it—space, time, matter, energy—will cease to exist.
(a) Show that the universe will continue to expand forever if the present density of matter δ (mass per unit volume) is less than the *critical density* $\delta_c = 3H^2/8\pi G$.* Hint: What is the escape velocity at a distance R from our galaxy, due to the matter inside a sphere of radius R centered on us?
(b) Estimate the value of the critical density δ_c, given that the gravitational constant G is approximately 6.67×10^{-20} km^3/(kg·s^2).

49 Newton's second law of motion $F = ma = m\,dv/dt$ can be written in the form $F = d/dt\,(mv)$ in terms of the *momentum* mv of a particle of mass m and velocity v, and remains valid in this form even if m is not constant, as assumed so far. Suppose a spherical raindrop falls through air saturated with water vapor, and assume that by condensation the mass of the raindrop increases at a rate proportional to its surface area, with c the constant of proportionality. If the initial radius and velocity of the raindrop are both zero, show that the drag exerted by the condensation of the water vapor has the effect of making the raindrop fall with acceleration $\frac{1}{4}g$. Hint: Show that $d/dr\,(r^3v) = (\delta/c)\,r^3g$, where r is the radius of the raindrop and δ is its density.

*It would be interesting to know the ultimate fate of the universe, but unfortunately no one knows the present density of matter.

6 DEFINITE INTEGRALS

6.1 INTRODUCTION

At the beginning of Chapter 2 we described calculus as the study of methods for calculating two important quantities associated with curves, namely,

1 Slopes of tangent lines to curves, and
2 Areas of regions bounded by curves.

Of course, this description gives an oversimplified view of the subject, for it emphasizes calculus as a tool in the service of geometry but says nothing about its indispensable role in the study of science. Nevertheless, it explains the traditional division of calculus into two distinct parts: differential calculus, which deals with slopes of tangent lines, and integral calculus, which is concerned with areas.

The problem of areas was of great interest to the ancient Greeks. They knew a good deal about the areas of triangles, circles, and related configurations, but any other figure presented a new and usually insoluble problem. Archimedes was able to apply a technique called the *method of exhaustion* to calculate the area of a segment of a parabola, and also to calculate a few other particular geometric quantities. But for almost 2000 years this handful of calculations by Archimedes stood as the isolated achievement of a great genius, unmatchable by others. However, by the middle of the seventeenth century several European thinkers—most notably Fermat and Pascal—began to push the method of exhaustion beyond the point where Archimedes had left it. The decisive breakthrough was achieved a little later by Newton and Leibniz, who showed that if a quantity can be computed by exhaustion, then it can also be computed much more easily by using antiderivatives. This crucial discovery is called the *Fundamental Theorem of Calculus*. It binds together the two parts of the subject, and is undoubtedly (as we have said before) the most important single fact in the whole of mathematics.

This is the path we follow in the present chapter. Since calculations will seem to play a prominent part in our work, it is even more necessary than usual for students to keep firmly in mind that the underlying ideas are more important than the calculations.

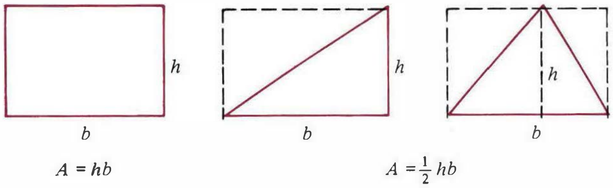


Figure 6.1

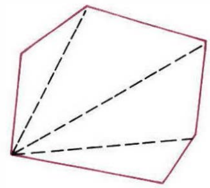


Figure 6.2

Figure 6.3

6.2 THE PROBLEM OF AREAS

Every rectangle and every triangle has a number associated with it called its *area*. The area of a rectangle is defined to be the product of its height and its base, and the area of a triangle is one-half the product of the height and the base (Fig. 6.1). Since a polygon can always be decomposed into triangles (Fig. 6.2), its area is the sum of the areas of these triangles.

The circle is a more difficult figure. The Greeks solved the problem of finding its area in a very natural way. First, they approximated this area by inscribing a square (Fig. 6.3). Then they improved the approximation step by step by doubling and redoubling the number of sides, that is, by inscribing a regular octagon, then a regular 16-gon, and so on. The areas of these inscribed polygons evidently approach the exact area of the circle more and more closely. This idea yields the familiar formula

$$A = \pi r^2 \tag{1}$$

for the area A of a circle in terms of its radius r. The details of the reasoning are as follows. Suppose that the circle has inscribed in it a regular polygon with a large number of sides (Fig. 6.4). Each of the small isosceles triangles shown in the figure has area $\frac{1}{2}hb$, and the sum of these areas equals the area of the polygon, which closely approximates the area of the circle. If p denotes the perimeter of the polygon, then we see that

$$\begin{aligned} A_{\text{polygon}} &= \tfrac{1}{2}hb + \tfrac{1}{2}hb + \cdots + \tfrac{1}{2}hb \\ &= \tfrac{1}{2}h(b + b + \cdots + b) = \tfrac{1}{2}hp. \end{aligned}$$

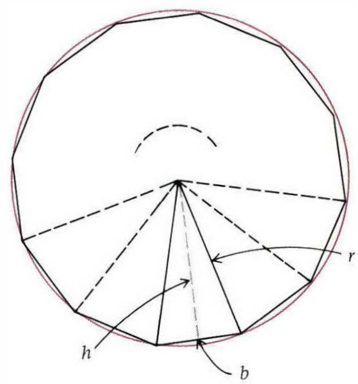


Figure 6.4

Now let c be the circumference of the circle, so that $c = 2\pi r$ by the definition of π.* Then, as the number of sides of the polygon increases, h approaches r (in symbols, $h \to r$), $p \to c$, and therefore

*That is, π is defined to be the ratio of the circumference to the diameter, so $\pi = c/2r$ and therefore $c = 2\pi r$.

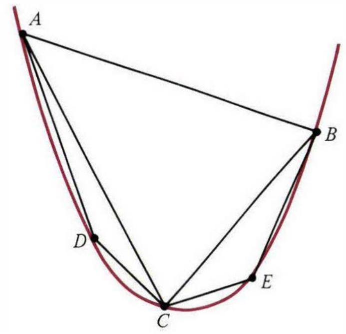


Figure 6.5

$$A_{\text{polygon}} = \tfrac{1}{2}hp \to \tfrac{1}{2}rc = \tfrac{1}{2}r(2\pi r) = \pi r^2,$$

which establishes (1). The phrase "method of exhaustion" is clearly a good description of this process, because the area of the circle is "exhausted" by the areas of the inscribed polygons.

We next examine the procedure by which Archimedes calculated the area of a parabolic segment, that is, the area of the part of the parabola in Fig. 6.5 bounded by the arbitrary chord AB and the arc $ADCEB$. There is no convenient way to inscribe regular polygons in this figure, so Archimedes used triangles instead. His first approximation was the triangle ABC, where the vertex C is chosen as that point where the tangent to the parabola is parallel to AB. His second approximation was obtained by adding to the triangle ABC the two triangles ACD and BCE, where the vertex D is the point where the tangent is parallel to AC and the vertex E is the point where the tangent is parallel to BC. To obtain his third approximation, he inscribed triangles in the same way in each of the four regions still not included (one such region is that between the arc CE and the chord CE), so his third approximation was the sum of the areas of the triangles ABC, ACD, BCE, and the four new triangles. By continuing to exhaust the parabolic segment in this way, he was able to show that its area is exactly four-thirds the area of the first triangle ABC. The details of his argument are a bit complicated; and since our interest here is mainly in the idea of the method of exhaustion, we omit these details.

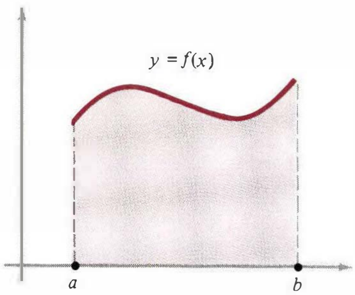


Figure 6.6

The general problem before us is that of finding the area of a region with a curved boundary. However, most of our work will be concentrated on a special case of this general area problem—namely, finding the area under the graph of a function $y = f(x)$ between two vertical lines $x = a$ and $x = b$, as shown in Fig. 6.6. Such a region has a boundary that is curved only along its upper edge, and is therefore much easier to work with. A knowledge of this special case is often enough to enable us to cope with more complicated regions. To understand how this is possible, notice in Fig. 6.7 that the area of a region whose entire boundary is curved can often be obtained by subtracting the area under its lower edge from the area under its upper edge, where each of the latter areas is of the special type shown in Fig. 6.6.

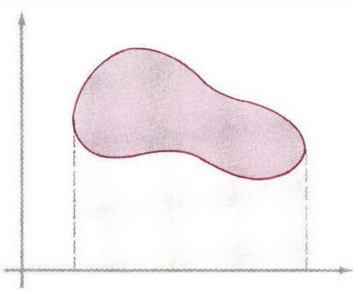


Figure 6.7

In Section 6.4 and thereafter, we will denote an area of the type shown in Fig. 6.6 by the standard symbol

$$\int_a^b f(x)\,dx, \tag{2}$$

which is read "the definite integral from a to b of $f(x)\,dx$." The reason for this notation will become clear in Section 6.4. For the present, however, we warn students in advance not to confuse the definite integral (2) with the indefinite integral (or antiderivative)

$$\int f(x)\,dx \tag{3}$$

introduced in Chapter 5. In spite of the fact that these two integrals have the same family name and look very much alike, they are totally different entities: The definite integral (2) is a number, and the indefinite integral (3) is a function (or a collection of functions).

At first sight it might appear that the problem of calculating areas is a matter of geometry and nothing more—interesting to mathematicians, perhaps, but with no practical uses in the real world outside of mathematics. This is not the case at all. It will become clear in the next chapter that many important concepts and problems in physics and engineering depend on exactly the same kinds of ideas as those used in calculating areas. As examples we mention the concepts of work and energy in physics, and also the engineering problem of finding the total force acting against the face of a dam due to the pressure of the water in a reservoir. Finding areas is therefore much more than merely a game mathematicians play for their own diversion. Nevertheless, for the sake of clarity we confine our attention in this chapter to the area problem itself, and in Chapter 7 we begin to sample the immense range of applications of the underlying idea.

Remark 1 As a matter of historical interest, it appears that the first person to find the exact area of a figure bounded by curves was Hippocrates of Chios, the most famous Greek mathematician of the fifth century B.C. To understand what he did, consider the circle shown in Fig. 6.8, with the points A, B, C, D at the ends of the horizontal and vertical diameters. Using C as a center, describe the circular arc AEB connecting A and B. The crescent-shaped figure bounded by the arcs ADB and AEB is called a *lune of Hippocrates* (*luna* is Latin for "moon"), after the man who made the remarkable discovery that its area is exactly equal to the area of the shaded square whose side is the radius of the circle. Thus Hippocrates "squared the lune," even though he was unable to square the circle itself.*

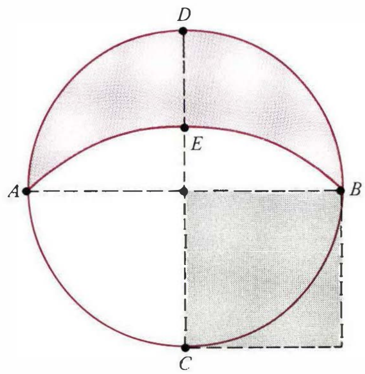


Figure 6.8 Squaring the lune.

Remark 2 Most of us remember from school that the numerical value of π is approximately 3.14, and some of us even remember a more accurate approximation, $\pi \cong 3.14159$. Also, in one of his treatises Archimedes derived his famous inequality

$$3\tfrac{10}{71} < \pi < 3\tfrac{1}{7} = \tfrac{22}{7},$$

which is the basis for the rough but widely used approximation $\pi \cong \frac{22}{7}$. Where do these values come from?

The number π was defined above as the ratio of the circumference of a circle to its diameter. As we saw, this yields the formula $A = \pi r^2$, which tells us that π is also the area of the unit circle (circle with unit radius) $x^2 + y^2 = 1$. The problem of computing π therefore amounts to the problem of finding the area of the unit circle.

To accomplish this, let p_n and P_n be n-sided regular polygons, with p_n inscribed in the unit circle and P_n circumscribed around it, as shown in Fig. 6.9. To find the areas of these polygons, it suffices to find the areas of the isosceles triangles making up p_n and P_n and then to multiply by n. If θ is half the vertex angle, then θ is clearly the same for both isosceles triangles; and using degree measure, we have

$$\theta = \frac{1}{2} \cdot \frac{360°}{n} = \frac{180°}{n}.$$

*Hippocrates' exceedingly beautiful (but easy to understand) proof is given in the Appendix at the end of this chapter.

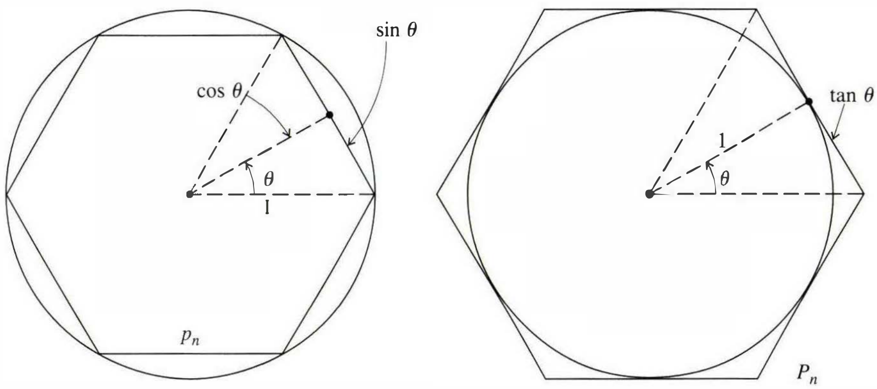


Figure 6.9

By examining the figure we see that the area of p_n is

$$A(p_n) = n \cdot 2 \cdot \tfrac{1}{2} \sin \theta \cos \theta = \frac{n}{2} \sin 2\theta = \frac{n}{2} \sin \frac{360°}{n} \tag{4}$$

and the area of P_n is

$$A(P_n) = n \cdot 2 \cdot \tfrac{1}{2} \tan \theta = n \tan \frac{180°}{n}. \tag{5}$$

By substituting convenient values of n in formulas (4) and (5), and using a calculator, we easily fill in the values shown in the adjoining table. Because

$$A(p_n) \leq \pi \leq A(P_n)$$

for all n, it is clear that $\pi \cong 3.14159$, correct to five decimal places. In Section 14.4 we describe other methods that have made it possible to compute π to more than 500,000 decimal places.

n	$A(p_n)$	$A(P_n)$
6	2.598076	3.464102
12	3.000000	3.215390
24	3.105829	3.159660
48	3.132629	3.146086
96	3.139350	3.142715
200	3.141076	3.141851
400	3.141463	3.141657
800	3.141560	3.141609
1600	3.141585	3.141597
3200	3.141591	3.141594
6400	3.141592	3.141593

6.3 THE SIGMA NOTATION AND CERTAIN SPECIAL SUMS

In order to clarify our discussion of definite integrals in the next section, we introduce here a standard mathematical notation used for abbreviating long sums. This is called the *sigma notation,* because it uses the Greek letter Σ (sigma). In the Greek alphabet the letter Σ corresponds to our letter S, which is the first letter of the word "sum." This helps us to remember the purpose of the sigma notation, which is to suggest the idea of summation or addition.

Thus, if $a_1, a_2, \ldots, a_n$ are any given numbers, their sum is denoted by

$$\sum_{k=1}^{n} a_k. \tag{1}$$

This symbol is read "the sum from $k = 1$ to n of a_k." The idea compressed in (1) is that we are to write down each of the numbers a_k as the subscript k varies from 1 to n (namely, $a_1, a_2, \ldots, a_n$) and then add all these numbers together:

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n.$$

We write $k = 1$ below the Σ in (1), and n above it, to say that the sum starts with the term a_k with k replaced by 1, and stops with the term a_k with k replaced by n. The letter k used as the subscript here is called the *index of summation.* Any other letter (i or j, for instance) would do just as well. Thus,

$$\sum_{k=1}^{5} k^3, \qquad \sum_{i=1}^{5} i^3, \qquad \text{and} \qquad \sum_{j=1}^{5} j^3$$

all represent the same sum, namely, $1^3 + 2^3 + 3^3 + 4^3 + 5^3 = 225$.

We give a few additional specific examples of the sigma notation:

$$\sum_{k=1}^{3} \frac{k}{k^2+1} = \frac{1}{1^2+1} + \frac{2}{2^2+1} + \frac{3}{3^2+1};$$

$$\sum_{k=1}^{4} (-1)^{k+1} \frac{1}{k^2} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2};$$

$$\sum_{k=1}^{n} k = 1 + 2 + \cdots + n;$$

$$\sum_{k=1}^{n} 2k = 2 + 4 + \cdots + 2n;$$

$$\sum_{k=1}^{n} (2k-1) = 1 + 3 + \cdots + (2n-1).$$

Notice particularly in the second sum the use of the factor $(-1)^{k+1}$ to produce the alternating signs $+, -, +, -$ as the index of summation k takes the values 1, 2, 3, 4. The last three of these sums are evidently the sum of the first n positive integers, the sum of the first n even numbers, and the sum of the first n odd numbers.

The following are some formulas from elementary algebra that will be needed in the next section:

$$\sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}; \tag{2}$$

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}; \tag{3}$$

$$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2. \tag{4}$$

These formulas can be proved by the method of mathematical induction. However, an easier way to establish (2) is to write the sum once in the natural order, as shown, and then again in reverse order,

$$s = 1 + \quad 2 \quad + \cdots + n,$$

$$s = n + (n-1) + \cdots + 1.$$

By adding these equations and noticing that each column on the right adds up to $n + 1$ and there are n columns, we get $2s = n(n + 1)$, from which (2) follows at once.

There is yet another way of proving (2) that is worth knowing about because it can easily be adapted to yield (3) and (4) as well, and further formulas of the same type. It depends on the simple fact that $(k + 1)^2 = k^2 + 2k + 1$, or equivalently

$$(k+1)^2 - k^2 = 2k + 1. \tag{5}$$

If we let $k = 1, 2, 3, \ldots, n$ in (5), and write the resulting equations one below the other, we obtain

$$2^2 - 1^2 = 2 \cdot 1 + 1,$$

$$3^2 - 2^2 = 2 \cdot 2 + 1,$$

$$4^2 - 3^2 = 2 \cdot 3 + 1,$$
$$\cdots$$
$$(n+1)^2 - n^2 = 2 \cdot n + 1.$$

On the left here we have 2^2 and -2^2, 3^2 and -3^2, . . . , n^2 and $-n^2$. Therefore, when these equations are added with due attention to the cancellations on the left, the result is

$$(n+1)^2 - 1^2 = 2\left(\sum_{k=1}^{n} k\right) + n;$$

and solving for the sum in parentheses yields (2):

$$\sum_{k=1}^{n} k = \tfrac{1}{2}[(n+1)^2 - 1^2 - n] = \tfrac{1}{2}[n^2 + n]$$
$$= \frac{n(n+1)}{2}.$$

PROBLEMS

1 Find the numerical value of

(a) $\sum_{i=1}^{5} i^2$; (b) $\sum_{j=1}^{5} 2^j$; (c) $\sum_{k=50}^{53} k$;

(d) $\sum_{k=1}^{8} (-1)^k$; (e) $\sum_{i=1}^{500} (-1)^i$; (f) $\sum_{j=1}^{300} 5$;

(g) $\sum_{k=0}^{6} \cos 2\pi k$.

Hint: Write the sums out, and examine them carefully.

2 Use the sigma notation to write the following sums compactly:
(a) $3 + 9 + 27 + 81$;
(b) $3 - 5 + 7 - 9 + 11 - 13$;
(c) $\frac{1}{5} + \frac{1}{10} + \frac{1}{15} + \cdots + \frac{1}{45}$;
(d) $1 + 2 + 2^2 + \cdots + 2^{200}$;
(e) $a^5 + a^6 + a^7 + \cdots + a^{10}$;
(f) $\frac{1}{4} - \frac{1}{6} + \frac{1}{8} - \frac{1}{10} + \frac{1}{12} - \frac{1}{14}$;
(g) $1 + 2^2 + 3^3 + 256$.

3 Prove formula (3) by using the expansion $(k+1)^3 = k^3 + 3k^2 + 3k + 1$ and the method suggested in the text.

4 Prove formula (4) similarly, by using the expansion $(k+1)^4 = k^4 + 4k^3 + 6k^2 + 4k + 1$.

5 Use (2), (3), and (4) to find closed formulas for the sum of the first $n - 1$ (instead of the first n) integers, squares, and cubes*:
(a) $1 + 2 + \cdots + (n-1) = ?$
(b) $1^2 + 2^2 + \cdots + (n-1)^2 = ?$
(c) $1^3 + 2^3 + \cdots + (n-1)^3 = ?$

*The indicated sums in (2), (3), and (4) are called *open* because the three-dot notation is used to suggest many terms that are present but not written. In contrast to this, the formulas on the right sides of these equations are called *closed*.

6 Use the method suggested in the text to discover and prove closed formulas for (a) $1^4 + 2^4 + \cdots + n^4$; (b) $1^5 + 2^5 + \cdots + n^5$.

7 By doing a little arithmetic we see that

$$1^3 + 2^3 = (1+2)^2,$$
$$1^3 + 2^3 + 3^3 = (1+2+3)^2,$$

and

$$1^3 + 2^3 + 3^3 + 4^3 = (1+2+3+4)^2.$$

Show that

$$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$$

for every positive integer n.

8 There is a wonderful geometric proof of formula (4) that was known to the Arab mathematicians more than a thousand years ago. It depends on the square shown in Fig. 6.10, which is constructed as follows. Beginning at the point O, lay off successive segments of lengths 1, 2, 3, etc., and finally one of length n extending up to the point A. Do the same on a line OB perpendicular to OA, so that

$$OA = OB = 1 + 2 + \cdots + n$$
$$= \frac{n(n+1)}{2}.\dagger$$

†This formula for the sum of the first n positive integers is proved and used in the writings of Archimedes, and was presumably discovered by him. This formula was therefore known to the Arab mathematicians of the Middle Ages, who translated, honored, and preserved the works of Archimedes during those dark centuries when most Europeans could not read or write and knew nothing of mathematics, and those few who could read and write lived in monasteries and were submerged in piety.

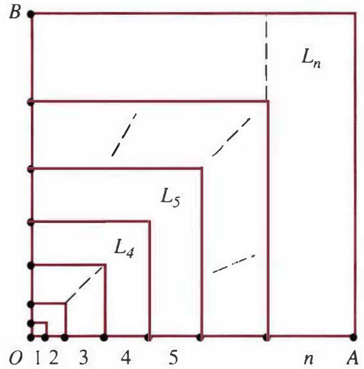


Figure 6.10

The area S of the square is therefore given by the formula

$$S = \left[\frac{n(n+1)}{2}\right]^2.$$

However, the square is the sum of n L-shaped regions, as indicated:

$$S = L_1 + L_2 + \cdots + L_n.$$

Use the fact that L_n can be split into the two rectangles in the figure to show that

$$L_n = n^3,$$

so that

$$S = 1^3 + 2^3 + \cdots + n^3,$$

and thereby complete this proof of formula (4).

The results of the next two problems will be needed in Section 6.5.

9 (a) Use the product formula

$$\sin m\theta \cos n\theta = \tfrac{1}{2}[\sin (m+n)\theta + \sin (m-n)\theta]$$

(Additional Problem 63 in Chapter 1) to show that

$$2 \sin \tfrac{1}{2}x \cos kx = \sin (k + \tfrac{1}{2})x - \sin (k - \tfrac{1}{2})x.$$

(b) By adding the identities in part (a) for $k = 1, 2, \ldots, n$ and exploiting cancellations as in the text, establish the formula

$$\sum_{k=1}^{n} \cos kx = \frac{\sin (n + \frac{1}{2})x - \sin \frac{1}{2}x}{2 \sin \frac{1}{2}x},$$

where x is not an integer multiple of 2π.

(c) Use the product formula in (a) again to write the sum in (b) in the form

$$\sum_{k=1}^{n} \cos kx = \frac{\sin \frac{1}{2}nx \cos \frac{1}{2}(n+1)x}{\sin \frac{1}{2}x}.$$

10 Use the method of Problem 9 to establish the corresponding formula

$$\sum_{k=1}^{n} \sin kx = \frac{\sin \frac{1}{2}nx \sin \frac{1}{2}(n+1)x}{\sin \frac{1}{2}x},$$

where x is not an integer multiple of 2π.

6.4 THE AREA UNDER A CURVE. DEFINITE INTEGRALS. RIEMANN

We begin by restating the problem we are trying to solve. Let $y = f(x)$ be a given nonnegative function defined on a closed interval $a \le x \le b$, as shown in Fig. 6.11. How do we calculate the area of the shaded region in the figure, that is, the area of the region under the graph, above the x-axis, and between the vertical lines $x = a$ and $x = b$?

Closed intervals like the one mentioned here will occur quite often in our discussion, so we use the briefer notation $[a, b]$. Also, most of the functions we study will be continuous. The reader will recall that this means the following: From the intuitive point of view, the graph consists of a single piece, with no gaps or holes; and more precisely, for each point c in $[a, b]$ we must have

$$\lim_{x \to c} f(x) = f(c).$$

Such a function has several basic properties that we wish to recognize explicitly: It is bounded, in the sense that there exists a constant K such that $|f(x)| \le K$ for all x in $[a, b]$; and it assumes maximum and minimum values, in the sense that the graph has a highest point and a lowest point.*

We return to Fig. 6.11, with the specific assumption that the function $y = f(x)$ is continuous on $[a, b]$. How do we find the area of the shaded region? If we take

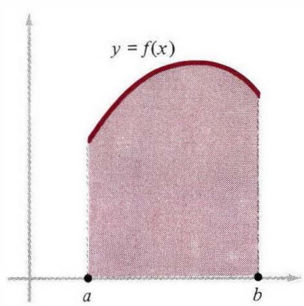


Figure 6.11

*See the discussion of the Extreme Value Theorem in Section 2.6.

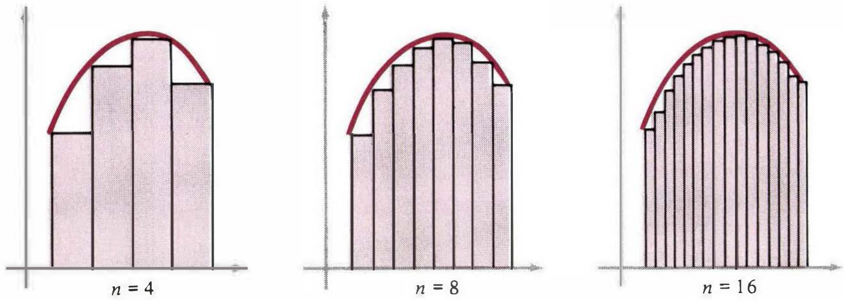


Figure 6.12 Approximating the area.

the nature of this region into account—that is, the fact that only the upper edge is curved—then the method of exhaustion suggests the following approximation procedure using thin rectangles.

Let n be a positive integer and divide the interval $[a, b]$ into n equal subintervals. Using each subinterval as a base, construct the tallest rectangle that lies entirely under the graph. Write down the sum s_n of the areas of all these thin rectangles. This sum approximates the area under the graph, and the approximation is improved by taking larger values of n, or equivalently, by dividing $[a, b]$ into a larger number of smaller subintervals. Finally, calculate the exact area under the graph by finding the limiting value approached by the approximating sums s_n as n approaches infinity:

$$\text{area of region} = \lim_{n\to\infty} s_n. \tag{1}$$

The effect of this procedure is suggested in Fig. 6.12, showing a larger and larger number of thinner and thinner rectangles.

We now describe this idea with greater precision by introducing some suitable notation.

Again, let n be a positive integer and divide the interval $[a, b]$ into n equal subintervals by inserting $n - 1$ equally spaced points of division $x_1, x_2, \ldots, x_{n-1}$ between a and b. If we denote a by x_0 and b by x_n, then the endpoints of these subintervals are

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b, \tag{2}$$

and the subintervals themselves are

$$[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n], \tag{3}$$

as shown in Fig. 6.13. We denote the length of the kth subinterval by Δx_k, so

$$\Delta x_k = x_k - x_{k-1}. \tag{4}$$

Since the subintervals are equal in length, it is clear that $\Delta x_k = (b - a)/n$. Let m_k denote the minimum value of $f(x)$ on the kth subinterval $[x_{k-1}, x_k]$. Then this minimum value is assumed at some point $\bar{x}_k$ in the subinterval:

$$f(\bar{x}_k) = m_k, \qquad x_{k-1} \le \bar{x}_k \le x_k.$$

For the particular curve shown in Fig. 6.13, $\bar{x}_k$ is easily seen to be the left endpoint of the subinterval when the curve is rising and the right endpoint when it is falling. Since the area of each inscribed rectangle is the product of its height and its base, the approximating sum s_n of the areas of all these rectangles is clearly

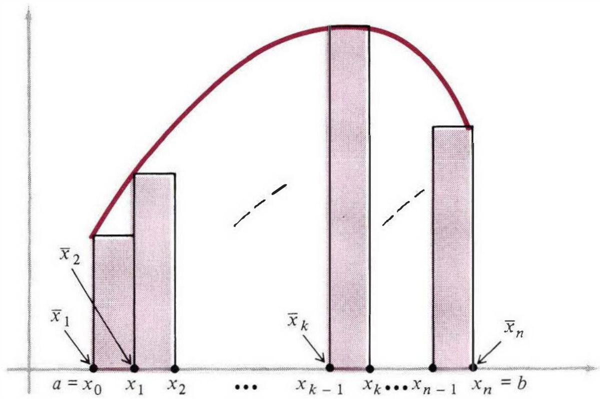


Figure 6.13 Using lower sums.

$$s_n = f(\bar{x}_1)\,\Delta x_1 + f(\bar{x}_2)\,\Delta x_2 + \cdots + f(\bar{x}_k)\,\Delta x_k + \cdots + f(\bar{x}_n)\,\Delta x_n.$$

If we use the sigma notation to abbreviate this sum, we get

$$s_n = \sum_{k=1}^{n} f(\bar{x}_k)\,\Delta x_k, \tag{5}$$

and (1) becomes

$$\text{area of region} = \lim_{n\to\infty} \sum_{k=1}^{n} f(\bar{x}_k)\,\Delta x_k. \tag{6}$$

This formula is all right as far as it goes, but from several points of view it is inconvenient and unduly restrictive. We broaden its scope and deepen its meaning in a series of remarks.

Remark 1 It is not necessary that the subintervals (3) must be equal in length. In fact, the underlying theory is greatly simplified if this restriction is removed. We therefore allow the subintervals (3) to be *equal or unequal* in length, so that the increments (4) may be different from one another. In formula (6), it is now no longer enough to require that n approaches infinity; we must also require that the length of the longest subinterval approaches zero. Since the latter condition includes the former, we replace (6) by

$$\text{area of region} = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(\bar{x}_k)\,\Delta x_k, \tag{7}$$

where $\max \Delta x_k$ denotes the length of the longest subinterval.

Remark 2 The sum (5) is called a *lower sum* because it uses inscribed rectangles and approximates the area of the region from below. We can also approximate the area from above, as follows. Roughly speaking, we now use each subinterval as a base, as before, but this time we construct the shortest rectangle whose top lies entirely above the curve.

To express this in symbols, let M_k denote the maximum value of $f(x)$ on the kth subinterval $[x_{k-1}, x_k]$. As before, this maximum value is assumed at some point $\bar{\bar{x}}_k$ in the subinterval:

$$f(\bar{\bar{x}}_k) = M_k, \qquad x_{k-1} \le \bar{\bar{x}}_k \le x_k.$$

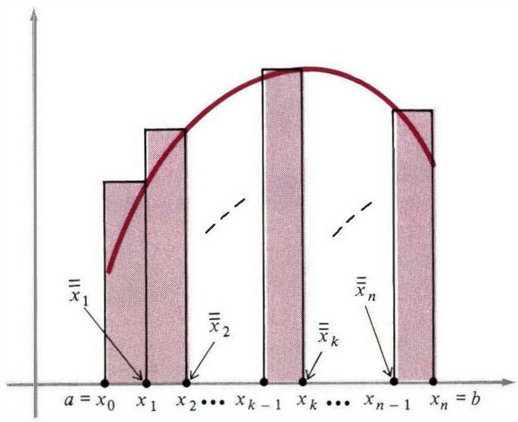


Figure 6.14 Using upper sums.

The sum of the areas of the circumscribed rectangles is therefore

$$S_n = \sum_{k=1}^{n} f(\bar{\bar{x}}_k)\,\Delta x_k. \tag{8}$$

This is called an *upper sum* because it approximates the area of the region from above, as shown in Fig. 6.14. Geometric intuition tells us that the area of our region can just as well be obtained as the limit of upper sums, so we have

$$\text{area of region} = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(\bar{\bar{x}}_k)\,\Delta x_k. \tag{9}$$

However, entirely apart from intuition—which is sometimes misleading—it can be rigorously proved as a theorem of pure mathematics that the limits in (7) and (9) both exist and have the same value for any continuous function.†

Further, if x_k^* is taken to be *any* point in the kth subinterval $[x_{k-1}, x_k]$, then we clearly have

$$s_n \le \sum_{k=1}^{n} f(x_k^*)\,\Delta x_k \le S_n.$$

It therefore follows from the theorem just stated that both (7) and (9) can be replaced by the formula

$$\text{area of region} = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\,\Delta x_k, \tag{10}$$

where the only restriction placed on x_k^* is that $x_{k-1} \le x_k^* \le x_k$.

Remark 3 The limit in (10)—or in (7) or (9)—is symbolized by the standard Leibniz notation

$$\int_a^b f(x)\,dx, \tag{11}$$

†The details of this proof are not appropriate for an introductory course in calculus, where most students already have quite enough to think about. However, for the sake of the few exceptionally skeptical and tenacious students who might be interested in pursuing the matter, these details are given in Appendix A.5.

which is read (as we said in Section 6.2) "the *definite integral* from a to b of $f(x)\,dx$." If we write down the definition of (11),

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\,\Delta x_k, \tag{12}$$

then every part of the symbol on the left side is intended to remind us of the corresponding part of the approximating sum on the right side. The *integral sign* $\int$ is an elongated letter S, as in "sum," chosen because of the similarity between a definite integral and a sum of small quantities; the passage to the limit in (12) is suggested by replacing the letter Σ by the symbol $\int$. Also, the usual symbol Δ for an increment is replaced by the letter d to remind us of this limit operation, just as in the Leibniz notation dy/dx for the derivative. Thus, with the passage to the limit in (12),

$$\begin{array}{ccc} \displaystyle\sum_{k=1}^{n} & \text{becomes} & \displaystyle\int_a^b, \\ f(x_k^*) & \text{becomes} & f(x), \\ \Delta x_k & \text{becomes} & dx. \end{array}$$

The numbers a and b attached to the integral sign are called the *lower* and *upper limits of integration.*† Limits of integration are always present in a definite integral, and help distinguish it from the similar-appearing but very different indefinite integral

$$\int f(x)\,dx.$$

The function $f(x)$ in (11) is called the *integrand*—the thing being integrated—and the variable x is the *variable of integration.* The role of the dx as an important intuitive component of definite integrals will become much clearer in the next chapter.

Remark 4 In our discussion so far, we have adopted the naive but reasonable attitude that the area of the region under the graph clearly exists, and that all we have to do is devise a method for computing it. However, the following example shows that the situation is more complicated than this.

Consider the function $f(x)$ defined on [0, 1] by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational,} \\ 1 & \text{if } x \text{ is irrational.} \end{cases}$$

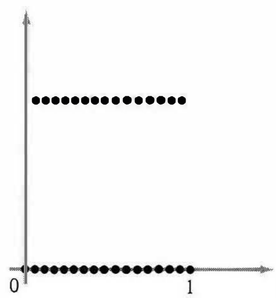
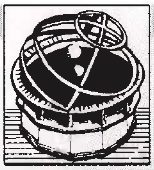


Figure 6.15

The graph is suggested in Fig. 6.15, and the very discontinuous nature of this function is shown by the fact that at least one irrational number lies between every pair of rationals and at least one rational number lies between every pair of irrationals. What is the area of the region under this graph? It is quite easy to see that every lower sum is 0 and every upper sum is 1, so the area calculated by (7) is 0 and the area calculated by (9) is 1. Also, the limit on the right of (12)

†Here the word "limit" has nothing to do with the limit concepts that are the basis of calculus. It is used in its loose, everyday sense, meaning "border" or "boundary." The limits of integration tell us where the integration begins and where it ends; they specify the left and right endpoints of the interval over which the integration is carried out, and remind us of $k = 1$ and $k = n$ in the approximating sum.



NOTE ON RIEMANN

No great mind of the past has exerted a deeper influence on the mathematics of the modern era than Bernhard Riemann (1826–1866), the son of a poor country minister in northern Germany. This influence was especially profound in geometry, number theory, and the theory of functions.

From 1846 to 1851 he studied at the universities of Berlin and Göttingen. In 1851 he obtained his doctorate at Göttingen under Carl Friedrich Gauss—at that time universally considered to be the greatest mathematician in the world—with his celebrated dissertation, "Foundations for a General Theory of Functions of a Complex Variable." In this path-breaking work he set forth an approach that is now familiar to all students of this subject, based on general principles and geometric ideas rather than formulas and calculations.

In number theory his only published work was a brief but exceedingly profound paper of less than 10 pages devoted to the distribution of the prime numbers. This mighty effort started tidal waves in several branches of pure mathematics, and its influence will probably still be felt a thousand years from now. Every advanced course on number theory given at any university in the world today is saturated with Riemann's ideas.

In 1854 he was appointed *Privatdozent* (unpaid lecturer) at Göttingen, which at that time was the necessary first step on the academic ladder. Before he could be appointed, however, he was required to present a trial lecture to the faculty. It was the custom for the candidate to offer three titles, and the head of his department usually accepted the first. Riemann rashly listed as his third topic the foundations of geometry, a subject on which he was unprepared but which Gauss had been turning over in his mind for 60 years. Naturally, Gauss was curious to see how this particular candidate would cope with such a challenge, and to Riemann's dismay he designated this as the subject of the lecture. Riemann quickly tore himself away from his other interests at the time—"my investigations of the connection between electricity, magnetism, light, and gravitation"—and wrote his lecture in the next 2 months. The result was one of the great classical masterpieces of mathematics, and probably the most important scientific lecture ever given. In it he greatly extended Gauss's own investigations of 30 years earlier and created a vast generalization of all known geometries which is now known as *Riemannian Geometry*. Some 50 years later these ideas turned out to be indispensable tools for Albert Einstein in his creation of the General Theory of Relativity. It is recorded that even Gauss—who had seen everything, mathematically speaking, had thought of most of it himself, and was almost impossible to impress—was surprised and enthusiastic.

We have merely scratched the surface of Riemann's great influence on the history of mathematics. In view of this it is quite surprising that his collected works fill only one average-sized volume. However, his writings were brief and powerful and pregnant with meaning for future generations, not routine publications of routine research that is dead the moment the printer's ink dries.

His short life was plagued by ill health aggravated by the abominable climate of northern Germany. He died of tuberculosis in Italy at the age of only 39, trying to escape the cold and rain and ice of Göttingen. His gravestone was incorporated into the wall of a village cemetery in Italy, and in 1906 his remains could no longer be found. However, he lives almost everywhere in modern mathematics.

does not exist. Does the concept of area have any meaning in a situation like this?

This bizarre example suggests the following indirect but more logical approach to the problem of area. If we are given a bounded nonnegative function $f(x)$ defined but not necessarily continuous on $[a, b]$, we begin by examining the limit on the right of (12). If this limit exists, then we define its value to be the *area* of the region under the graph, and we say that the function $f(x)$ is *integrable* on $[a, b]$. And if this limit does not exist, then it is meaningless to speak of the area of the region. Almost all the functions we encounter in practice are continuous, and the theorem stated in Remark 2 guarantees that every continuous function is integrable, so these logical fine points will have little practical significance for most of our work. Nevertheless, these issues are interesting and important from the point of view of the theory of calculus, and students should be aware of them even though we choose not to emphasize them.

The definite integral which is defined here is often called the *Riemann integral,* in honor of the nineteenth-century German mathematician who was the first to give a careful discussion of integrals of discontinuous functions. Also, the sums on the right of (12) are often called *Riemann sums.*

6.5 THE COMPUTATION OF AREAS AS LIMITS

The concepts discussed in Section 6.4 suggest an actual procedure for calculating areas. We now examine how this procedure works in a few specific cases.

Example 1 Consider the function $y = f(x) = x$ on the interval $[0, b]$. The region under this graph (Fig. 6.16) is a triangle with height b and base b, so its area is obviously $b^2/2$. However, it is of some interest to verify that our limit process gives the same result, but more important, to understand *how* the limit process gives this result.

Let n be a large positive integer and divide the interval $[0, b]$ into n equal subintervals by means of $n - 1$ equally spaced points

$$x_1 = \frac{b}{n}, \qquad x_2 = \frac{2b}{n}, \qquad \ldots, \qquad x_{n-1} = \frac{(n-1)b}{n}. \tag{1}$$

The bases of the rectangles are $\Delta x_k = b/n$, and if we use upper sums as shown in Fig. 6.16, then the heights of the rectangles are

$$f(x_1) = \frac{b}{n}, \qquad f(x_2) = \frac{2b}{n}, \qquad \ldots, \qquad f(x_n) = \frac{nb}{n},$$

and we have

$$\begin{aligned} S_n &= \left(\frac{b}{n}\right)\left(\frac{b}{n}\right) + \left(\frac{2b}{n}\right)\left(\frac{b}{n}\right) + \cdots + \left(\frac{nb}{n}\right)\left(\frac{b}{n}\right) \\ &= \frac{b^2}{n^2}(1 + 2 + \cdots + n). \end{aligned}$$

By using formula (2) in Section 6.3, we can write this as

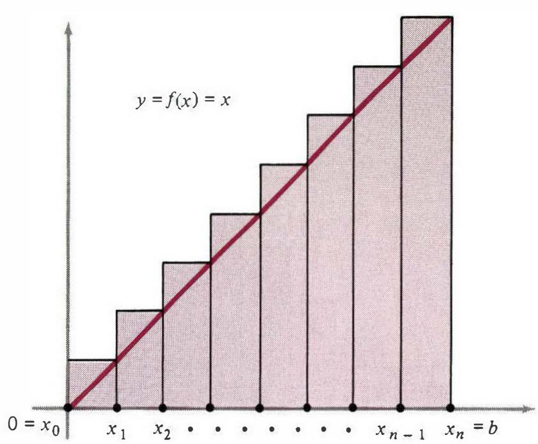


Figure 6.16

$$S_n = \frac{b^2}{n^2} \cdot \frac{n(n+1)}{2} = \frac{b^2}{2} \cdot \frac{n}{n} \cdot \frac{n+1}{n} = \frac{b^2}{2}\left(1 + \frac{1}{n}\right).$$

We therefore conclude that

$$\text{area of region} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{b^2}{2}\left(1 + \frac{1}{n}\right) = \frac{b^2}{2},$$

which we knew at the beginning. In the notation of definite integrals, this result is

$$\int_0^b x\,dx = \frac{b^2}{2}. \tag{2}$$

In this example we chose to use equal subintervals and upper sums. There was no compulsion to make these choices; our motive was only to make the calculations as easy as possible.

Example 2 Now consider the function $y = f(x) = x^2$ on the interval $[0, b]$, as shown in Fig. 6.17. Let n be a large positive integer and again divide the interval $[0, b]$ into n equal subintervals of length $\Delta x_k = b/n$ by using the points of division (1). We again use upper sums S_n, so the heights of the successive rectangles are easily seen to be

$$f(x_1) = \left(\frac{b}{n}\right)^2, \qquad f(x_2) = \left(\frac{2b}{n}\right)^2, \qquad \ldots, \qquad f(x_n) = \left(\frac{nb}{n}\right)^2,$$

and we have

$$\begin{aligned} S_n &= \left(\frac{b}{n}\right)^2\left(\frac{b}{n}\right) + \left(\frac{2b}{n}\right)^2\left(\frac{b}{n}\right) + \cdots + \left(\frac{nb}{n}\right)^2\left(\frac{b}{n}\right) \\ &= \frac{b^3}{n^3}(1^2 + 2^2 + \cdots + n^2). \end{aligned}$$

This time we use formula (3) in Section 6.3 to write

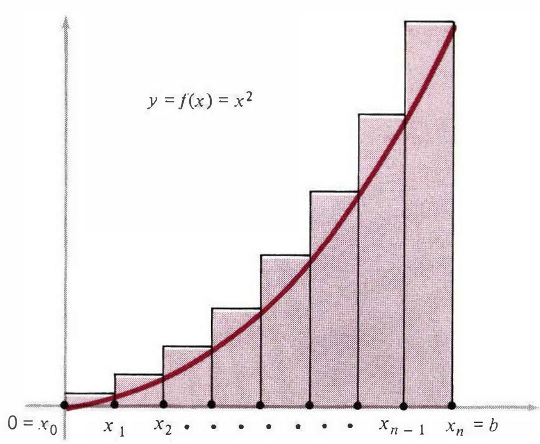


Figure 6.17

$$S_n = \frac{b^3}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{b^3}{6} \cdot \frac{n}{n} \cdot \frac{n+1}{n} \cdot \frac{2n+1}{n}$$
$$= \frac{b^3}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right).$$

As $n \to \infty$ this clearly yields

$$\text{area of region} = \lim_{n\to\infty} S_n = \frac{b^3}{3},$$

or equivalently,

$$\int_0^b x^2\, dx = \frac{b^3}{3}. \tag{3}$$

This calculation produces a result which we did *not* know at the beginning.

In Problem 1 we ask students to show in the same way that

$$\int_0^b x^3\, dx = \frac{b^4}{4}. \tag{4}$$

It is natural to conjecture from (2), (3), and (4) that the formula

$$\int_0^b x^n\, dx = \frac{b^{n+1}}{n+1} \tag{5}$$

is probably true for *all* positive integers $n = 1, 2, 3, \ldots$. The validity of (5) was established for the cases $n = 3, 4, \ldots, 9$ by the Italian mathematician Cavalieri in 1635 and 1647, but his laborious geometric methods bogged down at $n = 10$. A few years later Fermat discovered a beautiful argument that proves (5) at one stroke for all positive integers. This argument is somewhat aside from our main purpose here; it can be found in Section B.5 of the book *Calculus Gems* mentioned earlier.

Example 3 Next, we find the area under the cosine curve $y = \cos x$ from $x = 0$ to $x = b$, where $0 < b \le \pi/2$ (Fig. 6.18). Again we let n be a large positive integer and divide the interval $[0, b]$ into n equal subintervals of length $\Delta x_k = b/n$ by using the points of division (1). This time we use lower sums s_n, and since the function is decreasing, the points $\bar{x}_k$ are the right endpoints of the subintervals. The heights of the successive rectangles are therefore

$$\cos\frac{b}{n}, \qquad \cos\frac{2b}{n}, \qquad \ldots, \qquad \cos\frac{nb}{n},$$

and we have

$$s_n = \left(\cos\frac{b}{n}\right)\left(\frac{b}{n}\right) + \left(\cos\frac{2b}{n}\right)\left(\frac{b}{n}\right) + \cdots + \left(\cos\frac{nb}{n}\right)\left(\frac{b}{n}\right) = \frac{b}{n}\sum_{k=1}^{n}\cos\frac{kb}{n}.$$

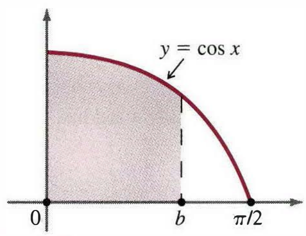


Figure 6.18

To calculate the limit of this as $n \to \infty$, we use the formula of Problem 9 in Section 6.3,

$$\sum_{k=1}^{n} \cos kx = \frac{\sin \frac{1}{2}nx \cos \frac{1}{2}(n+1)x}{\sin \frac{1}{2}x},$$

with $x = b/n$. We therefore have

$$\text{area of region} = \lim_{n\to\infty} s_n$$

$$= \lim_{n\to\infty} \frac{b}{n} \cdot \frac{\sin \frac{1}{2}b \cos\left[\frac{(n+1)b}{2n}\right]}{\sin (b/2n)}. \tag{6}$$

To calculate this limit we begin by observing that

$$\cos\left[\frac{(n+1)b}{2n}\right] = \cos\left(1 + \frac{1}{n}\right)\frac{b}{2} \to \cos\frac{b}{2} \quad \text{as} \quad n \to \infty,$$

since the cosine is continuous. Next, if we put $\theta = b/2n$, then $\theta \to 0$ as $n \to \infty$, and by using the limit (3) in Section 2.5 we see that

$$\frac{b}{n} \cdot \frac{1}{\sin (b/2n)} = 2 \cdot \frac{b/2n}{\sin (b/2n)} = 2 \cdot \frac{\theta}{\sin \theta} \to 2 \quad \text{as} \quad n \to \infty.$$

These facts enable us to write (6) as

$$\text{area of region} = \lim_{n\to\infty} s_n = 2 \sin\frac{b}{2} \cos\frac{b}{2} = \sin b,$$

or equivalently,

$$\int_0^b \cos x \, dx = \sin b.$$

PROBLEMS

1 Use upper sums to show that the area under the graph of $y = x^3$ over the interval $[0, b]$ is $b^4/4$.

2 Find the area under the graph of $y = x$ over the interval $[0, b]$ by using lower sums instead of the upper sums of Example 1.

3 Find the area under the graph of $y = x^2$ over the interval $[0, b]$ by using lower sums instead of the upper sums of Example 2.

4 Solve Problem 1 by using lower sums instead of upper sums.

5 As we know, every parabola with vertex at the origin which opens upward has an equation of the form $y = ax^2$. It is easy to see from Example 2 that

$$\int_0^b ax^2 \, dx = a\frac{b^3}{3}.$$

Use this to prove the theorem of Archimedes stated in Section 6.2 for the special case in which the chord AB is perpendicular to the axis of the parabola.

6 Find the area under the curve $y = \sin x$ from $x = 0$ to $x = b$, where $0 < b < \pi$. Hint: Use equal subintervals, take the points x_k^* to be the right endpoints of the subintervals, and apply Problem 10 in Section 6.3.

6.6 THE FUNDAMENTAL THEOREM OF CALCULUS

As our main achievement so far in this chapter, we have formulated a rather complicated definition of the definite integral of a continuous function as the limit of approximating sums,

$$\int_a^b f(x)\, dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\, \Delta x_k. \tag{1}$$

We have also considered several examples of the use of this definition in calculating the values of certain simple integrals, such as

$$\int_0^b x \, dx = \frac{b^2}{2}, \qquad \int_0^b x^2 \, dx = \frac{b^3}{3}, \qquad \text{and} \qquad \int_0^b x^3 \, dx = \frac{b^4}{4}. \tag{2}$$

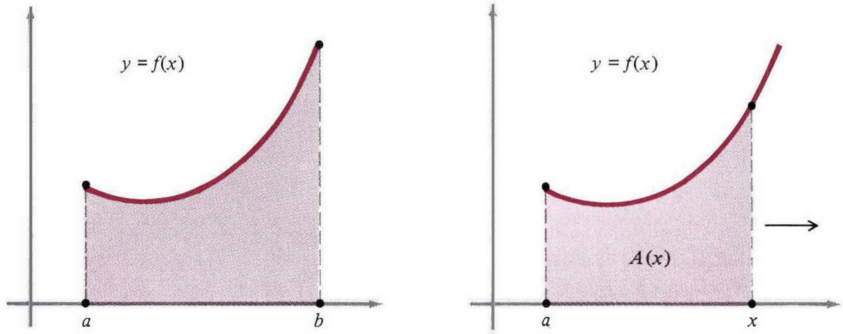


Figure 6.19

These calculations have had two purposes: to emphasize the essential nature of the integral by giving students some direct experience with approximating sums, and also to suggest the severe limitations of this method as a practical tool for evaluating integrals. Thus, for example, how can we possibly use limits of sums to find the numerical values of such complicated integrals as

$$\int_0^1 \frac{x^4\,dx}{\sqrt[3]{7+x^5}} \quad \text{and} \quad \int_1^2 \left(1+\frac{1}{x}\right)^4 \frac{dx}{x^2}? \tag{3}$$

This is clearly out of the question, so where do we go from here? What is evidently needed is a much more efficient and powerful method of computing integrals, and we find this method in the ideas of Newton and Leibniz.

The Newton-Leibniz approach to the problem of calculating the integral (1) depends on an idea that seems paradoxical at first sight. In order to solve this problem, we replace it by an apparently harder problem. Instead of asking for the *fixed* area on the left in Fig. 6.19, we ask for the *variable* area produced when the edge on the right side of the figure is considered to be moveable, so that the area is a function of x, as suggested on the right in Fig. 6.19. If this area function is denoted by $A(x)$, then clearly $A(a) = 0$ and $A(b)$ is the fixed area on the left in the figure. Our aim is to find an explicit formula for $A(x)$, and then to determine the desired fixed area by setting $x = b$. There are several steps in this process, which we consider separately for the sake of clarity.

STEP 1 We begin by establishing the crucial fact that

$$\frac{dA}{dx} = f(x). \tag{4}$$

This says that *the rate of change of the area A with respect to x is equal to the length of the right edge of the region.* To prove this statement, we must appeal to the definition of the derivative,

$$\frac{dA}{dx} = \lim_{\Delta x \to 0} \frac{A(x+\Delta x) - A(x)}{\Delta x}.$$

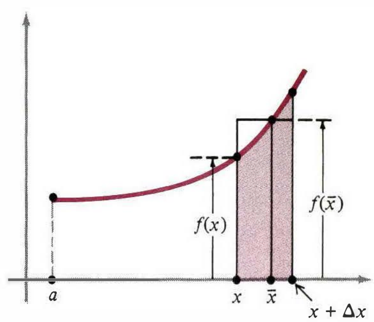


Figure 6.20

Now $A(x)$ is the area under the graph between a and x, and $A(x + \Delta x)$ is the area between a and $x + \Delta x$. Hence the numerator $A(x + \Delta x) - A(x)$ is the area between x and $x + \Delta x$ (see the shaded region in Fig. 6.20). It is easy to see that this area is exactly equal to the area of a rectangle with the same base whose height is $f(\bar{x})$, where $\bar{x}$ is a suitably chosen point between x and $x + \Delta x$.* This enables us to complete the proof of (4) as follows:

$$\frac{dA}{dx} = \lim_{\Delta x \to 0} \frac{A(x + \Delta x) - A(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(\bar{x})\,\Delta x}{\Delta x}$$
$$= \lim_{\Delta x \to 0} f(\bar{x}) = f(x),$$

since $f(x)$ is continuous. To explain the last step here in a bit more detail, we point out that $\Delta x \to 0$ is equivalent to $x + \Delta x \to x$; since $\bar{x}$ is caught between x and $x + \Delta x$, we also have $\bar{x} \to x$, and the continuity of the function now yields the conclusion that $f(\bar{x}) \to f(x)$.

STEP 2 Equation (4) makes it possible for us to achieve our goal of finding a formula for the area function $A(x)$. The reasoning goes this way. By (4), $A(x)$ is one of the antiderivatives of $f(x)$. But if $F(x)$ is *any* antiderivative of $f(x)$, then we know from Chapter 5 that

$$A(x) = F(x) + c \tag{5}$$

for some value of the constant c. To determine c, we put $x = a$ in (5) and obtain $A(a) = F(a) + c$; but since $A(a) = 0$, this yields $c = -F(a)$. Therefore

$$A(x) = F(x) - F(a) \tag{6}$$

is the desired formula.

STEP 3 All that remains is to observe that

$$\int_a^b f(x)\,dx = A(b) = F(b) - F(a),$$

by (6) and the meaning of $A(x)$.

We summarize our conclusions by formally stating the Fundamental Theorem of Calculus:

If $f(x)$ is continuous on a closed interval $[a, b]$, and if $F(x)$ is any antiderivative of $f(x)$, so that $(d/dx)\,F(x) = f(x)$ or equivalently

$$\int f(x)\,dx = F(x), \tag{7}$$

then

$$\int_a^b f(x)\,dx = F(b) - F(a). \tag{8}$$

This theorem transforms the difficult problem of evaluating definite integrals by calculating limits of sums into the much easier problem of finding antiderivatives. To find the value of $\int_a^b f(x)\,dx$, we therefore no longer have to think about sums at all; we merely find an antiderivative $F(x)$ in any way we can—by in-

*When this statement is expressed in formal language, it is called the *First Mean Value Theorem of Integral Calculus*. Loosely speaking, if the top of the rectangle is at just the right level, then the part of the area protruding above it exactly balances the deficiency below it.

spection, routine calculation, ingenious calculation, or looking it up in a book—and then compute the number $F(b) - F(a)$.

For instance, in Section 6.5 we used a good deal of algebraic ingenuity to obtain the formulas (2). Now, with the aid of the Fundamental Theorem, we see these formulas as obvious consequences of the following simple facts:

$$\int x\,dx = \frac{x^2}{2}, \qquad \int x^2\,dx = \frac{x^3}{3}, \qquad \text{and} \qquad \int x^3\,dx = \frac{x^4}{4}.$$

More generally, for any exponent $n > 0$ we clearly have

$$\int_a^b x^n\,dx = \frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1}, \qquad \text{because} \qquad \int x^n\,dx = \frac{x^{n+1}}{n+1}.$$

Remark 1 In the process of working problems, it is often convenient to use the *bracket symbol,*

$$F(x)\Big]_a^b = F(b) - F(a), \tag{9}$$

which is read "$F(x)$ bracket a, b." This symbol means exactly what (9) says it does: To find its value, we write the value of $F(x)$ when x has the upper value b, and subtract the value of $F(x)$ when x has the lower value a. For example, $x^2\big]_3^4 = 4^2 - 3^2 = 16 - 9 = 7$. By using this notation, (8) can be written in the form

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b.$$

Remark 2 It should be clear from this discussion that *any* antiderivative of $f(x)$ will do in (8). In case students are in doubt about this, they should recall that if $F(x)$ is one antiderivative, then any other can be obtained by adding a suitable constant c to form $F(x) + c$; and since

$$F(x) + c\Big]_a^b = [F(b) + c] - [F(a) + c] = F(b) - F(a),$$

the constant c has no effect on the result. We may therefore ignore constants of integration when finding antiderivatives for the purpose of computing definite integrals. (Nevertheless, these constants of integration remain indispensable when we are working with differential equations, as we saw in Section 5.4.)

Example 1 Evaluate each of the following definite integrals:

(a) $\int_{-1}^2 x^4\,dx$; (b) $\int_1^{16} \frac{dx}{\sqrt{x}}$; (c) $\int_8^{27} \sqrt[3]{x}\,dx$; (d) $\int_{13}^{14} (x-13)^{10}\,dx$.

Solution In each case an antiderivative is easy to find by inspection:

(a) $\displaystyle\int_{-1}^2 x^4\,dx = \frac{1}{5}x^5\Big]_{-1}^2 = \frac{1}{5}[32 - (-1)] = \frac{33}{5}$;

(b) $\displaystyle\int_1^{16} \frac{dx}{\sqrt{x}} = 2\sqrt{x}\Big]_1^{16} = 2(4-1) = 6$;

$$\text{(c)} \int_8^{27} \sqrt[3]{x}\, dx = \frac{3}{4} x^{4/3}\Big]_8^{27} = \frac{3}{4}(81 - 16) = \frac{195}{4};$$

$$\text{(d)} \int_{13}^{14} (x - 13)^{10}\, dx = \frac{1}{11}(x - 13)^{11}\Big]_{13}^{14} = \frac{1}{11}(1 - 0) = \frac{1}{11}.$$

The Fundamental Theorem establishes a strong connection between definite integrals and antiderivatives. This connection has made it customary to use the integral sign to denote an antiderivative, as in (7), and to replace the word "antiderivative" by the term "indefinite integral." The reader is familiar with these usages from Chapter 5. From this point on we will often drop the adjective (indefinite, definite) and use the word "integral" alone to refer to either the function (7) or the number (8), relying on the context and the reader's understanding of what is going on in order to avoid confusion. As an infallible aid in keeping track of which is which, we emphasize that a definite integral always has limits of integration attached to it, and that an indefinite integral never has such limits.

We apologize again for any confusion that may be caused by using such similar notations, $\int f(x)\, dx$ and $\int_a^b f(x)\, dx$, for such very different concepts. However, these notations have been with us for 300 years, and trying to change them now would be as futile as asking the wind not to blow. [Some years ago, one author tried to introduce the notation $A[f(x)]$ for the antiderivative to replace $\int f(x)\, dx$. His book disappeared from view faster than yesterday's newspaper.] Rather, it is the responsibility of students to read the symbols $\int f(x)\, dx$ and $\int_a^b f(x)\, dx$ carefully. We all read words carefully, and distinguish between such similar-appearing words as "peak" and "peek," "venal" and "venial," "manor" and "manner." Mathematics must be read with even more care.

From our experience in Chapter 5, we know—or can calculate—many indefinite integrals, and therefore many definite integrals are now within our reach. In particular, the definite integrals (3) are not at all difficult to compute, as we now show.

Example 2 Evaluate

$$\int_0^1 \frac{x^4\, dx}{\sqrt[3]{7 + x^5}}.$$

Solution For the sake of clarity, we consider separately the problem of finding the indefinite integral. The substitution

$$u = 7 + x^5, \qquad du = 5x^4\, dx$$

yields

$$\int \frac{x^4\, dx}{\sqrt[3]{7 + x^5}} = \int (7 + x^5)^{-1/3} x^4\, dx = \int u^{-1/3}\left(\frac{1}{5}\, du\right) = \frac{1}{5}\int u^{-1/3}\, du$$

$$= \frac{1}{5} \cdot \frac{3}{2} u^{2/3}$$

$$= \frac{3}{10}(7 + x^5)^{2/3}.$$

By the Fundamental Theorem we therefore have

$$\int_0^1 \frac{x^4\,dx}{\sqrt[3]{7+x^5}} = \frac{3}{10}(7+x^5)^{2/3}\Big]_0^1 = \frac{3}{10}(4-7^{2/3}) = \frac{3}{10}(4-\sqrt[3]{49}).$$

Example 3 Evaluate

$$\int_1^2 \left(1+\frac{1}{x}\right)^4 \frac{dx}{x^2}.$$

Solution Here we have

$$u = 1+\frac{1}{x}, \qquad du = -\frac{dx}{x^2},$$

so

$$\int \left(1+\frac{1}{x}\right)^4 \frac{dx}{x^2} = \int u^4(-du) = -\frac{1}{5}u^5$$
$$= -\frac{1}{5}\left(1+\frac{1}{x}\right)^5.$$

The Fundamental Theorem now yields

$$\int_1^2 \left(1+\frac{1}{x}\right)^4 \frac{dx}{x^2} = -\frac{1}{5}\left(1+\frac{1}{x}\right)^5\Big]_1^2$$
$$= -\frac{1}{5}\left(\frac{243}{32}-32\right) = \frac{781}{160}.$$

Example 4 Find the area under the curve $y = \cos x$ from $x = 0$ to $x = b$, where $0 < b \le \pi/2$.

Solution This area (see Fig. 6.18) is given by the definite integral

$$\int_0^b \cos x\,dx.$$

But the indefinite integral of $\cos x$ is familiar to us,

$$\int \cos x\,dx = \sin x,$$

so we immediately have

$$\int_0^b \cos x\,dx = \sin x\Big]_0^b = \sin b - \sin 0 = \sin b.$$

A comparison with Example 3 in Section 6.5 demonstrates the power of the Fundamental Theorem with particular clarity. The calculation in Example 3 was difficult and depended on an obscure trigonometric identity, whereas the calculation here is very easy indeed—but only because the Fundamental Theorem is available to us and we know a little about indefinite integrals.

Remark 3 Newton and Leibniz are commonly credited with discovering calculus at about the same time but independently of each other. Yet the concepts of the derivative as the slope of the tangent, and the definite integral as the area under a curve, were familiar to many thinkers who preceded them. Under these circumstances, why are Newton and Leibniz given the lion's share of the credit for creating this new branch of mathematics, which played such a central role in the rise of science as the dominant feature of Western civilization? Mostly because they were the principal discoverers of the Fundamental Theorem of Calculus. They, and they alone, understood its importance and began to construct the necessary supporting machinery, and also applied it with spectacular success to problems in science and geometry.

Nevertheless, historians of science have traced the roots of the Fundamental Theorem back to hints in the earlier geometric work of Barrow and Pascal, whose writings are known to have influenced Newton and Leibniz. As Newton said in one of his rare moments of self-deprecation—he was not a modest man—"If I have seen farther, it is by standing on the shoulders of giants." One of these giants was Fermat, who was the first to prove the area formula stated in Fig. 6.21. This suggests—as we look back with 20-20 hindsight—that he must therefore have known the Fundamental Theorem itself, which seems such a short step away. But unfortunately he failed to notice it.

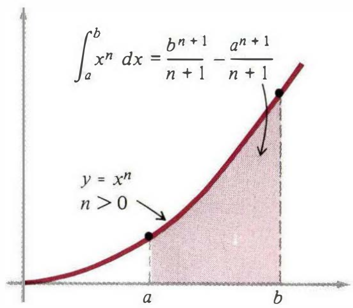


Figure 6.21

The Fundamental Theorem of Calculus is unquestionably one of the greatest achievements of the human mind. It is also one of the most influential, when we consider how much of the subsequent development of mathematics and the physical sciences depends upon it. Before it was discovered, from the time of Archimedes in the third century B.C. to the time of Fermat in the middle of the seventeenth century, problems of finding areas, volumes, and lengths of curves were so difficult that only people of genius could hope to solve them, and there are very few of these in any generation. But now, equipped with the great arsenal of systematic methods that Newton and Leibniz and their followers built on the foundation of the Fundamental Theorem, we will see in the following chapters that these problems are open to all of us.

PROBLEMS

A sketch is a necessary part of the solution of almost any problem involving a geometric quantity, and students should form the habit of drawing one as a matter of routine. If drawn with reasonable (but not excessive) care, such a sketch can help us avoid errors by reminding us of what we are doing, and often acts as a valuable source of ideas.

1 Use integration to find the area of the triangle bounded by the line $y = 2x$, the x-axis, and the line $x = 3$. Check your answer by elementary geometry.

2 Use integration to find the area of the triangle bounded by the axes and the line $3x + 2y = 6$. Check your answer by elementary geometry.

In Problems 3–5, find the area between each parabola and the x-axis.

3 $x^2 + y = 4$.

4 $4x^2 + 9y = 36$.

5 $4x^2 + 12y = 24x$.

In Problems 6–10, each curve has one arch above the x-axis. Find the area of the region under the arch.

6 $y = -x^3 + 4x$.

7 $y = x^3 - 9x$.

8 $y = 2x^2 - x^3$.

9 $y = x^4 - 6x^2 + 8$.

10 $y = x^3 - 5x^2 + 2x + 8$.

In Problems 11–21, find the area bounded by the given curve, the x-axis, and the given vertical lines.

11 $y = x^2$, $x = -2$ and $x = 3$.

12 $y = x^3$, $x = 0$ and $x = 2$.

13 $y = 3x^2 + x + 2$, $x = 1$ and $x = 2$.

14 $y = x^2 - 3x$, $x = -3$ and $x = -1$.

15 $y = 2x + \dfrac{1}{x^2}$, $x = 1$ and $x = 3$.

16 $y = \dfrac{1}{\sqrt{x+3}}$, $x = 1$ and $x = 6$.

17 $y = 3x^2 + 2$, $x = 0$ and $x = 3$.

18 $y = 2x + 3$, $x = 0$ and $x = 3$.

19 $y = \sqrt{2x+3}$, $x = -1$ and $x = 3$.

20 $y = \dfrac{1}{\sqrt{2x+3}}$, $x = -1$ and $x = 3$.

21 $y = \dfrac{1}{(2x+3)^2}$, $x = -1$ and $x = 3$.

22 If n is positive, then

$$\int_{-1}^{1} x^n\,dx = \frac{x^{n+1}}{n+1}\bigg]_{-1}^{1}.$$

Why is this calculation incorrect if n is a negative number $\neq -1$?

In Problems 23–42, find the value of each definite integral.

23 $\displaystyle\int_{-1/3}^{2/3} \frac{dx}{\sqrt{3x+2}}$.

24 $\displaystyle\int_{0}^{1} (2x+3)\,dx$.

25 $\displaystyle\int_{-1}^{0} 7x^6\,dx$.

26 $\displaystyle\int_{1}^{4} \sqrt{x}\,dx$.

27 $\displaystyle\int_{0}^{2} \sqrt{4x+1}\,dx$.

28 $\displaystyle\int_{-1}^{2} (x+1)^2\,dx$.

29 $\displaystyle\int_{2a}^{3a} \frac{x\,dx}{(x^2-a^2)^2}$.

30 $\displaystyle\int_{0}^{2b} \frac{x\,dx}{\sqrt{x^2+b^2}}$.

31 $\displaystyle\int_{0}^{1} (x-x^2)\,dx$.

32 $\displaystyle\int_{-1}^{2} (1+x)(2-x)\,dx$.

33 $\displaystyle\int_{0}^{a} (a^2x-x^3)\,dx$.

34 $\displaystyle\int_{0}^{1} (x+1)^9\,dx$.

35 $\displaystyle\int_{0}^{b} (\sqrt{b}-\sqrt{x})^2\,dx$.

36 $\displaystyle\int_{0}^{1} x^2(1-x^2)\,dx$.

37 $\displaystyle\int_{0}^{1} x^2(1-x)^2\,dx$.

38 $\displaystyle\int_{1}^{2} \left(x+\frac{1}{x}\right)^2 dx$.

39 $\displaystyle\int_{0}^{\pi/4} \sin 4x\,dx$.

40 $\displaystyle\int_{\pi/6}^{\pi/4} \sin t\,dt$.

41 $\displaystyle\int_{0}^{\pi/2} (2\sin\theta + \cos\theta)\,d\theta$.

42 $\displaystyle\int_{0}^{\pi/3} \frac{\sin\theta}{\cos^2\theta}\,d\theta$.

6.7 PROPERTIES OF DEFINITE INTEGRALS

ALGEBRAIC AND GEOMETRIC AREAS

In the previous sections we considered the area of the region under the curve $y = f(x)$ between $x = a$ and $x = b$, and two assumptions were more or less explicit: (1) $f(x) \geq 0$ throughout the interval, and (2) $a < b$. However, the formula defining the definite integral as the limit of approximating sums, namely,

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\,\Delta x_k, \tag{1}$$

is independent of these assumptions.

For example, suppose that the curve lies below the x-axis, as shown on the left in Fig. 6.22. In this case we would hesitate to speak of the region "under the curve," but we can certainly describe it as the region "bounded by the curve and the x-axis, between $x = a$ and $x = b$." Each term of the sum (1) is clearly negative because $f(x_k^*) < 0$. Accordingly, $f(x_k^*)\,\Delta x_k$ is the negative of the area of the shaded rectangle, the integral is the negative of the area of the region, and consequently

$$\text{area of the region} = -\int_a^b f(x)\,dx.$$

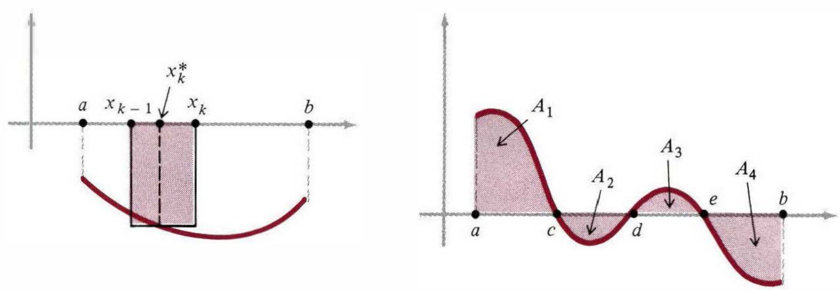


Figure 6.22

Similarly, if the curve lies partly above the x-axis and partly below it, as shown on the right in Fig. 6.22, then the integral (1) can be thought of as a sum of positive and negative terms, corresponding to parts of the region lying above and below the x-axis:

$$\int_a^b f(x)\,dx = A_1 - A_2 + A_3 - A_4, \tag{2}$$

where the areas A_1, A_2, A_3, A_4 are understood to be positive. The integral (2) is often called the *algebraic area* of the region bounded by the curve and the x-axis, because it counts areas of regions above the x-axis with a positive sign and areas of regions below the x-axis with a negative sign.* The actual area of the region bounded by the curve and the x-axis, with each part counted as a positive number, is called the *geometric area:*

$$A_1 + A_2 + A_3 + A_4 = \int_a^c - \int_c^d + \int_d^e - \int_e^b. \tag{3}$$

To find the geometric area, we must sketch the graph, locate the crossing points, and calculate each integral on the right of (3) separately so that they can be combined with the correct signs.

MISCELLANEOUS PROPERTIES

If we drop the condition $a < b$ and instead assume that $a > b$, we can still retain the purely numerical definition (1) for the definite integral. The only change is that as we traverse the interval from a to b the increments Δx_k are negative. This yields the equation

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx, \tag{4}$$

which is valid for all numbers a and b ($a \neq b$). Also, since (4) says that interchanging the limits of integration changes the sign of the integral, it is natural to take the equation

$$\int_a^a f(x)\,dx = 0 \tag{5}$$

as the definition of the integral on the left.

If $a < b$, and if c is any number between a and b, it is easy to see from (1) that

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx. \tag{6}$$

Properties (4) and (5) allow us to conclude that (6) is true for any three numbers a, b, c, regardless of their relation to one another.

We list several further properties of definite integrals that follow in a routine way from the definition (1):

*The discussion in Section 6.6 leading to the Fundamental Theorem of Calculus extends without essential change to integrals of this type. An alternative proof of the Fundamental Theorem, based on entirely different ideas, is given in Appendix A.6.

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx; \tag{7}$$

$$\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx; \tag{8}$$

$$\text{if } f(x) \le g(x) \text{ on } [a, b], \qquad \text{then} \qquad \int_a^b f(x)\,dx \le \int_a^b g(x)\,dx. \tag{9}$$

In words, property (7) says that a constant factor can be moved across the integral sign, and (8) says that the integral of a sum is the sum of the separate integrals.

VARIABLE LIMITS OF INTEGRATION

We have used x as the "variable of integration" in writing the definite integral

$$\int_a^b f(x)\,dx. \tag{10}$$

However, (10) is a fixed number whose value does not depend on which letter is used for this variable. Instead of (10), we could equally well write

$$\int_a^b f(t)\,dt, \qquad \int_a^b f(u)\,du,$$

or any similar expression, and the meaning would be the same. Letters used in this way are often called *dummy variables.*

In most situations it doesn't matter what letters are used, as long as the ideas are clearly understood. However, sometimes we wish to construct a new function $F(x)$ by integrating a given function $f(x)$ from a fixed lower limit to a *variable* upper limit, as in

$$F(x) = \int_a^x f(x)\,dx. \tag{11}$$

It is evident that this usage can be confusing, because the letter x is used with two different meanings on the right: as the upper limit of integration above the integral sign, and as a dummy variable behind the integral sign. For this reason, it is customary to write (11) in the form

$$F(x) = \int_a^x f(t)\,dt, \tag{12}$$

with t used as the dummy variable in place of x.

The function $F(x)$ defined by (12) has two properties that make it important. First, it exists whenever the integrand is continuous on the interval between a and x. And second, we proved in Section 6.6 that the derivative of this function is simply the value of the integrand at the upper limit:

$$\frac{d}{dx}F(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x). \tag{13}$$

This provides a satisfactory theoretical solution of the problem of finding an indefinite integral for any given continuous function $f(x)$. As a practical matter, it may be very difficult—or even impossible—to calculate

$$\int f(x)\,dx = F(x)$$

in any recognizable form involving familiar functions. But even if we can't find a formula for $F(x)$, it is at least some consolation to know that in principle an indefinite integral of a continuous function always exists, namely, the function defined by (12).

Example 1 The problem of finding an explicit formula for the indefinite integral

$$\int \frac{dx}{\sqrt[3]{x^{10}+1}} = F(x)$$

is beyond our reach now, and will always be beyond our reach. However, if we don't require an explicit formula, but only a well-defined function, then

$$F(x) = \int_0^x \frac{dt}{\sqrt[3]{t^{10}+1}}$$

will do.

Example 2 Let us try to calculate

$$\frac{d}{dx}\left(\int_0^x \frac{dt}{1+t^2}\right).$$

At this stage of our work we have no way of carrying out the integration to find a formula for the function in parentheses so that this function can be differentiated. But this doesn't matter. By (13) we immediately have

$$\frac{d}{dx}\left(\int_0^x \frac{dt}{1+t^2}\right) = \frac{1}{1+x^2},$$

so no preliminary integration is necessary before the differentiation can be carried out.

PROBLEMS

1 In each of the following cases, compute the geometric area of the region bounded by the x-axis and the given curves:
(a) $y = 3x - x^2$, $x = 1$, $x = 4$;
(b) $y = x^2 - 2x$, $x = 1$, $x = 4$;
(c) $y = 4 + 4x^3$, $x = -2$, $x = 1$;
(d) $y = x - \dfrac{8}{x^2}$, $x = 1$, $x = 4$.

2 Find the area bounded by the axes and the given curve:
(a) $y = \sqrt{4 - x}$; (b) $\sqrt{x} + \sqrt{y} = \sqrt{a}$.

3 Find the area bounded by $y^2 = x^3$ and $x = 4$.

4 Find the area enclosed by the loop of $y^2 = x(x-4)^2$.

5 If $a < c < b$ and $f(x) \geq 0$ on $[a, b]$, draw a suitable picture and explain why equation (6) is an obvious relation among areas.

6 If $f(x) \geq 0$ on $[a, b]$ and $c > 0$, draw a suitable picture and explain why equation (7) is an obvious statement about areas. Do the same for equations (8) and (9) if both $f(x)$ and $g(x)$ are nonnegative on $[a, b]$.

7 If $f(x)$ is an *even* function, that is, if $f(-x) = f(x)$, show geometrically or otherwise that

$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx.$$

8 Verify the equation in Problem 7 by calculating the following integrals of even functions:

$$\int_{-2}^{2} x^2\,dx \qquad \text{and} \qquad \int_{-19}^{19} (1 + x^{24})\,dx.$$

9 If $f(x)$ is an *odd* function, that is, if $f(-x) = -f(x)$, show geometrically or otherwise that

$$\int_{-a}^{a} f(x)\,dx = 0.$$

10 Verify the equation in Problem 9 by computing the following integrals of odd functions:

$$\int_{-2}^{2} x^5\,dx \qquad \text{and} \qquad \int_{-7}^{7} \frac{x\,dx}{\sqrt{x^2 + 11}}.$$

11 The graph of $y = x^2$, $x \geq 0$, can be considered to be the graph of $x = \sqrt{y}$, $y \geq 0$. Show by geometry that this implies the validity of the equation

$$\int_0^a x^2\,dx + \int_0^{a^2} \sqrt{y}\,dy = a^3, \qquad a > 0.$$

Check this by calculating the integrals.

12 Generalize Problem 11 by finding and checking a similar equation for $y = x^n$, where n is any positive number.

13 Use the known area of a circle to find the value of the integral

$$\int_{-a}^{a} \sqrt{a^2 - x^2}\,dx.$$

14 The graph of the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad a > b > 0,$$

is called an *ellipse*. Sketch it, and use the result of Problem 13 to find the enclosed area.

15 Show that

(a) $\dfrac{d}{dx}\displaystyle\int_x^b f(t)\,dt = -f(x)$;

(b) $\dfrac{d}{dx}\displaystyle\int_a^{u(x)} f(t)\,dt = f(u(x))\,\dfrac{du}{dx}$.

16 In each of the following, compute the indicated derivative:

(a) $\dfrac{d}{dx}\displaystyle\int_1^{x+2} \frac{dt}{t}$; (b) $\dfrac{d}{dx}\displaystyle\int_{2x}^{5} t^3\,dt$;

(c) $\dfrac{d}{dx}\displaystyle\int_1^{x} \frac{dt}{1+t}$; (d) $\dfrac{d}{dx}\displaystyle\int_x^{1} \frac{dt}{1+t^4}$;

(e) $\dfrac{d}{dx}\displaystyle\int_1^{x^2} \frac{dt}{\sqrt{t + \sqrt{t+1}}}$.

CHAPTER 6 REVIEW: CONCEPTS, METHODS

Define, state, or think through the following.

1 Sigma notation.
2 Special sums.
3 Area under a curve; lower sums and upper sums.
4 Definite integral as a limit of sums.
5 Limits of integration, integrand, variable of integration.
6 Integrable function.
7 Every continuous function is integrable.
8 Fundamental Theorem of Calculus.
9 Indefinite integral.
10 Algebraic and geometric areas.
11 Variable limit of integration.

ADDITIONAL PROBLEMS FOR CHAPTER 6

SECTION 6.5

1 Show that

$$\int_0^b \sqrt{x}\,dx = \tfrac{2}{3}b^{3/2}$$

by taking $x_k = k^2b/n^2$ and $x_k^* = x_k$ in formula (12) of Section 6.4. Notice that this problem illustrates the calculation of an integral as a limit by using subintervals of different lengths.

*2 Show that

$$\int_1^b \frac{1}{x^2}\,dx = 1 - \frac{1}{b}$$

by using equal subintervals and taking $x_k^* = \sqrt{x_{k-1}x_k}$ in formula (12) of Section 6.4. (Why is $x_{k-1} < x_k^* < x_k$?) Hint: It will be necessary to use a variation of the idea behind the formula

$$\begin{aligned}\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)} &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots \\ &\quad + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}.\end{aligned}$$

*3 Show that

$$\int_1^b \frac{1}{\sqrt{x}}\,dx = 2(\sqrt{b} - 1)$$

by using equal subintervals and taking

$$x_k^* = \frac{x_{k-1} + x_k + \sqrt{2x_{k-1}x_k}}{4} = \left(\frac{\sqrt{x_{k-1}} + \sqrt{x_k}}{2}\right)^2$$

in formula (12) of Section 6.4. Remember to check that $x_{k-1} \le x_k^* \le x_k$.

SECTION 6.6

4 Find the area between each parabola and the x-axis:
(a) $x^2 + 3y = 9$;
(b) $3x^2 + 4y = 48$;
(c) $x^2 + 4x + 2y = 0$.

5 The part of the curve $b^2y = 4h(bx - x^2)$ that lies above the x-axis forms a parabolic arch with height h and base b. Sketch the graph and check these statements. Use integration to show that the area under this arch is two-thirds the area of the rectangle with the same height and base.

6 Each curve has one arch above the x-axis. Find the area under this arch.
(a) $y = 10 - x - 2x^2$.
(b) $y = -x^3 - 4x^2 - 4x$.
(c) $y = x^3 + 2x^2 - 8x$.
(d) $y = x^4 - 6x^2 + 9$.
(e) $y = x\sqrt{1 - x}$.

7 Find the area bounded by the given curve, the x-axis, and the given vertical lines:
(a) $y = x^2 + 2x + 1$, $x = -1$ and $x = 1$;
(b) $y = \sqrt{x + 2}$, $x = 2$ and $x = 7$;
(c) $y = \sqrt[3]{3 - x}$, $x = -5$ and $x = 3$;
(d) $y = x\sqrt{5 - x^2}$, $x = 0$ and $x = \sqrt{5}$;
(e) $y = \dfrac{x}{(x^2 + 1)^2}$, $x = 0$ and $x = 3$.

8 Find the value of each definite integral:
(a) $\int_0^1 x(x^2 + 2)^3\, dx$; (b) $\int_{-1}^0 3x^2(3 + x^3)^2\, dx$;
(c) $\int_0^a x\sqrt{a^2 - x^2}\, dx$; (d) $\int_0^a x\sqrt{a^2 + x^2}\, dx$;
(e) $\int_{-2}^4 (8 - 4x + x^2)\, dx$;
(f) $\int_8^{27} (2x^{-2/3} + 8x^{1/3})\, dx$;
(g) $\int_0^1 \sqrt{9 - 8x}\, dx$;
(h) $\int_2^3 \dfrac{dx}{(3x - 5)^{5/2}}$;
(i) $\int_0^{\sqrt{3}} \dfrac{x\, dx}{\sqrt{4 - x^2}}$;
(j) $\int_0^2 \sqrt{1 + x^3}\, x^2\, dx$;
(k) $\int_0^b (b^{2/3} - x^{2/3})^3\, dx$.

SECTION 6.7

9 In each of the following cases, compute the geometric area of the region bounded by the x-axis and the given curves:
(a) $y = 6 - 3x^2$, $x = 0$, $x = 2$;
(b) $y = x^2 + 2x$, $x = -3$, $x = 0$;
(c) $y = x^2 - x - 2$, $x = 1$, $x = 3$;
(d) $y = x^3 - 3x$, $x = -2$, $x = 3$.

10 In each of the following cases, compute both the algebraic and geometric areas of the region bounded by the x-axis and the given curves:
(a) $y = 3x^5 - x^3$, $x = -1$, $x = 1$;
(b) $y = (x^2 - 4)(9 - x^2)$.

11 Compute
(a) $\dfrac{d}{dx}\int_0^{x^4} \dfrac{dt}{1 + t}$; (b) $\dfrac{d}{dx}\int_1^{1+x^2} \dfrac{dt}{t}$;
(c) $\dfrac{d}{dx}\int_0^{x^3} \dfrac{dt}{\sqrt{3t + 7}}$; (d) $\dfrac{d}{dx}\int_0^{x^5} \dfrac{t\, dt}{\sqrt{1 + t^2}}$.

12 Verify the results obtained in parts (c) and (d) of Problem 11 by actually carrying out the integration and then differentiating.

APPENDIX: THE LUNES OF HIPPOCRATES

According to one tradition, Hippocrates of Chios (ca. 430 B.C.)—not to be confused with his better-known contemporary, the physician Hippocrates of Cos—was originally a merchant whose goods were stolen by pirates.* He then went to Athens, where he lived for many years, studied mathematics, and compiled a book on the elements of geometry that strongly influenced Euclid more than a century later.

*Aristotle, who rarely missed a chance to express his scorn for mathematicians, gives a more demeaning account of Hippocrates' misfortune. "It is well known," he wrote with relish, "that people brilliant in one particular field may be quite foolish in most other things. Thus Hippocrates, though skilled in geometry, was so stupid and spineless that he let a tax collector of Byzantium cheat him out of a fortune." This, from the man who asserted that heavier bodies fall to the ground more rapidly and that men have more teeth than women.

We recall Hippocrates' discovery as stated in Section 6.2: The lune (crescent-shaped region) in Fig. 6.23 bounded by the circular arcs ADB and AEB (the latter having C as its center) has an area exactly equal to the area of the shaded square whose side is the radius of the circle. (Hippocrates also found the areas of two other kinds of lunes, but we do not discuss these here.)

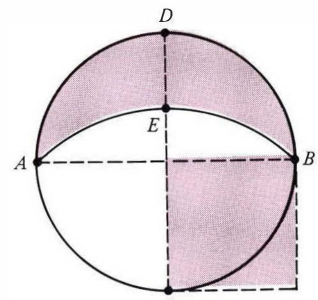


Figure 6.23

This astonishing theorem seems to be the earliest precise determination of the area of a region bounded by curves. Its proof is simple but ingenious and depends on the last of the following three geometric facts, each of which implies the next: (a) The areas of two circles are to each other as the squares of the radii (Fig. 6.24); (b) sectors of two circles with equal central angles are to each other as the squares of the radii (Fig. 6.25); (c) segments of two circles with equal central angles are to each other as the squares of the radii (Fig. 6.26). We shall need (c) in the special case of right angles at the center.

The proof of Hippocrates' theorem now proceeds as follows. Redraw the lune as shown in Fig. 6.27. The chords joining D with A and B are tangent to the arc AEB and divide the lune into three regions with areas a_1, a_2, a_3. If the radius of the smaller circle is denoted by r, then the Pythagorean theorem tells us that the radius of the larger circle is $\sqrt{2}r$. It is easy to see that a_1 and a_2 are equal segments of the smaller circle and that a_4 is a segment of the larger circle, all with right angles at the center. We now use statement (c) to infer that

$$\frac{a_1}{a_4} = \frac{r^2}{(\sqrt{2}r)^2} = \frac{1}{2}.$$

This yields

$$a_1 = \tfrac{1}{2}a_4 \qquad \text{and} \qquad a_2 = \tfrac{1}{2}a_4,$$

so

$$a_1 + a_2 = a_4.$$

It now follows that

$$\begin{aligned} \text{area of lune} &= a_1 + a_2 + a_3 \\ &= a_4 + a_3 \\ &= \text{area of triangle } ABD \\ &= r^2 = \text{area of square } OBFC, \end{aligned}$$

and the argument is complete.

Hippocrates was a contemporary of Pericles, the great political and cultural leader of Athens during its Golden Age. But nothing Pericles achieved has the

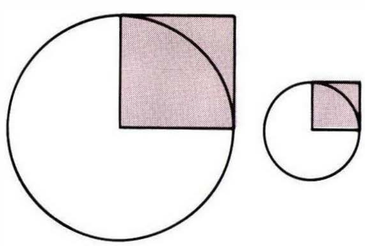


Figure 6.24

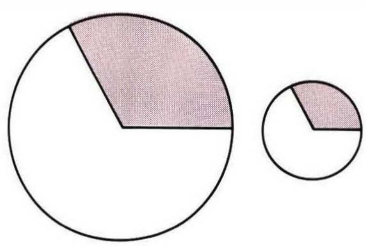


Figure 6.25

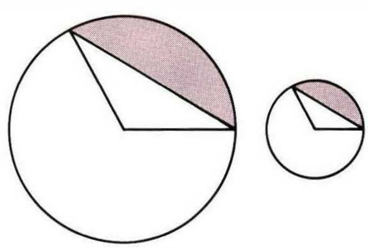


Figure 6.26

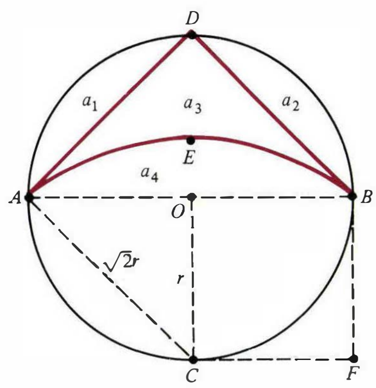


Figure 6.27

enduring quality of this beautiful geometric discovery; even the Parthenon, whose design and construction he supervised, is crumbling away. The reasoning of Hippocrates is a paragon of mathematical proof, untouched by time: In a few elegant steps it converts something easy to understand but difficult to believe into something impossible to doubt.

7 APPLICATIONS OF INTEGRATION

7.1 INTRODUCTION. THE INTUITIVE MEANING OF INTEGRATION

In Chapter 6 we accomplished two major purposes. First, we approximated the area under a given curve by certain sums and found the exact area by forming the limit of these sums. And second, we learned how to calculate the numerical value of this limit by using the much more powerful method provided by the Fundamental Theorem of Calculus. Almost the whole content of Chapter 6 can be compressed into the following statement: If $f(x)$ is continuous on $[a, b]$, then

$$\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\, \Delta x_k = \int_a^b f(x)\, dx$$

$$= F(x)\Big]_a^b = F(b) - F(a), \tag{1}$$

where $F(x)$ is any indefinite integral of $f(x)$.

There are many other quantities in geometry and physics that can be treated in essentially the same way. Among these are volumes, arc lengths, surface areas, and such basic physical quantities as the work done by a variable force acting over a given distance. In each case the process is the same: An interval of the independent variable is divided into small subintervals, the quantity in question is approximated by certain corresponding sums, and the limit of these sums yields the exact value of the quantity in the form of a definite integral—which is then evaluated by means of the Fundamental Theorem.

Once we have seen the details of this limit-of-sums process being carried out for the area under a curve, as was done in Chapter 6, it is unnecessary and boring to think through these details over and over again for each new quantity that we meet. The notation needed for this is complicated and repetitive, and actually impedes the intuitive understanding that we wish to cultivate.

In this spirit, we turn briefly to Fig. 7.1 and consider the easy, intuitive way of constructing the definite integral in (1). We think of the area under the curve as composed of a great many thin vertical rectangular strips. The typical strip shown in the figure has height y and width dx, and therefore area

$$dA = y\, dx = f(x)\, dx, \tag{2}$$

since $y = f(x)$. This area is called the *differential element of area,* or simply *the element of area*; it is located at an arbitrary position within the region, and this position is specified by a value of x between a and b. We now think of the total

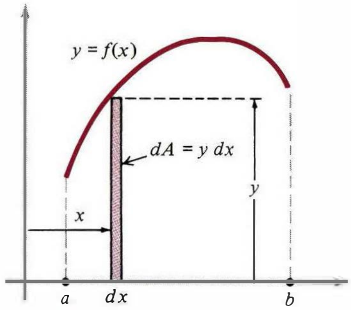


Figure 7.1

area A of the region as the result of adding up these elements of area dA as our typical strip sweeps across the region. This act of addition or summation can be symbolized by writing

$$A = \int dA. \tag{3}$$

Since the element of area sweeps across the region as x increases from a to b, we can express the idea in (3) with greater precision by writing

$$A = \int dA = \int y\,dx = \int_a^b f(x)\,dx. \tag{4}$$

We reach a true definite integral only in the last step in (4), where the variable of integration and the limits of integration become visibly present. In this way we glide smoothly over the messy details and set up the definite integral for the area directly, without having to think about limits of sums at all.

From this point of view, integration is the act of calculating the whole of a quantity by cutting it up into a great many convenient small pieces and then adding up these pieces. It is this intuitive Leibnizian approach to the process of integration that we intend to illustrate and reinforce in the following sections.

7.2 THE AREA BETWEEN TWO CURVES

Suppose we are given two curves $y = f(x)$ and $y = g(x)$, as shown in Fig. 7.2, with points of intersection at $x = a$ and $x = b$ and with the first curve lying above the second on the interval $[a, b]$. In setting up an integral for the area between these curves, it is natural to use thin vertical strips as indicated. The height of such a strip is the distance $f(x) - g(x)$ from the lower curve to the upper at the position x, and its base is dx. The element of area is therefore

$$dA = [f(x) - g(x)]\,dx,$$

and the total area is

$$A = \int dA = \int_a^b [f(x) - g(x)]\,dx. \tag{1}$$

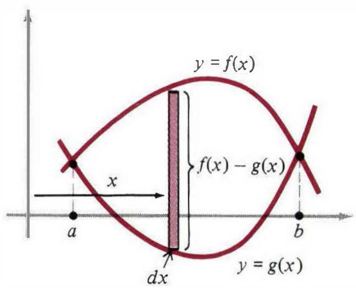


Figure 7.2

We integrate from the smaller limit of integration a to the larger b so that the increment (or differential) dx will be positive. It should also be pointed out that a and b are the values of x for which the two functions yield the same y's; that is, they are the solutions of the equation $f(x) = g(x)$, and to find them we solve this equation.

We urge students not to be satisfied with merely memorizing formula (1) and applying it mechanically to area problems. Our aim is the mastery of a method, and this aim is better served by thinking geometrically and constructing the needed formula from scratch for each individual problem. The method applies equally well to finding areas by using thin horizontal strips, which are often more convenient. In this case the width of a typical strip will be dy, and the total area will be found by integrating with respect to y.

As an aid to students we give an outline of the steps that should be followed in finding an area by integration.

STEP 1 Sketch the region whose area is to be found. Write down on the sketch the equations of the bounding curves and find their points of intersection.

STEP 2 Decide whether to use thin vertical strips that have width dx or thin horizontal strips that have width dy, and draw a typical strip on the sketch.

STEP 3 By looking at the sketch and using the equations of the bounding curves, write down the area dA of the typical strip as the product of its length and its width. Express dA entirely in terms of the variable (x or y) appearing in the width.

STEP 4 Integrate dA between appropriate x or y limits, these limits being found by examining the sketch.

Example 1 The region bounded by the curves $y = x^2$ and $y = 4$ is shown in Fig. 7.3. If we use vertical strips, then the length of our typical strip is $4 - x^2$ and its area is $dA = (4 - x^2)\,dx$. The total area of the region is therefore

$$\int_{-2}^{2} (4 - x^2)\,dx = 4x - \tfrac{1}{3}x^3\Big]_{-2}^{2}$$
$$= (8 - \tfrac{8}{3}) - (-8 + \tfrac{8}{3}) = \tfrac{32}{3}.$$

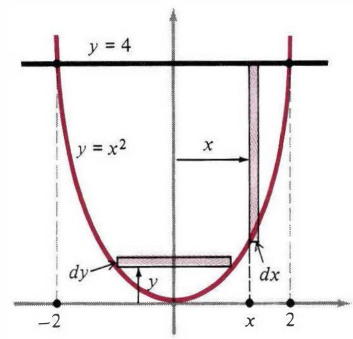


Figure 7.3

We urge students to use symmetry whenever possible in order to simplify the calculations. In this case the left-right symmetry of the figure suggests that we integrate only from $x = 0$ to $x = 2$ to find the right half of the area, and then double the result to obtain the total area:

$$2\int_{0}^{2} (4 - x^2)\,dx = 2(4x - \tfrac{1}{3}x^3)\Big]_{0}^{2} = 2(8 - \tfrac{8}{3}) = \tfrac{32}{3}.$$

As this calculation shows, it is often an advantage (only a slight advantage in this case) to have 0 as one of the limits of integration.

If we decide to use horizontal strips, then the length of the strip is the value of x (in terms of y) at the right end minus the value of x at the left end. This is $\sqrt{y} - (-\sqrt{y})$, so $dA = [\sqrt{y} - (-\sqrt{y})]\,dy = 2\sqrt{y}\,dy$ and the total area is

$$\int_{0}^{4} 2\sqrt{y}\,dy = \tfrac{4}{3}y^{3/2}\Big]_{0}^{4} = \tfrac{32}{3}.$$

The answer is the same as before, which is not surprising but is nevertheless reassuring.

We emphasize once again how important a good sketch is for understanding and carrying out these procedures.

Example 2 The region bounded by the curves $y = 3 - x^2$ and $y = x + 1$ is shown in Fig. 7.4. We find where the curves intersect by solving the equations simultaneously. We do this by equating the y's, which gives

$$3 - x^2 = x + 1,$$
$$x^2 + x - 2 = 0,$$
$$(x + 2)(x - 1) = 0,$$
$$x = -2, 1.$$

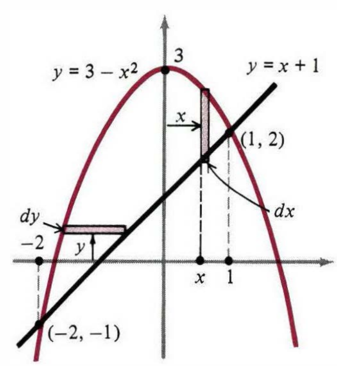


Figure 7.4

The points of intersection are thus $(-2, -1)$ and $(1, 2)$. The length of the indicated vertical strip is $(3 - x^2) - (x + 1) = 2 - x^2 - x$ so the area of the region is found by integrating the element of area $dA = (2 - x^2 - x)\,dx$ as x goes from -2 to 1,

$$\int_{-2}^{1} (2 - x^2 - x)\,dx = (2x - \tfrac{1}{3}x^3 - \tfrac{1}{2}x^2)\Big]_{-2}^{1}$$
$$= (2 - \tfrac{1}{3} - \tfrac{1}{2}) - (-4 + \tfrac{8}{3} - 2) = 4\tfrac{1}{2}.$$

It is inconvenient to use horizontal strips in this problem, because a horizontal strip clearly reaches from the left half of the parabola to the line if $y < 2$ and from the left half of the parabola to the right half if $y > 2$, and this means that different formulas for dA must be used according as $y < 2$ or $y > 2$.

Example 3 Find the area of the region bounded by the curves $y = \cos x$ and $y = \sin 2x$ on the interval $0 \le x \le \pi/2$.

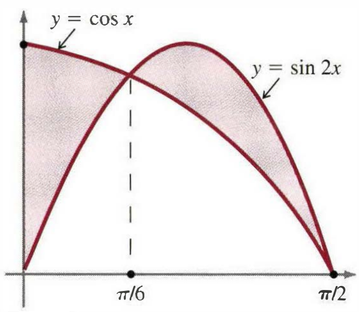


Figure 7.5

Solution The curves are shown in Fig. 7.5, and the region—consisting of two parts—is shaded. The main feature of this example is that the curves cross each other, so that first one curve, and then the other, is the "upper" curve. To deal with this, we must begin by finding exactly where the curves cross, which means we must solve the equation $\cos x = \sin 2x$. We do this by writing

$$\cos x = 2 \sin x \cos x, \qquad \sin x = \tfrac{1}{2}, \qquad x = \frac{\pi}{6}.$$

Accordingly,

$$dA = \begin{cases} (\cos x - \sin 2x)\,dx & \text{for } 0 \le x \le \dfrac{\pi}{6}, \\ (\sin 2x - \cos x)\,dx & \text{for } \dfrac{\pi}{6} \le x \le \dfrac{\pi}{2}. \end{cases}$$

The desired area is therefore

$$\int_0^{\pi/6} (\cos x - \sin 2x)\,dx + \int_{\pi/6}^{\pi/2} (\sin 2x - \cos x)\,dx$$
$$= (\sin x + \tfrac{1}{2}\cos 2x)\Big]_0^{\pi/6} + (-\tfrac{1}{2}\cos 2x - \sin x)\Big]_{\pi/6}^{\pi/2}$$
$$= (\tfrac{1}{2} + \tfrac{1}{4} - 0 - \tfrac{1}{2}) + (\tfrac{1}{2} - 1 + \tfrac{1}{4} + \tfrac{1}{2}) = \tfrac{1}{2}.$$

PROBLEMS

In Problems 1–19, sketch the curves and find the areas of the regions they bound.

1 $y = x^2$, $y = 2x$.
2 $y = x^2$, $x = y^2$.
3 $y = x^2 + 2$, $y = 4 - x^2$.
4 $y = 4x^3 + 3x^2 + 2$, $y = 2$.
5 $y = x^2 - 2x$, $y = 3$.
6 $y = x^3 - 3x$, $y = x$ $(x \ge 0)$.
7 $y = x^4 - 4x^2$, $y = -4$.
8 $y = x^3 - 4x$, $y = 5x$ $(x \ge 0)$.
*9 $y = 2x + \dfrac{9}{x^2}$, $y = -2x + 13$.
10 $x = y^2$, $y = x + 3$, $y = -2$, $y = 1$.
11 $y = x^2 - 4x$, $y = 2x$.
12 $y = x^3$, $y = 2x - x^2$.
13 $x = y^2$, $x = 2y + 3$.
14 $x = y^2$, $x = 2y$.
15 $y = x^2 + 1$, $y = 3 - x^2$, $x = -2$, $x = 2$.
16 $y = x^4 - 4$, $y = 3x^2$.
17 $x = 8 - y^2$, $x = y^2 - 8$.

18 $y = x^3$, $x = y^3$.
19 $y = x^3$, $y = 32\sqrt{x}$.
20 Find the area in Example 2 by integrating with respect to y, first with one integrand from $y = -1$ to $y = 2$, and then with another integrand from $y = 2$ to $y = 3$.
21 Find in two ways the area under $y = x^2$ from $x = 0$ to $x = 4$.
22 Find in two ways the area under $y = x^3$ from $x = 0$ to $x = 2$.
23 Find the area bounded by
(a) the x-axis and $y = x^2 - x^3$;
(b) the y-axis and $x = 2y - y^2$.
24 The area between $x = y^2$ and $x = 4$ is divided into two equal parts by the line $x = a$. Find a.
*25 Find the area between $y = x^3$ and its tangent at $x = 1$.
26 Find the area above the x-axis bounded by $y = 1/x^2$, $x = 1$, and $x = b$, where b is some number greater than 1. The result will depend on b. What happens to this area as $b \to \infty$?
27 Solve Problem 26 with $y = 1/x^2$ replaced by $y = 1/\sqrt{x}$.
28 Solve Problem 26 with $y = 1/x^2$ replaced by $y = 1/x^p$, where p is a fixed positive number greater than 1. What happens if p is a fixed positive number less than 1?
29 In each case find the area of the region bounded by the given curves over the stated interval:
(a) $y = \sin x$ and $y = \cos x$, $0 \le x \le \pi/2$;
(b) $y = \sin x$ and $y = \cos 2x$, $0 \le x \le \pi/4$;
(c) $y = x$ and $y = \sin x$, $-\pi/4 \le x \le \pi/2$.
30 Sketch the graph of $y = \sin^4 x \cos x$ on the interval $0 \le x \le \pi/2$, and find the area of the region between the curve and the x-axis.
31 A rectangle with sides parallel to the axes has one vertex at the origin and the opposite vertex on the curve $y = ax^n$ at the point where $x = b$ ($a > 0$, $n > 0$, and $b > 0$). Show that the fraction of the area of the rectangle that lies below the curve depends on n but is independent of a and b.
32 By calculating the areas involved by integration, verify the theorem of Archimedes stated in Section 6.2 for the parabolic segment cut off from the parabola $y = x^2$ by the line $y = -x + 2$. Hint: Begin by finding the points A, B, and C in Fig. 6.5.
*33 The shaded region inside the square of side a shown in Fig. 7.6 consists of all points that are closer to the center of the square than to its boundary. Find the area of this region.

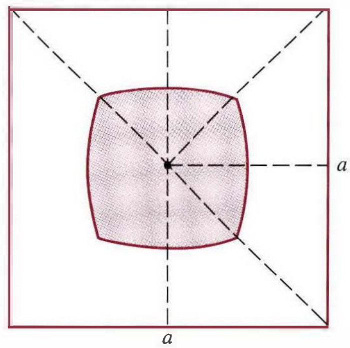


Figure 7.6

7.3 VOLUMES: THE DISK METHOD

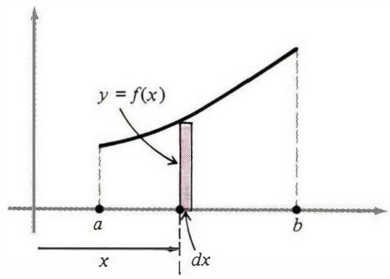
If the region under a curve $y = f(x)$ between $x = a$ and $x = b$ is revolved about the x-axis, it generates a three-dimensional figure called a *solid of revolution*. The symmetrical shape of this solid makes its volume easy to compute.

The situation is illustrated in Fig. 7.7. On the left we show the region itself, together with a typical thin vertical strip of thickness dx whose base lies on the x-axis. When the region is revolved about the x-axis, this strip generates a thin circular disk shaped like a coin, as shown on the right, with radius $y = f(x)$ and thickness dx. The volume of this disk is our *element of volume* dV. Since the disk is a cylinder, its volume is clearly the area of the circular face times the thickness,

$$dV = \pi y^2\, dx = \pi f(x)^2\, dx. \tag{1}$$

We now imagine that the solid of revolution is filled with a very large number of very thin disks like this, so that the total volume is the sum of all the elements of volume as our typical disk sweeps through the solid from left to right, that is, as x increases from a to b:

$$V = \int dV = \int \pi y^2\, dx = \int_a^b \pi f(x)^2\, dx. \tag{2}$$



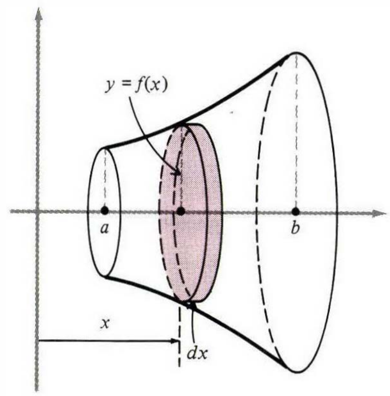


Figure 7.7

This is another fundamental formula that students should *not* memorize. Instead, it is much better to understand it so clearly that memorization is unnecessary.

Some students may feel that formula (2) cannot give the *exact* volume of the solid, because it doesn't take into account the volume of the small "peeling" around the outside of the disk in Fig. 7.7. However, just as in the calculation of areas, this slight apparent error visible in the figure—due to using disks instead of actual slices—disappears as a consequence of the limit process that is part of the meaning of the integral sign. We can therefore calculate volumes using formula (2) and have full confidence that our results will be exactly correct, not merely approximations.

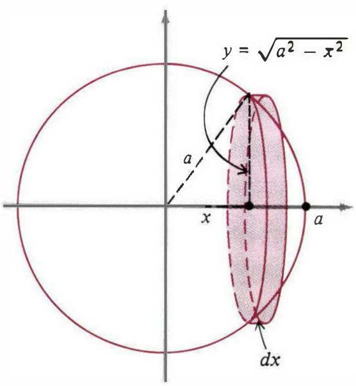


Figure 7.8

Example 1 A sphere can be thought of as the solid of revolution generated by revolving a semicircle about its diameter (Fig. 7.8). If the equation of the semicircle is $x^2 + y^2 = a^2$, $y \geq 0$, then $y = \sqrt{a^2 - x^2}$ and the element of volume is

$$dV = \pi y^2\,dx = \pi(a^2 - x^2)\,dx.$$

By using the left-right symmetry of the sphere, we can find its total volume by integrating dV from $x = 0$ to $x = a$ and multiplying by 2:

$$\begin{aligned} V = 2\int_0^a \pi(a^2 - x^2)\,dx &= 2\pi\left(a^2x - \tfrac{1}{3}x^3\right)\Big]_0^a \\ &= \tfrac{4}{3}\pi a^3. \end{aligned} \tag{3}$$

This result confirms the well-known (but little-understood) formula from elementary geometry. If we integrate dV only from $x = a - h$ to $x = a$, we obtain the formula for the volume of a *segment* of a sphere of thickness h,

$$\begin{aligned} V = \int_{a-h}^a \pi(a^2 - x^2)\,dx &= \pi\left(a^2x - \tfrac{1}{3}x^3\right)\Big]_{a-h}^a \\ &= \pi\{\tfrac{2}{3}a^3 - [a^2(a-h) - \tfrac{1}{3}(a-h)^3]\} \\ &= \pi h^2(a - \tfrac{1}{3}h), \end{aligned}$$

after some algebraic simplification. It should be noticed that this formula reduces to (3) when $h = 2a$.

Example 2 Another important formula from elementary geometry states that a cone of height h and radius of base r has volume $V = \frac{1}{3}\pi r^2 h$; or equivalently, the volume is one-third the volume of the circumscribed cylinder. To obtain this formula by integration, and thereby to understand the origin of the factor $\frac{1}{3}$, we think of the cone as the solid of revolution generated by revolving the right triangle shown in the first quadrant of Fig. 7.9 about its base on the x-axis. The hypotenuse of this triangle is clearly part of the straight line $y = (r/h)x$, so the element of volume is

$$dV = \pi y^2\, dx = \frac{\pi r^2}{h^2} x^2\, dx.$$

We now obtain the total volume by integrating dV from $x = 0$ to $x = h$,

$$V = \int_0^h \frac{\pi r^2}{h^2} x^2\, dx = \frac{\pi r^2}{h^2} \cdot \frac{1}{3} x^3 \bigg]_0^h = \tfrac{1}{3}\pi r^2 h. \tag{4}$$

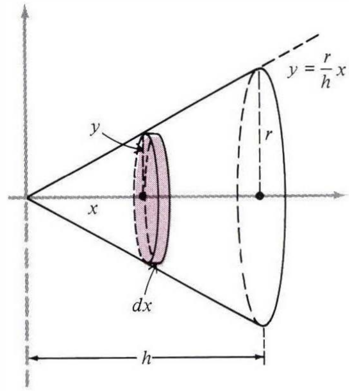


Figure 7.9

For obvious reasons, the method of these examples is usually called the *disk method.* The same idea can be applied to solids of other types, in which the element of volume is not necessarily a circular disk. Suppose that each cross section of a solid made by a plane perpendicular to a certain line is a triangle or square or some other geometric figure whose area is easy to find. Then our element of volume dV is the product of this area and the thickness of a thin slice, and we can calculate the total volume of the solid by the *method of moving slices* as suggested in Fig. 7.10:

$$dV = A(x)\, dx, \qquad V = \int dV = \int_a^b A(x)\, dx.$$

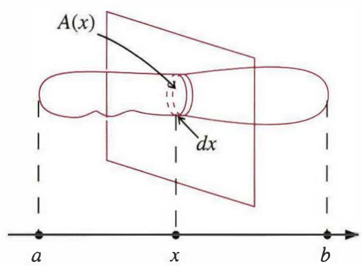


Figure 7.10

Example 3 A wedge is cut from the base of a cylinder of radius a by a plane passing through a diameter of the base and inclined at an angle of 45° to the base. To find the volume of this wedge, we first draw a careful sketch (Fig. 7.11). A slice perpendicular to the edge of the wedge, as shown, has a triangular face. By using the notation established in the figure, we see that the volume of this slice is

$$\begin{aligned} dV &= \tfrac{1}{2}\sqrt{a^2 - y^2} \cdot \sqrt{a^2 - y^2}\, dy \\ &= \tfrac{1}{2}(a^2 - y^2)\, dy, \end{aligned}$$

so the volume of the wedge is

$$\begin{aligned} V &= 2\int_0^a \tfrac{1}{2}(a^2 - y^2)\, dy = a^2 y - \tfrac{1}{3}y^3 \bigg]_0^a \\ &= \tfrac{2}{3}a^3. \end{aligned}$$

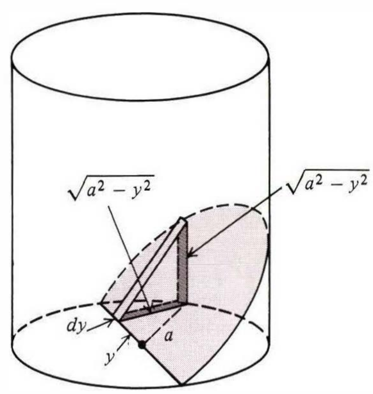


Figure 7.11

A vertical slice parallel to the edge of the wedge evidently has a rectangular face (students should draw their own sketches). If x is the distance from the edge to this slice, then with careful thought we can see that this time the element of volume is

$$dV = 2x\sqrt{a^2 - x^2}\, dx,$$

and therefore

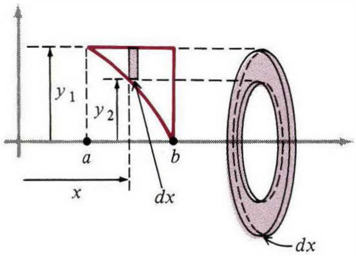


Figure 7.12

$$V = \int_0^a 2x\sqrt{a^2 - x^2}\, dx$$

$$= -\tfrac{2}{3}(a^2 - x^2)^{3/2}\Big]_0^a = \tfrac{2}{3}a^3,$$

as before.

Remark 1 The following minor variation of the disk method is often useful and is necessary for many of the problems at the end of this section. Suppose the strip being revolved about an axis is separated from this axis by a certain distance, as suggested on the left in Fig. 7.12. In this case the element of volume generated by the strip is a disk with a hole in it—what might be described as a *washer* (this washer is moved out to the right in the figure for the sake of clarity). The volume of this washer is the volume of the disk minus the volume of the hole,

$$dV = \pi(y_1{}^2 - y_2{}^2)\, dx.$$

The total volume of the solid of revolution is therefore

$$V = \int dV = \int_a^b \pi(y_1{}^2 - y_2{}^2)\, dx,$$

where y_1 and y_2, the outer and inner radii of the washer, are determined as functions of x from the given conditions of the problem. This procedure for calculating volumes is called—naturally enough—the *washer method.* It applies to solids of revolution that have hollow spaces inside them.

Remark 2 Both of the formulas obtained in Examples 1 and 2 can be vividly expressed and easily remembered by stating the volumes of the cone and sphere as simple fractional parts of the volumes of the circumscribed cylinders (Fig. 7.13).

Remark 3 Formula (3) for the volume of a sphere was discovered by Archimedes in the third century B.C. Since he used a very beautiful and ingenious early form of integration, we give his argument in the Appendix at the end of this chapter for those who wish to examine it.

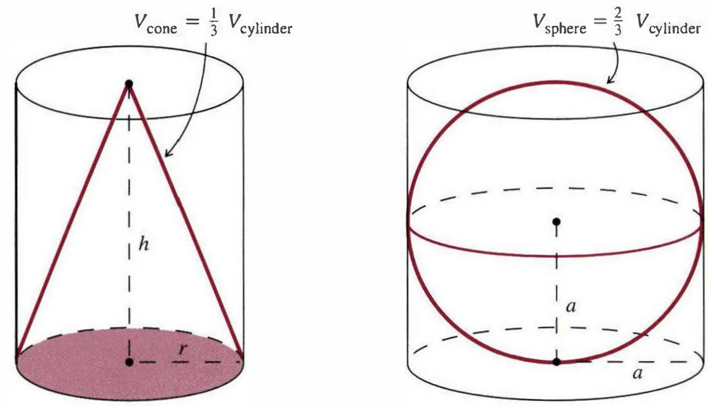


Figure 7.13

PROBLEMS

1 Find the volume of the solid of revolution generated when the area bounded by the given curves is revolved about the x-axis:
(a) $y = \sqrt{x}$, $y = 0$, $x = 4$;
(b) $y = 2x - x^2$, $y = 0$;
(c) $y^3 = x$, $y = 0$, $x = 1$;
(d) $y = x$, $y = 1$, $x = 0$;
(e) $x = 2y - y^2$, $x = 0$;
(f) $x^{2/3} + y^{2/3} = a^{2/3}$, first quadrant.

2 Problem 14 in Section 6.7 is concerned with the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad a > b > 0.$$

If the area inside this ellipse is revolved about the x-axis, the resulting solid (which resembles a football) is called a *prolate spheroid*. Find its volume. [If $a < b$, the solid is called an *oblate spheroid*. Observe that the volume formula is the same regardless of how a and b are related to each other, and also that it reduces to formula (3) when $b = a$.]

3 The horizontal cross section of a certain pyramid at a distance x down from the top is a square of side $(b/h)x$, where h is the height of the pyramid and b is the side of the base. Show that the volume of the pyramid is one-third the area of the base times the height.

4 A horn-shaped solid is generated by a moving circle perpendicular to the y-axis whose diameter lies in the xy-plane and extends from $y = 27x^3$ to $y = x^3$. Find the volume of this solid between $y = 0$ and $y = 8$.

5 The square bounded by the axes and the lines $x = 2$, $y = 2$ is cut into two parts by the curve $y^2 = 2x$. Show that these parts generate equal volumes when revolved about the x-axis.

6 The two areas described in Problem 5 are revolved about the line $x = 2$. Find the volumes generated.

7 A tent consists of canvas stretched from a circular base of radius a to a vertical semicircular rod fastened to the base at the ends of a diameter. Find the volume of this tent.

8 The base of a solid is a quadrant of a circle of radius a. Each cross section perpendicular to one edge of the base is a semicircle whose diameter lies in the base. Find the volume.

9 The base of a certain solid is the circle $x^2 + y^2 = a^2$. Each plane perpendicular to the x-axis intersects the solid in a square cross section with one side in the base of the solid. Find its volume.

10 If the area bounded by the parabola $y = H - (H/R^2)x^2$ and the x-axis is revolved about the y-axis, the resulting bullet-shaped solid is a segment of a paraboloid of revolution with height H and radius of base R. Show that its volume is half the volume of the circumscribing cylinder.

11 If the circle $(x - b)^2 + y^2 = a^2$ $(0 < a < b)$ is revolved about the y-axis, it generates a doughnut-shaped solid called a *torus*. Find the volume of this torus by the washer method. Hint: If necessary, use the result of Problem 13 in Section 6.7. (Notice the remarkable fact that the volume of the torus is the product of the area of the circle and the distance traveled by its center as it revolves about the y-axis.)

12 Find the volume of the solid formed by revolving the area inside the curve $x^2 + y^4 = 1$ about (a) the x-axis; (b) the y-axis.

13 The base of a certain solid is an equilateral triangle of side a, with one vertex at the origin and an altitude along the x-axis. Each plane perpendicular to the x-axis intersects the solid in a square cross section with one side in the base of the solid. Find the volume.

14 Each plane perpendicular to the x-axis intersects a certain solid in a circular cross section whose diameter lies in the xy-plane and extends from $x^2 = 4y$ to $y^2 = 4x$. The solid lies between the points of intersection of these curves. Find its volume.

15 The base of a certain solid is the region bounded by the parabola $x^2 = 4y$ and the line $y = 9$, and each cross section perpendicular to the y-axis is a square with one side in the base. If a plane perpendicular to the y-axis cuts this solid in half, how far from the origin is this plane?

***16** Two great circles lying in planes that are perpendicular to each other are drawn on a wooden sphere of radius a. Part of the sphere is then shaved off in such a way that each cross section of the remaining solid that is perpendicular to the common diameter of the two great circles is a square whose vertices lie on these circles. Find the volume of this solid.

***17** The axes of two cylinders, each of radius a, intersect at right angles. Find the common volume. Hint: Drawing this figure is nine-tenths of the problem. To do this, draw the usual x-axis and y-axis, and also a z-axis "coming out of the paper," so that the xz-plane is horizontal. Let the axis of one cylinder be the x-axis and the axis of the other the z-axis, but draw only the quarter of the first cylinder that lies in front of the xy-plane and above the xz-plane, and the quarter of the second that lies to the right of the yz-plane and above the xz-plane. Their intersection is one-eighth the total volume and is not difficult to sketch. Now consider horizontal cross sections.

18 Consider the area in the first quadrant under the curve $x^2y^3 = 1$ and to the right of $x = 1$. By integrating from $x = 1$ to $x = b$ and then letting $b \to \infty$, show that this area is infinite, but that on revolving it about the x-axis we obtain a finite volume.

19 A birdbath 4 in deep has the shape of a segment of a sphere of diameter 16 in. It contains a decorative lead cannonball of diameter 6 in. If the birdbath is filled with water to a depth of x inches, how much water does it contain?

20 In Example 3, find the volume of the wedge if the plane through a diameter of the base is inclined at an angle of 30° to the base.

21 *Cavalieri's principle* states that if two solids have the same height H, and if sections made by planes parallel to the bases and at the same distance x from these bases always have equal areas $A(x)$ and $B(x)$, then the two solids have equal volumes.†

(a) Prove this by integration.

(b) Use this principle and the formula for the volume of a cone (Example 2) to derive the formula for the volume of a sphere (Example 1) by examining Fig. 7.14. (The figure on the right is a cylinder with two conical hollows in it whose common vertex is at the center.)

22 Water evaporates from an open bowl of unspecified shape at a rate proportional to the area of the water surface; that is,

$$\frac{dV}{dt} = -cA(h),$$

where V is the volume of water, $A(h)$ is the area of the water surface when the depth is h, and c is a positive constant.

†Bonaventura Cavalieri (1598–1647), the Italian mathematician mentioned in Section 6.5, was a disciple of Galileo and clearly had many excellent ideas of his own.

(a) Show that

$$\frac{dh}{dt} = -c,$$

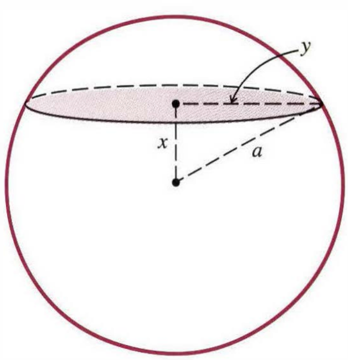
so that the water level drops at a constant rate regardless of the shape of the bowl. Hint: $V = \int_0^h A(x)\,dx$.

(b) If $h = h_0$ when $t = 0$, when will the bowl be empty?

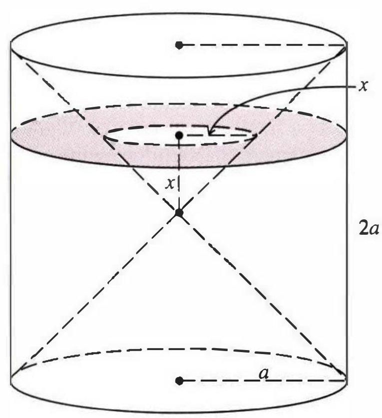
23 A cylindrical drinking glass of height h and radius of base a is full of water. The glass is tilted, and the water spilled, until the remaining water just covers the bottom of the glass. Use rectangular cross sections to calculate how much water remains in the glass. Hint: $\int_{-a}^{a} \sqrt{a^2 - x^2}\,dx = \frac{1}{2}\pi a^2$, because the integral represents the area of a semicircle of radius a.

24 Solve the preceding problem without calculation, by using only common sense and geometry.

25 The *clepsydra,* or ancient water clock, was a bowl from which water was allowed to escape through a small hole in the bottom. It was often used in Greek and Roman courts to time the speeches of lawyers in order to keep them from talking too much. Let $y = f(x)$ be a curve that rises from the origin into the first quadrant of the xy-plane, and assume the clepsydra has the shape of the surface obtained by revolving this curve about the y-axis. According to *Torricelli's law,* water flows out through the hole in the bottom of the clepsydra at the speed it would acquire in falling freely from the water level to the level of the hole. Find what the function $y = f(x)$ must be in order to guarantee that the water level falls at a constant rate. Hint: By equations (6) and (7) in Section 5.5, Torricelli's law implies that the exit speed of the water through the hole is proportional to the square root of the depth of the water.



area $= \pi y^2$
$= \pi(a^2 - x^2)$



area $= \pi a^2 - \pi x^2$
$= \pi(a^2 - x^2)$

Figure 7.14

7.4 VOLUMES: THE METHOD OF CYLINDRICAL SHELLS

There is another method of finding volumes that is often more convenient than those described in Section 7.3.

To understand this method, consider the region shown on the left in Fig. 7.15, that is, the region in the first quadrant bounded by the axes and the indicated curve $y = f(x)$. If this region is revolved about the x-axis, then the thin vertical strip in the figure generates a disk, and we can calculate the total volume of the solid by adding up (or integrating) the volumes of these disks from $x = 0$ to $x = b$. This, of course, is the disk method described in Section 7.3. However, if the region is revolved about the y-axis, as shown in the center of the figure, then we get an entirely different solid of revolution and the vertical strip generates a thin-walled cylindrical shell. This shell can be thought of as resembling a soup can whose top and bottom have been removed, or perhaps a thin-walled cardboard mailing tube. Its volume dV is essentially the area of the inner cylindrical surface $(2\pi xy)$* times the thickness of the wall (dx), so

$$dV = 2\pi xy\, dx. \tag{1}$$

As the radius x of this shell increases from $x = 0$ to $x = b$, we can see from Fig. 7.15 that the resulting series of cylindrical shells fills the solid of revolution from the axis outward, in much the same way as the cylindrical growth layers in the trunk of a tree fill the trunk from the axis outward. The total volume of this solid is therefore the sum—or integral—of the elements of volume dV,

$$V = \int dV = \int 2\pi xy\, dx = \int_0^b 2\pi x f(x)\, dx, \tag{2}$$

since $y = f(x)$. In principle, this volume V can also be found by using horizontal disks generated by thin horizontal strips; however, this could turn out to be difficult, since the given equation $y = f(x)$ would have to be solved for x in terms of y.

Just as in our other applications of integration, formulas (1) and (2) give brief expression to a complex process involving limits of sums; and as usual, we omit the details of this process in the interests of clarity.

Also as usual, we suggest that students would be wise not to memorize formula (2). This formula is somewhat similar to the corresponding formula for the disk method, and students who try to memorize them and use them without think-

*Remember: The curved surface area of a cylinder of radius x and height y is obtained by "unrolling" the cylinder so that it becomes a rectangle of base $2\pi x$ and height y; see the right side of Fig. 7.15.

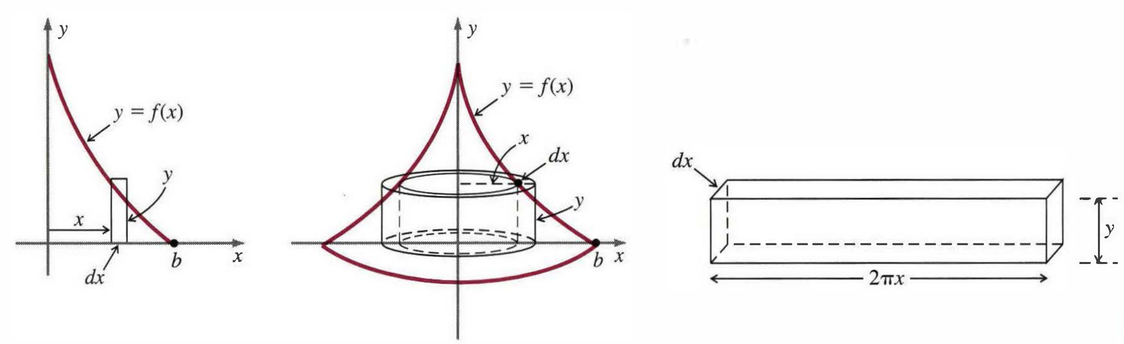


Figure 7.15

ing about their meaning will almost certainly confuse them and come to grief. It is better to sketch a figure and construct (1) directly from the visible evidence of this figure, and then form (2) by integration. Also, this approach has the further advantage that we are not tied to any particular notation, and can easily adapt the basic idea to solids of revolution about various axes.

Example 1 In Example 1 of Section 7.3 we calculated the volume of a sphere by the disk method. We now solve the same problem by the shell method (see Fig. 7.16). The volume of the shell shown in the figure is

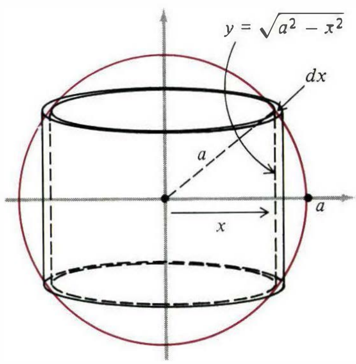


Figure 7.16

$$\begin{aligned} dV &= 2\pi x(2y)\,dx \\ &= 4\pi x\sqrt{a^2 - x^2}\,dx. \end{aligned}$$

The volume of the sphere is therefore

$$\begin{aligned} V &= 4\pi\int_0^a x\sqrt{a^2 - x^2}\,dx = 4\pi(-\tfrac{1}{3})(a^2 - x^2)^{3/2}\Big]_0^a \\ &= -\frac{4\pi}{3}(a^2 - x^2)^{3/2}\Big]_0^a = \tfrac{4}{3}\pi a^3. \end{aligned}$$

In this connection we can profitably consider a related problem: If a vertical hole of diameter a is bored through the center of the sphere, find the remaining volume. For this, it clearly suffices to integrate dV as the radius x of the shell increases from $x = a/2$ to $x = a$, so

$$\begin{aligned} V &= 4\pi\int_{a/2}^a x\sqrt{a^2 - x^2}\,dx = -\frac{4\pi}{3}(a^2 - x^2)^{3/2}\Big]_{a/2}^a \\ &= \frac{4\pi}{3}\left(\frac{3}{4}a^2\right)^{3/2} = \frac{4\pi}{3}\left(\frac{3\sqrt{3}}{8}a^3\right) = \frac{\sqrt{3}}{2}\pi a^3. \end{aligned}$$

This problem could be solved by the washer method, but the shell method is much more convenient.

Example 2 The region in the first quadrant above $y = x^2$ and below $y = 2 - x^2$ is revolved about the y-axis (Fig. 7.17). To find the volume by the shell method, we observe that the height of our typical shell is $y = (2 - x^2) - x^2 = 2 - 2x^2$, so

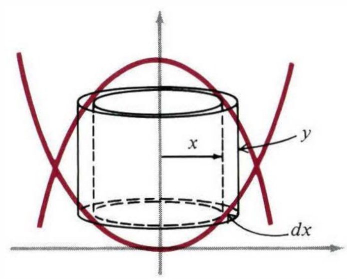


Figure 7.17

$$\begin{aligned} dV &= 2\pi xy\,dx = 2\pi x(2 - 2x^2)\,dx \\ &= 4\pi(x - x^3)\,dx; \end{aligned}$$

and since the curves intersect at $x = \pm 1$, we have

$$\begin{aligned} V &= 4\pi\int_0^1 (x - x^3)\,dx \\ &= 4\pi(\tfrac{1}{2}x^2 - \tfrac{1}{4}x^4)\Big]_0^1 = \pi. \end{aligned}$$

Students often wish to set up this integral incorrectly by integrating from $x = -1$ to $x = 1$. The reason why this is incorrect can be seen by understanding from the geometry that our typical shell sweeps through the solid from the axis outward: x is the *radius* of the shell, and this radius increases from 0 to 1, not from -1 to 1.

Notice that if we attempt to solve this problem by the disk method, then it is necessary to calculate two separate integrals—one referring to the volume below the points of intersection of the two curves, and the other to the volume above.

Example 3 *Blood flow.* The great artery of the human body—the aorta—is a tube about as large as the base of an average human thumb. The heart pumps blood through it so powerfully that blood particles near the center move at speeds of about 50 cm/s (20 in/s). On the other hand, blood is a viscous liquid, and near the artery wall the blood tends to stick to the wall, and its speed there is essentially zero. The problem of calculating the total flow under these circumstances requires integration by the method of cylindrical shells.

We begin with the very simple idea that if a liquid flows through a cylindrical tube with constant speed s_0, then the volume of liquid passing a fixed point per unit time (the *flow* F) is $s_0 A$, where A is the area of a cross section of the tube (Fig. 7.18).

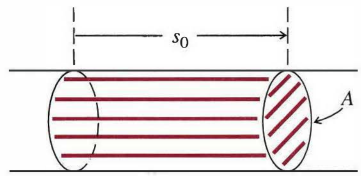


Figure 7.18

However, we know that the flow of blood in an artery in the human body is much more complicated than this. Let us assume that the artery is a cylindrical tube with radius R and length L (Fig. 7.19). Because of the viscosity mentioned above, the flow of blood takes place in thin cylindrical layers, with the blood in each layer moving at approximately constant speed and the blood in different layers moving at different speeds. In this so-called *laminar flow* the blood moves slowly near the artery wall and faster near the center, as suggested in Fig. 7.19, so that the inner layers slip past the outer ones (Fig. 7.20).

The precise relation between the speed s and the distance r from the center is given by the formula

$$s = \frac{P}{4\eta L}(R^2 - r^2), \tag{3}$$

where P is the pressure difference between the ends of the artery and η (eta) is the viscosity of blood.* We notice that this formula gives zero speed if $r = R$ and

*Formula (3) can be derived from general principles in the theory of viscous fluid dynamics. The details can be found on pp. 39–41 of R. L. Whitmore, *Rheology of the Circulation* (Pergamon Press, 1968).

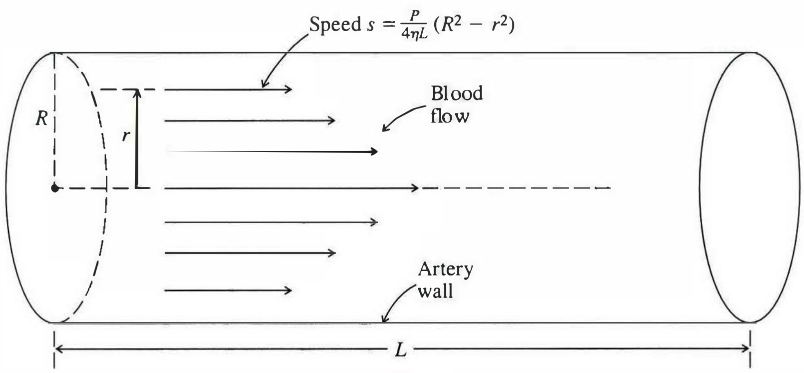


Figure 7.19

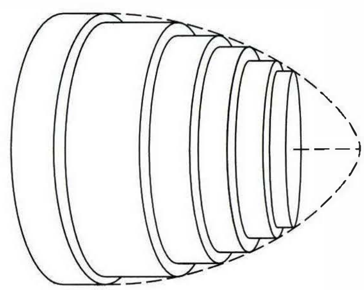


Figure 7.20

maximum speed $PR^2/4\eta L$ if $r = 0$. It is customary to measure R, r, and L in centimeters (cm), P in dynes/cm^2, and η in dyne-s/cm^2, so that s is measured in cm/s. A typical value for R in the human body is $R = 0.2$ cm, and a realistic value for the constant $P/4\eta L$ is 500. With these values formula (3) becomes

$$\begin{aligned} s &= 500(0.2^2 - r^2) \\ &= 20 - 500r^2 \qquad \text{cm/s.} \end{aligned} \tag{4}$$

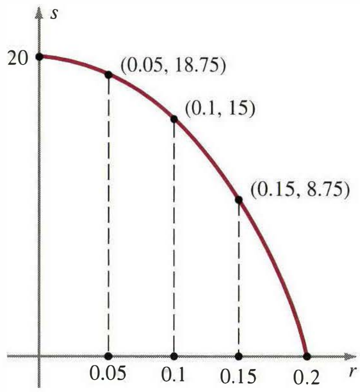


Figure 7.21

The graph of this function is part of a parabola (Fig. 7.21), and this graph shows how quickly the speed of a blood particle approaches zero as its position approaches the wall of the artery. Thus, at the center the speed s is 20 cm/s, but when $r = 0.15$ we see that s is only $20 - 500(0.15)^2 = 8.75$ cm/s.

Now, to calculate the *flow* F (the total volume of blood passing a fixed point per unit time) we write down the element of flow dF in a thin cylindrical shell of radius r and thickness dr:

$$\begin{aligned} dF &= s \cdot 2\pi r\, dr \\ &= \frac{P}{4\eta L}(R^2 - r^2) \cdot 2\pi r\, dr \\ &= \frac{\pi P}{2\eta L}(R^2 r - r^3)\, dr. \end{aligned}$$

All that remains is to add together these elements of flow over all the shells, that is, to integrate from 0 to R:

$$\begin{aligned} F = \int dF &= \int_0^R \frac{\pi P}{2\eta L}(R^2 r - r^3)\, dr \\ &= \frac{\pi P}{2\eta L}\int_0^R (R^2 r - r^3)\, dr \\ &= \frac{\pi P}{2\eta L}\left(\tfrac{1}{2}R^2 r^2 - \tfrac{1}{4}r^4\right)\Big]_0^R = \frac{\pi P}{8\eta L}R^4. \end{aligned}$$

This formula,

$$F = \frac{\pi P}{8\eta L}R^4, \tag{5}$$

is called *Poiseuille's law* in the field of cardiovascular physiology. It shows that the flow is proportional to the fourth power of the radius of the artery, so doubling this radius increases the flow by a factor of 16.*

*Jean Poiseuille (1799–1869) was a French physician-physiologist whose experimental studies of the flow of liquids through thin glass tubes have rarely been equaled since his classical paper of 1846. By varying one parameter and holding all the others fixed, he learned from his experiments that P is directly proportional to the length L, to the viscosity η, to the flow F, and inversely proportional to the fourth power R^4 of the radius R, so that

$$P = k \cdot \frac{L\eta F}{R^4}.$$

Of course, the fact that Poiseuille's constant of proportionality k equals $8/\pi$ can only be learned from the mathematics. See p. 134 of H. S. Badeer, *Cardiovascular Physiology* (Karger Publishing Co., 1984). For the somewhat surprising role of Isaac Newton in this subject, see pp. 50–51 of A. C. Burton, *Physiology and Biophysics of the Circulation* (Year Book Medical Publishers, 1965).

PROBLEMS

1 Solve the problem of the sphere with the hole bored through it (Example 1) by the washer method.

2 Solve the problem in Example 2 by the disk method.

In Problems 3–8, sketch the region bounded by the given curves and use the shell method to find the volume of the solid generated by revolving this region about the given axis.

3 $y = \sqrt{x}$, $x = 4$, $y = 0$; the y-axis.

4 $x^2 = 4y$, $y = 4$; the x-axis.

5 $y = x^3$, $x = 3$, $y = 0$; the y-axis.

6 $x = y^2$, $x^2 = 8y$; the x-axis.

7 $y = \dfrac{1}{x}$, $x = a$, $x = b$ $(0 < a < b)$, $y = 0$; the y-axis.

8 $y = x^2$, $y = \frac{1}{4}(3x^2 + 1)$; the y-axis.

9 The region bounded by $y = x/\sqrt{x^3 + 8}$, the x-axis, and the line $x = 2$ is revolved about the y-axis. Find the volume of the solid generated in this way. (Observe that the washer method is not practical in this problem.)

10 A hole of radius $\sqrt{3}$ is bored through the center of a sphere of radius 2. Find the volume removed.

11 Consider the region in the first quadrant bounded by $y = 4 - x^2$ and the axes.

(a) Use both the disk method and the shell method to find the volume of the solid generated when this region is revolved about the y-axis.

*(b) Use both methods to find the volume of the solid generated when this region is revolved about the x-axis.

12 Let r and h be positive numbers. The region bounded by the line $x/r + y/h = 1$ and the axes is revolved about the y-axis. Use the shell method to obtain the standard formula for the volume of a cone.

13 A spherical ring is the solid that remains after drilling a hole through the center of a solid sphere. If the sphere has radius a and the ring has height h, prove the remarkable fact that the volume of the ring depends on h but not on a.

14 The parabola $a^2y = bx^2$, $0 \le y \le b$, is revolved about the y-axis. Use the shell method to show that the volume of the resulting paraboloid is one-half the volume of the cylinder with the same height and base.

15 The region in the first quadrant above $y = 3x^2$ and below $y = 4 - 6x^2$ is revolved about the y-axis. Find the volume generated in this way.

16 For the artery of radius 0.2 cm (about $\frac{1}{12}$ in) described by equation (4), calculate the flow F and also the total volume of blood passing a fixed point in 1 hour.

17 High blood pressure is one of the consequences of narrowing of the arteries by unwelcome fatty deposits, because to maintain the same flow to the tissues the heart must pump harder. If P_0 and R_0 are normal values of the pressure and radius for a particular artery, and the abnormal values due to narrowing are P and R, show that for the flow to remain constant we must have

$$\frac{P}{P_0} = \left(\frac{R_0}{R}\right)^4.$$

If the radius of the artery is reduced to seven-eighths of its normal value, how much is the pressure increased?

18 A cylindrical can partly filled with water is rotated about its axis with constant angular velocity ω. As the rotation proceeds, the water level rises along the wall and sinks in the center to form the concave surface shown in Fig. 7.22. Show that this surface has the shape of the surface of revolution formed by revolving the parabola

$$y = \frac{\omega^2}{2g}x^2 + h$$

about its axis, where g (as usual) is the acceleration due to gravity. Hint: The centripetal force acting on a particle of water of mass m at the free surface is $mx\omega^2$, where x is the distance from the axis, and this is the resultant according to the parallelogram law (as shown in the figure) of the downward gravitational force mg and the reaction force R normal to the surface which is due to other nearby particles of water.

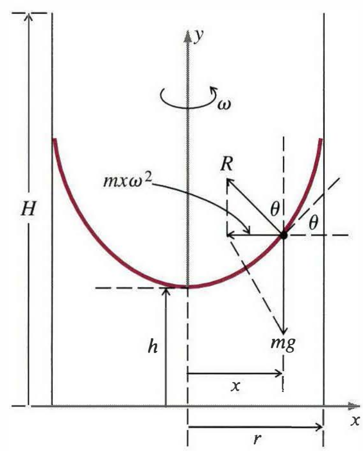


Figure 7.22

(a) Let V_0 be the given volume of water, and by calculating V_0 by the method of cylindrical shells, express the depth h as a function of ω.

(b) If the can rotates faster and faster, then either the bottom will be exposed, as in Fig. 7.23*a*, or water will begin to spill out the top, as in Fig. 7.23*b*. If the can is originally half full, which happens first?

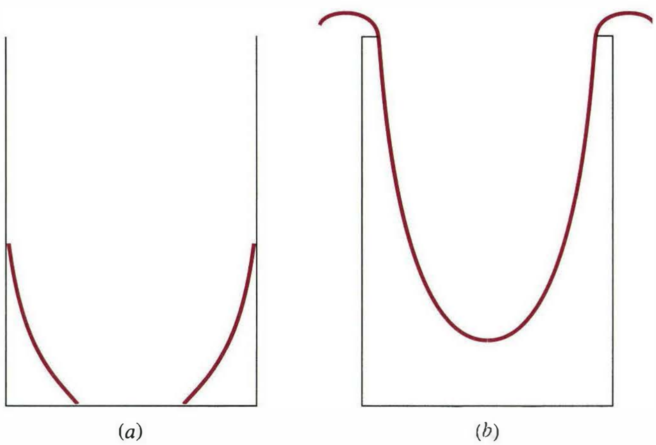


Figure 7.23

7.5 ARC LENGTH

An *arc* is the part of a curve that lies between two specific points A and B, as shown on the left in Fig. 7.24. Physically, the length of an arc is a very simple concept. Mathematically, it is somewhat more complicated. From the physical point of view, we merely bend a piece of string to fit the curve from A to B, mark the points corresponding to A and B, straighten out the string, and measure its length with a ruler.

This process can be carried out by means of an approximation procedure that lends itself to mathematical treatment, as follows. Divide the arc AB into n parts by using points $P_0 = A, P_1, P_2, \ldots, P_n = B$; place pins at these points; and let the string stretch in short straight-line paths from each pin to the next. We illustrate this idea on the right in Fig. 7.24 with $n = 3$. The part of this string between A and B is evidently shorter than the arc, since a straight line is the shortest distance between two points. However, if we take larger and larger values of n, and at the same time require that the pins be placed closer and closer together, then the length of the string should approach the length of the arc. We now ex-

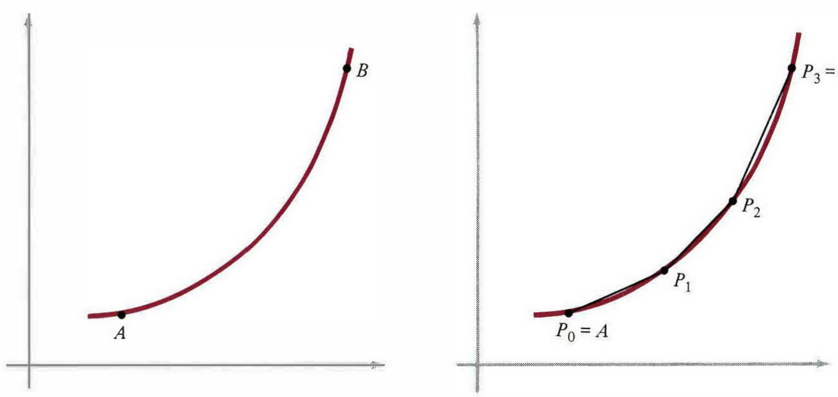


Figure 7.24

press this idea in mathematical language and derive a practical method of calculating arc length by integration.

Let us assume that the arc under discussion is the graph of a continuous function $y = f(x)$ for $a \leq x \leq b$. We partition the interval $[a, b]$ into n subintervals by using points $x_0 = a, x_1, \ldots, x_{k-1}, x_k, \ldots, x_n = b$, as shown in Fig. 7.25. We let P_k be the point (x_k, y_k), where $y_k = f(x_k)$. The total length of the polygonal path $P_0P_1 \cdots P_{k-1}P_k \cdots P_n$ is the sum of the lengths of the chords joining each point to the next. If we write

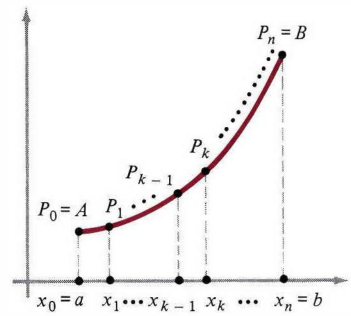


Figure 7.25

$$\Delta x_k = x_k - x_{k-1} \qquad \text{and} \qquad \Delta y_k = y_k - y_{k-1}, \qquad k = 1, 2, \ldots, n,$$

then it is clear by the Pythagorean theorem that we have

$$\begin{aligned} \text{length of } k\text{th chord} &= \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} \\ &= \sqrt{\left[1 + \frac{(\Delta y_k)^2}{(\Delta x_k)^2}\right](\Delta x_k)^2} \\ &= \sqrt{1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2}\, \Delta x_k, \end{aligned} \tag{1}$$

by first factoring $(\Delta x_k)^2$ out of the sum and then out of the square root sign. We now assume that $y = f(x)$ is not only continuous but also differentiable. This permits us to replace the ratio inside the radical, which is the slope of the chord joining P_{k-1} to P_k, by the value of the derivative at some point x_k^* between x_{k-1} and x_k:

$$\frac{\Delta y_k}{\Delta x_k} = f'(x_k^*), \qquad x_{k-1} < x_k^* < x_k.$$

The justification for this step lies in the fact that the chord is parallel to the tangent at some point on the curve between P_{k-1} and P_k.[†] This enables us to write (1) as

$$\text{length of } k\text{th chord} = \sqrt{1 + [f'(x_k^*)]^2}\, \Delta x_k,$$

so the total length of the polygonal path is

$$\sum_{k=1}^{n} \sqrt{1 + [f'(x_k^*)]^2}\, \Delta x_k. \tag{2}$$

We now obtain our conclusion by forming the limit of these sums as n approaches infinity and the length of the longest subinterval approaches zero:

$$\begin{aligned} \text{length of arc } AB &= \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} \sqrt{1 + [f'(x_k^*)]^2}\, \Delta x_k \\ &= \int_a^b \sqrt{1 + [f'(x)]^2}\, dx, \end{aligned} \tag{3}$$

provided $f'(x)$ is continuous so that this integral exists.

At first sight, formula (3) may appear to be rather hard to keep in mind. However, if we use the Leibniz notation dy/dx instead of $f'(x)$, then the following intuitive approach makes this formula much easier to grasp and remember. Let the

[†]This highly plausible assertion is called the *Mean Value Theorem*. This theorem is one of the cornerstones of the theory of calculus and is discussed in Section 2.6.

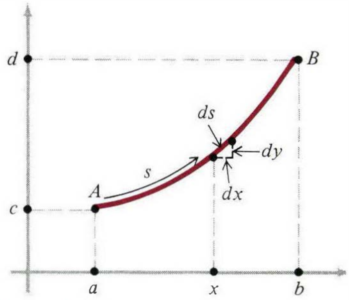


Figure 7.26

letter s denote the variable arc length from A to a variable point on the curve, as shown in Fig. 7.26. Let s be allowed to increase by a small amount ds, so that ds is the differential *element of arc length*, and let dx and dy be the corresponding changes in x and y. We think of ds as so small that this part of the curve is virtually straight, and therefore ds is the hypotenuse of a tiny right triangle called the *differential triangle*. For this triangle the Pythagorean theorem says that

$$ds^2 = dx^2 + dy^2, \tag{4}$$

and this simple equation is the source of all wisdom in the calculation of arc lengths.† If we solve (4) for ds, then factor dx^2 out of the sum and remove it from the square root sign, we clearly get

$$\begin{aligned} ds &= \sqrt{dx^2 + dy^2} \\ &= \sqrt{\left(1 + \frac{dy^2}{dx^2}\right) dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx. \end{aligned} \tag{5}$$

We now touch again on the basic theme of this chapter and point out that the total length of the arc AB can be thought of as the sum—or integral—of all the elements of arc ds as ds sweeps along the curve from A to B. In view of (5), this yields

$$\text{length of arc } AB = \int ds = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx, \tag{6}$$

which is (3). This formula tells us that x is the variable of integration and that y is to be treated as a function of x. However, it is sometimes more convenient to use y as the variable of integration and treat x as a function of y. In this case we replace (5) by

$$\begin{aligned} ds &= \sqrt{dx^2 + dy^2} \\ &= \sqrt{\left(\frac{dx^2}{dy^2} + 1\right) dy^2} = \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy, \end{aligned} \tag{7}$$

which is obtained by factoring dy instead of dx out of the square root sign. With y as the variable of integration, the integral for the length of the arc AB is then

$$\int ds = \int_c^d \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy, \tag{8}$$

which is sometimes easier to evaluate than (6).

Most mathematicians remember formulas (6) and (8) not by memorizing them as they stand, but instead by starting with (4) and mentally performing as needed the simple manipulations in (5) and (7). This way, the whole package of ideas is harder to forget, and we win a small skirmish in the constant struggle of memory against forgetting.

Example Find the length of the curve $y^2 = 4x^3$ between the points $(0, 0)$ and $(2, 4\sqrt{2})$.

†Parentheses are usually omitted in writing squares of differentials. Thus, ds^2 means $(ds)^2$ and not $d(s^2)$, etc.

Solution This curve is shown in Fig. 7.27, and the arc in question is the indicated piece of the curve in the first quadrant. If we solve for y, then we get

$$y = 2x^{3/2}, \qquad \text{so} \qquad \frac{dy}{dx} = 3x^{1/2}.$$

Formula (6) now yields

$$\text{length of arc} = \int_0^2 \sqrt{1 + 9x}\, dx = \frac{1}{9}\int_0^2 (1 + 9x)^{1/2}\, 9dx$$

$$= \tfrac{1}{9}\cdot\tfrac{2}{3}(1 + 9x)^{3/2}\Big]_0^2 = \tfrac{2}{27}(19\sqrt{19} - 1).$$

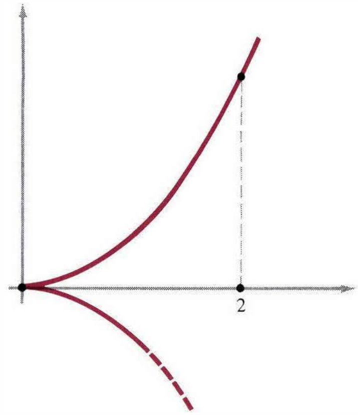


Figure 7.27

This calculation should serve as a warning, for if we try to find the length of an arc on almost any familiar curve, then because of the presence of the square root in (6) the resulting integral will probably be impossible for us to work out. At this stage we must choose our problems very carefully in order for the integrals to be computable. This should also make us aware of our urgent need for more integration techniques. Filling this need is the main purpose of the next three chapters.

Remark 1 It is possible to give an example of a continuous curve $y = f(x)$, $a \leq x \leq b$, that does not have a length. This very surprising fact suggests that the underlying theory of arc length is more complicated than it seems.† In the preceding discussion we found it necessary to assume that the function $y = f(x)$ has a continuous derivative. Such a curve is called a *smooth curve,* and the word "arc" is usually restricted to mean a piece of a curve with this property. A smooth curve is often described geometrically by saying that it has a "continuously turning tangent."

Remark 2 Some students may have the impression that equations (4) and (5) —which are equivalent to each other—are only approximately correct, because the differential triangle in Fig. 7.26 is only a "quasi-triangle" whose "hypotenuse" is not even a straight-line segment. But this is not the case. These equations are exactly correct, as the following argument shows. We know that (3) is valid, so the arc length s in Fig. 7.26 can be written as

$$s = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt,$$

using t as the dummy variable of integration. It is clear that s is a function of the upper limit x; and if we calculate the derivative of this function by using formula (13) in Section 6.7, then we get

$$\frac{ds}{dx} = \sqrt{1 + [f'(x)]^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2},$$

which is equivalent to (5).

†For examples and a few additional ideas, see Appendix A.7.

PROBLEMS

In Problems 1–8, find the length of the specified arc of the given curve.

1 $y^2 = x^3$ between (0, 0) and (4, 8).

2 $y = \frac{1}{4}x^4 + \frac{1}{8x^2}$, $1 \le x \le 2$.

3 $y = \frac{1}{3}x^3 + \frac{1}{4x}$, $1 \le x \le 3$.

4 $y = \frac{1}{3}\sqrt{x}(3 - x)$, $0 \le x \le 3$.

5 $x = \frac{1}{2}y^3 + \frac{1}{6y}$, $1 \le y \le 3$.

6 $y = \frac{5}{12}x^{6/5} - \frac{5}{8}x^{4/5}$, $1 \le x \le 32$.

7 $y = \frac{1}{3}(2 + x^2)^{3/2}$, $0 \le x \le 3$.

8 $y = \frac{2}{3}(1 + x^2)^{3/2}$, $0 \le x \le 3$.

9 Let A, B, C be positive constants. Show that the length of an arc of the curve $y = A(B + Cx^2)^{3/2}$ can be calculated by means of an integral not involving a square root if
(a) $A = \frac{2}{3}$ and $B^2C = 1$, in which case the curve is $y = (2/3B^3)(B^3 + x^2)^{3/2}$; or
(b) $B = 2$ and $3A\sqrt{C} = 1$, in which case the curve is $y = (1/3\sqrt{C})(2 + Cx^2)^{3/2}$.

Show that each of these curves includes Problems 7 and 8 as special cases.

10 The curve $x^{2/3} + y^{2/3} = a^{2/3}$ is called an *astroid* or a *hypocycloid of four cusps*. Sketch it and find its total length.

11 If $0 < a < b$ and n is not equal to 1 or -1, show that the length of

$$y = \frac{x^{n+1}}{n+1} + \frac{1}{4(n-1)} \cdot \frac{1}{x^{n-1}}$$

between $x = a$ and $x = b$ can be calculated by means of an integral not involving a square root. Notice that Problems 2 and 3 are special cases of this result.

12 In each case set up the integral for the arc length, but do not attempt to evaluate it (these integrals are beyond our capacity at the present stage of our work):
(a) $y = \sqrt{x}$, $1 \le x \le 4$;
(b) $y = x^2$, $0 \le x \le 1$;
(c) $y = x^3$, $0 \le x \le 1$;
(d) the part of $y = -x^2 + 4x - 3$ lying above the x-axis.

7.6 THE AREA OF A SURFACE OF REVOLUTION

Let us consider a smooth curve lying above the x-axis, as shown on the left in Fig. 7.24. When this curve is revolved about the x-axis, it generates a *surface of revolution.* We now set ourselves the problem of calculating the area of such a surface.

For reasons that will become clear, we begin by considering a very simple surface of revolution, the curved lateral part of a cone whose base has radius r and whose slant height is L. If this cone is cut down the side from the vertex to the base—that is, along a generator—and laid out flat, as shown in Fig. 7.28, then we get a sector of a circle of radius L whose curved edge has length $2\pi r$, and the lateral area A of the cone equals the area of this sector. It is geometrically clear that the ratio of the area of the sector to the complete area of the circle equals the ratio of the length of the curved edge to the complete circumference of the circle, that is

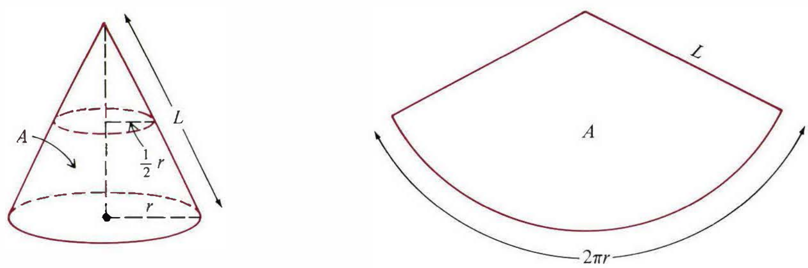


Figure 7.28

$$\frac{A}{\pi L^2} = \frac{2\pi r}{2\pi L}, \qquad \text{so} \qquad A = \pi r L.$$

The lateral surface of the cone can evidently be thought of as the surface of revolution swept out by a generator as it revolves around the axis. If the formula is written as

$$A = L \cdot 2\pi(\tfrac{1}{2}r),$$

then we see that the lateral area of a cone equals the product of the length of a generator and the distance traveled by the midpoint on its journey around the axis.

Next, we generalize slightly and find the area of the surface of revolution generated when a line segment of length L is revolved about an axis at a distance r from its midpoint.* This area is the lateral area of a *frustum* of a cone, as shown in Fig. 7.29. If we denote this area by A, then A is the difference between the lateral areas of the two cones in the figure, so

$$A = \pi r_1 L_1 - \pi r_2 L_2 = \pi(r_1 L_1 - r_2 L_2).$$

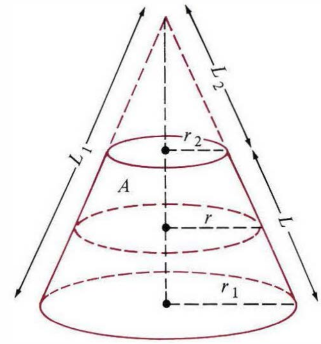


Figure 7.29

By similar triangles, it is clear that

$$\frac{L_2}{r_2} = \frac{L_1}{r_1} \qquad \text{or} \qquad r_1 L_2 = r_2 L_1.$$

With the aid of a bit of algebraic ingenuity, this enables us to write A in the form

$$\begin{aligned} A &= \pi(r_1 L_1 - r_1 L_2 + r_2 L_1 - r_2 L_2) = \pi[r_1(L_1 - L_2) + r_2(L_1 - L_2)] \\ &= \pi(L_1 - L_2)(r_1 + r_2) = (L_1 - L_2) \cdot 2\pi\left(\frac{r_1 + r_2}{2}\right) = L \cdot 2\pi r. \end{aligned}$$

We therefore conclude that in this case as well, the area of the surface of revolution equals the product of the length of the segment and the distance traveled by the midpoint on its journey around the axis.

We now apply these ideas to the general area problem stated at the beginning of this section. Our approach will be intuitive and geometric.

We begin by approximating the smooth curve $y = f(x)$ by a polygonal path consisting of many short line segments connecting nearby points on the curve, as shown on the left in Fig. 7.30. The surface generated by revolving the curve

*In the case of a cone, one end of the segment lies on the axis and forms the vertex of the cone.

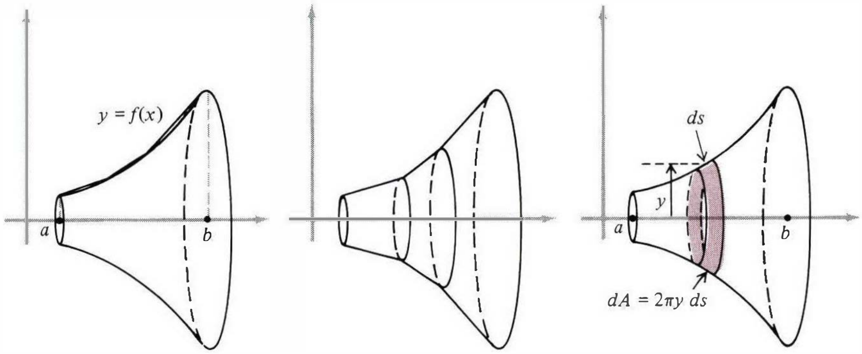


Figure 7.30

about the x-axis will have approximately the same area as the surface generated by revolving this polygonal path about the x-axis (Fig. 7.30, center). The latter surface is evidently made up of a number of pieces, each of which is shaped like a frustum of a cone. This situation suggests the fundamental idea illustrated on the right in the figure. If the element of arc length ds is revolved about the x-axis, then it generates a thin ribbonlike element of area dA; and if the midpoint of ds is at a distance y from the x-axis, then the above discussion tells us that

$$dA = 2\pi y\, ds = 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx.$$

We obtain the total area A of the surface by forming the sum—or integral—of all the elements of area dA as dA sweeps along the complete surface,

$$A = \int dA = \int 2\pi y\, ds = \int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx,$$

where y is assumed to be known as a function of x [$y = f(x)$]. If we choose instead to revolve our curve about the y-axis, and thereby to generate an entirely different surface of revolution, then in the same way its area is given by

$$A = \int 2\pi x\, ds.$$

The underlying idea in both of these formulas can be expressed by writing

$$A = \int 2\pi(\text{radius of revolution})\, ds.$$

In using this formula to perform an actual calculation, the element of arc length ds must be written in terms of a convenient variable of integration and appropriate limits of integration must be provided.

Example Find the surface area of a sphere of radius a.

Solution The surface of this sphere can be considered as the surface of revolution generated by revolving the semicircle $y = \sqrt{a^2 - x^2}$ about the x-axis (Fig. 7.31). Since

$$\frac{dy}{dx} = \frac{d}{dx}(a^2 - x^2)^{1/2} = \frac{-x}{\sqrt{a^2 - x^2}},$$

and ds sweeps along the arc of the circle in the first quadrant as x increases from 0 to a, we can use the left-right symmetry of the figure and write

$$\begin{aligned} A &= \int 2\pi y\, ds = 2\int_0^a 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \\ &= 4\pi\int_0^a \sqrt{a^2 - x^2}\sqrt{1+\frac{x^2}{a^2 - x^2}}\, dx \\ &= 4\pi\int_0^a a\, dx = 4\pi a^2. \end{aligned}$$

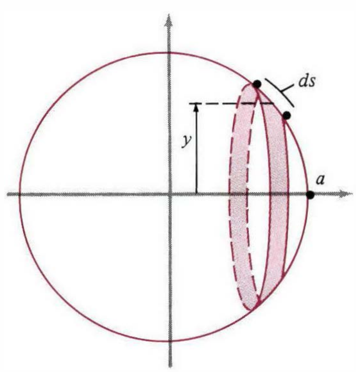


Figure 7.31

It is also possible to use y as the variable of integration, because ds also sweeps along the arc in the first quadrant as y increases from 0 to a. The calculation is a little more complicated, but it may be instructive for students to see how it works. Since $x = \sqrt{a^2 - y^2}$ in the first quadrant, we have

$$\frac{dx}{dy} = \frac{d}{dy}(a^2 - y^2)^{1/2} = \frac{-y}{\sqrt{a^2 - y^2}},$$

and therefore

$$\begin{aligned} A &= \int 2\pi y\, ds = 2\int_0^a 2\pi y \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy \\ &= 4\pi \int_0^a y\sqrt{\frac{y^2}{a^2 - y^2} + 1}\, dy = 4\pi a \int_0^a \frac{y\, dy}{\sqrt{a^2 - y^2}} \\ &= 4\pi a(-\tfrac{1}{2}) \int_0^a (a^2 - y^2)^{-1/2}(-2y\, dy) = 4\pi a(-\tfrac{1}{2})2\sqrt{a^2 - y^2}\Big]_0^a \\ &= 4\pi a^2, \end{aligned}$$

as before.

Remark In addition to discovering the volume of a sphere, Archimedes also found its surface area by means of a brilliant piece of insight that links these two quantities to each other. His idea was to split up the solid sphere into a large number of small "pyramids," as follows. Imagine that the surface of our sphere of radius a is divided into many tiny "triangles," as suggested in Fig. 7.32. Of course, these little figures are not actually triangles, since there are no straight lines on the surface of a sphere. However, they are so small that each figure is nearly flat and they are nearly triangles. Let each such triangle be used as the base of a pyramid of height a whose vertex is the center of the sphere. If A_k is the area of the base of our tiny pyramid and V_k is its volume, for $k = 1, 2, \ldots, n$, then we know that $V_k = \frac{1}{3}A_k a$. (The fact that the volume of a pyramid is one-third the area of the base times the height was discovered by Democritus two centuries before the time of Archimedes.) By adding these equations for $k = 1, 2, \ldots, n$, we obtain

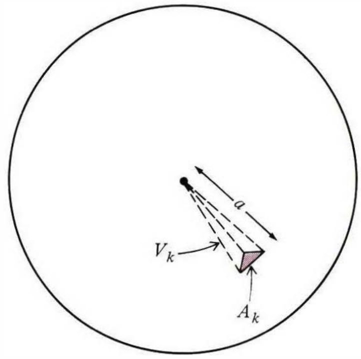


Figure 7.32

$$\sum_{k=1}^{n} V_k = \sum_{k=1}^{n} \tfrac{1}{3}A_k a = \frac{1}{3}\left(\sum_{k=1}^{n} A_k\right) a.$$

Since all our pyramids fill the solid sphere, this tells us that the volume V and surface area A of the sphere are related by the equation

$$V = \tfrac{1}{3}Aa.$$

But now Archimedes' discovery that $V = \frac{4}{3}\pi a^3$ enables us to write this equation in the form

$$\tfrac{4}{3}\pi a^3 = \tfrac{1}{3}Aa,$$

so

$$A = 4\pi a^2,$$

just as in the example.

PROBLEMS

In Problems 1–6, find the area of the surface of revolution generated by revolving the given arc about the indicated axis.

1 $y = \dfrac{1}{4}x^4 + \dfrac{1}{8x^2}$, $1 \le x \le 2$, the y-axis.

2 $y = \frac{1}{3}\sqrt{x}(3 - x)$, $0 \le x \le 3$, the x-axis.

3 $y = \frac{1}{3}(2 + x^2)^{3/2}$, $0 \le x \le 2$, the y-axis.

4 $y = x^2$, $0 \le x \le 2$, the y-axis.

5 $y = x^3$, $0 \le x \le 1$, the x-axis.

6 $y = 2\sqrt{x}$, $2 \le x \le 8$, the x-axis.

7 The arc of the parabola $x^2 = 4py$ between $(0, 0)$ and $(2p, p)$ is revolved about the y-axis. Find the area of the surface of revolution (a) by integrating with respect to x; (b) by integrating with respect to y.

8 The loop of $9y^2 = x(3 - x)^2$ is revolved about the y-axis. Find the area of the surface generated in this way.

9 Find the area of the surface generated by revolving the astroid (or hypocycloid of four cusps) $x^{2/3} + y^{2/3} = a^{2/3}$ about the y-axis.

10 Consider a cylinder circumscribed about a sphere of radius a. Let two planes perpendicular to the axis of the cylinder intersect the sphere. If these planes are at a distance h apart, show that the area of the *spherical zone* that lies between them on the sphere is $2\pi ah$. (It is a remarkable fact that this is the same as the area between these planes on the lateral surface of the cylinder. Note also that if $h = 2a$ this result yields the formula for the total surface area of the sphere.)

11 If a curve lies above the x-axis, its *moment* around the x-axis is defined to be $\int y\,ds$, where this integration is extended over the complete length of the curve. The moment of the curve around the y-axis is $\int x\,ds$. The point $(\bar{x}, \bar{y})$ is called the *centroid* of the curve (Fig. 7.33) if its coordinates are defined by

$$\bar{x} = \frac{\int x\,ds}{\int ds} = \frac{\int x\,ds}{\text{length of curve}}$$

and

$$\bar{y} = \frac{\int y\,ds}{\int ds} = \frac{\int y\,ds}{\text{length of curve}}.$$

If the curve is thought of as a uniform metal wire, its centroid is its *center of gravity*, or balancing point. Find the centroid of the semicircle $x^2 + y^2 = a^2$, $y \ge 0$.

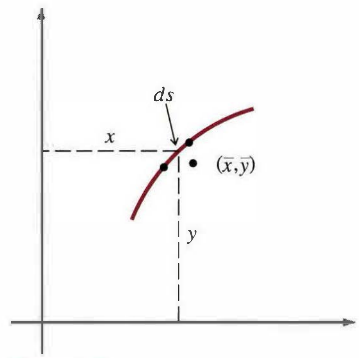


Figure 7.33

12 In Problem 11, let the curve be revolved about the x-axis and show that the area of the surface of revolution equals the length of the curve times the distance traveled by the centroid on its journey around the x-axis.

13 Use the theorem proved in Problem 12 to find
(a) $\bar{y}$ for the semicircular arc in Problem 11;
(b) the surface area of the *torus* (a fancy name for a doughnut) obtained by revolving the circle $x^2 + (y - b)^2 = a^2$, $a \le b$, about the x-axis.

7.7 WORK AND ENERGY

It is a common experience that in moving an object against a force acting on it, as in lifting a heavy stone, we have the sensation of expending effort or doing work. Even before we define the physical concept of work, we are convinced that it takes twice as much work to lift a 20-lb stone a given distance as it does to lift a 10-lb stone, and also that the work done in lifting a stone 3 ft is three times that done in lifting it 1 ft. These ideas point the way to our basic definition: If a constant force F acts through a distance d, then the *work* done during this process is the product of the force and the distance through which it acts,

$$\text{work} = \text{force} \cdot \text{distance}$$

or

$$W = F \cdot d. \tag{1}$$

It is understood here that the force acts in the direction of the motion.

As we know, the "weight" of an object is the force with which the object is attracted to the earth by gravity. For a given object moving at or near the surface of the earth, this force remains essentially constant in magnitude and is always directed toward the center of the earth. Thus, if a box of groceries weighing 20 lb is lifted 3 ft from the floor and placed on a table, then definition (1) tells us that 60 ft-lb of work are done; but if the box is then carried into another room and placed on a shelf without raising it or lowering it, then this action accomplishes no work because the box was moved a distance zero in the direction of the force. And if a tractor drags a boulder 18 in by applying a constant force of 2 tons, then the tractor does 36 in-tons (or 3 ft-tons) of work.*

This definition is satisfactory as long as the force F is constant. However, many forces do not remain constant during the process of performing work. In a situation like this we divide the process into many small parts and calculate the total work by integrating the elements of work corresponding to these parts.

This idea is illustrated by the operation of stretching a spring, as follows.

Example 1 A certain spring has a natural length of 16 in. When it is stretched x inches beyond its natural length, *Hooke's law* states that the spring pulls back with a restoring force of $F = kx$ pounds, where k is a constant. The constant of proportionality k is called the *spring constant,* and can be thought of as a measure of the stiffness of the spring. For the spring under discussion, 8 lb of force are required to hold it stretched 2 in. How much work is done in stretching this spring from its natural length to a length of 24 in?

Solution First, the fact that $F = 8$ when $x = 2$ allows us to find k. We have $8 = k \cdot 2$, so $k = 4$ and $F = 4x$. To clarify our ideas, we draw a picture of the spring in its unstretched condition, and also after it has been stretched x inches (Fig. 7.34). Now, if we imagine that the spring is stretched a very small additional distance dx, then the force changes very little over this increment of distance and can be treated as essentially constant. The work done against the pull of the spring over this increment of distance is

$$dW = F\,dx = 4x\,dx, \tag{2}$$

and the total work done during the complete stretching process is

$$W = \int dW = \int F\,dx = \int_0^8 4x\,dx = 2x^2\Big]_0^8 = 128 \text{ in-lb},$$

since x increases from 0 to 8 as the length of the spring increases from 16 to 24.

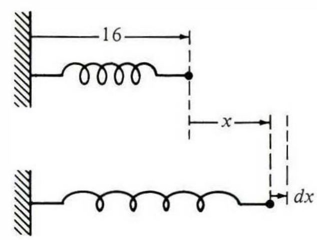


Figure 7.34

In a similar way, we can consider the work done by any variable force that acts in a given direction as its point of application moves in this direction. If we coordinatize the line of action by introducing an x-axis, and if the point of

*If force is measured in pounds and distance in feet, work is measured in foot-pounds. This is the English or engineering system. In science the terms are as follows. In the centimeter-gram-second (cgs) system, the unit of force is the *dyne,* defined as the force that imparts an acceleration of 1 cm/s^2 to a mass of 1 gram, while in the meter-kilogram-second (mks) system, the unit of force is the *newton,* defined as the force that imparts an acceleration of 1 m/s^2 to a mass of 1 kilogram. The corresponding units of work are the dyne-centimeter, or *erg,* and the newton-meter, or *joule.* For conversion 1 ft-lb $\cong$ 1.356 joules and 1 joule $= 10^7$ ergs.

application of the variable force $F(x)$ moves from $x = a$ to $x = b$, then $dW = F(x)\,dx$ is the *element of work* and

$$W = \int dW = \int_a^b F(x)\,dx \tag{3}$$

gives the total work done during the process. This formula can be taken either as a definition or as a natural method of computing the work in accordance with the way of thinking described in Example 1. In our next example the same idea is applied to a different situation.

Example 2 According to Newton's law of gravitation, any two particles of matter of masses M and m attract each other with a force F whose magnitude is directly proportional to the product of the masses and inversely proportional to the square of the distance r between them,

$$F = G\frac{Mm}{r^2},$$

where G is the so-called *constant of gravitation.* If M is fixed at the origin, how much work is required to move m from $r = a$ to $r = b$, where $a < b$?

Solution The element of work is

$$dW = F\,dr = GMm\,\frac{dr}{r^2}, \tag{4}$$

so the total work is

$$W = \int dW = GMm\int_a^b \frac{dr}{r^2} = GMm\left(-\frac{1}{r}\right)\Big]_a^b = GMm\left(\frac{1}{a} - \frac{1}{b}\right).$$

If we think of the final position $r = b$ as being chosen farther and farther away, so that $b \to \infty$, then the work W approaches the limiting value GMm/a. This quantity is the work which must be done against the force of attraction to move m from $r = a$ to an infinite distance, that is, to separate the masses completely; it is called the *potential* of the two particles.

Each of the preceding examples is concerned with a variable force acting through a given distance. Our next example is very different. It involves a process in which the parts of a body—in this case, drops of water—are moved different distances against a constant force, and the total work is calculated as the sum of the various bits of work associated with the various parts.

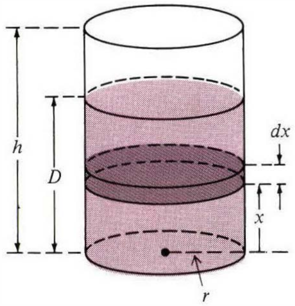


Figure 7.35

Example 3 Consider a cylindrical tank of radius r and height h, filled with water to a depth D (Fig. 7.35). How much work is done in pumping the water out over the rim of the tank? (As usual, we denote the weight-density of water, that is, the weight per unit volume, by w.)

Solution The essence of this problem is the fact that each drop of water must be lifted from its initial position up to the rim of the tank and dumped over the side. The work done in this process is the same for all drops which are the same distance below the rim. This suggests that we consider all the water located in a thin horizontal layer of thickness dx at a height x above the bottom of the tank, that we write down the element of work dW needed to lift this entire layer up to the rim of the tank, and that we calculate the total work in our usual way, by adding (or integrating) these elements of work as x increases from 0 to D, so that our typical layer sweeps through all the water in the tank. It is clear from the figure that the volume of the layer is $\pi r^2\,dx$, so its weight is $w\pi r^2\,dx$, and the work done in lifting this layer through the distance $h - x$ to the top of the tank is

$$dW = w\pi r^2\,dx \cdot (h - x). \tag{5}$$

The total work done in pumping out all the water is therefore

$$W = \int dW = w\pi r^2 \int_0^D (h - x)\,dx$$

$$= w\pi r^2(hx - \tfrac{1}{2}x^2)\Big]_0^D = w\pi r^2(hD - \tfrac{1}{2}D^2).$$

We repeat: The crux of the method in this example is the fact that all drops of water in our typical layer are essentially the same distance below the rim of the tank, and can therefore be treated together in calculating the work.

Students should observe that the use of definition (1) in a suitable form is the key to each of these examples. Specifically, formulas (2), (4), and (5) are simply the versions of (1) that are appropriate in each case.

We devote the rest of this section to a brief discussion of the important concept of energy.

Consider a variable force F that acts on a particle of mass m over a given distance along a straight line, which we take to be the x-axis. This force not only does work, but also imparts an acceleration dv/dt to the particle in accordance with Newton's second law of motion,

$$F = m\frac{dv}{dt}, \qquad \text{where } v = dx/dt. \tag{6}$$

This acceleration produced by the force changes the velocity v of m, and therefore also changes its *kinetic energy*—or energy due to motion—which is defined by the formula

$$\text{kinetic energy} = \tfrac{1}{2}mv^2.$$

We are now in a position to prove the following important theorem of mechanics:

The work done by the force F during the process described above equals the change in the kinetic energy of the particle; and in particular, if the particle starts from rest, then the work done on it equals the kinetic energy it attains.

The proof is easy. We begin by writing (6) in the form

$$F = m\frac{dv}{dt} = m\frac{dv}{dx}\frac{dx}{dt} = mv\frac{dv}{dx}.$$

Formula (3) now yields

$$W = \int_a^b F\,dx = \int_a^b mv\frac{dv}{dx}\,dx = \int_{v_a}^{v_b} mv\,dv$$

$$= \tfrac{1}{2}mv^2\Big]_{v_a}^{v_b} = \tfrac{1}{2}mv_b^2 - \tfrac{1}{2}mv_a^2, \tag{7}$$

so the work W equals the change in the kinetic energy, as stated.

Remark In certain types of physical situations—but not in all—it is possible to introduce the concept of *potential energy.* We do this very briefly as follows. In using formula (3) for the calculation in (7), we tacitly assumed that the unspecified force F is a continuous function depending only on the coordinate x over the interval $a \le x \le b$, say $F = F(x)$. (Notice that a frictional force does not have this property; for it depends not only on the location of the particle m, but also on the direction in which it is moving.) By the discussion at the end of Section 6.7, this assumption guarantees that there exists a function $V(x)$ such that $dV/dx = -F(x)$. We can therefore evaluate the work W in (7) in another way, as follows:

$$W = \int_a^b F(x)\,dx = \int_b^a -F(x)\,dx = V(x)\Big]_b^a$$
$$= V(a) - V(b). \tag{8}$$

This enables us to write (7) as

$$\tfrac{1}{2}mv_b^2 - \tfrac{1}{2}mv_a^2 = V(a) - V(b)$$

or

$$\tfrac{1}{2}mv_b^2 + V(b) = \tfrac{1}{2}mv_a^2 + V(a). \tag{9}$$

On the left side of (9) we drop the subscript and replace $V(b)$ by $V(x)$ in order to emphasize that v and $V(x)$ are considered to be variables; and on the right side we hold v_a and $V(a)$ fixed. Equation (9) now takes the form

$$\tfrac{1}{2}mv^2 + V(x) = \tfrac{1}{2}mv_a^2 + V(a) = E, \tag{10}$$

where the constant E is called the *total energy* of the particle. The function $V(x)$ is called the *potential energy* of the particle, and (10) states that the sum of the kinetic energy and potential energy is constant. This is the *law of conservation of energy,* which is one of the basic principles of classical physics.

We see from (10) that if $F(x)$ does work and thereby increases kinetic energy, it does so at the expense of potential energy and can therefore be viewed as converting potential energy into an equal amount of kinetic energy.

We point out that the definition of $V(x)$ means that this function is determined only to within an additive constant, so in any specific situation the state of zero potential energy can be chosen to suit our convenience. Also, students may wonder about the mild trickery with algebraic signs that takes place in the definition of $V(x)$ and in the calculation (8). The purpose of this is to guarantee the appearance of plus signs instead of minus signs in (10) so that we can speak of the sum of the kinetic and potential energies as being constant instead of their difference.

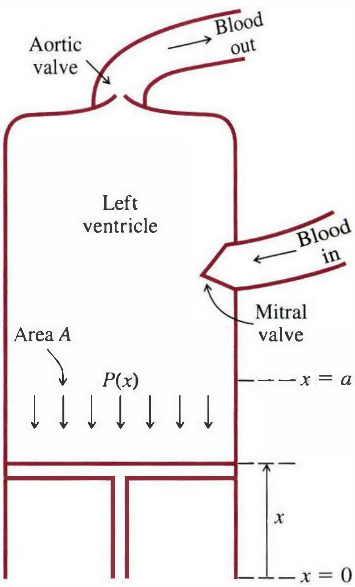


Figure 7.36

Example 4 *The working heart.* From the point of view of physics, the human heart is a pump. Blood enters the left ventricle through the mitral valve (Fig. 7.36) and is then pumped out to the body through the aortic valve as the heart muscle contracts. During each contraction the pressure exerted by the heart wall increases in a roughly linear manner from a diastolic pressure of about 80 mm Hg (millimeters of mercury) to a systolic pressure of about 120 mm Hg. We shall calculate the work done in the left ventricle during one heartbeat, assuming—a realistic figure—that the volume of this ventricle decreases by about 75 cm^3 during one contraction. We shall need to know that 100 mm Hg $\cong$ 1.33×10^5 dynes/cm^2.

For convenience in working with the idea of a pump, we imagine the heart's action to be carried on by the movement of a piston from $x = 0$ to $x = a$, as shown in the figure, instead of by muscular contraction. If A is the area of the piston head, then $aA = 75$. The pressure $P(x)$ against which the piston works is easily seen from Fig. 7.37 to be

$$P(x) = \frac{40}{a}x + 80.$$

We now put all this together and observe that the variable force exerted by the piston during one upward stroke is $P(x)A$, and the work done during this stroke is

$$\begin{aligned} W = \int_0^a P(x)A\,dx &= A\int_0^a \left(\frac{40}{a}x + 80\right)dx \\ &= A\left(\frac{20}{a}x^2 + 80x\right)\bigg]_0^a = 100aA \\ &\cong (1.33 \times 10^5 \text{ dynes/cm}^2) \cdot (75 \text{ cm}^3) \\ &\cong 10^7 \text{ dyne-cm} \\ &\cong 1 \text{ joule} \\ &\cong 0.74 \text{ ft-lb}. \end{aligned}$$

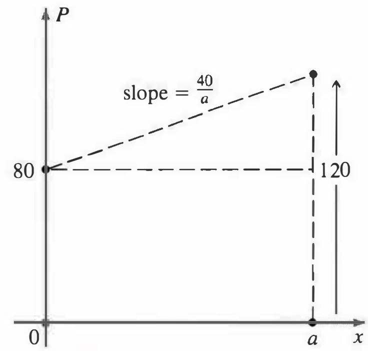


Figure 7.37

In the case of a person weighing 120 lb who has a pulse rate of 60, we can quickly learn by punching a few keys on our calculator that the heart does enough work in a 24-hour day to lift the person through a vertical distance of more than 500 ft. The human heart is a remarkable organ and is shockingly underappreciated!

PROBLEMS

1 A spring has a natural length of 10 in, and a 12-lb force stretches it $\frac{1}{2}$ in. Find the work done in stretching the spring from 10 in to 18 in.

2 A spring has a natural length of 12 in, and a 45-lb force stretches it to 15 in. Find the work done in stretching it from 15 in to 19 in.

3 A spring supporting a railroad car has a natural length of 12 in, and a force of 8000 lb compresses it $\frac{1}{2}$ in. Find the work done in compressing it from 12 in to 9 in. (Hooke's law is valid for compressing springs as well as for stretching them.)

4 Find the natural length of a spring if the work done in stretching it from a length of 2 ft to a length of 3 ft is one-fourth the work done in stretching it from 3 ft to 5 ft.

5 A bucket weighing 5 lb when empty is loaded with 60 lb of sand. Unfortunately there is a hole in the bucket, and sand leaks out uniformly at such a rate that a third of the sand is lost when the bucket has been lifted 10 ft. Find the work done in lifting the bucket this distance.

6 A cable 100 ft long that weighs 4 lb/ft is hanging from a windlass. How much work is done in winding it up?

7 Solve Problem 6 if a 300-lb weight is attached to the free end of the cable.

8 A 5-lb monkey is attached to the end of a 30-ft hanging chain that weighs 0.2 lb/ft. It climbs the chain to the top. How much work does it do?

9 Gas in a cylindrical chamber moves a piston by expanding or contracting. Let the cross-sectional area of

the cylinder be A, and let its variable volume and length be V and x (Fig. 7.38). If p is the pressure of the gas, then the force the gas exerts on the piston is pA.

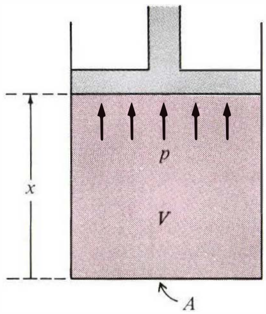


Figure 7.38

(a) If the gas expands from a volume V_1 to a volume V_2, show that the work done by the gas on the piston is

$$W = \int_{V_1}^{V_2} p\,dV.$$

(b) If a force is exerted on the piston to compress the gas from a volume V_1 to a volume V_2, show that the work done on the gas is

$$W = -\int_{V_1}^{V_2} p\,dV.$$

10 If air is compressed or expanded without any loss or gain of heat but with a possible change of temperature, then it obeys the *adiabatic* gas law $pV^{1.4} = c$, where c is a constant. If a cylinder contains 243 in^3 of air at a pressure of 14 lb/in^2, find the work done by the piston on the air in compressing it adiabatically to a volume of 32 in^3. (If this air is compressed slowly so that the heat generated is allowed to escape and the temperature remains constant, the compression is said to be *isothermal.* In this case the pressure and volume are related by *Boyle's law* $pV = c$, and in trying to calculate the work we are led to an integral of the form $\int dV/V$, which is beyond our reach. One of the main purposes of Chapter 8 is to enable us to cope with integrals of this kind, which are important in many applications.)

11 Consider a cylindrical buoy of cross-sectional area 8 ft^2 which is floating upright in water whose weight-density is $w = 62.5$ lb/ft^3. According to Archimedes' principle, a floating body is acted on by an upward buoyant force equal to the weight of the displaced water, and in a state of equilibrium this upward force balances the downward force acting on the body due to gravity.
(a) Show that there is an upward force of $62.5(8x)$ lb acting on the buoy when it is held x feet down from its equilibrium position.
(b) How much work is done in pushing the buoy 1 ft down from its equilibrium position?

***12** A conical buoy that weighs B pounds floats upright in water with its vertex a ft below the surface. A crane on a dock lifts the buoy until its vertex just clears the surface. How much work is done? Hint: When the crane has lifted the buoy x ft, then the force required to hold it in this position is the weight of the buoy minus the upward buoyant force due to the water still displaced, and this can be expressed as a function of x.

13 If an iron ball is attracted to a magnet by a force of $F = 15/x^2$ pounds when the ball is x feet from the magnet, find the work done in pulling the ball away from the magnet from a point where $x = 2$ to a point where $x = 6$.

14 According to Coulomb's law, two electrons repel each other with a force that is inversely proportional to the square of the distance between them. Suppose one electron is held fixed at the origin on the x-axis. Find the work done in moving a second electron along the x-axis from $x = 2$ to $x = 1$. From $x = a$ to $x = b$, where $0 < b < a$.

15 If two particles of matter of masses M and m are a units apart, how much work must be done to move them twice as far apart?

16 If R is the radius of the earth (about 4000 mi) and g is the acceleration due to gravity at the surface of the earth, then the force of attraction exerted by the earth on a body of mass m is $F = mgR^2/r^2$, where r is the distance from m to the center of the earth. If this body weighs 100 lb at the surface of the earth, what does it weigh at an altitude of 1000 mi? At an altitude of 4000 mi? How much work is required to lift it from the surface to an altitude of 1000 mi?

17 Generalize Problem 16 by finding how much work must be done by a rocket on a satellite of mass m in lifting it to an altitude h above the surface of the earth.

18 Suppose that a hole is drilled straight through the center of the earth, and that a body of mass m is dropped into this hole. As the body falls, the force of attraction exerted on it by the earth is $F = mgr/R$, where r is the distance from m to the center of the earth. (The reason behind this law of force will become clear in a later chapter.) Find the work done by the earth in pulling m from the surface down to the center.

19 A conical tank 10 ft deep and 8 ft across the top is full of water. Find the work done in pumping the water over the top of a nearby 12-ft fence.

20 Find the work done in Problem 19 if the tank is initially filled only to a depth of 5 ft and if the water is pumped just to the top of the tank and over the edge.

21 A spherical tank of radius a is at the top of a tower with its bottom at a distance h above the ground. How much work is needed to fill the tank with water pumped from ground level?

22 A great conical mound of height h is built by the slaves of an oriental monarch, to commemorate a victory over the barbarians. If the slaves simply heap up uniform material found at ground level, and if the total weight of the finished mound is M, show that the work they do is $\frac{1}{4}hM$.

23 The Great Pyramid of Egypt is perhaps the greatest single building ever erected by man. It was originally 482 ft high with a square base 765 ft on a side, and it covered an area large enough so that St. Peter's in Rome, the cathedrals of Milan and Florence, Westminster Abbey, and St. Paul's Cathedral in London could all be grouped within it. It contained enough stone to build a wall 1 ft thick and 7 ft high all the way around France. The Greek historian Herodotus said that it was built in 20 years by the labor of 100,000 men. Calculate the plausibility of this assertion as follows: Assume that the Great Pyramid is made of stone that weighs 150 lb/ft^3, that each laborer worked 10 hours per day for 350 days each year, and that each laborer did 200 ft-lb of effective work per hour in lifting stones from ground level to their final positions in the pyramid.† If Herodotus' figure of 20 years is correct, approximately how many laborers were needed?

24 Geologists who study mountain building are able to calculate the energy needed to lift a mountain up from sea level. In the case of Mt. Everest, assume the mountain has the shape of a cone of height 30,000 ft and radius of base 60,000 ft, with uniform density 150 lb/ft^3.

(a) How much work was required to build Mt. Everest if all of its constituent rock was initially at sea level?

(b) The atomic bomb at Hiroshima released energy equivalent to 20,000 tons of TNT, and some hydrogen bombs tested in the 1950s had energy 500 times as great, on the order of 10 megatons of TNT. How does the work needed to build Mt. Everest compare with the energy of a 10-megaton bomb? (10 megatons is about 3×10^{15} ft-lb.)

25 If the same amount of work done on two particles starting from rest causes one to move twice as fast as the other, how are their masses related?

26 When the mass m of a particle is constant, Newton's second law of motion

$$F = ma = m\frac{dv}{dt} \qquad (*)$$

can be written in the less familiar form

$$F = \frac{d}{dt}(mv), \qquad (**)$$

which remains valid even when m is not constant. According to *Einstein's special theory of relativity,* the mass is indeed not constant: It increases as the velocity v increases, and is determined as a function of v by the formula

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}},$$

where c is the velocity of light (approximately 186,000 mi/s, or 300,000 km/s) and m_0 is the so-called *rest mass.* When this expression for m is inserted in $(**)$, the result is *Einstein's law of motion*

$$F = m_0\frac{d}{dt}\left(\frac{v}{\sqrt{1 - v^2/c^2}}\right). \qquad (***)$$

If v is small compared with c so that $\sqrt{1 - v^2/c^2}$ is close to 1, then Newton's law $(*)$ is a very close approximation to Einstein's law. This is what happens for all problems of classical physics (in fact, this is almost a definition of classical physics). However, if v is an appreciable fraction of c, as in most phenomena of atomic physics, then the two laws differ considerably, and all the experimental evidence supports Einstein's version.

(a) Show that a particle acted on by a constant force F can never achieve the velocity of light, no matter how long the force acts.‡ Hint: Integration of $(***)$ gives $Ft/m_0 = v/\sqrt{1 - v^2/c^2}$ if the particle starts from rest.

(b) By differentiation write $(***)$ in the form

$$F = \frac{m_0 a}{(1 - v^2/c^2)^{3/2}}$$

where $a = dv/dt,$ and thereby obtain another way of comparing Einstein's law with Newton's law $(*)$.

27 (This problem continues the ideas of Problem 26.) Consider a particle of rest mass m_0 that starts from rest at the origin on the x-axis and moves in the positive direction under the influence of a positive force F. Einstein's law of motion [part (b) of Problem 26] tells us that a is also positive, so the velocity is increasing. If the energy E of the particle is understood to be the work done on it by F, show that E is related to the increase in the mass, which is $M = m - m_0$, by *Einstein's*

†If 200 ft-lb per hour seems too small a figure, remember that much time and strength were spent in quarrying, cutting, and transporting the stones.

‡This is in sharp contrast with the implication of Newton's law $(*)$, that if the force F is constant, then the acceleration a is also constant, so the velocity increases at a constant rate, and can therefore be made greater than c if the force acts long enough.

famous equation $E = Mc^2$. Hint: Write $a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$ and calculate $E = \int_0^x F\,dx$. [The crux of Einstein's equation—not mentioned in this problem—is the much deeper fact that the rest mass m_0 also has energy associated with it, in the amount m_0c^2. This can be thought of as the "energy of being" of the particle, in the sense that mass possesses energy just by virtue of existing. If we add this to the kinetic energy calculated in the problem, we obtain the complete Einstein equation: total energy $= Mc^2 + m_0c^2 = (m - m_0 + m_0)c^2 = mc^2$.]

7.8 HYDROSTATIC FORCE

In the previous sections of this chapter we have seen how integration can be used to answer many natural questions that arise in geometry and basic physics.

In this section we undertake a brief excursion into the science of *hydrostatics,* which is concerned with the behavior of liquids at rest. In particular, we calculate the force exerted outward against the walls of an open container by water at rest inside the container. The containers we consider can be anything from a small fishbowl to the reservoir behind a gigantic dam. We do not undertake this excursion for its own sake, but rather because it provides an additional excellent illustration of the main theme of this chapter—the idea that the whole of a quantity can be calculated by dividing it into many convenient small pieces and adding up these pieces by means of integration.

If a tank with a rectangular bottom and vertical sides is filled with water to a depth h (Fig. 7.39), then the force exerted downward on the bottom is equal to the weight of the water contained in the tank. If A is the area of the bottom, then this force is given by the formula

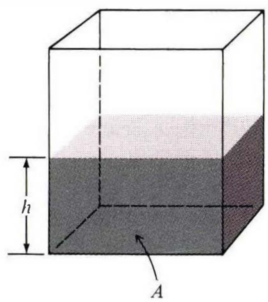


Figure 7.39

$$F = whA, \tag{1}$$

where w is the weight-density of the water, which is approximately 62.5 lb/ft^3, or $\frac{1}{32}$ ton/ft^3. It is obviously necessary for the units of measurement in (1) to be compatible. In our work we measure h in feet, A in square feet, and w in pounds or tons per cubic foot. The force F is then expressed in pounds or tons.

If we divide (1) by A, then the resulting quantity

$$p = wh \tag{2}$$

is the *pressure,* or *force per unit area,* exerted by the water on the bottom of the tank. The pressure at a given depth h below the surface can therefore be thought of as the weight of a column of water h units high that rests on a horizontal base whose area is 1 square unit. Formula (2) is quite remarkable, for it states that the pressure is proportional to the depth alone, and that the size and shape of the container are completely irrelevant. For example, at a depth of 4 ft in a swimming pool the pressure is the same as it is at a depth of 4 ft in a nearby lake (namely, 250 lb/ft^2) regardless of the size of the lake; and we find the same pressure at the bottom of a vertical glass tube 1 inch in diameter if we plug the bottom with a cork and fill it with 4 ft of water. Furthermore, it can be verified experimentally that at any point in a liquid the pressure is the same in all directions. This means that a flat plate below the surface has the same pressure acting on one face at a given depth whether it is placed horizontally, vertically, or at an angle, and this pressure is normal (perpendicular) to the face of the plate. As skin divers know from personal experience, the water pressure on the eardrums depends only on how deep they are, and not at all on the angle at which the head is tilted.

In order to find the total force exerted by the water against the bottom of the tank in Fig. 7.39, it is enough to multiply the pressure at the bottom by the area of the base,

$$F = pA,$$

which is merely formula (1). It is more difficult to find the force against one of the sides, because the pressure is not constant there but increases as the depth increases. Instead of pursuing this particular problem, we consider a more general situation.

In Fig. 7.40 we show a flat plate of unspecified shape submerged vertically in a body of water. To find the total force exerted by the water against one face of this plate, we imagine this face to be divided into a large number of thin horizontal strips. The typical strip shown in the figure is at a depth h below the surface. Its width dh is so small compared with h that the pressure is essentially constant over the entire strip, and has the value $p = wh$. The area of the strip is $dA = x\,dh$, so the *element of force* dF acting against the strip is given by

$$dF = p\,dA = wh \cdot x\,dh.$$

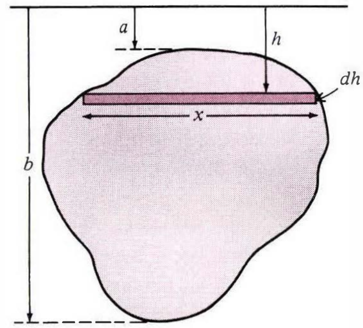


Figure 7.40

The total force F acting against the whole face of the plate is now obtained by integrating these elements of force as our typical strip sweeps across the plate from top to bottom,

$$F = \int dF = \int_a^b wh \cdot x\,dh. \tag{3}$$

In order to carry out the indicated integration in a specific problem, it is necessary to know x as a function of h, and this is determined geometrically from the shape of the plate. As in the preceding sections of this chapter, it is better to understand and apply the ideas used in constructing formula (3) than to try to memorize this formula and use it without thinking. We repeat the crux of the method: Thin horizontal strips are used because the pressure can be treated as essentially constant over all of such a strip, and the force acting against this strip is then simply the pressure times the area.

Example 1 A vertical gate in a dam has the shape of a square 4 ft on a side, the upper edge being 2 ft below the surface of the water (Fig. 7.41). Find the total force this gate must withstand.

Solution In this case $x = 4$ and h increases from 2 to 6, so

$$F = \int_2^6 wh \cdot 4\,dh = 2wh^2\Big]_2^6 = 2w \cdot 32 = 2 \text{ tons}.$$

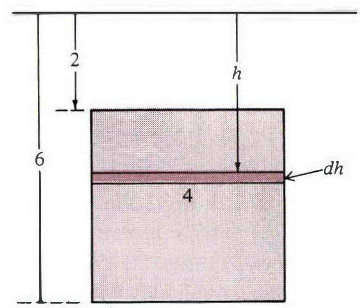


Figure 7.41

Example 2 A triangular dam in a ditch is 10 ft across the top and 6 ft deep (Fig. 7.42). Find the force of the water against this dam when the water is at the top and ready to spill over.

Solution By similar triangles we see that

$$\frac{x}{10} = \frac{6-h}{6}, \qquad \text{so} \qquad x = \tfrac{5}{3}(6-h).$$

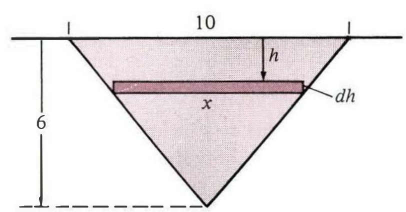


Figure 7.42

Since h increases from 0 to 6, we have

$$F = \int_0^6 wh \cdot \tfrac{5}{3}(6 - h)\, dh = \tfrac{5}{3}w(3h^2 - \tfrac{1}{3}h^3)\Big]_0^6$$

$$= 60w = 1\tfrac{7}{8} \text{ tons} = 3750 \text{ lb.}$$

PROBLEMS

In Problems 1–4, it is assumed that the face of a dam adjacent to the water is vertical and has the stated shape. In each case find the total force against the dam.

1 A rectangle 150 ft wide and 12 ft high; water 8 ft deep.

2 An isosceles trapezoid 200 ft wide at the top, 100 ft wide at the bottom, and 20 ft high; reservoir full of water.

3 An isosceles triangle 60 ft wide at the top and 20 ft high in the center; reservoir full of water.

4 An isosceles trapezoid 90 ft across the top, 60 ft across the bottom, and 20 ft high; water 12 ft deep.

In Problems 5–8, it is assumed that a vertical gate in the face of a dam has the stated shape. In each case find the total force of the water against the gate.

5 A triangle 4 ft wide at the top and 5 ft high, with upper edge 1 ft below the water surface.

6 An isosceles trapezoid 6 ft wide at the top, 8 ft wide at the bottom, and 6 ft high, with upper edge 4 ft below the water surface.

7 A triangle 4 ft wide at the bottom and 4 ft high, with the upper vertex 2 ft below the water surface.

8 A semicircle 4 ft in diameter with its diameter at the water surface.

9 A cylindrical barrel 4 ft high and 3 ft in diameter stands upright and is half filled with oil that weighs 50 lb/ft^3. What is the total force of the oil against the lateral wall of the barrel?

10 If the barrel in Problem 9 lies on its side, what is the force of the oil against one of the circular ends?

11 A rectangular gate in a vertical dam is 5 ft wide and 6 ft high. Find the force against this gate when the water level is 8 ft above its top. How much higher must the water rise to double the force?

12 The vertical ends of a water trough are isosceles triangles with base 3 ft and height 2 ft. Find the force against one end when the trough is full of water.

13 The end of a swimming pool is a rectangle inclined 45° to the horizontal. If the edge at the surface is 12 ft long and the submerged edge 10 ft long, find the force the water exerts against this rectangle.

*14 A rectangular tank is filled with two nonmixing liquids whose densities are w_1 and w_2, where $w_1 < w_2$. In one side of the tank there is a square window $3\sqrt{2}$ ft on a side with one of its diagonals vertical and the upper vertex 1 ft below the surface, and with the other diagonal on the boundary between the liquids. Find the force the liquids exert against the window.

CHAPTER 7 REVIEW: CONCEPTS, METHODS

Define, state, or think through the following.

1 Area by vertical or horizontal strips.

2 Volume by disks, moving cross sections, or washers.

3 Volume by cylindrical shells.

4 Arc length by $ds = \sqrt{dx^2 + dy^2}$.

5 Area of surface of revolution.

6 Work by $dW = F\, dx$.

7 Kinetic and potential energy.

8 Pressure.

9 Hydrostatic force by $dF = p\, dA$.

ADDITIONAL PROBLEMS FOR CHAPTER 7

SECTION 7.2

In Problems 1–13, sketch the curves and find the areas of the regions they bound.

1 $y = x^2$, $y = x$.

2 $x = 3y + y^2$, $x + y + 3 = 0$.

3 $y = x^4 - 2x^2$, $y = 2x^2$.

4 $y^2 = x^3$, $x = 4$.

5 $y = x^2 - 2x - 3$, $y = 2x + 2$.

*6 $y = \dfrac{2}{\sqrt{x + 2}}$, $x + 3y - 5 = 0$.

7 $y = 6x - x^2$, $y = x$.
8 $y^2 = 4x$, $2x - y = 4$.
9 $y^2 = 2x$, $x - y = 4$.
10 $y = 4 - x^2$, $y = 4 - 4x$.
11 $y^2 = -4x + 4$, $y^2 = -2x + 4$.
12 $y = 9 - x^2$, $y = x^2$.
13 $y = 9 - x^2$, $(x + 3)^2 = -4y$.
14 Find the complete area enclosed by $y^2 = 9x^2 - x^4$.
15 Find the area bounded by $y = x^2$, $y = 4$, $y = 2 - x$.
16 Find $c > 0$ so that the area bounded by $y = x^2 - c$ and $y = c - x^2$ equals 9.
*17 Find the area of the region in the second quadrant bounded by the x-axis and the parabolas $y = x^2$, $y = \sqrt{x + 18}$.
*18 Find the area between $4y = x^3$ and its tangent at $x = -2$.

SECTION 7.3

19 Find the volume of the solid of revolution generated when the area bounded by the given curves is revolved about the x-axis:
(a) $y = 2 - x^2$, $y = 1$;
(b) $y = 3x - x^2$, $y = x$;
(c) $y^2 = 4x$, $y = x$;
(d) $y = x^2 + 3$, $y = 4$;
(e) $\sqrt{x} + \sqrt{y} = \sqrt{a}$, $x = 0$, $y = 0$.
20 Find the volume generated by revolving the area bounded by $x = y^2$ and $x = 4$ about
(a) the x-axis; (b) the y-axis;
(c) the line $y = 2$; (d) the line $x = 4$;
(e) the line $x = -1$.
21 Find the volume generated by revolving the area bounded by $x = 4y - y^2$ and $x = 0$ about
(a) the y-axis; (b) the x-axis.
22 Each plane perpendicular to the x-axis intersects a certain solid in a circular cross section whose diameter lies in the xy-plane and extends from $y = x^2$ to $y = 8 - x^2$. The solid lies between the points of intersection of these curves. Find its volume.
23 The base of a certain solid is the circle $x^2 + y^2 = a^2$. Each plane perpendicular to the x-axis intersects the solid in a cross section that is an isosceles right triangle with one leg in the base of the solid. Find the volume.
24 The base of a certain solid is the area bounded by $x^2 = 4ay$ and $y = a$. Each cross section perpendicular to the y-axis is an equilateral triangle with one side lying in the base. Find the volume of the solid.
25 A plane which is perpendicular to the x-axis and contains a circle of radius x^2 moves from $x = a$ to $x = b$. If the center of the circle moves along a curve $y = f(x)$, find the volume of the solid the circle generates.
*26 A solid is generated by revolving about the x-axis the area bounded by a curve $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$. Its volume is $\pi(b^3 - b^2a)$ for all $b > a$. Find $f(x)$.
27 Find the volume generated by revolving the area bounded by the curves $x^2 = 4ay$, $y = a$, $x = 0$ about (a) the y-axis; (b) the x-axis; (c) the line $y = a$.
28 Let R be a region of area A in a horizontal plane, and suppose that R is bounded by a closed curve C that does not intersect itself. Let P be a point whose height above this plane is h, and form a generalized "cone" by drawing segments connecting P to the points of C. Show that the volume of this cone is $V = \frac{1}{3}Ah$. Hint: If $A(x)$ is the area of the horizontal cross section at a height x above the plane, observe that $A(x) = [(h - x)^2/h^2]A$.
29 A line passes through a vertex of a square of side a and is perpendicular to the plane in which the square lies. As this vertex moves a distance h along the line, the square turns through a complete revolution with the line as the axis. Find the volume of the screw-shaped solid the square generates. What is the volume if the square turns through two complete revolutions while moving the same distance along the line?
30 The square bounded by the axes and the lines $x = 1$, $y = 1$ is cut into two parts by the curve $y = x^n$, where n is a positive constant. Find the value of n for which these two parts generate equal volumes when revolved about the y-axis.
*31 Two oblique circular cylinders of equal height h have a circle of radius a as a common lower base and their upper bases are tangent to each other. Find the common volume.

SECTION 7.4

In Problems 32–37, sketch the region bounded by the given curves and use the shell method to find the volume of the solid generated by revolving this region about the given axis.

32 $y = \sqrt[3]{x}$, $x = 8$, $y = 0$; the y-axis.
33 $x = y^2 - 4y$, $x = 0$; the x-axis.
34 $y = 5x - x^2$, $y = 0$; the y-axis.
35 $x = y^3 + 1$, $y + 2x = 2$, $y = 1$; the x-axis.
36 $y = x^2$, $y = x^3$; the y-axis.
37 $2x - y = 12$, $x - 2y = 3$, $x = 4$; the y-axis.
38 The region bounded by the given curves is revolved about the y-axis. Find the volume of the solid of revolution by using both the shell method and the washer method.
(a) $y = 4x - x^2$, $y = 0$.
(b) $y = x^3$, $x = 2$, $y = 0$.
39 The region in the first quadrant between $y = 3x^2$ and $y = \frac{11}{4}x^2 + 1$ is revolved about the y-axis. Find the volume generated in this way.
40 The region bounded by $y^2 = 4x$ and $y = x$ is revolved about the x-axis. Find the volume generated in this way (a) by the shell method; (b) by the washer method.

*41 Consider the torus generated by revolving the circle $(x-b)^2 + y^2 = a^2$ $(0 < a < b)$ about the y-axis. Use the shell method to show that the volume of this torus equals the area of the circle times the distance traveled by its center during the revolution. Hint: At the right moment, change the variable of integration from x to $z = x - b$.

42 Find the volume generated by revolving about the y-axis the region bounded by $y = (x-1)(x-2)(x-3)$ and the x-axis between $x = 1$ and $x = 2$.

SECTION 7.5

In Problems 43–49, find the length of the specified arc of the given curve.

43 $9y^2 = 4x^3$ between $(0, 0)$ and $(3, 2\sqrt{3})$.

44 $y = \dfrac{1}{8}x^4 + \dfrac{1}{4x^2}$, $1 \le x \le 2$.

45 $y = \dfrac{1}{6}x^3 + \dfrac{1}{2x}$, $1 \le x \le 3$.

46 $x = \dfrac{1}{10}y^5 + \dfrac{1}{6y^3}$, $1 \le y \le 2$.

47 $y = \dfrac{1}{24}x^3 + \dfrac{2}{x}$, $2 \le x \le 4$.

48 $y = \frac{1}{6}\sqrt{x}(4x - 3)$, $1 \le x \le 9$.

49 $y = \frac{5}{48}(1 + 4x^{4/5})^{3/2}$, $1 \le x \le 32$.

50 Let A and B be positive constants. If $0 < a < b$, show that the problem of finding the length of the arc of the curve

$$y = Ax^3 + \frac{B}{x}$$

for $a \le x \le b$ leads to the integral

$$\int_a^b (3Ax^2 + Bx^{-2})\,dx$$

if $AB = \frac{1}{12}$.

51 Let A and B be positive constants. If $0 < a < b$, find a simple condition relating A and B that makes it possible to calculate the length of the arc of the curve

$$y = Ax^4 + \frac{B}{x^2}$$

between $x = a$ and $x = b$ by means of an integral not involving a square root.

52 Solve Problem 51 for the curve

$$y = Ax^5 + \frac{B}{x^3}.$$

SECTION 7.6

In Problems 53–55, find the area of the surface of revolution generated by revolving the given arc about the indicated axis.

53 $y = \frac{2}{3}(1 + x^2)^{3/2}$, $0 \le x \le 3$, the y-axis.

54 $y = \frac{2}{3}x^{3/2} - \frac{1}{2}x^{1/2}$, $0 \le x \le 4$, the y-axis.

55 $y = 2\sqrt{15 - x}$, $0 \le x \le 15$, the x-axis.

56 The loop of $18y^2 = x(6-x)^2$ is revolved about the x-axis. Find the area of the surface generated in this way.

57 Sketch the graph of $8a^2y^2 = x^2(a^2 - x^2)$ and find the area of the surface generated when this curve is revolved about the x-axis.

SECTION 7.7

58 A 4-lb force will stretch a spring 6 in. How much work is done in stretching it 3 ft?

59 A spring pulls back with a force of 7 lb when it is stretched from its natural length of 12 in to a length of 13 in. How much work is required to compress it from a length of 11 in to a length of 7 in?

60 Show that the work done in stretching a spring of natural length L from a length a to a length b $(L < a < b)$ is equal to the amount of the stretch $(b - a)$ times the tension in the spring when its length is $\frac{1}{2}(a + b)$.

61 A bag of sand is lifted at the constant rate of 3 ft/s for 10 seconds. At the beginning the bag contains 100 lb of sand, but the sand leaks out at the rate of 4.5 lb/s. How much work is done in lifting this bag?

62 If a certain gas in a cylinder obeys an adiabatic gas law of the form $pV^{5/3} = c$, and if it initially occupies 64 in^3 at a pressure of 128 lb/in^2, find the work it does against the piston in expanding to 8 times its initial volume.

63 Find the work done in compressing 1024 ft^3 of air at a pressure of 27 lb/in^2 down to 243 ft^3 if the air obeys the adiabatic gas law $pV^{1.4} = c$.

64 Generalize Problem 63 by finding the work done in compressing air of initial volume V_1 and pressure p_1 down to a volume V_2, assuming the adiabatic gas law $pV^{1.4} = c$.

*65 A conical buoy that weighs B pounds floats upright in water with its vertex a feet below the surface. If the top of the buoy is $\frac{1}{3}a$ feet out of the water, how much work is done in pushing the buoy down until its top is just at the surface of the water?

*66 A spherical buoy of radius a feet that weighs B pounds has exactly the weight-density w of water, so that it floats with its top just touching the surface. A crane on a dock lifts the buoy until it just clears the water. How much work is done?

67 If two electrons are held fixed at the points $x = 0$ and $x = -1$ on the x-axis, find the work done in moving a third electron along the x-axis from $x = 4$ to $x = 1$.

68 Imagine a *very* deep mine shaft, of depth $D = \frac{1}{2}R$, extending halfway down to the center of the earth (ignore all practical difficulties caused by the internal constitution of the earth). A person whose weight is w at the surface is lifted from the bottom of the shaft to the top.

Under the assumption that the weight remains constant during the journey, the work done would be wD. Show that the work done during this process is actually $\frac{3}{4}wD$, by taking into account the fact that the force of gravity below the surface of the earth is proportional to the distance from its center.

69 A tank has the shape of the paraboloid of revolution obtained by revolving $y = x^2$ $(0 \le x \le \sqrt{5})$ about the y-axis. If it is full of water, how much work is required to empty it by pumping all the water out over the edge?

70 Let a cylindrical barrel of diameter 3 ft and height 5 ft be filled to a depth of 2 ft with water and then, above the water, with 2 additional ft of oil that weighs 50 lb/ft^3. Find the work done in pumping the water and oil over the edge of the barrel.

71 A hemispherical tank of radius 8 ft is full of water. If a hole is punched in the bottom, find the work done by gravity in emptying the tank.

72 Two cables are hanging side by side from the ceiling of a gymnasium. The first is an elastic cable of length L and the second is inelastic and has length $2L$. As two gymnasts of equal weight climb down these cables, the weight of the first stretches his cable to a total length of $2L$. Show that when the two gymnasts climb back up to the ceiling, the first does only $\frac{3}{4}$ of the work done by the second.

SECTION 7.8

73 Find the force due to water pressure against a rectangular floodgate 10 ft wide and 8 ft deep whose upper edge is at the surface of the water.

74 Find the force against the lower half of the floodgate in Problem 73.

In Problems 75 and 76, it is assumed that a vertical gate in the face of a dam has the stated shape. In each case find the total force on the gate.

75 A triangle 6 ft wide and 4 ft high, with upper edge at the water surface.

76 A triangle with base B and height H, with its vertex at the water surface.

77 A rectangular canal lock is 30 ft wide. When the water is 20 ft deep, what is the force of the water against the lock?

78 A rudder has the shape of an isosceles right triangle whose equal legs are 2 ft long. It is submerged vertically in water with one of the equal legs vertical and the other horizontal, and with the horizontal leg 3 ft below the surface and the opposite vertex 1 ft below the surface. Find the force of the water against one face of the rudder.

79 A rectangular gate in a dam has width 10 ft and height 8 ft. Find the force against the gate when the water level is 20 ft above its top.

80 Assume that the gate in Problem 79 cannot withstand a force greater than 100 tons. How high must the water be above the top of the gate in order to break through?

*81 The vertical end of a vat is a segment of a parabola opening upward which is 4 ft across the top and 8 ft deep. What is the force against this end when the vat is full of beer weighing 60 lb/ft^3?

APPENDIX: ARCHIMEDES AND THE VOLUME OF A SPHERE

To understand how Archimedes discovered the volume of a sphere, it is necessary to know a little about the level of knowledge from which he started.

As he states in one of his treatises, it was Democritus two centuries earlier who discovered that the volume of a cone is one-third the volume of a cylinder with the same height and the same base.† We will need this fact.

Also, the Greeks knew a little analytic geometry, but without our notation. They were acquainted with the idea that a locus in a plane can be studied by considering the distances from a moving point to two perpendicular lines, and if the sum of the squares of these distances is constant, they knew that the locus is a circle. In our notation, this condition amounts to the equation $x^2 + y^2 = a^2$.

†Democritus (about 460–370 B.C.) was the founder of the atomic theory of matter and the greatest philosopher of physical science among the ancient Greeks. He wrote at least 75 works on almost every conceivable subject, from physics and mathematics to logic, ethics, magnets, fevers, diets, agriculture, law, "the sacred writings in Babylon," "the right use of history," and even the growth of animal horns, spiders and their webs, and the eyes of owls. Of these works there remain only a few hundred fragments quoted by later writers; for example: "It is better to examine one's own faults than those of others," and "I would rather discover one cause than gain the kingdom of Persia." Plato hated him and was jealous of him, and Aristotle praised his genius—two weighty recommendations.

Further, Archimedes himself virtually created Greek physics. It is well known that he discovered the law of floating bodies. More important for our present purposes, he also discovered the principle of the lever and many facts about centers of gravity.

We are now ready to follow Archimedes in his search for the volume of a sphere. He considered the sphere to be generated by revolving a circle about its diameter. In modern notation we start with the circle

$$x^2 + y^2 = 2ax, \tag{1}$$

which has radius a and is tangent to the y-axis at the origin. This circle is shown on the left in Fig. 7.43, which is almost identical with Archimedes' original figure. Equation (1) contains the term y^2, and since πy^2 is the area of the variable cross section of the sphere x units to the right of the origin, it is natural to multiply through by π and write (1) in the form

$$\pi x^2 + \pi y^2 = \pi 2ax. \tag{2}$$

This leads us to interpret πx^2 as the area of the variable cross section of the cone generated by revolving about the x-axis the right triangle under the line $y = x$ between $x = 0$ and $x = 2a$. This in turn suggests that we seek a similar interpretation for the term $\pi 2ax$ on the right side of (2). If we persist in this search, we might perhaps think of multiplying by $2a$ and thus rewriting (2) as

$$2a(\pi x^2 + \pi y^2) = x\pi(2a)^2. \tag{3}$$

The motivation for this change clearly lies in the fact that $\pi(2a)^2$ can now be interpreted as the area of the cross section of the cylinder with the same height and base as the cone.

We therefore have on the left in Fig. 7.43 three circular disks viewed edge on, of areas πy^2, πx^2, and $\pi(2a)^2$, which are the intersections of a single plane with three solids of revolution. This plane is perpendicular to the x-axis at a distance x units to the right of the origin, and the solids are the sphere, the cone, and the cylinder, as indicated in the figure.

On the left side of equation (3) the sum of the first two areas is multiplied by $2a$, and on the right side the third area is multiplied by x. This observation led

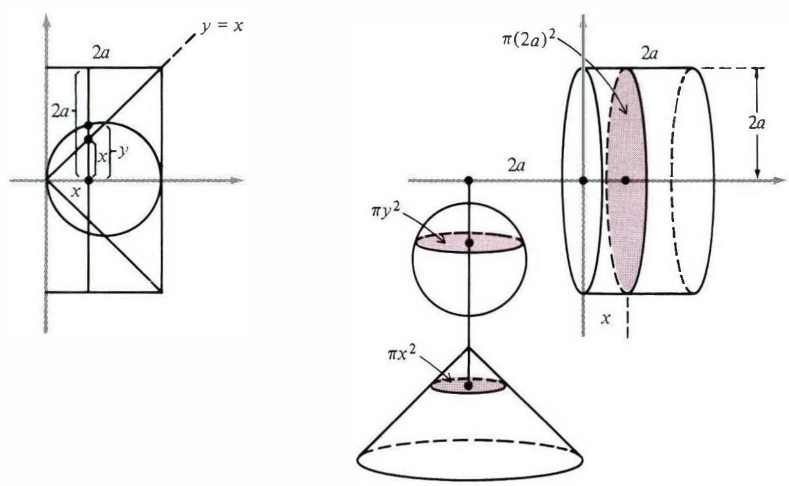


Figure 7.43 Archimedes' balancing argument.

Archimedes to the following great idea, as shown on the right in the figure. He left the disk with radius $2a$ where it is, in a vertical position x units to the right of the origin; and he moved the disks with radii y and x to a point $2a$ units to the left of the origin, where he hung them in a horizontal position with their centers (centers of gravity) under this point, suspended by a weightless string. The purpose of this maneuver can be understood only if we think of the x-axis as a lever and the origin as its fulcrum or balancing point. It can now be seen that equation (3) deals with moments. (A *moment* is the product of the suspended weight and the length of the lever arm.) From this point of view, equation (3) states that the combined moments of the two disks on the left equals the moment of the single disk on the right, and so, by Archimedes' own principle of the lever, this lever is in equilibrium.

We now carry out the final step of the reasoning. As x increases from 0 to $2a$, the three cross sections sweep through their respective solids and fill these solids. Since the three cross sections are in equilibrium throughout this process, the solids themselves are also in equilibrium. Let V denote the volume of the sphere, which was unknown until Archimedes finished this calculation. If we use Democritus' formula for the volume of the cone, and also the volume of the cylinder and the obvious location of its center of gravity, then the equilibrium of the solids in the positions shown in the figure yields the equation

$$2a[\tfrac{1}{3}\pi(2a)^2(2a) + V] = a\pi(2a)^2(2a). \tag{4}$$

It is now easy to solve (4) for V and obtain

$$V = \tfrac{4}{3}\pi a^3.$$

The ideas discussed here were created by a man who has been described—with good reason—as "the greatest genius of the ancient world." Indeed, nowhere can one find a more striking display of intellectual power combined with imagination of the highest order. Archimedes himself was so pleased with his discovery that he asked for a figure to be cut on his tombstone showing a sphere inscribed in a cylinder. And it was done.*

*The Roman orator Cicero wrote the following about two centuries after the death of Archimedes:

> I shall call up from the dust on which he drew his figures [the dust on the ground was the blackboard of the ancient mathematicians] an obscure, insignificant citizen of Syracuse, Archimedes. When I was quaestor I sought out his grave, which was unknown to the Syracusans (as they totally denied its existence), and found it enclosed all round and covered with brambles and thickets; for I remembered certain doggerel lines inscribed, as I had heard, upon his tomb, which stated that a sphere along with a cylinder had been set up on top of his grave. Accordingly, after taking a good look all round (for there are a great quantity of graves at the Agrigentine Gate), I noticed a small column rising a little above the bushes, on which there was the figure of a sphere and a cylinder. And so I at once said to the Syracusans (I had their leading men with me) that I believed it was the very thing of which I was in search. Slaves were sent in with sickles who cleared the ground of obstacles, and when a passage to the place was opened we approached the pedestal fronting us; the epigram was traceable with about half the lines legible, as the latter portion was worn away. [Cicero's *Tusculan Disputations* (Loeb Classical Library, p. 491).]

The present writer visited Syracuse in 1987, and if there had been any hope of finding this tomb—more than 2000 years later—he would have stayed until he found it.

Archimedes (287–212 B.C.) died in the conquest of Syracuse by the Romans during the Second Punic (Carthaginian) War. In the general confusion following the fall of the city, he was found concentrating on some diagrams he had drawn in the sand, and was killed by a marauding soldier who did not know who he was. In one version of the story he said to the intruder, who came too close, "Do not disturb my circles," whereupon the enraged soldier ran a sword through his body.

8 EXPONENTIAL AND LOGARITHM FUNCTIONS

8.1 INTRODUCTION

Our main purpose in this chapter is to learn how to work successfully with the indefinite integral

$$\int \frac{dx}{x}. \tag{1}$$

As we shall see, this purpose compels us to study the special exponential and logarithm functions

$$y = e^x \qquad \text{and} \qquad y = \log_e x. \tag{2}$$

The letter e used in these functions denotes the most important special number in mathematics after π. In decimal form it is an infinite nonrepeating decimal that is known to hundreds of thousands of decimal places; the first few digits are

$$2.71828\ldots$$

The ultimate reason for our interest in these matters is that the integral (1) and the functions (2) arise in a great variety of problems involving population growth, radioactive decay, chemical reaction rates, electric circuits, and many other phenomena in physics, chemistry, biology, geology, and virtually every science that uses quantitative methods, including meteorology, oceanography, and even archaeology. This integral and these functions are also indispensable in many branches of pure mathematics.

In order to reach a clear understanding of why the number e and the functions (2) matter so much, it is desirable to broaden the context a bit and consider the more general exponential and logarithm functions

$$y = a^x \qquad \text{and} \qquad y = \log_a x,$$

where a is a positive constant $\neq 1$. This is where we begin, and by adopting this approach we hope to make it perfectly clear that we choose a equal to e for compelling reasons of convenience and simplicity.

8.2 REVIEW OF EXPONENTS AND LOGARITHMS

Students who have managed to get this far in this book certainly have a working grasp of exponents, and perhaps also of logarithms as defined in terms of exponents. Nevertheless, we briefly review the main definitions and facts as they appear in the traditional approach.

We consider expressions of the form a^x where $a > 0$ and x is any real number. It is easy to explain exactly what a^x means if x is an integer n, and we assume students understand this explanation. The following is a brief reminder:

$$\text{If } n > 0, \text{ then } a^n = a \cdot a \cdots a \ (n \text{ factors}), \qquad a^0 = 1, \qquad a^{-n} = \frac{1}{a^n};$$

$$a^m a^n = a^{m+n}, \qquad \text{e.g.,} \qquad a^2 a^3 = (a \cdot a)(a \cdot a \cdot a) = a \cdot a \cdot a \cdot a \cdot a = a^5;$$

$$\frac{a^m}{a^n} = a^{m-n}, \qquad \text{e.g.,} \qquad \frac{a^5}{a^3} = \frac{a \cdot a \cdot a \cdot a \cdot a}{a \cdot a \cdot a} = \frac{\cancel{a} \cdot \cancel{a} \cdot a}{a \cdot a \cdot a} \cdot \frac{a \cdot a}{1} = a \cdot a = a^2;$$

$$(a^m)^n = a^{mn}, \qquad \text{e.g.,} \qquad (a^3)^2 = (a \cdot a \cdot a)(a \cdot a \cdot a) = a \cdot a \cdot a \cdot a \cdot a \cdot a = a^6.$$

Next, in Section 3.5 we summarized the meaning of fractional exponents, and we repeat the essence of this summary here. If $r = p/q$ is a fraction in lowest terms with $q > 0$, then by definition

$$a^r = a^{p/q} = (\sqrt[q]{a})^p, \tag{1}$$

where $\sqrt[q]{a}$ is the unique positive number whose qth power is a.

If the exponent x is an irrational number, then difficulties appear that students might not notice if we didn't mention them. For instance, what is meant by the expression $2^{\sqrt{2}}$? Clearly, it doesn't make sense to multiply 2 by itself $\sqrt{2}$ times. Also, since $\sqrt{2}$ can't be written as a fraction, definition (1) is useless. Is $2^{\sqrt{2}}$ really a definite number with a specific value? The answer is Yes, but this is not at all obvious. A natural way to proceed is to use the fact that any irrational number can be approximated as closely as we please by rational numbers. We can therefore define a^x by

$$a^x = \lim_{r \to x} a^r,$$

where r approaches x through rational values. This way of defining a^x when x is irrational is satisfactory from the logical point of view; however, it is a long and tedious chore to prove rigorously that everything works out as we expect and that the familiar laws of exponents remain valid. We skip over these boring details and merely state the final result, that the laws of exponents continue to hold in the following form:

$$a^{x_1} a^{x_2} = a^{x_1 + x_2}, \qquad \frac{a^{x_1}}{a^{x_2}} = a^{x_1 - x_2}, \qquad (a^{x_1})^{x_2} = a^{x_1 x_2},$$

where x_1 and x_2 are arbitrary real numbers.

The next natural step in this development is to examine the properties of the general *exponential function* $y = a^x$. Here again we simply state the important facts without making any attempt to discuss the logical details of how these facts are established. As above, we assume that a is a positive constant, and also that $a \neq 1$. The case $a = 1$ is of no interest because $1^x = 1$ for all x. Let us suppose first that $a > 1$. Then $y = a^x$ is a continuous function of x; it is increasing; its values are all positive; and it has the further obvious properties that

$$\lim_{x \to -\infty} a^x = 0 \qquad \text{and} \qquad \lim_{x \to \infty} a^x = \infty. \tag{2}$$

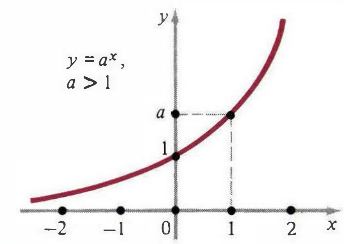


Figure 8.1

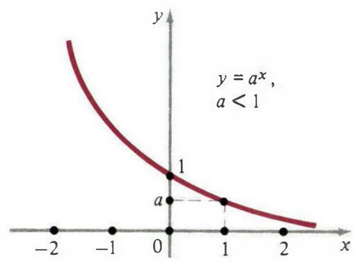


Figure 8.2

To sketch the graph, we plot a few points corresponding to several integral values of x, both positive and negative, and then connect these points by a smooth curve, as shown in Fig. 8.1. If $a < 1$, then $y = a^x$ is a decreasing function and its graph has the shape shown in Fig. 8.2.

When this much information about exponents is known or assumed, it is very easy to define logarithms and obtain some of their properties. On the most primitive level, a logarithm is an exponent. Thus, the fact that $100 = 10^2$ says that 2 is the logarithm of 100 to the base 10 (written $2 = \log_{10} 100$); and $4 = 64^{1/3}$ says that $\frac{1}{3}$ is the logarithm of 4 to the base 64 ($\frac{1}{3} = \log_{64} 4$).

More generally, the properties of exponents discussed above show clearly that if a is a positive constant $\neq 1$, then to each positive x there corresponds a unique y such that $x = a^y$. This y is written in the form $y = \log_a x$, and is called the *logarithm* of x to the *base a*. Accordingly,

$$y = \log_a x \qquad \text{has the same meaning as} \qquad x = a^y, \tag{3}$$

in the sense that each equation expresses the same relation between x and y, with the first written in a form solved for y and the second in a form solved for x. We can state this somewhat differently by saying that the symbol "$\log_a$" is created for the specific purpose of enabling us to solve $x = a^y$ for y in terms of x.

The basic properties of logarithms are direct translations of corresponding properties of exponents. Thus, if $x_1 = a^{y_1}$ and $x_2 = a^{y_2}$, then $x_1x_2 = a^{y_1}a^{y_2} = a^{y_1+y_2}$. But $y_1 = \log_a x_1$ and $y_2 = \log_a x_2$, so we have

$$\log_a x_1x_2 = \log_a x_1 + \log_a x_2.$$

Similarly,

$$\log_a \frac{x_1}{x_2} = \log_a x_1 - \log_a x_2$$

and

$$\log_a x^b = b \log_a x,$$

where b is any real number. Further, (3) tells us that

$$a^{\log_a x} = x \qquad \text{and} \qquad \log_a a^x = x.$$

We note also that the particular facts

$$\log_a 1 = 0 \qquad \text{and} \qquad \log_a a = 1$$

are equivalent to $1 = a^0$ and $a = a^1$.

In studying the *logarithm function*

$$y = \log_a x, \tag{4}$$

we consciously think of x and y as variables instead of mere numbers. Our starting point is the fact that (4) is equivalent to $x = a^y$. It is clear from this that x must be positive in order for y to exist, so (4) is defined only for $x > 0$. The graph of (4) is easy to obtain from the graph of $x = a^y$ by interchanging the axes, as we show in Fig. 8.3 for the case $a > 1$.* In this case $y = \log_a x$ is evidently an

*By "interchanging the axes" we mean the following: on the left in Fig. 8.3, imagine the two axes and the curve to be three pieces of stiff wire glued together into a rigid frame; lift this frame off the page and flip it over in space so that the x- and y-axes are revolved into their normal positions; and finally, return the frame to the page as it appears on the right in the figure.

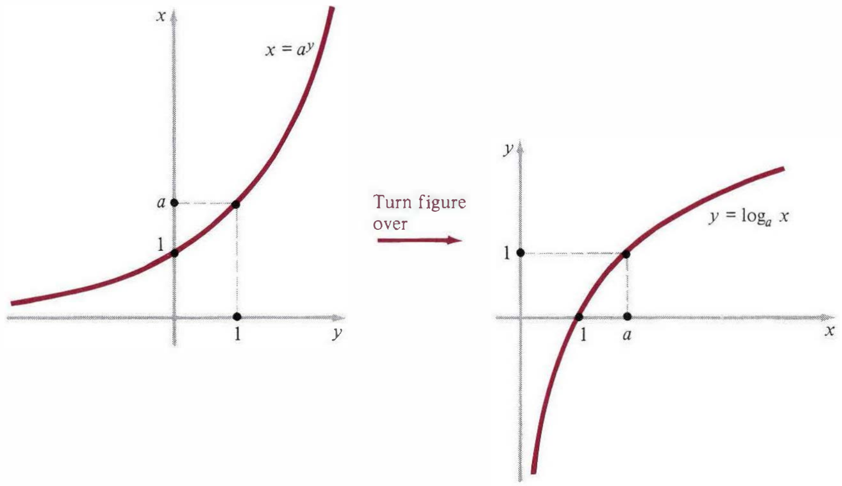


Figure 8.3

increasing continuous function of x. The features of this function that correspond to the properties (2) are

$$\lim_{x\to 0+} \log_a x = -\infty \qquad \text{and} \qquad \lim_{x\to\infty} \log_a x = \infty.$$

The most convenient logarithm for actual numerical calculations is the logarithm to the base 10, the so-called *common logarithm.* Common logarithms were once widely used by engineers and scientists and students in high school trigonometry courses, but such uses have virtually disappeared in these days of calculators and computers. However, modern technological changes in the way people do calculations have had no influence whatever on the importance of the logarithm *as a function*; it remains indispensable in the theoretical parts of mathematics and its applications, and these theoretical uses are what concern us in this chapter.

PROBLEMS

1 Express in terms of logarithms:
(a) $4^2 = 16$; (b) $3^4 = 81$;
(c) $81^{0.5} = 9$; (d) $32^{4/5} = 16$.

2 Express in terms of exponents:
(a) $\log_{10} 10 = 1$; (b) $\log_2 8 = 3$;
(c) $\log_5 \frac{1}{25} = -2$; (d) $\log_6 216 = 3$.

3 Evaluate:
(a) $\log_{10} 10{,}000$; (b) $\log_2 64$;
(c) $\log_{10} 0.0001$; (d) $\log_8 4$.

4 Solve for x:
(a) $\log_4 x = 3.5$; (b) $\log_8 x = \frac{5}{3}$;
(c) $\log_3 x = 5$; (d) $\log_{32} x = 0.6$.

5 Find the base a:
(a) $\log_a 4 = 0.4$; (b) $\log_a 8 = -\frac{3}{4}$;
(c) $\log_a 36 = 2$; (d) $\log_a 7 = \frac{1}{2}$.

6 If $y = \log_a (x + \sqrt{x^2 - 1})$, show that $x = \frac{1}{2}(a^y + a^{-y})$.

7 Show that $\log_a (x + \sqrt{x^2 - 1}) = -\log_a (x - \sqrt{x^2 - 1})$.

8 The magnitude M of an earthquake on the Richter scale is a number that ranges from $M = 0$ for the smallest earthquake that can be detected by instruments to $M = 8.9$ for the greatest known earthquake. M is given by the empirical formula

$$M = \frac{2}{3} \log_{10} \frac{E}{E_0},$$

where E is the energy released by the earthquake in kilowatthours and $E_0 = 7 \times 10^{-3}$.*

(a) Suppose the magnitudes of two earthquakes differ by 1 on the Richter scale. Show that the ratio of the energy of the larger earthquake to that of the smaller is $10^{3/2} \cong 31.62$.

(b) How much energy is released by an earthquake of magnitude 6?

(c) A city whose population is 300,000 uses about 3×10^5 kilowatthours (kWh) of electric energy every day. If the energy of an earthquake could somehow be transformed into electric energy, how many days' supply for this city would be provided by the earthquake in part (b)?

(d) The great Alaskan earthquake of 1964 had a Richter magnitude of 8.4. Answer the question in part (c) for this earthquake.

9 In chemistry the pH of a solution is defined by the formula $\mathrm{pH} = -\log_{10}[H^+]$, where $[H^+]$ denotes the hydrogen ion concentration as measured in moles per liter.† (One mole—or gram molecular weight—of a substance consists of 6×10^{23} molecules of the substance.) The value of $[H^+]$ for pure water is found by experiment to be 1.00×10^{-7}.

(a) What is the pH of pure water?

(b) A solution is called *acidic* or *basic* (*alkaline*) according as its value of $[H^+]$ is greater or less than that for pure water. What pH's characterize acidic and basic solutions?

10 Show that the number $\log_3 2$ is irrational. Hint: Assume the contrary, that $\log_3 2 = p/q$ where p and q are positive integers, and express this in terms of exponents. Can an integral power of 3 equal an integral power of 2?

11 Find the flaw in the following "proof" that $\frac{1}{2} < \frac{1}{4}$: multiply both sides of the inequality $1 < 2$ by $\log \frac{1}{2}$ to get, successively,

$$1 \cdot \log \tfrac{1}{2} < 2 \cdot \log \tfrac{1}{2},$$
$$\log \tfrac{1}{2} < \log (\tfrac{1}{2})^2,$$
$$\log \tfrac{1}{2} < \log \tfrac{1}{4},$$
$$\tfrac{1}{2} < \tfrac{1}{4}.$$

12 A prime number is an integer $p > 1$ that has no factors except itself and 1. The first few primes are $p = 2, 3, 5, 7, 11, \ldots$. In 1992 the largest known prime (discovered by people who like to play with supercomputers) was $2^{756,839} - 1$.

(a) When written out in decimal form, how many digits will this number have? Hint: Solve the equation $10^x = 2^{756,839}$.

(b) How many pages of this book will be needed to print this number? (One page holds about 4600 digits.)

*Charles Richter (1900–1985) was a professor of seismology at CalTech. He invented his magnitude measure in 1935. The following are Richter magnitudes, and numbers of people killed, in other memorable earthquakes of recent history:

8.9—Japan: Mar. 2, 1933, 2990 dead;
8.3—San Francisco, Calif.: Apr. 18, 1906, 2000 dead;
8.2—Tangshan, China: July 28, 1976, 242,000–800,000 dead;
8.1—Mexico City: Sept. 19, 1985, 9500 dead;
7.7—Peru: May 31, 1970, 66,794 dead;
7.2—Italy: Nov. 23, 1980, 3000 dead;
6.9—Armenia: Dec. 7, 1988, 25,000 dead.

†The symbol "pH" is an abbreviation of the French expression *puissance d'Hydrogène* (power of hydrogen).

8.3 THE NUMBER e AND THE FUNCTION $y = e^x$

The number e is often defined by the limit

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n. \tag{1}$$

This definition has the advantage of brevity but the serious disadvantage of shedding no light whatever on the significance of this crucial number. We prefer to define e differently, in a manner that reveals as clearly as possible why this number is so important. We then obtain (1) later, as merely one among many explicit formulas for e that can be used in a variety of ways.

Our aim in this section is to study a function $y = f(x)$ that is unchanged by differentiation:

$$\frac{d}{dx} f(x) = f(x). \tag{2}$$

It is far from obvious that any such function exists [we don't count the trivial case $f(x) = 0$]. As we shall see, the desired function turns out to be one of the exponential functions $y = a^x$ for $a > 1$. The central meaning of the number e can now be stated as follows: It is the specific value of the base a that causes the function $f(x) = a^x$ to have the property (2). In this way we understand what purpose e serves. However, we must still give a satisfactory definition and show as simply as possible that this definition accomplishes the stated purpose.

Let us calculate the derivative of $f(x) = a^x$ and see what happens. As usual when differentiating a new type of function, we go back to the definition of the derivative,

$$\frac{d}{dx} f(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

It will be convenient here to denote the increment by the single letter h instead of the familiar Δx (Fig. 8.4):

$$\frac{d}{dx} a^x = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} = \lim_{h \to 0} \frac{a^x a^h - a^x}{h}$$

$$= \lim_{h \to 0} \left(a^x \frac{a^h - 1}{h} \right) = a^x \left(\lim_{h \to 0} \frac{a^h - 1}{h} \right). \tag{3}$$

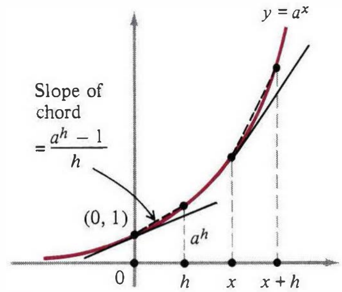


Figure 8.4

As Fig. 8.4 shows, the quantity in parentheses on the right side of (3) is the slope of the tangent line to the curve $y = a^x$ at the point (0, 1). If this slope equals 1, then the right side of (3) reduces to a^x and this particular function a^x has the property (2). This brings us to our definition: e is the specific value of the base a that produces this result, that is,

$$e \text{ is the number for which } \lim_{h \to 0} \frac{e^h - 1}{h} = 1. \tag{4}$$

We can obtain considerable insight into the nature of the number e by sketching $y = a^x$ for the cases $a = 1.5$, $a = 2$, $a = 3$, and $a = 10$, as shown in Fig. 8.5. These curves tell us that as the base a increases continuously from numbers close to 1 to larger numbers, the slope of the tangent to $y = a^x$ at the point (0, 1) increases continuously from values close to 0 to larger values, and therefore this slope is exactly equal to 1 for some intermediate value of a. This intermediate value is e; and as we hope students will agree, it is geometrically clear from these remarks that e exists. Next, we plot the points on the first three of these curves corresponding to $x = 1$ in order to stress the fact that the slopes of the chords joining these points to (0, 1) are $\frac{1}{2}$, 1, and 2. This is conclusive geometric evidence that the slope of the tangent at (0, 1) is <1 for the cases $a = 1.5$ and $a = 2$, and plausible evidence that this slope is >1 for the case $a = 3$; and therefore e is certainly >2 and probably <3.

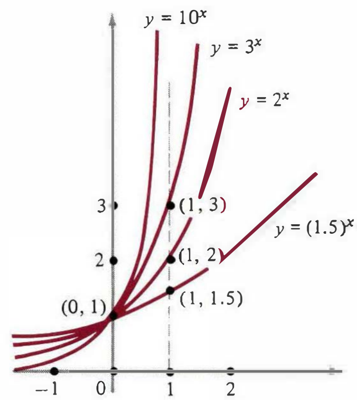


Figure 8.5

In Fig. 8.6 we show the graph of $y = e^x$ with emphasis placed on its defining characteristic: It is the single member of the family of exponential functions $y = a^x$ $(a > 1)$ whose tangent line at the point (0, 1) has slope 1. The function $y = e^x$ is often called *the* exponential function, to distinguish it from its comparatively unimportant relatives.

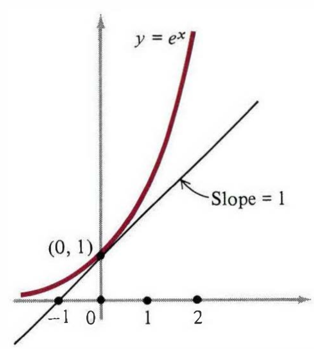


Figure 8.6

We can investigate the number e more closely by noting that (4) tells us that

$$\frac{e^h - 1}{h} \text{ is approximately equal to } 1,$$

and that this approximation gets better and better as h approaches 0. Thus, by simple manipulations we obtain

$$\frac{e^h - 1}{h} \cong 1, \qquad e^h - 1 \cong h, \qquad e^h \cong 1 + h, \qquad e \cong (1 + h)^{1/h},$$

and finally,

$$e = \lim_{h\to 0} (1 + h)^{1/h}. \tag{5}$$

In words, this says that e is the limit of 1 plus a small number, raised to the power of the reciprocal of the small number, as that small number approaches 0. If we write $h = 1/n$ where n is understood to be a positive integer that $\to \infty$ as $h \to 0$, then (5) yields

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n,$$

which is (1). This formula enables us to compute rough approximations to e fairly easily, as the following table shows:

n	$\left(1 + \frac{1}{n}\right)^n$
1	2
2	$\frac{9}{4} = 2\frac{1}{4} = 2.25$
3	$\frac{64}{27} = 2\frac{10}{27} = 2.370$
4	$\frac{625}{256} = 2\frac{113}{256} = 2.441$

However, this is a slow process and the value of e has been computed to great accuracy by other and more efficient methods. To 15 decimal places it is

$$e = 2.718281828459045\ldots.^*$$

The number e, like the number π, is woven inseparably into the fabric of both nature and mathematics. Many remarkable properties of e have been discovered over the centuries. For example, e is irrational; indeed, it is not even a root of any polynomial equation with rational coefficients.

However, we must not forget our original purpose in this section, which was to study a function that is unchanged by differentiation. We have now made a good start on this task, in the sense that we have explained the meaning of the following statement and established its validity:

$$\frac{d}{dx} e^x = e^x. \tag{6}$$

An equivalent statement is that $y = e^x$ satisfies the differential equation

$$\frac{dy}{dx} = y.$$

Every function $y = ce^x$ also satisfies this equation, because

*Many people remember this much of e by grouping the digits this way,

2.7 1828 1828 45 90 45,

in order to visualize the repeated 1828 followed by 45, then twice 45, then 45 again.

$$\frac{dy}{dx} = \frac{d}{dx}(ce^x) = c\frac{d}{dx}e^x = ce^x = y.$$

Further, we assert that these are the *only* functions that are unchanged by differentiation. To prove this, suppose that $y = f(x)$ is any function with this property. Then by the quotient rule,

$$\frac{d}{dx}\left[\frac{f(x)}{e^x}\right] = \frac{e^x f'(x) - f(x)e^x}{e^{2x}} = \frac{e^x f(x) - f(x)e^x}{e^{2x}} = 0.$$

This implies that $f(x)/e^x = c$ for some constant c, so $f(x) = ce^x$, as stated.

By the chain rule, (6) generalizes immediately to

$$\frac{d}{dx}e^u = e^u\frac{du}{dx}, \tag{7}$$

where $u = u(x)$ is understood to be any differentiable function of x.

Example 1 In view of (7), the following derivatives are obvious:

$$\frac{d}{dx}e^{4x} = 4e^{4x}, \qquad \frac{d}{dx}e^{x^2} = 2xe^{x^2}, \qquad \frac{d}{dx}e^{1/x} = \left(-\frac{1}{x^2}\right)e^{1/x}.$$

If we write (7) in differential form, as $d(e^u) = e^u\,du$, then by reading this backwards we obtain the integration formula

$$\int e^u\,du = e^u + c. \tag{8}$$

Example 2 To integrate $\int e^{5x}\,dx$, we write

$$\int e^{5x}\,dx = \tfrac{1}{5}\int e^{5x}\,d(5x) = \tfrac{1}{5}e^{5x} + c,$$

where $5x$ plays the role of u in formula (8). This problem is so simple that there is no need to make explicit use of the method of substitution. It suffices to keep in mind what (8) says and make minor adjustments accordingly, as indicated.

Example 3 The integral

$$\int \frac{9xe^{\sqrt{3x^2+2}}\,dx}{\sqrt{3x^2+2}}$$

is more complicated. Our only hope is that (8) will see us through, so we write

$$u = \sqrt{3x^2+2} = (3x^2+2)^{1/2}$$

and

$$du = \tfrac{1}{2}(3x^2+2)^{-1/2}\,6x\,dx = \frac{3x\,dx}{\sqrt{3x^2+2}}.$$

This substitution (or change of variable) enables us to express the given integral in a much simpler form, and thereby to finish the calculation,

$$\int \frac{9xe^{\sqrt{3x^2+2}}\,dx}{\sqrt{3x^2+2}} = 3\int e^u\,du = 3e^u + c = 3e^{\sqrt{3x^2+2}} + c.$$

Students should observe that the complicated appearance of the given integral is only a disguise concealing the relatively simple form displayed in (8). Learning the art of integration is mostly learning to see the underlying form through the disguise.

Example 4 *Continuously compounded interest.* If P dollars is deposited in a bank that pays an interest rate of 8 percent per year, compounded semiannually, then after t years the accumulated amount is

$$A = P(1 + 0.04)^{2t}.$$

More generally, if the interest rate is $100x$ percent ($x = 0.08$ for 8 percent), and if this interest is compounded n times a year, then after t years the accumulated amount is

$$A = P\left(1 + \frac{x}{n}\right)^{nt}.$$

If n is now increased indefinitely, so that the interest is compounded more and more frequently, then we approach the limiting case of continuously compounded interest. To find the formula for A under these circumstances, we observe that (5) yields

$$\left(1 + \frac{x}{n}\right)^{nt} = \left[\left(1 + \frac{x}{n}\right)^{n/x}\right]^{xt} \to e^{xt},$$

so

$$A = Pe^{xt}. \tag{9}$$

Ordinary compound interest produces growth in spurts or jumps at the end of each interest period. In contrast to this, we see from (9) that continuously compounded interest produces steady continuous growth of a type called *exponential growth.* In Sections 8.5 and 8.6 we discuss many additional examples of exponential growth as it occurs in the natural sciences.

Remark 1 The function e^x grows very rapidly as x increases; in fact, it grows faster than x^p for any fixed positive exponent p, no matter how large, in the sense that

$$\lim_{x\to\infty} \frac{e^x}{x^p} = \infty.$$

An outline of a proof for the case in which p is a positive integer n is given in Additional Problems 18 to 20.

Remark 2 We have deduced the existence of the limits in (1) and (5) from the definition of e given in (4). However, this definition itself is highly geometric in nature, and some mathematicians might be inclined to dismiss our entire approach to these ideas as "reasoning by wishful thinking." To mollify such critics, and also for the occasional students who might be interested, we provide an independent proof of the existence of these limits in Appendix A.8.

PROBLEMS

In Problems 1–10, find the derivative dy/dx of the given function.

1 $y = \frac{1}{2}(e^x + e^{-x})$.

2 $y = \frac{1}{2}(e^x - e^{-x})$.

3 $y = x^2e^x$.

4 $y = x^2e^{-x^2}$.

5 $y = e^{e^x}$.

6 $y = x^e + e^x$.

7 $y = \dfrac{ax - 1}{a^2}e^{ax}$.

8 $y = (3x + 1)e^{-3x}$.

9 $y = (2x^2 - 2x + 1)e^{2x}$.

10 $y = e^{1/x^2} + 1/e^{x^2}$.

Evaluate the integrals in Problems 11–16.

11 $\int e^{3x}\,dx$.

12 $\int xe^{-x^2}\,dx$.

13 $\displaystyle\int e^{(1/5)x}\,dx$.

14 $\displaystyle\int \frac{3dx}{e^{2x}}$.

15 $\displaystyle\int 6x^2e^{x^3}\,dx$.

16 $\displaystyle\int \frac{e^{\sqrt{x}}\,dx}{\sqrt{x}}$.

17 Sketch the graph of each of the following functions and find its maximum and minimum points and points of inflection:
(a) $y = e^{-x^2}$; (b) $y = xe^{x/3}$.

18 Find the base of the largest rectangle that rests on the x-axis and has its upper vertices on the curve $y = e^{-x^2}$.

19 Sketch the curve $y = \frac{1}{2}(e^x + e^{-x})$ and find its length from $x = 0$ to $x = b$ $(b > 0)$.

20 The arc in Problem 19 is revolved about the x-axis. Find the area of the surface of revolution generated in this way.

21 If a particle moves on the x-axis in such a way that its position x at time t is given by $x = Ae^{kt} + Be^{-kt}$, where A, B, and k are constants, show that the particle is repelled from the origin with a force proportional to its distance from the origin. Hint: Use Newton's second law of motion, $F = ma$.

22 If the tangent to $y = e^x$ at the point $x = x_0$ intersects the x-axis at $x = x_1$, show that $x_0 - x_1 = 1$.

23 Graph $y = e^{-x}$, find the area under this curve from $x = 0$ to $x = b$ $(b > 0)$, and find the limit approached by this area as $b \to \infty$.

24 Verify that $y = e^{-x}$ and $y = e^{2x}$ are both solutions of the differential equation $y'' - y' - 2y = 0$.

25 Evaluate the following limits:

(a) $\displaystyle\lim_{n\to\infty}\left(1 + \frac{1}{2n}\right)^{2n}$;

(b) $\displaystyle\lim_{n\to\infty}\left(1 + \frac{1}{3n+1}\right)^{3n+1}$;

(c) $\displaystyle\lim_{n\to\infty}\left(1 + \frac{1}{n^2}\right)^{n^2}$;

(d) $\displaystyle\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^{2n}$;

(e) $\displaystyle\lim_{n\to\infty}\left(1 + \frac{1}{2n}\right)^{n}$.

26 Use the argument in Example 4 to obtain the formula

$$e^x = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n.$$

27 Use the Intermediate Value Theorem (Section 2.6) to show that the equation $x + e^x = 0$ has a root. Why does this equation have only one root?

28 Use Newton's method (Section 4.6) to calculate the root of the equation in the preceding problem correct to six decimal places.

29 When Benjamin Franklin died in 1790, it was found that he had directed in his will for the sum of \$10,000 to be given jointly to the cities of Philadelphia and Boston, to be invested at compound interest for the benefit of the people of those cities. In a recent TV broadcast by CNN it was stated that in 1990, 200 years later, the accumulated amount would have been 90 billion dollars (\$90,000,000,000) if invested at a "reasonable" rate of interest. What rate of interest would have been required to achieve this astounding result?

8.4 THE NATURAL LOGARITHM FUNCTION $y = \ln x$. EULER

Logarithms to the base 10—common logarithms—are often taught in high school, starting with the following familiar definition: For any positive number x, $\log_{10} x$ is that number y such that $x = 10^y$. In just the same way, for any positive number x, $\log_e x$ is defined to be that number y such that $x = e^y$. This is illustrated on the left in Fig. 8.7.

The number $\log_e x$ is called the *natural logarithm of x*, for reasons that will become clear in Remark 2. In deference to standard practice at this level, we denote this number by the simpler notation $\ln x$, pronounced "ell enn ex." Thus,

$$y = \ln x \text{ has the same meaning as } x = e^y,$$

in the sense that we are dealing here with a single equation, first written in a form solved for y and then written in a form solved for x. The graph of $y = \ln x$

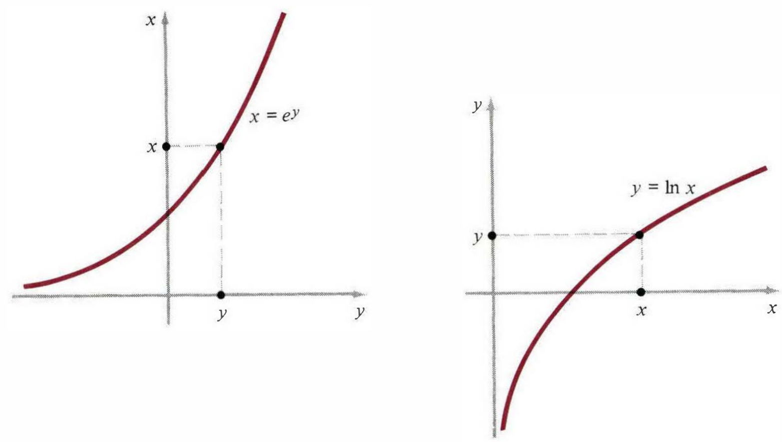


Figure 8.7

is obtained by simply turning over the graph of $x = e^y$ so as to interchange the positions of the axes (Fig. 8.7, right). Just as in Section 8.2, the natural logarithm function $y = \ln x$ is defined only for positive values of x and has the following familiar properties:

$$\ln x_1x_2 = \ln x_1 + \ln x_2 \qquad \text{and} \qquad \ln \frac{x_1}{x_2} = \ln x_1 - \ln x_2;$$

$$\ln x^b = b \ln x;$$

$$e^{\ln x} = x \qquad \text{and} \qquad \ln e^x = x;$$

$$\lim_{x \to 0+} \ln x = -\infty \qquad \text{and} \qquad \lim_{x \to \infty} \ln x = \infty.$$

Also, $\ln 1 = 0$ and $\ln e = 1$.

We can compute the derivative dy/dx of the function $y = \ln x$ very easily, by differentiating $x = e^y$ implicitly with respect to x:

$$1 = e^y \frac{dy}{dx}, \qquad \text{so} \qquad \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}.$$

This yields the formula

$$\frac{d}{dx} \ln x = \frac{1}{x},$$

and we immediately have the chain rule extension

$$\frac{d}{dx} \ln u = \frac{1}{u} \frac{du}{dx}, \tag{1}$$

where u is understood to be any differentiable function of x.

Example 1 As direct applications of (1) we have

$$\frac{d}{dx} \ln (3x + 1) = \frac{1}{3x + 1} \frac{d(3x + 1)}{dx} = \frac{3}{3x + 1},$$

$$\frac{d}{dx} \ln (1 - x^2) = \frac{1}{1 - x^2} \frac{d(1 - x^2)}{dx} = \frac{-2x}{1 - x^2},$$

$$\frac{d}{dx}\ln\left(\frac{3x}{2x+1}\right) = \frac{1}{3x/(2x+1)}\,\frac{(2x+1)\cdot 3 - 3x\cdot 2}{(2x+1)^2}$$
$$= \frac{1}{x(2x+1)}.$$

We point out that the last calculation can be simplified by first writing $\ln[3x/(2x+1)] = \ln 3 + \ln x - \ln(2x+1)$, so that

$$\frac{d}{dx}\ln\left(\frac{3x}{2x+1}\right) = \frac{1}{x} - \frac{2}{2x+1} = \frac{1}{x(2x+1)}.$$

The differential version of (1) is $d(\ln u) = du/u$, which leads at once to the main formula of this chapter,

$$\int \frac{du}{u} = \ln u + c. \tag{2}$$

It is understood in (2) that u is positive, because only in this case does $\ln u$ have a meaning. However, it is easy to see that the integrand can always be written with a positive denominator, by juggling the signs. Thus, if $u < 0$ we can write

$$\int \frac{du}{u} = \int \frac{d(-u)}{-u} = \ln(-u) + c. \tag{3}$$

Many writers cover all cases by writing (2) in the form

$$\int \frac{du}{u} = \ln|u| + c.$$

However, we shall not do this, for the reason that most of the applications require a quick transition from logs to exponentials, and the presence of the absolute value sign interferes with the smooth operation of this process. We prefer to use (2) as it is, and to remember as we do this that u must be positive. In situations where u is negative, we easily make the minor adjustments indicated in (3).

Students will recall that the fundamental integration formula

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + c, \qquad n \neq -1,$$

failed to cover one exceptional case, namely, $n = -1$. Formula (2) now fills this gap, since it tells us that

$$\int u^{-1}\,du = \int \frac{du}{u} = \ln u + c.$$

Example 2 The following applications of (2) are easy to carry out by inspection:

$$\int \frac{dx}{x+1} = \ln(x+1) + c,$$
$$\int \frac{dx}{1-2x} = -\frac{1}{2}\int \frac{-2dx}{1-2x} = -\frac{1}{2}\ln(1-2x) + c,$$
$$\int \frac{3x^3\,dx}{x^4+1} = \frac{3}{4}\int \frac{4x^3\,dx}{x^4+1} = \frac{3}{4}\ln(x^4+1) + c.$$

In more complicated problems it is desirable to make an explicit substitution or change of variable, in order to diminish the likelihood of accidental error.

In Section 5.4 we discussed the method of separation of variables for solving differential equations. The equation

$$\frac{dy}{dx} = ky \tag{4}$$

is one of the simplest and most important to which this method can be applied. We give the details of this procedure here because the same ideas will be used over and over again in the next two sections, and the sooner students become thoroughly familiar with them, the better:

$$\frac{dy}{y} = k\,dx, \qquad \int \frac{dy}{y} = \int k\,dx, \qquad \ln y = kx + c_1,$$

$$y = e^{kx+c_1} = e^{c_1}e^{kx},$$

and finally,

$$y = ce^{kx},$$

where c is simply a more convenient notation for the constant e^{c_1}. From our point of view, the exponential and logarithm functions find their main reason for being in the fact that they enable us to solve the differential equation (4) in this smooth and straightforward manner. It is also clear from the calculations just given that these functions go together like the two sides of a coin: you can't spend one side without also spending the other.

The next two sections are filled with many far-reaching applications of equation (4) to various fields of science. We hope students will agree that these applications fully justify the attention we have given to this differential equation and to the functions that are necessary for solving it.

Remark 1 We know that $\ln x \to \infty$ as $x \to \infty$. This property of the logarithm is illustrated on the right in Fig. 8.7. However, the graph of $y = \ln x$ rises very slowly, since it is the mirror image of the rapidly rising graph of $x = e^y$. Just how slowly $y = \ln x$ increases can be understood by noticing that it doesn't reach the level $y = 10$ until $x = e^{10} \cong 22{,}000$. The fact that $\ln x$ grows more slowly than x can be expressed by writing

$$\lim_{x\to\infty} \frac{\ln x}{x} = 0. \tag{5}$$

We might try to estimate more accurately how slowly $\ln x$ grows by comparing it with an even smaller function than x, say $\sqrt{x}$ or $\sqrt[3]{x}$. The remarkable fact is that $\ln x$ grows more slowly than *any* positive power of x:

$$\lim_{x\to\infty} \frac{\ln x}{x^p} = 0, \tag{6}$$

where p is any positive constant. Proofs of (5) and (6) are indicated in Problem 13 and Additional Problem 26.

Remark 2 We mention here another way of seeing—with additional clarity—how the number e arises in calculus. The idea is to calculate the derivative of

$\log_a x$ as if we were doing this for the very first time in history, in an exploratory spirit, without any preconception of what the base a "ought" to be. We begin by applying the definition of the derivative,

$$\frac{d}{dx}\log_a x = \lim_{\Delta x \to 0} \frac{\log_a(x + \Delta x) - \log_a x}{\Delta x}. \tag{7}$$

Our next step is to manipulate the expression following the limit sign into a more convenient form by using the properties of logarithms discussed in Section 8.2,

$$\begin{aligned}\frac{\log_a(x + \Delta x) - \log_a x}{\Delta x} &= \frac{1}{\Delta x}\log_a\left(\frac{x + \Delta x}{x}\right)\\ &= \frac{1}{\Delta x}\log_a\left(1 + \frac{\Delta x}{x}\right)\\ &= \frac{1}{x}\frac{x}{\Delta x}\log_a\left(1 + \frac{\Delta x}{x}\right)\\ &= \frac{1}{x}\log_a\left(1 + \frac{\Delta x}{x}\right)^{x/\Delta x}.\end{aligned}$$

The definition (7) now yields

$$\begin{aligned}\frac{d}{dx}\log_a x &= \lim_{\Delta x \to 0}\left[\frac{1}{x}\log_a\left(1 + \frac{\Delta x}{x}\right)^{x/\Delta x}\right]\\ &= \frac{1}{x}\lim_{\Delta x \to 0}\left[\log_a\left(1 + \frac{\Delta x}{x}\right)^{x/\Delta x}\right]\\ &= \frac{1}{x}\log_a\left[\lim_{\Delta x \to 0}\left(1 + \frac{\Delta x}{x}\right)^{x/\Delta x}\right].\end{aligned}$$

If we maintain our spirit of research, then the distinctive limit in brackets here attracts our attention. It is natural to simplify its structure a bit by putting $h = \Delta x/x$, and to recognize that $\Delta x \longrightarrow 0$ is equivalent to $h \longrightarrow 0$. We now define a new mathematical constant e by means of the resulting limit,

$$e = \lim_{h \to 0}(1 + h)^{1/h}, \tag{8}$$

and we at once obtain

$$\frac{d}{dx}\log_a x = \frac{1}{x}\log_a e. \tag{9}$$

One of our continuing purposes in calculus—though students may find this hard to believe—is to make the formulas we work with as simple as possible. Since $\log_e e = 1$, it is clear that (9) takes its simplest form if the base a is chosen to be the number e:

$$\frac{d}{dx}\log_e x = \frac{1}{x}. \tag{10}$$

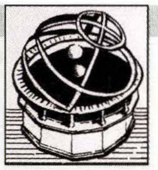
The function $\log_e x$ (or $\ln x$) is called the "natural" logarithm because formula (10) makes it the most convenient logarithm to use in calculus and its applications.

The ideas described here are those by means of which the Swiss mathematician Euler (pronounced "OIL-er") essentially discovered both e and the functions $\ln x$ and e^x in the early eighteenth century.

Remark 3 Students should be informed that some writers define the function $\ln x$ by the formula

$$\ln x = \int_1^x \frac{dt}{t}. \tag{11}$$

These writers are then committed to deriving all the properties of the logarithm from the properties of this integral. Also, it is necessary to define the exponential function in terms of the logarithm instead of the other way around. This approach to the ideas of this chapter has its merits from the point of view of the theory of calculus. However, for most students, exponents come before logarithms as naturally as milk comes before cheese; and regardless of the fine points of logic, it is bound to seem perverse and unnatural to begin our subject with (11)—however much it may delight the soul of a mathematician.



NOTE ON EULER

Leonard Euler (1707–1783) was Switzerland's foremost scientist and one of the three greatest mathematicians of modern times—the other two being Gauss and Riemann.

He was perhaps the most prolific author of all time in any field. From 1727 to 1783 his writings poured forth in a seemingly endless flood, constantly adding knowledge to every known branch of pure and applied mathematics, and also to many that were not known until he created them. He averaged about 800 printed pages a year throughout his long life, and yet he almost always had something worthwhile to say and never seems long-winded. The publication of his complete works was started in 1911, and the end is not yet in sight. This edition was planned to include 887 titles in 72 volumes. However, since that time extensive new deposits of previously unknown manuscripts have been unearthed. It is now estimated that more than 100 large volumes will be required for completion of the project, well into the twenty-first century. Euler evidently wrote mathematics with the ease and fluency of a skilled speaker discoursing on subjects with which he is intimately familiar. His writings are models of relaxed clarity. He never condensed, and he reveled in the rich abundance of his ideas and the vast scope of his interests. The French physicist Arago, in speaking of Euler's incomparable mathematical facility, remarked that "He calculated without apparent effort, as men breathe, or as eagles sustain themselves in the wind." He suffered total blindness during the last 17 years of his life, but with the aid of his powerful memory and fertile imagination, and with helpers to write his books and scientific papers from dictation, he actually increased his already prodigious output of work.

Euler was a native of the city of Basel in Switzerland and a student of John Bernoulli at the University—himself one of the most eminent mathematicians of the time—but he soon outstripped his teacher. His working life was spent as a member of the Academies of Science at Berlin and St. Petersburg, and most of his papers were published in the journals of these organizations. After the launching of calculus by Newton and Leibniz in the seventeenth century, mathematics developed rapidly but without much order or coherence. Euler tamed this mathematical wilderness as the explorers and settlers tamed the wilderness that became the United States of America. He was also a man of broad culture, well versed in the classical languages and literatures (he knew the *Aeneid* by heart), many modern languages, physiology, medicine, botany, geography, and the entire body of physical science as it was known in his time. However, he had little talent for metaphysics or disputation, and came out second best in many good-natured verbal encounters with Voltaire at the court of Frederick the Great. His personal life was as placid and uneventful as is possible for a man with 13 children.

Though he was not himself a teacher, Euler has had a deeper influence on the teaching of mathematics than any other person. This came about chiefly through his three great treatises: *Introductio in Analysin Infinitorum* (1748); *Institutiones Calculi Differentialis* (1755); and *Institutiones Calculi Integralis* (1768–1794). There is considerable truth in the old saying that all elementary and advanced calculus textbooks since 1748 are essentially copies of Euler or copies of copies of Euler. These works summed up and codified the discoveries of his predecessors, and are full of Euler's own ideas. He extended and perfected plane and solid analytic geometry, introduced the analytic approach to trigonometry, and was responsible for the modern treatment of the functions $\ln x$ ($= \log_e x$) and e^x. He created a consistent theory of logarithms of negative and imaginary numbers and discovered that $\ln x$ has an infinite number of values. It was through his work that the symbols e, π, and i ($= \sqrt{-1}$) became common currency for all mathematicians, and it was he who linked them together in the astonishing equation $e^{\pi i} = -1$. This is merely a special case (put $\theta = \pi$) of his famous formula $e^{i\theta} = \cos\theta + i\sin\theta$—see Section 14.8—which connects the exponential and trigonometric functions and is absolutely indispensable in higher analysis.* Among his other contributions to standard mathematical notation were $\sin x$, $\cos x$, the use of $f(x)$ for an unspecified function, and the use of Σ for summation.† Good notations are important, but the ideas behind them are what really count, and in this respect, Euler's fertility was almost beyond belief. He preferred concrete special problems to the general theories in vogue today, and his unique insight into the connections among apparently unrelated formulas blazed many trails into new fields of mathematics which he left for his successors to cultivate.

He was the first and greatest master of infinite series and infinite products, and his works are crammed with striking discoveries in these fields. James Bernoulli (John's older brother) found the sums of several infinite series, but he was not able to find the sum of the reciprocals of the squares, $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$. He wrote, "If someone should succeed in finding this sum, and will tell me about it, I shall be much obliged to him." In 1736, long after James's death, Euler made the wonderful discovery that

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6}.$$

He also found the sums of the reciprocals of the fourth and sixth powers,

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots = 1 + \frac{1}{16} + \frac{1}{81} + \cdots = \frac{\pi^4}{90}$$

and

$$1 + \frac{1}{2^6} + \frac{1}{3^6} + \cdots = 1 + \frac{1}{64} + \frac{1}{729} + \cdots = \frac{\pi^6}{945}.$$

When John heard about these feats, he wrote, "If only my brother were alive now."‡ Few would believe that these formulas are related—as they are—to Wallis's infinite product (1656),

$$\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots.$$

Euler was the first to explain this in a satisfactory way, in terms of his infinite product expansion of the sine,

$$\frac{\sin x}{x} = \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right) \cdots.$$

The ideas described here are explained more fully in the appendices to Chapters 13 and 14.

His work in all departments of mathematics strongly influenced the further development of this subject through the next two centuries. He contributed many important ideas to differential equations, including substantial parts of the theory of second-order linear equations and the method of solution by power series. He gave the first systematic discussion of the calculus of variations, which he founded on his basic differential equation for a minimizing curve. He introduced the number now known as ***Euler's constant,***

$$\gamma = \lim_{n \to \infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n\right) = 0.5772\ldots,$$

which is the most important special number in mathematics after π and e (see Section 13.6). He discovered the integral defining the gamma function,

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt,$$

which is often the first of the so-called ***higher transcendental functions*** that students meet beyond the level of calculus, and he developed many of its applications and special properties. He also worked with Fourier series, encountered the Bessel functions in his study of the vibrations of a stretched circular membrane, and applied Laplace transforms to solve differential equations—all before Fourier, Bessel, and Laplace were born. In almost every direction people traveled in exploring the world of classical mathematics, they met Euler coming back—for he had been that way before them.

*An even more astonishing consequence of his formula is the fact that an imaginary power of an imaginary number can be real, in particular $i^i = e^{-\pi/2}$; for if we put $\theta = \pi/2$, we obtain $e^{\pi i/2} = i$, so

$$i^i = (e^{\pi i/2})^i = e^{\pi i^2/2} = e^{-\pi/2}.$$

Euler further showed that i^i has infinitely many values, of which this calculation produces only one.

†See F. Cajori, *A History of Mathematical Notations* (Open Court, 1929).

‡The world is still waiting—more than 250 years later—for someone to discover the sum of the reciprocals of the cubes.

Just as William Shakespeare's business was the writing of plays for the use of his theatrical company, Euler's business was mathematical research and publication for the academies that employed him; and each was the greatest master of his business that the world has ever seen. We know that it is almost impossible to speak English without using Shakespeare's words and phrases; and in the same way, it is almost impossible to think about mathematics without using Euler's thoughts.*

*For students who wish to enter into the mind of this great mathematician and experience some of his most interesting work in number theory at first hand—and in a context not requiring much previous knowledge—we recommend Chapter VI of G. Polya's fine book, *Induction and Analogy in Mathematics* (Princeton University Press, 1954).

PROBLEMS

1 Simplify each of the following:
(a) $e^{\ln 2}$; (b) $\ln e^3$; (c) $e^{-\ln x}$;
(d) $\ln e^{1/x}$; (e) $\ln (1/e^x)$; (f) $e^{\ln(1/x)}$;
(g) $e^{-\ln(1/x)}$; (h) $e^{\ln 3+\ln x}$; (i) $\ln e^{\ln 1}$;
(j) $\ln e\sqrt[3]{e}$; (k) $e^{\ln 4-\ln 3}$; (l) $\ln (\ln e)$;
(m) $e^{3 \ln x+2 \ln y}$; (n) $e^{3 \ln 2}$; (o) $e^{3+\ln 2}$;
(p) $e^{x+2 \ln x}$.

2 Find dy/dx in each case:
(a) $y = \ln (3x + 2)$; (b) $y = \ln (x^2 + 1)$;
(c) $y = \ln (e^x + 1)$; (d) $y = \ln (e^x)^3$;
(e) $y = x \ln x - x$; (f) $y = \ln x^2$;
(g) $y = (\ln x)^2$; (h) $y = \ln (3x^2 - 4x + 5)$;
(i) $y = \dfrac{\ln x}{x}$; (j) $y = \ln (\ln x)$;
(k) $y = \ln (x + \sqrt{x^2 + 1})$.

3 Find dy/dx in each case:
(a) $\ln xy + 2x - 3y = 4$; (b) $\ln \dfrac{y}{x} - xy = 2$.

4 Find dy/dx in each case. Whenever possible, use properties of logarithms to simplify the function before differentiating. See (a) and (b).
(a) $y = \ln (x\sqrt{x^2 + 1}) = \ln x + \frac{1}{2} \ln (x^2 + 1)$.
(b) $y = \ln \sqrt{\dfrac{x - 1}{x + 1}} = \dfrac{1}{2} [\ln (x - 1) - \ln (x + 1)]$.
(c) $y = \ln (3x - 2)^4$.
(d) $y = \ln \left(\dfrac{2x + 1}{x + 2}\right)$.
(e) $y = 3 \ln x^4$.
(f) $y = \ln \dfrac{1}{x}$.
(g) $y = 3 \ln 152x$.
(h) $y = 5 \ln 21x + 4 \ln 37x$.
(i) $y = \ln \sqrt[3]{x^6 + 1}$.
(j) $y = \dfrac{1}{3} \ln \dfrac{x^3}{x^3 + 1}$.
(k) $y = \ln [(3x - 7)^4(2x + 5)^3]$.

5 Integrate each of the following:
(a) $\displaystyle\int \frac{dx}{3x + 1}$; (b) $\displaystyle\int \frac{x\,dx}{3x^2 + 2}$;
(c) $\displaystyle\int \frac{3x^2 + 2}{x}\,dx$; (d) $\displaystyle\int \frac{x + 1}{x}\,dx$;
(e) $\displaystyle\int \frac{x\,dx}{x + 1}$; (f) $\displaystyle\int \frac{x\,dx}{x^2 + 1}$;
(g) $\displaystyle\int \frac{x\,dx}{3 - 2x^2}$; (h) $\displaystyle\int \frac{(2x - 1)\,dx}{x(x - 1)}$;
(i) $\displaystyle\int \frac{\ln x\,dx}{x}$; (j) $\displaystyle\int \frac{dx}{x \ln x}$;
(k) $\displaystyle\int \frac{dx}{\sqrt{x}(\sqrt{x} + 1)}$; (l) $\displaystyle\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$.

6 If c is a positive constant, show that the equation $cx + \ln x = 0$ has exactly one solution. Hint: Sketch the graph of $y = cx + \ln x$ with special attention to the behavior of dy/dx.

7 Show that the equation $x = \ln x$ has no solution
(a) by minimizing $y = x - \ln x$;
(b) geometrically, by considering the graphs of $y = x$ and $y = \ln x$.

8 Find the length of the curve $y = \frac{1}{2}x^2 - \frac{1}{4} \ln x$ between $x = 1$ and $x = 8$.

9 Sketch the graph of $y = x^2 - 18 \ln x$. Locate all maxima, minima, and points of inflection.

10 The area under $y = e^{-x}$ from $x = 0$ to $x = \ln 3$ is revolved about the x-axis. Find the volume generated in this way.

11 The area under $y = 1/\sqrt{x}$ from $x = 1$ to $x = 4$ is revolved about the x-axis. Find the volume generated in this way.

12 Show that the area under $y = 1/x$ from $x = a$ to $x = b$ $(0 < a < b)$ is the same as the area under this curve from $x = ka$ to $x = kb$ for any $k > 0$.

13 Prove that

$$\lim_{x\to\infty} \frac{\ln x}{x} = 0$$

by first showing that for $x > 1$

$$\ln x = \int_1^x \frac{dt}{t} \le \int_1^x \frac{dt}{\sqrt{t}} = 2(\sqrt{x} - 1).$$

Hint: Compare the graphs of $y = 1/t$ and $y = 1/\sqrt{t}$ for $t \ge 1$.

14 Use the result of Problem 13 to show that

$$\lim_{x \to 0+} x \ln x = 0.$$

Hint: Change the variable to $u = 1/x$.

15 Use the result of Problem 14 to sketch the graph of $y = x \ln x$ for all $x > 0$. Locate its minimum and verify that the graph is concave up everywhere.

16 Sketch the graph of $y = (\ln x)/x$ for all $x > 0$, and locate its maximum and point of inflection.

17 The speed at which a signal is transmitted along a cable on the bottom of the ocean is proportional to $x^2 \ln 1/x$, where x is the ratio of the radius of the core of the cable to the radius of the entire cable. What value of x maximizes the speed of transmission?

18 *Logarithmic differentiation* is a technique for computing the derivative of a function like

$$y = \sqrt[3]{(x+1)(x-2)(2x+7)},$$

which is fairly complicated but whose logarithm can be written in a much simpler form:

$$\ln y = \tfrac{1}{3}[\ln (x+1) + \ln (x-2) + \ln (2x+7)].$$

Find dy/dx by differentiating this equation implicitly with respect to x.

19 Use the method of Problem 18 to find dy/dx if

(a) $y = \dfrac{e^x(x^2 - 1)}{\sqrt{6x - 2}}$; (b) $y = \sqrt[5]{\dfrac{x^2 + 3}{x + 5}}$.

20 The method of logarithmic differentiation (see Problem 18) can also be used to differentiate functions like $y = x^x$, where both the base and the exponent are variable. Thus, we can write

$$\ln y = \ln x^x = x \ln x,$$

or equivalently,

$$y = e^{x \ln x}.$$

Find dy/dx from both equations and use this derivative to find the minimum value of $y = x^x$ for $x > 0$. Sketch the graph.

21 Use the method of Problem 20 to find dy/dx if

(a) $y = x^{x^x}$; (b) $y = \sqrt[x]{x} = x^{1/x}$.

Sketch the graph of the function in (b) and find its maximum value.

22 In Problem 21(b), the behavior of the function $y = \sqrt[x]{x}$ for large x shows that

$$\lim_{n \to \infty} \sqrt[n]{n} = 1.$$

Find the limits of the following expressions as $n \to \infty$:

(a) $(\ln n)^{1/n}$; (b) $(n \ln n)^{1/n}$;

(c) $\left(\dfrac{\ln n}{n}\right)^{1/n}$; (d) $\left(\dfrac{n}{e^n}\right)^{1/n}$.

23 Obtain the limit formula $\lim_{x \to 0} (1 + x)^{1/x} = e$ by using the fact that

$$(1 + x)^{1/x} = e^{\ln(1+x)/x} = e^{[\ln(1+x) - \ln 1]/x}.$$

24 If a is a positive number, show that

$$\lim_{x \to 0} \frac{a^x - 1}{x} = \ln a.$$

Hint: The limit is a value of a certain derivative.

25 Show that

$$\lim_{n \to \infty} n(\sqrt[n]{a} - 1) = \ln a.$$

Hint: Put $x = 1/n$ in Problem 24.

8.5 APPLICATIONS. POPULATION GROWTH AND RADIOACTIVE DECAY

As we emphasized in Section 8.1, our main purpose in this chapter is to develop the mathematical machinery that is necessary for treating a variety of related applications. This machinery is now in place, and the time has come to see what it can do.

Example 1 *Population growth.* Consider a laboratory culture of bacteria with unlimited food and no enemies. If $N = N(t)$ denotes the number of bacteria present at time t, it is natural to assume that the rate of change of N is proportional to N itself.* If the number of bacteria present at the beginning is N_0, and this number doubles after 2 hours (the "doubling time"), how many are there after 6 hours? After t hours?

*Briefly, we expect twice as many "births" in a given short interval of time when twice as many bacteria are present.

Solution Even though bacteria come in units and are not continuously divisible, there are so many present, and they are produced at such tiny time intervals, that it is reasonable to treat $N(t)$ as a continuous, even differentiable, function. The assumed law of growth tells us that

$$\frac{dN}{dt} = kN \qquad (k > 0), \tag{1}$$

or, separating variables,

$$\frac{dN}{N} = k\,dt.$$

Integration yields

$$\ln N = kt + c. \tag{2}$$

To determine the value of the constant of integration c, we use the fact that initially (at $t = 0$) we have $N = N_0$. Thus, in equation (2) we have $\ln N_0 = 0 + c$ or $c = \ln N_0$, so (2) becomes

$$\ln N = kt + \ln N_0$$

or

$$\ln N - \ln N_0 = kt, \qquad \ln \frac{N}{N_0} = kt, \qquad \frac{N}{N_0} = e^{kt},$$

and therefore

$$N = N_0 e^{kt}. \tag{3}$$

To find k we use the fact that the population doubles in 2 hours. This gives

$$2N_0 = N_0 e^{2k}, \qquad e^{2k} = 2, \qquad 2k = \ln 2, \qquad k = \tfrac{1}{2} \ln 2,$$

so (3) becomes

$$N = N_0 e^{(t \ln 2)/2}, \tag{4}$$

which gives the population after t hours. Finally, putting $t = 6$ in (4) gives $N = N_0 e^{3 \ln 2} = N_0 e^{\ln 8} = 8N_0$, so the population increases by a factor of 8 in 6 hours.

The situation just described is another example of *exponential growth.* This type of growth is characterized by a function of the form (3) where the constant k is positive.

Example 2 *Radioactive decay.* After 3 days, 50 percent of the radioactivity produced by a nuclear explosion has disappeared. How long does it take for 99 percent of this radioactivity to disappear?

Solution We assume for the sake of simplicity that the radioactivity is entirely due to a single radioactive substance. This substance undergoes *radioactive decay* into nonradioactive substances by means of the spontaneous decomposition of its atoms, at a steady rate that is a characteristic property of the substance itself. Each such decomposition is accompanied by a small burst of radiation, and these bursts are detected and counted by Geiger counters. We are not concerned here with the inner complexities of these remarkable events, but only with the fact that the rate of change of the mass of our substance is negative and is proportional at each moment to the mass of the substance at that moment.* This statement means that if $x = x(t)$ is the mass of the radioactive substance at time t, then

$$\frac{dx}{dt} = -kx \qquad (k > 0), \tag{5}$$

where the minus sign says that x is decreasing. The positive constant k is called the *rate constant*; it clearly measures the speed of the decay process. As before, we separate the variables and integrate,

$$\frac{dx}{x} = -k\,dt, \qquad \ln x = -kt + c. \tag{6}$$

If x_0 is the amount of the substance produced by the explosion, so that $x = x_0$ when $t = 0$, then we see that $c = \ln x_0$, so (6) becomes

$$\ln x = -kt + \ln x_0$$

or

$$\ln x - \ln x_0 = -kt, \qquad \ln \frac{x}{x_0} = -kt, \qquad \frac{x}{x_0} = e^{-kt},$$

and consequently

$$x = x_0 e^{-kt}. \tag{7}$$

In principle at least, x is never zero, because the exponential e^{-kt} never vanishes. It is therefore inappropriate to speak of the "total lifetime" of a radioactive substance. However, it is both convenient and customary to use the concept of half-life: The *half-life* of a radioactive substance is the time required for the substance to decay to half its original amount (Fig. 8.8). If we denote the half-life by T, then (7) yields $\frac{1}{2}x_0 = x_0e^{-kT}$, so $e^{kT} = 2$ and

$$kT = \ln 2. \tag{8}$$

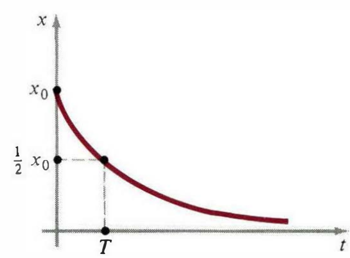


Figure 8.8

This equation relates the half-life to the rate constant k, and enables us to find either if the other is known.

In the specific problem we started with, 50 percent of the radioactivity disappears in 3 days. This tells us that the half-life of the substance is 3 days, so by (8) we see that $3k = \ln 2$ or $k = \frac{1}{3}\ln 2$; and in this particular case, (7) becomes

$$x = x_0 e^{-(t \ln 2)/3}.$$

The disappearance of 99 percent of the radioactivity means that 1 percent remains, and therefore $x = \frac{1}{100}x_0$. This happens when t satisfies the equation

$$\tfrac{1}{100}x_0 = x_0 e^{-(t \ln 2)/3},$$

which is equivalent to

$$e^{(t \ln 2)/3} = 100 \qquad \text{or} \qquad \frac{t \ln 2}{3} = \ln 100.$$

Finally, by using tables of natural logarithms (or a calculator) we find that

*Thus, if the mass of our substance were doubled, we would expect to lose twice as many atoms by decomposition in a given short interval of time.

$$t = \frac{3 \ln 100}{\ln 2} = \frac{6 \ln 10}{\ln 2} \cong 20 \text{ days.}$$

It should be understood that this example is greatly oversimplified, because an actual nuclear explosion produces many different radioactive by-products with half-lives varying from a fraction of a second to many years. Thus polonium 212 (3 ten-millionths of a second) and krypton 91 (10 seconds) would disappear almost immediately, whereas strontium 90 (28 years) lingers for decades and contributes substantially to the dangers of nuclear fallout.*

The situation just discussed is an example of *exponential decay*. This phrase refers only to the form of the function (7) and the manner in which the quantity x diminishes, and not necessarily to the idea that something is disintegrating.

Remark The concepts explained in Example 2 are the basis for a scientific tool of fairly recent development which has been of great significance for geology and archaeology. In essence, radioactive elements occurring in nature (with known half-lives) can be used to assign dates to events that took place from a few thousand to a few billion years ago. For example, the common isotope of uranium (uranium 238) decays through several stages into helium and an isotope of lead (lead 206), with a half-life of 4.5 billion years. When rock containing uranium is in a molten state, as in lava flowing from the mouth of a volcano, the lead created by this decay process is dispersed by currents in the lava; but after the rock solidifies, the lead is locked in place and steadily accumulates alongside the parent uranium. A piece of granite can be analyzed to determine the ratio of lead to uranium, and this ratio permits an estimate of the time that has elapsed since the critical moment when the granite crystallized. Several methods of age determination involving the decay of thorium and the isotopes of uranium into the various isotopes of lead are in current use. Another method depends on the decay of potassium into argon, with a half-life of 1.3 billion years; and yet another, preferred for dating the oldest rocks, is based on the decay of rubidium into strontium, with a half-life of 50 billion years. These studies are complex and susceptible to errors of many kinds; but they can often be checked against one another, and are capable of yielding reliable dates for many events in geological history linked to the formation of igneous rocks. Rocks tens of millions of years old are quite young, ages ranging into hundreds of millions of years are common, and the oldest rocks yet discovered are upward of 3 billion years old. This of course is a lower limit for the age of the earth's crust, and so for the age of the earth itself. Other investigations, using various types of astronomical data, age determinations for minerals in meteorites, and so on, have suggested a probable age for the earth of about 4.5 billion years.†

These radioactive elements decay so slowly that the methods of age determination based on them are not suitable for dating events that took place relatively

*For students who have not met these ideas before, the number following the name of each of the chemical elements mentioned is the *mass number* (= total number of protons and neutrons in the nucleus) of the particular isotope referred to. For example, strontium as it occurs in nature has four stable isotopes of mass numbers (in the order of their abundance) 88, 86, 87, 84. Several unstable isotopes are produced in nuclear reactions, of which strontium 90 is the best known.

†For a full discussion of these matters, as well as many other methods and results of the science of geochronology, see F. E. Zeuner, *Dating the Past*, 4th ed. (Methuen, 1958).

recently. This gap was filled by Willard Libby's discovery in the late 1940s of *radiocarbon,* a radioactive isotope of carbon (carbon 14) with a half-life of about 5600 years. By 1950 Libby and his associates had developed the technique of *radiocarbon dating,* which added a second hand to the slow-moving geological clocks just described and made it possible to date events in the later stages of the ice age and some of the movements and activities of prehistoric people. The contributions of this technique to late Pleistocene geology and archaeology have been spectacular.

In brief outline, the facts and principles involved are these. Radiocarbon is produced in the upper atmosphere by the action of cosmic ray neutrons on nitrogen. This radiocarbon is oxidized to carbon dioxide, which in turn is mixed by the winds with the nonradioactive carbon dioxide already present. Since radiocarbon is constantly being formed and constantly decomposing back into nitrogen, its proportion to ordinary carbon in the atmosphere has long since reached an equilibrium state. All air-breathing plants incorporate this proportion of radiocarbon into their tissues, as do the animals that eat these plants. This proportion remains constant as long as a plant or animal lives; but when it dies it ceases to absorb new radiocarbon, while the supply it has at the time of death continues the steady process of decay. Thus, if a piece of old wood has half the radioactivity of a living tree, it lived about 5600 years ago, and if it has only one-fourth this radioactivity, it lived about 11,200 years ago. This principle provides a method for dating any ancient object of organic origin, for instance, wood, charcoal, vegetable fiber, flesh, skin, bone, or horn. The reliability of the method has been verified by applying it to the heartwood of giant sequoia trees whose growth rings record 3000 to 4000 years of life, and to furniture from Egyptian tombs whose age is also known independently. There are technical difficulties, but the method is now felt to be capable of reasonable accuracy as long as the periods of time involved are not too great (up to about 50,000 years).

Radiocarbon dating has been applied to thousands of samples, and laboratories for carrying on this work number in the dozens. Among the more interesting age estimates are these: linen wrappings from the Dead Sea scrolls of the Book of Isaiah, recently found in a cave in Palestine and thought to be first or second century B.C., 1917 $\pm$ 200 years; charcoal from the Lascaux cave in southern France, site of the remarkable prehistoric paintings, 15,516 $\pm$ 900 years; charcoal from the prehistoric monument at Stonehenge, in southern England, 3798 $\pm$ 275 years; charcoal from a tree burned at the time of the volcanic explosion that formed Crater Lake in Oregon, 6453 $\pm$ 250 years. Campsites of ancient people throughout the western hemisphere have been dated by using pieces of charcoal, fiber sandals, fragments of burned bison bone, and the like. The results suggest that human beings did not arrive in the New World until about the period of the last Ice Age, some 11,500 years ago, when the level of the water in the oceans was substantially lower than it now is and they could have walked across the Bering Straits from Siberia to Alaska.*

*Libby won the 1960 Nobel Prize for chemistry as a consequence of the work described here. His own account of the method, with its pitfalls and conclusions, can be found in his book *Radiocarbon Dating,* 2nd ed. (Univ. of Chicago Press, 1955).

PROBLEMS

1 The bacteria in a certain culture increase according to the law $dN/dt = kN$. If $N = 2000$ at the beginning and $N = 4000$ when $t = 3$, find (a) the value of N when $t = 1$; and (b) the value of t when $N = 48{,}000$.

2 If the rate of increase of the population of a country is 3 percent per year, by what factor does it increase every 10 years? What percentage increase will double the population every 10 years?

3 Sleepyville has 5 times the population of Boomtown. The first is growing at the rate of 2 percent per year, and the second at 10 percent per year. In how many years will they have equal populations?

4 It is often assumed that $\frac{1}{3}$ acre of land is needed to provide food for one person. It is also estimated that there are 10 billion acres of arable land in the world, and therefore a maximum population of 30 billion people can be sustained if no other sources of food are known. The total world population at the beginning of 1970 was 3.6 billion. Assuming that the population continues to increase at the rate of 2 percent per year, when will the maximum population be reached? What will be the population in the year 2000?

5 The half-life of radium is 1620 years. What percentage of a given quantity of radium will remain after 100 years?

6 Cobalt 60, with a half-life of 5.3 years, is extensively used in medical radiology. How long does it take for 90 percent of a given quantity to decay?

7 In a certain chemical reaction a compound C decomposes at a rate proportional to the amount of C that remains. It is found by experiment that 8 g of C diminish to 4 g in 2 hours. At what time will only 1 g be left?

8 "A fool and his money are soon parted." One particular fool loses money in gambling at a rate (in dollars per hour) equal to one-third of the amount he has at any given time. How long will it take him to lose half of his original stake?

9 A cylindrical tank of radius 4 ft and height 10 ft, with its axis vertical, is full of water but has a small hole in the bottom. Assuming that water squirts out of the hole at a speed proportional to the pressure at the bottom of the tank, and that one-fifth of the water leaks out in the first hour, find a formula for the depth of the water left in the tank after t hours.

10 According to *Lambert's law of absorption,* the percentage of incident light absorbed by a thin layer of translucent material is proportional to the thickness of the layer. If sunlight falling vertically on ocean water is reduced to one-half its initial intensity I_0 at a depth of 10 m, show that the formula

$$I = I_0 e^{-(x \ln 2)/10}$$

gives the intensity I at a depth of x meters.

11 According to *Newton's law of cooling,* a body at temperature T cools at a rate proportional to the difference between T and the temperature of the surrounding air. A vat of boiling soup at 100°C is brought into a room where the air is 20°C, and is left to cool. After 1 hour its temperature is 60°C. How much additional time is required for it to cool to 30°C?

12 Consider a column of air of cross-sectional area 1 in^2 extending from sea level up to "infinity." The atmospheric pressure p at an altitude h above sea level is the weight of the air in this column above the altitude h. Assuming that the density of the air is proportional to the pressure (this is a consequence of Boyle's law $pV = k$ at constant temperature), show that p satisfies the differential equation

$$\frac{dp}{dh} = -cp,$$

where c is a positive constant, and deduce that

$$p = p_0 e^{-ch},$$

where p_0 is the atmospheric pressure at sea level. Hint: If h increases by a small amount dh and dp is the corresponding change in p (see Fig. 8.9), then $-dp$ is the weight of the air in the small portion of the column whose height is dh; and this weight is the density times the volume, so $-dp = (cp)(1 \cdot dh)$.

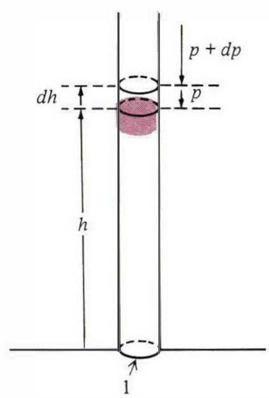


Figure 8.9

13 The radiocarbon in living wood decays at the rate of 15.30 disintegrations per minute (dpm) per gram of contained carbon. Using 5600 years as the half-life of radiocarbon, estimate the age of each of the following specimens discovered by archaeologists and tested for radioactivity in 1950:

(a) a piece of a chair leg from the tomb of King Tutankhamen, 10.14 dpm;

(b) a piece of a beam of a house built in Babylon during the reign of King Hammurabi, 9.52 dpm;
(c) dung of a giant sloth found 6 ft 4 in under the surface of the ground inside Gypsum Cave in Nevada, 4.17 dpm;
(d) a hardwood atlatl (spear-thrower) found in Leonard Rock Shelter in Nevada, 6.42 dpm.

14 Suppose that two chemical substances in solution react together to form a compound. If the reaction occurs by means of the collision and interaction of the molecules of the substances, then we expect the rate of formation of the compound to be proportional to the number of collisions per unit time, which in turn is jointly proportional to the amounts of the substances that are untransformed. A chemical reaction that proceeds in this manner is called a *second-order reaction,* and this law of reaction is often referred to as the *law of mass action.*† Consider a second-order reaction in which x grams of the compound contain ax grams of the first substance and bx grams of the second, where $a + b = 1$. If there are aA grams of the first substance present initially, and bB grams of the second, then the law of mass action says that

$$\frac{dx}{dt} = k(aA - ax)(bB - bx) = kab(A - x)(B - x).$$

If $A \neq B$, show that

$$\frac{B(A-x)}{A(B-x)} = e^{kab(A-B)t} \qquad (*)$$

provides a solution for which $x = 0$ when $t = 0$.‡ Hint: Take the logarithm of both sides and differentiate with respect to t.

15 In Problem 14, find $\lim_{t\to\infty} x(t)$
(a) by solving equation (*) for x as an explicit function of t and using this function;
(b) by merely inspecting equation (*).

16 A switch is suddenly closed in an electric circuit, connecting a battery of voltage E to a resistance R and inductance L in series (Fig. 8.10). The battery causes a variable current $I = I(t)$ to flow in the circuit. By elementary physics, the voltage drop across the resistance is RI and across the inductance is $L\,dI/dt$, and the sum of these two voltage drops must equal the applied voltage E:

$$L\frac{dI}{dt} + RI = E.^{\S}$$

By separating the variables and integrating, and using the fact that $I = 0$ when $t = 0$, find the current I as a function of t. Graph this function.

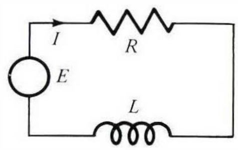


Figure 8.10

17 Consider a given quantity of gas that undergoes an adiabatic expansion or compression, which means that no heat is gained or lost during the process. The French scientist Poisson showed in 1823 that the pressure and volume of this gas satisfy the differential equation

$$\frac{dp}{p} + \gamma\frac{dV}{V} = 0,$$

where γ is a constant whose value depends on whether the gas is monatomic, diatomic, etc.¶ Integrate this equation to obtain

$$pV^{\gamma} = c.$$

This is called *Poisson's gas equation* or the *adiabatic gas law,* and is of fundamental importance in meteorology.

†For a first-order reaction, see Problem 7.

‡In Chapter 10 we develop a method for discovering this solution.

§Students who are unfamiliar with electric circuits may find it helpful to think of the current I as analogous to the rate of flow of water in a pipe. The battery plays the role of a pump producing pressure (voltage) that causes the water to flow. The resistance is analogous to friction in the pipe, which opposes the flow by producing a drop in the pressure; and the inductance opposes any change in the flow by producing a drop in pressure if the flow is increasing, and an increase in pressure if the flow is decreasing.

¶For more details on the physical background, see pp. 275–276 of R. A. Millikan, D. Roller, and E. C. Watson, *Mechanics, Molecular Physics, Heat, and Sound* (The M.I.T. Press, 1965).

8.6 MORE APPLICATIONS. INHIBITED POPULATION GROWTH, etc.

As the reader is certainly aware, the problem of realistically analyzing the growth of a population is not adequately dealt with in Example 1 of Section 8.5. The difficulty with this discussion is that the basic equation,

$$\frac{dN}{dt} = kN \qquad (k > 0),$$

describes only the simplest ideal situation, in which the inner impulse of the population to expand is given a completely free rein; it does not take into account any of the inhibiting factors that put a ceiling on the possible size of a real population. It is obvious, for example, that the human population of the earth can never expand to the stage where there will be only a small fraction of an acre of usable land per person. Long before the point is reached at which the whole surface of the earth becomes a teeming slum, the rate of population growth will be forced down; social, psychological, and economic effects will depress the birthrate, and there will also be an increase in the death rate due to the starvation, disease, and warfare that are the inescapable companions of overpopulation. In our next example we try to recognize some of these factors, and thereby mirror reality a little more closely.

Example 1 *Inhibited population growth.* Consider a small colony of rabbits of population N_0 that is "planted" at time $t = 0$ on a grassy island where they have no enemies. When the population $N = N(t)$ is small, it tends to grow at a rate proportional to itself; but when it becomes larger, there is more and more competition for the limited food and living space, and N grows at a smaller rate. If N_1 is the largest population the island can support, and if the rate of growth of the population N is assumed to be jointly proportional to N and to $N_1 - N$, so that

$$\frac{dN}{dt} = kN(N_1 - N) \qquad (k > 0), \tag{1}$$

find N as a function of t.

Solution It should be noticed explicitly at the outset that N increases slowly—that is, dN/dt is small—when N is small, and also when N is large but close to N_1, so that $N_1 - N$ is small. To solve (1), we separate variables and integrate,

$$\int \frac{dN}{N(N_1 - N)} = \int k\,dt. \tag{2}$$

The calculation of the integral on the left side of (2) requires the easily verified algebraic fact that

$$\frac{1}{N(N_1 - N)} = \frac{1}{N_1}\left(\frac{1}{N} + \frac{1}{N_1 - N}\right). \tag{3}$$

With the aid of (3), we can write (2) in the form

$$\frac{1}{N_1}\left(\int \frac{dN}{N} + \int \frac{dN}{N_1 - N}\right) = \int k\,dt,$$

which yields

$$\frac{1}{N_1}[\ln N - \ln (N_1 - N)] = kt + c_1$$

or

$$\frac{1}{N_1} \ln \frac{N}{N_1 - N} = kt + c_1.$$

If we multiply through by N_1, this becomes

$$\ln \frac{N}{N_1 - N} = N_1kt + c,$$

where $c = N_1c_1$. Since $N = N_0$ when $t = 0$, we see that $c = \ln [N_0/(N_1 - N_0)]$, so we have

$$\ln \frac{N}{N_1 - N} = N_1kt + \ln \frac{N_0}{N_1 - N_0},$$

which is equivalent to

$$\frac{N}{N_1 - N} = \frac{N_0}{N_1 - N_0} e^{N_1kt}.$$

We solve this equation for N by writing

$$N(N_1 - N_0) = N_0N_1e^{N_1kt} - NN_0e^{N_1kt},$$

$$N[N_0e^{N_1kt} + (N_1 - N_0)] = N_0N_1e^{N_1kt},$$

and

$$N = \frac{N_0N_1e^{N_1kt}}{N_0e^{N_1kt} + (N_1 - N_0)}.$$

We can write this in a more convenient form, and thereby obtain our final result, by multiplying the numerator and denominator on the right by e^{-N_1kt}:

$$N = \frac{N_0N_1}{N_0 + (N_1 - N_0)e^{-N_1kt}}. \tag{4}$$

It should be observed that (4) gives $N = N_0$ when $t = 0$, and also that $N \to N_1$ as $t \to \infty$, as we expect. The graph of (4) is shown in Fig. 8.11. In ecology and mathematical biology this curve is called the *inhibited growth curve,* or sometimes the *sigmoid growth curve.*

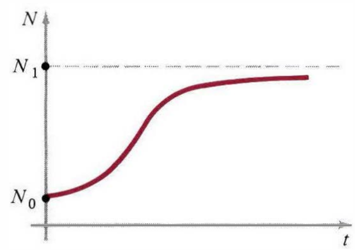


Figure 8.11

In Example 1 of Section 5.5, we discussed the idealized problem of a freely falling body, in which we ignored the effect of air resistance and assumed that the only force acting on the body was the force of gravity. We are now in a position to improve our discussion of this problem by taking air resistance into account.

Example 2 *Falling body with air resistance.* Consider a stone of mass m that is dropped from rest from a great height in the earth's atmosphere. If the only forces acting on the stone are the earth's gravitational attraction mg (where g is the acceleration due to gravity, assumed to be constant) and a retarding force due to air resistance, which is assumed to be proportional to the velocity v, find v as a function of the time t.

Solution Let s be the distance the stone falls in time t, so that the velocity $v = ds/dt$ and the acceleration $a = dv/dt = d^2s/dt^2$. There are two forces acting on the falling stone, a downward force mg due to gravity, and an upward force kv due to air resistance, where k is a positive constant. Newton's second law of motion $F = ma$ says that the total force acting on the stone at any moment equals the product of its mass and its acceleration. With our assumptions, the equation $ma = F$ becomes

$$m\frac{dv}{dt} = mg - kv,$$

or dividing through by m,

$$\frac{dv}{dt} = g - cv, \tag{5}$$

where $c = k/m$. We solve (5) by separating variables and integrating, which gives

$$\int \frac{dv}{g - cv} = \int dt$$

or

$$-\frac{1}{c} \ln (g - cv) = t + c_1;$$

and by changing the notation for constants in a familiar way, we can write this in the form

$$\ln (g - cv) = -ct + c_2$$

or

$$g - cv = c_3 e^{-ct}. \tag{6}$$

The initial condition $v = 0$ when $t = 0$ tells us that $c_3 = g$, so (6) becomes

$$g - cv = ge^{-ct}$$

or

$$v = \frac{g}{c}(1 - e^{-ct}). \tag{7}$$

Since c is positive, this formula tells us that $v \to g/c$ as $t \to \infty$. It is a surprising fact that the velocity of our falling stone does not increase indefinitely, but instead approaches a finite limiting value. This limiting value of v is called the *terminal velocity*. If we differentiate (7), we find that the acceleration is given by the formula $a = ge^{-ct}$, so $a \to 0$ as $t \to \infty$. From the physical point of view, this means that as time goes on the air resistance tends to balance out the force of gravity, so that the total force acting on the stone approaches zero.

Our next example is typical of many problems involving continuously changing mixtures.

Example 3 *Mixing.* Brine containing 2 lb of salt per gallon flows into a tank that initially holds 200 gal of water in which 100 lb of salt are dissolved. If the brine enters the tank at the rate of 10 gal/min, and if the mixture (which is kept uniform by stirring) flows out at the same rate, how much salt is in the tank after 20 minutes? After 100 minutes?

Solution Let x be the number of pounds of salt in the tank after t minutes. The key to thinking about this problem is the following fact:

$$\text{rate of change of } x = \text{rate at which salt enters tank} - \text{rate at which salt leaves tank}. \tag{8}$$

It is clear that salt enters the tank at the rate of $2 \cdot 10 = 20$ lb/min. The concentration of salt at any time is $x/200$ lb/gal, so the rate at which it leaves the tank is $(x/200) \cdot 10 = x/20$ lb/min. Accordingly, (8) becomes

$$\frac{dx}{dt} = 20 - \frac{x}{20} = \frac{400 - x}{20}.$$

By the familiar process of separating variables and integrating, and using the initial condition $x = 100$ when $t = 0$, we obtain

$$x = 400 - 300e^{-t/20}. \tag{9}$$

(As usual when we omit computational details, students should carry these details through for themselves.) By using a calculator, we now find that $x = 289.7$ when $t = 20$, and that $x = 398.0$ when $t = 100$. Also, it is obvious from (9) that $x \to 400$ as $t \to \infty$.

PROBLEMS

1 In Example 1, what is the population when its rate of growth is largest?

2 In a genetics experiment, 50 fruit flies are placed in a glass jar that will support a maximum population of 1000 flies. If 30 days later the population has grown to 200 flies, when will the fly population reach half of the jar's capacity?

3 Let x be the number of people in a community of total population x_1 who have heard a certain rumor t days after the rumor was launched. Common sense suggests that the rate of increase of x, that is, the rate at which this rumor spreads through the community, is proportional to the frequency of contact between those who have heard the rumor and those who have not, and this in turn is jointly proportional to the number of people who have heard the rumor and the number of those who have not. This yields the differential equation

$$\frac{dx}{dt} = cx(x_1 - x),$$

where c is a constant expressing the level of social activity. If the rumor is initially imparted to x_0 individuals ($x = x_0$ when $t = 0$), find x as a function of t. Use this function to show that $x \to x_1$ as $t \to \infty$. Sketch the graph.

4 Rework Example 2 under the more general assumption that the initial velocity is v_0. Show that the terminal velocity is still g/c, and therefore does not depend on v_0. Convince yourself that this is reasonable.

5 A motorboat moving in still water is resisted by the water with a force proportional to its velocity v. Show that the velocity t seconds after the power is shut off is given by the formula $v = v_0e^{-ct}$, where c is a positive constant and v_0 is the velocity at the moment the power is shut off. Also, if s is the distance the boat coasts in time t, find s as a function of t and sketch the graph of this function. Hint: Use Newton's second law of motion.

6 Consider the situation described in Problem 5, with the difference that the resisting force is proportional to the square of the velocity v. Find v and s as functions of t, and sketch the graph of the latter function.

7 By the result of Problem 5, the distance s approaches a finite limit as t increases; but in Problem 6 this distance becomes infinite. Because the resisting force seems to be greater in the second case, we would expect the distance traveled to be less than in the first case. Explain this seeming contradiction.

8 A tank initially contains 400 gal of brine in which 100 lb of salt are dissolved. Pure water is run into the tank at the rate of 20 gal/min, and the mixture (which is kept uniform by stirring) is drained off at the same rate. How many pounds of salt remain in the tank after 30 minutes?

9 Rework Problem 8 if instead of pure water, brine containing $\frac{1}{10}$ lb of salt per gallon is run into the tank at 20 gal/min, the mixture being drained off at the same rate.

10 A country has 5 billion dollars of paper money in circulation. Each day 30 million dollars is brought into the banks for deposit and the same amount is paid out. Because of a change of regime, the government decides to issue new paper money displaying pictures of different people, so whenever the old money comes into the banks it is destroyed and replaced by the new money. How long will it take for the paper money in circulation to become 90 percent new?

CHAPTER 8 REVIEW: CONCEPTS, FORMULAS

Define and think through the following.

1 Exponential and logarithm functions.

2 Definition of e. The exponential function $y = e^x$.

3 $\dfrac{d}{dx} e^u = e^u \dfrac{du}{dx}$ and $\int e^u\, du = e^u + c$.

4 The natural logarithm function $y = \ln x$.

5 $\dfrac{d}{dx} \ln u = \dfrac{1}{u}\dfrac{du}{dx}$ and $\int \dfrac{du}{u} = \ln u + c$.

6 Exponential growth and decay.

ADDITIONAL PROBLEMS FOR CHAPTER 8

SECTION 8.3

In Problems 1–6, find the derivative dy/dx of the given function.

1 $y = e^{\sqrt{1-x^2}}$.
2 $y = (1 - e^{3x})^2$.
3 $y = e^{x^2-2x+1}$.
4 $y = (e^{4x} - 3)^3$.
5 $y = e^{\sqrt{x}} + \sqrt{e^x}$.
6 $y = \sqrt{e^{2x} + 2x}$.

Evaluate the integrals in Problems 7–11.

7 $\int e^{-3x}\,dx$.
8 $\int e^{ax+b}\,dx$.
9 $\int \frac{e^{1/x}\,dx}{x^2}$.
10 $\int \frac{4\,dx}{\sqrt{e^x}}$.
11 $\int \frac{e^x\,dx}{\sqrt{e^x+1}}$.

*12 Find the area between $y = e^x$ and the chord $y = ex - x + 1$.

13 Find the point on the graph of $y = e^{ax}$ at which the tangent line passes through the origin.

14 Evaluate the following limits:

(a) $\lim_{n\to\infty}\left(1 + \frac{1}{4n+2}\right)^{4n+9}$;

(b) $\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^{n-2}$; (c) $\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^{3n}$;

(d) $\lim_{n\to\infty}\left(1 + \frac{1}{3n}\right)^{n}$; (e) $\lim_{n\to\infty}\left(1 + \frac{1}{2n^2}\right)^{2n}$.

15 Verify that $y = e^{x^2}$ is a solution of the differential equation $y'' - 2xy' - 2y = 0$.

16 Verify that $y = (e^{2x} - 1)/(e^{2x} + 1)$ is a solution of the differential equation $dy/dx = 1 - y^2$.

17 The area under $y = e^x$ from $x = 0$ to $x = 3$ is revolved about the x-axis. Find the volume generated in this way.

*18 Prove that for all $x > 0$ and all positive integers n,

$$e^x > 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!},$$

where the symbol $n!$ (read "n factorial") denotes the product $1 \cdot 2 \cdot 3 \cdots n$. Hint: Since $e^t > 1$ for $t > 0$,

$$e^x = 1 + \int_0^x e^t\,dt > 1 + \int_0^x dt = 1 + x,$$

$$e^x = 1 + \int_0^x e^t\,dt > 1 + \int_0^x (1 + t)\,dt$$

$$= 1 + x + \frac{x^2}{2},$$

and so on.

*19 If n is any given positive integer, prove that $e^x > x^n$ for all sufficiently large values of x. Hint: Use Problem 18 for $n + 1$.

*20 Prove that

$$\lim_{x\to\infty}\frac{e^x}{x^n} = \infty$$

for any positive integer n.

*21 If n is a positive integer, show that $y = x^n e^{-x}$ assumes its maximum value at $x = n$, so that its values at $x = n - 1$ and $x = n + 1$ are less than the maximum. Use this fact to show that

$$\left(\frac{n+1}{n}\right)^n < e < \left(\frac{n}{n-1}\right)^n;$$

and use this in turn to show that

$$\left(1 + \frac{1}{n}\right)^n < e < \left(1 + \frac{1}{n}\right)^{n+1}$$

for every n. When $n = 5$, the second inequality here yields $e < 3$. Verify this.

SECTION 8.4

22 Find dy/dx in each case:

(a) $y = x \ln x^2 - 2x$; (b) $y = \frac{1}{2}\ln(x^2 + 2x)$;
(c) $y = x^2 \ln x$; (d) $y = \ln(5x^4 - 7x^3 + 3)$;
(e) $y = \frac{\ln x}{x^2}$; (f) $y = \ln x^5$;
(g) $y = (\ln x)^5$; (h) $y = \frac{1}{\ln x}$;
(i) $y = \sqrt{\ln x}$.

23 Find dy/dx in each case:

(a) $3x - y^2 + \ln xy = 1$;
(b) $x^2 + \ln\frac{x}{y} + 3y + 2 = 0$.

24 Find dy/dx in each case:

(a) $y = \ln\sqrt{x}$; (b) $y = \ln x\sqrt[3]{x}$;
(c) $y = \ln\left(\frac{x^2+4}{2x+3}\right)$; (d) $y = \ln\sqrt{2x^3 - 4x}$;
(e) $y = \ln(x + 1)^5$; (f) $y = \ln(x^2\sqrt{x^4 + 1})$;
(g) $y = \ln\frac{x}{3 - 2x}$;
(h) $y = \ln\sqrt[3]{6x^2 + 3x}$;
(i) $y = \ln\sqrt{\frac{4 + x^2}{4 - x^2}}$;
(j) $y = \ln\left(\frac{x}{1 + \sqrt{1 + x^2}}\right)$;
(k) $y = x\sqrt{x^2 - 3} - 3\ln(x + \sqrt{x^2 - 3})$;
(l) $y = -\frac{1}{2}\ln\left(\frac{2 + \sqrt{x^2 + 4}}{x}\right)$.

25 Integrate each of the following:

(a) $\int \frac{dx}{1+2x}$; (b) $\int \frac{dx}{1-3x}$;

(c) $\int_0^1 \frac{x^2\,dx}{2-x^3}$; (d) $\int_0^3 \frac{x\,dx}{x^2+1}$;

(e) $\int_0^6 \frac{x\,dx}{x+3}$; (f) $\int \frac{dx}{x\sqrt{\ln x}}$;

(g) $\int_0^8 \frac{x^{1/3}\,dx}{1+3x^{4/3}}$; (h) $\int \frac{x\,dx}{1-x^2}$;

(i) $\int_0^2 \frac{\ln(x+1)\,dx}{x+1}$; (j) $\int \frac{(2x-1)\,dx}{3x^2-3x+7}$;

(k) $\int \frac{e^x\,dx}{e^x+1}$; (l) $\int \frac{(2x+3)\,dx}{(x+1)(x+2)}$;

(m) $\int \frac{(\ln x)^2\,dx}{x}$; (n) $\int \frac{\ln\sqrt{x}\,dx}{x}$;

(o) $\int \frac{\ln(\ln x)\,dx}{x\ln x}$; (p) $\int \frac{1}{x}\ln\left(\frac{1}{x}\right)dx$.

26 If p is a positive constant, show that

$$\lim_{x\to\infty}\frac{\ln x}{x^p}=0.$$

Hint: Replace x by the variable $y=x^p$.

27 If a and b are positive constants, show that

$$\lim_{x\to\infty}\frac{(\ln x)^a}{x^b}=0.$$

28 In Problem 27, find the largest value of

$$y=\frac{(\ln x)^a}{x^b}\qquad\text{for } x\geq 1.$$

29 If a is a positive constant, find the length of the curve

$$y=\frac{x^2}{2a}-\frac{a}{4}\ln x$$

between $x=1$ and $x=2$. For what value of a is this length a minimum?

30 If a and b are positive constants, find the length of the curve

$$\frac{y}{b}=\left(\frac{x}{a}\right)^2-\frac{1}{8}\left(\frac{a^2}{b^2}\right)\ln\frac{x}{a}$$

from $x=a$ to $x=3a$.

31 Use the fact that $a=e^{\ln a}$ to find dy/dx in each of the following cases:

(a) $y=10^x$; (b) $y=3^x$;

(c) $y=\pi^x$; (d) $y=7^{3x}$;

(e) $y=6^{x^2-2x}$; (f) $y=5^{\sqrt{x}}$.

32 Use the idea of Problem 31 to integrate each of the following:

(a) $\int_0^1 2^x\,dx$; (b) $\int_0^1 10^x\,dx$;

(c) $\int_1^{\sqrt{2}} x3^{-x^2}\,dx$; (d) $\int_0^1 7^{2x-1}\,dx$;

(e) $\int 3^{-x}\,dx$; (f) $\int x9^{2x^2}\,dx$;

(g) $\int_0^1 5^{-3x}\,dx$; (h) $\int \frac{10^{\sqrt{x}}\,dx}{\sqrt{x}}$.

33 Sketch the graph of $y=x^2/5^x$, and locate its maximum and two points of inflection.

34 (a) In changing logarithms from the base a to the base b, one needs the equations $\log_b x=(\log_b a)\cdot(\log_a x)$ and $(\log_a b)(\log_b a)=1$. Prove them.

(b) Compute $\int \frac{dx}{x\log_{10} x}$.

(c) For each choice of the constant $a>1$, show that $y=(\log_a x)/x$ has a maximum at $x=e$ and a point of inflection at $x=e\sqrt{e}$. Sketch the graph.

35 Find dy/dx if

(a) $y=(\ln x)^x$; (b) $y=x^{\ln x}$;

(c) $y=(\ln x)^{\ln x}$; (d) $y=x^{\sqrt{x}}$;

(e) $y=x^{\sqrt[3]{x}}$.

SECTION 8.5

36 The number of bacteria in a culture doubles every hour. How long does it take for a thousand bacteria to produce a billion?

37 The world population at the beginning of 1970 was 3.6 billion. The weight of the earth is 6586×10^{18} tons. If the population of the world continues increasing at a rate of 2 percent per year, and if the average person weighs 120 lb, in what year will the weight of all the people equal the weight of the earth?

38 Cesium 137 is used in medical and industrial radiology. Estimate its half-life if 20 percent decays in 10 years.

39 In a certain chemical reaction a substance S decomposes at a rate proportional to the amount of S not decomposed. If 25 g of this substance is reduced to 10 g in 4 hours, when will 21 g be decomposed?

40 A certain object cools from 120°F to 95°F in half an hour when surrounded by air whose temperature is 70°F. Use Newton's law of cooling to find its temperature at the end of another half hour.

41 A cup of coffee is made with boiling water at 212°F and taken into a room whose air temperature is 72°F. After 20 minutes it has cooled to 100°F. What is its temperature after cooling for a full hour?

42 Assume that the atmospheric pressure p is related to the altitude h above sea level by the differential equation

$$\frac{dp}{dh}=-cp,$$

where c is a positive constant. If p is 15 lb/in^2 at sea level and 10 lb/in^2 at 10,000 ft, find the atmospheric pressure at the top of Mount Everest, where $h\cong 30{,}000$ ft.

43 A rocket of total mass m is traveling with velocity v in a distant region of space where the force of gravity is negligible. Its thrust is provided by burning an appropriate fuel and expelling the exhaust products backward at a constant velocity a relative to the rocket. The mass m is therefore variable, and Newton's second law of motion is

$$F = \frac{d}{dt}(mv),$$

which in this case becomes

$$\left(-\frac{dm}{dt}\right)(a - v) = \frac{d}{dt}(mv).$$

(a) Show that $m\dfrac{dv}{dt} = -a\dfrac{dm}{dt}$.

(b) Use part (a) to show that $\dfrac{dm}{dv} = -\dfrac{1}{a}m$.

(c) Use part (b) to show that $m = m_0e^{-v/a}$ if $v = 0$ and $m = m_0$ when $t = 0$.

(d) The mass m clearly diminishes as the flight progresses, so the velocity v increases. If m_1 is the mass of the initial fuel supply and $\bar{v}$ is the maximum velocity, show that

$$\bar{v} = a \ln \frac{m_0}{m_0 - m_1}.$$

Notice that $m_0 - m_1$ is the so-called *structural mass* of the rocket, i.e., its mass exclusive of fuel.

44 The presence of a certain antibiotic destroys a type of bacteria at a rate jointly proportional to the number N of bacteria and the amount of antibiotic. If there were no antibiotic present, the bacteria would grow at a rate proportional to their number. Assume that the amount of antibiotic is 0 at $t = 0$ and increases at a constant rate. Construct a suitable differential equation for N, solve this equation, and sketch the solution.

45 Assume for the sake of simplicity that uranium 238 decays directly into lead 206 with a half-life of $T = 4.5$ billion years.

(a) If a given quantity of just-solidified volcanic rock contains x_0 atoms of uranium and no lead, show that t years later there are $x = x_0e^{-kt}$ atoms of uranium and $y = x_0(1 - e^{-kt})$ atoms of lead, where $kT = \ln 2$.

(b) If we can measure the ratio $r = y/x$ in an ancient volcanic rock, and if we have reasonable grounds for believing that all the lead comes from uranium that was locked in the rock when it solidified, then we can calculate the age of the rock with a fair degree of confidence. Show that this age is given by the formula

$$t = \frac{1}{k}\ln(1 + r) = \frac{T}{\ln 2}\ln(1 + r) \cong \frac{Tr}{\ln 2}$$

when r is small. Hint: Examine the graph of $\ln(1 + r)$ for small values of r.

(c) In a certain rock, r is found to be 0.082. Show that this rock may be about 530 million years old.

46 In the branch of psychology called *psychophysics*, an attempt is made to establish a quantitative connection between the sensation S experienced by a person and the stimulus R that causes this sensation, as in the sensation of heaviness produced by a weight held in the hand. If a small change dR in the stimulus from R to $R + dR$ produces a corresponding change dS in the sensation, then dS is not proportional to dR. Thus, if a weight we hold in our hand is increased from 5 lb to 6 lb, we detect much more of a difference in heaviness than when it is increased from 20 lb to 21 lb. The *Fechner-Weber law* was first formulated by E. H. Weber in 1834 and expounded in detail by G. T. Fechner in 1860, and it played a substantial role in early experimental psychology through the influence of Wilhelm Wundt. This law states that dS is proportional, not to the actual amount dR the stimulus is changed, but to the relative amount it is changed,

$$dS = k\frac{dR}{R}.$$

Find S as a function of R if $S = 0$ when $R = 1$.

SECTION 8.6

47 A flu epidemic hits a city and spreads at a rate jointly proportional to the number of people who are infected and the number of those who are not. If the number of people stricken grows from 10 percent to 20 percent of the population in the first 10 days, how many more days will be required for half the population to be infected?

***48** *Volterra's prey-predator equations* describe an ecological community of the following kind. On an island with plenty of grass, there live x rabbits (the prey) and y foxes (the predator). The number of encounters per unit time between rabbits and foxes is proportional to the product xy of their populations. The rabbits tend to increase at a rate proportional to their number and to decrease at a rate proportional to the product xy. The foxes tend to decrease at a rate proportional to their number and to increase at a rate proportional to xy. This gives the system of differential equations

$$\frac{dx}{dt} = ax - bxy, \qquad \frac{dy}{dt} = -cy + dxy,$$

where a, b, c, d are positive constants.

(a) Show that $x = c/d$ and $y = a/b$ is a solution of the system. These are called the *equilibrium populations*.

(b) Show that any solution $x = x(t)$, $y = y(t)$ satisfies the equation $(x^c e^{-dx})(y^a e^{-by}) = k$, where k is a positive constant. Hint: Eliminate dt from the system by division, separate variables, and integrate.

(c) Use the equation in part (b) to show that neither $x(t)$ nor $y(t)$ can $\to \infty$ as $t \to \infty$.

49 Consider a falling body of mass m and assume that the retarding force due to air resistance is proportional to the square of the velocity. If the body falls from rest, find a formula for the velocity in terms of the distance fallen, and thereby find the terminal velocity in this case. Hint: $dv/dt = (dv/ds)(ds/dt) = v\,dv/ds$.

***50** A torpedo is traveling at a velocity of 60 km/h at the moment it runs out of fuel. If the water resists its motion with a force proportional to the velocity v, and if 1 km of travel reduces v to 40 km/h, find the distance s the torpedo coasts in t hours, and also the total distance it coasts.

51 Brine containing 1 lb of salt per gallon flows at the rate of 10 gal/min into a tank initially filled with 120 gal of pure water. If the concentration is kept uniform by stirring, and the mixture flows out at the same rate, when will the tank contain 40 lb of salt? When will it contain 100 lb of salt?

52 A large tank initially contains 45 lb of salt dissolved in 50 gal of water. Pure water flows in at the rate of 3 gal/min, and the mixture (which is kept uniform by stirring) flows out at the rate of 2 gal/min. When will the tank contain 5 lb of salt? How many gallons of water will be in the tank at that time?

53 An aquarium contains 10 gal of polluted water. A filter is attached to this aquarium which drains off the polluted water at the rate of 5 gal/h and replaces it at the same rate by pure water. How long does it take to reduce the pollution to half its initial level?

54 Let ϵ be a small positive number. The differential equation

$$\frac{dN}{dt} = kN^{1+\epsilon},$$

where k is a positive constant, is called the *doomsday equation* because the "growth term" $kN^{1+\epsilon}$ is slightly larger than that for normal—or natural—exponential growth (that is, kN).

(a) Solve this equation if $N = N_0$ when $t = 0$.

(b) Show that there is a finite time $t = t_0$ such that $N \to \infty$ as $t \to t_0$.

(c) The fast-growing population of a certain inhabited planet has a growth term $kN^{1.1}$. If there are 5 billion people initially and 6 billion people 10 years later, when is doomsday?

(d) In part (c), when is doomsday if the growth term is $kN^{1.01}$ and the other conditions are unchanged? (No matter how small the positive number ϵ may be, there *is* an inescapable doomsday.)

9 TRIGONOMETRIC FUNCTIONS

9.1 REVIEW OF TRIGONOMETRY

We continue the program started in Chapter 8 of extending the scope of our work to include broader and broader classes of functions, this time the full range of trigonometric functions. In science, we have already pointed out that these functions are indispensable tools for the study of periodic phenomena of all kinds, ranging from the back-and-forth movement of the bob of a pendulum clock to the revolution of the planets in their orbits around the sun. And in mathematics —as we shall see in Chapter 10—almost all of the more advanced methods of integration lean heavily on the trigonometric functions and their properties.

We assume that students have studied trigonometry in high school. Also, in Section 1.7 we provided a brief account of a few of the simpler properties of the sine and cosine functions. Nevertheless, no matter how well the basic facts have been learned, they are easy to forget unless they are needed and used on a day-to-day basis, which they will be through most of the rest of this book. We therefore devote this section to a review of the subject from the beginning, ignoring the fact that students already have a little knowledge of sines and cosines. There are a number of fundamental formulas built into this exposition, and these are so important for the purposes of calculus that students should relearn them systematically and thoroughly. Even though our treatment is very concise, it is essentially self-contained; and hard-working students who have little previous experience with trigonometry should be able to get along comfortably in the following chapters with only what they find in these pages.

RADIAN MEASURE

The most common unit for measuring angles is the degree (1 right angle = 90 degrees = 90°). However, the standard unit for angle measurement in calculus is the *radian.* One radian is the angle which, placed at the center of a circle, subtends an arc whose length equals the radius (Fig. 9.1, left). More generally, the number of radians in an arbitrary central angle (Fig. 9.1, center) is defined to be the ratio of the length of the subtended arc to the radius, $\theta = s/r$, so that $s = r\theta$. Since the circumference of the circle is $c = 2\pi r$, a complete central angle of 360° is equivalent to $2\pi r/r = 2\pi$ radians. Thus,

$$2\pi \text{ radians} = 360^\circ, \qquad \pi \text{ radians} = 180^\circ,$$

$$1 \text{ radian} = \frac{180}{\pi} \cong 57.296^\circ, \qquad 1^\circ = \frac{\pi}{180} \cong 0.0175 \text{ radian}.$$

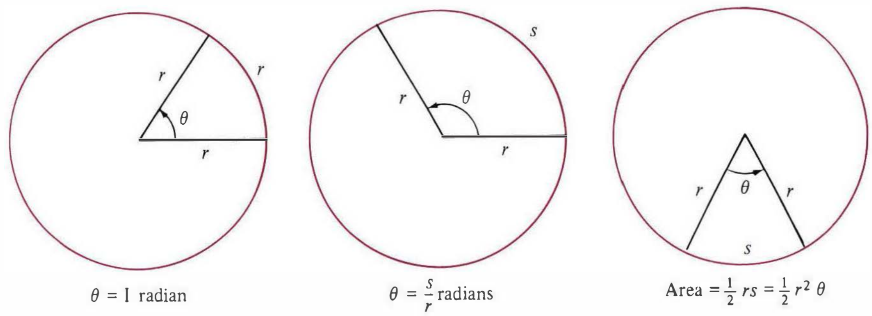


Figure 9.1

Further, 90° = $\pi/2$, 60° = $\pi/3$, 45° = $\pi/4$, and 30° = $\pi/6$, where we follow the convention of omitting the word "radian" in using radian measure.

Just as the calculus of logarithms is simplified by using the base *e*, the calculus of the trigonometric functions is simplified by using radian measure. We will point out the specific reason for this in Section 9.2. Throughout our work we will use radian measure routinely and mention degrees only occasionally.

It is sometimes useful to know that the area *A* of the sector whose central angle is θ (Fig. 9.1, right) is given by the formula

$$A = \tfrac{1}{2}rs = \tfrac{1}{2}r^2\theta,$$

since $s = r\theta$. This is easy to prove by using the fact that the area of the sector is to the area of the circle as the arc s is to the circumference:

$$\frac{A}{\pi r^2} = \frac{s}{2\pi r}, \qquad \text{so} \qquad A = \frac{1}{2}rs.$$

This is easy to remember by thinking of the sector as if it were a triangle with height r and base s.

THE TRIGONOMETRIC FUNCTIONS

Consider the unit circle in the xy-plane (Fig. 9.2). If θ is a positive number, let the radius OP start in the position OA and revolve counterclockwise through θ radians. Thus, $\theta = \pi$ produces half a revolution and $\theta = 2\pi$ produces a complete revolution, both counterclockwise. If θ is negative, we let OP revolve clockwise through $-\theta$ radians. See Fig. 9.3. In this way, each real number θ (positive, negative, or zero) determines a unique position of OP in Fig. 9.2, and therefore a unique point $P = (x, y)$ with the property that $x^2 + y^2 = 1$. The sine and cosine of θ are now defined by

$$\sin\theta = y \qquad \text{and} \qquad \cos\theta = x.$$

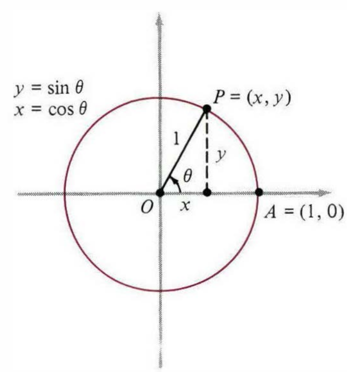


Figure 9.2

It is evident from the definition that $-1 \le \sin\theta \le 1$, and similarly for $\cos\theta$; and the algebraic signs of these quantities depend on which quadrant of the plane the point P happens to lie in. For every θ, the numbers θ and $\theta + 2\pi$ clearly determine the same point P, so

$$\sin(\theta + 2\pi) = \sin\theta \qquad \text{and} \qquad \cos(\theta + 2\pi) = \cos\theta.$$

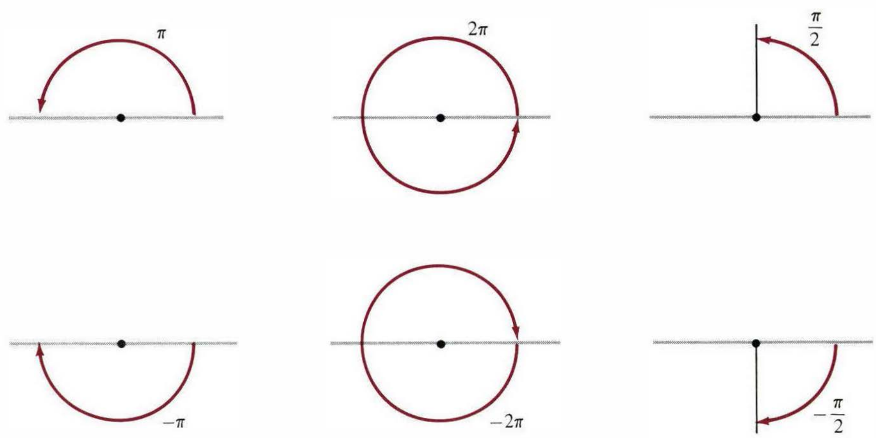


Figure 9.3

Thus the values of sin θ and cos θ repeat when θ increases by 2π. We express this property of sin θ and cos θ by saying that these functions are *periodic* with *period* 2π.

The remaining four trigonometric functions—the tangent, cotangent, secant, and cosecant—are defined by

$$\tan\theta = \frac{y}{x}, \qquad \cot\theta = \frac{x}{y}, \qquad \sec\theta = \frac{1}{x}, \qquad \csc\theta = \frac{1}{y}.$$

The sine and cosine are the basic functions, and the others can be expressed in terms of these two [see identities (1) to (4) below].

When θ is a positive number $<\pi/2$, the right triangle interpretations of the sine, cosine, and tangent are as follows (see Fig. 9.4):

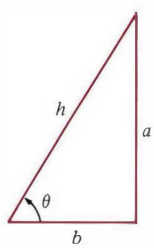


Figure 9.4

$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{a}{h},$$

$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{b}{h},$$

$$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{a}{b}.$$

We have drawn the right triangle here with base angle equal to the angle θ shown in Fig. 9.2, and the validity of these statements rests on the similarity of the two triangles in the figures (since $\sin\theta = y = y/1$, etc.). In the equivalent forms

$$a = h\sin\theta, \qquad b = h\cos\theta, \qquad a = b\tan\theta,$$

the right triangle interpretations have many uses in physics and geometry. Nevertheless, the purposes of calculus require that θ be an unrestricted real variable, and for this reason the unit circle definitions are preferable.

IDENTITIES

Several simple relations among our functions are direct consequences of the definitions:

$$\boxed{\begin{aligned} \tan\theta &= \frac{\sin\theta}{\cos\theta}, && (1)\\ \cot\theta &= \frac{\cos\theta}{\sin\theta}, && (2)\\ \sec\theta &= \frac{1}{\cos\theta}, && (3)\\ \csc\theta &= \frac{1}{\sin\theta}, && (4)\\ \tan\theta &= \frac{1}{\cot\theta}. && (5) \end{aligned}}$$

Altogether, there are 21 fundamental identities that express the main properties of the trigonomeric functions and constitute the core of the subject. These identities fall into several natural groups, and are therefore easier to remember than we might expect. We emphasize these groups by enclosing them in boxes.

Our next identities state the effect of replacing θ by $-\theta$. From Fig. 9.5 and the obvious fact that the endpoints of the two radii lie on the same vertical line for all values of θ, we at once have the first two of the identities

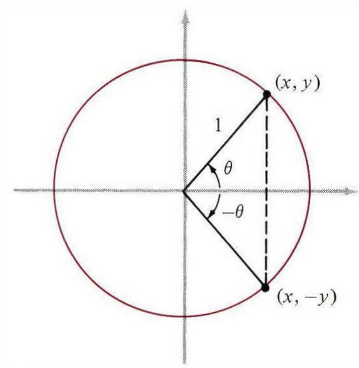


Figure 9.5

$$\boxed{\begin{aligned} \sin(-\theta) &= -\sin\theta, && (6)\\ \cos(-\theta) &= \cos\theta, && (7)\\ \tan(-\theta) &= -\tan\theta, && (8) \end{aligned}}$$

The third follows easily from (1) combined with (6) and (7).*

Our next group consists of three equivalent versions of the equation $x^2 + y^2 = 1$. Before stating these, we must explain that the symbols $\sin^2\theta$ and $\cos^2\theta$ are standard notations for the numbers $(\sin\theta)^2$ and $(\cos\theta)^2$. If we write $x^2 + y^2 = 1$ in the form $y^2 + x^2 = 1$, then this yields the first of the identities

$$\boxed{\begin{aligned} \sin^2\theta + \cos^2\theta &= 1, && (9)\\ \tan^2\theta + 1 &= \sec^2\theta, && (10)\\ 1 + \cot^2\theta &= \csc^2\theta. && (11) \end{aligned}}$$

The second and third in this group are obtained by dividing (9) first by $\cos^2\theta$, and then by $\sin^2\theta$.

For obvious reasons, the following are called the *addition formulas*:

$$\boxed{\begin{aligned} \sin(\theta+\phi) &= \sin\theta\cos\phi + \cos\theta\sin\phi, && (12)\\ \cos(\theta+\phi) &= \cos\theta\cos\phi - \sin\theta\sin\phi, && (13)\\ \tan(\theta+\phi) &= \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}. && (14) \end{aligned}}$$

*It is clear that there are similar identities for the cotangent, secant, and cosecant. However, these are of little significance; and in keeping with our purpose of presenting a stripped-down version of trigonometry, we ignore them.

We indicate in Problem 10 a method for proving the first two of these—which are not obvious and ought to be memorized—and the third follows from the first two by a straightforward argument. Write

$$\tan(\theta+\phi) = \frac{\sin(\theta+\phi)}{\cos(\theta+\phi)} = \frac{\sin\theta\cos\phi+\cos\theta\sin\phi}{\cos\theta\cos\phi-\sin\theta\sin\phi}.$$

Now, by dividing both numerator and denominator on the right by $\cos\theta\cos\phi$, we obtain

$$\tan(\theta+\phi) = \frac{\sin\theta/\cos\theta+\sin\phi/\cos\phi}{1-(\sin\theta/\cos\theta)(\sin\phi/\cos\phi)},$$

which is essentially (14). The corresponding *subtraction formulas* are

$$\sin(\theta-\phi) = \sin\theta\cos\phi - \cos\theta\sin\phi, \tag{15}$$

$$\cos(\theta-\phi) = \cos\theta\cos\phi + \sin\theta\sin\phi, \tag{16}$$

$$\tan(\theta-\phi) = \frac{\tan\theta-\tan\phi}{1+\tan\theta\tan\phi}. \tag{17}$$

These follow directly from the addition formulas by replacing ϕ by $-\phi$ and using (6), (7), and (8).

The *double-angle formulas* are

$$\sin 2\theta = 2\sin\theta\cos\theta, \tag{18}$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta. \tag{19}$$

These are the special cases of (12) and (13) obtained by replacing ϕ by θ. (There is also an obvious double-angle formula for the tangent; but this is of minor importance and we omit it.)

The *half-angle formulas* are

$$2\cos^2\theta = 1+\cos 2\theta, \tag{20}$$

$$2\sin^2\theta = 1-\cos 2\theta. \tag{21}$$

These are easy to prove by writing (9) and (19) together, as

$$\cos^2\theta+\sin^2\theta = 1,$$

$$\cos^2\theta-\sin^2\theta = \cos 2\theta.$$

Adding yields (20), and subtracting yields (21).

VALUES

If we keep firmly in mind the definitions of $\sin\theta$, $\cos\theta$, and $\tan\theta$, then there are several first-quadrant values of θ for which the exact values of these functions are easy to find. All that is necessary is to remember the Pythagorean theorem and look carefully at the three parts of Fig. 9.6:

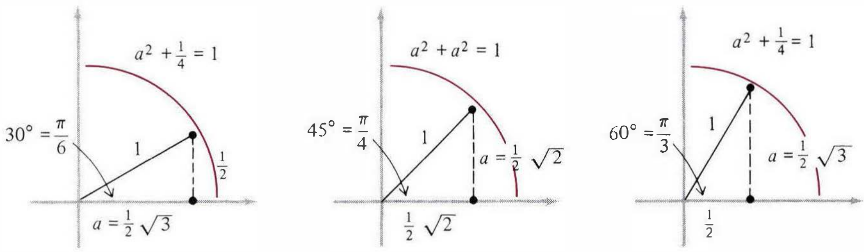


Figure 9.6

$$\sin\frac{\pi}{6} = \frac{1}{2} \qquad \sin\frac{\pi}{4} = \frac{1}{2}\sqrt{2} \qquad \sin\frac{\pi}{3} = \frac{1}{2}\sqrt{3}$$

$$\cos\frac{\pi}{6} = \frac{1}{2}\sqrt{3} \qquad \cos\frac{\pi}{4} = \frac{1}{2}\sqrt{2} \qquad \cos\frac{\pi}{3} = \frac{1}{2}$$

$$\tan\frac{\pi}{6} = \frac{\frac{1}{2}}{\frac{1}{2}\sqrt{3}} = \frac{1}{3}\sqrt{3} \qquad \tan\frac{\pi}{4} = 1 \qquad \tan\frac{\pi}{3} = \frac{\frac{1}{2}\sqrt{3}}{\frac{1}{2}} = \sqrt{3}$$

Also, an inspection of Fig. 9.2 with OP in various positions gives us similar information for the cases $\theta = 0$, $\pi/2$, π, $3\pi/2$, and 2π (the entry $*$ means that the quantity is undefined):

$$\sin 0 = 0 \qquad \sin\frac{\pi}{2} = 1 \qquad \sin\pi = 0 \qquad \sin\frac{3\pi}{2} = -1 \qquad \sin 2\pi = 0$$

$$\cos 0 = 1 \qquad \cos\frac{\pi}{2} = 0 \qquad \cos\pi = -1 \qquad \cos\frac{3\pi}{2} = 0 \qquad \cos 2\pi = 1$$

$$\tan 0 = 0 \qquad \tan\frac{\pi}{2} = * \qquad \tan\pi = 0 \qquad \tan\frac{3\pi}{2} = * \qquad \tan 2\pi = 0$$

In our subsequent work, facts of this kind will often be needed at a moment's notice. They are best learned, not by an effort of memory, but rather by an act of understanding—knowing the definitions of the trigonometric functions and visualizing (or quickly sketching) appropriate pictures. We also emphasize the way the algebraic signs of our functions vary from one quadrant to another. The facts are obvious from the definitions and Fig. 9.2, and are stated in the following table:

Quadrant	*1*	*2*	*3*	*4*
$\sin\theta$	+	+	−	−
$\cos\theta$	+	−	−	+
$\tan\theta$	+	−	+	−

GRAPHS

The graph of $\sin\theta$ is easy to sketch by looking at Fig. 9.2 and following the way y varies as θ increases from 0 to 2π, that is, as the radius swings around through one complete counterclockwise revolution. It is clear that $\sin\theta$ starts at 0, increases to 1, decreases to 0, decreases further to -1, and increases to 0. This gives one complete cycle of $\sin\theta$, as shown on the left in Fig. 9.7. The complete graph (on the right in Fig. 9.7) consists of infinitely many repetitions of this cycle, to the right and to the left. The graph of $\cos\theta$ can be sketched in essentially

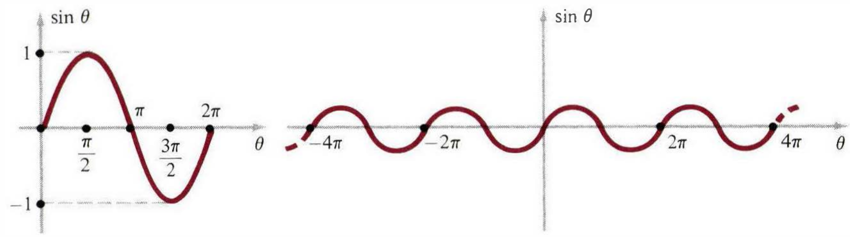


Figure 9.7

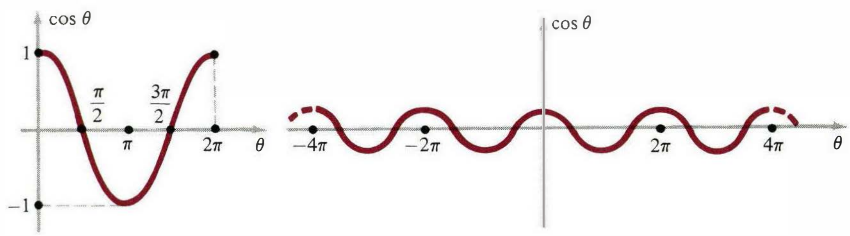


Figure 9.8

the same way (Fig. 9.8), the main difference being that $\cos\theta$ starts at 1, decreases to 0, decreases further to -1, increases to 0, and increases further to 1.

The graph of $\tan\theta$ is quite different from the graphs of $\sin\theta$ and $\cos\theta$. We point out first that $\tan\theta$ is periodic with period π:

$$\tan(\theta+\pi) = \frac{\sin(\theta+\pi)}{\cos(\theta+\pi)} = \frac{-\sin\theta}{-\cos\theta} = \tan\theta.$$

This permits us to get the full range of values of $\tan\theta$ by visualizing the ratio y/x in Fig. 9.2 and allowing θ to increase from $-\pi/2$ to $\pi/2$. The result is the central curve shown in Fig. 9.9, and the complete graph of $\tan\theta$ consists of infinitely many repetitions of this curve to the right and to the left. The fact that $\tan\theta \to \infty$ as $\theta \to \pi/2$ (from the left) is often loosely expressed by writing $\tan \pi/2 = \infty$.

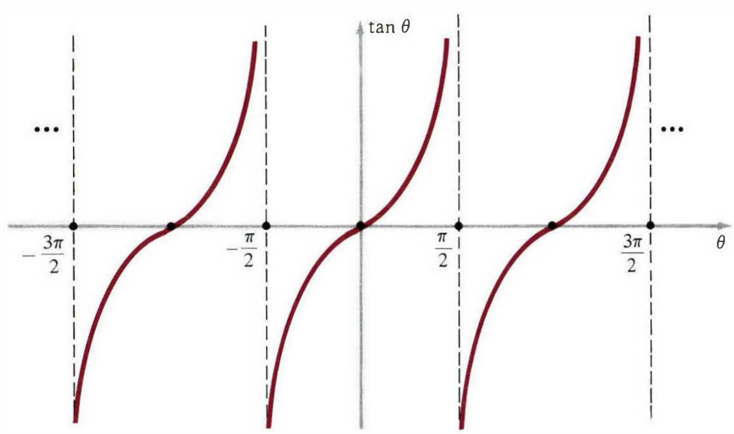


Figure 9.9

LAW OF COSINES

This is a useful tool in a variety of situations in mathematics and physics. It expresses the third side of a triangle (Fig. 9.10) in terms of two given sides a and b and the included angle θ:

$$c^2 = a^2 + b^2 - 2ab \cos \theta.$$

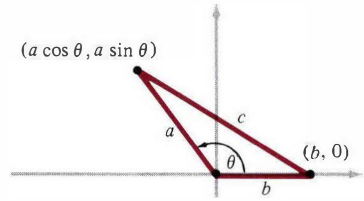


Figure 9.10

The proof is routine if we place the triangle in the xy-plane as shown in the figure and apply the distance formula to the vertices $(a \cos \theta, a \sin \theta)$ and $(b, 0)$. The square of the side c is clearly

$$\begin{aligned} c^2 &= (a \cos \theta - b)^2 + (a \sin \theta - 0)^2 \\ &= a^2(\cos^2 \theta + \sin^2 \theta) + b^2 - 2ab \cos \theta \\ &= a^2 + b^2 - 2ab \cos \theta, \end{aligned}$$

and the argument is complete. An important application of the law of cosines is made in Problem 10, where it is used to prove identity (16), and thereby identities (12) and (13).

PROBLEMS

1 Convert from degrees to radians:
(a) 15°; (b) 105°; (c) 120°;
(d) 75°; (e) 150°; (f) 135°;
(g) 225°; (h) 210°; (i) 630°;
(j) 900°.

2 Convert from radians to degrees:
(a) $5\pi/3$; (b) $7\pi/6$; (c) $2\pi/9$;
(d) $3\pi/2$; (e) $4\pi/3$; (f) 3π;
(g) $7\pi/15$; (h) $\pi/36$; (i) $\pi/5$;
(j) $25\pi/3$.

3 A decorative garden is to have the shape of a circular sector of radius r and central angle θ. If the perimeter is fixed in advance, what value of θ will maximize the area of the garden?

4 Find the values of $\sin \theta$, $\cos \theta$, and $\tan \theta$ when θ equals
(a) $-\pi/6$; (b) $3\pi/4$; (c) $4\pi/3$;
(d) $-5\pi/4$; (e) $2\pi/3$; (f) 17π;
(g) -102π.

5 If the base of an isosceles triangle is 10, express its area A as a function of the vertex angle θ.

6 If the height of an isosceles triangle is h, express its perimeter p as a function of the base angle θ.

7 Express the height H of a flagpole in terms of the length L of its shadow and the angle of elevation θ of the sun.

8 A hunter sits on a platform built in a tree 30 m above the ground. He sees a tiger at an angle of 30° below the horizontal. How far is he from the tiger?

9 Sketch the graph of
(a) $\sin 2\theta$ (hint: this curve runs through one complete cycle as 2θ increases from 0 to 2π);
(b) $3 \sin 2\theta$; (c) $\sin 4\theta$;
(d) $\sin \frac{1}{2}\theta$; (e) $2 \cos 3\theta$.

10 In this problem we outline a method of proving identities (12) and (13) by first establishing (16). Figure 9.11 shows the unit circle with two arbitrary angles θ and ϕ and their corresponding points $P_\theta = (\cos \theta, \sin \theta)$ and $P_\phi = (\cos \phi, \sin \phi)$.

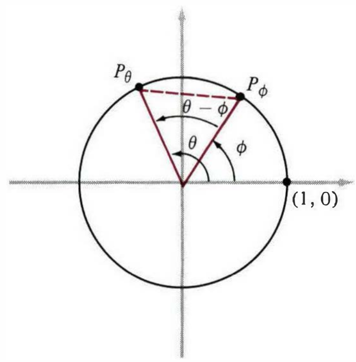


Figure 9.11

(a) Calculate the square of the distance between these points in two ways, by using the distance formula and the law of cosines, and thus prove identity (16),

$$\cos(\theta - \phi) = \cos \theta \cos \phi + \sin \theta \sin \phi.$$

(b) Use part (a) to prove identity (13),

$$\cos(\theta + \phi) = \cos \theta \cos \phi - \sin \theta \sin \phi.$$

(c) Use part (a) to show that $\cos(\pi/2 - \phi) = \sin\phi$.
(d) Use part (c) to show that $\sin(\pi/2 - \phi) = \cos\phi$. Hint: Replace ϕ by $\pi/2 - \phi$.
(e) Use parts (a), (c), and (d) to prove identity (12),

$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi.$$

Hint: $\sin(\theta + \phi) = \cos[\pi/2 - (\theta + \phi)]$
$= \cos[(\pi/2 - \theta) - \phi] = \cdots$.

11 Derive formulas for $\sin 3\theta$ in terms of $\sin\theta$, and for $\cos 3\theta$ in terms of $\cos\theta$.

12 Derive a formula for $\cos 4\theta$ in terms of $\cos\theta$.

13 Derive a formula for $\sin 4\theta$ in terms of $\sin\theta$ and $\cos\theta$.

14 If a and b are any constants, show that there exist constants A and B with the property that $a\sin\theta + b\cos\theta$ can be written in the form $A\sin(\theta + B)$.

15 Compute $\sin 15°$ by using
(a) $15° = 45° - 30°$; (b) $15° = \frac{1}{2}(30°)$.

16 Find all solutions of the given equation in the interval $0 \le \theta \le 2\pi$:
(a) $\sin\theta = 0$; (b) $\sin\theta = 1$;
(c) $\sin\theta = -1$.

17 Find all solutions of the given equation in the interval $0 \le \theta \le 2\pi$:
(a) $\cos\theta = 1$; (b) $\cos\theta = 0$;
(c) $\cos\theta = -1$.

18 Find all values of θ in the interval $0 \le \theta \le 2\pi$ for which
(a) $\sin\theta = -\frac{1}{2}$; (b) $\cos\theta = \frac{1}{2}\sqrt{2}$;
(c) $\sin\theta = \frac{1}{2}\sqrt{3}$; (d) $\sin\theta = \cos\theta$.

19 Find all solutions of each of the following equations:
(a) $\sin 3\theta = \frac{1}{2}\sqrt{2}$; (b) $\cos\frac{1}{5}\theta = -1$;
(c) $\sin 5\theta = -\frac{1}{2}$.

20 Show that
(a) $\sin\theta\sin\phi = \frac{1}{2}[\cos(\theta - \phi) - \cos(\theta + \phi)]$;
(b) $\cos\theta\cos\phi = \frac{1}{2}[\cos(\theta - \phi) + \cos(\theta + \phi)]$;
(c) $\sin\theta\cos\phi = \frac{1}{2}[\sin(\theta + \phi) + \sin(\theta - \phi)]$.

21 Show that
(a) $\sin\theta + \sin\phi = 2\sin\left(\frac{\theta + \phi}{2}\right)\cos\left(\frac{\theta - \phi}{2}\right)$;
(b) $\sin\theta - \sin\phi = 2\cos\left(\frac{\theta + \phi}{2}\right)\sin\left(\frac{\theta - \phi}{2}\right)$;
(c) $\cos\theta + \cos\phi = 2\cos\left(\frac{\theta + \phi}{2}\right)\cos\left(\frac{\theta - \phi}{2}\right)$;
(d) $\cos\theta - \cos\phi = -2\sin\left(\frac{\theta + \phi}{2}\right)\sin\left(\frac{\theta - \phi}{2}\right)$.

Hint: These identities can be established laboriously, by working from the right sides to the left sides, or easily, by an ingenious use of Problem 20.

Establish the identities in Problems 22–32.

22 $\sin 2\theta = \dfrac{2\tan\theta}{1 + \tan^2\theta}$.

23 $\cos 2\theta = \dfrac{1 - \tan^2\theta}{1 + \tan^2\theta}$.

24 $\tan\theta = \dfrac{\sin\theta + \sin 2\theta}{1 + \cos\theta + \cos 2\theta}$.

25 $\tan\theta = \dfrac{\sin 2\theta}{1 + \cos 2\theta}$.

26 $\cot\theta = \dfrac{\sin 2\theta}{1 - \cos 2\theta}$.

27 $\tan^2\theta = \dfrac{1 - \cos 2\theta}{1 + \cos 2\theta}$.

28 $\tan\theta\tan\frac{1}{2}\theta = \sec\theta - 1$.

29 $\tan\frac{1}{2}\theta = \dfrac{\sin\theta}{1 + \cos\theta}$.

30 $\dfrac{1 + \sin\theta + \cos\theta}{1 + \sin\theta - \cos\theta} = \cot\frac{1}{2}\theta$.

31 $1 - 4\sin^4\theta = \cos 2\theta\,(1 + 2\sin^2\theta)$.

32 $\tan\frac{1}{2}\theta + \cot\frac{1}{2}\theta = 2\csc\theta$.

33 Sketch the graph of the function
(a) $y = \sin\dfrac{1}{x}$; (b) $y = x\sin\dfrac{1}{x}$;
(c) $y = x^2\sin\dfrac{1}{x}$.

Notice particularly that each of these functions is undefined at $x = 0$.

34 In Problem 33, supplement the definition of each function by specifying that $y = 0$ when $x = 0$. With these changes, show that at the point $x = 0$ the function (a) is discontinuous, the function (b) is continuous but not differentiable, and the function (c) is differentiable.

35 If one side and the opposite angle of a triangle are fixed, and the other two sides are variable, use the law of cosines to show that the area is a maximum when the triangle is isosceles. (Can you prove this by using geometry alone?)

***36** A conical paper cup is formed from a circular sheet of paper by cutting out a circular sector and joining the two straight edges of the remaining piece. What should the central angle of the sector be in order to maximize the volume of the cup?

37 A light hangs above the center of a circular table whose radius is 3 ft. The illumination at any point on the table is directly proportional to the sine of the angle between the table and the ray of light to that point, and inversely proportional to the square of the distance from the point to the light source. How high should the light be hung in order to maximize the illumination at the edge of the table?

***38** A heavy spherical ball is lowered carefully into a full conical wine glass whose depth is a and whose generating angle (between the axis and a generator) is α. Show that the greatest overflow occurs when the radius of the ball is

$$\frac{a\sin\alpha}{\sin\alpha + \cos 2\alpha}.$$

9.2 THE DERIVATIVES OF THE SINE AND COSINE

The calculus of the trigonometric functions begins with two of the most important formulas in all of mathematics,

$$\frac{d}{dx}\sin x = \cos x \tag{1}$$

and

$$\frac{d}{dx}\cos x = -\sin x. \tag{2}$$

We have already discussed these formulas in Section 3.4. However, no harm will be done—and perhaps much good—if we pretend that we have never heard of them before and discuss them all over again from the beginning. We emphasize that the letter x used here is simply an ordinary real variable, as in any function $y = f(x)$; and if it is thought of as an angle, then this angle is always to be understood in radian measure.

Formulas (1) and (2) can be proved by straightforward applications of the definition of the derivative,

$$\frac{d}{dx}f(x) = \lim_{\Delta x\to 0}\frac{f(x+\Delta x) - f(x)}{\Delta x}. \tag{3}$$

In order to carry through these calculations, it turns out that we need to know the following two special limits,

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1 \tag{4}$$

and

$$\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta} = 0.^* \tag{5}$$

The validity of these statements can be understood directly from geometry, by thinking of θ as a small angle and looking at the unit circle on the left in Fig. 9.12, where the definitions given in the previous section tell us that $PQ = \sin\theta$, $PR = \theta$, and $QR = 1 - \cos\theta$. It is easy to see that the ratio $(\sin\theta)/\theta = PQ/PR$ is <1 and close to 1, and it visibly approaches 1 as θ approaches 0. This behavior of the ratio $(\sin\theta)/\theta$ is further emphasized by the magnified version of

*These limits were both established in Section 2.5. For the sake of variety, we offer supporting arguments here that are somewhat different from those used before.

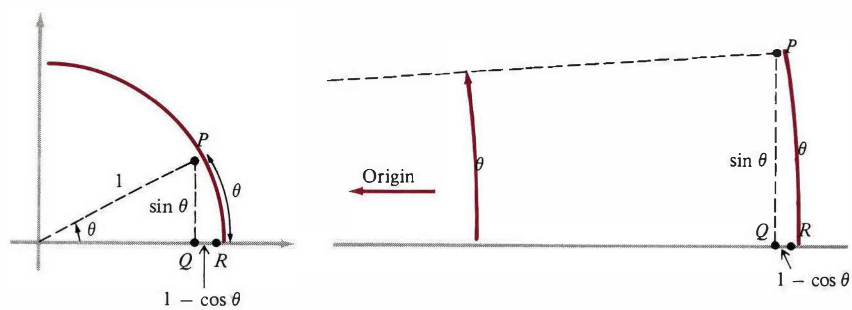


Figure 9.12

part of the picture that is shown on the right in Fig. 9.12, where θ is taken to be very small and the origin is understood to be several feet to the left. The same type of "proof by inspection" can also be applied to (5). This time the figure tells us that the ratio $(1 - \cos\theta)/\theta = QR/PR$ is a small number that clearly approaches 0 as θ approaches 0.

To establish formula (1), we apply (3) to the function $f(x) = \sin x$,

$$\frac{d}{dx}\sin x = \lim_{\Delta x \to 0} \frac{\sin(x + \Delta x) - \sin x}{\Delta x}.$$

Since $\sin(x + \Delta x) = \sin x \cos \Delta x + \cos x \sin \Delta x$, we exercise a little ingenuity and write

$$\begin{aligned}\frac{d}{dx}\sin x &= \lim_{\Delta x \to 0} \frac{\sin x \cos \Delta x + \cos x \sin \Delta x - \sin x}{\Delta x}\\ &= \lim_{\Delta x \to 0}\left[\cos x\left(\frac{\sin \Delta x}{\Delta x}\right) - \sin x\left(\frac{1 - \cos \Delta x}{\Delta x}\right)\right].\end{aligned}$$

Using (4) and (5) with θ replaced by Δx now yields

$$\frac{d}{dx}\sin x = (\cos x)\cdot 1 - (\sin x)\cdot 0 = \cos x,$$

which concludes the proof of (1). To prove formula (2), we begin with

$$\frac{d}{dx}\cos x = \lim_{\Delta x \to 0} \frac{\cos(x + \Delta x) - \cos x}{\Delta x}.$$

Since $\cos(x + \Delta x) = \cos x \cos \Delta x - \sin x \sin \Delta x$, we have

$$\begin{aligned}\frac{d}{dx}\cos x &= \lim_{\Delta x \to 0} \frac{\cos x \cos \Delta x - \sin x \sin \Delta x - \cos x}{\Delta x}\\ &= \lim_{\Delta x \to 0}\left[-\sin x\left(\frac{\sin \Delta x}{\Delta x}\right) - \cos x\left(\frac{1 - \cos \Delta x}{\Delta x}\right)\right]\\ &= (-\sin x)\cdot 1 - (\cos x)\cdot 0 = -\sin x,\end{aligned}$$

and the proof of (2) is complete. The addition formulas for the sine and cosine obviously play essential roles in these arguments, and this is their main use in mathematics.

We now generalize (1) and (2) by means of the chain rule, and obtain the extremely useful formulas

$$\frac{d}{dx}\sin u = \cos u\frac{du}{dx} \tag{6}$$

and

$$\frac{d}{dx}\cos u = -\sin u\frac{du}{dx}. \tag{7}$$

As usual, u is understood to be any differentiable function of x.

Example 1 Find the derivative of each of the following functions:
(a) $y = \sin 5x$; (b) $y = \sin\sqrt{x}$; (c) $y = \cos(2 - 3x^4)$.

Solution For (a), we use (6) with $u = 5x$, so

$$\frac{dy}{dx} = \cos 5x \frac{d}{dx}(5x) = 5 \cos 5x.$$

For (b), $u = \sqrt{x} = x^{1/2}$, so

$$\frac{dy}{dx} = \cos \sqrt{x} \frac{d}{dx}(x^{1/2}) = \frac{1}{2\sqrt{x}} \cos \sqrt{x}.$$

For (c), we use (7) with $u = 2 - 3x^4$, so

$$\frac{dy}{dx} = -\sin (2 - 3x^4) \frac{d}{dx}(2 - 3x^4) = 12x^3 \sin (2 - 3x^4).$$

Students must learn to use formulas (6) and (7) in combination with all previous rules of differentiation. In this connection it is necessary to remember the standard notation for powers of trigonometric functions: $\sin^n x$ means $(\sin x)^n$. There is one exception to this usage, for $(\sin x)^{-1}$ is *never* written $\sin^{-1} x$; the latter expression is reserved exclusively for the inverse sine function discussed in Section 9.5.

Example 2 Find the derivative of each of the following functions:
(a) $y = \sin^3 4x$; (b) $y = e^{\cos x}$;
(c) $y = \ln (\sin x)$; (d) $y = \sin (\ln x)$.

Solution

(a) $$\frac{dy}{dx} = 3(\sin 4x)^2 \frac{d}{dx}(\sin 4x) = 3(\sin 4x)^2(\cos 4x) \cdot 4$$
$$= 12 \sin^2 4x \cos 4x.$$

(b) $$\frac{dy}{dx} = e^{\cos x} \frac{d}{dx}(\cos x) = -\sin x \, e^{\cos x}.$$

(c) $$\frac{dy}{dx} = \frac{1}{\sin x} \frac{d}{dx}(\sin x) = \frac{\cos x}{\sin x} = \cot x.$$

(d) $$\frac{dy}{dx} = \cos (\ln x) \frac{d}{dx}(\ln x) = \frac{\cos (\ln x)}{x}.$$

Example 3 Show that $(d/dx)(\frac{1}{3} \cos^3 x - \cos x) = \sin^3 x$.

Solution

$$\frac{d}{dx}\left(\frac{1}{3} \cos^3 x - \cos x\right) = \frac{1}{3} \cdot 3 \cos^2 x(-\sin x) + \sin x$$
$$= \sin x(1 - \cos^2 x) = \sin^3 x.$$

Remark 1 Formulas (6) and (7) enable us to understand very clearly why radian measure is preferred to degree measure when working with the trigonometric functions in calculus. Let $\sin x^\circ$ and $\cos x^\circ$ denote the sine and cosine of an angle of x degrees. We know that x degrees equals $\pi x/180$ radians, so

$$\sin x^\circ = \sin \frac{\pi x}{180}.$$

By formula (6),

$$\frac{d}{dx}\sin x^\circ = \frac{d}{dx}\sin\frac{\pi x}{180} = \frac{\pi}{180}\cos\frac{\pi x}{180}$$

or

$$\frac{d}{dx}\sin x^\circ = \frac{\pi}{180}\cos x^\circ;$$

and similarly

$$\frac{d}{dx}\cos x^\circ = -\frac{\pi}{180}\sin x^\circ.$$

These formulas make it obvious why we use radian measure routinely in calculus—for the sake of simplicity, in order to avoid the repeated occurrence of the nuisance factor $\pi/180$.

Remark 2 There is another way of obtaining the basic derivative formulas (1) and (2) for the sine and cosine. This alternate approach has the advantage of providing direct insight into why these formulas are true. We start with the definitions from Section 9.1 (see Fig. 9.13),

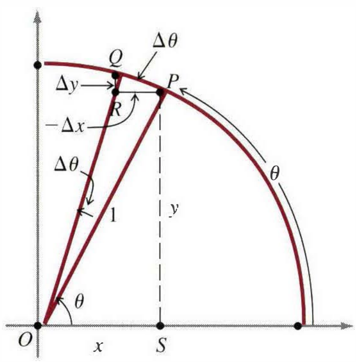


Figure 9.13

$$y = \sin\theta \qquad \text{and} \qquad x = \cos\theta,$$

in which we use θ for the independent variable because x and y are the coordinates of P. We change θ by a small amount $\Delta\theta$ and examine the resulting changes Δy and Δx in y and x, as shown in the figure. If we think of PQR as a tiny "right triangle" with hypotenuse" $\Delta\theta$, then the crux of the present argument is the fact that the triangles PQR and POS are similar. By using proportional sides of similar triangles we obtain the approximate equations

$$\frac{\Delta y}{\Delta\theta} \cong \frac{x}{1} = x \qquad \text{and} \qquad \frac{-\Delta x}{\Delta\theta} \cong \frac{y}{1} = y.$$

As $\Delta\theta$ approaches 0, these approximations get better and better, and we conclude that

$$\frac{dy}{d\theta} = \lim_{\Delta\theta\to 0}\frac{\Delta y}{\Delta\theta} = x \qquad \text{and} \qquad \frac{dx}{d\theta} = \lim_{\Delta\theta\to 0}\frac{\Delta x}{\Delta\theta} = -y,$$

or equivalently,

$$\frac{d}{d\theta}\sin\theta = \cos\theta \qquad \text{and} \qquad \frac{d}{d\theta}\cos\theta = -\sin\theta.$$

This geometric reasoning may be somewhat lacking in the precision of the traditional proofs of formulas (1) and (2) given at the beginning of this section. Nevertheless, it has the great merits of simplicity and directness.

PROBLEMS

In each of Problems 1–18, find the derivative dy/dx of the given function.

1 $y = \sin(3x - 2)$.

2 $y = \cos(1 - 7x)$.

3 $y = 3\sin 16x$.

4 $y = \sin^2 x$.

5 $y = \sin x^2$.

6 $y = \sin^2 6x$.

7 $y = 5 \sin 3x + 3 \cos 5x$.
8 $y = \sin^2 x + \cos^2 x$.
9 $y = x \sin x$.
10 $y = x^3 \sin 3x$.
11 $y = \sin^2 3x \cos^2 3x$.
12 $y = \cos^4 x - \sin^4 x$.
13 $y = \frac{1}{5} \sin^5 x - \frac{2}{3} \sin^3 x + \sin x$.
14 $y = \sin(\sin x)$.
15 $y = e^{2x} \sin 3x$.
16 $y = \sin(\ln x^2)$.
17 $y = \ln(\cos x)$.
18 $y = e^{x^2 + \sin x}$.
19 If a is a positive constant, verify that $y = c_1 \sin ax + c_2 \cdot \cos ax$ is a solution of the differential equation

$$\frac{d^2y}{dx^2} + a^2 y = 0$$

for every choice of the constants c_1 and c_2. (In the Additional Problems we outline a proof of the important fact that every solution of this differential equation has the stated form. For this reason, $y = c_1 \sin ax + c_2 \cdot \cos ax$ is called the *general solution* of the differential equation.)
20 Show that $(d/dx) \cos x = -\sin x$ by using the identity $\cos x = \sin(\pi/2 - x)$ and formula (6).
21 Find the angle at which the curve $y = \frac{1}{3} \sin 3x$ crosses the x-axis.
22 Sketch the graph of $y = \sin x + \cos x$ on the interval $0 \le x \le 2\pi$, and find the maximum height of this curve above the x-axis.
23 Find the maximum height of the curve $y = 4 \sin x - 3 \cos x$ above the x-axis.
24 Obtain the second of the following identities by differentiating the first: $\sin 2x = 2 \sin x \cos x$, $\cos 2x = \cos^2 x - \sin^2 x$.
25 Obtain the second of the following identities by differentiating the first: $\sin 3x = 3 \sin x - 4 \sin^3 x$, $\cos 3x = 4 \cos^3 x - 3 \cos x$.
26 Obtain the second of the following identities by differentiating the first with respect to either of the variables, keeping the other fixed:

$$\sin(x + y) = \sin x \cos y + \cos x \sin y,$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y.$$

27 Show that $y = \sin x$ and $y = \tan x$ have the same tangent at $x = 0$.
28 Show that the function $y = x + \sin x$ $(x \ge 0)$ has no maximum or minimum values even though there are many points where $dy/dx = 0$. Sketch the graph.
29 A regular polygon with n sides is inscribed in a circle of radius r.
(a) Show that the perimeter of this polygon is $p_n = 2nr \cdot \sin(\pi/n)$.
(b) Find $\lim_{n\to\infty} p_n$, and verify by elementary geometry that your answer is correct.
30 If a, b, c are constants with $ab \neq 0$, show that the graph of $y = a \sin(bx + c)$ is always concave toward the x-axis and that its points of inflection are the points where it crosses the x-axis.
31 Sketch the graphs of $y = \sin x$ and $y = \sin 2x$ together on a single set of axes. These curves have many points of intersection. Find the smallest positive x-coordinate of such a point, and calculate the acute angle at which the curves intersect at this point. Hint: See identity (17) in Section 9.1.
32 The functions $f(x) = \sin x$ and $g(x) = \cos x$ have the following properties: (a) $f'(x) = g(x)$; (b) $g'(x) = -f(x)$; (c) $f(0) = 0$; (d) $g(0) = 1$. If $F(x)$ and $G(x)$ is any pair of functions with the same properties, prove that $F(x) = \sin x$ and $G(x) = \cos x$. Hint: Show that

$$[F(x) - f(x)]^2 + [G(x) - g(x)]^2 = \text{a constant},$$

and find the value of this constant. [This problem has a very remarkable meaning: The functions $\sin x$ and $\cos x$ are *completely described* by properties (a) to (d), and therefore the total nature of these functions—everything that is now known about them or ever will be known—is implicitly contained in these four simple properties.]

In each of Problems 33–43, find the value of the indicated limit.

33 $\lim_{x\to 0} \frac{\tan x}{x}$.
34 $\lim_{x\to 0} \frac{\sin 3x}{x}$.
35 $\lim_{x\to 0} \frac{\tan x}{\sin x}$.
36 $\lim_{x\to 0} \frac{\sin 3x}{\sin 5x}$.
37 $\lim_{x\to 0} \tan 3x \csc 6x$.
38 $\lim_{x\to 0} \frac{1 - \cos x}{x^2}$.
39 $\lim_{x\to\infty} x \sin \frac{1}{x}$.
40 $\lim_{x\to\infty} 3x \tan \frac{\pi}{x}$.
41 $\lim_{x\to 0} \frac{2x^2 + 2x}{\sin 2x}$.
42 $\lim_{x\to 0} \sin 3x \cot 5x$.
43 $\lim_{x\to\pi/2} \frac{\cos x}{x - \pi/2}$.
44 The graph of $y = \cos\theta$ has an obvious maximum at the point corresponding to $\theta = 0$, so the tangent is horizontal there. Show that this fact is equivalent to the limit (5).

9.3 THE INTEGRALS OF THE SINE AND COSINE. THE NEEDLE PROBLEM

The differential versions of formulas (7) and (6) in the previous section are

$$d(\cos u) = -\sin u\, du \qquad \text{and} \qquad d(\sin u) = \cos u\, du.$$

These immediately yield the integration formulas

$$\int \sin u\, du = -\cos u + c \tag{1}$$

and

$$\int \cos u\, du = \sin u + c. \tag{2}$$

Example 1 Evaluate $\int \cos 5x\, dx$.

Solution Let $u = 5x$. Then $du = 5\, dx$, $dx = \frac{1}{5}\, du$, and formula (2) gives

$$\int \cos 5x\, dx = \int \cos u \cdot (\tfrac{1}{5}\, du) = \tfrac{1}{5} \int \cos u\, du$$
$$= \tfrac{1}{5} \sin u + c = \tfrac{1}{5} \sin 5x + c.$$

After a little practice, it will be easy for students to make this kind of substitution mentally. In fact, we can dispense with the new variable u altogether, and compress this solution to the following simple steps:

$$\int \cos 5x\, dx = \tfrac{1}{5} \int \cos 5x\, d(5x) = \tfrac{1}{5} \sin 5x + c.$$

Example 2 Evaluate $\int 7x \sin (2 - 9x^2)\, dx$.

Solution Let $u = 2 - 9x^2$. Then $du = -18x\, dx$, $x\, dx = -\frac{1}{18}\, du$, and

$$\int 7x \sin (2 - 9x^2)\, dx = \int 7 \sin u \cdot (-\tfrac{1}{18}\, du)$$
$$= -\tfrac{7}{18} \int \sin u\, du$$
$$= \tfrac{7}{18} \cos u + c = \tfrac{7}{18} \cos (2 - 9x^2) + c.$$

Here the auxiliary variable u plays an important part in our work. It not only emphasizes the need to apply formula (1), but also helps us keep track of the various coefficients and algebraic signs that appear in the calculation—and therefore helps us avoid mistakes.

Example 3 Compute the definite integral

$$\int_{\pi/6}^{\pi/4} \frac{\cos 2x\, dx}{\sin^3 2x}.$$

Solution We begin by finding the indefinite integral. Since $d(\sin 2x) = 2 \cos 2x\, dx$, we put $u = \sin 2x$. This gives $du = 2 \cos 2x\, dx$, so

$$\int \frac{\cos 2x\, dx}{\sin^3 2x} = \int \frac{\frac{1}{2}\, du}{u^3} = \frac{1}{2}\int u^{-3}\, du = -\frac{1}{4}u^{-2} = -\frac{1}{4 \sin^2 2x}.$$

We remind students that the constant of integration can always be ignored in computing definite integrals, and for this reason we don't bother to write it here. The Fundamental Theorem of Calculus now permits us to complete the solution by writing

$$\int_{\pi/6}^{\pi/4} \frac{\cos 2x\,dx}{\sin^3 2x} = -\frac{1}{4\sin^2 2x}\bigg]_{\pi/6}^{\pi/4} = -\frac{1}{4} - \left(-\frac{1}{3}\right) = \frac{1}{12}.$$

In our next example we discuss an application of these methods to a famous problem about probability that was invented by the French scientist Buffon in the early eighteenth century.

Example 4 *Buffon's needle problem.* A needle 2 in long is tossed at random onto a floor made of boards 2 in wide. What is the probability that the needle falls across one of the cracks?

Solution We begin with a brief digression to explain our use of the word "probability." In mathematics this word means a numerical measure of the likelihood that a certain event will occur. As an example, consider the rectangle shown on the left in Fig. 9.14, in which a portion of the figure is shaded. If a point is chosen at random in this rectangle, for instance by making the rectangle into a target and throwing a dart blindfolded, then the probability of choosing a shaded point is $\frac{1}{4}$. We assume here that each point is just as likely to be chosen as any other, and this number expresses the fact that the proportion of shaded points among all points in the rectangle is $\frac{1}{4}$. In the second rectangle the probability of choosing a shaded point is $\frac{1}{8}$, and in the third rectangle it is $\frac{3}{8}$. We take it as self-evident that the probability of choosing a shaded point equals the ratio of the shaded area to the total area.

Let us now return to the needle problem. We describe the position in which the needle falls on the floor by the two variables x and θ shown in Fig. 9.15; x is the distance OP from the midpoint of the needle to the nearest crack, and θ is the smallest angle between OP and the needle. A toss of the needle amounts to a random choice of the variables x and θ in the intervals

$$0 \le x \le 1 \qquad \text{and} \qquad 0 \le \theta \le \frac{\pi}{2}, \tag{3}$$

and this in turn amounts to a random choice of a point in the rectangle shown in Fig. 9.16. Furthermore, a close inspection of Fig. 9.15 shows that the event we are interested in, namely, that the needle falls across a crack, corresponds to the inequality

$$x < \cos\theta. \tag{4}$$

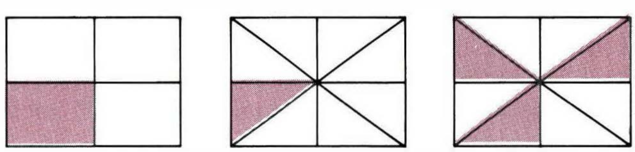


Figure 9.14

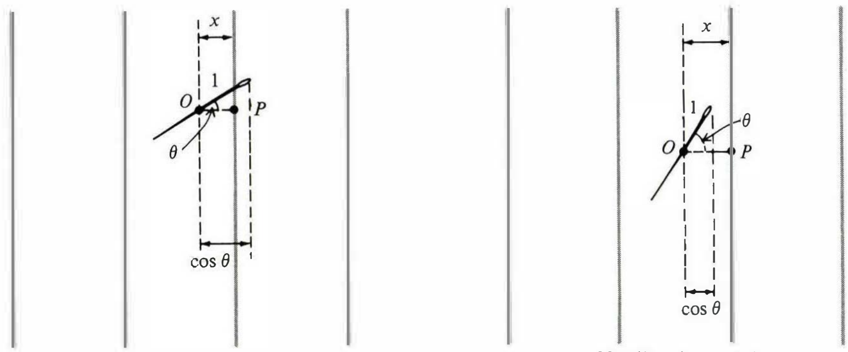


Figure 9.15

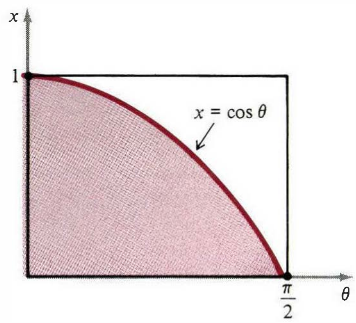


Figure 9.16

This inequality describes the shaded region in Fig. 9.16 under the graph of $x = \cos\,\theta$. We therefore conclude that the probability of the needle falling across a crack equals the following ratio of areas:

$$\frac{\text{area under curve}}{\text{area of rectangle}} = \frac{\int_0^{\pi/2} \cos\,\theta\,d\theta}{\pi/2} = \frac{1}{\pi/2} = \frac{2}{\pi}, \tag{5}$$

which is slightly less than $\frac{2}{3}$. This calculation can be extended at once to the more general situation in which d is the distance between adjacent cracks and the length of the needle is $L \leq d$. The inequalities (3) are replaced by

$$0 \leq x \leq \frac{1}{2}d \qquad \text{and} \qquad 0 \leq \theta \leq \frac{\pi}{2},$$

and (4) becomes

$$x < \tfrac{1}{2}L\,\cos\,\theta.$$

In this case the probability of the needle falling across a crack is easily seen to be

$$\frac{\text{area under curve}}{\text{area of rectangle}} = \frac{\int_0^{\pi/2} \frac{1}{2}L\,\cos\,\theta\,d\theta}{(\frac{1}{2}d)(\pi/2)} = \frac{2L}{\pi d}. \tag{6}$$

(Students should draw their own sketch for this case similar to Fig. 9.16, and in particular should notice the reason for the restriction $L \leq d$.)

Remark We obtained these conclusions about the probability of success in the needle experiment by pure reason alone, without any appeal to experience. However, the "sequence of trials" approach to the concept of probability has some interesting implications for the needle problem. In the case of the 2-in needle and the 2-in floorboards, let us actually perform the experiment of tossing the needle onto the floor a large number of times, say n times, where $n = 100$ or $n = 1000$ depending on our ability to tolerate boredom. Let us also keep careful count of the number k of times the needle falls across a crack. Then the abstract probability that the needle falls across a crack on any one toss should be closely approximated by the ratio k/n, and this approximation should improve as n increases. Roughly speaking, this means that

$$\lim_{n\to\infty} \frac{k}{n} = \frac{2}{\pi},$$

so we should have

$$\frac{k}{n} \cong \frac{2}{\pi};$$

and solving this approximate equation for π yields

$$\pi \cong \frac{2n}{k}$$

for large values of n. In principle, therefore, this provides an experimental method of calculating π. In fact, however, this method is not capable of much accuracy because of the inherent errors that appear in all measurements. We will discuss practical methods for computing π to very great accuracy in Chapter 13.

PROBLEMS

Evaluate the indefinite integrals in Problems 1–20.

1 $\int \sin 5x \, dx$.

2 $\int \cos (2x - 5) \, dx$.

3 $\int \sin (1 - 9x) \, dx$.

4 $\int (3 \cos 2x - 2 \sin 3x) \, dx$.

5 $\int 2 \sin x \cos x \, dx$.

6 $\int \cos^2 x \sin x \, dx$.

7 $\int \sin^3 2x \cos 2x \, dx$.

8 $\int \sin x \cos x (\sin x + \cos x) \, dx$.

9 $\int \sin^7 \frac{1}{2}x \cos \frac{1}{2}x \, dx$.

10 $\int 4x \sin x^2 \, dx$.

11 $\displaystyle\int \frac{\sin \sqrt{x} \, dx}{\sqrt{x}}$.

12 $\displaystyle\int \frac{\cos (\ln x) \, dx}{x}$.

13 $\int \cos (\sin 2x) \cos 2x \, dx$.

14 $\displaystyle\int \frac{\cos x \, dx}{\sin^2 x}$.

15 $\displaystyle\int \frac{\sin [(2x - 1)/3] \, dx}{\cos^2 [(2x - 1)/3]}$.

16 $\displaystyle\int \frac{\cos x \, dx}{\sin x}$.

17 $\displaystyle\int \frac{\sin x \, dx}{\cos x}$.

18 $\displaystyle\int \frac{\cos 3x \, dx}{\sqrt{\sin 3x}}$.

19 $\int (2x + 1) \cos (x^2 + x) \, dx$.

20 $\int (x + \cos x)^4 (1 - \sin x) \, dx$.

Evaluate the definite integrals in Problems 21–24.

21 $\displaystyle\int_0^{\pi/5} \sin 5x \, dx$.

22 $\displaystyle\int_{-\pi/6}^{2\pi/3} \cos 3x \, dx$.

23 $\displaystyle\int_{\pi/4}^{\pi/2} \frac{\cos x \, dx}{\sin^2 x}$.

24 $\displaystyle\int_0^{\sqrt{\pi}} x \cos x^2 \, dx$.

25 Find the area under one arch of $y = \sin 3x$.

26 In the first quadrant, the y-axis and the curves $y = \sin x$ and $y = \cos x$ bound a "triangle-shaped" region. Find its area.

27 Find the area under one arch of $y = 3 \cos 2x$.

28 Find the area under one arch of $y = 6 \sin \frac{1}{2}x$ and above the line $y = 3$.

29 Find the volume generated by revolving about the x-axis the region under $y = \sin x$ and between $x = 0$ and $x = \pi$. Hint: Remember the half-angle formula $2 \sin^2 x = 1 - \cos 2x$.

30 Consider the region between $y = \sin x$ and the x-axis for $0 \le x \le \pi/2$. For what constant c does the line $x = c$ divide this region into two parts of equal areas?

31 Anticipate the results of the next section by deriving the following differentiation formulas:

$$\frac{d}{dx} \tan x = \sec^2 x;$$

$$\frac{d}{dx} \cot x = -\csc^2 x;$$

$$\frac{d}{dx} \sec x = \sec x \tan x;$$

$$\frac{d}{dx} \csc x = -\csc x \cot x.$$

Hint: Express each function in terms of $\sin x$ and $\cos x$.

32 Obtain the following integration formulas from the differentiation formulas in Problem 31:

$$\int \sec^2 x \, dx = \tan x + c;$$

$$\int \csc^2 x \, dx = -\cot x + c;$$

$$\int \sec x \tan x \, dx = \sec x + c;$$

$$\int \csc x \cot x \, dx = -\csc x + c.$$

9.4 THE DERIVATIVES OF THE OTHER FOUR FUNCTIONS

The results of Problem 31 in Section 9.3 enable us to complete our list of formulas for differentiating the trigonometric functions:

$$\frac{d}{dx}\tan u = \sec^2 u\,\frac{du}{dx}; \tag{1}$$

$$\frac{d}{dx}\cot u = -\csc^2 u\,\frac{du}{dx}; \tag{2}$$

$$\frac{d}{dx}\sec u = \sec u\tan u\,\frac{du}{dx}; \tag{3}$$

$$\frac{d}{dx}\csc u = -\csc u\cot u\,\frac{du}{dx}. \tag{4}$$

These formulas are quite easy to remember if we notice that the derivative of each cofunction (cot, csc) can be obtained from the derivative of the corresponding function (tan, sec) by (a) inserting a minus sign, and (b) replacing each function by its cofunction. Thus, formula (2) is obtained from formula (1) by inserting a minus sign, replacing $\tan u$ by $\cot u$, and replacing $\sec u$ by $\csc u$. In view of this rule, it is only necessary to memorize formulas (1) and (3), because the rule immediately produces the other two.

Example 1 Find dy/dx if $y = \tan^3 4x$.

Solution Since $y = \tan^3 4x = (\tan 4x)^3$, the power rule gives

$$\frac{dy}{dx} = 3(\tan 4x)^2 \cdot \frac{d}{dx}\tan 4x.$$

By formula (1) with $u = 4x$,

$$\frac{d}{dx}\tan 4x = (\sec^2 4x)(4),$$

and by putting the various pieces together we obtain

$$\frac{dy}{dx} = 12\tan^2 4x\sec^2 4x.$$

Example 2 Find dy/dx if $y = \cot(1 - 3x)$.

Solution By formula (2) with $u = 1 - 3x$,

$$\frac{dy}{dx} = -\csc^2(1 - 3x)\cdot(-3) = 3\csc^2(1 - 3x).$$

The differentiation formulas (1) to (4) immediately produce four new integration formulas:

$$\int \sec^2 u\,du = \tan u + c; \tag{5}$$

$$\int \csc^2 u\,du = -\cot u + c; \tag{6}$$

$$\int \sec u\tan u\,du = \sec u + c; \tag{7}$$

$$\int \csc u \cot u \, du = -\csc u + c. \tag{8}$$

Example 3 Calculate $\int \sec 3x \tan 3x \, dx$.

Solution This reminds us of (7) with $u = 3x$, so we write

$$\int \sec 3x \tan 3x \, dx = \frac{1}{3} \int \sec 3x \tan 3x \, d(3x) = \frac{1}{3} \sec 3x + c.$$

In this problem the structure of the integral is clear enough so that there is no real need to make an explicit change of variable.

Example 4 Evaluate $\int 3x \sec^2 x^2 \, dx$.

Solution This reminds us of (5) with $u = x^2$. Since $du = 2x\,dx$ and $x\,dx = \frac{1}{2}du$, we have

$$\int 3x \sec^2 x^2 \, dx = 3 \int \sec^2 u \cdot \left(\frac{1}{2} du\right) = \frac{3}{2} \int \sec^2 u \, du$$
$$= \tfrac{3}{2} \tan u + c = \tfrac{3}{2} \tan x^2 + c.$$

Here we use the auxiliary variable u as insurance against error. After students have acquired a bit of experience with problems of this type, they will prefer to carry out the integration directly, by inspection.

Example 5 Calculate $\int \tan^2 2x \, dx$.

Solution This integral doesn't resemble any of our types. However, the trigonometric identity $\tan^2 2x + 1 = \sec^2 2x$ connects our problem with formula (5). Once this fact is noticed, we easily write

$$\int \tan^2 2x \, dx = \int (\sec^2 2x - 1) \, dx = \int \sec^2 2x \, dx - \int dx$$
$$= \frac{1}{2} \int \sec^2 2x \, d(2x) - \int dx = \frac{1}{2} \tan 2x - x + c.$$

PROBLEMS

In each of Problems 1–12, calculate dy/dx.

1 $y = \tan 4x^2$.
2 $y = \cot 4x$.
3 $y = \tan^2 (\sin x)$.
4 $y = 3 \cot (1 - x^3)$.
5 $y = \sec^2 x - \tan^2 x$.
6 $y = 2 \sec 3x$.
7 $y = 4 \csc (-6x)$.
8 $y = (\cot x + \csc x)^2$.
9 $y = \sqrt{\csc 2x}$.
10 $y = \cot (\cos x)$.
11 $y = e^{\tan x}$.
12 $y = \ln (\csc x)$.

Evaluate the integral in each of Problems 13–20.

13 $\int \csc^2 6x \, dx$.
14 $\int_0^{\pi/8} \sec^2 2x \, dx$.
15 $\int \frac{dx}{\sin^2 2x}$.
16 $\int \sec^2 \frac{1}{3}x \, dx$.
17 $\int \tan^4 x \sec^2 x \, dx$.
18 $\int_0^{\pi/6} \sec 2x \tan 2x \, dx$.
19 $\int \cot 7x \csc 7x \, dx$.
20 $\int \sec^7 x \tan x \, dx$.
21 Find the area bounded by the curve $y = \tan x \sec^2 x$, the x-axis, and the line $x = \pi/4$.
22 Find the area in the first quadrant bounded by $y = \sec^2 x$, $y = 8 \cos x$, and the y-axis.

23 Find the area in the first quadrant bounded by $y = \sec^2 x$, $y = 2\tan^2 x$, and the y-axis.

24 The region bounded by the curve $y = \tan x$, the x-axis, and the line $x = \pi/3$ is revolved about the x-axis. Find the volume of the solid of revolution generated in this way.

25 Sketch the graph of the function $y = \tan x + \cot x$ on the interval $0 < x < \pi/2$ and find its minimum value.

26 Solve Problem 25 without calculus, by using the identity

$$\tan x + \cot x = \frac{2}{\sin 2x}.$$

***27** Sketch the graph of the function $y = 8 \csc x - 4 \cot x$ on the interval $0 < x \le \pi/2$ and find its minimum value. Is there a point of inflection?

***28** The classic corridor problem (Problem 29 in Section 4.3) can be expressed as follows. If two corridors of widths a and b meet at right angles (Fig. 9.17), then the

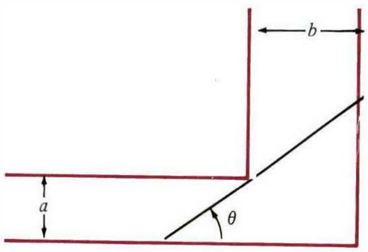


Figure 9.17

length of the longest thin rod that can be moved in a horizontal position around the corner is the length of the shortest line segment placed like the one in the figure. Find this length by using the angle θ as the independent variable.

29 A revolving light 6 mi offshore from a straight shoreline makes 4 revolutions per minute. How fast is the spot of light moving along the shore at the instant when the beam makes an angle of 30° with the shoreline?

***30** A rope with a ring at one end is looped over two pegs in a horizontal line. The free end is passed through the ring and has a weight suspended from it, so that the rope is held taut. If the rope slips freely through the ring and over the pegs, then the weight will descend as far as possible in order to minimize its potential energy. Find the angle formed at the bottom of the loop.

31 (Arterial branching) In the flow of blood through a human artery as discussed in Example 3 of Section 7.4, the physical resistance to the flow is called *vascular resistance.* This quantity is denoted by R and defined to be the ratio of the driving pressure P to the flow F:

$$R = \frac{P}{F}.^{\dagger}$$

By rearranging Poiseuille's law in the example referred to, we find that

$$R = \frac{8\eta L}{\pi r^4} = k\frac{L}{r^4},$$

where L and r are the length and radius of the artery and $k = 8\eta/\pi$ is a constant determined by the viscosity η of the blood. Figure 9.18 shows an artery with radius r_1 branching into a smaller artery with radius r_2.

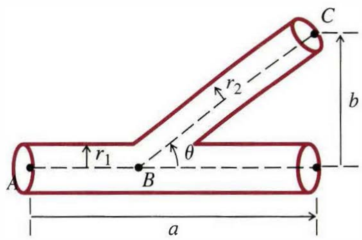


Figure 9.18

(a) Show that the total resistance along the path ABC is

$$R = k\frac{a - b\cot\theta}{r_1^4} + k\frac{b\csc\theta}{r_2^4}.$$

(b) Show that the resistance along the path ABC is minimized when

$$\cos\theta = \left(\frac{r_2}{r_1}\right)^4.$$

(c) In part (b), find the value of θ to the nearest degree if the radius of the branch artery is four-fifths the radius of the larger artery.

† For students who know a bit about electric circuits, we point out that this concept of vascular resistance, when rewritten in the form $F = P/R$, is precisely analogous to Ohm's law $I = E/R$, which relates the current I in a circuit to the electromotive force E and the electrical resistance R.

9.5 THE INVERSE TRIGONOMETRIC FUNCTIONS

Our attention in this section is focused on the two integration formulas

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x \tag{1}$$

and

$$\int \frac{dx}{1+x^2} = \tan^{-1} x. \tag{2}$$

The unfamiliar functions on the right sides of these equations will be fully explained below. They are called *inverse trigonometric functions*, and are created expressly to enable us to calculate the integrals on the left. These functions have other uses, but this is their primary purpose, the main justification for their existence.

Before we start at the beginning and give a careful and orderly description of these functions, we pause briefly to understand in a rough way how they originate. The difficulty with the integral on the left of (1) is caused by the awkward expression $\sqrt{1-x^2}$ in the denominator. If we consider this obstacle for a moment, the inside quantity $1 - x^2$ might make us think of the trigonometric expression $1 - \sin^2 \theta$, which of course equals $\cos^2 \theta$. Thus, if we write

$$x = \sin \theta, \tag{3}$$

then we have $\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = \cos\theta$, and the square root sign disappears. But we also have $dx = \cos\theta \, d\theta$, so we can unravel our troublesome integral as follows:

$$\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{\cos\theta \, d\theta}{\cos\theta} = \int d\theta = \theta. \tag{4}$$

The process of solving (3) for θ in terms of x is symbolized by writing $\theta = \sin^{-1} x$, so (4) yields (1). A similar analysis can be applied to (2), but these remarks are perhaps enough to make our point about the way the inverse trigonometric functions arise—they are forced upon us by the need to calculate certain integrals. Now for the details that make these functions respectable.

THE INVERSE SINE

We know that $\sin \pi/6 = \frac{1}{2}$. Thus, if we are asked to find an angle (in radian measure) whose sine is $\frac{1}{2}$, we can answer at once that $\pi/6$ is such an angle. We are also aware that there are many other angles with this property.

As we have just seen, it is necessary in calculus to have a symbol to denote an angle whose sine is a given number x. There are two such symbols in everyday use,

$$\sin^{-1} x \qquad \text{and} \qquad \arcsin x.$$

These notations are fully equivalent to each other and can be used interchangeably, though we shall confine ourselves to the first. The first is read "the inverse sine of x," and the second "the arc sine of x," and both mean "an angle whose sine is x." It is essential to understand that in the symbol $\sin^{-1} x$, the -1 is *not* an exponent, and therefore $\sin^{-1} x$ *never* means $1/(\sin x)$. We discuss the reason for this seemingly strange notation in Remark 2.

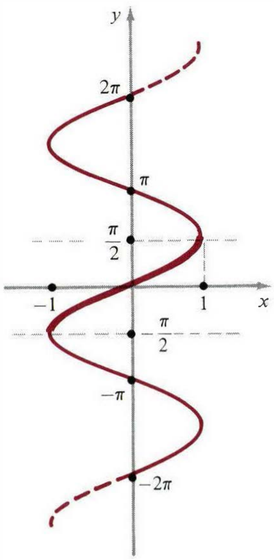


Figure 9.19

These ideas can be summarized as follows: The formulas

$$x = \sin y \qquad \text{and} \qquad y = \sin^{-1} x$$

mean exactly the same thing, in the sense that

$$x = 3y \qquad \text{and} \qquad y = \tfrac{1}{3}x$$

mean exactly the same thing. In each case the equation is first written in a form solved for x, and then (the same equation!) in a form solved for y.

In order to sketch the graph of $y = \sin^{-1} x$, it suffices to sketch $x = \sin y$ with y treated as the independent variable—on the horizontal axis—and then to turn the picture over, returning the axes to their customary positions (Fig. 9.19). It is clear that y exists only when x lies in the interval $-1 \le x \le 1$. However, for any such x there are infinitely many corresponding y's, and this situation cannot be allowed if $y = \sin^{-1} x$ is to be considered a function. (Recall that a function is single-valued by the very definition of the concept.) We deal with this difficulty by means of a universally understood agreement: The only values of $y = \sin^{-1} x$ we consider are those that lie in the interval $-\pi/2 \le y \le \pi/2$, and this restriction is henceforth part of the meaning of the symbol $y = \sin^{-1} x$. The graph of the function $y = \sin^{-1} x$ (it is truly a function now, because of the restriction just described) is the heavy portion of the curve in Fig. 9.19.

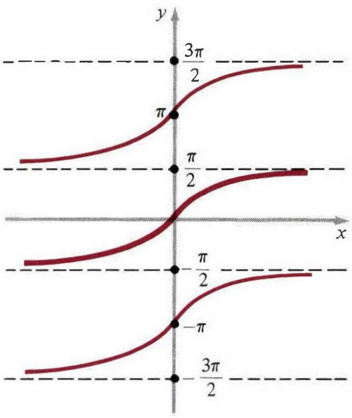


Figure 9.20

THE INVERSE TANGENT

The function $y = \tan^{-1} x$ (the other notation here is $y = \arctan x$) is defined in essentially the same way:

$$y = \tan^{-1} x \qquad \text{means} \qquad x = \tan y \qquad \text{and} \qquad -\frac{\pi}{2} < y < \frac{\pi}{2}.$$

The symbol $\tan^{-1} x$ is read "the inverse tangent of x," and it means "the angle (in the specified interval) whose tangent is x." The graph of the function $y = \tan^{-1} x$ is the heavy curve in Fig. 9.20.

We now calculate the derivative dy/dx of the function $y = \sin^{-1} x$ by differentiating

$$x = \sin y$$

implicitly with respect to x. The result is

$$1 = \cos y \frac{dy}{dx},$$

so

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - \sin^2 y}} = \frac{1}{\sqrt{1 - x^2}}.$$

We choose the positive square root here because $y = \sin^{-1} x$ is clearly an increasing function (see Fig. 9.19). This result can be written in the form

$$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1 - x^2}}, \tag{5}$$

where $-1 < x < 1$. In just the same way we find the derivative of $y = \tan^{-1} x$ by differentiating

$$x = \tan y$$

implicitly with respect to x. This gives

$$1 = \sec^2 y \frac{dy}{dx},$$

so

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2}.$$

We therefore have

$$\frac{d}{dx} \tan^{-1} x = \frac{1}{1 + x^2} \tag{6}$$

for all x.

Formulas (5) and (6) are the facts that lead to the main tools of this section. First, we have the chain rule extensions of these formulas, which greatly broaden their scope:

$$\frac{d}{dx} \sin^{-1} u = \frac{1}{\sqrt{1 - u^2}} \frac{du}{dx} \tag{7}$$

and

$$\frac{d}{dx} \tan^{-1} u = \frac{1}{1 + u^2} \frac{du}{dx}. \tag{8}$$

As usual, u is understood to be any differentiable function of x.

Example 1 Find dy/dx for each of the following functions:
(a) $y = \sin^{-1} 4x$; (b) $y = \sin^{-1} x^3$; (c) $y = \frac{1}{3} \tan^{-1} (3x - 5)$.

Solution For (a), we use (7) with $u = 4x$, so

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - (4x)^2}} \frac{d}{dx}(4x) = \frac{4}{\sqrt{1 - 16x^2}}.$$

For (b), we use (7) with $u = x^3$, so

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - (x^3)^2}} \frac{d}{dx}(x^3) = \frac{3x^2}{\sqrt{1 - x^6}}.$$

For (c), we use (8) with $u = 3x - 5$, so

$$\frac{dy}{dx} = \frac{1}{3} \frac{1}{1 + (3x - 5)^2} \frac{d}{dx}(3x - 5) = \frac{1}{1 + (3x - 5)^2}.$$

Even more important for our future work are the integration formulas equivalent to (7) and (8):

$$\int \frac{du}{\sqrt{1 - u^2}} = \sin^{-1} u + c \tag{9}$$

and

$$\int \frac{du}{1+u^2} = \tan^{-1} u + c. \tag{10}$$

These formulas are indispensable tools for integral calculus, and all by themselves amply justify the study of trigonometry.

Example 2 Calculate each of the following integrals:

(a) $\int \frac{dx}{\sqrt{1-9x^2}}$; (b) $\int \frac{dx}{1+25x^2}$; (c) $\int \frac{5x^2dx}{1+4x^6}$.

Solution (a) Put $u = 3x$. Then $du = 3dx$, and by (9)

$$\int \frac{dx}{\sqrt{1-9x^2}} = \int \frac{\frac{1}{3}du}{\sqrt{1-u^2}} = \frac{1}{3}\sin^{-1} u + c = \frac{1}{3}\sin^{-1} 3x + c.$$

(b) Put $u = 5x$. Then $du = 5dx$, and by (10)

$$\int \frac{dx}{1+25x^2} = \int \frac{\frac{1}{5}du}{1+u^2} = \frac{1}{5}\tan^{-1} u + c = \frac{1}{5}\tan^{-1} 5x + c.$$

(c) To get started here we must notice that $4x^6 = (2x^3)^2$. This suggests putting $u = 2x^3$. Then $du = 6x^2\,dx$, and by (10)

$$\int \frac{5x^2\,dx}{1+4x^6} = 5\int \frac{\frac{1}{6}du}{1+u^2} = \frac{5}{6}\tan^{-1} u + c = \frac{5}{6}\tan^{-1} 2x^3 + c.$$

The crucial feature of this integral is clearly the presence of x^2 in the numerator, for without this factor the method we have used would be unworkable.

Remark 1 As students doubtless suspect, four other inverse trigonometric functions can be defined if we wish to do so. However, these functions are not really needed for the purpose of integration. We can illustrate this point by observing that if $u > 0$ then

$$\int \frac{du}{u\sqrt{u^2-1}} = \int \frac{du}{u\sqrt{u^2(1-1/u^2)}} = \int \frac{du}{u^2\sqrt{1-(1/u)^2}}$$
$$= -\int \frac{d(1/u)}{\sqrt{1-(1/u)^2}} = -\sin^{-1}\frac{1}{u} + c.$$

(If $u < 0$, the factor u^2 under the radical in the second step comes out of the radical as $-u$.) This integral is a standard type that many writers integrate by using the inverse secant—which this calculation clearly shows to be superfluous. The fact of the matter is that the inverse sine and the inverse tangent suffice for all our purposes for calculating integrals, so for the sake of simplicity we ignore the other inverse trigonometric functions. (The notation $\cos^{-1} x$ will occasionally be used, but only for convenience in designating the angle between 0 and π whose cosine is x, where x is a number between 1 and -1.)

Remark 2 Suppose that a variable x is a function of a variable y as shown on the left in Fig. 9.21. In this case, not only does each y (in a certain interval) de-

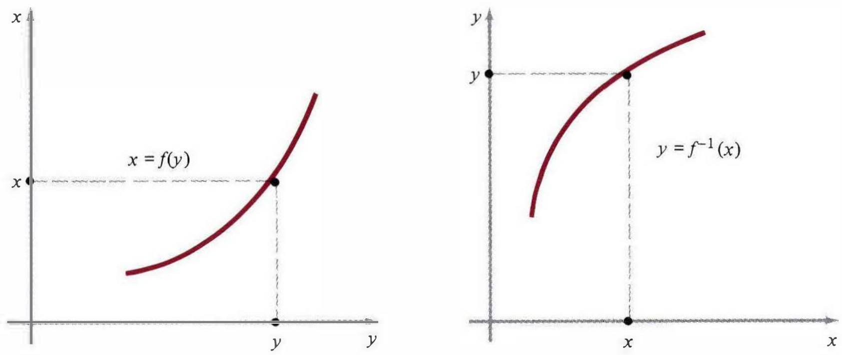


Figure 9.21

termine a unique x, but also each x determines a unique y. Thus, y is also a function of x. If the given function is written $x = f(y)$, then the second function is often called the *inverse function* of the first and denoted by the symbol $y = f^{-1}(x)$ [read "f inverse of x"]. The graph of $y = f^{-1}(x)$ is simply the graph of $x = f(y)$ turned over as shown on the right in Fig. 9.21, so that the axes are returned to their normal positions. The essence of this situation is that when two functions are related in this way, then each undoes what the other one does, in the sense that

$$f^{-1}(f(y)) = y \qquad \text{and} \qquad f(f^{-1}(x)) = x.$$

It is this reciprocal relation that is suggested by the word "inverse" and the symbol f^{-1}. We have encountered inverse functions in Chapter 8 and also in this section, but we have no special need to develop the subject in detail. We do point out, however, that any increasing or decreasing function $x = f(y)$ obviously has an inverse; and it can be proved that if either function has a nonzero derivative at a point, then so does the other and

$$\frac{dy}{dx} = \frac{1}{dx/dy}.$$

Here again—as in the case of the chain rule—we have a situation in which the Leibniz fractional notation for derivatives strongly suggests a true theorem in the guise of a simple manipulation of differentials.

Remark 3 Formula (10) leads rather quickly (though unrigorously) to the famous Leibniz formula

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots, \tag{11}$$

which connects the number π with the odd numbers 1, 3, 5, 7, ... To see this connection, we begin with the formula from elementary algebra for the sum of a geometric series,

$$1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r}. \tag{12}$$

(The reader will perhaps recall from high school algebra that this formula is valid for $|r| < 1$, but here we pay little attention to such cautionary details.) If r in (12) is replaced by $-t^2$ and the resulting equation is reversed, we get

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + \cdots. \tag{13}$$

We now apply (10) to obtain

$$\tan^{-1} x = \int_0^x \frac{dt}{1+t^2} = \int_0^x [1 - t^2 + t^4 - t^6 + \cdots]\, dt$$
$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots,$$

which yields Leibniz's formula (11) when $x = 1$. These ideas and the legitimacy of these procedures will be studied much more carefully in Chapters 13 and 14.

PROBLEMS

1 Given that $\theta = \sin^{-1}(-\frac{1}{2})$, find $\cos\theta$, $\tan\theta$, $\cot\theta$, $\sec\theta$, $\csc\theta$.

2 Given that $\theta = \tan^{-1}\sqrt{3}$, find $\sin\theta$, $\cos\theta$, $\cot\theta$, $\sec\theta$, $\csc\theta$.

3 Evaluate each of the following:
(a) $\sin^{-1} 1 - \sin^{-1}(-1)$; (b) $\tan^{-1} 1 - \tan^{-1}(-1)$;
(c) $\sin(\sin^{-1} 0.123)$; (d) $\cos(\sin^{-1} 0.6)$;
(e) $\sin(2\sin^{-1} 0.6)$; (f) $\tan^{-1}(\tan \pi/7)$;
(g) $\sin^{-1}(\sin 5\pi/6)$; (h) $\tan^{-1}(\tan[-3\pi/4])$.

Find dy/dx in each of Problems 4–13.

4 $y = \sin^{-1}\frac{1}{2}x$.

5 $y = \frac{1}{5}\tan^{-1}\frac{1}{5}x$.

6 $y = \frac{1}{2}\tan^{-1} x^2$.

7 $y = \sin^{-1}\dfrac{x-1}{x+1}$.

8 $y = \tan^{-1}\dfrac{x-1}{x+1}$.

9 $y = x\sin^{-1} x + \sqrt{1-x^2}$.

10 $y = x\tan^{-1} x - \ln\sqrt{1+x^2}$.

11 $y = x(\sin^{-1} x)^2 - 2x + 2\sqrt{1-x^2}\sin^{-1} x$.

12 $y = \frac{1}{2}(\sin^{-1} x + x\sqrt{1-x^2})$.

13 $y = \tan^{-1}\dfrac{4\sin x}{3+5\cos x}$.

14 If a is a positive constant, show that

$$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} + c$$

and

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + c.$$

These simple generalizations of formulas (9) and (10) are often more convenient in applications.

Evaluate the integrals in Problems 15–25.

15 $\displaystyle\int_0^{1/2} \frac{dx}{\sqrt{1-x^2}}$.

16 $\displaystyle\int_{-1}^{1} \frac{dx}{1+x^2}$.

17 $\displaystyle\int \frac{dx}{\sqrt{1-4x^2}}$.

18 $\displaystyle\int \frac{dx}{1+3x^2}$.

19 $\displaystyle\int_0^{1/2} \frac{dx}{1+4x^2}$.

20 $\displaystyle\int \frac{x\,dx}{1+4x^4}$.

21 $\displaystyle\int \frac{dx}{\sqrt{9-4x^2}}$.

22 $\displaystyle\int \frac{dx}{\sqrt{16-9x^2}}$.

23 $\displaystyle\int \frac{dx}{4+9x^2}$.

24 $\displaystyle\int_{\sqrt{2}}^{2} \frac{dx}{x\sqrt{x^2-1}}$.

25 $\displaystyle\int_{-2}^{-\sqrt{2}} \frac{dx}{x\sqrt{x^2-1}}$.

26 A picture hangs on a wall with its base a feet above the level of an observer's eye. If the picture is b feet high and the observer stands x feet from the wall, show that the angle θ subtended by the picture is given by the formula

$$\theta = \tan^{-1}\frac{a+b}{x} - \tan^{-1}\frac{a}{x}.$$

What value of x maximizes this angle?

27 The points (1, 2) and (2, 1) in the first quadrant are joined by two segments to a variable point $(0, y)$ on the y-axis, where $y < 3$. If θ denotes the angle between these segments, what is the largest value θ can have?

28 A balloon is released at eye level and rises at the rate of 5 ft/s. An observer 50 ft away watches the balloon rise. How fast is the angle of elevation increasing 6 seconds after the moment of release?

29 The top of a 15-ft ladder is sliding down a wall. When the base of the ladder is 9 ft from the wall, it is sliding away at the rate of 3 ft/s. (a) What is the angle between the wall and the ladder at that moment? (b) How fast is the angle increasing at that moment?

30 Sketch the curve $y = 1/(1+x^2)$. Find the area of the region under this curve between $x = 0$ and $x = b$, where b is a positive constant. Find the limit of this area as $b \to \infty$.

31 Comment on the legitimacy of the formula

$$\int_0^3 \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} 3.$$

32 Sketch the curve $y = 1/\sqrt{1-x^2}$ on the interval $0 \le x < 1$. Find the area under this curve between $x = 0$ and $x = b$ where $0 < b < 1$. Find the limit of this area as $b \to 1$.

33 The circle $x^2 + y^2 = a^2$ is revolved about the line $y = a$. Find the area of the resulting surface of revolution.

9.6 SIMPLE HARMONIC MOTION. THE PENDULUM

Most people understand that sound is vibration, and for this reason alone the study of vibrations is an important part of science. But vibrations—or oscillations, or waves, or periodic phenomena generally—are much more pervasive than this. They appear in many contexts having little to do with sound, for instance in connection with radio waves, light waves, alternating electric currents, the vibration of atoms in crystals, etc. The study of vibrations in this broader sense is clearly one of the most fundamental themes of physical science, and in any such study sines and cosines play a central role.

One of the simplest types of vibrations occurs when an object or point moves back and forth along a straight line (the x-axis) in such a way that its acceleration is always proportional to its position and is directed in the opposite sense:

$$\frac{d^2x}{dt^2} = -kx, \qquad k > 0. \tag{1}$$

Motion of this kind is called *simple harmonic motion.* To emphasize that the constant k is positive, it is customary to write $k = a^2$ with $a > 0$. The differential equation (1) then takes the form

$$\frac{d^2x}{dt^2} + a^2x = 0. \tag{2}$$

It is easy to see that any function of the form

$$x = A \sin(at + b), \qquad A \neq 0, \tag{3}$$

satisfies equation (2).* We merely calculate

$$\frac{dx}{dt} = Aa\cos(at+b) \qquad \text{and} \qquad \frac{d^2x}{dt^2} = -Aa^2\sin(at+b) = -a^2x,$$

and observe that

$$\frac{d^2x}{dt^2} + a^2x = 0.$$

It is equally true, though not so easy to see, that every nontrivial solution of (2) can be written in the form (3). We will demonstrate this in Remarks 1 and 2, but meanwhile we take it for granted.

Since the function $\sin(at + b)$ oscillates between -1 and 1, the function (3) oscillates between $-|A|$ and $|A|$. The number $|A|$ is called the *amplitude* of the motion (Fig. 9.22). Also, since the sine is periodic with period 2π, $\sin(at + b)$ is periodic with period $2\pi/a$, because this is the amount t must increase in order to increase $at + b$ by 2π. This number $T = 2\pi/a$ is called the *period* of the motion, and is the time required for one complete cycle. If we measure t in seconds,

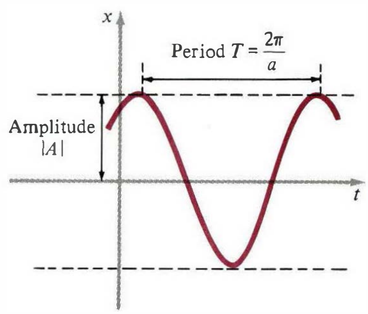


Figure 9.22

*We add the condition $A \neq 0$ to avoid the trivial case in which x is identically zero and consequently there is no motion.

then the number f of cycles per second satisfies the equation $fT = 1$, and is therefore the reciprocal of the period,

$$f = \frac{1}{T} = \frac{a}{2\pi}.$$

This number is called the *frequency* of the motion.

Another equivalent form of the general solution (3) that is often useful is

$$x = A \cos (at + b). \tag{4}$$

This is easily seen from the fact that b in (3) is an arbitrary constant, and can therefore be replaced by the equally arbitrary constant $b + \pi/2$. This gives

$$x = A \sin \left(at + b + \frac{\pi}{2}\right) = A \cos (at + b),$$

since $\sin (\theta + \pi/2) = \cos \theta$.

There are two main interpretations of simple harmonic motion, one geometric and the other physical.

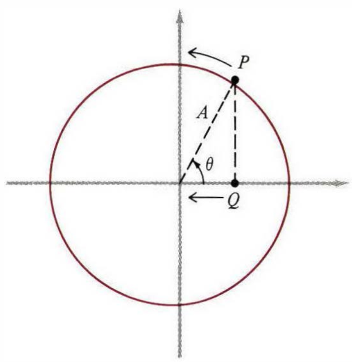


Figure 9.23

The geometric meaning can be understood by considering a point P that moves with constant angular velocity around a circle of radius A (Fig. 9.23). If this constant angular velocity is denoted by a, then

$$\frac{d\theta}{dt} = a \qquad \text{and therefore} \qquad \theta = at + b,$$

where b is the value of θ when $t = 0$. If Q is the projection of P on the x-axis, then its x-coordinate is

$$x = A \cos \theta = A \cos (at + b).$$

This shows that Q moves back and forth along the x-axis in simple harmonic motion as P moves steadily around the circle in uniform circular motion, and any simple harmonic motion can be visualized in this way.

The physical meaning appears when we think of equation (1) as describing the motion of a body of mass m rather then merely a point. Newton's second law of motion says that $F = ma$, so equation (1) becomes

$$\frac{1}{m} F = -kx \qquad \text{or} \qquad F = -kmx.$$

A force F of this kind is called a *restoring force,* because its magnitude is proportional to the displacement x and it always acts to pull the body back toward the equilibrium position $x = 0$. We discuss the idea more fully in our first two examples.

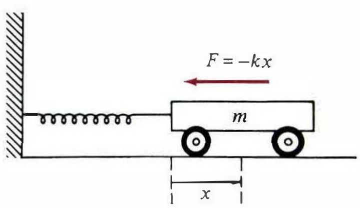


Figure 9.24

Example 1 Consider a cart of mass m attached to a nearby wall by means of a spring (Fig. 9.24). The spring exerts no force when the cart is at its equilibrium position $x = 0$. If the cart is pulled aside to a position x, then the spring exerts a restoring force $F = -kx$, where k is a positive constant whose magnitude is a measure of the stiffness of the spring (see Example 1 in Section 7.7). Suppose that the cart is pulled out to the position $x = x_0$ and released without any initial velocity at time $t = 0$. Discuss its subsequent motion if friction and air resistance are negligible.

Solution We are assuming that the only force acting on the cart is the restoring force $F = -kx$, so by Newton's second law of motion we have

$$m\frac{d^2x}{dt^2} = -kx \qquad \text{or} \qquad \frac{d^2x}{dt^2} + \frac{k}{m}x = 0.$$

It is convenient to write this equation as

$$\frac{d^2x}{dt^2} + a^2x = 0,$$

where $a = \sqrt{k/m}$. The form of the general solution we prefer here is

$$x = c_1 \sin at + c_2 \cos at, \tag{5}$$

which can be obtained by expanding either (3) or (4). The initial conditions

$$x = x_0 \qquad \text{and} \qquad v = \frac{dx}{dt} = 0 \qquad \text{when} \qquad t = 0$$

imply that $c_2 = x_0$ and $c_1 = 0$, so (5) becomes

$$x = x_0 \cos at.$$

It is clear from this that the cart moves in simple harmonic motion with period $T = 2\pi/a = 2\pi\sqrt{m/k}$ and frequency

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}. \tag{6}$$

We see from (6) that the frequency of this vibration increases if the stiffness k of the spring is increased, and decreases if the mass m of the cart is increased, as our common sense would have led us to expect.

Example 2 Suppose that a tunnel is bored straight through the center of the earth from one side to the other, and that a body of mass m is dropped into this tunnel. Assuming as usual that the earth is a perfect sphere of uniform density and radius R of about 4000 mi, the effect of gravity is such that the body is attracted toward the center of the earth by a force F proportional to its distance x from the center (Fig. 9.25).* Show that the body traverses the tunnel from one end to the other and back again with simple harmonic motion, and calculate the period of this motion.

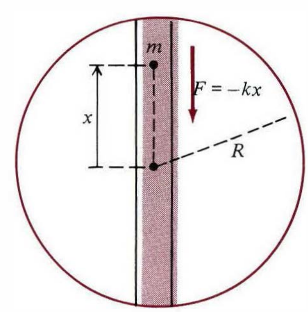


Figure 9.25

Solution Clearly $F = -kx$ for a suitable constant k. To find the value of this constant we use the fact that $F = -mg$ at the surface of the earth, where $x = R$, so

$$-mg = -kR \qquad \text{or} \qquad k = \frac{mg}{R}.$$

Newton's second law of motion therefore takes the form

$$m\frac{d^2x}{dt^2} = -\frac{mg}{R}x \qquad \text{or} \qquad \frac{d^2x}{dt^2} + \frac{g}{R}x = 0.$$

*The reason for this law of force will be explained in a later chapter, in connection with triple integrals in spherical coordinates.

No further discussion is needed in order to conclude that this is simple harmonic motion with period $2\pi\sqrt{R/g}$. A sequence of easy approximate calculations gives

$$2\pi\sqrt{\frac{R}{g}} \cong 6.3\sqrt{\frac{4000 \cdot 5280 \text{ ft}}{32 \text{ ft/s}^2}} \cong 6.3\sqrt{\frac{500 \cdot 5280}{4}} \text{ s}$$

$$\cong 6.3\sqrt{\frac{500 \cdot 5280}{4 \cdot 3600}} \text{ min} \cong 6.3\sqrt{200} \text{ min}$$

$$\cong \tfrac{19}{3} \cdot 14 \text{ min} \cong 89 \text{ min.}$$

The period is of course the total time required for a round trip through the tunnel to the other side of the earth and back again. A one-way trip requires only about 45 minutes, and the journey to the center of the earth only about 22 minutes.

Example 3 A pendulum consists of a bob (a weight) suspended at the end of a light string and allowed to swing back and forth under the action of gravity. As usual, we idealize the situation and consider a particle of mass m at the end of a weightless string of length L (Fig. 9.26). Find the period of this pendulum under the assumption that its oscillations are small.

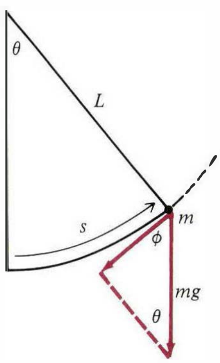


Figure 9.26

Solution The downward force of gravity on the bob is mg, and this has a component $mg \cos \phi = mg \sin \theta$ tangent to the path. Since $s = L\theta$, the tangential acceleration of the bob is

$$\frac{d^2s}{dt^2} = \frac{d^2(L\theta)}{dt^2} = L\frac{d^2\theta}{dt^2},$$

and Newton's second law applied to the motion of the bob along its circular path is

$$mL\frac{d^2\theta}{dt^2} = -mg \sin \theta \qquad \text{or} \qquad \frac{d^2\theta}{dt^2} + \frac{g}{L}\sin \theta = 0. \tag{7}$$

The presence of $\sin \theta$ makes this differential equation impossible to solve exactly, and the motion is not simple harmonic. However, for small oscillations we recall that $\sin \theta$ is approximately equal to θ, so (7) becomes (approximately)

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0.$$

This equation tells us that the angular motion is approximately simple harmonic with period $2\pi\sqrt{L/g}$. When these ideas are analyzed in more detail, it turns out that the period of this oscillation actually depends on the amplitude of the motion, and this is the source of the so-called *circular error* in pendulum clocks.

Remark 1 We return to the matter of proving that (3) is indeed the general solution of (2). By Problem 19 in Section 9.2 we know that every nontrivial solution of (2) has the form

$$x = c_1 \sin at + c_2 \cos at, \tag{8}$$

where the constants c_1 and c_2 are not both zero. To write (8) in the form (3), we begin by setting $A = \sqrt{c_1^2 + c_2^2}$. Then the point $(c_1/A, c_2/A)$ is a point on the unit circle, and therefore there is an angle b such that

$$\cos b = \frac{c_1}{A} \qquad \text{and} \qquad \sin b = \frac{c_2}{A}.$$

These equations now enable us to write (8) as

$$\begin{aligned} x &= A(\sin at \cos b + \cos at \sin b) \\ &= A \sin (at + b), \end{aligned}$$

by the addition formula for the sine.

Remark 2 It is also possible to obtain (3) directly from (2), as follows. If we write

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}, \tag{9}$$

then (2) becomes

$$v\frac{dv}{dx} + a^2x = 0 \qquad \text{or} \qquad v\,dv + a^2x\,dx = 0,$$

and by integrating we get

$$v^2 + a^2x^2 = \text{a constant} \qquad \text{or} \qquad v^2 + a^2x^2 = a^2A^2,$$

where A is the positive value of x at which $v = 0$. This yields

$$\frac{dx}{dt} = v = \pm a\sqrt{A^2 - x^2} \qquad \text{or} \qquad \frac{dx}{\sqrt{A^2 - x^2}} = \pm a\,dt,$$

where the choice of sign here depends on whether the velocity v is positive or negative at the moment. We suppose for definiteness that $v > 0$ and integrate again to obtain

$$\sin^{-1}\frac{x}{A} = at + b \qquad \text{or} \qquad \frac{x}{A} = \sin (at + b),$$

so

$$x = A \sin (at + b),$$

which is (3).

PROBLEMS

1 In each of the following motions calculate the amplitude and period by rewriting in the form $x = A \sin (at + b)$.
(a) $x = 5 \sin t - 5 \cos t$;
(b) $x = \sqrt{3} \cos 3t - \sin 3t$;
(c) $x = \sin t + \cos t$;
(d) $x = 2\sqrt{3} \sin 2t - 2 \cos 2t$.

2 In any simple harmonic motion of the form (3), show that the velocity v is related to the position x by the equation

$$v^2 = a^2(A^2 - x^2).$$

Deduce that the speed is greatest when the body passes through its equilibrium position, and is zero at the ends of the interval, where the body reverses the direction of its motion.

3 In Example 1, suppose the spring is stretched 3 in by a force of 6 lb. If the cart weighs 12 lb, and if it is pulled out 4 in from its equilibrium position and struck a sudden blow sending it back toward its equilibrium position at a velocity of 3 ft/s, find the amplitude and period of the resulting simple harmonic motion. Hint: Recall that mass is weight divided by g.

4 A body in simple harmonic motion passes through its equilibrium position at $t = 0, 1, 2, \ldots$. Find a position function of the form (3) if $v = dx/dt = -3$ when $t = 0$.

5 Suppose that a straight tunnel is bored through the earth between any two points on the surface. If tracks are laid, then—neglecting friction—a train placed in the tunnel at one end will roll through the earth under its own weight, stop at the other end, and return. Show that the time required for a complete round trip is the same for all such tunnels, and estimate its value.

***6** A spherical buoy of radius r floats half submerged in water. If it is depressed slightly, Archimedes' principle tells us that a restoring force equal to the weight of the displaced water presses it upward; and if it is released, it will bob up and down. Show that if the friction of the water is negligible, then the motion will be simple harmonic, and find its period.

7 People who manufacture grandfather clocks have a professional interest in pendulums that take 1 second for each swing and thus have a period of 2 seconds. Estimate the length of such a pendulum.

9.7 (OPTIONAL) HYPERBOLIC FUNCTIONS

The hyperbolic functions are certain combinations of exponential functions that occur in various applications, with properties similar to those of the trigonometric functions. The reason for the name will be made clear below.

The two basic hyperbolic functions are the *hyperbolic sine* and *hyperbolic cosine,* defined by

$$\sinh x = \tfrac{1}{2}(e^x - e^{-x}) \qquad \text{and} \qquad \cosh x = \tfrac{1}{2}(e^x + e^{-x}). \tag{1}$$

The symbol "sinh" is pronounced "cinch," rhyming with "pinch," and "cosh" rhymes with "gosh." Just as in trigonometry, inspection of the definitions shows at once that these functions have the simple properties

$$\sinh(-x) = -\sinh x \qquad \text{and} \qquad \cosh(-x) = \cosh x. \tag{2}$$

On the other hand—in contrast to the sine and cosine—these functions are not periodic. The other four hyperbolic functions are less often used and are defined by identities analogous to those of trigonometry:

$$\tanh x = \frac{\sinh x}{\cosh x}, \qquad \operatorname{sech} x = \frac{1}{\cosh x},$$

$$\coth x = \frac{1}{\tanh x}, \qquad \operatorname{csch} x = \frac{1}{\sinh x}.$$

There are numerous relations among these functions that are similar to the trigonometric identities. Also, their derivatives and integrals resemble those of the corresponding trigonometric functions. However, there is a different pattern of algebraic signs, and close attention is needed in order to avoid making mistakes. Since these identities and formulas are so easily confused with those of trigonometry—which are much more important—we do not recommend that students make an effort to learn them.

IDENTITIES

Among the many identities, we mention only the ones most frequently used:

$$\cosh^2 x - \sinh^2 x = 1, \tag{3}$$

$$\cosh^2 x + \sinh^2 x = \cosh 2x, \tag{4}$$

$$\sinh (x + y) = \sinh x \cosh y + \cosh x \sinh y, \tag{5}$$

$$\cosh (x + y) = \cosh x \cosh y + \sinh x \sinh y. \tag{6}$$

Each of these, and many others, can be proved directly from the definitions (1). Thus, in the case of (3),

$$\begin{aligned}\cosh^2 x - \sinh^2 x &= \tfrac{1}{4}(e^x + e^{-x})^2 - \tfrac{1}{4}(e^x - e^{-x})^2\\ &= \tfrac{1}{4}[e^{2x} + 2 + e^{-2x} - (e^{2x} - 2 + e^{-2x})]\\ &= 1.\end{aligned}$$

This identity can also be established by multiplying together

$$e^x = \cosh x + \sinh x \qquad \text{and} \qquad e^{-x} = \cosh x - \sinh x,$$

which are the results of adding and subtracting equations (1).

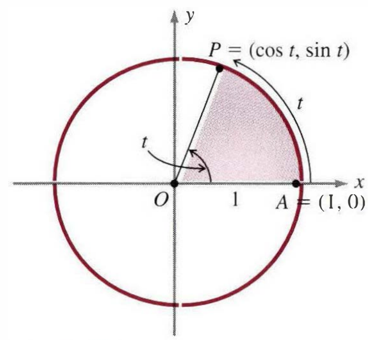


Figure 9.27

WHY THE NAME?

If t is any real number, then the point $P = (\cos t, \sin t)$ lies on the unit circle $x^2 + y^2 = 1$, because

$$\cos^2 t + \sin^2 t = 1.$$

In fact (see Fig. 9.27), the definitions of the sine and cosine allow us to interpret t as the angle AOP from OA to OP, where the angle is understood to be measured in radians. For this reason the sine and cosine are often called *circular functions.*

Similarly, for any real number t the point $P = (\cosh t, \sinh t)$ lies on the right branch of the curve $x^2 - y^2 = 1$ (see Fig. 9.28), because

$$\cosh^2 t - \sinh^2 t = 1$$

and $\cosh t > 0$. We shall see later that this curve is called a hyperbola, and accordingly $\cosh t$ and $\sinh t$ are called *hyperbolic functions.* This time the variable t is not the angle AOP. However, it turns out (see Problem 31) that t is twice the area of the shaded hyperbolic sector in the figure, just as in the trigonometric case t is twice the area of the shaded circular sector in Fig. 9.27.

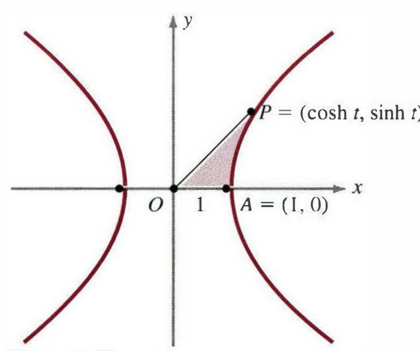


Figure 9.28

DERIVATIVES AND INTEGRALS

The derivatives are similar to those of the trigonometric functions, but not exactly the same:

$$\frac{d}{dx} \sinh u = \cosh u \frac{du}{dx}, \tag{7}$$

$$\frac{d}{dx} \cosh u = \sinh u \frac{du}{dx}, \tag{8}$$

$$\frac{d}{dx} \tanh u = \operatorname{sech}^2 u \frac{du}{dx}. \tag{9}$$

The first, for example, comes from

$$\frac{d}{dx} \sinh x = \frac{1}{2} \frac{d}{dx} (e^x - e^{-x}) = \tfrac{1}{2}(e^x + e^{-x}) = \cosh x.$$

By turning these around we have the integration formulas

$$\int \cosh u\,du = \sinh u + c, \tag{10}$$

$$\int \sinh u\,du = \cosh u + c, \tag{11}$$

and so on. In the case of $\tanh u$ we have

$$\int \tanh u\,du = \int \frac{\sinh u\,du}{\cosh u} = \int \frac{d(\cosh u)}{\cosh u} = \ln(\cosh u) + c. \tag{12}$$

GRAPHS

We begin with $y = \cosh x$. First, the fact that $\cosh(-x) = \cosh x$ shows that the graph is symmetric about the y-axis, and it crosses the y-axis at the point $(0, 1)$ because

$$\cosh 0 = \tfrac{1}{2}(e^0 + e^{-0}) = 1.$$

We see that

$$\frac{d}{dx}\cosh x = \sinh x = \tfrac{1}{2}(e^x - e^{-x}) = 0$$

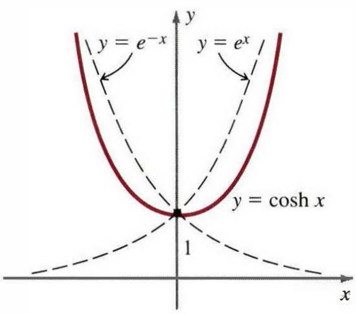


Figure 9.29

if and only if $x = 0$, so the graph has a horizontal tangent at the point $(0, 1)$ and at no other point. Since

$$\begin{aligned}\frac{d^2}{dx^2}\cosh x &= \frac{d}{dx}\sinh x \\ &= \cosh x = \tfrac{1}{2}(e^x + e^{-x}) > 0,\end{aligned}$$

the graph is concave up everywhere and therefore has the appearance shown in Fig. 9.29. This graph can also be obtained by geometrically adding the two curves $y = e^x$ and $y = e^{-x}$—that is, by adding the two y's for each x—and taking half of each resulting y-value.

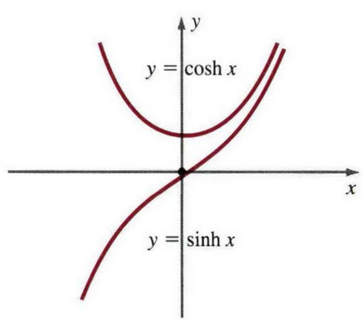


Figure 9.30

Now for the graph of $y = \sinh x$. This graph (Fig. 9.30) passes through the origin since

$$\sinh 0 = \tfrac{1}{2}(e^0 - e^{-0}) = 0,$$

and the identity $\sinh(-x) = -\sinh x$ shows that it is symmetric about the origin. The graph is rising at every point because

$$\frac{d}{dx}\sinh x = \cosh x = \tfrac{1}{2}(e^x + e^{-x}) > 0.$$

The fact that

$$\begin{aligned}\frac{d^2}{dx^2}\sinh x &= \frac{d}{dx}\cosh x = \sinh x \\ &= \tfrac{1}{2}(e^x - e^{-x}) \quad \text{is} \quad \begin{cases} > 0 & \text{for } x > 0, \\ < 0 & \text{for } x < 0, \end{cases}\end{aligned}$$

shows that the graph is concave up for $x > 0$ and concave down for $x < 0$. The point $(0, 0)$ is the only point of inflection. Figure 9.30 shows a comparison of the two graphs. The graph of $y = \cosh x$ lies above the graph of $y = \sinh x$ because

$$\cosh x - \sinh x = \tfrac{1}{2}(e^x + e^{-x}) - \tfrac{1}{2}(e^x - e^{-x}) = e^{-x} > 0,$$

and their difference $e^{-x} \to 0$ as $x \to \infty$.

INVERSE FUNCTIONS

The inverse hyperbolic functions can be expressed in terms of logarithms, and therefore present nothing new. We illustrate with the hyperbolic sine. The graph is rising at all points (Fig. 9.30), so there exists an inverse

$$y = \sinh^{-1} x$$

which is obtained by solving

$$x = \sinh y = \tfrac{1}{2}(e^y - e^{-y})$$

for y in terms of x. This can be written as

$$2x = e^y - e^{-y} \qquad \text{or} \qquad e^{2y} - 2xe^y - 1 = 0,$$

which is a quadratic equation in e^y. Solving this by the quadratic formula gives

$$\begin{aligned} e^y &= \frac{2x \pm \sqrt{4x^2 + 4}}{2} \\ &= x + \sqrt{x^2 + 1}, \end{aligned}$$

where the minus sign is discarded because it gives a negative value to e^y. By solving for y we now obtain

$$y = \sinh^{-1} x = \ln(x + \sqrt{x^2 + 1}). \tag{13}$$

The derivative can be found by differentiating this logarithm. A better way is to differentiate $\sinh y = x$ implicitly:

$$\cosh y \frac{dy}{dx} = 1 \qquad \text{or} \qquad \frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1 + \sinh^2 y}} = \frac{1}{\sqrt{1 + x^2}}.$$

This gives the formulas

$$\frac{d}{dx} \sinh^{-1} u = \frac{1}{\sqrt{1 + u^2}} \frac{du}{dx} \tag{14}$$

and

$$\int \frac{du}{\sqrt{1 + u^2}} = \sinh^{-1} u + c. \tag{15}$$

We ask the student in the problems to provide a similar discussion of the inverse function $y = \tanh^{-1} x$.

THE CATENARY

The most famous application of the hyperbolic functions is to the following classical problem: Determine the exact shape of the curve assumed by a flexible chain or cable of uniform density which is suspended between two points and hangs

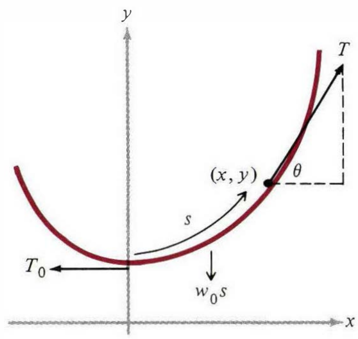


Figure 9.31

under its own weight. This curve is called a *catenary*, from the Latin word for chain, *catena*.

Let the y-axis pass through the lowest point of the chain (Fig. 9.31), let s be the arc length from this point to a variable point (x, y), and let w_0 be the linear density (weight per unit length) of the chain. We obtain the differential equation of the catenary from the fact that the part of the chain between the lowest point and (x, y) is in static equilibrium under the action of three forces: the tension T_0 at the lowest point; the variable tension T at (x, y), which acts in the direction of the tangent because of the flexibility of the chain; and a downward force $w_0 s$ equal to the weight of the chain between these two points.

Equating the horizontal component of T to T_0 and the vertical component of T to the weight of the chain gives

$$T \cos \theta = T_0 \qquad \text{and} \qquad T \sin \theta = w_0 s,$$

and by dividing we eliminate T and get $\tan \theta = w_0 s/T_0$ or (since $\tan \theta = dy/dx$)

$$\frac{dy}{dx} = as, \qquad \text{where} \qquad a = \frac{w_0}{T_0}.$$

We next eliminate the variable s by differentiating with respect to x,

$$\frac{d^2y}{dx^2} = a\frac{ds}{dx} = a\frac{\sqrt{dx^2 + dy^2}}{dx}$$

$$= a\sqrt{1 + \left(\frac{dy}{dx}\right)^2}. \tag{16}$$

This is the differential equation of the catenary.

We now solve equation (16) by two successive integrations. This process is facilitated by introducing the auxiliary variable $m = dy/dx$, so that (16) becomes

$$\frac{dm}{dx} = a\sqrt{1 + m^2}.$$

On separating variables and integrating we get

$$\int \frac{dm}{\sqrt{1 + m^2}} = a \int dx,$$

and by (15) this yields

$$\sinh^{-1} m = ax + c_1.$$

Since $m = 0$ when $x = 0$ (why?), we see that $c_1 = 0$, so $\sinh^{-1} m = ax$ or

$$m = \sinh ax.$$

But $m = dy/dx$, so we have another differential equation to solve:

$$\frac{dy}{dx} = \sinh ax,$$

$$dy = \sinh ax\, dx,$$

$$y = \int \sinh ax\, dx$$

If we now place the origin of the coordinate system at just the right level so that $y = 1/a$ when $x = 0$, then $c_2 = 0$ and our equation for the catenary takes its final form,

$$y = \frac{1}{a} \cosh ax. \tag{17}$$

Equation (17) reveals the precise mathematical nature of the catenary and is the basis for a number of practical applications to problems such as the sag of telephone lines and the design of suspension bridges like the Golden Gate Bridge in San Francisco.

PROBLEMS

1 Find the exact numerical value of
(a) $\sinh(\ln 2)$; (b) $\cosh(-\ln 3)$; (c) $\tanh(2 \ln 3)$.

Establish the identities in Problems 2–11.

2 $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$.
3 $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$.
4 $\sinh(x - y) = \sinh x \cosh y - \cosh x \sinh y$.
5 $\cosh(x - y) = \cosh x \cosh y - \sinh x \sinh y$.
6 $\sinh 2x = 2 \sinh x \cosh x$.
7 $\cosh 2x = \cosh^2 x + \sinh^2 x$.
8 $2 \cosh^2 x = \cosh 2x + 1$.
9 $2 \sinh^2 x = \cosh 2x - 1$.
10 $\tanh^2 x + \operatorname{sech}^2 x = 1$.
11 $\coth^2 x - 1 = \operatorname{csch}^2 x$.
12 Show that $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$ for any positive integer n.

In Problems 13–18, find dy/dx.

13 $y = \sinh x^3$.
14 $y = \cosh(5x - 3)$.
15 $y = \ln(\tanh 3x)$.
16 $y = \sinh^4 3x$.
17 $y = \cosh^2 5x - \sinh^2 5x$.
18 $y = \tanh x^2$.

In Problems 19–24, find the integral.

19 $\int \sinh(5x - 3)\, dx$.
20 $\displaystyle\int \frac{\sinh 3x}{1 + \cosh 3x}\, dx$.
21 $\int \sqrt{1 + \cosh x}\, dx$.
22 $\displaystyle\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\, dx$ [compare with formula (12)].
23 $\int \tanh^2 x\, dx$.
24 $\int (\sinh^2 x + \cosh^2 x)\, dx$.
25 Sketch the graph of $y = \tanh x$ by merely inspecting Fig. 9.30.
26 Use $\tanh y = x$ to express the inverse function $y = \tanh^{-1} x$ in the form

$$y = \tanh^{-1} x = \frac{1}{2} \ln \frac{1 + x}{1 - x}.$$

27 Use Problem 26 to show in two ways that

$$\frac{d}{dx} \tanh^{-1} x = \frac{1}{1 - x^2},$$

so that

$$\int \frac{dx}{1 - x^2} = \tanh^{-1} x.$$

28 Use formula (15) to show that

$$\int_0^3 \frac{dx}{\sqrt{x^2 + 9}} = \ln(1 + \sqrt{2}).$$

29 Find the area under the catenary $y = \dfrac{1}{a} \cosh ax$ from $x = 0$ to $x = 1/a$.
30 If an object moving on the x-axis is *repelled* from the origin (instead of being attracted to it) with a force proportional to its position x, then its motion satisfies the differential equation

$$\frac{d^2x}{dt^2} = a^2 x$$

where a is a positive constant. Show that for any constants c_1 and c_2 the function

$$x = c_1 \sinh at + c_2 \cosh at$$

is a solution of this differential equation.
(a) Find the solution of

$$\frac{d^2x}{dt^2} = 9x$$

that satisfies the initial conditions $x(0) = 2$ and $x'(0) = 1$.
(b) Find the solution of

$$4 \frac{d^2x}{dt^2} = x$$

that satisfies the initial conditions $x(0) = 1$ and $x'(0) = 2$.

31 Show that the area $A(t)$ of the shaded hyperbolic sector in Fig. 9.28 is $\frac{1}{2}t$, so that $t = 2A(t)$, as stated. Hint: Begin by observing that

$$A(t) = \tfrac{1}{2}\cosh t \sinh t - \int_1^{\cosh t} \sqrt{x^2-1}\,dx.$$

Then show that $A'(t) = \frac{1}{2}$ by using various identities and the rule for differentiating integrals given in Section 6.7. Finally, use the fact that $A(0) = 0$.

CHAPTER 9 REVIEW: DEFINITIONS, FORMULAS

Think through and learn the following.

1 Basic trigonometry (see front endpaper).
2 The derivatives of the six trigonometric functions, and the corresponding integral formulas.
3 The definitions of $\sin^{-1} x$ and $\tan^{-1} x$.
4 The derivatives of $\sin^{-1} x$ and $\tan^{-1} x$, and the corresponding integral formulas.
5 The differential equation $\dfrac{d^2x}{dt^2} + a^2x = 0$ of simple harmonic motion, and its solutions.
6 The definitions of $\sinh x$ and $\cosh x$.

ADDITIONAL PROBLEMS FOR CHAPTER 9

SECTION 9.2

In each of Problems 1–18, find the derivative dy/dx of the given function.

1 $y = \sin(1-9x)$.
2 $y = 7\cos(7x-13)$.
3 $y = \cos^2 x$.
4 $y = \cos x^2$.
5 $y = \cos^2 5x$.
6 $y = 5\sin(1-18x)$.
7 $y = \cos^2 3x - \sin^2 3x$.
8 $y = \cos^2 9x + \sin^2 9x$.
9 $y = x^2 \cos x$.
10 $y = \dfrac{\sin x}{x}$.
11 $y = x\sin x + \cos x$.
12 $y = \sqrt{1+\sin 2x}$.
13 $y = \cos(\cos x)$.
14 $y = e^{\sin^2 x}$.
15 $y = \cos(\sin x)$.
16 $y = \ln(x \sin x)$.
17 $y = \sin(e^{\ln x})$.
18 $y = \ln[\sin(\ln x)]$.

19 Consider the differential equation

$$\frac{d^2y}{dx^2} + a^2y = 0,$$

where a is a positive constant. Use the following steps to prove that every solution of this equation has the form

$$y = c_1 \sin ax + c_2 \cos ax$$

for a suitable choice of the constants c_1 and c_2.

(a) If $y = g(x)$ and $y = h(x)$ are solutions, show that every linear combination $y = c_1\, g(x) + c_2\, h(x)$ is also a solution.

(b) If $y = f(x)$ is a solution, show that

$$a^2[f(x)]^2 + [f'(x)]^2 = \text{a constant}.$$

Deduce that if $y = f(x)$ is a solution such that $f(0) = f'(0) = 0$, then $f(x) = 0$ for all x.

(c) If $y = f(x)$ is any solution, show that

$$f(x) = c_1 \sin ax + c_2 \cos ax$$

for a suitable choice of the constants c_1 and c_2. Hint: Apply part (b) to

$$f(x) - \frac{1}{a}f'(0)\sin ax - f(0)\cos ax.$$

20 Use Problem 21(b) in Section 9.1 to give another proof of the formula $(d/dx)\sin x = \cos x$.

21 Give another proof of the limit (4) in Section 9.2 by the following steps: If θ is a small positive angle $(0 < \theta < \pi/2)$ in the unit circle shown in Fig. 9.32, then
(a) area $\Delta OPQ <$ area sector $OPQ <$ area ΔOQR;
(b) $\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta$;

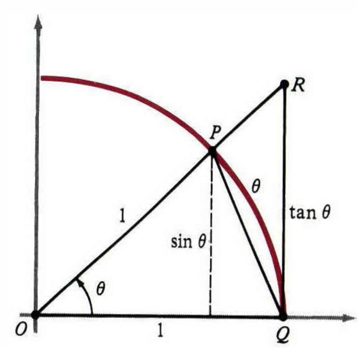


Figure 9.32

(c) $1 < \dfrac{\theta}{\sin\theta} < \dfrac{1}{\cos\theta}$; (d) $1 > \dfrac{\sin\theta}{\theta} > \cos\theta$.

*22 Figure 9.33 shows the familiar mechanism of a piston (which moves back and forth in a cylinder) attached at a point P to a connecting rod of length b which in turn is attached to a point Q on a crankshaft that rotates in a circle of radius a with center at O.

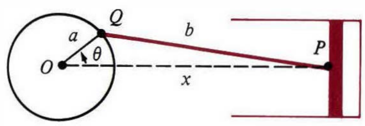


Figure 9.33

(a) Find dx/dt, the velocity of the piston, in terms of $d\theta/dt$, the angular velocity of the crankshaft. Hint: Use the law of cosines.
(b) If the angular velocity of the crankshaft is denoted by the customary symbol ω, show that the speed of the piston is $\omega \cdot OR$, where R is the point in which the line PQ intersects the line through O perpendicular to OP.

*23 A given fixed circle has radius a. A second circle has its center on the given circle, and the arc of the second circle that lies inside the given circle has length s. Show that s has its largest value when a suitable angle θ satisfies the equation $\cot \theta = \theta$.

24 A heavy block of weight W is to be dragged along a flat table by a force F whose line of action is inclined at an angle θ to the line of motion, as shown in Fig. 9.34. The motion is resisted by a frictional force μN which is proportional to the normal force $N = W - F \cdot \sin \theta$ with which the block presses perpendicularly against the surface of the table (μ is a constant called the coefficient of friction). The block moves when the forward component of F equals the frictional resistance, i.e., when $F \cos \theta = \mu(W - F \sin \theta)$. Find the direction and magnitude of the smallest force F that will move the block.

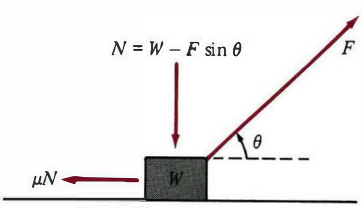


Figure 9.34

In each of Problems 25–36, find the value of the indicated limit.

25 $\lim_{x\to 0} \dfrac{\tan^3 x}{x^2}$.

26 $\lim_{x\to 0} \dfrac{\sin x}{2x}$.

27 $\lim_{x\to \pi} \dfrac{\sin x}{\pi - x}$.

28 $\lim_{x\to 0} \dfrac{\sin^2 x}{x}$.

29 $\lim_{x\to 0} \dfrac{x + \tan x}{\sin x}$.

30 $\lim_{x\to 0} \dfrac{\tan 3x}{4x}$.

31 $\lim_{x\to 0} \dfrac{2x}{\sin 3x}$.

32 $\lim_{x\to 0} x \cot 3x$.

33 $\lim_{x\to 0} \dfrac{\sin 2x}{3x^2 + x}$.

34 $\lim_{x\to 0} x \csc^2 \sqrt{3x}$.

35 $\lim_{x\to 2} \dfrac{\cos \pi/x}{x - 2}$.

36 $\lim_{x\to \pi} \dfrac{\sin 2x}{\pi - x}$.

SECTION 9.3

Evaluate the indefinite integrals in Problems 37–54.

37 $\int \cos 3x \, dx$.

38 $\int \sin (7x + 1) \, dx$.

39 $\int \cos (1 - \frac{1}{2}x) \, dx$.

40 $\int \cos^2 7x \sin 7x \, dx$.

41 $\int \sin^5 3x \cos 3x \, dx$.

42 $\int \cos^2 \frac{3}{5}x \sin \frac{3}{5}x \, dx$.

43 $\int (2 - \cos^2 3x) \sin 3x \, dx$.

44 $\int 3 \sin x \sin 2x \, dx$.

45 $\int x^2 \cos x^3 \, dx$.

46 $\int \sqrt{x} \sin x^{3/2} \, dx$.

47 $\int \sin (\cos 2x) \sin 2x \, dx$.

48 $\int \sqrt{\cos 2x} \sin 2x \, dx$.

49 $\displaystyle\int \frac{\cos 4x \, dx}{\sin^2 4x}$.

50 $\displaystyle\int \frac{\sin x \, dx}{\cos^5 x}$.

51 $\displaystyle\int \frac{\sin x \, dx}{(3 + 2 \cos x)^2}$.

52 $\int \sqrt{1 + \sin 2x} \cos 2x \, dx$.

53 $\displaystyle\int \frac{\cos 5x \, dx}{\sqrt{7 - \sin 5x}}$.

54 $\int (1 + 4 \sin 8x)^7 \cos 8x \, dx$.

Evaluate the definite integrals in Problems 55–58.

55 $\displaystyle\int_0^{\pi/14} \cos 7x \, dx$.

56 $\displaystyle\int_0^{\pi/18} \sin 6x \, dx$.

57 $\displaystyle\int_0^{\pi/6} \frac{\sin 2x \, dx}{\cos^2 2x}$.

58 $\displaystyle\int_0^{\sqrt[5]{\pi}} 10 x^4 \sin x^5 \, dx$.

59 Find the area bounded by $y = \sin x$ and $y = \cos x$ between the first two positive values of x at which these curves intersect.

60 Find the area bounded by $y = 1 - \cos 2x$ and $y = \cos x - 1$ between $x = 0$ and $x = 2\pi$.

61 Find the area bounded by $y = 4 - 3 \sin 2x$ and $y = 2 \cos 5x - 3$ between $x = 0$ and $x = 3\pi$.

62 Show that if m and n are positive integers, then

$$\int_0^{2\pi} \sin mx \sin nx \, dx = \begin{cases} 0 & \text{if } m \neq n, \\ \pi & \text{if } m = n, \end{cases}$$

$$\int_0^{2\pi} \cos mx \cos nx\, dx = \begin{cases} 0 & \text{if } m \neq n, \\ \pi & \text{if } m = n, \end{cases}$$

$$\int_0^{2\pi} \sin mx \cos nx\, dx = 0.$$

Hint: See Problem 20 in Section 9.1. (These facts are very important in the theory of Fourier series, which is one of the most useful parts of advanced mathematics from the point of view of applications to science.)

*63 In this problem we ask students to establish the formula

$$\int_a^b \sin x\, dx = \cos a - \cos b \qquad (*)$$

directly from the limit definition of the integral, without making any use of the Fundamental Theorem of Calculus.

(a) Show that

$$\sin x + \sin 2x + \cdots + \sin nx = \frac{\cos \frac{1}{2}x - \cos (n + \frac{1}{2})x}{2 \sin \frac{1}{2}x}.$$

Hint: Write down the identity $2 \sin \theta \sin \phi = \cos(\theta - \phi) - \cos(\theta + \phi)$ for the n cases in which the pair (θ, ϕ) is taken to be $(x, \frac{1}{2}x)$, $(2x, \frac{1}{2}x)$, $\ldots$, $(nx, \frac{1}{2}x)$, and add.

(b) For $b > 0$, the limit definition of the integral gives

$$\int_0^b \sin x\, dx = \lim_{n \to \infty} \sum_{k=1}^{n} \left(\sin \frac{kb}{n}\right) \cdot \frac{b}{n} = \lim_{n \to \infty} \frac{b}{n} \sum_{k=1}^{n} \sin \frac{kb}{n}.$$

Use part (a) with $x = b/n$ to show that the value of this limit is $1 - \cos b$.

(c) Use simple area arguments to show that the result in part (b) is also valid for the cases $b = 0$ and $b < 0$.

(d) Use parts (b) and (c) to establish $(*)$.

*64 Establish the formula

$$\int_a^b \cos x\, dx = \sin b - \sin a \qquad (**)$$

by a line of reasoning similar to that in Problem 63.

65 Give another proof of formula $(**)$ in Problem 64 by using the following reasoning: If the graph of $y = \cos x$ is moved a distance $\pi/2$ to the right, it is translated into the graph of $y = \sin x$; the integral in $(**)$, which represents the area between the curve $y = \cos x$ and the x-axis from $x = a$ to $x = b$, can therefore be written as another integral representing the area between the curve $y = \sin x$ and the x-axis from $x = a + \pi/2$ to $x = b + \pi/2$.

SECTION 9.4

In each of Problems 66–79, calculate dy/dx.

66 $y = \cot(2 - 5x)$.
67 $y = 4 \tan 3x$.
68 $y = \frac{1}{4} \sec^4 x$.
69 $y = \sqrt{\cot 2x}$.
70 $y = \csc(1 - 2x)$.
71 $y = \sec^4 x - \tan^4 x$.
72 $y = 2x + \tan 2x$.
73 $y = \cot^2 5x$.
74 $y = \sec^3 x$.
75 $y = x \tan \dfrac{1}{x}$.
76 $y = \cot(\ln x)$.
77 $y = \sqrt{\sec \sqrt{x}}$.
78 $y = \csc^3 x + \csc x^3$.
79 $y = \tan(\tan x)$.

Evaluate the integral in each of Problems 80–87.

80 $\displaystyle\int \frac{dx}{\cos^2 5x}$.
81 $\int \csc \frac{1}{3}x \cot \frac{1}{3}x\, dx$.
82 $\displaystyle\int_{\pi/6}^{\pi/4} \csc^2 x \cot x\, dx$.
83 $\int \csc^2 3x\, dx$.
84 $\int (2 + 5 \tan x)^7 \sec^2 x\, dx$.
85 $\int \csc^4 x \cot x\, dx$.
86 $\int \sqrt{\cot x} \csc^2 x\, dx$.
87 $\int \cot^3 x \csc^2 x\, dx$.

88 The region under the curve $y = \sec x$ between $x = 0$ and $x = \pi/4$ is revolved about the x-axis. Find the volume of the solid of revolution generated in this way.

89 Solve Problem 88 for the curve $y = \sec^2 x$.

90 Sketch the graph of the function $y = \frac{1}{3} \tan 2x + \cot 2x$ on the interval $0 < x < \pi/4$ and find its minimum value.

91 A racing car is moving around a circular track at a constant speed of 100 km/h. There is a bright light at the center of the track and a straight fence tangent to the track at a point T. How fast is the shadow of the car moving along the fence when the car is $\frac{1}{8}$ lap beyond T?

*92 In Problem 18 of Section 4.4 students were asked to show that the volume of the smallest cone that can be circumscribed about a given sphere of radius a is exactly twice the volume of the sphere. Solve this problem by trigonometric methods, by taking the generating angle of a circumscribed cone (half the vertex angle) as the independent variable.

SECTION 9.5

93 Evaluate each of the following:
(a) $\tan^{-1}(-\sqrt{3})$; (b) $\sin^{-1} \frac{1}{2}\sqrt{3}$;
(c) $4 \sin^{-1}(-\frac{1}{2}\sqrt{2})$; (d) $\sin(\sin^{-1} 0.7)$;
(e) $\sin^{-1}(\sin 0.7)$; (f) $\tan^{-1}(\tan[-1])$;
(g) $\sin^{-1}(\cos \pi/6)$.

94 If the base b and area A of a triangle are fixed, use geometry alone to find the base angles if the angle opposite the base has its largest value.

Find dy/dx in each of Problems 95–103.

95 $y = \sin^{-1} \frac{1}{5}x$.

96 $y = \frac{1}{2} \tan^{-1} \frac{1}{2}x$.

97 $y = \frac{1}{5} \tan^{-1} x^5$.

98 $y = \sqrt{x} - \tan^{-1} \sqrt{x}$.

99 $y = \tan^{-1} \sqrt{x^2 - 1}$.

100 $y = -\sin^{-1} \dfrac{1}{x}$.

101 $y = \tan^{-1} x + \ln\sqrt{1 + x^2}$.

102 $y = a \sin^{-1} \dfrac{x}{a} + \sqrt{a^2 - x^2}$.

103 $y = \sqrt{x^2 - 1} - \tan^{-1} \sqrt{x^2 - 1}$.

Evaluate the integrals in Problems 104–112.

104 $\displaystyle\int_0^{\sqrt{3}} \frac{dx}{1 + x^2}$.

105 $\displaystyle\int_{-1/2}^{(1/2)\sqrt{3}} \frac{dx}{\sqrt{1 - x^2}}$.

106 $\displaystyle\int \frac{dx}{\sqrt{1 - 16x^2}}$.

107 $\displaystyle\int \frac{dx}{1 + 5x^2}$.

108 $\displaystyle\int_{1/\sqrt{3}}^{1} \frac{dx}{x\sqrt{4x^2 - 1}}$.

109 $\displaystyle\int \frac{dx}{\sqrt{25 - 4x^2}}$.

110 $\displaystyle\int \frac{dx}{49 + 36x^2}$.

111 $\displaystyle\int \frac{x^3 dx}{1 + x^8}$.

112 $\displaystyle\int \frac{15x^4\, dx}{\sqrt{1 - x^{10}}}$.

113 A billboard is perpendicular to a straight road, and its nearest edge is 18 ft from the road. The billboard is 54 ft wide. As a motorist approaches the billboard along the road, at what point does he see the billboard in the widest angle?

114 An airplane at an altitude of 7 mi and a speed of 500 mi/h is flying directly away from an observer on the ground. What is the rate of change of the angle of elevation when the airplane is over a point 4 mi away from the observer?

115 A woman is walking along a sidewalk at the rate of 6 ft/s. A police car spotlight 30 ft from the sidewalk follows her as she walks. At what rate is the spotlight turning when the woman is 40 ft past the point on the sidewalk nearest the light?

SECTION 9.6

116 With reference to Example 1, recall the definitions of kinetic and potential energy given in Section 7.7.

(a) Show that the potential energy V of the cart is $\frac{1}{2}kx^2$, where it is understood that $V = 0$ when $x = 0$.

(b) Show directly from Newton's second law of motion

$$m \frac{d^2x}{dt^2} = -kx$$

that the sum of the kinetic and potential energies of the cart is constant. Hint: Use equation (9) in Section 9.6.

(c) Express the total energy E of the cart in terms of its initial position x_0 and initial velocity v_0.

(d) Express the total energy E of the cart in terms of the amplitude A and frequency f of the vibration.

117 A block of wood 6 in on an edge and weighing 4 lb floats upright in water. If the block is depressed slightly and released, find its period of oscillation assuming that the friction of the water is negligible. Hint: Use $w = 62.5$ lb/ft^3 for the density of water.

118 A body in simple harmonic motion has amplitude A and period T. Find its maximum velocity.

119 Find the amplitude and frequency of the simple harmonic motion $x = 3 \sin 2t + 4 \cos 2t$.

120 If the period of a simple harmonic motion is $2\pi/3$, find a position function of the form (3) that satisfies the conditions $x = 1$ and $v = dx/dt = 3$ when $t = 0$.

***121** Let the pendulum in Example 3 be pulled to one side through an angle α and released. Use the principle of conservation of energy to show that the period T of oscillation is given by the formula

$$T = 4\sqrt{\frac{L}{2g}} \int_0^{\alpha} \frac{d\theta}{\sqrt{\cos\theta - \cos\alpha}}.$$

10 METHODS OF INTEGRATION

10.1 INTRODUCTION. THE BASIC FORMULAS

If we start with the constants and the seven familiar functions x, e^x, $\ln x$, $\sin x$, $\cos x$, $\sin^{-1} x$, and $\tan^{-1} x$, and go on to build all possible finite combinations of these by applying the algebraic operations and the process of forming a function of a function, then we generate the class of *elementary functions.* Thus,

$$\ln\left[\frac{\tan^{-1}(x^2 + 35x^3)}{e^x + \sin\sqrt{x^3 + 1}}\right]$$

is an elementary function. These functions are often said to have *closed form,* because they can be written down in explicit formulas involving only a finite number of familiar functions.

It is clear that the problem of calculating the derivative of an elementary function can always be solved by a systematic application of the rules developed in the preceding chapters, and this derivative is always an elementary function. However, the inverse problem of integration—which in general is much more important—is very different and has no such clear-cut solution.

As we know, the problem of calculating the indefinite integral of a function $f(x)$,

$$\int f(x)\,dx = F(x), \tag{1}$$

is equivalent to finding a function $F(x)$ such that

$$\frac{d}{dx}F(x) = f(x). \tag{2}$$

It is true that we have succeeded in integrating a good many elementary functions by inverting differentiation formulas. But this doesn't carry us very far, because it amounts to little more than calculating the integral (1) by knowing the answer (2) in advance.

The fact of the matter is this: There does not exist any systematic procedure that can always be applied to any elementary function and leads step by step to a guaranteed answer in closed form. Indeed, there may not even *be* an answer. For example, the function $f(x) = e^{-x^2}$ looks simple enough, but its integral

$$\int e^{-x^2}\,dx \tag{3}$$

cannot be calculated within the class of elementary functions. This assertion is more than merely a report on the present inability of mathematicians to integrate (3); it is a statement of a deep theorem, to the effect that no elementary function exists whose derivative is e^{-x^2}.*

If all this sounds discouraging, it shouldn't be. There is much more that can be done in the way of integration than we have suggested so far, and it is very important for students to acquire a certain amount of technical skill in carrying out integrations whenever they *are* possible. The fact that integration must be considered as more of an art than a systematic process really makes it more interesting than differentiation. It is more like solving puzzles, because there is less certainty and more scope for individual ingenuity. Many students find this an agreeable change from the cut-and-dried routines that make some parts of mathematics rather dull.

Since integration is differentiation read backwards, our starting point must be a short table of standard types of integrals obtained by inverting differentiation formulas as we have done in the previous chapters. Much more extensive tables than the one given below are available in libraries, and with the aid of these tables most of the problems in this chapter can be solved by merely looking them up. However, students should realize that if they follow such a course they will defeat the intended purpose of developing their own skills. For this reason we make no use of integral tables beyond the short list of 15 formulas given below. Instead, we urge students to concentrate their efforts on gaining a clear understanding of the various methods of integration and learning how to apply them.

In addition to the method of substitution, which is already familiar to us, there are three principal methods of integration to be studied in this chapter: reduction to trigonometric integrals, decomposition into partial fractions, and integration by parts. These methods enable us to transform a given integral in many ways. The object of these transformations is always to break up the given integral into a sum of simpler parts that can be integrated at once by means of familiar formulas. Students should therefore be certain that they have thoroughly memorized all the following basic formulas. These formulas should be so well learned that when one of them is needed it pops into the mind almost involuntarily, like the name of a friend.

1 $\displaystyle\int u^n\,du = \frac{u^{n+1}}{n+1} + c \qquad (n \neq -1).$

2 $\displaystyle\int \frac{du}{u} = \ln u + c.$

*Let there be no misunderstanding. The indefinite integral (3) *does* exist, because the function $F(x)$ defined by

$$F(x) = \int_0^x e^{-t^2}\,dt$$

is a perfectly respectable function with the property that

$$\frac{d}{dx}F(x) = e^{-x^2}.$$

[See equations (12) and (13) in Section 6.7.] The difficulty is that it can be proved that there is no way of expressing $F(x)$ as an elementary function. Some of the facts in this interesting part of calculus are described in Appendix A.9.

3 $\int e^u\,du = e^u + c.$

4 $\int \cos u\,du = \sin u + c.$

5 $\int \sin u\,du = -\cos u + c.$

6 $\int \sec^2 u\,du = \tan u + c.$

7 $\int \csc^2 u\,du = -\cot u + c.$

8 $\int \sec u \tan u\,du = \sec u + c.$

9 $\int \csc u \cot u\,du = -\csc u + c.$

10 $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\frac{u}{a} + c.$

11 $\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + c.$

12 $\int \tan u\,du = -\ln(\cos u) + c.$

13 $\int \cot u\,du = \ln(\sin u) + c.$

14 $\int \sec u\,du = \ln(\sec u + \tan u) + c.$

15 $\int \csc u\,du = -\ln(\csc u + \cot u) + c.$

The last four formulas are new, and complete our list of the integrals of the six trigonometric functions. Formulas 12 and 13 can be found by a straightforward process:

$$\int \tan u\,du = \int \frac{\sin u\,du}{\cos u} = -\int \frac{d(\cos u)}{\cos u} = -\ln(\cos u) + c$$

and

$$\int \cot u\,du = \int \frac{\cos u\,du}{\sin u} = \int \frac{d(\sin u)}{\sin u} = \ln(\sin u) + c.$$

Many people find that the easiest way to remember these two formulas is to think of the process by which they are obtained. Formula 14 can be found by an ingenious trick: If we multiply the integrand by $1 = (\sec u + \tan u)/(\sec u + \tan u)$, then we obtain

$$\int \sec u\,du = \int \frac{(\sec u + \tan u)\sec u\,du}{\sec u + \tan u} = \int \frac{(\sec^2 u + \sec u \tan u)\,du}{\sec u + \tan u}$$

$$= \int \frac{d(\sec u + \tan u)}{\sec u + \tan u} = \ln(\sec u + \tan u) + c.$$

A similar trick yields formula 15.

We repeat: These 15 formulas constitute the foundation on which we build throughout this chapter, and they must be at our fingertips. Take 20 or 30 minutes to memorize them. And then tomorrow, when they have been partially forgotten, memorize them again. And so on. The effort will be well rewarded.

10.2 THE METHOD OF SUBSTITUTION

In the method of substitution we introduce the auxiliary variable u as a new symbol for part of the integrand in the hope that its differential du will account for some other part and thereby reduce the complete integral to an easily recognizable form. Success in the use of this method depends on choosing a fruitful substitution, and this in turn depends on the ability to see at a glance that part of the integrand is the derivative of some other part.

We give several examples to help students review the procedure and make certain that they fully understand it.

Example 1 Find $\int xe^{-x^2}\,dx$.

Solution If we put $u = -x^2$, then $du = -2x\,dx$, $x\,dx = -\frac{1}{2}\,du$, and therefore

$$\int xe^{-x^2}\,dx = -\tfrac{1}{2}\int e^u\,du = -\tfrac{1}{2}e^u = -\tfrac{1}{2}e^{-x^2} + c.$$

It will be noticed that we insert the constant of integration only in the last step. Strictly speaking, this is incorrect; but we willingly commit this minor error in order to avoid cluttering up the previous steps with repeated c's. We also point out that this integral is easy to calculate even though the similar integral $\int e^{-x^2}\,dx$ is impossible. The reason for this is clearly the presence of the factor x, which is essentially (that is, up to a constant factor) the derivative of the exponent $-x^2$.

Example 2 Find

$$\int \frac{\cos x\,dx}{\sqrt{1+\sin x}}.$$

Solution Here we notice that $\cos x\,dx$ is the differential of $\sin x$, and also of $1 + \sin x$. Thus, if we put $u = 1 + \sin x$, then $du = \cos x\,dx$ and

$$\int \frac{\cos x\,dx}{\sqrt{1+\sin x}} = \int \frac{du}{\sqrt{u}} = \int u^{-1/2}\,du$$

$$= \frac{u^{1/2}}{\frac{1}{2}} = 2\sqrt{u} = 2\sqrt{1+\sin x} + c.$$

Example 3 Find

$$\int \frac{dx}{x\ln x}.$$

Solution The fact that dx/x is the differential of $\ln x$ suggests the substitution $u = \ln x$, so $du = dx/x$ and

$$\int \frac{dx}{x\ln x} = \int \frac{du}{u} = \ln u = \ln(\ln x) + c.$$

Example 4 Find

$$\int \frac{dx}{\sqrt{9-4x^2}}.$$

Solution Since $4x^2 = (2x)^2$ we put $u = 2x$, so that $du = 2dx$, $dx = \frac{1}{2}\,du$, and

$$\int \frac{dx}{\sqrt{9-4x^2}} = \frac{1}{2}\int \frac{du}{\sqrt{9-u^2}} = \frac{1}{2}\sin^{-1}\frac{u}{3} = \frac{1}{2}\sin^{-1}\frac{2x}{3} + c.$$

Example 5 Find

$$\int \frac{x\,dx}{\sqrt{9-4x^2}}.$$

Solution Here the fact that the x in the numerator is essentially the derivative of the expression $9 - 4x^2$ inside the radical suggests the substitution $u = 9 - 4x^2$. Then $du = -8x\,dx$, and

$$\int \frac{x\,dx}{\sqrt{9-4x^2}} = -\frac{1}{8}\int \frac{du}{\sqrt{u}} = -\frac{1}{8}\int u^{-1/2}\,du$$

$$= -\frac{1}{8}\frac{u^{1/2}}{\frac{1}{2}} = -\frac{1}{4}\sqrt{u} = -\frac{1}{4}\sqrt{9-4x^2} + c.$$

In any particular integration problem the choice of the substitution is a matter of trial and error guided by experience. If our first substitution doesn't work, we should feel no hesitation about discarding it and trying another. Example 5 is similar in appearance to Example 4 and it might be thought that the same substitution will work again, but in fact—as we have seen—it requires an entirely different substitution.

We remind students of the summary of the method of substitution given at the end of Section 5.3. Also, we repeat the justification of the method given there because we now wish to extend this method to cover the case of definite integrals as well.

We start with a complicated integral of the form

$$\int f[g(x)]g'(x)\,dx. \tag{1}$$

If we put $u = g(x)$, then $du = g'(x)\,dx$ and the integral takes the new form

$$\int f(u)\,du.$$

If we can integrate this, so that

$$\int f(u)\,du = F(u) + c, \tag{2}$$

then since $u = g(x)$ we ought to be able to integrate (1) by writing

$$\int f[g(x)]g'(x)\,dx = F[g(x)] + c. \tag{3}$$

All that is needed to justify our procedure is to notice that (3) is a correct result, because

$$\frac{d}{dx} F[g(x)] = F'[g(x)]g'(x) = f[g(x)]g'(x)$$

by the chain rule.

The method of substitution applies to definite integrals as well as indefinite integrals. The crucial requirement is that the limits of integration must be suitably changed when the substitution is made. This can be expressed as follows:

$$\int_a^b f[g(x)]g'(x)\,dx = \int_c^d f(u)\,du,$$

where $c = g(a)$ and $d = g(b)$. The proof uses (2) and (3) and two applications of the Fundamental Theorem of Calculus,

$$\int_a^b f[g(x)]g'(x)\,dx = F[g(b)] - F[g(a)]$$

$$= F(d) - F(c) = \int_c^d f(u)\,du.$$

Thus, once the original integral is changed into a simpler integral in the variable u, the numerical evaluation can be carried out entirely in terms of u, provided the limits of integration are also correctly changed.

Example 6 Compute

$$\int_0^{\pi/3} \frac{\sin x\,dx}{\cos^2 x}.$$

Solution We put $u = \cos x$, so that $du = -\sin x\,dx$. Observe that $u = 1$ when $x = 0$ and $u = \frac{1}{2}$ when $x = \pi/3$. By changing both the variable of integration and the limits of integration we obtain

$$\int_0^{\pi/3} \frac{\sin x\,dx}{\cos^2 x} = \int_1^{1/2} \frac{-du}{u^2} = \frac{1}{u}\Big]_1^{1/2} = 2 - 1 = 1.$$

This technique removes the necessity of returning to the original variable in order to make the final numerical evaluation.

PROBLEMS

Find the following integrals.

1 $\int \sqrt{3 - 2x}\,dx.$

2 $\int \frac{2x\,dx}{(4x^2 - 1)^2}.$

3 $\int \frac{\ln x\,dx}{x[1 + (\ln x)^2]}.$

4 $\int \cos x\, e^{\sin x}\,dx.$

5 $\int \sin 2x\,dx.$

6 $\int \frac{x\,dx}{\sqrt{16 - x^4}}.$

7 $\int \cot(3x - 1)\,dx.$

8 $\int \sin x \cos x\,dx.$

9 $\int x\sqrt{x^2 + 1}\,dx.$

10 $\int \frac{dx}{x + 2}.$

11 $\int e^{5x}\,dx.$

12 $\int x \cos x^2\,dx.$

13 $\int \csc^2(3x + 2)\,dx.$

14 $\int \frac{dx}{x^2 + 16}.$

15 $\int_{-3}^{1} \frac{dx}{\sqrt{3 - 2x}}.$

16 $\int (x^3 + 1)^2\,dx.$

17 $\int \frac{\sin x\,dx}{\sqrt{1 - \cos x}}.$

18 $\int \frac{(2x + 1)\,dx}{x^2 + x + 2}.$

19 $\int \frac{e^{\tan^{-1} x}}{1 + x^2}\,dx.$

20 $\int \frac{\sin \sqrt{x}}{\sqrt{x}}\,dx.$

21 $\int \sec 5x \tan 5x\, dx$.

22 $\int \frac{dx}{x\sqrt{\ln x}}$.

23 $\int \frac{\ln x\, dx}{x}$.

24 $\int \frac{\sin x\, dx}{\cos^2 x}$.

25 $\int_0^{\pi/2} \frac{\cos x\, dx}{1 + \sin x}$.

26 $\int \cos 3x\, dx$.

27 $\int \frac{e^x\, dx}{\sqrt{1 - e^{2x}}}$.

28 $\int \frac{dx}{\cos 2x}$.

29 $\int \sin^2 x \cos x\, dx$.

30 $\int_0^3 \tan^2 \frac{1}{3}x \sec^2 \frac{1}{3}x\, dx$.

31 $\int \frac{e^x\, dx}{1 + e^x}$.

32 $\int \frac{\cos (\ln x)\, dx}{x}$.

33 $\int \tan 3x\, dx$.

34 $\int \frac{\sec^2 x\, dx}{\sqrt{1 + \tan x}}$.

35 $\int \frac{4x\, dx}{\sqrt{x^2 + 1}}$.

36 $\int \frac{e^{\sqrt{x}}\, dx}{\sqrt{x}}$.

37 $\int \frac{e^x\, dx}{1 + e^{2x}}$.

38 $\int \frac{\sin^{-1} x\, dx}{\sqrt{1 - x^2}}$.

39 $\int (e^x + 1)^6 e^x\, dx$.

40 $\int 6x^2 e^{-x^3}\, dx$.

41 $\int \sec^2 5x\, dx$.

42 $\int \cot 4x\, dx$.

43 $\int \csc 2x \cot 2x\, dx$.

44 $\int_2^3 \frac{2x\, dx}{x^2 - 3}$.

Compute each of the following definite integrals by making a suitable substitution and changing the limits of integration.

45 $\int_1^2 \frac{(2x + 1)\, dx}{\sqrt{x^2 + x + 2}}$.

46 $\int_0^{\pi/4} \tan^2 x \sec^2 x\, dx$.

47 $\int_1^e \frac{\sqrt{\ln x}\, dx}{x}$.

48 $\int_0^{\pi/3} \sec^3 x \tan x\, dx$.

49 Each of the following integrals is easy to compute for a particular value of n. Find this value and carry out the integration. For example, $\int x^n \sin x^2\, dx$ is easily computed for $n = 1$:

$$\int x \sin x^2\, dx = -\tfrac{1}{2} \cos x^2 + c.$$

(a) $\int x^n e^{x^4}\, dx$. (b) $\int x^n \cos x^3\, dx$.
(c) $\int x^n \ln x\, dx$. (d) $\int x^n \sec^2 \sqrt{x}\, dx$.

50 The derivation given in the text for formula 14 is somewhat tainted by rabbit-out-of-the-hat trickery. Derive this formula in a more reasonable way by using

$$\int \sec u\, du = \int \frac{du}{\cos u} = \int \frac{\cos u\, du}{\cos^2 u} = \int \frac{\cos u\, du}{1 - \sin^2 u}$$

to write the given integral as an integral of the form $\int du/(1 - u^2)$, and then use

$$\frac{1}{1 - u^2} = \frac{1}{2}\left(\frac{1}{1 + u} + \frac{1}{1 - u}\right).$$

51 Give a similar derivation for formula 15.

10.3 CERTAIN TRIGONOMETRIC INTEGRALS

In the next two sections we discuss several methods for reducing a given integral to one involving trigonometric functions. It will therefore be useful to increase our ability to calculate such trigonometric integrals.

A power of a trigonometric function multiplied by its differential is easy to integrate. Thus,

$$\int \sin^3 x \cos x\, dx = \int \sin^3 x\, d(\sin x) = \tfrac{1}{4} \sin^4 x + c$$

and

$$\int \tan^2 x \sec^2 x\, dx = \int \tan^2 x\, d(\tan x) = \tfrac{1}{3} \tan^3 x + c.$$

Other trigonometric integrals can often be reduced to problems of this type by using appropriate trigonometric identities.

We begin by considering integrals of the form

$$\int \sin^m x \cos^n x\, dx, \tag{1}$$

where one of the exponents is an odd positive integer. If n is odd, we factor out $\cos x\,dx$, which is $d(\sin x)$; and since an even power of $\cos x$ remains, we can use the identity $\cos^2 x = 1 - \sin^2 x$ to express the remaining part of the integrand entirely in terms of $\sin x$. And if m is odd, we factor out $\sin x\,dx$, which is $-d(\cos x)$, and use the identity $\sin^2 x = 1 - \cos^2 x$ in a similar way. The following two examples illustrate the procedure.

Example 1

$$\begin{aligned}\int \sin^2 x \cos^3 x\,dx &= \int \sin^2 x \cos^2 x \cos x\,dx\\ &= \int \sin^2 x(1 - \sin^2 x)\,d(\sin x)\\ &= \int (\sin^2 x - \sin^4 x)\,d(\sin x)\\ &= \tfrac{1}{3}\sin^3 x - \tfrac{1}{5}\sin^5 x + c.\end{aligned}$$

Example 2

$$\begin{aligned}\int \sin^3 x\,dx &= \int \sin^2 x \sin x\,dx\\ &= -\int (1 - \cos^2 x)\,d(\cos x)\\ &= -\cos x + \tfrac{1}{3}\cos^3 x + c.\end{aligned}$$

If one of the exponents in (1) is an odd positive integer that is quite large, it may be necessary to use the binomial theorem, and in such a case an explicit use of the method of substitution may be desirable for the sake of clarity. For instance, every odd positive power of $\cos x$, whether large or small, has the form

$$\cos^{2n+1} x = \cos^{2n} x \cos x = (\cos^2 x)^n \cos x = (1 - \sin^2 x)^n \cos x,$$

where n is a nonnegative integer. If we put $u = \sin x$ and $du = \cos x\,dx$, then

$$\begin{aligned}\int \cos^{2n+1} x\,dx &= \int (1 - \sin^2 x)^n \cos x\,dx\\ &= \int (1 - u^2)^n\,du.\end{aligned}$$

If necessary, the expression $(1 - u^2)^n$ can now be expanded by applying the binomial theorem, and the resulting polynomial in u is easy to integrate term by term.

If both exponents in (1) are nonnegative even integers, then it is necessary to change the form of the integrand by using the half-angle formulas

$$\cos^2\theta = \tfrac{1}{2}(1 + \cos 2\theta) \qquad \text{and} \qquad \sin^2\theta = \tfrac{1}{2}(1 - \cos 2\theta). \tag{2}$$

We hope students have thoroughly memorized these important formulas, but if they are forgotten they can easily be recovered by adding and subtracting the identities

$$\cos^2\theta + \sin^2\theta = 1,$$
$$\cos^2\theta - \sin^2\theta = \cos 2\theta.$$

The uses of (2) are shown in the following examples.

Example 3 The half-angle formula for the cosine enables us to write

$$\int \cos^2 x\,dx = \tfrac{1}{2}\int (1+\cos 2x)\,dx = \tfrac{1}{2}\int dx + \tfrac{1}{2}\int \cos 2x\,dx$$
$$= \tfrac{1}{2}x + \tfrac{1}{4}\int \cos 2x\,d(2x) = \tfrac{1}{2}x + \tfrac{1}{4}\sin 2x + c.$$

If we wish to express this result in terms of the variable x (instead of $2x$), we use the double-angle formula $\sin 2x = 2\sin x\cos x$ and write

$$\int \cos^2 x\,dx = \tfrac{1}{2}x + \tfrac{1}{2}\sin x\cos x + c.$$

Example 4 Two successive applications of the half-angle formula for the cosine give

$$\cos^4 x = (\cos^2 x)^2 = \tfrac{1}{4}(1+\cos 2x)^2 = \tfrac{1}{4}(1 + 2\cos 2x + \cos^2 2x)$$
$$= \tfrac{1}{4}[1 + 2\cos 2x + \tfrac{1}{2}(1+\cos 4x)]$$
$$= \tfrac{3}{8} + \tfrac{1}{2}\cos 2x + \tfrac{1}{8}\cos 4x,$$

so

$$\int \cos^4 x\,dx = \tfrac{3}{8}x + \tfrac{1}{4}\sin 2x + \tfrac{1}{32}\sin 4x + c.$$

As these examples show, the value of the half-angle formulas (2) for this work lies in the fact that they allow us to reduce the exponent by a factor of $\frac{1}{2}$ at the expense of multiplying the angle by 2, which is a considerable advantage purchased at very low cost.

Example 5 By using both of the half-angle formulas we get

$$\int \sin^2 x\cos^2 x\,dx = \int \frac{1-\cos 2x}{2}\cdot\frac{1+\cos 2x}{2}\,dx$$
$$= \tfrac{1}{4}\int (1-\cos^2 2x)\,dx = \tfrac{1}{4}\int [1 - \tfrac{1}{2}(1+\cos 4x)]\,dx$$
$$= \tfrac{1}{8}\int dx - \tfrac{1}{8}\int \cos 4x\,dx = \tfrac{1}{8}x - \tfrac{1}{32}\sin 4x + c.$$

We can also find this integral by combining the results of Examples 3 and 4:

$$\int \sin^2 x\cos^2 x\,dx = \int (1-\cos^2 x)\cos^2 x\,dx$$
$$= \int \cos^2 x\,dx - \int \cos^4 x\,dx$$
$$= \tfrac{1}{2}x + \tfrac{1}{4}\sin 2x - \tfrac{3}{8}x - \tfrac{1}{4}\sin 2x - \tfrac{1}{32}\sin 4x$$
$$= \tfrac{1}{8}x - \tfrac{1}{32}\sin 4x + c.$$

We next consider integrals of the form

$$\int \tan^m x \sec^n x \, dx,$$

where n is an even positive integer or m is an odd positive integer. Our work is based on the fact that $d(\tan x) = \sec^2 x \, dx$ and $d(\sec x) = \sec x \tan x \, dx$, and we exploit the identity $\tan^2 x + 1 = \sec^2 x$. An example illustrating each case will be enough to show the general method.

Example 6

$$\begin{aligned}\int \tan^4 x \sec^6 x \, dx &= \int \tan^4 x \sec^4 x \sec^2 x \, dx \\ &= \int \tan^4 x \, (\tan^2 x + 1)^2 \, d(\tan x) \\ &= \int \tan^4 x \, (\tan^4 x + 2 \tan^2 x + 1) \, d(\tan x) \\ &= \int (\tan^8 x + 2 \tan^6 x + \tan^4 x) \, d(\tan x) \\ &= \tfrac{1}{9} \tan^9 x + \tfrac{2}{7} \tan^7 x + \tfrac{1}{5} \tan^5 x + c.\end{aligned}$$

Example 7

$$\begin{aligned}\int \tan^3 x \sec^5 x \, dx &= \int \tan^2 x \sec^4 x \sec x \tan x \, dx \\ &= \int (\sec^2 x - 1) \sec^4 x \, d(\sec x) \\ &= \int (\sec^6 x - \sec^4 x) \, d(\sec x) \\ &= \tfrac{1}{7} \sec^7 x - \tfrac{1}{5} \sec^5 x + c.\end{aligned}$$

In essentially the same way we can handle integrals of the form

$$\int \cot^m x \csc^n x \, dx,$$

where n is an even positive integer or m is an odd positive integer. Our tools in these cases are the formulas $d(\cot x) = -\csc^2 x \, dx$ and $d(\csc x) = -\csc x \cot x \cdot dx$, and when necessary we use the identity $1 + \cot^2 x = \csc^2 x$.

Another approach to trigonometric integrals that is sometimes useful is to express each function occurring in the integral in terms of sines and cosines alone.

Example 8 We already know from our work with derivatives that

$$\int \sec x \tan x \, dx = \sec x + c.$$

However, this formula can also be obtained directly, by writing

$$\int \sec x \tan x \, dx = \int \frac{1}{\cos x} \frac{\sin x}{\cos x} \, dx = \int \frac{\sin x \, dx}{\cos^2 x}.$$

If we now put $u = \cos x$ and $du = -\sin x\, dx$, then we get

$$\int \sec x \tan x\, dx = \int \frac{\sin x\, dx}{\cos^2 x}$$
$$= \int \frac{-du}{u^2} = \frac{1}{u} = \frac{1}{\cos x} = \sec x + c.$$

PROBLEMS

Find each of the following integrals.

1 $\int \sin^2 x\, dx.$

2 $\int \sin^4 x\, dx.$

3 $\int \cos^6 x\, dx.$

4 $\int \cos^2 3x\, dx.$

5 $\int \sin^3 x \cos^2 x\, dx.$

6 $\int \sin^2 x \cos^5 x\, dx.$

7 $\int \cos^3 x\, dx.$

8 $\int_0^{\pi/2} \sin^3 x \cos^3 x\, dx.$

9 $\int \sqrt{\sin x} \cos^3 x\, dx.$

10 $\int \sin^3 5x \cos 5x\, dx.$

11 $\int \sin^2 3x \cos^2 3x\, dx.$

12 $\int \dfrac{dx}{\sin x \cos x}.$

13 $\int_0^{\pi/4} \sec^4 x\, dx.$

14 $\int \dfrac{dx}{\cos^2 x}.$

15 $\int \tan^5 x \sec^3 x\, dx.$

16 $\int \csc^4 x\, dx.$

17 $\int \cot^2 x\, dx.$

18 $\int \cot^3 x\, dx.$

19 $\int \dfrac{dx}{\sin^2 4x}.$

20 $\int \cot^2 5x \csc^4 5x\, dx.$

21 $\int \dfrac{1 + \cos 2x}{\sin^2 2x}\, dx.$

22 $\int \tan^2 x \cos x\, dx.$

23 $\int \sin 3x \cot 3x\, dx.$

24 Find $\int \tan x\, dx$ (which we already know) by the method of Example 7.

25 Use the identity $\tan^2 x = \sec^2 x - 1$ to find
(a) $\int \tan^2 x\, dx$, $\int \tan^4 x\, dx$, $\int \tan^6 x\, dx$;
(b) $\int \tan^3 x\, dx$, $\int \tan^5 x\, dx$, $\int \tan^7 x\, dx$.

26 If n is any positive integer ≥ 2, show that

$$\int \tan^n x\, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\, dx.$$

This is called a *reduction formula*, because it reduces the problem of integrating $\tan^n x$ to the problem of integrating $\tan^{n-2} x$.

27 Find the volume of the solid of revolution generated when the indicated region under each of the following curves is revolved about the x-axis:
(a) $y = \sin x$, $0 \leq x \leq \pi$;
(b) $y = \sec x$, $0 \leq x \leq \pi/4$;
(c) $y = \tan 2x$, $0 \leq x \leq \pi/8$;
(d) $y = \cos^2 x$, $\pi/2 \leq x \leq \pi$.

28 Find the length of the curve $y = \ln(\cos x)$ between $x = 0$ and $x = \pi/4$.

29 Find $\int \sec^3 x\, dx$ by exploiting the observation that $\sec^3 x$ will clearly appear in the derivative of $\sec x \tan x$.

30 Find $\int \csc^3 x\, dx$ by adapting the idea suggested for Problem 29.

10.4 TRIGONOMETRIC SUBSTITUTIONS

An integral involving one of the radical expressions $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, $\sqrt{x^2 - a^2}$ (where a is a positive constant) can often be transformed into a familiar trigonometric integral by using a suitable trigonometric substitution or change of variable.

There are three cases, which depend on the trigonometric identities

$$1 - \sin^2 \theta = \cos^2 \theta, \tag{1}$$
$$1 + \tan^2 \theta = \sec^2 \theta, \tag{2}$$
$$\sec^2 \theta - 1 = \tan^2 \theta. \tag{3}$$

If the given integral involves $\sqrt{a^2 - x^2}$, then changing the variable from x to θ by writing

$$x = a \sin \theta \qquad \text{replaces} \qquad \sqrt{a^2 - x^2} \qquad \text{by} \qquad a \cos \theta, \tag{4}$$

because $a^2 - x^2 = a^2 - a^2 \sin^2 \theta = a^2(1 - \sin^2 \theta) = a^2 \cos^2 \theta$. Similarly, if the given integral involves $\sqrt{a^2 + x^2}$, then by identity (2) we see that the substitution

$$x = a \tan \theta \qquad \text{replaces} \qquad \sqrt{a^2 + x^2} \qquad \text{by} \qquad a \sec \theta, \tag{5}$$

because $a^2 + x^2 = a^2 + a^2 \tan^2 \theta = a^2(1 + \tan^2 \theta) = a^2 \sec^2 \theta$; and if it involves $\sqrt{x^2 - a^2}$, then by identity (3) the substitution

$$x = a \sec \theta \qquad \text{replaces} \qquad \sqrt{x^2 - a^2} \qquad \text{by} \qquad a \tan \theta, \tag{6}$$

because $x^2 - a^2 = a^2 \sec^2 \theta - a^2 = a^2(\sec^2 \theta - 1) = a^2 \tan^2 \theta$. We illustrate these procedures as follows.

Example 1 Find

$$\int \frac{\sqrt{a^2 - x^2}}{x}\, dx.$$

Solution This integral is of the first type, so we write

$$x = a \sin \theta, \qquad dx = a \cos \theta \, d\theta, \qquad \sqrt{a^2 - x^2} = a \cos \theta.$$

Then

$$\begin{aligned}
\int \frac{\sqrt{a^2 - x^2}}{x}\, dx &= \int \frac{a \cos \theta}{a \sin \theta}\, a \cos \theta \, d\theta = a \int \frac{\cos^2 \theta}{\sin \theta}\, d\theta \\
&= a \int \frac{1 - \sin^2 \theta}{\sin \theta}\, d\theta = a \int (\csc \theta - \sin \theta)\, d\theta \\
&= -a \ln (\csc \theta + \cot \theta) + a \cos \theta.
\end{aligned} \tag{7}$$

This completes the integration, and we now must write the answer in terms of the original variable x. We do this quickly and easily by drawing a right triangle (Fig. 10.1) whose sides are labeled in the simplest way that is consistent with the equation $x = a \sin \theta$ or $\sin \theta = x/a$. This figure tells us at once that

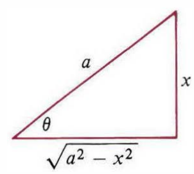


Figure 10.1

$$\csc \theta = \frac{a}{x}, \qquad \cot \theta = \frac{\sqrt{a^2 - x^2}}{x}, \qquad \text{and} \qquad \cos \theta = \frac{\sqrt{a^2 - x^2}}{a},$$

so from (7) we have

$$\int \frac{\sqrt{a^2 - x^2}}{x}\, dx = \sqrt{a^2 - x^2} - a \ln \left(\frac{a + \sqrt{a^2 - x^2}}{x} \right) + c.$$

Example 2 Find

$$\int \frac{dx}{\sqrt{a^2 + x^2}}.$$

Solution Here we have an integral of the second type, so we write

$$x = a \tan \theta, \qquad dx = a \sec^2 \theta \, d\theta, \qquad \sqrt{a^2 + x^2} = a \sec \theta.$$

This yields

$$\begin{aligned}
\int \frac{dx}{\sqrt{a^2 + x^2}} = \int \frac{a \sec^2 \theta \, d\theta}{a \sec \theta} &= \int \sec \theta \, d\theta \\
&= \ln (\sec \theta + \tan \theta).
\end{aligned} \tag{8}$$

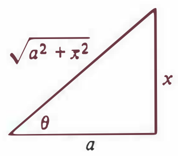


Figure 10.2

The substitution equation $x = a \tan \theta$ or $\tan \theta = x/a$ is pictured in Fig. 10.2, and from this figure we obtain

$$\sec \theta = \frac{\sqrt{a^2 + x^2}}{a} \quad \text{and} \quad \tan \theta = \frac{x}{a}.$$

We therefore continue the calculation in (8) by writing

$$\int \frac{dx}{\sqrt{a^2 + x^2}} = \ln \left(\frac{\sqrt{a^2 + x^2} + x}{a} \right) + c' \tag{9}$$

$$= \ln (\sqrt{a^2 + x^2} + x) + c. \tag{10}$$

Students will notice that since

$$\ln \left(\frac{\sqrt{a^2 + x^2} + x}{a} \right) = \ln (\sqrt{a^2 + x^2} + x) - \ln a,$$

the constant $-\ln a$ has been grouped together with the constant of integration c', and the quantity $-\ln a + c'$ is then rewritten as c. Usually we don't bother to make notational distinctions between one constant of integration and another, because all are completely arbitrary; but we do so here in the hope of clarifying the transition from (9) to (10).

Example 3 Find

$$\int \frac{\sqrt{x^2 - a^2}}{x} dx.$$

Solution This integral is of the third type, so we write

$$x = a \sec \theta, \qquad dx = a \sec \theta \tan \theta \, d\theta, \qquad \sqrt{x^2 - a^2} = a \tan \theta.$$

Then

$$\int \frac{\sqrt{x^2 - a^2}}{x} dx = \int \frac{a \tan \theta}{a \sec \theta} a \sec \theta \tan \theta \, d\theta$$

$$= a \int \tan^2 \theta \, d\theta = a \int (\sec^2 \theta - 1) \, d\theta$$

$$= a \tan \theta - a\theta.$$

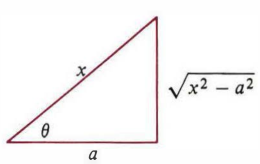


Figure 10.3

In this case our substitution equation $\sec \theta = x/a$ is portrayed in Fig. 10.3, which tells us that

$$\tan \theta = \frac{\sqrt{x^2 - a^2}}{a} \quad \text{and} \quad \theta = \tan^{-1} \frac{\sqrt{x^2 - a^2}}{a}.$$

The desired integral can therefore be written as

$$\int \frac{\sqrt{x^2 - a^2}}{x} dx = \sqrt{x^2 - a^2} - a \tan^{-1} \frac{\sqrt{x^2 - a^2}}{a} + c.$$

There is one feature of these calculations that we have not taken into account. In (4) we tacitly wrote

$$\sqrt{1 - \sin^2 \theta} = \cos \theta$$

without checking the correctness of the algebraic sign. This was careless, because $\cos \theta$ is sometimes negative and sometimes positive. However, the variable θ, which in this case is $\sin^{-1} x/a$, is restricted to the interval $-\pi/2 \leq \theta \leq \pi/2$, and on this interval $\cos \theta$ is nonnegative, as we assumed. Similar comments apply to the substitutions (5) and (6).

Example 4 As a concrete illustration of the use of these methods, we determine the equation of the *tractrix*. This famous curve can be defined as follows: It is the path of an object dragged along a horizontal plane by a string of constant length when the other end of the string moves along a straight line in the plane. (The word "tractrix" comes from the Latin *tractere,* meaning "to drag.")

Suppose the plane is the xy-plane and the object starts at the point $(a, 0)$ with the other end of the string at the origin. If this end moves up the y-axis as shown on the left in Fig. 10.4, then the string is always tangent to the curve, and the length of the tangent between the y-axis and the point of contact is always equal to a. The slope of the tangent is therefore given by the formula

$$\frac{dy}{dx} = -\frac{\sqrt{a^2 - x^2}}{x};$$

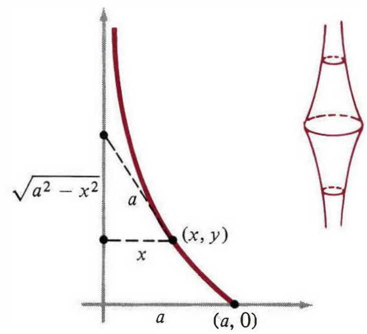


Figure 10.4

and by separating the variables and using the result of Example 1, we have

$$y = -\int \frac{\sqrt{a^2 - x^2}}{x}\, dx = a \ln \left(\frac{a + \sqrt{a^2 - x^2}}{x}\right) - \sqrt{a^2 - x^2} + c.$$

Since $y = 0$ when $x = a$, we see that $c = 0$, so

$$y = a \ln \left(\frac{a + \sqrt{a^2 - x^2}}{x}\right) - \sqrt{a^2 - x^2}$$

is the equation of the tractrix, or at least of the part shown in the figure.

If the end of the string moves down the y-axis, then another part of the curve is generated; and if these two parts are revolved about the y-axis, the resulting "double-trumpet" surface shown on the right in Fig. 10.4 is called a *pseudosphere.* In the branch of mathematics concerned with the geometry of curved surfaces, the pseudosphere is a model for Lobachevsky's version of non-Euclidean geometry. It is a surface of constant negative curvature, and the sum of the angles of any triangle on the surface is less than 180°.

Another famous curve whose equation can be determined by these methods of integration is the *catenary,* which is the curve assumed by a flexible chain or cable hanging between two fixed points. The details are a bit complicated, so we give a derivation in Appendix 1 at the end of this chapter for students who have chosen to omit the optional Section 9.7.

The substitution procedures described in this section can be given a general justification or proof similar to that provided in Section 10.2. Students who are interested in such matters will find the details in Appendix A.10.

PROBLEMS

Find each of the following integrals.

1 $\displaystyle\int \frac{\sqrt{a^2 - x^2}}{x^2}\,dx.$

2 $\displaystyle\int \frac{x^2\,dx}{\sqrt{4 - x^2}}.$

3 $\displaystyle\int \frac{dx}{(a^2 + x^2)^2}.$

4 $\displaystyle\int \frac{dx}{x^2\sqrt{a^2 + x^2}}.$

5 $\displaystyle\int \frac{x^3\,dx}{\sqrt{9 - x^2}}.$

6 $\displaystyle\int \frac{dx}{x\sqrt{a^2 - x^2}}.$

7 $\displaystyle\int \frac{dx}{x\sqrt{a^2 + x^2}}.$

8 $\displaystyle\int \frac{dx}{x + x^3}.$

9 $\displaystyle\int \frac{dx}{\sqrt{x^2 - a^2}}.$

10 $\displaystyle\int \frac{dx}{x^3\sqrt{x^2 - a^2}}.$

11 $\displaystyle\int \sqrt{a^2 + x^2}\,dx.$*

12 $\displaystyle\int \frac{x^3\,dx}{a^2 + x^2}.$

13 $\displaystyle\int \frac{dx}{a^2 - x^2}.$

14 $\displaystyle\int \frac{dx}{(a^2 - x^2)^{3/2}}.$

15 $\displaystyle\int \frac{\sqrt{a^2 + x^2}}{x}\,dx.$

16 $\displaystyle\int x^3\sqrt{a^2 + x^2}\,dx.$

17 $\displaystyle\int \frac{\sqrt{x^2 - a^2}}{x^2}\,dx.$

18 $\displaystyle\int \frac{dx}{(x^2 - a^2)^{3/2}}.$

19 $\displaystyle\int x^2\sqrt{a^2 - x^2}\,dx.$

20 $\displaystyle\int (1 - 4x^2)^{3/2}\,dx.$

The following integrals would normally be found in a different way, but this time work them out by using trigonometric substitutions.

21 $\displaystyle\int \frac{x\,dx}{\sqrt{4 - x^2}}.$

22 $\displaystyle\int \frac{x\,dx}{(a^2 - x^2)^{3/2}}.$

23 $\displaystyle\int \frac{dx}{a^2 + x^2}.$

24 $\displaystyle\int \frac{x\,dx}{4 + x^2}.$

*Hint: See Problem 29 in Section 10.3.

25 $\displaystyle\int x\sqrt{9 - x^2}\,dx.$

26 $\displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}}.$

27 $\displaystyle\int \frac{x\,dx}{\sqrt{9 + x^2}}.$

28 $\displaystyle\int \frac{x\,dx}{\sqrt{x^2 - 4}}.$

29 Use integration to show that the area of a circle of radius a is πa^2.

30 In a circle of radius a, a chord b units from the center cuts off a chunk of the circle called a *segment*. Find a formula for the area of this segment.

31 If the circle $(x - b)^2 + y^2 = a^2$ $(0 < a < b)$ is revolved about the y-axis, the resulting solid of revolution is called a *torus* (see Problem 11 in Section 7.3). Use the shell method to find the volume of this torus.

32 Find the length of the parabola $y = x^2$ between $x = 0$ and $x = 1$. Hint: Use the result of Problem 29 in Section 10.3.

33 Find the length of the curve $y = \ln x$ between $x = 1$ and $x = \sqrt{8}$.

34 The given region under each of the following curves is revolved about the x-axis. Find the volume of the solid of revolution.

(a) $y = \dfrac{x^{3/2}}{\sqrt{x^2 + 4}}$ between $x = 0$ and $x = 4$.

(b) $y = \dfrac{1}{x^2 + 1}$ between $x = 0$ and $x = 1$.

(c) $y = \sqrt[4]{4 - x^2}$ between $x = 1$ and $x = 2$.

35 The curve $\frac{1}{2}x^2 + y^2 = 1$ is an ellipse. Sketch the graph and show that its complete length equals the length of one cycle of $y = \sin x$. (This integral is a so-called *elliptic integral*, and is known to be impossible to evaluate in terms of elementary functions. For more details see Appendix A.9.)

10.5 COMPLETING THE SQUARE

In Section 10.4 we used trigonometric substitutions to calculate integrals containing $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, and $\sqrt{x^2 - a^2}$. By the algebraic device of completing the square, we can extend these methods to integrals involving general quadratic polynomials and their square roots, that is, expressions of the form $ax^2 + bx + c$ and $\sqrt{ax^2 + bx + c}$. We remind students that the process of completing the square is based on the simple fact that

$$(x + A)^2 = x^2 + 2Ax + A^2;$$

this tells us that the right side is a perfect square (the square of $x + A$) because its constant term is in the square of half the coefficient of x.

Example 1 Find

$$\int \frac{(x+2)\,dx}{\sqrt{3+2x-x^2}}.$$

Solution Since the coefficient of the term x^2 under the radical is negative, we place the terms containing x in parentheses preceded by a minus sign, leaving space for completing the square,

$$\begin{aligned} 3+2x-x^2 = 3-(x^2-2x+\quad) &= 4-(x^2-2x+1) \\ &= 4-(x-1)^2 = a^2-u^2, \end{aligned}$$

where $u = x - 1$ and $a = 2$. Since $x = u + 1$, we have $dx = du$ and $x + 2 = u + 3$, and therefore

$$\begin{aligned} \int \frac{(x+2)\,dx}{\sqrt{3+2x-x^2}} &= \int \frac{(u+3)\,du}{\sqrt{a^2-u^2}} = \int \frac{u\,du}{\sqrt{a^2-u^2}} + 3\int \frac{du}{\sqrt{a^2-u^2}} \\ &= -\sqrt{a^2-u^2} + 3\sin^{-1}\frac{u}{a} \\ &= -\sqrt{3+2x-x^2} + 3\sin^{-1}\left(\frac{x-1}{2}\right) + c. \end{aligned}$$

Example 2 Find

$$\int \frac{dx}{x^2+2x+10}.$$

Solution We complete the square on the terms containing x, and write

$$\begin{aligned} x^2+2x+10 = (x^2+2x+\quad)+10 &= (x^2+2x+1)+9 \\ &= (x+1)^2+9 = u^2+a^2, \end{aligned}$$

where $u = x + 1$ and $a = 3$. We now have $du = dx$ or $dx = du$, so

$$\begin{aligned} \int \frac{dx}{x^2+2x+10} &= \int \frac{du}{u^2+a^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} \\ &= \frac{1}{3}\tan^{-1}\left(\frac{x+1}{3}\right) + c. \end{aligned}$$

Example 3 Find

$$\int \frac{x\,dx}{\sqrt{x^2-2x+5}}.$$

Solution We write

$$\begin{aligned} x^2-2x+5 = (x^2-2x+\quad)+5 &= (x^2-2x+1)+4 \\ &= (x-1)^2+4 = u^2+a^2, \end{aligned}$$

where $u = x - 1$ and $a = 2$. Then $x = u + 1$, $dx = du$, and we have

$$\int \frac{x\,dx}{\sqrt{x^2-2x+5}} = \int \frac{(u+1)\,du}{\sqrt{u^2+a^2}} = \int \frac{u\,du}{\sqrt{u^2+a^2}} + \int \frac{du}{\sqrt{u^2+a^2}}.$$

The second integral here is the one considered in Example 2 in Section 10.4, so we have

$$\int \frac{du}{\sqrt{u^2 + a^2}} = \ln\,(u + \sqrt{u^2 + a^2}),$$

and therefore

$$\begin{aligned}\int \frac{x\,dx}{\sqrt{x^2 - 2x + 5}} &= \sqrt{u^2 + a^2} + \ln\,(u + \sqrt{u^2 + a^2}) \\ &= \sqrt{x^2 - 2x + 5} + \ln\,(x - 1 + \sqrt{x^2 - 2x + 5}) + c.\end{aligned}$$

Example 4 Find

$$\int \frac{dx}{\sqrt{x^2 - 4x - 5}}.$$

Solution Here we have

$$\begin{aligned}x^2 - 4x - 5 = (x^2 - 4x + \quad) - 5 &= (x^2 - 4x + 4) - 9 \\ &= (x - 2)^2 - 9 = u^2 - a^2,\end{aligned}$$

where $u = x - 2$ and $a = 3$. By using the result of Problem 9 in Section 10.4 (or by quickly working out the necessary formula again by putting $u = a \sec \theta$) we complete the calculation as follows:

$$\begin{aligned}\int \frac{dx}{\sqrt{x^2 - 4x - 5}} = \int \frac{du}{\sqrt{u^2 - a^2}} &= \ln\,(u + \sqrt{u^2 - a^2}) \\ &= \ln\,(x - 2 + \sqrt{x^2 - 4x - 5}) + c.\end{aligned}$$

If an integral involves the square root of a third-, fourth-, or higher-degree polynomial, then it can be proved that there does not exist any general method for carrying out the integration. A few integrals of this kind are discussed in Appendix A.9.

PROBLEMS

Calculate the following integrals.

1 $\displaystyle\int \frac{dx}{\sqrt{2x - x^2}}$.

2 $\displaystyle\int \frac{dx}{\sqrt{5 + 4x - x^2}}$.

3 $\displaystyle\int \frac{dx}{x^2 + 4x + 5}$.

4 $\displaystyle\int \frac{dx}{x^2 - x + 1}$.

5 $\displaystyle\int \frac{(x + 1)\,dx}{\sqrt{2x - x^2}}$.

6 $\displaystyle\int \frac{(x + 3)\,dx}{\sqrt{5 + 4x - x^2}}$.

7 $\displaystyle\int \frac{x^2\,dx}{\sqrt{6x - x^2}}$.

8 $\displaystyle\int \frac{(x - 1)\,dx}{\sqrt{x^2 + 4x + 5}}$.

9 $\displaystyle\int \frac{(x + 7)\,dx}{x^2 + 2x + 5}$.

10 $\displaystyle\int \frac{\sqrt{x^2 + 2x - 3}}{x + 1}\,dx$.

11 $\displaystyle\int \frac{dx}{\sqrt{x^2 - 2x - 8}}$.

12 $\displaystyle\int \frac{dx}{\sqrt{5 + 3x - 2x^2}}$.

13 $\displaystyle\int \frac{dx}{\sqrt{4x^2 + 4x + 17}}$.

14 $\displaystyle\int \frac{(4x + 3)\,dx}{(x^2 - 2x + 2)^{3/2}}$.

15 $\displaystyle\int \frac{dx}{(x^2 - 2x - 3)^{3/2}}$.

16 $\displaystyle\int \frac{dx}{(x + 2)\sqrt{x^2 + 4x + 3}}$.

10.6 THE METHOD OF PARTIAL FRACTIONS

We recall that a rational function is a quotient of two polynomials. By taking the denominator of such a quotient to be 1, we see that the polynomials themselves are included among the rational functions. As we know, the simple rational functions

$$2x+1, \quad \frac{1}{x^2}, \quad \frac{1}{x}, \quad \frac{x}{x^2+1}, \quad \text{and} \quad \frac{1}{x^2+1}$$

have the following integrals:

$$x^2+x, \quad -\frac{1}{x}, \quad \ln x, \quad \tfrac{1}{2}\ln(x^2+1), \quad \text{and} \quad \tan^{-1} x.$$

Our purpose in this section is to describe a systematic procedure for computing the integral of any rational function, and we shall find that this integral can always be expressed in terms of polynomials, rational functions, logarithms, and inverse tangents. The basic idea is to break up a given rational function into a sum of simpler fractions (called *partial fractions*) which can be integrated by methods discussed earlier.

A rational function is called *proper* if the degree of the numerator is less than the degree of the denominator. Otherwise, it is said to be *improper*. For example,

$$\frac{x}{(x-1)(x+2)^2} \quad \text{and} \quad \frac{x^2+2}{x(x^2-9)}$$

are proper, while

$$\frac{x^4}{x^4-1} \quad \text{and} \quad \frac{2x^3-3x^2+2x-4}{x^2+4}$$

are improper. If we have to integrate an improper rational function, it is essential to begin by performing long division until we reach a remainder whose degree is less than that of the denominator. We illustrate with the second improper rational function just mentioned. Long division yields

$$\begin{array}{r|l}
 & 2x \quad - 3 \\ \hline
x^2+4 & 2x^3 - 3x^2 + 2x - 4 \\
 & \underline{2x^3 \qquad\quad + 8x} \\
 & \quad - 3x^2 - 6x - 4 \\
 & \quad \underline{- 3x^2 \qquad - 12} \\
 & \qquad\quad - 6x + 8
\end{array}$$

This means that the rational function in question can be written in the form

$$\frac{2x^3-3x^2+2x-4}{x^2+4} = 2x-3+\frac{-6x+8}{x^2+4}. \tag{1}$$

By applying this process, any improper rational function $P(x)/Q(x)$ can be expressed as the sum of a polynomial and a proper rational function,

$$\frac{P(x)}{Q(x)} = \text{polynomial} + \frac{R(x)}{Q(x)}, \tag{2}$$

where the degree of $R(x)$ is less than the degree of $Q(x)$. In the particular case of (1), this decomposition by means of long division enables us to carry out the integration quite easily, by writing

$$\int \frac{2x^3 - 3x^2 + 2x - 4}{x^2 + 4}\,dx = x^2 - 3x - 6\int \frac{x\,dx}{x^2 + 4} + 8\int \frac{dx}{x^2 + 4}$$

$$= x^2 - 3x - 3\ln(x^2 + 4) + 4\tan^{-1}\frac{x}{2} + c.$$

In the general case (2), these remarks tell us that we can restrict our attention to proper rational functions, since the integration of polynomials is always easy. This restriction is not only convenient, but also necessary, because it is *only* to proper rational functions that the following discussions apply.

In elementary algebra we learned how to combine fractions over a common denominator. We must now learn how to reverse this process and split a given fraction into a sum of fractions having simpler denominators. This procedure is called *decomposition into partial fractions.*

Example 1 It is clear that

$$\frac{3}{x - 1} + \frac{2}{x + 3} = \frac{3(x + 3) + 2(x - 1)}{(x - 1)(x + 3)} = \frac{5x + 7}{(x - 1)(x + 3)}. \tag{3}$$

In the reverse process we start with the right side of (3) as our given rational function and seek constants A and B such that

$$\frac{5x + 7}{(x - 1)(x + 3)} = \frac{A}{x - 1} + \frac{B}{x + 3}. \tag{4}$$

(For the sake of understanding the method, let us pretend for a moment that we don't know that $A = 3$ and $B = 2$ will work.) If we clear fractions in (4) by multiplying through by $(x - 1)(x + 3)$, we get

$$5x + 7 = A(x + 3) + B(x - 1) \tag{5}$$

or

$$5x + 7 = (A + B)x + (3A - B). \tag{6}$$

Since (6) is to be an identity in x, we can find A and B by equating coefficients of like powers of x. This gives a system of two equations in the two unknowns A and B,

$$\begin{cases} A + B = 5 \\ 3A - B = 7, \end{cases} \qquad \text{whose solution is} \qquad A = 3,\ B = 2.$$

There is another convenient way to find A and B, by using (5) directly. Since (5) must hold for all x, it must hold in particular for $x = 1$ (which removes B) and for $x = -3$ (which removes A). Briefly,

$$\begin{array}{llll} x = 1: & 5 + 7 = A(1 + 3) + 0, & 4A = 12, & A = 3; \\ x = -3: & -15 + 7 = 0 + B(-3 - 1), & -4B = -8, & B = 2. \end{array}$$

This method is faster than it looks, and can be carried out by inspection. Whichever method we use to find A and B, (4) becomes

$$\frac{5x + 7}{(x - 1)(x + 3)} = \frac{3}{x - 1} + \frac{2}{x + 3},$$

and this is the partial fractions decomposition of the rational function on the left. Of course, the purpose of this decomposition is to enable us to integrate the given function,

$$\int \frac{5x+7}{(x-1)(x+3)}\,dx = \int \left(\frac{3}{x-1} + \frac{2}{x+3}\right) dx$$
$$= 3 \ln (x-1) + 2 \ln (x+3) + c.$$

The type of expansion used in (4) works in just the same way under more general circumstances, as follows: Let $P(x)/Q(x)$ be a proper rational function whose denominator $Q(x)$ is an nth-degree polynomial. If $Q(x)$ can be factored completely into *distinct linear factors* $x - r_1, x - r_2, \dots, x - r_n$, then there exist n constants $A_1, A_2, \dots, A_n$ such that

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x-r_1} + \frac{A_2}{x-r_2} + \cdots + \frac{A_n}{x-r_n}. \tag{7}$$

The constants in the numerators can be determined by either of the methods suggested in Example 1; and when this is done, the partial fractions decomposition (7) provides an easy way to integrate the given rational function.

Example 2 Find

$$\int \frac{6x^2 + 14x - 20}{x^3 - 4x}\,dx.$$

Solution We factor the denominator by writing $x^3 - 4x = x(x^2 - 4) = x(x+2)(x-2)$. Accordingly, we have a decomposition of the form

$$\frac{6x^2 + 14x - 20}{x^3 - 4x} = \frac{6x^2 + 14x - 20}{x(x+2)(x-2)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-2} \tag{8}$$

for certain constants A, B, C. To find these constants we clear fractions in (8), which yields

$$6x^2 + 14x - 20 = A(x+2)(x-2) + Bx(x-2) + Cx(x+2).$$

By setting $x = 0, -2, 2$ (this is the second method in Example 1), we easily see that $A = 5$, $B = -3$, $C = 4$, so (8) becomes

$$\frac{6x^2 + 14x - 20}{x^3 - 4x} = \frac{5}{x} - \frac{3}{x+2} + \frac{4}{x-2}.$$

We therefore have

$$\int \frac{6x^2 + 14x - 20}{x^3 - 4x}\,dx = 5 \ln x - 3 \ln (x+2) + 4 \ln (x-2) + c.$$

In theory, every polynomial $Q(x)$ with real coefficients can be factored completely into real linear and quadratic factors, some of which may be repeated.* In practice, this factorization is hard to carry out for polynomials of degree 3 or more, except in special cases. Nevertheless, let us assume this has been done, and let us see how the decomposition (7) must be altered to take account of the most general circumstances that can arise.

*This statement is a consequence of the *Fundamental Theorem of Algebra,* which is discussed in Section 14.8.

If a linear factor $x - r$ occurs with multiplicity m, then the corresponding term $A/(x - r)$ in the decomposition (7) must be replaced by a sum of the form

$$\frac{B_1}{x - r} + \frac{B_2}{(x - r)^2} + \cdots + \frac{B_m}{(x - r)^m}.$$

A quadratic factor $x^2 + bx + c$ of multiplicity 1 gives rise to a single term

$$\frac{Ax + B}{x^2 + bx + c},$$

and if this quadratic factor occurs with multiplicity m, then it gives rise to a sum of the form

$$\frac{A_1x + B_1}{x^2 + bx + c} + \frac{A_2x + B_2}{(x^2 + bx + c)^2} + \cdots + \frac{A_mx + B_m}{(x^2 + bx + c)^m}.$$

This is the whole story, and the theory guarantees that every proper rational function can be expanded into a sum of partial fractions in the manner described above.*

Example 3 Find

$$\int \frac{3x^3 - 4x^2 - 3x + 2}{x^4 - x^2}\, dx.$$

Solution We have

$$\frac{3x^3 - 4x^2 - 3x + 2}{x^4 - x^2} = \frac{3x^3 - 4x^2 - 3x + 2}{x^2(x + 1)(x - 1)}$$

$$= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x + 1} + \frac{D}{x - 1}.$$

Clearing fractions gives the identity

$$3x^3 - 4x^2 - 3x + 2 = Ax(x + 1)(x - 1) + B(x + 1)(x - 1) + Cx^2(x - 1) + Dx^2(x + 1).$$

Now put

$x = 0$: $\quad 2 = -B, \quad B = -2;$

$x = 1$: $\quad -2 = 2D, \quad D = -1;$

$x = -1$: $\quad -2 = -2C, \quad C = 1.$

Equating coefficients of x^3 gives

$$3 = A + C + D, \quad \text{so} \quad A = 3.$$

Our partial fractions decomposition is therefore

$$\frac{3x^3 - 4x^2 - 3x + 2}{x^4 - x^2} = \frac{3}{x} - \frac{2}{x^2} + \frac{1}{x + 1} - \frac{1}{x - 1},$$

so

$$\int \frac{3x^3 - 4x^2 - 3x + 2}{x^4 - x^2}\, dx = 3 \ln x + \frac{2}{x} + \ln (x + 1) - \ln (x - 1) + c.$$

*This statement is called the *Partial Fractions Theorem*; it is proved in Appendix A.11. Students will notice that the above description of the partial fractions decomposition assumes that the highest power of x in $Q(x)$ has coefficient 1; this can always be arranged by a minor algebraic adjustment.

Example 4 Find

$$\int \frac{2x^3 + x^2 + 2x - 1}{x^4 - 1}\,dx.$$

Solution We have

$$\frac{2x^3 + x^2 + 2x - 1}{x^4 - 1} = \frac{2x^3 + x^2 + 2x - 1}{(x + 1)(x - 1)(x^2 + 1)}$$

$$= \frac{A}{x + 1} + \frac{B}{x - 1} + \frac{Cx + D}{x^2 + 1},$$

so

$$2x^3 + x^2 + 2x - 1 = A(x - 1)(x^2 + 1) + B(x + 1)(x^2 + 1) + Cx(x^2 - 1) + D(x^2 - 1).$$

Now put

$x = 1$:	$4 = 4B,$	$B = 1;$
$x = -1$:	$-4 = -4A,$	$A = 1;$
$x = 0$:	$-1 = -A + B - D,$	$D = 1.$

Equating coefficients of x^3 gives

$$2 = A + B + C, \quad \text{so} \quad C = 0.$$

Our partial fractions decomposition is therefore

$$\frac{2x^3 + x^2 + 2x - 1}{x^4 - 1} = \frac{1}{x + 1} + \frac{1}{x - 1} + \frac{1}{x^2 + 1},$$

so

$$\int \frac{2x^3 + x^2 + 2x - 1}{x^4 - 1}\,dx = \ln(x + 1) + \ln(x - 1) + \tan^{-1} x + c.$$

As a final comment, we point out that all the partial fractions that can possibly arise have the form

$$\frac{A}{(x - r)^n} \quad \text{or} \quad \frac{Ax + B}{(x^2 + bx + c)^n}, \qquad n = 1, 2, 3, \ldots .$$

Functions of the first type can be integrated by using the substitution $u = x - r$, and it is clear that the results are always logarithms or rational functions. A function of the second type in which the quadratic polynomial $x^2 + bx + c$ has no real linear factors, that is, in which the roots of $x^2 + bx + c = 0$ are imaginary, can be integrated by completing the square and making a suitable substitution. When this is done, we get integrals of the form

$$\int \frac{u\,du}{(u^2 + k^2)^n}, \qquad \int \frac{du}{(u^2 + k^2)^n}.$$

The first of these is $\frac{1}{2}\ln(u^2 + k^2)$ if $n = 1$, and $(u^2 + k^2)^{1-n}/2(1 - n)$ if $n > 1$. When $n = 1$, the second integral is calculated by the formula

$$\int \frac{du}{u^2 + k^2} = \frac{1}{k}\tan^{-1}\frac{u}{k}.$$

The case $n > 1$ can be reduced to the case $n = 1$ by repeated application of the *reduction formula*

$$\int \frac{du}{(u^2+k^2)^n} = \frac{1}{2k^2(n-1)} \cdot \frac{u}{(u^2+k^2)^{n-1}} + \frac{2n-3}{2k^2(n-1)} \int \frac{du}{(u^2+k^2)^{n-1}}. \tag{9}$$

We state this complicated formula for the sole purpose of showing that the only functions that arise from the indicated reduction procedure are rational functions and inverse tangents. The formula itself can either be verified by differentiation or obtained from scratch by the methods of the next section.

This discussion shows that the integral of every rational function can be expressed in terms of polynomials, rational functions, logarithms, and inverse tangents. The detailed work can be very laborious, but at least the path that must be followed is clearly visible.

PROBLEMS

1 Express each of the following improper rational functions as the sum of a polynomial and a proper rational function, and integrate:
(a) $\dfrac{x^2}{x-1}$; (b) $\dfrac{x^3}{3x+2}$; (c) $\dfrac{x^3}{x^2+1}$;
(d) $\dfrac{x+3}{x+2}$; (e) $\dfrac{x^2-1}{x^2+1}$.

Find each of the following integrals.

2 $\displaystyle\int \frac{12x-17}{(x-1)(x-2)}\,dx.$

3 $\displaystyle\int \frac{14x-12}{2x^2-2x-12}\,dx.$

4 $\displaystyle\int \frac{10-2x}{x^2+5x}\,dx.$

5 $\displaystyle\int \frac{2x+21}{x^2-7x}\,dx.$

6 $\displaystyle\int \frac{9x^2-24x+6}{x^3-5x^2+6x}\,dx.$

7 $\displaystyle\int \frac{x^2+46x-48}{x^3+5x^2-24x}\,dx.$

8 $\displaystyle\int \frac{16x^2+3x-7}{x^3-x}\,dx.$

9 $\displaystyle\int \frac{4x^2+11x-117}{x^3+10x^2-39x}\,dx.$

10 $\displaystyle\int \frac{6x^2-9x+9}{x^3-3x^2}\,dx.$

11 $\displaystyle\int \frac{-4x^2-5x-3}{x^3+2x^2+x}\,dx.$

12 $\displaystyle\int \frac{4x^2+2x+4}{x^3+4x}\,dx.$

13 $\displaystyle\int \frac{3x^2-x+4}{x^3+2x^2+2x}\,dx.$

14 $\displaystyle\int \frac{x^4}{x^2+4}\,dx.$

15 $\displaystyle\int \frac{x^4+3x^2-4x+5}{(x-1)^2(x^2+1)}\,dx.$

16 $\displaystyle\int \frac{x^2+2x}{(x+1)^2}\,dx.$

17 $\displaystyle\int \frac{x^2}{x+2}\,dx.$

18 $\displaystyle\int \frac{x+1}{x-1}\,dx.$

19 $\displaystyle\int \frac{x^2+1}{x+2}\,dx.$

20 $\displaystyle\int \frac{x^3-3x^2+2x-3}{x^2+1}\,dx.$

21 $\displaystyle\int \frac{\cos\theta}{\sin^2\theta+3\sin\theta-4}\,d\theta.$

22 $\displaystyle\int \frac{16\sec^2\theta}{\tan^3\theta-4\tan^2\theta}\,d\theta.$

23 $\displaystyle\int \frac{e^x}{e^{2x}-4}\,dx.$

24 $\displaystyle\int \frac{1}{1+e^x}\,dx.$

25 Use partial fractions to obtain the formula

$$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\frac{a+x}{a-x}.$$

Also calculate this integral by trigonometric substitution, and verify that the two answers agree.

26 Find
(a) $\displaystyle\int \frac{3\sin\theta\,d\theta}{\cos^2\theta-\cos\theta-2}$; (b) $\displaystyle\int \frac{5e^t\,dt}{e^{2t}+e^t-6}$.

27 In Problem 14 of Section 8.5 it is stated that the differential equation

$$\frac{dx}{dt} = kab(A-x)(B-x), \qquad A \neq B,$$

has

$$\frac{B(A-x)}{A(B-x)} = e^{kab(A-B)t}$$

as a solution for which $x = 0$ when $t = 0$. Derive this solution by using partial fractions.

28 Verify the reduction formula (9) by differentiating the first term on the right.

29 Suppose that a given population can be divided into two

groups: those who have a certain infectious disease, and those who do not have it but can catch it by having contact with an infected person. If x and y are the proportions of infected and uninfected people, then $x + y = 1$. Assume (1) that the disease spreads by the contacts just mentioned between sick people and well people, (2) that the rate of spread dx/dt is proportional to the number of such contacts, and (3) that the two such contacts, and (3) that the two groups mingle freely with each other, so that the number of contacts is jointly proportional to x and y. If $x = x_0$ when $t = 0$, find x as a function of t, sketch the graph, and use this function to show that ultimately the disease will spread through the entire population. When the formula for the derivative of a product (the

10.7 INTEGRATION BY PARTS

When the formula for the derivative of a product (the product rule) is written in the notation of differentials, it is

$$d(uv) = u\,dv + v\,du \qquad \text{or} \qquad u\,dv = d(uv) - v\,du,$$

and by integrating we obtain

$$\int u\,dv = uv - \int v\,du. \tag{1}$$

This formula provides a method of finding $\int u\,dv$ if the second integral $\int v\,du$ is easier to calculate than the first. The method is called *integration by parts,* and it often works when all other methods fail.

Example 1 Find $\int x \cos x\,dx$.

Solution If we put

$$u = x, \qquad dv = \cos x\,dx,$$

then

$$du = dx, \qquad v = \sin x,$$

and (1) gives

$$\int x \cos x\,dx = x \sin x - \int \sin x\,dx.$$

This is good luck, because the integral on the right is easy. We therefore have

$$\int x \cos x\,dx = x \sin x + \cos x + c.$$

It is worth noticing that in this example we could have chosen u and dv differently. If we put

$$u = \cos x, \qquad dv = x\,dx,$$

then

$$du = -\sin x\,dx, \qquad v = \tfrac{1}{2}x^2,$$

and (1) gives

$$\int x \cos x\,dx = \tfrac{1}{2}x^2 \cos x + \tfrac{1}{2}\int x^2 \sin x\,dx.$$

This equation is true, but it is completely worthless as a means of solving our problem, because the second integral is harder than the first. We urge students to learn from experience, and to use trial and error as intelligently as possible in choosing u and dv. Also, students should feel free to abandon a choice that doesn't seem to work, and quickly go on to another choice that offers more hope of success.

The method of integration by parts applies particularly well to products of different types of functions, like $x \cos x$ in Example 1, which is a product of a polynomial and a trigonometric function. In using the method, the given differential must be thought of as a product $u \cdot dv$. The part called dv must be something we can integrate, and the part called u should usually be something that is simplified by differentiation, as in our next example.

Example 2 Find $\int \ln x \, dx$.

Solution Here our only choice is

$$u = \ln x, \qquad dv = dx,$$

so

$$u = \frac{dx}{x}, \qquad v = x,$$

and we have

$$\int \ln x \, dx = x \ln x - \int x \frac{dx}{x} = x \ln x - x + c.$$

In some cases it is necessary to carry out two or more integrations by parts in succession.

Example 3 Find $\int x^2 e^x \, dx$.

Solution If we put

$$u = x^2, \qquad dv = e^x \, dx,$$

then

$$du = 2x \, dx, \qquad v = e^x,$$

and (1) gives

$$\int x^2 e^x \, dx = x^2 e^x - 2\int xe^x \, dx. \tag{2}$$

Here the second integral is easier than the first, so we are encouraged to continue in the same way. When the second integral is integrated by parts with

$$u = x, \qquad dv = e^x \, dx,$$

so that

$$du = dx, \qquad v = e^x,$$

then we get

$$\int xe^x \, dx = xe^x - \int e^x \, dx$$

$$= xe^x - e^x.$$

When this is inserted in (2), our final result is

$$\int x^2e^x\,dx = x^2e^x - 2xe^x + 2e^x + c.$$

It sometimes happens that the integral we start with appears a second time during the integration by parts, in which case it is often possible to solve for this integral by elementary algebra.

Example 4 Find $\int e^x \cos x\,dx$.

Solution For convenience we denote this integral by J. If we put

$$u = e^x, \qquad dv = \cos x\,dx,$$

then

$$du = e^x\,dx, \qquad v = \sin x,$$

and (1) yields

$$J = e^x \sin x - \int e^x \sin x\,dx. \tag{3}$$

Now we come to the interesting part of this problem. Even though the new integral is no easier than the old, it turns out to be fruitful to apply the same method again to the new integral. Thus, we put

$$u = e^x, \qquad dv = \sin x\,dx,$$

so that

$$du = e^x\,dx, \qquad v = -\cos x,$$

and obtain

$$\int e^x \sin x\,dx = -e^x \cos x + \int e^x \cos x\,dx. \tag{4}$$

The integral on the right is J again, so (4) can be written

$$\int e^x \sin x\,dx = -e^x \cos x + J. \tag{5}$$

In spite of appearances, we are not going in a circle, because substituting (5) in (3) gives

$$J = e^x \sin x + e^x \cos x - J.$$

It is now easy to solve for J by writing

$$2J = e^x \sin x + e^x \cos x \qquad \text{or} \qquad J = \tfrac{1}{2}(e^x \sin x + e^x \cos x),$$

and all that remains is to insert the constant of integration:

$$\int e^x \cos x\,dx = \tfrac{1}{2}e^x(\sin x + \cos x) + c.$$

The method of this example is often used to make an integral depend on a simpler integral of the same type, and thus to obtain a convenient *reduction formula*, by repeated use of which the given integral can easily be calculated.

Example 5 Find a reduction formula for $J_n = \int \sin^n x\, dx$.

Solution We integrate by parts with

$$u = \sin^{n-1} x, \qquad dv = \sin x\, dx,$$

so that

$$du = (n-1)\sin^{n-2} x \cos x\, dx, \qquad v = -\cos x,$$

and therefore

$$\begin{aligned} J_n &= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \cos^2 x\, dx \\ &= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x(1 - \sin^2 x)\, dx \\ &= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\, dx - (n-1)\int \sin^n x\, dx \\ &= -\sin^{n-1} x \cos x + (n-1)\, J_{n-2} - (n-1)\, J_n. \end{aligned}$$

We now transpose the term involving J_n and obtain

$$nJ_n = -\sin^{n-1} x \cos x + (n-1)J_{n-2},$$

so that

$$J_n = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n} J_{n-2},$$

or equivalently,

$$\int \sin^n x\, dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x\, dx. \tag{6}$$

The reduction formula (6) allows us to reduce the exponent on $\sin x$ by 2. By repeated application of this formula we can therefore ultimately reduce J_n to J_0 or J_1, according as n is even or odd. But both of these are easy:

$$J_0 = \int \sin^0 x\, dx = \int dx = x \qquad \text{and} \qquad J_1 = \int \sin x\, dx = -\cos x.$$

For example, with $n = 4$ we get

$$\int \sin^4 x\, dx = -\tfrac{1}{4}\sin^3 x \cos x + \tfrac{3}{4}\int \sin^2 x\, dx,$$

and with $n = 2$,

$$\begin{aligned} \int \sin^2 x\, dx &= -\tfrac{1}{2}\sin x \cos x + \tfrac{1}{2}\int dx \\ &= -\tfrac{1}{2}\sin x \cos x + \tfrac{1}{2}x. \end{aligned}$$

Therefore,

$$\begin{aligned} \int \sin^4 x\, dx &= -\tfrac{1}{4}\sin^3 x \cos x + \tfrac{3}{4}(-\tfrac{1}{2}\sin x \cos x + \tfrac{1}{2}x) \\ &= -\tfrac{1}{4}\sin^3 x \cos x - \tfrac{3}{8}\sin x \cos x + \tfrac{3}{8}x + c. \end{aligned}$$

The same result can be achieved by earlier techniques depending on repeated use of the half-angle formulas, but our present methods are more efficient for large exponents. In our next example we illustrate another way in which the reduction formula (6) can be used.

Example 6 Calculate

$$\int_0^{\pi/2} \sin^8 x\,dx.$$

Solution For convenience we write

$$I_n = \int_0^{\pi/2} \sin^n x\,dx.$$

By formula (6) we have

$$I_n = -\frac{1}{n}\sin^{n-1} x \cos x\Big]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx,$$

so

$$I_n = \frac{n-1}{n} I_{n-2}.$$

We apply this formula with $n = 8$, then repeat with $n = 6$, $n = 4$, $n = 2$:

$$I_8 = \tfrac{7}{8}I_6 = \tfrac{7}{8}\cdot\tfrac{5}{6}I_4 = \tfrac{7}{8}\cdot\tfrac{5}{6}\cdot\tfrac{3}{4}I_2 = \tfrac{7}{8}\cdot\tfrac{5}{6}\cdot\tfrac{3}{4}\cdot\tfrac{1}{2}I_0.$$

Therefore

$$\int_0^{\pi/2} \sin^8 x\,dx = \frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\int_0^{\pi/2} dx = \frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{35\pi}{256}.$$

Remark 1 The reduction formula (6) can also be used to establish one of the most fascinating formulas of mathematics, *Wallis's infinite product* for $\pi/2$:

$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdots.$$

For the details of the proof, see Appendix 2 at the end of the chapter.

Remark 2 In Section 9.5 we stated *Leibniz's formula* for $\pi/4$,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$$

For students who are interested in little-known corners of the history of mathematics, we describe in Appendix 3 at the end of the chapter how Leibniz himself discovered his formula by a very ingenious application of integration by parts.

At this point we have described all the standard methods of integration that the student is expected to be acquainted with. A few additional techniques of minor importance remain, and two of these are briefly sketched in the problems at the end of Appendix A.9; but for most practical purposes we have reached the end of this particular road.

PROBLEMS

Find each of the following integrals by the method of integration by parts.

1 $\int x \ln x\, dx$.
2 $\int \tan^{-1} x\, dx$.
3 $\int x \tan^{-1} x\, dx$.
4 $\int xe^{ax}\, dx$.
5 $\int e^x \sin x\, dx$.
6 $\int e^{ax} \cos bx\, dx$.
7 $\int \sqrt{1-x^2}\, dx$.
8 $\int \sin^{-1} x\, dx$.
9 $\displaystyle\int x \sin^{-1} x\, dx$.
10 $\displaystyle\int_0^{\pi/2} x \sin x\, dx$.
11 $\displaystyle\int x \cos(3x-2)\, dx$.
12 $\displaystyle\int \frac{\tan^{-1} x}{x^2}\, dx$.
13 $\int x \sec^2 x\, dx$.
14 $\int \sin(\ln x)\, dx$.
15 $\int \ln(a^2+x^2)\, dx$.
16 $\int x^2 \ln(x+1)\, dx$.
17 $\displaystyle\int \frac{\ln x}{x}\, dx$.
18 $\displaystyle\int (\ln x)^2\, dx$.

19 The region under the curve $y = \cos x$ between $x = 0$ and $x = \pi/2$ is revolved about the y-axis. Find the volume of the resulting solid.

20 Find $\int(\sin^{-1} x)^2\, dx$. Hint: Make the substitution $y = \sin^{-1} x$.

21 If $P(x)$ is a polynomial, show that

$$\int P(x)e^x\, dx = (P - P' + P'' - P''' + \cdots)e^x.$$

In the next two problems, derive the given reduction formula and apply it to the indicated special case(s).

22 (a) $\displaystyle\int \cos^n x\, dx = \frac{1}{n} \sin x \cos^{n-1} x + \frac{n-1}{n}\int \cos^{n-2} x\, dx.$

(b) $\displaystyle\int_0^{\pi/2} \cos^7 x\, dx.$

(c) $\displaystyle\int_0^{\pi/2} \cos^8 x\, dx.$

23 (a) $\int(\ln x)^n\, dx = x(\ln x)^n - n\int(\ln x)^{n-1}\, dx$.

(b) $\int(\ln x)^5\, dx$.

24 The region under the curve $y = \sin x$ between $x = 0$ and $x = \pi$ is revolved about the y-axis. Find the volume of the resulting solid (a) by the shell method; and (b) by the washer method.

25 The curve in Problem 24 is revolved about the x-axis. Find the area of the resulting surface of revolution.

26 (The volcanic ash problem) When a volcano erupts, the cloud of ejected ash gradually settles onto the surface of the nearby land. The depth of the deposited layer of ash decreases with distance from the volcano. Assume that the depth of the ash r feet from the volcano is ae^{-br} feet, where a and b are positive constants.

(a) Find the total volume of ash that falls within a distance c of the volcano. Hint: What is the element of volume dV of ash that falls on a narrow ring of width dr and inner radius r centered on the volcano?

(b) What is the limit of this volume as $c \to \infty$?

10.8 A MIXED BAG. STRATEGY FOR DEALING WITH INTEGRALS OF MISCELLANEOUS TYPES

As the student understands by now, differentiation is straightforward but integration is not. In finding the derivative of a function it is obvious which formula must be applied. But it may not be at all obvious which method should be used to integrate a given function.

Since the problems in each section of this chapter have focused on the methods of that section, it has usually been clear what method to use on a given integral. Generally speaking, the methods at our disposal now are direct substitution, trigonometric substitution, partial fractions, and parts. But what if an integral is met out of context, with no obvious clue as to how to work it out? In this section we try to suggest a strategy for this common situation.

An essential prerequisite is a knowledge of the basic integration formulas. For the sake of emphasis, we repeat the list given in Section 10.1, together with three additional formulas arising from our work in this chapter. As we pointed out earlier, the first 15 formulas should be memorized, and we hope students will take our advice seriously this time. It is useful to know them all, but the last three (marked with an asterisk) need not be memorized since they are easy to derive, as follows. Formulas 16 and 17 are immediate from the simple partial fractions decompositions

$$\frac{1}{x^2 - a^2} = \frac{1}{(x+a)(x-a)} = \frac{1}{2a}\left[\frac{1}{x-a} - \frac{1}{x+a}\right]$$

and

$$\frac{1}{a^2 - x^2} = \frac{1}{(a+x)(a-x)} = \frac{1}{2a}\left[\frac{1}{a+x} + \frac{1}{a-x}\right].$$

These decompositions can easily be understood by mentally recombining the terms in brackets with the aid of a common denominator; we then see directly what the constant factor outside the brackets must be. Formula 18 is almost immediate from the trigonometric substitutions $x = a \tan \theta$ and $x = a \sec \theta$, respectively. In this list of formulas we use x instead of u as the variable of integration—since the usefulness of the u-notation is now thoroughly familiar to us—and for the sake of simplicity we omit the constant of integration.

1 $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} \qquad (n \neq -1).$

2 $\displaystyle\int \frac{dx}{x} = \ln x.$

3 $\displaystyle\int e^x\,dx = e^x.$

4 $\displaystyle\int \cos x\,dx = \sin x.$

5 $\displaystyle\int \sin x\,dx = -\cos x.$

6 $\displaystyle\int \sec^2 x\,dx = \tan x.$

7 $\displaystyle\int \csc^2 x\,dx = -\cot x.$

8 $\displaystyle\int \sec x \tan x\,dx = \sec x.$

9 $\displaystyle\int \csc x \cot x\,dx = -\csc x.$

10 $\displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a}.$

11 $\displaystyle\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}.$

12 $\displaystyle\int \tan x\,dx = -\ln(\cos x).$

13 $\displaystyle\int \cot x\,dx = \ln(\sin x).$

14 $\displaystyle\int \sec x\,dx = \ln(\sec x + \tan x).$

15 $\displaystyle\int \csc x\,dx = -\ln(\csc x + \cot x).$

*16 $\displaystyle\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right).$

*17 $\displaystyle\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right).$

*18 $\displaystyle\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln(x + \sqrt{x^2 \pm a^2}).$

These formulas constitute our arsenal of weapons for attacking integrals, and it is up to us to decide which to use in a particular case. If we do not see what to do immediately, the following strategy may be helpful.

STRATEGY FOR INTEGRATION

1 *Simplify the integrand.* The use of algebraic or trigonometric identities will sometimes simplify the integrand and make a method of integration obvious. For example:

$$\int \sqrt{x}(\sqrt{x} + \sqrt[3]{x})\, dx = \int (x + x^{5/6})\, dx;$$

$$\int (\sin x + \cos x)^2\, dx = \int (\sin^2 x + 2 \sin x \cos x + \cos^2 x)\, dx$$

$$= \int (1 + 2 \sin x \cos x)\, dx;$$

$$\int \frac{1 - \tan^2 x}{\sec^2 x}\, dx = \int (1 - \tan^2 x) \cos^2 x\, dx$$

$$= \int \left(1 - \frac{\sin^2 x}{\cos^2 x}\right) \cos^2 x\, dx$$

$$= \int (\cos^2 x - \sin^2 x)\, dx = \int \cos 2x\, dx.$$

In the second problem, if we fail to notice that $\sin^2 x + \cos^2 x = 1$, and instead integrate $\sin^2 x$ and $\cos^2 x$ separately, then we can still solve the problem, but we have missed an opportunity to do things the easy way. A similar remark applies to the third problem, with its use of the double-angle formula for the cosine.

2 *Look for an obvious substitution.* Try to find some function $u = g(x)$ in the integrand whose differential $du = g'(x)\, dx$ is also present, apart from a constant factor. For example, in

$$\int \frac{x\, dx}{4 - x^2}$$

we notice that if $u = 4 - x^2$, then $du = -2x\, dx$ and $x\, dx = -\frac{1}{2}\, du$. It is therefore much simpler to use this substitution than to use partial fractions or the trigonometric substitution $x = 2 \sin \theta$, each of which also works but takes longer to carry out.

3 *Classify the integrand.* This is the heart of the matter. If Steps 1 and 2 have not helped, then we turn to a more careful examination of the form of the integrand $f(x)$.

(a) If $f(x)$ is (or can be written as) a product of powers of $\sin x$ and $\cos x$, or $\tan x$ and $\sec x$, or $\cot x$ and $\csc x$, then the methods of Section 10.3 can be used.*

*A special method for integrating *any* rational function of $\sin x$ and $\cos x$ is described in Appendix A.9 at the end of the book. This method will not be needed for any of the review problems at the end of this section.

(b) If $f(x)$ involves $\sqrt{a^2 \pm x^2}$ or $\sqrt{x^2 \pm a^2}$, or powers of these expressions, use the trigonometric substitutions of Section 10.4.
(c) If $f(x)$ is a rational function, use partial fractions as explained in Section 10.6—unless there is a better way for a particular integral.
(d) If $f(x)$ is a product of functions of different types, try integration by parts. As we have seen in Section 10.7, this method also works for many individual inverse functions like $\ln x$, $\sin^{-1} x$, and $\tan^{-1} x$.
(e) Be observant, thoughtful, flexible and persistent—all of which are of course easier said than done. If a method doesn't work, be ready to try another. Sometimes several methods work. Keep your options open and do things the easy way—if any. And remember that doing a problem more than one way is a good learning experience.

Our purpose in the following examples is to try to suggest possible lines of attack by "thinking out loud." We are interested mainly in brainstorming these integrals, and in most cases we will not work out all the details to the final answer.

Example 1 $\displaystyle\int \frac{x^2}{x^6 - 1}\,dx.$

Comments Since the integrand is a rational function, partial fractions will work. This requires factoring $x^6 - 1$ into $(x^3 + 1)(x^3 - 1) = (x + 1)(x^2 - x + 1) \cdot (x - 1)(x^2 + x + 1)$ and then finding constants A, B, C, D, E, F such that

$$\frac{x^2}{x^6 - 1} = \frac{A}{x + 1} + \frac{Bx + C}{x^2 - x + 1} + \frac{D}{x - 1} + \frac{Ex + F}{x^2 + x + 1}.$$

We can do this if we must, but actually carrying out this work is not an attractive prospect.

Let us probe in a different direction. A much more promising method is to notice that x^6 is the square of x^3 and that the numerator of the integrand is almost the derivative of x^3. Accordingly, if we put $u = x^3$, then $du = 3x^2\,dx$, $x^2\,dx = \frac{1}{3}\,du$, and the integral becomes

$$\frac{1}{3}\int \frac{du}{u^2 - 1} = \frac{1}{6}\ln\left(\frac{u - 1}{u + 1}\right) = \frac{1}{6}\ln\left(\frac{x^3 - 1}{x^3 + 1}\right),$$

by formula 16.

Example 2 $\displaystyle\int \frac{x^2}{1 + x^2}\,dx.$

Comments The trigonometric substitution $x = \tan\theta$ will work. Partial fractions will also work, but since the integrand is an improper rational function, we must begin with long division. However, an easier way to accomplish the result without actually carrying out the long division is simply to add and subtract 1 in the numerator,

$$\int \frac{x^2}{1 + x^2}\,dx = \int \left(\frac{x^2 + 1 - 1}{1 + x^2}\right) dx = \int \left(1 - \frac{1}{1 + x^2}\right) dx$$
$$= x - \tan^{-1} x.$$

Example 3 $\displaystyle\int \frac{e^{2x}\,dx}{e^x - 1}.$

Comments We begin by noticing that $e^{2x}\,dx = e^x(e^x\,dx) = e^x\,d(e^x)$. This suggests that we put $u = e^x$, so that $e^{2x}\,dx = u\,du$ and the integral can be written

$$\begin{aligned}\int \frac{u\,du}{u-1} &= \int \frac{u-1+1}{u-1}\,du = \int \left(1 + \frac{1}{u-1}\right) du \\ &= u + \ln(u-1) = e^x + \ln(e^x - 1).\end{aligned}$$

By subtracting and adding 1 here we employ a slight variation of the idea used in Example 2.

Example 4 $\displaystyle\int \frac{4x+1}{1+x^2}\,dx.$

Comments The numerator is nearly (but not quite) the derivative of the denominator. This suggests that we break the integrand into a sum and rearrange the constants to achieve this desirable condition:

$$\begin{aligned}\int \frac{4x+1}{1+x^2}\,dx &= \int \left(2 \cdot \frac{2x}{1+x^2} + \frac{1}{1+x^2}\right) dx \\ &= 2\int \frac{2x\,dx}{1+x^2} + \int \frac{dx}{1+x^2} = 2\ln(1+x^2) + \tan^{-1} x.\end{aligned}$$

Example 5 $\displaystyle\int \frac{2x+6}{x^2+7x+10}\,dx.$

Comments In Example 4 we arranged part of the numerator to be the derivative of the denominator. A similar purpose here suggests that we write

$$\begin{aligned}\int \frac{2x+6}{x^2+7x+10}\,dx &= \int \frac{(2x+7)-1}{x^2+7x+10}\,dx \\ &= \int \frac{(2x+7)\,dx}{x^2+7x+10} - \int \frac{dx}{x^2+7x+10}.\end{aligned}$$

The first of these integrals has been arranged to be $\ln(x^2 + 7x + 10)$, and we can easily work out the second by factoring the denominator into $(x+2)(x+5)$ and using partial fractions.

Example 6 $\displaystyle\int \frac{x^5\,dx}{(1+x^2)^4}.$

Comments The trigonometric substitution $x = \tan\theta$ will work. Partial fractions will also work, but if we try this there will be eight unknown constants to find. We hope for something better.

Let us try the substitution $u = 1 + x^2$. Our only reason for this is that it simplifies the denominator to u^4. Then $du = 2x\,dx$, and we have

$$\int \frac{x^5\,dx}{(1+x^2)^4} = \int \frac{(x^2)^2\,(x\,dx)}{(1+x^2)^4} = \frac{1}{2}\int \frac{(u-1)^2\,du}{u^4}$$
$$= \frac{1}{2}\int \frac{u^2-2u+1}{u^4}\,du = \frac{1}{2}\int (u^{-2}-2u^{-3}+u^{-4})\,du,$$

which is easy.

Example 7 $\displaystyle\int \frac{dx}{x(\ln x)^2}$.

Comments We notice at once that the differential of $\ln x$ is dx/x. We therefore put $u = \ln x$, so that $du = dx/x$ and

$$\int \frac{dx}{x(\ln x)^2} = \int \frac{du}{u^2} = -\frac{1}{u} = -\frac{1}{\ln x}.$$

Example 8 $\displaystyle\int \frac{x\,dx}{\sqrt[3]{x+1}}$.

Comments This requires a so-called *rationalizing substitution,* that is, one that eliminates the radical. We put $u = \sqrt[3]{x+1}$, so that $u^3 = x+1$, $3u^2\,du = dx$, and $x = u^3 - 1$. We can now write

$$\int \frac{x\,dx}{\sqrt[3]{x+1}} = \int \frac{(u^3-1)3u^2\,du}{u} = \int (3u^4 - 3u)\,du,$$

which is easy.

Example 9 $\displaystyle\int \sqrt{\frac{1+x}{1-x}}\,dx$.

Comments The rationalizing substitution

$$u = \sqrt{\frac{1+x}{1-x}}$$

will work here, but the result is a messy rational function. A better idea is to multiply both numerator and denominator by $\sqrt{1+x}$, which gives

$$\int \sqrt{\frac{1+x}{1-x}}\,dx = \int \sqrt{\frac{1+x}{1-x}}\cdot\frac{\sqrt{1+x}}{\sqrt{1+x}}\,dx = \int \frac{1+x}{\sqrt{1-x^2}}\,dx$$
$$= \int \frac{dx}{\sqrt{1-x^2}} + \int \frac{x\,dx}{\sqrt{1-x^2}} = \sin^{-1} x - \sqrt{1-x^2}.$$

Example 10 $\displaystyle\int \frac{1}{1+\cos x}\,dx$.

Comments This time we multiply both numerator and denominator by $1 - \cos x$ to obtain a somewhat different application of the idea in Example 9:

$$\int \frac{1}{1+\cos x}\,dx = \int \frac{1}{1+\cos x}\cdot\frac{1-\cos x}{1-\cos x}\,dx = \int \frac{1-\cos x}{1-\cos^2 x}\,dx$$
$$= \int \frac{1-\cos x}{\sin^2 x}\,dx = \int \csc^2 x\,dx - \int \frac{\cos x\,dx}{\sin^2 x}$$
$$= -\cot x + \frac{1}{\sin x}.$$

Example 11 $\int e^{\sqrt{x}}\,dx.$

Comments It is natural to try the substitution $u = \sqrt{x}$, even though we have no idea what is likely to happen. Then $u^2 = x$, $2u\,du = dx$, and we have

$$\int e^{\sqrt{x}}\,dx = \int 2\,ue^u\,du.$$

This integral is now an obvious candidate for integration by parts.

The following list of problems contains integrals of all the types we have encountered, arranged in random order so that students can test their diagnostic powers.

PROBLEMS

Find the following integrals.

1 $\int \frac{x\,dx}{\sqrt{1-x^2}}.$

2 $\int x^4 \ln x\,dx.$

3 $\int \sin^2 x \cos^5 x\,dx.$

4 $\int \frac{dx}{x^3+4x}.$

5 $\int \frac{\sqrt{1+\ln x}}{x\ln x}\,dx.$

6 $\int (e^{3x})^4 e^x\,dx.$

7 $\int \sin\sqrt{x}\,dx.$

8 $\int \frac{x^3}{x^4-1}\,dx.$

9 $\int \cos x \tan x\,dx.$

10 $\int \frac{\cos x}{1+\sin^2 x}\,dx.$

11 $\int x\sin^2 x\,dx.$

12 $\int \frac{\ln x + \sqrt{x}}{x}\,dx.$

13 $\int \frac{e^{2x}}{1+e^x}\,dx.$

14 $\int \frac{\ln(x+1)}{x^2}\,dx.$

15 $\int \frac{x^2\,dx}{\sqrt[3]{x-1}}.$

16 $\int \sin x\cos(\cos x)\,dx.$

17 $\int \frac{\tan^{-1}\sqrt{x}}{\sqrt{x}}\,dx.$

18 $\int \sec^4 x\,dx.$

19 $\int \frac{3x+5}{x-2}\,dx.$

20 $\int (1+\sqrt{x})^8\,dx.$

21 $\int \frac{\sqrt{4-x^2}}{x}\,dx.$

22 $\int \frac{dx}{e^{2x}+5e^x}.$

23 $\int x^5 e^{-x^3}\,dx.$

24 $\int (e^x+1)^2\,dx.$

25 $\int \frac{x}{(x+3)^2}\,dx.$

26 $\int x\sqrt[3]{x+5}\,dx.$

27 $\int \frac{x}{x^4+2x^2+10}\,dx.$

28 $\int x^2\sqrt{x^3-4}\,dx.$

29 $\int \frac{x\ln x}{\sqrt{x^2-1}}\,dx.$

30 $\int \frac{\sin 2x}{\sqrt{4-\cos^4 x}}\,dx.$

31 $\int x^2 \sin x^3\,dx.$

32 $\int x\sec x\tan x\,dx.$

33 $\int \frac{x}{(x^2+1)(x^2+4)}\,dx.$

34 $\int \frac{dx}{x\sqrt{2x-16}}.$

35 $\int \frac{x^2\,dx}{(x-1)^3}.$

36 $\int x^3 \ln x\,dx.$

37 $\int \tan^3 x\sec^4 x\,dx.$

38 $\int \left(e^x - \frac{1}{e^x}\right)^2 dx.$

39 $\int \frac{x}{1-x^2+\sqrt{1-x^2}}\,dx.$

40 $\int x^3 e^{-2x}\,dx.$

41 $\int \sin^2 x\cos^4 x\,dx.$

42 $\int \frac{x\,dx}{\sqrt{1-4x^2}}.$

43 $\int \frac{\sqrt{x-1}}{x+3}\,dx$

44 $\int \frac{x^3}{16+x^8}\,dx.$

45 $\int \ln(x^2+3)\,dx.$

46 $\int \frac{e^{\tan^{-1} x}}{1+x^2}\,dx.$

47 $\int e^{5x} \cos 3x \, dx.$

48 $\int \frac{x+1}{x^2 - 2x + 2} \, dx.$

49 $\int \frac{x \, dx}{x^4 - 2x^2 - 3}.$

50 $\int e^{x+e^x} \, dx.$

51 $\int \frac{x^4 + 1}{x^5 + 5x + 3} \, dx.$

52 $\int x \sin^{-1} (x^2) \, dx.$

53 $\int \frac{dx}{x + 7 + 5\sqrt{x+1}}.$

54 $\int \frac{dx}{x^3 + x^2 + x + 1}.$

55 $\int \frac{\sin x \, dx}{1 + 3 \cos^2 x}.$

56 $\int \frac{2x + 3}{x^2 + 1} \, dx.$

57 $\int \frac{x^3}{(x+1)^8} \, dx.$

58 $\int \sin (\ln x) \, dx.$

59 $\int \tan^6 x \, dx.$

60 $\int \frac{\sin x + \cos x}{\sin x - \cos x} \, dx.$

61 $\int \frac{x}{x^2 + 5x + 6} \, dx.$

62 $\int \frac{dx}{\sqrt{1 - 4x^2}}.$

63 $\int \ln \sqrt{2x - 1} \, dx.$

64 $\int \frac{4 \, dx}{x^2 + 4x + 20}.$

65 $\int \sqrt{\frac{1+x}{x-1}} \, dx.$

66 $\int \frac{x}{\sqrt{16 - x^4}} \, dx.$

67 $\int \sin^2 5x \cos^2 5x \, dx.$

68 $\int \frac{dx}{x^2 + 5x - 6}.$

69 $\int \cot x \ln (\sin x) \, dx.$

70 $\int \frac{dx}{e^{3x} - e^x}.$

71 $\int \cot^3 2x \csc^3 2x \, dx.$

72 $\int \frac{e^x + 1}{e^x - 1} \, dx.$

73 $\int \ln (2x + x^2) \, dx.$

74 $\int \frac{dx}{\sin 4x}.$

75 $\int \sqrt[3]{x}(1 - \sqrt{x}) \, dx.$

76 $\int \frac{e^x \, dx}{e^{2x} - 1}.$

77 $\int \ln (1 + x^2) \, dx.$

78 $\int \frac{x^4}{(x^5 + 1)^3} \, dx.$

79 $\int x \tan^2 x \, dx.$

80 $\int \frac{\tan^{-1} 2x}{1 + 4x^2} \, dx.$

81 $\int \sec^7 x \tan x \, dx.$

82 $\int \frac{dx}{x^2 + 5x + 6}.$

83 $\int x \sin^{-1} x \, dx.$

84 $\int \frac{1 + \cos^2 x}{1 - \cos^2 x} \, dx.$

85 $\int \frac{x^3}{1 + x^8} \, dx.$

86 $\int \frac{\tan^{-1} x}{x^2} \, dx.$

87 $\int \frac{\sec^2 x \, dx}{\sqrt{\sec^2 x - 1}}.$

88 $\int \frac{dx}{1 + 2e^x - e^{-x}}.$

89 $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} \, dx.$

90 $\int \frac{\sqrt{x^2 + 9}}{x} \, dx.$

91 $\int \frac{\ln x^{10}}{x} \, dx.$

92 $\int \frac{dx}{\cos 5x}.$

93 $\int \frac{\sqrt{x - 2}}{x + 2} \, dx.$

94 $\int \frac{1}{\sqrt{x} + \sqrt{x + 1}} \, dx.$

95 $\int \frac{dx}{\sin^2 x}.$

96 $\int \frac{dx}{\sqrt{9x^2 + 12x - 5}}.$

97 $\int \sqrt{(1 + 3x)(1 - 3x)} \, dx.$

98 $\int \sin^3 x \cos^2 x \, dx.$

99 $\int \frac{2x + 5}{x^2 + 5x + 6} \, dx.$

100 $\int \sin^2 x \cos^3 x \, dx.$

101 $\int \frac{x}{x^4 - 2x^2 - 3} \, dx.$

102 $\int \sqrt{1 + \sqrt{1 + \sqrt{x}}} \, dx.$

103 $\int \frac{x^4}{x^3 - 1} \, dx.$

104 $\int x \sec^2 x \, dx.$

105 $\int \frac{\sin^3 x}{\cos^5 x} \, dx.$

106 $\int \sin 2x \cos x \, dx.$

107 $\int \sin x \cos 2x \, dx.$

108 $\int \frac{4}{x^4 - 1} \, dx.$

109 $\int \frac{x \, dx}{\sqrt[3]{x - 1}}.$

110 $\int \frac{dx}{e^x + 5 + 4e^{-x}}.$

111 $\int \ln (ax + b) \, dx.$

112 $\int \frac{\cos^4 x}{\sin^2 x} \, dx.$

113 $\int e^x \cos (e^x) \, dx.$

114 $\int \frac{dx}{3x^2 - 13x + 4}.$

115 $\int \sqrt{x} \ln x \, dx.$

116 $\int \frac{\cos x}{\sin^2 x - 2 \sin x + 3} \, dx.$

117 $\int (\tan x + \cot x)^2 \, dx.$

118 $\int \ln (x\sqrt{x}) \, dx.$

119 $\int \frac{32x + 80}{(x - 1)(x + 3)^2} \, dx.$

120 $\int \frac{dx}{a^2 + b^2x^2}.$

121 $\int \ln (1 - \sqrt{x}) \, dx.$

122 $\int \frac{x^2 \, dx}{(x^2 + 1)^3}.$

123 $\int x \tan^{-1} (x - 1) \, dx.$

124 $\int \frac{dx}{2x^2 - 2x + 1}.$

125 $\int \frac{-x^2}{\sqrt{1 - x^2}} \, dx.$

10.9 NUMERICAL INTEGRATION. SIMPSON'S RULE

From the point of view of the theorist, the main value of calculus is intellectual; it helps us comprehend the underlying connections among natural phenomena. However, anyone who uses calculus as a practical tool in science or engineering must occasionally face the question of how the theory can be applied to yield useful methods for performing actual numerical calculations.

In this section we consider the problem of computing the numerical value of a definite integral

$$\int_a^b f(x)\,dx \tag{1}$$

in decimal form to any desired degree of accuracy. In order to find the value of (1) by using the formula

$$\int_a^b f(x)\,dx = F(b) - F(a), \tag{2}$$

we must be able to find an indefinite integral $F(x)$ and we must be able to evaluate it at both $x = a$ and $x = b$. When this is not possible, formula (2) is useless. This approach fails even for such simple-looking integrals as

$$\int_0^\pi \sqrt{\sin x}\,dx \quad \text{and} \quad \int_1^5 \frac{e^x}{x}\,dx,$$

because there are no elementary functions whose derivatives are $\sqrt{\sin x}$ and e^x/x (see Appendix A.9).

Our purpose here is to describe two methods of computing the numerical value of (1) as accurately as we wish by simple procedures that can be applied regardless of whether an indefinite integral can be found or not. The formulas we develop use only simple arithmetic and the values of $f(x)$ at a finite number of points in the interval $[a, b]$. In comparison with the use of the approximating sums that are used in defining the integral (see Section 6.4), the formulas of this section are more efficient in the sense that they give much better accuracy for the same amount of computational labor.

THE TRAPEZOIDAL RULE

Let the interval $[a, b]$ be divided into n equal parts by points $x_0, x_1, \ldots, x_n$ from $x_0 = a$ to $x_n = b$. Let $y_0, y_1, \ldots, y_n$ be the corresponding values of $y = f(x)$. We then approximate the area between $y = f(x)$ and the x-axis, for $x_{k-1} \le x \le x_k$, by the trapezoid whose upper edge is the segment joining the points (x_{k-1}, y_{k-1}) and (x_k, y_k) [see Fig. 10.5]. The area of this trapezoid is clearly

$$\tfrac{1}{2}(y_{k-1} + y_k)(x_k - x_{k-1}). \tag{3}$$

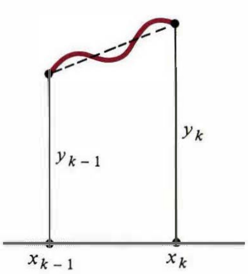


Figure 10.5

If we write

$$\Delta x = x_k - x_{k-1} = \frac{b-a}{n}, \tag{4}$$

then adding the expressions (3) for $k = 1, 2, \ldots, n$ gives the approximation formula

$$\int_a^b f(x)\,dx \cong (\tfrac{1}{2}y_0 + y_1 + y_2 + \cdots + y_{n-1} + \tfrac{1}{2}y_n)\,\Delta x,$$

because each of the y's except the first and the last occurs twice. This formula is called the *trapezoidal rule.*

Example 1 Use the trapezoidal rule with $n = 4$ to calculate an approximate value for the integral

$$\int_0^1 \sqrt{1 - x^3}\,dx.$$

Here $y = f(x) = \sqrt{1 - x^3}$ and $x_0 = 0$, $x_1 = \frac{1}{4}$, $x_2 = \frac{1}{2}$, $x_3 = \frac{3}{4}$, $x_4 = 1$. We can compute the y's easily by using a calculator:

$$
\begin{aligned}
y_0 &= 1, \\
y_1 &= \sqrt{\tfrac{63}{64}} = \sqrt{0.984} = 0.992, \\
y_2 &= \sqrt{\tfrac{7}{8}} = \sqrt{0.875} = 0.935, \\
y_3 &= \sqrt{\tfrac{37}{64}} = \sqrt{0.578} = 0.760, \\
y_4 &= 0.
\end{aligned}
$$

By the trapezoidal rule, we therefore have

$$\int_0^1 \sqrt{1 - x^3}\, dx \cong \tfrac{1}{4}(0.500 + 0.992 + 0.935 + 0.760 + 0.000) = 0.797.$$

SIMPSON'S RULE*

Our second method is based on a more ingenious device than approximating each small piece of the curve by a line segment; this time we approximate each piece by a portion of a parabola that "fits" the curve in a manner to be described.

Again we divide the interval $[a, b]$ into n equal parts, but now we require that n be an *even* integer. Consider the first three points x_0, x_1, x_2 and the corresponding points on the curve $y = f(x)$, as shown in Fig. 10.6. If these points are not collinear, then there is a unique parabola that has vertical axis and that passes through all three points. To see this, recall that the equation of any parabola with vertical axis has the form $y = P(x)$ where $P(x)$ is a quadratic polynomial, and observe that this polynomial can always be written in the form

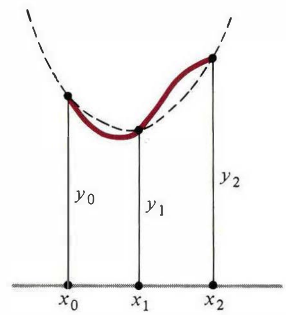


Figure 10.6

$$P(x) = a + b(x - x_1) + c(x - x_1)^2. \tag{5}$$

We choose the constants a, b, c to make the parabola pass through the three points under consideration, as indicated in the figure. Three conditions are necessary for this:

$$
\text{at } x = \begin{cases}
x_0, & a + b(x_0 - x_1) + c(x_0 - x_1)^2 = y_0; \quad (6) \\
x_1, & a = y_1; \\
x_2, & a + b(x_2 - x_1) + c(x_2 - x_1)^2 = y_2. \quad (7)
\end{cases}
$$

Equations (6) and (7) can be solved for the constants b and c. However, it is more convenient to use the definition (4) of Δx and the fact that $a = y_1$ to write these equations in the form

$$
\begin{aligned}
-b\,\Delta x + c\,\Delta x^2 &= y_0 - y_1, \\
b\,\Delta x + c\,\Delta x^2 &= y_2 - y_1,
\end{aligned}
$$

from which we obtain

$$2c\,\Delta x^2 = y_0 - 2y_1 + y_2. \tag{8}$$

*Thomas Simpson (1710–1761), an English mathematics teacher whose name is wrongly attached to the rule that bears his name, was in his earlier years a professional astrologer and confidence man (one of his escapades forced him to flee to another town). His eventual success as a writer of elementary mathematics textbooks was greatly helped by accusations of plagiarism. This success enabled him to escape from poverty and leave his shady past behind him.

We now think of the parabola (5) as a close approximation to the curve $y = f(x)$ on the interval $[x_0, x_2]$, and we compute this part of the integral (1) accordingly,

$$\int_{x_0}^{x_2} f(x)\,dx \cong \int_{x_0}^{x_2} [a + b(x - x_1) + c(x - x_1)^2]\,dx$$
$$= \left[ax + \tfrac{1}{2}b(x - x_1)^2 + \tfrac{1}{3}c(x - x_1)^3\right]_{x_0}^{x_2}.$$

When this is evaluated in terms of Δx, we obtain

$$2a\,\Delta x + \tfrac{2}{3}c\,\Delta x^3.$$

By using (8) and the fact that $a = y_1$, we can write this in the form

$$2y_1\,\Delta x + \tfrac{1}{3}(y_0 - 2y_1 + y_2)\,\Delta x = \tfrac{1}{3}(y_0 + 4y_1 + y_2)\,\Delta x.$$

The same procedure can be applied to each of the intervals $[x_2, x_4]$, $[x_4, x_6]$, ... When the results are all added together, we get the approximation formula

$$\int_a^b f(x)\,dx \cong \tfrac{1}{3}(y_0 + 4y_1 + 2y_2 + \cdots + 4y_{n-1} + y_n)\,\Delta x,$$

which is called *Simpson's rule*. We specifically point out the structure of the expression in parentheses: y_0 and y_n occur with coefficient 1, the remaining y's with even subscripts occur with coefficient 2, and the y's with odd subscripts occur with coefficient 4.

Example 2 Use Simpson's rule with $n = 4$ to calculate an approximate value for the integral

$$\int_0^2 \frac{dx}{1 + x^4}.$$

This time we have $y = f(x) = 1/(1 + x^4)$ and $x_0 = 0$, $x_1 = \frac{1}{2}$, $x_2 = 1$, $x_3 = \frac{3}{2}$, $x_4 = 2$. A simple table helps to keep the computations in order:

$y_0 = 1$	$y_0 = 1.000$
$y_1 = \frac{16}{17} = 0.941$	$4y_1 = 3.764$
$y_2 = \frac{1}{2} = 0.500$	$2y_2 = 1.000$
$y_3 = \frac{16}{97} = 0.165$	$4y_3 = 0.660$
$y_4 = \frac{1}{17} = 0.059$	$y_4 = 0.059$
	sum $= 6.483$

Simpson's rule now yields

$$\int_0^2 \frac{dx}{1 + x^4} \cong \tfrac{1}{6}(6.483) = 1.081.$$

Sometimes data is obtained from a scientific experiment with equally spaced observations. If this data represents isolated values of a function whose analytic expression is not known, then it may be wished to obtain an approximation to the integral of this function over the range of observation. Simpson's rule can be used in such a situation.

Example 3 If the experimental data is

x	0	0.5	1	1.5	2
y	1.0000	1.6487	2.7183	4.4817	7.3891

then

$$\int_0^2 y\,dx \cong \tfrac{1}{6}[1 + 4(1.6487) + 2(2.7183) + 4(4.4817) + 7.3891]$$
$$= 6.3912.$$

As a matter of fact, $y = e^x$ was the function used to generate this table of values, so the value of the integral is $e^2 - 1 = 6.3890560989$ to 10 decimal places.

Any serious study of a method of approximate calculation must include a detailed estimate of the magnitude of the error committed so that definite knowledge is available of the level of accuracy attained. We do not pursue this matter very far here, but merely state that the error in Simpson's rule is known to be at most

$$\frac{M(b-a)}{180}\,\Delta x^4, \tag{9}$$

where M is the maximum value of $f^{(4)}(x)$ on $[a, b]$. Derivations of this bound for the error can be found in books on numerical analysis. The power of Δx that appears in (9) tells us that if we reduce the width Δx by a factor of 10 (using 10 times as many subintervals), then we expect the maximum error to shrink by a factor of $10^4 = 10{,}000$. If we replace Δx in (9) by $(b - a)/n$, the bound (9) takes the form

$$\frac{M(b-a)^5}{180n^4}. \tag{10}$$

This formula enables us to impose a previously determined bound on the error by specifying a suitable value for n.

Example 3 *(continued)* We see that the actual error in the above calculation is about 0.0021 when $n = 4$. What value of n will guarantee that the error will be at most 0.0001?

In this case, assuming $f(x) = e^x$ really was the function underlying our data, then $f^{(4)}(x) = e^x$ and $M = e^2$. By (10) we therefore have

$$\frac{e^2 \cdot 2^5}{180n^4} = 0.0001,$$

so

$$n^4 = \frac{e^2 \cdot 2^5}{180}\,10^4 \qquad \text{or} \qquad n \cong 10.7.$$

Any integer $n \geq 11$ will therefore provide this level of accuracy.

Students who own calculators and enjoy working with them should note that the methods and problems of this section—and also of others to come, especially Section 14.5—provide plenty of raw material for these calculator enthusiasts.

PROBLEMS

1 Clearly,

$$\int_0^1 \sqrt{x}\,dx = \tfrac{2}{3} = 0.666\ldots.$$

Calculate the value of this integral approximately with $n = 4$ by using
(a) the trapezoidal rule (recall that $\sqrt{2} = 1.414\ldots$ and $\sqrt{3} = 1.732\ldots$);
(b) Simpson's rule.
Since the two rules are almost equally easy to apply, and Simpson's rule is usually more accurate, the trapezoidal rule is rarely used in practical computations.

2 Clearly,

$$\int_0^{\pi} \sin x\,dx = 2.$$

Calculate the value of this integral approximately by using Simpson's rule with $n = 4$.

3 The exact value of

$$\int_0^{\pi} \sqrt{\sin x}\,dx$$

is not known. Find its approximate value by using Simpson's rule with $n = 4$.

4 The exact value of

$$\int_1^5 \frac{e^x}{x}\,dx$$

is not known. Use Simpson's rule when $n = 4$ to find its approximate value.

5 The exact value of

$$\int_0^2 e^{-x^2}\,dx$$

is not known, but to 10 decimal places it is 0.8820813908. Calculate this integral approximately by using Simpson's rule with $n = 4$.

6 Find an approximate value for ln 2 by using the fact that

$$\ln 2 = \int_1^2 \frac{dx}{x}$$

and applying Simpson's rule with $n = 4$. (To 10 decimal places, ln 2 = 0.6931471806.)

7 Use the formula

$$\frac{\pi}{4} = \int_0^1 \frac{dx}{1 + x^2}$$

to find an approximate value for π by using Simpson's rule with $n = 4$. (To 10 decimal places, $\pi = 3.1415926536$.)

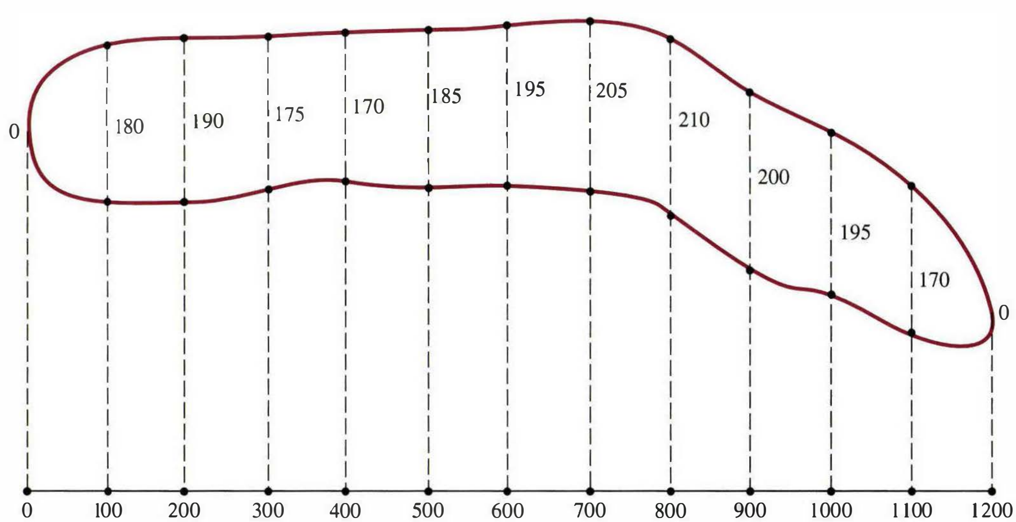


Figure 10.7 A dogleg fairway on a golf course.

8 In Example 3, what positive integers n will guarantee that the error is at most 0.000001?

9 The width, in feet, at equally spaced points along the fairway of a hole on a golf course is given in Fig. 10.7. The management wishes to estimate the number of square yards of the fairway as a basis for deciding how long it should take a groundskeeper to mow it. Use Simpson's rule to provide such an estimate.

10 Suppose that the three points on the curve in the derivation of Simpson's rule are collinear. Use (8) to show that in this case $c = 0$, and conclude that under this assumption the curve through the points is a straight line instead of a parabola.

11 Simpson's rule is designed to be exactly correct if $f(x)$ is a quadratic polynomial. It is a remarkable fact that it also gives an exact result for cubic polynomials. Prove this. Hint: Notice that it suffices to establish the statement for $n = 2$; then prove it in this case for the function $f(x) = x^3$; then extend it to any cubic polynomial.

12 Use formula (9) to prove the statement in Problem 11.

CHAPTER 10 REVIEW: FORMULAS, METHODS

Think through and learn the following.

1 The 15 basic formulas (write them down from memory).
2 Method of substitution.
3 Integrals of the form

$$\int \sin^m x \cos^n x\, dx, \qquad \int \tan^m x \sec^n x\, dx,$$
$$\int \cot^m x \csc^n x\, dx.$$

4 The trigonometric substitutions $x = a \sin \theta$, $x = a \tan \theta$, $x = a \sec \theta$.
5 Completing the square: $(x + A)^2 = x^2 + 2Ax + A^2$.
6 Method of partial fractions.
7 Integration by parts.
8 Simpson's rule.

ADDITIONAL PROBLEMS FOR CHAPTER 10

SECTION 10.2

Find each of the following integrals.

1 $\int \sqrt{3x + 5}\, dx$.

2 $\int \frac{(\ln x)^6\, dx}{x}$.

3 $\int \frac{6x\, dx}{1 + 3x^2}$.

4 $\int \frac{e^{1/x}\, dx}{x^2}$.

5 $\int \cos(1 - 5x)\, dx$.

6 $\int \sin x \sin(\cos x)\, dx$.

7 $\int \frac{\sec\sqrt{x}\tan\sqrt{x}\, dx}{\sqrt{x}}$.

8 $\int \frac{x^3\, dx}{\sqrt{1 - x^8}}$.

9 $\int \frac{2x\, dx}{1 + x^4}$.

10 $\int \frac{x^2 + 5}{x^2 + 4}\, dx$.

11 $\int \cot 4x\, dx$.

12 $\int \frac{dx}{\sin 2x}$.

13 $\int \frac{dx}{x(\ln x)^2}$.

14 $\int \frac{dx}{3 - x}$.

15 $\int \frac{\sec^2 x\, dx}{\tan x}$.

16 $\int 10x^4 e^{x^5}\, dx$.

17 $\int \sin\left(\frac{3x - 5}{2}\right) dx$.

18 $\int \csc^2(2 - x)\, dx$.

19 $\int 6x^2 \cot x^3 \csc x^3\, dx$.

20 $\int \frac{\sec^2 x\, dx}{\sqrt{1 - \tan^2 x}}$.

21 $\int \frac{dx}{x[1 + (\ln x)^2]}$.

22 $\int \cot \pi x\, dx$.

23 $\int \frac{dx}{(3x + 5)^2}$.

24 $\int \tan x \sec^4 x\, dx$.

25 $\int \frac{dx}{3 - 2x}$.

26 $\int \frac{(e^x + 2x)\, dx}{e^x + x^2 - 2}$.

27 $\int x^2 \cos(1 + x^3)\, dx$.

28 $\int \sin(2 - x)\, dx$.

29 $\int x \csc^2(x^2 + 1)\, dx$.

30 $\int \frac{dx}{\sqrt{3 - 4x^2}}$.

31 $\int \frac{\cos x\, dx}{1 + \sin^2 x}$.

32 $\int \frac{dx}{1 + 4x^2}$.

33 $\int \frac{dx}{\tan 2x}$.

34 $\int (\csc x - 1)^2\, dx$.

35 $\int \frac{\tan^{-1} x\, dx}{1 + x^2}$.

36 $\int \sqrt[3]{3x - 2}\, dx$.

37 $\int \frac{dx}{2x + 1}$.

38 $\int \frac{(e^x - e^{-x})\, dx}{e^x + e^{-x}}$.

39 $\int e^{x/3}\,dx$.

40 $\int \frac{dx}{\sec 2x}$.

41 $\int \frac{\sec^2(\sin x)\,dx}{\sec x}$.

42 $\int (\csc x - \cot x)\csc x\,dx$.

43 $\int \frac{dx}{\sqrt{1-25x^2}}$.

44 $\int \frac{dx}{16+25x^2}$.

45 $\int \frac{\sec x \tan x\,dx}{1+\sec^2 x}$.

46 $\int (1+\sec x)^2\,dx$.

47 $\int \frac{(\ln x)^2\,dx}{x}$.

48 $\int \frac{\cos x\,dx}{\sin^2 x}$.

49 $\int \frac{\sin x\,dx}{1+\cos x}$.

50 $\int \frac{6\csc^2 x\,dx}{1-3\cot x}$.

51 $\int \frac{dx}{e^{3x}}$.

52 $\int e^x \cos e^x\,dx$.

53 $\int \frac{\sin(\ln x)\,dx}{x}$.

54 $\int \frac{\csc^2\sqrt{x}\,dx}{\sqrt{x}}$.

55 $\int \frac{\csc 1/x \cot 1/x\,dx}{x^2}$.

56 $\int \frac{4dx}{3+4x^2}$.

57 $\int \frac{e^{2x}\,dx}{1+e^{4x}}$.

58 $\int \frac{x\,dx}{\sin x^2}$.

59 $\int x^3\sqrt{2+x^4}\,dx$.

60 $\int \frac{x\,dx}{\sqrt{2-x^2}}$.

61 $\int \frac{(1+e^x)\,dx}{e^x+x}$.

62 $\int xe^{x^2}\,dx$.

63 $\int \frac{2dx}{\sqrt{e^x}}$.

64 $\int x\sin(1-x^2)\,dx$.

65 $\int \frac{dx}{\sin^2 x}$.

66 $\int \frac{dx}{\sqrt{4-9x^2}}$.

67 $\int x\tan x^2\,dx$.

68 $\int \frac{\sec^2 x\,dx}{\sqrt{\tan x}}$.

69 $\int \frac{x\,dx}{1+x^2}$.

70 $\int 2e^{2x}\,dx$.

71 $\int xe^{3x^2-2}\,dx$.

72 $\int 3x^2\sin x^3\,dx$.

73 $\int \sec x(\sec x+\tan x)\,dx$.

74 $\int \frac{x^2\,dx}{9+x^6}$.

75 $\int x^{2/3}\sqrt{1+x^{5/3}}\,dx$.

76 $\int \frac{4x^3\,dx}{1+x^4}$.

77 $\int \sec^2 x\, e^{\tan x}\,dx$.

78 $\int x\sec^2 x^2\,dx$.

79 $\int (1+\cos x)^4 \sin x\,dx$.

80 $\int \frac{(1+\cos x)\,dx}{x+\sin x}$.

81 $\int \cos(\tan x)\sec^2 x\,dx$.

82 $\int \frac{\csc^2(\ln x)\,dx}{x}$.

Compute each of the following definite integrals by making a suitable substitution and changing the limits of integration.

83 $\int_0^{\sqrt{2}/2} \frac{2x\,dx}{\sqrt{1-x^4}}$.

84 $\int_0^{\sqrt{\pi}} x\sin x^2\,dx$.

85 $\int_{\pi/8}^{\pi/4} \cot 2x\csc^2 2x\,dx$.

86 $\int_0^{\pi/2} \frac{\cos x\,dx}{1+\sin^2 x}$.

87 $\int_0^4 2x\sqrt{x^2+9}\,dx$.

88 $\int_0^3 \frac{x\,dx}{\sqrt{x^2+16}}$.

SECTION 10.3

Calculate each of the following integrals.

89 $\int \sin^2 5x\,dx$.

90 $\int \cos^4 3x\,dx$.

91 $\int \cos^2 7x\,dx$.

92 $\int \sin^6 x\,dx$.

93 $\int \sin^5 x\cos^2 x\,dx$.

94 $\int \sin^5 x\,dx$.

95 $\int \cos^3 4x\,dx$.

96 $\int \cos^3 2x\sin 2x\,dx$.

97 $\int \frac{\cos^3 x\,dx}{\sin^4 x}$.

98 $\int \frac{\sin^5 x\,dx}{\sqrt{\cos x}}$.

99 $\int \sin^{3/5} x\cos x\,dx$.

100 $\int \sin^2 x\cos^4 x\,dx$.

101 $\int \sec^6 x\,dx$.

102 $\int \frac{dx}{\cos^4 x}$.

103 $\int \tan^3 x\sec^7 x\,dx$.

104 $\int \cot^4 x\,dx$.

105 $\int \cot^5 x\,dx$.

106 $\int \frac{dx}{\sin^4 3x}$.

107 $\int (\sec 3x+\csc 3x)^2\,dx$.

108 $\int \frac{dx}{\sec x\tan x}$.

SECTION 10.4

Find each of the following integrals.

109 $\int \sqrt{3-x^2}\,dx$.

110 $\int \frac{dx}{(a^2+x^2)^{3/2}}$.

111 $\int \frac{x^2\,dx}{a^2+x^2}$.

112 $\int \frac{\sqrt{4-9x^2}}{x}\,dx$.

113 $\int x^3\sqrt{a^2-x^2}\,dx$.

114 $\int \frac{x^3\,dx}{\sqrt{a^2+x^2}}$.

115 $\int \frac{\sqrt{a^2+x^2}}{x^2}\,dx.$

116 $\int \frac{dx}{x^2\sqrt{a^2-x^2}}.$

117 $\int \frac{dx}{x^4\sqrt{a^2-x^2}}.$

118 $\int \frac{dx}{x^2+x^4}.$

119 $\int \frac{x^2\,dx}{(a^2+x^2)^2}.$

120 $\int x^3(a^2-x^2)^{3/2}\,dx.$

121 $\int \frac{dx}{x^2\sqrt{x^2-9}}.$

122 $\int \sqrt{x^2-1}\,dx.$

123 $\int \frac{dx}{(1-9x^2)^{3/2}}.$

124 $\int \frac{x^2\,dx}{\sqrt{a^2+x^2}}.$

125 $\int \frac{dx}{x\sqrt{9+4x^2}}.$

126 $\int \frac{dx}{\sqrt{9-(x-1)^2}}.$

127 $\int \frac{x^2\,dx}{(a^2-x^2)^{3/2}}.$

128 $\int \frac{dx}{x^4\sqrt{a^2+x^2}}.$

129 $\int \frac{x^2\,dx}{(a^2+x^2)^{3/2}}.$

130 $\int \frac{\sqrt{a^2-x^2}}{x^4}\,dx.$

131 $\int \frac{x^2\,dx}{\sqrt{x^2-a^2}}.$

132 $\int \frac{x^3\,dx}{(x^2-a^2)^{3/2}}.$

SECTION 10.5

Calculate each of the following integrals.

133 $\int \frac{dx}{\sqrt{65-8x-x^2}}.$

134 $\int \frac{dx}{\sqrt{1+4x-x^2}}.$

135 $\int \frac{dx}{5x^2+10x+15}.$

136 $\int \frac{(3x-5)\,dx}{x^2+2x+2}.$

137 $\int \frac{dx}{\sqrt{2+2x-3x^2}}.$

138 $\int \frac{(1-x)\,dx}{\sqrt{8+2x-x^2}}.$

139 $\int \frac{x^2\,dx}{\sqrt{2x-x^2}}.$

140 $\int \frac{x\,dx}{\sqrt{x^2-4x+5}}.$

141 $\int \frac{dx}{3x^2-6x+15}.$

142 $\int \frac{(3x+4)\,dx}{\sqrt{2x+x^2}}.$

143 $\int \frac{dx}{(x-1)\sqrt{x^2-2x-3}}.$

144 $\int \frac{(2x-5)\,dx}{\sqrt{4x-x^2}}.$

145 $\int \frac{(3x+7)\,dx}{\sqrt{x^2+4x+8}}.$

146 $\int \sqrt{x^2+2x+2}\,dx.$

147 $\int \frac{(2x-3)\,dx}{(x^2+2x-3)^{3/2}}.$

148 $\int \sqrt{x^2-2x}\,dx.$

SECTION 10.6

Find each of the following integrals.

149 $\int \frac{16x+69}{x^2-x-12}\,dx.$

150 $\int \frac{3x-56}{x^2+3x-28}\,dx.$

151 $\int \frac{-8x-16}{4x^2-1}\,dx.$

152 $\int \frac{12x-63}{x^2-3x}\,dx.$

153 $\int \frac{3x^2-10x-60}{x^3+x^2-12x}\,dx.$

154 $\int \frac{8x^2+55x-25}{x^3-25x}\,dx.$

155 $\int \frac{-2x^2-18x+18}{x^3-9x}\,dx.$

156 $\int \frac{4x^2-2x-108}{x^3+5x^2-36x}\,dx.$

157 $\int \frac{-3x^3+x^2+2x+3}{x^4+x^3}\,dx.$

158 $\int \frac{9x^2-35x+28}{x^3-4x^2+4x}\,dx.$

159 $\int \frac{x^2-5x-8}{x^3+4x^2+8x}\,dx.$

160 $\int \frac{3x^2-5x+4}{x^3-x^2+x-1}\,dx.$

SECTION 10.7

Calculate the integrals in Problems 161–176 by the method of integration by parts.

161 $\int x^2\tan^{-1}x\,dx.$

162 $\int x^2\cos x\,dx.$

163 $\int \cos(\ln x)\,dx.$

164 $\int x\sin^2 x\,dx.$

165 $\int x^3\cos x\,dx.$

166 $\int \sqrt{1+x^2}\,dx.$

167 $\int \frac{\ln x\,dx}{(x+1)^2}.$

168 $\int \frac{xe^x\,dx}{(x+1)^2}.$

169 $\int \frac{x^3\,dx}{\sqrt{1+x^2}}.$

170 $\int x(x+3)^{10}\,dx.$

171 $\int e^{ax}\sin bx\,dx.$

172 $\int x^n\ln x\,dx\ (n\neq -1).$

173 $\int \frac{x\,dx}{e^x}.$

174 $\int x^2\sin x\,dx.$

175 $\int x^3e^{-2x}\,dx.$

176 $\int \ln(x+\sqrt{x^2+a^2})\,dx.$

177 Find the area under the curve $y=\sin\sqrt{x}$ from $x=0$ to $x=\pi^2$.

178 Calculate the integral $\int \frac{x^3}{\sqrt{1+x^2}}\,dx$ by using the identity

$$\frac{x^3}{\sqrt{1+x^2}} = \frac{x(1+x^2-1)}{\sqrt{1+x^2}} = x\sqrt{1+x^2} - \frac{x}{\sqrt{1+x^2}}.$$

Make sure your answer agrees with the result of Problem 169.

179 Calculate the integral $\int_0^a x^2\sqrt{a-x}\,dx$ (a) by using the substitution $u = \sqrt{a-x}$; (b) by parts.

180 Use integration by parts to show that

$$\int \sqrt{a^2-x^2}\,dx = x\sqrt{a^2-x^2} + \int \frac{x^2}{\sqrt{a^2-x^2}}\,dx.$$

Write $x^2 = -(-x^2) = -(a^2-x^2-a^2)$ in the numerator of the second integral and thereby obtain the formula

$$\int \sqrt{a^2-x^2}\,dx = \tfrac{1}{2}x\sqrt{a^2-x^2} + \tfrac{1}{2}a^2 \int \frac{dx}{\sqrt{a^2-x^2}}$$
$$= \tfrac{1}{2}x\sqrt{a^2-x^2} + \tfrac{1}{2}a^2 \sin^{-1}\frac{x}{a} + c.$$

181 Use the method of Problem 180 to obtain the formula

$$\int (a^2-x^2)^n\,dx = \frac{x(a^2-x^2)^n}{2n+1} + \frac{2a^2n}{2n+1}\int (a^2-x^2)^{n-1}\,dx.$$

***182** Use the idea of Problem 181 to obtain formula (9) in Section 10.6,

$$\int \frac{dx}{(a^2+x^2)^n} = \frac{1}{2a^2(n-1)} \cdot \frac{x}{(a^2+x^2)^{n-1}} + \frac{2n-3}{2a^2(n-1)}\int \frac{dx}{(a^2+x^2)^{n-1}}.$$

In the next three problems, derive the given reduction formula and apply it to the indicated special case.

183 (a) $\int x^m(\ln x)^n\,dx = \dfrac{x^{m+1}(\ln x)^n}{m+1} - \dfrac{n}{m+1}\int x^m(\ln x)^{n-1}\,dx.$

(b) $\int x^5(\ln x)^3\,dx.$

184 (a) $\int x^n e^{ax}\,dx = \dfrac{1}{a}x^n e^{ax} - \dfrac{n}{a}\int x^{n-1}e^{ax}\,dx.$

(b) $\int x^3 e^{-2x}\,dx.$

185 (a) $\int \sec^n x\,dx = \dfrac{1}{n-1}\sec^{n-2}x\tan x + \dfrac{n-2}{n-1}\int \sec^{n-2}x\,dx.$

(b) $\int \sec^3 x\,dx$ (see Problem 29 in Section 10.3).

APPENDIX 1: THE CATENARY, OR CURVE OF A HANGING CHAIN

As a specific example of the use of the methods of integration discussed in Section 10.4, we solve the classical problem of determining the exact shape of the curve assumed by a flexible chain (or cable, or rope) of uniform density which is suspended between two points and hangs under its own weight. This curve is called a *catenary*, from the Latin word for chain, *catena.**

Let the y-axis pass through the lowest point of the chain (Fig. 10.8), let s be the arc length from this point to a variable point (x, y), and let w_0 be the linear density (weight per unit length) of the chain. We obtain the differential equation of the catenary from the fact that the part of the chain between the lowest point and (x, y) is in static equilibrium under the action of three forces: the tension T_0 at the lowest point; the variable tension T at (x, y), which acts in the direction of the tangent because of the flexibility of the chain; and a downward force $w_0 s$ equal to the weight of the chain between these two points.

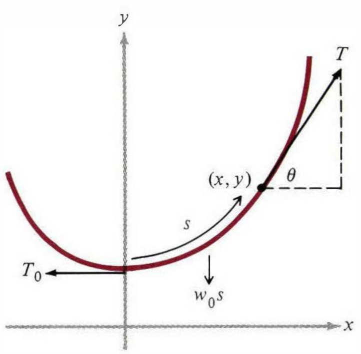


Figure 10.8

Equating the horizontal component of T to T_0 and the vertical component of T to the weight of the chain gives

$$T\cos\theta = T_0 \qquad \text{and} \qquad T\sin\theta = w_0 s,$$

and by dividing we eliminate T and get $\tan\theta = w_0 s/T_0$ or

$$\frac{dy}{dx} = as, \qquad \text{where} \qquad a = \frac{w_0}{T_0}.$$

*The catenary problem is also solved in the optional Section 9.7 by using methods depending on hyperbolic functions. The solution given here does not depend on these methods and can therefore be understood by students who have omitted that optional section.

We next eliminate the variable s by differentiating with respect to x,

$$\frac{d^2y}{dx^2} = a\frac{ds}{dx} = a\sqrt{1 + \left(\frac{dy}{dx}\right)^2}. \tag{1}$$

This is the differential equation of the catenary.

We now solve equation (1) by two successive integrations. This process is facilitated by introducing the auxiliary variable $p = dy/dx$, so that (1) becomes

$$\frac{dp}{dx} = a\sqrt{1 + p^2}.$$

On separating variables and integrating, we get

$$\int \frac{dp}{\sqrt{1 + p^2}} = \int a\,dx. \tag{2}$$

To calculate the integral on the left, we make the trigonometric substitution $p = \tan\phi$, so that $dp = \sec^2\phi\,d\phi$ and $\sqrt{1 + p^2} = \sec\phi$. Then

$$\int \frac{dp}{\sqrt{1 + p^2}} = \int \frac{\sec^2\phi\,d\phi}{\sec\phi} = \int \sec\phi\,d\phi$$
$$= \ln(\sec\phi + \tan\phi) = \ln(\sqrt{1 + p^2} + p),$$

so (2) becomes

$$\ln(\sqrt{1 + p^2} + p) = ax + c_1.$$

Since $p = 0$ when $x = 0$, we see that $c_1 = 0$, so

$$\ln(\sqrt{1 + p^2} + p) = ax.$$

It is easy to solve this equation for p, which yields

$$\frac{dy}{dx} = p = \frac{1}{2}(e^{ax} - e^{-ax}),$$

and by integrating we obtain

$$y = \frac{1}{2a}(e^{ax} + e^{-ax}) + c_2.$$

If we now place the origin of the coordinate system in Fig. 10.8 at just the right level so that $y = 1/a$ when $x = 0$, then $c_2 = 0$ and our equation takes its final form,

$$y = \frac{1}{2a}(e^{ax} + e^{-ax}). \tag{3}$$

Equation (3) reveals the precise mathematical nature of the catenary and can be used as the basis for further investigations of its properties.*

The problem of finding the true shape of the catenary was proposed by James Bernoulli in 1690. Galileo had speculated long before that the curve was a parabola, but Huygens had shown in 1646 (at the age of 17), largely by physical reasoning, that this is not correct, without, however, shedding any light on what the shape might be. Bernoulli's challenge produced quick results, for in 1691 Leibniz, Huygens (now aged 62), and James's brother John all published independent solutions of the problem. John Bernoulli was ex-

*The hyperbolic cosine defined in Section 9.7 enables us to write the function (3) in the form

$$y = \frac{1}{a}\cosh ax.$$

ceedingly pleased that he had been successful in solving the problem, while his brother James, who proposed it, had failed. The taste of victory was still sweet 27 years later, as we see from this passage in a letter John wrote in 1718:

> The efforts of my brother were without success. For my part, I was more fortunate, for I found the skill (I say it without boasting; why should I conceal the truth?) to solve it in full. . . . It is true that it cost me study that robbed me of rest for an entire night. It was a great achievement for those days and for the slight age and experience I then had. The next morning, filled with joy, I ran to my brother, who was struggling miserably with this Gordian knot without getting anywhere, always thinking like Galileo that the catenary was a parabola. Stop! Stop! I say to him, don't torture yourself any more trying to prove the identity of the catenary with the parabola, since it is entirely false.

However, James evened the score by proving in the same year of 1691 that of all possible shapes a chain hanging between two fixed points might have, the catenary has the lowest center of gravity, and therefore the smallest potential energy. This was a very significant discovery, because it was the first hint of the profound idea that in some mysterious way the actual configurations of nature are those that minimize potential energy.

APPENDIX 2: WALLIS'S PRODUCT $\frac{\pi}{2}=\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdots$

As an application of integration by parts in Section 10.7, we obtained the following reduction formula:

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx. \tag{1}$$

This formula leads in an elementary but ingenious way to a very remarkable expression for the number $\pi/2$ as an infinite product,

$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdots\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}\cdots. \tag{2}$$

This expression was discovered by the English mathematician John Wallis in 1656 and is called *Wallis's product*. Apart from its intrinsic interest, formula (2) underlies other important developments in both pure and applied mathematics, so we prove it here.

If we define I_n by

$$I_n = \int_0^{\pi/2} \sin^n x\,dx,$$

then (1) tells us that

$$I_n = \frac{n-1}{n} I_{n-2}. \tag{3}$$

It is clear that

$$I_0 = \int_0^{\pi/2} dx = \frac{\pi}{2} \qquad \text{and} \qquad I_1 = \int_0^{\pi/2} \sin x\,dx = 1.$$

We now distinguish the cases of even and odd subscripts, and use (3) to calculate I_{2n} and I_{2n+1}, as follows:

$$\begin{aligned} I_{2n} &= \frac{2n-1}{2n} I_{2n-2} = \frac{2n-1}{2n}\cdot\frac{2n-3}{2n-2} I_{2n-4} \\ &= \cdots = \frac{2n-1}{2n}\cdot\frac{2n-3}{2n-2}\cdot\frac{2n-5}{2n-4}\cdots\frac{1}{2} I_0 \\ &= \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2n-1}{2n}\cdot\frac{\pi}{2}; \end{aligned} \tag{4}$$

and

$$I_{2n+1} = \frac{2n}{2n+1} I_{2n-1} = \frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} I_{2n-3}$$

$$= \cdots = \frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} \cdot \frac{2n-4}{2n-3} \cdots \frac{2}{3} I_1$$

$$= \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdots \frac{2n}{2n+1}. \tag{5}$$

As the next link in the chain of this reasoning, we need the fact that the ratio of these two quantities approaches 1 as $n \to \infty$,

$$\frac{I_{2n}}{I_{2n+1}} \to 1. \tag{6}$$

To establish this, we begin by noticing that on the interval $0 \le x \le \pi/2$ we have $0 \le \sin x \le 1$, and therefore

$$0 \le \sin^{2n+2} x \le \sin^{2n+1} x \le \sin^{2n} x.$$

This implies that

$$0 < \int_0^{\pi/2} \sin^{2n+2} x \, dx \le \int_0^{\pi/2} \sin^{2n+1} x \, dx \le \int_0^{\pi/2} \sin^{2n} x \, dx,$$

or equivalently,

$$0 < I_{2n+2} \le I_{2n+1} \le I_{2n}. \tag{7}$$

If we divide through by I_{2n} and use the fact that by (3) we have

$$\frac{I_{2n+2}}{I_{2n}} = \frac{2n+1}{2n+2},$$

then (7) yields

$$\frac{2n+1}{2n+2} \le \frac{I_{2n+1}}{I_{2n}} \le 1.$$

This implies that

$$\frac{I_{2n+1}}{I_{2n}} \to 1 \quad \text{as} \quad n \to \infty,$$

and this is equivalent to (6).

The final steps of the argument are as follows. On dividing (5) by (4), we obtain

$$\frac{I_{2n+1}}{I_{2n}} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots \frac{2n}{2n-1} \cdot \frac{2n}{2n+1} \cdot \frac{2}{\pi},$$

so

$$\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots \frac{2n}{2n-1} \cdot \frac{2n}{2n+1} \left(\frac{I_{2n}}{I_{2n+1}} \right).$$

On forming the limit as $n \to \infty$ and using (6), we obtain

$$\frac{\pi}{2} = \lim_{n \to \infty} \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots \frac{2n}{2n-1} \cdot \frac{2n}{2n+1},$$

and this is what (2) means.

We also remark that Wallis's product (2) is equivalent to the formula

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{4^2}\right)\left(1-\frac{1}{6^2}\right)\cdots=\frac{2}{\pi}. \tag{8}$$

This is easy to see if we write each number in parentheses on the left in factored form. This gives

$$\left(1-\frac{1}{2}\right)\left(1+\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\left(1+\frac{1}{4}\right)\left(1-\frac{1}{6}\right)\left(1+\frac{1}{6}\right)\cdots=\frac{2}{\pi}$$

or

$$\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{3}{4}\cdot\frac{5}{4}\cdot\frac{5}{6}\cdot\frac{7}{6}\cdots=\frac{2}{\pi},$$

which is clearly equivalent to (2). Formula (8) will reappear in Appendix 1 at the end of Chapter 13 as a special case of another even more wonderful formula.*

APPENDIX 3: HOW LEIBNIZ DISCOVERED HIS FORMULA $\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots$

The area of the quarter-circle of radius 1 shown in Fig. 10.9 is obviously $\pi/4$. We follow Liebniz and calculate this area in a different way. The part that we actually calculate is the area A of the circular segment cut off by the chord OT, because the remainder of the quarter-circle is clearly an isosceles right triangle of area $\frac{1}{2}$.

We obtain the stated area A by integrating the sliverlike elements of area OPQ, where the arc PQ is considered to be so small that it is virtually straight. We think of OPQ as a triangle whose base is the segment PQ of length ds and whose height is the perpendicular distance OR from the vertex O to the base PQ extended. The two similar right triangles in the figure tell us that

$$\frac{ds}{dx}=\frac{OS}{OR} \quad \text{or} \quad OR\,ds = OS\,dx,$$

so the area dA of OPQ is

$$dA=\tfrac{1}{2}OR\,ds=\tfrac{1}{2}OS\,dx=\tfrac{1}{2}y\,dx,$$

*Wallis was Savilian Professor of Geometry at Oxford for 54 years, from 1649 until his death in 1703 at the age of 87, and played an important part in forming the climate of thought in which Newton flourished. He introduced negative and fractional exponents as well as the now-standard symbol ∞ for infinity, and was the first to treat conic sections as plane curves of the second degree. His infinite product stimulated his friend Lord Brouncker (first president of the Royal Society) to discover the astonishing formula

$$\frac{4}{\pi}=1+\cfrac{1^2}{2+\cfrac{3^2}{2+\cfrac{5^2}{2+\cfrac{7^2}{2+\cfrac{9^2}{2+\cdots}}}}},$$

from which the theory of continued fractions later arose. [No one knows how Brouncker made this discovery, but a proof based on the work of Euler in the next century is given in the chapter on Brouncker in J. L. Coolidge's *The Mathematics of Great Amateurs* (Oxford University Press, 1949).] Among the activities of Wallis's later years was a lively quarrel with the famous philosopher Hobbes, who was under the impression that he had succeeded in squaring the circle and published his erroneous proof. Wallis promptly refuted it, but Hobbes was both arrogant and too ignorant to understand the refutation, and defended himself with a barrage of additional errors, as if a question about the validity of a mathematical proof could be settled by rhetoric and invective.

where y denotes the length of the segment OS. The element of area dA sweeps across the circular segment in question as x increases from 0 to 1, so

$$A = \int dA = \frac{1}{2}\int_0^1 y\,dx;$$

and integrating by parts in order to reverse the roles of x and y gives

$$A = \frac{1}{2}\,xy\Big]_0^1 - \frac{1}{2}\int_0^1 x\,dy = \frac{1}{2} - \frac{1}{2}\int_0^1 x\,dy, \tag{1}$$

where the limits on the two integrals are understood to be $y = 0$ and $y = 1$. To continue the calculation, we observe that since

$$y = \tan \tfrac{1}{2}\phi \qquad \text{and} \qquad x = 1 - \cos\phi = 2\sin^2 \tfrac{1}{2}\phi,$$

the trigonometric identity

$$\tan^2 \tfrac{1}{2}\phi = \frac{\sin^2 \frac{1}{2}\phi}{\cos^2 \frac{1}{2}\phi} = \sin^2 \tfrac{1}{2}\phi \sec^2 \tfrac{1}{2}\phi = \sin^2 \tfrac{1}{2}\phi(1 + \tan^2 \tfrac{1}{2}\phi)$$

yields

$$\frac{x}{2} = \frac{y^2}{1 + y^2}.$$

The version of the geometric series given in formula (13) in Section 9.5 enables us to write this as

$$\frac{x}{2} = y^2(1 - y^2 + y^4 - y^6 + \cdots) = y^2 - y^4 + y^6 - y^8 + \cdots,$$

so (1) becomes

$$\begin{aligned} A &= \frac{1}{2} - \int_0^1 (y^2 - y^4 + y^6 - y^8 + \cdots)\,dy \\ &= \frac{1}{2} - \left[\frac{1}{3}y^3 - \frac{1}{5}y^5 + \frac{1}{7}y^7 - \frac{1}{9}y^9 + \cdots\right]_0^1 \\ &= \frac{1}{2} - \left(\frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \cdots\right) \\ &= \frac{1}{2} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots. \end{aligned}$$

When $\frac{1}{2}$ is added to this to account for the area of the isosceles right triangle, and the result is equated to the known area $\pi/4$ of the quarter-circle, we have Leibniz's formula

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$$

Is it any wonder that he took great pleasure and pride in this discovery for the rest of his life?

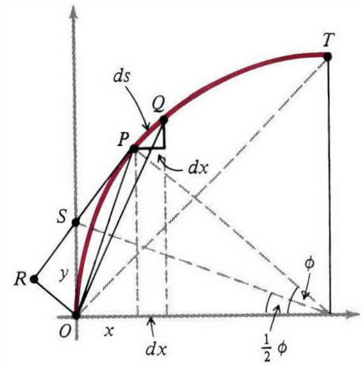


Figure 10.9

11 FURTHER APPLICATIONS OF INTEGRATION

11.1 THE CENTER OF MASS OF A DISCRETE SYSTEM

Most of the ideas in this chapter are based on the simple physical concept of center of gravity. As we shall see, this concept has implications for geometry, and it turns out to be possible to use it to arrive at a reasonable notion of what ought to be meant by the "center" of a general geometric figure. In this introductory section we confine ourselves to describing the concepts, and make no use of integration.

We begin by considering two children of weights w_1 and w_2 sitting at distances d_1 and d_2 from the fulcrum of a seesaw (Fig. 11.1). As we know, each child can increase the tendency of his or her weight to turn one end down by moving farther out from the fulcrum, and the seesaw balances when

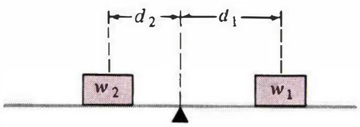


Figure 11.1

$$w_1 d_1 = w_2 d_2. \tag{1}$$

This principle was discovered by Archimedes, and is known as the *law of the lever.* If we establish a horizontal x-axis with its origin at the fulcrum and the positive direction to the right, then (1) can be written in the form

$$w_1 x_1 + w_2 x_2 = 0 \qquad \text{or} \qquad \sum_{k=1}^{2} w_k x_k = 0,$$

where $x_1 = d_1$ and $x_2 = -d_2$.

We now extend this discussion by considering the x-axis as a weightless horizontal rod that pivots at the point p, as shown in Fig. 11.2, and we assume that n weights w_k are placed at points x_k, where $k = 1, 2, \ldots, n$. By Archimedes' law, this system of weights will exactly balance, or be in equilibrium about p, if

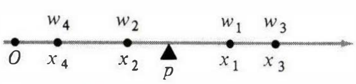


Figure 11.2

$$\sum w_k(x_k - p) = 0.$$

More generally, whether this system is in equilibrium or not, the sum $\Sigma w_k(x_k - p)$ measures the tendency of the system to turn in a clockwise direction about the pivot point p. This sum is called the *moment* of the system about p, and the system is in equilibrium if this moment is zero. If the weights w_k and their positions x_k are given in some arbitrary manner, and if we are free to move the pivot point p, then it is easy to find a point $p = \bar{x}$ at which the system will balance, that is, with the property that the moment of the system about $\bar{x}$ is zero. The required condition is

$$\sum w_k(x_k - \bar{x}) = 0.$$

This is equivalent to

$$\sum w_k x_k - \sum w_k \bar{x} = 0 \qquad \text{or} \qquad \bar{x} \sum w_k = \sum w_k x_k,$$

so

$$\bar{x} = \frac{\Sigma w_k x_k}{\Sigma w_k}. \tag{2}$$

This balancing point $\bar{x}$ is called the *center of gravity* of the given system of weights.

We now recall that the weight of a body at the surface of the earth is simply the force exerted on the body by the gravitational attraction of the earth, and is therefore given by Newton's formula $F = mg$, where m is the mass of the body and g is the acceleration due to gravity (approximately 32 feet per second per second or 9.80 meters per second per second). In the above discussion this means that $w_k = m_k g$, where m_k is the mass of the kth body. Formula (2) can therefore be written as

$$\bar{x} = \frac{\Sigma m_k g x_k}{\Sigma m_k g} = \frac{\Sigma m_k x_k}{\Sigma m_k}. \tag{3}$$

With the influence of gravity removed from the discussion in this way, and the weights w_k in (2) replaced by the masses m_k in (3), it is customary to call the point $\bar{x}$ the *center of mass* of the system.

It is easy to extend these ideas to a two-dimensional system of masses m_k located at points (x_k, y_k) in a horizontal xy-plane (Fig. 11.3). We define the *moment* of this system about the y-axis by

$$M_y = \sum m_k x_k, \tag{4}$$

which is the sum of each of the masses multiplied by its signed distance from the y-axis. If we think of the xy-plane as a weightless horizontal tray, as suggested by the figure, then in physical language the condition $M_y = 0$ means that this tray—with the given distribution of masses—will balance if it rests on a knife-edge along the y-axis. Similarly, the moment of the system about the x-axis is defined by

$$M_x = \sum m_k y_k. \tag{5}$$

Students should carefully observe the interchange of x's and y's in formulas (4) and (5); to compute M_y we use the x_k's, and to compute M_x we use the y_k's. If we denote the total mass of all the particles in the system by m, so that

$$m = \sum m_k,$$

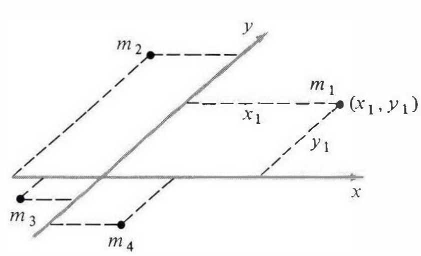


Figure 11.3

then the *center of mass* of the system is defined to be the point $(\bar{x}, \bar{y})$, where

$$\bar{x} = \frac{\Sigma m_k x_k}{\Sigma m_k} = \frac{M_y}{m} \tag{6}$$

and

$$\bar{y} = \frac{\Sigma m_k y_k}{\Sigma m_k} = \frac{M_x}{m}. \tag{7}$$

The center of mass of our system can be interpreted in two ways. First, if equations (6) and (7) are written in the form

$$m\bar{x} = M_y \qquad \text{and} \qquad m\bar{y} = M_x,$$

then we see that $(\bar{x}, \bar{y})$ is the point at which the entire mass m of the system can be concentrated without changing the total moment about either axis. The second interpretation depends on writing (6) and (7) in the form

$$\sum m_k(x_k - \bar{x}) = 0 \qquad \text{and} \qquad \sum m_k(y_k - \bar{y}) = 0.$$

If we think of our system in the way described, as a distribution of masses on a weightless horizontal tray, then these equations tell us that the tray will balance if it rests on a knife-edge along the line $x = \bar{x}$ parallel to the y-axis, and also along the line $y = \bar{y}$ parallel to the x-axis. These conditions imply that the tray will balance if it rests on a knife-edge along *any* line through $(\bar{x}, \bar{y})$. It will therefore also balance if supported by a sharp nail *precisely at* the point $(\bar{x}, \bar{y})$.

In the preceding discussion, the xy-coordinate system in Fig. 11.3 provides a frame of reference that is useful for developing the ideas. However, it is clear from the physical meaning of the center of mass that the location of this point is determined by the masses themselves and their individual positions, and does not depend on the particular coordinate system that is used to describe these positions. As a practical consequence, this fact tells us that in any specific situation we are free to choose any coordinate system that seems convenient under the circumstances.

Remark The "center of population" of the United States has been described as the point at which a life-sized flat map of the whole country would balance on a pin if all Americans weighed the same. The location of this point has been calculated from the data in each census. In 1790 it was a few miles east of Baltimore. It has been moving westward ever since; in 1980 it crossed the Mississippi River; and in 1990 it was about 25 mi southwest of St. Louis (see Fig. 11.4). It is interesting to speculate on what changes in the position of this point would be produced by "weighing" Americans according to age, or wealth, or education, instead of treating them as interchangeable units.

11.2 CENTROIDS

The ideas discussed in Section 11.1 apply to discrete systems of particles located at a finite number of points in a plane. We now consider how integration can be used to generalize these ideas to a continuous distribution of mass throughout a region R in the xy-plane, as shown in Fig. 11.5.

We shall think of R as a thin sheet of homogeneous material—say, a uniform metal plate—whose density δ (= mass per unit area) is constant. To define the

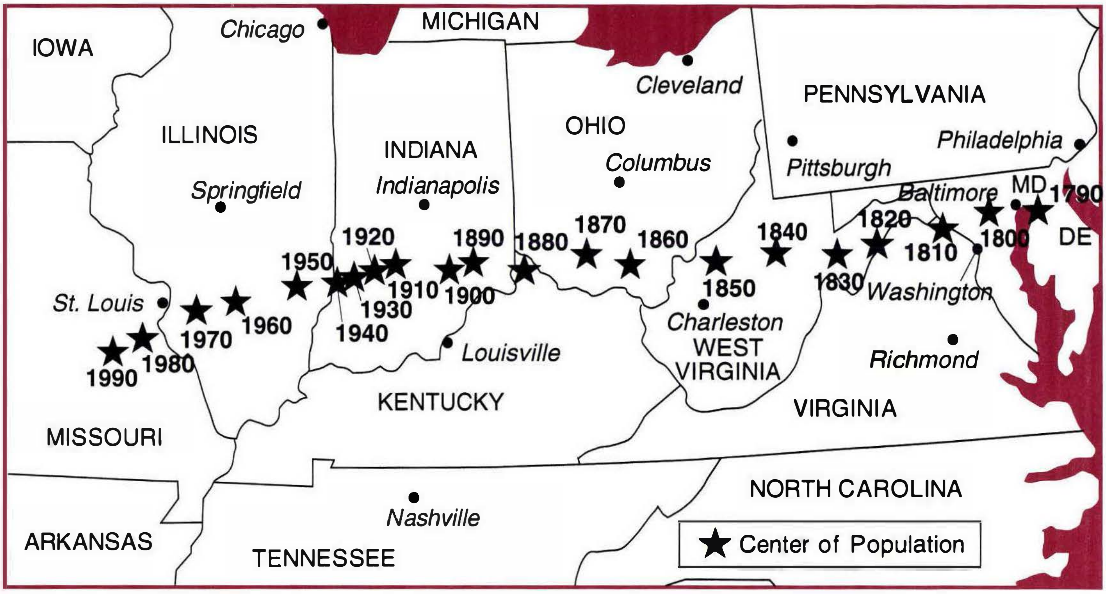


Figure 11.4 Center of population, 1790 to 1990. (*Courtesy U.S. Bureau of the Census.*)

moment of this plate about the y-axis, we consider a thin vertical strip of height $f(x)$ and width dx, whose position in the region is specified by the variable x (Fig. 11.5, left). The area of this strip is $f(x)\,dx$ and its mass is $\delta f(x)\,dx$; and since all of its mass is essentially at the same distance x from the y-axis, its moment about this axis is $x\delta f(x)\,dx$. The total moment of the plate about the y-axis is therefore obtained by allowing the strip to sweep across the region, and by integrating—or adding together—all these small contributions to the moment as x increases from a to b,

$$M_y = \int_a^b x\delta f(x)\,dx. \tag{1}$$

This formula can be derived by laboriously constructing approximating sums and then forming the limit of these sums, which leads to (1) by means of the definition of the integral. However, we prefer to continue in the spirit of Chapter 7,

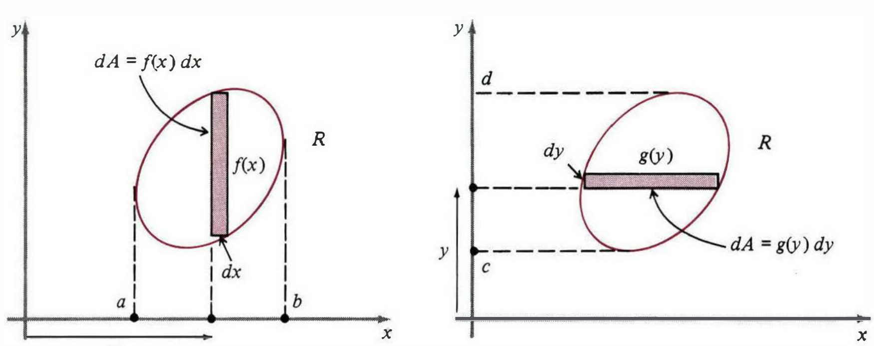


Figure 11.5

and the preceding discussion provides yet another illustration of the power of the Leibnizian approach to integration as we described it in Section 7.1.

Similarly, the moment of the plate about the x-axis is obtained by considering a thin horizontal strip of length $g(y)$ and width dy (Fig. 11.5, right), and is given by the formula

$$M_x = \int_c^d y\delta g(y)\,dy.$$

The total mass of the plate can evidently be expressed in two ways,

$$m = \int_a^b \delta f(x)\,dx = \int_c^d \delta g(y)\,dy.$$

The *center of mass* $(\bar{x}, \bar{y})$ of the plate is now defined by

$$\bar{x} = \frac{\int_a^b x\delta f(x)\,dx}{\int_a^b \delta f(x)\,dx} = \frac{M_y}{m}$$

and

$$\bar{y} = \frac{\int_c^d y\delta g(y)\,dy}{\int_c^d \delta g(y)\,dy} = \frac{M_x}{m}.$$

The reader should observe particularly that these formulas generalize (6) and (7) in the preceding section from the discrete case to the continuous case. Also, from the point of view of geometry they have the following remarkable feature. Since the density δ is assumed to be constant, it can be factored out of the integrals and removed by cancellation, and the formulas for $\bar{x}$ and $\bar{y}$ become

$$\bar{x} = \frac{\int_a^b xf(x)\,dx}{\int_a^b f(x)\,dx} \quad \text{and} \quad \bar{y} = \frac{\int_c^d yg(y)\,dy}{\int_c^d g(y)\,dy} \tag{2}$$

Each denominator here is clearly the total area of the region, and the numerators can be thought of as the moments of this area about the y-axis and x-axis, respectively. The center of mass is therefore determined solely by the geometric configuration of the region R and does not depend on the density of any mass that this region may contain, as long as this density is constant. For this reason the point $(\bar{x}, \bar{y})$ is called the *centroid* of the region, meaning "point like a center." The examples and problems that follow will make it clear that this terminology is well suited to the geometric concept it is meant to describe.

It will be convenient for our work in the next section if we simplify formulas (2) even further. In the case of $\bar{x}$, the area of the thin vertical strip is an element of area in the sense of Sections 7.1 and 7.2, so we write it as $dA = f(x)\,dx$; and in the case of $\bar{y}$, we similarly have $dA = g(y)\,dy$ for the area of the thin horizontal strip. Formulas (2) can therefore be written in the streamlined form

$$\bar{x} = \frac{\int x\,dA}{\int dA} \quad \text{and} \quad \bar{y} = \frac{\int y\,dA}{\int dA}. \tag{3}$$

We emphasize that each dA in these formulas is understood to be the area of a thin strip parallel to the appropriate axis, in order to guarantee that all points in the strip will be essentially at the same distance from this axis. It is also understood here that the process of integration expressed by these symbols is extended over the region under discussion. The limits of integration are omitted deliberately, and don't really need to be written down unless we are performing actual calculations in a specific case.

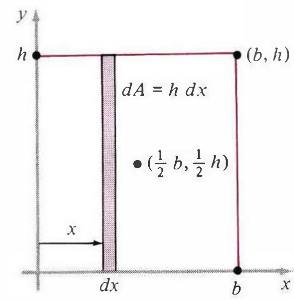


Figure 11.6

Example 1 Find the centroid of a rectangle.

Solution If the rectangle has height h and base b, then we can place the coordinate system so that the origin is at the lower left corner and the point (b, h) is at the upper right corner, as shown in Fig. 11.6. Since the area of this rectangle is hb, we have

$$\bar{x} = \frac{\int_0^b x \cdot h\,dx}{hb} = \frac{1}{hb}\left[\frac{1}{2}hx^2\right]_0^b = \frac{1}{hb}\left[\frac{1}{2}hb^2\right] = \frac{1}{2}b.$$

In just the same way we find that $\bar{y} = \frac{1}{2}h$, so the centroid is the point $(\frac{1}{2}b, \frac{1}{2}h)$, which is clearly the center of the rectangle.

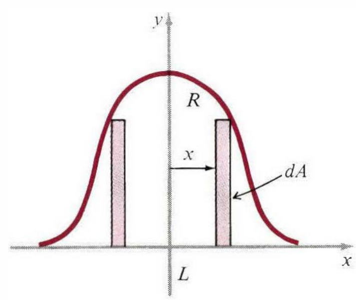


Figure 11.7

In general, it appears that the centroid of a region must lie on a line of symmetry of the region, if such a line exists. This is easily seen to be true, as follows. If L is a line of symmetry of a region R, then we can choose this line to be the y-axis (Fig. 11.7), and we wish to convince ourselves that $\bar{x} = 0$. If dA is an arbitrary thin vertical element of area at position x, then by symmetry there is a corresponding element of area at position $-x$; and since $x\,dA + (-x)\,dA = 0$, we have

$$\int x\,dA = 0, \qquad \text{and therefore} \qquad \bar{x} = \frac{\int x\,dA}{\int dA} = 0.$$

Further, if a region has two distinct lines of symmetry, then the conclusion we have just reached tells us that the centroid must lie on both lines and is therefore the point of intersection of these lines. Accordingly, in every case where a geometric figure has a "center" in the usual sense of the word, this center *is* the centroid. However, as our next example shows, centroids are easily calculated for many regions that are not ordinarily considered to have centers at all. From this point of view, the centroid of a region is a far-reaching generalization of the concept of the center of a geometric figure.

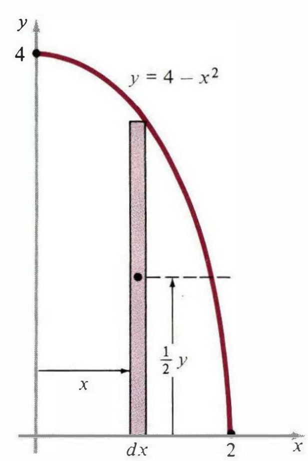


Figure 11.8

Example 2 Find the centroid of the region in the first quadrant bounded by the axes and the curve $y = 4 - x^2$ (Fig. 11.8).

Solution By using the vertical strip in the figure, we see that the area of the region is

$$A = \int dA = \int_0^2 (4 - x^2)\,dx = \left[4x - \frac{1}{3}x^3\right]_0^2 = \frac{16}{3},$$

so

$$\bar{x} = \frac{\int x\,dA}{A} = \frac{3}{16}\int_0^2 x(4-x^2)\,dx = \frac{3}{16}\left[2x^2 - \frac{1}{4}x^4\right]_0^2 = \frac{3}{4}.$$

Similarly, using a horizontal strip not shown in the figure, we have

$$\bar{y} = \frac{\int y\,dA}{A} = \frac{3}{16}\int_0^4 y\sqrt{4-y}\,dy.$$

To evaluate this integral we make the substitution $u = 4 - y$, so that $y = 4 - u$ and $dy = -du$, and we also change the limits of integration from $y = 0, 4$ to $u = 4, 0$:

$$\begin{aligned}\bar{y} &= \frac{3}{16}\int_0^4 y\sqrt{4-y}\,dy = \frac{3}{16}\int_4^0 u^{1/2}(4-u)(-du)\\ &= \frac{3}{16}\int_0^4 (4u^{1/2} - u^{3/2})\,du = \frac{3}{16}\left[\frac{8}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right]_0^4\\ &= \frac{3}{16}\left(\frac{64}{3} - \frac{64}{5}\right) = \frac{8}{5}.\end{aligned}$$

The integration here is a bit complicated because it uses a horizontal strip, and this forces us to solve the equation of the curve for x in terms of y. We therefore describe an alternative method for computing $\bar{y}$ that uses the vertical strip shown in the figure and the result of Example 1. Since the centroid of this rectangular strip is located at its center, the moment of the strip about the x-axis is $\frac{1}{2}y\,dA = \frac{1}{2}y^2\,dx$, and therefore

$$\begin{aligned}\bar{y} &= \frac{\int \frac{1}{2}y^2\,dx}{A} = \frac{3}{32}\int_0^2 (4-x^2)^2\,dx = \frac{3}{32}\int_0^2 (16 - 8x^2 + x^4)\,dx\\ &= \frac{3}{32}\left[16x - \frac{8}{3}x^3 + \frac{1}{5}x^5\right]_0^2 = \frac{3}{32}\left[32 - \frac{64}{3} + \frac{32}{5}\right] = \frac{8}{5},\end{aligned}$$

as before.

One more word about centroids. We have discussed centroids of plane regions. We can just as easily speak of the centroid of an arc in the xy-plane or of a region in three-dimensional space. The definitions and formulas are so similar to what we have already done that we won't burden students with detailed explanations. However, we do remark that in finding the centroid of an arc (Fig. 11.9) it may be helpful to think of the arc as a piece of curved wire of constant density 1 (= mass per unit length), so that the mass of a portion of the wire is simply its length. With ds understood to be the element of arc length in the sense of Section 7.5, we therefore have

$$\bar{x} = \frac{\int x\,ds}{\int ds} \quad \text{and} \quad \bar{y} = \frac{\int y\,ds}{\int ds}. \tag{4}$$

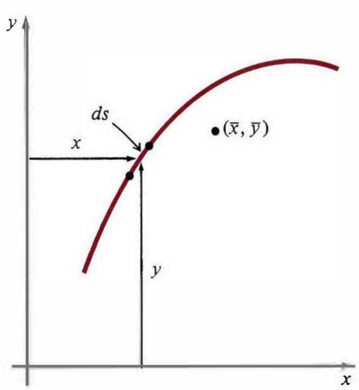


Figure 11.9

Each denominator here is the total length of the arc, and the numerators are the moments of the arc about the y-axis and x-axis, respectively.

PROBLEMS

Find the centroid of the plane region R that is bounded by:

1 $y = x^2$, $y = 0$, $x = 2$.
2 $y = 4x - x^2$ and $y = x$.
3 $y = \sqrt{a^2 - x^2}$ and $y = 0$.
4 $y = \sin x$ and $y = 0$ $(0 \le x \le \pi)$.
5 $x^2 = ay$ and $y = a$.
6 $x^2 = ay$ and $y^2 = ax$.
7 $y = \sqrt[3]{x}$, $y = 0$, $x = 8$.
8 $x^2 + y^2 = a^2$ and $x + y = a$ (first quadrant).
9 $x^2 + y^2 = a^2$, $x = a$, $y = a$.
10 $y = 1/x$, $y = 0$, $x = 1$, and $x = 2$.
11 Find the centroid of the first-quadrant part of the curve $x^{2/3} + y^{2/3} = a^{2/3}$.
12 Find the centroid of the semicircular arc $y = \sqrt{a^2 - x^2}$.
13 The semicircle under $y = \sqrt{b^2 - x^2}$ is removed from the semicircle under $y = \sqrt{a^2 - x^2}$, where $b < a$. Find the centroid of the remaining region. Find the limit of $\bar{y}$ as $b \to a$, and compare with the result of Problem 12.
14 Let $y = f(x)$ be a nonnegative function defined on the interval $a \le x \le b$. If the region bounded by this curve, the x-axis, and the lines $x = a$, $x = b$ is revolved about the x-axis, show that the resulting solid of revolution has its centroid on the x-axis with

$$\bar{x} = \frac{\int_a^b x f(x)^2\,dx}{\int_a^b f(x)^2\,dx}.$$

15 Use the result of Problem 14 to find the centroid of (a) a cone with height h and radius of base r; and (b) a hemisphere of radius a.
16 It is known from elementary geometry that the three medians of a triangle intersect at a point that is two-thirds of the way from each vertex to the midpoint of the opposite side.* Show that this point is the centroid of the triangle. Hint: Place the axes so that the vertices are (0, 0), $(a, 0)$, and (b, c). It suffices to find the centroid $(\bar{x}, \bar{y})$ and show that this point lies on the median from (b, c). Why is this so? Why does physical intuition tell us to expect this result?

*See Problem 35 in Section 1.2.

11.3 THE THEOREMS OF PAPPUS

Two beautiful geometric theorems connecting centroids with solids and surfaces of revolution were discovered in the fourth century A.D. by Pappus of Alexandria, the last of the great Greek mathematicians.

First Theorem of Pappus Consider a plane region that lies completely on one side of a line in its plane. If this region is revolved about the line as an axis, then the volume of the solid generated in this way equals the product of the area of the region and the distance traveled around the axis by its centroid.

This is easily proved by the following argument. Let the axis of revolution be the x-axis, as shown in Fig. 11.10. Then the distance $\bar{y}$ of the centroid from this axis is defined by

$$\bar{y} = \frac{\int y\,dA}{\int dA} = \frac{\int y\,dA}{A},$$

which is equivalent to

$$A\bar{y} = \int y\,dA$$

or

$$A \cdot 2\pi\bar{y} = \int 2\pi y\,dA.$$

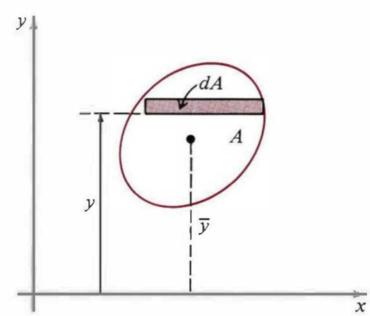


Figure 11.10

All that is needed now is to observe that this equation is precisely the assertion of the theorem, because $2\pi\bar{y}$ is the distance traveled by the centroid and the integral on the right is the volume of the solid as calculated by the shell method.

Example 1 Find the volume of the torus (doughnut) generated by revolving a circle of radius a about a line in its plane at a distance b from its center, where $b > a$ (Fig. 11.11).

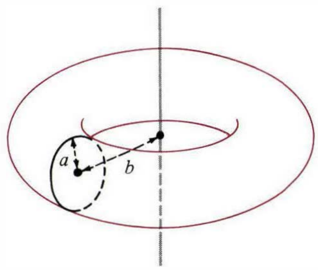


Figure 11.11

Solution The centroid of the circle is its center, and this travels a distance $2\pi b$ around the axis. The area of the circle is πa^2, so by the first theorem of Pappus the volume of the torus is

$$V = \pi a^2 \cdot 2\pi b = 2\pi^2 a^2 b.$$

(See Problem 31 in Section 10.4.)

Second Theorem of Pappus Consider an arc of a plane curve that lies completely on one side of a line in its plane. If this arc is revolved about the line as an axis, then the area of the surface generated in this way equals the product of the length of the arc and the distance traveled around the axis by its centroid.

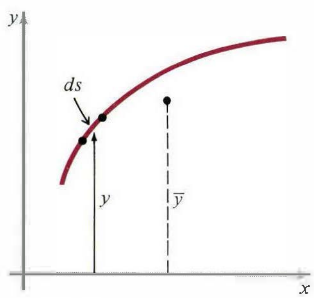


Figure 11.12

The proof is similar to that given above. Again we take the axis to be the x-axis (Fig. 11.12), and we start with the definition of the distance $\bar{y}$ from this axis to the centroid of the arc,

$$\bar{y} = \frac{\int y\, ds}{\int ds} = \frac{\int y\, ds}{s},$$

which is equivalent to

$$s\bar{y} = \int y\, ds$$

or

$$s \cdot 2\pi\bar{y} = \int 2\pi y\, ds.$$

And again this is exactly the assertion of the theorem, because the integral on the right is the area of the surface of revolution.

Example 2 With the aid of this theorem it is easy to see that the surface area of the torus described in Example 1 is

$$A = 2\pi a \cdot 2\pi b = 4\pi^2 ab.$$

Apart from their aesthetic appeal, the theorems of Pappus are useful in two ways. When centroids are known from symmetry considerations—as in the examples—we can use these theorems to find volumes and areas. And also, when volumes and areas are known, we can often use these theorems in reverse to determine the locations of centroids. Both types of applications are illustrated in the following problems.

PROBLEMS

1 Use the known formulas $V = \frac{4}{3}\pi a^3$ and $A = 4\pi a^2$ for the volume and surface area of a sphere of radius a to locate the centroid of (a) the semicircular region under $y = \sqrt{a^2 - x^2}$; (b) the arc $y = \sqrt{a^2 - x^2}$. Compare with Problems 3 and 12 in Section 11.2.

2 Use the centroids found in Problem 1 to find the volume and surface area generated when the semicircular region and the arc are revolved about the line $y = a$.

3 By Problem 10 in Section 7.5, the total length of the curve $x^{2/3} + y^{2/3} = a^{2/3}$ is $6a$. Use this fact and the result of Problem 11 in Section 11.2 to find the area of the surface generated by revolving this curve about (a) the x-axis; (b) the line $x + y = a$.

4 A square with side a is revolved about an axis lying in its plane which intersects it at one of its vertices but at no other points. What should be the position of the axis to yield the largest volume for the resulting solid of revolution? What is this largest volume? What is the corresponding surface area?

5 A regular hexagon with side a is revolved about one of its sides. What is the volume of the resulting solid of revolution? What is the area of the surface of this solid?

6 The regular hexagon in Problem 5 is revolved about an axis through a vertex which is perpendicular to the line from the center to that vertex. Find the volume and surface area of the resulting solid of revolution.

7 Use Pappus' first theorem to find (a) the volume of a cylinder with height h and radius of base r; (b) the volume of a cone with height h and radius of base r.

8 It is known from elementary geometry that πrL is the lateral area of a cone of base radius r and slant height L. Obtain this formula as a consequence of Pappus' second theorem.

11.4 MOMENT OF INERTIA

Consider a rigid body rotating about a fixed axis. For example, the body might be a solid sphere spinning about a diameter, or a solid cube swinging back and forth like a pendulum about a horizontal axis along one of its edges. In order to study motions of this kind, it is necessary to introduce the concept of the *moment of inertia* of the body about the axis. Our purpose in the next few paragraphs is not only to define this concept, but also to explain its intuitive meaning so that students can understand why it matters.

When a rigid body moves in a straight line, all its constituent particles move in the same direction with the same velocity. On the other hand, when a rigid body rotates about an axis, its constituent particles move around circles of different sizes and have different velocities, and for this reason we expect the problem of describing the body's motion to be more difficult. Fortunately, however, this situation is simpler than it seems, and it turns out to be possible to study rotating bodies by using ideas and formulas that are completely analogous to those already familiar for the case of linear motion.

We begin with a brief review of the linear formulas. Consider a particle of mass m moving in a straight line (Fig. 11.13). If its position is given by the variable s, then $v = ds/dt$ and $a = dv/dt$ are its velocity and acceleration. A force F acting on the particle is related to the acceleration by Newton's second law of motion.

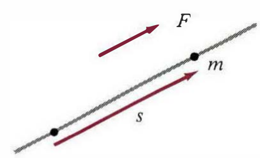


Figure 11.13

$$F = ma \qquad \text{or} \qquad a = \frac{1}{m}F. \tag{1}$$

The second form of this equation is useful for its clear expression of the idea that the acceleration of the particle is caused by the force and is proportional to this force. This form also helps us think of the mass m of the particle as a measure of its capacity to resist acceleration, because if the force F is the same and m increases, then a decreases.

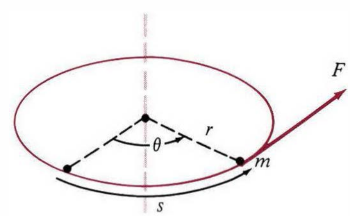


Figure 11.14

Now consider a particle of mass m rotating around a fixed axis in a circle of radius r (Fig. 11.14). If its angular position is given by the angle θ as measured from some fixed direction, then $\omega = d\theta/dt$ and $\alpha = d\omega/dt$ are its *angular velocity* and *angular acceleration*. These rotational quantities are related to the corresponding linear quantities s, v, and a, as measured along the circular path, by means of the equations $s = r\theta$, $v = r\omega$, and $a = r\alpha$. The twisting effect of the tangential force F is measured by its *torque* $T = Fr$, which is the product of the force and the distance from its line of action to the axis. We have seen that force produces linear acceleration in accordance with equation (1). In just the same way, torque produces angular acceleration in accordance with the corresponding equation

$$T = I\alpha, \tag{2}$$

where the constant of proportionality I is called the *moment of inertia*. I can be thought of as a measure of the capacity of the system to resist angular acceleration, and in this sense it is the rotational analog of mass.

These remarks describe the conceptual role of the moment of inertia. To discover what its definition must be in order to fit it for this role, we transform (2) by replacing T by Fr and α by a/r, and then we replace F by ma:

$$Fr = I\frac{a}{r}, \qquad mar = I\frac{a}{r}.$$

The last equation tells us that I must be defined by the formula

$$I = mr^2. \tag{3}$$

In this section we are mainly concerned with learning how to use integration to calculate the moment of inertia, about a given axis, of a uniform thin sheet of material of constant density δ (= mass per unit area). It may be helpful to think of such a sheet as a thin plate of homogeneous metal. Our method is to imagine the plate divided into a large number of small pieces in such a way that each piece can be treated as a particle to which formula (3) can be applied. We then find the total moment of inertia by integrating—or adding together—the individual moments of inertia of all these pieces.

Example 1 A uniform thin rectangular plate has sides a and b and density δ. Find its moment of inertia about an axis that bisects the two sides of length a (Fig. 11.15).

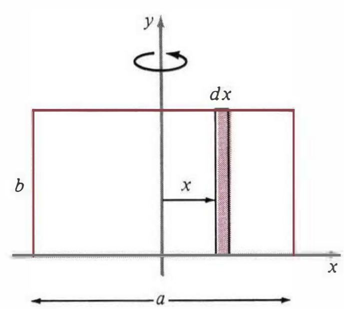


Figure 11.15

Solution Introduce coordinate axes as indicated in the figure, with the y-axis as the axis of rotation. We concentrate our attention on the thin vertical strip shown in the figure because all of its points are essentially at the same distance x from the axis of rotation. The moment of inertia of the strip about the axis is $x^2 \cdot \delta b\,dx$, so the total moment of inertia of the plate is

$$I = \int_{-a/2}^{a/2} x^2 \cdot \delta b\,dx = \delta b\left[\frac{1}{3}x^3\right]_{-a/2}^{a/2}$$

$$= \delta b\left[\frac{1}{24}a^3 - \left(-\frac{1}{24}a^3\right)\right] = \frac{1}{12}\delta a^3 b. \tag{4}$$

It is customary to write the moment of inertia in a form that displays the total mass M. In this case $M = \delta ab$, so

$$I = \tfrac{1}{12}Ma^2.$$

Two thin vertical strips that are symmetrically placed with respect to the axis of rotation have the same moment of inertia. In equation (4) we could therefore have written the integral in the form

$$I = 2\int_0^{a/2} x^2 \cdot \delta b \, dx = \cdots,$$

which makes possible a slightly simpler calculation.

Example 2 A uniform thin circular plate has radius a and mass M. Find its moment of inertia about a diameter.

Solution Introduce coordinate axes as shown in Fig. 11.16. If the density of the plate is denoted by δ, then the moment of inertia of the indicated strip about the y-axis is $x^2 \cdot \delta 2y \, dx = x^2 \cdot \delta 2\sqrt{a^2 - x^2} \, dx$, so the total moment of inertia is

$$I = 2\int_0^a x^2 \cdot \delta 2\sqrt{a^2 - x^2} \, dx = 4\delta \int_0^a x^2\sqrt{a^2 - x^2} \, dx.$$

To evaluate this integral we make the trigonometric substitution $x = a \sin \theta$, so that $dx = a \cos \theta \, d\theta$ and

$$I = 4\delta a^4 \int_0^{\pi/2} \sin^2 \theta \cos^2 \theta \, d\theta = \tfrac{1}{4}\delta\pi a^4 = \tfrac{1}{4}Ma^2.$$

(As always, students should verify the omitted details of this calculation for themselves.)

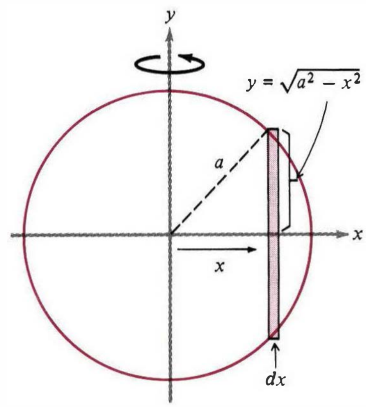


Figure 11.16

Example 3 Find the moment of inertia of the circular plate in Example 2 about an axis through the center and perpendicular to the plate.

Solution This time the axis is to be imagined as protruding out of the page from the center of the circle (Fig. 11.17), and we divide the area into thin rings with centers at the center of the circle, as shown. The total moment of inertia is therefore

$$I = \int_0^a r^2 \cdot \delta 2\pi r \, dr = 2\pi\delta\left[\frac{1}{4}r^4\right]_0^a$$
$$= \tfrac{1}{2}\delta\pi a^4 = \tfrac{1}{2}Ma^2.$$

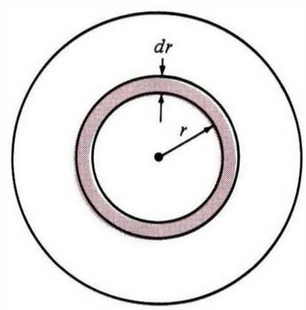


Figure 11.17

Remark 1 We recall that a particle of mass m moving with velocity v has kinetic energy given by the formula

$$\text{K.E.} = \tfrac{1}{2}mv^2,$$

and also that this energy is the amount of work that must be done on the particle to bring it to a stop. On the other hand, if the particle rotates in a circle of radius r, then $v = r\omega$ and we have

$$\text{K.E.} = \tfrac{1}{2}mr^2\omega^2 = \tfrac{1}{2}I\omega^2,$$

and again this is the work required to stop the rotating particle. By comparing these formulas we reinforce the idea that the moment of inertia I plays the same role in rotational motion as is played by the mass m in linear motion.

Remark 2 In addition to its importance in connection with the physics of rotating bodies, the moment of inertia also has significant applications in structural engineering, where it is found that the stiffness of a beam is proportional to the moment of inertia of a cross section of the beam about a horizontal axis through its centroid. This fact is exploited in the design of the familiar steel girders called "I-beams," where flanges at the top and bottom of the beam—as in the letter I—increase the moment of inertia and hence the stiffness of the beam.

PROBLEMS

1 A uniform thin rectangular plate of mass M has sides a and b. Find its moment of inertia about one of the sides of length b.

2 A uniform thin plate of mass M is bounded by the curve $y = \cos x$ and the x-axis between $x = -\pi/2$ and $x = \pi/2$. Find its moment of inertia about the y-axis.

3 Find the moment of inertia of a uniform thin triangular plate of mass M, height h, and base b about its base.

4 Find the moment of inertia of the triangular plate in Problem 3 about an axis parallel to its base and passing through the opposite vertex.

5 A uniform thin circular plate has radius a and mass M. Find its moment of inertia about an axis tangent to the plate.

6 Find the moment of inertia of a uniform straight wire of mass M and length a about an axis perpendicular to the wire at one end.

7 A uniform wire of mass M is bent into a circle of radius a. Find its moment of inertia about a diameter.

8 Find the moment of inertia of a uniform solid cylinder of mass M, height h, and radius a about its axis. Hint: Use the shell method.

9 Find the moment of inertia of a uniform solid cone of mass M, height h, and radius of base a about its axis.

10 Find the moment of inertia of a uniform solid sphere of mass M and radius a about a diameter.

11 If the moment of inertia of a body of mass M about a given axis is $I = Mr^2$, then the number r is called the *radius of gyration* of the body about that axis. This is the distance from the axis at which all the mass of the body could be concentrated at a single point without changing its moment of inertia. Referring to Problems 8–10, find the radius of gyration about the indicated axis of (a) the cylinder; (b) the cone; (c) the sphere.

CHAPTER 11 REVIEW: DEFINITIONS, CONCEPTS

Think through the following.

1 Moment about an axis.
2 Center of mass.
3 Centroid.
4 Pappus' theorems.
5 Moment of inertia.

ADDITIONAL PROBLEMS FOR CHAPTER 11

SECTION 11.1

1 Consider the plane distribution of particles whose center of mass $(\bar{x}, \bar{y})$ is defined by equations (6) and (7) in Section 11.1. If $Ax + By + C = 0$ is any line in the plane, then we may suppose (introducing a factor if necessary) that $A^2 + B^2 = 1$; and by Additional Problem 21 in Chapter 1 we see that the signed distance from this line to (x_k, y_k) is

$$d_k = Ax_k + By_k + C,$$

this being positive on one side of the line and negative on the other.

(a) Show that the entire mass $m = \Sigma m_k$ of the system can be concentrated at the center of mass $(\bar{x}, \bar{y})$ without changing the total moment $\Sigma m_k d_k$ about the arbitrary line.

(b) Use part (a) to show that the total moment is zero about every line through $(\bar{x}, \bar{y})$.

2 Consider again the plane distribution of particles discussed in Problem 1.

(a) If the axes are translated as shown in Fig. 11.18, then the old coordinates and the new coordinates of a fixed point P are connected by the transformation equations

$$x = X + a, \qquad y = Y + b.$$

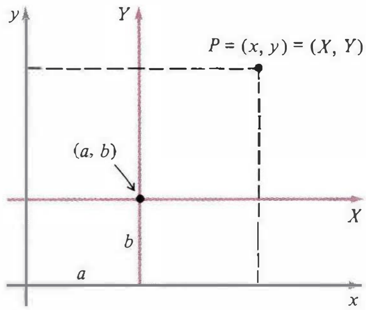


Figure 11.18

Calculate the center of mass in the new coordinate system, and show that it is the same point as before.

(b) If the axes are rotated through an angle θ as shown in Fig. 11.19, then the old coordinates and the new

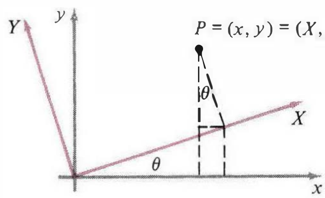


Figure 11.19

coordinates of a fixed point P are connected by the transformation equations

$$x = X\cos\theta - Y\sin\theta, \qquad y = X\sin\theta + Y\cos\theta.$$

Show that the center of mass as calculated in the new coordinate system is the same point as before.

(c) Deduce that the location of the center of mass is independent of the coordinate system that is used to calculate it.

SECTION 11.2

3 Find the centroid of the plane region R that is bounded by
(a) $y = x^2$ and $y = x$;
(b) $y = 2x^2$ and $y = x^2 + 1$;
(c) $y = 2x - x^2$ and $y = 0$;
(d) $y = x - x^4$ and $y = 0$;
(e) $y^3 = x^2$ and $y = 2$;
(f) $y = x^3$ and $y = 4x$ $(x \geq 0)$;
(g) $y = e^x$, $y = -e^x$, $x = 0$, $x = 1$.

*4 Find the centroid of the part of the curve $y = x^2$ that lies between $x = 0$ and $x = b$.

SECTION 11.3

5 Consider a rectangle with height $2a$ and base $2b$ placed in the xy-plane with its sides parallel to the axes and its center at the point $(0, c)$, where $c \geq \sqrt{a^2 + b^2}$. If this rectangle is rotated counterclockwise through an angle θ about the point $(0, c)$ and then revolved about the x-axis, show that the volume and surface area of the resulting solid of revolution are the same for all values of θ. What are they?

6 A regular hexagon inscribed in the circle $x^2 + y^2 = 1$ has one of its vertices at the point $(1, 0)$. If this hexagon is revolved about the line $3x + 4y = 25$, find the volume and surface area of the resulting solid of revolution.

SECTION 11.4

7 Show that the moment of inertia of a uniform thin plate in the xy-plane about an axis perpendicular to this plane at the origin is equal to the sum of its moments of inertia about the two coordinate axes. Use this fact to find the moment of inertia of a uniform thin square plate of mass M and side a about an axis through its center and perpendicular to its plane.

8 A uniform thin plate of mass M has the curve

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

as its boundary. Use the method of Problem 7 to find its moment of inertia about an axis through the origin and perpendicular to its plane.

9 Consider a uniform thin plate of mass M in the xy-plane. Let I be its moment of inertia about a line L in this plane, and let I_0 be its moment of inertia about a parallel line L_0 through the centroid. Show that

$$I = I_0 + Md^2,$$

where d is the distance between L and L_0 (this is called the *parallel axis theorem*). Hint: Place the coordinate system so that L_0 is the y-axis and L is the line $x = d$.

*10 Consider a uniform solid body of mass M in three-dimensional space. Let I be its moment of inertia about a line L, and let I_0 be its moment of inertia about a parallel line L_0 through the centroid. Then the *parallel axis theorem* stated in Problem 9 holds in exactly the same form:

$$I = I_0 + Md^2,$$

where d is the distance between L and L_0. Establish this fact, and apply it to find the moment of inertia of (a) a uniform solid sphere of mass M and radius a about a tangent; (b) a uniform solid cube of mass M and edge a about an edge. Hint: See Problem 7.

12 INDETERMINATE FORMS AND IMPROPER INTEGRALS

12.1 INTRODUCTION. THE MEAN VALUE THEOREM REVISITED

In the next few chapters we will need better methods for computing limits than any we have available now. Accordingly, our main purposes in the first part of this chapter are to understand the types of limit problems that lie ahead and to acquire the tools that will enable us to solve these problems with maximum efficiency.

In Section 2.5 we saw that the limit of a quotient is the quotient of the limits, in the following sense: If

$$\lim_{x \to a} f(x) = L \qquad \text{and} \qquad \lim_{x \to a} g(x) = M,$$

then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M}, \tag{1}$$

provided that $M \neq 0$. Unfortunately, however, it is a fact of life that many of the most important limits are of the form (1) in which both $L = 0$ and $M = 0$. When this happens, formula (1) is useless for calculating the value of the limit and this limit is said to have the *indeterminate form* 0/0 at $x = a$. The expression "indeterminate form" is used because in this case the limit on the left of (1) may very well exist, but nothing can be concluded about its value without further investigation. This is shown by the four examples

$$\frac{x}{x}, \quad \frac{x^2}{x}, \quad \frac{x}{x^3}, \quad \frac{x \sin 1/x}{x},$$

each of which is a quotient of two functions that both approach zero as $x \to 0$. We see from these examples—by canceling x's from the numerators and denominators—that such a quotient may have the limit 1, or 0, or ∞, or it may have no limit at all, finite or infinite.

Indeterminate forms can sometimes be evaluated by using simple algebraic devices. For example,

$$\lim_{x \to 2} \frac{3x^2 - 7x + 2}{x^2 + 5x - 14} \tag{2}$$

has the indeterminate form 0/0, and this limit is easy to calculate by factoring and canceling,

$$\lim_{x\to 2}\frac{3x^2-7x+2}{x^2+5x-14}=\lim_{x\to 2}\frac{(x-2)(3x-1)}{(x-2)(x+7)}=\lim_{x\to 2}\frac{3x-1}{x+7}=\frac{5}{9}.$$

In other cases, more complicated methods are required. Thus, the limit

$$\lim_{x\to 0}\frac{\sin x}{x} \tag{3}$$

is another indeterminate form of the type 0/0, and in Secions 2.5 and 9.2 geometric arguments were used to show that the value of this important limit is 1. In this connection we point out the suggestive fact that the limit (3) can also be evaluated by noticing that it is the derivative of the function $\sin x$ at $x = 0$:

$$\begin{aligned}\lim_{x\to 0}\frac{\sin x}{x}&=\lim_{x\to 0}\frac{\sin x-\sin 0}{x-0}\\&=\frac{d}{dx}\sin x\Big]_{x=0}=\cos x]_{x=0}=\cos 0=1.\end{aligned}$$

Indeed, every derivative

$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a} \tag{4}$$

is an indeterminate form of the type 0/0, since both numerator and denominator of the fraction on the right approach zero as x approaches a.*

These remarks suggest that there is a close but hidden connection between indeterminate forms and derivatives. And so there is. But to understand this connection, it is first necessary to recall the Mean Value Theorem.

As we learned in Section 2.6, this theorem states that if a function $y = f(x)$ is defined and continuous on a closed interval $a \le x \le b$, and differentiable at each point of the interior $a < x < b$, then there exists at least one number c between a and b for which

$$f'(c)=\frac{f(b)-f(a)}{b-a},$$

or equivalently,

$$f(b)-f(a)=f'(c)(b-a).$$

This assertion is best understood in geometric language (see Fig. 12.1); it says that at some point on the graph between the endpoints, the tangent line is parallel to the chord joining these endpoints. From this point of view the theorem seems obviously true and is difficult to doubt; but in fact, as we saw in Section 2.6, it is a rather deep theorem whose validity depends in a crucial way on the stated hypotheses.

In most of our work we try to avoid dwelling on the theoretical parts of calculus. Here, however, we must make an exception, because the central fact of this chapter (L'Hospital's rule, in the next section†) cannot be understood unless we know what the Mean Value Theorem says.

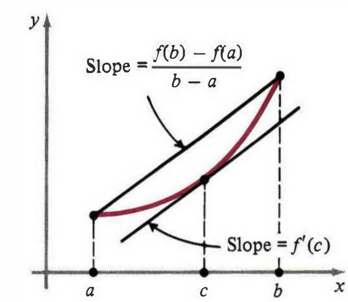


Figure 12.1

*Students should examine formula (4) together with a suitable sketch, in order to convince themselves that this formula can be taken as the definition of the derivative of an arbitrary function $f(x)$ at a point $x = a$. We have not had occasion to use this version of the definition before, but it will be particularly convenient for our work in the present chapter.

†L'Hospital is pronounced "LOW-pee-tal."

12.2 THE INDETERMINATE FORM 0/0. L'HOSPITAL'S RULE

We remarked earlier that there is a close connection between indeterminate forms and derivatives. We begin to explore this connection with the following simple theorem: *If $f(x)$ and $g(x)$ are both equal to zero at $x = a$ and have derivatives there, then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)} = \left.\frac{f'(x)}{g'(x)}\right]_{x=a}, \tag{1}$$

provided that $g'(a) \neq 0$. To prove this, it suffices to use $f(a) = 0$ and $g(a) = 0$ to write

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{[f(x) - f(a)]/(x - a)}{[g(x) - g(a)]/(x - a)} \to \frac{f'(a)}{g'(a)},$$

as stated.

As examples of the use of (1), we easily find the limits (2) and (3) in Section 12.1,

$$\lim_{x\to 2} \frac{3x^2 - 7x + 2}{x^2 + 5x - 14} = \left.\frac{6x - 7}{2x + 5}\right]_{x=2} = \frac{5}{9} \tag{2}$$

and

$$\lim_{x\to 0} \frac{\sin x}{x} = \left.\frac{\cos x}{1}\right]_{x=0} = \cos 0 = 1. \tag{3}$$

As another example, we have

$$\lim_{x\to 0} \frac{\tan 6x}{e^{2x} - 1} = \left.\frac{6 \sec^2 6x}{2e^{2x}}\right]_{x=0} = \frac{6}{2} = 3, \tag{4}$$

a result that would have been hard to find in any other way.

Formula (1) requires the existence of the derivatives of the functions $f(x)$ and $g(x)$ at the single point $x = a$. At other points these functions need not have derivatives, nor indeed even be continuous. However, if the derivatives exist in an interval about a and are continuous at a, then we can obtain formula (1) in another way, by applying the Mean Value Theorem separately to the numerator and denominator,

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(c_1)(x - a)}{g'(c_2)(x - a)} = \frac{f'(c_1)}{g'(c_2)} \to \frac{f'(a)}{g'(a)}, \tag{5}$$

as $x \to a$. Here c_1 and c_2 lie between x and a, so both approach a as $x \to a$.

What purpose is served by giving a second alternative proof of formula (1) when the first proof is perfectly satisfactory? The point is this: Formula (1) is a good tool to have, but is still only of limited value, because it often happens in the problems we consider that $f'(a) = g'(a) = 0$, and in this case the right side of (1) is meaningless. However, we can use our second proof to get around this difficulty as follows. Suppose that c_1 and c_2 in (5) can be taken equal to one another so that the first part of (5) can be written as

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(c)(x - a)}{g'(c)(x - a)} = \frac{f'(c)}{g'(c)}, \tag{6}$$

where c is between x and a. Then in forming the limit as $x \to a$, (6) permits us to replace the quotient

$$\frac{f(x)}{g(x)} \quad \text{by the quotient} \quad \frac{f'(x)}{g'(x)}.$$

L'Hospital's rule states that under certain easily satisfied conditions this replacement is legitimate, that is,

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}, \tag{7}$$

provided the limit on the right exists. Students should remember that $f(a) = g(a) = 0$ is assumed here, and we also mention that even though ordinary two-sided limits are usually intended in (7), one-sided limits are allowed.

It may be helpful to students if we now give a formal statement of our main result.

Theorem (L'Hospital's Rule) Let a be a real number and let $f(x)$ and $g(x)$ be functions that are differentiable on some open interval containing a. Assume also that $g'(x) \neq 0$ on this interval, except perhaps at the point a itself. If $f(a) = 0$ and $g(a) = 0$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}, \tag{7}$$

provided the limit on the right exists.[*]

L'Hospital's rule is named after the French mathematician—a pupil of John Bernoulli—who published it in his book *Analyse des infiniment petits* (182 pp., Paris, 1696), which was the first calculus textbook and enjoyed wide popularity and influence.

Example 1 At the beginning of this section we evaluated the limits (2), (3), and (4) by using formula (1). These limits can also be found by using L'Hospital's rule (7):

$$\lim_{x\to 2}\frac{3x^2 - 7x + 2}{x^2 + 5x - 14} = \lim_{x\to 2}\frac{6x - 7}{2x + 5} = \frac{5}{9},$$

$$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{\cos x}{1} = 1,$$

$$\lim_{x\to 0}\frac{\tan 6x}{e^{2x} - 1} = \lim_{x\to 0}\frac{6\sec^2 6x}{2e^{2x}} = 3.$$

The reason (7) works so smoothly in these problems is that in each case the second limit exists and is easy to find by inspection, since the functions involved

[*]For those students who are interested in the proof of L'Hospital's rule, we here explain as briefly as possible the details of the reasoning that underlies (7). We assume—as stated—that $f(a) = g(a) = 0$, that x approaches a from one side or the other, and that on that side the functions $f(x)$ and $g(x)$ satisfy the following three conditions: (i) Both are continuous on some closed interval I having a as an endpoint; (ii) both are differentiable in the interior of I; and (iii) $g'(x) \neq 0$ in the interior of I. With these hypotheses, (6) is an immediate consequence of a technical extension of the Mean Value Theorem known as the Generalized Mean Value Theorem; and if x is now allowed to approach a from the side under consideration, then (7) follows from (6) as indicated above. Those tenacious students who like to nail everything down will find a proof of the Generalized Mean Value Theorem in Appendix A.4.

are continuous. The point we wish to make here is that whatever (1) can do, (7) can do just as easily; and as the next example shows, (7) is much more powerful and often works easily when (1) doesn't work at all.

Example 2 L'Hospital's rule (7) proves its value in limit problems like

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2}.$$

Here formula (1) is useless, as we see from the failure of the attempted calculation

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{\sin x}{2x}\bigg]_{x=0} = \frac{0}{0}.$$

The reason for this failure is that (1) assumes that $g'(a) \neq 0$, and this condition is not satisfied here. However, by (7) we have

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \lim_{x \to 0} \frac{\sin x}{2x},$$

if the second limit exists. But this second limit is again of the form 0/0, so L'Hospital's rule applies a second time and permits us to continue and reach the correct answer,

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \lim_{x \to 0} \frac{\sin x}{2x} = \lim_{x \to 0} \frac{\cos x}{2} = \frac{1}{2}.$$

Another limit that behaves in this way is

$$\begin{aligned} \lim_{x \to 0} \frac{\sqrt{x+1} - (1 + \frac{1}{2}x)}{x^2} &= \lim_{x \to 0} \frac{\frac{1}{2}(x+1)^{-1/2} - \frac{1}{2}}{2x} \\ &= \lim_{x \to 0} \frac{-\frac{1}{4}(x+1)^{-3/2}}{2} = -\frac{1}{8}. \end{aligned}$$

The limits in Example 2 illustrate the great advantage L'Hospital's rule (7) has over formula (1): It is valid whenever the limit on the right exists, regardless of whether $g'(a)$ is zero or not. Thus, as these problems show, if $f'(a) = g'(a) = 0$, then we have another indeterminate form 0/0 and we can apply L'Hospital's rule a second time,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} = \lim_{x \to a} \frac{f''(x)}{g''(x)},$$

provided the last-written limit exists. As a practical matter, the functions we encounter in this book satisfy the conditions needed for L'Hospital's rule. We therefore apply the rule almost routinely, by continuing to differentiate the numerator and denominator separately as long as we still get the form 0/0 at $x = a$. As soon as one or the other (or both) of these derivatives is different from zero at $x = a$, we must stop differentiating and hope to evaluate the last limit by some direct method.

Example 3 A careless attempt to apply L'Hospital's rule may yield an incorrect result, as in the calculation

$$\lim_{x\to 0}\frac{\sin 4x}{2x+3}=\lim_{x\to 0}\frac{4\cos 4x}{2}=\frac{4}{2}=2.$$

The correct answer is

$$\lim_{x\to 0}\frac{\sin 4x}{2x+3}=\frac{0}{3}=0.$$

In this problem the numerator and denominator of the given quotient are not both equal to zero at $x=0$, so L'Hospital's rule is not applicable.

Our methods work in just the same way for limits in which $x\to\infty$; that is, if $f(x)\to 0$ and $g(x)\to 0$ as $x\to\infty$, then

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{f'(x)}{g'(x)},\tag{8}$$

if the limit on the right exists. To see this, we put $x=1/t$ and observe that $t\to 0+$ (recall that this notation means that t approaches zero from the right). Briefly, L'Hospital's rule (7) now gives

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{t\to 0+}\frac{f(1/t)}{g(1/t)}=\lim_{t\to 0+}\frac{f'(x)\,dx/dt}{g'(x)\,dx/dt},$$

which yields (8) after dx/dt is canceled.

Finally, in both forms of L'Hospital's rule, as expressed in formulas (7) and (8), it is easy to see that the procedure remains valid if the value of the limit on the right is ∞ or $-\infty$.

PROBLEMS

Find the following limits.

1 $\displaystyle\lim_{x\to 0}\frac{\sin 3x}{\sin x}$.

2 $\displaystyle\lim_{x\to 1}\frac{\ln x}{x-1}$.

3 $\displaystyle\lim_{x\to 2}\frac{x-2}{6x^2-10x-4}$.

4 $\displaystyle\lim_{x\to 0}\frac{e^x-e^{-x}}{\sin 5x}$.

5 $\displaystyle\lim_{x\to 0}\frac{\sqrt{x+9}-3}{x}$.

6 $\displaystyle\lim_{x\to 1}\frac{4x^3-5x+1}{\ln x}$.

7 $\displaystyle\lim_{x\to 0}\frac{\sqrt[3]{x+1}-(1+\frac{1}{3}x)}{x^2}$.

8 $\displaystyle\lim_{x\to 0}\frac{\sin^{-1}3x}{x}$.

9 $\displaystyle\lim_{x\to 0}\frac{e^x-1-x}{x^2}$.

10 $\displaystyle\lim_{x\to 0}\frac{e^x-1-x}{1-\cos\pi x}$.

11 $\displaystyle\lim_{x\to 0}\frac{x^3}{\sin x-x}$.

12 $\displaystyle\lim_{x\to 0}\frac{e^{2x}-1}{\sin 5x}$.

13 $\displaystyle\lim_{x\to 0}\frac{3x}{\tan x}$.

14 $\displaystyle\lim_{x\to\pi/4}\frac{\ln(\tan x)}{\sin x-\cos x}$.

15 $\displaystyle\lim_{x\to 0}\frac{\sin^2 x+8x}{e^{2x}-1}$.

16 $\displaystyle\lim_{x\to 6}\frac{\sqrt{x-2}-2}{x^2-36}$.

17 $\displaystyle\lim_{x\to 0}\frac{x-\sin x}{x-\tan x}$.

18 $\displaystyle\lim_{x\to\pi}\frac{\ln(\cos 2x)}{(x-\pi)^2}$.

19 $\displaystyle\lim_{x\to\infty}\frac{1/x}{\sin\pi/x}$.

20 $\displaystyle\lim_{x\to 0}\frac{a^x-b^x}{x}$.

21 $\displaystyle\lim_{x\to 0}\frac{\tan 2x-2x}{x-\sin x}$.

22 $\displaystyle\lim_{x\to 0}\frac{\sin x^3}{\sin^3 x}$.

23 $\displaystyle\lim_{x\to 0}\frac{\sin 2x-2\sin x}{\sin 3x-3\sin x}$.

24 $\displaystyle\lim_{x\to 1}\frac{\sqrt[4]{x}-1}{\sqrt[5]{x}-1}$.

25 $\displaystyle\lim_{x\to 0}\frac{(e^x-1)^3}{(x-2)e^x+x+2}$.

26 In Fig. 12.2, P is a point on a circle with center O and radius a. The segment AQ equals the arc AP, and the line PQ intersects the line OA at B. Show that OB approaches $2a$ as P approaches A along the circle. Hint: ΔQAB is similar to ΔPRB.

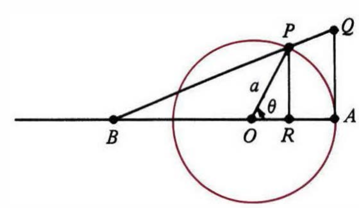


Figure 12.2

27 In Problem 26, let $f(\theta)$ be the area of the triangle ARP and let $g(\theta)$ be the area of the region that remains after the triangle ORP is removed from the sector OAP. Find formulas for the functions $f(\theta)$ and $g(\theta)$ and evaluate $\lim_{\theta\to 0} f(\theta)/g(\theta)$.

28 L'Hospital's rule (7) works in just the same way if the conditions $f(a) = g(a) = 0$ are replaced by the conditions $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$. Explain. Use this idea to evaluate

$$\lim_{x\to 0} \frac{(1+x)^{1/x} - e}{x}.$$

29 The formula

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

is one version of the definition of the derivative. By treating the right side as an indeterminate form, derive this formula from L'Hospital's rule.

12.3 OTHER INDETERMINATE FORMS

For certain applications it is important to know that L'Hospital's rule remains valid for indeterminate forms of the type ∞/∞. That is, if the numerator and denominator of the quotient $f(x)/g(x)$ both become infinite as $x \to a$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}, \tag{1}$$

provided the limit on the right exists. The argument is a bit tricky, and is sketched in Remark 2 so that those who wish to skip it can conveniently do so. Just as in Section 12.2, one-sided limits are allowed and (1) extends immediately to the case in which $x \to \infty$; also, it remains valid if the limit on the right is ∞ or $-\infty$.

Example 1 Show that

$$\lim_{x\to\infty} \frac{x^p}{e^x} = 0 \tag{2}$$

for every constant p.

Solution We begin by observing that if $p \le 0$, then this limit is not an indeterminate form and its value is easily seen to be zero. On the other hand, when $p > 0$, the limit is clearly an indeterminate form of the type ∞/∞. L'Hospital's rule (1) for the case in which $x \to \infty$ therefore gives

$$\lim_{x\to\infty} \frac{x^p}{e^x} = \lim_{x\to\infty} \frac{px^{p-1}}{e^x},$$

if the limit on the right exists; and if this process is continued step by step, we can reduce the exponent for x to zero or a negative number, and the desired conclusion (2) now follows from the above remark about this case. This example gives us important insight into the nature of the exponential function: as $x \to \infty$, e^x increases faster than any positive power of x, however large, and therefore faster than any polynomial.

Example 2 Show that

$$\lim_{x\to\infty} \frac{\ln x}{x^p} = 0 \tag{3}$$

for every constant $p > 0$.

Solution This limit is clearly an indeterminate form of the type ∞/∞, so by L'Hospital's rule we have

$$\lim_{x\to\infty} \frac{\ln x}{x^p} = \lim_{x\to\infty} \frac{1/x}{px^{p-1}} = \lim_{x\to\infty} \frac{1}{px^p} = 0.$$

Expressed in words, (3) tells us that as $x \to \infty$, $\ln x$ increases more slowly than any positive power of x, however small.

We have discussed the limits (2) and (3) before, by clumsier special methods, in Sections 8.3 and 8.4. But our present treatment of these important facts is clearly preferable, because the powerful method of analysis based on L'Hospital's rule extends easily to many similar limits.

The expressions

$$0 \cdot \infty, \qquad \infty - \infty, \qquad 0^0, \qquad \infty^0, \qquad 1^\infty$$

symbolize other types of indeterminate forms that sometimes arise. The product $f(x)g(x)$ where one factor approaches zero and the other becomes infinite $(0 \cdot \infty)$ can be reduced to 0/0 or ∞/∞ by putting the reciprocal of one factor in the denominator. The difference between two functions which are both becoming infinite $(\infty - \infty)$ can often be manipulated into a more convenient form. A power $y = f(x)^{g(x)}$ that produces an indeterminate form of one of the other types is best handled by taking logarithms,

$$\ln y = \ln f(x)^{g(x)} = g(x) \ln f(x). \tag{4}$$

This reduces the problem to the more familiar form $0 \cdot \infty$; and since $y = e^{\ln y}$, we then use the continuity of the exponential function to infer that $\lim y = \lim e^{\ln y} = e^{\lim \ln y}$. These generalities are illustrated in the following examples.

Example 3 Evaluate

$$\lim_{x\to 0+} x \ln x. \tag{5}$$

Solution Here x is required to approach zero from the positive side because $\ln x$ is defined only for positive x's. Since $\ln x \to -\infty$ as $x \to 0+$, it is clear that (5) is an indeterminate form of the type $0 \cdot \infty$. The value of this limit is not obvious, because as $x \to 0+$, we cannot tell whether the product $x \ln x$ is influenced more by the smallness of x or by the largeness (in absolute value) of $\ln x$. However, we can easily convert the limit into an indeterminate form of the type ∞/∞ and apply L'Hospital's rule (1), as follows:

$$\lim_{x\to 0+} x \ln x = \lim_{x\to 0+} \frac{\ln x}{1/x} = \lim_{x\to 0+} \frac{1/x}{-1/x^2} = \lim_{x\to 0+} (-x) = 0.$$

Thus, the smallness of x turns out to dominate the behavior of the product $x \ln x$ near $x = 0$.

Example 4 Evaluate

$$\lim_{x\to \pi/2} (\sec x - \tan x). \tag{6}$$

Solution This is of the type $\infty - \infty$. We convert it into an indeterminate form of the type 0/0 and apply L'Hospital's rule,

$$\lim_{x\to\pi/2} (\sec x - \tan x) = \lim_{x\to\pi/2} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right)$$

$$= \lim_{x\to\pi/2} \frac{1 - \sin x}{\cos x} = \lim_{x\to\pi/2} \frac{-\cos x}{-\sin x} = 0.$$

Example 5 Find $\lim_{x\to 0+} x^x$.

Solution This limit is of the type 0^0, and we reduce it to the simpler type $0 \cdot \infty$ by taking the logarithm. To do this most conveniently, we write $y = x^x$ and observe that

$$\ln y = \ln x^x = x \ln x \to 0 \qquad \text{as} \qquad x \to 0+,$$

by Example 3. This tells us that

$$x^x = y = e^{\ln y} \to e^0 = 1,$$

by the continuity of the exponential function. Therefore we have

$$\lim_{x\to 0+} x^x = 1. \tag{7}$$

Example 6 Find $\lim_{x\to\infty} x^{1/x}$.

Solution This limit is of the type ∞^0. We write $y = x^{1/x}$ and observe that

$$\ln y = \ln x^{1/x} = \frac{\ln x}{x} \to 0 \qquad \text{as} \qquad x \to \infty,$$

by Example 2. This tells us that

$$x^{1/x} = y = e^{\ln y} \to e^0 = 1,$$

or equivalently,

$$\lim_{x\to\infty} x^{1/x} = 1. \tag{8}$$

Example 7 Show that

$$\lim_{x\to 0} (1 + ax)^{1/x} = e^a \tag{9}$$

for every constant a.

Solution If $a = 0$ this limit is not an indeterminate form, and the statement is clearly true because each side has the value 1. If $a \neq 0$, the limit is an indeterminate form of the type 1^∞. In this case we write $y = (1 + ax)^{1/x}$ and observe that

$$\lim_{x\to 0} \ln y = \lim_{x\to 0} \frac{\ln (1 + ax)}{x} = \lim_{x\to 0} \frac{a/(1 + ax)}{1} = a.$$

This implies that

$$(1 + ax)^{1/x} = y = e^{\ln y} \to e^a,$$

which is (9).

Remark 1 *The L'Hospital habit.* Like any mathematical procedure, L'Hospital's rule should be used intelligently, and not purely mechanically. We should try to control the bad habit of automatically applying L'Hospital's rule to every limit problem that comes up. Often there is an easier way, for instance, the use of familiar limits or simple algebraic transformations.

(a) The limit

$$\lim_{x\to\infty} \frac{6x^5 - 2}{2x^5 + 3x^2 + 4}$$

is of the type ∞/∞, and can be found by repeated use of L'Hospital's rule. But it is much simpler to divide both numerator and denominator by x^5 and write

$$\frac{6x^5 - 2}{2x^5 + 3x^2 + 4} = \frac{6 - 2/x^5}{2 + 3/x^3 + 4/x^5} \to \frac{6 - 0}{2 + 0 + 0} = 3.$$

(b) The limit

$$\lim_{x\to 0} \frac{\sin^3 x}{x^3}$$

is of the type 0/0. L'Hospital's rule can be applied, and works, but it is much easier to notice that

$$\frac{\sin^3 x}{x^3} = \left(\frac{\sin x}{x}\right)^3 \to 1^3 = 1,$$

because we already know that $(\sin x)/x \to 1$ as $x \to 0$.

(c) The limit

$$\lim_{x\to\infty} \frac{\sqrt{x^2 + 1}}{x}$$

is of the type ∞/∞, and L'Hospital's rule gives

$$\lim_{x\to\infty} \frac{\sqrt{x^2 + 1}}{x} = \lim_{x\to\infty} \frac{x/\sqrt{x^2 + 1}}{1} = \lim_{x\to\infty} \frac{x}{\sqrt{x^2 + 1}} = \frac{1}{\lim\limits_{x\to\infty} (\sqrt{x^2 + 1}/x)}.$$

This brings us back to the limit we started with, and gets us nowhere. However, it is very easy to insert the denominator into the radical and write

$$\frac{\sqrt{x^2 + 1}}{x} = \sqrt{\frac{x^2 + 1}{x^2}} = \sqrt{1 + \frac{1}{x^2}} \to \sqrt{1 + 0} = 1.$$

Remark 2 The argument for L'Hospital's rule (1) in the case ∞/∞ can be briefly sketched as follows. Let $f(x)$ and $g(x)$ both become infinite as $x \to a$ from one side or the other, and suppose that $f'(x)/g'(x) \to L$. We want to show that also $f(x)/g(x) \to L$. For $\bar{x}$ near enough to a on the side under consideration (see Fig. 12.3), $f'(x)/g'(x)$ can be made as close as we please to L between $\bar{x}$ and a. If x is between $\bar{x}$ and a, and if $f(x)$ and $g(x)$ are assumed to satisfy the simple conditions (i) to (iii) stated in the footnote in Section 12.2, then

Figure 12.3

$$\frac{f(x) - f(\bar{x})}{g(x) - g(\bar{x})} = \frac{f'(c)}{g'(c)}$$

for some c between x and $\bar{x}$. Since c is also between $\bar{x}$ and a, we know that

$f'(c)/g'(c)$ is close to L. Now hold $\bar{x}$ fixed and let $x \to a$. Then $f(x)$ and $g(x)$ grow very large, $f(\bar{x})/f(x)$ and $g(\bar{x})/g(x)$ become very small, and

$$\frac{f(x) - f(\bar{x})}{g(x) - g(\bar{x})} = \frac{f(x)}{g(x)}\left[\frac{1 - f(\bar{x})/f(x)}{1 - g(\bar{x})/g(x)}\right]$$

is close to $f(x)/g(x)$. It follows that $f(x)/g(x)$ is close to $f'(c)/g'(c)$, which in turn is close to L, so $f(x)/g(x)$ is itself close to L when x is close to a, and this is what we wanted to establish.

PROBLEMS

Evaluate the following limits by any method.

1 $\lim\limits_{x\to\infty} \dfrac{18x^3}{3 + 2x^2 - 6x^3}$.

2 $\lim\limits_{x\to\infty} \dfrac{\ln(\ln x)}{\ln x}$.

3 $\lim\limits_{x\to\pi/2} \dfrac{\tan x}{1 + \sec x}$.

4 $\lim\limits_{x\to\infty} \dfrac{\ln x^2}{\sqrt{x}}$.

5 $\lim\limits_{x\to\pi/2} \dfrac{\tan x}{\tan 3x}$.

6 $\lim\limits_{x\to\infty} \dfrac{x^2}{e^{3x}}$.

7 $\lim\limits_{x\to 0+} \dfrac{\ln x}{\csc x}$.

8 $\lim\limits_{x\to\infty} \dfrac{(\ln x)^{10}}{x}$.

9 $\lim\limits_{x\to 0+} \dfrac{\ln(\sin^2 x)}{\ln x}$.

10 $\lim\limits_{x\to 0} x \cot x$.

11 $\lim\limits_{x\to\infty} x \sin \dfrac{1}{x}$.

12 $\lim\limits_{x\to\pi/2} (\pi - 2x)\tan x$.

13 $\lim\limits_{x\to\infty} (x^2 - 1)e^{-x^2}$.

14 $\lim\limits_{x\to\infty} x^3 e^{-x}$.

15 $\lim\limits_{x\to\infty} e^{-x}\ln x$.

16 $\lim\limits_{x\to\infty} x\left(\dfrac{\pi}{2} - \tan^{-1} x\right)$.

17 $\lim\limits_{x\to 0+} \sin x \ln x$.

18 $\lim\limits_{x\to 0} x^2 \csc(5\sin^2 x)$.

19 $\lim\limits_{x\to\infty} \left(\dfrac{x^2}{x-1} - \dfrac{x^2}{x+1}\right)$.

20 $\lim\limits_{x\to 0} \left(\dfrac{1}{x} - \dfrac{1}{\sin x}\right)$.

21 $\lim\limits_{x\to 0} \left(\dfrac{1}{x} - \dfrac{1}{e^x - 1}\right)$.

22 $\lim\limits_{x\to 1} \left(\dfrac{1}{x-1} - \dfrac{1}{\ln x}\right)$.

23 $\lim\limits_{x\to 0+} (\sin x)^x$.

24 $\lim\limits_{x\to 0+} (\tan x)^{\sin x}$.

25 $\lim\limits_{x\to 0+} x^{\tan x}$.

26 $\lim\limits_{x\to 0+} x^{x^2}$.

27 $\lim\limits_{x\to 0+} (e^x - 1)^x$.

28 $\lim\limits_{x\to 0+} x^{\ln(1+x)}$.

29 $\lim\limits_{x\to 0+} (\sin x)^{\sin x}$.

30 $\lim\limits_{x\to 0} (1 - \cos x)^{1-\cos x}$.

31 $\lim\limits_{x\to\infty} (\ln x)^{1/x}$.

32 $\lim\limits_{x\to\pi/2-} (\tan x)^{\cos x}$.

33 $\lim\limits_{x\to\infty} (1 + e^{ax})^{1/x}$, $a > 0$.

34 $\lim\limits_{x\to\infty} (x + e^x)^{2/x}$.

35 $\lim\limits_{x\to\infty} (1 + ax)^{1/x}$, $a > 0$.

36 $\lim\limits_{x\to\infty} (1 + x^{100})^{1/x}$.

37 $\lim\limits_{x\to 0} (\cos x)^{1/x}$.

38 $\lim\limits_{x\to\pi/2} (\sin x)^{\tan x}$.

39 $\lim\limits_{x\to 1} x^{1/(1-x^2)}$.

40 $\lim\limits_{x\to 0+} (\cos\sqrt{x})^{1/x}$.

41 In spite of the evidence piled up in Problems 23–30, indeterminate forms of the type 0^0 do not always have the value 1. Show this by calculating

$$\lim_{x\to 0+} x^{p/\ln x},$$

where p is a nonzero constant.

42 (a) Sketch the graph of the function $y = f(x)$ defined by

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

(b) Show that $\lim\limits_{x\to 0} x^{-n}e^{-1/x^2} = 0$ for every positive integer n.

(c) Show that $f(x)$ as defined in (a) has an nth derivative $f^{(n)}(x)$ for every positive integer n and every $x \neq 0$. [We do not ask for a general formula for $f^{(n)}(x)$, but students should carry the calculations far enough to show that $f^{(n)}(x)$ is always given by a formula of a certain form, involving certain constant coefficients.]

(d) Use parts (b) and (c) to show that $f^{(n)}(0) = 0$ for every positive integer n.

***43** As $x \to 0+$, show that

$$\cot x - \frac{1}{x} \to 0 \qquad \text{and} \qquad \cot x + \frac{1}{x} \to \infty,$$

but that

$$\left(\cot x - \frac{1}{x}\right)\left(\cot x + \frac{1}{x}\right) \to -\frac{2}{3}.$$

44 Use (4) in the text to explain why 1^{0}, 0^1, and 0^∞ are not indeterminate forms.

12.4 IMPROPER INTEGRALS

When we write down an ordinary definite integral as defined in Chapter 6,

$$\int_a^b f(x)\,dx, \tag{1}$$

we assume that the limits of integration are finite numbers and that the integrand $f(x)$ is continuous on the bounded interval $a \le x \le b$. If $f(x) \ge 0$, we are thoroughly familiar with the idea that the integral (1) represents the area of the region shown on the left in Fig. 12.4.

In the next section and in Chapter 13 it will be necessary to consider so-called *improper integrals* of the form

$$\int_a^\infty f(x)\,dx, \tag{2}$$

in which the upper limit is infinite and the integrand $f(x)$ is assumed to be continuous on the unbounded interval $a \le x < \infty$.* We define the integral (2) in the natural way suggested on the right in Fig. 12.4; that is, we integrate from a to a finite but variable upper limit t, and then we allow t to approach ∞ and define (2) by

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx.$$

If this limit exists and has a finite value, then the improper integral is said to *converge* or to be *convergent*, and this value is assigned to it. Otherwise, the integral is called *divergent*. If $f(x) \ge 0$, then (2) can be thought of as the area of the unbounded region on the right in Fig. 12.4. In this case the area of the region is finite or infinite according as the improper integral (2) converges or diverges.

Example 1

$$\int_0^\infty e^{-x}\,dx = \lim_{t\to\infty}\int_0^t e^{-x}\,dx = \lim_{t\to\infty}\,[-e^{-x}]_0^t = \lim_{t\to\infty}\left(-\frac{1}{e^t}+1\right) = 1.$$

This improper integral converges, because the limit exists and is finite.

*The word "improper" is used because of the "impropriety" at the upper limit of integration. If we wish, we can speak of (1) as a *proper integral* because it has no improprieties, but this is neither necessary nor customary.

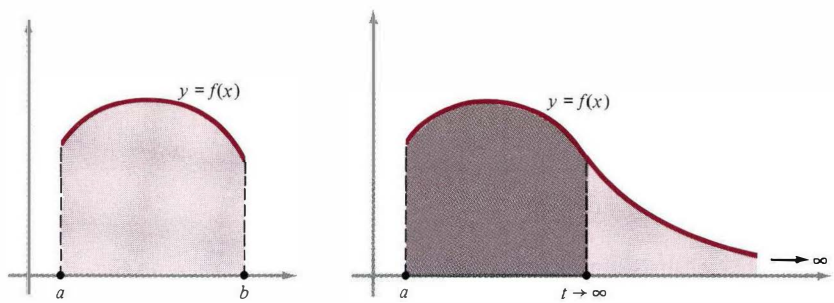


Figure 12.4

Students often tend to abbreviate this calculation by writing

$$\int_0^\infty e^{-x}\,dx = [-e^{-x}]_0^\infty = -\frac{1}{e^\infty} + 1 = 1,$$

instead of writing out the limits as we have done in Example 1. This shorthand rarely causes any real difficulties. However, in our work in this section we will always write out the limits for the sake of emphasizing that improper integrals are *defined* as limits.

Example 2

$$\int_1^\infty \frac{dx}{x^2} = \lim_{t\to\infty}\int_1^t \frac{dx}{x^2} = \lim_{t\to\infty}\left[-\frac{1}{x}\right]_1^t = \lim_{t\to\infty}\left(-\frac{1}{t}+1\right) = 1.$$

This improper integral also converges.

Example 3

$$\int_1^\infty \frac{dx}{x} = \lim_{t\to\infty}\int_1^t \frac{dx}{x} = \lim_{t\to\infty}\,[\ln x]_1^t = \lim_{t\to\infty}\ln t = \infty.$$

This integral diverges, because the limit is infinite.

Example 4

$$\int_0^\infty \cos x\,dx = \lim_{t\to\infty}\int_0^t \cos x\,dx = \lim_{t\to\infty}\sin t,$$

which does not exist. This integral diverges, because the limit does not exist.

Our next example generalizes Examples 2 and 3 and contains specific information that will be needed in Chapter 13.

Example 5 If p is a positive constant, show that the improper integral

$$\int_1^\infty \frac{dx}{x^p}$$

converges if $p > 1$ and diverges if $p \le 1$.

Solution The case $p = 1$ is settled in Example 3, so we assume that $p \neq 1$. In this case we have

$$\int_1^\infty \frac{dx}{x^p} = \lim_{t\to\infty}\int_1^t \frac{dx}{x^p} = \lim_{t\to\infty}\left[\frac{x^{1-p}}{1-p}\right]_1^t$$

$$= \lim_{t\to\infty}\left[\frac{t^{1-p}-1}{1-p}\right] = \begin{cases} \dfrac{1}{p-1} & \text{if } p > 1 \\ \infty & \text{if } p < 1, \end{cases}$$

and this completes the proof.

We consider the geometric meaning of Example 5 by examining Fig. 12.5. The fundamental facts are these. The integral

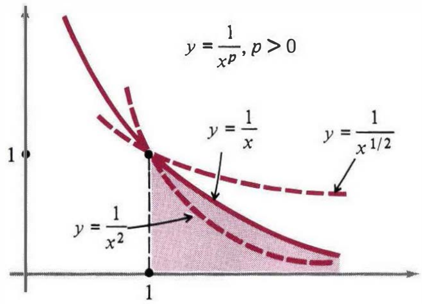


Figure 12.5

$$\int_1^\infty \frac{1}{x^2}\,dx$$

converges because the curve approaches the x-axis "rapidly enough" as $x \to \infty$. And integrals such as

$$\int_1^\infty \frac{1}{x}\,dx \qquad \text{and} \qquad \int_1^\infty \frac{1}{x^{1/2}}\,dx$$

diverge because the curves *do not* approach the x-axis "rapidly enough" as $x \to \infty$. When the exponent p is allowed to decrease through values greater than 1 (for instance, $p = 4, 3, 2, 1.5$, etc.), then it is easy to see that the corresponding graph of $y = 1/x^p$ to the right of $x = 1$ rises; also, the calculation shows that the area of the unbounded region under this graph increases but remains finite. When p reaches 1 this area suddenly becomes infinite, and it remains infinite for all values of $p < 1$. It is indeed remarkable that a region of infinite extent can have a finite area, as happens here when $p > 1$. We will comment further on this phenomenon in Remark 1 below.

Another type of improper integral arises when the integrand $f(x)$ is continuous on a bounded interval of the form $a \leq x < b$ but becomes infinite as x approaches b, as shown in Fig. 12.6. In this case we can integrate from a to a variable upper limit t which is less than b. This integral is a function of t, and we can now ask whether this function approaches a limit as $t \to b$. If so, we use this limit as the definition of the improper integral of $f(x)$ from a to b,

$$\int_a^b f(x)\,dx = \lim_{t \to b} \int_a^t f(x)\,dx.$$

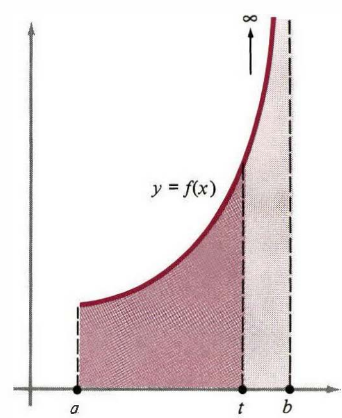


Figure 12.6

As before, this integral is called *convergent* if the limit exists and is finite, and *divergent* otherwise.

Example 6

$$\int_0^1 \frac{dx}{\sqrt{1-x}} = \lim_{t \to 1} \int_0^t \frac{dx}{\sqrt{1-x}} = \lim_{t \to 1} [-2\sqrt{1-x}]_0^t$$

$$= \lim_{t \to 1} [-2\sqrt{1-t} + 2] = 2.$$

This improper integral clearly converges.

There are several other types of improper integrals which we mention only briefly because the ideas are essentially the same as those already described.

If the impropriety of an integral occurs at the lower limit, we use t as the lower limit and then let $t \to -\infty$ or $t \to a$, as the case may be. If the integrand misbehaves at several points, then the improper integral—if it exists—is obtained by dividing the original interval into subintervals.

Finally, if $f(x)$ is continuous on the entire real line, then we write, *by definition,*

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx,$$

where convergence for the improper integral on the left means that both integrals on the right converge. An integral from $-\infty$ to ∞ can be split at any convenient finite point just as well as at the point $x = 0$.

Example 7

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{dx}{1+x^2} &= \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2} \\ &= \lim_{t\to-\infty} \int_{t}^{0} \frac{dx}{1+x^2} + \lim_{t\to\infty} \int_{0}^{t} \frac{dx}{1+x^2} \\ &= \lim_{t\to-\infty} [\tan^{-1} x]_t^0 + \lim_{t\to\infty} [\tan^{-1} x]_0^t \\ &= \lim_{t\to-\infty} (-\tan^{-1} t) + \lim_{t\to\infty} \tan^{-1} t = -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2} = \pi.\end{aligned}$$

Remark 1 Students may still be skeptical that a region of infinite extent can have a finite area. If so, then the following example may help. Consider the region under the curve $y = 1/2^x$ for $0 \le x < \infty$. This region is shaded in Fig. 12.7, and clearly has a smaller area than the combined area of all the rectangles shown in the figure. But these rectangles have base 1 and heights $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$, so their combined area is exactly 2, as indicated:

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots = 2.^*$$

It follows that the shaded region—of infinite extent!—has finite area less than 2. The area of this region can even be computed exactly; since $2^x = (e^{\ln 2})^x = e^{x \ln 2}$, it is

*This is an infinite geometric series of a kind often studied in high school algebra courses. We discuss these series in much more detail in Chapter 13.

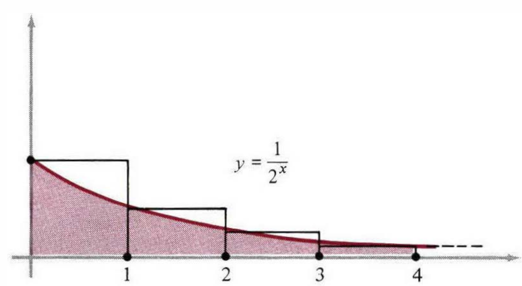


Figure 12.7

$$\int_0^\infty \frac{dx}{2^x} = \lim_{t\to\infty}\int_0^t \frac{dx}{2^x} = \lim_{t\to\infty}\int_0^t e^{-x\ln 2}\,dx$$
$$= \lim_{t\to\infty}\left[-\frac{1}{\ln 2}e^{-x\ln 2}\right]_0^t$$
$$= \lim_{t\to\infty}\left[-\frac{1}{\ln 2}\cdot\frac{1}{2^t}+\frac{1}{\ln 2}\right] = \frac{1}{\ln 2}$$
$$\cong 1.4427.$$

Remark 2 Generally speaking, improper integrals play a more substantial role in higher mathematics than they do in calculus. We mention two important examples—which we do not pursue any further in this book—to give students some idea of what we're talking about.

(a) The improper integral

$$\Gamma(p) = \int_0^\infty x^{p-1}e^{-x}\,dx$$

(the symbol on the left is capital *gamma* in the Greek alphabet) is called the *gamma function*. This is a very interesting function which is studied in advanced calculus and elsewhere. It has innumerable applications to physics as well as to geometry, number theory, and other parts of pure mathematics.

(b) The improper integral

$$F(p) = \int_0^\infty e^{-px}f(x)\,dx$$

has many significant applications to electric circuits, vibrating membranes, and heat conduction, and to the solution of certain types of differential equations. It is a function of p associated with the given function $f(x)$, and is called the *Laplace transform of* $f(x)$.*

*See Chapter 9 in the author's book *Differential Equations with Applications and Historical Notes*, 2nd ed. (McGraw-Hill, 1991).

PROBLEMS

In each of the following problems, determine whether or not the improper integral converges, and find its value if it does.

1 $\int_3^\infty e^{-2x}\,dx.$

2 $\int_0^\infty \frac{dx}{(1+x)^3}.$

3 $\int_8^\infty \frac{dx}{x^{4/3}}.$

4 $\int_0^\infty \sin x\,dx.$

5 $\int_1^\infty \frac{1}{x^2}\sin\frac{1}{x}\,dx.$

6 $\int_e^\infty \frac{dx}{x\ln x}.$

7 $\int_e^\infty \frac{dx}{x(\ln x)^2}.$

8 $\int_0^\infty e^{-x}\cos x\,dx.$

9 $\int_0^\infty (x-1)e^{-x}\,dx.$

10 $\int_1^\infty \left(\frac{1}{\sqrt{x}}-\frac{1}{\sqrt{x+3}}\right)dx.$

11 $\int_1^\infty \frac{dx}{x(x+2)}.$

12 $\int_3^\infty \frac{dx}{x\sqrt{16+x^2}}.$

13 $\int_0^2 \frac{\ln x\,dx}{\sqrt{x}}.$

14 $\int_0^2 \frac{dx}{4-x^2}.$

15 $\int_{-\infty}^\infty |x|e^{-x^2}\,dx.$

16 $\int_{-\infty}^\infty e^{-x}\cos x\,dx.$

17 Let p be a positive constant. Determine the values of p for which the improper integral

$$\int_0^1 \frac{dx}{x^p}$$

is convergent, and those for which it is divergent.

18 Consider the region under the graph of $y = 1/x$ for $x \geq 1$. Even though this region has infinite area, show that the solid of revolution obtained by revolving this region about the x-axis has finite volume, and compute this volume.

19 Consider the region in the first quadrant under the curve $y = 1/(x+1)^3$. Find the volume of the solid of revolution generated by revolving this region about (a) the x-axis; (b) the y-axis.

20 The region under the curve $y = 4/(3x^{3/4})$ for $x \geq 1$ is revolved about the x-axis. Find the volume of the solid of revolution generated in this way.

21 Show that the surface area of the solid of revolution described in Problem 20 is infinite. As a result of these calculations, we see that a container in the shape of this surface can be filled with paint (it has finite volume), but nevertheless its inner surface cannot be painted (it has infinite surface area). Hint: Use the obvious inequality

$$\frac{1}{x^{3/4}}\sqrt{1+\frac{1}{x^{7/2}}} > \frac{1}{x^{3/4}}$$

to show that

$$\text{surface area} > \frac{8\pi}{3}\int_1^\infty \frac{dx}{x^{3/4}} = \infty.$$

22 If $a > 0$ and the graph of $y = ax^2 + bx + c$ lies entirely above the x-axis, show that

$$\int_{-\infty}^{\infty} \frac{dx}{ax^2+bx+c} = \frac{2\pi}{\sqrt{4ac-b^2}}.$$

23 (A comparison test) Let $f(x)$ and $g(x)$ be continuous functions with the property that $0 \leq f(x) \leq g(x)$ for $a \leq x < \infty$. Show that

(a) if $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ also converges;

(b) if $\int_a^\infty f(x)\,dx$ diverges, then $\int_a^\infty g(x)\,dx$ also diverges.

24 Use the comparison test in Problem 23 to determine whether each of the following integrals converges or diverges:

(a) $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x^3+5}}$; (b) $\displaystyle\int_2^\infty (x^6-1)^{-1/7}\,dx$;

(c) $\displaystyle\int_2^\infty \frac{\cos^4 5x}{x^3}\,dx$; *(d) $\displaystyle\int_e^\infty \frac{\ln x}{x^2}\,dx$.

12.5 THE NORMAL DISTRIBUTION. GAUSS

Suppose a measurement or experiment is performed many times, and that its result is a number. We can think, for example, of weighing the babies born in a certain hospital during a given year, or of measuring the annual rainfall in a certain city over a number of years. Suppose the possible results of our measurement or experiment are numbers x that lie in an interval $a \leq x \leq b$. To record our results we can divide the interval $[a, b]$ into n subintervals of equal length, say $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, and then count the number of times m_k that our result is a number between x_{k-1} and x_k. When this way of arranging the data is represented by a step function whose height is m_k over the kth subinterval, the resulting graph is called a *histogram*.

In Fig. 12.8 the birth weight data in the table on the left—taken from genuine vital statistics—is displayed in the histogram on the right. The total number of babies born in this hospital in this year was 2555. To find the average birth weight directly, we would have to calculate the number

$$\frac{\text{sum of all birth weights}}{\text{total number of babies}}.$$

But our table doesn't provide individual birth weights, so without access to the original data this calculation is beyond our power. However, by using the midpoint of each weight interval, we find that the sum of all the birth weights is approximately

$$\begin{aligned}(1.5)(12) + (2.5)(18) + (3.5)(46) + (4.5)(158) \\ + (5.5)(422) + (6.5)(828) + (7.5)(491) + (8.5)(429) \\ + (9.5)(133) + (10.5)(18) = 17{,}419.5 \text{ lb.} \qquad (1)\end{aligned}$$

The average birth weight, also called the *mean,* is therefore approximately $17{,}419.5/2555 = 6.82$ lb. In the histogram each term of the sum (1) is the product of the x-coordinate of the midpoint of a subinterval and the area of the corresponding rectangle.

Birth weights, lb	*Number of babies*	*% of total*
1–2	12	0.5
2–3	18	0.7
3–4	46	1.8
4–5	158	6.2
5–6	422	16.5
6–7	828	32.4
7–8	491	19.2
8–9	429	16.8
9–10	133	5.2
10–11	18	0.7

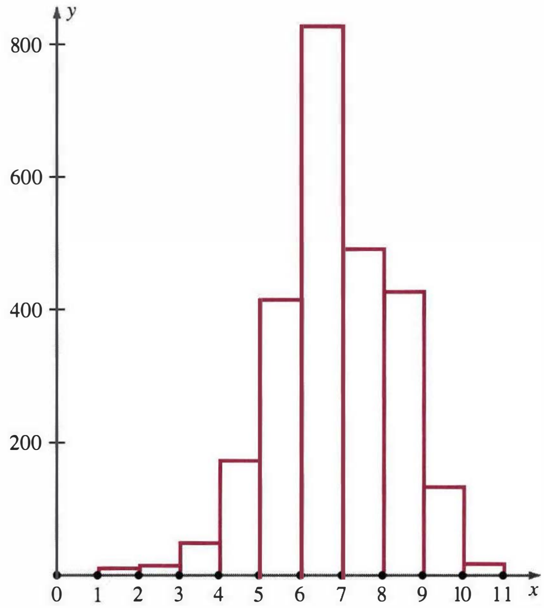


Figure 12.8

If we reconstruct our histogram by using a larger and larger number of smaller and smaller subintervals, we expect the graph to approach the graph of a smooth function $f(x)$. We can now adjust the unit of length along the vertical axis so that the total area under the curve is 1. This gives a function $y = f(x)$ called the *frequency density*. This function has two characteristic properties:

$$f(x) \geq 0 \qquad \text{and} \qquad \int_a^b f(x)\,dx = 1. \tag{2}$$

Also, if $a \leq c < d \leq b$, then the integral

$$\int_c^d f(x)\,dx \tag{3}$$

gives the ratio of the number of times the measurement produces a value between c and d to the total number of measurements, that is, the relative frequency of the result $c \leq x \leq d$. In the same way, $f(x)\,dx$ can be thought of as the proportion of results that lie between x and $x + dx$. From this point of view, the integral (3) can be interpreted as the probability that a randomly chosen measurement will have a result between c and d, and $f(x)$ is then called a *probability density function.*

In order to gain further insight into these concepts, let us for a moment think of $f(x)$ as the mass density function of a rod of total mass 1 that lies along the x-axis between $x = a$ and $x = b$. Then $f(x)\,dx$ is the element of mass, $xf(x)\,dx$ is the moment of this element of mass about the origin, and the integral

$$\bar{x} = \int_a^b xf(x)\,dx \tag{4}$$

is the center of mass of the rod, since $\int_a^b f(x)\,dx = 1$. Also, the integral

$$I = \int_a^b (x - \bar{x})^2 f(x)\,dx \tag{5}$$

is the moment of inertia of the rod about the line $x = \bar{x}$ as axis. We know from our experience in Chapter 11 that this quantity is small if most elements of mass are nestled close to the axis, and larger otherwise.

In the case of a general probability density $f(x)$ with properties (2), the integral corresponding to (4),

$$m = \int_a^b xf(x)\,dx,$$

is called the *mean*. As we know, the mean m is the point on the x-axis where the region under the probability density graph, if it were made out of cardboard and placed in a horizontal position, would balance on the line $x = m$. The square root of the integral corresponding to (5),

$$\sigma = \sqrt{\int_a^b (x-m)^2 f(x)\,dx},$$

is called the *standard deviation*. If σ is small, the results of our measurements cluster around the mean m; and if σ is large, then a significant portion of these results are farther away from m.

In the general mathematical theory of probability of which these ideas are only a hint, it is customary to consider probability densities that are defined for *all x*, so that no limitations are placed on the possible results of the measurement or experiment under consideration. A *probability density* is then defined to be any function that satisfies the conditions

$$f(x) \geq 0 \qquad \text{and} \qquad \int_{-\infty}^{\infty} f(x)\,dx = 1, \tag{6}$$

and the *mean m* and *standard deviation* σ are defined by

$$m = \int_{-\infty}^{\infty} xf(x)\,dx \qquad \text{and} \qquad \sigma^2 = \int_{-\infty}^{\infty} (x-m)^2 f(x)\,dx. \tag{7}$$

Of course, these integrals are improper integrals in the sense discussed in Section 12.4.

SEVERAL IMPORTANT IMPROPER INTEGRALS

To reach our goal of understanding the normal distribution we must first consider several properties of the function $y = f(x) = e^{-x^2}$, whose bell-shaped graph is sketched in Fig. 12.9. We begin by pointing out that this function is even, which means that $f(-x) = f(x)$, so the graph is symmetric about the y-axis. Also, the values of the function are all positive, it has a maximum $y = 1$ at $x = 0$, and the graph has two points of inflection at $x = \pm\frac{1}{2}\sqrt{2}$ (check this by calculating y''). It is clear that

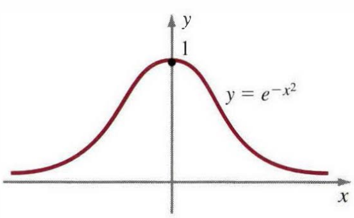


Figure 12.9

$$\lim_{x\to\pm\infty} e^{-x^2} = 0, \tag{8}$$

because $e^{-x^2} = 1/e^{x^2}$ and $e^{x^2} \to \infty$ as $x \to \pm\infty$. Also

$$\lim_{x\to\pm\infty} xe^{-x^2} = 0, \tag{9}$$

because for $|x| > 1$ we have $|xe^{-x^2}| = |x|e^{-x^2} < x^2e^{-x^2}$, and we know that $\lim_{x\to\pm\infty} x^2e^{-x^2} = \lim_{z\to\infty} ze^{-z} = 0$ by Example 1 in Section 12.3.

It is a remarkable fact that the area under the curve $y = e^{-x^2}$ has the finite value

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}, \tag{10}$$

because

$$\int_{0}^{\infty} e^{-x^2}\,dx = \tfrac{1}{2}\sqrt{\pi}. \tag{11}$$

This astonishing formula connecting e and π is best established by using double integration in polar coordinates. The details of the proof are given in Example 3 of Section 20.4, but for the present we simply accept it.

Next, we use the definition of an improper integral to write

$$\int_0^\infty xe^{-x^2}\,dx = \lim_{t\to\infty}\int_0^t xe^{-x^2}\,dx = \lim_{t\to\infty}[-\tfrac{1}{2}e^{-x^2}]_0^t$$

$$= \lim_{t\to\infty}(\tfrac{1}{2} - \tfrac{1}{2}e^{-t^2}) = \tfrac{1}{2}.$$

Here we used (8). Similarly $\int_{-\infty}^{0} xe^{-x^2}\,dx = -\frac{1}{2}$, so by putting these two integrals together we obtain

$$\int_{-\infty}^{\infty} xe^{-x^2}\,dx = 0.^* \tag{12}$$

Finally, an integration by parts with $u = x$, $dv = xe^{-x^2}\,dx$ gives

$$\int x^2e^{-x^2}\,dx = -\frac{1}{2}xe^{-x^2} + \frac{1}{2}\int e^{-x^2}\,dx,$$

so

$$\int_0^t x^2e^{-x^2}\,dx = -\frac{1}{2}te^{-t^2} + \frac{1}{2}\int_0^t e^{-x^2}\,dx.$$

By (9) and (11) we now have

$$\int_0^\infty x^2e^{-x^2}\,dx = \lim_{t\to\infty}\int_0^t x^2e^{-x^2}\,dx$$

$$= \lim_{t\to\infty}\left(-\frac{1}{2}te^{-t^2}\right) + \lim_{t\to\infty}\frac{1}{2}\int_0^t e^{-x^2}\,dx$$

$$= 0 + \frac{1}{2}\int_0^\infty e^{-x^2}\,dx = \tfrac{1}{4}\sqrt{\pi}.$$

Since the integrand $x^2e^{-x^2}$ is an even function, we conclude that

$$\int_{-\infty}^{\infty} x^2e^{-x^2}\,dx = \tfrac{1}{2}\sqrt{\pi}. \tag{13}$$

THE NORMAL CURVE

Let m be any number and σ any positive number. Then the function

*This result also follows without calculation by observing that the integrand is an odd function, that is, it has the property $f(-x) = -f(x)$.

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-m)^2/2\sigma^2} \tag{14}$$

is called the *normal* (or *Gaussian*) *probability density function* with mean m and standard deviation σ. Since clearly $f(x) > 0$ for all x, to verify what is implicitly stated here we must show that

$$\int_{-\infty}^{\infty} f(x)\, dx = 1, \tag{15}$$

$$\int_{-\infty}^{\infty} xf(x)\, dx = m, \tag{16}$$

and

$$\int_{-\infty}^{\infty} (x-m)^2 f(x)\, dx = \sigma^2. \tag{17}$$

To prove these facts we use the change of variable $t = (x - m)/\sigma\sqrt{2}$, so that t varies from $-\infty$ to ∞ as x varies from $-\infty$ to ∞ and

$$x = m + \sigma\sqrt{2}\, t, \qquad dx = \sigma\sqrt{2}\, dt, \qquad f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-t^2}.$$

By using (10), (12), and (13) we establish (15), (16), and (17) as follows:

$$\int_{-\infty}^{\infty} f(x)\, dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2}\sigma\sqrt{2}\, dt = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2}\, dt = 1,$$

$$\begin{aligned} \int_{-\infty}^{\infty} xf(x)\, dx &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (m + \sigma\sqrt{2}t)e^{-t^2}\sigma\sqrt{2}\, dt \\ &= \frac{m}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2}\, dt + \sigma\sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} te^{-t^2}\, dt = m, \end{aligned}$$

and

$$\begin{aligned} \int_{-\infty}^{\infty} (x-m)^2 f(x)\, dx &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} 2\sigma^2 t^2 e^{-t^2}\sigma\sqrt{2}\, dt \\ &= \frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} t^2 e^{-t^2}\, dt = \sigma^2. \end{aligned}$$

The graph of (14) is called the *normal* (or *Gaussian*) *curve* with mean m and standard deviation σ. It is symmetric about the line $x = m$, because the function (14) has the same values for $x_1 = m + a$ and $x_2 = m - a$. Also, the curve is bell-shaped, and the function assumes its maximum value of $1/\sigma\sqrt{2\pi} = 0.399/\sigma$ at $x = m$. Further, the curve has two points of inflection at the points $x = m + \sigma$ and $x = m - \sigma$. To see this we calculate

$$f'(x) = -\frac{x-m}{\sigma^3\sqrt{2\pi}} e^{-(x-m)^2/2\sigma^2}$$

and

$$\begin{aligned} f''(x) &= -\frac{1}{\sigma^3\sqrt{2\pi}} e^{-(x-m)^2/2\sigma^2} + \frac{(x-m)^2}{\sigma^5\sqrt{2\pi}} e^{-(x-m)^2/2\sigma^2} \\ &= \frac{1}{\sigma^3\sqrt{2\pi}} \left[\left(\frac{x-m}{\sigma}\right)^2 - 1\right] e^{-(x-m)^2/2\sigma^2}. \end{aligned}$$

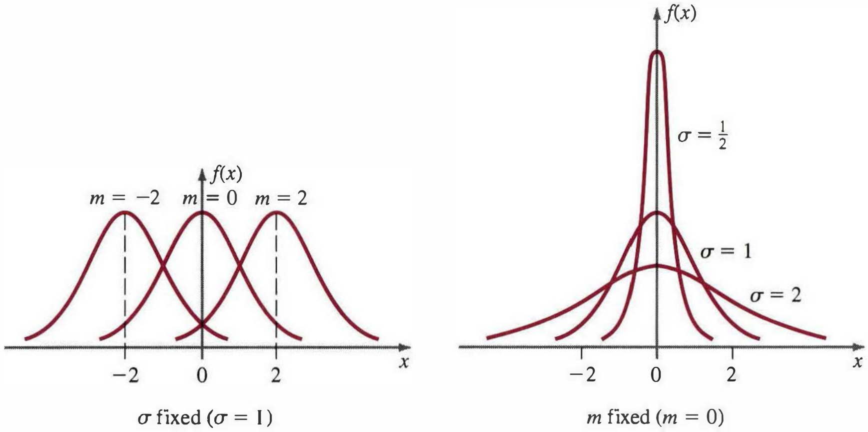


Figure 12.10 Changes in $f(x)$ as m varies and as σ varies.

This formula tells us that the second derivative is positive for $|x - m| > \sigma$ and negative for $|x - m| < \sigma$, which proves the statement about points of inflection.

Normal curves with $\sigma = 1$ and $m = 0, 2, -2$ are shown on the left in Fig. 12.10, and with $m = 0$ and $\sigma = \frac{1}{2}, 1, 2$ on the right. We observe that these curves are wide and flat for large σ, and narrow and peaked for small σ. For the special case in which $m = 0$ and $\sigma = 1$, we obtain the important *standard normal probability density*

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}. \tag{18}$$

The graph of this function is shown in Fig. 12.11. We notice that for $-1 \leq x \leq 1$ (within one standard deviation of the mean) we obtain 68.2 percent of the area under the curve, and for $-2 \leq x \leq 2$ (within two standard deviations of the mean) we obtain 95.4 percent of the area under the curve. It is an interesting fact that these percentages hold for the areas under all normal curves within one or two standard deviations of the mean.

When $f(x)$ is any probability density, the function of t defined by

$$F(t) = \int_{-\infty}^{t} f(x)\, dx$$

is called its *distribution function.* According to our previous interpretation, $F(t)$ is the probability that x lies in the interval $(-\infty, t]$. In particular, the *normal dis-*

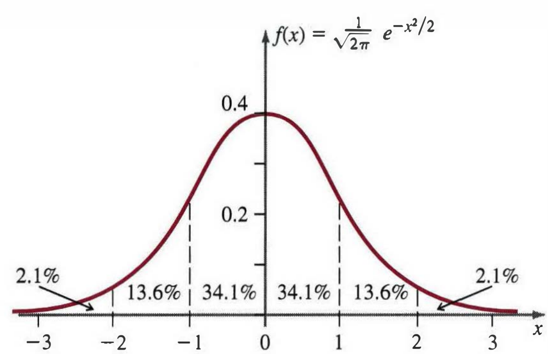


Figure 12.11 The standard normal curve ($m = 0$, $\sigma = 1$)

tribution function (or simply the *normal distribution*) with mean m and standard deviation σ is the function

$$F(t) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{t} e^{-(x-m)^2/2\sigma^2}\, dx. \tag{19}$$

In the simplest special case, in which $m = 0$ and $\sigma = 1$, it is customary to denote this by

$$\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-x^2/2}\, dx, \tag{20}$$

and to refer to it as the *standard normal distribution*. Tables have been constructed for the function $\Phi(t)$ by the methods of numerical integration, and these tables can be used to solve many problems in science and mathematics involving probability and statistics. Students who wish to explore these important ideas are urged to take an advanced course on mathematical probability.

We have hinted at a procedure here, and it might be helpful to give a brief explanation of how this procedure works. To say that the quantity x is *normally distributed* means that its density function is well approximated by (14) for suitable choices of m and σ. The probability that x lies in the interval $a \le x \le b$ is denoted by $P(a \le x \le b)$ and is given by

$$P(a \le x \le b) = \frac{1}{\sigma\sqrt{2\pi}} \int_{a}^{b} e^{-(x-m)^2/2\sigma^2}\, dx. \tag{21}$$

If we make the substitution $t = (x - m)/\sigma$, then a and b become

$$a' = \frac{a-m}{\sigma} \qquad \text{and} \qquad b' = \frac{b-m}{\sigma},$$

and the integral just written is transformed into

$$\begin{aligned} P(a \le x \le b) = P(a' \le t \le b') &= \frac{1}{\sqrt{2\pi}} \int_{a'}^{b'} e^{-t^2/2}\, dt \\ &= \Phi(b') - \Phi(a'). \end{aligned}$$

This quantity can now be calculated by using tables to look up the numerical values of $\Phi(b')$ and $\Phi(a')$.

Many phenomena in science and society are normally distributed, and can therefore be modeled and calculated by using this machinery—for instance the heights of men of the same age in a large population, the speeds of molecules in a gas, the results of measuring a physical quantity many times, and so on.

Example 1 The mean annual rainfall in New York City is 42 in. The annual rainfall over many years is closely approximated by the normal density function with $m = 42$ and standard deviation $\sigma = 2$,

$$f(x) = \frac{1}{2\sqrt{2\pi}} e^{-(x-42)^2/8}.$$

A sketch of this normal curve is shown in Fig. 12.12. Use this information to compute the proportion of years with rainfall between (a) 40 and 44 in; (b) 38 and 46 in.

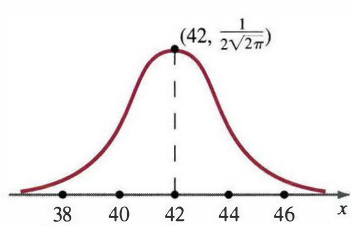


Figure 12.12

Solution (a) The proportion of years with rainfall between 40 and 44 in is

$$\frac{1}{2\sqrt{2\pi}}\int_{40}^{44} e^{-(x-42)^2/8}\,dx.$$

With the change of variable $t = (x - 42)/2$—and access to a table of values of $\Phi(t)$—this becomes

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\int_{-1}^{1} e^{-t^2/2}\,dt &= \Phi(1) - \Phi(-1)\\ &= 0.8413 - 0.1587 \cong 0.6826.\end{aligned}$$

(b) Similarly, the proportion of years with rainfall between 38 and 46 in is (with the same change of variable)

$$\begin{aligned}\frac{1}{2\sqrt{2\pi}}\int_{38}^{46} e^{-(x-42)^2/8}\,dx &= \frac{1}{\sqrt{2\pi}}\int_{-2}^{2} e^{-t^2/2}\,dt\\ &= \Phi(2) - \Phi(-2)\\ &= 0.9772 - 0.0228 \cong 0.9544.\end{aligned}$$

Example 2 An examination is sometimes considered to have done its job of spreading student grades fairly if the frequency histogram of grades can be approximated by a normal density function. Some teachers who go to this trouble then use this histogram and approximating curve to estimate m and σ, and assign the letter grade A to grades greater than $m + \sigma$, B to grades between m and $m + \sigma$, C to grades between $m - \sigma$ and m, D to grades between $m - 2\sigma$ and $m - \sigma$, and F to grades below $m - 2\sigma$. This is what is meant (or used to be meant) by grading *on the curve*. This approach to calculating grades is probably almost extinct in the modern era of grade inflation.

Remark 1 How does it happen that these probability discussions are saturated with various forms of the function e^{-x^2}? We attempt to answer this question by showing how the normal probability density function (14) can be derived from simple and reasonable assumptions.

Consider the experiment of a marksman repeatedly shooting at a target whose bull's-eye is the origin of the xy-plane (Fig. 12.13), and suppose that we are only interested in the x-coordinates of the points of impact. These x-coordinates provide an ideally simple example of quantities distributed in the pattern we wish to examine, being bunched together around $x = 0$ and tapering off symmetrically to the sides.

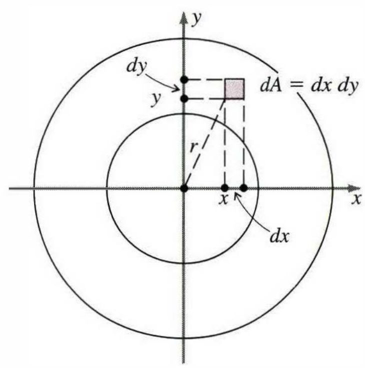


Figure 12.13

If $f(x)$ is the probability density function of these x-coordinates, then $f(x)\,dx$ is the probability for any particular shot that its x-coordinate lies in the interval from x to $x + dx$. Similarly the probability of the y-coordinate lying in the interval from y to $y + dy$ is $g(y)\,dy$, where $g(y)$ is the probability density in the y-direction. Now, assuming that the x- and y-deviations from the bull's-eye are independent of each other, then the product of the two probabilities,

$$[f(x)\,dx][g(y)\,dy] = f(x)g(y)\,dx\,dy = f(x)g(y)\,dA,$$

is the probability that the bullet hits the element of area dA shown in the figure. Assuming further that the experiment possesses circular symmetry, this probability will be the same for any equal element of area at the same distance r in any direction from the bull's-eye. This amounts to assuming that $f(x)g(y)$ is a function only of r^2,

$$f(x)g(y) = h(r^2), \tag{22}$$

where $r^2 = x^2 + y^2$.

Differentiating both sides of (22) first with respect to x and then with respect to y gives

$$f'(x)g(y) = h'(r^2) \cdot 2x \quad \text{and} \quad f(x)g'(y) = h'(r^2) \cdot 2y.$$

By eliminating $h'(r^2)$ from these equations we obtain

$$\frac{f'(x)g(y)}{2x} = \frac{f(x)g'(y)}{2y}$$

or

$$\frac{f'(x)}{2xf(x)} = \frac{g'(y)}{2yg(y)}. \tag{23}$$

Since the left side is a function of x alone, and the right side is a function of y alone, (23) implies that both sides are constant, in particular

$$\frac{f'(x)}{2xf(x)} = c \quad \text{or} \quad \frac{f'(x)}{f(x)} = 2cx.$$

Integration now gives

$$\ln f(x) = cx^2 + d$$

or

$$f(x) = e^d \cdot e^{cx^2} = De^{cx^2} \tag{24}$$

where $D = e^d$. But $f(x)$ is a probability density function, so we must have

$$\int_{-\infty}^{\infty} f(x)\, dx = 1, \tag{25}$$

and this implies that c must be negative. We are free to write c in the form $c = -1/2\sigma^2$ for a positive constant σ, and (24) now becomes

$$f(x) = De^{-x^2/2\sigma^2}.$$

By integrating this from $-\infty$ to ∞, changing the variable of integration from x to $t = x/\sigma\sqrt{2}$, and using (10) and (25), we obtain

$$D\int_{-\infty}^{\infty} e^{-x^2/2\sigma^2}\, dx = D\sigma\sqrt{2}\int_{-\infty}^{\infty} e^{-t^2}\, dt = D\sigma\sqrt{2}\sqrt{\pi} = 1.$$

Therefore $D = 1/\sigma\sqrt{2\pi}$ and our function takes its final form,

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-x^2/2\sigma^2},$$

which is the normal probability density (14) with mean $m = 0$.

Remark 2 Formula (11) can be thought of as a special case (let $a \to \infty$) of a little-known formula called *Laplace's continued fraction*:

$$\int_0^a e^{-x^2}\,dx = \frac{1}{2}\sqrt{\pi} - \cfrac{e^{-a^2}}{2a + \cfrac{1}{a + \cfrac{2}{2a + \cfrac{3}{a + \cfrac{4}{2a + \cdots}}}}}.$$

We mention this not because it has any practical importance for us in our present work, but rather because it would be unconscionable—almost immoral—to deprive the student of the opportunity of seeing one of the most wonderful and beautiful individual facts in the whole of mathematics.*

*See p. 357 of H. S. Wall, *Analytic Theory of Continued Fractions* (Van Nostrand, 1948; reprinted by Dover, 1967). In this reference Laplace's formula is given among the special cases of the continued fraction of Gauss for the quotient of two hypergeometric functions. Hypergeometric functions are discussed on p. 200 and in Problem 1 on p. 203 of the present writer's book, *Differential Equations with Applications and Historical Notes* (McGraw-Hill, 2nd ed., 1991).

NOTE ON GAUSS

The German Carl Friedrich Gauss (1777–1855) was the greatest of all mathematicians and perhaps the most richly gifted genius of whom there is any record. The profound creative activity of this awe-inspiring figure embraced all of mathematics—geometry, number theory, algebra, and analysis—as well as physics and astronomy. The fame of the town of Göttingen in Germany as the leading center of mathematics in the world until the 1930s dates from the time of Gauss, who was professor there and director of the Göttingen astronomical observatory. His fundamental contribution to the concept of normal distributions arose from his work on the theory of errors in making physical measurements.* It is strange and ironic that this colossal figure in the intellectual history of western civilization should be almost entirely unknown among most educated people.

*For more information on this, see pp. 78–83 of *Carl Friedrich Gauss* by Tord Hall (MIT Press, 1970); or pp. 138–140 of *Gauss: A Biographical Study* by W. K. Bühler (Springer-Verlag, 1981). A brief general account of Gauss's life and work can be found in Section A.25 of the present writer's book, *Calculus Gems* (McGraw-Hill, 1992).

PROBLEMS

1 Find the value of the constant k for which each of the following is a probability density function on $(-\infty, \infty)$:

$$ke^{-|x|}, \qquad \frac{k}{1+x^2}, \qquad kxe^{-x^2}.$$

2 Several functions $f(x)$ are defined by the following expressions for $0 \le x \le 1$ and are identically zero for all other values of x: 1, $3x^2$, $5x^4$, $e - e^x$, $\pi/2 \sin \pi x$. Verify that each is a probability density function.

3 Verify that each of the following is a probability density function and find its mean m:
(a) $f(x) = \frac{1}{4}(x+1)$ for $0 \le x \le 2$ and 0 elsewhere;
(b) $f(x) = \frac{3}{8}(1 - \frac{1}{4}x^2)$ for $-2 \le x \le 2$ and 0 elsewhere.

4 Verify that the function defined by $f(x) = \frac{1}{2}x$ for $0 \le x \le 2$ and 0 elsewhere is a probability density function and find its mean m and standard deviation σ.

5 In Example 2, use Fig. 12.12 to estimate the percentages of students who receive the grades A, B, C, D, F.

CHAPTER 12 REVIEW: DEFINITIONS, CONCEPTS

Think through the following.

1 Mean Value Theorem.
2 L'Hospital's rule for 0/0.
3 L'Hospital's rule for other indeterminate forms.
4 Improper integrals.
5 Probability density function.
6 Mean and standard deviation.
7 Normal probability density.
8 Standard normal probability density.

ADDITIONAL PROBLEMS FOR CHAPTER 12

SECTION 12.2

Find the following limits.

1 $\lim_{x\to 0} \frac{\sin 5x}{\sin 2x}$.

2 $\lim_{x\to 2} \frac{\ln (x-1)}{x-2}$.

3 $\lim_{x\to 5} \frac{x^2+x-30}{\sqrt{x-1}-2}$.

4 $\lim_{x\to 1} \frac{\sin \pi x}{1-x^2}$.

5 $\lim_{x\to 4} \frac{x-4}{\sqrt[3]{x+4}-2}$.

6 $\lim_{x\to -3} \frac{x^2+2x-3}{2x^2+3x-9}$.

7 $\lim_{x\to 2} \frac{\tan (2x-4)}{x^3-8}$.

8 $\lim_{x\to 1} \frac{x^3+x^2-2}{\ln x}$.

9 $\lim_{x\to 0} \frac{\sqrt[5]{x+1}-(1+\frac{1}{5}x)}{3x^2}$.

10 $\lim_{x\to 0} \frac{\sqrt[4]{x+16}-(2+\frac{1}{32}x)}{x^2}$.

11 $\lim_{x\to 3+} \frac{\ln (x-2)}{(x-3)^2}$.

12 $\lim_{x\to 0} \frac{x \sin (\sin x)}{1-\cos (\sin x)}$.

13 $\lim_{x\to 0} \frac{\sin x^3}{x-\sin x}$.

14 $\lim_{x\to 0} \frac{e^{x^2}-1}{x \sin x}$.

15 $\lim_{x\to \infty} \frac{e^{3/x}-1}{\sin 1/x}$.

16 $\lim_{x\to 0+} \frac{\tan^{-1} x}{1-\cos 2x}$.

17 $\lim_{x\to 0} \frac{1-\cos x}{x \sin x}$.

18 $\lim_{x\to 16+} \frac{\sqrt[4]{x-16}}{\sqrt[4]{x}-2}$.

19 $\lim_{x\to 0+} \frac{\sin^{-1} x}{\sin^2 3x}$.

20 $\lim_{x\to 0} \frac{2\cos x-2+x^2}{3x^4}$.

21 $\lim_{x\to \pi/2} \frac{1-\sin x}{\cos x}$.

22 $\lim_{x\to 0} \frac{2x}{\tan^{-1} x}$.

23 $\lim_{x\to 2} \frac{3\sqrt[3]{x-1}-x-1}{3(x-2)^2}$.

24 $\lim_{x\to 1} \frac{\ln x}{x^2-x}$.

25 $\lim_{x\to 0} \frac{\sin^2 x+2\cos x-2}{\cos^2 x-x\sin x-1}$.

26 $\lim_{x\to 0} \frac{\sin x-\tan x}{x^3}$.

27 $\lim_{x\to 0} \frac{\cos x-1+\frac{1}{2}x^2}{x^4}$.

28 $\lim_{x\to 0} \frac{\sin x^2-\sin^2 x}{x^4}$.

29 $\lim_{x\to 0} \frac{e^x+e^{-x}-2}{1-\cos 4x}$.

30 $\lim_{x\to \pi} \frac{1+\cos x}{(x-\pi)^2}$.

31 $\lim_{x\to 1} \frac{x^3+3e^{1-x}-4}{x-\ln x-1}$.

32 $\lim_{x\to 0} \frac{x-\sin x}{x \tan x}$.

33 $\lim_{x\to 0} \frac{x^2 \tan x}{\tan x - x}$.

34 $\lim_{x\to 0} \frac{1-\cos^2 x}{x^2}$.

35 $\lim_{x\to 0} \frac{\ln (x+1)}{e^{3x}-1}$.

36 $\lim_{x\to 0} \frac{1-\cos 2\sqrt{a}\, x}{2x^2}$.

37 $\lim_{x\to 1} \frac{x^{10}-1}{x^9-1}$.

38 $\lim_{x\to 0} \frac{x-\sin x}{1-\cos x}$.

39 $\lim_{x\to 0} \frac{x-\tan^{-1} x}{x^3}$.

40 $\lim_{x\to \pi} \frac{\sin^2 x}{1+\cos 5x}$.

41 $\lim_{x\to \infty} \frac{\tan^2 (1/x)}{\ln^2 (1+4/x)}$.

42 Consider the circular sector of radius 1 shown in Fig. 12.14. The point C is the intersection of the tangent lines at A and B. If $f(\theta)$ is the area of the triangle ABC and $g(\theta)$ is the area of the region that remains when the triangle OAB is removed from the sector, evaluate $\lim_{\theta\to 0} f(\theta)/g(\theta)$.

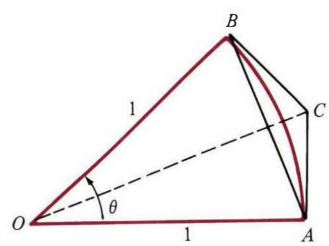


Figure 12.14

43 Show that

$$\lim_{x\to 0} \frac{x^2 \sin (1/x)}{x}$$

is an indeterminate form of the type 0/0 that exists but cannot be evaluated by L'Hospital's rule. What is the value of this limit? Does this example show that L'Hospital's rule is false?

pital's rule is false?

44 Use L'Hospital's rule to establish the following formulas for the direct calculation of the second derivative:

(a) $f''(x) = \lim_{h \to 0} \dfrac{f(x+2h) - 2f(x+h) + f(x)}{h^2}$;

(b) $f''(x) = \lim_{h \to 0} \dfrac{f(x+h) - 2f(x) + f(x-h)}{h^2}$.

45 If n is a positive integer, show that

$$\lim_{x \to 1} \frac{nx^{n+1} - (n+1)x^n + 1}{(x-1)^2} = \frac{n(n+1)}{2}.$$

(For the meaning of this rather strange-appearing result, see Problem 46.)

46 If n is a positive integer and $x \neq 1$, the formula

$$1 + x + x^2 + x^3 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x} = \frac{x^{n+1} - 1}{x - 1}$$

is familiar from high school algebra. Differentiate it to obtain

$$1 + 2x + 3x^2 + \cdots + nx^{n-1} = \frac{nx^{n+1} - (n+1)x^n + 1}{(x-1)^2}, \quad (*)$$

and then take limits as $x \to 1$ and use Problem 45 to derive the formula

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

*47 Multiply equation (*) in Problem 46 by x, differentiate, etc., and thereby derive the formula

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

SECTION 12.3

Evaluate the following limits by any method.

48 $\lim_{x \to \infty} \dfrac{3x^2 + 9}{x + e^x}$.

49 $\lim_{x \to \frac{1}{2}-} \dfrac{\ln(1 - 2x)}{\tan \pi x}$.

50 $\lim_{x \to 3\pi/2} \dfrac{2 + \sec x}{\tan x}$.

51 $\lim_{x \to \infty} \dfrac{\ln x^{100}}{\sqrt[5]{x}}$.

52 $\lim_{x \to \infty} \dfrac{x + \ln x}{x \ln x}$.

53 $\lim_{x \to 0+} \dfrac{\ln x}{\cot x}$.

54 $\lim_{x \to 0+} \dfrac{\ln(\sin x)}{\ln(\sin 2x)}$.

55 $\lim_{x \to \infty} \dfrac{\ln x}{e^{2x}}$.

56 $\lim_{x \to \infty} \dfrac{\ln(\ln x)}{\sqrt{x}}$.

57 $\lim_{x \to \infty} \dfrac{xe^x}{e^{x^2}}$.

58 $\lim_{x \to 0+} x^2 \ln x$.

59 $\lim_{x \to 0+} x^p \ln x$, $p > 0$.

60 $\lim_{x \to 0+} x^2 e^{1/x}$.

61 $\lim_{x \to \infty} x \sin \dfrac{p}{x}$, $p \neq 0$.

62 $\lim_{x \to 0+} \tan x \ln x$.

63 $\lim_{x \to \pi/2} \left(x - \dfrac{\pi}{2}\right) \tan 3x$.

64 $\lim_{x \to \pi/2} (2x - \pi) \sec x$.

65 $\lim_{x \to \pi/2} \tan x \ln(\sin x)$.

66 $\lim_{x \to \infty} x(e^{1/x} - 1)$.

67 $\lim_{x \to 0+} \sin x \ln(\sin x)$.

68 $\lim_{x \to 0} \left(\dfrac{1}{x^2} - \dfrac{1}{x \sin x}\right)$.

69 $\lim_{x \to 0} \left(\dfrac{1}{1 - \cos x} - \dfrac{2}{x^2}\right)$.

70 $\lim_{x \to 0} \left[\dfrac{1 + x}{\ln(1 + x)} - \dfrac{1}{x}\right]$.

71 $\lim_{x \to 0} \left[\dfrac{1}{\ln(1 + x)} - \dfrac{1}{e^x - 1}\right]$.

72 $\lim_{x \to \infty} (x - \sqrt{x^2 + x})$.

73 $\lim_{x \to 0+} x^{\sin x}$.

74 $\lim_{x \to 0+} (\sin x)^{\tan x}$.

75 $\lim_{x \to 0+} (e^x - 1)^{\sin x}$.

76 $\lim_{x \to 1+} (x^2 - 1)^{x-1}$.

77 $\lim_{x \to \pi/2-} (\cos x)^{\cos x}$.

78 $\lim_{x \to \pi/4-} (1 - \tan x)^{1 - \tan x}$.

79 $\lim_{x \to 0+} (x + \sin x)^{\tan x}$.

80 $\lim_{x \to 1+} (\ln x)^{\sin(x-1)}$.

81 $\lim_{x \to 0+} [\ln(1 + x)]^x$.

82 $\lim_{x \to 0+} x^{ax^b}$, $b > 0$.

83 $\lim_{x \to 0+} x^{x^x}$ $[x^{x^x} = x^{(x^x)}]$.

84 $\lim_{x \to \infty} (x + e^{ax})^{b/x}$.

85 $\lim_{x \to \infty} (1 + x^p)^{1/x}$, $p > 0$.

86 $\lim_{x \to \infty} (1 + x)^{e^{-x}}$.

87 $\lim_{x \to 0+} (1 + \csc x)^{\sin^2 x}$.

88 $\lim_{x \to 0} \left(1 + \dfrac{1}{x}\right)^x$.

89 $\lim_{x \to 0+} (\csc x)^x$.

90 $\lim_{x \to 0+} (\cot x)^x$.

91 $\lim_{x \to \infty} x^{\ln(1 + 1/x)}$.

92 $\lim_{x \to 0} (1 - 2x)^{3/x}$.

93 $\lim_{x \to \infty} \left(1 + \dfrac{2}{x}\right)^x$.

94 $\lim_{x \to \infty} \left(1 + \dfrac{1}{x}\right)^{5x}$.

95 $\lim_{x \to 0} (e^x + 3x)^{1/x}$.

96 $\lim_{x \to 0} (1 + 2x)^{\cot x}$.

97 $\lim_{x \to 0} (1 + 3x)^{\csc x}$.

98 $\lim_{x \to 0} (\cos 2x)^{1/x^2}$.

99 Show that

$$\lim_{x \to \infty} \frac{x + \sin x}{x}$$

is an indeterminate form of the type ∞/∞ that exists but cannot be found by L'Hospital's rule. What is the value of this limit?

100 Find the value a must have if

$$\lim_{x\to\infty}\left(\frac{x+a}{x-a}\right)^x = 4.$$

SECTION 12.4

Determine whether or not each of the following integrals converges, and find its value if it does.

101 $\displaystyle\int_2^\infty e^{-3x}\,dx.$

102 $\displaystyle\int_5^\infty \frac{dx}{x^3}.$

103 $\displaystyle\int_4^\infty \frac{dx}{x\sqrt{x}}.$

104 $\displaystyle\int_0^\infty \frac{x^2\,dx}{x^3+1}.$

105 $\displaystyle\int_0^\infty e^{-x}\sin x\,dx.$

106 $\displaystyle\int_0^\infty xe^{-x}\,dx.$

107 $\displaystyle\int_0^\infty \frac{x\,dx}{x^4+1}.$

108 $\displaystyle\int_1^\infty xe^{-x^2}\,dx.$

109 $\displaystyle\int_2^\infty \frac{dx}{4+x^2}.$

110 $\displaystyle\int_2^\infty \frac{dx}{x^2-1}.$

111 $\displaystyle\int_0^\infty \frac{x^2\,dx}{e^{x^3}}.$

112 $\displaystyle\int_e^\infty \frac{\ln x}{x}\,dx.$

113 $\displaystyle\int_e^\infty \frac{dx}{x\ln x\sqrt{\ln x}}.$

114 $\displaystyle\int_0^\infty \frac{dx}{\sqrt[3]{e^x}}.$

115 $\displaystyle\int_0^{\pi/2} \frac{dx}{1-\sin x}.$

116 $\displaystyle\int_0^{\pi/2} \frac{dx}{\sin x}.$

117 $\displaystyle\int_0^2 \frac{\ln x}{x}\,dx.$

118 $\displaystyle\int_0^4 \frac{8dx}{\sqrt{16-x^2}}.$

119 $\displaystyle\int_0^3 \frac{x\,dx}{\sqrt{9-x^2}}.$

120 Let p be a positive constant. Determine the values of p for which the improper integral

$$\int_0^1 \frac{dx}{(1-x)^p}$$

is convergent, and those for which it is divergent.

121 Show that the region in the first quadrant under the curve $y = 1/(x+1)^2$ has a finite area but does not have a centroid.

122 If x is a positive constant, show that

$$\int_0^\infty e^{-a^2x^2}\,dx = \frac{1}{a}\int_0^\infty e^{-x^2}\,dx.$$

Without performing any actual integrations, use this fact to show that the centroid of the region between the curve $y = e^{-a^2x^2}$ and the x-axis is $(0, \sqrt{2}/4)$.

13 INFINITE SERIES OF CONSTANTS

13.1 WHAT IS AN INFINITE SERIES?

We have touched briefly on this subject several times before, but now the time has come to confront it directly.

An *infinite series*, or simply a *series*, is an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots, \tag{1}$$

where the three dots at the end indicate that the terms continue indefinitely. In other words, there are infinitely many numbers a_n (one for each positive integer n) and (1) is the indicated sum of this infinite collection of terms. The number a_n is called the *nth term* of the series, and is usually some simple function of n. We include it in (1) if we wish to make an explicit statement of the law of formation of the terms. However, if this law of formation is clear from the context, we can write (1) more informally as

$$a_1 + a_2 + a_3 + \cdots \qquad \text{or} \qquad a_1 + a_2 + \cdots.$$

We will often use the sigma notation of Section 6.3 to write the series (1) in the compact form

$$\sum_{n=1}^{\infty} a_n.$$

The is read "the sum from $n = 1$ to infinity of a_n."

Needless to say, it is quite impossible to perform the operation of addition an infinite number of times—life isn't long enough—so (1) cannot be interpreted literally and its meaning must be approached in a subtler way. It was one of the great achievements of nineteenth-century mathematics to discover that a perfectly reasonable and satisfactory meaning can be given to (1) by using the concept of the limit of a sequence. If we exercise suitable caution, this meaning allows us to work with infinite series just as easily as if they involved only a finite number of terms. In many cases we will actually be able to find the number that is the exact sum of the series, and these sums often turn out to be very surprising indeed.

We will get to all this in the following sections, but first we briefly consider a few of the many natural ways in which infinite series arise in mathematics.

A We usually assume that we understand the real number system, in particular, what is meant by an infinite decimal. However, it is often overlooked that an infinite decimal is *defined* as an infinite series,

$$.a_1a_2a_3 \ldots a_n \ldots = \frac{a_1}{10} + \frac{a_2}{10^2} + \frac{a_3}{10^3} + \cdots + \frac{a_n}{10^n} + \cdots, \tag{2}$$

where each of the a's is understood to be one of the ten digits 0, 1, 2, . . . , 9.

Everyone knows that

$$\tfrac{1}{3} = 0.333\ldots, \tag{3}$$

but not everyone is sure why, or even what this means. This is not at all surprising, because (3) cannot be fully understood without some acquaintance with infinite series, enough to use (2) to evaluate the right side of (3). We will discuss this and other related issues in Section 13.3.

B The elementary long division of $1 - x$ into 1, i.e., $\dfrac{1}{1-x}$,

$$\begin{array}{r|l} & 1 + x + x^2 + \cdots \\ \hline 1 - x & 1 \\ & \underline{1 - x} \\ & \quad x \\ & \quad \underline{x - x^2} \\ & \qquad x^2 \\ & \qquad \underline{x^2 - x^3} \\ & \qquad\quad x^3 \cdots, \end{array}$$

tells us that

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^{n-1} + \frac{x^n}{1-x}. \tag{4}$$

This process can be carried out to as many steps as we wish, and it is natural to wonder how the function on the left of (4) is related to the infinite series that seems to be forming on the right. That is, is it true that

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots ? \tag{5}$$

Many readers have seen this series before, in connection with geometric progressions in elementary algebra. The series on the right of (5) is usually called the *power series expansion* of the function on the left, because it contains steadily increasing powers of x.

C Other important power series expansions are now readily available to us, even though full verifications are not. For example, if we replace x by $-x$ in (5) we obtain

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots; \tag{6}$$

and if we now replace x by x^2 in (6), the result is

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots. \tag{7}$$

The left sides of (6) and (7) remind us of the familiar formulas

$$\int \frac{dx}{1+x} = \ln(1+x) \qquad \text{and} \qquad \int \frac{dx}{1+x^2} = \tan^{-1} x.$$

By integrating the right sides of (6) and (7) as if they were polynomials (but remember: they are *not* polynomials!), we obtain the power series expansions

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \tag{8}$$

and

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots. \tag{9}$$

If these formulas are valid for $x = 1$—and this is a very big "if"—then by putting $x = 1$ and reversing the resulting equations we get

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2 \tag{10}$$

and

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}. \tag{11}$$

Equation (11), connecting π with the positive odd numbers, was discovered by Leibniz in 1673 (Appendix 3 at the end of Chapter 10), and was one of the most beautiful mathematical discoveries of the seventeenth century. It made a deep impression on the minds of the earliest workers in the field of calculus, as it does on us.

We emphasize that these derivations of (8), (9), (10) and (11) *do not* constitute acceptable mathematical proofs, because the validity of the procedures used in obtaining them has not been established. At this stage and with only these supporting arguments, they have only the status of conjectures. It must be remembered that there is a very wide chasm in mathematics between conjecture and actual knowledge, and we wish to *know*.

D Finally, infinite series arise in a very insistent way in the study of differential equations. To see how this happens, let us consider the simple equation

$$\frac{dy}{dx} = y. \tag{12}$$

This equation asks for a function which is unchanged by differentiation, and we know that $y = ce^x$ is such a function for every constant c; in fact, we know that these are the *only* functions with this property (Section 8.3). But to emphasize the point we wish to make, let us pretend that we don't know any solutions and try to guess one. Since polynomials are the simplest functions of all, we might try one of these first. But we have no idea what degree to choose for this hoped-for polynomial solution. This suggests the use of a bit of creative vagueness, so we leave the degree unspecified and try to find coefficients $a_0, a_1, a_2, a_3, a_4, \ldots$ so that

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots \tag{13}$$

will be a solution of (12). By differentiating (13) term by term we obtain

$$\frac{dy}{dx} = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots; \tag{14}$$

and substituting (13) and (14) in (12) gives

$$a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots. \tag{15}$$

If we now equate coefficients of equal powers of x in (15), we get

$$a_1 = a_0, \qquad 2a_2 = a_1, \qquad 3a_3 = a_2, \qquad 4a_4 = a_3, \qquad \ldots,$$

so

$$a_1 = a_0, \qquad a_2 = \frac{a_1}{2} = \frac{a_0}{2}, \qquad a_3 = \frac{a_2}{3} = \frac{a_0}{2 \cdot 3}, \qquad a_4 = \frac{a_3}{4} = \frac{a_0}{2 \cdot 3 \cdot 4}, \qquad \ldots. \tag{16}$$

At this point we remind students of the factorial notation introduced in Section 3.6. If n is a positive integer, we write

$$n! = 1 \cdot 2 \cdot 3 \cdots n \tag{17}$$

and call this n *factorial*. Thus, $1! = 1$, $2! = 1 \cdot 2 = 2$, $3! = 1 \cdot 2 \cdot 3 = 6$, $4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$, $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$, and so on. The definition (17) is meaningless in the case $n = 0$, but for many reasons it is customary to define $0!$ by $0! = 1$. We shall be using factorials often in the next few chapters, so there will be ample opportunity for students to become thoroughly familiar with this notation.

Returning to our problem, we can use the factorial notation to write equations (16) as

$$a_1 = a_0, \qquad a_2 = \frac{a_0}{2!}, \qquad a_3 = \frac{a_0}{3!}, \qquad a_4 = \frac{a_0}{4!}, \qquad \ldots.$$

Our tentative solution (13) of the differential equation (12) therefore becomes

$$\begin{aligned} y &= a_0 + a_0x + \frac{a_0}{2!}x^2 + \frac{a_0}{3!}x^3 + \frac{a_0}{4!}x^4 + \cdots \\ &= a_0\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right), \end{aligned}$$

where a_0 is an arbitrary constant. On comparing this with the known solutions ce^x, we are led to the natural conjecture that the exponential function e^x equals the infinite series shown in parentheses:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots. \tag{18}$$

It turns out that this formula—the power series expansion of e^x—is indeed true. In fact, for all values of x we can calculate e^x as accurately as we please from the series by taking enough terms, and this is how numerical tables for e^x are constructed. However, students should clearly understand that this discussion is merely suggestive, and is by no means a valid proof. Proofs will come later.

In attempting to solve other differential equations in this way, we are led to other series, some representing known familiar functions but many representing previously unknown functions. As an example of the former, we know from Section 9.6 that the important differential equation

$$\frac{d^2y}{dx^2} + y = 0 \tag{19}$$

has the general solution

$$y = c_1 \sin x + c_2 \cos x.$$

But if we try to solve (19) by means of power series, in the manner suggested by the above discussion, then we obtain the power series expansions of the sine and cosine,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \tag{20}$$

and

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots. \tag{21}$$

The validity of these two expansions will be proved in Section 13.4, and this approach to solving differential equations will be explored further in Chapter 14.

In working with infinite series, as in almost any part of calculus, there are many things that we want to be able to do freely with the tools we are studying. The role of the theory is mostly to justify the various procedures that are necessary for carrying out our purposes—such procedures, for example, as the term-by-term differentiation and integration used above. This situation was well expressed by the famous financier J. P. Morgan in describing the role of lawyers in his business operations. "I don't hire lawyers to tell me what I can't do," he said. "I hire them to find legal ways for me to do what I want to do."

Our overall aim in this chapter and the next is to establish power series as familiar and reliable tools. The main reason for this is that we want to be able to accept the power series that arise from differential equations as legitimate and fully satisfactory solutions of these equations.

Of course, every power series is an infinite series of functions. But when x is given a specific numerical value, as in obtaining (10) and (11) above, then the power series becomes an infinite series of constants. We therefore undertake a careful study of series of constants in this chapter to provide a solid foundation for our work with power series in the next chapter.

Nevertheless, our interest in series is not confined to their practical value for applications, and in the course of our work we will touch on many fascinating topics in pure mathematics that are well worth studying for their own sake. Thus, we will see that series are linked to some of the most interesting parts of the theory of numbers, concerning prime numbers, irrational numbers, the nature of the constants e and π, and similar matters. We wish to keep the structure of this chapter as simple as possible and still put the full richness of the subject within easy reach of the interested reader. For this reason we place most of this optional material in the appendices at the end of the chapter, where it can be examined or not according to the wishes of the individual student.

13.2 CONVERGENT SEQUENCES

Any reasonably satisfactory study of series must be based on a careful definition of convergence for sequences. However, the behavior of most sequences is easy to understand without elaborate explanations, and a genuine theory of convergent sequences would be an unwelcome obstacle blocking our way to the main concepts of this chapter. We will therefore discuss sequences rather briefly, and try to steer a middle course between excessive informality and tedious detail.

If to each positive integer n there corresponds a definite number x_n, then the x_n's are said to form a *sequence*. We think of the x_n's as arranged in the order of their subscripts,

$$x_1, x_2, \ldots, x_n, \ldots,$$

and we often abbreviate this array to $\{x_n\}$. It is clear that a sequence is nothing but a function defined for all positive integers n, with the emphasis placed on the subscript notation x_n instead of the function notation $x(n)$. The numbers constituting a sequence are called its *terms*. Thus, x_1 and x_2 are the first and second terms of the given sequence, and x_n is the nth term.*

Example 1 In each of the following we define a sequence $\{x_n\}$ by giving a formula for its nth term:

(a) $x_n = 1$, that is, $1, 1, 1, \ldots$;

(b) $x_n = [1 - (-1)^n]/2$, that is, $1, 0, 1, 0, \ldots$;

(c) $x_n = 1/n$, that is $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$;

(d) $x_n = (n - 1)/n$, that is $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$;

(e) $x_n = (-1)^{n+1}/n$, that is, $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots$;

(f) $x_n = 1 + \dfrac{1}{2} + \dfrac{1}{4} + \cdots + \dfrac{1}{2^{n-1}}$;

(g) $x_n = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \cdots + \dfrac{1}{n}$;

(h) $x_n = \left(1 + \dfrac{1}{n}\right)^n$.

A sequence like (a), in which all the terms are equal, is called a *constant sequence*. Not every sequence has a simple formula, or even any formula at all. This is shown by the sequence $\{d_n\}$, where d_n is the nth digit after the decimal point in the decimal expansion of π.

It is sometimes convenient to relax the definition and allow a sequence to start with the zeroth term x_0, or even with the second or third term x_2 or x_3, instead of requiring it to begin with the first term x_1. One reason for this is that we want to include sequences like that defined by $x_n = 1/\ln n$, where x_1 is meaningless. In any case, we continue to call the term with subscript n the nth term.

A sequence $\{x_n\}$ is said to be *bounded* if there are two numbers A and B such that $A \le x_n \le B$ for every n, and in this case A is called a *lower bound* and B an

*The words "sequence" and "series" have essentially the same meaning in ordinary speech, but in mathematics their meanings are quite distinct. A *sequence* is merely an infinite list of numbers arranged in order, with a first, a second, and so on, whereas a *series* is an infinite sum of numbers.

upper bound for the sequence. A sequence that is not bounded is said to be *unbounded.* In Example 1, it is easy to see that sequences (a) to (f) are bounded, but it is less obvious that (g) is not (hint: $\frac{1}{3}+\frac{1}{4}>\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$, $\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}>\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{1}{2}$, and so on). The sequence (h) is also bounded, but this is not evident on inspection and will be established below.

Our main interest is in the concept of the limit of a sequence. Roughly speaking, this refers to the fact that certain sequences $\{x_n\}$ have the property that the numbers x_n get closer and closer to some real number L as n increases. Another way of stating this is to say that $|x_n - L|$ gets smaller as n gets larger. As an illustration, consider the sequence $\{x_n\}$ whose nth term is $x_n = (n-1)/n$:

$$0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots .$$

These numbers seem to "approach" the number 1 as we move farther and farther out in the sequence. As a matter of fact, for each n we have

$$|x_n - 1| = \left|\frac{n-1}{n} - 1\right| = \left|-\frac{1}{n}\right| = \frac{1}{n};$$

and the number $1/n$, and therefore $|x_n - 1|$, *can be made as small as we please by taking n sufficiently large.* We express this behavior by saying that the sequence *has the limit* 1, and we write

$$\lim_{n\to\infty} \frac{n-1}{n} = 1.$$

It is helpful to visualize this behavior in the manner suggested by Fig. 13.1.

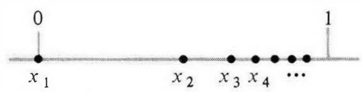


Figure 13.1

The general definition is as follows. A sequence $\{x_n\}$ is said to have a number L as *limit* if for each positive number ϵ there exists a positive integer n_0 with the property that

$$|x_n - L| < \epsilon \qquad \text{for all} \qquad n \ge n_0. \tag{1}$$

When L is related to $\{x_n\}$ in this way, we write

$$\lim_{n\to\infty} x_n = L, \qquad \text{or more briefly,} \qquad \lim x_n = L,$$

and we say that x_n *converges to L.* This is also expressed by saying that x_n *approaches L as n becomes infinite,* which we can write as

$$x_n \to L \qquad \text{as} \qquad n \to \infty.$$

This notation is often abbreviated even further, to $x_n \to L$.

This definition requires that each ϵ, no matter how small, must have at least one corresponding n_0 that "works" for it in the sense expressed by (1). In general, we expect that for smaller ϵ's, larger n_0's will be needed; that is, when the required measure of closeness is made smaller, we must go farther out in the sequence to satisfy it.

A sequence is said to *converge* or to be *convergent* if it has a limit. A convergent sequence cannot have two different limits, because it is not possible for x_n to be as close as we please to both of two different numbers for all sufficiently large n's.

A convergent sequence is bounded, but not all bounded sequences are convergent. The sequence 1, 0, 1, 0, . . . of Example 1(b) is a bounded sequence that is not convergent.

It is not always easy to decide whether a given sequence converges, and if it does, what its limit is. The following facts are often useful in problems of this kind: If $x_n \to L$ and $y_n \to M$, then

$$\lim (x_n + y_n) = L + M, \qquad \lim (x_n - y_n) = L - M, \qquad \lim x_n y_n = LM,$$

and, with the additional assumption that $M \neq 0$,

$$\lim \frac{x_n}{y_n} = \frac{L}{M}.$$

These facts can be rigorously proved by carefully using the definition and the properties of inequalities. We omit the details. By using these rules, we can easily perform such feats as calculating

$$\lim \frac{2n^3 + n - 5}{7n^3 - 2n^2 + 4} = \lim \frac{2 + 1/n^2 - 5/n^3}{7 - 2/n + 4/n^3} = \frac{2 + 0 - 0}{7 - 0 + 0} = \frac{2}{7},$$

where the essential first step is to divide both numerator and denominator by the highest power of n occurring in the denominator.

The usual intuitive idea of convergence—that $x_n \to L$ means that x_n can be made "as close as we please" to L by taking n "sufficiently large"—is natural and necessary, and is the way most mathematicians really think about this concept. Accordingly, in most of our work with sequences we shall rely on common sense to tell us how much detail is needed to make an argument convincing.

Example 2 If $|x| < 1$, then $\lim x^n = 0$. Most people are willing to accept this on the grounds that "a number numerically less than 1 which is raised to higher and higher powers gets smaller and smaller." But if a more detailed argument is desired, it can be given as follows. The assertion is clear if $x = 0$, so assume that $0 < |x| < 1$. Then $|x| = 1/(1 + a)$ for some $a > 0$, so by the binomial theorem we have

$$\frac{1}{|x^n|} = \frac{1}{|x|^n} = (1 + a)^n = 1 + na + \text{positive terms} > na.$$

We see from this that $|x^n| < 1/na$; and since $1/na \to 0$, we clearly have $x^n \to 0$.

Example 3 For every x, $\lim x^n/n! = 0$. This is not at all obvious, because even though $n!$ increases rapidly as n grows, for large values of x it is quite conceivable that x^n might grow even more rapidly. To demonstrate that $x^n/n! \to 0$ as $n \to \infty$, we may assume that $x > 0$ (why is this permissible?). To start the argument we choose a fixed positive integer m so large that $x/m < \frac{1}{2}$, and then we put $a = x^m/m!$. For any integer $n > m$ we write $n = m + k$ and observe that

$$0 < \frac{x^n}{n!} = a \cdot \frac{x}{m+1} \cdot \frac{x}{m+2} \cdots \frac{x}{m+k} < a\left(\frac{1}{2}\right)^k.$$

As $n \to \infty$, k also $\to \infty$, so $a(\frac{1}{2})^k \to 0$, and we conclude that $x^n/n! \to 0$.

Example 4 The fact that $\lim (\sqrt{n+1} - \sqrt{n}) = 0$ will probably seem reasonable after a little thought (it only says that $\sqrt{n}$ is nearly equal to $\sqrt{n+1}$ for large n), but a definitive argument may not be so easy to find. Such an argument can be constructed by writing the quantity $\sqrt{n+1}-\sqrt{n}$ as a fraction with denominator 1 and rationalizing the numerator, as follows:

$$\begin{aligned}\frac{\sqrt{n+1}-\sqrt{n}}{1} &= \frac{\sqrt{n+1}-\sqrt{n}}{1}\cdot\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} \\ &= \frac{1}{\sqrt{n+1}+\sqrt{n}} \to 0.\end{aligned}$$

In working with sequences in connection with infinite series, we will often need to be able to recognize that a sequence is convergent even though we know nothing about the numerical value of the limit. In such a case we cannot make any direct use of the definition of a limit. We now discuss a very important method for handling such situations.

A sequence $\{x_n\}$ is said to be *increasing* if

$$x_1 \le x_2 \le x_3 \le \cdots \le x_n \le x_{n+1} \le \cdots,$$

that is, if each term is greater than or equal to the one that precedes it.* Among the sequences listed in Example 1, (a), (d), (f), and (g) are clearly increasing; (h) is also increasing, but this is not obvious on inspection.

Increasing sequences are pleasant to work with because their convergence behavior is particularly easy to determine. We have the following simple criterion: *An increasing sequence converges if and only if it is bounded.* This criterion is not only simple, but also extremely important, because the theory of convergent series given in the rest of this chapter stems directly from it.

This criterion is quite easy to establish. Imagine the terms of the sequence plotted on the real line, as shown in Fig. 13.2, with each term to the right of (or on) its predecessor. If the sequence is unbounded, then its terms simply march off the page, and the sequence clearly cannot converge. This proves half of the criterion, the "only if" part. To establish the other half, we assume that the sequence is bounded with an upper bound B, as shown in the figure, and we must produce a limit for the sequence. Very briefly, we see geometrically that the x_n's, which move steadily to the right and yet cannot penetrate the barrier at B, must "pile up" at some point $L \le B$, so L is the limit of the sequence and the sequence converges to L.†

This convergence criterion has many important applications, one of which is given in the following example. For this we will need the formula for the sum of a geometric progression,

$$1 + x + x^2 + \cdots + x^{n-1} = \frac{1-x^n}{1-x}, \qquad x \ne 1. \tag{2}$$

This formula is merely a rearrangement of equation (4) in Section 13.1. It can also be established in another way, by dividing $x-1$ into x^n-1.

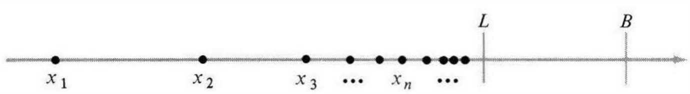


Figure 13.2 A bounded increasing sequence.

*Some writers require the terms of an increasing sequence to satisfy the strict inequality $x_n < x_{n+1}$ for all n. However, our definition allows an increasing sequence to be stationary, in the sense that adjacent terms may be equal.

†A much more detailed argument can be given here, but we have no wish to strain the patience of reasonable people. See Appendix A.1 for some of the details.

Example 5 Our purpose here is to prove that

$$\lim_{n\to\infty}\left(1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}\right)=e. \tag{3}$$

We accomplish this by discussing together the two closely related sequences $\{x_n\}$ and $\{y_n\}$ defined by

$$x_n=\left(1+\frac{1}{n}\right)^n \quad\text{and}\quad y_n=1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}.$$

We will demonstrate that both of these sequences are increasing and bounded, and therefore convergent, and furthermore that they converge to the same limit. Our first step is to show that $\{x_n\}$ is increasing and bounded. By the binomial theorem, x_n can be expressed as the following sum of $n+1$ terms:

$$\begin{aligned}x_n=\left(1+\frac{1}{n}\right)^n&=1+n\cdot\frac{1}{n}+\frac{n(n-1)}{2!}\cdot\frac{1}{n^2}+\frac{n(n-1)(n-2)}{3!}\cdot\frac{1}{n^3}+\cdots\\&\qquad+\frac{n(n-1)\cdots[n-(n-1)]}{n!}\cdot\frac{1}{n^n}\\&=1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\cdots\\&\qquad+\frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right).\end{aligned} \tag{4}$$

As we pass from x_n to x_{n+1} by replacing n by $n+1$, it is easy to see from this sum that each term after $1+1$ increases, and also that another term is added, so $x_n < x_{n+1}$. Further, a term-by-term comparison of (4) with y_n shows that $x_n \le y_n$. By applying formula (2), we see that the y_n's have 3 as an upper bound,

$$\begin{aligned}y_n&=1+1+\frac{1}{2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{2\cdot 3\cdots n}\\&\le 1+1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}}=1+2\left(1-\frac{1}{2^n}\right)<3,\end{aligned}$$

and therefore the x_n's also have 3 as an upper bound. Since $\{x_n\}$ is an increasing sequence with 3 as an upper bound, we know that it converges. Its limit is of course the number e, which was introduced in a somewhat different way in Section 8.3,

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e. \tag{5}$$

Since $x_n \le y_n < y_{n+1} < 3$, we see that $\{y_n\}$ is also a bounded increasing sequence which approaches a limit $y \ge e$. All that remains is to show that $y \le e$, because this will yield our main conclusion that $y=e$. The argument is as follows. If $m < n$ and we consider only the first $m+1$ terms of (4), then we have

$$\begin{aligned}&1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\cdots\\&\qquad+\frac{1}{m!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{m-1}{n}\right)<x_n<e.\end{aligned}$$

If m is held fixed and n is allowed to increase, then we obtain

$$y_m = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{m!} \leq e,$$

so $y \leq e$. We conclude from this that $y = e$, or

$$\lim_{n\to\infty}\left(1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!}\right) = e,$$

which is (3). We observe that

$$\lim_{n\to\infty}\left(1 - \frac{1}{n}\right)^n = \frac{1}{e}, \tag{6}$$

because this limit can be written as

$$\lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right)^{n+1} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{n+1} = \lim_{n\to\infty}\frac{n/(n+1)}{(1+1/n)^n} = \frac{1}{e}.$$

The additional fact that

$$\lim_{n\to\infty}\left(1 - \frac{1}{n}\right)^{-n} = e \tag{7}$$

is an immediate consequence of (6).

Example 6 Most students will recall that a *prime number*, or simply a *prime*, can be defined as an integer $p > 1$ that has no positive factors (or divisors) except 1 and p. These numbers form one of the most interesting of all sequences,

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, \ldots. \tag{8}$$

Indeed, the fact that there are infinitely many of them, so that they actually do constitute a sequence, is itself a famous theorem of number theory proved by Euclid. The sequence (8) is clearly not convergent, and it may appear that the concept of a convergent sequence has little or no relevance to the primes. However, this impression is quite wrong, for students who wish to pursue the subject will find that the convergence behavior of certain sequences is very close to the heart of the modern theory of prime numbers. We support this remark by stating without proof the following very profound theorem about the approximate size of the nth prime: If p_n denotes the nth prime number, then p_n is "asymptotically equal" to $n \ln n$, in the sense that

$$\lim_{n\to\infty}\frac{p_n}{n \ln n} = 1. \tag{9}$$

Readers who are interested in these matters are urged to take a course in number theory. More information about (9) can be found in Section B.16 of the author's book, *Calculus Gems* (McGraw-Hill, 1992).

PROBLEMS

1 State whether each of the indicated sequences converges or diverges, and if it converges, find its limit:

(a) $\sqrt[3]{n}$; (b) $\dfrac{1+(-1)^n}{n}$;

(c) $\sin\dfrac{\pi}{5n}$; (d) $\dfrac{10^{10^{10}}\sqrt{n}}{n+1}$;

(e) $\dfrac{3^n}{2^n + 10^{10}}$; (f) $\dfrac{\sqrt{n+2}}{2\sqrt{n}}$;

(g) $\ln(n+1) - \ln n$; (h) $\dfrac{n^2}{\sqrt{4n^4+5}}$;

(i) $\dfrac{1}{n} - \dfrac{1}{n+1}$; (j) $\cos n\pi$;

(k) $\cos \dfrac{(2n+1)^2\pi}{2}$; (l) $\dfrac{5n^3 - 2n}{n^4 + 3n^2 - 10}$;

(m) $n^{(-1)^n}$; (n) $\dfrac{\sqrt{n}\sin(n!e^n)}{n+1}$;

(o) $n \sin \dfrac{\pi}{n}$; (p) $\dfrac{(1-\sqrt{n})(3+\sqrt{n})}{4n+5}$.

2 Show that $n!/n^n \to 0$. Hint: Write it out, and look.

3 The limits of many sequences can be found by replacing the discrete variable n by a continuous variable x and applying L'Hospital's rule for the case $x \to \infty$. Use this method to show that

(a) $\dfrac{\ln n}{n} \to 0$; (b) $\sqrt[n]{n} \to 1$;

(c) if $|a| < 1$; then $na^n \to 0$;

(d) if k is any positive integer, then $n^k/e^n \to 0$;

(e) if a is any real number, then $(1 + a/n)^n \to e^a$.

4 State whether each of the indicated sequences converges or diverges, and if it converges, find its limit:

(a) $3^{3/n}$; (b) $e^{-10/n}$;

(c) $n/2^n$; (d) $\dfrac{\ln(n+1)}{n}$;

(e) $n^2/3^n$; (f) $n^{1/(n+1)}$;

(g) $(n+10)^{1/(n+10)}$; (h) $n^2 \sin n\pi$;

(i) $n^2 \cos n\pi$.

5 Find $\lim x_n$ if

(a) $x_n = \sqrt{n}(\sqrt{n+a} - \sqrt{n})$;

(b) $x_n = n\left[\left(a + \dfrac{1}{n}\right)^4 - a^4\right]$.

6 If $0 < a < b$, show that $\lim \sqrt[n]{a^n + b^n} = b$.

7 If $f(x) = \lim_{n\to\infty} (2/\pi)\tan^{-1} nx$, show that $f(x) = x/|x|$ if $x \neq 0$ and $f(0) = 0$. Sketch the graph of this function.

8 If the terms of a sequence $\{x_n\}$ are positive numbers, show that:

(a) the sequence is increasing if $x_{n+1}/x_n \geq 1$ for all n;

(b) the sequence is decreasing if $x_{n+1}/x_n \leq 1$ for all n.[†]

9 Use Problem 8 to show that $\lim x_n$ exists if

(a) $x_n = \dfrac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}$;

(b) $x_n = \dfrac{1}{n^2}\left[\dfrac{2 \cdot 4 \cdot 6 \cdots (2n)}{1 \cdot 3 \cdot 5 \cdots (2n-1)}\right]$;

(c) $x_n = \dfrac{1}{n}\left[\dfrac{2 \cdot 4 \cdot 6 \cdots (2n)}{1 \cdot 3 \cdot 5 \cdots (2n-1)}\right]^2$.

10 Find the value of

(a) $\lim \dfrac{(n+1)^n}{n^{n+1}}$;

(b) $\lim \dfrac{(n+1)\ln n - n\ln(n+1)}{\ln n}$.

11 Show that

$$\left[\frac{n+1}{n^2} + \frac{(n+1)^2}{n^3} + \cdots + \frac{(n+1)^n}{n^{n+1}}\right] \to e - 1.$$

12 Show that

(a) $\left(1 + \dfrac{1}{2n+3}\right)^{2n+3} \to e$;

(b) $\left(1 + \dfrac{1}{n^2}\right)^{n^2} \to e$; (c) $\left(1 + \dfrac{1}{n}\right)^{2n} \to e^2$;

(d) $\left(1 + \dfrac{1}{n^2}\right)^{n} \to 1$; (e) $\left(1 + \dfrac{1}{2n}\right)^{n} \to \sqrt{e}$.

*13 Consider a suitable number of circles of equal size packed in n rows inside an equilateral triangle, as shown in Fig. 13.3. If c_n denotes the number of these circles, then it is clear from the geometry of the situation that $c_1 = 1$, $c_2 = 1 + 2$, $c_3 = 1 + 2 + 3$, and so on. If A is the area of the triangle and A_n is the combined area of the c_n circles, show that

$$\lim_{n\to\infty} \frac{A_n}{A} = \frac{\pi}{2\sqrt{3}}.$$

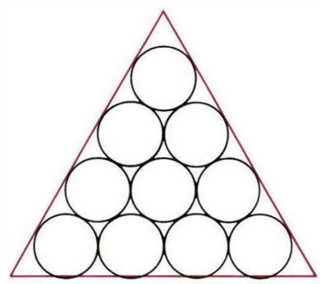


Figure 13.3

14 The sequence

$$1,\ 1 + \tfrac{1}{2},\ 1 + \frac{1}{2 + \frac{1}{2}},\ \ldots$$

can be defined recursively by $x_1 = 1$, $x_{n+1} = 1 + \dfrac{1}{1 + x_n}$ for $n \geq 1$. Assuming that $\lim x_n = L$ exists, show that $L = \sqrt{2}$. This gives the *continued fraction expansion*

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}.$$

[†]Naturally, a sequence $\{x_n\}$ is said to be *decreasing* if

$$x_1 \geq x_2 \geq x_3 \geq \cdots \geq x_n \geq x_{n+1} \geq \cdots,$$

i.e., if each term is less than or equal to the one that precedes it.

13.3 CONVERGENT AND DIVERGENT SERIES

Most people are familiar with the fact that

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots = 2. \tag{1}$$

However, since we cannot add infinitely many numbers in the same way that we can add finitely many, the meaning of (1) is evidently quite different from the meaning of a statement like

$$1 + 2 + 3 + 4 = 10.$$

What (1) really means is that the sequence of partial sums on the left, that is, the sequence of numbers

$$\begin{aligned}
&1,\\
&1 + \tfrac{1}{2} = 1\tfrac{1}{2},\\
&1 + \tfrac{1}{2} + \tfrac{1}{4} = 1\tfrac{3}{4},\\
&1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} = 1\tfrac{7}{8},\\
&\quad \ldots,
\end{aligned}$$

converges to the number 2 on the right. This suggests the approach we adopt for the general case.

If $a_1, a_2, \ldots, a_n, \ldots$ is a sequence of numbers, then the expression

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots \tag{2}$$

is called an *infinite series,* or simply a *series,* and the a_n's are called its *terms.* We emphasize that until a meaning is assigned to it by a suitable definition, the expression (2) is merely a formal collection of symbols arranged in a certain way, because the indicated operation of adding infinitely many numbers has no meaning in itself. To attach a numerical value to (2) in a natural and useful way, as suggested in the preceding paragraph, we form the sequence of *partial sums*

$$\begin{aligned}
s_1 &= a_1,\\
s_2 &= a_1 + a_2,\\
&\ldots\\
s_n &= a_1 + a_2 + \cdots + a_n,\\
&\ldots.
\end{aligned}$$

The series (2) is said to *converge,* or to be *convergent,* if the sequence $\{s_n\}$ converges; and if $\lim s_n = s$, then we say that the series *converges to* s or that s is the *sum* of the series, and we express this by writing

$$a_1 + a_2 + \cdots + a_n + \cdots = s \qquad \text{or} \qquad \sum_{n=1}^{\infty} a_n = s.$$

If the series does not converge, then we say that it *diverges* or is *divergent,* and no sum is assigned to it.

At this point a few remarks about notation and usage are in order. As we indicated, the statement that the series $\sum_{n=1}^{\infty} a_n$ converges to the sum s is usually written $\sum_{n=1}^{\infty} a_n = s$. Thus, the notation $\sum_{n=1}^{\infty} a_n$ is used with a dual meaning: to specify a series regardless of convergence or divergence, and also (if the series converges) to denote its sum. Students will find that this ambiguity causes no difficulty in practice.

Another matter concerns the indexing (or numbering) of the terms. It is often more natural to number the terms of a series beginning with $n = 0$; that is, we write some series in the form

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + \cdots + a_n + \cdots$$

(and in this case we also write $s_n = a_0 + a_1 + \cdots + a_n$). For example, the series on the left of (1) can be written as

$$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} \qquad \text{or} \qquad \sum_{n=0}^{\infty} \frac{1}{2^n},$$

but the latter form is somewhat neater. It is a triviality that any general statement about series written as $\sum_{n=1}^{\infty} a_n$ has an exact analog for series written as $\sum_{n=0}^{\infty} a_n$. For this reason, when no ambiguity is likely or when the distinction is immaterial, we often omit the limits of summation and for the sake of simplicity write $\sum a_n$ instead of $\sum_{n=1}^{\infty} a_n$ or $\sum_{n=0}^{\infty} a_n$. These remarks also apply to series of the form $\sum_{n=k}^{\infty} a_n$ for any integer $k \geq 2$.

We now briefly consider several fundamental examples.

Example 1 Probably the simplest and most important of all infinite series is the familiar *geometric series*

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots. \tag{3}$$

By a slight alteration of equation (2) in Section 13.2 (replace n by $n + 1$), the nth partial sum of this series is given by the closed formula

$$s_n = 1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x} \tag{4}$$

if $x \neq 1$. If $|x| < 1$, we see from this that $s_n \to 1/(1 - x)$, so for these x's we have

$$1 + x + x^2 + \cdots + x^n + \cdots = \frac{1}{1 - x}. \tag{5}$$

The series (3) therefore converges to the sum $1/(1 - x)$ for $|x| < 1$, and is readily seen to diverge for all other values of x. This answers the question raised in part B of Section 13.1.

It is now easy to understand the full meaning of formula (1): the series on the left is a geometric series with $x = \frac{1}{2}$, so by (5) we have

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} + \cdots = \frac{1}{1 - \frac{1}{2}} = 2.$$

Similarly,

$$1 + \frac{2}{5} + \frac{4}{25} + \cdots + \left(\frac{2}{5}\right)^n + \cdots = \frac{1}{1 - \frac{2}{5}} = \frac{5}{3}$$

and

$$1 - \frac{2}{3} + \frac{4}{9} - \cdots + \left(-\frac{2}{3}\right)^n + \cdots = \frac{1}{1 - (-\frac{2}{3})} = \frac{3}{5}.$$

Further, if we write the repeating decimal 0.333 . . . as an infinite series and apply (5) at the right stage, then we get the result mentioned in Section 13.1,

$$\begin{aligned} 0.333\ldots &= \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots \\ &= \frac{3}{10}\left[1 + \frac{1}{10} + \left(\frac{1}{10}\right)^2 + \cdots\right] \\ &= \frac{\frac{3}{10}}{1 - \frac{1}{10}} = \frac{3}{10}\left(\frac{10}{9}\right) = \frac{1}{3}. \end{aligned}$$

Example 2 Another series whose behavior is particularly simple is

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots = 1.$$

To establish convergence and verify that the sum is 1, we use an ingenious trick due to Leibniz and observe that

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

This enables us to write the nth partial sum as

$$\begin{aligned} s_n &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1}, \end{aligned}$$

which makes it obvious that $s_n \to 1$. Any series whose nth partial sum collapses in this way into a closed formula is called a *telescopic series.*

As these examples suggest, the most direct method for studying the convergence behavior of a series is to find a closed formula for its nth partial sum. The main disadvantage of this approach is that it rarely works (first the good news, then the bad news!), because it is usually impossible to find such a formula. It is this situation that forces us to rely mostly on various indirect methods for establishing the convergence or divergence of series.

The main indirect method rests on the convergence criterion for sequences discussed in Section 13.2, that is, on the fact that an increasing sequence converges if and only if it is bounded. Thus, if the terms of our series are all nonnegative numbers, then we clearly have $s_n \le s_n + a_{n+1} = s_{n+1}$ for every n, and therefore the s_n's form an increasing sequence. It follows in this case that the sequence $\{s_n\}$ of partial sums—*and with it the series*—converges if and only if the s_n's have an upper bound. Our next example furnishes a good illustration of the use of this simple but important idea.

Example 3 The *harmonic series*

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots \tag{6}$$

diverges because its partial sums are unbounded, as we saw in Example 1(g) at the beginning of Section 13.2. To establish this in a bit more detail, let m be a positive integer and choose $n > 2^{m+1}$. Then

$$s_n > 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2^{m+1}}$$

$$= \left(1 + \frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \cdots + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^m + 1} + \cdots + \frac{1}{2^{m+1}}\right)$$

$$> \frac{1}{2} + 2 \cdot \frac{1}{4} + 4 \cdot \frac{1}{8} + \cdots + 2^m \cdot \frac{1}{2^{m+1}} = (m+1)\frac{1}{2}.$$

This proves that s_n can be made larger than the sum of any number of $\frac{1}{2}$'s and therefore as large as we please, by taking n large enough, so the s_n's are unbounded and (6) diverges. A series that behaves in this way is often said to *diverge to infinity,* and we express this behavior by writing

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots = \infty.$$

A great many interesting series—some convergent and others divergent—can be obtained from the harmonic series by thinning it out, that is, by deleting terms according to a systematic pattern. For instance, if we remove all terms except reciprocals of powers of 2, what remains is the convergent geometric series

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \cdots;$$

and if we remove all terms except reciprocals of primes, then—as we shall see in a later section—the resulting series diverges,

$$\sum \frac{1}{p_n} = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \cdots = \infty.$$

The simplest general principle that is useful in deciding whether a series converges or not is the *nth term test*:

If the series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots$$

converges, then $a_n \to 0$ as $n \to \infty$; or equivalently, if a_n does not approach zero as $n \to \infty$, then the series must necessarily diverge.

To prove this, we merely observe that $a_n = s_n - s_{n-1} \to s - s = 0$. This result shows that $a_n \to 0$ is a necessary condition for convergence, in the sense that it follows from the convergence of the series Σa_n. Unfortunately, however, it is not a sufficient condition; that is, it does not imply the convergence of the series. This is easy to see by considering the harmonic series $\Sigma 1/n$, which diverges even though $1/n \to 0$. The nth term test is essentially a divergence test. As examples of its use, we mention the series

$$\sum_{n=1}^{\infty} (-1)^{n+1} = 1 - 1 + 1 - 1 + \cdots$$

and

$$\sum_{n=1}^{\infty} \frac{n}{n+1} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots .$$

The first diverges because the sequence $(-1)^{n+1}$ does not converge at all, and so cannot converge to zero, and the second diverges because $n/(n+1) \to 1 \neq 0$.

Remark *Repeating decimals.* The procedure for converting any rational number a/b (in lowest terms) into its decimal expansion is well known: Divide b into a. Let us carry out this procedure in the case of the rational number $\frac{22}{7}$, which is often used as the simplest rational approximation to π, correct to two decimal places:

$$\begin{array}{r l}
 & \;3.142857142857\ldots \\
7 & \overline{)22.0000000\ldots\ldots} \\
 & \underline{21} \\
 & \;\;1\,0 \\
 & \;\;\;\underline{7} \\
 & \;\;\;30 \\
 & \;\;\;\underline{28} \\
 & \;\;\;\;20 \\
 & \;\;\;\;\underline{14} \\
 & \;\;\;\;\;60 \\
 & \;\;\;\;\;\underline{56} \\
 & \;\;\;\;\;\;40 \\
 & \;\;\;\;\;\;\underline{35} \\
 & \;\;\;\;\;\;\;50 \\
 & \;\;\;\;\;\;\;\underline{49} \\
 & \;\;\;\;\;\;\;\;10 \cdots .
\end{array}$$

The successive remainders here are 1, 3, 2, 6, 4, 5, 1; as soon as 1 appears a second time, the cycle begins all over again and generates the repeating block of digits 142857. This example illustrates—and almost proves—the fact that *the decimal expansion of any rational number is repeating.** The proof of this general statement consists of little more than noticing the phenomena displayed in the example. When b is divided into a, the remainder at each stage is one of the numbers $0, 1, 2, \ldots, b-1$. Since there are only a finite number of possible values for these remainders, some remainder necessarily appears a second time, and the division process repeats from that point on to give a repeating decimal. We also note that if the remainder 0 appears, then the decimal terminates, but a terminating decimal can always be thought of as repeating, as in $0.25 = 0.25000\ldots.$

The converse of this statement is also true: *Any repeating decimal is the expansion of a rational number.* To see why this is so, let us examine a typical repeating decimal, say $3.7222\ldots.$ If we split off the nonrepeating part, write the repeating part using powers of 10, and use formula (5) at the proper stage, then we obtain

*Certain rational numbers have two distinct decimal expansions, e.g., $\frac{1}{4} = 0.25000\ldots = 0.24999\ldots.$ This situation is analyzed in Problems 14 and 15 following.

$$\begin{aligned}3.7222\ldots &= \frac{37}{10} + 0.0222\ldots = \frac{37}{10} + \frac{2}{10^2} + \frac{2}{10^3} + \frac{2}{10^4} + \cdots \\ &= \frac{37}{10} + \frac{2}{100}\left[1 + \frac{1}{10} + \left(\frac{1}{10}\right)^2 + \cdots\right] = \frac{37}{10} + \frac{\frac{2}{100}}{1 - \frac{1}{10}} = \frac{37}{10} + \frac{2}{100}\cdot\frac{10}{9} \\ &= \frac{37}{10} + \frac{2}{90} = \frac{335}{90} = \frac{67}{18},\end{aligned}$$

which is a rational number expressed as a fraction in lowest terms. It is evident that a similar procedure works equally well for any repeating decimal, so the statement at the beginning of this paragraph is clearly true.

We can summarize our results by saying that *the rational numbers are exactly those real numbers whose decimal expansions are repeating*. Equivalently, *the irrational numbers are exactly those real numbers whose decimal expansions are nonrepeating.*

PROBLEMS

1 There is nothing to prevent us from forming the geometric series $1 + x + x^2 + \cdots$ for any real number x. Show that this series diverges whenever $|x| \geq 1$.

2 Determine whether each of the following geometric series is convergent or divergent, and if convergent find its sum:

(a) $1 + \frac{1}{4} + \frac{1}{16} + \cdots$; (b) $9 + 3 + 1 + \cdots$;
(c) $2 + \frac{3}{2} + \cdots$; (d) $\frac{1}{2} + \frac{1}{3} + \cdots$;
(e) $\frac{1}{4} - \frac{1}{20} + \cdots$; (f) $\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n$;
(g) $\sum_{n=0}^{\infty}\left(-\frac{4}{5}\right)^n$; (h) $\sum_{n=0}^{\infty}\left(\frac{5\sqrt{2}}{7}\right)^n$;
(i) $\sum_{n=0}^{\infty}\left(\frac{8}{5\sqrt{3}}\right)^n$; (j) $\sum_{n=0}^{\infty}\frac{1}{(3-\sqrt{5})^n}$;
(k) $\sum_{n=0}^{\infty}7\left(-\frac{4}{7}\right)^n$; (l) $\sum_{n=0}^{\infty}\frac{2^n}{5^{n/2}}$.

3 Show that formula (4) is essentially equivalent to the following factorization formula of elementary algebra:

$$x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + \cdots + x + 1).$$

Give an independent verification of this factorization formula.

4 A certain rubber ball is dropped from a height of H ft. Each time it bounces it rises to a height rh, where h is the height of the previous bounce. Find the total distance the ball travels.

5 Which of the following series are convergent and which are divergent?

(a) $\sin \pi + \sin 2\pi + \cdots + \sin n\pi + \cdots$.
(b) $\sin\frac{\pi}{2} + \sin\frac{2\pi}{2} + \cdots + \sin\frac{n\pi}{2} + \cdots$.
(c) $\cos \pi + \cos 2\pi + \cdots + \cos n\pi + \cdots$.
(d) $\cos\frac{\pi}{2} + \cos\frac{3\pi}{2} + \cdots + \cos\frac{(2n-1)\pi}{2} + \cdots$.
(e) $\ln\sqrt{3} + \ln\sqrt[3]{3} + \ln\sqrt[4]{3} + \cdots + \ln\sqrt[n]{3} + \cdots$.
(f) $\ln\sqrt{3} + \ln\sqrt[4]{3} + \ln\sqrt[8]{3} + \cdots + \ln\sqrt[2^n]{3} + \cdots$.
(g) $\frac{1}{10+3} + \frac{2}{10+6} + \frac{3}{10+9} + \cdots + \frac{n}{10+3n} + \cdots$.
(h) $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots$.

6 Convert each of the following repeating decimals into a fraction (in lowest terms):

(a) $0.777\ldots$; (b) $0.151515\ldots$;
(c) $0.639639639\ldots$; (d) $2.3070707\ldots$.

7 If a and b are digits, show that

(a) $0.aaa\ldots = \frac{a}{9}$; (b) $0.ababab\ldots = \frac{10a+b}{99}$.

8 The decimal $0.101001000100001\ldots$, in which the 1's are followed by successively longer chains of 0's, looks as if it is nonrepeating, and therefore defines an irrational number. Construct an argument that converts this impression into a certainty. Hint: Assume that the decimal is repeating.

9 (The fly problem) Two bicyclists start 20 mi apart and head toward each other, each pedaling at a steady 10 mi/h. At the same time a fly traveling 40 mi/h starts from the front wheel of one bicycle and flies to the front wheel of the other, then turns around and flies back to the front wheel of the first, and continues back and forth in this manner until the bicycles collide and he is crushed between the wheels. How far has the fly flown? The hard way to solve this problem is to express the total distance as an infinite series and find its sum. There is also an easy way. Do it both ways.*

10 Find the sum of each of the following series:

(a) $\frac{1+4}{9}+\frac{1+8}{27}+\frac{1+16}{81}+\cdots;$

(b) $18-6+2-\frac{2}{3}+\cdots;$

(c) $\frac{\sin\theta}{2}+\frac{\sin^2\theta}{4}+\frac{\sin^3\theta}{8}+\cdots;$

(d) $\frac{1}{2+x^2}+\frac{1}{(2+x^2)^2}+\frac{1}{(2+x^2)^3}+\cdots.$

11 Describe all convergent series of integers.

12 In the series $\frac{1}{3}-\frac{2}{5}+\frac{3}{7}-\frac{4}{9}+\cdots$, the numerators are the successive positive integers, the denominators are the successive odd numbers starting with 3, and the signs alternate.
(a) Write the series using the sigma notation.
(b) Show that the series diverges.

13 Express each of the following numbers as a repeating decimal:
(a) $\frac{3}{5}$; (b) $\frac{5}{3}$; (c) $\frac{27}{25}$; (d) $\frac{27}{24}$; (e) $\frac{27}{26}$.

14 Show that a positive rational number a/b (in lowest terms) has a terminating decimal expansion if and only if the positive integer b has the form $b = 2^m5^n$, where the exponents m and n are nonnegative integers. Check this statement against the results of Problem 13.

*There is a famous anecdote about the first time one of the most brilliant scientists of the twentieth century heard this problem, and how he reacted to it. See P. R. Halmos, "The Legend of John von Neumann," *Amer. Math. Monthly*, **80** (1973), pp. 386–387.

15 A terminating decimal such as $\frac{3}{8} = 0.375 = 0.375000\ldots$ can also be written as a repeating decimal ending in an infinite chain of 9's if the last nonzero digit is decreased by one unit, as in $0.375000\ldots = 0.374999\ldots$. Prove this by using formula (5).

16 Show that the number

$$0.1234567891011121314151617\ldots,$$

in which all the positive integers are written down in order after the decimal point, is irrational.

17 Unrestrained deficit spending by the federal government inflates the nation's money supply and leads in directions that politicians prefer not to think about. However, much of the money spent by the government is spent in turn by those receiving it, those receiving this twice-spent money spend some of it in their turn, and so on indefinitely. This produces a chain reaction that economists call the *multiplier effect*, and results in much greater total spending (and therefore total income) than the government's original expenditure. Suppose the original expenditure is E dollars, and that each recipient of spent or respent money spends $100c$ percent of it and saves $100s$ percent. The numbers c and s are called the *propensity to consume* and the *propensity to save;* both numbers are between 0 and 1, and $c + s = 1$, since all money is either spent or saved. In this way the income of the entire country is increased by kE dollars, where the factor k is called the *multiplier*. These basic concepts of macroeconomics were introduced by the English economist John Maynard Keynes (1883–1946). Show that $k = 1/s > 1$. For example, if $c = 0.9$ and $s = 0.1$, then $k = 10$, and for every \$1 spent by the government, the national income is increased by \$10.

13.4 GENERAL PROPERTIES OF CONVERGENT SERIES

It is interesting to observe that both the harmonic series

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$$

and the geometric series

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$$

have positive terms that decrease toward zero, and yet the first diverges while the second converges. This suggests the subtlety we will encounter as we penetrate further into the study of infinite series.

One of the attractions of our subject is that it offers so many results that stir the imagination and stimulate curiosity. For instance, it seems reasonably clear from the preceding observation that a series of positive terms will converge if its terms decrease "rapidly enough." This is true. For example, we next see that the divergent harmonic series can be made to converge by squaring the positive integers in the denominators, which makes the terms themselves smaller.

Example 1 The series of the reciprocals of the squares,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots$$
$$= 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots,$$

converges with sum ≤ 2. This follows at once from the fact that the partial sums form an increasing sequence with 2 as an upper bound:

$$\begin{aligned} s_n &= 1 + \frac{1}{2\cdot 2} + \frac{1}{3\cdot 3} + \cdots + \frac{1}{n\cdot n} \\ &< 1 + \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{(n-1)n} \\ &= 1 + \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) \\ &= 2 - \frac{1}{n} < 2. \end{aligned}$$

Further, it is known that the sum of this series is $\pi^2/6$:

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots = \frac{\pi^2}{6}.$$

This very remarkable fact was discovered by Euler in 1736. His method of discovery (which uses the power series expansion of $\sin x$—also discovered by him) is described in Appendix 1 at the end of this chapter.

Another way to change the harmonic series into a convergent series is to change the signs of alternate terms. This produces a series whose sum (another astonishing fact!) is ln 2:

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots = \ln 2.$$

In Section 13.1 we obtained this formula from the power series expansion of the function $\ln(1 + x)$, which will not be solidly established until Chapter 14. However, in Section 13.6 it will be possible to give an easy rigorous proof by an entirely different method.

Example 2 If we recall that $0! = 1$ and $1! = 1$, then it is clear that the series

$$\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

has partial sums $s_0 = 1$, $s_1 = 2$, and, for $n \ge 2$,

$$s_n = 1 + 1 + \frac{1}{2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{2\cdot 3\cdots n}.$$

If each factor in the denominators is replaced by 2, then we see that

$$\begin{aligned} s_n &\le 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} \\ &= 1 + 2\left(1 - \frac{1}{2^n}\right) = 3 - \frac{1}{2^{n-1}} < 3, \end{aligned}$$

so the series converges with sum ≤ 3. By Example 5 in Section 13.2 we know that the sum of this series is actually e:

$$\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots = e. \tag{1}$$

We will use this fact below to prove that e is an irrational number.

These examples and those in the previous section provide a small supply of specific series of known convergence behavior, where this behavior is decidable by rather elementary means. The value of these familiar series for determining the behavior of new series by various methods of comparison will begin to appear in Section 13.5. First, however, there are several simple properties of convergent series in general that need to be mentioned explicitly.

The effective use of infinite series rests on our freedom to manipulate them by the various processes of algebra. However, we will soon see that carelessness can easily lead to confusion and disaster. It is therefore of prime importance to know exactly which operations are permissible and which are traps for the unwary.

If $\Sigma_{n=1}^{\infty} a_n$ converges to s, we write

$$a_1 + a_2 + \cdots + a_n + \cdots = s \tag{2}$$

and call s the "sum" of the series. This well-established terminology is perhaps unfortunate, for it tends to foster the belief that an infinite series can be treated as if it were an ordinary finite sum. In reality, of course, s is not obtained simply by addition, but is the limit of a sequence of finite sums, and the properties of series must be based on this definition and not on any tempting but misleading analogy. As we shall see, many properties of finite sums do carry over to series, but we must always be careful not to assume this without proof, because some do not.

As an example of the pitfalls that lie around us, consider the familiar fact that rearranging the order of the terms of a finite sum has no effect on the numerical value of that sum. In contrast to this, in Problem 10 we ask students to see for themselves that the sum of a convergent infinite series can be altered by writing its terms—exactly the same terms!—in a different order. This astounding (and fascinating) behavior illustrates the need for caution. It also emphasizes the delicacy of the concepts we are working with, and gives us fair warning that we cannot hope to study infinite series successfully without giving a reasonable amount of attention to the underlying theory.

We begin by pointing out that in dealing with finite sums we can freely insert or remove parentheses, as in the expressions

$$1 - 1 + 1 = (1 - 1) + 1 = 1 - (1 - 1) = 1,$$

but this is not true for infinite series. For instance, the series $1 - 1 + 1 - 1 + \cdots$ clearly diverges, but

$$(1 - 1) + (1 - 1) + \cdots = 0 + 0 + \cdots$$

converges to 0, and

$$1 - (1 - 1) - (1 - 1) - \cdots = 1 - 0 - 0 - \cdots$$

converges to 1.* These examples show that the insertion or removal of parentheses can change the nature of an infinite series. However, in the case of a convergent series like (2), any series obtained from it by inserting parentheses, such as

$$a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6) + \cdots,$$

still converges and has the same sum. The reason for this is that the partial sums of the new series form a subsequence of the original sequence of partial sums, and therefore necessarily converge to the same limit. In the same way, we see that parentheses can be removed if the resulting series converges.

We next remark that if $a_1 + a_2 + \cdots$ converges to s, then $a_1 + 0 + a_2 + 0 + \cdots$ also converges and has the same sum, because the two sequences of partial sums are $s_1, s_2, \ldots$ and $s_1, s_1, s_2, s_2, \ldots$, and the repetitions in the latter do not interfere with its convergence to s. Similarly, any finite number of 0's can be inserted or removed anywhere in a series without affecting its convergence behavior or (if it converges) its sum.

It is important to observe that when two convergent series are added term by term, the resulting series converges to the expected sum; that is, if $\Sigma_{n=1}^{\infty} a_n = s$ and $\Sigma_{n=1}^{\infty} b_n = t$, then $\Sigma_{n=1}^{\infty}(a_n + b_n) = s + t$. This is easy to prove, for if s_n and t_n are the partial sums, then

$$\begin{aligned}(a_1 + b_1) + (a_2 + b_2) + \cdots + (a_n + b_n)& \\ &= (a_1 + a_2 + \cdots + a_n) + (b_1 + b_2 + \cdots + b_n) \\ &= s_n + t_n \to s + t.\end{aligned}$$

Similarly, $\Sigma_{n=1}^{\infty}(a_n - b_n) = s - t$ and $\Sigma_{n=1}^{\infty} ca_n = cs$ for any constant c. It is also convenient to know that if

$$a_1 + a_2 + \cdots = s,$$

then

$$a_0 + a_1 + a_2 + \cdots = a_0 + s \qquad \text{and} \qquad a_2 + a_3 + \cdots = s - a_1.$$

The first statement is clear from the fact that

$$\lim (a_0 + a_1 + a_2 + \cdots + a_n) = \lim a_0 + \lim (a_1 + a_2 + \cdots + a_n) = a_0 + s,$$

and the second follows in the same way. Thus, any finite number of terms can be added or subtracted at the beginning of a convergent series without disturbing its convergence, and the sums of the various series are related in the expected ways.

We now use several of the properties of series discussed above to prove the following theorem of Euler: *e is irrational.*

*Let us rearrange these numbers into the bizarre calculation

$$\begin{aligned}0 = 0 + 0 + 0 + \cdots &= (1 - 1) + (1 - 1) + (1 - 1) + \cdots \\ &= 1 - 1 + 1 - 1 + 1 - 1 + \cdots \\ &= 1 - (1 - 1) - (1 - 1) - \cdots \\ &= 1 - 0 - 0 - \cdots \\ &= 1.\end{aligned}$$

Guidobaldo del Monte (1545–1607), patron and friend of Galileo, thought that this result proves the existence of God, because "something has been created out of nothing."

Our starting point is equation (1),

$$e = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} + \cdots,$$

from which it follows that the number

$$e - 1 - 1 - \frac{1}{2!} - \cdots - \frac{1}{n!} = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots \tag{3}$$

is positive for every positive integer n. We assume that e is rational, so that $e = p/q$ for certain positive integers p and q, and we deduce a contradiction from this assumption. Let n in (3) be chosen so large that $n > q$, and define a number a by

$$a = n!\left[e - 1 - 1 - \frac{1}{2!} - \cdots - \frac{1}{n!}\right].$$

Since q divides $n!$, a is a positive integer. However, (3) implies that

$$\begin{aligned} a &= n!\left[\frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots\right] \\ &= \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \cdots \\ &< \frac{1}{n+1} + \frac{1}{(n+1)^2} + \cdots \\ &= \frac{1}{n+1}\left[1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \cdots\right] \\ &= \frac{1}{n+1} \cdot \frac{1}{1 - 1/(n+1)} = \frac{1}{n}. \end{aligned}$$

This contradiction (there is no positive integer $< 1/n$) completes the argument.

Further information about irrational numbers (π is irrational, etc.) is given in Appendix 2 at the end of this chapter.

PROBLEMS

1 If Σa_n converges and Σb_n diverges, show that $\Sigma(a_n + b_n)$ diverges. Hint: Assume that it converges and deduce a contradiction.

2 Decide whether each of the following series converges or diverges, and give convincing reasons for your answers:

(a) $\dfrac{1}{500} + \dfrac{1}{505} + \dfrac{1}{510} + \cdots;$

(b) $\sum\left[\dfrac{2}{n} - \left(\dfrac{3}{4}\right)^n\right];$ (c) $\sum\left(\dfrac{1}{3^n} + \dfrac{1}{4^n}\right);$

(d) $\sum\left[\dfrac{2}{n(n+1)} - \dfrac{100}{n!}\right];$ (e) $\sum 2^{-1/n};$

(f) $\sum \dfrac{1}{\ln 2^n};$ (g) $\sum \dfrac{1}{2n^2};$

(h) $\sum \cos\dfrac{(2n+1)\pi}{2};$ (i) $\sum \cos\dfrac{n\pi}{4}.$

3 For each of the following series, find the values for x for which the series converges and express the sum as a simple function of x:

(a) $ax + ax^3 + ax^5 + \cdots,\ a \neq 0;$

(b) $\dfrac{1}{x} + \dfrac{1}{x^2} + \dfrac{1}{x^3} + \cdots;$

(c) $x + \dfrac{x}{1+x} + \dfrac{x}{(1+x)^2} + \cdots;$

(d) $\ln x + (\ln x)^2 + (\ln x)^3 + \cdots.$

4 Show that

$$x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \cdots$$

converges for all x and find its sum.

5 Show that $\Sigma\, 1/e^n$ converges but $\Sigma\, 1/(e^{\ln n})$ diverges. For what values of x does Σe^{nx} converge?

6 Show that

(a) $\displaystyle\sum_{n=1}^{\infty} [\tan^{-1}(n+1) - \tan^{-1} n] = \pi/4$;

(b) $\displaystyle\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n}\right) = \infty$.

7 If $f(n) \to L$, show that

$$\sum_{n=1}^{\infty} [f(n) - f(n+1)] = f(1) - L$$

and use this to establish the indicated sums of the following telescopic series:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} = \frac{1}{2}$;

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{2n+1}{n(n+1)} = 1$;

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2} = 1$;

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(4n-1)(4n+3)} = \frac{1}{12}$.

8 It follows from Example 1 that the series

$$\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \cdots,$$

where the denominators are the squares of the successive primes, converges. Why?

9 A *decimal* $a_0.a_1a_2 \ldots a_n \ldots$ is simply an abbreviated way of writing the infinite series

$$\sum_{n=0}^{\infty} \frac{a_n}{10^n} = a_0 + \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots,$$

where it is understood that a_0 is an arbitrary integer and each of the a_n's for $n \geq 1$ is one of the digits 0, 1, 2, . . . , 9. Show that every decimal converges.

10 Consider the series

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \tfrac{1}{7} - \tfrac{1}{8} + \cdots = \ln 2, \quad (*)$$

and write under it, as follows, the result of multiplying through by the factor $\frac{1}{2}$:

$$\tfrac{1}{2} \qquad - \tfrac{1}{4} \qquad + \tfrac{1}{6} \qquad - \tfrac{1}{8} + \cdots = \tfrac{1}{2}\ln 2.$$

Now add, combining the terms placed in vertical columns, to obtain the series

$$1 + \tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{5} + \tfrac{1}{7} - \tfrac{1}{4} + + - \cdots = \tfrac{3}{2}\ln 2. \quad (**)$$

Satisfy yourself (a) that $(**)$ is valid; (b) that the series can be produced by rearranging the terms of the series $(*)$, so that the first two positive terms of $(*)$ are followed by the first negative term, then the next two positive terms by the second negative term, etc; and (c) that the value of the sum of the series $(*)$ has been mysteriously multiplied in this way by the factor $\frac{3}{2}$.†

***11** In this problem we ask the student to give a solid proof of the validity of the power series expansions

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \quad (*)$$

and

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots. \quad (**)$$

The machinery consists of the familiar formulas

$$\int_0^x \sin t\, dt = 1 - \cos x \qquad \text{and} \qquad \int_0^x \cos t\, dt = \sin x,$$

and also the following general property of definite integrals:

$$\text{if } a < b \text{ and } f(x) \leq g(x), \text{ then } \int_a^b f(x)\, dx \leq \int_a^b g(x)\, dx.$$

(a) In the inequality $\cos x \leq 1$, replace x by t and integrate both sides of $\cos t \leq 1$ from 0 to a fixed positive number x to obtain

$$\sin x \leq x.$$

(b) In the same way, use the result of part (a) to obtain

$$1 - \cos x \leq \frac{x^2}{2} \qquad \text{or} \qquad \cos x \geq 1 - \frac{x^2}{2}.$$

(c) In the same way, use the result of part (b) to obtain

$$\sin x \geq x - \frac{x^3}{3!}.$$

(d) By continuing this process indefinitely, generate the two sets of inequalities

$$\begin{array}{ll}
\sin x \leq x & \cos x \leq 1 \\[6pt]
\sin x \geq x - \dfrac{x^3}{3!} & \cos x \geq 1 - \dfrac{x^2}{2!} \\[10pt]
\sin x \leq x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} & \cos x \leq 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} \\[10pt]
\sin x \geq x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} & \cos x \geq 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} \\[6pt]
\cdots & \cdots.
\end{array}$$

†This phenomenon will be explored from a different point of view, and we hope clarified, in Sections 13.6 and 13.8.

(e) To complete the proofs of (*) and (**) for the positive value of x under consideration, show that it suffices to know that $x^n/n! \to 0$ as $n \to \infty$, which is Example 3 in Section 13.2

(f) Finally, show that (*) and (**) are also valid for $x < 0$.

13.5 SERIES OF NONNEGATIVE TERMS. COMPARISON TESTS

The easiest infinite series to work with are those whose terms are all nonnegative numbers. The reason for this—as we saw in Section 13.3—is that the total theory of these series can be expressed by the following simple statement: *If $a_n \geq 0$, then the series Σa_n converges if and only if its sequence $\{s_n\}$ of partial sums is bounded.*

Thus, in order to establish the convergence of a series of nonnegative terms, it suffices to show that its terms approach zero fast enough to keep the partial sums bounded. But how fast is "fast enough"? One answer to this question can be stated informally as follows: at least as fast as the terms of a known convergent series of nonnegative terms. This idea is contained in a formal statement called the *comparison test*: *If $0 \leq a_n \leq b_n$, then*

Σa_n converges if Σb_n converges;
Σb_n diverges if Σa_n diverges.

The proof is easy. The first step is to notice that if s_n and t_n are the partial sums of Σa_n and Σb_n, then the assumption yields

$$s_n = a_1 + a_2 + \cdots + a_n \leq b_1 + b_2 + \cdots + b_n = t_n.$$

Our conclusion now follows at once from this inequality and the statement in the preceding paragraph, for if the t_n's are bounded, then so are the s_n's, and if the s_n's are unbounded, then so are the t_n's.

Example 1 The comparison test is easy to apply to the series

$$\sum_{n=1}^{\infty} \frac{1}{3^n + 1} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{\ln n}.$$

The first series converges, because

$$\frac{1}{3^n + 1} \leq \frac{1}{3^n}$$

and $\Sigma\, 1/3^n$ converges; and the second diverges, because

$$\frac{1}{n} \leq \frac{1}{\ln n} \tag{1}$$

and $\Sigma 1/n$ diverges. [To verify (1) in the equivalent form $\ln n \leq n$, recall that the graph of $y = \ln x$ lies below the graph of $y = x$.]

It is worth remarking here that we can disregard any finite number of terms at the beginning of a series if we are interested only in deciding whether that series converges or diverges.* This tells us that the condition $0 \leq a_n \leq b_n$ for the comparison test need not hold for all n, but only for all n from some point on.

*On the other hand, if we are interested in the sum of a convergent series, then obviously we must take all of its terms into account.

As an illustration, suppose we want to show that $\Sigma(n+1)/n^n$ converges by comparison with $\Sigma 1/n^2$. The inequality

$$\frac{n+1}{n^n} \le \frac{1}{n^2}$$

is not true for all n, but it is true for all $n \ge 4$. The series therefore converges by comparison with the convergent series $\Sigma 1/n^2$.

The comparison test is very simple in principle, but in complicated cases it can be difficult to establish the necessary inequality between the nth terms of the two series being compared. Since limits are often easier to work with than inequalities, the following *limit comparison test* is a more convenient tool for studying many series: *If Σa_n and Σb_n are series with positive terms such that*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1, \tag{2}$$

then either both series converge or both series diverge. To establish this, we observe that (2) implies that for all sufficiently large n we have

$$\frac{1}{2} \le \frac{a_n}{b_n} \le 2$$

or

$$\tfrac{1}{2} b_n \le a_n \le 2b_n. \tag{3}$$

Since the convergence behavior of a series is not affected by multiplying each of its terms by the same nonzero constant, our conclusion is an easy consequence of the inequalities (3) and the comparison test as extended in the preceding paragraph. Thus, for instance, if Σb_n converges, then $\Sigma 2b_n$ converges and, by the second inequality in (3), Σa_n also converges; etc.

Example 2 The series $\Sigma(n+2)/(2n^3-3)$ converges, because $\Sigma 1/2n^2$ converges and

$$\frac{(n+2)/(2n^3-3)}{1/2n^2} = \frac{2n^3+4n^2}{2n^3-3} \to 1 \qquad \text{as} \qquad n \to \infty;$$

and $\Sigma \sin(1/n)$ diverges, because $\Sigma 1/n$ diverges and

$$\frac{\sin 1/n}{1/n} \to 1 \qquad \text{as} \qquad n \to \infty.$$

[Recall that $\lim_{x\to 0} (\sin x)/x = 1$; see Section 9.2.]

The limit comparison test is slightly more convenient to use if condition (2) is replaced by

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L,$$

where $0 < L < \infty$. The proof is essentially the same and will not be repeated.

Example 2 shows that in using the limit comparison test we must try to guess the probable behavior of Σa_n by estimating the "order of magnitude" of the nth term a_n. That is, we must try to judge whether a_n is approximately equal to a constant multiple of the nth term of some familiar series whose convergence behavior is known to us, such as

$$\sum x^n, \quad \sum \frac{1}{n}, \quad \sum \frac{1}{n^2}, \quad \text{or} \quad \sum \frac{1}{n!}.$$

To apply this method effectively, it is clearly desirable to have at our disposal a "stockpile" of comparison series of known behavior. Our next example provides a family of series that is especially valuable for this purpose. We emphasize once again that the limit comparison test is used *only if the terms of the series being tested are all positive numbers.*

Example 3 If p is a positive constant, then the *p-series*

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots \tag{4}$$

diverges if $p \le 1$ and converges if $p > 1$.

To establish this, we notice first that if $p \le 1$, then $n^p \le n$ or $1/n \le 1/n^p$, so (4) diverges by comparison with the harmonic series $\Sigma 1/n$. We now prove that (4) converges if $p > 1$ by showing that its partial sums have an upper bound. Let n be given and choose m so that $n < 2^m$. Then

$$\begin{aligned} s_n \le s_{2^m - 1} &= 1 + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \cdots + \frac{1}{7^p}\right) \\ &\qquad + \cdots + \left[\frac{1}{(2^{m-1})^p} + \cdots + \frac{1}{(2^m - 1)^p}\right] \\ &\le 1 + \frac{2}{2^p} + \frac{4}{4^p} + \cdots + \frac{2^{m-1}}{(2^{m-1})^p}. \end{aligned}$$

If we put $a = 1/2^{p-1}$, then $a < 1$ since $p > 1$, and

$$s_n \le 1 + a + a^2 + \cdots + a^{m-1} = \frac{1 - a^m}{1 - a} < \frac{1}{1 - a}.$$

This provides an upper bound for the s_n's, and the argument is complete.

As an illustration of the use of this family of series, we see that

$$\sum \frac{1}{\sqrt{n^3 + 3}}$$

converges, because the p-series (with $p = \frac{3}{2}$) $\Sigma 1/n^{3/2}$ converges and

$$\frac{1/\sqrt{n^3 + 3}}{1/n^{3/2}} = \sqrt{\frac{n^3}{n^3 + 3}} \to 1.$$

It is worth noticing that $\Sigma 1/n^p$ does not necessarily converge if p is a variable $>$ 1. This is shown by the series

$$\sum \frac{1}{n^{1+1/n}},$$

which diverges because $\Sigma 1/n$ diverges and

$$\frac{1/n^{1+1/n}}{1/n} = \frac{1}{\sqrt[n]{n}} \to 1.$$

[Recall that $\lim_{n\to\infty} n^{1/n} = 1$ by Problem 3(b) in Section 13.2.]

We conclude this section with some observations on the process of rearranging the terms of a series, which was briefly discussed in Section 13.4. Suppose that Σa_n is a convergent series of nonnegative terms whose sum is s, and form a new series Σb_n by rearranging the a_n's in any way. For instance, Σb_n might be the series

$$a_{10} + a_3 + a_5 + a_1 + a_6 + a_2 + \cdots.$$

Let n be a given positive integer and consider the nth partial sum $t_n = b_1 + b_2 + \cdots + b_n$ of the new series. Since each b is some a, there exists an m with the property that each term in t_n is one of the terms in $s_m = a_1 + a_2 + \cdots + a_m$. This tells us that $t_n \le s_m \le s$, so Σb_n converges to a sum $t \le s$. On the other hand, the first series is also a rearrangement of the second, so by the same reasoning we have $s \le t$, and therefore $t = s$. This proves that *if a convergent series of nonnegative terms is rearranged in any manner, then the resulting series also converges and has the same sum.* If this conclusion seems rather obvious and trivial to students, let them recall from Problem 10 in Section 13.4 that it isn't true if we drop the assumption that the terms of the given series are nonnegative numbers.

PROBLEMS

1 Establish the convergence or divergence of the following series by using the comparison test:

(a) $\sum \frac{1}{\sqrt{n(n+1)}}$; (b) $\sum \frac{1}{\sqrt{n^2(n+1)}}$;

(c) $\sum \frac{1}{n^n}$; (d) $\sum \frac{1}{(\ln n)^n}$;

(e) $\sum \frac{1}{n^{\ln n}}$; (f) $\sum \frac{n+1}{n(n-1)}$;

(g) $\sum \frac{(2n+3)^n}{n^{2n}}$; (h) $\sum \left(\frac{n}{n+1}\right)^{n^2}$.

Determine by any method whether each of the following series converges or diverges.

2 $\sum \frac{3}{n^2+1}$.

3 $\sum \frac{1+3n^2}{n^3+700}$.

4 $\sum \sin \frac{1}{n^2}$.

5 $\sum \cos \frac{1}{n^2}$.

6 $\sum \frac{1}{3^n+9}$.

7 $\sum \frac{1}{(1+1/n)^n}$.

8 $\sum \frac{\sqrt{n}}{n^2+5}$.

9 $\sum \frac{\ln n}{n}$.

10 $\sum \frac{3n+2}{n} \cdot \frac{4^n}{5^n+1}$.

11 $\sum \frac{1}{n+\sqrt{n}}$.

12 $\sum \frac{\ln n}{n^3}$.

13 $\sum \frac{1000}{\sqrt[3]{n+1}\,\sqrt[4]{n^3+5}}$.

14 $\sum \frac{n^2}{n^2+100}$.

15 $\sum \frac{1}{n10^n}$.

16 $\sum \frac{1}{5000n}$.

17 $\sum \frac{n^2+3n-7}{n^3-2n+5}$.

18 $\sum \frac{\sqrt[3]{n+2}}{\sqrt[4]{n^3+3}\,\sqrt[5]{n^3+5}}$.

19 $\sum \frac{n^2}{n^5-\pi}$.

20 $\sum \frac{3+\cos n}{n^2}$.

21 $\sum \ln(1+1/n^p),\ p>0$.

22 $\sum \frac{\sqrt{n+1}-\sqrt{n}}{n}$.

23 $\sum (1-\cos 1/n)$.

24 $\sum \sqrt{n}\ln \frac{n+1}{n}$.

25 $\sum \frac{1}{n^p}\left(1+\frac{1}{2^p}+\cdots+\frac{1}{n^p}\right),\ p>0$.

26 If Σa_n is a convergent series with nonnegative terms, show that $\Sigma a_n{}^2$ also converges. With the same hypotheses, show by examples that $\Sigma\sqrt{a_n}$ is sometimes convergent and sometimes divergent.

27 If p is a positive constant, show that

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^p} = 1 + \frac{1}{3^p} + \frac{1}{5^p} + \cdots$$

converges if $p > 1$ and diverges if $p \le 1$.

28 Show that $\Sigma 1/n$ diverges by comparing it with the divergent series $\Sigma \ln(1 + 1/n)$ of Problem 6(b) in Section 13.4. Hint: Compare the graphs of the functions $y = x$ and $y = \ln(1 + x)$ for $x > 0$.

29 Use the idea of Problem 28 to show that

$$\sum_{n=1}^{\infty} \ln \frac{(n+1)^2}{n(n+2)}$$

converges. Also, find the sum of this series.

13.6 THE INTEGRAL TEST. EULER'S CONSTANT

Among the simplest infinite series are those whose terms form a decreasing sequence of positive numbers. In this section we study certain series of this type by means of improper integrals of the form

$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx. \tag{1}$$

We recall that the integral on the left is said to be *convergent* if the limit on the right exists (as a finite number), and in this case the value of the integral is by definition the value of the limit. If this limit does not exist, then the integral is called *divergent*. There is an obvious analogy between (1) and the corresponding definition for series,

$$\sum_{n=1}^{\infty} a_n = \lim_{k\to\infty} \sum_{n=1}^{k} a_n.$$

Our purpose is to exploit this analogy by using integrals to obtain information about series.

Consider a series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots \tag{2}$$

whose terms are positive and decreasing. In most cases the nth term a_n is a function of n given by a simple formula, $a_n = f(n)$. Suppose that the function $y = f(x)$ obtained by substituting the continuous variable x in place of the discrete variable n is a decreasing function of x for $x \geq 1$, as shown in Fig. 13.4. On the left in this figure we see that the rectangles of areas $a_1, a_2, \ldots, a_n$ have a greater combined area than the area under the curve from $x = 1$ to $x = n + 1$, so

$$a_1 + a_2 + \cdots + a_n \geq \int_1^{n+1} f(x)\,dx \geq \int_1^n f(x)\,dx. \tag{3}$$

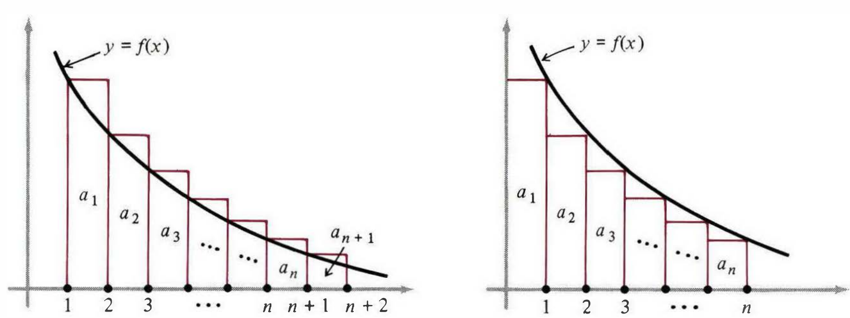


Figure 13.4

On the right side of the figure we make the rectangles face to the left, so that they lie under the curve. If we momentarily ignore the first rectangle, with area a_1, then we see that

$$a_2 + a_3 + \cdots + a_n \leq \int_1^n f(x)\,dx;$$

and including a_1 gives

$$a_1 + a_2 + \cdots + a_n \leq a_1 + \int_1^n f(x)\,dx. \tag{4}$$

By combining (3) and (4), we obtain

$$\int_1^n f(x)\,dx \leq a_1 + a_2 + \cdots + a_n \leq a_1 + \int_1^n f(x)\,dx. \tag{5}$$

The point of all this is that the inequalities (5) enable us to establish the *integral test*:

If $f(x)$ is a positive decreasing function for $x \geq 1$ with the property that $f(n) = a_n$ for each positive integer n, then the series and integral

$$\sum_{n=1}^{\infty} a_n \qquad \text{and} \qquad \int_1^{\infty} f(x)\,dx$$

*converge or diverge together.**

The argument is easy, for if the series converges, then the inequality on the left of (5) shows that the integral does also; and if the integral converges, then the inequality on the right shows that the series also converges.

Example 1 *The p-series revisited.* If p is a positive constant, then we know from Section 13.5 that the p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots \tag{6}$$

converges if $p > 1$ and diverges if $p \leq 1$. It is of some interest to give another proof of this as an illustration of the integral test. Since $a_n = 1/n^p$, we consider the function $f(x) = 1/x^p$ (which clearly satisfies all the stated conditions) and examine the integral

$$\int_1^{\infty} \frac{dx}{x^p}.$$

If $p = 1$, this integral diverges, because

$$\int_1^{\infty} \frac{dx}{x} = \lim_{b\to\infty} \int_1^b \frac{dx}{x} = \lim_{b\to\infty} \ln b = \infty.$$

*This test is often called the *Cauchy integral test*, after its discoverer, the eminent nineteenth-century French mathematician Augustin Louis Cauchy (pronounced "Ko-shee"). In mathematical productivity Cauchy (1789–1857) was surpassed only by Euler, and his collected works fill 27 fat volumes. He was a prolific contributor to number theory, algebra, and many branches of physics. However, his most important achievements were in the field of analysis. Together with his contemporaries Gauss and Abel, he was a pioneer in the rigorous treatment of limits, continuous functions, derivatives, integrals, infinite series, and differential equations.

If $p \neq 1$, then

$$\int_1^\infty \frac{dx}{x^p} = \lim_{b\to\infty} \int_1^b x^{-p}\,dx = \lim_{b\to\infty} \left(\frac{b^{1-p}-1}{1-p}\right),$$

and the issue of convergence hangs on the behavior of b^{1-p} as $b \to \infty$. If $p < 1$, so that $1 - p > 0$, then $b^{1-p} \to \infty$ and the integral diverges. If $p > 1$, so that $1 - p < 0$, then $b^{1-p} \to 0$ and the integral converges. By the integral test we now conclude again that the p-series (6) converges if $p > 1$ and diverges if $p \leq 1$.

It is clear that the integral test holds for any interval of the form $x \geq k$, not just for $x \geq 1$. We make use of this remark in our next example, which deals with a class of series whose behavior is not revealed by any of our previous tests.

Example 2 The terms of the series

$$\sum_{n=2}^\infty \frac{1}{n \ln n} \tag{7}$$

decrease faster than those of the harmonic series. Nevertheless, it is easy to see by the integral test that (7) diverges, for

$$\int_2^\infty \frac{dx}{x\ln x} = \lim_{b\to\infty} \int_2^b \frac{dx}{x\ln x} = \lim_{b\to\infty} [\ln \ln x]_2^b$$

$$= \lim_{b\to\infty} (\ln \ln b - \ln \ln 2) = \infty.$$

More generally, if p is a positive constant, then

$$\sum_{n=2}^\infty \frac{1}{n(\ln n)^p}$$

converges if $p > 1$ and diverges if $p \leq 1$; for if $p \neq 1$, we have

$$\int_2^\infty \frac{dx}{x(\ln x)^p} = \lim_{b\to\infty} \int_2^b \frac{dx}{x(\ln x)^p} = \lim_{b\to\infty} \left[\frac{(\ln x)^{1-p}}{1-p}\right]_2^b$$

$$= \lim_{b\to\infty} \left[\frac{(\ln b)^{1-p} - (\ln 2)^{1-p}}{1-p}\right],$$

and this limit exists if and only if $p > 1$.*

We now return to the series (2) and squeeze some additional information out of the inequalities (5). By subtracting the integral that occurs on the left, these inequalities can be written as

$$0 \leq a_1 + a_2 + \cdots + a_n - \int_1^n f(x)\,dx \leq a_1, \tag{8}$$

*The series of this example are called *Abel's series,* after the great Norwegian mathematician Niels Henrik Abel, who first investigated them and determined their convergence behavior. Abel (1802–1829) died of tuberculosis at the age of 26 before the publication of his many brilliant discoveries made him world famous among mathematicians. His most memorable achievement was his proof (mentioned on p. 106) of the impossibility of solving the general fifth-degree equation by means of radicals. He also contributed to the rigorous theory of infinite series, and his discovery of elliptic and other transcendental functions launched a new era in mathematical analysis.

and this serves to focus our attention on the quantity in the middle. If we denote this quantity by $F(n)$ so that

$$F(n) = a_1 + a_2 + \cdots + a_n - \int_1^n f(x)\,dx,$$

then (8) becomes

$$0 \le F(n) \le a_1.$$

From our present point of view, the key to this situation is the fact that $\{F(n)\}$ is a decreasing sequence. This follows from the calculation

$$\begin{aligned} F(n) - F(n+1) &= \left[a_1 + a_2 + \cdots + a_n - \int_1^n f(x)\,dx\right] \\ &\qquad - \left[a_1 + a_2 + \cdots + a_{n+1} - \int_1^{n+1} f(x)\,dx\right] \\ &= \int_n^{n+1} f(x)\,dx - a_{n+1} \ge 0, \end{aligned}$$

where the reason for the last-written inequality can be understood by examining the left side of Fig. 13.4. Since any decreasing sequence of nonnegative numbers converges, the limit

$$L = \lim_{n\to\infty} F(n) = \lim_{n\to\infty}\left[a_1 + a_2 + \cdots + a_n - \int_1^n f(x)\,dx\right] \tag{9}$$

exists and satisfies the inequalities $0 \le L \le a_1$.

As our main application of these ideas, we deduce the existence of the important limit

$$\lim_{n\to\infty}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n\right). \tag{10}$$

This is easily seen to be the special case of (9) in which $a_n = 1/n$ and $f(x) = 1/x$, because

$$\int_1^n \frac{dx}{x} = \ln x\,\Big]_1^n = \ln n.$$

The value of the limit (10) is usually denoted by the Greek letter γ (*gamma*), and is called *Euler's constant*:

$$\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n\right). \tag{11}$$

This constant occurs quite frequently in several parts of advanced calculus, especially in the theory of the gamma function, and is, along with π and e, one of the most important special numbers of mathematics. Its numerical value, $\gamma = 0.57721\ 56649\ 01532\ 86060\ldots$, has been calculated to many hundreds of decimal places. Nevertheless, no one knows whether γ is rational or irrational.

In order to describe some of the uses of Euler's constant, it is convenient to introduce a notation which has been widely accepted in twentieth-century mathematics. Let $\{a_n\}$ and $\{b_n\}$ be two sequences, and suppose that $b_n > 0$. We say that "a_n is little-oh of b_n," and symbolize this by writing

$$a_n = o(b_n),$$

if $a_n/b_n \to 0$. In particular, $a_n = o(1)$ means that $a_n \to 0$. An equation of the form $a_n = b_n + o(1)$ means that $a_n - b_n = o(1)$, so a_n and b_n differ by a quantity that approaches zero as $n \to \infty$. In our work we will use the symbol $o(1)$ to mean any sequence that approaches zero as $n \to \infty$, as in the calculation $[a + o(1)] + 2[b + o(1)] = a + 2b + o(1)$.

With the aid of this notation, (11) can be written in the form

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} = \ln n + \gamma + o(1). \tag{12}$$

Since $\ln n \to \infty$ as $n \to \infty$, this formula displays in a very transparent way the reason for the divergence of the harmonic series. It is also useful for many other purposes, as the following examples show.

Example 3 We can use (12) to give a simple proof of the formula

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2. \tag{13}$$

Let s_n be the nth partial sum of this series, and observe that

$$\begin{aligned}
s_{2n} &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n} \\
&= \left(1 + \frac{1}{3} + \cdots + \frac{1}{2n-1}\right) - \left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}\right) \\
&= \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2n}\right) - 2\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}\right) \\
&= \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2n}\right) - \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) \\
&= [\ln 2n + \gamma + o(1)] - [\ln n + \gamma + o(1)] \\
&= \ln 2 + o(1) \to \ln 2.
\end{aligned}$$

The odd partial sums approach the same limit, because

$$s_{2n+1} = s_{2n} + \frac{1}{2n+1} \to \lim s_{2n} = \ln 2,$$

so the proof of (13) is complete. We emphasize that this method establishes (13) on the basis of (12) alone, without making any use of the power series expansion of $\ln (1 + x)$ as given in Section 13.1.

Example 4 We can also use (12) to obtain the remarkable formula

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + + - \cdots = \frac{3}{2} \ln 2, \tag{14}$$

which was the subject of Problem 10 in Section 13.4. The method is similar to that of Example 3. If s_n is the nth partial sum of (14), then, since $2n$ is the nth even number and $2n - 1$ is the nth odd number, we have

$$s_{3n} = \left(1 + \frac{1}{3} - \frac{1}{2}\right) + \left(\frac{1}{5} + \frac{1}{7} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{4n-3} + \frac{1}{4n-1} - \frac{1}{2n}\right)$$
$$= \left(1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots + \frac{1}{4n-1}\right) - \left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}\right)$$
$$= \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{4n}\right) - \left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{4n}\right) - \left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}\right)$$
$$= \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{4n}\right) - \frac{1}{2}\left(1 + \frac{1}{2} + \cdots + \frac{1}{2n}\right) - \frac{1}{2}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)$$
$$= [\ln 4n + \gamma + o(1)] - \tfrac{1}{2}[\ln 2n + \gamma + o(1)] - \tfrac{1}{2}[\ln n + \gamma + o(1)]$$
$$= \ln 4n - \frac{1}{2}\ln 2n^2 + o(1) = \ln \frac{4n}{\sqrt{2}\,n} + o(1)$$
$$= \ln 2^{3/2} + o(1) \to \ln 2^{3/2} = \tfrac{3}{2}\ln 2.$$

It is easy to see that the partial sums

$$s_{3n+1} = s_{3n} + \frac{1}{4n+1} \qquad \text{and} \qquad s_{3n+2} = s_{3n} + \frac{1}{4n+1} + \frac{1}{4n+3}$$

approach the same limit, so the proof of (14) is complete.

Remark The basic idea of the integral test is to compare sums with integrals by looking at their geometric meanings in terms of areas. This idea can also be used to prove the divergence of the series of the reciprocals of the primes, as mentioned in Section 13.3:

$$\sum \frac{1}{p_n} = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \cdots = \infty.$$

This proof is a bit complicated; and since it is not essential to the main line of thought in this chapter, we place it in Appendix 3 at the end of the chapter.

PROBLEMS

Use the integral test to determine whether each of the following series converges or diverges.

1 $\displaystyle\sum_{n=1}^{\infty} \frac{n}{e^{n^2}}$. **2** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. **3** $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$.

4 $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$. **5** $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^4+1}$. **6** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n+1}$.

7 $\displaystyle\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{1+n^2}$. **8** $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$.

9 (a) The series $\sum_{n=3}^{\infty} (\ln n)/n$ diverges by comparison with the harmonic series, since

$$\frac{1}{n} \le \frac{\ln n}{n}$$

for $n \ge 3$. Establish this divergence by means of the integral test.

(b) If p is a positive constant, show that $\sum_{n=3}^{\infty} (\ln n)/n^p$ converges if $p > 1$ and diverges if $p \le 1$.

10 (a) Use the integral test to show that the series $\sum_{n=1}^{\infty} n/e^n$ converges.

(b) What is the sum of the series in (a)? Hint: Assume that the geometric series $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots = 1/(1-x)$ can legitimately be differentiated term by term on the interval $-1 < x < 1$.

11 The curve in Fig. 13.5 is the graph of $y = 1/x$. Convince yourself that the combined area of all the infinitely many shaded regions is Euler's constant γ. By inspecting the

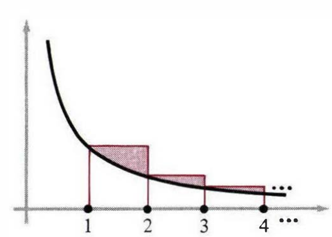


Figure 13.5

figure, show that the value of γ is between $\frac{1}{2}$ and 1, and is only slightly larger than $\frac{1}{2}$.

12 Use (12) to show that

$$1 + \tfrac{1}{2} - \tfrac{2}{3} + \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{2}{6} + + - \cdots = \ln 3.$$

13 If $\{x_n\}$ is the sequence defined by

$$x_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n},$$

then $x_n \to \ln 2$ because

$$x_n = \frac{1}{n} \cdot \frac{1}{1 + 1/n} + \frac{1}{n} \cdot \frac{1}{1 + 2/n} + \cdots + \frac{1}{n} \cdot \frac{1}{1 + n/n}$$
$$\to \int_0^1 \frac{dx}{1+x} = \ln 2.$$

Establish this fact by using formula (12).

14 The harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

diverges very slowly. To grasp *how* slowly, use (12) to show that in order to get s_n to exceed
(a) 10, we must add about 12,000 terms;
(b) 20, we must add about 272,405,000 terms;
(c) 184, we must add about 4.56×10^{79} terms.
The last number is somewhat larger than the estimated total number of elementary particles in the entire universe, which is about 2.36×10^{79}.*

*The English astronomer Sir Arthur Eddington believed he had shown that the number of these particles is precisely $\frac{3}{2} \times 136 \times 2^{256}$, whose value is about 2.36×10^{79}. Most modern astronomers think Eddington's number is nonsense; on the other hand, few of them (if any) seem able to criticize his ideas—so in the time-honored tradition, they dismiss him as irrelevant. More detail on these matters can be found in Sir Edmund Whittaker's *From Euclid to Eddington* (Dover, 1958).

13.7 THE RATIO TEST AND ROOT TEST

In the case of the geometric series Σr^n with $r > 0$, the ratio a_{n+1}/a_n of the $(n+1)$st term to the nth term has the constant value r, since

$$\frac{a_{n+1}}{a_n} = \frac{r^{r+1}}{r^n} = r. \tag{1}$$

We know that this series converges if $r < 1$, essentially because for these r's, condition (1) guarantees that the terms decrease rapidly. Analogy leads us to expect that any series Σa_n of positive terms will also converge if the ratio a_{n+1}/a_n is small for large n, even though this ratio may not have a constant value.

These ideas are made precise in the *ratio test*:

If Σa_n is a series of positive terms such that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L, \tag{2}$$

then

(a) *if $L < 1$, the series converges*;
(b) *if $L > 1$, the series diverges*;
(c) *if $L = 1$, the test is inconclusive.*

To establish (a), we assume that $L < 1$ and choose any number r between L and 1 so that $L < r < 1$. Then the meaning of (2) tells us that there exists an n_0 such that $a_{n+1}/a_n \le r$ for all $n \ge n_0$, which is equivalent to

$$\frac{a_{n+1}}{a_n} \le \frac{r^{n+1}}{r^n} = r \qquad \text{or} \qquad \frac{a_{n+1}}{r^{n+1}} \le \frac{a_n}{r^n} \qquad \text{for } n \ge n_0.$$

This says that the sequence $\{a_n/r^n\}$ is decreasing for $n \ge n_0$; in particular, $a_n/r^n \le a_{n_0}/r^{n_0}$ for $n \ge n_0$. Thus, if we put $K = a_{n_0}/r^{n_0}$, then we have

$$a_n \le Kr^n \qquad \text{for } n \ge n_0. \tag{3}$$

But ΣKr^n converges because $r < 1$, and therefore, by the comparison test, (3) implies that Σa_n converges. To prove (b), we simply observe that $L > 1$ implies that $a_{n+1}/a_n \ge 1$, or equivalently $a_{n+1} \ge a_n$, from some point on, so a_n cannot approach zero, and by the nth term test we know that the series diverges. Part (c) says that if $L = 1$, then no conclusion can be drawn, that is, sometimes the series converges and sometimes it diverges. To demonstrate this, we consider the p-series $\Sigma 1/n^p$. It is clear that for all values of p we have

$$\frac{a_{n+1}}{a_n} = \frac{n^p}{(n+1)^p} = \left(\frac{n}{n+1}\right)^p \to 1,$$

and yet this series converges if $p > 1$ and diverges if $p \leq 1$.

The ratio test is especially useful for handling series whose nth term a_n is given by a formula that involves various products, since even though a_n itself may be complicated, the ratio a_{n+1}/a_n can often be simplified by cancellations.

Example 1 We know that the series

$$\sum_{n=0}^{\infty} \frac{1}{n!}$$

converges by the argument given in Section 13.4. The ratio test yields the same conclusion much more easily, because

$$\begin{aligned} L = \lim \frac{a_{n+1}}{a_n} &= \lim \frac{1/(n+1)!}{1/n!} \\ &= \lim \frac{n!}{(n+1)!} = \lim \frac{1}{n+1} = 0. \end{aligned}$$

Since $L < 1$, the series converges.

Students should notice our use of the equation $(n + 1)! = (n + 1)n!$ in this example, because this fact will often be needed in our future work.

Example 2 In the case of the series

$$\sum_{n=0}^{\infty} \frac{3^n}{n!}, \tag{4}$$

it is easy to see that

$$L = \lim \frac{a_{n+1}}{a_n} = \lim \frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n} = \lim \frac{3}{n+1} = 0.$$

Again we have $L < 1$, so the series converges.

Since the nth term of any convergent series approaches zero, and therefore the convergence of (4) tells us that $3^n/n! \to 0$, we know that $n!$ increases faster than 3^n as $n \to \infty$. Students should try to develop an intuitive feeling for the relative rates of growth of expressions like these as an aid in forming quick but reliable judgments about the probable behavior of series. In this connection we point out here that the numerator 3^n of the nth term of series (4) contributes the 3 to the numerator of the ratio a_{n+1}/a_n after simplification, and that the $n!$ in the denominator contributes the $n + 1$ to the denominator of this ratio.

Example 3 For the series

$$\sum_{n=1}^{\infty} \frac{n^{10}}{3^n},$$

we have

$$L = \lim \frac{a_{n+1}}{a_n} = \lim \frac{(n+1)^{10}}{3^{n+1}} \cdot \frac{3^n}{n^{10}}$$

$$= \lim \left(1 + \frac{1}{n}\right)^{10} \cdot \frac{1}{3} = \frac{1}{3}.$$

Again we have $L < 1$, so the series converges by the ratio test.

In this example the convergence of the series tells us that 3^n grows faster than n^{10}, and we see from the calculation of L that the series behaves like the geometric series with $r = \frac{1}{3}$. We also observe that the polynomial factor n^{10} contributes the factor 1 to the calculation of L, so no such polynomial factor ever has any effect on the outcome of the ratio test.

Example 4 The remark preceding Example 1 is illustrated with special clarity by the series

$$\frac{1}{2} + \frac{1 \cdot 3}{2 \cdot 5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 5 \cdot 8} + \cdots + \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 5 \cdot 8 \cdots (3n-1)} + \cdots.$$

Here the cancellation of factors yields

$$L = \lim \frac{a_{n+1}}{a_n} = \lim \frac{1 \cdot 3 \cdots (2n-1)(2n+1)}{2 \cdot 5 \cdots (3n-1)(3n+2)} \cdot \frac{2 \cdot 5 \cdots (3n-1)}{1 \cdot 3 \cdots (2n-1)}$$

$$= \lim \frac{2n+1}{3n+2} = \frac{2}{3},$$

and the series converges because $L < 1$.

We now discuss the so-called *root test,* which is another convenient tool for studying the convergence behavior of series.

Suppose that Σa_n is a series of nonnegative terms with the property that from some point on we have

$$a_n \le r^n, \qquad \text{where } 0 < r < 1. \tag{5}$$

The geometric series Σr^n clearly converges, so Σa_n also converges by the comparison test. The fact that the inequalities (5) can be written in the form

$$\sqrt[n]{a_n} \le r < 1 \tag{6}$$

brings us to a convenient statement of the *root test*:

If Σa_n is a series of nonnegative terms such that

$$\lim_{n\to\infty} \sqrt[n]{a_n} = L, \tag{7}$$

then

(a) *if $L < 1$, the series converges;*
(b) *if $L > 1$, the series diverges;*
(c) *if $L = 1$, the test is inconclusive.*

The proof rests on the preceding remarks. For (a), if $L < 1$ and r is any number such that $L < r < 1$, then the meaning of (7) tells us that (6) holds for all sufficiently large n's, so Σa_n converges. For (b), if $L > 1$, then $\sqrt[n]{a_n} \geq 1$ from some point on, so $a_n \geq 1$ for all sufficiently large n's, and the series diverges because a_n does not approach zero. Finally, we establish (c) by observing that $L = 1$ for both the divergent series $\Sigma\, 1/n$ and the convergent series $\Sigma\, 1/n^2$, since $\sqrt[n]{n} \to 1$ as $n \to \infty$.

Example 5 In the case of the series

$$\sum \frac{1}{(\ln n)^n},$$

we have

$$L = \lim \sqrt[n]{a_n} = \lim \frac{1}{\ln n} = 0.$$

Since $L < 1$, the series converges.

In general, it is clear that the root test is most likely to be useful for treating series in which a_n is complicated but $\sqrt[n]{a_n}$ is simple, so that $\lim \sqrt[n]{a_n}$ is easy to compute. However, the practical value of the root test is outweighed by its theoretical significance, and this appears mainly in the advanced theory of power series.

Remark 1 The ratio test and the root test were first stated and correctly proved by Cauchy in 1821, as part of the earliest satisfactory exposition of the basic concepts of the theory of series.

Remark 2 We have seen that the ratio test is inconclusive when $\lim a_{n+1}/a_n = 1$, but this is far from the end of the story. If $a_{n+1}/a_n \to 1$ from above, then we have $a_{n+1}/a_n \geq 1$ or $a_{n+1} \geq a_n$, and Σa_n certainly diverges, because a_n does not approach zero. But if $a_{n+1}/a_n \to 1$ from below, then there are several more delicate tests that are capable of yielding additional information. The curious reader will find some of these tests discussed in Appendix A.12.

PROBLEMS

Use the ratio test to determine the behavior of the following series.

1 $\sum \frac{n}{2^n}$.

2 $\sum \frac{n^2}{2^n}$.

3 $\sum \frac{n^n}{2^n}$.

4 $\sum \frac{n!}{2^n}$.

5 $\sum \frac{n!}{n^n}$.

6 $\sum \frac{n!}{(2n)!}$.

7 $\sum \frac{(n!)^2}{(2n)!}$.

8 $\sum \frac{(n!)^3}{(2n)!}$.

9 $\sum \frac{3 \cdot 5 \cdots (2n+1)}{n!}$.

10 $\sum \frac{2^{2n}}{(2n+1)!}$.

11 $\sum \frac{2^{3n}}{3^{2n}}$.

12 $\sum \frac{(2n+2)}{3^n (n!)^2}$.

13 $1 + \frac{1 \cdot 3}{2!} + \frac{1 \cdot 3 \cdot 5}{3!} + \cdots + \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!} + \cdots$.

14 $\sum \frac{(n!)^2 3^n}{(2n)!}$.

15 Let Σa_n be a series of positive terms with the following property: There exists a number $r < 1$ and a positive integer n_0 such that $a_{n+1}/a_n \leq r$ for all $n \geq n_0$. Show that Σa_n converges even though $\lim a_{n+1}/a_n$ may not exist.

Use the root test to determine the behavior of the following series.

16 $\sum (\sqrt[n]{n} - 1)^n$.

17 $\sum \sqrt{n}\left(\frac{2n-1}{n+13}\right)^n$.

18 $\sum \frac{e^n}{n^n}$.

19 $\sum \left(\frac{n+1}{n}\right)^{3n} \cdot \frac{1}{3n}$.

20 $\sum e^{2n}\left(\frac{n}{n+1}\right)^{n^2}$.

21 $\sum e^n\left(\frac{n}{n+1}\right)^{n^2}$.

22 $\sum \frac{n^3}{(\ln 2)^n}$.

23 $\sum \frac{n^{10}}{(\ln 3)^n}$.

13.8 THE ALTERNATING SERIES TEST. ABSOLUTE CONVERGENCE

Most of our attention so far has been directed at series of positive terms. We now wish to consider series with both positive and negative terms. The simplest are those whose terms are alternately positive and negative. These are called *alternating series,* and can be written in the form

$$\sum_{n=1}^{\infty} (-1)^{n+1}a_n = a_1 - a_2 + a_3 - a_4 + \cdots, \tag{1}$$

where the a_n's are all positive numbers. As examples that are already familiar from our previous work we mention the ln 2 series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2, \tag{2}$$

and also the series

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}, \tag{3}$$

whose sum was discovered by Leibniz in 1673 (Appendix 3, Chapter 10).

It is easy to see that both of the alternating series (2) and (3) have the property that the a_n's form a decreasing sequence that approaches zero:

(i) $a_1 \geq a_2 \geq a_3 \geq \cdots$;
(ii) $a_n \to 0$.

In 1705 Leibniz noticed that these two simple conditions are enough to guarantee that *any* alternating series (1) converges. This fact is called the *alternating series test.*

The essence of the situation lies in the back-and-forth movement of the partial sums of the series (1) under the stated hypotheses, as illustrated in Fig. 13.6. To locate the partial sums $s_1, s_2, s_3, \ldots$, we start at the origin and go to the right a distance a_1 to reach s_1, then go left a smaller distance a_2 to reach s_2, then go right the still smaller distance a_3 to reach s_3, and so on. The behavior of these

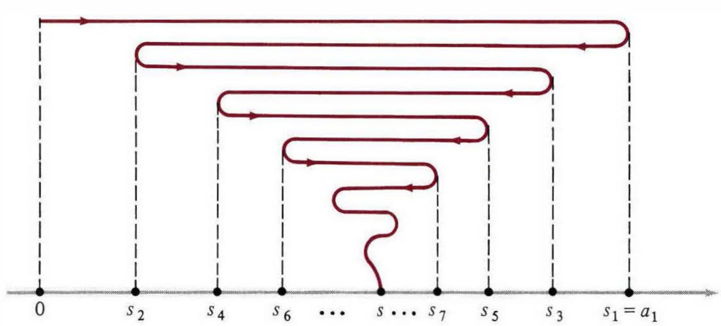


Figure 13.6 The alternating series test.

partial sums is similar to that of a swinging pendulum that oscillates back and forth and slowly approaches an equilibrium position which represents the sum s of the series. We suggest that students keep this figure in mind while reading the proof in the following paragraph.

Now for the details of the argument. A typical even partial sum s_{2n} can be written in two ways, as

$$\begin{aligned} s_{2n} &= (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2n-1} - a_{2n}) \\ &= a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2n-2} - a_{2n-1}) - a_{2n}, \end{aligned}$$

where each expression in parentheses is nonnegative because the a_n's form a decreasing sequence. The first way of writing s_{2n} displays it as the sum of n nonnegative terms, so $s_{2n} \le s_{2n+2}$ and the even partial sums form an increasing sequence, as shown in the figure. The second way of writing s_{2n} shows that $s_{2n} \le a_1$, so the s_{2n}'s have an upper bound. Since every bounded increasing sequence converges, there exists a number s such that

$$\lim_{n\to\infty} s_{2n} = s.$$

But the odd partial sums approach the same limit, because

$$\begin{aligned} s_{2n+1} &= a_1 - a_2 + a_3 - a_4 + \cdots - a_{2n} + a_{2n+1} \\ &= s_{2n} + a_{2n+1}, \end{aligned}$$

and therefore

$$\lim_{n\to\infty} s_{2n+1} = \lim_{n\to\infty} s_{2n} + \lim_{n\to\infty} a_{2n+1} = s + 0 = s,$$

since $a_{2n+1} \to 0$ as $n \to \infty$. This tells us that the sequence $\{s_n\}$ of *all* the partial sums converges to the limit s, and therefore the alternating series (1) converges to the sum s under the stated conditions.

Example 1 The alternating series test clearly implies the convergence of series (2) and (3),

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2 \qquad \text{and} \qquad 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4},$$

because $1/n$ and $1/(2n-1)$ both decrease to zero. However, this test gives us no information at all about the sums of these series. Students will recall that the first of these indicated sums was established earlier by two very different methods, one conjectural and the other solidly rigorous, whereas the second sum is still only a conjecture.*

Example 2 Determine the convergence behavior of the alternating series

$$\text{(a)} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{1000 + 5n}; \qquad \text{(b)} \sum_{n=2}^{\infty} (-1)^{n+1} \frac{\ln n}{n}.$$

Solution (a) Even though this series is alternating, we nevertheless have $a_n = n/(1000 + 5n) \to \frac{1}{5}$ as $n \to \infty$, so the series diverges by the nth term test.

*The method of Leibniz in Appendix 3 of Chapter 10 assumes the validity of term-by-term integration of geometric series. This procedure *is* valid, but we don't yet know it with certainty.

(b) To prove that this series converges by using the alternating series test, we must show that $a_n = (\ln n)/n$ decreases to zero. We know that $(\ln n)/n \to 0$ by Problem 3(a) in Section 13.2. To demonstrate that the a_n's are decreasing, we note that the function

$$f(x) = \frac{\ln x}{x} \qquad \text{has derivative} \qquad f'(x) = \frac{1 - \ln x}{x^2}.$$

This derivative is negative for $x > e$, so $f(x)$ is a decreasing function for $x > e$, and therefore $a_n \geq a_{n+1}$ for $n \geq 3$. (As usual in considering the matter of convergence, we can disregard the first few terms of the series.) In this case—and in others—we may be convinced that a_n decreases to zero without feeling any need for a detailed verification. However, if there is any doubt at all, we should be prepared to supply such a verification.

ABSOLUTE CONVERGENCE

Why is it that the alternating harmonic series

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots$$

converges, even though the harmonic series

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \cdots$$

diverges? The essential reason for the divergence of the harmonic series is that its terms don't decrease quite fast enough, as do the terms of the convergent series $\Sigma 1/2^n$, for example. The partial sums of $\Sigma 1/n$ consist of many small terms that add up to a large total, whereas the terms of $\Sigma 1/2^n$ decrease so fast that no sum of any large number of them can even reach 2.

In contrast to the relatively simple behavior of these series, the alternating harmonic series converges, not only because its terms get small, but also because the well-placed minus signs prevent the partial sums from growing too large and permit them to approach a finite limit.

Some series with terms of mixed signs do not need the assistance of minus signs for convergence, but converge because of the smallness of their terms alone; they would still converge even if all the minus signs were replaced by plus signs. Series of this kind are especially important and are called absolutely convergent; that is, a series Σa_n is said to be *absolutely convergent* if $\Sigma|a_n|$ converges.

These remarks suggest that absolute convergence is a stronger property than ordinary convergence, in the sense that *absolute convergence implies convergence.* This is true and is easy to prove, as follows. Suppose that Σa_n is an absolutely convergent series, so that $\Sigma|a_n|$ converges. The inequalities $0 \leq a_n + |a_n| \leq 2|a_n|$ are clearly valid, because $a_n + |a_n|$ equals 0 or $2|a_n|$ according as $a_n < 0$ or $a_n \geq 0$; and since $\Sigma 2|a_n|$ converges, we know that $\Sigma(a_n + |a_n|)$ also converges by the comparison test. Since $\Sigma(a_n + |a_n|)$ and $\Sigma|a_n|$ both converge, so does their difference, which is Σa_n.

When we try to establish the convergence of a series whose terms have mixed signs, testing for absolute convergence is a good first step, because (as we have just seen) this implies convergence. To do this, we merely change all minus signs to plus signs and test the resulting series of nonnegative terms. We remind students that all our previous tests—the comparison tests, integral test, ratio test, and root test—apply only to series of positive (or nonnegative) terms, and are therefore essentially tests for absolute convergence.

Example 3 Test the following series for absolute convergence, and also for convergence:

(a) $1 - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + \cdots;$

(b) $1 \pm \dfrac{1}{2^2} \pm \dfrac{1}{3^2} \pm \dfrac{1}{4^2} \pm \cdots;$

(c) $1 - \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} - \dfrac{1}{\sqrt{4}} + \cdots.$

Solution (a) Here the series $\Sigma|a_n|$ of absolute values is $\Sigma 1/n^2$, which converges. This tells us that series (a) is absolutely convergent, and therefore convergent. This series also converges by the alternating series test.

(b) The intent here is that the plus or minus signs are to be inserted in any manner, either at random or according to some systematic pattern. In either case, it is clear that, just as in part (a), the series is absolutely convergent, and therefore convergent. However, without the concept of absolute convergence we would have no means of determining the convergence behavior of this series.

(c) In this case the series of absolute values is $\Sigma 1/\sqrt{n}$, which is a divergent p-series. Series (c) is therefore not absolutely convergent. Nevertheless, this series is clearly convergent by the alternating series test.

Remark 1 A convergent series that is not absolutely convergent is said to be *conditionally convergent*. As examples we mention series (2) and (3), and also the series in Example 3(c). All series of this kind are capable of startling but fascinating misbehavior, and should be labeled "handle with care." For instance, any such series can be made to converge to any given number as its sum, or even to diverge, by suitably changing the order of its terms without changing the terms themselves. On the other hand, any absolutely convergent series can be rearranged in any manner without changing either its convergence behavior or its sum. In an earlier section we said, "The effective use of infinite series rests on our freedom to manipulate them by the various processes of algebra." Generally speaking, this freedom is available only when we are working with absolutely convergent series. These issues and others that are not normally part of a first course are discussed in detail in Appendix A.13.

Remark 2 The only test we have that establishes convergence rather than absolute convergence is the alternating series test. Several other tests of this kind are presented in Appendix A.14.

Remark 3 Students may have noticed that almost all of our work in the preceding sections of this chapter has been devoted to showing whether a series converges or diverges, and very little to finding the sum of a convergent series. A good reason for this is that the second problem is usually much harder than the first. Thus, the convergence of the series of the reciprocals of the squares is quite easy to establish, but the exact value of its sum, namely, the fact that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6},$$

is far from obvious and can only be discovered by great ingenuity. Further, the convergence of the series of the reciprocals of the cubes is equally easy to establish, but its sum has *never* been discovered and remains to this day one of the more tantalizing unsolved problems of mathematics:

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots = ?$$

A second, and perhaps even more important reason for not pressing harder on the problem of finding the sum of a convergent series is that this problem will be constantly before us in the next chapter. That is, we shall be "representing" a given function by a certain kind of series, and we will always give careful attention to the matter of proving that this series actually converges to the given function as its sum. In this way, the sums of many convergent series of constants will be known to us as a minor consequence of our work on convergent series of functions.

PROBLEMS

Classify each of the following series as absolutely convergent, conditionally convergent, or divergent.

1 $\sum (-1)^{n+1} \frac{1}{\sqrt{n+10}}$. **2** $\sum (-1)^{n+1} \frac{1}{n\sqrt{n}}$.

3 $\sum (-1)^{n+1} \frac{n}{n+2}$. **4** $\sum (-1)^{n+1} \frac{\sqrt[3]{n}}{\sqrt{n}}$.

5 $\sum (-1)^{n+1} \frac{2^n}{n!}$. **6** $\sum (-1)^{n+1} \frac{1}{3^{1/n}}$.

7 $\sum (-1)^{n+1} \frac{n^3}{1+n^5}$. **8** $\sum (-1)^{n+1} \frac{1}{\ln (n+2)}$.

9 $\sum (-1)^{n+1} \frac{\sqrt{n}}{\ln n}$. **10** $\sum (-1)^{n+1} \ln \frac{1}{n}$.

11 $\sum (-1)^{n+1} \frac{\ln n}{\sqrt{n}}$. **12** $\sum (-1)^{2n+1} \frac{1}{\sqrt{n}}$.

13 $\sum (-1)^{n+1} \frac{\sqrt{n}}{n+3}$. **14** $\sum (-1)^{n+1} \frac{1}{5n}$.

15 $\sum (-1)^{n+1} \frac{2}{3n^2}$. **16** $\sum (-1)^{n+1} \sin n\pi$.

17 $\sum (-1)^{n+1} \frac{\sin^2 n}{n^2}$. **18** $\sum (-1)^{n+1} \frac{\sin^2 n}{n^{5/2}}$.

19 $\sum (-1)^{n+1} \frac{\sin^2 n}{n^{9/2}}$. **20** $\sum (-1)^{n+1} \ln \sqrt[n]{n}$.

21 $\sum (-1)^{n+1} \frac{1}{\sqrt[5]{n^4}}$. **22** $\sum (-1)^{n+1} \frac{n^4 3^n}{n!}$.

23 $1 - \frac{1}{2} + \frac{1}{3!} - \frac{1}{4} + \frac{1}{5!} - \frac{1}{6} + \cdots$.

24 $1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \cdots$.

25 $1 + \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \cdots$.

26 We know that if $\{a_n\}$ is a decreasing sequence of positive numbers such that $a_n \to 0$, then the alternating series (1) converges to some number s as its sum. Show that s lies between consecutive partial sums s_n and s_{n+1} and that $|s_n - s| \le a_{n+1}$.

27 State whether each of the following is true or false:
(a) every convergent alternating series is conditionally convergent;
(b) every absolutely convergent series is convergent;
(c) every convergent series is absolutely convergent;
(d) every alternating series converges;
(e) if Σa_n is conditionally convergent, then $\Sigma |a_n|$ diverges;

(f) if $\Sigma|a_n|$ diverges, then Σa_n is conditionally convergent.

28 If the a_n's are all positive numbers, show that the series $-a_1 + a_2 - a_3 + a_4 - \cdots$ converges if and only if the series $a_1 - a_2 + a_3 - a_4 + \cdots$ converges. [This shows that starting an alternating series with a positive term, as in (1), is merely a convenience, not a necessity.]

29 If Σa_n and Σb_n are absolutely convergent, show that
(a) $\Sigma(a_n + b_n)$ is absolutely convergent;
(b) Σca_n is absolutely convergent for any constant c.

30 If $\Sigma a_n{}^2$ and $\Sigma b_n{}^2$ converge, show that $\Sigma a_n b_n$ is absolutely convergent. Hint: $(a - b)^2 \geq 0$, so $2ab \leq a^2 + b^2$.

31 Use Problem 30 to show that if $\Sigma a_n{}^2$ converges, then $\Sigma a_n/n$ is absolutely convergent.

32 In using the alternating series test, it is a common error to check only that $a_n \to 0$; but this is not enough, and convergence cannot be deduced unless both of conditions (i) and (ii) are verified. Demonstrate this fact in the case of the alternating series

$$\tfrac{2}{1} - \tfrac{1}{1} + \tfrac{2}{2} - \tfrac{1}{2} + \tfrac{2}{3} - \tfrac{1}{3} + \cdots$$

by showing that
(a) $a_n \to 0$;
(b) the a_n's do not form a decreasing sequence;
(c) the series diverges (hint: consider s_{2n}).

33 If s is any given number, show that the alternating harmonic series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ can be rearranged (that is, its terms can be written in a different order) in such a way that the resulting series converges to s. Hint: Take just enough positive terms to get above s, then just enough negative terms to get below s, etc.

CHAPTER 13 REVIEW: DEFINITIONS, CONCEPTS, TESTS

Think through the following.

1 Limit of a sequence.
2 Convergent sequence.
3 An increasing sequence converges if and only if it is bounded.
4 Convergence and sum for infinite series.
5 Geometric series.
6 Harmonic series.
7 The nth term test for divergence.
8 Repeating and nonrepeating decimals.
9 The series for e.
10 The limit comparison test.
11 The p-series.
12 The integral test.
13 Euler's constant.
14 The ratio test.
15 The root test.
16 The alternating series test.
17 Absolute convergence.
18 Absolute convergence implies convergence.
19 Conditional convergence.

ADDITIONAL PROBLEMS FOR CHAPTER 13

SECTION 13.2

1 Find $\lim x_n$ if

(a) $x_n = \left(1 - \dfrac{1}{2}\right)\left(1 - \dfrac{1}{3}\right)\cdots\left(1 - \dfrac{1}{n}\right)$;

(b) $x_n = \left(1 - \dfrac{1}{2^2}\right)\left(1 - \dfrac{1}{3^2}\right)\cdots\left(1 - \dfrac{1}{n^2}\right)$;

(c) $x_n = \dfrac{1}{n^2 + 1} + \dfrac{1}{n^2 + 2} + \cdots + \dfrac{1}{n^2 + n}$;

(d) $x_n = \dfrac{1}{\sqrt{n^2 + 1}} + \dfrac{1}{\sqrt{n^2 + 2}} + \cdots + \dfrac{1}{\sqrt{n^2 + n}}$.

2 If $f(x) = \lim_{n\to\infty} [\lim_{m\to\infty} (2/\pi) \tan^{-1} (m \sin^2 [n!\pi x])]$, show that $f(x) = 0$ when x is rational, and 1 when x is irrational.

3 If $f(x) = \lim_{n\to\infty} [\lim_{m\to\infty} \cos^m(n!\pi x)]$, show that $f(x) = 1$ when x is rational, and 0 when x is irrational.

4 Find the value of each of the following limits:

(a) $\displaystyle\lim_{n\to\infty} \frac{1}{x^n + x^{-n}}$ $(x > 0)$; (b) $\displaystyle\lim_{n\to\infty} \frac{x^{n+1} + n}{x^n + 2n}$.

5 For each sequence $\{x_n\}$ whose nth term is given, verify that the first three terms are $1, \frac{1}{2}, \frac{1}{3}$ and find the fourth term:

(a) $x_n = \dfrac{1}{n}$; (b) $x_n = \dfrac{1}{2n^3 - 12n^2 + 23n - 12}$;

(c) $x_n = \dfrac{1}{n2^{(n-1)(n-2)(n-3)}}$.

*6 If a is any given number, define a sequence $\{x_n\}$ (by constructing a suitable formula for x_n in terms of n) which has the property that $x_1 = 1$, $x_2 = \frac{1}{2}$, $x_3 = \frac{1}{3}$, and $x_4 = a$.

*7 The so-called *Fibonacci sequence* 1, 1, 2, 3, 5, 8, 13, . . . is defined recursively by putting $x_1 = 1$, $x_2 = 1$, and $x_n = x_{n-2} + x_{n-1}$ for $n > 2$.† Find a formula for x_n in terms of n. Hint: Make the ingenious guess that x_n has the form $\alpha A^n + \beta B^n$ for suitable values of α, β, A, B; then determine A and B so that the recursion formula is true for all α's and β's; and finally, find α and β so that $x_1 = 1$ and $x_2 = 1$.

8 If $\{x_n\}$ is the Fibonacci sequence defined in Problem 7, show that $\lim x_{n+1}/x_n = (1 + \sqrt{5})/2$.

9 The sequence $\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots$ can be defined recursively by putting $x_1 = \sqrt{2}$ and $x_{n+1} = \sqrt{2x_n}$ for $n \geq 1$.
(a) Use mathematical induction to prove that $x_n < x_{n+1} < 2$ for every n.‡ This shows that the sequence is increasing and has 2 as an upper bound, and therefore converges to a limit $x \leq 2$.
(b) Show that $x = 2$ by using the recursion formula.
(c) Show that $x = 2$ by finding an explicit formula for x_n in terms of n.

10 The sequence $\sqrt{2}, \sqrt{2 + \sqrt{2}}, \sqrt{2 + \sqrt{2 + \sqrt{2}}}$, . . . can be defined recursively by putting $x_1 = \sqrt{2}$ and $x_{n+1} = \sqrt{2 + x_n}$ for $n \geq 1$. Show that it is increasing with 2 as an upper bound, and find its limit.

*11 If $a > 0$, then the sequence $\sqrt{a}, \sqrt{a + \sqrt{a}}, \sqrt{a + \sqrt{a + \sqrt{a}}}, \ldots$ can be defined recursively as in Problem 10. Show that it converges and find its limit.

12 Let $f(x)$ be an increasing continuous function on the interval $0 \leq x \leq 1$. Define two sequences $\{a_n\}$ and $\{b_n\}$ by

$$a_n = \frac{1}{n}\sum_{k=0}^{n-1} f\left(\frac{k}{n}\right), \qquad b_n = \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right).$$

(a) Show that

$$a_n \leq \int_0^1 f(x)\,dx \leq b_n$$

and

$$0 \leq \int_0^1 f(x)\,dx - a_n \leq \frac{f(1) - f(0)}{n}.$$

(b) Show that both sequences converge to the limit $\int_0^1 f(x)\,dx$.
(c) State a corresponding fact for the interval $a \leq x \leq b$.

*13 Use Problem 12 to obtain the following limits:

(a) $\dfrac{1}{n^2} + \dfrac{2}{n^2} + \cdots + \dfrac{n}{n^2} \to \dfrac{1}{2}$;

(b) $\dfrac{1^2}{n^3} + \dfrac{2^2}{n^3} + \cdots + \dfrac{n^2}{n^3} \to \dfrac{1}{3}$;

(c) $\dfrac{1}{n+1} + \dfrac{1}{n+2} + \cdots + \dfrac{1}{n+n} \to \ln 2$;

(d) $\dfrac{1}{n+1} + \dfrac{1}{n+2} + \cdots + \dfrac{1}{kn} \to \ln k$;

(e) $\dfrac{n}{n^2+1^2} + \dfrac{n}{n^2+2^2} + \cdots + \dfrac{n}{n^2+n^2} \to \dfrac{\pi}{4}$;

(f) $\dfrac{n}{(n+1)^2} + \dfrac{n}{(n+2)^2} + \cdots + \dfrac{n}{(n+n)^2} \to \dfrac{1}{2}$;

(g) $\dfrac{1}{n}\left(\sin\dfrac{\pi}{n} + \sin\dfrac{2\pi}{n} + \cdots + \sin\dfrac{n\pi}{n}\right) \to \dfrac{2}{\pi}$;

(h) $\dfrac{1}{n}\left(\sin^2\dfrac{\pi}{n} + \sin^2\dfrac{2\pi}{n} + \cdots + \sin^2\dfrac{n\pi}{n}\right) \to \dfrac{1}{2}$;

(i) $\dfrac{1}{n}(\sqrt[n]{e} + \sqrt[n]{e^2} + \cdots + \sqrt[n]{e^n}) \to e - 1$;

(j) $\dfrac{1}{\sqrt{n^2+1^2}} + \dfrac{1}{\sqrt{n^2+2^2}} + \cdots + \dfrac{1}{\sqrt{n^2+n^2}} \to \ln(1 + \sqrt{2})$;

(k) $\dfrac{1}{n}\left(\ln\dfrac{1}{n} + \ln\dfrac{2}{n} + \cdots + \ln\dfrac{n}{n}\right) \to -1$.

14 Use part (k) of Problem 13 to show that $\dfrac{\sqrt[n]{n!}}{n} \to \dfrac{1}{e}$.

*15 Use Problem 14 to show that

(a) $\dfrac{1}{n}\sqrt[n]{(n+1)(n+2)\cdots(n+n)} \to \dfrac{4}{e}$;

(b) $\dfrac{1}{n}\sqrt[n]{(2n+1)(2n+2)\cdots(2n+n)} \to \dfrac{27}{4e}$.

*16 If $x_n \to x$, then the sequence of the arithmetic means of the x_n's also converges to x; that is,

†Fibonacci, or Leonardo of Pisa (ca. 1170–1230), was an Italian businessman who traveled extensively in the Middle East and was chiefly responsible for introducing the Hindu-Arabic numerals (i.e., 1, 2, 3, . . .) into Europe. He encountered his sequence in a problem about the progeny of rabbits. It has since been applied extensively (and eccentrically) to religion, art, the shapes of seashells, etc., etc. Fibonacci's problem was this: Start a rabbit colony with a pair of newborn rabbits, one male and one female. Suppose it takes a newborn pair 1 month to grow to sexual maturity and 1 more month to produce a litter. Assuming that no rabbits die and that each litter consists of one male and one female, find the number of pairs in the colony after n months. Answer: the nth term of the Fibonacci sequence—think about it.

‡Recall that the principle of mathematical induction asserts the following: A statement $S(n)$ which is meaningful (in the sense of being either true or false) for each positive integer n is true for all n if (i) $S(1)$ is true; and (ii) $S(n)$ implies $S(n + 1)$. This principle is discussed in detail in Appendix B.2.

$$y_n = \frac{x_1 + x_2 + \cdots + x_n}{n} \to x.$$

Prove this in two steps, as follows.
(a) Begin by assuming that $x = 0$, find a positive integer n_0 such that $|x_n| < \epsilon/2$ for all $n \geq n_0$, and use the fact that for these n's we have

$$|y_n| \leq \frac{|x_1 + x_2 + \cdots + x_{n_0 - 1}|}{n} + \frac{|x_{n_0}| + \cdots + |x_n|}{n}$$
$$< \frac{a}{n} + \frac{\epsilon}{2},$$

where $a = |x_1 + x_2 + \cdots + x_{n_0-1}|$ is a constant.
(b) In the general case, where $x = 0$ is not assumed, use the fact that since $x_n - x \to 0$, we can infer from part (a) that

$$y_n - x = \frac{(x_1 - x) + (x_2 - x) + \cdots + (x_n - x)}{n}$$
$$\to 0.$$

17 Use Problem 16 to show that

(a) $$\frac{1 + \frac{1}{2} + \cdots + 1/n}{n} \to 0;$$

(b) $$\frac{1 + \sqrt{2} + \sqrt[3]{3} + \cdots + \sqrt[n]{n}}{n} \to 1.$$

*18 If $\{x_n\}$ is a sequence of positive numbers such that $x_{n+1}/x_n \to r$, then we also have $\sqrt[n]{x_n} \to r$. Prove this as follows: Put $y_n = \ln x_{n+1}/x_n$; show that

$$\frac{y_1 + y_2 + \cdots + y_{n-1}}{n} = \ln \sqrt[n]{x_n} - \ln \sqrt[n]{x_1};$$

and apply Problem 16.

19 Use Problem 18 to show that $\sqrt[n]{n!}/n \to 1/e$.

*20 Wallis's product, which can be expressed in the form

$$\frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots \frac{2n}{2n-1} \cdot \frac{2n}{2n+1} \to \frac{\pi}{2},$$

is proved in Appendix 2 at the end of Chapter 10. Since $2n/(2n + 1) \to 1$, this can also be written as

$$\frac{2^2}{3^2} \cdot \frac{4^2}{5^2} \cdot \frac{6^2}{7^2} \cdots \frac{(2n-2)^2}{(2n-1)^2} \cdot 2n \to \frac{\pi}{2}.$$

By taking square roots and multiplying numerator and denominator by $2 \cdot 4 \cdot 6 \cdots (2n - 2)$, establish the formula

$$\lim_{n\to\infty} \frac{(n!)^2 2^{2n}}{(2n)!\sqrt{n}} = \sqrt{\pi},$$

which is needed in Problem 21.

*21 It is a remarkable fact that the function $f(n) = \sqrt{2\pi n}\, n^n e^{-n}$ is a good approximation to $n!$ for large n, in the sense that the relative error approaches zero:

$$\lim_{n\to\infty} \frac{f(n) - n!}{n!} = \lim_{n\to\infty} \left[\frac{f(n)}{n!} - 1\right] = 0.$$

This is equivalent to the statement that

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}\, n^n e^{-n}} = 1,$$

which is known as *Stirling's formula*.† In addition to its intrinsic interest, this formula is a useful tool (in statistics and the theory of probability) for the approximate numerical calculation of $n!$ when n is large. Prove Stirling's formula by verifying the following statements:
(a) $2/(2n + 1) \leq \ln (1 + 1/n)$ (hint: compare the area under the curve $y = 1/x$ from $x = n$ to $x = n + 1$ with the area of the trapezoid whose top is tangent to the curve at $x = n + \frac{1}{2}$);

(b) $$e \leq \left(1 + \frac{1}{n}\right)^{n+1/2}$$

$$\left[\text{hint: } \left(n + \frac{1}{2}\right) \ln\left(1 + \frac{1}{n}\right) \leq \ln\left(1 + \frac{1}{n}\right)^{n+1/2}\right];$$

(c) the area A under $y = \ln x$ from $x = 1$ to $x = n$ is

$$\int_1^n \ln x\, dx = n \ln n - n + 1 = \ln\left(\frac{n}{e}\right)^n + 1;$$

(d) the number x_n defined by

$$x_n = \frac{(n/e)^n \sqrt{n}}{n!}$$

is ≤ 1 [hint: compare the area A in part (c) with the area $B = 1 + \ln n! - \ln \sqrt{n}$ of the following figure: Divide the interval from $x = 1$ to $x = n$ into subintervals by the points $\frac{3}{2}, \frac{5}{2}, \ldots, n - \frac{1}{2}$; on the first and last subintervals construct rectangles with heights 2 and $\ln n$; and on the remaining subintervals construct trapezoids whose tops are tangent to the curve $y = \ln x$ at $x = 2, 3, \ldots, n - 1$];
(e) $\{x_n\}$ is an increasing sequence which is bounded by part (d), so $\lim x_n$ exists;
(f) $\lim x_n = \lim \dfrac{x_n^2}{x_{2n}} = \dfrac{1}{\sqrt{2\pi}}$

[hint: use the formula established in Problem 20];
(g) part (f) implies Stirling's formula.

SECTIONS 13.3 AND 13.4

22 If $\Sigma_{n=1}^{\infty} a_n = s$, what is the sum of the series $\Sigma_{n=1}^{\infty}(a_n + a_{n+1})$?

†James Stirling (1692–1770) began his career by being expelled from Oxford for supporting the defunct Stuart dynasty, and ended it as the successful manager of a mining company. In his salad days he was a friend of Newton, and wrote an essay on infinite series in which he almost discovered the formula that bears his name.

23 For what values of x is

$$\frac{1}{x} + \frac{2}{x^3} + \frac{4}{x^5} + \cdots = \frac{x}{x^2 - 2}$$

valid?

24 Find the values of x for which

$$\frac{x}{1+x} - \left(\frac{x}{1+x}\right)^2 + \left(\frac{x}{1+x}\right)^3 - \cdots$$

converges. What is its sum?

***25** By finding a closed formula for the nth partial sum s_n, show that the series $\Sigma_{n=1}^{\infty} nx^n$ converges to $x/(1-x)^2$ when $|x| < 1$ and diverges otherwise.

26 Find the sum of the series $\frac{1}{3} + \frac{2}{9} + \frac{3}{27} + \cdots$.

***27** Use the fact that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ to show that

(a) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$;

(b) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$;

(c) $\frac{1}{1^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{13^2} + \frac{1}{17^2} + \cdots = \frac{\pi^2}{9}$.

28 Show that

$$\int_0^{\infty} \frac{x\,dx}{e^x - 1} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

by expressing the integrand as a geometric series and integrating term by term.

***29** Show that

(a) $\frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} + \cdots = \ln 2$;

(b) $\frac{1}{2\cdot 3} + \frac{1}{4\cdot 5} + \frac{1}{6\cdot 7} + \cdots = 1 - \ln 2$;

(c) $\frac{1}{1\cdot 2} - \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} - \frac{1}{4\cdot 5} + \cdots$
$= 2\ln 2 - 1$.

30 Show that

(a) $\sum_{n=1}^{\infty} \frac{n}{(n+1)!} = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots = 1$;

(b) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \frac{1}{4}$;

(c) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)\cdots(n+k)} = \frac{1}{k\cdot k!}$.

31 Find the sum of

(a) $\sum_{n=2}^{\infty} \ln\left(1 - \frac{1}{n^2}\right)$;

(b) $\sum_{n=1}^{\infty} \frac{1}{(n+1)\sqrt{n} + n\sqrt{n+1}}$.

***32** If $f(n) \to L$, show that

$$\sum_{n=1}^{\infty} [f(n) - f(n+2)] = f(1) + f(2) - 2L$$

and use this to establish the following statements:

(a) $\frac{1}{1\cdot 3} + \frac{1}{2\cdot 4} + \frac{1}{3\cdot 5} + \frac{1}{4\cdot 6} + \cdots = \frac{3}{4}$;

(b) $\frac{1}{1\cdot 3} - \frac{1}{2\cdot 4} + \frac{1}{3\cdot 5} - \frac{1}{4\cdot 6} + \cdots = \frac{1}{4}$.

***33** If $a_1, a_2, a_3, \ldots$ are the positive integers whose decimal representations do not contain the digit 5, show that $\Sigma 1/a_n$ converges and has sum < 90.

***34** Figure 13.7 shows the region bounded by two circles

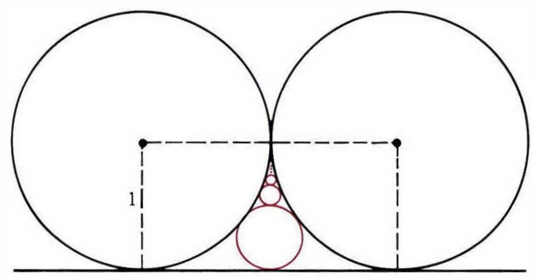


Figure 13.7

of radius 1 that are tangent to each other and by a straight line tangent to both. A sequence of smaller circles, each having the largest possible radius, is inscribed in the region in the manner shown in the figure. It is clear from the geometry of the situation that the lengths of the diameters of these smaller circles are the terms of a convergent series whose sum is 1. Show that this series is

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots.$$

SECTION 13.5

35 Determine whether each of the following series converges or diverges:

(a) $\sum \frac{2}{n^2+n}$; (b) $\sum \frac{n^2+31}{10{,}000n^3}$;

(c) $\sum \frac{1}{(n+3)^2}$; (d) $\sum \frac{n}{\sqrt{n^2+2}}$;

(e) $\sum \frac{1}{[1+(n-1)/n]^n}$; (f) $\sum \frac{\sqrt{n}}{n+5}$;

(g) $\sum \frac{\tan^{-1} n}{n^3}$; (h) $\sum \frac{3}{2+\sqrt{n}}$;

(i) $\sum \frac{(3n+1)^3}{(n^3+2)^2}$; (j) $\sum \frac{1}{\sqrt[3]{3n^2+1}}$;

(k) $\sum \dfrac{1}{\sqrt{n(n+1)(n+2)}}$;

(l) $\sum \dfrac{2n+3}{n \cdot 3^n}$; (m) $\sum \dfrac{5n-7}{(n+5)n!}$;

(n) $\sum \sqrt{\sin^3 \dfrac{1}{n}}$; (o) $\sum \left(\dfrac{n+1}{2n}\right)^n$;

(p) $\sum \left(\dfrac{n^2-1}{n^3+3}\right)^{1/3}$; (q) $\sum \dfrac{\sqrt{n+1}}{n^{n+1/2}}$;

(r) $\sum \dfrac{2^n+3^n}{3^n+4^n}$; (s) $\sum (1-e^{-1/n})^n$;

(t) $\sum \dfrac{(n+1)^n}{n^{n+1}}$; (u) $\sum \sin^2 \pi\left(n+\dfrac{1}{n}\right)$;

(v) $\sum \dfrac{[\ln (n+1)]^n}{n^{n+1}}$.

36 If $a > 1$, show that $\Sigma 1/a^{\ln n}$ diverges if $a \le e$ and converges if $a > e$.

37 Prove that any convergent series of positive terms can be rearranged so that its terms form a decreasing sequence.

38 If p is any positive constant, show that $\Sigma 1/(\ln n)^p$ diverges.

39 Show that the series

$$\sum \frac{1}{(\ln n)^{\ln n}} \quad \text{and} \quad \sum \frac{1}{(\ln \ln n)^{\ln n}}$$

are both convergent. Hint: Express $(\ln n)^{\ln n}$ as a power of n.

40 Show that

$$\sum \frac{1}{(\ln n)^{\ln \ln n}}$$

diverges. Hint: $(\ln \ln n)^2 \le \ln n$ for large n (why?).

41 If $a_n \ge 0$ and Σa_n converges, and if $\{b_n\}$ is a bounded sequence of nonnegative numbers, prove that $\Sigma a_n b_n$ also converges. Use the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ to show that this is false if the assumptions $a_n \ge 0$ and $b_n \ge 0$ are dropped.

42 If Σa_n and Σb_n are series of nonnegative terms such that $\Sigma a_n{}^2$ and $\Sigma b_n{}^2$ both converge, show that $\Sigma a_n b_n$ also converges.

SECTION 13.6

43 Show that

$$\sum_{n=3}^{\infty} \frac{1}{n \ln n \ln \ln n}$$

diverges, and also that if p is a positive constant, then

$$\sum_{n=3}^{\infty} \frac{1}{n \ln n (\ln \ln n)^p}$$

converges if $p > 1$ and diverges if $p \le 1$.

44 If k is any integer > 1, show that

$$\left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{kn}\right) \to \ln k.$$

45 The sum of the convergent series $\Sigma_{n=1}^{\infty} 1/n^3$ is not known. However, if this sum is denoted by s, show that

(a) $\dfrac{1}{1^3} + \dfrac{1}{3^3} + \dfrac{1}{5^3} + \cdots = \dfrac{7}{8}s$;

(b) $\dfrac{1}{1^3} - \dfrac{1}{2^3} + \dfrac{1}{3^3} - \dfrac{1}{4^3} + \cdots = \dfrac{3}{4}s$.

*46 For $p > 1$, the sum of the p-series $\Sigma_{n=1}^{\infty} 1/n^p$ is a function of p called the *zeta function* (the symbol ζ is the Greek letter *zeta*) and denoted by $\zeta(p)$; that is,

$$\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots.$$

Euler discovered that $\zeta(2) = \pi^2/6$, $\zeta(4) = \pi^4/90$, and $\zeta(6) = \pi^6/945$ (see the Appendix at the end of Chapter 14), but the value of $\zeta(p)$ is not known when p is odd.

(a) Use the inequalities (5) in Section 13.6 to show that the zeta function satisfies the inequalities

$$\frac{1}{p-1} \le \zeta(p) \le \frac{p}{p-1}$$

and

$$1 \le \zeta(p) \le \frac{p}{p-1}.$$

(b) Show that $\lim_{p \to 1} \zeta(p) = \infty$ and $\lim_{p \to \infty} \zeta(p) = 1$.

(c) Show that $\lim\limits_{a \to 0+} a \sum\limits_{n=1}^{\infty} \dfrac{1}{n^{1+a}} = 1$.

*47 Let k be an integer > 1 and show that

$$\sum_{n=1}^{\infty} \frac{a_n(k)}{n} = \ln k,$$

where $a_n(k)$ is defined by

$$a_n(k) = \begin{cases} 1 & \text{if } n \text{ is not a multiple of } k, \\ -(k-1) & \text{if } n \text{ is a multiple of } k. \end{cases}$$

*48 The *Cauchy condensation test* states that if $a_1, a_2, \ldots, a_n, \ldots$ is a decreasing sequence of positive numbers, then the two series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \cdots$$

and

$$\sum_{n=0}^{\infty} 2^n a_{2^n} = a_1 + 2a_2 + 4a_4 + 8a_8 + \cdots$$

converge or diverge together. (This statement is called the *condensation test* because it says that a rather small proportion of the terms of the first series determines its convergence behavior.)

(a) Prove the condensation test. Hint: If s_n and t_n are the partial sums, group the terms of the first series into blocks to show that $s_n \le t_m$ if $n \le 2^m$, and $t_m \le 2s_n$ if $2^m \le n$.

(b) Use the condensation test to show that the series

$$\sum \frac{1}{n} \quad \text{and} \quad \sum \frac{1}{n \ln n}$$

diverge, and that the series

$$\sum \frac{1}{n^p} \quad \text{and} \quad \sum \frac{1}{n(\ln n)^p}$$

converge if $p > 1$.

49 Prove that

$$1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1} = \frac{1}{2}\ln n + \ln 2 + \frac{1}{2}\gamma + o(1).$$

***50** Show that

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(2n+1)} = 2 - 2\ln 2;$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(4n^2-1)} = 2\ln 2 - 1;$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(9n^2-1)} = \frac{3}{2}(\ln 3 - 1);$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(16n^2-1)} = 3\ln 2 - 2;$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(36n^2-1)} = \frac{3}{2}\ln 3 + 2\ln 2 - 3;$

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(4n^2-1)^2} = \frac{1}{8};$

(g) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(4n^2-1)^2} = \frac{3}{2} - 2\ln 2.$

SECTION 13.7

Use the ratio test to determine the behavior of the following series.

51 $\displaystyle\sum \frac{n}{e^n}.$

52 $\displaystyle\sum n^{1000}(\tfrac{2}{3})^n.$

53 $\displaystyle\sum \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}.$

54 $\displaystyle\sum \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{1 \cdot 4 \cdot 7 \cdots (3n-2)}.$

55 $\displaystyle\sum \frac{1 \cdot 6 \cdot 11 \cdots (5n-4)}{2 \cdot 6 \cdot 10 \cdots (4n-2)}.$

56 $\displaystyle\sum \frac{1000^n}{n!}.$

57 $\displaystyle\sum \frac{(n+3)!}{n!3^n}.$

58 $\displaystyle\sum \frac{2^{2n}}{2 \cdot 4 \cdot 6 \cdots (2n)}.$

59 $\displaystyle\frac{1}{2} + \frac{1 \cdot 4}{2 \cdot 4} + \frac{1 \cdot 4 \cdot 7}{2 \cdot 4 \cdot 6} + \cdots + \frac{1 \cdot 4 \cdot 7 \cdots (3n-2)}{2 \cdot 4 \cdot 6 \cdots (2n)} + \cdots.$

60 (a) Show that the ratio test fails for the series

$$\sum \frac{1}{2^{n+(-1)^n}}.$$

(b) Show that the root test succeeds for the series in part (a) and tells us that this series converges. (Thus, the root test works in some cases where the ratio test fails. Even more can be said, for Additional Problem 18 asserts that if $\{a_n\}$ is any sequence of positive numbers, then

$$\frac{a_{n+1}}{a_n} \to L \quad \text{implies} \quad \sqrt[n]{a_n} \to L.$$

In principle, therefore, the root test is more powerful than the ratio test.)

61 Consider the series

$$\sum_{n=1}^{\infty} a_n = a + b + a^2 + b^2 + a^3 + b^3 + \cdots,$$

where $0 < a < b < 1$. Show that the ratio test fails, and establish convergence by using the root test.

SECTION 13.8

62 Show that the series

$$\frac{1}{\sqrt{2}-1} - \frac{1}{\sqrt{2}+1} + \frac{1}{\sqrt{3}-1} - \frac{1}{\sqrt{3}+1} + \cdots$$

diverges. Does this contradict the alternating series test?

63 Use the alternating series test to prove the existence of Euler's constant γ as follows.

(a) Show that the series

$$1 - \int_1^2 \frac{dx}{x} + \frac{1}{2} - \int_2^3 \frac{dx}{x} + \frac{1}{3} - \int_3^4 \frac{dx}{x} + \frac{1}{4} - \int_4^5 \frac{dx}{x} + \cdots$$

converges.

(b) If the sum of the series in (a) is denoted by γ, show that

$$s_{2n-1} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n,$$

so that

$$\lim_{n\to\infty}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n\right) = \gamma.$$

64 Use the alternating series test to show that the improper integral

$$\int_0^\infty \frac{\sin x}{x}\,dx$$

converges. Hint: Sketch the graph of $y = (\sin x)/x$ for $x > 0$ and observe that it consists of an infinite number of parts, each covering an interval of length π and lying alternately above and below the x-axis; and then express the integral as an alternating series,

$$\int_0^\infty \frac{\sin x}{x}\,dx = \int_0^\pi \frac{\sin x}{x}\,dx + \int_\pi^{2\pi} \frac{\sin x}{x}\,dx + \cdots$$
$$= a_1 - a_2 + a_3 - a_4 + \cdots.$$

***65** Show that $1 - \frac{1}{4} + \frac{1}{7} - \frac{1}{10} + \cdots = \frac{1}{3}\ln 2 + \pi/3\sqrt{3}$.

***66** Show that

$$\sum_{n=2}^{\infty} (-1)^n \frac{\ln n}{n} = \gamma \ln 2 - \frac{1}{2}(\ln 2)^2.$$

Hint: See equation (9) in Section 13.6.

APPENDIX 1: EULER'S DISCOVERY OF THE FORMULA $\sum_1^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$

In Section 13.4 we encountered Euler's formula for the sum of the reciprocals of the squares,

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6}. \tag{1}$$

Our purpose in this appendix is to understand the heuristic reasoning that led Euler to this wonderful discovery.

We begin with some simple algebra. If a and b are $\neq 0$, then it is clear that these numbers are the roots of the equation

$$\left(1 - \frac{x}{a}\right)\left(1 - \frac{x}{b}\right) = 0. \tag{2}$$

This equation can also be written in the form

$$1 - \left(\frac{1}{a} + \frac{1}{b}\right)x + \frac{1}{ab}x^2 = 0, \tag{3}$$

in which it is evident that the negative of the coefficient of x is the sum of the reciprocals of the roots. If we replace x by x^2, and a and b by a^2 and b^2, then (2) and (3) become

$$\left(1 - \frac{x^2}{a^2}\right)\left(1 - \frac{x^2}{b^2}\right) = 0 \tag{4}$$

and

$$1 - \left(\frac{1}{a^2} + \frac{1}{b^2}\right)x^2 + \frac{1}{a^2b^2}x^4 = 0. \tag{5}$$

The roots of (4) are plainly $\pm a$ and $\pm b$, and (5) is the same equation in polynomial form, from which we see that the negative of the coefficient of x^2 is the sum of the reciprocals of the squares of the positive roots. This pattern persists as we move to equations of higher degree, for

$$\left(1 - \frac{x^2}{a^2}\right)\left(1 - \frac{x^2}{b^2}\right)\left(1 - \frac{x^2}{c^2}\right) = 0$$

(whose roots are obviously $\pm a$, $\pm b$, and $\pm c$) can be written as

$$1 - \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)x^2 + \left(\frac{1}{a^2b^2} + \frac{1}{a^2c^2} + \frac{1}{b^2c^2}\right)x^4 - \frac{1}{a^2b^2c^2}x^6 = 0,$$

and so on.

Let us now consider the transcendental equation

$$\sin x = 0$$

or

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = 0.$$

This can be thought of as "a polynomial equation of infinite degree" with an infinite number of roots $0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$. The root 0 can be removed by dividing by x, which gives the equation

$$\frac{\sin x}{x} = 0$$

or

$$1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots = 0,$$

with roots $\pm\pi, \pm 2\pi, \pm 3\pi, \ldots$. In the light of our knowledge of the roots of this equation, the situation in the previous paragraph suggests that the infinite series

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$$

can be written as an "infinite product",

$$\frac{\sin x}{x} = \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)\cdots. \tag{6}$$

Further, our analogy also suggests that

$$\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \cdots = \frac{1}{3!},$$

from which Euler's formula (1) follows at once. As an additional observation, it is interesting to note that if we put $x = \pi/2$ in (6), we find that

$$\begin{aligned}\frac{2}{\pi} &= \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{4^2}\right)\left(1 - \frac{1}{6^2}\right)\cdots \\ &= \left(\frac{1}{2}\cdot\frac{3}{2}\right)\left(\frac{3}{4}\cdot\frac{5}{4}\right)\left(\frac{5}{6}\cdot\frac{7}{6}\right)\cdots,\end{aligned}$$

which is equivalent to

$$\frac{\pi}{2} = \left(\frac{2}{1}\cdot\frac{2}{3}\right)\left(\frac{4}{3}\cdot\frac{4}{5}\right)\left(\frac{6}{5}\cdot\frac{6}{7}\right)\cdots.$$

This is *Wallis's product,* which was rigorously proved in Appendix 2 at the end of Chapter 10.

These daring speculations are characteristic of Euler's unique genius, but we hope that no students will suppose that they carry the force of rigorous proof. It will be seen that the crux of the matter is the question of the meaning and validity of (6), which is known as *Euler's infinite product for the sine.* The shortcomings of this discussion invite the construction of a general theory of infinite products, within which formulas like (6) can take their place as firmly established facts. This aim is achieved in more advanced fields of mathematics.

APPENDIX 2: MORE ABOUT IRRATIONAL NUMBERS. π IS IRRATIONAL

Readers who have never thought about the matter before may wonder why we care about irrational numbers. In order to understand this, let us assume for a moment that the only numbers we have are the rationals—which, after all, are the only numbers ever used in making scientific measurements. Under these circumstances the symbol $\sqrt{2}$ has no meaning, since there is no rational number whose square is 2. One consequence of this is that the circle $x^2 + y^2 = 4$ and the straight line $y = x$ through its center do not intersect; that is, in spite of appearances, there is no point that lies on both, because both curves are discontinuous in the sense of having many missing points, and each threads its way through a gap in the other. This suggests that the system of rational numbers is an inadequate tool for representing the continuous objects of geometry and the continuous motions of physics. In addition, without irrational numbers most sequences and series would not converge and most integrals would not exist; and since it is also true that e and π would be meaningless (we prove below that π is irrational), the enormous and intricate structure of mathematical analysis would collapse into a heap of rubbish so insignificant as to be hardly worth sweeping up. As a practical matter, it is clear that if the irrationals did not exist, it would be necessary to invent them. It was the ancient Greeks who discovered that irrational numbers are indispensable in geometry, and this was one of their more important contributions to civilization.

In Section 13.4 we proved that e is irrational by assuming the contrary and constructing a number a which was then shown to be a positive integer < 1—an obvious impossibility. This strategy is also the key to the proofs of the following two theorems, but the details are somewhat more complicated.

We shall need a few properties of the function $f(x)$ defined by

$$f(x) = \frac{x^n(1-x)^n}{n!} = \frac{1}{n!}\sum_{k=n}^{2n} c_k x^k, \tag{1}$$

where the c_k's are certain integers and n is a positive integer to be specified later. First, it is clear that if $0 < x < 1$, then we have

$$0 < f(x) < \frac{1}{n!}. \tag{2}$$

Next, $f(0) = 0$ and $f^{(m)}(0) = 0$ if $m < n$ or $m > 2n$; also if $n \le m \le 2n$, then

$$f^{(m)}(0) = \frac{m!}{n!}c_m,$$

and this number is an integer. Thus, $f(x)$ and all its derivatives have integral values at $x = 0$. Since $f(1 - x) = f(x)$, the same is true at $x = 1$.

Theorem 1 *e^r is irrational for every rational number $r \neq 0$.*

Proof If $r = p/q$ and e^r is rational, then so is $(e^r)^q = e^p$. Also, if e^{-p} is rational, so is e^p. It therefore suffices to prove that e^p is irrational for every positive integer p.

Assume that $e^p = a/b$ for certain positive integers a and b. We define $f(x)$ by (1) and $F(x)$ by

$$F(x) = p^{2n}f(x) - p^{2n-1}f'(x) + p^{2n-2}f''(x) - \cdots - pf^{(2n-1)}(x) + f^{(2n)}(x), \tag{3}$$

and we observe that $F(0)$ and $F(1)$ are integers. Next,

$$\frac{d}{dx}[e^{px}F(x)] = e^{px}[F'(x) + pF(x)] = p^{2n+1}e^{px}f(x), \tag{4}$$

where the last equality is obtained from a detailed examination of $F'(x) + pF(x)$ based on (3). Equation (4) shows that

$$b\int_0^1 p^{2n+1}e^{px}f(x)\,dx = b[e^{px}F(x)]_0^1 = aF(1) - bF(0),$$

which is an integer. However, (2) implies that

$$0 < b\int_0^1 p^{2n+1}e^{px}f(x)\,dx < \frac{bp^{2n+1}e^p}{n!} = bpe^p\,\frac{(p^2)^n}{n!};$$

and since the expression on the right $\to 0$ as $n \to \infty$ (by Example 3 in Section 13.2), it follows that the integer $aF(1) - bF(0)$ has the property that

$$0 < aF(1) - bF(0) < 1$$

if n is large enough. Since there is no positive integer < 1, this contradiction completes the proof.

If we say that a point (x, y) in the plane is a *rational point* whenever both x and y are rational numbers, then this theorem asserts that the curve $y = e^x$ traverses the plane in such a way that it misses all rational points except $(0, 1)$. An equivalent statement is that $y = \ln x$ misses all rational points except $(1, 0)$, so $\ln 2, \ln 3, \ldots$ are all irrational. It can also be proved that $y = \sin x$ misses all rational points except $(0, 0)$, and that $y = \cos x$ misses all rational points except $(0, 1)$.* Each of these theorems implies that π is irrational, since $\sin \pi = 0$ and $\cos \pi = -1$. However, we prefer to prove the irrationality of π by the following more direct argument.

Theorem 2 *π is irrational.*

Proof It is clearly sufficient to prove that π^2 is irrational, so we assume the contrary, that $\pi^2 = a/b$ for certain positive integers a and b. We again define $f(x)$ by (1), but this time we put

$$F(x) = b^n[\pi^{2n}f(x) - \pi^{2n-2}f''(x) + \pi^{2n-4}f^{(4)}(x) - \cdots + (-1)^n f^{(2n)}(x)], \tag{5}$$

and again we observe that $F(0)$ and $F(1)$ are integers. A calculation based on (5) shows that

$$\frac{d}{dx}[F'(x)\sin \pi x - \pi F(x)\cos \pi x] = [F''(x) + \pi^2 F(x)]\sin \pi x$$
$$= b^n\pi^{2n+2}f(x)\sin \pi x = \pi^2 a^n f(x)\sin \pi x,$$

so

$$\int_0^1 \pi a^n f(x)\sin \pi x\,dx = \left[\frac{F'(x)\sin \pi x}{\pi} - F(x)\cos \pi x\right]_0^1 = F(1) + F(0),$$

which is an integer. But (2) implies that

$$0 < \int_0^1 \pi a^n f(x)\sin \pi x\,dx < \frac{\pi a^n}{n!} < 1$$

if n is large enough; and this contradiction—that $F(1) + F(0)$ is a positive integer < 1—concludes the proof.

The underlying method of proof in Theorems 1 and 2 was devised by the French mathematician Hermite in 1873, but the details of the latter argument were first published by Niven in 1947.

*The details can be found in Chapter II of I. Niven's excellent book, *Irrational Numbers* (Wiley, 1956).

APPENDIX 3: THE SERIES $\Sigma 1/p_n$ OF THE RECIPROCALS OF THE PRIMES

About 300 B.C. Euclid gave the classical proof of the fact that there exist infinitely many prime numbers. About 2000 years after the time of Euclid, in 1737, Euler discovered two fundamentally different new proofs, and the methods he used laid the foundations of a new branch of mathematics that is now called *analytic number theory.*

In order to understand Euler's ideas, we begin by recalling that the harmonic series

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}+\cdots$$

diverges. On the other hand, we know that for any exponent $s>1$, the series

$$1+\frac{1}{2^s}+\frac{1}{3^s}+\cdots+\frac{1}{n^s}+\cdots$$

converges, and the so-called *zeta function* is defined to be the sum of this series,

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=1+\frac{1}{2^s}+\frac{1}{3^s}+\cdots,$$

considered as a function of the variable s.*

Euler's basic discovery was a remarkable identity connecting the zeta function with the prime numbers,

$$\zeta(s)=\prod_p \frac{1}{1-1/p^s}, \tag{1}$$

where the expression on the right denotes the product of the numbers $1/(1-p^{-s})$ for all primes $p = 2, 3, 5, 7, 11, \ldots$, that is, where

$$\prod_p \frac{1}{1-1/p^s}=\frac{1}{1-1/2^s}\cdot\frac{1}{1-1/3^s}\cdot\frac{1}{1-1/5^s}\cdot\frac{1}{1-1/7^s}\cdots.$$

To see how the identity (1) arises, we recall that the geometric series $1/(1-x)=1+x+x^2+\cdots$ is valid for $|x|<1$, so for each prime p we have

$$\frac{1}{1-1/p^s}=1+\frac{1}{p^s}+\frac{1}{p^{2s}}+\frac{1}{p^{3s}}+\cdots.$$

Without stopping to justify the process, we now multiply these series together for all primes p, remembering that each integer $n>1$ is uniquely expressible as a product of powers of different primes. This yields

$$\begin{aligned}\prod_p \frac{1}{1-1/p^s}&=\prod_p\left(1+\frac{1}{p^s}+\frac{1}{p^{2s}}+\frac{1}{p^{3s}}+\cdots\right)\\&=1+\frac{1}{2^s}+\frac{1}{3^s}+\cdots+\frac{1}{n^s}+\cdots\\&=\sum_{n=1}^{\infty}\frac{1}{n^s}=\zeta(s),\end{aligned}$$

which is the identity (1).

One of Euler's arguments is based on (1) and goes this way. We begin by observing that if there were only a finite number of primes, then the product on the right side of (1) would be an ordinary finite product and would clearly have a finite value for every $s>0$, even for $s=1$. However, the value of the left side of (1) for $s=1$ is the harmonic series,

*We denote the independent variable by s (instead of p, as in Section 13.5) in order to retain the notation that is customary in the theory of numbers.

$$\zeta(1) = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots,$$

which diverges to infinity. This argument by contradiction, which can be made into a rigorous proof, shows that there must be an infinite number of primes. Euler's second argument rests on his discovery that *the series of the reciprocals of the primes diverges*,

$$\sum \frac{1}{p_n} = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \cdots = \infty; \tag{2}$$

for if there were only a finite number of primes, it is obvious that this series couldn't possibly diverge.

The proof of (2) that we give here starts with the geometric series

$$\frac{1}{1 - \frac{1}{2}} = 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots,$$

$$\frac{1}{1 - \frac{1}{3}} = 1 + \frac{1}{3} + \frac{1}{3^2} + \cdots,$$

$$\frac{1}{1 - \frac{1}{5}} = 1 + \frac{1}{5} + \frac{1}{5^2} + \cdots,$$

$$\cdots$$

$$\frac{1}{1 - 1/p_n} = 1 + \frac{1}{p_n} + \frac{1}{p_n{}^2} + \cdots.$$

If we multiply these series together by forming a new series whose terms are all possible products of one term selected from each of the series on the right, then this new series converges in any order to the product of the numbers on the left.* Since every integer greater than 1 is uniquely expressible as a product of powers of different primes, the product of these series is the series of the reciprocals of all positive integers whose prime factors are $\leq p_n$. In particular, all positive integers $\leq p_n$ have this property, so

$$\frac{1}{1 - \frac{1}{2}} \cdot \frac{1}{1 - \frac{1}{3}} \cdots \frac{1}{1 - 1/p_n} \geq \sum_{k=1}^{p_n} \frac{1}{k}$$

$$> \int_1^{p_n+1} \frac{dx}{x} = \ln(p_n + 1) > \ln p_n.$$

(It is in the transition here from the sum to the integral that we use the ideas of Section 13.6.) It follows that

$$\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right) \cdots \left(1 - \frac{1}{p_n}\right) < \frac{1}{\ln p_n},$$

and taking logarithms of both sides yields

$$\sum_{k=1}^{n} \ln\left(1 - \frac{1}{p_k}\right) < -\ln \ln p_n. \tag{3}$$

We next show that

$$-\frac{2}{p_k} < \ln\left(1 - \frac{1}{p_k}\right), \tag{4}$$

for when this is applied to (3), we get

*This statement follows from one of the theorems proved in Appendix A.13.

$$-2\sum_{k=1}^{n}\frac{1}{p_k} < -\ln \ln p_n$$

or

$$\sum_{k=1}^{n}\frac{1}{p_k} > \frac{1}{2}\ln \ln p_n,$$

and our conclusion that $\Sigma 1/p_n$ diverges will follow from the fact that $\ln \ln p_n \to \infty$. To establish (4) and complete the argument, it suffices to observe that the line $y = 2x$ lies below the curve $y = \ln(1 + x)$ on the interval $-\frac{1}{2} \le x \le 0$ and that every prime is ≥ 2.*

*For other proofs of (2), see I. Niven, *Amer. Math. Monthly*, **78** (1971), pp. 272–273; and C. V. Eynden, *Amer. Math. Monthly*, **87** (1980), pp. 394–397.

14 POWER SERIES

14.1 INTRODUCTION

We have remarked before that polynomials are the simplest functions of all, and power series can be thought of as polynomials of infinite degree. An expansion of a function $f(x)$ in a power series,

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots,$$

is therefore a way of expressing $f(x)$ by means of functions of a particularly simple kind.

Our introductory investigations in Section 13.1 led us to the following important power series expansions:

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots, \qquad \text{valid for } -1 < x < 1; \tag{1}$$

$$\ln (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots, \qquad \text{validity unknown for the moment;} \tag{2}$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots, \qquad \text{validity unknown for the moment;} \tag{3}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \qquad \text{validity unknown for the moment.} \tag{4}$$

We have rigorously proved formula (1), but at this stage of our work the others are supported only by the plausibility arguments given in Section 13.1, which are suggestive but not conclusive. What is needed for converting plausibility into certainty is a method for approximating a function by polynomials, together with a useful expression for the error committed by making such an approximation. This method is provided below by Taylor's formula with remainder. The remainder—or expression for the error—will enable us to obtain definite information about the x's for which expansions like (2), (3), and (4) are valid. We will also be able to use polynomial approximations for computing numerical values of functions to any previously specified degree of accuracy.

Students of calculus do not always understand that infinite series are primarily tools for the study of functions. For instance, in Problem 11 of Section 13.4 we proved the validity of the power series expansions of the sine and cosine,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \qquad \text{and} \qquad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots, \tag{5}$$

for all values of x; but since these functions were presumably well known beforehand, it may not be clear what purpose is served by expressing them in this form. Expansions of known functions have their own importance, especially in the computation of numerical values for these functions. However, in advanced work it often happens that an unknown power series arises from some other source, perhaps as a solution of a differential equation. In such a case the series is used to *define* the otherwise unknown function which is its sum, so the series itself is the only tool we have for investigating the properties of this function. This situation can best be understood by supposing that the basic facts about $\sin x$ and $\cos x$—their continuity, the identities they satisfy, their properties with respect to differentiation and integration, etc.—could only be discovered by examining the series given above. For these familiar functions such a tortuous process is of course unnecessary, but for many important functions of higher mathematics there is no practical alternative.

Thus, we discussed series of constants in Chapter 13 as a prelude to studying power series in this chapter. And our primary motive in studying power series is to learn what we can about the sums of such series. Also, once the expansions (2), (3), and (4) are firmly established, it will be pleasant to know that so many familiar but very different functions share the property of being expressible in the single pattern provided by power series. These functions are therefore—in a manner of speaking—"brothers under the skin." This kinship is the central theme of the branch of advanced mathematics called complex analysis.

Meanwhile, we emphasize that the power series expansions (1) to (5) are among the most important formulas in all of mathematics. In addition to fully understanding them—and this is one of the main purposes of the present chapter—students should also memorize them.

14.2 THE INTERVAL OF CONVERGENCE

To start at the beginning, a *power series* is a series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots, \tag{1}$$

where the coefficients a_n are constants and x is a variable. In view of what we have said in the preceding section, it should be clearly understood here that this series stands on its own and is not assumed to be associated with any given function. Since power series are almost always indexed from $n = 0$ to $n = \infty$, we often simplify the notation by writing (1) as $\Sigma a_n x^n$.

The geometric series

$$\sum x^n = 1 + x + x^2 + \cdots \tag{2}$$

is evidently the simplest power series. We know that this series converges for $|x| < 1$ and diverges for $|x| \geq 1$. In general, we expect a power series to converge for some values of x and to diverge for others.

We will clearly be very interested in knowing the x's for which a given power series $\Sigma a_n x^n$ converges. For any such x, the sum of the series is a number whose value depends on x and is therefore a function of x. If we denote this function by $f(x)$, then $f(x)$ can be thought of as defined by the equation

$$f(x) = \sum a_n x^n. \tag{3}$$

Sometimes a power series has a known function as its sum. For example, if $|x| < 1$, we know that the series (2) has $1/(1 - x)$ as its sum. However, in general there is no reason to expect that the sum of a convergent power series will turn out to be a function that we can recognize from our previous experience.

We will be concerned with two major groups of questions. First, what properties does the function $f(x)$ defined by (3) have? Is it continuous? Is it differentiable? If it *is* differentiable, can its derivative be calculated by differentiating (3) term by term? And second—turning the whole situation around—if a function $f(x)$ is given beforehand, under what circumstances does it have a power series expansion of the form (3)? How can this expansion be calculated, and what can be said about the x's for which the expansion is valid? These are some of the issues we study in the next few sections.

We begin by determining the nature of the set of points at which an arbitrary power series converges.

First, a few examples. It is clear that every power series converges for $x = 0$. Some series converge *only* for this value of x, for instance,

$$\sum_{n=1}^{\infty} n^n x^n = x + 2^2x^2 + 3^3x^3 + 4^4x^4 + \cdots.^* \tag{4}$$

To see this, it suffices to observe that for any $x \neq 0$ we have $|nx| > 1$ if n is large enough, and therefore the nth term $(nx)^n$ does not approach zero and the series cannot converge. At the opposite extreme are series like

$$\sum \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \tag{5}$$

which converges for all values of x. We establish this by showing that (5) is absolutely convergent for every x, and this is easy to prove by the ratio test:

$$\frac{|x^{n+1}/(n+1)!|}{|x^n/n!|} = \frac{|x|^{n+1}}{(n+1)!} \cdot \frac{n!}{|x|^n} = \frac{|x|}{n+1} \to 0.$$

(When using the ratio test, it is necessary to test for absolute convergence, because this test applies only to series of positive terms.) As a simple example that lies between these extremes, we have the geometric series (2), which converges on the interval $|x| < 1$ and diverges everywhere else.

Our task is to discover the structure of the set of all x's for which a given power series converges. The examples discussed above show that there are at least three possibilities: This set may consist of the single point $x = 0$, or the entire real line, or a finite interval centered at the origin. We will prove that these are the *only* possibilities.

In order to establish this, we need the following lemma:

If a power series $\Sigma a_n x^n$ converges at x_1, $x_1 \neq 0$, then it converges absolutely at all x with $|x| < |x_1|$; and if it diverges at x_1, then it diverges at all x with $|x| > |x_1|$.

The proof is quite easy. If $\Sigma a_n x_1{}^n$ converges, then $a_n x_1{}^n \to 0$. In particular, if n is sufficiently large, we have $|a_n x_1{}^n| < 1$, and therefore

$$|a_n x^n| = |a_n x_1{}^n| \left|\frac{x}{x_1}\right|^n < r^n, \tag{6}$$

*This series is indexed from $n = 1$ to $n = \infty$ because n^n has no meaning when $n = 0$.

where $r = |x/x_1|$. Now suppose that $|x| < |x_1|$. Then we have

$$r = \left|\frac{x}{x_1}\right| < 1,$$

and the inequality (6) tells us that $\Sigma|a_n x^n|$ converges by comparison with the convergent geometric series Σr^n. This proves the first statement. To prove the second statement, we assume that $\Sigma a_n x_1{}^n$ diverges. Then $\Sigma a_n x^n$ cannot converge for any x with $|x| > |x_1|$, for if it does, then what has just been proved implies the absolute convergence—and therefore the convergence—of $\Sigma a_n x_1{}^n$, and this contradicts our assumption.

We are now in a position to state and prove the main facts about the convergence behavior of an arbitrary power series.

Given a power series $\Sigma a_n x^n$, precisely one of the following is true:

(i) *The series converges only for $x = 0$.*
(ii) *The series is absolutely convergent for all x.*
(iii) *There exists a positive real number R such that the series is absolutely convergent for $|x| < R$ and divergent for $|x| > R$.*

The argument goes this way. If (i) is not true, then the series converges for some $x_1 \neq 0$, and by the lemma we know that the positive number $r = |x_1|$ has the property that $\Sigma a_n x^n$ is absolutely convergent on the interval $-r < x < r$. Let S be the set of all positive numbers r with this property. If S is unbounded, then (ii) is true. Now suppose that neither (i) nor (ii) is true, so that S is a nonempty set of positive numbers which has an upper bound. By the basic completeness property of the real number system, S has a *least* upper bound. This means that there is a smallest number R such that $r \leq R$ for all r's in S.* It is now easy to see that R has the properties stated in (iii).

The positive real number R in case (iii) is called the *radius of convergence* of the power series. We have seen that the series converges absolutely at every point of the open interval $(-R, R)$, and diverges outside the closed interval $[-R, R]$. No general statement can be made about the behavior of the series at the endpoints R and $-R$. There are examples of series that converge at both endpoints, or diverge at both, or converge at one and diverge at the other; to find out what happens for any particular series, we must test each endpoint separately. The set of all x's for which a power series converges is called its *interval of convergence.* These ideas are illustrated in Fig. 14.1.

It is customary to put $R = 0$ when the series converges only for $x = 0$, and to put $R = \infty$ when it converges for all x. This convention allows us to cover all possibilities (except endpoint behavior) in a single statement:

Every power series $\Sigma a_n x^n$ has a radius of convergence R, where $0 \leq R \leq \infty$, with the property that the series converges absolutely if $|x| < R$ and diverges if $|x| > R$.

It should be noticed that if $R = 0$, then no x satisfies the condition $|x| < R$, and if $R = \infty$, then no x satisfies the condition $|x| > R$, so in both of these cases our general statement is true by default.

*See Appendix A.1 for a general discussion of least upper bounds.

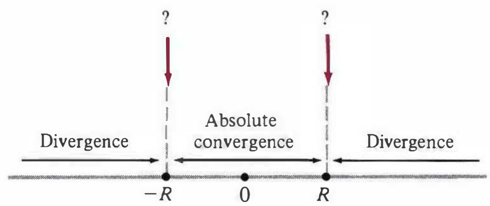


Figure 14.1 The interval of convergence.

If a power series $\Sigma a_n x^n$ is given, how do we find its interval of convergence?

The first step is to find the radius of convergence R. There is a simple formula for R that works in many situations:

$$R = \lim \left| \frac{a_n}{a_{n+1}} \right|, \tag{7}$$

provided this limit exists—and has ∞ as an allowed value. This follows directly from the ratio test, because the series converges absolutely, or diverges, according as the number

$$\lim \frac{|a_{n+1}x^{n+1}|}{|a_n x^n|} = \lim \left| \frac{a_{n+1}}{a_n} \right| \cdot |x| = \frac{|x|}{\lim |a_n/a_{n+1}|}$$

is < 1 or > 1, that is, according as

$$|x| < \lim \left| \frac{a_n}{a_{n+1}} \right| \qquad \text{or} \qquad |x| > \lim \left| \frac{a_n}{a_{n+1}} \right|.$$

This establishes (7).

The second step is to test the behavior of the series at the endpoints.

Example 1 Find the interval of convergence of the series

$$\sum \frac{x^n}{n^2} = x + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \cdots.$$

As indicated, this series is understood to be indexed from $n = 1$ to $n = \infty$, because $a_n = 1/n^2$ has no meaning for $n = 0$.

Solution Here we have

$$\left| \frac{a_n}{a_{n+1}} \right| = \frac{1/n^2}{1/(n+1)^2} = \frac{(n+1)^2}{n^2} \to 1,$$

so $R = 1$. In this example the series converges at both endpoints: At $x = 1$ the series is $\Sigma 1/n^2$, which is a convergent p-series, and at $x = -1$ it is $\Sigma(-1)^n/n^2$, which converges by the alternating series test. The interval of convergence is therefore the entire closed interval $[-1, 1]$.

Example 2 Find the interval of convergence of the series

$$\sum \frac{n+2}{3^n} x^n = 2 + \frac{3}{3}x + \frac{4}{3^2}x^2 + \cdots.$$

Solution This time we have

$$\left| \frac{a_n}{a_{n+1}} \right| = \frac{n+2}{3^n} \cdot \frac{3^{n+1}}{n+3} = \frac{n+2}{n+3} \cdot 3 \to 3,$$

so $R = 3$. In this case the series diverges at both endpoints: At $x = 3$ it becomes $2 + 3 + 4 + \cdots$, and at $x = -3$ it becomes $2 - 3 + 4 - \cdots$. The interval of convergence is therefore $(-3, 3)$.

Example 3 Find the interval of convergence of the series

$$\sum \frac{x^n}{n+1} = 1 + \frac{x}{2} + \frac{x^2}{3} + \cdots.$$

Solution Here we have

$$\left|\frac{a_n}{a_{n+1}}\right| = \frac{1/(n+1)}{1/(n+2)} = \frac{n+2}{n+1} \to 1,$$

so $R = 1$. At $x = 1$ the series is the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \cdots$, which diverges, and at $x = -1$ it is the alternating harmonic series $1 - \frac{1}{2} + \frac{1}{3} - \cdots$, which converges. The interval of convergence is therefore $[-1, 1)$.

Example 4 Find the interval of convergence of the series

$$\sum (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots. \tag{8}$$

Solution Formula (7) does not apply directly because half of the coefficients of this power series are zero. Nevertheless, the series can be written in the form

$$1 - \frac{y}{2!} + \frac{y^2}{4!} - \cdots,$$

where $y = x^2$. Our formula can be applied to this series and yields

$$\left|\frac{a_n}{a_{n+1}}\right| = \frac{1/(2n)!}{1/(2n+2)!} = \frac{(2n+2)!}{(2n)!} = (2n+1)(2n+2) \to \infty,$$

so $R = \infty$ for the y-series. It follows that the x-series also converges for every x, so the desired interval of convergence for (8) is $(-\infty, \infty)$.

If a is a real number, the series

$$\sum_{n=0}^{\infty} a_n(x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots \tag{9}$$

is called a *power series in* $x - a$. For the sake of emphasis, the special case (1), in which $a = 0$, is often called a *power series in* x. If we put $z = x - a$, then (9) becomes $\Sigma a_n z^n$, which is a power series in z. If we can determine the interval of convergence of this latter series by the methods previously described, then this information can be used to find the interval of convergence of (9). For instance, if $\Sigma a_n z^n$ has $[-R, R)$ as its interval of convergence, then it converges for $-R \le z < R$. This means that $-R \le x - a < R$ or $a - R \le x < a + R$. We then say that $[a - R, a + R)$ is the *interval of convergence*, and that R is the *radius of convergence*, of the power series (9).

For the sake of simplicity of notation, we shall confine most of our detailed discussions to power series in x.

PROBLEMS

Find the interval of convergence of each of the following power series.

1 $\sum \frac{n}{4^n} x^n$.

2 $\sum n!x^n$.

3 $\sum \frac{n!}{100^n} x^n$.

4 $\sum \frac{2^n}{n^2} x^n$.

5 $\sum \frac{x^n}{n+4}$.

6 $\sum (-1)^{n+1} \frac{x^n}{\sqrt{n}}$.

7 $\sum \frac{x^n}{n^3+1}$.

8 $\sum \frac{2^n}{(2n)!} x^{2n}$.

9 $\sum \frac{(3n)!}{(2n)!} x^n$.

10 $\sum \frac{x^n}{n(n+1)}$.

11 $\sum \frac{x^{2n+1}}{(-3)^n}$.

12 $\sum \frac{x^n}{n2^n}$.

13 $\sum \frac{(-1)^n}{2n+1} x^{2n+1}$.

14 $\sum n^2 x^n$.

15 $\sum \frac{(-2)^n}{n} x^n$.

16 $\sum \frac{(-1)^{n+1}}{\sqrt[3]{n}\, 2^n} x^n$.

17 $\sum \frac{x^n}{\ln n}$.

18 $\sum \frac{\ln n}{n} x^n$.

19 $\sum \frac{(-1)^n}{n(\ln n)^2} x^n$.

20 $\sum \frac{3^n}{n4^n} x^n$.

21 $\sum \frac{n^2}{2^n} (x-4)^n$.

22 $\sum \frac{3^n}{n^2+1} (x-1)^n$.

23 $\sum \frac{10^n}{(2n)!} (x-7)^n$.

24 $\sum \frac{(x-3)^n}{n^2 2^n}$.

25 $\sum \frac{(2n)!}{n!} (x-10)^n$.

26 $\sum \frac{(-1)^{n+1}}{n \ln n} (x-3)^n$.

27 $\sum \frac{\ln n}{e^n} (x-e)^n$.

28 $\sum \frac{n^2}{2^{3n}} (x+2)^n$.

29 Find the radius of convergence of
(a) the hypergeometric series

$$1 + \frac{ab}{c} x + \frac{a(a+1)b(b+1)}{2!c(c+1)} x^2 + \frac{a(a+1)(a+2)b(b+1)(b+2)}{3!c(c+1)(c+2)} x^3 + \cdots;$$

(b) the Bessel function

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots.$$

30 Give an example of a power series with $R = \pi$.

31 If infinitely many coefficients of a power series are nonzero integers, show that $R \leq 1$.

14.3 DIFFERENTIATION AND INTEGRATION OF POWER SERIES

Consider a power series $\Sigma a_n x^n$ with positive radius of convergence R. We saw in Section 14.2 that this series can be used to define a function $f(x)$ whose domain of definition is the interval of convergence of the series. Specifically, for each x in this interval we define $f(x)$ to be the sum of the series,

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots. \tag{1}$$

This relation between the series and the function is often expressed by saying that $\Sigma a_n x^n$ is a *power series expansion* of $f(x)$. For example, we know that if $|x| < 1$, then

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \cdots, \tag{2}$$

because the geometric series $\Sigma(-1)^n x^n$ converges and has the sum $1/(1+x)$. Accordingly, the function $f(x) = 1/(1+x)$ has the series on the right of (2) as a power series expansion.

Polynomials, which are *finite* sums of terms of the form $a_n x^n$, are very simple functions. They are continuous everywhere, and can be differentiated and integrated term by term. Even though the sum of a power series can be a much more complicated function, it is still simple enough to share these three properties with polynomials inside the interval of convergence.

We give the following formal statement of these very important facts:

(i) *The function $f(x)$ defined by* (1) *is continuous on the open interval* $(-R, R)$.

(ii) *The function $f(x)$ is differentiable on $(-R, R)$, and its derivative is given by the formula*

$$f'(x) = a_1 + 2a_2x + 3a_3x^2 + \cdots + na_nx^{n-1} + \cdots. \tag{3}$$

(iii) *If x is any point in $(-R, R)$, then*

$$\int_0^x f(t)\,dt = a_0x + \frac{1}{2}a_1x^2 + \frac{1}{3}a_2x^3 + \cdots + \frac{1}{n+1}a_nx^{n+1} + \cdots. \tag{4}$$

The proofs of these statements depend on a special kind of convergence called *uniform convergence.* The details can be found in Appendix A.15.

Several comments are in order.

First, we observe that if (ii) is applied to the function $f'(x)$ in (3), then it follows that $f'(x)$ is itself differentiable. This in turn implies that $f''(x)$ is differentiable, and so on. Thus, the original function $f(x)$ has derivatives of all orders. We can summarize the situation this way: *In the interior of its interval of convergence, a power series defines an infinitely differentiable function whose derivatives can be calculated by differentiating the series term by term.* The term-by-term differentiation can be emphasized by writing (3) as

$$\frac{d}{dx}\left(\sum a_nx^n\right) = \sum \frac{d}{dx}(a_nx^n).$$

It is worth knowing that the term-by-term differentiability of a convergent series of functions is usually false; it is true here only because we are dealing with a special kind of series. As a simple example of the failure of this property, we mention the series $\Sigma_{n=1}^{\infty} (\sin nx)/n^2$, which is absolutely convergent for all x by comparison with the convergent series $\Sigma 1/n^2$, because $|(\sin nx)/n^2| \le 1/n^2$. The difficulty arises with the term-by-term differentiated series $\Sigma(\cos nx)/n$, because this series diverges for $x = 0$.

In the case of (iii), if we prefer to avoid using the dummy variable t, then (4) can be written in the form

$$\int f(x)\,dx = a_0x + \frac{1}{2}a_1x^2 + \frac{1}{3}a_2x^3 + \cdots + \frac{1}{n+1}a_nx^{n+1} + \cdots, \tag{5}$$

provided we find the indefinite integral on the left that equals zero when $x = 0$. The term-by-term integration of power series can be emphasized by writing (5) as

$$\int\left(\sum a_nx^n\right)dx = \sum\left(\int a_nx^n\,dx\right).$$

We also point out that it is part of the meaning of (3) and (4) that the differentiated and integrated series on the right sides of these equations converge on the interval $(-R, R)$. We shall prove this at the end of this section. However, before doing so we give several examples of the practical value of the procedures discussed here.

Example 1 Find a power series expansion of $\ln(1 + x)$.

Solution Our starting point is the fact that the derivative of $\ln(1+x)$ is $1/(1+x)$, and for $|x|<1$ this function has the power series expansion given by (2). Now, using (5) and the fact that $\ln(1+x)$ equals zero when $x=0$, and integrating (2) term by term, we at once obtain

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^n \frac{x^{n+1}}{n+1} + \cdots$$
$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}.$$

We know from the preceding discussion that this expansion is valid for $|x|<1$. Further, we know from Section 13.6 that it is also valid for $x=1$, but our present methods give no information on this matter.

Example 2 Find a power series expansion of $\tan^{-1} x$.

Solution The derivative of this function is $1/(1+x^2)$, and we see by replacing x by x^2 in (2) that

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - \cdots + (-1)^n x^{2n} + \cdots$$

if $|x|<1$. Using (4) this time, we get

$$\tan^{-1} x = \int_0^x \frac{dt}{1+t^2} = \int_0^x (1 - t^2 + t^4 - t^6 + \cdots)\, dt$$
$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

for $|x|<1$. Again, our present methods give no information about what happens at the endpoints $x = \pm 1$.

Example 3 Find a power series expansion of e^x.

Solution In Section 8.3 we proved that e^x is the only function that equals its own derivative everywhere and has the value 1 at $x=0$. To construct a power series equal to its own derivative, we use the fact that when such a series is differentiated, the degree of each term drops by 1. We therefore want each term to be the derivative of the one that follows it. Starting with 1 as the constant term, the next should be x, then $\frac{1}{2}x^2$, then $\frac{1}{2\cdot 3}x^3$, and so on. This produces the series

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots, \tag{6}$$

which converges for all x because

$$R = \lim \frac{1/n!}{1/(n+1)!} = \lim (n+1) = \infty.$$

We have constructed the series (6) so that its sum is unchanged by differentiation and has the value 1 at $x=0$. In view of the above remark, this establishes the validity of the expansion

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

for all x.

Example 4 Find power series expansions of $1/(1 - x)^2$ and $1/(1 - x)^3$.

Solution We begin by noticing that

$$\frac{1}{(1 - x)^2} = \frac{d}{dx}\left(\frac{1}{1 - x}\right).$$

The next steps are to expand $1/(1 - x)$ into the power series Σx^n for $|x| < 1$, and then to differentiate this series term by term:

$$\begin{aligned}\frac{1}{(1 - x)^2} &= \frac{d}{dx}(1 + x + x^2 + \cdots + x^n + \cdots)\\ &= 1 + 2x + 3x^2 + \cdots + nx^{n-1} + \cdots\\ &= \sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty}(n + 1)x^n.\end{aligned}$$

Another differentiation yields

$$\begin{aligned}\frac{2}{(1 - x)^3} &= \frac{d}{dx}\left[\frac{1}{(1 - x)^2}\right]\\ &= \frac{d}{dx}(1 + 2x + 3x^2 + \cdots + nx^{n-1} + \cdots)\\ &= 2 + 3 \cdot 2x + 4 \cdot 3x^2 + \cdots + n(n - 1)x^{n-2} + \cdots,\end{aligned}$$

so

$$\begin{aligned}\frac{1}{(1 - x)^3} &= \frac{1}{2}[2 + 3 \cdot 2x + 4 \cdot 3x^2 + \cdots + n(n - 1)x^{n-2} + \cdots]\\ &= \sum_{n=2}^{\infty}\frac{n(n - 1)}{2}x^{n-2} = \sum_{n=0}^{\infty}\frac{(n + 2)(n + 1)}{2}x^n.\end{aligned}$$

Example 5 Find the sum of the series

$$x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots = \sum_{n=1}^{\infty}\frac{x^n}{n}.$$

Solution It is easy to see that $R = 1$, so the series converges to some function $f(x)$ for $|x| < 1$. Differentiating this series clearly simplifies it to

$$f'(x) = 1 + x + x^2 + \cdots + x^n + \cdots = \frac{1}{1 - x}.$$

Since $f(0) = 0$, integration now yields

$$f(x) = -\ln(1 - x).$$

Example 6 Find the sum of the series

$$x + 4x^2 + 9x^3 + \cdots = \sum_{n=1}^{\infty} n^2 x^n.$$

Solution Again we clearly have $R = 1$, so the series converges to some function $f(x)$ for $|x| < 1$. We can write

$$f(x) = x + 4x^2 + 9x^3 + \cdots + n^2x^n + \cdots = xg(x),$$

where

$$g(x) = 1 + 2^2x + 3^2x^2 + \cdots + n^2x^{n-1} + \cdots.$$

At this point we notice that

$$g(x) = \frac{d}{dx}(x + 2x^2 + 3x^3 + \cdots + nx^n + \cdots)$$

$$= \frac{d}{dx}[x(1 + 2x + 3x^2 + \cdots + nx^{n-1} + \cdots)].$$

By Example 4,

$$1 + 2x + 3x^2 + \cdots + nx^{n-1} + \cdots = \frac{1}{(1-x)^2},$$

so

$$g(x) = \frac{d}{dx}\left[\frac{x}{(1-x)^2}\right] = \frac{1+x}{(1-x)^3}$$

and

$$f(x) = \frac{x+x^2}{(1-x)^3}.$$

In conclusion, we return to the unfinished business of showing that series (3) and (4) converge on the interval $(-R, R)$.

The proof for (4) is easy: Since $\Sigma|a_nx^n|$ converges and

$$\left|\frac{a_nx^n}{n+1}\right| \le |a_nx^n|,$$

the comparison test implies that $\sum \left|\frac{a_nx^n}{n+1}\right|$ converges, and therefore

$$x\sum \frac{a_nx^n}{n+1} = \sum \frac{1}{n+1}a_nx^{n+1}$$

also converges.

The proof for (3) is a bit more complicated. Let x be a point in the interval $(-R, R)$ and choose $\epsilon > 0$ so that $|x| + \epsilon < R$. Since $|x| + \epsilon$ is in the interval, $\Sigma|a_n(|x| + \epsilon)^n|$ converges. In Problem 7 students are asked to show that the inequality

$$|nx^{n-1}| \le (|x| + \epsilon)^n$$

is true for all sufficiently large n's. This implies that

$$|na_nx^{n-1}| \le |a_n(|x| + \epsilon)^n|,$$

so the series $\Sigma|na_nx^{n-1}|$ converges by the comparison test.

PROBLEMS

1 Find power series expansions for the following functions, and determine the values of x for which these expansions are valid:
(a) $\dfrac{1}{(1+x)^2}$; (b) $\dfrac{1}{(1+x)^3}$.

2 Show that

$$\sum_{n=0}^{\infty} \frac{(n+1)(n+2)(n+3)}{6} x^n = \frac{1}{(1-x)^4}.$$

3 Find the sum of each of the following series:
(a) $x + \dfrac{x^3}{3} + \dfrac{x^5}{5} + \cdots + \dfrac{x^{2n+1}}{2n+1} + \cdots$;
(b) $1 + \dfrac{x}{2!} + \dfrac{x^2}{3!} + \cdots + \dfrac{x^{n-1}}{n!} + \cdots$;
(c) $x + 2x^2 + 3x^3 + \cdots + nx^n + \cdots$;
(d) $x + 2x^3 + 3x^5 + \cdots + nx^{2n-1} + \cdots$.

4 Show that

$$\sum_{n=1}^{\infty} \frac{x^n}{n^2} = -\int_0^x \frac{\ln(1-t)}{t}\,dt.$$

5 Show that the Bessel function

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots$$

satisfies the differential equation $xy'' + y' + xy = 0$.

6 Obtain the series

$$\ln(x + \sqrt{1+x^2}) = x - \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} + \cdots$$

by integrating another series.

7 If $\epsilon > 0$, show that the inequality $|nx^{n-1}| \le (|x| + \epsilon)^n$ is true for all sufficiently large n's. Hint: $n^{1/n}|x|^{1-(1/n)} \to |x|$.

8 On p. 41-7 in Vol. 1 of his *Lectures On Physics* (Addison-Wesley, 1964), Richard Feynman (Nobel Prize, 1965) writes:

> Thus the average energy is
>
> $$\langle E \rangle = \frac{\hbar\omega(0 + x + 2x^2 + 3x^3 + \cdots)}{1 + x + x^2 + \cdots}.$$
>
> Now the two sums which appear here we shall leave for the reader to play with and have some fun with. When we are all finished summing and substituting for x in the sum, we should get—if we make no mistakes in the sum—
>
> $$\langle E \rangle = \frac{\hbar\omega}{e^{\hbar\omega/kT} - 1}.$$
>
> This, then, was the first quantum-mechanical formula ever known, or ever discussed, and it was the beautiful culmination of decades of puzzlement.

Use Problem 3(c) to have the fun that Feynman recommends. (It is not necessary to know that $\hbar$ is Planck's constant, but it *is* necessary to substitute $x = e^{-\hbar\omega/kT}$.) A few sentences on, Feynman writes: "This expression should, of course, approach kT as $\omega \to 0$. See if you can prove that it does—learn how to do the mathematics." Prove that it does.

14.4 TAYLOR SERIES AND TAYLOR'S FORMULA

We have solved the problem of determining the general nature of the function that is the sum of a convergent power series: Inside the interval of convergence, it is a continuous function with derivatives of all orders. We now investigate the converse problem of starting with a given infinitely differentiable function and expanding it in a power series. In Section 14.3 we established several such expansions for a few special functions with particularly simple derivatives. Our purpose here is to consider a method of much greater generality.

It may seem that the coefficients of a convergent power series are not connected with one another in any necessary way. In fact, however, they are bound together by an invisible chain, which we now make visible.

To this end, let us assume that a function $f(x)$ is the sum of a power series with positive radius of convergence,

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots, \qquad R > 0. \tag{1}$$

By the results of Section 14.3, repeated term-by-term differentiation is legitimate and yields

$$f'(x) = a_1 + 2a_2x + 3a_3x^2 + \cdots,$$

$$f''(x) = 1 \cdot 2a_2 + 2 \cdot 3a_3x + 3 \cdot 4a_4x^2 + \cdots,$$

$$f'''(x) = 1 \cdot 2 \cdot 3a_3 + 2 \cdot 3 \cdot 4a_4x + 3 \cdot 4 \cdot 5a_5x^2 + \cdots,$$

and in general,

$$f^{(n)}(x) = n!a_n + \text{terms containing } x \text{ as a factor.}$$

We know that these series expansions of the derivatives are valid on the open interval $|x| < R$. By putting $x = 0$ in these equations we obtain

$$f(0) = a_0, \qquad f'(0) = a_1, \qquad f''(0) = 1 \cdot 2a_2,$$

$$f'''(0) = 1 \cdot 2 \cdot 3a_3, \qquad \ldots, \qquad f^{(n)}(0) = n!a_n,$$

so

$$a_0 = f(0), \qquad a_1 = f'(0),$$

$$a_2 = \frac{f''(0)}{2!}, \qquad a_3 = \frac{f'''(0)}{3!}, \qquad \ldots, \qquad a_n = \frac{f^{(n)}(0)}{n!}. \tag{2}$$

These are very remarkable formulas, for they tell us that *if* $f(x)$ has a power series expansion of the form (1), *then* its coefficients must be the numbers given by (2). The series (1) therefore becomes

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots. \tag{3}$$

The power series on the right of (3) is called the *Taylor series* of $f(x)$ [at $x = 0$]. The following conclusion is implicit in this discussion:

If a function is represented by a power series with positive radius of convergence, then there is only one such series and it must be the Taylor series of the function.

Briefly, power series expansions are unique, because (2) tells us that the coefficients are uniquely determined by the function itself. If we use the standard conventions mentioned earlier, that $0! = 1$ and that the zeroth derivative of $f(x)$ is $f(x)$ itself [$f^{(0)}(x) = f(x)$], then (3) can be written as

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n. \tag{4}$$

The numbers $a_n = f^{(n)}(0)/n!$ are called the *Taylor coefficients* of $f(x)$.*

Equation (3) was true in the preceding discussion because we started with a convergent series whose sum is denoted by $f(x)$. We now start with a function $f(x)$ that has derivatives of all orders throughout some open interval I containing

*Brook Taylor (1685–1731) was secretary of the Royal Society and an enthusiastic supporter of Newton in his acrimonious controversy with Leibniz and the Bernoullis about the invention of calculus. Taylor published his power series expansion of a function in 1715, but only as a formula and without any consideration at all of the issue of convergence. This expansion had already been published by John Bernoulli in 1694. Taylor was fully aware of this fact, but chose to ignore it out of partisan malice. In those days science roused passions that today we see only in politics and religion.

the point $x = 0$. We can form the Taylor series on the right of (3) and ask the question, Is equation (3) a valid expansion of $f(x)$ on the interval I? Lest there be any misunderstanding, we state as clearly as possible that this equation is *not* always valid, and whether it is or not depends entirely on the individual nature of the function $f(x)$. In Remark 2 we will give an example of an infinitely differentiable function whose Taylor series converges everywhere, *but not to the value of the function,* so in this particular case equation (3) is false.

In order to put our ideas on a firm foundation, we proceed as follows. Break off the Taylor series on the right side of (3) at the term containing x^n and define the *remainder* $R_n(x)$ by the equation

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + R_n(x). \tag{5}$$

Then the Taylor series on the right side of (3) converges to the function $f(x)$ precisely when

$$\lim_{n\to\infty} R_n(x) = 0. \tag{6}$$

These equations don't really solve anything, because $R_n(x)$ is defined to be whatever it takes to make (5) true, and (6) is merely the meaning of the statement that the Taylor series converges to the function. This approach is useful only if we can show that $R_n(x)$ can be expressed in a form that makes it feasible to try to prove (6) in the case of particular functions. We emphasize that (6) is not always true, because (3) is not always true. The most convenient general formula for $R_n(x)$ is

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}, \tag{7}$$

where c is some number between 0 and x. When $R_n(x)$ is expressed this way, (5) is called *Taylor's formula with derivative remainder.** The proof of (7) that we give is fairly technical, and is placed in Remark 3 so that students who wish to skip it can easily do so.

To summarize, *Taylor's formula with derivative remainder* states that under the above assumptions the function $f(x)$ can be written in the form

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1},$$

where c is some number between 0 and x.

We now give several illustrations of the use of (6) and (7). First, however, we observe that

$$\lim_{n\to\infty} \frac{|x|^n}{n!} = 0 \tag{8}$$

for every x, because the series $\Sigma x^n/n!$ is absolutely convergent everywhere. We shall need this fact in our first two examples.

Example 1 Find the Taylor series for the function $f(x) = e^x$, and use (6) and (7) to prove that it converges to e^x for every x.

Solution We clearly have

*There are other forms of the remainder that we do not discuss.

$$f(x) = e^x, \qquad f(0) = 1;$$
$$f'(x) = e^x, \qquad f'(0) = 1;$$
$$f''(x) = e^x, \qquad f''(0) = 1;$$

and so on. By substituting in (3) we obtain

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}; \tag{9}$$

and to prove the validity of this expansion by our present methods, we examine the remainder $R_n(x)$. For any x, the maximum value of the exponential function on the interval from 0 to x is easily seen to be its value M at the right endpoint. By (7) and (8) we therefore have

$$|R_n(x)| = \left| \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} \right| = \left| \frac{e^c}{(n+1)!} x^{n+1} \right| \le M \frac{|x|^{n+1}}{(n+1)!} \to 0,$$

so (9) is valid for all x. Of course, we established (9) in another way in Section 14.3.

Example 2 Find the Taylor series for $f(x) = \sin x$, and use (6) and (7) to prove that it converges to $\sin x$ for every x.

Solution We can arrange our work as follows:

$$f(x) = \sin x, \qquad f(0) = 0;$$
$$f'(x) = \cos x, \qquad f'(0) = 1;$$
$$f''(x) = -\sin x, \qquad f''(0) = 0;$$
$$f'''(x) = -\cos x, \qquad f'''(0) = -1;$$

and the subsequent derivatives follow this same pattern. By substituting in (3), we obtain

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}. \tag{10}$$

So far, all we know is that *if* $\sin x$ has a power series expansion, *then* that expansion must be (10). To prove that (10) is actually true for every x, we use (6) and (7). Since either

$$|f^{(n+1)}(x)| = |\sin x| \qquad \text{or} \qquad |f^{(n+1)}(x)| = |\cos x|,$$

it is clear that $|f^{(n+1)}(c)| \le 1$ for every number c. Therefore by (7) and (8) we have

$$|R_n(x)| = |f^{(n+1)}(c)| \frac{|x|^{n+1}}{(n+1)!} \le \frac{|x|^{n+1}}{(n+1)!} \to 0,$$

so (10) is true for all x. Students will remember that we proved (10) in Problem 11 of Section 13.4 by a very different method, one requiring considerable ingenuity. Our present method has the advantage of being straightforward and systematic.

Example 3 Find the Taylor series for $\cos x$, and prove that it converges to $\cos x$ for every x.

Solution We could proceed directly, as in Example 2. Instead, this is left for students to carry out in the problems, and we obtain the desired series by differentiating (10),

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}. \tag{11}$$

The validity of this expansion is guaranteed by the results of Section 14.3. Since we have found *a* power series expansion for $\cos x$, we know by the discussion of uniqueness given above that this series must be the Taylor series.

The three series established in these examples can be used to find power series expansions for many other functions. Thus, since (9) is true for every x, a power series representation for e^{-x^2} can be found by substituting $-x^2$ for x. This yields

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots + (-1)^n \frac{x^{2n}}{n!} + \cdots. \tag{12}$$

We can apply this formula to evaluate the definite integral

$$\int_0^1 e^{-x^2}\, dx,$$

even though the corresponding indefinite integral cannot be calculated in terms of elementary functions. Term-by-term integration of (12) gives

$$\int_0^1 e^{-x^2}\, dx = \left[x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \cdots\right]_0^1$$

$$= 1 - \frac{1}{3} + \frac{1}{5 \cdot 2!} - \frac{1}{7 \cdot 3!} + \cdots.$$

This expression for the value of the integral as a series of constants is exact, and can be used to obtain this value in decimal form to any desired degree of accuracy.

Example 4 *The inverse tangent series and the computation of π.* In Example 2 of Section 14.3 we gave a rigorous proof that the expansion

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \tag{13}$$

is valid for $-1 < x < 1$, but this proof gave no information about what happens at the endpoints $x = \pm 1$. Taylor's formula is not much help here either, because it is very laborious to calculate the successive derivatives of the function $f(x) = \tan^{-1} x$. To give a rigorous proof that (13) is actually valid on the closed interval $-1 \le x \le 1$, we adopt a completely different—and much more elementary—approach.

We begin with the algebraic identity

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^{n-1} + \frac{x^n}{1-x}, \qquad x \neq 1,$$

which can easily be verified by multiplying both sides by $1 - x$. Replacing x by $-t$ gives

$$\frac{1}{1+t} = 1 - t + t^2 - \cdots + (-1)^{n-1}t^{n-1} + \frac{(-1)^n t^n}{1+t},$$

with the restriction that $t \neq -1$. When t is now replaced by t^2, this becomes

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - \cdots + (-1)^{n-1}t^{2n-2} + \frac{(-1)^n t^{2n}}{1+t^2},$$

without restriction. By integrating both sides of this from $t = 0$ to $t = x$, we obtain

$$\tan^{-1} x = \int_0^x \frac{dt}{1+t^2} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^{n-1}\frac{x^{2n-1}}{2n-1} + R_n(x), \qquad (14)$$

where

$$R_n(x) = (-1)^n \int_0^x \frac{t^{2n}}{1+t^2}\, dt.$$

We now prove that $R_n(x) \to 0$ as $n \to \infty$, provided only that $-1 \leq x \leq 1$. But this is easy, because the inequality $1 + t^2 \geq 1$ allows us to write

$$|R_n(x)| = \left| \int_0^x \frac{t^{2n}}{1+t^2}\, dt \right| \leq \left| \int_0^x t^{2n}\, dt \right| = \frac{|x|^{2n+1}}{2n+1} \leq \frac{1}{2n+1} \to 0.$$

The fact that $R_n(x) \to 0$ for the stated values of x enables us to conclude from (14) that

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots, \qquad -1 \leq x \leq 1. \qquad (15)$$

Leibniz's famous formula

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

now follows at once from (15) on putting $x = 1$. In principle, Leibniz's formula can be used for computing the numerical value of π, but as a practical matter, the series converges so slowly that this method is of little value. A more efficient procedure is to use the formula

$$\frac{\pi}{4} = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}, \qquad (16)$$

and then to compute the two terms on the right by means of (15). [To establish (16), notice that if $A = \tan^{-1}\frac{1}{2}$ and $B = \tan^{-1}\frac{1}{3}$, then

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{6}} = 1.]$$

However, most of the extended computations of π have been based on the formula

$$\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}, \qquad (17)$$

which was discovered in 1706 by John Machin, a Scottish astronomer. We shall not pursue the details of these matters any further, but instead simply mention that π has been computed by these methods to more than 500,000 decimal places, of which the first twenty are

$$\pi = 3.14159 \quad 26535 \quad 89793 \quad 23846 \quad \ldots .$$

Further information can be found in P. Beckmann, *A History of* π (Golem Press, 1971), especially pp. 140–141 and 180–181.*

Remark 1 Given a function $f(x)$ that is infinitely differentiable in some interval containing the point $x = 0$, we have examined the possibility of expanding this function as a power series in x. More generally, if $f(x)$ is infinitely differentiable in some interval containing the point $x = a$, we can ask about its possible expansion as a power series in $x - a$,

$$f(x) = \sum_{n=0}^{\infty} a_n(x - a)^n = a_0 + a_1(x - a) + a_2(x - a)^2 + \cdots . \tag{18}$$

This problem is equivalent to the first, because (18) is the same as the series $g(w) = a_0 + a_1w + a_2w^2 + \cdots$, where $w = x - a$ and $g(w) = f(x)$, and therefore no separate discussion is required. Since $g^{(n)}(0) = f^{(n)}(a)$, the *Taylor series of* $f(x)$ *in powers of* $x - a$ *(or at* $x = a$*)* is

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x - a)^n \\ &= f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots . \end{aligned} \tag{19}$$

Some writers refer to (3), which is the special case of (19) corresponding to $a = 0$, as *Maclaurin's series.* However, this custom has no historical justification and is rapidly being abandoned. Whenever we use the phrase "Taylor series" without qualification, we always mean "Taylor series in powers of x," or at $x = 0$.

The fundamental tool for proving the validity of the Taylor series expansion (19) for a specific function $f(x)$ is a slightly expanded version of *Taylor's formula with remainder* as discussed above. This expanded version states that $f(x)$ can be written in the form

$$\begin{aligned} f(x) = f(a) + f'(a)(x - a) &+ \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 \\ &+ \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n + R_n(x), \end{aligned} \tag{20}$$

where the remainder $R_n(x)$ is given by the formula

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n + 1)!}(x - a)^{n+1} \tag{21}$$

and c is some number between a and x. And just as in the simpler but equivalent case expressed by (5), (6), and (7), the series on the right of (19) converges

*More recently, π has been computed to more than 100 million decimal places. For an account of this see the article on the Indian mathematical genius Srinivasa Ramanujan (1887–1920) in the February 1988 issue of *Scientific American.*

to $f(x)$ precisely when

$$\lim_{n\to\infty} R_n(x) = 0. \tag{22}$$

The polynomial part of (20) is called the *nth-degree Taylor polynomial at $x = a$.* We notice that the remainder term (21) is very similar in form to the terms in the Taylor polynomial, except that $f^{(n+1)}(x)$ is evaluated at c instead of a. All we can say about c is that it lies somewhere between a and x. The expression (21) for the remainder is often called *Lagrange's remainder formula.**

Example 5 Use (21) and (22) to prove the validity of the following Taylor series expansion of $f(x) = \sin x$ about the point $a = \pi/2$:

$$\sin x = 1 - \frac{1}{2!}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(x - \frac{\pi}{2}\right)^4 - \frac{1}{6!}\left(x - \frac{\pi}{2}\right)^6 + \cdots. \tag{23}$$

Solution We start by calculating the successive derivatives of $f(x) = \sin x$ and finding their values at $a = \pi/2$:

$$\begin{aligned}
f(x) &= \sin x, & f\left(\frac{\pi}{2}\right) &= \sin\frac{\pi}{2} = 1;\\
f'(x) &= \cos x, & f'\left(\frac{\pi}{2}\right) &= \cos\frac{\pi}{2} = 0;\\
f''(x) &= -\sin x, & f''\left(\frac{\pi}{2}\right) &= -\sin\frac{\pi}{2} = -1;\\
f'''(x) &= -\cos x, & f'''\left(\frac{\pi}{2}\right) &= -\cos\frac{\pi}{2} = 0;\\
f^{(4)}(x) &= \sin x, & f^{(4)}\left(\frac{\pi}{2}\right) &= \sin\frac{\pi}{2} = 1;
\end{aligned}$$

and so on. These results show that in this case the Taylor series (19) is just the series (23). We must now prove that this series converges to $\sin x$ for all x. But Lagrange's formula gives

$$|R_n(x)| = \left|\frac{f^{(n+1)}(c)}{(n+1)!}\left(x - \frac{\pi}{2}\right)^{n+1}\right| \le \frac{|x - \pi/2|^{n+1}}{(n+1)!},$$

because $f^{(n+1)}(c)$ is either $\pm\sin c$ or $\pm\cos c$, and therefore $|f^{(n+1)}(c)| \le 1$. By using (8) we now conclude that

$$\lim_{n\to\infty} R_n(x) = 0,$$

and this proves the validity of (23).

---

*Joseph Louis Lagrange (1736–1813) was the foremost French mathematician of the late eighteenth and early nineteenth centuries. In his early life he made outstanding discoveries in the calculus of variations, number theory, and analytical mechanics. His genius for generalization and analysis was most fully revealed in his great treatise *Mécanique Analytique* (1788). In this masterpiece he unified general mechanics and made of it, as the Irish scientist Sir William Hamilton later said, "a kind of scientific poem." Among the enduring legacies of this work for the mathematical physics of our own time are Lagrange's generalized coordinates, Lagrange's equations of motion, and the concept of potential energy. [For some details about these matters, see pp. 526–532 of George F. Simmons, *Differential Equations,* 2nd ed. (McGraw-Hill, 1991).] His later life was shadowed by a profound melancholy in which he lost his taste for mathematics and science and was sadly skeptical of the grandiose bureaucratic schemes of politicians for reforming human nature and relieving human misery. "If you wish to see the human mind truly great," he said, "enter Newton's study when he is decomposing sunlight with his prism or unveiling the system of the world with his mathematics."

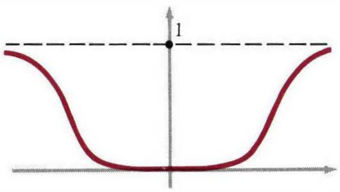


Figure 14.2

Remark 2 In Problem 42 of Section 12.3 we asked students to consider the function $f(x)$ defined for all real numbers x by

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0, \\ 0 & x = 0. \end{cases}$$

This function is continuous and has derivatives of all orders for all values of x. Furthermore, every derivative vanishes at $x = 0$, that is, $f^{(n)}(0) = 0$ for every positive integer n. This means that the graph of the function (Fig. 14.2) is extremely flat at the origin—we might even say "infinitely flat." For this function, Taylor's formula (5) becomes

$$f(x) = 0 + 0 + \cdots + 0 + R_n(x).$$

The Taylor series of the function is therefore the series

$$0 + 0 + \cdots + 0 + \cdots,$$

which converges for every x but converges to $f(x)$ only for $x = 0$. Thus, even though a function has derivatives of all orders everywhere, it still is not necessarily represented by its Taylor series; and if we wish to establish the validity of such a representation, we must invoke solid additional arguments of some kind, as illustrated in the examples discussed above.

Remark 3 We now give a proof of formula (7) for the remainder $R_n(x)$. First, we define a function $S_n(x)$ by writing

$$R_n(x) = S_n(x)x^{n+1}$$

for $x \neq 0$. Next, we hold x fixed and define a function $F(t)$ for $0 \leq t \leq x$ (or $x \leq t \leq 0$) by writing

$$F(t) = f(x) - f(t) - f'(t)(x-t) - \frac{f''(t)}{2!}(x-t)^2 - \cdots - \frac{f^{(n)}(t)}{n!}(x-t)^n - S_n(x)(x-t)^{n+1}.$$

Equation (5) shows that $F(0) = 0$. Also, it is obvious that $F(x) = 0$. It follows that $F'(c) = 0$ for some number c between 0 and x.* By differentiating $F(t)$ with respect to t, canceling, and replacing t by c, we get

$$F'(c) = -\frac{f^{(n+1)}(c)}{n!}(x-c)^n + S_n(x)(n+1)(x-c)^n = 0,$$

so

$$S_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}$$

and the proof of (7) is complete.

*In words, if the graph of our function $F(t)$ touches the t-axis at two points, then it must have a horizontal tangent somewhere in between. This inference rests on the Mean Value Theorem.

PROBLEMS

1 Use (3) to find the Taylor series of each of the following functions, and then use (6) and (7) to prove that each of these expansions is valid for all x:
(a) $\cos x$; (b) e^{-x}; (c) e^{3x}.

2 Obtain the series (11) for $\cos x$ from the series (10) for $\sin x$ by integrating term by term. Hint: Remember that the indefinite integral on the left must equal zero when $x = 0$.

3 Obtain the series (10) for $\sin x$ by differentiating the series (11) for $\cos x$.

4 Find the Taylor series expansion of each of the following functions (hint: any power series that converges to a function in an interval about $x = 0$ must be the Taylor series of that function):
(a) x^2e^x; (b) xe^{-3x};
(c) $\cos \sqrt{x}$; (d) $x \sin 5x$;
(e) $\sin x^2$;
(f) $\cos^2 x$ [hint: $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$];
(g) $\sin^2 x$; (h) $1 - 7 \sin^2 x$;
(i) $f(x) = \dfrac{\sin x}{x}$ for $x \neq 0, f(0) = 1$.

In Problems 5–9, use any method to obtain each of the given Taylor series expansions as far as indicated, without worrying about convergence.

5 $\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots$.

6 $\sec^2 x = 1 + x^2 + \frac{2}{3}x^4 + \cdots$.

7 $\ln (\cos x) = -\dfrac{x^2}{2} - \dfrac{x^4}{12} - \dfrac{x^6}{45} - \cdots$.

8 $\ln (1 + \sin x) = x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{12}x^4 + \cdots$.

9 $\ln (1 + e^x) = \ln 2 + \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{192}x^4 + \cdots$.

10 Show that a polynomial $P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ is its own Taylor series.

11 If $P(x)$ is a polynomial of degree n and a is any number, show that

$$P(x) = P(a) + \frac{P'(a)}{1!}(x - a) + \cdots + \frac{P^{(n)}(a)}{n!}(x - a)^n.$$

12 Find the Taylor series expansion of
(a) $3x^2 - 5x + 7$ in powers of $x - 1$;
(b) x^3 in powers of $x + 2$.
Check the expansions in (a) and (b) by using algebra.

13 (a) Let p be an arbitrary constant and use (3) to obtain the *binomial series*

$$(1 + x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots + \frac{p(p-1)(p-2)\cdots(p-n+1)}{n!}x^n + \cdots.$$

(b) Observe that this series terminates and is a polynomial whenever p is a nonnegative integer, and only in this case. In all other cases, show that this series has radius of convergence $R = 1$. The fact that the expansion in (a) is valid for $|x| < 1$ is not easy to establish by the methods of this section; a different type of proof is outlined in Additional Problem 9 at the end of this chapter.

14 Use Problem 13 to write the Taylor series expansion of $(1 - x^2)^{-1/2}$, and integrate this to get the Taylor series for $\sin^{-1} x$.

15 Find power series representations for

(a) $\displaystyle\int \frac{\sin x}{x}\,dx$; (b) $\displaystyle\int \sqrt{1 + x^3}\,dx$;

(c) $\displaystyle\int \frac{dx}{\sqrt{1 + x^4}}$.

16 Prove formula (17) by putting $A = \tan^{-1} \frac{1}{5}$, $B = \tan^{-1} \frac{1}{239}$, and computing successively $\tan 2A$, $\tan 4A$, and $\tan (4A - B)$.

17 Calculate the numerical value of π from each of formulas (16) and (17) by using the first four nonzero terms of the series (15) to calculate the inverse tangents. Round off your answers to five decimal places.

18 Use Lagrange's remainder formula to prove the validity of the following Taylor series expansions for all x:

(a) $$\sin x = \frac{1}{2} + \frac{1}{2}\sqrt{3}\left(x - \frac{\pi}{6}\right) - \frac{1}{2}\cdot\frac{1}{2!}\left(x - \frac{\pi}{6}\right)^2 - \frac{1}{2}\sqrt{3}\cdot\frac{1}{3!}\left(x - \frac{\pi}{6}\right)^3 + \frac{1}{2}\cdot\frac{1}{4!}\left(x - \frac{\pi}{6}\right)^4 + \cdots;$$

(b) $$\cos x = \frac{1}{2}\sqrt{2} - \frac{1}{2}\sqrt{2}\left(x - \frac{\pi}{4}\right) - \frac{1}{2}\sqrt{2}\cdot\frac{1}{2!}\left(x - \frac{\pi}{4}\right)^2 + \frac{1}{2}\sqrt{2}\cdot\frac{1}{3!}\left(x - \frac{\pi}{4}\right)^3 + \frac{1}{2}\sqrt{2}\cdot\frac{1}{4!}\left(x - \frac{\pi}{4}\right)^4 + \cdots;$$

(c) $$\sin x = \frac{1}{2}\sqrt{3} + \frac{1}{2}\left(x - \frac{\pi}{3}\right) - \frac{1}{2}\sqrt{3}\cdot\frac{1}{2!}\left(x - \frac{\pi}{3}\right)^2 - \frac{1}{2}\cdot\frac{1}{3!}\left(x - \frac{\pi}{3}\right)^3 + \frac{1}{2}\sqrt{3}\cdot\frac{1}{4!}\left(x - \frac{\pi}{3}\right)^4 + \cdots;$$

(d) $$\cos x = -\left(x - \frac{\pi}{2}\right) + \frac{1}{3!}\left(x - \frac{\pi}{2}\right)^3 - \frac{1}{5!}\left(x - \frac{\pi}{2}\right)^5 + \cdots.$$

14.5 COMPUTATIONS USING TAYLOR'S FORMULA

How are the values of $\sin x$ computed? In everyday work, we simply push a couple of buttons on our calculator and the desired values are displayed. But where do the values come from that the engineers put into the calculator in the first place? Is there a little old man in the calculator factory with very sharp eyes, a superb protractor, and a space-age-quality ruler who draws the angles with great care, measures the opposite side and hypotenuse, and then computes the sine? Not very likely. This method might give two or even three decimal places correctly, but not the many decimal places available from a good calculator.

In this section we describe a method for obtaining any value of $\sin x$ with great accuracy. More generally, we will show how to use the Taylor polynomials of Section 14.4 to find the values of such important functions as $\sin x$, $\cos x$, e^x, and $\ln x$ to as many decimal places as we please. And in doing this, we will freely use the great power of calculators for quickly performing simple but tedious arithmetical calculations.

THE USE OF LAGRANGE'S REMAINDER FORMULA

In the previous section we proved that for all x,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots.$$

In particular, if we put $x = 1$ we obtain the following familiar formula for e as the sum of an infinite series:

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots.$$

This means that e can be approximated to any degree of accuracy by using a suitable partial sum,

$$e \cong 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}. \tag{1}$$

n	$1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$
0	1.000000000
1	2.000000000
2	2.500000000
3	2.666666667
4	2.708333334
5	2.716666667
9	2.718281526
10	2.718281801

The value of n in this sum is at our disposal: the larger we take n, the more accurate the approximation will be. This is shown in the adjoining table, where we have used (1) to approximate e for various values of n. The value of e to nine-decimal-place accuracy is 2.718281828. This level of accuracy is achieved with $n = 12$.

Now suppose we want to approximate e to a previously specified level of accuracy. It would then be important to determine how large the value of n should be to guarantee the desired accuracy. For example, how large should n be to guarantee an error that is less than 0.000005? Lagrange's remainder formula is the basic tool for answering such questions.

According to Lagrange's formula (21) in Section 14.4, if $f(x)$ is approximated by its nth-degree Taylor polynomial at $x = a$, then the absolute value of the error $R_n(x)$ is

$$|R_n(x)| = \left|\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}\right| = \frac{|f^{(n+1)}(c)|}{(n+1)!}|x-a|^{n+1}. \tag{2}$$

In this formula, c is an unknown number between a and x, so the value of $f^{(n+1)}(c)$ usually cannot be determined exactly. However, we saw in Section 14.4

that it is often possible to find an upper bound for $|f^{(n+1)}(c)|$, that is, to find a number M such that $|f^{(n+1)}(c)| \leq M$. In such a case it follows from (2) that

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}, \tag{3}$$

so we have a convenient upper bound for the magnitude of the error $R_n(x)$. In the following examples we give several applications of these ideas, but first we introduce some terminology intended to clarify the concept of the accuracy of approximations.

We say that an approximation is *accurate to n decimal places* if the magnitude of the error is less than 0.5×10^{-n}. This means that the largest possible error cannot change the figure in the nth decimal place. The following table will be helpful for understanding this idea.

Level of accuracy	*Magnitude of error is less than*
1 decimal place	$0.05 = 0.5 \times 10^{-1}$
2 decimal places	$0.005 = 0.5 \times 10^{-2}$
3 decimal places	$0.0005 = 0.5 \times 10^{-3}$
4 decimal places	$0.00005 = 0.5 \times 10^{-4}$

Example 1 Use (1) to find an approximation for e that is accurate to four decimal places.

Solution By Taylor's formula with remainder we have

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \frac{e^c}{(n+1)!}x^{n+1},$$

where c is between 0 and x. For $x = 1$ this gives

$$e = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} + \frac{e^c}{(n+1)!},$$

where c is between 0 and 1. Thus the magnitude of the error in the approximation (1) is

$$|R_n| = \left|\frac{e^c}{(n+1)!}\right| = \frac{e^c}{(n+1)!},$$

where $0 < c < 1$. Even though we do not know the exact value of c, nevertheless $c < 1$ tells us that $e^c < e^1 = e$, so

$$|R_n| < \frac{e}{(n+1)!}. \tag{4}$$

Furthermore, even though we do not know the exact value of e (this is what we are trying to approximate!), we do know that $e < 3$. This allows us to replace (4) by

$$|R_n| < \frac{3}{(n+1)!}. \tag{5}$$

We are now on solid ground: If we choose n so that

$$|R_n| < \frac{3}{(n+1)!} < 0.5 \times 10^{-4} = 0.00005, \tag{6}$$

then the approximation (1) is guaranteed to be accurate to four decimal places. A suitable value of n can be found by trial and error by using a calculator. Thus, we can calculate $3/(n + 1)!$ for $n = 0, 1, 2, \ldots$ until we find a value of n satisfying (6). For $n = 7$ and $n = 8$ we have

$$\frac{3}{(7+1)!} \cong 0.000074 \quad \text{and} \quad \frac{3}{(8+1)!} \cong 0.000008,$$

so $n = 8$ is the first positive integer satisfying (6). Therefore, to four-decimal-place accuracy we have

$$e \cong 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!},$$

which a calculator shows is approximately 2.71827876985. We conclude that $e \cong 2.7183$ to four-decimal-place accuracy.

APPROXIMATING SINES AND COSINES

Example 2 Use the Taylor series for $\sin x$ at $a = 0$ to approximate $\sin 5°$ to five-decimal-place accuracy.

Solution In the Taylor series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots,$$

we remind the student that the angle x is assumed to be measured in radians. Since $5° = \pi/36$ radians, we have

$$\sin 5° = \sin \frac{\pi}{36} = \frac{\pi}{36} - \frac{(\pi/36)^3}{3!} + \frac{(\pi/36)^5}{5!} - \frac{(\pi/36)^7}{7!} + \cdots. \tag{7}$$

We must now decide how many terms of this series to keep in order to guarantee five-decimal-place accuracy.

If we write $f(x) = \sin x$, then by (2) the magnitude of the error committed when $\sin x$ is approximated by its nth-degree Taylor polynomial is

$$|R_n| = \frac{|f^{(n+1)}(c)|}{(n+1)!}|x|^{n+1},$$

where c is some number between 0 and x. Since $f^{(n+1)}(c)$ is either $\pm\sin c$ or $\pm\cos c$, it is clear that $|f^{(n+1)}(c)| \le 1$, and therefore

$$|R_n| \le \frac{|x|^{n+1}}{(n+1)!}.$$

In particular, if $x = \pi/36$ then

$$|R_n| \le \frac{(\pi/36)^{n+1}}{(n+1)!}.$$

This shows that we will achieve five-decimal-place accuracy if

$$\frac{(\pi/36)^{n+1}}{(n+1)!} < 0.5 \times 10^{-5} = 0.000005. \tag{8}$$

By experimenting with the help of a calculator, we find that for $n = 2$ and $n = 3$ we have

$$\frac{(\pi/36)^3}{3!} \cong 0.0001 \qquad \text{and} \qquad \frac{(\pi/36)^4}{4!} \cong 0.000002,$$

so $n = 3$ is the first positive integer satisfying (8). This shows that in (7) we only need to keep terms up to the third power to guarantee five-decimal-place accuracy. We can therefore write

$$\sin 5^\circ \cong \frac{\pi}{36} - \frac{(\pi/36)^3}{3!} \cong 0.08716,$$

in full confidence that five decimal places are correct.

To approximate the value of a function $f(x)$ at a point x_0 by using a Taylor series in $x - a$, two factors must be kept in mind when choosing the location of the point a. First, it should be fairly easy to evaluate $f(x)$ and its derivatives at a, since these values are necessary for constructing the Taylor series. And second, a should be as close as possible to x_0, since the series will converge more rapidly when $x_0 - a$ is small; that is, fewer terms will be required in a partial sum to obtain a specified level of accuracy.

Example 3 Approximate $\cos 93^\circ$ to six-decimal-place accuracy.

Solution We choose $a = \pi/2$ $(= 90^\circ)$, because $\cos x$ and its derivatives are easy to evaluate at this point and this value of a is quite close to the point $x_0 = 93^\circ = \frac{31}{60}\pi$ where we want to make the approximation. From Problem 18(d) in the previous section, the Taylor series for $\cos x$ at $a = \pi/2$ is

$$\cos x = -\left(x - \frac{\pi}{2}\right) + \frac{1}{3!}\left(x - \frac{\pi}{2}\right)^3 - \frac{1}{5!}\left(x - \frac{\pi}{2}\right)^5 + \cdots.$$

For $x = \frac{31}{60}\pi$ this becomes

$$\cos 93^\circ = \cos \frac{31}{60}\pi = -\frac{\pi}{60} + \frac{1}{3!}\left(\frac{\pi}{60}\right)^3 - \frac{1}{5!}\left(\frac{\pi}{60}\right)^5 + \cdots. \tag{9}$$

We must now decide how many terms of this series to keep for six-decimal-place accuracy.

If we write $f(x) = \cos x$, then the magnitude of the error committed in approximating $\cos x$ by its nth-degree Taylor polynomial at $a = \pi/2$ is

$$|R_n| = \frac{|f^{(n+1)}(c)|}{(n+1)!}\left|x - \frac{\pi}{2}\right|^{n+1},$$

where c is some number between $\pi/2$ and x. Just as in Example 2 we have $|f^{(n+1)}(c)| \le 1$, so

$$|R_n| \le \frac{|x - \pi/2|^{n+1}}{(n+1)!}.$$

In particular, if $x = \frac{31}{60}\pi$ then

$$|R_n| \le \frac{(\pi/60)^{n+1}}{(n+1)!}.$$

We will have six-decimal-place accuracy if

$$\frac{(\pi/60)^{n+1}}{(n+1)!} < 0.5 \times 10^{-6} = 0.0000005. \tag{10}$$

For the cases $n = 2$ and $n = 3$, our calculator tells us that

$$\frac{(\pi/60)^3}{3!} \cong 0.00002 \qquad \text{and} \qquad \frac{(\pi/60)^4}{4!} \cong 0.0000003,$$

so $n = 3$ is the first positive integer that satisfies (10). Thus, in (9) we only need to keep terms up to the third power to be sure of six-decimal-place accuracy:

$$\cos 93^\circ = -\frac{\pi}{60} + \frac{1}{3!}\left(\frac{\pi}{60}\right)^3 \cong -0.052336.$$

APPROXIMATING LOGARITHMS

The approximation of natural logarithms begins with the familiar series

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \tag{11}$$

which is valid for $-1 < x \le 1$ (Example 1 in Section 14.3). This series itself is of little value for computation because it converges so slowly. However, if we replace x by $-x$ we obtain

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots \tag{12}$$

for $-1 \le x < 1$, and by subtracting (12) from (11) we get

$$\ln\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots\right) \tag{13}$$

for $-1 < x < 1$. This series can be used for computing the natural logarithm of any positive number y by putting

$$y = \frac{1+x}{1-x}, \tag{14}$$

or equivalently

$$x = \frac{y-1}{y+1}. \tag{15}$$

If we sketch the graph of the function (14) [Fig. 14.3], we see at once that $y > 0$ corresponds to $-1 < x < 1$. Thus, for example, to compute ln 2 we put $y = 2$ in (15), which gives $x = \frac{1}{3}$. Substituting this value in (13) yields

$$\ln 2 = 2\left[\frac{1}{3} + \frac{1}{3}\left(\frac{1}{3}\right)^3 + \frac{1}{5}\left(\frac{1}{3}\right)^5 + \frac{1}{7}\left(\frac{1}{3}\right)^7 + \cdots\right].$$

If we now calculate the sum of the four terms shown here and round off to five decimal places, we get

$$\ln 2 \cong 0.69313,$$

which compares favorably with the calculator value $\ln 2 \cong 0.69314718056$.

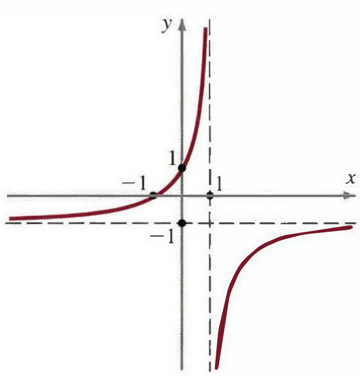


Figure 14.3

PROBLEMS

It is to be understood that these are all calculator problems, so we will not repeat the symbol ▦ that signals this fact.

1 Use the inequality (5) to find a value of n that guarantees that (1) will approximate e to
(a) five-decimal-place accuracy;
(b) ten-decimal-place accuracy.

2 Use $x = -1$ in the Taylor series for e^x at $a = 0$ to approximate $1/e$ to four-decimal-place accuracy.

3 Use $x = \frac{1}{2}$ in the Taylor series for e^x at $a = 0$ to approximate $\sqrt{e}$ to five-decimal-place accuracy.

4 Use the Taylor series for $\sin x$ at $a = 0$ to approximate $\sin 3°$ to six-decimal-place accuracy.

5 Use the Taylor series for $\cos x$ at $a = 0$ to approximate $\cos(\pi/15)$ to six-decimal-place accuracy.

6 Use an appropriate Taylor series for $\sin x$ to approximate $\sin 84°$ to six-decimal-place accuracy.

7 Use an appropriate Taylor series for $\cos x$ to approximate $\cos 32°$ to six-decimal-place accuracy.

8 Use an appropriate Taylor series for $\sin x$ to approximate $\sin 50°$ to six-decimal-place accuracy.

9 Find the lowest-degree Taylor polynomial for $\sin x$ at $a = 0$ that guarantees five-place-accuracy on the interval $|x| \le \pi/4$.

10 If R_n is the nth remainder for $\sin x$ at $a = 0$, show that
(a) $|R_3| < 0.5 \times 10^{-5}$ for $|x| < 0.22 \cong 12.6°$;
(b) $|R_5| < 0.5 \times 10^{-5}$ for $|x| < 0.59 \cong 33.8°$;
(c) $|R_7| < 0.5 \times 10^{-5}$ for $|x| < 1.06 \cong 60.7°$;
(d) $|R_9| < 0.5 \times 10^{-5}$ for $|x| < 1.61 \cong 92.2°$.

11 If R_n is the nth remainder for $\cos x$ at $a = 0$, show that
(a) $|R_2| < 0.5 \times 10^{-5}$ for $|x| < 0.10 \cong 5.7°$;
(b) $|R_4| < 0.5 \times 10^{-5}$ for $|x| < 0.39 \cong 22.3°$;
(c) $|R_6| < 0.5 \times 10^{-5}$ for $|x| < 0.81 \cong 46.4°$;
(d) $|R_8| < 0.5 \times 10^{-5}$ for $|x| < 1.33 \cong 76.2°$.

12 Use the first two nonzero terms of the series (13) to approximate $\ln 1.25$. Round off your answer to four decimal places. Repeat the calculation by using the first four nonzero terms of the series.

13 Repeat the preceding problem for $\ln 3$ by using the first two, four, five, and six nonzero terms of the series, each time rounding off your answer to four decimal places.

14 Find an interval of values centered on $x = 0$ within which $\sin x$ can be approximated by $x - \dfrac{x^3}{3!}$ with four-decimal-place accuracy guaranteed. Find a corresponding interval for approximating $\sin x$ by $x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!}$ with the same accuracy.

15 Find an upper bound for the error in the approximation of $\cos x$ by $1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!}$ on the interval $|x| < 0.4$.

14.6 APPLICATIONS TO DIFFERENTIAL EQUATIONS

One very large part of mathematics where power series can be put to effective use is in the solution of differential equations. We hinted at this in Section 13.1, and now we explore the possibilities a little more fully.

Example 1 We begin by considering the equation

$$y' = y, \tag{1}$$

which we discussed in an informal way in Section 13.1. We assume that this equation has a power series solution of the form

$$y = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots \tag{2}$$

that converges for $|x| < R$ with $R > 0$. We know from Section 14.3 that a power series can be differentiated term by term in its interval of convergence, so

$$y' = a_1 + 2a_2x + 3a_3x^2 + \cdots + (n+1)a_{n+1}x^n + \cdots. \tag{3}$$

If two power series are equal on an interval, the ideas of Section 14.4 tell us that the corresponding coefficients must also be equal. Therefore, since $y' = y$, the series (2) and (3) must have equal coefficients:

$$a_1 = a_0, \quad 2a_2 = a_1, \quad 3a_3 = a_2, \quad \ldots, \quad (n+1)a_{n+1} = a_n, \quad \ldots.$$

These equations enable us to express each a_n in terms of a_0:

$$a_1 = a_0, \quad a_2 = \frac{a_1}{2} = \frac{a_0}{2}, \quad a_3 = \frac{a_2}{3} = \frac{a_0}{2 \cdot 3}, \quad \ldots, \quad a_n = \frac{a_0}{n!}, \quad \ldots.$$

When these coefficients are inserted in (2), we obtain our power series solution,

$$y = a_0\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots\right), \tag{4}$$

where no condition is imposed on a_0. It is important to understand that so far this solution is only tentative, because we have no guarantee that (1) actually has a power series solution of the form (2). The above argument shows only that *if* (1) has such a solution, *then* that solution must be (4). However, it follows at once from the ratio test that the series in (4) converges for all x, so the term-by-term differentiation is valid and (4) really is a solution of (1). In this case we easily recognize the series in (4) as the power series expansion of e^x, so (4) can be written as

$$y = a_0 e^x.$$

Needless to say, we can get this solution directly from (1) by separating variables and integrating. Nevertheless, it is important to realize that (4) would still be a perfectly respectable solution even if (1) were unsolvable by elementary methods and the series in (4) could not be recognized as the expansion of a familiar function.

Example 2 We next attack the second-order equation

$$y'' + y = 0 \tag{5}$$

by the same method. That is, we assume the equation has a solution in the form of a power series

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots = \sum_{n=0}^{\infty} a_nx^n \tag{6}$$

with radius of convergence $R > 0$. It is permissible to differentiate this series term by term, so

$$y' = a_1 + 2a_2x + 3a_3x^2 + \cdots = \sum_{n=1}^{\infty} na_nx^{n-1}$$

and

$$y'' = 2a_2 + 2 \cdot 3a_3x + 3 \cdot 4a_4x^2 + \cdots = \sum_{n=2}^{\infty} (n-1)na_nx^{n-2}.$$

In order to combine the series for y and y'' more easily, we write y'' as follows:

$$y'' = \sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2}x^n. \tag{7}$$

In effect, we replace n in the first formula for y'' by $n + 2$ and start the summation at 0 instead of 2. Now, adding the series (6) and (7) in accordance with the differential equation (5), we obtain

$$\sum_{n=0}^{\infty} [(n+1)(n+2)a_{n+2} + a_n]x^n = 0. \tag{8}$$

By thinking of the right side of this as a power series whose coefficients are all zero, we see that all the coefficients on the left must also be zero:

$$(n+1)(n+2)a_{n+2} + a_n = 0$$

or

$$a_{n+2} = -\frac{a_n}{(n+1)(n+2)}, \tag{9}$$

where $n = 0, 1, 2, 3, \ldots$. This is called the *recursion formula* for the coefficients a_n, because it enables us to calculate them step by step starting with a_0. We can see more clearly what this means if we write these equations out as follows:

$$\begin{aligned}
n &= 0, & a_2 &= -\frac{a_0}{1\cdot 2} = -\frac{a_0}{2!};\\
n &= 1, & a_3 &= -\frac{a_1}{2\cdot 3} = -\frac{a_1}{3!};\\
n &= 2, & a_4 &= -\frac{a_2}{3\cdot 4} = \frac{a_0}{4!};\\
n &= 3, & a_5 &= -\frac{a_3}{4\cdot 5} = \frac{a_1}{5!};\\
n &= 4, & a_6 &= -\frac{a_4}{5\cdot 6} = -\frac{a_0}{6!};\\
n &= 5, & a_7 &= -\frac{a_5}{6\cdot 7} = -\frac{a_1}{7!};
\end{aligned}$$

and so on. The emerging pattern is now clear:

$$\text{for even coefficients,} \qquad a_{2n} = (-1)^n \frac{a_0}{(2n)!};$$

$$\text{for odd coefficients,} \qquad a_{2n+1} = (-1)^n \frac{a_1}{(2n+1)!}.$$

By putting these coefficients back into the power series (6), we can write the solution of (5) as

$$\begin{aligned}
y &= a_0 + a_1 x - \frac{a_0}{2!}x^2 - \frac{a_1}{3!}x^3 + \frac{a_0}{4!}x^4 + \frac{a_1}{5!}x^5 - \frac{a_0}{6!}x^6 - \frac{a_1}{7!}x^7 + \cdots \\
&= a_0\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right] + a_1\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right].
\end{aligned} \tag{10}$$

Again, all we have shown is that *if* (5) has a solution of the form (6), *then* that solution must be (10). And again, since both series in (10) converge by the ratio test, term-by-term differentiation of these series is valid and (10) really is a solution of (5), without any restrictions on the constants a_0 and a_1. In particular, we see that each of the bracketed series in (10) taken individually is a solution of (5), by first putting $a_0 = 1$ and $a_1 = 0$, and then $a_0 = 0$ and $a_1 = 1$.

In this problem luck is on our side, because we recognize the two series in (10) as the Taylor series expansions of $\cos x$ and $\sin x$. This solution of (5) is therefore the general solution that we already know from our earlier work, namely, $y = a_0 \cos x + a_1 \sin x$, where a_0 and a_1 are arbitrary constants.*

*Problem 19 in Section 9.2.

However, in most cases we won't be this lucky. Our solution will be a power series, but only rarely will we be able to recognize it as the power series expansion of any function that we know. The student should realize that we have attached names to only a few of the infinite number of functions that exist out there in the universe of mathematics, and it is extremely unlikely that any given differential equation will have solutions that can be expressed in terms of these familiar named functions. Our next example provides an illustration of this.

Example 3 The differential equation

$$xy'' + y' + xy = 0 \tag{11}$$

has applications in mathematical physics that justify giving it serious special attention. To solve it, we assume a power series solution

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots = \sum_{n=0}^{\infty} a_nx^n \tag{12}$$

with positive radius of convergence. Then

$$y' = a_1 + 2a_2x + 3a_3x^2 + \cdots = \sum_{n=1}^{\infty} na_nx^{n-1}$$

and

$$y'' = 2a_2 + 2 \cdot 3a_3x + 3 \cdot 4a_4x^2 + \cdots = \sum_{n=2}^{\infty} (n-1)na_nx^{n-2}.$$

The terms on the left side of (11) can now be written

$$xy'' = 2a_2x + 2 \cdot 3a_3x^2 + 3 \cdot 4a_4x^3 + \cdots = \sum_{n=2}^{\infty} (n-1)na_nx^{n-1},$$

$$y' = a_1 + 2a_2x + 3a_3x^2 + \cdots = \sum_{n=1}^{\infty} na_nx^{n-1}$$

$$= a_1 + \sum_{n=2}^{\infty} na_nx^{n-1},$$

and

$$xy = a_0x + a_1x^2 + a_2x^3 + \cdots = \sum_{n=0}^{\infty} a_nx^{n+1}$$

$$= \sum_{n=2}^{\infty} a_{n-2}x^{n-1}.$$

If we add these three series in accordance with the differential equation (11), the result is

$$a_1 + \sum_{n=2}^{\infty} [(n-1)na_n + na_n + a_{n-2}]x^{n-1} = 0.$$

All the coefficients on the left must be zero, so $a_1 = 0$ and (with a slight simplification) we have the recursion formula

$$n^2a_n + a_{n-2} = 0$$

or

$$a_n = -\frac{a_{n-2}}{n^2} \tag{13}$$

for $n \geq 2$. Since $a_1 = 0$, repeated application of (13) tells us that $a_n = 0$ for every odd subscript. We now calculate the nonzero coefficients a_{2n} as follows:

$$a_0, \qquad a_2 = -\frac{a_0}{2^2}, \qquad a_4 = -\frac{a_2}{4^2} = \frac{a_0}{2^2 \cdot 4^2},$$

$$a_6 = -\frac{a_4}{6^2} = -\frac{a_0}{2^2 \cdot 4^2 \cdot 6^2}, \qquad \ldots, \qquad a_{2n} = (-1)^n \frac{a_0}{2^{2n}(n!)^2},$$

since $2^2 \cdot 4^2 \cdot 6^2 \cdots (2n)^2 = (2 \cdot 1)^2(2 \cdot 2)^2(2 \cdot 3)^2 \cdots (2 \cdot n)^2 = 2^{2n}(n!)^2$. With this information we can now write the solution (12) as

$$\begin{aligned} y &= a_0\left[1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots\right] \\ &= a_0 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^{2n}(n!)^2}. \end{aligned}$$

This series is very similar to the cosine series in (10). It can be obtained from the cosine series by replacing each odd factor in the denominators by the next larger even number. Its sum is a useful special function that is studied in advanced mathematics. This function is denoted by $J_0(x)$ and called the *Bessel function of order* 0 [see Problems 29(b) in Section 14.2 and 5 in Section 14.3].[*]

[*]Friedrich Wilhelm Bessel (1784–1846) was a famous German astronomer and an intimate friend of the great mathematician Gauss, with whom he corresponded for many years. He was the first man to determine accurately the distance of a fixed star: his parallax measurement of 1838 yielded a distance for the star 61 Cygni of 11 light-years, or about 360,000 times the diameter of the earth's orbit. In 1844 he discovered that Sirius, the brightest star in the sky, has a traveling companion and is therefore what is now known as a *binary star*. This Companion of Sirius, with the size of a planet but the mass of a star, and consequently a density many thousands of times the density of water, is one of the most interesting objects in the universe. It was the first white dwarf star to be discovered (see Section 5.5) and occupies a special place in modern theories of stellar evolution.

PROBLEMS

1 Consider the following differential equations:
(a) $y' = 2xy$;
(b) $y' + y = 1$.
In each case, find a power series solution of the form $\Sigma a_n x^n$, try to recognize the resulting series as the expansion of a familiar function, and verify your conclusion by solving the equation directly.

2 Consider the following differential equations:
(a) $xy' = y$;
(b) $x^2 y' = y$.
In each case, find a power series solution of the form $\Sigma a_n x^n$, solve the equation directly, and explain any discrepancies that arise.

3 The differential equations considered in the text and preceding problems are all linear, which means essentially that the dependent variable y and its derivatives occur only to the first power. The equation

$$y' = 1 + y^2 \tag{$*$}$$

is nonlinear, and it is easy to see directly that $y = \tan x$ is the particular solution for which $y(0) = 0$. Show that

$$\tan x = x + \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5 + \cdots$$

by assuming a solution for equation ($*$) in the form of a power series $\Sigma a_n x^n$ and finding the a_n's in two ways:

(a) by the method of the examples in the text (note particularly how the nonlinearity of the equation complicates the formulas);

(b) by differentiating equation (*) repeatedly to obtain

$$y'' = 2yy', \qquad y''' = 2yy'' + 2(y')^2, \qquad \ldots,$$

and using the formula $a_n = f^{(n)}(0)/n!$.

4 Solve the equation

$$y' = x - y, \qquad y(0) = 0$$

by each of the methods suggested in Problem 3. What familiar function does the resulting series represent?

5 Find a power series solution of $xy'' - y = 0$. For what x does this series converge?

14.7 (OPTIONAL) OPERATIONS ON POWER SERIES

In Section 14.4 we obtained Taylor series for various functions by using the formula $a_n = f^{(n)}(0)/n!$ to find the coefficients. But computing successive derivatives can be difficult and discouraging work if no simple pattern emerges. We can easily appreciate this fact by finding the seventh derivative of such functions as $\tan x$ or $x^5/(1 - x^4)$, because the calculations visibly sink us deeper into the bog with every step. For some functions we were able to establish the validity of their Taylor expansions by proving that $R_n(x) \to 0$ as $n \to \infty$, but this also can be difficult. To avoid such problems, we now discuss several algebraic methods for obtaining valid new Taylor expansions from ones that are already known.

Before we begin, we remind students that power series expansions are unique. This means that if a function $f(x)$ can be expressed as the sum of a power series *by any method,* then this series must be the Taylor series of $f(x)$. For example, we know from Section 13.3 that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots, \qquad |x| < 1; \tag{1}$$

and by first replacing x by x^4 and then multiplying through by x^5, we find that

$$\frac{1}{1-x^4} = 1 + x^4 + x^8 + x^{12} + \cdots, \qquad |x| < 1,$$

and

$$\frac{x^5}{1-x^4} = x^5 + x^9 + x^{13} + x^{17} + \cdots, \qquad |x| < 1.$$

We have deliberately avoided the very laborious task of using the formula $a_n = f^{(n)}(0)/n!$ to verify that the three series on the right are actually the Taylor series of the functions on the left. But these verifications aren't necessary, because this conclusion follows automatically from the uniqueness principle stated above.

MULTIPLICATION

Suppose we are given two power series expansions,

$$f(x) = \Sigma a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \tag{2}$$

and

$$g(x) = \Sigma b_n x^n = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots, \tag{3}$$

both valid on an interval $|x| < R$. If we ignore the question of convergence for a moment, then we can multiply these series in the same way we multiply two polynomials. That is, we systematically multiply each term of the first series into all the terms of the second series and then collect terms involving the same power of x. First, the term-by-term multiplication—

$$\begin{array}{ll} a_0: & a_0b_0 + a_0b_1x + a_0b_2x^2 + a_0b_3x^3 + \cdots \\ a_1x: & \qquad\quad a_1b_0x + a_1b_1x^2 + a_1b_2x^3 + \cdots \\ a_2x^2: & \qquad\qquad\qquad a_2b_0x^2 + a_2b_1x^3 + \cdots \\ a_3x^3: & \qquad\qquad\qquad\qquad\quad a_3b_0x^3 + \cdots \\ & \qquad\qquad\qquad\qquad \cdots . \end{array}$$

By adding these columns, we obtain the power series

$$a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2 + (a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0)x^3 + \cdots . \tag{4}$$

The form of the coefficient of x^n in (4) is evident: The subscripts of the a's increase as the subscripts of the b's decrease, and their sum remains constant and equals the exponent n on x^n. Briefly, we have multiplied (2) and (3) to obtain

$$f(x)g(x) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right)x^n. \tag{5}$$

We assert that this *product* of the series (2) and (3) actually converges on the interval $|x| < R$ to the product of the functions $f(x)$ and $g(x)$, as indicated by (5). The proof is not easy, and depends on the absolute convergence of the two series in the given interval. The details can be found in Appendix A.13.

Example 1 Find the Taylor series for $e^x \sin x$.

Solution We know that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \tag{6}$$

and

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots . \tag{7}$$

Our work can be arranged as follows:

$$\begin{aligned} e^x \sin x &= \left(1 + x + \frac{x^2}{2} + \cdots\right)\left(x - \frac{x^3}{6} + \cdots\right) \\ &= x \qquad - \frac{x^3}{6} + \cdots \\ &\qquad + x^2 \qquad\qquad - \frac{x^4}{6} + \cdots \\ &\qquad\qquad + \frac{x^3}{2} \qquad\qquad - \frac{x^5}{12} + \cdots \\ &= x + x^2 + \tfrac{1}{3}x^3 + \cdots . \end{aligned}$$

Since the two given series (6) and (7) converge for all x, the same is true of the product series. It is rarely easy—and usually quite impossible—to recognize the formula for the general term of the product series in this process.

Example 2 Find the Taylor series for $[\ln(1-x)]/(x-1)$.

Solution We know that

$$\ln(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right), \qquad |x| < 1,$$

so

$$\begin{aligned}\frac{\ln(1-x)}{x-1} &= \left(\frac{1}{1-x}\right)[-\ln(1-x)] \\ &= (1 + x + x^2 + \cdots)\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right) \\ &= x + (1 + \tfrac{1}{2})x^2 + (1 + \tfrac{1}{2} + \tfrac{1}{3})x^3 + \cdots \\ &= \sum_{n=1}^{\infty}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)x^n, \qquad |x| < 1.\end{aligned}$$

This is one of the rare cases where the general term of the product series is easy to recognize.

DIVISION

Two power series can be divided by the long division process that is used in elementary algebra for dividing polynomials. Since we are working with power series, the terms are of course arranged in order of increasing exponents, instead of in order of decreasing exponents as is usual with polynomials. In particular cases, however, one or both of the given series may be a polynomial.

Example 3 Find the Taylor series for $\tan x$ by dividing the series for $\sin x$ by the series for $\cos x$.

Solution We have

$$\begin{array}{r|l} & x + \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5 + \cdots \\ \hline 1 - \dfrac{x^2}{2} + \dfrac{x^4}{24} - \cdots & x - \dfrac{x^3}{6} + \dfrac{x^5}{120} - \cdots \\ & x - \dfrac{x^3}{2} + \dfrac{x^5}{24} - \cdots \\ \cline{2-2} & \quad \tfrac{1}{3}x^3 - \tfrac{1}{30}x^5 + \cdots \\ & \quad \tfrac{1}{3}x^3 - \tfrac{1}{6}x^5 + \cdots \\ \cline{2-2} & \qquad \tfrac{2}{15}x^5 + \cdots, \end{array}$$

so

$$\tan x = x + \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5 + \cdots. \tag{8}$$

It can be shown that this expansion is valid on the interval $|x| < \pi/2$. Since this is the largest interval with center $x = 0$ on which the denominator $\cos x$ is nonzero, the series (8) has radius of convergence $R = \pi/2$. The problem of discovering a formula for the general term of this series was solved by Euler in 1748. His solution depends on the ideas in the next section and is outlined in the Appendix at the end of this chapter.

The actual process of dividing one power series by another is clearly not very difficult. The theory that justifies this process is given in Appendix A.16.

SUBSTITUTION

This is a method we have already used: If a power series

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots \tag{9}$$

converges for $|x| < R$ and if $|g(x)| < R$, then we can certainly find $f[g(x)]$ by substituting $g(x)$ for x in (9). For example, there is no problem in using the series (1), (6), and (7) to obtain

$$\begin{aligned}\frac{1}{1+2x^2} = \frac{1}{1-(-2x^2)} &= 1 + (-2x^2) + (-2x^2)^2 + \cdots \\ &= 1 - 2x^2 + 4x^4 - \cdots, \qquad |2x^2| < 1;\end{aligned}$$

$$\begin{aligned}e^{x^4} &= 1 + x^4 + \frac{(x^4)^2}{2!} + \frac{(x^4)^3}{3!} + \cdots \\ &= 1 + x^4 + \frac{x^8}{2!} + \frac{x^{12}}{3!} + \cdots, \qquad \text{all } x;\end{aligned}$$

and

$$\begin{aligned}\sin 3x &= 3x - \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} - \cdots \\ &= 3x - \frac{27}{3!}x^3 + \frac{243}{5!}x^5 - \cdots, \qquad \text{all } x.\end{aligned}$$

In these examples we have substituted the simple functions $g(x) = -2x^2$, x^4, and $3x$ into appropriate power series, but much more is possible. Under suitable conditions, we can actually substitute one power series into another! Thus, suppose that the function $g(x)$ is given by a power series,

$$g(x) = b_0 + b_1 x + b_2 x^2 + \cdots, \tag{10}$$

and substitute this entire series for x in (9),

$$\begin{aligned}f[g(x)] &= a_0 + a_1 g(x) + a_2[g(x)]^2 + \cdots \\ &= a_0 + a_1[b_0 + b_1 x + \cdots] + a_2[b_0 + b_1 x + \cdots]^2 + \cdots.\end{aligned} \tag{11}$$

Again, this is perfectly legitimate as long as $|g(x)| < R$. However, the series on the right of (11) can now be converted into a power series in x by squaring, cubing, etc., and collecting like powers of x, and it can be proved that the power series formed in this way converges to $f[g(x)]$ whenever (10) is absolutely convergent and $|g(x)| < R$.*

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*See p. 180 of K. Knopp, *Theory and Application of Infinite Series* (Hafner, 1951).

Example 4 Apply the method of substitution to find the Taylor series of $e^{\sin x}$ up to the term containing x^4.

Solution We can use (6) and (7) to write

$$\begin{aligned} e^{\sin x} &= 1 + \left(x - \frac{x^3}{6} + \cdots\right) + \frac{1}{2}\left(x - \frac{x^3}{6} + \cdots\right)^2 \\ &\qquad + \frac{1}{6}\left(x - \frac{x^3}{6} + \cdots\right)^3 + \frac{1}{24}\left(x - \frac{x^3}{6} + \cdots\right)^4 + \cdots \\ &= 1 + \left(x - \frac{x^3}{6} + \cdots\right) + \frac{1}{2}\left(x^2 - \frac{1}{3}x^4 + \cdots\right) + \frac{1}{6}(x^3 + \cdots) \\ &\qquad + \frac{1}{24}(x^4 + \cdots) + \cdots \\ &= 1 + x + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \cdots. \end{aligned}$$

If we try to apply this method to $e^{\cos x}$, we find that infinitely many terms contribute to the formation of each coefficient, which is a difficult situation to deal with. For this reason, the method of substitution is not a practical tool unless $b_0 = 0$ in the series (10).

EVEN AND ODD FUNCTIONS

A function $f(x)$ defined on an interval $|x| < R$ is said to be *even* if $f(-x) = f(x)$, and *odd* if $f(-x) = -f(x)$. The Taylor series of the even function $\cos x$ contains only even powers of x, and the Taylor series of the odd function $\sin x$ contains only odd powers of x. These facts are special cases of a general principle: If $f(x)$ is an even function, then its Taylor series has the form

$$a_0 + a_2x^2 + a_4x^4 + a_6x^6 + \cdots;$$

and if $f(x)$ is an odd function, then its Taylor series has the form

$$a_1x + a_3x^3 + a_5x^5 + a_7x^7 + \cdots.$$

That is, the Taylor series of an even (odd) function contains only even (odd) powers of x. This is very easy to prove from the uniqueness of power series expansions.* As another example of this phenomenon, we know beforehand that the series on the right side of (8) contains only odd powers of x, because $\tan x$ is an odd function.

Many functions are even and many are odd, but most are neither. However, every function $f(x)$ defined on an interval $|x| < R$ can be expressed as the sum of an even function and an odd function:

$$f(x) = \tfrac{1}{2}[f(x) + f(-x)] + \tfrac{1}{2}[f(x) - f(-x)] = g(x) + h(x),$$

where—as is easily verified—

*We have only to point out that if $f(x) = \Sigma a_nx^n$ is even, then $\Sigma a_nx^n = \Sigma(-1)^na_nx^n$, so by uniqueness we have $a_n = (-1)^na_n$, and therefore $a_n = -a_n$ or $a_n = 0$ if n is odd. Similar reasoning applies if $f(x)$ is odd.

$$g(x) = \tfrac{1}{2}[f(x) + f(-x)] \qquad \text{is even}$$

and

$$h(x) = \tfrac{1}{2}[f(x) - f(-x)] \qquad \text{is odd.}$$

Further, if $f(x) = \Sigma_{n=0}^{\infty} a_n x^n$, then

$$g(x) = \sum_{n=0}^{\infty} a_{2n} x^{2n} \qquad \text{and} \qquad h(x) = \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1}.$$

Thus, the Taylor series of $f(x)$ splits into two power series, one with even exponents representing the even part of $f(x)$ and one with odd exponents representing the odd part.

Example 5 The even part and the odd part of

$$f(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

are

$$g(x) = \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

and

$$h(x) = \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots.$$

These are the two hyperbolic functions $\cosh x$ and $\sinh x$ that were defined in Section 9.7.

Example 6 Find the sum of the series

$$\frac{x^2}{2} + \frac{x^4}{4} + \cdots.$$

Solution This is the even part of the familiar series

$$f(x) = -\ln(1 - x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots.$$

The sum of the given series is therefore

$$\begin{aligned}\tfrac{1}{2}[f(x) + f(-x)] &= \tfrac{1}{2}[-\ln(1 - x) - \ln(1 + x)] \\ &= -\tfrac{1}{2}\ln(1 - x^2) = -\ln\sqrt{1 - x^2}.\end{aligned}$$

PROBLEMS

1 In Example 1, continue the calculation and find the terms of the product series as far as the term containing x^6.

In Problems 2–13, use multiplication to show that the given function has the indicated power series expansion.

2 $\dfrac{\sin x}{1 - x} = x + x^2 + \dfrac{5}{6}x^3 + \dfrac{5}{6}x^4 + \dfrac{101}{120}x^5 + \cdots.$

3 $e^{x+x^2} = 1 + x + \frac{3}{2}x^2 + \frac{7}{6}x^3 + \frac{25}{24}x^4 + \cdots.$

4 $e^x \cos x = 1 + x - \frac{1}{3}x^3 - \frac{1}{6}x^4 - \frac{1}{30}x^5 + \cdots.$

5 $\dfrac{\tan^{-1} x}{1-x} = x + x^2 + \dfrac{2}{3}x^3 + \dfrac{2}{3}x^4 + \dfrac{13}{15}x^5 + \cdots$.

6 $\ln^2 (1-x) = x^2 + x^3 + \frac{11}{12}x^4 + \frac{5}{6}x^5 + \cdots$.

7 $\dfrac{\cos x}{1-x} = 1 + x + \dfrac{1}{2}x^2 + \dfrac{1}{2}x^3 + \dfrac{13}{24}x^4 + \dfrac{13}{24}x^5 + \cdots$.

8 $\tan^2 x = x^2 + \frac{2}{3}x^4 + \frac{17}{45}x^6 + \cdots$.

9 $\dfrac{e^x}{2+x} = \dfrac{1}{2} + \dfrac{1}{4}x + \dfrac{1}{8}x^2 + \dfrac{1}{48}x^3 + \dfrac{1}{96}x^4 + \cdots$.

10 $\sqrt{1+x}\ln(1+x) = x - \frac{1}{24}x^3 + \frac{1}{24}x^4 - \frac{71}{1920}x^5 + \cdots$.

11 $e^{-x}\tan x = x - x^2 + \frac{5}{6}x^3 - \frac{1}{2}x^4 + \frac{41}{120}x^5 + \cdots$.

12 $\dfrac{1-x}{1-x^3} = \dfrac{1}{1+x+x^2} = 1 - x + x^3 - x^4 + x^6 - x^7 + x^9 - x^{10} + \cdots$.

13 $\dfrac{\sin x}{1+x} = x - x^2 + \dfrac{5}{6}x^3 - \dfrac{5}{6}x^4 + \dfrac{101}{120}x^5 + \cdots$.

14 By squaring the series for $\sin x$ and $\cos x$, show that $\sin^2 x + \cos^2 x = 1$, at least as far as the x^6 term.

15 If $f(x) = \sum_{n=0}^{\infty} a_n x^n$, use multiplication of series to show that

$$\frac{1}{1-x}f(x) = \sum_{n=0}^{\infty}(a_0 + a_1 + \cdots + a_n)x^n.$$

Use this result to write down the series in Problems 2, 5, and 7 by inspection.

16 Use Problem 15 to find the sum of the series $\sum_{n=0}^{\infty}(n+1)x^n$.

17 The binomial series expansion of $1/\sqrt{1-x}$ is

$$\frac{1}{\sqrt{1-x}} = 1 + \frac{1}{2}x + \frac{1\cdot 3}{2^2\cdot 2!}x^2 + \frac{1\cdot 3\cdot 5}{2^3\cdot 3!}x^3 + \frac{1\cdot 3\cdot 5\cdot 7}{2^4\cdot 4!}x^4 + \cdots.$$

Check this by squaring the series and showing that the result is $1 + x + x^2 + x^3 + x^4 + \cdots$, at least as far as the x^4 term.

18 In Example 3, continue the calculation and find the series for $\tan x$ as far as the term containing x^7.

19 Use division to obtain the series expansions given in Problems 2, 5, 7, 9, 12, and 13.

In Problems 20–27, use division to obtain the given expansions.

20 $\dfrac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$.

21 $\dfrac{1-x}{1-x+x^2} = 1 - x^2 - x^3 + x^5 + x^6 - x^8 - x^9 + \cdots$.

22 $\dfrac{x^2}{1-x+x^2-x^3} = x^2 + x^3 + x^6 + x^7 + \cdots$.

23 $\dfrac{x}{\sin x} = 1 + \dfrac{1}{6}x^2 + \dfrac{7}{360}x^4 + \cdots$.

24 $\dfrac{1}{1+x+x^2+x^3+\cdots} = 1 - x$.

25 $\dfrac{\sin x}{\ln(1+x)} = 1 + \dfrac{1}{2}x - \dfrac{1}{4}x^2 - \dfrac{1}{24}x^3 + \cdots$.

26 $\sec x = \dfrac{1}{\cos x} = 1 + \dfrac{1}{2}x^2 + \dfrac{5}{24}x^4 + \cdots$.

27 $\dfrac{\sin^{-1} x}{\cos x} = x + \dfrac{2}{3}x^3 + \dfrac{11}{30}x^5 + \cdots$.

In Problems 28 and 29, use the method of substitution to find the given Taylor series.

28 $\ln(1+\sin x) = x - \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots$.

29 $\dfrac{1}{1-x^2e^x} = 1 + x^2 + x^3 + \dfrac{3}{2}x^4 + \dfrac{13}{6}x^5 + \dfrac{73}{24}x^6 + \cdots$.

30 Use substitution and the fact that $\sec x = 1/\cos x = 1/[1-(1-\cos x)]$ to find the Taylor series for $\sec x$ up to the term containing x^6. What is the radius of convergence?

31 Use multiplication and the result of Problem 30 to find the Taylor series for $\tan x$ up to the term containing x^7.

32 (a) Find the Taylor series for $\sec^2 x$ as far as the term containing x^6 by expanding $\sec^2 x = 1/\cos^2 x = 1/(1-\sin^2 x)$ as a geometric series in $\sin^2 x$.
(b) Find the same series by squaring the series found in Problem 30.
(c) Find the same series by differentiating the series found in Problem 31.

33 Show that a function $f(x)$ defined on an interval $|x| < R$ can be expressed *in only one way* as the sum of an even function $g(x)$ and an odd function $h(x)$.

34 Find the sum of the series $x + \dfrac{x^3}{3} + \dfrac{x^5}{5} + \cdots$.

35 Calculate each of the following limits by first finding the Taylor series of the given function:

(a) $\displaystyle\lim_{x\to 0}\frac{1-\cos x}{x^2}$; (b) $\displaystyle\lim_{x\to 0}\frac{x-\sin x}{x^3}$.

In this way, the use of Taylor series often provides a convenient alternative to the use of L'Hospital's rule.

36 Find the sum of each of the following series:
(a) $x + x^2 - x^3 + x^4 + x^5 - x^6 + + - \cdots$;
(b) $x^2 + x^3 + x^4 - x^5 + x^6 + x^7 + x^8 - x^9 + + + - \cdots$;
(c) $\dfrac{x}{2!} + \dfrac{x^2}{3!} + \dfrac{x^3}{4!} + \dfrac{x^4}{5!} + \cdots$;
(d) $1 + \dfrac{x^4}{4!} + \dfrac{x^8}{8!} + \dfrac{x^{12}}{12!} + \cdots$.

37 Calculate $f^{(7)}(0)$ if $f(x) = \tan x$ and use this to verify the coefficient of x^7 in the expansion found in Problem 18.

14.8 (OPTIONAL) COMPLEX NUMBERS AND EULER'S FORMULA

Consider the three familiar power series expansions

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots, \tag{1}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots, \tag{2}$$

and

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots. \tag{3}$$

The second and third of these series seem to be parts of the first series in some nonobvious way that involves changes in the signs of some of the terms. This in turn suggests that the three functions on the left are probably related to one another. There is indeed such a relation, which is known as *Euler's formula*:

$$e^{ix} = \cos x + i \sin x, \tag{4}$$

where $i = \sqrt{-1}$ is the so-called *imaginary unit*. This formula turns out to be one of the most important facts in the whole of mathematics, with implications that deeply influence mathematics itself and also many of its applications, particularly in the fields of physics and electrical engineering. A full explanation of Euler's formula would require us to develop a fairly complete theory of complex numbers and functions of a complex variable. With apologies, we leave this task to a more advanced course, and instead briefly outline a few of the necessary ideas in a very incomplete way that at least has the merit of lending a little plausibility to formula (4).

Up to this point, all of our work in this book has taken place in the context of the real number system. Nevertheless, the real numbers do have a serious deficiency—not every polynomial equation has a solution. Thus, the quadratic equation $x^2 + 1 = 0$ has no solution in the real number system because there is no real number whose square is -1. This deficiency was so crippling that several hundred years ago mathematicians felt the need to use the seemingly contradictory symbol $\sqrt{-1}$ to signify a solution of $x^2 + 1 = 0$. This symbol was later denoted by the letter i, and was thought of as an imaginary or fictitious number that could be manipulated algebraically just like an ordinary real number, except that $i^2 = -1$. Any qualms these early mathematicians may have felt about the puzzling nature of this "number" were set aside because it was too useful to ignore. Thus, for example, the equation $x^2 + 1 = 0$ was factored by writing it in the equivalent forms $x^2 - i^2 = 0$ or $(x + i)(x - i) = 0$, and its solutions were exhibited as the numbers $x = \pm i$.

Without entering into the details that would be needed to give mathematical respectability to our discussion, we now simply describe the *complex numbers* as all formal expressions $a + bi$, where a and b are real numbers and i is the *imaginary unit* for which $i^2 = -1$. The complex numbers take on the character of a legitimate number system by means of the following general rule for performing calculations: In adding, multiplying, and dividing, follow all the familiar rules of elementary algebra and then simplify wherever possible by using the equation $i^2 = -1$ to remove all powers of i higher than the first, as in

$$i^3 = i^2 \cdot i = (-1) \cdot i = -i, \qquad i^4 = i^2 \cdot i^2 = (-1)(-1) = 1, \qquad i^5 = i^4 \cdot i = i, \tag{5}$$

and so on.

The complex number $a + bi$ can be identified with the real number a if $b = 0$, so the complex number system constitutes an enlargement of the real number system. Not only does the equation $x^2 + 1 = 0$ acquire the two solutions i and $-i$ in this way, but also every quadratic equation $ax^2 + bx + c = 0$ acquires the two familiar solutions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

which are real and distinct if $b^2 - 4ac > 0$, real and equal if $b^2 - 4ac = 0$, and complex and distinct if $b^2 - 4ac < 0$. For example, the equation $x^2 - 6x + 13 = 0$ has the distinct complex roots

$$x = \frac{6 \pm \sqrt{36 - 52}}{2} = \frac{6 \pm \sqrt{-16}}{2} = \frac{6 \pm 4\sqrt{-1}}{2}$$
$$= \frac{6 \pm 4i}{2} = 3 \pm 2i.$$

Much more than this is true: Every polynomial equation of the form

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0,$$

where n is a positive integer and the a's are arbitrary real numbers with $a_n \neq 0$, has exactly n roots (some of which may be equal to one another) among the complex numbers. Moreover, this is still true even if the coefficients are complex. This fact is known as the *fundamental theorem of algebra.* It shows that there is no need to construct further enlargements of the complex number system in order to solve all polynomial equations with complex coefficients.*

We now return to our original purpose, which was to gain a little insight into why Euler's formula (4) is true.

A perfectly satisfactory theory of power series can be constructed in which the variable is permitted to be a complex number instead of merely a real number. Within this theory, all of the series (1), (2), and (3) converge for all complex values of the variable. If we replace x in (1) by ix and use (5), then we obtain

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \cdots$$
$$= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \cdots$$
$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right),$$

which gives Euler's formula

$$e^{ix} = \cos x + i \sin x.$$

If we now replace x by $-x$ and use the fact that $\cos(-x) = \cos x$ and $\sin(-x) = -\sin x$, then this becomes

$$e^{-ix} = \cos x - i \sin x;$$

*There are many proofs of this important theorem, of varying levels of sophistication. See, for example, pp. 269–271 of R. Courant and H. Robbins, *What Is Mathematics?* (Oxford University Press, 1941).

and by first adding and then subtracting, we at once obtain

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \qquad \text{and} \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i}.$$

These formulas have many uses in advanced mathematics. A particularly interesting application is sketched in the Appendix at the end of this chapter. We also point out that if we put $x = \pi$ in Euler's formula, then we get

$$e^{\pi i} = \cos \pi + i \sin \pi$$

or

$$e^{\pi i} = -1.$$

This beautiful equation, connecting the mysterious and pervasive numbers π, e, and i, is one of the most remarkable facts in mathematics.

The ideas of this section are sketched in such a cursory fashion that they are bound to seem more suggestive than convincing. The eminent British mathematician E. C. Titchmarsh once remarked, "I met a man recently who told me that, so far from believing in the square root of minus one, he did not even believe in minus one. This is at any rate a consistent attitude." There is only one way to lift these concepts from the status of reasonable speculations to the realm of certainty, and this is to undertake a careful study of the theory of functions of a complex variable, also known as Complex Analysis. This subject is one of the richest and most rewarding branches of mathematics, and we heartily recommend it.

CHAPTER 14 REVIEW: CONCEPTS, FORMULAS, METHODS

Think through the following, and memorize the main expansions.

1 Power series, radius and interval of convergence.
2 Differentiation and integration of power series.
3 Taylor series.
4 Taylor's formula with derivative remainder.
5 Taylor series expansions of $\frac{1}{1+x}$, $\ln(1+x)$, $\tan^{-1} x$, e^x, $\sin x$, $\cos x$.

ADDITIONAL PROBLEMS FOR CHAPTER 14

SECTION 14.2

1 Consider a power series $\Sigma a_n x^n$ and assume that $\lim \sqrt[n]{|a_n|}$ exists, with ∞ as an allowed value. Show that the radius of convergence R of the series is given by the formula

$$R = \frac{1}{\lim \sqrt[n]{|a_n|}}.$$

Use this formula to find the radius of convergence of

(a) $\sum \frac{x^n}{n^n}$; (b) $\sum \frac{1}{(\ln n)^n} x^n$;

(c) $\sum \frac{n^{n^2}}{(n+1)^{n^2}} x^n$.

2 If the radius of convergence of $\Sigma a_n x^n$ can be calculated from formula (7) in Section 14.2, show that it can also be calculated from the formula in Problem 1. (Hint: See Additional Problem 18 in Chapter 13.) Show that the latter formula is more powerful than the former by considering the series

$$\sum_{n=1}^{\infty} \frac{x^n}{2^{n+(-1)^{n+1}}} = \frac{x}{2^2} + \frac{x^2}{2^1} + \frac{x^3}{2^4} + \frac{x^4}{2^3} + \cdots.$$

*3 If a power series converges conditionally at a point x_1, or diverges in such a way that its terms are bounded, show that x_1 must be an endpoint of the interval of convergence.

4 Use Problem 3 to find by inspection the radius of convergence of

(a) $1 + \frac{x}{3} + \frac{x^2}{3^2} + \frac{x^3}{3^3} + \cdots$;

(b) $x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \cdots;$

(c) $\Sigma x^{n!} = x + x + x^2 + x^6 + x^{24} + \cdots;$

(d) $1 + \frac{x}{2} + \frac{x^2}{3^2} + \frac{x^3}{2^3} + \frac{x^4}{3^4} + \frac{x^5}{2^5} + \frac{x^6}{3^6} + \cdots;$

(e) $\Sigma[2^{(-1)^n}x]^n = 1 + \frac{x}{2} + 2^2x^2 + \frac{x^3}{2^3} + 2^4x^4 + \cdots.$

5 Find the interval of convergence of $\Sigma a_n x^n$ if its coefficients are chosen in order from among the numbers 2, 3, . . . , 12 by throwing a pair of dice.

SECTION 14.3

6 Consider a power series $\Sigma a_n x^n$ in which the coefficients repeat cyclically, $a_{n+k} = a_n$. Show that

(a) $R = 1$;

(b) the sum is

$$\frac{a_0 + a_1x + a_2x^2 + \cdots + a_{k-1}x^{k-1}}{1 - x^k}.$$

7 Find the sum of each of the following series:

(a) $x - \frac{x^3}{3^2} + \frac{x^5}{5^2} - \frac{x^7}{7^2} + \cdots;$

(b) $\frac{x^2}{1\cdot 2} - \frac{x^3}{2\cdot 3} + \frac{x^4}{3\cdot 4} - \frac{x^5}{4\cdot 5} + \cdots;$

(c) $1 + 4x + 9x^2 + 16x^3 + 25x^4 + \cdots;$

(d) $\frac{x^4}{4} + \frac{x^8}{8} + \frac{x^{12}}{12} + \frac{x^{16}}{16} + \cdots;$

(e) $x + 2^3x^2 + 3^3x^3 + 4^3x^4 + \cdots;$

(f) $4 + 5x + 6x^2 + 7x^3 + \cdots.$

SECTION 14.4

8 Use any method to obtain each of the following Taylor series expansions as far as indicated:

(a) $e^{\sin x} = 1 + x + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \cdots;$

(b) $\frac{1}{1+e^x} = \frac{1}{2} - \frac{1}{4}x + \frac{1}{48}x^3 + \cdots;$

(c) $e^{x^2-x} = 1 - x + \frac{3}{2}x^2 - \frac{7}{6}x^3 + \frac{25}{24}x^4 + \cdots.$

9 Consider the binomial series

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(-1)(p-2)}{3!}x^3 + \cdots + \frac{p(p-1)(p-2)\cdots(p-n+1)}{n!}x^n + \cdots,$$

where p is an arbitrary constant. In Problem 13 of Section 14.4 the series on the right was obtained as the Taylor series of the function $(1+x)^p$, and we also saw that this series converges for $|x| < 1$. We here outline a sequence of steps to prove that the series on the right actually converges *to the function on the left* for these values of x.

(a) Let $f(x)$ denote the sum of the series for $|x| < 1$, calculate $f'(x)$ and $xf'(x)$, and show that

$$(1+x)f'(x) = pf(x).$$

(b) Define $g(x)$ by

$$g(x) = \frac{f(x)}{(1+x)^p}$$

and use part (a) to show that $g'(x) = 0$ for $|x| < 1$, so that $g(x) = c$ for some constant c.

(c) Show that $c = 1$ in part (b), and conclude that

$$(1+x)^p = f(x).$$

SECTION 14.7

10 If $f_1(x) = \Sigma_{n=1}^{\infty} nx^n$, calculate $(1-x)f_1(x)$, and use the result to find a closed formula for $f_1(x)$.

11 Use the idea of Problem 10 to find a closed formula for $f_2(x) = \Sigma_{n=1}^{\infty} n(n+1)x^n$.

12 Use the idea of Problems 10 and 11 to find a closed formula for $f_3(x) = \Sigma_{n=1}^{\infty} n(n+1)(n+2)x^n$.

*13 In the notation of Problems 10 to 12, show that

$$\sum_{n=1}^{\infty} n^2x^n = f_2(x) - f_1(x)$$

and

$$\sum_{n=1}^{\infty} n^3x^n = f_3(x) - 3f_2(x) + f_1(x).$$

Using these ideas as a starting point, devise a proof of the following theorem: If $p(n)$ is a polynomial in n, then $f(x) = \Sigma_{n=0}^{\infty} p(n)x^n$ is a rational function.

*14 Show that $1/\sqrt{1 - 2xt + t^2} = \Sigma_{n=1}^{\infty} P_n(x)t^n$, where $P_n(x)$ is a polynomial of degree n, by substituting $h = 2xt - t^2$ in the binomial series for $1/\sqrt{1-h}$ (see Problem 17 in Section 14.7). Find $P_0(x)$, $P_1(x)$, $P_2(x)$, and $P_3(x)$. The polynomials $P_n(x)$ are called the *Legendre polynomials*; they are important in mathematical physics, for instance, in the study of heat flow in solid spheres.

15 Calculate the following limits by using Taylor series:

(a) $\lim_{x\to 0} \frac{x\cos x - \sin x}{x^2 \tan x}$; (b) $\lim_{x\to 0} \frac{\sin x - \tan x}{\sin^2 x}$;

(c) $\lim_{x\to 0} \frac{\sqrt{1+x^2} + \cos x - 2}{x^4}$.

APPENDIX: THE BERNOULLI NUMBERS AND SOME WONDERFUL DISCOVERIES OF EULER

In this Appendix we derive several formulas discovered by Euler that rank among the most elegant truths in the whole of mathematics. We use the word "derive" instead of "prove" because some of our arguments are rather formal and require more advanced ideas than we can provide here to become fully rigorous in the sense demanded by modern concepts of mathematical proof. However, the mere fact that we are not able here to seal every crack in the reasoning seems a flimsy excuse for denying students an opportunity to glimpse some of the wonders that can be found in this part of calculus. For those who wish to dig deeper, full proofs are given in the treatise by K. Knopp mentioned in Section 14.7.

THE BERNOULLI NUMBERS

Since

$$\frac{e^x - 1}{x} = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \cdots$$

for $x \neq 0$, and this power series has the value 1 at $x = 0$, the reciprocal function $x/(e^x - 1)$ has a power series expansion valid in some neighborhood of the origin if the value of this function is defined to be 1 at $x = 0$. We write this series in the form

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n = B_0 + B_1 x + \frac{B_2}{2!} x^2 + \cdots. \tag{1}$$

The numbers B_n defined in this way are called the *Bernoulli numbers,* and it is clear that $B_0 = 1$. It is easy to see that

$$\frac{x}{e^x - 1} = \frac{x}{2}\left(\frac{e^x + 1}{e^x - 1} - 1\right) = -\frac{x}{2} + \frac{x}{2} \cdot \frac{e^x + 1}{e^x - 1}. \tag{2}$$

A routine check shows that the second term on the right is an even function, so $B_1 = -\frac{1}{2}$ and $B_n = 0$ if n is odd and > 1. If we write (1) in the form

$$\left(\frac{B_0}{0!} + \frac{B_1}{1!} x + \frac{B_2}{2!} x^2 + \cdots\right)\left(\frac{1}{1!} + \frac{x}{2!} + \frac{x^2}{3!} + \cdots\right) = 1,$$

then it is clear that the coefficient of x^{n-1} in the product on the left equals zero if $n > 1$. By the rule for multiplying power series, this yields

$$\frac{B_0}{0!} \cdot \frac{1}{n!} + \frac{B_1}{1!} \cdot \frac{1}{(n-1)!} + \frac{B_2}{2!} \cdot \frac{1}{(n-2)!} + \cdots + \frac{B_{n-1}}{(n-1)!} \cdot \frac{1}{1!} = 0,$$

and by multiplying through by $n!$ we obtain

$$\frac{n!}{0!n!} B_0 + \frac{n!}{1!(n-1)!} B_1 + \frac{n!}{2!(n-2)!} B_2 + \cdots + \frac{n!}{(n-1)!1!} B_{n-1} = 0. \tag{3}$$

This equation can also be written more briefly as

$$\binom{n}{0} B_0 + \binom{n}{1} B_1 + \binom{n}{2} B_2 + \cdots + \binom{n}{n-1} B_{n-1} = 0$$

or

$$\sum_{k=0}^{n-1} \binom{n}{k} B_k = 0,$$

where $\binom{n}{k}$ is the binomial coefficient $n!/[k!(n-k)!]$. By taking $n = 3, 5, 7, 9, 11, \ldots$ in (3) and doing a little arithmetic, we easily find that

$$B_2 = \tfrac{1}{6}, \qquad B_4 = -\tfrac{1}{30}, \qquad B_6 = \tfrac{1}{42}, \qquad B_8 = -\tfrac{1}{30}, \qquad B_{10} = \tfrac{5}{66}, \qquad \ldots .$$

These calculations can be continued recursively as far as we please, and so all the Bernoulli numbers can be considered as known, even though considerable labor may be required to make any particular one of them visibly present. We also point out that it is obvious from (3) and the mode of calculation that every B_n is rational.

THE POWER SERIES FOR THE TANGENT

We now begin to explore the uses of these numbers.

In equation (2) we move the term $-x/2$ to the left and use the fact that

$$\frac{x}{2} \cdot \frac{e^x + 1}{e^x - 1} = \frac{x}{2} \cdot \frac{e^{x/2} + e^{-x/2}}{e^{x/2} - e^{-x/2}}$$

to obtain

$$\frac{x}{2} \cdot \frac{e^{x/2} + e^{-x/2}}{e^{x/2} - e^{-x/2}} = \sum_{n=0}^{\infty} \frac{B_{2n}}{(2n)!} x^{2n}. \tag{4}$$

On the left side of this, we now replace x by $2ix$, which yields

$$\frac{2ix}{2} \cdot \frac{e^{ix} + e^{-ix}}{e^{ix} - e^{-ix}} = x \frac{(e^{ix} + e^{-ix})/2}{(e^{ix} - e^{-ix})2i} = x \cot x,$$

by the formulas for $\sin x$ and $\cos x$ that were derived in Section 14.8. Making the same substitution on the right side of (4) gives

$$\sum_{n=0}^{\infty} \frac{B_{2n}}{(2n)!} (2ix)^{2n} = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n} B_{2n}}{(2n)!} x^{2n},$$

so

$$x \cot x = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n} B_{2n}}{(2n)!} x^{2n}. \tag{5}$$

The trigonometric identity $\tan x = \cot x - 2 \cot 2x$ now enables us to use (5) to write

$$\begin{aligned} \tan x &= \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n} B_{2n}}{(2n)!} x^{2n-1} - 2 \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n} B_{2n}}{(2n)!} (2x)^{2n-1} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n} B_{2n}}{(2n)!} x^{2n-1} - \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n} B_{2n}}{(2n)!} 2^{2n} x^{2n-1} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n} B_{2n}}{(2n)!} (1 - 2^{2n}) x^{2n-1}, \end{aligned}$$

so

$$\tan x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^{2n}(2^{2n} - 1) B_{2n}}{(2n)!} x^{2n-1}.$$

This is the full power series for $\tan x$ that was encountered several times in truncated form in Section 14.7. Based on our knowledge of the Bernoulli numbers, the first few terms of this series are easy to calculate explicitly,

$$\tan x = x + \tfrac{1}{3} x^3 + \tfrac{2}{15} x^5 + \tfrac{17}{315} x^7 + \tfrac{62}{2835} x^9 + \cdots .$$

THE PARTIAL FRACTIONS EXPANSION OF THE COTANGENT

By using entirely different methods, Euler discovered another remarkable expansion of the cotangent: If x is not an integer, then

$$\pi \cot \pi x = \frac{1}{x} + 2x \sum_{n=1}^{\infty} \frac{1}{x^2 - n^2}. \tag{6}$$

We will examine this formula from two very different points of view, and give two derivations.

First, it is quite easy to see that (6) is analogous to the expansion of a rational function in partial fractions. For instance, if we consider the rational function $(2x + 1)/(x^2 - 3x + 2)$ and notice that the denominator has zeros 1 and 2 and can therefore be factored into $(x - 1)(x - 2)$, then this leads to the expansion

$$\frac{2x+1}{x^2 - 3x + 2} = \frac{2x + 1}{(x-1)(x-2)} = \frac{c_1}{x-1} + \frac{c_2}{x-2}$$

for certain constants c_1 and c_2. The constant c_1 can now be determined by multiplying through by $x - 1$ and allowing x to approach 1, and similarly for c_2. Formally, (6) can be obtained in much the same way by noticing that $\cot \pi x = \cos \pi x / \sin \pi x$ has a denominator with zeros $0, \pm 1, \pm 2, \ldots$, and should therefore be expressible in the form

$$\cot \pi x = \frac{a}{x} + \sum_{n=1}^{\infty} \left(\frac{b_n}{x - n} + \frac{c_n}{x + n} \right). \tag{7}$$

From this, the constants a, b_n, and c_n can be found by the procedure suggested (they are all equal to $1/\pi$), and (7) can then be rearranged to yield (6). For reasons that will now be obvious, it is customary to refer to (6) as the *partial fractions expansion of the cotangent*. The main gap in this suggestive but rather tentative derivation is of course the fact that we have no prior guarantee that an expansion of the form (7) is possible.

Another way of approaching (6) is to begin with the infinite product (6) in Appendix 1 at the end of Chapter 13:

$$\frac{\sin x}{x} = \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right) \cdots \left(1 - \frac{x^2}{n^2\pi^2}\right) \cdots.$$

If we take the logarithm of both sides to obtain

$$\ln \frac{\sin x}{x} = \sum_{n=1}^{\infty} \ln \left(1 - \frac{x^2}{n^2 \pi^2}\right),$$

and then differentiate, the result is easily seen to be

$$\cot x - \frac{1}{x} = \sum_{n=1}^{\infty} \frac{-2x}{n^2\pi^2 - x^2}$$

or

$$\cot x = \frac{1}{x} + 2x \sum_{n=1}^{\infty} \frac{1}{x^2 - n^2 \pi^2};$$

and replacing x by πx and then multiplying through by πx yields

$$\pi x \cot \pi x = 1 + 2x^2 \sum_{n=1}^{\infty} \frac{1}{x^2 - n^2}, \tag{8}$$

which is equivalent to (6).

EULER'S FORMULA FOR $\Sigma 1/n^{2k}$

We now obtain a major payoff from (5) and (8) by replacing x by πx in (5) and equating the two expressions for $\pi x \cot \pi x$,

$$1+\sum_{n=1}^{\infty}\frac{-2x^2}{n^2-x^2}=1+\sum_{k=1}^{\infty}(-1)^k\frac{2^{2k}B_{2k}}{(2k)!}(\pi x)^{2k}, \tag{9}$$

where we use k as the index of summation on the right for reasons that will appear in a moment. Each term of the series on the left is easy to expand in a geometric series,

$$\frac{-2x^2}{n^2-x^2}=-2\frac{x^2/n^2}{1-x^2/n^2}=-2\sum_{k=1}^{\infty}\left(\frac{x^2}{n^2}\right)^k=-2\sum_{k=1}^{\infty}\frac{x^{2k}}{n^{2k}},$$

so (9) can be written as

$$1+\sum_{n=1}^{\infty}\left(-2\sum_{k=1}^{\infty}\frac{x^{2k}}{n^{2k}}\right)=1+\sum_{k=1}^{\infty}(-1)^k\frac{2^{2k}B_{2k}}{(2k)!}\pi^{2k}x^{2k}.$$

We now interchange the order of summation on the left and obtain

$$1+\sum_{k=1}^{\infty}\left(-2\sum_{n=1}^{\infty}\frac{1}{n^{2k}}\right)x^{2k}=1+\sum_{k=1}^{\infty}(-1)^k\frac{2^{2k}B_{2k}}{(2k)!}\pi^{2k}x^{2k},$$

and equating the coefficients of x^{2k} yields

$$\sum_{n=1}^{\infty}\frac{1}{n^{2k}}=(-1)^{k-1}\frac{2^{2k}B_{2k}}{2(2k)!}\pi^{2k}$$

for each positive integer k. In particular, for $k = 1, 2, 3$ we get

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6},\qquad \sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90},\qquad \sum_{n=1}^{\infty}\frac{1}{n^6}=\frac{\pi^6}{945}.$$

It is very remarkable that for more than 250 years there has been no progress whatever toward finding the exact sum of any one of the series

$$\sum_{n=1}^{\infty}\frac{1}{n^3},\qquad \sum_{n=1}^{\infty}\frac{1}{n^5},\qquad \sum_{n=1}^{\infty}\frac{1}{n^7},\qquad \ldots .$$

Perhaps a second Euler is needed for this breakthrough, but none is in sight.

15 CONIC SECTIONS

15.1 INTRODUCTION. SECTIONS OF A CONE

In order to understand the central ideas of Chapters 13 and 14, it was necessary to pay close attention to the precise wording of definitions and to the details of proofs, so the level of mathematical rigor in those chapters was rather high. However, we now turn to work that is mostly geometric in nature. We shall therefore rely much more heavily on reasoning based on spatial intuition and the kind of insight that can be obtained from carefully drawn figures.

Consider a circle C. Let A be the line through the center of C perpendicular to the plane of C, and let V be a point on A not in the plane of C, as shown in Fig. 15.1. Let P be a point on C, and draw the infinite straight line through P that also passes through V. As P moves around C, the line PV sweeps out a *right circular cone* with *axis* A and *vertex* V. Each of the lines PV is called a *generator* of the cone, and the angle α between the axis and any generator is called the *vertex angle*. The cone shown in Fig. 15.1 has a vertical axis, and the upper and lower portions of the cone that meet at the vertex are called the *nappes* of the cone.* In elementary geometry a cone is usually understood to be a solid figure occupying the bounded region of space that lies between V and the plane of C and is inside the surface we have just described. However, in the present context the cone is this surface itself, and is understood to consist of both nappes, extending to infinity in both directions.

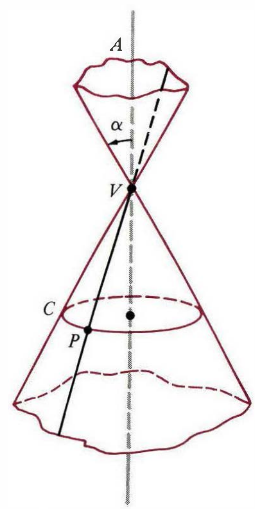


Figure 15.1

The curves obtained by slicing a cone with a plane that does not pass through the vertex are called *conic sections,* or simply *conics*. If the slicing plane is parallel to a generator, the conic is called a *parabola*. Otherwise, the conic is called an *ellipse* or a *hyperbola,* depending on whether the plane cuts just one or both nappes. The hyperbola is to be thought of as a single curve consisting of two "branches," one on each nappe. These three curves are illustrated in Fig. 15.2.

The three curves shown in Fig. 15.2 can be described in another way. Imagine a source of light placed at V and a circular ring placed at C. Then the shadow cast by the ring on a plane will be a parabola, an ellipse, or one branch of a hyperbola, depending on the steepness of the plane. If the plane is parallel to one of the lines joining V to C, we get a parabolic shadow; the shadow will be an ellipse if the plane is less steep than this, and part of a hyperbola if it is more steep.

*"Nappe" is from the French word *nappe,* meaning a sheet of something, perhaps cloth, as in "napkin" or "napery" (household linen).

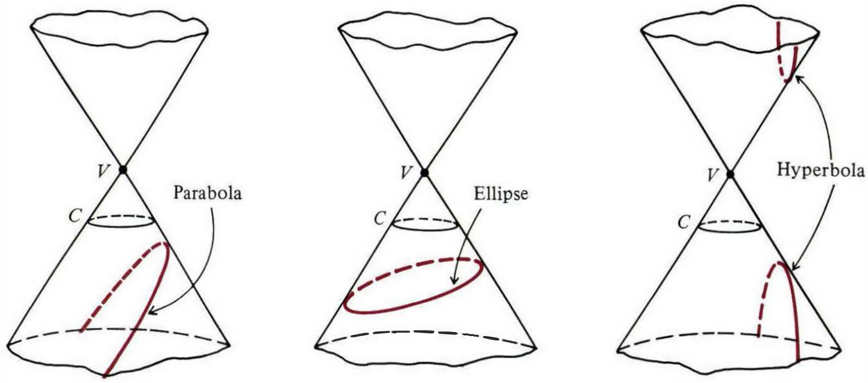


Figure 15.2

It should be noted that if we move each of the slicing planes in Fig. 15.2 parallel to itself until it passes through the vertex, then we get three so-called *degenerate* conic sections, namely, a single straight line, a point, and a pair of intersecting straight lines.

Many important discoveries in both pure mathematics and science have been linked to the conic sections. The classical Greeks—Archimedes, Apollonius and others— studied these beautiful curves for the sheer pleasure of it, as a form of play, without any thought of their possible uses. The first applications appeared almost 2000 years later, at the beginning of the seventeenth century. About the year 1604 Galileo discovered that if a projectile is fired horizontally from the top of a tower and is assumed to be acted on only by the force of gravity—that is, if air resistance and other complicating factors are ignored— then the path of the projectile will be a parabola. One of the great events in the history of astronomy occurred only a few years later, in 1609, when Kepler published his discovery that the orbit of Mars is an ellipse and then went on to suggest that all the planets move in elliptical orbits. And about 60 years after this, Newton was able to prove mathematically that an elliptical planetary orbit implies, and is implied by, an inverse square law of gravitational attraction. This led Newton to formulate and publish (in 1687) his famous theory of universal gravitation as the explanation of the mechanism of the solar system, which has been described as the greatest contribution to science ever made by one man. These developments took place hundreds of years ago, but the study of conic sections is far from outdated even today. Indeed, these curves are important tools for present-day explorations of outer space, and also for research into the behavior of atomic particles. Artificial satellites move around the earth in elliptical orbits, and the path of an alpha particle moving in the electric field of an atomic nucleus is a hyperbola. These examples and many others show that the importance of conic sections, both historically and in modern times, is difficult to exaggerate.

We shall be studying the conic sections as plane curves. For this purpose it is convenient to make use of equivalent definitions that refer only to the plane in which the curves lie and depend on special points in this plane called *foci* (*focus* is the singular). An ellipse can be defined as the set of all points in the plane the sum of whose distances d_1 and d_2 from two fixed points F and F' (the foci) is constant, as shown on the left in Fig. 15.3. A hyperbola is the set of all points for which the difference $|d_1 - d_2|$ is constant. And a parabola is the set of all

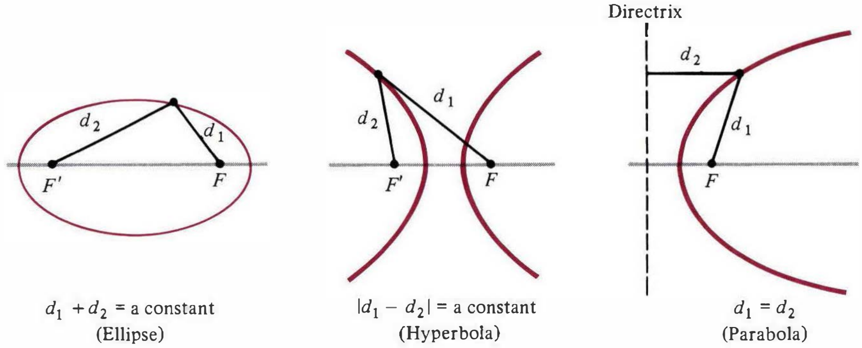


Figure 15.3

points for which the distance to a fixed point F (the focus) equals the distance to a fixed line (called the *directrix*).

There is a simple and elegant argument which shows that the focal property of an ellipse follows from its definition as a section of a cone. This proof uses the two spheres shown in Fig. 15.4, which are internally tangent to the cone along the horizontal circles C_1 and C_2, and are also tangent to the slicing plane at the points F and F'. If P is an arbitrary point on the ellipse, we must show that the sum of the distances $PF + PF'$ is constant in the sense that it does not depend on the particular position of P. To see this, we notice that if Q and R are the points on C_1 and C_2 that lie on the generator through P, then $PF = PQ$ and $PF' = PR$, because any two tangents to a sphere drawn from a common external point have the same length. It follows that $PF + PF' = PQ + PR = QR$; and the argument is completed by observing that QR, as the distance from C_1 to C_2 down a generator, has the same value for every position of P.

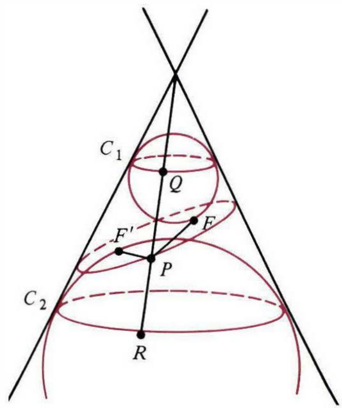


Figure 15.4 The focal property of an ellipse.

With slight modifications this proof also works for the hyperbola and the parabola. In the case of the hyperbola, we use one sphere in each portion of the cone, with both spheres tangent to the slicing plane. And for the parabola we use one sphere tangent to the slicing plane. The focus of the parabola is this point of tangency, and its directrix is the line of intersection of the slicing plane with the plane of the circle along which the sphere is internally tangent to the cone. Students should use these hints to draw suitable pictures and prove for themselves that the focal properties of the hyperbola and the parabola can be derived from their definitions as sections of a cone.

15.2 ANOTHER LOOK AT CIRCLES AND PARABOLAS

Circles and parabolas were discussed fairly thoroughly in Chapter 1. However, that was a long time ago, and it may be helpful to give a very brief review of the main facts in order to assist students in fitting these topics into the context of the present chapter.

CIRCLES

Referring to Fig. 15.4, we see at once that a circle can be thought of as the special case of an ellipse obtained by taking the slicing plane perpendicular to the axis of the cone, so that the foci coincide. Nevertheless, for several reasons it is convenient to discuss circles separately, and to reserve the word "ellipse" for the case in which the foci are two distinct points.

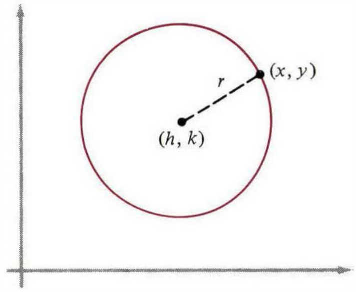


Figure 15.5 A circle.

A *circle,* therefore—as we very well know—can be defined as a plane curve consisting of the set of all points at a given fixed distance (called the *radius*) from a given fixed point (called the *center*). If $r > 0$ is the radius and (h, k) is the center, and if (x, y) is an arbitrary point on the circle (see Fig. 15.5), then by using the distance formula we can write the defining condition as

$$\sqrt{(x - h)^2 + (y - k)^2} = r$$

or

$$(x - h)^2 + (y - k)^2 = r^2, \tag{1}$$

which is the equation of the circle in standard form. By squaring the terms on the left of (1) and rearranging, this equation can be written in the form

$$x^2 + y^2 + Ax + By + C = 0. \tag{2}$$

Conversely, by completing the square on the x and y terms, any equation of the form (2) can be written in the form (1), and therefore represents a circle if $r^2 > 0$. As students will remember, there is a slight difficulty with this procedure as a result of the fact that the constant r^2 on the right of (1) may turn out to be zero or a negative number. In these cases, (1) can be thought of as the equation of a single point or the empty set.

PARABOLAS

As we saw in Section 15.1, a parabola can be defined as a plane curve consisting of the set of all points P that are equally distant from a given fixed point F and a given fixed line d, as shown on the left in Fig. 15.6. The fixed point is called the *focus,* and the fixed line is called the *directrix.* To find a simple equation for this curve, we introduce the coordinate system shown on the right in the figure, in which the focus is the point $F = (0, p)$, where p is a positive number, and the directrix is the line $y = -p$. If $P = (x, y)$ is an arbitrary point on the parabola, then by using the distance formula the defining condition can be written as

$$\sqrt{x^2 + (y - p)^2} = y + p. \tag{3}$$

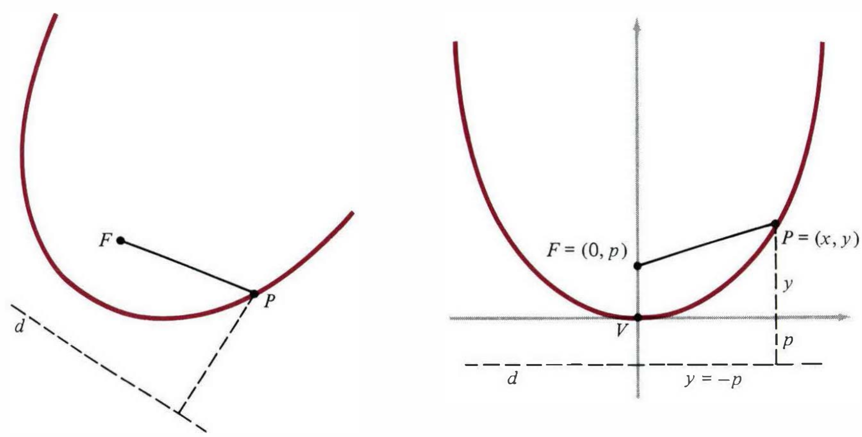


Figure 15.6 A parabola.

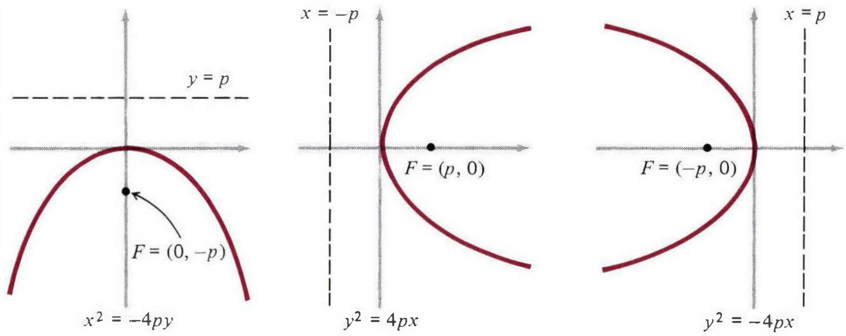


Figure 15.7

By squaring and simplifying we get

$$x^2 + y^2 - 2py + p^2 = y^2 + 2py + p^2$$

or

$$x^2 = 4py. \tag{4}$$

Conversely, by reversing the steps, it can be shown that (3) can be derived from (4). Equation (4) is therefore the equation of this particular parabola in standard form. The line through the focus perpendicular to the directrix is called the *axis* of the parabola, and the point V where the parabola intersects the axis is called the *vertex*. For the parabola (4), the axis is clearly the y-axis, and the vertex is the origin.

If we change the position of the parabola relative to the coordinate axes, we naturally change its equation. Three other simple positions, each with its corresponding equation, are shown in Fig. 15.7. Students should verify the correctness of all three equations. We emphasize that the constant p is always understood to be a positive number; geometrically, it is the distance from the vertex to the focus, and also from the vertex to the directrix.

We illustrate a further point about parabolas by considering the equation

$$x^2 - 8x - y + 19 = 0.$$

If we write this as $x^2 - 8x = y - 19$ and complete the square on x, then the result is

$$(x - 4)^2 = y - 3.$$

If we now introduce new variables x' and y' by writing

$$x' = x - 4,$$

$$y' = y - 3,$$

then our equation becomes

$$x'^2 = y'.$$

The graph of this equation is clearly a parabola with vertical axis whose vertex lies at the origin in the x', y' coordinate system, and this origin is located at the point (4, 3) in the x, y system, as shown in Fig. 15.8. In exactly the same way, any equation of the form

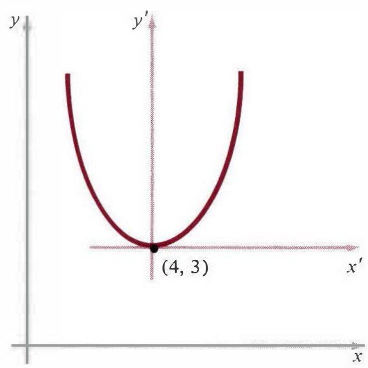


Figure 15.8

$$x^2 + Ax + By + C = 0, \qquad B \neq 0, \tag{5}$$

represents a parabola with vertical axis. The vertex of this parabola is easily located by completing the square on x, and in this way the equation can be written in the form

$$(x-h)^2 = 4p(y-k) \qquad \text{or} \qquad (x-h)^2 = -4p(y-k),$$

where the point (h, k) is the vertex.* Similarly, any equation of the form

$$y^2 + Ax + By + C = 0, \qquad A \neq 0,$$

represents a parabola with horizontal axis, and the geometric features of this parabola can be discovered by completing the square on y and writing the equation as

$$(y-k)^2 = 4p(x-h) \qquad \text{or} \qquad (y-k)^2 = -4p(x-h).$$

We conclude this section by describing the so-called *reflection property* of parabolas. Consider the tangent line at a point $P = (x, y)$ on the parabola $y^2 = 4px$ (Fig. 15.9), where $F = (p, 0)$ is the focus. As shown in the figure, let α be the angle between the tangent and the segment FP, and let β be the angle between the tangent and the horizontal line through P. In Problem 9, students are asked to prove that $\alpha = \beta$.

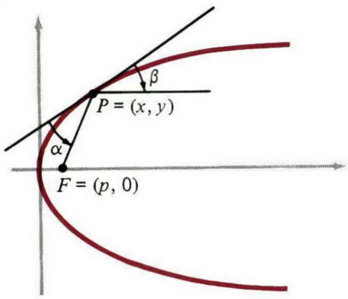


Figure 15.9 The reflection property.

This geometric property of parabolas has many applications. For example, it is used in the design of mirrors for searchlights. To construct such a mirror, revolve the parabola about its axis to form a surface of revolution, then coat the inside with silver paint to make a reflecting surface. If a source of light is placed at F, each ray will be reflected along a line parallel to the axis to form a beam of parallel rays. The same principle is used in a more important way in the design of mirrors for reflecting telescopes and solar furnaces, where rays of light that are parallel to the axis and come in toward the mirror are reflected in to the focus. This reflection property of parabolas also underlies the design of radar antennas and radio telescopes.

*We point out here that if $B = 0$ is allowed in (5), then the graph of the equation can be one straight line, or two parallel lines, or the empty set. For the particular equation $x^2 - 2x - k = 0$, or equivalently $(x-1)^2 = k + 1$, these cases correspond to $k = -1$, $k > -1$, and $k < -1$.

PROBLEMS

1 For each of the following equations, determine the nature of the graph by completing the square:
(a) $x^2 + y^2 - 2x - 6y - 15 = 0$;
(b) $x^2 + y^2 + 4x - 18y + 88 = 0$;
(c) $x^2 + y^2 - 10x + 2y + 26 = 0$;
(d) $x^2 + y^2 - 16x + 12y + 96 = 0$;
(e) $x^2 + y^2 + 6x - 14y + 58 = 0$;
(f) $x^2 + y^2 + 14x + 10y + 95 = 0$.

2 If $0 < a < b$, find the radius r and center (h, k) of the circle that passes through the points $(0, a)$ and $(0, b)$ and is tangent to the x-axis at a point to the right of the origin.

3 For each of the following parabolas, find the vertex, focus, and directrix:
(a) $x^2 + 4x - 4y = 0$;
(b) $y^2 - 8x - 2y + 25 = 0$;
(c) $x^2 + 4x + 16y - 76 = 0$;
(d) $y^2 + 12x - 2y + 25 = 0$;
(e) $y = x^2 + 2x + 3$.

4 A searchlight reflector is designed as stated in the text. If it is 2 ft deep and the opening is 5 ft across, find the focus.

5 Find the equation of the parabolic arch with base b and height h that is shown in Fig. 15.10.

6 Show that the area of the parabolic segment in Fig. 15.10 is $\frac{2}{3}hb$. (Notice that this area is four-thirds the area of the triangle with the same base and height, a fact that was discovered and proved by Archimedes.)

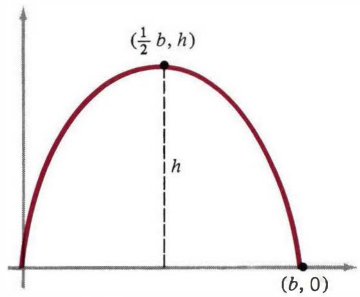


Figure 15.10

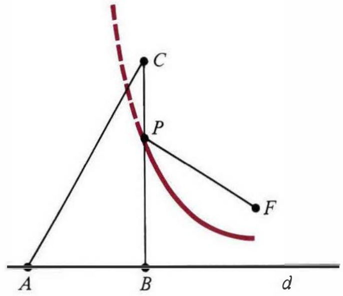


Figure 15.11

7 If the parabolic segment in Fig. 15.10 is revolved about its axis, show that the volume of the resulting solid of revolution is three-halves the volume of the inscribed cone.

8 A parabola with focus F and directrix d that are given and marked on a sheet of paper can be constructed as follows (see Fig. 15.11). On a drafting board, fasten a ruler to the paper with its edge along d, and place the short leg AB of a draftsman's triangle ABC against the edge of the ruler. At the opposite vertex C of the triangle fasten one end of a piece of string whose length is the same as that of the long leg BC of the triangle, and fasten the other end of the string at F. If a pencil point at P keeps the string taut, as shown in the figure, then the point of the pencil draws part of a parabola as the triangle slides along the ruler. Explain why this construction works.

9 Prove that $\alpha = \beta$ in Fig. 15.9. Hint: Extend FP through P and use the subtraction formula for the tangent to show that $\tan \alpha = \tan \beta$.

10 Show that the lines tangent to a parabola at the ends of a focal chord (a chord through the focus) intersect at right angles.

*11 Show that the lines tangent to a parabola at the ends of a focal chord intersect on the directrix.

12 Let C be a circle of radius r_0 and L a line that lies in the same plane and does not intersect C. Show that the centers of all circles that do not surround C and are tangent to both C and L lie on a parabola (Fig. 15.12). State the location of the focus and directrix of this parabola.

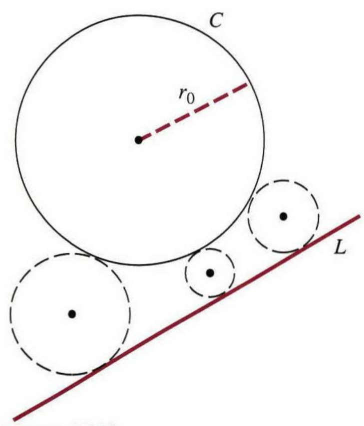


Figure 15.12

15.3 ELLIPSES

In Section 15.2 we gave "set of all points" definitions for both circles and parabolas. It is also possible to give "locus" definitions, in which each curve is defined —and thought of—as the path of a moving point that satisfies a certain condition as it moves. This language has the advantage of greater pictorial vividness. Thus, a parabola can be defined as the locus of a point that moves in such a way that it maintains equal distances from a given fixed point and a given fixed line.

Similarly, in accordance with Section 15.1, we can define an *ellipse* as the locus of a point P that moves in such a way that the sum of its distances from two fixed points F and F' is constant, as shown on the left in Fig. 15.13. To simplify later equations, we denote this constant by $2a$ and write the defining condition as

$$PF + PF' = 2a. \tag{1}$$

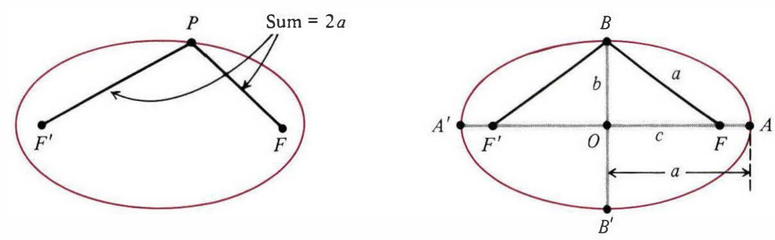


Figure 15.13

The two points F and F' are called the *foci* (plural of *focus*) of the ellipse because of the reflection property discussed in Remark 1 below. Since circles are not considered to be ellipses in this discussion, F and F' are understood to be two *distinct* points: $F \neq F'$.

The definition provides an easy way to draw an ellipse on a sheet of paper. We begin by fastening the paper to a drawing board with two tacks placed at F and F'. Next, we tie the ends of a piece of string to the tacks and pull the string taut with the point of a pencil. It is clear that if the pencil is moved around while the string is kept taut, then its point draws an ellipse. Because of this construction, the defining condition (1) is often called the *string property* of an ellipse.

We now introduce several standard notations for the dimensions of an ellipse. It is easy to see from the definition that the curve is symmetric with respect to the line through the foci, and also with respect to the perpendicular bisector of the segment FF'. On the right in Fig. 15.13 the segment AA' is called the *major axis* and the segment BB' is called the *minor axis* of the ellipse, and the point O where these axes intersect is called the *center*. The two points A and A' at the ends of the major axis are called the *vertices* of the ellipse. We denote the length of the minor axis by $2b$ and the distance between the foci by $2c$. It is clear that $BF = a$, because $BF + BF' = 2a$ and $BF = BF'$, so

$$a^2 = b^2 + c^2. \tag{2}$$

Since $AF + AF' = 2a$ and $AF' = FA'$, we see that $AA' = 2a$, so the length of the major axis is $2a$. The numbers a and b are called the *semimajor axis* and the *semiminor axis.*

It is easy to see from equation (2) that $b < a$. If b is very small compared with a, so that the ellipse is long and thin, then (2) shows that c is nearly as large as a, and the foci are near the ends of the major axis; and if b is nearly as large as a, so that the ellipse is nearly circular, then c is small, and the foci are close to the center. The ratio c/a is called the *eccentricity* of the ellipse and is denoted by e:

$$e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a}. \tag{3}$$

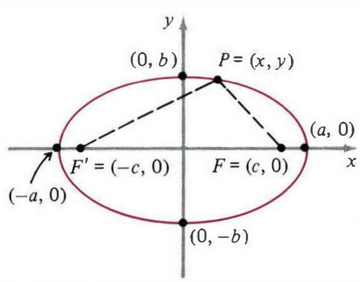


Figure 15.14

Notice that $0 < e < 1$. Nearly circular ellipses have eccentricity near 0, and long, thin ellipses have eccentricity near 1.

In order to find a simple equation for the ellipse, we take the x-axis along the segment FF' and the y-axis as the perpendicular bisector of this segment. Then the foci are $F = (c, 0)$ and $F' = (-c, 0)$, as shown in Fig. 15.14, and the defining condition (1) yields

$$\sqrt{(x - c)^2 + y^2} + \sqrt{(x + c)^2 + y^2} = 2a \tag{4}$$

as the equation of the curve.

To simplify this equation, we follow the usual procedure for eliminating radicals, namely, solve for one of the radicals and square. If we move the first radical in (4) over to the right side, square both sides, and simplify, then we obtain

$$PF = \sqrt{(x-c)^2 + y^2} = a - \frac{c}{a}x \tag{5}$$

and

$$PF' = \sqrt{(x+c)^2 + y^2} = a + \frac{c}{a}x, \tag{6}$$

where (6) follows from (5) because $PF' = 2a - PF$. By squaring again and simplifying, either of these equations gives

$$\left(\frac{a^2 - c^2}{a^2}\right)x^2 + y^2 = a^2 - c^2$$

or

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1. \tag{7}$$

By using (2) to simplify (7) still further, we now put the equation into its final form,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{8}$$

This argument shows that (8) is satisfied if (4) is. It can be shown, conversely, that (4) is satisfied if (8) is, but we omit the details. Equation (8) is therefore the standard form of the equation of the ellipse shown in Fig. 15.14.

We pause briefly to point out that equation (8) easily yields most of the simpler geometric features of the ellipse that are visible in Fig. 15.14. (i) If $y = 0$, then the equation tells us that $x = \pm a$, and if $x = 0$, then $y = \pm b$, so the curve crosses the x and y axes at the four points $(\pm a, 0)$ and $(0, \pm b)$. (ii) Since both terms x^2/a^2 and y^2/b^2 are nonnegative and their sum is 1, it follows that neither of them can be greater than 1, so $|x| \leq a$ and $|y| \leq b$. This means that the whole ellipse is contained in the rectangle whose sides are $x = \pm a$ and $y = \pm b$, and is therefore—unlike the parabola—a bounded curve. (iii) If (x, y) satisfies the equation, then so do $(x, -y)$ and $(-x, y)$, so the curve is symmetric with respect to both the x-axis and the y-axis. This tells us that to graph the complete curve it suffices to sketch the graph in the first quadrant and then extend it to the other quadrants by symmetry. The left-right symmetry of the ellipse that is so obvious from equation (8) is really rather remarkable, because most people contemplating Fig. 15.2 for the first time feel quite sure that an ellipse should be an egg-shaped oval which has a "small end" at the part of the ellipse nearest the vertex of the cone and a "big end" at the part farthest from this vertex—but of course this is not true.

We consider again formulas (5) and (6) for the right and left *focal radii* PF and PF', which can be written as

$$PF = a - \frac{c}{a}x = e\left[\frac{a}{e} - x\right] \tag{9}$$

and

$$PF' = a + \frac{c}{a}x = e\left(\frac{a}{e} + x\right) = e\left[x - \left(-\frac{a}{e}\right)\right], \tag{10}$$

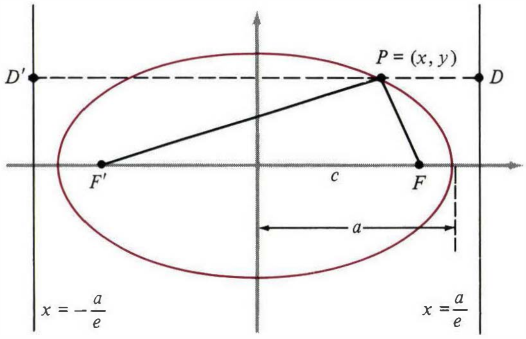


Figure 15.15

where $e = c/a$ is the eccentricity defined earlier. The quantities in brackets can be interpreted (see Fig. 15.15) as the distances PD and PD' from P to the lines $x = a/e$ and $x = -a/e$, respectively. Formulas (9) and (10) can therefore be written in the form

$$\frac{PF}{PD} = e \quad \text{and} \quad \frac{PF'}{PD'} = e. \tag{11}$$

Each of the lines $x = a/e$ and $x = -a/e$ is called a *directrix* of the ellipse. Equations (11) show that *an ellipse can be characterized as the locus of a point that moves in such a way that the ratio of its distance from a fixed point (a focus) to its distance from a fixed line (the corresponding directrix) equals a constant* $e < 1$. We shall see in Chapter 16 and elsewhere that this way of characterizing ellipses is often very useful.

Example 1 Identify the graph of $16x^2 + 25y^2 = 400$ as an ellipse, and find its vertices, foci, eccentricity, and directrices. Sketch the graph.

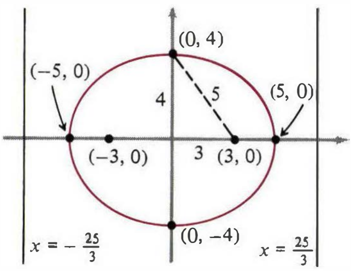


Figure 15.16

Solution First, we divide by 400 to convert the equation into the standard form

$$\frac{x^2}{25} + \frac{y^2}{16} = 1,$$

which on comparison with (8) tells us that the graph is an ellipse. Since $a^2 = 25$ and $b^2 = 16$, we have $a = 5$ and $b = 4$, so the vertices are $(\pm 5, 0)$ and the ends of the minor axis are $(0, \pm 4)$, as shown in Fig. 15.16. Next, $c^2 = a^2 - b^2 = 25 - 16 = 9$, so $c = 3$ and the foci are $(\pm 3, 0)$. Finally, the eccentricity is $e = c/a = \frac{3}{5}$, and the directrices are the vertical lines $x = \pm a/e = \pm\frac{25}{3}$.

In the above discussion it is assumed that the ellipse has its center at the origin and its foci on the x-axis. However, if its center is the origin and its foci lie on the y-axis, then its major axis is vertical and the roles of x and y are interchanged.

Example 2 Show that $9x^2 + 4y^2 = 36$ represents an ellipse, and find its vertices, foci, eccentricity, and directrices. Sketch the graph.

Solution As before, we divide by 36 to convert the given equation into the recognizable standard form

$$\frac{x^2}{4} + \frac{y^2}{9} = 1.$$

Observe that here *the denominator of the y term is larger,* so we have an ellipse whose major axis is vertical. The semimajor and semiminor axes are clearly $a = 3$ and $b = 2$, so the vertices (see Fig. 15.17) are $(0, \pm 3)$ and the ends of the minor axis are $(\pm 2, 0)$. Since $c^2 = a^2 - b^2 = 9 - 4 = 5$, $c = \sqrt{5}$, and the foci are the points $(0, \sqrt{5})$ and $(0, -\sqrt{5})$ on the y-axis. The eccentricity is $e = c/a = \sqrt{5}/3$, and the directrices are the horizontal lines $y = \pm a/e = \pm 9/\sqrt{5} = \pm\frac{9}{5}\sqrt{5}$.

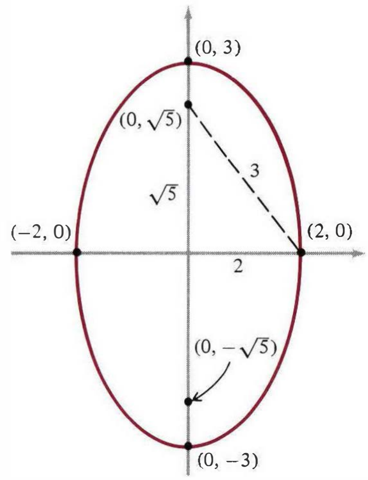


Figure 15.17

Examples 1 and 2 illustrate the fact that if we have an equation of the form

$$\frac{x^2}{(\quad)^2} + \frac{y^2}{(\quad)^2} = 1$$

with unequal denominators, then the equation represents an ellipse, and the question of whether the foci and major axis lie on the x-axis or the y-axis is determined by which denominator is larger.

In equation (8), x and y are the horizontal and vertical displacements from the axes of the ellipse to the point $P = (x, y)$. If the center is the point (h, k) instead of the origin, then these displacements are $x - h$ and $y - k$, and the equation of the ellipse becomes

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1. \tag{12}$$

Example 3 Show that $4x^2 + 16y^2 - 24x - 32y = 12$ is the equation of an ellipse, and find its vertices, foci, eccentricity, and directrices. Sketch the graph.

Solution The equation can be written as

$$4(x^2 - 6x) + 16(y^2 - 2y) = 12.$$

Completing the squares inside the parentheses, we obtain

$$4(x-3)^2 + 16(y-1)^2 = 64$$

or

$$\frac{(x-3)^2}{16} + \frac{(y-1)^2}{4} = 1.$$

Comparison with (12) shows that this represents an ellipse with center (3, 1), horizontal major axis, and semiaxes $a = 4$, $b = 2$, so the vertices (Fig. 15.18) are the points (7, 1), (−1, 1) and the ends of the minor axis are (3, 3), (3, −1). The foci are a distance $c = \sqrt{a^2 - b^2} = \sqrt{12} = 2\sqrt{3}$ to the right and left of the center, and are therefore the points $(3 \pm 2\sqrt{3}, 1)$. The eccentricity is $e = c/a = \frac{1}{2}\sqrt{3}$, and the directrices are vertical lines at a distance $a/e = 8/\sqrt{3} = \frac{8}{3}\sqrt{3}$ to the right and left of the center. Their equations are $x = 3 \pm \frac{8}{3}\sqrt{3}$.

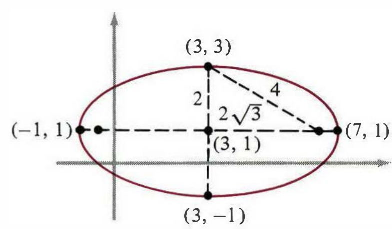


Figure 15.18

Remark 1 Like parabolas, ellipses also have a remarkable reflection property. Let P be a point on an ellipse with foci F and F', and let T be the tangent at P, as shown in Fig. 15.19. If T makes angles α and β with the two focal radii PF and PF', then $\alpha = \beta$. Students are asked to prove this in Problem 9.

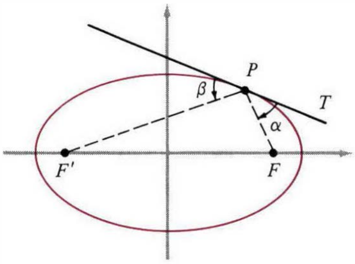


Figure 15.19

This reflection property has no important scientific applications like those we saw in the case of parabolas, but there is at least one mildly amusing consequence. Let the ellipse in the figure be revolved about its major axis to form a surface of revolution, and imagine that a room is built with its walls and ceiling having the shape of the upper part of this surface, with the two foci about shoulder height above the floor. Then a whisper uttered at one focus can be clearly heard a considerable distance away at the other focus even though it is inaudible at intermediate points, because the sound waves bounce off the walls and are reflected to the second focus, and furthermore arrive together because they all travel the same distance. There actually exist several rooms of this kind—known as *whispering galleries*—in certain American museums of science and in the castles of a few eccentric European monarchs.

A less frivolous application is in the new treatment for kidney stones called *lithotripsy* (from the Greek *lithos,* stone + *tripsis,* a rubbing or pounding). An ellipsoidal reflector is placed in such a position that the offending kidney stone is at one focus. High-intensity sound waves generated at the other focus are reflected harmlessly through the patient's body and are concentrated at the stone, which they pound into powder. The patient is spared having to go through surgery and recovers in a few days.

Remark 2 Except for small perturbations resulting from the influence of the other planets, each planet in the solar system revolves around the sun in an elliptical orbit with the sun at one focus. As we pointed out in Section 15.1, this phenomenon was discovered empirically by Kepler in the early seventeenth century, and was explained mathematically by Newton in the later decades of the same century. We shall give a detailed treatment of Newton's ideas at the end of Chapter 17.

Mercury	.21	Saturn	.06
Venus	.01	Uranus	.05
Earth	.02	Neptune	.01
Mars	.09	Pluto	.25
Jupiter	.05		

Figure 15.20 Eccentricities of planetary orbits.

Most of the planets, including the earth, have orbits that are nearly circular. This can be seen from the eccentricities given in the table in Fig. 15.20. Mercury, however, has a rather eccentric orbit, with $e = 0.21$, as does Pluto, with $e = 0.25$. Other bodies in the solar system have even more eccentric orbits, for instance, the flying mountains known as asteroids. Thus the asteroid Icarus, which was discovered at Mount Palomar in 1949 and is about 1 mi in diameter, has an orbit so eccentric, with $e = 0.83$, that at its closest approach to the sun (Fig. 15.21) it is halfway between the sun and the orbit of Mercury, and at its farthest it is out beyond the orbit of the earth.*

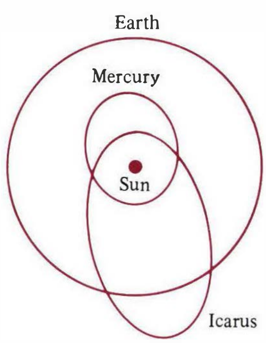


Figure 15.21

One of the most interesting objects in the solar system is Halley's Comet, which has eccentricity $e = 0.98$ and an orbit (Fig. 15.22) about 7 astronomical units wide by 35 astronomical units long. [One astronomical unit (AU) is the semimajor axis of the earth's orbit, approximately 93 million miles or 150 million kilometers.] The period of revolution of this comet around the sun is about 76 years. It appeared in 1910, and again in 1985–1986. It was observed in 1682, and the astronomer Edmund Halley (Newton's friend) successfully predicted its return in 1758, many years after his own death in 1742. This was one of the most convincing successes of Newton's theory of gravitation. At its closest approach,

*The surface temperature of Icarus has been estimated at about 900°F at its closest approach to the sun. Arthur C. Clarke has used this fact as the basis for a fine story, "Summertime on Icarus," in his collection *The Nine Billion Names of God* (New American Library, 1974). See also Chapter 2, "The Little Planets," in Fletcher G. Watson's *Between the Planets* (Doubleday Anchor Books, 1962), especially p. 29.

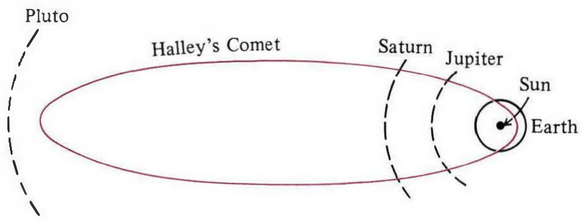


Figure 15.22 Orbit of Halley's Comet, drawn approximately to scale.

Halley's Comet is only 0.59 AU away from the sun. Its previous visits to the near neighborhood of the sun have been traced back step by step by means of historical records to the year 11 B.C., and perhaps even earlier.*

*For a more detailed account of these remarkable events, see P. L. Brown, *Comets, Meteorites and Men* (Taplinger, 1974); or N. Calder, *The Comet Is Coming!* (Viking, 1980).

PROBLEMS

1 Find the equation of the ellipse
(a) with foci at $(\pm 2, 0)$ and major axis of length 10;
(b) with foci at $(0, \pm 4)$ and minor axis of length 12;
(c) with major and minor axes of lengths 4 and 3, respectively, center at the origin, and foci on the y-axis;
(d) with foci at $(\pm 3, 0)$ and eccentricity $e = \frac{3}{4}$;
(e) with eccentricity $e = \frac{1}{2}$, center at the origin, and the ends of the major axis at $(0, \pm 6)$;
(f) with eccentricity $e = \frac{1}{5}$ and the ends of the minor axis at $(0, \pm 10)$.

2 Find the equation of the ellipse
(a) with vertices $(6, 2)$, $(-4, 2)$ and minor axis of length 6;
(b) with major axis 8 units long, and foci at $(6, 3)$ and $(2, 3)$;
(c) with minor axis 6 units long, and foci at $(1, 0)$ and $(1, 6)$;
(d) with eccentricity $e = \frac{3}{4}$, and ends of the major axis at $(10, 1)$ and $(-6, 1)$.

3 Find the center, vertices, foci, and eccentricity of each of the following ellipses:
(a) $25x^2 + 9y^2 = 225$;
(b) $x^2 + 4y^2 = 4$;
(c) $2(x + 2)^2 + (y - 1)^2 = 2$;
(d) $x^2 + 4y^2 - 2x = 0$;
(e) $4x^2 + 9y^2 - 16x + 18y = 11$;
(f) $x^2 + 2y^2 - 8y = 0$.

4 Consider an equation of the form

$$Ax^2 + By^2 + Cx + Dy + E = 0,$$

where A and B are both positive or both negative and $A \neq B$. Show that the graph is an ellipse, a single point, or the empty set.

5 Write down the integrals that give (a) the first-quadrant area of the circle $x^2 + y^2 = a^2$, and (b) the first-quadrant area of the ellipse $x^2/a^2 + y^2/b^2 = 1$. Show that the second integral is b/a times the first, and in this way obtain the area of the ellipse from the known area of the circle.

6 If a cylindrical drinking glass with some water in it is tilted slightly, as in Fig. 15.23, then the surface of the water is no longer a circle but looks like an ellipse. Prove that it really is an ellipse by modifying the argument used in Fig. 15.4.

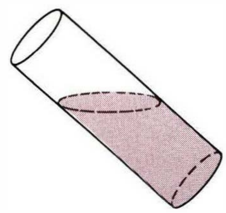


Figure 15.23

7 Find the volume of the solid of revolution obtained by revolving the ellipse $x^2/a^2 + y^2/b^2 = 1$ about
(a) the x-axis;
(b) the y-axis.
If $a > b$, the first solid is called a *prolate spheroid* and the second an *oblate spheroid.*

8 The base of a solid is the region bounded by an ellipse with semiaxes 5 and 3. Find the volume of the solid if each cross section in a plane perpendicular to the major axis is
(a) a square;
(b) an equilateral triangle.

9 Prove that $\alpha = \beta$ in Fig. 15.19. Hint: Extend FP and $F'P$ through P and show that $\tan \alpha = \tan \beta$ by using the subtraction formula for the tangent.

10 Consider two ellipses with the same eccentricity e, both centered at the origin and both with major axis on the x-axis. Suppose that their equations are

$$\frac{x^2}{a_1^2}+\frac{y^2}{b_1^2}=1 \quad \text{and} \quad \frac{x^2}{a_2^2}+\frac{y^2}{b_2^2}=1.$$

Show that these ellipses are similar in the sense that
(a) there exists a constant k such that

$$\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}=k;$$

(b) if a half-line from the origin O intersects the first ellipse at P_1 and the second at P_2, then

$$\frac{OP_1}{OP_2}=k.$$

11 Show that the line tangent to the ellipse $x^2/a^2 + y^2/b^2 = 1$ at a point $P_1 = (x_1, y_1)$ is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1.$$

12 If tangent lines to the ellipse $x^2/25 + y^2/16 = 1$ intersect the y-axis at (0, 8), find the points of tangency.

13 If tangent lines to the ellipse $x^2/a^2 + y^2/b^2 = 1$ intersect the y-axis at $(0, d)$, where $d > b$, find the points of tangency.

14 Let F be a point which is inside a given circle but is not the center C. Consider a point P that moves in such a way as to be equidistant from F and the circle. Show that the path of P is an ellipse.

15 Show that the point on an ellipse that is closest to a focus is the end of the major axis nearest that focus, and also that the point on the ellipse farthest from this focus is the other end of the major axis.

16 The *apogee* of an earth satellite is its maximum altitude above the surface of the earth during orbit, and its *perigee* is its minimum altitude during orbit.* If R is the radius of the earth, use Problem 15 to show that if a satellite has an elliptical earth orbit with the center of the earth at one focus and semimajor axis a, then

$$2a = 2R + \text{apogee} + \text{perigee}.$$

17 The point of the orbit of a planet nearest the sun is called the *perihelion,* and the point farthest from the sun is called the *aphelion.* If the ratio of the earth's distance from the sun at perihelion to its distance at aphelion is $\frac{29}{30}$, find the eccentricity of the earth's orbit.

18 A line segment moves with one end A on the y-axis and the other end B on the x-axis. A point P fixed on the segment is a units from A and b units from B. Find the equation of the path of P.

19 (a) Show that the length of the part of the ellipse $x^2/a^2 + y^2/b^2 = 1$ $(a > b)$ that lies in the first quadrant is

$$\int_0^a \sqrt{\frac{a^2-e^2x^2}{a^2-x^2}}\,dx.$$

(b) Use the change of variable $x = a \sin\theta$ to transform the integral in (a) into

$$a\int_0^{\pi/2} \sqrt{1-e^2\sin^2\theta}\,d\theta.$$

This is called a *complete elliptic integral of the second kind,* and cannot be evaluated by means of elementary functions.

20 Let P be a point on the ellipse $x^2/a^2 + y^2/b^2 = 1$ that does not lie on either axis. If $a > b$, show without using calculus that the distance from P to the origin is greater than b and less than a.

21 There are exactly two lines with given slope m that are tangent to the ellipse $x^2/a^2 + y^2/b^2 = 1$. Find their equations.

22 Consider two circles centered at the origin with radii a and b, where $b < a$. Draw a half-line from the origin intersecting the smaller circle at Q and the larger circle at R. If the horizontal line through Q and the vertical line through R intersect at $P = (x, y)$, show that P lies on the ellipse $x^2/a^2 + y^2/b^2 = 1$. Hint: Let the coordinates of Q and R be (q, y) and (x, r), and use the fact that Q and R lie on a line through the origin.

23 Let C_1 and C_2 be circles in the same plane with radii r_1 and r_2. Assume that $r_1 > r_2$ and that C_1 surrounds C_2, but that C_1 and C_2 are not concentric. Show that the centers of all circles that lie between C_1 and C_2 and are tangent to both lie on an ellipse whose foci are the centers of C_1 and C_2 (Fig. 15.24). What is the length of the major axis?

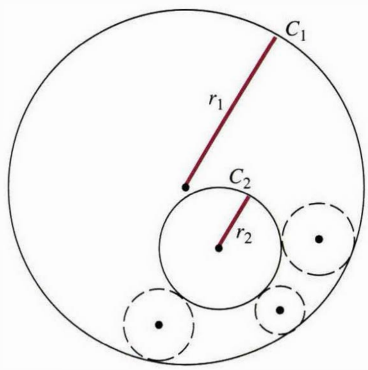


Figure 15.24

*These words are also used to mean the corresponding points of the orbit.

15.4 HYPERBOLAS

The ideas of Section 15.1 allow us to define a *hyperbola* as the locus of a point P that moves in such a way that the difference of its distances from two fixed points F and F' (called the *foci*) is constant. If this constant is denoted by $2a$, with $a > 0$, then a little thought will show that the locus consists of two *branches*, as shown in Fig. 15.25, where the right branch is the locus of the equation $PF' - PF = 2a$ and the left branch is the locus of the equation $PF - PF' = 2a$. The defining condition for the complete hyperbola can therefore be written as

$$PF' - PF = \pm 2a. \qquad (1)$$

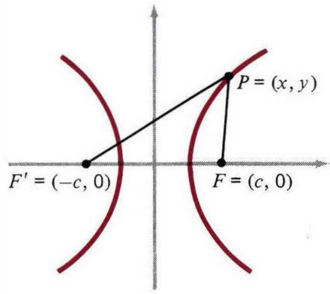


Figure 15.25 A hyperbola.

To find a simple equation for the hyperbola, we take the x-axis along the segment FF' and the y-axis as the perpendicular bisector of this segment. If $2c$ denotes the distance between F and F', then $F = (c, 0)$ and $F' = (-c, 0)$, as shown in Fig. 15.25, and (1) becomes

$$\sqrt{(x + c)^2 + y^2} - \sqrt{(x - c)^2 + y^2} = \pm 2a.$$

By moving the second radical to the right side, squaring, and simplifying, we obtain the focal radius formulas

$$PF = \sqrt{(x - c)^2 + y^2} = \pm\left(\frac{c}{a}x - a\right) \qquad (2)$$

and

$$PF' = \sqrt{(x + c)^2 + y^2} = \pm\left(\frac{c}{a}x + a\right), \qquad (3)$$

where (3) follows from (2) because $PF' = \pm 2a + PF$. As in (1), the plus signs here correspond to the right branch of the curve, and the minus signs to the left branch. By squaring and simplifying, either of these equations gives

$$\left(\frac{c^2 - a^2}{a^2}\right)x^2 - y^2 = c^2 - a^2$$

or

$$\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1. \qquad (4)$$

To simplify this equation still further, we begin by observing that in the triangle $PF'F$ with P on the right branch we have $PF' < PF + FF'$, because one side of a triangle is less than the sum of the other two sides. Therefore $PF' - PF < FF'$, or $2a < 2c$, so $a < c$ and $c^2 - a^2$ is a positive number which we denote by b^2,

$$b^2 = c^2 - a^2. \qquad (5)$$

This enables us to write (4) as

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \qquad (6)$$

which is the standard form of the equation of the hyperbola shown in Fig. 15.25.

We now turn to a careful consideration of equation (6) and the light it sheds on the nature of the hyperbola it represents. Our discussion will reveal several additional features of this curve that are not obvious from the definition and that are indicated in greater detail in Fig. 15.26.

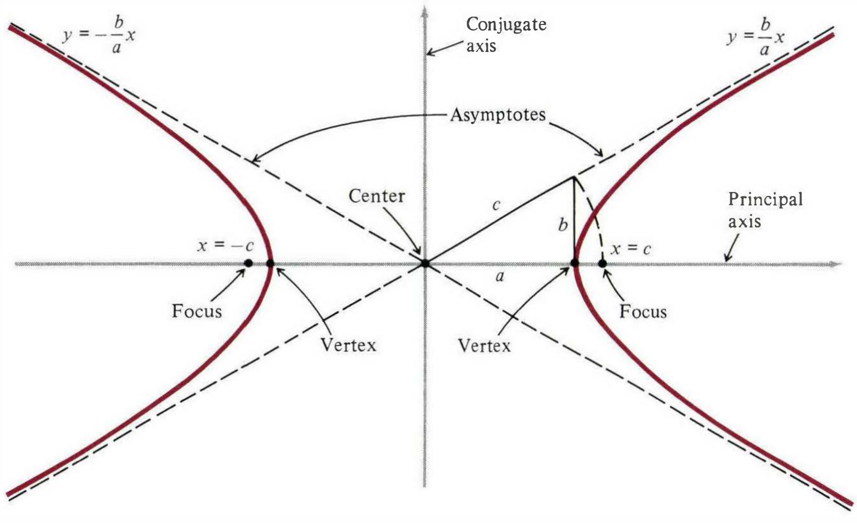


Figure 15.26 Features of a hyperbola.

Since the equation contains only even powers of x and y, the hyperbola is symmetric with respect to both coordinate axes. They are therefore called the *axes* of the curve, and their intersection is called the *center*. The left-right, up-down symmetry is perhaps the only feature of the hyperbola that is easy to see directly from the definition.

When $y = 0$, the equation gives $x = \pm a$, but when $x = 0$, y is imaginary. Therefore the axis through the foci, called the *principal axis,* intersects the curve at two points called the *vertices,* which are located at a distance a on each side of the center; but the other axis, called the *conjugate axis,* does not intersect the curve at all. The hyperbola thus consists of two separate parts, its symmetrical *branches,* on opposite sides of the conjugate axis.

These facts are easier to see if equation (6) is solved for y,

$$y = \pm\frac{b}{a}\sqrt{x^2 - a^2}. \tag{7}$$

This formula shows that there are no points of the graph in the vertical strip $-a < x < a$, because for these x's the quantity inside the radical is negative. When $x = \pm a$, (7) yields $y = 0$; these two points are the vertices. And now, as x increases from a or decreases from $-a$, we get two distinct values of y that increase numerically as x moves farther to the right or left; this behavior produces the upper and lower arms of each branch of the curve.

A very significant feature of the graph can be observed by writing (7) in the form

$$y = \pm\frac{b}{a}x\sqrt{1 - \frac{a^2}{x^2}}. \tag{8}$$

When x is numerically large, the quantity inside the radical in (8) is nearly 1, and for this reason it appears that the hyperbola is very close to the pair of straight lines

$$y = \pm\frac{b}{a}x. \tag{9}$$

We can verify this guess as follows. In the first quadrant, if x is large, then the vertical distance from the hyperbola up to the corresponding line is

$$\begin{aligned}\frac{b}{a}x - \frac{b}{a}\sqrt{x^2 - a^2} &= \frac{b}{a}(x - \sqrt{x^2 - a^2}) \\ &= \frac{b}{a}\,\frac{(x - \sqrt{x^2 - a^2})(x + \sqrt{x^2 - a^2})}{x + \sqrt{x^2 - a^2}} \\ &= \frac{ab}{x + \sqrt{x^2 - a^2}}.\end{aligned}$$

This clearly approaches zero as $x \to \infty$. The lines (9) are therefore called the *asymptotes* of the hyperbola. The asymptotes provide a convenient guide for sketching a hyperbola whose equation is given: Simply plot the vertices, draw the asymptotes, and fill in the two branches of the curve in a reasonable way, as suggested by the figure.

The triangle shown in the first quadrant of Fig. 15.26 is a convenient mnemonic device for remembering the main geometric features of a hyperbola. Its base a is the distance from the center to the vertex on the right; its height b is the distance from this vertex up to the asymptote in the first quadrant, whose slope is b/a; and since (5) tells us that

$$c^2 = a^2 + b^2,$$

the hypotenuse c of this triangle is also the distance from the center to a focus.

The ratio c/a is called the *eccentricity* of the hyperbola, and is denoted by e:

$$e = \frac{c}{a} = \frac{\sqrt{a^2 + b^2}}{a} = \sqrt{1 + \left(\frac{b}{a}\right)^2}.$$

It is clear that $e > 1$. When e is near 1, then b is small compared with a, and the hyperbola lies in a small angle between the asymptotes. When e is large, then b is large compared with a, the angle between the asymptotes is large, and the hyperbola is rather flat at the vertices.

To understand the significance of the eccentricity, we consider again formulas (2) and (3) for the right and left focal radii PF and PF'. These formulas can be written as

$$PF = \pm(ex - a) = \pm e\left[x - \frac{a}{e}\right] \tag{10}$$

and

$$PF' = \pm(ex + a) = \pm e\left(x + \frac{a}{e}\right) = \pm e\left[x - \left(-\frac{a}{e}\right)\right], \tag{11}$$

where the plus signs apply to the right branch of the curve (see Fig. 15.27) and the minus signs to the left branch. If P lies on the right branch, as shown in the figure, then the quantities in brackets can be interpreted as the distances PD and PD' from P to the lines $x = a/e$ and $x = -a/e$, respectively. The same statement is true if P lies on the left branch, if the effect of the minus signs is properly taken into account. Therefore, in all cases formulas (10) and (11) can be written in the form

$$\frac{PF}{PD} = e \qquad \text{and} \qquad \frac{PF'}{PD'} = e. \tag{12}$$

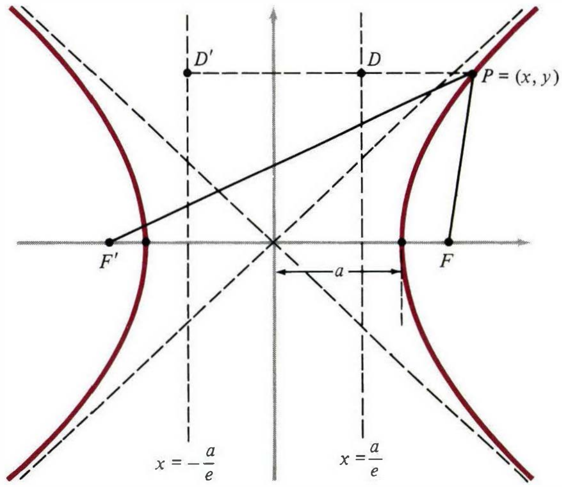


Figure 15.27

Each of the lines $x = a/e$ and $x = -a/e$ is called a *directrix* of the hyperbola. Equations (12) show that *a hyperbola can be characterized as the locus of a point that moves in such a way that the ratio of its distance from a fixed point (a focus) to its distance from a fixed line (the corresponding directrix) equals a constant* $e > 1$. Just as in the case of ellipses, this way of characterizing hyperbolas will be needed in our future work.

By interchanging the roles of x and y in the preceding discussion, we find that the equation

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \tag{13}$$

represents a hyperbola with vertical principal axis, vertices at $(0, \pm a)$, and foci at $(0, \pm c)$, where $c^2 = a^2 + b^2$. This time the asymptotes are the lines

$$y = \pm \frac{a}{b} x,$$

as we easily see by writing (13) in a form solved for y,

$$y = \pm \frac{a}{b} \sqrt{x^2 + b^2} = \pm \frac{a}{b} x \sqrt{1 + \frac{b^2}{x^2}}.$$

Notice that the axis containing the foci of a hyperbola is not determined by the relative size of a and b, as it was in the case of an ellipse, but rather by which term is subtracted from which in the standard form of the equation. The numbers a and b can therefore be of any relative size. In particular they can be equal, in which case the asymptotes are perpendicular to each other and the hyperbola is called *rectangular*. The equations

$$x^2 - y^2 = a^2 \qquad \text{and} \qquad y^2 - x^2 = a^2$$

represent rectangular hyperbolas.

Example 1 Find the equation of the hyperbola with foci $(\pm 6, 0)$ and the lines $5y = \pm 2\sqrt{5}\, x$ as asymptotes.

Solution First, the location of the foci tells us that the principal axis is the x-axis. We see that $c = 6$ and $b/a = \frac{2}{5}\sqrt{5}$, so $a = (\sqrt{5}/2)b$. Since $a^2 + b^2 = c^2$, we have $\frac{5}{4}b^2 + b^2 = 36$, so $b^2 = \frac{4}{9} \cdot 36 = 16$ and $a^2 = \frac{5}{4}b^2 = \frac{5}{4} \cdot 16 = 20$. This shows that

$$\frac{x^2}{20} - \frac{y^2}{16} = 1$$

is the equation of the hyperbola.

Example 2 Determine the principal axis of the hyperbola $6y^2 - 9x^2 = 36$ and find its vertices, foci, and asymptotes.

Solution The equation can be put in the standard form

$$\frac{y^2}{6} - \frac{x^2}{4} = 1,$$

so the principal axis is the y-axis, $a^2 = 6$, $b^2 = 4$, and $c^2 = a^2 + b^2 = 10$. Hence the vertices are $(0, \pm\sqrt{6})$, the foci are $(0, \pm\sqrt{10})$, and the asymptotes are $y = \pm(\sqrt{6}/2)x$.

Just as in the case of the ellipse, we can easily write the equation of a hyperbola with center (h, k) and principal axis parallel to one of the coordinate axes. The equation is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \quad \text{or} \quad \frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1,$$

according as the principal axis is horizontal or vertical. This suggests that we consider equations of the form

$$Ax^2 + By^2 + Cx + Dy + E = 0,$$

where A and B have opposite signs. Such an equation will usually represent a hyperbola, but in certain special cases it may represent a pair of intersecting straight lines. The next example illustrates these possibilities.

Example 3 Identify the graph of

$$16x^2 - 9y^2 - 64x - 18y + E = 0$$

for various values of E.

Solution The procedure is to complete the square on the x and y terms, which yields

$$16(x^2 - 4x) - 9(y^2 + 2y) = -E$$

and

$$16(x-2)^2 - 9(y+1)^2 = 55 - E.$$

There are now three cases.

CASE 1 $55 - E > 0$; for example, $E = -89$, so that $55 - E = 144$. In this case we have

$$\frac{(x-2)^2}{9} - \frac{(y+1)^2}{16} = 1,$$

which is a hyperbola with center $(2, -1)$ and horizontal principal axis.

CASE 2 $55 - E < 0$; for example, $E = 199$, so that $55 - E = -144$. Here we have

$$\frac{(y+1)^2}{16} - \frac{(x-2)^2}{9} = 1,$$

which is a hyperbola with center $(2, -1)$ and vertical principal axis.

CASE 3 $55 - E = 0$; $E = 55$. This time our equation becomes

$$16(x-2)^2 - 9(y+1)^2 = 0$$

or

$$4(x-2) = \pm 3(y+1).$$

This represents the two lines

$$y + 1 = \pm\tfrac{4}{3}(x-2),$$

which are the asymptotes in the first two cases.

Remark 1 Hyperbolas have the following reflection property: The tangent line at any point P on a hyperbola bisects the angle between the focal radii PF and PF'. This means that $\alpha = \beta$ in the notation of Fig. 15.28 (see Problem 21). As a consequence of this, if the hyperbola is revolved about its principal axis to form a surface of revolution, and if the convex sides of each part are silvered to make them reflecting surfaces, then any ray of light that approaches a convex side along a line pointing toward a focus (Fig. 15.28, right) is reflected toward the other focus.

This property of hyperbolas is the essential principle in the design of reflecting telescopes of the Cassegrain type (Fig. 15.29). As the figure shows, one focus of the hyperbolic mirror is at the focus of the parabolic mirror and the other is at the vertex of the parabolic mirror, where an eyepiece or camera is located.

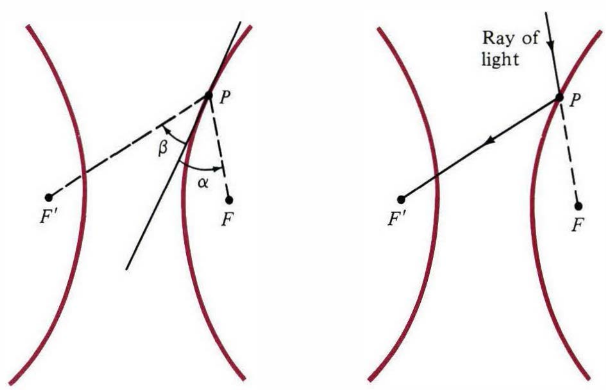


Figure 15.28 The reflection property.

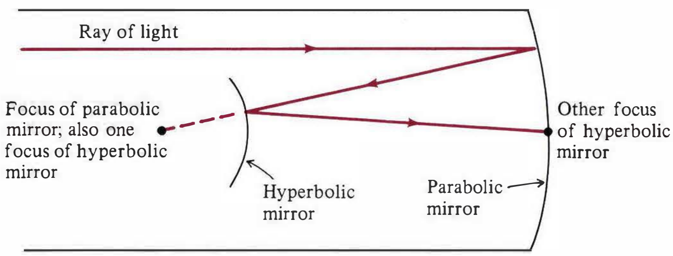


Figure 15.29 Design of Cassegrain telescope.

Faint parallel rays of starlight are therefore reflected off the parabolic mirror toward its focus, then are intercepted by the hyperbolic mirror and reflected back toward the eyepiece or camera.

Remark 2 There are two kinds of comets. Some are permanent members of the solar system, like Halley's Comet described in Section 15.3, and travel forever around the sun in elliptical orbits with the sun at one focus. Others enter the solar system at high speeds from outer space, swing around the sun in hyperbolic orbits with the sun at one focus, and then escape into outer space again. The crucial factor is the total energy E of the comet itself, which is the sum of the kinetic energy due to its motion and the potential energy due to the gravitational attraction of the sun. It turns out that if $E < 0$, the orbit is an ellipse, and if $E > 0$, the orbit is a hyperbola. (The case $E = 0$ corresponds to a parabolic orbit, but this is exceedingly unlikely.)

PROBLEMS

In Problems 1–8, sketch the graph of the given hyperbola and find the vertices, foci, asymptotes, eccentricity, and directrices.

1 $\dfrac{x^2}{4} - \dfrac{y^2}{9} = 1.$

2 $\dfrac{y^2}{36} - \dfrac{x^2}{16} = 1.$

3 $\dfrac{y^2}{4} - \dfrac{x^2}{9} = 1.$

4 $\dfrac{x^2}{25} - \dfrac{y^2}{16} = 1.$

5 $4y^2 - x^2 = 16.$

6 $x^2 - 3y^2 = 12.$

7 $y^2 - x^2 = 1.$

8 $x^2 - 9y^2 = 1.$

In Problems 9–16, find the equation of the hyperbola determined by the given conditions.

9 Foci $(0, \pm 5)$, vertex $(0, 3)$.

10 Vertices $(\pm 3, 0)$, focus $(5, 0)$.

11 Vertices $(\pm 3, 0)$, asymptote $y = 2x$.

12 Foci $(-1, 8)$ and $(-1, -2)$, vertex $(-1, 7)$.

13 Foci $(\pm 8, 0)$, $e = \frac{4}{3}$.

14 Vertices $(0, \pm 5)$, $e = 2$.

15 Vertices $(\pm 6, 0)$, directrix $x = 4$.

16 Foci $(1, 1)$ and $(-1, -1)$, difference of focal radii ± 2.

In Problems 17–20, identify the graph of the given equation as in the discussion of Example 3.

17 $16x^2 - 3y^2 - 32x - 12y - 44 = 0.$

18 $9y^2 - 7x^2 + 72y - 70x - 94 = 0.$

19 $36x^2 - 25y^2 + 144x - 50y + 119 = 0.$

20 $11y^2 - 12x^2 + 88y + 72x + 300 = 0.$

21 Show that $\alpha = \beta$ in Fig. 15.26.

22 Let F be a point which is outside a given circle. Consider a point P that moves in such a way as to be equidistant from F and the circle. Show that the path of P is one branch of a hyperbola.

23 Suppose that an ellipse and a hyperbola are *confocal*, that is, have the same foci F and F'. Use the reflection properties of the two curves to give a purely geometric proof that they are perpendicular to each other at every point P of intersection.

24 (a) Show that

$$\frac{x^2}{25 - k} + \frac{y^2}{16 - k} = 1$$

represents an ellipse if $k < 16$ and a hyperbola if $16 < k < 25$, and that all these curves are confocal.

(b) Find the first-quadrant point of intersection of the curves given by $k = 0$ and $k = 20$, and find the tangent line to each curve at this point.

(c) Show that the tangent lines in part (b) are perpendicular to each other by showing that the product of their slopes is -1.

25 Consider two hyperbolas with the same eccentricity e, both centered at the origin and both with principal axis on the x-axis. Suppose that their equations are

$$\frac{x^2}{a_1^2} - \frac{y^2}{b_1^2} = 1 \quad \text{and} \quad \frac{x^2}{a_2^2} - \frac{y^2}{b_2^2} = 1.$$

Show that these hyperbolas are similar in the sense that (a) there exists a constant k such that

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = k;$$

(b) if a half-line from the origin O intersects the first hyperbola at P_1 and the second at P_2, then

$$\frac{OP_1}{OP_2} = k.$$

26 Find the locus of the centers of all circles that are tangent to the y-axis and cut off a segment of length $2a$ on the x-axis.

27 Let F and F' be two points on a sheet of paper whose distance apart is $2c$. Take a piece of string and tie a knot K in it so that the difference between the lengths of the two parts into which K divides the string is $2a$, where $0 < a < c$. Tie the ends of the string to two tacks placed at F and F', and loop the string around the point of a pencil, as shown in Fig. 15.30. If the string is held taut and the knot K is carefully pulled, show that the pencil at P draws one branch of a hyperbola.

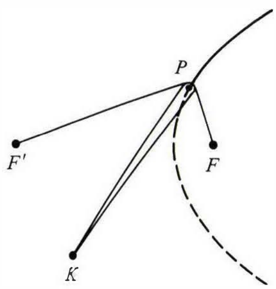


Figure 15.30

28 If two hyperbolas $x^2/a^2 - y^2/b^2 = 1$ and $y^2/A^2 - x^2/B^2 = 1$ have the same asymptotes, show that their eccentricities e and E are related by the equation

$$\frac{1}{e^2} + \frac{1}{E^2} = 1.$$

29 Show that the line tangent to the hyperbola $x^2/a^2 - y^2/b^2 = 1$ at the point $P_1 = (x_1, y_1)$ is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1.$$

30 If tangent lines to the hyperbola $x^2/25 - y^2/16 = 1$ intersect the y-axis at $(0, 8)$, find the points of tangency.

31 If tangent lines to the hyperbola $x^2/a^2 - y^2/b^2 = 1$ intersect the y-axis at $(0, d)$, find the points of tangency.

32 A line through a point P on a hyperbola and parallel to the nearest asymptote intersects the nearest directrix at Q. If F is the corresponding focus, show that $PQ = PF$.

33 Let C_1 and C_2 be circles in the same plane with different radii r_1 and r_2. Assume that C_1 and C_2 do not intersect and that neither surrounds the other. Show that the centers of all circles that are outside both, do not surround either, and are tangent to both, lie on one branch of a hyperbola whose foci are the centers of C_1 and C_2 (Fig. 15.31).

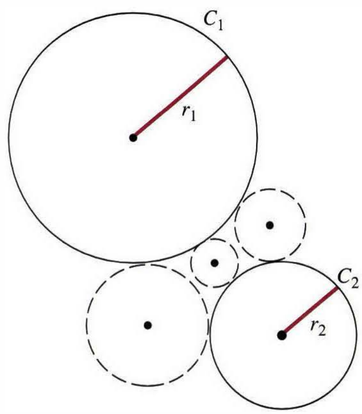


Figure 15.31

15.5 THE FOCUS-DIRECTRIX-ECCENTRICITY DEFINITIONS

Students have already seen that there are several distinct but equivalent ways of defining the conic sections, each with its own merits. We began with the definition by means of a given cone and a slicing plane that cuts through the cone more or less steeply, yielding our three types of curves by varying the degree of steepness. This three-dimensional approach is vivid and geometric, and provides a clear visual impression of what the curves look like. However, for the purpose of obtaining Cartesian equations for use in precise quantitative studies, we needed two-dimensional characterizations, and for this the focal properties discussed at the end of Section 15.1 turned out to be convenient. The concepts of eccentricity and directrix emerged in the course of our detailed work on ellipses and hyperbolas, and we saw that each of these curves can be given yet another two-dimensional characterization by means of a focus, a directrix, and an eccentricity. Our purpose in this brief section is to show that all three of the conic sections—parabolas, ellipses, and hyperbolas—can in this way be given unified definitions that depend directly on our original concept of these curves as sections of a cone.*

Our discussion is based on Fig. 15.32, which shows a cone with vertex angle α and a slicing plane with tilting angle β. This tilting angle can be defined as the angle between the axis of the cone and a normal line to the plane, but it plays its main role in our argument as the indicated acute angle of the right triangle PQD. The figure is drawn to illustrate the case of an ellipse, but the argument is valid for the other cases as well.

We begin at the beginning. Let there be inscribed in the cone a sphere which is tangent to the slicing plane at a point F, and tangent to the cone along a circle C. If d is the line in which the slicing plane intersects the plane of the circle C, we shall prove that the conic section has F as its focus and d as its directrix, and the facts about the eccentricity will emerge in the course of our discussion.

To this end, let P be a point on the conic section, let Q be the point where the line through P and parallel to the axis of the cone intersects the plane of C, let R be the point where the generator through P intersects C, and let D be the foot of the perpendicular from P to the line d. Then PR and PF are two segments which are tangent to the sphere from the same point P, and therefore have the same length,

$$PR = PF. \tag{1}$$

*For reasons that will soon be clear, circles must be excluded from this discussion, because the necessary geometric constructions are not possible when the slicing plane is perpendicular to the axis of the cone.

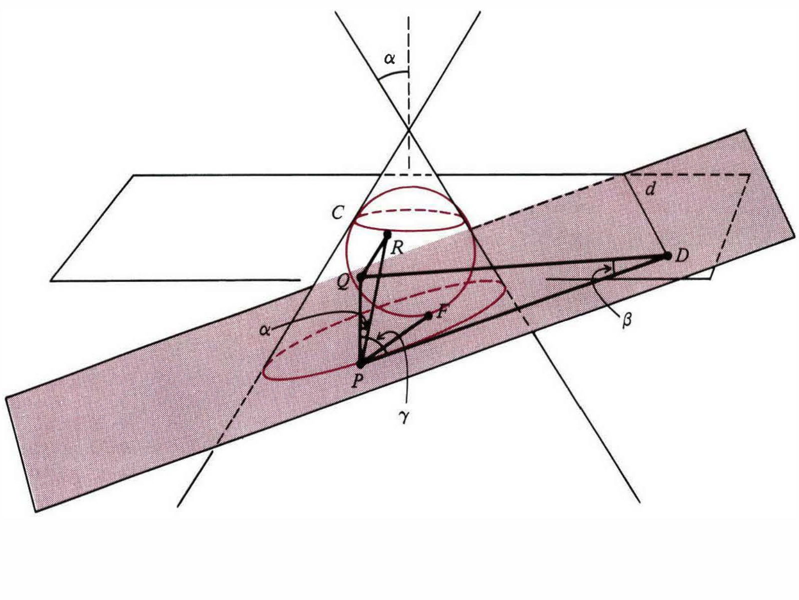


Figure 15.32

Also, from the right triangle PQR we have

$$PQ = PR \cos\alpha;$$

and from the right triangle PQD we have

$$PQ = PD \sin\beta.$$

It follows that

$$PR\cos\alpha = PD\sin\beta,$$

so

$$\frac{PR}{PD} = \frac{\sin\beta}{\cos\alpha}.$$

In view of (1) this means that

$$\frac{PF}{PD} = \frac{\sin\beta}{\cos\alpha}.$$

This can be written in the slightly more convenient form

$$\frac{PF}{PD} = \frac{\cos\gamma}{\cos\alpha}, \tag{2}$$

where γ is the other acute angle in the right triangle PQD. If we now define the eccentricity e by

$$e = \frac{\cos\gamma}{\cos\alpha},$$

then this number is constant for a given cone and a given slicing plane, and (2) becomes

$$\frac{PF}{PD} = e \begin{cases} < 1 & \text{for an ellipse,} \\ = 1 & \text{for a parabola,} \\ > 1 & \text{for a hyperbola,} \end{cases}$$

where the statements on the right are easily verified by inspecting the figure. Thus, for a parabola, we see that PD is parallel to a generator of the cone, so $\gamma = \alpha$ and $e = 1$; for an ellipse, we have $\gamma > \alpha$, so $\cos\gamma < \cos\alpha$ and $e < 1$; and for a hyperbola, we have $\gamma < \alpha$, so $\cos\gamma > \cos\alpha$ and $e > 1$.

The words "parabola," "ellipse," and "hyperbola" come from three Greek words meaning "a comparison," "a deficiency," and "an excess," referring to the fact that for the corresponding curves we have $e = 1$, $e < 1$, and $e > 1$. One should also compare these words with the words "parable," "ellipsis," and "hyperbole" in modern English.

15.6 (OPTIONAL) SECOND-DEGREE EQUATIONS. ROTATION OF AXES

The general equation of the second degree in x and y is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \tag{1}$$

where at least one of the coefficients A, B, C is different from zero. The latter requirement, of course, guarantees that the degree of the equation really is 2, rather than 1 or 0. In the preceding sections we have found that circles, parabolas, ellipses, and hyperbolas are all curves whose equations are special cases of (1). Thus, for example, the circle

$$(x-h)^2 + (y-k)^2 = r^2$$

can be obtained from (1) by taking

$$A = C = 1, \qquad B = 0, \qquad D = -2h, \qquad E = -2k,$$
$$F = h^2 + k^2 - r^2,$$

and the parabola

$$x^2 = 4py$$

by taking

$$A = 1, \qquad E = -4p, \qquad B = C = D = F = 0.$$

In addition to the conic sections mentioned here, we have also noted various "exceptional cases" that can arise as graphs of (1) from special choices of the coefficients. Thus, the graph of

$$x^2 + y^2 = 0$$

is a point, and the graph of

$$x^2 + y^2 + 1 = 0$$

is the empty set. Further, the graph of

$$x^2 = 0$$

is a single line, namely, the y-axis, and the graph of

$$x^2 - y^2 = 0, \qquad \text{or equivalently} \qquad (x+y)(x-y) = 0,$$

is a pair of lines, namely, $x + y = 0$ and $x - y = 0$. Our purpose in this section is to investigate the full range of possibilities of the curves represented by (1). Briefly, we shall find that the eight graphs we have just listed exhaust all possibilities:

The graph of every second-degree equation of the form (1) *is a circle, a parabola, an ellipse, a hyperbola, a point, the empty set, a single line, or a pair of lines.*

The main problem before us is posed by the so-called *mixed term Bxy* in (1), because when this term is present we have no idea how to identify the graph. No such terms have arisen in our previous work on the conic sections. The reason for this is that in every case we have been careful to choose the coordinate axes in a simple and natural position, so that at least one axis is parallel to an axis of symmetry of the curve under discussion. In order to see what can happen when a curve is placed in a skew position relative to the axes, let us find the equation of the hyperbola (see Fig. 15.33) with foci $F = (2, 2)$ and $F' = (-2, -2)$, where $PF' - PF = \pm 4$. We have

$$\sqrt{(x+2)^2 + (y+2)^2} - \sqrt{(x-2)^2 + (y-2)^2} = \pm 4,$$

and when we move the second radical to the right side, square, solve for the radical that still remains, and square again, this reduces to

$$xy = 2. \tag{2}$$

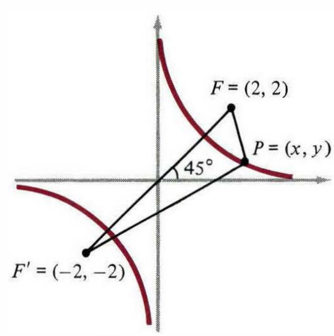


Figure 15.33

This is really a very simple equation, but nevertheless it does provide a special case of (1) in which the mixed term is present. The asymptotes of the hyperbola (2) are evidently the x- and y-axes, and its principal axis is the line $y = x$, which makes a 45° angle with the x-axis. It will become clear that a mixed term is present only when a curve is "tilted" in this way with respect to the coordinate axes, and also that this term can be removed by rotating the axes to "untilt" the curve. In the case of (2), it is easy to see by looking at the figure that this curve can be untilted by rotating the axes through a 45° angle in the counterclockwise direction.

To construct the machinery that is necessary for carrying out an arbitrary rotation of axes, we start with the xy-system and rotate these axes counterclockwise through an angle θ to obtain the $x'y'$-system, as shown in Fig. 15.34. A point P in the plane will then have two pairs of rectangular coordinates, (x, y) and (x', y'). To see how these coordinates are related, we observe from the figure that

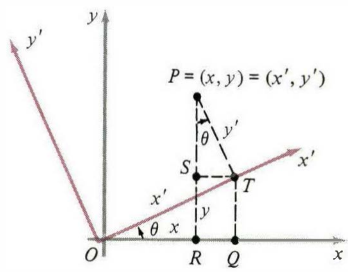


Figure 15.34 Rotation of axes.

$$x = OR = OQ - RQ = OQ - ST$$
$$= x' \cos\theta - y' \sin\theta$$

and

$$y = RP = RS + SP = QT + SP$$
$$= x' \sin\theta + y' \cos\theta.$$

We write these equations together for convenient reference,

$$\begin{aligned} x &= x' \cos\theta - y' \sin\theta, \\ y &= x' \sin\theta + y' \cos\theta; \end{aligned} \tag{3}$$

they are called the *equations for rotation of axes*. For example, if $\theta = 45°$, then, since $\sin 45° = \cos 45° = \frac{1}{2}\sqrt{2} = 1/\sqrt{2}$, we have

$$x = \frac{x' - y'}{\sqrt{2}}, \qquad y = \frac{x' + y'}{\sqrt{2}}. \tag{4}$$

And for another example, if $\theta = 30°$, then since $\sin 30° = \frac{1}{2}$ and $\cos 30° = \frac{1}{2}\sqrt{3}$, we have

$$x = \frac{\sqrt{3}x' - y'}{2}, \qquad y = \frac{x' + \sqrt{3}y'}{2}. \tag{5}$$

As a simple illustration of the use of these equations, we substitute (4) into (2) and obtain

$$\frac{x'^2 - y'^2}{2} = 2 \qquad \text{or} \qquad \frac{x'^2}{4} - \frac{y'^2}{4} = 1.$$

This is immediately recognizable as a rectangular hyperbola whose principal axis is the x'-axis. Of course, we already knew this from the way (2) was obtained. However, if we had started with (2) without knowing anything about the nature of its graph, then this procedure for removing the mixed term would have enabled us to identify the curve without difficulty.

In the case of equation (2), the 45° rotation represented by equations (4) worked. But how could we have known this in advance? Can we be sure that a suitable rotation will always remove the xy term if one is present? And if so, how do we find a suitable angle of rotation?

To answer these questions we return to the general second-degree equation (1),

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

and we apply the general rotation (3) through an unspecified angle θ, which yields

$$A(x' \cos \theta - y' \sin \theta)^2 + B(x' \cos \theta - y' \sin \theta)(x' \sin \theta + y' \cos \theta) + C(x' \sin \theta + y' \cos \theta)^2 + D(x' \cos \theta - y' \sin \theta) + E(x' \sin \theta + y' \cos \theta) + F = 0.$$

When we collect coefficients for the various terms, we get a new equation of the same form,

$$A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0, \tag{6}$$

with new coefficients related to the old ones by the following formulas:

$$\begin{aligned} A' &= A \cos^2 \theta + B \sin \theta \cos \theta + C \sin^2 \theta, \\ B' &= -2A \sin \theta \cos \theta + B(\cos^2 \theta - \sin^2 \theta) + 2C \sin \theta \cos \theta, \\ C' &= A \sin^2 \theta - B \sin \theta \cos \theta + C \cos^2 \theta, \\ D' &= D \cos \theta + E \sin \theta, \\ E' &= -D \sin \theta + E \cos \theta, \\ F' &= F. \end{aligned} \tag{7}$$

We have written down all these formulas for future reference, but for the moment we are only interested in B'. If we start out with a second-degree equation (1) in which the mixed term is present, $B \neq 0$, then we can always find an angle θ of rotation such that the new mixed term is eliminated. To find a suitable angle θ, we simply put $B' = 0$ in (7) and solve for θ. To do this most easily, we use the double-angle formulas

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

and

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$$

to write

$$B' = B \cos 2\theta + (C - A) \sin 2\theta.$$

Then $B' = 0$ if we choose θ so that

$$\cot 2\theta = \frac{A - C}{B}. \tag{8}$$

Since we are assuming that $B \neq 0$, it is clear that this is always possible, and furthermore that θ can always be chosen in the first quadrant, $0 < \theta < \pi/2$.

Example 1 Determine the nature of the curve whose equation is

$$4x^2 + 2xy + 4y^2 = 15. \tag{9}$$

Solution Here we have $A = 4$, $B = 2$, and $C = 4$. The mixed term will be removed by choosing θ according to (8), which in this case gives

$$\cot 2\theta = 0, \qquad 2\theta = 90°, \qquad \theta = 45°.$$

We therefore substitute equations (4) into (9) and obtain

$$5x'^2 + 3y'^2 = 15 \qquad \text{or} \qquad \frac{x'^2}{3} + \frac{y'^2}{5} = 1$$

after simplification. This is clearly an ellipse with its foci on the new y'-axis.

We observe that when $B \neq 0$ and $A = C$, a rotation through 45° is always appropriate.

Example 2 Determine the nature of the curve whose equation is

$$11x^2 + 10\sqrt{3}xy + y^2 - 32 = 0. \tag{10}$$

Solution Here we have

$$\cot 2\theta = \frac{11 - 1}{10\sqrt{3}} = \frac{1}{\sqrt{3}}, \qquad 2\theta = 60°, \qquad \theta = 30°.$$

In this case we use equations (5), which transform (10) into

$$16x'^2 - 4y'^2 - 32 = 0 \qquad \text{or} \qquad \frac{x'^2}{2} - \frac{y'^2}{8} = 1$$

after simplification. This is a hyperbola with its principal axis along the new x'-axis.

We now return to our original problem of classifying all possible graphs of the second-degree equation (1). Since the axes can always be rotated to eliminate the mixed term, there is no loss of generality in assuming that this has been done. We are therefore confronted by equation (6) with $B' = 0$, and we drop the primes to simplify the notation,

$$Ax^2 + Cy^2 + Dx + Ey + F = 0. \tag{11}$$

Our experience in the preceding sections enables us to distinguish four cases and to be certain that there are no others. The graph of equation (11), with $B = 0$, is:

1 A circle if $A = C \neq 0$. In special cases the graph can be a single point or the empty set.
2 An ellipse if A and C are both positive or both negative and $A \neq C$. Again, in special cases the graph can be a single point or the empty set.
3 A hyperbola if A and C have opposite signs. In special cases the graph can be a pair of intersecting straight lines.
4 A parabola if either $A = 0$ or $C = 0$ (but not both). In special cases the graph can be one straight line, or two parallel lines, or the empty set.

In Chapter 16 we will meet many equations of the third degree, fourth degree, etc. The exhaustive list given here of possible graphs of second-degree equations is relatively simple and stands in sharp contrast to the wilderness of bizarre curves that awaits us in connection with these higher-degree equations.

PROBLEMS

In Problems 1–11, determine and carry out a suitable rotation of axes to eliminate the mixed term, find the new equation, and identify the curve.

1 $5x^2 - 6xy + 5y^2 = 8$.

2 $x^2 - 2xy + y^2 + x + y = \sqrt{2}$.

3 $2x^2 + 4\sqrt{3}xy - 2y^2 = 8$.

4 $11x^2 + 4\sqrt{3}xy + 7y^2 - 65 = 0$.

5 $x^2 + 2xy + y^2 + 8x - 8y = 0$.

6 $x^2 - 3xy + y^2 = 10$.

7 $3x^2 + 2xy + 3y^2 = 8$.

8 $x^2 + 2\sqrt{3}xy + 3y^2 + 2\sqrt{3}x - 2y = 0$.

9 $5x^2 + 4\sqrt{3}xy + 9y^2 = 33$.

10 $3x^2 + 4\sqrt{3}xy - y^2 = 30$.

*11 $6x^2 - 6xy + 14y^2 = 5$.

12 An ellipse has foci $(1, 0)$ and $(0, \sqrt{3})$ and passes through the point $(-1, 0)$. Use the focus definition to find its equation. Through what angle should the axes be rotated to eliminate the mixed term from this equation?

13 Use equations (7) to show that $B^2 - 4AC = B'^2 - 4A'C'$. For this reason, the number $B^2 - 4AC$ is said to be *invariant under rotations*.

14 The number $B^2 - 4AC$ is called the *discriminant* of equation (1). To understand why, rotate the axes to remove the mixed term and use Problem 13 to show that

$$B^2 - 4AC \begin{cases} < 0 & \text{for circles and ellipses,} \\ = 0 & \text{for parabolas,} \\ > 0 & \text{for hyperbolas,} \end{cases}$$

where the various special cases are considered to belong to the appropriate categories.

15 Verify the statement in Problem 14 for Problems 1–11.

16 As a check on the equations for rotation of axes, show that the equation of a circle centered at the origin, $x^2 + y^2 = r^2$, is unchanged in form when the axes are rotated through an arbitrary angle θ.

17 If a rotation of axes through an angle θ is followed by a rotation through an angle ϕ, this obviously amounts to a rotation through the angle $\theta + \phi$. Use the formulas

$$x = x' \cos\theta - y' \sin\theta$$
$$y = x' \sin\theta + y' \cos\theta$$

and

$$x' = x'' \cos\phi - y'' \sin\phi$$
$$y' = x'' \sin\phi + y'' \cos\phi$$

to show that

$$x = x'' \cos(\theta + \phi) - y'' \sin(\theta + \phi)$$
$$y = x'' \sin(\theta + \phi) + y'' \cos(\theta + \phi).$$

18 Show that a 45° rotation of axes transforms the equation $x^4 + 6x^2y^2 + y^4 = 32$ into $x'^4 + y'^4 = 16$. Sketch the curve and both sets of axes.

CHAPTER 15 REVIEW: DEFINITIONS, PROPERTIES

Think through the following.

1 Conic sections from a cone.

2 Parabola, focus, directrix, axis, vertex.

3 Equation of parabola—standard form.

4 Reflection property of parabolas.

5 Ellipse, foci, major and minor axes, eccentricity.

6 Equation of ellipse—standard form.

7 Reflection property of ellipses.

8 Hyperbola, foci, principal axis, asymptotes, eccentricity.

9 Equation of hyperbola—standard form.

10 Reflection property of hyperbolas.

11 Focus-directrix-eccentricity definitions.

ADDITIONAL PROBLEMS FOR CHAPTER 15

SECTION 15.2

1 The chord of a parabola through the focus and perpendicular to the axis is called the *latus rectum.* If (as usual) p is the distance between the vertex and the focus, find the length of the latus rectum.

2 Find the equation of the circle tangent to the directrix of the parabola $x^2 = 4py$ with the focus of the parabola as its center. What are the points of intersection of the parabola and the circle?

3 Show that every upward-opening parabola with focus at the origin has an equation of the form $x^2 = 4p(y + p)$, and every downward-opening parabola with focus at the origin has an equation of the form $x^2 = -4\bar{p}(y - \bar{p})$, where p and $\bar{p}$ are positive constants.

4 Show that every upward-opening parabola with focus at the origin intersects at right angles every downward-opening parabola with focus at the origin.

5 Let P be a point on the parabola $x^2 = 4py$ other than the vertex. If Q and R are the points at which the tangent and normal at P intersect the axis of the parabola, and if S is the foot of the perpendicular from P to this axis, then the segments QS and RS are called the *subtangent* and *subnormal.*

(a) Show that the vertex V bisects the subtangent.
(b) Show that the subnormal has constant length $2p$.
(c) Show that P and R are the same distance from the focus F.
(d) Show that P and Q are the same distance from the focus F, so that F bisects the segment QR.
(e) If the tangent at P intersects the directrix at a point T, show that PFT is a right angle.
(f) If the tangent at P intersects the tangent at V (the x-axis) at a point U, show that PUF is a right angle.

6 Show that the reflection property of parabolas follows easily (without calculation) from part (d) of Problem 5.

7 Show that the vertex is the point on a parabola that is closest to the focus.

SECTION 15.3

8 The reflection property of ellipses is an easy consequence (without calculation) of the following geometric property of ellipses. In Fig. 15.35, let the line T be tangent at P to the ellipse with foci F and F'. Let G and G' be the reflections in T of F and F', so that T is the perpendicular bisector of the segments FG and $F'G'$. Then, as suggested by the figure, the segments FG' and $F'G$ intersect at P. To prove this, let Q be any point on T different from P and verify that

(a) $FP + PF' < FQ + QF'$;
(b) $FP + PG' < FQ + QG'$;
(c) P lies on the segment FG'.

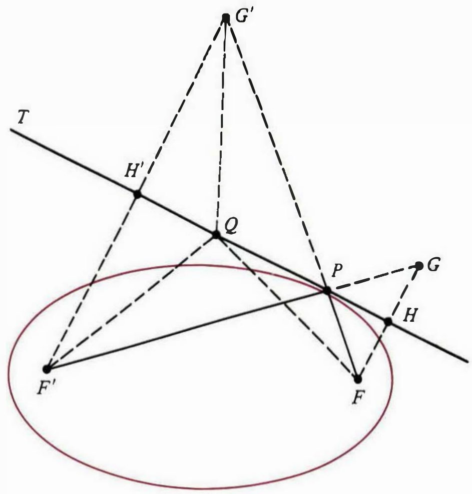


Figure 15.35

Similarly, P lies on the segment $F'G$, so it is the point of intersection of these segments. Finally, use this fact to infer that the angles FPH and $F'PH'$ are equal.

9 A barrel has the shape of a prolate spheroid with its ends cut off by planes through the foci. If the barrel is 4 ft high and the diameter of its top (and bottom) is 2 ft, find the volume.

10 A *latus rectum* of an ellipse is a chord through a focus and perpendicular to the major axis. If the equation of the ellipse is $x^2/a^2 + y^2/b^2 = 1$ with $a > b$, show that

(a) the length of a latus rectum is $2b^2/a$;
(b) the slope of the tangent line at the upper end of the latus rectum to the right of the y-axis is $-e$;
(c) the tangent line in part (b) intersects the corresponding directrix on the x-axis.

11 Show that every parabola of the form $y = Ax^2$ intersects every ellipse of the form $x^2 + 2y^2 = B$ at right angles.

12 Suppose that the tangent to an ellipse at a point P intersects a directrix at a point Q. If F is the corresponding focus, show that PFQ is a right angle.

13 Let P be a point on the ellipse $x^2/a^2 + y^2/b^2 = 1$ and let Q be the point where the tangent at P intersects the line $x = -a$. If A is the point $(a, 0)$, show that AP is parallel to OQ.

14 Show that the product of the distances from the foci of an ellipse to a tangent has the same value for all tangents. Hint: Use Additional Problem 21 in Chapter 1.

SECTION 15.4

15 Show that the product of the distances from the foci of a hyperbola to a tangent has the same value for all tangents.

16 Show that the product of the distances from a point P on a hyperbola to the asymptotes has the same value for all P's.

17 Just as for an ellipse, a *latus rectum* of a hyperbola $x^2/a^2 - y^2/b^2 = 1$ is a chord through a focus and perpendicular to the line on which the foci lie. Show that
(a) the length of a latus rectum is $2b^2/a$;
(b) the slope of the tangent line at the upper end of the latus rectum to the right of the origin is e;
(c) the tangent line in (b) intersects the corresponding directrix on the x-axis.

18 Suppose that the tangent to a hyperbola at a point P intersects the nearest directrix at a point Q. If F is the corresponding focus, show that PFQ is a right angle.

19 Show that the distance from either focus of the hyperbola $x^2/a^2 - y^2/b^2 = 1$ to either asymptote is b.

20 Sketch the graphs of the equations

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \pm 1$$

on a single coordinate system. These two hyperbolas have the same asymptotes, and their four branches "enclose" a region that stretches out to infinity in four directions. Determine whether the area of this region is finite or infinite.

21 Let c be a given positive number.
(a) Show that for all positive values of h the ellipses

$$\frac{x^2}{c^2 + h} + \frac{y^2}{h} = 1$$

have the same foci $(\pm c, 0)$.
(b) Show that for all positive values of $k < c^2$ the hyperbolas

$$\frac{x^2}{c^2 - k} - \frac{y^2}{k} = 1$$

have the same foci as the ellipses in part (a).
(c) If $P_1 = (x_1, y_1)$ is a point of intersection of one of the ellipses in (a) with one of the hyperbolas in (b), show that the tangents to the two curves at this point are perpendicular.

22 Show that a line through a focus of a hyperbola and perpendicular to an asymptote intersects the asymptote on the corresponding directrix.

16 POLAR COORDINATES

16.1 THE POLAR COORDINATE SYSTEM

As we know, a coordinate system in the plane allows us to associate an ordered pair of numbers with each point in the plane. This simple but powerful idea enables us to study many problems of geometry—especially the properties of curves—by the methods of algebra and calculus. Up to this stage of our work we have considered only the rectangular (or Cartesian) coordinate system, in which the emphasis is placed on the distances of a point from two perpendicular axes. However, it often happens that a curve appears to have a special affinity for the origin, like the path of a planet whose journey around its orbit is determined by the central attracting force of the sun. Such a curve is often best described as the path of a moving point whose position is specified by its direction from the origin and its distance out from the origin. This is exactly what polar coordinates do, as we now explain.

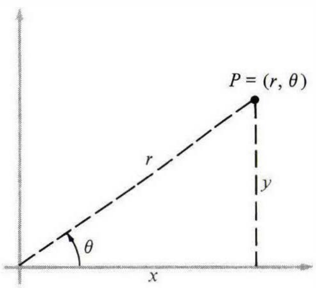


Figure 16.1 Polar coordinates.

A point is located by means of its distance and direction from the origin, as shown in Fig. 16.1. Direction is specified by an angle θ (in radians), measured from the positive x-axis. This angle is understood to be described in the counterclockwise sense if θ is positive and in the clockwise sense if θ is negative, just as in trigonometry. Distance is given by the directed distance r, measured out from the origin along the terminal side of the angle θ. The two numbers r and θ, written in this order as an ordered pair (r, θ), are called *polar coordinates* of the point. The direction $\theta = 0$ (the positive x-axis) is called the *polar axis*.

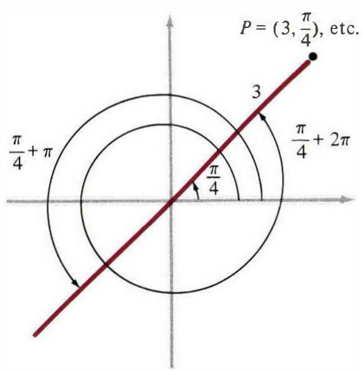


Figure 16.2

Every point has many pairs of polar coordinates. For instance, the point P in Fig. 16.2 has polar coordinates $(3, \pi/4)$, but it also has polar coordinates $(3, \pi/4 + 2\pi)$, $(3, \pi/4 - 4\pi)$, etc. Any multiple of 2π added to or subtracted from the θ-coordinate of a point yields another angle with the same terminal side, and therefore another θ-coordinate of the same point.

The term "directed distance" is intended to suggest that we often meet situations in which r is negative. In this case it is understood that instead of moving out from the origin in the direction indicated by the terminal side of θ, we move *back through the origin* a distance $-r$ in the opposite direction. Thus, another pair of polar coordinates for the point P in Fig. 16.2 is $(-3, \pi/4 + \pi)$. In Fig. 16.3 we plot the two points $Q = (2, \pi/6)$ and $R = (-2, \pi/6)$.

The value $r = 0$ specifies the origin, regardless of the value of θ. For instance, the pairs $(0, 0)$, $(0, \pi/2)$, $(0, -\pi/4)$ are all polar coordinates of the origin.

The fact that a point does not determine a unique pair of polar coordinates is a nuisance, but only a minor nuisance. Nevertheless, it *is* true that when any particular pair of polar coordinates is given, this pair determines the corresponding point without any ambiguity.

Even though it is incorrect to speak of *the* polar coordinates of a point because they are not unique, this error of usage is very common and is tolerated for the sake of euphony.

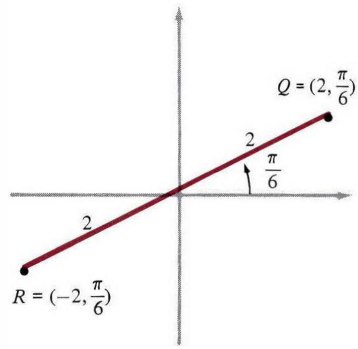


Figure 16.3

It is important to know the connection between rectangular and polar coordinates. Figure 16.1 shows at once that

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta. \tag{1}$$

When r and θ are known, these equations tell us how to find x and y. We also have the equations

$$r^2 = x^2 + y^2 \qquad \text{and} \qquad \tan\theta = \frac{y}{x}, \tag{2}$$

which enable us to find r and θ when x and y are known. In using these equations, it is necessary to take a little care to make sure that the sign of r and the choice of θ are consistent with the quadrant in which the given point (x, y) lies.

Example 1 The rectangular coordinates of a point are $(-1, \sqrt{3})$. Find a pair of polar coordinates for this point.

Solution We have

$$r = \pm\sqrt{1+3} = \pm 2 \qquad \text{and} \qquad \tan\theta = -\sqrt{3}.$$

Since the point is in the second quadrant, we can use our knowledge of trigonometry to choose $r = 2$ and $\theta = 2\pi/3$, so one pair of polar coordinates for the point is $(2, 2\pi/3)$. Another acceptable pair with a negative value of r is $(-2, -\pi/3)$. Students should plot the point and have a clear visual understanding of each of these statements, as suggested by Fig. 16.4.

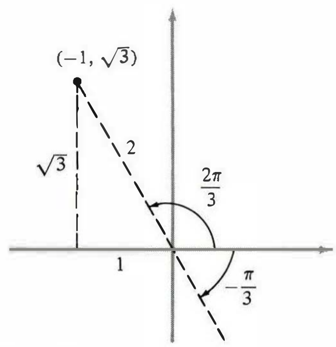


Figure 16.4

Just as in the case of rectangular coordinates, the *graph* of a polar equation

$$F(r, \theta) = 0 \tag{3}$$

is the set of all points $P = (r, \theta)$ whose polar coordinates satisfy the equation. Since the point P has many different pairs of coordinates, it is necessary to state explicitly that P lies on the graph if *any one* of its many different pairs of coordinates satisfies the equation.

Example 2 Show that the points $(1, \pi/2)$ and $(0, \pi/2)$ both lie on the graph of $r = \sin^2\theta$.

Solution The point $(1, \pi/2)$ lies on the graph because the given coordinates satisfy the equation: $1 = \sin^2 \pi/2$. On the other hand, the point $(0, \pi/2)$ lies on the graph even though $0 \neq \sin^2 \pi/2$. The reason for this seemingly strange behavior is that $(0, 0)$ is also a pair of coordinates for the same point, and $0 = \sin^2 0$.

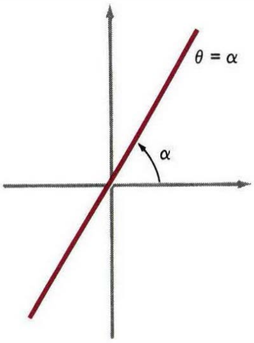


Figure 16.5

In most of the situations we will encounter, equation (3) can be solved for r and takes the form

$$r = f(\theta). \tag{4}$$

If the function $f(\theta)$ is reasonably simple, the graph is fairly easy to sketch. We merely choose a convenient sequence of values for θ, each determining its own direction from the origin, and compute the corresponding values of r that tell us how far out to go in each of these directions. We begin by discussing the simplest possible equations.

Example 3 The equation $\theta = \alpha$, where α is a constant, has as its graph the line through the origin that makes an angle α with the positive x-axis (Fig. 16.5).

Example 4 The equation $r = a$, where a is a positive constant, has as its graph the circle with center at the origin and radius a (Fig. 16.6).

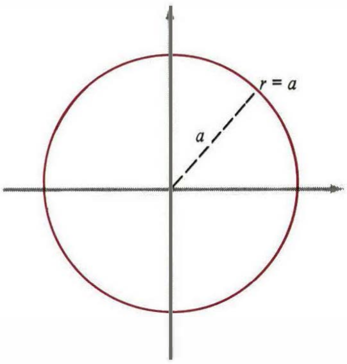


Figure 16.6

Our next example is more complicated, and serves to introduce several important methods.

Example 5 The graph of $r = 2 \cos \theta$ is another circle, but this is not obvious. One way to try to get an idea of the shape of an unknown polar graph is to compute a short table of selected values and plot the corresponding points, as shown in Fig. 16.7.

A better procedure than computing values and plotting points is to sketch the graph as the path of a moving point, by direct analysis of the polar equation, as follows. When $\theta = 0$, $r = 2 \cos 0 = 2$. As θ increases through the first quadrant, from 0 to $\pi/2$, $2 \cos \theta$ decreases from 2 to 0, and we obtain the upper part of the curve shown in Fig. 16.7. As θ increases from $\pi/2$ to π, $2 \cos \theta$ decreases from 0 to -2, and the lower part of the curve is traced out. As θ increases from π to $3\pi/2$, the upper part of the curve is retraced, and as θ increases from $3\pi/2$ to 2π, the lower part is retraced.

θ	r
0	2
$\pi/6$	$\sqrt{3}$
$\pi/4$	$\sqrt{2}$
$\pi/3$	1
$\pi/2$	0
$2\pi/3$	-1
$3\pi/4$	$-\sqrt{2}$
$5\pi/6$	$-\sqrt{3}$
π	-2

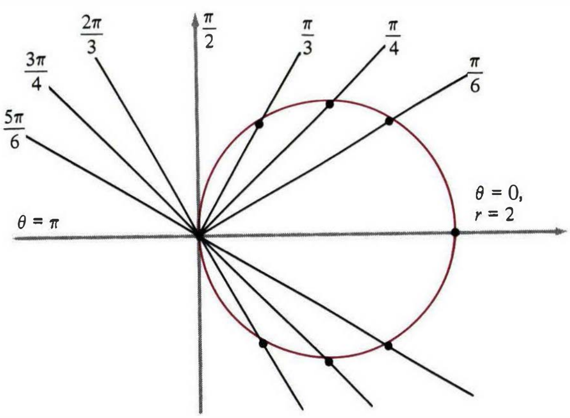


Figure 16.7

It is clear that the resulting graph is some kind of oval, perhaps even a circle. To verify that it really is a circle, we find and recognize the rectangular equation of the curve. To accomplish this, we multiply the given equation $r = 2 \cos \theta$ by r and use the change-of-variable equations (1) and (2) to write

$$\begin{aligned} r^2 &= 2r \cos \theta, \\ x^2 + y^2 &= 2x, \\ x^2 - 2x + y^2 &= 0, \\ (x-1)^2 + y^2 &= 1. \end{aligned}$$

This last equation tells us that the graph is a circle with center (1, 0) and radius 1. It should be pointed out that multiplying the given equation by r introduces the origin as a point on the graph; however, since this point is already on the graph, nothing is changed.

The method illustrated here, sketching a polar graph by direct examination of the polar equation $r = f(\theta)$, will often be important in our future work. Briefly, the process is this: We imagine a radius swinging around the origin in the counterclockwise direction, with our curve being traced out by a point attached to this turning radius which is free to move toward the origin or away from it in accordance with the behavior of the function $f(\theta)$. In many of our examples and problems, $f(\theta)$ will be a simple expression involving the trigonometric functions $\sin \theta$ or $\cos \theta$. In these circumstances it will clearly be very useful to have a solid grasp of the way these functions vary as the radius makes one complete revolution, that is, as θ increases from 0 to $\pi/2$, then from $\pi/2$ to π, π to $3\pi/2$, and $3\pi/2$ to 2π.

PROBLEMS

1 Find the rectangular coordinates of the points with the given polar coordinates, and plot the points:
(a) $(2, \pi/4)$; (b) $(4, -\pi/3)$;
(c) $(0, -\pi)$; (d) $(-1, 7\pi/6)$;
(e) $(2, -\pi/2)$; (f) $(4, 3\pi/4)$;
(g) $(3, \pi)$; (h) $(-6, -\pi/4)$;
(i) $(1, 0)$; (j) $(0, 1)$;
(k) $(2, -5\pi/3)$; (l) $(13, \tan^{-1} \frac{12}{5})$;
(m) $(-4, 11\pi/6)$; (n) $(3, -3\pi/2)$.

2 Find two pairs of polar coordinates, with r's having opposite signs, for the points with the following rectangular coordinates:
(a) $(-2, 2)$; (b) $(4, 0)$;
(c) $(2\sqrt{3}, 2)$; (d) $(2, 2\sqrt{3})$;
(e) $(\sqrt{3}, 1)$; (f) $(0, 4)$;
(g) $(-3, -3)$; (h) $(5, 5)$;
(i) $(0, -2)$; (j) $(-\sqrt{3}, 1)$;
(k) $(5, -12)$; (l) $(-3, 4)$;
(m) $(-1, 0)$; (n) $(1, 2)$.

3 A regular pentagon is inscribed in the circle $r = 1$ with one vertex on the positive x-axis. Find the polar coordinates of all the vertices.

4 Show that the point $(3, 3\pi/4)$ lies on the curve $r = 3 \sin 2\theta$.

5 Show that the point $(3, 3\pi/2)$ lies on the curve $r^2 = 9 \sin \theta$.

6 Sketch each of the following curves and show that each is a circle by finding the equivalent rectangular equation:
(a) $r = 6 \sin \theta$;
(b) $r = -8 \cos \theta$;
(c) $r = -4 \sin \theta$.

7 Sketch the curve $r = 4 (\sin \theta + \cos \theta)$, and identify it by finding the equivalent rectangular equation.

8 Show that the graph of $r = 2a \cos \theta + 2b \sin \theta$ is either a single point or a circle through the origin. Find the center and radius of the circle.

9 Sketch and identify each of the following graphs:
(a) $r = 2 \csc \theta$; (b) $r = 4 \sec \theta$;
(c) $r = -3 \csc \theta$; (d) $r = -2 \sec \theta$.

16.2 MORE GRAPHS OF POLAR EQUATIONS

We continue our program of getting better acquainted with polar graphs. In this section we concentrate particularly on sketching polar equations $r = f(\theta)$ of the type mentioned earlier, where $f(\theta)$ involves $\sin\theta$ or $\cos\theta$ in some simple way.

We again emphasize the change in point of view that is necessary for sketching polar equations. With rectangular coordinates and $y = f(x)$, we are accustomed to the idea of a point x moving along the horizontal axis and y as the directed distance measured up or down to the corresponding point (x, y) in the plane. We think in terms of "left-right" and "up-down."

With polar coordinates and $r = f(\theta)$, however, we must think of the angle θ swinging around like the hand of a clock turning in the wrong direction. For each θ we measure out from the origin a directed distance $f(\theta)$, and our moving point is farther out or closer in according as $f(\theta)$ is larger or smaller. We must think in terms of "around and around" and "in and out."

Example 1 The curve $r = a(1 + \cos\theta)$ with $a > 0$ is called a *cardioid.* When $\theta = 0$, $\cos\theta = 1$ and $r = 2a$. As θ increases from 0 to $\pi/2$ and on to π, $\cos\theta$ decreases from 1 to 0 to -1, so r decreases steadily from $2a$ to a to 0. This is shown in the upper half of Fig. 16.8. As θ continues to increase through the third and fourth quadrants, we see that $\cos\theta$, and with it r, retraces its values in reverse order, reaching $\cos\theta = 1$ and $r = 2a$ at $\theta = 2\pi$. Since $\cos\theta$ is periodic with period 2π, values of θ less than 0 or greater than 2π give points already sketched. The complete cardioid shown in the figure is evidently symmetric about the x-axis. The strange name this curve bears is accounted for by its fancied resemblance to a heart.

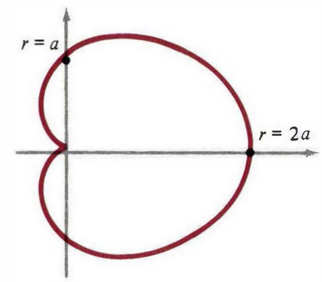


Figure 16.8 A cardioid.

When facing a polar equation, it is a natural temptation to try to return to familiar ground by converting immediately to rectangular coordinates. This is accomplished by using the transformation equations mentioned in Section 16.1,

$$r^2 = x^2 + y^2, \qquad \sin\theta = \frac{y}{r}, \qquad \cos\theta = \frac{x}{r}, \qquad \tan\theta = \frac{y}{x}.$$

In the case of the cardioid discussed in Example 1, its equation $r = a(1 + \cos\theta)$ becomes

$$r = a\left(1 + \frac{x}{r}\right), \qquad r^2 = a(r + x), \qquad x^2 + y^2 - ax = ar,$$

and finally,

$$(x^2 + y^2 - ax)^2 = a^2(x^2 + y^2).$$

This rectangular equation of the cardioid doesn't really tell us much. Clearly, it is better in this case to think exclusively in the language of polar coordinates. Nevertheless, there is a certain interest in seeing that the cardioid is a fourth-degree curve, in contrast to the second-degree curves discussed in Chapter 15.

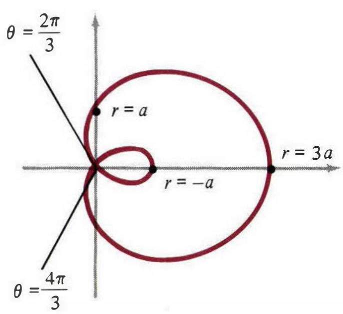


Figure 16.9 A limaçon.

Example 2 The curve $r = a(1 + 2\cos\theta)$ with $a > 0$ is called a *limaçon* (French for "snail"). When $\theta = 0$, $r = 3a$. As θ increases, r decreases, becoming 0 when $2\cos\theta = -1$, that is, when $\theta = 2\pi/3$. As θ continues increasing to π, r continues to decrease through negative values from 0 to $-a$, and the point whose movement we are following traces the lower half of the inner loop shown in Fig. 16.9.

Just as in Example 1, as θ continues to increase through the third and fourth quadrants, r retraces its values in reverse order; the inner loop is completed at $\theta = 4\pi/3$, and the outer loop is completed at $\theta = 2\pi$.

The curves in Examples 1 and 2 are both symmetric about the x-axis. We always have this kind of symmetry when r is a function only of $\cos\theta$, because of the identity $\cos(-\theta) = \cos\theta$. Similarly, if r is a function only of $\sin\theta$, then the curve is symmetric about the y-axis, because of the identity $\sin(\pi - \theta) = \sin\theta$.

We sometimes encounter curves whose equations have the form $r^2 = f(\theta)$. In this case, if θ is an angle for which $f(\theta) < 0$, then there is no corresponding point on the graph, because we must have $r^2 \geq 0$. But if θ is an angle for which $f(\theta) > 0$, then there are *two* corresponding points on the graph, with $r = \pm\sqrt{f(\theta)}$. These points are equally far from the origin in opposite directions, so the graph of $r^2 = f(\theta)$ is always symmetric with respect to the origin.

Example 3 The curve $r^2 = 2a^2 \cos 2\theta$ is called a *lemniscate*. For each θ there are two r's,

$$r = \pm\sqrt{2}a\sqrt{\cos 2\theta}. \tag{1}$$

As θ increases from 0 to $\pi/4$, 2θ increases from 0 to $\pi/2$ and $\cos 2\theta$ decreases from 1 to 0. Accordingly, the two r's in (1) simultaneously trace out the two parts of the curve shown on the left in Fig. 16.10. As θ continues to increase through the second half of the first quadrant and the first half of the second quadrant, 2θ varies through the second and third quadrants and $\cos 2\theta$ is negative, so there is no graph for these θ's. Through the second half of the second quadrant, $\cos 2\theta$ is positive again, and the two r's given by (1) simultaneously complete the two loops begun on the left in the figure. Further investigation reveals that no additional points are obtained, and the complete lemniscate is shown on the right. The name of this curve comes from the Latin word *lemniscus,* meaning a ribbon tied into a bow in the form of a figure eight.*

*The lemniscate was introduced by James Bernoulli in 1694. It played a considerable role in some of the early work of Gauss (in 1797) and Abel (in 1826) on elliptic functions and ruler-and-compass constructions in geometry. See M. Rosen, "Abel's Theorem on the Lemniscate," *Amer. Math. Monthly,* **88** (1981), pp. 387–395.

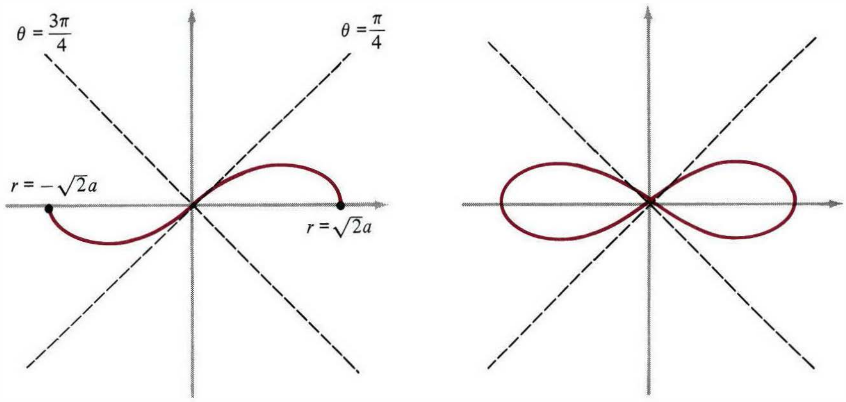


Figure 16.10 A lemniscate.

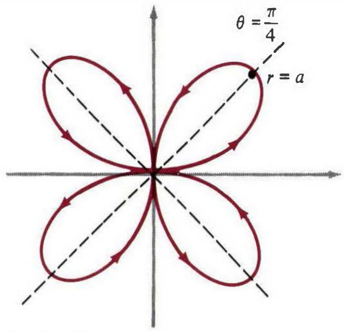


Figure 16.11 A four-leaved rose.

Example 4 The curve $r = a \sin 2\theta$ with $a > 0$ is called a *four-leaved rose,* for reasons that will become clear. To sketch it, we observe that as θ increases from 0 to $\pi/4$, 2θ increases from 0 to $\pi/2$ and r increases from 0 to a; and as θ increases from $\pi/4$ to $\pi/2$, 2θ increases from $\pi/2$ to π and r decreases from a to 0. This gives the leaf in the first quadrant (Fig. 16.11). Values of θ between $\pi/2$ and π (2θ between π and 2π) yield negative r's which trace out the leaf in the fourth quadrant; those between π and $3\pi/2$ (2θ between 2π and 3π) yield positive r's which trace out the leaf in the third quadrant; and those between $3\pi/2$ and 2π (2θ between 3π and 4π) produce negative r's and the leaf in the second quadrant.

We sometimes need to find the points of intersection of two curves that are defined by polar equations. It is natural to try to do this by solving the equations simultaneously. Unfortunately, this may not give *all* the points of intersection. The reason for this is that a point can lie on each of two curves and yet not have a pair of polar coordinates that satisfies both equations simultaneously. An extreme example of this behavior is provided by the two equations

$$r = 1 + \cos^2 \theta \qquad \text{and} \qquad r = -1 - \cos^2 \theta,$$

whose graphs are identical. In this case there are no simultaneous solutions because all the first r's are positive and all the second r's are negative, and yet there are infinitely many points of intersection.

What can be done about finding intersections? The most sensible approach is to depend on drawing good enough graphs of both equations on a single figure to see whether there are any points of intersection. When there are, it is usually possible to find the polar coordinates of these points either by solving simultaneous equations or else by observing where the points are by direct inspection of the figure.

Remark People who enjoy geometry in school usually take special pleasure in construction problems. As students will perhaps recall, the Greek mathematicians of antiquity learned how to perform a great variety of intricate constructions with only ruler and compass allowed as tools for drawing straight lines and circles: For instance, an angle can be bisected; a segment can be trisected; the perpendicular bisector of a segment can be drawn; regular polygons with n sides, where $n = 3, 4, 5, 6$, can be constructed; etc. All of these constructions and many more have been known since the time of Euclid and Archimedes, and the details form an important part of the study of plane geometry.*

The creation of geometric constructions with ruler and compass alone, when considered as an intellectual game played according to clearly understood rules, was certainly one of the most fascinating and enduring games ever invented. The complicated constructions that turn out to be possible must be seen to be believed. Nevertheless, after ingenious and persistent efforts extending over more than 2000 years, there were three classical Greek construction problems that still remained unsolved at the beginning of the nineteenth century. These problems were:

*See Chapters III, V, and IX of H. Tietze, *Famous Problems of Mathematics* (Graylock Press, 1965).

1 *To trisect an angle,* that is, to divide a given angle into three equal parts
2 *To double a cube,* that is, to construct the edge of a cube with twice the volume of a given cube
3 *To square a circle,* that is, to construct a square whose area equals that of a given circle

In the course of the nineteenth century all three constructions were conclusively proved to be impossible under the stated conditions.

The traditional restriction to the use of ruler and compass alone seems to have originated with the ancient Greek philosophers, but the Greek mathematicians themselves did not hesitate to use other tools. In particular, they invented various bizarre curves for the specific purpose of solving one or another of the classical construction problems. Some of these curves are described in the problems that follow.

PROBLEMS

1 The following curves are also called *cardioids*. Sketch them, observing the way the position of the curve changes as the form of the equation changes.*
(a) $r = a(1 - \cos\theta)$. (b) $r = a(1 + \sin\theta)$.
(c) $r = a(1 - \sin\theta)$.

2 All curves of the form $r = a \pm b\cos\theta$ or $r = a \pm b\sin\theta$ with $a, b > 0$ are called *limaçons*. If $a > b$, the graph is a single loop. If $a < b$, the graph consists of a smaller loop inside a larger one, as in Fig. 16.9. Sketch the following limaçons:
(a) $r = 3 + 2\cos\theta$; (b) $r = 1 + 2\sin\theta$;
(c) $r = 1 - \sqrt{2}\cos\theta$; (d) $r = 5 - 3\sin\theta$.

3 Sketch the lemniscate $r^2 = 2a^2\sin 2\theta$.

4 Sketch the graphs of the following polar equations:
(a) $r = 2a\cos\theta$; (b) $r = 2a\sin\theta$;
(c) $r = 2 - \cos\theta$; (d) $r = 2 + \cos\theta$;
(e) $r^2 = \cos\theta$; (f) $r = 4\sin^2\theta$;
(g) $r = \cos 2\theta$; (h) $r = 1 + \sin 2\theta$;
(i) $r = 2 + \sin 2\theta$; (j) $r = \cos\frac{1}{2}\theta$;
(k) $r = \sin\frac{1}{2}\theta$; (l) $r = 2\sin^2\frac{1}{2}\theta$;
(m) $r = 1 + 2\sin\theta$; (n) $r = \tan\theta$;
(o) $r = \cot\theta$; (p) $r = \sin 3\theta$;
(q) $r = \cos 3\theta$.

5 Transform the given rectangular equation into an equivalent polar equation:
(a) $x = 5$; (b) $y = -3$;
(c) $x^2 + y^2 = 9$; (d) $x^2 - y^2 = 9$;
(e) $y = x^2$; (f) $xy = 1$;
(g) $y^2 = x(x^2 - y^2)$; (h) $y^2 = x^2\left(\frac{2 + x}{2 - x}\right)$.

*Unless the contrary is explicitly stated, it is customary to assume that the constant a that occurs in polar equations like these is a positive number.

6 Transform the given polar equation into an equivalent rectangular equation:
(a) $r = 2$; (b) $\theta = \pi/4$;
(c) $r\cos\theta = 3$; (d) $r = 4\sin\theta$;
(e) the limaçon of Example 2, $r = a(1 + 2\cos\theta)$;
(f) the lemniscate of Example 3, $r^2 = 2a^2\cos 2\theta$;
(g) the rose of Example 4, $r = a\sin 2\theta$;
(h) $r = \tan\theta$; (i) $r^2 = \cos 4\theta$.

7 Let a be a positive number and consider the points $F = (a, 0)$ and $F' = (-a, 0)$. The lemniscate $r^2 = 2a^2\cos 2\theta$ has the following simple geometric property: It is the set of all points P such that the product of the distances PF and PF' equals a^2. Prove this by first finding the rectangular equation of the curve and then transforming this equation into its polar form $r^2 = 2a^2\cos 2\theta$.

8 Use the formula $y = r\sin\theta$ to find the largest value of y on
(a) the cardioid $r = 2(1 + \cos\theta)$;
(b) the lemniscate $r^2 = 8\cos 2\theta$.

9 Use the formula $x = r\cos\theta$ to find the polar coordinates of the points on the cardioid $r = 2(1 + \cos\theta)$ with the smallest x-coordinate. What is this smallest x-coordinate?

10 Find all points of intersection of each pair of curves:
(a) $r = 4\cos\theta$, $r = 4\sqrt{3}\sin\theta$;
(b) $r = \sqrt{2}\sin\theta$, $r^2 = \cos 2\theta$;
(c) $r^2 = 4\cos 2\theta$, $r^2 = 4\sin 2\theta$;
(d) $r = 1 - \cos\theta$, $r = \cos\theta$;
(e) $r = a$, $r = 3a\sin\theta$;
(f) $r = a$, $r^2 = 2a^2\sin 2\theta$.

11 A line segment of length $2a$ slides in such a way that one end is always on the x-axis and the other end is always on the y-axis. Find the polar equation of the lo-

cus of the point P in which a line from the origin perpendicular to the moving segment intersects the segment.

*12 Consider a circle of diameter $2a$ that is tangent to the y-axis at the origin (Fig. 16.12). Let OA be the diameter that lies along the x-axis, AB a segment tangent to the circle at A, and C the point at which OB intersects the circle. If $P = (r, \theta)$ lies on OB in such a position that $OP = CB$, find the polar equation of the locus of P. This curve is called a *cissoid*, meaning "ivy-shaped" —or, more precisely, the *cissoid of Diocles* (Greek, second century B.C.).†

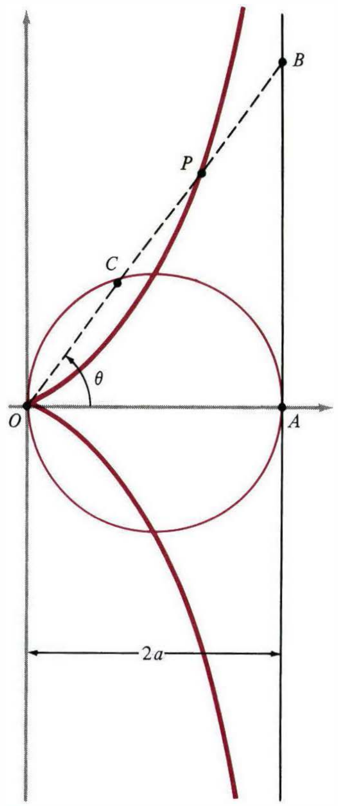


Figure 16.12 A cissoid.

13 Find the rectangular equation of the cissoid in Problem 12.

14 Show that the line $x = 2a$ is an asymptote of the cissoid in Problem 12.

15 Let a and b be given positive numbers and consider the line whose rectangular equation is $x = a$ and whose polar equation is $r \cos \theta = a$ or $r = a \sec \theta$ (Fig. 16.13). The line OA in the figure intersects the line $x = a$ at the point A, and P is a distance b beyond A. The locus of P is called a *conchoid*, meaning "shell-shaped"—or, more precisely, the *conchoid of Nicomedes* (Greek, third century B.C.). The polar equation of the conchoid is clearly $r = a \sec \theta + b$. Find its rectangular equation.‡

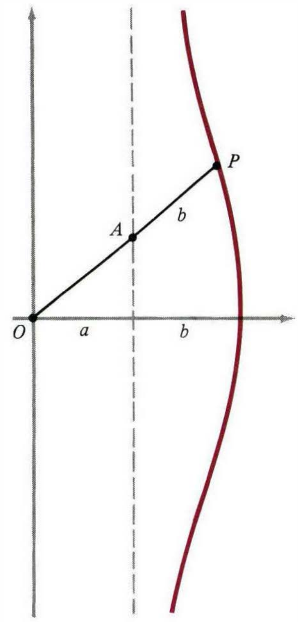


Figure 16.13 A conchoid.

16 If a rotation of the polar axis through a specified angle transforms one polar equation into another, then their graphs are clearly congruent. Show that a counterclockwise rotation of the polar axis through an angle α can be accomplished by replacing θ by $\theta + \alpha$. Use this method with suitable choices of α to show that

(a) the cardioids in Problem 1 are congruent to the cardioid $r = a(1 + \cos \theta)$ discussed in Example 1;

(b) the limaçon $r = a \pm b \sin \theta$ in Problem 2 is congruent to the limaçon $r = a \pm b \cos \theta$;

(c) the lemniscate $r^2 = 2a^2 \sin 2\theta$ in Problem 3 is congruent to the lemniscate $r^2 = 2a^2 \cos 2\theta$ discussed in Example 3.

†In the Additional Problems we explain how the cissoid can be used to solve the problem of doubling a cube.

‡In the Additional Problems we explain how the conchoid can be used to solve the problem of trisecting an angle.

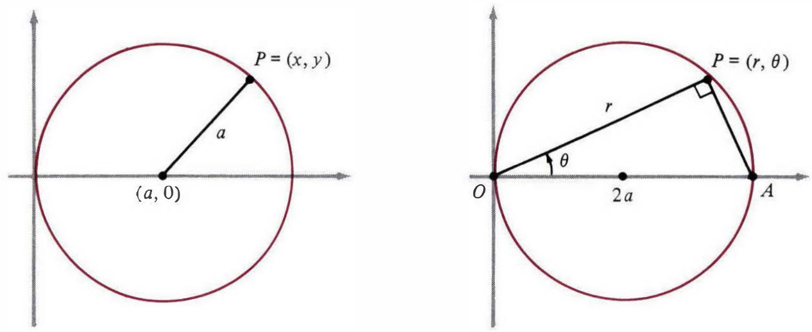


Figure 16.14

16.3 POLAR EQUATIONS OF CIRCLES, CONICS, AND SPIRALS

We have already had considerable experience in transforming the rectangular equation of a given curve into an equivalent polar equation for the same curve. Our basic tools for this procedure are the transformation equations listed in Section 16.2.

Consider, for example, the circle (Fig. 16.14, left) with center $(a, 0)$ and radius a:

$$(x - a)^2 + y^2 = a^2 \qquad \text{or} \qquad x^2 + y^2 = 2ax. \tag{1}$$

Since $x^2 + y^2 = r^2$ and $x = r \cos \theta$, this equation becomes

$$r^2 = 2ar \cos \theta,$$

which is equivalent to

$$r = 2a \cos \theta \tag{2}$$

because the origin $r = 0$ lies on the graph of (2).

This example illustrates one way to find the polar equation of a curve, namely, transform its rectangular equation into polar coordinates. Another method that is better whenever it is feasible is to obtain the polar equation directly from some characteristic geometric property of the curve. In the case of the circle just discussed, we use the fact that the angle OPA in the figure on the right is a right angle. Since OPA is a right triangle with r the adjacent side to the acute angle θ, we clearly have

$$r = 2a \cos \theta,$$

which of course is the same equation previously obtained, but derived in a very different way.

We shall use this second and more natural method to find the polar equations of various curves in the following examples.

Example 1 Find the polar equation of the circle with radius a and center at the point C with polar coordinates (b, α), where b is assumed to be positive.

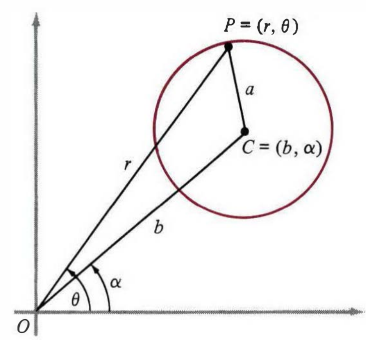


Figure 16.15

Solution Let $P = (r, \theta)$ be any point on the circle, as shown in Fig. 16.15, and apply the law of cosines to the triangle OPC to obtain

$$a^2 = r^2 + b^2 - 2br \cos (\theta - \alpha).$$

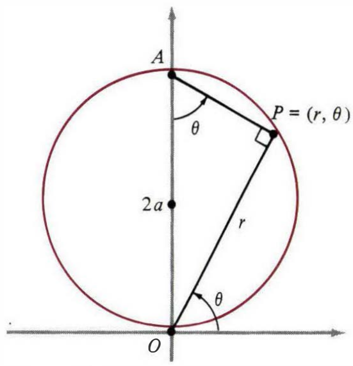


Figure 16.16

This is the polar equation of the circle. For circles that pass through the origin we have $b = a$, and the equation can be written as

$$r = 2a \cos (\theta - \alpha). \tag{3}$$

In particular, when $\alpha = 0$, then (3) reduces to (2), and when $\alpha = \pi/2$, so that the center lies on the y-axis, then $\cos (\theta - \pi/2) = \sin \theta$, and (3) reduces to

$$r = 2a \sin \theta. \tag{4}$$

In this case the right triangle OPA in Fig. 16.16 provides a more direct geometric way of obtaining (4), since here r is the opposite side to the acute angle θ.

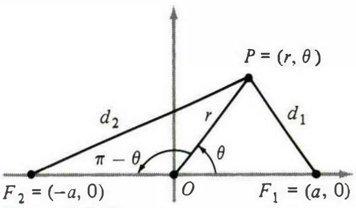


Figure 16.17

Example 2 Let F_1 and F_2 be the two points whose rectangular coordinates are $(a, 0)$ and $(-a, 0)$, as shown in Fig. 16.17. If b is a positive constant, find the polar equation of the locus of a point P that moves in such a way that the product of its distances from F_1 and F_2 is b^2.

Solution If $P = (r, \theta)$ is an arbitrary point on the curve, then the defining condition is

$$d_1 d_2 = b^2 \qquad \text{or} \qquad d_1{}^2 d_2{}^2 = b^4,$$

where $d_1 = PF_1$ and $d_2 = PF_2$. We apply the law of cosines twice, first to the triangle OPF_1,

$$d_1{}^2 = r^2 + a^2 - 2ar \cos \theta, \tag{5}$$

and then to the triangle OPF_2,

$$d_2{}^2 = r^2 + a^2 - 2ar \cos (\pi - \theta). \tag{6}$$

Since $\cos (\pi - \theta) = -\cos \theta$, we can write (6) as

$$d_2{}^2 = r^2 + a^2 + 2ar \cos \theta, \tag{7}$$

and by multiplying (5) and (7) we obtain

$$d_1{}^2 d_2{}^2 = (r^2 + a^2)^2 - (2ar \cos \theta)^2$$

or

$$b^4 = r^4 + a^4 + 2a^2r^2(1 - 2 \cos^2 \theta).$$

The trigonometric identity $2 \cos^2 \theta = 1 + \cos 2\theta$ permits us to write this equation as

$$b^4 = r^4 + a^4 - 2a^2r^2 \cos 2\theta. \tag{8}$$

In the special case $b = a$, the curve passes through the origin, and the equation takes the much simpler form

$$r^2 = 2a^2 \cos 2\theta. \tag{9}$$

We recognize this as the equation of the lemniscate discussed in Example 3 of Section 16.2. When $b > a$, the curve consists of a single loop, but when $b < a$ it breaks into two separate loops. The cases $b < a$ and $b = a$ are illustrated in

Fig. 16.18, along with two cases of $b > a$. Collectively, these curves are called the *ovals of Cassini.**

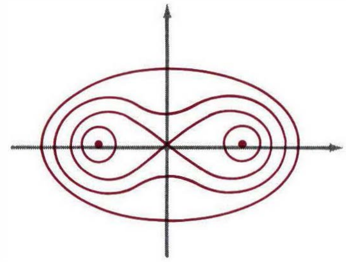


Figure 16.18 The ovals of Cassini.

Polar coordinates are particularly well suited to working with conic sections, as we see in the next example.

Example 3 Find the polar equation of the conic section with eccentricity e if the focus is at the origin and the corresponding directrix is the line $x = -p$ to the left of the origin.

Solution With the notation of Fig. 16.19, the focus-directrix-eccentricity characterization of the conic section is

$$\frac{PF}{PD} = e \qquad \text{or} \qquad PF = e \cdot PD. \tag{10}$$

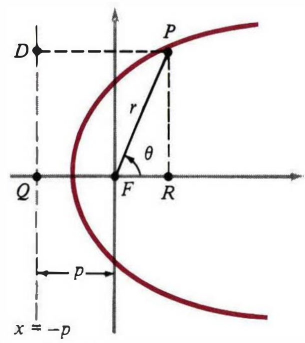


Figure 16.19

We recall that the curve is an ellipse, a parabola, or a hyperbola according as $e < 1$, $e = 1$, or $e > 1$. By examining the figure, we see that $PF = r$ and

$$\begin{aligned} PD = QR &= QF + FR \\ &= p + r\cos\theta, \end{aligned}$$

so (10) is

$$r = e(p + r\cos\theta).$$

This is easily solved for r, which gives

$$r = \frac{ep}{1 - e\cos\theta} \tag{11}$$

as the polar equation of our conic section.

We give two concrete illustrations of the ideas in Example 3.

Example 4 Find the polar equation of the conic with eccentricity $\frac{1}{3}$, focus at the origin, and directrix $x = -4$.

Solution We merely substitute $e = \frac{1}{3}$ and $p = 4$ in equation (11), which yields

$$r = \frac{\frac{1}{3}(4)}{1 - \frac{1}{3}\cos\theta} = \frac{4}{3 - \cos\theta}.$$

The curve is an ellipse. Observe that the denominator here is never zero, so r is bounded for all θ's.

*The Italian astronomer Giovanni Domenico Cassini thought of these ovals in 1680 in connection with his efforts to understand the relative motions of the earth and the sun. He proposed them as alternatives to Kepler's ellipses before Newton settled the matter with his theory of the solar system in 1687. Cassini discovered several of the satellites of Saturn, and also the so-called *Cassini division* in Saturn's ring, thereby showing that this ring consists of more than one piece.

Example 5 Given the conic with equation

$$r = \frac{25}{4 - 5 \cos \theta},$$

find the eccentricity, locate the directrix, and identify the curve.

Solution We begin by dividing numerator and denominator by 4 to put the equation in the exact form of (11),

$$r = \frac{\frac{25}{4}}{1 - \frac{5}{4} \cos \theta}.$$

This tells us that $e = \frac{5}{4}$ and $ep = \frac{25}{4}$, so $p = 5$. The directrix is the line $x = -5$, and the curve is a hyperbola. Observe that the denominator here is zero when $\cos \theta = \frac{4}{5}$, so r becomes infinite near these directions.

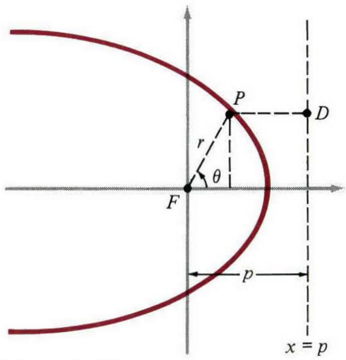


Figure 16.20

In connection with Example 3, it is worth pointing out that if the directrix is the line $x = p$ to the right of the origin, as in Fig. 16.20, then $PD = p - r \cos \theta$. The equation $PF = e \cdot PD$ now has the form

$$r = e(p - r \cos \theta),$$

and instead of (11) we have

$$r = \frac{ep}{1 + e \cos \theta}.$$

Polar coordinates are very convenient for describing certain spirals.

Example 6 The *spiral of Archimedes* (Fig. 16.21) can be defined as the locus of a point P that starts at the origin and moves outward at a constant speed along a radius which, in turn, is rotating counterclockwise at a constant speed from its initial position along the polar axis, where both motions start at the same time.* Since both r and θ are proportional to the time t measured from the beginning of the motions, r is proportional to θ and the polar equation of the spiral is $r = a\theta$, where a is a positive constant. In the figure, it is assumed that θ starts at zero and increases into positive values, as implied by the definition. However, if we wish to allow θ to be negative, then there is another part to the spiral which we have deliberately not sketched for the sake of keeping the figure uncluttered.

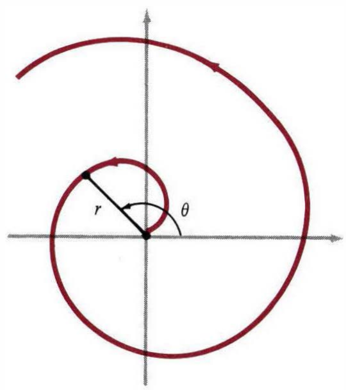


Figure 16.21 The spiral of Archimedes.

Example 7 In the spiral discussed in Example 6, r is directly proportional to θ, $r = a\theta$. We now consider the case in which r is inversely proportional to θ,

$$r = \frac{a}{\theta} \qquad \text{or} \qquad r\theta = a, \tag{12}$$

where a is a positive constant. For positive values of θ, the graph is the curve shown in Fig. 16.22; it is called a *hyperbolic spiral* because of the resemblance of $r\theta = a$ to the equation $xy = a$, which represents a hyperbola in rectangular coordinates.

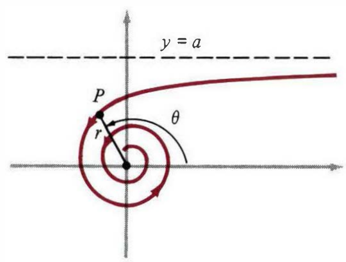


Figure 16.22 A hyperbolic spiral.

*These are almost the same words which Archimedes himself uses to define his spiral. See his treatise "On Spirals" in *The Works of Archimedes,* T. L. Heath, ed. (Dover, n.d.), especially p. 154.

The essential features of the graph are easy to see by considering $r = a/\theta$. When $\theta = 0$, there is no r; when θ is small and positive, r is large and positive; and as θ increases to ∞, r decreases to 0. This tells us that a variable point P on the graph comes in from infinity and winds around the origin in the counterclockwise direction in an infinite number of shrinking coils as θ increases indefinitely. To understand the behavior of this curve for small positive θ's, we need to think about what happens to

$$y = r \sin \theta = a \frac{\sin \theta}{\theta}.$$

We know that

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1,$$

and therefore

$$\lim_{\theta \to 0} y = \lim_{\theta \to 0} a \frac{\sin \theta}{\theta} = a.$$

It follows that the line $y = a$ is an asymptote of the curve, as shown in the figure.

If θ is allowed to be negative, we get another part of the curve, which again we do not sketch in order to avoid cluttering up the figure. The nature of this other part is easily understood by observing that if r and θ are replaced by $-r$ and $-\theta$, then equation (12) is unaltered. This means that for every point (r, θ) on the curve, the point $(-r, -\theta)$, which is symmetrically located with respect to the y-axis, is also on the curve. Thus, the other part is a second mirror-image spiral that winds in to the origin in the clockwise direction as $\theta \to -\infty$.

PROBLEMS

In Problems 1–6, find the polar equation of the circle determined by the stated conditions.

1 Center $(4, \pi/6)$, radius 3.
2 Center $(-3, \pi/3)$, radius 4.
3 Center $(5, 0)$, radius 5.
4 Center $(3, \pi/2)$, radius 3.
5 Center on the line $\theta = \pi/3$ and passing through $(6, \pi/2)$ and $(0, 0)$.
6 Center $(5, \pi/4)$ and passing through $(8, 0)$.
7 A line is drawn from the origin perpendicular to a tangent to the circle $r = 2a \cos \theta$. Find the equation of the locus of the point of intersection and sketch the curve.
8 Find the rectangular equation of the ovals of Cassini (Example 2) by direct use of the condition $d_1{}^2 d_2{}^2 = b^4$.
9 The largest and smallest values of r on the lemniscate (9) are clearly $r = \sqrt{2}a$ and $r = 0$. Find the largest and smallest values of r on the ovals of Cassini (8)
(a) in the one-loop case $b = 2a$;
(b) in the two-loop case $b = \frac{1}{2}a$.
10 The equation $r = 4/(3 - \cos \theta)$ in Example 4 represents an ellipse with one focus at the origin. Sketch the curve, find both of its directrices, and locate the center.
11 If a conic section with eccentricity e has focus at the origin and directrix $y = -p$ below the origin, show that its polar equation is $r = ep/(1 - e \sin \theta)$. What is the polar equation if the directrix is the line $y = p$ above the origin?

Find the eccentricity of each of the following conic sections (in Problems 12–15) and sketch the curve.

12 $r = \dfrac{6}{1 - \cos \theta}$.
13 $r = \dfrac{10}{2 - \cos \theta}$.
14 $r = \dfrac{4}{2 + 4 \cos \theta}$.
15 $r = \dfrac{18}{6 + \cos \theta}$.
16 One focus of a hyperbola with eccentricity $e = \frac{4}{3}$ is at the origin, and the corresponding directrix is the line $x = 7$ (or $r \cos \theta = 7$). Find the polar equation and the polar coordinates of the second focus and center, and sketch the curve.
17 When $e > 1$, the equation $r = ep/(1 - e \cos \theta)$ represents a hyperbola. Use this equation to determine the slopes of the asymptotes.

18 Transform the equation $r = ep/(1 - e \cos \theta)$ into rectangular coordinates. Use the facts established in Chapter 15 about the equations of conics in rectangular coordinates to show that the given equation represents a parabola if $e = 1$, an ellipse if $0 < e < 1$, and a hyperbola if $e > 1$.

19 If $e < 1$, use calculus to find the polar coordinates of the point on the ellipse $r = ep/(1 - e \cos \theta)$ that is
(a) farthest from the origin;
(b) closest to the origin.

20 Find the length of the major axis of the ellipse in Problem 19. Also find the polar coordinates of its center.

***21** In his attempts to trisect an angle, Hippias of Elis, a leading sophist of the time of Socrates, invented a new curve, as follows. Consider a square $OABC$ of side a located in the first quadrant of the xy-plane (Fig. 16.23). Let OA rotate clockwise about O at a constant speed to the position OC. In the same time, let AB move downward at a constant speed to the position OC. The *quadratrix APG* is the locus of the point P of intersection of the turning radius OD and the moving segment EF.†

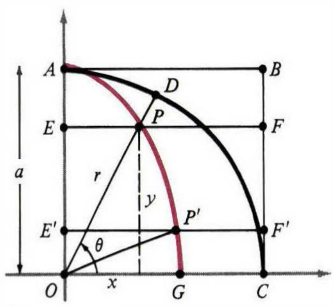


Figure 16.23 The quadratrix.

(a) Find the rectangular equation of the quadratrix [hint: $y/a = \theta/(\pi/2)$; why?].
(b) Find the polar equation of the quadratrix.
(c) Use part (b) to show that $OG = 2a/\pi$.
(d) Pappus of Alexandria (fourth century A.D.) proved geometrically that $ADC/OC = OC/OG$. Show that this is equivalent to the result stated in part (c).
(e) Verify the validity of the following procedure for using the quadratrix as a tool for trisecting an arbitrary acute angle θ: Construct the point E' that trisects OE, so that $OE' = \frac{1}{3}OE$; draw $E'F'$ parallel to OC, and let P' be the point where this line intersects the quadratrix; draw OP' and conclude that $\angle COP' = \frac{1}{3}\theta$.‡ [Part (e) requires that the segment OE be trisected by the point E'. Figure 16.24 shows how this point can be produced by a Euclidean (ruler-and-compass) construction: Measure off any length b three times in any direction OQ, join R to E, and draw the line parallel to RE through the first point of division, intersecting OE at E'.]

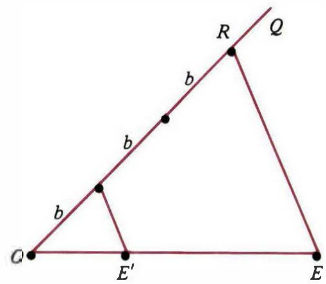


Figure 16.24 Trisection of segment OE.

***22** In Problem 21, let $a = \pi/2$, so that the rectangular equation of the quadratrix is $x = y \cot y$. Interchange x and y so that the equation becomes $y = x \cot x$, and use division of power series to obtain Newton's result that the area under the quadratrix $y = x \cot x$ from 0 to $x \leq \pi/2$ is

$$x - \tfrac{1}{9}x^3 - \tfrac{1}{225}x^5 - \tfrac{2}{6615}x^7 - \cdots.$$

23 Verify the validity of the following method for trisecting an angle AOB by using the spiral of Archimedes, $r = a\theta$ (Fig. 16.25):

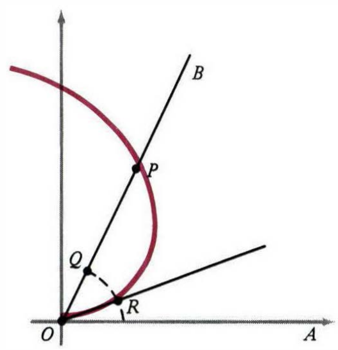


Figure 16.25 Trisection of angle AOB.

(a) let OB intersect the spiral at P, and construct the point Q that trisects OP, so that $OQ = \frac{1}{3}OP$;
(b) construct the circle with center O and radius OQ, and let this circle intersect the spiral at R;
(c) draw OR and conclude that $\angle AOR = \frac{1}{3}\angle AOB$.§

†The point G is not defined as part of the quadratrix because OA reaches OC at the same moment AB reaches OC, so there is no point of intersection. However, G is the limiting position of the points on the quadratrix that approach the x-axis.

‡In the Additional Problems we show how the quadratrix can also be used to square a circle.

§In the Additional Problems we show how the spiral of Archimedes can also be used to square a circle.

16.4 ARC LENGTH AND TANGENT LINES

Consider a curve whose polar equation is $r = f(\theta)$, and let s denote arc length measured along the curve from a specified point in a specified direction (Fig. 16.26). By Section 7.5 we know that the differential element of arc length ds is given by the formula

$$ds^2 = dx^2 + dy^2.$$

But $x = r\cos\theta$ and $y = r\sin\theta$, and by differentiating with respect to θ by the product rule, we obtain

$$\frac{dx}{d\theta} = -r\sin\theta + \cos\theta\,\frac{dr}{d\theta} \quad \text{and} \quad \frac{dy}{d\theta} = r\cos\theta + \sin\theta\,\frac{dr}{d\theta},$$

or equivalently, in the notation of differentials,

$$dx = -r\sin\theta\,d\theta + \cos\theta\,dr \quad \text{and} \quad dy = r\cos\theta\,d\theta + \sin\theta\,dr. \tag{1}$$

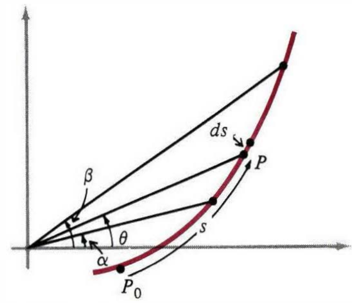


Figure 16.26

It follows from these formulas that

$$\begin{aligned} ds^2 &= dx^2 + dy^2 \\ &= r^2\sin^2\theta\,d\theta^2 - 2r\sin\theta\cos\theta\,dr\,d\theta + \cos^2\theta\,dr^2 \\ &\qquad + r^2\cos^2\theta\,d\theta^2 + 2r\sin\theta\cos\theta\,dr\,d\theta + \sin^2\theta\,dr^2 \\ &= r^2\,d\theta^2 + dr^2. \end{aligned}$$

Thus, we have

$$ds^2 = r^2\,d\theta^2 + dr^2 \tag{2}$$

or

$$ds = \sqrt{r^2\,d\theta^2 + dr^2} = \sqrt{\left(r^2 + \frac{dr^2}{d\theta^2}\right)d\theta^2} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta.$$

This formula enables us to compute arc lengths of polar curves by integration, as suggested by the figure:

$$\text{arc length from } \theta = \alpha \text{ to } \theta = \beta \text{ equals } \int ds = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta.$$

Example 1 Find the total length of the cardioid $r = a(1 - \cos\theta)$.

Solution This curve is quite familiar to us and is shown in Fig. 16.29. From the equation of the curve, we have $dr = a\sin\theta\,d\theta$, so formula (2) gives

$$\begin{aligned} ds^2 &= a^2(1-\cos\theta)^2\,d\theta^2 + a^2\sin^2\theta\,d\theta^2 \\ &= a^2[(1-\cos\theta)^2 + \sin^2\theta]\,d\theta^2 \\ &= 2a^2(1-\cos\theta)\,d\theta^2. \end{aligned}$$

Therefore

$$\begin{aligned} ds &= \sqrt{2}a\sqrt{1-\cos\theta}\,d\theta \\ &= 2a\left|\sin\tfrac{1}{2}\theta\right|\,d\theta, \end{aligned}$$

since $1 - \cos\theta = 2\sin^2\frac{1}{2}\theta$. We know that $\sin\frac{1}{2}\theta \geq 0$ for $0 \leq \theta \leq 2\pi$, so we can drop the absolute value signs and write

$$s = \int ds = \int_0^{2\pi} 2a \sin \frac{1}{2}\theta \, d\theta$$

$$= -4a \cos \frac{1}{2}\theta \Big]_0^{2\pi} = 4a - (-4a) = 8a.$$

The symmetry of this curve about the horizontal axis tells us that we can also obtain the total length by integrating from 0 to π and multiplying by 2,

$$s = 2 \int_0^{\pi} 2a \sin \frac{1}{2}\theta \, d\theta = -8a \cos \frac{1}{2}\theta \Big]_0^{\pi}$$

$$= 0 - (-8a) = 8a.$$

As a matter of routine, we should accustom ourselves to simplifying the calculation of integrals as much as possible by exploiting whatever symmetry is available.

The above formula for ds in polar coordinates can also be used to find areas of surfaces of revolution, as explained in Section 7.6.

Example 2 Find the area of the surface generated when the lemniscate $r^2 = 2a^2 \cos 2\theta$ is revolved about the x-axis.

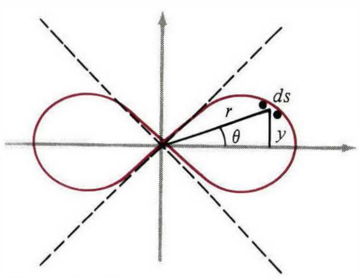


Figure 16.27

Solution An element of arc length ds (Fig. 16.27) generates an element of surface area

$$dA = 2\pi y \, ds,$$

where

$$y = r \sin \theta \qquad \text{and} \qquad ds = \sqrt{r^2 \, d\theta^2 + dr^2},$$

so

$$dA = 2\pi r \sin \theta \sqrt{r^2 \, d\theta^2 + dr^2} = 2\pi \sin \theta \sqrt{r^4 \, d\theta^2 + r^2 \, dr^2}. \qquad (3)$$

From the equation of the curve we have

$$r \, dr = -2a^2 \sin 2\theta \, d\theta,$$

so

$$r^4 \, d\theta^2 + r^2 \, dr^2 = (4a^4 \cos^2 2\theta + 4a^4 \sin^2 2\theta) \, d\theta^2$$

$$= 4a^4 \, d\theta^2$$

and (3) becomes

$$dA = 4\pi a^2 \sin \theta \, d\theta.$$

The total surface area is twice the area of the right half, which is generated as ds moves along the part of the lemniscate in the first quadrant, that is, as θ increases from 0 to $\pi/4$. We therefore have

$$A = 2 \int_0^{\pi/4} 4\pi a^2 \sin \theta \, d\theta = -8\pi a^2 \cos \theta \Big]_0^{\pi/4}$$

$$= -8\pi a^2 \left(\frac{\sqrt{2}}{2} - 1 \right) = 4\pi a^2 (2 - \sqrt{2}).$$

When working with rectangular coordinates, we specify the direction of a curve $y = f(x)$ at a point by the angle α from the positive x-axis to the tangent line. However, in the case of a polar curve $r = f(\theta)$, it is easier to work with the angle ψ (psi) from the radius vector to the tangent line, as shown in Fig. 16.28. We see from the figure that $\alpha = \theta + \psi$, so $\psi = \alpha - \theta$; and since $\tan \alpha = dy/dx$ and $\tan \theta = y/x$, the subtraction formula for the tangent gives

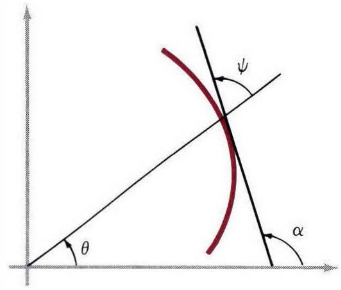


Figure 16.28

$$\begin{aligned}\tan \psi &= \tan (\alpha - \theta) \\ &= \frac{\tan \alpha - \tan \theta}{1 + \tan \alpha \tan \theta} \\ &= \frac{dy/dx - y/x}{1 + (dy/dx) \cdot (y/x)} \\ &= \frac{x\,dy - y\,dx}{x\,dx + y\,dy}. \qquad (4)\end{aligned}$$

The reason why ψ is a convenient angle to use with polar coordinates is that (4) can be put into a very simple form. First, the fact that $x^2 + y^2 = r^2$ tells us that $x\,dx + y\,dy = r\,dr$. Next, from (1) we obtain

$$\begin{aligned}x\,dy - y\,dx &= r^2 \cos^2 \theta\, d\theta + r \sin \theta \cos \theta\, dr + r^2 \sin^2 \theta\, d\theta - r \sin \theta \cos \theta\, dr \\ &= r^2\, d\theta.\end{aligned}$$

By substituting these expressions into (4), we find that

$$\tan \psi = \frac{r\,d\theta}{dr} = \frac{r}{dr/d\theta}. \qquad (5)$$

This formula is the basic tool for working with tangent lines to polar curves.

Example 3 Find the angle ψ for the cardioid $r = a(1 - \cos \theta)$.

Solution This curve is shown in Fig. 16.29. The equation of the curve gives

$$\frac{dr}{d\theta} = a \sin \theta,$$

so

$$\begin{aligned}\tan \psi = \frac{r}{dr/d\theta} &= \frac{a(1 - \cos \theta)}{a \sin \theta} \\ &= \frac{2 \sin^2 \frac{1}{2}\theta}{2 \sin \frac{1}{2}\theta \cos \frac{1}{2}\theta} \\ &= \tan \frac{1}{2}\theta.\end{aligned}$$

We therefore have $\psi = \frac{1}{2}\theta$, and as θ increases from 0 to 2π, ψ increases from 0 to π, as indicated in the figure.

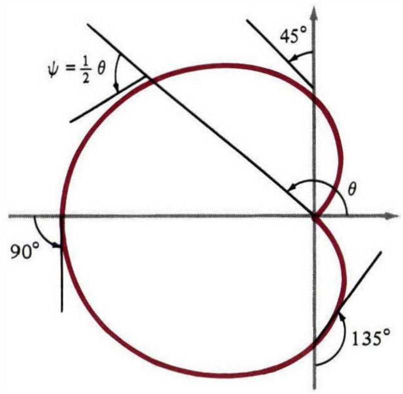


Figure 16.29

As another example of the use of the formula for tan ψ, we consider an interesting curve called the *exponential spiral.*

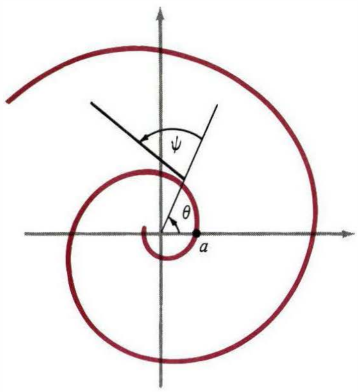


Figure 16.30 The equiangular spiral.

Example 4 Find the angle ψ for the curve $r = ae^{b\theta}$, where $a > 0$ and $b \neq 0$.

Solution If $b > 0$, we see that r increases as θ increases, as shown in Fig. 16.30. Further, it is clear that $r \to \infty$ as $\theta \to \infty$ and $r \to 0$ as $\theta \to -\infty$. The distinctive feature of this curve is that ψ is constant, because

$$\tan \psi = \frac{r}{dr/d\theta} = \frac{ae^{b\theta}}{abe^{b\theta}} = \frac{1}{b}.$$

This enables us to find ψ in the form $\psi = \tan^{-1}(1/b)$. If $b < 0$, the curve spirals in to the origin instead of outward as θ increases. The curve $r = ae^{b\theta}$ is sometimes called the *equiangular spiral* because of the constancy of ψ.

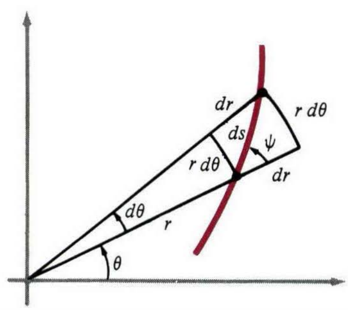


Figure 16.31

The two main facts of this section, formulas (2) and (5), are easy to remember by using Fig. 16.31 as a mnemonic device. In this figure we have an arc of length ds joining two points with polar coordinates r, θ and $r + dr$, $\theta + d\theta$. The outer part of the figure is approximately a rectangle, and the "differential triangle" on the right is approximately a right triangle with hypotenuse ds and with $r\,d\theta$ and dr as the legs opposite and adjacent to the angle ψ. The formulas

$$ds^2 = r^2\,d\theta^2 + dr^2$$

and

$$\tan \psi = \frac{r\,d\theta}{dr}$$

are now self-evident from this triangle, by the theorem of Pythagoras and the right triangle definition of the tangent. Needless to say, this way of reasoning is not a proof, but it is very useful nevertheless. It is also a good example of the true Leibnizian spirit in calculus, in the sense repeatedly explained in Chapter 7.

PROBLEMS

1 For the spiral of Archimedes $r = a\theta$ ($\theta \geq 0$), show that $\psi = 45°$ when $\theta = 1$ radian, and also that $\psi \to 90°$ as the spiral winds on around the origin in the counterclockwise direction. Sketch the curve and show the angle ψ for the direction $\theta = 1$ radian.

2 If two curves $r = f_1(\theta)$ and $r = f_2(\theta)$ intersect at a point other than the origin for a common value of θ, show by examining a figure that the angle γ between their tangents can be found from the formula

$$\tan \gamma = \frac{\tan \psi_2 - \tan \psi_1}{1 + \tan \psi_2 \tan \psi_1}.$$

Under what circumstances will the two curves intersect orthogonally (at right angles)?

3 Sketch each of the following pairs of curves on a single figure and use the result of Problem 2 to show that they intersect orthogonally:
(a) $r = 2a \sin \theta$, $r = 2b \cos \theta$;
(b) $r = a(1 + \cos \theta)$, $r = b(1 - \cos \theta)$, except at the origin;
(c) $r = a/(1 - \cos \theta)$, $r = b/(1 + \cos \theta)$;
(d) $r = a/(1 - \cos \theta)$, $r = a(1 - \cos \theta)$;
(e) $r^2 = 2a^2 \cos 2\theta$, $r^2 = 2b^2 \sin 2\theta$, except at the origin.

4 Show that the spirals $r = \theta$ and $r = 1/\theta$ intersect orthogonally at $\theta = 1$.

5 Find the area of the surface generated by revolving the cardioid $r = a(1 - \cos \theta)$ about the x-axis.

6 The lemniscate $r^2 = 2a^2 \cos 2\theta$ is revolved about the y-axis. Find the area of the surface of revolution generated in this way.

7 Consider the tangent at a point P on the spiral $r = a\theta$ ($\theta \geq 0$), and let the line OT which is perpendicular to OP at the origin O meet this tangent at T (Fig. 16.32). Show that the segment OT equals the circular arc ASP with center O which is drawn from the polar axis to the point P.

8 At what angle does the lemniscate $r^2 = 2a^2 \cos 2\theta$ intersect the circle $r = a$?

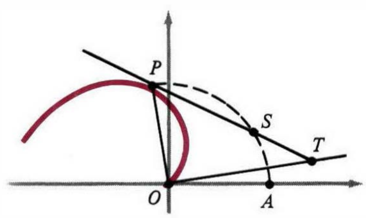


Figure 16.32

9 On the upper half of the parabola $r = a/(1 - \cos \theta)$, show that $\tan \psi = -\tan \frac{1}{2}\theta$, and conclude that $\psi = \pi - \frac{1}{2}\theta$. Show that this establishes the following reflection property: The tangent at any point on the parabola makes equal angles with the horizontal through that point and the line from the origin through the point.

10 Show that the part of the hyperbolic spiral $r\theta = 1$ from $\theta = \pi/2$ to $\theta = \infty$ has infinite length.

11 Find the surface area generated by revolving the circle $r = 2a \cos \theta$ about the line
(a) $\theta = 0$;
(b) $\theta = \pi/2$.

12 Find the length of one turn of the spiral $r = \theta$, from $\theta = 0$ to $\theta = 2\pi$.

13 If a curve $r = f(\theta)$ has the property that ψ is a constant $\neq \pi/2$, show that the curve must be the exponential spiral $r = ae^{b\theta}$.

14 Use integration to find the circumference of the circle $r = 2a \cos \theta$.

15 Show that if a point moves at constant speed along the exponential spiral $r = ae^{b\theta}$, then the radius r changes at a constant rate.

*16 If a curve $r = f(\theta)$ has the property that $\psi = \frac{1}{2}\theta$, show that the curve must be the cardioid $r = a(1 - \cos \theta)$.

17 Find the length of the exponential spiral $r = e^{-\theta}$ from $r = 1$ to the origin.

18 Sketch the exponential spiral $r = ae^{b\theta}$ for the case in which a is positive and b is negative, and show that the arc length from $\theta = 0$ to $\theta = \infty$ is equal to the length of the part of the tangent at $\theta = 0$ that is cut off by the x- and y-axes.

*19 Show that the length L of the right loop of the lemniscate $r^2 = 2a^2 \cos 2\theta$ can be expressed in the form

$$L = \sqrt{2}a \int_{-\pi/4}^{\pi/4} \frac{d\theta}{\sqrt{\cos 2\theta}} = \sqrt{2}a \int_{-\pi/4}^{\pi/4} \frac{d\theta}{\sqrt{1 - 2 \sin^2 \theta}}.$$

Introduce the new variable $u = \tan \theta$ and show that

$$\sin^2 \theta = \frac{u^2}{1 + u^2} \quad \text{and} \quad d\theta = \frac{du}{1 + u^2},$$

and that therefore

$$L = \sqrt{2}a \int_{-1}^{1} \frac{du}{\sqrt{1 - u^4}}.^\dagger$$

†This is a special elliptic integral that played a large part in the investigations of Gauss mentioned in a footnote of Section 16.2. See also Problem 16 in Appendix A.9.

16.5 AREAS IN POLAR COORDINATES

Our problem here is to find the area A of the region bounded by a polar curve $r = f(\theta)$ and two half-lines $\theta = \alpha$ and $\theta = \beta$, as shown in Fig. 16.33. Our approach is modeled on the "differential element of area" idea of Section 7.1.

In working with areas in rectangular coordinates, we use thin rectangular strips and rely on the fact that the area of a rectangle equals length times width. Here we need the fact (Fig. 16.34) that the area of a sector of a circle of radius r and central angle θ (measured in radians) is $\frac{1}{2}r^2\theta$. In Fig. 16.33 our element of area dA is the area of the very thin sector with radius r and central angle $d\theta$, so

$$dA = \tfrac{1}{2}r^2\,d\theta. \tag{1}$$

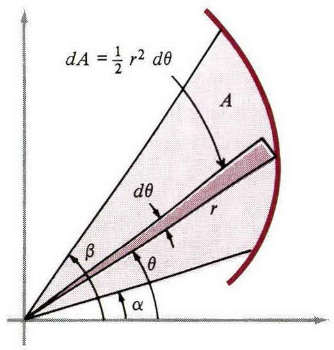


Figure 16.33

In the manner of Section 7.1, we think of the total area A as the result of adding up these elements of area dA as our thin sector sweeps across the region, that is, as θ increases from α to β:

$$A = \int dA = \int_\alpha^\beta \frac{1}{2}\,r^2\,d\theta. \tag{2}$$

Again, the essence of the process of integration is that we calculate the whole of a quantity by cutting it up into a great many convenient small pieces and then adding up these pieces.

We shall give a more mathematically sophisticated approach to formula (2) in Remark 2. First, however, we illustrate its use in several examples. As students will observe in these examples, it is always essential in solving area problems to have a good idea of what the curve looks like, because the correct limits of integration will be determined from the figure.

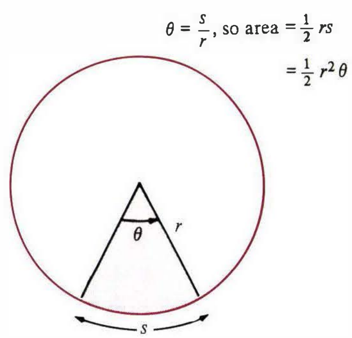


Figure 16.34

Example 1 Use integration to find the area of the circle $r = 2a\cos\theta$.

Solution The complete circle (Fig. 16.35) is swept out as θ increases from $-\pi/2$ to $\pi/2$. By symmetry we can integrate from 0 to $\pi/2$ and multiply by 2,

$$\begin{aligned} A &= 2\int_0^{\pi/2} \frac{1}{2}\,r^2\,d\theta = 2\int_0^{\pi/2} \frac{1}{2}\cdot 4a^2\cos^2\theta\,d\theta \\ &= 4a^2\int_0^{\pi/2} \frac{1}{2}\,(1+\cos 2\theta)\,d\theta \\ &= 2a^2\left(\theta + \frac{1}{2}\sin 2\theta\right)\Big]_0^{\pi/2} = \pi a^2. \end{aligned}$$

Naturally, we expected this answer because our circle has radius a, but it is reassuring to obtain a familiar result by a new method.

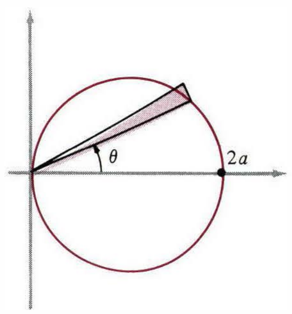


Figure 16.35

Example 2 Find the total area enclosed by the lemniscate $r^2 = 2a^2\cos 2\theta$ (Fig. 16.36).

Solution By symmetry, we calculate the area of the first quadrant part and multiply by 4:

$$\begin{aligned} A &= 4\int_0^{\pi/4} \frac{1}{2}\,r^2\,d\theta = 4\int_0^{\pi/4} \frac{1}{2}\cdot 2a^2\cos 2\theta\,d\theta \\ &= 4a^2\int_0^{\pi/4} \cos 2\theta\,d\theta = 2a^2\sin 2\theta\Big]_0^{\pi/4} = 2a^2. \end{aligned}$$

This problem provides a good illustration of the value of exploiting symmetry; for if we carelessly integrate all the way around from 0 to 2π, forgetting that r^2 is sometimes positive and sometimes negative, then our final answer turns out to be 0, which is obviously wrong.

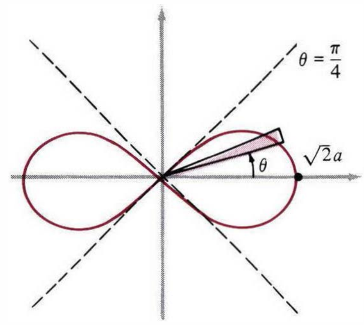


Figure 16.36

Example 3 Find the area inside the circle $r = 6a \cos \theta$ and outside the cardioid $r = 2a(1 + \cos \theta)$.

Solution By equating the r's and solving for θ, we see that the curves intersect in the first quadrant at $\theta = \pi/3$, as shown in Fig. 16.37. The indicated element of area is

$$\begin{aligned} dA &= \tfrac{1}{2}(r_{\text{circle}})^2\, d\theta - \tfrac{1}{2}(r_{\text{cardioid}})^2\, d\theta \\ &= \tfrac{1}{2}[(r_{\text{circle}})^2 - (r_{\text{cardioid}})^2]\, d\theta \\ &= \tfrac{1}{2}[36a^2 \cos^2 \theta - 4a^2(1 + \cos \theta)^2]\, d\theta \\ &= 2a^2(8 \cos^2 \theta - 1 - 2 \cos \theta)\, d\theta. \end{aligned}$$

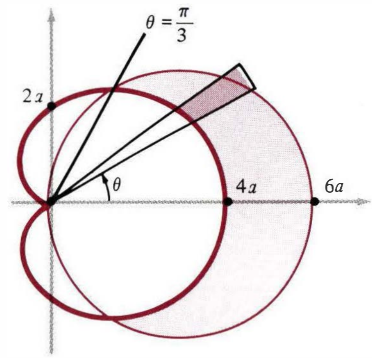


Figure 16.37

By symmetry, the area we seek is double the first quadrant area, so

$$\begin{aligned} A &= 2\int_0^{\pi/3} 2a^2(8 \cos^2 \theta - 1 - 2 \cos \theta)\, d\theta \\ &= 4a^2 \int_0^{\pi/3} [4(1 + \cos 2\theta) - 1 - 2 \cos \theta]\, d\theta \\ &= 4a^2 \Big[3\theta + 2 \sin 2\theta - 2 \sin \theta\Big]_0^{\pi/3} = 4\pi a^2. \end{aligned}$$

Remark 1 The ideas of this section have an important application to the astronomy of the solar system. Consider a point P moving along a polar curve $r = f(\theta)$. We can think of P as a planet moving along its orbit, with the sun at the origin. If A is the area swept out by the radius OP from a fixed direction α to a variable direction θ, as shown in Fig. 16.38, then we have

$$dA = \tfrac{1}{2} r^2\, d\theta.$$

If both A and θ are thought of as functions of time t, then we see that

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt}.$$

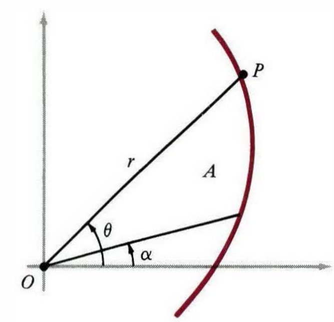


Figure 16.38

The derivative dA/dt is, of course, the rate of change of the area A. Kepler's second law of planetary motion states that a planet moves in such a way that the radius joining the planet to the sun sweeps out area at a constant rate. This means that dA/dt is constant, which in turn means that

$$r^2 \frac{d\theta}{dt} = \text{a constant} \tag{3}$$

for any given planet. Thus, for example, if a planet's orbit takes it in twice as close to the origin, then its angular velocity $d\theta/dt$ must increase by a factor of 4.

This fact has far-reaching implications which we shall examine more thoroughly in the last section of the next chapter.

Remark 2 We briefly reconsider formula (2) for the area A shown in Fig. 16.33. Our purpose is to remind students of the point of view developed in Section 6.4, namely, that a definite integral is defined to be a limit of approximating sums. As usual, we begin with a subdivision of the interval of integration $[\alpha, \beta]$:

$$\alpha = \theta_0 < \theta_1 < \cdots < \theta_n = \beta.$$

For each $k = 1, 2, \ldots, n$, let m_k and M_k be the minimum and maximum values of $f(\theta)$ on the kth subinterval $[\theta_{k-1}, \theta_k]$ of length $\Delta\theta_k = \theta_k - \theta_{k-1}$. Also, let ΔA_k be the area within the curve $r = f(\theta)$ corresponding to this subinterval. In Fig. 16.39 we show the area ΔA_k squeezed between the areas of the inscribed sector with radius $r = m_k$ and the circumscribed sector with radius $r = M_k$. We therefore have

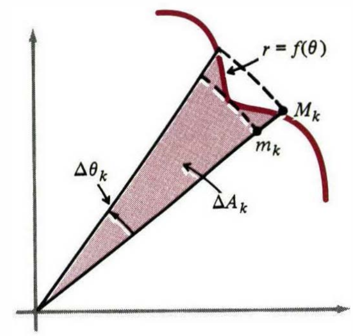


Figure 16.39

$$\tfrac{1}{2}m_k^2\,\Delta\theta_k \le \Delta A_k \le \tfrac{1}{2}M_k^2\,\Delta\theta_k.$$

By adding these inequalities from $k = 1$ to $k = n$, we obtain

$$\sum_{k=1}^{n} \frac{1}{2} m_k^2\,\Delta\theta_k \le A \le \sum_{k=1}^{n} \frac{1}{2} M_k^2\,\Delta\theta_k,$$

because A is the sum of the ΔA_k's. We now vary the subdivision in the manner described in Section 6.4, so that max $\Delta\theta_k \to 0$. Then each of these sums approaches the definite integral

$$\int_\alpha^\beta \frac{1}{2} f(\theta)^2\,d\theta$$

for any continuous function $r = f(\theta)$, and since the area A is squeezed between the sums, we legitimately conclude that

$$A = \int_\alpha^\beta \frac{1}{2} f(\theta)^2\,d\theta = \int_\alpha^\beta \frac{1}{2} r^2\,d\theta,$$

which is (2).

PROBLEMS

1. Find the area enclosed by the cardioid $r = a(1 + \cos\theta)$.
2. Find the area between the parabola $r = 8/(1 - \cos\theta)$ and the y-axis.
3. Show that the area inside the first turn of the spiral $r = a\theta$, that is, for $0 \le \theta \le 2\pi$, equals one-third the area of the circle that passes through the endpoint of this turn and has center at the origin.*
4. Find the total area inside the rose $r = \sin 2\theta$.
5. Find the total area inside the rose $r = \sin 3\theta$.
6. Find the area inside the smaller loop of the limaçon $r = 1 + 2\cos\theta$.
7. Find the area between the two loops of the limaçon $r = 1 + 2\cos\theta$.
8. Find the area between the circle $r = 2a\cos\theta$ and the line $y = x$
   (a) by integration;
   (b) by elementary geometry.
9. Find the area that lies inside both curves $r = a\cos\theta$ and $r = a(1 - \cos\theta)$.
10. Find the area that lies inside both curves $r = a$ and $r^2 = 2a^2\cos 2\theta$.
11. Use integration in polar coordinates to show that the area of the rectangle bounded by $x = 0$, $y = 0$, $x = a$, $y = b$ is ab.
12. Find the area common to the two circles $r = 2a\cos\theta$ and $r = 2b\sin\theta$.

*This statement and the result of Problem 7 in Section 16.4 are the main theorems proved by Archimedes in his treatise *On Spirals* (Propositions 20 and 24).

13 Show that the area between the cissoid $r = 2a$ (sec $\theta -$ cos θ) and its asymptote $x = 2a$ (see Problem 12 in Section 16.2) is 3 times the area of the generating circle.

14 In equation (3), show that the value of the constant is $2A_e/T$, where A_e is the area of the elliptical orbit and T is the time required for the planet to go once around its orbit.

CHAPTER 16 REVIEW: CONCEPTS, FORMULAS

Think through the following.

1 Equations connecting rectangular and polar coordinates.
2 Graphs of polar equations: circles, cardioids, lemniscates, spirals.
3 Arc length formula: $ds^2 = r^2\, d\theta^2 + dr^2$.
4 Tangent line formula: $\tan \psi = r\, d\theta/dr$.
5 Area formula: $dA = \frac{1}{2}r^2\, d\theta$.

ADDITIONAL PROBLEMS FOR CHAPTER 16

SECTION 16.2

1 Transform the given rectangular equation into an equivalent polar equation:
(a) $y = 4x$;
(b) $4x^2 + 9y^2 = 36$;
(c) $x^2 + y^2 - 2x + 4y = 0$;
(d) $2x - 5y = 3$;
(e) $y^2 = 4x$;
(f) $x^2 + y^2 - 4y = \sqrt{x^2 + y^2}$;
(g) $x^3 + y^3 = 12xy$.

2 Transform the given polar equation into an equivalent rectangular equation:
(a) $r = -3$; (b) $\theta = 3\pi/4$;
(c) $r \sin \theta = -5$; (d) $r = 2 \sec \theta$;
(e) $r^2 = \sin 2\theta$; (f) $r = \cos 2\theta$;
(g) $r = \sin 3\theta$; (h) $r = \cos 3\theta$;
(i) $r^2 = \sin^2 \theta \tan \theta$.

3 Find all points of intersection of each pair of curves:
(a) $r \sin \theta = a,\ r \cos \theta = a$;
(b) $r = a(1 + \cos \theta),\ r = a(1 - \sin \theta)$;
(c) $r = a \cos 2\theta,\ 4r \cos \theta = \sqrt{3}a$;
(d) $r \sin \theta = 3,\ r = 6 \sin \theta$;
(e) $r = 1 + \cos \theta,\ r^2 = \frac{1}{2} \cos \theta$;
(f) $r = a,\ r^2 = 2a^2 \cos 2\theta$;
(g) $r = a(1 + \sin \theta),\ r = 2a \cos \theta$;
(h) $r \cos \theta = 2,\ r \sin \theta + 2\sqrt{3} = 0$;
(i) $r = 2 \sin^2 \theta,\ r = -2$;
(j) $r \cos \theta = 1,\ r = 2 \cos \theta + 1$;
(k) $r = 1 + \cos \theta,\ r = 3 \cos \theta$;
(l) $r = a(1 + \cos \theta),\ r = a(1 + \sin \theta)$;
(m) $r = a \sin 2\theta,\ r = a(1 - \cos 2\theta)$.

4 Diocles invented his cissoid to solve the classical Greek problem of doubling a cube, that is, constructing a second cube whose volume is twice that of a given cube. If OA in Problem 12 of Section 16.2 is the edge of the given cube, then the edge of the second cube must have length $\sqrt[3]{2}\, OA$. Verify that the following construction produces such a length. Let D be a point on the positive y-axis such that $OD = 2OA$, let AD intersect the cissoid at E, and extend OE to intersect the asymptote $x = 2a$ at E. Then $(AF)^3 = 2(OA)^3$, so AF is the required length.* Hint: If $E = (x, y)$, show that $y^3 = 2x^3$.

5 Verify that the following construction, using the conchoid discussed in Problem 15 of Section 16.2, trisects the given angle BOP (Fig. 16.40). Draw any line $x = a$, and let A be the point where it intersects OP. Form

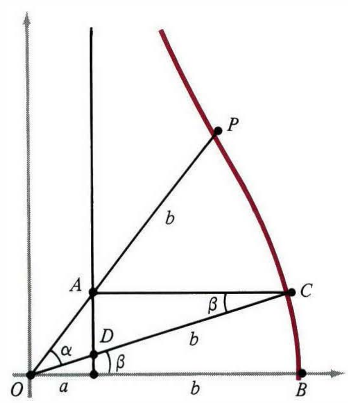


Figure 16.40

*Hippocrates of Chios (see Section 6.2) reduced the problem of doubling a cube of edge a to the problem of constructing two mean proportionals x and y between a and $2a$:

$$\frac{a}{x} = \frac{x}{y} = \frac{y}{2a}.$$

From these equations we have $x^2 = ay$ and $xy = 2a^2$, and eliminating y we find that $x^3 = 2a^3$. Thus x is the edge of a cube that has twice the volume of the first cube. This value is the x-coordinate of the point of intersection of the parabola $x^2 = ay$ and the hyperbola $xy = 2a^2$, so the problem of doubling a cube is solved if we allow ourselves to use these curves as tools. Historians of mathematics believe that the conic sections may have arisen in just this way.

the conchoid determined by the numbers a and b, where $b = 2OA$. If C is the point where the horizontal line through A intersects this conchoid, let D be the point where OC intersects the line $x = a$, so that $DC = b = 2OA$. Let α be the angle POC and β the angle BOC, so that the given angle BOP is $\alpha + \beta$. Then by the law of sines we have

$$\frac{OA}{\sin \beta} = \frac{AC}{\sin \alpha},$$

so

$$\frac{OA}{\sin \beta} = \frac{2\ OA \cos \beta}{\sin \alpha}$$

and $\sin \alpha = 2 \sin \beta \cos \beta = \sin 2\beta$. Therefore $\alpha = 2\beta$ and $\angle BOP = 3\beta$, so $\beta = \frac{1}{3}\angle BOP$ and the line OC trisects the given angle.

6 A general conchoid can be defined as follows. Let O be a fixed point and C a given curve. On the line OA from O to a point A on C, continue to a point P, where AP is a positive constant b. Then the locus of P is called the *conchoid of C with pole O and constant b*. If C is a straight line and O is any point not on C, we get the conchoid of Nicomedes (Problem 15 in Section 16.2) as a special case. Show that a conchoid of a circle $r = a \cos \theta$ with pole O at the origin and constant b is the limaçon $r = a \cos \theta + b$, so that conchoids include limaçons—and therefore cardioids—as special cases.†

SECTION 16.3

7 Verify the validity of the following procedure for squaring a circle of a radius a by using the quadratrix defined in Problem 21 of Section 16.3:

(a) By part (d) of the problem just mentioned,

$$\frac{OG}{OC} = \frac{OC}{ADC},$$

so a segment whose length is $\frac{1}{4}$ the circumference of our circle can be constructed as the third proportional to the segments OG and OC. (In an equation of the form $c/d = d/x$, x is called the *third proportional* to the given segments of lengths c and d, and x can be constructed by ruler and compass as indicated in Fig. 16.41.)

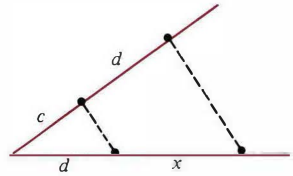


Figure 16.41

(b) By part (a), a segment can be constructed whose length b is $\frac{1}{2}$ the circumference of our circle, with ab the area of the circle. The side s of a square whose area is ab can now be constructed as the mean proportional to a and b. (In an equation of the form $a/s = s/b$, s is called the *mean proportional* to a and b, and can be constructed by ruler and compass as indicated in Fig. 16.42.)

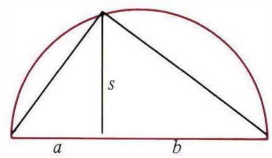


Figure 16.42

8 Verify the validity of the following procedure for squaring a circle of radius a by using the spiral of Archimedes:

(a) draw the circle with center O and radius a, and superimpose the spiral $r = a\theta$ with constant of proportionality equal to the radius of this circle (Fig. 16.43);

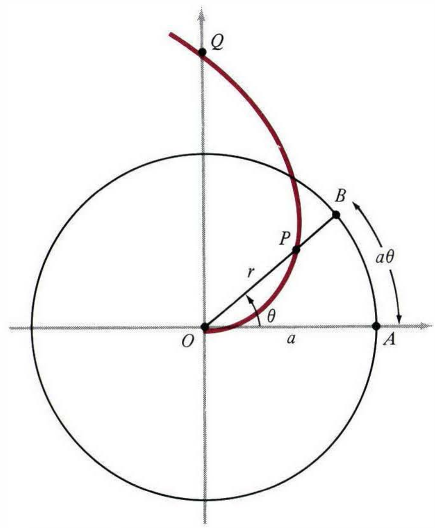


Figure 16.43

†The limaçon appears to have been named by the French mathematician Roberval, who used it as an example in one of his writings in the 1630s and called it the "limaçon de monsieur Pascal." This refers to Étienne Pascal, father of the famous Blaise Pascal. Étienne was a friend and correspondent of Mersenne, which made it easy for his son to enter the elite intellectual circles of France at an early age.

(b) since OP and the arc AB both have length $a\theta$ and are therefore equal, it follows that the radius OQ for $\theta = \pi/2$ is a segment whose length is $\frac{1}{4}$ the circumference of our circle;

(c) the squaring of the circle is now completed exactly as in part (b) of Problem 7.

SECTION 16.4

9 For the hyperbolic spiral $r\theta = a$ ($\theta > 0$), show that $\psi = 135°$ when $\theta = 1$ radian, and also that $\psi \to 90°$ as the spiral winds on around the origin in the counter clockwise direction. Sketch the curve and show the angle ψ for the direction $\theta = 1$ radian.

10 If a point moves along a polar curve $r = f(\theta)$ at a constant speed, and is also moving away from the origin at a constant speed, show that the curve must be the exponential spiral $r = ae^{b\theta}$.

*11 Show that the arc length of one leaf of each of the following roses equals the total arc length of the corresponding ellipse (but do not try to evaluate the integrals involved, because it cannot be done):
(a) $r = 2 \sin 2\theta$, $x^2 + 4y^2 = 1$;
(b) $r = 6 \cos 3\theta$, $x^2 + 9y^2 = 9$.

*12 The distances r_1 and r_2 from the foci to any point on an ellipse satisfy the equation

$$r_1 + r_2 = \text{a constant.}$$

By differentiating both sides of this equation with respect to arc length s and interpreting the result in terms of differentials, show that the tangent at any point on the ellipse makes equal angles with the lines to the foci.

13 Establish the reflection property of parabolas by an adaptation of the argument used in Problem 12.

14 Consider the part of the lemniscate $r^2 = 2a^2 \cos 2\theta$ that lies in the first quadrant, and show that at any point on this curve the angle between the radial direction and the outward normal is 2θ.

15 Find the angle at which the circles $r = a$ and $r = 2a \cos \theta$ intersect.

16 Show that the length of an arc of the exponential spiral $r = ae^{b\theta}$ is proportional to the difference of the radii at its ends.

17 Find the length of the spiral $r = a\theta^2$ from $\theta = 0$ to $\theta = 2\pi$. Sketch the curve.

18 Find the length of the curve $r = a \sin^3 \frac{1}{3}\theta$ from $\theta = 0$ to $\theta = 3\pi/2$. What is the total length of this curve?

SECTION 16.5

*19 Suppose that a polar curve $r = r(\theta)$, $\alpha \le \theta \le \beta$, also has a representation as a rectangular curve $y = y(x)$, $a \le x \le b$, as shown in Fig. 16.44. (It is convenient here to

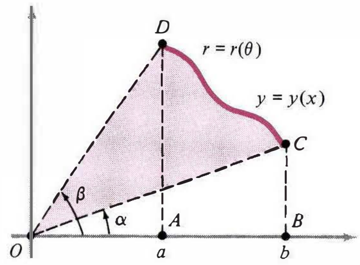


Figure 16.44

denote each function and its dependent variable by the same letter.) From the point of view of rectangular coordinates, the area of the region OCD is

$$\begin{aligned} A_{OCD} &= A_{OAD} + A_{ABCD} - A_{OBC} \\ &= \tfrac{1}{2}r^2(\beta) \sin \beta \cos \beta + \int_a^b y\, dx \\ &\qquad - \tfrac{1}{2}r^2(\alpha) \sin \alpha \cos \alpha \\ &= \tfrac{1}{4}\Big[r^2(\theta) \sin 2\theta\Big]_\alpha^\beta \\ &\qquad + \int_\beta^\alpha r(\theta) \sin \theta[r'(\theta) \cos \theta - r(\theta) \sin \theta]\, d\theta. \end{aligned}$$

By integrating by parts at the right moment, show that this formula reduces to

$$A_{OCD} = \int_\alpha^\beta \frac{1}{2} r^2\, d\theta.$$

20 Find the area inside the circle $r = a$ and outside the cardioid $r = a(1 - \cos \theta)$.

21 Find the area outside the circle $r = 4 \cos \theta$ and inside the limaçon $r = 1 + 2 \cos \theta$.

22 Find the area outside the circle $r = a$ and inside the circle $r = 2a \cos \theta$.

23 Show that the area inside the first turn of the exponential spiral $r = ae^{b\theta}$ is $(r_2{}^2 - r_1{}^2)/4b$, where r_1 is the initial radius and r_2 is the terminal radius.

24 Find the area inside one loop of the curve $r^2 = a^2 \sin \theta$.

25 Find the total area outside the circle $r = a$ and inside the lemniscate $r^2 = 2a^2 \cos 2\theta$.

17 PARAMETRIC EQUATIONS. VECTORS IN THE PLANE

17.1 PARAMETRIC EQUATIONS OF CURVES

When we think of a curve as the path of a moving point, it is often more convenient to study the curve by using two simple equations for x and y in terms of a third independent variable t,

$$x = f(t) \qquad \text{and} \qquad y = g(t), \tag{1}$$

than by using a single more complicated equation of the form

$$F(x, y) = 0. \tag{2}$$

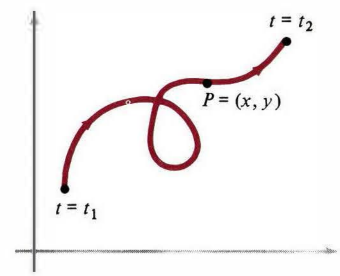
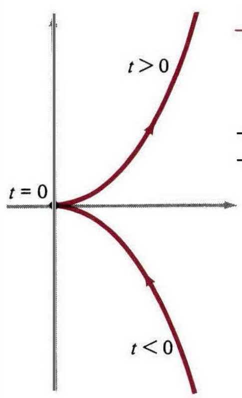


Figure 17.1

In physical problems we often consider a moving point, and t is understood to be the time measured from the moment at which the motion begins. The point P whose coordinates are x and y then traces out the curve as t traverses some definite interval, say $t_1 \leq t \leq t_2$. This provides not only a description of the path on which the point moves, but also information about the direction of its motion and its location on the path for various values of t, as suggested in Fig. 17.1. The third variable in terms of which x and y are expressed is called a *parameter* (from the Greek *para*, meaning "together," and *meter*, meaning "measure"), and equations (1) are called parametric equations of the curve. If we want the rectangular equation of the curve in the form (2), we must eliminate the parameter from equations (1).



t	x	y
0	0	0
1	1	1
2	4	8
−1	1	−1
−2	4	−8

Figure 17.2

Example 1 Sketch the curve $x = t^2$, $y = t^3$ and find its rectangular equation.

Solution We can plot a few points by calculating x and y for several values of t, as indicated by the table in Fig. 17.2. A few calculations are worthwhile to give us something concrete to start with. However, it is more profitable to study how x and y vary as t varies, instead of merely plotting points. Here we see that as t increases from 0 to ∞, x and y both start at 0 and increase into positive values, but y increases faster than x. This means that for large t's the point $P = (x, y)$ moves away from the x-axis faster than from the y-axis, as shown. For negative t's x is still positive, but y is negative, so this part of the curve is a reflection about the x-axis of the upper part which we have just described. The general shape of the curve can be discovered from the behavior of the slope of the tangent dy/dx, which can easily be calculated as a function of t by dividing $dy = 3t^2\,dt$ by $dx = 2t\,dt$:

$$\frac{dy}{dx} = \frac{3t^2\,dt}{2t\,dt} = \frac{3}{2}t \to \begin{cases} 0 & \text{as } t \to 0, \\ \infty & \text{as } t \to \infty, \\ -\infty & \text{as } t \to -\infty. \end{cases}$$

Finally, we notice from the parametric equations that the square of y equals the cube of x, so

$$y^2 = x^3 \qquad \text{or} \qquad y = x^{3/2}$$

is the rectangular equation of the curve.

As we remarked earlier, the use of parametric equations is very natural if we think of a curve as the path of a moving point whose position depends on the time t measured from some convenient initial moment.

Example 2 Let a projectile be fired from the origin at time $t = 0$ with an initial velocity of magnitude v_0 ft/s (or m/s) and direction given by the angle of elevation α (Fig. 17.3), and assume that the only force acting on the projectile is the force of gravity. Discuss the subsequent motion.

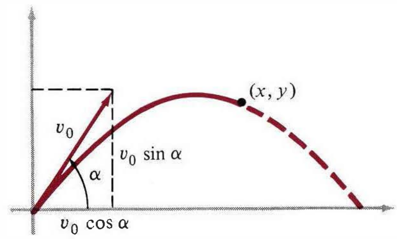


Figure 17.3

Solution We consider the x- and y-components of the acceleration separately. Since the force of gravity acts downward, we have

$$a_x = \frac{dv_x}{dt} = 0 \qquad \text{and} \qquad a_y = \frac{dv_y}{dt} = -g,$$

where $g = 32$ ft/s^2 (or 9.80 m/s^2) is the acceleration due to gravity. Therefore

$$v_x = c_1 \qquad \text{and} \qquad v_y = -gt + c_2$$

for certain constants c_1 and c_2. But when $t = 0$, we have $v_x = v_0 \cos \alpha$ and $v_y = v_0 \sin \alpha$, and consequently

$$v_x = \frac{dx}{dt} = v_0 \cos \alpha \qquad \text{and} \qquad v_y = \frac{dy}{dt} = -gt + v_0 \sin \alpha.$$

Another integration yields

$$x = (v_0 \cos \alpha)t + c_3 \qquad \text{and} \qquad y = -\tfrac{1}{2}gt^2 + (v_0 \sin \alpha)t + c_4.$$

But $x = y = 0$ when $t = 0$, so $c_3 = c_4 = 0$ and

$$x = (v_0 \cos \alpha)t, \qquad y = -\tfrac{1}{2}gt^2 + (v_0 \sin \alpha)t. \tag{3}$$

These are parametric equations for the path of the projectile. We can use equations (3) to show that the projectile follows a parabolic path. To do this, we eliminate the parameter by solving the first equation for t and substituting in the second:

$$t = \frac{x}{v_0 \cos \alpha},$$

$$\begin{aligned} y &= -\frac{1}{2}g \cdot \frac{x^2}{v_0^2 \cos^2 \alpha} + (v_0 \sin \alpha) \cdot \frac{x}{v_0 \cos \alpha} \\ &= -\frac{g}{2v_0^2 \cos^2 \alpha}x^2 + (\tan \alpha)\,x. \end{aligned}$$

The fact that y is a quadratic function of x shows that the point (x, y) moves on a parabola. Further properties of this motion are developed in Problem 13.

In any motion problem like that discussed in Example 2, it is natural to use the time t as the parameter. However, in problems that are more concerned with geometry than with physics, the most convenient parameter is likely to have some geometric significance. In Example 1, for instance, it is perfectly possible to think of the parameter t as a pure variable, without any connotation at all. On the other hand, in this example we have $y/x = t^3/t^2 = t$, so we can also think of t as the slope of the radial line from the origin to a variable point on the curve, and this certainly lends additional vividness to our conception of the way the curve is traced out as the parameter varies. Needless to say, there is nothing sacred about the letter t, and we are always free to use any letter we wish as a parameter.

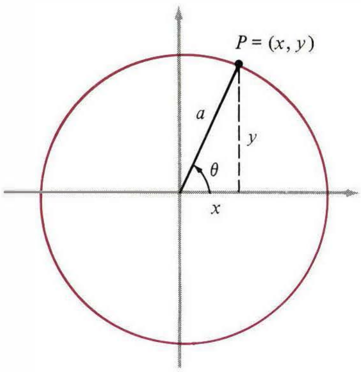


Figure 17.4

Example 3 Consider the circle shown in Fig. 17.4, with radius a and center at the origin. It is easy to see that

$$x = a \cos \theta, \qquad y = a \sin \theta \tag{4}$$

are parametric equations for this circle, where θ is the indicated central angle. As θ varies from 0 to 2π, the point $P = (x, y)$ starts at $(a, 0)$ and moves once around the circle in the counterclockwise direction. If we didn't already know the rectangular equation of this circle, we could obtain it from the identity $\cos^2 \theta + \sin^2 \theta = 1$, which yields

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1 \qquad \text{or} \qquad x^2 + y^2 = a^2. \tag{5}$$

But students should notice that this equation is a static thing, and in passing from (4) to (5) we lose our sense of the circle as a curve traversed by a moving point.

Example 4 The ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

shown in Fig. 17.5 can be parametrized as follows. Since

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1,$$

there exists an angle θ such that $\cos \theta = x/a$ and $\sin \theta = y/b$, so

$$x = a \cos \theta, \qquad y = b \sin \theta.$$

As θ varies from 0 to 2π, the point $P = (x, y)$ starts at $(a, 0)$ and moves once around the ellipse in the counterclockwise direction. Observe from the figure that θ is not the central angle of the point $P = (x, y)$; instead, it is the central angle of the points A and B on the two circles, one circumscribed about the ellipse and the other inscribed in the ellipse, and P is the intersection of the vertical line through A and the horizontal line through B.

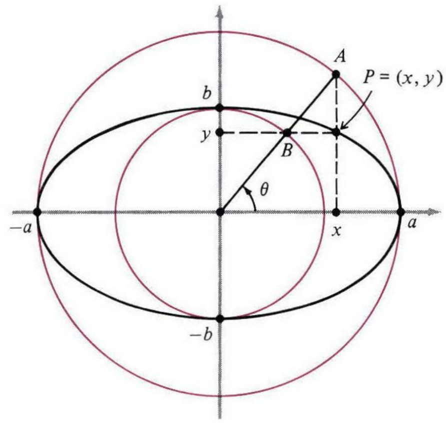


Figure 17.5

Example 5 The parabola $x^2 = 4py$ (Fig. 17.6) can be parametrized in many ways. One method is to use the slope of the tangent at (x, y) as parameter,

$$t = \frac{dy}{dx}.$$

Since

$$2x = 4p\,\frac{dy}{dx} \qquad \text{or} \qquad \frac{dy}{dx} = \frac{x}{2p},$$

the parametric equations in this case are

$$x = 2pt, \qquad y = pt^2.$$

Another method is to use as parameter the number

$$m = \frac{y}{x},$$

which is the slope of the radial line to (x, y). Here we have

$$y = mx \qquad \text{and} \qquad x^2 = 4py = 4pmx,$$

so

$$x = 4pm, \qquad y = 4pm^2$$

are the parametric equations. In each case the entire parabola is traced out as the parameter increases from $-\infty$ to ∞.

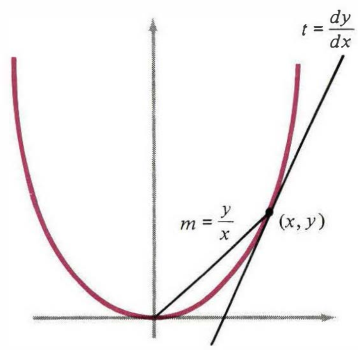


Figure 17.6

Our next example illustrates the fact that a parametric curve is often only a part of the corresponding rectangular curve.

Example 6 Sketch the curve $x = \cos^2 (\pi/2)t$, $y = \sin^2 (\pi/2)t$ and find its rectangular equation.

Solution Since $\cos^2 (\pi/2)t + \sin^2 (\pi/2)t = 1$, the point $P = (x, y)$ moves on the straight line $x + y = 1$ (Fig. 17.7). But neither x nor y can be negative, so the point is confined to that portion of this line that lies in the first quadrant. It is

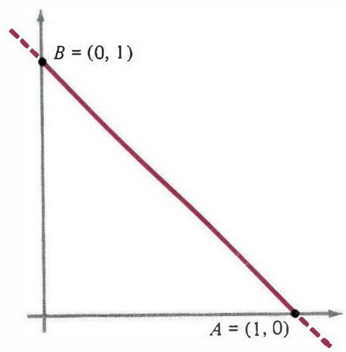


Figure 17.7

easy to see that the point is at $A = (1, 0)$ when $t = 0$; that it moves to $B = (0, 1)$ as t increases from 0 to 1; that it moves back to A as t increases from 1 to 2; and so on.

Example 7 In Example 6 we discussed a parametric curve consisting of only part of a straight line. We now show that the parametric equations

$$x = t - 1, \qquad y = 2t + 3 \tag{6}$$

represent *all* of a straight line. By eliminating t from these equations, that is, by multiplying the first by 2 and subtracting the second, we obtain

$$2x - y = -5. \tag{7}$$

Thus all points (x, y) satisfying (6) also satisfy (7), which is the equation of a straight line. Conversely, given a point (x, y) satisfying (7), let $t = 1 + x$ [we obtain this by solving the first equation in (6) for t]. Then

$$x = t - 1,$$

and from (7) we have

$$y = 2x + 5 = 2(t - 1) + 5 = 2t + 3,$$

so the point (x, y) lies on the parametric curve (6), as we wished to show.

Our previous ways of representing curves, by rectangular coordinates and by polar coordinates, are easy to fit into our present system of parametric representation. Thus, if we have a curve $y = f(x)$, then we can write

$$y = f(x) \qquad \text{and} \qquad x = x,$$

so that x itself is used as the parameter. Also, a curve that is given in polar coordinates by the polar equation $r = F(\theta)$ can be viewed as a parametric curve with parameter θ. To see this, we use the transformation equations $x = r \cos \theta$ and $y = r \sin \theta$ to write

$$x = F(\theta) \cos \theta, \qquad y = F(\theta) \sin \theta.$$

For example, the spiral of Archimedes $r = a\theta$ becomes

$$x = a\theta \cos \theta, \qquad y = a\theta \sin \theta;$$

and the cardioid $r = a(1 + \cos \theta)$ can be expressed as

$$x = a(\cos \theta + \cos^2 \theta), \qquad y = a(\sin \theta + \sin \theta \cos \theta).$$

PROBLEMS

1 In each case, sketch the curve represented by the given parametric equations, describe the way the point (x, y) moves as t varies from large negative to large positive values, and find the rectangular equation:
(a) $x = 1 + t$, $y = 1 - t$;
(b) $x = -1 + 2t$, $y = 2 + 4t$.

2 If x and y are linear functions of t,

$$x = x_0 + at, \qquad y = y_0 + bt,$$

show that the graph is always a complete straight line unless both $a = 0$ and $b = 0$. Can every straight line be represented in this way?

3 Sketch the graph of $x = 1 - t^2$, $y = 2 + t^2$. Describe the way the point (x, y) moves as t varies from large negative to large positive values, and find the rectangular equation of the curve.

For each of the following pairs of parametric equations (in

Problems 4–9), sketch the curve and find its rectangular equation.

4 $x = 3 \cos t$, $y = 2 \sin t$.
5 $x = 1 + t^2$, $y = 3 - t$.
6 $x = \sin t$, $y = -3 + 2 \cos t$.
7 $x = \sec t$, $y = \tan t$.
8 $x = t^3$, $y = 1 - t^2$.
9 $x = \sin t$, $y = \cos 2t$.
10 Sketch the graph represented by

$$x = t + \frac{1}{t}, \qquad y = t - \frac{1}{t}$$

and find its rectangular equation. Hint: Square and subtract.

11 Are the parametric curves

$$x = t + \frac{1}{t}, \qquad y = t - \frac{1}{t}$$

and

$$x = e^t + e^{-t}, \qquad y = e^t - e^{-t}$$

identical? Explain.

12 We know by Example 3 that the unit circle $x^2 + y^2 = 1$ (Fig. 17.8) can be parametrized by the equations

$$x = \cos \theta, \qquad y = \sin\theta, \qquad 0 \leq \theta \leq 2\pi.$$

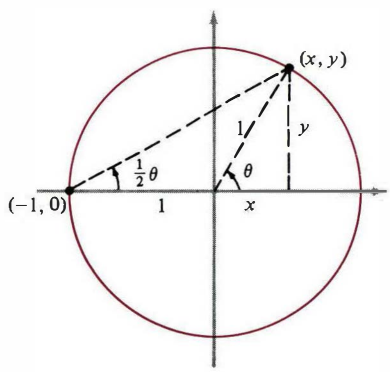


Figure 17.8

A very different parametrization can be obtained by using as the parameter the tangent of the angle $\frac{1}{2}\theta$ shown in the figure,

$$t = \frac{y}{1 + x}.$$

(a) By first eliminating y from this equation and $x^2 + y^2 = 1$, show that the equations

$$x = \frac{1 - t^2}{1 + t^2}, \qquad y = \frac{2t}{1 + t^2}, \qquad -\infty < t < \infty,$$

parametrize the entire circle except for the point $(-1, 0)$.

(b) A point (x, y) in the plane such that both x and y are rational numbers is called a *rational point*. Show that for rational values of t the equations in (a) give all rational points on the unit circle, except the point $(-1, 0)$.*

13 Consider the motion of the projectile described in Example 2.

(a) Show that the maximum height of the projectile is

$$y_{\max} = \frac{v_0^2 \sin^2 \alpha}{2g}.$$

(b) Show that the range R of the projectile, i.e., the distance from the origin to the point where the projectile reaches the x-axis on its descent, is given by the formula

$$R = \frac{v_0^2}{g} \sin 2\alpha.$$

(c) What angle of elevation α produces the maximum range?

(d) Show that doubling the magnitude of the initial velocity multiplies both the maximum height and the range by a factor of 4.

14 The *witch* is a bell-shaped curve that can be defined as follows. Consider the circle of radius a which is tangent to the x-axis at the origin (Fig. 17.9). The variable line OA through the origin intersects the line $y = 2a$ at the point A and the circle at the point B. The point P is the intersection of the vertical line through A and the horizontal line through B, and the witch is the locus of P as OA varies. Find parametric equations for this curve

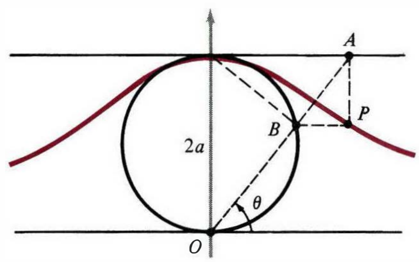


Figure 17.9 The witch.

*In general it is very difficult to find rational points on curves, and it is quite remarkable that we are able to do so for the case of the unit circle. For instance, it has been asserted that if n is any integer > 2, the only rational points on the curve $x^n + y^n = 1$ are those for which $x = 0$ or $y = 0$. This statement is called *Fermat's last theorem*. It has been proved for many particular values of n over the past 350 years, but in its full generality it remains to this day one of the most famous (and intractable) unsolved problems of mathematics. See the last footnote in Section 1.4.

by using as parameter the angle θ from the positive x-axis to the line OA. Also find its rectangular equation.†

15 The *involute* of a circle is the curve traced out by the point at the end of a thread as the thread is held taut and unwound from a fixed spool, as shown in Fig. 17.10. If the center of the spool is placed at the origin and its radius is a, and if the thread begins to unwind at the point $A = (a, 0)$, find parametric equations for the involute by using the angle AOT shown in the figure as the parameter θ.

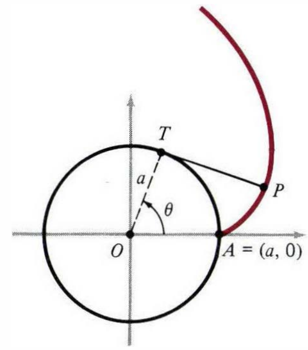


Figure 17.10 The involute of a circle.

*16 The *folium of Descartes,* shown in Fig 17.11, is the graph of the equation $x^3 + y^3 = 3axy$.‡

(a) Introduce the parameter $t = y/x$ and find parametric equations for the curve.

(b) Use the equations found in (a) to show that the line $x + y + a = 0$ is an asymptote, by showing that $x + y \to -a$ as $t \to -1$.

(c) The folium is clearly symmetric about the line $y = x$, because interchanging x and y leaves the equation unaltered. Use this, together with the geometric meaning of t and the results of (a) and (b), to verify as much as possible of the general nature of the curve as suggested by the figure. In particular, decide how various parts of the curve are traced out as t varies over various ranges of values.

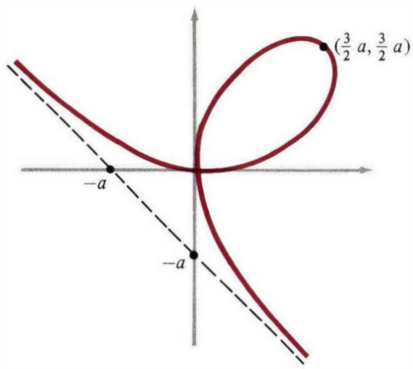


Figure 17.11 The folium of Descartes.

†The witch is sometimes called the *witch of Agnesi* after the Italian mathematician Maria Agnesi (1718–1799), who referred to it in her book on calculus published in 1748. This curve had previously been called the *versoria,* a Latin word meaning "a guy rope attached to a sail," but she apparently confused this word with a different Latin word, *versiera,* meaning a "witch," and the name has stuck.

‡The word *folium* means "leaf" in Latin. This curve was originally used by Descartes as a challenge to Fermat to find its tangent line at an arbitrary point. Fermat succeeded immediately, much to Descartes's dismay.

17.2 THE CYCLOID AND OTHER SIMILAR CURVES

The *cycloid* is the curve traced out by a point on the circumference of a circle when the circle rolls along a straight line in its own plane, as shown in Fig. 17.12. We shall see that this curve has many remarkable geometric and physical properties.

The only convenient way of representing a cycloid is by means of parametric equations. We assume that the rolling circle has radius a and that it rolls along the x-axis, starting from a position in which the center of the circle is on the positive y-axis. The curve is the locus of the point P on the circle which is located at the origin O when the center C is on the y-axis. The angle θ in the figure is the angle through which the radius CP turns as the circle rolls to a new position. If x and y are the coordinates of P, then the rolling of the circle implies that $OB =$ arc $BP = a\theta$, so $x = OB - AB = OB - PQ = a\theta - a\sin\theta = a(\theta - \sin\theta)$. Also, $y = BC - QC = a - a\cos\theta = a(1 - \cos\theta)$. The cycloid therefore has the parametric representation

$$x = a(\theta - \sin\theta), \qquad y = a(1 - \cos\theta). \tag{1}$$

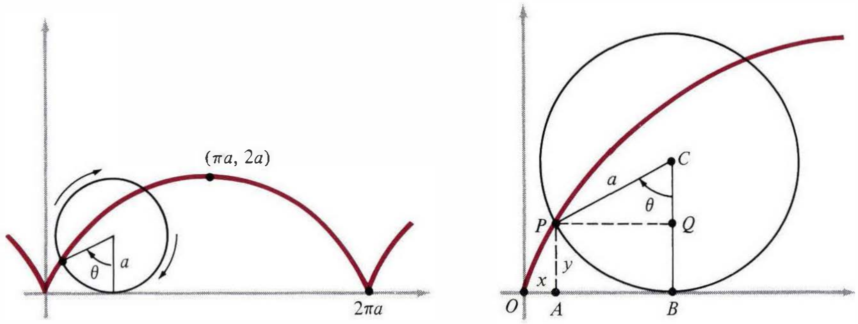


Figure 17.12 The cycloid.

It is clear from Fig. 17.12 that y is a function of x, but it is also clear from equations (1) that it is not possible to find a simple formula for this function. The cycloid is one of many curves for which the parametric equations are much simpler and easier to work with than the rectangular equation.

From equations (1) we have

$$y' = \frac{dy}{dx} = \frac{a \sin \theta \, d\theta}{a(1 - \cos \theta) \, d\theta} = \frac{\sin \theta}{1 - \cos \theta}$$

$$= \frac{2 \sin \frac{1}{2}\theta \cos \frac{1}{2}\theta}{2 \sin^2 \frac{1}{2}\theta} = \cot \tfrac{1}{2}\theta. \tag{2}$$

We observe that the derivative y' is not defined for $\theta = 0, \pm 2\pi, \pm 4\pi$, etc. These values of θ correspond to the points where the cycloid touches the x-axis; these points are called *cusps*. The tangent to the cycloid is vertical at the cusps.

In the following examples we establish the main geometric properties of the cycloid.

Example 1 Show that the area under one arch of the cycloid is three times the area of the rolling circle (*Torricelli's Theorem*).

Solution One arch is traced out as the circle turns through one complete revolution. The usual area integral can therefore be written as follows, using the parameter θ as the variable of integration:

$$A = \int_0^{2\pi a} y \, dx = \int_0^{2\pi} y \frac{dx}{d\theta} \, d\theta = \int_0^{2\pi} a(1 - \cos \theta) a(1 - \cos \theta) \, d\theta$$

$$= a^2 \int_0^{2\pi} (1 - \cos \theta)^2 \, d\theta = a^2 \int_0^{2\pi} (1 - 2 \cos \theta + \cos^2 \theta) \, d\theta$$

$$= a^2 \int_0^{2\pi} (1 + \cos^2 \theta) \, d\theta = a^2 \int_0^{2\pi} d\theta + a^2 \int_0^{2\pi} \tfrac{1}{2}(1 + \cos 2\theta) \, d\theta = 3\pi a^2.$$

Example 2 Show that the length of one arch of the cycloid is four times the diameter of the rolling circle (*Wren's Theorem*).

Solution Since $dx = a(1 - \cos \theta) \, d\theta$ and $dy = a \sin \theta \, d\theta$, the element of arc length ds is given by

$$ds^2 = dx^2 + dy^2 = a^2[(1 - \cos\theta)^2 + \sin^2\theta]\,d\theta^2$$
$$= 2a^2[1 - \cos\theta]\,d\theta^2 = 4a^2 \sin^2 \tfrac{1}{2}\theta\,d\theta^2,$$

so

$$ds = 2a \sin \tfrac{1}{2}\theta\,d\theta.$$

The length of one arch is therefore

$$L = \int ds = \int_0^{2\pi} 2a \sin \tfrac{1}{2}\theta\,d\theta = -4a \cos \tfrac{1}{2}\theta\Big]_0^{2\pi} = 8a.$$

Example 3 Show that the tangent to the cycloid at the point P in Fig. 17.12 passes through the top of the rolling circle.

Solution The point at the top of the circle has coordinates $(a\theta, 2a)$. The slope of the tangent at P is given by (2). The equation of the tangent at P is therefore

$$y - a(1 - \cos\theta) = \frac{\sin\theta}{1 - \cos\theta}(x - a\theta + a\sin\theta).$$

We substitute $x = a\theta$ in this equation and solve for y, which gives

$$y = a(1 - \cos\theta) + \frac{\sin\theta}{1 - \cos\theta} \cdot a\sin\theta = \frac{a(1 - \cos\theta)^2 + a\sin^2\theta}{1 - \cos\theta} = 2a.$$

This shows that the tangent at P does indeed pass through the point $(a\theta, 2a)$ at the top of the circle.

Galileo seems to have been the first to notice the cycloid and investigate its properties, in the early 1600s. He didn't actually discover any of these properties, but he gave the curve its name and recommended its study to his friends, including Mersenne in Paris. Mersenne informed Descartes and others about it, and in 1638 Descartes found a construction for the tangent which is equivalent to the property given in Example 3. In 1644 Galileo's disciple Torricelli (who invented the barometer) published his discovery of the area under one arch. The length of one arch was discovered in 1658 by the great English architect Christopher Wren.* The list of famous men who have worked on the cycloid will be continued, but first we consider some other related curves.

If a circle rolls on the *inside* of a fixed circle, the locus of a point on the rolling circle is called a *hypocycloid*. If a circle rolls on the *outside* of a fixed circle, the locus of a point on the rolling circle is called an *epicycloid*.†

We show how to represent a hypocycloid parametrically. Let the fixed circle have radius a and the rolling circle radius b, where $b < a$. Let the fixed circle have its center at the origin (Fig. 17.13), and let the smaller rolling circle start in a position internally tangent to the fixed circle at the point A on the positive

*Wren was an astronomer and a mathematician—in fact, Savilian Professor of Astronomy at Oxford—before the Great Fire of London in 1666 gave him his opportunity to build St. Paul's Cathedral, as well as dozens of smaller churches throughout the city.

†The distinction between these words is easy to remember because the Greek prefix *hypo* means under or beneath, as in "hypodermic," and *epi* means on or above, as in "epicenter."

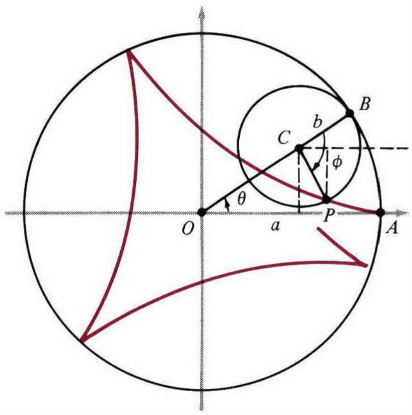


Figure 17.13 The hypocycloid.

x-axis. We consider the point P on the rolling circle that was initially at A. With θ and ϕ as shown in the figure, the rolling of the small circle implies that the arcs AB and BP are equal: $a\theta = b\phi$. We can then see that the coordinates of P are

$$x = (a - b)\cos\theta + b\cos(\phi - \theta),$$

$$y = (a - b)\sin\theta - b\sin(\phi - \theta).$$

But $\phi - \theta = [(a - b)/b]\theta$, so the parametric equations of the hypocycloid are

$$\begin{aligned} x &= (a - b)\cos\theta + b\cos\frac{a - b}{b}\theta, \\ y &= (a - b)\sin\theta - b\sin\frac{a - b}{b}\theta. \end{aligned} \tag{3}$$

The arc length along the fixed circle between successive cusps of the hypocycloid is $2\pi b$. If $2\pi a$ is an integral multiple of $2\pi b$, so that a/b is an integer n, then the hypocycloid has n cusps and the point P returns to A after the smaller circle rolls off its circumference n times on the fixed circle. We leave it to students to decide when P will return to A if a/b is a rational number but not an integer, for example, if $a/b = \frac{5}{2}$. A discussion of the case in which a/b is irrational is beyond the scope of this book; it suffices to say that as the smaller circle rolls around and around indefinitely, the cusps of the resulting hypocycloid are evenly and densely distributed on the fixed circle.*

The parametric equations of a hypocycloid of four cusps can be written in a very simple form by using some trigonometric identities. If $a = 4b$, equations (3) become

$$x = 3b\cos\theta + b\cos 3\theta, \qquad y = 3b\sin\theta - b\sin 3\theta.$$

*The curious reader will find additional information in Theorem 439 of G. H. Hardy and E. M. Wright, *Introduction to the Theory of Numbers* (Oxford, 1954); or in Theorem 6.3 of I. Niven, *Irrational Numbers* (Wiley, 1956).

But

$$\cos 3\theta = \cos(2\theta + \theta) = \cos 2\theta \cos\theta - \sin 2\theta \sin\theta$$
$$= (2\cos^2\theta - 1)\cos\theta - 2\sin^2\theta\cos\theta$$
$$= [2\cos^2\theta - 1 - 2(1 - \cos^2\theta)]\cos\theta$$
$$= 4\cos^3\theta - 3\cos\theta,$$

and a similar calculation yields

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta.$$

Our parametric equations therefore become

$$x = 4b\cos^3\theta = a\cos^3\theta, \qquad y = 4b\sin^3\theta = a\sin^3\theta. \tag{4}$$

From these equations it is easy to obtain the corresponding rectangular equation,

$$x^{2/3} + y^{2/3} = a^{2/3}. \tag{5}$$

Because of its appearance (Fig. 17.14), a hypocycloid of four cusps is often called an *astroid.*

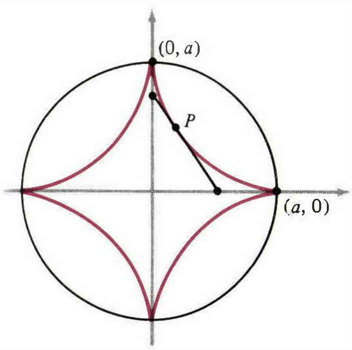


Figure 17.14 The astroid.

Example 4 Consider the tangent to the astroid at a point P in the first quadrant. Show that the part of this tangent which is cut off by the coordinate axes has constant length, independent of the position of P.

Solution By equations (4), the slope of the tangent is

$$y' = \frac{dy}{dx} = \frac{3a\sin^2\theta\cos\theta\,d\theta}{-3a\cos^2\theta\sin\theta\,d\theta} = -\tan\theta,$$

so the equation of the tangent is

$$y - a\sin^3\theta = -\tan\theta\,(x - a\cos^3\theta).$$

We find the x-intercept by putting $y = 0$ and solving for x,

$$x = a\cos^3\theta + a\sin^2\theta\cos\theta = a\cos\theta.$$

Similarly, the y-intercept is $y = a\sin\theta$. The length of the part of the tangent cut off by the axes is therefore

$$\sqrt{a^2\cos^2\theta + a^2\sin^2\theta} = a,$$

which is constant.

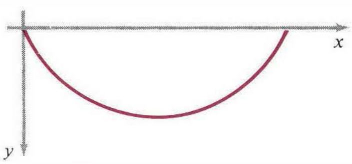


Figure 17.15

We now return to the cycloid discussed earlier, and reflect both it and the y-axis about the x-axis, as shown in Fig. 17.15. The parametric equations (1) are still valid, and the resulting curve has several interesting physical properties, which we now describe and analyze.

In 1696 John Bernoulli conceived and solved the now famous *brachistochrone problem.* He published the problem (but not the solution) as a challenge to other mathematicians of the time. The problem is this: Among all smooth curves in a vertical plane that join a given point P_0 to a given lower point P_1 not directly below it, find that particular curve along which a particle will slide down from P_0 to P_1 in the shortest possible time.* We can think of the particle as a bead of

*The word "brachistochrone" comes from two Greek words meaning "shortest time."

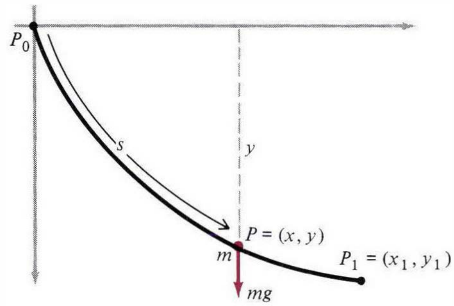


Figure 17.16

mass m sliding down an ideal frictionless wire, with the downward force of gravity mg as the only force acting on the bead.

If we assume that the points P_0 and P_1 lie at the origin and at (x_1, y_1) in the first quadrant, as shown in Fig. 17.16, then Bernoulli's problem can be stated in mathematical language as follows. The bead is released from rest at P_0, so its initial velocity and initial kinetic energy are zero. The work done by gravity in pulling it down from the origin to an arbitrary point $P = (x, y)$ is mgy. This must equal the increase in the kinetic energy of the bead as it slides down the wire to this point, so

$$\tfrac{1}{2}mv^2 = mgy,$$

and therefore

$$v = \frac{ds}{dt} = \sqrt{2gy}. \tag{6}$$

This can be written as

$$dt = \frac{ds}{\sqrt{2gy}} = \frac{\sqrt{dx^2 + dy^2}}{\sqrt{2gy}} = \frac{\sqrt{1 + (dy/dx)^2}\, dx}{\sqrt{2gy}}. \tag{7}$$

The total time T_1 required for the bead to slide down the wire from P_0 to P_1 will depend on the shape of the wire as specified by its equation $y = f(x)$; it is given by

$$T_1 = \int dt = \int_0^{x_1} \sqrt{\frac{1 + (y')^2}{2gy}}\, dx. \tag{8}$$

The brachistochrone problem therefore amounts to this: to find the particular curve $y = f(x)$ that passes through P_0 and P_1 and minimizes the value of the integral (8).

Since the straight line joining P_0 and P_1 is clearly the shortest path, we might guess that this line also yields the shortest time. However, a moment's consideration of the possibilities will make us more skeptical about this conjecture. There might be an advantage in having the bead slide down more steeply at first, thereby increasing its speed more quickly at the beginning of the motion; for with a faster start, it is reasonable to suppose that the bead might reach P_1 in a shorter time, even though it travels over a longer path. And this is the way it turns out: The brachistochrone curve is an arc of a cycloid through P_0 and P_1 with a cusp at the origin.

Leibniz and Newton, as well as John Bernoulli and his older brother James, solved the problem.* John's solution, which is very ingenious but rather specialized in the methods it uses, is given in the Appendix at the end of this chapter. The cycloid was well known to all these men through the earlier work of the great Dutch scientist Huygens on pendulum clocks (see below). When John found that the cycloid is also the solution of his brachistochrone problem, he was astounded and delighted. He wrote: "With justice we admire Huygens because he first discovered that a heavy particle slides down to the bottom of a cycloid in the same time, no matter where it starts. But you [his readers] will be petrified with astonishment when I say that this very same cycloid, the tautochrone of Huygens, is also the brachistochrone we are seeking."†

Huygens was a profound student of the theory of the pendulum, and in fact was the inventor of the pendulum clock. He was very well aware of the theoretical flaw in such a clock, which is due to the fact that the period of oscillation of a pendulum is not strictly independent of the amplitude of the swing.‡ We can express this flaw in another way by saying that if a bead is released on a frictionless circular wire in a vertical plane, then the time the bead takes to slide down to the bottom will depend on the height of the starting point. Huygens wondered what would happen if the circular wire were replaced by one having the shape of an inverted cycloidal arch. But he did more than merely wonder, for he then went on to make the remarkable discovery referred to in the passage previously quoted, that for a wire of this shape the bead will slide down from any point to the bottom in exactly the same time, no matter where it is released (Fig. 17.17). This is the *tautochrone* ("equal time") *property* of the cycloid, and we now prove it by using the formulas given above.

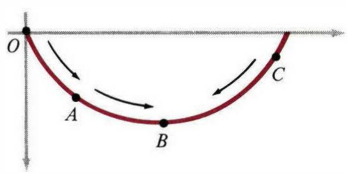


Figure 17.17 Beads released on the cycloidal wire at O, A, C will reach B in the same amount of time.

If we write (8) in the equivalent form

$$T_1 = \int \sqrt{\frac{dx^2 + dy^2}{2gy}}$$

and substitute equations (1) into this, we obtain

$$T_1 = \int_0^{\theta_1} \sqrt{\frac{2a^2(1 - \cos\theta)}{2ag(1 - \cos\theta)}}\, d\theta = \theta_1 \sqrt{\frac{a}{g}}$$

as the time required for the bead to slide down a cycloidal wire from P_0 to P_1. The time needed for the bead to reach the bottom of this wire is the value of T_1 when $\theta_1 = \pi$, namely, $\pi\sqrt{a/g}$. Huygens' tautochrone property amounts to the statement that the bead will reach the bottom in exactly the same time if it starts at any intermediate point (x_0, y_0). To prove this, we replace (6) by

$$v = \frac{ds}{dt} = \sqrt{2g(y - y_0)}.$$

*Newton published his solution anonymously. When John Bernoulli saw it, he wryly remarked, "I recognize the lion by his track."

†For an English translation of Bernoulli's writings on this subject, see pp. 644–655 of D. E. Smith, *A Source Book in Mathematics* (Macmillan, 1929). Bernoulli's vivid, enthusiastic, personal style is in sharp contrast to the dead, gray, impersonal style of most of the writing in scientific journals nowadays.

‡See the remark about the "circular error" in Example 3 in Section 9.6.

The total time required for the bead to slide down to the bottom is therefore

$$T = \int_{\theta_0}^{\pi} \sqrt{\frac{2a^2(1 - \cos\theta)}{2ag(\cos\theta_0 - \cos\theta)}}\, d\theta = \sqrt{\frac{a}{g}} \int_{\theta_0}^{\pi} \sqrt{\frac{1 - \cos\theta}{\cos\theta_0 - \cos\theta}}\, d\theta$$

$$= \sqrt{\frac{a}{g}} \int_{\theta_0}^{\pi} \frac{\sin\frac{1}{2}\theta\, d\theta}{\sqrt{\cos^2\frac{1}{2}\theta_0 - \cos^2\frac{1}{2}\theta}}, \qquad (9)$$

where the last step makes use of the trigonometric identities $2\sin^2\theta = 1 - \cos 2\theta$ and $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$. If we now use the substitution

$$u = \frac{\cos\frac{1}{2}\theta}{\cos\frac{1}{2}\theta_0}, \qquad du = -\frac{1}{2}\,\frac{\sin\frac{1}{2}\theta\, d\theta}{\cos\frac{1}{2}\theta_0},$$

then the integral (9) becomes

$$T = -2\sqrt{\frac{a}{g}} \int_1^0 \frac{du}{\sqrt{1 - u^2}} = 2\sqrt{\frac{a}{g}}\, \sin^{-1} u \Big]_0^1 = \pi\sqrt{\frac{a}{g}}.$$

This shows that T has the same value as before and is therefore independent of the starting point, and the argument is complete.*

Once Huygens established the tautochrone property of the cycloid, a further problem presented itself: How could he arrange for a pendulum in a clock to move along a cycloidal, rather than a circular, path? Here he made a further beautiful discovery. If we suspend from the point P at the cusp between two equal inverted cycloidal semiarches a flexible pendulum whose length equals the length of one of the semiarches (Fig. 17.18), then the bob will draw up as it swings to the side in such a way that its path is another cycloid.†

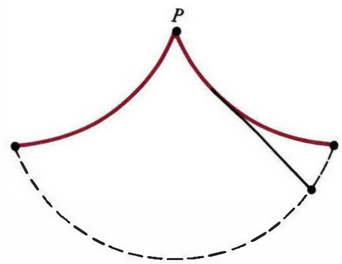


Figure 17.18 A flexible pendulum constrained by cycloidal jaws swings along another cycloid.

*Instead of a "frictionless" bead sliding down a cycloidal wire, we can use a "frictionless" steel ball rolling down a cycloidal channel. A piece of eighteenth century furniture built to demonstrate this can be seen in the Science Museum in Florence, Italy. The fine woodworker John H. Lewis of Colorado Springs has built a work of art based on this principle, with two parallel cycloidal channels for two balls; when released from different positions at the same moment, the balls roll to the bottom in the same time.

†For a proof of this statement, see Section B.23 of the author's *Calculus Gems* (McGraw-Hill, 1992).

PROBLEMS

1 Find the rectangular equation of the cycloid by eliminating θ from the parametric equations (1). Observe how hopeless it is to try to solve this for y as a simple function of x.

2 Show that for the cycloid (1) the second derivative is given by $y'' = dy'/dx = -a/y^2$. Observe that this fact implies that the cycloid is concave down between the cusps, as shown in Fig. 17.12.

3 Use the equation of the normal to the cycloid at P (in Fig. 17.12) to show that this normal passes through the point B at the bottom of the rolling circle. Also, obtain this conclusion from the result of Example 3 by using elementary geometry.

4 Assume that the circle in Fig. 17.12 rolls to the right along the x-axis at a constant speed, with the center C moving at v_0 units per second. (a) Find the rates of change of the coordinates x and y of the point P. (b) What is the greatest rate of increase of x, and where is P when this is attained? (c) What is the greatest rate of increase of y, and for what value of θ is this attained?

5 If a polygon $ABCD$ rolls (awkwardly) on a straight line $A'D'$, as shown in Fig. 17.19, then the point A will trace

out in succession several arcs of circles with centers B', C', D'. The tangent to any such arc is evidently perpendicular to the line joining the point of tangency to the corresponding center. Therefore, if the rolling circle that generates a cycloid is thought of as a polygon with an infinite number of sides, then the tangent to the cycloid at any point is the line perpendicular to the line joining the point of tangency to the bottom of the rolling circle. This is Descartes's method for finding the tangent at any point of a cycloid. Verify that it is correct.

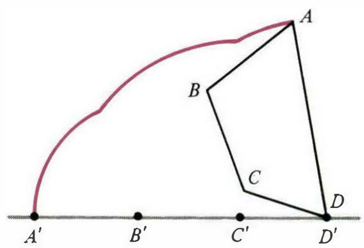


Figure 17.19

6 Find the area inside the astroid (5).

7 Find the total length of the astroid (5).

8 Find the area of the surface generated by revolving the astroid (5) about the x-axis.

9 Show that the hypocycloid of two cusps, with $a = 2b$, is simply the diameter of the fixed circle that lies along the x-axis. In this case, if the center C of the rolling circle moves around with constant angular velocity ω, so that $d\theta/dt = \omega$, show that P moves back and forth on the x-axis with simple harmonic motion (Section 9.6) of period $2\pi/\omega$ and maximum speed $a\omega$.

10 If the astroid (4) is generated by the small circle rolling around counterclockwise with constant angular velocity ω, find the position of the point P in the first quadrant for which y is increasing most rapidly.

11 The hypocycloid of three cusps, with $a = 3b$, is called a *deltoid*. Sketch this curve, find its parametric equations, and find its total length.

12 Find parametric equations for the epicycloid generated by a circle of radius b rolling on the outside of a fixed circle of radius a. Use a figure similar to Fig. 17.13, where the fixed circle has its center at the origin and the point P is initially at $(a, 0)$.

13 Show that the equations in Problem 12 can be obtained from equations (3) in the text by replacing b by $-b$.

14 The epicycloid of two cusps, with $a = 2b$, is called a *nephroid* (meaning "kidney-shaped"). Sketch this curve, find its parametric equations, and calculate its total length.

17.3 VECTOR ALGEBRA. THE UNIT VECTORS $\mathbf{i}$ AND $\mathbf{j}$

A physical quantity such as mass, temperature, or kinetic energy is completely determined by a single real number that specifies its magnitude. These are called *scalar quantities,* or simply *scalars.* In contrast to this, other entities called *vector quantities,* or *vectors,* possess both magnitude and direction. As examples we mention velocities, forces, and displacements.

Example 1 Let us briefly consider the case of velocity. When we discuss a point moving along a straight line, we can specify its position by means of a coordinate, which may be positive or negative, and the velocity of the moving point is the derivative of this coordinate with respect to time, that is, the rate of change of position. Direction is certainly important in such a discussion, but in this simple one-dimensional case all questions about direction are easy to handle by using positive and negative numbers.

However, to specify the velocity of a point moving along a curved path in the plane, it is essential to give both the speed of the point (the rate at which it traverses distance) and the direction of its motion. This combination of two ingredients is the *velocity vector,* or simply the *velocity,* of the moving point. It is natural to represent this vector (see Fig. 17.20) by an arrow or directed line segment $\mathbf{v}$ whose tail is placed at the current position of the point, whose length is the speed in some agreed system of measurement, and whose direction is the direction of motion.

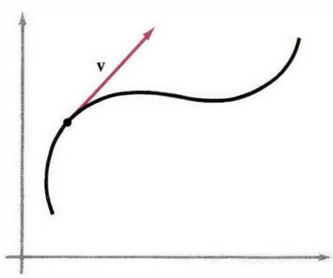


Figure 17.20

Example 2 A force applied to an object is also a vector quantity, whose magnitude is the strength of the force and whose direction is the direction in which the force acts. For instance, the gravitational force **F** exerted by the earth on a circling artificial satellite (Fig. 17.21) is directed toward the center of the earth and its magnitude is proportional to $1/r^2$, where r is the distance from the satellite to the center of the earth.

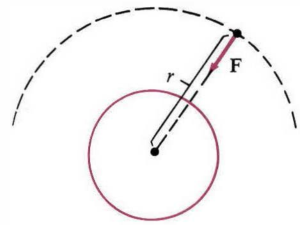


Figure 17.21

From the mathematical point of view, we don't merely *represent* a vector by a directed line segment; we say that a vector *is* a directed line segment. This frees us to develop the algebra of vectors independently of any particular physical interpretation.

As we have indicated, vectors are often denoted in print by boldface type. A good substitute for this in the case of handwritten work is to use letters with arrows over them. Thus, **v** and $\vec{v}$ denote the same vector. Also, if a vector extends from a point P to a point Q, we can place an arrowhead at Q and denote the vector by $\overrightarrow{PQ}$ (Fig. 17.22). We then call P the *tail* or *initial point* and Q the *head* or *terminal point* of the vector. The vector $\overrightarrow{PQ}$ can be thought of as representing the *displacement* of a point along the line segment from P to Q, that is, the path taken by a point as it moves from P to Q. Such vectors describe the relative positions of points. The length or magnitude of a vector $\overrightarrow{PQ}$ is denoted by the symbol $|\overrightarrow{PQ}|$; this notation is used because the length of a vector is in many ways similar to the absolute value of a real number.

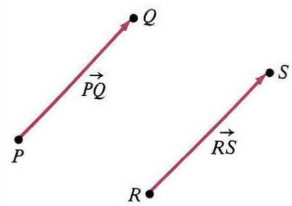


Figure 17.22 $\overrightarrow{PQ} = \overrightarrow{RS}$

Two vectors $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ are said to be *equal*, and we write $\overrightarrow{PQ} = \overrightarrow{RS}$, if they have the same length and direction (Fig. 17.22). This definition of equality enables us to move a vector from one position to another without changing it, as long as its length and direction are unaltered. Thus, the vectors shown in Fig. 17.23 are all equal to each other; in other words, they are the same vector in different positions. The *position vector* of a point P in the coordinate plane is the vector $\overrightarrow{OP}$ from the origin O to the point P (Fig. 17.24). Such vectors describe the positions of points relative to the origin. As suggested in Fig. 17.23, any vector **A** can be placed with its tail at the origin, and thereby becomes the position vector of the point P that lies at its head.

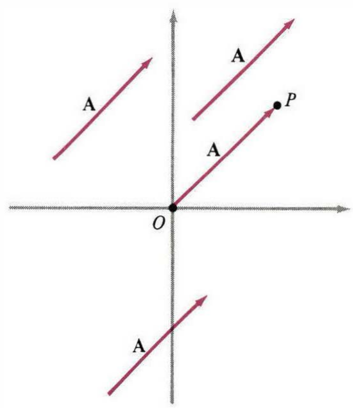


Figure 17.23 The same vector in different positions.

We shall discuss two algebraic operations on vectors. The first operation is that of adding two vectors to get another vector, and the second is that of multiplying a vector by a number to get another vector. In any discussion involving vectors, it is customary to refer to numbers as *scalars*, and this second operation is usually called *scalar multiplication*.

First, addition. Suppose a vector $\mathbf{A} = \overrightarrow{PQ}$ represents the displacement of a point along the line segment from P to Q. As shown in Fig. 17.25, when the dis-

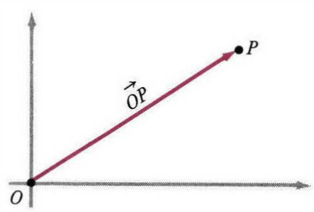


Figure 17.24 The position vector of **P**.

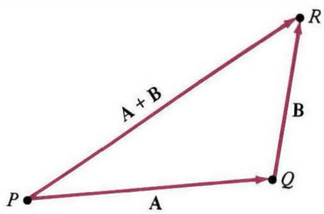


Figure 17.25 Addition.

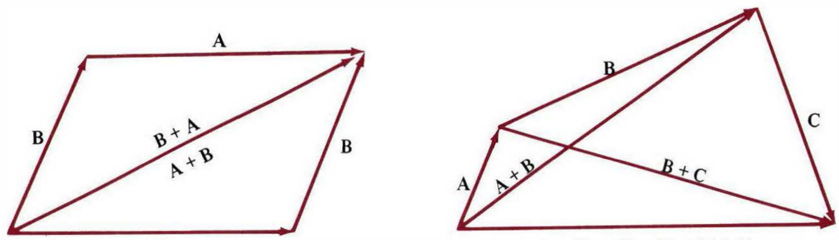


Figure 17.26 The commutative and associative laws.

placement $\mathbf{A} = \overrightarrow{PQ}$ is followed by a displacement $\mathbf{B} = \overrightarrow{QR}$, the final result is equivalent to the single displacement $\overrightarrow{PR}$. It is therefore natural to think of $\overrightarrow{PR}$ as the sum of $\overrightarrow{PQ}$ and $\overrightarrow{QR}$, and to write

$$\overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{QR}.$$

This suggests the definition we adopt for vector addition: If **A** and **B** are any two vectors, we add them as shown in the figure, by placing the tail of **B** at the head of **A**; the vector from the tail of **A** to the head of **B** is then written $\mathbf{A} + \mathbf{B}$ and called the *sum* of **A** and **B**. Figure 17.26 shows that addition is commutative and associative,

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \qquad \text{and} \qquad \mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}.$$

The associative law enables us to omit parentheses, writing $\mathbf{A} + \mathbf{B} + \mathbf{C}$ for $\mathbf{A} + (\mathbf{B} + \mathbf{C})$.

These ideas suggest another equivalent way to find the sum of two vectors **A** and **B**. If we place their tails together, as in Fig. 17.27, and form the parallelogram with **A** and **B** as adjacent sides, then $\mathbf{A} + \mathbf{B}$ is the vector from the common tail to the opposite vertex. This shows that our definition of addition is well suited to working with forces in physics; for if **A** and **B** are interpreted as two forces acting at their common tail, then it is known from experiment that $\mathbf{A} + \mathbf{B}$ is the *resultant force,* that is, the single force that produces the same effect as the two combined forces. This is called the *parallelogram rule,* for the addition of forces and also for vector addition.

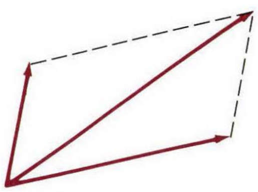


Figure 17.27

Example 3 Velocities are also combined by the parallelogram rule. For instance, a man in a canoe wishes to paddle across a river to the point on the other bank directly opposite to his starting point (Fig. 17.28). The river flows at 3 mi/h, and he can paddle at 6 mi/h. In what direction should he aim his canoe?

Solution His actual velocity is the vector sum of the velocity of the water and his velocity relative to the water. For this sum to be perpendicular to the bank, he must aim his canoe upstream at an angle θ for which $\sin \theta = \frac{3}{6} = \frac{1}{2}$, so $\theta = 30°$.

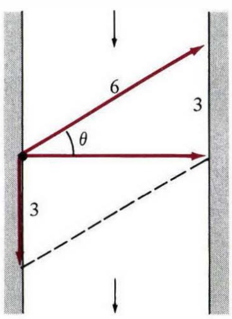


Figure 17.28

Now for scalar multiplication. If we add a vector **A** to itself, we obtain a vector in the same direction but twice as long, and it is natural to write this as $\mathbf{A} + \mathbf{A} = 2\mathbf{A}$. By a natural extension, if c is any real number, then $c\mathbf{A}$ is defined to be the vector which is $|c|$ times as long as **A**, in the same direction as **A** if c is positive and in the opposite direction if c is negative (Fig. 17.29). A vector of zero length is denoted by $\mathbf{0}$ and called the *zero vector*; this vector has no direction. Evidently $1 \cdot \mathbf{A} = \mathbf{A}$ and $0 \cdot \mathbf{A} = \mathbf{0}$. The properties

$$c(d\mathbf{A}) = (cd)\mathbf{A},$$
$$(c + d)\mathbf{A} = c\mathbf{A} + d\mathbf{A},$$
$$c(\mathbf{A} + \mathbf{B}) = c\mathbf{A} + c\mathbf{B}$$

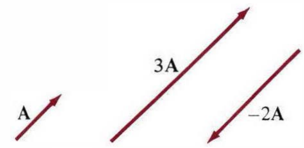


Figure 17.29 Scalar multiplication.

are valid and easy to establish, but we shall not pause to discuss them in detail. It is worth noting, however, that a proof of the last property for the case $c > 0$ is implicit in Fig. 17.30. Also, we agree that the factor c can be written on either side of the vector, $c\mathbf{A} = \mathbf{A}c$; we will not employ this clumsy usage very often, but it is occasionally convenient.

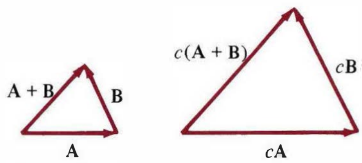


Figure 17.30

The vector $(-1) \cdot \mathbf{B}$ is written $-\mathbf{B}$; it is evidently a vector equal in length to $\mathbf{B}$ but having the opposite direction. Just as in elementary algebra, $\mathbf{A} + (-\mathbf{B})$ is written $\mathbf{A} - \mathbf{B}$. There is a simple geometric construction for $\mathbf{A} - \mathbf{B}$, resulting from the fact that $\mathbf{A} - \mathbf{B}$ is what must be added to $\mathbf{B}$ to give $\mathbf{A}$: When $\mathbf{A}$ and $\mathbf{B}$ are placed so that their tails coincide, then $\mathbf{A} - \mathbf{B}$ is the vector from the head of $\mathbf{B}$ to the head of $\mathbf{A}$ (Fig. 17.31).

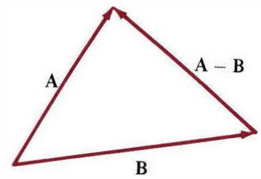


Figure 17.31 Subtraction.

Since the laws governing addition and scalar multiplication of vectors are identical with those that we know from elementary algebra, we are justified in using the familiar rules of algebra to solve linear equations involving vectors. The following examples illustrate the efficiency of these procedures for solving certain types of geometric problems.

Example 4 In Fig. 17.32, the ratio of the segment AP to the segment AB is t, where $0 < t < 1$. Express the vector $\mathbf{R}$ in terms of $\mathbf{A}$, $\mathbf{B}$, and t.

Solution The vector $\overrightarrow{AB}$ is $\mathbf{B} - \mathbf{A}$, and the vector $\overrightarrow{AP}$ is $t(\mathbf{B} - \mathbf{A})$. Since $\mathbf{R} = \mathbf{A} + \overrightarrow{AP}$, we have

$$\mathbf{R} = \mathbf{A} + t(\mathbf{B} - \mathbf{A}) = (1 - t)\mathbf{A} + t\mathbf{B}.$$

In particular, if P is the midpoint of AB so that $t = \frac{1}{2}$, then

$$\mathbf{R} = \tfrac{1}{2}\mathbf{A} + \tfrac{1}{2}\mathbf{B} = \tfrac{1}{2}(\mathbf{A} + \mathbf{B}).$$

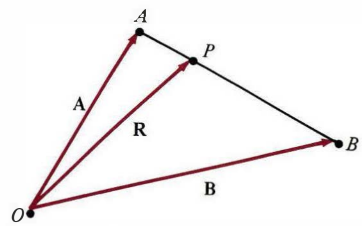


Figure 17.32

Example 5 Use vector methods to show that the three medians of any triangle intersect at a point which is two-thirds of the way from each vertex to the midpoint of the opposite side.*

Solution Let A, B, C be the vertices of a triangle (Fig. 17.33), and let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ be the vectors from an outside point O to these vertices. If M is the midpoint of BC, then $\overrightarrow{OM} = \frac{1}{2}(\mathbf{B} + \mathbf{C})$, $\overrightarrow{AM} = \overrightarrow{OM} - \mathbf{A} = \frac{1}{2}(\mathbf{B} + \mathbf{C}) - \mathbf{A}$, and if P is the point two-thirds of the way from A to M, then we have

$$\begin{aligned}\overrightarrow{OP} &= \mathbf{A} + \tfrac{2}{3}\overrightarrow{AM} = \mathbf{A} + \tfrac{2}{3}[\tfrac{1}{2}(\mathbf{B} + \mathbf{C}) - \mathbf{A}] \\ &= \tfrac{1}{3}\mathbf{A} + \tfrac{1}{3}(\mathbf{B} + \mathbf{C}) = \tfrac{1}{3}(\mathbf{A} + \mathbf{B} + \mathbf{C}). \qquad (1)\end{aligned}$$

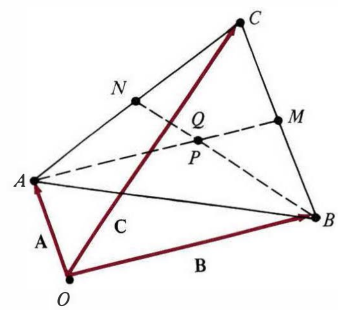


Figure 17.33

*Recall that a *median* of a triangle is a segment drawn from a vertex to the midpoint of the opposite side.

Similarly, if N is the midpoint of AC and Q is two-thirds of the way from B to N, then we see that

$$\begin{aligned}\overrightarrow{OQ} &= \mathbf{B} + \tfrac{2}{3}\overrightarrow{BN} = \mathbf{B} + \tfrac{2}{3}(\overrightarrow{ON} - \mathbf{B})\\ &= \mathbf{B} + \tfrac{2}{3}[\tfrac{1}{2}(\mathbf{A} + \mathbf{C}) - \mathbf{B}] = \tfrac{1}{3}(\mathbf{A} + \mathbf{B} + \mathbf{C}).\end{aligned} \tag{2}$$

Comparison of (1) and (2) shows that the two points P and Q coincide, and in the same way we obtain the same point yet again if we go two-thirds of the way from C to the midpoint of AB. This completes the proof. We can also draw our conclusion more elegantly by observing that since (1) is symmetric in the three vectors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, we clearly get the same point no matter which midpoint we start with.

A vector of length 1 is called a *unit vector*. It is easy to see that if we divide any nonzero vector $\mathbf{A}$ by its own length, we obtain a unit vector $\mathbf{A}/|\mathbf{A}|$ in the same direction. This simple fact is surprisingly useful.

When we are working with vectors in the coordinate plane, it is often convenient to use the standard unit vectors $\mathbf{i}$ and $\mathbf{j}$; as shown in Fig. 17.34, $\mathbf{i}$ points in the direction of the positive x-axis and $\mathbf{j}$ points in the direction of the positive y-axis. We have seen that any vector $\mathbf{A}$ in the xy-plane can be placed with its tail at the origin, and in this way becomes the position vector $\overrightarrow{OP}$ of the point P at its head. If P has coordinates a_1 and a_2, then the vectors $a_1\mathbf{i}$ and $a_2\mathbf{j}$ run from the origin to the points a_1 and a_2 on the axes, and by the parallelogram rule we have

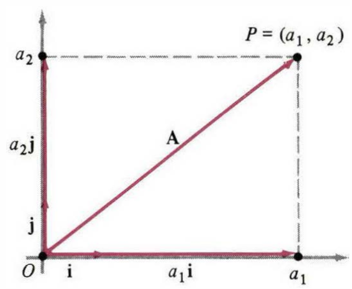


Figure 17.34

$$\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j}. \tag{3}$$

The number a_1 in (3) is called the *x-component* or **i**-*component* of the vector $\mathbf{A}$, and a_2 is called its *y-component* or **j**-*component*. These components are scalars, and should be distinguished from the vector components $a_1\mathbf{i}$ and $a_2\mathbf{j}$. By the Pythagorean theorem, we clearly have

$$|A| = \sqrt{a_1^2 + a_2^2}.$$

Formula (3) tells us that every vector in the plane is a *linear combination* of $\mathbf{i}$ and $\mathbf{j}$. The value of this formula is based on the fact that such linear combinations can be manipulated by the ordinary rules of algebra. Thus, if

$$\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j} \quad \text{and} \quad \mathbf{B} = b_1\mathbf{i} + b_2\mathbf{j},$$

then

$$\begin{aligned}\mathbf{A} + \mathbf{B} &= (a_1\mathbf{i} + a_2\mathbf{j}) + (b_1\mathbf{i} + b_2\mathbf{j})\\ &= (a_1 + b_1)\mathbf{i} + (a_2 + b_2)\mathbf{j}.\end{aligned}$$

Also,

$$c\mathbf{A} = c(a_1\mathbf{i} + a_2\mathbf{j}) = (ca_1)\mathbf{i} + (ca_2)\mathbf{j}.$$

Example 6 If $\mathbf{A} = 3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{B} = 2\mathbf{i} - 5\mathbf{j}$, find $|\mathbf{A}|$ and express $3\mathbf{A} - 4\mathbf{B}$ in terms of $\mathbf{i}$ and $\mathbf{j}$.

Solution Clearly, $|\mathbf{A}| = \sqrt{9 + 16} = 5$ and

$$\begin{aligned}3\mathbf{A} - 4\mathbf{B} &= 3(3\mathbf{i} + 4\mathbf{j}) - 4(2\mathbf{i} - 5\mathbf{j})\\ &= \mathbf{i} + 32\mathbf{j}.\end{aligned}$$

PROBLEMS

1 For each of the following pairs of vectors **A** and **B**, find $|\mathbf{A}|$, $3\mathbf{A} - 5\mathbf{B}$, and $6\mathbf{A} + \mathbf{B}$:
(a) $\mathbf{A} = \mathbf{i} - 3\mathbf{j}$, $\mathbf{B} = -2\mathbf{i} + 5\mathbf{j}$;
(b) $\mathbf{A} = -7\mathbf{i} + 2\mathbf{j}$, $\mathbf{B} = 3\mathbf{i} + 2\mathbf{j}$;
(c) $\mathbf{A} = -6\mathbf{j}$, $\mathbf{B} = -2\mathbf{i} + 3\mathbf{j}$;
(d) $\mathbf{A} = 3\mathbf{i} + 5\mathbf{j}$, $\mathbf{B} = 2\mathbf{i} - 8\mathbf{j}$.

2 For each of the following pairs of points P and Q, find the vector $\overrightarrow{PQ}$ in terms of **i** and **j**:
(a) $P = (-5, 0)$, $Q = (1, 3)$;
(b) $P = (-1, -4)$, $Q = (-2, 3)$;
(c) $P = (1, -5)$, $Q = (6, 4)$;
(d) $P = (0, 2)$, $Q = (3, -5)$.

3 For any three points P, Q, R in the plane, we have

$$\overrightarrow{PQ} + \overrightarrow{QR} + \overrightarrow{RP} = \mathbf{0}.$$

Why? Verify this for the special case $P = (2, -4)$, $Q = (-3, 5)$, $R = (-4, 0)$ by expressing each vector in terms **i** and **j** and carrying out the addition.

4 Find a vector in the same direction as $6\mathbf{i} - 2\mathbf{j}$ that has (a) three times its length; (b) half its length.

5 For each of the following vectors **A**, find two unit vectors parallel to **A**:
(a) $\mathbf{A} = 3\mathbf{i} - 4\mathbf{j}$;
(b) $\mathbf{A} = -5\mathbf{i} + 12\mathbf{j}$;
(c) $\mathbf{A} = 5\mathbf{i} - 7\mathbf{j}$;
(d) $\mathbf{A} = 24\mathbf{i} - 7\mathbf{j}$.

6 Find a vector of length 3 which has (a) the same direction as $5\mathbf{i} - 2\mathbf{j}$; (b) the opposite direction to $4\mathbf{i} + 5\mathbf{j}$.

7 Find two vectors of length 26 and slope $\frac{5}{12}$.

8 Find a unit vector which, if its tail is placed at the point $(4, 4)$ on $x^2 = 4y$, is normal to the curve and points toward the positive y-axis.

9 If **A** is a nonzero vector, we know that $\mathbf{A}/|\mathbf{A}|$ is a unit vector with the same direction as **A**. Use this fact to write down a vector that bisects the angle between two nonzero vectors **A** and **B** whose tails coincide.

10 Three vectors are drawn from the vertices of a triangle to the midpoints of the opposite sides. Show that the sum of these vectors is zero.

11 Use vector methods to show that the diagonals of a parallelogram bisect each other.

12 Use vector methods to show that a line from a vertex of a parallelogram to the midpoint of a nonadjacent side trisects a diagonal.

13 Solve the canoe problem in Example 3 when the current and canoe speeds are 2 and $2\sqrt{2} \cong 2.8$ mi/h, respectively.

14 If the velocity of the wind is $\mathbf{v}_w$ and an airplane flies with velocity $\mathbf{v}_a$ relative to the air, then the velocity of the plane relative to the ground is

$$\mathbf{v}_g = \mathbf{v}_w + \mathbf{v}_a.$$

The vectors $\mathbf{v}_a$ and $\mathbf{v}_g$ are called the *apparent velocity* and the *true velocity*, respectively.
(a) If the wind is blowing from the northeast at 60 mi/h and the pilot wishes to fly straight east at 600 mi/h, what should be the plane's apparent velocity?
(b) Repeat part (a) if the pilot wishes to fly southeast at 600 mi/h.

17.4 DERIVATIVES OF VECTOR FUNCTIONS. VELOCITY AND ACCELERATION

In Section 17.3 we became acquainted with the *algebra* of vectors. In the rest of this chapter we shall be interested in problems of motion, and this requires us to work with the *calculus* of vectors. When vectors and calculus are allowed to interact with each other, the result is a mathematical discipline of great power and efficiency for studying multidimensional problems of geometry and physics. This vector calculus—usually called *vector analysis*—is one of the major topics of advanced courses in calculus. In this chapter we can only introduce the subject and discuss a few of the classic applications, culminating in a treatment of Kepler's laws of planetary motion and Newton's law of universal gravitation.

We begin by pointing out the connection between vectors and the parametric equations of curves discussed in the first two sections of this chapter.

Suppose a point $P = (x, y)$ moves along a curve in the xy-plane, and suppose further that we know its position at any time t (Fig. 17.35). This means that the coordinates x and y are known as functions of the scalar variable t, so that

$$x = x(t) \qquad \text{and} \qquad y = y(t).$$

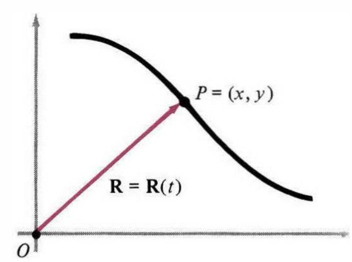


Figure 17.35

These are parametric equations for the path in terms of the time parameter t.* A more concise description of the motion is obtained by using the position vector of the moving point,

$$\mathbf{R} = \overrightarrow{OP} = x(t)\mathbf{i} + y(t)\mathbf{j}.$$

We can emphasize that $\mathbf{R}$ is a vector function of t by writing $\mathbf{R} = \mathbf{R}(t)$. Thus the study of a pair of parametric equations is equivalent to the study of a single vector function, and, as we shall see, the latter is often much more effective at revealing the essence of what is going on.

Just as in ordinary calculus, $\mathbf{R}(t)$ is said to be *continuous* at $t = t_0$ if

$$\lim_{t \to t_0} \mathbf{R}(t) = \mathbf{R}(t_0), \tag{1}$$

which means that $|\mathbf{R}(t) - \mathbf{R}(t_0)|$ can be made as small as we please by taking t sufficiently close to t_0. It follows easily from (1) that

$$\mathbf{R}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} \tag{2}$$

is continuous if and only if $x(t)$ and $y(t)$ are both continuous.

We define the derivative of the vector function $\mathbf{R}(t)$ exactly as might be expected. When t changes to $t + \Delta t$, the change in $\mathbf{R}$ is $\Delta\mathbf{R} = \mathbf{R}(t + \Delta t) - \mathbf{R}(t)$, and the *derivative* of $\mathbf{R}(t)$ with respect to t is defined as the limit

$$\frac{d\mathbf{R}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta\mathbf{R}}{\Delta t}. \tag{3}$$

That is, we divide the vector $\Delta\mathbf{R}$ by Δt and then find the limit of the new vector $\Delta\mathbf{R}/\Delta t$ as $\Delta t \to 0$. This vector will approach a limit if and only if its head approaches a limiting position, and this happens if and only if each of its components approaches a limit. It is clear that in terms of components we have

$$\begin{aligned}\frac{\Delta\mathbf{R}}{\Delta t} &= \frac{\mathbf{R}(t + \Delta t) - \mathbf{R}(t)}{\Delta t} \\ &= \frac{x(t + \Delta t) - x(t)}{\Delta t}\,\mathbf{i} + \frac{y(t + \Delta t) - y(t)}{\Delta t}\,\mathbf{j}.\end{aligned}$$

Thus, if the definition (3) is applied to (2), we see at once that $\mathbf{R}(t)$ is differentiable if and only if $x(t)$ and $y(t)$ are, and in this case

$$\frac{d\mathbf{R}}{dt} = \frac{dx}{dt}\,\mathbf{i} + \frac{dy}{dt}\,\mathbf{j}. \tag{4}$$

As in ordinary calculus, we often write $\mathbf{R}'(t)$ for $d\mathbf{R}/dt$ and $\mathbf{R}''(t)$ for $d^2\mathbf{R}/dt^2$.

Several familiar differentiation rules can now be extended to vector functions. One of the most important is this: If a vector function is multiplied by a scalar function, and if both can be differentiated, then their product can be differentiated according to the rule

$$\frac{d}{dt}(u\mathbf{R}) = u\,\frac{d\mathbf{R}}{dt} + \mathbf{R}\,\frac{du}{dt}.$$

*In Section 17.1 we wrote the parametric equations of a curve as $x = f(t)$, $y = g(t)$. However, it is more direct and convenient to use the same letter for the function as for the dependent variable, as we do here.

This is proved in just the same way as the product rule for two scalar functions. Also, the rule for the sum of two vector functions is just what we expect, the derivative of a constant vector function is the vector $\mathbf{0}$, and the chain rule is valid.

It is important to understand the meaning of the derivative $d\mathbf{R}/dt$ *as a vector*, and not merely in terms of its components as given by (4). To do this, we follow the geometric meaning of the various steps expressed in the definition (3). First, the change Δt of the independent variable t carries the position vector from $\mathbf{R}(t)$ to $\mathbf{R}(t + \Delta t)$, as shown in Fig. 17.36. The vector $\Delta\mathbf{R} = \mathbf{R}(t + \Delta t) - \mathbf{R}(t)$ is directed along the chord from the head of $\mathbf{R}(t)$ to the head of $\mathbf{R}(t + \Delta t)$. Dividing $\Delta\mathbf{R}$ by the scalar Δt changes its length and produces another vector $\Delta\mathbf{R}/\Delta t$ parallel to $\Delta\mathbf{R}$. Since the limiting direction of the chord as $\Delta t \to 0$ is the direction of the tangent, the derivative $d\mathbf{R}/dt$ is tangent to the path at the head of $\mathbf{R}$. As we know, every vector can be thought of as having its tail at the origin, but in Fig. 17.36 we place the tail of $d\mathbf{R}/dt$ at the head of $\mathbf{R}$ in order better to visualize what is happening.

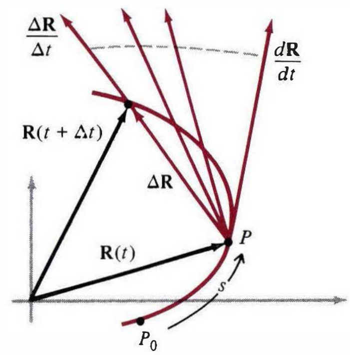


Figure 17.36

To interpret the length of the vector $d\mathbf{R}/dt$, let s be the length of the curve from a fixed point P_0 given by $t = t_0$ to the variable point P given by t, where $t \geq t_0$. By (4) we have

$$\left|\frac{d\mathbf{R}}{dt}\right| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \frac{\sqrt{dx^2 + dy^2}}{dt} = \frac{ds}{dt}. \tag{5}$$

Since t is time in the present discussion, the derivative ds/dt is the rate at which the moving point P traverses distance, that is, its speed.

These observations tell us that the vector $d\mathbf{R}/dt$ has as its direction and length the direction and speed of our moving point. It is therefore natural to adopt the following formal definitions. Just as in the case of one-dimensional motion, we define the *velocity* $\mathbf{v}$ of a moving point as the rate of change of its position,

$$\mathbf{v} = \frac{d\mathbf{R}}{dt},$$

and the *speed* v as the magnitude of the velocity,

$$v = |\mathbf{v}| = \left|\frac{d\mathbf{R}}{dt}\right|.$$

Example 1 If $\mathbf{R} = (4 \cos 2t)\mathbf{i} + (3 \sin 2t)\mathbf{j}$, find the path of the moving point, the velocity $\mathbf{v}$, and the points on the path where the speed v is greatest and least.

Solution The curve has parametric equations $x = 4 \cos 2t$, $y = 3 \sin 2t$, so the path is the ellipse shown in Fig. 17.37,

$$\frac{x^2}{16} + \frac{y^2}{9} = 1.$$

The point $P = (x, y)$ moves around this ellipse in the counterclockwise direction, as indicated by the arrows in the figure. The velocity is

$$\mathbf{v} = (-8 \sin 2t)\mathbf{i} + (6 \cos 2t)\mathbf{j}, \tag{6}$$

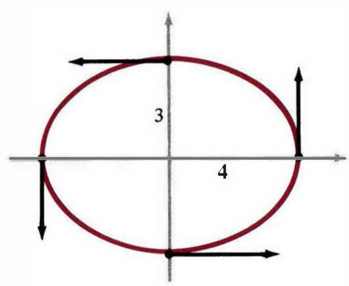


Figure 17.37

and the speed is

$$v = |\mathbf{v}| = (64 \sin^2 2t + 36 \cos^2 2t)^{1/2} = (28 \sin^2 2t + 36)^{1/2}.$$

It is clear from this formula for v that the smallest speed is 6, and this occurs when $\sin 2t = 0$, and by the given formula for **R**, this happens when P is at either end of the major axis. The greatest speed is 8, and this occurs when $\sin 2t = 1$, so that $\cos 2t = 0$, that is, at either end of the minor axis.

Just as the velocity **v** of our moving point is the rate of change of its position, the *acceleration* **a** is the rate of change of its velocity,

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{R}}{dt^2}.$$

Thus, our present concepts of velocity and acceleration are direct extensions of more limited versions of these concepts from our earlier studies of one-dimensional motion.

If the moving point P is the location of a moving physical object, and can therefore be thought of as a particle of mass m moving under the action of an applied force **F**, then *Newton's second law of motion* states that

$$\mathbf{F} = m\mathbf{a}. \tag{7}$$

This vector form of Newton's law shows that the force and acceleration vectors both have the same direction. Since we visualize the force **F** as being applied to the particle, so that its tail is at P, it is customary also to place the tail of **a** at P, as shown in Fig. 17.38. Both **F** and **a** usually point toward the concave side of the curve, but in exceptional cases they may be tangent to the curve.

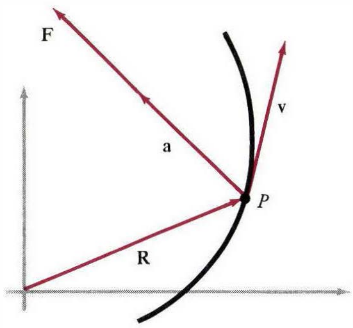


Figure 17.38 $\mathbf{F} = m\mathbf{a}$

Example 1 (continued) To find the acceleration **a** of the motion given by $\mathbf{R} = (4 \cos 2t)\mathbf{i} + (3 \sin 2t)\mathbf{j}$, we have only to differentiate the velocity (6) with respect to t,

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = (-16 \cos 2t)\mathbf{i} + (-12 \sin 2t)\mathbf{j}.$$

Since this can be written in the form

$$\begin{aligned}\mathbf{a} &= -4[(4 \cos 2t)\mathbf{i} + (3 \sin 2t)\mathbf{j}] \\ &= -4\mathbf{R},\end{aligned}$$

the acceleration vector is always directed toward the center of the elliptical path.

A simple but important situation is that in which a particle travels at constant speed around a circular path.

Example 2 *Uniform circular motion.* A particle of mass m moves counterclockwise around the circle $x^2 + y^2 = r^2$ with constant speed v. Find the acceleration of the particle and the force needed to produce this motion.

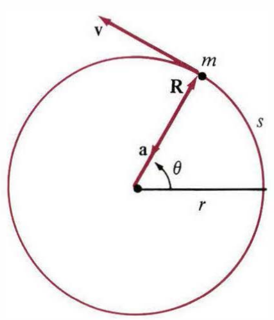


Figure 17.39

Solution By using the notation in Fig. 17.39, the path can be written as

$$\mathbf{R} = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j}, \tag{8}$$

with θ as the parameter. Since $s = r\theta$, we have

$$v = \frac{ds}{dt} = r\frac{d\theta}{dt},$$

and therefore $d\theta/dt = v/r$. This enables us to find the velocity and acceleration from (8) by using the chain rule:

$$\begin{aligned}\mathbf{v} &= \frac{d\mathbf{R}}{dt} = \frac{d\mathbf{R}}{d\theta} \cdot \frac{d\theta}{dt} \\ &= [(-r \sin \theta)\mathbf{i} + (r \cos \theta)\mathbf{j}] \cdot \frac{v}{r} \\ &= v[(-\sin \theta)\mathbf{i} + (\cos \theta)\mathbf{j}];\end{aligned}$$

and

$$\begin{aligned}\mathbf{a} &= \frac{d\mathbf{v}}{dt} = \frac{d\mathbf{v}}{d\theta} \cdot \frac{d\theta}{dt} \\ &= v[(-\cos \theta)\mathbf{i} + (-\sin \theta)\mathbf{j}] \cdot \frac{v}{r} \\ &= -\frac{v^2}{r}[(\cos \theta)\mathbf{i} + (\sin \theta)\mathbf{j}].\end{aligned}$$

By multiplying and dividing by r, it is easy to see that $\mathbf{a} = -(v^2/r^2)\mathbf{R}$. This tells us that the acceleration vector $\mathbf{a}$ points in toward the center of the circle and has magnitude

$$|\mathbf{a}| = \frac{v^2}{r^2}|\mathbf{R}| = \frac{v^2}{r}.$$

By Newton's law (7), the force $\mathbf{F}$ needed to produce this motion must point toward the center of the circle and have constant magnitude mv^2/r. Such a force is called a *centripetal force*.

It is obvious by now that the time t is a parameter of fundamental importance for studying the motion of a point P along a curved path. Another important parameter is the arc length s, measured along the curve from a fixed point P_0 to P, as shown in Fig. 17.40. We now consider $\mathbf{R}$ as a function of s and examine the meaning of the derivative $d\mathbf{R}/ds$. If P moves along the curve to Q when s changes to $s + \Delta s$, then

$$\frac{\Delta \mathbf{R}}{\Delta s} = \frac{\overrightarrow{PQ}}{\Delta s}$$

is a vector in the direction of the chord from P to Q whose length is

$$\frac{PQ}{\Delta s} = \frac{\text{chord}}{\text{arc}}.$$

When $\Delta s \to 0$, the direction of the chord approaches the direction of the tangent and the ratio of the chord to the arc approaches 1. Therefore the vector $\mathbf{T}$, which is defined by

$$\mathbf{T} = \frac{d\mathbf{R}}{ds} = \lim_{\Delta s \to 0} \frac{\Delta \mathbf{R}}{\Delta s},$$

is a vector of unit length which is tangent to the curve at P and points in the direction of increasing s. $\mathbf{T}$ is called the *unit tangent vector*.

To clarify our first use of $\mathbf{T}$, we recall that the formulas for the velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ in terms of their components are

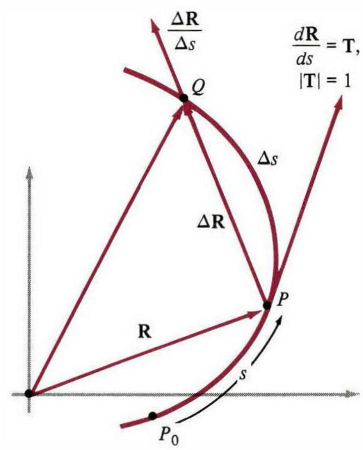


Figure 17.40

$$\mathbf{v} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} \qquad \text{and} \qquad \mathbf{a} = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j}. \tag{9}$$

These formulas are convenient for calculation, but they don't contribute much to our intuitive understanding of the nature of the vectors **v** and **a**. However, the chain rule enables us to write the velocity **v** in the form

$$\mathbf{v} = \frac{d\mathbf{R}}{dt} = \frac{d\mathbf{R}}{ds}\frac{ds}{dt} = \mathbf{T}\frac{ds}{dt}. \tag{10}$$

For the purpose of conveying insight, this formula is much superior to the first of formulas (9), because in (10) the direction of **v** is given by **T** and its magnitude is given by ds/dt, and the meaning of each is visible at a glance. Our main aim in the next two sections is to obtain a corresponding formula for the acceleration **a**.

Remark In most of our work we restrict ourselves to parametrized curves $\mathbf{R} = \mathbf{R}(t)$ that are *smooth,* in the sense that the derivative $\mathbf{R}'(t)$ is continuous and nonzero at every point. In principle, the continuity of $\mathbf{R}'(t)$ enables us to find s as a function of t from formula (5),

$$s = \int_{t_0}^{t} |\mathbf{R}'(t)|\, dt = \int_{t_0}^{t} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

And since

$$\frac{ds}{dt} = |\mathbf{R}'(t)| > 0,$$

the function $s = s(t)$ is strictly increasing and therefore has an inverse function $t = t(s)$. This permits us to introduce s as a parameter for the curve,

$$\mathbf{R} = \mathbf{R}(t) = \mathbf{R}[t(s)].$$

However, in most cases it is difficult or impossible to carry out these calculations. The integral for s may be hard to evaluate; and even if $s = s(t)$ is known explicitly, it may be hard to find the inverse function $t = t(s)$. Fortunately these difficulties are not a serious obstacle, because there is seldom any real need to have **R** expressed explicitly as a function of s. It is the *idea* of using arc length as a parameter that is important for understanding motion along curves—as in the preceding paragraph—and not the actual act of doing so in specific problems.

PROBLEMS

1 Give a geometric description of the locus of the head of **R** if $\mathbf{R} = \mathbf{A} + t\mathbf{B}$, where neither **A** nor **B** is **0** and **B** is not parallel to **A**. Draw a sketch.

2 What is the locus of the head of **R** if $\mathbf{R} = at\mathbf{i} + b(1 - t)\mathbf{j}$, where a and b are nonzero constants?

3 Show that the locus of the head of $\mathbf{R} = t\mathbf{i} + (mt + b)\mathbf{j}$ is the line $y = mx + b$.

4 What is the locus of the head of $\mathbf{R} = (t + 1)\mathbf{i} + (t^2 + 2t + 3)\mathbf{j}$?

In Problems 5–9, **R** is the position of a moving point at time t. In each case compute the velocity, acceleration, and speed.

5 $\mathbf{R} = (t^2 + 1)\mathbf{i} + (t - 1)\mathbf{j}$.

6 $\mathbf{R} = t^2\mathbf{i} + t^3\mathbf{j}$.

7 $\mathbf{R} = t\mathbf{i} + (t^3 - 3t)\mathbf{j}$.

8 $\mathbf{R} = (\cos 2t)\mathbf{i} + (\sin t)\mathbf{j}$.

9 $\mathbf{R} = (\tan t)\mathbf{i} + (\sec t)\mathbf{j}$.

10 If the position vector of a moving particle is $\mathbf{R} = (a \cos kt)\mathbf{i} + (b \sin kt)\mathbf{j}$, where a, b, k are positive con-

stants, then the particle moves on the ellipse $x^2/a^2 + y^2/b^2 = 1$. Show that $\mathbf{a} = -k^2\mathbf{R}$ and describe the force $\mathbf{F}$ that produces such a motion.

11 If the acceleration of a moving particle is $\mathbf{a} = a\mathbf{j}$, where a is a constant, find $\mathbf{R}$ by two successive integrations with respect to t and show that the path is a parabola, a straight line, or a single point.

12 If a moving particle is acted on by no force, so that $\mathbf{a} = \mathbf{0}$, show that the particle moves with constant speed along a straight line. This is *Newton's first law of motion.*

17.5 CURVATURE AND THE UNIT NORMAL VECTOR

In Section 17.4 we expressed the velocity $\mathbf{v}$ of our moving point P in terms of the unit tangent vector $\mathbf{T}$ shown in Fig. 17.41, where $\mathbf{T}$ was obtained as the derivative of the position vector $\mathbf{R}$ with respect to arc length s,

$$\mathbf{T} = \frac{d\mathbf{R}}{ds}.$$

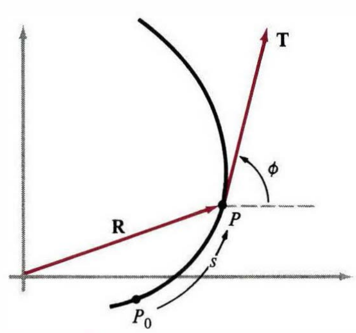


Figure 17.41

As our first step toward the general acceleration formula derived in Section 17.6, we must now analyze the derivative of $\mathbf{T}$ with respect to s, and this requires us to examine the purely geometric concept of the "curvature" of a curve.

If we consult our intuitive feelings about the notion of curvature, most of us will agree that a straight line does not curve at all; that is, it has zero curvature. Also, a circle has the same curvature at every point, and a small circle has greater curvature than a large one, as suggested in Fig. 17.42. In the case of a nonuniform curve like the one on the right, the curvature ought to be smaller where the curve is relatively straight and larger where the curve bends more sharply.

These opinions are based on the idea that curvature at a point ought to measure how rapidly the direction of a curve is changing at that point with respect to distance along the curve. Since direction is specified by the angle ϕ from the x-axis to the tangent line (Fig. 17.41), we consider this angle as a function of the arc length s and define the *curvature* k to be the rate of change of ϕ with respect to s,

$$k = \frac{d\phi}{ds}. \tag{1}$$

The curvature can be either positive or negative, and it may be zero in certain cases. Since $k > 0$ means that ϕ is increasing as s increases, it is clear that this means that the curve turns away to the left of the tangent as we move along the curve in the positive direction. Similarly, $k < 0$ means the curve turns away to the right of the tangent.

It is obvious from the definition (1) that the curvature of a straight line is zero, since ϕ does not change as we move along the line. In the case of a circle of radius a (Fig. 17.43), we have

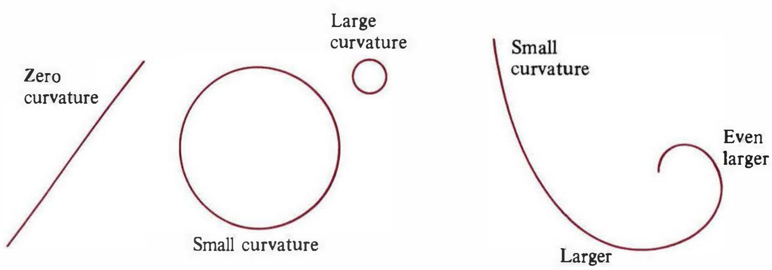


Figure 17.42 The meaning of curvature.

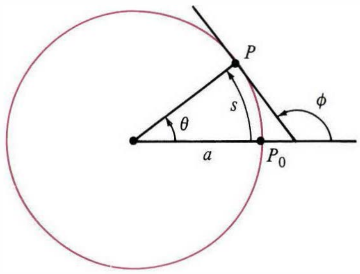


Figure 17.43

$$\phi = \theta + \frac{\pi}{2},$$

and by using the fact that $\theta = s/a$, we easily see that the curvature is

$$k = \frac{d\phi}{ds} = \frac{d\theta}{ds} = \frac{1}{a}. \tag{2}$$

Since the curvature of a circle is clearly constant, we can also obtain the result (2) by observing that a complete revolution amounts to a change in direction of 2π radians over a curve of length $2\pi a$, so

$$k = \frac{d\phi}{ds} = \frac{2\pi}{2\pi a} = \frac{1}{a}.$$

We note in passing that formula (2) shows that smaller circles have larger curvatures, as indicated in Fig. 17.42.

Apart from these very simple cases, the actual calculation of the curvature is carried out by various rather complicated formulas, depending on how the curve is defined.

The simplest situation is that in which the curve is the graph of a function $y = f(x)$. Since $\tan\phi = dy/dx$, we have

$$\phi = \tan^{-1}\frac{dy}{dx} \qquad \text{and} \qquad d\phi = \frac{d^2y/dx^2}{1 + (dy/dx)^2}\,dx.$$

Also, the expression $ds = \sqrt{dx^2 + dy^2}$ for the differential of arc length gives

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2} dx. \tag{3}$$

On dividing $d\phi$ by ds, we see that in this case the curvature is given by the formula

$$k = \frac{d\phi}{ds} = \frac{d^2y/dx^2}{[1 + (dy/dx)^2]^{3/2}}. \tag{4}$$

In Section 4.2 we used the sign of the second derivative d^2y/dx^2 to find out which direction the curve is bending, concave up or concave down. Formula (4) gives us this information and much more—it tells us precisely *how much* the curve is bending.

Example 1 Show that the curvature of the parabola $y = x^2$ is greatest at the vertex.

Solution We are familiar enough with the general shape of parabolas (Fig. 17.44) to accept this statement without difficulty, because the curve visibly flattens out as $|x| \to \infty$. To verify it by calculation, we use the fact that $dy/dx = 2x$ and $d^2y/dx^2 = 2$ to write

$$k = \frac{d^2y/dx^2}{[1 + (dy/dx)^2]^{3/2}} = \frac{2}{(1 + 4x^2)^{3/2}}.$$

It is clear that this quantity has its greatest value when $x = 0$, which is at the vertex, and also that $k \to 0$ as $|x| \to \infty$. To illustrate how quickly the curve flattens out as we move away from the vertex, we notice that at the point (2, 4)—which is fairly close to the vertex—we have

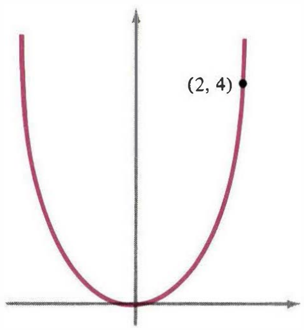


Figure 17.44

$$k = \frac{2}{(1 + 16)^{3/2}} < \frac{2}{16^{3/2}} = \frac{1}{32}.$$

Thus, at this point the parabola is flatter than a circle of radius 32, which is quite surprising.

If a curve is defined by parametric equations $x = x(t)$ and $y = y(t)$, then its curvature is computed from a slightly different formula. This time we start with

$$\phi = \tan^{-1} \frac{dy/dt}{dx/dt} \tag{5}$$

and

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]^{1/2} dt. \tag{6}$$

The calculations leading to the curvature formula are a bit more complicated because of the quotient in (5), and the result is

$$\begin{aligned} k = \frac{d\phi}{ds} &= \frac{(dx/dt)(d^2y/dt^2) - (dy/dt)(d^2x/dt^2)}{[(dx/dt)^2 + (dy/dt)^2]^{3/2}} \\ &= \frac{x'y'' - y'x''}{[(x')^2 + (y')^2]^{3/2}}. \end{aligned} \tag{7}$$

Students will notice that (7) includes (4) as a special case, when a curve $y = f(x)$ is thought of as a parametric curve $x = x$, $y = f(x)$, with x replacing t as the parameter.

There is a slight difficulty with signs that should be mentioned. By choosing the positive square root in both (3) and (6), we are assuming that the direction of increasing arc length s is the same as the direction in which the parameter increases. If this is not the case in applying (4) or (7) to a specific problem, then it is necessary to change the sign to get the actual curvature.

Example 2 Show that a circle of radius a has curvature $1/a$ by using the parametric equations $x = a \cos \theta$, $y = a \sin \theta$.

Solution We apply formula (7) with the understanding that primes denote derivatives with respect to θ. First we calculate

$$\begin{aligned} x' &= -a \sin \theta, & y' &= a \cos \theta, \\ x'' &= -a \cos \theta, & y'' &= -a \sin \theta. \end{aligned}$$

Formula (7) now gives

$$k = \frac{a^2 \sin^2 \theta + a^2 \cos^2 \theta}{(a^2 \sin^2 \theta + a^2 \cos^2 \theta)^{3/2}} = \frac{1}{a},$$

as expected.

Now that we understand the concept of curvature, we are ready to deal quickly with the main problem of this section, which is to analyze the derivative of the unit tangent vector $\mathbf{T}$ with respect to s.

We begin by observing that in terms of the slope angle ϕ (Fig. 17.45) we have

$$\mathbf{T} = \mathbf{i} \cos \phi + \mathbf{j} \sin \phi,$$

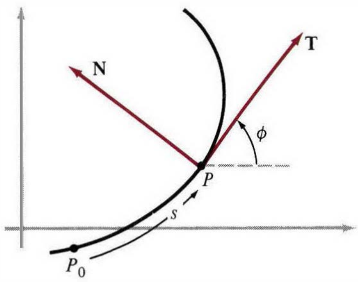


Figure 17.45

so

$$\frac{d\mathbf{T}}{d\phi} = -\mathbf{i}\sin\phi + \mathbf{j}\cos\phi. \tag{8}$$

The derivative (8) is clearly a unit vector, because its length is

$$\left|\frac{d\mathbf{T}}{d\phi}\right| = \sqrt{\sin^2\phi + \cos^2\phi} = 1.$$

Also, it is perpendicular to $\mathbf{T}$, because its slope is

$$\frac{\cos\phi}{-\sin\phi} = -\frac{1}{\tan\phi},$$

which is the negative reciprocal of the slope of $\mathbf{T}$. In fact, the derivative (8) is the *unit normal vector* $\mathbf{N}$ shown in the figure,

$$\frac{d\mathbf{T}}{d\phi} = \mathbf{N}, \tag{9}$$

where $\mathbf{N}$ is obtained by rotating $\mathbf{T}$ through an angle $\pi/2$ in the counterclockwise direction. This is established by comparing (8) with

$$\mathbf{N} = \mathbf{i}\cos\left(\phi + \frac{\pi}{2}\right) + \mathbf{j}\sin\left(\phi + \frac{\pi}{2}\right) = -\mathbf{i}\sin\phi + \mathbf{j}\cos\phi.$$

By using the chain rule together with (1) and (9), we now easily obtain the main result of this section,

$$\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}}{d\phi}\frac{d\phi}{ds} = \mathbf{N}k.$$

It should be clear at this stage why it was necessary to discuss curvature in analyzing the meaning of $d\mathbf{T}/ds$: Since $\mathbf{T}$ has constant length, only its direction changes as s varies, and this is what brings us to the curvature. We also point out that regardless of whether k is positive or negative, $\mathbf{N}k$ always points toward the concave side of the curve.

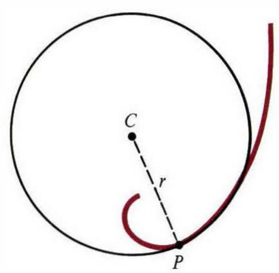


Figure 17.46 The circle of curvature.

Remark Let P be a point on a curve at which the curvature k is not zero, and draw the normal toward the concave side of the curve, as shown in Fig. 17.46. Every circle through P whose center lies on this normal will be tangent to the curve at P. That particular circle whose curvature is equal to $|k|$ is called the *circle of curvature*. Also, the center C of this circle is called the *center of curvature*, and its radius r is called the *radius of curvature*. We know from (2) that in the case of a circle, the radius is the reciprocal of the curvature, so the radius of curvature is given by the formula

$$r = \frac{1}{|k|} = \frac{[1 + (dy/dx)^2]^{3/2}}{|d^2y/dx^2|},$$

if the curve is the graph of a function $y = f(x)$. A similar formula holds for a parametric curve. As P moves along the given curve, the locus of the corresponding center of curvature C is called the *evolute* of the given curve.*

*Some remarkable applications of the theory of evolutes to cycloids are given in Section B.23 of the author's *Calculus Gems* (McGraw-Hill, 1992).

PROBLEMS

1 Find the curvature of the given curve as a function of x or t:
(a) $y = \sqrt{x}$;
(b) $y = \ln \sec x$;
(c) $y = x + \dfrac{1}{x}$;
(d) $x = e^t \sin t$, $y = e^t \cos t$;
(e) $x = t^2$, $y = \ln t$.

2 Find the radius of curvature of the given curve at the given point:
(a) $x = t^2$, $y = t^3$ at $t = 2$;
(b) $x = e^t$, $y = e^{-t}$ at $t = 0$;
(c) $y = \dfrac{1}{x}$ at $(1, 1)$;
(d) $x = \tan t$, $y = \cot t$ at $t = \pi/4$.

3 In each case find the largest value (if any) of the curvature:
(a) $y = \sin x$;
(b) $y = \frac{1}{3}x^3$;
(c) $y = \ln x$.

4 Carry out the details of establishing the parametric curvature formula (7).

5 Find the curvature of the circle $x^2 + y^2 = a^2$ by applying formula (4) separately to $y = \sqrt{a^2 - x^2}$ and $y = -\sqrt{a^2 - x^2}$. What difficulty arises, and how can it be fixed?

6 For the curve $y = e^x$, find the radius of curvature and the equation of the circle of curvature at the point (0, 1). Sketch the curve and this circle. Use the equation of the circle to calculate the values of dy/dx and d^2y/dx^2 at the point (0, 1), and verify that these derivatives have the same values there as the corresponding derivatives of $y = e^x$.

7 At what point on the curve $y = e^x$ is the radius of curvature smallest? What is this smallest radius?

8 We know that if $y = f(x)$ is a straight line, then $k = 0$. Show, conversely, that if $k = 0$, then $y = f(x)$ is a straight line.

9 Find the largest value of the radius of curvature on the first quadrant part of the hypocycloid of four cusps $x = a \cos^3 \theta$, $y = a \sin^3 \theta$. Where does the radius have this largest value?

10 By Problem 15 in Section 17.1, the equations

$$x = \cos \theta + \theta \sin \theta,$$
$$y = \sin \theta - \theta \cos \theta$$

represent the involute of a circle of radius 1. Find the curvature at any point.

11 Find the radius of curvature of the cycloid

$$x = a(\theta - \sin \theta),$$
$$y = a(1 - \cos \theta)$$

at any point.

12 (a) Sketch the ellipse $x = a \cos \theta$, $y = b \sin \theta$, where $0 < b < a$, and find its curvature k at an arbitrary point.
(b) Without calculating $dk/d\theta$, use the formula in (a) to show that k has its largest values at the ends of the major axis and its smallest values at the ends of the minor axis. Show that these values are a/b^2 and b/a^2, respectively. Notice that if $b = a$, then we have a circle, and both of these formulas give $k = 1/a$, as they should.†

***13** Let a be a positive number and consider the curve $y = x^a$ for $x > 0$. Show that the curvature approaches a finite limit as $x \to 0$ if $a \le \frac{1}{2}$ or $a = 1$ or $a \ge 2$, and only in these cases.

†For curves in general, a point where the curvature has a maximum or minimum value is called a *vertex*. By problem 12, an ellipse has four vertices. An ellipse is a special case of an *oval*, which is a convex closed curve whose parametric equations $x = x(t)$, $y = y(t)$ have continuous second derivatives. The famous *four vertex theorem* of differential geometry states that every oval has at least four vertices.

17.6 TANGENTIAL AND NORMAL COMPONENTS OF ACCELERATION

Consider a moving particle whose position at time t is given by the parametric equations $x = x(t)$ and $y = y(t)$. The position vector of this particle is $\mathbf{R} = x\mathbf{i} + y\mathbf{j}$, and its velocity and acceleration are

$$\mathbf{v} = \frac{d\mathbf{R}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}, \qquad \mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j}. \tag{1}$$

Unfortunately, the $\mathbf{i}$- and $\mathbf{j}$-components of these vectors have no physical meaning, because they depend on the coordinate system, and the choice of the coordinate system is arbitrary; it is not determined by the intrinsic nature of the motion itself. However, we saw in Section 17.4 that the velocity can also be written as

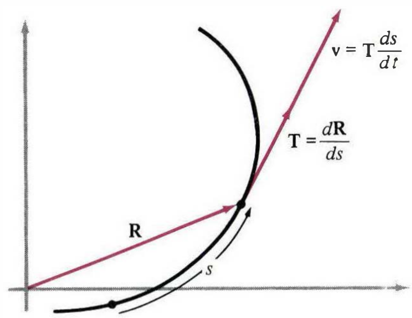


Figure 17.47

$$\mathbf{v} = \frac{d\mathbf{R}}{dt} = \frac{d\mathbf{R}}{ds}\frac{ds}{dt}$$

or

$$\mathbf{v} = \mathbf{T}\,\frac{ds}{dt}, \tag{2}$$

where **T** is the unit tangent vector (Fig. 17.47). This expression for the velocity *does* have physical meaning, because **T** gives the direction of the motion and ds/dt gives its magnitude, the speed.

To obtain a similar revealing expression for the acceleration, we differentiate (2) with respect to t,

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \mathbf{T}\,\frac{d^2s}{dt^2} + \frac{ds}{dt}\,\frac{d\mathbf{T}}{dt}. \tag{3}$$

By Section 17.5 we know that

$$\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds}\,\frac{ds}{dt} = \frac{d\mathbf{T}}{d\phi}\,\frac{d\phi}{ds}\,\frac{ds}{dt}$$

$$= \mathbf{N}k\,\frac{ds}{dt}, \tag{4}$$

where k is the curvature and **N** is the unit normal vector shown in Fig. 17.48. When (4) is substituted in (3) we get our fundamental result,

$$\mathbf{a} = \mathbf{T}\,\frac{d^2s}{dt^2} + \mathbf{N}k\left(\frac{ds}{dt}\right)^2. \tag{5}$$

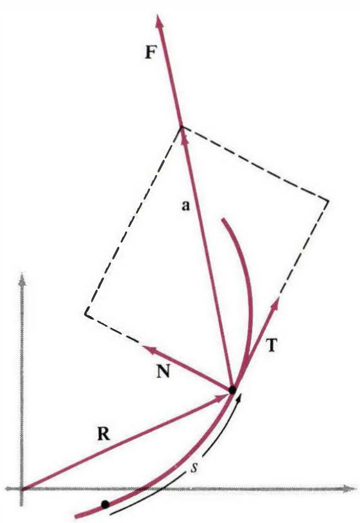


Figure 17.48

This is an important equation in mechanics. The vectors **T** and **N** serve as reference unit vectors much like **i** and **j**. They enable us to resolve the acceleration into two "natural" components, in the direction of the motion and normal to this direction, in contrast to the arbitrary components given by the second of equations (1). The *tangential component*, d^2s/dt^2, is simply the derivative of the speed $v = ds/dt$ of the particle along its path. The *normal component*, $k(ds/dt)^2 = kv^2$, has magnitude

$$|k|v^2 = \frac{v^2}{r}, \tag{6}$$

where r is the radius of curvature. It is clear from (5) that when $k \neq 0$ and the particle is actually moving, the acceleration is always directed toward the concave side of the curve.

Of course, the great importance of the acceleration lies in the fact that when a particle of mass m is acted on by a force **F**, it moves in accordance with Newton's second law of motion $\mathbf{F} = m\mathbf{a}$. The vectors **F** and **a** therefore have the same direction, as shown in the figure, and this fits with our intuitive understanding that when a force changes the direction of a moving particle, it pulls the particle away from the direction of the tangent toward the concave side of the path.

Since the curvature k is available whenever the curve is given in parametric form, the tangential and normal components of acceleration can be calculated from (5). However, it is often more efficient to use the following procedure. The acceleration vector is the same whether it is expressed in terms of **i**- and **j**-components or tangential and normal components, so

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} = a_t\mathbf{T} + a_n\mathbf{N},$$

where the a's have the obvious meanings. This tells us that

$$|\mathbf{a}|^2 = a_x^2 + a_y^2 = a_t^2 + a_n^2,$$

so

$$a_n = \sqrt{|\mathbf{a}|^2 - a_t^2} \tag{7}$$

where $a_t = d^2s/dt^2$.

Example 1 If a particle moves along the curve whose parametric equations are

$$x = \cos t + t \sin t, \quad y = \sin t - t \cos t,$$

find the tangential and normal components of acceleration.

Solution The position vector is

$$\mathbf{R} = (\cos t + t \sin t)\mathbf{i} + (\sin t - t \cos t)\mathbf{j},$$

so

$$\begin{aligned}\mathbf{v} &= \frac{d\mathbf{R}}{dt}\\ &= (-\sin t + t \cos t + \sin t)\mathbf{i} + (\cos t + t \sin t - \cos t)\mathbf{j}\\ &= (t \cos t)\mathbf{i} + (t \sin t)\mathbf{i}\end{aligned}$$

and

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = (-t \sin t + \cos t)\mathbf{i} + (t \cos t + \sin t)\mathbf{j}.$$

The speed $v = ds/dt$ is given by

$$\frac{ds}{dt} = |\mathbf{v}| = \sqrt{(t \cos t)^2 + (t \sin t)^2} = t,$$

so the tangential component of acceleration is

$$a_t = \frac{d^2s}{dt^2} = \frac{d}{dt}t = 1.$$

The normal component of acceleration can be computed directly, by finding k and using $a_n = k(ds/dt)^2$. However, it is easier to use (7), which gives

$$a_n = \sqrt{|\mathbf{a}|^2 - a_t^2}$$
$$= \sqrt{(-t \sin t + \cos t)^2 + (t \cos t + \sin t)^2 - 1} = t.$$

Example 2 If a particle of mass m moves around a circular path of radius r with constant speed $v = ds/dt$, then d^2s/dt^2 is zero. By equations (5) and (6), the acceleration is directed toward the center of the circle and has magnitude v^2/r. Further, the centripetal force acting on the particle has magnitude

$$F_1 = \frac{mv^2}{r}. \tag{8}$$

Thus, for an automobile going around a given unbanked curve, it takes four times as much normal force between the tires and the road to "hold the road" at 60 mi/h as at 30 mi/h, and the required force is doubled again if the radius is halved. These are the results about uniform circular motion that we obtained in Example 2 of Section 17.4.

Now suppose that our particle is an artificial satellite in a circular orbit around the earth, as shown in Fig. 17.49. If M is the mass of the earth, then Newton's law of gravitation tells us that the force of attraction which the earth exerts on the satellite has magnitude

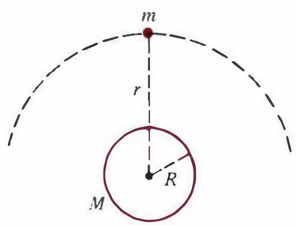


Figure 17.49

$$F_2 = G\frac{Mm}{r^2}, \tag{9}$$

where r is the distance from the satellite to the center of the earth and G is a constant of proportionality called the constant of gravitation. We know that the weight of the satellite is the force which gravity exerts on it at the surface of the earth, and this is mg. Therefore, if R denotes the radius of the earth, then $F_2 = mg$ when $r = R$, so (9) tells us that

$$mg = G\frac{Mm}{R^2} \quad \text{or} \quad GM = gR^2.$$

This enables us to write (9) in the more convenient form

$$F_2 = \frac{gR^2m}{r^2}. \tag{10}$$

For a satellite in stable circular orbit, the centripetal force is precisely equal to the gravitational force, so $F_1 = F_2$ and

$$v^2 = \frac{gR^2}{r}. \tag{11}$$

This formula gives the speed at which a satellite must move in order to maintain a circular orbit at a specified distance r from the center of the earth.

We make two observations about formula (11). First, if our satellite is moving in a circular orbit at a relatively low altitude above the surface of the earth, then $r \cong R$ and $v \cong \sqrt{gR}$. This orbital speed is approximately 5 mi/s, which should be compared with the escape speed of $\sqrt{2gR}$ or 7 mi/s that we calculated in Example 3 of Section 5.5.

Second, we consider a communications relay satellite that is placed in a circular orbit around the earth and has a period of revolution of $T = 24$ hours. This is a so-called *synchronous orbit,* in which the satellite moves with the turning earth and appears to hang motionless in the sky. If r is the radius of this orbit, then the orbital speed is $v = 2\pi r/T$, and when this is substituted in (11) we find that

$$r^3 = \frac{gR^2T^2}{4\pi^2}.$$

With suitable adjustments applied to the values $g = 32$ ft/s^2, $R = 4000$ mi, and $T = 24$ h, we easily find that r is approximately 26,000 mi, which means that the satellite must be about 22,000 mi above the surface of the earth. Such satellites were first conceived in 1945 by the famous science fiction writer Arthur C. Clarke, and are the crucial links in our present-day worldwide television communications.

Of course, the ideas discussed in this example assume a circular path, which is approximately true for some satellites. We shall give a detailed treatment of elliptical orbits in Section 17.7.

PROBLEMS

1 In Example 1, find a_n the other way, by first calculating k and then using $a_n = k(ds/dt)^2$.

In Problems 2–7, find the velocity and acceleration vectors, then find the speed and the tangential and normal components of the acceleration.

2 $\mathbf{R} = (2t - 5)\mathbf{i} + (t^2 + 3)\mathbf{j}$.

3 $\mathbf{R} = a \cos \omega t\, \mathbf{i} + a \sin \omega t\, \mathbf{j}$, where a and ω are positive constants.

4 $\mathbf{R} = \cos t^2\, \mathbf{i} + \sin t^2\, \mathbf{j}$.

5 $\mathbf{R} = e^t \cos t\, \mathbf{i} + e^t \sin t\, \mathbf{j}$.

6 $\mathbf{R} = t \cos t\, \mathbf{i} + t \sin t\, \mathbf{j}$.

7 $\mathbf{R} = 2 \ln (t^2 + 1)\mathbf{i} + (2t - 4 \tan^{-1} t)\mathbf{j}$.

In Problems 8 and 9, find the normal component of the acceleration for the given values of t.

8 $\mathbf{R} = a \cos t\, \mathbf{i} + b \sin t\, \mathbf{j}$; $t = 0, \pi/2$.

9 $\mathbf{R} = t\, \mathbf{i} + \sin t\, \mathbf{j}$; $t = 0, \pi/2$.

10 Deduce from equation (5) that
(a) the path of a moving particle will be a straight line if the normal component of acceleration is zero;
(b) if the speed of a moving particle is constant, then the force is always directed along the normal;
(c) if the force acting on a moving particle is always directed along the normal, then the speed is constant.

11 A road has the shape of the parabola $120y = x^2$. A truck is loaded in such a way that it will tip over if the normal component of its acceleration exceeds 30. What speeds will guarantee disaster for the truck as it swings around the vertex of the parabola?

12 If a particle moves along a path whose curvature k is never zero, how must the speed be adjusted if the normal component of the acceleration is to be held at a constant magnitude?

17.7 KEPLER'S LAWS AND NEWTON'S LAW OF GRAVITATION

As we know, Isaac Newton conceived the basic ideas of calculus in the years 1665 and 1666 (at age 22 and 23) for the purpose of helping him to understand the movements of the planets against the background of the fixed stars. In order to appreciate what was involved in this achievement, we briefly recall the main stages in the development of astronomical thinking up to his time.

The ancient Greeks constructed an elaborate mathematical model to account for the complicated movements of the sun, moon, and planets as viewed from the earth. A combination of uniform circular motions was used to describe the motion of each body about the earth. It was very natural for them—as it is for all people—to adopt the geocentric point of view that the earth is fixed at the center of the universe and everything else moves around it. Also, they were in-

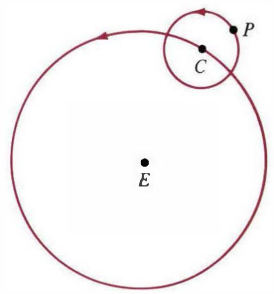


Figure 17.50 An epicycle.

fluenced by the semimystical Pythagorean belief that nothing but motion at constant speed in a perfect circle is worthy of a celestial body.

In this Greek model, each planet P moves uniformly around a small circle (called an *epicycle*) with center C, and at the same time C moves uniformly around a larger circle centered at the earth E, as shown in Fig. 17.50. The radius of each circle and the angular speeds of P and C around the centers C and E are chosen to match the observed motion of the planet as closely as possible. This theory of epicycles was given its definitive form in Ptolemy's massive treatise *Almagest* in the second century A.D., and the theory itself is called the *Ptolemaic system.*

The next great step forward was taken by the Polish astronomer Copernicus. Shortly before his death in 1543, when he was presumably almost beyond the reach of a wrathful Church, he at last allowed the publication of his heretical book, *On the Revolution of the Celestial Spheres.* This work changed the Ptolemaic point of view by placing the sun, instead of the earth, at the center of each primary circle. Nevertheless, this *heliocentric system* was of much greater cultural than scientific importance. It enlarged the consciousness of many educated Europeans by giving them a better understanding of their place in the scheme of things, but it also kept the clumsy machinery of Ptolemy's circles whose centers move around on other circles.

It was Johannes Kepler (1571–1630) who finally eliminated this jumble of circles. Kepler was the assistant of the wealthy Danish astronomer Tycho Brahe, and when Brahe died in 1601, Kepler inherited the great masses of raw data they had accumulated on the positions of the planets at various times. Kepler worked incessantly on this material for 20 years, and at last succeeded in distilling from it his three beautifully simple laws of planetary motion, which were the climax of thousands of years of purely observational astronomy:

1 The orbit of each planet is an ellipse with the sun at one focus.
2 The line segment joining a planet to the sun sweeps out equal areas in equal times. See Fig. 17.51.
3 The *square* of the period of revolution of a planet is proportional to the *cube* of the semimajor axis of the planet's elliptical orbit. That is, if T is the time required for a planet to make one complete revolution about the sun and a is the semimajor axis shown in the figure, then the ratio T^2/a^3 is the same for all planets in the solar system.

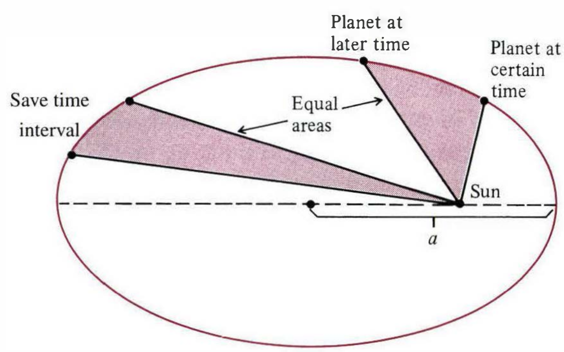


Figure 17.51 Kepler's second law.

From Kepler's point of view, these were empirical statements that fitted the data, and he had no idea of why they might be true or how they might be related to one another. In short, there was no theory to provide a context within which they could be understood.

Newton created such a theory. In the 1660s he discovered how to derive the inverse square law from Kepler's laws by mathematical reasoning, and also how to derive Kepler's laws from the inverse square law. We recall that *Newton's inverse square law of universal gravitation* states that any two particles of matter in the universe attract each other with a force directed along the line between them and of magnitude

$$G\frac{Mm}{r^2}, \tag{1}$$

where M and m are the masses of the particles, r is the distance between them, and G is a constant of nature called the gravitational constant. With this simple, clean, clear law as the unifying principle of his thinking, Newton published his theory of gravitation in 1687 in his *Principia Mathematica*. In this one book—perhaps the greatest of all scientific treatises—his success in using mathematical methods to explain the most diverse natural phenomena was so profound and far-reaching that he essentially created the sciences of physics and astronomy where only a handful of disconnected observations and simple inferences had existed before. These achievements launched the modern age of science and technology and radically altered the direction of human history.

We now derive Kepler's laws of planetary motion from Newton's law of gravitation, and to this end we discuss the motion of a small particle of mass m (a planet) under the attraction of a fixed large particle of mass M (the sun).

For problems involving a moving particle in which the force acting on it is always directed along the line from the particle to a fixed point, it is usually simplest to resolve the velocity, acceleration, and force into components along and perpendicular to this line. We therefore place the fixed particle M at the origin of a polar coordinate system (Fig. 17.52) and express the position vector of the moving particle m in the form

$$\mathbf{R} = r\mathbf{u}_r, \tag{2}$$

where $\mathbf{u}_r$ is the unit vector in the direction of $\mathbf{R}$. It is clear that

$$\mathbf{u}_r = \mathbf{i}\cos\theta + \mathbf{j}\sin\theta, \tag{3}$$

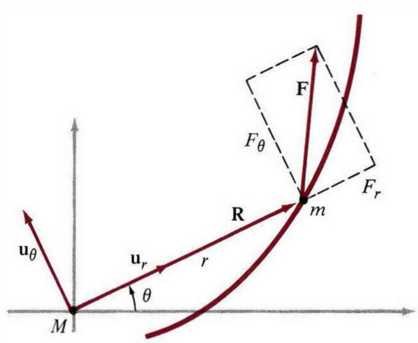


Figure 17.52

and also that the corresponding unit vector $\mathbf{u}_\theta$, perpendicular to $\mathbf{u}_r$ in the direction of increasing θ, is given by

$$\mathbf{u}_\theta = -\mathbf{i}\sin\theta + \mathbf{j}\cos\theta. \tag{4}$$

It is easy to see by componentwise differentiation that

$$\frac{d\mathbf{u}_r}{d\theta} = \mathbf{u}_\theta \qquad \text{and} \qquad \frac{d\mathbf{u}_\theta}{d\theta} = -\mathbf{u}_r. \tag{5}$$

Thus, differentiating $\mathbf{u}_r$ and $\mathbf{u}_\theta$ with respect to θ has the effect of rotating these vectors 90° in the counterclockwise direction. We shall need the derivatives of $\mathbf{u}_r$ and $\mathbf{u}_\theta$ with respect to the time t. By means of the chain rule we at once obtain the formulas

$$\frac{d\mathbf{u}_r}{dt} = \frac{d\mathbf{u}_r}{d\theta}\frac{d\theta}{dt} = \mathbf{u}_\theta\frac{d\theta}{dt} \qquad \text{and} \qquad \frac{d\mathbf{u}_\theta}{dt} = \frac{d\mathbf{u}_\theta}{d\theta}\frac{d\theta}{dt} = -\mathbf{u}_r\frac{d\theta}{dt}, \tag{6}$$

which are essential for computing the velocity and acceleration vectors $\mathbf{v}$ and $\mathbf{a}$.

Direct calculation from (2) now yields

$$\mathbf{v} = \frac{d\mathbf{R}}{dt} = r\frac{d\mathbf{u}_r}{dt} + \mathbf{u}_r\frac{dr}{dt} = r\frac{d\theta}{dt}\mathbf{u}_\theta + \frac{dr}{dt}\mathbf{u}_r \tag{7}$$

and

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{dr}{dt}\frac{d\theta}{dt}\mathbf{u}_\theta + r\frac{d^2\theta}{dt^2}\mathbf{u}_\theta + r\frac{d\theta}{dt}\frac{d\mathbf{u}_\theta}{dt} + \frac{d^2r}{dt^2}\mathbf{u}_r + \frac{dr}{dt}\frac{d\mathbf{u}_r}{dt};$$

and by keeping formulas (6) in mind and rearranging, the latter equation can be written in the form

$$\mathbf{a} = \left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right)\mathbf{u}_\theta + \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\mathbf{u}_r. \tag{8}$$

If the force $\mathbf{F}$ acting on m is written as

$$\mathbf{F} = F_\theta\mathbf{u}_\theta + F_r\mathbf{u}_r, \tag{9}$$

then, from (8) and (9) and Newton's second law of motion $m\mathbf{a} = \mathbf{F}$, we get

$$m\left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right) = F_\theta \qquad \text{and} \qquad m\left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right] = F_r. \tag{10}$$

These differential equations govern the motion of the particle m and are called the *equations of motion*; they are valid regardless of the nature of the force $\mathbf{F}$. Our next task is to extract the desired conclusions from these equations by making suitable assumptions about the direction and magnitude of $\mathbf{F}$.

CENTRAL FORCES AND KEPLER'S SECOND LAW

$\mathbf{F}$ is called a *central force* if it has no component perpendicular to $\mathbf{R}$, that is, if $F_\theta = 0$. Under this assumption the first of equations (10) becomes

$$r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} = 0.$$

On multiplying through by r, we obtain

$$r^2\frac{d^2\theta}{dt^2} + 2r\frac{dr}{dt}\frac{d\theta}{dt} = 0$$

or

$$\frac{d}{dt}\left(r^2 \frac{d\theta}{dt}\right) = 0,$$

so

$$r^2 \frac{d\theta}{dt} = h \tag{11}$$

for some constant h. We shall assume that h is positive, or equivalently that $d\theta/dt$ is positive, which evidently means that m is moving around the origin in a counterclockwise direction.

If $A = A(t)$ is the area swept out by $\mathbf{R}$ from some fixed position of reference so that $dA = \frac{1}{2}r^2\, d\theta$, then (11) implies that

$$dA = \frac{1}{2}\left(r^2 \frac{d\theta}{dt}\right) dt = \tfrac{1}{2}h\, dt.$$

On integrating this from t_1 to t_2, we get

$$A(t_2) - A(t_1) = \tfrac{1}{2}h(t_2 - t_1). \tag{12}$$

This yields Kepler's second law: The line segment joining the sun to a planet sweeps out equal areas in equal intervals of time.

CENTRAL GRAVITATIONAL FORCES AND KEPLER'S FIRST LAW

We now specialize even further, and assume that $\mathbf{F}$ is a central attractive force whose magnitude is given by the inverse square law (1), so that

$$F_r = -G\frac{Mm}{r^2}. \tag{13}$$

If we write (13) in the slightly simpler form

$$F_r = -\frac{km}{r^2}$$

where $k = GM$, then the second of equations (10) becomes

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -\frac{k}{r^2}. \tag{14}$$

The next step in this line of thought is difficult to motivate, because it involves considerable technical ingenuity, but we will try. Our purpose is to use the differential equation (14) to obtain the equation of the orbit in the polar form $r = f(\theta)$, so we want to eliminate t from (14) and consider θ as the independent variable. Also, we want r to be the dependent variable, but if (11) is used to put (14) in the form

$$\frac{d^2r}{dt^2} - \frac{h^2}{r^3} = -\frac{k}{r^2}, \tag{15}$$

then the presence of powers of $1/r$ suggests that it might be temporarily convenient to introduce a new dependent variable $z = 1/r$.

To accomplish these various aims, we must first express d^2r/dt^2 in terms of $d^2z/d\theta^2$, by calculating

$$\frac{dr}{dt} = \frac{d}{dt}\left(\frac{1}{z}\right) = -\frac{1}{z^2}\frac{dz}{dt} = -\frac{1}{z^2}\frac{dz}{d\theta}\frac{d\theta}{dt}$$

$$= -\frac{1}{z^2}\frac{dz}{d\theta}\frac{h}{r^2} = -h\frac{dz}{d\theta}$$

and

$$\frac{d^2r}{dt^2} = -h\frac{d}{dt}\left(\frac{dz}{d\theta}\right) = -h\frac{d}{d\theta}\left(\frac{dz}{d\theta}\right)\frac{d\theta}{dt}$$

$$= -h\frac{d^2z}{d\theta^2}\frac{h}{r^2} = -h^2z^2\frac{d^2z}{d\theta^2}.$$

When the latter expression is inserted in (15), and r is replaced by $1/z$, we get

$$-h^2z^2\frac{d^2z}{d\theta^2} - h^2z^3 = -kz^2$$

or

$$\frac{d^2z}{d\theta^2} + z = \frac{k}{h^2}. \tag{16}$$

To solve this equation, we observe that, except for the constant term on the right, it is the differential equation of simple harmonic motion discussed in Section 9.6. To eliminate the constant term, we put

$$w = z - \frac{k}{h^2},$$

so that $d^2w/d\theta^2 = d^2z/d\theta^2$ and (16) becomes

$$\frac{d^2w}{d\theta^2} + w = 0.$$

As we know, the general solution of this familiar equation is

$$w = A\sin\theta + B\cos\theta,$$

so

$$z = A\sin\theta + B\cos\theta + \frac{k}{h^2}. \tag{17}$$

For the sake of simplicity, we now shift the direction of the polar axis in such a way that r is minimal (that is, m is closest to the origin) when $\theta = 0$. This means that z is to be maximal in this direction, so

$$\frac{dz}{d\theta} = 0 \qquad \text{and} \qquad \frac{d^2z}{d\theta^2} < 0$$

when $\theta = 0$. By calculating $dz/d\theta$ and $d^2z/d\theta^2$ from (17), we easily see that these conditions imply that $A = 0$ and $B > 0$. If we now replace z by $1/r$, then (17) can be written as

$$r = \frac{1}{k/h^2 + B\cos\theta} = \frac{h^2/k}{1 + (Bh^2/k)\cos\theta};$$

and if we put $e = Bh^2/k$, then our equation for the orbit becomes

$$r = \frac{h^2/k}{1 + e\cos\theta}, \tag{18}$$

where e is a positive constant.

We recall from Section 16.3 that (18) is the polar equation of a conic section with focus at the origin and vertical directrix to the right; and furthermore, that this conic section is an ellipse, a parabola, or a hyperbola according as $e < 1$, $e = 1$, or $e > 1$. Since the planets remain in the solar system and do not move infinitely far away from the sun, the ellipse is the only possibility. This yields Kepler's first law: The orbit of each planet is an ellipse with the sun at one focus.*

KEPLER'S THIRD LAW

We now restrict ourselves to the case in which m has an elliptic orbit (Fig. 17.53) whose polar and rectangular equations are (18) and

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

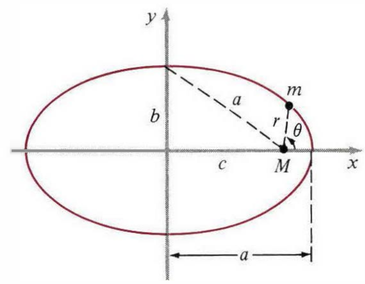


Figure 17.53

We know that $e = c/a$ and $c^2 = a^2 - b^2$, so $e^2 = (a^2 - b^2)/a^2$ and

$$b^2 = a^2(1 - e^2). \tag{19}$$

In astronomy the semimajor axis a of the elliptical orbit is called the *mean distance,* because it is one-half the sum of the least and greatest values of r. These are the values of r corresponding to $\theta = 0$ and $\theta = \pi$ in (18), so by (18) and (19) we have

$$a = \frac{1}{2}\left(\frac{h^2/k}{1+e} + \frac{h^2/k}{1-e}\right) = \frac{h^2}{k(1-e^2)} = \frac{h^2a^2}{kb^2},$$

which yields

$$b^2 = \frac{h^2a}{k}. \tag{20}$$

If T is the period of m (that is, the time required for one complete revolution in its orbit), then, since the area of the ellipse is πab, it follows from (12) that $\pi ab = \frac{1}{2}hT$, so $T = 2\pi ab/h$. By using (20), we now obtain

$$T^2 = \frac{4\pi^2a^2b^2}{h^2} = \left(\frac{4\pi^2}{k}\right)a^3. \tag{21}$$

Since the constant $k = GM$ depends on the central attracting mass M but not on m, (21) holds for all the planets in our solar system and we have Kepler's third law: The squares of the periods of revolution of the planets are proportional to the cubes of their mean distances.

*In the discussion of equation (17) we have ignored the possibility that r might have a constant value and therefore not be minimal in any direction, so that z has a constant value and is not maximal in any direction. This happens when both $A = 0$ and $B = 0$, so that $z = k/h^2$ and $r = h^2/k$. Under these circumstances we have a circular orbit with radius h^2/k, and this can be included under equation (18) by allowing the possibility that $e = 0$. However, we saw in Section 17.6 that a circular orbit of given radius requires a certain precise orbital speed, which is infinitely unlikely for an actual planet and can be disregarded as a genuine possibility.

As we explained in Section 15.3, the standard unit of distance among astronomers who work with the solar system is the *astronomical unit*. This is the mean distance from the earth to the sun, which is approximately 93,000,000 mi, or 150,000,000 km. Equation (21) takes the more convenient form

$$T^2 = a^3 \tag{22}$$

when time is measured in years and distance in astronomical units. The reason for this, of course, is that 1 year is by definition the period of revolution of the earth in its orbit, so that with these units of measurement $T = 1$ when $a = 1$.

We would like to point out that the mathematical theory discussed in this section is just the beginning of what Newton accomplished, and constitutes only a first approximation to the full story of planetary motion. For instance, we have assumed that only the sun and *one* planet are present. But actually, of course, all the other planets are present as well, and each exerts its own independent gravitational force on the planet under consideration. These additional influences introduce what are called *perturbations* into the idealized elliptical orbit derived here, and the main purpose of the science of celestial mechanics is to take all these complexities into account. One of the great events of nineteenth century astronomy arose in just this way, namely, the discovery of the planet Neptune by Adams and Leverrier, through their attempts to explain the relatively large deviations of Uranus from its Keplerian orbit.*

Also, we have assumed that the sun and planet under discussion are particles, that is, points at which mass is concentrated. In fact, of course, they are extended bodies with substantial dimensions. One of Newton's most remarkable achievements was to prove that the sun and planets behave like particles under the inverse square law of attraction. We will prove this statement ourselves in Chapter 20, where we study three-dimensional integrals.

Newton's enormous success revived and greatly intensified the almost-forgotten Greek belief that it is possible to understand the universe in a rational way. This new confidence in its own intellectual powers permanently altered humanity's perception of itself, and over the past 300 years almost every department of human life has felt its consequences.

*For the details of this dramatic story, see pp. 820–839 of *The World of Mathematics,* James R. Newman, ed. (Simon and Schuster, 1956).

PROBLEMS

1 Newton himself did not know the value of the constant of gravitation G. This was determined by means of a classic experiment in 1789 by the English scientist Henry Cavendish. Once G is known, explain how equation (21), written in the form

$$T^2 = \left(\frac{4\pi^2}{GM}\right) a^3,$$

might be used to calculate the mass of the sun.

2 What is the period of revolution T (in years) of a planet whose mean distance from the sun is
(a) twice that of the earth?
(b) three times that of the earth?
(c) 25 times that of the earth?

*3 Kepler's first two laws, in the form of equations (11) and (18), imply that m is attracted toward the origin with a force whose magnitude is inversely proportional to the square of r. This was Newton's fundamental discovery, for it caused him to propound his law of gravitation and investigate its consequences. Prove this by assuming (11) and (18) and verifying the following statements:
(a) $F_\theta = 0$;

(b) $\dfrac{dr}{dt} = \dfrac{ke}{h}\sin\theta$;

(c) $\dfrac{d^2r}{dt^2} = \dfrac{ke\cos\theta}{r^2}$;

(d) $F_r = -\dfrac{mk}{r^2} = -G\dfrac{Mm}{r^2}$.

*4 Use formula (17) to show that the speed v of a planet at any point of its orbit is given by

$$v^2 = k\left(\frac{2}{r} - \frac{1}{a}\right).$$

5 Suppose that the earth explodes into fragments which fly off at the same speed in different directions into orbits of their own. Use Kepler's third law and the result of Problem 4 to show that all fragments that do not fall into the sun or escape from the solar system will reunite later at the same point.

CHAPTER 17 REVIEW: CONCEPTS, FORMULAS

Think through the following.

1 Parametric curve.
2 Cycloid: area and length.
3 Brachistochrone problem.
4 Tautochrone property.
5 Scalars and vectors.
6 Parallelogram rule.
7 Velocity $\mathbf{v} = d\mathbf{R}/dt$.
8 Acceleration $\mathbf{a} = d\mathbf{v}/dt = d^2\mathbf{R}/dt^2$.
9 Newton's second law of motion: $\mathbf{F} = m\mathbf{a}$.
10 Curvature: definition and formula.
11 Kepler's laws of planetary motion.
12 Newton's law of gravitation.

ADDITIONAL PROBLEMS FOR CHAPTER 17

SECTION 17.1

1 Find the area of the loop of the folium of Descartes shown in Fig. 17.11. Hint: Use the polar equation of the folium and evaluate the area integral with the aid of the substitution $u = \tan\theta$.

2 Find parametric equations for the right loop of the lemniscate $r^2 = 2a^2\cos 2\theta$ by using the slope of the radial line $t = y/x$ as parameter. How can the left loop be represented?

SECTION 17.2

3 Consider the cycloid discussed in Section 17.2.
(a) Find the volume of the solid generated by revolving the region under one arch about the x-axis.
(b) Find the area of the surface generated by revolving one arch about the x-axis.

*4 Let a be fixed and consider the hypocycloid of n cusps, so that $a = nb$. Find the total length L_n of this curve, and also the limit approached by L_n as $n \to \infty$.

*5 Find the length of one arch of the epicycloid generated by a circle of radius b rolling on the outside of a fixed circle of radius a.[†]

*6 Let a be fixed and consider the epicycloid of n cusps, so that $a = nb$. Find the total length L_n of this curve, and also the limit approached by L_n as $n \to \infty$.

*7 Consider an ideal pendulum consisting of a particle of mass m at the end of a weightless string of length L (Fig. 17.54). If it is pulled aside through an angle α and released, show that its period of oscillation T can be expressed in the form

$$T = 4\sqrt{\frac{L}{g}}\int_0^1 \frac{du}{\sqrt{(1-u^2)(1-k^2u^2)}},$$

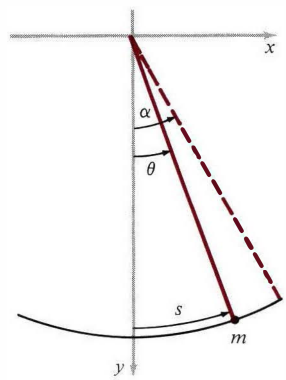


Figure 17.54 An ideal pendulum.

[†]Newton discovered this length, and obtained Wren's Theorem (Example 2 in Section 17.2) by letting $a \to \infty$. See Book I of the *Principia,* Prop. 48 and Cor. 2 to Prop. 52. It is interesting to try to bridge the gap between Newton's language and our own.

where $k = \sin \frac{1}{2}\alpha$ and $u = (1/k) \sin \frac{1}{2}\theta$. Hint: $x = L \sin \theta$ and $y = L \cos \theta$, and $\frac{1}{2}mv^2 = mg(L \cos \theta - L \cos \alpha)$. This integral is called a *complete elliptic integral of the first kind*, and it cannot be evaluated by means of elementary functions. When α is small, so that k^2 is very small, we have the approximation

$$T \cong 4\sqrt{\frac{L}{g}} \sin^{-1} u \Big]_0^1 = 2\pi\sqrt{\frac{L}{g}},$$

as in Example 3 in Section 9.6.

*8 Consider a wire bent into the shape of a cycloid with parametric equations $x = a(\theta - \sin \theta)$ and $y = a(1 - \cos \theta)$, and invert it as in Fig. 17.15. If a bead is released on the wire and slides without friction under the influence of gravity alone, show that its velocity v satisfies the equation

$$4av^2 = g(s_0{}^2 - s^2),$$

where s_0 and s are the arc lengths from the lowest point to the bead's initial position and its position at any later time, respectively. By differentiation obtain the equation

$$\frac{d^2s}{dt^2} + \frac{g}{4a} s = 0,$$

which shows that the bead moves in simple harmonic motion. Use the ideas of Section 9.6 to find s as a function of t, determine the period of the motion, and observe that this establishes in another way the tautochrone property of the cycloid proved in Section 17.2.

SECTION 17.3

9 Use vector methods to show that the line joining the midpoints of two sides of a triangle is parallel to the third side and half its length.

10 Generalize Problem 9 by using vector methods to show that the line joining the midpoints of the nonparallel sides of a trapezoid is parallel to the parallel sides and half the sum of their lengths.

SECTION 17.5

11 Locate the points on the curve $y = \frac{1}{4}x^4$ where the radius of curvature is smallest. What is this smallest radius?

12 Show that the radius of curvature at any point (x, y) on the hypocycloid of four cusps $x^{2/3} + y^{2/3} = a^{2/3}$ is three times the distance from the origin to the line which is tangent to the curve at (x, y).

SECTION 17.7

13 The Cavendish value for G is $6.7 \times 10^{-8}\ \text{cm}^3 \cdot \text{g}^{-1} \cdot \text{s}^{-2}$ when mass is measured in grams, distance in centimeters, and time in seconds. In Example 2 in Section 17.6 we used the fact that $GM_e = gR^2$, where M_e is the mass of the earth. Calculate M_e (approximately) in grams by using the values $g = 980\ \text{cm/s}^2$ and $R = 6.37 \times 10^8$ cm.

14 With the notation of Section 17.7, the inverse square law can be written as

$$\mathbf{F} = -\frac{km}{r^2}\mathbf{u}_r$$

where $k = GM$, and since $m\mathbf{a} = \mathbf{F}$, we have

$$\frac{d\mathbf{v}}{dt} = -\frac{k}{r^2}\mathbf{u}_r.$$

Verify the following steps to obtain another derivation of Kepler's first law from the inverse square law:

(a) Use (6) and (11) to write

$$\frac{d\mathbf{v}}{dt} = -\frac{k}{r^2}\left(-\frac{1}{d\theta/dt}\frac{d\mathbf{u}_\theta}{dt}\right) = \frac{k}{h}\frac{d\mathbf{u}_\theta}{dt}.$$

(b) Integrate the equation in (a) to obtain

$$\mathbf{v} = \frac{k}{h}\mathbf{u}_\theta + \left(v_0 - \frac{k}{h}\right)\mathbf{j},$$

by assuming initial conditions in the following form (Fig. 17.55): At $t = 0$, m has its closest approach to the origin and crosses the polar axis at the point $\mathbf{R} = r_0\mathbf{i}$ with velocity $\mathbf{v} = v_0\mathbf{j}$.

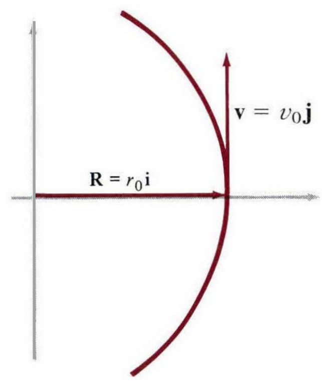


Figure 17.55

(c) Equate $\mathbf{u}_\theta$-components of the equation in (b) to obtain

$$r\frac{d\theta}{dt} = \frac{k}{h} + \left(v_0 - \frac{k}{h}\right)\cos \theta,$$

and use (11) to write this in the form

$$\frac{h}{r} = \frac{k}{h} + \left(v_0 - \frac{k}{h}\right)\cos \theta.$$

(d) Solve the equation in (c) for r to obtain

$$r = \frac{h^2/k}{1 + (v_0 h/k - 1)\cos\theta}.$$

(e) Use (7) and (11) to show that $h = r_0 v_0$, and write the equation in (d) in the form

$$r = \frac{(r_0 v_0)^2/k}{1 + (r_0 v_0^2/k - 1)\cos\theta} = \frac{(r_0 v_0)^2/k}{1 + e\cos\theta},$$

where $e = r_0 v_0^2/k - 1$.

(f) Observe that the equation in (c) represents

A circle if $r_0 v_0^2 = GM$;
An ellipse if $GM < r_0 v_0^2 < 2GM$;
A parabola if $r_0 v_0^2 = 2GM$;
A hyperbola if $r_0 v_0^2 > 2GM$.

APPENDIX: BERNOULLI'S SOLUTION OF THE BRACHISTOCHRONE PROBLEM

As explained in Section 17.2, we begin with a point P_0 and a lower point P_1, and we seek the shape of the curved wire joining these points down which a bead will slide without friction in the shortest possible time.

We start by considering an apparently unrelated problem in optics. Figure 17.56 illustrates a situation in which a ray of light travels from A to P with constant velocity v_1, and then, entering a denser medium, travels from P to B with a smaller velocity v_2. In terms of the notation in the figure, the total time T required for the journey is given by

$$T = \frac{\sqrt{a^2 + x^2}}{v_1} + \frac{\sqrt{b^2 + (c - x)^2}}{v_2}.$$

If we assume that this ray of light is able to select its path from A to B in such a way as to minimize T, then $dT/dx = 0$, and with a little work we see that the minimizing path is characterized by the equation

$$\frac{\sin\alpha_1}{v_1} = \frac{\sin\alpha_2}{v_2}.$$

This is *Snell's law of refraction.** The assumption that light travels from one point to another along the path requiring the shortest time is called *Fermat's principle of least time.* This principle not only provides a rational basis for Snell's law—which is an experimental fact—but also can be applied to find the path of a ray of light through a medium of variable density, where in general light will travel along curves instead of straight lines. In Fig. 17.57(a) we have a stratified optical medium. In the individual layers the velocity of light is constant, but the velocity decreases from each layer to the one below it. As the descending ray of light passes from layer to layer, it is refracted more and more toward

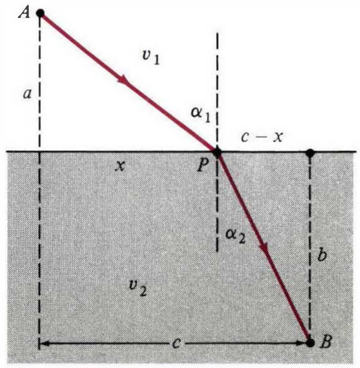


Figure 17.56 The refraction of light.

*See Example 4 in Section 4.4.

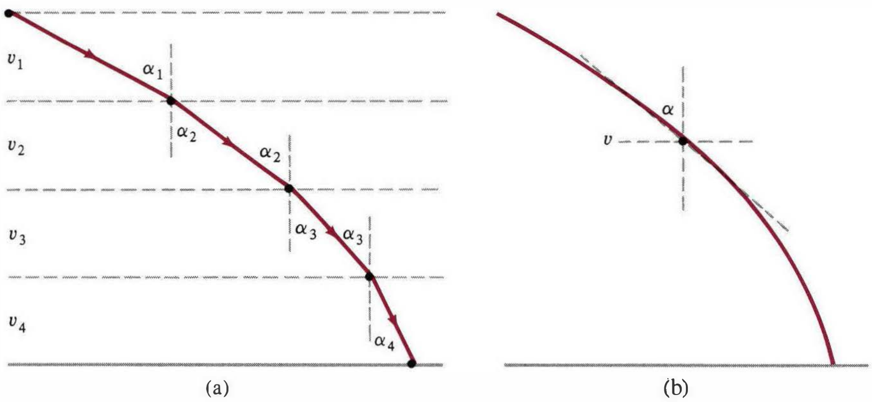


Figure 17.57 Refraction in other optical media.

the vertical, and when Snell's law is applied to the boundaries between the layers, we obtain

$$\frac{\sin \alpha_1}{v_1} = \frac{\sin \alpha_2}{v_2} = \frac{\sin \alpha_3}{v_3} = \frac{\sin \alpha_4}{v_4}.$$

If we next allow these layers to grow thinner and more numerous, then in the limit the velocity of light decreases continuously as the ray descends, and we conclude that

$$\frac{\sin \alpha}{v} = \text{a constant}.$$

This situation is indicated in Fig. 17.57(b); it is approximately what happens to a ray of sunlight falling on the earth as it slows in descending through atmosphere of increasing density.

Returning now to the brachistochrone problem, we introduce a coordinate system as in Fig. 17.58 and assume that the bead (like the ray of light) is capable of selecting the path down which it will slide from P_0 to P_1 in the shortest possible time. The argument given above yields

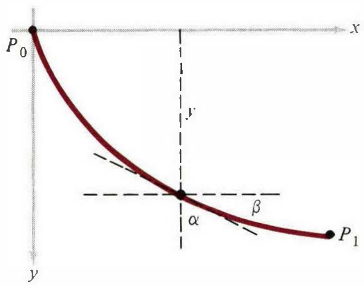


Figure 17.58

$$\frac{\sin \alpha}{v} = \text{a constant}. \tag{1}$$

If the bead has mass m, so that mg is the downward force that gravity exerts on it, then the fact that the work done by gravity in pulling the bead down the wire equals the increase in the kinetic energy of the bead tells us that $mgy = \frac{1}{2}mv^2$. This gives

$$v = \sqrt{2gy}. \tag{2}$$

From the geometry of the situation we also have

$$\sin \alpha = \cos \beta = \frac{1}{\sec \beta} = \frac{1}{\sqrt{1 + \tan^2 \beta}} = \frac{1}{\sqrt{1 + (y')^2}}. \tag{3}$$

On combining equations (1), (2), and (3)—obtained from optics, mechanics, and calculus—we get

$$y[1 + (y')^2] = c \tag{4}$$

as the differential equation of the brachistochrone.

We now complete our discussion, and discover what curve the brachistochrone actually is, by solving equation (4). When y' is replaced by dy/dx and the variables are separated, (4) becomes

$$dx = \sqrt{\frac{y}{c - y}}\, dy,$$

so

$$x = \int \sqrt{\frac{y}{c - y}}\, dy.$$

We evaluate this integral by starting with the algebraic substitution $u^2 = y/(c - y)$, so that

$$y = \frac{cu^2}{1 + u^2} \qquad \text{and} \qquad dy = \frac{2cu}{(1 + u^2)^2}\, du.$$

Then

$$x = \int \frac{2cu^2}{(1 + u^2)^2}\, du,$$

and the trigonometric substitution $u = \tan \phi$, $du = \sec^2 \phi\, d\phi$ enables us to write this as

$$
\begin{aligned}
x &= \int \frac{2c \tan^2 \phi \sec^2 \phi}{(1 + \tan^2 \phi)^2}\, d\phi \\
&= 2c \int \frac{\tan^2 \phi}{\sec^2 \phi}\, d\phi = 2c \int \sin^2 \phi \, d\phi \\
&= c \int (1 - \cos 2\phi)\, d\phi = \frac{1}{2} c(2\phi - \sin 2\phi).
\end{aligned}
$$

The constant of integration here is zero because $y = 0$ when $\phi = 0$, and since P_0 is at the origin, we also want to have $x = 0$ when $\phi = 0$. The formula for y gives

$$y = \frac{c \tan^2 \phi}{\sec^2 \phi} = c \sin^2 \phi = \frac{1}{2} c(1 - \cos 2\phi).$$

We now simplify our equations by writing $a = \frac{1}{2}c$ and $\theta = 2\phi$, which yields

$$x = a(\theta - \sin \theta), \qquad y = a(1 - \cos \theta).$$

These are the standard parametric equations of the cycloid with a cusp at the origin. We note that there is a single value of a that makes the first inverted arch of this cycloid pass through the point P_1 in Fig. 17.58; for if a is allowed to increase from 0 to ∞, then the arch inflates, sweeps over the first quadrant of the plane, and clearly passes through P_1 for a single suitably chosen value of a.

18 VECTORS IN THREE-DIMENSIONAL SPACE. SURFACES

18.1 COORDINATES AND VECTORS IN THREE-DIMENSIONAL SPACE

In the preceding seventeen chapters we have discussed many aspects of the calculus of functions of a *single* variable. The geometry of these functions is two-dimensional because the graph of a function of a single variable is a curve in the plane. Most of the remainder of this book is concerned with the calculus of functions of *several* (two or more) independent variables. The geometry of functions of two variables is three-dimensional, because in general the graph of such a function is a curved surface in space.

In this chapter we discuss the analytic geometry of three-dimensional space. Our treatment will emphasize vector algebra, partly because this approach provides a more direct and intuitive understanding of the equations of lines and planes, and partly because the concepts of dot and cross products as developed in the next two sections are indispensable in many other parts of mathematics and physics.

Rectangular coordinates in the plane can be generalized in a natural way to rectangular coordinates in space. The position of a point in space is described by giving its location relative to three mutually perpendicular *coordinate axes* passing through the *origin* O. We always draw the x-, y-, and z-axes as shown in Fig. 18.1, with equal units of length on all three axes and with arrows indicating the

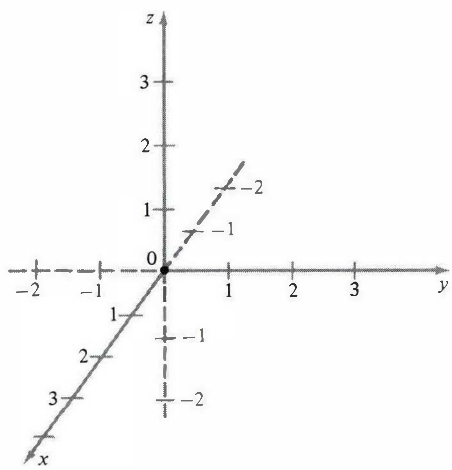


Figure 18.1 Coordinate axes.

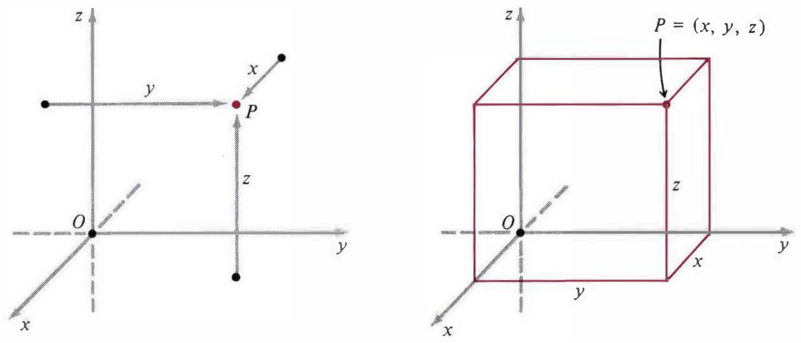


Figure 18.2 Locating a point by its rectangular coordinates.

positive directions. Each pair of axes determines a *coordinate plane*: the x-axis and y-axis determine the xy-plane, etc. The configuration of axes in this figure is called *right-handed,* because if the thumb of the right hand points in the direction of the positive z-axis, then the curl of the fingers gives the positive direction of rotation in the xy-plane, from the positive x-axis to the positive y-axis.

Since many people have trouble visualizing space figures from plane drawings, we point out that Fig. 18.1 can be thought of as part of a rectangular room drawn in perspective, with the origin O at the far left corner of the floor. The xy-plane is the floor, and has the normal appearance of the xy-plane if we look down on it from a point on the positive z-axis; the yz-plane is the back wall of the room, in the plane of the paper; and the xz-plane is the wall on the left side of the room.

A point P in space (see Fig. 18.2) is said to have *rectangular* (or *Cartesian*) *coordinates* x, y, z if:

x is its signed distance from the yz-plane;

y is its signed distance from the xz-plane;

z is its signed distance from the xy-plane.

Just as in plane analytic geometry, we write $P = (x, y, z)$ and identify the point P with the ordered triple of its coordinates. On the right in the figure we attempt to strengthen the illusion of three dimensions by completing the box that has O and P as opposite vertices.*

The three coordinate planes divide all of space into eight cells called *octants.* The cell emphasized in Fig. 18.2, where x, y, and z are all positive numbers, is called the *first octant.* (No one bothers to number the other seven octants.)

Even before plunging into a general study of the equations of lines and planes in Section 18.4, we can notice a few obvious facts. The xy-plane is the set of all points $(x, y, 0)$; it consists precisely of those points in space whose z-coordinate is 0, so its equation is

$$z = 0.$$

Similarly, the equation of the yz-plane is $x = 0$, and the equation of the xz-plane is $y = 0$.

*The technical term for the object shown on the right is "rectangular parallelepiped." We prefer the simpler word "box."

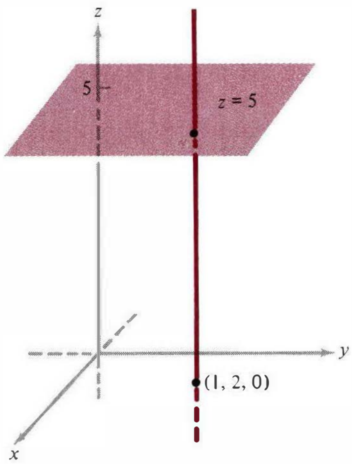


Figure 18.3 Horizontal plane.

The z-axis is the set of all points $(0, 0, z)$. It is therefore represented by the pair of equations

$$x = 0, \qquad y = 0. \tag{1}$$

These are the equations of the yz-plane and the xz-plane, respectively, so equations (1) taken together characterize the z-axis as the intersection of these two coordinate planes. Similarly, the equations of the x-axis are $y = 0$, $z = 0$; and the equations of the y-axis are $x = 0$, $z = 0$.

There is nothing special about the number 0 in these remarks. For instance, the equation of the horizontal plane 5 units above the xy-plane is $z = 5$; and the equations of the vertical line that passes through the point $(1, 2, 0)$ in the xy-plane are $x = 1$, $y = 2$. See Fig. 18.3.

Almost all of the ideas about vectors that were presented in Section 17.3 are valid in three-dimensional space and require no further discussion. This remark applies to the concept of a vector, to the definition of equality for vectors, and to the definitions of addition and scalar multiplication. In all this material there is no need at all to suppose that the vectors lie in a plane.

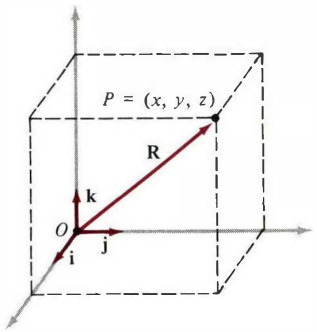


Figure 18.4 The position vector of P.

The only real difference is that a vector in space has three components rather than two. In computing with vectors in the plane, we used the unit vectors **i** and **j** in the positive x- and y-directions. In order to compute with vectors in three-dimensional space, we introduce a third unit vector **k** in the positive z-direction, as shown in Fig. 18.4. If $P = (x, y, z)$ is any point in space, the position vector $\mathbf{R} = \overrightarrow{OP}$ can be written in the form

$$\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k},$$

and the numbers x, y, and z are called its **i**-, **j**-, and **k**-*components.*

The *length* of the vector **R** is given by the formula

$$|\mathbf{R}| = \sqrt{x^2 + y^2 + z^2}. \tag{2}$$

This can be proved by a double application of the theorem of Pythagoras, as illustrated in Fig. 18.5:

$$\begin{aligned} |\mathbf{R}|^2 = OP^2 &= OB^2 + BP^2 \\ &= OA^2 + AB^2 + BP^2 \\ &= x^2 + y^2 + z^2. \end{aligned}$$

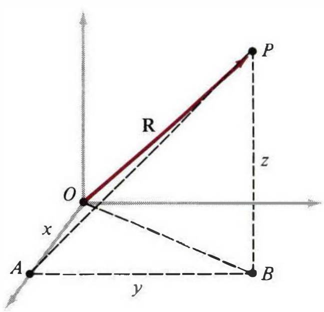


Figure 18.5

If $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ are any two points in space (Fig. 18.6), the distance between them is the length of the vector $\overrightarrow{P_1P_2}$ from P_1 to P_2. Since

$$\begin{aligned} \overrightarrow{P_1P_2} = \mathbf{R}_2 - \mathbf{R}_1 &= (x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}) - (x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}) \\ &= (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}, \end{aligned}$$

we can use (2) to obtain

$$|\overrightarrow{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}. \tag{3}$$

This is the important *distance formula*; it has many uses.

Since a sphere is the set of all points P at a given distance r from a given fixed point P_0 (Fig. 18.7), the equation of a sphere can be written as

$$|\overrightarrow{P_0P}| = r. \tag{4}$$

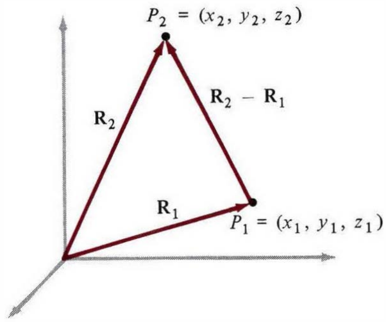


Figure 18.6

If $P = (x, y, z)$ and $P_0 = (x_0, y_0, z_0)$, then (3) enables us to write (4) in the equivalent form

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2. \tag{5}$$

This is the standard equation of the sphere with center $P_0 = (x_0, y_0, z_0)$ and radius r.

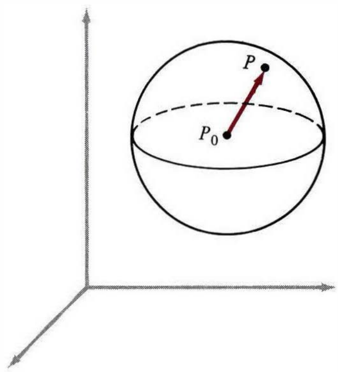


Figure 18.7 Sphere.

Example If we complete the squares in the equation

$$x^2 + y^2 + z^2 + 4x - 2y - 6z + 8 = 0, \tag{6}$$

it becomes

$$(x + 2)^2 + (y - 1)^2 + (z - 3)^2 = 6.$$

By comparing this with (5), we see at once that (6) is the equation of the sphere with center $(-2, 1, 3)$ and radius $\sqrt{6}$.

PROBLEMS

1 Sketch the box with the vertices $(1, -1, 0)$, $(1, 4, 0)$, $(-2, 4, 0)$, $(-2, -1, 0)$, $(1, -1, 5)$, $(1, 4, 5)$, $(-2, 4, 5)$, $(-2, -1, 5)$. Write down the equations of the faces and edges of the box that pass through the vertex $(1, 4, 5)$.

2 Sketch the box bounded by the planes $x = 1, x = 3, y = 0, y = 4, z = 1, z = 5$. Write down the vertices.

3 Sketch the tetrahedron whose base vertices are $(3, 4, 0)$, $(3, -4, 0)$, and $(-5, 4, 0)$ and whose fourth vertex is $(0, 0, 6)$. Use the fact that the volume of a tetrahedron is one-third the area of the base times the height to find the volume of this tetrahedron.

4 Sketch the straight lines whose equations are given:
(a) $x = 2, z = 3$; (b) $y = 1, z = 4$;
(c) $x = -3, y = 1$.

5 Describe the graph of the equation
(a) $xy = 0$; (b) $xyz = 0$.

6 Describe and sketch the locus of all points $P = (x, y, z)$ that satisfy the given pairs of simultaneous equations:
(a) $x^2 + y^2 = 4, z = 3$;
(b) $x = 4, z = 4y^2$;
(c) $y = x, x = 5$;
(d) $z = -x^2, y = 0$.

7 Find the point on the y-axis which is equidistant from $(2, 5, -3)$ and $(-3, 6, 1)$.

8 Find and simplify the equation of the locus of all points that are equidistant from $(7, 0, -4)$ and $(-3, 2, 2)$. Describe this locus in geometric language.

9 Write the equation of the sphere with radius 7 and center on the positive z-axis, if the sphere is tangent to the plane $z = 0$.

10 Find the equation of the sphere with center $(3, -2, 5)$ which is
(a) tangent to the xy-plane;
(b) tangent to the yz-plane;
(c) tangent to the xz-plane.

11 Identify the graph of each of the following equations, and if it is a sphere, give its center and radius:
(a) $x^2 + y^2 + z^2 + 2x - 6y - 10z + 26 = 0$;
(b) $x^2 + y^2 + z^2 - 10x + 2y - 6z + 35 = 0$;

(c) $2x^2 + 2y^2 + 2z^2 - 16x + 8y + 4z + 49 = 0$;
(d) $x^2 + y^2 + z^2 + 2x - 14y - 6z + 59 = 0$;
(e) $4x^2 + 4y^2 + 4z^2 - 16x + 24y + 52 = 0$.

12 A point P moves in such a way that it is always twice as far from (3, 2, 0) as from (3, 2, 6). Show that the locus of P is a sphere and find its center and radius.

13 If $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$, use vectors to show that the coordinates of the midpoint of the segment P_1P_2 are
$x = \frac{1}{2}(x_1 + x_2)$, $y = \frac{1}{2}(y_1 + y_2)$, $z = \frac{1}{2}(z_1 + z_2)$.

14 Find the equation of the sphere that has the two given points as ends of a diameter:
(a) $(6, 2, -1), (-2, 4, 3)$; (b) $(0, 1, -7), (-6, 7, 3)$.

15 Show that the triangle with vertices $(4, 3, 6), (-2, 0, 8)$, and $(1, 5, 0)$ is a right triangle. Find its area.

16 For each of the following pairs of points, find the vector from the first point to the second, and also the distance between the points:
(a) $(2, 0, -3), (5, 1, 2)$; (b) $(-2, 1, 7), (1, -4, 2)$;
(c) $(8, 3, 6), (2, -2, 0)$; (d) $(1, 5, -3), (9, 7, 1)$.

17 Find the vector from the origin O to the intersection of the medians of the triangle whose vertices are $A = (3, 2, 2)$, $B = (-1, 0, 4)$, and $C = (5, 3, -2)$.

18 If **A**, **B**, **C** are any three distinct vectors, their endpoints form a triangle. Find the position vector of the intersection of the medians of this triangle.

***19** If **A**, **B**, **C**, **D** are any four distinct vectors, their endpoints form a tetrahedron. Show that the four lines joining each vertex to the intersection of the medians of the opposite face are concurrent, and find the position vector of their common point.

18.2 THE DOT PRODUCT OF TWO VECTORS

Up to this point in our work we have not defined the product of two vectors **A** and **B**. There are two different ways of doing this, both of which have important uses in geometry and physics. Since there is no reason to choose one of these definitions in preference to the other, we keep both, using a dot for one definition and a cross for the other. The *dot product* (or *scalar product*) of **A** and **B** is denoted by $\mathbf{A} \cdot \mathbf{B}$ and is a number. The *cross product* (or *vector product*) is denoted by $\mathbf{A} \times \mathbf{B}$ and is a vector. These two kinds of multiplication are totally different. We discuss the first in this section and the second in Section 18.3.

The *dot product* $\mathbf{A} \cdot \mathbf{B}$ of two vectors **A** and **B** is defined to be the product of their lengths and the cosine of the angle between them. This definition can be written as

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos \theta, \tag{1}$$

where θ $(0 \le \theta \le \pi)$ is the angle between **A** and **B** when they are placed so that their tails coincide (Fig. 18.8). It is clear from the definition that $\mathbf{A} \cdot \mathbf{B}$ is a *scalar* (or number), not a vector.

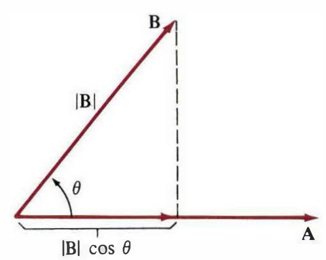


Figure 18.8 Scalar projection.

As Fig. 18.8 shows, the number $|\mathbf{B}| \cos \theta$ is the *scalar projection* of **B** on **A**, denoted by $\text{proj}_{\mathbf{A}}\, \mathbf{B}$. Definition (1) can therefore be interpreted geometrically as follows:

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= |\mathbf{A}|(|\mathbf{B}| \cos \theta) = |\mathbf{A}| \text{ proj}_{\mathbf{A}}\, \mathbf{B} \\ &= (\text{length of } \mathbf{A}) \times (\text{scalar projection of } \mathbf{B} \text{ on } \mathbf{A}). \end{aligned}$$

By interchanging the roles of **A** and **B**, we also have

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= |\mathbf{B}|(|\mathbf{A}| \cos \theta) = |\mathbf{B}| \text{ proj}_{\mathbf{B}}\, \mathbf{A} \\ &= (\text{length of } \mathbf{B}) \times (\text{scalar projection of } \mathbf{A} \text{ on } \mathbf{B}). \end{aligned}$$

The *vector projection* of **B** on **A** is also indicated in the figure. Both types of projections are useful in applications.

It is easy to see from the definition (1) that the dot product has the properties

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}, \quad \text{the commutative law,} \tag{2}$$

and

$$(c\mathbf{A}) \cdot \mathbf{B} = c(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \cdot (c\mathbf{B}). \tag{3}$$

It also has the property

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}, \qquad \text{the distributive law,} \tag{4}$$

but this is not quite as evident as (2) and (3). To establish (4), we observe from Fig 18.9 that

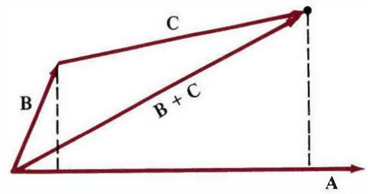


Figure 18.9

$$\begin{aligned}\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) &= |\mathbf{A}|[\text{proj}_{\mathbf{A}}(\mathbf{B} + \mathbf{C})] \\ &= |\mathbf{A}|(\text{proj}_{\mathbf{A}}\, \mathbf{B} + \text{proj}_{\mathbf{A}}\, \mathbf{C}) \\ &= |\mathbf{A}|\text{proj}_{\mathbf{A}}\, \mathbf{B} + |\mathbf{A}|\text{proj}_{\mathbf{A}}\, \mathbf{C} \\ &= \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}.\end{aligned}$$

If we combine (4) with the commutative law (2), we also have

$$(\mathbf{A} + \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot \mathbf{C} + \mathbf{B} \cdot \mathbf{C}. \tag{5}$$

Properties (4) and (5) permit us to multiply out sums of vectors by the ordinary procedures of elementary algebra, as in

$$(\mathbf{A} + \mathbf{B}) \cdot (\mathbf{C} + \mathbf{D}) = \mathbf{A} \cdot \mathbf{C} + \mathbf{A} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{C} + \mathbf{B} \cdot \mathbf{D}.$$

Another simple consequence of the definition (1) is the fact that

$$\mathbf{A} \cdot \mathbf{A} = |\mathbf{A}|^2 \tag{6}$$

for any vector **A**.

Example 1 In the notation of Fig. 18.10, the cosine law of trigonometry states that

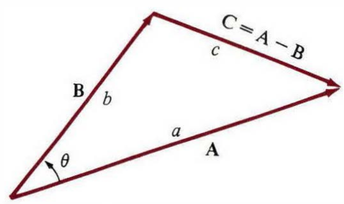


Figure 18.10

$$c^2 = a^2 + b^2 - 2ab \cos \theta.$$

This can be proved very easily by using property (6) to write

$$\begin{aligned}c^2 = |\mathbf{C}|^2 = |\mathbf{A} - \mathbf{B}|^2 &= (\mathbf{A} - \mathbf{B}) \cdot (\mathbf{A} - \mathbf{B}) \\ &= \mathbf{A} \cdot \mathbf{A} + \mathbf{B} \cdot \mathbf{B} - 2\mathbf{A} \cdot \mathbf{B} \\ &= |\mathbf{A}|^2 + |\mathbf{B}|^2 - 2\mathbf{A} \cdot \mathbf{B} \\ &= a^2 + b^2 - 2ab \cos \theta.\end{aligned}$$

If we apply the definition (1) to the mutually perpendicular unit vectors **i**, **j**, and **k** introduced in Section 18.1, we obtain

$$\begin{aligned}\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} &= 1, \\ \mathbf{i} \cdot \mathbf{j} = \mathbf{i} \cdot \mathbf{k} = \mathbf{j} \cdot \mathbf{k} &= 0.\end{aligned} \tag{7}$$

These facts enable us to find a convenient formula for computing the dot product of any two vectors given in **i**, **j**, **k** form,

$$\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \qquad \text{and} \qquad \mathbf{B} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}.$$

If we expand $\mathbf{A} \cdot \mathbf{B}$ by using (7) together with the general properties previously discussed, we get

$$\mathbf{A} \cdot \mathbf{B} = a_1b_1 + a_2b_2 + a_3b_3, \tag{8}$$

since six of the nine terms in the expansion vanish. Thus, to compute $\mathbf{A} \cdot \mathbf{B}$, we simply multiply their respective **i**-, **j**-, and **k**-components, and add.

If **A** and **B** are nonzero vectors, the definition (1) can be written in the form

$$\cos\theta = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}||\mathbf{B}|}. \tag{9}$$

This formula displays the main significance of the dot product in geometry: It provides a simple way to find the angle between two vectors and, in particular, to decide when two vectors are perpendicular. Indeed, if we agree that the zero vector is perpendicular to every vector, then by (9) we see at once that

$$\mathbf{A} \perp \mathbf{B} \qquad \text{if and only if} \qquad \mathbf{A} \cdot \mathbf{B} = 0.$$

Formula (8) makes it possible for us to use the dot product in these ways as a convenient computational tool.

Example 2 Find the cosine of the angle θ between the vectors $\mathbf{A} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $\mathbf{B} = -3\mathbf{i} + 4\mathbf{j}$.

Solution It is clear that

$$|\mathbf{A}| = \sqrt{1+4+4} = 3, \qquad |\mathbf{B}| = \sqrt{9+16} = 5, \qquad \mathbf{A} \cdot \mathbf{B} = -3 + 8 + 0 = 5.$$

Therefore by (9) we have

$$\cos\theta = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}||\mathbf{B}|} = \frac{5}{3 \cdot 5} = \frac{1}{3}.$$

If we want the angle θ itself, we can use a calculator to find that $\theta \cong 70.5°$.

Example 3 Compute the cosine of the angle θ between **A** and **B** if $\mathbf{A} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ and $\mathbf{B} = -\mathbf{i} + c\mathbf{k}$, and find a value of c for which $\mathbf{A} \perp \mathbf{B}$.

Solution We have

$$|\mathbf{A}| = \sqrt{1+4+4} = 3, \qquad |\mathbf{B}| = \sqrt{1+c^2}, \qquad \mathbf{A} \cdot \mathbf{B} = -1 + 2c,$$

so

$$\cos\theta = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}||\mathbf{B}|} = \frac{2c-1}{3\sqrt{1+c^2}}.$$

When $c = \frac{1}{2}$, this quantity has the value 0, and hence the vectors are perpendicular.

The simplest physical illustration of the use of the dot product is furnished by the concept of work. We recall that the work W done by a constant force F exerted along the line of motion in moving a particle through a distance d is given by $W = Fd$. But what if the force is a constant vector **F** pointing in some direction other than the line of motion from P to Q, as shown in Fig. 18.11? Only the vector component of **F** in the direction of the line of motion does work, so in this case we have

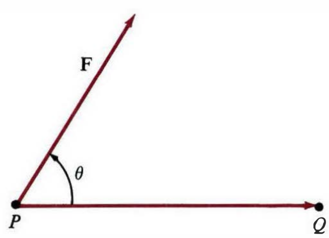


Figure 18.11

$$W = (|\mathbf{F}|\cos\theta)|\overrightarrow{PQ}| = |\mathbf{F}||\overrightarrow{PQ}|\cos\theta = \mathbf{F} \cdot \overrightarrow{PQ},$$

that is,

$$W = \mathbf{F} \cdot \overrightarrow{PQ}. \tag{10}$$

In more advanced treatments of the physical uses of vectors, it is often necessary to calculate the work done by variable forces whose points of application move along curved paths, and formula (10) is the starting point for all such applications.

PROBLEMS

1 Show that the vectors

$$\mathbf{A} = \mathbf{i} + 3\mathbf{j} + 4\mathbf{k},$$
$$\mathbf{B} = 4\mathbf{i} + 4\mathbf{j} - 4\mathbf{k},$$

are perpendicular.

2 Show that

$$\mathbf{A} = \mathbf{i} - 2\mathbf{j} + \mathbf{k},$$
$$\mathbf{B} = \mathbf{i} - \mathbf{k},$$
$$\mathbf{C} = \mathbf{i} + \mathbf{j} + \mathbf{k},$$

are mutually perpendicular.

3 Find the angle between each of the given pairs of vectors:
(a) $\mathbf{A} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$, $\mathbf{B} = -\mathbf{i} + \mathbf{j} + 2\mathbf{k}$;
(b) $\mathbf{A} = \mathbf{i}$, $\mathbf{B} = \mathbf{i} + \mathbf{j}$;
(c) $\mathbf{A} = 3\mathbf{i} + 4\mathbf{j}$, $\mathbf{B} = 4\mathbf{i} - 3\mathbf{j} + 9\mathbf{k}$.

4 Use dot products to show that the given three points are the vertices of a right triangle. Which is the vertex of the right angle?
(a) $P = (1, 7, 3)$, $Q = (0, 7, -1)$, $R = (-1, 6, 2)$.
(b) $P = (2, -5, -2)$, $Q = (-1, -2, 2)$, $R = (4, 1, -5)$.
(c) $P = (2, 7, -2)$, $Q = (0, 4, -1)$, $R = (1, 4, 1)$.

5 Show that the vectors

$$\mathbf{A} = \mathbf{i} - 3\mathbf{j} - 5\mathbf{k},$$
$$\mathbf{B} = 2\mathbf{i} - \mathbf{j} + \mathbf{k},$$
$$\mathbf{C} = 3\mathbf{i} - 4\mathbf{j} - 4\mathbf{k},$$

form the sides of a right triangle if placed in the proper positions.

6 Find the angle θ between a diagonal of a cube and
(a) an adjacent edge;
(b) an adjacent diagonal of a face.

7 Let $\mathbf{A}$ be a nonzero vector, and suppose that $\mathbf{B}$ and $\mathbf{C}$ are two vectors such that $\mathbf{A} \cdot \mathbf{B} = \mathbf{A} \cdot \mathbf{C}$. Is it legitimate to cancel $\mathbf{A}$ from both sides of this equation and conclude that $\mathbf{B} = \mathbf{C}$? Explain.

8 Find a value of c for which the given vectors will be perpendicular:
(a) $3\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$, $2\mathbf{i} + 4\mathbf{j} + c\mathbf{k}$;
(b) $\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{i} + \mathbf{j} + c\mathbf{k}$.

9 If $a = |\mathbf{A}|$ and $b = |\mathbf{B}|$, show that the vector $b\mathbf{A} + a\mathbf{B}$ bisects the angle between $\mathbf{A}$ and $\mathbf{B}$.

10 With the notation of Problem 9, show that $b\mathbf{A} + a\mathbf{B}$ and $b\mathbf{A} - a\mathbf{B}$ are perpendicular.

11 Use the dot product to prove that an angle inscribed in a semicircle is a right angle. Hint: With the notation in Fig. 18.12, calculate $(\mathbf{A} + \mathbf{B}) \cdot (\mathbf{A} - \mathbf{B})$.

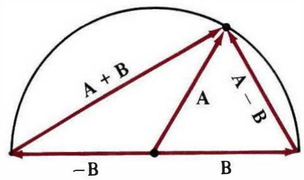


Figure 18.12

12 If $Q = (1, -1, 7)$, find the points $P = (0, c, c)$ on the line $z = y$ in the yz-plane such that the vector $\overrightarrow{OP}$ is perpendicular to the vector $\overrightarrow{PQ}$.

13 If $\mathbf{A} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $\mathbf{B} = 4\mathbf{i} - 2\mathbf{k}$, find the vector component of $\mathbf{A}$ along $\mathbf{B}$. Solve this problem by finding a general formula for the vector component of $\mathbf{A}$ along $\mathbf{B}$ if $\mathbf{A}$ and $\mathbf{B}$ are any two vectors.

14 Use vector methods to show that the distance from a point (x_0, y_0) to a line $ax + by + c = 0$ (both in the xy-plane) is

$$\frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}.$$

15 Use property (6) to prove the *parallelogram law* of elementary geometry: The sum of the squares of the diagonals of a parallelogram equals the sum of the squares of the four sides. Hint: With the notation of Fig. 18.13, use (6) to expand $|\mathbf{A} + \mathbf{B}|^2 + |\mathbf{A} - \mathbf{B}|^2$.

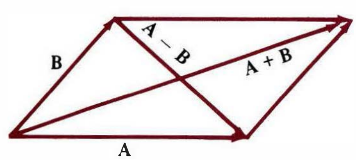


Figure 18.13 The parallelogram law.

16 For the triangle OAB in Fig. 18.14, the law of cosines states that

$$|\mathbf{A} - \mathbf{B}|^2 = |\mathbf{A}|^2 + |\mathbf{B}|^2 - 2|\mathbf{A}||\mathbf{B}| \cos \theta.$$

Give another proof of formula (8) by solving this equation for $|\mathbf{A}||\mathbf{B}| \cos \theta$ and simplifying the result.

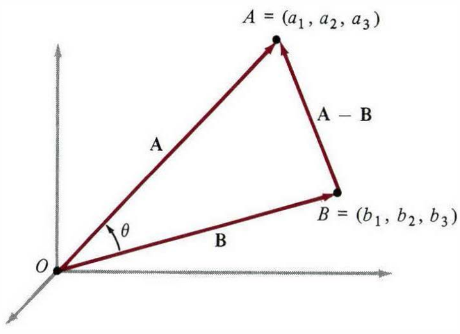


Figure 18.14

17 If a vector $\mathbf{V} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ makes angles α, β, and γ with the positive x-, y-, and z-axes (Fig. 18.15), then these angles are called the *direction angles* and $\cos \alpha$, $\cos \beta$, and $\cos \gamma$ are called the *direction cosines* of $\mathbf{V}$. Show that
(a) $(a\mathbf{i} + b\mathbf{j} + c\mathbf{k})/\sqrt{a^2 + b^2 + c^2}$ is a unit vector having the same direction as $\mathbf{V}$;
(b) $\cos \alpha = \dfrac{a}{\sqrt{a^2 + b^2 + c^2}}$,
$\cos \beta = \dfrac{b}{\sqrt{a^2 + b^2 + c^2}}$,
$\cos \gamma = \dfrac{c}{\sqrt{a^2 + b^2 + c^2}}$;
(c) $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$.

18 How many lines through the origin make angles of 45° with both the positive x-axis and the positive y-axis?

19 How many lines through the origin make angles of 60° with both the positive x-axis and the positive y-axis? What angles do they make with the positive z-axis?

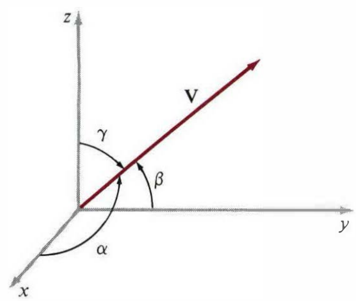


Figure 18.15 Direction angles.

20 Find the work done by the force $\mathbf{F}$ when its point of application moves from P to Q:
(a) $\mathbf{F} = 2\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}$, $P = (1, 2, -2)$, $Q = (3, -1, 1)$;
(b) $\mathbf{F} = 3\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$, $P = (-1, 2, 3)$, $Q = (1, 2, -1)$.

21 Find the work done by a force $\mathbf{F} = -c\mathbf{k}$ when its point of application moves from $P = (x_1, y_1, z_1)$ to $Q = (x_2, y_2, z_2)$.

22 Find the work done by a constant force $\mathbf{F}$ if its point of application moves around a closed polygonal path.

18.3 THE CROSS PRODUCT OF TWO VECTORS

Many problems in geometry require us to find a vector that is perpendicular to each of two given vectors $\mathbf{A}$ and $\mathbf{B}$. A routine way of doing this is provided by the *cross product* (or *vector product*) of $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A} \times \mathbf{B}$. This cross product is very different from the dot product $\mathbf{A} \cdot \mathbf{B}$ — for one thing, $\mathbf{A} \times \mathbf{B}$ is a vector, while $\mathbf{A} \cdot \mathbf{B}$ is a scalar. First we define this new product, then we describe its algebraic properties so that we can compute it with reasonable ease, and finally we illustrate some of its uses.

Consider two nonzero vectors $\mathbf{A}$ and $\mathbf{B}$. Suppose that one of these vectors is translated, if necessary, so that their tails coincide, and let θ be the angle from $\mathbf{A}$ to $\mathbf{B}$ (*not* from $\mathbf{B}$ to $\mathbf{A}$), with $0 \le \theta \le \pi$. If $\mathbf{A}$ and $\mathbf{B}$ are not parallel, so that $0 < \theta < \pi$, then these two vectors determine a plane, as shown in Fig. 18.16. We now choose the unit vector $\mathbf{n}$ which is normal (perpendicular) to this plane and whose direction is determined by the *right-hand thumb rule*. This means that if the right hand is placed so that the thumb is perpendicular to the plane of $\mathbf{A}$ and $\mathbf{B}$ and the fingers curl from $\mathbf{A}$ to $\mathbf{B}$ in the direction of the angle θ, then $\mathbf{n}$ points in the same direction as the thumb of this hand. This gives the direction of the vector $\mathbf{A} \times \mathbf{B}$ that we are defining. Not only do the vectors $\mathbf{A}$ and $\mathbf{B}$ determine

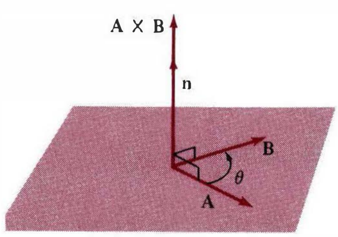


Figure 18.16

the plane under consideration, but they also determine a parallelogram in this plane, of area $|\mathbf{A}||\mathbf{B}|\sin\theta$ (see Fig. 18.17). We take the area of this parallelogram as the magnitude of the vector $\mathbf{A}\times\mathbf{B}$. With these preliminaries, we can now state the definition of the cross product of $\mathbf{A}$ and $\mathbf{B}$, in this order, as follows:

$$\mathbf{A}\times\mathbf{B} = |\mathbf{A}||\mathbf{B}|\sin\theta\,\mathbf{n}. \tag{1}$$

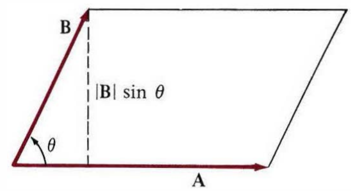


Figure 18.17

Observe that if $\mathbf{A}$ or $\mathbf{B}$ is $\mathbf{0}$, or if $\mathbf{A}$ and $\mathbf{B}$ are parallel, then they do not determine a plane, and hence the unit normal vector $\mathbf{n}$ is not defined. But in these cases $|\mathbf{A}| = 0$ or $|\mathbf{B}| = 0$, or $\sin\theta = 0$, so by (1) we have $\mathbf{A}\times\mathbf{B} = \mathbf{0}$ and the determination of $\mathbf{n}$ is not necessary. If we agree that the zero vector is to be considered as parallel to every vector, then it is easy to see that

$$\mathbf{A} \text{ is parallel to } \mathbf{B} \text{ if and only if } \mathbf{A}\times\mathbf{B} = \mathbf{0}.$$

In particular, we have

$$\mathbf{A}\times\mathbf{A} = \mathbf{0}$$

for every $\mathbf{A}$. If instead of $\mathbf{A}\times\mathbf{B}$ we consider $\mathbf{B}\times\mathbf{A}$, then the direction of the angle θ is reversed, and we must flip the right hand over so that the thumb points in the opposite direction. This means that $\mathbf{n}$ is replaced by $-\mathbf{n}$, and therefore

$$\mathbf{B}\times\mathbf{A} = -\mathbf{A}\times\mathbf{B}. \tag{2}$$

This shows that the cross product is not commutative, and we must pay close attention to the order of the factors.

If we keep (2) in mind and apply the definition (1) to the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ (Fig. 18.18), then we easily see that

$$\begin{aligned} \mathbf{i}\times\mathbf{j} &= -\mathbf{j}\times\mathbf{i} = \mathbf{k}, \\ \mathbf{j}\times\mathbf{k} &= -\mathbf{k}\times\mathbf{j} = \mathbf{i}, \\ \mathbf{k}\times\mathbf{i} &= -\mathbf{i}\times\mathbf{k} = \mathbf{j}, \end{aligned} \tag{3}$$

and also that

$$\mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = \mathbf{0}.$$

For example, the right-hand thumb rule says that the direction of $\mathbf{i}\times\mathbf{j}$ is the same as the direction of $\mathbf{k}$. But the area of the parallelogram determined by $\mathbf{i}$ and $\mathbf{j}$ is 1, and since $\mathbf{k}$ itself has length 1, we have

$$\mathbf{i}\times\mathbf{j} = \mathbf{k}.$$

The products (3) are easy to remember by visualizing the figure. Another way to remember them is to arrange $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ in cyclic order,

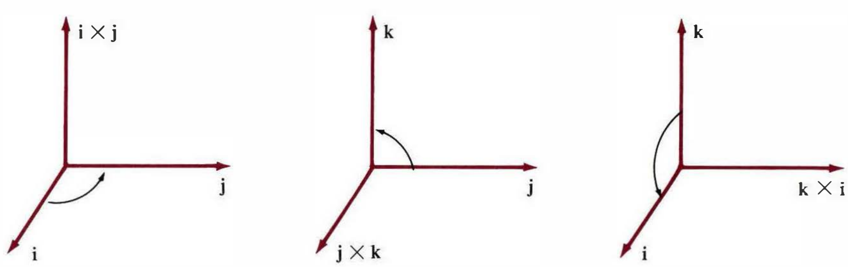


Figure 18.18

$$\mathbf{i} \to \mathbf{j} \to \mathbf{k},$$

and to observe that

(each unit vector) × (the next one) = (the third one).

Our next objective is to develop a convenient formula for calculating $\mathbf{A} \times \mathbf{B}$ in terms of the components of $\mathbf{A}$ and $\mathbf{B}$, where

$$\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \quad \text{and} \quad \mathbf{B} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}. \tag{4}$$

In order to multiply out the product

$$\mathbf{A} \times \mathbf{B} = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) = ?$$

we need to know that the cross product possesses the following algebraic properties:

$$(c\mathbf{A}) \times \mathbf{B} = c(\mathbf{A} \times \mathbf{B}) = \mathbf{A} \times (c\mathbf{B}), \tag{5}$$

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}, \tag{6}$$

$$(\mathbf{A} + \mathbf{B}) \times \mathbf{C} = \mathbf{A} \times \mathbf{C} + \mathbf{B} \times \mathbf{C}. \tag{7}$$

Property (5) is easily established directly from the definition (1). Property (7) follows from (6) by using (2),

$$\begin{aligned}(\mathbf{A} + \mathbf{B}) \times \mathbf{C} &= -[\mathbf{C} \times (\mathbf{A} + \mathbf{B})] \\ &= -(\mathbf{C} \times \mathbf{A} + \mathbf{C} \times \mathbf{B}) \\ &= -\mathbf{C} \times \mathbf{A} - \mathbf{C} \times \mathbf{B} \\ &= \mathbf{A} \times \mathbf{C} + \mathbf{B} \times \mathbf{C}.\end{aligned}$$

The real difficulty here is with the distributive law (6). There is no simple proof of this fact; and rather than hold up our progress by pausing to insert a complicated proof here, we simply take (6) for granted and continue on to our immediate objective. A proof of (6) is given in Remark 2 for the use of any students who may wish to examine it.

We continue with our task of multiplying out the cross product of the vectors (4). Remembering to pay close attention to the order of the factors, we have

$$\begin{aligned}\mathbf{A} \times \mathbf{B} &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\ &= a_1\mathbf{i} \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) + a_2\mathbf{j} \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\ &\qquad + a_3\mathbf{k} \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})\end{aligned}$$

so

$$\begin{aligned}\mathbf{A} \times \mathbf{B} = a_1b_1\mathbf{i} \times \mathbf{i} + a_1b_2\mathbf{i} \times \mathbf{j} + a_1b_3\mathbf{i} \times \mathbf{k} \\ + a_2b_1\mathbf{j} \times \mathbf{i} + a_2b_2\mathbf{j} \times \mathbf{j} + a_2b_3\mathbf{j} \times \mathbf{k} \\ + a_3b_1\mathbf{k} \times \mathbf{i} + a_3b_2\mathbf{k} \times \mathbf{j} + a_3b_3\mathbf{k} \times \mathbf{k}.\end{aligned}$$

By using (3), we now obtain the rather awkward formula

$$\mathbf{A} \times \mathbf{B} = \mathbf{i}(a_2b_3 - a_3b_2) - \mathbf{j}(a_1b_3 - a_3b_1) + \mathbf{k}(a_1b_2 - a_2b_1). \tag{8}$$

(The slightly strange way of writing the signs here has a purpose that will become clear below.)

It is not necessary to memorize formula (8), because there is an equivalent version involving determinants that is easy to remember. We recall that a determinant of order 2 is defined by

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1.$$

For example,

$$\begin{vmatrix} 3 & -2 \\ 4 & 5 \end{vmatrix} = 3 \cdot 5 - (-2) \cdot 4 = 23.$$

A determinant of order 3 can be defined in terms of determinants of order 2:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}. \tag{9}$$

Here we see that each number in the first row on the left is multiplied by the determinant of order 2 that remains when that number's row and column are deleted. We particularly notice the minus sign attached to the middle term on the right side of formula (9).

Even though a determinant of order 3 can be expanded along any row or column, we use only expansions along the first row, as in (9). For example,

$$\begin{aligned} \begin{vmatrix} 3 & 2 & -1 \\ 4 & 3 & 3 \\ -2 & 7 & 1 \end{vmatrix} &= 3 \begin{vmatrix} 3 & 3 \\ 7 & 1 \end{vmatrix} - 2 \begin{vmatrix} 4 & 3 \\ -2 & 1 \end{vmatrix} + (-1) \begin{vmatrix} 4 & 3 \\ -2 & 7 \end{vmatrix} \\ &= 3(3 \cdot 1 - 3 \cdot 7) - 2[4 \cdot 1 - 3 \cdot (-2)] + (-1)[4 \cdot 7 - 3 \cdot (-2)] \\ &= -54 - 20 - 34 = -108. \end{aligned}$$

Formula (8) for the vector product of $\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{B} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ is clearly equivalent to

$$\mathbf{A} \times \mathbf{B} = \mathbf{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}. \tag{10}$$

Motivated by (9), we now write (10) in the form

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}. \tag{11}$$

This is the concise and easily remembered formula for $\mathbf{A} \times \mathbf{B}$ that we have been seeking. The "symbolic determinant" here is to be evaluated by expanding along its first row, just as in equation (9). We emphasize that the components of the first vector $\mathbf{A}$ in $\mathbf{A} \times \mathbf{B}$ form the second row of the determinant in (11), and that the components of the *second* vector $\mathbf{B}$ form the *third* row of this determinant.*

*Some authors *define* $\mathbf{A} \times \mathbf{B}$ by formula (11). This approach has several disadvantages, one of which is that considerable effort is needed before the geometric nature of $\mathbf{A} \times \mathbf{B}$ (that is, its length and direction) can be understood. We prefer to define $\mathbf{A} \times \mathbf{B}$ directly, in terms of its length and direction, and to consider formula (11) as simply a convenient tool for making calculations. Definitions of vector operations that avoid dependence on explicit representations of vectors in terms of any particular coordinate system are called *invariant* or *coordinate-free*.

Example 1 Calculate the cross product of $\mathbf{A} = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ and $\mathbf{B} = \mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$.

Solution By formula (11) we have

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 4 \\ 1 & 5 & -3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} -1 & 4 \\ 5 & -3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 2 & 4 \\ 1 & -3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 2 & -1 \\ 1 & 5 \end{vmatrix}$$
$$= -17\mathbf{i} + 10\mathbf{j} + 11\mathbf{k}.$$

As a routine check to help guard ourselves against computational errors, we observe that our answer is perpendicular to $\mathbf{A}$ because $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{A} = -34 - 10 + 44 = 0$, and is perpendicular to $\mathbf{B}$ because $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{B} = -17 + 50 - 33 = 0$.

Example 2 Find all unit vectors perpendicular to both of the vectors $\mathbf{A} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{B} = -4\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$.

Solution Since $\mathbf{A} \times \mathbf{B}$ is automatically perpendicular to both $\mathbf{A}$ and $\mathbf{B}$, we compute

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ -4 & 3 & -5 \end{vmatrix} = \mathbf{i}\begin{vmatrix} -1 & 3 \\ 3 & -5 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 2 & 3 \\ -4 & -5 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 2 & -1 \\ -4 & 3 \end{vmatrix}$$
$$= -4\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}.$$

We next convert this into a unit vector in the same direction by dividing by its own length, which is $\sqrt{16 + 4 + 4} = \sqrt{24} = 2\sqrt{6}$:

$$\frac{-4\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}}{2\sqrt{6}} = \frac{-2\mathbf{i} - \mathbf{j} + \mathbf{k}}{\sqrt{6}}.$$

And finally, we introduce a plus-or-minus sign,

$$\pm\frac{-2\mathbf{i} - \mathbf{j} + \mathbf{k}}{\sqrt{6}},$$

because there are two possible directions.

Example 3 Find the area of the triangle whose vertices are $P = (2, -1, 3)$, $Q = (1, 2, 4)$, and $R = (3, 1, 1)$.

Solution Two sides of the triangle are represented by the vectors

$$\mathbf{A} = \overrightarrow{PQ} = (1 - 2)\mathbf{i} + (2 + 1)\mathbf{j} + (4 - 3)\mathbf{k} = -\mathbf{i} + 3\mathbf{j} + \mathbf{k},$$
$$\mathbf{B} = \overrightarrow{PR} = (3 - 2)\mathbf{i} + (1 + 1)\mathbf{j} + (1 - 3)\mathbf{k} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}.$$

The vector

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 3 & 1 \\ 1 & 2 & -2 \end{vmatrix} = -8\mathbf{i} - \mathbf{j} - 5\mathbf{k}$$

has magnitude $\sqrt{64 + 1 + 25} = \sqrt{90} = 3\sqrt{10}$, and this is equal to the area of the parallelogram with $\mathbf{A} = \overrightarrow{PQ}$ and $\mathbf{B} = \overrightarrow{PR}$ as adjacent sides. The area of the given triangle is clearly half the area of this parallelogram, and is therefore $\frac{3}{2}\sqrt{10}$.

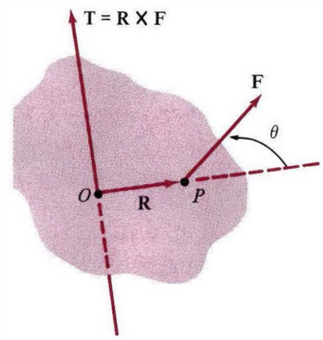


Figure 18.19 Torque vector.

Remark 1 The cross product arises quite naturally in many situations in physics. For example, if a force **F** is applied to a body at a point P (Fig. 18.19), and if **R** is the vector from a fixed origin O to P, then this force tends to rotate the body about an axis through O and perpendicular to the plane of **R** and **F**. The *torque vector* **T** defined by

$$\mathbf{T} = \mathbf{R} \times \mathbf{F}$$

specifies the direction and magnitude of this rotational effect, since $|\mathbf{R}||\mathbf{F}| \sin \theta$ is the moment of the force about the axis, namely, the product of the length of the lever arm and the scalar component of **F** perpendicular to **R**.

As another example, we mention the force **F** exerted on a moving charged particle by a magnetic field **B**. It turns out that

$$\mathbf{F} = q\mathbf{V} \times \mathbf{B},$$

where **V** is the velocity of the charged particle and q is the magnitude of its charge. This is the primary fact that causes the aurora borealis, or northern lights, which are produced by blasts of charged particles from the sun streaming through the magnetic field of the earth. This basic principle of electromagnetism also underlies the design and operation of cyclotrons and TV sets.

Remark 2 We now return to the problem of establishing the distributive law (6). We prove (6) only for unit vectors **A**, because once this has been done, an application of (5) allows us to obtain (6) immediately for vectors **A** of arbitrary length.

With a unit vector **A** and an arbitrary vector **V**, $\mathbf{A} \times \mathbf{V}$ can be constructed by performing the following two operations, shown on the left side of Fig. 18.20: First, project **V** on the plane perpendicular to **A** to obtain a vector $\mathbf{V}'$ of length $|\mathbf{V}| \sin \theta$; then rotate $\mathbf{V}'$ in this plane through an angle of 90° in the positive direction to obtain $\mathbf{V}''$, which is $\mathbf{A} \times \mathbf{V}$ since **A** is a unit vector. Each of these operations transforms a triangle into a triangle; so if we start with the three vectors **B**, **C**, and $\mathbf{B} + \mathbf{C}$ shown on the right, the final three vectors $\mathbf{B}''$, $\mathbf{C}''$, and $(\mathbf{B} + \mathbf{C})''$ still form a triangle, and therefore $(\mathbf{B} + \mathbf{C})'' = \mathbf{B}'' + \mathbf{C}''$. But this means that

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C},$$

and the argument is complete.

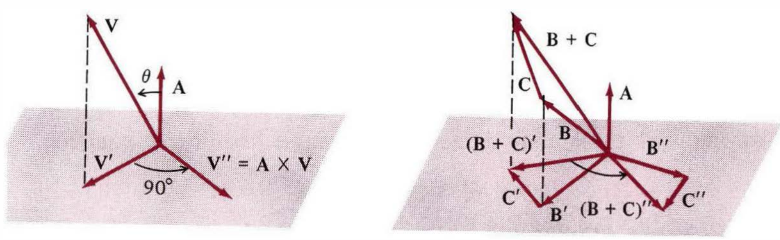


Figure 18.20 The distributive law.

PROBLEMS

1 Calculate $\mathbf{A} \times \mathbf{B}$ and check the result by showing that it is perpendicular to both $\mathbf{A}$ and $\mathbf{B}$:
(a) $\mathbf{A} = 3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$, $\mathbf{B} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$;
(b) $\mathbf{A} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\mathbf{B} = \mathbf{i} + \mathbf{j} + \mathbf{k}$;
(c) $\mathbf{A} = 5\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}$, $\mathbf{B} = -3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$;
(d) $\mathbf{A} = \mathbf{i}$, $\mathbf{B} = \mathbf{i} + \mathbf{j}$.

2 Find a vector $\mathbf{N}$ perpendicular to the plane of the three points $P = (1, -1, 4)$, $Q = (2, 0, 1)$, $R = (0, 2, 3)$.

3 Find the area of the triangle PQR in Problem 2.

4 Find the distance from the origin to the plane in Problem 2 by finding the scalar projection of $\overrightarrow{OP}$ along the vector $\mathbf{N}$.

5 If $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = 0$, what can be concluded about the configuration of $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$?

6 Show that $|\mathbf{A} \times \mathbf{B}|^2 = |\mathbf{A}|^2|\mathbf{B}|^2 - (\mathbf{A} \cdot \mathbf{B})^2$.

7 Show that the cross product is not associative by showing that

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \neq (\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$$

for the three vectors $\mathbf{A} = \mathbf{i} + \mathbf{j}$, $\mathbf{B} = \mathbf{j}$, $\mathbf{C} = \mathbf{k}$.

8 Show that the cross product of each pair of the following vectors is parallel to the third: $\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, $\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{i} - \mathbf{k}$. What does this tell us about the configuration of the vectors?

9 Show that if $\mathbf{A}$ is a nonzero vector and $\mathbf{A} \times \mathbf{B} = \mathbf{A} \times \mathbf{C}$, then $\mathbf{B} = \mathbf{C}$ is not necessarily true.

10 If $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are mutually perpendicular, show that $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{0}$.

11 Let P_1 and Q_1 be two points on a line L_1, and let P_2 and Q_2 be two points on a line L_2. If L_1 and L_2 are not parallel, then the perpendicular distance d between them is the absolute value of the scalar projection of $\overrightarrow{P_1P_2}$ on a unit vector that is perpendicular to both lines. Why?
(a) Show that

$$d = \left| \overrightarrow{P_1P_2} \cdot \frac{\overrightarrow{P_1Q_1} \times \overrightarrow{P_2Q_2}}{|\overrightarrow{P_1Q_1} \times \overrightarrow{P_2Q_2}|} \right|.$$

(b) Find d if L_1 is the line determined by $P_1 = (-1, 1, 1)$ and $Q_1 = (1, 0, 0)$, and L_2 is the line determined by $P_2 = (3, 1, 0)$ and $Q_2 = (4, 5, -1)$.

12 If $\mathbf{A} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ is normal to one plane and $\mathbf{B} = -\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ is normal to another plane, do the planes necessarily intersect? Give a reason for your answer. If they do intersect, find a vector parallel to their line of intersection.

18.4 LINES AND PLANES

Since all the machinery of vector *algebra* is now in place, it might be expected that we would next turn to the *calculus* of vector functions in three-dimensional space. However, a full study of this subject belongs to a later course in advanced calculus or vector analysis, and is not part of our purpose in this book.

Nevertheless, on a few occasions we will need to consider the position vector $\mathbf{R}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ of a point P that moves along a space curve, as shown in Fig. 18.21. The derivative of this function is defined in the obvious way,

$$\frac{d\mathbf{R}}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{R}(t + \Delta t) - \mathbf{R}(t)}{\Delta t},$$

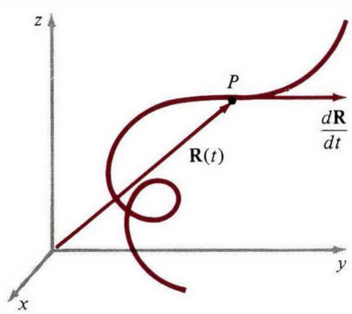


Figure 18.21 A space curve.

and has all the properties we expect on the basis of our experience in Chapter 17. In particular, $d\mathbf{R}/dt$ is tangent to the path at the point P, and is the velocity of P if the parameter t is time, and the unit tangent vector if t is arc length.

With these brief remarks we put aside the calculus of vector functions, and turn to the main subject of the rest of this chapter, namely, the analytic geometry of lines, planes, and curved surfaces in three-dimensional space. We shall find that the vector algebra discussed in the preceding sections is a very valuable tool for this work.

As we know, in plane analytic geometry a single first-degree equation,

$$ax + by + c = 0,$$

is the equation of a straight line (assuming that a and b are not both zero). However, we shall see that in the geometry of three dimensions such an equation represents a plane, and therefore it is not possible to represent a line in space by any single first-degree equation.

We begin with the study of lines. A line in space can be given geometrically in three ways: as the line through two points, as the intersection of two planes, or as the line through a point in a specified direction. The third way is the most important for us.

Suppose L is the line in space that passes through a given point $P_0 = (x_0, y_0, z_0)$ and is parallel to a given nonzero vector

$$\mathbf{V} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k},$$

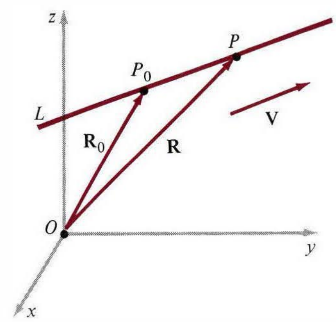


Figure 18.22 A line in space.

as shown in Fig. 18.22. Then another point $P = (x, y, z)$ lies on the line L if and only if the vector $\overrightarrow{P_0P}$ is parallel to the vector $\mathbf{V}$. That is, P lies on L if and only if $\overrightarrow{P_0P}$ is a scalar multiple of $\mathbf{V}$, so that

$$\overrightarrow{P_0P} = t\mathbf{V} \tag{1}$$

for some real number t. If $\mathbf{R}_0 = \overrightarrow{OP_0}$ and $\mathbf{R} = \overrightarrow{OP}$ are the position vectors of P_0 and P, then $\overrightarrow{P_0P} = \mathbf{R} - \mathbf{R}_0$ and (1) gives

$$\mathbf{R} = \mathbf{R}_0 + t\mathbf{V}, \tag{2}$$

which is the *vector equation* of L. As t varies from $-\infty$ to ∞, the point P traverses the entire infinite line L, moving in the direction of $\mathbf{V}$.

If we write (2) in the form

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k} + t(a\mathbf{i} + b\mathbf{j} + c\mathbf{k})$$

and equate the coefficients of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, we get the three scalar equations

$$\begin{aligned} x &= x_0 + at, \\ y &= y_0 + bt, \\ z &= z_0 + ct. \end{aligned} \tag{3}$$

These are the *parametric equations* of the line L through the point $P_0 = (x_0, y_0, z_0)$ and parallel to the vector $\mathbf{V} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$. Observe that the parametric equations of a straight line are not unique. The numbers x_0, y_0, and z_0 can be replaced by the coordinates of any other point on L, and a, b, and c can be replaced by the components of any other nonzero vector parallel to L, and the resulting parametric equations will be completely equivalent to equations (3) in the sense that they describe the same line.

In order to obtain the Cartesian equations of the line, we eliminate the parameter from equations (3) by equating the three expressions obtained by solving for t. This gives

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}. \tag{4}$$

These are called the *symmetric equations* of the line L. If any one of the constants a, b, c is zero in a denominator of (4), then the corresponding numerator must also be zero. This is easy to see from the parametric form (3), which shows, for example, that if

$$x = x_0 + at \qquad \text{and} \qquad a = 0,$$

then $x = x_0$. Thus, when one of the denominators in (4) vanishes, we interpret this as meaning that the corresponding numerator must also vanish. With this interpretation, equations (4) can always be used, even though division by zero is normally forbidden.

Example 1 A line L goes through the points $P_0 = (3, -2, 1)$ and $P_1 = (5, 1, 0)$. Find the parametric equations and the symmetric equations of L. Also find the points at which this line pierces the three coordinate planes.

Solution The line L is parallel to the vector $\overrightarrow{P_0P_1} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$, so by using P_0 as the known point on the line, equations (3) give the parametric equations

$$x = 3 + 2t,$$
$$y = -2 + 3t,$$
$$z = 1 - t.$$

By eliminating t, we obtain the symmetric equations

$$\frac{x-3}{2} = \frac{y+2}{3} = \frac{z-1}{-1}.$$

To find the point at which L pierces the xy-plane, we set $z = 0$ in the third parametric equation and see that $t = 1$. With this value of t, $x = 5$ and $y = 1$, so the point is $(5, 1, 0)$. Similarly, $x = 0$ implies that $t = -\frac{3}{2}$, so the point in the yz-plane is $(0, -\frac{13}{2}, \frac{5}{2})$; and $y = 0$ implies $t = \frac{2}{3}$, so the point in the xz-plane is $(\frac{13}{3}, 0, \frac{1}{3})$.

Now we turn to the study of planes. A plane can also be characterized in several ways: as the plane through three noncollinear points, as the plane through a line and a point not on the line, or as the plane through a point and perpendicular to a specified direction. Again, the third approach is the most convenient for us.

Consider the plane that passes through a given point $P_0 = (x_0, y_0, z_0)$ and is perpendicular to a given nonzero vector

$$\mathbf{N} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}, \tag{5}$$

as shown in Fig. 18.23. Another point $P = (x, y, z)$ lies on this plane if and only if the vector P_0P is perpendicular to the vector $\mathbf{N}$, which means that

$$\mathbf{N} \cdot \overrightarrow{P_0P} = 0. \tag{6}$$

If $\mathbf{R}_0 = \overrightarrow{OP_0}$ and $\mathbf{R} = \overrightarrow{OP}$ are the position vectors of P_0 and P, so that $\overrightarrow{P_0P} = \mathbf{R} - \mathbf{R}_0$, then (6) becomes

$$\mathbf{N} \cdot (\mathbf{R} - \mathbf{R}_0) = 0. \tag{7}$$

This is the *vector equation* of the plane under discussion.

Since $\mathbf{R} - \mathbf{R}_0 = (x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}$, (7) can be written out in the scalar form

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0. \tag{8}$$

This is the *Cartesian equation* of the plane through the point $P_0 = (x_0, y_0, z_0)$ with normal vector $\mathbf{N} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$. For example, the equation of the plane through $P_0 = (5, -3, 1)$ with normal vector $\mathbf{N} = 4\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ is

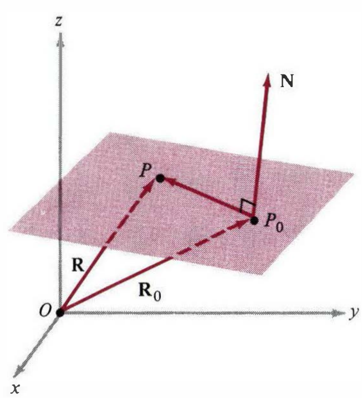


Figure 18.23 A plane in space.

$$4(x - 5) + 3(y + 3) - 2(z - 1) = 0$$

or

$$4x + 3y - 2z = 9.$$

Observe that the coefficients of x, y, and z in the last equation are the components of the normal vector. This is always the case, for equation (8) can be written in the form

$$ax + by + cz = d, \tag{9}$$

where $d = ax_0 + by_0 + cz_0$; and the coefficients of x, y, and z in this equation are clearly the components of the normal vector (5). Conversely, every *linear equation* in x, y, and z of the form (9) represents a plane with normal vector $\mathbf{N} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ if the coefficients a, b, and c are not all zero. To see this, we notice that if (for instance) $a \neq 0$, then this permits us to choose y_0 and z_0 arbitrarily and solve the equation $ax_0 + by_0 + cz_0 = d$ for x_0. With these values, (9) can be written as

$$ax + by + cz = ax_0 + by_0 + cz_0$$

or

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0,$$

and this is immediately recognizable as the equation of the plane through (x_0, y_0, z_0) with normal vector $\mathbf{N} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.

Example 2 Find an equation for the plane through the three points $P_0 = (3, 2, -1)$, $P_1 = (1, -1, 3)$, and $P_2 = (3, -2, 4)$.

Solution To use equation (8), we must find a vector $\mathbf{N}$ that is normal to the plane. This is easy to do by using the cross product. We compute

$$\mathbf{N} = \overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & -3 & 4 \\ 0 & -4 & 5 \end{vmatrix} = \mathbf{i} + 10\mathbf{j} + 8\mathbf{k}.$$

Since $\overrightarrow{P_0P_1}$ and $\overrightarrow{P_0P_2}$ lie in the plane, their cross product $\mathbf{N}$ is normal to the plane. Using equation (8) with P_0 as the given point, our plane has the equation

$$(x - 3) + 10(y - 2) + 8(z + 1) = 0$$

or

$$x + 10y + 8z = 15,$$

after simplification.

Example 3 Find the point at which the line

$$\frac{x-2}{1} = \frac{y+3}{2} = \frac{z-4}{2}$$

pierces the plane $x + 2y + 2z = 22$.

Solution To find parametric equations for the line, we introduce t as the common ratio in the given symmetric equations,

$$\frac{x-2}{1} = \frac{y+3}{2} = \frac{z-4}{2} = t,$$

which gives

$$x = 2 + t, \qquad y = -3 + 2t, \qquad z = 4 + 2t.$$

We want the value of t for which the variable point (x, y, z) on the line lies on the given plane. By substituting these equations into the equation of the plane, we obtain

$$(2 + t) + 2(-3 + 2t) + 2(4 + 2t) = 22,$$

so $t = 2$ at the point where the line pierces the plane. By substituting $t = 2$ back in the parametric equations of the line, we find that the desired point is $(4, 1, 8)$.

Example 4 Find the cosine of the angle between the two planes $x + 4y - 4z = 9$ and $x + 2y + 2z = -3$. Also, find parametric equations for the line of intersection of these planes.

Solution Clearly the angle θ between two planes is the angle between their normals (Fig. 18.24). By inspecting the equations of the given planes, we see at once that their normals are

$$\mathbf{N}_1 = \mathbf{i} + 4\mathbf{j} - 4\mathbf{k}, \qquad \mathbf{N}_2 = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}.$$

We therefore use the dot product to obtain

$$\cos\theta = \frac{\mathbf{N}_1 \cdot \mathbf{N}_2}{|\mathbf{N}_1||\mathbf{N}_2|} = \frac{1}{3\sqrt{33}}.$$

From this we can find the angle θ if we wish, by tables or otherwise.

To find parametric equations for the line of intersection, we need a vector $\mathbf{V}$ parallel to this line and a point on the line. We find $\mathbf{V}$ by computing the cross product of $\mathbf{N}_1$ and $\mathbf{N}_2$,

$$\mathbf{V} = \mathbf{N}_1 \times \mathbf{N}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & -4 \\ 1 & 2 & 2 \end{vmatrix} = 16\mathbf{i} - 6\mathbf{j} - 2\mathbf{k}.$$

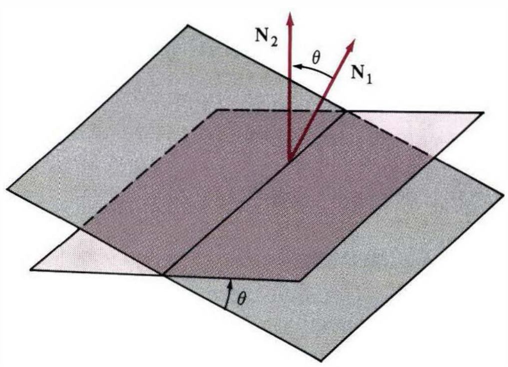


Figure 18.24

Since any vector parallel to the line will do, we divide by 2 and use the slightly simpler vector $8\mathbf{i} - 3\mathbf{j} - \mathbf{k}$. To find a point on the line, we can set $z = 0$ and solve the resulting system in the unknowns x and y,

$$x + 4y = 9,$$
$$x + 2y = -3.$$

This yields $x = -15$, $y = 6$. The desired point is therefore $(-15, 6, 0)$, and the parametric equations of the line are

$$x = -15 + 8t,$$
$$y = 6 - 3t,$$
$$z = -t.$$

We repeat that there is nothing unique about these equations, for we could have found a point on the line in many other ways and there are many different vectors parallel to the line.

As we remarked at the beginning of this section, any two intersecting planes determine a straight line in space. The equations of the two planes are satisfied simultaneously only by points on the line of intersection. From this point of view, a pair of linear equations considered as a simultaneous system can be interpreted as representing a line, namely, the line of intersection of the two planes represented by the individual equations. (Of course, the planes must actually intersect, and not be parallel or identical.) Thus, in Example 4 the pair of simultaneous equations

$$x + 4y - 4z = 9,$$
$$x + 2y + 2z = -3,$$

represents the line discussed in that example. We also point out that the symmetric equations (4) are equivalent to the three simultaneous equations

$$b(x - x_0) - a(y - y_0) = 0,$$
$$c(x - x_0) - a(z - z_0) = 0,$$
$$c(y - y_0) - b(z - z_0) = 0.$$

These are the equations of three planes that intersect in the line L represented by (4). The first has normal vector $b\mathbf{i} - a\mathbf{j}$, which is parallel to the xy-plane, so the first plane is perpendicular to the xy-plane. Similarly, the second plane is perpendicular to the xz-plane and the third is perpendicular to the yz-plane. Any pair of these equations represents the line L, which is the intersection of the corresponding pair of planes.

PROBLEMS

1 Label each of the following statements as true or false:
(a) Two planes perpendicular to a line are parallel.
(b) Two lines perpendicular to a third line are parallel.
(c) Two planes parallel to a third plane are parallel.
(d) Two lines perpendicular to a plane are parallel.
(e) Two planes parallel to a line are parallel.
(f) Two lines parallel to a third line are parallel.
(g) Two planes perpendicular to a third plane are parallel.
(h) Two lines parallel to a plane are parallel.

2 What conclusion can be drawn about the lines

$$\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c},$$

$$\frac{x-x_0}{A}=\frac{y-y_0}{B}=\frac{z-z_0}{C}$$

if $aA+bB+cC=0$?

3 What conclusion can be drawn about the lines

$$\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c},$$

$$\frac{x-x_1}{A}=\frac{y-y_1}{B}=\frac{z-z_1}{C}$$

if $a/A=b/B=c/C$?

4 Write symmetric equations for the line through the point $(3, 0, -2)$ and parallel to
(a) the vector $4\mathbf{i}-3\mathbf{j}+5\mathbf{k}$;
(b) the line $(x+1)/7=(y-2)/2=z/(-3)$;
(c) the x-axis.

5 Write parametric equations for the line through the point $(2, -1, -3)$ and parallel to
(a) the vector $\mathbf{i}+4\mathbf{j}-2\mathbf{k}$;
(b) the line $x/3=(y+7)/(-1)=(z-3)/6$;
(c) the line $x=2t-3$, $y=3-2t$, $z=5t-4$.

6 Write symmetric equations for the line through the points
(a) $(2, -1, 3)$ and $(5, 2, -2)$;
(b) $(7, 3, -1)$ and $(3, -1, 3)$.

7 Write parametric equations for the line through the points
(a) $(2, 0, 3)$ and $(-1, 3, 5)$;
(b) $(4, 2, -1)$ and $(0, 2, -1)$.

8 If L is the line through the points $(-6, 6, -4)$ and $(12, -6, 2)$, find the points where L pierces the coordinate planes.

9 Show that the lines

$$x=1+t, \qquad y=2t, \qquad z=1+3t$$

and

$$x=3s, \qquad y=2s, \qquad z=2+s$$

intersect, and find their point of intersection.

10 Find the distance between the lines

(a) $$\frac{x-2}{-1}=\frac{y-3}{4}=\frac{z}{2}$$
and
$$\frac{x+1}{1}=\frac{y-2}{0}=\frac{z+1}{2};$$

(b) $x=2t-4, \quad y=4-t, \quad z=-2t-1$
and $x=4t-5, \quad y=-3t+5, \quad z=-5t+5$.

11 Find the distance from the origin to the line

$$\frac{x-4}{3}=\frac{y-2}{4}=\frac{4-z}{5}.$$

12 (a) As a function of t, find the distance D from the point $P_0=(1, 2, 3)$ to a variable point on the line

$$x=3+t, \qquad y=2+t, \qquad z=1+t.$$

(b) By differentiating, find the value of t that minimizes D, find the actual minimum distance, and find the corresponding point P_1 on the line.
(c) Verify that the vector $\overrightarrow{P_0P_1}$ is perpendicular to the line.

13 Find the equation of the plane that contains the point $(1, 3, 1)$ and the line $x=t$, $y=t$, $z=t+2$.

14 Find symmetric equations for the line of intersection of each of the following pairs of planes:
(a) $2x+y+z=0$, $3x+4y-z=10$;
(b) $2x+3y+5z=21$, $3x-2y+z=12$.

15 Use vector methods to show that the distance D from the point (x_0, y_0, z_0) to the plane $ax+by+cz+d=0$ is given by the formula

$$D=\frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}.$$

16 Show that the planes $ax+by+cz+d_1=0$ and $ax+by+cz+d_2=0$ are parallel, and that the distance between them is

$$\frac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}}.$$

17 Find the distance between the planes $x-2y+4z=1$ and $2x-4y+8z=-14$.

18 Find an equation for the plane that is parallel to the plane $2x-5y+3z=7$ and passes through the point $(5, 2, 3)$.

19 Consider the sphere of radius 3 with its center at the origin. The plane tangent to this sphere at $(1, 2, 2)$ intersects the x-axis at a point P. Find the coordinates of P.

20 Find the value of the parameter t for which the planes

$$3x-4y+2z+9=0,$$
$$3x+4y-tz+7=0,$$

are perpendicular.

21 Verify that the planes

$$2x+3y+4z-1=0,$$
$$x-2y+3z-4=0,$$

intersect in a line L. Find symmetric equations for L in two ways:
(a) by finding two points on L and going on from there;
(b) by first eliminating x from the given equations, and then y, to find two planes through L that are perpendicular to the yz-plane and the xz-plane, respectively, and then solving each of these equations for z and equating all the z's.

22 Show that the two sets of equations

$$\frac{x-4}{3}=\frac{y-6}{4}=\frac{z+9}{-12}$$

and

$$\frac{x-1}{-6}=\frac{y-2}{-8}=\frac{z-3}{24}$$

represent the same straight line.

23 Let p_1 and p_2 be two planes that intersect in a line L and have equations

$$a_1x+b_1y+c_1z+d_1=0,$$

$$a_2x+b_2y+c_2z+d_2=0.$$

If k is a constant, show that

$$(a_1x+b_1y+c_1z+d_1)+k(a_2x+b_2y+c_2z+d_2)=0$$

is the equation of a plane containing L. For various values of k, this equation represents every member of the family of all planes containing L, with one exception. What is this exception?

24 Find the equation of the plane that contains the intersection of the planes $2x+3y-z=1$ and $3x-y+5z=2$ and passes through $(1, 4, 1)$.

25 Find the equation of the plane that contains the intersection of the planes $x-2y-5z=3$ and $5x+y-z=1$ and is parallel to $4x+3y+4z+7=0$.

26 Find the coordinates of the point P at which the line

$$\frac{x-1}{3}=\frac{y+3}{4}=\frac{z-3}{2}$$

pierces the plane $3x+4y+5z=76$.

27 Show that the line

$$\frac{x+8}{9}=\frac{y-10}{-4}=\frac{z-9}{-6}$$

lies in the plane $2x-3y+5z=-1$.

28 Show that the line of intersection of the planes

$$x+y-z=0 \quad \text{and} \quad x-y-5z+7=0$$

is parallel to the line

$$\frac{x+3}{3}=\frac{y-1}{-2}=\frac{z-5}{1}.$$

29 Find the cosine of the angle between the given planes:
(a) $2x-y+2z=3$, $3x+2y-6z=7$;
(b) $5x-3y+2z=3$, $x+3y+2z=-11$.

30 Show that the single equation

$$(2x+y-z-3)^2+(x+2y-3z+5)^2=0$$

represents a straight line in space. (But this equation is of the second degree.)

18.5 CYLINDERS AND SURFACES OF REVOLUTION

We know that the graph of an equation $f(x, y)=0$ is usually a curve in the xy-plane. In just the same way, the graph of an equation

$$F(x, y, z)=0 \tag{1}$$

is usually a surface in xyz-space. The simplest surfaces are planes, and we saw in Section 18.4 that the equation of a plane is a linear equation that can be written in the form

$$ax+by+cz+d=0;$$

that is, it contains only first-degree terms in the variables x, y, and z. In this section and the next we examine a few other simple surfaces containing terms of higher degree that often appear in multivariable calculus.

Cylinders are the next surfaces after planes in order of complexity. To understand what these surfaces are, we consider a plane curve C and a line L not parallel to the plane of C. By a *cylinder* we mean the geometric figure in space that is generated (or swept out) by a straight line moving parallel to L and passing through C (Fig. 18.25).* The moving line is called the *generator* of the cylinder. The cylinder can be thought of as consisting of infinitely many parallel lines,

*This concept includes the familiar *right circular* cylinders of elementary geometry, for which the curve C is a circle and the line L is perpendicular to the plane of the circle. In geometry the adjectives are often omitted, because no other kinds of cylinders are considered. However, it should be noticed that when C is itself a straight line, the cylinder is a plane, so cylinders also include planes as special cases.

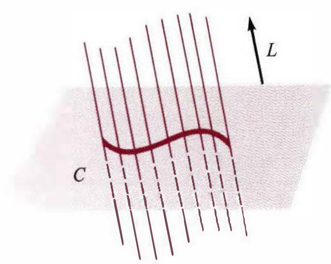


Figure 18.25 A general cylinder.

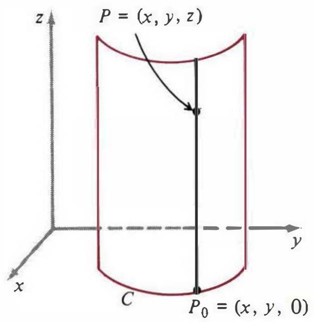


Figure 18.26

called *rulings,* corresponding to various positions of the generator. This is suggested in the figure.

For example, suppose that the given curve C is the curve

$$f(x, y) = 0 \tag{2}$$

in the xy-plane, and let the generator be parallel to the z-axis, as shown in Fig. 18.26. *Then exactly the same equation* (2) *is the equation of the cylinder in three-dimensional space.* The reason for this is that the point $P = (x, y, z)$ lies on the cylinder if and only if the point $P_0 = (x, y, 0)$ lies on the curve C, and this happens if and only if $f(x, y) = 0$. The essential feature of (2) as the equation of the cylinder is that it is an equation of the form (1) from which the variable z is missing. To express this in another way, the fact that we are dealing with a cylinder whose rulings are parallel to the z-axis means that for a point $P = (x, y, z)$, the value of z has no bearing on whether P lies on the cylinder or not; and since only the variables x and y are relevant to this issue, only the variables x and y can be present in the equation of the cylinder—that is, z must be missing from this equation.

Example 1 Sketch the cylinder

$$\frac{x^2}{9} + \frac{y^2}{4} = 1.$$

Solution This appears to be the equation of an ellipse in the xy-plane. However, it is stated that this is a cylinder, and since the variable z is missing from the equation, the rulings of this cylinder are parallel to the z-axis. In Fig. 18.27, the ellipse in the xy-plane is drawn first, then two vertical rulings, then a horizontal elliptical cross section above the xy-plane. In spite of the limitations of our figure (which we hope students will try to overcome by an active use of imagination), it should be remembered that all rulings on a cylinder extend to infinity in both directions. This surface is called an *elliptic cylinder.*

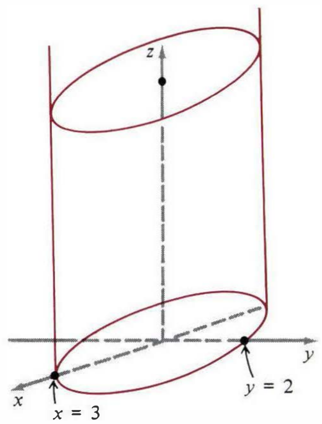


Figure 18.27 Elliptic cylinder.

It is clear that this discussion can be carried through for a cylinder with rulings parallel to any coordinate axis. We therefore have the conclusion that *any equation in rectangular coordinates x, y, z with one variable missing represents a cylinder whose rulings are parallel to the axis corresponding to the missing variable.*

Example 2 Sketch the cylinder $z = x^2$.

Solution In the xz-plane, this is the equation of a parabola with vertex at the origin that opens in the positive z-direction. However, we know that we are dealing with a cylinder, and since the variable y is missing from the equation, the rulings of this cylinder are parallel to the y-axis. In Fig. 18.28 the parabola in the xz-plane is drawn first, then several rulings, and then a second parabolic cross section located to the right of the xz-plane. This surface can be described as a *parabolic cylinder.*

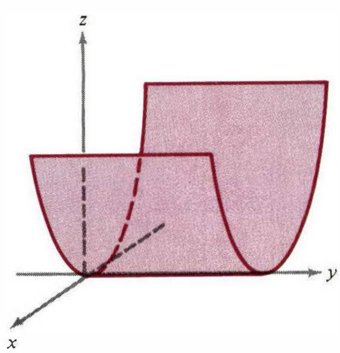


Figure 18.28 Parabolic cylinder.

Another way to generate a surface by using a plane curve C is to revolve the curve (in space) about a line L in its plane. The resulting surface is called a *sur-*

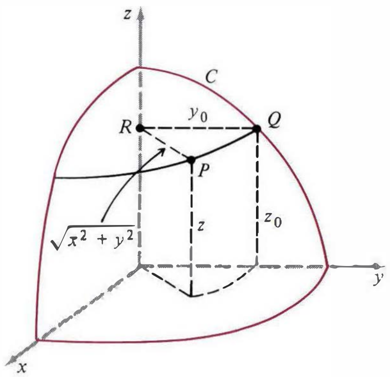


Figure 18.29

face of revolution with *axis L*. In Chapter 7 we became acquainted with surfaces of revolution by calculating their areas as an application of definite integrals. We now consider the equations of these surfaces.

Suppose, for example, that the curve C lies in the yz-plane and has equation

$$f(y, z) = 0. \tag{3}$$

As this curve is revolved about the z-axis, a typical point $P = (x, y, z)$ on the resulting surface comes from a point Q on C, as shown in Fig. 18.29. Since Q lies on C, its coordinates (y_0, z_0) satisfy (3),

$$f(y_0, z_0) = 0. \tag{4}$$

But the relation of P to Q tells us that $z_0 = z$ and $y_0 = \sqrt{x^2 + y^2}$, so (4) yields

$$f(\sqrt{x^2 + y^2}, z) = 0 \tag{5}$$

as the equation of the surface of revolution. Briefly, as Q swings out to the point P on the surface, the distances QR and PR to the z-axis are equal, and we get equation (5) by replacing y in (3) by $\sqrt{x^2 + y^2}$. Equation (5) assumes that $y \geq 0$ on C. If y is positive on some parts of C and negative on others, we must replace y in (3) by $\pm\sqrt{x^2 + y^2}$ to get

$$f(\pm\sqrt{x^2 + y^2}, z) = 0$$

as the equation of the complete surface. The awkward radical with its plus-or-minus sign can usually be eliminated by squaring.

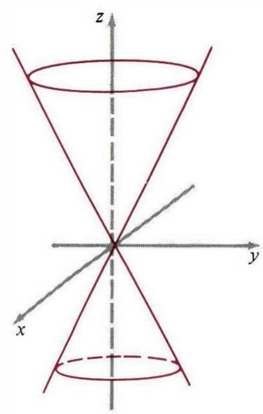


Figure 18.30 Cone.

Example 3 If the line $z = 3y$ in the yz-plane is revolved about the z-axis, the resulting surface of the revolution is clearly a right circular cone of two nappes with vertex at the origin and axis the z-axis (Fig. 18.30). To get the equation of this cone, we replace y in the equation $z = 3y$ by $\pm\sqrt{x^2 + y^2}$ and then rationalize by squaring:

$$z = \pm 3\sqrt{x^2 + y^2}, \qquad z^2 = 9(x^2 + y^2).$$

If we had merely replaced y by $\sqrt{x^2 + y^2}$ to obtain

$$z = 3\sqrt{x^2 + y^2},$$

we would have had the equation of only the upper nappe of the cone, the part where $z \geq 0$.

In essentially the same way, we can obtain equations for surfaces of revolution with the x-axis or the y-axis as the axis of symmetry.

PROBLEMS

Sketch the cylinders whose equations are given in Problems 1–8. If a cylinder has an obvious name, state it.

1 $y = x^2$.
2 $y^2 + 4z^2 = 16$.
3 $x = \sin y$.
4 $xz = 4$.
5 $x + 3z = 6$.
6 $x^2 + z^2 = 9$.
7 $x = \tan y$, $-\pi/2 < y < \pi/2$.
8 $y = e^x$.
9 The rulings of a cylinder are parallel to the y-axis. Its intersection with the xz-plane is a circle with center $(0, 0, a)$ and radius a. Sketch the cylinder and find its equation.
10 The rulings of a cylinder are parallel to the x-axis. Its intersection with the yz-plane is a parabola with vertex at $(0, 0, 0)$ and focus at $(0, 0, -p)$. Sketch the cylinder and find its equation.
11 Find the equation of the surface of revolution generated by revolving the curve $z = e^{-y^2}$ about
(a) the z-axis; (b) the y-axis.
Sketch both surfaces.
12 Find the equation of the surface of revolution generated by revolving the circle $(y - b)^2 + z^2 = a^2$ $(a < b)$ about
(a) the z-axis; (b) the y-axis.
Sketch both surfaces.
13 In each of the following, write the equation for the surface of revolution generated by revolving the given curve about the indicated axis, and sketch the surface:
(a) $y = z^2$, the y-axis;
(b) $9x^2 + 4y^2 = 36$, the y-axis;
(c) $z = 4 - x^2$, the z-axis;
(d) $x = y^2$, the x-axis.
14 Any direction in space not parallel to the xy-plane can be specified by a vector of the form $\mathbf{V} = a\mathbf{i} + b\mathbf{j} + \mathbf{k}$ (why?). If a curve C in the xy-plane has the equation $f(x, y) = 0$, show that the equation of the cylinder generated by a moving line that is parallel to $\mathbf{V}$ and passes through C (Fig. 18.31) is

$$f(x - az, y - bz) = 0.$$

Hint: Write the symmetric equations of the line through a point $(x_0, y_0, 0)$ on C and parallel to $\mathbf{V}$.

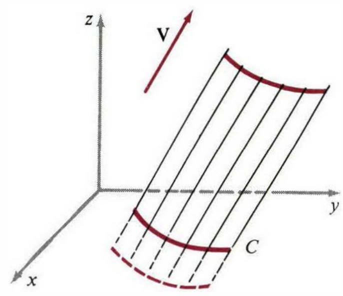


Figure 18.31

15 Find the equation of the cylinder generated by a line through the circle $x^2 + y^2 = 6x$ in the xy-plane that moves parallel to the vector $\mathbf{V} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$.
16 Find the equation of the cylinder generated by a line through the parabola $y = x^2$ in the xy-plane that moves parallel to the vector $\mathbf{V} = -2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$.

18.6 QUADRIC SURFACES

In Section 15.6 we learned that the graph of a second-degree equation in the variables x and y is always a conic section—a parabola, an ellipse, a hyperbola, or perhaps some degenerate form of one of these curves, such as a point, the empty set, or a pair of straight lines.

In three-dimensional space the most general equation of the second degree is

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0. \tag{1}$$

We assume that not all of the coefficients $A, B, \ldots, F$ are zero, so that the degree of the equation is really 2 instead of 1 or 0. The graph of such an equation is called a *quadric surface*. We have already encountered several quadric surfaces, such as spheres and parabolic, elliptic, and hyperbolic cylinders, but there are a number of others as well. Indeed, if we set aside the familiar case of cylinders, then by suitable rotations and translations of the coordinate axes—which we do not discuss—it is possible to simplify any equation of the form (1) and thereby show that there are exactly six distinct kinds of nondegenerate quadric surfaces:

1 The ellipsoid.
2 The hyperboloid of one sheet.
3 The hyperboloid of two sheets.
4 The elliptic cone.
5 The elliptic paraboloid.
6 The hyperbolic paraboloid.

In the following we give an example of each type of surface in which the equation appears in as simple a form as possible.

Students should become familiar with these surfaces and their equations, and in particular should try to understand how the shape of each surface is related to the special features of its equation. For the purpose of visualizing and sketching a surface, it is often useful to examine its *sections,* which are the curves of intersection of the surface with planes

$$x = k, \qquad y = k, \qquad z = k$$

parallel to the coordinate planes. We point out explicitly that every second-degree section of every quadric surface is a conic section. Sections that are closed curves are usually the easiest to sketch, and therefore we look for elliptic sections and sketch these first. Symmetry considerations should also be kept in mind.

In the following examples, the numbers a, b, and c are all assumed to be positive. We comment informally rather than exhaustively on the surface considered in each example.

Example 1 The *ellipsoid*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{2}$$

is shown in Fig. 18.32. Since only even powers of x, y, and z occur in the equation, this surface is symmetric about each coordinate plane. The sections in the xz- and yz-planes are the ellipses

$$\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1, \qquad \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

with a common vertical axis. The section in a horizontal plane $z = k$ is the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{k^2}{c^2},$$

and this decreases in size as k varies from 0 to c or $-c$. The numbers a, b, and c are the intercepts on the coordinate axes, and are called the *semiaxes.* If two of the semiaxes are equal, the ellipsoid is called a *spheroid*—an *oblate* spheroid

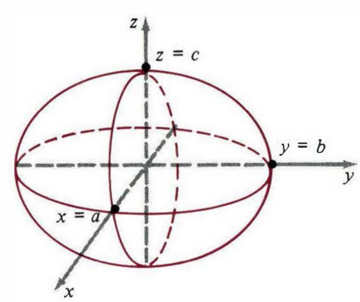


Figure 18.32 Ellipsoid.

if it is flattened like a "flying saucer," and a *prolate* spheroid if it is elongated like a football. Of course, if $a = b = c$, then the ellipsoid is a sphere.

Example 2 The graph of the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \tag{3}$$

is a *hyperboloid of one sheet* (Fig. 18.33). If we write the equation in the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2} + 1, \tag{4}$$

then we see that all its horizontal sections in planes $z = k$ are ellipses, and that these ellipses grow larger as their planes move up or down from the xy-plane, the smallest ellipse being the one in the xy-plane. The section of the surface in the yz-plane is the hyperbola

$$\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

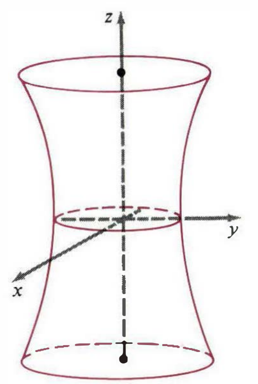


Figure 18.33 Hyperboloid of one sheet.

It is this hyperbola that binds together the horizontal elliptical sections into a smooth surface. The phrase "of one sheet" is used because this surface consists of one piece, in contrast to the hyperboloid discussed in the next example, which consists of two pieces. Observe that equation (3) is obtained from (2) by changing the sign of the third term on the left; we get the same kind of surface no matter which of these terms has its sign changed.

Example 3 The *hyperboloid of two sheets*

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{5}$$

is shown in Fig. 18.34. This equation is obtained from (2) by changing the signs of the first two terms on the left. (The reason for this choice is explained below.) If we write the equation in the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2} - 1, \tag{6}$$

then we see that all its horizontal sections in planes $z = k$ with $k \geq c$ or $k \leq -c$ are ellipses or single points, while sections in planes $z = k$ with $|k| < c$ are empty. The section in the yz-plane is the hyperbola

$$\frac{z^2}{c^2} - \frac{y^2}{b^2} = 1,$$

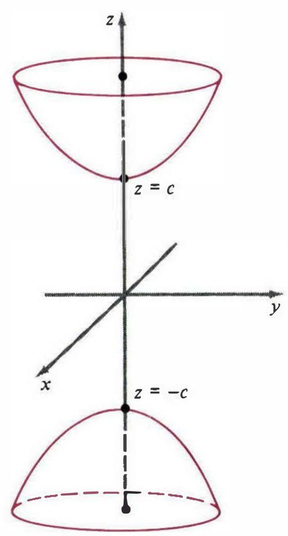


Figure 18.34 Hyperboloid of two sheets.

and it is this hyperbola that unifies the horizontal sections into a smooth surface—of "two sheets." Observe that (6) is identical with (4) except for the presence of the minus sign on the right, and it is this sign that makes all the difference between the surfaces in these two examples; for the right side of (4) is positive for all z's, whereas the right side of (6) is negative for $|z| < c$.

Example 4 The graph of the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2} \tag{7}$$

is an *elliptic cone* (Fig. 18.35). This surface intersects the xz-plane and the yz-plane in the pairs of intersecting straight lines

$$z = \pm\frac{c}{a}x \qquad \text{and} \qquad z = \pm\frac{c}{b}y,$$

respectively. It intersects the xy-plane at the origin alone. All horizontal sections in planes $z = k$ with $k \neq 0$ are ellipses. (In Chapter 15 it was convenient to distinguish circles from ellipses; here we include circles among the ellipses.) It is clear from the form of (7) that if (x, y, z) is a point on the surface, then (tx, ty, tz) is also on the surface for any number t. This tells us that the entire surface can be thought of as generated by a moving line through the origin O and a variable point P on any horizontal elliptical section. When $a = b$, the cone is the familiar right circular cone.

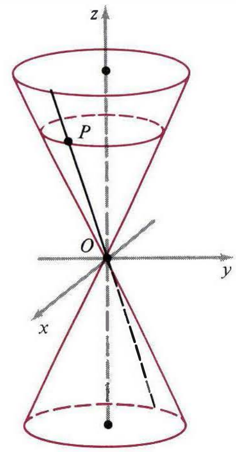


Figure 18.35 Elliptic cone.

Example 5 The *elliptic paraboloid*

$$z = ax^2 + by^2. \tag{8}$$

is shown in Fig. 18.36. The vertical sections of this surface in the xz-plane and yz-plane are the parabolas

$$z = ax^2 \qquad \text{and} \qquad z = by^2,$$

respectively. The horizontal section in the plane $z = k$ is an ellipse if $k > 0$, the origin alone if $k = 0$, and empty if $k < 0$.

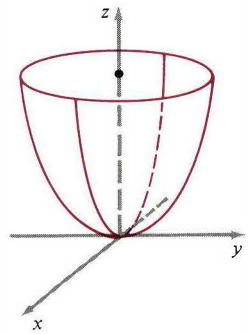


Figure 18.36 Elliptic paraboloid.

Example 6 In Fig. 18.37 we sketch the *hyperbolic paraboloid*

$$z = by^2 - ax^2. \tag{9}$$

The section in the yz-plane is the parabola $z = by^2$ opening upward, and that in the xz-plane is the parabola $z = -ax^2$ opening downward. In all planes $y = k$ parallel to the xz-plane, the sections are downward-opening parabolas that are identical with one another and can be thought of as hanging from their vertices at various points along the parabola $z = by^2$; this is emphasized in the way we

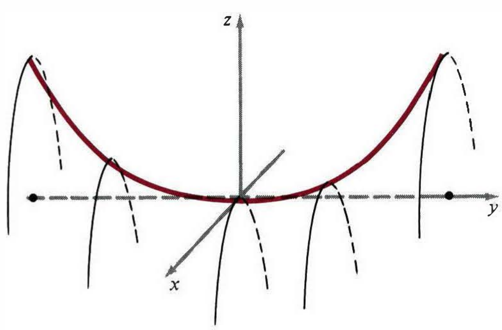


Figure 18.37 Hyperbolic paraboloid.

have drawn the figure. Near the origin the surface rises in the y-direction and falls in the x-direction, and thus has the general shape of a saddle or a mountain pass. For this reason, the surface is often called a *saddle surface,* with the origin as the *saddle point.* It is clear from (9) that in the horizontal plane $z = k$, the section is a hyperbola with principal axis in the y-direction if $k > 0$, and a hyperbola with principal axis in the x-direction if $k < 0$; if $k = 0$, the section is a pair of intersecting straight lines through the origin.

PROBLEMS

Sketch and identify the surfaces in Problems 1–14.

1 $2x^2 + y^2 + 4z^2 = 16$.
2 $z^2 = 4(x^2 + y^2)$.
3 $z = 4(x^2 + y^2)$.
4 $x^2 + z^2 - 4y^2 = 4$.
5 $y^2 - 4x^2 - 9z^2 = 36$.
6 $z = 4 - 2x^2 - 3y^2$.
7 $z = x^2 - 2y^2$.
8 $x^2 = y^2 + 4z^2$.
9 $x^2 - 4y^2 - z^2 = 4$.
10 $x^2 + 9y^2 - 4z^2 = 36$.
11 $36x^2 + 4y^2 + 9z^2 = 36$.
12 $y = 4 - x^2 - 2y^2$.
13 $z + 4x^2 = y^2$.
14 $x^2 + y^2 - z^2 - 2x - 4y + 1 = 0$.

15 Find the points at which the line

$$\frac{x-6}{3} = \frac{y+2}{-6} = \frac{z-2}{4}$$

pierces the ellipsoid

$$\frac{x^2}{81} + \frac{y^2}{36} + \frac{z^2}{9} = 1.$$

16 Show that the plane $2x - 2z - y = 10$ intersects the paraboloid

$$2z = \frac{x^2}{9} + \frac{y^2}{4}$$

at a single point, and find the point.

17 (a) Consider the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

and find the area $A(k)$ of the elliptical section in the horizontal plane $z = k$. Hint: Recall that πAB is the area of an ellipse with semiaxes A and B.

(b) Use the formula found in (a) to calculate the volume of the ellipsoid by integration.

18 Consider the elliptic paraboloid $z = ax^2 + by^2$, and use integration to show that the volume of the segment cut off by the plane $z = k$ ($k > 0$) is half the area of its base times its height.

19 Show that the projection on the xy-plane of the curve of intersection of the surfaces $z = 1 - x^2$ and $z = x^2 + y^2$ is an ellipse. Hint: What does it mean geometrically to eliminate z from these equations?

20 Show that the projection on the yz-plane of the curve of intersection of the plane $x = 2y$ and the paraboloid $x = y^2 + z^2$ is a circle.

21 Show that the projection on the xy-plane of the intersection of the paraboloids $z = 3x^2 + 5y^2$ and $z = 8 - 5x^2 - 3y^2$ is a circle.

22 The two equations

$$x^2 + 3y^2 - z^2 + 3x = 0,$$
$$2x^2 + 6y^2 - 2z^2 - 4y = 3,$$

when taken together as a simultaneous system, define the space curve in which the corresponding surfaces intersect. Show that this curve lies in a plane. Hint: Project onto a coordinate plane.

23 Use the methods of Section 15.6 to discover the nature of the graph of $z = xy$. Sketch the surface.

A *ruled surface* is a surface S with the property that for each point P on S there is a straight line through P that lies entirely on S. All cones and cylinders are ruled surfaces, while ellipsoids, hyperboloids of two sheets, and elliptic paraboloids obviously are not. It is very surprising that all hyperboloids of one sheet and all hyperbolic paraboloids are ruled surfaces.

24 Show that the hyperboloid of one sheet $x^2 + y^2 - z^2 = 1$ is a ruled surface, as follows:

(a) The section of the surface in the xy-plane is the circle C whose equation is $x^2 + y^2 = 1$. Let $P_0 = (x_0, y_0, 0)$ be a point on C, and show that the line L whose equations are

$$x = x_0 + y_0t, \qquad y = y_0 - x_0t, \qquad z = t,$$

passes through P_0 and lies entirely on the surface.

(b) If $P = (x, y, z)$ is an arbitrary point on the surface, show that the line L in part (a) passes through P for a suitable point $P_0 = (x_0, y_0, 0)$. Thus, as P_0 moves around C, the lines L cover the surface.*

*The family of lines $x = x_0 + y_0t$, $y = y_0 - x_0t$, $z = -t$ also covers the surface, and for this reason the hyperboloid of one sheet is often called a *doubly ruled* surface.

25 Show that the hyperbolic paraboloid $z = y^2 - x^2$ is a ruled surface by showing that if $P_0 = (x_0, y_0, y_0{}^2 - x_0{}^2)$ is any point on the surface, then the line

$$x = x_0 + t, \qquad y = y_0 + t,$$
$$z = (y_0{}^2 - x_0{}^2) + 2(y_0 - x_0)t$$

passes through P_0 and lies entirely on the surface.*

The two families of straight lines constituting the doubly ruled surfaces discussed in Problems 24 and 25 are shown in Fig. 18.38.

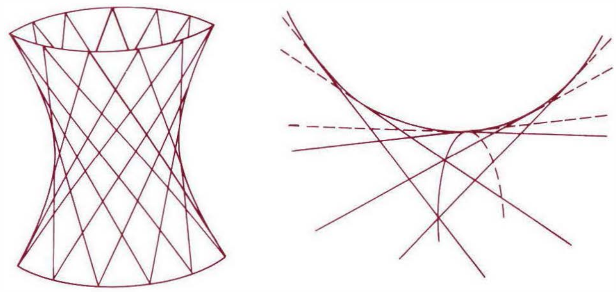


Figure 18.38 Doubly ruled surfaces.

*The family of lines $x = x_0 + t$, $y = y_0 - t$, $z = (y_0{}^2 - x_0{}^2) - 2(y_0 + x_0)t$ also covers the surface, so the hyperbolic paraboloid is also doubly ruled.

18.7 CYLINDRICAL AND SPHERICAL COORDINATES

In plane analytic geometry we used a rectangular coordinate system for some types of problems and a polar coordinate system for others. We saw that there are many situations in which one system is more convenient than the other. The same is true for the study of geometry and calculus in space. We now describe two other three-dimensional coordinate systems, in addition to the now-familiar rectangular coordinate system, that are often useful for dealing with special kinds of problems.

Consider a point P in space whose rectangular coordinates are (x, y, z). The *cylindrical coordinates* of this point are obtained by replacing x and y with the corresponding polar coordinates r and θ, and allowing z to remain unchanged. That is, we place a z-axis on top of a polar coordinate system and describe the location of a point in space by the three coordinates (r, θ, z). We will always assume that this cylindrical coordinate system is superimposed on a rectangular coordinate system in the manner shown in Fig. 18.39, so that the transformation equations connecting the two sets of coordinates of a given point are

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z,$$

and

$$r^2 = x^2 + y^2, \qquad \tan\theta = \frac{y}{x}, \qquad z = z.$$

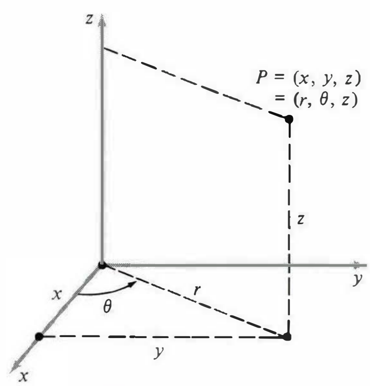


Figure 18.39 Cylindrical coordinates.

It is easy to see that the graph of the equation $r =$ a constant is a right circular cylinder whose axis is the z-axis; this is the reason for the term "cylindrical coordinates." Similarly, the graph of $\theta =$ a constant is a plane containing the z-axis, and the graph of $z =$ a constant is a horizontal plane.

Example 1 Find cylindrical coordinates for the points P_1 and P_2 whose rectangular coordinates are $(3, 3, 7)$ and $(2\sqrt{3}, 2, 5)$, respectively.

Solution For P_1 we have $r = \sqrt{9 + 9} = 3\sqrt{2}$, $\tan\theta = 1$, $z = 7$, so a set of cylindrical coordinates is $(3\sqrt{2}, \pi/4, 7)$. For P_2 we have $r = \sqrt{12 + 4} = 4$, $\tan\theta = 1/\sqrt{3} = \frac{1}{3}\sqrt{3}$, $z = 5$, so a set of cylindrical coordinates is $(4, \pi/6, 5)$.

Example 2 Describe the surfaces

(a) $r + z = 3$, and

(b) $r(2 \cos \theta + 5 \sin \theta) + 3z = 0$.

Solution (a) The intersection of the surface $r + z = 3$ with the yz-plane is the straight line $y + z = 3$, because $r = y$ in the yz-plane. But θ is missing from the given equation, so the desired surface is symmetric about the z-axis, and is therefore the cone generated by revolving the line $y + z = 3$ about the z-axis. More generally, it follows from our discussion of surfaces of revolution in Section 18.5 that if a curve $f(y, z) = 0$ is revolved about the z-axis, then the cylindrical equation of the resulting surface is $f(r, z) = 0$.

(b) Since $r \cos \theta = x$ and $r \sin \theta = y$, the given equation transforms into $2x + 5y + 3z = 0$, which is the plane through the origin with normal vector $\mathbf{N} = 2\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}$.

Example 3 Find a cylindrical equation for (a) the spheroid $x^2 + y^2 + 2z^2 = 4$, and (b) the hyperbolic paraboloid $z = x^2 - y^2$.

Solution The equation in (a) transforms at once into $r^2 + 2z^2 = 4$. For (b), we have

$$\begin{aligned} z &= x^2 - y^2 \\ &= r^2 \cos^2 \theta - r^2 \sin^2 \theta = r^2 (\cos^2 \theta - \sin^2 \theta) \\ &= r^2 \cos 2\theta, \end{aligned}$$

so $z = r^2 \cos 2\theta$ is the desired equation.

In physics, cylindrical coordinates are particularly convenient for studying situations in which there is axial symmetry, that is, symmetry about a line in space. As examples we mention two important classes of problems: those dealing with the flow of heat in solid cylindrical rods, and those concerned with the movements of a vibrating circular membrane—for instance, a drumhead.

Again consider a point P in space whose rectangular coordinates are (x, y, z). The *spherical coordinates* of P are the numbers (ρ, ϕ, θ) shown in Fig. 18.40. Here ρ (the Greek letter *rho*) is the distance from the origin O to P, so $\rho \geq 0$. The angle ϕ is the angle down from the positive z-axis to the radial line OP, and it is understood that ϕ is restricted to the interval $0 \leq \phi \leq \pi$. Finally, the angle θ has exactly the same meaning in spherical coordinates as it has in cylindrical coordinates; that is, θ is the angle from the positive x-axis to the line OP', where P' is the projection of P on the xy-plane. It is clear from the figure that $OP' = \rho \sin \phi$, and since $x = OP' \cos \theta$ and $y = OP' \sin \theta$, we have the transformation equations

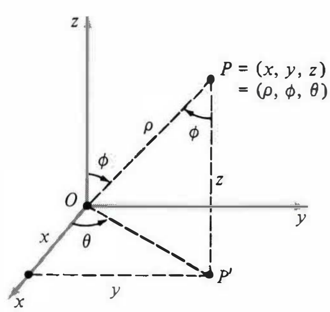


Figure 18.40 Spherical coordinates.

$$x = \rho \sin \phi \cos \theta, \qquad y = \rho \sin \phi \sin \theta, \qquad z = \rho \cos \phi,$$

and

$$\rho^2 = x^2 + y^2 + z^2, \qquad \tan \phi = \frac{\sqrt{x^2 + y^2}}{z}, \qquad \tan \theta = \frac{y}{x}.$$

The term "spherical coordinates" is used because the graph of the equation $\rho = a$ (constant) is a sphere with center at the origin and radius a. The graph of $\phi = \alpha$ (constant) is the upper nappe of a cone with vertex at the origin and vertex angle α, if $0 < \alpha < \pi/2$. The graph of $\theta = \theta_0$ (constant) is a plane containing the z-axis, just as in cylindrical coordinates.

Example 4 Find an equation in spherical coordinates for the sphere $x^2 + y^2 + z^2 - 2az = 0$, where $a > 0$.

Solution Since $\rho^2 = x^2 + y^2 + z^2$ and $z = \rho \cos \phi$, the given equation can be written as

$$\rho^2 - 2a\rho \cos \phi = 0 \qquad \text{or} \qquad \rho(\rho - 2a \cos \phi) = 0.$$

The graph of this equation is the graph of $\rho = 0$ together with the graph of $\rho - 2a \cos \phi = 0$. But the graph of $\rho = 0$ (namely, the origin) is part of the graph of $\rho = 2a \cos \phi$, so the desired equation is

$$\rho = 2a \cos \phi.$$

This is the sphere of radius a that is tangent to the xy-plane at the origin, as shown in Fig. 18.41.

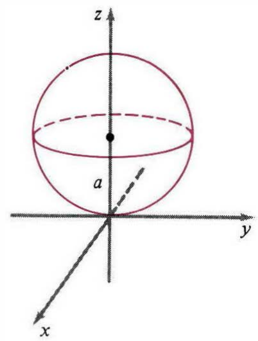


Figure 18.41

Example 5 What is the graph of the spherical equation $\rho = 2a \sin \phi$?

Solution The variable θ is missing from this equation, so we have a surface of revolution about the z-axis. In the yz-plane the equation $\rho = 2a \sin \phi$ represents a circle of radius a, as shown in Fig. 18.42. Since the graph we are seeking is obtained by revolving this circle about the z-axis, this graph is a torus (doughnut) in which the hole has radius zero.

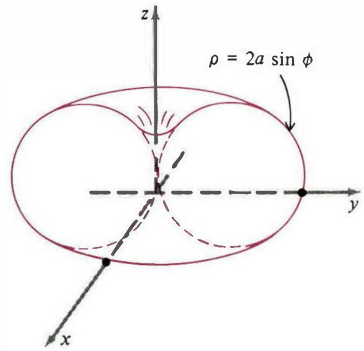


Figure 18.42

There are many physical uses of spherical coordinates, ranging from problems about heat conduction to problems in the theory of gravitation. We shall discuss some of these applications in Chapter 20.

PROBLEMS

1 Find a set of cylindrical coordinates for the point whose rectangular coordinates are
(a) $(2, 2, -1)$; (b) $(1, -\sqrt{3}, 7)$;
(c) $(3, \sqrt{3}, 2)$; (d) $(3, 6, 5)$.

2 Find the rectangular coordinates of the point with cylindrical coordinates
(a) $(\sqrt{2}, \pi/4, -2)$; (b) $(\sqrt{3}, 5\pi/6, 11)$;
(c) $(1, 1, 1)$; (d) $(2, \pi/3, \pi)$.

3 Find a set of spherical coordinates for the point whose rectangular coordinates are
(a) $(1, 1, \sqrt{6})$; (b) $(1, -1, -\sqrt{6})$;
(c) $(1, 1, \sqrt{2})$; (d) $(0, -1, \sqrt{3})$.

4 Find the rectangular coordinates of the point with spherical coordinates
(a) $(3, \pi/2, \pi/2)$; (b) $(4, \pi/2, \pi)$;
(c) $(4, \pi/3, \pi/3)$; (d) $(4, 2\pi/3, \pi/3)$.

In Problems 5–11, find a cylindrical equation for the surface whose rectangular equation is given. Sketch the surface.

5 $x^2 + y^2 + z^2 = 16$. 6 $x^2 + y^2 = 6z$.
7 $x^2 + y^2 = z^2$. 8 $x^2 - y^2 = 3$.
9 $x^2 + y^2 - 2y = 0$. 10 $x^2 + y^2 - 4x = 0$.
11 $x^2 + y^2 = 9$.

12 Find a cylindrical equation for the surface whose rectangular equation is $z^2(x^2 - y^2) = 4xy$.

In Problems 13–18, find a spherical equation for the surface whose rectangular equation is given. Sketch the surface.

13 $x^2 + y^2 + z^2 = 16$.

14 $x^2 + y^2 + z^2 + 4z = 0$.
15 $x^2 + y^2 + z^2 - 6z = 0$.
16 $x^2 + y^2 = 9$.
17 $z = 4 - x^2 - y^2$.
18 $(x^2 + y^2 + z^2)^3 = (x^2 + y^2)^2$.

CHAPTER 18 REVIEW: DEFINITIONS, EQUATIONS

Think through the following.

1 Rectangular (or Cartesian) coordinates.
2 Equation of a sphere.
3 Dot product: definition and formula.
4 Cross product: definition and formula.
5 Equations of a line: parametric and symmetric.
6 Equation of a plane.
7 Cylinder.
8 Surface of revolution.
9 The six quadric surfaces: graphs and equations.
10 Cylindrical coordinates.
11 Spherical coordinates.

19 PARTIAL DERIVATIVES

19.1 FUNCTIONS OF SEVERAL VARIABLES

Many of the functions that arise in mathematics and its applications involve two or more independent variables. We have already met functions of this kind in our study of solid analytic geometry. Thus, the equation $z = x^2 - y^2$ is the equation of a certain saddle surface, but it also defines z as a function of the two variables x and y, and the surface can be thought of as the graph of this function.

We usually denote an arbitrary function of the two variables x and y by writing $z = f(x, y)$, and we can visualize such a function by sketching—or imagining—its graph in xyz-space, as suggested in Fig. 19.1. In this figure, $P = (x, y)$ is a "suitable" point in the xy-plane—that is, a point in the domain D of the function—and z is the directed distance up or down to the corresponding point on the surface. This surface is thought of as lying "over" the domain D, even though part of it may actually be below the xy-plane.

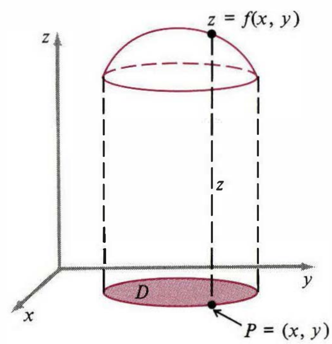


Figure 19.1 A surface in space.

By an obvious extension of the notation used here, $w = f(x, y, z, t, u, v)$ is a function of the six variables displayed in parentheses. For example, if the temperature T at a point P inside a solid iron sphere depends on the three rectangular coordinates x, y, and z of P, then we write $T = f(x, y, z)$; and if we also allow for the possibility that the temperature at a given point may vary with the time t, then T is a function of all four variables, $T = f(x, y, z, t)$.

In this chapter we shall see that the main themes of single-variable differential calculus—derivatives, rates of change, chain rule computations, maximum-minimum problems, and differential equations—can all be extended to functions of several variables. However, students should be prepared for the fact that there are striking differences between single-variable calculus and multivariable calculus. Since most of these differences already show up in functions of only two independent variables, we usually emphasize this case, and refer more briefly to functions of three or more variables. In the next chapter we turn to the integral calculus of functions of several variables.

DOMAIN

Just as in our previous work, the *domain* (or *domain of definition*) of a function $z = f(x, y)$ is the set of all points $P = (x, y)$ in the xy-plane for which there exists a corresponding z, and similarly for functions defined in xyz-space, $xyzt$-space, etc. Most of the functions we deal with are defined by formulas, and in these cases the domain is understood to be the largest set of points for which the formula makes sense. For example, the domain of

$$z = f(x, y) = \frac{1}{x - y}$$

is understood to be the set of all points (x, y) with $x \neq y$, that is, all points in the xy-plane that do not lie on the line $y = x$. The domain of

$$z = g(x, y) = \sqrt{9 - x^2 - y^2}$$

is the set of all points (x, y) for which $9 - x^2 - y^2 \geq 0$, that is, the circular disk $x^2 + y^2 \leq 9$ of radius 3 with center at the origin. And the domain of

$$w = h(x, y, z) = \frac{2x + 3y + 4z}{x^2 + y^2 + z^2}$$

is the set of all points (x, y, z) for which $x^2 + y^2 + z^2 \neq 0$, that is, all points of xyz-space except the origin.

In discussing a general function $z = f(x, y)$, we shall often require that this function be defined at a certain point P_0 and throughout some *neighborhood* of this point. This means that the domain of $f(x, y)$ must include not only P_0 itself, but also every point "sufficiently close" to P_0, that is, every point in some small circular disk centered on P_0. Similar remarks apply to functions defined in xyz-space, etc.

CONTINUITY

There are several places in this chapter where it will be necessary to mention continuity in order to state things correctly. This concept extends in a natural way from the one-variable case to functions $f(x, y)$, as follows.

A function $f(x, y)$ is said to be *continuous* at a point (x_0, y_0) in its domain if its value $f(x, y)$ can be made as close as we please to $f(x_0, y_0)$ by taking the point (x, y) close enough to (x_0, y_0), that is, if $|f(x, y) - f(x_0, y_0)|$ can be made as small as we please by making both $|x - x_0|$ and $|y - y_0|$ small enough. For example, $f(x, y) = xy$ is continuous at any point (x_0, y_0), because

$$\begin{aligned} |xy - x_0y_0| &= |xy - xy_0 + xy_0 - x_0y_0| \\ &= |x(y - y_0) + y_0(x - x_0)| \\ &\leq |x||y - y_0| + |y_0||x - x_0|, \end{aligned}$$

and it is easy to see that the quantity last written can be made as small as we please by making both $|x - x_0|$ and $|y - y_0|$ small enough.

On the other hand, the function defined by

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0), \end{cases} \tag{1}$$

is not continuous at the origin $(0, 0)$. For, if we let (x, y) approach $(0, 0)$ along a line $y = mx$ with $m \neq 0$, then

$$f(x, y) = \frac{mx^2}{x^2 + m^2x^2} = \frac{m}{1 + m^2}, \tag{2}$$

which is a nonzero constant, and these values cannot be made as close as we please to $f(0, 0) = 0$ by making (x, y) close enough to $(0, 0)$. To express this in another way, (2) shows that the values of the function approach different limiting values as the point (x, y) approaches the origin from different directions, and this is impossible if the function is continuous at the origin.

We shall not pursue the details of this topic any further, beyond making the rather loose statement that any finite combination of elementary functions is continuous at each point of its domain. Also, continuity is defined in essentially the same way for functions of three or more variables.

LEVEL CURVES

Many simple functions $z = f(x, y)$ have graphs that are much too difficult to sketch. Fortunately there is another way to understand and express the geometric nature of such a function.

The basic idea comes from the art of the mapmaker. In mapping terrain with valleys, hills, and mountains, it is common practice to draw curves joining points of constant elevation. When these curves are included on a map and properly labeled, the resulting topographical map enables an experienced user to obtain a clear mental picture of the contours of the land in three-dimensional space from this two-dimensional representation.

We can do the same thing to portray a function $z = f(x, y)$ of two variables. For any value c that $f(x, y)$ assumes, we can sketch the curve

$$f(x, y) = c$$

in the xy-plane, as shown in Fig. 19.2. Such a curve is called a *level curve*; it lies in the domain of the function, and on it $z = f(x, y)$ has the constant value c.

A collection of level curves is called a *contour map*; it can give a good idea of the shape of the graph, and is the next best thing to a three-dimensional sketch. For instance, the graph of $z = xy$ is difficult—though not impossible—to draw. However, a reasonably clear idea of the shape of this graph is given by the contour map shown in Fig. 19.3, which is easy to draw. Each level curve $xy = c$ is a hyperbola in the first and third quadrants if $c > 0$, a hyperbola in the second and fourth quadrants if $c < 0$, and the two axes taken together if $c = 0$. We ascend the surface as we leave the origin going into the first and third quadrants,

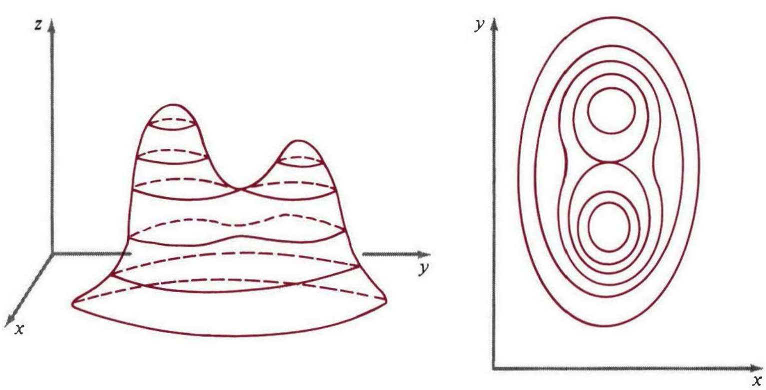


Figure 19.2 Level curves.

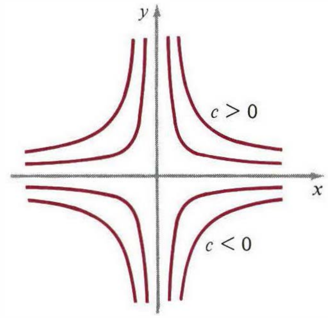


Figure 19.3

and descend it as we leave going into the second and fourth quadrants, and in this way we see that the origin is the saddle point of a saddle surface. Students should try to use this figure to visualize the shape of the surface as it appears in three-dimensional space, looking down on it from above.

LEVEL SURFACES

Drawing graphs for functions of two variables is often difficult, but drawing graphs for functions of three variables is always impossible. We would need a visible space of four dimensions to contain such a graph, and no such space is available.

However, the concept of level curves suggests a way to visualize the behavior of a function $w = f(x, y, z)$ of three variables: examine its *level surfaces.* These are the surfaces

$$f(x, y, z) = c \tag{3}$$

for various values of the constant c. Of course, level surfaces can be hard to draw, but a knowledge of what they are can help us form a useful intuitive idea of the nature of the function. In Fig. 19.4 we present a schematic view of three adjacent level surfaces of the form (3) for three values of the constant c, where $c_1 < c_2 < c_3$. As a point $P = (x, y, z)$ moves along the lowest surface, the value of $w = f(x, y, z)$ is constantly equal to c_1; but as this point hops to the next surface above it, the value of the function increases to c_2; and so on.

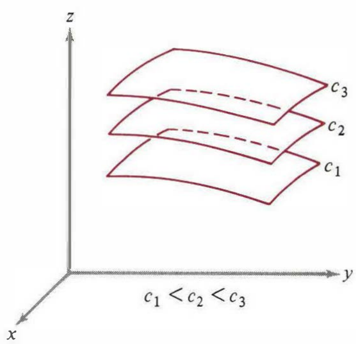


Figure 19.4 Level surfaces.

We consider two simple examples. In the case of the function $w = x + 2y + 3z$, the level surfaces are easily seen to be the planes

$$x + 2y + 3z = c$$

with normal vector $\mathbf{N} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$; and for $w = \sqrt{x^2 + y^2 + z^2}$, the level surfaces are the concentric spheres

$$x^2 + y^2 + z^2 = c^2.$$

In applications, if the function $w = f(x, y, z)$ represents the temperature at the point $P = (x, y, z)$, then the level surfaces are called *isothermal surfaces*; if it represents potential, they are called *equipotential surfaces.*

HIGHER DIMENSIONS

Level surfaces have a certain limited value, but in a sense they avoid the real question: How do we go about trying to obtain an intuitive understanding of the behavior of functions of three or more variables?

Briefly, what we do is work by analogy with the one- and two-variable cases. For example, there is nothing to prevent us from considering the set of all quadruples of numbers such as $(2, -3, 1, 4)$ as forming a perfectly legitimate four-dimensional space, with an origin $(0, 0, 0, 0)$, four coordinate axes, and a satisfactory concept of the distance from an arbitrary point (x, y, z, w) to the origin,

$$d = \sqrt{x^2 + y^2 + z^2 + w^2}.$$

We can now consider the graph of a function

$$w = f(x, y, z)$$

as forming a three-dimensional "surface" in this four-dimensional space, with the domain D of the function lying in the three-dimensional "coordinate plane" consisting of all points of the form $(x, y, z, 0)$.

In a similar way, if n is any positive integer we can think of the graph of a function of n variables as forming an n-dimensional "surface" in $(n + 1)$-dimensional space. It is true that for $n \geq 3$ we can no longer draw pictures, but we can still bolster our intuition by using geometric language, and we can still think geometrically, but in a looser way. However, as we move further away from the kind of mathematics that we can study and understand by drawing pictures, it is necessary to give more attention to the algebraic and analytic aspects of what we are doing, in order to avoid being misled by words and analogies. Nevertheless, the words, analogies, and geometric intuition remain indispensable, for they suggest worthwhile things to think about and prevent us from feeling totally lost among abstractions.

PROBLEMS

In Problems 1–12, find the domain of the given function.

1 $f(x, y) = \dfrac{xy}{y - 2x}$.

2 $f(x, y) = \dfrac{1}{x} + \dfrac{1}{y}$.

3 $f(x, y) = \sqrt{xy}$.

4 $f(x, y) = \dfrac{1}{(e^x + e^y)^2}$.

5 $f(x, y) = \ln(y - 3x)$.

6 $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$.

7 $f(x, y, z) = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$.

8 $f(x, y, z) = \dfrac{z}{4x^2 - y^2}$.

9 $f(x, y, z) = \sqrt{16 - x^2 - y^2 - z^2}$.

10 $f(x, y, z) = \dfrac{1}{xyz}$.

11 $f(x, y, z) = xy \ln z + 3 \tan \frac{1}{2}z$.

12 $f(x, y, z) = \ln(x^2 + y^2 + z^2 - 1)$.

13 Show that the function defined by

$$f(x, y) = \begin{cases} \dfrac{xy}{\sqrt{x^2 + y^2}} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0), \end{cases}$$

is continuous at the origin. Hint: Use $x = r \cos \theta$ and $y = r \sin \theta$ to transform to polar coordinates.

In Problems 14–24, represent the given function by drawing a few level curves, and try to visualize the surface from the resulting contour map.

14 $z = x^2 + y^2$.

15 $z = x^2 + 2y^2$.

16 $z = x + y$.

17 $z = x - y$.

18 $z = 2x - y$.

19 $z = x^2 - y$.

20 $z = x^3 - y$.

21 $z = y/x$.

22 $z = y/x^2$.

23 $z = x^2 - y^2$.

24 $z = \sqrt{x^2 - y^2}$.

In each of the following problems, sketch a few level surfaces for the given function and use these to estimate the general direction in which the values of the function increase.

25 $w = \dfrac{x^2}{4} + \dfrac{y^2}{9} + \dfrac{z^2}{16}$.

26 $w = \dfrac{1}{x^2 + y^2 + z^2}$.

27 $w = 2x - 5y + 3z$.

28 $w = x^2 + y^2 - z^2$.

19.2 PARTIAL DERIVATIVES

Suppose that $y = f(x)$ is a function of only one variable. We know that its derivative, defined by

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x},$$

can be interpreted as the rate of change of y with respect to x. In the case of a function $z = f(x, y)$ of two variables, we shall need similar mathematical machinery for working with the rate at which z changes as both x and y vary. The key idea is to allow only one variable to change at a time, while holding the other fixed. For functions of more than two variables, we vary one of them while holding *all* the others fixed. Specifically, we differentiate with respect to only one variable at a time, regarding all the others as constants, and this gives us one derivative corresponding to each of the independent variables. These individual derivatives are the constituents from which we build the more complicated machinery that will be needed later.

To return to our function $z = f(x, y)$ of two variables, we first hold y fixed and let x vary. The rate of change of z with respect to x is denoted by $\partial z/\partial x$ and defined by

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}.$$

This limit (if it exists) is called the *partial derivative of z with respect to x*, and is read "partial z, partial x." The most commonly used notations for this partial derivative are

$$\frac{\partial z}{\partial x}, \qquad z_x, \qquad \frac{\partial f}{\partial x}, \qquad f_x, \qquad f_x(x, y),$$

and we shall use all of these from time to time in order to help students become accustomed to them. The symbol ∂ in the notation $\partial z/\partial x$ is called the "roundback d" or "curly d"; it is used to emphasize that there are other independent variables present during the process of differentiating with respect to x.

Similarly, if x is held fixed and y is allowed to vary, then the *partial derivative of z with respect to y* is defined by

$$\frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y},$$

and the standard notations in this case are

$$\frac{\partial z}{\partial y}, \qquad z_y, \qquad \frac{\partial f}{\partial y}, \qquad f_y, \qquad f_y(x, y).$$

The actual calculation of partial derivatives for most functions is very easy: Treat every independent variable except the one we are interested in as if it were a constant, and apply the familiar rules.

Example 1 Calculate the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ of the function $f(x, y) = x^3 - 3x^2y^3 + y^2$.

Solution To find the partial of f with respect to x, we think of y as a constant and differentiate in the usual way,

$$\frac{\partial f}{\partial x} = 3x^2 - 6xy^3.$$

When we regard x as a constant and differentiate with respect to y, we obtain

$$\frac{\partial f}{\partial y} = -9x^2y^2 + 2y.$$

The notations $f_x(x, y)$ and $f_y(x, y)$ are useful for indicating the values of partial derivatives at specific points.

Example 2 (a) If $f(x, y) = xy^2 + x^3$, then

$$f_x(x, y) = y^2 + 3x^2, \qquad f_y(x, y) = 2xy,$$
$$f_x(2, 1) = 13, \qquad f_y(2, 1) = 4.$$

In the other notation, the numerical values given here by the simple and convenient symbols $f_x(2, 1)$ and $f_y(2, 1)$ would have to be written more clumsily as

$$\left(\frac{\partial f}{\partial x}\right)_{(2,1)} \quad \text{and} \quad \left(\frac{\partial f}{\partial y}\right)_{(2,1)}.$$

(b) If $g(x, y) = xe^{xy^2}$, then

$$g_x(x, y) = xy^2e^{xy^2} + e^{xy^2}, \qquad g_y(x, y) = 2x^2ye^{xy^2}.$$

(c) If $h(x, y) = \sin x^2 \cos 3y$, then

$$h_x(x, y) = 2x \cos x^2 \cos 3y, \qquad h_y(x, y) = -3 \sin x^2 \sin 3y.$$

These examples illustrate the fact that the partial derivatives of a function of x and y are themselves functions of x and y.

These ideas and notations apply just as easily to functions of any number of variables.

Example 3 If $w = f(x, y, z, u, v) = xy^2 + 2x^3 + xyz + zu + \tan uv$, then

$$\frac{\partial w}{\partial x} = y^2 + 6x^2 + yz, \qquad \frac{\partial w}{\partial y} = 2xy + xz, \qquad \frac{\partial w}{\partial z} = xy + u,$$

$$\frac{\partial w}{\partial u} = z + v\sec^2 uv, \qquad \frac{\partial w}{\partial v} = u \sec^2 uv.$$

In the one-variable case, we know that the derivative dy/dx can legitimately be thought of as a fraction, the quotient of the differentials dy and dx. The notation $\partial z/\partial x$ for the partial derivative $f_x(x, y)$ suggests that something similar might be done with ∂z and ∂x. However, it is not possible to treat partial derivatives as fractions. We give an example to emphasize this point.

Example 4 The *ideal gas law* states that for a given quantity of gas, the pressure p, volume V, and absolute temperature T are connected by the equation $pV = nRT$, where n is the number of moles of gas in the sample and R is a constant. Show that

$$\frac{\partial p}{\partial V}\frac{\partial V}{\partial T}\frac{\partial T}{\partial p} = -1.$$

Solution Since

$$p = \frac{nRT}{V}, \qquad V = \frac{nRT}{p}, \qquad T = \frac{pV}{nR},$$

we have

$$\frac{\partial p}{\partial V} = -\frac{nRT}{V^2}, \qquad \frac{\partial V}{\partial T} = \frac{nR}{p}, \qquad \frac{\partial T}{\partial p} = \frac{V}{nR}.$$

It follows that

$$\frac{\partial p}{\partial V}\frac{\partial V}{\partial T}\frac{\partial T}{\partial p} = \left(-\frac{nRT}{V^2}\right)\frac{nR}{p}\frac{V}{nR} = -\frac{nRT}{pV} = -1.$$

The fact that this result is -1 instead of $+1$ shows that we cannot treat the partial derivatives on the left as fractions.

When we are working with a function $z = f(x, y)$ of only two variables, the partial derivatives have the following simple geometric interpretation. The graph of this function is a surface, as shown in Fig. 19.5. Let (x_0, y_0) be a given point in the xy-plane, with (x_0, y_0, z_0) the corresponding point on the surface. To hold y fixed at the value y_0 means to intersect the surface with the plane $y = y_0$, and the intersection is the curve

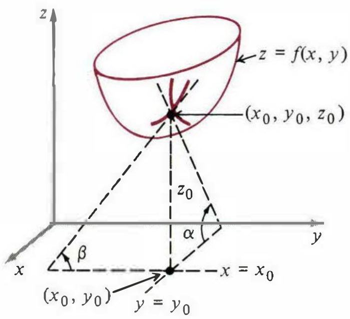


Figure 19.5

$$z = f(x, y_0)$$

in that plane. The number

$$\left(\frac{\partial z}{\partial x}\right)_{(x_0,y_0)} = f_x(x_0, y_0)$$

is the slope of the tangent line to this curve at $x = x_0$. Thus, in the figure we have

$$\tan\alpha = \left(\frac{\partial z}{\partial x}\right)_{(x_0,y_0)} = f_x(x_0, y_0).$$

Similarly, the intersection of the surface with the plane $x = x_0$ is the curve

$$z = f(x_0, y),$$

and the other partial derivative is the slope of the tangent to this curve at $y = y_0$,

$$\tan\beta = \left(\frac{\partial z}{\partial y}\right)_{(x_0,y_0)} = f_y(x_0, y_0).$$

No such interpretation is available when there are more than two independent variables.

We remarked that for a function $z = f(x, y)$ of two variables, the partial derivatives f_x and f_y are also functions of two variables, and may themselves have partial derivatives. As we might expect, these *second-order partial derivatives* are denoted by several symbols. If we start with the first derivatives

$$\frac{\partial f}{\partial x} = f_x \quad \text{and} \quad \frac{\partial f}{\partial y} = f_y,$$

then the derivatives with respect to x are

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}f_x = f_{xx}$$

and

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial x}f_y = f_{yx};$$

and the derivatives with respect to y are

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial}{\partial y}f_x = f_{xy}$$

and

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} f_y = f_{yy}.$$

This notation may seem a bit confusing at first, but it is actually quite reasonable. Observe that in f_{yx} we differentiate first with respect to the "inside" variable y, then with respect to the "outside" variable x. This is the natural order, since f_{yx} ought to mean $(f_y)_x$. Thus, in the symbols f_{yx} and f_{xy}, the subscript letters accumulate from left to right, because this is the order in which the differentiations are performed. For the same reason, in the symbols

$$\frac{\partial^2 f}{\partial x\,\partial y} \quad \text{and} \quad \frac{\partial^2 f}{\partial y\,\partial x},$$

it is natural for the letters indicating the variable of differentiation to accumulate from right to left: first y, then x in the first of these; and first x, then y in the second.

The *pure* second partial derivatives,

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} \quad \text{and} \quad f_{yy} = \frac{\partial^2 f}{\partial y^2},$$

don't represent anything really new. Each is found by holding one variable constant and differentiating twice with respect to the other variable, and each gives the rate of change of the rate of change of f in the direction of one of the axes.

Example 5 If $f(x, y) = x^3 e^{5y} + y \sin 2x$, then

$$f_x = 3x^2 e^{5y} + 2y \cos 2x, \qquad f_y = 5x^3 e^{5y} + \sin 2x,$$
$$f_{xx} = 6xe^{5y} - 4y \sin 2x, \qquad f_{yy} = 25x^3 e^{5y}.$$

On the other hand, the *mixed* second partial derivatives,

$$f_{xy} = \frac{\partial^2 f}{\partial y\,\partial x} \quad \text{and} \quad f_{yx} = \frac{\partial^2 f}{\partial x\,\partial y},$$

represent new ideas. The mixed partial derivative f_{xy} gives the rate of change in the y-direction of the rate of change of f in the x-direction, and f_{yx} gives the rate of change in the x-direction of the rate of change of f in the y-direction. It is not at all clear how these two mixed partials are related to each other, if indeed they are related at all.

Example 5 (continued) For the function being considered, $f(x, y) = x^3 e^{5y} + y \sin 2x$, we easily see that

$$f_x = 3x^2 e^{5y} + 2y \cos 2x, \qquad f_y = 5x^3 e^{5y} + \sin 2x,$$
$$f_{xy} = 15x^2 e^{5y} + 2 \cos 2x, \qquad f_{yx} = 15x^2 e^{5y} + 2 \cos 2x.$$

For the particular function considered in this example, we obviously have

$$f_{xy} = f_{yx}, \tag{1}$$

or equivalently,

$$\frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 f}{\partial x\,\partial y},$$

so the order of differentiation seems to be unimportant—at least in this case. But this is not an accident, and (1) is true for almost all functions that normally arise in applications. More precisely, *if both f_{xy} and f_{yx} exist for all points near (x_0, y_0) and are continuous at (x_0, y_0), then*

$$f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0).$$

A proof of this statement is given in Appendix A.17.

Partial derivatives of order greater than two, as well as higher-order derivatives of functions of more than two variables, are defined in the obvious way. For example, if $w = f(x, y, z)$, then

$$\frac{\partial^3 f}{\partial x\, \partial y\, \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y\, \partial z}\right) = (f_{zy})_x = f_{zyx},$$

$$\frac{\partial^4 f}{\partial z\, \partial y\, \partial x^2} = \frac{\partial}{\partial z}\left(\frac{\partial^3 f}{\partial y\, \partial x^2}\right) = (f_{xxy})_z = f_{xxyz},$$

etc. In general, with suitable continuity, it is immaterial in what order a sequence of partial differentiations is carried out, for by (1) we can reverse the order of any two successive differentiations. For example, $f_{xxyz} = f_{xyxz} = f_{xyzx} = f_{yxzx} = f_{yzxx}$.

PROBLEMS

In Problems 1–14, find $\partial z/\partial x$ and $\partial z/\partial y$.

1 $z = 2x + 3y$. **2** $z = 5x^2y$.

3 $z = \dfrac{2y^2}{3x + 1}$. **4** $z = y \cos x$.

5 $z = x^2 \sin y$. **6** $z = \tan 3x + \cot 4y$.

7 $z = x \tan 2y + y \tan 3x$. **8** $z = \sin xy$.

9 $z = \cos(3x - y)$. **10** $z = xye^{xy}$.

11 $z = e^x \sin y$. **12** $z = \tan^{-1} \dfrac{x}{y}$.

13 $z = e^y \ln x^2$. **14** $z = \ln(3x + y^2)$.

In Problems 15–18, find the partial derivatives with respect to x, y, and z.

15 $w = x^2y^5z^7$. **16** $w = \sin^{-1} \dfrac{z}{xy}$.

17 $w = x \ln \dfrac{y}{z}$. **18** $w = e^{x^2+y^3+z^4}$.

19 Consider the surface $z = 2x^2 + y^2$.
(a) The plane $y = 3$ intersects the surface in a curve. Find the equations of the tangent line to this curve at $x = 2$.
(b) The plane $x = 2$ intersects the surface in a curve. Find the equations of the tangent line to this curve at $y = 3$.

20 Consider the surface $z = x^2/(y^2 - 3)$.
(a) The plane $y = 2$ intersects the surface in a curve. Find the equations of the tangent line to this curve at $x = 3$.
(b) The plane $x = 3$ intersects the surface in a curve. Find the equations of the tangent line to this curve at $y = 2$.

21 Show that all of the following functions $z = f(x, y)$ satisfy the equation $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 0$:
(a) $z = \dfrac{x}{y}$; (b) $z = \dfrac{x}{x + y}$;
(c) $z = \ln \dfrac{2y^2}{x^2}$; (d) $z = \dfrac{xy^2}{x^3 + y^3}$.

22 If $z = ye^{x/y}$, show that $xz_x + yz_y = z$.

23 If $z = x^5 - 2x^4y + 5x^2y^3$, show that

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 5z.$$

In Problems 24–28, verify that $\partial^2 z/\partial x\, \partial y = \partial^2 z/\partial y\, \partial x$.

24 $z = \tan^{-1} \dfrac{x}{y}$.

25 $z = \ln(x + 5y)$.

26 $z = e^{xy} \cos(y - 2x)$.

27 $z = f(x)g(y)$.

28 $z = x^3 \tan 2x \csc 3y^4 \sin^{-1} \sqrt{x^2 + 1}$.

29 Show that each of the following functions satisfies *Laplace's equation* $\partial^2 f/\partial x^2 + \partial^2 f/\partial y^2 = 0$:
(a) $f(x, y) = \ln(x^2 + y^2)$; (b) $f(x, y) = e^x \sin y$;
(c) $f(x, y) = e^{-3x} \cos 3y$; (d) $f(x, y) = \tan^{-1} \dfrac{y}{x}$.

30 Show that each of the following functions satisfies the *wave equation* $a^2\,\partial^2 f/\partial x^2 = \partial^2 f/\partial t^2$:
(a) $f(x, t) = (x + at)^3$; (b) $f(x, t) = (x - at)^5$;
(c) $f(x, t) = \sin(x + at)$; (d) $f(x, t) = e^{x-at}$.

31 Find a function $f(x, y)$ such that

$$\frac{\partial f}{\partial x} = 3y^2 - 2x\cos y \quad \text{and} \quad \frac{\partial f}{\partial y} = 6xy + x^2 \sin y + 2.$$

It is sometimes convenient to define functions by integrals of the form

$$F(x) = \int_a^b f(x, y)\,dy.$$

It is usually possible to calculate the derivative $F'(x)$ of such a function by "differentiating under the integral sign":

$$F'(x) = \frac{d}{dx}\int_a^b f(x, y)\,dy = \int_a^b \left[\frac{\partial}{\partial x} f(x, y)\right] dy.^*$$

32 Verify the formula just stated in the following cases:
(a) $a = 0$, $b = 1$, $f(x, y) = x + y$;
(b) $a = 0$, $b = 1$, $f(x, y) = x^3y^2 + x^2y^3$;
(c) $a = 0$, $b = \pi$, $f(x, y) = \sin xy$.

*A proof is given in Appendix A.18.

19.3 THE TANGENT PLANE TO A SURFACE

The concept of a tangent plane to a surface corresponds to the concept of a tangent line to a curve. Geometrically, the tangent plane to a surface at a point is the plane that "best approximates" the surface near the point. It will be necessary for us to think rather carefully about what this means, because—as we shall see in Sections 19.5 and 19.6—weighty practical consequences depend on it.

Consider a surface $z = f(x, y)$, as shown in Fig. 19.6. As we pointed out in Section 19.2, the plane $y = y_0$ intersects this surface in a curve C_1 whose equation is

$$z = f(x, y_0),$$

and the plane $x = x_0$ intersects it in a curve C_2 whose equation is

$$z = f(x_0, y);$$

and the slopes of the tangent lines to these curves at the point $P_0 = (x_0, y_0, z_0)$ are the partial derivatives

$$f_x(x_0, y_0) \quad \text{and} \quad f_y(x_0, y_0). \tag{1}$$

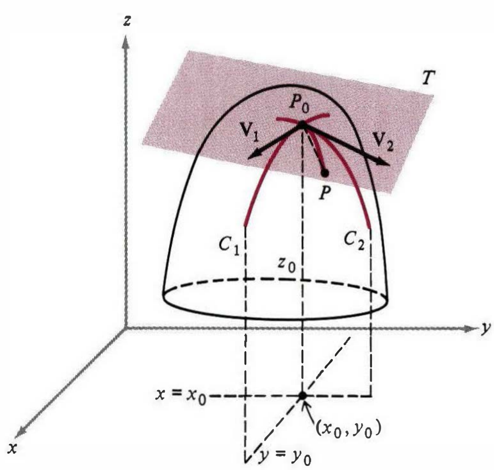


Figure 19.6 The tangent plane.

These two tangent lines determine a plane, and, as Fig. 19.6 suggests, if the surface is sufficiently smooth near P_0, then this plane will be tangent to the surface at P_0.

It is important to be quite clear about what we mean by a tangent plane, so we give a definition. In this context, where P_0 is a point on a surface $z = f(x, y)$, let T be a plane through P_0 and let P be any other point on the surface. If, as P approaches P_0 along the surface, the angle between the segment P_0P and the plane T approaches zero, then T is called the *tangent plane* to the surface at P_0.

It is easy to see that a surface need not have a tangent plane at a point P_0. A very simple example is provided by the half-cone $z = \sqrt{x^2 + y^2}$ shown in Fig. 19.7. It is clear that no plane is tangent to this surface at the origin. In this case the curves C_1 and C_2 have no tangent lines at the origin, and the partial derivatives (1) do not exist there. However, even when the curves C_1 and C_2 are smooth enough to have tangent lines at P_0, the surface may still not have a tangent plane at P_0, because of nonsmooth behavior near P_0 in the regions between C_1 and C_2. In Section 19.4 we discuss a vital lemma to the effect that this cannot happen if the partial derivatives $f_x(x, y)$ and $f_y(x, y)$ exist at all points in some neighborhood of (x_0, y_0) and are continuous at (x_0, y_0) itself.

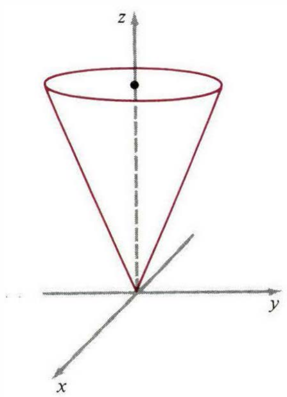


Figure 19.7

Meanwhile, we assume that the tangent plane exists at P_0, and we develop a method of finding its equation. Since the point $P_0 = (x_0, y_0, z_0)$ lies on this tangent plane, we know that the equation has the form

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0, \tag{2}$$

where $N = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is any normal vector. It remains to find N, and to do this we use the cross product of two vectors $\mathbf{V}_1$ and $\mathbf{V}_2$ that are tangent to the curves C_1 and C_2 at P_0 (see Fig. 19.6). To find $\mathbf{V}_1$, we use the fact that along the tangent line to C_1, an increase of 1 unit in x produces a change $f_x(x_0, y_0)$ in z, while y does not change at all. Thus, the vector

$$\mathbf{V}_1 = \mathbf{i} + 0 \cdot \mathbf{j} + f_x(x_0, y_0)\mathbf{k}$$

is tangent to C_1 at P_0. Similarly, the vector

$$\mathbf{V}_2 = 0 \cdot \mathbf{i} + \mathbf{j} + f_y(x_0, y_0)\mathbf{k}$$

is tangent to C_2 at P_0. Since $\mathbf{V}_1$ and $\mathbf{V}_2$ lie in the tangent plane, we are now able to obtain our normal vector $\mathbf{N}$ by calculating

$$\mathbf{N} = \mathbf{V}_2 \times \mathbf{V}_1 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & f_y(x_0, y_0) \\ 1 & 0 & f_x(x_0, y_0) \end{vmatrix} = f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j} - \mathbf{k}. \tag{3}$$

(The order of factors in this cross product is chosen only for convenience, to produce one minus sign in the result instead of two.) When the components of (3) are inserted in (2), we see that the desired equation is

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0,$$

or equivalently,

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0). \tag{4}$$

Example 1 Find the tangent plane to the surface

$$z = f(x, y) = 2xy^3 - 5x^2$$

at the point (3, 2, 3).

Solution The first step should be to check that this point actually lies on the given surface, and we assume that this has been done. Here we have $f_x = 2y^3 - 10x$ and $f_y = 6xy^2$, so $f_x(3, 2) = -14$ and $f_y(3, 2) = 72$. The equation of the tangent plane is therefore

$$z - 3 = -14(x - 3) + 72(y - 2).$$

Tangent planes to surfaces where z is not explicitly given as a function of x and y will be discussed in Section 19.5. However, we can get a preliminary idea of what to expect by applying our present method to simple cases.

Example 2 Find the tangent plane to the sphere

$$x^2 + y^2 + z^2 = 14 \tag{5}$$

at the point (1, 2, 3).

Solution Even though this sphere is not a surface of the form $z = f(x, y)$, it can be thought of as a combination of two such surfaces, the upper and lower hemispheres. By solving (5) for z, we see that the upper hemisphere is given by

$$z = f(x, y) = \sqrt{14 - x^2 - y^2},$$

so

$$f_x = \frac{-x}{\sqrt{14 - x^2 - y^2}} \quad \text{and} \quad f_y = \frac{-y}{\sqrt{14 - x^2 - y^2}}.$$

These formulas give

$$f_x(1, 2) = -\tfrac{1}{3} \quad \text{and} \quad f_y(1, 2) = -\tfrac{2}{3},$$

so the equation of the tangent plane is

$$z - 3 = -\tfrac{1}{3}(x - 1) - \tfrac{2}{3}(y - 2),$$

or

$$x + 2y + 3z = 14.$$

In this example we solved equation (5) explicitly for z, then proceeded as before. An alternative method that is often easier is to assume that the given equation defines z implicitly as a function of x and y, and to find the partial derivatives by implicit differentiation. With this method we use equation (4) in the slightly different form

$$z - z_0 = \left(\frac{\partial z}{\partial x}\right)_{P_0}(x - x_0) + \left(\frac{\partial z}{\partial y}\right)_{P_0}(y - y_0), \tag{6}$$

where the coefficients are written this way because $\partial z/\partial x$ and $\partial z/\partial y$ need not depend only on x and y.

Example 3 To find the tangent plane of Example 2 by the method just suggested, we first hold y fixed and differentiate (5) implicitly with respect to x, which gives

$$2x + 2z\frac{\partial z}{\partial x} = 0,$$

so $\partial z/\partial x = -x/z$. Similarly, $\partial z/\partial y = -y/z$. At the point $P_0 = (1, 2, 3)$, these partial derivatives have the numerical values

$$\left(\frac{\partial z}{\partial x}\right)_{P_0} = -\frac{1}{3} \quad \text{and} \quad \left(\frac{\partial z}{\partial y}\right)_{P_0} = -\frac{2}{3},$$

so by (6) the tangent plane is

$$z - 3 = -\tfrac{1}{3}(x - 1) - \tfrac{2}{3}(y - 2),$$

just as before. Of course, this method is of particular value when the equation of the surface is difficult or impossible to solve for z.

PROBLEMS

In Problems 1–10, find an equation for the tangent plane to the given surface at the indicated point.

1 $z = (x^2 + y^2)^2$, $(1, 2, 25)$.

2 $z = 4xy$, $(4, \frac{1}{4}, 4)$.

3 $z = \sin x + \sin 2y + \sin 3(x + y)$, $(0, 0, 0)$.

4 $z = x^2 + xy + y^2 - 10y + 5$, $(3, 2, 4)$.

5 $z = x^2 - 2y^2$, $(3, 2, 1)$.

6 $z = \dfrac{2x + y}{x - 2y}$, $(3, 1, 7)$.

7 $z = e^y \cos x$, $(0, 0, 1)$.

8 $z = \tan^{-1}\dfrac{x}{y}$, $\left(4, 4, \dfrac{\pi}{4}\right)$.

9 $xy^2 + yz^2 + zx^2 = 25$, $(1, 2, 3)$

10 $z^3 + xyz = 33$, $(1, 2, 3)$.

11 Let $P_0 = (x_0, y_0, z_0)$ with $z_0 > 0$ be a point on the sphere

$$x^2 + y^2 + z^2 = a^2.$$

Show that the tangent plane at this point is perpendicular to the radius vector to the point, in agreement with the definition given in geometry.

12 Use implicit differentiation to show that the equation of the tangent plane to the sphere

$$x^2 + y^2 + z^2 = a^2$$

at the point $P_0 = (x_0, y_0, z_0)$ is $x_0x + y_0y + z_0z = a^2$.

13 Use implicit differentiation to find the equation of the tangent plane to the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

at the point $P_0 = (x_0, y_0, z_0)$.

14 Let a be a positive constant and consider the tangent plane to the surface $xyz = a$ at a point in the first octant. Show that the tetrahedron formed by this plane and the coordinate planes has constant volume, independent of the point of tangency. What is this volume?

15 The *angle between two surfaces* at a common point is the smallest positive angle between the normals to these surfaces at this point. Find the angle between $z = e^{xy} - 1$ and $z = \ln\sqrt{x^2 + y^2}$ at $(0, 1, 0)$.

16 If $P_0 = (x_0, y_0, z_0)$ is a point on the curve of intersection of two surfaces $z = f(x, y)$ and $z = g(x, y)$, devise a method for finding a tangent vector to this curve at P_0. Apply this method to find a vector tangent to the curve of intersection of the cone $z^2 = 3x^2 + 4y^2$ and the plane $3x - 2y + z = 8$ at the point $P_0 = (2, 1, 4)$.

17 If a surface has an equation of the form $z = xf(x/y)$, show that all of its tangent planes have a common point. What is this point?

18 If $P_0 = (x_0, y_0, z_0)$ is a point on the cone $z^2 = a(x^2 + y^2)$ other than the vertex, show that the tangent plane at P_0 has $z_0z = a(x_0x + y_0y)$ as its equation. Conclude that every such plane passes through the vertex. Show that the normal line at P_0 has

$$x = x_0 + ax_0t, \qquad y = y_0 + ay_0t, \qquad z = z_0 - z_0t$$

as parametric equations.

19 On the cone in Problem 18, consider all points of fixed height h above the xy-plane and draw normal lines at these points. Show that the points where these lines intersect the xy-plane form a circle, and find the radius of this circle.

20 Let normal lines be drawn at all points on the surface $z = ax^2 + by^2$ which are at a given fixed height h above the xy-plane, and find the equation of the curve in which these lines intersect the xy-plane.

19.4 INCREMENTS AND DIFFERENTIALS. THE FUNDAMENTAL LEMMA

Most of calculus can be understood by using geometric intuition mixed with a little common sense, without getting bogged down in the underlying theory of the subject. In a few places, however, this theory is inescapable, because without it there is no way to grasp what is going on in the main developments of the subject itself. This is true for infinite series and the theory of convergence. It is also true for the topics of the next two sections—directional derivatives and the chain rule—which cannot be understood without a certain degree of attention to the theoretical issues that we now briefly discuss.

In order to see what these issues are, we begin by considering a function $y = f(x)$ of one variable that has a derivative at a point x_0. If Δx is an increment that carries x_0 to a nearby point $x_0 + \Delta x$ (see Fig. 19.8), we are interested in the corresponding increment in y,

$$\Delta y = f(x_0 + \Delta x) - f(x_0).$$

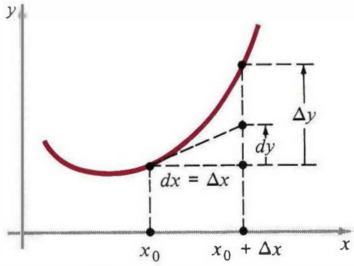


Figure 19.8 Differentials dx and dy.

The definition of the derivative $f'(x_0)$ is

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}, \tag{1}$$

and this can be written in the equivalent form

$$\frac{\Delta y}{\Delta x} = f'(x_0) + \epsilon,$$

where $\epsilon \to 0$ as $\Delta x \to 0$. Accordingly, with no further hypotheses than the assumed existence of the derivative (1), we can write the increment Δy in the form

$$\Delta y = f'(x_0)\,\Delta x + \epsilon\,\Delta x, \qquad \text{where } \epsilon \to 0 \text{ as } \Delta x \to 0. \tag{2}$$

The situation is entirely different for a function of two (or more) variables, as we now explain.

Consider a function $z = f(x, y)$ and let (x_0, y_0) be a point at which the partial derivatives $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ both exist. The increment in z produced by moving from (x_0, y_0) to a nearby point $(x_0 + \Delta x, y_0 + \Delta y)$ is

$$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0),$$

as shown in Fig. 19.9. In order to develop the tools we shall need in Sections 19.5 and 19.6, it will be necessary to express Δz in a form analogous to (2),

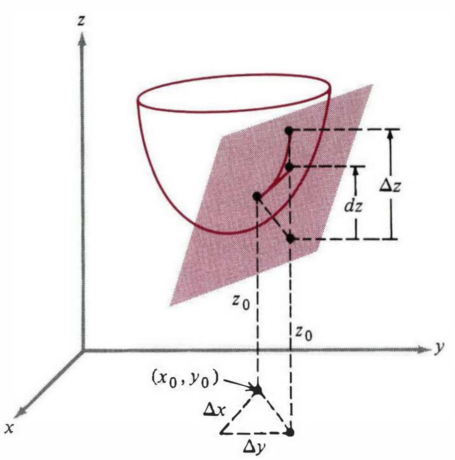


Figure 19.9 The differential dz.

$$\Delta z = f_x(x_0, y_0)\,\Delta x + f_y(x_0, y_0)\,\Delta y + \epsilon_1\,\Delta x + \epsilon_2\,\Delta y, \tag{3}$$

where ϵ_1 and $\epsilon_2 \to 0$ as Δx and $\Delta y \to 0$. Unfortunately, in sharp contrast to the one-variable case, the mere existence of the partial derivatives f_x and f_y at (x_0, y_0) is not enough to guarantee the validity of (3). Sufficient conditions for this conclusion are given in the

Fundamental Lemma *Suppose that a function $z = f(x, y)$ and its partial derivatives f_x and f_y are defined at a point (x_0, y_0), and also throughout some neighborhood of this point. Suppose further that f_x and f_y are continuous at (x_0, y_0). Then the increment Δz can be expressed in the form* (3), *where ϵ_1 and $\epsilon_2 \to 0$ as Δx and $\Delta y \to 0$.*

This statement is called a *lemma* for the usual reason: its significance lies not in itself, but rather in the use that can be made of it elsewhere. A proof is given in Appendix A.19.

We do not wish to linger on these matters, but nevertheless a few brief remarks are in order.

Remark 1 In the case of a function of one variable, (1) and (2) are equivalent, and if either condition holds it is customary to denote Δx by dx and to write $dy = f'(x_0)\,dx$, so that dy is the change in y along the tangent line. A function $z = f(x, y)$ for which $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ both exist is said to be *differentiable* at (x_0, y_0) if the conclusion of the lemma is valid—so that more is required than merely the existence of the partial derivatives. In this case—and *only* in this case!—we denote Δx and Δy by dx and dy, and we define the *differential* dz by*

$$dz = f_x(x_0, y_0)\,dx + f_y(x_0, y_0)\,dy.$$

Under these circumstances it can be proved that the surface $z = f(x, y)$ has a tangent plane at (x_0, y_0, z_0) and that *dz is the change in z* along this plane, as suggested in Fig. 19.9. The differential dz is usually written in the equivalent forms

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \qquad \text{or} \qquad df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

Remark 2 A function $z = f(x, y)$ which is differentiable at a point is automatically continuous there. This follows at once from (3), which shows that $\Delta z \to 0$ if Δx and $\Delta y \to 0$. In the single-variable case, we know that if a function has a derivative at a point, then it is necessarily continuous there. However, this is not true for functions of more than one variable: the mere existence of the partial derivatives f_x and f_y at a point does not imply the continuity of $f(x, y)$ at that point. This is shown by the example of the bizarre function discussed in Section 19.1, for which $f_x(0, 0) = f_y(0, 0) = 0$ and yet the function is discontinuous at $(0, 0)$.

The concepts of a differentiable function and its differential, and also the Fundamental Lemma, can be extended in an obvious way to functions of any finite number of variables. This would involve much additional writing but no new ideas, and we shall not burden the reader with the details.

*Sometimes dz is called the *total differential.*

19.5 DIRECTIONAL DERIVATIVES AND THE GRADIENT

Let $f(x, y, z)$ be a function (of *three* variables!) defined throughout some region of three-dimensional space, and let P be a point in this region. At what rate does f change as we move away from P in a specified direction? In the directions of the positive x-, y-, and z-axes, we know that the rates of change of f are given by the partial derivatives $\partial f/\partial x$, $\partial f/\partial y$, and $\partial f/\partial z$. But how do we calculate the rate of change of f if we move away from P in a direction that is not a coordinate direction? In analyzing this problem, we will encounter the very important concept of the gradient of a function.

Suppose that the point P under consideration has coordinates x, y, and z, so that $P = (x, y, z)$; let $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ be the position vector of P, and let the specified direction be given by a unit vector $\mathbf{u}$, as shown in Fig. 19.10. If we move away from P in this direction to a nearby point $Q = (x + \Delta x, y + \Delta y, z + \Delta z)$, then the function f will change by an amount Δf. If we now divide this change Δf by the distance $\Delta s = |\Delta \mathbf{R}|$ between P and Q, then the quotient $\Delta f/\Delta s$ is the average rate of change of f (with respect to distance) as we move from P to Q. For instance, if the value of f at P is the temperature at this point, then $\Delta f/\Delta s$ is the average rate of change of temperature along the segment PQ. The limiting value of $\Delta f/\Delta s$ as Q approaches P, namely,

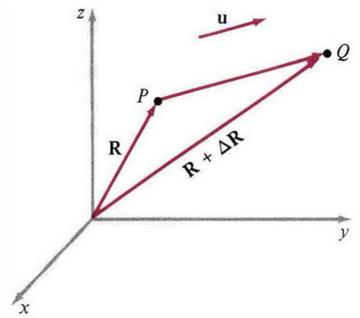


Figure 19.10

$$\frac{df}{ds} = \lim_{\Delta s \to 0} \frac{\Delta f}{\Delta s},$$

is called the *derivative of f at the point P in the direction* **u**, or simply the *directional derivative* of f. In the case of the temperature function, df/ds represents the instantaneous rate of change of temperature with respect to distance—roughly speaking, how fast it is getting hotter—at the point P as we move away from P in the direction specified by **u**.

This is all very well, but how do we actually calculate df/ds in a specific case? To discover how to do this, we assume that $f(x, y, z)$ has continuous partial derivatives with respect to x, y, and z. Indeed, to avoid the tedious repetition of hypotheses, we make this a blanket assumption for every function we discuss, unless we explicitly state otherwise. With this, the Fundamental Lemma enables us to write Δf in the form

$$\Delta f = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \frac{\partial f}{\partial z}\Delta z + \epsilon_1 \Delta x + \epsilon_2 \Delta y + \epsilon_3 \Delta z, \tag{1}$$

where $\epsilon_1, \epsilon_2, \epsilon_3 \to 0$ as Δx, Δy, and $\Delta z \to 0$, that is, as $\Delta s \to 0$. Dividing (1) by Δs now gives

$$\frac{\Delta f}{\Delta s} = \frac{\partial f}{\partial x}\frac{\Delta x}{\Delta s} + \frac{\partial f}{\partial y}\frac{\Delta y}{\Delta s} + \frac{\partial f}{\partial z}\frac{\Delta z}{\Delta s} + \epsilon_1 \frac{\Delta x}{\Delta s} + \epsilon_2 \frac{\Delta y}{\Delta s} + \epsilon_3 \frac{\Delta z}{\Delta s}, \tag{2}$$

and by taking the limit as $\Delta s \to 0$, we see that the last three terms in (2) approach zero and we obtain the formula

$$\frac{df}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} + \frac{\partial f}{\partial z}\frac{dz}{ds}. \tag{3}$$

This formula should be recognized as a special kind of chain rule, in the sense that as we move along the line through P and parallel to **u**, f is a function of x, y, and z, where x, y, and z are in turn functions of the distance s, and (3) shows how to differentiate f with respect to s.

We observe that the first factor in each product on the right of (3) depends only on the function f and the coordinates of the point P at which the partial derivatives of f are evaluated, while the second factor in each product is independent of f and depends only on the direction in which df/ds is being calculated. These facts suggest that the right side of (3) ought to be thought of—and written—as the dot product of two vectors, as follows:

$$\begin{aligned}\frac{df}{ds} &= \left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right) \cdot \left(\frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j} + \frac{dz}{ds}\mathbf{k}\right) \\ &= \left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right) \cdot \frac{d\mathbf{R}}{ds}. \end{aligned} \tag{4}$$

The first factor here is a vector called the *gradient* of f. It is denoted by the symbol grad f, so that by definition

$$\text{grad } f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}. \tag{5}$$

With this notation, (4) can be written as

$$\frac{df}{ds} = (\text{grad } f) \cdot \frac{d\mathbf{R}}{ds}. \tag{6}$$

But we know that $d\mathbf{R}/ds$ is a unit vector, and since it has the same direction as $\mathbf{u}$, it equals $\mathbf{u}$. Formula (6) is therefore equivalent to

$$\frac{df}{ds} = (\text{grad } f) \cdot \mathbf{u}. \tag{7}$$

This tells us how to calculate df/ds, because (5) is presumably simple to compute from the given function f, and then to evaluate at the given point P, and the dot product (7) of two known vectors is easy to find.

For a given function f and a given point P, grad f is a fixed vector which can be placed so that its tail lies at P. We also place the tail of $\mathbf{u}$ at P, as shown in Fig. 19.11. To understand the significance of grad f, we use the definition of the dot product and the fact that $\mathbf{u}$ is a unit vector to write (7) in the form

$$\frac{df}{ds} = |\text{grad } f| \cos\theta, \tag{8}$$

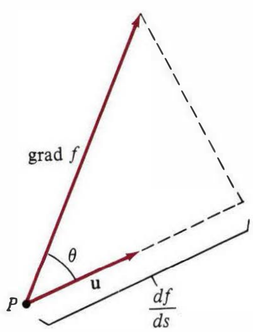


Figure 19.11 Directional derivative.

where θ is the angle between grad f and $\mathbf{u}$. Since the direction of $\mathbf{u}$ can be chosen to suit our convenience, (8) immediately yields the first fundamental property of the gradient:

Property 1 The directional derivative df/ds in any given direction is the scalar projection of grad f in that direction (see Fig. 19.11).

In this sense, the single vector grad f contains within itself the directional derivatives of f at P in all possible directions.

Next, if $\mathbf{u}$ is chosen to point in the same direction as grad f, so that $\theta = 0$ and $\cos 0 = 1$, then (8) shows that df/ds has its maximum value—that is, f increases most rapidly—in this direction. Also, this maximum value equals $|\text{grad } f|$. These remarks give the next two fundamental properties of the gradient:

Property 2 The vector grad f points in the direction in which f increases most rapidly.

Property 3 The length of the vector grad f is the maximum rate of increase of f.

As these remarks show, even though formulas (7) and (8) are equivalent, they play very different roles in our thinking, for we use (7) to calculate df/ds and (8) to understand the intuitive meaning of the vector grad f.

Example 1 If $f(x, y, z) = x^2 - y + z^2$, find the directional derivative df/ds at the point (1, 2, 1) in the direction of the vector $4\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$.

Solution At the point (1, 2, 1), we have grad $f = 2x\mathbf{i} - \mathbf{j} + 2z\mathbf{k} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$. We obtain a unit vector $\mathbf{u}$ in the desired direction by dividing the given vector by its own length,

$$\mathbf{u} = \frac{4\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}}{\sqrt{16 + 4 + 16}} = \frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}.$$

Formula (7) now gives

$$\begin{aligned}\frac{df}{ds} &= (\text{grad } f) \cdot \mathbf{u} \\ &= (2\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \cdot (\tfrac{2}{3}\mathbf{i} - \tfrac{1}{3}\mathbf{j} + \tfrac{2}{3}\mathbf{k}) = 3.\end{aligned}$$

Thus, the function f is increasing at the rate of 3 units per unit distance as we leave (1, 2, 1) in the given direction.

Example 2 Let the temperature of the air at points in space be given by the function $f(x, y, z) = x^2 - y + z^2$. A mosquito located at (1, 2, 1) wishes to get cool as soon as possible. In what direction should it fly?

Solution We saw in Example 1 that grad $f = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ at the point (1, 2, 1). Since the direction of grad f is that in which the temperature increases most rapidly, the mosquito should fly in the opposite direction, that of $-\text{grad} f = -2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$.

The fourth fundamental property of the gradient is useful in geometry. In order to explain what it is, we denote the point under consideration by $P_0 = (x_0, y_0, z_0)$ to emphasize that it is fixed in this discussion, and we let c_0 be the value of our function f at the point P_0. Then the set of all points in space at which $f(x, y, z)$ has the same value c_0 constitutes, in general, a level surface through P_0 whose equation is $f(x, y, z) = c_0$. We wish to show that the vector grad f is normal (perpendicular) to this level surface at the point P_0, as suggested on the left in Fig. 19.12. To this end, we consider a curve that lies on the surface and passes through P_0. If we move to a nearby point Q on this curve and measure s along the curve, then $\Delta f = 0$ because f has the same value at all points on the surface, and therefore $df/ds = 0$ at P_0 in the direction of the tangent to the curve. Formula (6) remains valid and implies that

$$(\text{grad } f) \cdot \frac{d\mathbf{R}}{ds} = 0, \tag{9}$$

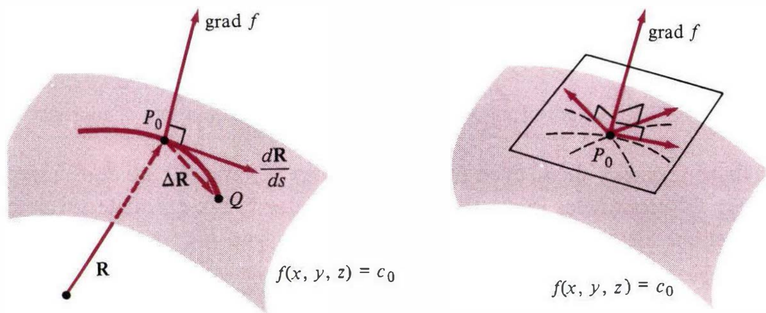


Figure 19.12 Gradient is normal to level surface.

where $d\mathbf{R}/ds$ is the unit tangent vector to the curve at P_0. The vanishing of the dot product in (9) tells us that grad f is perpendicular to this tangent vector. But the same reasoning applies to every curve on the surface that passes through P_0, so grad f is perpendicular to the tangent vectors to all these curves (Fig. 19.12, right). Since these tangent vectors determine the tangent plane at P_0, and being normal to the surface means being normal to this tangent plane, we have:

Property 4 The gradient of a function $f(x, y, z)$ at a point P_0 is normal to the level surface of f that passes through P_0.

In the context of this discussion, we point out that the equation of any surface can be written in the form $f(x, y, z) = c_0$, and can therefore be regarded as a level surface of the function $f(x, y, z)$. If $P_0 = (x_0, y_0, z_0)$ is a point on this surface, then Property 4 tells us that the vector

$$\mathbf{N} = \text{grad } f = \left(\frac{\partial f}{\partial x}\right)_{P_0}\mathbf{i} + \left(\frac{\partial f}{\partial y}\right)_{P_0}\mathbf{j} + \left(\frac{\partial f}{\partial z}\right)_{P_0}\mathbf{k}$$

is normal to the tangent plane at P_0, so if $\mathbf{N} \neq \mathbf{0}$, the equation of this tangent plane is

$$\left(\frac{\partial f}{\partial x}\right)_{P_0}(x - x_0) + \left(\frac{\partial f}{\partial y}\right)_{P_0}(y - y_0) + \left(\frac{\partial f}{\partial z}\right)_{P_0}(z - z_0) = 0. \tag{10}$$

We observe that this equation includes equation (4) in Section 19.3 as a special case; for if the surface is given in the form $z = g(x, y)$, then this can be written as $g(x, y) - z = 0$, so the surface is a level surface of the function $f(x, y, z) = g(x, y) - z$, and this makes the coefficients in (10) equal to $g_x(x_0, y_0)$, $g_y(x_0, y_0)$, -1.

Example 3 Find the equation of the tangent plane to the surface $xy^2z^3 = 12$ at the point $(3, -2, 1)$.

Solution This surface is a level surface of the function $f(x, y, z) = xy^2z^3$. The vector grad f at the point $(3, -2, 1)$ is normal to the surface at this point. This vector is

$$\begin{aligned}\text{grad } f &= y^2z^3\mathbf{i} + 2xyz^3\mathbf{j} + 3xy^2z^2\mathbf{k}\\ &= 4\mathbf{i} - 12\mathbf{j} + 36\mathbf{k} = 4(\mathbf{i} - 3\mathbf{j} + 9\mathbf{k}).\end{aligned}$$

Therefore the equation of the tangent plane is

$$(x-3)-3(y+2)+9(z-1)=0$$

or

$$x-3y+9z=18.$$

Remark 1 The main uses of directional derivatives and gradients are found in the geometry and physics of three-dimensional space. However, these concepts can also be defined in two dimensions, and they have similar (but thinner) properties. Thus, a curve $f(x, y) = c_0$ can be thought of as a level curve of the function $z = f(x, y)$; and if the gradient of this function is defined by

$$\text{grad } f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j},$$

then the value of this gradient at a point $P_0 = (x_0, y_0)$ on the curve is a vector that is normal to the curve.

Remark 2 The gradient of a function $f(x, y, z)$ can be written in "operational form" as

$$\text{grad } f = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) f.$$

The *del operator* preceding the function f is usually denoted by the symbol ∇ (an inverted delta, read "del") so that

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}.$$

This del operator is similar to, but more complicated than, the familiar differentiation operator d/dx. When del is applied to a function f, it produces a vector, namely, the vector grad f. In this notation, formulas (5), (6), and (7) become

$$\text{grad } f = \nabla f, \qquad \frac{df}{ds} = \nabla f \cdot \frac{d\mathbf{R}}{ds}, \qquad \text{and} \qquad \frac{df}{ds} = \nabla f \cdot \mathbf{u}.$$

We will make very substantial use of the operator ∇ in Chapter 21.

PROBLEMS

1 Find the gradient of f at P if
   (a) $f(x, y, z) = xy + xz + yz$, $P = (-1, 3, 5)$;
   (b) $f(x, y, z) = e^{xy} \cos z$, $P = (0, 2, 0)$;
   (c) $f(x, y, z) = \ln (x^2 + y^2 + z^2)$, $P = (1, 2, -2)$;
   (d) $f(x, y, z) = xy/z$, $P = (2, -1, 5)$.

2 Find the directional derivative of f at P in the direction of the given vector:
   (a) $f(x, y, z) = xy^2 + x^2z + yz$, $P = (1, 1, 2)$, $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$;
   (b) $f(x, y, z) = \ln (x^2 + y^2 + z^2)$, $P = (0, 0, 1)$, vector from P to $(2, 2, 0)$;
   (c) $f(x, y, z) = x \sin y + y \sin z + z \sin x$, $P = (1, 0, 0)$, $2\sqrt{3}\mathbf{i} + 2\mathbf{j}$;
   (d) $f(x, y, z) = xye^z + yze^x$, $P = (1, 0, 0)$, vector from P to $(2, 2, 1)$.

3 Find the maximum value of the directional derivative of f at P, and the direction in which it occurs:
   (a) $f(x, y, z) = \sin xy + \cos yz$, $P = (-3, 0, 7)$;
   (b) $f(x, y, z) = e^x \cos y + e^y \cos z + e^z \cos x$, $P = (0, 0, 0)$;
   (c) $f(x, y, z) = 2xyz + y^2 + z^2$, $P = (2, 1, 1)$;
   (d) $f(x, y, z) = e^{xyz}$, $P = (2, 1, 1)$.

4 In what direction should one travel, starting at the origin, to obtain the most rapid rate of decrease of the function

$$f(x, y, z) = (2 - x - y)^3 + (3x + 2y - z + 1)^2?$$

5 Find the unit vectors normal to the surface $xyz = 4$ at the point $(2, -2, -1)$.

6 If $f(x, y, z) = x^2 + 4y^2 - 8z$, find df/ds at $(4, 1, 0)$ (a) along the line $(x-4)/2 = (y-1)/1 = z/(-2)$ in the direction of decreasing x; (b) along the normal to the plane $3(x-4) - (y-1) + 2z = 0$ in the direction of increasing x; (c) in the direction in which f increases most rapidly.

7 Suppose that the temperature T at a point $P = (x, y, z)$ is given by $T = 2x^2 - y^2 + 4z^2$. Find the rate of change of T at the point $(1, -2, 1)$ in the direction of the vector $4\mathbf{i} - \mathbf{j} + 2\mathbf{k}$. In what direction does T increase most rapidly at this point? What is this maximum rate of increase?

8 Find the tangent plane and normal line to the hyperboloid $x^2 + y^2 - z^2 = 5$ at the point $(4, 5, 6)$.

9 Show that the tangent plane to the quadric surface $ax^2 + by^2 + cz^2 = d$ at the point (x_0, y_0, z_0) has $ax_0x + by_0y + cz_0z = d$ as its equation.

10 Show that the del operator has the following properties that demonstrate its close similarity to the differentiation operator d/dx:
(a) $\nabla(f + g) = \nabla f + \nabla g$;
(b) $\nabla(fg) = f\nabla g + g\nabla f$;
(c) $\nabla\left(\dfrac{f}{g}\right) = \dfrac{g\nabla f - f\nabla g}{g^2}$;
(d) $\nabla f^n = nf^{n-1}\nabla f$.

19.6 THE CHAIN RULE FOR PARTIAL DERIVATIVES

The single-variable chain rule for ordinary derivatives tells us how to differentiate composite functions. It says that if w is a function of x where x is in turn a function of a third variable t, say $w = f(x)$ where $x = g(t)$, then

$$\frac{dw}{dt} = \frac{dw}{dx}\frac{dx}{dt}. \tag{1}$$

We know from ample experience that this is an indispensable tool of calculus; it is used more frequently than any other differentiation rule.

The simplest multivariable chain rule involves a function $w = f(x, y)$ of two variables x and y, where x and y are each functions of another variable t, $x = g(t)$ and $y = h(t)$. Then w is a function of t,

$$w = f[g(t), h(t)] = F(t),$$

and we shall prove that the derivative of this composite function is given by the formula

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}. \tag{2}$$

This is the *chain rule* for this situation.

The proof of (2) is easy. We begin by changing t to $t + \Delta t$, where $\Delta t \neq 0$. This increment in t produces increments Δx and Δy in x and y, which in turn produce an increment Δw in w. Since all the functions we discuss are assumed to have continuous partial derivatives, the Fundamental Lemma enables us to write Δw in the form

$$\Delta w = \frac{\partial w}{\partial x}\Delta x + \frac{\partial w}{\partial y}\Delta y + \epsilon_1\,\Delta x + \epsilon_2\,\Delta y, \tag{3}$$

where ϵ_1 and $\epsilon_2 \to 0$ as Δx and $\Delta y \to 0$. On dividing (3) by Δt, we obtain

$$\frac{\Delta w}{\Delta t} = \frac{\partial w}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial w}{\partial y}\frac{\Delta y}{\Delta t} + \epsilon_1\frac{\Delta x}{\Delta t} + \epsilon_2\frac{\Delta y}{\Delta t}. \tag{4}$$

If we now form the limit as $\Delta t \to 0$, then Δx and Δy also $\to 0$, so ϵ_1 and $\epsilon_2 \to 0$, and (4) immediately yields (2).

Example 1 If $w = 3x^2 + 2xy - y^2$ where $x = \cos t$ and $y = \sin t$, find dw/dt.

Solution Formula (2) tells us that

$$\frac{dw}{dt} = (6x + 2y)(-\sin t) + (2x - 2y)\cos t.$$

By substituting $x = \cos t$ and $y = \sin t$, we can express this in terms of t alone,

$$\begin{aligned}\frac{dw}{dt} &= (6\cos t + 2\sin t)(-\sin t) + (2\cos t - 2\sin t)(\cos t)\\ &= -6\sin t\cos t - 2\sin^2 t + 2\cos^2 t - 2\sin t\cos t\\ &= 2(\cos^2 t - \sin^2 t) - 8\sin t\cos t = 2\cos 2t - 4\sin 2t.\end{aligned}$$

We can check this result by first substituting and then differentiating, which gives

$$w = 3\cos^2 t + 2\sin t\cos t - \sin^2 t$$

and

$$\begin{aligned}\frac{dw}{dt} &= 6\cos t(-\sin t) + 2\sin t(-\sin t) + 2\cos^2 t - 2\sin t\cos t\\ &= 2(\cos^2 t - \sin^2 t) - 8\sin t\cos t = 2\cos 2t - 4\sin 2t,\end{aligned}$$

as before.

In the situation of formula (2), it is convenient to call w the *dependent variable*, x and y the *intermediate variables*, and t the *independent variable*. We notice that the right side of (2) has two terms, one for each intermediate variable, and that each of these terms resembles the right side of the single-variable chain rule (1).

Formula (2) extends in an obvious way to any number of intermediate variables. For instance, if $w = f(x, y, z)$ where x, y, and z are each functions of t, then

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}. \tag{5}$$

The proof of this is essentially the same as the proof of (2), except that it uses the Fundamental Lemma for three variables instead of two.

Further, x, y, and z here need not be functions of only one independent variable, but can be functions of two or more variables. Thus, if x, y, and z are each functions of the variables t and u, then w is also a function of t and u, and its partial derivatives are given by

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t} \tag{6}$$

and

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}. \tag{7}$$

We use roundback d's everywhere here because every function depends on more than one variable. It is necessary to be very clear about the meanings of the letters in formulas like these. For example, on the left side of (6), w is considered a function of t and u, while on the right side it is considered a function of x, y, and z. The proofs are the same as before, and all of these formulas—(2), (5), (6), (7), and their extensions to any number of intermediate and independent variables—are collectively called the *chain rule.*

In Section 19.4 we defined the differential dw of a function $w = f(x, y, z)$ by the formula

$$dw = \frac{\partial w}{\partial x}\,dx + \frac{\partial w}{\partial y}\,dy + \frac{\partial w}{\partial z}\,dz. \tag{8}$$

The chain rule (5) tells us that if x, y, z are themselves functions of a single independent variable t, then it is permissible to calculate dw/dt by formally dividing (8) by dt. Similarly, if x, y, z are functions of the independent variables t, u and we want to calculate $\partial w/\partial t$, then the chain rule (6) tells us that we can find $\partial w/\partial t$ by dividing (8) by dt and writing roundback ∂'s in place of ordinary d's to show that there is another independent variable present which is being held fixed.

The individual terms on the right side of (8) are sometimes called the *partial differentials* of w with respect to x, y, z. From this point of view, the quantity dw defined by (8) deserves the name *total differential,* as we remarked in Section 19.4.

Example 2 A function of several variables is said to be *homogeneous of degree* n if multiplying each variable by t (where $t > 0$) has the same effect as multiplying the original function by t^n. Thus, $f(x, y)$ is homogeneous of degree n if

$$f(tx, ty) = t^n f(x, y). \tag{9}$$

For example, $f(x, y) = x^2 + 3xy$ is homogeneous of degree 2, because $f(tx, ty) = (tx)^2 + 3(tx)(ty) = t^2(x^2 + 3xy) = t^2 f(x, y)$. Similarly, $f(x, y) = (x + y)/(x - y)$ is homogeneous of degree 0, $f(x, y) = (xy - x^2e^{x/y})/y$ is homogeneous of degree 1, and $f(x, y, z) = \sqrt{x^3 - 3xy^2 + 2z^3}$ is homogeneous of degree $\frac{3}{2}$. Most functions, for instance $f(x, y) = y^2 + x \sin y$, are not homogeneous at all.

There is a theorem of Euler about homogeneous functions that has several important applications: *If $f(x, y)$ is homogeneous of degree n, then*

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x, y). \tag{10}$$

To prove this, we hold x and y fixed and differentiate both sides of (9) with respect to t. We can clarify this process by writing $u = tx$ and $v = ty$, so that (9) becomes

$$f(u, v) = t^n f(x, y).$$

Then by using the chain rule to differentiate with respect to t, we obtain

$$\frac{\partial f}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial t} = nt^{n-1}f(x, y)$$

or

$$x\frac{\partial f}{\partial u} + y\frac{\partial f}{\partial v} = nt^{n-1}f(x, y),$$

and putting $t = 1$ yields (10). Similarly, if $f(x_1, x_2, \ldots, x_m)$ is homogeneous of degree n, then the same argument shows that

$$x_1\frac{\partial f}{\partial x_1} + x_2\frac{\partial f}{\partial x_2} + \cdots + x_m\frac{\partial f}{\partial x_m} = nf(x_1, x_2, \ldots, x_m).$$

Euler's theorem has some interesting consequences for economics. As an example, suppose that $f(x, y)$ is the production (measured in dollars) of x units of capital and y units of labor. If the amounts of capital and labor are doubled, then it is reasonable to expect that the resulting production will also double, that is, that $f(2x, 2y) = 2f(x, y)$. More generally, we expect that

$$f(tx, ty) = tf(x, y),$$

so the production function is homogeneous of degree 1. [In economics, this property of $f(x, y)$ is called *constant returns to scale.*] Euler's theorem now says that

$$f(x, y) = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}. \tag{11}$$

The partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ are called the *marginal product of capital* and the *marginal product of labor,* respectively. In this language, (11) is a theorem of quantitative economics whose verbal statement is, "The total value of production equals the cost of capital plus the cost of labor if each is paid for at the rate of its marginal product." Under these circumstances there are no surplus earnings, and in the real world this is a very bad thing.*

Example 3 Many applications of the chain rule involve calculating the effect on some equation or expression when new variables are introduced. As an illustration of a method that will be useful for solving the wave equation in Section 19.9, we now solve the partial differential equation

$$a\frac{\partial w}{\partial x} = \frac{\partial w}{\partial y}, \qquad a \neq 0. \tag{12}$$

That is, we find the most general function $w = f(x, y)$ that satisfies this equation. To do this, we introduce new independent variables u, v by writing

$$u = x + ay, \qquad v = x - ay. \tag{13}$$

We think of w as a function of u and v,

$$w = F(u, v),$$

and we find the uv-equation equivalent to (12) by using the chain rule to write

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial w}{\partial u} + \frac{\partial w}{\partial v},$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y} = a\frac{\partial w}{\partial u} - a\frac{\partial w}{\partial v}.$$

By substituting in these expressions, we see that (12) transforms into the partial differential equation

$$2a\frac{\partial w}{\partial v} = 0 \qquad \text{or} \qquad \frac{\partial w}{\partial v} = 0.$$

*For further information on these matters, see pp. 81–84 of J. M. Henderson and R. E Quandt, *Microeconomic Theory* (McGraw-Hill, 1971); or Chapter 12, "Homogeneous Functions and Euler's Theorem," in D. E. James and C. D. Throsby, *Quantitative Methods in Economics* (Wiley, 1973). For an application of Euler's theorem to advanced theoretical mechanics, see p. 531 of the present writer's text, *Differential Equations,* 2nd ed. (McGraw-Hill, 1991).

This equation is very easy to solve, because it says that the function $w = F(u, v)$ is constant when u is held fixed and v is allowed to vary, and therefore is a function of u alone. This means that our desired solution of (12) is

$$w = g(u) = g(x + ay),$$

where $g(u)$ is a *completely arbitrary* (continuously differentiable) function of u. We apologize to students for introducing "out of the blue" the apparently unmotivated transformation equations (13). However, some of the developments of Section 19.9 will make this procedure seem fairly natural.

Example 4 Partial derivatives are the main mathematical tools used in thermodynamics. It is the universal practice in this science to avoid confusion by using subscripts on partial derivatives to specify the variable (or variables) held fixed in the differentiation. Thus, if $w = F(x, y)$ then $\partial w/\partial x$ would be denoted by

$$\left(\frac{\partial w}{\partial x}\right)_y.$$

This notation tells us that w is being thought of as a function of x and y, and that y is held fixed and x is the variable of differentiation. This usage may seem superfluous, but the following situation—which is quite common in thermodynamics—shows that it is not.

If $w = f(x, y)$ where y is a function $g(x, t)$ of x and another variable t, so that w is a composite function of x and t, we find its partial derivative with respect to x.

This is a typical chain rule situation with x and y the intermediate variables and x and t the independent variables:

$$w = f(x, y) \qquad \text{where} \quad \begin{cases} x = x, \\ y = g(x, t). \end{cases}$$

The chain rule therefore gives

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial x}, \tag{14}$$

so

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial x}. \tag{15}$$

Unfortunately this equation contains two partial derivatives of w with respect to x. By an effort of thought one can keep in mind that $\partial w/\partial x$ on the left of (15) is the derivative of the composite function, while $\partial w/\partial x$ on the right is the derivative of $w = f(x, y)$. Nevertheless, this ambiguous notation invites confusion and is contrary to the overall spirit of mathematical symbols, which are intended to make it easy to be correct with a minimum of thought. However, if we use the subscript notation of thermodynamics, then (14) can be written as

$$\left(\frac{\partial w}{\partial x}\right)_t = \left(\frac{\partial w}{\partial x}\right)_y\left(\frac{\partial x}{\partial x}\right)_t + \left(\frac{\partial w}{\partial y}\right)_x\left(\frac{\partial y}{\partial x}\right)_t.$$

Since $(\partial x/\partial x)_t = 1$, this becomes

$$\left(\frac{\partial w}{\partial x}\right)_t = \left(\frac{\partial w}{\partial x}\right)_y + \left(\frac{\partial w}{\partial y}\right)_x\left(\frac{\partial y}{\partial x}\right)_t, \tag{16}$$

which is somewhat clumsy but much less vulnerable to misunderstanding than (15).*

*Students whose main interest is physics may wish to read discussions of these matters in some of the standard treatises. See, for example, p. 19 of Enrico Fermi, *Thermodynamics* (Dover, 1956); p. 28 of Philip M. Morse, *Thermal Physics* (W. A. Benjamin, 1969); or pp. 30–33, 52–55 of F. W. Sears, *Thermodynamics* (Addison-Wesley, 1953).

PROBLEMS

In Problems 1–4, find dw/dt in two ways, (a) by using the chain rule and then expressing everything in terms of t, and (b) by first substituting and then differentiating.

1 $w = e^{x^2+y^2}$, $x = \cos t$, $y = \sin t$.

2 $w = xy + yz + zx$, $x = 3t^2$, $y = e^t$, $z = e^{-t}$.

3 $w = \dfrac{3xy}{x^2 - y^2}$, $x = t^2$, $y = 3t$.

4 $w = \ln(x^4 + 2x^2y + 3y^2)$, $x = t$, $y = 2t^2$.

In Problems 5 and 6, find $\partial w/\partial t$ and $\partial w/\partial u$ by the chain rule and check your answers by using a different method.

5 $w = x^2 + y^2$, $x = t^2 - u^2$, $y = 2tu$.

6 $w = \dfrac{x}{x^2 + y^2}$, $x = t\cos u$, $y = t\sin u$.

7 If f is any (continuously differentiable) function, show that $w = f(x^2 - y^2)$ is a solution of the partial differential equation

$$y\frac{\partial w}{\partial x} + x\frac{\partial w}{\partial y} = 0.$$

Hint: Write $w = f(u)$ where $u = x^2 - y^2$ and apply the chain rule with only one intermediate variable.

8 If a and b are constants and $w = f(ax + by)$, show that

$$b\frac{\partial w}{\partial x} = a\frac{\partial w}{\partial y}.$$

9 If $w = f(x^2 - y^2, y^2 - x^2)$, show that

$$y\frac{\partial w}{\partial x} + x\frac{\partial w}{\partial y} = 0.$$

10 If $w = f\left(\dfrac{y - x}{xy}, \dfrac{z - y}{yz}\right)$, show that

$$x^2\frac{\partial w}{\partial x} + y^2\frac{\partial w}{\partial y} + z^2\frac{\partial w}{\partial z} = 0.$$

11 The differential dw of a function $w = f(x, y, z)$ is defined by (8) only for the case in which x, y, z are independent variables. If x, y, z are not independent, but instead are functions of independent variables t, u, then dw must be defined by

$$dw = \frac{\partial w}{\partial t}dt + \frac{\partial w}{\partial u}du.$$

Show that the definition implies (8), so that (8) remains valid regardless of whether x, y, z are independent or not.

12 If u and v are both functions of x, y, z, show that
(a) $d(c) = 0$, c a constant;
(b) $d(cu) = c\,du$;
(c) $d(u + v) = du + dv$;
(d) $d(uv) = u\,dv + v\,du$;
(e) $d\left(\dfrac{u}{v}\right) = \dfrac{v\,du - u\,dv}{v^2}$;
(f) if $w = f(u)$, then $dw = f'(u)\,du$.

13 Verify Euler's theorem (10) for each of the following functions:
(a) $f(x, y) = xy^2 + x^2y - y^3$;
(b) $f(x, y) = e^{x/y}$;
(c) $f(x, y) = \sqrt{x^2 + y^2}$;
(d) $f(x, y) = \dfrac{\sqrt{x + y}}{x}$.

14 If $w = f(x, y)$ where $x = r\cos\theta$ and $y = r\sin\theta$, show that

$$\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = \left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2.$$

15 If α is a constant and $w = f(x, y)$ where $x = u\cos\alpha - v\sin\alpha$ and $y = u\sin\alpha + v\cos\alpha$, show that

$$\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = \left(\frac{\partial w}{\partial u}\right)^2 + \left(\frac{\partial w}{\partial v}\right)^2.$$

16 If $w = f(x, y)$ where $x = e^u\cos v$ and $y = e^u\sin v$, show that

$$\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = e^{-2u}\left[\left(\frac{\partial w}{\partial u}\right)^2 + \left(\frac{\partial w}{\partial v}\right)^2\right].$$

17 Obtain formula (16) by using differentials, as follows:
(a) Write

$$dw = \left(\frac{\partial w}{\partial x}\right)_y dx + \left(\frac{\partial w}{\partial y}\right)_x dy,$$

$$dw = \left(\frac{\partial w}{\partial x}\right)_t dx + \left(\frac{\partial w}{\partial t}\right)_x dt,$$

$$dy = \left(\frac{\partial y}{\partial x}\right)_t dx + \left(\frac{\partial y}{\partial t}\right)_x dt.$$

(b) Substitute dy from the third formula into the first, and compare the result with the second.

*18 Let the pressure, volume, and temperature of a given quantity of a certain gas be denoted (as usual) by p, V, T. These variables are not independent, but are connected by an equation of the general form

$$f(p, V, T) = 0,$$

which is called the *equation of state.* This equation determines any one of the variables as a function of the other two.
(a) By calculating dp and dV, and eliminating dV, show that

$$\left(\frac{\partial p}{\partial V}\right)_T = \frac{1}{(\partial V/\partial p)_T},$$

$$\left(\frac{\partial p}{\partial V}\right)_T \left(\frac{\partial V}{\partial T}\right)_p + \left(\frac{\partial p}{\partial T}\right)_V = 0,$$

$$\left(\frac{\partial p}{\partial V}\right)_T \left(\frac{\partial V}{\partial T}\right)_p \left(\frac{\partial T}{\partial p}\right)_V = -1.$$

(b) If the internal energy E of the quantity of gas under discussion is a function of V and T, then, since T is a function of p and V, E is indirectly a function of p and V. Show that

$$\left(\frac{\partial E}{\partial V}\right)_p = \left(\frac{\partial E}{\partial V}\right)_T + \left(\frac{\partial E}{\partial T}\right)_V \left(\frac{\partial T}{\partial V}\right)_p.$$

It is sometimes necessary to work with functions of the form

$$w = F(u, v, x) = \int_u^v f(x, y)\, dy,$$

where $u = u(x)$ and $v = v(x)$ are functions of x. The chain rule yields

$$\frac{dw}{dx} = \frac{\partial w}{\partial u}\frac{du}{dx} + \frac{\partial w}{\partial v}\frac{dv}{dx} + \frac{\partial w}{\partial x},$$

and by applying the Fundamental Theorem of Calculus and differentiating under the integral sign (Problem 32 in Section 19.2), we obtain

$$\frac{d}{dx}\int_u^v f(x, y)\, dy$$
$$= -f(x, u)\frac{du}{dx} + f(x, v)\frac{dv}{dx} + \int_u^v \left[\frac{\partial}{\partial x} f(x, y)\right] dy.$$

This is known as *Leibniz's formula.*

19 Verify Leibniz's formula in the following cases:
(a) $u = x$, $v = x^2$, $f(x, y) = x + y$;
(b) $u = x$, $v = x^2$, $f(x, y) = x^3y^2 + x^2y^3$;
(c) $u = x$, $v = x^2$, $f(x, y) = \ln y$.

19.7 MAXIMUM AND MINIMUM PROBLEMS

In the case of functions of a single variable, one of the main applications of derivatives is to the study of maxima and minima. In Chapter 4 we developed various tests involving first and second derivatives, and we used these tests for graphing functions and attacking a wide variety of geometric and physical problems. Maximum and minimum problems for functions of two or more variables can be much more complicated. We confine ourselves here to an introduction to such problems, including a two-variable version of the second derivative test (Remark 3, Section 4.2).

Suppose that a function $z = f(x, y)$ has a maximum value at a point $P_0 = (x_0, y_0)$ in the interior of its domain. This means that $f(x, y)$ is defined, and also $f(x, y) \le f(x_0, y_0)$, throughout some neighborhood of P_0, as shown on the left in Fig. 19.13.* If we hold y fixed at the value y_0, then $z = f(x, y_0)$ is a function of x alone, and since it has a maximum value at $x = x_0$, its derivative must be zero there, as in Chapter 4. That is, $\partial z/\partial x = 0$ at this point. In just the same way, $\partial z/\partial y = 0$ at this point. The equations

$$\frac{\partial z}{\partial x} = 0 \quad \text{and} \quad \frac{\partial z}{\partial y} = 0 \tag{1}$$

*In this discussion we are considering only a so-called *relative* (or *local*) maximum, which takes into account only points near to P_0, but for simplicity we drop the adjective.

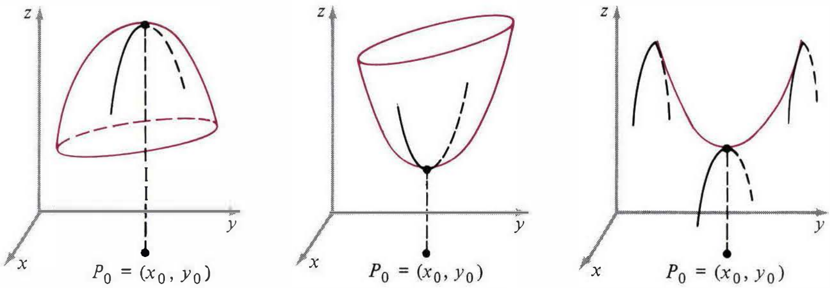


Figure 19.13

are therefore two equations in two unknowns that are satisfied at the maximum point (x_0, y_0). In many cases we can solve these equations simultaneously to find the point (x_0, y_0), and thus the actual maximum value of the function.

Exactly the same considerations apply to the minimum value shown in the center of the figure. However, when we try to locate maximum or minimum values of a function by solving equations (1), it is necessary to keep in mind that these equations are also satisfied at a saddle point like that shown on the right, where the function has a maximum in one direction and a minimum in some other direction. Equations (1) mean only that the tangent plane is horizontal, and it is then up to us to decide what significance this fact has.

By analogy with our earlier definition in Chapter 4 for functions of one variable, we call a point (x_0, y_0) where both partial derivatives are zero a *critical point* of $f(x, y)$.

Example 1 Find the dimensions of the rectangular box with open top and a fixed volume of 4 ft^3 which has the smallest possible surface area.

Solution If x and y are the edges of the base, and z is the height, then the area is

$$A = xy + 2xz + 2yz.$$

Since $xyz = 4$, we have $z = 4/xy$, and the area to be minimized can be expressed as a function of the two variables x and y,

$$A = xy + \frac{8}{y} + \frac{8}{x}. \tag{2}$$

We seek a critical point of this function, that is, a point where

$$\frac{\partial A}{\partial x} = y - \frac{8}{x^2} = 0, \qquad \frac{\partial A}{\partial y} = x - \frac{8}{y^2} = 0.$$

To solve these equations simultaneously, we first write them as

$$x^2y = 8, \qquad xy^2 = 8.$$

Dividing gives $x/y = 1$, so $y = x$ and either equation becomes $x^3 = 8$. Therefore $x = y = 2$, and it follows from this that $z = 1$, so the box with minimum surface area has a square base and a height one-half the edge of the base.

In this example it is geometrically clear that the critical point (2, 2) is actually a minimum point, and not a maximum or saddle point. However, in a more complicated situation we might find a critical point and yet be completely unable to state its nature, based on commonsense considerations alone. A useful tool for classifying critical points is provided by the *second derivative test*:

If $f(x, y)$ has continuous second partial derivatives in a neighborhood of a critical point (x_0, y_0), and if a number D (called the discriminant) is defined by

$$D = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - [f_{xy}(x_0, y_0)]^2, \tag{3}$$

then (x_0, y_0) is

(i) *a maximum point if $D > 0$ and $f_{xx}(x_0, y_0) < 0$;*
(ii) *a minimum point if $D > 0$ and $f_{xx}(x_0, y_0) > 0$;*
(iii) *a saddle point if $D < 0$.*

Further, if $D = 0$, then no conclusion can be drawn, and any of the behaviors described in (i) *to* (iii) *can occur.*

A complete proof of this theorem requires machinery that is not available to us. We refer interested students to more advanced books.*

Example 1 (continued) As an illustration of the use of the second derivative test, we apply it to verify that the critical point (2, 2) found in Example 1 is a minimum point of the function (2). Here we have

$$A_{xx} = \frac{16}{x^3}, \qquad A_{yy} = \frac{16}{y^3}, \qquad A_{xy} = 1,$$

so the discriminant (3) has the value $D = 2 \cdot 2 - 1^2 = 3 > 0$. Since A_{xx} is also positive at the point (2, 2), the test tells us that this critical point is indeed a minimum point, as claimed.

Example 2 Find the critical points of the function

$$z = 3x^2 + 2xy + y^2 + 10x + 2y + 1,$$

and use the second derivative test to classify them.

Solution Here we have

$$\frac{\partial z}{\partial x} = 6x + 2y + 10 = 0 \qquad \text{and} \qquad \frac{\partial z}{\partial y} = 2x + 2y + 2 = 0,$$

so the system of equations we must solve is

$$3x + y = -5,$$
$$x + y = -1.$$

By simple manipulations we easily see that $x = -2$ and $y = 1$, so there is a single critical point $(-2, 1)$. At this point we have

$$D = z_{xx}z_{yy} - z_{xy}{}^2 = 6 \cdot 2 - 2^2 = 8 > 0,$$

and since $z_{xx} = 6 > 0$, the critical point is a minimum point.

*See, for example, pp. 157–159 of R. C. Buck, *Advanced Calculus* (McGraw-Hill, 1978).

Success in finding maximum and minimum points for a function $z = f(x, y)$ clearly depends on our ability to solve the two simultaneous equations $f_x = 0$ and $f_y = 0$. In Examples 1 and 2 these equations were very easy to solve. However, as students can readily imagine, there are many complicated situations that arise in which routine methods of solving simultaneous equations are quite useless. The only general advice we can give is to try to solve one of the equations for one of the unknowns in terms of the other, substitute this in the second equation, and try to solve the result. Apart from this, make good guesses and be ingenious —advice that is easier to give than to follow!

PROBLEMS

In Problems 1–8, find the critical points and classify them by means of the second derivative test.

1 $z = 5x^2 - 3xy + y^2 - 15x - y + 2$.

2 $z = 2x^2 + xy + 3y^2 + 10x - 9y + 11$.

3 $z = x^5 + y^4 - 5x - 32y - 3$.

4 $z = x^2 + y^3 - 6xy$.

5 $z = x^2y + 3xy - 3x^2 - 4x + 2y$.

6 $z = 3xy^2 + y^2 - 3x - 6y + 7$.

7 $z = x^3 + y^3 + 3xy + 5$.

8 $z = xy(2x + 4y + 1)$.

9 For each of the following functions $z = f(x, y)$, show that f_x, f_y, and D are all 0 at the origin. Also show that at the origin (a) has a minimum, (b) has a maximum, and (c) has a saddle point.
(a) $f(x, y) = x^4 + y^4$.
(b) $f(x, y) = -x^4 - y^4$.
(c) $f(x, y) = x^3y^3$.

10 Show that a rectangular box with a top and fixed volume has the smallest surface area if it is a cube.

11 Show that a rectangular box with a top and fixed surface area has the largest volume if it is a cube.

12 If the equations $f(x) = 0$ and $f'(x) = 0$ have no common roots, show that any critical points of $z = yf(x) + g(x)$ must be saddle points.

13 If the sum of three numbers x, y, and z is 12, what must these numbers be for the product of x, y^2, and z^3 to be as large as possible?

*14 The function $z = (y - x^2)(y - 2x^2)$ has a saddle point at (0, 0).
(a) Verify that (0, 0) is the only critical point.
(b) Show that the second derivative test fails to establish that this critical point is a saddle point.
(c) Show that this critical point is a saddle point by direct examination of the sign of the function near (0, 0).
(d) It seems reasonable to suppose that a critical point $P_0 = (x_0, y_0)$ of $z = f(x, y)$ will necessarily be a minimum point for this surface if every vertical section of the surface through P_0 has P_0 as a minimum point. Show that this idea is false by examining the vertical section of $z = (y - x^2)(y - 2x^2)$ in the plane $y = mx$.

15 A rectangular box has three faces in the coordinate planes and one vertex $P = (x, y, z)$ in the first octant on the plane $ax + by + cz = 1$. Find the volume of the largest such box.

16 Solve Problem 15 if P lies on the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. Hint: Use implicit differentiation.

17 Solve Problem 15 if P lies on the paraboloid $z = 1 - x^2 - y^2$.

18 If $z = f(s, t)$ is the square of the distance between a variable point on the line

$$x = -2 + 4s, \qquad y = 3 + s, \qquad z = -1 + 5s$$

and a variable point on the line

$$x = -1 - 2t, \qquad y = 3t, \qquad z = 3 + t,$$

show that this function has one critical point which is a minimum. In this way find the distance between the lines.

19 Find the distance from the origin to the plane $x + 2y + 3z = 14$. Hint: Minimize $w = x^2 + y^2 + z^2$ by treating y and z as the independent variables.

20 The sides of an open rectangular box cost twice as much per square foot as the base. Find the relative dimensions of the largest box that can be made for a given cost.

21 Find the equation of the plane through (2, 2, 1) that cuts off the smallest volume from the first octant.

22 If (x_1, y_1), (x_2, y_2), and (x_3, y_3) are the vertices of a triangle, find the point (x, y) such that the sum of the squares of its distances from the vertices is as small as possible.

23 If α, β, γ are the angles of a triangle, find the maximum value of $\sin\alpha + \sin\beta + \sin\gamma$.

*24 Among all triangles with given fixed perimeter, show that the equilateral triangle has the largest area. Hint: Let the perimeter be $2s$, use Heron's formula $A = \sqrt{s(s-a)(s-b)(s-c)}$, and maximize $\ln A$. (Can you solve this problem without calculation, by merely thinking about it?)

*25 Among all triangles inscribed in a given circle, show that the equilateral triangle has the greatest area. Hint:

If α, β, γ are the central angles subtending the sides of the triangle, so that $\alpha + \beta + \gamma = 2\pi$, observe that the area of the triangle is a constant multiple of $\sin\alpha + \sin\beta - \sin(\alpha + \beta)$.

26 When an electric current of magnitude I flows through a wire of resistance R, the heat generated is proportional to I^2R. Two terminals are connected by three wires of resistances R_1, R_2, R_3. A given current flowing between the terminals will divide in such a way as to minimize the total heat produced. Show that the currents I_1, I_2, I_3 in the three wires will satisfy the equations $I_1R_1 = I_2R_2 = I_3R_3$.

***27** A pentagon consists of an isosceles triangle on top of a rectangle. If the perimeter P is fixed, find the dimensions of the rectangle and the height of the triangle that yield the maximum area.

***28** Show that the surface $z = (2x^2 + y^2)e^{1-x^2-y^2}$ looks like two mountain peaks joined by two ridges with a hollow depression between them.

29 A laboratory scientist performs an experiment n times and obtains n pairs of data,

$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n).$$

The theory underlying her experiment suggests that these points should lie on a straight line $y = mx + b$, but they do not because of experimental error. She then determines the line that gives the "best fit" for the data in the sense of the *method of least squares* (see Problem 30 in Section 4.4): She chooses m and b to minimize the sum of the squares of the vertical deviations (Fig. 19.14),

$$S = S(m, b) = \sum_{i=1}^{n} (mx_i + b - y_i)^2.$$

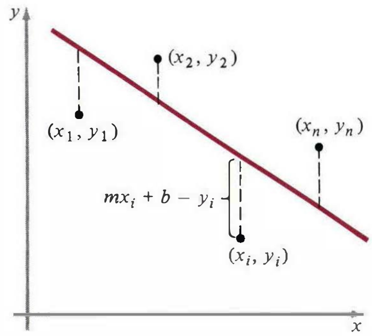


Figure 19.14

Show that m and b are determined as the simultaneous solution of the equations

$$m\sum x_i^2 + b\sum x_i = \sum x_iy_i,$$
$$m\sum x_i + nb = \sum y_i.$$

30 Use the method of least squares explained in Problem 29 to find the line that best fits the data (1, 1.7), (2, 1.8), (3, 2.3), (4, 3.2).

***31** Use the second derivative test to verify that S in Problem 29 is actually minimized by the stated values of m and b. Hint: It is necessary to use the fact that $(\Sigma x_i)^2 < n\Sigma x_i^2$ unless the x_i's are all equal; as a start toward establishing this, show that the maximum value of $f(x, y, z) = x + y + z$ on the sphere $x^2 + y^2 + z^2 = a^2$ is $\sqrt{3}a$, and conclude that $(x + y + z)^2 \le 3(x^2 + y^2 + z^2)$ for any three numbers x, y, z.

19.8 CONSTRAINED MAXIMA AND MINIMA. LAGRANGE MULTIPLIERS

In this section we explain the method of Lagrange multipliers by means of intuitive ideas that depend on the geometric meaning of gradients. This method is used for maximizing or minimizing functions of several variables subject to one or more constraints. It is an important tool in economics, differential geometry, and advanced theoretical mechanics.

We begin with the simplest case, that of two variables and one constraint.

In Section 19.7 we learned how to calculate maximum and minimum values of a function $z = f(x, y)$ of two independent variables x and y. However, in many problems x and y are not independent, but instead are connected by a *side condition* or *constraint* in the form of an equation

$$g(x, y) = 0. \tag{1}$$

In Chapter 4 we became thoroughly familiar with situations of this kind. The following is a simple illustration.

Example 1 Find the dimensions of the rectangle of maximum area that can be inscribed in a semicircle of radius a.

Solution It is clear from Fig. 19.15 that the problem is to maximize the function

$$A = 2xy \tag{2}$$

subject to the constraint

$$x^2 + y^2 = a^2. \tag{3}$$

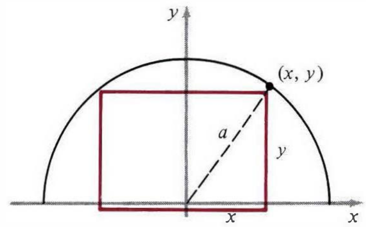


Figure 19.15

In Example 3 of Section 4.3 we solved this problem by using the constraint (3) to express A as a function of only one variable,

$$y = \sqrt{a^2 - x^2}, \qquad \text{so that} \qquad A = 2x\sqrt{a^2 - x^2}.$$

We then calculated dA/dx, set it equal to zero, solved the resulting equation, and so on. This example will be continued after some remarks and explanations.

The procedure we have just described works well enough for this problem, but as a general method it has two defects. First, in this particular case equation (3) is easy to solve for y, but in another problem the constraint (1) might be so complicated that it would be difficult or impossible to solve. The other defect lies in the fact that even though the variables x and y play equal roles in the problem, they are handled differently in the solution: We singled out one variable, x, to be the independent variable, and the other, y, to be the dependent variable. It is often more convenient, and certainly more elegant, to treat such problems in a symmetric form, in which no preference is given to any one of the variables over the others.*

We now return to the general problem of maximizing a function $f(x, y)$ subject to the constraint $g(x, y) = 0$. To understand what is going on, we sketch the graph of $g(x, y) = 0$ (Fig. 19.16) together with several level curves $f(x, y) = c$ of the function $f(x, y)$, noting the direction in which c increases. In the figure, for instance, we suppose that $c_1 < c_2 < c_3 < c_4$. To find the maximum value of $f(x, y)$ along the curve $g(x, y) = 0$, we look for the largest c for which $f(x, y) = c$ intersects $g(x, y) = 0$. At such an intersection point (P_0 in the figure) the two curves have the same tangent line, so they also have the same normal line. But the vectors

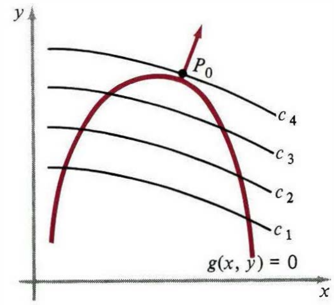


Figure 19.16

$$\text{grad} f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

and

$$\text{grad}\, g = \frac{\partial g}{\partial x}\mathbf{i} + \frac{\partial g}{\partial y}\mathbf{j}$$

are normal to these curves, and are therefore parallel to each other at the point P_0. Hence one vector is a scalar multiple of the other at P_0, that is,

$$\text{grad} f = \lambda\, \text{grad}\, g \tag{4}$$

for some number λ. (This argument assumes that grad $g \neq \mathbf{0}$ at P_0, so that the curve $g(x, y) = 0$ actually has a tangent at this point.)

*The great physicist Einstein once said—probably in a fit of impatience with mathematicians and their ways—that "Elegance is for tailors," but he was wrong. For mathematicians and theoretical physicists alike, the aesthetic factor in their thinking is as indispensable as the senses of taste and smell are for a master chef.

The vector equation (4), together with $g(x, y) = 0$, yields the three scalar equations

$$\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x}, \qquad \frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y}, \qquad g(x, y) = 0. \tag{5}$$

Accordingly, we have three equations that we can try to solve simultaneously for the three unknowns x, y, and λ. The points (x, y) that we find are the only possible locations for the maximum (or minimum) values of $f(x, y)$ with the constraint $g(x, y) = 0$. The corresponding values of λ may emerge from the process of solving (5), but they are usually not of much interest to us. The final step is to calculate the value of $f(x, y)$ at each of the solution points (x, y) in order to distinguish maximum values from minimum values.

The *method of Lagrange multipliers* is simply the following handy device for obtaining equations (5): Define a function $L(x, y, \lambda)$ of the three variables x, y, and λ by

$$L(x, y, \lambda) = f(x, y) - \lambda g(x, y), \tag{6}$$

and observe that equations (5) are equivalent, in the same order, to the equations

$$\frac{\partial L}{\partial x} = 0, \qquad \frac{\partial L}{\partial y} = 0, \qquad \frac{\partial L}{\partial \lambda} = 0. \tag{7}$$

The variable λ is called the *Lagrange multiplier*. Thus, to find the *constrained* maximum or minimum values of $f(x, y)$ with the constraint $g(x, y) = 0$, we look for the *unconstrained* (or *free*) maximum or minimum values of the function L defined by (6). We emphasize that this method has two major features that can be of practical value, and are often important for theoretical work: It does not disturb the symmetry of the problem by making an arbitrary choice of the independent variable, and it removes the constraint at the small expense of introducing λ as another variable.

Example 1 (continued) To solve the inscribed rectangle problem by this new method, we first express the constraint (3) in the form $x^2 + y^2 - a^2 = 0$ and then write down the function

$$L = 2xy - \lambda(x^2 + y^2 - a^2).$$

The equations (7) are

$$\frac{\partial L}{\partial x} = 2y - 2\lambda x = 0, \tag{8}$$

$$\frac{\partial L}{\partial y} = 2x - 2\lambda y = 0, \tag{9}$$

$$\frac{\partial L}{\partial \lambda} = -(x^2 + y^2 - a^2) = 0. \tag{10}$$

Equations (8) and (9) yield $y = \lambda x$ and $x = \lambda y$, and substituting in (10) gives

$$\lambda^2(x^2 + y^2) = a^2.$$

But (10) tells us that $x^2 + y^2 = a^2$, so $\lambda^2 = 1$ and $\lambda = \pm 1$. The value $\lambda = -1$ would imply that $y = -x$, which is impossible because both x and y are positive numbers, so $\lambda = 1$ and $y = x$. This gives the shape of the largest inscribed rectangle, namely, twice as long as it is wide, because

$$\text{length} = 2x = 2y = 2(\text{width}).$$

If we want the actual dimensions of this largest rectangle, we substitute $y = x$ into $x^2 + y^2 = a^2$ to find that $x = y = \frac{1}{2}\sqrt{2}a$, so the length $= 2x = \sqrt{2}a$ and the width $= y = \frac{1}{2}\sqrt{2}a$.

One of the merits of the method of Lagrange multipliers is that it extends very easily to situations with more variables or more constraints. For instance, to maximize $f(x, y, z)$ subject to the constraint $g(x, y, z) = 0$, the gradient

$$\text{grad}\, f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

must be normal to the surface $g(x, y, z) = 0$ (Fig. 19.17), so grad f must be parallel to grad g, and again we have

$$\text{grad}\, f = \lambda\, \text{grad}\, g.$$

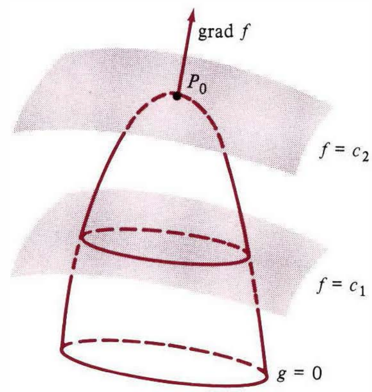


Figure 19.17

The four equations

$$\frac{\partial f}{\partial x} = \lambda\frac{\partial g}{\partial x}, \qquad \frac{\partial f}{\partial y} = \lambda\frac{\partial g}{\partial y}, \qquad \frac{\partial f}{\partial z} = \lambda\frac{\partial g}{\partial z}, \qquad g(x, y, z) = 0,$$

in the four unknowns x, y, z, λ are again equivalent to the simpler equations

$$\frac{\partial L}{\partial x} = 0, \qquad \frac{\partial L}{\partial y} = 0, \qquad \frac{\partial L}{\partial z} = 0, \qquad \frac{\partial L}{\partial \lambda} = 0,$$

where $L = f(x, y, z) - \lambda g(x, y, z)$.

Similarly, suppose we want to maximize or minimize $f(x, y, z)$ subject to *two* constraints $g(x, y, z) = 0$ and $h(x, y, z) = 0$. Each constraint defines a surface, and in general these two surfaces have a curve of intersection. As before, a point P_0 where $f(x, y, z)$ has a maximum or minimum value on this curve is a point where a level surface of f is tangent to the curve, that is, a point where grad f is normal to the curve (Fig. 19.18). But the vectors grad g and grad h determine the normal plane to the curve at P_0, and since grad f lies in this plane, there must be scalars λ and μ (*two* Lagrange multipliers this time) with the property that

$$\text{grad}\, f = \lambda\, \text{grad}\, g + \mu\, \text{grad}\, h.$$

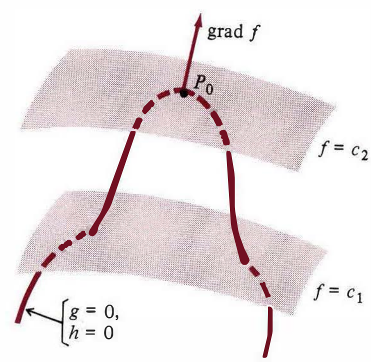


Figure 19.18

(This argument assumes that grad $g \neq \mathbf{0}$ and grad $h \neq \mathbf{0}$, and that these vectors are not parallel.) Just as before, this vector equation and the two constraint equations are easily seen to be equivalent to the following five equations in five unknowns:

$$\frac{\partial L}{\partial x} = 0, \qquad \frac{\partial L}{\partial y} = 0, \qquad \frac{\partial L}{\partial z} = 0, \qquad \frac{\partial L}{\partial \lambda} = 0, \qquad \frac{\partial L}{\partial \mu} = 0,$$

where $L = f(x, y, z) - \lambda g(x, y, z) - \mu h(x, y, z)$.

We illustrate these methods with two examples.

Example 2 Find the point on the plane $x + 2y + 3z = 6$ that is closest to the origin.

Solution We want to minimize the distance $\sqrt{x^2 + y^2 + z^2}$ subject to the constraint $x + 2y + 3z - 6 = 0$. If the distance is a minimum, its square is a minimum, so we simplify the calculations a bit by minimizing $x^2 + y^2 + z^2$ with the same constraint. Let

$$L = x^2 + y^2 + z^2 - \lambda(x + 2y + 3z - 6).$$

Then the equations we must solve are

$$\frac{\partial L}{\partial x} = 2x - \lambda = 0,$$

$$\frac{\partial L}{\partial y} = 2y - 2\lambda = 0,$$

$$\frac{\partial L}{\partial z} = 2z - 3\lambda = 0,$$

$$\frac{\partial L}{\partial \lambda} = -(x + 2y + 3z - 6) = 0.$$

Substituting the values for x, y, and z from the first three equations into the fourth gives

$$\tfrac{1}{2}\lambda + 2\lambda + \tfrac{9}{2}\lambda = 6 \qquad \text{or} \qquad \tfrac{14}{2}\lambda = 6 \qquad \text{or} \qquad \lambda = \tfrac{6}{7}.$$

It now follows that $x = \frac{3}{7}$, $y = \frac{6}{7}$, and $z = \frac{9}{7}$, so the desired point is $(\frac{3}{7}, \frac{6}{7}, \frac{9}{7})$.

Example 3 Find the point on the line of intersection of the planes $x + y + z = 1$ and $3x + 2y + z = 6$ that is closest to the origin.

This time we want to minimize $x^2 + y^2 + z^2$ subject to the two constraints $x + y + z - 1 = 0$ and $3x + 2y + z - 6 = 0$. If we write

$$L = x^2 + y^2 + z^2 - \lambda(x + y + z - 1) - \mu(3x + 2y + z - 6),$$

then our equations are

$$\frac{\partial L}{\partial x} = 2x - \lambda - 3\mu = 0,$$

$$\frac{\partial L}{\partial y} = 2y - \lambda - 2\mu = 0,$$

$$\frac{\partial L}{\partial z} = 2z - \lambda - \mu = 0,$$

$$\frac{\partial L}{\partial \lambda} = -(x + y + z - 1) = 0,$$

$$\frac{\partial L}{\partial \mu} = -(3x + 2y + z - 6) = 0.$$

The first three equations give

$$x = \tfrac{1}{2}(\lambda + 3\mu), \qquad y = \tfrac{1}{2}(\lambda + 2\mu), \qquad z = \tfrac{1}{2}(\lambda + \mu).$$

When these expressions are substituted in the fourth and fifth equations and the results are simplified, we get

$$3\lambda + 6\mu = 2,$$

$$3\lambda + 7\mu = 6,$$

so $\mu = 4$ and $\lambda = -\frac{22}{3}$. These values give $x = \frac{7}{3}$, $y = \frac{1}{3}$, $z = -\frac{5}{3}$, so the desired point is $(\frac{7}{3}, \frac{1}{3}, -\frac{5}{3})$.

Remark 1 In economics, Lagrange multipliers are used to analyze the problem of maximizing the total production of a manufacturing firm subject to the constraint of fixed available resources. For example, let

$$P = f(x, y) = Ax^{\alpha}y^{\beta}, \qquad \alpha + \beta = 1,$$

be the production (measured in dollars) resulting from x units of capital and y units of labor. Then this function—known to economists as the *Cobb-Douglas production function*—is homogeneous of degree 1 in the sense explained in Example 2 of Section 19.6. If the cost of each unit of capital is a dollars, and of each unit of labor is b dollars, and if a total of c dollars is available to cover the combined costs of capital and labor, then we want to maximize the production $P = f(x, y)$ subject to the constraint $ax + by = c$. In Problem 23 we ask students to show that production is maximized when $x = \alpha c/a$ and $y = \beta c/b$.*

Remark 2 At the beginning of this section we said that Lagrange multipliers also have applications to differential geometry and advanced theoretical mechanics. These applications are too complicated to describe here, but the details can be found on pp. 521–523 and 529 of the present writer's text, *Differential Equations,* 2nd ed. (McGraw-Hill, 1991).

*For further details, interested students can look up Cobb-Douglas production functions in the indexes of the books mentioned in the first footnote of Section 19.6.

PROBLEMS

Solve all of the following problems by Lagrange multipliers.

1 A rectangle with sides parallel to the axes is inscribed in the region bounded by the axes and the line $x + 2y = 2$. Find the maximum area of this rectangle.

2 Find the rectangle of maximum perimeter (with sides parallel to the axes) that can be inscribed in the ellipse $x^2 + 4y^2 = 4$.

3 Find the rectangle of maximum area (with sides parallel to the axes) that can be inscribed in the ellipse $x^2 + 4y^2 = 4$.

*4 On each of the following curves, find the points that are closest to the origin and those that are farthest from the origin:
(a) $x^2 + xy + y^2 = 3$; (b) $x^4 + 3xy + y^4 = 2$.

5 If a cylinder has fixed volume V_0, find the relation between the height h and the radius of the base r that minimizes the surface area.

6 Find the maximum and minimum values of $f(x, y) = 2x^2 + y + y^2$ on the circle $x^2 + y^2 = 1$.

7 Find the maximum and minimum values of $f(x, y) = x^2 - xy + y^2$ on the circle $x^2 + y^2 = 1$.

8 Find the ellipse $x^2/a^2 + y^2/b^2 = 1$ that passes through the point (4, 1) and has the smallest area. Hint: The area of this ellipse is πab.

*9 Find the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ that passes through the point (1, 2, 3) and has the smallest volume. Hint: The volume of this ellipsoid is $\frac{4}{3}\pi abc$.

10 Find the maximum value of $f(x, y, z) = 2x + 2y - z$ on the sphere $x^2 + y^2 + z^2 = 4$.

11 Find the minimum value of $f(x, y, z) = x^2 + 2y^2 + 3z^2$ on the plane $x - y - z = 1$.

12 Find the maximum value of $f(x, y, z) = x + y + z$ on the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

13 Find the maximum volume of a rectangular box if the sum of the lengths of its edges is $12a$.

14 Find the maximum volume of a rectangular box if the sum of the areas of its faces is $6a^2$.

15 (a) Show that of all triangles inscribed in a given circle, the equilateral triangle has the maximum perimeter. Hint: If a is the radius of the circle and α, β, γ are the central angles subtending the three sides, what is the perimeter?

(b) In part (a), show that the inscribed equilateral triangle also has the maximum area.

16 Find the point on the line of intersection of the planes $x + 2y + z = 1$ and $-3x - y + 2z = 4$ that is closest to the origin.

17 (a) Find the point on the plane $ax + by + cz + d = 0$ that is closest to the origin, and use this information to write down a formula for the distance from the origin to the plane.
(b) Adapt the method used in part (a) to show that the distance from an arbitrary point (x_0, y_0, z_0) to the given plane is

$$\frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

18 (a) Show that the triangle with the greatest area A for a given perimeter is equilateral. Hint: If x, y, z are the sides, then $A = \sqrt{s(s-x)(s-y)(s-z)}$, where $2s = x + y + z$.
(b) Show that the triangle with the smallest perimeter for a given area is equilateral.

19 If the sum of n positive numbers $x_1, x_2, \ldots, x_n$ has a fixed value s, show that their product $x_1x_2 \cdots x_n$ has s^n/n^n as its maximum value, and conclude from this that the geometric mean of n positive numbers can never exceed their arithmetic mean:

$$\sqrt[n]{x_1x_2\cdots x_n} \le \frac{x_1 + x_2 + \cdots + x_n}{n}.$$

***20** (a) Find the maximum value of

$$\sum_{i=1}^{n} x_i y_i$$

with the constraints

$$\sum_{i=1}^{n} x_i^2 = 1 \qquad \text{and} \qquad \sum_{i=1}^{n} y_i^2 = 1.$$

(b) Use part (a) to prove that for any numbers $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$,

$$\sum_{i=1}^{n} a_ib_i \le \left(\sum_{i=1}^{n} a_i^2\right)^{1/2}\left(\sum_{i=1}^{n} b_i^2\right)^{1/2}.$$

Hint: Put

$$x_i = \frac{a_i}{\left(\sum_{i=1}^{n} a_i^2\right)^{1/2}} \qquad \text{and} \qquad y_i = \frac{b_i}{\left(\sum_{i=1}^{n} b_i^2\right)^{1/2}}.$$

The inequality in (b) is an important fact in higher mathematics called the *Schwarz inequality.*

***21** Use the method of Problem 20 to establish *Hölder's inequality*: If $1/p + 1/q = 1$ and the a_i's and b_i's are nonnegative numbers, then

$$\sum_{i=1}^{n} a_ib_i \le \left(\sum_{i=1}^{n} a_i^p\right)^{1/p}\left(\sum_{i=1}^{n} b_i^q\right)^{1/q}.$$

Observe that when $p = q = 2$, Hölder's inequality reduces to the Schwarz inequality.

22 Refer back to Example 4 in Section 4.4, and the notation in Fig. 4.28, to obtain Snell's law of refraction by minimizing the total time of travel,

$$T = \frac{a}{v_a \cos\alpha} + \frac{b}{v_w \cos\beta},$$

subject to the constraint $a \tan\alpha + b \tan\beta =$ a constant.

23 Show that to maximize the Cobb-Douglas production function $P = f(x, y) = Ax^\alpha y^\beta$ $(\alpha + \beta = 1)$ subject to the constraint of fixed total costs, $ax + by = c$, we must put $x = \alpha c/a$ and $y = \beta c/b$.

19.9 (OPTIONAL) LAPLACE'S EQUATION, THE HEAT EQUATION, AND THE WAVE EQUATION. LAPLACE AND FOURIER

A very large part of mathematical physics is concerned with three classic partial differential equations: *Laplace's equation,*

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = 0; \tag{1}$$

the *heat equation,*

$$a^2\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) = \frac{\partial w}{\partial t}; \tag{2}$$

and the *wave equation,*

$$a^2\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) = \frac{\partial^2 w}{\partial t^2}. \tag{3}$$

As the notation indicates, in (2) and (3) the variable w is understood to be a function of the time t and the three space coordinates x, y, z of a point P, and in (1) w depends only on x, y, z and is independent of t. The quantity a is a constant. Each of our three equations also has simpler two- and one-dimensional versions, depending on whether two space coordinates are present, or only one. Thus,

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0 \tag{4}$$

is the two-dimensional Laplace equation, and

$$a^2 \frac{\partial^2 w}{\partial x^2} = \frac{\partial w}{\partial t} \qquad \text{and} \qquad a^2 \frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 w}{\partial t^2}$$

are the one-dimensional heat equation and wave equation.

A full study of these equations can occupy years, because their physical meaning is extraordinarily rich, and also much concentrated thought is necessary to master the various branches of advanced mathematics that are needed to solve and interpret them. In this section we consider several aspects of these equations that do not require too much technical background.

LAPLACE'S EQUATION

If a number of particles of masses $m_1, m_2, \ldots, m_n$, attracting according to the inverse square law of gravitation, are placed at points $P_1, P_2, \ldots, P_n$, then the *potential* due to these particles at any point P (that is, the work done against their attractive forces in moving a unit mass from P to an infinite distance) is

$$w = \frac{Gm_1}{PP_1} + \frac{Gm_2}{PP_2} + \cdots + \frac{Gm_n}{PP_n}, \tag{5}$$

where G is the gravitational constant.* If the points $P, P_1, P_2, \ldots, P_n$ have rectangular coordinates (x, y, z), (x_1, y_1, z_1), $(x_2, y_2, z_2), \ldots, (x_n, y_n, z_n)$, so that

$$PP_1 = \sqrt{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2},$$

with similar expressions for the other distances, then it is quite easy to verify that the potential w satisfies Laplace's equation (1). This equation does not involve either the particular masses or the coordinates of the points at which they are located, so it is satisfied by the potential produced in empty space by an arbitrary discrete or continuous distribution of mass.

The function w defined by (5) is called a *gravitational potential.* If we work instead with electrically charged particles of charges $q_1, q_2, \ldots, q_n$, then their *electrostatic potential* has the same form as (5) with the m's replaced by q's and G by Coulomb's constant, so it also satisfies Laplace's equation. In fact, this equation has such a wide variety of applications that its study is a branch of mathematics in its own right, known as *potential theory.*

THE HEAT EQUATION

When we study the flow of heat in thermally conducting bodies, we encounter an entirely different type of problem leading to a partial differential equation.

*See Example 2 in Section 7.7. In this example we show that if two particles of masses M and m are separated by a distance a, then the work done in separating them to an infinite distance is GMm/a.

In the interior of a body where heat is flowing from one region to another, the temperature generally varies from point to point at any one time, and from time to time at any one point. Thus, the temperature w is a function of the space coordinates x, y, z and the time t, say $w = f(x, y, z, t)$. The precise form of this function naturally depends on the shape of the body, the thermal characteristics of its material, the initial distribution of temperature, and the conditions maintained on the surface of the body. The French physicist-mathematician Fourier studied this problem in his classic treatise of 1822, *Théorie Analytique de la Chaleur (Analytic Theory of Heat).* He used physical principles to show that the temperature function w must satisfy the heat equation (2).* We shall retrace his reasoning in a simple one-dimensional situation, and thereby derive the one-dimensional heat equation.

The following are the physical principles that will be needed.

(a) Heat flows in the direction of decreasing temperature, that is, from hot regions to cold regions.

(b) The rate at which heat flows across an area is proportional to the area and to the rate of change of temperature with respect to distance in a direction perpendicular to the area. (This proportionality factor is denoted by k and called the *thermal conductivity* of the substance.)

(c) The quantity of heat gained or lost by a body when its temperature changes, that is, the change in its thermal energy, is proportional to the mass of the body and to the change of temperature. (This proportionality factor is denoted by c and called the *specific heat* of the substance.)

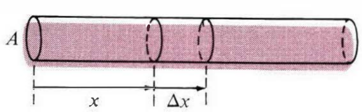


Figure 19.19

We now consider the flow of heat in a thin cylindrical rod of cross-sectional area A (Fig. 19.19) whose lateral surface is perfectly insulated so that no heat flows through it. This use of the word "thin" means that the temperature is assumed to be uniform on any cross section, and is therefore a function only of the time and the position of the cross section, say $w = f(x, t)$. We examine the rate of change of the heat contained in a thin slice of the rod between the positions x and $x + \Delta x$.

If ρ is the density of the rod, that is, its mass per unit volume, then the mass of the slice is

$$\Delta m = \rho A \, \Delta x.$$

Furthermore, if Δw is the temperature change at the point x in a small time interval Δt, then (c) tells us that the quantity of heat stored in the slice in this time interval is

$$\Delta H = c \, \Delta m \, \Delta w = c\rho A \, \Delta x \, \Delta w,$$

so the rate at which heat is being stored is approximately

$$\frac{\Delta H}{\Delta t} = c\rho A \, \Delta x \frac{\Delta w}{\Delta t}. \tag{6}$$

We assume that no heat is generated inside the slice — for instance, by chemical or electrical processes — so that the slice gains heat only by means of the flow

*The same partial differential equation also describes a more general class of diffusion processes, and is sometimes called the *diffusion equation.*

of heat through its faces. By (b) the rate at which heat flows into the slice through the left face is

$$-kA \left.\frac{\partial w}{\partial x}\right|_{x}.$$

The negative sign here is chosen in accordance with (a), so that this quantity will be positive if $\partial w/\partial x$ is negative. Similarly, the rate at which heat flows into the slice through the right face is

$$kA \left.\frac{\partial w}{\partial x}\right|_{x+\Delta x},$$

so the total rate at which heat flows into the slice is

$$kA \left.\frac{\partial w}{\partial x}\right|_{x+\Delta x} - kA \left.\frac{\partial w}{\partial x}\right|_{x}. \tag{7}$$

If we equate the expressions (6) and (7), the result is

$$kA \left.\frac{\partial w}{\partial x}\right|_{x+\Delta x} - kA \left.\frac{\partial w}{\partial x}\right|_{x} = c\rho\, A\, \Delta x \frac{\Delta w}{\Delta t},$$

or

$$\frac{k}{c\rho}\left[\frac{\partial w/\partial x|_{x+\Delta x} - \partial w/\partial x|_{x}}{\Delta x}\right] = \frac{\Delta w}{\Delta t}.$$

Finally, by letting Δx and $\Delta t \to 0$ we obtain the desired equation,

$$a^2 \frac{\partial^2 w}{\partial x^2} = \frac{\partial w}{\partial t},$$

where $a^2 = k/c\rho$. This is the physical reasoning that leads to the one-dimensional heat equation. The three-dimensional equation (2) can be derived in essentially the same way.

THE WAVE EQUATION

All phenomena of wave propagation, for example, of light or sound or radio waves, are governed by the wave equation (3). We shall consider the simple case of a one-dimensional wave described by the one-dimensional wave equation

$$a^2 \frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 w}{\partial t^2}. \tag{8}$$

Such a wave involves some property $w = f(x, t)$, such as the position of a particle, the intensity of an electric field, or the pressure in a column of air, that depends not only on the position x but also on the time t.

In order to understand the connection between waves and equation (8), we consider a function $w = F(x - at)$. At $t = 0$, it defines the curve $w = F(x)$, and at any later time $t = t_1$, it defines the curve $w = F(x - at_1)$. It is easy to see that these curves are identical except that the latter is translated to the right through a distance at_1, and therefore with velocity

$$v = \frac{at_1}{t_1} = a.$$

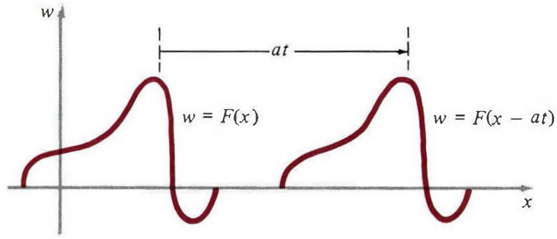


Figure 19.20 A traveling wave.

This shows that the function $w = F(x - at)$ represents a traveling wave that moves to the right with velocity a, as suggested in Fig. 19.20. If we assume that $w = F(u)$ has a second derivative, then by the chain rule applied to $w = F(u)$ where $u = x - at$, we have

$$\frac{\partial w}{\partial x} = F'(u), \qquad \frac{\partial w}{\partial t} = F'(u)\cdot(-a) = -aF'(u),$$

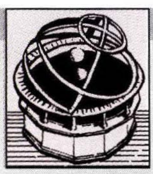
$$\frac{\partial^2 w}{\partial x^2} = F''(u), \qquad \frac{\partial^2 w}{\partial t^2} = -aF''(u)\cdot(-a) = a^2F''(u).$$

It is clear from this that $w = F(x - at)$ satisfies the one-dimensional wave equation (8).

Similarly, the function $w = G(x + at)$ represents a traveling wave that moves to the left with velocity a, and it is equally easy to show that this function is a solution of (8). By the linearity of differentiation, it follows that the sum

$$w = F(x - at) + G(x + at) \tag{9}$$

is also a solution. In fact, it can be shown (see Problem 8) that if F and G are *arbitrary* twice-differentiable functions, then (9) is the *general* solution of (8), in the sense that every solution of (8) has the form (9). It is fairly clear that the function (9) represents the most general one-dimensional wave, and this result confirms it.

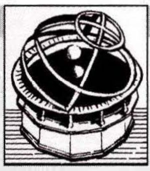


NOTE ON LAPLACE

Pierre Simon de Laplace (1749–1827) was a French mathematician and theoretical astronomer who was so famous in his own time that he was known as the Newton of France. His main scientific interests throughout his life were celestial mechanics and the theory of probability.

At the age of 24 he was already deeply engaged in the detailed application of Newton's law of gravitation to the solar system as a whole, in which the planets and their satellites are not governed by the sun alone but interact with one another in a bewildering variety of ways. Even Newton had been of the opinion that divine intervention would occasionally be needed to prevent this complex mechanism from degenerating into chaos. Laplace decided to seek reassurance elsewhere, and succeeded in proving that the ideal solar system of mathematics is a stable dynamical system that will endure unchanged for all time. This achievement was only one of the long series of triumphs recorded in his monumental treatise ***Mécanique Céleste*** (published in five volumes from 1799 to 1825), which summed up the work on gravitation of several generations of illustrious mathematicians. Many anecdotes are associated with this work. One of the best known describes the occasion on which Napoleon tried to get a rise out of Laplace by protesting that he had written a huge book on the system of the world without once mentioning God as the author of the universe. Laplace is supposed to have replied, "Sire, I had no need of that hypothesis." The principal legacy of the *Mécanique Céleste* to later generations lay in Laplace's wholesale development of potential theory, with its far-reaching implications for a dozen different branches of physical science ranging from gravitation and fluid mechanics to electromagnetism and atomic physics. Even though the concept of the potential is due to Lagrange, Laplace exploited it so extensively that ever since his time the fundamental differential equation of potential theory has been known as Laplace's equation.

His other masterpiece was the treatise *Théorie Analytique des Probabilités* (1812), in which he incorporated his own discoveries in probability from the preceding 40 years. This book is generally agreed to be the greatest contribution to this part of mathematics ever made by one man. In the introduction he says, "At bottom, the theory of probability is only common sense reduced to calculation." This may be so, but the following 700 pages of intricate analysis—in which he freely used Laplace transforms, generating functions, and many other highly nontrivial tools—has been said by some to surpass in complexity even the *Mécanique Céleste.*



NOTE ON FOURIER

Jean Baptiste Joseph Fourier (1768–1830), an excellent mathematical physicist, was a friend of Napoleon (so far as such people have friends) and accompanied his master to Egypt in 1798. On his return he became prefect (governor) of the district of Isère in southeastern France, and in this capacity built the first real road from Grenoble to Turin. He also befriended the boy Champollion, who later deciphered the Rosetta Stone as the first long step toward understanding the hieroglyphic writing of the ancient Egyptians.

During these years he worked on the theory of the conduction of heat, and in 1822 published his famous *Théorie Analytique de la Chaleur,* in which he made extensive use of the series that now bear his name. These series were of profound significance in connection with the evolution of the concept of a function. The general attitude at that time was to call $f(x)$ a function if it could be represented by a single expression like a polynomial, a finite combination of elementary functions, a power series $\sum_{n=0}^{\infty} a_n x^n$, or a trigonometric series of the form

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

If the graph of $f(x)$ were "arbitrary"—for example, a polygonal line with a number of corners and even a few gaps—then $f(x)$ would not have been accepted as a genuine function. Fourier claimed that "arbitrary" graphs can be represented by trigonometric series and should therefore be treated as legitimate functions, and it came as a shock to many that he turned out to be right. It was a long time before these issues were completely clarified, and it was no accident that the definition of a function that is now almost universally used was first formulated by Dirichlet in 1837 in a research paper on the theory of Fourier series. Also, the classical definition of the definite integral due to Riemann was first given in his fundamental paper of 1854 on the subject of Fourier series. Indeed, many of the most important mathematical discoveries of the nineteenth century are directly linked to the theory of Fourier series, and the applications of this subject to mathematical physics have been scarcely less profound.

Fourier himself is one of the fortunate few: his name has become rooted in all civilized languages as an adjective that is well known to physical scientists and mathematicians in every part of the world.

PROBLEMS

1 (a) Verify that the function

$$w = \frac{1}{\sqrt{(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2}}$$

satisfies Laplace's equation (1).

(b) A differential equation is called *linear* if the sum of two solutions is a solution, and any constant times a solution is a solution. Verify that Laplace's equation (1) is linear, and conclude from part (a) that the potential (5) is a solution.

2 (a) Determine whether or not the function

$$w = \frac{1}{\sqrt{(x-x_1)^2 + (y-y_1)^2}}$$

is a solution of Laplace's equation (4) in two dimensions.

(b) Show that the function $w = \ln[(x-x_1)^2 + (y-y_1)^2]$ is a solution of Laplace's equation (4) in two dimensions.

3 Verify that each of the following functions satisfies Laplace's equation (1):

(a) $w = x^2 + 2y^2 - 3z^2$;

(b) $w = x^2 - y^2 + 57z$;

(c) $w = 4z^3 - 6(x^2 + y^2)z$;

(d) $w = e^{-2x} \sin 2y + 3z$;

(e) $w = e^{3x}e^{4y} \cos 5z$;

(f) $w = e^{13x} \sin 12y \cos 5z$.

*4 If $w = f(x, y)$ is transformed into $w = F(r, \theta)$ by the equations $x = r\cos\theta$, $y = r\sin\theta$ (these are the transformation equations from rectangular to polar coordinates), show that the two-dimensional Laplace equation

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0$$

becomes

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} = 0.$$

5 Use Problem 4 to show that each of the functions $w_1 = r^n \sin n\theta$ and $w_2 = r^n \cos n\theta$ satisfies Laplace's equation in two dimensions.

6 Suppose that a solution $w = f(x, t)$ of the one-dimensional heat equation

$$a^2 \frac{\partial^2 w}{\partial x^2} = \frac{\partial w}{\partial t}$$

has the form $f(x, t) = g(x)h(t)$, that is, is the product of a function of x and a function of t.

(a) Show that $a^2 g''(x)h(t) = g(x)h'(t)$.

(b) Part (a) implies that

$$\frac{g''(x)}{g(x)} = \frac{h'(t)}{a^2 h(t)},$$

where the left side is a function of x alone and the right side is a function of t alone. Deduce that there is a constant λ such that $g''/g = \lambda$ and $h'/(a^2h) = \lambda$.

(c) Assume that the constant λ in part (b) is negative, and can therefore be written in the form $\lambda = -k^2$ for some positive number k. Show that $g(x) = c_1 \sin kx + c_2 \cos kx$ is a solution of the equation $g''/g = -k^2$ for every choice of the constants c_1 and c_2.

(d) Show that $h(t) = ce^{-a^2k^2t}$ is a solution of the equation $h'/(a^2h) = -k^2$ for every choice of the constant c. Thus all of the functions

$$w = f(x, t) = ce^{-a^2k^2t}(c_1 \sin kx + c_2 \cos kx)$$

are solutions of the heat equation, and any sum of such solutions is a solution.

7 A steady-state solution of the heat equation (2) is one that does not depend on t, and in this case the heat equation reduces to Laplace's equation (1). Solve the one-dimensional Laplace equation.

*8 Use the chain rule to show that under the change of variables specified by $u = x - at$, $v = x + at$, the equation

$$a^2 \frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 w}{\partial t^2} \quad \text{becomes} \quad \frac{\partial^2 w}{\partial u\, \partial v} = 0.$$

Hint: See Example 3 in Section 19.6. Use this result to show that $w = F(x - at) + G(x + at)$ is the most general solution of the one-dimensional wave equation.

19.10 (OPTIONAL) IMPLICIT FUNCTIONS

In Section 3.5 we stated that when we are given an equation

$$F(x, y) = 0, \tag{1}$$

there usually exists at least one function

$$y = f(x) \tag{2}$$

that "solves" (1), in the sense that (2) reduces (1) to an identity in x. With the idea in mind that y in (1) stands for this function of x, we then differentiated the identity (1) with respect to x and went on to solve the resulting equation for dy/dx, calling the process "implicit differentiation." For instance, if we have the equation

$$x^2y^5 - 2xy + 1 = 0, \tag{3}$$

then by differentiating with respect to x we obtain

$$x^2 \cdot 5y^4 \frac{dy}{dx} + 2xy^5 - 2x \frac{dy}{dx} - 2y = 0, \tag{4}$$

so

$$\frac{dy}{dx} = \frac{2y - 2xy^5}{5x^2y^4 - 2x}. \tag{5}$$

Most students feel slightly uncomfortable about implicit differentiation, and with good reason. For one thing, in this particular case we have no idea whether (3) actually defines y as a function of x or not; and if it doesn't, then the subsequent calculation leading to (5) has no meaning at all. Also, the procedure itself is a bit clumsy, because it requires us to keep in mind the different roles played by the variables x and y. We are now in a position to clarify the meaning of this process, and also to give a precise statement of the conditions under which an equation of the form (1) defines a differentiable function (2).

We broaden the discussion slightly, and instead of (1) consider an equation of the form

$$F(x, y) = c, \tag{6}$$

whose graph is a level curve of the function $z = F(x, y)$. For example, the graph of

$$x^2 + y^2 = 1 \tag{7}$$

is a circle about the origin (Fig. 19.21, left), and this is a level curve of the function $F(x, y) = x^2 + y^2$. Generally, as in this case, the graph of (6) will be some sort of curve that is not the graph of a single function. However, even though the entire graph of (7) is not the graph of a single function, it is clear that every point (x_0, y_0) on this graph with $y_0 \neq 0$ lies on a *portion* of the graph that *is* the graph of a function—indeed, of a differentiable function. Specifically, if $y_0 > 0$ then (x_0, y_0) lies on the graph of the function

$$y = f_1(x) = \sqrt{1 - x^2}, \tag{8}$$

and if $y_0 < 0$ then (x_0, y_0) lies on the graph of the function

$$y = f_2(x) = -\sqrt{1 - x^2}. \tag{9}$$

Similarly, the graph of (6) might consist of the graphs of two or more differentiable functions $y = f(x)$, as suggested on the right in the figure.

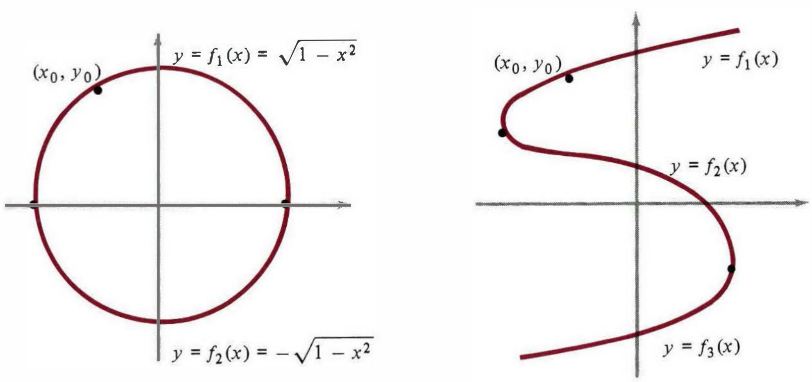


Figure 19.21

We next point out that the function $z = F(x, y)$ has the constant value c along the graph of any such function $y = f(x)$,

$$z = F[x, f(x)] = c.$$

As usual, we assume that $F(x, y)$ has continuous partial derivatives, so it is permissible to write

$$\frac{dz}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0,$$

or equivalently,

$$\frac{dz}{dx} = F_x(x, y) + F_y(x, y)\frac{dy}{dx} = 0. \tag{10}$$

The middle term here is just the chain rule evaluation of dz/dx when $z = F(x, y)$ and y is a function of x, and the result is zero because z is constant as a function of x. If $F_y(x, y) \neq 0$, equation (10) can be solved for dy/dx,

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)}. \tag{11}$$

In the language of Section 3.5, any differentiable function $y = f(x)$ with the property that

$$F[x, f(x)] = c$$

is an *implicit function* defined by (6), and (11) provides a general formula for the derivative of such a function.

If we apply formula (11) to equation (7), where $F(x, y) = x^2 + y^2$, we obtain

$$F_x = 2x \qquad \text{and} \qquad F_y = 2y, \qquad \text{so} \qquad \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{x}{y}, \qquad y \neq 0. \tag{12}$$

In this case we know from (8) and (9) that (7) actually determines two implicit functions $y = f(x)$, so the calculations (12) are legitimate and apply to either function as long as we avoid points where $y = 0$. However, suppose that instead of (7) we have one of the equations

$$x^2 + y^2 = -1 \qquad \text{or} \qquad x^2 + y^2 = 0. \tag{13}$$

By acting blindly without thinking, we can write down the calculations (12) just as before and "find" dy/dx. The difficulty with this is obvious: Since the graph of (13) is either empty or consists of a single point, no implicit function $y = f(x)$ exists, and these calculations would be nothing more than a kind of mathematical doubletalk, which seems to be saying something but really says nothing at all.

In order to avoid committing such nonsense, it is necessary to have definite knowledge that implicit functions exist. This is the purpose of the

Implicit Function Theorem *Let $F(x, y)$ have continuous partial derivatives throughout some neighborhood of a point (x_0, y_0), and assume that $F(x_0, y_0) = c$ and $F_y(x_0, y_0) \neq 0$. Then there is an interval I about x_0 with the property that there exists exactly one differentiable function $y = f(x)$ defined on I such that $y_0 = f(x_0)$ and*

$$F[x, f(x)] = c.$$

Further, the derivative of this function is given by the formula

$$\frac{dy}{dx} = -\frac{F_x}{F_y},$$

and is therefore continuous.

It should be understood that this theorem is a purely theoretical statement to the effect that the specified implicit function $y = f(x)$ does in fact exist, and it has no bearing on the issue of whether a simple formula can be found for this function. A proof is given in Appendix A.20.

Example 1 We consider once more the equation mentioned earlier,

$$F(x, y) = x^2y^5 - 2xy + 1 = 0. \tag{3}$$

It is clear that the point (1, 1) lies on the graph, so the graph is not empty. Since $F_x = 2xy^5 - 2y$ and $F_y = 5x^2y^4 - 2x$, our theorem guarantees that equation (3) determines an implicit function $y = f(x)$ about any point of the graph where $F_y = 5x^2y^4 - 2x \neq 0$, for instance the point (1, 1). It is instructive to write down equation (10) for this case,

$$(2xy^5 - 2y) + (5x^2y^4 - 2x)\frac{dy}{dx} = 0, \tag{14}$$

and to compare the result with (4), where implicit differentiation is carried out by the old method. Equation (14) evidently yields

$$\frac{dy}{dx} = \frac{2y - 2xy^5}{5x^2y^4 - 2x},$$

just as before.

The simplicity of our present method is even more clearly visible when there are three variables in the given equation.

Thus, suppose an equation $F(x, y, z) = c$ defines a certain implicit function $z = f(x, y)$, and let us find $\partial z/\partial x$ in terms of the function $F(x, y, z)$. The equations

$$w = F(x, y, z),$$

$$x = x, \qquad y = y, \qquad z = f(x, y),$$

give w as a composite function of x and y. Also,

$$w = F[x, y, f(x, y)] = c,$$

so if we differentiate this with respect to x, the chain rule yields

$$\frac{\partial w}{\partial x} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0.$$

We therefore obtain

$$\frac{\partial z}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z}, \tag{15}$$

and this formula is valid wherever $\partial F/\partial z \neq 0$. In just the same way, we also have

$$\frac{\partial z}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial z}. \tag{16}$$

As students have surely guessed, there is also an Implicit Function Theorem that covers this situation. Briefly, it says that if $\partial F/\partial z \neq 0$ at a point (x_0, y_0, z_0) on a surface $F(x, y, z) = c$, then in a neighborhood of this point the surface defines a unique implicit function $z = f(x, y)$ such that $z_0 = f(x_0, y_0)$, and furthermore the partial derivatives of this function are given by (15) and (16).

Example 2 It is easy to verify that the point $(1, 2, -1)$ lies on the graph of the equation

$$x^2z + yz^5 + 2xy^3 = 13, \tag{17}$$

so this graph is not empty. If the equation defines an implicit function $z = f(x, y)$ in a neighborhood of this point, then we can calculate $\partial z/\partial x$ by implicit differentiation in the old way. This means we differentiate (17) implicitly with respect to x, thinking of y as a constant, which gives

$$x^2\frac{\partial z}{\partial x} + 2xz + y \cdot 5z^4\frac{\partial z}{\partial x} + 2y^3 = 0,$$

so

$$\frac{\partial z}{\partial x} = -\frac{2xz + 2y^3}{x^2 + 5yz^4}.$$

This procedure is unsatisfactory because we don't know in the beginning whether any such function $z = f(x, y)$ actually exists—after all, (17) is a fifth-degree equation in z—and also because in the implicit differentiation each of the three variables has to be treated in a different way, and it is quite easy to lose track of what is going on. Our present ideas provide a much better method. We have

$$F(x, y, z) = x^2z + yz^5 + 2xy^3,$$

so

$$\frac{\partial F}{\partial x} = 2xz + 2y^3 \qquad \text{and} \qquad \frac{\partial F}{\partial z} = x^2 + 5yz^4.$$

It is easy to see that $\partial F/\partial z = 11 \neq 0$ at $(1, 2, -1)$, so the Implicit Function Theorem guarantees that $z = f(x, y)$ exists. Also, by (15) we have

$$\frac{\partial z}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z} = -\frac{2xz + 2y^3}{x^2 + 5yz^4},$$

which avoids the messy implicit differentiation.

Remark The two-variable version of the Implicit Function Theorem enables us to complete a long-standing piece of unfinished business. In the earlier chapters of this book we gave quite a bit of attention to the important problem of finding the inverse function of a given function $g(y) = x$, in other words, the problem of solving the equation

$$F(x, y) = g(y) - x = 0 \tag{18}$$

for the variable y. Specifically, this is the way the familiar functions $y = \ln x$, $y = \sin^{-1} x$, and $y = \tan^{-1} x$ were defined. Each of these inverse functions was discussed earlier in an *ad hoc* but perfectly legitimate way. We are now in a position to draw the general inference that when $g(y)$ has a continuous derivative and $\partial F/\partial y = g'(y) \neq 0$, then (18) can indeed be solved for y, $y = f(x)$, and also that this function has a continuous derivative given by

$$\frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y} = -\frac{-1}{g'(y)} = \frac{1}{dx/dy}.$$

This completes the line of thought that was briefly described in Remark 2 of Section 9.5.

PROBLEMS

In Problems 1–6, use formula (11) to compute dy/dx.

1 $y^2 - 3x^2 - 1 = 0.$
2 $x^6 + 2y^4 = 1.$
3 $x \sin y = x + y.$
4 $\sin y + \tan y = x^2 + x^3.$
5 $e^{xy} = 2xy^2.$
6 $e^x \sin y = e^y \cos x.$

In Problems 7–10, use formulas (15) and (16) to compute $\partial z/\partial x$ and $\partial z/\partial y$.

7 $\ln z = z + 2y - 3x.$
8 $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = 9.$
9 $z = xy \sin xz.$
10 $\sin xy + \sin yz + \sin xz = 1.$

11 Use formulas (15) and (16) to find the largest value of z on the ellipsoid $2x^2 + 3y^2 + z^2 + yz - xz = 1$.

12 The folium of Descartes (Problem 16 in Section 17.1) has $x^3 + y^3 = 3axy$ as its equation. Use formula (11) to find the highest point on the loop.

13 If $F(x, y)$ has continuous second partial derivatives and the equation $F(x, y) = c$ defines $y = f(x)$ as a twice-differentiable function, show that if $F_y \neq 0$,

$$\frac{d^2y}{dx^2} = -\frac{F_{xx}F_y^2 - 2F_{xy}F_xF_y + F_{yy}F_x^2}{F_y^3}.$$

14 Compute d^2y/dx^2 by using Problem 13 if
(a) $x^4y^5 = 1$; (b) $e^y = x + y$.

CHAPTER 19 REVIEW: DEFINITIONS, METHODS

Think through the following.

1 Domain, continuity, and level curves for $z = f(x, y)$.
2 Definition and geometric meaning of partial derivatives of $z = f(x, y)$.
3 Equality of mixed partial derivatives.
4 Equation of tangent plane to $z = f(x, y)$.
5 Directional derivative and gradient.
6 The del operator.
7 The chain rule.
8 Method of Lagrange multipliers for constrained maxima and minima.

20 MULTIPLE INTEGRALS

20.1 VOLUMES AS ITERATED INTEGRALS

A continuous function $f(x, y)$ of two variables can be integrated over a plane region R in much the same way that a continuous function of one variable can be integrated over an interval. The result is a number called the *double integral* of $f(x, y)$ over R and denoted by

$$\iint_R f(x, y)\, dA \qquad \text{or} \qquad \iint_R f(x, y)\, dx\, dy.$$

A different but closely related concept is that of an *iterated* (or *repeated*) *integral*. We discuss iterated integrals in this section, and in the next section return to the topic of double integrals and explain what they are and how they are related to iterated integrals.

In Section 7.3 we discussed the "method of moving slices" for finding volumes. Thus, if $A(x)$ is the area of the section cut from a solid by a plane perpendicular to the x-axis at a distance x from the origin, then the formula

$$V = \int_a^b A(x)\, dx \tag{1}$$

gives the volume of the solid between the planes $x = a$ and $x = b$. The essence of this formula lies in the idea that

$$dV = A(x)\, dx$$

is the volume of a thin slice of the solid of thickness dx. The total volume (1) is then found by adding together (or integrating) these elements of volume as our typical slice sweeps through the complete solid, that is, as x increases from a to b.

However, if the section itself has curved boundaries—as happens in many cases—then the determination of $A(x)$ also requires integration. For instance, the section shown in Fig. 20.1 extends from the xy-plane $z = 0$ up to the curved surface $z = f(x, y)$. By considering x to be arbitrary but momentarily fixed between a and b, we see that the area of this section is

$$A(x) = \int_{y_1(x)}^{y_2(x)} f(x, y)\, dy, \tag{2}$$

where $y = y_1(x)$ and $y = y_2(x)$ are the equations of the curves that bound the base

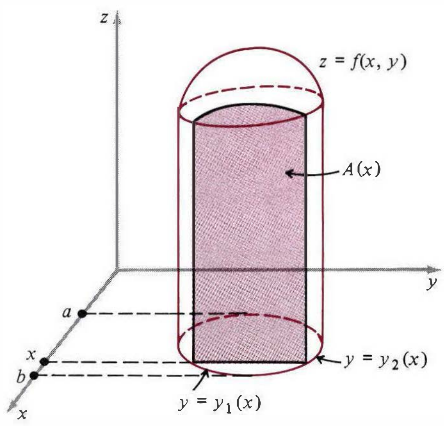


Figure 20.1

on the left and right. To find the total volume V, we now insert (2) in (1) and obtain the *iterated integral*

$$V = \int_a^b \left[\int_{y_1(x)}^{y_2(x)} f(x, y)\, dy \right] dx. \tag{3}$$

Students should notice particularly that in (3) we first integrate $f(x, y)$ with respect to y, holding x fixed. The limits of integration depend on this fixed but arbitrary value of x, and so does the resulting value of the inner integral. This inner integral is precisely the function $A(x)$ given by (2), which we then integrate with respect to x from a to b to obtain the iterated integral (3). To summarize, we start with a positive function $f(x, y)$ of two variables; we first "integrate y out," which gives a function of x alone; and then we "integrate x out," which gives a number—the volume of the solid.

On the other hand, in some cases it may be more convenient to cut the solid by a plane perpendicular to the y-axis and to form the iterated integral in the other order, first integrating x and then y,

$$V = \int_c^d \left[\int_{x_1(y)}^{x_2(y)} f(x, y)\, dx \right] dy. \tag{4}$$

These two possible orders of integration are suggested in Fig. 20.2, representing the base of the solid, with (3) shown on the left and (4) on the right. The iterated integrals (3) and (4) are usually written without brackets, as

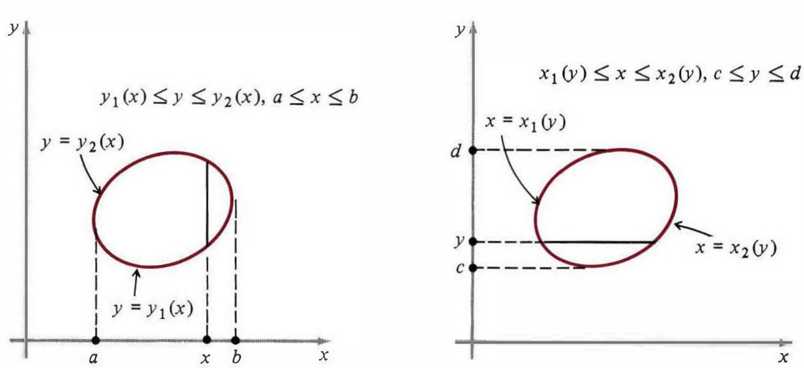


Figure 20.2

$$\int_a^b \int_{y_1(x)}^{y_2(x)} f(x, y)\, dy\, dx \qquad \text{and} \qquad \int_c^d \int_{x_1(y)}^{x_2(y)} f(x, y)\, dx\, dy;$$

however, we can always retain the brackets for additional clarity if we wish to do so. The order in which the integrations are carried out (first with respect to y and then with respect to x, or the reverse) is determined by the order in which the differentials dx and dy are written in these iterated integrals: *We always work from the inside out.*

Example 1 Use an iterated integral to find the volume of the tetrahedron bounded by the coordinate planes and the plane $x + y + z = 1$.

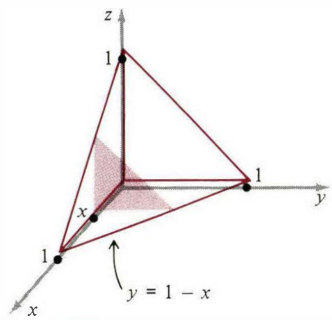


Figure 20.3

Solution The section in the plane x = a constant is the triangle shown in Fig. 20.3, with base extending from $y = 0$ to the line $y = 1 - x$. Its area is

$$A(x) = \int_0^{1-x} z\, dy = \int_0^{1-x} (1 - x - y)\, dy.$$

We now find the desired volume by integrating this from $x = 0$ to $x = 1$,

$$V = \int_0^1 \int_0^{1-x} (1 - x - y)\, dy\, dx = \int_0^1 \left[y - xy - \frac{1}{2} y^2 \right]_0^{1-x} dx$$

$$= \int_0^1 \left(\frac{1}{2} - x + \frac{1}{2} x^2 \right) dx = \frac{1}{6}.$$

The correctness of this result can be verified by elementary geometry, from the fact that the volume of any tetrahedron is one-third the area of the base times the height.

Example 2 Determine the region in the xy-plane over which the iterated integral

$$\int_{-1}^2 \int_{x^2}^4 f(x, y)\, dy\, dx$$

extends.

Solution In the inner integral, with x fixed between -1 and 2, y varies from the curve $y = x^2$ up to the line $y = 4$ (see Fig. 20.4). In the second integration x increases from -1 to 2. The region is that shown in the figure, and is bounded by the curve $y = x^2$ and the lines $y = 4$ and $x = -1$. Students should notice particularly how we determine what the region is by examining the limits of integration.

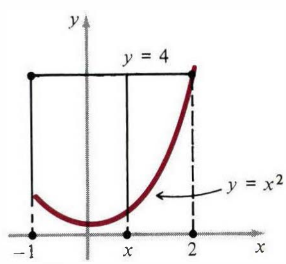


Figure 20.4

Example 3 The iterated integral

$$\int_0^1 \int_{x^2}^x 2y\, dy\, dx \tag{5}$$

extends over a certain region in the xy-plane. Write an equivalent integral with the order of integration reversed, and evaluate both integrals.

Solution We see that the given integral extends over the region shown in Fig. 20.5, between the curves $y = x^2$ and $y = x$, where $0 \le x \le 1$. With the order of integration reversed, y is first held fixed between $y = 0$ and $y = 1$, and x increases from $x = y$ to $x = y^{1/2}$. The required integral is therefore

$$\int_0^1 \int_y^{y^{1/2}} 2y\,dx\,dy = \int_0^1 [2xy]_y^{y^{1/2}}\,dy = \int_0^1 (2y^{3/2} - 2y^2)\,dy = \frac{2}{15}.$$

The given integral (5) has the same value,

$$\int_0^1 \int_{x^2}^{x} 2y\,dy\,dx = \int_0^1 [y^2]_{x^2}^{x}\,dx = \int_0^1 (x^2 - x^4)\,dx = \frac{2}{15},$$

because both iterated integrals give the volume of a certain solid, and this volume must be the same regardless of how it is calculated. In computational problems of this kind, we are naturally free to use any methods of integration we wish from our past experience—trigonometric substitution, integration by parts, etc.

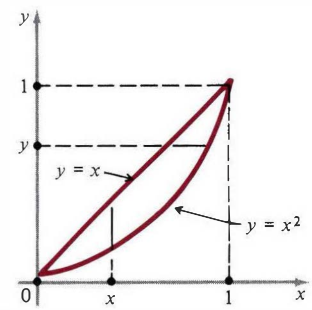


Figure 20.5

PROBLEMS

Determine the regions over which the iterated integrals in Problems 1 and 2 extend.

1 $\displaystyle\int_0^1 \int_0^y f(x, y)\,dx\,dy.$ **2** $\displaystyle\int_0^4 \int_0^{\sqrt{x}} f(x, y)\,dy\,dx.$

Evaluate each of the iterated integrals in Problems 3–14. Also sketch the region R over which the integral extends.

3 $\displaystyle\int_0^1 \int_{x^2}^{x} (2x + 2y)\,dy\,dx.$ **4** $\displaystyle\int_0^1 \int_0^1 xy^2\,dy\,dx.$

5 $\displaystyle\int_0^4 \int_0^y 3\sqrt{y^2 + 9}\,dx\,dy.$ **6** $\displaystyle\int_1^2 \int_{y^2}^{y^3} dx\,dy.$

7 $\displaystyle\int_0^{\pi/2} \int_0^{\cos x} y\,dy\,dx.$ **8** $\displaystyle\int_1^{e^3} \int_0^{1/y} e^{xy}\,dx\,dy.$

9 $\displaystyle\int_1^3 \int_0^{\ln y} ye^x\,dx\,dy.$ **10** $\displaystyle\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} y\,dx\,dy.$

11 $\displaystyle\int_0^\pi \int_0^x x\cos y\,dy\,dx.$ **12** $\displaystyle\int_0^\pi \int_0^{\sin x} y^2\,dy\,dx.$

13 $\displaystyle\int_1^2 \int_x^{2x} \frac{dy\,dx}{(x + y)^2}.$

14 $\displaystyle\int_0^\pi \int_0^{\pi - y} \sin(x + y)\,dx\,dy.$

In Problems 15–18, write an equivalent iterated integral with the order of integration reversed.

15 $\displaystyle\int_0^1 \int_y^1 f(x, y)\,dx\,dy.$

16 $\displaystyle\int_0^1 \int_0^{\sqrt{2-2x^2}} f(x, y)\,dy\,dx.$

17 $\displaystyle\int_1^2 \int_{e^y}^{e^2} f(x, y)\,dx\,dy.$

18 $\displaystyle\int_{-2}^2 \int_{1-\sqrt{2-x}}^{\frac{1}{2}x} f(x, y)\,dy\,dx.$

In Problems 19–24, write an equivalent iterated integral with the order of integration reversed, and evaluate both integrals.

19 $\displaystyle\int_0^1 \int_{\sqrt{y}}^1 2x^3\,dx\,dy.$ **20** $\displaystyle\int_0^2 \int_0^{4-x^2} 2xy\,dy\,dx.$

21 $\displaystyle\int_0^2 \int_0^1 (5 - 2x - y)\,dy\,dx.$

22 $\displaystyle\int_1^{e^3} \int_{\ln y}^3 dx\,dy.$ **23** $\displaystyle\int_{-5}^4 \int_{2-\sqrt{4-y}}^{\frac{1}{3}(y+2)} dx\,dy.$

24 $\displaystyle\int_0^{\sqrt{2}} \int_{-\sqrt{4-2x^2}}^{\sqrt{4-2x^2}} x\,dy\,dx.$

In Problems 25–28, use iterated integrals to find the volumes of the given regions of space. Sketch each region.

25 The region in the first octant bounded by the coordinate planes and the plane

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1,$$

where a, b, c are positive numbers.

26 The region in the first octant bounded by the plane $x + y = 1$ and the cylinder $z = 1 - x^2$.

27 The region in the first octant bounded by the plane $y = x$ and the cylinder $z = 4 - y^2$.

28 The region in the first octant bounded by the surface $z = 4 - x - y^2$.

***29** Use any method to find the volume of the region bounded by the surface $x^{2/3} + y^{2/3} + z^{2/3} = a^{2/3}$.

20.2 DOUBLE INTEGRALS AND ITERATED INTEGRALS

The double integral of a function of two variables is the two-dimensional analog of the definite integral of a function of one variable. It is convenient here to call this latter type of integral a *single integral,* in contrast to the term *double integral.*

As we know, the value of the single integral $\int_a^b f(x)\,dx$ is determined by the function $f(x)$ and the interval $[a, b]$. In the case of a double integral, the interval $[a, b]$ is replaced by a region R in the xy-plane, and the double integral of $f(x, y)$ over R is denoted by the symbol

$$\iint_R f(x, y)\,dA. \tag{1}$$

The reason for the dA notation will be explained below.

We recall that in Section 6.4 a single integral was defined as the limit of certain sums. We now define the double integral (1) in much the same way.

Consider a continuous function $f(x, y)$ defined on a region R in the xy-plane. It is necessary to assume that R is *bounded,* in the sense that it can be enclosed in a sufficiently large rectangle and doesn't go off to infinity in any direction; otherwise, just as in the case of a single integral where a or b is infinite, the double integral will be *improper.*

We begin by covering R with a network of lines parallel to the axes, as shown in Fig. 20.6, where the distances between consecutive parallel lines are permitted to be equal or unequal. These lines divide the plane into many small rectangles. Some rectangles will lie entirely or partly outside of R, and these we ignore. Other rectangles will lie entirely inside R, and if there are n of these altogether—we assume there is at least one—then we number them in any order from 1 to n, denoting by ΔA_k the area of the kth rectangle. We now choose an arbitrary point (x_k, y_k) in the kth rectangle and form the sum

$$\sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k. \tag{2}$$

Finally, suppose that many more parallel lines are added to produce a network that divides the given rectangles into even smaller rectangles, and consider the sum (2) corresponding to this finer partition of the plane. If these sums approach a unique limit as n becomes infinite and the maximum diagonal of the rectangles (that is, the longest diagonal of any of the rectangles) approaches zero—in-

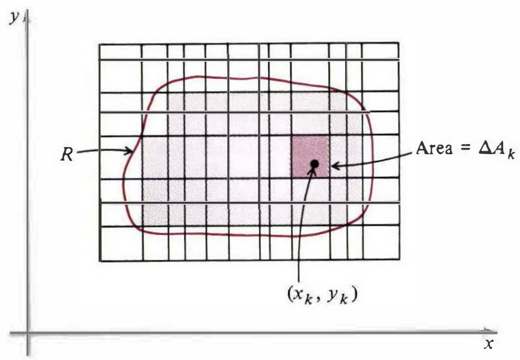


Figure 20.6

dependent of the choice of dividing lines and the points (x_k, y_k) in the rectangles —then the double integral (1) is defined to be this limit:

$$\iint_R f(x, y)\, dA = \lim \sum_{k=1}^{n} f(x_k, y_k)\, \Delta A_k. \tag{3}$$

So far, it appears that the definition (3) differs very little from the corresponding definition of a single integral. However, there are certain technical difficulties in two dimensions that do not arise in one dimension. For one thing, plane regions can be much more complicated than intervals $[a, b]$. Nevertheless, the existence of double integrals can be rigorously proved under assumptions that are general enough for all practical purposes. In particular, it is enough to assume that the regions we consider contain their boundaries and that these boundaries consist of a finite number of smooth curves.

We shall not attempt a careful theoretical treatment of double integrals. This is a difficult subject, and is best left to courses in advanced calculus.* Instead, we prefer to emphasize the intuitive meaning of double integrals, and to concentrate our attention on their geometric and physical applications.

As an illustration of this point of view, suppose that $z = f(x, y)$ is the equation of a surface in xyz-space that lies above the region R, so that $f(x, y) > 0$ in R, as shown in Fig. 20.7. Then $f(x_k, y_k)\, \Delta A_k$ is approximately the volume (height times area of base) of the thin column in the figure; the sum (2) is the sum of many such volumes and therefore approximates the total volume of the solid under the surface; and the limit (3), which is the double integral

$$\iint_R f(x, y)\, dA, \tag{1}$$

gives the exact volume of this solid.†

*Even at this level, one needs an advanced calculus course of the traditional kind. For example, see Philip Franklin, *A Treatise on Advanced Calculus* (Wiley, 1940); or Angus E. Taylor, *Advanced Calculus* (Ginn, 1955).

†The double integral (1) is actually the volume of a *region* in three-dimensional space, but it seems to be more natural to speak of the volume of a *solid*.

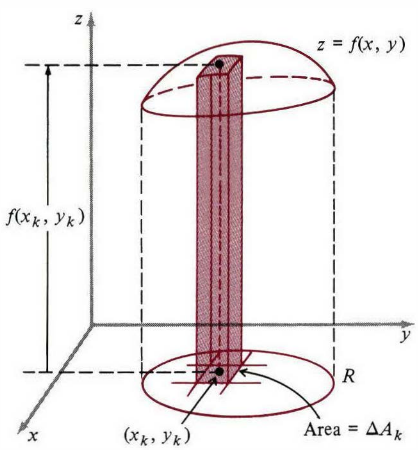


Figure 20.7

It is clear that if $f(x, y)$ has a constant value, say $f(x, y) = c$, then

$$\iint_R f(x, y)\, dA = cA,$$

where A is the area of the region R. In particular, if $f(x, y) = 1$ we have

$$\iint_R dA = A.$$

We also point out that in the definition (3) there is no requirement that $f(x, y)$ must be positive. If $f(x, y)$ takes both positive and negative values, then the double integral represents an *algebraic volume* instead of a geometric volume; that is, the volume between the surface $z = f(x, y)$ and the xy-plane counts positively when $f(x, y) > 0$ and negatively when $f(x, y) < 0$.

Since the area of a rectangle with sides parallel to the axes can be written as $\Delta A = \Delta x\, \Delta y$, it is reasonable to use

$$\iint_R f(x, y)\, dx\, dy \tag{4}$$

as an alternative notation for the double integral (1). In this form the double integral resembles an iterated integral, and in fact, as we next explain, when the region R has a certain simple shape the double integral (1) is always equal to a suitably chosen iterated integral. This equality often misleads students into thinking that double integrals are essentially the same as iterated integrals, but they are not. We shall say more below about the distinction between these two types of integrals.

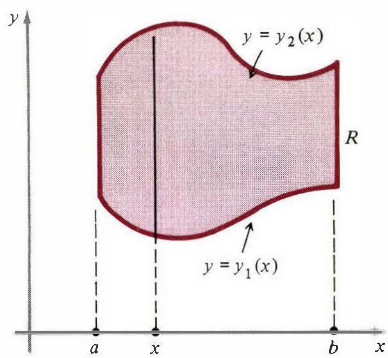


Figure 20.8

A region R is called *vertically simple* if it can be described by inequalities of the form

$$a \le x \le b, \qquad y_1(x) \le y \le y_2(x), \tag{5}$$

where $y = y_1(x)$ and $y = y_2(x)$ are continuous functions on $[a, b]$. A region of this kind is shown in Fig. 20.8. Similarly, a region R is called *horizontally simple* if it can be described by inequalities of the form

$$c \le y \le d, \qquad x_1(y) \le x \le x_2(y), \tag{6}$$

where $x = x_1(y)$ and $x = x_2(y)$ are continuous functions on $[c, d]$. The region in Fig. 20.9 has this property.

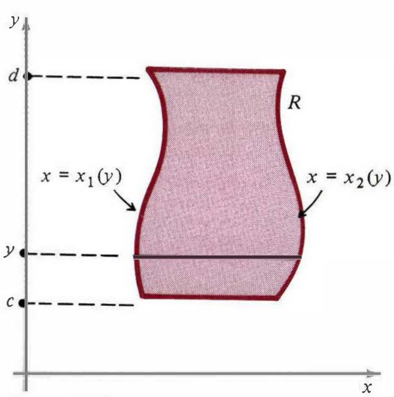


Figure 20.9

The following are the basic facts about the use of iterated integrals to compute double integrals: *if R is the vertically simple region given by* (5), *then*

$$\iint_R f(x, y)\, dA = \int_a^b \int_{y_1(x)}^{y_2(x)} f(x, y)\, dy\, dx; \tag{7}$$

and if R is the horizontally simple region given by (6), *then*

$$\iint_R f(x, y)\, dA = \int_c^d \int_{x_1(y)}^{x_2(y)} f(x, y)\, dx\, dy. \tag{8}$$

In addition to their obvious practical value for the computation of double integrals, these equations also serve to clarify the conceptual distinction between double integrals and iterated integrals. A double integral is a number associated with a function $f(x, y)$ and a region R, and this number exists and has a meaning independently of any particular method of computing it. On the other hand, an iterated integral is a double integral *plus a built-in computational procedure.* Thus, every iterated integral is a double integral, but not vice versa.

Example 1 Compute the double integral $\iint_R 2xy\,dA$ in two different ways, where R is the region bounded by the parabola $x = y^2$ and the straight line $y = x$.

Solution *It is essential to always sketch the region R of integration before trying to evaluate a double integral.* In this case the region is shown in Fig. 20.10. It is clear that R is vertically simple with $a = 0$, $b = 1$, $y_1(x) = x$, $y_2(x) = x^{1/2}$, so by (7)

$$\iint_R 2xy\,dA = \int_0^1 \int_x^{x^{1/2}} 2xy\,dy\,dx = \int_0^1 [xy^2]_x^{x^{1/2}}\,dx$$

$$= \int_0^1 (x^2 - x^3)\,dx = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$

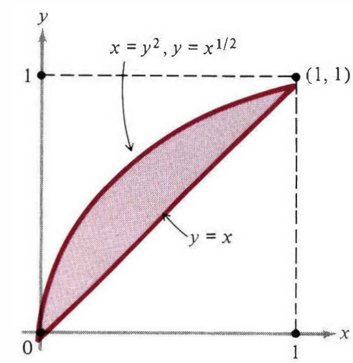


Figure 20.10

The region R is also horizontally simple with $c = 0$, $d = 1$, $x_1(y) = y^2$, $x_2(y) = y$, so by (8)

$$\iint_R 2xy\,dA = \int_0^1 \int_{y^2}^{y} 2xy\,dx\,dy = \int_0^1 [x^2y]_{y^2}^{y}\,dy$$

$$= \int_0^1 (y^3 - y^5)\,dy = \frac{1}{4} - \frac{1}{6} = \frac{1}{12}.$$

Example 2 Compute $\iint_R (1 + 2x)\,dA$, where R is the region bounded by $x = y^2$ and $x - y = 2$.

Solution This region is shown in Fig. 20.11. In order to integrate first with respect to y and then with respect to x, we would need to compute two separate integrals, one to the left of the line $x = 1$ and the other to the right, because the limits of the y-integration are different in these two parts of the region:

$$\iint_R (1 + 2x)\,dA = \int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} (1 + 2x)\,dy\,dx + \int_1^4 \int_{x-2}^{\sqrt{x}} (1 + 2x)\,dy\,dx.$$

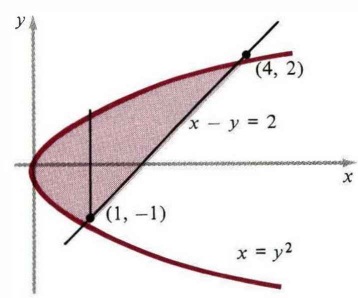


Figure 20.11

The other order is easier, and yields

$$\iint_R (1 + 2x)\,dA = \int_{-1}^2 \int_{y^2}^{y+2} (1 + 2x)\,dx\,dy = \int_{-1}^2 [x + x^2]_{y^2}^{y+2}\,dy$$

$$= \int_{-1}^2 (6 + 5y - y^4)\,dy = \frac{189}{10}.$$

Example 2 shows that even when the region R is both vertically and horizontally simple, it may be easier to integrate in one order than in the other, and we naturally prefer to do things in the easiest way. Sometimes the choice of the order of integration is determined by the nature of the integrand $f(x, y)$, for it may be difficult—or even impossible—to compute an integral in one order, but easy to do so *if the order of integration is reversed.*

Example 3 Compute

$$\int_0^1 \int_{2y}^2 4e^{x^2}\,dx\,dy.$$

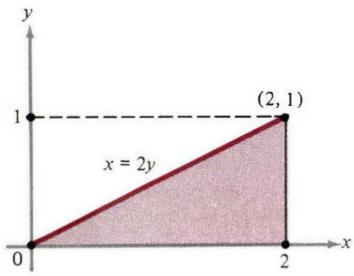


Figure 20.12

Solution We cannot integrate in this order because $\int e^{x^2}\,dx$ is not an elementary function. We therefore try the other order. This requires us to sketch the region R by examining the limits on the given iterated integral. R is shown in Fig. 20.12, and in the other order the underlying double integral has the value

$$\iint_R 4e^{x^2}\,dA = \int_0^2 \int_0^{x/2} 4e^{x^2}\,dy\,dx = \int_0^2 \left[4ye^{x^2}\right]_0^{x/2} dx$$

$$= \int_0^2 2xe^{x^2}\,dx = e^{x^2}\Big]_0^2 = e^4 - 1.$$

PROBLEMS

In Problems 1–6, use double integrals to find the areas of the regions bounded by the given curves and lines.

1 The parabola $x = y^2$ and the line $y = x - 2$.

2 The parabola $y = x - x^2$ and the line $x + y = 0$.

3 The axes and the line $2x + y = 2a$ $(a > 0)$.

4 The y-axis, the line $y = 3x$, and the line $y = 6$.

5 The x-axis, the curve $y = e^{-x}$, and the lines $x = 0$, $x = a$ $(a > 0)$.

6 The parabolas $y = x^2$ and $y = 2x - x^2$.

In Problems 7–10, find the volumes above the xy-plane bounded by the given surfaces.

7 The paraboloid $z = x^2 + y^2$ and the planes $x = \pm 1$, $y = \pm 1$.

8 The cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 2$.

9 The cylinder $y = 4 - x^2$ and the planes $y = 3x$, $z = x + 4$.

10 The cylinder $x^2 + y^2 = a^2$ and the paraboloid $az = x^2 + y^2$.

11 Find the volume of the solid bounded by the coordinate planes, the planes $x = 2$ and $y = 5$, and the surface $2z = xy$.

12 Find the volume of the solid in the first octant bounded by the cylinder $4y = x^2$ and the planes $x = 0$, $z = 0$, $y = 4$, and $x - y + 2z = 2$.

In Problems 13–16, set up a double integral whose value is the stated volume, express this double integral in two ways as an iterated integral, and evaluate one of these.

13 The volume under the plane $z = 2y$ and above the first-quadrant region bounded by $y = 0$, $x = 2$, $x^2 + y^2 = 16$.

14 The volume under the plane $z = x + y$ and above the first-quadrant region inside the ellipse $9x^2 + 4y^2 = 36$.

15 The volume under the cylinder $x = z^2$ and above the region in the xy-plane bounded by $x = 0$ and $y^2 + 9x = 9$.

16 The volume in the first octant bounded by the cylinder $z = 4 - y^2$ and the planes $x = 0$, $y = 0$, $z = 0$, $3x + 4y = 12$.

17 Calculate the value of $\iint_R x\,dA$ if R is the first-quadrant part of the ring between the circles $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$, where $a < b$. Do this two ways, corresponding to the two possible orders of integration.

18 Compute the double integral in Example 2 in the other order, requiring two separate iterated integrals.

20.3 PHYSICAL APPLICATIONS OF DOUBLE INTEGRALS

We have seen that the double integral

$$\iint_R f(x, y)\,dA \tag{1}$$

gives the volume of a certain solid if $f(x, y) \geq 0$. This integral has many other useful interpretations that arise by making special choices of the function $f(x, y)$. Before we discuss these, it will be convenient to return to the way of thinking about integration that was described and extensively illustrated in Chapter 7.

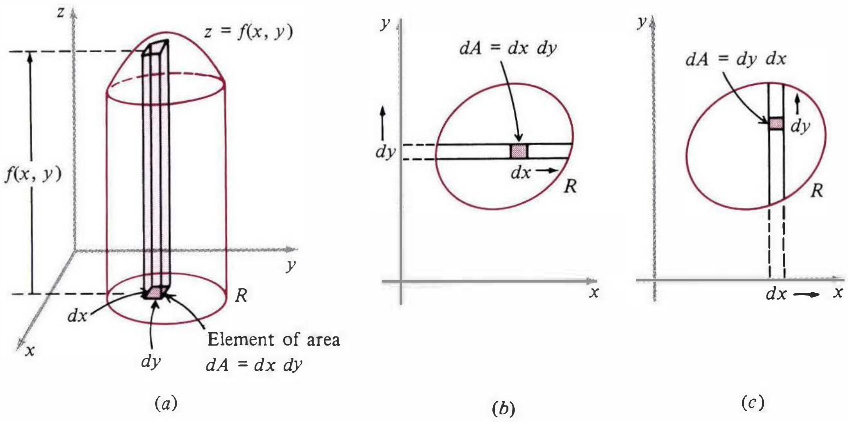


Figure 20.13 Two orders of integration.

The limit-of-sums definition of (1) that was given in Section 20.2 is necessary from the point of view of logic and mathematical legitimacy. However, for working with applications it is better to think of the volume given by (1) as composed of infinitely many infinitely thin columns, as suggested in Fig. 20.13*a*. A typical column stands on an infinitely small rectangular *element of area* dA whose sides are dx and dy, so that

$$dA = dx\,dy = dy\,dx. \tag{2}$$

The height of this column is $f(x, y)$, so its volume is

$$dV = f(x, y)\,dA.$$

The total volume V is now obtained by adding together—or integrating—all of these infinitely small elements of volume,

$$V = \iint dV = \iint_R f(x, y)\,dA. \tag{3}$$

We understand here that the complete double integral (1) is produced by allowing dA to sweep in any manner over the whole of the region R. In parts (*b*) and (*c*) of Fig. 20.13 we indicate the two ways of calculating (3) as an iterated integral: in (*b*), we first allow dA to move across R along a thin horizontal strip, corresponding to integrating first x and then y; and in (*c*), we first allow dA to move across R along a thin vertical strip, integrating first y and then x. As suggested by formula (2), the double integral (3) can be written in either of the forms

$$\iint_R f(x, y)\,dx\,dy \quad \text{or} \quad \iint_R f(x, y)\,dy\,dx,$$

depending on which iterated integral we wish to consider; and to apply these ideas to a particular problem, all that remains is to insert suitable limits of integration and carry out the calculations.

This description of the intuitive meaning of the double integral (1) expresses the essence of the Leibniz approach to integration: to find the whole of a quantity, imagine it to be judiciously divided into a great many small pieces, and then add these pieces together. This is the unifying theme of the following applications, and also of many further developments in the rest of this chapter. And here again, as so often before, the superb Leibniz notation almost does our thinking for us.

In Chapter 11 we discussed the concepts of moment, center of mass, and moment of inertia for a thin plate of homogeneous material that occupies a given region R in the xy-plane. The word "homogeneous" meant that the density δ of the material (= mass per unit area) was assumed to be constant, that is, to have the same value at every point $P = (x, y)$ in R. We are now in a position to allow δ to be a function of x and y, $\delta = \delta(x, y)$, so that thin plates of varying density can be brought within the scope of our methods.

I. MASS

If $\delta = \delta(x, y)$ is the density of our thin plate, then $\delta(x, y)\,dA$ is the mass contained in the element of area dA, and the total mass of the plate is

$$M = \iint_R \delta(x, y)\,dA. \tag{4}$$

II. MOMENT

The moment of the element of mass $\delta(x, y)\,dA$ with respect to the x-axis is the mass multiplied by the "lever arm" y, namely, $y\delta(x, y)\,dA$, and the total moment of the plate with respect to the x-axis is

$$M_x = \iint_R y\delta(x, y)\,dA. \tag{5}$$

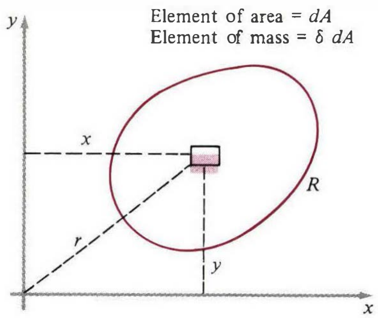


Figure 20.14

See Fig. 20.14. Similarly, the total moment with respect to the y-axis is

$$M_y = \iint_R x\delta(x, y)\,dA. \tag{6}$$

III. CENTER OF MASS

This is the point $(\bar{x}, \bar{y})$ whose coordinates are defined by

$$\bar{x} = \frac{M_y}{M} = \frac{\iint_R x\delta(x, y)\,dA}{\iint_R \delta(x, y)\,dA} \tag{7}$$

and

$$\bar{y} = \frac{M_x}{M} = \frac{\iint_R y\delta(x, y)\,dA}{\iint_R \delta(x, y)\,dA}. \tag{8}$$

Physically, this is the point at which the total mass of the plate could be concentrated without changing its moment with respect to either axis. When the density δ is constant so that the mass of the plate is uniformly distributed, then the δ's can be removed from the integrals in (7) and (8) and canceled away. In this case the center of mass becomes the geometric center of the region R, and for this reason is usually called the *centroid*.

IV. MOMENT OF INERTIA

When the square of the lever arm distance is used instead of its first power [as in (5) and (6)], we get the moment of inertia of the plate about the corresponding axis. Thus, the moment of inertia I_x about the x-axis is defined by

$$I_x = \iint_R y^2 \delta(x, y)\, dA. \tag{9}$$

Similarly, the moment of inertia I_y about the y-axis is

$$I_y = \iint_R x^2 \delta(x, y)\, dA. \tag{10}$$

Also of interest is the moment of inertia of the plate about the z-axis. This is often called the *polar moment of inertia,* and is defined by

$$I_z = \iint_R r^2 \delta(x, y)\, dA, \tag{11}$$

where $r^2 = x^2 + y^2$. As we explained in Section 11.4, the moment of inertia of a body about an axis is its capacity to resist angular acceleration about that axis; this quantity plays the same role in rotational motion as mass does in linear motion.

Students should explicitly notice that in each of the formulas (4), (5), (6), (9), (10), (11) we obtain the total quantity under discussion by adding together—or integrating—the "infinitesimal" parts of it associated with the element of area dA, as dA sweeps over the region R.

Example A thin plate of material of variable density occupies the square R whose vertices are $(0, 0)$, $(a, 0)$, (a, a), $(0, a)$. The density at a point $P = (x, y)$ is the product of the distances from P to the axes, $\delta = xy$. Find the mass of the plate, its center of mass, and its moment of inertia about the x-axis.

Solution A sketch of the situation is shown in Fig. 20.15. We have

$$M = \iint_R \delta\, dA = \int_0^a \int_0^a xy\, dy\, dx = \int_0^a \left[\frac{1}{2} xy^2\right]_0^a dx$$

$$= \frac{1}{2} a^2 \int_0^a x\, dx = \frac{1}{4} a^4.$$

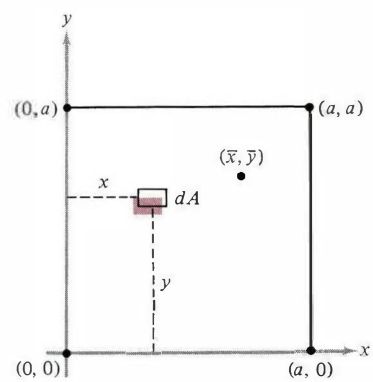


Figure 20.15

The x-coordinate of the center of mass is

$$\bar{x} = \frac{M_y}{M} = \frac{4}{a^4} \iint_R x\delta\, dA = \frac{4}{a^4} \int_0^a \int_0^a x^2 y\, dy\, dx$$

$$= \frac{4}{a^4} \int_0^a \left[\frac{1}{2} x^2 y^2\right]_0^a dx = \frac{2}{a^2} \int_0^a x^2\, dx = \frac{2}{3} a,$$

and by symmetry we have $\bar{x} = \bar{y} = \frac{2}{3}a$. The desired moment of inertia is

$$I_x = \iint_R y^2 \delta\, dA = \int_0^a \int_0^a xy^3\, dy\, dx = \int_0^a \left[\frac{1}{4} xy^4\right]_0^a dx$$

$$= \frac{1}{4} a^4 \int_0^a x\, dx = \frac{1}{8} a^6.$$

It is customary to express the moment of inertia of a body about an axis in terms of its total mass M, which in this case gives

$$I_x = \tfrac{1}{2} M a^2.$$

Remark 1 We emphasize that the symbols dA, dx, and dy in formula (2) do *not* designate differentials in the sense discussed in Section 19.4. Instead, they are merely notational aids that enable us to write down appropriate double integrals directly, without repeatedly going back to the complicated limit-of-sums definitions of these integrals.

Remark 2 A surprising application of our present ideas is given in the Appendix at the end of this chapter, where Euler's formula

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

is obtained by evaluating a certain double integral.

PROBLEMS

In Problems 1–8, find the total mass M and the center of mass $(\bar{x}, \bar{y})$ of the thin plate of material that lies in the given region R and has the given density δ. Use symmetry wherever possible to simplify calculations.

1 R is the square with vertices $(0, 0)$, $(a, 0)$, (a, a), $(0, a)$; $\delta = x + y$.

2 R is the first-quadrant region bounded by the axes and the circle $x^2 + y^2 = 1$; $\delta = xy$.

3 R is the region bounded by the parabola $x = y^2$ and the line $x = 4$; $\delta = x$.

4 R is the region bounded by the axes and the line $x + y = a$; $\delta = x^2 + y^2$.

5 R is the region bounded by $x = 0$ and the right half of the circle $x^2 + y^2 = a^2$; $\delta = x$.

6 R is the region bounded by the parabola $y = x^2$ and the line $y = x$; $\delta = \sqrt{x}$.

7 R is the region between $y = \sin x$ and the x-axis from $x = 0$ to $x = \pi$; $\delta = x$.

8 R is the region bounded by the parabola $y = x^2$ and the line $y = x + 2$; $\delta = x^2$.

9 Find the moment of inertia I_x for the square plate considered in the text if the density δ is constant.

10 Show that $I_z = I_x + I_y$. Use this and the result of Problem 9 to find the moment of inertia of a uniform (constant density) cube of edge a and mass M about one of its edges.

11 If the density δ is constant, find the moment of inertia I_x of the thin triangular plate bounded by the line $x + y = a$ and the axes $x = 0$, $y = 0$.

12 Solve Problem 11 for the triangular plate bounded by the lines $x + y = a$, $x = a$, $y = a$.

13 Solve Problem 11 if the density is $\delta = xy$.

14 Find the polar moment of inertia I_z of the circular plate bounded by $x^2 + y^2 = a^2$ if the density δ is constant.

20.4 DOUBLE INTEGRALS IN POLAR COORDINATES

It is often more convenient to describe the boundaries of a region by using polar coordinates r, θ than by using rectangular coordinates x, y. In these circumstances we can usually save ourselves a lot of work by expressing a double integral

$$\iint_R f(x, y)\, dA \tag{1}$$

in terms of polar coordinates. The integrand is easy to transform by using the equations $x = r\cos\theta$, $y = r\sin\theta$ to write $f(x, y)$ as a function of r and θ,

$$f(x, y) = f(r\cos\theta, r\sin\theta).$$

For example, if $f(x, y) = x^2 + y^2$, this becomes $(r\cos\theta)^2 + (r\sin\theta)^2 = r^2(\cos^2\theta + \sin^2\theta) = r^2$. But what do we do about the element of area dA?

The answer to this question is suggested by Fig. 20.16. We recall that the element of area in rectangular coordinates,

$$dA = dx\, dy,$$

is intended to remind us of the small rectangles with sides parallel to the axes that were used to define the double integral (1) in Section 20.2. In working with polar coordinates it is natural to subdivide the plane in another way, by a series of circles with centers at the origin and a series of rays emanating from the origin. These circles and rays form many small cells that resemble rectangles, as shown by the shaded part of the figure. The double integral (1) can now be given an equivalent definition by means of a limit-of-sums process that uses these small "polar rectangles." However, we omit the details and use Fig. 20.16 only to suggest the line of thought we should follow, as we now explain.

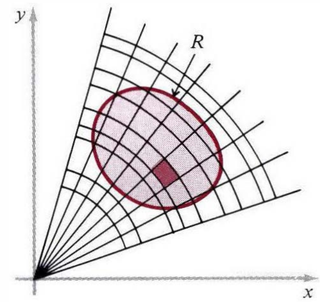


Figure 20.16

The element of area $dA = dx\, dy$ in rectangular coordinates is the area of the small rectangle swept out by an increase dx in x and an increase dy in y (see Fig. 20.17*a*). Figure 20.16 suggests the approach to be used with polar coordinates: If r increases to $r + dr$ and θ increases to $\theta + d\theta$ (Fig. 20.17*b*), then a small polar rectangle is swept out whose sides are dr, the change in r, and $r\, d\theta$.* The area of the small polar rectangle is therefore approximately

$$dA = (dr)(r\, d\theta) = r\, dr\, d\theta. \tag{2}$$

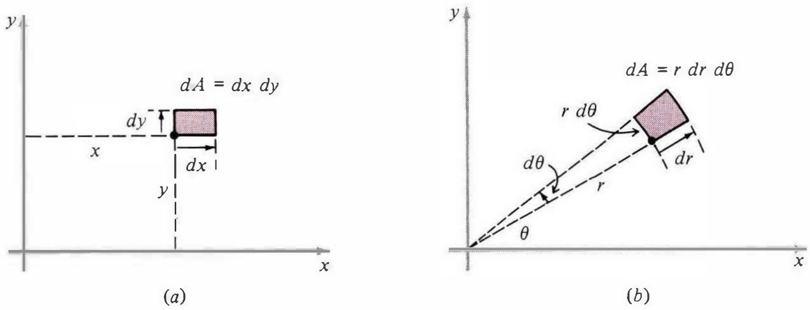


Figure 20.17 The polar element of area.

This is the basic formula of this section. It gives the element of area in polar coordinates, and it enables us to write the double integral (1) in polar form, as

$$\iint_R f(x, y)\, dA = \iint_R f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta. \tag{3}$$

Many of the regions R we deal with are *radially simple*, in the sense that they can be described by inequalities of the form

$$\alpha \le \theta \le \beta, \qquad r_1(\theta) \le r \le r_2(\theta).$$

Figure 20.18 shows a region of this kind, and also suggests how the figure can be used to write the double integral (3) as an iterated integral,

$$\iint_R f(x, y)\, dA = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta.$$

Here we integrate first r and then θ, working from the inside out as always. We visualize the element of area dA as first moving out across R along the indicated radial strip, from the inner curve $r = r_1(\theta)$ to the outer curve $r = r_2(\theta)$. The resulting strip is then rotated from $\theta = \alpha$ to $\theta = \beta$ in order to sweep

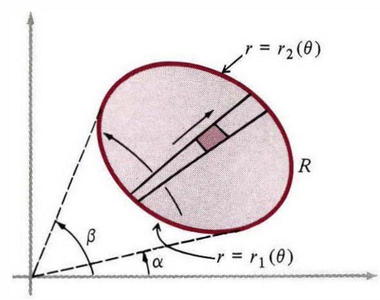


Figure 20.18

*The second side of this polar rectangle is a short arc of a circle of radius r that is cut off by a central angle $d\theta$, and its length s is given by the formula $s = r \cdot d\theta$, *because the angle is measured in radians.*

over all of R. Iterated integrals can also be set up in the other order, but these are seldom used.

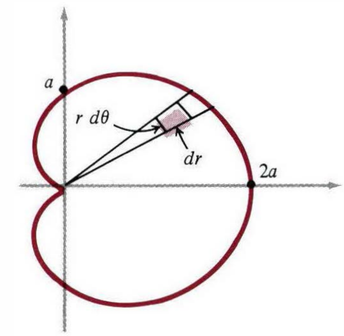


Figure 20.19

Example 1 Find the area of the region R enclosed by the cardioid $r = a(1 + \cos\theta)$.

Solution This cardioid is shown in Fig. 20.19, and we find the area by integrating the element of area $dA = r\,dr\,d\theta$ over the region,

$$A = \iint_R dA = \iint_R r\,dr\,d\theta.$$

For fixed θ, we allow r to increase from $r = 0$ to $r = a(1 + \cos\theta)$. As usual, we exploit all available symmetry, so we next allow θ to increase from 0 to π and obtain the total area by multiplying by 2,

$$\begin{aligned} A &= 2\int_0^\pi \int_0^{a(1+\cos\theta)} r\,dr\,d\theta = 2\int_0^\pi \left[\frac{1}{2}r^2\right]_0^{a(1+\cos\theta)} d\theta \\ &= 2\int_0^\pi \frac{1}{2}a^2(1+\cos\theta)^2\,d\theta = a^2\int_0^\pi (1 + 2\cos\theta + \cos^2\theta)\,d\theta \\ &= a^2\int_0^\pi \left(1 + 2\cos\theta + \frac{1}{2}[1 + \cos 2\theta]\right) d\theta \\ &= a^2\left[\theta + 2\sin\theta + \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_0^\pi = \frac{3}{2}\pi a^2. \end{aligned}$$

This problem can also be solved by the method of Section 16.5, which would have started with the third integral in our calculation. However, our present method has much greater flexibility. It allows us, for example, to find the centroid of the region R by thinking of it as a thin plate of material of constant density $\delta = 1$. It is clear by symmetry that $\bar{y} = 0$, and we find $\bar{x}$ by writing

$$\bar{x} = \frac{M_y}{M} = \frac{2}{3\pi a^2}\iint_R x\,dA.$$

We ask students to complete the details of this calculation in Problem 25.

Example 2 Derive the formula for the volume of a sphere of radius a by our present methods.

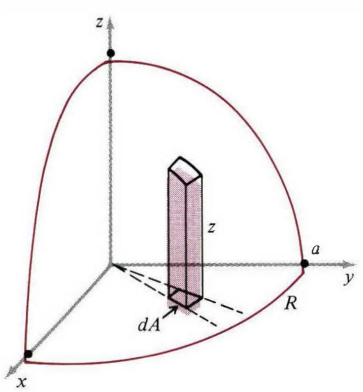


Figure 20.20

Solution If the sphere has center at the origin, its equation is $x^2 + y^2 + z^2 = a^2$ or $r^2 + z^2 = a^2$, and the equation of the upper hemisphere is $z = \sqrt{a^2 - r^2}$. By symmetry, we calculate the volume in the first octant (Fig. 20.20) and multiply by 8. The region R over which we integrate is defined by $0 \le \theta \le \pi/2$ and $0 \le r \le a$, so

$$\begin{aligned} V &= 8\iint_R z\,dA = 8\int_0^{\pi/2}\int_0^a \sqrt{a^2 - r^2}\,r\,dr\,d\theta \\ &= 8\int_0^{\pi/2}\int_0^a -\frac{1}{2}(a^2 - r^2)^{1/2}(-2r\,dr)\,d\theta = -4\int_0^{\pi/2}\left[\frac{2}{3}(a^2 - r^2)^{3/2}\right]_0^a d\theta \\ &= -4\int_0^{\pi/2}\left(-\frac{2}{3}a^3\right) d\theta = \frac{4}{3}\pi a^3. \end{aligned}$$

Students should notice particularly how the presence of the r in the inner integral makes this calculation work out smoothly.

Example 3 As we learned in Section 12.5, the improper integral

$$\int_0^\infty e^{-x^2}\,dx$$

is important in the theory of probability and elsewhere. We shall find its value by a clever device that depends on an improper double integral in polar coordinates. Write

$$I = \int_0^\infty e^{-y^2}\,dy.$$

Since it doesn't matter what letter we use for the variable of integration, we have

$$I^2 = \left(\int_0^\infty e^{-x^2}\,dx\right)\left(\int_0^\infty e^{-y^2}\,dy\right).$$

By moving the first factor past the second integral sign, this can be written in the form

$$\begin{aligned} I^2 &= \int_0^\infty \left(\int_0^\infty e^{-x^2}\,dx\right) e^{-y^2}\,dy = \int_0^\infty \left(\int_0^\infty e^{-x^2}e^{-y^2}\,dx\right) dy \\ &= \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy. \end{aligned}$$

This double integral is extended over the entire first quadrant of the xy-plane. In polar coordinates it becomes

$$I^2 = \int_0^{\pi/2}\int_0^\infty e^{-r^2}\,r\,dr\,d\theta = \int_0^{\pi/2}\left[-\frac{1}{2}e^{-r^2}\right]_0^\infty d\theta = \int_0^{\pi/2}\frac{1}{2}\,d\theta = \frac{\pi}{4},$$

so $I = \frac{1}{2}\sqrt{\pi}$ or

$$\int_0^\infty e^{-x^2}\,dx = \frac{1}{2}\sqrt{\pi}. \tag{4}$$

This formula is especially remarkable because it is known that the indefinite integral

$$\int e^{-x^2}\,dx$$

is impossible to express as an elementary function.*

*There is a famous story about the nineteenth-century Scottish physicist Lord Kelvin. "Do you know what a mathematician is?" Kelvin once asked a class. He stepped to the blackboard and wrote

$$\int_{-\infty}^\infty e^{-x^2}\,dx = \sqrt{\pi},$$

which is clearly equivalent to (4). "A mathematician," he continued, "is one to whom *that* is as obvious as twice two makes four is to you." As a matter of fact, this formula is *not* obvious, either to the present writer or to any of the many mathematicians he has known. The conclusion seems to be that Kelvin was both showing off and trying to put down his class in a rather mean-spirited way.

PROBLEMS

In Problems 1–13, use double integrals in polar coordinates to find the areas of the indicated regions.

1 The circle $r = a$.

2 The circle $r = 2a \cos \theta$.

3 The region common to the circles $r = a$ and $r = 2a \cos \theta$.

4 One loop of $r = a \cos 2\theta$.

5 The right loop of the lemniscate $r^2 = 2a^2 \cos 2\theta$.

6 The region inside the curve $r = 2 + \sin 3\theta$.

7 The region inside the lemniscate $r^2 = 2a^2 \cos 2\theta$ and outside the circle $r = a$.

8 The region inside $r = \tan \theta$ and between $\theta = 0$ and $\theta = \pi/4$.

9 The region inside the cardioid $r = a(1 + \cos \theta)$ and outside the circle $r = a$.

10 The region inside the circle $r = a$ and outside the cardioid $r = a(1 + \cos \theta)$.

11 The region inside the cardioid $r = 2a(1 + \cos \theta)$ and outside the circle $r = 3a$.

*12 The region between $r = \pi/4$ and $r = \pi/2$, between $r = \theta$ and $r = \frac{1}{2}\theta$ $(\theta \geq 0)$.

13 The region inside the cardioid $r = 1 + \cos \theta$ and to the right of the line $x = \frac{3}{4}$.

*14 If R is the region bounded by the lines $y = x$, $y = 0$, $x = 1$, evaluate the double integral

$$\iint_R \frac{dx\, dy}{(1 + x^2 + y^2)^{3/2}}$$

by changing to polar coordinates.

15 Evaluate the integral

$$\int_0^{2a} \int_0^{\sqrt{2ax-x^2}} (x^2 + y^2)\, dy\, dx$$

by changing to polar coordinates.

In Problems 16–22, write the given integral in the form

$$\int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} z\, r\, dr\, d\theta.$$

16 $\displaystyle\int_0^2 \int_0^{\sqrt{4-x^2}} z\, dy\, dx.$

17 $\displaystyle\int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} z\, dy\, dx.$

18 $\displaystyle\int_{-1}^0 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} z\, dx\, dy.$

19 $\displaystyle\int_0^1 \int_{x^2}^{x} z\, dy\, dx.$

20 $\displaystyle\int_0^4 \int_0^{\sqrt{4-(x-2)^2}} z\, dy\, dx.$

21 $\displaystyle\int_0^{\sqrt{2}/2} \int_y^{\sqrt{1-y^2}} z\, dx\, dy.$

22 $\displaystyle\int_0^2 \int_0^{\sqrt{2y-y^2}} z\, dx\, dy.$

23 A cylindrical hole of radius b is drilled through the center of a sphere of radius a.

(a) Find the volume of the hole. Notice that this formula gives the volume of the sphere when $b = a$.

(b) Find the volume of the ring-shaped solid that remains. Express this volume in terms of the height h of the ring. Notice the remarkable fact that this volume depends only on h, and not on either the radius a of the sphere or the radius b of the hole.

*24 Find the centroid of the region enclosed by the loop of $r = a \cos 2\theta$ that lies in the first and fourth quadrants.

25 Find the centroid of the region enclosed by the cardioid $r = a(1 + \cos \theta)$.

*26 Find the centroid of the region enclosed by the right loop of the lemniscate $r^2 = 2a^2 \cos 2\theta$.

27 Find the centroid of the semicircular disk $x^2 + y^2 \leq a^2$, $y \geq 0$.

28 Find the volume of the solid cone $0 \leq z \leq h(a - r)/a$.

29 Find the volume of the solid under the cone $z = 2a - r$ whose base is bounded by the cardioid $r = a(1 + \cos \theta)$.

30 Find the volume cut out of the sphere $x^2 + y^2 + z^2 = 4a^2$ by the cylinder $x^2 + y^2 = 2ax$.

31 For the solid bounded by the xy-plane, the cylinder $x^2 + y^2 = a^2$, the paraboloid $z = b(x^2 + y^2)$ with $b > 0$, find (a) the volume, (b) the centroid.

32 Find the polar moment of inertia I_z of the circular plate bounded by $r = a$ if the density δ is constant. (Compare this very easy calculation with the work needed to solve the same problem using rectangular coordinates, in Problem 14 in Section 20.3.)

33 Find the polar moment of inertia I_z of a thin plate of constant density δ that has the shape of the circle $r = 2a \cos \theta$.

34 Solve Problem 33 for a plate that has the shape of the cardioid $r = a(1 + \cos \theta)$.

35 A thin plate is bounded by the circle $r = a$ and has density $\delta = a^2/(a^2 + r^2)$. Find its mass M and polar moment of inertia I_z.

36 The center of a circle of radius $2a$ lies on a circle of radius a. Find the centroid of the region between the two circles.

37 A thin plate of constant density δ has the shape of a circular sector of radius a and central angle 2α. Find the moment of inertia about the bisector of the angle.

38 Find the centroid of the circular sector described in Problem 37. Obtain the result of Problem 27 as a special case of this.

*39 Use the fact that $\int_0^\infty e^{-x^2}\, dx = \frac{1}{2}\sqrt{\pi}$ to show that

(a) $\displaystyle\int_0^\infty e^{-2x^2}\, dx = \frac{1}{4}\sqrt{2\pi};$

(b) $\displaystyle\int_0^\infty e^{-3x^2}\, dx = \frac{1}{6}\sqrt{3\pi};$

(c) $\int_0^\infty e^{-4x^2}\,dx = \frac{1}{4}\sqrt{\pi}$;

(d) $\int_0^{\pi/2} \frac{e^{-\tan^2 x}}{\cos^2 x}\,dx = \frac{1}{2}\sqrt{\pi}$;

(e) $\int_0^\infty \frac{e^{-x}}{\sqrt{x}}\,dx = \sqrt{\pi}$;

(f) $\int_0^\infty x^2 e^{-x^2}\,dx = \frac{1}{4}\sqrt{\pi}$;

(g) $\int_0^\infty \sqrt{x}\,e^{-x}\,dx = \frac{1}{2}\sqrt{\pi}$;

(h) $\int_0^1 \frac{dx}{\sqrt{-\ln x}} = \sqrt{\pi}$;

(i) $\int_0^1 \sqrt{-\ln x}\,dx = \frac{1}{2}\sqrt{\pi}$.

40 Use the method of Example 3 to evaluate

$$\int_0^\infty \int_0^\infty \frac{dx\,dy}{(1 + x^2 + y^2)^2}.$$

41 There is a slight difficulty with the calculation of I^2 in Example 3, because we have not discussed improper double integrals. In this problem we outline a somewhat less cavalier approach to formula (4). In Fig. 20.21 we show a quadrant of a circle of radius a, which is inside a square of side a, which in turn is inside a quadrant of a circle of radius $\sqrt{2}a$. Denote these regions by R_1, R_2, R_3.

(a) Show that

$$\iint_{R_1} e^{-r^2}\,dA = \frac{\pi}{4}(1 - e^{-a^2})$$

and

$$\iint_{R_3} e^{-r^2}\,dA = \frac{\pi}{4}(1 - e^{-2a^2}).$$

(b) Show that

$$\iint_{R_2} e^{-r^2}\,dA = \int_0^a \int_0^a e^{-(x^2+y^2)}\,dx\,dy = \left(\int_0^a e^{-x^2}\,dx\right)^2.$$

(c) Use (a) and (b) to show that

$$\frac{\pi}{4}(1 - e^{-a^2}) < \left(\int_0^a e^{-x^2}\,dx\right)^2 < \frac{\pi}{4}(1 - e^{-2a^2}).$$

(d) Use (c) to conclude that

$$\int_0^\infty e^{-x^2}\,dx = \frac{1}{2}\sqrt{\pi}.$$

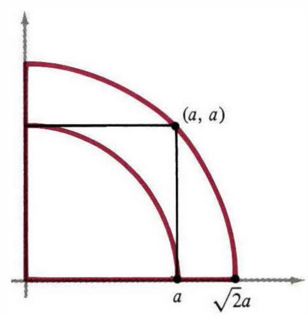


Figure 20.21

20.5 TRIPLE INTEGRALS

The definition of a triple integral follows the same pattern of ideas that was used to define a double integral in Section 20.2. We shall therefore confine ourselves to a very brief explanation.

A triple integral involves a function $f(x, y, z)$ defined on a three-dimensional region R. We divide R into many small rectangular boxes (and parts of boxes) by planes parallel to the coordinate planes, and we denote the volume of the kth box that lies wholly inside R by ΔV_k. Next, we evaluate the function at a point (x_k, y_k, z_k) in the kth box and form the product $f(x_k, y_k, z_k)\,\Delta V_k$. Finally, we form the sum of these products over all the boxes that lie inside R,

$$\sum_{k=1}^{n} f(x_k, y_k, z_k)\,\Delta V_k.$$

The triple integral of $f(x, y, z)$ over R is now defined to be the limit of these sums as n becomes infinite and the maximum diagonal of the boxes (that is, the longest diagonal of any of the boxes) approaches zero,

$$\iiint_R f(x, y, z)\,dV = \lim \sum_{k=1}^{n} f(x_k, y_k, z_k)\,\Delta V_k. \tag{1}$$

Sometimes we use the alternative notation

$$\iiint_R f(x, y, z)\, dx\, dy\, dz, \tag{2}$$

with no implication intended about the order of integration. This arises from the fact that since the volume of a box with faces parallel to the coordinate planes can be written as $\Delta V = \Delta x\, \Delta y\, \Delta z$, we have the element of volume formula

$$dV = dx\, dy\, dz. \tag{3}$$

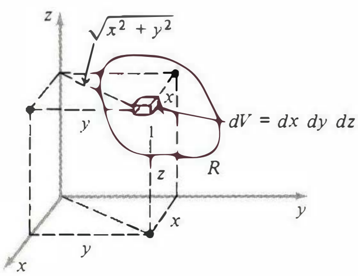


Figure 20.22

Figure 20.22 suggests the way the triple integral can be formed directly from the function $f(x, y, z)$ and the element of volume dV, in the manner explained in the previous two sections: that is, we multiply dV by $f(x, y, z)$ and integrate (or add together) the quantities $f(x, y, z)\, dV$ as the element of volume dV sweeps over the entire region R. As before, this way of thinking is merely a convenient abbreviation of the complex limit-of-sums process that constitutes the actual definition of the triple integral.

The main theoretical fact is that the triple integral (1) [or (2)] exists if $f(x, y, z)$ is continuous and the boundary of R is reasonably well behaved. We shall not pursue this issue any further. And the main practical fact is that triple integrals can often be calculated as iterated integrals.

Before we discuss iterated triple integrals, we quickly extend the ideas of Section 20.3 to the present context. First, if the region R is thought of as a solid body of variable density $\delta = \delta(x, y, z)$ [= mass per unit volume], then δdV is the element of mass—that is, the mass contained in the element of volume—and the total mass is

$$M = \iiint_R \delta dV.$$

Similar considerations lead to formulas for the moments with respect to the various coordinate planes, denoted by M_{yz}, M_{xz}, and M_{xy}; and also to formulas for the moments of inertia about the various axes, denoted by I_x, I_y, and I_z. These formulas (see Fig. 20.22) are

$$M_{yz} = \iiint_R x\ \delta dV, \qquad M_{xz} = \iiint_R y\ \delta dV, \qquad M_{xy} = \iiint_R z\ \delta dV;$$

and

$$I_x = \iiint_R (y^2 + z^2)\ \delta dV, \qquad I_y = \iiint_R (x^2 + z^2)\ \delta dV, \qquad I_z = \iiint_R (x^2 + y^2)\ \delta dV.$$

Also, the equations

$$\bar{x} = \frac{M_{yz}}{M}, \qquad \bar{y} = \frac{M_{xz}}{M}, \qquad \bar{z} = \frac{M_{xy}}{M},$$

define the center of mass of the body, or the centroid if δ is constant.

Just as we did with double integrals, we usually evaluate triple integrals by iteration. For example, if R is described by inequalities of the form

$$a \le x \le b, \qquad y_1(x) \le y \le y_2(x), \qquad z_1(x, y) \le z \le z_2(x, y),$$

as shown in Fig. 20.23, then

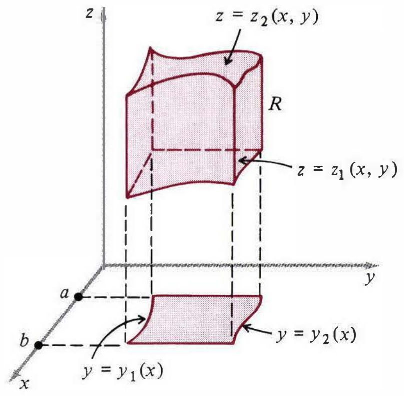


Figure 20.23

$$\iiint_R f(x, y, z)\, dV = \int_a^b \left[\int_{y_1(x)}^{y_2(x)} \left(\int_{z_1(x,y)}^{z_2(x,y)} f(x, y, z)\, dz \right) dy \right] dx.$$

We usually omit the parentheses and brackets, and write this in the form

$$\int_a^b \int_{y_1(x)}^{y_2(x)} \int_{z_1(x,y)}^{z_2(x,y)} f(x, y, z)\, dz\, dy\, dx.$$

As always, we integrate from the inside out, here integrating first with respect to z, then with respect to y, and finally with respect to x. Other orders of integration are often possible, and the order we choose in any specific problem is determined by a little foresight and our preference for easy calculations over hard ones.

Example 1 Find the centroid of the tetrahedron bounded by the coordinate planes and the plane $x + y + z = 1$.

Solution We can treat the tetrahedron (Fig. 20.24) as a solid of density $\delta = 1$ so that mass equals volume. By geometry the volume of the tetrahedron is $V = \frac{1}{6}$, and $\bar{z}$ is defined by

$$\bar{z} = \frac{1}{V} \iiint_R z\, dV.$$

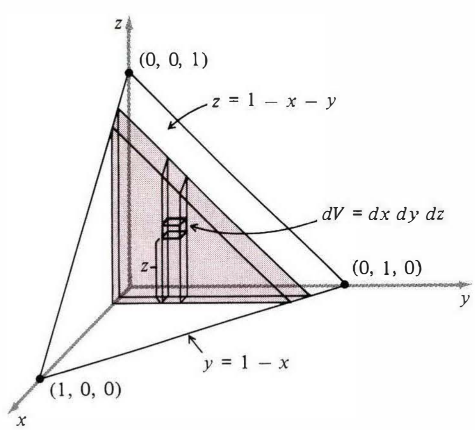


Figure 20.24

If we integrate first z, then y, then x, this means that we must write

$$\bar{z} = \frac{1}{V} \iiint_R z\,dz\,dy\,dx,$$

with suitable limits of integration inserted. To find the z-limits we use the indicated equation of the slanting plane and imagine that the element of volume shown in the figure—like an elevator car in an elevator shaft—moves up from $z = 0$ to $z = 1 - x - y$. Next, the resulting column generates a slice by moving across the solid from left to right, from $y = 0$ to $y = 1 - x$. And finally, the slice moves through the solid from back to front, from $x = 0$ to $x = 1$. Thus,

$$\begin{aligned}
\bar{z} &= \frac{1}{\frac{1}{6}} \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z\,dz\,dy\,dx = 6 \int_0^1 \int_0^{1-x} \left[\frac{1}{2} z^2\right]_0^{1-x-y} dy\,dx \\
&= 3 \int_0^1 \int_0^{1-x} (1 - x - y)^2\,dy\,dx = 3 \int_0^1 \left[-\frac{1}{3}(1 - x - y)^3\right]_0^{1-x} dx \\
&= \int_0^1 (1 - x)^3\,dx = -\frac{1}{4}(1 - x)^4\Big]_0^1 = \frac{1}{4}.
\end{aligned}$$

By the symmetry of the situation we see that the centroid is the point $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. We could also have found $\bar{z}$ by integrating in any other order, for instance, first x, then y, then z,

$$\bar{z} = \frac{1}{\frac{1}{6}} \int_0^1 \int_0^{1-z} \int_0^{1-y-z} z\,dx\,dy\,dz,$$

where the limits of integration are determined as they are above, that is, by examining the figure. Students should verify that this integral gives the same result as before.

Example 2 Use a triple integral to find the volume of the sphere $x^2 + y^2 + z^2 = a^2$.

Solution The total volume is 8 times the volume in the first octant, so by integrating in the order z, y, x (see Fig. 20.25) we have

$$\begin{aligned}
V &= 8 \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dz\,dy\,dx \\
&= 8 \int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2 - y^2}\,dy\,dx. \qquad (4)
\end{aligned}$$

To calculate the inner integral here, we use the method of trigonometric substitution with $y = A \sin \theta$, $dy = A \cos \theta\,d\theta$ to obtain the auxiliary formula

$$\begin{aligned}
\int_0^A \sqrt{A^2 - y^2}\,dy &= A^2 \int_0^{\pi/2} \cos^2 \theta\,d\theta = \frac{1}{2} A^2 \int_0^{\pi/2} (1 + \cos 2\theta)\,d\theta \\
&= \frac{1}{2} A^2 \left[\theta + \frac{1}{2} \sin 2\theta\right]_0^{\pi/2} = \frac{1}{4} \pi A^2.
\end{aligned}$$

With $A = \sqrt{a^2 - x^2}$, this enables us to write (4) as

$$V = 8 \int_0^a \frac{1}{4} \pi(a^2 - x^2)\,dx = 2\pi \left[a^2 x - \frac{1}{3} x^3\right]_0^a = \frac{4}{3} \pi a^3,$$

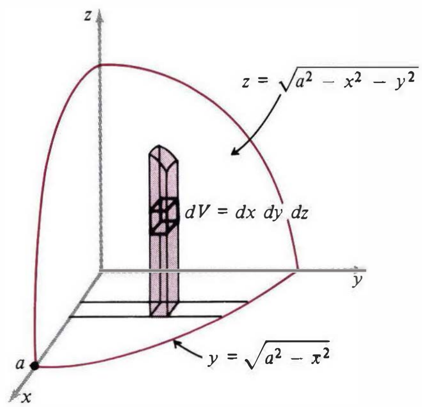


Figure 20.25

and the calculation is complete. Of course, we are thoroughly familiar with this result, which we have already obtained by a number of different methods. Our purpose here is to provide another illustration of the technique of triple integration.

PROBLEMS

In Problems 1–10, evaluate the given iterated integral.

1 $\displaystyle\int_0^1 \int_0^{x^2} \int_0^{xy^3} 18x^3y^2z\, dz\, dy\, dx.$

2 $\displaystyle\int_0^1 \int_{y^2}^1 \int_0^{1-x} x\, dz\, dx\, dy.$

3 $\displaystyle\int_0^a \int_0^b \int_0^c \sin\frac{\pi x}{a}\, dz\, dy\, dx.$

4 $\displaystyle\int_0^1 \int_0^{1-y} \int_0^{x^2+y^2} y\, dz\, dx\, dy.$

5 $\displaystyle\int_0^2 \int_0^\pi \int_0^{\ln 4} x^3 \cos\frac{y}{2}\, e^z\, dz\, dy\, dx.$

6 $\displaystyle\int_0^1 \int_0^{1-x} \int_0^{2-x} xyz\, dz\, dy\, dx.$

7 $\displaystyle\int_0^2 \int_0^{\sqrt{4-z^2}} \int_{y^2+z^2-4}^{4-y^2-z^2} dx\, dy\, dz.$

8 $\displaystyle\int_0^1 \int_0^{\sqrt{3}z} \int_0^{\sqrt{3(y^2+z^2)}} xyz\sqrt{x^2+y^2+z^2}\, dx\, dy\, dz.$

9 $\displaystyle\int_0^2 \int_0^{\sqrt{4-y^2}} \int_0^{4-x^2-y^2} y\, dz\, dx\, dy.$

10 $\displaystyle\int_0^1 \int_1^{2y} \int_0^x (x+2z)\, dz\, dx\, dy.$

11 Change the order of integration by putting suitable limits on the right side:

$$\int_0^a \int_0^x \int_0^y f(x, y, z)\, dz\, dy\, dx = \iiint f(x, y, z)\, dx\, dy\, dz.$$

12 Evaluate both integrals in Problem 11 if $f(x, y, z) = 1$.

13 Evaluate both integrals in Problem 11 if $f(x, y, z) = x$.

14 Evaluate both integrals in Problem 11 if $f(x, y, z) = yz$.

15 Change the order of integration by putting suitable limits on the right side:

$$\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^{1-x^2-y^2} f(x, y, z)\, dz\, dy\, dx = \iiint f(x, y, z)\, dx\, dy\, dz.$$

16 Same directions as Problem 15:

$$\int_0^6 \int_0^{6-x} \int_0^{6-x-y} f(x, y, z)\, dz\, dy\, dx = \iiint f(x, y, z)\, dx\, dy\, dz.$$

In Problems 17–24, use triple integration to find the volumes of the given regions.

17 The region in the first octant bounded by the cylinder $x = 4 - y^2$ and the planes $y = z$, $x = 0$, $z = 0$.

18 The region above the xy-plane bounded by the surfaces $z^2 = 16y$, $z^2 = y$, $y = x$, $y = 4$, and $x = 0$.

19 The region bounded by the paraboloids $z = x^2 + 9y^2$ and $z = 18 - x^2 - 9y^2$.

20 The region bounded by the paraboloids $z = 8 - x^2 - y^2$ and $z = x^2 + 3y^2$.

21 The region bounded by the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

22 The region bounded by the cylinder $z = 4 - y^2$ and the paraboloid $z = x^2 + 3y^2$.

23 The tetrahedron bounded by the coordinate planes and the plane
$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1,$$
where a, b, c are positive numbers.

24 The region bounded by the cylinder $x^2+y^2=4x$, the xy-plane, and the paraboloid $4z=x^2+y^2$.

25 The density of a cube is proportional to the square of the distance from one corner. Show that the mass is what it would be if the density were constant and equal to the original density at another corner adjacent to the first.

26 If the density $\delta=xy$, find the moment with respect to the xy-plane of the part of the sphere $x^2+y^2+z^2\le a^2$ that lies in the first octant.

27 The cube bounded by the coordinate planes and the planes $x=a$, $y=a$, $z=a$ has density $\delta=cz$ where c is a constant. Find its moment of inertia I_z about the z-axis.

***28** Show that
$$\int_0^a\int_0^b\int_0^c \cos(x+y+z)\,dz\,dy\,dx$$
$$=8\sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2}\cos\frac{a+b+c}{2}.$$

***29** Show that the four-dimensional "sphere" $x^2+y^2+z^2+u^2=a^2$ has volume
$$V=16\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}}\int_0^{\sqrt{a^2-x^2-y^2-z^2}}du\,dz\,dy\,dx$$
$$=\frac{1}{2}\pi^2a^4.$$
Hint: Notice that the inner triple integral is the volume of the first octant of a three-dimensional sphere of radius $\sqrt{a^2-x^2}$.

***30** Use the result of Problem 29 to find the volume of the five-dimensional "sphere" $x^2+y^2+z^2+u^2+v^2=a^2$.

20.6 CYLINDRICAL COORDINATES

If a solid has axial symmetry—that is, symmetry about a line in space—it is often convenient to place its axis of symmetry on the z-axis and use cylindrical coordinates r, θ, z (Fig. 20.26) for the calculation of triple integrals. Instead of the element of volume in rectangular coordinates,
$$dV=dx\,dy\,dz,$$
we use the element of volume in cylindrical coordinates,
$$dV=r\,dr\,d\theta\,dz. \tag{1}$$

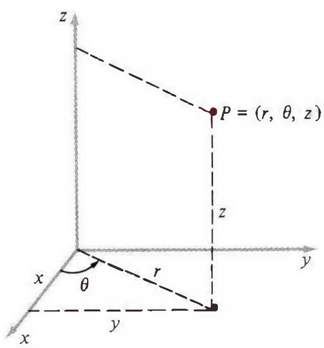


Figure 20.26 Cylindrical coordinates.

It is easy to understand this formula by starting at a point (r,θ,z) and giving the coordinates small increments dr, $d\theta$, dz. These increments sweep out a small cell in space which is approximately a rectangular box with edges $r\,d\theta$, dr, and dz, as shown in Fig. 20.27, and dV as given by (1) is simply the product of these edges. Triple integrals now have the form
$$\iiint_R f(x,y,z)\,dV=\iiint_R f(r\cos\theta,r\sin\theta,z)\,r\,dr\,d\theta\,dz.$$

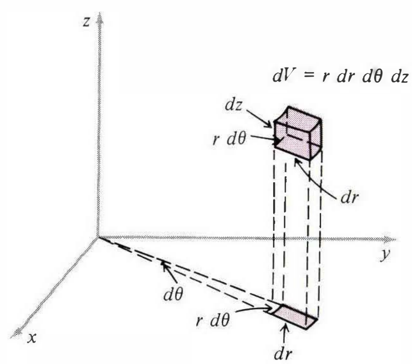


Figure 20.27 The cylindrical element of volume.

We can often calculate such an integral by writing it as an iterated integral, in the manner illustrated in the following examples.

Example 1 Use a triple integral in cylindrical coordinates to find the moment of inertia of a uniform solid cylinder of height h, base radius a, and mass M about its axis.

Solution Place the cylinder in the position shown in Fig. 20.28. The word "uniform" in this context means that the density δ is constant. The mass contained in dV is δdV, and the moment of inertia of this mass about the z-axis is $r^2\ \delta dV$. The total moment of inertia of the cylinder about its axis is therefore

$$\iiint_R r^2\ \delta dV = \iiint_R r^2\ \delta\ r\ dr\ d\theta\ dz$$

$$= \delta \int_0^{2\pi} \int_0^a \int_0^h r^3\ dz\ dr\ d\theta$$

$$= \delta h \int_0^{2\pi} \int_0^a r^3\ dr\ d\theta = \delta h \cdot \frac{1}{4} a^4 \int_0^{2\pi} d\theta$$

$$= \delta \cdot \frac{1}{2} \pi a^4 h = \frac{1}{2} Ma^2,$$

since $M = \delta \cdot \pi a^2 h$. That the limits on these integrals are all constants is a consequence of the fact that cylindrical coordinates are perfectly suited to this problem.

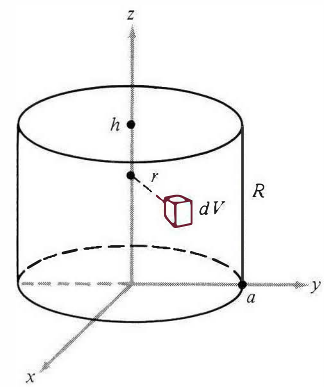


Figure 20.28

Example 2 Use a triple integral in cylindrical coordinates to find the volume of the sphere $x^2 + y^2 + z^2 = a^2$.

Solution The cylindrical equation of the sphere is $r^2 + z^2 = a^2$, so the equation of the upper hemisphere is $z = \sqrt{a^2 - r^2}$. We multiply the volume above the xy-plane by 2, and find this volume by integrating in the order z, r, θ, as suggested in Fig. 20.29:

$$V = 2 \int_0^{2\pi} \int_0^a \int_0^{\sqrt{a^2-r^2}} r\ dz\ dr\ d\theta = 2 \int_0^{2\pi} \int_0^a r\sqrt{a^2 - r^2}\ dr\ d\theta$$

$$= 2 \int_0^{2\pi} \left[-\frac{1}{3}(a^2 - r^2)^{3/2} \right]_0^a d\theta = 2 \int_0^{2\pi} \frac{1}{3} a^3\ d\theta = \frac{4}{3} \pi a^3.$$

Of course, we obtain the same result as in Example 2 in Section 20.5, but the calculation is much easier here because cylindrical coordinates are better than rectangular coordinates for working with spheres.

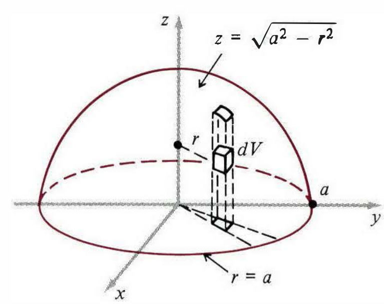


Figure 20.29

Example 3 Find the moment of inertia of a uniform solid sphere of radius a and mass M about a diameter.

Solution We may assume that our present sphere occupies the region bounded by the sphere $r^2 + z^2 = a^2$ in Example 2. If the constant density is denoted by δ, then the moment of inertia about the z-axis is

$$I_z = 2\int_0^{2\pi}\int_0^a\int_0^{\sqrt{a^2-r^2}} r^2 \cdot \delta r\,dz\,dr\,d\theta = 2\delta\int_0^{2\pi}\int_0^a r^3\sqrt{a^2-r^2}\,dr\,d\theta$$

$$= \delta \cdot 4\pi\int_0^a r^3\sqrt{a^2-r^2}\,dr.$$

(In the last step here we integrated out of the indicated order for the purpose of disposing of the simple θ-integral so that we could concentrate our attention on the harder r-integral. Students will become accustomed to this type of shortcut.) To evaluate this integral we use the substitution $r = a \sin \phi$, $dr = a \cos \phi\, d\phi$ to write

$$\int r^3\sqrt{a^2-r^2}\,dr = a^5\int \sin^3\phi\cos^2\phi\,d\phi$$

$$= a^5\int (\cos^2\phi - \cos^4\phi)\sin\phi\,d\phi$$

$$= a^5\left(\frac{1}{5}\cos^5\phi - \frac{1}{3}\cos^3\phi\right).$$

This gives

$$I_z = \delta \cdot 4\pi a^5\left[\frac{1}{5}\cos^5\phi - \frac{1}{3}\cos^3\phi\right]_0^{\pi/2} = \delta \cdot \frac{8}{15}\pi a^5 = \frac{2}{5}Ma^2,$$

since $M = \delta \cdot \frac{4}{3}\pi a^3$.

PROBLEMS

Use cylindrical coordinates to solve the following problems.

1 Find the volume of the solid bounded above by the paraboloid $z = 1 - x^2 - y^2$ and below by the xy-plane.

2 Find the mass of the solid in Problem 1 if the density is
(a) proportional to the distance from the xy-plane, $\delta = cz$;
(b) proportional to the distance from the z-axis, $\delta = cr$;
(c) proportional to the square of the distance from the origin, $\delta = c(r^2 + z^2)$.

3 A uniform solid cone of height h and base radius a rests on the xy-plane with its vertex on the positive z-axis. Find its center of mass.

4 If the mass of the cone in the preceding problem is M, find its moment of inertia I_z about the z-axis
(a) by integrating first with respect to z;
(b) by integrating first with respect to r.

5 A cylindrical hole of radius b is bored through the center of a uniform solid sphere of radius a. If the density is denoted by δ, find the mass of the ring-shaped solid that remains, and also its moment of inertia about the axis of the hole. Notice that this result generalizes the result of Example 3.

6 A wedge is cut from a uniform solid cylinder of radius a by a plane tangent to the base and inclined at a 45° angle to the base. Find its moment of inertia about the axis of the cylinder.

7 A uniform solid cone has height h, radius of base a, and mass M. Find its moment of inertia about an axis through the vertex and parallel to the base. Hint: Let the cone have its vertex at the origin and its axis on the z-axis, and find I_x.

8 A uniform solid cone has height h, radius of base a, and mass M. Find its moment of inertia about a diameter of the base. Hint: Let the cone have its base in the xy-plane and its axis on the z-axis, and find I_x.

9 A uniform solid hemisphere is bounded above by the sphere $x^2 + y^2 + z^2 = a^2$ and below by the xy-plane. Find its center of mass. (The result of this problem is a theorem of Archimedes.)

10 Find the mass of a cylindrical solid of height h and base radius a if the density at a point is proportional to the distance from the axis of the cylinder.

11 A cylindrical hole of radius a is bored through the center of a solid sphere of radius $2a$. Find the volume of the hole.

12 Find the volume of the region bounded above by the plane $z = 2x$ and below by the paraboloid $z = x^2 + y^2$.

13 Find the volume of the solid bounded above by the plane $z = x$ and below by the paraboloid $z = x^2 + y^2$.

14 Find the mass of the solid in Problem 13 if the density at each point is proportional to the square of the distance from the z-axis.

15 Find the volume of the region bounded above by the plane $z = x + y$, below by the xy-plane, and on the sides by the cylinder $x^2 + y^2 = a^2$ and the planes $x = a$, $y = a$.

*16 Find the volume of the region bounded above by the paraboloid $z = x^2 + y^2$, below by the xy-plane, and on the side by the hyperboloid

$$x^2 + y^2 = 1 + \frac{z^2}{4}.$$

17 Find the volume of the region bounded above and below by the sphere $x^2 + y^2 + z^2 = 4a^2$ and on the side by the cylinder $(x - a)^2 + y^2 = a^2$.

*18 If the region in Problem 17 is filled with matter of constant density $\delta = 1$, find the moment of inertia of this solid about the z-axis.

19 Find the moment of inertia of a uniform solid cylinder of radius a and mass M about a generator. Hint: Place the cylinder so that a generator lies on the z-axis.

20 Find the volume of the region bounded above by the sphere $x^2 + y^2 + z^2 = 2a^2$ and below by the paraboloid $az = x^2 + y^2$.

*21 Find the moment of inertia of a uniform solid sphere of radius a and mass M about a tangent line. Hint: Place the sphere with its center at the origin and let the tangent line be the line of intersection of the planes $x = a$, $y = 0$.

22 Find the volume of the region inside the cylinder $r = a \sin\theta$ which is bounded above by the sphere $x^2 + y^2 + z^2 = a^2$ and below by the upper half of the ellipsoid $x^2/a^2 + y^2/a^2 + z^2/b^2 = 1$ where $b < a$.

23 Find the volume of the region bounded above by the sphere $x^2 + y^2 + z^2 = a^2$ and below by the cone $z = r \cot\alpha$. Use this result to find the volume of a hemisphere of radius a.

*24 Find the volume of the spherical segment of height h which is cut from a sphere of radius a by a plane at a distance $a - h$ from the center.

20.7 SPHERICAL COORDINATES. GRAVITATIONAL ATTRACTION

Just as cylindrical coordinates help us deal with problems involving symmetry about a line, spherical coordinates are designed to fit situations with symmetry about a point, as in the case of a solid sphere whose density is proportional to the distance from its center. We became acquainted with the spherical coordinates ρ, ϕ, θ (see Fig. 20.30) in Section 18.7. We now put them to use in the calculation of certain triple integrals.

In order to express a triple integral

$$\iiint_R f(x, y, z)\, dV$$

in spherical coordinates, we need to be able to write x, y, z as functions of ρ, ϕ, θ. This is easy to do by simply looking at Fig. 20.30:

$$z = \rho\cos\phi,$$

$$r = \rho\sin\phi,$$

$$x = \rho\sin\phi\cos\theta,$$

$$y = \rho\sin\phi\sin\theta.$$

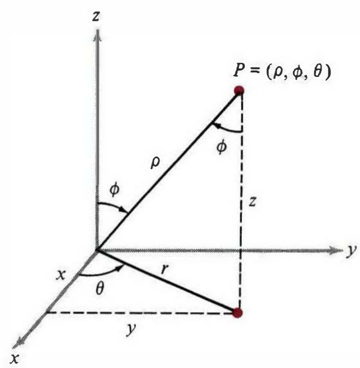


Figure 20.30 Spherical coordinates.

We must now find a formula for the element of volume dV in terms of ρ, ϕ, θ. To do this we start at a point $P = (\rho, \phi, \theta)$ and give small increments $d\rho$, $d\phi$, $d\theta$ to its spherical coordinates. As we see in Fig. 20.31, the displacement of P in the ρ-direction has length $d\rho$, that in the ϕ-direction has length $\rho\, d\phi$, and that in the θ-direction has length $\rho \sin\phi\, d\theta$. These three lengths are the edges of the "spherical box" shown in the figure, so the volume of this box is $(d\rho)\,(\rho\, d\phi)\,(\rho \sin\phi\, d\theta)$ and we have

$$dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta.$$

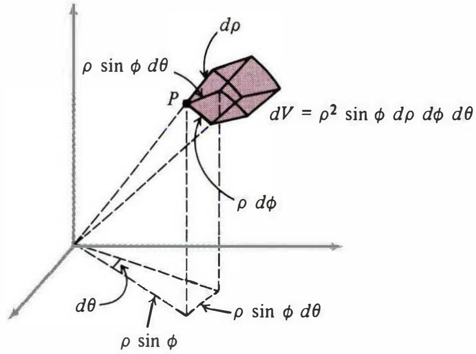


Figure 20.31 The spherical element of volume.

To calculate a triple integral in spherical coordinates we therefore write

$$\iiint_R f(x, y, z)\, dV = \iiint_R f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi)\rho^2 \sin \phi \, d\rho \, d\phi \, d\theta.$$

In any particular problem we try to express this as an iterated integral in such a way that dV sweeps over the region R in a convenient manner. In most cases the nature of the region R will suggest an appropriate order of integration, together with corresponding limits of integration.

Example 1 Use a triple integral in spherical coordinates to find the volume of the sphere $x^2 + y^2 + z^2 = a^2$.

Solution The equation of this sphere in spherical coordinates is $\rho = a$. We calculate the integral

$$V = \iiint_R dV = \iiint_R \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

by integrating in the order ρ, ϕ, θ. The first integration, as ρ increases from 0 to a, adds the elements of volume dV to give the volume of the "spike" shown in Fig. 20.32; the second, as ϕ increases from 0 to π, adds the volumes of these spikes to give the volume of the wedge in the figure; and the third, as θ increases from 0 to 2π, adds the volumes of these wedges around the z-axis to give the volume of the entire sphere. The actual calculation is

$$\begin{aligned} V &= \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\ &= \left[\int_0^a \rho^2 \, d\rho\right]\left[\int_0^{\pi} \sin \phi \, d\phi\right]\left[\int_0^{2\pi} d\theta\right] \\ &= \frac{1}{3} a^3 \cdot 2 \cdot 2\pi = \frac{4}{3}\pi a^3, \end{aligned}$$

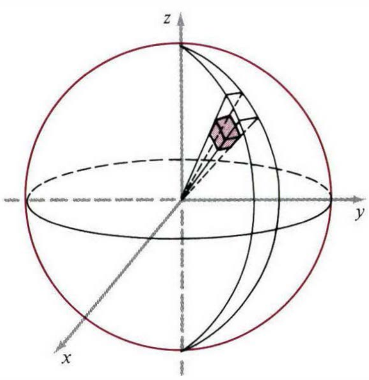


Figure 20.32

as expected. This problem is perfectly suited to spherical coordinates, as we see from the simplicity of this calculation compared with those given in the corresponding examples in Sections 20.5 and 20.6.

Example 2 Find the centroid of the region bounded by the sphere $\rho = a$ and the cone $\phi = \alpha$.

Solution This region (Fig. 20.33) is shaped like a filled ice cream cone. Its volume is

$$V = \int_0^{2\pi}\int_0^{\alpha}\int_0^{a} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$
$$= \frac{1}{3}a^3 \cdot 2\pi \int_0^{\alpha} \sin\phi \, d\phi$$
$$= \frac{2}{3}\pi a^3(1 - \cos\alpha).$$

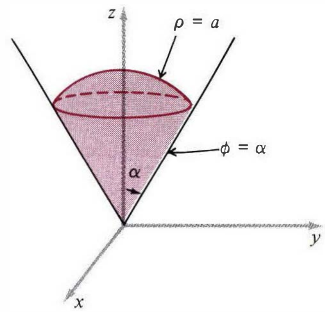


Figure 20.33

As a check, this gives $\frac{4}{3}\pi a^3$ as the volume of the sphere when $\alpha = \pi$. Now for the centroid. It is clear by symmetry that $\bar{x} = \bar{y} = 0$. To find $\bar{z}$, we must first find the moment of the region with respect to the xy-plane,

$$M_{xy} = \iiint_R z \, dV = \int_0^{2\pi}\int_0^{\alpha}\int_0^{a} (\rho\cos\phi)\, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$
$$= \frac{1}{2}\pi a^4 \int_0^{\alpha} \sin\phi\cos\phi \, d\phi$$
$$= \frac{1}{4}\pi a^4 \sin^2\alpha.$$

Finally, we have

$$\bar{z} = \frac{M_{xy}}{V} = \frac{3}{2\pi a^3(1-\cos\alpha)} \cdot \frac{1}{4}\pi a^4 \sin^2\alpha = \frac{3}{8}a(1+\cos\alpha).$$

When $\alpha = \pi/2$ this specializes to $\bar{z} = \frac{3}{8}a$, which is the result of Problem 9 in Section 20.6.

In our next example we discuss an idea with important implications for several branches of physical science.

Example 3 *The gravitational attraction of a thin spherical shell.* Suppose that matter of total mass M is uniformly distributed on the surface of a sphere of radius a centered at the origin (Fig. 20.34). Show that the gravitational force $\mathbf{F}$ exerted by this thin spherical shell on a particle of mass m located at a point (0, 0, b), with $b > a$, is exactly what it would be if all the mass of the shell were concentrated at its center. That is, show that

$$|\mathbf{F}| = G\frac{Mm}{b^2}, \tag{1}$$

where G is the constant of gravitation.

Solution By symmetry it is clear that the vector $\mathbf{F}$ is directed downward, so $\mathbf{F} = F_z\mathbf{k}$ where F_z is negative. By Fig. 20.31 the element of area on the surface of the sphere is

$$dA = a^2 \sin\phi \, d\phi \, d\theta; \tag{2}$$

and since the mass per unit area on the surface is $M/4\pi a^2$, the mass contained in dA is

$$dM = \frac{M}{4\pi}\sin\phi \, d\phi \, d\theta.$$

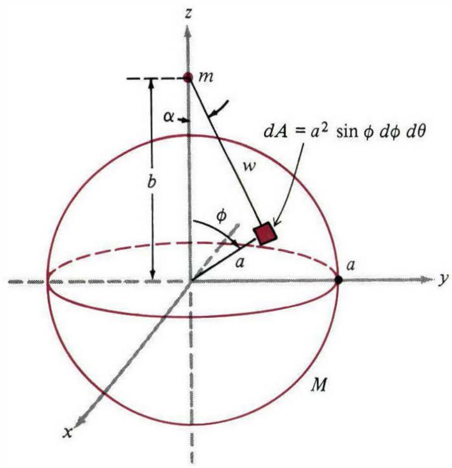


Figure 20.34

Newton's law of gravitation states that the magnitude of the force this element of mass exerts on m (see Fig. 20.34) is

$$G\frac{dM \cdot m}{w^2} = \frac{GMm}{4\pi w^2} \sin \phi \, d\phi \, d\theta,$$

with downward component

$$G\frac{dM \cdot m}{w^2} \cos \alpha = \frac{GMm}{4\pi w^2} \cos \alpha \sin \phi \, d\phi \, d\theta.$$

We now find the magnitude of the total force the shell exerts on m by integrating this expression over the surface of the sphere,

$$|\mathbf{F}| = \int_0^{2\pi} \int_0^{\pi} \frac{GMm}{4\pi w^2} \cos \alpha \sin \phi \, d\phi \, d\theta$$

$$= \frac{GMm}{2} \int_0^{\pi} \frac{1}{w^2} \cos \alpha \sin \phi \, d\phi. \tag{3}$$

To calculate this integral we change the variable of integration from ϕ to w and integrate from $w = b - a$ to $w = b + a$ (see the figure). The reason for this strategy will become clear as we proceed. To accomplish the necessary transformation of the integral in (3), we first use the law of cosines to write

$$w^2 = a^2 + b^2 - 2ab \cos \phi, \tag{4}$$

so

$$2w \, dw = 2ab \sin \phi \, d\phi$$

or

$$\sin \phi \, d\phi = \frac{w \, dw}{ab}. \tag{5}$$

To write $\cos \alpha$ as a function of w, we use the fact that

$$w \cos \alpha + a \cos \phi = b$$

or

$$\cos \alpha = \frac{b - a \cos \phi}{w}.$$

With the aid of (4), this becomes

$$\cos\alpha = \frac{b - [(a^2 + b^2 - w^2)/2b]}{w} = \frac{b^2 - a^2 + w^2}{2bw}. \tag{6}$$

When (5) and (6) are substituted in (3), we obtain

$$\begin{aligned}|\mathbf{F}| &= \frac{GMm}{2}\int_{b-a}^{b+a} \frac{1}{w^2}\left(\frac{b^2 - a^2 + w^2}{2bw}\right)\frac{w\,dw}{ab} \\ &= \frac{GMm}{4ab^2}\int_{b-a}^{b+a}\left(\frac{b^2 - a^2}{w^2} + 1\right)dw. \end{aligned}\tag{7}$$

The value of the integral here is

$$\begin{aligned}\left[-\frac{(b^2 - a^2)}{w} + w\right]_{b-a}^{b+a} &= [-(b - a) + (b + a) + (b + a) - (b - a)] \\ &= 4a,\end{aligned}$$

so (7) becomes

$$|\mathbf{F}| = \frac{GMm}{4ab^2}\cdot 4a = G\frac{Mm}{b^2},$$

and the proof of (1) is complete.

The conclusion reached in this example implies one of Newton's greatest theorems in mathematical astronomy: *Under the inverse square law of gravitation, a uniform solid sphere attracts an outside particle as if its mass were concentrated at its center*; for such a sphere can be thought of as if it were composed of a great many concentric thin spherical shells, like the layers of an onion, and each shell attracts in this way. Indeed, our discussion proves even more, namely, that the same statement holds for a solid sphere of variable density, provided that the density depends only on the distance from the center. Newton's theorem shows that in computing the mutual gravitational attraction of various bodies in the solar system, like the sun, the earth, and the moon, it is legitimate to replace these huge bodies by equal point masses—that is, particles—located at their centers. It is believed by some historians of science that Newton delayed the publication of his theory of the solar system for 20 years until he was able to prove this theorem.

PROBLEMS

Use spherical coordinates to solve the following problems.

1 If the region in Example 2 is filled with matter of constant density δ, find the moment of inertia of the resulting solid about the z-axis. Use this result to show that the moment of inertia of a uniform solid sphere of radius a and mass M about a diameter is $\frac{2}{5}Ma^2$

2 In Example 2, $\bar{z} \to \frac{3}{4}a$ as $\alpha \to 0$. Explain this, in view of the fact that the region approaches a line segment as $\alpha \to 0$ and the centroid of a line segment is its midpoint.

3 Find the volume of the torus $\rho = 2a \sin\phi$ (see Fig. 18.42).

4 If $0 < b < a$ and $0 < \alpha < \pi$, find the volume of the region bounded by the concentric spheres $\rho = b$, $\rho = a$ and the cone $\phi = \alpha$.

5 Find the centroid of the hemispherical shell $0 < b \le \rho \le a$, $z \ge 0$.

6 Find the moment of inertia about the z-axis of the shell in Problem 5 if it is a solid of constant density δ.

7 A wedge is cut from a solid sphere of radius a by two planes that intersect on a diameter. If α is the angle between the planes, find the volume of the wedge.

8 Find the mass of a solid sphere of radius a if the density at each point equals the distance from the surface.

*9 Use a triple integral (in spherical coordinates) to verify that the volume of a cone of height h and base radius r is $\frac{1}{3}\pi r^2 h$.

10 If the density of a solid sphere of radius a is proportional to the distance from the center, $\delta = c\rho$, show that its mass is $c\pi a^4$.

11 Let n be a nonnegative constant, and consider a solid sphere of radius a centered at the origin whose density is proportional to the nth power of the distance from the center, $\delta = c\rho^n$.
(a) Find the moment of inertia of this sphere about the z-axis.
(b) Show how the result obtained in (a) yields the conclusion that the moment of inertia, about a diameter, of a uniform solid sphere of radius a and mass M is $\frac{2}{5}Ma^2$.

12 In Problem 11, allow the exponent n to be negative and determine what restriction must be placed on n if the mass of the sphere is to be finite. Hint: Find the mass between concentric spheres $\rho = b$ and $\rho = a$ with $0 < b < a$, and then let $b \to 0$.

13 Sketch the region bounded by the surface $\rho = a(1 - \cos\phi)$, and find its volume.

14 Find the mass of a solid sphere of radius a centered at the origin if the density at a point P equals the product of the distances from P to the origin and to the z-axis.

15 Consider a solid sphere of radius a centered at the origin with variable density $\delta = \delta(\rho, \phi, \theta)$.
(a) Set up an iterated integral for the mass M with the integrations in the order θ, ϕ, ρ.
(b) Simplify the integral in (a) as much as possible for the special case in which the density is a function of ρ alone, say $\delta(\rho, \phi, \theta) = f(\rho)$.
(c) Show that the formula in (b) can be obtained directly by using thin spherical shells, without any use of iterated integrals.

16 Apply formula (2) to find the area of the polar cap on a sphere of radius a which is defined by $0 \le \phi \le \alpha$, and use this result to find the total surface area of the sphere.

17 In Example 3, assume that the particle m lies inside the spherical shell, so that $b < a$, and show that in this case the integral in (7) has the value zero. This proves the remarkable fact that a uniform thin spherical shell of matter exerts no gravitational force whatever on bodies located inside its cavity. Further, the same conclusion is also true for any nonthin spherical shell in which matter of variable density fills the space between two concentric spheres, provided that the density depends only on the distance from their common center.

18 Assume that the earth is spherical and of constant density, and imagine that a small tunnel is bored through the center. Neglecting the effect of this removal of matter, show that the gravitational attraction of the earth on a particle in the tunnel is *directly* proportional to the distance from the particle to the center of the earth. Is this necessarily true if the density is variable but depends only on the distance from the center?

19 Assume that the region discussed in Example 2 is filled with matter of constant density δ, and find the gravitational attraction it exerts on a particle of mass m placed at the origin.

20 If the rounded top is cut off the solid in Problem 19, leaving a cone of height $h = a\cos\alpha$, what now is the gravitational attraction exerted on a particle of mass m placed at the origin?

21 Assume that matter of constant density δ (= mass per unit area) is spread over the entire xy-plane and that a particle of mass m is located at the point $(0, 0, b)$ on the z-axis. Show that the gravitational attraction exerted by the planar mass on the particle is given by the following improper integral in polar coordinates,

$$\iint_R \frac{GM\delta b}{(r^2 + b^2)^{3/2}}\, r\, dr\, d\theta,$$

where R is the entire xy-plane. Evaluate this integral by computing it over a circle of radius a centered at the origin and then letting $a \to \infty$. Notice the remarkable fact that the value of this integral does not depend on b, so that the attractive force of the infinite plane on the particle is independent of the distance from the plane. Why is this obvious without calculation?

20.8 AREAS OF CURVED SURFACES. LEGENDRE'S FORMULA

In Section 7.6 we discussed the problem of finding the area of a surface of revolution. We now consider the area problem for more general curved surfaces, specifically, those that have equations of the form

$$z = f(x, y),$$

where both partial derivatives $f_x(x, y)$ and $f_y(x, y)$ are continuous functions.

The method we describe rests on the simple fact that if two planes intersect at an angle γ (see Fig. 20.35), then all areas in one plane are multiplied by $\cos\gamma$ when projected on the other,

$$A = S\cos\gamma.$$

This is clearly true for the area of a rectangle with one side parallel to the line of intersection of the planes, and it follows for other regions by a limiting process. In just the same way, we project an element of surface area dS down from the given curved surface $z = f(x, y)$ onto an element of area dA in the xy-plane, as shown in Fig. 20.36. Here we have

$$dA = dS \cos \gamma,$$

where γ is the angle between the vertical line in the figure and the upward-pointing normal to the surface. This equation yields

$$dS = \frac{dA}{\cos \gamma},$$

so the total area of the curved surface is given by the formula

$$S = \iint dS = \iint_R \frac{dA}{\cos \gamma}, \tag{1}$$

where R is the region in the xy-plane that lies under the part of the surface $z = f(x, y)$ whose area we wish to find. The element of area dA in Fig. 20.36 is drawn without any special shape, because the double integral (1) is sometimes used with rectangular coordinates and sometimes with polar coordinates.

In order to make (1) into a practical tool for actual calculations, we need a formula for $\cos \gamma$. We find this formula from the fact that the vector $f_x\mathbf{i} + f_y\mathbf{j} - \mathbf{k}$ is normal to the surface, as we saw in Section 19.3. This particular normal vector points downward, because its $\mathbf{k}$-component is negative. If we reverse the direction and divide by the length, then we see that the vector

$$\mathbf{u} = \frac{-f_x\mathbf{i} - f_y\mathbf{j} + \mathbf{k}}{\sqrt{f_x^2 + f_y^2 + 1}}$$

is the upward-pointing unit normal, and therefore $\cos \gamma$ is its $\mathbf{k}$-component,

$$\cos \gamma = \frac{1}{\sqrt{f_x^2 + f_y^2 + 1}}.$$

This enables us to write (1) in the form

$$S = \iint_R \sqrt{f_x^2 + f_y^2 + 1}\, dA, \tag{2}$$

which is the basic formula of this section.

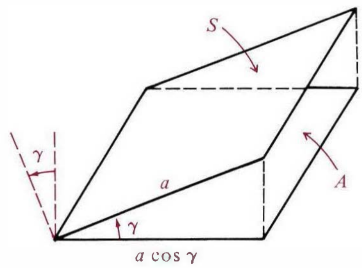


Figure 20.35

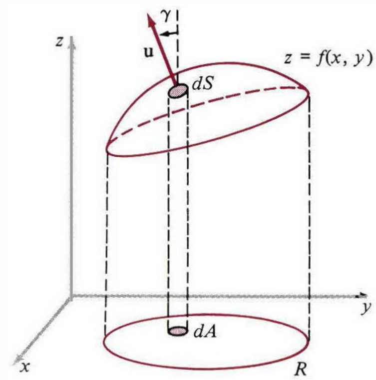


Figure 20.36

Example 1 Find the area of the upper half of the sphere $x^2 + y^2 + z^2 = a^2$ (Fig. 20.37).

Solution The upper hemisphere is represented by the equation $z = \sqrt{a^2 - x^2 - y^2}$, so we have

$$f_x = \frac{-x}{\sqrt{a^2 - x^2 - y^2}},$$

with a similar formula for f_y. The integrand in (2) is therefore

$$\left(\frac{x^2}{a^2 - x^2 - y^2} + \frac{y^2}{a^2 - x^2 - y^2} + 1\right)^{1/2} = \frac{a}{\sqrt{a^2 - x^2 - y^2}},$$

so the area of the hemisphere is

$$S = \iint_R \frac{a}{\sqrt{a^2 - x^2 - y^2}}\, dA, \tag{3}$$

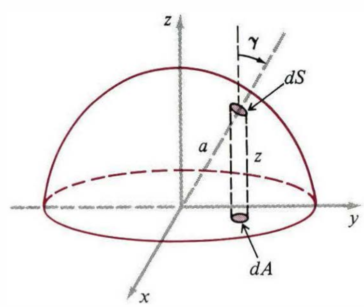


Figure 20.37

where R is the region in the xy-plane bounded by the circle $x^2 + y^2 = a^2$. [It is worth noticing that in this particular case the figure tells us directly that $\cos \gamma = z/a$, so the integrand in (1) is

$$\frac{1}{\cos \gamma} = \frac{a}{z} = \frac{a}{\sqrt{a^2 - x^2 - y^2}}$$

and the integral (3) can be written down at once, without calculation.] We now evaluate the integral (3) by introducing polar coordinates,

$$S = a \int_0^{2\pi} \int_0^a \frac{r\,dr\,d\theta}{\sqrt{a^2 - r^2}} = a \int_0^{2\pi} \left[-\sqrt{a^2 - r^2} \right]_0^a d\theta$$

$$= a^2 \int_0^{2\pi} d\theta = 2\pi a^2.$$

This result is in agreement with Archimedes' formula from elementary geometry, which states that the surface area of a sphere of radius a is $4\pi a^2$.

Example 2 Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies inside the sphere $x^2 + y^2 + z^2 = 6$.

Solution The boundary of the base region R is the projection on the xy-plane of the curve of intersection of the two surfaces. See Fig. 20.38. This is most easily determined by writing the surfaces in cylindrical coordinates, $z = r^2$ and $r^2 + z^2 = 6$. When z is eliminated, we find that the boundary of R is the circle $r^2 = 2$ or $r = \sqrt{2}$. In this case we have $f(x, y) = x^2 + y^2$, so $f_x = 2x$ and $f_y = 2y$, and therefore the desired surface area is

$$S = \iint_R \sqrt{4x^2 + 4y^2 + 1}\, dA.$$

Again we carry out the calculation by using polar coordinates, which gives

$$S = \int_0^{2\pi} \int_0^{\sqrt{2}} \sqrt{4r^2 + 1}\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{1}{12}(4r^2 + 1)^{3/2} \right]_0^{\sqrt{2}} d\theta$$

$$= 2\pi \cdot \frac{1}{12}(27 - 1) = \frac{13}{3}\pi.$$

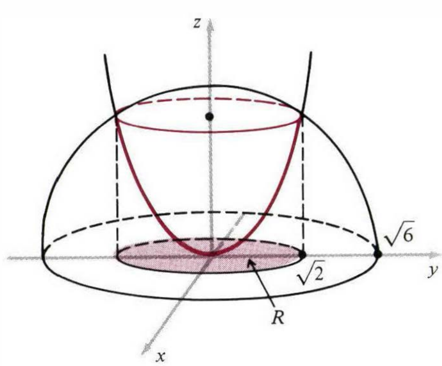


Figure 20.38

Formulas (1) and (2) are the standard formulas of calculus for actually finding the areas of specific curved surfaces. They work well, and we hope they seem reasonable to students. Nevertheless, the *theory* of surface area is very difficult, in particular, the problem of giving a fully satisfactory definition of the concept itself. This problem has occupied the attention of mathematicians for almost a hundred years, and research on these matters continues to this day. Anyone who wishes to understand the nature of the difficulty should study the classic example of H. A. Schwarz (1890), which jolted the mathematical world of the time out of its complacency. Schwarz's example is a simple and familiar curved surface whose area can be computed in several equally reasonable ways to yield wildly different results.*

*This example is described in many places. See, for instance, p. 204 of D. V. Widder, *Advanced Calculus*, 2nd ed. (Prentice-Hall, 1961).

PROBLEMS

Solve Problems 1–6 by using the ideas of this section but without integration.

1 Find the area of the triangle cut from the plane $x + 2y + 3z = 6$ by the coordinate planes.

2 Find the area above the xy-plane cut from the cone $z^2 = x^2 + y^2$ by the cylinder $x^2 + y^2 = 2ax$.

3 Find the area cut from the plane $x + y + z = 7$ by the cylinder $x^2 + y^2 = a^2$

4 Find the area cut from the plane $z = by$ by the cylinder $x^2 + y^2 = a^2$.

5 Find the area of the part of the cone $z^2 = x^2 + y^2$ that lies between the xy-plane and the plane $2z + y = 3$. Hint: What is the area of an ellipse?

6 In Problem 5, find the area of the ellipse in which the plane intersects the cone.

7 Find the area of the part of the sphere $x^2 + y^2 + z^2 = a^2$ that lies above the xy-plane and inside the cylinder $x^2 + y^2 = ax$.

8 In Problem 7, find the area of the part of the cylinder above the xy-plane that lies inside the sphere. Hint: Find $\int h\, ds$, where h is the height of the cylinder and ds is the element of arc length in the xy-plane.

9 Find the area cut from the paraboloid $z = x^2 + y^2$ by the plane $z = 1$.

10 Find the area of the part of the surface $z^2 = 2xy$ that lies above the xy-plane and is bounded by the planes $x = 0$, $x = 2$ and $y = 0$, $y = 1$.

11 Find the area cut from the saddle surface $az = x^2 - y^2$ by the cylinder $x^2 + y^2 = a^2$.

12 Find the area of the part of the sphere $x^2 + y^2 + z^2 = 2a^2$ that lies inside the upper half of the cone $z^2 = x^2 + y^2$.

13 If R is any region in the xy-plane, show that the area of the part of the paraboloid $z = ax^2 + by^2$ that lies above R is equal to the area of the part of the saddle surface $z = ax^2 - by^2$ that lies above (or below) R. Show that this statement is also true for the pairs of surfaces $z = x^2 + y^2$, $z = 2xy$ and $z = \ln (x^2 + y^2)$, $z = 2 \tan^{-1} x/y$.

14 Find the area of the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the xy-plane.

*15 Find the area of the part of the cylinder $z = 1 - x^3$ that is cut out by the planes $y = 0$, $z = 0$, and $y = ax$ where $a > 0$.

16 Find the area of the part of the cylinder $x^2 + y^2 = a^2$ that is cut out by the cylinder $y^2/a^2 + z^2/b^2 = 1$.

17 Find the area of the part of the cylinder $x^2 + z^2 = a^2$ that lies in the first octant and between the planes $y = 3x$ and $y = 5x$.

18 Find the area of the part of the cylinder $y^2 + z^2 = a^2$ that lies inside the cylinder $x^2 + y^2 = a^2$.

19 The cylinder $r^2 = 4 \cos 2\theta$ intersects the xy-plane in a lemniscate. Find the area of the part of the paraboloid $4z = x^2 + y^2$ that lies inside this cylinder.

*20 In Section 7.6 we used the formula

$$S = \int_a^b 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

to find the area of the surface of revolution obtained when the curve $y = f(x)$ is revolved about the x-axis. Show that our new method is consistent with the old, by deriving this formula from (2). Hint: The equation of the surface of revolution is $y^2 + z^2 = f(x)^2$.

*21 A *spherical triangle* is the figure on the surface of a sphere which is bounded by arcs of three great circles (Fig. 20.39). If a is the radius of the sphere, then *Legendre's formula*† for the area of such a triangle is

†A. M. Legendre (1752–1833) was an able French mathematician who had the bad luck to see most of his life's work rendered obsolete by the discoveries of younger and more brilliant men. In spite of this, he retained his amiable and generous disposition.

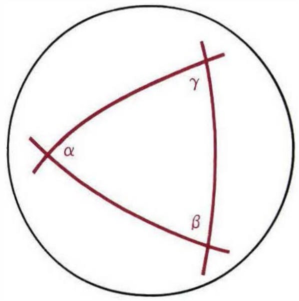


Figure 20.39 A spherical triangle.

$$S = a^2(\alpha + \beta + \gamma - \pi),$$

where α, β, γ are the angles between the sides. (The quantity $\alpha + \beta + \gamma - \pi$ is called the *spherical excess* of the triangle.) Thus, on a given sphere the area of a triangle depends only on its angles. Prove Legendre's formula by the following steps.

(a) In the special triangle cut from the sphere $x^2 + y^2 + z^2 = a^2$ by the planes $y = 0$, $y = x \tan \alpha$, $z = x \tan \mu$, show that the angles between the sides are α, $\pi/2$, and β, where $\cos \beta = \sin \alpha \sin \mu$. Hint: Use vectors.

(b) Show that the projections on the xy-plane of the sides of the right triangle in (a) are $\theta = 0$, $\theta = \alpha$, and $r = a/\sqrt{1 + \tan^2 \mu \cos^2 \theta}$, and that the area of this triangle is

$$S = a^2 \int_0^\alpha \left[1 - \frac{\sin \mu \cos \theta}{\sqrt{1 - \sin^2 \mu \sin^2 \theta}}\right] d\theta.$$

(c) Carry out the integration in (b) and thereby show that the area of the right triangle is

$$a^2\left(\alpha + \beta - \frac{\pi}{2}\right).$$

(d) Complete the proof of Legendre's formula by dividing an arbitrary triangle into two right triangles and using (c).

Apply Legendre's formula to show that the area of the complete sphere is $4\pi a^2$. Hint: Divide the surface into convenient triangles.

CHAPTER 20 REVIEW: METHODS, FORMULAS

Think through the following.

1 Iterated integrals and double integrals: $dA = dx\, dy$.

2 Mass, moment, center of mass, and moment of inertia for thin plates of variable density.

3 Polar coordinates: $dA = r\, dr\, d\theta$.

4 Triple integrals: $dV = dx\, dy\, dz$.

5 Cylindrical coordinates: $dV = r\, dr\, d\theta\, dz$.

6 Spherical coordinates: $dV = \rho^2 \sin \phi\, d\rho\, d\phi\, d\theta$.*

7 $dS = \dfrac{dA}{\cos \gamma} = \sqrt{f_x^2 + f_y^2 + 1}\, dA$.

*In Appendix A.21 we describe some general ideas that provide a unified view of the three formulas $dA = r\, dr\, d\theta$, $dV = r\, dr\, d\theta\, dz$, and $dV = \rho^2 \sin \phi\, d\rho\, d\phi\, d\theta$.

APPENDIX: EULER'S FORMULA $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ BY DOUBLE INTEGRATION

The geometric series $1/(1 - r) = 1 + r + r^2 + \cdots$ enables us to write

$$\begin{aligned}\int_0^1\int_0^1 \frac{dx\, dy}{1 - xy} &= \int_0^1\int_0^1 (1 + xy + x^2y^2 + \cdots)\, dx\, dy \\ &= \int_0^1 \left(x + \frac{1}{2}x^2y + \frac{1}{3}x^3y^2 + \cdots\right)\Big]_0^1 dy \\ &= \int_0^1 \left(1 + \frac{y}{2} + \frac{y^2}{3} + \cdots\right) dy \\ &= \left(y + \frac{y^2}{2^2} + \frac{y^3}{3^2} + \cdots\right)\Big]_0^1 = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots.\end{aligned}$$

The sum of Euler's series $\Sigma 1/n^2$ is therefore the value of the double integral

$$I = \int_0^1\int_0^1 \frac{dx\, dy}{1 - xy}.$$

We evaluate this integral—and thereby determine the sum of the series—by means of a rotation of the coordinate system through the angle $\theta = \pi/4$.

If we rotate the xy-system into the uv-system through an arbitrary angle θ, as shown in Fig. 20.40, then the transformation equations are

$$x = u \cos \theta - v \sin \theta,$$

$$y = u \sin \theta + v \cos \theta.$$

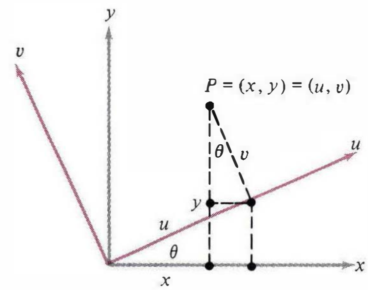


Figure 20.40

When $\theta = \pi/4$ these equations become

$$x = \tfrac{1}{2}\sqrt{2}(u - v),$$

$$y = \tfrac{1}{2}\sqrt{2}(u + v),$$

so we have

$$xy = \frac{1}{2}(u^2 - v^2) \qquad \text{and} \qquad 1 - xy = \frac{2 - u^2 + v^2}{2}.$$

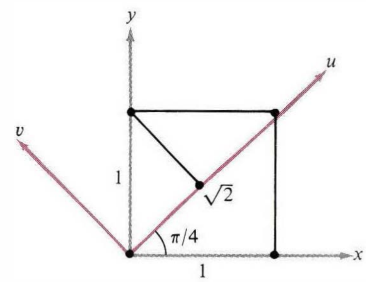


Figure 20.41

By inspecting Fig. 20.41, we see that the integral I can be written in the form

$$I = 4\int_0^{\sqrt{2}/2}\int_0^u \frac{dv\,du}{2 - u^2 + v^2} + 4\int_{\sqrt{2}/2}^{\sqrt{2}}\int_0^{\sqrt{2}-u} \frac{dv\,du}{2 - u^2 + v^2}.$$

If we denote the integrals on the right by I_1 and I_2, then

$$I_1 = 4\int_0^{\sqrt{2}/2}\left[\int_0^u \frac{dv}{2 - u^2 + v^2}\right] du.$$

$$= 4\int_0^{\sqrt{2}/2}\left[\frac{1}{\sqrt{2 - u^2}}\tan^{-1}\left(\frac{v}{\sqrt{2 - u^2}}\right)\right]_0^u du$$

$$= 4\int_0^{\sqrt{2}/2}\frac{1}{\sqrt{2 - u^2}}\tan^{-1}\left(\frac{v}{\sqrt{2 - u^2}}\right) du.$$

To continue the calculation, we use the substitution

$$u = \sqrt{2}\sin\theta, \qquad \sqrt{2 - u^2} = \sqrt{2}\cos\theta, \qquad du = \sqrt{2}\cos\theta\,d\theta,$$

$$\tan^{-1}\left(\frac{u}{\sqrt{2 - u^2}}\right) = \tan^{-1}\left(\frac{\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta}\right) = \theta.$$

Then

$$I_1 = 4\int_0^{\pi/6}\frac{1}{\sqrt{2}\cos\theta}\cdot\theta\cdot\sqrt{2}\cos\theta\,d\theta = 2\theta^2\Big]_0^{\pi/6} = \frac{\pi^2}{18}.$$

To calculate I_2 we write

$$I_2 = 4\int_{\sqrt{2}/2}^{\sqrt{2}}\left[\int_0^{\sqrt{2}-u}\frac{dv}{2 - u^2 + v^2}\right] du$$

$$= 4\int_{\sqrt{2}/2}^{\sqrt{2}}\left[\frac{1}{\sqrt{2 - u^2}}\tan^{-1}\left(\frac{v}{\sqrt{2 - u^2}}\right)\right]_0^{\sqrt{2}-u} du$$

$$= 4\int_{\sqrt{2}/2}^{\sqrt{2}}\frac{1}{\sqrt{2 - u^2}}\tan^{-1}\left(\frac{\sqrt{2} - u}{\sqrt{2 - u^2}}\right) du.$$

To continue the calculation, we use the same substitution as before, with the additional fact that

$$\begin{aligned}
\tan^{-1}\left(\frac{\sqrt{2}-u}{\sqrt{2-u^2}}\right) &= \tan^{-1}\left(\frac{\sqrt{2}-\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta}\right) = \tan^{-1}\left(\frac{1-\sin\theta}{\cos\theta}\right) \\
&= \tan^{-1}\left(\frac{\cos\theta}{1+\sin\theta}\right) = \tan^{-1}\left(\frac{\sin(\pi/2-\theta)}{1+\cos(\pi/2-\theta)}\right) \\
&= \tan^{-1}\left(\frac{2\sin\frac{1}{2}(\pi/2-\theta)\cos\frac{1}{2}(\pi/2-\theta)}{2\cos^2\frac{1}{2}(\pi/2-\theta)}\right) = \frac{1}{2}\left(\frac{\pi}{2}-\theta\right).
\end{aligned}$$

This enables us to write

$$\begin{aligned}
I_2 &= 4\int_{\pi/6}^{\pi/2} \frac{1}{\sqrt{2}\cos\theta}\left(\frac{\pi}{4}-\frac{1}{2}\theta\right)\sqrt{2}\cos\theta\, d\theta = 4\left[\frac{\pi}{4}\theta - \frac{1}{4}\theta^2\right]_{\pi/6}^{\pi/2} \\
&= 4\left[\left(\frac{\pi^2}{8}-\frac{\pi^2}{16}\right)-\left(\frac{\pi^2}{24}-\frac{\pi^2}{144}\right)\right] = \frac{\pi^2}{9}.
\end{aligned}$$

We complete the calculation by putting these results together,

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = I = I_1 + I_2 = \frac{\pi^2}{18}+\frac{\pi^2}{9} = \frac{\pi^2}{6}.$$

It is interesting to observe that

$$\int_0^1\int_0^1\int_0^1 \frac{dx\,dy\,dz}{1-xyz} = \sum_{n=1}^{\infty}\frac{1}{n^3},$$

so that any person who can evaluate this triple integral will thereby discover the sum of the series on the right—which has remained an unsolved problem since Euler first raised the question in 1736.

21 LINE AND SURFACE INTEGRALS. GREEN'S THEOREM, GAUSS'S THEOREM, AND STOKES' THEOREM

21.1 LINE INTEGRALS IN THE PLANE

This chapter brings together into a unified package several topics in multivariable calculus that are important for physical science and engineering, as well as for mathematics itself. The main focal points of our work are the concepts of line integral and surface integral, which provide yet other ways (in addition to double and triple integrals) of extending ordinary integration to higher dimensions. Line integrals are used, for example, to compute the work done by a variable force in moving a particle along a curved path from one point to another. In their origin and applications, these integrals are therefore associated with mathematical physics as much as they are with mathematics. The main result of the first part of this chapter (Green's Theorem) uses partial derivatives to establish a connecting link between line integrals and double integrals, and this in turn enables us to distinguish those vector fields that have potential energy functions from those that do not. Here again, as so often in our earlier work, mathematics and physics constitute a seamless fabric in which neither ingredient has much meaning or value without the other.

Throughout this chapter we assume that the functions under discussion have all the continuity and differentiability properties that are needed in any given situation.

Our first problem is to formulate a satisfactory concept of work. If we push a particle along a straight path with a constant force $\mathbf{F}$ (constant in both direction and magnitude), then we know that the work done by this force is the product of the component of $\mathbf{F}$ in the direction of motion and the distance the particle moves. It is convenient to use the dot product to write this in the form

$$W = \mathbf{F} \cdot \Delta\mathbf{R}, \tag{1}$$

where $\Delta\mathbf{R}$ is the vector from the initial position of the particle to its final position (Fig. 21.1). Now suppose that the force $\mathbf{F}$ is not constant, but instead is a

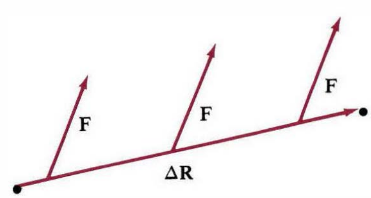


Figure 21.1

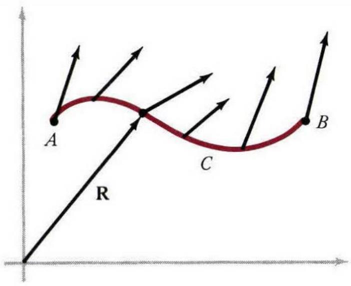


Figure 21.2

vector function that varies from point to point throughout a certain region of the plane, say

$$\mathbf{F} = \mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}. \tag{2}$$

Suppose also that this variable force pushes a particle along a smooth curve C in the plane (Fig. 21.2), where C has parametric equations

$$x = x(t) \quad \text{and} \quad y = y(t), \quad t_1 \leq t \leq t_2. \tag{3}$$

What is the work done by this force as the point of application moves along the curve from the initial point A to the final point B?

Before answering this question, we remark that the vector-valued function (2) is usually called a *force field*. More generally, a *vector field* in the plane is any vector-valued function that associates a vector with each point (x, y) in a certain plane region R. In this context a function whose values are numbers (scalars) is called a *scalar field*. For example, the function $f(x, y) = x^2y^3$ is a scalar field defined on the entire xy-plane. Every scalar field $f(x, y)$ gives rise to the corresponding vector field

$$\nabla f(x, y) = \text{grad}\, f(x, y) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}.$$

(Remember that the symbol ∇f is pronounced "del f.") This is called the *gradient field* of f; its intuitive meaning was described in Section 19.5. For the function just mentioned, we have $\nabla f = 2xy^3\mathbf{i} + 3x^2y^2\mathbf{j}$. Some vector fields are gradient fields, but most are not. We shall see in the next section that those vector fields that are also gradient fields are of special importance.

We now return to the problem of calculating the work done by the variable force

$$\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j} \tag{2}$$

along the smooth curve C. This leads to a new kind of integral called a line integral and denoted by

$$\int_C \mathbf{F} \cdot d\mathbf{R} \quad \text{or} \quad \int_C M(x, y)\, dx + N(x, y)\, dy.$$

We begin the definition by approximating the curve by a polygonal path as shown in Fig. 21.3. That is, choose points $P_0 = A, P_1, P_2, \ldots, P_{n-1}, P_n = B$ along C in this order, let $\mathbf{R}_k$ be the position vector of P_k, and define the n incremental vectors shown in the figure by $\Delta\mathbf{R}_k = \mathbf{R}_{k+1} - \mathbf{R}_k$, where $k = 0, 1, \ldots, n-1$. If we now denote by $\mathbf{F}_k$ the value of the vector function $\mathbf{F}$ at P_k and form the sum

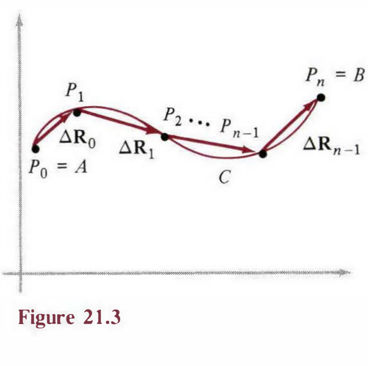


Figure 21.3

$$\sum_{k=0}^{n-1} \mathbf{F}_k \cdot \Delta\mathbf{R}_k, \tag{4}$$

then the *line integral of* $\mathbf{F}$ *along* C is defined to be the limit of sums of this form, and we write

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \lim \sum_{k=0}^{n-1} \mathbf{F}_k \cdot \Delta\mathbf{R}_k. \tag{5}$$

In this limit the polygonal paths are understood to approximate the curve C more and more closely, in the sense that the number of points of division increases and the maximum length of the incremental vectors approaches zero.*

The idea behind the definition (5) is that $\mathbf{F}$ (being continuous) is almost constant along the short path segment $\Delta\mathbf{R}_k$, so by formula (1) we see that $\mathbf{F}_k \cdot \Delta\mathbf{R}_k$ is approximately the work done by $\mathbf{F}$ along the corresponding part of the curve, and therefore the sum (4) is approximately the work done by $\mathbf{F}$ along the entire curve C, with the limit (5) giving the exact value of this work.

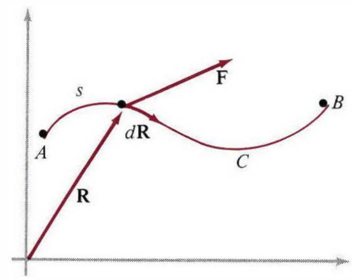


Figure 21.4

A quick intuitive way of constructing the line integral (5) is illustrated in Fig 21.4. If $d\mathbf{R}$ is the element of displacement along C, then the corresponding element of work done by $\mathbf{F}$ is $dW = \mathbf{F} \cdot d\mathbf{R}$. The total work is now obtained by integrating (or adding together) these elements of work along the entire curve C,

$$W = \int dW = \int_C \mathbf{F} \cdot d\mathbf{R}. \tag{6}$$

For additional insight into the meaning of this formula, we think of the position vector $\mathbf{R}$ as a function of the arc length s measured from the initial point A. Since we know that $d\mathbf{R}/ds$ is the unit tangent vector $\mathbf{T}$ (Section 17.4), we can write

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \int_C \mathbf{F} \cdot \frac{d\mathbf{R}}{ds}\, ds = \int_C \mathbf{F} \cdot \mathbf{T}\, ds. \tag{7}$$

The line integral (6) can therefore be thought of as the integral of the tangential component of $\mathbf{F}$ along the curve C. It can be seen from (7) that line integrals include ordinary integrals as special cases; for if the curve C lies along the x-axis between $x = a$ and $x = b$, and if $\mathbf{F} = f(x)\mathbf{i}$, then (7) reduces to $\int_a^b f(x)\, dx$.

If the variable vector $\mathbf{F}$ is given by $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$, then since $\mathbf{R} = x\mathbf{i} + y\mathbf{j}$ and $d\mathbf{R} = dx\mathbf{i} + dy\mathbf{j}$, the formula for computing the dot product of two vectors yields

$$\mathbf{F} \cdot d\mathbf{R} = M(x, y)\, dx + N(x, y)\, dy.$$

The line integral (6) can therefore be written in the form

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \int_C M(x, y)\, dx + N(x, y)\, dy.$$

The parametric representation $x = x(t)$ and $y = y(t)$, $t_1 \le t \le t_2$, for the curve C allows us to express everything here in terms of t,

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{R} &= \int_C M(x, y)\, dx + N(x, y)\, dy \\ &= \int_{t_1}^{t_2} \left[M(x, y)\frac{dx}{dt} + N(x, y)\frac{dy}{dt}\right] dt.\end{aligned}$$

This is an ordinary single integral with t as the variable of integration, and it can be evaluated in the usual way.

So much for generalities. We now turn our attention to getting some practice in the actual calculation of line integrals.

*The term *line integral* for the limit (5) is perhaps unfortunate, because the curve C need not be a straight line segment. *Curve integral* would be more appropriate, but the terminology is well established and cannot be changed now.

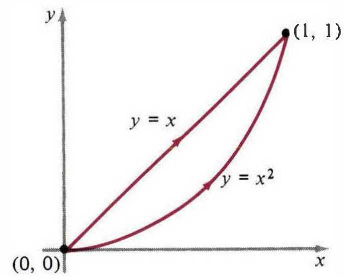


Figure 21.5

Example 1 Evaluate the line integral

$$I = \int_C x^2y\,dx + (x - 2y)\,dy,$$

where C is the segment of the parabola $y = x^2$ from (0, 0) to (1, 1) (see Fig. 21.5).

Solution We may parametrize the curve in any way that seems convenient. (It is not difficult to show that the value of the line integral does not depend on what parametric equations are used, provided that the orientation or direction is kept the same.) The simplest parametric representation of this curve is

$$x = t, \qquad y = t^2, \qquad \text{where } 0 \le t \le 1.$$

Here we have $dx = dt$ and $dy = 2t\,dt$, so the line integral is

$$I = \int_0^1 t^2 \cdot t^2\,dt + (t - 2t^2)2t\,dt$$

$$= \int_0^1 [t^4 + 2t^2 - 4t^3]\,dt = \left[\frac{1}{5}t^5 + \frac{2}{3}t^3 - t^4\right]_0^1 = -\frac{2}{15}.$$

To illustrate the fact that the value of the line integral is independent of the choice of parameter, let us use the representation

$$x = \sin t, \qquad y = \sin^2 t, \qquad \text{where } 0 \le t \le \pi/2.$$

This time we have $dx = \cos t\,dt$ and $dy = 2\sin t\cos t\,dt$, so

$$I = \int_0^{\pi/2} \sin^2 t \cdot \sin^2 t \cdot \cos t\,dt + (\sin t - 2\sin^2 t)2\sin t\cos t\,dt$$

$$= \int_0^{\pi/2} [\sin^4 t + 2\sin^2 t - 4\sin^3 t]\cos t\,dt$$

$$= \left[\frac{1}{5}\sin^5 t + \frac{2}{3}\sin^3 t - \sin^4 t\right]_0^{\pi/2} = -\frac{2}{15},$$

as before.

Every curve C that we use with line integrals is understood to have a direction, from its initial point to its final point. Even though the value of a line integral does not depend on the parameter, it *does* depend on the direction. If $-C$ denotes the same curve traversed in the opposite direction, then we have

$$\int_{-C} \mathbf{F}\cdot d\mathbf{R} = -\int_C \mathbf{F}\cdot d\mathbf{R},$$

or equivalently,

$$\int_{-C} M\,dx + N\,dy = -\int_C M\,dx + N\,dy.$$

That is, integrating in the opposite direction changes the sign of the integral. This can be seen at once from Fig. 21.3 and the definition (5), because the directions of all the incremental vectors $\Delta\mathbf{R}_k$ are reversed.

Example 2 Evaluate the line integral.

$$I = \int_C x^2y\,dx + (x - 2y)\,dy,$$

where C is the straight line segment $y = x$ from (0, 0) to (1, 1).

Solution This is the same integrand as in Example 1, and the initial and final points of the curve are the same, but the curve itself is different (see Fig. 21.5). Using x as the parameter, so that the parametric equations are $x = x$ and $y = x$, we have $dx = dx$ and $dy = dx$, so

$$I = \int_0^1 x^2 \cdot x\,dx + (x - 2x)\,dx$$

$$= \int_0^1 [x^3 - x]\,dx = \left[\frac{1}{4}x^4 - \frac{1}{2}x^2\right]_0^1 = -\frac{1}{4},$$

which we observe is different from the value $-\frac{2}{15}$ obtained along the parabolic path.

The integral in this example can be written as

$$\int_C \mathbf{F} \cdot d\mathbf{R}, \qquad \text{where } \mathbf{F} = x^2y\mathbf{i} + (x - 2y)\mathbf{j}.$$

If $\mathbf{F}$ is thought of as a force field, then the work done by $\mathbf{F}$ in moving a particle from (0, 0) to (1, 1) is different for the two curves in Examples 1 and 2. This illustrates the fact that in general the line integral of a given vector field from one given point to another depends on the choice of the curve, and has different values for different curves.

If a curve C consists of a finite number of smooth curves joined at corners, then we say that C is a *piecewise smooth curve*, or a *path*. The value of a line integral along C is then defined as the sum of its values along the smooth pieces of C. This is illustrated in the first part of our next example.

Example 3 Evaluate the line integral

$$\int_C y\,dx + (x + 2y)\,dy$$

from (1, 0) to (0, 1), where C is (a) the broken line from (1, 0) to (1, 1) to (0, 1); (b) the arc of the circle $x = \cos t$, $y = \sin t$; (c) the straight line segment $y = 1 - x$. See Fig. 21.6.

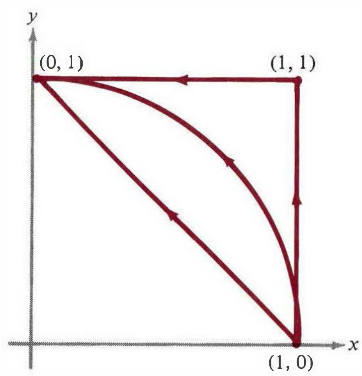


Figure 21.6

Solution (a) Along the segment from (1, 0) to (1, 1) we have $x = 1$ and $dx = 0$; and along the segment from (1, 1) to (0, 1) we have $y = 1$ and $dy = 0$. Since the complete line integral is the sum of the line integrals along each of the segments, we have

$$\int_C y\,dx + (x + 2y)\,dy = \int_0^1 (1 + 2y)\,dy + \int_1^0 dx$$

$$= \left[y + y^2\right]_0^1 + x\Big]_1^0 = 1.$$

(b) Here we have $x = \cos t$ and $y = \sin t$ for $0 \le t \le \pi/2$, so $dx = -\sin t\, dt$ and $dy = \cos t$ dt, and therefore

$$\int_C y\,dx + (x + 2y)\,dy = \int_0^{\pi/2} -\sin^2 t\,dt + (\cos t + 2\sin t)\cos t\,dt$$
$$= \int_0^{\pi/2} (\cos^2 t - \sin^2 t + 2\sin t\cos t)\,dt$$
$$= \int_0^{\pi/2} (\cos 2t + 2\sin t\cos t)\,dt$$
$$= \left[\frac{1}{2}\sin 2t + \sin^2 t\right]_0^{\pi/2} = 1.$$

(c) To integrate along the segment $y = 1 - x$ we can use x as the parameter, so that $dy = -dx$. Since x varies from 1 to 0 along this path, the integral is

$$\int_C y\,dx + (x + 2y)\,dy = \int_1^0 (1 - x)\,dx + [x + 2(1 - x)](-dx)$$
$$= \int_1^0 (-1)\,dx = 1.$$

In this example all three line integrals have the same value, and we might suspect that perhaps with this integrand we get the same value for *any* path from (1, 0) to (0, 1). This is indeed true, as we shall see in Section 21.2, where we investigate the underlying reasons why some line integrals from one point to another have values that are independent of the path of integration, and others do not.

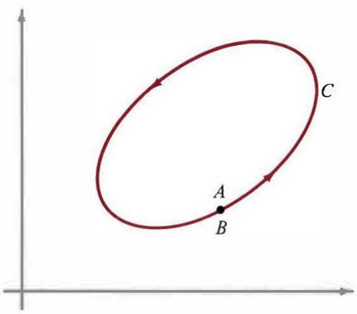


Figure 21.7

It will often be necessary to consider situations in which the path of integration C is a *closed* curve, which means that the final point B is the same as the initial point A (Fig. 21.7). For the sake of emphasis, in this case a line integral is usually written with a small circle on the integral sign, as in

$$\oint_C \mathbf{F}\cdot d\mathbf{R} \qquad \text{or} \qquad \oint_C M\,dx + N\,dy.$$

Example 4 Calculate $\oint_C \mathbf{F}\cdot d\mathbf{R}$, where $\mathbf{F} = y\mathbf{i} + 2x\mathbf{j}$ and C is the circle $x^2 + y^2 = 1$ described counterclockwise from $A = (1, 0)$ back to the same point (Fig. 21.8).

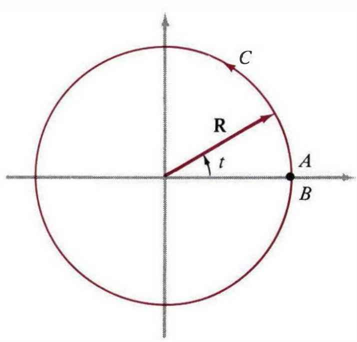


Figure 21.8

Solution A simple parametric representation is $x = \cos t$ and $y = \sin t$, where the counterclockwise orientation means that t increases from 0 to 2π. Since $\mathbf{R} = x\mathbf{i} + y\mathbf{j} = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$, we have

$$d\mathbf{R} = (-\sin t\,\mathbf{i} + \cos t\,\mathbf{j})\,dt,$$

and therefore

$$\mathbf{F}\cdot d\mathbf{R} = (\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j})\cdot(-\sin t\,\mathbf{i} + \cos t\,\mathbf{j})\,dt$$
$$= (2\cos^2 t - \sin^2 t)\,dt$$
$$= \left(\frac{1}{2} + \frac{3}{2}\cos 2t\right)dt,$$

by the half-angle formulas. It now follows that

$$\oint_C \mathbf{F} \cdot d\mathbf{R} = \int_0^{2\pi} \left(\frac{1}{2} + \frac{3}{2} \cos 2t \right) dt$$

$$= \left[\frac{1}{2} t + \frac{3}{4} \sin 2t \right]_0^{2\pi} = \pi.$$

PROBLEMS

1 Evaluate the line integral $I = \int_C xy^2\, dx - (x + y)\, dy$, where C is
(a) the straight line segment from (0, 0) to (1, 2);
(b) the parabolic path $y = 2x^2$ from (0, 0) to (1, 2);
(c) the broken line from (0, 0) to (1, 0) to (1, 2).
Sketch all paths.

2 Evaluate the line integral $I = \int_C x^2y\, dx - xy^2\, dy$, where C is the broken line joining the points (0, 0), (1, 1), (2, 1) in this order.

3 Evaluate $\int_C dx/y + dy/x$, where C is the part of the hyperbola $xy = 4$ from (1, 4) to (4, 1).

4 Show that (a) $\int_C (x^2 - 2y)\, dx = -\frac{2}{3}$, (b) $\int_C 2xy^2\, dy = \frac{1}{2}$, and (c) $\int_C (x^2 - 2y)\, dx + 2xy^2\, dy = -\frac{1}{6}$, if C is the straight line segment $y = x$, $0 \le x \le 1$.

5 Show that $\int_C (x^2 + 3xy)\, dx + (3x^2 - 2y^2)\, dy = -\frac{71}{12}$, if C is the segment of the parabola $x = t$, $y = t^2$ from $t = 1$ to $t = 2$.

6 Evaluate $\int_C (dx + dy)/(x^2 + y^2)$, where C is the upper half of the circle $x^2 + y^2 = a^2$ from $(a, 0)$ to $(-a, 0)$.

7 Find the values of the line integral $\int_C (x - y)\, dx + \sqrt{x}\, dy$ along the following paths C from (0, 0) to (1, 1): (a) $x = t$, $y = t$; (b) $x = t$, $y = t^2$; (c) $x = t^2$, $y = t$; (d) $x = t$, $y = t^3$. Sketch all paths.

8 Show that $\int_C (x^2 + y^2)\, dx = -\frac{2}{3}$, if C is the broken line from (0, 0) to (1, 1) to (0, 1).

9 Evaluate $\int_C x\, dx + x^2\, dy$ from $(-1, 0)$ to $(1, 0)$
(a) along the x-axis;
(b) along the semicircle $y = \sqrt{1 - x^2}$;
(c) along the broken line from $(-1, 0)$ to (0, 1) to (1, 1) to (1, 0).
Sketch all paths.

10 Evaluate $\oint_C (3x + 4y)\, dx + (2x + 3y^2)\, dy$, where C is the circle $x^2 + y^2 = 4$ traversed counterclockwise from (2, 0).

11 Find the values of the line integral $\int_C 2xy\, dx + (x^2 + y^2)\, dy$ along the following paths C from (0, 0) to (1, 1): (a) $y = x$; (b) $y = x^2$; (c) $x = y^2$; (d) $y = x^3$; (e) $x = y^3$; (f) the broken line from (0, 0) to (1, 0) to (1, 1). Sketch all paths.

12 Find the values of the line integral $\int_C x^2\, dx + y^2\, dy$ along the following paths C from (0, 1) to (1, 0): (a) the circular arc $x = \cos t$, $y = \sin t$; (b) the straight line segment; (c) the segment of the parabola $y = 1 - x^2$.

13 If $\mathbf{F} = (y\mathbf{i} - x\mathbf{j})/(x^2 + y^2)$, find $\int_C \mathbf{F} \cdot d\mathbf{R}$ from $(-1, 0)$ to (1, 0)
(a) along the semicircle $y = \sqrt{1 - x^2}$;
(b) along the broken line from $(-1, 0)$ to (0, 1) to (1, 1) to (1, 0).

14 Compute $\oint_C \mathbf{F} \cdot d\mathbf{R}$ if $\mathbf{F} = (x + y)\mathbf{i} + (y^2 - x)\mathbf{j}$, where C is the closed curve that begins at (1, 0), proceeds along the upper half of the unit circle to $(-1, 0)$, and returns to (1, 0) along the x-axis.

15 Evaluate $\int_C y\sqrt{y}\, dx + x\sqrt{y}\, dy$, where C is the part of the curve $x^2 = y^3$ from (1, 1) to (8, 4).

16 If $\mathbf{F} = (2x + y)\mathbf{i} + (3x - 2y)\mathbf{j}$, evaluate $\int_C \mathbf{F} \cdot d\mathbf{R}$ along
(a) the straight line from (0, 0) to (1, 1);
(b) the parabola $y = x^2$ from (0, 0) to (1, 1);
(c) $y = \sin(\pi/2)\, x$ from (0, 0) to (1, 1);
(d) $x = y^n$ $(n > 0)$ from (0, 0) to (1, 1).

17 Show that $\oint_C (-y\, dx + x\, dy)/(x^2 + y^2) = 2\pi$ where C is the circle $x^2 + y^2 = a^2$ traversed counterclockwise from $(a, 0)$.

18 If $\mathbf{F} = xy\mathbf{i} + (y^2 + 1)\mathbf{j}$, calculate $\int_C \mathbf{F} \cdot d\mathbf{R}$, where C is
(a) the line segment from (0, 0) to (1, 1);
(b) the broken line from (0, 0) to (1, 0) to (1, 1);
(c) the parabola $x = y^2$ from (0, 0) to (1, 1).

19 Find the values of $\int_C y\, dx + x\, dy$ along the following paths C from $(-a, 0)$ to $(a, 0)$:
(a) the upper half of the circle $x^2 + y^2 = a^2$;
(b) the broken line from $(-a, 0)$ to $(-a, a)$ to (a, a) to $(a, 0)$;
(c) the straight line segment joining the points.

20 Evaluate $\int_C xy^2\, dx + x^3y\, dy$, where C is the broken line consisting of the segments from $(-1, -1)$ to $(2, -1)$ and from $(2, -1)$ to (2, 4).

21 A particle is moved around a square path from (0, 0) to (1, 0) to (1, 1) to (0, 1) to (0, 0) under the action of the force field $\mathbf{F} = (2x + y)\mathbf{i} + (x + 4y)\mathbf{j}$. Find the work done.

22 Calculate $\oint_C 2xy\, dx + (x^2 + y^2)\, dy$, where C is the boundary of the semicircular region $x^2 + y^2 \le 1$, $y \ge 0$ described counterclockwise.

21.2 INDEPENDENCE OF PATH. CONSERVATIVE FIELDS

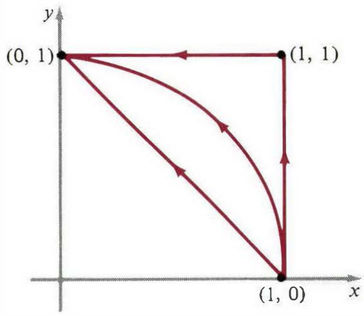


Figure 21.9

In Example 3 of Section 21.1, we calculated the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{R} \tag{1}$$

of the vector field

$$\mathbf{F}(x, y) = y\mathbf{i} + (x + 2y)\mathbf{j} \tag{2}$$

along each of the three different paths from (1, 0) to (0, 1) shown in Fig. 21.9, and we obtained the same value 1 for the integral along all of these paths. This was not an accident. The underlying reason for this result is the fact that the vector field (2) is the *gradient of a scalar field,* namely, of the function

$$f(x, y) = xy + y^2, \tag{3}$$

because clearly

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} = y\mathbf{i} + (x + 2y)\mathbf{j} = \mathbf{F}.$$

To understand the significance of this statement, recall from Section 19.5 that in multivariable calculus the gradient plays a similar role to that of the derivative in single-variable calculus. The Fundamental Theorem of (single-variable) Calculus can be expressed in the form

$$\int_a^b f'(x)\, dx = f(b) - f(a),$$

where $f(x)$ is a function of a single variable. The corresponding result here is

$$\int_C \nabla f \cdot d\mathbf{R} = f(B) - f(A), \tag{4}$$

where $f(x, y)$ is a function of two variables (a scalar field) and A and B are the initial and final points of the path C. For example, since the vector field (2) is the gradient of the scalar field (3), that is, $\mathbf{F} = \nabla f$, formula (4) tells us that the value of the line integral (1) along any of the paths C shown in Fig. 21.9 is

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \int_C \nabla f \cdot d\mathbf{R} = f(0, 1) - f(1, 0) = 1 - 0 = 1,$$

without calculation.

Formula (4) is called the *Fundamental Theorem of Calculus for Line Integrals.* We can state this theorem more precisely as follows:

If a vector field $\mathbf{F}$ *is the gradient of some scalar field* f *in a region* R*, so that* $\mathbf{F} = \nabla f$ *in* R*, and if* C *is any piecewise smooth curve in* R *with initial and final points* A *and* B*, then*

$$\int_C \mathbf{F} \cdot d\mathbf{R} = f(B) - f(A). \tag{5}$$

To prove this, suppose that C is smooth with parametric equations $x = x(t)$ and $y = y(t)$, $a \le t \le b$. Then

$$\int_C \mathbf{F}\cdot d\mathbf{R} = \int_C \nabla f\cdot d\mathbf{R} = \int_a^b \left[\nabla f\cdot \frac{d\mathbf{R}}{dt}\right] dt = \int_a^b \left[\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}\right] dt$$
$$= \int_a^b \frac{d}{dt} f[x(t), y(t)]\, dt$$
$$= f[x(b), y(b)] - f[(x(a), y(a)]$$
$$= f(B) - f(A).$$

The crucial steps here depend on the multivariable chain rule (Section 19.6) and the one-variable Fundamental Theorem of Calculus. The argument for piecewise smooth curves now follows at once by applying (5) to each smooth piece separately, adding, and canceling the function values at the corners.

This theorem has several layers of meaning. We begin by illustrating its usefulness for evaluating line integrals.

Example 1 Compute the line integral of the vector field $\mathbf{F} = y\cos xy\ \mathbf{i} + x\cos xy\ \mathbf{j}$ along the parabolic path $y = x^2$ from (0, 0) to (1, 1).

Solution For this path it is natural to use x as the parameter, where x varies from 0 to 1. Since $dy = 2x\, dx$, we have

$$\mathbf{F}\cdot d\mathbf{R} = y\cos xy\, dx + x\cos xy\, dy$$
$$= x^2\cos x^3\, dx + x\cos x^3\, 2x\, dx$$
$$= 3x^2\cos x^3\, dx,$$

so

$$\int_C \mathbf{F}\cdot d\mathbf{R} = \int_0^1 3x^2\cos x^3\, dx = \sin x^3\Big]_0^1 = \sin 1 - \sin 0 = \sin 1.$$

This straightforward calculation of the line integral is easy to carry out, but a much easier method is now available. The first step is to notice that the vector field $\mathbf{F}$ is the gradient of the scalar field $f(x, y) = \sin xy$. (Students should verify this.) With this fact in hand, all that remains is to apply formula (5):

$$\int_C \mathbf{F}\cdot d\mathbf{R} = \sin xy\Big]_{(0,0)}^{(1,1)} = \sin 1 - \sin 0 = \sin 1.$$

The great advantage of this method is that no attention at all needs to be paid to the actual path of integration from the first point to the second.

As this example shows, the Fundamental Theorem can sometimes be used in the practical task of evaluating line integrals. Nevertheless, its main importance is theoretical. First, we point out that the right side of (5) depends only on the points A and B, and not at all on the path C that joins them. The line integral on the left side of (5) therefore has the same value for all paths C from A to B. This can be expressed by saying that *the line integral of a gradient field is independent of the path.* Next, it is clear from formula (5) that if C is a closed path, so that the final point B is the same as the initial point A, then $f(B) - f(A) = 0$ and therefore the line integral is zero. That is, *if* $\mathbf{F}$ *is a gradient field, then*

$$\oint_C \mathbf{F}\cdot d\mathbf{R} = 0$$

for every closed path C.

These arguments show that

$$\text{Gradient field} \begin{array}{l} \nearrow \text{ integral is independent of path} \\ \searrow \text{ integral around closed path is zero} \end{array}$$

(The symbol $\Rightarrow$ means "implies.") Actually, these three properties are equivalent, in the sense that each implies the other two.

To begin the demonstration of equivalence, suppose that the line integral of the vector field $\mathbf{F}$ is independent of the path. We shall prove that the integral of $\mathbf{F}$ around a closed path is zero. To see why this is so, we examine Fig. 21.10, in which two points A and B are chosen on the closed path C. These points divide C into paths C_1 from A to B and C_2 from B to A. Since C_1 and $-C_2$ are both paths from A to B, the assumption of independence of path implies that

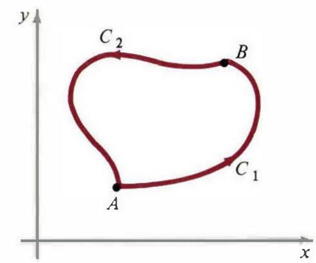


Figure 21.10

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{R} = \int_{-C_2} \mathbf{F} \cdot d\mathbf{R} = -\int_{C_2} \mathbf{F} \cdot d\mathbf{R}.$$

It follows from this that

$$\oint_C \mathbf{F} \cdot d\mathbf{R} = \int_{C_1} \mathbf{F} \cdot d\mathbf{R} + \int_{C_2} \mathbf{F} \cdot d\mathbf{R} = 0,$$

as asserted. Conversely, if we assume that the integral around every closed path is zero, then we can easily reverse this argument to show that the integral from A to B is independent of the path.

To complete the proof of the equivalence of the three properties, it suffices to show that if $\mathbf{F}$ is a vector field whose line integral is independent of path, then $\mathbf{F} = \nabla f$ for some scalar field f. To do this, we choose a fixed point (x_0, y_0) in the region under discussion and let (x, y) be an arbitrary point in this region. Given any path C from (x_0, y_0) to (x, y) [we assume there is such a path], we *define* the function $f(x, y)$ by means of the formula

$$f(x, y) = \int_C \mathbf{F} \cdot d\mathbf{R} = \int_{(x_0, y_0)}^{(x, y)} \mathbf{F} \cdot d\mathbf{R}.$$

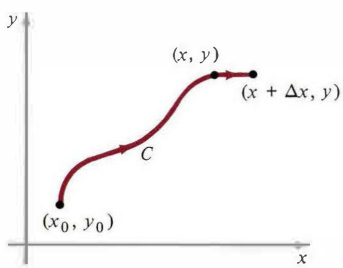


Figure 21.11

See Fig. 21.11. Because of the hypothesis of independence of path, the value of this integral depends only on the point (x, y) and not on the path C, and therefore provides an unambiguous definition for $f(x, y)$. To verify that $\nabla f = \mathbf{F}$, we suppose that the vector field $\mathbf{F}$ has the usual form, $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$, so that

$$f(x, y) = \int_{(x_0, y_0)}^{(x, y)} M\,dx + N\,dy.$$

To show that $\partial f/\partial x = M$, we hold y fixed and move along the straight path from (x, y) to $(x + \Delta x, y)$, as shown in the figure. Since $dy = 0$ on this short path increment, we have

$$f(x + \Delta x, y) - f(x, y) = \int_x^{x+\Delta x} M\,dx,$$

so

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_x^{x+\Delta x} M\,dx = M,$$

by the Fundamental Theorem of Calculus. Similarly, $\partial f/\partial y = N$, so $\nabla f = \mathbf{F}$ and the argument is complete.

As we suggested earlier, the main significance of these ideas lies in their applications to physics. In order to understand what is involved, let us suppose that $\mathbf{F}$ is a force field and that a particle of mass m is moved by this force along a curved path C from a point A to a point B. Let the path be parametrized by the time t, with parametric equations $x = x(t)$ and $y = y(t)$, $t_1 \le t \le t_2$. Then the work done by $\mathbf{F}$ in moving the particle along this path is

$$W = \int_C \mathbf{F} \cdot d\mathbf{R} = \int_{t_1}^{t_2} \left[\mathbf{F} \cdot \frac{d\mathbf{R}}{dt}\right] dt. \tag{6}$$

According to Newton's second law of motion we have

$$\mathbf{F} = m \frac{d\mathbf{v}}{dt},$$

where $\mathbf{v} = d\mathbf{R}/dt$ is the velocity. If v denotes the speed, so that $v = |\mathbf{v}|$, then we can write the integrand in (6) as

$$\mathbf{F} \cdot \frac{d\mathbf{R}}{dt} = m \frac{d\mathbf{v}}{dt} \cdot \mathbf{v} = \frac{1}{2} m \frac{d}{dt} (\mathbf{v} \cdot \mathbf{v}) = \frac{1}{2} m \frac{d}{dt} (v^2).^*$$

Therefore (6) becomes

$$W = \frac{1}{2} m \int_{t_1}^{t_2} \frac{d}{dt} (v^2)\, dt = \frac{1}{2} m v^2 \bigg]_{t_1}^{t_2} = \frac{1}{2} m v_B{}^2 - \frac{1}{2} m v_A{}^2, \tag{7}$$

where v_A and v_B are the initial and final speeds, that is, the speeds at the points A and B. Since $\frac{1}{2}mv^2$ is the kinetic energy of the particle, (7) says that *the work done equals the change in kinetic energy.* (A similar discussion for the case of linear motion is given in Section 7.7.)

We continue this line of thought to its natural conclusion. The force field $\mathbf{F}$ is called *conservative* if it is the gradient of a scalar field. For reasons that will appear in a moment, it is customary in this context to introduce a minus sign and write $\mathbf{F} = -\nabla V$, so that V increases most rapidly in the direction opposite to $\mathbf{F}$. The function $V(x, y)$ is then called the *potential energy.* This function is just the negative of what we have been denoting by f. It exists if and only if $\mathbf{F}$ is a gradient field, and when it does, the Fundamental Theorem (5) tells us that

$$W = \int_C \mathbf{F} \cdot d\mathbf{R} = -\int_C \nabla V \cdot d\mathbf{R} = -[V(B) - V(A)] = V(A) - V(B), \tag{8}$$

where A and B are the initial and final points of the arbitrary path C. If we now equate (7) and (8), we get

$$V(A) - V(B) = \frac{1}{2} m v_B{}^2 - \frac{1}{2} m v_A{}^2$$

or

$$\frac{1}{2} m v_A{}^2 + V(A) = \frac{1}{2} m v_B{}^2 + V(B). \tag{9}$$

*In this calculation we use the product rule for the derivative of the dot product of two vector functions of t. This is easy to prove from formula (8) in Section 18.2.

(We now see that the minus sign is introduced into the definition of potential energy in order to make the signs here come out right.) Equation (9) says that the sum of the kinetic energy and the potential energy is the same at the initial point as it is at the final point. Since these points are arbitrary, *the total energy is constant.* This is the *law of conservation of energy,* which is one of the basic principles of classical physics. This law is true in any conservative force field, such as the earth's gravitational field or the electric field produced by any distribution of electric charge.

Our work has demonstrated that a force field is conservative if and only if it satisfies any one of the following equivalent conditions:

$$\begin{array}{ccc} & \nearrow\!\!\!\!\swarrow & \text{integral is independent of path} \\ \text{Gradient field} & & \Updownarrow \\ & \searrow\!\!\!\!\nwarrow & \text{integral around closed path is zero} \end{array}$$

The importance of these fields justifies turning our attention to the practical problem of determining whether a given force field is or is not conservative. Since any vector field can be thought of as a force field, these remarks apply to vector fields in general.

Example 2 Show that the vector field $\mathbf{F} = xy\mathbf{i} + xy^2\mathbf{j}$ is not conservative.

Solution One way of doing this is to show that $\int_C \mathbf{F} \cdot d\mathbf{R}$ *does* depend on the path. We choose two convenient points, say (0, 0) and (1, 1), and integrate from the first to the second along any two convenient different paths, as shown in Fig. 21.12. First, along the line $y = x$ we have

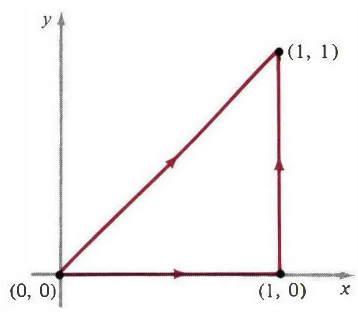


Figure 21.12

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \int_C xy\,dx + xy^2\,dy = \int_0^1 (x^2 + x^3)\,dx = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}.$$

On the other hand, along the broken line from (0, 0) to (1, 0) to (1, 1) we have

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \int_0^1 0 \cdot dx + \int_0^1 y^2\,dy = \frac{1}{3}.$$

Since the values of these integrals are not equal, the field is not conservative. In probing for unequal line integrals in this way, we are perfectly free to choose paths that make the calculations easy. (Of course, if these two line integrals had turned out to be equal, this would not have precluded unequal results for two other line integrals, so nothing would have been proved one way or the other.)

Another method is to assume that the field *is* conservative, so that $\mathbf{F} = \nabla f$ for some function $f(x, y)$, and to deduce a contradiction from this assumption. The assumption means that there exists a function f such that $\partial f/\partial x = xy$ and $\partial f/\partial y = xy^2$. But this is impossible, because the mixed partial derivatives would then be

$$\frac{\partial^2 f}{\partial y\,\partial x} = x \qquad \text{and} \qquad \frac{\partial^2 f}{\partial x\,\partial y} = y^2,$$

which are obviously not equal, whereas the theory of partial derivatives tells us that these derivatives must be equal. This contradiction tells us that no f exists, so $\mathbf{F}$ is not conservative.

The reasoning used in the second method of this example depends on the equality of mixed partial derivatives,

$$\frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 f}{\partial x\,\partial y}, \tag{10}$$

which is valid in any region where both derivatives are continuous (Section 19.2). This reasoning can be extended as follows: if

$$\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j} \tag{11}$$

is a conservative vector field, so that an f exists with the property that $\nabla f = \mathbf{F}$ or

$$\frac{\partial f}{\partial x} = M \quad \text{and} \quad \frac{\partial f}{\partial y} = N,$$

then by (10) we know that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \tag{12}$$

Condition (12) is therefore necessary for a vector field to be conservative, and we have seen how this fact can be used. But is it also sufficient? That is, does (12) guarantee that (11) is conservative? We investigate this question in Section 21.3.

PROBLEMS

In Problems 1–4, use both methods of Example 2 to show that the vector field is not conservative.

1 $\mathbf{F} = y\mathbf{i} - x\mathbf{j}$.

2 $\mathbf{F} = x(y - 1)\mathbf{i} + x\mathbf{j}$.

3 $\mathbf{F} = x^3y\mathbf{i} + xy^2\mathbf{j}$.

4 $\mathbf{F} = \dfrac{y\mathbf{i} + y\mathbf{j}}{x^2 + y^2}$.

In Problems 5–8, show that the given line integrals are not independent of path by integrating along two different paths from (0, 0) to (1, 1).

5 $\int_C 2xy\,dx + (y^2 - x^2)\,dy$.

6 $\int_C 2xy\,dx + (y - x^2)\,dy$.

7 $\int_C (x^2 - y^3)\,dx + 3xy^2\,dy$.

8 $\int_C (x - y)\,dx + (x + y)\,dy$.

9 Show that

$$\int_{(-2,1)}^{(1,4)} 2xy\,dx + x^2\,dy$$

is independent of the path, and evaluate the integral by
(a) using formula (5);
(b) integrating along any convenient path.

10 Show that

$$\int_{(-1,0)}^{(1,\pi)} \sin y\,dx + x\cos y\,dy$$

is independent of the path, and evaluate the integral by
(a) using formula (5);
(b) integrating along any convenient path.

In Problems 11–16, show that the integral is independent of the path and use any method to evaluate it.

11 $\int_{(-2,-1)}^{(1,5)} 2y\,dx + 2x\,dy$.

12 $\int_{(0,0)}^{(4,5)} y^2e^x\,dx + 2ye^x\,dy$.

13 $\int_{(0,0)}^{(\pi/2,1)} e^y\cos x\,dx + e^y\sin x\,dy$.

14 $\int_{(-1,1)}^{(2,3)} 3x^2y^2\,dx + 2x^3y\,dy$.

15 $\int_{(-2,1)}^{(4,1)} 2xy\,dx + (x^2 + y^2)\,dy$.

16 $\int_{(0,0)}^{(1,1)} (x + y)\,dx + x\,dy$.

17 Suppose that a particle of mass m moves in the xy-plane under the influence of the constant gravitational force $\mathbf{F} = -mg\mathbf{j}$. If the particle moves from (x_1, y_1) to (x_2, y_2) along a path C, show that the work done by $\mathbf{F}$ is

$$W = mg(y_1 - y_2),$$

regardless of the path.

21.3 GREEN'S THEOREM

As we said at the beginning of this chapter, Green's Theorem establishes an important link between line integrals and double integrals. Our purpose in this section is to reveal the nature of this link.

Consider a vector field

$$\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j} \tag{1}$$

defined on a certain region in the xy-plane. We now take up the question of whether the condition

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{2}$$

is sufficient to guarantee that $\mathbf{F}$ is conservative, that is, that $\mathbf{F}$ is the gradient of some scalar field f. In the light of what we have learned in Section 21.2, this is equivalent to asking whether condition (2) implies that the integral of $\mathbf{F}$ around every closed path is zero. We shall use our investigation of this question as a means of discovering Green's Theorem, which we will then prove and apply in various ways.

The simplest type of closed path C is a rectangular path like the one shown in Fig. 21.13. We shall calculate the integral of $\mathbf{F}$ around this path and see what is needed to make its value zero. Integrating counterclockwise as shown, and beginning with the path segment on the lower edge of the rectangular region R, we have

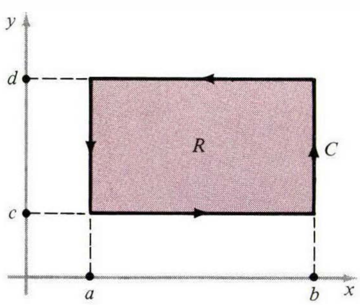


Figure 21.13

$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{R} &= \oint_C M(x, y)\,dx + N(x, y)\,dy \\ &= \int_a^b M(x, c)\,dx + \int_c^d N(b, y)\,dy + \int_b^a M(x, d)\,dx + \int_d^c N(a, y)\,dy \\ &= \int_c^d [N(b, y) - N(a, y)]\,dy - \int_a^b [M(x, d) - M(x, c)]\,dx. \end{aligned} \tag{3}$$

We next make an ingenious application of the Fundamental Theorem of Calculus to write these two integrands as

$$N(b, y) - N(a, y) = N(x, y)\Big]_{x=a}^{x=b} = \int_a^b \frac{\partial N}{\partial x}\,dx$$

and

$$M(x, d) - M(x, c) = M(x, y)\Big]_{y=c}^{y=d} = \int_c^d \frac{\partial M}{\partial y}\,dy.$$

This enables us to write (3) in the form

$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{R} &= \oint_C M\,dx + N\,dy \\ &= \int_c^d \int_a^b \frac{\partial N}{\partial x}\,dx\,dy - \int_a^b \int_c^d \frac{\partial M}{\partial y}\,dy\,dx.\end{aligned}$$

These iterated integrals can be written as double integrals over the region R enclosed by C, so we have

$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{R} &= \oint_C M\,dx + N\,dy \\ &= \iint_R \frac{\partial N}{\partial x}\,dA - \iint_R \frac{\partial M}{\partial y}\,dA = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dA.\end{aligned} \tag{4}$$

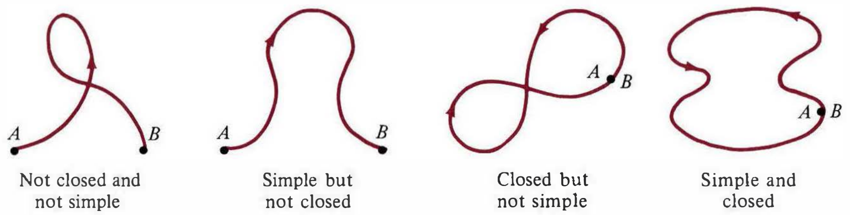


Figure 21.14 Various types of curves.

Now we can see what is happening. Condition (2) implies that this double integral is zero, so $\oint_C \mathbf{F} \cdot d\mathbf{R} = 0$. It is tempting to infer from this that condition (2) implies $\mathbf{F}$ is conservative. However, this inference requires that $\oint_C \mathbf{F} \cdot d\mathbf{R} = 0$ for *every* closed path C, and we have demonstrated this only for rectangular paths like the one in Fig. 21.13.

If we pluck out the essence of this argument, we see that it lies in equation (4), which we can write in the form

$$\oint_C M\,dx + N\,dy = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dA. \tag{5}$$

This statement, that a line integral around a closed curve equals a certain double integral over the region inside the curve, is called *Green's Theorem,* after the English mathematical physicist George Green.*

Strictly speaking, Green's Theorem is not merely equation (5), but rather a fairly careful statement of conditions under which (5) is valid. To state such conditions, it is necessary to introduce the concept of a simple closed curve. We already know that a closed curve is one for which the final point B is the same as the initial point A. A plane curve is said to be *simple* if it does not intersect itself anywhere between its endpoints (Fig. 21.14). Unless the contrary is explicitly stated, we assume that simple closed curves are *positively* oriented, which means that they are traversed in such a way that their interiors are always on the left, as shown on the right in the figure.

Green's Theorem can now be stated as follows:

If C is a piecewise smooth, simple closed curve that bounds a region R, and if $M(x, y)$ and $N(x, y)$ are continuous and have continuous partial derivatives along C and throughout R, then

$$\oint_C M\,dx + N\,dy = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dA. \tag{5}$$

We have proved (5) only for rectangular regions R of the kind shown in Fig. 21.13. We now give a similar argument for the case in which R is both vertically simple and horizontally simple, in the sense described in Section 20.2. Then we shall indicate how to extend the theorem to more general regions.

---

*Green (1793–1841) was obliged to leave school at an early age to work in his father's bakery, and consequently had little formal education. By assiduous study in his spare time, he taught himself mathematics and physics from library books, particularly Laplace's *Mécanique Céleste.* In 1828 he published locally at his own expense his most important work, *Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism.* Although Green's Theorem (in an equivalent form) appeared in this pamphlet, little notice was taken until the pamphlet was republished in 1846, five years after his death, and thereby came to the attention of scientists who had the knowledge to appreciate its merits.

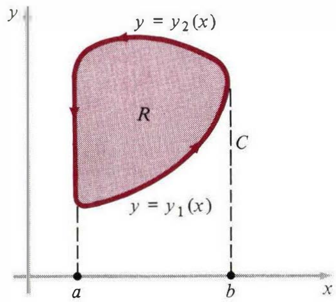


Figure 21.15

Since R is assumed to be vertically simple (Fig. 21.15), its boundary C can be thought of as consisting of a lower curve $y = y_1(x)$ and an upper curve $y = y_2(x)$, possibly separated by vertical segments on the sides. The integral $\int M(x, y)\,dx$ over any part of C that consists of vertical segments is zero, since $dx = 0$ on such a segment. We therefore have

$$\oint_C M(x, y)\,dx = \int_a^b M[x, y_1(x)]\,dx + \int_b^a M[x, y_2(x)]\,dx, \tag{6}$$

where the lower curve is traced from left to right and the upper curve from right to left. By the Fundamental Theorem of Calculus, (6) can be written as

$$\begin{aligned} \oint_C M\,dx &= \int_a^b \{M[x, y_1(x)] - M[x, y_2(x)]\}\,dx \\ &= \int_a^b \Big[-M(x, y) \Big]_{y=y_1(x)}^{y=y_2(x)} dx \\ &= \int_a^b \int_{y_1(x)}^{y_2(x)} -\frac{\partial M}{\partial y}\,dy\,dx = \iint_R -\frac{\partial M}{\partial y}\,dA. \end{aligned} \tag{7}$$

But R is also assumed to be horizontally simple, and a similar argument, which we ask students to give in Problem 22, shows that

$$\oint_C N\,dy = \iint_R \frac{\partial N}{\partial x}\,dA. \tag{8}$$

We now obtain Green's Theorem (5) for the region R by adding (7) and (8).

A complete and rigorous proof of Green's Theorem is beyond the scope of this book. Nevertheless, it is quite easy to extend the argument to cover any region R that can be subdivided into a finite number of regions $R_1, R_2, \ldots, R_n$ that are both vertically and horizontally simple. The validity of Green's Theorem for R then follows from its validity for each of the regions $R_1, R_2, \ldots, R_n$.

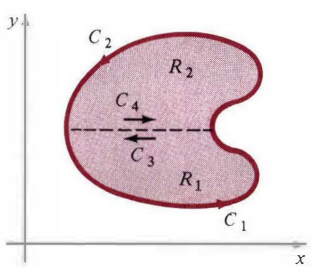


Figure 21.16

For example, the region R in Fig. 21.16 can be subdivided into the regions R_1 and R_2 by introducing the indicated cut, which becomes part of the boundary of R_1 when traced from right to left (C_3), and part of the boundary of R_2 when traced from left to right (C_4). By applying Green's Theorem separately to R_1 and R_2, we get

$$\oint_{C_1+C_3} M\,dx + N\,dy = \iint_{R_1} \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dA$$

and

$$\oint_{C_2+C_4} M\,dx + N\,dy = \iint_{R_2} \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dA.$$

If we add these two equations the result is

$$\oint_{C_1+C_2} M\,dx + N\,dy = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dA,$$

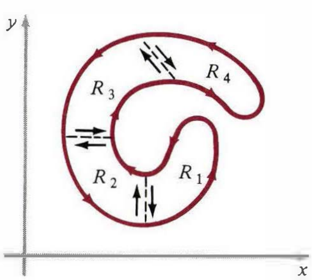


Figure 21.17

which is Green's Theorem for the region R. This occurs because the two line integrals along C_3 and C_4 cancel each other, since C_3 and C_4 are the same curve traced in opposite directions. Similarly, Green's Theorem can be extended to the region in Fig. 21.17 by subdividing it into the four simpler regions shown in the figure.

Example 1 Evaluate the line integral

$$I = \oint_C (3x - y)\,dx + (x + 5y)\,dy$$

around the unit circle $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$.

Solution The straightforward calculation of this integral gives

$$I = \int_0^{2\pi} [(3\cos t - \sin t)(-\sin t) + (\cos t + 5\sin t)(\cos t)]\,dt$$

$$= \int_0^{2\pi} [2\sin t\cos t + 1]\,dt = \Big[\sin^2 t + t\Big]_0^{2\pi} = 2\pi.$$

This is easy enough, but Green's Theorem makes it even easier. Since $M = 3x - y$ and $N = x + 5y$, we have

$$\frac{\partial M}{\partial y} = -1 \qquad \text{and} \qquad \frac{\partial N}{\partial x} = 1,$$

so

$$I = \iint_R [1 - (-1)]\,dA$$

$$= 2\iint_R dA = 2(\text{area of circle}) = 2\pi.$$

Example 2 Evaluate the line integral

$$I = \oint_C (2y + \sqrt{1 + x^5})\,dx + (5x - e^{y^2})\,dy$$

around the circle $x^2 + y^2 = 4$.

Solution The actual calculation of this integral looks like a very forbidding task, but Green's Theorem provides another way. since $M = 2y + \sqrt{1 + x^5}$ and $N = 5x - e^{y^2}$,

$$\frac{\partial M}{\partial y} = 2 \qquad \text{and} \qquad \frac{\partial N}{\partial x} = 5.$$

Therefore

$$I = \iint_R (5 - 2)\,dA = 3\iint_R dA = 3(\text{area of circle}) = 3(4\pi) = 12\pi,$$

since R is a circular disk of radius 2.

Example 3 If R is any region to which Green's Theorem is applicable, show that the area A of R is given by the formula

$$A = \frac{1}{2}\oint_C -y\,dx + x\,dy. \tag{9}$$

Solution Since $M = -y$ and $N = x$, and therefore

$$\frac{\partial M}{\partial y} = -1 \qquad \text{and} \qquad \frac{\partial N}{\partial x} = 1,$$

Green's Theorem yields

$$\oint_C -y\,dx + x\,dy = \iint_R [1-(-1)]\,dA = 2\iint_R dA = 2A,$$

as stated.

Example 4 Use formula (9) to find the area bounded by the ellipse $x^2/a^2 + y^2/b^2 = 1$.

Solution We can parametrize the ellipse by $x = a\cos t$, $y = b\sin t$, where $0 \le t \le 2\pi$. Then formula (9) yields

$$\begin{aligned} A &= \frac{1}{2}\int_0^{2\pi} [(-b\sin t)(-a\sin t) + (a\cos t)(b\cos t)]\,dt \\ &= \frac{1}{2}\int_0^{2\pi} ab\,dt = \pi ab. \end{aligned}$$

Our original problem in this section was to determine whether the condition

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{2}$$

is sufficient to guarantee that the vector field

$$F = M(x, y)\mathbf{i} + N(x, y)\mathbf{j} \tag{1}$$

is conservative. Green's Theorem provides the solution. For if C is any simple closed path in the domain of $\mathbf{F}$, and if the region enclosed by C is also in the domain, then Green's Theorem tells us that

$$\oint_C \mathbf{F}\cdot d\mathbf{R} = \oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA.$$

By using this equation we see that if $\partial M/\partial y = \partial N/\partial x$ then the double integral is zero, and therefore the line integral is zero. If the line integral is zero around every simple closed path, then it is also zero around every closed path, and this proves that $\mathbf{F}$ is conservative. We emphasize that for this reasoning to work, the region enclosed by C must lie entirely in the domain of $\mathbf{F}$. A convenient way to guarantee this is to require that the domain of $\mathbf{F}$ must be *simply connected*, which means that the inside of every simple closed path in the domain also lies in the domain. Roughly speaking, the domain of $\mathbf{F}$ is not allowed to have any holes. In Fig. 21.18 we show regions with one, two, and three holes, respectively; the points inside the inner curves do not belong to the regions R, so these regions are not simply connected. Our overall conclusion can be stated as follows:

If the domain of definition of the vector field $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ is simply connected, then $\mathbf{F}$ is conservative if and only if the condition $\partial M/\partial y = \partial N/\partial x$ is satisfied.

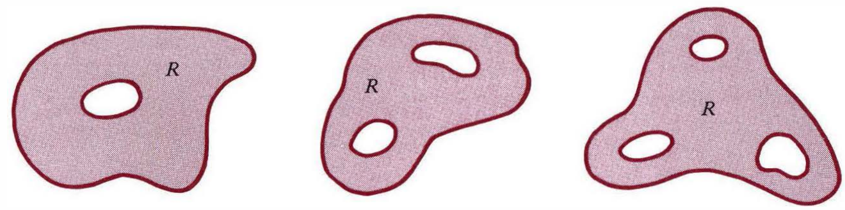


Figure 21.18 Regions not simply connected.

One final question: If a given vector field $\mathbf{F}$ is known to be conservative, so that $\mathbf{F} = \nabla f$ for some function $f(x, y)$, how do we find f? Such a function is called a *potential function*, or simply a *potential*, for $\mathbf{F}$.* One way is by inspection, but this only works in simple cases. A more systematic method is illustrated in the following example. As the student will see, it amounts to integrating the equations

$$\frac{\partial f}{\partial x} = M, \qquad \frac{\partial f}{\partial y} = N,$$

and condition (2) guarantees that this can be done.

Example 5 Find a potential f for the vector field

$$\mathbf{F} = (y^2 + 1)\mathbf{i} + 2xy\mathbf{j}.$$

Solution Here we have $M = y^2 + 1$ and $N = 2xy$. It is easy to verify that $\partial M/\partial y = \partial N/\partial x$, and therefore f exists and our only problem is to find it. We know that

$$\frac{\partial f}{\partial x} = y^2 + 1 \qquad \text{and} \qquad \frac{\partial f}{\partial y} = 2xy. \tag{10}$$

In computing $\partial f/\partial x$, we differentiate with respect to x while holding y constant, so by integrating the first of equations (10) with respect to x, we obtain $f = xy^2 + x + g(y)$, where $g(y)$ is a function of y that is yet to be determined. By differentiating with respect to y, we see that $\partial f/\partial y = 2xy + g'(y)$, and by comparing this with the second of equations (10) we conclude that $g'(y) = 0$. It follows that $g(y)$ is a constant C that can be chosen arbitrarily, and therefore the potential we are seeking is $f(x, y) = xy^2 + x + C$. It is easy to check this result by verifying that $\nabla f = \mathbf{F}$.

*Recall that for physical reasons the *potential energy* associated with a force field $\mathbf{F}$ is any scalar function V (if one exists) such that $\mathbf{F} = -\nabla V$. The concepts of potential and potential energy are closely related but not identical.

PROBLEMS

In Problems 1–4, evaluate the line integrals directly, and also by Green's Theorem.

1 $\oint_C (xy - y^2)\,dx + xy^2\,dy$, where C is the simple closed path formed by $y = 0$, $x = 1$, $y = x$.

2 $\oint_C x\,dx + xy^2\,dy$, where C is the simple closed path formed by $y = x^2$ and $y = x$.

3 $\oint_C 1/y\,dx + 1/x\,dy$, where C is the simple closed path formed by $y = x$, $y = 4$, $x = 1$.

4 $\oint_C y^2\,dx + x^2\,dy$, where C is the simple closed path formed by $y = 0$, $x = 1$, $y = 1$, $x = 0$.

In Problems 5–12, use Green's Theorem to compute the given line integrals.

5 $\oint_C xy\,dx + (x + y)\,dy$, where C is the closed path (obviously simple) formed by $y = 0$, $x = 0$, $y = 1$, $x = -1$.

6 $\oint_C -xy/(1 + x)\,dx + \ln(1 + x)\,dy$, where C is the closed path formed by $y = 0$, $x + 2y = 4$, $x = 0$.

7 $\oint_C -x^2y/(1 + x^2)\,dx + \tan^{-1}x\,dy$, where C is the closed path formed by $y = 0$, $x = 1$, $y = 1$, $x = 0$.

8 $\oint_C x\,dx + xy\,dy$, where C is the closed path formed by $y = 0$, $x^2 + y^2 = 1$ $(x, y \geq 0)$, $x = 0$.

9 $\oint_C (e^{x^3} + y^2)\,dx + (x + \sqrt{1 + y^7})\,dy$, where C is the closed path formed by $y = 0$, $x = 1$, $y = x$.

10 $\oint_C -y^3\,dx + x^3\,dy$, where C is the closed path formed by $y = x^3$ and $y = x$.

11 $\oint_C (-y^2 + \tan^{-1}x)\,dx + \ln y\,dy$, where C is the closed path formed by $y = x^2$ and $x = y^2$.

12 $\oint_C (x^2 - y)\,dx + x\,dy$, where C is the circle $x^2 + y^2 = 9$.

In Problems 13–20, use formula (9) to find the area bounded by the given curves.

13 $y = 3x$ and $y^2 = 9x$.

14 $y = 0$, $x + y = a$ $(a > 0)$, $x = 0$.

15 The x-axis and one arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$.

16 $y = x^2$ and $x = y^3$.

17 $x = a\cos^3\theta$ and $y = a\sin^3\theta$, $0 \leq \theta \leq 2\pi$ (an astroid or hypocycloid of four cusps).

18 $y = x^2$ and $x = y^2$.

19 The x-axis and one arch of $y = \sin x$.

20 $9y = x$, $xy = 1$, $y = x$.

21 The loop of the folium of Descartes (with Cartesian equation $x^3 + y^3 = 3axy$) is shown in Fig. 17.11. In Problem 16 of Section 17.1 we asked students to introduce the parameter $t = y/x$ and obtain the parametric equations

$$x = \frac{3at}{1 + t^3}, \qquad y = \frac{3at^2}{1 + t^3}.$$

Use formula (9) to find the area of the loop. Hint: The part of the loop below the line $y = x$ is traced out as t increases from 0 to 1.

22 Give the details of the argument establishing formula (8) for the case in which R is horizontally simple.

In Problems 23–28, verify that the given vector field is conservative and find a potential for it.

23 $\mathbf{F} = y^3\mathbf{i} + 3xy^2\mathbf{j}$.

24 $\mathbf{F} = e^y\cos x\mathbf{i} + e^y\sin x\mathbf{j}$.

25 $\mathbf{F} = (ye^{xy} - 2x)\mathbf{i} + (xe^{xy} + 2y)\mathbf{j}$.

26 $\mathbf{F} = y\cos xy\mathbf{i} + x\cos xy\mathbf{j}$.

27 $\mathbf{F} = (\sin y - y\sin x)\mathbf{i} + (x\cos y + \cos x)\mathbf{j}$.

28 $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$.

29 Let C_1, C_2, and C_3 be the simple closed curves shown in Fig. 21.19, and let R be the region inside C_1 and outside C_2 and C_3. Assume that $M(x, y)$ and $N(x, y)$ are continuous and have continuous partial derivatives in R and along all the curves. Show that Green's Theorem

$$\oint_C M\,dx + N\,dy = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dA$$

remains valid in this case, provided that C is understood to be the *total* boundary of R, consisting of C_1, C_2, and C_3 positively oriented as shown in the figure. (The curves C_2 and C_3 are oriented clockwise, but nevertheless the orientation is positive because they are traversed in such a way that the region R remains on the left.)

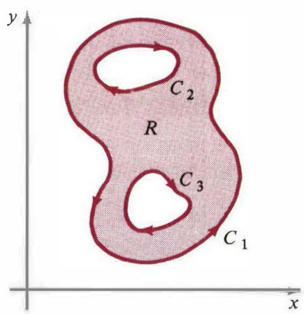


Figure 21.19

30 Can Green's Theorem be used to evaluate the line integral

$$\oint_C \frac{-y}{x^2 + y^2}\,dx + \frac{x}{x^2 + y^2}\,dy,$$

(a) where C is the circle $x^2 + y^2 = 1$?
(b) where C is the triangle with vertices $(1, 0)$, $(1, 2)$, $(2, 2)$?

31 If C_1 is the circle $x^2 + y^2 = 1$ and C_2 is an arbitrary simple closed path containing C_1, as shown in Fig. 21.20, use the idea of Problem 29 to show that

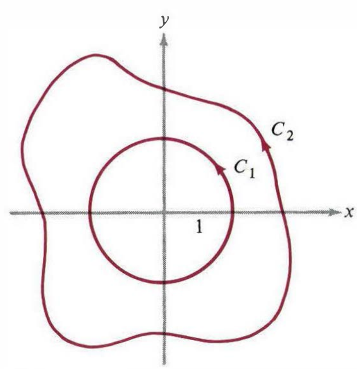


Figure 21.20

$$\oint_{C_2} \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$
$$= \oint_{C_1} \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy,$$

and evaluate the integral on the left by calculating the integral on the right.

21.4 SURFACE INTEGRALS AND GAUSS'S THEOREM

In this section and the next, we move out of the plane into three-dimensional space and give a brief intuitive introduction to the two fundamental integral theorems of vector analysis. These theorems are roughly similar to each other, for both make assertions of the following kind:

> *The integral of a certain function over the boundary of a region is equal to the integral of a related function over the region itself.*

It is possible to spend considerable time analyzing such purely mathematical issues as what is meant by a region and its boundary, but in this short sketch we shall proceed informally and concentrate instead on the physical meaning of what we are doing.

The concept of *gradient,* as we presented it in Chapter 19, applies only to scalar fields, that is, functions whose values are numbers. The gradient of a scalar field $f(x, y, z)$ is a vector field that represents the rate of change of f, because at any point its component in a given direction is the directional derivative of f in that direction. Our purpose here is to consider the more complicated problem of describing the rate of change of a *vector* field. There are two fundamental tools for measuring the rate of change of a vector field: the *divergence* and the *curl.*

We recall that the gradient of a scalar field $f(x, y, z)$ is defined by

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k},$$

where the symbol ∇ ("del") represents the vector differential operator

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}.$$

If $\mathbf{F} = L\mathbf{i} + M\mathbf{j} + N\mathbf{k}$ is a given vector field, we can apply ∇ to $\mathbf{F}$ in two ways, by using the dot and cross products. We interpret the dot product of ∇ and $\mathbf{F}$ to mean

$$\nabla \cdot \mathbf{F} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \cdot (L\mathbf{i} + M\mathbf{j} + N\mathbf{k})$$
$$= \frac{\partial L}{\partial x} + \frac{\partial M}{\partial y} + \frac{\partial N}{\partial z}.$$

This scalar quantity is called the *divergence* of $\mathbf{F}$ and is often denoted by div $\mathbf{F}$, so that

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial L}{\partial x} + \frac{\partial M}{\partial y} + \frac{\partial N}{\partial z}. \tag{1}$$

The cross product of ∇ and $\mathbf{F}$ is interpreted to mean*

*Remember formula (11) in Section 18.3.

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ L & M & N \end{vmatrix} = \left(\frac{\partial N}{\partial y} - \frac{\partial M}{\partial z}\right)\mathbf{i} + \left(\frac{\partial L}{\partial z} - \frac{\partial N}{\partial x}\right)\mathbf{j} + \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right)\mathbf{k}.$$

This vector quantity is called the *curl* of $\mathbf{F}$ and is often denoted by curl $\mathbf{F}$, so that

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F}. \tag{2}$$

Example 1 Compute the divergence and curl of the vector field $\mathbf{F} = 2x^2y\mathbf{i} + 3xz^3\mathbf{j} + xy^2z^2\mathbf{k}$.

Solution By using formulas (1) and (2) we at once obtain

$$\begin{aligned} \text{div } \mathbf{F} = \nabla \cdot \mathbf{F} &= \frac{\partial}{\partial x}(2x^2y) + \frac{\partial}{\partial y}(3xz^3) + \frac{\partial}{\partial z}(xy^2z^2) \\ &= 4xy + 2xy^2z \end{aligned}$$

and

$$\begin{aligned} \text{curl } \mathbf{F} = \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x^2y & 3xz^3 & xy^2z^2 \end{vmatrix} \\ &= (2xyz^2 - 9xz^2)\mathbf{i} + (-y^2z^2)\mathbf{j} + (3z^3 - 2x^2)\mathbf{k}. \end{aligned}$$

There is clearly no difficulty about performing routine calculations of this kind. The real questions are, What do they mean and what is their value? Our purpose in the rest of this section is to explore the meaning of the divergence, and to do this we need the concept of flux.

THE MEANING OF THE DIVERGENCE

We shall use an example from hydrodynamics to motivate the ideas. Suppose that a stream of fluid (gas or liquid) is flowing through a region of space. At a given point (x, y, z), let its density be the scalar function $\delta = \delta(x, y, z)$ and its velocity the vector function $\mathbf{v} = \mathbf{v}(x, y, z)$, and consider the vector field $\mathbf{F} = \delta\mathbf{v}$. Now consider a small flat patch of surface inside the fluid, with area ΔA and unit normal vector $\mathbf{n}$, as shown in Fig. 21.21. If we think of this patch as a piece of screen or netting, so that the fluid can move through it without hindrance, we wish to find an expression for the amount of fluid that flows through the patch per unit time. It is clear from the figure that the fluid passing through the patch in a small time interval Δt forms a small tube of approximate volume $(\mathbf{v}\ \Delta t) \cdot \mathbf{n}\ \Delta A$, and the approximate mass of the fluid in this tube is $\delta(\mathbf{v}\ \Delta t) \cdot \mathbf{n}\ \Delta A$.* The approximate mass of fluid crossing the area ΔA per unit time is therefore $\delta\mathbf{v} \cdot \mathbf{n}\ \Delta A$ or $\mathbf{F} \cdot \mathbf{n}\ \Delta A$. This is called the *flux* of the vector field $\mathbf{F}$ through the area ΔA.

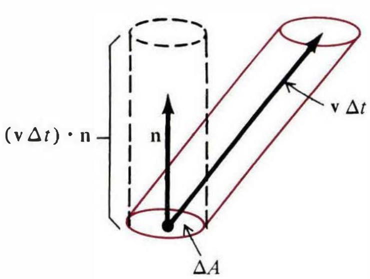


Figure 21.21

We now put forward an alternative definition for the divergence of $\mathbf{F}$ and then show that this new definition agrees with the one given above in formula (1). The

*We assume in this discussion that all functions are continuous, so when ΔA and Δt are very small, the vector $\mathbf{v}$ changes very little in direction or magnitude from one point of ΔA to another, and the density δ changes very little from one point of the tube to another.

purpose of this maneuver is to arrive at a way of thinking about the divergence that conveys an intuitive understanding of what it means.

Consider a point $P = (x, y, z)$ at the center of a small rectangular box with edges Δx, Δy, Δz, as shown in Fig. 21.22. We compute the total flux of the vector field $\mathbf{F}$ outward through the six faces of this box (i.e., on each face we choose $\mathbf{n}$ to be the outward unit normal). We then divide this total flux by the volume $\Delta V = \Delta x\, \Delta y\, \Delta z$ of the box, and form the limit of this flux per unit volume as the dimensions of the box approach zero. This is our new definition for the divergence of $\mathbf{F}$ at the point $P = (x, y, z)$:

$$\operatorname{div} \mathbf{F} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} (\text{flux of } \mathbf{F} \text{ out through the faces}). \tag{3}$$

Physically, this represents the mass of fluid that emerges from a small element of volume containing the point P, per unit time per unit volume.

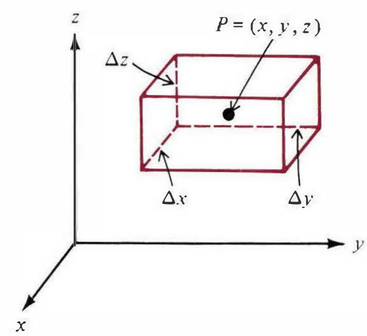


Figure 21.22

To show that this definition agrees with formula (1), we carry out a rough calculation of the limit (3), where $\mathbf{F} = L\mathbf{i} + M\mathbf{j} + N\mathbf{k}$. On the front face of the box in Fig. 21.22 we see that the outward unit normal is $\mathbf{i}$, so the flux out through this face is approximately $L(x + \frac{1}{2}\Delta x, y, z)\, \Delta y\, \Delta z$. Since the outward unit normal on the back face is $-\mathbf{i}$, the flux out through this face is approximately $-L(x - \frac{1}{2}\Delta x, y, z)\, \Delta y\, \Delta z$, and therefore the combined flux out through the front and back faces is approximately

$$[L(x + \tfrac{1}{2}\Delta x, y, z) - L(x - \tfrac{1}{2}\Delta x, y, z)]\, \Delta y\, \Delta z.$$

Similarly, the faces in the y-direction and z-direction contribute flux of approximate amounts

$$[M(x, y + \tfrac{1}{2}\Delta y, z) - M(x, y - \tfrac{1}{2}\Delta y, z)]\, \Delta x\, \Delta z$$

and

$$[N(x, y, z + \tfrac{1}{2}\Delta z) - N(x, y, z - \tfrac{1}{2}\Delta z)]\, \Delta x\, \Delta y.$$

We next divide the sum of these three quantities—the total flux out through all the faces of the box—by $\Delta V = \Delta x\, \Delta y\, \Delta z$ to obtain

$$\frac{L(x + \frac{1}{2}\Delta x, y, z) - L(x - \frac{1}{2}\Delta x, y, z)}{\Delta x} + \frac{M(x, y + \frac{1}{2}\Delta y, z) - M(x, y - \frac{1}{2}\Delta y, z)}{\Delta y} + \frac{N(x, y, z + \frac{1}{2}\Delta z) - N(x, y, z - \frac{1}{2}\Delta z)}{\Delta z}.$$

Finally, if we take the limit of this expression as $\Delta x, \Delta y, \Delta z \to 0$, then formula (3) yields the earlier definition (1), as stated,*

$$\operatorname{div} \mathbf{F} = \frac{\partial L}{\partial x} + \frac{\partial M}{\partial y} + \frac{\partial N}{\partial z}.$$

This result permits us to consider (3) as the basic definition of the divergence and (1) as merely a formula for computing it in rectangular coordinates.

*Here we use a slightly different way of defining the derivative of a function. See Additional Problem 9 in Chapter 2.

SURFACE INTEGRALS

Let S be a smooth surface and $f(x, y, z)$ a continuous function defined on S. The *surface integral* of f over S is denoted by

$$\iint_S f(x, y, z)\, dA, \tag{4}$$

and is defined as a limit of sums in the following way. We begin by subdividing the surface into n small pieces with areas $\Delta A_1, \Delta A_2, \ldots, \Delta A_n$. We next choose a point (x_i, y_i, z_i) on the ith piece, find the value $f(x_i, y_i, z_i)$ of the function at this point, multiply this value by the area ΔA_i to obtain the product $f(x_i, y_i, z_i)\, \Delta A_i$, and form the sum of these products,

$$\sum_{i=1}^{n} f(x_i, y_i, z_i)\, \Delta A_i. \tag{5}$$

Finally, we let n tend to infinity in such a way that the largest diameter of the pieces approaches zero; that is, we carry out a sequence of subdivisions of the surface S into smaller and smaller pieces, each time constructing a sum of the form (5). If these sums approach a limiting value, independent of the way the subdivisions are formed and the way the points (x_i, y_i, z_i) are chosen, then this limit is the definition of the surface integral (4):

$$\iint_S f(x, y, z)\, dA = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i, y_i, z_i)\, \Delta A_i.$$

It may be encouraging to students to know that only rarely do we actually evaluate a surface integral. It is the *concept* of these integrals that is important, because they provide a convenient language for expressing certain basic ideas of mathematics and physics.

To see what a surface integral can represent, we return to our example from hydrodynamics. Consider a fluid flowing through a certain region of space, and let $\delta = \delta(x, y, z)$ and $\mathbf{v} = \mathbf{v}(x, y, z)$ be its density and velocity, as before. Suppose that S is a smooth surface lying inside the region, and think of S as a curved piece of screen or netting that permits the fluid to pass through it without any resistance (Fig. 21.23). As we saw in our previous discussion, the mass of fluid crossing a surface element of area dA and unit normal $\mathbf{n}$ per unit time, is $\delta\mathbf{v} \cdot \mathbf{n}\, dA$ or $\mathbf{F} \cdot \mathbf{n}\, dA$, where $\mathbf{F} = \delta\mathbf{v}$. Accordingly, the surface integral

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dA \tag{6}$$

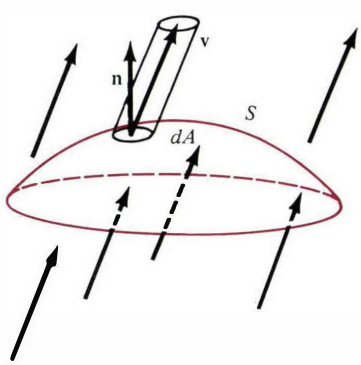


Figure 21.23

gives the rate of flow of the fluid through the entire surface S in terms of mass per unit time. This is called the *flux* of $\mathbf{F}$ through S.

More generally, if $\mathbf{F}$ is any vector field whatever, the surface integral (6) is still called the *flux* of $\mathbf{F}$ through the surface S. The physical meaning of this integral clearly depends on the nature of the physical quantity represented by $\mathbf{F}$. A variety of interpretations and applications arise by letting $\mathbf{F}$ be a vector field related to heat flow, or gravitation, or electricity, or magnetism. Hydrodynamics is only one of many subjects in which these concepts are useful.

We restricted ourselves to a smooth surface in the above discussion in order to guarantee that the unit normal vector $\mathbf{n}$ will be a continuous function of the position of its tail, and this in turn is necessary in order to guarantee that the integrand $\mathbf{F} \cdot \mathbf{n}$ in (6) is a continuous scalar function. A surface S is called *piecewise smooth* if it consists of a finite number of smooth pieces. The surfaces we work with are understood to be piecewise smooth, and the value of an integral of the form (6) over such a surface is defined to be the sum of its values over the smooth pieces.

GAUSS'S THEOREM

Surface integrals like (6) take on special importance when they are extended over closed surfaces. A surface S is said to be *closed* if it is the boundary of a bounded region of space. As examples we mention the surfaces of a sphere, a cube, a cylinder, and a tetrahedron.

Gauss's Theorem (also called the *Divergence Theorem*) states that

The flux of a vector field $\mathbf{F}$ *out through a closed surface* S *equals the integral of the divergence of* $\mathbf{F}$ *over the region* R *bounded by* S,

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dA = \iiint_R \operatorname{div} \mathbf{F}\, dV. \tag{7}$$

This is a rather crude statement, without any of the hypotheses or carefully formulated restrictions that characterize most respectable mathematical theorems. We shall provide an equally crude "proof"—which, however, has the great merit of showing at a glance why the theorem is true.

First, we use planes parallel to the coordinate planes to subdivide the region R into a great many small rectangular boxes of the kind shown in Fig. 21.24 (we ignore the incomplete boxes that do not lie wholly inside R). For the box in the figure, with volume ΔV, definition (3) tells us that the outward flux of $\mathbf{F}$ over the faces is given by the approximate formula

$$\text{flux of } \mathbf{F} \text{ over faces} \cong (\operatorname{div} \mathbf{F})\, \Delta V. \tag{8}$$

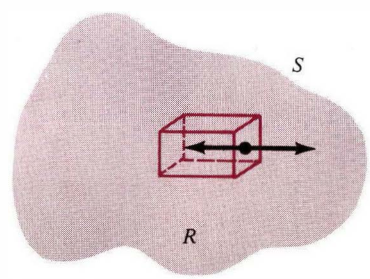


Figure 21.24

We now observe that the outward flux of $\mathbf{F}$ through the surface S is approximately equal to the total flux over all the faces of all the boxes, since for two adjacent boxes the outward flux from one through their common face precisely cancels the outward flux from the other through the same face, leaving only the flux through all the exterior faces. In view of (8), this tells us that

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dA \cong \Sigma(\operatorname{div} \mathbf{F})\, \Delta V.$$

Finally, by using the fact that the sum on the right is an approximating sum for the triple integral of the divergence of $\mathbf{F}$ over R, we obtain (7) by taking smaller and smaller subdivisions of R.

Example 2 Make a direct calculation of the flux of the vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ out through the surface of the cylinder whose lateral surface is $x^2 + y^2 = a^2$ and whose bottom and top are $z = 0$ and $z = b$. Also find this flux by applying Gauss's Theorem.

Solution On the lateral surface L we have $\mathbf{n} = (x\mathbf{i} + y\mathbf{j})/a$, so the flux over L is

$$\iint_L \mathbf{F}\cdot\mathbf{n}\,dA = \iint_L \frac{x^2+y^2}{a}\,dA = \iint_L a\,dA = a(2\pi ab) = 2\pi a^2 b.$$

On the top T we have $\mathbf{n} = \mathbf{k}$, so on T, $\mathbf{F}\cdot\mathbf{n} = z = b$ and the flux is

$$\iint_T \mathbf{F}\cdot\mathbf{n}\,dA = \iint_T b\,dA = b(\pi a^2) = \pi a^2 b.$$

On the bottom B we have $\mathbf{n} = -\mathbf{k}$, so on B, $\mathbf{F}\cdot\mathbf{n} = -z = 0$ and the flux is

$$\iint_B \mathbf{F}\cdot\mathbf{n}\,dA = \iint_B 0\,dA = 0.$$

Accordingly, the flux over the whole surface is $2\pi a^2 b + \pi a^2 b + 0 = 3\pi a^2 b$. To find this flux by applying Gauss's Theorem, we have only to notice that

$$\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3,$$

and therefore

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dA = \iiint_R \operatorname{div}\mathbf{F}\,dV = \iiint_R 3\,dV = 3(\text{volume}) = 3\pi a^2 b.$$

In Example 2 the integrations were too easy to present much of a technical challenge. We next consider a similar problem in which actual calculations are needed.

Example 3 Let S be the surface of the cylinder described in Example 2, and find the surface integral of the function $x^2 z$ over S.

Solution As before, S is piecewise smooth and we must integrate separately over the lateral area L, the top T, and the bottom B:

$$\iint_S x^2 z\,dA = \iint_L x^2 z\,dA + \iint_T x^2 z\,dA + \iint_B x^2 z\,dA.$$

The third integral here is clearly zero, because $z = 0$ on B. For the first integral we have (using cylindrical coordinates as shown in Fig. 21.25)

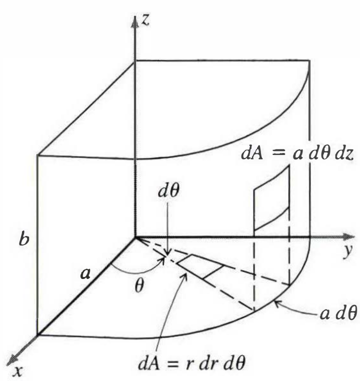


Figure 21.25

$$\begin{aligned}\iint_L x^2 z\,dA &= \int_0^b \int_0^{2\pi} (a\cos\theta)^2\, z(a\,d\theta\,dz)\\ &= a^3 \int_0^b \int_0^{2\pi} z\cos^2\theta\,d\theta\,dz\\ &= a^3 \int_0^b z\left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi} dz = \pi a^3 \int_0^b z\,dz = \frac{1}{2}\pi a^3 b^2.\end{aligned}$$

For the second integral we use $dA = r\,dr\,d\theta$, $x = r\cos\theta$, $z = b$, so

$$\begin{aligned}\iint_T x^2 z\,dA &= \int_0^{2\pi}\int_0^a (r\cos\theta)^2 b(r\,dr\,d\theta)\\ &= \frac{1}{4}a^4 b \int_0^{2\pi} \cos^2\theta\,d\theta\\ &= \frac{1}{4}a^4 b\left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi} = \frac{1}{4}\pi a^4 b.\end{aligned}$$

The total surface integral is therefore

$$\iint_S x^2 z \, dA = \frac{1}{2} \pi a^3 b^2 + \frac{1}{4} \pi a^4 b + 0$$
$$= \frac{1}{4} \pi a^3 b(2b + a).$$

Finally, we consider a problem in which spherical coordinates are used for calculating a surface integral.

Example 4 Let S be the surface of the solid (Fig. 21.26) bounded below by the xy-plane and above by the upper half of the sphere $x^2 + y^2 + z^2 = a^2$. Find the flux of the vector field $\mathbf{F} = z\mathbf{k}$ out through S by direct calculation, and also by using Gauss's Theorem.

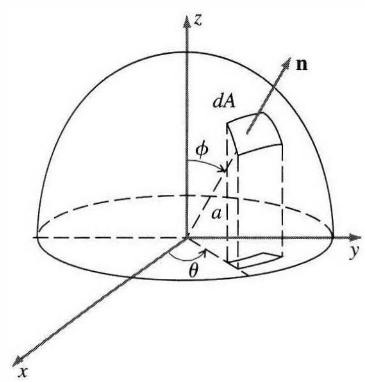


Figure 21.26

Solution Let B denote the bottom of the solid in the xy-plane, and T the hemispherical top. On B the unit normal vector $\mathbf{n}$ is given by $\mathbf{n} = -\mathbf{k}$, and on T we have $\mathbf{n} = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})/a$, so

$$\mathbf{F} \cdot \mathbf{n} = \begin{cases} -z = 0 & \text{on } B, \\ \dfrac{z^2}{a} = \dfrac{(a \cos \phi)^2}{a} = a \cos^2 \phi & \text{on } T. \end{cases}$$

This shows that the flux through B is zero, and since the element of area on T is $dA = (a\, d\phi)(a \sin \phi \, d\theta) = a^2 \sin \phi \, d\phi \, d\theta$, the total flux out through S is given by

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dA = \iint_T (a \cos^2 \phi)(a^2 \sin \phi \, d\phi \, d\theta)$$
$$= a^3 \int_0^{2\pi} \int_0^{\pi/2} \cos^2 \phi \sin \phi \, d\phi \, d\theta$$
$$= a^3 \int_0^{2\pi} \left[-\frac{1}{3} \cos^3 \phi \right]_0^{\pi/2} d\theta = a^3 \int_0^{2\pi} \frac{1}{3} \, d\theta = \frac{2}{3} \pi a^3.$$

To find this flux by using Gauss's Theorem, we merely observe that since $\operatorname{div} \mathbf{F} = 1$ we have

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dA = \iiint_R \operatorname{div} \mathbf{F} \, dV = \iiint_R dV = \text{volume} = \frac{2}{3} \pi a^3.$$

Gauss's Theorem is a profound theorem of mathematical analysis, with a wealth of important applications to many of the physical sciences. The cursory sketch of these ideas that we have given here—together with a similar sketch of Stokes' Theorem in the next section—is perhaps as far as an introductory calculus course should go in this direction. Students who wish to learn more are encouraged to continue and take advanced courses (vector analysis, potential theory, mathematical physics, etc.) in which these themes are fully developed.

PROBLEMS

1 Find the divergence of the vector field $\mathbf{F}$ if
(a) $\mathbf{F} = (y - z)\mathbf{i} + (z - x)\mathbf{j} + (x - y)\mathbf{k}$;
(b) $\mathbf{F} = (2z^2 - \sin e^y)\mathbf{i} + xy\mathbf{j} - xz\mathbf{k}$;
(c) $\mathbf{F} = xy\mathbf{i} + xz^3\mathbf{j} + (2z - yz)\mathbf{k}$;
(d) $\mathbf{F} = e^x \sin y\mathbf{i} + e^x \cos y\mathbf{j} + e^z \sin x\mathbf{k}$;
(e) $\mathbf{F} = \frac{x}{r}\mathbf{i} + \frac{y}{r}\mathbf{j} + \frac{z}{r}\mathbf{k}$, where $r = \sqrt{x^2 + y^2 + z^2}$.

In Problems 2–6, use Gauss's Theorem to find the flux of the given vector field over the given surface S.

2 $\mathbf{F} = x\mathbf{i} - y\mathbf{j} + z\mathbf{k}$; S is the surface of the cylinder in Example 2.

3 $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$; S is the surface of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

4 $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$; S is the surface of the tetrahedron formed by the plane $x/a + y/b + z/c = 1 (a, b, c > 0)$ and the coordinate planes.

5 $\mathbf{F} = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$; S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$.

6 $\mathbf{F} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$; S is the surface of the cone $z^2 = m^2(x^2 + y^2)$, $0 \le z \le h$.

7 If $\mathbf{R}$ is the position vector $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = \sqrt{x^2 + y^2 + z^2}$ is its length, find the divergence of the central force field $\mathbf{F} = f(r)(\mathbf{R}/r)$, where $f(r)$ is an arbitrary differentiable function.

8 Find the flux of the vector field defined in Problem 7 over the sphere $x^2 + y^2 + z^2 = a^2$ if
(a) $f(r) = r$; (b) $f(r) = \frac{1}{r^2}$.

9 If n is a positive number and $f(r) = 1/r^n$ in Problem 7, show that the divergence of the force field $\mathbf{F}$ is zero if $n = 2$, and only in this case.

10 Verify Gauss's Theorem for the vector field $\mathbf{F} = 2z\mathbf{i} + (x - y)\mathbf{j} + (2xy + z)\mathbf{k}$ and the rectangular box whose faces are $x = 0$, $x = 1$, $y = 0$, $y = 2$, $z = 0$, $z = 3$.

11 Use Gauss's Theorem to find the flux of $\mathbf{F}$ over the surface of the box in Problem 10 if
(a) $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$;
(b) $\mathbf{F} = xz\mathbf{i} + xy\mathbf{j} + yz\mathbf{k}$.

12 If $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + \mathbf{k}$, find the value of $\iint_S \mathbf{F} \cdot \mathbf{n}\, dA$ over
(a) the surface of the cube whose faces are $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$;
(b) the surface of the sphere $x^2 + y^2 + z^2 = 4$;
(c) the part of the plane $x + 2y + 3z = 6$ that lies in the first octant, where $\mathbf{n}$ points upward.

13 If S is a closed surface, show that the flux of the position vector $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ out through S is $3V$, where V is the volume of the region bounded by S. What is the flux through S of $\mathbf{F} = 27x\mathbf{i} - 11y\mathbf{j} + 4z\mathbf{k}$?

14 Use Gauss's Theorem to find the flux of the vector field $\mathbf{F} = x^2\mathbf{i} - 2xy\mathbf{j} + xyz^2\mathbf{k}$ out through the surface of the solid bounded by $z = 0$ and $z = \sqrt{a^2 - x^2 - y^2}$.

15 Let S be the surface of the sphere $x^2 + y^2 + z^2 = a^2$. If $\mathbf{F} = r^2(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$, where $r^2 = x^2 + y^2 + z^2$, find the flux of $\mathbf{F}$ out through S by direct calculation, and also by using Gauss's Theorem.

16 If S is the surface of the cube whose vertices are (0, 0, 0), (1, 0, 0), (1, 1, 0), (0, 1, 0), (0, 0, 1), (1, 0, 1), (1, 1, 1), and (0, 1, 1), verify Gauss's Theorem for the vector field $\mathbf{F} = xz\mathbf{i} + y^2\mathbf{j} + x\mathbf{k}$.

17 If S is the surface of the tetrahedron bounded by the coordinate planes and the plane $2x + 2y + z = 6$, verify Gauss's Theorem for the vector field $\mathbf{F} = x\mathbf{i} + y^2\mathbf{j} + z\mathbf{k}$.

18 If S is the surface of the solid bounded by $z = 0$ and $z = \sqrt{a^2 - x^2 - y^2}$, verify Gauss's Theorem for the vector field $\mathbf{F} = 2xz\mathbf{i} + yz\mathbf{j} + z^2\mathbf{k}$.

19 Let S be the surface of the solid bounded by the cylinder $x^2 + y^2 = 4$, the plane $x + z = 4$, and the xy-plane. Find the flux out through S of the vector field $\mathbf{F} = (x^2 + e^y)\mathbf{i} + (xy - \tan z)\mathbf{j} + \sin x\,\mathbf{k}$.

21.5 STOKES' THEOREM

Stokes' Theorem is an extension of Green's Theorem to three dimensions, involving curved surfaces and their boundaries rather than plane regions and their boundaries. Sir George Stokes (1819–1903) was an eminent British mathematical physicist. He introduced the theorem known by his name in an examination question for students at Cambridge University in 1854. A fairly full account of Stokes' personality and scientific work can be found in G. E. Hutchinson, *The Enchanted Voyage* (Yale University Press, 1962). We shall state the theorem after a few preliminaries that will help us understand its meaning.

Suppose that $\mathbf{F} = L\mathbf{i} + M\mathbf{j} + N\mathbf{k}$ is a vector field defined in a certain region of space. It will be convenient in this section to think of $\mathbf{F}$ as the velocity field of a flowing fluid. Suppose also that C is a curve that lies in the region and is specified by certain parametric equations. The line integral of $\mathbf{F}$ along C, denoted by

$$\int_C \mathbf{F} \cdot d\mathbf{R} \qquad \text{or} \qquad \int_C L\,dx + M\,dy + N\,dz,$$

is defined and calculated in just the same way as in two dimensions, and requires no further explanation. If C is a closed curve, as shown in Fig. 21.27, the line integral is usually written as

$$\oint_C \mathbf{F} \cdot d\mathbf{R}.$$

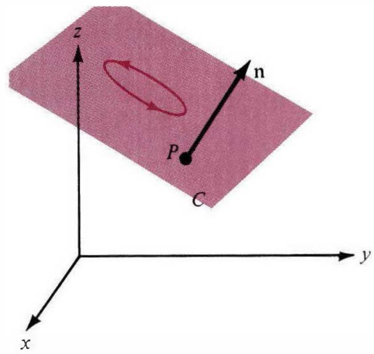


Figure 21.27

This integral measures the tendency of the fluid to circulate or swirl around C, and is called the *circulation* of $\mathbf{F}$ around C.

Now suppose that C is a small simple closed curve that lies in a plane with unit normal vector $\mathbf{n}$, where the direction of $\mathbf{n}$ is related to the direction of C by the right-hand thumb rule, and let P be a point inside C (Fig. 21.27). If ΔA is the area of the region enclosed by C, then

$$\frac{1}{\Delta A}\oint_C \mathbf{F} \cdot d\mathbf{R}$$

can be thought of as the circulation of $\mathbf{F}$ per unit area around P, and the limit

$$\lim_{\Delta A \to 0} \frac{1}{\Delta A}\oint_C \mathbf{F} \cdot d\mathbf{R}$$

is called the *circulation density* of $\mathbf{F}$ at P around $\mathbf{n}$. The point of these remarks is that this concept is closely related to the curl of the vector field $\mathbf{F}$, which was defined in Section 21.4 by

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ L & M & N \end{vmatrix}. \tag{1}$$

In fact, it can be shown that the curl of $\mathbf{F}$ has the property that at any point its component in a given direction $\mathbf{n}$ is precisely the circulation density of $\mathbf{F}$ around $\mathbf{n}$,

$$(\text{curl } \mathbf{F}) \cdot \mathbf{n} = \lim_{\Delta A \to 0} \frac{1}{\Delta A}\int_C \mathbf{F} \cdot d\mathbf{R}. \tag{2}$$

The proof of (2) is fairly complicated and will not be given here; it can be found in any good book on vector analysis.

We can visualize the meaning of (2) in a concrete way if we imagine a small paddle wheel placed in the flowing fluid at the point P with its axis pointing in the direction of $\mathbf{n}$ (Fig. 21.28). The circulation of the fluid around $\mathbf{n}$ will cause the paddle wheel to turn, and the speed at which it spins will be proportional to the circulation density. The paddle wheel will spin fastest when it points in the direction in which the circulation density is largest, and (2) tells us that this happens when $\mathbf{n}$ points in the same direction as curl $\mathbf{F}$. We conclude that at each point of space the vector curl $\mathbf{F}$ has the direction in which the circulation density is largest, with magnitude equal to this largest circulation density. The pad-

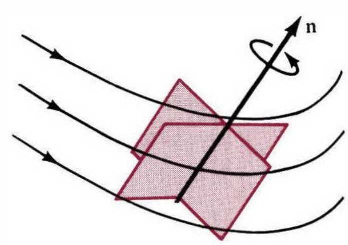


Figure 21.28

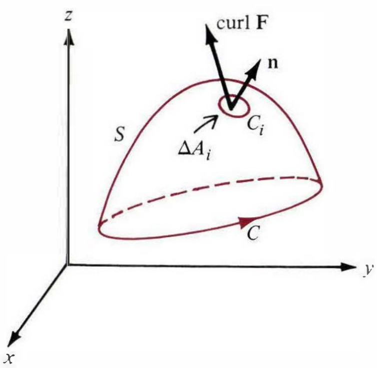


Figure 21.29

dle wheel we have described here can therefore be thought of as an imaginary instrument for sensing the direction and magnitude of the curl.

We are now ready for the main theorem of this section. *Stokes' Theorem* asserts the following (see Fig. 21.29):

If S is a surface in space with boundary curve C, then the circulation of a vector field **F** *around C is equal to the integral over S of the normal component of the curl of* **F**,

$$\oint_C \mathbf{F} \cdot d\mathbf{R} = \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n} \, dA. \tag{3}$$

Just as in the case of Gauss's Theorem in the preceding section, we choose to keep this statement as simple as possible and not complicate it with the hypotheses and restrictions that would be needed to convert it into a genuine mathematical theorem. For instance, it is necessary to assume that S is a two-sided (orientable) surface with the direction of the unit normal vector **n** related to the direction of C by the right-hand thumb rule, as shown in the figure. Also, of course, **n** must be continuous, **F** must be continuous, L, M, and N must have continuous partial derivatives, and so on.

This rough, intuitive version of Stokes' Theorem has a rough, intuitive "proof" based on equation (2). First, we subdivide the surface S into a large number of small patches with areas ΔA_i and boundary curves C_i. By applying (2) to the ith patch we obtain the approximate equation

$$\oint_{C_i} \mathbf{F} \cdot d\mathbf{R} \cong (\text{curl } \mathbf{F}) \cdot \mathbf{n} \, \Delta A_i.$$

If we add the left sides of these equations for all curves C_i, then the line integrals over all interior common boundaries cancel, being calculated once in each direction, leaving only the line integral around the exterior boundary C (see Fig. 21.30). This gives

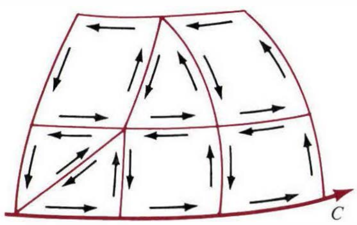


Figure 21.30

$$\oint_C \mathbf{F} \cdot d\mathbf{R} \cong \sum (\text{curl } \mathbf{F}) \cdot \mathbf{n} \, \Delta A_i, \tag{4}$$

and by taking smaller and smaller subdivisions we obtain (3), since the sums on the right side of (4) are approximating sums for the surface integral on the right side of (3).

Example 1 If the surface S is a region R lying flat in the xy-plane, then $\mathbf{n} = \mathbf{k}$ and by (1) we see that

$$(\text{curl } \mathbf{F}) \cdot \mathbf{n} = \frac{\partial M}{\partial x} - \frac{\partial L}{\partial y},$$

so (3) reduces to

$$\oint_C L\, dx + M\, dy = \iint_R \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} \right) dA.$$

This is Green's Theorem (Section 21.3), which is thus a special case of Stokes' Theorem.

Example 2 Evaluate the line integral

$$I = \oint_C y^3z^2\,dx + 3xy^2z^2\,dy + 2xy^3z\,dz$$

around the closed curve C whose vector equation is $\mathbf{R} = a \sin t\mathbf{i} + b\cos t\mathbf{j} + c\cos t\mathbf{k}$, $0 \le t \le 2\pi$, where $abc \neq 0$.

Solution This integral is the circulation around C of the vector field $\mathbf{F} = y^3z^2\mathbf{i} + 3xy^2z^2\mathbf{j} + 2xy^3z\mathbf{k}$. An easy calculation shows that curl $\mathbf{F} = \nabla \times \mathbf{F} = \mathbf{0}$. If S is any surface whose boundary is C, then the right side of (3) has the value 0 in this case, and therefore Stokes' Theorem tells us that $I = 0$.

Also, for this particular $\mathbf{F}$ and C it is not too difficult to verify Stokes' Theorem by calculating I directly. This gives

$$\begin{aligned} I = \int_0^{2\pi} [&(b^3\cos^3 t)(c^2\cos^2 t)(a\cos t) \\ &+ 3(a\sin t)(b^2\cos^2 t)(c^2\cos^2 t)(-b\sin t) \\ &+ 2(a\sin t)(b^3\cos^3 t)(c\cos t)(-c\sin t)]\,dt \\ = ab^3c^2 &\int_0^{2\pi} [\cos^6 t - 5\cos^4 t\sin^2 t]\,dt = ab^3c^2 \sin t\cos^5 t\Big]_0^{2\pi} = 0. \end{aligned}$$

In Section 21.2 we proved that three properties of vector fields in the plane are equivalent to one another. Stokes' Theorem makes it possible to extend these ideas in a natural way to three-dimensional space. Specifically, if $\mathbf{F}$ is a vector field defined in a simply connected region of space, then any one of the following four properties implies the remaining three:*

(a) $\oint_C \mathbf{F}\cdot d\mathbf{R} = 0$ for every simple closed curve C.
(b) $\int_C \mathbf{F}\cdot d\mathbf{R}$ is independent of the path.
(c) $\mathbf{F}$ is a gradient field, i.e., $\mathbf{F} = \nabla f$ for some scalar field f.
(d) curl $\mathbf{F} = \mathbf{0}$.

The equivalence of (a), (b), and (c) is established in just the same way as in two dimensions; the fact that (c) implies (d) is a straightforward calculation; and Stokes' Theorem enables us to show very easily that (d) implies (a). A vector field with any one of these properties is said to be *conservative* or *irrotational* [because of (d)].

For students who desire a fuller explanation of the reasons underlying the equivalence of these four properties, we offer the following details of the arguments.

To understand the equivalence of (a) and (b) we examine Fig. 21.31, in which C_1 and C_2 are two paths from A to B and C is the simple closed curve formed by tracing out C_1 and $-C_2$ in this order, where $-C_2$ means C_2 traced in the opposite direction. For these paths, property (b) tells us that

$$\int_{C_1} \mathbf{F}\cdot d\mathbf{R} = \int_{C_2} \mathbf{F}\cdot d\mathbf{R},$$

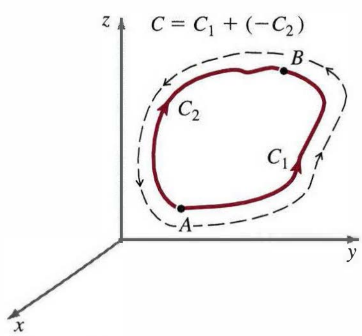


Figure 21.31

*A region in three-dimensional space is said to be *simply connected* if every simple closed curve in the region can be shrunk continuously to a point without leaving the region.

which is equivalent to

$$\int_{C_1} \mathbf{F}\cdot d\mathbf{R} - \int_{C_2} \mathbf{F}\cdot d\mathbf{R} = 0,$$

or

$$\int_{C_1} \mathbf{F}\cdot d\mathbf{R} + \int_{-C_2} \mathbf{F}\cdot d\mathbf{R} = 0,$$

or

$$\oint_{C} \mathbf{F}\cdot d\mathbf{R} = 0.$$

This shows that (b) implies (a), and the reasoning is clearly reversible.

To understand why (c) implies (b), we use the above notation and write

$$\begin{aligned}\int_{C_1} \mathbf{F}\cdot d\mathbf{R} &= \int_{C_1} \nabla f\cdot d\mathbf{R}\\ &= \int_{C_1}\left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right)\cdot(dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})\\ &= \int_{C_1}\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz\\ &= \int_{C_1} df = f(B) - f(A).\end{aligned}$$

Our conclusion now follows from the fact that the expression last written depends only on the points A and B, and not at all on the path of integration. (A slightly more detailed treatment of this reasoning for the two-dimensional case is given in Section 21.2.)

To show that (b) implies (c), we must use independence of path to construct a potential function $f(x, y, z)$. This is easy to do by choosing a fixed point (x_0, y_0, z_0) and integrating $\mathbf{F}$ along any path from (x_0, y_0, z_0) to a variable point (x, y, z), as suggested in Fig. 21.32. Since the value of the integral is independent of the choice of path, this integral is a function only of the point (x, y, z), and defines our potential function:

$$f(x, y, z) = \int_{(x_0,y_0,z_0)}^{(x,y,z)} \mathbf{F}\cdot d\mathbf{R}.$$

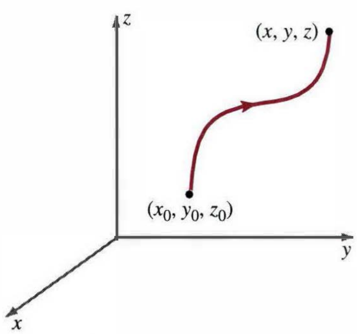


Figure 21.32

The next step is to show that $\nabla f = \mathbf{F}$ by using the calculations given for the two-dimensional case in Section 21.2, but we do not repeat these details.

To prove that (c) implies (d) by the "straightforward calculation" mentioned above, we have only to write

$$\begin{aligned}\text{curl } \mathbf{F} = \text{curl } \nabla f &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z}\end{vmatrix}\\ &= \mathbf{i}\left(\frac{\partial^2 f}{\partial y\partial z} - \frac{\partial^2 f}{\partial z\partial y}\right) + \mathbf{j}\left(\frac{\partial^2 f}{\partial z\partial x} - \frac{\partial^2 f}{\partial x\partial z}\right) + \mathbf{k}\left(\frac{\partial^2 f}{\partial x\partial y} - \frac{\partial^2 f}{\partial y\partial x}\right),\end{aligned}$$

because this expression vanishes by the equality of the mixed partial derivatives.

To establish the final implication, that (d) implies (a), we consider a simple closed curve C, as shown in Fig. 21.33. Since our region is simply connected, C can be shrunk continuously to a point without leaving the region. In this shrinking process, C sweeps out a surface S, and by Stokes' Theorem we have

$$\oint_C \mathbf{F} \cdot d\mathbf{R} = \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n} \, dA.$$

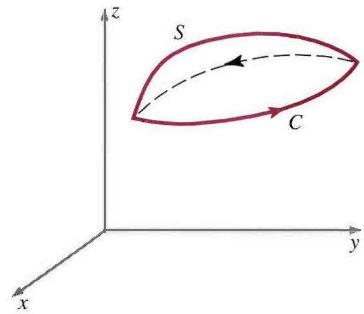


Figure 21.33

The integral on the right equals 0 because of our assumption that curl $\mathbf{F} = \mathbf{0}$, and this tells us that

$$\oint_C \mathbf{F} \cdot d\mathbf{R} = 0,$$

which completes the argument.

One final remark: The relations among properties (a) through (d) will not be truly understood until we reach the stage at which the implications described above can be grasped as an organic whole and recalled in a few seconds of thought.

We have seen that Gauss's Theorem relates an integral over a closed surface to a corresponding volume integral over the region of space enclosed by the surface, and Stokes' Theorem relates an integral around a closed curve to a corresponding surface integral over any surface bounded by the curve. As we suggested at the beginning of Section 21.4, these statements are very similar and are presumably somehow connected with each other. It turns out that both are special cases of a powerful theorem of modern analysis called the *generalized Stokes Theorem.* Students who wish to understand these relationships must study the theory of differential forms.

PROBLEMS

1 If S is a closed surface that lies in a region of space in which a vector field $\mathbf{F}$ is defined, show that

$$\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n} \, dA = 0.$$

2 If the vector field $\mathbf{F} = -y\mathbf{i} + x\mathbf{j} + 0\mathbf{k}$ is the velocity field of a flowing fluid, sketch enough of this field in the xy-plane (that is, sketch enough of the velocity vectors attached to various points) to understand the nature of the motion. Then calculate curl $\mathbf{F}$, let C be the circle $\mathbf{R} = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j} + 0\mathbf{k}$ $(0 \le \theta \le 2\pi)$ in the xy-plane, and verify the formula

$$(\text{curl } \mathbf{F}) \cdot \mathbf{n} = \frac{1}{A}\oint_C \mathbf{F} \cdot d\mathbf{R}$$

for this circle and its interior in the xy-plane.

3 Repeat Problem 2 for the vector fields
(a) $\mathbf{F} = ax\mathbf{j}$, where a is a positive constant;
(b) $\mathbf{F} = f(r)\mathbf{R}$, where r is the length of the position vector $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $f(r)$ is an arbitrary differentiable function.

In Problems 4–10, apply Stokes' Theorem to find $\oint_C \mathbf{F} \cdot d\mathbf{R}$ for the given $\mathbf{F}$ and the given C. In each case let C be oriented counterclockwise as seen from above.

4 $\mathbf{F} = y(x - z)\mathbf{i} + (2x^2 + z^2)\mathbf{j} + y^3 \cos xz\,\mathbf{k}$; C is the boundary of the square $0 \le x \le 2$, $0 \le y \le 2$, $z = 5$.

5 $\mathbf{F} = (z - y)\mathbf{i} + y\mathbf{j} + x\mathbf{k}$; C is the intersection of the top half of the sphere $x^2 + y^2 + z^2 = 4$ with the cylinder $r = 2\cos\theta$.

6 $\mathbf{F} = y\mathbf{i} + (x + y)\mathbf{j} + (x + y + z)\mathbf{k}$; C is the ellipse in which the plane $z = x$ intersects the cylinder $x^2 + y^2 = 1$.

7 $\mathbf{F} = (y - x)\mathbf{i} + (x - z)\mathbf{j} + (x - y)\mathbf{k}$; C is the boundary of the triangular part of the plane $x + y + 2z = 2$ that lies in the first octant.

8 $\mathbf{F} = (3y + z)\mathbf{i} + (\sin y - 3x)\mathbf{j} + (e^z + x)\mathbf{k}$; C is the circle $x^2 + y^2 = 1$, $z = 5$.

9 $\mathbf{F} = 2z\mathbf{i} + 6x\mathbf{j} - 3y\mathbf{k}$; C is the ellipse in which the plane $z = y + 1$ intersects the cylinder $x^2 + y^2 = 1$.

10 $\mathbf{F} = e^{x^2}\mathbf{i} + (x + z)\sin y^3\mathbf{j} + (y^2 - x^2 + 2yz)\mathbf{k}$; C is the boundary of the triangular part of the plane $x + y + z = 3$ that lies in the first octant.

In Problems 11–14, verify Stokes' Theorem for the given **F**, *S*, and *C*.

11 $\mathbf{F} = (z - y)\mathbf{i} + (x + z)\mathbf{j} - (x + y)\mathbf{k}$, S is the part of the paraboloid $z = 9 - x^2 - y^2$ that lies above the xy-plane, and C is its boundary circle $x^2 + y^2 = 9$ in the xy-plane, oriented counterclockwise as seen from above.

12 $\mathbf{F} = xy\mathbf{i} + yz\mathbf{j} + zx\mathbf{k}$, S is the part of the plane $x + y + z = 1$ that lies in the first octant, and C is its boundary oriented counterclockwise as seen from above.

13 $\mathbf{F} = y\mathbf{i} - x\mathbf{j}$, S is the top half of the sphere $x^2 + y^2 + z^2 = 4$, and C is its boundary circle $x^2 + y^2 = 4$ in the xy-plane, oriented counterclockwise as seen from above.

14 $\mathbf{F} = (x + y)\mathbf{i} + (y + z)\mathbf{j} + (z + x)\mathbf{k}$, S is the elliptical disk $x^2/a^2 + y^2/b^2 \le 1$, $z = 0$, and C is its boundary oriented counterclockwise as seen from above.

15 Let S be the top half of the ellipsoid $x^2 + y^2 + z^2/9 = 1$, oriented so that $\mathbf{n}$ is directed upward. If $\mathbf{F} = x^3\mathbf{i} + y^4\mathbf{j} + z^3 \sin xy\,\mathbf{k}$, evaluate

$$\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dA$$

by replacing S by a simpler surface with the same boundary.

16 Repeat Problem 15 if $\mathbf{F} = xz^2\mathbf{i} + x^3\mathbf{j} + \cos xz\mathbf{k}$ and S is the top half of the ellipsoid $x^2 + y^2 + 4z^2 = 1$ with $\mathbf{n}$ directed upward.

21.6 MAXWELL'S EQUATIONS. A FINAL THOUGHT

To gain a slight glimpse of the significance of the ideas of this chapter, we look very briefly at the famous equations formulated in the 1860s by James Clerk Maxwell (1831–1879). These equations are remarkable because they contain a complete theory of everything that was then known or would later become known about electricity and magnetism. Maxwell was the greatest theoretical physicist of the nineteenth century, and an excellent account of his life and work is given by James R. Newman in *Science and Sensibility,* vol. 1, pp. 139–193 (Simon and Schuster, 1961).

In Maxwell's theory there are two vector fields defined at every point in space: an electric field **E** and a magnetic field **B**. The electric field is produced by charged particles (electrons, protons, etc.) that may be moving or stationary, and the magnetic field by moving charged particles.

All known phenomena involving electromagnetism can be explained and understood by means of *Maxwell's equations*:

1 $\nabla \cdot \mathbf{E} = \dfrac{q}{\epsilon_0}.$

2 $\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}.$

3 $\nabla \cdot \mathbf{B} = 0.$

4 $c^2\nabla \times \mathbf{B} = \dfrac{\mathbf{j}}{\epsilon_0} + \dfrac{\partial \mathbf{E}}{\partial t}.$

Here q is the charge density, ϵ_0 is a constant, c is the velocity of light, and $\mathbf{j}$ is the current density (not to be confused with the unit vector in the direction of the y-axis). We make no attempt to discuss the meaning of these four equations, but we do point out that the first two make statements about the divergence and curl of **E**, and the second two about the divergence and curl of B. Equivalent verbal statements of Maxwell's equations are given by Richard Feynman (Nobel

Prize, 1965) on p. 18-2 in vol. 2 of his *Lectures on Physics* (Addison-Wesley, 1964):

1′ Flux of **E** through a closed surface $= \dfrac{\text{charge inside}}{\epsilon_0}$.

2′ Line integral of **E** around a loop $= -\dfrac{\partial}{\partial t}$ (flux of **B** through the loop).

3′ Flux of **B** through a closed surface $= 0$.

4′ c^2 (integral of **B** around a loop) $= \dfrac{\text{current through the loop}}{\epsilon_0} + \dfrac{\partial}{\partial t}$ (flux of **E** through the loop).

By a "loop," Feynman means what we have called a simple closed curve. The fact that these verbal statements are indeed equivalent to Maxwell's equations 1 to 4 depends on Gauss's Theorem and Stokes' Theorem. This is perhaps easier to grasp when these verbal statements are expressed in terms of line and surface integrals:

1″ $\displaystyle\iint_S \mathbf{E}\cdot\mathbf{n}\,dA = \frac{Q}{\epsilon_0}$ (S is a closed surface).

2″ $\displaystyle\oint_C \mathbf{E}\cdot d\mathbf{R} = -\frac{\partial}{\partial t}\iint_S \mathbf{B}\cdot\mathbf{n}\,dA$ (C is a simple closed curve and S is a surface bounded by C).

3″ $\displaystyle\iint_S \mathbf{B}\cdot\mathbf{n}\,dA = 0$ (S is a closed surface).

4″ $\displaystyle c^2\oint_C \mathbf{B}\cdot d\mathbf{R} = \frac{1}{\epsilon_0}\iint_S \mathbf{j}\cdot\mathbf{n}\,dA + \frac{\partial}{\partial t}\iint_S \mathbf{E}\cdot\mathbf{n}\,dA$ (C is a simple closed curve and S is a surface bounded by C).

Our only purpose in mentioning these matters is to try to make it perfectly clear to the student that the mathematics we have been doing in this chapter has profoundly important applications in physical science. Feynman devotes the first 21 chapters in vol. 2 of his *Lectures* to the meaning and implications of Maxwell's equations. At one point he memorably remarks:

> From a long view of the history of mankind—seen from, say, ten thousand years from now—there can be little doubt that the most significant event of the 19th century will be judged as Maxwell's discovery of the laws of electrodynamics. The American Civil War will pale into provincial insignificance in comparison with this important scientific event of the same decade.

In making this provocative comment, perhaps Feynman was carried away by his ebullient enthusiasm—but perhaps not.

CHAPTER 21 REVIEW: CONCEPTS, THEOREMS

Think through the following.

1 Line integral.
2 Fundamental Theorem of Calculus for line integrals.
3 Conservative field in the plane: three equivalent properties.
4 Green's Theorem.
5 Divergence of a vector field: definition, formula.
6 Surface integral.
7 Gauss's Theorem (or the Divergence Theorem).
8 Curl of a vector field: definition, meaning.
9 Stokes' Theorem.
10 Conservative (or irrotational) field in space: four equivalent properties.

APPENDIX

A THE THEORY OF CALCULUS

I mean the word proof not in the sense of the lawyers, who set two half proofs equal to a whole one, but in the sense of the mathematician, where $\frac{1}{2}$ proof $= 0$ and it is demanded for proof that every doubt becomes impossible.

Carl Friedrich Gauss

Certitude is not the test of certainty. We have been cocksure of many things that were not so.

Oliver Wendell Holmes

A.1 THE REAL NUMBER SYSTEM

When considered for its own sake, and apart from any uses it may have, the real number system appears as an intricate intellectual structure whose endless complexities are of interest mainly to mathematicians. However, from the practical point of view, it is the foundation on which virtually all other branches of mathematics rest, and as such, it underlies every quantitative aspect of civilized life.

Most of us learn in school how to use real numbers for counting, measurement, and solving algebraic problems. Nevertheless, no matter how much skill of this kind we develop, few of us ever confront the question of just what the real numbers *are*. Our purpose here is to answer this question as briefly and clearly as possible. In doing so, we will also provide an adequate basis for the capsule discussions of the theory of calculus that are given in the following sections.

There are several ways to introduce the real number system. We adopt the most efficient of these—the axiomatic approach—in which we start with the real numbers themselves as given undefined objects possessing certain simple properties that we use as axioms. This means we assume there exists a set R of objects, called *real numbers*, that satisfy the 10 axioms listed in the following pages. All further properties of real numbers, regardless of how profound they may be, are ultimately provable as logical consequences of these. The axioms fall into three natural groups. Those of the first group are stated in terms of the two operations $+$ and $\cdot$, addition and multiplication, which can be applied to any pair x and y of real numbers to produce their sum $x + y$ and their product $x \cdot y$ (also denoted more simply by xy).

Algebra Axioms

1 Commutative laws: $x + y = y + x$, $xy = yx$.
2 Associative laws: $x + (y + z) = (x + y) + z$, $x(yz) = (xy)z$.

4 Existence of identity elements: There exist two distinct real numbers, denoted by 0 and 1, such that $0 + x = x + 0 = x$ and $1 \cdot x = x \cdot 1 = x$ for every x.
5 Existence of negatives: For each x there exists a unique y such that $x + y = y + x = 0$.
6 Existence of reciprocals: For each $x \neq 0$ there exists a unique z such that $xz = zx = 1$.

The number y in axiom 5 is customarily denoted by $-x$, and z in axiom 6 by $1/x$ or x^{-1}. Subtraction and division can now be defined by $x - y = x + (-y)$ and $x/y = x(1/y)$. All the usual laws of elementary algebra can be deduced from these axioms and these definitions. We illustrate this process by giving three very brief proofs.

Example 1 (i) $x + y = x + z$ implies $y = z$ (the cancellation law of addition). *Proof*: Since $x + y = x + z$, $(-x) + (x + y) = (-x) + (x + z)$; by axiom 2, $[(-x) + x] + y = [(-x) + x] + z$; by axiom 5, $0 + y = 0 + z$; and by axiom 4, $y = z$.

(ii) $x \cdot 0 = 0$. *Proof*: Axiom 4 gives $0 + 1 = 1$, so $x(0 + 1) = x \cdot 1$; by axiom 3, $x \cdot 0 + x \cdot 1 = x \cdot 1$; by axiom 4, $x \cdot 0 + x = x = 0 + x$; by axiom 1, $x + x \cdot 0 = x + 0$; and by (i), $x \cdot 0 = 0$.

(iii) $(-1)(-1) = 1$. *Proof*: Axiom 5 gives $1 + (-1) = 0$, so on multiplying by -1 and using axioms 3, 4, and (ii) we obtain $(-1) + (-1)(-1) = 0$; and adding 1 to both sides of this yields $(-1)(-1) = 1$ after careful reduction.

The next group of axioms enables us to establish an order relation in the real number system. It is convenient to introduce this relation indirectly, by basing it on a concept of positiveness. This means we assume there exists in R a special subset P, called the set of positive numbers, that satisfies the three axioms listed below. The statement that a number x is in the set P is symbolized by writing $0 < x$, or equivalently $x > 0$.

Order Axioms

7 For each x, one and only one of the following possibilities is true: $x = 0$, $x > 0$, $-x > 0$.
8 If x and y are positive, so is $x + y$.
9 If x and y are positive, so is xy.

We now introduce the familiar order relations $<$ and $>$ as follows: $x < y$ is defined to mean $y - x > 0$, and $x > y$ is equivalent to $y < x$. As usual, $x \leq y$ means that $x < y$ or $x = y$, and $x \geq y$ is equivalent to $y \leq x$. All the customary rules for working with inequalities can be proved as theorems on the basis of these axioms and definitions.

Example 2 It is quite easy to show that for any real numbers x and y one and only one of these possibilities is true: $x = y$, $x < y$, $x > y$ (proof: apply axiom 7 to the number $y - x$). We next consider the proofs of the following familiar facts:

If $x < y$ and $y < z$, then $x < z$.

If $x > 0$ and $y < z$, then $xy < xz$.

If $x < 0$ and $y < z$, then $xy > xz$.

If $x < y$ then $x + z < y + z$ for any z.

The definitions allow us to express these statements in equivalent forms that are more convenient from the point of view of providing proofs:

If $y - x > 0$ and $z - y > 0$, then $z - x > 0$.

If $x > 0$ and $z - y > 0$, then $xz - xy > 0$.

If $-x > 0$ and $z - y > 0$, then $xy - xz > 0$.

If $y - x > 0$, then $(y + z) - (x + z) > 0$ for any z.

The first of these assertions is an obvious consequence of axiom 8, the second and third follow directly from axiom 9, and the fourth is trivial, since $(y + z) - (x + z) = y - x$.

The program of carefully deducing all the algebraic and order properties of R from axioms 1 to 9 is rather long and boring, and no useful purpose would be served by pursuing this aspect of the matter any further. It is quite enough for students to understand that this program can be carried out, and we omit the details.

The nine axioms given above do not fully determine the real number system. This is most easily seen by noticing that the set Q of all rational numbers is a number system different from R that also satisfies all nine axioms. Of course, the difference between Q and R is simply that Q lacks the irrationals, which any workable number system ought to have. One more axiom is needed to guarantee that R is free from this defect, or equivalently that the real number system has no "gaps" or "holes."

Two preliminary definitions are necessary before our final axiom can be stated. Both refer to an arbitrary set S of real numbers. A real number b is called an *upper bound* for S if $x \leq b$ for every x in S. Further, a real number b_0 is called a *least upper bound* for S if (i) b_0 is an upper bound for S, and (ii) $b_0 \leq b$ for every upper bound b of S. A set has many upper bounds if it has one, but it can have only one least upper bound. The proof is easy: If b_0 and b_1 are both least upper bounds for S, then $b_0 \leq b_1$ (since b_0 is a least upper bound and b_1 is an upper bound) and $b_1 \leq b_0$ (since b_1 is a least upper bound and b_0 is an upper bound), so $b_0 = b_1$. This argument permits us to speak of *the* least upper bound of S. These concepts can be visualized in the usual way, as suggested by Fig. A.1.

Example 3 The set of all positive integers has no upper bound. If S is the closed interval $0 \leq x \leq 1$, then the numbers 1, 2, 3.74, and 513 (among others) are all upper bounds of S, and 1 is its least upper bound. The same statements are true if S is the open interval $0 < x < 1$. In the first case, the least upper bound 1 belongs to the set S, but in the second case it does not. The set S consisting of all numbers in the sequence

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots$$

also has 1 as its least upper bound.

The following is the final axiom for the real number system R.

Least Upper Bound Axiom

10 Every nonempty set of real numbers that has an upper bound also has a least upper bound.

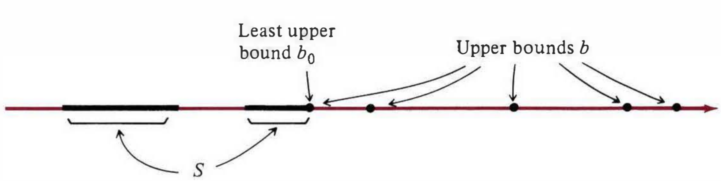


Figure A.1

This axiom guarantees that the real number system has the property of "completeness" or "continuity" that is absolutely essential for the development of calculus. The best way to grasp the significance of this axiom is to observe that it is not true for the set Q of rational numbers: If S is taken to be the set of positive rationals r such that $r^2 < 2$, then S has an upper bound in Q but does not have a least upper bound in Q (the least upper bound of S in R is $\sqrt{2}$, but this number is not in Q).

Remark 1 We have implied, but not actually stated, that the 10 axioms given here completely characterize the real number system R. The meaning of this statement can be clarified by formulating our ideas on a more abstract level, as follows. In modern algebra a set of objects that satisfies axioms 1 to 6 is called a *field.* There are many different fields, some finite and others infinite. The simplest consists of the two elements 0 and 1 alone, with addition and multiplication defined by

$$0 + 0 = 0, \qquad 0 + 1 = 1 + 0 = 1, \qquad 1 + 1 = 0,$$

$$0 \cdot 0 = 0, \qquad 0 \cdot 1 = 1 \cdot 0 = 0, \qquad 1 \cdot 1 = 1.$$

A field that satisfies the additional axioms 7 to 9 is called an *ordered field.* Both Q and R are ordered fields, but there are also a number of others. It can be proved that an ordered field must have infinitely many distinct elements, so some fields—including the two-element field just mentioned—cannot be ordered. We use axiom 10 to narrow our scope still further, and an ordered field that satisfies this axiom is called a *complete ordered field.* It can be proved that any two complete ordered fields are abstractly identical in a very precise sense, so there is really only one, namely, R.* It is therefore possible to define a real number very simply, as an element in a complete ordered field. However, it is clear that no such definition can be considered satisfactory without a good deal of preliminary explanation and proof.

Remark 2 There may be a few exceedingly skeptical readers who find themselves thinking thoughts like these: "What this writer says sounds reasonable enough, *provided the real number system R exists in the first place.* But how do we know that it does? After all, this number system is not a physical object that can be seen and touched, but a creation of the mind—like a unicorn—and perhaps we deceive ourselves by supposing that it exists."

There are two ways to answer this objection. One is to give a concrete definition of R as the set of all infinite decimals, with the usual agreement that such decimals as 0.25000 . . . and 0.24999 . . . are to be considered equal. Addition, multiplication, and the set of positive numbers must now be given satisfactory definitions, and in this scheme of things our axioms 1 to 10 become theorems whose proofs lean heavily on these definitions. This program is surprisingly difficult to carry out.†

A second approach is to use the much more basic positive integers as a given supply of building materials for the explicit step-by-step construction of the real number system —first the integers, then the rationals, and finally the reals. This time the axioms 1 to 10 appear as theorems that can be deduced from assumed properties of the positive integers.‡

We do not encourage students to investigate these matters any further, for there is no part of mathematics more tedious and unrewarding than the detailed construction of the real number system by either of these methods.

*For a fuller discussion, with proofs (or sketches of proofs), see pp. 1–8 in the first volume of E. Hille's *Analytic Function Theory* (Ginn and Co., 1959).

†See Chapter 1 of J. F. Ritt's *Theory of Functions* (King's Crown Press, 1947).

‡The classical source for this construction is E. Landau, *Foundations of Analysis* (Chelsea, 1951).

A.2 THEOREMS ABOUT LIMITS

We begin by recalling the definition of the limit of a function from Section 2.5. Consider a function $f(x)$ that is defined for values of x arbitrarily close to a point a on the x-axis but not necessarily at a itself. Another way to express this requirement is to say that there are x's in the domain of the function that satisfy the inequalities $0 < |x - a| < \delta$ for every positive number δ. Under these circumstances, the statement that

$$\lim_{x \to a} f(x) = L$$

is defined to mean the following: For each positive number ϵ there exists a positive number δ with the property that

$$|f(x) - L| < \epsilon$$

for every x in the domain of the function that satisfies the inequalities

$$0 < |x - a| < \delta.$$

In the hope of clarifying the meaning of this definition, we examine the way it is used in a simple special case. It is obvious by inspection that

$$\lim_{x \to 1} (3x - 1) = 2. \tag{1}$$

However, to prove this by using the definition, we must start with an arbitrary positive number ϵ and find a $\delta > 0$ that "works" for this ϵ, in the sense that

$$0 < |x - 1| < \delta \qquad \text{implies} \qquad |(3x - 1) - 2| < \epsilon. \tag{2}$$

But the last inequality here is the same as $|3x - 3| < \epsilon$ or $-\epsilon < 3x - 3 < \epsilon$; and after division by 3 this becomes $-\frac{1}{3}\epsilon < x - 1 < \frac{1}{3}\epsilon$. This suggests that $\delta = \frac{1}{3}\epsilon$ might work. To show that it does, we observe that if $0 < |x - 1| < \frac{1}{3}\epsilon$ then $-\frac{1}{3}\epsilon < x - 1 < \frac{1}{3}\epsilon$, which implies $-\epsilon < 3x - 3 < \epsilon$ or $|(3x - 1) - 2| < \epsilon$. Thus, for any $\epsilon > 0$ the number $\delta = \frac{1}{3}\epsilon$ actually has the property stated in (2). The requirement of the definition is therefore satisfied, and (1) is proved.

It is natural to object to this procedure and to feel that carefully proving a transparent statement like (1) is empty mumbo-jumbo and a waste of time. However, the point is this: (1) is obviously true and doesn't really need a proof, but many important limits are far from obvious and cannot be dealt with by simple inspection. For instance, it is no exaggeration to say that large parts of advanced mathematics would disappear like a puff of smoke without the ideas and methods that depend on the vital limits

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \qquad \text{and} \qquad \lim_{x \to 0} (1 + x)^{1/x} = e.$$

(The fundamental constant denoted by e is officially introduced in Chapter 8; its approximate value is 2.71828.) We clearly need powerful tools to cope with limits like these, not vague ideas and fuzzy concepts. We have proved (1) not for its own sake, but in order to illustrate the use of the definition of the limit of a function. This definition is not intended to be merely a passive description in the sense of most dictionary definitions. On the contrary, it is a sharp-edged instrument of proof that is capable of being manipulated effectively in complex and subtle arguments where sloppy thinking brings nothing but confusion. We have two purposes in the theorems and proofs given below: first, to establish the results themselves, and thereby provide a solid logical foundation for all of our work that depends on limits of functions; and second, to further illustrate the use of the definition in the machinery of formal proofs.

Our first theorem states a fact that most people take for granted without fully realizing it, namely, that a function $f(x)$ cannot approach two different limits as x approaches a.

Theorem 1 *If* $\lim_{x\to a} f(x) = L_1$ *and* $\lim_{x\to a} f(x) = L_2$, *then* $L_1 = L_2$.

Proof Our method of proof is to show that the assumption $L_1 \neq L_2$ leads to the absurd conclusion $|L_1 - L_2| < |L_1 - L_2|$. We therefore assume that $L_1 \neq L_2$, so that $|L_1 - L_2|$ is positive, and we let ϵ be the positive number $\frac{1}{2}|L_1 - L_2|$. By the first hypothesis there exists a number $\delta_1 > 0$ such that

$$0 < |x - a| < \delta_1 \qquad \text{implies} \qquad |f(x) - L_1| < \epsilon,$$

and by the second hypothesis there exists a number $\delta_2 > 0$ such that

$$0 < |x - a| < \delta_2 \qquad \text{implies} \qquad |f(x) - L_2| < \epsilon.$$

Define δ to be the smaller of the numbers δ_1 and δ_2. Then $0 < |x - a| < \delta$ implies both

$$|f(x) - L_1| < \epsilon \qquad \text{and} \qquad |f(x) - L_2| < \epsilon,$$

and therefore

$$\begin{aligned} |L_1 - L_2| &= |[L_1 - f(x)] + [f(x) - L_2]| \\ &\leq |L_1 - f(x)| + |f(x) - L_2| \\ &< \epsilon + \epsilon = 2\epsilon = |L_1 - L_2|. \end{aligned}$$

This contradiction—that the number $|L_1 - L_2|$ is less than itself—shows that it cannot be true that $|L_1 - L_2|$ is positive, so $L_1 = L_2$.

Theorem 2 *If* $f(x) = x$, *then* $\lim_{x\to a} f(x) = a$; *that is,*

$$\lim_{x\to a} x = a.$$

Proof Let $\epsilon > 0$ be given, and choose $\delta = \epsilon$. Then $0 < |x - a| < \delta = \epsilon$ implies that $|f(x) - a| < \epsilon$, since $f(x) = x$.

Theorem 3 *If* $f(x) = c$, *where* c *is a constant, then* $\lim_{x\to a} f(x) = c$; *that is,*

$$\lim_{x\to a} c = c.$$

Proof Since $|f(x) - c| = |c - c| = 0$ for all x, any $\delta > 0$ will do, because $|f(x) - c|$ will be $< \epsilon$ for any given $\epsilon > 0$ and all x.

Theorem 4 *If* $\lim_{x\to a} f(x) = L$ *and* $\lim_{x\to a} g(x) = M$, *then*

(i) $\lim_{x\to a} [f(x) + g(x)] = L + M$;
(ii) $\lim_{x\to a} [f(x) - g(x)] = L - M$; *and*
(iii) $\lim_{x\to a} f(x)g(x) = LM$.

Proof For (i), let $\epsilon > 0$ be given, let $\delta_1 > 0$ be a number such that

$$0 < |x - a| < \delta_1 \qquad \text{implies} \qquad |f(x) - L| < \tfrac{1}{2}\epsilon,$$

and let $\delta_2 > 0$ be a number such that

$$0 < |x - a| < \delta_2 \qquad \text{implies} \qquad |g(x) - M| < \tfrac{1}{2}\epsilon.$$

Define δ to be the smaller of the numbers δ_1 and δ_2. Then $0 < |x - a| < \delta$ implies

$$\begin{aligned}|[f(x) + g(x)] - (L + M)| &= |[f(x) - L] + [g(x) - M]| \\ &\le |f(x) - L| + |g(x) - M| \\ &< \tfrac{1}{2}\epsilon + \tfrac{1}{2}\epsilon = \epsilon,\end{aligned}$$

and this proves (i).

The argument for (ii) is almost identical with that just given and will be omitted.

In proving (iii), we wish to make the difference $f(x)g(x) - LM$ depend on the differences $f(x) - L$ and $g(x) - M$. This can be accomplished by subtracting and adding $f(x)M$, as follows:

$$\begin{aligned}|f(x)g(x) - LM| &= |[f(x)g(x) - f(x)M] + [f(x)M - LM]| \\ &\le |f(x)g(x) - f(x)M| + |f(x)M - LM| \\ &= |f(x)||g(x) - M| + |M||f(x) - L| \\ &\le |f(x)||g(x) - M| + (|M| + 1)|f(x) - L|.\end{aligned}$$

Let $\epsilon > 0$ be given. We know that there exist positive numbers δ_1, δ_2, δ_3 such that

$0 < |x - a| < \delta_1$ implies $|f(x) - L| < 1$, which in turn implies $|f(x)| < |L| + 1$;

$0 < |x - a| < \delta_2$ implies $|g(x) - M| < \dfrac{1}{2}\epsilon\left(\dfrac{1}{|L| + 1}\right)$;

$0 < |x - a| < \delta_3$ implies $|f(x) - L| < \dfrac{1}{2}\epsilon\left(\dfrac{1}{|M| + 1}\right)$.

Define δ to be the smallest of the numbers δ_2, δ_2, δ_3. Then $0 < |x - a| < \delta$ implies

$$|f(x)g(x) - LM| < \tfrac{1}{2}\epsilon + \tfrac{1}{2}\epsilon = \epsilon,$$

and the proof of (iii) is complete.

Theorem 5 *If* $\lim_{x\to a} f(x) = L$ *and* $\lim_{x\to a} g(x) = M$ *where* $M \neq 0$, *then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}.$$

Proof By Theorem 4 [part (iii)] and the fact that

$$\frac{f(x)}{g(x)} = f(x) \cdot \frac{1}{g(x)},$$

it suffices to prove that

$$\lim_{x\to a} \frac{1}{g(x)} = \frac{1}{M}.$$

We begin with the fact that if $g(x) \neq 0$, then

$$\left|\frac{1}{g(x)} - \frac{1}{M}\right| = \frac{|g(x) - M|}{|M||g(x)|}.$$

Choose $\delta_1 > 0$ so that

$$0 < |x - a| < \delta_1 \quad \text{implies} \quad |g(x) - M| < \tfrac{1}{2}|M|.$$

For these x's we have

$$|g(x)| > \frac{1}{2}|M| \qquad \text{or} \qquad \frac{1}{|g(x)|} < \frac{2}{|M|},$$

and therefore

$$\left|\frac{1}{g(x)} - \frac{1}{M}\right| < \frac{2}{|M|^2}|g(x) - M|.$$

Let $\epsilon > 0$ be given and choose $\delta_2 > 0$ so that

$$0 < |x - a| < \delta_2 \qquad \text{implies} \qquad |g(x) - M| < \frac{|M|^2}{2}\epsilon.$$

We now define δ to be the smaller of the numbers δ_1 and δ_2 and observe that

$$0 < |x - a| < \delta \qquad \text{implies} \qquad \left|\frac{1}{g(x)} - \frac{1}{M}\right| < \frac{2}{|M|^2} \cdot \frac{|M|^2}{2}\epsilon = \epsilon,$$

and this concludes the argument.

Theorem 6 *If there exists a positive number p with the property that*

$$g(x) \le f(x) \le h(x)$$

for all x that satisfy the inequalities $0 < |x - a| < p$, *and if* $\lim_{x\to a} g(x) = L$ *and* $\lim_{x\to a} h(x) = L$, *then*

$$\lim_{x\to a} f(x) = L.$$

Proof This statement is sometimes called the "squeeze theorem," because it says that a function squeezed between two functions approaching the same limit L must also approach L (see Fig. A.2). For the proof, let $\epsilon > 0$ be given, and choose positive numbers δ_1 and δ_2 so that

$$0 < |x - a| < \delta_1 \qquad \text{implies} \qquad L - \epsilon < g(x) < L + \epsilon$$

and

$$0 < |x - a| < \delta_2 \qquad \text{implies} \qquad L - \epsilon < h(x) < L + \epsilon.$$

Define δ to be the smallest of the numbers p, δ_1, δ_2. Then $0 < |x - a| < \delta$ implies

$$L - \epsilon < g(x) \le f(x) \le h(x) < L + \epsilon,$$

so $|f(x) - L| < \epsilon$ and the proof is complete.

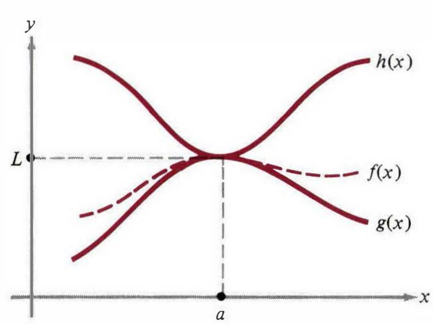


Figure A.2

We continue with the proofs of a few simple facts about continuous functions that follow almost immediately from these theorems about limits. First, however, let us recall that a function $f(x)$ is said to be *continuous at a point a* if

$$\lim_{x \to a} f(x) = f(a).$$

It is sometimes convenient to use the epsilon-delta version of this statement: for each $\epsilon > 0$ there exists a $\delta > 0$ with the property that

$$|f(x) - f(a)| < \epsilon$$

for every x in the domain of the function that satisfies the inequality

$$|x - a| < \delta.$$

Theorem 7 *If $f(x)$ and $g(x)$ are continuous at a point a, then $f(x) + g(x)$, $f(x) - g(x)$, and $f(x)g(x)$ are also continuous at a. Further, $f(x)/g(x)$ is continuous at a if $g(a) \neq 0$.*

Proof We prove only the statement about $f(x) + g(x)$, the other arguments being similar. Since $f(x)$ and $g(x)$ are continuous at a, we have

$$\lim_{x \to a} f(x) = f(a) \qquad \text{and} \qquad \lim_{x \to a} g(x) = g(a).$$

Part (i) of Theorem 4 now guarantees that

$$\lim_{x \to a} [f(x) + g(x)] = f(a) + g(a),$$

and this proves that $f(x) + g(x)$ is continuous at a.

Theorem 8 *The functions $f(x) = x$ and $g(x) = c$, where c is a constant, are continuous for all values of x.*

Proof These statements follow at once from Theorems 2 and 3.

Theorem 9 *Any polynomial*

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \tag{3}$$

is continuous for all values of x.

Proof By Theorem 8 and the multiplication part of Theorem 7, each of the following functions is continuous for all values of x: x, $x^2 = x \cdot x$, $x^3 = x \cdot x^2, \ldots, x^k$ for any positive integer k, and cx^k where c is any constant. Since the constant term a_0 is continuous, this tells us that each term of (3) is continuous for all values of x, and we obtain the conclusion by repeated application of the addition part of Theorem 7.

Theorem 10 *Any rational function*

$$R(x) = \frac{P(x)}{Q(x)},$$

where $P(x)$ and $Q(x)$ are polynomials, is continuous for all values of x for which $Q(x) \neq 0$.

Proof This is an immediate consequence of Theorem 9 and the division part of Theorem 7.

We conclude this section by proving that "a continuous function of a continuous function is continuous."

Theorem 11 *If $g(x)$ is continuous at a and $f(x)$ is continuous at $g(a)$, then the composite function $f(g(x))$ is continuous at a.*

Proof Let $\epsilon > 0$ be given. Since $f(x)$ is continuous at $g(a)$, we know that there exists a $\delta_1 > 0$ such that

$$|f(g(x)) - f(g(a))| < \epsilon \tag{4}$$

if

$$|g(x) - g(a)| < \delta_1. \tag{5}$$

But $g(x)$ is continuous at a, so there exists a $\delta > 0$ such that $|x - a| < \delta$ implies $|g(x) - g(a)| < \delta_1$. We therefore see that $|x - a| < \delta$ implies (5), which in turn implies (4), and this is all that is needed to complete the proof.

A.3 SOME DEEPER PROPERTIES OF CONTINUOUS FUNCTIONS

We recall that a *closed interval* $[a, b]$ on the x-axis is an interval which includes its endpoints a and b. A function is said to be *continuous on a closed interval* if it is defined and continuous at each point of the interval. Functions of this kind have several important properties that we now discuss and prove.

Theorem 1 (Boundedness Theorem) *Let $f(x)$ be a function continuous on a closed interval $[a, b]$. Then $f(x)$ is bounded on $[a, b]$; that is, there exists a number C with the property that $|f(x)| \le C$ for all x in $[a, b]$.*

A good way to study a theorem like this critically is to see what happens if the hypotheses are weakened or removed. In Theorem 1 there are two main hypotheses: (1) the interval $[a, b]$ is closed; and (2) the function $f(x)$ is continuous at each point of the interval. We show by examples that if either hypothesis is weakened, then the conclusion of the theorem can be false.

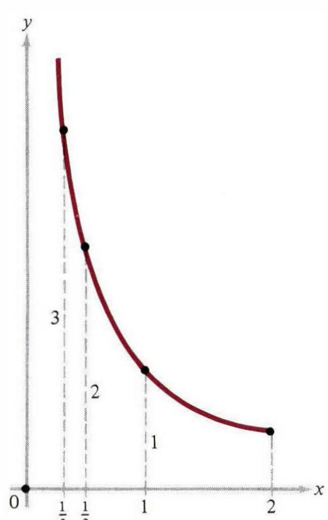


Figure A.3

Example 1 The function $f(x) = 1/x$ is clearly continuous on the closed interval $[1, 2]$, so according to Theorem 1, $f(x)$ should be bounded on this interval. Indeed, a bound C is easy to find:

$$|f(x)| \le 1 \text{ for all } x \text{ in } [1, 2].$$

Further (see Fig. A.3), $f(x)$ is continuous on the closed interval $[1/n, 2]$ for any positive integer n, and in this case the number n is a bound:

$$|f(x)| \le n \text{ for all } x \text{ in } [1/n, 2].$$

On the other hand, $f(x)$ is also continuous on the nonclosed interval $(0, 2]$, but $f(x)$ is not bounded on this interval. For, no matter how large a value of C we take, there are points in the interval for which $f(x) > C$; specifically, if $0 < x < 1/C$, then $f(x) = 1/x > C$. This shows that the hypothesis requiring that the interval $[a, b]$ be closed is necessary.

We now extend the definition of $f(x)$ to include the point $x = 0$, by putting

$$f(x) = \begin{cases} 1/x & \text{if } 0 < x \le 2, \\ 0 & \text{if } x = 0. \end{cases}$$

This function is defined on the entire closed interval $[0, 2]$, and it is unbounded on this interval for the same reason. This time the conclusion of Theorem 1 is false because the function $f(x)$ is not continuous at each point of the closed interval: it is discontinuous at the single point $x = 0$.

These remarks show that the hypotheses of Theorem 1 cannot be weakened, and the following proof demonstrates that with both hypotheses in place the conclusion of the theorem is inescapable.

*Proof of Theorem 1** Our proof uses the fact that a nonempty set of real numbers with an upper bound necessarily has a least upper bound (see Appendix A.1). Let S be the set of all points c in $[a, b]$ with the property that $f(x)$ is bounded on $[a, c]$. It is clear that S is nonempty and has b as an upper bound, and therefore has a least upper bound which we denote by c_0. We claim that $c_0 = b$. To establish this, suppose that $c_0 < b$. Since $f(x)$ is continuous at $x = c_0$, it is easy to see that $f(x)$ is bounded on $[c_0 - \epsilon, c_0 + \epsilon]$ for some $\epsilon > 0$. Since $f(x)$ is also bounded on $[a, c_0 - \epsilon]$, it is clearly bounded on $[a, c_0 + \epsilon]$. This contradicts the fact that c_0 is the least upper bound of S, so $c_0 = b$. This tells us that $f(x)$ is bounded on $[a, c]$ for every $c < b$. One more step is needed to finish the proof. Since $f(x)$ is continuous at $x = b$, it is bounded on some closed interval $[b - \epsilon, b]$. By what we just proved, $f(x)$ is also bounded on $[a, b - \epsilon]$, so it is bounded on all of $[a, b]$.

If a function $f(x)$ is bounded on $[a, b]$, then its range—the set of all its values—has an upper bound and a lower bound. If M and m are the least upper bound and greatest lower bound of the range, then

$$m \leq f(x) \leq M \text{ for all } x \text{ in } [a,b].$$

For bounded functions in general, the numbers M and m need not belong to the range. However, our next theorem asserts that if $f(x)$ is continuous, then both numbers M and m are actually assumed as values of the function.

Theorem 2 (Extreme Value Theorem) *Let $f(x)$ be a function continuous on a closed interval $[a, b]$. Then $f(x)$ assumes a maximum value M and a minimum value m; that is, there exist points x_1 and x_2 in $[a, b]$ such that*

$$f(x_1) \leq f(x) \leq f(x_2)$$

for all x in $[a, b]$.

This statement is intuitively clear if we think of a continuous function on a closed interval as one whose graph consists of a single continuous piece, without any gaps or holes; for as we move along the curve from the left endpoint $(a, f(a))$ to the right endpoint $(b, f(b)$, we feel compelled to believe that there must be a high point on the curve where $f(x)$ has its maximum value and a low point where $f(x)$ has its minimum value. This is true, but the situation is again very delicate, because if either hypothesis is weakened—even slightly—then the conclusion of the theorem can be false.

Example 2 Consider the function $f(x)$ defined by $f(x) = x$ on the nonclosed interval $[0, 1)$, and also the function $g(x)$ defined by

$$g(x) = \begin{cases} x & \text{if } 0 \leq x < 1, \\ 0 & \text{if } 1 \leq x \leq 2 \end{cases}$$

on the closed interval $[0, 2]$. Both functions are shown in Fig. A.4. The function $f(x)$ does not assume a maximum value even though it is continuous on the interval $[0, 1)$, because

*Some of the details of the proofs in this Appendix are left for students to fill in.

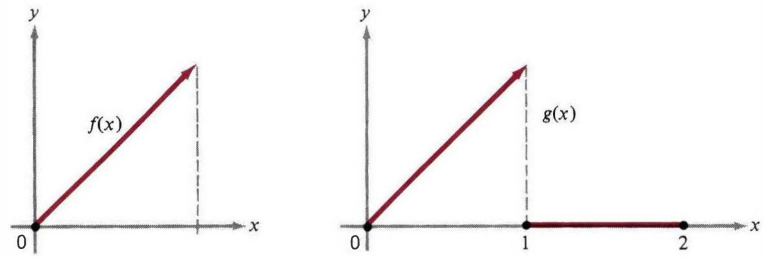


Figure A.4

this interval is not closed; and the function $g(x)$ does not assume a maximum value even though the interval [0, 2] is closed, because $g(x)$ is discontinuous at the single point $x = 1$. In each case the values of the function get close to the number 1 (which is the least upper bound M of the range) as $x \to 1$ from the left, but there is no point where the function actually *has* the value 1.

Proof of Theorem 2 We prove the statement about assuming a maximum value. By Theorem 1, $f(x)$ is bounded on $[a, b]$, so the range has an upper bound and therefore a least upper bound M. We must show that there exists a point x_2 in $[a, b]$ such that $f(x_2) = M$. Suppose there is no such point, that is, suppose that $f(x) < M$ for all x in $[a, b]$. Then $M - f(x)$ is positive on $[a, b]$, the function

$$g(x) = \frac{1}{M - f(x)}$$

is continuous on $[a, b]$, and Theorem 1 implies that this function is bounded. This means that there exists a number C such that

$$\frac{1}{M - f(x)} \leq C$$

for all x in $[a, b]$, so

$$\frac{1}{C} \leq M - f(x) \qquad \text{or} \qquad f(x) \leq M - \frac{1}{C}.$$

This contradicts the fact that M is the least upper bound of the set of all $f(x)$'s and we are thereby forced to the desired conclusion: there exists at least one point x_2 in $[a, b]$ for which $f(x_2) = M$. The statement that $f(x)$ assumes a minimum value at some point x_1 is proved similarly.

The Extreme Value Theorem says that a function continuous on a closed interval actually takes on a maximum value and a minimum value. There is a companion to this theorem which states that such a function also takes on every value between its maximum and minimum values. Thus, a function continuous on a closed interval has a range which is itself a closed interval. To put it another way, such a function does not skip any values. We begin with a preliminary theorem that has many applications of its own (see Section 4.6).

Theorem 3 *Let $f(x)$ be a function continuous on a closed interval $[a, b]$. If $f(a)$ and $f(b)$ have opposite signs, that is, if*

$$f(a) < 0 < f(b) \qquad \textit{or} \qquad f(a) > 0 > f(b),$$

then there exists a point c between a and b such that $f(c) = 0$.

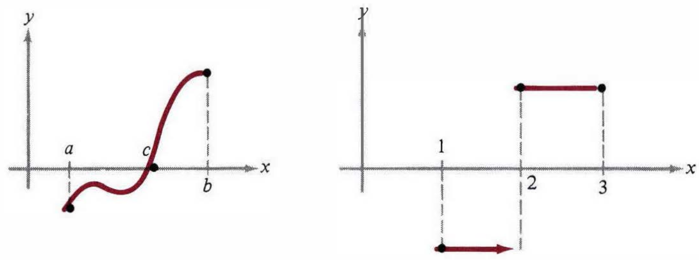


Figure A.5

This says—in effect—that the graph of a function continuous on a closed interval cannot get from one side of the x-axis to the other side without actually crossing this axis at a definite point (Fig. A.5, left). However, this conclusion can be false if the function fails to be continuous even at a single point. This is shown (Fig. A.5, right) by the function $f(x)$ defined on the interval $[1, 3]$ by

$$f(x) = \begin{cases} -1 & \text{if } 1 \le x < 2, \\ 1 & \text{if } 2 \le x \le 3. \end{cases}$$

Proof of Theorem 3 Suppose first that $f(a) < 0 < f(b)$. Since $f(a) < 0$ and $f(x)$ is continuous at $x = a$, there exists a number d in the open interval (a, b) such that $f(x)$ is negative on $[a, d)$. Let c be the least upper bound of the set of all such d's, and observe that $f(x)$ is negative for all $x < c$. It cannot be true that $f(c) > 0$, for by continuity this would imply that $f(x)$ is positive on some interval $(c - \epsilon, c]$, contrary to what we have just observed. Also, it cannot be true that $f(c) < 0$, for by continuity this would imply that $f(x)$ is negative on some interval $[a, c + \epsilon)$, contrary to the definition of c. We conclude that $f(c) = 0$. The argument for the other case is similar.

Theorem 4 (The Intermediate Value Theorem) *Let $f(x)$ be a function continuous on a closed interval $[a, b]$. If M and m are the maximum and minimum values of $f(x)$ on $[a, b]$, and if C is any number between M and m so that $m < C < M$, then there exists a point c in $[a, b]$ such that $f(c) = C$.*

Proof The function $g(x) = f(x) - C$ is also continuous on $[a, b]$. If x_1 and x_2 are points in $[a, b]$ at which $f(x_1) = m$ and $f(x_2) = M$, then $g(x)$ is negative at x_1 and positive at x_2:

$$g(x_1) = f(x_1) - C = m - C < 0$$

and

$$g(x_2) = f(x_2) - C = M - C > 0.$$

By Theorem 3, there exists a point c between x_1 and x_2 (and therefore in $[a, b]$) such that $g(c) = 0$. But this means $f(c) - C = 0$ or $f(c) = C$.

As another consequence of Theorem 3, we have

Theorem 5 *Let $f(x)$ be a function continuous on the closed unit interval $[0, 1]$ which has the further property that its values also lie in this interval (Fig. A.6). Then there exists at least one point c in $[0, 1]$ such that $f(c) = c$.*

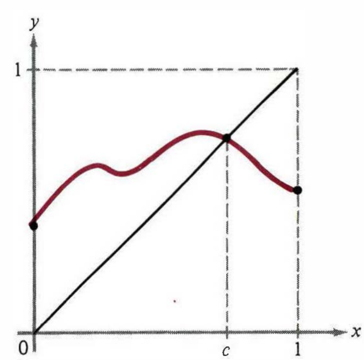


Figure A.6

Proof The function $g(x) = f(x) - x$ is continuous on $[0, 1]$ and has the property that $g(0) = f(0) - 0 = f(0) \geq 0$ and $g(1) = f(1) - 1 \leq 0$. By Theorem 3, there exists a point in c in $[0, 1]$ such that $g(c) = f(c) - c = 0$, so $f(c) = c$.

A function $f(x)$ with the properties assumed here is often called a *continuous mapping of the interval* [0, 1] *into itself,* and the point c is called a *fixed point* of this mapping. Theorem 5 is a special case of a famous and far-reaching theorem of modern mathematics called *Brouwer's fixed point theorem,* which asserts that continuous mappings of certain very general spaces into themselves always have fixed points.

A.4 THE MEAN VALUE THEOREM

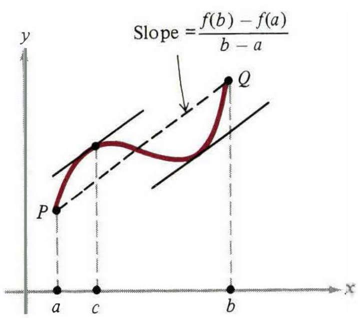


Figure A.7

This theorem is one of the most useful facts in the theoretical part of calculus. In geometric language, it is easy to state and intuitively plausible. It asserts that between any two points P and Q on the graph of a differentiable function there must exist at least one point where the tangent line is parallel to the chord joining P and Q, as shown in Fig. A.7. For the curve in the figure there are two such points. There may be many or there may be only one, but the theorem guarantees that there must always be at least one such point. By using the notation in the figure, we can express the statement of the theorem analytically by saying that there exists at least one number c between a and b ($a < c < b$) with the property that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

The significance of the Mean Value Theorem lies not in itself but in its consequences, for it provides a convenient way of getting a grip on many theoretical facts of practical importance. This will become clear in Theorems 3 and 4, and also in later sections of this appendix.

A rigorous proof of the Mean Value Theorem is usually developed in the following way. We begin by establishing the special case of the theorem in which the points P and Q both lie on the x-axis:

Theorem 1 (Rolle's Theorem*) *If a function $f(x)$ is continuous on the closed interval $a \le x \le b$ and differentiable in the open interval $a < x < b$, and if $f(a) = f(b) = 0$, then there exists at least one number c between a and b with the property that $f'(c) = 0$.*

This theorem says that if a differentiable curve touches the x-axis at two points, then there must be at least one point on the curve between these points at which the tangent is horizontal (Fig. A.8). Equivalently, the zeros of a differentiable function are always separated by zeros of its derivative.

Economists have a maxim, "There is no such thing as a free lunch." For us—in the realm of pure mathematics—this means we cannot get something for nothing; or in other words, strong conclusions require strong hypotheses. The conclusion of Rolle's Theorem depends heavily on its hypotheses, and the following examples show that these hypotheses cannot be weakened without destroying the conclusion.

Example 1 The function

$$f(x) = \begin{cases} x & 0 \le x \le 1, \\ 2 - x & 1 \le x \le 2 \end{cases}$$

*Michel Rolle (1652–1719) was an otherwise obscure French mathematician whose name is inseparably linked to one of the principal foundation stones of the theory of calculus. This is deliciously ironic, because, having studied the emerging subject and finding it unconvincing, he vigorously attacked it in the French Academy of Sciences as a bundle of ingenious fallacies. When his friends convinced him that things were not quite as bad as that, the opposition faction collapsed and the new analysis began a century of rapid and luxuriant growth in continental Europe. Rolle stated a polynomial version of his theorem in an almost-forgotten book on the algebra of solving equations (1691).

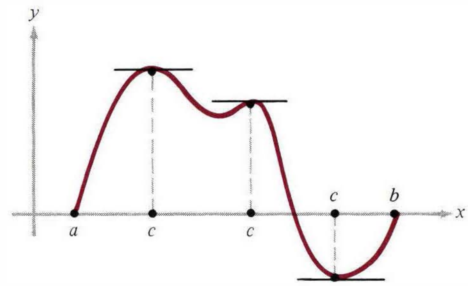


Figure A.8

(see Fig. A.9) is zero at $x = 0$ and $x = 2$ and is continuous on the closed interval $0 \le x \le 2$. It is differentiable in the open interval $0 < x < 2$, except at the single point $x = 1$, where the derivative does not exist. The derivative $f'(x)$ is clearly not zero at any point in the interval, and this failure of the conclusion of Rolle's Theorem arises from the fact that the function fails to be differentiable at a single crucial point.

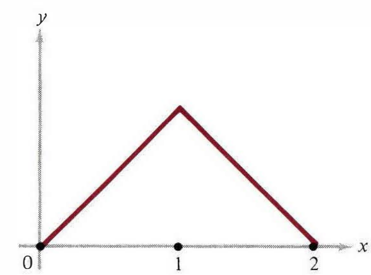


Figure A.9

Example 2 The function

$$f(x) = \begin{cases} x & 0 \le x < 1, \\ 0 & x = 1 \end{cases}$$

(see Fig. A.10) is zero at $x = 0$ and $x = 1$ and is differentiable in the open interval $0 < x < 1$. It is continuous on the closed interval $0 \le x \le 1$, except at the single point $x = 1$. The derivative $f'(x)$ is not zero at any point in the interval, and in this case the failure of the conclusion of Rolle's Theorem arises from the discontinuity of the function at a single point.

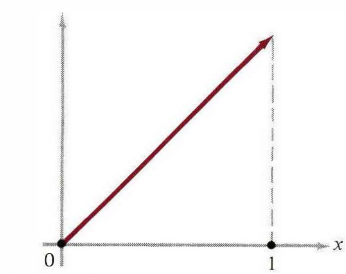


Figure A.10

Proof of Theorem 1 By Theorem 2 in Appendix A.3, our continuity hypothesis implies that $f(x)$ assumes a maximum value M and a minimum value m on $[a, b]$. The fact that $f(x)$ is zero at the endpoints a and b tells us that $m \le 0 \le M$. If $f(x)$ is zero at every point of $[a, b]$, then clearly $f'(c) = 0$ for every c in (a, b), and in this trivial case the conclusion is true. We may therefore suppose that the function assumes nonzero values, so either $M > 0$ or $m < 0$ (or perhaps both). We first consider the case in which $M > 0$. If c is a point at which $f(c) = M$, then $a < c < b$ because the function is zero at the endpoints a and b. Since $f(x)$ is differentiable in the open interval $a < x < b$, the derivative

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \tag{1}$$

exists.* It is part of the meaning of (1) that this limit must exist and have the same value when x approaches c from the left and from the right. If x approaches c from the left, we have

$$x - c < 0 \qquad \text{and} \qquad f(x) - f(c) \le 0,$$

where the second inequality follows from the fact that $f(c) = M$ is a maximum value. This implies that

*Equation (1) is clearly an equivalent way of writing

$$f'(c) = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x}.$$

$$f'(c) = \lim_{x \to c-} \frac{f(x) - f(c)}{x - c} \geq 0. \tag{2}$$

Similarly, if x approaches c from the right, we have

$$x - c > 0 \qquad \text{and} \qquad f(x) - f(c) \leq 0,$$

so

$$f'(c) = \lim_{x \to c+} \frac{f(x) - f(c)}{x - c} \leq 0. \tag{3}$$

We conclude from (2) and (3) that $f'(c) = 0$, as asserted. If $M = 0$, then $m < 0$, and this case can be treated by a similar argument.

Our main theorem can now be stated as follows (see Fig. A.7).

Theorem 2 (Mean Value Theorem) *If a function $f(x)$ is continuous on the closed interval $a \leq x \leq b$ and differentiable in the open interval $a < x < b$, then there exists at least one number c between a and b with the property that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}. \tag{4}$$

Proof It is easy to see that the equation of the chord joining P and Q in Fig. A.7 is

$$y = f(a) + \left[\frac{f(b) - f(a)}{b - a}\right](x - a).$$

The function

$$F(x) = f(x) - f(a) - \left[\frac{f(b) - f(a)}{b - a}\right](x - a) \tag{5}$$

is therefore the vertical distance from the chord up to the graph of $y = f(x)$. It is easy to see that the function (5) satisfies the hypotheses of Theorem 1, so there exists a point c between a and b with the property that $F'(c) = 0$. But this is equivalent to

$$f'(c) - \frac{f(b) - f(a)}{b - a} = 0,$$

which in turn is equivalent to (4), so the proof is complete.

We now consider some of the applications of this theorem.

It is clear that the derivative of a constant function is zero. Is the converse true? That is, if the derivative of a function is zero on an interval, is the function necessarily constant on that interval? At the beginning of Section 5.3 we encountered an important piece of reasoning about indefinite integrals in which this converse was needed, and we took it for granted. We are now in a position to prove it by using the Mean Value Theorem.

Theorem 3 *If a function $f(x)$ is continuous on a closed interval I, and if $f'(x)$ exists and is zero in the interior of I, then $f(x)$ is constant on I.*

Proof To say that $f(x)$ is constant on I means that it has only a single value there. To prove that this is the case, suppose it has two different values, say $f(a) \neq f(b)$ for $a < b$

in I. Then the Mean Value Theorem implies that for some c between a and b we have

$$f'(c) = \frac{f(b) - f(a)}{b - a} \neq 0.$$

But this cannot be true, since $f'(x) = 0$ at all points in the interior of I. This contradiction shows that $f(x)$ cannot have different values in I, and is therefore constant on I, as we wished to prove.

At the beginning of Chapter 4 we based our work on curve sketching on the "intuitively obvious" fact that a function is increasing or decreasing according as its derivative is positive or negative. The Mean Value Theorem makes it possible to give a rigorous proof of this.

Theorem 4 *Let $f(x)$ be a function continuous on a closed interval I and differentiable in the interior of I. If $f'(x) > 0$ in the interior of I, then $f(x)$ is increasing on I. Similarly, if $f'(x) < 0$ in the interior of I, then $f(x)$ is decreasing on I.*

Proof We shall prove only the first statement, in which we assume that $f'(x) > 0$ in the interior of I. For any two points $a < b$ in I, the Mean Value Theorem tells us that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

for some c between a and b. But $f'(c) > 0$, so the fraction on the right side of this equation is positive. Since $b - a$ is positive, it follows that $f(b) - f(a)$ is also positive, so $f(a) < f(b)$ and consequently $f(x)$ is increasing on I.

Finally, we use Rolle's Theorem to prove a technical extension of the Mean Value Theorem that is needed for establishing L'Hospital's rule in Chapter 12.

Theorem 5 (Generalized Mean Value Theorem) *Let $f(x)$ and $g(x)$ be continuous on the closed interval $a \leq x \leq b$ and differentiable in the open interval $a < x < b$, and assume further that $g'(x) \neq 0$ for $a < x < b$. Then there exists at least one number c between a and b with the property that*

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}. \tag{6}$$

Proof We begin by noticing that if $g(a) = g(b)$, then by Rolle's Theorem $g'(x)$ vanishes at some point between a and b, contrary to hypothesis. Therefore $g(a) \neq g(b)$, and the right side of (6) makes sense. To prove the theorem, consider the function

$$F(x) = [f(b) - f(a)][g(x) - g(a)] - [f(x) - f(a)][g(b) - g(a)].$$

It is easy to see that this function satisfies the hypotheses of Rolle's Theorem, so there exists a point c between a and b with the property that $F'(c) = 0$. But this is equivalent to

$$[f(b) - f(a)]g'(c) - f'(c)[g(b) - g(a)] = 0,$$

which is equivalent to (6).

Students should notice that this theorem reduces to Theorem 2 if $g(x) = x$.

A.5 THE INTEGRABILITY OF CONTINUOUS FUNCTIONS

In Section 6.4 the definite integral of a function over an interval was defined by means of a complicated passage to the limit, as follows.

We start with an arbitrary bounded function $f(x)$ defined on a closed interval $[a, b]$. We subdivide this interval into n equal or unequal subintervals by inserting $n - 1$ points of division $x_1, x_2, \ldots, x_{n-1}$, so that

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b. \tag{1}$$

These points are said to constitute a *partition* P of $[a, b]$ into the subintervals

$$[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n].$$

If $\Delta x_k = x_k - x_{k-1}$ is the length of the kth subinterval, then the length of the longest subinterval is called the *norm* of the partition and is denoted by the symbol $\|P\|$,

$$\|P\| = max\ \{\Delta x_1, \Delta x_2, \ldots, \Delta x_n\}.$$

In each of the subintervals $[x_{k-1}, x_k]$ we choose an arbitrary point x_k^*. We now multiply the value of the function $f(x)$ at the point x_k^* by the length Δx_k of the corresponding subinterval and form the sum of these products as the subscript k varies from 1 to n,

$$\sum_{k=1}^{n} f(x_k^*)\ \Delta x_k. \tag{2}$$

For each positive integer n we consider all possible partitions (1) and all possible choices of the points x_k^*, and therefore all possible values of the sum (2). If there exists a number I such that the sum (2) approaches I as $n \to \infty$ and $\|P\| \to 0$, regardless of how the partitions P are formed and the points x_k^* are chosen, then we call this number I the *definite integral* (or briefly the *integral*) of $f(x)$ on $[a, b]$ and denote it by the symbol

$$I = \int_a^b f(x)\ dx.$$

Under these circumstances the function $f(x)$ is said to be *integrable* on $[a, b]$. It is customary to express these ideas by writing

$$\int_a^b f(x)\ dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x_k^*)\ \Delta x_k, \tag{3}$$

where there is no need to specify that $n \to \infty$ because this is implied by the stronger condition $\|P\| \to 0$.

As we said at the beginning, the limit operation in (3) is quite complicated and bears only a superficial resemblance to such straightforward limits as

$$\lim_{x\to 2} (x^2 + 1) = 5 \qquad \text{and} \qquad \lim_{n\to\infty} \left(2 + \frac{1}{n}\right) = 2.$$

In each of these cases we consider the behavior of a certain function in terms of the behavior of an independent variable, but (3) does not lend itself to this way of thinking. We could try to use $\|P\|$ as an independent variable, and describe the limit in terms of the idea expressed by the symbol $\|P\| \to 0$. But this is difficult, because the sum (2) is not a single-valued function of the quantity $\|P\|$; to a given value of $\|P\|$ there correspond an infinite number of different partitions P and an infinite number of ways of choosing the points x_k^*, and therefore an infinite number of values of the sum (2).

The complexity of the limit operation in (3) is a considerable inconvenience when it comes to giving rigorous proofs of theorems. The cumbersome notation required for such proofs forces the reasoning itself to be awkward and clumsy. For this reason, it is customary in modern treatments of the theory of integration to define the definite integral in a very different way, one which avoids appealing to any kind of passage to the limit. We now describe this more convenient approach and use it to prove our main theorem.

We therefore ignore our previous definition and begin all over again at the beginning, with an arbitrary bounded function $f(x)$ defined on a closed interval $[a, b]$. Since $f(x)$ is bounded, it has a greatest lower bound m and a least upper bound M. If P is any given partition of $[a, b]$, we denote by m_k and M_k the greatest lower bound and least upper bound of $f(x)$ on the kth subinterval $[x_{k-1}, x_k]$. (If $f(x)$ were assumed to be continuous on $[a, b]$, then by Theorem 2 in Appendix A.3 the m's and M's would be minimum values and maximum values of the function. But we are not assuming continuity at this stage, so we must work instead with greatest lower bounds and least upper bounds.) We now form the *lower sum*

$$s_P = \sum_{k=1}^{n} m_k \, \Delta x_k$$

and the *upper sum*

$$S_P = \sum_{k=1}^{n} M_k \, \Delta x_k.$$

It is obvious that $s_P \leq S_P$. Further, we have the important

Lemma *Every lower sum is less than or equal to every upper sum; that is, if P_1 and P_2 are any two partitions of $[a, b]$, then $s_{P_1} \leq S_{P_2}$.*

Proof It is east to see that if a single point is added to a partition, then the lower sum is unchanged or increases and the upper sum is unchanged or decreases; and the same is true if any finite number of points are added to produce a refinement of the given partition. We now apply this fact to the new partition P_3 which is formed from the points of P_1 and P_2 taken together. Since P_3 is clearly a refinement of both P_1 and P_2, it follows that

$$s_{P_1} \leq s_{P_3} \leq S_{P_3} \leq S_{P_2},$$

which completes the argument.

Among other things, this lemma tells us that every upper sum is an upper bound for the set of all lower sums, and that every lower sum is a lower bound for the set of all upper sums. We can therefore form the least upper bound of all possible lower sums, which is called the *lower integral* and denoted by

$$\underline{I} = \underline{\int_a^b} f(x) \, dx.$$

Similarly, the greatest lower bound of all upper sums is called the *upper integral* and denoted by

$$\bar{I} = \overline{\int_a^b} f(x) \, dx.$$

At this point we make a further application of the lemma to conclude that

$$\underline{I} \leq \bar{I}.$$

Accordingly, every bounded function defined on a closed interval has a lower integral and an upper integral, and these two integrals are defined without making any appeal to the concept of a limit. If the lower and upper integrals coincide, then we call their common value the *integral* of $f(x)$ on $[a, b]$ and denote it by the usual symbol,

$$I = \int_a^b f(x) \, dx;$$

and in this case the function $f(x)$ is said to be *integrable* on $[a, b]$. On the other hand, it is quite possible to have $\underline{I} < \overline{I}$, in which case $f(x)$ is *not* integrable. The function described in Remark 4 of Section 6.4 provides a good example of this recalcitrant behavior.

We now come to our main theorem, which guarantees that most of the functions we meet in practice are integrable. First, a bit of new terminology that will be useful in the proof. If $f(x)$ is a bounded function defined on an interval $[a, b]$, and if m and M are its greatest lower bound and least upper bound on this interval, then the difference $M - m$ is called the *oscillation of* $f(x)$ on $[a, b]$.

Theorem *If a function $f(x)$ is continuous on a closed interval $[a, b]$, then it is integrable on $[a, b]$.*

Proof Consider a partition P of $[a, b]$ into subintervals $[x_{k-1}, x_k]$, and form the lower and upper sums

$$s_P = \sum_{k=1}^{n} m_k \,\Delta x_k \qquad \text{and} \qquad S_P = \sum_{k=1}^{n} M_k \,\Delta x_k.$$

The difference between these sums is

$$S_P - s_P = \sum_{k=1}^{n} (M_k - m_k)\,\Delta x_k, \tag{4}$$

where $M_k - m_k$ is the oscillation of $f(x)$ on the kth subinterval $[x_{k-1}, x_k]$. If we can show that the difference (4) can be made as small as we please by choosing a suitable partition P, then this will clearly be enough to prove the theorem. We accomplish this in the following way. Let ϵ be a given small positive number. If it can be shown that there exists a partition P such that the oscillation of the function is less than $\epsilon/(b - a)$ on every subinterval, that is

$$M_k - m_k < \frac{\epsilon}{b-a} \qquad \text{for} \qquad k = 1, 2, \ldots, n,$$

then it will follow that

$$S_P - s_P = \sum_{k=1}^{n} (M_k - m_k)\,\Delta x_k < \frac{\epsilon}{b-a} \sum_{k=1}^{n} \Delta x_k = \frac{\epsilon}{b-a}(b - a) = \epsilon.$$

Since ϵ can be made as small as we please, this will complete the proof.

We must therefore prove the existence of a partition P with the required property. If we simplify the notation by writing $\epsilon_1 = \epsilon/(b - a)$, so that ϵ_1 is perceived as merely another positive number that can be made as small as we please, then this property of the partition P can be stated as follows: The oscillation of the continuous function $f(x)$ on every subinterval of the partition must be less than ϵ_1.*

We give an indirect proof, that is, we assume that for at least one number $\epsilon_1 > 0$ no partition of the desired type exists, and we show that this assumption leads to a contradiction. Let c be the midpoint of $[a, b]$. Then no partition of the desired type exists for at least one of the two subintervals $[a, c]$ and $[c, b]$, for if each of these subintervals has such a partition, then the full interval $[a, b]$ also has. Let $[a_1, b_1]$ be that half of $[a, b]$ with no such partition; and if both halves have no such partition, let $[a_1, b_1]$ be the left half, $[a, c]$. Now bisect $[a_1, b_1]$, and in the same way produce one of its halves, say $[a_2, b_2]$, with no such partition; and continue the process indefinitely. We observe that the

*This fact about a continuous function defined on a closed interval is usually referred to in the literature as the *Theorem on Uniform Continuity*.

oscillation of $f(x)$ on the nth subinterval $[a_n, b_n]$ is at least ϵ_1, and also that the length of this subinterval is $(b - a)/2^n$. Let a_0 be the least upper bound of the set of left endpoints $a_1, a_2, a_3, \ldots$ of this nested sequence of subintervals. Then a_0 certainly lies in the interval $[a, b]$; and by the continuity of $f(x)$ at a_0, there exists an interval $(a_0 - \delta, a_0 + \delta)$ in which the oscillation of $f(x)$ is less than ϵ_1. However, if n is large enough, the interval $[a_n, b_n]$ lies wholly within the interval $(a_0 - \delta, a_0 + \delta)$, and therefore the oscillation of $f(x)$ on $[a_n, b_n]$ must also be less than ϵ_1, contradicting our previous inference that the oscillation of $f(x)$ on $[a_n, b_n]$ is at least ϵ_1. This contradiction finally concludes the proof of the theorem.

If students wonder whether a discontinuous function can be integrable, the answer is Yes. The function whose graph is shown in Fig. A.11 provides an example of this assertion. It is defined on the closed interval [0, 1], and its values are

$$\frac{1}{2} \text{ for } 0 \le x < \frac{1}{2},$$

$$\frac{3}{4} \text{ for } \frac{1}{2} \le x < \frac{3}{4},$$

$$\frac{7}{8} \text{ for } \frac{3}{4} \le x < \frac{7}{8},$$

$$\ldots$$

$$1 \text{ for } x = 1.$$

This function has an infinite number of points of discontinuity, but it also has the property of being *nondecreasing,* in the sense that $x_1 < x_2$ implies $f(x_1) \le f(x_2)$, and any such function is integrable on any closed interval $[a, b]$. Students are invited to prove this for themselves by noticing that in this case the difference (4) can be written as

$$\begin{aligned} S_P - s_P &= \sum_{k=1}^{n} (M_k - m_k)\,\Delta x_k \\ &\le \|P\| \sum_{k=1}^{n} (M_k - m_k) = \|P\|[f(b) - f(a)]. \end{aligned}$$

The set of all integrable functions can be characterized in a simple and absolutely precise way, but we do not pursue this matter any further here.

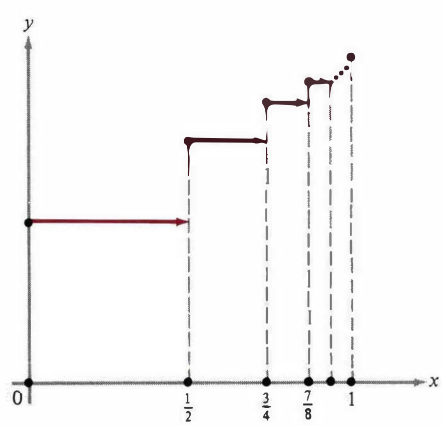


Figure A.11

A.6 ANOTHER PROOF OF THE FUNDAMENTAL THEOREM OF CALCULUS

The proof given here uses the Mean Value Theorem established in Appendix A.4 and assumes that students understand the concepts developed in Appendix A.5.

To set the stage for the argument, we consider a function $f(x)$ that is continuous on a closed interval $[a, b]$. If $F(x)$ is any function such that $F'(x) = f(x)$, we must prove that

$$\int_a^b f(x)\,dx = F(b) - F(a). \tag{1}$$

We accomplish this by showing that the number on the right of (1) lies between the lower sum and the upper sum associated with an arbitrary partition

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b \tag{2}$$

of the interval $[a, b]$.

The reasoning is as follows. The function $F(x)$ satisfies the hypotheses of the Mean Value Theorem on each subinterval of the partition (2). This theorem therefore guarantees the existence of points $x_1^*, x_2^*, \ldots, x_n^*$ in these subintervals such that

$$F(x_1) - F(a) = F'(x_1^*)(x_1 - a) = f(x_1^*)\,\Delta x_1,$$

$$F(x_2) - F(x_1) = F'(x_2^*)(x_2 - x_1) = f(x_2^*)\,\Delta x_2,$$

$$\cdots$$

$$F(b) - F(x_{n-1}) = F'(x_n^*)(b - x_{n-1}) = f(x_n^*)\,\Delta x_n.$$

If we add these equations and take advantage of the cancellations on the left, we get

$$F(b) - F(a) = \sum_{k=1}^{n} f(x_k^*)\,\Delta x_k. \tag{3}$$

The right side of (3) clearly lies between the lower sum and the upper sum associated with the partition (2), so the proof is complete.

A.7 CONTINUOUS CURVES WITH NO LENGTH

In Section 7.5 we introduced the concept of the length of a curve, and we suggested there that the theory of arc length is more complicated than it seems. We now offer a few further thoughts on this subject, with some examples to illustrate the bizarre situations that can arise.

Let AB be the graph of a function $y = f(x)$ that is continuous on a closed interval $[a, b]$. We connect A and B by a broken line whose vertices lie on the graph and use this line to define the length of the curve, as follows. We partition the interval $[a, b]$ into n subintervals by inserting $n - 1$ points $x_1, x_2, \ldots, x_{n-1}$, with $x_0 = a$ and $x_n = b$, as shown in Fig. A.12. Let P_k be the point (x_k, y_k), where $y_k = f(x_k)$, and let L_n be the length of the inscribed polygonal line:

$$L_n = AP_1 + P_1P_2 + \cdots + P_{n-1}B.$$

If for all partitions of $[a, b]$—or equivalently, all choices of the points $P_1, P_2, \ldots, P_{n-1}$—the set of lengths L_n is bounded, then the curve is said to be *rectifiable* and the least upper bound L of the set of L_n's is called the *length* of the curve.

On the other hand, if the set of all lengths L_n is unbounded, then the curve is said to be *nonrectifiable* and has no length. Most of the continuous curves we encounter are rectifiable, but it is not true that every continuous curve has this property. Since this contradicts the intuitive preconceptions most people have about curves, we shall give an example below of a continuous curve that has no length.

First, however, a bit of theory, which we hope will clarify the ideas. We begin by considering the case of a curve that rises as we move along it from left to right; that is, we

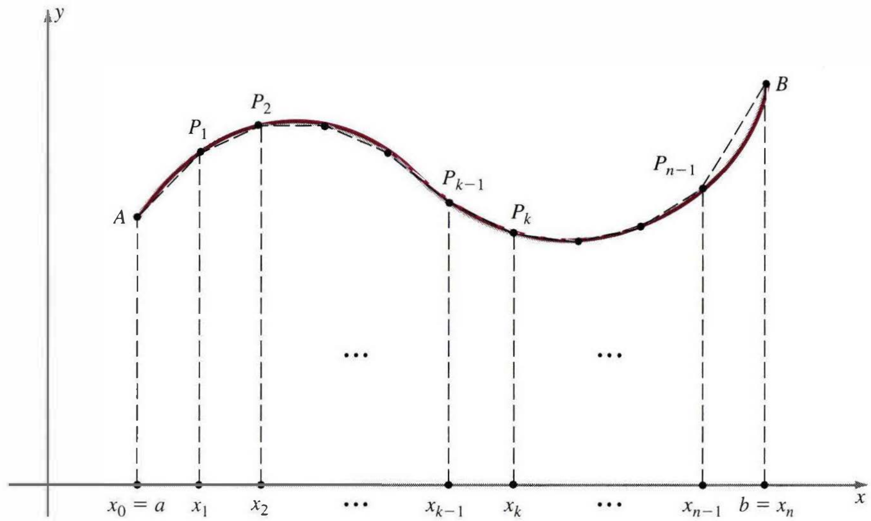


Figure A.12

assume that the continuous function $f(x)$ is increasing (Fig. A.13). If we introduce the notation

$$\Delta x_k = x_k - x_{k-1}, \qquad \Delta y_k = y_k - y_{k-1},$$

then we see that

$$P_{k-1}P_k < \Delta x_k + \Delta y_k,$$

and by adding we obtain

$$L_n = AP_1 + P_1P_2 + \cdots + P_{n-1}B < AC + CB.$$

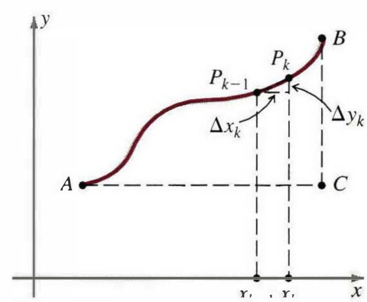


Figure A.13

This upper bound is independent of the number and location of the points P_k, so the curve is rectifiable. A curve that falls as we move from left to right—the graph of a decreasing function—is easily seen to be rectifiable in the same way.

A curve that rises and falls a finite number of times is clearly rectifiable, because the sum of the bounds for the parts is a bound for the whole. This is equivalent to saying that the graph of $y = f(x)$ is rectifiable if it has a finite number of local maxima and minima. Nonrectifiable curves must therefore be highly oscillatory, with an infinite number of wiggles.

It is convenient to express this idea in a more precise way. For a general curve the number Δy_k can be positive or negative, but always

$$|\Delta y_k| < P_{k-1}P_k \le \Delta x_k + |\Delta y_k|,$$

and by adding we see that

$$|\Delta y_1| + \cdots + |\Delta y_n| < L_n \le b - a + |\Delta y_1| + \cdots + |\Delta y_n|.$$

If the set of all sums on the left, that is, the numbers

$$V_n = |\Delta y_1| + \cdots + |\Delta y_n|,$$

is bounded for all possible partitions of the interval $[a, b]$, then the function $y = f(x)$ is said to be of *bounded variation* on the interval. We see from this discussion that since $V_n < L_n$, if the set of L_n's is bounded, then the set of V_n's is also bounded; and since $L_n \le b - a + V_n$, if the set of V_n's is bounded, then the set of L_n's is also bounded. We therefore have

Theorem 1 *The graph of $y = f(x)$ is rectifiable on an interval $[a, b]$ if and only if $f(x)$ is of bounded variation on this interval.*

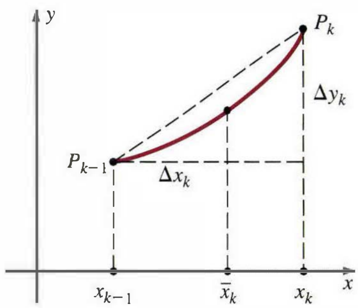


Figure A.14

If the set of V_n's is bounded, its least upper bound V is called the *total variation* of the function $f(x)$ on the interval $[a, b]$. V is a measure of the total amount of rising and falling of the graph. As examples, for the function $y = x^2$ on $-2 \le x \le 3$ we have $V = 13$, since the graph falls four units and rises nine units; and for $y = \cos x$ on $0 \le x \le 2\pi$ we have $V = 4$. Each curve therefore has a length, but this length may be difficult to calculate.

Our blanket assumption is that the function $y = f(x)$ is continuous on the closed interval $a \le x \le b$. Let us now make the further assumption that $f(x)$ has a derivative on the open interval $a < x < b$. Then the Mean Value Theorem tells us (Fig. A.14) that there exists a point $\bar{x}_k$, $x_{k-1} < \bar{x}_k < x_k$, with the property that

$$f'(\bar{x}_k) = \frac{\Delta y_k}{\Delta x_k},$$

so

$$|\Delta y_k| = |f'(\bar{x}_k)|\,\Delta x_k.$$

We can now prove

Theorem 2 *If $y = f(x)$ has a bounded derivative on the open interval $a < x < b$, then its graph is rectifiable.*

The proof is easy: If $|f'(x)| \le M$, then $|\Delta y_k| \le M\Delta x_k$ and

$$\begin{aligned} V_n &= |\Delta y_1| + \cdots + |\Delta y_n| \\ &\le M(\Delta x_1 + \Delta x_2 + \cdots + \Delta x_n) \\ &= M(b - a). \end{aligned}$$

The function is therefore of bounded variation, and by Theorem 1 its graph is rectifiable.

Now for some instructive examples.

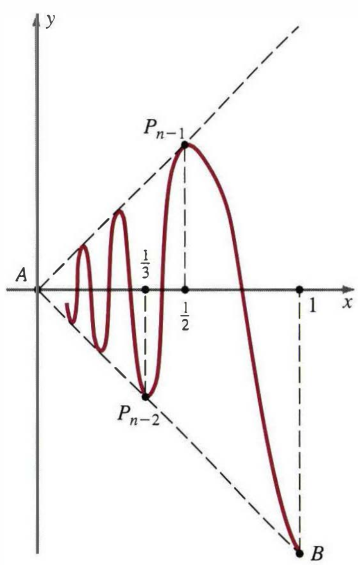


Figure A.15

Example 1 Define $y = f(x)$ on $0 \le x \le 1$ by

$$f(x) = \begin{cases} x \cos \pi/x & \text{if } 0 < x \le 1, \\ 0 & \text{if } x = 0. \end{cases}$$

The graph of this function is shown in Fig. A.15; it clearly has an infinite number of wiggles as it approaches its left endpoint. We take the points $A, P_1, \ldots, P_{n-1}, B$ to be the points on the graph corresponding to $x = 0, 1/n, 1/(n-1), \ldots, 1/2, 1$. We know that

$$f\left(\frac{1}{k}\right) = \frac{1}{k}\cos k\pi = \frac{(-1)^k}{k},$$

so

$$\begin{aligned} V_n &= |\Delta y_1| + \cdots + |\Delta y_n| \\ &= \frac{1}{n} + \left(\frac{1}{n} + \frac{1}{n-1}\right) + \cdots + \left(\frac{1}{3} + \frac{1}{2}\right) + \left(\frac{1}{2} + 1\right). \end{aligned}$$

The right side of this is greater than the sum

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

It is known from the elementary theory of infinite series that these sums are unbounded as n increases. The function $f(x)$ is therefore not of bounded variation, and by Theorem 1 its graph is not rectifiable.

Example 2 We next consider a close relative of the function discussed in Example 1. This time we define $y = f(x)$ on the interval $0 \le x \le 1$ by

$$f(x) = \begin{cases} x^2 \cos \pi/x & \text{if } 0 < x \le 1, \\ 0 & \text{if } x = 0. \end{cases}$$

The graph is shown in Fig. A.16, and it also has a infinite number of wiggles. The derivative of this function in the open interval is

$$f'(x) = \pi \sin \frac{\pi}{x} + 2x \cos \frac{\pi}{x}.$$

This derivative is easily seen to be bounded,

$$|f'(x)| \le \pi + 2,$$

so by Theorem 2 its graph is rectifiable. Even though we know this curve has a length, the problem of finding its value is far beyond our capabilities.

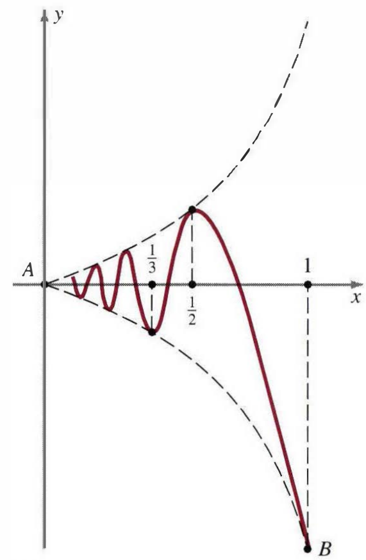


Figure A.16

A.8 THE EXISTENCE OF $e = \lim_{h\to 0} (1 + h)^{1/h}$

In this discussion we begin by defining e to be the limit

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n. \tag{1}$$

We then carefully extend this formula step by step until we reach the more general conclusion that

$$e = \lim_{h\to 0} (1 + h)^{1/h}, \tag{2}$$

where h is allowed to approach 0 in any manner whatever, through rational or irrational, positive or negative, values.

Our first task is to prove the existence of the limit (1), and thereby to legitimize this definition of e. By the binomial theorem the quantity

$$x_n = \left(1 + \frac{1}{n}\right)^n$$

can be express. ' as the following sum of $n + 1$ terms,

$$1 + n \cdot \frac{1}{n} + \frac{n(n-1)}{1 \cdot 2} \cdot \frac{1}{n^2} + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} \cdot \frac{1}{n^3} + \cdots + \frac{1}{n^n}$$

$$= 1 + 1 + \frac{1}{1 \cdot 2}\left(1 - \frac{1}{n}\right) + \frac{1}{1 \cdot 2 \cdot 3}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \cdots + \frac{1}{n^n}. \tag{3}$$

As n increases, the number of terms in this sum increases, and also each term after the second increases. This shows that

$$x_1 < x_2 < x_3 < \cdots < x_n < x_{n+1} < \cdots. \tag{4}$$

Also, the expansion (3) tells us that

$$x_n < 1 + 1 + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \cdots + \frac{1}{1 \cdot 2 \cdot 3 \cdots n}$$

$$< 1 + 1 + \left(\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}\right) < 1 + 1 + 1 = 3, \tag{5}$$

since the expression in parentheses is part of the familiar geometric series

$$\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} + \cdots = 1.$$

By (4) and (5) the x_n's steadily increase but always remain < 3, so they necessarily approach a limiting value. In the present context this limiting value is e by definition. This argument proves (1), and also we clearly have

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n+1}\right)^{n+1},$$

which will be needed below.

We next consider the limit (2) for the special case in which h is required to approach 0 through positive values. When $h < 1$, there exists a unique positive integer n such that

$$n \le \frac{1}{h} < n + 1.$$

This implies that

$$\left(1 + \frac{1}{n+1}\right)^n < (1+h)^{1/h} < \left(1 + \frac{1}{n}\right)^{n+1},$$

which in turn can be written as

$$\frac{[1 + 1/(n+1)]^{n+1}}{1 + 1/(n+1)} < (1+h)^{1/h} < \left(1 + \frac{1}{n}\right)^n\left(1 + \frac{1}{n}\right). \tag{6}$$

As $h \to 0$, $n \to \infty$ and the first and third terms of the inequality (6) approach e. Since $(1 + h)^{1/h}$ is caught between the two, it must have the same limit, and therefore (2) is proved for the case in which $h \to 0$ through positive values.

We conclude our analysis by establishing (2) for the case in which h approaches 0 through negative values. If we put $h = -k$, then

$$\begin{aligned}(1+h)^{1/h} &= (1-k)^{-1/k} = \left(\frac{1}{1-k}\right)^{1/k} \\ &= \left(1 + \frac{k}{1-k}\right)^{1/k} = \left(1 + \frac{k}{1-k}\right)^{(1-k)/k}\left(1 + \frac{k}{1-k}\right) \to e \cdot 1 = e,\end{aligned}$$

by the result of the previous paragraph.

A.9 FUNCTIONS THAT CANNOT BE INTEGRATED

In spite of the many successes achieved by the methods of Chapter 10, certain integrals have always resisted every attempt to express them in terms of elementary functions: for instance,

$$\int e^{-x^2}\,dx, \qquad \int \frac{e^x}{x}\,dx, \qquad \int \cos x^2\,dx,$$

$$\int \frac{dx}{\ln x}, \qquad \int \sqrt{\sin x}\,dx, \qquad \int \frac{\sin x}{x}\,dx.$$

There are also the so-called *elliptic integrals,* of which

$$\int \sqrt{1 - x^3}\,dx \qquad \text{and} \qquad \int \frac{dx}{\sqrt{1 - x^4}}$$

are examples.* In the nineteenth century it was finally proved, by the great French mathematician Liouville and his followers, that the problem of working out these integrals in terms of elementary functions is not merely difficult—it is actually impossible.

The full depth of Liouville's ideas cannot be plumbed in a calculus course.† Nevertheless, it is quite possible to gain some impression of how these ideas work without necessarily undertaking a long program of preliminary study.‡

Among other things, Liouville discovered and proved the following theorem:

If $f(x)$ and $g(x)$ are rational functions and $g(x)$ is not a constant, and if $\int f(x)e^{g(x)}\,dx$ is an elementary function, then this integral must have the form

$$\int f(x)e^{g(x)}\,dx = Re^{g(x)}$$

for some rational function R.

We illustrate the value of this theorem by using it to prove that the integral

$$\int \frac{e^x}{x}\,dx \tag{1}$$

is not elementary (that is, cannot be expressed in terms of elementary functions). Suppose, on the contrary, that this integral *is* elementary. Then by Liouville's theorem we know that

$$\int \frac{e^x}{x}\,dx = Re^x$$

for some rational function R. But this means that

$$\frac{e^x}{x} = \frac{d}{dx}(Re^x) \qquad \text{or} \qquad \frac{e^x}{x} = Re^x + R'e^x,$$

so

$$\frac{1}{x} = R + R'. \tag{2}$$

Since R is rational, it can be written in the form $R = P/Q$, where P and Q are polynomials with no common factor. We know that

$$R' = \frac{QP' - PQ'}{Q^2},$$

so (2) becomes

$$\frac{1}{x} = \frac{P}{Q} + \frac{QP' - PQ'}{Q^2},$$

which is equivalent to

$$Q^2 = PQx + x(QP' - PQ')$$

or

*In general, an elliptic integral is any integral of the form $\int R(x, y)\,dx$, where $R(x, y)$ is a rational function of the two variables x, y and where y is the square root of a polynomial of the third or fourth degree in x. The name *elliptic integral* is used because an integral of this type arises in the problem of finding the circumference of an ellipse.

†Liouville's theory is expounded in full in the monograph by J. F. Ritt, *Integration in Finite Terms* (Columbia University Press, 1948).

‡See D. G. Mead's article "Integration," *Amer. Math. Monthly*, **68** (1961), pp. 152–156.

$$Q(Q - Px - P'x) = -PQ'x. \tag{3}$$

Our purpose is to deduce a contradiction from (3), and we proceed as follows. Let x^n be the highest power of x that can be factored out of the polynomial Q, so that $Q(x) = x^n Q_1(x)$ where $Q_1(x)$ is a polynomial such that $Q_1(0) \neq 0$. We first observe that $n > 0$; for if $n = 0$, so that $Q(0) \neq 0$, then $x = 0$ reduces the right side of (3) to zero but not the left side, which cannot happen because (3) is an identity in x. This implies two facts that we need in order to obtain our final contradiction. First, $P(0) \neq 0$, because P and Q have no common factor and therefore x cannot be a factor of P. Second, we have

$$\begin{aligned} Q'(x) &= x^n Q_1'(x) + nx^{n-1}Q_1(x) \\ &= x^{n-1}[xQ_1'(x) + nQ_1(x)]; \end{aligned}$$

and since the polynomial in brackets has a nonzero value when $x = 0$, we know that x^{n-1} is the highest power of x that can be factored out of Q'. These two facts taken together tell us that x^n is the highest power of x that can be factored out of the polynomial on the right side of (3), whereas x^{n+1} can be factored out of the left side. This contradiction brings us to the conclusion that (2) is impossible, so the integral (1) is not elementary.

Remark 1 We know from our work in Section 6.7 that for any continuous integrand the definite integral

$$F(x) = \int_0^x f(t)\, dt \tag{4}$$

exists and has the property that

$$\frac{d}{dx} F(x) = f(x). \tag{5}$$

Since (5) is equivalent to

$$\int f(x)\, dx = F(x),$$

we see that the indefinite integral of every continuous function exists. However, this fact has nothing to do with the issue of whether the integral can be expressed in terms of the elementary functions. When such an expression is not possible, formula (4) can be thought of as providing a legitimate and sometimes useful method for creating new functions. For example, the nonelementary function of x defined by

$$\frac{1}{\sqrt{2\pi}} \int_0^x e^{-t^2/2}\, dt$$

has important applications in the theory of probability, and for this reason it has been studied and tabulated and has thereby acquired a certain status as a "known function."

Remark 2 It is easy to see that the integral

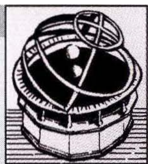
$$\int \frac{e^x}{x}\, dx \qquad \text{becomes} \qquad \int \frac{dt}{\ln t}$$

under the substitution $t = e^x$; for $x = \ln t$, $dx = dt/t$, and therefore

$$\int \frac{e^x}{x}\, dx = \int \frac{t}{\ln t} \frac{dt}{t} = \int \frac{dt}{\ln t}.$$

Since we know that the first integral is not elementary, it is clear that the second integral is also not elementary. This is worth noticing because the function of x defined by

$$\int_2^x \frac{dt}{\ln t} \tag{6}$$



NOTE ON LIOUVILLE

Joseph Liouville (1809–1882) was a highly respected professor at the Collège de France in Paris and the founder and editor of the *Journal des Mathématiques Pures et Appliquées,* a famous periodical that played an important role in French mathematical life through the latter part of the nineteenth century. For some reason, however, his own remarkable achievements as a creative mathematician have not received the appreciation they deserve.

His theory of integrals of elementary functions as described above was perhaps the most original of all his achievements, for in it he proved that many familiar integrals are not merely difficult to work out, but actually impossible.

The fascinating and difficult theory of transcendental numbers is another important branch of mathematics that originated in Liouville's work. The irrationality of π and e (that is, the fact that these numbers are not roots of any linear equation $ax + b = 0$ whose coefficients are integers) had been proved in the eighteenth century by Lambert and Euler. In 1844 Liouville showed that e is also not a root of any quadratic equation with integral coefficients. This led him to conjecture that e is *transcendental,* which means that it does not satisfy any polynomial equation

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

with integral coefficients. His efforts to prove this failed, but his ideas contributed to Hermite's success in 1873 and then to Lindemann's 1882 proof that π is also transcendental. Lindemann's result showed at last that the age-old problem of squaring the circle by a ruler-and-compass construction is impossible. One of the great mathematical achievements of modern times was Gelfond's 1929 proof that e^{π} is transcendental, but nothing is yet known about the nature of any of the numbers $\pi + e$, πe, or π^e. Liouville also discovered a sufficient condition for transcendence and used it in 1851 to produce the first examples of real numbers that are provably transcendental. One of these is[*]

$$\sum_{n=1}^{\infty} \frac{1}{10^{n!}} = \frac{1}{10^1} + \frac{1}{10^2} + \frac{1}{10^6} + \cdots = 0.11000100\ldots.$$

His methods here have also led to extensive further work in the twentieth century.

The ancient Greek philosopher-scientist Democritus said, "I would rather discover one cause than be King of Persia." What Liouville accomplished was certainly better than being King of Persia, or being any king or political leader whatsoever. He was a thinker whose work will live as long as people care about beautiful ideas.

[*]For the details of this, see pp. 288–290 of the present writer's book, *Calculus Gems* (McGraw-Hill, 1992).

is of great importance in the theory of prime numbers, and the behavior of this function for large values of x has been studied exhaustively for more than a century.[*] [The lower limit of integration in (6) is chosen to be 2 in order to avoid the point $t = 1$, where $\ln t = 0$.]

[*]See pp. 2–4 of H. M. Edwards, *Riemann's Zeta Function* (Academic Press, 1974).

PROBLEMS

1 Consider an integral of the form $\int R(\sin x, \cos x)\, dx$, where the integrand is a rational function of $\sin x$ and $\cos x$. Show that the substitution

$$z = \tan \tfrac{1}{2}x$$

converts this integral into the integral of a rational function of z, which can then be worked out by routine procedures. Hint: Show that

$$\sec^2 \frac{1}{2}x = 1 + z^2, \qquad \cos x = \cos 2\left(\frac{1}{2}x\right) = \frac{1 - z^2}{1 + z^2},$$

$$\sin x = \frac{2z}{1 + z^2}, \qquad \text{and} \qquad dx = \frac{2dz}{1 + z^2}.$$

2 Use the method of Problem 1 to find

(a) $\displaystyle\int \frac{dx}{2 + \cos x}$; (b) $\displaystyle\int \frac{\sin x\, dx}{2 + \sin x}$.

3 Use the method of Problem 1 to find
(a) $\int \sec x\, dx$; (b) $\int \tan x\, dx$.
Express your answers in the usual form [i.e., $\ln(\sec x + \tan x)$ and $-\ln(\cos x)$].

4 Use the method of Problem 1 to obtain the following formulas

(a) $\displaystyle\int \frac{dx}{a + b\sin x} = \int \frac{2dz}{az^2 + 2bz + a}$;

(b) $\displaystyle\int \frac{dx}{a + b\sin x + c\cos x} = \int \frac{2dz}{(a - c)z^2 + 2bz + (a + c)}$;

(c) $\displaystyle\int \frac{\sin x\, dx}{1 + \sin x} = \int \frac{4z\, dz}{(1 + z)^2(1 + z^2)}$;

(d) $\displaystyle\int \frac{\cos x\, dx}{1 + \cos x} = \int \frac{(1 - z^2)\, dz}{1 + z^2}$.

A *rationalizing substitution* is a change of variable that eliminates radicals or fractional exponents. Find the following integrals by using this idea.

5 $\displaystyle\int \frac{dx}{1+\sqrt{x}}$. Hint: Put $u = \sqrt{x}$.

6 $\displaystyle\int \frac{\sqrt{x} + 1}{\sqrt{x} - 1}\, dx$.

7 $\displaystyle\int \frac{dx}{\sqrt{x} + \sqrt[3]{x}}$. Hint: Put $u = \sqrt[6]{x}$.

8 $\displaystyle\int \frac{3\sqrt{x}\, dx}{4(1 + x^{3/4})}$.

9 $\displaystyle\int \frac{\sqrt{x}}{1 + x}\, dx$.

10 $\displaystyle\int \frac{x^{2/3}}{1 + x}\, dx$.

11 $\displaystyle\int \frac{\sqrt[4]{x}}{1 + \sqrt{x}}\, dx$.

12 $\displaystyle\int \frac{dx}{x(1 - \sqrt[4]{x})}$.

13 $\displaystyle\int \frac{\sqrt{x + 2}}{x + 3}\, dx$.

14 $\displaystyle\int \frac{\sqrt[3]{x + 1}}{x}\, dx$.

15 The special elliptic integral

$$\int \frac{du}{\sqrt{(1 - u^2)(1 - k^2u^2)}}$$

is called the *elliptic integral of the first kind*. Show that each of the following integrals can be brought into this form by means of the indicated substitution:

(a) $\displaystyle\int \frac{dx}{\sqrt{1 - k^2\sin^2 x}} = \int \frac{du}{\sqrt{(1 - u^2)(1 - k^2u^2)}}$, $u = \sin x$;

(b) $\displaystyle\int \frac{dx}{\sqrt{\cos 2x}} = \int \frac{du}{\sqrt{(1 - u^2)(1 - 2u^2)}}$, $u = \sin x$;

(c) $\displaystyle\int \frac{dx}{\sqrt{\cos x}} = 2\int \frac{du}{\sqrt{(1 - u^2)(1 - 2u^2)}}$, $u = \sin \frac{1}{2}x$;

(d) $\displaystyle\int \frac{dx}{\sqrt{\cos x - \cos\alpha}} = \sqrt{2}k\int \frac{du}{\sqrt{(1 - u^2)(1 - k^2u^2)}}$,

$u = \sin \frac{1}{2}x$ and $k = \csc \frac{1}{2}\alpha$.

16 Consider the integral in part (b) of Problem 15,

$$\int \frac{dx}{\sqrt{\cos 2x}} = \int \frac{dx}{\sqrt{1 - 2\sin^2 x}}.$$

Show that the substitution $u = \tan x$ transforms this integral into the special elliptic integral

$$\int \frac{du}{\sqrt{1 - u^4}}.$$

17 If p and q are rational numbers, show that the integral

$$\int x^p(1 - x)^q\, dx \qquad (*)$$

is elementary in each of the following cases:
(a) p is an integer (hint: if $q = m/n$ with $n > 0$, put $1 - x = u^n$);
(b) q is an integer;
(c) $p + q$ is an integer $\left[\text{hint: } \displaystyle\int x^p(1 - x)^q\, dx = \int x^{p+q}\left(\frac{1 - x}{x}\right)^q dx\right]$.

The Russian mathematician Chebyshev proved that these are the only cases for which the integral (*) is elementary.† Accordingly,

$$\int \sqrt{x}\,\sqrt[3]{1 - x}\, dx, \qquad \int \sqrt[3]{x}\,\sqrt{1 - x}\, dx, \qquad \int \sqrt[3]{x - x^2}\, dx$$

are not elementary.

18 Use the theorem of Chebyshev stated in Problem 17 to prove that each of the following integrals is not elementary:
(a) $\int \sqrt{1 - x^3}\, dx$;
(b) $\int \sqrt{1 - x^4}\, dx$;
(c) $\int \sqrt{1 - x^n}\, dx$, where n is any integer > 2;

†See Ritt, p. 37.

(d) $\int \frac{dx}{\sqrt{1-x^n}}$, where n is any integer > 2.

19 Use Problem 17 to prove that
(a) $\int \sqrt{\sin x}\, dx$ is not elementary (hint: put $u = \sin^2 x$);
(b) $\int \sin^p x\, dx$, where p is a rational number, is elementary if and only if p is an integer;
(c) $\int \sin^p x \cos^q x\, dx$, where p and q are rational numbers, is elementary if and only if p or q is an odd integer or $p + q$ is an even integer.

A.10 THE VALIDITY OF INTEGRATION BY INVERSE SUBSTITUTION

In making the direct substitutions discussed in Section 10.2, our procedure was to put $u = g(x)$ where $g(x)$ was part of the integrand. For this method to work, we had to have $du = g'(x)\, dx$ as another part of the integrand, and this meant that altogether the integrand had to have a rather special form.

A much more natural way to change the variable in an integral $\int f(x)\, dx$ is to introduce a new variable u by writing $x = h(u)$ and $dx = h'(u)\, du$, where $h(u)$ is some function that is suggested by the form of the integral. This means that if we translate the given integral from the x-notation to the u-notation by writing

$$\int f(x)\, dx = \int f[h(u)]h'(u)\, du = \int g(u)\, du, \tag{1}$$

where $g(u) = f[h(u)]h'(u)$, then we hope the integral on the right will be easy to calculate. In fact, if

$$\int g(u)\, du = G(u), \tag{2}$$

then we expect to have

$$\int f(x)\, dx = G[k(x)], \tag{3}$$

where $u = k(x)$ is the inverse function of $x = h(u)$.* This process is called *inverse substitution.* It is a very useful method, *if* we can find $G(u)$ and *if* we know the inverse function $u = k(x)$. These remarks constitute a general description of what is going on in the method of trigonometric substitutions.

We can prove the validity of inverse substitution as follows. The point is this: In direct substitution as discussed in Section 10.2 we used the integral transformation (1) in the other direction, to calculate $\int g(u)\, du$. We showed that if

$$\int f(x)\, dx = F(x),$$

then

$$\int g(u)\, du = F[h(u)].$$

Thus, in the present context, where we also have (2), it follows that

$$F[h(u)] = G(u) + c$$

for some constant c. But this says that

$$F(x) = G[k(x)] + c,$$

and therefore $G[k(x)]$ is an integral of $f(x)$, as claimed in (3).

Even more, we can use the same method for dealing with definite integrals if the limits of integration are correctly changed; that is,

*That is, $u = k(x)$ is the result of solving $x = h(u)$ for u in terms of x. The concept of an inverse function is discussed in Remark 2 in Section 9.5.

$$\int_a^b f(x)\,dx = \int_c^d g(u)\,du,$$

where $c = k(a)$ and $d = k(b)$. This can be established very easily by thinking of it in the other direction, as

$$\int_c^d g(u)\,du = \int_a^b f(x)\,dx,$$

where $a = h(c)$ and $b = h(d)$, because this second version was proved in Section 10.2.

A.11 PROOF OF THE PARTIAL FRACTIONS THEOREM

Our purpose here is to establish the validity of the partial fractions decomposition as stated in a piecemeal manner in Section 10.6. We are considering a rational function $P(x)/Q(x)$, and we assume that $Q(x)$ is a polynomial of degree n that is completely factored into real linear and quadratic factors of various multiplicities. In the beginning we do not assume that $P(x)/Q(x)$ is proper. This enables us to understand more clearly the significance of this assumption when it becomes necessary to make it. Our basic tool is the following lemma about the removal of a linear factor from the denominator.

Lemma *Let $x - r$ be a linear factor of $Q(x)$ of multiplicity 1 so that $Q(x) = (x - r)Q_1(x)$ with $Q_1(r) \neq 0$. Then $P(x)/Q(x)$ can be written in the form*

$$\frac{P(x)}{Q(x)} = \frac{P(x)}{(x - r)Q_1(x)} = \frac{A}{x - r} + \frac{P_1(x)}{Q_1(x)}, \tag{1}$$

where A is a constant and $P_1(x)$ is a polynomial such that $P_1(x)/Q_1(x)$ is a proper rational function whenever $P(x)/Q(x)$ is. The constant A can be calculated from either of the formulas

$$A = \frac{P(r)}{Q_1(r)} = \frac{P(r)}{Q'(r)}. \tag{2}$$

Proof We must find a suitable A and $P_1(x)$, and we do this by letting (1) suggest what their definitions ought to be. With these definitions we then show that (1) is valid.

By combining the fractions on the right side of (1), we see that A and $P_1(x)$ must be chosen so that the numerators are identical,

$$P(x) = AQ_1(x) + (x - r)P_1(x). \tag{3}$$

Since this is to be an identity, it must hold in particular for $x = r$. This gives $P(r) = AQ_1(r) + 0$, so we put

$$A = \frac{P(r)}{Q_1(r)}. \tag{4}$$

This is a legitimate definition because $Q_1(r) \neq 0$. Since

$$Q(x) = (x - r)Q_1(x) \qquad \text{and} \qquad Q'(x) = (x - r)Q_1'(x) + Q_1(x),$$

we see that $Q'(r) = Q_1(r)$, and this establishes the second formula for A stated in (2). Using the formula for A given by (4), we now solve (3) for $P_1(x)$,

$$\begin{aligned} P_1(x) &= \frac{P(x) - AQ_1(x)}{x - r} = \frac{P(x) - [P(r)/Q_1(r)]Q_1(x)}{x - r} \\ &= \frac{1}{Q_1(r)} \frac{P(x)Q_1(r) - P(r)Q_1(x)}{x - r}. \end{aligned} \tag{5}$$

We adopt this as our definition of $P_1(x)$. It may appear that this function is not a polynomial. However, the numerator of this fraction is clearly a polynomial that has the value 0 for $x = r$, so by the factor theorem of algebra it has $x - r$ as a factor. The common factor $x - r$ can now be canceled from the numerator and denominator, and we conclude that $P_1(x)$ is indeed a polynomial. We now show that (1) is valid when A and $P_1(x)$ are defined as they are above:

$$\begin{aligned}\frac{A}{x-r}+\frac{P_1(x)}{Q_1(x)} &= \frac{AQ_1(x)+(x-r)P_1(x)}{(x-r)Q_1(x)} \\ &= \frac{[P(r)/Q_1(r)]Q_1(x)+[1/Q_1(r)][P(x)Q_1(r)-P(r)Q_1(x)]}{(x-r)Q_1(x)} \\ &= \frac{P(x)}{(x-r)Q_1(x)}.\end{aligned}$$

Finally, the statement that $P_1(x)/Q_1(x)$ is proper whenever $P(x)/Q(x)$ is proper follows from (3) by using the fact that the degree of $Q_1(x)$ is $n - 1$; for if the degree of $P_1(x)$ is $\geq n - 1$, then (3) shows that the degree of $P(x)$ is $\geq n$.

This lemma enables us to do everything we wish with respect to splitting off partial fractions generated by linear factors of $Q(x)$. At this stage we specifically assume that $P(x)/Q(x)$ is proper, so that each time the lemma is applied the residual rational function $P_1(x)/Q_1(x)$ will also be proper.

We first observe that if $Q(x)$ can be factored completely into distinct linear factors, so that

$$Q(x) = (x - r_1)(x - r_2) \cdots (x - r_n),$$

then

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x-r_1}+\frac{A_2}{x-r_2}+\cdots+\frac{A_n}{x-r_n},$$

for we can remove the factors from the denominator one at a time in accordance with the lemma. At the last step the residual denominator is $x - r_n$, and since the numerator is necessarily of lower degree, this numerator must be a constant.

Suppose next that $x - r$ is a linear factor of $Q(x)$ of multiplicity m, so that $Q(x) = (x - r)^m Q_1(x)$ with $Q_1(r) \neq 0$. To cope with this situation we apply the lemma repeatedly in a slightly different way. First, by (1) we have

$$\frac{P(x)}{(x-r)Q_1(x)} = \frac{B_m}{x-r}+\frac{P_1(x)}{Q_1(x)}.$$

Dividing through by $x - r$ and applying (1) again yields

$$\frac{P(x)}{(x-r)^2Q_1(x)} = \frac{B_m}{(x-r)^2}+\frac{P_1(x)}{(x-r)Q_1(x)} = \frac{B_m}{(x-r)^2}+\frac{B_{m-1}}{x-r}+\frac{P_2(x)}{Q_1(x)}.$$

By continuing in this way, we find in the end that

$$\begin{aligned}\frac{P(x)}{Q(x)} = \frac{P(x)}{(x-r)^mQ_1(x)} &= \frac{B_m}{(x-r)^m}+\cdots+\frac{B_1}{x-r}+\frac{P_m(x)}{Q_1(x)} \\ &= \frac{P_m(x)}{Q_1(x)}+\frac{B_1}{x-r}+\cdots+\frac{B_m}{(x-r)^m}.\end{aligned}$$

In this manner we strip off all the linear factors from the denominator of our proper rational function $P(x)/Q(x)$ and generate the corresponding partial fractions as described in Section 10.6.

The rest of the proof requires an acquaintance with complex numbers, because the imaginary zeros of a real polynomial come in conjugate pairs and this fact plays an essential role in the argument. Before we begin, it is necessary to observe that our fundamental lemma works in just the same way if the number r happens to be imaginary.

Now suppose that $x^2 + bx + c$ is a quadratic factor of $Q(x)$ of multiplicity 1 which is irreducible in the sense that $b^2 - 4c < 0$, so that the roots r_1 and r_2 of the equation $x^2 + bx + c = 0$ are conjugate complex numbers.* Then

$$Q(x) = (x^2 + bx + c)Q_2(x) = (x - r_1)(x - r_2)Q_2(x),$$

and by two applications of our lemma we can find constants A_1 and A_2 and a polynomial $P_2(x)$ such that

$$\frac{P(x)}{Q(x)} = \frac{P(x)}{(x - r_1)(x - r_2)Q_2(x)} = \frac{A_1}{x - r_1} + \frac{A_2}{x - r_2} + \frac{P_2(x)}{Q_2(x)}.$$

By using (2) we see that

$$A_1 = \frac{P(r_1)}{Q'(r_1)} \quad \text{and} \quad A_2 = \frac{P(r_2)}{Q'(r_2)},$$

and these formulas imply that A_1 and A_2 are also conjugate complex numbers. By combining the corresponding partial fractions, we can now write

$$\frac{P(x)}{Q(x)} = \frac{(A_1 + A_2)x - (A_1r_2 + A_2r_1)}{(x - r_1)(x - r_2)} + \frac{P_2(x)}{Q_2(x)} = \frac{Ax + B}{x^2 + bx + c} + \frac{P_2(x)}{Q_2(x)},$$

where the numbers $A = A_1 + A_2$ and $B = -(A_1r_2 + A_2r_1)$ are real because r_1, r_2 and A_1, A_2 are conjugate pairs of complex numbers. Also, we know from the last expression that $P_2(x)$ is a real polynomial. If the factor $x^2 + bx + c$ occurs with multiplicity $m > 1$, then we simply remove it over and over in the way used above with repeated linear factors. This produces exactly the partial fractions decomposition described in Section 10.6.

When these procedures have been applied to each of the real linear and quadratic factors of $Q(x)$, and all the corresponding partial fractions have been stripped away, then there is nothing left of $Q(x)$, the decomposition is complete, and the partial fractions theorem is fully proved.

Up to this point we have said nothing about uniqueness, but it is worth remarking that a proper rational function can be decomposed into partial fractions in only one way. This will follow at once from our overall discussion if we can show in the lemma that the expansion (1) is unique. But this is easy; for if we assume two forms for this expansion,

$$\frac{A}{x - r} + \frac{P_1(x)}{Q_1(x)} = \frac{B}{x - r} + \frac{P_2(x)}{Q_1(x)},$$

then we have

$$AQ_1(x) + (x - r)P_1(x) = BQ_1(x) + (x - r)P_2(x).$$

By letting $x \to r$, we see from this that $B = A$, so A in (1) is unique; and this implies that $P_1(x)$ in (1) is also unique.

*If $x^2 + bx + c$ were not irreducible, it would already have been factored into real linear factors in the "complete" factorization of $Q(x)$ previously mentioned.

A.12 THE EXTENDED RATIO TESTS OF RAABE AND GAUSS

The convergence tests we discuss in this appendix are more delicate than the ratio test and enable us to arrive at a definite conclusion for many series with the property that $a_{n+1}/a_n \to 1$ from below. We begin with the following general theorem of Kummer.*

Theorem 1 *Assume that $a_n > 0$, $b_n > 0$, and $\Sigma 1/b_n$ diverges. If*

$$\lim \left(b_n - \frac{a_{n+1}}{a_n} \cdot b_{n+1} \right) = L,$$

then Σa_n converges if $L > 0$ and diverges if $L < 0$.

Proof If $L > 0$, then

$$b_n - \frac{a_{n+1}}{a_n} \cdot b_{n+1} \geq h > 0$$

for all $n \geq$ some n_0, so

$$a_n b_n - a_{n+1} b_{n+1} \geq h a_n > 0 \tag{1}$$

for these n's. This shows that $\{a_n b_n\}$ is a decreasing sequence of positive numbers for $n \geq n_0$, so $K = \lim a_n b_n$ exists. It is now clear that $\Sigma_{n=n_0}^{\infty}(a_n b_n - a_{n+1} b_{n+1})$ is a convergent telescopic series (with sum $a_{n_0} b_{n_0} - K$), so by (1) and the comparison test we conclude that $\Sigma h a_n$ converges, and therefore Σa_n also converges.

Next, if $L < 0$ we have

$$a_n b_n - a_{n+1} b_{n+1} \leq 0$$

for all $n \geq$ some n_0, so $\{a_n b_n\}$ is an increasing sequence of positive numbers for these n's. It follows that

$$a_n b_n \geq a_{n_0} b_{n_0} \qquad \text{or} \qquad a_n \geq (a_{n_0} b_{n_0}) \cdot \frac{1}{b_n}$$

for $n \geq n_0$, so Σa_n diverges because $\Sigma 1/b_n$ diverges.

Students will observe that if we take $b_n = 1$ in Kummer's theorem, we obtain the ratio test. As another application we deduce *Raabe's test.*†

Theorem 2 *If $a_n > 0$ and*

*The German mathematician Ernst Eduard Kummer (1810–1893) is remembered mainly for his work on the arithmetic of algebraic number fields, by means of which he proved Fermat's last theorem for many prime exponents. He also contributed to geometry (the entity known as *Kummer's surface* was much later found by Eddington to be related to Dirac's theory of the electron) and extended Gauss's work on hypergeometric series. He was a good-humored and rather easygoing man with a ready (and sometimes racy) wit. He taught at Breslau until 1855, when the death of Gauss dislocated the mathematical map of Europe. Dirichlet succeeded Gauss at Göttingen, and Kummer replaced Dirichlet at Berlin.

†Joseph Ludwig Raabe (1801–1859) was born of poor parents in Galicia and studied in Vienna. When cholera swept that city in 1831 he moved to Zürich. In 1833 the Austrian embassy in Bern demanded that the government in Zürich return Raabe to Austria, because he had broken Austrian law by taking a position with the University of Zürich. This ludicrous demand was very sensibly ignored, and Raabe spent the rest of his life in various posts at the University. He was a man of unusual modesty and was considered a very gifted teacher. He is now known only for the convergence test discussed here, but he also worked on the summation of series, on systems of linear differential equations, and on the problem of the motion of the center of gravity of the planets.

$$\frac{a_{n+1}}{a_n} = 1 - \frac{A}{n} + \frac{A_n}{n} \tag{2}$$

where $A_n \to 0$, then Σa_n converges if $A > 1$ and diverges if $A < 1$.

Proof Take $b_n = n$ in Kummer's theorem. Then

$$\begin{aligned}\lim\left(b_n - \frac{a_{n+1}}{a_n} \cdot b_{n+1}\right) &= \lim\left[n - \left(1 - \frac{A}{n} + \frac{A_n}{n}\right)(n+1)\right] \\ &= \lim\left[-1 + \frac{A(n+1)}{n} - \frac{A_n(n+1)}{n}\right] \\ &= A - 1,\end{aligned}$$

and by Kummer's theorem it follows that Σa_n converges if $A > 1$ and diverges if $A < 1$.

For practical purposes, it is worth noting that Raabe's test can be formulated more conveniently as follows: If $a_n > 0$ and

$$\lim n\left(1 - \frac{a_{n+1}}{a_n}\right) = A, \tag{3}$$

then Σa_n converges if $A > 1$ and diverges if $A < 1$. To prove this, it suffices to express (3) in the equivalent form

$$n\left(1 - \frac{a_{n+1}}{a_n}\right) = A - A_n \tag{4}$$

where $A_n \to 0$, since (4) is merely another way of writing (2).

Example 1 If we apply the ratio test to the series

$$\frac{1}{2} + \frac{1 \cdot 3}{2 \cdot 4} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} + \cdots + \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} + \cdots, \tag{5}$$

then we find that

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1)}{2 \cdot 4 \cdots (2n)(2n+2)} \cdot \frac{2 \cdot 4 \cdots (2n)}{1 \cdot 3 \cdots (2n-1)} \\ &= \frac{2n+1}{2n+2} \to 1 \text{ from below,}\end{aligned}$$

so the test fails. however

$$n\left(1 - \frac{a_{n+1}}{a_n}\right) = n\left(1 - \frac{2n+1}{2n+2}\right) = \frac{n}{2n+2} \to \frac{1}{2},$$

so (5) diverges by Raabe's test.

Example 2 Now consider the related series in which each term is squared,

$$\left[\frac{1}{2}\right]^2 + \left[\frac{1 \cdot 3}{2 \cdot 4}\right]^2 + \left[\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right]^2 + \cdots + \left[\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}\right]^2 + \cdots. \tag{6}$$

Here we see that

$$\frac{a_{n+1}}{a_n} = \frac{(2n+1)^2}{(2n+2)^2} \to 1 \text{ from below,}$$

so the ratio test fails again. Furthermore, we also have

$$n\left(1-\frac{a_{n+1}}{a_n}\right)=n\left(1-\frac{4n^2+4n+1}{4n^2+8n+4}\right)=\frac{4n^2+3n}{4n^2+8n+4}\to 1,$$

so even Raabe's test fails in this case

When $A = 1$ in Raabe's test, we turn to *Gauss's test.*

Theorem 3 *If $a_n > 0$ and*

$$\frac{a_{n+1}}{a_n}=1-\frac{A}{n}+\frac{A_n}{n^{1+c}}$$

where $c > 0$ and A_n is bounded as $n\to\infty$, then Σa_n converges if $A > 1$ and diverges if $A \le 1$.

Proof If $A \neq 1$, the statement follows from Raabe's test, since $A_n/n^c \to 0$. We therefore confine our attention to the case $A = 1$. Take $b_n = n \ln n$ in Kummer's theorem. Then

$$\begin{aligned}\lim\left(b_n-\frac{a_{n+1}}{a_n}\cdot b_{n+1}\right)&=\lim\left[n\ln n-\left(1-\frac{1}{n}+\frac{A_n}{n^{1+c}}\right)(n+1)\ln(n+1)\right]\\&=\lim\left[n\ln n-\frac{(n^2-1)}{n}\ln(n+1)-\frac{(n+1)}{n}\cdot\frac{A_n\ln(n+1)}{n^c}\right]\\&=\lim\left[n\ln\left(\frac{n}{n+1}\right)+\frac{\ln(n+1)}{n}-\frac{(n+1)}{n}\cdot\frac{A_n\ln(n+1)}{n^c}\right]\\&=-1+0-0=-1,\end{aligned}$$

and the divergence of the series in this case is a consequence of Kummer's theorem.

Gauss actually expressed his test in a specialized form adapted to series in which a_{n+1}/a_n is a quotient of two polynomials having the same term of highest degree. This version is also known as *Gauss's test.*

Theorem 4 *If $a_n > 0$ and*

$$\frac{a_{n+1}}{a_n}=\frac{n^k+\alpha n^{k-1}+\cdots}{n^k+\beta n^{k-1}+\cdots},\tag{7}$$

then Σa_n converges if $\beta-\alpha > 1$ and diverges if $\beta - \alpha \le 1$.

Proof If the quotient on the right of (7) is worked out by long division, we get

$$\frac{a_{n+1}}{a_n}=1-\frac{\beta-\alpha}{n}+\frac{A_n}{n^2},$$

where A_n is a quotient of the form

$$\frac{\gamma n^{k-2}+\cdots}{n^{k-2}+\cdots}$$

and is therefore clearly bounded as $n\to\infty$. The statement now follows from Theorem 3 with $c = 1$.

Example 3 For a simple application we consider the series (6), for which Raabe's test failed. Here we have

$$\frac{a_{n+1}}{a_n}=\frac{4n^2+4n+1}{4n^2+8n+4}=\frac{n^2+n+\frac{1}{4}}{n^2+2n+1},$$

so $\beta-\alpha = 2 - 1 = 1$ and the series diverges by Gauss's test.

Example 4 Gauss's original purpose in devising his test was to study the important *hypergeometric series*

$$1+\sum_{n=1}^{\infty}\frac{a(a+1)\cdots(a+n-1)b(b+1)\cdots(b+n-1)}{n!c\,(c+1)\cdots(c+n-1)}x^n \tag{8}$$

when $x = 1$:

$$1+\frac{a\cdot b}{1\cdot c}+\frac{a(a+1)b(b+1)}{1\cdot 2c(c+1)}+\cdots$$
$$+\frac{a(a+1)\cdots(a+n-1)b(b+1)\cdots(b+n-1)}{n!c(c+1)\cdots(c+n-1)}+\cdots. \tag{9}$$

Here we assume that none of the constants a, b, c is zero or a negative integer. This condition on a and b keeps the series from terminating, while that on c avoids division by zero. The ratio

$$\frac{a_{n+1}}{a_n}=\frac{(a+n)(b+n)}{(n+1)(c+n)}=\frac{n^2+(a+b)\,n+ab}{n^2+(c+1)n+c}$$

is positive for all sufficiently large n, so the terms of (9) ultimately have the same sign. Any such series can be treated by Gauss's test (or the ratio test, or Raabe's test); and since in this case $\beta-\alpha=(c+1)-(a+b)$, we see that (9) converges if $c>a+b$ and diverges if $c\le a+b$.

The hypergeometric series (8) is extremely interesting and versatile, and is capable of representing virtually every function that occurs in elementary analysis.* Here we confine ourselves to remarking that when $a = 1$ and $b = c$ it reduces to the ordinary geometric series—hence its name.

*See Problem 1 on p. 203 of the present writer's book, *Differential Equations*, 2nd ed. (McGraw-Hill, 1991).

PROBLEMS

1 Show that

$$\sum_{n=1}^{\infty}\left[\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\right]^3$$

converges. More generally, let k be an arbitrary positive integer and show that

$$\sum_{n=1}^{\infty}\left[\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\right]^k$$

converges if $k > 2$ and diverges if $k \le 2$.

2 Determine the convergence behavior of the following series:

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{4\cdot 6\cdot 8\cdots(2n+2)}$;

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{2\cdot 7\cdot 12\cdots(5n-3)}{6\cdot 11\cdot 16\cdots(5n+1)}$;

(c) $\displaystyle\sum_{n=1}^{\infty}\frac{2\cdot 4\cdot 6\cdots(2n)}{3\cdot 5\cdot 7\cdots(2n+1)}$;

(d) $\displaystyle\sum_{n=1}^{\infty}\frac{2\cdot 4\cdot 6\cdots(2n)}{5\cdot 7\cdot 9\cdots(2n+3)}$;

(e) $\displaystyle\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\cdot\frac{1}{n}$.

3 Find the positive integers k for which the following series converge:

(a) $\displaystyle\sum_{n=1}^{\infty}\left[\frac{2\cdot 4\cdot 6\cdots(2n)}{3\cdot 5\cdot 7\cdots(2n+1)}\right]^k$;

(b) $\displaystyle\sum_{n=1}^{\infty}\left[\frac{2\cdot 4\cdot 6\cdots(2n)}{5\cdot 7\cdot 9\cdots(2n+3)}\right]^k$;

(c) $\displaystyle\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\cdot\frac{1}{n^k}$.

4 Determine the values of a and b (where neither is zero or a negative integer) for which the following series converge:

(a) $$\frac{a}{b} + \frac{a(a+1)}{b(b+1)} + \cdots + \frac{a(a+1)\cdots(a+n-1)}{b(b+1)\cdots(b+n-1)} + \cdots;$$

(b) $$\frac{a}{b} + \frac{a(a+2)}{b(b+2)} + \cdots + \frac{a(a+2)\cdots(a+2n-2)}{b(b+2)\cdots(b+2n-2)} + \cdots.$$

A.13 ABSOLUTE VS. CONDITIONAL CONVERGENCE

We begin with two simple examples:

$$\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \tag{1}$$

and

$$\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots. \tag{2}$$

As we saw in Section 13.8, there is an important distinction between these series. By the alternating series test, each is convergent as it stands. However, if we change the signs of all the negative terms—that is, if we replace each term by its absolute value—then the series become

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots;$$

and the first of these altered series now diverges, while the second still converges.

It was this phenomenon that led us in Section 13.8 to make the following definition: A series Σa_n is said to be *absolutely convergent* if $\Sigma|a_n|$ converges. Thus, (2) is absolutely convergent but (1) is not. The careful reader will notice that this definition in itself says nothing about the convergence of Σa_n. However, we proved in Section 13.8 that absolute convergence does indeed imply ordinary convergence.

The series (1) shows that the converse of this theorem is false, that is, convergence does not imply absolute convergence. Absolute convergence is therefore a stronger property than ordinary convergence, and we shall see that absolutely convergent series have several important properties which they do not share with convergent series in general. This brings us to another definition that was given but not pursued very far in Section 13.8: A series that is convergent but not absolutely convergent is said to be *conditionally convergent.* Our present purpose is to establish some of the general properties of absolutely convergent series, and also to emphasize the sharp contrast between these series and those which are only conditionally convergent. For instance, we shall see that rearranging the terms of an absolutely convergent series has no effect on its behavior or its sum, but that rearranging a conditionally convergent series can have a drastic effect.

It is convenient to begin this program by considering an arbitrary series Σa_n and defining p_n and q_n by

$$p_n = \frac{|a_n| + a_n}{2} \quad \text{and} \quad q_n = \frac{|a_n| - a_n}{2}. \tag{3}$$

It is clear that $p_n = a_n$ and $q_n = 0$ if $a_n > 0$, and that $p_n = 0$ and $q_n = -a_n$ if $a_n < 0$. Accordingly, if the given series is a mixture of positive and negative terms, then we can think of Σp_n as consisting of the positive terms of Σa_n, and of Σq_n as consisting of the negatives of its negative terms. This is not quite correct because many p_n's and q_n's can be zero, but it does provide a point of view which is useful for understanding the following theorems.

Theorem 1 *Consider a series Σa_n and define p_n and q_n by (3). If Σa_n converges conditionally, then Σp_n and Σq_n both diverge; and if Σa_n converges absolutely, then Σp_n and Σq_n both converge and the sums of these series are related by the equation $\Sigma a_n = \Sigma p_n - \Sigma q_n$.*

Proof It is clear from (3) that $a_n = p_n - q_n$ and $|a_n| = p_n + q_n$. Our basic tools are these equations and the fact that convergent series can be added or subtracted term by term.

To prove the first statement, we assume that Σa_n converges and $\Sigma |a_n|$ diverges. If Σq_n converges, then the equation $p_n = a_n + q_n$ tells us that Σp_n must also converge. Similarly, if Σp_n converges, the equation $q_n = p_n - a_n$ tells us that Σq_n also converges. Thus, if either Σp_n or Σq_n converges, both must converge; and in this case the equation $|a_n| = p_n + q_n$ implies that $\Sigma |a_n|$ converges—contrary to the assumption. This proves that the conditional convergence of Σa_n implies that Σp_n and Σq_n both diverge. To establish the second statement, we assume that $\Sigma |a_n|$ converges. We know that Σa_n also converges, so equations (3) imply that Σp_n and Σq_n both converge. It follows from this that

$$\sum p_n - \sum q_n = \sum (p_n - q_n) = \sum a_n,$$

and the proof is complete.

The first part of this theorem is illustrated by the conditionally convergent series (1), in which Σp_n and Σq_n are the divergent series

$$1 + 0 + \frac{1}{3} + 0 + \frac{1}{5} + \cdots \qquad \text{and} \qquad 0 + \frac{1}{2} + 0 + \frac{1}{4} + \cdots.$$

In the case of the absolutely convergent series (2), Σp_n and Σq_n are

$$1 + 0 + \frac{1}{3^2} + 0 + \frac{1}{5^2} + \cdots \qquad \text{and} \qquad 0 + \frac{1}{2^2} + 0 + \frac{1}{4^2} + \cdots,$$

both of which are convergent. Briefly, Theorem 1 tells us that the convergence of an absolutely convergent series is due to the smallness of its terms, while that of a conditionally convergent series is due not only to the smallness of its terms but also to cancellations between its positive and negative terms.

In the last paragraph of Section 13.5 we proved a theorem about rearranging a convergent series of nonnegative terms. We now extend this theorem to the case of absolutely convergent series.

Theorem 2 *If Σa_n is an absolutely convergent series with sum s, and if the a_n's are rearranged in any way to form a new series Σb_n, then this new series is also absolutely convergent with sum s.*

Proof The series $\Sigma |a_n|$ is convergent and consists of nonnegative terms. Since the b_n's are just the a_n's in a different order, it follows from the theorem just mentioned that $\Sigma |b_n|$ also converges, and therefore Σb_n is absolutely convergent. If $\Sigma b_n = t$, then Theorem 1 allows us to write

$$s = \sum a_n = \sum p_n - \sum q_n \tag{4}$$

and

$$t = \sum b_n = \sum P_n - \sum Q_n, \tag{5}$$

where each of the constituent series on the right is convergent and consists of nonnegative terms. But the P_n's and Q_n's are simply the p_n's and q_n's in a different order, so by another application of the theorem in Section 13.5 we have $\Sigma P_n = \Sigma p_n$ and $\Sigma Q_n = \Sigma q_n$. The fact that $t = s$ now follows at once from (4) and (5).

This theorem was proved in 1837 by Dirichlet, who discovered the phenomenon discussed in Problem 10 of Section 13.4 and was the first to understand the significance of absolutely convergent series.

In striking contrast to the behavior of absolutely convergent series as stated in Theorem 2, we find that the sum of a conditionally convergent series depends in an essential way on the order of its terms, and that the value of this sum can be changed at will by a suitable rearrangement of these terms. This fact was discovered and proved in 1854 by the great German mathematician Riemann, and is known as *Riemann's rearrangement theorem.* It can be formulated as follows.

Theorem 3 *Let Σa_n be a conditionally convergent series. Then its terms can be rearranged to yield a convergent series whose sum is an arbitrary preassigned number, or a series that diverges to ∞, or a series that diverges to $-\infty$.*

Proof The idea of the proof is surprisingly simple. We begin by using Theorem 1 to form the two divergent series of nonnegative terms Σp_n and Σq_n.

To establish the first statement, let s be any number and construct a rearrangement of the given series as follows. Start by writing down p's in order until the partial sum

$$p_1 + p_2 + \cdots + p_{n_1}$$

is first $\geq s$; next, continue with $-q$'s until the total partial sum

$$p_1 + p_2 + \cdots + p_{n_1} - q_1 - q_2 - \cdots - q_{m_1}$$

is first $\leq s$; then continue with p's until the total partial sum

$$p_1 + \cdots + p_{n_1} - q_1 - \cdots - q_{m_1} + p_{n_1+1} + \cdots + p_{n_2}$$

is first $\geq s$; and so on. The possibility of each of these steps is guaranteed by the divergence of Σp_n and Σq_n; and the resulting rearrangement of Σa_n converges to s because $p_n \to 0$ and $q_n \to 0$.

In order to make the rearrangement diverge to ∞, it suffices to write down enough p's to yield

$$p_1 + p_2 + \cdots + p_{n_1} \geq 1,$$

then to insert $-q_1$, then to continue with p's until

$$p_1 + \cdots + p_{n_1} - q_1 + p_{n_1+1} + \cdots + p_{n_2} \geq 2,$$

then to insert $-q_2$, and so on. We can produce divergence to $-\infty$ by a similar construction.

One of the principal applications of Theorem 2 relates to the multiplication of series. In this connection it is notationally convenient to index the terms of the series we consider by $n = 0, 1, 2, \ldots$. If we multiply two series

$$a_0 + a_1 + \cdots + a_n + \cdots \qquad \text{and} \qquad b_0 + b_1 + \cdots + b_n + \cdots \tag{6}$$

by forming all possible products a_ib_j (as in the case of finite sums), then we obtain the following doubly infinite array:

$$\begin{array}{ccccc} a_0b_0 & a_0b_1 & a_0b_2 & a_0b_3 & \cdots \\ a_1b_0 & a_1b_1 & a_1b_2 & a_1b_3 & \cdots \\ a_2b_0 & a_2b_1 & a_2b_2 & a_2b_3 & \cdots \\ a_3b_0 & a_3b_1 & a_3b_2 & a_3b_3 & \cdots \\ & & \cdots & & \end{array} \tag{7}$$

There are many ways of arranging these products into a single infinite series, of which two are important. The first is to group them by diagonals, as indicated by the arrows:

$$a_0b_0 + (a_0b_1 + a_1b_0) + (a_0b_2 + a_1b_1 + a_2b_0) + \cdots. \tag{8}$$

This series can be defined as $\Sigma_{n=0}^{\infty} c_n$, where

$$c_n = a_0b_n + a_1b_{n-1} + \cdots + a_nb_0.$$

It is called the *product* (or sometimes the *Cauchy product*) of the two series (6), and is particularly useful in working with power series.

A second method of arranging (7) into a series is by squares, as suggested by the broken lines:

$$a_0b_0 + (a_0b_1 + a_1b_1 + a_1b_0) + (a_0b_2 + a_1b_2 + a_2b_2 + a_2b_1 + a_2b_0) + \cdots. \tag{9}$$

The advantage of this arrangement is that the nth partial sum s_n of (9) is given by

$$s_n = (a_0 + a_1 + \cdots + a_n)(b_0 + b_1 + \cdots + b_n), \tag{10}$$

and this is useful in proving a preliminary fact about the multiplication of series.

Theorem 4 *If the two series* (6) *have nonnegative terms and converge to s and t, then their product* (8) *converges to st.*

Proof It is clear from (10) that (9) converges to st. Now denote the series (8) and (9) *without parentheses* by (8′) and (9′). The series (9′) of nonnegative terms still converges to st; for if m is an integer such that $n^2 \leq m \leq (n+1)^2$, then the mth partial sum of (9′) lies between s_{n-1} and s_n, and both of these converge to st. By Theorem 2, the terms of (9′) can be rearranged to yield (8′) without changing the sum st; and when parentheses are suitably inserted, we see that (8) converges to st.

The force of this result can best be appreciated by observing that the product of the convergent series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}} = \frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \cdots \tag{11}$$

with itself does not converge at all. We ask students to convince themselves of this in Problem 9.

We now extend Theorem 4 to the case of absolutely convergent series.

Theorem 5 *If the series $\Sigma_{n=0}^{\infty} a_n$ and $\Sigma_{n=0}^{\infty} b_n$ are absolutely convergent, with sums s and t, then their product*

$$\sum_{n=0}^{\infty} (a_0b_n + a_1b_{n-1} + \cdots a_nb_0) = a_0b_0 + (a_0b_1 + a_1b_0) + (a_0b_2 + a_1b_1 + a_2b_0) + \cdots + (a_0b_n + a_1b_{n-1} + \cdots + a_nb_0) + \cdots \tag{12}$$

is also absolutely convergent, with sum st.

Proof The series $\Sigma_{n=0}^{\infty} |a_n|$ and $\Sigma_{n=0}^{\infty} |b_n|$ are convergent and have nonnegative terms, so by the proof of Theorem 4 we see that the series

$$\begin{aligned} &|a_0||b_0| + |a_0||b_1| + |a_1||b_0| \\ &\quad + |a_0||b_2| + |a_1||b_1| + |a_2||b_0| + \cdots \\ &\quad + |a_0||b_n| + |a_1||b_{n-1}| + \cdots + |a_n||b_0| + \cdots \\ &\qquad = |a_0b_0| + |a_0b_1| + |a_1b_0| \\ &\qquad\quad + |a_0b_2| + |a_1b_1| + |a_2b_0| + \cdots \\ &\qquad\quad + |a_0b_n| + |a_1b_{n-1}| + \cdots + |a_nb_0| + \cdots \end{aligned} \tag{13}$$

converges, and therefore

$$a_0b_0 + a_0b_1 + a_1b_0 + \cdots + a_0b_n + \cdots + a_nb_0 + \cdots \tag{14}$$

is absolutely convergent. It follows from Theorem 2 that the sum of (14) will not change if we rearrange its terms and write it as

$$\begin{aligned} &a_0b_0 + a_0b_1 + a_1b_1 + a_1b_0 \\ &\quad + a_0b_2 + a_1b_2 + a_2b_2 + a_2b_1 + a_2b_0 + \cdots. \end{aligned} \tag{15}$$

We now observe that the sum of the first $(n+1)^2$ terms of (15) is $(a_0 + a_1 + \cdots + a_n)(b_0 + b_1 + \cdots + b_n)$, so it is clear that (15), and with it (14), converges to st. Since (12) is obtained by suitably inserting parentheses in (14), we see that (12) also converges to st. All that remains is to show that (12) converges absolutely; but this follows by the comparison test from the inequality

$$|a_0b_n + a_1b_{n-1} + \cdots + a_nb_0| \le |a_0 b_n| + |a_1b_{n-1}| + \cdots + |a_nb_0|$$

and the fact that the series

$$\begin{aligned} &|a_0b_0| + (|a_0b_1| + |a_1b_0|) + \cdots \\ &\quad + (|a_0b_n| + |a_1b_{n-1}| + \cdots + |a_nb_0|) + \cdots, \end{aligned}$$

obtained from (13) by inserting parentheses, is convergent.

This theorem shows that the absolute convergence of both of the given series is a sufficient condition for the convergence of their product to st. It is an interesting fact that this conclusion can also be obtained from the weaker hypothesis that only one of the two series is absolutely convergent (the example following Theorem 4 shows that the product of two conditionally convergent series need not converge at all!). A proof of this result is outlined in Problem 10.

PROBLEMS

1 Use the formula $1 + \frac{1}{2} + \cdots + 1/n = \ln n + \gamma + o(1)$ to establish the validity of the stated sums of the following rearrangements of the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2$:

(a) $1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{2} - \frac{1}{4} - \frac{1}{6} + \cdots = \ln 2$;

(b) $1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{2} - \frac{1}{4} + \cdots = \frac{1}{2} \ln 6$;

(c) $1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \cdots = \frac{1}{2} \ln 2$;

(d) $1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} + \frac{1}{3} - \frac{1}{10} - \frac{1}{12} - \frac{1}{14} - \frac{1}{16} + \cdots = 0$.

2 A series Σa_n is sometimes said to be *unconditionally convergent* if it converges and every rearrangement converges to the same sum. Show that Σa_n converges unconditionally if and only if it converges absolutely.

3 If Σa_n is absolutely convergent, prove that Σa_n^2 converges.

4 If $a_1 + a_2 + a_3 + a_4 + \cdots$ is absolutely convergent, prove that $a_1 + a_2 + a_3 + a_4 + \cdots = (a_1 + a_3 + \cdots) + (a_2 + a_4 + \cdots)$. Is this necessarily true for any convergent series?

5 Let Σa_n be a convergent series with sum s. If Σb_n is a rearrangement in which no term of the first series is moved more than n_0 places from its original position, prove that Σb_n still converges to s.

6 What is the sum of

$$1 + \tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{5} - \tfrac{1}{4} + \tfrac{1}{7} - \tfrac{1}{6} + \cdots ?$$

*7 Prove that the series

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots = \ln 2$$

converges to $\ln 2\sqrt{p/q}$ if its terms are rearranged by writing the first p positive terms, then the first q negative terms, then the next p positive terms, then the next q negative terms, etc.

*8 We know that the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ diverges but the alternating harmonic series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ converges.

(a) Show that $1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + + - \cdots$ diverges.

(b) If the signs are changed in the harmonic series in such a way that p positive signs are followed by q negative signs, then p positive signs by q negative signs, etc., show that the resulting series diverges if $p \neq q$ and converges if $p = q$.

9 Show that the product of the series (11) with itself diverges.

*10 If $\Sigma_{n=0}^{\infty} a_n$ and $\Sigma_{n=0}^{\infty} b_n$ converge to s and t, and if $\Sigma_{n=0}^{\infty} a_n$ converges absolutely, prove that their product

$$\sum_{n=0}^{\infty} (a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0)$$

converges to st (this result is known as *Mertens' theorem*[†]). Hint: Put $s_n = a_0 + \cdots + a_n$, $t_n = b_0 + \cdots + b_n$, and $\alpha_n = t_n - t$; show that the nth partial sum of the product can be written as

$$s_n t + a_0\alpha_n + a_1\alpha_{n-1} + \cdots + a_n\alpha_0;$$

and prove that if β_n is defined by

$$\begin{aligned}\beta_n &= a_0\alpha_n + a_1\alpha_{n-1} + \cdots + a_n\alpha_0 \\ &= \alpha_0 a_n + \alpha_1 a_{n-1} + \cdots + \alpha_n a_0,\end{aligned}$$

then $\beta_n \to 0$.

[†]Franz Mertens (1840–1927) was born in Poland, studied at Berlin under Kummer and Kronecker, and taught at Cracow, Graz, and Vienna. He retained extraordinary vigor of mind and body to an advanced age, and wrote the last of his more than 100 research papers at the age of 86. His main interest was analytic number theory, where he was a master in the use of elementary methods to simplify difficult proofs. He discovered the theorem given here in connection with his success (after Euler had failed) in proving the convergence of the series $\Sigma\,(\pm 1/p_n)$, where the p_n's are the primes and the signs are + or − according as p_n is of the form $4n + 1$ or $4n + 3$.

A.14 DIRICHLET'S TEST

With one exception, all of our convergence tests in Chapter 13 are tests that apply only to series of positive (or nonnegative) terms; that is, they are tests for absolute convergence. This exception is the alternating series test, which says that the series

$$b_1 - b_2 + b_3 - b_4 + \cdots \tag{1}$$

converges if the b_n's form a decreasing sequence of positive numbers and $b_n \to 0$. We can think of (1) as generated by multiplying the terms of the series $1 - 1 + 1 - 1 + \cdots$ by the terms of the sequence $b_1, b_2, b_3, b_4, \ldots$. From this point of view it is natural (and profitable) to generalize by considering a series

$$a_1 + a_2 + \cdots + a_n + \cdots \tag{2}$$

and sequence

$$b_1, b_2, \ldots, b_n, \ldots, \tag{3}$$

and the problem is to find conditions under which the series

$$a_1 b_1 + a_2 b_2 + \cdots + a_n b_n + \cdots \tag{4}$$

converges. It is obvious that if (2) is absolutely convergent and (3) is bounded, then (4) is also absolutely convergent. Our purpose here, however, is to obtain criteria for the convergence of (4) that are not merely tests for absolute convergence.

To accomplish this, we need *Abel's partial summation formula*: If $s_n = a_1 + a_2 + \cdots + a_n$, then

$$a_1b_1 + a_2b_2 + \cdots + a_nb_n = s_1(b_1 - b_2) + s_2(b_2 - b_3) + \cdots + s_{n-1}(b_{n-1} - b_n) + s_nb_n. \tag{5}$$

The proof is easy. Since $a_1 = s_1$ and $a_n = s_n - s_{n-1}$ for $n > 1$, we have

$$\begin{aligned} a_1b_1 &= s_1b_1 \\ a_2b_2 &= s_2b_2 - s_1b_2 \\ a_3b_3 &= s_3b_3 - s_2b_3 \\ &\cdots \\ a_{n-1}b_{n-1} &= s_{n-1}b_{n-1} - s_{n-2}b_{n-1} \\ a_nb_n &= s_nb_n - s_{n-1}b_n. \end{aligned}$$

On adding these equations and suitably grouping the terms on the right, we obtain (5). This result enables us to establish *Dirichlet's test*.

Theorem 1 *If the series* (2) *has bounded partial sums, and if* (3) *is a decreasing sequence of positive numbers such that* $b_n \to 0$, *then the series* (4) *converges.*

Proof If we put $S_n = a_1b_1 + a_2b_2 + \cdots + a_nb_n$, then (5) tells us that

$$S_n = T_n + s_nb_n, \tag{6}$$

where

$$T_n = s_1(b_1 - b_2) + s_2(b_2 - b_3) + \cdots + s_{n-1}(b_{n-1} - b_n).$$

We must prove that $\lim S_n$ exists, and we do this by showing that $\lim T_n$ and $\lim s_nb_n$ both exist. Our first assumption says that there is a constant M such that $|s_n| \leq M$ for every n, so $|s_nb_n| \leq Mb_n$; and since $b_n \to 0$, we conclude that $s_nb_n \to 0$. Next, T_n is the $(n-1)$st partial sum of the series

$$s_1(b_1 - b_2) + s_2(b_2 - b_3) + \cdots, \tag{7}$$

so $\lim T_n$ will certainly exist if (7) converges. To establish this, it suffices to show that (7) is absolutely convergent. We now use the assumption that the b_n's are positive and decreasing, which yields

$$\begin{aligned} &|s_1(b_1 - b_2)| + |s_2(b_2 - b_3)| + \cdots + |s_{n-1}(b_{n-1} - b_n)| \\ &\quad \leq M(b_1 - b_2) + M(b_2 - b_3) + \cdots + M(b_{n-1} - b_n) \\ &\quad = M(b_1 - b_n) \leq Mb_1. \end{aligned}$$

This implies that (7) is absolutely convergent, and the proof is complete.

In order to make effective use of Dirichlet's test, we must be acquainted with a few series having bounded partial sums. Naturally, all convergent series have this property, but many that are divergent also have it. Perhaps the simplest is $1 - 1 + 1 - 1 + \cdots$; and from this we see at once that the alternating series test is an immediate consequence of Theorem 1.

Dirichlet's test is particularly useful in demonstrating the convergence of certain trigonometric series, and here we encounter some further series that do not converge but nevertheless have bounded partial sums.

As an example, we show that if x is any number $\neq k\pi$, where $k = 0, \pm 1, \pm 2, \ldots$, then the series

$$\cos x + \frac{\cos 3x}{3} + \frac{\cos 5x}{5} + \cdots + \frac{\cos (2n-1)x}{2n-1} + \cdots \tag{8}$$

converges. Here the b_n's can be taken as $1, \frac{1}{3}, \frac{1}{5}, \ldots$, so it suffices to prove that the partial sums of

$$\cos x + \cos 3x + \cos 5x + \cdots + \cos (2n-1)x + \cdots \tag{9}$$

are bounded. To do this, we use the trigonometric identity

$$2 \cos a \sin b = \sin (a+b) - \sin (a-b).$$

By putting $b = x$ and $a = x, 3x, 5x, \ldots, (2n-1)x$, we obtain

$$2 \cos x \sin x = \sin 2x - 0$$

$$2 \cos 3x \sin x = \sin 4x - \sin 2x$$

$$2 \cos 5x \sin x = \sin 6x - \sin 4x$$

$$\cdots$$

$$2 \cos (2n-1)x \sin x = \sin 2nx - \sin(2n-2)x;$$

and adding yields

$$2 \sin x[\cos x + \cos 3x + \cdots + \cos (2n-1)x] = \sin 2nx$$

or

$$\cos x + \cos 3x + \cdots + \cos (2n-1)x = \frac{\sin 2nx}{2 \sin x}.$$

But $|\sin 2nx| \leq 1$, so

$$|\cos x + \cos 3x + \cdots + \cos (2n-1)x| \leq \frac{1}{2|\sin x|}. \tag{10}$$

This proves that the partial sums of (9) are bounded, so (8) converges. It should now be apparent why we assumed that $x \neq k\pi$: We must have $\sin x \neq 0$ in (10). Actually, of course, it is obvious that (8) diverges if $x = k\pi$.

PROBLEMS

1 Let p be a positive constant. It is clear from the discussion in the text that

$$\cos x + \frac{\cos 3x}{3^p} + \frac{\cos 5x}{5^p} + \cdots$$

converges for every $x \neq k\pi$; and when $x = k\pi$, the series converges if $p > 1$ and diverges if $p \leq 1$. Use the identity

$$2 \sin a \sin b = \cos (a-b) - \cos (a+b)$$

to show that

$$\sin x + \frac{\sin 3x}{3^p} + \frac{\sin 5x}{5^p} + \cdots$$

converges for all x.

2 Prove the identities

$$2\sin\frac{x}{2}(\sin x + \sin 2x + \cdots + \sin nx) = \cos\frac{x}{2} - \cos\frac{(2n+1)x}{2}$$

and

$$2\sin\frac{x}{2}(\cos x + \cos 2x + \cdots + \cos nx) = \sin\frac{(2n+1)x}{2} - \sin\frac{x}{2}.$$

Let $b_1, b_2, \ldots, b_n, \ldots$ be a decreasing sequence of positive numbers such that $b_n \to 0$, and use these identities to show that

(a) $\Sigma_{n=1}^{\infty} b_n \sin nx$ converges for all x;

(b) $\Sigma_{n=1}^{\infty} b_n \cos nx$ converges for all $x \neq 2k\pi$.

3 Investigate the convergence behavior of

$$\frac{\cos 3x}{2} + \frac{\cos 6x}{3} + \frac{\cos 9x}{4} + \cdots.$$

4 Show that the following series converge:

$$1 + \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \cdots,$$

$$\frac{1}{\ln 2} + \frac{1}{\ln 3} - \frac{1}{\ln 4} - \frac{1}{\ln 5} + \frac{1}{\ln 6} + \frac{1}{\ln 7} - \cdots,$$

and

$$\frac{3}{1} - \frac{4}{2} + \frac{3}{3} - \frac{2}{4} + \frac{3}{5} - \frac{4}{6} + \frac{3}{7} - \frac{2}{8} + \cdots.$$

5 A series of the form $\Sigma_{n=1}^{\infty} a_n/n^x$ is called a *Dirichlet series*. If it converges for $x = x_0$, show that it also converges for every $x > x_0$.

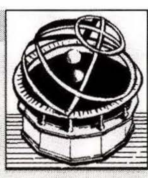
6 Prove *Abel's test*: If the series (2) converges and (3) is a bounded increasing or decreasing sequence, then the series (4) also converges. (Note the relation of this statement to Dirichlet's test—more is assumed about the series, but less about the sequence. It is almost certain that Abel knew Dirichlet's test.)

7 Use Abel's test to show that the following series converge:

(a) $\cos 1 + \frac{1}{3}\cos\frac{1}{2} - \frac{1}{2}\cos\frac{1}{3} + \frac{1}{5}\cos\frac{1}{4} + \frac{1}{7}\cos\frac{1}{5} - \frac{1}{4}\cos\frac{1}{6} + \cdots$;

(b) $2 + \dfrac{3}{2^2} - \dfrac{4}{3^2} - \dfrac{5}{4^2} + \dfrac{6}{5^2} + \dfrac{7}{6^2} - \cdots$.

8 If Σa_n converges, so do $\Sigma a_n/n$, $\Sigma a_n/\ln n$, $\Sigma a_n \cos 1/n$, $\Sigma a_n \sin 1/n$, $\Sigma(1 + 1/n)a_n$, $\Sigma(1 + 1/n)^n a_n$, and $\Sigma\sqrt[n]{n}\, a_n$. Why?



NOTE ON DIRICHLET

P. G. L. Dirichlet (1805–1859) was a German mathematician who made many contributions of lasting value to analysis and number theory. One of his earliest achievements was a milestone in analysis: In 1829 he gave the first satisfactory proof that certain specific types of functions are actually the sums of their Fourier series. Previous work in this field had consisted wholly of the uncritical manipulation of formulas; Dirichlet transformed the subject into genuine mathematics in the modern sense. As a by-product of this research, he also contributed greatly to the correct understanding of the nature of a function, and gave the definition which is now most often used, namely, that y is a function of x when to each value of x in a given interval there corresponds a unique value of y. He added that it does not matter whether y depends on x according to some "formula" or "law" or "mathematical operation," and he emphasized this by giving the example of the function of x which has the value 1 for all rational x's and the value 0 for all irrational x's.

Perhaps his greatest works were two long memoirs of 1837 and 1839 in which he made very remarkable applications of analysis to the theory of numbers. It was in the first of these that he proved his wonderful theorem that there are an infinite number of primes in any arithmetic progression of the form $a + nb$, where a and b are positive integers with no common factor. His discoveries about absolutely convergent series also appeared in 1837. His convergence test, discussed above, was published posthumously in his *Lectures on Number Theory* (1863). These lectures went through many editions and had a very wide influence.

In later life Dirichlet became a friend and disciple of Gauss, and also a friend and advisor of Riemann, whom he helped in a small way with his doctoral dissertation. In 1855, after lecturing at Berlin for many years, he succeeded Gauss in the professorship at Göttingen.

A.15 UNIFORM CONVERGENCE FOR POWER SERIES

Consider a power series $\Sigma a_n x^n$ with positive radius of convergence R, and let $f(x)$ be its sum. Our purpose here is to prove that $f(x)$ is continuous and differentiable on $(-R, R)$, and also that its derivative and integral can legitimately be calculated by differentiating and integrating the series term by term.

Let $S_n(x)$ be the nth partial sum of the series, so that

$$S_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n.$$

We write

$$f(x) = S_n(x) + R_n(x)$$

and call $R_n(x)$ the *remainder.* Evidently,

$$R_n(x) = a_{n+1}x^{n+1} + a_{n+2}x^{n+2} + \cdots.$$

For each x in the interval of convergence, we know that $R_n(x) \to 0$ as $n \to \infty$; that is, for any given $\epsilon > 0$ we have

$$|R_n(x)| < \epsilon \text{ for sufficiently large } n,\ n \geq n_0. \tag{1}$$

We emphasize here that this is true for each x individually, and is merely an equivalent way of expressing the fact that $\Sigma a_n x^n$ converges to $f(x)$. However, much more can be said: Throughout any given closed interval inside the interval of convergence, say $|x| \leq |x_1| < R$, (1) holds for all x simultaneously. Since $R_n(x) = f(x) - S_n(x)$, we can express this in another way by saying that throughout the given closed interval, $S_n(x)$ can be made to approximate $f(x)$ as closely as we please by taking n large enough.

To prove this, we observe that for every x in the given closed interval $|x| \leq |x_1| < R$, we have

$$\begin{aligned} |R_n(x)| &= |a_{n+1}x^{n+1} + a_{n+2}x^{n+2} + \cdots| \\ &\leq |a_{n+1}x^{n+1}| + |a_{n+2}x^{n+2}| + \cdots \\ &\leq |a_{n+1}x_1{}^{n+1}| + |a_{n+2}x_1{}^{n+2}| + \cdots. \end{aligned}$$

The argument is completed by using the fact that the last-written sum can be made $< \epsilon$ by taking n large enough, $n \geq n_0$, because of the absolute convergence of $\Sigma a_n x_1{}^n$. The point is, that the same n_0 works for all x's in the given closed interval. The conclusion proved here, that $R_n(x)$ can be made small *independently of x in the given closed interval,* is expressed by saying that the series is *uniformly convergent* in this interval.

CONTINUITY OF THE SUM

We will prove that $f(x)$ is continuous at each interior point x_0 of the interval of convergence. Consider a closed subinterval $|x| \leq |x_1| < R$ containing x_0 in its interior. If $\epsilon > 0$ is given, then by uniform convergence we can find an n with the property that $|R_n(x)| < \epsilon$ for all x's in the subinterval. Since the polynomial $S_n(x)$ is continuous at x_0, we can find $\delta > 0$ so small that $|x - x_0| < \delta$ implies x lies in the subinterval and $|S_n(x) - S_n(x_0)| < \epsilon$. By putting these conditions together we find that $|x - x_0| < \delta$ implies

$$\begin{aligned} |f(x) - f(x_0)| &= |[S_n(x) + R_n(x)] - [S_n(x_0) + R_n(x_0)]| \\ &= |[S_n(x) - S_n(x_0)] + R_n(x) - R_n(x_0)| \\ &\leq |S_n(x) - S_n(x_0)| + |R_n(x)| + |R_n(x_0)| \\ &< \epsilon + \epsilon + \epsilon = 3\epsilon. \end{aligned}$$

Since 3ϵ can be taken as small as we please, this proves that $f(x)$ is continuous at x_0.

INTEGRATING TERM BY TERM

We have just proved that

$$f(x) = \sum a_n x^n \tag{2}$$

is continuous on $(-R, R)$. We can therefore integrate this function between limits a and b that lie inside the interval,

$$\int_a^b f(x)\,dx = \int_a^b \left(\sum a_n x^n\right) dx. \tag{3}$$

Our purpose here is to show that the right side of this can legitimately be integrated term by term,

$$\int_a^b \left(\sum a_n x^n\right) dx = \sum \int_a^b a_n x^n\,dx.$$

In words, the integral of the sum equals the sum of the integrals. An equivalent statement is that (3) can be written as

$$\int_a^b f(x)\,dx = \sum \int_a^b a_n x^n\,dx. \tag{4}$$

To prove this, we begin by observing that since $S_n(x)$ is a polynomial, and therefore continuous everywhere, all three of the functions in

$$f(x) = S_n(x) + R_n(x)$$

are continuous on $(-R, R)$. This allows us to write

$$\int_a^b f(x)\,dx = \int_a^b S_n(x)\,dx + \int_a^b R_n(x)\,dx. \tag{5}$$

Since any polynomial can be integrated term by term, the first integral on the right of (5) is

$$\begin{aligned}\int_a^b S_n(x)\,dx &= \int_a^b (a_0 + a_1x + a_2x^2 + \cdots + a_nx^n)\,dx \\ &= \int_a^b a_0\,dx + \int_a^b a_1x\,dx + \int_a^b a_2x^2\,dx + \cdots + \int_a^b a_nx^n\,dx.\end{aligned}$$

To prove (4), it therefore suffices to show that

$$\int_a^b R_n(x)\,dx \to 0 \qquad \text{as } n \to \infty.$$

For this we use uniform convergence, which tells us that if $\epsilon > 0$ is given and $|x| \le |x_1| < R$ is a closed subinterval of $(-R, R)$ that contains both a and b, then $|R_n(x)| < \epsilon$ for all x in the subinterval if n is large enough. All that remains is to write

$$\left|\int_a^b R_n(x)\,dx\right| \le \int_a^b |R_n(x)|\,dx < \epsilon|b - a|,$$

and to notice that this can be made as small as we wish.

As a special case of (4) we take the limits 0 and x instead of a and b, and obtain

$$\begin{aligned}\int_0^x f(t)\,dt &= \sum \frac{1}{n+1} a_nx^{n+1} \\ &= a_0x + \frac{1}{2}a_1x^2 + \frac{1}{3}a_2x^3 + \cdots + \frac{1}{n+1}a_nx^{n+1} + \cdots,\end{aligned} \tag{6}$$

where the "dummy variable" t is used in the integral for the reason explained in Section 6.7.

DIFFERENTIATING TERM BY TERM

We now prove that the function $f(x)$ in (2) is not only continuous but also differentiable, and that its derivative can be calculated by differentiating (2) term by term,

$$f'(x) = \sum na_n x^{n-1}. \tag{7}$$

To do this, we begin by recalling from Section 14.3 that the series on the right of (7) converges on $(-R, R)$. If we denote its sum by $g(x)$, so that

$$g(x) = \sum na_n x^{n-1} = a_1 + 2a_2x + 3a_3x^2 + \cdots + na_nx^{n-1} + \cdots,$$

then (6) tells us that

$$\begin{aligned}\int_0^x g(t)\,dt &= a_1x + a_2x^2 + a_3x^3 + \cdots \\ &= f(x) - a_0.\end{aligned}$$

Since the left side of this has a derivative, so does the right side, and by differentiating we obtain

$$f'(x) = g(x) = \sum na_n x^{n-1},$$

as required.

A.16 DIVISION OF POWER SERIES

In order to justify the division of power series as described in Section 14.7, it suffices to justify dividing a power series into 1. To see this, we have only to notice that

$$\frac{\Sigma a_n x^n}{\Sigma b_n x^n} = (\Sigma a_n x^n) \cdot \left(\frac{1}{\Sigma b_n x^n}\right);$$

for this tells us that if we can expand $1/(\Sigma b_n x^n)$ in a power series with positive radius of convergence, then we can achieve our purpose by multiplying this series by $\Sigma a_n x^n$. It is clearly necessary to assume that $b_0 \neq 0$ (why?). We may assume that $b_0 = 1$ without any loss of generality, because if b_0 has any other nonzero value, we simply factor it out, leaving the leading coefficient equal to 1:

$$\frac{1}{b_0 + b_1x + b_2x^2 + \cdots} = \frac{1}{b_0} \cdot \frac{1}{1 + (b_1/b_0)x + (b_2/b_0)x^2 + \cdots}.$$

In view of these remarks, we direct our efforts at proving the following statement:

If $\Sigma b_n x^n$ has $b_0 = 1$ and positive radius of convergence R, then $1/(\Sigma b_n x^n)$ can be expanded in a power series $\Sigma c_n x^n$ which also has positive radius of convergence.

We begin by determining the c_n's. The condition $1/(\Sigma b_n x^n) = \Sigma c_n x^n$ means that $(\Sigma b_n x^n)(\Sigma c_n x^n) = 1$, so

$$b_0c_0 + (b_0c_1 + b_1c_0)x + (b_0c_2 + b_1c_1 + b_2c_0)x^2 + \cdots + (b_0c_n + b_1c_{n-1} + \cdots + b_nc_0)x^n + \cdots = 1,$$

and therefore

$$b_0c_0 = 1, \qquad b_0c_1 + b_1c_0 = 0, \qquad b_0c_2 + b_1c_1 + b_2c_0 = 0,$$
$$\ldots, \qquad b_0c_n + b_1c_{n-1} + \cdots + b_nc_0 = 0, \ldots.$$

Since $b_0 = 1$, these equations determine the c_n's recursively:

$$c_0 = 1, \qquad c_1 = -b_1c_0, \qquad c_2 = -b_1c_1 - b_2c_0,$$
$$\ldots, \qquad c_n = -b_1c_{n-1} - b_2c_{n-2} - \cdots - b_nc_0, \ldots.$$

All that remains is to prove that the power series $\Sigma c_n x^n$ with these coefficients has positive radius of convergence, and for this it suffices to show that the series converges for at least one nonzero x. This we now do.

Let r be any number such that $0 < r < R$, so that $\Sigma b_n r^n$ converges. Then there exists a constant $K \geq 1$ with the property that $|b_n r^n| \leq K$ or $|b_n| \leq K/r^n$ for all n. It now follows that

$$|c_0| = 1 \leq K,$$

$$|c_1| = |b_1c_0| = |b_1| \leq \frac{K}{r},$$

$$|c_2| \leq |b_1c_1| + |b_2c_0| \leq \frac{K}{r}\cdot\frac{K}{r} + \frac{K}{r^2}\cdot K = \frac{2K^2}{r^2},$$

$$|c_3| \leq |b_1c_2| + |b_2c_1| + |b_3c_0| \leq \frac{K}{r}\cdot\frac{2K^2}{r^2} + \frac{K}{r^2}\cdot\frac{K}{r} + \frac{K}{r^3}\cdot K$$
$$\leq (2+1+1)\frac{K^3}{r^3} = \frac{4K^3}{r^3} = \frac{2^2K^3}{r^3},$$

since $K^2 < K^3$. In general,

$$|c_n| \leq |b_1c_{n-1}| + |b_2c_{n-2}| + \cdots + |b_nc_0|$$
$$\leq \frac{K}{r}\cdot\frac{2^{n-2}K^{n-1}}{r^{n-1}} + \frac{K}{r^2}\cdot\frac{2^{n-3}K^{n-2}}{r^{n-2}} + \cdots + \frac{K}{r^n}\cdot K$$
$$\leq (2^{n-2} + 2^{n-3} + \cdots + 1 + 1)\frac{K^n}{r^n} = \frac{2^{n-1}K^n}{r^n} \leq \frac{2^nK^n}{r^n}.$$

We therefore have $|c_nx^n| \leq |(2K/r)x|^n$, so the series Σc_nx^n is absolutely convergent—and therefore convergent—for any x that satisfies the condition $|x| < r/2K$. This shows that Σc_nx^n has nonzero radius of convergence, and the argument is complete.

A.17 THE EQUALITY OF MIXED PARTIAL DERIVATIVES

We shall prove that

$$f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0), \tag{1}$$

under the assumption that both mixed partials f_{xy} and f_{yx} exist at all points near (x_0, y_0) and are continuous at (x_0, y_0).

Let Δx and Δy be so small that f_{xy} and f_{yx} exist throughout the rectangle with vertices (x_0, y_0), $(x_0 + \Delta x, y_0)$, $(x_0, y_0 + \Delta y)$, $(x_0 + \Delta x, y_0 + \Delta y)$(see Fig. A.17). We carry out the proof by applying the Mean Value Theorem (Appendix A.4) in various ways to the expression

$$D = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0 + \Delta x, y_0) - f(x_0, y_0 + \Delta y) + f(x_0, y_0). \tag{2}$$

We begin by considering the function

$$F(x) = f(x, y_0 + \Delta y) - f(x, y_0),$$

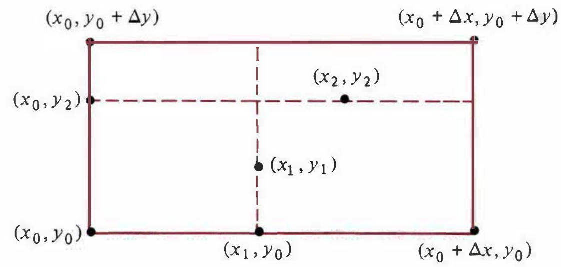


Figure A.17

where Δy is held fixed. The expression (2) can be written in the form

$$\begin{aligned} D &= [f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0 + \Delta x, y_0)] - [f(x_0, y_0 + \Delta y) - f(x_0, y_0)] \\ &= F(x_0 + \Delta x) - F(x_0), \end{aligned}$$

so by applying the Mean Value Theorem we obtain

$$\begin{aligned} D &= \Delta x\, F'(x_1) \\ &= \Delta x[f_x(x_1, y_0 + \Delta y) - f_x(x_1, y_0)], \end{aligned}$$

where x_1 is some number between x_0 and $x_0 + \Delta x$. Since $f_x(x_1, y)$ is differentiable as a function of y, we can apply the Mean Value Theorem again to obtain

$$D = \Delta x\, \Delta y\, f_{xy}(x_1, y_1), \tag{3}$$

where y_1 lies between y_0 and $y_0 + \Delta y$.

We now start all over again with the function

$$G(y) = f(x_0 + \Delta x, y) - f(x_0, y),$$

where Δx is held fixed. The expression (2) can be written in the form

$$\begin{aligned} D &= [f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y)] - [f(x_0 + \Delta x, y_0) - f(x_0, y_0)] \\ &= G(y_0 + \Delta y) - G(y_0). \end{aligned}$$

Just as before, by two applications of the Mean Value Theorem we find that

$$\begin{aligned} D = \Delta y\, G'(y_2) &= \Delta y[f_y(x_0 + \Delta x, y_2) - f_y(x_0, y_2)] \\ &= \Delta y\, \Delta x\, f_{yx}(x_2, y_2), \end{aligned} \tag{4}$$

where y_2 lies between y_0 and $y_0 + \Delta y$ and x_2 lies between x_0 and $x_0 + \Delta x$.

Finally, by equating (3) and (4) we see that

$$f_{xy}(x_1, y_1) = f_{yx}(x_2, y_2). \tag{5}$$

Now let Δx and Δy approach zero, so that the rectangle shrinks toward the point (x_0, y_0). As this happens, it is clear that the points (x_1, y_1) and (x_2, y_2), which lie inside the rectangle, approach (x_0, y_0), and we obtain our conclusion (1) from (5) and the continuity of f_{xy} and f_{yx} at (x_0, y_0).

A.18 DIFFERENTIATION UNDER THE INTEGRAL SIGN

In Problems 32 in Section 19.2 and 19 in Section 19.6, we used the formula

$$\frac{d}{dx}\int_a^b f(x, y)\, dy = \int_a^b f_x(x, y)\, dy. \tag{1}$$

Our purpose here is to prove this under the assumption that $f(x, y)$ and its partial derivative $f_x(x, y)$ are both continuous functions on the closed rectangle $x_0 \le x \le x_1$, $a \le y \le b$.

It is convenient to write

$$F(x) = \int_a^b f(x, y)\, dy.$$

If x and $x + \Delta x$ both lie in the interval $x_0 \leq x \leq x_1$, then

$$\begin{aligned} F(x + \Delta x) - F(x) &= \int_a^b f(x + \Delta x, y)\, dy - \int_a^b f(x, y)\, dy \\ &= \int_a^b [f(x + \Delta x, y) - f(x, y)]\, dy. \end{aligned}$$

Next, the Mean Value Theorem enables us to write this integrand in the form

$$f(x + \Delta x, y) - f(x, y) = \Delta x\, f_x(\bar{x}, y),$$

where $\bar{x}$ lies between x and $x + \Delta x$. Further, since $f_x(x, y)$ is assumed to be continuous on the closed rectangle, it can be shown that the absolute value of the difference

$$f_x(\bar{x}, y) - f_x(x, y)$$

is less than a positive number ϵ which is independent of x and y and approaches zero with Δx.* By putting these ingredients together, we obtain

$$\begin{aligned} \left| \frac{F(x + \Delta x) - F(x)}{\Delta x} - \int_a^b f_x(x, y)\, dy \right| &= \left| \int_a^b [f_x(\bar{x}, y) - f_x(x, y)]\, dy \right| \\ &\leq \int_a^b |f_x(\bar{x}, y) - f_x(x, y)|\, dy \\ &< \int_a^b \epsilon\, dy = \epsilon(b - a). \end{aligned}$$

If we now let Δx approach zero, then ϵ also approaches zero, and we have

$$\lim_{\Delta x \to 0} \frac{F(x + \Delta x) - F(x)}{\Delta x} = \int_a^b f_x(x, y)\, dy,$$

which is (1).

A.19 A PROOF OF THE FUNDAMENTAL LEMMA

This lemma is stated and discussed in Section 19.4, and is the linchpin that holds together the main tools of Chapter 19. It has to do with a function $z = f(x, y)$ whose partial derivatives f_x and f_y exist at (x_0, y_0) and at all nearby points, and are continuous at the point (x_0, y_0) itself. The assertion of the lemma is that under these conditions the increment

$$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$$

can be expressed in the form

$$\Delta z = f_x(x_0, y_0)\, \Delta x + f_y(x_0, y_0)\, \Delta y + \epsilon_1\, \Delta x + \epsilon_2\, \Delta y, \tag{1}$$

where ϵ_1 and ϵ_2 are quantities that $\to 0$ as Δx and $\Delta y \to 0$.

To prove this statement, we analyze the change Δz in two steps, as shown in Fig. A.18, first changing x alone and moving from (x_0, y_0) to $(x_0 + \Delta x, y_0)$, and then changing y alone and moving from $(x_0 + \Delta x, y_0)$ to $(x_0 + \Delta x, y_0 + \Delta y)$. We denote the first change in z by $\Delta_1 z$, so that

$$\Delta_1 z = f(x_0 + \Delta x, y_0) - f(x_0, y_0).$$

By the Mean Value Theorem we can write this as

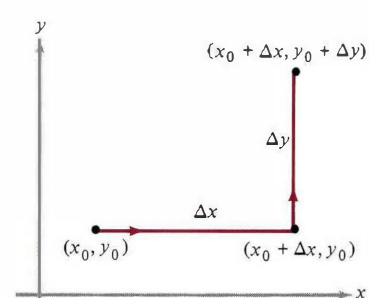


Figure A.18

*This is the two-dimensional analog of the property of uniform continuity that we discussed and proved in Appendix A.5.

$$\Delta_1 z = \Delta x\, f_x(x_1, y_0), \tag{2}$$

where x_1 is between x_0 and $x_0 + \Delta x$. Similarly, if we denote the second part of the change in z by $\Delta_2 z$, so that

$$\Delta_2 z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0 + \Delta x, y_0),$$

then

$$\Delta_2 z = \Delta y\, f_y(x_0 + \Delta x, y_1), \tag{3}$$

where y_1 is between y_0 and $y_0 + \Delta y$. As Δx and $\Delta y \to 0$, $x_1 \to x_0$ and $y_1 \to y_0$. By the assumed continuity of f_x and f_y at (x_0, y_0), we can therefore write

$$f_x(x_1, y_0) = f_x(x_0, y_0) + \epsilon_1$$

and

$$f_y(x_0 + \Delta x, y_1) = f_y(x_0, y_0) + \epsilon_2,$$

where ϵ_1 and $\epsilon_2 \to 0$ as Δx and $\Delta y \to 0$. This permits us to write (2) and (3) as

$$\Delta_1 z = f_x(x_0, y_0)\Delta x + \epsilon_1\, \Delta x \tag{4}$$

and

$$\Delta_2 z = f_y(x_0, y_0)\, \Delta y + \epsilon_2\, \Delta y. \tag{5}$$

Since $\Delta z = \Delta_1 z + \Delta_2 z$, we now complete the proof by adding (4) and (5) to obtain (1).

A.20 A PROOF OF THE IMPLICIT FUNCTION THEOREM

We stated and discussed the Implicit Function Theorem in Section 19.10, and our purpose here is to provide a proof.

To restate the situation, we assume that $F(x, y)$ has continuous partial derivatives throughout some neighborhood of a point $P_0 = (x_0, y_0)$, and also that $F(x_0, y_0) = c$ and $F_y(x_0, y_0) \neq 0$. We shall prove that there exists a rectangle centered on P_0 within which the graph of $F(x, y) = c$ is the graph of a single differentiable function $y = f(x)$, and also that the derivative of this function is given by the formula

$$\frac{dy}{dx} = -\frac{F_x}{F_y}. \tag{1}$$

First, we know from the Fundamental Lemma (Section 19.4) that since F has continuous partial derivatives in the neighborhood mentioned above, F itself is continuous in this neighborhood.

Let us suppose, for definiteness, that $F_y > 0$ at P_0. (A similar proof can be constructed if $F_y < 0$ at P_0.) We observe that if $F_y > 0$ along a vertical segment, then F is an increasing function of y along that segment. It follows that no value of F (such as $F = c$, in which we are interested) can be taken on more than once on such a segment.

We begin by constructing a rectangle R_0 centered on P_0 within and on the boundary of which F and F_y are continuous and $F_y > 0$. This is possible because of the continuity of the functions. Along any vertical segment across R_0 the function F is an increasing function of y. By the Intermediate Value Theorem (Appendix A.3) we have $F = c$ on this segment if and only if $f < c$ at the lower end and $F > c$ at the upper end. Consider, for example, the vertical segment P_1P_2 through P_0. Since $F = c$ at P_0, we have $F < c$ at P_1, and by continuity we have $F < c$ in some neighborhood of P_1 on the lower edge of R_0. Similarly, we have $F > c$ in some neighborhood of P_2 on the upper edge of R_0. A vertical segment close enough to P_1P_2 will have its ends in both neighborhoods and will therefore intersect the graph of $F = c$ in exactly one point.

It is evident from this that we can shrink the base of R_0, if necessary, to form a new rectangle R centered on P_0 such that $F < c$ on its lower edge and $F > c$ on its upper edge. See Fig. A.19. Inside this rectangle there is one y that corresponds to each x in such a way that $F(x, y) = c$, and this defines our function $y = f(x)$. It is clear that this function is continuous, because the height of R_0 can be taken as small as we please to begin with.

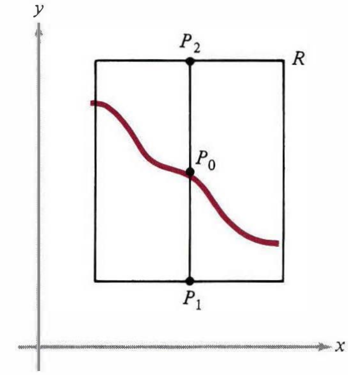


Figure A.19

To establish differentiability and formula (1), we consider a change Δx and the corresponding change $\Delta y = f(x_0 + \Delta x) - f(x_0)$. Since the new point $(x_0 + \Delta x, y_0 + \Delta y)$ is still on the graph of $F(x, y) = c$, we have

$$F(x_0, y_0) = c \qquad \text{and} \qquad F(x_0 + \Delta x, y_0 + \Delta y) = c, \qquad \text{so} \qquad \Delta F = 0.$$

But the Fundamental Lemma gives

$$\Delta F = F_x \,\Delta x + F_y \,\Delta y + \epsilon_1 \,\Delta x + \epsilon_2 \,\Delta y,$$

where $\epsilon_1, \epsilon_2 \to 0$ as Δx and $\Delta y \to 0$, and therefore

$$0 = F_x + F_y \frac{\Delta y}{\Delta x} + \epsilon_1 + \epsilon_2 \frac{\Delta y}{\Delta x}$$

or

$$\frac{\Delta y}{\Delta x} = -\frac{F_x + \epsilon_1}{F_y + \epsilon_2}. \tag{2}$$

We know that $y = f(x)$ is continuous. If $\Delta x \to 0$, it therefore follows that $\Delta y \to 0$, so ϵ_1, $\epsilon_2 \to 0$ and (2) implies that $y = f(x)$ is differentiable with derivative given by (1). Finally, this proof of differentiability applies in just the same way to any other point on the graph inside the rectangle R.

A.21 CHANGE OF VARIABLES IN MULTIPLE INTEGRALS. JACOBIANS

Our basic tools for integrating in polar, cylindrical, and spherical coordinates are the formulas

$$dA = r\,dr\,d\theta, \qquad dV = r\,dr\,d\theta\,dz, \qquad \text{and} \qquad dV = \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta, \tag{1}$$

for the elements of area and volume in these three coordinate systems. However, the justifications we gave in Chapter 20 were purely intuitive and geometric. Our purpose in this brief final appendix is to describe a broader theoretical setting within which these formulas can be understood as merely different aspects of a single idea.

The problem that we now consider is the following: What happens to a multiple integral

$$\iint_R \cdots \int f(x, y, \ldots)\,dx\,dy \cdots$$

if we change the variables from $x, y, \ldots$ to $u, v, \ldots$?

We know the answer to this question in the case of a single variable: If $f(x)$ is continuous and the function $x = x(u)$ has a continuous derivative, then

$$\int_a^b f(x)\,dx = \int_c^d f[x(u)]\,\frac{dx}{du}\,du, \tag{2}$$

where $a = x(c)$ and $b = x(d)$. As an example of the use of this formula, we point out that the trigonometric substitution $x = \sin\theta$, $dx = \cos\theta\,d\theta$ enables us to write

$$\int_0^1 \sqrt{1 - x^2}\,dx = \int_0^{\pi/2} \cos\theta \cdot \cos\theta\,d\theta = \int_0^{\pi/2} \frac{1}{2}(1 + \cos 2\theta)\,d\theta = \frac{\pi}{4}.$$

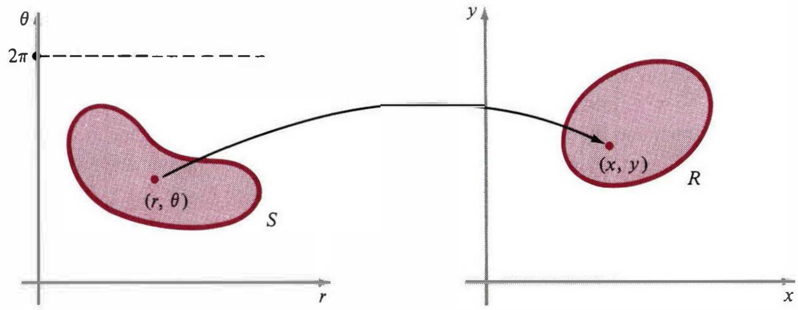


Figure A.20

Students should observe particularly that the change of variable in this calculation is accompanied by a corresponding change of the interval of integration.

Our only similar experience in the two-variable case is with changing double integrals from rectangular to polar coordinates by using the transformation equations

$$x = r\cos\theta, \qquad y = r\sin\theta. \tag{3}$$

Up to this stage we have interpreted these equations as expressing the rectangular coordinates of a given point in terms of its polar coordinates. However, they can also be interpreted as defining a *transformation* or *mapping* that carries points (r, θ) in the $r\theta$-plane over to points (x, y) in the xy-plane. That is, if a point (r, θ) is given, then equations (3) determine the corresponding point (x, y), as suggested in Fig. A.20. Further, in order to make this correspondence one-to-one, it is customary to restrict the point (r, θ) to lie in the part of the $r\theta$-plane specified by the inequalities $0 \le r, 0 \le \theta < 2\pi$.

From this point of view, the formula for changing a double integral into polar coordinates [formula (3) in Section 20.4] can be written as

$$\iint_R f(x, y)\, dx\, dy = \iint_S f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta. \tag{4}$$

Thus, we are allowed to substitute $x = r\cos\theta$ and $y = r\sin\theta$ in the integral on the left, but we must then replace $dx\,dy$ by $r\,dr\,d\theta$ and R by the corresponding region S in the $r\theta$-plane. In our previous work we made no mention of the region S, but instead—and equivalently—changed the limits of integration on iterated integrals to describe the same region R in terms of polar coordinates.

Formula (4) is a special case of a very general formula for changing variables in double integrals. The detailed proof is beyond the scope of this book, but at least we can state the result. First we need a definition. Consider a pair of functions of two variables,

$$x = x(u, v), \qquad y = y(u, v), \tag{5}$$

and assume that they have continuous partial derivatives. The *Jacobian* of these functions is the determinant defined by

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.^{*} \tag{6}$$

*Determinants of this form were first discussed by the German mathematician C. G. J. Jacobi (1804–1851). He did important work in the theory of elliptic functions, and applied his discoveries in astonishing ways to the theory of numbers. He also created a new and fruitful approach to theoretical dynamics. The Hamilton-Jacobi equations are part of the standard equipment of every student of mathematical physics. Also, Jacobi uttered the following magnificent and unforgettable defense of science for its own sake: "The sole aim of science is the honor of the human mind, and from this point of view a question about numbers is as important as a question about the system of the world."

This is often called a functional determinant, because it is a function of the variables u and v. As an example, we see that the Jacobian of the polar coordinates transformation (3) is

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

The general change of variables formula for double integrals can now be stated as follows: *If* (5) *is a one-to-one transformation of a region S in the uv-plane onto a region R in the xy-plane, and if the Jacobian* (6) *is positive, then*

$$\iint_R f(x, y)\, dx\, dy = \iint_S f[x(u, v), y(u, v)] \frac{\partial(x, y)}{\partial(u, v)}\, du\, dv. \tag{7}$$

Since r is the Jacobian of the polar coordinates transformation (3), it is clear that (4) is a special case of (7). Further, we can think of (7) as a two-dimensional extension of (2), with the derivative dx/du being replaced by the Jacobian $\partial(x, y)/\partial(u, v)$.

Formula (7) in turn can be extended to triple integrals. First, we define the *Jacobian* of the transformation

$$\begin{cases} x = x(u, v, w), \\ y = y(u, v, w), \\ z = z(u, v, w), \end{cases} \quad \text{by} \quad \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}.$$

Then, under the similar assumptions, we have the following extension of (7):

$$\iiint_R f(x, y, z)\, dx\, dy\, dz = \iiint_S F(u, v, w) \frac{\partial(x, y, z)}{\partial(u, v, w)}\, du\, dv\, dw, \tag{8}$$

where $F(u, v, w) = f[x(u, v, w), y(u, v, w), z(u, v, w)]$. The main thing to notice here is that

$$dx\, dy\, dz \quad \text{is replaced by} \quad \frac{\partial(x, y, z)}{\partial(u, v, w)}\, du\, dv\, dw.$$

Two important special cases of (8) are those of *cylindrical coordinates,*

$$\iiint_R f(x, y, z)\, dx\, dy\, dz = \iiint_S F(r, \theta, z) r\, dr\, d\theta\, dz,$$

where $F(r; \theta, z) = f(r\cos\theta, r\sin\theta, z)$; and *spherical coordinates,*

$$\iiint_R f(x, y, z)\, dx\, dy\, dz = \iiint_S F(\rho, \phi, \theta)\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta,$$

where $F(\rho, \phi, \theta) = f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)$. We leave it to the student to verify the spherical coordinates formula by using the transformation equations

$$\begin{cases} x = \rho\sin\phi\cos\theta, \\ y = \rho\sin\phi\sin\theta, \\ z = \rho\cos\phi, \end{cases}$$

to calculate the Jacobian

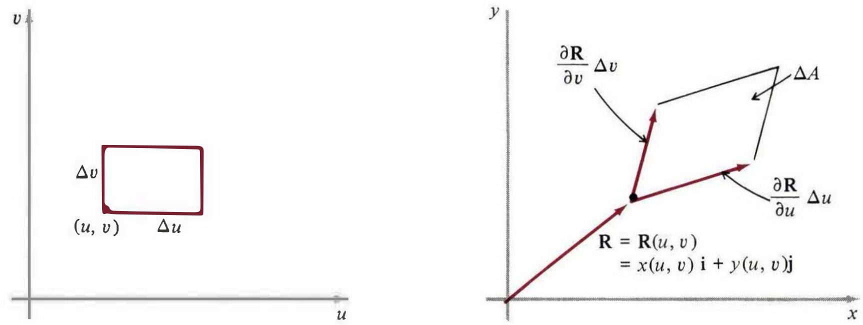


Figure A.21

$$\frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix} = \rho^2\sin\phi.$$

It is in this way that we can understand a little more fully what lies behind formulas (1).

One question remains, and for the sake of simplicity we state it only for the two-variable case: What is the underlying reason for the presence of the Jacobian on the right side of formula (7)? We now give a very brief intuitive explanation of this by means of vectors. In the uv-plane the equations $u = a$ constant and $v = a$ constant determine a network of straight lines parallel to the axes, whereas in the xy-plane these equations determine a network of intersecting curves. A small rectangle in the uv-plane with sides Δu and Δv corresponds to a small parallelogram in the xy-plane (see Fig. A.21) with sides that can be written in vector form as

$$\frac{\partial \mathbf{R}}{\partial u}\Delta u = \left(\frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j}\right)\Delta u$$

and

$$\frac{\partial \mathbf{R}}{\partial v}\Delta v = \left(\frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j}\right)\Delta v,$$

approximately. In calculating the integral on the left side of (7) as a limit of sums, it is natural to abandon the usual rectangular cells and instead use these small parallelograms. If we denote by ΔA the area of the parallelogram in the figure, then ΔA equals the magnitude of the cross product of the two vectors given above. Since this cross product is

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & 0 \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & 0 \end{vmatrix} \Delta u\,\Delta v = \left[\frac{\partial(x, y)}{\partial(u, v)}\Delta u\,\Delta v\right]\mathbf{k},$$

we have

$$\Delta A = \frac{\partial(x, y)}{\partial(u, v)}\Delta u\,\Delta v. \tag{9}$$

This shows that the Jacobian plays the role of a local magnification factor for areas. Further, these remarks constitute a sketch of a proof of (7), because all that remains to establish (7) is to form the integral on the left side as a limit of sums and make use of (9).

APPENDIX

B A FEW REVIEW TOPICS

B.1 THE BINOMIAL THEOREM

The binomial theorem is a general formula for the expanded n-factor product

$$(a+b)^n = (a+b)(a+b)\cdots(a+b). \tag{1}$$

If we compute a few cases by repeated laborious multiplication, we find that

$$\begin{aligned}
(a+b)^1 &= a+b,\\
(a+b)^2 &= a^2+2ab+b^2,\\
(a+b)^3 &= a^3+3a^2b+3ab^2+b^3,\\
(a+b)^4 &= a^4+4a^3b+6a^2b^2+4ab^3+b^4,\\
(a+b)^5 &= a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5.
\end{aligned}$$

It is clear that the expansion of (1) begins with a^n and ends with b^n, and also that the intermediate terms involve steadily decreasing powers of a and steadily increasing powers of b so that the sum of the two exponents is exactly n in each term. What is not so clear is the way the coefficients are calculated. To anticipate our final result, the expansion in question (the *binomial theorem*) is

$$\begin{aligned}
(a+b)^n = a^n + na^{n-1}b &+ \frac{n(n-1)}{2}a^{n-2}b^2\\
&+ \frac{n(n-1)(n-2)}{2\cdot 3}a^{n-3}b^3 + \cdots\\
&+ \frac{n(n-1)(n-2)\cdots(n-k+1)}{1\cdot 2\cdot 3\cdots k}a^{n-k}b^k\\
&+ \cdots + b^n.
\end{aligned} \tag{2}$$

Our purpose is to understand the reasons behind the form of these coefficients. The best way to do this is to take a short detour through the closely related topics of permutations and combinations.

Before starting this detour, we remind students that if n is a positive integer, then the product of all the positive integers up to n is denoted by $n!$, called *n factorial*:

$$n! = 1\cdot 2\cdot 3\cdots n.$$

Thus, $1! = 1$, $2! = 1\cdot 2 = 2$, $3! = 1\cdot 2\cdot 3 = 6$, $4! = 1\cdot 2\cdot 3\cdot 4 = 24$, etc. For reasons that will appear below, we define $0!$ to be 1. These numbers increase very rapidly, as we see by doing a little arithmetic:

$$5! = 120, \qquad 6! = 720, \qquad 7! = 5040, \qquad 8! = 40{,}320,$$
$$9! = 362{,}880, \qquad 10! = 3{,}628{,}800.$$

Further, with the aid of a calculator we learn that

$$20! \cong 2.433 \times 10^{18} \qquad \text{and} \qquad 40! \cong 8.159 \times 10^{47}.$$

Any product of consecutive positive integers can easily be written in terms of factorials. For example,

$$6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = \frac{10!}{5!}.$$

In general, if $k < n$, then

$$(k + 1)(k + 2) \cdots n = \frac{n!}{k!}.$$

PERMUTATIONS

We now discuss certain methods of counting that are useful in many applications of mathematics.

The reasoning on which our work is based can be illustrated by a simple example. Consider a journey from a city A through a city B to a city C. Suppose it is possible to go from A to B by 3 different routes and from B to C by 5 different routes. Then the total number of different routes from A through B to C is $3 \cdot 5 = 15$; for we can go from A to B in any one of 3 ways, and for each of these ways there are 5 ways of going on from B to C.

The basic principle here is this: If two successive independent decisions are to be made, and if there are c_1 choices for the first and c_2 choices for the second, then the total number of ways of making these two decisions is the product $c_1 c_2$. It is clear that the same principle is valid for any number of successive independent decisions.

The following is our main application of this idea. Given n distinct objects, in how many ways can we arrange them in order, that is, with a first, a second, a third, and so on? The answer is easy. There are n choices for the first object. After the first object is chosen, there are $n - 1$ choices for the second, then $n - 2$ choices for the third, etc. By the basic principle stated above, the total number of orderings is therefore

$$n(n - 1)(n - 2) \cdots 2 \cdot 1 = n!.$$

Each ordering of a set of objects is called a *permutation* of those objects. We have reached the following conclusion:

The number of permutations of n objects is $n!$.

Example 1 (a) There are $5! = 120$ ways of arranging 5 books on a shelf. (b) There are $9! = 362{,}880$ possible batting orders for the 9 players on a baseball team. (c) There are $52! \cong 8.066 \times 10^{67}$ ways of shuffling a deck of 52 cards.

We next consider a slight generalization. Suppose again that we have n distinct objects. This time we ask how many ways k of them can be chosen in order. Each such ordering is called a *permutation of n objects taken k at a time*, and the total number of these permutations is denoted by $P(n, k)$. There are evidently n choices for the first, $n - 1$ choices for the second, $n - 2$ choices for the third, and $n - (k - 1) = n - k + 1$ choices for the kth. The total number of these permutations is therefore

$$P(n, k) = n(n - 1)(n - 2) \cdots (n - k + 1).$$

If we write this number in terms of factorials, then our conclusion can be formulated as follows:

The number of permutations of n objects taken k at a time is

$$P(n, k) = n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!}.$$

Example 2 (a) If we have 7 books and only 3 spaces on a bookshelf, then the number of ways of filling these spaces with the available books (counting the order of the books) is

$$P(7, 3) = \frac{7!}{(7-3)!} = \frac{7!}{4!} = 7 \cdot 6 \cdot 5 = 210.$$

(b) The number of ways (counting the order of the cards) in which a 5-card poker hand can be dealt from a deck of 52 cards is

$$P(52, 5) = \frac{52!}{(52-5)!} = \frac{52!}{47!} = 52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 = 311{,}875{,}200.$$

Of course, the order of the cards in a poker hand is immaterial to the value of the hand, so the number of distinct poker hands is a considerably smaller number. We take account of this below, in our discussion of combinations.

COMBINATIONS

A set of k objects chosen from a given set of n objects, without regard to the order in which they are arranged, is called a *combination of n objects taken k at a time*. The total number of such combinations is sometimes denoted by $C(n, k)$, but more frequently by $\binom{n}{k}$. For reasons to be explained, the numbers $\binom{n}{k}$ are called *binomial coefficients*.

Permutations and combinations are related in a simple way. Each permutation of n objects taken k at a time consists of a choice of k objects (a combination) followed by an ordering of these k objects. But there are $\binom{n}{k}$ ways to choose k objects, and then $k!$ ways to arrange them in order, so

$$P(n, k) = \binom{n}{k} \cdot k! \qquad \text{or} \qquad \binom{n}{k} = \frac{P(n, k)}{k!}.$$

Our formula for $P(n, k)$ now yields the following conclusion:

The number of combinations of n objects taken k at a time is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

The binomial coefficients have many properties, of which we mention only a few:

$$\binom{n}{0} = \binom{n}{n} = \frac{n!}{0!n!} = 1, \qquad \binom{n}{1} = \binom{n}{n-1} = \frac{n!}{1!(n-1)!} = n,$$

and

$$\binom{n}{k} = \binom{n}{n-k}.$$

The last fact can be established easily from the formula, or, more directly, by simply observing that a choice of k objects from a set of n objects is equivalent to a choice of the $n-k$ objects that are left behind.

Example 3 (a) The number of committees of 3 people that can be chosen from a group of 8 people is

$$\binom{8}{3} = \frac{8!}{3!5!} = \frac{8 \cdot 7 \cdot 6}{2 \cdot 3} = 56.$$

(b) A certain governmental commission is to consist of 2 economists and 3 engineers. If 6 economists and 5 engineers are candidates for the appointments, how many different commissions are possible? From the 6 economists, 2 can be chosen in $\binom{6}{2}$ ways; and from the 5 engineers, 3 can be chosen in $\binom{5}{3}$ ways. The number of possible commissions is therefore

$$\binom{6}{2}\binom{5}{3} = \frac{6\,!}{2!4!} \cdot \frac{5!}{3!2!} = \frac{6 \cdot 5}{2} \cdot \frac{5 \cdot 4}{2} = 150.$$

(c) The number of different 5-card poker hands that can be dealt from a deck of 52 cards is

$$\binom{52}{5} = \frac{52!}{5!47!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{2 \cdot 3 \cdot 4 \cdot 5} = 2{,}598{,}960.$$

THE BINOMIAL THEOREM

To establish the binomial theorem (2), all that is necessary is to look at (1) and observe that each term of the expansion can be thought of as the product of n letters, one taken from each factor of the product

$$(a + b)(a + b) \cdots (a + b), \qquad n \text{ factors.}$$

Thus, a product $a^{n-k}b^k$ is obtained by choosing k b's and the rest a's. The number of ways this can be done is $\binom{n}{k}$. The coefficient of $a^{n-k}b^k$ on the right side of (2) is therefore $\binom{n}{k}$, and the proof is complete.

PROBLEMS

1 Write in terms of factorials
(a) $5 \cdot 6 \cdot 7 \cdot 8 \cdot 9$; (b) $22 \cdot 21 \cdot 20 \cdot 19 \cdot 18 \cdot 17$;
(c) $\dfrac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$.

2 Compute
(a) $\dfrac{8!}{5!}$; (b) $\dfrac{11!}{8!}$; (c) $\dfrac{15!}{3!12!}$;
(d) $\dfrac{25!}{4!21!}$; (e) $P(22, 2)$; (f) $P(7, 5)$.

3 If 6 horses run in a race, how many different orders of finishing are there? How many possibilities are there for the first 3 places (win, place, and show)?

4 A club has 10 members. In how many ways can a president, a vice president, and a secretary be chosen?

5 How many batting orders are possible for a baseball team if the 4 best hitters are the first 4 to bat?

6 How many batting orders are possible for a baseball team if the fielders are the first 3 to bat and the pitcher bats last?

7 How many 10-digit numbers can be formed from all 10 digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 if 0 is not allowed as the first digit?

8 How many 5-digit numbers can be formed from the 10 digits if the first digit cannot be 0 and no repetitions are allowed? If repetitions are allowed?

9 How many license plates can be made using 7 symbols, of which the first 3 are different letters of the alphabet and the last 4 are digits of which the first cannot be 0?

10 How many ways can 3 history books and 4 physics books be put on a shelf if books on the same subject must be kept together? If the history books must be kept together but the physics books need not be?

11 How many different signals can be made from 5 different flags if each signal consists of 5 flags placed one above the other on a flagpole? If each signal consists of 3 flags? If each signal consists of one or more flags?

12 In how many ways can 6 people be seated in a row of 6 chairs?

13 In how many ways can 6 people be seated in a row of 8 chairs?

14 In how many ways can 6 people be seated in a row of 6 chairs
(a) if a certain 2 insist on sitting next to each other?
(b) if a certain 2 refuse to sit next to each other?

15 In how many ways can 3 men and 3 women be seated in a row of 6 chairs if the men and women alternate?

16 Compute

(a) $\binom{100}{2}$; (b) $\binom{150}{3}$; (c) $\binom{10}{5}$.

17 A contractor employs 10 workers. In how many ways can he choose 4 of them to do a certain job?

18 In how many ways can 12 jurors be selected from a panel of 20 eligible citizens?

19 At a meeting, 28 people all shake hands with one another. How many handshakes are there?

20 In an examination a student has a choice of any 10 out of 12 questions. In how many ways can she choose her questions?

21 From a committee of 10, in how many ways can one choose a subcommittee of 4 and another subcommittee of 3, with no overlap?

22 How many committees of 4 can be chosen from a group of 12 men? How many of these will include a specific man? How many will exclude this man?

23 In how many ways can a committee of 5 men and 4 women be chosen from a club of 10 men and 7 women?

24 In how many ways can we select a committee of 5 from a group of 11
(a) if 2 of the group insist on serving together or not at all?
(b) if 2 of the group refuse to serve together?

25 In how many different orders can we shelve sets of 5 books, each set consisting of 3 history books and 2 chemistry books, if the books are to be chosen from a set of 9 history books and 7 chemistry books?

26 Three bags contain 8 black, 6 white, and 10 red marbles, respectively. In how many ways can we choose 6 black, 4 white, and 7 red marbles?

27 From a deck of 52 cards, how many 5-card poker hands are flushes (all cards of the same suit)? How many are full houses (3 cards of one kind together with a pair of another kind)?

28 How many lines are determined by 12 points in a plane if no 3 of the points are collinear?

29 How many triangles are determined by 13 points in a plane if no 3 of the points are collinear?

30 How many lines are determined by m points in a plane if k of them (where $k < m$) lie on the same line and, except for these, no 3 of the points are collinear?

31 How many planes are determined by 9 points in space if no 4 of the points are coplanar?

32 How many rectangles are formed by 5 vertical lines intersecting 8 horizontal lines?

33 How many diagonals can be drawn in a regular polygon with n sides? (A diagonal is a segment joining 2 nonadjacent vertices.)

34 Use the binomial theorem to expand each of the following:
(a) $(2x - y)^7$; (b) $(3a + 2b)^6$;
(c) $(x^2y - 3z^3)^5$.

35 Find the term of
(a) $(x - 4)^{15}$ that involves x^{11};
(b) $(2x - 3)^9$ that involves x^4;
(c) $(a^2 - b^3)^{14}$ that involves a^{10}.

B.2 MATHEMATICAL INDUCTION

Discoveries in mathematics are sometimes made by carefully examining empirical evidence. As an illustration, let us try to find a formula for the sum of the first n odd numbers, where n is any positive integer. We compute:

$$\begin{aligned}
&\text{for } n = 1, && 1 = 1 && = 1^2,\\
&\text{for } n = 2, && 1 + 3 = 4 && = 2^2,\\
&\text{for } n = 3, && 1 + 3 + 5 = 9 && = 3^2,\\
&\text{for } n = 4, && 1 + 3 + 5 + 7 = 16 && = 4^2,\\
&\text{for } n = 5, && 1 + 3 + 5 + 7 + 9 = 25 && = 5^2.
\end{aligned}$$

The pattern emerging from this evidence seems to suggest that the value of the sum always equals the square of the number in terms in the sum. Since $2n - 1$ is the nth odd number, we can formulate this conjecture as follows:

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2 \tag{1}$$

for every positive integer n.

The evidence for (1) is suggestive but far from conclusive. If we continue to test our conjecture for $n = 6, 7, 8$, and so on, and if it continues to hold up for these additional values of n, then this will certainly increase our confidence that (1) is probably true for every positive integer n. However, verifications of this kind can never constitute a proof, no matter how far they may be carried. If we verify (1) for all values of n up to $n = 1000$, then the logical possibility still remains that (1) might fail to be true for $n = 1001$.* There is an infinite chasm between "probably true" and "absolutely certain." What is needed is a logical argument proving that (1) is *always* true, for *all* values of n, beyond any doubt whatsoever. This is what the method of proof by mathematical induction accomplishes. We explain this method of reasoning by showing how it works in the case of formula (1), and then we state it as a formal principle.

Example 1 To prove (1) by mathematical induction, we begin by observing that this formula is true for $n = 1$, because it reduces to $1 = 1^2$. (We already knew this.) We next prove that if k is a value of n for which (1) is true, then (1) is necessarily also true for the next integer, $n = k + 1$. Thus, we assume that (1) is true for $n = k$,

$$1 + 3 + 5 + \cdots + (2k - 1) = k^2. \tag{2}$$

With the aid of this hypothesis we try to prove that (1) is also true for $n = k + 1$,

$$1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) = (k + 1)^2. \tag{3}$$

(The next-to-the-last term on the left here is displayed for the sake of clarity in our next step.) By using (2) we see that the left side of (3) can be written as

$$\begin{aligned} 1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) &= k^2 + (2k + 1) \\ &= (k + 1)^2, \end{aligned}$$

so (3) is true if (2) is true. But this is enough to guarantee that (1) is actually true for all n. To see this, suppose we wish to assure ourselves that (1) is true for some specific value of n, say $n = 37$. The reasoning is as follows: We know by actual computation that (1) is true for $n = 1$; since it is true for $n = 1$, the argument just given tells us that it is also true for $n = 2$; since it is true for $n = 2$, it must be true for $n = 3$; and so on, up to $n = 37$ (or any other value of n).

Our main principle is little more than a distillation of the essence of this example.

Principle of Mathematical Induction *Let $S(n)$ be a proposition depending on a positive integer n.† Suppose that each of the following conditions is known to be satisfied.*

I *$S(1)$ is true.*

II *If $S(n)$ is assumed to be true for an integer $n = k$, then it is necessarily true for the next integer, $n = k + 1$.*

Under these circumstances it follows that $S(n)$ is true for every positive integer n.

*As a trivial illustration of this point, the equation

$$n^2 - 1 = (n + 1)(n - 1) + [(n - 1)(n - 2) \cdots (n - 1000)]$$

is clearly true for the first thousand values of n, because the bracketed expression is zero, and yet it is false for $n = 1001, 1002, \ldots$.

†This means that for each specific value of n, $S(n)$ is a statement that is either true or false, without any ambiguity.

Briefly, if we write down the propositions $S(n)$ in order,

$$S(1), S(2), S(3), \ldots,$$

then the process of verification is started by I, and II is a link from each to the next guaranteeing that the process continues without end.

The idea of induction can be illustrated in many nonmathematical ways. For instance, imagine a row of dominoes standing on end. Suppose they are spaced in such a way that if any one of them falls, then it will knock over the next one. Suppose further that we actually knock over the first domino. In this situation we know that all the dominoes will fall. Our knowledge is based on two facts, which are closely analogous to I and II:

(i) The first domino *does* fall, because we knock it over.
(ii) *If* any domino falls, *then* it will knock over the next one.

We must be careful with the meaning of (ii); it does not state that any domino actually does fall, only that each domino is related to the next one in a certain way.

We continue with two additional examples of the method, in which we establish two formulas that are needed in Chapter 6. These are formulas for the sum of the first n positive integers, and for the sum of the first n squares:

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}, \tag{4}$$

and

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}. \tag{5}$$

Several of the remarks and problems that follow are concerned with the natural question of how such formulas can be discovered and understood. For the moment, however, we confine our attention to proving them by the method of mathematical induction.

Example 2 To prove (4) by induction, we start by verifying I in this case, that is, we note that (4) is obviously true for $n = 1$:

$$1 = \frac{1 \cdot 2}{2}.$$

To verify II, we begin by assuming (4) for $n = k$,

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}, \tag{6}$$

and we hope to prove (4) for $n = k + 1$,

$$1 + 2 + 3 + \cdots + k + (k+1) = \frac{(k+1)(k+2)}{2}. \tag{7}$$

By using (6) we can write the left side of (7) as

$$\begin{aligned} 1 + 2 + 3 + \cdots + k + (k+1) &= \frac{k(k+1)}{2} + (k+1) \\ &= (k+1)\left(\frac{k}{2} + 1\right) \\ &= \frac{(k+1)(k+2)}{2}. \end{aligned}$$

Condition II is therefore satisfied, so by induction (4) is valid for all positive integers n.

Example 3 The proof of (5) is just as easy. To verify I, put $n = 1$:

$$1^2 = \frac{1 \cdot 2 \cdot 3}{6}.$$

To verify II, we must assume

$$1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

and use this to prove

$$1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}.$$

The details are routine,

$$\begin{aligned} 1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= (k+1)\left[\frac{k(2k+1)}{6} + (k+1)\right] \\ &= (k+1)\left[\frac{2k^2 + 7k + 6}{6}\right] \\ &= \frac{(k+1)(k+2)(2k+3)}{6}, \end{aligned}$$

so by induction the proof of (5) is complete.

Remark 1 Mathematical induction is a venerable method of proof that every student of mathematics ought to understand. Our purpose here has been to explain this method, and also to illustrate its use by proving two formulas [(4) and (5)] that are necessary for other parts of our work. However, much remains to be said.

Proofs by induction produce belief without insight, and are therefore fundamentally unsatisfying. It is important to know that a mathematical theorem *is* true, but it is often more important to understand *why* it is true. There are other proofs of formulas (1), (4), and (5) which convey much more insight into these formulas, and which also suggest how they might have been discovered in the first place. We begin with (4).

If we denote the sum of the integers from 1 to n by S, so that

$$S = 1 + 2 + \cdots + (n-1) + n,$$

then it might occur to us to write this sum in the reverse order, as

$$S = n + (n-1) + \cdots + 2 + 1.$$

If we now notice that the two first terms on the right add up to $n + 1$, and also the two second terms, and so on, then it is natural to add these two equations together to get

$$2S = n(n+1) \quad \text{or} \quad S = \frac{n(n+1)}{2}.$$

These ideas serve to discover formula (4) and also to prove it, simultaneously.

We next turn to formula (1), which again we discover and prove at a single stroke. Consider the sum of the first n odd numbers,

$$1 + 3 + 5 + \cdots + (2n-1).$$

We notice certain obvious gaps in this sum, where the even numbers ought to be. If we fill in these gaps, and at the same time compensate for this filling in, then, using (4), we easily obtain

$$\begin{aligned}1 + 3 + 5 + \cdots + (2n - 1) &= (1 + 2 + 3 + \cdots + 2n) - (2 + 4 + \cdots + 2n)\\ &= (1 + 2 + 3 + \cdots + 2n) - 2(1 + 2 + \cdots + n)\\ &= \frac{2n(2n + 1)}{2} - 2 \cdot \frac{n(n + 1)}{2}\\ &= 2n^2 + n - n^2 - n = n^2,\end{aligned}$$

which is (1).

We have discovered and proved the formula for the sum of the first n positive integers,

$$1 + 2 + 3 + \cdots + n = \frac{n(n + 1)}{2}. \tag{4}$$

It is somewhat more difficult to discover (5), that is, a formula for the sum of the first n squares,

$$1^2 + 2^2 + 3^2 + \cdots + n^2.$$

We know the answer by Example 3, but let us disregard this for a moment and try to think how we might discover it. It is natural to consider the two sums together:

n	1	2	3	4	5	6	$\cdots$
$1 + 2 + \cdots + n$	1	3	6	10	15	21	$\cdots$
$1^2 + 2^2 + \cdots + n^2$	1	5	14	30	55	91	$\cdots$

How are these sums related? It might occur to us to consider their ratio:

n	1	2	3	4	5	6	$\cdots$
$\dfrac{1 + 2 + \cdots + n}{1^2 + 2^2 + \cdots + n^2}$	1	$\frac{3}{5}$	$\frac{3}{7}$	$\frac{1}{3}$	$\frac{3}{11}$	$\frac{3}{13}$	$\cdots$

If we write these ratios in the form

$$\frac{3}{3} \quad \frac{3}{5} \quad \frac{3}{7} \quad \frac{3}{9} \quad \frac{3}{11} \quad \frac{3}{13} \quad \cdots,$$

then it is difficult to miss the pattern that emerges. It seems clear that

$$\frac{1 + 2 + \cdots + n}{1^2 + 2^2 + \cdots + n^2} = \frac{3}{2n + 1},$$

and by using (4) we easily find that

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n + 1)(2n + 1)}{6}. \tag{5}$$

This is certainly not a proof of (5). Nevertheless, it presents us with a plausible conjecture which we can then try to prove by induction, as we have done in Example 3.

Remark 2 There is another very ingenious way of discovering (5) that also constitutes a proof. It begins with the expansion

$$(k + 1)^3 = k^3 + 3k^2 + 3k + 1,$$

expressed in the more convenient form

$$(k + 1)^3 - k^3 = 3k^2 + 3k + 1.$$

If we write down this identity for $k = 1, 2, \ldots, n$ and add, then by taking advantage of wholesale cancellations we find that

$$
\begin{aligned}
2^3 - 1^3 &= 3 \cdot 1^2 + 3 \cdot 1 + 1 \\
3^3 - 2^3 &= 3 \cdot 2^2 + 3 \cdot 2 + 1 \\
&\cdots \\
(n+1)^3 - n^3 &= 3 \cdot n^2 + 3 \cdot n + 1 \\
\hline
(n+1)^3 - 1^3 &= 3(1^2 + 2^2 + \cdots + n^2) + 3(1 + 2 + \cdots + n) + n
\end{aligned}
$$

This enables us to obtain a formula for the sum of the squares in terms of our known formula (4) for the sum $1 + 2 + \cdots + n$:

$$
\begin{aligned}
1^2 + 2^2 + \cdots + n^2 &= \tfrac{1}{3}[n^3 + 3n^2 + 3n - \tfrac{3}{2}n(n+1) - n] \\
&= \tfrac{1}{6}(2n^3 + 6n^2 + 6n - 3n^2 - 3n - 2n) \\
&= \frac{n}{6}(2n^2 + 3n + 1) \\
&= \frac{n(n+1)(2n+1)}{6}.
\end{aligned}
$$

The idea of this proof is due to the great French writer-scientist-mathematician-theologian Blaise Pascal. It can be extended quite easily to yield the sum of the first n cubes,

$$1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2, \tag{8}$$

the sum of the first n fourth powers, and so on indefinitely.

Remark 3 Mathematical induction as a method of demonstrative proof originated in the work of Pascal on the binomial coefficients. The interested reader will find this work described and quoted in vol. 1 of G. Polya's remarkable book, *Mathematical Discovery* (Wiley, 1962), pp. 73–75.

PROBLEMS

1 Use (1) and (4) to find a formula for each of the following:
(a) $2 + 4 + 6 + \cdots + 2n$;
(b) $(n+1) + (n+2) + (n+3) + \cdots + 3n$;
(c) $1 + 3 + 5 + \cdots + (4n - 1)$;
(d) $(2n+1) + (2n+3) + (2n+5) + \cdots + (4n-1)$;
(e) $3 + 8 + 13 + \cdots + (5n - 2)$.

2 Discover a formula for $1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2$ by using its relation to the sum $1^2 + 2^2 + 3^2 + \cdots + (2n)^2$.

3 Prove each of the following by induction:
(a) $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \dfrac{1}{3 \cdot 4} + \cdots + \dfrac{1}{n(n+1)} = \dfrac{n}{n+1}$;
(b) $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \dfrac{n(n+1)(n+2)}{3}$;
(c) $\dfrac{1}{1 \cdot 3} + \dfrac{1}{3 \cdot 5} + \dfrac{1}{5 \cdot 7} + \cdots + \dfrac{1}{(2n-1)(2n+1)} = \dfrac{n}{2n+1}$;
(d) $1 \cdot 3 + 3 \cdot 5 + 5 \cdot 7 + \cdots + (2n-1)(2n+1) = \dfrac{n(4n^2 + 6n - 1)}{3}$.

Prove (a) without using mathematical induction, by means of the algebraic identity

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1};$$

also, devise a similar method of proving (c).

4 Prove each of the following by induction:
(a) $1 + \dfrac{1}{2} + \dfrac{1}{4} + \cdots + \dfrac{1}{2^n} = 2 - \dfrac{1}{2^n}$;
(b) $1 + r + r^2 + \cdots + r^n = \dfrac{1 - r^{n+1}}{1 - r} \quad (r \neq 1)$;
(c) $\dfrac{1}{2} + \dfrac{2}{2^2} + \dfrac{3}{2^3} + \cdots + \dfrac{n}{2^n} = 2 - \dfrac{(n+2)}{2^n}$;
(d) $r + 2r^2 + 3r^3 + \cdots + nr^n = \dfrac{r - (n+1)r^{n+1} + nr^{n+2}}{(1-r)^2} \quad (r \neq 1)$;
(e) $\dfrac{1}{n} + \dfrac{1}{n+1} + \dfrac{1}{n+2} + \cdots + \dfrac{1}{2n-1} = 1 - \dfrac{1}{2} + \dfrac{1}{3} - \dfrac{1}{4} + \cdots + \dfrac{1}{2n-1}$;

(f) $\dfrac{1}{1+x} + \dfrac{2}{1+x^2} + \dfrac{4}{1+x^4} + \cdots + \dfrac{2^n}{1+x^{2^n}} =$

$$\frac{1}{x-1} + \frac{2^{n+1}}{1-x^{2^{n+1}}} \qquad (x \neq \pm 1);$$

(g) $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\dfrac{1}{2}n(n+1)\right]^2;$

(h) $1^4 + 2^4 + 3^4 + \cdots + n^4 =$

$$\frac{1}{30}n(n+1)(6n^3 + 9n^2 + n - 1).$$

5 Use the method of Remark 2 to discover and prove the formulas in parts (g) and (h) of Problem 4.

6 In each of the following, guess the general law suggested by the given facts and prove it by induction:

(a) $1 = 1,$
$1 - 4 = -(1 + 2),$
$1 - 4 + 9 = 1 + 2 + 3,$
$1 - 4 + 9 - 16 = -(1 + 2 + 3 + 4);$

(b) $1 - \frac{1}{2} = \frac{1}{2},$
$(1 - \frac{1}{2})(1 - \frac{1}{3}) = \frac{1}{3},$
$(1 - \frac{1}{2})(1 - \frac{1}{3})(1 - \frac{1}{4}) = \frac{1}{4},$
$(1 - \frac{1}{2})(1 - \frac{1}{3})(1 - \frac{1}{4})(1 - \frac{1}{5}) = \frac{1}{5}.$

7 Guess the formulas that simplify the following products, and prove them by induction:

(a) $\left(1 - \dfrac{1}{4}\right)\left(1 - \dfrac{1}{9}\right)\left(1 - \dfrac{1}{16}\right)\cdots\left(1 - \dfrac{1}{n^2}\right);$

(b) $(1 - x)(1 + x)(1 + x^2)(1 + x^4)\cdots(1 + x^{2^n}).$

8 Let $S(n)$ be the following statement:

$$1 + 2 + 3 + \cdots + n = \frac{(n-1)(n+2)}{2}.$$

(a) Prove that if $S(n)$ is true for $n = k$, then it is also true for $n = k + 1$.

(b) Criticize the assertion, "By induction we therefore know that $S(n)$ is true for all positive integers n."

ANSWERS TO ODD-NUMBERED PROBLEMS

CHAPTER 1

Section 1.2, p. 8

1. (a) Rational; (b) integer, rational; (c) integer, rational; (d) rational; (e) integer, rational; (f) irrational; (g) integer, rational; (h) irrational; (i) rational; (j) rational.
3. 11.
5. $\pi - 3$.
7. $5 - x$.
9. $x^2 + 10$.
11. $3x^2 - 1$.
13. (a) $x < 0$ and $x > 1$; (b) $-2 < x < 1$; (c) $x < -7$ and $x > 3$; (d) $-\frac{3}{2} < x < 1$; (e) $-3 < x < \frac{1}{2}$; (f) all x.
15. (a) $x > 0$; (b) $-2 < x < 0$ and $x > 2$; (c) $x < -1$ and $x > 3$; (d) $x < -1$, $0 < x < 1$, and $x > 3$.
17. $a = b$.
19. (a) Vertical; (b) horizontal; (c) horizontal; (d) vertical; (e) horizontal; (f) vertical; (g) vertical; (h) horizontal.
21. (a) $5\sqrt{2}$; (b) $\sqrt{13}$; (c) $\sqrt{89}$; (d) $|a - b|\sqrt{2}$.
27. Center $(-2, \frac{3}{2})$, radius $\frac{1}{2}\sqrt{113}$.
29. $(-1, -1)$.
31. Symmetric with respect to the straight line through the origin that bisects the first and third quadrants.
33. $\frac{1}{2}\sqrt{2}\, h$.

Section 1.3, p. 14

1. (a) $-\frac{2}{7}$; (b) $\frac{8}{3}$; (c) $\frac{1}{6}$; (d) -1; (e) 0; (f) 10.
5. (a) Yes; (b) no; (c) no; (d) yes.
7. (a) $y = -4x + 5$; (b) $3x + 7y = 2$; (c) $2x - 3y = 12$; (d) $y = -4$; (e) $x = 1$; (f) $x + 3y + 2 = 0$; (g) $x + 2y = 11$; (h) $3y - 2x = 17$; (i) $x + 2y = 9$; (j) $x + y = 1$.
9. (a) $\dfrac{x}{-3} + \dfrac{y}{-5} = 1$; (b) $\dfrac{x}{-8} + \dfrac{y}{3} = 1$; (c) $\dfrac{x}{1} + \dfrac{y}{6} = 1$; (d) $\dfrac{x}{\frac{9}{2}} + \dfrac{y}{-3} = 1$.
11. $(\frac{19}{5}, -\frac{11}{10})$.
13. $F = \frac{9}{5}C + 32$ or $C = \frac{5}{9}(F - 32)$.

Section 1.4, p. 22

1. (a) $(x - 4)^2 + (y - 6)^2 = 9$; (b) $(x + 3)^2 + (y - 7)^2 = 5$; (c) $(x + 5)^2 + (y + 9)^2 = 49$; (d) $(x - 1)^2 + (y + 6)^2 = 2$; (e) $(x - a)^2 + y^2 = a^2$ or $x^2 + y^2 = 2ax$; (f) $x^2 + (y - a)^2 = a^2$ or $x^2 + y^2 = 2ay$.
3. (a) Circle, center (2, 2) and radius $2\sqrt{2}$; (b) point (9, 7); (c) circle, center $(-4, -5)$ and radius 1; (d) circle, center $(-\frac{3}{2}, 4)$ and radius 3; (e) empty; (f) point $(\frac{1}{2}\sqrt{2}, -\frac{1}{2}\sqrt{2})$; (g) circle, center $(8, -3)$ and radius 11.
5. Distinct real roots, $b^2 - 4ac > 0$; equal real roots, $b^2 - 4ac = 0$; no real roots, $b^2 - 4ac < 0$.
7. $y = \pm 2\sqrt{2x + 4}$.
9. (a) $y^2 = -12x$; (b) $x^2 = 4y$; (c) $y^2 = 8x$; (d) $3x^2 = -4y$; (e) $y^2 + 12x + 12 = 0$; (f) $x^2 - 6x - 8y + 17 = 0$.
11. 20 ft.

Section 1.5, p. 28

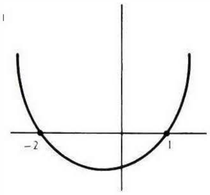
1. (a) 42; (b) 17; (c) -3; (d) 32; (e) $5a^2 + 30a + 42$; (f) $125t^2 - 3$.
3. 5.
5. $2x + h$.
7. $-\dfrac{1}{x(x + h)}$.
9. $f(1) = 0, f(2) = 2, f(3) = 10, f(0) = -2, f(-1) = -10, f(-2) = -30$.
13. (a) $x \geq 0$; (b) $x \leq 0$; (c) all x; (d) $x \leq -2$, $x \geq 2$; (e) all x except 2, -2; (f) all x; (g) $x \leq -2$, $x \geq 1$; (h) $x < -2$, $x > 1$; (i) $-3 \leq x \leq 1$; (j) $x \leq 0$, $x > 2$.
15. $f(0) = 0$, $f(1)$ does not exist, $f(2) = 2$, $f(3) = \frac{3}{2}$, $f(f(3)) = 3$. In the last part, it is tacitly understood that x is restricted to those values for which $f(f(x))$ exists: that is, $x \neq 1$.
17. $f(0) = 1$, $f(1)$ does not exist, $f(2) = -1$, $f(f(2)) = \frac{1}{2}$, $f(f(f(2))) = 2$.
19. $f(x_1)f(x_2) = f(x_1 + x_2)$.
21. No; it is true if and only if $ad + b = bc + d$.
23. (a) $a = 4$, $b = -5$, $c = 3$.
25. $y = -x + \sqrt{3 - x^2}$ and $y = -x - \sqrt{3 - x^2}$.
27. $A = \frac{1}{4}x\sqrt{16 - x^2}$.
29. $A = x\sqrt{4a^2 - x^2}$.
31. (a) Yes, $A = \dfrac{c^2}{4\pi}$; (b) yes, $A = \frac{1}{16}p^2$; (c) no.
33. $V = 2\pi r^2\sqrt{a^2 - r^2}$, $A = 2\pi r^2 + 4\pi r\sqrt{a^2 - r^2}$; $V = \frac{1}{4}\pi(4a^2h - h^3)$, $A = \frac{1}{2}\pi(4a^2 - h^2) + \pi h\sqrt{4a^2 - h^2}$.

35. $A = 2\pi r^2 + \dfrac{2V}{r}$.

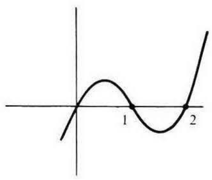
37. (a) Largest area = 625 ft^2, both sides = 25 ft; (b) $A = 100x - 2x^2 = 1250 - 2(x - 25)^2$, largest area = 1250 ft^2, sides = 25 and 50 ft.

Section 1.6, p. 37

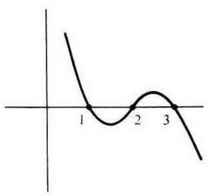
1. (a)



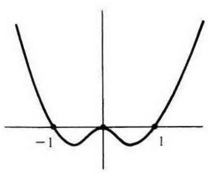
(b)



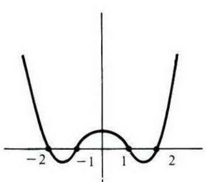
(c)



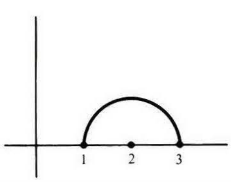
(d)



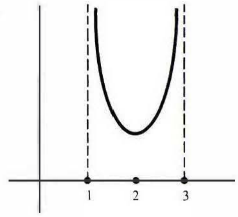
(e)



3. (a)



(b)



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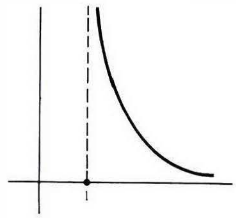

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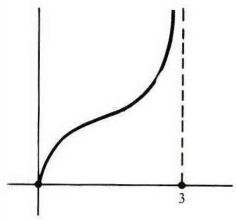

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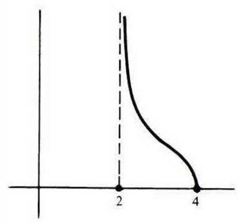

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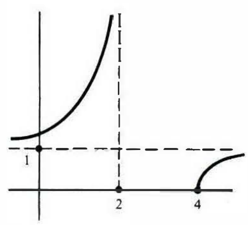
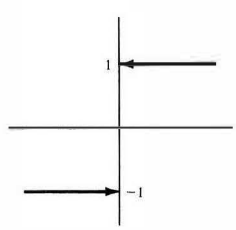
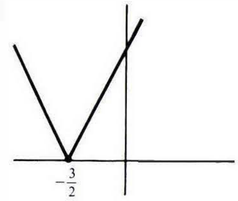


5. (a)



(b)



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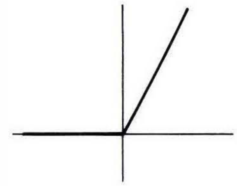

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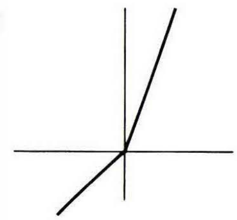

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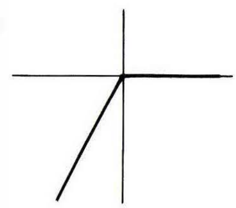
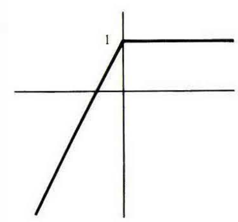
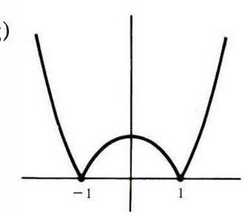


(f)



(g)



7. Only (b).

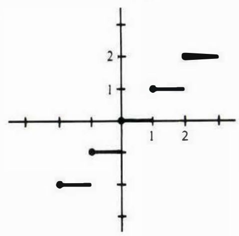
Section 1.7, p. 45

1. (a) $\dfrac{\pi}{12}$; (b) $\dfrac{5\pi}{6}$; (c) $\dfrac{25\pi}{3}$; (d) $-\dfrac{\pi}{5}$; (e) $-\dfrac{11\pi}{18}$; (f) $\dfrac{7\pi}{180}$.

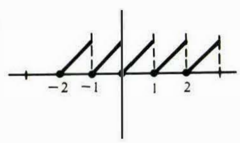
3. (a) $-\frac{1}{2}$; (b) $\frac{1}{2}\sqrt{3}$; (c) $-\frac{1}{2}\sqrt{3}$; (d) $\frac{1}{2}\sqrt{2}$; (e) $-\frac{1}{2}$; (f) $-\frac{1}{2}\sqrt{2}$.
7. (a) $\sin \dfrac{\pi}{2}$; (b) $\sin 0$; (c) $-\sin \dfrac{\pi}{3}$; (d) $-\sin \dfrac{\pi}{3}$; (e) $\cos 0$; (f) $\cos \dfrac{\pi}{4}$; (g) $-\cos \dfrac{\pi}{5}$; (h) $\sin \dfrac{\pi}{2}$; (i) $\cos \dfrac{\pi}{3}$.
11. $\sin 15° = \frac{1}{2}\sqrt{2-\sqrt{3}}$; $\cos 15° = \frac{1}{2}\sqrt{2+\sqrt{3}}$.
13. (a) $\cos \dfrac{\pi}{4} = \dfrac{1}{2}\sqrt{2}$; (b) $\cos \dfrac{3\pi}{4} = -\dfrac{1}{2}\sqrt{2}$.
15. (a) $\sin \dfrac{2\pi}{3} = \dfrac{\sqrt{3}}{2}$; (b) $\cos \dfrac{5\pi}{4} = -\dfrac{1}{2}\sqrt{2}$; (c) $\sin \dfrac{17\pi}{6} = \dfrac{1}{2}$.
19. $\frac{1}{4}\sqrt{2}(\sqrt{3}+1)$.

Additional Problems, p. 47

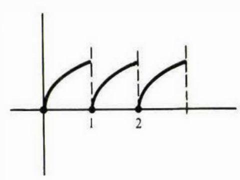
9. No, to both questions.
15. $(y_1 - y_2)x + (x_2 - x_1)y = x_2y_1 - x_1y_2$.
19. (a) $\left(b, \dfrac{ab-b^2}{c}\right)$; (b) $\left(\dfrac{a}{2}, \dfrac{b^2+c^2-ab}{2c}\right)$; (c) $\left(\dfrac{a+b}{3}, \dfrac{c}{3}\right)$.
23. (a) $x - 7y + 5 = 0$, $7x + y - 15 = 0$; (b) $x = (1 \pm \sqrt{2})y$.
25. $|b| \le 2\sqrt{10}$.
27. (a) $(x - \frac{4}{3}a)^2 + y^2 = \frac{4}{9}a^2$; (b) $(x^2+y^2)^2 = 2a^2(x^2 - y^2)$.
31. $7x + y = 10$ and $x - y + 2 = 0$.
33. $y = -2x + 2$; $(0, 2)$ and $(\frac{4}{5}, \frac{2}{5})$.
35. $x^2 + y^2 - 2xy - 4x - 4y + 4 = 0$.
37. The line is $x = 2pm$.
41. No.
43. $g(x) = x^3$.
45. $V = \frac{1}{2}Ar - \pi r^3$.
47. $V = \dfrac{2}{3}\pi a\left(\dfrac{r^4}{r^2 - a^2}\right)$.
49. $\alpha = \dfrac{d}{ad-bc}$, $\beta = \dfrac{-b}{ad-bc}$, $\gamma = \dfrac{-c}{ad-bc}$, $\delta = \dfrac{a}{ad-bc}$.
51. $(x-1)(x-2)\cdots(x-n)$; $x^n + 1$; x^n.
53. (a) Odd; (b) even; (c) even; (d) odd; (e) neither; (f) odd; (g) neither; (h) neither.
55. (a) Even; (b) even; (c) odd.
57. $y = 275(x-1)(x-2) - \sqrt{3}(x-1)(x-3) + \dfrac{\pi}{2}(x-2)(x-3)$.
59. (a)



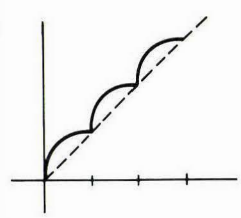
(b)



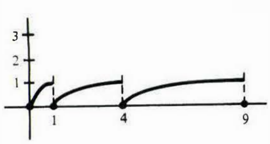
(c)



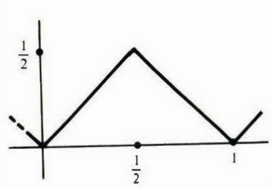
(d)



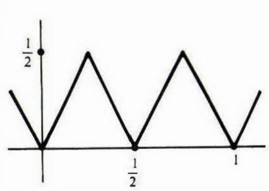
(e)



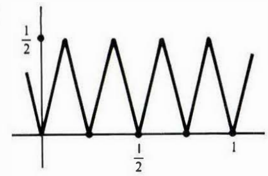
61. (a)



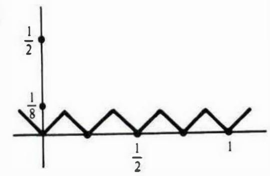
(b)



(c)



(d)



67. $\frac{1}{4}h^2$.

CHAPTER 2

Section 2.2, p. 57

1. (a) $4x + y + 4 = 0$; (b) $8x - y = 16$; (c) $8x - y = 16$.
5. (a) $2x_0 - 4$; (b) $2x_0 - 2$; (c) $4x_0$; (d) $2x_0$.
7. $8x + y + 7 = 0$.
11. $y = 4x + 1$, $y = -4x + 25$.

Section 2.3, p. 62

3. $-16x$.
5. -72.
7. $-10 + 30x$.
9. $6y + 7$.
11. $3500 - 14x$.
13. $10x + 25$.
15. $-32x - 40$.
17. $(0, 6)$.
19. $(3, 0)$.
21. $(10,100)$.
23. $5 - 3x^2$.
25. $6x^2 - 6x + 6$.
27. $1 + \dfrac{1}{x^2}$.
29. $\dfrac{1}{(x+1)^2}$.
31. $\dfrac{-2}{x^3}$.
33. $\dfrac{-2x}{(x^2+1)^2}$.
35. $\dfrac{-2(x^2+1)}{(x^2-1)^2}$.
37. $\dfrac{1}{2\sqrt{x-1}}$.
39. (b) Area = 2.
43. $g'(0) = 0$; $y = 3$.
45. $(1, 1)$.

Section 2.4, p. 67

1. $v = 6t - 12$; (a) $t = 2$, (b) $t > 2$.
3. $v = 4t + 28$; (a) $t = -7$, (b) $t > -7$.
5. $v = 14t$; (a) $t = 0$, (b) $t > 0$.
7. $v = 8t - 24$; (a) $t = 3$, (b) $t > 3$.
9. (a) 7 seconds; (b) 48 ft/s; (c) 176 ft/s; (d) 224 ft/s.
11. (a) 12; (b) 6; (c) 18.
13. 10 seconds.
15. (a) 3200 gal/min; (b) 2400 gal/min.

17. dr/dt decreases as r increases.

Section 2.5, p. 73

1. 15.
3. -5.
5. 3.
7. -3.
9. 4.
11. 5.
13. 0
15. $\frac{4}{3}$.
17. (a) 6; (b) 4; (c) -2; (d) 0; (e) does not exist; (f) $\frac{1}{4}$.
19. (a) 5; (b) $\frac{1}{2}$; (c) 0; (d) 1; (e) 1; (f) $\frac{1}{3}$; (g) $\frac{2}{3}$.
23. See Fig. 2.20.
25. (a) Limit $= 1$; (b) $x \cong 0.4$.

Section 2.6, p. 79

1. (a) None; (b) $1, -1$; (c) 1; (d) all $x < 0$; (e) all $x \le 0$; (f) none; (g) 3, -4; (h) none. [Remember that a function is automatically discontinuous at every point not in its domain; thus, $1/x$ is discontinuous at $x = 0$ even though it is a continuous function.]
3. $\frac{3}{2}$.
5. 3.
7. $\frac{5}{4}$.
9. 1.
15. (a) $\frac{1}{2}$ at $x = \pi/6$; (b) $\frac{1}{2}\sqrt{2}$ at $x = \pi/4$; (c) 1 at $x = \pi/2$.
17. (a) Yes; (b) yes.
19. (a) Yes, at $x = 0$; (b) yes, at $x = 0$; (c) no; (d) yes, at $x = 0$.
21. (a) No; (b) yes, at $x = 0$.
23. No maximum, minimum $= 1$.
25. Maximum $= 1$, no minimum.
27. Maximum $= -3$, minimum $= -8$.
29. No maximum, minimum $= 2$.

Additional Problems, p. 81

1. $b = -6$.
5. (b) Drop the perpendicular from P to a point A on the axis of the parabola. Draw the circle whose center is the vertex V and which passes through A. Let B be the second point at which this circle intersects the axis, and draw the line PB. This line will be tangent to the parabola at P.
7. (a) $x = 0$; (b) $x = \pm 2$; (c) $x = \frac{3}{2}$; (d) differentiable at all points.
13. $m = 2a$, $b = -a^2$.
15. When $t = \frac{3}{4}$; 8 ft/s.
19. Does not exist.
21. -5.
23. Does not exist.
25. Does not exist.
27. 2.
29. 2.
31. -3.
33. $\frac{1}{7}$.
35. -5.
37. $\frac{1}{2}$.
39. 4.
41. $3a/2$.
43. 1.
45. 0.
47. Does not exist.
49. Does not exist.
51. 3.
53. 0.
55. 0.
57. 1.
59. $\lim_{x\to 0+} f(x)$, $\lim_{x\to 0-} f(x)$, and $\lim_{x\to 0} f(x)$ do not exist.
61. Because there are rationals as close as we please to every irrational, and irrationals as close as we please to every rational.
63. Slope $\cong 0.693$.

CHAPTER 3

Section 3.1, p. 87

1. (a) $54x^8$; (b) 0; (c) $-60x^3$; (d) $1500x^{99}(x^{400} + 1)$; (e) $2x - 6$; (f) $x^4 + x^3 + x^2 + x + 1$; (g) $4x^3 + 3x^2 + 2x + 1$; (h) $5x^4 - 40x^3 + 120x^2 - 160x + 80$; (i) $12x(x^{10} + x^4 - x - 1)$; (j) $18x^2 - 6x + 4$.
3. (a) $v = -6 + 6t$, $a = 6$; (b) $v = -9 + 18t^2$, $a = 36t$; (c) $v = 18t - 12$, $a = 18$.
5. $y = 7x - 10$.
7. $(1, -2)$ and $(-\frac{1}{3}, -\frac{22}{27})$.
11. $a = 1, 3$.
13. $\left(-\dfrac{b}{2a}, c - \dfrac{b^2}{4a}\right)$.
15. (a) $3ac < b^2$; (b) $3ac = b^2$; (c) $3ac > b^2$.
17. $y - a^3 = 3a^2(x - a)$; all $a \neq 0$.
19. $y = 12x - 16$ and $y = 3x + 2$.
23. $(-1, -2)$.

Section 3.2, p. 91

1. $2x$.
3. $15x^4 + 57x^2 + 6$.
5. $18x^2 + 2x - 1$.
7. $72x^5 + 20x^4 + 6x + 1$.
9. $\dfrac{-2}{(x-1)^2}$.
11. $\dfrac{4x^3 + 12x^2 - 1}{(x+2)^2}$.
13. $\dfrac{3 - 6x^2}{(1 + 2x^2)^2}$.
15. $\dfrac{-4x}{(1 + x^2)^2}$.
17. $\dfrac{-4x}{(x^2 - 1)^2}$.
19. $\dfrac{-3}{(2x - 3)^2}$.
21. $\dfrac{-x^2 + 2x + 3}{(x^2 + 2x + 1)^2}$.
23. $\dfrac{10x^3 - 36x^2 + 42x}{(5x - 7)^2}$.
25. $\dfrac{288x^{10} - 360x^5}{(24x^5 - 5)^2}$.
27. $\dfrac{-x^2 - 2x + 1}{x^2(x - 1)^2}$.
29. $\dfrac{-4}{x^2}$.
31. $\dfrac{2x^4 - 2}{x^3}$.
33. $\dfrac{-(x + 30)}{x^3}$.
35. $(3, 2)$ and $(-3, -2)$.
37. Two; $(-3 \pm \sqrt{5})/2$.
39. $4x + 5y = 13$, $5x - 4y = 6$.
41. $2y = x + 2$.
43. $x - 3y = 2$, $3x + y = 6$.
45. Area $= 1$.
47. $(0, 2)$, $(\pm 1, 1)$.

Section 3.3, p. 97

1. $\dfrac{10}{(2 - 5x)^3}$.
3. $6(x + 2)(x^2 + 4x - 1)^2$.
5. $(13 - 8x)(5 - x)^2(4 + x)^4$.
7. $\dfrac{-12}{(3x + 1)^5}$.
9. $-36(1 - 6x)^5$.
11. $\dfrac{12(x^3 - 1)^3}{x^{13}}$.
13. $16(2x + 1)^3[1 + (2x + 1)^4]$.
15. $\dfrac{-5(x^5 - 5)(x^5 - 1)^3}{x^{26}}$.
17. $4(x^5 - 3x)^3 \cdot (5x^4 - 3)$.
19. $6(x + x^2 - 2x^5)^5 \cdot (1 + 2x - 10x^4)$.
21. $\dfrac{4x}{(12 - x^2)^3}$.
23. $7(x^2 + 3x - 5)^6 \cdot (2x + 3)$.
25. $-6(3x^2 - 5x + 2)^{-7} \cdot (6x - 5)$.
27. $4(5x + 3)^3(4x - 3)^6(55x + 6)$.
29. $\dfrac{2x(x^2 + 9)}{(9 - x^2)^3}$.
31. $2(2x - 3)^7(3x^2 - x + 2)^9(84x^2 - 108x + 31)$.
33. $\dfrac{-2(2t - 1)^2(t^2 - 2t - 9)}{(t^2 + 3)^3}$.
35. $\dfrac{72}{(5 - 4t)^4}$.
37. $5(2x^2 + 5x - 3)^4(4x + 5)$.
39. $\dfrac{20(3x + 1)^3}{(1 - 2x)^5}$.
41. $50x - 55$.
43. $y = 16x - 15$.
45. (a) $3u^2 \dfrac{du}{dx}$; (b) $4(2u - 1)\dfrac{du}{dx}$;

(c) $4u(u^2-2)\dfrac{du}{dx}$.

Section 3.4, p. 101

1. $5\cos(5x-2)$.
3. $-\cos(\cos x)\cdot\sin x$.
5. $3\sin^2 x\cdot\cos x$.
7. $\dfrac{1}{1+\cos x}$.
9. $3x^2\cos x^3$.
11. $5\sec^2 5x$.
13. $\sec^2(\sin x)\cdot\cos x$.
15. $8x(1+\tan^2 x^2)\tan x^2\cdot\sec^2 x^2$.
17. $15(\cos 3x+\sin 5x)$.
19. $-45(5x-3)^2\sin 3(5x-3)^3$.
21. $\dfrac{\cos x}{(1-\sin x)^2}$.
23. $\dfrac{3}{1+\cos 3x}$.
25. $\dfrac{-x\sin x-\cos x}{x^2}$.
27. $3x^2\sin\dfrac{1}{x^2}-2\cos\dfrac{1}{x^2}$.
29. $6\sin x\cos x\,(2-\cos^2 x)^2$.
31. $\cos(\tan x)\cdot\sec^2 x$.
35. $2n\pi+\dfrac{7\pi}{6}$ or $2n\pi-\dfrac{\pi}{6}$, n an integer.
37. (a) The particle starts at $s=A$ when $t=0$, moves to $s=-A$ when $t=\pi/k$, and moves back to $s=A$ when $t=2\pi/k$. This oscillatory motion continues with period $2\pi/k$. (b) $v=-Ak\sin kt$. (c) $v=0$ when $x=\pm A$ and $|v|$ has its largest value Ak when $s=0$. (d) $a=-Ak^2\cos kt=-k^2 s$.

Section 3.5, p. 107

1. $-\dfrac{3x^2}{4y^2}$.
3. $\dfrac{1}{1-7y^6}$.
5. $\dfrac{3x^2-4y}{3y^2+4x}$.
7. $-\sqrt{\dfrac{y}{x}}$.
9. $-\sqrt[3]{\dfrac{y}{x}}$.
11. $\dfrac{2}{3x^2}$.
13. $\dfrac{\pm 3}{2\sqrt{3x-1}}$.
15. $\dfrac{-2}{(1+x)^2}$.
17. $\dfrac{\pm 9x}{2\sqrt{36-9x^2}}$.
19. $\frac{4}{5}x^{-1/5}$.
21. $-\frac{3}{4}x^{-7/4}$.
23. $\frac{6}{5}x^{-3/5}$.
25. $\dfrac{3(x^3-16)}{4x^3}\sqrt[4]{\dfrac{x^2}{x^3+8}}$.
27. $-\dfrac{9}{2}\dfrac{(x+2)^{1/2}}{(x-1)^{5/2}}$.
29. (a) $x+4y=7$; (b) $x+2y=4$; (c) $x+3y=0$; (d) $x+3y=19$.
33. (a) $2x^{3/2}$; (b) $2x^{5/2}$.
35. $(\sqrt[3]{2},\sqrt[3]{4})$.
37. $\dfrac{\cos x}{3y^2+2y}$.
39. $\dfrac{y}{\cos y-x}$.
41. $\dfrac{\cos\sqrt{x}}{2\sqrt{x}}$.
43. $\dfrac{\tan\sqrt{x}\sec^2\sqrt{x}}{\sqrt{x}}$.
45. $\dfrac{5\sin 2x}{\sqrt{6-5\cos 2x}}$.
47. $-\dfrac{3\sec^2(3x-1)^{-1/2}}{2(3x-1)^{3/2}}$.

Section 3.6, p. 110

1. (a) 8, 0, 0, 0; (b) $16x-11$, 16, 0, 0; (c) $24x^2+14x-1$, $48x+14$, 48, 0; (d) $4x^3-39x^2+10x+3$, $12x^2-78x+10$, $24x-78$, 24; (e) $\frac{5}{2}x^{3/2}$, $\frac{15}{4}x^{1/2}$, $\frac{15}{8}x^{-1/2}$, $-\frac{15}{16}x^{-3/2}$.
3. (a) $n!(1-x)^{-(n+1)}$; (b) $(-1)^n n!\,3^n(1+3x)^{-(n+1)}$; (c) $(-1)^{n+1}n!(1+x)^{-(n+1)}$.
5. $-\dfrac{(n-1)a^n x^{n-2}}{y^{2n-1}}$.
7. (a) $t=\frac{1}{2}$, $s=0$, $v=12$; (b) $t=4$, $s=32$, $v=6$; (c) $t=1$, $s=6$, $v=-3$.
9. $3,\frac{1}{3}$.
13. (a) $\sin x,\cos x$; (b) $-\sin x,-\cos x$; (c) $\sin x,\cos x$; (d) $-\cos x,\sin x$.

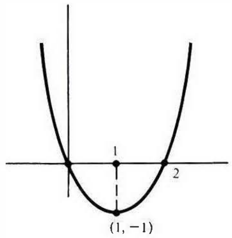
Additional Problems, p. 111

1. $(-1, 10)$ and $(3, -22)$.
3. $(1, 2)$ and $(-1, -2)$; the smallest slope = 1, at $(0, 0)$.
5. Slope $=4x^3-4x$; $x=0,\pm 1$; $-1<x<0$, $x>1$.
7. $a=1$, $b=1$, $c=0$.
9. $a=1$, $b=0$, $c=-1$.
13. $a=1$, $b=-2$, $c=2$, $d=-1$.
17. $(6, 9)$, $(-2, 1)$, $(-4, 4)$.
19. (a) $\dfrac{-4x}{(x^2-1)^2}$; (b) $\dfrac{-4(x+1)}{(x-1)^3}$; (c) $\dfrac{x(4-x^3)}{(x^3+2)^2}$; (d) $\dfrac{-2x^2-6x-11}{(x^2+x-4)^2}$; (e) $\dfrac{x^2(3-x^2)}{(1-x^2)^2}$; (f) $\dfrac{-2}{(1+x)^2}$; (g) $\dfrac{18x^4-24x^3-9}{(x-1)^2}$; (h) $\dfrac{-10(x+3)}{(x-2)^3}$.
23. (a) $(x+2)(x+3)+(x+1)(x+3)+(x+1)(x+2)$; (b) $(x^3+3x^2)(x^4+4)\times(2x+2)+(x^2+2x)(x^4+4)\times(3x^2+6x)+(x^2+2x)(x^3+3x^2)\times(4x^3)$.
25. $(0, 10\sqrt{5})$, $(\pm 3, \sqrt{5})$.
27. $(2, -2)$ and $(-10, \frac{2}{5})$.
29. (a) $-6(1+2x)^2(4-5x)^5(15x+1)$; (b) $10x(x^2+1)^9(x^2-1)^{14}(5x^2+1)$; (c) $\dfrac{-2x(2x^2-19)}{(16+x^2)^4}$; (d) $-3x^6(3-2x)^2\times(4x-9)(32x^2-96x+63)$.
31. (a) $y=(x^4+1)^3$; (b) $y=2(x^6+1)^6$.
33. (a) $3\sin(1-3x)$; (b) $-7x^6\cos(1-x^7)$; (c) $\sin(\cos x)\cdot\sin x$; (d) $\sin[\sin(\cos x)]\cdot\cos(\cos x)\cdot\sin x$; (e) $-4\cos^3 x\cdot\sin x$; (f) $90x(1-3x^2)^2\cos^4(1-3x^2)^3\cdot\sin(1-3x^2)^3$; (g) $\dfrac{1}{1-\sin x}$; (h) $15\sin^4 3x\cdot\cos 3x$; (i) $-4x^3\sin x^4$; (j) $15\sin x\cdot\cos^4 x\cdot(1-\cos^5 x)^2$; (k) $-3\sec^2(1-3x)$; (l) $-24x^2\tan^3(1-2x^3)\cdot\sec^2(1-2x^3)$; (m) $-\sin(\tan x)\cdot\sec^2 x$; (n) $-\cos[\cos(\tan x)]\cdot\sin(\tan x)\cdot\sec^2 x$; (o) $20x^4\tan^3 x^5\cdot\sec^2 x^5$.
35. $2n\pi\pm\dfrac{\pi}{2}$, n an integer.
39. (a) $y\to 0$ as $x\to\pm\infty$; (c) $2x\sin\dfrac{1}{x}-\cos\dfrac{1}{x}$; (d) 0.
41. (a) $\dfrac{6x+5}{3(3x^2+5x-1)^{2/3}}$; (b) $\dfrac{2}{5x^{3/5}}$; (c) $1, -\frac{1}{4}$; (d) $-\dfrac{x^{1/2}}{(8-x^{3/2})^{1/3}}$.
43. (a) $y=10x-19$; (b) $x-4y+9=0$; (c) $12x-13y=11$; (d) $y=-2x-15$.
47. (a) $-2(1+3x)^{-5/3}$; (b) $-\dfrac{x+4}{4(x+1)^{5/2}}$; (c) $-\frac{4}{25}x^{-6/5}$; (d) $\frac{35}{4}x^{3/2}$; (e) $-\frac{1}{4}x^{-3/2}+\frac{3}{4}x^{-5/2}$; (f) $20(x^2+1)(x^2+4)^{1/2}$.
53. (a) $-20\cos x+x\sin x$; (b) $-3^{62}\sin 3x$.

CHAPTER 4

Section 4.1, p. 119

1.



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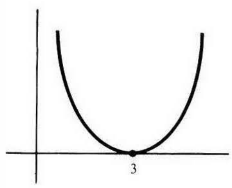

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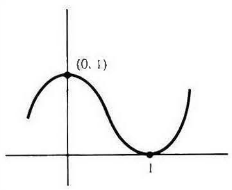

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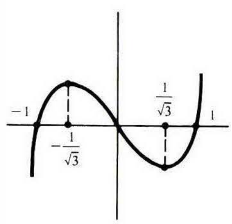

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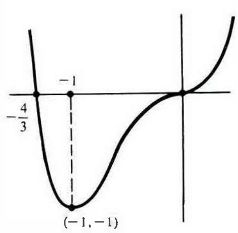

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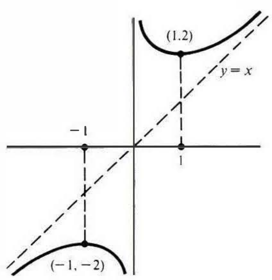

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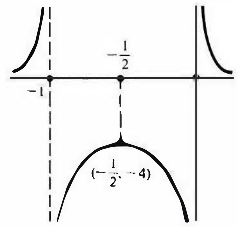

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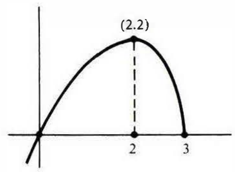

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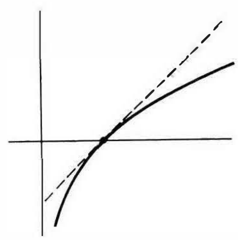

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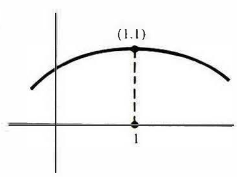

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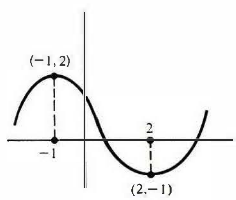

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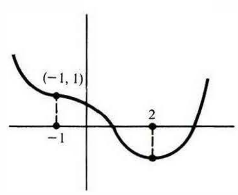

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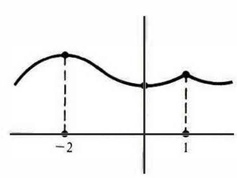


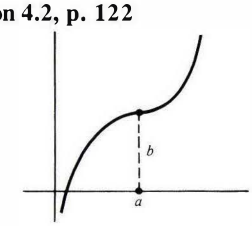
23. $(0, 3)$, $(\pi, -1)$, $(2\pi, 3)$ and $(\frac{2}{3}\pi, -\frac{3}{2})$, $(\frac{4}{3}\pi, -\frac{3}{2})$.

27. max $= \frac{3}{2}\sqrt{3}$ at $x = \dfrac{\pi}{6}$; min $= -\frac{3}{2}\sqrt{3}$ at $x = \dfrac{5\pi}{6}$.

29. 125.

Section 4.2, p. 122

1.



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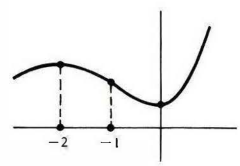

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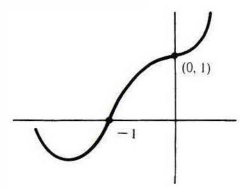

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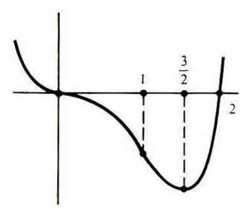

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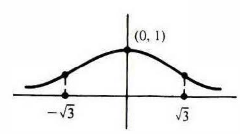

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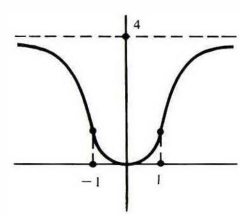

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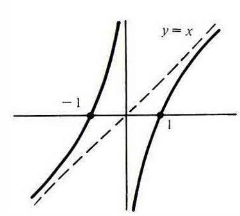

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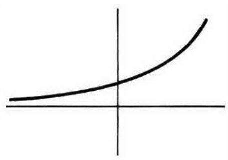

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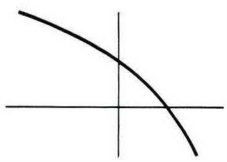
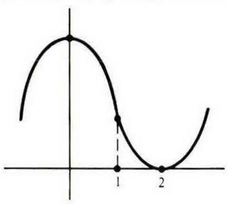


17. (2, 0). 19. $a = 3$.

23. (a) $a > 0$; (b) $a < 0$.

25. (a)



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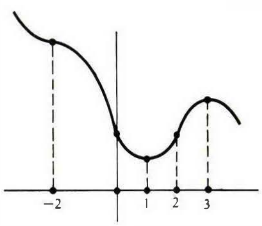

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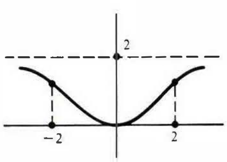

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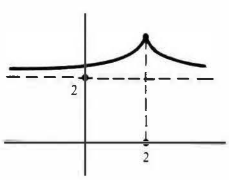


27. (a) PI $x = \pi$, CU $0 < x < \pi$, CD $\pi < x < 2\pi$; (b) PI $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$, CU $0 < x < \frac{\pi}{4}$, CD $\frac{\pi}{4} < x < \frac{3\pi}{4}$, CU $\frac{3\pi}{4} < x < \frac{5\pi}{4}$, CD $\frac{5\pi}{4} < x < \frac{7\pi}{4}$, CU $\frac{7\pi}{4} < x < 2\pi$; (c) PI $x = \pi$, CD $0 < x < \pi$, CU $\pi < x < 2\pi$.

Section 4.3, p. 129

1. $\frac{1}{2}$. 7. a/b.
9. \$8.50. 11. 2 P.M.; 30 mi.
13. 4, 4. 15. 108.
17. 4 by 8 in. 19. 1.
21. 24 in. 23. $\sqrt{3}$.
25. $\frac{3}{2}a$. 27. 1.
29. $(a^{2/3} + b^{2/3})^{3/2}$ (convince yourself that this and the preceding problem are essentially the same).
31. $v_{\min} = \sqrt[4]{\frac{4g\sigma}{\delta}}$, $\lambda = 2\pi\sqrt{\frac{\sigma}{g\delta}}$.
33. $\frac{1}{2}(B + H)^2$.

Section 4.4, p. 137

1. 1. 3. $\frac{2}{3}R$.
5. 4 by 4 in. 7. $\frac{1}{2}$.
9. (1, 1).
11. (a) $\frac{3}{2}$ mi; (b) 1 h and 44 min; (c) 8 min longer.
13. $15/\sqrt{10}$ mi/h.
19. 1. 21. $a = 2$.
23. $x = \frac{1}{2}\sqrt{2}\,a$. 25. (a) 0; (b) 1.
27. (2, 4). 29. $A_{\max} = \frac{3}{4}\sqrt{3}$.
31. The spider should walk straight to the midpoint of a side not containing S or B, then straight on to B.

Section 4.5, p. 142

1. (a) 120π ft^2/s; (b) 240π ft^2/s.
3. $2/\pi$ ft/min.
5. 4 ft/s at each of the stated moments.
7. 3 ft/s. 9. $4\frac{1}{3}$ ft/s.
13. 52 mi/h.
15. $\frac{1}{3}$ lb/in^2 per min.
17. (a) $\dfrac{1}{\pi}$ in/min; (b) $\dfrac{1}{2\sqrt[3]{2}\,\pi}$ in/min.
19. $\dfrac{1}{5\pi}$ in/s.
21. $2/(\sqrt[3]{2} - 1) \cong 7.69$ h.
23. $40\sqrt{2}$ ft/min.

Section 4.6, p. 146

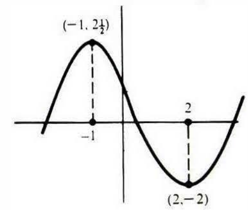
3. 0.618034. 5. 2.154435.
7. 1.305407 ft.
11. 0.918247 and 2.863580.
13. (a) 1.236123; (b) 0.876726.

Section 4.7, p. 155

1. \$43/unit; \$43.03.
3. \$80/unit; \$80.14.
5. \$193.89; $x = 89$.
7. \$0.85; $x = 232$.
9. \$0.31; $x = 93$. 11. 200.
13. 5000. 15. 345.
17. (a) $p = 110 - \frac{1}{20}x$; (b) \$55.
19. \$160. 21. \$16.
23. \$573.33; $x = 20$.
25. 636.

Additional Problems, p. 156

1.



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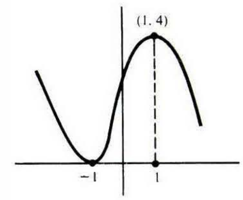

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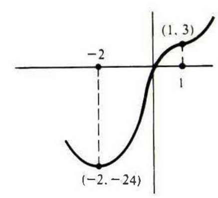

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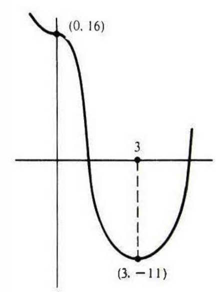

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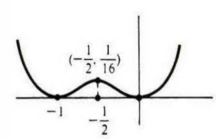

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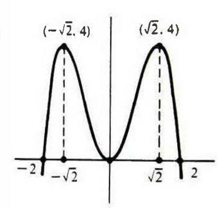
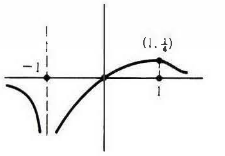
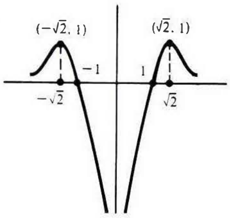
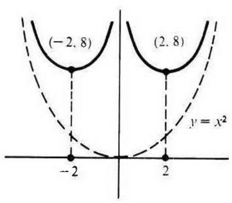
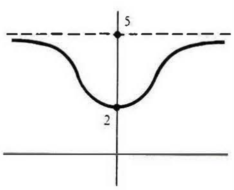
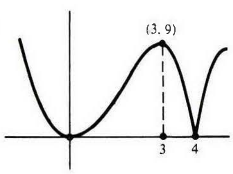


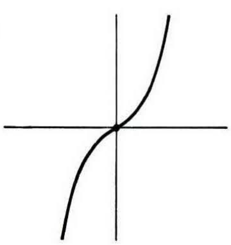
13. 

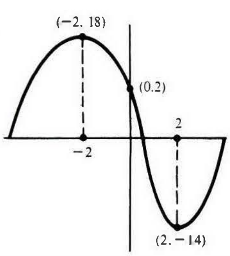
15. 

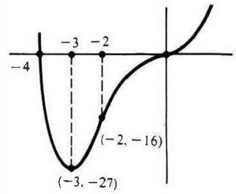
17. 

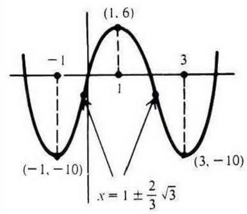
19. 

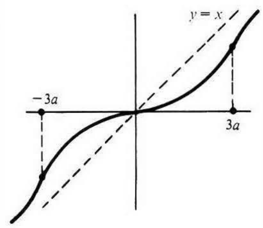
21. 

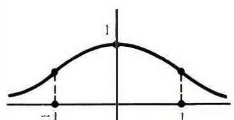
23. 

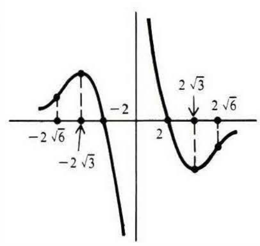
25. 

27. 

29. 

31. 

33. 

35. 

37. (a) Point of inflection at $x = 1$; (b) points of inflection at $x = 1, 2$; (c) points of inflection at $x = -2, 0, 1, 2, 3$.
39. $a = -3$. 41. $\frac{1}{3}\sqrt{3}$.
43. $x = 21, y = 35$.
45. $18 = 16 + 2$. 47. 5, 5, 5.
51. $\frac{1}{3}\sqrt{3}\, a, \frac{2}{3}b$. 59. 2 in.
63. 4000 knives at a price of \$18 apiece.
65. 20 days.
67. (a) 120 ft; (b) 312 ft.
69. $\frac{1}{4}$. 71. $\sqrt{2}$.
73. $\pi/4$. 75. 4.
77. $4/\pi$.
79. A square with side $\frac{1}{6}(a + b - \sqrt{a^2 - ab + b^2})$.
81. $\sqrt{2}$.
83. 3 by 6 by 12 inches.
85. $a - \dfrac{bs}{\sqrt{r^2 - s^2}}$ m, if this number is positive.
87. $x^2 + y^2 = 32$.
89. (3, 3).
91. (5, 0) and (−5, 0).
99. (a) 12 ft/s; (b) 3 ft/s.
101. $1/\pi$ ft/min. 103. At least 9 ft.
107. $\dfrac{dy}{dt} = \dfrac{ax}{\sqrt{x^2 + r^2}}$ in/s.
109. 0.32 lb/min.
111. 144π m^3/min.
115. Decreasing 1 in^2/min.
117. When $t = \dfrac{R_0\sqrt{b} - r_0\sqrt{a}}{a\sqrt{a} - b\sqrt{b}}$.
119. (a) 3.316625; (b) 1.903778; (c) 2.087798.
123. 1.856636 in approximately.
125. \$42. 127. 30.

CHAPTER 5

Section 5.2, p. 169

1. $(63x^8 - 15x^4)\, dx$.
3. $\dfrac{(2x - 3x^3)\, dx}{\sqrt{1 - x^2}}$.
5. $\dfrac{(2 - x)\, dx}{\sqrt{4x - x^2}}$.
7. $(2x^{-1/3} + 2x^{-4/5} - 17)\, dx$.
9. $\dfrac{(15x^2 + 8x)\, dx}{2\sqrt{3x + 2}}$.
11. $\dfrac{dy}{dx} = \dfrac{-15(3u^2 - 2u + 1)x^2(x^3 + 2)^4}{(u^2 - u)^2}$.
13. $\Delta A = 2\pi r\, \Delta r + \pi\, \Delta r^2$ and $dA = 2\pi r\, dr = 2\pi r\, \Delta r$, since $dr = \Delta r$. Imagine that the thin circular ring is cut across and unrolled—with slight distortions—into a long thin rectangle with length $2\pi r$ and width Δr.
15. 12, 12.048064.
17. 16.167, 16.166236.
19. 26.75, 26.749612.
21. 6.019, 6.018462.

23. 0.849, 0.848048.
25. 1.037 mi. 27. 125.66 ft.

Section 5.3, p. 177

1. $\frac{1}{2}x^2 + x + c$.
3. $\frac{1}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{5}x^5 + c$.
5. $2\sqrt{x} + c$. 7. $\frac{4}{7}x^{7/4} + c$.
9. $\frac{3}{2}x^{2/3} + c$.
11. $\frac{2}{3}x^{3/2} - 2\sqrt{x} + c$.
13. $6\sqrt{x} + \frac{4}{3}x^{3/2} + c$.
15. $\frac{1}{3}x^3 + \frac{1}{6}x^6 + c$.
17. $7x + \frac{1}{2}x^2 + c$.
19 $\frac{2}{3}x^6 + 3x^2 - 5x + c$.
21. $-2x^{-2} + c$.
23. $\frac{2}{3}x^{3/2} - 4x^{7/2} - 3x^{-1} + c$.
25. $9x^{1/3} - 16x^{1/4} + c$.
27. $2\sqrt{x} - \frac{1}{10}x^{10/3} + c$.
29. $x^4 - 4x^2 + 17x + c$.
31. $\frac{3}{10}x^{10/3} + \frac{12}{7}x^{7/3} + 3x^{4/3} + c$.
33. $\frac{8}{3}x^{3/2} - \frac{24}{7}x^{7/2} + \frac{18}{11}x^{11/2} + c$.
35. $\frac{1}{5}x^{500} + c$.
37. $\frac{1}{6}(3 + 4x)^{3/2} + c$.
39. $-\frac{1}{2}(2x - 3)^{-1} + c$.
41. $-\frac{1}{4}\sqrt{5 - 4x^2} + c$.
43. $\frac{8}{5}(1 + \sqrt{x})^{5/4} + c$.
45. $\frac{1}{3}(1 + x^2)^{3/2} + c$.
47. $\frac{1}{6}(7 - x)^{-6} + c$.
49. $-\frac{1}{3}(2 - x^2)^{3/2} + c$.
51. $2\sqrt{x^3 - 5} + c$.
53. $\frac{1}{110}(10x + 10)^{11} + c$.
55. $\frac{1}{15}(3x^2 + 4)^{5/2} + c$.
57. $4\sqrt{3x^3 - x + 2} + c$.
59. $\frac{1}{5}x^{15} + c$. 61. $y = x^3 + 2$.
63. (a) $\frac{1}{2}\sin 2x + c$; (b) $-\frac{1}{5}\cos 5x + c$; (c) $2\sin 2x - 3\cos 5x + c$; (d) $-\frac{1}{2}\cos 2x + \frac{1}{5}\sin 5x + c$.
65. (b) $\frac{1}{2}x - \frac{1}{4}\sin 2x + c$.
67. (a) $\frac{1}{5}\sin^5 x + c$; (b) $-\frac{1}{6}\cos^6 x + c$; (c) $\sin(\sin x) + c$.
69. (a) $\frac{1}{2}\sin^2 x + c$; (b) $-\frac{1}{2}\cos^2 x + c$. They differ by a constant.

Section 5.4, p. 181

1. $y = 2x^3 + 2x^2 - 5x + c$.
3. $y = 6x^4 + 6x^3 - 4x^2 + 3x + c$.
5. $3y^2 - 4y^{3/2} = 3x^2 + 4x^{3/2} + c$.
7. $y = -\dfrac{1}{x} + \dfrac{1}{2}x^2 + c$.
9. $y = \dfrac{1}{5 - x^2}$.
11. $y = \sqrt{\dfrac{2x^2 - 31}{33 - 2x^2}}$.
13. $3\sqrt{y} = x\sqrt{x} - 3$.

Section 5.5, p. 187

1. 8 s; velocity = -128 ft/s and speed = 128 ft/s.
3. $v = -32t + 128$; $s = -16t^2 + 128t$.
5. $10\sqrt{10}$ s. 7. 40 ft/s.
9. 96 ft/s.
11. $v_0^2/64$ ft; 96 ft/s.

Additional Problems, p. 188

1. $\frac{3}{5}x^5 - \frac{7}{4}x^4 + 10x + c$.
3. $\frac{1}{3}x^3 - \frac{3}{2}x^2 + x - 4\sqrt{x} + c$.
5. $\frac{1}{4}x^4 + \frac{2}{3}x^3 + \frac{1}{2}x^2 + c$.
7. $17x^3 - 27x^4 + c$.
9. $6x - \frac{2}{3}x^{3/2} - \frac{1}{2}x^2 + c$.
11. $-\frac{2}{9}(2 - 3x)^{3/2} + c$.
13. $\frac{1}{825}(5x + 2)^{165} + c$.
15. $5\sqrt{1 + x^2} + c$.
17. $\frac{1}{3}\sqrt{2x^3 - 1} + c$.
19. $\frac{3}{4}(x^2 - 2x + 3)^{2/3} + c$.
21. $-\frac{3}{2}\sqrt[3]{2 - x^2} + c$.
23. $-\dfrac{1}{3}\left(1 + \dfrac{1}{x}\right)^3 + c$.
25. $\frac{3}{7}(x + 1)^{7/3} + c$.
27. $\frac{3}{7}(1 + x)^{7/3} - \frac{3}{4}(1 + x)^{4/3} + c$.
29. $\frac{2}{11}(x^3 + x + 32)^{11/2} + c$.
31. $\frac{1}{28}(x^3 - 1)^{4/3}(4x^3 + 3) + c$.
33. (a) $y = \sqrt{\dfrac{7x^2 - 3}{3x^2 + 13}}$; (b) $3\sqrt{y - 4} = (x - 1)^{3/2} - 2$.
35. $x^2 - y^2 = c$.
39. (a) 25 s, 1200 ft/s; (b) $25\sqrt{2} \cong 35$ s, $800\sqrt{2} \cong 1120$ ft/s.
41. 30 m/s.
43. (a) 44 ft; (b) 680 ft.
45. About 1.86 mi. About 0.36 in.
47. About 179,427 mi/s.

CHAPTER 6

Section 6.3, p. 196

1. (a) 55; (b) 62; (c) 206; (d) 0; (e) 0; (f) 1500; (g) 7.
5. (a) $\dfrac{(n-1)n}{2}$; (b) $\dfrac{(n-1)n(2n-1)}{6}$; (c) $\left[\dfrac{(n-1)n}{2}\right]^2$.

Section 6.6, p. 212

1. 9. 3. $\frac{32}{3}$.
5. 12. 7. $\frac{81}{4}$.
9. $48\sqrt{2}/5$. 11. $\frac{35}{3}$.
13. $\frac{21}{2}$. 15. $\frac{26}{3}$.
17. 33. 19. $\frac{26}{3}$.
21. $\frac{4}{9}$. 23. $\frac{2}{3}$.
25. 1. 27. $\frac{13}{3}$.
29. $\frac{5}{48}a^{-2}$. 31. $\frac{1}{6}$.
33. $a^4/4$. 35. $b^2/6$.
37. $\frac{1}{30}$. 39. $\frac{1}{2}$.
41. 3.

Section 6.7, p. 216

1. (a) $\frac{31}{6}$; (b) $\frac{22}{3}$; (c) 19; (d) $\frac{13}{2}$.
3. $\frac{128}{5}$. 13. $\pi a^2/2$.

Additional Problems, p. 217

7. (a) $\frac{8}{3}$; (b) $\frac{38}{3}$; (c) 12; (d) $5\sqrt{5}/3$; (e) $\frac{9}{20}$.
9. (a) $4(2\sqrt{2} - 1)$; (b) $\frac{8}{3}$; (c) 3; (d) $\frac{35}{4}$.
11. (a) $\dfrac{4x^3}{1 + x^4}$; (b) $\dfrac{2x}{1 + x^2}$; (c) $\dfrac{3x^2}{\sqrt{3x^3 + 7}}$; (d) $\dfrac{5x^9}{\sqrt{1 + x^{10}}}$.

CHAPTER 7

Section 7.2, p. 224

1. $\frac{4}{3}$. 3. $\frac{8}{3}$.
5. $\frac{32}{3}$. 7. $\frac{64}{15}\sqrt{2}$.
9. 4. 11. 36.
13. $\frac{32}{3}$. 15. 8.
17. $\frac{128}{3}\sqrt{2}$. 19. $\frac{320}{3}$.
21. $\frac{64}{3}$. 23. (a) $\frac{1}{12}$; (b) $\frac{4}{3}$.
25. $\frac{27}{4}$.
27. $2(\sqrt{b} - 1) \to \infty$ as $b \to \infty$.
29. (a) $2(\sqrt{2} - 1)$; (b) $\frac{1}{2}(3\sqrt{3} - \sqrt{2} - 3)$; (c) $\dfrac{5\pi^2}{32} + \dfrac{1}{2}\sqrt{2} - 2$.
33. $\dfrac{4\sqrt{2} - 5}{3}a^2 \cong (0.21895)a^2$.

Section 7.3, p. 229

1. (a) 8π; (b) $16\pi/15$; (c) $3\pi/5$; (d) $2\pi/3$; (e) $8\pi/3$; (f) $16\pi a^3/105$.
5. 4π. 7. $\frac{4}{3}a^3$.
9. $\frac{16}{3}a^3$. 11. $2\pi^2a^2b$.
13. $\frac{1}{6}\sqrt{3}\,a^3$. 15. $9\sqrt{2}/2$.
17. $\frac{16}{3}a^3$. 19. $5\pi x^2$ in^3.
21. (a) The volumes are $\int_0^H A(x)\,dx$ and $\int_0^H B(x)\,dx$, which are equal because $A(x) = B(x)$ for every x.

23. $\frac{1}{2}\pi a^2h$. 25. $y = ax^4$.

Section 7.4, p. 235

3. $128\pi/5$. 5. $486\pi/5$.
7. $2\pi(b-a)$. 9. $\frac{8\pi}{3}(2-\sqrt{2})$.
11. (a) 8π; (b) $256\pi/15$.
13. $\pi h^3/6$. 15. $8\pi/9$.
17. By a factor of 1.71, approximately.

Section 7.5, p. 240

1. $\frac{8}{27}(10\sqrt{10}-1)$.
3. $\frac{53}{6}$. 5. $\frac{118}{9}$.
7. 12.

Section 7.6, p. 244

1. $253\pi/20$. 3. 12π.
5. $\frac{\pi}{27}(10\sqrt{10}-1)$.
7. $\frac{8\pi}{3}(2\sqrt{2}-1)p^2$.
9. $\frac{12}{5}\pi a^2$. 11. $\left(0, \frac{2}{\pi}a\right)$.
13. (a) $\bar{y} = \frac{2}{\pi}a$; (b) area $= 4\pi^2ab$.

Section 7.7, p. 249

1. 64 ft-lb. 3. 6000 ft-lb.
5. 550 ft-lb. 7. 50,000 ft-lb.
11. (b) 250 ft-lb. 13. 5 ft-lb.
15. $GMm/2a$. 17. $mgRh/(R+h)$.
19. $240\pi w$ ft-lb. 21. $\frac{4}{3}\pi a^3w(h+a)$.
23. About 118,500.
25. $m_1 = \frac{1}{4}m_2$.

Section 7.8, p. 254

1. 150 tons. 3. 125 tons.
5. $\frac{5}{6}$ ton. 7. $\frac{7}{6}$ tons.
9. 300π lb.
11. $10\frac{5}{16}$ tons; 11 ft.
13. $300\sqrt{2}w$ or approximately 13.2 tons.

Additional Problems, p. 254

1. $\frac{1}{6}$. 3. $\frac{128}{15}$.
5. 36. 7. $\frac{125}{6}$.
9. 18. 11. $\frac{8}{3}$.
13. 64. 15. $\frac{37}{6}$.
17. $\frac{136}{3}$.
19. (a) $56\pi/15$; (b) $56\pi/15$; (c) $32\pi/3$; (d) $48\pi/5$; (e) $\pi a^3/15$.
21. (a) $512\pi/15$; (b) $128\pi/3$.
23. $\frac{8}{3}a^3$. 25. $\frac{1}{5}\pi(b^5-a^5)$.
27. (a) $2\pi a^3$; (b) $\frac{8}{5}\pi a^3$; (c) $\frac{16}{15}\pi a^3$.
29. a^2h. 31. $\frac{4}{3}a^2h$.
33. $128\pi/3$. 35. $11\pi/15$.
37. $135\pi/2$. 39. 2π.
43. $\frac{14}{3}$. 45. $\frac{14}{3}$.
47. $\frac{17}{6}$. 49. $\frac{495}{8}$.
51. $AB = \frac{1}{32}$. 53. 90π.
55. 168π. 57. $\pi a^2/2$.
59. 7 ft-lb. 61. 2325 ft-lb.
63. $945 \cdot 2^{13}$ ft-lb.
65. $\frac{67}{324}aB$.
67. $\frac{21}{20}c$, where c is the constant of proportionality.
69. $125\pi w/6$. 71. $\frac{10}{3}\cdot 8^3\pi w$ ft-lb.
73. 10 tons. 75. $\frac{1}{2}$ ton.
77. 187.5 tons. 79. 60 tons.
81. 4096 lb.

CHAPTER 8

Section 8.2, p. 263

1. (a) $\log_4 16 = 2$; (b) $\log_3 81 = 4$; (c) $\log_{81} 9 = 0.5$; (d) $\log_{32} 16 = \frac{4}{5}$.
3. (a) 4; (b) 6; (c) -4; (d) $\frac{2}{3}$.
5. (a) $a = 32$; (b) $a = \frac{1}{16}$; (c) $a = 6$; (d) $a = 49$.
9. (a) 7; (b) acidic pH < 7, basic pH > 7.

Section 8.3, p. 269

1. $\frac{1}{2}(e^x - e^{-x})$. 3. $(x^2+2x)e^x$.
5. $e^{e^x}e^x$. 7. xe^{ax}.
9. $4x^2e^{2x}$. 11. $\frac{1}{3}e^{3x}+c$.
13. $5e^{x/5}+c$. 15. $2e^{x^3}+c$.
17. (a) Max. pt. (0, 1), no min. pt., pts. of infl. $\left(\pm\frac{1}{2}\sqrt{2}, \frac{1}{\sqrt{e}}\right)$; (b) no max. pt., min. pt. $\left(-3, -\frac{3}{e}\right)$, pt. of infl. $\left(-6, -\frac{6}{e^2}\right)$.
19. $\frac{1}{2}(e^b - e^{-b})$.
23. Area $= 1 - e^{-b} \to 1$ as $b \to \infty$.
25. (a) e; (b) e; (c) e; (d) e^2; (e) $\sqrt{e}$.
29. 8%.

Section 8.4, p. 276

1. (a) 2; (b) 3; (c) $1/x$; (d) $1/x$; (e) $-x$; (f) $1/x$; (g) x; (h) $3x$; (i) 0; (j) $\frac{4}{3}$; (k) $\frac{4}{3}$; (l) 0; (m) x^3y^2; (n) 8; (o) $2e^3$; (p) x^2e^x.
3. (a) $\frac{y(1+2x)}{x(3y-1)}$; (b) $\frac{y(1+xy)}{x(1-xy)}$.
5. (a) $\frac{1}{3}\ln(3x+1)+c$; (b) $\frac{1}{6}\ln(3x^2+2)+c$; (c) $\frac{3}{2}x^2 + 2\ln x + c$; (d) $x + \ln x + c$; (e) $x - \ln(x+1)+c$; (f) $\frac{1}{2}\ln(x^2+1)+c$; (g) $-\frac{1}{4}\ln(3-2x^2)+c$; (h) $\ln x(x-1)+c$; (i) $\frac{1}{2}(\ln x)^2+c$; (j) $\ln(\ln x)+c$; (k) $2\ln(\sqrt{x}+1)+c$; (l) $\ln(e^x+e^{-x})+c$.
9. No max., no pt. of infl., min. $(3, 9-18\ln 3) \cong (3, -11)$.
11. $\pi\ln 4$. 15. min. $\left(\frac{1}{e}, -\frac{1}{e}\right)$.
17. $1/\sqrt{e}$.
19. (a) $y\left(1 + \frac{2x}{x^2-1} - \frac{3}{6x-2}\right)$; (b) $\frac{y}{5}\left(\frac{2x}{x^2+3} - \frac{1}{x+5}\right)$.
21. (a) $x^{x^x}x^x\left[\frac{1}{x} + (\ln x)(1+\ln x)\right]$; (b) $\sqrt[x]{x}\left(\frac{1-\ln x}{x^2}\right)$. Max $= \sqrt[e]{e}$.

Section 8.5, p. 282

1. (a) 2520; (b) 13.8.
3. 20.1. 5. 95.8 percent.
7. When $t = 6$. Can you solve this problem without calculation, by merely thinking about it?
9. $x = 10(\frac{4}{5})^t$. 11. 2 more hours.
13. (a) About 3330 years (1380 B.C.); (b) about 3850 years (1900 B.C.); (c) about 10,510 years; (d) about 7010 years.
15. $x \to A$ if $A < B$; $x \to B$ if $A > B$.

Section 8.6, p. 287

1. $\frac{1}{2}N_1$.
3. $x = \frac{x_0x_1}{x_0 + (x_1 - x_0)e^{-cx_1t}}$.
5. $s = \frac{v_0}{c}(1 - e^{-ct})$.
7. When $v < 1$, the resisting force in the second case becomes very small.
9. About 53.4 lb.

Additional Problems, p. 288

1. $-xe^{\sqrt{1-x^2}}/\sqrt{1-x^2}$.
3. $(2x-2)e^{x^2-2x+1}$.
5. $\frac{e^{\sqrt{x}}}{2\sqrt{x}} + \frac{1}{2}\sqrt{e^x}$.
7. $-\frac{1}{3}e^{-3x}+c$. 9. $-e^{1/x}+c$.
11. $2\sqrt{e^x+1}+c$.

13. $(1/a, e)$. 17. $\frac{\pi}{2}(e^6 - 1)$.
23. (a) $\dfrac{y(3x+1)}{x(2y^2-1)}$; (b) $\dfrac{y(2x^2+1)}{x(1-3y)}$.
25. (a) $\frac{1}{2}\ln(1+2x)+c$; (b) $-\frac{1}{3}\ln(1-3x)+c$; (c) $\frac{1}{3}\ln 2$; (d) $\frac{1}{2}\ln 10$; (e) $\ln 3$; (f) $2\sqrt{\ln x}+c$; (g) $\frac{1}{2}\ln 7$; (h) $-\frac{1}{2}\ln(1-x^2)+c$; (i) $\frac{1}{2}(\ln 3)^2$; (j) $\frac{1}{3}\ln(3x^2-3x+7)+c$; (k) $\ln(e^x+1)+c$; (l) $\ln(x+1)(x+2)+c$; (m) $\frac{1}{3}(\ln x)^3+c$; (n) $\frac{1}{4}(\ln x)^2+c$; (o) $\frac{1}{2}[\ln(\ln x)]^2+c$; (p) $-\frac{1}{2}(\ln x)^2+c$.
29. $\dfrac{3}{2a}+\dfrac{a}{4}\ln 2$; $a=\sqrt{\dfrac{6}{\ln 2}}$.
31. (a) $(\ln 10)10^x$; (b) $(\ln 3)3^x$; (c) $(\ln \pi)\pi^x$; (d) $(3\ln 7)7^{3x}$; (e) $(\ln 6)(2x-2)6^{x^2-2x}$; (f) $\left(\dfrac{\ln 5}{2\sqrt{x}}\right)5^{\sqrt{x}}$.
33. Max. at $x=\dfrac{2}{\ln 5}$; pts. of infl. at $x=\dfrac{2\pm\sqrt{2}}{\ln 5}$.
35. (a) $(\ln x)^x\left[\dfrac{1}{\ln x}+\ln(\ln x)\right]$; (b) $(2\ln x)x^{\ln x-1}$; (c) $\dfrac{(\ln x)^{\ln x}}{x}\times[1+\ln(\ln x)]$; (d) $\dfrac{x^{\sqrt{x}}}{\sqrt{x}}(1+\frac{1}{2}\ln x)$; (e) $\dfrac{x^{\sqrt[3]{x}}}{x^{2/3}}(1+\frac{1}{3}\ln x)$.
37. In the year 3524, approximately.
39. In 4 more hours.
41. 73.12°F.
47. 17 more days.
49. $v^2=\dfrac{g}{c}(1-e^{-2cs})$; $v\to\sqrt{\dfrac{g}{c}}$ as $s\to\infty$.
51. When $t = 4.86$ min; when $t = 21.50$ min.
53. About 1.39 h.

CHAPTER 9

Section 9.1, p. 299

1. (a) $\pi/12$; (b) $7\pi/12$; (c) $2\pi/3$; (d) $5\pi/12$; (e) $5\pi/6$; (f) $3\pi/4$; (g) $5\pi/4$; (h) $7\pi/6$; (i) $7\pi/2$; (j) 5π.
3. $\theta = 2$ radians.
5. $A = 25\cot\frac{1}{2}\theta$.
7. $H = L\tan\theta$.
11. $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$, $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.
13. $\sin 4\theta = (4\sin\theta - 8\sin^3\theta)\cos\theta$.
15. (a) $\frac{1}{4}(\sqrt{6}-\sqrt{2})$; (b) $\frac{1}{2}\sqrt{2-\sqrt{3}}$. Show that these numbers are equal.
17. (a) $0, 2\pi$; (b) $\pi/2, 3\pi/2$; (c) π.
19. (a) $\theta=\dfrac{\pi}{4}+\dfrac{2\pi n}{3}$ or $\dfrac{\pi}{12}+\dfrac{2\pi n}{3}$ for all integers n; (b) $\theta=(1+2n)5\pi$ for all integers n; (c) $\theta=\dfrac{7\pi}{30}+\dfrac{2\pi n}{5}$ or $\dfrac{11\pi}{30}+\dfrac{2\pi n}{5}$ for all integers n.
35. For a proof by geometry, use the fact that the vertex opposite the fixed side must lie on a circle of which this side is a chord.
37. $\frac{3}{2}\sqrt{2}$ ft.

Section 9.2, p. 304

1. $3\cos(3x-2)$.
3. $48\cos 16x$. 5. $2x\cos x^2$.
7. $15(\cos 3x - \sin 5x)$.
9. $x\cos x + \sin x$.
11. $\frac{3}{2}\sin 12x$. 13. $\cos^5 x$.
15. $3e^{2x}\cos 3x + 2e^{2x}\sin 3x$.
17. $-\tan x$. 21. 45°.
23. 5.
31. $\pi/3$, tangent = 3.
33. 1. 35. 1.
37. $\frac{1}{2}$. 39. 1.
41. 1. 43. -1.

Section 9.3, p. 309

1. $-\frac{1}{5}\cos 5x + c$.
3. $\frac{1}{9}\cos(1-9x)+c$.
5. $\sin^2 x + c$ or $-\cos^2 x + c$ or $-\frac{1}{2}\cos 2x + c$.
7. $\frac{1}{8}\sin^4 2x + c$. 9. $\frac{1}{4}\sin^8\frac{1}{2}x + c$.
11. $-2\cos\sqrt{x}+c$.
13. $\frac{1}{2}\sin(\sin 2x)+c$.
15. $\frac{3}{2}\sec\left(\dfrac{2x-1}{3}\right)+c$.
17. $-\ln(\cos x)+c$.
19. $\sin(x^2+x)+c$.
21. $\frac{2}{5}$. 23. $\sqrt{2}-1$.
25. $\frac{2}{3}$. 27. 3.
29. $\frac{1}{2}\pi^2$.

Section 9.4, p. 311

1. $8x\sec^2 4x^2$.
3. $2\tan(\sin x)\cdot\sec^2(\sin x)\cdot\cos x$.
5. 0.
7. $24\csc(-6x)\cot(-6x)$.
9. $-\sqrt{\csc 2x}\cot 2x$.
11. $\sec^2 x\, e^{\tan x}$.
13. $-\frac{1}{6}\cot 6x + c$.
15. $-\frac{1}{2}\cot 2x + c$.
17. $\frac{1}{5}\tan^5 x + c$.
19. $-\frac{1}{7}\csc 7x + c$.
21. $\frac{1}{2}$. 23. $\frac{1}{2}(\pi-2)$.
25. 2. 27. $4\sqrt{3}$; no.
29. 3.2π mi/s. 31. (c) 66°.

Section 9.5, p. 318

1. $\frac{1}{2}\sqrt{3}, -\frac{1}{3}\sqrt{3}, -\sqrt{3}, \frac{2}{3}\sqrt{3}, -2$.
3. (a) π; (b) $\pi/2$; (c) 0.123; (d) 0.8; (e) 0.96; (f) $\pi/7$; (g) $\pi/6$; (h) $\pi/4$.
5. $1/(25+x^2)$.
7. $\dfrac{1}{\sqrt{x}(x+1)}$. 9. $\sin^{-1} x$.
11. $(\sin^{-1} x)^2$. 13. $\dfrac{4}{5+3\cos x}$.
15. $\pi/6$. 17. $\frac{1}{2}\sin^{-1} 2x + c$.
19. $\pi/8$. 21. $\frac{1}{2}\sin^{-1}\frac{2}{3}x + c$.
23. $\frac{1}{6}\tan^{-1}\frac{3}{2}x + c$.
25. $-\pi/12$. 27. $\pi/4$.
29. (a) $\sin^{-1}\frac{3}{5}$; (b) $\frac{1}{4}$ rad/s.
31. The formula is invalid, because the integrand $1/\sqrt{1-x^2}$ is discontinuous at the point $x = 1$ in the interval of integration.
33. $4\pi^2a^2$.

Section 9.6, p. 323

1. (a) $x=5\sqrt{2}\sin\left(t-\dfrac{\pi}{4}\right)$, $A=5\sqrt{2}$, $T=2\pi$; (b) $x=2\sin\left(3t+\dfrac{2\pi}{3}\right)$, $A=2$, $T=\dfrac{2\pi}{3}$; (c) $x=\sqrt{2}\sin\left(t+\dfrac{\pi}{4}\right)$, $A=\sqrt{2}$, $T=2\pi$; (d) $x=4\sin\left(2t-\dfrac{\pi}{6}\right)$, $A=4$, $T=\pi$.
3. $A=\dfrac{\sqrt{145}}{2}$ in; $T=\dfrac{\pi}{4}$.
5. $T=2\pi\sqrt{\dfrac{R}{g}}\cong 89$ min.
7. About 39 in.

Section 9.7, p. 329

1. (a) $\frac{3}{4}$; (b) $\frac{5}{3}$; (c) $\frac{40}{41}$.
13. $3x^2\cosh x^3$.
15. $6\,\mathrm{csch}\, 6x$. 17. 0.
19. $\frac{1}{5}\cosh(5x-3)+c$.
21. $2\sqrt{2}\sinh\frac{1}{2}x + c$.

23. $x - \tanh x + c$.

29. $\dfrac{1}{a^2}\sinh 1$.

Additional Problems, p. 330

1. $-9\cos(1-9x)$.
3. $-2\sin x\cos x = -\sin 2x$.
5. $-10\sin 5x\cos 5x = -5\sin 10x$.
7. $-6\sin 6x$.
9. $-x^2\sin x + 2x\cos x$.
11. $x\cos x$.
13. $(\sin x)[\sin(\cos x)]$.
15. $-(\cos x)[\sin(\sin x)]$.
17. $\cos x$.
25. 0.
27. 1.
29. 2.
31. $\frac{2}{3}$.
33. 2.
35. $\pi/4$.
37. $\frac{1}{3}\sin 3x + c$.
39. $-2\sin(1-\frac{1}{2}x) + c$.
41. $\frac{1}{18}\sin^6 3x + c$.
43. $-\frac{2}{3}\cos 3x + \frac{1}{9}\cos^3 3x + c$.
45. $\frac{1}{3}\sin x^3 + c$.
47. $\frac{1}{2}\cos(\cos 2x) + c$.
49. $-\frac{1}{4}\csc 4x + c$.
51. $\dfrac{1}{2(3+2\cos x)} + c$.
53. $-\frac{2}{5}\sqrt{7-\sin 5x} + c$.
55. $\frac{1}{7}$.
57. $\frac{1}{2}$.
59. $2\sqrt{2}$.
61. 21π.
67. $12\sec^2 3x$.
69. $\dfrac{-\csc^2 2x}{\sqrt{\cot 2x}}$.
71. $4\sec^2 x\tan x$.
73. $-10\cot 5x\csc^2 5x$.
75. $\tan\dfrac{1}{x} - \dfrac{1}{x}\sec^2\dfrac{1}{x}$.
77. $\dfrac{\sqrt{\sec\sqrt{x}+\tan\sqrt{x}}}{4\sqrt{x}}$.
79. $\sec^2 x\sec^2(\tan x)$.
81. $-3\csc\frac{1}{3}x + c$.
83. $-\frac{1}{3}\cot 3x + c$.
85. $-\frac{1}{4}\csc^4 x + c$.
87. $-\frac{1}{4}\cot^4 x + c$.
89. $4\pi/3$.
91. 300 km/h.
93. (a) $-\pi/3$; (b) $\pi/3$; (c) $-\pi$; (d) 0.7; (e) 0.7; (f) -1; (g) $\pi/3$.
95. $\dfrac{1}{\sqrt{25-x^2}}$.
97. $\dfrac{x^4}{1+x^{10}}$.
99. $\dfrac{1}{x\sqrt{x^2-1}}$.
101. $\dfrac{1+x}{1+x^2}$.
103. $\dfrac{\sqrt{x^2-1}}{x}$.
105. $\pi/2$.
107. $\dfrac{1}{\sqrt{5}}\tan^{-1}\sqrt{5}x + c$.
109. $\frac{1}{2}\sin^{-1}\frac{2}{5}x + c$.
111. $\frac{1}{4}\tan^{-1}x^4 + c$.
113. 36 ft from the point on the road closest to the billboard.
115. $\frac{18}{250}$ rad/s.
117. $T = 2\pi/\sqrt{2w} \cong 0.56$ s.
119. $A = 5, f = 1/\pi$.

CHAPTER 10

Add a constant of integration to the answer for each indefinite integral in this chapter.

Section 10.2, p. 339

1. $-\frac{1}{3}(3-2x)^{3/2}$.
3. $\frac{1}{2}\ln[1+(\ln x)^2]$.
5. $-\frac{1}{2}\cos 2x$.
7. $\frac{1}{3}\ln[\sin(3x-1)]$.
9. $\frac{1}{3}(x^2+1)^{3/2}$.
11. $\frac{1}{5}e^{5x}$.
13. $-\frac{1}{3}\cot(3x+2)$.
15. 2.
17. $2\sqrt{1-\cos x}$.
19. $e^{\tan^{-1}x}$.
21. $\frac{1}{5}\sec 5x$.
23. $\frac{1}{2}(\ln x)^2$.
25. $\ln 2$.
27. $\sin^{-1}e^x$.
29. $\frac{1}{3}\sin^3 x$.
31. $\ln(1+e^x)$.
33. $-\frac{1}{3}\ln(\cos 3x)$.
35. $4\sqrt{x^2+1}$.
37. $\tan^{-1}e^x$.
39. $\frac{1}{7}(e^x+1)^7$.
41. $\frac{1}{5}\tan 5x$.
43. $-\frac{1}{2}\csc 2x$.
45. $4(\sqrt{2}-1)$.
47. $\frac{2}{3}$.
49. (a) $n=3$, $\frac{1}{4}e^{x^4}$; (b) $n=2$, $\frac{1}{3}\sin x^3$; (c) $n=-1$, $\frac{1}{2}(\ln x)^2$; (d) $n=-\frac{1}{2}$, $2\tan\sqrt{x}$.

Section 10.3, p. 344

1. $\frac{1}{2}x - \frac{1}{4}\sin 2x$.
3. $\frac{5}{16}x + \frac{1}{4}\sin 2x + \frac{3}{64}\sin 4x - \frac{1}{48}\sin^3 2x$.
5. $-\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x$.
7. $\sin x - \frac{1}{3}\sin^3 x$.
9. $\frac{2}{3}\sin^{3/2}x - \frac{2}{7}\sin^{7/2}x$.
11. $\frac{1}{8}x - \frac{1}{96}\sin 12x$.
13. $\frac{4}{3}$.
15. $\frac{1}{7}\sec^7 x - \frac{2}{5}\sec^5 x + \frac{1}{3}\sec^3 x$.
17. $-\cot x - x$.
19. $-\frac{1}{4}\cot 4x$.
21. $-\frac{1}{2}\cot 2x - \frac{1}{2}\csc 2x$.
23. $\frac{1}{3}\sin 3x$.
25. (a) $\tan x - x$, $\frac{1}{3}\tan^3 x - \tan x + x$, $\frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \tan x - x$; (b) $\frac{1}{2}\tan^2 x + \ln(\cos x)$, $\frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x - \ln(\cos x)$, $\frac{1}{6}\tan^6 x - \frac{1}{4}\tan^4 x + \frac{1}{2}\tan^2 x + \ln(\cos x)$.
27. (a) $\pi^2/2$; (b) π; (c) $(4\pi-\pi^2)/8$; (d) $3\pi^2/16$.
29. $\frac{1}{2}[\sec x\tan x + \ln(\sec x + \tan x)]$.

Section 10.4, p. 348

1. $-\sin^{-1}\dfrac{x}{a} - \dfrac{\sqrt{a^2-x^2}}{x}$.
3. $\dfrac{1}{2a^3}\tan^{-1}\dfrac{x}{a} + \dfrac{x}{2a^2(a^2+x^2)}$.
5. $-\frac{1}{3}\sqrt{9-x^2}(x^2+18)$.
7. $\dfrac{1}{a}\ln\left(\dfrac{x}{a+\sqrt{a^2+x^2}}\right)$.
9. $\ln(x+\sqrt{x^2-a^2})$.
11. $\frac{1}{2}x\sqrt{a^2+x^2} + \frac{1}{2}a^2\ln(x+\sqrt{a^2+x^2})$.
13. $\dfrac{1}{2a}\ln\dfrac{a+x}{a-x}$.
15. $\sqrt{a^2+x^2} - a\ln\left(\dfrac{a+\sqrt{a^2+x^2}}{x}\right)$.
17. $\ln(x+\sqrt{x^2-a^2}) - \dfrac{\sqrt{x^2-a^2}}{x}$.
19. $\dfrac{1}{8}\left[a^4\sin^{-1}\dfrac{x}{a} + \sqrt{a^2-x^2}(2a^3-a^2x)\right]$.
21. $-\sqrt{4-x^2}$.
23. $\dfrac{1}{a}\tan^{-1}\dfrac{x}{a}$.
25. $-\frac{1}{3}(9-x^2)^{3/2}$.
27. $\sqrt{9+x^2}$.
31. $2\pi^2ba^2$.
33. $3-\sqrt{2}+\ln(1+\frac{1}{2}\sqrt{2})$.

Section 10.5, p. 350

1. $\sin^{-1}(x-1)$.
3. $\tan^{-1}(x+2)$.
5. $-\sqrt{2x-x^2} + 2\sin^{-1}(x-1)$.
7. $\frac{27}{2}\sin^{-1}\left(\dfrac{x-3}{3}\right) - 6\sqrt{6x-x^2} - \frac{1}{2}(x-3)\sqrt{6x-x^2}$.
9. $\frac{1}{2}\ln(x^2+2x+5) + 3\tan^{-1}\left(\dfrac{x+1}{2}\right)$.
11. $\ln(x-1+\sqrt{x^2-2x-8})$.
13. $\frac{1}{2}\ln(2x+1+\sqrt{4x^2+4x+17})$.
15. $-\dfrac{x-1}{4\sqrt{x^2-2x-3}}$.

Section 10.6, p. 356

1. (a) $x + 1 + \dfrac{1}{x-1}$, $\frac{1}{2}x^2 + x + \ln(x-1)$; (b) $\frac{1}{3}x^2 - \frac{2}{9}x + \frac{4}{27} - \dfrac{\frac{8}{27}}{3x+2}$, $\frac{1}{9}x^3 - \frac{1}{9}x^2 + \frac{4}{27}x - \frac{8}{81}\ln(3x+2)$; (c) $x - \dfrac{x}{x^2+1}$, $\frac{1}{2}x^2 - \frac{1}{2}\ln(x^2+1)$; (d) $1 + \dfrac{1}{x+2}$, $x + \ln(x+2)$; (e) $1 - \dfrac{2}{x^2+1}$, $x - 2\tan^{-1} x$.
3. $3\ln(x-3) + 4\ln(x+2)$.
5. $5\ln(x-7) - 3\ln x$.
7. $2\ln x - 4\ln(x+8) + 3\ln(x-3)$.
9. $3\ln x + 2\ln(x+13) - \ln(x-3)$.
11. $-\ln(x+1) - \dfrac{2}{x+1} - 3\ln x$.
13. $2\ln x + \frac{1}{2}\ln(x^2+2x+2) - 6\tan^{-1}(x+1)$.
15. $x + \frac{1}{2}\ln(x-1) - \dfrac{5}{2x-2} + \frac{3}{4}\ln(x^2+1) + 2\tan^{-1} x$.
17. $\frac{1}{2}x^2 - 2x + 4\ln(x+2)$.
19. $\frac{1}{2}x^2 - 2x + 5\ln(x+2)$.
21. $\dfrac{1}{5}\ln\left(\dfrac{\sin\theta - 1}{\sin\theta + 4}\right)$.
23. $\dfrac{1}{4}\ln\left(\dfrac{e^x - 2}{e^x + 2}\right)$.
29. $x = \dfrac{x_0}{x_0 + (1 - x_0)e^{-kt}}$.

Section 10.7, p. 362

1. $\frac{1}{2}x^2\ln x - \frac{1}{4}x^2$.
3. $\frac{1}{2}x^2\tan^{-1} x - \frac{1}{2}x + \frac{1}{2}\tan^{-1} x$.
5. $\frac{1}{2}e^x(\sin x - \cos x)$.
7. $\frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x$.
9. $\frac{1}{2}x^2\sin^{-1} x - \frac{1}{4}\sin^{-1} x + \frac{1}{4}\sqrt{1-x^2}$.
11. $\frac{1}{3}x\sin(3x-2) + \frac{1}{9}\cos(3x-2)$.
13. $x\tan x + \ln(\cos x)$.
15. $x\ln(a^2+x^2) - 2x + 2a\tan^{-1}\dfrac{x}{a}$.
17. $\frac{1}{2}(\ln x)^2$.
19. $\pi(\pi - 2)$.
23. (b) $x(\ln x)^5 - 5x(\ln x)^4 + 20x(\ln x)^3 - 60x(\ln x)^2 + 120x\ln x - 120x$.
25. $2\pi[\sqrt{2} + \ln(\sqrt{2} + 1)]$.

Section 10.8, p. 368

1. $-\sqrt{1-x^2}$.
3. $\frac{1}{3}\sin^3 x - \frac{2}{5}\sin^5 x + \frac{1}{7}\sin^7 x$.
5. $2\sqrt{1+\ln x} + \ln\left(\dfrac{\sqrt{1+\ln x} - 1}{\sqrt{1+\ln x} + 1}\right)$.
7. $-2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x}$.
9. $-\cos x$.
11. $\frac{1}{4}(x^2 - x\sin 2x - \frac{1}{2}\cos 2x)$.
13. $e^x - \ln(e^x + 1)$.
15. $\frac{3}{8}(x-1)^{8/3} + \frac{6}{5}(x-1)^{5/3} + \frac{3}{2}(x-1)^{2/3}$.
17. $2\sqrt{x}\tan^{-1}\sqrt{x} - \ln(1+x)$.
19. $3x + 11\ln(x-2)$.
21. $2\ln\left(\dfrac{2 - \sqrt{4-x^2}}{x}\right) + \sqrt{4-x^2}$.
23. $-\frac{1}{3}e^{-x^3}(x^3+1)$.
25. $\ln(x+3) + \dfrac{3}{x+3}$.
27. $\dfrac{1}{6}\tan^{-1}\left(\dfrac{x^2+1}{3}\right)$.
29. $\ln x\sqrt{x^2-1} - \sqrt{x^2-1} + \tan^{-1}\sqrt{x^2-1}$.
31. $-\frac{1}{3}\cos x^3$.
33. $\dfrac{1}{6}\ln\left(\dfrac{x^2+1}{x^2+4}\right)$.
35. $\ln(x-1) - \dfrac{2}{x-1} - \dfrac{1}{2(x-1)^2}$.
37. $\frac{1}{6}\tan^6 x + \frac{1}{4}\tan^4 x$.
39. $-\ln(1 + \sqrt{1-x^2})$.
41. $\frac{1}{16}(x - \frac{1}{4}\sin 4x + \frac{1}{3}\sin^3 2x)$.
43. $2\sqrt{x-1} - 4\tan^{-1}\left(\dfrac{\sqrt{x-1}}{2}\right)$.
45. $x\ln(x^2+3) - 2x + 2\sqrt{3}\tan^{-1}\dfrac{x}{\sqrt{3}}$.
47. $\dfrac{e^{5x}}{34}(3\sin 3x + 5\cos 3x)$.
49. $\dfrac{1}{8}\ln\left(\dfrac{x^2-3}{x^2+1}\right)$.
51. $\frac{1}{5}\ln(x^5 + 5x + 3)$.
53. $-4\ln(\sqrt{x+1} + 2) + 6\ln(\sqrt{x+1} + 3)$.
55. $-\dfrac{1}{\sqrt{3}}\tan^{-1}(\sqrt{3}\cos x)$.
57. $-\frac{1}{4}(x+1)^{-4} + \frac{3}{5}(x+1)^{-5} - \frac{1}{2}(x+1)^{-6} + \frac{1}{7}(x+1)^{-7}$.
59. $\frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \tan x - x$.
61. $-2\ln(x+2) + 3\ln(x+3)$.
63. $x\ln\sqrt{2x-1} - \frac{1}{2}x - \frac{1}{4}\ln(2x-1)$.
65. $\sin^{-1} x - \sqrt{1-x^2}$.
67. $\frac{1}{8}x - \frac{1}{160}\sin 20x$.
69. $\frac{1}{2}[\ln(\sin x)]^2$.
71. $\frac{1}{6}\csc^3 2x - \frac{1}{10}\csc^5 2x$.
73. $x\ln(2x + x^2) - 2x + 2\ln(x+2)$.
75. $\frac{3}{4}x^{4/3} - \frac{6}{11}x^{11/6}$.
77. $x\ln(1+x^2) - 2x + 2\tan^{-1} x$.
79. $x\tan x - \frac{1}{2}x^2 + \ln(\cos x)$.
81. $\frac{1}{7}\sec^7 x$.
83. $\frac{1}{4}(2x^2\sin^{-1} x - \sin^{-1} x + x\sqrt{1-x^2})$.
85. $\frac{1}{4}\tan^{-1} x^4$.
87. $\ln(\tan x)$.
89. $2e^{\sqrt{x}}$.
91. $5(\ln x)^2$.
93. $2\sqrt{x-2} - 4\tan^{-1}(\frac{1}{2}\sqrt{x-2})$.
95. $-\cot x$.
97. $\frac{1}{6}(\sin^{-1} 3x + 3x\sqrt{1-9x^2})$.
99. $\ln(x^2 + 5x + 6)$.
101. $\dfrac{1}{8}\ln\left(\dfrac{x^2-3}{x^2+1}\right)$.
103. $\frac{1}{2}x^2 + \frac{1}{3}\ln(x-1) - \frac{1}{6}\ln(x^2 + x + 1) + \dfrac{1}{\sqrt{3}}\tan^{-1}\left(\dfrac{2x+1}{\sqrt{3}}\right)$.
105. $\frac{1}{4}\sec^4 x - \frac{1}{2}\sec^2 x$.
107. $\cos x - \frac{2}{3}\cos^3 x$.
109. $\frac{3}{5}(x-1)^{5/3} + \frac{3}{2}(x-1)^{2/3}$.
111. $\dfrac{1}{a}(ax+b)\ln(ax+b) - x$.
113. $\sin e^x$.
115. $\frac{2}{9}x^{3/2}(3\ln x - 2)$.
117. $\tan x - \cot x$.
119. $7\ln\left(\dfrac{x-1}{x+3}\right) - \dfrac{4}{x+3}$.
121. $(x-1)\ln(1-\sqrt{x}) - \frac{1}{2}x - \sqrt{x}$.
123. $\frac{1}{2}(x-1)^2\tan^{-1}(x-1) - \frac{1}{2}(x-1) + (x - \frac{1}{2})\tan^{-1}(x-1) - \frac{1}{2}\ln[(x-1)^2 + 1]$.
125. $\frac{1}{2}(x\sqrt{1-x^2} - \sin^{-1} x)$.

Section 10.9, p. 374

1. (a) 0.643; (b) 0.656.
3. 2.2845.
5. 0.881.
7. 3.14156.
9. About 23,630 yd^2.

Additional Problems, p. 375

1. $\frac{2}{9}(3x+5)^{3/2}$. 3. $\ln(1+3x^2)$.
5. $-\frac{1}{5}\sin(1-5x)$.
7. $2\sec\sqrt{x}$. 9. $\tan^{-1}x^2$.
11. $\frac{1}{4}\ln(\sin 4x)$. 13. $-1/\ln x$.
15. $\ln(\tan x)$.
17. $-\dfrac{2}{3}\cos\left(\dfrac{3x-5}{2}\right)$.
19. $-2\csc x^3$. 21. $\tan^{-1}(\ln x)$.
23. $-\dfrac{1}{3(3x+5)}$.
25. $-\frac{1}{2}\ln(3-2x)$.
27. $\frac{1}{3}\sin(1+x^3)$.
29. $-\frac{1}{2}\cot(x^2+1)$.
31. $\tan^{-1}(\sin x)$.
33. $\frac{1}{2}\ln(\sin 2x)$. 35. $\frac{1}{2}(\tan^{-1}x)^2$.
37. $\frac{1}{2}\ln(2x+1)$.
39. $3e^{x/3}$. 41. $\tan(\sin x)$.
43. $\frac{1}{5}\sin^{-1}5x$. 45. $\tan^{-1}(\sec x)$.
47. $\frac{1}{3}(\ln x)^3$.
49. $-\ln(1+\cos x)$.
51. $-\frac{1}{3}e^{-3x}$. 53. $-\cos(\ln x)$.
55. $\csc\dfrac{1}{x}$. 57. $\frac{1}{2}\tan^{-1}e^{2x}$.
59. $\frac{1}{6}(2+x^4)^{3/2}$. 61. $\ln(e^x+x)$.
63. $-4/\sqrt{e^x}$. 65. $-\cot x$.
67. $-\frac{1}{2}\ln(\cos x^2)$.
69. $\frac{1}{2}\ln(1+x^2)$.
71. $\frac{1}{6}e^{3x^2-2}$. 73. $\tan x+\sec x$.
75. $\frac{2}{5}(1+x^{5/3})^{3/2}$.
77. $e^{\tan x}$.
79. $-\frac{1}{5}(1+\cos x)^5$.
81. $\sin(\tan x)$. 83. $\pi/6$.
85. $\frac{1}{4}$. 87. $\frac{196}{3}$.
89. $\frac{1}{2}x-\frac{1}{20}\sin 10x$.
91. $\frac{1}{2}x+\frac{1}{28}\sin 14x$.
93. $-\frac{1}{3}\cos^3x+\frac{2}{5}\cos^5x-\frac{1}{7}\cos^7x$.
95. $\frac{1}{4}\sin 4x-\frac{1}{12}\sin^3 4x$.
97. $\csc x-\frac{1}{3}\csc^3x$.
99. $\frac{5}{8}\sin^{8/5}x$.
101. $\frac{1}{5}\tan^5x+\frac{2}{3}\tan^3x+\tan x$.
103. $\frac{1}{9}\sec^9x-\frac{1}{7}\sec^7x$.
105. $-\frac{1}{4}\cot^4x+\frac{1}{2}\cot^2x+\ln(\sin x)$.
107. $\frac{1}{3}\tan 3x-\frac{1}{3}\cot 3x-\frac{2}{3}\ln(\csc 6x+\cot 6x)$.
109. $\dfrac{3}{2}\sin^{-1}\dfrac{x}{\sqrt{3}}+\dfrac{1}{2}x\sqrt{3-x^2}$.
111. $x-a\tan^{-1}\dfrac{x}{a}$.
113. $-\frac{1}{15}(a^2-x^2)^{3/2}(3x^2+2a^2)$.
115. $\ln(x+\sqrt{a^2+x^2})-\dfrac{\sqrt{a^2+x^2}}{x}$.
117. $-\dfrac{1}{3a^4x^3}\sqrt{a^2-x^2}(2x^2+a^2)$.
119. $\dfrac{1}{2a}\tan^{-1}\dfrac{x}{a}-\dfrac{x}{2(a^2+x^2)}$.
121. $\dfrac{\sqrt{x^2-9}}{9x}$. 123. $\dfrac{x}{\sqrt{1-9x^2}}$.
125. $-\dfrac{1}{3}\ln\left(\dfrac{3+\sqrt{9+4x^2}}{2x}\right)$.
127. $\dfrac{x}{\sqrt{a^2-x^2}}-\sin^{-1}\dfrac{x}{a}$.
129. $\ln(x+\sqrt{a^2+x^2})-\dfrac{x}{\sqrt{a^2+x^2}}$.
131. $\frac{1}{2}x\sqrt{x^2-a^2}+\frac{1}{2}a^2\ln(x+\sqrt{x^2-a^2})$.
133. $\sin^{-1}\left(\dfrac{x+4}{9}\right)$.
135. $\frac{1}{10}\sqrt{2}\tan^{-1}\left(\dfrac{x+1}{\sqrt{2}}\right)$.
137. $\dfrac{1}{\sqrt{3}}\sin^{-1}\left(\dfrac{3x-1}{\sqrt{7}}\right)$.
139. $\frac{3}{2}\sin^{-1}(x-1)-2\sqrt{2x-x^2}-\frac{1}{2}(x-1)\sqrt{2x-x^2}$.
141. $\dfrac{1}{6}\tan^{-1}\left(\dfrac{x-1}{2}\right)$.
143. $-\dfrac{1}{2}\sin^{-1}\left(\dfrac{2}{x-1}\right)$.
145. $3\sqrt{x^2+4x+8}+\ln(x+2+\sqrt{x^2+4x+8})$.
147. $\dfrac{5x-3}{4\sqrt{x^2+2x-3}}$.
149. $19\ln(x-4)-3\ln(x+3)$.
151. $3\ln(2x+1)-5\ln(2x-1)$.
153. $5\ln x+\ln(x+4)-3\ln(x-3)$.
155. $-2\ln x+3\ln(x+3)-3\ln(x-3)$.
157. $2\ln x+\dfrac{1}{x}-\dfrac{3}{2x^2}-5\ln(x+1)$.
159. $-\ln x+\ln(x^2+4x+8)-\dfrac{5}{2}\tan^{-1}\left(\dfrac{x+2}{2}\right)$.
161. $\frac{1}{3}x^3\tan^{-1}x-\frac{1}{6}x^2+\frac{1}{6}\ln(1+x^2)$.
163. $\frac{1}{2}x[\cos(\ln x)+\sin(\ln x)]$.
165. $x^3\sin x+3x^2\cos x-6x\sin x-6\cos x$.
167. $-\dfrac{\ln x}{x+1}+\ln x-\ln(x+1)$.
169. $\frac{1}{3}\sqrt{1+x^2}(x^2-2)$.
171. $\dfrac{e^{ax}}{a^2+b^2}(a\sin bx-b\cos bx)$.
173. $-xe^{-x}-e^{-x}$.
175. $-\frac{1}{2}x^3e^{-2x}-\frac{3}{4}x^2e^{-2x}-\frac{3}{4}xe^{-2x}-\frac{3}{8}e^{-2x}$.
177. 2π. 179. $\frac{16}{105}a^{7/2}$.
183. (b) $\dfrac{x^6(\ln x)^3}{6}-\dfrac{x^6(\ln x)^2}{12}+\dfrac{x^6\ln x}{36}-\dfrac{x^6}{216}$.
185. (b) $\frac{1}{2}[\sec x\tan x+\ln(\sec x+\tan x)]$.

CHAPTER 11

Section 11.2, p. 391

1. $(\frac{3}{2},\frac{6}{5})$. 3. $\left(0,\dfrac{4}{3\pi}a\right)$.
5. $(0,\frac{3}{5}a)$. 7. $(\frac{32}{7},\frac{4}{5})$.
9. $\left(\dfrac{2}{3(4-\pi)}a,\dfrac{2}{3(4-\pi)}a\right)$.
11. $(\frac{2}{5}a,\frac{2}{5}a)$
13. $\dfrac{4}{3\pi}\dfrac{a^2+ab+b^2}{a+b}$; this $\to\dfrac{2}{\pi}a$ as $b\to a$.
15. (a) On the axis, a distance $\frac{1}{4}h$ from the center of the base; (b) on the axis, a distance $\frac{3}{8}a$ from the center of the base.

Section 11.3, p. 393

1. (a) $\left(0,\dfrac{4}{3\pi}a\right)$; (b) $\left(0,\dfrac{2}{\pi}a\right)$.
3. (a) $\frac{12}{5}\pi a^2$; (b) $6\sqrt{2}\pi a^2$.
5. $\frac{9}{2}\pi a^3$; $6\sqrt{3}\pi a^2$.
7. (a) πr^2h; (b) $\frac{1}{3}\pi r^2h$.

Section 11.4, p. 396

1. $\frac{1}{3}Ma^2$. 3. $\frac{1}{6}Mh^2$.
5. $\frac{5}{4}Ma^2$. 7. $\frac{1}{2}Ma^2$.
9. $\frac{3}{10}Ma^2$.
11. (a) $\frac{1}{2}\sqrt{2}a \cong 0.707a$; (b) $\frac{1}{10}\sqrt{30}a \cong 0.548a$; (c) $\frac{1}{5}\sqrt{10}a \cong 0.632a$.

Additional Problems, p. 396

3. (a) $(\frac{1}{2}, \frac{2}{5})$; (b) $(0, \frac{4}{5})$; (c) $(1, \frac{2}{5})$; (d) $(\frac{5}{9}, \frac{5}{27})$; (e) $(0, \frac{10}{7})$; (f) $(\frac{16}{15}, \frac{64}{21})$; (g) $\left(\frac{1}{e-1}, 0\right)$.
5. $8\pi abc$; $8\pi(a + b)c$.
7. $\frac{1}{6}Ma^2$.

CHAPTER 12

Section 12.2, p. 403

1. 3. 3. $\frac{1}{14}$.
5. $\frac{1}{6}$. 7. $-\frac{1}{9}$.
9. $\frac{1}{2}$. 11. -6.
13. 3. 15. 4.
17. $-\frac{1}{2}$. 19. $1/\pi$.
21. 16. 23. $\frac{1}{4}$.
25. 6.
27. $f(\theta) = \frac{1}{2}a^2(\sin\theta - \sin\theta\cos\theta)$, $g(\theta) = \frac{1}{2}a^2(\theta - \sin\theta\cos\theta)$; limit $= \frac{3}{4}$.

Section 12.3, p. 408

1. -3. 3. 1.
5. 3. 7. 0.
9. 2. 11. 1.
13. 0. 15. 0.
17. 0. 19. 2.
21. $\frac{1}{2}$. 23. 1.
25. 1. 27. 1.
29. 1. 31. 1.
33. e^a. 35. 1.
37. 1. 39. $1/\sqrt{e}$.
41. e^p.

Section 12.4, p. 413

1. $1/(2e^6)$. 3. $\frac{3}{2}$.
5. $1 - \cos 1$. 7. 1.
9. 0. 11. $\ln\sqrt{3}$.
13. $\sqrt{2}(\ln 4 - 4)$.
15. 1.
17. Converges if $p < 1$, diverges if $p \geq 1$.
19. (a) $\pi/5$; (b) π.

Section 12.5, p. 423

1. $k = \dfrac{1}{2}$, $k = \dfrac{1}{\pi}$, no k.
3. (a) $m = \frac{7}{6}$; (b) $m = 0$.
5. 16, 34, 34, 14, 2 percent.

Additional Problems, p. 424

1. $\frac{5}{2}$. 3. 44.
5. 12. 7. $\frac{1}{6}$.
9. $-\frac{2}{75}$. 11. ∞.
13. 6. 15. 3.
17. $\frac{1}{2}$. 19. ∞.
21. 0. 23. $-\frac{1}{9}$.
25. 0. 27. $\frac{1}{24}$.
29. $\frac{1}{8}$. 31. 9.
33. 3. 35. $\frac{1}{3}$.
37. $\frac{10}{9}$. 39. $\frac{1}{3}$.
41. $\frac{1}{16}$.
43. 0. No; instead, it emphasizes the logical point that L'Hospital's rule makes a definite statement only when the limit on the right exists.
49. 0. 51. 0.
53. 0. 55. 0.
57. 0. 59. 0.
61. p. 63. $-\frac{1}{3}$.
65. 0. 67. 0.
69. $\frac{1}{6}$. 71 1.
73. 1 75. 1.
77. 1. 79. 1.
81. 1. 83. $-\infty$.
85. 1. 87. 1.
89. 1. 91. 1.
93. e^2. 95. e^4.
97. e^3. 99. 1.
101. $1/(3e^6)$. 103. 1.
105. $\frac{1}{2}$. 107. $\pi/4$.
109. $\pi/8$. 111. $\frac{1}{3}$.
113. 2. 115. Diverges.
117. Diverges. 119. 3.

CHAPTER 13

Section 13.2, p. 437

1. (a) D; (b) C, 0; (c) C, 0; (d) C, 0; (e) D; (f) C, $\frac{1}{2}$; (g) C, 0; (h) C, $\frac{1}{2}$; (i) C, 0; (j) D; (k) C, 0; (l) C, 0; (m) D; (n) C, 0; (o) C, π; (p) C, $-\frac{1}{4}$.
5. (a) $a/2$; (b) $4a^3$.
9. A decreasing sequence of positive numbers converges.

Section 13.3, p. 444

5. (a) C; (b) D; (c) D; (d) C; (e) D; (f) C; (g) D; (h) D.
9. 40 mi.
11. All terms must be zero from some point on.
13. (a) 0.6000 . . . ; (b) 1.666 . . . ; (c) 1.08000 . . . ; (d) 1.125000 . . . ; (e) 1.0384615384615 . . . .

Section 13.4, p. 449

3. (a) $|x| < 1$, $ax/(1 - x^2)$; (b) $|x| > 1$, $1/(x - 1)$; (c) $|1 + x| > 1$, $1 + x$ (also, if $x = 0$ the sum is 0); (d) $e^{-1} < x < e$, $\ln x/(1 - \ln x)$.
5. $x < 0$.

Section 13.5, p. 454

1. (a) D; (b) C; (c) C; (d) C; (e) C; (f) D; (g) C; (h) C.
3. D. 5. D.
7. D. 9. D.
11. D. 13. C.
15. C. 17. D.
19. C.
21. C if $p > 1$, D if $p \leq 1$.
23. C.
25. C if $p > 1$, D if $p \leq 1$.
29. ln 2.

Section 13.6, p. 460

1. C. 3. D.
5. C. 7. C.

Section 13.7, p. 464

1. C. 3. D.
5. C. 7. C.
9. D. 11. C.
13. D. 17. D.
19. C. 21. D.
23. C.

Section 13.8, p. 469

1. CC. 3. D.
5. AC. 7. AC.
9. D. 11. CC.
13. CC. 15. AC.
17. AC. 19. AC.
21. CC. 23. D.
25. CC.
27. (a) F; (b) T; (c) F; (d) F; (e) T; (f) F.

Additional Problems, p. 470

1. (a) 0; (b) $\frac{1}{2}$; (c) 0; (d) 1.
5. (a) $\frac{1}{4}$; (b) $\frac{1}{16}$; (c) $\frac{1}{256}$.
7. $x_n = \dfrac{A^n - B^n}{\sqrt{5}}$, where A and B are the positive and negative roots of $x^2 - x - 1 = 0$.

11. $\dfrac{1 + \sqrt{1 + 4a}}{2}$.
23. $|x| > \sqrt{2}$.
25. $s_n = \dfrac{x(1 - x^n)}{(1 - x)^2} - \dfrac{nx^{n+1}}{1 - x}$.
31. (a) $-\ln 2$; (b) 1.
35. (a) C; (b) D; (c) C; (d) D; (e) C; (f) D; (g) C; (h) D; (i) C; (j) D; (k) C; (l) C; (m) C; (n) C; (o) C; (p) D; (q) C; (r) C; (s) C; (t) D; (u) C; (v) C.
51. C.
53. Inconclusive.
55. D.
57. C.
59. D.

CHAPTER 14

Section 14.2, p. 489
1. $(-4, 4)$.
3. $R = 0$.
5. $[-1, 1)$.
7. $[-1, 1]$.
9. $R = 0$.
11. $(-\sqrt{3}, \sqrt{3})$.
13. $[-1, 1]$.
15. $(-\frac{1}{2}, \frac{1}{2}]$.
17. $[-1, 1)$.
19. $[-1, 1]$.
21. $(2, 6)$.
23. $R = \infty$.
25. $R = 0$.
27. $(0, 2e)$.
29. (a) $R = 1$; (b) $R = \infty$.

Section 14.3, p. 494
1. (a) $\Sigma(-1)^{n+1} nx^{n-1}$, $|x| < 1$; (b) $\Sigma(-1)^n \dfrac{(n + 2)(n + 1)}{2} x^n$, $|x| < 1$.
3. (a) $\dfrac{1}{2} \ln \dfrac{1 + x}{1 - x}$; (b) $f(x) = \dfrac{e^x - 1}{x}$ if $x \neq 0$, $f(0) = 1$; (c) $\dfrac{x}{(1 - x)^2}$; (d) $\dfrac{x}{(1 - x^2)^2}$.

Section 14.4, p. 503
15. (a) $x - \dfrac{x^3}{3 \cdot 3!} + \dfrac{x^5}{5 \cdot 5!} - \cdots$; (b) $x + \frac{1}{8}x^4 - \frac{1}{56}x^7 + \cdots$; (c) $x - \frac{1}{10}x^5 + \frac{1}{24}x^9 - \cdots$.
17. 3.14085; 3.14159.

Section 14.5, p. 509
1. (a) 9; (b) 13.
3. 1.64872.
5. 0.978148.
7. 0.848048.
9. $\sin x \cong x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!}$.
13. 1.0833, 1.0981, 1.0985, 1.0986.
15. 0.000006.

Section 14.6, p. 513
1. (a) $y = a_0\left(1 + x^2 + \dfrac{x^4}{2!} + \dfrac{x^6}{3!} + \dfrac{x^8}{4!} + \cdots\right) = a_0e^{x^2}$; (b) $y = a_0 - (a_0 - 1)x + \dfrac{(a_0 - 1)}{2!}x^2 - \dfrac{(a_0 - 1)}{3\,!}x^3 + \cdots = 1 + (a_0 - 1) \times \left(1 - x + \dfrac{x^2}{2!} - \dfrac{x^3}{3!} + \cdots\right) = 1 + (a_0 - 1)e^{-x}$.
5. $y = a_1\left[x + \dfrac{x^2}{1!2!} + \dfrac{x^3}{2!3!} + \dfrac{x^4}{3!4!} + \cdots\right] = a_1 \displaystyle\sum_{n=1}^{\infty} \dfrac{x^n}{(n - 1)!n!}$; converges for all x.

Section 14.7, p. 519
1. $x + x^2 + \frac{1}{3}x^3 - \frac{1}{30}x^5 - \frac{1}{90}x^6 + \cdots$.
31. $x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \cdots$; $R = \pi/2$.
35. (a) $\frac{1}{2}$; (b) $\frac{1}{6}$.
37. 272.

Additional Problems, p. 523
1. (a) ∞; (b) ∞; (c) e.
5. $(-1, 1)$.
7. (a) $\displaystyle\int_0^x \frac{\tan^{-1} t}{t}\, dt$; (b) $(1 + x) \ln (1 + x) - x$; (c) $\dfrac{1 + x}{(1 - x)^3}$; (d) $-\frac{1}{4} \ln (1 - x^4)$; (e) $\dfrac{x + 4x^2 + x^3}{(1 - x)^4}$; (f) $\dfrac{4 - 3x}{(1 - x)^2}$.
11. $f_2(x) = 2x/(1 - x)^3$.
15. (a) $-\frac{1}{3}$; (b) 0; (c) $-\frac{1}{12}$.

CHAPTER 15

Section 15.2, p. 534
1. (a) Circle, center $(1, 3)$ and radius 5; (b) empty set; (c) point $(5, -1)$; (d) circle, center $(8, -6)$ and radius 2; (e) point $(-3, 7)$; (f) empty set.
3. (a) $(-2, -1)$, $(-2, 0)$, $y = -2$; (b) $(3, 1)$, $(5, 1)$, $x = 1$; (c) $(-2, 5)$, $(-2, 1)$, $y = 9$; (d) $(-2, 1)$, $(-5, 1)$, $x = 1$; (e) $(-1, 2)$, $(-1, \frac{9}{4})$, $y = \frac{7}{4}$.
5. $b^2y = 4hx(b - x)$.

Section 15.3, p. 541
1. (a) $x^2/25 + y^2/21 = 1$; (b) $x^2/36 + y^2/52 = 1$; (c) $4x^2/9 + y^2/4 = 1$; (d) $x^2/16 + y^2/7 = 1$; (e) $x^2/27 + y^2/36 = 1$; (f) $24x^2/2500 + y^2/100 = 1$.
3. (a) $(0, 0)$, $(0, \pm 5)$, $(0, \pm 4)$, $e = \frac{4}{5}$; (b) $(0, 0)$, $(\pm 2, 0)$, $(\pm\sqrt{3}, 0)$, $e = \sqrt{3}/2$; (c) $(-2, 1)$, $(-2, 1 \pm \sqrt{2})$, $(-2, 2)$ and $(-2, 0)$, $e = \sqrt{2}/2$; (d) $(1, 0)$, $(2, 0)$ and $(0, 0)$, $(1 \pm \frac{1}{2}\sqrt{3}, 0)$, $e = \sqrt{3}/2$; (e) $(2, -1)$, $(5, -1)$ and $(-1, -1)$, $(2 \pm \sqrt{5}, -1)$, $e = \sqrt{5}/3$; (f) $(0, 2)$, $(\pm 2\sqrt{2}, 2)$, $(\pm 2, 2)$, $e = \sqrt{2}/2$.
7. (a) $\frac{4}{3}\pi ab^2$; (b) $\frac{4}{3}\pi a^2b$.
13. $\left(\pm\dfrac{a}{d}\sqrt{d^2 - b^2}, \dfrac{b^2}{d}\right)$.
17. $\frac{1}{59}$.
21. $y = mx \pm \sqrt{b^2 + a^2m^2}$.
23. $r_1 + r_2$.

Section 15.4, p. 549
1. $(\pm 2, 0)$, $(\pm\sqrt{13}, 0)$, $2y = \pm 3x$, $e = \sqrt{13}/2$, $x = \pm 4\sqrt{13}$.
3. $(0, \pm 2)$, $(0, \pm\sqrt{13})$, $3y = \pm 2x$, $e = \sqrt{13}/2$, $y = \pm 4/\sqrt{13}$.
5. $(0, \pm 2)$, $(0, \pm 2\sqrt{5})$, $2y = \pm x$, $e = \sqrt{5}$, $y = \pm 2/\sqrt{5}$.
7. $(0, \pm 1)$, $(0, \pm\sqrt{2})$, $y = \pm x$, $e = \sqrt{2}$, $\sqrt{2}y = \pm x$.
9. $y^2/9 - x^2/16 = 1$.
11. $x^2/9 - y^2/36 = 1$.
13. $x^2/36 - y^2/28 = 1$.
15. $x^2/36 - y^2/45 = 1$.
17. Hyperbola with center $(1, -2)$ and horizontal principal axis.
19. Two straight lines $5(y + 1) = \pm 6(x + 2)$.
31. $\left(\pm\dfrac{a}{d}\sqrt{b^2 + d^2}, -\dfrac{b^2}{d}\right)$.

Section 15.6, p. 557
1. $\theta = 45°$, $x'^2/4 + y'^2 = 1$, ellipse.
3. $\theta = 30°$, $y'^2/2 - x'^2/2 = 1$, hyperbola.
5. $\theta = 45°$, $x'^2 = 4\sqrt{2}y'$, parabola.
7. $\theta = 45°$, $x'^2/2 + y'^2/4 = 1$, ellipse.
9. $\theta = 60°$, $x'^2/3 + y'^2/11 = 1$, ellipse.
11. $\theta = \sin^{-1} 1/\sqrt{10}$, $x'^2 + 3y'^2 = 1$, ellipse.

Additional Problems, p. 558

1. $4p$.
9. $\frac{4}{3}\pi(2 + \sqrt{17})$ ft^3.

CHAPTER 16

Section 16.1, p. 563

1. (a) $(\sqrt{2}, \sqrt{2})$; (b) $(2, -2\sqrt{3})$; (c) $(0, 0)$; (d) $(\frac{1}{2}\sqrt{3}, \frac{1}{2})$; (e) $(0, -2)$; (f) $(-2\sqrt{2}, 2\sqrt{2})$; (g) $(-3, 0)$; (h) $(-3\sqrt{2}, 3\sqrt{2})$; (i) $(1, 0)$; (j) $(0, 0)$; (k) $(1, \sqrt{3})$; (l) $(5, 12)$; (m) $(-2\sqrt{3}, 2)$; (n) $(0, 3)$.
3. $(1, 0)$, $(1, 2\pi/5)$, $(1, 4\pi/5)$, $(1, 6\pi/5)$, $(1, 8\pi/5)$.
7. $(x-2)^2 + (y-2)^2 = 8$; circle with center $(2, 2)$ and radius $2\sqrt{2}$.
9. (a) Line $y = 2$; (b) line $x = 4$; (c) line $y = -3$; (d) line $x = -2$.

Section 16.2, p. 567

5. (a) $r = 5 \sec \theta$; (b) $r = -3 \csc \theta$; (c) $r = 3$; (d) $r^2 = 9 \sec 2\theta$; (e) $r = \tan \theta \sec \theta$; (f) $r^2 = 2 \csc 2\theta$; (g) $r = \dfrac{\sin^2 \theta}{\cos \theta \cos 2\theta}$; (h) $r = \dfrac{2 \cos \theta}{\tan^2 \theta - 1}$.
9. $(1, 2\pi/3)$ and $(1, 4\pi/3)$; $-\frac{1}{2}$.
11. $r = a \sin 2\theta$.
13. $x^3 = y^2(2a - x)$.
15. $(x - a)(x^2 + y^2) = b^2x^2$.

Section 16.3, p. 573

1. $9 = r^2 + 16 - 8r \cos(\theta - \pi/6)$.
3. $r = 10 \cos \theta$.
5. $r = 4\sqrt{3} \cos(\theta - \pi/3)$.
7. $r = a(1 + \cos \theta)$.
9. (a) $\sqrt{5}a$, $\sqrt{3}a$; (b) $\frac{1}{2}\sqrt{5}a$, $\frac{1}{2}\sqrt{3}a$.
11. $r = ep/(1 + e \sin \theta)$.
13. $e = \frac{1}{2}$.
15. $e = \frac{1}{6}$.
17. $\pm\sqrt{e^2 - 1}$.
19. (a) $\left(\dfrac{ep}{1-e}, 0\right)$; (b) $\left(\dfrac{ep}{1+e}, \pi\right)$.
21. (a) $x = y \cot \dfrac{\pi y}{2a}$; (b) $r = \dfrac{2a}{\pi}\theta \csc \theta$.

Section 16.4, p. 579

5. $\frac{32}{5}\pi a^2$.
11. (a) $4\pi a^2$; (b) $4\pi^2 a^2$.
17. $\sqrt{2}$.

Section 16.5, p. 582

1. $\frac{3}{2}\pi a^2$.
5. $\pi/4$.
7. $\pi + 3\sqrt{3}$.
9. $a^2\left(\dfrac{7\pi}{12} - \sqrt{3}\right)$.

Additional Problems, p. 583

1. (a) $\tan \theta = 4$; (b) $r^2 = 36/(4 + 5 \times \sin^2 \theta)$; (c) $r = 2 \cos \theta - 4 \sin \theta$; (d) $r = 3/(2 \cos \theta - 5 \sin \theta)$; (e) $r = 4 \cot \theta \csc \theta$; (f) $r = 1 + 4 \sin \theta$; (g) $r = 6 \sin 2\theta/(\sin^3 \theta + \cos^3 \theta)$.
3. (a) $(\sqrt{2}a, \pi/4)$; (b) the origin and $\left(\dfrac{2-\sqrt{2}}{2}a, \dfrac{3\pi}{4}\right)$, $\left(\dfrac{2+\sqrt{2}}{2}a, -\dfrac{\pi}{4}\right)$; (c) $(a/2, \pm\pi/6)$; (d) $(3\sqrt{2}, \pi/4)$, $(3\sqrt{2}, 3\pi/4)$; (e) the origin and $(\frac{1}{2}, 2\pi/3)$, $(\frac{1}{2}, 4\pi/3)$; (f) $(\pm a, \pi/6)$, $(\pm a, -\pi/6)$; (g) the origin and $(8a/5, \sin^{-1} 3/5)$; (h) $(4, -\pi/3)$; (i) $(\pm 2, \pi/2)$; (j) $(2, \pm\pi/3)$, $(-1, \pi)$; (k) the origin and $(\frac{3}{2}, \pm\pi/3)$; (l) the origin and $\left(\dfrac{2+\sqrt{2}}{2}a, \dfrac{\pi}{4}\right)$, $\left(\dfrac{2-\sqrt{2}}{2}a, \dfrac{5\pi}{4}\right)$; (m) the origin and $(\pm a, \pi/4)$, $(\pm a, 3\pi/4)$.
15. Larger angle $= 2\pi/3$.
17. $\frac{1}{3}a[(4 + 4\pi^2)^{3/2} - 8]$.
21. $\sqrt{3} - \frac{1}{3}\pi$.
25. $\frac{1}{3}a^2(3\sqrt{3} - \pi)$.

CHAPTER 17

Section 17.1, p. 590

1. (a) $x + y = 2$; (b) $2x - y = -4$.
3. $x + y = 3$.
5. $x - 1 = (y - 3)^2$.
7. $x^2 - y^2 = 1$.
9. $y = 1 - 2x^2$.
11. No; the second is part of the first.
13. (c) 45°.
15. $x = a \cos \theta + a\theta \sin \theta$, $y = a \sin \theta - a\theta \cos \theta$.

Section 17.2, p. 599

1. $a \sin^{-1} \dfrac{\sqrt{2ay - y^2}}{a} = \sqrt{2ay - y^2} + x$.
7. 6a.
11. $x = 2b \cos \theta + b \cos 2\theta$, $y = 2b \sin \theta - b \sin 2\theta$; $\frac{16}{3}a$.

Section 17.3, p. 605

1. (a) $\sqrt{10}$, $13\mathbf{i} - 34\mathbf{j}$, $4\mathbf{i} - 13\mathbf{j}$; (b) $\sqrt{53}$, $-36\mathbf{i} - 4\mathbf{j}$, $-39\mathbf{i} + 14\mathbf{j}$; (c) 6, $10\mathbf{i} - 33\mathbf{j}$, $-2\mathbf{i} - 33\mathbf{j}$; (d) $\sqrt{34}$, $-\mathbf{i} + 55\mathbf{j}$, $20\mathbf{i} + 22\mathbf{j}$.
5. (a) $\pm\dfrac{(3\mathbf{i} - 4\mathbf{j})}{5}$; (b) $\pm\dfrac{(-5\mathbf{i} + 12\mathbf{j})}{13}$; (c) $\pm\dfrac{(5\mathbf{i} - 7\mathbf{j})}{\sqrt{74}}$; (d) $\pm\dfrac{(24\mathbf{i} - 7\mathbf{j})}{25}$.
7. $\pm 2(12\mathbf{i} + 5\mathbf{j})$.
9. $\dfrac{\mathbf{A}}{|\mathbf{A}|} + \dfrac{\mathbf{B}}{|\mathbf{B}|}$.
13. $\theta = 45°$.

Section 17.4, p. 610

1. The line through the head of **A** which is parallel to **B**.
5. $2t\mathbf{i} + \mathbf{j}$, $2\mathbf{i}$, $\sqrt{4t^2 + 1}$.
7. $\mathbf{i} + (3t^2 - 3)\mathbf{j}$, $6t\mathbf{j}$, $\sqrt{1 + 9(t^2 - 1)^2}$.
9. $\sec^2 t\mathbf{i} + \sec t \tan t\mathbf{j}$, $2 \sec^2 t \tan t \times \mathbf{i} + (\sec^3 t + \sec t \tan^2 t)\mathbf{j}$, $|\sec t|\sqrt{2 \sec^2 t - 1}$.
11. $\mathbf{R} = \frac{1}{2}at^2\mathbf{j} + \mathbf{v}_0 t + \mathbf{R}_0$.

Section 17.5, p. 615

1. (a) $-2/(1 + 4x)^{3/2}$; (b) $\cos x$; (c) $2x^3/(2x^4 - 2x^2 + 1)^{3/2}$; (d) $-\frac{1}{2}\sqrt{2}e^{-t}$; (e) $-4t^2/(4t^4 + 1)^{3/2}$.
3. (a) 1; (b) $\frac{1}{18}5^{5/4}6^{1/2}$; (c) none.
5. If (as usual) s increases on the circle in the counterclockwise direction, then k as calculated from (4) has the wrong sign on the upper half-circle, because s increases in the direction of decreasing x. Change the sign of this result to get $1/a$ on both halves of the circle.
7. $(-\frac{1}{2} \ln 2, \frac{1}{2}\sqrt{2})$, $\frac{3}{2}\sqrt{3}$.
9. $\frac{3}{2}a$ at $\theta = \pi/4$.
11. $4a \sin \frac{1}{2}\theta$.

Section 17.6, p. 619

3. $(-a\omega \sin \omega t)\mathbf{i} + (a\omega \cos \omega t)\mathbf{j}$, $(-a\omega^2 \cos \omega t)\mathbf{i} + (-a\omega^2 \sin \omega t)\mathbf{j}$, $a\omega$, 0, $a\omega^2$.
5. $(-e^t \sin t + e^t \cos t)\mathbf{i} + (e^t \cos t + e^t \sin t)\mathbf{j}$, $(-2e^t \sin t)\mathbf{i} + (2e^t \cos t)\mathbf{j}$, $\sqrt{2}e^t$, $\sqrt{2}e^t$, $\sqrt{2}e^t$.
7. $\dfrac{4t}{t^2 + 1}\mathbf{i} + \dfrac{2t^2 - 2}{t^2 + 1}\mathbf{j}$, $\dfrac{4(1 - t^2)}{(t^2 + 1)^2}\mathbf{i} + \dfrac{8t}{(t^2 + 1)^2}\mathbf{j}$, 2, 0, $4/(t^2 + 1)$.

9. $a_n = 0$ when $t = 0$, $a_n = -1$ when $t = \pi/2$.
11. $v > 30\sqrt{2}$.

Section 17.7, p. 626

1. Since $M = \left(\frac{4\pi^2}{G}\right)\left(\frac{a^3}{T^2}\right)$, determine the ratio a^3/T^2 for any particular planet (for instance, the earth) and proceed with the arithmetic.

Additional Problems, p. 627

1. $3a^2/2$.
3. (a) $5\pi^2a^3$; (b) $\frac{64}{3}\pi a^2$.
5. $8b + \frac{8b^2}{a}$.
11. Smallest radius $= 9/(7\sqrt[6]{28})$, at $t = \pm\sqrt[6]{\frac{2}{7}}$.
13. Approximately 5.94×10^{27} g.

CHAPTER 18

Section 18.1, p. 635

1. Faces: $x = 1, y = 4, z = 5$. Edges: $x = 1, y = 4$; $y = 4, z = 5$; $x = 1, z = 5$.
3. 64.
5. (a) The yz-plane and the xz-plane taken together; (b) all three coordinate planes taken together.
7. (0, 4, 0).
9. $x^2 + y^2 + (z - 7)^2 = 49$ or $x^2 + y^2 + z^2 = 14z$.
11. (a) The sphere with center $(-1, 3, 5)$ and radius 3; (b) the point $(5, -1, 3)$; (c) the empty set; (d) the point $(-1, 7, 3)$; (e) the sphere with center $(2, -3, 0)$ and radius $\frac{1}{2}$.
15. $\frac{49}{2}$.
17. $\frac{1}{3}(7\mathbf{i} + 5\mathbf{j} + 4\mathbf{k})$.
19. $\frac{1}{4}(\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D})$.

Section 18.2, p. 639

3. (a) 60°; (b) 45°; (c) 90°.
7. No. 13. $2\mathbf{i} - \mathbf{k}$;
19. Two; 45° and 135°.
21. $c(z_1 - z_2)$.

Section 18.3, p. 646

1. (a) $14\mathbf{i} + 7\mathbf{j}$; (b) $3\mathbf{i} - 3\mathbf{j}$; (c) $2\mathbf{i} - 14\mathbf{j} - 22\mathbf{k}$; (d) $\mathbf{k}$.
3. $2\sqrt{6}$.
5. Assuming that their tails coincide, all three vectors lie in a plane.
11. (b) $11/\sqrt{107}$.

Section 18.4, p. 651

1. (a) T; (b) F; (c) T; (d) T; (e) F; (f) T; (g) F; (h) F.
3. They are parallel.
5. (a) $x = 2 + t, y = -1 + 4t, z = -3 - 2t$; (b) $x = 2 + 3t, y = -1 - t, z = -3 + 6t$; (c) $x = 2 + 2t, y = -1 - 2t, z = -3 + 5t$.
7. (a) $x = 2 + 3t, y = -3t, z = 3 - 2t$; (b) $x = 4 + 4t, y = 2, z = -1$.
9. $(\frac{3}{2}, 1, \frac{5}{2})$. 11. 6.
13. $2x - y - z + 2 = 0$.
17. $8/\sqrt{21}$. 19. (9, 0, 0).
21. $\frac{x-2}{17} = \frac{y+1}{-2} = \frac{z}{-7}$.
23. The second plane.
25. $4x + 3y + 4z + 2 = 0$.
29. (a) $\frac{8}{21}$; (b) 0.

Section 18.5, p. 656

1. Parabolic cylinder.
5. Plane.
9. $x^2 + (z - a)^2 = a^2$.
11. (a) $z = e^{-(x^2+y^2)}$; (b) $x^2 + z^2 = e^{-2y^2}$.
13. (a) $y = x^2 + z^2$; (b) $9(x^2 + z^2) + 4y^2 = 36$; (c) $z = 4 - x^2 - y^2$; (d) $x = y^2 + z^2$.
15. $(x - 2z)^2 + (y - 3z)^2 - 6(x - 2z) = 0$.

Section 18.6, p. 660

1. Ellipsoid.
3. Circular paraboloid.
5. Hyperboloid of two sheets.
7. Hyperbolic paraboloid.
9. Hyperboloid of two sheets.
11. Ellipsoid.
13. Hyperbolic paraboloid.
15. (6, −2, 2), (3, 4, −2).
17. (a) $A(k) = \pi ab\left(1 - \frac{k^2}{c^2}\right)$; (b) $\frac{4}{3}\pi abc$.
23. Hyperbolic paraboloid.

Section 18.7, p. 663

1. (a) $(2\sqrt{2}, \pi/4, 1)$; (b) $(2, -\pi/3, 7)$; (c) $(2\sqrt{3}, \pi/6, 2)$; (d) $(3\sqrt{5}, \tan^{-1} 2, 5)$.
3. (a) $(2\sqrt{2}, \pi/6, \pi/4)$; (b) $(2\sqrt{2}, 5\pi/6, -\pi/4)$; (c) $(2, \pi/4, \pi/4)$; (d) $(2, \pi/6, -\pi/2)$.
5. $r^2 + z^2 = 16$. 7. $r^2 = z^2$.
9. $r = 2 \sin \theta$. 11. $r = 3$.
13. $\rho = 4$. 15. $\rho = 6 \cos \phi$.
17. $\rho^2 \sin^2 \phi + \rho \cos \phi = 4$.

CHAPTER 19

Section 19.1, p. 669

1. The entire plane except the line $y = 2x$.
3. The first and third quadrants, including the axes.
5. The part of the plane above the line $y = 3x$.
7. All of xyz-space except the origin.
9. The solid sphere $x^2 + y^2 + z^2 \leq 16$.
11. All of xyz-space for which $z > 0$ except the planes $z = \pi, 3\pi, \ldots$.
25. Away from the origin.
27. In the positive x, negative y, positive z direction.

Section 19.2, p. 674

1. 2, 3.
3. $-6y^2/(3x + 1)^2$, $4y/(3x + 1)$.
5. $2x \sin y$, $x^2 \cos y$.
7. $\tan 2y + 3y \sec^2 3x$, $2x \sec^2 2y + \tan 3x$.
9. $-3 \sin (3x - y)$, $\sin (3x - y)$.
11. $e^x \sin y$, $e^x \cos y$.
13. $2e^y/x$, $e^y \ln x^2$.
15. $2xy^5z^7$, $5x^2y^4z^7$, $7x^2y^5z^6$.
17. $\ln \frac{y}{z}, \frac{x}{y}, -\frac{x}{z}$.
19. (a) $y = 3, z = 8x + 1$; (b) $x = 2, z = 6y - 1$.
31. $f(x, y) = 3xy^2 - x^2 \cos y + 2y$.

Section 19.3, p. 678

1. $z - 25 = 20(x - 1) + 40(y - 2)$.
3. $z = 4x + 5y$.
5. $z - 1 = 6(x - 3) - 8(y - 2)$.
7. $z - 1 = y$.
9. $10x + 13y + 13z = 75$.
13. $x_0x/a^2 + y_0y/b^2 + z_0z/c^2 = 1$.
15. 60°. 17. The origin.
19. $h(1 + a)/\sqrt{a}$.

Section 19.5, p. 685

1. (a) $8\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$; (b) $2\mathbf{i}$; (c) $\frac{2}{9}(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$; (d) $\frac{1}{5}(-\mathbf{i} + 2\mathbf{j} + \frac{2}{5}\mathbf{k})$.
3. (a) $3, -\mathbf{j}$; (b) $\sqrt{3}, \mathbf{i} + \mathbf{j} + \mathbf{k}$; (c) $2\sqrt{19}, \mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$; (d) $3e^2$, $\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$.
5. $\pm(\mathbf{i} + \mathbf{j} - 2\mathbf{k})/\sqrt{6}$.
7. $56/\sqrt{21}$; $\mathbf{i} + \mathbf{j} + 2\mathbf{k}$; $4/\sqrt{6}$.

Section 19.6, p. 691

1. 0.
3. $-9(t^2+9)/(t^2-9)^2$.
5. $4t^3+4tu^2, 4u^3+4t^2u$.

Section 19.7, p. 695

1. (3, 5), a minimum.
3. (1, 2), a minimum; (−1, 2), a saddle point.
5. (−1, −2) and (−2, 8), both saddle points.
7. (0, 0), a saddle point; (−1, −1), a maximum.
13. 2, 4, 6.
15. $1/(27abc)$.
17. $\frac{1}{8}$.
19. $\sqrt{14}$.
21. $x/6+y/6+z/3=1$.
23. $3\sqrt{3}/2$.
27. Base of rectangle $=(2-\sqrt{3})P$; height of rectangle $=\frac{1}{6}(3-\sqrt{3})P$; height of triangle $=\frac{1}{6}(2\sqrt{3}-3)P$.

Section 19.8, p. 701

1. $\frac{1}{2}$.
3. Corner in first quadrant is $(\sqrt{2}, \frac{1}{2}\sqrt{2})$.
5. $2r=h$.
7. $\frac{3}{2}, \frac{1}{2}$.
9. $x^2/3+y^2/12+z^2/27=1$.
11. $\frac{6}{11}$.
13. a^3.
17. (a) $|d|/\sqrt{a^2+b^2+c^2}$.

Section 19.9, p. 707

7. $w=c_1x+c_2$.

Section 19.10, p. 713

1. $3x/y$.
3. $(1-\sin y)/(x\cos y-1)$.
5. $(ye^{xy}-2y^2)/(4xy-xe^{xy})$.
7. $3z/(z-1), 2z/(1-z)$.
9. $\dfrac{xyz\cos xz+y\sin xz}{1-x^2y\cos xz}, \dfrac{x\sin xz}{1-x^2y\cos xz}$.
11. $\sqrt{\frac{24}{19}}$.

CHAPTER 20

Section 20.1, p. 717

1. The triangle bounded by $x=0$, $y=1$, $y=x$.
3. $\frac{3}{10}$.
5. 98.
7. $\pi/8$.
9. $\frac{14}{3}$.
11. π.
13. $\frac{1}{6}\ln 2$.
15. $\int_0^1\int_0^x f(x,y)\,dy\,dx$.
17. $\int_e^{e^2}\int_1^{\ln x} f(x,y)\,dy\,dx$.
19. $\int_0^1\int_0^{x^2} 2x^3\,dy\,dx=\frac{1}{3}$.
21. $\int_0^1\int_0^2 (5-2x-y)\,dx\,dy=5$.
23. $\int_{-1}^2\int_{3x-2}^{4x-x^2} dy\,dx=\frac{9}{2}$.
25. $\frac{1}{6}abc$.
27. 4.
29. $\frac{4}{35}\pi a^3$.

Section 20.2, p. 722

1. $\frac{9}{2}$.
3. a^2.
5. $1-e^{-a}$.
7. $\frac{8}{3}$.
9. $\frac{625}{12}$.
11. $\frac{25}{2}$.
13. $\frac{40}{3}$.
15. $\frac{3}{4}\pi$.
17. $\frac{1}{3}(b^3-a^3)$.

Section 20.3, p. 726

1. $M=a^3$; $\bar{x}=\bar{y}=\frac{7}{12}a$.
3. $M=\frac{128}{5}$; $\bar{x}=\frac{20}{7}$, $\bar{y}=0$.
5. $M=\frac{2}{3}a^3$; $\bar{x}=\frac{3}{16}\pi a$, $\bar{y}=0$.
7. $M=\pi$; $\bar{x}=(\pi^2-4)/\pi$, $\bar{y}=\pi/8$.
9. $\frac{1}{3}Ma^2$.
11. $\frac{1}{6}Ma^2$.
13. $\frac{1}{5}Ma^2$.

Section 20.4, p. 730

1. πa^2.
3. $\frac{1}{6}(4\pi-3\sqrt{3})a^2$.
5. a^2.
7. $\frac{1}{3}(3\sqrt{3}-\pi)a^2$.
9. $\frac{1}{4}(8+\pi)a^2$.
11. $\frac{1}{2}(9\sqrt{3}-2\pi)a^2$.
13. $\frac{1}{16}(9\sqrt{3}+8\pi)$.
15. $\frac{3}{4}\pi a^4$.
17. $\int_{-\pi/2}^{\pi/2}\int_0^3 z\,r\,dr\,d\theta$.
19. $\int_0^{\pi/4}\int_0^{\tan\theta\sec\theta} z\,r\,dr\,d\theta$.
21. $\int_0^{\pi/4}\int_0^1 z\,r\,dr\,d\theta$.
23. (a) $\frac{4}{3}\pi[a^3-(a^2-b^2)^{3/2}]$; (b) $\frac{4}{3}\pi(a^2-b^2)^{3/2}=\frac{1}{6}\pi h^3$.
25. $\bar{x}=\frac{5}{6}a$, $\bar{y}=0$.
27. $\bar{x}=0$, $\bar{y}=\dfrac{4}{3\pi}a$.
29. $\frac{4}{3}\pi a^3$.
31. $\frac{1}{2}\pi a^4 b$; $\bar{x}=\bar{y}=0$, $\bar{z}=\frac{1}{3}a^2b$.
33. $\frac{3}{2}Ma^2$.
35. $Ma^2\left(\dfrac{1-\ln 2}{\ln 2}\right)$.
37. $\frac{1}{4}Ma^2\left(1-\dfrac{\sin 2\alpha}{2\alpha}\right)$.

Section 20.5, p. 735

1. $\frac{1}{24}$.
3. $2abc/\pi$.
5. 24.
7. 4π.
9. $\frac{64}{15}$.
11. $\int_0^a\int_z^a\int_y^z f(x,y,z)\,dx\,dy\,dz$.
13. $a^4/8$.
15. $\int_0^1\int_{-\sqrt{1-z}}^{\sqrt{1-z}}\int_{-\sqrt{1-y^2-z}}^{\sqrt{1-y^2-z}} f(x,y,z)\,dx\,dy\,dz$.
17. 4.
19. 27π.
21. $\frac{4}{3}\pi abc$.
23. $\frac{1}{6}abc$.
27. $\frac{2}{3}Ma^2$.

Section 20.6, p. 738

1. $\pi/2$.
3. $\bar{x}=\bar{y}=0$, $\bar{z}=\frac{1}{4}h$.
5. $\frac{1}{5}M(2a^2+3b^2)$.
7. $\frac{3}{20}M(a^2+4h^2)$.
9. $\bar{x}=\bar{y}=0$, $\bar{z}=\frac{3}{8}a$.
11. $\frac{4}{3}(8-3\sqrt{3})\pi a^3$.
13. $\pi/32$.
15. $a^3/3$.
17. $\frac{16}{9}a^3(3\pi-4)$.
19. $\frac{3}{2}Ma^2$.
21. $\frac{7}{5}Ma^2$.
23. $\frac{2}{3}\pi a^3(1-\cos\alpha)$.

Section 20.7, p. 743

1. $\delta\cdot\frac{2}{5}\pi a^5(\frac{2}{3}-\cos\alpha+\frac{1}{3}\cos^3\alpha)=\frac{3}{5}Ma^2[\frac{2}{3}-\frac{1}{3}\cos\alpha\,(1+\cos\alpha)]$.
3. $2\pi^2a^3$.
5. $\dfrac{3}{8}\left(\dfrac{a^4-b^4}{a^3-b^3}\right)$.
7. $\frac{2}{3}\alpha a^3$.
11. $\dfrac{c8\pi a^{n+5}}{3(n+5)}$.
13. $\frac{8}{3}\pi a^3$.
15. (a) $M=\int_0^a\int_0^\pi\int_0^{2\pi}\delta(\rho,\phi,\theta)\,\rho^2\sin\phi\,d\theta\,d\phi\,d\rho$; (b) $M=4\pi\int_0^a\rho^2 f(\rho)\,d\rho$.
19. $\pi Gm\delta a\sin^2\alpha$.
21. $2\pi Gm\,\delta$.

Section 20.8, p. 747

1. $3\sqrt{14}$.
3. $\pi a^2\sqrt{3}$.
5. $2\pi\sqrt{6}$.
7. $a^2(\pi-2)$.
9. $\frac{1}{6}\pi(5\sqrt{5}-1)$.
11. $\frac{1}{6}\pi a^2(5\sqrt{5}-1)$.
15. $\frac{1}{12}a[3\sqrt{10}+\ln(3+\sqrt{10})]$.
17. $2a^2$.
19. $\frac{4}{9}(20-3\pi)$.

CHAPTER 21

Section 21.1, p. 757

1. (a) -2; (b) $-\frac{8}{3}$; (c) -4.
3. 0.
7. (a) $\frac{2}{3}$; (b) $\frac{29}{30}$; (c) $\frac{1}{3}$; (d) $\frac{31}{28}$.
9. (a) 0; (b) 0; (c) $-\frac{2}{3}$.
11. $\frac{4}{3}$ for all paths.
13. π for both paths.
15. $\frac{105}{2}$.
19. 0 along all paths.
21. 0.

Section 21.2, p. 763

9. 0.
11. 6.
13. e.
15. 12.

Section 21.3, p. 769

1. $\frac{1}{12}$.
3. $2 \ln 4 - \frac{15}{4}$.
5. $\frac{3}{2}$.
7. 1.
9. $\frac{1}{6}$.
11. $\frac{3}{10}$.
13. $\frac{1}{2}$.
15. $3\pi a^2$.
17. $\frac{3}{8}\pi a^2$.
19. 2.
21. $\frac{3}{2}a^2$.
23. xy^3.
25. $e^{xy} - x^2 + y^2$.
27. $x \sin y + y \cos x$.
31. 2π.

Section 21.4, p. 778

1. (a) 0; (b) 0; (c) 2; (d) $e^z \sin x$; (e) $2/r$.
3. $4\pi abc$.
5. $\frac{12}{5}\pi a^5$.
7. $2\dfrac{f(r)}{r} + f'(r)$.
11. (a) 36; (b) 18.
13. $20V$.
15. $4\pi a^5$.
17. $\frac{63}{2}$ for both integrals.
19. -12π.

Section 21.5, p. 783

5. π.
7. -1.
9. 4π.
11. 18π.
13. -8π.
15. 0.

APPENDIX A

Section A.9, p. 815

5. $2\sqrt{x} - 2 \ln (1 + \sqrt{x})$.
7. $2\sqrt{x} - 3\sqrt[3]{x} + 6\sqrt[6]{x} - 6 \ln (\sqrt[6]{x} + 1)$.
9. $2\sqrt{x} - 2 \tan^{-1} \sqrt{x}$.
11. $\frac{4}{3}x^{3/4} - 4\sqrt[4]{x} + 4 \tan^{-1} \sqrt[4]{x}$.
13. $2\sqrt{x+2} - 2 \tan^{-1} \sqrt{x+2}$.

Section A.12, p. 824

3. (a) $k \geq 3$; (b) all k; (c) all k.

Section A.14, p. 832

3. C if $x \neq \frac{2}{3}k\pi$, D if $x = \frac{2}{3}k\pi$.

APPENDIX B

Section B.1, p. 848

1. (a) $\dfrac{9!}{4!}$; (b) $\dfrac{22!}{16!}$; (c) $\dfrac{52!}{5!47!}$.
3. 720; 120.
5. 2880.
7. $9 \cdot 9! = 3{,}265{,}920$.
9. 140,400,000.
11. 120; 60; 325.
13. 20,160.
15. $2(3!3!) = 72$.
17. 210.
19. 378.
21. 4200.
23. 8820.
25. 211,680.
27. $4\binom{13}{5} = 5148$; $\left[13\binom{4}{3}\right]\left[12\binom{4}{2}\right] = 3744$.
29. 286.
31. 84.
33. $\binom{n}{2} - n$.
35. (a) $349{,}440x^{11}$; (b) $-489{,}888x^4$; (c) $-2002a^{10}b^{27}$.

Section B.2, p. 854

1. (a) $n(n+1)$; (b) $n(4n+1)$; (c) $4n^2$; (d) $3n^2$; (e) $\frac{1}{2}n(5n+1)$.
7. (a) $\dfrac{n+1}{2n}$, $n \geq 2$; (b) $1 - x^{2^{n+1}}$, $n \geq 0$.

INDEX

Abel, N. H., 106, 456, 457, 565
  partial summation formula, 831
  series, 457
  test, 833
Abscissa, 6
Absolute:
  maximum, 116
  value, 4
Absolute convergence, 825
Absolutely convergent series, 467
Absorption, Lambert's law of, 282
Acceleration, 65, 182, 608
  angular, 394
  due to gravity, 183
Accurate to *n* decimal places, 505
Adams, J. C., 626
Addition formulas, 42, 295
Adiabatic gas law, 250, 28[illegible]
Age of rocks, 280
Agnesi, M., 592
Air resistance, 285
Algebra, fundamental theorem of, 353, 522
Algebraic area, 214
Algebraic function, 27
Algebraic volume, 720
Alternating series, 465
Alternating series test, 465
Amplitude, 319
Analysis:
  complex, 523
  vector, 605
Analytic geometry, 15
Analytic trigonometry, 39
Angle of inclination, 12
Angle between two surfaces, 678
Angles, direction, 640
Angular acceleration, 394
Angular velocity, 394
Antiderivative, 171
Aphelion, 542
Apogee, 542
Apollonius, 520
Approximation, tangent line, 166
Arago, F., 274
Arc, 236
  centroid of, 390
  length, 236
    element of, 238
Archimedes, 66, 190, 192, 193, 206, 225, 228, 243, 257, 384, 530, 534, 572, 738
  principle in hydrostatics, 250, 324
  principle of the lever, 259, 384
  spiral of, 572, 579
  on surface area of sphere, 228, 243
  on volume of sphere, 257
Area, 191
  algebraic, 214
  of circle, 191
  of curved surface, 744
  geometric, 214
  of parabolic segment, 192
Area:
  element of, 221
  problem of, 51
Aristotle, 218, 257
  on Hippocrates' misfortune, 218
Arithmetic mean, 702
Arrow diagram, 24
Arterial branching, 312
Astroid (*see* Hypocycloid, of four cusps)
Astronomical unit, 626
Asymptote of a curve, 33
Atmospheric pressure, 282
Average cost, 148
Average velocity, 63
Axiom, least upper bound, 789
Axis(es):
  of a cone, 529
  coordinate, 632
  of a hyperbola, 544
  of a parabola, 18, 533
  polar, 560
Axis(es) *(Cont.)*:
  rotation of, 554

Badeer, H. S., 234
Barrow, Isaac, 212
Bartlett's Familiar Quotations, 30
Beckmann, P., 500
Bernoulli, James, 67, 169, 275, 379, 495, 565, 598
Bernoulli, John, 67, 169, 274, 379, 401, 495, 596, 598
  on Euler's feats, 275
  joy on solving the catenary problem, 379–380
  solution of brachistochrone problem, 596–598, 629
Bernoulli numbers, 525
Bessel equation, 494, 512–513
Bessel, F. W., 275, 513
Bessel function, 489, 494, 513
Big Bang, 189
Big Crunch, 189
Binomial coefficients, 847
Binomial series, 503
Binomial theorem, 84, 110, 845
Black holes, 187
Blood flow, 233
Bound:
  lower, 432
  upper, 433
Bounded sequence, 432
Bounded variation, function of, 809
Boundedness theorem, 796
Boyle's law, 250
Brachistochrone, differential equation of, 630
Brachistochrone problem, 596–598, 629
Bracket symbol, 209
Brahe, Tycho, 620
Branches of hyperbola, 543, 544
Brouncker's continued fraction, 382

Brouwer's fixed point theorem, 800
Brown, P. L., 541
Buck, R. C., 694
Buffon's needle problem, 307
Bühler, W. K., 423
Burton, A. C., 234

Cajori, F., 275
Calculus:
  differential, 51
  fundamental theorem of, 52, 190, 208, 212, 758, 808
    for line integrals, 758
  infinitesimal, 169
  integral, 51
  Leibnizian spirit in, 578
Calder, N., 541
Canceling, 35
Cardano, G., 106
Cardioid, 564, 567, 575, 577
Cartesian:
  coordinates, 15
  equation of a plane, 648
  plane, 15
Cassegrain telescope, 548
Cassini, G. D., 571
  ovals of, 571
Catenary, 327, 328, 347, 378–380
Cauchy, A. L., 456, 464
Cauchy condensation test, 474
Cauchy integral test, 456
Cauchy product of two series, 828
Cavalieri, Bonaventura, 205, 230
Cavalieri's principle, 230
Cavendish, Henry, 626
Center:
  of circle, 532
  of curvature, 614
  of ellipse, 536
  of gravity, 244, 385
  of a hyperbola, 544
  of mass, 385, 386, 388, 724
  of population, 386
Central force, 622
Central gravitational force, 623
Centripetal force, 609
Centroid, 244, 388, 724
  of an arc, 390
Certainty, Ultimate, 119
Certitude, 787
Chain rule, 93, 686, 688
Champollion, J., 707
Chebyshev, P., 816
Church, wrathful, 620
Cicero, Marcus Tullius, 259
Circle, 16, 532
  area of, 191
  center of, 532
  of curvature, 614
Circle *(Cont.)*:
  equation of, 16, 532
  involute of, 592
  parametric equations for, 588
  polar equation of, 569
  radius of, 532
Circular error, 322, 598
Circular functions, 325
Circular motion, uniform, 608
Circulation, 779
Circulation density, 779
Cissoid of Diocles, 568, 583
Clarke, Arthur C., 540, 619
Classical physics, 251
Clepsydra, 230
Closed curve, 756
Closed interval, 4
CNN, 269
Cobb-Douglas production function, 701
Coefficient, binomial, 847
Coefficient of friction, 331
Coefficients of $f(x)$, Taylor, 495
Combination, 847
Comets, 549
Common logarithm, 263
Communications satellite, 618
Comparison test, 451
  for improper integrals, 414
  limit, 452
Complete elliptic integral, 542
Complete ordered field, 790
Completing the square, 17, 348
Complex analysis, 484, 523
Complex number, 521
Composite function, 25, 93
Computation of π, 498
Concave down, 120
Concave up, 120
Conchoid:
  general, 584
  of Nicomedes, 568, 583, 584
Condensation test, Cauchy's, 474
Conditional convergence, 825
Conditionally convergent series, 468
Cone:
  axis of, 529
  elliptic, 659
  generator of, 529
  right circular, 529
  vertex of, 529
  vertex angle of, 529
Conic section, 529
  degenerate, 530
Conics, 529
Conjugate axis of hyperbola, 544
Conservation of energy, law of, 248, 762
Conservative field, 761, 781
Constant, Euler's, 275, 458, 460
Constant of integration, 172
Constraint, 696
Construction problems, 566
Continued fraction:
  Brouncker's, 382
  expansion of $\sqrt{2}$, 438
  Laplace's, 422
Continuity, uniform, 839
  theorem on, 806
Continuous function, 75, 666, 795
Continuous at a point, 75
Continuous vector, 606
Continuously compounded interest, 268
Contour map, 667
Convergence:
  absolute, 825
  conditional, 825
  interval of, 486, 488
  radius of, 486, 488
  of series, 439
  unconditional, 830
  uniform, 490, 834
    for power series, 834
Convergent improper integral, 409, 411, 455
Convergent sequence, 433
Convergent series, 439
Coolidge, J. L., 382
Cooling, Newton's law of, 282
Coordinate axes, 632
Coordinate plane, 6, 633
Coordinate system, rectangular, 5
Coordinates:
  Cartesian, 15, 633
  cylindrical, 661
  polar, 560
  rectangular, 633
  spherical, 662
Copernicus, Nicolaus, 620
Corners, 118
Corridor problem, 130, 312
Cosecant, 294
Cosine, 293
  of an acute angle, 38
  of an arbitrary angle, 40
  direction, 640
  graphs of, 42, 298
  hyperbolic, 324
  power series expansion of, 431, 483
  series, 431, 450
  Taylor series for, 503
  values of, 42
Cosines, law of, 299, 637
Cost:
  average, 148
  function, 146
  marginal, 147
Cotangent, 294
  Euler's partial fractions expansion of, 526

Coulomb's law, 250
Courant, R., 522
Critical point, 116, 693
Critical value, 116
Cross product, 640
Curl, 772
Curvature, 611
  center of, 614
  circle of, 614
  radius of, 613
Curve:
  asymptote of, 33
  closed, 756
  level, 667
  nonrectifiable, 808
  normal (Gaussian), 418
  piecewise smooth, 755
  rectifiable, 808
  simple, 765
  simple closed, 765
    positively oriented, 765
  smooth, 239, 610
Curvilinear motion, 181
Cusps, 118
Cycloid, 592, 593
  parametric equations for, 592
Cylinder, 653
  generator of, 653
  right circular, 653
Cylindrical coordinates, 661
  element of volume in, 736, 739
Cylindrical shells, 231

Decay:
  exponential, 280
  radioactive, 278
Decimal, 450, 790
  infinite, 428
  repeating, 443
Decomposition into partial fractions, 352
Decreasing function, 77, 115
Definite integral, 192, 201, 804
  Leibniz notation for, 200
Definition of *e*, 265
Degenerate conic section, 530
Degree of polynomial, 27
Del operator, 685
Delta notation, 55
Deltoid curve, 600
Demand curve, 152
Demand, elasticity of, 153
Demand function, 152
Democritus, 243, 257, 815
Dependent variable, 24
Derivative, 58
  directional, 681
  left, 62
  mixed second partial, 673
  partial, 670
Derivative *(Cont.)*:
    second-order, 672
  right, 62
  second, 107
  of vector, 606
Descartes René, 15, 20, 136, 594, 600
  dismay of, 592
  folium of, 106, 107, 592, 627, 770
  refuted by Fermat, 136
  on Snell's law, 136
Determinants, 643
Difference quotient, 58
Differentiable function, 58, 680
Differentiable at a point, 58
Differential, 164, 680
  partial, 688
  total, 680, 688
Differential calculus, 51
Differential equations, 178
  general solution of, 179
  order of, 178
  particular solution of, 179
Differential, partial, 688
Differential triangle, 238
Differentiation, 58, 83
  implicit, 103, 708
  logarithmic, 277
Diffusion equation, 704
Dirac, P. A. M., 821
Direction angles, 640
Direction cosines, 640
Directional derivative, 681
Directrix, 17
  of ellipse, 538
  of hyperbola, 546
  of parabola, 531, 532
Dirichlet, P. G. L., 707, 821, 833
Dirichlet's test, 831
Dirichlet's theorem on primes, 833
Discontinuity:
  of a function, 36
  infinite, 34
Discriminant, 557
Disk method, 227
Distance formula, 7, 634, 652
Distance from a point to a line, 48, 160
Distribution:
  normal, 420
  standard normal, 420
Distribution function, 419
  normal, 419
Divergence, 771
  of series, 439
  of vector field, 771, 773
Divergence theorem, 775
Divergent improper integral, 409, 411, 455
Divergent series, 439
Divine intervention, 706
Division of series, 516
Double-angle formulas, 42
Domain, 23, 665
Doomsday equation, 291
Dot product, 636
Double integral, 718
Double-angle formulas, 296
Doubly ruled surface, 660
Dummy variable, 215
Dyne, 245

e, 269
  definition of, 265
  irrationality of, 448
  series for, 447
  transcendence of, 815
e^x, power series expansion of, 430, 491
Eccentricity:
  of ellipse, 536
  of hyperbola, 545
Eddington, Arthur, 461
Edwards, H. M., 815
Egypt, Great Pyramid of, 251
Einstein, Albert, 183, 202, 697
  on elegance, 697
  equation $E = Mc^2$, 251–252
  law of motion, 251
Elastic, 153, 154
Elasticity of demand, 153
Electric circuit, 283
Electrostatic potential, 703
Element:
  of arc length, 238
  of area, 221
    in polar coordinates, 727
  of force, 253
  of volume, 225
    in cylindrical coordinates, 736, 739
  of work, 246
Elementary function, 334
Ellipse, 217, 529, 535
  area of, 217, 541
  center of, 536
  directrix of, 538
  eccentricity of, 536
  equation of, 537
  focal radii of, 537
  foci of, 536
  latus rectum of, 558
  major axis of, 536
  minor axis of, 536
  parametric equations for, 588
  reflection property of, 539
  semimajor axis of, 536
  semiminor axis of, 536
  string property of, 536
  vertices of, 536
Ellipsoid, 657
Elliptic cone, 659
Elliptic cylinder, 654

Elliptic integral, 348, 579, 813
 complete, 542
 of the first kind, 628, 816
Elliptic paraboloid, 659
Endpoints, 4, 118
Energy:
 kinetic, 247, 395, 761
 law of conservation of, 248, 762
 potential, 247, 761, 769
 total, 248
Epicycloid, 594
Epsilon-delta definition, 70
Equation(s):
 Bessel, 494, 512–513
 of a circle, 16, 532
 differential (*see* Differential equations)
 diffusion, 704
 doomsday, 291
 general linear, 13
 of a graph, 15
 heat, 702
 intercept, 15
 Laplace's, 674, 702, 703
 of a line, 11
 Maxwell's, 784
 one-dimensional heat, 704
 parametric, 586
  for circle, 588
  for cycloid, 592
  for ellipse, 588
  of line, 647
 of plane, 648
  Cartesian, 648
  vector, 648
 point-slope, 12
 slope-intercept, 13
 of a sphere, 635
 of state, 692
 symmetric, of line, 647
 vector, of line, 647
 Volterra's prey-predator, 290
 wave, 675, 702
  one-dimensional, 705
Equiangular spiral, 578
Equipotential surfaces, 668
Erg, 245
Escape velocity, 186
Euclid, 47, 480
 on primes, 437
Euler, L., 169, 274, 446, 448, 456, 474, 480, 525, 750, 815, 830
 constant, 275, 458, 460
 discovery of *e*, 273
 elegant truths of, 525
 expansion of cotangent, 527
 formula, 521
  for $\Sigma\ 1/n^2$, 476, 748
  for $\Sigma\ 1/n^{2k}$, 528
 identity, 480
Euler *(Cont.)*:
 infinite product for sine, 477
 irrationality of *e*, 449
 quoted: on fabric of world, 136
 series $\Sigma\ 1/n^2$, 748
 theorem: on homogeneous functions, 688
Even function, 50, 216, 518
Evolute, 614
Exhaustion, method of, 190
Expansion, power series, 428, 483
 of cos x, 431, 483
 of e^x, 430, 491
 of $f(x)$, 489
 of sin x, 431, 483
 of $\tan^{-1} x$, 491
Exponential decay, 280
Exponential function:
 general, 261
 the, 265
Exponential growth, 268, 278
Exponential spiral, 578
Exponents, fractional, 104, 261
Extreme value theorem, 77, 197, 797
Eynden, C. V., 482

Factorial, 109, 430, 845
Factorial notation, 109, 430
Fechner-Weber law, 290
Fermat, P., 15, 21, 52, 53, 136, 190, 212, 592
 concept of tangent, 53
 and Descartes, 592
 last theorem, 21, 591, 821
 principle of least time, 136, 629
 theorem, 78
Fermi, Enrico, 691
Feynman, Richard, 494, 784–785
Fibonacci, 471
Fibonacci sequence, 471
Field (algebraic), 790
 ordered, 790
  complete, 790
Field (in vector analysis):
 conservative, 761, 781
 force, 752
 gradient, 752
  of scalar, 758
 irrotational, 781
 scalar, 752
 vector, 752
First law of motion, Newton's, 182, 611
First Mean Value Theorem of Integral Calculus, 208
First-order reaction, 283
Fixed point theorem, Brouwer's, 800
Flux:
 through surface, 774
 of vector field, 772
Focal chord of a parabola, 49
Focal radii of ellipse, 537
Focus, 17
 of ellipse, 536
 of hyperbola, 543
 of parabola, 532
Focus-directrix-eccentricity definition, 550
Folium of Descartes, 106, 107, 592, 627, 770
Force:
 central, 622
  gravitational, 623
 centripetal, 609
 of gravity, 183
 restoring, 320
Force field, 752
Formula(s):
 Abel's partial summation, 831
 addition, 42, 295
 distance, 7, 634, 652
 double-angle, 42, 296
 Euler's 476, 521
  for $\Sigma\ 1/n^2$, 476, 748
  for $\Sigma\ 1/n^{2k}$, 528
 half-angle, 42, 46, 296, 341
 Heron's, 695
 Legendre's, 747
 Leibniz, 317, 361, 692
 midpoint, 8
 quadratic, 22
 recursion, 511
 reduction, 344, 356, 360
 Stirling's, 472
 subtraction, 45, 296
Four squares theorem, 21
Four vertex theorem, 615
Four-leaved rose, 566
Fourier, J., 275, 704, 707
Fractional exponents, 104, 261
Franklin, Benjamin, 269
Franklin, P., 719
Frequency, 320
Frequency density, 415
Function, 23
 algebraic, 27
 Bessel, 489, 494, 513
 of bounded variation, 809
 circular, 325
 composite, 25, 93
 continuous, 75, 666, 795
 cost, 146
 decreasing, 77, 115
 demand, 152
 differentiable, 58, 680
 discontinuity of, 36
 distribution, 419
 domain of, 23
 elementary, 334
 even, 216, 518
  and odd, 50, 518

Function *(Cont.)*:
exponential, 265
of a function, 26
gamma, 413
general exponential, 261
general logarithm, 262
graph of, 24
homogeneous, 688
hyperbolic, 325
implicit, 102, 710
improper rational, 351
increasing, 77, 115
integrable, 202, 806
inverse, 317
inverse hyperbolic, 327
limit of, 68
linear, 29
normal distribution, 419
odd, 217, 518
period of, 43
periodic, 43, 294
potential, 769
price, 152
probability density, 415
normal (Gaussian), 418
profit, 151
proper rational, 351
quadratic, 29
rational, 27, 351
improper, 351
proper, 351
revenue, 151
transcendental, 27
trigonometric, 292
inverse, 313
zero of, 31
zeta, 474, 480
Fundamental lemma, 680
Fundamental theorem of algebra, 353, 522
Fundamental theorem of calculus, 52, 190, 208, 212, 758, 808
for line integrals, 758

Galileo, 1, 230, 379, 448, 530, 594
Gamma function, 413
Gas law:
adiabatic, 161, 250, 283
ideal, 671
Gauss, C. F., 202, 274, 423, 456, 513, 565, 579, 821, 833
on lawyers, 787
Gauss's test, 823
Gauss's theorem on divergence, 755
Gelfond, A. O., 815
General concoid, 584
General exponential function, 261
General linear equation, 13
General logarithm function, 262
General solution, 179
Generalized mean value theorem, 401, 803
Generalized Stokes theorem, 783
Generator:
of a cone, 529
of a cylinder, 653
Geometric area, 214
Geometric mean, 702
Geometric series, 440
Geometry, analytic, 15
Grade inflation, 421
Gradient, 682, 771
Gradient:
field, 752
of a scalar field, 758
Graph:
of an equation, 15
of a function, 24
Graph(s):
of cosine, 42, 298
of polar equation, 561
of sine, 42, 298
of tangent, 298
Gravitation:
constant of, 246
Newton's law of, 185, 246, 621, 706
Gravitational attraction of spherical shell, 741
Gravitational force, central, 623
Gravitational potential, 703
Gravity:
acceleration due to, 183
center of, 244, 385
force of, 183
Great Plague of London, 66
Great Pyramid of Egypt, 251
Green, George, 765
Green's theorem, 765, 780
Growth:
exponential, 268, 278
population, 277
inhibited, 284
Growth curve, inhibited or sigmoid, 285
Gyration, radius of, 396

Half-angle formulas, 42, 46, 296, 341
Half-life, 279
Hall, Tord, 423
Halley, Edmund, 540
Halley's Comet, 540
Halmos, P. R., 445
Hamilton, Sir W., 501
Hardy, G. H., 595
Harmonic series, 441
Heart, working, 248
Heat equation, 702
one-dimensional, 704
Heath, T. L., 572
Heilbroner, R. L., 154
Heliocentric system, 620
Henderson, J. M., 689
Hermite, C., 479, 815
Heron of Alexandria, 135
Heron's formula, 695
Hille, E., 790
Hippocrates of Chios, 193, 218, 583
lune of, 193
Histogram, 414
Hobbes, Thomas, 382
Hölder's inequality, 702
Holmes, Oliver Wendell:
on certitude, 787
on Descartes, 20
Homogeneous function, Euler's theorem on, 688
Hooke's law, 245
Horizontally simple, 720
Hubble's law, 189
Hungarian proverb, 118
Hutchinson, G. Evelyn, 778
Huygens, Christiaan, 379, 598
and tautochrone, 598
Hydrostatics, 252
Hyperbola, 529, 543
asymptote of, 545
axes of, 544
branch of, 543, 544
center of, 544
conjugate axis of, 544
directrix of, 546
eccentricity of, 545
focus of, 543
latus rectum of, 559
principal axis of, 544
rectangular, 546
reflection property of, 548
vertices of, 544
Hyperbolic functions, 325
inverse, 327
Hyperbolic paraboloid, 659
Hyperbolic spiral, 572
Hyperboloid:
of one sheet, 658
of two sheets, 658
Hypergeometric series, 489, 824
Hypocycloid, 594
of four cusps, 114, 240, 596

i, 521
i-, **j**-, **k**-component, 634
I-beam, 396
Ideal gas law, 671
Identities, trigonometric, 41–42
Imaginary unit, 521
Implicit differentiation, 103, 708
Implicit function, 102, 710
Implicit function theorem, 710, 840
Improper integral, 409, 455
comparison test for, 414

Improper integral *(Cont.)*:
convergent, 409, 411, 455
divergent, 409, 411, 455
Inclination, 12
Increasing function, 77, 115
Increasing sequence, 435
Increment, 55
Indefinite integral, 171–172, 192
Independent variable, 24
Indeterminate form, 398, 404, 405
Index of refraction, 136
Induction, mathematical, 471
principle of, 850
Inelastic, 153, 154
Inequalities, 3
Inequality:
Hölder's, 702
Schwarz's, 160, 702
Inertia:
moment of, 393, 394, 724
polar moment of, 725
Infinite decimal, 428
Infinite discontinuity, 34
Infinite product for the sine, Euler's, 477
Infinite product, Wallis's, 275
Infinite series (*see* Series)
Infinitesimal, 168
Infinitesimal calculus, 169
Infinity, 5
Inflection, point of, 120
Inhibited growth curve, 285
Inhibited population growth, 284
Initial condition, 180
Instantaneous velocity, 64
Integers, 2
Integrable function, 202, 806
Integral, 171, 804, 805
complete elliptic, 542
definite, 192, 201, 804
double, 718
elliptic, 348, 579, 813
of the first kind, 628, 816
improper, 409, 411, 455
comparison test for, 414
convergent, 409, 411, 455
divergent, 409, 411, 455
indefinite, 171–172, 192
iterated, 715
line, 752
fundamental theorem of calculus for, 758
lower, 805
Riemann, 203, 707
sign, 171, 201
surface, 774
test, 456
Cauchy, 456
triple, 731
upper, 805
Integral calculus, 51
first mean value theorem of, 208
Integrand, 171, 201
Integration, 171
constant of, 172
Leibniz approach to, 723
limits of, 201
numerical, 369
by parts, 357
by substitution, 174, 176
variable of, 201
Intercept equation, 15
Interest, continuously compounded, 268
Interior point, 5
Intermediate value theorem, 78, 143, 799
Interval, 4
of convergence, 486, 488
Inverse function, 317
Inverse hyperbolic function, 327
Inverse sine, 313
Inverse square law of universal gravitation, Newton's, 621
Inverse substitution, 817
Inverse tangent, 314
Inverse tangent series, 498
Inverse trigonometric functions, 313
Involute of a circle, 592
Irrational number, 2, 444
Irrotational field, 781
Isothermal surfaces, 668
Iterated integral, 715

Jacobi, C. G. J., 842
Jacobian, 842
James, D. E., 689
Jesus, horoscope of, 106
Jointly proportional, 130
Joule, 245

Kelvin, Lord (W. Thomson), 729
Kepler, Johannes, 530, 540, 571, 605, 620, 621
first law, 625
laws of planetary motion, 620
problem of, 159
second law, 623
of planetary motion, 581
third law, 625
Keynes, J. M., 154, 445
Kinetic energy, 247, 395, 761
Knopp, K., 517, 525
Kummer, E. E., 821
Kummer's theorem, 821

Lagrange, J. L., 169, 501
Lagrange multiplier, 698
Lagrange's remainder formula, 501
Lambert, J. H., 815
Lambert's law of absorption, 282
Landau, E., 790
Laplace, P. S., 275, 706, 765
Laplace transform, 413
Laplace's continued fraction, 422
Laplace's equation, 674, 702, 703
Latus rectum:
of ellipse, 558
of hyperbola, 559
of parabola, 558
Law:
of absorption, 282
adiabatic gas, 250, 283
Boyle's, 250
of conservation of energy, 248, 762
of cooling, Newton's, 282
of cosines, 299, 637
Coulomb's, 250
Fechner-Weber, 290
of gravitation, Newton's, 185, 706
Hooke's, 245
ideal gas, 671
Kepler's:
first, 625
second, 623
of planetary motion, 581
third, 625
of the lever, Archimedes', 384
of mass action, 283
of motion:
Einstein's, 251
Newton's first, 182
second, 182, 251, 608, 761
Ohm's, 312
Parallelogram, 639
Poiseuille's, 234, 312
of reflection, 134–135
of refraction, Snell's, 135, 629, 702
of sines, 50
Torricelli's, 230
Lawyers, Gauss on, 787
Least squares, 159
method of, 138, 696
Least time, Fermat's principle of, 136, 629
Least upper bound, 789
Least upper bound axiom, 789
Legendre, A. M., 747
Legendre's formula, 747
Legendre polynomials, 524
Leibniz, G. W., 21, 52, 67, 169, 171, 190, 207, 211, 274, 361, 382, 429, 441, 465, 466, 495, 598
alternating series test, 465
approach to calculus, 168–169, 222
approach to integration, 723
formula, 317, 361, 499, 692
for differentiating integrals, 692
for $\pi/4$, 317, 361, 499
notation, 60, 61
notation for definite integral, 200

Leibniz *(Cont.)*:
notation for derivatives, 60–61
series, 317, 441, 499
use of differentials, 168
Leibnizian myths, 168
Leibnizian spirit in Calculus, 222, 578
Lemniscate, 48, 565, 576, 580
Length of vector, 634
Leonardo of Pisa, 471
Level curve, 667
Level surface, 668
Lever, Archimedes' principle of, 259, 384
Leverrier, U. J., 626
Lewis, J. H., 599
L'Hospital, G.F.A. de, 401
L'Hospital's rule, 401
Libby, W., 281
Limaçon, 564, 567
Limit:
of a function, 68
of a sequence, 433
Limit comparison test, 452
Limit laws, 72
Limit-of-sums process, 199–201
Limits of integration, 201
Lindemann, F., 815
Line:
equation of, 11
parametric equations of, 647
slope of, 11
symmetric equations of, 647
vector equations of, 647
Line integral, 752
fundamental theorem of calculus for, 758
Linear approximation, 166
Linear equation, general, 13
Linear function, 29
Liouville, J., 813, 815
Liouville's theorem, 815
Lithotripsy, 540
Little old man, 504
Little-oh notation, 458
Lobachevsky, N., 347
Locus, 20
Logarithm, 262
common, 263
natural, 269
Logarithm function, general, 262
Logarithm series, 490–491, 508
Logarithmic differentiation, 277
Long division, 351
Lord Brouncker, 382
Lower bound, 432
Lower limits of integration, 201
Lower sum, 199
Lune of Hippocrates, 193

Machin, J., 500
Machine diagram, 24
Maclaurin's series, 500
Magic, in mathematics, 176
Major axis of ellipse, 536
Marginal analysis, 151
Marginal cost, 147
Marginal profit, 151
Marginal Q, 66
Marginal revenue, 151
Mass, 724
center of, 384, 385, 386, 388
Mass action, law of, 283
Mathematical induction, 471
principle of, 850
Maximum, 116
absolute, 116
at endpoints, cusps, and corners, 118
relative, 692
Maxwell, James Clerk, 784
Maxwell's equations, 784
Mead, D. G., 813
Mean, 416
arithmetic and geometric, 702
proportional, 584
Mean value theorem, 76, 171, 237, 399, 802
generalized, 803
Mechanics, Newtonian, 182
Median, 10
Mersenne, Marin, 584, 594
Mertens, F., 830
Method:
of cylindrical shells, 231
disk, 227
of exhaustion, 190
of least squares, 138, 696
of moving slices, 227
of partial fractions, 351
of substitution, 337
washer, 228
Microeconomics, 146
Midpoint formulas, 8
Millikan, R. A., 283
Minimum, 116
at endpoints, cusps, and corners, 118
Minor axis of ellipse, 536
Mixed second partial derivative, 673
Mixing, 286
Moment, 244, 259, 384, 724
of inertia, 393, 394, 724
polar, 725
Momentum, 189
Monte, Guidobaldo del, 448
Morgan, J. P., 431
Morse, P. M., 691
Motion:
curvilinear, 181
Einstein's law of, 251
Newton's:
first law of, 182, 611
Motion *(Cont.)*:
second law of, 182, 251, 608, 761
rectilinear, 181
simple harmonic, 319
uniform circular, 608
Mountain building, 251
Moving slices, method of, 227
Multiplier effect, 445

Napoleon, 706, 707
Nappes, 529
Natural logarithm, 269
Natural number, 2
Needle problem, Buffon's, 307
Neighborhood, 666
Nephroid curve, 600
Neumann, J. von, 445
Neutron stars, 187
Newman, J. R., 626, 784
newton (unit), 245
Newton, Isaac, 21, 52, 66, 190, 207, 211, 234, 274, 472, 495, 530, 540, 571, 574, 598, 605, 619, 626, 627, 743
area under quadratrix, 574, 584
and Bernoulli, John, 598
and Fermat, 21
first law of motion, 182, 611
and Halley, 540
inverse square law of universal gravitation, 621
law of cooling, 282
law of gravitation, 185, 246, 706
mechanics, 182
method of approximate solution, 144
second law of motion, 182, 251, 608, 761
theorem on attraction of spheres, 741
Niven, I., 479, 482, 595
Norm of partition, 804
Normal, 88
Normal curve (Gaussian), 418
Normal distribution, 420
function, 420
Normal probability density function, 418
Notation, factorial, 109, 430
*n*th term test, 442
*n*th-degree Taylor polynomial, 501
Number(s),
Bernoulli, 525
complex, 521
irrational, 2, 444
natural, 2
prime, 437
rational, 2, 444
real, 787
transcendental, 815
Numerical integration, 369

Oblate spheroid, 229, 541, 657

Odd function, 50, 217, 518
One-dimensional heat equation, 704
One-dimensional wave equation, 705
Open interval, 4
Orbit, synchronous, 619
Orbital speed, 618
Ordered field, 790
  complete, 790
Ordinate, 6
Origin, 5, 632
Oscillation of a function, 806
Ovals of Cassini, 571

p-series, 453, 456
Paddle wheel, 779
Paint paradox, 414
Pappus of Alexandria, 391, 574
Pappus' theorems, 391–392
Parabola, 17, 529
  axis of, 18, 533
  directrix of, 531, 532
  equation of, 18, 533
  focal chord of, 49
  focus of, 532
  latus rectum of, 558
  reflection property of, 88, 534
  vertex of, 18, 533
Parabolic cylinder, 654
Paraboloid:
  elliptic, 659
  hyperbolic, 659
Parallel axis theorem, 397
Parallelogram law, 639
Parallelogram rule, 602
Parameter, 586
Parametric equations, 586
  for circle, 588
  for cycloid, 592
  for ellipse, 588
  of line, 647
Parthenon, 220
Partial derivative, 670
  mixed second, 673
  second-order, 672
Partial differential, 688
Partial fractions, 351
  decomposition into, 352
  expansion of cotangent, 526
  method of, 351
  theorem, 354, 818
Partial sums, 439
Particle, 181
Particular solution, 179
Partition, 804
  norm of, 804
Parts, integration by, 357
Pascal, Blaise, 21, 190, 212, 584, 854
Pascal, Etienne, 584
Path, 755
Pendulum, 322, 627
  period of, 322
Perfect competitor, 155
Pericles, 219
Perigee, 542
Perihelion, 542
Period, 319, 294
  of a function, 43
Periodic function, 43
Permutation, 846
pH, 264
pi (π):
  computation of, 498
  definition of, 191
  experimental method of calculating, 309
  irrationality of, 479
  Leibniz's formula for $\pi/4$, 317, 499
  value of, 500
  Wallis's product for $\pi/2$, 361, 380
Piecewise smooth curve, 755
Piecewise smooth surface, 775
Plane:
  Cartesian, 15
    equation of, 648
  coordinate, 633
  equation of, 648
  tangent, 676
  vector equation of, 648
Planetary motion, Kepler's laws of, 581, 620
Plato, hatred of Democritus, 257
Point:
  critical, 693
  of inflection, 120
  interior, 5
  surface, 660
Point-slope equation, 12
Poiseuille, Jean, 234
Poiseuille's law, 234, 312
Poisson, S., 283
Poisson's gas equation, 283
Poker hands, 848
Polar axis, 560
Polar coordinates, 560
  element of area in, 727
Polar equation:
  of circle, 569
  of conic section, 571
  graph of, 561
Polar moment of inertia, 725
Polya, G., 129, 276, 854
Polynomial, 26
  degree of, 27
  Legendre, 524
Pope, Alexander, 67
Population, center of, 386
Population growth, 277
  inhibited, 284
Population, world, 289
Positive integer, 2
Potential, 246, 769
Potential, electrostatic, 703
Potential energy, 247, 761, 769
Potential function, 769
Potential, gravitational, 703
Power rule, 95
Power series, 484
  expansion, 428, 483
    of cos x, 431, 483
    of e^x, 430
    of $f(x)$, 489
    of sin x, 431, 483
    of $\tan^{-1} x$, 491
  interval of convergence, 486
  properties of, 490
  radius of convergence, 486
  for tan x, 526
  uniform convergence, 834
  in $x - a$, 488
Powers of belief, 181
Pressure, 252
  atmospheric, 283
Prey-predator equations, Volterra's, 290
Price function, 152
Prime number, 437
Primes, series of reciprocals of, 480
Principle, Archimedes, 324
Principal axis of a hyperbola, 544
Principle of mathematical induction, 850
Probability, 307
Probability density, 416
  standard normal, 419
Probability density function, 415
  normal (Gaussian), 418
Problem(s):
  of areas, 51
  brachistochrone, 596
  construction, 566
  of tangents, 51
Product:
  cross, 640
  dot, 636
  of numerical series, 827–829
  of power series, 514–516
  of series, 515
Product rule, 88, 112
Product, Wallis's, 275, 361, 380, 472, 477
Profit function, 151
Profit, marginal, 151
Projection:
  scalar, 636
  vector, 636
Prolate spheroid, 229, 541, 657
Proper rational function, 351
Properties of power series, 490
Proportional:
  mean, 584
  third, 584

Pseudosphere, 347
Ptolemaic system, 620
Ptolemy, 620
Pythagoras, 9
Pythagorean theorem, 7, 47, 578

Quadrant, 6
Quadratic formula, 22
Quadratic function, 29
Quadratrix, 574
Quadric surface, 656
Quandt, R. E., 689
Quotient rule, 90

Raabe, J. L., 821
Raabe's test, 821
Radially simple, 727
Radian, 39, 292
Radioactive decay, 278
Radiocarbon dating, 281
Radius:
  of circle, 532
  of convergence, 486, 488
  of curvature, 614
  of gyration, 396
Ramanujan, Srinivasa, 500
Range, 23
Rate of change, 65, 139
Ratio test, 461
Rational function, 27, 351
  improper, 351
  proper, 351
Rational number, 2, 444
Rational point, 479
Rationalizing substitution, 816
Reaction:
  first-order, 283
  second-order, 283
Real line, 3
Real numbers, 787
Reciprocals of primes, series of, 480
Rectangular (or Cartesian) coordinates, 633
Rectangular coordinate system, 5
Rectangular coordinates in space, 632
Rectangular hyperbola, 546
Rectilinear motion, 181
Recursion formula, 511
Reduction formula, 344, 356, 360
Reflection, law of, 134–135
Reflection property:
  of ellipse, 539
  of hyperbola, 548
  of parabola, 88, 534
Refraction:
  index of, 136
  Snell's law of, 135, 629, 702
Region, simply connected, 781
Relative maximum, 692
Remainder formula, 496, 500, 501
Repeated integral (*see* Iterated integral)
Repeating decimal, 443
Rest mass, 251
Restoring force, 320
Revenue function, 151
Revenue, marginal, 151
Revolution:
  solid of, 225
  surface of, 240, 655
Rhetoric and invective, 382
Richter, Charles, 264
Richter scale, 263
Riemann, B., 202, 274, 707, 827, 833
  integral, 203
  rearrangement theorem, 827
Riemann sums, 203
Riemannian Geometry, 202
Right circular cone, 529
Right circular cylinder, 653
Right triangle trigonometry, 39
Right-hand thumb rule, 640
Ritt, J. F., 790, 813, 816
Robbins, H., 522
Roberval, G., 584
Robinson, A., 169
Rolle, Michel, 800
Rolle's theorem, 800
Roller, D., 283
Root test, 463
Rose, four-leaved, 566
Rosen, M., 565
Rotation of axes, 554
Routine research, 202
Rule:
  parallelogram, 602
  trapezoidal, 370
Ruled surface, 660
Ruler-and-compass constructions, 566–567
Rulings on surface, 654
Rumor, spread of, 287

Saddle point, 660
Saddle surface, 660
Scalar, 600
Scalar field, 752
Scalar projection, 636
Schwarz, H. A., 747
Schwarz's inequality, 160, 702
Sears, F. W., 691
Secant, 294
Second derivative, 107
Second derivative test, 122, 694
Second law of motion, Newton's, 182, 251, 608, 761
Second partial derivative, mixed, 673
Second-order partial derivatives, 672
Second-order reaction, 283
Segment of a sphere, volume of, 226
Semimajor axis of ellipse, 536
Semiminor axis of ellipse, 536
Separation of variables, 179, 272
Sequence, 432
  bounded, 432
  convergent, 433
  decreasing, 438
  Fibonacci, 471
  increasing, 435
  limit, 433
  lower bound of, 432
  of partial sums, 439
  unbounded, 433
  upper bound of, 433
Series, 427, 439
  Abel's, 457
  absolutely convergent, 467
  alternating, 465
  binomial, 503
  conditionally convergent, 468
  convergent, 439
  cosine, 431, 450
  divergent, 439
  division of, 516, 836
  for e, 447
  geometric, 440
  harmonic, 441
  hypergeometric, 489, 824
  infinite, 427, 439
  interval of convergence of, 486
  inverse tangent, 498
  Leibniz's, 317, 441, 499
  ln 2, 459
  logarithm, 490–491
  Maclaurin's, 500
  multiplication of, 514, 827–829
  p-, 453, 456
  partial sums of, 439
  power, 484
    in $x - a$, 488
  product of, 515
  sine, 497
  sum of, 439
  for tangent, 526
  Taylor, 495
    for $\cos x$, 498
    for e^x, 496
    for even and odd functions, 518
    of $f(x)$, 495
      in powers of $x - a$, 500
    for $\sin x$, 497
  telescopic, 441
Shakespeare, William, 276
Shell method, 231
Sigma notation, 194
Sigmoid growth curve, 285
Simmons, G. F., 2, 47, 205, 413, 423, 437, 501, 599, 614, 689, 701, 815, 824
Simple closed curve, 765
  positively oriented, 765

Simple curve, 765
Simple harmonic motion, 319
Simply connected region, 781
Simpson, T., 371
Simpson's rule, 372
Sine, 293
  of an acute angle, 38
  of an arbitrary angle, 40
  Euler's infinite product for, 477
  graphs of, 42, 298
  hyperbolic, 324
  inverse, 313
  power series expansion of, 431, 483
  values of, 42
Sines, law of, 50
Skepticism, 94
Slope of a line, 11
Slope-intercept equation, 13
Smith, D. E., 598
Smooth curve, 239, 610
Snell, W., 135
Snell's law of refraction, 135, 629, 702
Solid of revolution, 225
Speed, 64, 182, 607
  orbital, 618
Sphere:
  equation of, 635
  surface area of, 228, 243
  volume of, 257, 728, 734, 737, 740
"Sphere":
  five-dimensional, 736
  four-dimensional, 736
Spherical coordinates, 662
Spherical shell, gravitational attraction of, 741
Spherical triangle, 747
Spherical zone, 244
Spheroid:
  oblate, 229, 541, 657
  prolate, 229, 541, 657
Spiral:
  of Archimedes, 572, 579
  equiangular, 578
  exponential, 578
  hyperbolic, 572
Spread of infectious disease, 357
Spring constant, 245
Square, completing the, 348
Squeeze theorem, 794
Standard deviation, 416
Standard normal distribution, 420
Standard normal probability density, 419
Stirling, J., 472
Stirling's formula, 472
Stokes, Sir George, 778
Stokes' theorem, 780
String property of ellipse, 536
Subnormal, 558
Substitution:
  inverse, 817
  method of, 337
  rationalizing, 816
  trigonometric, 344
Subtangent, 558
Subtraction formulas, 45, 296
Sum(s):
  lower, 199
  partial, 439
  Riemann, 203
  of a series, 439
  upper, 200
Surface:
  area of a sphere, 243
  doubly ruled, 660
  equipotential, 668
  flux through, 774
  isothermal, 668
  level, 668
  piecewise smooth, 775
  quadric, 656
  of revolution, 240, 655
  ruled, 660
  rulings on, 654
  saddle, 660
Surface integral, 774
Symmetric equations of line, 647
Synchronous orbit, 619
System:
  heliocentic, 620
  Ptolemaic, 620

Tan x, power series for, 526
Tan^{-1} x, power series for, 498
Tangent, 294
  graphs of, 298
  inverse, 314
  power series for, 526
  problem of, 51
Tangent line, 53
  approximation, 166
Tangent plane, 676
Taste of victory, 380
Tautochrone property, 598, 628
Taylor, A. E., 719
Taylor, Brook, 495
Taylor:
  coefficients, of $f(x)$, 495
  formula with derivative remainder, 496, 500
  polynomial, nth-degree, 501
  series:
    for cos x, 498
    for e^x, 496
    for even and odd functions, 518
    of $f(x)$, 495
      in powers of $x - a$, 500
    for sin x, 497
Telescopes of Cassegrain type, 548
Telescopic series, 441
Terminal velocity, 286
Test:
  Abel's, 833
  alternating series, 465
  Cauchy:
    condensation, 474
    integral, 456
  comparison, 451
    limit, 452
  Dirichlet's, 831
  Gauss's, 823
  integral, 456
  limit comparison, 452
  nth term, 442
  Raabe's, 821
  ratio, 461
  root, 463
  second derivative, 122, 694
Thales, 9, 16
Theorem:
  of Algebra, fundamental, 353, 522
  binomial, 84, 110, 845
  boundedness, 796
  Brouwer's, 800
  of Calculus, fundamental, 190, 208, 212, 758
  divergence, 775
  extreme value, 77, 197, 797
  Fermat's, 78
    last, 21, 591, 821
  four squares, 21
  four vertex, 615
  Gauss's, 775
  Green's, 765, 780
  implicit function, 710, 840
  intermediate value, 78, 143, 799
  Kummer's, 821
  mean value, 76, 171, 237, 399, 802
    generalized, 401, 803
  parallel axis, 397
  partial fractions, 354, 818
  Pythagorean, 7, 47, 578
  Riemann's rearrangement, 827
  Rolle's, 800
  squeeze, 794
  Stokes', 780
  Torricelli's, 593
  uniform continuity, 806
  Wren's, 593, 627
Thinking out loud, 365
Third proportional, 584
Three-step rule, 58
Throsby, C. D., 689
Tietze, H., 106, 566
Titchmarsh, E. C., 523
Torque, 394
  vector, 645

Torricelli, Evangelista, 594
Torricelli's law, 230
Torricelli's theorem, 593
Torus, 229, 392
Total differential, 680
Total energy, 248
Tractrix, 347
Transcendence of e, 815
Transcendental function, 27
Transcendental number, 815
Trapezoidal rule, 370
Triangle, spherical, 747
Trigonometric functions, 292
  inverse, 313
Trigonometric substitution, 344
Trigonometry:
  analytic, 39
  right triangle, 39
Triple integral, 731
Tunnel through center of earth, 321

Ultimate Certainty, 119
Unbounded, 433
Unconditional convergence, 830
Uniform circular motion, 608
Uniform continuity, 839
  theorem on, 806
Uniform convergence, 490, 834
  for power series, 834
Universal gravitation, Newton's inverse square law of, 621
Upper bound, 433, 789
Upper bound, least, 789
  axiom, 789
Upper limits of integration, 201
Upper sum, 200

Value, 23
  critical, 116
  maximum, 116
  minimum, 116
  of π, 500
Values:
  of cosine, 42
Values *(Cont.)*:
  of sine, 42
Variable(s):
  dependent, 24
  dummy, 214
  independent, 24
  of integration, 201
  separation of, 272
Vascular resistance, 312
Vector, 600
  continuous, 606
  derivative of, 606
  field, 752
    flux of, 772
  length of, 634
  torque, 645
Vector analysis, 605
Vector equation:
  of line, 647
  of plane, 548
Vector projection, 636
Velocity, 64, 182, 607
  angular, 394
  average, 63
  escape, 186
  instantaneous, 64
  terminal, 286
Vertex:
  angle of a cone, 529
  of a cone, 529
  of a parabola, 18, 533
Vertically simple, 720
Vertices:
  of ellipse, 536
  of a hyperbola, 544
Victory, taste of, 380
Volcanic ash problem, 362
Volterra's prey-predator equations, 290
Volume, 719
  algebraic, 720
  element of, 255
  geometric, 720
  of a segment of a sphere, 226
  of a sphere, 728, 734, 737, 740
Wall, H. S., 423
Wallis, John, 382
Wallis's product, 275, 361, 380, 472, 477
Washer method, 228
Watson, E. C., 283
Watson, Fletcher G., 540
Wave equation, 675, 702
  one-dimensional, 705
Weber, E. H., 290
Weight, 183, 245
Whispering gallery, 540
White dwarfs, 187
Whitehead, A. N., 105
Whitmore, R. L., 233
Whittaker, Sir Edmund, 461
Widder, D. V., 747
Wiles, Andrew, 21
Witch, 591
Work, 244, 638, 751
  element of, 246
Working heart, 248
World population, 289
Wrathful Church, 620
Wren, Christopher, 594
Wren's theorem, 593, 627
Wright, E. M., 595
Wundt, W., 290

x-axis, 5
x-coordinate, 6
x-intercept, 15
xy-plane, 6

y-axis, 5
y-coordinate, 6
y-intercept, 13

Zero of a function, 31
Zeta function, 474, 480
Zeuner, F. E., 280
Zone, spherical, 244

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WITH ANALYTIC
GEOMETRY